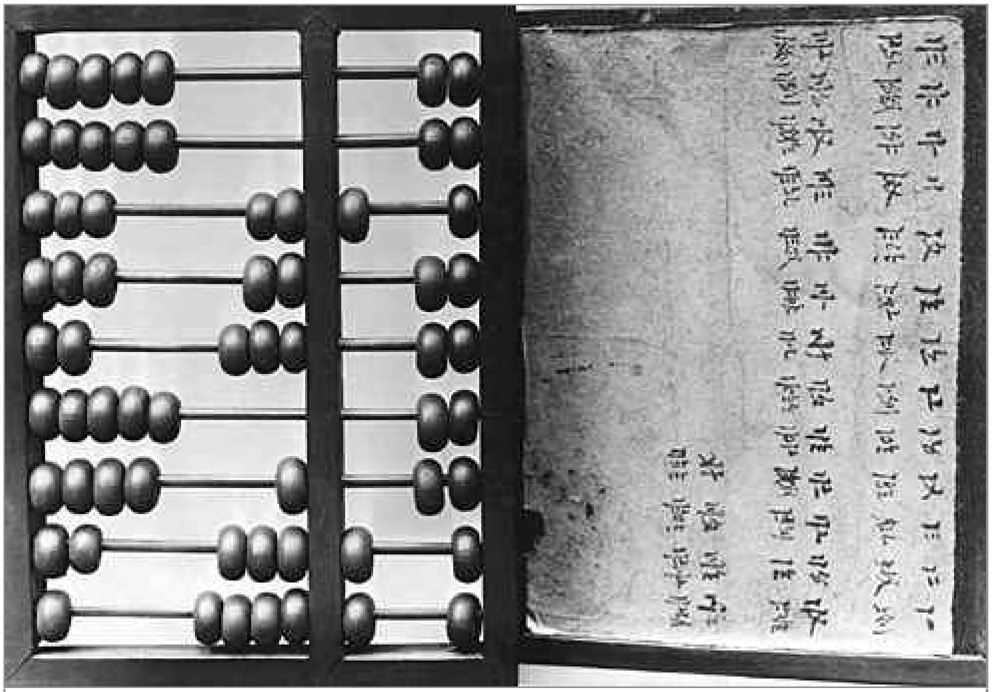
Arithmetic

I INTRODUCTION

Arithmetic, literally, the art of counting. The word comes from the Greek *arithmētikē,* which combines the ideas of two words: *arithmos,* meaning “number”, and *technē,* referring to an art or skill.



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Abacus

The abacus is an instrument used to perform arithmetic calculations. Developed in ancient times, the abacus is still used in China, Japan, and Korea.

The numbers used for counting are called the *positive integers.* These are generated by adding 1 to each number in an unending series. Different civilizations throughout history have developed different kinds of number systems. One of the most common is the one used in all modern cultures, in which objects are counted in groups of ten. Called the base 10, or *decimal,* system, it is the one that is used in this article.

In the base 10 system, integers are represented by digits expressing various powers of 10. For example, take the number 1534. Every digit in this number has its own place value, and these increase by another power of 10 as they move to the left. The first place value is a unit value (here, 4 × 1); the second is 10 (here, 3 × 10, or 30); the third place value is 10 × 10, or 100 (here, 5 × 100, or 500); and the fourth place value is 10 × 10 × 10, or 1,000 (here, 1 × 1,000, or 1,000).

II FUNDAMENTAL DEFINITIONS AND LAWS

Arithmetic is concerned with the ways in which numbers can be combined through addition, subtraction, multiplication, and division. The word *number* also includes negative numbers, fractions, and algebraic irrational numbers. The arithmetic laws of addition, multiplication, and distribution are as those for algebra.

A Addition

The arithmetic operation of addition is indicated by the plus sign (+) and is a means of counting by increments greater than 1. For example, four apples and five apples could be added by putting them together and then counting them individually from 1 to 9. Addition, however, makes it possible to add, or compute, sums more readily. The simplest combinations of sums must be memorized. The following table represents the sums of any two digits from 0 to 9:

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

To find the sum of any two numbers between 0 and 9, locate the first of the numbers in the vertical column at the left of the table and the other number in the horizontal row at the top. The sum is the number in the body of the table at the intersection of the particular row and column that have been selected.

Arithmetic also makes it possible to add long lists of large numbers by applying simple rules that make the work quite easy. For example, it would be possible to add 27 + 32 + 49 by first counting to 27, then counting 32 more times after 27, and then another 49 times beyond that to get the total, or sum. But if the numbers are listed so that all the units are in one column, all the tens are in another column, and so on, the addition is relatively simple:

tens	units	
2	7	(addend)
3	2	(addend)
+4	9	(addend)
		(sum)

First the units (7 + 2 + 9) are added; they total 18. Then the digits in the tens place (2 + 3 + 4) are added; they total 9, but this means 9 *tens,* or 90. This can now be written as

```
  27
  32
 +49
 ----
  18  (total of the units)
  90  (total of the tens)
 ----
 108
```

In the last step, the total of the units is added to the total of the tens. This procedure may be shortened by carrying the 1 of 18, which stands for one ten, over to the tens column and adding it directly with the digits there.

```
  1   (carried from the18
 27     of the units total)
 32
+49
----
  8
```

The digits in the tens column are added, and the sum 108 is the result. Similarly, in adding numbers with three or more places, numbers may be carried to the hundreds, thousands, or higher places.

B Subtraction

The arithmetic operation of subtraction is indicated by the minus sign (-) and is the opposite, or *inverse,* operation to addition. Again, it is possible to subtract 23 from 66 by counting backwards from 66 or by taking away 23 items from a collection of 66, until one reaches the remainder of 43. The rules of arithmetic for subtraction, however, provide a much simpler method for obtaining the answer. First the numbers are aligned under one another, units under units, tens under tens:

$$\begin{array}{rl} 66 & \text{(minuend)} \\ -\,23 & \text{(subtrahend)} \\ \hline 43 & \text{(remainder)} \end{array}$$

Subtraction is a bit more complicated if, in any column, the digits of the subtrahend are larger than those of the minuend. This can be handled in a manner analogous to carrying, but known as borrowing in subtraction. For example, to subtract 46 from 92, a 1 can be borrowed from the tens column—that is, from the 9 of 92—leaving 8 in the tens column.The 10 is brought over to the units column and added to the 2 already there, giving 12 (notated $^{1}2$) in that column from which 6 can now be subtracted:

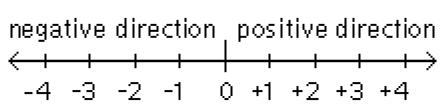
$$\begin{array}{rl} {}^{8}\cancel{9}{}^{1}2 & \text{(the 1 has been borrowed} \\ -\,4\;6 & \text{from the 9 in the tens column)} \\ \hline 6 & \end{array}$$

The subtraction is completed by carrying out the difference in the tens column, that is, by taking 4 away from 8 to give 4; the answer is, therefore, 46.

C Negative Numbers

As long as smaller numbers are being subtracted from larger ones, carrying out the operation of subtraction in arithmetic is not difficult. But if the subtrahend is larger than the minuend, the only way to produce an answer, or remainder, is to introduce the concept of negative numbers.

Negative numbers may be most easily visualized by considering the familiar numbers of arithmetic, the positive integers, arranged in a line and increasing in a positive direction. Negative numbers are reached by proceeding away from 0 in the opposite direction. The following number line represents both positive and negative numbers:



If 4 is subtracted from 2, written symbolically as 2 - 4, four units are counted backwards from +2, and -2 is reached—the difference when 4 is subtracted from 2. Rules for basic arithmetic operations involving negative numbers are given at the end of this section, following the laws of basic arithmetic operations.

D Multiplication

The arithmetic operation of multiplication is indicated by the times sign (×). Sometimes a dot is used to indicate the multiplication of two or more numbers, and in some cases parentheses are used. For example, 3 × 4, 3 · 4, and (3)(4) all indicate the product of 3 times 4. Multiplication is simply repeated addition. The expression 3 × 4 means that 3 is to be added to itself four times; or, similarly, that 4 is to be added to itself three times. In either case, the answer is the same. When large numbers are involved, however, such repeated addition is tedious; thankfully arithmetic provides procedures for simplifying the operations. As in addition, it is necessary to know the basic multiples of the integers between 0 and 9, which are provided in the following table:

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

To find the product of any two numbers between 0 and 9, locate the first of the numbers in the vertical column at the left of the table and the second number in the horizontal row at the top of the table. The product is the number in the body of the table at the intersection of the column and row in question. For example, 7 × 8 = 56.

In order to multiply two numbers, the units, tens, hundreds, and so on, are aligned, as in the following example.

$$\begin{array}{rl} 386 & \text{(multiplicand)} \\ \times\quad 4 & \text{(multiplier)} \\ \hline & \text{(product)} \end{array}$$

Each digit of the multiplicand is then multiplied by the multiplier, in this case, 4. The results of all these multiplications are then added together, giving the product 1,544:

$$\begin{array}{rl} 386 & \\ \times\quad 4 & \\ \hline 24 & (4 \times 6 \text{ in the units place}) \\ 320 & (4 \times 8 \text{ in the tens place, or } 4 \times 80) \\ 1200 & (4 \times 3 \text{ in the hundreds place, or } 4 \times 300) \\ \hline 1544 & \end{array}$$

This operation can be shortened by carrying tens and hundreds, as in addition:

$$\begin{array}{r} {}^{2} \\ 386 \\ \times\quad 4 \\ \hline 4 \end{array}$$

The 6 is multiplied by 4, giving 24; the 4 is written in the units place in the product and the 2, which stands for 2 tens, or 20, is carried. The 8 in the tens place is now multiplied by 4, giving 32, to which

2 is added, giving 34. (Actually, 80 has been multiplied by 4, giving 320, and the 20, which had been carried, is then added to 320.) The 4 is written down in the tens place, and the 3 (which stands for 3 hundreds) is carried and placed over the hundreds column. To finish, the 3 in the hundreds place of the multiplicand is multiplied by 4, giving 12 (actually 1,200), and the 3 that was carried is added to give 15 (actually 1,500):

```
  32
  386
×   4
-----
 1544
```

For larger multipliers, an analogous procedure may be followed. For example, the following product is begun by carrying out multiplication by the unit 3 of the multiplier:

```
  22
  579 (multiplicand)
×  63 (multiplier)
-----
 1737 (partial product)
```

The 9 is next multiplied by 6, giving 54, and the 4 is placed under the tens place of the partial product. This is done because the digit 6 in the multiplier is in the tens place and actually represents 60. The 5 is carried as usual. The 7 in the tens place of the multiplicand is multiplied by the 6, giving 42, and the carried 5 is added, giving 47 (really 6 times 70, plus 50, giving 470). Now the 7 is written in the hundreds place (directly to the left of the 4 in the tens place), and the 4 is carried. The rest of the multiplication is straightforward, and the total product is obtained by adding the two partial products:

```
   45
   579 (multiplicand)
 ×  63 (multiplier)
 -----
  1737 (partial product)
 3474  (partial product)
 -----
 36477 (total product)
```

For each digit in the multiplier there will be a line for each partial product obtained as the units, tens, hundreds, and so on are multiplied out. Each successive partial product is placed one digit to the left of the partial product above it, corresponding to the position of the digit being multiplied in the multiplier. Thus, in the above example, the partial product of 6 times 579 yields 3,474, and the 4 in the digits place must be written in the tens place for the partial product. This is because the multiplier is really 60, not 6; the entire partial product is actually a multiple of 10, 34,740, which would, of course, produce the same alignment:

```
   579
 ×  63
 -----
  1737
 34740
 -----
 36477
```

Superscript (raised) numbers are employed to notate the multiplication of a number by itself. For example, $5 \cdot 5$ is notated 5^2 and is termed “five squared”. Similarly, $5 \cdot 5 \cdot 5 = 5^3$, or “five cubed”.

E Division

The arithmetic operation of division is the opposite, or *inverse,* operation to multiplication. Using the example of 12 divided by 4, division may be indicated by the division sign ($12 \div 4$), a bar ($\frac{12}{4}$), a slash (12/4), or the notation $4\overline{)12}$.

Division is the arithmetic operation that determines how many times a given number is contained in another as a multiple. For example, 4 is contained three times in 12; thus, 12 divided by 4 is 3, or $\frac{12}{4}$ is 3, or

$$\begin{array}{r} 3 \\ 4\overline{)12} \end{array} \qquad \begin{array}{r} \text{quotient} \\ \text{divisor}\overline{)\text{dividend}}. \end{array}$$

Many divisions may be carried out by inspection, but more complicated cases require a procedure known as long division. For example, in the quotient 3,918 ÷ 6, or $6\overline{)3918}$, the number 6 will not divide 3 one or more times, but 6 will go into 39 at least six times and less than seven times (6 × 6 is 36). The number 6 is therefore written in the quotient over the 9 in the dividend, 3,918, and 36 (the product of the divisor, 6, and the first number of the quotient, 6) is written under the 39 and subtracted from it, yielding 3.

What has actually been determined thus far in this process of long division is that 3,918 contains 600 6s, and that 318 is left when these are taken away. Long division is a repetition of a cycle of simple operations, an iterative process, or algorithm, that is much used in modern computing science. Thus in the next step of the division process, the 1 in the dividend is brought down and written to the right of the 3 to give 31:

$$\begin{array}{r} 6 \\ 6\overline{)3918} \\ \underline{-36} \\ 31 \end{array}$$

The 6 in the divisor will go into 31 at least five times and less than six times (6 × 5 is 30). The 5 is then written above the 1 in the dividend. The product, 5 × 6, or 30, is placed under the 31 and subtracted from it, yielding 1. The 8 from the dividend is brought down and written to the right of the 1 to give 18. The 6 in the divisor will go into 18 exactly three times, so the number 3 is written in the quotient above the 8 in the dividend:

$$\begin{array}{r} 653 \\ 6\overline{)3918} \\ \underline{-36} \\ 31 \\ \underline{-30} \\ 18 \\ \underline{-18} \\ 00 \end{array}$$

The answer to how many times 6 will divide 3,918 is 653. This can be easily verified by multiplying 6 × 653; this proof yields the dividend 3,918.

E1 Remainders

Had the dividend been 3,923, the last subtraction would not have yielded 00, but would have left a remainder of 5. Such a remainder must then be incorporated into the answer as a fraction of the divisor; in this case 6, leaving a fractional remainder of $\frac{5}{6}$:

$$\begin{array}{r} 653\,\frac{5}{6} \\ 6\overline{)3923}\phantom{\,\frac{5}{6}} \\ \underline{36}\phantom{\,\frac{5}{6}} \\ 32\phantom{\,\frac{5}{6}} \\ \underline{30}\phantom{\,\frac{5}{6}} \\ 23\phantom{\,\frac{5}{6}} \\ \underline{18}\phantom{\,\frac{5}{6}} \\ 5\phantom{\,\frac{5}{6}} \end{array}$$

Again, it is easily verified, by multiplication (see below, under fractions), that $6 \times 653\frac{5}{6}$ is 3,923.

E2 Long Division

Long division is an arithmetic procedure whereby larger divisors may be dealt with in carrying out divisions. The method is basically analogous to the shorter method for division described above. For example, in dividing 3,626 by 25

$$25\overline{)3626}$$

25 will go into 36 once; therefore, a 1 is placed over the 6 in 36. The 25 is multiplied by 1, and this product (25) is written under the 36 and subtracted from it, yielding 11. The 2 is then brought down to give 112:

$$\begin{array}{r} 1 \\ 25\overline{)3626} \\ -25 \\ \hline 112 \end{array}$$

Because 25 will go into 112 at most four times, a 4 is written next to the 1 in the quotient, over the 2 in the dividend. The divisor is multiplied by 4, and the product, 100, is written under the 112 and subtracted from it, yielding 12. The 6 is brought down to give 126. Because 25 will go into 126 five times, with a remainder of 1, the answer is

$$\begin{array}{r} 145\frac{1}{25} \\ 25\overline{)3626} \\ -25 \\ \hline 112 \\ -100 \\ \hline 126 \\ -125 \\ \hline 1 \end{array}$$

F Operations with Negative Numbers

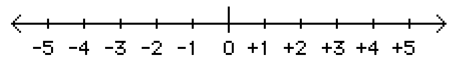
Before it is possible to deal adequately with arithmetic operations involving negative numbers, the concept of *absolute value* must be introduced. Given any number, whether positive or negative, the absolute value of that number is the value of the number regardless of its sign. Thus, the absolute value of +5 is 5, and the absolute value of -5 is also 5. In symbolic notation, the absolute value of any number *a* is written as $|a|$ and defined as follows: The absolute value of *a* is *a* if *a* is positive, and the absolute value of *a* is *-a* if *a* is negative. In the following discussion, inequality signs are used. The symbol > means “greater than”; ≥ means “greater than or equal to”. Similarly, < means “less than”; ≤ means “less than or equal to”. The symbol ≠ means “not equal to”. For example, the expression $a \geq 0$ is read “*a* is greater than or equal to zero”.

$$|a| = a \text{ if } a \geq 0;\ \ |7| = |7|$$
$$|a| = -a \text{ if } a < 0;\ \ |-9| = -(-9) = +9$$

By referring once again to the number line, which includes both positive and negative numbers, it is easy to see that the sum of any two negative numbers will be negative and that the sum will simply be the total of the absolute values of the two numbers, but negative:

$$(-5) + (-6) = -11$$

If, however, the signs of the two numbers are opposite, then it is necessary to proceed more cautiously.



To add -4 and +2, for example, it is clear that by beginning at -4 and adding 2 by moving two units to the right in the positive direction, the result will be -2 (the answer is the difference between the absolute values of the two numbers with the result taking the sign of the larger). The result must be negative because in terms of absolute value, |-4| > |+2|. The answer, however, is obtained by calculating the difference between the absolute values 4 and 2. Similarly, to add -3 and +7, beginning at -3 and moving seven units to the right produces +4. This may be conveniently summarized in two rules.

F1 Rule 1

If two numbers of like sign are to be added, the absolute values of the two numbers are added, and the answer is given the sign of the numbers in question:

$$(+6)+(+7)=13; (-4)+(-8)=-12$$

F2 Rule 2

If two numbers of different sign are to be added, their absolute values are ascertained, and the smaller number is subtracted from the larger. The result is given the sign of the number with the larger absolute value:

$$(-5)+(+6)=+1; (-8)+(+5)=-3$$

The subtraction of negative numbers is easier to visualize by referring again to the number line. Subtracting one number from another indicates a reversal of direction from addition. To subtract 5 from 7, for example, begin at 7 and move five units in the negative direction, reaching +2 as the answer. Similarly, to subtract -2 from 5, start at 5 and move in the opposite direction from -2, that is to say, in the positive direction, two units to +7. Subtraction basically indicates that one is to change the sign, or direction, of the number in question. For example,

$$6-(-3)=6+(+3)=+9$$

Subtraction problems are best converted to problems in addition; following rules 1 and 2 above, the addition is then carried out as indicated. For subtraction, the following rule applies:

F3 Rule 3

Whenever subtraction is indicated, change the subtraction to addition after changing the sign of the number following the subtraction sign, and then add, using the above rules for addition:

$$(+4)-(+7)=(+4)+(-7)=-3$$
$$(-4)-(-6)=(-4)+(+6)=+2$$

Multiplication of negative numbers is carried out using the following two conventions.

F4 Rule 4

To multiply two numbers of the same sign, multiply their absolute values and give the resulting product a positive sign:

$$(+7)(+2) = +14$$
$$(-6)(-5) = +30$$

F5 Rule 5

To multiply two numbers of different sign, multiply their absolute values and give the resulting product a negative sign:

$$(-3)(+6) = -18; \quad (+5)(-3) = -15$$

Division is the inverse operation of multiplication; therefore, dividing signed numbers involves rules similar to those for multiplication.

F6 Rule 6

To divide two numbers of the same sign, divide their absolute values and give the resulting quotient a positive sign:

$$(-6)/(-3) = +2; \quad (+4)/(+2) = +2$$

F7 Rule 7

To divide two numbers of different sign, divide their absolute values and give the resulting quotient a negative sign:

$$(10)/(-5) = -2; \quad (-9)/(+3) = -3$$

If a number carries no explicit sign, it is assumed to be positive.

III THEORY OF DIVISORS

Before discussing fractions, some additional remarks must be made about several other classes of numbers. An even number is any number divisible by 2. An odd number is any number not divisible by 2. A prime number is any integer greater than 1, that is divisible only by itself and 1. Examples of prime numbers are 2, 3, 5, 7, 11, 13, 17, 19 .... The only even prime number is 2. All integers that are not prime are said to be *composite*, and all composite numbers can be written as the product of prime numbers.

A The Fundamental Theorem of Arithmetic

"Every integer greater than 1 that is not a prime number is the product of one and only one set of prime numbers". This theorem was first proved by the German mathematician Carl Friedrich Gauss. Given a number such as 14, the theorem states that it may be written uniquely as the product of primes, in this case, $14 = 2 \cdot 7$. Similarly, $50 = 2 \cdot 5 \cdot 5 = 2 \cdot 5^2$. By factoring numbers into their prime components, both their least common multiple and their greatest common divisor can be found.

B Lowest Common Multiple

The lowest common multiple (LCM) of any two (or more) numbers is the lowest number into which each number will divide evenly. The LCM will contain the maximum number of prime factors of all primes appearing in each of the given numbers. For example, to find the LCM for the three numbers 27, 63, and 75, each number is first factored: $27 = 3^3$, $63 = 3^2 \cdot 7$, and $75 = 3 \cdot 5^2$. The LCM must contain at least factors of 3^3, 7, and 5^2; thus, $3^3 \cdot 7 \cdot 5^2 = 4{,}725$ is the least number into which 27, 63, and 75 will all divide evenly.

C Greatest Common Divisor

The largest factor common to a given set of numbers is called the highest common factor (HCF) or greatest common divisor (GCD). For example, given 9, 15, and 27, the GCD is 3, which can readily be seen by examining the prime factorization of each number: $9 = 3^2$, $15 = 3 \cdot 5$, $27 = 3^3$; the only factor common to each number is 3.

IV FRACTIONS

Numbers representing parts of a whole are called rational numbers, or fractions. Simple fractions are familiar: $\frac{1}{4}$ of a pie plus $\frac{1}{2}$ of a pie is $\frac{3}{4}$ of a pie, and so on. In general, fractions may be expressed as the quotient of two integers *a* and *b:*

$$\frac{a \text{ (numerator)}}{b \text{ (denominator)}}$$

A fraction is said to be reduced to its lowest terms if neither the numerator nor the denominator has a factor in common. For example, $\frac{6}{8}$ is not a fraction reduced to lowest terms because both 6 and 8 contain factors of 2: $\frac{6}{8} = (2 \cdot 3)/(2 \cdot 4)$; $\frac{3}{4}$, however, is a fraction reduced to lowest terms.

A Cancellation Law of Multiplication

If a/c *is* a fraction, and $b \neq 0$, then $(a \cdot b)/(c \cdot b) = a/c$. Two types of fraction exist, proper and improper. A *proper fraction* is one in which the numerator is less than the denominator; $\frac{2}{3}$, $-\frac{7}{8}$, and $\frac{16}{19}$ are all proper fractions. An *improper fraction* is one in which the numerator is larger than the denominator; $\frac{3}{2}$, $-\frac{8}{4}$, and $\frac{7}{3}$ are improper fractions. Improper fractions can be converted to mixed fractions or whole numbers (for example, $\frac{3}{2} = 1\frac{1}{2}$, $-\frac{8}{4} = -2$, and $\frac{7}{3} = 2\frac{1}{3}$) by dividing the numerator by the denominator and expressing any remainder as a fraction of the denominator.

B Adding and Subtracting Fractions

To add or subtract fractions the denominators of which are the same, the numerators are added or subtracted, just as for integers, and the result is expressed as a fraction of the denominator. The result is reduced to lowest terms. For example,

$$\frac{2}{3} + \frac{4}{3} = \frac{2+4}{3} = \frac{6}{3} = 2$$

$$\frac{5}{8} - \frac{1}{8} = \frac{5-1}{8} = \frac{4}{8} = \frac{1}{2}$$

Only fractions of equal denominators may be added or subtracted as they stand. If the denominators are unequal, a common denominator must be found. The lowest common denominator (LCD), in fact, will always be the lowest common multiple (LCM) of the denominators in question. For example, in the expression $\frac{2}{3} + \frac{3}{4}$, the denominators 3 and 4 are different, but the LCM of 3 and 4 is 12. Thus, to add $\frac{2}{3}$ and $\frac{3}{4}$, the fractions must be changed into those with 12 as a common denominator. Because both the numerator and denominator can be multiplied by the same number without changing the value of the fractions, it is apparent that

$$\frac{2}{3} = \frac{2 \cdot 4}{3 \cdot 4} = \frac{8}{12}; \frac{3}{4} = \frac{3 \cdot 3}{4 \cdot 3} = \frac{9}{12}$$

Now it is possible to add:

$$\frac{2}{3} + \frac{3}{4} = \frac{8}{12} + \frac{9}{12} = \frac{8+9}{12} = \frac{17}{12} = 1\frac{5}{12}$$

Similarly,

$$\frac{7}{10} - \frac{3}{5} = \frac{7}{10} - \frac{3 \cdot 2}{5 \cdot 2} = \frac{7}{10} - \frac{6}{10} = \frac{1}{10}$$

C Multiplying Fractions

Given two fractions $\frac{a}{b}$ and $\frac{c}{d}$, multiplication is straightforward: numerators are multiplied together and denominators are multiplied together, so that

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

For example,

$$\frac{6}{5} \cdot \frac{2}{3} = \frac{12}{15} = \frac{4}{5}$$

The answer must always be reduced to the lowest terms. Similarly,

$$(-\frac{2}{3})(\frac{4}{5}) = -\frac{8}{15}$$

The rules for multiplying signed fractions are the same as those for multiplying signed integers. The same is true for the rules governing addition, subtraction, and division of signed fractions.

D Dividing Fractions

The division of fractions is most easily understood in terms of reciprocals. Given any number *a,* and $a \neq 0$, the reciprocal of *a* is $\frac{1}{a}$. This is also called the *inverse* of *a* and may be written as a^{-1}. Note that *a*

$(\frac{1}{a}) = 1$.

To carry out the following division

$$\frac{a}{b} \div \frac{c}{d}, \text{ or } \frac{\frac{a}{b}}{\frac{c}{d}}$$

both the numerator and the denominator can be multiplied by the same nonzero number without affecting the value of the original number. Both numerator and denominator in the above example are multiplied by $\frac{d}{c}$:

$$\frac{\frac{a}{b}\cdot\frac{d}{c}}{\frac{c}{d}\cdot\frac{d}{c}} = \frac{\frac{a}{b}\cdot\frac{d}{c}}{\frac{cd}{cd}} = \frac{\frac{a}{b}\cdot\frac{d}{c}}{1} = \frac{ad}{bc}$$

This division is equivalent to multiplying the numerator $\frac{a}{b}$ by the reciprocal of the denominator $\frac{c}{d}$, that is by $\frac{d}{c}$.

The following two examples illustrate the general procedure outlined above:

$$\frac{2}{3} \div \frac{5}{8} = \frac{\frac{2}{3}}{\frac{5}{8}} = \frac{2}{3}\cdot\frac{8}{5} = \frac{16}{15} = 1\frac{1}{15}$$

Note that $\frac{8}{5}$ is the reciprocal of the denominator $\frac{5}{8}$.

$$\frac{2}{3} \div (-\frac{4}{5}) = \frac{\frac{2}{3}}{-\frac{4}{5}} = (\frac{2}{3})(-\frac{5}{4}) = -\frac{10}{12} = -\frac{5}{6}$$

Note that -5/4 is the reciprocal of $-\frac{4}{5}$.

In general, division by a fraction is equivalent to multiplication by its reciprocal:

$$\frac{1}{\frac{a}{b}} = \frac{1(b/a)}{\frac{a}{b}(b/a)} = \frac{1(b/a)}{\frac{ab}{ab}} = \frac{1(b/a)}{1} = \frac{b}{a}$$

For example,

$$1 \div \frac{3}{2} = 1\cdot\frac{2}{3} = \frac{2}{3}$$

Zero (0) has no reciprocal. This follows because the reciprocal of any number *a* is a number $(\frac{1}{a})$ such that $a(\frac{1}{a}) = 1$. To say that zero has no reciprocal is to say that no number $(\frac{1}{a})$ exists such that $0(\frac{1}{a}) = 1$. This is obvious, since no number exists that can be multiplied by zero to give 1.

V DECIMALS

The idea of place values can be extended to accommodate fractions. Instead of writing $\frac{2}{10}$, or two-tenths, a decimal point (.) can be introduced so that 0.2 can represent the fraction. Just as places to the left of the decimal point represent units, tens, hundreds, and so on, those to the right of the decimal represent places for tenths ($\frac{1}{10}$), hundredths ($\frac{1}{100}$), thousandths (1/1000), and so forth. These are decreasing powers of ten, which can be represented as 10^{-1}, 10^{-2}, 10^{-3}, and so on. In general, a number such as 5,428.632 is called a *decimal fraction* in which 0.632 represents

$$\underbrace{6(10^{-1})}_{\text{tenths}} + \underbrace{3(10^{-2})}_{\text{hundredths}} + \underbrace{2(10^{-3})}_{\text{thousandths}}$$

This number is read as follows: “five thousand four hundred and twenty-eight point six three two”.

A Addition and Subtraction

In a number system based on the number 10, because places to both the left and right of the decimal point represent powers of ten, addition and subtraction of decimal fractions are handled in the same manner as described for integers. The decimal point, however, must always be aligned, so that tens are under tens, units under units, tenths under tenths, and so on, ensuring that each value is being added to or subtracted from a similar value at every step. For example, to add 365.289 and 32.4, the decimal points are aligned, and then the numbers are added beginning at the right and moving to the left.

$$\begin{array}{r} 365.289 \\ +32.4 \\ \hline \end{array}$$

The addition of zeros to the right of a number does not change the number, but filling in the places helps ensure that the same number of places exist to the right of the decimal point for all numbers being added or subtracted:

$$\begin{array}{r} 365.289 \\ +32.400 \\ \hline 397.689 \end{array}$$

Subtraction with decimals proceeds in much the same way as addition: zeros are added to give both numbers the same number of places to the right of the decimal point. The numbers are then subtracted as usual.

B Multiplication

Multiplying decimals is similar to multiplying integers, except that the values of the decimal points must be kept in mind. For example, in multiplying 0.3 and 0.5, visualizing the process involved is easier if each of these decimals is converted to common fractions and then multiplied.

$$\frac{3}{10} \times \frac{5}{10} = \frac{3 \cdot 5}{10 \cdot 10} = \frac{15}{100} = 0.15$$

The 15, of course, is the direct value of the product of 3 and 5, but the placement of the decimal point is determined by the fact that 0.3 and 0.5 each represent one decimal place, so a total of two decimal places must be represented in the answer; thus, the product is 0.15.

Another example will serve to make this concept even clearer. Multiply 0.2 and 0.3. The product of the two numbers is 6, but since 0.2 and 0.3 each have one decimal place, the product must have a total of two decimal places. This can be achieved by placing the decimal point two places to the left in the product: 0.06.

Decimal fractions are multiplied as if they were integers. The decimal point is then placed at the appropriate place in the product so that the number of places there is the same as the total number of decimal places in the multiplier and multiplicand. For example:

$$\begin{array}{r} 0.259 \\ \times 0.12 \\ \hline 518 \\ 259 \\ \hline 3108 \end{array}$$

Because the multiplicand has three decimal places and the multiplier has two, the product must have a total of five decimal places. Counting five places to the left from the digit furthest to the right (digit 8 of the final product), the answer must be 0.03108.

One final example will also illustrate the importance of counting the proper number of decimal places in determining products of decimal fractions. Multiply .001 and .002. The multiplication of the digits is simple: the answer is 2. There are, however, three decimal places in the multiplier, and three in the multiplicand, giving a total of six places must be preserved in the product. Six places must be counted to the left of the 2, giving the answer 0.000002.

C Division

Like multiplication, the division of decimal fractions follows the same procedures used to divide integers, except that care must be taken to determine the proper placement of decimal points in the quotients. Dividing a decimal fraction by a whole number is straightforward: the decimal point is placed in the quotient directly above the decimal point in the dividend, and the division is performed as usual.

$$\begin{array}{r} 3.38 \\ 7\overline{)23.66} \\ -21 \\ \hline 26 \\ -21 \\ \hline 56 \\ -56 \\ \hline 00 \end{array}$$

In cases where the divisor is also a decimal fraction, the problem is converted to one in which the divisor is an integer; division may then proceed as in the above example. To divide 14 by 0.7, for example, the divisor is converted to an integer by multiplying it by 10: (0.7)(10) = 7. The dividend must then be multiplied by an equal amount, that is by 10, to preserve the value of the division. This may be understood more easily by considering what division means. In the above example the division can be rewritten as a fraction, 14/0.7. Multiplying both fixed numerator and denominator by the same amount will not change the value of the fraction:

$$\frac{14}{0.7} = \frac{14(10)}{0.7(10)} = \frac{140}{7}$$

Similarly, the division of 2.675 by 0.23 can be considered in the form 2.675/0.23. This can be converted to a division involving an integer divisor, 23, if both numerator and denominator are multiplied by 100:

$$\frac{2.675}{0.23} = \frac{2.675\,(100)}{0.23\,(100)} = \frac{267.5}{23}$$

In summary, any division of one decimal fraction by another may be converted into a straightforward division with an integer divisor by moving the decimal point as many places to the right in the divisor as is necessary to make it an integer. The decimal point in the dividend is then moved an equal number of places to the right, and zeros are added if necessary. The division is carried out as usual, the decimal point in the quotient being placed directly over the new decimal point in the dividend.

For example, in the division $.002\overline{)21.5}$, the decimal point in the divisor is moved three places to the right, giving the integer 2. The decimal point in the dividend is moved three places to the right as well, giving $2\overline{)21500}$. The division is then carried out as usual. The quotient in this case is 10,750.