

Mathematics

I INTRODUCTION

Mathematics, study of relationships among quantities, magnitudes, and properties and of logical operations by which unknown quantities, magnitudes, and properties may be deduced. In the past mathematics was regarded as the science of quantity, whether of magnitudes, as in geometry, or of numbers, as in arithmetic, or the generalization of these two fields, as in algebra. Towards the middle of the 19th century mathematics came to be regarded increasingly as the science of relations, or as the science that draws necessary conclusions. This latter view encompasses mathematical or symbolic logic— the science of using symbols to provide an exact theory of logical deduction and inference based on definitions, axioms, postulates, and rules for transforming primitive elements into more complex relations and theorems.

This brief survey of the history of mathematics traces the evolution of mathematical ideas and concepts, beginning in prehistory. Indeed, mathematics is nearly as old as humanity itself: evidence of a sense of geometry and interest in geometric pattern has been found in the designs of prehistoric pottery and textiles and in cave paintings. Primitive counting systems were almost certainly based on using the fingers of one or both hands, as evidenced by the predominance of the numbers 5 and 10 as the bases for most number systems today.

II ANCIENT MATHEMATICS

CIVILIZATION	NUMERIC VALUE											
	1	2	3	5	10	20	21	50	100	500	1000	10,000
Babylonian	𐎶	𐎶𐎶	𐎶𐎶𐎶	𐎶𐎶𐎶𐎶	<	<<	<<𐎶	𐎶𐎶𐎶𐎶	𐎶𐎶𐎶𐎶𐎶			
	>	>>	>>>	>>>𐎶	•	••	••𐎶	𐎶𐎶𐎶				
Egyptian Hieroglyphic	I	II	III	IIII II	𐀀	𐀀𐀀	𐀀𐀀𐀀	𐀀𐀀𐀀𐀀𐀀𐀀	𐀀𐀀𐀀𐀀𐀀𐀀	𐀀𐀀𐀀𐀀𐀀𐀀𐀀	𐀀𐀀𐀀𐀀𐀀𐀀𐀀𐀀	𐀀𐀀𐀀𐀀𐀀𐀀𐀀𐀀𐀀𐀀𐀀
	I	II	III	𐀀	𐀀	𐀀		𐀀	𐀀	𐀀	𐀀	𐀀
Egyptian Hieratic	I	II	III	𐀀	𐀀	𐀀		𐀀	𐀀	𐀀	𐀀	𐀀
	I	II	III	𐀀	𐀀	𐀀	𐀀𐀀𐀀	𐀀	𐀀	𐀀	𐀀	𐀀
Greek Herodianic	I	II	III	𐀀	𐀀	𐀀𐀀	𐀀𐀀𐀀	𐀀	𐀀	𐀀	𐀀	𐀀
	I	II	III	𐀀	𐀀	𐀀𐀀	𐀀𐀀𐀀	𐀀	𐀀	𐀀	𐀀	𐀀
Roman	I	II	III	V or 𐀀	X	XX	XXI	L or 𐀀	𐀀, C, or 𐀀	𐀀, 𐀀 𐀀, 𐀀	𐀀𐀀𐀀𐀀 𐀀 𐀀	𐀀𐀀
	I	II	III	V or 𐀀	X	XX	XXI	L or 𐀀	𐀀, C, or 𐀀	𐀀, 𐀀 𐀀, 𐀀	𐀀𐀀𐀀𐀀 𐀀 𐀀	𐀀𐀀

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Ancient Mathematical Characters

The earliest records of advanced, organized mathematics date back to the ancient Mesopotamian

country of Babylonia and to Egypt of the 3rd millennium BC. There mathematics was dominated by arithmetic, with an emphasis on measurement and calculation in geometry and with no trace of later mathematical concepts such as axioms or proofs.

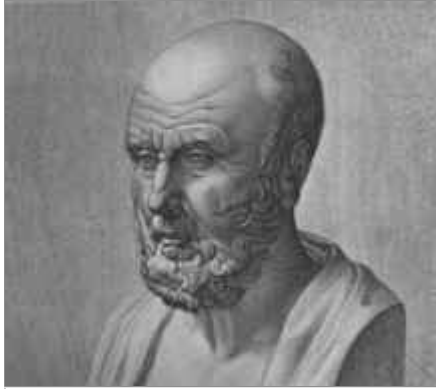
The earliest Egyptian texts, composed about 1800 BC, reveal a decimal numeration system with separate symbols for the successive powers of 10 (1, 10, 100, and so forth), just as in the system used by the Romans. Numbers were represented by writing down the symbol for 1, 10, 100, and so on, as many times as the unit was in a given number. For example, the symbol for 1 was written five times to represent the number 5, the symbol for 10 was written six times to represent the number 60, and the symbol for 100 was written three times to represent the number 300. Together, these symbols represented the number 365. Addition was done by totalling separately the units, 10s, 100s, and so forth in the numbers to be added. Multiplication was based on successive doublings, and division was based on the inverse of this process.

The Egyptians used sums of unit fractions ($\frac{1}{n}$), supplemented by the fraction $\frac{2}{3}$, to express all other fractions. For example, the fraction $\frac{2}{7}$ was the sum of the fractions $\frac{1}{4}$ and $\frac{1}{28}$. Using this system, the Egyptians were able to solve all problems of arithmetic that involved fractions, as well as some elementary problems in algebra. In geometry, the Egyptians arrived at correct rules for finding areas of triangles, rectangles, and trapezoids, and for finding volumes of figures such as bricks, cylinders, and, of course, pyramids. To find the area of a circle, the Egyptians used the square on $\frac{8}{9}$ of the diameter of the circle, a value close to the value of the ratio known as pi, but actually about 3.16 rather than pi's value of about 3.14.

The Babylonian system of numeration was quite different from the Egyptian system. In the Babylonian system, using clay tablets consisting of various wedge-shaped marks, a single wedge indicated 1 and an arrow-like wedge stood for 10 (see table). Numbers up through 59 were formed from these symbols through an additive process, as in Egyptian mathematics. The number 60, however, was represented by the same symbol as 1, and from this point on a positional symbol was used. That is, the value of one of the first 59 numerals depended henceforth on its position in the total numeral. For example, a numeral consisting of a symbol for 2 followed by one for 27 and ending in one for 10 stood for $2 \times 60^2 + 27 \times 60 + 10$. This principle was extended to the representation of fractions as well, so that the above sequence of numbers could equally well represent $2 \times 60 + 27 + 10 \times (\frac{1}{60})$, or $2 + 27 \times (\frac{1}{60}) + 10 \times (\frac{1}{60}^2)$. With this *sexagesimal* system (base 60), as it is called, the Babylonians had as convenient a numerical system as the decimal (base 10) system.

The Babylonians in time developed a sophisticated mathematics by which they could find the positive roots of any quadratic equation. They could even find the roots of certain cubic equations. The Babylonians had a variety of tables, including tables for multiplication and division, tables of squares, and tables of compound interest. They could solve complicated problems using Pythagoras' theorem; one of their tables contains integer solutions to the Pythagorean equation, $a^2 + b^2 = c^2$, arranged so that c^2/a^2 decreases steadily from 2 to about $\frac{4}{3}$. The Babylonians were also able to sum not only arithmetic and some geometric series, but also sequences of squares. They also arrived at a good approximation for $\sqrt{2}$. In geometry, they calculated the area of rectangles, triangles, and trapezoids, as well as the volumes of simple shapes such as bricks and cylinders. However, the Babylonians did not arrive at the correct formula for the volume of a pyramid.

A Greek Mathematics



Culver Pictures

Hippocrates

In addition to his considerable medical work, Hippocrates was an accomplished mathematician. In the 4th century BC, he discovered that the areas of crescent-shaped figures bounded by arcs of circles are equal to areas of certain triangles.

The Greeks adopted elements of mathematics from both the Babylonians and the Egyptians. The new element in Greek mathematics, however, was the invention of an abstract mathematics founded on a logical structure of definitions, axioms, and proofs. According to later Greek accounts, this development began in the 6th century BC with Thales of Miletus and Pythagoras of Samos, the latter a religious leader who taught the importance of studying numbers in order to understand the world. Some of his disciples made important discoveries about the theory of numbers and geometry, all of which were attributed to Pythagoras.

In the 5th century BC, some of the great geometers were the atomist philosopher Democritus of Abdera, who discovered the correct formula for the volume of a pyramid, and Hippocrates of Kos, who discovered that the areas of crescent-shaped figures bounded by arcs of circles are equal to areas of certain triangles. This discovery is related to the famous problem of squaring the circle—that is, constructing a square equal in area to a given circle. Two other famous mathematical problems that originated during the century were those of trisecting an angle and doubling a cube—that is, constructing a cube the volume of which is double that of a given cube. All of these problems were solved, and in a variety of ways, all involving the use of instruments more complicated than a straight-edge and a geometrical compass. Not until the 19th century was it shown that the three problems mentioned above could never have been solved using those instruments alone.

In the latter part of the 5th century BC, an unknown mathematician discovered that no unit of length would measure both the side and diagonal of a square. That is, the two lengths are *incommensurable*. This means that no counting numbers n and m exist whose ratio expresses the relationship of the side to the diagonal. Since the Greeks considered only the counting numbers (1, 2, 3, and so on) as numbers, they had no numerical way to express this ratio of diagonal to side. (This ratio, $\sqrt{2}$, would today be called *irrational*.) As a consequence the Pythagorean theory of ratio, based on numbers, had to be abandoned and a new, nonnumerical theory introduced. This was done by the 4th-century BC mathematician Eudoxus of Cnidus, whose solution may be found in the *Elements* of Euclid. Eudoxus

also discovered a method for rigorously proving statements about areas and volumes by successive approximations.



Archimedes

Archimedes made extensive contributions to theoretical mathematics, in particular geometry. Through his study of conic sections he derived formulas for the areas of circles and parabolas, and his work became the basis for the development of calculus in the 17th century.

Euclid was a mathematician and teacher who worked at the famed Museum of Alexandria and who also wrote on optics, astronomy, and music. The 13 books that make up his *Elements* contain much of the basic mathematical knowledge discovered up to the end of the 4th century BC on the geometry of polygons and the circle, the theory of numbers, the theory of incommensurables, solid geometry, and the elementary theory of areas and volumes.

The century that followed Euclid was marked by mathematical brilliance, as displayed in the works of Archimedes of Syracuse and a younger contemporary, Apollonius of Perga. Archimedes used a method of discovery, based on theoretically weighing infinitely thin slices of figures, to find the areas and volumes of figures arising from the conic sections. These conic sections had been discovered by a pupil of Eudoxus named Menaechmus, and they were the subject of a treatise by Euclid, but Archimedes' writings on them are the earliest to survive. Archimedes also investigated centres of gravity and the stability of various solids floating in water. Much of his work is part of the tradition that led, in the 17th century, to the discovery of the calculus. Archimedes was killed by a Roman soldier during the sack of Syracuse. His younger contemporary, Apollonius, produced an eight-book treatise on the conic sections that established the names of the sections: ellipse, parabola, and hyperbola. It also provided the basic treatment of their geometry until the time of the French philosopher and scientist René Descartes in the 17th century.

After Euclid, Archimedes, and Apollonius, Greece produced no geometers of comparable stature. The writings of Hero of Alexandria in the 1st century AD show how elements of both the Babylonian and Egyptian mensurational, arithmetic traditions survived alongside the logical edifices of the great geometers. Very much in the same tradition, but concerned with much more difficult problems, are the books of Diophantus of Alexandria in the 3rd century AD. They deal with finding rational solutions to kinds of problems that lead immediately to equations in several unknowns. Such equations are now called Diophantine equations and are the subject of Diophantine Analysis.

B Applied Mathematics in Greece

Paralleling the studies described in pure mathematics were studies made in optics, mechanics, and astronomy. Many of the greatest mathematical writers, such as Euclid and Archimedes, also wrote on astronomical topics. Shortly after the time of Apollonius, Greek astronomers adopted the Babylonian system for recording fractions and, at about the same time, composed tables of chords in a circle. For a circle of some fixed radius, such tables give the length of the chords subtending a sequence of arcs increasing by some fixed amount. They are equivalent to a modern sine table, and their composition marks the beginnings of trigonometry. In the earliest such tables—those of Hipparchus in about 150 BC—the arcs increased in steps of $7\frac{1}{2}^\circ$, from 0° to 180° . By the time of the astronomer Ptolemy in the 2nd century AD Greek mastery of numerical procedures had progressed to the point where Ptolemy was able to include in his *Almagest* a table of chords in a circle for steps of $\frac{1}{2}^\circ$, which, although expressed sexagesimally, is accurate to about five decimal places.

In the meantime, methods were developed for solving problems involving plane triangles, and a theorem—named after the astronomer Menelaus of Alexandria—was established for finding the lengths of certain arcs on a sphere when other arcs are known. These advances gave Greek astronomers what they needed to solve the problems of spherical astronomy and to develop an astronomical system that held sway until the time of the German astronomer Johannes Kepler.

III MEDIEVAL AND RENAISSANCE MATHEMATICS

Following the time of Ptolemy, a tradition of study of the mathematical masterpieces of the preceding centuries was established in various centres of Greek learning. The preservation of such works as have survived to modern times began with this tradition. It was continued in the Islamic world, where original developments based on these masterpieces first appeared. The earliest original developments based on these masterpieces, however, did not appear at such centres of tradition but in the Islamic world.

A Islamic and Indian Mathematics

After a century of expansion, in which the religion of Islam spread from its beginnings in the Arabian Peninsula to dominate an area extending from Spain to the borders of China, Muslims began to acquire the results of the “foreign sciences”. At centres such as the House of Wisdom in Baghdad, supported by the ruling caliphs and wealthy individuals, translators produced Arabic versions of Greek and Indian mathematical works.

By the year 900 the acquisition was complete, and Muslim scholars began to build on what they had acquired. Thus mathematicians extended the Hindu decimal positional system of arithmetic from whole numbers to include decimal fractions, and the 12th-century Persian mathematician Omar Khayyam generalized Hindu methods for extracting square and cube roots to include fourth, fifth, and higher roots. In algebra, al-Karaji completed Muhammad al-Khwarizmi's algebra of polynomials to include even polynomials with an infinite number of terms. (Al-Khwarizmi's name, incidentally, is the

source of the word *algorithm*, and the title of one of his books is the source of the word *algebra*.) Geometers such as Ibrahim ibn Sinan continued Archimedes' investigations of areas and volumes, and Kamal al-Din and others applied the theory of conic sections to solve optical problems. Using the Hindu sine function and Menelaus' theorem, mathematicians from Habas al-Hasib to Nasir ad-Din at-Tusi created the mathematical disciplines of plane and spherical trigonometry. These did not become mathematical disciplines in the West until the publication of *De Triangulis Omnimodibus* by the German astronomer Regiomontanus.

Finally, a number of Muslim mathematicians made important discoveries in the theory of numbers, while others explained a variety of numerical methods for solving equations. The Latin West acquired much of this learning during the 12th century, the great century of translation. Together with translations of the Greek classics, these Muslim works were responsible for the growth of mathematics in the West during the late Middle Ages. Italian mathematicians such as Leonardo Fibonacci and Luca Pacioli, one of the many 15th-century writers on algebra and arithmetic for merchants, depended heavily on Arabic sources for their knowledge.

IV WESTERN RENAISSANCE MATHEMATICS

Although the late medieval period saw some fruitful mathematical considerations of problems of infinity by writers such as Nicole Oresme, it was not until the early 16th century that a truly important mathematical discovery was made in the West. The discovery, an algebraic formula for the solution of both the cubic and quartic equations, was published in 1545 by the Italian mathematician Gerolamo Cardano in his *Ars Magna*. The discovery drew the attention of mathematicians to complex numbers and stimulated a search for solutions to equations of degree higher than four. It was this search, in turn, that led to the first work on group theory at the end of the 18th century, and to the French mathematician Évariste Galois's theory of equations in the early 19th century.

The 16th century also saw the beginnings of modern algebraic and mathematical symbols, as well as the remarkable work on the solution of equations by the French mathematician François Viète. His writings influenced many mathematicians of the following century, including Pierre de Fermat in France and Isaac Newton in England.

V MATHEMATICS SINCE THE 16TH CENTURY

Europeans dominated in the development of mathematics after the Renaissance.

A 17th Century



Hulton Deutsch

René Descartes

René Descartes founded analytic geometry, which uses algebra to represent geometric lines and curves in terms of axes and coordinates. He also contributed to the theory of equations.

During the 17th century, the greatest advances were made in mathematics since the time of Archimedes and Apollonius. The century opened with the discovery by the Scottish mathematician John Napier of logarithms, whose continued utility prompted the French astronomer Pierre Simon Laplace to remark, almost two centuries later, that Napier, by halving the labours of astronomers, had doubled their lifetimes.

The science of number theory, which had lain dormant since the medieval period, illustrates the 17th-century advances built on ancient learning. It was Diophantus' *Arithmetica* that stimulated Fermat to advance the theory of numbers greatly. His most important conjecture in the field, written in the margin of his copy of the *Arithmetica*, was that no solutions exist to $a^n + b^n = c^n$ for positive integers a , b , and c when n is greater than 2. This conjecture, known as Fermat's Last Theorem, stimulated much important work in algebra and number theory; not until 1995 was the theorem proved satisfactorily, by Andrew Wiles with help from Richard Taylor.



Corbis

Gottfried Leibniz

The 17th-century thinker Gottfried Leibniz made many contributions to mathematics. He formulated the theory of calculus slightly after, though independently to, Newton, and gave it the notation that we use today. He also made advances in the area of symbolic logic.

Two important developments in pure geometry occurred during the century. The first was the publication, in *Discourse on Method* (1637) by Descartes, of his discovery of analytic geometry, which showed how to use the algebra that had developed since the Renaissance to investigate the geometry of curves. (Fermat made the same discovery but did not publish it.) This book, together with short treatises that had been published with it, stimulated and provided the basis for Isaac Newton's mathematical work in the 1660s. The second development in geometry was the publication by the French engineer Gérard Desargues in 1639 of his discovery of projective geometry. Although the work was much appreciated by Descartes and the French philosopher and scientist Blaise Pascal, its eccentric terminology and the excitement of the earlier publication of analytic geometry delayed the development of its ideas until the early 19th century and the works of the French mathematician Jean Victor Poncelet.

Another major step in mathematics in the 17th century was the beginning of probability theory in the correspondence of Pascal and Fermat on a problem in gambling, called the problem of points. This unpublished work stimulated the Dutch scientist Christiaan Huygens to publish a small tract on probabilities in dice games, which was reprinted by the Swiss mathematician Jakob Bernoulli in his *Art of Conjecturing*. Both Bernoulli and the French mathematician Abraham De Moivre, in his *Doctrine of Chances* in 1718, applied the newly discovered calculus to make rapid advances in the theory, which by then had important applications in the rapidly developing insurance industry.

Without question, however, the crowning mathematical event of the 17th century was Newton's discovery, between 1664 and 1666, of differential and integral calculus. In making this discovery, Newton built on earlier work by his fellow Englishmen, John Wallis and Isaac Barrow, as well as on work of such Continental mathematicians as Descartes, Francesco Bonaventura Cavalieri, Johann van Waveren Hudde, and Gilles Personne de Roberval. About eight years later than Newton, who had not yet published his discovery, the German Gottfried Wilhelm Leibniz rediscovered calculus and published first, in 1684 and 1686. Leibniz's notation systems, such as dx , are used today in calculus.

B 18th Century



Culver Pictures

Leonhard Euler

Although hindered by loss of sight, Leonhard Euler was an important contributor to both pure and applied mathematics. Euler is best known for his analytical treatment of mathematics and his discussion of concepts in calculus, but he is also noted for his work in acoustics, mechanics, astronomy, and optics.

The remainder of the 17th century and a good part of the 18th were taken up by the work of disciples of Newton and Leibniz, who applied their ideas to solving a variety of problems in physics, astronomy, and engineering. In the course of doing so they also created new areas of mathematics. For example, Johann and Jakob Bernoulli invented the calculus of variations, and French mathematician Gaspard Monge invented differential geometry. Also in France, Joseph Louis Lagrange gave a purely analytic treatment of mechanics in his great *Analytical Mechanics* (1788), in which he stated the famous Lagrange equations for a dynamical system. He contributed to differential equations and number theory, as well, and originated the theory of groups. His contemporary, Laplace, wrote *The Analytic Theory of Probabilities* (1812) and the classic *Celestial Mechanics* (1799-1825), which earned him the title of the “French Newton”.

The greatest mathematician of the 18th century was Leonhard Euler, a Swiss, who made basic contributions to calculus and to all other branches of mathematics, as well as to the applications of mathematics. He wrote textbooks on calculus, mechanics, and algebra that became models of style for writing in these areas. The success of Euler and other mathematicians in using calculus to solve mathematical and physical problems, however, only accentuated their failure to develop a satisfactory justification of its basic ideas. That is, Newton's own accounts were based on kinematics and velocities, Leibniz's explanation was based on infinitesimals, and Lagrange's treatment was purely algebraic and founded on the idea of infinite series. All these systems were unsatisfactory when measured against the logical standards of Greek geometry, and the problem was not resolved until the following century.

C 19th Century

In 1821 a French mathematician, Augustin Louis Cauchy, succeeded in giving a logically satisfactory approach to calculus. He based his approach only on finite quantities and the idea of a limit. This solution posed another problem, however, that of a logical definition of “real number”. Although Cauchy's explanation of calculus rested on this idea, it was not Cauchy but the German mathematician Julius W. R. Dedekind who found a satisfactory definition of real numbers in terms of the rational numbers. This definition is still taught, but other definitions were given at the same time by the German mathematicians Georg Cantor and Karl T. W. Weierstrass. A further important problem, which arose out of the problem—first stated in the 18th century—of describing the motion of a vibrating string, was that of defining what is meant by function. Euler, Lagrange, and the French mathematician Jean-Baptiste Fourier all contributed to the solution, but it was the German mathematician Peter G. L. Dirichlet who proposed the definition in terms of a correspondence between elements of the domain and the range. This is the definition that is found in texts today.

In addition to firming the foundations of analysis, as the techniques of the calculus were by then called, mathematicians of the 19th century made great advances in the subject. Early in the century, Carl Friedrich Gauss gave a satisfactory explanation of complex numbers, and these numbers then formed a whole new field for analysis, one that was developed in the work of Cauchy, Weierstrass, and the German mathematician Georg F. B. Riemann. Another important advance in analysis was Fourier's study of infinite sums whose terms are trigonometric functions. Known today as Fourier series, they are still powerful tools in pure and applied mathematics. In addition, the investigation of which functions could be equal to Fourier series led Cantor to the study of infinite sets and to an arithmetic

of infinite numbers. Cantor's theory, which was considered quite abstract and even attacked as a "disease from which mathematics will soon recover", now forms part of the foundations of mathematics and has more recently found applications in the study of turbulent flow in fluids.

A further 19th-century discovery that was considered apparently abstract and useless at the time was non-Euclidean geometry. In non-Euclidean geometry, more than one parallel can be drawn to a given line through a given point not on the line. Evidently this was discovered first by Gauss, but Gauss was fearful of the controversy that might result from publication. The same results were rediscovered independently and published by the Russian mathematician Nikolay Ivanovich Lobachevsky and the Hungarian János Bolyai. Non-Euclidean geometries were studied in a very general setting by Riemann with his invention of manifolds and, since the work of Einstein in the 20th century, they have also found applications in physics.

Gauss was one of the greatest mathematicians who ever lived. Diaries from his youth show that this infant prodigy had already made important discoveries in number theory, an area in which his book *Disquisitiones Arithmeticae* (1801) marks the beginning of the modern era. While only 18, Gauss discovered that a regular polygon with m sides can be constructed by straight-edge and compass when m is a power of two times distinct primes of the form $2^n + 1$. In his doctoral dissertation he gave the first satisfactory proof of the fundamental theorem of algebra. He often combined scientific and mathematical investigations. Examples include his development of statistical methods along with his investigations of the orbit of a newly discovered planetoid; his founding work in the field of potential theory, along with the study of magnetism; and his study of the geometry of curved surfaces in tandem with his investigations of surveying.

Of more importance for algebra itself than Gauss's proof of its fundamental theorem was the transformation of the subject during the 19th century from a study of polynomials to a study of the structure of algebraic systems. A major step in this direction was the invention of symbolic algebra in England by George Peacock. Another was the discovery of algebraic systems that have many, but not all, of the properties of the real numbers. Such systems include the quaternions of the Irish mathematician William Rowan Hamilton, the vector analysis of the American mathematician and physicist J. Willard Gibbs, and the ordered n -dimensional spaces of the German mathematician Hermann Günther Grassmann. A third major step was the development of group theory, from its beginnings in the work of Lagrange. Galois applied this work deeply to provide a theory of when polynomials may be solved by an algebraic formula.

Just as Descartes had applied the algebra of his time to the study of geometry, so the German mathematician Felix Klein and the Norwegian mathematician Marius Sophus Lie applied the algebra of the 19th century. Klein applied it to the classification of geometries in terms of their groups of transformations (the so-called Erlanger Programm), and Lie applied it to a geometric theory of differential equations by means of continuous groups of transformations known as Lie groups. In the 20th century, algebra has also been applied to a general form of geometry known as topology.

Another subject that was transformed in the 19th century, notably by English mathematician George Boole's *Laws of Thought* (1854) and Cantor's set theory, was the foundations of mathematics. Towards the end of the century, however, a series of paradoxes was discovered in Cantor's theory.

One such paradox, found by English mathematician Bertrand Russell, aimed at the very concept of a set. Mathematicians responded by constructing set theories sufficiently restrictive to keep the paradoxes from arising, but they left open the question of whether other paradoxes might arise in these restricted theories—that is, whether the theories were consistent. As of the present time, only relative consistency proofs have been given—that is, theory A is consistent if theory B is consistent.) Particularly disturbing is the result, proved in 1931 by the American logician Kurt Gödel, that in any axiom system sophisticated enough to be interesting to most mathematicians, it is possible to frame propositions whose truth cannot be decided within the system.

VI CURRENT MATHEMATICS

At the International Conference of Mathematicians held in Paris in 1900, the German mathematician David Hilbert spoke to the assembly. Hilbert was a professor at Göttingen, the former academic home of Gauss and Riemann. He had contributed to most areas of mathematics, from his classic *Foundations of Geometry* (1899) to the jointly authored *Methods of Mathematical Physics*. Hilbert's address at Göttingen was a survey of 23 mathematical problems that he felt would guide the work being done in mathematics during the coming century. These problems have indeed stimulated a great deal of the mathematical research of the century. When news breaks that another of the “Hilbert problems” has been solved, mathematicians all over the world await the details of the story with impatience.

Important as these problems have been, an event that Hilbert could not have foreseen seems destined to play an even greater role in the future development of mathematics—namely, the invention of the programmable digital computer. Although the roots of the computer go back to the geared calculators of Pascal and Leibniz in the 17th century, it was Charles Babbage in 19th-century England who designed a machine that could automatically perform computations based on a programme of instructions stored on cards or tape. Babbage's imagination outran the technology of his day, and it was not until the invention of the relay, then of the vacuum tube, and then of the transistor, that large-scale, programmed computation became feasible. This development has given great impetus to areas of mathematics such as numerical analysis and finite mathematics. It has suggested new areas for mathematical investigation, such as the study of algorithms. It has also become a powerful tool in areas as diverse as number theory, differential equations, and abstract algebra. In addition, the computer has made possible the solution of several long-standing problems in mathematics, such as the four-colour problem first proposed in the mid-19th century. The theorem stated that four colours are sufficient to colour any map, given that any two countries with a contiguous boundary require different colours. The theorem was finally proved in 1976 by means of a large-scale computer at the University of Illinois.

Mathematical knowledge in the modern world is advancing at a faster rate than ever before. Theories that were once separate have been incorporated into theories that are both more comprehensive and more abstract. Although many important problems have been solved, other hardy perennials, such as the Riemann hypothesis, remain, and new and equally challenging problems arise. Even the most abstract mathematics seems to be finding applications.

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