Matrix Theory

INTRODUCTION

Matrix Theory, a branch of pure mathematics, introduced by Arthur Cayley in 1858, associated with the solution of systems of linear equations, which arise naturally in science, engineering, and social sciences.

An $m \times n$ matrix is an array of mn numbers arranged in m rows and n columns, and enclosed in brackets. For example,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

are 2 × 3 and 3 × 2 matrices. The entries in a matrix can belong to various mathematical systems such as integers, rational, real, or complex numbers. The entry in the i-th row and j-th column of a matrix **A** is denoted by a_{ij} or $(\mathbf{A})_{ij}$.

An $m \times n$ matrix stores mn pieces of information a_{ij}, indexed by two parameters i, j. For instance, if m countries each export n commodities, then a_{ij} could be the amount of the j-th commodity exported by the i-th country in a given year, so each row or column of **A** represents a particular country or commodity.

The need to manipulate this information leads to an algebraic theory in which the basic operations of arithmetic are applied to matrices. If **A** and **B** are both $m \times n$ matrices, their sum **A** + **B** is obtained by adding their corresponding entries, that is, $(\mathbf{A} + \mathbf{B})_{ij} = a_{ij} + b_{ij}$. For example,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{pmatrix}$$
$$= \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

The *difference* **A** - **B** is defined similarly by $(\mathbf{A} - \mathbf{B})_{ij} = a_{ij} - b_{ij}$. (Matrices of different shapes cannot be added or subtracted.) Thus if **A** and **B** represent exports for consecutive years, then **A** + **B** represents exports over the two-year period, and if **C** represents imports during the first year, then **A** - **C** represents net exports for that year.

If **A** is an $m \times n$ matrix, and **B** is an $n \times s$ matrix, their product **AB** is an $m \times s$ matrix with $(\mathbf{AB})_{ij}$ formed from the i-th row of **A** and the j-th column of **B** by $(\mathbf{AB})_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$. For instance:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} =$$
$$\begin{pmatrix} (1\times1)+(2\times3)+(3\times5) & (1\times2)+(2\times4)+(3\times6) \\ (4\times1)+(5\times3)+(6\times5) & (4\times2)+(5\times4)+(6\times6) \end{pmatrix}$$
$$= \begin{pmatrix} 22 & 28 \\ 49 & 64 \end{pmatrix}$$

In our export example, if **D** is an $n \times 1$ matrix (or column vector) whose entries are the costs per unit amount of the n commodities, then **AD** is an $m \times 1$ matrix whose entries are the values of the exports of the m countries.

A square matrix is an $n \times n$ matrix for some n. If **A** and **B** are both $n \times n$ matrices, then **A** + **B**,**A** - **B**, **AB** and **BA** all exist and are also $n \times n$ matrices. The algebra of square matrices resembles the algebra of numbers in many ways (though **AB** may differ from **BA**). For instance the $n \times n$ identity matrix **I** defined by:

$$(\mathbf{I})_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

satisfies **IA** = **AI** = **A** for all $n \times n$ matrices **A,** so it behaves like the number 1. Each $n \times n$ matrix **A** has a number called its determinant det(A): if $n = 1$ then det(**A**) = a_{11}, and if $n > 1$ then:

$$\det(\mathbf{A}) = a_{11}\mathbf{D}_1 - a_{12}\mathbf{D}_2 + \ldots + (-1)^{n+1} a_{1n}\mathbf{D}_n$$

where $\mathbf{D}_j$ (called a minor of **A**) is the determinant of the $(n-1) \times (n-1)$ matrix formed by deleting the first row and the j-th column of **A.** If det(**A**) ≠ 0 then **A** has an *inverse* matrix $\mathbf{A}^{-1}$ satisfying $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

II SIMULTANEOUS EQUATIONS

An important application of matrices is in the solution of simultaneous linear equations. Given m equations in n unknowns $x_1, \ldots, x_n$, say

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2, \\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m, \end{gathered}$$

let **A** be the $m \times n$ matrix with $(\mathbf{A})_{ij} = a_{ij}$ ($i = 1, \ldots, m$, $j = 1, \ldots, n$), and let

$$\mathbf{X} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

The equations may be written in matrix form as **AX** = **B**, and solved (where possible) by manipulating this equation. For instance, if $m = n$ and det(**A**) ≠ 0 there is a unique solution $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.

III GEOMETRY

Matrices have important applications in geometry. A point in ordinary 3-dimensional space can be specified by 3 numbers the x, y, and z coordinates. This means that a point can be represented by a simple column vector and a set of points as a set of column vectors. Transformations, such as rotation around a point, reflection in a plane, and scaling can all be performed by the multiplication and addition of matrices. These procedures can be generalized to more abstract cases of n-dimensional space by increasing the size of the matrices involved.

IV FURTHER MATRIX NOTATION

The transpose, $\mathbf{A}^t$, of matrix **A** is formed by interchanging its rows and columns, that is $(\mathbf{A}^t)_{ij} = a_{ji}$ for all i, j. A square matrix is orthogonal if $\mathbf{A}^t\mathbf{A} = \mathbf{I}$.

The adjoint, **A***, of a matrix **A** is formed by reversing the sign of any imaginary numbers in the elements of $\mathbf{A}^t$ (this is known as making the complex conjugate). A matrix is unitary if **A*A** = **I**. Unitary matrices are important in physics, specifically quantum theory, as they support conservation laws.