Complex Number

Complex Number, an expression of the form $a + bi$ where a and b are real numbers and i is the square root of minus one. They can be added, subtracted, multiplied and divided and have the algebraic structure of what is called a field in mathematics. In engineering and physics, complex numbers are used extensively to describe electric circuits and electromagnetic waves. The number i appears explicitly in the famous Schrödinger wave equation which is fundamental to the quantum theory of the atom. Complex analysis, which combines complex numbers with ideas from calculus, has been widely applied to subjects as different as the theory of numbers and the design of aeroplane wings.

Historically, complex numbers arose in the search for solutions to equations such as $x^2 = -1$. There is no real number x whose square is -1 and early mathematicians would have said this equation has no solution. However, by the mid-16th century Cardano and his contemporaries were experimenting with solutions of equations that involved the square roots of negative numbers. Cardano suggested that the real number 40 could be expressed as $(5 + \sqrt{-15})(5- \sqrt{-15})$. Euler introduced the modern symbol i for $\sqrt{-1}$ in 1777 and wrote down his famous relationship $e^{\pi i} = -1$ which connects four of the fundamental numbers of mathematics. For his doctoral dissertation in 1799, Gauss proved his famous Fundamental Theorem of Algebra, which says that every polynomial with complex coefficients has a complex root. The study of complex functions was continued by Cauchy who in 1825 generalized the real definite integral of calculus to functions of a complex variable.

For a complex number $a + bi$, a is called the real part and b is called the imaginary part. The complex number $-2 + 3i$ has real part -2 and imaginary part 3. Addition of complex numbers is performed by adding the real and imaginary parts separately. To add $1 + 4i$ and $2 - 2i$, add the real parts 1 and 2 and then the imaginary parts 4 and -2 to obtain the complex number $3 + 2i$. The general rule for addition is $(a + bi) + (c + di) = (a + c) + (b + d)i$.

Multiplication of complex numbers is based on $i \cdot i = -1$ and the assumption that multiplication distributes over addition. This gives the rule $(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$ from which it follows that $(1 + 4i)(2 - 2i) = 10 + 6i$. If $z = a + bi$ is any complex number, then the *complex conjugate* of z is $z^* = a - bi$ and the absolute value or modulus of z is $|z| = \sqrt{(a^2 + b^2)}$. For example, the complex conjugate of $1 + 4i$ is $1 - 4i$ and the modulus of $1 + 4i$ is $\sqrt{(1^2 + 4^2)} = \sqrt{17}$. A basic relationship connecting absolute value and complex conjugate is $z \cdot z^* = |z|^2$.

In the same way that real numbers can be thought of as points on a line, complex numbers can be thought of as points on a plane. The number $a + bi$ is identified with the point in a plane with x coordinate a and y coordinate b. The points $1 + 4i$ and $2 - 2i$ are plotted in figure 1 and correspond to the points (1,4) and (2,-2). One of the first people to think of complex numbers geometrically as points in the plane was Argand in 1806 and for this reason, figure 1 is sometimes referred to as an Argand diagram. If a complex number in the plane is thought of as a vector joining the origin to that point, then addition of complex numbers corresponds to standard vector addition. Figure 1 shows the complex number $3 + 2i$ obtained as the sum of the vectors $1 + 4i$ and $2 - 2i$.

Since points in the plane can be written in terms of the polar coordinates r and θ, every complex number z can be written in the form $z = r(\cos\theta + i\sin\theta)$. Here, r is the modulus or distance to the origin and θ is the argument of z or the angle z makes with the x axis. If $z = r(\cos\theta + i\sin\theta)$ and $w = s(\cos\varnothing + i\sin\varnothing)$ are two complex numbers in polar form, then their product in polar form is given by $zw = rs(\cos(\theta + \varnothing) + i\sin(\theta + \varnothing))$. This has a simple geometric interpretation that is illustrated in figure 2.

There are many polynomial equations that have no real solutions, such as $x^2 - 2x + 2 = 0$. However, if x is allowed to be complex, $x^2 + 1 = 0$ has the solutions $x = \pm i$ and $x^2 - 2x + 2 = 0$ has the solutions $x = 1 \pm i$. Gauss's impressive achievement was to show that every nontrivial polynomial with complex coefficients has at least one complex root. From this, it follows that every complex polynomial of degree n must have exactly n roots, not necessarily distinct. Consequently, every complex polynomial of degree n can be written as a product of exactly n linear factors.