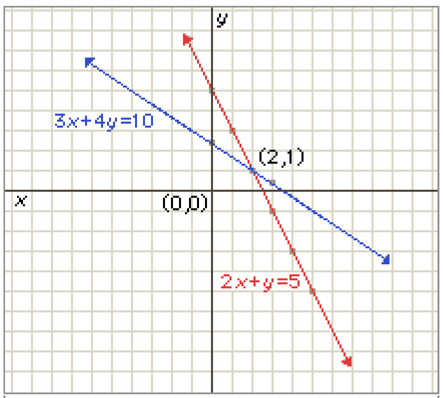
Algebra

I INTRODUCTION

Algebra, branch of mathematics in which symbols represent relationships. Classical algebra grew out of methods of solving equations; it represents numbers with symbols that combine according to the basic arithmetical operations of addition, subtraction, multiplication, division, and the extraction of roots. However, arithmetic cannot generalize mathematical relations such as Pythagoras' theorem, which states that in any right-angled triangle, the area of the square drawn on the hypotenuse is equal to the sum of the areas of the squares drawn on the other two sides. Arithmetic can produce only specific instances of these relations (for example, 3, 4, and 5, where $3^2 + 4^2 = 5^2$). Algebra, by contrast, can make a purely general statement that fulfils the conditions of the theorem: $a^2 + b^2 = c^2$. Any number multiplied by itself is termed squared, and is indicated by a superscript number 2. For example, 3×3 is notated 3^2; similarly, $a \times a$ is equivalent to a^2.



Solving and Graphing Simultaneous Equations

When solving more than one equation at a time, we are interested in finding the set of all solutions that will satisfy both equations. An easy way to find this set of solutions for the linear equations $3x + 4y = 10$ and $2x + y = 5$ is to express one variable in terms of the other. In this case, the variable y is isolated in the second equation, which may be rewritten $y = 5 - 2x$. If we substitute this new expression of y into the first equation, we get: $3x + 4(\mathbf{5 - 2x}) = 10$. Now there is only one variable and the equation may be solved. The solution, $x = 2$, may then be substituted into both equations, which yields a value of $y = 1$. Thus, the set of solutions that satisfies both equations is (2,1). Graphically, any values that satisfy both equations will result in an intersection of the lines (see graph).

Modern algebra has evolved from classical algebra by increasing its attention to the structures within mathematics. Mathematicians consider modern algebra to be a set of objects with rules for connecting or relating them. As such, in its most general form, algebra may fairly be described as the language of mathematics.

II HISTORY

The history of algebra began in ancient Egypt and Babylonia, where people learned to solve linear ($ax = b$) and quadratic ($ax^2 + bx = c$) equations, as well as indeterminate equations such as $x^2 + y^2 = z^2$, whereby several unknowns are involved. The ancient Babylonians solved arbitrary quadratic equations by essentially the same procedures taught today. They could also solve some indeterminate equations.

The Alexandrian mathematicians Hero of Alexandria and Diophantus continued the traditions of Egypt and Babylon, but Diophantus' book *Arithmetica* is on a much higher level and gives many surprising solutions to difficult indeterminate equations. This ancient knowledge of solutions of equations in turn found a home early in the Islamic world, where it was known as the “science of restoration and balancing”. (The Arabic word for restoration, *al-jabru,* is the root of the word “algebra”.) In the 9th century, the Arab mathematician al-Khwarizmi wrote one of the first Arabic algebras, a systematic exposé of the basic theory of equations, with both examples and proofs. By the end of the 9th century, the Egyptian mathematician Abu Kamil had stated and proved the basic laws and identities of algebra and solved such complicated problems as finding x, y, and z such that $x + y + z = 10$, $x^2 + y^2 = z^2$, and $xz = y^2$.

Ancient civilizations wrote out algebraic expressions using only occasional abbreviations, but by medieval times Islamic mathematicians were able to talk about arbitrarily high powers of the unknown x, and, without yet using modern symbolism, work out the basic algebra of polynomials. This included the ability to multiply, divide, and find square roots of polynomials as well as a knowledge of the binomial theorem. The Persian mathematician, astronomer, and poet Omar Khayyam showed how to express roots of cubic equations by means of line segments obtained by intersecting conic sections, but he could not find a formula for the roots. A Latin translation of Al-Khwarizmi's *Algebra* appeared in the 12th century. In the early 13th century, the Italian mathematician Leonardo Fibonacci achieved a close approximation to the solution of the cubic equation $x^3 + 2x^2 + cx = d$. Because Fibonacci had travelled in Islamic countries, he probably used an Arabic method of successive approximations.

Early in the 16th century, the Italian mathematicians Scipione del Ferro, Niccolò Tartaglia, and Gerolamo Cardano solved the general cubic equation in terms of the constants appearing in the equation. Tartaglia and Cardano's pupil, Ludovico Ferrari, soon found an exact solution to equations of the fourth degree, and as a result, mathematicians for the next several centuries tried to find a formula for the roots of equations of degree five, or higher. Early in the 19th century, however, the Norwegian mathematician Niels Abel and the French mathematician Évariste Galois proved that no such formula exists.

An important development in algebra in the 16th century was the introduction of symbols for the unknown and for algebraic powers and operations. As a result of this development, Book III of *La Géometrie* (1637), written by the French philosopher and mathematician René Descartes, looks much like a modern algebra text. Descartes's most significant contribution to mathematics, however, was his discovery of analytic geometry, which reduces the solution of geometric problems to the solution of algebraic ones. His geometry text also contained the essentials of a course on the theory of equations, including his so-called rule of signs for counting the number of what Descartes called the “true” (positive) and “false” (negative) roots of an equation. Work continued through the 18th century on the theory of equations, and in 1799 the German mathematician Carl Friedrich Gauss published the proof showing that every polynomial equation has at least one root in the complex plane (*see* Number: *Complex Numbers*).

By the time of Gauss, algebra had entered its modern phase. Attention shifted from solving polynomial equations to studying the structure of abstract mathematical systems whose axioms were based on the behaviour of mathematical objects, such as complex numbers, that mathematicians encountered when studying polynomial equations. Two examples of such systems are groups and quaternions, which share some of the properties of number systems but also depart from them in important ways. Groups began as systems of permutations and combinations of roots of polynomials, but went on to become one of the chief unifying concepts of 19th-century mathematics. Important contributions to their study were made by the French mathematicians Galois and Augustin Cauchy, the British mathematician Arthur Cayley, and the Norwegian mathematicians Niels Abel and Sophus Lie. Quaternions were discovered by British mathematician and astronomer William Rowan Hamilton, who extended the arithmetic of complex numbers to quaternions; while complex numbers are of the form $a + bi$, quaternions are of the form $a + bi + cj + dk$.

Immediately after Hamilton's discovery, the German mathematician Hermann Grassmann began investigating vectors. Despite its abstract character, American physicist J. W. Gibbs recognized in vector algebra a system of great utility for physicists, just as Hamilton had recognized the usefulness of quaternions. The widespread influence of this abstract approach led George Boole to write *The Laws of Thought* (1854), an algebraic treatment of basic logic. Since that time, modern algebra—also called abstract algebra—has continued to develop. Important new results have been discovered, and the subject has found applications in all branches of mathematics and in many of the sciences as well.

III SYMBOLS AND SPECIAL TERMS

The symbols of algebra include numbers, letters, and signs that indicate various arithmetic operations. Numbers are, of course, constants, but letters can represent either constants or variables. Letters that are used to represent constants are generally taken from the beginning of the alphabet; those used to represent variables are taken from the end.

A Operations and the Grouping of Symbols

The grouping of algebraic symbols and the sequence of arithmetic operations rely on grouping symbols to ensure that the language of algebra is clearly read. Grouping symbols include parentheses ( ), brackets [ ], braces { }, and horizontal bars—also called vincula—that are used most often for division and roots, as in the following:

$$\frac{ax+b}{c-dy} \qquad \sqrt{b^2-4ac}$$

The basic operational signs of algebra are familiar from arithmetic: addition (+), subtraction (-), multiplication (×), and division (÷). In the case of multiplication, the “×” is often omitted or replaced by a dot, as in $a \cdot b$. A group of consecutive symbols, such as *abc,* indicates the product of *a, b,* and *c.* Division is commonly indicated by bars, as in the preceding example. A virgule, or slash (/), is also used to separate the numerator, above the line, from the denominator, below the line, of a fraction, but care must be taken to group the terms appropriately. For example, $ax + b/c - dy$ indicates that ax and dy are separate terms, as is b/c, whereas $(ax + b)/(c - dy)$ correctly represents the fraction

$$\frac{ax+b}{c-dy}$$

B Order of Operations

Multiplications are performed first, then divisions, followed by additions, and then subtractions. Grouping symbols indicate the order in which operations are to be performed—that is, carry out all operations within a group first, beginning with the innermost group. For example

$$\left\{2\left[3+(6\cdot 5+2)\right]\right\} = \left\{2\left[3+(30+2)\right]\right\} =$$
$$\left\{2\left[3+(32)\right]\right\} = \left\{2\left[35\right]\right\} = 70$$

C Special Definitions

Any statement involving the equality relation (=) is called an equation. An equation is called an “identity” if the equality is true for all values of its variables: thus $(x + y)^2 = x^2 + 2xy + z^2$ is an identity. If the equation is true for some values of its variables and false for others, the equation is conditional. A term is any algebraic expression consisting only of products of constants and variables; $2x$, $-a$, $\frac{1}{4}s^4x$, and $x^2(2zy)^3$ are all examples of terms. The numerical part of a term is called its coefficient. The coefficients of each term above are, respectively, 2, -1, $\frac{1}{4}$, and 8 (the last term may be rewritten $8x^2(zy)^3$).

An expression containing one term is called a monomial; two terms, a binomial; and three terms, a trinomial. A polynomial is any finite sum (or difference) of terms. For example, a general polynomial of degree n might be expressed as

$$a_0 + a_1x^1 + a_2x^2 + \ldots + a_nx^n$$

In this context, “degree” refers to the largest exponent of the variables in a polynomial. For example, if the largest exponent of a variable is 3, as in $ax^3 + bx^2 + cx$, the polynomial is said to be of degree three. Similarly, the expression $x^n + x^{n-1} + x^{n-2}$ is of degree n.

A linear equation in one variable is a polynomial equation of degree one—that is, of the form $ax + b = 0$. These are called “linear equations” because they represent the equation of straight lines in analytic geometry.

A quadratic equation in one variable is a polynomial equation of degree two—that is, of the form $ax^2 + bx + c = 0$.

A prime number is any integer (whole number) that can be evenly divided only by itself and by the number 1. Thus, 2, 3, 5, 7, 11, and 13 are all prime numbers.

Powers of a number are formed by successively multiplying the number by itself. The term *a* raised to the third power, for example, can be expressed as $a \cdot a \cdot a$ or a^3.

The prime factors of any number are those factors to which it can be reduced such that the number is expressed only as the product of primes and their powers. For example, the prime factors of 15 are 3 and 5. Similarly, because $60 = 2^2 \times 3 \times 5$, the prime factors of 60 are 2, 3, and 5.

IV OPERATIONS WITH POLYNOMIALS

In operating with polynomials, the assumption is that the usual laws of the arithmetic of numbers hold. In arithmetic, the numbers used are the set of real numbers. These consist of the rational numbers and the irrational numbers. Rational numbers are those that can be represented as the ratio of two integers; they include such fractions as $\frac{1}{2}$, $\frac{3}{4}$, and so on, as well as the integers themselves (including 0), and the negatives of all these. The irrational numbers are those that cannot be so represented; they include such numbers as $\sqrt{2}$, which require an infinite sequence of digits to be written out as a decimal. These too include negative as well as positive values. Arithmetic alone cannot go beyond the real numbers, but algebra and geometry can include complex numbers.

A Laws of Addition

A1. The sum of any two real numbers *a* and *b* is again a real number, denoted $a + b$. The real numbers are closed under the operations of addition, subtraction, multiplication, division, and the extraction of roots; this means that applying any of these operations to real numbers yields a quantity that also is a real number.

A2. No matter how terms are grouped in carrying out additions, the sum will always be the same: $(a + b) + c = a + (b + c)$. This is called the associative law of addition.

A3. Given any real number *a,* there is a real number zero (0) called the additive identity, such that $a + 0 = 0 + a = a$.

A4. Given any real number *a,* there is a number $(-a)$, called the additive inverse of *a,* such that $(a) + (-a) = 0$.

A5. No matter in what order addition is carried out, the sum will always be the same: $a + b = b + a$. This is called the commutative law of addition.

Any set of numbers obeying laws A1 to A4 is said to form a group. If the set also obeys A5, it is said to be an Abelian, or commutative, group.

B Laws of Multiplication

Laws similar to those for addition also apply to multiplication. Special attention should be given to the multiplicative identity and inverse, M3 and M4.

M1. The product of any two real numbers *a* and *b* is again a real number, denoted $a \cdot b$ or *ab.*

M2. No matter how terms are grouped in carrying out multiplications, the product will always be the same: $(ab)c = a(bc)$. This is called the associative law of multiplication.

M3. Given any real number *a,* there is a number one (1) called the multiplicative identity, such that $a(1) = 1(a) = a$.

M4. Given any nonzero real number *a,* there is a number (a^{-1}), or $(1/a)$, called the multiplicative inverse, such that $a(a^{-1}) = (a^{-1})a = 1$.

M5. No matter in what order multiplication is carried out, the product will always be the same: $ab = ba$. This is called the commutative law of multiplication.

Any set of elements obeying these five laws is said to be an Abelian, or commutative, group under multiplication. The set of all real numbers, excluding zero—because division by zero is inadmissible—forms such a commutative group under multiplication.

C Distributive Laws

Another important property of the set of real numbers links addition and multiplication in two distributive laws, as follows:

D1. $a(b + c) = ab + ac$

D2. $(b + c)a = ba + ca$

Any set of elements with an equality relation, for which two operations (such as addition and multiplication) are defined, that obeys all the laws for addition A1-A5, the laws for multiplication M1-M5, and the distributive laws D1 and D2, constitutes a field.

V MULTIPLICATION OF POLYNOMIALS

The following is a simple example of the product of a binomial and a monomial:

$$(ax + b)(cx^2) = acx^3 + bcx^2$$

This same principle—multiplying each term of the one polynomial by each term of the other—is directly extended to polynomials of any number of terms. For example, the product of a binomial and a trinomial is carried out as follows:

$$(ax^3 + bx^2 - cx)(dx + e) = adx^4 + aex^3 + bdx^3 + bex^2 - cdx^2 - cex$$

After such operations have been performed, all terms of the same degree should be combined whenever possible to simplify the entire expression:

$$= adx^4 + (ae + bd)x^3 + (be - cd)x^2 - cex$$

VI FACTORING POLYNOMIALS

Given a complicated algebraic expression, it is often useful to factor it into the product of several simpler terms. For example, $2x^3 + 8x^2y$ can be factored, or rewritten, as $2x^2(x + 4y)$. Determining the factors of a given polynomial may be a simple matter of inspection or may require trial and error. Not all polynomials, however, can be factored using real-number coefficients, and these are called prime polynomials.

Some common factorizations are given in the following examples

Trinomials of the form:

$$x^2 + 2xy + y^2 = (x + y)(x + y) = (x + y)^2$$
$$x^2 - 2xy + y^2 = (x - y)(x - y) = (x - y)^2$$
$$25x^2 + 20xy + 4y^2 = (5x + 2y)^2$$

The difference of two squares

$$x^2 - y^2 = (x + y)(x - y)$$
$$25x^2 - 16y^2 = (5x + 4y)(5x - 4y)$$

Trinomials of the form:

$$x^2 + (a + b)x + ab = (x + a)(x + b)$$
$$x^2 + 7x + 10 = (x + 5)(x + 2)$$

Sums and differences of cubes:

$$x^3 \pm y^3 = (x \pm y)(x^2 \mp xy + y^2)$$
$$x^3 + 8y^3 = (x + 2y)(x^2 - 2xy + 4y^2)$$

Grouping may often be useful in factoring; terms that are similar are grouped wherever possible, as in the following example:

$$\begin{aligned} 2x^2z + x^2y - 6xz - 3xy &= x^2(2z + y) - 3x(2z + y) \\ &= (x^2 - 3x)(2z + y) \\ &= x(x - 3)(2z + y) \end{aligned}$$

VII HIGHEST COMMON FACTORS

Given a polynomial, it is frequently important to isolate the greatest common factor from each term of the polynomial. For example, in the expression $9x^3 + 18x^2$, the number 9 is a factor of both terms, as is x^2. After factoring, $9x^2(x + 2)$ is obtained, and $9x^2$ is the greatest common factor for all terms of the original polynomial (in this case a binomial). Similarly, for the trinomial $6a^2x^3 + 9abx + 15cx^2$, the number 3 is the largest numerical factor common to 6, 9, and 15, and x is the largest variable factor common to all three terms. Thus, the greatest common factor of the trinomial is $3x$.

VIII LOWEST COMMON MULTIPLES

Finding lowest common multiples is useful in combining algebraic fractions. The procedure is analogous to that used to combine ordinary fractions in arithmetic. To combine two or more fractions, the denominators must be the same; the most direct way to produce common denominators is simply to multiply all the denominators together. For example

$$\frac{a}{b}+\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{d}+\frac{c}{d}\cdot\frac{b}{b}=\frac{ad+bc}{bd}$$

However, bd may not be the lowest common denominator. For example

$$\frac{2}{3}+\frac{1}{6}=\frac{2\cdot 6}{3\cdot 6}+\frac{1\cdot 3}{6\cdot 3}=\frac{12+3}{18}=\frac{15}{18}$$

However, 18 is only one possible common denominator; the lowest common denominator is 6:

$$\frac{2}{3}\cdot\frac{2}{2}+\frac{1}{6}=\frac{4}{6}+\frac{1}{6}=\frac{5}{6}$$

In algebra, the problem of finding lowest common multiples of denominators is similar. Given several algebraic expressions, the lowest common multiple is the expression of lowest degree and least coefficient that can be divided exactly (without remainder) by each of the expressions. Thus, to find a common multiple of the terms $2x^2y$, $30x^2y^2$, and $9ay^3$, all three expressions could simply be multiplied together, and it would be easy to show that $(2x^2y\ (30x^2y^2\ (9ay^3)$ is exactly divisible by each of the three terms; however, this would not be the lowest common multiple. To determine which is the lowest, each of the terms is reduced to its prime factors. For the numerical coefficients 2, 30, and 9, the prime factors are 2, 2·3·5, and 3·3, respectively; the lowest common multiple for the numerical coefficients must therefore be 2·3·3·5, or 90, as this is the product of the minimal set of prime factors required by all three numbers. Similarly, because the constant a appears only once, it too must be a factor. Of the variables, x^2 and y^3 are required, so that the lowest common multiple of the three terms is $90ax^2y^3$. Each term will exactly divide into this expression.

IX SOLUTION OF EQUATIONS

Given an equation, algebra proceeds to supply solutions based on the general idea of the identity $a = a$. As long as the same arithmetic or algebraic procedure is applied simultaneously to both sides of the equation, the equality remains unaffected. The basic strategy is to isolate the unknown term on one side of the equation and the solution on the other. For example, to solve the linear equation in one unknown

$$5x + 6 = 3x + 12$$

the variable terms are isolated on one side and the constant terms on the other. The term $3x$ can be removed from the right side by subtracting; $3x$ must then be subtracted from the left side as well:

$$\begin{array}{rcr} 5x+6 & = & 3x+12 \\ -3x & & -3x \\ \hline 2x+6 & = & 12 \end{array}$$

The number 6 is then subtracted from both sides:

$$\begin{array}{rcr} 2x + 6 = & & 12 \\ -\ 6 & & -6 \\ \hline 2x & = & 6 \end{array}$$

To isolate *x* on the left side, both sides of the equation are divided by 2:

$$\frac{2x}{2} = \frac{6}{2}$$

and the solution then follows directly: $x = 3$. This can easily be verified by substituting the solution value $x = 3$ back into the original equation:

$$\begin{aligned} 5x + 6 &= 3x + 12 \\ 5(3) + 6 &= 3(3) + 12 \\ 15 + 6 &= 9 + 12 \\ 21 &= 21 \end{aligned}$$

A Solution of Quadratic Equations

Given any quadratic equation of the general form

$$ax^2 + bx + c = 0$$

a number of approaches are possible depending on the specific nature of the equation in question. If the equation can be factored, then the solution is straightforward. For instance:

$$x^2 - 3x = 10$$

First the equation is put into the standard form

$$x^2 - 3x - 10 = 0$$

which can be factored as follows:

$$(x - 5)(x + 2) = 0$$

This condition can be met only when either of the individual factors is zero—that is, when $x = 5$ or $x = -2$. That these are the solutions to the equation may again be verified by substitution.

X METHOD OF COMPLETING THE SQUARE

If, on inspection, no obvious means of factoring the equation directly can be found, an alternative might exist. For example, in the equation

$$4x^2 + 12x = 7$$

the expression $4x^2 + 12x$ could be factored as a perfect square if it were $4x^2 + 12x + 9$, which equals $(2x + 3)^2$. This can easily be achieved by adding 9 to the left side of the equation. The same amount must then, of course, be added to the right side:

$$4x^2 + 12x + 9 = 7 + 9$$
$$(2x + 3)^2 = 16$$

This can be reduced to

$$(2x + 3) = \sqrt{16}$$

That is:

$$2x + 3 = +4$$

or

$$2x + 3 = -4$$

because 16 has the two square roots +4 and -4. The first equation leads to the solution $x = \frac{1}{2}$ (because $2x + 3 = 4$, $2x = 1$ [subtracting 3 from both sides], and $x = \frac{1}{2}$ [dividing both sides by 2]). The second equation leads to the solution $x = -7/2$, or $x = -3\frac{1}{2}$. Both solutions can be verified as before, by substituting the values in question back into the equation.

A The Quadratic Formula

Any quadratic equation of the form

$$ax^2 + bx + c = 0$$

can be solved using the quadratic formula. In all cases the two solutions of x are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For example, to find the roots of

$$x^2 - 4x = -3$$

the equation is first put into the standard form

$$x^2 - 4x + 3 = 0$$

As a result, $a = 1$, $b = -4$, and $c = 3$. These terms are then substituted into the quadratic formula

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(3)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16 - 12}}{2}$$

$$= \frac{4 \pm \sqrt{4}}{2} = \frac{4 \pm 2}{2} = 3 \text{ and } 1$$

B Solution of Two Simultaneous Equations

Algebra frequently has to solve not just a single equation but several at the same time. The problem is to find the set of all solutions that simultaneously satisfies all equations. The equations to be solved are called simultaneous equations, and specific algebraic techniques can be used to solve them. For example, given the two linear equations in two unknowns

$$3x + 4y = 10 \qquad (1)$$

$$2x + y = 5 \qquad (2)$$

a simple solution exists: the variable y in equation (2) is isolated ($y = 5 - 2x$), and then this value of y is substituted into equation (1)

$$3x + 4(5 - 2x) = 10$$

This reduces the problem to one involving the single linear unknown x, and it follows that

$$3x + 20 - 8x = 10$$

or

$$-5x = -10$$

so that

$$x = 2$$

When this value is substituted into either equation (1) or (2), it follows that

$$y = 1$$

The solution is obtained more quickly by observing that, if both sides of equation (2) are multiplied by 4, then

$$3x + 4y = 10 \qquad (1)$$
$$8x + 4y = 20 \qquad (2)$$

If equation (1) is subtracted from equation (2), then $5x = 10$, or $x = 2$, as before.

A matrix is a compact way of expressing simultaneous equations, and the above methods lead on to matrix theory and the solution of linear equations in any number of unknowns.