Calculus

I INTRODUCTION

Calculus, branch of mathematics concerned with rates of change, gradients of curves, maximum and minimum values of functions, and the calculation of lengths, areas, and volumes. It is widely used, especially in science and engineering, wherever continuously varying quantities occur.

II HISTORICAL DEVELOPMENT

Corbis

Gottfried Leibniz

The 17th-century thinker Gottfried Leibniz made many contributions to mathematics. He formulated the theory of calculus slightly after, though independently to, Newton, and gave it the notation that we use today. He also made advances in the area of symbolic logic.

Calculus derives from ancient Greek geometry. Democritus calculated the volumes of pyramids and cones, probably by regarding them as consisting of infinitely many cross-sections of infinitesimal (infinitely small) thickness, and Eudoxus and Archimedes used the “method of exhaustion”, finding the area of a circle by approximating it arbitrarily closely with inscribed polygons. However, difficulties with irrational numbers and the paradoxes of Zeno prevented a systematic theory developing. In the early 17th century Cavalieri and Torricelli extended the use of infinitesimals, while Descartes and Fermat used algebra for determining areas and tangents (integration and differentiation, in modern terms). Fermat and Barrow knew these two processes were closely related, and Newton (in the 1660s) and Leibniz (in the 1670s) proved the Fundamental Theorem of Calculus, that they are mutually inverse. Newton's discoveries, motivated by his theory of gravitation, preceded Leibniz's, but his delays in publishing them caused bitter priority disputes, and Leibniz's notation was eventually adopted.

The 18th century saw widespread applications of calculus, but imprecise use of infinite and infinitesimal quantities and geometric intuition still caused confusion and controversy about its foundations, the philosopher Berkeley being a notable critic. The 19th-century analysts replaced these vague notions with firm foundations based on finite quantities: Bolzano and Cauchy defined limits and derivatives precisely, Cauchy and Riemann did likewise for integrals, and Dedekind and Weierstrass for real numbers. For instance, it was now understood that differentiable functions are continuous, and continuous functions integrable, but both converses fail. In the 20th century, non-standard analysis belatedly legitimized infinitesimals, while the development of computers increased the applicability of calculus.

III DIFFERENTIAL CALCULUS

DERIVATIVES AND INDEFINITE INTEGRALS

In the table below, the letters u and v stand for any functions of x, and a and m are constants. An arbitrary constant of integration should be added to each of the integrals.

DERIVATIVES	INDEFINITE INTEGRALS	DERIVATIVES	INDEFINITE INTEGRALS
1. $\frac{dx}{dx} = 1$	1. $\int dx = x$	9. $\frac{d}{dx}\cos x = -\sin x$	9. $\int \cos x\, dx = \sin x$
2. $\frac{d}{dx}(au) = a\frac{du}{dx}$	2. $\int au\, dx = a\int u\, dx$	10. $\frac{d}{dx}\tan x = \sec^2 x$	10. $\int \tan x\, dx = \ln\lvert\sec x\rvert$
3. $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$	3. $\int (u+v)dx = \int u\, dx + \int v\, dx$	11. $\frac{d}{dx}\cot x = -\csc^2 x$	11. $\int \cot x\, dx = \ln\lvert\sin x\rvert$
4. $\frac{d}{dx}x^m = mx^{m-1}$	4. $\int x^m\, dx = \frac{x^{m+1}}{m+1} \quad (m \neq -1)$	12. $\frac{d}{dx}\sec x = \tan x \sec x$	12. $\int \sec x\, dx = \ln\lvert\sec x + \tan x\rvert$
5. $\frac{d}{dx}\ln x = \frac{1}{x}$	5. $\int \frac{dx}{d} = \ln\lvert x\rvert$	13. $\frac{d}{dx}\csc x = -\cot x \csc x$	13. $\int \csc x\, dx = \ln\lvert\csc x - \cot x\rvert$
6. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	6. $\int u\frac{dv}{dx}\, dx = uv - \int v\frac{du}{dx}\, dx$	14. $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$	14. $\int \frac{dx}{1+x^2} = \arctan x$
7. $\frac{d}{dx}e^x = e^x$	7. $\int e^x\, dx = e^x$	15. $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$	15. $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x$
8. $\frac{d}{dx}\sin x = \cos x$	8. $\int \sin x\, dx = -\cos x$	16. $\frac{d}{dx}\text{arcsec}\, x = \frac{1}{x\sqrt{x^2-1}}$	16. $\int \frac{dx}{x\sqrt{x^2-1}} = \text{arcsec}\, x$

Derivatives and Indefinite Integrals of Common Functions

Differential calculus is concerned with rates of change. Suppose that two variables x and y are related by an equation $y = f(x)$ for some function f, indicating how the value of y depends on the value of x. For instance, x could represent time, and y the distance travelled by some moving object at time x. A small change h in x, from a value x_0 to $x_0 + h$, induces a change k in y from $y_0 = f(x_0)$ to $y_0 + k = f(x_0 + h)$; thus $k = f(x_0 + h) - f(x_0)$, and the ratio k/h represents the average rate of change of y as x increases from x_0 to $x_0 + h$. The graph of the function $y = f(x)$ is a curve in the xy-plane, and k/h is the gradient of the line AB through the points $A = (x_0, y_0)$ and $B = (x_0 + h, y_0 + k)$ on this curve; this is shown in Figure 1, where $h = AC$ and $k = CB$, so k/h is the tangent of the angle BAC.

If h approaches 0, with x_0 fixed, then k/h approaches the instantaneous rate of change of y at x_0; geometrically, B approaches A along the graph of $y = f(x)$, and the line AB approaches the tangent AT to the graph at A, so k/h approaches the gradient of the tangent (and hence of the curve) at A. We therefore define the derivative $f'(x_0)$ of the function $y = f(x)$ at x_0 to be the value (or limit) which k/h approaches as h approaches 0, written:

$$f'(x_0) = \lim_{h\to 0}\frac{k}{h} = \lim_{h\to 0}\frac{f(x_0+h) - f(x_0)}{h};$$

This represents both the rate of change of y and the gradient of the graph at A. When x is time and y is distance, for example, the derivative represents instantaneous velocity. Positive, negative, or zero values of $f'(x_0)$ respectively indicate that $f(x)$ is increasing, decreasing, or stationary at x_0. The derivative is a new function $f'(x)$ of x, sometimes denoted by dy/dx, df/dx or Df. For example, let $y = f(x) = x^2$, so the graph is a parabola. Then

$$\begin{gathered} k = f(x_0 + h) - f(x_0) = \\ (x_0 + h)^2 - x_0^2 = \\ (x_0^2 + 2x_0 h + h^2) - x_0^2 = \\ 2x_0 h + h^2 \end{gathered}$$

so $k/h = 2x_0 + h$, which approaches $2x_0$ as $h \to 0$. Thus the gradient when $x = x_0$ is $2x_0$, and the derivative of $f(x) = x^2$ is $f'(x) = 2x$. Similarly x^m has derivative mx^{m-1} for each fixed m. The derivatives of all commonly occurring functions are known: see the table for some examples.

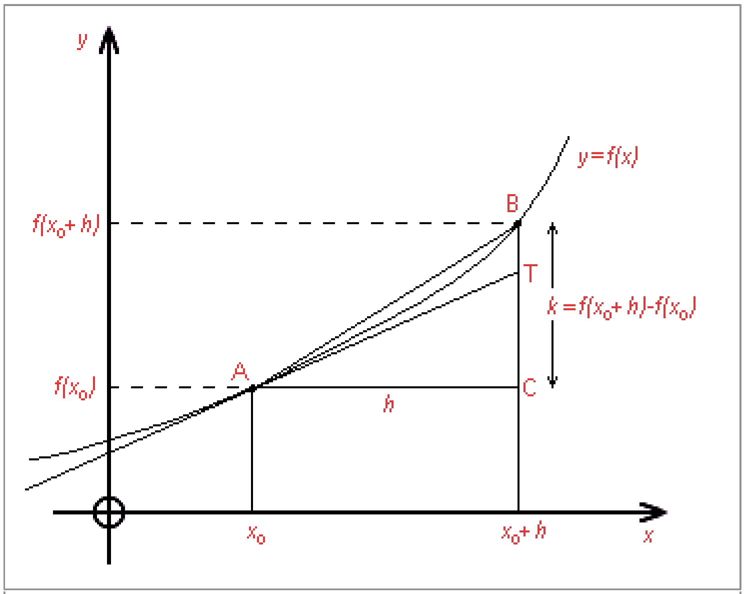

Figure 1: The Gradient of a Curve

The gradient, or slope, of any curve at a particular point is defined as the gradient of the tangent— that is, of the line that just touches the curve at that point. Thus, in the diagram above, the gradient of the curve at the point A is the gradient of the line AT, which just touches the curve at A. This gradient can be approximated by that of the straight line AB, which joins A and B, a nearby point on the curve. The gradient of AB is k / h. As B is allowed to approach A, both k and h approach 0, but their ratio approaches a fixed value, which is the slope of AT. Differential calculus is concerned with calculating the gradient at any point from the equation of the curve, $y = f(x)$.

Some words of caution are needed here. Firstly, to find the derivative we make h small (positive or negative), but never zero: this would give $k/h = 0/0$, which is meaningless. Secondly, not every function f has a derivative at each x_0, since k/h need not approach a limit as $h \to 0$. For instance, $f(x) = |x|$ has no derivative at $x_0 = 0$, since k/h is 1 or -1 as $h > 0$ or $h < 0$; geometrically, the graph has a corner (and hence no tangent) at $A = (0,0)$. Thirdly, although the notation dy/dx suggests the ratio of two numbers dy and dx (denoting infinitesimal changes in y and x), it is really a single number, the limit of a ratio k/h as both terms approach 0.

Differentiation is the process of calculating derivatives. If a function f is formed by combining two functions u and v, its derivative f' can be obtained from u and v by simple rules; for instance the derivative of a sum is the sum of their derivatives, that is, if $f = u + v$ (meaning that $f(x) = u(x) + v(x)$ for all x) then $f' = u' + v'$, and a similar rule $(u - v)' = u' - v'$ applies to differences. If a function is multiplied by a constant, then so is its derivative, that is, $(cu)' = cu'$ for any constant c. The rules for products and quotients are less obvious: if $f = uv$ then $f' = uv' + u'v$, and if $f = u/v$ then $f' = (u'v - uv')/v^2$ provided $v(x) \neq 0$.

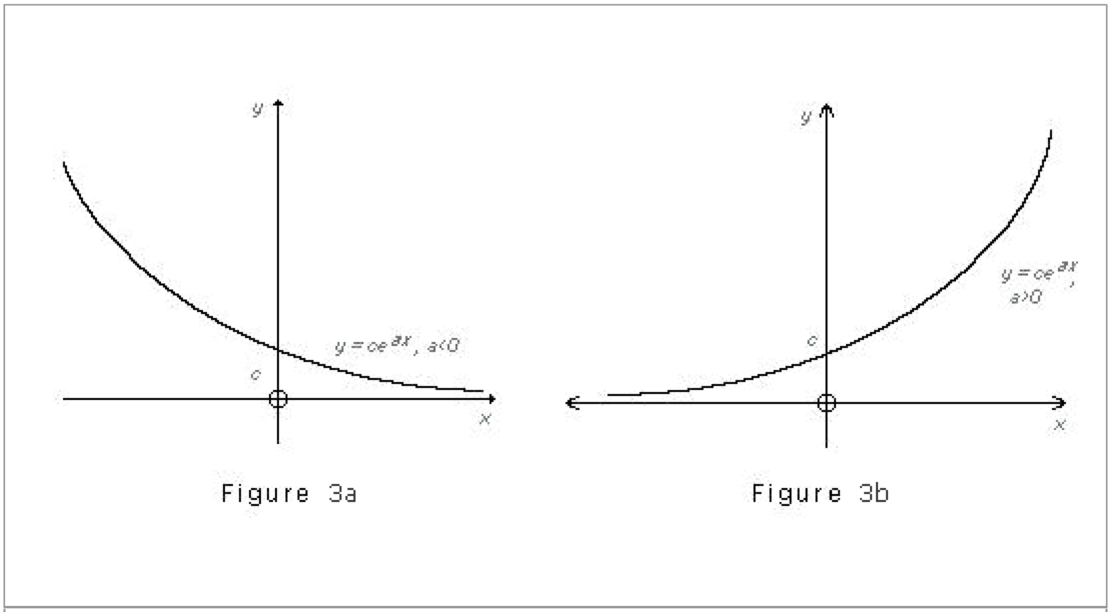

Figure 3: Exponential Decay and Growth

An exponential variation is one whose rate is always proportional to the value of the quantity varying. In (3a), the change is a decrease, or *exponential decay.* An example of this is radioactive decay, in which the number of atoms disintegrating in each second decreases as the total number of atoms present decreases. In (3b), the change is an *exponential growth.* An example of this is a “population explosion”, in which the number of offspring in each generation is proportional to the total population at that moment. Both types of change are governed by the same function, $y = ce^{ax}$, but in the two cases a is less than or greater than 0, respectively.

Using these rules, quite complicated functions can be differentiated: for instance x^2 and x^5 have derivatives $2x$ and $5x^4$, so the function $3x^2 - 4x^5$ has derivative $(3x^2 - 4x^5)' = (3x^2)' - (4x^5)' = 3.(x^2)' - 4.(x^5)' = 3.(2x) - 4.(5x^4) = 6x - 20x^4$. More generally, any polynomial $f(x) = a_0 + a_1x + \ldots + a^n x_n$ has derivative $f'(x) = a_1 + 2a_2x + \ldots + na_nx^{n-1}$; in particular, constant functions have derivative 0.

If $y = u(z)$ and $z = v(x)$, so that y depends on z and z depends on x, then $y = u(v(x))$, so y depends on x, written $y = f(x)$ where f is the composition of u and v; the chain rule states that $dy/dx = (dy/dz).(dz/dx)$, or equivalently $f'(x) = u'(v(x)).\ v'(x)$. For instance, if $y = e^z$ where $e = 2.718\ldots$ is the exponential constant, and $z = ax$ where a is any constant, then $y = e^{ax}$; now $dy/dz = e^z$ (see table) and $dz/dx = a$, so $dy/dx = ae^{ax}$.

Many problems can be formulated and solved using derivatives. For example, let y be the amount present in a sample of radioactive material at time x. According to theory and observation, the sample decays at a rate proportional to the amount remaining, that is, $dy/dx = ay$ for some negative constant a. To find y in terms of x, we therefore need a function $y = f(x)$ such that $dy/dx = ay$ for all x. The most general such function is $y = ce^{ax}$ where c is a constant. Since $e^0 = 1$ we have $y = c$ when $x = 0$, so c is the initial amount present (at time $x = 0$). Since $a<0$ we have $e^{ax} \to 0$ as x increases, so $y \to 0$, confirming that the sample gradually decays to nothing. This is an example of exponential decay, shown in figure 3a. If a is a positive constant, we obtain the same solution $y = ce^{ax}$, but as time progresses y now increases rapidly (since e^{ax} does when $a>0$); this is exponential growth, shown in figure 3b and observed in nuclear explosions and certain animal communities, where growth-rate is proportional to population.

IV INTEGRAL CALCULUS

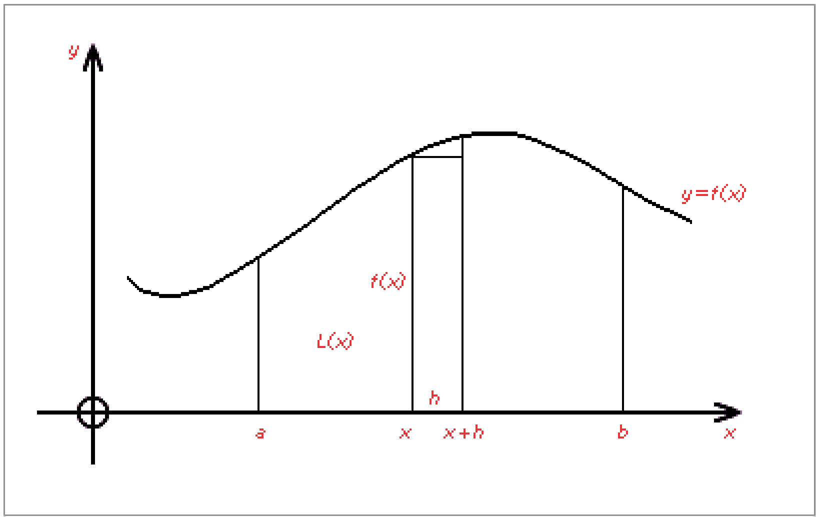

Figure 4: Integral Calculus

The calculation of the area under a curve is a classic example of the use of integral calculus. Here the area between the curve and the x-axis from $x = a$ to $x = b$ is approximately equal to the sum of a large number of thin rectangles like the one shown. This has an area approximately equal to $f(x)$ times h. As h is made smaller, the rectangles become thinner and more numerous, and their total area approaches ever more closely to the true area under the curve. Integral calculus is concerned with deriving the exact value of the area from a knowledge of the equation describing the curve, $y = f(x)$.

Integral calculus involves the inverse process to differentiation, called integration. Given a function f, we seek a function F with derivative $F' = f$; this is an integral or antiderivative of f, written $F(x) = \int f(x)dx$ or simply $F = \int f\,dx$ (a notation we will explain later). Tables of derivatives can be used for integration: thus x^2 has derivative $2x$, so $2x$ has x^2 as an integral. If F is any integral of f, the most general integral of f is $F + c$, where c is an arbitrary constant called the constant of integration; this is because a constant has derivative 0, so $(F + c)' = F' + c' = f + 0 = f$. Thus $\int 2x dx = x^2 + c$, for instance.

The basic rules for integrating compound functions resemble those for differentiation. The integral of a sum or difference is the sum or difference of their integrals, and likewise for multiplication by a constant. Thus $x = \frac{1}{2}.2x$ has integral $\frac{1}{2}x^2$, and similarly $\int x^m\,dx = x^{m+1}/(m+1)$ for any $m \neq -1$. (We exclude $m = -1$ to avoid dividing by 0; the natural logarithm $\ln|x|$ is an integral of $x^{-1} = 1/x$ for any $x \neq 0$.) Integration is generally harder than differentiation, but many of the more familiar functions can be integrated by these and other rules (see the table).

A classic application of integration is to calculate areas. Let A be the area of the region between the graph of a function $y = f(x)$ and the x-axis, for $a \le x \le b$. For simplicity, assume that $f(x) \ge 0$ between a and b. For each $x \ge a$, let $L(x)$ be the area of this region to the left of x, so we need to find $A = L(b)$. First we differentiate $L(x)$. If h is a small change in x, the region below the graph between x and $x + h$ is approximately a rectangle of height $f(x)$ and width h (see figure 4); the corresponding change $k = L(x + h) - L(x)$ in area is therefore approximately $f(x)h$, so k/h is approximately $f(x)$. As $h \to 0$ these approximations become more exact, so $k/h \to f(x)$ and hence $L'(x) = f(x)$. Thus L is an integral of f, so if we know any integral F of f then $L = F + c$ for some constant c. Now $L(a) = 0$ (since the region to the left of x vanishes when $x = a$), so $c = -F(a)$ and hence $L(x) = F(x) - F(a)$ for all $x \ge a$. In particular, $A = L(b) = F(b) - F(a)$, written

$$A = \int_a^b f(x)\,dx = \left[F(x)\right]_a^b$$

This is the Fundamental Theorem of Calculus, valid whenever f is continuous between a and b, provided we assign negative areas to any regions below the x-axis, where $f(x) < 0$. (Continuity means that $f(x) \to f(x_0)$ as $x \to x_0$, so f has an unbroken graph.) For example $f(x) = x^2$ has integral $F(x) = x^3/3$, so

$$A = \int_a^b x^2 dx = [x^3/3]_a^b = (b^3 - a^3)/3;$$

and with this formula it is possible to work out a large number of useful quantities. For example, the volume of a cone of height h and radius r can be found by evaluating the expression $\int \pi\,(rx/h)^2\,dx$ between the limits $x = 0$ and $x = 1$; this is because the radius at a distance x below the apex of the cone is rx/h and the cross-sectional area is $\pi\,(rx/h)^2$. The result is $\pi r^2 h/3$.

Here $\int_a^b f(x)dx$ is a definite integral of f; this is a number, whereas the indefinite integral $\int f(x)dx$ is a function $F(x)$ (more precisely, a set of functions $F(x) + c$). The symbol $\int$ (a 17th-century S) suggests summation of areas $f(x)dx$ of infinitely many rectangles of height $f(x)$ and infinitesimal width dx; more precisely, $\int_a^b f(x)dx$ is the limit of a sum of finitely many rectangular areas, as their widths approach 0.

The derivative $dy/dx = f'(x)$ of a function $y = f(x)$ can be differentiated again to obtain a second derivative, denoted by d^2y/dx^2, $f''(x)$ or D^2f. If x is time and y is distance travelled, for instance, so that dy/dx is velocity v, then $d^2y/dx^2 = dv/dx$ is rate of change of velocity, that is, acceleration. By Newton's second law of motion, a body of constant mass m subject to a force F undergoes an acceleration a satisfying $F = ma$. For example, if the body falls under the gravitational force $F = mg$ (where g is the gravitational field strength) then $ma = F = mg$ implies $a = g$, so $dv/dx = g$. Integrating, we have $v = gx + c$ where c is constant; putting $x = 0$ shows that c is the initial velocity. Integrating $dy/dx = v = gx + c$, we have $y = \frac{1}{2}gx^2 + cx + b$ where b is constant; putting $x = 0$ shows that b is the initial value of y.

Higher derivatives $f^{(n)}(x) = d^n y/dx^n = D^n f$ of $f(x)$ are found by successively differentiating n times. Taylor's Theorem states that if $f(x)$ can be represented as a power series $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$ (where $a_0, a_1, \ldots$ are constants), then $a_n = f^{(n)}(0)/n!$ where $0! = 1$ and $n! = 1 \times 2 \times 3 \times \ldots$

$\times\ n$ for all $n \geq 1$. Most commonly used functions can be represented as power series; for instance if $f(x) = e^x$ then $f^{(n)}(x) = e^x$ for all n, so $f^{(n)}(0) = e^0 = 1$ and hence:

$$e^x = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

V PARTIAL DERIVATIVES

Functions of several variables can also have derivatives. Let $z = f(x,y)$, so z depends on x and y. Temporarily holding y constant we can regard z as a function of x, and differentiating this gives the partial derivative $\partial z/\partial x = \partial f/\partial x$; similarly, keeping x constant and differentiating with respect to y we obtain $\partial z/\partial y = \partial f/\partial y$. For instance, if $z = x^2 - xy + 3y^2$ then $\partial z/\partial x = 2x - y$ and $\partial z/\partial y = -x + 6y$. Geometrically, an equation $z = f(x,y)$ defines a surface in three-dimensional space; if the x- and y-axes are horizontal and the z-axis is vertical, then $\partial z/\partial x$ and $\partial z/\partial y$ represent the gradients of this surface at the point (x,y,z) in the directions of the x- and y-axes. Partial derivatives can also be calculated for functions of more than two variables, by keeping all but one variable temporarily constant; higher partial derivatives can be defined by repeating this operation. Partial derivatives are important in applied mathematics, where functions often depend on several variables such as space and time.