Geometry

I INTRODUCTION

Geometry (Greek *geō,* “Earth”; *metrein,* “to measure”), branch of mathematics that deals with the properties of space. In its most elementary form geometry is concerned with such metrical problems as determining the areas and diameters of two-dimensional figures and the surface areas and volumes of solids. Other fields of geometry include analytic geometry, descriptive geometry, topology, the geometry of spaces having four or more dimensions, fractal geometry, and non-Euclidean geometry.

II EARLY DEMONSTRATIVE GEOMETRY

The derivation of the term “geometry” is an accurate description of the works of the early geometers, who were concerned with such problems as measuring the size of fields and laying out accurate right angles for the corners of buildings. This type of empirical geometry, which flourished in ancient Egypt, Sumer, and Babylonia, was refined and systematized by the Greeks. The great achievement of Greek geometry was to show how the various truths of empirical geometry follow logically from a small number of self-evident axioms or postulates. Though we know little with certainty about the school of philosophers who grew up around Pythagoras of Samos in the 6th century BC, they may have been the first to begin this systematic study of geometry.

Typical of the postulates that were developed and accepted by Greek mathematicians is this statement: “A straight line is the shortest distance between two points.” From these axioms, a number of theorems about the properties of points, lines, angles, curves, and planes can be logically deduced. These theorems include: “The sum of the interior angles of any triangle is equal to the sum of two right angles”, and “The square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the other two sides” (known as Pythagoras' theorem). The demonstrative geometry of the Greeks, which dealt chiefly with polygons and circles and corresponding three-dimensional figures, was rigorously drawn up by the Greek mathematician Euclid, in his book *The Elements.* Euclid's text, in spite of its imperfections, has served as a basic textbook in geometry almost to the present day.

III EARLY GEOMETRICAL PROBLEMS

The Greeks introduced construction problems, which require a certain line or figure to be constructed by the use of a straight-edge and compasses alone. Simple examples are the construction of a line that will be twice as long as another line or of a line that will divide a given angle into two equal angles. Three famous construction problems dating from the time of the ancient Greeks resisted the efforts of many generations of mathematicians to solve them: duplicating the cube (constructing a cube double the volume of a given cube); squaring the circle (constructing a square equal in area to a given circle); and trisecting the angle (dividing a given angle into three equal parts). None of these constructions is possible with a straight-edge and compasses alone, but the proofs required advances in mathematics that were not made until the 18th and 19th centuries, and the impossibility of squaring the circle was not finally proved until 1882.

The Greeks, particularly Apollonius of Perga, made a study of the family of curves known as conic sections and discovered many of their fundamental properties. The conic sections are important in many fields of physical science; for example, the orbits of the planets around the Sun are basically conic sections.

Archimedes, one of the greatest of Greek scientists, made a number of important contributions to geometry. He devised ways to measure the areas of a number of curved figures and the surface areas and volumes of solids bounded by curved surfaces, such as paraboloids and cylinders. He also worked out a method for approximating the value of pi (π)—the ratio between the diameter and circumference of a circle—and stated that numerically it lay between $3\frac{1}{7}$ and $3\frac{10}{71}$.

IV ANALYTIC GEOMETRY

Geometry advanced little from the end of the Greek era to the end of the Middle Ages. The next great stride in the science was taken by the French philosopher and mathematician René Descartes, who published a geometrical essay in conjunction with his epoch-making treatise *A Discourse on Method* (1637). This work forged a link between geometry and algebra by showing how to apply the methods of one discipline to the other. This is the basis of analytic geometry, in which geometric figures are represented by algebraic expressions, a subject that underlies much modern work in geometry.

Another important development of the 17th century was the investigation of the properties of geometrical figures that do not vary when the figures are projected from one plane to another. A simple example of a theorem in projective geometry is illustrated in figure 1. If points *A, B, C* and *a, b, c* are placed anywhere on a conic section, such as a circle, and the points interconnected *A* to *b* and *c, B* to *c* and *a,* and *C* to *b* and *a,* the three points at which the corresponding lines intersect will lie in a straight line. Similarly, if any six tangents are drawn to a conic section, as in figure 2, and lines are drawn connecting the opposite intersections of these tangents, the connecting lines will meet at a single point. This theorem is said to be projective, since it is equally true for all the conic sections, and the sections themselves may be transformed into one another by suitable projections, as in figure 3, which shows the projection of a circle as an ellipse in another plane.

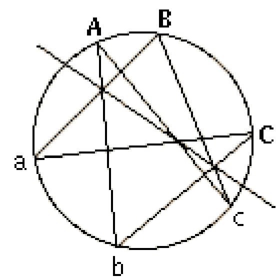

Figure 1

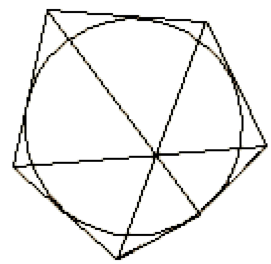

Figure 2

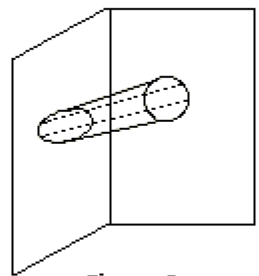

Figure 3

V MODERN DEVELOPMENTS

Geometry took a radical turn in the 19th century. The mathematicians Carl Friedrich Gauss, Nikolay Lobachevsky, János Bolyai, and G. F. B. Riemann independently developed consistent systems of non-Euclidean geometry. These systems arose from work concerned with Euclid's so-called “parallel postulate”, which states that through any point one and only one straight line can be drawn that is parallel to a given straight line. Denying this postulate led to bizarre and non-intuitive yet consistent models of space. This way of regarding axioms as arbitrary rather than absolute had far-reaching consequences for mathematics.

Around this time the British mathematician Arthur Cayley developed the geometry of spaces with more than three dimensions. A line is a one-dimensional space. If each point in the line is replaced with a line drawn perpendicular to it, then a two-dimensional space, a plane, is formed. Similarly, if each point in a plane is replaced with a line drawn perpendicular to it, then a three-dimensional space is formed. Furthermore, if each point in a three-dimensional space is replaced with a line drawn perpendicular to it, then a four-dimensional space is formed. Though this is obviously physically impossible and mentally upsetting, it is conceptually sound. The use of concepts involving more than three dimensions has had a number of important applications in the physical sciences, particularly in the development of theories of relativity.

Analytical methods may also be used to investigate regular geometrical figures in four or more dimensions and to compare them with similar figures in three or fewer dimensions. Such geometry is called structural geometry. A simple example of this approach to geometry is the definition of the simplest geometrical figure that can be drawn in spaces of zero, one, two, three, four, or more dimensions. In the first four of these spaces, the figures are the familiar point, line, triangle, and tetrahedron. In a space of four dimensions the simplest figure can be shown to be composed of five points as vertices, ten line segments as edges, ten triangles as faces, and five tetrahedra. A tetrahedron similarly analysed is composed of four vertices, six line segments, and four triangles; see figure 4.

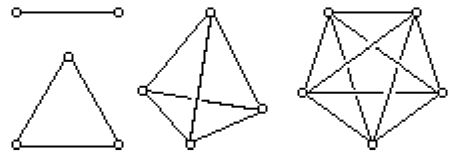

Figure 4

Another dimensional concept, that of fractional dimensions, also arose in the 19th century. In the 1970s the concept was developed as fractal geometry.