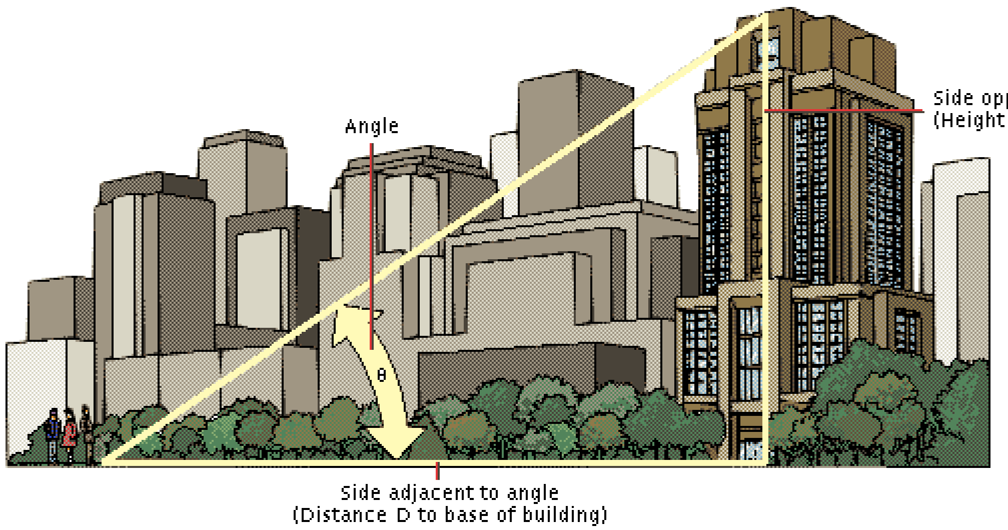
Trigonometry

INTRODUCTION

Trigonometry, branch of mathematics that deals with the relationships between the sides and angles of triangles and with the properties and applications of the trigonometric functions of angles. The two branches of trigonometry are *plane trigonometry,* which deals with figures lying wholly in a single plane, and *spherical trigonometry,* which deals with triangles that are sections of the surface of a sphere.



Using Trigonometry to Find the Height of a Building

To estimate the height, H, of a building, measure the distance, D, from the point of observation to the base of the building and the angle, θ (theta), shown in the diagram. The ratio of the height H to the distance D is equal to the trigonometric function tangent θ ($H/D = \tan\theta$). To calculate H, multiply tangent θ by the distance D ($H = D \tan\theta$). The angle can be roughly estimated by pointing one arm at the base of the building and the other arm at the roof and judging whether the angle formed is close to 15°, 30°, 45°, 60°, or 75°. The angle can be estimated more accurately with a protractor and a plumb bob made of a pencil hanging from a string. Hang the plumb bob from the zero point in the middle of the straight edge of the protractor. Sight along the edge of the protractor at the roof of the building. Measure the angle formed by the straight edge of the protractor and the plumb bob. Subtract this angle from 90°.

The earliest applications of trigonometry were in the fields of navigation, surveying, and astronomy, in which the main problem generally was to determine an inaccessible distance, such as the distance between the Earth and the Moon, or of a distance that could not be measured directly, such as the distance across a large lake. Other applications of trigonometry are found in physics, chemistry, and almost all branches of engineering, particularly in the study of periodic phenomena, such as sound waves or the flow of alternating current.

II PLANE TRIGONOMETRY

The concept of the trigonometric angle is basic to the study of trigonometry. A trigonometric angle is generated by a rotating ray. The rays *OA* and *OB* (figures 1a, 1b, and 1c) are considered originally coincident at *OA,* which is called the initial side. The ray *OB* then rotates to a final position called the terminal side. An angle and its measure are considered positive if they are generated by anticlockwise rotation in the plane, and negative if they are generated by clockwise rotation. Two trigonometric angles are equal if their rotations are in the same direction and of the same magnitude.

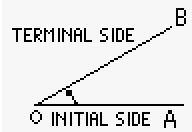


FIGURE 1A

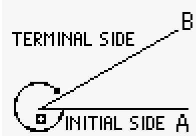


FIGURE 1B

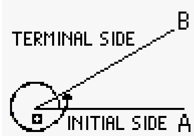


FIGURE 1C

An angular unit of measure is usually defined as an angle with a vertex at the centre of a circle and with sides that subtend, or cut off, a certain part of the circumference, shown as *s* in figure 2.

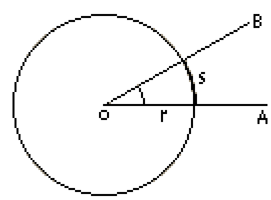


Figure 2

If the subtended arc *s* (*AB*) is equal to one-fourth of the total circumference *C,* that is, $s = \frac{1}{4}C$, so that *OA* is perpendicular to *OB,* the angular unit is a *right angle.* If $s = \frac{1}{2}C$, so that the points *A, O,* and *B* are on a straight line, the angular unit is a *straight angle.* If $s = 1/360C$, the angular unit is one degree. If $s = \frac{1}{2\pi}C$, so that the subtended arc is equal to the radius of the circle, the angular unit is a radian. By equating the various values of *C,* it follows that

$$1 \text{ straight angle} = 2 \text{ right angles} = 180 \text{ degrees} = \pi \text{ radians}$$

Each degree is subdivided into 60 equal parts called *minutes,* and each minute is subdivided into 60 equal parts called *seconds.* For finer measurements, decimal parts of a second may be used. Radian measurements smaller than a radian are expressed in decimals. The symbol for degree is °; for minutes, ‘; and for seconds, ". For radian measures either the abbreviation rad or no symbol at all may be used. Thus

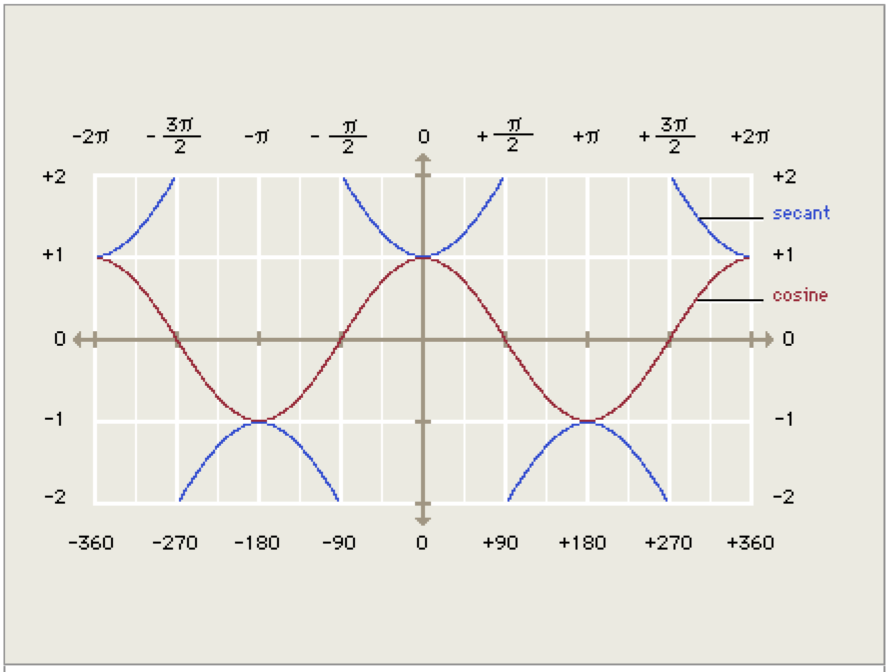
$$61°28'42.14" = 1.073 \text{ rad} = 1.073$$

The angular unit radian is understood in the last entry. (The notation 42".14 may be used instead of 42.14" to indicate decimal parts of seconds.)

By convention, a trigonometric angle is labelled with the Greek letter theta (θ). If the angle θ is given in radians, then the formula $s = r\theta$ may be used to find the length of the arc s; if θ is given in degrees, then

$$s = \frac{\pi r}{180}\theta$$

A Trigonometric Functions



Secant and Cosine Curves

The graph of the cosine of an angle against the value of the angle is identical to the graph of the sine, except that it is shifted. For example, cos 180° = sin 270° = –1. The secant is the reciprocal of the cosine—that is, 1 divided by the cosine.

Trigonometric functions are unitless values that vary with the size of an angle. An angle placed in a rectangular coordinate plane is said to be in standard position if its vertex coincides with the origin and its initial side coincides with the positive x-axis.

In figure 3, the point P lies on a line that starts at the origin at an angle θ from the positive x-axis. Each of the coordinates x and y may be positive or negative, depending on the quadrant (I, II, III, or IV) in which the point P lies; x may be zero, if P is on the y-axis, or y may be zero, if P is on the x-axis. The distance r is necessarily positive and is equal to $\sqrt{(x^2 + y^2)}$, in accordance with Pythagoras' theorem.

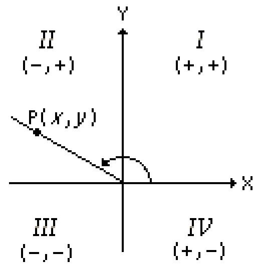


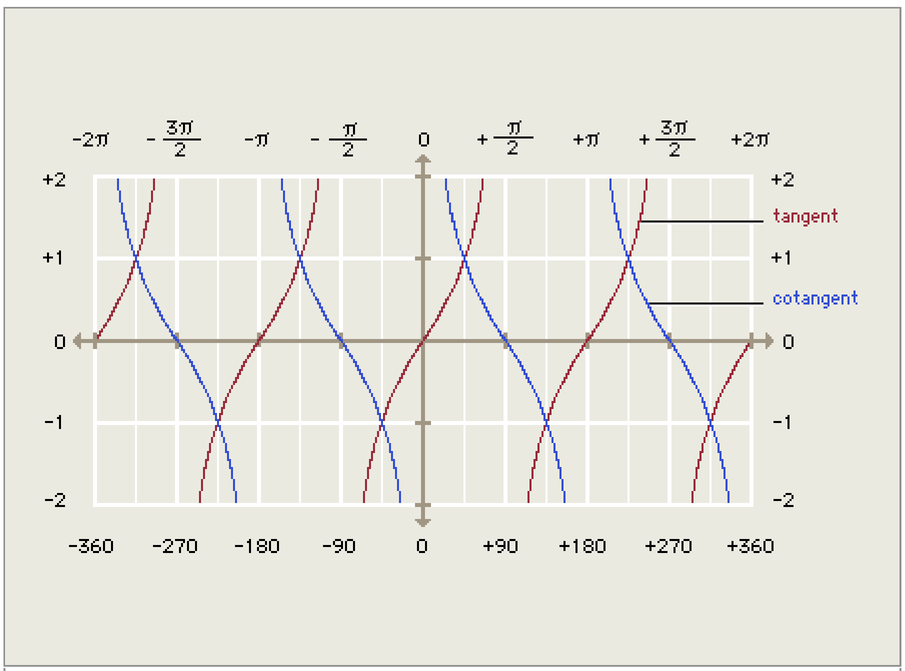
Figure 3

The six commonly used trigonometric functions are defined as follows:

sine (sin) of angle θ	$= \sin\theta = \frac{y}{r}$
cosine (cos) of angle θ	$= \cos\theta = \frac{x}{r}$
tangent (tan) of angle θ	$= \tan\theta = \frac{y}{x}$
cotangent (cot) of angle θ	$= \cot\theta = \frac{x}{y}$
secant (sec) of angle θ	$= \sec\theta = \frac{r}{x}$
cosecant (csc) of angle θ	$= \csc\theta = \frac{r}{y}$

Since x and y do not change if 2π radians are added to the angle—that is, 360° are added—it is clear that $\sin(\theta + 2\pi) = \sin\theta$. Similar statements hold for the five other functions. By definition, three of these functions are reciprocals of the three others, that is,

$$\cot\theta = \frac{1}{\tan\theta};\ \sec\theta = \frac{1}{\cos\theta};\ \csc\theta = \frac{1}{\sin\theta}$$



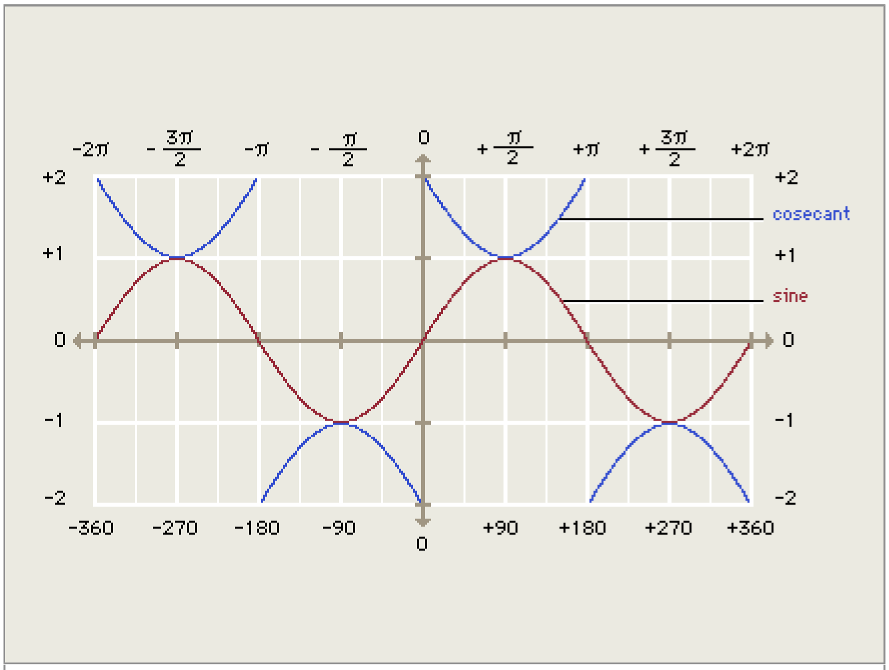
Tangent and Cotangent Curves

The tangent of an angle increases from 0 when the angle is 0° to infinity when the angle is +90° (+π/2 radians). Beyond this angle, the tangent has a large negative, apporaching 0 again at +180° (+π radians). The cotangent of an angle is the reciprocal of the tangent—that is, 1 divided by the tangent.

If point *P,* in the definition of the general trigonometric function, is on the *y*-axis, *x* is 0; therefore, because division by zero is inadmissible in mathematics, the tangent and secant of such angles as 90°, 270°, and -270° are not defined. If *P* is on the *x*-axis, *y* is 0; in this case, the cotangent and cosecant of such angles as 0°, 180°, and -180° are not defined. All angles have sines and cosines, because *r* is never equal to 0.

Since *r* is greater than or equal to *x* or *y,* the values of sin θ and cos θ range from -1 to +1; tan θ and cot θ are unlimited, assuming any real value; sec θ and cosec θ may be either equal to or greater than 1, or equal to or less than -1.

It is readily shown that the value of a trigonometric function of an angle does not depend on the length of *r* because the ratios depend on only the angle.



Cosecant and Sine Curves

A sine curve is the graph of the sine of an angle against the angle itself. The value of the sine of the angle is here plotted vertically, and ranges up to a maximum of 1 and down to a minimum of –1. The cosecant of an angle is the reciprocal of the sine—that is, 1 divided by the sine. It is always either greater than 1 or less than –1.

If θ is one of the acute angles of a right triangle, the definitions of the trigonometric functions given above can be applied to θ as follows. Imagine the vertex *A* is placed at the intersection of the *x*-axis and *y*-axis in figure 3, that *AC* extends along the positive *x*-axis, and that *B* is the point *P,* so that *AB* = *AP* = *r.* Then sin θ = *y*/*r* = *a*/*c,* and so on, as follows:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}$$
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}$$
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b}$$
$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{b}{a}$$
$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{c}{b}$$
$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{c}{a}$$

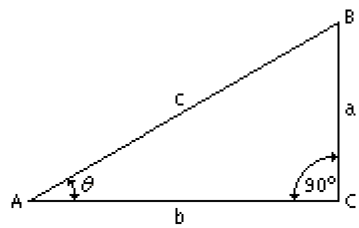


Figure 4

The numerical values of the trigonometric functions of a few angles can be readily obtained. For example, in an isosceles right triangle we know that θ = 45° and $b = a$. We also know that $c^2 = b^2 + a^2$, from Pythagoras' theorem, so we can deduce that $c^2 = 2a^2$ or $c = a\sqrt{2}$. Therefore:

$$\sin 45° = \cos 45° = \frac{1}{\sqrt{2}};\ \tan 45° = \cot 45°$$
$$= 1;\ \text{and } \sec 45° = \csc 45° = \sqrt{2}$$

The numerical values of the trigonometric functions of any angle can be determined approximately by drawing the angle in standard position with a ruler, compass, and protractor; by measuring *x, y,* and *r;* and then by calculating the appropriate ratios. Actually, it is necessary to calculate the values of sin θ and cos θ for only a few selected angles, because the values for other angles and for the other functions may be found by using one or more of the trigonometric identities that are listed below.

B Trigonometric Identities

The following formulas, called identities, which show the relationships between the trigonometric functions, hold for all values of the angle θ, or of two angles, θ and φ, for which the functions involved are:

I. $\sin\theta \csc\theta = \cos\theta \sec\theta = \tan\theta \cot\theta = 1$

II. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}, \cot\theta = \dfrac{\cos\theta}{\sin\theta}$

III. $\sin^2\theta + \cos^2\theta = \sec^2\theta - \tan^2\theta = \csc^2\theta - \cot^2\theta = 1$

IV. $\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$

$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$

$\tan(\theta \pm \phi) = \dfrac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}$

V. $\sin\theta = -\sin(-\theta), \cos\theta = \cos(-\theta)$

$\sin\theta = -\sin(\theta - 180°) = \cos(\theta - 90°)$

$\cos\theta = -\cos(\theta - 180°) = -\sin(\theta - 90°)$

By repeated use of one or more of the formulas in group V, which are known as reduction formulas, sin θ and cos θ can be expressed for any value of θ, in terms of the sine and cosine of angles between 0° and 90°. By use of the formulas in groups I and II, the values of tan θ, cot θ, sec θ, and cosec θ may be found from the values of sin θ and cos θ. It is therefore sufficient to tabulate the values of sin θ and cos θ for values of θ between 0° and 90°; in practice, to avoid tedious calculations, the values of the other four functions are also made available in tabulations for the same range of θ.

The variation of the values of the trigonometric functions for different angles may be represented by graphs, as in the accompanying figures. It is readily ascertained from these curves that each of the trigonometric functions is *periodic,* that is, the value of each is repeated at regular intervals called periods. The period of all the functions, except the tangent and the cotangent, is 360°, or 2 π radians. Tangent and cotangent have a period of 180°, or π radians.

C Inverse Functions

The statement *y* is the sine of θ, or *y* = sin θ is equivalent to the statement θ is an angle, the sine of which is equal to *y,* written symbolically as θ = arc sin *y* or, alternatively, as θ = $\sin^{-1} y$. The inverse functions, arc cos *y,* arc tan *y,* arc cot *y,* arc sec *y,* arc cosec *y,* are similarly defined. In the statement *y* = sin θ, or θ = arc sin *y,* a given value of θ will determine infinitely many values of *y.* Remember that sin 30° = sin 150° = sin (30° + 360°) = sin (150° + 360°). . .= $\frac{1}{2}$; therefore, if θ = arc sin $\frac{1}{2}$, then θ = 30° + *n*360° and θ = 150° + *n*360°, in which *n* is any integer, positive, negative, or zero. The value 30° is designated the basic or *principal value* of arc sin $\frac{1}{2}$. When used in this sense, the term *arc* generally is written with a capital *A.* Although custom is not uniform, the principal value of Arc sin *y,* Arc cos *y,* Arc tan *y,* Arc cot *y,* Arc sec *y,* or Arc cosec *y* commonly is defined to be the angle between 0° and 90° if *y* is positive; and, if *y* is negative, by the inequalities

$$-90° \leq \text{Arc sin } y, \text{Arc tan } y < 0°$$
$$90° < \text{Arc cos } y, \text{Arc cot } y \leq 180°$$
$$-180° \leq \text{Arc sec } y, \text{Arc csc} \leq -90°$$

D The General Triangle

Practical applications of trigonometry often involve determining distances that cannot be measured directly. Such a problem may be solved by making the required distance one side of a triangle, measuring other sides or angles of the triangle, and then applying the formulas below.

If *A, B, C* are the three angles of a triangle, and *a, b, c* the respective opposite sides, it may be proved that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \text{ (Law of Sines)}$$

$$a^2 = b^2 + c^2 - 2bc \cos A \text{ (Law of Cosines)}$$

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)} \text{ (Law of Tangents)}$$

The cosine and tangent laws can each be given two other forms by rotating the letters *a, b, c* and *A, B, C.*

These three relationships can be used to solve any triangle, that is, the unknown sides or angles can be found when given either: one side and two angles; two sides and the included angle; two sides and an angle opposite one of them (usually there are two triangles in this case); or all three sides.

III SPHERICAL TRIGONOMETRY

Spherical trigonometry, which is used principally in navigation and astronomy, is concerned with spherical triangles, that is, figures that are arcs of great circles on the surface of a sphere. The spherical triangle, like the plane triangle, has six elements, the three sides *a, b, c* and the angles *A, B, C.* But the three sides of the spherical triangle are angular as well as linear magnitudes, being arcs of great circles on the surface of a sphere and measured by the angle subtended at the centre. The triangle is completely determined when any three of its six elements are given, since relations exist between the various parts by means of which unknown elements may be found.

Spherical trigonometry is of great importance in the theory of stereographic projection and in geodesy. It is also the basis of the chief calculations of astronomy; for example, the solution of the so-called astronomical triangle is involved in finding the latitude and longitude of a place, the time of day, the position of a star, and various other data.

IV HISTORY

The history of trigonometry goes back to the earliest recorded mathematics in Egypt and Babylon. The Babylonians established the measurement of angles in degrees, minutes, and seconds. Not until the time of the Greeks, however, did any considerable amount of trigonometry exist. In the 2nd century BC the astronomer Hipparchus compiled a trigonometric table for solving triangles. Starting with $7\frac{1}{2}°$ and going up to 180° by steps of $7\frac{1}{2}°$, the table gave for each angle the length of the chord subtending that angle in a circle of a fixed radius *r.* Such a table is equivalent to a sine table. The value that Hipparchus used for *r* is not certain, but 300 years later the astronomer Ptolemy used *r* = 60 because the Hellenistic Greeks had adopted the Babylonian base-60 (sexagesimal) system of numbers.

In his great astronomical handbook, *The Almagest,* Ptolemy provided a table of chords for steps of $\frac{1}{2}°$, from 0° to 180°, that is accurate to 1/3600 of a unit. He also explained his method for constructing his table of chords, and in the course of the book he gave many examples of how to use the table to find unknown parts of triangles from known parts. Ptolemy provided what is now known as Menelaus's theorem for solving spherical triangles, and for several centuries his trigonometry was the primary introduction to the subject for any astronomer. At perhaps the same time as Ptolemy, Indian astronomers had developed a trigonometric system based on the sine function rather than the chord function of the Greeks. This sine function, unlike the modern one, was not a ratio but simply the length of the side opposite the angle in a right triangle of fixed hypotenuse. The Indians used various values for the hypotenuse.

Late in the 8th century, Muslim astronomers inherited both the Greek and the Indian traditions, but they seem to have preferred the sine function. By the end of the 10th century they had completed the sine and the five other functions and had discovered and proved several basic theorems of trigonometry for both plane and spherical triangles. Several mathematicians suggested using $r = 1$ instead of $r = 60$; this produces the modern values of the trigonometric functions. The Muslims also introduced the polar triangle for spherical triangles. All of these discoveries were applied both for astronomical purposes and as an aid in astronomical time-keeping and in finding the direction of Mecca for the five daily prayers required by Muslim law. Muslim scientists also produced tables of great precision. For example, their tables of the sine and tangent, constructed for steps of 1/60 of a degree, were accurate for better than one part in 700 million. Finally, the great astronomer Nasir ad-Din at- Tusi wrote the *Book of the Transversal Figure,* which was the first treatment of plane and spherical trigonometry as independent mathematical sciences.

The Latin West became acquainted with Muslim trigonometry through translations of Arabic astronomy handbooks, beginning in the 12th century. The first major Western work on the subject was written by the German astronomer and mathematician Johann Müller, known as Regiomontanus. In the next century the German astronomer Georges Joachim, known as Rheticus, introduced the modern conception of trigonometric functions as ratios instead of as the lengths of certain lines. The French mathematician François Viète introduced the polar triangle into spherical trigonometry, and stated the multiple-angle formulas for $\sin(n\theta)$ and $\cos(n\theta)$ in terms of the powers of $\sin(\theta)$ and $\cos(\theta)$.

Trigonometric calculations were greatly aided by the Scottish mathematician John Napier, who invented logarithms early in the 17th century. He also invented some mnemonic laws for solving spherical triangles, and some proportions (called Napier's analogies) for solving oblique spherical triangles.

Almost exactly one half century after Napier's publication of his logarithms, Isaac Newton invented the differential and integral calculus. One of the foundations of this work was Newton's representation of many functions as infinite series in the powers of x. Thus Newton found the series $\sin(x)$ and similar series for $\cos(x)$ and $\tan(x)$. With the invention of calculus, the trigonometric functions were taken over into analysis, where they still play important roles in both pure and applied mathematics.

Finally, in the 18th century the Swiss mathematician Leonhard Euler defined the trigonometric functions in terms of complex numbers. This made the whole subject of trigonometry just one of the many applications of complex numbers, and showed that the basic laws of trigonometry were simply consequences of the arithmetic of these numbers.