Engineering Mathematics - III

Second Year Engineering

Semester – IV

Computer Engineering/IT

According to New Revised Credit System Syllabus of Savitribai Phule Pune University, Pune.

EFFECTIVE FROM ACADEMIC YEAR JUNE 2016

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Igniting Minds

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Second Year Engineering (Semester – IV)

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First Edition : 2019

Published By : Gigatech Publishing House

631/32, Budhwar Peth, Office No. 105, First Floor,
Shan Bramha Complex, Pune – 411 002.
Phone No. 7757042853

ISBN : 978-93-88480-10-9

Price : ₹ 450/-

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Preface

This Book Engineering Mathematics-III is intended to be a textbook for students of Second Year Engineering In most sciences, one generation years down what another has built and what one has established another undoes. In Engineering Mathematics-III, each generation adds a new story to the old structure. Keeping this in mind, this book is written to have a better introduction of the Engineering Mathematics-III. This book is presented with simple but exact explanation of subject matter, application of each topic to real life, engineering problems, large number of illustrative examples followed by well graded exercise. We have tried to be rigorous and precise in presenting the concepts in very simple manner. We hope that the students will not only learn some powerful concepts, but also will develop their ability to understand the concept and apply it properly to solve engineering problems. We feel that faculty member will also enjoy reading this book which is enriched with application of each topic.

Acknowledgment

The authors acknowledge the help of colleagues and friends for the warm relationship which provides a source of energy in our endeavours.

We are grateful to our family members for the encouragement and constant cooperation and assistance in creation of this book.

We are pleased to acknowledge the encouragement from Management and other authorities during the write up of this book.

We are certainly thankful to the students of engineering who are a constant source of our enthusiasm and encouragement in our endeavours.

We are also thankful to **Gigatech Publishing House TEAM** for their continuous support, hard work and patience in preparing this book.

SYLLABUS

Unit - I : Linear Differential Equations (LDE) and Applications (09 Hrs.)

LDE of nth order with constant coefficients, Method of variation of parameters, Cauchy's & Legendre's DE, Simultaneous & Symmetric simultaneous DE. Modelling of Electrical circuits.

Unit – II : Transforms (09 Hrs.)

Fourier Transform (FT): Complex exponential form of Fourier series, Fourier integral theorem, Fourier Sine & Cosine integrals, Fourier transform, Fourier Sine and Cosine transforms and their inverses.

Z - Transform (ZT) : Introduction, Definition, Standard properties, ZT of standard sequences and their inverses. Solution of difference equations.

Unit – III : Statistics (09 Hrs.)

Measures of central tendency, Standard deviation, Coefficient of variation, Moments, Skewness and Kurtosis, Curve fitting of straight line, parabola and related curves, Correlation and Regression, Reliability of Regression Estimates.

Unit – IV : Probability and Probability Distributions (09 Hrs.)

Probability, Theorems on Probability, Bayes Theorem, Random variables, Mathematical Expectation, Probability density function, Probability distributions: Binomial, Poisson, Normal and Hypergometric, Test of Hypothesis: Chi-Square test, t-distribution.

Unit – V : Vector Calculus (09 Hrs.)

Vector differention, Gradient, Divergence and Curl, Directional derivative, Solenoid and Irrigational fields. Vector identities. Line, Surace and Volume integrals, Green's Lemma, Gauss's Divergence theorem and Stoke's theorem.

Unit - VI : Complex Variables (09 Hrs.)

Functions of Complex variables, Analytic functions, Cauchy-Riemann equations, Conformal mapping, Bilinear transformation, Cauchy's integral theorem, Cauchy's integral formula, Laurent's series and Residue theorem.

Recommended by SPPU Text Books and Reference Books

Text Books :

1. Erwin Kreyszig, “Advanced Engineering Mathematics”, 9e, (Wiley India).
2. Peter V. O'Neil, “Advanced Engineering Mathematics”, 7e, (Cengage Learning).

Reference Books :

1. M. D. Greenberg, “Advanced Engineering Mathematics”, 2e, Pearson Education.
2. Wylie C.R. & Barrett L.C. “Advanced Engineering Mathematics”, McGraw-Hill, Inc.
3. B. S. Grewal, “Higher Engineering Mathematics”, Khanna Publication, Delhi.
4. P. N. Wartikar & J. N. Wartikar, “Applied Mathematics (Volumes I and II)”, Pune Vidyarthi Griha Prakashan, Pune.
5. B.V. Ramana, “Higher Engineering Mathematics”, Tata McGraw-Hill.
6. Thomas L. Harman, James Dabney and Norman Richert, “Advanced Engineering Mathematics with MATLAB”, 2e, Brooks/Cole, Thomson Learning.

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LIST OF REQUIRED FORMULAE

1. Natural numbers = N = {1, 2, 3, 4, ….. }
2. Integers = I = {0, ± 1, ± 2, ± 3, ± 4, ….. }
3. Rational numbers = Q = $\left\{ \frac{p}{q} \text{ , } q \neq 0, p, q \in I \right\}$
4. Irrational numbers = I_r = $\left\{ \sqrt[n]{a} \text{ , where a is a rational number} \right\}$
5. Real numbers = {Q} ∪ {I_r}
6. $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$
7. Any equation of n^{th} degree always possesses n roots.
8. Nature of roots of algebraic equations
   i) Roots may be real and distinct
   ii) Roots may be real and repeated
   iii) Roots may be Imaginary and distinct
   iv) Roots may be Imaginary and repeated
9. Roots of quadratic equation $ax^2 + bx + c = 0$

   are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
10. $a^2 - b^2 = (a + b)(a - b)$
11. $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
12. $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
13. $a^2 + b^2 = (a + ib)(a - ib)$
14. $x^3 = 0 \Rightarrow x = 0, 0, 0$
15. $x^3 - ax^2 = 0 \Rightarrow x^2 (x - a) = 0 \Rightarrow x = 0, 0, a$
16. $(a + b)^2 = a^2 + 2ab + b^2$
17. $(a - b)^2 = a^2 - 2ab + b^2$
18. $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
19. $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$
20. $\frac{1}{1 + p} = (1 + p)^{-1} = 1 - p + p^2 - p^3 + --$
21. $\frac{1}{1 - p} = (1 - p)^{-1} = 1 + p + p^2 + p^3 + p^4 + + +$
22. Differentiation : where $D = \frac{d}{dx}$
   i) D (cosx) = – sinx
   ii) D (sinx) = cosx

iii) $D(\tan x) = \sec^2 x$

iv) $D(\cot x) = -\operatorname{cosec}^2 x$

v) $D(\sec x) = \sec x \tan x$

vi) $D(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

vii) $D(\tan^{-1} x) = \dfrac{1}{1 + x^2}$

viii) $D(\cot^{-1} x) = -\dfrac{1}{1 + x^2}$

ix) $D(\log x) = \dfrac{1}{x}$

x) $d(e^{ax}) = a\, e^{ax}$

xi) $D(uv) = udv + vdu$

xii) $D\left(\dfrac{u}{v}\right) = \dfrac{vdu - udv}{v^2}$

xiii) $D(a^x) = a^x \log a$

xiv) $D(\cosh x) = \sinh x$

xv) $D(\sinh x) = \cosh x$

xvi) $D[f(x)] = f'(x)\, dx$, where $D = \dfrac{d}{dx}$

23. Integration :

i) $\int \cosh x\, dx = \sinh x + C$

ii) $\int \sinh x\, dx = \cosh x + C$

iii) $\int e^{mx} dx = \dfrac{e^{mx}}{m} + C$

iv) $\int \sin x\, dx = -\cos x + C$

v) $\int \cos x\, dx = \sin x + C$

vi) $\int \sec x\, dx = \log(\sec x + \tan x) + C$

vii) $\int \tan x\, dx = \log \sec x + C$

viii) $\int \operatorname{cosec} x\, dx = \log(\operatorname{cosec} x - \cot x) + C$

ix) $\int \cot x\, dx = \log(\sin x) + C$

x) $\int \sec x \tan x\, dx = \sec x + C$

xi) $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + C$

xii) $\int \log x\, dx = (x \log x - x) + C$

xiii) $\int uv\, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots$ where "dash" denotes the derivative and "suffix" denotes the order of integration.

xiv) $\int e^{ax} \cos(bx + c)\, dx = \dfrac{e^{ax}}{a^2 + b^2}[a \cos(bx + c) + b \sin(bx + c)]$

xv) $\int e^{ax} \sin(bx + c)\, dx = \dfrac{e^{ax}}{a^2 + b^2} [a \sin (bx + c) - b \cos (bx + c)]$

xvi) $\int \operatorname{cosec}^2 x\, dx = - \cot x + C$

xvii) $\int \sec^2 x\, dx = \tan x + C$

xviii) $\dfrac{1}{D} f(x) = \int f(x)\, dx + C$

xix) $\int \dfrac{f'(x)}{f(x)}\, dx = \log f(x) + C$

xx) $\int [f(x)]^n f'(x)\, dx = \dfrac{[f(x)]^{n+1}}{n+1} + C$

xxi) $\int uv\, dx = u \left[\int v\, dx\right] - \int \left[(du) \int v\, dx\right] dx$

24. Hyperbolic functions / trigonometric functions

i) $\cosh x = \dfrac{e^x + e^{-x}}{2}$

ii) $\sinh x = \dfrac{e^x - e^{-x}}{2}$

iii) $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$

iv) $\sin x = \dfrac{1}{2i} (e^{ix} - e^{-ix})$

v) $\cosh (0) = 1$

vi) $\sinh (0) = 0$

vii) $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

viii) $\tan x = \dfrac{[e^{ix} - e^{-ix}]}{i\,[e^{ix} + e^{-ix}]}$

ix) $\cosh ax = \dfrac{e^{ax} + e^{-ax}}{2}$

x) $\sinh ax = \dfrac{e^{ax} - e^{-ax}}{2}$

xi) $\tanh ax = \dfrac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}$

25. Binomial expansion :

$(a + b)^n = a^n + n\, c_1\, a^{n-1} b + n\, c_2\, a^{n-2} b^2 + n\, c_3\, a^{n-3} b^3 + + + b^n$

26. Even and odd functions :

i) Even function if $f(-x) = f(x) \quad \forall x$

ii) odd function if $f(-x) = -f(x) \quad \forall x$

iii) (odd function) (odd function) = even function

iv) (even function) (even function) = even function

v) (odd function (even function) = odd function

vi) $f(x) = \sin x, \sin^3 x, \sin^5 x, \ldots$.. odd functions

vii) $f(x) = \cos x, \cos^2 x, \cos^3 x, \cos^4 x, \ldots$.. even functions

viii) $f(x) = \sin^2 x, \sin^4 x, \sin^6 x, \ldots$.. even functions

ix) $\int_{-a}^{a} f(x)\, dx = 0$, $f(x)$ = odd function

x) $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$, f(x) = even function

27. Series :

i) $\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \ldots.$

ii) $\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \ldots.$

iii) $\sinh x = x + \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \ldots.$

iv) $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots.$

v) $e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + +$

vi) $e^{-x} = 1 - \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} - \frac{x^3}{\lfloor 3} + + + \ldots.$

vii) $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + -$

viii) $\log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} -$

ix) Geometric series, $a + ar + ar^2 + ar^3 + + + ar^{n-1} + + \infty = \frac{a}{1-r}$, $|r| < 1$

28. Modulus :

i) If x is a real number, $|x| = x, \quad x \geq 0$

$= -x, x < 0$

ii) $|x| < a \Rightarrow -a < x < a$

iii) $|x| > a \Rightarrow -\infty < x < -a$ and $a < x < \infty$

29. Trigonometry :

i) $\sin 2x = 2\sin x \cos x$

ii) $\cos^2 x = \frac{1 + \cos 2x}{2}$, $\sin^2 x = \frac{1 - \cos 2x}{2}$

iii) $2\cos A \cos B = \cos(A + B) + \cos(A - B)$

iv) $2\sin A \sin B = \cos(A - B) - \cos(A + B)$

v) $2\sin A \cos B = \sin(A + B) + \sin(A - B)$

vi) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

vii) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

viii) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

ix) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

x) $\sin(0) = 0$, $\cos(0) = 1$, $\tan(0) = 0$

xi) $\sin(-x) = -\sin x$, $\tan(-x) = -\tan x$

xii) $\cos(-x) = \cos x$

xiii) $\sin^2\theta + \cos^2\theta = 1$

xiv) $\sec^2\theta = 1 + \tan^2\theta$

xv) $\mathrm{cosec}^2\theta = 1 + \cot^2\theta$

30. Geometry :

i) $y = 0$, X – axis

ii) $x = 0$, Y – axis

iii) $x = a$, a line parallel to $x = 0$

iv) $y = b$, a line parallel to $y = 0$

where a, b are some constants.

v) $y = mx$ is a equation of line through (0,0)

vi) A line through (x_1,y_1) and (x_2, y_2) is

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

vii) $x^2 + y^2 = a^2$ is a circle whose centre (0, 0) and radius a units

viii) $(x - h)^2 + (y - k)^2 = r^2$, is a circle with centre (h, k) and radius r units

ix) $y^2 = 4ax$, $y^2 = -4ax$, $x^2 = 4ay$, $x^2 = -4ay$ represents parabolas through (0,0)

x) $ax + by + c = 0$, represents a general line cutting x, y axes at $\left(-\frac{c}{a}, 0\right)$ and $\left(0, -\frac{c}{b}\right)$

xi) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, a ellipse

31. Complex numbers :

i) $|z| = 1 \Rightarrow x^2 + y^2 = 1$, a circle with centre (0,0) rad = 1

ii) $|z + 1| = 1 \Rightarrow (x + 1)^2 + (y - 0)^2 = 1^2$, a circle with centre (–1,0), rad = 1

iii) $|z + i| = 1 \Rightarrow (x - 0)^2 + (y + 1)^2 = 1^2$, a circle with centre (0, –1), rad = 1

iv) $z = x + iy$, cartesion form

v) $|z| = \sqrt{x^2 + y^2}$

vi) $\bar{z} = x - iy$

vii) $z = re^{i\theta}$, polar form

viii) $e^{i\theta} = \cos\theta + i \sin\theta$

ix) $e^{-i\theta} = \cos\theta - i \sin\theta$

x) $|e^{i\theta}| = 1$

xi) $|e^{-i\theta}| = 1$

xii) $|z| < a$, is an interior of circle $|z| = a$

xiii) $|z| \geq a$, is an exterior of circle $|z| = a$

32. Proper and Partial Fractions :

i) If in a fraction $\frac{P}{Q}$, degree of Q > degree of P, then $\frac{P}{Q}$ is called proper fraction, otherwise improper fractions.

ii) Proper fraction can be resolved into partial fractions as below

a) $\frac{P}{(x-m)(x+n)} = \frac{A}{x-m} + \frac{B}{x+n}$

b) $\frac{P}{(x-m)^2 (x+n)(x+r)^3} = \frac{A}{x-m} + \frac{B}{(x-m)^2} + \frac{C}{x+n} + \frac{D}{x+r} + \frac{E}{(x+r)^2} + \frac{F}{(x+r)^3}$

c) $\frac{P}{(x+n)(x^2+1)} = \frac{A}{x+n} + \frac{Bx+C}{x^2+1}$

33. Factorial : for $n \in N$, set of natural numbers

i) $\lfloor n = n! = n(n-1)(n-2) \ldots\ldots 3 \cdot 2 \cdot 1 = 1 \cdot 2 \cdot 3 \cdot 4 \ldots (n-2)(n-1)n$

ii) $0! = 1$

iii) $n\,c_1 = 1$

iv) $n\,c_2 = \frac{n(n-1)}{2!}$

v) $n\,c_3 = \frac{n(n-1)(n-2)}{3!}$

vi) $n\,c_r = \frac{n!}{r!\,(n-r)!}$

34. Three Dimensional Geometry :

i) $ax + by + cz = d$, represents a plane

ii) $y = 0, z = 0, x = 0$ represents xz, xy, yz planes respectively.

iii) $y = a, z = a, x = a$ represents planes parallel to xz, xy, yz respectively.

iv) $\frac{x - x_1}{a} = \frac{y - y1}{b} = \frac{z - z_1}{c}$ represents a line through (x_1, y_1, z_1) and whose directional ratios are a, b, c

v) The directional ratios and a line through (x_1, y_1, z_1) and (x_2, y_2, z_2) are $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$.

vi) $\overline{ox}, \overline{oy}, \overline{oz}$ are vectors in positive direction and $ox', oy'\ oz'$ are negative vectors of ox, oy, oz respectively.

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CONTENTS

Multiple Choice Questions (MCQ's)

Solved University Question Papers

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Unit
1

Linear Differential Equations (LDE) and Applications

Syllabus :

LDE of n^{th} order with constant coefficients, Method of variation of parameters, Cauchy's and Legendre's DE, Simultaneous and Symmetric simultaneous DE. Modeling of Electrical circuits.

1.1 Linear Differential Equations :

- **Definition :**

Linear differential equations are those in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together. Thus the general linear differential equation of the n^{th} order is of the form :

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + P_n y = x,$$

Where $P_1, P_2, \ldots P_n$ and $\times$ are functions of x only.

Linear differential equations with constant co–efficient are of the form :

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_n y = X \qquad \ldots(1)$$

Where $a_1, a_2, \ldots a_n$ are constants. Such equations are most important in the study of electro–mechanical vibrations and other engineering applications.

The general solution of a differential equation

(1) is $\qquad y = y_c + y_p = \text{C.F.} + \text{P. I.}$

In order to solve the equation (1) we have to first find complimentary function (c.f.) and then the particular Integral(P. I.)

- **Operator D :**

Put $\frac{d}{dx}$ = D, then the equation (1) can be written in the symbolic form

$$(D^n + a_1D^{n-1} + \ldots + a_n)\, y = x. \text{ i.e . } f(D)\, y = X. \qquad \ldots(2)$$

Where $f(D) = D^n + a_1D^{n-1} + a_2D^{n-2} + .. + a_n$.

The symbol D stands for the operation of differentiation and can be treated much the same as an algebraic quantity i.e. f(D) can be factorized by ordinary rules of algebra and the factors may be taken in any order.

- **Rules for finding the complimentary function :**

To solve the equation $\frac{d^ny}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_n y = 0$ …(3)

Where $a_1, a_2 \ldots a_n$ are constants.

The equation (3) in symbolic form is

$$(D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots + a_n)\, y = 0 \Rightarrow f(D)y = 0 \qquad \ldots(4)$$

Its symbolic co–efficient equated to zero. f(D) = 0

i.e. $D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots + a_n = 0$ is called the Auxiliary Equation (A.E.) let $m_1, m_2 \ldots m_n$ be its roots.

Case : I

If all the roots be real and distinct then eg.(4) is equivalent to

$$(D - m_1)\,(D - m_2) \ldots (D - m_n)\, y = 0 \qquad \ldots(5)$$

Now (5) will be satisfied by the solution of

$$(D - m_n)\, y = 0 \text{ i.e. } \frac{dy}{dx} - m_ny = 0, \text{ I. F.} = e^{-m_nx}$$

∴ its solution is $y.\, e^{-m_nx} = c_n$ i.e $y = c_ne^{m_nx}$.

Similarly since the factors in (5) can be taken in any order it will be satisfied by the solutions of

$$(D - m_1)\, y = 0, (D - m_2)\, y = 0 \text{ etc.}$$

i.e. $y = c_1e^{m_1x}$ $y = c_2e^{m_2x}$ ..etc.

Thus the complete solution of (3) is

$$y = c_1e^{m_1x} + c_2e^{m_2x} + \ldots + c_ne^{m_nx}.$$

Case : II

The roots are real and equal

Let $m_1 = m_2$ be two real and equal roots then

$f(D)y = 0$ is $(D - m_1)\,(D - m_1)\, y = 0$

Put $(D - m_1)y = u$ …(i)

$\therefore \quad (D - m_1)\, u = o,\ u = c_1 e^{m_1 x}.$

$$\frac{dy}{dx} - m_1 y = c_1 e^{m_1 x}.$$

$$\text{I.F.} = e^{\int -m_1 dx} = e^{-m_1 x}.$$

its solution is $\quad y.e^{-m_1 x} = \int c_1 e^{m_1 x}.\, e^{-m_1 x} dx + c_2$

$\therefore \quad ye^{-m_1 x} = c_1 x + c_2 \quad$ or $\quad y = (c_1 x + c_2) e^{m_1 x}.$

or $\quad y = (c_1 + c_2 x) e^{m_1 x}.$

if three roots are equal i.e. $m_1 = m_2 = m_3$

Then $\quad y = (c_1 + c_2 x + c_3 x^2) e^{m_1 x}$

in general i. e if 'n' roots are equal $m_1 = m_2 = m_3 = \ldots = m_n$ then its solution is $\quad y = (c_1 + c_2 x + c_3 x^2 + \ldots + c_n x^{n-1})\, e^{m_1 x}.$

Case : III

- **The roots are imaginary and distinct :**

  Let $\quad m_1 = \alpha + i\beta$

  and $\quad m_2 = \alpha - i\,\beta$ be two imaginary

And distinct roots then $(D - m_1)(D - m_2) y = o$

$\therefore \quad y = c_1 e^{m_1 x} + c_2 e^{-m_2 x} = c_1 e^{\alpha + i\beta} + c_2 e^{\alpha - i\beta}$

$$y = c_1 e^{\alpha x} e^{i\beta x} + c_2 e^{\alpha x} e^{-i\beta x}$$

$$y = e^{\alpha x}\,[c_1 (\cos \beta x + i \sin \beta x) + c_2 (\cos \beta x - i \sin \beta x)]$$

$$y = e^{\alpha x}\,[c_1 + c_2) \cos \beta x + (ic_1 - ic_2) \sin \beta x]$$

$$y = e^{\alpha x}\,[A \cos \beta x + B \sin \beta x]$$

where $\quad A = c_1 + c_2 \quad$ and $\quad B = i\,(c_1 - c_2).$

Similarly $\quad m_1 = \alpha_1 + i\beta_1,\ m_2 = \alpha_1 - i\beta_1,\quad m_3 = \alpha_2 + i\beta_2$

$m_4 = \alpha_2 - i\beta_2$, are imaginary and distinct roots. Its solution is

$$y = e^{\alpha_1 x}(c_1 \cos \beta_1 x + c_2 \sin \beta_1 x) + e^{\propto_2 x}(c_3 \cos \beta_2 x + c_4 \sin \beta_2 x)$$

Case : IV

- **The roots are imaginary and equal :**

  Let $\quad m_1 = m_2 = \alpha + i\beta$

  and $\quad m_3 = m_4 = \alpha - i\beta$

  be four imaginary and equal roots then its solution is

$$y = e^{\alpha x}\,[(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x].$$

Illustrative Examples

Example : 1

Solve $2\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} - 12y = 0$

Solution :

Put $\dfrac{d}{dx} = D$ the above equation.

Becomes $(2D^2 + 5D - 12)y = o$

The auxiliary equation is $2m^2 + 5m - 12 = 0$

$$\therefore \quad 2m^2 + 8m - 3m - 12 = 0$$

$$2m(m + 4) - 3(m + 4) = 0$$

$$(2m - 3)(m + 4) = 0$$

$\therefore \quad m = \dfrac{3}{2}, -4$ are real and distinct roots, its solution is

$$y = c_1 e^{\frac{3}{2}x} + c_2 e^{-4x}.$$

Example : 2

Solve $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + y = 0$

Solution :

Put $\dfrac{d}{dx} = D$ then

$(D^2 - 4D + 1)y = 0.$ Here the auxiliary equation is

$$m^2 - 4m + 1 = 0 \quad \rightarrow \quad m = \frac{4 \pm \sqrt{16 - 4}}{2}$$

$\therefore \quad m = 2 \pm \sqrt{3}$ The roots are real and distinct its solution is

$$y = c_1 e^{(2+\sqrt{3})x} + c_2 e^{(2-\sqrt{3})x}.$$

Example : 3

Solve $\dfrac{d^2y}{dx^2} + (a + b)\dfrac{dy}{dx} + aby = 0$ (where a, b are real).

Solution :

Put $\dfrac{d}{dx} = D$ then the above D.E. becomes

$$(D^2 + (a + b)D + ab)y = 0.$$

The auxiliary equation is

$$m^2 + (a + b)\,m + ab = 0$$
$$\Rightarrow \quad m^2 + am + bm + ab = 0$$
$$m(m + a) + b(m + a) = 0$$
$$\Rightarrow \quad (m + a)(m + b) = 0$$

$\therefore$ $m = -a, -b$ the roots are real and distinct.

Its solution is $y = c_1 e^{-ax} + c_2 e^{-bx}$.

Example : 4

Solve $\dfrac{d^3y}{dx^3} + 10\dfrac{d^2y}{dx^2} + 31\dfrac{dy}{dx} + 30y = 0$

Solution :

Put $\dfrac{d}{dx} = D$ the above D.E. becomes

$$(D^3 + 10D^2 + 31D + 30)y = 0$$

The auxiliary equation is $(m^3 + 10m^2 + 31\,m + 30) = 0$

-2	1	10	31	30
		-2	-16	-30
-3	1	8	15	00
		-3	-15	
-5	1	5	00	
		-5		
	1	0		

Fig. 1.1

By synthetic division we find the roots of the above equation.

$\therefore$ $m = -2, -3, -5$ are real and distinct roots its solution is

$$y = c_1 e^{-2x} + c_2 e^{-3x} + c_3 e^{-5x}$$

Example : 5

Solve $\dfrac{d^3y}{dx^3} - 7\dfrac{dy}{dx} - 6y = 0$

Solution :

Put $\dfrac{d}{dx} = D$ the above D.E. becomes

$$(D^3 - 7D - 6)y = 0$$

Its auxiliary equation is

$$(m^3 - 7m - 6) = 0$$

-1	1	0	-7	-6
		-1	1	6
-2	1	-1	-6	0
		-2	6	
3	1	-3	0	
		3		
	1	0		

Fig. 1.2

By synthetic division we find the roots of this equation. The roots are

$m = -1, -2, 3$ real and distinct its solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$$

Example : 6

Solve $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - y = 0$

Solution :

Put $\dfrac{d}{dx} = D$ the above D.E. becomes

$$(D^2 - 2D + 1)y = 0$$

Its auxiliary equation is

$$m^2 - 2m + 1 = 0$$

The roots of this equation are m = 1, 1 real and equal, its solution is

$$y = (c_1 + c_2x)e^x$$

Example : 7

Solve $\dfrac{d^3y}{dx^3} + 6\dfrac{d^2y}{dx^2} + 9\dfrac{dy}{dx} = 0$

Solution :

Put $\dfrac{d}{dx} = D$ the given D.E. is

$$(D^3 + 6D^2 + 9D)y = 0$$

Its auxiliary equation is

$$(m^3 + 6m^2 + 9m) = 0.$$

The roots of this equation are m = 0, – 3, – 3 real,

two roots are equal one root is distinct, its solution is

$$y = c_1e^{ox} + (c_2 + c_3x)\, e^{-3x}$$

Example : 8

Solve $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 12y = 0$

Solution :

Put $\dfrac{d}{dx} = D$ the given D.E. is

$$(D^3 + D^2 - 2D + 12)y = 0$$

Its auxiliary equation is

$$m^3 + m^2 - 2m + 12 = 0$$

The factors of this equation are

$$(m + 3)(m^2 - 2m + 4) = 0$$

The roots are

$$m = -3,\ 1 \pm i\sqrt{3},$$

One root is real and two roots are imaginary and distinct.

$$m^2 - 2m + 4 = 0$$

$$a = 1,\ b = -2,\ c = 4$$

$$m = \frac{2 \pm \sqrt{4 - 16}}{2}$$

$$m = \frac{2 \pm \sqrt{-12}}{2}$$

$$m = 1 \pm i\sqrt{3}$$

Its solution is $y = c_1e^{-3x} + e^{x}(c_2\cos\sqrt{3}\,x + c_3\sin\sqrt{3}\,x)$

Example : 9

Solve $\frac{d^4y}{dx^4} + 13\frac{d^2y}{dx^2} + 36y = 0$

Solution :

Put $\frac{d}{dx} = D$ the given D.E. is

$$(D^4 + 13D^2 + 36)y = 0$$

Its auxiliary equation is

$$m^4 + 13m^2 + 36 = 0$$

The factors of

$$(m^2 + 4)(m^2 + 9) = 0$$

$$\Rightarrow m = \pm 2i, \pm 3i$$

The roots are imaginary and distinct.

Its solution is $y = c_1\cos 2x + c_2\sin 2x + c_3\cos 3x + c_4\sin 3x$.

Example : 10

Solve $\frac{d^4y}{dx^4} - a^4y = 0$ (a is real)

Solution :

Put $\frac{d}{dx} = D$ the given D.E. is

$$(D^4 - a^4)y = 0$$

Its auxiliary equation is

$$(m^4 - a^4) = 0$$

The factors are

$$(m^2 - a^2)(m^2 + a^2) = 0$$

The roots of this equation are

$$m = \pm a, \pm ai,$$

Two roots are real and two roots are imaginary and distinct.

Its solution is

$$y = c_1e^{ax} + c_2e^{-ax} + c_3\cos ax + c_4\sin ax$$

Example : 11

Solve $\dfrac{d^4y}{dx^4} + a^4y = 0$

Solution :

Put $\dfrac{d}{dx} = D$ the given D.E. is

$$(D^4 + a^4)y = 0$$

Its auxiliary equation is $m^4 + a^4 = 0$

This equation can be written as $(m^2)^2 + (a^2)^2 + 2m^2a^2 - 2m^2a^2 = 0$

For making perfect square add and subtract by $2m^2a^2$.

$\therefore \quad (m^2 + a^2)^2 - (\sqrt{2}ma)^2 = 0$ the factors are

$$(m^2 + a^2 - \sqrt{2}ma)\,(m^2 + a^2 + \sqrt{2}ma) = 0.$$

The roots are imaginary and distinct

$$m = \frac{\sqrt{2}a \pm \sqrt{2a^2 - 4a^2}}{2}, \qquad m = \frac{-\sqrt{2}a \pm \sqrt{2a^2 - 4a^2}}{2}$$

$$m = \frac{\sqrt{2}a \pm \sqrt{-2a^2}}{2}, \qquad m = \frac{-\sqrt{2}a \pm \sqrt{2a^2}}{2}$$

$$m = \frac{a}{\sqrt{2}} \pm i\frac{a}{\sqrt{2}}, \qquad m = -\frac{a}{\sqrt{2}} \pm i\frac{a}{\sqrt{2}}$$

The solution is

$$y = e^{\frac{a}{\sqrt{2}}x}\left(c_1\cos\frac{a}{\sqrt{2}}x + c_2\sin\frac{a}{\sqrt{2}}x\right) + e^{-\frac{a}{\sqrt{2}}x}\left(c_3\cos\frac{a}{\sqrt{2}}x + c_4\sin\frac{a}{\sqrt{2}}x\right)$$

Example : 12

Solve $\left(\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y\right)\left(\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y\right) = 0$

Solution :

Put $\dfrac{d}{dx} = D$ the given D.E. is

$$(D^2 - 2D + 2)^2y = 0$$

Its auxiliary equation is $(m^2 - 2m + 2)^2 = 0$

The roots are $m = 1 \pm i, 1 \pm i$ imaginary and equal

Its solution is

$$y = e^x[(c_1 + c_2x)\cos x + (c_3 + c_4x)\sin x].$$

Example : 13

Solve $\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} + y = 0$

Solution :

Put $\frac{d}{dx} = D$ the given D.E. is

$$(D^4 + D^2 + 1)y = 0$$

Its auxiliary equation is

$$m^4 + m^2 + 1 = 0$$

This equation can be written as

$$(m^2)^2 + m^2 + 1 + m^2 - m^2 = 0$$

$(m^2 + 1)^2 - m^2 = 0$ the factors are

$$(m^2 + 1 - m)(m^2 + 1 + m) = 0$$

The roots of this equation are imaginary and distinct.

$$m = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$m = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

The solution is

$$y = e^{-\frac{x}{2}}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right) + e^{\frac{x}{2}}\left(c_3\cos\frac{\sqrt{3}}{2}x + c_4\sin\frac{\sqrt{3}}{2}x\right)$$

Example : 14

Solve $\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0$

Solution :

Put $\frac{d}{dx} = D$ the given D.E. is

$$(D^4 + 8D^2 + 16)y = 0$$

Its auxiliary equation is

$$m^4 + 8m^2 + 16 = 0$$

$$\therefore \quad (m^2 + 4)^2 = 0 \quad \Rightarrow \quad m = \pm 2i, \pm 2i$$

The roots are imaginary and equal

Its solution is $y = [(c_1 + c_2x)\cos 2x + (c_3 + c_4x)\sin 2x]$

Example : 15

Solve $\dfrac{d^6y}{dx^6} + 6\dfrac{d^4y}{dx^4} + 9\dfrac{d^2y}{dx^2} = 0$

Solution :

Put $\dfrac{d}{dx} = D$ the given D.E. is

$$(D^6 + 6D^4 + 9D^2)\,y = 0$$

Its auxiliary equation is

$$m^6 + 6m^4 + 9m^2 = 0$$

$$m^2(m^4 + 6m^2 + 9) = 0$$

$$\therefore \quad m^2(m^2+3)^2 = 0$$

$$\therefore \quad m = 0,\ 0,\ \pm i\sqrt{3},\ \pm i\sqrt{3},$$

Two roots are real and equal and four roots are imaginary and equal.

Its solution is $y = (c_1 + c_2x) + (c_3 + c_4x)\cos\sqrt{3}\,x + (c_5 + c_6x)\sin\sqrt{3}\,x.$

1.2 Particular Integral (P.I.) :

- **Definition :**

$\dfrac{1}{f(D)}$ X is that function of x, not containing arbitrary constants which when operated upon by f(D) gives X. i.e. $f(D)\left\{\dfrac{1}{f(D)}X\right\} = X$

Thus $\dfrac{1}{f(D)}X$ satisfies the equation $f(D)y = X$ and is, therefore, it is a particular Integral.

[obviously f(D) and $\dfrac{1}{f(D)}$ are inverse operators.]

1. General method :

i) Prove that $\dfrac{1}{D}X = \int X\,dx.$

Let $\dfrac{1}{D}X = y$ operating by D we get

$D\dfrac{1}{D}X = Dy$ i.e. $X = \dfrac{dy}{dx}$ Integrating both sides

w.r.t. x $y = \int X\,dx$ no constant being added

Thus $\dfrac{1}{D}X = \int X\,dx.$

ii) Prove that $\dfrac{1}{D-a}X = e^{ax}\int X e^{-ax}\,dx.$

Let $\dfrac{1}{D-a}X = y$ operating by $(D-a)$ we get

$$(D-a)\frac{1}{(D-a)}X = (D-a)\,y \quad \text{i.e. } X = \frac{dy}{dx} - ay$$

$$\therefore \quad \frac{dy}{dx} - ay = x$$ this is Leibnitz's linear differential equation.

$$\text{I.f.} = e^{\int -a dx} = e^{-ax}$$ its solution is

$$y.\,e^{-ax} = \int X e^{-ax}\,dx$$ no constant being added.

$$\text{Thus} \quad \frac{1}{D-a}X = e^{ax}\int X e^{-ax}\,dx.$$

$$\text{Similarly} \quad \frac{1}{D+a}X = e^{-ax}\int X e^{ax}\,dx.$$

iii) Method of partial fractions :

$$\text{let} \quad \frac{1}{f(D)}X = \frac{1}{(D-a_1)(D-a_2)\ldots(D-a_n)}X$$

$$= \left[\frac{A_1}{D-a_1} + \frac{A_2}{D-a_2} + \ldots + \frac{A_n}{D-a_n}\right]X.$$

$$\therefore \quad \frac{1}{f(D)}X = A_1e^{a_1x}\int Xe^{-a_1x}dx + A_2e^{a_2x}\int Xe^{-a_2x}dx + \ldots$$

$$+ A_ne^{a_nx}\int X e^{-a_nx}\,dx$$

iv) let $$\frac{1}{f(D)}X = \frac{1}{(D-a_1)(D-a_2)}X = \frac{1}{D-a_1}\left\{\frac{1}{D-a_2}\right\}X$$

$$\therefore \quad \frac{1}{f(D)}X = \frac{1}{D-a_1}\{e^{a_2x}\int Xe^{-a_2x}dx\}$$

$$= \frac{1}{D-a_1}X_1 = e^{a_1x}\int X_1e^{-a_1x}dx.$$

Illustrative Examples

Example : 1

Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \sin(e^x)$

Solution :

Put $\frac{d}{dx} = D$ then the given D.E. is

$$(D^2 + 3D + 2)y = \sin(e^x)$$

The auxiliary equation is

$$m^2 + 3m + 2 = 0$$

The roots are $m = -1, -2$ real and distinct.

$$\text{C. F.} = c_1e^{-x} + c_2e^{-2x}.$$

$$\text{P.I.} = \frac{1}{f(D)} X = \frac{1}{(D^2+3D+2)}\sin(e^x) = \frac{1}{(D+2)(D+1)}\sin(e^x)$$

$$= \frac{1}{(D+2)}\left\{\frac{1}{D+1}\sin(e^x)\right\} = \frac{1}{D+2}\left\{e^{-x}\int \sin(e^x)\, e^x\, dx\right\}$$

Put $e^x = t,\ e^x\, dx = dt$

$$= \frac{1}{D+2}\{e^{-x}\int \sin t\, dt\} = \frac{1}{D+2}\{e^{-x}(-\cos t)\}$$

$$= -\frac{1}{D+2} \quad e^{-x}\cos(e^x) = e^{-2x}\int e^{-x}\cos(e^x)\, e^{2x}\, dx$$

$$= -e^{-2x}\int \cos(e^x)\, e^x\, dx = -e^{-2x}\int \cos t\, dt$$

Put $e^x = t$

$\therefore$ $e^x\, dx = dt$

$$\text{P.I.} = -e^{-2x}\sin t = -e^{-2x}\sin(e^x)$$

The general solution is $y = \text{C.F.} + \text{P.I.}$

$$\boxed{y = c_1e^{-x} + c_2e^{-2x} - e^{-2x}\sin(e^x)}$$

Example : 2

Solve $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 2\log x + \frac{1}{x} + \frac{1}{x^2}$

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^2 - D - 2)y = 2\log x + \frac{1}{x} + \frac{1}{x^2}$$

The auxiliary equation is

$$m^2 - m - 2 = 0$$

$\therefore$ $m^2 - 2m + m - 2 = 0 \Rightarrow m(m-2) + 1(m-2) = 0 \Rightarrow (m+1)(m-2) = 0$

$\therefore$ $m = -1, 2$

The roots are real and distinct.

$$\text{C. F.} = c_1e^{-x} + c_2e^{2x}.$$

$$\text{P.I.} = \frac{1}{D^2 - D - 2}\left\{2\log x\frac{1}{x} + \frac{1}{x^2}\right\}$$

$$= \frac{1}{(D-2)}\left\{\frac{1}{D+1}\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right)\right\}$$

$$= \frac{1}{D-2}\left\{e^{-x}\int\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right)e^x\, dx\right\}$$

$$= \frac{1}{D-2}\left\{e^{-x}\int\left(2\log x + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x} - \frac{1}{x}\right)e^x\, dx\right\}$$

Add and subtract by $\frac{1}{x}$

$$= \frac{1}{D-2} \left\{ e^{-x} \int \left[\left(2\log x - \frac{1}{x}\right) + \left(\frac{2}{x} + \frac{1}{x^2}\right)\right] e^x \, dx \right\}$$

$$= \frac{1}{D-2} \quad e^{-x} . e^{x} \left(2\log x - \frac{1}{x}\right) \quad \left[\int e^x(f(x) + f'(x))\, dx = e^x f(x)\right]$$

$$= e^{2x} \int \left(2\log x - \frac{1}{x}\right) e^{-2x} \, dx = e^{2x}\left[\int 2\log x.\, e^{-2x}\, dx \int \frac{1}{x} e^{-2x}\, dx\right]$$

$$= e^{2x} \left[\int 2\log x . e^{-2x} - e^{-2x} \log x - \int 2\, e^{-2x} \log x \, dx\right]$$

$$= e^{2x} (- e^{-2x} \log x) = - \log x.$$

$\therefore$ The G.S is y = C.F. + P. I.

$$\boxed{y = c_1 e^{-x} + c_2 e^{2x} - \log x}$$

Example : 3

Solve $\frac{d^2y}{dx^2} - y = \frac{2}{1 + e^x}$

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^2 - 1)y = \frac{2}{1 + e^x}.$$

Its auxiliary equation is

$$(m^2 - 1) = 0$$

$\therefore$ $m = \pm 1$ are real and distinct roots.

$$C.F. = c_1 e^x + c_2 e^{-x}.$$

$$P.I. = \frac{1}{D^2 - 1} \frac{2}{1 + e^x}$$

$$= \frac{1}{D-1} \left\{\frac{1}{D+1} \frac{2}{1+e^x}\right\} = 2\frac{1}{D-1}\left\{e^{-x}\int \frac{e^x dx}{1+e^x}\right\}$$

$$P.I. = 2\frac{1}{D-1} \{e^{-x} \log (1 + e^x)\}$$

$$= 2\, e^x \int e^{-x} \log (1 + e^x)\, e^{-x}\, dx = 2\, e^{-x} \int e^{-2x} \log (1 + e^x)\, dx$$

$$= 2\, e^x \left\{\log (1 + e^x). \frac{e^{-2x}}{-2} - \int \frac{e^x}{1+e^x} . \frac{e^{-2x}}{-2}\, dx\right\}$$

$$= - e^{-x} \log (1 + e^x) + e^x \int \frac{e^{-x}}{1 + e^x} . dx$$

$$= - e^{-x} \log (1 + e^x) + e^x \int \left(\frac{1}{e^x} - \frac{1}{1+e^x}\right) dx$$

$$= -e^{-x}\log(1+e^x) + e^x\int\left(e^{-x} - \frac{(1+e^x-e^x)}{1+e^x}\right)dx$$

$$= -e^{-x}\log(1+e^x) + e^x\int\left(e^{-x} - 1 + \frac{e^x}{1+e^x}\right)dx$$

$$= -e^{-x}\log(1+e^x) + e^x\left(\frac{e^{-x}}{-1} - x + \log(1+e^x)\right)$$

$$= -e^{-x}\log(1+e^x) - 1 - xe^x + e^x\log(1+e^x)$$

$$\text{P.I.} = (e^x - e^{-x})\log(1+e^x) - xe^x - 1.$$

The complete solution is $y = \text{C.F.} + \text{P.I.}$

$$\boxed{y = c_1e^x + c_2e^{-x} + (e^x - e^{-x})\log(1+e^x) - xe^x - 1}$$

Example : 4

Solve $\dfrac{d^2y}{dx^2} + 7\dfrac{dy}{dx} + 12y = e^{-3x}\sec^2x(1+2\tan x)$

Solution :

Put $\dfrac{d}{dx} = D$ then the D.E. is

$$(D^2 + 7D + 12)y = e^{-3x}\sec^2x(1+2\tan x) .$$

Its auxiliary equation is

$$m^2 + 7m + 12 = 0$$

$$m^2 + 4m + 3m + 12 = 0 \Rightarrow m(m+4) + 3(m+4) = 0$$

$\therefore$ $(m+3)(m+4) = 0 \Rightarrow m = -3, -4$ are real and distinct roots.

$$\text{C. F.} = c_1e^{-3x} + c_2e^{-4x}.$$

$$\text{P.I.} = \frac{1}{D^2+7D+12}e^{-3x}\sec^2x(1+2\tan x) \quad \left\{\int e^x\left[f(x)+f'(x)\right]dx = e^xf(x)\right\}$$

$$= \frac{1}{(D+4)}\left\{\frac{1}{D+3}e^{-3x}\sec^2x(1+2\tan x)\right\}$$

$$= \frac{1}{D+4}\left\{e^{-3x}\int e^{-3x}\sec^2x(1+2\tan x)e^{3x}dx\right\}$$

$$= \frac{1}{D+4}e^{-3x}\int(1+2\tan x)\sec^2x\,dx$$

Put $\tan x = t$

$\sec^2x\,dx = dt$

$$= \frac{1}{D+4}e^{-3x}\int(1+2t)\,dx$$

$$= \frac{1}{D+4}\{e^{-3x}(t+t^2)\} = \frac{1}{D+4}\{e^{-3x}(\tan x + \tan^2x)\}$$

$$= e^{-4x}\int e^{-3x}(\tan x + \tan^2x)e^{4x}dx \quad (1+\tan^2x = \sec^2x)$$

$$= e^{-4x}\int e^{-3}\,[\tan x + \tan^2 x]\, e^{-4x}\, dx$$
$$= e^{-4x}\int e^{x}\,[\tan x - 1 + \mathrm{see}^2 x]\, dx$$
$$= e^{-4x}.e^{x}\,(\tan x - 1) = e^{-3x}\,(\tan x - 1)$$

The general solution is $y = \text{C.F.} + \text{P.I.}$

$$\boxed{y = c_1e^{-3x} + c_2e^{-4x} + e^{-3x}\,(\tan x - 1)}$$

Example : 5

Solve $\dfrac{d^2y}{dx^2} - y = e^{-x}\sin(e^{-x}) + \cos(e^{-x})$

Solution :

Put $\dfrac{d}{dx} = D$ then the D.E. is

$$(D^2 - 1)y = e^{-x}\sin(e^{-x}) + \cos(e^{-x}).$$

Its auxiliary equation is

$$m^2 - 1 = 0$$

$\therefore$ $m = \underline{+}\,1$

are real and distinct roots.

$$\text{C. F.} = c_1e^{x} + c_2e^{-x}.$$

$$\text{P.I.} = \frac{1}{D^2 - 1}\{e^{-x}\sin(e^{-x}) + \cos(e^{-x})\}e^{x}\;[f(x) + f'(x)]\;dx = e^{x} f(x)$$
$$= \frac{1}{D - 1}\;\frac{1}{D + 1}[\cos(e^{-x}) + e^{-x}\sin(e^{-x})]$$
$$= \frac{1}{D - 1}\;e^{-x}\int e^{x}\,[\cos(e^{-x}) + e^{-x}\sin(e^{-x})]\,dx$$
$$= \frac{1}{D - 1}\;\{e^{-x}\,e^{x}\cos(e^{-x})\}$$
$$= \frac{1}{D - 1}\;\cos(e^{-x})$$
$$= e^{x}\int\cos(e^{-x})\,e^{-x}\,dx \text{ put } e^{-x} = t \quad e^{-x}\,dx = -\,dt$$
$$= e^{x}\int\cos t - dt = -\,e^{x}\sin t = -\,e^{x}\sin(e^{-x})$$

$\therefore$ The complete solution of the given different equation is

$$y = \text{C.F.} + \text{P.I.}$$

$$\boxed{y = c_1e^{x} + c_2e^{-x} - e^{x}\sin(e^{-x})}$$

1.3 Particular Integral by short cut Methods :

1. i) let $X = e^{ax}$ then its P. I. is given by

$$\frac{1}{f(D)}\,e^{ax} = \frac{1}{f(a)}\,e^{ax} \quad \text{put } D = a, \text{ provided } f(a) \neq 0$$

If f(a) = 0 the above formula fails.

i.e. a case of failure :

In case of failure multiply the numerator by x(or Independent variable) L. H. S. of operator $\frac{1}{f(D)}$ and then differentiate f(D) w.r.t.D

In general $\frac{1}{(D-a)^n} e^{ax} = \frac{x^n}{n!} e^{ax}$.

ii) let $X = a^x$ then its P.I. is given by $(M^N = e^{N\log M})$

$$\frac{1}{f(D)} a^x = \frac{1}{f(D)} e^{(\log a)x} = \frac{1}{f(\log a)} a^x \text{ provided } f(\log a) \neq 0$$

Or $\frac{1}{f(D)} a^{\pm x} = \frac{1}{f(\pm \log a)} a^x$ provided $f(\pm \log a) \neq 0$

Put $D = \pm \log a$.

If $f(\pm \log a) = 0$ i.e. a case of failure, follow above procedure.

iii) let X = k where k = constant, then its P. I.

is given by $\frac{1}{f(D)} k = k. \frac{1}{f(D)} e^{ox} = k \frac{1}{f(0)} e^{ox}$ $(1 = e^{ox})$ put D = 0

provided $f(O) \neq 0$, if $f(0) = 0$ i.e. a case of failure follow above procedure.

Illustrative Examples

Example : 1

Solve $\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 8\frac{dy}{dx} - 4y = 4e^{2x} + 2^x + 3$

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^3 + 5D^2 + 8D - 4)y = 4e^{2x} + 2^x + 3$$

The auxiliary equation is

$$m^3 - 5m^2 + 8m - 4 = 0$$

$$\Rightarrow (m-1)(m-2)^2 = 0$$

$\therefore$ m = 1, 2, 2 the roots are real one root is different two roots are equal.

C. F. = $c_1e^x + (c_2 + c_3x) e^{2x}$.

P. I. = $\frac{1}{D^3 - 5D^2 + 8D - 4} 4e^{2x} + \frac{1}{D^3 - 5D^2 + 8D - 4} 2^x + \frac{1}{D^3 - 5D^2 + 8D - 4} \cdot 3$

$$\frac{1}{D^3 - 5D^2 + 8D - 4} 4e^{2x} = 4.x \frac{1}{3D^2 - 10D + 8} e^{2x} = 4x.x \frac{1}{6D - 10} e^{2x} = 4x^2 \cdot \frac{1}{2} e^{2x} = 2x^2e^{2x}$$

$$\frac{1}{D^3-5D^2+8D-4}2^x = \frac{1}{D^3-5D^2+8D-4}e^{(\log 2)x} = \frac{1}{(\log 2)^3-5(\log 2)^2+8\log 2-4}2^x$$

$$\frac{1}{D^3-5D^2+8D-4}3 = 3\frac{1}{D^3-5D^2+8D-4}e^{ox} = 3\frac{1}{0-0+0-4}e^{ox} = \frac{-3}{4}\ (e^{ox}=1)$$

$$\therefore \quad \text{P. I.} = 2x^2e^{2x} + \frac{1}{(\log 3)^3-5(\log 2)^2+8\log 2-4}2^x\frac{-3}{4}$$

The complete solution is y = C.F. + P. I.

$$\boxed{y = c_1e^x + (c_2+c_3x)e^{2x} + 2x^2e^{2x} + \frac{1}{(\log 2)^3-5(\log 2)^2+8\log 2-4}2^x - \frac{3}{4}}$$

Example : 2

Solve $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = e^x\cosh 2x.$

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^2-D-6)y = e^x\cosh 2x$$

The auxiliary equation is

$$m^2-m-6 = 0$$

$$\Rightarrow (m+2)(m-3) = 0$$

$\therefore$ $m = -2, 3$ are real and distinct roots.

$$\text{C. F.} = c_1e^{-2x} + c_2e^{3x}.$$

$$\text{P. I.} = \frac{1}{D^2-D-6}e^x\cosh 2x = \frac{1}{D^2-D-6}e^x\left(\frac{e^{2x}+e^{-2x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^2-D-6}e^{3x} + \frac{1}{D^2-D-6}e^{-x}\right]$$

$$= \frac{1}{2}\left[x.\frac{1}{2D-1}e^{3x} + \frac{1}{1+1-6}e^{-x}\right]$$

$$\text{P.I} = \frac{1}{2}\left[x\frac{1}{5}e^{3x} - \frac{1}{4}e^{-x}\right]$$

The complete solution is y = C.F. + P. I.

$$\boxed{y = c_1e^{-2x} + c_2e^{3x} + \frac{x}{10}e^{3x} - \frac{1}{8}e^{-x}}$$

Example : 3

Solve $\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = 2\cosh^2(2x).$

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^3 - 4D)y = 2\cosh^2(2x)$$

Its auxiliary equation is

$$m^3 - 4m = 0$$

$$\Rightarrow \quad m(m^2 - 4) = 0$$

$\therefore$ $m = 0, \pm 2$, the roots are real and distinct.

$$\text{C. F.} = c_1 + c_2e^{2x} + c_3e^{-2x}.$$

$$\cosh^2 2x = e^x\left(\frac{e^{2x} + e^{-2x}}{2}\right)^2 = \frac{1}{4}(e^{4x} + e^{-4x} + 2)$$

$$\text{P. I.} = \frac{1}{D^3 - 4D}\, 2\cosh^2 2x$$

$$\text{P. I.} = \frac{1}{D^3 - 4D}\, 2.\, \frac{1}{4}(e^{4x} + e^{-4x} + 2)$$

$$= \frac{1}{2}\left[\frac{1}{D^3 - 4D}\; e^{4x} + \frac{1}{D^3 - 4D}\; e^{-4x} + \frac{1}{D^3 - 4D}\; 2.e^{ox}\right] \quad (e^{ox} = 1)$$

$$= \frac{1}{2}\left[\frac{1}{(4)^3 - 4(4)}\, e^{4x} + \frac{1}{(-4)^3 - 4(-4)}\, e^{-4x} + 2^x\, \frac{1}{3D^2 - 4}\, e^{ox}\right]$$

$$= \frac{1}{2}\left[\frac{1}{48}\, e^{4x} - \frac{1}{48}\, e^{-4x} + 2x\, \frac{1}{0 - 4}\, e^{ox}\right]$$

$$= \frac{1}{48}\left(\frac{e^{4x} - e^{-4x}}{2}\right) - \frac{2}{4}\, x.$$

$$\text{P. I.} = \frac{1}{48}\sinh 4x - \frac{x}{2}.$$

The complete solution is y = C.F. + P.I.

$$\boxed{y = c_1 + c_2e^{2x} + c_3e^{-2x} + \frac{1}{48}\sinh 4x - \frac{x}{2}}$$

Example : 4

Solve $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} - 9y = 5^x - \log 2.$

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^2 + 6D + 9)y = 5^x - \log 2.$$

Its auxiliary equation is

$$m^2 + 6m + 9 = 0$$

$$\Rightarrow (m+3)^2 = 0$$

$$\therefore \quad m = -3, -3 \quad \text{the roots are real and equal.}$$

$$\text{C. F.} = (c_1 + c_2x)e^{-3x}.$$

$$\text{P. I.} = \frac{1}{D^2+6D+9}(5^x - \log 2) = \frac{1}{(D+3)^2}e^{(\log 5)x} - \frac{1}{(D+3)^2}(\log 2).e^{0x}$$

$$\text{P. I.} = \frac{1}{(\log 5+3)^2}5^x - \frac{1}{9}\log 2$$

The complete solution is y = C.F. + P.I.

$$\boxed{y = (c_1 + c_2x)e^{-3x} + \frac{1}{(\log 5+3)^2}5^x - \frac{1}{9}\log 2}$$

Example : 5

Solve $\frac{d^3y}{dx^3} + 3\frac{dy}{dx} = \cosh 2x \sinh 3x.$

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^3 + 3D)y = \cosh 2x \sinh 3x.$$

Its auxiliary equation is

$$m^3 + 3m = 0$$

$$\Rightarrow (m^2+3) = 0$$

$$\therefore \quad m = 0, \underline{+}\, i\sqrt{3}$$

The roots are one is real and two imaginary and distinct.

$$\text{C. F.} = c_1 + c_2\cos\sqrt{3}\,x + c_3\sin\sqrt{3}\,x.$$

$$\text{P. I.} = \frac{1}{D^3+3D}\cosh 2x\sinh 3x, \quad \cosh 2x\sinh 3x = \left(\frac{e^{2x}+e^{-2x}}{2}\right)\left(\frac{e^{3x}-e^{-3x}}{2}\right)$$

$$= \frac{1}{4}(e^{5x} - e^{-x} + e^{x} - e^{-5x})$$

$$= \frac{1}{D^3+3D}\,\frac{1}{4}(e^{5x} - e^{-5x} + e^{x} - e^{-x})$$

$$= \frac{1}{4}\left[\frac{1}{D^3+3D}e^{5x} - \frac{1}{D^3+3D}e^{-5x} + \frac{1}{D^3+3D}e^{x} - \frac{1}{D^3+3D}e^{-x}\right]$$

$$= \frac{1}{4}\left[\frac{1}{(5)^3+3(5)}e^{5x} - \frac{1}{(-5)^3+3(-5)}e^{-5x} + \frac{1}{(1)^3+3(1)}e^{x} - \frac{1}{(-1)^3+3(-1)}e^{-x}\right]$$

$$= \frac{1}{4}\left[\frac{1}{140}e^{5x} + \frac{1}{140}e^{-5x} + \frac{1}{4}e^{x} + \frac{1}{4}e^{-x}\right]$$

$$= \frac{1}{280}\left(\frac{e^{5x}+e^{-5x}}{2}\right) + \frac{1}{8}\left(\frac{e^{x}+e^{-x}}{2}\right)$$

$$\text{P.I.} = \frac{1}{280}\cosh 5x + \frac{1}{8}\cosh x.$$

The complete solution is y = C.F. + P.I.

$$\boxed{y = c_1 + c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x + \frac{1}{280}\cosh 5x + \frac{1}{8}\cosh x}$$

2. Let X = sin(ax + b) or cos(ax + b) then its particular integral is given by

$$\frac{1}{f(D^2)} \begin{matrix}\sin\\ \cos\end{matrix} (ax+b) = \frac{1}{f(-a^2)} \begin{matrix}\sin\\ \cos\end{matrix} (ax+b)$$

Put $D^2 = -(a^2)$ provided $f(-a^2) = 0$.

If $f(-a^2) \neq 0$ the above formula fails.

i.e. case of failure, In case of failure multiply the numerator by x(or Independent Variable) and differentiate f(D) with respect to D.

$$\frac{1}{D^2+a^2}\sin(ax+b) = x.\frac{1}{2D}\sin(ax+b)$$

$$\frac{x}{2}\int \sin(ax+b)\,dx = -\frac{x}{2a}\cos(ax+b)$$

$$\frac{1}{D^2+a^2}\cos(ax+b) = x.\frac{1}{2D}\cos(ax+b)$$

$$\frac{x}{2}\int \cos(ax+b)\,dx = \frac{x}{2a}\sin(ax+b).$$

Illustrative Examples

Example : 1

Solve $\frac{d^3y}{dx^3} + y = \sin(2x+3)$

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^3+1)y = \sin(2x+3)$$

Its auxiliary equation is

$$(m^3+1) = 0$$

$$(m+1)(m^2-m+1) = 0$$

$$\therefore \quad m = -1, \frac{1}{2} \pm i\frac{\sqrt{3}}{2},$$

One root is real and two roots are imaginary and distinct.

$$\text{C. F.} = c_1e^{-x} + e^{\frac{x}{2}}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right)$$

$$\text{P. I.} = \frac{1}{D^3+1}\sin(2x+3) = \frac{1}{D^2D+1}\sin(2x+3)$$

Put $D^2 = -2^2 = -4$

$$\text{P. I.} = \frac{1}{-4D+1}\sin(2x+3) = \frac{(1+4D)}{(1-4D)(1+4D)}\sin(2x+3)$$

{Rationalizing 1 (1 – 4D) }

$$\text{P. I.} = \frac{(1+4D)}{1-16D^2}\sin(2x+3) = \frac{(1+4D)}{1-16(-4)}\sin(2x+3)$$

$$\text{P. I.} = \frac{1}{65}[\sin(2x+3) + 4D\sin(2x+3)]\ [D\sin(2x+3) = 2\cos(2x+3)]$$

$$\text{P. I.} = \frac{1}{65}[\sin(2x+3) + 8\cos(2x+3)]$$

The complete solution is y = C.F. + P. I.

$$\boxed{y = c_1e^{-x} + e^{\frac{x}{2}}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right) + \frac{1}{65}[\sin(2x+3) + 8\cos(2x+3)]}$$

Example : 2

Solve $\frac{d^3y}{dx^3} + \frac{dy}{dx} = 2\cos x$

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^3 + D)y = 2\cos x$$

Its auxiliary equation is

$$m^3 + m = 0$$

$$\Rightarrow m(m^2+1) = 0$$

$$\therefore m = 0, \pm i$$

One root is real and two roots are imaginary and distinct.

$$\text{C. F.} = c_1 + c_2\cos x + c_3\sin x.$$

$$\text{P. I.} = \frac{1}{D^3+D}2\cos x = 2x\frac{1}{3D^2+1}\cos x \text{ Put } D^2 = -1^2 = -1$$

$$\text{P. I.} = 2x\frac{1}{3(-1)+1}\cos x = 2x\frac{1}{-2}\cos x = -x\cos x.$$

The general solution is y = C. F. + P. I.

$$\boxed{y = c_1 + c_2\cos x + c_3\sin x - x\cos x}$$

Example : 3

Solve $\frac{d^2y}{dx^2} + y = \cos\frac{x}{2}\cos\frac{3}{2}x$

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^2 + 1)y = \cos\frac{x}{2}\cos\frac{3}{2}x$$

Its auxiliary equation is $m^2 + 1 = 0$

$\therefore \Rightarrow m = \underline{+}\, i$

The roots are imaginary distinct.

$$C.F. = c_1\cos x + c_2\sin x.$$

$$2\cos A\cos B = \cos(A + B) + \cos(A - B)$$

$$\cos\frac{x}{2}\cos\frac{3}{2}x = \frac{1}{2}\left[\cos\left(\frac{1}{2}+\frac{3}{2}\right)x + \cos\left(\frac{1}{2}-\frac{3}{2}\right)x\right]$$

$$= \frac{1}{2}[\cos 2x + \cos x]$$

$$P.I. = \frac{1}{D^2 + 1}\cos\frac{x}{2}\cos\frac{3}{2}x.$$

$$P.I. = \frac{1}{D^2 + 1}\,\frac{1}{2}\,[\cos 2x + \cos x]$$

$$= \frac{1}{2}\left[\frac{1}{D^2 + 1}\cos 2x + \frac{1}{D^2 + 1}\cos x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-4 + 1}\cos 2x + x\,\frac{1}{2D}\cos x\right]$$

$$P.I. = -\frac{1}{6}\cos 2x + \frac{x}{4}\sin x$$

The complete solution is y = C.F. + P. I.

$$\boxed{y = c_1\cos x + c_2\sin x - \frac{1}{6}\cos 2x + \frac{x}{4}\sin x}$$

Example : 4

Solve $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 8y = 2\sin^2 x \left(\sin^2\theta = \frac{1 - \cos 2\theta}{2}\right)$

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^2 + 6D + 8)y = 2\sin^2 x$$

Its auxiliary equation is

$$m^2 + 6m + 8 = 0$$

$$\Rightarrow (m + 2)(m + 4) = 0$$

$$\therefore \quad m = -2, -4$$

are real and roots are imaginary distinct.

$$\text{C. F.} = c_1e^{-2x} + c_2e^{-4x}$$

$$\text{P. I.} = \frac{1}{D^2 + 6D + 8} 2\sin^2 x = \frac{1}{D^2 + 6D + 8}(1 - \cos 2x)$$

$$= \frac{1}{D^2 + 6D + 8} e^{ox} - \frac{1}{D^2 + 6D + 8}\cos 2x$$

$$\text{P. I.} = \frac{1}{0 + 0 + 8} e^{ox} - \frac{1}{-4 + 6D + 8}\cos 2x$$

$$= \frac{1}{8} - \frac{1}{4 + 6D} \times \frac{4 - 6D}{4 - 6D}\cos 2x$$

$$\text{P. I.} = \frac{1}{8} - \frac{4 - 6D}{16 - 36D^2}\cos 2x$$

$$= \frac{1}{8} - \frac{1}{160}(4\cos 2x + 12\sin 2x)$$

The complete solution is y = C.F. + P. I.

$$\boxed{y = c_1e^{-2x} + c_2e^{-4x} + \frac{1}{8} - \frac{1}{40}(\cos 2x + 3\sin 2x)}$$

Example : 5

Solve $\left(\frac{d^2}{dt^2} + n^2\right)^2 x = f\cos(nt + \propto)$

Solution :

Put $\frac{d}{dt} = D$ then the D.E. is

$$(D^2 + n^2)^2 x = f\cos(nt + \propto)$$

Its auxiliary equation is

$$(m^2 + n^2)^2 = 0$$

$$\therefore \quad \Rightarrow \quad m = \underline{+}\, i\, n, \underline{+}\, i\, n$$

The roots are imaginary and equal.

$$\text{C. F.} = (c_1 + c_2t)\cos nt + (c_3 + c_4t)\sin nt.$$

$$\text{P. I.} = \frac{1}{(D^3 + n^2)^2} f\cos(nt + \propto) = \text{f.t.}\frac{1}{2(D^2 + n^2).2D}\cos(nt + \propto)$$

Put $D^2 = -n^2$

$$= \frac{f\,t}{4}.t.\frac{1}{3D^2 + n^2}\cos(nt + \propto)$$

$$\text{P. I.} = \frac{ft^2}{4}\frac{1}{-3n^2 + n^2}\cos(nt + \propto) = -\frac{ft^2}{8n^2}\cos(nt + \propto)$$

The complete solution is y = C.F. + P. I.

$$\boxed{y = (c_1 + c_2t)\cos nt + (c_3 + c_4t)\sin nt - \frac{ft^2}{8n^2}\cos(nt + \propto)}$$

Example : 6

Solve $\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 9y = \sin 2x \cos x$

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^2 + 10D + 9)y = \sin 2x \cos x$$

Its auxiliary equation is

$$(m^2 + 10m + 9) = 0$$

$$\Rightarrow (m + 1)(m + 9) = 0$$

$$\therefore m = -1, -9$$

The roots are real and distinct.

$$\text{C. F.} = c_1e^{-x} + c_2e^{-9x}$$

$$2\sin A \cos B = \sin(A + B) + \sin(A - B)$$

$$\sin 2x \cos x = \frac{1}{2}(\sin 3x + \sin x)$$

$$\text{P. I.} = \frac{1}{D^2 + 10D + 9}\sin 2x \cos x$$

$$\text{P. I.} = \frac{1}{D^2 + 10D + 9}\frac{1}{2}(\sin 3x+\sin x)$$

$$= \frac{1}{2}\left[\frac{1}{D^2 + 10D + 9}\sin 3x + \frac{1}{D^2 + 10D + 9}\sin x\right]$$

$$\text{P. I.} = \frac{1}{2}\left[\frac{1}{-9 + 10D + 9}\sin 3x + \frac{1}{-1 + 10D + 9}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{1}{10}\int \sin 3x\,dx + \frac{1}{8+ 10D}\sin x\right]$$

$$\text{P. I.} = \frac{1}{2}\left[-\frac{1}{30}\cos 3x + \frac{8 - 10D}{64 - 100D^2}\sin x\right]$$

$$= \frac{1}{2}\left[-\frac{\cos 3x}{30} + \frac{2}{164}(4\sin x - 5\cos x)\right]$$

$$\text{P. I.} = -\frac{\cos 3x}{60} + \frac{1}{164}(4\sin x - 5\cos x)]$$

The complete solution is y = C.F. + P. I.

$$\boxed{y = c_1e^{-x} + c_2e^{-9x} - \frac{\cos 3x}{60} + \frac{1}{164}(4\sin x - 5\cos x)}$$

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3) Let $X = x^m$ then its particular integral is given by $\frac{1}{f(D)} x^m = [f(D)]^{-1} x^m$.

Expanding $[f(D)]^{-1}$ binomially in ascending power of D up to D^m.

Binomial expansion :

i) $a^n + c_1^n a^{n-1} b + c_2^n a^{n-2} b^2 + – + c_r^n a^{n-r} b^r + ... + c_n^n b^n$.

Where $c_r^n = \frac{n!}{r!(n-r)!}$ $c_1^n = n$, $c_2^n = \frac{n(n-1)}{2!}$

ii) $\frac{1}{(1-t)} = (1-t)^{-1} = 1 + t + t^2 + t^3 + \ldots$

iii) $\frac{1}{(1+t)} = (1+t)^{-1} = 1 - t + t^2 - t^3 + \ldots$

Illustrative Examples

Example : 1

Solve $\frac{d^3y}{dx^3} + 8y = x^4 + 2x + 1$

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^3 + 8)y = x^4 + 2x + 1$$

Its auxiliary equation is

$$m^3 + 8 = 0$$

$$\Rightarrow (m + 2)(m^2 - 2m + 4) = 0$$

$$\therefore m = -2, 1 \pm i\sqrt{3}$$

one root is real and two roots are imaginary and distinct.

$$\text{C. F.} = c_1e^{-2x} + e^x(c_2\cos\sqrt{3}\,x + c_3\sin\sqrt{3}\,x)$$

$$\text{P. I.} = \frac{1}{D^3 + 8}(x^4 + 2x + 1) = \frac{1}{8\left(1 + \frac{D^3}{8}\right)}(x^4 + 2x + 1)$$

$$\text{P. I.} = \frac{1}{8}\left[1 + \frac{D^3}{8}\right]^{-1}(x^4 + 2x + 1)$$

$$= \frac{1}{8}\left[1 - \frac{D^3}{8} + \left(\frac{D^3}{8}\right)^2 + \ldots\right](x^4 + 2x + 1)$$

$$D(x^4 + 2x + 1) = 4x^3 + 2$$

$$D^2(x^4 + 2x + 1) = 12x^2$$

$$D^3(x^4 + 2x + 1) = 24x$$

$$D^4(x^4 + 2x + 1) = 24$$

$$D^5(x^4 + 2x + 1) = 0$$

$$\text{P. I.} = \frac{1}{8}[(x^4 + 2x + 1) - 24x + 0]$$

$$\text{P. I.} = \frac{1}{8}[(x^4 - 22x + 1)]$$

The complete solution is y = C.F. + P. I.

$$\boxed{y = c_1e^{-2x} + e^{x}(c_2\cos\sqrt{3}x + c_3\sin\sqrt{3}x) + \frac{1}{8}(x^4 - 22x + 1)}$$

Example : 2

Solve $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = 7x^2 + 9$

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^2 + 5D + 4)y = 7x^2 + 9$$

Its auxiliary equation is

$$m^2 + 5m + 4 = 0$$

$$\Rightarrow (m + 1)(m + 4) = 0$$

$$\therefore \quad m = -1, -4$$

The roots are real and distinct.

$$\text{C. F.} = c_1e^{-x} + c_2e^{-4x}$$

$$\text{P. I.} = \frac{1}{(D^2 + 5D + 4)}(7x^2 + 9) = \frac{1}{4}\left[1 + \left(\frac{D^2 + 5D}{4}\right)\right]^{-1}(7x^2 + 9)$$

$$\text{P. I.} = \frac{1}{4}\left[1 - \left(\frac{D^2 + 5D}{4}\right) + \left(\frac{D^2 + 5D}{4}\right)^2 + \ldots\right](7x^2 + 9)$$

$$\text{P. I.} = \frac{1}{4}\left[1 - \frac{5}{4}D + \frac{21}{16}D^2 + \ldots\right](7x^2 + 9)$$

$$D(7x^2 + 9) = 14x$$

$$D^2(7x^2 + 9) = 14$$

$$D^3(7x^2 + 9) = 0$$

$$\text{P. I.} = \frac{1}{4}\left[(7x^2+9)-\frac{5}{4}(14x)+\frac{21}{16}(14)-0\right]$$

$$\text{P. I.} = \frac{1}{4}\left[7x^2-\frac{35}{2}x+\frac{219}{8}\right]$$

The complete solution is y = C.F. + P. I.

$$\boxed{y = c_1e^{-x}+c_2e^{-4x}+\frac{1}{4}\left(7x^2-\frac{35}{2}x+\frac{219}{8}\right)}$$

Example : 3

Solve $\frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}-6\frac{dy}{dx}=1+x^2$

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^3-D^2-6D)y = 1+x^2$$

Its auxiliary equation is

$$m^3-m^2-6m = 0$$

$$\Rightarrow m(m^2-m-6) = 0$$

$$\Rightarrow m(m+2)(m-3) = 0$$

$$\therefore m = 0, -2, 3$$

The roots are real and distinct.

$$\text{C. F.} = c_1e^{0x}+c_2e^{-2x}+c_3e^{3x}$$

$$\text{P. I.} = \frac{1}{D^3-D^2-6D}(1+x^2) = -\frac{1}{6D}\left[1+\left(\frac{D^2-D}{6}\right)\right]^{-1}(1+x^2)$$

$$\text{P. I.} = -\frac{1}{6D}\left[1+\left(\frac{D^2-D}{6}\right)+\left(\frac{D^2-D}{6}\right)^2+\ldots\right](1+x^2)$$

$$\text{P. I.} = -\frac{1}{6D}\left[1-\frac{D}{6}+\frac{7}{36}D^2+\ldots\right](1+x^2)$$

$$D(1+x^2) = 2x$$

$$D^2(1+x^2) = 2$$

$$D^3(1+x^2) = 0$$

$$\text{P. I.} = -\frac{1}{6D}\left[(1+x^2)-\frac{1}{6}(2x)+\frac{7}{36}(2)+0\right]$$

$$\text{P. I.} = -\frac{1}{6}\int\left(x^2-\frac{1}{3}x+\frac{25}{18}\right)dx = -\frac{1}{6}\left(\frac{x^3}{3}-\frac{x^2}{6}+\frac{25}{18}x\right)$$

The complete solution is y = C.F. + P. I.

$$\boxed{y = c_1+c_2e^{-2x}+c_3e^{3x}-\frac{1}{6}\left(\frac{x^3}{3}-\frac{x^2}{6}+\frac{25}{18}x\right)}$$

Example : 4

Solve $\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = x^3 - 3x^2 + 1.$

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^2 - D + 1)y = x^3 - 3x^2 + 1$$

Its auxiliary equation is

$$m^2 - m + 1 = 0$$

$$\therefore \qquad \Rightarrow \quad m = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

The roots are imaginary and distinct.

$$\text{C. F.} = e^{\frac{x}{2}}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)$$

$$\begin{aligned}
\text{P. I.} &= \frac{1}{D^2 - D + 1}(x^3 - 3x^2 + 1)\\
&= [1 + (D^2 - D)]^{-1}(x^3 - 3x^2 + 1)\\
&= [1 - (D^2 - D) + (D^2 - D)^2 - (D^2 - D)^3 + \ldots](x^3 - 3x^2 + 1)\\
&= [1 + D + (0)\, D^2 - D^3 + \ldots](x^3 - 3x^2 + 1)
\end{aligned}$$

$$\begin{aligned}
D^2(x^3 - 3x^2 + 1) &= 6x - 6\\
D^3(x^3 - 3x^2 + 1) &= 6\\
D^4(x^3 - 3x^2 + 1) &= 0\\
&= (x^3 - 3x^2 + 1) + (3x^2 - 6x) + 0 - 6\\
\text{P. I.} &= x^3 - 6x - 5
\end{aligned}$$

The complete solution is y = C.F. + P. I.

$$\boxed{y = e^{\frac{x}{2}}(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x) + x^3 - 6x - 5}$$

Example : 5

Solve $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 4\,y = 3x^2 - 5x + 2$

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^3 - 2D + 4)y = 3x^2 - 5x + 2$$

Its auxiliary equation is

$$m^3 - 2m + 4 = 0$$

$$\Rightarrow (m + 2)(m^2 - 2m + 2) = 0$$

$$\therefore \quad m = -2, 1 \pm i$$

One root is real and two roots are imaginary and distinct.

$$\text{C. F.} = c_1e^{-2x} + e^x(c_2\cos x + c_3\sin x)$$

$$\text{P. I.} = \frac{1}{D^3 - 2D + 4}(3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left[1 + \left(\frac{D^3 - 2D}{4}\right)\right]^{-1}(3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left[1 - \left(\frac{D^3 - 2D}{4}\right) + \left(\frac{D^3 - 2D}{4}\right)^2 + \ldots\right](3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left[1 + \frac{D}{2} + \frac{D^2}{4} + \ldots\right](3x^2 - 5x + 2)$$

$$D(3x^2 - 5x + 2) = 6x - 5$$

$$D^2(3x^2 - 5x + 2) = 6$$

$$D^3(3x^2 - 5x + 2) = 0$$

$$= \frac{1}{4}[(3x^2 - 5x + 2) + (6x - 5) + 6 + 0]$$

$$= \frac{1}{4}[3x^2 + x + 3]$$

The complete solution is y = C.F. + P. I.

$$\boxed{y = c_1e^{-2x} + e^x(c_2\cos x + c_3\sin x) + \frac{1}{4}(3x^2 + x + 3)}$$

Example : 6

Solve $\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} = x^3$

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^4 - 2D^3 + D^2)y = x^3$$

Its auxiliary equation is

$$m^4 - 2m^3 + m^2 = 0$$

$$\therefore \quad m^2(m^2 - 2m + 1) = 0$$

$$\therefore \quad \Rightarrow \quad m = 0, 0, 1, 1$$

The roots are real and equal.

$$\text{C. F.} = c_1 + c_2x + (c_3 + c_4x)e^x$$

$$\text{P. I.} = \frac{1}{D^4 - 2D^3 + D^2} x^3$$

$$= \frac{1}{D^2} (1 - D)^{-2} x^3$$

$$= \frac{1}{D^2} [1 + 2D + 3D^2 + 4D^3 + \ldots..]x^3$$

$$Dx^3 = 3x^2$$

$$D^2x^3 = 6x$$

$$D^3x^3 = 6$$

$$D^4x^3 = 0$$

$$\text{P. I.} = \frac{1}{D^2} (x^3 + 6x^2 + 18x + 24 + 0)$$

$$\text{P. I.} = \frac{1}{D} \int(x^3 + 6x^2 + 18x + 24x)\, dx$$

$$= \frac{1}{D} \left(\frac{x^4}{4} + 6\,\frac{x^3}{3} + 18\,\frac{x^2}{2} + 24x\right)$$

$$\text{P. I.} = \int \left(\frac{x^4}{4} + 6\,\frac{x^3}{3} + 18\,\frac{x^2}{2} + 24\right) dx$$

$$= \frac{x^5}{20} + \frac{6}{3}\,\frac{x^4}{4} + \frac{18}{2}\,\frac{x^3}{3} + 24\,\frac{x^2}{2}$$

$$\text{P. I.} = \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2.$$

The complete solution is y = C.F. + P. I.

$$\boxed{y = (c_1 + c_2x) + (c_3 + c_4x)\, e^x + \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2}$$

4) Let $X = e^{ax}$ v where v is a function of x then its particular integral is given by

$$\frac{1}{f(D)} e^{ax}v = e^{ax} \frac{1}{f(D + a)} v$$

[i.e. D is replaced by D + a].

Illustrative Examples

Example : 1

Solve $\frac{d^2y}{dx^2} + 6\,\frac{dy}{dx} + 9y = \frac{e^{-3x}}{x^3}$.

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^2 + 6D + 9)y = \frac{e^{-3x}}{x^3}$$

Its auxiliary equation is

$$m^2 + 6m + 9 = 0$$

$$\therefore \quad m = -3, -3$$

The roots are real and equal.

$$\text{C. F.} = (c_1 + c_2x)\, e^{-3x}$$

$$\text{P. I.} = \frac{1}{D^2 + 6D + 9} \frac{e^{-3x}}{x^3} = e^{-3x} \frac{1}{(D-3)^2 + 6(D-3) + 9} x^{-3}$$

$$\text{P. I.} = e^{-3x} \frac{1}{D^2 - 6D + 9 + 6D - 18 + 9} x^{-3} = e^{-3x} \frac{1}{D^2} x^{-3} = e^{-3x} \iint x^{-3} (dx)^2$$

$$\text{P. I.} = e^{-3x} \int \frac{x^{-2}}{-2} dx = e^{-3x} \frac{x^{-1}}{2} = \frac{e^{-3x}}{2x}.$$

The complete solution of the given differential equation is

$$y = \text{C.F.} + \text{P. I.}$$

$$\boxed{y = (c_1 + c_2x)\, e^{-3x} + \frac{e^{-3x}}{2x}}$$

Example : 2

Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{-x} \log x$.

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^2 + 2D + 1)y = e^{-x} \log x$$

Its auxiliary equation is

$$m^2 + 2m + 1 = 0$$

$$\therefore \quad m = -1, -1$$

The roots are real and equal.

$$\text{C. F.} = (c_1 + c_2x)\, e^{-x}$$

$$\text{P. I.} = \frac{1}{D^2 + 2D + 1} e^{-x} \log x = e^{-x} \frac{1}{(D-1)^2 + 2(D-1) + 1} \log x$$

$$= e^{-x} \frac{1}{D^2 - 2D + 1 + 20 - 2 + 1} \log x$$

$$= e^{-x} \frac{1}{D^2} \log x = e^{-x} \iint \log x \, (dx)^2$$

Integration by parts formula

$$\left\{ \int f_1(x)\, f_2(x)\, dx = f_1(x) \int f_2(x)\, dx - \int \left(f_1'(x). \int f_2(x)\, dx\right) dx \right\}.$$

$$= e^{-x} \int \left(\log x.\ x - \int \frac{1}{x} .\ x\, dx\right) dx = e^{-x} \int (x \log x - x)\, dx$$

$$= e^{-x}\left[\log x .\ \frac{x^2}{2} - \int \frac{1}{x} . \frac{x^2}{2}\, dx - \frac{x^2}{2}\right]$$

$$= e^{-x}\left[\frac{x^2}{2} \log x - \frac{x^2}{4} - \frac{x^2}{2}\right] e^{-x} \frac{x^2}{2}\left(\log x - \frac{1}{2} - 1\right)$$

$$\text{P. I.} = \frac{x^2}{2}\, e^{-x}\left(\log x - \frac{3}{2}\right)$$

The complete solution is y = C.F. + P. I.

$$\boxed{y = (c_1 + c_2 x)\, e^{-x} + \frac{x^2}{2} e^{-x}\left(\log x - \frac{3}{2}\right)}$$

Example : 3

Solve $\frac{d^2y}{dx^2} - 4y = x^2 e^{3x}$.

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^2 - 4)y = x^2 e^{3x}$$

Its auxiliary equation is

$$m^2 - 4 = 0$$

$$\therefore \quad m = \pm 2$$

The roots are real and distinct.

$$\text{C. F.} = c_1 e^{2x} + c_2 e^{-2x}$$

$$\text{P. I.} = \frac{1}{D^2 - 4}\ x^2 e^{3x} = e^{3x} \frac{1}{(D+3)^2 - 4}\, x^2 = e^{3x} \frac{1}{D^2 + 6D + 9 - 4}\, x^2$$

$$= e^{3x} \frac{1}{5\left(1 + \frac{D^2 + 6D}{5}\right)}\, x^2 = \frac{e^{3x}}{5}\left[1 + \left(\frac{D^2 + 6D}{5}\right)\right]^{-1} x^2$$

$$= \frac{e^{3x}}{5}\left[1 - \left(\frac{D^2 + 6D}{5}\right) + \left(\frac{D^2 + 6D}{5}\right)^2 + \ldots\right] x^2$$

$$= \frac{e^{3x}}{5}\left[1 - \frac{6}{5} D + \frac{31}{25} D^2 + \ldots\right] x^2$$

$$Dx^2 = 2x$$

$$D^2 x^2 = 2$$

$$D^3 x^2 = 0$$

$$= \frac{e^{3x}}{5}\left[x^2 - \frac{6}{5}(2x) + \frac{31}{25}(2) + 0\right]$$

The general solution is y = C.F. + P. I.

$$\boxed{y = c_1e^{2x} + c_2e^{-2x} + \frac{e^{3x}}{5}\left(x^2 - \frac{12}{5}x + \frac{62}{25}\right)}$$

Example : 4

Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 8y = 12\,e^{-x}\sin x \sin 3x.$

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^2 + 4D + 8)y = 12e^{-x}\sin x \sin 3x$$

Its auxiliary equation is

$$m^2 + 4m + 8 = 0$$

$$\therefore \Rightarrow m = -2 \pm 2i$$

The roots are imaginary and distinct.

$$\text{C. F.} = e^{-2x}(c_1\cos 2x + c_2 \sin 2x)$$

$$\text{P. I.} = \frac{1}{D^2 + 4D + 8}12e^{-x}\sin x \sin 3x = 12e^{-x}\frac{1}{(D-1)^2 + 4(D-1) + 8}\sin x \sin 3x$$

$$\text{P. I.} = 12e^{-x}\frac{1}{D^2 - 2D + 1 + 4D - 4 + 8}\frac{1}{2}(\cos 2x - \cos 4x)$$

$$= \frac{12}{2}e^{-x}\left[\frac{1}{D^2 + 2D + 5}\cos 2x - \frac{1}{D^2 + 2D + 5}\cos 4x\right]$$

$$= 6\,e^{-x}\left[\frac{1}{-4 + 2D + 5}\cos 2x - \frac{1}{-16 + 2D + 5}\cos 4x\right]$$

$$= 6\,e^{-x}\left[\frac{1}{1 + 2D}\frac{(1-2D)}{(1-2D)}\cos 2x - \frac{1}{(2D-11)}\frac{(2D+11)}{(2D+11)}\cos 4x\right]$$

$$= 6\,e^{-x}\left[\frac{(1-2D)}{1-4D^2}\cos 2x - \frac{2D+11}{4D^2+121}\cos 4x\right]$$

$$= 6\,e^{-x}\left[\frac{1}{17}(\cos 2x + 4\sin 2x) + \frac{1}{185}(-8\sin 4x + 11\cos 4x)\right]$$

The complete solution is y = C.F. + P. I.

$$\boxed{y = e^{-2x}(c_1\cos 2x + c_2\sin 2x) + 6e^{-x}\left[\frac{1}{17}(\cos 2x + 4\sin 2x) + \frac{1}{185}(11\cos 4x - 8\sin 4x)\right]}$$

Example : 5

Solve $\frac{d^2y}{dx^2} + y = \sin x \cosh x.$

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^2 + 1)y = \sin x \cosh x.$$

Its auxiliary equation is

$$m^2 + 1 = 0$$

$$\therefore \quad m = \pm\, i$$

The roots are imaginary and distinct.

$$\text{C. F.} = c_1 \cos x + c_2 \sin x$$

$$\text{P. I.} = \frac{1}{D^2+1} \sin x \cosh x = \frac{1}{D^2+1} \sin x \, \frac{1}{2}(e^x + e^{-x})$$

$$= \frac{1}{2}\left[\frac{1}{D^2+1} e^x \sin x + \frac{1}{D^2+1} e^{-x} \sin x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{(D+1)^2+1} \sin x + e^{-x} \frac{1}{(D-1)^2+1} \sin x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{D^2+2D+2} \sin x + e^{-x} \frac{1}{D^2-2D+2} \sin x\right]$$

$$\text{P. I.} = \frac{1}{2}\left[e^x \frac{1}{-1+2D+2} \sin x + e^{-x} \frac{1}{-1-2D+2} \sin x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{1+2D} \frac{(1-2D)}{(1-2D)} \sin x + e^{-x} \frac{1}{1-2D} \frac{(1+2D)}{(1+2D)} \sin x\right]$$

$$= \frac{1}{2}\left[e^x \frac{(1-2D)}{(1-4D^2)} \sin x + e^{-x} \frac{(1+2D)}{(1-4D^2)} \sin x\right]$$

$$= \frac{1}{2}\left[e^x \frac{(\sin x - 2\cos x)}{5} \quad e^{-x} \frac{(\sin x + 2\cos x)}{5}\right]$$

$$= \frac{1}{5}\left[\left(\frac{e^x + e^{-x}}{2}\right) \sin x - \left(\frac{e^x - e^{-x}}{2}\right) 2\cos x\right]$$

$$\text{P. I.} = \frac{1}{5}[\sin x \cosh x - 2\cos x \sinh x]$$

The complete solution is y = C.F. + P. I.

$$\boxed{y = c_1 \cos x + c_2 \sin x + \frac{1}{5}(\sin x \cosh x - 2\cos x \sinh x)}$$

5) Let X = xv where v is a function of x then its particular integral is given by

$$\frac{1}{f(D)}\, xv = \left\{x - \frac{f'(D)}{f(D)}\right\} \frac{1}{f(D)}\, v \left(\left(e^{iax} = \frac{\cos ax}{R.P} + \frac{i \sin ax}{I.P}\right)\right.$$

Note : i) if $\frac{1}{f(D)} x^m \sin ax = \text{I. P. } \frac{1}{f(D)} x^m e^{iax}$

ii) if $\frac{1}{f(D)} x^m \cos ax = \text{R.P. } \frac{1}{f(D)} x^m e^{iax}$

Illustrative Examples

Example : 1

Solve $\frac{d^2y}{dx^2} - y = x\sin 3x$.

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^2 - 1)y = x\sin 3x.$$

Its auxiliary equation is

$$m^2 - 1 = 0$$

$$\therefore \quad m = \pm 1$$

The roots are real and distinct.

$$\text{C. F.} = c_1 e^x + c_2 e^{-x}$$

$$\text{P. I.} = \frac{1}{D^2 - 1} x\sin 3x = \left\{x - \frac{2D}{D^2 - 1}\right\}\frac{1}{D^2 - 1} \sin 3x$$

$$= \left\{x - \frac{2D}{D^2 - 1}\right\}\frac{1}{-9-1} \sin 3x = -\frac{1}{10}\left\{\sin 3x.x - \frac{2D}{D^2 - 1}\sin 3x\right\}$$

$$\text{P. I.} = -\frac{1}{10} x\sin 3x + \frac{2}{10}\frac{2D}{-9-1}\sin 3x = -\frac{x}{10}\sin 3x + \frac{1}{25} 3\cos 3x.$$

The complete solution is y = C.F. + P. I.

$$\boxed{y = c_1 e^x + c_2 e^{-x} - \frac{x}{10}\sin 3x + \frac{3}{25}\cos 3x.}$$

Example : 2

Solve $\frac{d^2y}{dx^2} - y = x^2\cos x$.

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^2 - 1)y = x^2\cos x.$$

Its auxiliary equation is

$$(m^2 - 1) = 0$$

$$\therefore \quad \Rightarrow m = \pm 1$$

The roots are real and distinct.

$$\text{C. F.} = c_1 e^x + c_2 e^{-x}$$

$$\text{P. I.} = \frac{1}{D^2 - 1} x^2\cos x = \text{R.P.} \frac{1}{D^2 - 1} x^2 e^{ix} \quad e^{ix} = \underset{(\text{R.p})}{\cos x} + \underset{(\text{I.p.})}{i\sin x}$$

$$= \text{R.P. } e^{ix} = \frac{1}{(D+i)^2 - 1} x^2 = \text{R.P. } e^{ix} = \frac{1}{(D+i)^2 - 1} x^2 = \text{R.P. } \frac{1}{D^2 + 2\,iD - 1 - 1}$$

$$= \text{R.P.} \frac{e^{ix}}{-2} \left[1 - \left(\frac{D^2 + 2iD}{2}\right)\right] x^2$$

$$Dx^2 = 2x$$

$$D^2x^2 = 2$$

$$D^3x^2 = 0$$

$$= \text{R.P.} \frac{e^{ix}}{-2} \left[1 + \left(\frac{D^2 + 2iD}{2}\right) + \left(\frac{D^2 + 2iD}{2}\right)^2 + \ldots\right] x^2$$

$$= \text{R.P.} \frac{e^{ix}}{-2} \left[1 + iD - \frac{D^2}{2} + \ldots\right] x^2$$

$$= \text{R.P.} \frac{e^{ix}}{-2} \left[x^2 + i2x - \frac{2}{2} + 0\right]$$

$$= \text{R.P.} - \frac{1}{2} (\cos x + i \sin x) [(x^2 - 1) + i2x]$$

$$= \text{R.P. } - \frac{1}{2} [((x^2 - 1) \text{ R.p. } \cos x - 2x \sin x) + i\,(2x\cos x + (x^2 - 1) \text{ I.p. } \sin x)]$$

$$\text{P. I.} = -\frac{1}{2} [(x^2 - 1) \cos x - 2x \sin x].$$

The complete solution is y = C.F. + P. I.

$$\boxed{y = c_1e^x + c_2e^{-x} - \frac{1}{2} [(x^2 - 1) \cos x - 2x \sin x]}$$

Example : 3

Solve $\frac{d^2y}{dx^2} + 4y = x\sin^2 x$.

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^2 + 4)y = x\sin^2 x$$

Its auxiliary equation is

$$m^2 + 4 = 0$$

$$\therefore \quad m = \pm 2\,i$$

are imaginary and distinct roots.

$$\text{C. F.} = c_1\cos 2x + c_2 \sin 2x$$

$$\text{P. I.} = \frac{1}{D^2 + 4} x\sin^2 x$$

$$= \frac{1}{D^2 + 4} x \left(\frac{1 - \cos 2x}{2}\right) = \frac{1}{2}\left[\frac{1}{D^2 + 4} x - \frac{1}{D^2 + 4} x \cos 2x\right]$$

Now $$\frac{1}{D^2+4}x = \frac{1}{4}\left[1+\frac{D^2}{4}\right]^{-1} x = \frac{1}{4}\left[1-\frac{D^2}{4}+\ldots\right] x = \frac{1}{4}(x-0) = \frac{x}{4}$$

$$\frac{1}{D^2+4}x\cos 2x = \left\{x-\frac{2D}{D^2+4}\right\}\frac{1}{D^2+4}\cos 2x = \left\{x-\frac{2D}{D^2+4}\right\}\frac{x}{4}\sin 2x$$

$$= \frac{x^2}{4}\sin 2x - \frac{2D}{D^2+4}\frac{x}{4}\sin 2x$$

$$= \frac{x^2}{4}\sin 2x - \frac{1}{2}\frac{1}{D^2+4}(\sin 2x + 2x\cos 2x)$$

$$\therefore \quad \frac{1}{D^2+4}x\cos 2x = \frac{x^2}{4}\sin 2x - \frac{1}{2}\frac{1}{D^2+4}\sin 2x - \frac{1}{D^2+4}x\cos 2x$$

$$(1+1)\frac{1}{D^2+4}x\cos 2x = \frac{x^2}{4}\sin 2x - \frac{1}{2}x.\frac{1}{2D}\sin 2x$$

$$2\frac{1}{D^2+4}x\cos 2x = \frac{x^2}{4}\sin 2x - \frac{x}{4}\left(-\frac{\cos 2x}{2}\right)$$

$$\frac{1}{D^2+4}x\cos 2x = \frac{x^2}{8}\sin 2x + \frac{x}{16}\cos 2x$$

$$\text{P. I.} = \frac{1}{2}\left[\frac{x^2}{4} - \frac{x^2}{8}\sin 2x - \frac{x}{16}\cos 2x\right].$$

The complete solution is y = C.F. + P. I.

$$\boxed{y = c_1\cos 2x + c_2\sin 2x + \frac{x^2}{8} - \frac{x^2}{16}\sin 2x - \frac{x}{32}\cos 2x}$$

Example : 4

Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = xe^{-x}\sinh x$.

Solution :

Put $$\frac{d}{dx} = D \text{ then the D.E. is}$$

$$(D^2+4D+4)y = xe^{-x}\sinh x$$

Its auxiliary equation is

$$m^2+4m+4 = 0$$

$$\therefore \quad m = -2, -2$$

are real and equal roots.

$$\text{C. F.} = (c_1+c_2x)e^{-2x}$$

$$\text{P. I.} = \frac{1}{D^2+4D+4}xe^{-x}\sinh x$$

$$= \frac{1}{(D+2)^2}x\,e^{-x}\frac{1}{2}(e^x-e^{-x})$$

$$= \frac{1}{2}\left[\frac{1}{(D+2)^2}x - \frac{1}{(D+2)^2}xe^{-2x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{4}\left(1+\frac{D}{2}\right)^{-2} x - e^{-2x}\frac{1}{[(D-2)+2]^2}x\right]$$

$$= \frac{1}{2}\left[\frac{1}{4}\left(1-2\frac{D}{2}+3\frac{D^2}{4}+\ldots\right)x - e^{-2x}\frac{1}{D^2}x\right]$$

$$= \frac{1}{2}\left[\frac{1}{4}(x-1+0) - e^{-2x}\iint x^2\,(dx)^2\right]$$

$$= \frac{1}{8}(x-1) - \frac{e^{-2x}}{2}\int\frac{x^3}{3}dx$$

$$= \frac{1}{8}(x-1) - \frac{e^{-2x}}{2}\,\frac{x^4}{12}$$

$$\text{P. I.} = \frac{1}{8}(x-1) - \frac{e^{-2x}}{24}x^4.$$

The complete solution is y = C.F. + P. I.

$$\boxed{y = (c_1 + c_2x)e^{-2x} + \frac{1}{8}(x-1) - \frac{x^4}{24}e^{-2x}}$$

Example : 5

Solve $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = xe^x \sin 2x.$

Solution :

Put $\dfrac{d}{dx} = D$ then the D.E. is

$$(D^2 - 2D + 1)y = xe^x \sin 2x$$

Its auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$\therefore \Rightarrow m = 1, 1$

are real and equal roots.

$$\text{C. F.} = (c_1 + c_2x)\,e^x$$

$$\text{P. I.} = \frac{1}{D^2-2D+1}e^x x\sin 2x = e^x\frac{1}{(D+1)^2-2(D+1)+1}x\sin 2x$$

$$\text{P. I.} = e^x\frac{1}{D^2+2D+1-2D-2+1}x\sin 2x = e^x\frac{1}{D^2}\,x\sin 2x$$

$$\text{P. I.} = e^x\left\{x-\frac{2D}{D^2}\right\}\frac{1}{D^2}\sin 2x = \left\{x-\frac{2D}{D^2}\right\}\frac{1}{-4}\sin 2x$$

$$= -\frac{e^x}{4}\left\{x\sin 2x - \frac{2D}{D^2}\sin 2x\right\}$$

$$= -\frac{e^x}{4}\left\{x\sin 2x - \frac{2.2}{-4}\cos 2x\right\}$$

$$\text{P. I.} = -\frac{e^x}{4}\{x\sin 2x + \cos 2x\}.$$

The complete solution is y = C.F. + P. I.

$$\boxed{y = (c_1 + c_2 x)\, e^x - \frac{e^x}{4}(x\sin 2x + \cos 2x)}$$

3) Method of variation of parameters :

Consider the second order linear differential equation of the form

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = x$$

Let its C.F. $= c_1 y_1 + c_2 y_2$. and P.I $= uy_1 + vy_2$

where u,v are unknown variable functions of x.

$$\therefore \quad u = -\int \frac{Xy_2}{W},\ v = \int \frac{Xy_1}{W} \quad \text{where } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

The complete solution is y = C.F. + P. I.

Illustrative Examples

Example : 1

Solve $\frac{d^2y}{dx^2} + a^2y = \tan ax$ by the method of variation of parameters.

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^2 + a^2)y = \tan ax$$

Its auxiliary equation is

$$m^2 + a^2 = 0$$

$$\therefore \quad m = \underline{+}\, 2i$$

are imaginary and distinct roots.

$$\text{C. F.} = c_1\cos ax + c_2\sin ax$$

$$\text{P. I.} = uy_1 + vy_2$$

Where $y_1 = \cos ax$, $y_2 = \sin ax$.

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a\cos^2 ax + a\sin^2 ax = a$$

$$u = -\int \frac{Xy_2}{W} = -\int \frac{\tan ax}{a}\sin ax\, dx = -\frac{1}{a}\int \frac{\sin ax}{\cos ax}\ dx$$

$$u = -\frac{1}{a}\int \frac{1-\cos^2 ax}{\cos ax}\, dx = -\frac{1}{a}\int (\sec ax - \cos ax)\, dx$$

$$u = -\frac{1}{a^2}[\log(\sec ax + \tan ax) - \sin ax]$$

$$v = \int \frac{Xy_1}{W} = -\int \frac{\tan ax}{a}\cos ax\, dx = \frac{1}{a}\int \frac{\sin ax}{\cos ax}\cos ax\, dx = -\frac{1}{a^2}\cos ax$$

$$\therefore \quad \text{P. I.} = uy_1 + vy_2 = -\frac{1}{a^2}[\log(\sec ax + \tan ax) - \sin ax]\cos ax - \frac{1}{a^2}\cos ax \sin ax$$

$$\text{P. I.} = -\frac{1}{a^2}\log(\sec ax + \tan ax)\cos ax$$

The complete solution is y = C.F. + P. I.

$$\boxed{y = c_1 \cos ax + c_2 \sin ax - \frac{1}{a^2}\cos ax \log(\sec ax + \tan ax)}$$

Example : 2

Solve $\frac{d^2y}{dx^2} + 4y = \frac{1}{1+\cos 2x}$ by using variation of parameters method.

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^2 + 4)y = \frac{1}{1+\cos 2x}$$

Its auxiliary equation is

$$m^2 + 4 = 0$$

$$\therefore \quad m = \underline{+}\, 2i$$

are imaginary and distinct roots.

$$\text{C. F.} = c_1 \cos 2x + c_2 \sin 2x$$

$$\text{P. I.} = uy_1 + vy_2$$

Where $y_1 = \cos 2x$, $y_2 = \sin 2x$.

$$\therefore \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x = 2$$

$$u = -\int \frac{Xy_2}{W} = -\int \frac{1}{1+\cos 2x}\,\frac{\sin 2x}{2}\, dx = \frac{1}{4}\log(1+\cos 2x)$$

$$v = \int \frac{Xy_1}{W} = \int \frac{1}{1+\cos 2x}\,\frac{\cos 2x}{2}\, dx \qquad \left(\cos^2\theta = \frac{1+\cos 2\theta}{2}\right)$$

$$v = \frac{1}{2}\int \frac{1+\cos 2x - 1}{1+\cos 2x}\, dx = \frac{1}{2}\int\left(1 - \frac{1}{1+\cos 2x}\right)\int dx$$

$$v = \frac{1}{2}\int\left(1-\frac{1}{2\cos^2 x}\right)dx = \frac{1}{2}\int\left(1-\frac{1}{2}\sec^2 x\right)dx$$

$$v = \frac{1}{2}\left(x-\frac{1}{2}\tan x\right) = \frac{x}{2}-\frac{1}{4}\tan x.$$

$$\text{P. I.} = uy_1 + vy_2 = \frac{1}{4}\log(1+\cos 2x).\cos 2x + \left(\frac{x}{2}-\frac{1}{4}\tan x\right)\sin 2x$$

The complete solution is y = C.F. + P. I.

$$\boxed{y = c_1\cos 2x + c_2\sin 2x + \frac{\cos 2x}{4}\log(1+\cos 2x) + \sin 2x\left(\frac{x}{2}-\frac{1}{4}\tan x\right)}$$

Example : 3

Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x\sin x$ by using method of variation of parameters.

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^2 - 2D)y = e^x\sin x$$

Its auxiliary equation is

$$m^2 - 2m = 0$$

$\therefore$ $m = 0, 2$

are real and distinct roots.

$$\text{C. F.} = c_1 + c_2 e^{2x}$$

$$\text{P. I.} = uy_1 + vy_2$$

Where $y_1 = 1,\ y_2 = e^{2x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix} = 2e^{2x}$$

$$u = -\int\frac{Xy_2}{W} = -\int\frac{e^x\sin x}{2e^{2x}}\,e^{2x}\,dx = -\frac{1}{2}\int e^x\sin x\,dx$$

$$u = -\frac{1}{2}\left[\frac{e^x}{1+1}(\sin x - \cos x)\right] = -\frac{e^x}{4}(\sin x - \cos x)$$

$$v = \int\frac{Xy_1}{W} = \int\frac{e^x\sin x}{2e^2x}\,1.dx = \frac{1}{2}\int e^{-x}\sin x\,dx = \frac{1}{2}\left[\frac{e^{-x}}{1+1}(-\sin x - \cos x)\right]$$

$$v = -\frac{e^{-x}}{4}(\sin x + \cos x)$$

$$\text{P. I.} = uy_1 + vy_2 = -\frac{e^x}{4}(\sin x - \cos x) - \frac{e^{-x}}{4}(\sin x + \cos x)\,e^{2x}$$

$$\text{P. I.} = -\frac{e^x}{4}(\sin x - \cos x + \sin x + \cos x) = -\frac{e^x}{4}2\sin x = -\frac{e^x}{2}\sin x$$

The complete solution is y = C.F. + P. I.

$$\boxed{y = c_1 + c_2 e^{2x} - \frac{e^x}{2} \sin x}$$

Example : 4

Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = e^x \cot x$ using variation of parameters.

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^2 - 2D + 2)y = e^x \cot x$$

Its auxiliary equation is

$$m^2 - 2m + 2 = 0$$

$$\therefore \quad m = 1 \pm i$$

The roots are imaginary and distinct.

$$\text{C. F.} = e^x (c_1 \cos x + c_2 \sin x)$$

$$\text{P. I.} = uy_1 + vy_2$$

Where $y_1 = e^x \cos x,\ y_2 = e^x \sin x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x(\cos x - \sin x) & e^x(\sin x + \cos x) \end{vmatrix} = e^{2x}(\cos^2 x + \sin^2 x) = e^{2x}$$

$$u = -\int \frac{Xy_2}{W} = -\int \frac{e^x \cot x}{e^{2x}} e^x \sin x\, dx = -\int \frac{\cos x}{\sin x} \sin x\, dx = -\sin x$$

$$v = \int \frac{Xy_1}{W} = \int \frac{e^x \cot x}{e^{2x}} e^x \cos x\, dx = \int \frac{\cos^2 x}{\sin x} dx = \int \frac{1 - \sin^2 x}{\sin x} dx$$

$$v = \int (\operatorname{cosec} x - \sin x)\, dx = -\log(\operatorname{cosec} x - \cot x) + \cos x$$

$$\text{P. I.} = uy_1 + vy_2 = -e^x \cos x \sin x - e^x \sin x \log(\operatorname{cosec} x - \cot x) + e^x \sin x \cos x$$

$$\text{P. I.} = -e^x \sin x \log(\operatorname{cosec} x - \cot x)$$

The complete solution is y = C.F. + P. I.

$$\boxed{y = e^x (c_1 \cos x + c_2 \sin x) - e^x \sin x \log(\operatorname{cosec} x - \cot x)}$$

Example : 5

Solve $\frac{d^2y}{dx^2} - y = \frac{1}{(1 + e^{-x})^2}$ using variation of parameters.

Solution :

Put $\frac{d}{dx} = D$ then the D.E. is

$$(D^2 - 1)y = \frac{1}{(1 + e^{-x})^2}$$

Its auxiliary equation is

$$m^2 - 1 = 0$$

$$\therefore \quad m = \pm 1$$

The roots are real and distinct.

$$\text{C. F.} = c_1 e^x + c_2 e^{-x}$$

$$\text{P. I.} = uy_1 + vy_2$$

Where $y_1 = e^x, \; y_2 = e^{-x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1 = -2$$

Where $t = 1 + e^{-x} \quad dt = -e^{-x}\,dx \qquad \therefore \quad e^{-x}\,dx = = dt$

$$u = -\int \frac{Xy_2}{W} = -\int \frac{1}{(1 + e^{-x})^2}\,\frac{e^{-x}}{-2}\,dx = \frac{1}{2}\int \frac{-dt}{t^2} = \frac{1}{2}\left(-\left(\frac{-1}{t}\right)\right)$$

$$= \frac{1}{2}\cdot\frac{1}{t} = \frac{1}{2}\cdot\frac{1}{1 + e^{-x}}$$

$$u = \frac{1}{2}\,\frac{1}{1 + e^{-x}}$$

$$v = \int \frac{Xy_1}{W} = \int \frac{1}{(1 + e^{-x})^2}\,\frac{e^{-x}}{-2}\,dx = -\frac{1}{2}\int \frac{e^x}{(1 + e^{-x})^2}\,dx$$

$$v = -\frac{1}{2}\int \frac{e^{2x}}{(e^x + 1)^2}\,e^x\,dx \text{ put } e^x + 1 = t \quad e^x\,dx = dt \text{ and } e^x = t - 1$$

$$v = -\frac{1}{2}\int \frac{(t - 1)^2}{t^2}\,dx = -\frac{1}{2}\int \frac{t^2 - 2t + 1}{t^2}\,dx = -\frac{1}{2}\int\left(1 - \frac{2}{t} + \frac{1}{t^2}\right)dt$$

$$v = -\frac{1}{2}\left[t - 2\log t - \frac{1}{t}\right] = -\frac{1}{2}\left[(1 + e^x) - 2\log(1 + e^x) - \frac{1}{1 + e^x}\right]$$

$$\text{P. I.} = uy_1 + vy_2 = \frac{1}{2}\cdot\frac{1}{1 + e^{-x}}\,e^x - \frac{1}{2}\left[1 + e^x - 2\log(1 + e^x) - \frac{1}{1 + e^x}\right]e^{-x}$$

The complete solution is y = C.F. + P. I.

$$\boxed{y = c_1 e^x + c_2 e^{-x} + \frac{1}{2}\,\frac{e^x}{1 + e^{-x}} - \frac{1}{2}\left[1 + e^x - 2\log(1 + e^x) - \frac{1}{1 + e^x}\right]e^{-x}}$$

1.4 Homogeneous Differential Equations :

1. Cauchy's homogeneous differential equation :

The differential equation of the form

$$x^n\frac{d^ny}{dn^n} + a_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_{n-1}\,x\,\frac{dy}{dx} + a_n\,y = X \qquad \ldots(i)$$

Where $a_1, a_2 \ldots a_n$ are constants and X is a function of x or constant. The above equation is known as n^{th} order Cauchy's homogeneous differential equation. For finding the solution of the above equation we change the independent variable z by taking the substitution

$$x = e^z \quad \text{then} \quad z = \log x, = \frac{dz}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{1}{x}$$

$$\therefore \quad x\frac{dy}{dx} = \frac{dy}{dz} = Dy \quad \text{where} \quad D = \frac{d}{dz}$$

Similarly $x^2\dfrac{d^2y}{dx^2} = D(D-1)y,\qquad x^3\dfrac{d^3y}{dx^3} = D(D-1)(D-2)y$ and soon.

Substituting these values in equation (i) and after

Simplification we get $(D^n + k_1D^{n-1} + k_2D^{n-2} + \ldots + k_n)y = X(z)$ …(ii)

This is linear differential equation by using previous methods we find solution of equation (ii) i.e.

$$y = \text{C.F.} + \text{P. I.}$$

Illustrative Examples

Example : 1

$$x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$$

Solution :

Put $x = e^z \qquad z = \log x$ then

$$x\frac{dy}{dx} = Dy \quad \text{where} \quad D = \frac{d}{dz}$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

The given differential equation we get

$$\therefore \quad D(D-1)(D-2)y + 2D(D-1)y + 2y = 10\left(e^z + \frac{1}{e^z}\right)$$

$$[D(D^2 - 3D + 2) + (2D^2 - 2D) + 2]\,y = 10(e^z + e^{-z})$$

$$(D^3 - 3D^2 + 2D + 2D^2 - 2D + 2)\,y = 10(e^z + e^{-z})$$

$$(D^3 - D^2 + 2)\,y = 10(e^z + e^{-z})$$

Its auxiliary equation is

$$m^3 - m^2 + 2 = 0$$

$\therefore \quad (m + 1)(m^2 - 2m + 2) = 0$

$$m = -1, 1 \pm i$$

one root is real, two roots are imaginary and distinct.

$$\text{C. F.} = c_1 e^{-z} + e^z (c_2 \cos z + c_3 \sin z)$$

$$\text{P. I.} = \frac{1}{D^3 - D^2 + 2} 10 (e^z + e^{-z})$$

$$= 10\left[\frac{1}{D^3 - D^2 + 2} e^z + \frac{1}{D^3 - D^2 + 2} e^{-z}\right]$$

$$\text{P. I.} = 10\left[\frac{1}{1 - 1 + 2} e^z + z \frac{1}{3D^2 - 2D} e^{-z}\right]$$

$$= 10\left[\frac{1}{2} e^z + \frac{z}{5} e^{-z}\right]$$

The complete solution is y = C.F. + P. I.

$$y = c_1 e^{-z} + e^z (c_2 \cos z + c_3 \sin z) + 5e^z + 2ze^{-z}.$$

$$\boxed{y = c_1 \frac{1}{x} + x\,[c_2 \cos(\log x) + c_3 \sin(\log x)] + 5x + \frac{2}{x} \log x}$$

Example : 2

$$u = r \frac{d}{dr}\left(r \frac{du}{dr}\right) + a r^3$$

Solution :

The D.E. can be written as

$$r^2 \frac{d^2u}{dr^2} + r \frac{du}{dr} - u = -ar^3$$

Put $\quad r = e^z, \quad z = \log r \quad r \frac{du}{dr} = Du, \quad D = \frac{d}{dz} \quad r^2 \frac{d^2u}{dr^2} = D(D - 1)\, u.$

$$D(D - 1)\, u + Du - u = -a\, e^{3z}$$

$$(D^2 - D + D - 1)\, u = -ae^{3z}$$

$$(D^2 - 1)u = -ae^{3z}$$

Its auxiliary equation is

$$m^2 - 1 = 0, \qquad m = \pm 1$$

The roots are real and distinct

$$\text{C.F.} = c_1 e^z + c_2 e^{-z}$$

$$\text{P.I.} = \frac{1}{D^2 - 1} (-ae^{3z}) = -a \frac{1}{9 - 1} e^{3z} = -\frac{9}{8} e^{3z}.$$

The complete solution is y = C.F. + P. I.

$$u = c_1 e^z + c_2 e^{-z} - \frac{a}{8} e^3 z$$

$$\boxed{u = c_1 r + c_2 \frac{1}{r} - \frac{a}{8} r^3}$$

Example : 3

$$\frac{d^2v}{dr^2} - \frac{1}{r}\frac{dv}{dr} = A + B\log r.$$

Solution :

We multiply r^2 to the given differential equation we get

$$r^2\frac{d^2v}{dr^2} - r\frac{dv}{dr} = A r^2 + B r^2 \log r.$$

Put $\quad r = e^z, \quad z = \log r$ then $r\frac{dv}{dr} = Dv, \quad D = \frac{d}{dz}$ $r^2\frac{d^2v}{dr^2} = D(D-1)v.$

$$\therefore \quad D(D-1)v - Dv = A e^{2z} + B e^{2z}.z$$

$$(D^2 - 2D)v = A e^{2z} + B e^{2z}.z$$

Its auxiliary equation is

$$m^2 - 2m = 0, \qquad m = 0, 2$$

The roots are real and distinct

$$\text{C.F.} = c_1 + c_2 e^{2z}$$

$$\text{P.I.} = \frac{1}{D^2 - 2D}[A e^{2z} + B e^{2z}z]$$

$$= A\frac{1}{D^2-2D}e^{2z} + B\frac{1}{D^2-2D}e^{2z}z.$$

$$\text{P.I.} = A z\frac{1}{2D-2}e^{2z} + B e^{2z}\frac{1}{(D+2)^2 - 2(D+2)}z$$

$$= A z\frac{1}{4-2}e^{2z} + B e^{2z}\frac{1}{D^2+4D+4-2D-4}z$$

$$= \frac{A}{2}z e^{2z} + B e^{2z}\frac{1}{D^2+2D}z$$

$$\text{P.I.} = \frac{A}{2}z e^{2z} + B e^{2z}\frac{1}{4}(z^2 - z)$$

Now,

$$\frac{1}{D^2+2D}z = \frac{1}{2D}\left[1+\frac{D}{2}\right]^{-1}z = \frac{1}{2D}\left[1-\frac{D}{2}+\ldots\right]z$$

$$= \frac{1}{2D}\left(z-\frac{1}{2}\right) = \frac{1}{2}\int\left(z-\frac{1}{2}\right)dz = \frac{1}{2}\left(\frac{z^2}{2}-\frac{z}{2}\right)$$

$$= \frac{1}{4}(z^2 - z)$$

The complete solution is v = C.F. + P. I.

$$v = c_1 + c_2e^{2z} + \frac{A}{2}ze^{2z} + \frac{B}{4}e^{2z}(z^2 - z)$$

$$\boxed{v = c_1 + c_2 r^2 \frac{A}{2}r^2\log r + \frac{B}{4}r^2\left((\log r)^2 - \log r\right)}$$

Example : 4

$$\left(\frac{d^2}{dx^2} - \frac{2}{x^2}\right)^2 y = 0$$

Solution :

The given D.E. can be written as

$$\left(\frac{d^2}{dx^2} - \frac{2}{x^2}\right)\left(\frac{d^2}{dx^2} - \frac{2}{x^2}\right)^2 y = 0 \quad \ldots(i)$$

Put $$\left(\frac{d^2}{dx^2} - \frac{2}{x^2}\right) y = u \quad \ldots(ii)$$

From (i) we get $$\left(\frac{d^2}{dx^2} - \frac{2}{x^2}\right) u = 0 \quad \text{multiply by } x^2 \text{ we get}$$

$$x^2 \frac{d^2u}{dx^2} - 2u = 0$$

Put $x = e^z$, $z = \log x$

then $x\dfrac{du}{dx} = Du$, $x^2 \dfrac{d^2u}{dx^2} = D(D-1)u$ where $D = \dfrac{d}{dz}$

$$\therefore \quad D(D-1)u - 2u = 0$$

$$\Rightarrow \quad (D^2 - D - 2)u = 0$$

Its auxiliary equation is

$$m^2 - m - 2 = 0, \Rightarrow m = -1, 2$$

The roots are real and distinct

$$u = c_1 e^{-z} + c_2 e^{2z} = c_1 \frac{1}{x} + c_2 x^2$$

From equation (ii) we have $\left(\dfrac{d^2}{dx^2} - \dfrac{2}{x^2}\right) y = c_1 \dfrac{1}{x} + c_2 x^2$

Multiply this equation by x^2, we get

$$x^2 \frac{d^2y}{dx^2} - 2y = c_1 x + c_2 x^4$$

Put $x = e^z$, $z = \log x$

then $x\dfrac{dy}{dx} = Dy$, $x^2 \dfrac{d^2y}{dx^2} = D(D-1)y$ where $D = \dfrac{d}{dz}$

The above equation becomes

$$\therefore \quad D(D-1)y - 2y = c_1 e^z + c_2 e^{4z}$$

$$(D^2 - D - 2)y = c_1 e^z + c_2 e^{4z}$$

Its auxiliary equation is

$$m^2 - m - 2 = 0, \Rightarrow m = -1, 2$$

The roots are real and distinct

$$\text{C.F.} = c_3 e^{-z} + c_4 e^{2z}$$

$$\text{P.I.} = \frac{1}{D^2 - D - 2}(c_1e^z + c_2e^{4z}) = c_1\frac{1}{D^2 - D - 2}e^z + c_2\frac{1}{D^2 - D - 2}e^{4z}$$

$$= c_1\frac{e^z}{-2} + c_2\frac{e^{4z}}{10}$$

The complete solution is y = C.F. + P. I.

$$y = c_3 e^{-z} + c_4 e^{2z} - \frac{c_1}{2}e^z + \frac{c_2}{10}e^{4z}$$

$$\boxed{y = -\frac{c_1}{2}x + \frac{c_2}{10}x^4 + c_3\frac{1}{x} + c_4x^2}$$

Example : 5

Find the equation of the curve which satisfies the differential equation

$$4x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + y = 0$$

and crosses X – axis at an angle of 60° at x = 1

Solution :

Put $x = e^z, \quad z = \log x$

then $x\frac{dy}{dx} = Dy, \quad \text{where} \quad D = \frac{d}{dz}$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

from the given differential equation, we get

$\therefore \quad 4D(D-1)y - 4Dy + y = 0$

$\Rightarrow \quad (4D^2 - 8D + 1)y = 0$

Its auxiliary equation is

$$4m^2 - 8m + 1 = 0, \qquad m = 1 \pm \frac{\sqrt{3}}{2}$$

The roots are real and distinct

$$y = c_1e^{\left(1 + \frac{\sqrt{3}}{2}\right)z} + c_2e^{\left(1 - \frac{\sqrt{3}}{2}\right)z}$$

$$y = c_1x^{\left(1 + \frac{\sqrt{3}}{2}\right)} + c_2x^{\left(1 - \frac{\sqrt{3}}{2}\right)}$$

$$= x\left(c_1x^{\frac{\sqrt{3}}{2}} + c_2x^{-\frac{\sqrt{3}}{2}}\right) \qquad \ldots(i)$$

given $\quad y = 0, \quad \frac{dy}{dx} = \tan\psi = \tan 60° = \sqrt{3} \quad \text{when} \quad x = 1$

$$\frac{dy}{dx} = \left(c_1x^{\frac{\sqrt{3}}{2}} + c_2x^{-\frac{\sqrt{3}}{2}}\right) + x\left(\frac{\sqrt{3}}{2}c_1x^{\left(\frac{\sqrt{3}}{2} - 1\right)} - \frac{\sqrt{3}}{2}c_2x^{\left(\frac{\sqrt{3}}{2} - 1\right)}\right) \qquad \ldots(ii)$$

From (i) $\quad 0 = c_1 + c_2 \Rightarrow c_2 = -c_1$

From (ii) $\quad \sqrt{3} = c_1 + c_2 + \frac{\sqrt{3}}{2} c_1 - \frac{\sqrt{3}}{2} c_2$

$\therefore \quad \sqrt{3} = c_1 - c_1 + \frac{\sqrt{3}}{2} c_1 + \frac{\sqrt{3}}{2} c_1 \Rightarrow \sqrt{3} = 2\,\frac{\sqrt{3}}{2} c_1 \Rightarrow c_1 = 1\; c_2 = -1$

The complete solution is

$$\boxed{y = x\left(x^{\frac{\sqrt{3}}{2}} - x^{\frac{-\sqrt{3}}{2}}\right)}$$

Example : 6

$$3x\frac{d^2y}{dx^2} + 3x^2\frac{dy}{dx} + xy = \sin(\log x)$$

Solution :

Divide by x we get

$$x^3\frac{d^2y}{dx^2} + 3^x\frac{dy}{dx} + y = \frac{1}{x}\sin(\log x)$$

Put $\quad x = e^z, \quad z = \log x$

then $\quad x\frac{dy}{dx} = Dy,$

$$x^2\frac{dy}{dx^2} = D(D-1)y$$

where $\quad D = \frac{d}{dz}$

The above D.E. becomes

$$\therefore \quad D(D-1)y + 3Dy + y = e^{-z}\sin z$$

$$(D^2 + 2D + 1)y = e^{-z}\sin z$$

Its auxiliary equation is

$$m^2 + 2m + 1 = 0,$$

$$\Rightarrow \quad m = -1, -1$$

The roots are real and equal.

$$\text{C.F.} = (c_3 + c_2 z)e^{-z}$$

$$\text{P.I.} = \frac{1}{D^2 + 2D + 1}e^{-z}\sin z$$

$$= e^{-z}\frac{1}{(D-1)^2 + 2(D-1) + 1}\sin z$$

$$\text{P.I.} = e^{-z}\frac{1}{D^2 - 2D + 1 + 2D - 2 + 1}\sin z$$

$$= e^{-z}\frac{1}{D^2}\sin z = e^{-z}\frac{1}{(-1^2)}\sin z$$

$$\text{P.I.} = -e^{-z}\sin z$$

The complete solution is y = C.F. + P. I.

$$y = (c_1 + c_2 z)\, e^{-z} - e^{-z}\sin z$$

$$\boxed{y = (c_1 + c_2 \log x)\frac{1}{x} - \frac{1}{x}\sin(\log x)}$$

Example : 7

Solve $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = x^2 e^x$

Solution :

Put $x = e^z \qquad z = \log x$ then

$$x\frac{dy}{dx} = Dy \quad \text{where} \quad D = \frac{d}{dz}$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$$\therefore \quad D(D-1)\,y + Dy - y = e^{2z}e^{e^z}$$

$$(D^2 - D + D - 1)\,y = e^{2z}e^{e^z}$$

$$(D^2 - 1)\,y = e^{2z}e^{e^z}$$

Its auxiliary equation is

$$m^2 - 1 = 0$$

$$\therefore \quad m = \pm 1$$

The roots are real and distinct.

$$\text{C. F.} = c_1 e^z + c_2 e^{-z}$$

$$\text{P. I.} = uy_1 + vy_2$$

Where $y_1 = e^z, \; y_2 = e^{-z}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^z & e^{-z} \\ e^z & -e^{-z} \end{vmatrix} = -1 - 1 = -2$$

$$u = -\int\frac{Xy_2}{W} = -\int\frac{e^{2z}e^{ez}}{-2}\,e^{-z}\,dz = \frac{1}{2}\int e^{e^z}e^z\,dz$$

Put $e^z = t \qquad e^z\,dz = dt$

$$u = \frac{1}{2}\int e^t\,dt = \frac{1}{2}e^t = \frac{1}{2}e^{ez}$$

$$v = -\int\frac{Xy_1}{W} = \int\frac{e^{2z}e^{ez}}{-2}\,e^z\,dz = -\frac{1}{2}\int t^2 e^t\,dt.$$

$$v = -\frac{1}{2}[\,t^2e^t - (2t)\,e^t + (2)e^t - 0] = -\frac{1}{2}(t^2 - 2t + 2)\,e^t$$

$$v = -\frac{1}{2}(e^{2z} - 2e^z + 2)\,e^{ez}.$$

The complete solution is y = C.F. + P. I.

$$y = c_1e^z + c_2e^{-z} + \frac{1}{2}e^{e^z}.\,e^z - \frac{1}{2}(e^{2z} - 2e^z + 2)\,e^{e^z}.\,e^{-z}.$$

$$\boxed{y = c_1x + c_2\frac{1}{x} + \frac{1}{2}xe^x - \frac{1}{2}(x^2 - 2x + 2)\frac{e^x}{x}}$$

1.5 Legendre's Homogeneous Differential Equation :

The differential equation of the form

$$(ax + b)^n\frac{d^ny}{dx^n} + a_1(ax + b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_{n-1}(ax + b)\frac{dy}{dx} + a_ny = X \qquad \ldots(i)$$

Where a_1, a_2, $a_3 \ldots a_n$, a, b are constants and x is a function of x or constant. The above equation is known as Legendre's homogeneous differential equation of n^{th} order. For finding the solution of the above equation we put

$$ax + b = e^z \Rightarrow x = \frac{1}{a}(e^z - b) \qquad z = \log(ax + b)$$

and $$\frac{dz}{dx} = \frac{a}{ax + b} \quad \text{then} \quad \frac{dy}{dx} = \frac{dy}{dz} - \frac{dz}{dz} = \frac{dy}{dz}\,\frac{a}{ax + b}$$

$$\therefore \quad (ax + b)\frac{dy}{dx} = a\frac{dy}{dz} = aDy \text{ where } D = \frac{d}{dz}$$

Similarly$$(ax + b)^2\frac{d^2y}{dx^2} = a^2\,D\,(D - 1)y$$

$$(ax + b)^3\frac{d^3y}{dx^3} = a^3\,D(D - 1)\,(D - 2)y \text{ and soon.}$$

From eqn. (i) we get $(D^n + 14D^{n-2} + k_2D^{n-2} + k_n)\,y = X(z)$

This is f (D) y = x L. D. E.

By using previous methods we find solution.

The complete solution is y = C.F. + P. I.

Illustrative Examples

Example : 1

Solve $(2x + 5)^2\frac{d^2y}{dx^2} - 6\,(2x + 5)\frac{dy}{dx} + 8y = 6x$

Solution :

Put $2x + 5 = e^z,\ z = \log(2x + 5)$

$$x = \frac{1}{2}(e^z - 5) \quad \text{then}$$

$$(2x + 5)\frac{dy}{dx} = 2\,Dy$$

$$(2x + 5)^2\frac{d^2y}{dx^2} = 2^2\,D\,(D - 1)y \;\text{ where } D = \frac{d}{dz}$$

The given D.E. becomes.

$$4D(D - 1)\,y - 6.\,2Dy + 8y = 6.\,\frac{1}{2}(e^z - 5)$$

$$(4D^2 - 4D - 12D + 8)y = 3(e^z - 5)$$

$$4(D^2 - 4D + 2)y = 3(e^z - 5)$$

$$= (D^2 - 4D + 2)\,y = \frac{3}{4}(e^z - 5)$$

Its auxiliary equation is.

$$m^2 - 4m + 2 = 0$$

$\therefore$ $m = 2 \pm \sqrt{2}$ the roots are real and distinct.

$$\text{C.F.} = c_1e^{(2+\sqrt{2})z} + c_2e^{(2-\sqrt{2})\,z}$$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 2}\,\frac{3}{4}(e^z - 5)$$

$$= \frac{3}{4}\left[\frac{1}{D^2 - 4D + 2}e^z - \frac{1}{D^2 - 4D + 2}5e^{oz}\right]$$

$$= \frac{3}{4}\left[\frac{1}{1 - 4 + 2}e^z - 5\frac{1}{0 - 0 + 2}e^{oz}\right]$$

$$= \frac{3}{4}\left(-e^z - \frac{5}{2}\right)$$

The complete solution is y = C.F. + P. I.

$$y = c_1\,e^{(2+\sqrt{2})z} + c_2\,e^{(2-\sqrt{2})z} - \frac{3}{4}\left(e^z + \frac{5}{2}\right)$$

$$\boxed{y = c_1(2x + 5)^{(2+\sqrt{2})} + c_2\,(2x + 5)^{(2-\sqrt{2})} - \frac{3}{4}\left[(2x + 5) + \frac{5}{2}\right]}$$

Example : 2

Solve $(3x + 2)^2\dfrac{d^2y}{dx^2} + 3\,(3x + 2)\dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1$

Solution :

Put $3x + 2 = e^z,\; z = \log\,(3x + 2)$

$$x = \frac{1}{3}(e^z - 2) \quad \text{then}$$

$$(3x+2)\frac{dy}{dx} = 3\,Dy$$

$$(3x+2)^2\frac{d^2y}{dx^2} = 9^1\,D\,(D-1)y \quad \text{where } D = \frac{d}{dz}$$

The given D.E. becomes.

$$9D(D-1)\,y + 3.\,3Dy - 36y = 3\left[\frac{1}{3}(e^z-2)\right]^2 + 4.\frac{1}{3}(e^z-2) + 1$$

$$9[D^2 - D + D - 4]y = \frac{3}{9}\,(e^{2z} - 4e^z + 4) + \frac{4}{3}\,(e^z - 2) + 1$$

$$(D^2-4)y = \frac{1}{27}\left(e^{2z} - \frac{1}{3}\right)$$

Its auxiliary equation is.

$$m^2 - 4 = 0$$

$$\therefore \quad m = \underline{+}\,2 \text{ the roots are real and distinct.}$$

$$\text{C.F.} = c_1e^{2z} + c_2e^{-2z}$$

$$\text{P.I.} = \frac{1}{D^2-4}\,\frac{1}{27}\left(e^{2z} - \frac{1}{3}\right)$$

$$= \frac{1}{27}\left[\frac{1}{D^2-4}\,e^{2z} - \frac{1}{D^2-4}\cdot\frac{1}{3}\,e^{oz}\right]$$

$$= \frac{1}{27}\left[z.\,\frac{1}{2D}\,e^{2z} - \frac{1}{0-4}\cdot\frac{1}{3}\,e^{oz}\right]$$

$$\text{P.I.} = \frac{1}{27}\left(\frac{z}{4}\,e^{2z} + \frac{1}{12}\right) = \frac{1}{108}\left(ze^{2z} + \frac{1}{3}\right)$$

The complete solution is y = C.F. + P. I.

$$y = c_1\,e^{2z} + c_2\,e^{-2z} + \frac{1}{108}\left(ze^{2z} + \frac{1}{3}\right)$$

$$\boxed{y = c_1(3x+2)^2 + c_2\,(3x+2)^{-2} + \frac{1}{108}\left[(3x+2)^2\log(3x+2) + \frac{1}{3}\right]}$$

Example : 3

Solve $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = \log(1+x)^4 + \cos[\log(1+x)]$

Solution :

Put $1 + x = e^z,\ z = \log(1+x)$

$$x = (e^z - 1) \qquad \text{then}$$

$$(1+x)\frac{dy}{dx} = Dy, \quad (1+x)^2\frac{d^2y}{dx^2} = D\,(D-1)y$$

The D.E. becomes.

$$D(D-1)\,y + Dy + y = 4z + \cos z$$

$$\therefore \quad [D^2 - D + D + 1]\,y = 4z + \cos z$$

$$\Rightarrow \quad (D^2 + 1)y = 4z + \cos z$$

Its auxiliary equation is.

$$m^2 + 1 = 0$$

$$\therefore \quad m = \pm i$$

The roots are imaginary and distinct.

$$\text{C.F.} = c_1\cos z + c_2\sin z.$$

$$\text{P.I.} = \frac{1}{D^2+1}(4z + \cos z) = +[1 + D^2]^{-1}.4z + \frac{1}{D^2+1}\cos z$$

$$+ 4[1 - D^2 + \ldots.]\,z + z\frac{1}{2D}\cos z = +4(z+0) + \frac{z}{2}\int \cos z\,dz$$

$$\text{P.I.} = 4z + \frac{z}{2}\sin z$$

The complete solution is y = C.F. + P. I.

$$y = c_1\cos z + c_2\sin z + 4z + \frac{z}{2}\sin z.$$

$$\boxed{y = c_1\cos(\log(1+x)) + c_2\sin(\log(1+x)) + \frac{\log(1+x)}{2}[\sin(\log(1+x)) + 8}$$

Example : 4

Solve $(x+1)^2\dfrac{d^2y}{dx^2} + (x+1)\dfrac{dy}{dx} = (2x+3)(2x+4)$ or

$= 2(2x^2 + 7x + 6)$

Solution :

Put $x + 1 = e^z$, $\therefore$ $x = e^z - 1$, $z = \log(1+x)$ then

$$(1+x)\frac{dy}{dx} = Dy, \quad (1+x)^2\frac{d^2y}{dx^2} = D(D-1)y$$

Where $D = \dfrac{d}{dz}$

The D.E. becomes.

$$D(D-1)\,y + Dy = 2[2(e^z-1)^2 + 7(e^z-1) + 6]$$

$$\therefore \quad (D^2 - D + D)\,y = 2[2(e^{2z} - 2e^z + 1) + 7e^z - 7 + 6]$$

$$D^2y = 2[2e^{2z} + 3e^z + 1]$$

Its auxiliary equation is

$$m^2 = 0$$

$\therefore \quad m = 0, 0$

The roots are real and equal.

$$\text{C.F.} = c_1 + c_2 z.$$

$$\text{P.I.} = \frac{1}{D^2}\,(2e^{2z} + 3e^z + 1) = 2\left[2\,\frac{1}{D^2}\,e^{2z} + 3\,\frac{1}{D^2}\,e^z + \frac{1}{D^2}\,e^{oz}\right]$$

$$\text{P.I.} = 2\left[2\,.\,\frac{1}{4}\,e^{2z} + 3e^z + z\,.\,\frac{1}{2D}\,e^{oz}\right] = 2\left[\frac{1}{2}\,e^{2z} + 3e^z + \frac{z}{2}\,.z\,.\frac{1}{1}\,e^{oz}\right]$$

$$\text{P.I.} = e^{2z} + 6e^z + z^2.$$

The complete solution is y = C.F. + P. I.

$$y = c_1 + c_2 z + e^{2z} + 6e^z + z^2$$

$$\boxed{y = c_1 + c_2 \log(1 + x) + (1 + x)^2 + 6(1 + x) + (\log(1 + x))^2}$$

Example : 5

$\left(\frac{d^2}{dx^2} - \frac{2}{(1 + x)^2}\right) y = (1 + x)^2$ using method of variation of parameters.

Solution :

Multiply by $(1 + x)^2$ to give D. E. we get

$$(1 + x)^2 \frac{d^2y}{dx^2} - 2y = (1 + x)^4$$

Put $1 + x = e^z$, $x = (e^z - 1)$ $z = \log(1 + x)$ then

$$(1 + x)\frac{dy}{dx} = Dy, \quad (1 + x)^2 \frac{d^2y}{dx^2} = D(D - 1)y \quad \text{where } D = \frac{d}{dz}$$

The given D. E. becomes

$$D(D - 1)\,y - 2y = e^{4z}$$

$$\therefore \quad (D^2 - D - 2)\,y = e^{4z}$$

Its auxiliary equation is

$$\Rightarrow \quad m^2 - m - 2 = 0$$

$$\Rightarrow \quad (m + 1)(m - 2) = 0$$

$$\therefore \quad m = -1, 2$$

The roots are real and distinct.

$$\text{C. F.} = c_1 e^{-z} + c_2 e^{2z}$$

$$\text{P. I.} = uy_1 + vy_2$$

Where $y_1 = e^{-z}$, $y_2 = e^{2z}$.

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-z} & e^{2z} \\ -e^{-z} & 2e^{2z} \end{vmatrix} = 2e^z + e^z = 3e^z$$

$$u = -\int \frac{Xy_2}{W} = -\int \frac{e^{4z}}{3e^{z}}\, e^{2z}\, dz = -\frac{1}{3}\int e^{5z}\, dz = -\frac{e^{5z}}{15}$$

$$v = \int \frac{Xy_1}{W} = \int \frac{e^{4z}}{3e^{z}}\, e^{-z}\, dz = \frac{1}{3}\int e^{2z}\, dz = \frac{1}{6}\, e^{2z}$$

$$\text{P. I.} = -\frac{e^{5z}}{15}\, e^{-z} + \frac{1}{6}\, e^{2z}.e^{2z} = -\frac{1}{15}\, e^{4z} + \frac{1}{6}\, e^{4z} = \frac{(-2+5)}{30}\, e^{4z}$$

$$\text{P. I.} = \frac{1}{10}\, e^{4z}.$$

The complete solution is y = C. F. + P. I.

$$y = c_1 e^{-z} + c_2 e^{2z} + \frac{1}{10}\, e^{4z}$$

$$\boxed{y = c_1\,(1+x)^{-1} + c_2\,(1+x)^2 + \frac{1}{10}\,(1+x)^4}$$

1.6 System of simultaneous linear differential equation with constant coefficients :

In several applied mathematics problems, there are more than one dependent variables. Each of which is a function of one independent variable usually say time t. The formulation of such problems leads to a system (or a family) of simultaneous L.D.E. with constant co–efficients, such a system can be solved by the method of elimination, Laplace transform method, method using matrices and short cut operator methods.

Here only the method of elimination is considered.

Note : The number of simultaneous equations in the system = Number of dependent variables.

1.7 Method of Elimination :

Consider a system of two O. D. E. in two dependent variables x and y and one independent variable t given by

$$f_1(D)x + g_1(D)\,y = h_1(t)$$

$$f_2(D)x + g_2(D)y = h_2(t)$$

Where f_1, f_2, g_1, g_2 are all functions of the differential operator $D = \frac{d}{dt}$.

1. Eliminate y from the given system, resulting in a D. E. exclusively in x alone.
2. Solve the D. E. for x.
3. Substituting x (obtained step 2) in a similar manner. Obtain a D. E. only in y.
4. Solve the D. E. obtained in 3 for y.
5. The solution of one variable is known. For calculating the solution of second variable we can use any one given D. E. so that its derivative term should not contain in this equation.

Illustrative Examples

Example : 1

The system of equations

$$\frac{dx}{dt} + y = \sin t, \quad \frac{dy}{dt} + 4x = \cos t \quad \text{given that,} \quad x = 0, y = 1 \text{ when } t = 0.$$

Solution : Put $\frac{d}{dt} = D$, the equations reduces to

$$Dx + y = \sin t \qquad \ldots(i)$$

$$4x + Dy = \cos t \qquad \ldots(ii)$$

Eliminating y from (i) and (ii) we get

$$D^2x + Dy = \cos t \ (D(\sin t) = \cos t)$$

$$4x + Dy = \cos t$$

$$- \quad - \qquad -$$

$$D^2 x - 4x + 0 = 0$$

$(D^2 - 4)\, x = 0$ its auxiliary equation is $m^2 - 4 = 0$

$\therefore \quad m = \pm 2$

The roots are real and distinct. then

$$x = c_1 e^{2t} + c_2 e^{-2t} \qquad \ldots(iii)$$

For calculating the solution of y variable we use equation

(1) We have $y = \sin t - \frac{dx}{dt}$

$$\therefore \quad y = \sin t - 2c_1e^{2t} + 2c_2e^{-2t}$$

$$y = -2c_1e^{2t} + 2c_2e^{-2t} + \sin t \qquad \ldots(iv)$$

Given $x = 0,\ y = 1$, when $t = 0$

From (iii) we get $0 = c_1 + c_2 \Rightarrow c_1 = -c_2$

From (iv) we get $1 = -2c_1 + 2c_2 + 0 \Rightarrow 1 = 2c_2 + 2c_2$

$$c_2 = \frac{1}{4},\ c_1 = -\frac{1}{4}$$

The complete solution is

$$\boxed{\begin{aligned} x &= -\frac{1}{4}e^{2t} + \frac{1}{4}e^{-2t} = -\frac{1}{2}\left(\frac{e^{2t} - e^{-2t}}{2}\right) = -\frac{1}{2}\sinh 2t. \\ y &= \frac{1}{2}e^{2t} + \frac{1}{2}e^{-2t} + \sin t = \left(\frac{e^{2t} + e^{-2t}}{2}\right) + \sin t = \cosh 2t + \sin t. \end{aligned}}$$

Example : 2

Solve the system of equations

$$\frac{dx}{dt} + y = e^t, \quad x - \frac{dy}{dt} = e^{-t} \quad \text{given that,} \quad x = 1, y = 0 \text{ when } t = 0.$$

Solution :

Put $\frac{d}{dt} = D$, the above equations reduces to

$$Dx + y = e^t \quad \ldots(i) \qquad \text{Eliminating y from (i) and (ii)}$$

$$x - Dy = e^{-t} \quad \ldots(ii) \qquad \text{we get}$$

$$D^2x + x + 0 = e^t + e^{-t}$$

$$D^2x + Dy = e^t \ (De^t = e^t)$$

$$x - Dy = e^{-t}$$

$$D^2x + x + 0 = e^t + e^{-t}$$

The auxiliary equation is $(m^2 + 1) = 0$

$\therefore \quad m = \pm i$

The roots are imaginary and distinct.

$$\text{C.F.} = c_1\cos t + c_2\sin t$$

$$\text{P.I.} = \frac{1}{D^2+1}(e^t + e^{-t})$$

$$= \frac{1}{D^2+1}e^t + \frac{1}{D^2+1}e^{-t} = \frac{1}{2}e^t + \frac{1}{2}e^{-t}$$

The complete solution is = x = C. F. + P. I.

$$x = c_1\cos t + c_2\sin t + \frac{1}{2}e^t + \frac{1}{2}e^{-t} \quad \ldots(iii)$$

For finding the solution of y variable we use equation (i) we have

$$y = e^t - \frac{dx}{dt}$$

$$y = e^t - [-c_1\sin t + c_2\cos t + \frac{1}{2}e^t - \frac{1}{2}e^{-t}]$$

$$y = -c_2\cos t + c_1\sin t + \frac{1}{2}e^t + \frac{1}{2}e^{-t}] \quad \ldots(iv)$$

given $x = 1, y = 0$, when $t = 0$

from (iii) we get $1 = c_1 + 0 + \frac{1}{2} + \frac{1}{2} \Rightarrow 1 = c_1 + 1 \Rightarrow c_1 = 0$

from (iv) we get $0 = -c_2 + 0 + \frac{1}{2} + \frac{1}{2} \Rightarrow 0 = -c_2 + 1 \Rightarrow c_2 = 1$

The complete solution is

$$\boxed{\begin{aligned} x &= \sin t + \left(\frac{e^t - e^{-t}}{2}\right) = \sin t + \cosh t \\ y &= -\cos t + \frac{1}{2}(e^t + e^{-t}) = -\cos t + \cosh t \end{aligned}}$$

Example : 3

In a heat exchange the temperatures u, v of two liquids satisfy the equations

$$4\frac{du}{dx} = v - u = 2\frac{dv}{dx} \quad \text{Solve the equations}$$

for u and v given that u = 20, v = 100 when x = 0.

Solution :

The given D. E. can be written as

$$4\frac{du}{dx} = v - u \text{ and } 2\frac{dv}{dx} = v - u$$

Put $\frac{d}{dx} = D$ the above equation reduces to

$$(4D + 1)u = v \qquad \text{…(i) eliminating v from (i) and (ii)}$$

$$(2D - 1)v = -u \qquad \text{…(ii) we get}$$

$$(2D - 1)(4D + 1)u = -u$$

$$(8D^2 + 2D - 4D - 1 + 1)u = 0$$

$$(8D^2 - 2D)u = 0$$

$$\therefore \quad (4D^2 - D)u = 0$$

Its auxiliary equation is

$$4m^2 - m = 0$$

$$\therefore \quad m = 0, \frac{1}{4}$$

The roots are real and distinct.

$$\therefore \quad u = c_1 + c_2 e^{\frac{1}{4}x} \qquad \text{…(iii)}$$

For finding the solution of v variable we use equation (i) we have

$$v = u + 4\frac{du}{dx}$$

$$v = c_1 + c_2 e^{\frac{1}{4}x} + 4\left(0 + c_2 \frac{1}{4} e^{\frac{1}{4}x}\right)$$

$$v = c_1 + 2c_2 e^{\frac{1}{4}x} \qquad \text{…(iv)}$$

Given u = 20, v = 100 when x = 0

From (iii) we get $20 = c_1 + c_2$ …(v)

From (iv) we get $100 = c_1 + 2c_2$ …(vi)

Taking (vi) – (v) we get

$$80 = c_2$$

$$-60 = c_1$$

The complete solution of u and v are.

$$\boxed{\begin{aligned} u &= -60 + 80\, e^{\frac{1}{4}x} \\ v &= -60 + 160\, e^{\frac{1}{4}x} \end{aligned}}$$

Example : 4

Solve the system of equations.

$$\frac{dx}{dt} + y = 7x,\ 3\frac{dx}{dt} - \frac{dy}{dt} = 5(3x - y).$$ Solve the equations.

Solution :

Put $\frac{d}{dx} = D$ the above D. E. reduces to

$$(D - 7)x + y = 0 \qquad \ldots(i)$$ eliminating from (i) and (ii) we get

$$(3D - 15)x - (D - 5)y = 0 \qquad \ldots(ii)$$

$$3(D - 5)x + (D - 5)(D - 7)x = 0 \Rightarrow (D^2 - 9D + 20)x = 0$$

Its auxiliary equation is

$$m^2 - 9m + 20 = 0$$

$$\therefore \quad m = 4, 5$$

The roots are real and distinct

$$\boxed{x = c_1e^{4t} + c_2e^{5t}}$$

For finding the solution of y variable we take equation (i) we have

$$y = 7x - \frac{dx}{dt}$$

$$\therefore \quad y = 7(c_1e^{4t} + c_2e^{5t}) - (4c_1e^{4t} + 5c_2e^{5t})$$

$$\boxed{y = 3c_1e^{4t} + 2c_2e^{5t}}$$

Example : 5

If $$\frac{dx}{dt} - wy = a\cos pt \text{ and } \frac{dy}{dt} + wx = a\sin pt$$

Find x and y in terms of t. $(p \neq w)$

Solution :

Put $\frac{d}{dt} = D$ the above D. E. reduces to

$$Dx - wy = a\cos pt \qquad \ldots(i)$$

$$wx + Dy = a\sin pt \qquad \ldots(ii)$$

Eliminating y variable from (i) and (ii) we get

$$\begin{array}{rcl} D^2x - wDy & = & -ap\,\text{sinpt} \\ w^2x + wDy & = & a\,w\,\text{sinpt} \\ \hline D^2x + w^2x + 0 & = & a\,(w-p)\,\text{sinpt} \end{array}$$

$$\therefore \quad (D^2 + w^2)\,x = a\,(w-p)\,\text{sinpt}$$

The auxiliary equation is

$$m^2 + w^2 = 0 \Rightarrow m = \pm\, i\,w$$

The roots are imaginary and distinct.

$$\text{C. F.} = c_1 \cos wt + c_2 \sin wt.$$

$$\text{P. I.} = \frac{1}{D^2 + w^2}\, a\,(w-p)\,\text{sinpt}$$

$$= a(w-p)\frac{1}{-p^2 + w^2}\,\text{sinpt (where } w \neq p)$$

$$\text{P. I.} = \frac{a\,(w-p)}{(w-p)\,(w+p)}\,\text{sinpt}$$

$$= \frac{a}{w+p}\,\text{sinpt.}$$

The complete solution is x = C. F. + P. I.

$$\boxed{x = c_1\cos wt + c_2 \sin wt + \frac{a}{w+p}\sin pt} \quad \ldots\text{(iii)}$$

For finding the solution of y variable we use equation (i) we have

$$wy = \frac{dx}{dt} - a\,\text{cospt}$$

$$\therefore \quad wy = -w\,c_1 \sin wt + w\,c_2 \cos wt + \frac{a}{w+p}\,p\cos pt - a\cos pt$$

$$y = c_2 \cos wt - c_1 \sin wt + \left[\frac{ap}{w+p}(+\cos pt) - a\cos pt\right]\frac{1}{w}$$

$$y = c_2 \cos wt - c_1 \sin wt - \frac{aw}{w(w+p)}\cos pt$$

$$\boxed{y = c_2 \cos wt - c_1 \sin wt - \frac{a}{w+p}\,\text{cospt}} \quad \ldots\text{(iv)}$$

(iii) and (iv) are the required solutions for x and y variables.

1.8 Simultaneous Total Differential Equations :

The simultaneous total differential equations in three variables are given by

$$P_1dx + Q_1\,dy + R_1\,dz = 0$$

$$P_2dx + Q_2\,dy + R_2\,dz = 0 \quad \ldots(1)$$

Where P_1, Q_1, R_1, P_2, Q_2, R_2 are in general functions of x, y, z. If these equations be integrable with the solutions say $F_1(x, y, z) = c_1$ and $F_2(x, y, z) = c_2$ then these together constitute the solution of the D. E. (1).

If, however one or both of the equations in (1) be non–integrable, we can write them in the symmetrical form as $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ …(2)

Where $P = Q_1R_2 - Q_2R_1$, $Q = R_1 P_2 - R_2 P_1$, $R = P_1Q_2 - P_2Q_1$.

Symmetrical simultaneous equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

a) Method of grouping :

Sometimes it is convenient to take two terms in these equations say $\frac{dx}{P} = \frac{dy}{Q}$.

When z is absent or can be cancelled from this equation. Leaving with a differential equation
in x and y only.

This on integration gives one solution. To obtain second solution of the system of equations or using, the first solution, say x may be eliminated from the equation $\frac{dy}{Q} = \frac{dz}{R}$ leading to a differential equation in y and z. which on integration and substituting for the first constant of integration from, the previous solution gives the second solution of the system of equations.

Illustrative Examples

Example : 1

Solve $\frac{dx}{y^2z} = \frac{dy}{x^2z} = \frac{dz}{y^2x}$

Solution :

Taking the first two terms in the equation and cancelling z: we have

$$\frac{dx}{y^2} = \frac{dy}{x^2} \Rightarrow x^2dx - y^2dy = 0$$

an equation in x and y only. Its integration is

$$x^3 - y^3 = c_1 \quad \text{…(i)}$$

Similarly taking the first and the last terms of the given equation and cancelling y^2, leads to the equation

$$\frac{dx}{z} = \frac{dz}{x} \quad \text{or} \quad x\,dx - z dz = 0$$

an equation in x and z only this has for its integral

$$x^2 - z^2 = c_2 \quad \text{…(ii)}$$

Equation (i) and (ii) are the general solutions of the given simultaneous equations.

Example : 2

Solve $\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{xyz - 2x^2}$

Solution :

Taking first two terms we have

$$\frac{dx}{x} = \frac{dy}{y}$$

Its integration is $\log x - \log y = \log c_1 \Rightarrow \log \frac{x}{y} = \log c_1$

$$\therefore \quad \frac{x}{y} = c_1 \quad \ldots(i)$$

Similarly taking first and third term we have

$$\frac{dx}{x} = \frac{dz}{x(yz - 2x)} \Rightarrow dx = \frac{dz}{yz - 2x}$$ for eliminating

y from the equation we take from solution (i)

$$y = \frac{x}{c_1}$$

$$\therefore \quad dx = \frac{dz}{\frac{x}{c_1} z - 2x} \Rightarrow \frac{x}{c_1} dx = \frac{dz}{z - 2c_1}$$

Its integration is $\frac{1}{c_1} \frac{x^2}{2} = \log (z - 2c_1) + c_2$

$$\frac{y}{x} \cdot \frac{x^2}{2} = \log \left(z - 2\frac{x}{y}\right) + c_2 \Rightarrow \frac{xy}{2} = \log \left(z - 2\frac{x}{y}\right) + c_2 \quad \ldots(ii)$$

Equation (i) and (ii) are the required solutions.

Example : 3

Solve $\frac{dx}{xz} = \frac{dy}{yz} = \frac{2dz}{(x + y)^2}$

Solution :

Taking first two terms we get

$$\frac{dx}{x} = \frac{dy}{y}$$

Its integration is $\frac{x}{y} = c_1 \quad \ldots(i)$

consider $\frac{dx + dy}{(x + y)z} = \frac{2dz}{(x + y)^2} \Rightarrow (x + y)(dx + dy) = 2z\,dz$

its integration is $(x + y)^2 - 2z^2 = c_2 \quad \ldots(ii)$

equation (i) and (ii) are the required solutions.

Example : 4

Solve $\dfrac{dx}{y+z} = \dfrac{dy}{z+x} = \dfrac{dz}{x+y}$

Solution :

Each term of the equation is equal to

$$\frac{dx-dy}{x-y} = \frac{dz-dx}{z-x}$$

Its integration is

$$\log(x-y) = \log(z-x) + \log c_1 \Rightarrow x - y = c_1(z-x) \qquad \ldots(i)$$

Next we consider

$$\frac{dx+dy+dz}{2(x+y+z)} = -\frac{dx-dy}{x-y}$$

$$\therefore \quad \frac{dx+dy+dz}{(x+y+z)} + 2\frac{dx-dy}{(x-y)} = 0 \qquad \text{its integration is}$$

$$\log(x+y+z) + 2\log(x-y) = \log c_2$$

$$\therefore \quad (x+y+z)(x-y)^2 = c_2 \qquad \ldots(ii)$$

Equations (i) and (ii) are the required solutions.

Example : 5

Solve $\dfrac{dx}{(x+y)z} = \dfrac{dy}{(x-y)\,z} = \dfrac{dz}{x^2+y^2}$

Solution :

Taking first two terms we have

$$\frac{dx}{x+y} = \frac{dy}{x-y} \Rightarrow x\,dx - y\,dx = x\,dy + y\,dy$$

$$\therefore \quad x\,dx - y\,dy = x\,dy + y\,dx = d(xy)$$

Its integration is $\dfrac{x^2}{2} - \dfrac{y^2}{2} = xy + c_1 \qquad \ldots(i)$

Next we consider $\dfrac{x\,dx - y\,dy}{(x^2+y^2)z} = \dfrac{dz}{x^2+y^2}$

$$x\,dx - y\,dy = z\,dz$$

its integration is $x^2 - y^2 - z^2 = c_2 \qquad \ldots(ii)$

Equations (i) and (ii) are the required solutions.

Example : 6

Solve $\dfrac{dx}{2x} = \dfrac{dy}{-y} = \dfrac{dz}{4xy^2 - 2z}$

Solution :

Consider first two terms we have

$$\frac{dx}{2x} = \frac{dy}{-y}$$

$$\therefore \qquad \frac{dx}{x} + \frac{2}{y}dy = 0$$

Its integration is

$$\log x + 2\log y = \log c_1 \Rightarrow xy^2 = c_1 \qquad \ldots(i)$$

Similarly we consider

$$\frac{dx}{2x} = \frac{dz}{4xy^2 - 2z} \Rightarrow \frac{dx}{2x} = \frac{dz}{2\,(2c_1 - z)}$$

Its integration is $\log x = -\log(2c_1 - z) + \log c_2$

$$\therefore \qquad \log x + \log(2c_1 - z) = \log c_2 \Rightarrow \log[x\,(2xy^2 - z)] = \log c_2$$

$$x(2xy^2 - z) = c_2 \qquad \ldots(ii)$$

Equations (i) and (ii) are required solutions.

Example : 7

Solve $\dfrac{dx}{y + zx} = \dfrac{dy}{-x - yz} = \dfrac{dz}{x^2 - y^2}$

Solution :

Each term of the equation is equal to

$$\frac{dx + dy}{(x - y)\,(z - 1)} = \frac{dz}{(x - y)\,(x + y)}$$

$$\Rightarrow (x + y)\,(dx + dy) = (z - 1)\,dz$$

$$\therefore \qquad x\,dx - y\,dy = x\,dy + y\,dx = d\,(xy)$$

On integration we get $\dfrac{(x + y)^2}{2} = \dfrac{z^2}{2} - z + c_1 \qquad \ldots(i)$

Similarly we consider

$$\frac{y\,dx + x\,dy}{y^2 - x^2} = -\frac{dz}{y^2 - x^2}$$

$$\Rightarrow \qquad d\,(xy) + dz = 0 \text{ this}$$

on integration gives $xy + z = c_2 \qquad \ldots(ii)$

Equations (i) and (ii) are the required solutions.

Example : 8

Solve $\dfrac{dx}{1} = \dfrac{dy}{3} = \dfrac{dz}{5z + \tan(y - 3x)}$

Solution :

Taking the first two terms we have

$$3dx = dy$$

This on integration gives

$$y - 3x = c_1 \qquad \ldots(i)$$

Similarly taking second and third terms and by using solution (i) we have

$$\frac{dy}{3} = \frac{dz}{5z + \tan c_1}$$

This on integration gives $\quad \frac{y}{3} = \frac{1}{5} \log (5z + \tan c_1) + c_2$

$$\therefore \quad \frac{y}{3} = \frac{1}{5} \log (5z + \tan (y - 3x)) + c_2 \qquad \ldots\text{(ii)}$$

Equations (i) and (ii) are the required solutions.

Example : 9

Solve $\dfrac{dx}{x^2 + y^2} = \dfrac{dy}{2\,xy} = \dfrac{dz}{(x + y)^3\, z^2}$

Solution :

Each term of the equation equals to

$$\frac{dx + dy}{(x + y)^2} = \frac{dx - dy}{(x - y)^2}$$

Its integration is $\quad -\frac{1}{x + y} = -\frac{1}{x - y} - c_1$

$$\therefore \quad \frac{1}{x + y} - \frac{1}{x - y} = c_1 \qquad \ldots\text{(i)}$$

Similarly we take

$$\frac{dx + dy}{(x + y)^2} = \frac{dz}{(x + y)^3\, z^2}$$

Its integration is $\quad \frac{(x + y)^2}{2} + \frac{1}{z} = c_2 \qquad \ldots\text{(ii)}$

Equations (i) and (ii) are the required solutions.

Example : 10

Solve $\dfrac{dx}{y} = \dfrac{dy}{-x} = \dfrac{dz}{x\, e^{x^2 + y^2}}$

Solution :

Taking first two terms we have

$$x\,dx + y\,dy = 0$$

Its integration is $\quad x^2 + y^2 = c_1 \qquad \ldots\text{(i)}$

Similarly we take

$$\frac{dy}{-x} = \frac{dz}{x\, e^{c_1}} \Rightarrow e^{c_1}\, y + z = c_2$$

$$y\, e^{\left(x^2 + y^2\right)} + z = c_2 \qquad \ldots\text{(ii)}$$

Equations (i) and (ii) are the required solutions.

b) Method of Multipliers :

if L, m, n are multipliers, which are not necessarily constants then we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{L\,dx + m\,dy + n\,dz}{LP + mQ + nR}$$

by proper choice of multipliers, they may be used to solve the simultaneous equations in various ways.

If it is possible to find a set of multipliers L, m, n in such a way that

$$LP + mQ + nR = 0,$$

Then $L\,dx + m\,dy + n\,dz = 0$ if this be exact we may find its integral as

$$f(x, y, z) = c_1 \qquad \text{…(i)}$$

This gives one solution of the simultaneous equations.

If it is also possible to the find another set of multipliers say L′, m′, n′, so that

$$L'P + m'Q + n'R = 0, \text{ then}$$

$$L'dx + m'dy + n'dz = 0 \text{ and its integral may be found as say}$$

$$\phi(x, y, z) = c_2 \qquad \text{…(ii)}$$

Thus (i) and (ii) constitute the solution of the given equations.

Illustrative Examples

Example : 1

$$\frac{dx}{mz - ny} = \frac{dy}{nx - Lz} = \frac{dz}{Ly - mx}$$

Solution :

Using the multipliers L, m, n each term of the equation is equal to

$$\frac{L\,dx + m\,dy + n\,dz}{L(mz - ny) + m(nx - Lz) + n(Ly - mx)} = \frac{L\,dx + m\,dy + n\,dz}{0} \qquad \text{…(i)}$$

$$L\,dx + m\,dy + n\,dz = 0$$

This on integration weget

$$Lx + my + nz = c_1$$

Another set of similar multipliers is given by x, y, z leading to the equation

$$x\,dx + y\,dy + z\,dz = 0 \text{ and has for its integral}$$

$$x^2 + y^2 + z^2 = c_2 \qquad \text{…(ii)}$$

Thus (i) and (ii) are solutions of the given equations.

Note : Sometimes, the term $\frac{L\,dx + m\,dy + n\,dz}{Lp + mQ + nR}$ formed from the terms in the given equation is found more convenient for solving the equations than the term or terms already existing in the equation in this case the numerator is the differential of Lp + mQ + nR.

Example : 2

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - xz} = \frac{dz}{z^2 - xy}$$

Solution :

Each term of the equation is equal to

$$\frac{dx - dy}{x - y} = \frac{dy - dz}{y - z} = \frac{dz - dx}{z - x} \qquad \text{…(i)}$$

and these new terms form more convenient simultaneous equations, than the given simultaneous equations. Thus taking the first two terms of equation (i) and as each is exact, we have on integration

$$\log(x - y) + \log(y - z) = \log c_1$$

Or

$$x - y = c_1 (y - z) \qquad \text{…(ii)}$$

Consider the result obtained by multiplying each term in the given equations by x, y, z respectively, then each term of the equation is equal to

$$[x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - yz - zx - xy)]$$

$$\frac{x\,dx + y\,dy + z\,dz}{x^3 + y^3 + z^3 - 3xyz} = \frac{dx + dy + dz}{x^2 + y^2 + z^2 - yz - zx - zy}$$

$$x\,dx + y\,dy + z\,dz = (x + y + z)(dx + dy + dz)$$

$$x^2 + y^2 + z^2 = (x + y + z)^2 + c_2 \qquad \text{…(iii)}$$

Equations (ii) and (iii) constitute the solution of the given equations.

Example : 3

$$\frac{dx}{x(2y^4 - x^4)} = \frac{dy}{y(z^4 - 2x^4)} = \frac{dz}{z(x^4 - y^4)}$$

Solution :

Using the multipliers x^3, y^3, z^3 to each term of the equation leads to

$$x^3\,dx + y^3\,dy + z^3\,dz = 0$$

its integration is

$$x^4 + y^4 + z^4 = c_1 \text{ …(i)}$$

Again we take the multipliers $\frac{1}{x}, \frac{1}{y}, \frac{2}{z}$ to each term of the equation leads to

$$\frac{dx}{x} + \frac{dy}{y} + \frac{2}{z}dz = 0$$

Its integration is

$$\log x + \log y + 2\log z = \log c_2 \quad \Rightarrow \quad xyz^2 = c_2 \qquad \text{…(ii)}$$

Equations (i) and (ii) are the solutions of the given equations.

Example : 4

$$\frac{x\,dx}{z^2 - 2yz - y^2} = \frac{dy}{y + z} = \frac{dz}{y - z}$$

Solution :

Taking second and third term we have $\frac{dy}{y + z} = \frac{dz}{y - z}$

this can be written as

$$y\,dy - z\,dz = d(yz)$$

its integration

$$y^2 - z^2 - 2yz = c_1 \qquad \ldots(i)$$

Next we take multipliers 1, y, z to each term of the equation leads to

$$x\,dx + y\,dy + z\,dz = 0$$

this on integration gives

$$x^2 + y^2 + z^2 = c_2 \qquad \ldots(ii)$$

Equations (i) and (ii) are the required solutions.

Example : 5

$$\frac{a\,dx}{(b-c)\,yz} = \frac{b\,dy}{(c-a)\,zx} = \frac{c\,dz}{(a-b)\,xy}$$

Solution :

Using the multipliers ax, by, cz to each term of the equation leads to

$$a^2x\,dx + b^2y\,dy + c^2z\,dz = 0$$

its integration is

$$a^2x^2 + b^2y^2 + c^2z^2 = c_1 \qquad \ldots(i)$$

next we take the multipliers x, y, z to each term of the equation leads to

$$ax\,dx + by\,dy + cz\,dz = 0$$

Its integration is

$$ax^2 + by^2 + cz^2 = c_2 \qquad \ldots(ii)$$

Equations (i) and (ii) are the required solutions.

Example : 6

$$\frac{dx}{y - zx} = \frac{dy}{x + yz} = \frac{dz}{x^2 + y^2}$$

Solution :

Using the multipliers y, x, – 1 to each term of the equation leads to

$$y\,dx + x\,dy - dz = 0$$

$$\therefore \quad d(xy) - dz = 0$$

its integration is

$$xy - z = c_1 \qquad \ldots(i)$$

next we take the multipliers x, – y, z to each term of the equation leads to

$$x\,dx - y\,dy + z\,dz = 0$$

Its integration gives

$$x^2 - y^2 + z^2 = c_2 \qquad \ldots(ii)$$

Equations (i) and (ii) are the required solutions.

Example : 7

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)}$$

Solution :

Using the multipliers x, y, z to each term of the equation leads to

$$x\,dx + y\,dy + z\,dz = 0$$

its integration is

$$x^2 + y^2 + z^2 = c_1 \quad \ldots(i)$$

Again we take the multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ to each term of the equation leads to

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Its integration is

$$\log x + \log y + \log z = \log c_2$$

$$xyz = c_2 \quad \ldots(ii)$$

Equations (i) and (ii) are the required solutions.

Example : 8

$$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$$

Solution :

Taking second and third terms we have

$$\frac{dy}{y} = \frac{dz}{z}$$

$$\Rightarrow \log y = \log z + \log c_1$$

$$\therefore \quad y = c_1 z \quad \ldots(i)$$

next we take the multipliers x, y, z each term of the equation leads to

$$\frac{x\,dx + y\,dy + z\,dz}{x(x^2 + y^2 + z^2)} = \frac{dy}{2xy}$$

its integration is

$$y = c_2 (x^2 + y^2 + z^2) \quad \ldots(ii)$$

Equations (i) and (ii) are the required solutions.

1.9 Modeling of Electrical Circuits :

Consider a closed and simple electrical circuit containing inductor, resistor, capacitor and some e.m.f. source.

q = Total charges in circuit at time t

I = Current flowing in circuit at time t

Then, $$I = \frac{dq}{dt}$$

Applying Kirchhoff′s Voltage Law to the above circuit we get linear differential equation which can be solved. We can find value of arbitrary constants by using initial conditions. Physical interpretation can be drawn from solution. This procedure is called modeling of electrical circuit.

1.9.1 Kirchhoff's Voltage Law :

The algebraic sum of all the voltage drops around an electric loop or circuit is equal to the resultant electromotive force (e.m.f.) in the circuit. We need the following laws.

1. (Voltage drop across inductance L) $= L\dfrac{dI}{dt}$
2. (Voltage drop across resistance R) $= R\,I$
3. (Voltage drop across capacitance C) $= \dfrac{q}{C}$

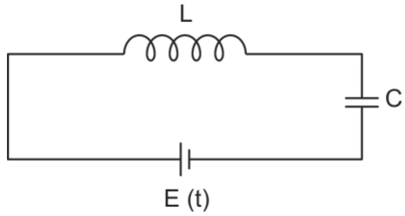

Fig.: 1.1

1. LC Circuit :

By KVL, $L\dfrac{dI}{dt} + \dfrac{q}{C} = E(t)$

But, $I = \dfrac{dq}{dt}$, so we get, $L\dfrac{d^2q}{dt^2} + \dfrac{q}{C} = E(t)$

2. LCR Circuit :

By KVL,

$$L\frac{dI}{dt} + RI + \frac{q}{C} = E(t)$$

But, $I = \dfrac{dq}{dt}$, so we get,

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E(t)$$

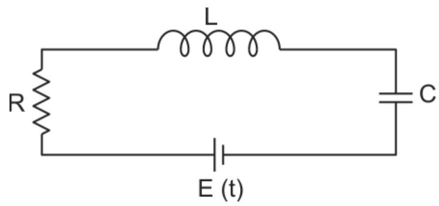

Fig.: 1. 2

Illustrative Examples

Example : 1

An inductor of 0.25 henries is connected in series with negligible resistance and a capacitor of 0.04 farads, a generator having alternative voltage given by 12 sin 10 t. Set up the differential equation and find the charge and current.

Solution : Given is LC circuit, By KVL we get,

$$L\frac{dI}{dt} + \frac{q}{C} = E$$

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = E \qquad \because \quad I = \frac{dq}{dt}$$

$$\frac{d^2q}{dt^2} + \frac{1}{LC}q = \frac{E}{L}$$

$$\left(D^2 + \frac{1}{LC}\right)q = \frac{E}{L}$$

Given that, $L = 0.25$ henries

$C = 0.04$ F

$$E = 12 \sin 10\,t$$

$$(D^2 + 100)\,q = 48 \sin 10\,t$$

AE is, $D^2 + 100 = 0$

$$(D - 10i)(D + 10i) = 0$$

$$D = \pm 10\,i$$

$$CF = C_1 \cos 10\,t + C_2 \sin 10\,t$$

$$PI = \frac{1}{D^2 + 100}\, 48 \sin 10\,t = \frac{1}{-100 + 100}\, 48 \sin 10\,t$$

$$= 48\,t \frac{1}{2D} \sin 10\,t = 24\,t \int \sin 10\,t\,dt$$

$$= 24\,t \frac{-\cos 10\,t}{10}$$

$$PI = -2.4\,t \cos 10\,t$$

C.S is $q = CF + PI$

$$q = c_1 \cos 10\,t + c_2 \sin 10\,t - 2.4\,t \cos 10\,t \qquad \ldots(1)$$

$$I = \frac{dq}{dt}$$

$$= -10c_1 \sin 10t + 10\,c_2 \cos 10t - 2.4\,(-10t \sin 10t + \cos 10t)$$

$$\mathbf{I = (-10\,c_1 + 24t) \sin 10\,t + (10\,c_2 - 2.4) \cos 10\,t \qquad \ldots(2)}$$

Example : 2

An uncharged condenser of capacity C charged by applying an e.m.f. of value $E \sin \frac{t}{\sqrt{LC}}$ through the leads of inductance L and of negligible resistance. The charge Q on the plate of condenser satisfies the differential equation : $\frac{d^2Q}{dt^2} + \frac{Q}{LC} = \frac{E}{L} \sin \frac{t}{\sqrt{LC}}$, prove that the charge at any time t is given by $Q = \frac{EC}{2}\left[\sin\frac{t}{\sqrt{LC}} - \frac{t}{\sqrt{LC}} \cos \frac{t}{\sqrt{LC}}\right]$

Solution :

Use, $D = \frac{d}{dt}$ in given differential equation, we get

$$\left(D^2 + \frac{1}{LC}\right) Q = \frac{E}{L} \sin \frac{t}{\sqrt{LC}}$$

Take $w^2 = \frac{1}{LC}$

$$(D^2 + w^2)\,Q = \frac{E}{L} \sin wt$$

AE is $D^2 + w^2 = 0$

$$(D - wi)(D + wi) = 0$$

$$D = \pm wi$$

$$CF = c_1 \cos wt + c_2 \sin wt$$

$$PI = \frac{1}{D^2 + w^2} \frac{E}{L} \sin wt$$

$$= \frac{E}{L} \frac{1}{D^2 + w^2} \sin wt$$

$$= \frac{E}{L} \frac{1}{-w^2 + w^2} \sin wt$$

$$= \frac{E}{L}\ t \frac{1}{2D} \sin wt$$

$$= \frac{E}{L}\ t \frac{1}{2} \int \sin wt\ dt$$

$$= \frac{E}{L} \frac{t}{2} \left(\frac{-\cos wt}{w}\right)$$

$$PI = -\frac{Et}{2Lw} \cos wt$$

The complete solution is

$$Q = CF + PI$$

$$Q = c_1 \cos wt + c_2 \sin wt - \frac{Et}{2Lw} \cos wt \qquad \ldots(1)$$

$$I = \frac{dQ}{dt}$$

$$= -c_1 w \sin wt + c_2 w \cos wt - \frac{E}{2Lw} [-wt \sin wt + \cos wt] \qquad \ldots(2)$$

At $t = 0$, $I = 0$ and $Q = 0$

(1) $\Rightarrow 0 = c_1 + 0 - 0 \Rightarrow c_1 = 0$

(2) $\Rightarrow 0 = -0 + c_2 w - \dfrac{E}{2\ Lw} [-0 + 1]$

$$\Rightarrow c_2 = \frac{E}{2\ Lw^2}$$

Put value of c_1 and c_2 in (1)

$$Q = \frac{E}{2\ Lw^2} \sin wt - \frac{Et}{2\ Lw} \cos wt$$

$$Q = \frac{E}{2\ Lw^2} [\sin wt - wt \cos wt]$$

$$\text{Now } w^2 = \frac{1}{LC}$$

$$\mathbf{Q} = \frac{\mathbf{EC}}{\mathbf{2}} \left[\sin \frac{t}{\sqrt{LC}} - \frac{t}{\sqrt{LC}} \cos \frac{t}{\sqrt{LC}}\right]$$

Example : 3

A capacitor of 10^{-3} farads is in the series with e.m.f. of 20 volts and an inductor of 0.4 Henry. At t = 0, the charge Q and current I are zero, find Q at any time t.

Solution :

By KVL,

$$L\frac{dI}{dt} + \frac{Q}{C} = E$$

$$\frac{dI}{dt} + \frac{1}{LC}Q = \frac{E}{L}$$

Given that , $C = 10^{-3}$ farads

$E = 20$ volts

$L = 0.4$ Henry

And, $I = \frac{dQ}{dt}$

$$\frac{d^2Q}{dt^2} + 2500\,Q = 50$$

Use, $D \equiv \frac{d}{dt}$

$$(D^2 + 2500)\,Q = 50$$

AE is, $D^2 + 2500 = 0$

$$(D - 50i)(D + 50i) = 0$$

$$D = \pm 50i$$

$$CF = c_1 \cos 50\,t + c_2 \sin 50\,t$$

$$PI = \frac{1}{D^2 + 2500}\,50$$

$$= \frac{1}{D^2 + 2500}\,50\,e^{0t} = \frac{1}{0 + 2500}\,50\,e^{0t}$$

$$PI = 0.02$$

The complete solution is,

$$Q = CF + PI$$

$$Q = c_1 \cos 50\,t + c_2 \sin 50\,t + 0.02 \qquad \ldots(1)$$

$$I = \frac{dQ}{dt}$$

$$= -50\,c_1 \sin 50\,t + 50\,c_2 \cos 50\,t \qquad \ldots(2)$$

Given that at t = 0, Q = I = 0

$$(1) \Rightarrow 0 = c_1 + 0 + 0.02, c_1 = -0.02$$

$$(2) \Rightarrow 0 = 0 + 50\, c_2, \Rightarrow c_2 = 0$$

$$(1) \Rightarrow Q = -0.02 \cos 50\, t + 0.02$$

$$\mathbf{Q = 0.02\, [1 - \cos 50\, t]}$$

Example : 4

In an L – C – R circuit, the charge q on a plate of a condenser is given by $L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = E \sin pt$. The circuit is tuned to resonance so that $p^2 = \frac{1}{LC}$. If initially the current j and the charge q be zero, show that for small value of $\frac{R}{L}$, the current in the circuit at time t is given by $\frac{Et}{2L} \sin pt$.

Solution :

Take $$D \equiv \frac{d}{dt}$$

$$L D^2 q + R D q + \frac{1}{C} q = E \sin pt$$

$$\left(LD^2 + RD + \frac{1}{C}\right) q = E \sin pt$$

AE is, $$LD^2 + RD + \frac{1}{C} = 0$$

$$D = \frac{1}{2L}\left[-R \pm \sqrt{R^2 - \frac{4L}{C}}\right] = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

As $\frac{R}{L}$ is small, so, to the first order in $\frac{R}{L}$.

$$D = -\frac{R}{2L} \pm i \frac{1}{\sqrt{LC}}$$

$$D = -\frac{R}{2L} \pm ip$$

$$CF = e^{\frac{-Rt}{2L}} [c_1 \cos pt + c_2 \sin pt]$$

$$CF = \left(1 - \frac{Rt}{2L}\right) (c_1 \cos pt + c_2 \sin pt)$$

$\because$ reject term in $\left(\frac{R}{L}\right)^2$ and higher powers

$$PI = \frac{1}{LD^2 + RD + \frac{1}{C}} E \sin pt$$

$$= E\frac{1}{-Lp^2 + RD + \frac{1}{C}}\ \sin,\ p = \frac{E}{R}\int \sin pt\, dt$$

$$PI = -\frac{E}{Rp}\cos pt$$

The complete solution is,

$$q = CF + PI$$

$$q = \left(1 - \frac{Rt}{2L}\right)(c_1 \cos pt + c_2 \sin pt) - \frac{E}{Rp}\cos pt \qquad \ldots(1)$$

$$j = \frac{dq}{dt}$$

$$= \left(1 - \frac{Rt}{2L}\right)(-c_1 p \sin pt + c_2 p \cos pt) - \frac{R}{2L}$$

$$(c_1 \cos pt + c_2 \sin pt) + \frac{E}{R}\sin pt \qquad \ldots(2)$$

Given that at $t = 0$, $q = 0$, $j = \frac{dq}{dt} = 0$

(1) $\Rightarrow\ 0 = c_1 - \frac{E}{Rp} \Rightarrow c_1 = \frac{E}{Rp}$

(2) $\Rightarrow\ 0 = c_2\, p - \frac{Rc_1}{2L}$, $\Rightarrow c_2 = \frac{Rc_1}{2Lp} \Rightarrow c_2 = \frac{E}{2Lp^2}$

Put value of c_1 and c_2 in (2)

$$j = \left(1 - \frac{Rt}{2L}\right)\left(-\frac{E}{Rp}\sin pt + \frac{E}{2Lp^2}\cos pt\right)p - \frac{R}{2L}\left(\frac{E}{Rp}\cos pt + \frac{E}{2Lp^2}\sin pt\right) + \frac{E}{R}\sin pt$$

$$\mathbf{j = \frac{Et}{2L}\sin pt}$$

Self Assessment Exercise 1.9

Ex.1 A circuit of an inductance L and condenser of capacity C is in series. An alternating e.m.f. Esin nt is applied to it at time $t = 0$, the initial current and charge on the condenser being zero. Find the current flowing in the circuit at any time t for : (i) $w \neq n$, (ii) $w = n$, where $w^2 = \frac{1}{LC}$.

Ans.: (i) $I = \frac{En}{(w^2 - n^2)\, L}[\cos nt - \cos wt]$ (ii) $I = \frac{E}{2L}\, t \sin wt$

Ex.2 A circuit consists of an inductance L, resistance R and condenser of capacity C in series. An alternating e.m.f. E sin wt is applied to it at time $t = 0$ where $w^2 = \frac{1}{LC}$. The initial current and charge on the condenser being zero. Find the charge at any time t if $R^2 < \frac{4L}{C}$

Ans. : $$q = e^{-\alpha t}\left[\frac{E}{Rw}\cos\beta t - \frac{E}{2\beta Lw}\sin\beta t\right] - \frac{E}{Rw}\cos wt$$

where $$\alpha = \frac{R}{2L} \text{ and } \beta = \frac{1}{2L}\sqrt{\frac{4L}{C} - R^2}$$

Ex.3 An e.m.f E sinpt is applied at t = 0 to a circuit containing a condenser C and inductance L in series the current I satisfies the equation : $L\frac{dI}{dt} + \frac{1}{C}\int I\,dt = E\sin pt$, where $I = -\frac{dq}{dt}$. If $p^2 = \frac{1}{LC}$ and initially the current and the charge are zero, find current at any time t.

Hint : Taking $I = -\frac{dq}{dt}$ in given differential equation we get,

$$-L\frac{d^2q}{dt^2} + \frac{1}{C}\int -\frac{dq}{dt}\,dt = E\sin pt$$

$$-L\frac{d^2q}{dt^2} - \frac{1}{C}q = E\sin pt$$

$$(D^2 + p^2)\,q = -\frac{E}{L}\sin pt$$

Finally, $$I = \frac{Et}{2L}\sin pt$$

Ex.4 A resistance of 50 ohms, an inductor of 2 henries and a 0.005 farad capacitor are in series with an e.m.f. of 40 volts and an open switch. Find the instantaneous charge and current after the switch is closed at t = 0, assuming that at that time the charge on the capacitor is 4 coulomb.

Ans. : $Q = 5.07\,e^{-5t} - 1.27\,e^{-20t} + 0.2$

$I = 25.4\,(e^{-20t} - e^{-5t})$

Descriptive Questions

Q. 1 $(D^3 + 3D^2 - 4)y = 6e^{-2x} + 4x^2$

Q. 2 $(D^2 - 4D + 4)y = 8(e^{2x} + \sin 2x + x^2)$

Q. 3 $(D^2 - 4)y = e^{3x}\,x^2$.

Q. 4 $(D^2 + 1)^2\,y = 24x\cos x$

Q. 5 $(D^2 - 2D + 4)^2\,y = x\,e^x\cos\left(\sqrt{3}\,x + \alpha\right)$

Q. 6 $\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y - 3x)}$

Q. 7 $\frac{dx}{x^2 + y^2} = \frac{dy}{2xy} = \frac{dz}{(x + y)^3 z}$

Q. 8 $\frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{2x - 3y}$

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Unit 2

Transforms

Syllabus :

Fourier Transform (FT) : *Complex exponential form of Fourier series, Fourier integral theorem, Fourier Sine & Cosine integrals, Fourier transform, Fourier Sine and Cosine transforms and their inverses.*

Z – Transform (ZT) : *Introduction, Definition, Standard properties, ZT of standard sequences and their inverses. Solution of difference equations.*

2.1 Fourier Transform (FT) :

2.1.1 Introduction :

Fourier transform comes from the study of Fourier series. In the study of Fourier series, complicated but periodic functions are written as the sum of simple waves mathematically represented by sines and cosines. The Fourier transform is an extension of the Fourier series that results when the period of the represented function is lengthened and allowed to approach infinity.

A Fourier transform when applied to a partial differential equation reduces the number of its independent variables by one. The theory of integral transforms afford mathematical devices through which solutions of numerous boundary value problems of engineering can be obtained e.g. conduction of heat, transverse vibrations of string, transverse oscillations of an elastic beam free and forced vibrations of a membrane etc.

2.1.2 Integral Transform :

The integral transform of a function f(x) denoted by I [f(x)] and is defined by.

$$I[f(x)] = \int_{x_1}^{x_2} f(x)\, K(\lambda,x)\, dx$$

Where K (λ,x) is called the kernel of the transform. The function f (x) is called the inverse transform of I [F(x)].

When K $(\lambda,x) = e^{-i\lambda x}$, we have the Fourier transform.

$$F(\lambda) = \int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x}\, dx$$

- **Fourier Integral :**

  Consider a function f(x) which satisfies the Dirichlets conditions in every interval $(-\infty,\infty)$ then

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda(u-x)}\, du d\lambda$$

2.1.3 Fourier Transform and Inverse Fourier Transform :

I) Fourier Transforms :

Fourier Transform of the function f(x) defined in the interval of $(-\infty,\infty)$ is defined as

$$\boxed{F(\lambda) = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du}$$ and the inverse Fourier transform of $F(\lambda)$ is defined as

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x}\, d\lambda}$$

II) Fourier cosine transform :

If f(x) is defined in the interval $0 < x < \infty$ then Fourier cosine transform of f(x) is defined as $\boxed{F_c(\lambda) = \int_{-\infty}^{\infty} f(u) \cos \lambda u\, du}$ and the inverse Fourier cosine transform of Fc (λ) is given by.

$$\boxed{f(x) = \frac{2}{\pi}\int_{0}^{\infty} Fc(\lambda) \cos \lambda x\, d\lambda}$$

III) Fourier sine transform :

If a function f(x) defined in the interval $0 < x < \infty$ then Fourier sine transform of f(x) is defined as.

$$\boxed{F_s(\lambda) = \int_{0}^{\infty} f(u) \sin \lambda u\, du}$$ and the inverse Fourier sine transform of $F_s(\lambda)$ is given by

$$\boxed{f(x) = \frac{2}{\pi}\int_{0}^{\infty} F_s(\lambda) \sin \lambda x\, d\lambda}$$

Note :

1. Results (II) of Fourier cosine transform and inverse Fourier cosine transform can be written in more symmetric form as.

$$\boxed{F_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u) \cos \lambda u \, du}$$

And $$\boxed{f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\lambda) \cos \lambda x \, d\lambda}$$

2. Similarly result (III) of Fourier sine transform and inverse Fourier sine transform can be written in more symmetric form as

$$\boxed{F_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u) \sin \lambda u \, du}$$

and $$\boxed{f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(\lambda) \sin \lambda x \, d\lambda}$$

Table of Fourier Transforms and Inverse Transforms.

Sr. No.	Name of the transform	Interval	Expression for the transform	Inverse transform
1.	Fourier	$-\infty < x < \infty$	$F(\lambda) = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du$	$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda) e^{-i\lambda x}\, d\lambda$
2.	Fourier cosine	$0 < x < \infty$	$F_c(\lambda) = \int_0^\infty f(u) \cos \lambda u \, du$	$f(x) = \frac{2}{\pi} \int_0^\infty F_c(\lambda) \cos \lambda x \, d\lambda$
3.	Fourier sine	$0 < x < \infty$	$F_s(\lambda) = \int_0^\infty f(u) \sin \lambda u \, du$	$f(x) = \frac{2}{\pi} \int_0^\infty F_s(\lambda) \sin \lambda x \, d\lambda$

Illustrations Examples

Type – I : Problems on Fourier Transform

Example : 1

Find the Fourier transform of $f(x) = e^{-|x|}$

Solution : Here $$f(x) = e^{-|x|}$$

Fourier transform of f(x) in the interval $-\infty < x < \infty$ is given by

$$F(\lambda) = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du$$

$$= \int_{-\infty}^{\infty} e^{-|u|} (\cos \lambda u - i \sin \lambda u)\, du$$

$$= \int_{-\infty}^{\infty} e^{-|u|} \cos \lambda u\, du - i \int_{-\infty}^{\infty} e^{-|u|} \sin \lambda u\, du$$

Since integrand in the first integral is even and second integral is odd

$$\therefore \quad F(\lambda) = 2 \int_{0}^{\infty} e^{u} \cos \lambda u\, du + 0$$

$$= 2 \left[\frac{e^{u}}{1+\lambda^2} (-\cos \lambda u + \lambda \sin \lambda u) \right]_0^{\infty} = \frac{\mathbf{2}}{\mathbf{1+\lambda^2}}$$

Example : 2

Find Fourier sine transform of $\frac{1}{x}$

Solution :

Here $\quad f(x) = \frac{1}{x}$

Fourier sine transform is since $F_s(\lambda) = \int_{0}^{\infty} f(u) \sin \lambda u\, du$

$$= \int_{0}^{\infty} \frac{1}{u} \sin \lambda u\, du$$

Put $\quad \lambda u = t, u = t/\lambda$

$\therefore \quad \lambda du = dt$

$du = dt/\lambda$

Also as limits $\quad u : 0$ to ∞

$\therefore \quad \Rightarrow \quad t : 0$ to ∞

$$\therefore \quad Fs(\lambda) = \int_{0}^{\infty} \frac{\lambda}{t} \sin t \frac{dt}{\lambda}$$

$$= \int_{0}^{\infty} \frac{\sin t}{t} dt = \frac{\boldsymbol{\pi}}{\mathbf{2}}$$

Example : 3

Find the Fourier cosine transform of $e^{-x} + e^{-2x}$, x > 0 **Nov. 2016**

Solution :

$f(x) = e^{-x} + e^{-2x}$, $o < x < \infty$

The Fourier cosine transform of f(x) is given by

$$F_c(\lambda) = \int_0^\infty f(u) \cos \lambda u \, du$$

$$= \int_0^\infty (e^{-u} + e^{-2u}) \cos \lambda u \, du$$

$$= \int_0^\infty e^{-u} \cos \lambda u \, du + \int_0^\infty e^{-2u} \cos \lambda u \, du$$

$$= \left[\frac{e^{-u}}{1+\lambda^2}(-\cos\lambda u + \lambda \sin \lambda u)\right]_0^\infty + \left[\frac{e^{-2u}}{4+\lambda^2}(-2\cos\lambda u + \lambda \sin \lambda u)\right]_0^\infty$$

$$= \left[0 - \frac{1}{1+\lambda^2}(-1)\right] + \left[0 - \frac{1}{4+\lambda^2}(-2)\right]$$

$$= \frac{1}{1+\lambda^2} + \frac{2}{4+\lambda^2} = \mathbf{\frac{6+3\lambda^2}{4+5\lambda^2+\lambda^4}}$$

Example : 4

Find the Fourier cosine transform of the function. **May 2016**

$$f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & 0 \quad x > a \end{cases}$$

Solution :

Fourier cosine transform of the function f(x) is given by

$$F_c(\lambda) = \int_0^\infty f(u) \cos \lambda u \, du$$

$$= \int_0^a \cos u \cos\lambda u \, du + \int_a^\infty (o) \cos\lambda u \, du$$

$$= \frac{1}{2}\int_0^a [\cos(\lambda+1)u + \cos(\lambda-1)u]du$$

$$= \frac{1}{2}\left[\frac{\sin(\lambda+1)u}{\lambda+1} + \frac{\sin(\lambda-1)u}{\lambda-1}\right]_0^a$$

$$= \mathbf{\frac{1}{2}\left[\frac{\sin(\lambda+1)a}{\lambda+1} + \frac{\sin(\lambda-1)a}{\lambda-1}\right]}$$

Example : 5

Find the Fourier sine transform of $\dfrac{e^{-ax}}{x}$

Solution :

Given : $f(x) = \dfrac{e^{-ax}}{x}$

Fourier sine transform is given by

$$F_s(x) = \int_0^\infty f(u) \sin \lambda u \, du$$

$$= \int_0^\infty \frac{e^{-au}}{u} \sin \lambda u \, du = I(\lambda) \text{ say} \qquad \ldots(1)$$

By DUIS

$$\frac{dI}{d\lambda} = \int_0^\infty \frac{\partial}{\partial \lambda}\left[\frac{e^{-au}}{u} \sin \lambda u\right] du$$

$$= \int_0^\infty \frac{e^{-au}}{u} \, u \cos \lambda u \, du$$

$$= \int_0^\infty e^{-au} \cos \lambda u \, du$$

$$= \left\{\frac{e^{-au}}{a^2 + \lambda^2} [\, -a \cos \lambda u + \lambda \sin \lambda u]\right\}_0^\infty$$

$$= \left\{0 - \frac{1}{a^2 + \lambda^2} (-a)\right\}$$

$$\frac{dI}{d\lambda} = \frac{a}{a^2 + \lambda^2}$$

Integrating we get

$$I(\lambda) = \int \frac{a}{a^2 + \lambda^2} d\lambda + A$$

$$I(\lambda) = \tan^{-1}\left(\frac{\lambda}{a}\right) + A \qquad \ldots.(2)$$

Put $\lambda = 0 \Rightarrow I(\lambda) = 0$ (from equation (1))

$\therefore$ from equation (2) , we have

$0 = 0 + A \Rightarrow A = 0$

$\therefore$ $\mathbf{I(\lambda) = F_s(\lambda) = \tan -1 \left(\frac{\lambda}{a}\right)}$

Type 2 : Problems on Fourier Integral representation and inverse Fourier transform

Example : 1

Using Fourier cosine integral of e^{-mx} (m > 0), prove that **Nov. 2015**

$$\int_0^\infty \frac{m\cos\lambda x\, d\lambda}{m^2+\lambda^2} = \frac{\pi}{2} e^{-mx}, \ (m > 0,\ x > 0)$$

Solution :

Taking Fourier cosine transform of

$$f(x) = e^{-mx}, \text{ we have}$$

$$F_c(\lambda) = \int_0^\infty f(u)\cos\lambda u\, du = \int_0^\infty e^{-mu}\cos\lambda u\, du$$

$$= \left[\frac{e^{-mu}}{m^2+\lambda^2}(-m\cos\lambda u + \lambda\sin\lambda u)\right]_0^\infty$$

$$= \left[(0) - \frac{1}{m^2+\lambda^2}(-m+0)\right]$$

$$Fc(\lambda) = \frac{m}{m^2+\lambda^2} \qquad \text{….(1)}$$

Now inverse cosine transform of

$F_c(\lambda)$ is given by

$$f(\lambda) = \frac{2}{\pi}\int_0^\infty F_c(\lambda)\cos\lambda x\, d\lambda$$

$$e^{-mx} = \frac{2}{\pi}\int_0^\infty \frac{m}{m^2+\lambda^2}\cos\lambda x\, d\lambda$$

$$\therefore \quad \int_0^\infty \frac{\mathbf{m\cos\lambda x}}{\mathbf{m^2+\lambda^2}}\, \mathbf{d\lambda} = \frac{\pi}{2}\, \mathbf{e^{-mx}}$$

Example : 2

Find the Fourier transform of

$$f(x) = \begin{cases} 1 & \text{for } |x| \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

Hence evaluate

$$\int_0^\infty \frac{\sin x}{x}\, dx$$

Solution : The Fourier transform of f(x) is

$$F(\lambda) = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du = \int_{-1}^{1} (1)\, e^{-i\lambda u}\, du$$

$$= \left[\frac{e^{-i\lambda u}}{-i\lambda}\right]_{-1}^{1} = \left[\frac{e^{-i\lambda}}{-i\lambda} - \frac{e^{i\lambda}}{-i\lambda}\right]$$

$$= \frac{e^{i\lambda} - e^{i\lambda}}{i\lambda} = \frac{2}{\lambda}\left[\frac{e^{-i\lambda} - e^{-i\lambda}}{2i}\right]$$

$$F(\lambda) = \frac{2}{\lambda}\sin\lambda = \frac{2\sin\lambda}{\lambda}$$

Now by the inverse Fourier transform,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\, e^{-i\lambda x}\, d\lambda$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin\lambda}{\lambda}\, e^{-i\lambda x}\, d\lambda$$

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin\lambda}{\lambda}\, e^{-i\lambda x}\, d\lambda$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} \frac{\sin\lambda}{\lambda}\, e^{-i\lambda x}\, d\lambda = \pi\, f(x) = \pi \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

Putting x = 0, we get

$$\int_{-\infty}^{\infty} \frac{\sin\lambda}{\lambda}\, dx = \pi$$

$$\therefore \quad 2\int_{0}^{\infty} \frac{\sin\lambda}{\lambda}\, d\lambda = \pi \quad (\because \text{ since the integrand is even})$$

$$\therefore \quad \int_{0}^{\infty} \frac{\sin\lambda}{\lambda}\, d\lambda = \frac{\pi}{2}$$

$$\therefore \quad \boldsymbol{\int_{0}^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}} \quad (\because \textbf{ variable is immaterial in definite integral})$$

Example : 3

Find the Fourier sine transform of $e^{-|x|}$. Hence show that $\int_{0}^{\infty} \frac{x \sin mx}{1 + x^2}\, dx = \frac{\pi}{2} e^{-m}$, m > 0

Solution :

Fourier sine transform of f(x) is

$$F_s(\lambda) = \int_0^\infty f(u)\sin\lambda u\, du$$

$$= \int_0^\infty e^{-u}\sin\lambda u\, du \quad (\therefore |x| = x, \text{ if } x>o)$$

$$= \left\{\frac{e^{-u}}{1+\lambda^2}(-\sin\lambda u - \lambda\cos\lambda u)\right\}_0^\infty$$

$$F_s(\lambda) = \frac{\lambda}{1+\lambda^2}$$

Inverse Fourier sine transform is

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\lambda)\sin\lambda x\, d\lambda$$

$$e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{\lambda}{1+\lambda^2}\sin\lambda x\, d\lambda, \quad x > 0$$

put x by m, we get

$$e^{-m} = \frac{2}{\pi}\int_0^\infty \frac{\lambda}{1+\lambda^2}\sin\lambda m\, d\lambda \quad m > 0$$

replace λ by x , we get

$$e^{-m} = \frac{2}{\pi}\int_0^\infty \frac{x\sin mx}{1+x^2}\, dx, \quad m > 0$$

$$\therefore \quad \int_0^\infty \frac{x\sin mx}{1+x^2}\, dx = \frac{\pi}{2}e^{-m}, \quad m > 0$$

Example : 4

By considering Fourier cosine transform of e^{-mx}, (m > 0) prove that. **May 2015**

$$\int_0^\infty \frac{\cos\lambda x}{x^2+m^2}\, d\lambda = \frac{\pi}{2m}e^{-mx} \quad m > 0, x > 0,$$

Solution : Fourier cosine transform of f(x) is

$$F_c(\lambda) = \int_0^\infty f(u)\cos\lambda u\, du = \int_0^\infty e^{-mu}\cos\lambda u\, du$$

$$= \left\{\frac{e^{-mu}}{m^2+\lambda^2}(-m\cos\lambda u + \lambda\sin\lambda u)\right\}_0^\infty$$

$$F_c(\lambda) = \frac{m}{m^2+\lambda^2}$$

Now Inverse Fourier cosine transform is

$$f(x) = \frac{2}{\pi}\int_0^\infty Fc\,(\lambda)\cos \lambda x\, d\lambda$$

$$e^{-mx} = \frac{2}{\pi}\int_0^\infty \frac{m}{m^2+\lambda^2}\cos \lambda x\, d\lambda$$

$$\Rightarrow \quad \boldsymbol{\int_0^\infty \frac{\cos \lambda x}{\lambda^2 + m^2}\, d\lambda = \frac{\pi}{2m}\, e^{-mx}}$$

Example : 5

Find the Fourier sine transform of the following function. **May 2018**

$$f(x) = \begin{cases} x & , \ 0 \le x \le 1 \\ 2 - x & , \ 1 < x \le 2 \\ 0 & , \ x > 2 \end{cases}$$

Solution : Fourier sine transform of f(x) is

$$F_s(\lambda) = \int_0^\infty f\,(u) \sin \lambda\, u\, du$$

$$= \int_0^1 u \sin \lambda\, u\, du + \int_1^2 (2 - u) \sin \lambda u\, du + \int_2^\infty (0) \sin \lambda u\, du$$

$$= \left\{(u)\frac{-\cos \lambda u}{\lambda} - 1\left(\frac{-\sin \lambda u}{\lambda^2}\right)\right\}_0^1 + \left\{(2-u)\left(\frac{-\cos \lambda u}{\lambda}\right) - (-1)\left(\frac{-\sin \lambda u}{\lambda^2}\right)\right\}_1^2$$

$$= \left\{\frac{-\cos\lambda}{\lambda} + \frac{\sin \lambda}{\lambda^2}\right\} + \left\{\frac{-\sin 2\lambda}{\lambda^2} + \frac{\cos \lambda}{\lambda} + \frac{\sin \lambda}{\lambda^2}\right\}$$

$$= \frac{2 \sin \lambda - \sin 2\lambda}{\lambda^2}$$

Type 3 : Fourier Integral Representation

Example : 1

Using Fourier integral representation show that. **Nov. 2017**

$$\int_0^\infty \frac{\lambda^3 \sin \lambda x}{\lambda^4+4}\, d\lambda = \frac{\pi}{2}\, e^{-x}\cos x,\ x > 0$$

Solution : Consider

$$f(x) = \frac{\pi}{2}\, e^{-x} \cos x, \ \ x > 0 \qquad \ldots(1)$$

Since the integral contains the term sin λx, so use Fourier sine transform

$$\therefore \quad F_s(\lambda) = \int_0^\infty f(u) \sin \lambda u \, du = \int_0^\infty \frac{\pi}{2} e^{-u} \cos u \sin \lambda u \, du$$

$$= \frac{\pi}{4} \int_0^\infty e^{-u} [\sin(\lambda+1)u + \sin(\lambda-1)u] \, du$$

$$= \frac{\pi}{4} \int_0^\infty e^{-u} \sin(\lambda+1)u \, du + \frac{\pi}{4} \int_0^\infty e^{-u} \sin(\lambda-u)u \, du$$

$$= \frac{\pi}{4} \left\{ \frac{e^{-u}}{1+(\lambda+1)^2} [-\sin(\lambda+1)u - (\lambda+1)\cos(\lambda+1)u] \right\}_0^\infty$$

$$= \frac{\pi}{4} \left\{ \frac{e^{-u}}{1+(\lambda-1)^2} [-\sin(\lambda-1)u - (\lambda-1)\cos(\lambda-1)u] \right\}_0^\infty$$

$$= \frac{\pi}{4} \left\{ \frac{\lambda+1}{1+(\lambda+1)^2} \right\} + \frac{\pi}{4} \left\{ \frac{\lambda-1}{1+(\lambda-1)^2} \right\}$$

$$= \frac{\pi}{4} \left\{ \frac{\lambda+1}{\lambda^2+2\lambda+2} + \frac{\lambda-1}{\lambda^2-2\lambda+2} \right\}$$

$$= \frac{\pi}{4} \left\{ \frac{2\lambda^3}{(\lambda^2+2)^2-4\lambda^2} \right\} = \frac{\pi}{2} \frac{\lambda^3}{\lambda^4+4}$$

Now by inverse Fourier sine transform

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(\lambda) \sin \lambda x \, d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\pi}{2} \frac{\lambda^3}{\lambda^4+4} \sin \lambda x \, d\lambda$$

$$f(x) = \int_0^\infty \frac{\lambda^3 \sin \lambda x}{\lambda^4+4} \, d\lambda \text{ by equation (1), we get}$$

$$\Rightarrow \quad \boldsymbol{\int_0^\infty \frac{\lambda^3 \sin \lambda x}{\lambda^4+4} \, d\lambda = \frac{\pi}{2} e^{-x} \cos x}$$

Example : 2

Using Fourier integral representation show that $\int_0^\infty \frac{2\cos \lambda x}{1+\lambda^2} d\lambda = \begin{cases} 0 & x < 0 \\ \pi e^{-x}, & x \geq 0 \end{cases}$

Solution :

Consider , $f(x) = \begin{cases} 0 & , \; x < 0 \\ \pi \, e^{-x} & , \; x \geq 0 \end{cases}$ ...(1)

Since integral contains the term $\cos \lambda x$, $x > 0$, so we can use Fourier cosine transform

$$\therefore \quad F_c(\lambda) = \int_0^\infty f(u)\cos\lambda u\, du$$

$$= \int_0^\infty \pi\, e^{-u}\cos\lambda u\, du \quad = \pi\int_0^\infty e^{-u}\cos\lambda u\, du$$

$$= \pi\left\{\frac{e^{-u}}{1+\lambda^2}\left[-\cos\lambda u + \lambda\sin\lambda u\right]\right\}_0^\infty$$

$$= \pi\left\{\frac{-1}{1+\lambda^2}(-1)\right\}$$

$$F_c(\lambda) = \frac{\pi}{1+\lambda^2}$$

Now, Inverse Fourier cosine transform of Fc (λ) is

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\lambda)\cos\lambda x\, d\lambda$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\pi}{1+\lambda^2}\cos\lambda x\, d\lambda$$

$$f(x) = \int_0^\infty \frac{2\cos\lambda x}{1+\lambda^2}\, d\lambda$$

By equation (1) we get

$$\int_0^\infty \frac{2\cos\lambda x}{1+\lambda^2}\, d\lambda = f(x) = \begin{cases} 0 & , \ x < 0 \\ \pi\, e^{-x} & , \ x \geq 0 \end{cases}$$

Type IV : Solution of Integral Equations

Example : 1

Solve the following integral equations.

a) $\displaystyle\int_0^\infty f(x)\sin\lambda x\, dx = \begin{cases} 1-\lambda, & 0 \leq \lambda \leq 1 \\ 0 \quad, & \lambda \geq 1 \end{cases}$

b) $\displaystyle\int_0^\infty f(x)\cos\lambda x\, dx = e^{-\lambda}, \lambda > 0$

Solution :

a) Since the term sin λx present in the integral, we can use Fourier sine transform.

$$F_s(\lambda) = \int_0^\infty f(u)\sin\lambda u\, du = \begin{cases} 1-\lambda, & 0 \leq \lambda \leq 1 \\ 0 \quad, & \lambda > 1 \end{cases} \Rightarrow F_s(\lambda) = \begin{cases} 1-\lambda, & 0 \leq \lambda \leq 1 \\ 0 \quad, & \lambda > 1 \end{cases}$$

Now inverse Fourier sine transform of

$F_s(\lambda)$ is
$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\lambda)\sin\lambda x\, d\lambda$$
$$= \frac{2}{\pi}\int_0^1 (1-\lambda)\sin\lambda x\, d\lambda + \frac{2}{\pi}\int_1^\infty (o)\sin\lambda x\, d\lambda$$
$$= \frac{2}{\pi}\left\{(1-\lambda)\left(\frac{-\cos\lambda x}{x}\right) - (-1)\left(\frac{-\sin\lambda x}{x^2}\right)\right\}_0^1$$
$$= \frac{2}{\pi}\left\{\frac{-\sin x}{x^2} + \frac{1}{x}\right\} = \frac{2}{\pi}\left\{\frac{x-\sin x}{x^2}\right\}$$

b) Since the term $\cos\lambda x$ present in the integral, we have to find inverse Fourier cosine transform.

Consider , $F_c(\lambda) = \int_0^\infty f(u)\cos\lambda u\, du = e^{-\lambda}, \lambda > 0 \Rightarrow F_c(\lambda) = e^{-\lambda}, \lambda > 0$

Now inverse Fourier consine transform of $F_c(\lambda)$ is
$$f(x) = \frac{2}{\pi}\int_0^\infty Fc(\lambda)\cos\lambda x\, d\lambda$$
$$= \frac{2}{\pi}\int_0^\infty e^{-\lambda}\cos\lambda x\, d\lambda$$
$$= \frac{2}{\pi}\left\{\frac{e^{-\lambda}}{1+x^2}(-\cos\lambda x + \sin\lambda x)\right\}_0^\infty$$
$$\mathbf{f(x) = \frac{2}{\pi}\left(\frac{1}{1+x^2}\right)}$$

Example : 2

Solve the integral equation $\int_0^\infty f(x)\cos\lambda x\, dx = \begin{cases} 1-\lambda & , 0 \le \lambda \le 1 \\ 0 & , \lambda > 1 \end{cases}$ and hence show that
$$\int_0^\infty \frac{\sin^2 z}{z^2}\, dz = \frac{\pi}{2}$$

Solution :

Since the term $\cos\lambda x$ present in the integral we have to find inverse Fourier cosine transform.

$$F_c(\lambda) = \int_0^\infty f(u)\cos\lambda u\, du = \begin{Bmatrix} 1-\lambda, & o \le \lambda \le 1 \\ & o, \lambda > 1 \end{Bmatrix} \Rightarrow F_c(\lambda) = \begin{cases} 1-\lambda & , 0 \le \lambda \le 1 \\ 0 & , \lambda > 1 \end{cases} \quad \ldots.(1)$$

Now inverse Fourier cosine transform of $F_c(\lambda)$ is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty Fc(\lambda)\cos\lambda x\, d\lambda$$

$$= \frac{2}{\pi}\int_0^1 (1-\lambda)\cos\lambda x\, d\lambda + \frac{2}{\pi}\int_1^\infty (0)\cos\lambda x\, d\lambda$$

$$= \frac{2}{\pi}\left\{(1-\lambda)\left(\frac{\sin\lambda x}{x}\right) - (-1)\left(-\frac{\cos\lambda x}{x^2}\right)\right\}_0^1$$

$$= \frac{2}{\pi}\left\{(1-\lambda)\left(\frac{\sin\lambda x}{x}\right) - \left(\frac{\cos\lambda x}{x^2}\right)\right\}_0^1$$

$$= \frac{2}{\pi}\left\{\frac{-\cos x}{x^2} + \frac{1}{x^2}\right\}$$

$$f(x) = \frac{2}{\pi}\left\{\frac{1-\cos x}{x^2}\right\} \qquad ....(2)$$

from equation (1) we have

$$F_c(\lambda) = \int_0^\infty f(u)\cos\lambda u\, du$$

$$= \int_0^\infty \frac{2}{\pi}\left(\frac{1-\cos u}{u^2}\right)\cos\lambda u\, du$$

$$= \frac{2}{\pi}\int_0^\infty \frac{2\sin^2\frac{u}{2}}{u^2}\cos\lambda u\, du$$

Put $\lambda = 0$, we get

$$[F_c(\lambda)]_{\lambda=0} = \frac{2}{\pi}\int_0^\infty \frac{2\sin^2\frac{u}{2}}{u^2}\, du$$

$$1 = \frac{2}{\pi}\int_0^\infty \frac{2\sin^2\frac{u}{2}}{u^2}\, du$$

Putting $\frac{u}{2} = z \Rightarrow u = 2z \Rightarrow du = 2\, dz$

$$\therefore \qquad 1 = \frac{2}{\pi}\int_0^\infty \frac{2\sin^2 z}{4z^2}\, 2\, dz$$

$$\Rightarrow \qquad \frac{\pi}{2} = \int_0^\infty \frac{\sin^2 z}{z^2}\, dz$$

Self–Assessment Exercise 2.1

Ex. 1. Find the Fourier transform of $f(x) = e^{-\frac{x^2}{2}}$ **Ans. :** $e^{-\lambda^2/2}$

Ex. 2. Find the Fourier sine transform of $\frac{e^{-ax}}{x}$ **Ans. :** $F_s(\lambda) = \tan^{-1}\frac{\lambda}{a}$

Ex. 3. Find the Fourier sine and cosine transforms of $f(x) = e^{-x}$ and hence show that

$$\int_0^\infty \frac{\cos mx}{1+x^2}\,dx = \frac{\pi}{2}e^{-m} \text{ and } \int_0^\infty \frac{x \sin mx}{1+x^2}\,dx = \frac{\pi}{2}\, e^{-m}$$

Ex. 4. Find the Fourier cosine integral representation for the following functions.

(i) $f(x) = \begin{cases} 1, & 0 \le \lambda \le 1 \\ 0, & \lambda > 1 \end{cases}$ **Ans. :** $f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda}{\lambda}\cos\lambda x\, d\lambda$

(ii) $f(x) = e^{-x} + e^{-2x}, x \ge 0$ **Ans. :** $f(x) = \frac{6}{\pi}\int_0^\infty \frac{(\lambda^2+2)}{\lambda^4+5\lambda^2+4}\cos\lambda x\, d\lambda$

Ex.5. Find the Fourier sine and cosine transform of $x\, e^{-ax}$

Ans. : $F_s(\lambda) = \frac{2a\lambda}{(a^2+\lambda^2)^2}$ $F_c(\lambda) = \frac{a^2-\lambda^2}{(a^2+\lambda^2)^2}$

Ex. 6. Find the Fourier sine and cosine transform of $f(x) = e^{-x} + e^{-2x}$

Ans. : $F_s(\lambda) = \frac{\lambda}{1+\lambda^2} + \frac{\lambda}{4+\lambda^2}$ $F_c(\lambda) = \frac{1}{1+\lambda^2} + \frac{2}{4+\lambda^2}$

Ex. 7. Solve the integral equation $\int_0^\infty f(x)\sin\lambda x\, dx = \begin{cases} 1, & 0 \le \lambda < 1 \\ 2, & 1 \le \lambda < 2 \\ 0, & \lambda \ge 2 \end{cases}$

Ans.: $f(x)\,\frac{(2 + 2\cos x - 4\cos 2x)}{\pi x}$

Ex. 8. Solve the integral equation $\int_0^\infty f(x)\sin\lambda u\, dx = e^{-\lambda}$ **Ans. :** $\frac{2}{\pi(1+x^2)}$

Ex. 9. Using the Fourier integral representation show that

(i) $\int_0^\infty \frac{\lambda\sin\lambda x}{1+\lambda^2}\,d\lambda = \frac{\pi}{2}e^{-x}, x > 0$ (ii) $\int_0^\infty \frac{\cos\lambda x}{1+\lambda^2}\,d\lambda = \frac{\pi}{2}\, e^{-x}, x \ge 0$

(iii) $\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda = \frac{\pi}{2}, 0 \le x \le 1$

Ex. 10 Find the Fourier transform of

$$F(x) = \begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

and hence evaluate $\int_0^\infty \frac{x\cos x - \sin x}{x^3}\cos\frac{x}{2}\,dx$ **Ans.:** $4\left(\frac{\sin\lambda - \lambda\cos\lambda}{\lambda^3}\right)$

Ex. 11 Find the Fourier transform of

$$f(x) = \begin{cases} 0 & 0 \le x < a \\ x & , a \le x \le b \\ 0 & x > b \end{cases}$$

Ans. : $\left(\dfrac{a \cos \lambda a - b \cos \lambda\, b}{\lambda}\right) + \left(\dfrac{\sin \lambda\, b - \sin \lambda\, a}{\lambda^2}\right)$

Ex. 12 Find the Fourier cosine transform of

$$f(x) = 2\, e^{-5x} + 5e^{-2x}$$

Ans. : $10\left(\dfrac{1}{\lambda^2 + 5} + \dfrac{1}{y^2 + 4}\right)$

Ex.13 Find the Fourier sine and cosine transform of

$$f(x) = \begin{cases} x^2 & 0 \le x < 1 \\ 0 \;, & x > 1 \end{cases}$$

Ans. : $\dfrac{1}{\lambda^3}\,[\, 2\lambda \sin \lambda - \lambda^2 \cos \lambda + 2 (\cos \lambda - 1)]$

$\dfrac{1}{\lambda^3}\,[\, 2\lambda \cos \lambda + \lambda^2 \sin \lambda - 2 \sin \lambda]$

2.2 Z – Transforms (ZT) :

Introduction :

Z transforms plays a key role in discrete analysis. The development of Communication Engineering is based on discrete analysis. Z transforms are required to study discrete analysis, whereas Laplace Transforms are required for continuous systems.

Z– transforms are useful in the study of control systems, communication systems, signal & linear time invariant systems and also for solving difference equations.

2.2.1 Prerequisites :

- **Sequence :**

  An arrangement of numbers with certain definite rule is called as a sequence.

1. The sequence a, a + d, a + 2d, …, a + (n – 1) d , …. is arithmetic progression / sequence with first term a and common difference is d.
2. The sequence a, ar, ar^2, ar^3 …. ar^n, ……. is geometric progression / sequence with first term a and common ratio is r.
3. The sequence $u_1, u_2, u_3, u_4, \ldots, u_n$ is called finite sequence.
4. The sequence $u_1, u_2, u_3, u_4, \ldots u_n, \ldots, \infty$ is called infinite sequence.

Ex. (i) 1, 5, 9, 13, 17, ..arithmetic sequence with first term a = 1, common difference d = 4

(ii) 2, 4, 8, 16, 32, …... geometric sequence with first term a = 2, common ratio r = 2

(iii) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ ….. is geometric sequence first term a = 1, common ratio $r = \frac{1}{2}$

- **The sequences are written in two ways :**

(i) The sequence written as $u_0, u_1, u_2, u_3 \ldots u_n$ is called as listing method

(ii) This sequence is also written in terms of its k^{th} term as $\{u_k\}$, where k = 0, 1, 2, 3, … ∞ and is called as k^{th} term method. The k^{th} term of the sequence is written as u_k or u (k) or f (k) or f_k.

(iii) The infinite sequence {f (k)} for at all k's is written as …. f (–2), f (–1), f (0), f (1), f (2), f (3), …..

(iv) The term f (0) is called as zeroth term or term at k = 0 of the sequence {f (k)}.

(v) The notation ↑ is used to denote the term at k = 0 or zeroth term of the sequence. If the zeroth term is not indicated in given sequence, we treat first term of the sequence as zeroth term.

Ex. 1. In the sequence

$$\ldots \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \underset{\uparrow}{1}, 2, 4, 8, \ldots$$

$f(0) = 1$, $f(1) = 2$, $f(2) = 4$, $f(3) = 8$, $f(-1) = \frac{1}{2}$, $f(-2) = \frac{1}{4}$, $f(-3) = \frac{1}{8}$, $f(-4) = \frac{1}{16}$

2. In the sequence

$$\ldots \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, \underset{\uparrow}{2}, 4, 8, \ldots$$

$f(0) = 2$, $f(1) = 4$, $f(2) = 8$, $f(-1) = 1$, $f(-2) = \frac{1}{2}$, $f(-3) = \frac{1}{4}$, $f(-4) = \frac{1}{8}$,

$f(-5) = \frac{1}{16}$

Note : Even through, above two sequences in example (1) and (2) have same terms but these sequences are not identical as their zeroth terms are different.

3. In the sequence 3, 9, 27, 81 zeroth term is not indicated and hence

$f(0) = 3, f(1) = 9, f(2) = 27, f(3) = 81 \ldots$

2.2.2 Linear Property :

If {f (k)} and {g (k)} be any two sequences having same number of terms, then

(i) $\{f(k) + g(k)\} = \{f(k)\} + \{g(k)\}$

(ii) $\{a\, f(k)\} = a\,\{f(k)\}$, where a is constant

(iii) $\{a\, f(k) + b\, g(k)\} = a\,\{f(k)\} + b\,\{g(k)\}$, where a,b are constants

2.3 Modulus of a Real Number :

If x is any real number

Then $|x| = x, x \geq 0.$

$\quad = -x, x < 0.$

(i) If $|x| < a$ then x lies between –a and a, i.e $-a < x < a$.

(ii) If $|x| \leq a$ then $-a \leq x \leq a$

(iii) $|x| > a$ means $-\infty < x < -a$ and $a < x < \infty$.

(iv) $|r| < 1$ means $-1 < r < 1$

2.4 Complex Number :

1. $z = x + iy$, $x, y \in R$, $i = \sqrt{-1}$ is called as a complex number
2. $\overline{z} = x - iy$ conjugate of z
3. $|z| = +\sqrt{x^2 + y^2}$ modulus of complex number z.
4. $|z_1 . z_2| = |z_1|\,|z_2|$
5. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
6. $|z| = |a|$ means $x^2 + y^2 = a^2$, a circle whose centre is (0, 0) and radius = a units.
7. $|z| < |a|$ means $x^2 + y^2 < a^2$, which is interior of circle $x^2 + y^2 = a^2$.
8. $|z| > |a|$ means $x^2 + y^2 > a^2$, which is exterior of circle $x^2 + y^2 = a^2$.
9. The complex number (x + iy) is represented by a point (x, y) on xoy plane.
10. $e^{i\theta} = \cos\theta + \sin\theta$, $e^{-i\theta} = \cos\theta - i\sin\theta$
11. $|e^{i\theta}| = 1$, $|e^{-i\theta}| = 1$
12. $\cos x = \frac{e^{ix} + e^{-ix}}{2}$
13. $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$
14. $\cosh x = \frac{e^{x} + e^{-x}}{2}$
15. $\sinh x = \frac{e^{x} - e^{-x}}{2}$

2.5 Series :

1. If $u_0, u_1, u_2, u_3, \ldots u_n \ldots u_\infty$ is a sequence then $u_0 + u_1 + u_2 + u_3 +++ u_n ++ u_\infty$ is called as a series and written as

$$u_0 + u_1 + u_2 + u_3 +++ u_n ++ u_\infty = \sum_{k=0}^{\infty} u_k$$

2. $u_0 + u_1 + u_2 + u_3 +++ u_n ++ u_\infty = \sum_{k=0}^{\infty} u_k$ is called as a infinite series.
3. $u_0 + u_1 + u_2 + u_3 +++ u_n = \sum_{k=0}^{n} u_k$ is called as finite series.
4. $\sum_{k=-\infty}^{\infty} u_k$ is also called as infinite series.

5. Convergence :

Let $\sum_{k=0}^{\infty} u_k$ be a infinite series, let $S_n = u_0 + u_1 + u_2 + u_3 + \cdots + u_n$ and if $\lim_{n \to \infty} S_n = L$, where L is a unique finite number then $\sum_{k=0}^{\infty} u_k$ is called as convergent series. Otherwise it is called as a divergent series.

6. The finite series $\sum_{k=0}^{n} u_k$ is always convergent.

7. If few terms are added or deleted from an infinite series, it does not change the convergence or divergence of series.

2.6 Geometric Series :

1. The series $a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n + \cdots \infty$ is called as a geometric series.

with first term = a and common ratio $r = \frac{ar}{a} = \frac{ar^2}{ar} = \frac{ar^3}{ar^2} = \frac{ar^n}{ar^{n-1}} = \cdots$

2. $a + ar + ar^2 + ar^3 + \cdots \infty = \frac{a}{1-r}$, $|r| < 1$

3. $|r| < 1$ means $-1 < r < 1$, which is called as condition of convergence of geometric series.

2.7 Z – Transform :

Let {f (k)} be a sequence and let z = x + iy, where $x, y \in R$, $i = \sqrt{-1}$ then $\sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$ is called as z Transform of sequence {f (k)} and denoted by Z {f (k)}.

$$\therefore \qquad Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

1. Z is called as Z – Transform operator.
2. Z {f (k)} exists only if the series on r.h.s converges.
3. Evaluation of the series on r.h.s gives a function in terms of z say F (z), which is a complex valued function.

$$\therefore \quad Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} = F(z)$$

4. $Z\{f(k) + g(k)\} = Z\{f(k)\} + Z\{g(k)\}$
5. $Z\{a\, f(k)\} = a\, Z\{f(k)\}$, where a is a constant.
6. $Z\{f(k)\} = -\cdots + f(-2)\, z^2 + f(-1)\, z' + f(0) + f(1)\, z^{-1} + f(2)\, z^{-2} + f(3)\, z^{-3} + \cdots$

Examples

1. **If** $\{f(k)\} = \left\{\frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \underset{\uparrow}{1}, 2, 4, 8\right\}$

Here $f(0) = 1,\ f(1) = 2,\ f(2) = 4,\ f(3) = 8,\ f(-1) = \frac{1}{2},\ f(-2) = \frac{1}{4},\ f(-3) = \frac{1}{8},$ $f(-4) = \frac{1}{16}$

$$\begin{aligned}
\text{Then } Z\{f(k)\} &= f(-4)\,z^4 + f(-3)\,z^3 + f(-2)\,z^2 + f(-1)\,z^1 + f(0)\,z^0 + f(1)\,z^{-1} \\
&\quad + f(2)\,z^{-2} + f(3)\,z^{-3} \\
&= \left(\frac{1}{16}\right)z^4 + \left(\frac{1}{8}\right)z^3 + \left(\frac{1}{4}\right)z^2 + \left(\frac{1}{2}\right)z + 1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} \\
&= \frac{1}{16}z^4 + \frac{1}{8}z^3 + \frac{1}{4}z^2 + \frac{1}{2}z + 1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3}
\end{aligned}$$

---

2. **If** $\{f(k)\} = \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, \underset{\uparrow}{2}, 4, 8$

Here $f(0) = 2,\ f(1) = 4,\ f(2) = 8,\ f(-1) = 1,\ f(-2) = \frac{1}{2},\ f(-3) = \frac{1}{4},\ f(-4) = \frac{1}{8},$ $f(-5) = \frac{1}{16}$

$$\begin{aligned}
\text{Then, } Z\{f(k)\} &= f(-5)\,z^5 + f(-4)\,z^4 + f(-3)\,z^3 + f(-2)\,z^2 + f(-1)\,z + f(0)\,z^0 \\
&\quad + f(1)\,z^{-1} + f(2)\,z^{-2} \\
&= \frac{1}{16}z^5 + \frac{1}{8}z^4 + \frac{1}{4}z^3 + \frac{1}{2}z^2 + z + 2 + \frac{4}{z} + \frac{8}{z^2}
\end{aligned}$$

---

3. **If** $\{f(k)\} = \left\{\frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8\right\}$

As zero[th] term is not indicated we select $f(0) = \frac{1}{16}$

Here $f(0) = \frac{1}{16},\ f(1) = \frac{1}{8},\ f(2) = \frac{1}{4},\ f(3) = \frac{1}{2},\ f(4) = 1,\ f(5) = 2,\ f(6) = 4,\ f(7) = 8$

$$\begin{aligned}
\text{Then } Z\{f(k)\} &= f(0)\,z^0 + f(1)\,z^{-1} + f(2)\,z^{-2} + f(3)\,z^{-3} + f(4)\,z^{-4} + f(5)\,z^{-5} \\
&\quad + f(6)\,z^{-6} + f(7) = z^{-7} \\
&= \frac{1}{16} + \frac{1}{8}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{2}z^{-3} + z^{-4} + 2z^{-5} + 4z^{-6} + 8z^{-7} \\
&= \frac{1}{16} + \frac{1}{8z} + \frac{1}{4z^2} + \frac{1}{2z^3} + \frac{1}{z^4} + \frac{2}{z^5} + \frac{4}{z^6} + \frac{8}{2^7}
\end{aligned}$$

4. If $f(k) = \left(\frac{1}{2}\right)^k$, $-2 \le k \le 2$.

Then $\{f(k)\} = \left\{\left(\frac{1}{2}\right)^{-2}, \left(\frac{1}{2}\right)^{-1}, \left(\frac{1}{2}\right)^{0}, \left(\frac{1}{2}\right)^{1}, \left(\frac{1}{2}\right)^{2}\right\} = \left\{4, 2, \underset{\uparrow}{1}, \frac{1}{2}, \frac{1}{4}\right\}$

$\therefore \quad Z\{f(k)\} = 4z^2 + 2z + 1 + \frac{1}{2z} + \frac{1}{4z^2}$

5. If $\{f(k)\} = \{3^k\}$, $0 \le k \le 3$.

Then $\{f(k)\} = \{1, 3^1, 3^2, 3^3\} = \left\{\underset{\uparrow}{1}\ 3, 9, 27\right\}$

$\therefore \quad Z\{f(k)\} = f(0)\,z^{-0} + f(1)\,z^{-1} + f(2)\,z^{-2} + f(3)\,z^{-3}$

$= 1 + \frac{3}{z} + \frac{9}{z^2} + \frac{27}{z^3}$

2.8 Z– Transforms of Standard Sequences :

1. If $f(k) = a^k$, where a is a constant and $k \ge 0$

We know,

$$z\{a^k\} = \sum_{k=-\infty}^{\infty} a^k z^{-k}$$

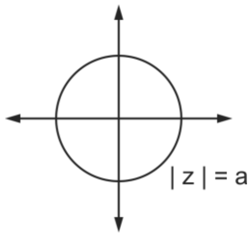

Fig.: 2.1

$$= \sum_{k=0}^{\infty} \frac{a^k}{z^k} = \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^k \quad \therefore K \ge 0$$

$$= \left(\frac{a}{z}\right)^0 + \left(\frac{a}{z}\right)^1 + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \left(\frac{a}{z}\right)^4 +++$$

$$= 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \left(\frac{a}{z}\right)^4 +++ \text{ which is a geometric series}$$

having first term = 1 and $r = \frac{a}{z}$.

$$= \frac{1}{1-\left(\frac{a}{z}\right)}, \frac{|a|}{|z|} < 1 = \frac{1}{\left(\frac{z-a}{z}\right)}, \frac{|a|}{|z|} < 1$$

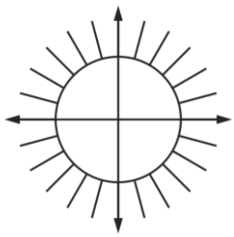

Fig.: 2.2

$$z\{a^k\} = \frac{z}{z-a}, |z| > |a|$$

$$\therefore \quad \boxed{Z\{a^k\} = \frac{z}{z-a}, |z| > |a|, k \ge 0}$$

Note that $|z| > |a|$ (figure 2) is called as Region of absolute Convergence (ROC). It means the formula for $z\{a^k\}$ above is true only for those values of x, y which lies out side circle $|z| = |a|$ figure (3.1), i.e. a circle whose centre is (0, 0) and radius = a units.

2. If f (k) = a^k , k < 0

We know, $$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} = \sum_{k=-\infty}^{-1} a^k z^{-k} = \sum_{k=-\infty}^{-1} \left(\frac{a}{z}\right)^k = \sum_{k=-1}^{-\infty} \left(\frac{a}{z}\right)^k$$

$$= \left(\frac{a}{z}\right)^{-1} + \left(\frac{a}{z}\right)^{-2} + \left(\frac{a}{z}\right)^{-3} + \left(\frac{a}{z}\right)^{-3} \left(\frac{a}{z}\right)^{-4} +++$$

$$= \left(\frac{z}{a}\right) + \left(\frac{z}{a}\right)^2 + \left(\frac{z}{a}\right)^3 + \left(\frac{z}{a}\right)^4 +++ a$$ which is a Geometric series with

first term $\frac{z}{a}$ and common ratio $\frac{z}{a}$

$$= \frac{\left(\frac{z}{a}\right)}{1-\left(\frac{z}{a}\right)}, \left|\frac{z}{a}\right| < 1 = \frac{\frac{z}{a}}{\frac{(a-z)}{a}} = -\left[\frac{z}{z-a}\right], \ |z| < |a|$$

$$\therefore \quad \boxed{Z\{a^k\} = -\left(\frac{z}{z-a}\right), |z| < |a|, k < 0}$$

---

3. If f (k) = $c^k \cos\alpha k$, where c and α are constants, k ≥ 0.

We know $$Z\{c^k \cos\alpha k\} = \sum_{k=-\infty}^{\infty} (c^k \cos\alpha\, k)\, z^{-k}$$

$$= \sum_{k=0}^{\infty} c^k \left[\frac{e^{i\alpha k} + e^{-i\alpha k}}{2}\right] z^{-k} \qquad \therefore k \geq 0$$

$$= \frac{1}{2}\sum_{k=0}^{\infty} c^k e^{i\alpha k} z^{-k} + \frac{1}{2}\sum_{k=0}^{\infty} c^k e^{-i\alpha k} z^{-k}$$

$$= \frac{1}{2}\sum_{k=0}^{\infty} (c\, e^{i\alpha} z^{-1})^k + \frac{1}{2}\sum_{k=0}^{\infty} (c\, e^{-i\alpha} z^{-1})^k$$

$$= \frac{1}{2}\left[1 + (c\, e^{i\alpha} z^{-1}) + (c\, e^{i\alpha} z^{-1})^2 + (c\, e^{i\alpha} z^{-1})^3 ++\right]$$

$$+ \frac{1}{2}\left[1 + (c\, e^{-i\alpha} z^{-1}) + (c\, e^{-i\alpha} z^{-1})^2 + (c\, e^{i\alpha} z^{-1})^3 +++\right],$$

Geometric series

$$= \frac{1}{2}\left[\frac{1}{1 - c\, e^{i\alpha} z^{-1}} + \frac{1}{1 - c\, e^{-i\alpha} z^{-1}}\right], \ |c\, e^{i\alpha} z^{-1}| < 1, \ |c\, e^{-i\alpha} z^{-1}| < 1$$

$$= \frac{1}{2}\left[\frac{z}{z - c\, e^{i\alpha}} + \frac{z}{z - c\, e^{-i\alpha}}\right], \qquad \frac{|c|\,|e^{i\alpha}|}{|z|} < 1, \ \frac{|c|\,|e^{-i\alpha}|}{|z|} < 1$$

$$= \frac{z}{2}\left[\frac{z - c\, e^{-i\alpha} + z - c\, e^{i\alpha}}{(z - c\, e^{i\alpha})\,(z - ce^{-i\alpha})}\right] \qquad |c| < |z|, |c| < |z|$$

$$= \frac{z(z - c\cos\alpha)}{z^2 - 2cz\cos\alpha + c^2}, \qquad |z| > |c|, k \geq 0$$

$$\therefore \boxed{Z\{c^k \cos\alpha k\} = \frac{z(z - c\cos\alpha)}{z^2 - 2\,c\,z\cos\alpha + c^2}, |z| > |c|, k \geq 0}$$

4. $Z\{a^{|k|}\} = \frac{z}{z-a} + \frac{az}{1-az}, \qquad |a| < |z| < \frac{1}{|a|}$

5. $Z\{c^k \sin\alpha k\} = \frac{cz\sin\alpha}{z^2 - 2cz\cos\alpha + c^2}, \; |z| > |c|, \; k \geq 0$

6. $Z\{c^k \cosh\alpha k\} = \frac{z(z - \cosh\alpha)}{z^2 - 2cz\cosh\alpha + c^2}, |z| > \max\{|ce^{\alpha}| \; | ce^{-\alpha}|\}, \; k \geq 0$

7. $Z\{c^k \sinh\alpha k\} = \frac{cz\sinh\alpha}{z^2 - 2cz\cosh\alpha + c^2}, |z| > \max\{|ce^{\alpha}| \; | ce^{-\alpha}|\}, \; k \geq 0$

8. $Z\{\cos\alpha k\} = \frac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1}, |z|>1, k \geq 0$

9. $Z\{\sin\alpha k\} = \frac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1}, |z|>1, k \geq 0$

10. **Unit step function :** A function $U(K) = 0, \quad k < 0$

$= 1, \quad k \geq 0$ is called as a unit step function

$$\boxed{Z\{U(k)\} = \frac{z}{z-1}, |z| > 1}$$

- **Dirac Delta Function/ Unit impulse Function :**

  A function defined as, $\delta(k) = 1, k = 0$

  $= 0, k \neq 0$ is called as a Dirac Delta Function.

11. $\boxed{Z\{\delta(k)\} = 1}$

12. $Z\{1^k\} = Z\{1\} = \frac{z}{z-1}, \quad |z| > 1, k \geq 0$

13. $Z\{(-a)^k\} = \frac{z}{z+a}, \quad |z| > |a|, \quad k \geq 0$

Illustrative Examples

Example : 1

Obtain Z transforms of (1) 2^{3k}, (2) 4^{-2k}, (3) a^{5k} for $k \geq 0$

Solution :

1. $$Z\{2^{3k}\} = [(2^3)^k]$$

$$= z[8^k] = \frac{z}{z-8}, |z| > 8, k \geq 0$$

2. $Z\{4^{-2k}\} = z\{(4^{-2})^k\}$

$= z\left(\frac{1}{16}\right)^k = \frac{z}{z-\frac{1}{16}}, \quad |z| > \frac{1}{16}, \ k \geq 0$

3. $Z\{a^{5k}\} = z\{(a^5)^k\} = \frac{z}{z-a^5}, \quad |z| > a^5, \ k \geq 0$

Example : 2

Obtain Z- Transforms of following sequences for, $k \geq 0$

1. $\cos\left(\frac{k\pi}{2} + \frac{\pi}{4}\right)$
2. $\sin(3k + 4)$
3. $2^k \cos(2k + 1)$
4. $4^k \sin(2k + 5)$
5. $2^k \cosh 6k$
6. $4^k \sinh 3k$

Solution :

1. We know $Z\{\cos\alpha k\} = \frac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1}$

$$Z\{\sin\alpha k\} = \frac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1}$$

$$\therefore Z\left\{\cos\left(\frac{k\pi}{2} + \frac{\pi}{4}\right)\right\} = Z\left\{\cos\frac{k\pi}{2}\cos\frac{\pi}{4} - \sin\frac{k\pi}{2}\sin\frac{\pi}{4}\right\}$$

$$= \frac{1}{\sqrt{2}}\left[Z\left(\cos\frac{k\pi}{2}\right) - Z\left(\sin\frac{k\pi}{2}\right)\right]$$

$$= \frac{1}{\sqrt{2}}\left[\frac{z\left(z-\cos\frac{\pi}{2}\right)}{z^2 - 2z\cos\frac{\pi}{2} + 1} - \frac{z\sin\frac{\pi}{2}}{z^2 - 2z\cos\frac{\pi}{2} + 1}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\frac{z^2}{z^2+1} - \frac{z}{z^2+1}\right] = \frac{z(z-1)}{\sqrt{2}(z^2+1)}, \quad |z| > 1$$

2. $Z\{\sin(3k+4)\} = Z\{\sin 3k \cos 4 + \cos 3k \sin 4\}$

$$= \cos 4 \, Z\{\sin 3k\} + \sin 4 \, Z\{\cos 3k\}$$

$$= \cos 4\left[\frac{z\sin 3}{z^2 - 2z\cos 3 + 1}\right] + \sin 4\left[\frac{z(z-\cos 3)}{z^2 - 2z\cos 3 + 1}\right]$$

$$= \frac{z\cos 4 \sin 3 + z^2 \sin 4 - z\sin 4\cos 3}{(z^2 - 2z\cos 3 + 1)}, \quad |z| > 1$$

3. $Z\{2^k \cos(2k+1)\} = Z\{2^k(\cos 2k \cos 1 - \sin 2k \sin 1)\}$

$$= \cos(1)\, Z\{2^k \cos 2k\} - \sin(1)\, Z\{2^k \sin 2k\}$$

$$= \cos(1)\left[\frac{z(z-2\cos 2)}{z^2 - 4z\cos 2 + 4}\right] - \sin(1)\left[\frac{2z\sin 2}{z^2 - 4z\cos 2 + 4}\right]$$

$$= \frac{z\cos(1)(z-2\cos 2) - 2z\sin(1)\sin 2}{(z^2 - 4z\cos 2 + 4)}, \quad |z| > 2$$

4. $Z[4^k \sin(2k+5)] = Z\{4^k[\sin 2k \cos 5 + \cos 2k \sin 5]\}$

$= \cos 5 Z\{4^k \sin 2k\} + \sin 5 Z\{4^k \cos 2k\}$

$= \dfrac{\cos 5\,[4z \sin 2]}{z^2 - 8z\cos 2 + 16} + \sin 5\left[\dfrac{z(z - 4\cos 2)}{z^2 - 8z\cos 2 + 16}\right]$

$= \dfrac{[4z\cos 5 \sin 2] + \sin 5\,[z(z-4\cos 2)]}{z^2 - 8z\cos 2 + 16}$, $|z| > 4$

5. $Z\{2^k \cosh 6k\} = \dfrac{z(z - 2\cosh 6)}{z^2 - 4z\cosh 6 + 4}$, $|z| > \max\{|2e^6|\ |2e^{-6}|\}, |z| > 2e^6, k \geq 0$

6. $Z\{2^k \sinh 3k\} = \dfrac{2z\sinh 3}{z^2 - 4z\cosh 3 + 4}$, $|z| > \max\{|2e^3|\ |2e^{-3}|\}$, $|z| > 2e^3, k \geq 0$

Example : 3

Obtain Z Transforms of following sequences for $k \geq 0$.

1. 2^{3k+1} 2. e^{mk+n} 3. e^{n-pk}

Solution :

1. $Z\{2^{3k+1}\} = Z\{2^{3k}\, 2^1\} = 2Z\{2^{3k}\} = 2Z\{8^k\}$

$= 2\left[\dfrac{z}{z-8}\right] = \dfrac{2z}{z-8}$, $|z| > 8$

2. $Z\{e^{mk+n}\} = Z\{e^{mk} \cdot e^n\}$

$= e^n Z\{e^{mk}\} = e^n Z\{(e^m)^k\}$

$= e^n\left[\dfrac{z}{z-e^m}\right]$, $|z| > e^m$

3. $Z\{e^{n-pk}\} = Z\{e^n \cdot e^{-pk}\}$

$= e^n Z\{e^{-pk}\} = e^n Z\{(e^{-p})^k\}$

$= \mathbf{e^n\left[\dfrac{z}{z - e^{-p}}\right]}$, $|z| > e^{-p}$

Example : 4

Obtain Z Transforms of following sequences

1. $Z\{4^{3k-1}\}$, $k < 0$ 2. $Z\{(1/2)^{|k|}$

Solution :

1. $Z\{4^{3k-1}\} = Z\{4^{3k} \cdot 4^{-1}\}$

$= \dfrac{1}{4} Z\{(64)^k\} = \dfrac{1}{4}\left[-\left(\dfrac{z}{z-64}\right)\right]$

$= \mathbf{-\dfrac{1}{4}\left[\dfrac{z}{z-64}\right]}$, $|z| < 64, k < 0$

2. $$Z\{(1/2)^{|k|}\} = \frac{z}{z-\frac{1}{2}} + \frac{\left(\frac{1}{2}\right)z}{1-\frac{1}{2}z}, \quad \left|\frac{1}{2}\right| < |z| < \frac{1}{\left|\frac{1}{2}\right|}$$

$$= \mathbf{\frac{2z}{2z-1} + \frac{z}{2-z}}, \quad \frac{1}{2} < |z| < 2$$

Example : 5

Obtain Z Transforms of following

1. $\{4^k + 6^k\}$, $k \geq 0$, 2. $\{2^k - 3^k\}$, $k \geq 0$

Solution :

1. $$Z\{4^k + 6^k\} = Z\{2^k\} + Z\{6^k\}$$

$$= \frac{z}{z-2} + \frac{z}{z-6}, \qquad |z| > 2, |z| > 6$$

2. $$Z\{2^k - 3^k\} = Z\{2^k\} - Z\{3^k\}$$

$$= \mathbf{\frac{z}{z-2} + \frac{z}{z-3}}, \qquad |z| > 3, |z| > 2$$

Example : 6

Obtain Z Transforms of following :

1. $Z\{2U(k)\}$, 2. $Z\{4\delta(k)\}$

Solution :

1. $$Z\{2U(k)\} = 2Z\{U(k)\}$$

$$= 2\left[\frac{z}{z-1}\right] = \mathbf{\frac{2z}{z-1}}$$

2. $$Z\{4\delta(k)\} = 4Z\{\delta(k)\} = \mathbf{4}$$

Example : 7

Show that :

$$Z\left\{\frac{1-\cos\left(\frac{k\pi}{2}\right)-\sin\left(\frac{k\pi}{2}\right)}{2}\right\} = \frac{z}{(z-1)(z^2+1)}, \qquad k \geq 0$$

Solution : L.H.S. $$= \frac{1}{2}Z\left\{1-\cos\left(\frac{k\pi}{2}\right)-\sin\left(\frac{k\pi}{2}\right)\right\}$$

$$= \frac{1}{2}\left\{\frac{z}{z-1} - \frac{z\left(z-\cos\frac{\pi}{2}\right)}{z^2-2z\cos\frac{\pi}{2}+1} - \frac{z\sin\frac{\pi}{2}}{z^2-2z\cos\frac{\pi}{2}+1}\right\}$$

$$= \frac{1}{2}\left[\frac{z}{z-1} - \frac{z^2}{z^2+1} - \frac{z}{z^2+1}\right]$$

$$= \frac{1}{2}\left[\frac{z(z^2+1) - z^2(z-1) - z(z-1)}{(z-1)(z^2+1)}\right]$$

$$= \frac{1}{2}\left[\frac{z^3 + z - z^3 + z^2 - z^2 + z}{(z-1)(z^2+1)}\right] = \frac{1}{2}\left[\frac{2z}{(z-1)(z^2+1)}\right]$$

$$= \left[\frac{\mathbf{z}}{\mathbf{(z-1)(z^2+1)}}\right] = \textbf{R.H.S.,} \quad |z| > 1, \; k \ge 0$$

Example : 8

Obtain $Z\{e^{-k}\sin 2k\}, k \ge 0$

Solution :

$$= Z\left\{\left(\frac{1}{e}\right)^k \sin 2k\right\}$$

$$= Z\{c^k \sin 2k\} \;, \text{ where } c = \frac{1}{e}$$

$$= \frac{cz\sin 2}{z^2 - 2cz\cos 2 + c^2} = \frac{\frac{1}{e}z\sin 2}{z^2 - 2z\left(\frac{1}{e}\right)\cos 2 + \frac{1}{e^2}}$$

$$= \frac{\mathbf{ez\sin 2}}{\mathbf{z^2e^2 - 2ze\cos 2 + 1}} \quad , \quad |z| > \frac{1}{e}$$

Example : 9

Obtain $Z\{e^{ik} - e^{-ik}\}$, $k \ge 0$

Solution : $Z[e^{ik} - e^{-ik}] = Z\left\{\frac{e^{ik} - e^{-ik}}{2i}\right\}(2i)$ $\quad \boxed{\sin x = \frac{e^{ix} - e^{-ix}}{2i}}$

$$= 2i\,Z\{\sin k\}$$

$$= 2i\left[\frac{z\sin(1)}{z^2 - 2z\cos(1) + 1}\right]$$

$$= \frac{\mathbf{2iz\sin(1)}}{\mathbf{z^2 - 2z\cos(1) + 1}} \quad , \quad |z| > 1 \quad , \quad k \ge 0$$

Example : 10

Show that $Z\{e^{k\theta} + e^{-k\theta}\} = \frac{2z[z - \cosh\theta]}{z^2 - 2z\cosh\theta + 1}$, $k \ge 0$

Solution : L.H.S. $= Z[e^{k\theta} + e^{-k\theta}] = Z\left\{2\left(\frac{e^{k\theta} + e^{-k\theta}}{2}\right)\right\}$

$$= Z\{2\cosh k\,\theta\} = Z\{2\cosh \theta k\}$$

$$= \mathbf{2}\left[\frac{\mathbf{z(z - \cosh\theta)}}{\mathbf{z^2 - 2z\cosh\theta + 1}}\right]$$

$$= \textbf{R.H.S.}$$

Example : 11

Obtain $z\left\{\frac{4^k}{k!}\right\}$, where $k \geq 0$

Solution :

$$Z\left\{\frac{4^k}{k!}\right\} = \sum_{k=0}^{\infty} \frac{4^k}{k!} z^{-k} = \sum_{k=0}^{\infty} \frac{(4/z)^k}{k!}$$

$$= \frac{(4/z)^0}{0!} + \frac{(4/z)^1}{1!} + \frac{(4/z)^2}{2!} + \frac{(4/z)^3}{3!} + \frac{(4/z)^4}{4!} +++$$

$$= 1 + \frac{1}{1!}\left(\frac{4}{z}\right) + \frac{(4/z)^2}{L^2} + \frac{(4/z)^3}{L^3} +++$$

$$= \mathbf{e^{4/z}} \qquad \because\ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} +++$$

Example : 12

Obtain $Z\left\{\left(\frac{1}{2}\right)^k\right\}$ for $k = 0, 2, 4, 6, \ldots\ldots$

Solution :

$$Z\left\{\left(\frac{1}{2}\right)^k\right\} = \sum_{k=0,2,4,6,\ldots} \left(\frac{1}{2}\right)^k z^{-k}$$

$$= \sum_{k=0,2,4,6,\ldots} \left(\frac{1}{2z}\right)^k$$

$$= \left(\frac{1}{2z}\right)^0 + \left(\frac{1}{2z}\right)^2 + \left(\frac{1}{2z}\right)^4 +++$$

$$= 1 + \left(\frac{1}{2z}\right)^2 + \left(\frac{1}{2z}\right)^4, \text{ a geometric series with first term 1 \& } r = \left(\frac{1}{2z}\right)^2$$

$$= \frac{1}{1-\left(\frac{1}{2z}\right)^2} = \frac{1}{1-\frac{1}{4z^2}} = \mathbf{\frac{4z^2}{4z^2-1}},\ \mathbf{|z^2| < \frac{1}{4}}$$

Example : 13

Obtain $Z\left\{\left(\frac{1}{4}\right)^k\right\}$, $k = 1, 3, 5, 7, \ldots\ldots$

Solution :

$$Z\left\{\left(\frac{1}{4}\right)^k\right\} = \sum_{k=1,3,5,\ldots} \left(\frac{1}{4}\right)^k z^{-k}$$

$$= \sum_{k=1,3,5,\ldots} \left(\frac{1}{4z}\right)^k$$

$$= \left(\frac{1}{4z}\right) + \left(\frac{1}{4z}\right)^3 + \left(\frac{1}{4z}\right)^5 +++ \text{ which is a geometric series with first term } \frac{1}{4z}, \ r = \left(\frac{1}{4z}\right)^2$$

$$= \frac{\left(\frac{1}{4z}\right)}{1-\left(\frac{1}{4z}\right)^2} = \frac{\left(\frac{1}{4z}\right)}{\frac{16z^2-1}{16z^2}} \qquad , \qquad \left|\frac{1}{16z^2}\right| > 1$$

$$= \mathbf{\frac{4z}{16z^2-1}} \qquad , \qquad |z^2| < \frac{1}{16}$$

Example : 14

Obtain $Z\left\{\left(\frac{1}{(k+1)!}\right)\right\}$, for $k \geq 0$

Solution : $Z\left\{\left(\frac{1}{(k+1)!}\right)\right\} = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} z^{-k}$

$$\because \qquad = 1 + \frac{z^{-1}}{2!} + \frac{z^{-2}}{3!} + \frac{z^{-3}}{4!} +++$$

$$= \frac{z}{z}\left[1 + \frac{z^{-1}}{2!} + \frac{z^{-2}}{3!} + \frac{z^{-3}}{4!} +++\right]$$

$$= z\left[\frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \frac{z^{-4}}{4!} ++\right]$$

$$= z\left[e^{z^{-1}} - 1\right] = \mathbf{z\left[e^{\frac{1}{z}} - 1\right]}$$

Example : 15

Obtain $Z\left\{\frac{1}{k!}\right\}$, $k \geq 0$

Solution : $Z\left\{\frac{1}{k!}\right\} = \sum_{k=0}^{1} \frac{1}{k!} z^{-k}$

$$= \frac{1}{0!} + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} ++++$$

$$= e^{z^{-1}} = \mathbf{e^{\frac{1}{z}}} \qquad \because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} +++$$

Example : 16

Obtain $Z\left\{\frac{1}{k}\right\}$, $k > 0$

Solution :

$$z\{f(k)\} = \sum_{k=1}^{\infty} \frac{z^{-k}}{k} = \sum_{k=1}^{\infty} f(k)\, z^{-k}$$

$$= \frac{z^{-1}}{1} + \frac{z^{-2}}{2} + \frac{z^{-3}}{3} + \frac{z^{-4}}{4} +++$$

$$= \mathbf{-\log(1 - z^{-1})} \qquad \because \log(1+x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \ldots.$$

Example : 17

Find Z transform of $\{ f(k) \}$, where $f(k) = \begin{cases} 4^k, & k < 0 \\ 2^k, & k \geq 0 \end{cases}$

Solution :

$$Z\{f(k)\} = \sum_{k=-\infty}^{-1} f_{(k)} z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} f_{(k)} z^{-k} + \sum_{k=0}^{\infty} f_{(k)} z^{-k}$$

$$= \sum_{k=-\infty}^{-1} 4^k z^{-k} + \sum_{k=0}^{\infty} 2^k z^{-k}$$

$$= -\left(\frac{z}{z-4}\right) + \left(\frac{z}{z-2}\right), \qquad |z| < 4, |z| > 2$$

$$= \mathbf{\frac{z}{z-2} - \frac{z}{z-4}}, \qquad 2 < |z| < 4$$

Example : 18

Find Z Transform of $\{ f(k) \}$

where $f(k) = \begin{cases} 2^k, & k < 0 \\ \left(\frac{1}{2}\right)^k, & k \geq 0 \end{cases}$

Solution :

$$Z\{f(k)\} = z\{2^k\}_{k<0} + z\left\{\left(\frac{1}{2}\right)^k\right\}_{k \geq 0}$$

$$= -\left(\frac{z}{z-2}\right) + \frac{z}{z - \frac{1}{2}}, \qquad |z| < 2, |z| > \frac{1}{2}$$

$$= \mathbf{\frac{2z}{2z-1} - \frac{z}{z-2}}, \qquad \frac{1}{2} < |z| < 2$$

Example : 19

Find Z transform of $\{f(k)\}$, where $f(k) = \begin{cases} \left(-\frac{1}{4}\right)^{k+1}, & k < 0 \\ \left(-\frac{1}{5}\right)^{k+1}, & k \geq 0 \end{cases}$

Solution :

$$Z\{f(k)\} = z\left\{\left(-\frac{1}{4}\right)^{k+1}\right\}_{k<0} + z\left\{\left(-\frac{1}{5}\right)^{k+1}\right\}_{k \geq 0},$$

$$= z\left\{\left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)^k\right\}_{k<0} + z\left\{\left(-\frac{1}{5}\right)\left(-\frac{1}{5}\right)^k\right\}_{k\ge 0}$$

$$= -\frac{1}{4} Z\left\{\left(-\frac{1}{4}\right)^k\right\} - \frac{1}{5} Z\left\{\left(-\frac{1}{5}\right)^k\right\} \quad ,$$

$$= -\frac{1}{4}\left[-\left(\frac{z}{z+\frac{1}{4}}\right)\right] - \frac{1}{5}\left[\left(\frac{z}{z+\frac{1}{5}}\right)\right] \quad , \quad |z| < \frac{1}{4}\ , |z| > \frac{1}{5}$$

$$= \frac{z}{4\left(z+\frac{1}{4}\right)} - \frac{z}{5\left(z+\frac{1}{5}\right)} \quad , \quad \frac{1}{5} < |z| < \frac{1}{4}$$

$$= \mathbf{\frac{z}{(4z+1)} - \frac{z}{(5z+1)}} \quad , \quad \frac{1}{5} < |z| < \frac{1}{4}$$

Example : 20

Obtain Z Transform of $\left\{\left(\frac{1}{3}\right)^{|k|}\right\}$ **May 2014**

Solution :

$$Z\left\{\left(\frac{1}{3}\right)^{|k|}\right\} = \frac{z}{z-\frac{1}{3}} + \frac{\left(\frac{1}{3}\right)z}{1-\frac{1}{3}z} \quad , \quad \left|\frac{1}{3}\right| < |z| < \frac{1}{\left|\frac{1}{3}\right|}$$

$$= \mathbf{\frac{3z}{3z-1} + \frac{z}{3-z}} \quad , \quad \left|\frac{1}{3}\right| < |z| < 3$$

Example : 21

Obtain Z Transform of $\left\{\left(\frac{2}{3}\right)^{|k|}\right\}$ **Dec. 2015**

Solution :

We know $$Z\left\{a^{|k|}\right\} = \frac{z}{z-a} + \frac{az}{1-az} \quad , \quad |a| < |z| < \left|\frac{1}{a}\right|$$

$$Z\left\{\left(\frac{2}{3}\right)^{|k|}\right\} = \frac{z}{z-\frac{2}{3}} + \frac{\frac{2}{3}z}{1-\frac{2}{3}z} \quad , \quad \left|\frac{2}{3}\right| < |z| < \frac{1}{\left|\frac{2}{3}\right|}$$

$$= \mathbf{\frac{3z}{3z-2} + \frac{2z}{3-2z}} \quad , \quad \frac{2}{3} < |z| < \frac{3}{2}$$

Example : 22

Obtain Z Transform of $Z\{4^k e^{-6k}\}$, $k \ge 0$

Solution :

$$Z\{4^k e^{-6k}\} = Z\left\{4^k\left(\frac{1}{e^6}\right)^k\right\}$$

$$= Z\left\{\left(\frac{4}{e^6}\right)^k\right\}$$

$$= \frac{\mathbf{z}}{\mathbf{z}-\left(\frac{\mathbf{4}}{\mathbf{e^6}}\right)} \quad , \quad |z| > \frac{4}{e^6}$$

Example : 23

Obtain Z Transform of $\{ 2^k e^{-2k} \}$, $k \geq 0$

Solution :

$$Z\{ 2^k e^{-2k} \} = Z\{ (2e^{-2})^k\}$$

$$= \frac{z}{z-2e^{-2}} \quad , \quad |z| > 2e^{-2}$$

$$= \frac{\mathbf{ze^2}}{\mathbf{ze^2 - 2}} \quad , \quad |z| > \frac{2}{e^2}$$

Self-Assessment Exercises 2.2

- **Obtain the Z Transform of Following Sequences :**

Ex.1 $e^k \sin 2k$ **Ans. :** $\frac{ez \sin 2}{z^2 - 2ez \cos 2 + e^2}$

Ex.2 $\cos\left(\frac{k\pi}{2}+\frac{\pi}{4}\right)$, $k \geq 0$ **Ans. :** $\frac{z(z-1)}{\sqrt{2}(z^2+1)}$

Ex.3 $\cosh\left[\frac{k\pi}{2}+\alpha\right]$, $k \geq 0$ **Ans. :** $\frac{z^2 \cosh\alpha \ - \ z \cosh\left(\frac{\pi}{2}-\alpha\right)}{z^2 - 2z \cosh\left(\frac{\pi}{2}\right)+1}$

Ex.4 $\sin(3k+6)$, $k \geq 0$ **Ans. :** $\frac{z(z \sin 5 - \sin 2)}{z^2 - 2z \cos 3 + 1}$

Ex.5 $3^{-2k} + 1$, $k \geq 0$ **Ans. :** $\frac{z}{z-\frac{1}{9}} + \frac{z}{z-1}$

Ex.6 $e^{2k} + 4$, $k \geq 0$ **Ans. :** $\frac{z}{z-e^2} + 4\left(\frac{z}{z-1}\right)$

Ex.7 $\frac{1}{5}[(-3)^k - (-4)^k]$, $k \geq 0$ **Ans. :** $\frac{z}{5(z^2+7z+12)}$

Ex.8 $\frac{1}{5}[(-3)^k - (-8)^k]$, $k \geq 0$ **Ans. :** $\frac{z}{z^2+11z+24}$

Ex.9 $\frac{1}{6}(-2)^k + \frac{11}{6}4^k$, $k \geq 0$ **Ans. :** $\frac{2z^2+3z}{z^2-2z-8}$

Ex.10 $e^{-ak}\sin(k\theta)$, $k \geq 0$ **Ans. :** $\dfrac{z\, e^{a} \sin\theta}{z^2 e^{2a} - 2z\, e^{a} \cos\theta + 1}$

Ex.11 $\dfrac{1}{(k+1)!}$, $k \geq 0$ **Ans. :** $z\left(e^{\frac{1}{z}} - 1\right)$

Ex.12 $\dfrac{1}{k!}$, $k \geq 0$ **Ans. :** $e^{\frac{1}{z}}$

Ex.13 $\cos\left(\dfrac{k\pi}{8} + \theta\right)$, $k \geq 0$ **Ans. :** $\dfrac{z^2 \cos\theta - z\cos\left(\frac{\pi}{8} - \theta\right)}{z^2 - 2z\cos\left(\frac{\pi}{8}\right) + 1}$

Ex.14 $\left(\dfrac{1}{2}\right)^{|k|}$ **Ans. :** $\dfrac{-3z}{(1-2z)(z-2)}$

Ex.15 $\dfrac{a^k}{k}$, $k > 0$ **Ans. :** $\log(1 - az^{-1})$

Ex.16 Obtain Z Transform of following sequences for $k \geq 0$

(i) $e^{-ak} \sin 8k$ (ii) $2^{2k} \cos 4k$ (iii) e^{-mk}

(iv) $4^k \cosh 2k$ (v) e^{mk+n} (vi) $2^k \cos\left(\dfrac{\pi k}{2} + \dfrac{\pi}{4}\right)$

(vii) $\{-6, -4, -2, \underset{\uparrow}{0}, 2, 4\}$ (viii) $\{2, 4, 6, 8\}$

(ix) $\left(\dfrac{1}{3}\right)^k$ (x) $e^{k\alpha}$ (xi) $\cosh\left(\dfrac{\pi k}{2}\right)$

(xii) $e^{-k} \sin k$

Ex.17 $\dfrac{(-3)^k}{k!}$, $k \geq 0$ **Ans.** : $e^{\frac{-3}{z}}$

Ex.18 $\cos\left(\dfrac{\pi k}{4} + \theta\right)$, $k \geq 0$ **Ans. :** $\dfrac{z}{\sqrt{2}}\left[\dfrac{(\sqrt{2}\,z - 1)\cos\theta - \sin\theta}{z^2 - z\sqrt{2} + 1}\right]$

Ex.19 $f(k) = 3\left(\dfrac{1}{4}\right)^k + 4\left(\dfrac{1}{5}\right)^k$, $k \geq 0$ **Ans. :** $\dfrac{12z}{4z-1} + \dfrac{20z}{5z-1}$, $|z| > \dfrac{1}{4}$

Ex.20 $(e^{-k} - e^{-2k})$, $k \geq 0$ **Ans. :** $\dfrac{ez}{ez-1} + \dfrac{e^2 z}{e^2 z - 1}$

2.9 Theorems on Z – Transforms :

2.9.1 Theorem 1 : Change of Scale:

If $Z\{f(k)\} = F(z)$ then $Z\{a^k f(k)\} = [F(z)]_{z \to \frac{z}{a}} = F\left(\dfrac{z}{a}\right)$

2.9.2 Theorem 2 : Multiplication by k :

If Z{ f (k) } = F(z) then

(i) $Z\{ k\, f(k) \} = \left(-z\frac{d}{dz}\right) F(z)$

(ii) $Z\{ k^2 f_{(k)} \} = \left(-z\frac{d}{dz}\right)^2 F(z)$

(iii) $Z\{ k^3 f(k) \} = \left(-z\frac{d}{dz}\right)^3 F(z)$

Note : (i) $\left(-z\frac{d}{dz}\right)^2 \neq (-1)^2 z^2 \left(\frac{d}{dz}\right)^2$

(ii) $\left(-z\frac{d}{dz}\right)^2 = -z\left[\frac{d}{dz}\left(-z\frac{d}{dz}\right)\right]$

2.9.3 Theorem 3 : Recurrence Formula :

If $f(k) = k^p$, where p is a positive integer then $Z\{ k^p \} = -z\frac{d}{dz}\left[Z(k)^{p-1} \right]$

2.9.4 Theorem 4 : Shifting Property :

(i) Shifting to the Right

$Z\{ f(k) \} = F(z)$ then $Z\{ f(k-n) \} = z^{-n} F(z)$, where $k \geq n$, n is a positive integer

(ii) Shifting to the Left

$Z\{ f(k+1) \} = z\,F(z) - z\,f(0)$

$Z\{ f(k+2) \} = z^2 F(z) - z^2 f(0) - z\,f(1)$

$Z\{ f(k+3) \} = z^3 F(z) - z^3 f(0) - z^2 f(1) - z\,f(2)$

Illustrative Examples

Example : 1

Obtain $Z\{ 4^k U(k) \}$

Solution :

We know, $Z\{U(k)\} = \frac{z}{z-1}$, $|z| > 1$

By change of scale property,

$z\{ 4^k U(k) \} = \left(\frac{z}{z-1}\right)_{z \to \frac{z}{4}} = \mathbf{\frac{z}{z-4}}$, $|z| > 4$

Example : 2

If $Z\{f(k)\} = \frac{3z+1}{(z-1)(z+1)}$, Find $Z\{2^k f(k)\}$

Solution :

$$Z\{f(k)\} = \frac{3z+1}{(z-1)(z+1)} = F(z)$$

$$Z\{2^k f(k)\} = \left[\frac{3z+1}{(z-1)(z+1)}\right]_{z \to \frac{z}{2}}$$

$$= \frac{\frac{3z}{2}+1}{\left(\frac{z}{2}-1\right)\left(\frac{z}{2}+1\right)} \quad \text{, by change of scale property}$$

$$= \mathbf{\frac{2(3z+2)}{(z-2)(z+2)}}$$

Example : 3

Obtain $Z\{k a^k\}$, $k \geq 0$

Solution :

We know, $Z\{a^k\} = \frac{z}{z-a}$, $|z| > |a|$

$$= F(z)$$

By "Multiplying by k" theorem,

$$Z\{k a^k\} = -z\frac{d}{dz}[F(z)]$$

$$= -z\frac{d}{dz}\left[\frac{z}{z-a}\right]$$

$$= -z\left[\frac{(z-a)(1)-z(1)}{(z-a)^2}\right]$$

$$= \frac{-z(-a)}{(z-a)^2} = \mathbf{\frac{az}{(z-a)^2}}\ \mathbf{, |z| > |a|}$$

Example : 4

Obtain $Z\{k\,U(k)\}$

Solution : $Z\{U(k)\} = \frac{z}{z-1}$, $|z| > 1$

By "Multiplying by k" theorem,

$$Z\{k\,U(k)\} = -z\frac{d}{dz}\left(\frac{z}{z-1}\right)$$

$$= \mathbf{\frac{z}{(z-1)^2}} \quad \mathbf{,\ |z| > 1}$$

Example : 5

Obtain Z { k } , $k \geq 0$

Solution :

$$Z\{k\} = Z\{k(1)\}$$

$$= Z\{k\,1^k\}$$

$$\text{Let } f(k) = 1^k$$

$$\therefore \quad Z\{f(k)\} = z\{1^k\} = \frac{z}{z-1} = F(z)$$

$$\therefore \quad Z\{k\,f(k)\} = -z\frac{d}{dz}\left[\frac{z}{z-1}\right], \text{ by multiplication theorem}$$

$$\therefore \quad \mathbf{Z\{k\}} = \frac{\mathbf{z}}{\mathbf{(z-1)^2}} \quad , \quad |z| > 1$$

Example : 6

Obtain Z { k^2 } , $k \geq 0$

Solution :

$$\text{We have} \quad Z\{1\} = \frac{z}{z-1} \quad , \quad |z| > 1$$

$$\therefore Z\{k^2\} = \left(-z\frac{d}{dz}\right)^2\left(\frac{z}{z-1}\right)$$

$$= -z\frac{d}{dz}\left[-z\frac{d}{dz}\left(\frac{z}{z-1}\right)\right]$$

$$= z\frac{d}{dz}\left[z\frac{d}{dz}\left(\frac{z}{z-1}\right)\right]$$

$$= z\frac{d}{dz}\left[z\left(\frac{z-1-z}{(z-1)^2}\right)\right] = z\frac{d}{dz}\left[-\frac{z}{(z-1)^2}\right]$$

$$= z\left(\frac{z+1}{(z-1)^3}\right) = \frac{\mathbf{z(z+1)}}{\mathbf{(z-1)^3}} \quad , \quad |z| > 1$$

Example : 7

Obtain Z { 4^{k-2} U (k – 2) }

Solution

$$\text{Let} \quad f(k-2) = 4^{k-2}\,U(k-2)$$

$$\therefore f(k) = 4^k\,U(k)$$

$$\therefore Z\{f(k)\} = Z\{4^k\,U(k)\}$$

$$= \left[\frac{z}{z-1}\right]_{z \to \frac{z}{4}} \quad \text{by change of scaler rule}$$

$$= \left[\frac{z}{z-4}\right] = F(z) \quad , \quad |z| > 4$$

By shifting theorem,

$$Z[f(k-2)] = z^{-2} F(z)$$

$$= z^{-2}\left[\frac{z}{z-4}\right]$$

$$\mathbf{Z\{4^{k-2} U(k-2)\} = \frac{1}{z(z-4)}} \quad , \quad |z| > 4$$

Example : 8

Obtain Z { 6^k U (k – 1) }

Solution

$$Z\{6^k U(k-1)\} = Z\{6^{k-1} 6\ U(k-1)\}$$

$$= 6Z\{6^{k-1} 6\ U(k-1)\}$$

$$= 6Z\{f(k-1)\}$$

$$= 6z^{-1} Z\{f(k)\} \quad , \quad \text{By shifting theorem}$$

$$= 6z^{-1}[Z\{6^k U(k)\}]$$

$$= 6z^{-1}\left[\frac{z}{z-1}\right]_{z \to \frac{z}{6}} \quad , \quad \text{by change of scale}$$

$$= 6z^{-1}\left[\frac{z}{z-6}\right] = \mathbf{\frac{6}{z-6}} \quad , \quad |z| > 6$$

Example : 9

Obtain Z { k^2 } , $k \geq 0$

Solution :

By Recurrence formula

$$Z\{k^p\} = -z \frac{d}{dz}[Z(k^{p-1})]$$

$$Z\{k^2\} = -z \frac{d}{dz}[Z(k)]$$

$$= -z \frac{d}{dz}\left[\frac{z}{(z-1)^2}\right] = \mathbf{\frac{z(z+1)}{(z-1)^3}}$$

Example : 10

Obtain Z{ (k + 2) 2^k } , $k \geq 0$

Solution :

$$Z\{(k+2)2^k\} = Z\{k\,2^k + 2\,2^k\}$$

$$= Z\{k\,2^k\} + 2z\{2^k\}$$

$$= -z\frac{d}{dz}\left(\frac{z}{z-2}\right) + 2\left(\frac{z}{z-2}\right)$$

$$= -z\left[\frac{(z-2-z)}{(z-2)^2}\right]+\frac{2z}{z-2}$$

$$= \frac{2z}{(z-2)^2}+\frac{2z}{z-2}$$

$$= \frac{2z\,[\,1+z-2]}{(z-2)^2}$$

$$= \mathbf{\frac{2z\,(z-1)}{(z-2)^2}} \quad , \quad |z| > 2\ ,\ k \geq 0$$

Example : 11

Obtain $Z\{\,k\,e^{-ak}\,\}$, $k \geq 0$

Solution :

We know,

$$Z\{\,a^k\,\} = \frac{z}{z-a}$$

$$Z\{e^{-ak}\} = \frac{z}{z-e^{-a}} = \frac{z\,e^a}{z\,e^a-1}$$

By "Multiplying by k" theorem,

$$Z\{\,k\,e^{-ak}\,\} = -z\frac{d}{dz}\left[\frac{z\,e^a}{z\,e^a-1}\right]$$

$$= -e^a\,z\left[\frac{(z\,e^a-1)\,(1)-z\,(e^a)}{(z\,e^a-1)^2}\right]$$

$$= \mathbf{\frac{z\,e^a}{(e^a\,z-1)^2}} \quad , \quad |z| > e^{-a}$$

Example : 12

Obtain $Z\{\,k\cos\theta k\,\}$, $k \geq 0$

Solution :

We know, $Z(\cos\theta k) = \dfrac{z\,(z-\cos\theta)}{z^2-2z\cos\theta+1}$

By "Multiplying by k" theorem,

$$Z\{\,k\cos(\theta k)\,\} = -z\frac{d}{dz}\left[\frac{z^2-z\cos\theta}{z^2-2z\cos\theta+1}\right]$$

$$= -z\left[\frac{(z^2-2z\cos\theta+1)(2z-\cos\theta)-(z^2-z\cos\theta)(2z-2\cos\theta)}{(z^2-2z\cos\theta+1)^2}\right]$$

$$= -z\left[\frac{-z^2\cos\theta+2z-\cos\theta}{(z^2-2z\cos\theta+1)^2}\right]$$

$$= \mathbf{\frac{z^3\cos\theta-2z^2+z\cos\theta}{(z^2-2z\cos\theta+1)^2}}$$

Example : 13

Obtain $Z\{ (k+1)\,4^k \}$, $k \geq 0$

Solution :

We know,

$$Z\{(k+1)\,4^k\} = Z\{k\,4^k + 4^k\}$$

$$= -z\,\frac{d}{dz}\left(\frac{z}{z-4}\right) + \left(\frac{z}{z-4}\right)$$

$$= -z\left[\frac{z-4-z}{(z-4)^2}\right] + \frac{z}{z-4}$$

$$= \frac{4z}{(z-4)^2} + \frac{z}{z-4}$$

$$= \frac{z\,[4] + [z^2 - 4z]}{(z-4)^2}$$

$$= \mathbf{\frac{z^2}{(z-4)^2}} \quad , \quad |z| > 4$$

Example : 14

Obtain $Z\{ k^2\ 4^{k-1}\,U(k-1) \}$

Solution :

We know,

Let $f(k-1) = 4^{k-1}\,U(k-1)$

$$F(k) = 4^k\,U(k)$$

$$Z(f(k)) = Z\{4^k\,U(k)\}$$

$$= \left[\frac{z}{z-1}\right]_{z \to \frac{z}{4}} \quad , \text{ by change of scale}$$

$$= \frac{z}{z-4}$$

$$= F(z)$$

But $Z\{f(k-1)\} = z^{-1}\ Z\{f(k)\}$, by shifting theorem

$$= z^{-1}\left[\frac{z}{z-4}\right]$$

$$= \frac{1}{z-4}$$

$\therefore$ $$z\{k^2 f(k+1)\} = \left(-z\,\frac{d}{dz}\right)^2 Z\{f(k-1)\} \quad , \text{ by Multiplication by k}$$

$$= -z\,\frac{d}{dz}\left[-z\,\frac{d}{dz}\left(\frac{1}{z-4}\right)\right]$$

$$= -z\,\frac{d}{dz}\left[-z\left(-\frac{1}{(z-4)^2}\right)\right]$$

$$= -z\,\frac{d}{dz}\left[\frac{z}{(z-4)^2}\right]$$

$$= -z\left[\frac{(z-4)^2(1) - z\,2(z-4)}{(z-4)^4}\right]$$

$$= -z\left[\frac{z-4-2z}{(z-4)^3}\right]$$

$$= -z\left[\frac{-4-z}{(z-4)^3}\right]$$

$$= \mathbf{\frac{z\,(z+4)}{(z-4)^3}}\ , \quad |z| > 4$$

Example : 15

Obtain $Z\{\,k\,e^{-k}\,\}$, $k \geq 0$

Solution :

We know $\quad Z\{\,e^{-k}\,\} = \dfrac{z}{z-e^{-1}} = \dfrac{z}{z-\dfrac{1}{e}}$

$$= \frac{ez}{ez-1} = F(z)\ ,\ |z| > \frac{1}{e}$$

$\therefore$ By "multiplication by k″ rule"

$$Z\{\,k\,e^{-k}\,\} = -z\,\frac{d}{dz}[\,F(z)\,]$$

$$= -z\,\frac{d}{dz}\left[\frac{ez}{ez-1}\right] = -z\left[\frac{(ez-1)\,(e) - ez\,(e)}{(ez-1)^2}\right]$$

$$= -z\left[\frac{e^2z - e - e^2z}{(ez-1)^2}\right] = \mathbf{\frac{ze}{(ez-1)^2}\ ,\ |z| > \frac{1}{e}}$$

Example : 16

Obtain $Z\{\,k^2\,U(k)\,\}$

Solution :

We know

$$Z\{\,U(k)\,\} = \frac{z}{z-1}\quad ,\ |z| > 1$$

$\therefore$ By "multiplication by k^2" rule

$$Z\{\,k^2\,U(k)\,\} = \left(-z\,\frac{d}{dz}\right)^2 F(z)$$

$$= \left(-z\,\frac{d}{dz}\right)^2\left(\frac{z}{z-1}\right)$$

$$= -z\,\frac{d}{dz}\left[-z\,\frac{d}{dz}\left(\frac{z}{z-1}\right)\right] = z\,\frac{d}{dz}\left[\,z\left(\frac{(z-1)-z}{(z-1)^2}\right)\right]$$

$$= z\frac{d}{dz}\left[z\left(\frac{-1}{(z-1)^2}\right)\right] = z\frac{d}{dz}\left[\left(\frac{-z}{(z-1)^2}\right)\right]$$

$$= -z\left[\frac{(z-1)^2(1) - z\,2\,(z-1)}{(z-1)^4}\right]$$

$$= -z\,(z-1)\left[\frac{z-1-2z}{(z-1)^4}\right]$$

$$= -z\left[\frac{-z-1}{(z-1)^3}\right] = \boldsymbol{\frac{z\,(z+1)}{(z-1)^3}} \quad , \quad |z| > 1$$

Example : 17

Obtain $Z\{\,k\,\delta(k)\,\}$

Solution :

We know

$$Z\{\,\delta(k)\,\} = 1 = F(z)$$

$$\therefore Z\{\,k\,\delta(k)\,\} = -z\frac{d}{dz}[\,F(z)\,]$$

$$= -z\frac{d}{dz}(1)$$

$$= \mathbf{0}$$

Self-Assessment Exercise 2.3

Find the Z transforms of the following for k ≥ 0.

Ex.1 $Z\{\,k^3\,\}$ **Ans. :** $\dfrac{z^3+4z^2+2}{(z-1)^4}$, $|z| > 1$

Ex.2 $Z\{\,k^4\,\}$ **Ans. :** $\dfrac{z^4+11z^3+11z^2+z}{(z-1)^5}$, $|z| > 1$

Ex.3 $Z\{\,k^2\,a^k\,\}$ **Ans. :** $\dfrac{az\,(z+a)}{(z-a)^3}$, $|z| > |a|$

Ex.4 $Z\{\,(k+1)^2\,\}$ **Ans. :** $\dfrac{z^2\,(z+1)}{(z-1)^3}$, $|z| > 1$

Ex.5 $Z\{\,k\,e^{ak}\,\}$ **Ans. :** $\dfrac{z\,e^a}{(z-e^a)^2}$, $|z| > |e^a|$

Ex.6 If $z\{\,f(k)\,\} = \dfrac{4z-1}{(z+3)(z-2)}$ then find $z\{\,4^k f(k)\,\}$ **Ans. :** $\dfrac{16\,(z-1)}{(z+12)(z-8)}$, $|z| > 12$

Ex.7 If $Z\{\,f(k)\,\} = \dfrac{z}{z-8}$ then find $z\{\,2^k f(k)\,\}$ **Ans. :** $\dfrac{z}{z-16}$, $|z| > 16$

Ex.8 $Z\{ k\,U(k) \}$ **Ans. :** $\dfrac{z}{(z-1)^2}$, $|z| > 1$

Ex.9 $Z\{ k\,4^k \}$ **Ans. :** $\dfrac{4z}{(z-4)^2}$, $|z| > 4$

Ex.10 $Z\{ 6^k\,U(k) \}$ **Ans. :** $\dfrac{z}{z-6}$, $|z| > 6$

Ex.11 $Z\{ 8^{k-2}\,U(k-2) \}$ **Ans. :** $\dfrac{1}{z(z-8)}$, $|z| > 8$

Ex.12 $Z\{ 4^k\,U(k-1) \}$ **Ans. :** $\dfrac{4}{z-4}$, $|z| > 4$

Ex.13 $Z\{ k\,3^k \}$ **Ans. :** $\dfrac{3z}{(z-3)^2}$, $|z| > 3$

Ex.14 $Z\{ k\,5^k \}$ **Ans. :** $\dfrac{5z}{(z-5)^2}$, $|z| > 5$

Ex.15 $Z\{ (1+k)\,8^k \}$ **Ans. :** $\dfrac{z^2}{(z-8)^2}$, $|z| > 8$

Ex.16 $Z\{ k^2\,6^k \}$ **Ans. :** $\dfrac{6z(z+6)}{(z-6)^3}$, $|z| > 6$

Ex.17 $Z\{ ke^{-3k} \}$ **Ans. :** $\dfrac{z\,e^3}{(ze^3-1)^2}$, $|z| > |e^{-3}|$

Ex.18 $Z\{ k^2e^{-ak} \}$ **Ans. :** $\dfrac{z\,e^{-a}(z+e^{-a})}{(z-e^{-a})^3}$ $|z| > |e^{-3}|$ **(May 2015)**

Ex.19 $Z\{ k^2\,5^k \}$ **Ans. :** $\dfrac{5z(z+5)}{(z-5)^3}$, $|z| > 5$

Ex.20 $Z\{ k^2\,e^{-4k} \}$ **Ans. :** $\dfrac{e^4 z(1+e^4 z)}{(e^4 z-1)^3}$, $|z| > |e^{-4}|$

2.10 Inverse Z Transforms :

If $Z\{ f(k) \} = F(z)$

then

$\{ f(k) \} = Z^{-1}[F(z)]$

$\therefore \quad Z^{-1}[F(z)] = \{ f(k) \}$

1. Z^{-1} is called as a Inverse Z Transform operator
2. $Z^{-1}[F(z) + G(z)] = Z^{-1}[F(z)] + Z^{-1}[G(z)]$
3. $Z^{-1}[a\,F(z)] = a\,Z^{-1}F(z)$, where a is a constant
4. $Z^{-1}\left[\dfrac{z}{z-a}\right] = \{ a^k \},\ k \geq 0,\quad |z| > |a|$

5. $Z^{-1}\left[\dfrac{z}{z+a}\right] = \{ (-a)^k \}, \quad k \ge 0, |z| > |a|$

6. $Z^{-1}\left[\dfrac{z}{z-1}\right] = \{ 1^k \} = \{ 1 \}, \quad k \ge 0, |z| > 1$

7. $Z^{-1}\left[\dfrac{z}{z-1}\right] = \{ U(k) \}, \quad |z| > 1$

8. $Z^{-1}\left[-\left(\dfrac{z}{z-a}\right)\right] = \{ a^k \}, \quad k < 0, |z| < |a|$

9. $Z^{-1}\left[\dfrac{1}{z-a}\right] = \{ a^{k-1} \}, \quad k \ge 0, |z| > |a|$

10. $Z^{-1}\left[\dfrac{z}{(z-a)^2}\right] = \{ k\, a^k \}, \quad k \ge 0, |z| > |a|$

11. $Z^{-1}\left[\dfrac{z}{(z-1)^2}\right] = \{ k \}, \quad k \ge 0, |z| > 1$

12. $Z^{-1}\left[\dfrac{z\,(z+1)}{(z-1)^3}\right] = \{ k^2 \}, \quad k \ge 0, |z| > 1$

Some useful Inverse Z – Transforms

Sr. No.	F(z)	$\{ f(k) = z^{-1} [F(z)] \}$
1	$\left(\dfrac{z}{z-a}\right)$	a^k , $k \ge 0$ $\lvert z \rvert > \lvert a \rvert$
2	$\left(\dfrac{z}{z-1}\right)$	1 , $k \ge 0$ $\lvert z \rvert > 1$
3	$\dfrac{z}{(z+a)}$	$(-a)^k$, $k \ge 0$ $\lvert z \rvert > \lvert a \rvert$
4	$-\left[\dfrac{z}{z-a}\right]$	a^k , $k < 0$ $\lvert z \rvert < \lvert a \rvert$
5	$\dfrac{1}{(z-a)}$	a^{k-1} , $k \ge 0$ $\lvert z \rvert > 1$
6	$\dfrac{1}{(z-a)}$	$-a^{k-1}$, $k < 0$ $\lvert z \rvert < \lvert a \rvert$
7	$\dfrac{z^2}{(z-a)^2}$	$(k+1)\, a^k$, $k \ge 0$ $\lvert z \rvert > \lvert a \rvert$
8	1	$\delta(k)$, $\forall k$
9	$\dfrac{z}{z-1}$	$U(k)$, $\forall k$ $\lvert z \rvert > 1$

2.10.1 Inverse Z–T by Using Standard Formulae :

Illustrative Examples

Example : 1

Obtain $Z^{-1}\left[\frac{z}{4z+1}\right] \quad |z| > \frac{1}{4}, \; k \geq 0$

Solution :

$$Z^{-1}\left[\frac{z}{4z+1}\right] = Z^{-1}\left[\frac{z}{4\left(z+\frac{1}{4}\right)}\right]$$

$$= \frac{1}{4} Z^{-1}\left[\frac{z}{z+\frac{1}{4}}\right]$$

$$= \frac{\mathbf{1}}{\mathbf{4}}\left\{\left(\frac{\mathbf{-1}}{\mathbf{4}}\right)^{\mathbf{k}}\right\} \quad , \quad |z| > \frac{1}{4}, \; k \geq 0$$

Example : 2

Obtain $Z^{-1}\left[\frac{2z}{2z-1}\right] \quad , \; k \geq 0$

Solution :

$$Z^{-1}\left[\frac{2z}{2z-1}\right] = Z^{-1}\left[\frac{2z}{2\left(z-\frac{1}{2}\right)}\right] = Z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right]$$

$$= \left\{\left(\frac{\mathbf{1}}{\mathbf{2}}\right)^{\mathbf{k}}\right\} \quad , \quad k \geq 0, \; |z| > \frac{1}{2}$$

Example : 3

Obtain $Z^{-1}\left[\frac{2z+3}{2z+1}\right] \quad , \; k \geq 0$

Solution :

$$Z^{-1}\left[\frac{2z+3}{2z+1}\right] = Z^{-1}\left[\frac{2z+1+2}{2z+1}\right] = Z^{-1}\left[1 + \frac{2}{2z+1}\right]$$

$$= Z^{-1}\left[1 + \frac{1}{\left(z+\frac{1}{2}\right)}\right]$$

$$= Z^{-1}[1] + z^{-1}\left[\frac{1}{\left(z+\frac{1}{2}\right)}\right]$$

$$= \boldsymbol{\delta(k) + \left(-\frac{1}{2}\right)^{k-1} \quad , \quad k \geq 0}$$

Example : 4

Obtain $Z^{-1}\left[\dfrac{z+2}{z-1}\right]$, $k \geq 0$

Solution :

$$= Z^{-1}\left[\frac{z-1+3}{z-1}\right]$$

$$= Z^{-1}\left[1+\frac{3}{z-1}\right]$$

$$= Z^{-1}[\,1\,] + 3z^{-1}\left[\frac{1}{z-1}\right]$$

$$= \boldsymbol{\delta\, k + 3\,(1)^{k-1}}$$

2.10.2 Method of Partial Fractions :

1. **Proper Fraction:** The fraction $\left[\dfrac{f(z)}{g(z)}\right]$ is called as proper fraction if degree of f(z) < degree of g(z).

2. **Improper Fraction :** The fraction $\left[\dfrac{f(z)}{g(z)}\right]$ is called as a proper fraction if degree of f(z) > degree of g(z).

3. The proper fraction can be resolved into partial fractions as below

$$\frac{f(z)}{(z-a)(z+b)} = \frac{A}{(z-a)} + \frac{B}{(z+b)}$$

where, A & B are constants and can be found.

Illustrative Examples

Example : 1

Find $Z^{-1}\left[\dfrac{z}{(z+2)\,(z-4)}\right]$, $|z| > 4$, $k \geq 0$

Solution :

Let $$F(z) = \frac{z}{(z+2)\,(z-4)}$$

$$\frac{F(z)}{z} = \frac{1}{(z+2)\,(z-4)}$$

Resolve $\dfrac{1}{(z+2)\,(z-4)}$ into partial fractions

$$\frac{1}{(z+2)\,(z-4)} = \frac{A}{(z+2)} + \frac{B}{(z-4)}$$

$$= \frac{A(z-4)+B(z+2)}{(z+2)\,(z-4)}$$

Comparing Numerators, we get

$$1 = A(z-4) + B(z+2)$$

Put $z - 4 = 0 \Rightarrow 1 = 0 + 6B \quad \therefore \boxed{B = \frac{1}{6}}$

Put $z + 2 = 0 \Rightarrow 1 = A(-2-4) \quad \therefore A = -\frac{1}{6}$

$$\therefore \quad \frac{1}{(z+2)(z-4)} = \frac{-\frac{1}{6}}{z+2} + \frac{\frac{1}{6}}{z-4}$$

$$= \frac{1}{6}\left[\frac{1}{z-4} - \frac{1}{z+2}\right]$$

$$\therefore \quad \frac{F(z)}{z} = \frac{1}{6}\left[\frac{1}{z-4} - \frac{1}{z+2}\right]$$

$$\therefore \quad F(z) = \frac{1}{6}\left[\frac{z}{z-4} - \frac{z}{z+2}\right], \ |z| > 4, \ |z| > 2$$

$$\therefore \quad \mathbf{Z^{-1}[F(z)] = \frac{1}{6}\{4^k - (-2)^k\}, \ |z| > 4, \ k \geq 0}$$

Example : 2

Obtain $Z^{-1}\left[\frac{3z^2+2z}{z^2-3z+2}\right]$, $k \geq 0$ **Dec. 2015**

Solution :

$$\text{Let } F(z) = \frac{3z^2 + 2z}{(z-2)(z-1)}$$

$$\therefore \quad \frac{F(z)}{z} = \frac{3z+2}{(z-2)(z-1)}$$

$$\text{Resolve} \quad \frac{3z+2}{(z-2)(z-1)} = \frac{8}{z-2} - \frac{5}{z-1}$$

$$\therefore \quad \frac{F(z)}{z} = \frac{3z+2}{(z-2)(z-1)}$$

$$= \frac{8}{z-2} - \frac{5}{z-1}$$

$$\therefore \quad F(z) = 8\left(\frac{z}{z-2}\right) - 5\left(\frac{z}{z-1}\right), \ |z| > 2, \ |z| > 1$$

$$\therefore \quad Z^{-1}[F(z)] = 8\, z^{-1}\left[\frac{z}{z-2}\right] - 5\, z^{-1}\left[\left(\frac{z}{z-1}\right)\right], \ k \geq 0$$

$$= \mathbf{8\{2^k\} - 5\{1^k\}}, \ k \geq 0$$

Example : 3

Obtain $Z^{-1}\left[\dfrac{13z}{20z^2 + 9z + 1}\right]$

Solution :

$$\text{Let } F(z) = \frac{13z}{20z^2 + 9z + 1}$$

$$\therefore \quad F(z) = \frac{13z}{(5z+1)(4z+1)}$$

$$\therefore \quad \frac{F(z)}{z} = \frac{13}{(5z+1)(4z+1)}$$

Resolve
$$\frac{13}{(4z+1)(5z+1)} = \frac{A}{4z+1} + \frac{B}{5z+1}$$
$$= \frac{A(5z+1) + B(4z+1)}{(4z+1)(5z+1)}$$

Comparing Numerators,

$$13 = A(5z+1) + B(4z+1)$$

Put

$$\left.\begin{aligned} 5z+1 &= 0 \\ z &= -\frac{1}{5} \end{aligned}\right\} \Rightarrow \quad 13 = A(0) + B\left(4\left(-\frac{1}{5}\right) + 1\right) = B\left(1 - \frac{4}{5}\right) = B\left(\frac{1}{5}\right)$$

$$\therefore \quad \boxed{B = 65}$$

Put

$$\left.\begin{aligned} 4z+1 &= 0 \\ z &= -\frac{1}{4} \end{aligned}\right\} \Rightarrow \quad 13 = A\left(-\frac{5}{4} + 1\right) + 0$$

$$= A\left(\frac{-5+4}{4}\right) = A\left(\frac{-1}{4}\right)$$

$$\therefore \quad \boxed{A = -52}$$

$$\therefore \quad \frac{13}{(4z+1)(5z+1)} = -\frac{52}{4z+1} + \frac{65}{5z+1}$$

$$\frac{F(z)}{z} = \frac{13}{\left(z + \frac{1}{5}\right)} - \frac{13}{\left(z + \frac{1}{4}\right)}$$

$$\therefore \quad Z^{-1}[F(z)] = Z^{-1}\left[\frac{13z}{z + \frac{1}{5}}\right] - Z^{-1}\left[\frac{13z}{z + \frac{1}{4}}\right]$$

$$= \mathbf{13}\left[\left(-\frac{\mathbf{1}}{\mathbf{5}}\right)^{\mathbf{k}} - \left(-\frac{\mathbf{1}}{\mathbf{4}}\right)^{\mathbf{k}}\right]$$

2.10.3 Residue Method :

1. Let F (z) be some function of z

2. **Simple pole :** If F (z) = ∞ at z = a then z = a is called as a simple pole of F(z).

3. **Multiple pole :** If a pole is repeated for more than once then the pole is called as a multiple pole.

4. Residue of F (z) at simple pole z = a} $= [(z-a)\,F(z)]_{z=a}$

5. If a pole z = b is repeated n times then

$$\left.\begin{matrix}\text{Residue of F (z) at}\\ \text{repeated pole z = b}\end{matrix}\right\} = \frac{1}{\lfloor n-1}\left[\frac{d^{n-1}}{dz^{n-1}}(z-b)^n F(z)\right]_{z=b}$$

2.10.4 Residue Theorem :

Let F (z) be a function of z, then Z^{-1} [f (z)] = sum of all residues at all poles of $[z^{k-1} F(z)]$

2.10.5 Working Procedure of Residue Theorem :

(i) Let F (z) be a given function of z.

(ii) Find all simple and multiple poles of F (z).

(iii) Construct a new function $[z^{k-1} F(z)]$.

(iv) Find all Residues of $[z^{k-1} F(z)]$ say $R_1, R_2, R_3, \ldots$

(v) $Z^{-1}[F(z)] = [R_1 + R_2 + R_3 + +]$.

Illustrative Examples

Example : 1

Obtain $Z^{-1}\left[\frac{2z}{(z-1)(z-2)}\right]$, $k \geq 0$

Solution :

Let $$F(z) = \left[\frac{2z}{(z-1)(z-2)}\right]$$

∴ F(z) have simple poles at z = 1, 2

∴ Let $$z^{k-1} F(z) = z^{k-1}\left[\frac{2z}{(z-1)(z-2)}\right] = \frac{2z^k}{(z-1)(z-2)}$$

$$R_1 = [R(z=1)] = \left((z-1)\left[\frac{2z^k}{(z-1)(z-2)}\right]\right)_{z=1}$$

$$= \left(\frac{2z^k}{z-2}\right)_{z=1}$$

$$= \frac{2(1)^k}{1-2} = -2(1)^k$$

$$R_2 = [R\,(z = 2)] = \left((z-2)\left[\frac{2\,z^k}{(z-1)(z-2)}\right]\right)_{z=2}$$

$$= \left(\frac{2\,z^k}{z-1}\right)_{z=2} = \frac{2\,(2)^k}{2-1} = 2\,(2)^k$$

$$\therefore \quad Z^{-1}\left[\frac{2\,z}{(z-1)(z-2)}\right] = R_1 + R_2 = -2(1)^k + 2\,(2^k)$$

$$= \mathbf{2\,[\,2\,]^k + [\,-1\,]^k = 2[2^k - 1]}$$

Example : 2

Obtain $Z^{-1}\left[\dfrac{36z}{12z^2 - 7z + 1}\right]$, $k \geq 0$

Solution :

$$\text{Let } F\,(z) = \frac{36z}{12z^2 - 7z + 1} = \frac{36z}{(4z-1)\,(3z-1)}$$

$$= \frac{3z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{3}\right)}$$

$\therefore$ F (z) have two simple poles at $z = \dfrac{1}{4}$, $z = \dfrac{1}{3}$

$$\therefore \quad z^{k-1}\,F\,(z) = \frac{3z^k}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{3}\right)}$$

$$\therefore \quad R_1 = \left[R\left(z = \frac{1}{4}\right)\right] = \left[\left(z-\frac{1}{4}\right)\frac{3z^k}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{3}\right)}\right]_{z=\frac{1}{4}}$$

$$= \frac{3\left(\frac{1}{4}\right)^k}{\left(-\frac{1}{12}\right)} = -36\left(\frac{1}{4}\right)^k$$

$$R_2 = \left[R\left(z = \frac{1}{3}\right)\right] = \frac{3\left(\frac{1}{3}\right)^k}{\left(\frac{1}{12}\right)} = 36\left(\frac{1}{3}\right)^k$$

$$\therefore \quad \mathbf{Z^{-1}\left[\frac{36z}{12z^2 - 7z + 1}\right] = 36\left[\left(\frac{1}{3}\right)^k - \left(\frac{1}{4}\right)^k\right]}$$

Example : 3

Obtain $Z^{-1}\left[\dfrac{z^2}{z^2 + 1}\right]$ **May. 2010**

Solution :

$$\text{Let} \quad F(z) = \frac{z^2}{z^2 + 1} = \frac{z^2}{(z+i)\,(z-i)}$$

$\therefore$ F(z) have two simple poles at z= i , –i

$$\therefore \quad z^{k-1} F(z) = \frac{z^{k-1} z^2}{(z^2+1)} = \frac{z^{k+1}}{(z+i)(z-i)}$$

$$\therefore \quad R_1 = [\text{ R at } z = i\ ] = \left[(z-i) \frac{z^{k+1}}{(z+i)(z-i)} \right]_{z=i} = \frac{(i)^{k+1}}{i+i} = \frac{1}{2i} (i)^{k+1}$$

$$\therefore \quad R_2 = [\text{ R at } z = -i\ ] = \left[(z+i) \frac{z^{k+1}}{(z+i)(z-i)} \right]_{z=-i} = -\frac{1}{2i} (-i)^{k+1}$$

$$\therefore \quad Z^{-1}\left[\frac{z^2}{z+1}\right] = \frac{1}{2i} (i)^{k+1} - \frac{1}{2i} (-i)^{k+1}$$

$$= \frac{1}{2i} (i)^k (i)^1 - \frac{1}{2i} (-i)^k (-i)^1$$

$$= \mathbf{\frac{1}{2}[(i)^k + (-i)^k]}$$

Example : 4

Obtain $Z^{-1}\left[\frac{z}{(z-4)(z+2)^2}\right]$ $k \geq 0$,

Solution : Let $F(z) = \frac{z}{(z-4)(z+2)^2}$

$\therefore$ F (z) have a simple pole at z = 4 & multiple pole at z = –2 (repeated twice)

$$\therefore \quad z^{k-1} F(z) = \frac{z^k}{(z-4)(z+2)^2}$$

$$\therefore \quad R_1 = [R\ (z = 4)] = \left[(z-4) \frac{z^k}{(z-4)(z+2)^2} \right]_{z=4}$$

$$= \left[\frac{z^k}{(z+2)^2} \right]_{z=4}$$

$$= \frac{4^k}{36}$$

$$\therefore \quad R_2 = [R\ (z = -2)] = [\text{ R at a pole repeated 2 times}]$$

$$= \frac{1}{\lfloor 2-1} \left[\frac{d^{2-1}}{dz^{2-1}} \left((z+2)^2 \frac{z^k}{(z-4)(z+2)^2} \right) \right]_{z=-2}$$

$$= \frac{1}{\lfloor 1} \left[\frac{d}{dz} \left(\frac{z^k}{z-4} \right) \right]_{z=-2}$$

$$= \left[\frac{(z-4)\, k\, z^{k-1} - z^k (1)}{(z-4)^2} \right]_{z=-2}$$

$$= \frac{(-2-4)\, k\, (-2)^{k-1} - (-2)^k}{36}$$

$$R_2 = \frac{-6k\,(-2)^{k-1} - (-2)^k}{36}$$

$$Z^{-1}\left[\frac{z}{(z-4)(z+2)^2}\right] = R_1 + R_2$$

$$= \frac{4^k}{36} + \frac{-6k(-2)^{k-1} - (-2)^k}{36}$$

$$= \mathbf{\frac{1}{36}\left[4^k - 6k(-2)^{k-1} - (-2)^k\right]}$$

Example : 5

Obtain $Z^{-1}\left[\frac{1}{(z-6)(z-7)}\right]$

Solution :

$$\text{Let } F(z) = \frac{1}{(z-6)(z-7)}$$

$\therefore$ F (z) have two simple poles at $z = 6, z = 7$

$$\therefore \quad z^{k-1} F(z) = \frac{z^{k-1}}{(z-6)(z-7)}$$

$$\therefore \quad R_1 = [R(z=6)] = \left[(z-6)\frac{z^{k-1}}{(z-6)(z-7)}\right]_{z=6}$$

$$= \left[\frac{6^{k-1}}{-1}\right]$$

$$= -6^{k-1}$$

$$\therefore \quad R_2 = [R(z=7)] = \left[(z-7)\frac{z^{k-1}}{(z-6)(z-7)}\right]_{z=7}$$

$$= \left[\frac{7^{k-1}}{1}\right]$$

$$= 7^{k-1}$$

$$\therefore \quad Z^{-1}\left[\frac{1}{(z-6)(z-7)}\right] = R_1 + R_2 = 7^{k-1} - 6^{k-1}$$

Example : 6

Obtain $Z^{-1}\left[\frac{1}{(z-3)(z-2)}\right]$, $|z| > 3$ **May 2007, Dec. 2017**

Solution :

$$\text{Let} \quad F(z) = \frac{1}{(z-3)(z-2)}$$

$$\therefore \quad z^{k-1} F(z) = \frac{z^{k-1}}{(z-2)(z-3)}$$

$\therefore$ F(z) have two simple poles at $z = 2, z = 3$

$$\therefore \quad z^{k-1} F(z) = \frac{z^{k-1}}{(z-2)(z-3)}$$

$$\therefore \quad R_1 = [R(z=2)] = \left(\frac{z^{k-1}}{z-3}\right)_{z=2} = \left(\frac{2^{k-1}}{2-3}\right) = -2^{k-1}$$

$$\therefore \quad R_2 = [R(z=3)] = \left(\frac{z^{k-1}}{z-2}\right)_{z=3}$$

$$= \left[\frac{3^{k-1}}{3-2}\right] = 3^{k-1}$$

$$\therefore \quad \mathbf{Z^{-1}\left[\frac{1}{(z-3)(z-2)}\right] = R_1 + R_2 = 3^{k-1} - 2^{k-1}}$$

Example : 7

Obtain $Z^{-1}\left[\dfrac{z^2}{z^2+4}\right]$

Solution :

$$z^2 + 4 = z^2 - (-4) = z^2 - i^2 4 = z^2 - (2i)^2 = (z+2i)(z-2i)$$

$$\therefore \quad \text{Let} \quad F(z) = \frac{z^2}{z^2+4} = \frac{z^2}{(z+2i)(z-2i)}$$

$\therefore$ F(z) have two simple poles at $z = 2i$, $z = -2i$

$$\therefore \quad z^{k-1} F(z) = \frac{z^{k-1} z^2}{(z+2i)(z-2i)} = \frac{z^{k+1}}{(z+2i)(z-2i)}$$

$$\therefore \quad R_1 = [R(z=2i)] = \left[(z-2i)\frac{z^{k+1}}{(z+2i)(z-2i)}\right]_{z=2i}$$

$$= \left[\frac{z^{k+1}}{(z+2i)}\right]_{z=2i}$$

$$= \left[\frac{(2i)^{k+1}}{2i+2i}\right]$$

$$= \frac{1}{4i}(2i)^{k+1}$$

$$R_2 = [R(z=-2i)] = \left[(z+2i)\frac{z^{k+1}}{(z-2i)(z+2i)}\right]_{z=-2i}$$

$$= \left[\frac{z^{k+1}}{(z-2i)}\right]_{z=-2i}$$

$$= \frac{(-2i)^{k+1}}{-2i-2i}$$

$$= -\frac{1}{4i}(-2i)^{k+1}$$

$$\therefore \quad Z^{-1}\left[\frac{z^2}{z^2+4}\right] = R_1 + R_2 = \left[\frac{1}{4i}(2i)^{k+1} - \frac{1}{4i}(-2i)^{k+1}\right]$$

$$= \mathbf{\frac{1}{4i}\left[(2i)^{k+1} - (-2i)^{k+1}\right]}$$

Example : 8

Obtain $Z^{-1}\left[\dfrac{z}{z^3 - z^2 + z - 1}\right]$

Solution :

$$Z^{-1}\left[\frac{z}{z^3 - z^2 + z - 1}\right] = Z^{-1}\left[\frac{z}{z^2(z-1) + 1(z-1)}\right]$$

$$= Z^{-1}\left[\frac{z}{(z-1)(z^2+1)}\right]$$

$$= Z^{-1}\left[\frac{z}{(z-1)(z+i)(z-i)}\right]$$

Let $F(z) = \dfrac{z}{(z-1)(z+i)(z-i)}$

$\therefore$ F(z) have simple poles at $z = 1, i, -i$

$$\therefore \quad z^{k-1} F(z) = \frac{z^k}{(z-1)(z+i)(z-i)}$$

$$\therefore \quad R_1 = [R(z=1)] = \left[(z-1)\frac{z^k}{(z-1)(z+i)(z-i)}\right]_{z=1}$$

$$= \frac{(1)^k}{(1+i)(1-i)} = \frac{(1)^k}{1-i^2} = \frac{(1)^k}{2}$$

$$\therefore \quad R_2 = [R(z=i)] = \left[(z-i)\frac{z^k}{(z-1)(z+i)(z-i)}\right]_{z=i}$$

$$= \left[\frac{z^k}{(z-1)(z+i)}\right]_{z=i} = \frac{(i)^k}{(i-1)(i+i)}$$

$$= \frac{(i)^k}{2i(i-1)} = \frac{(i)^k}{2i^2 - 2i} = \frac{i^k}{-2i-2}$$

$$= \frac{i^k}{-2(i+1)}$$

$$\therefore \quad R_3 = [R(z=-i)] = \left[(z+i)\frac{z^k}{(z-1)(z+i)(z-i)}\right]_{z=-i}$$

$$= \frac{(-i)^k}{(-i-1)(-i-i)} = \frac{(-i)^k}{-2i(-i-1)}$$

$$= \frac{(-i)^k}{2i^2 + 2i} = \frac{(-i)^k}{-2+2i} = \frac{(-i)^k}{2(i-1)}$$

$$\therefore \quad Z^{-1}\left[\frac{1}{(z^3 - z^2 + z - 1)}\right] = R_1 + R_2 + R_3$$

$$= \frac{(1^k)}{2} + \frac{i^k}{-2(i+1)} + \frac{(-i)^k}{2(1-i)}$$

$$= \mathbf{\frac{1}{2} + \frac{i^k}{-2(i+1)} + \frac{(-i)^k}{2(1-i)}}$$

Example : 9

Obtain $Z^{-1}\left[\left(\frac{z}{z-1}\right)^3\right]$

Solution :

Let, $$F(z) = \left(\frac{z}{z-1}\right)^3 = \frac{z^3}{(z-1)^3}$$

$\therefore$ F (z) have a triple pole at z = 1

$$\therefore \quad z^{k-1} F(z) = z^{k-1}\left(\frac{z^3}{(z-1)^3}\right) = \frac{z^{k+2}}{(z-1)^3}$$

$\therefore$ R = [Residue at z = 1] (multiple pole repeated 3 times)

$$= \frac{1}{\lfloor 3-1}\left[\frac{d^{3-1}}{dz^{3-1}}\left((z-1)^3 \frac{z^{k+2}}{(z-1)^3}\right)\right]_{z=1}$$

$$= \frac{1}{2!}\left[\frac{d^2}{dz^2}\left(z^{(k+2)}\right)\right]_{z=1}$$

$$= \frac{1}{2}\left[\frac{d}{dz}(k+2)\, z^{k+1}\right]_{z=1}$$

$$= \frac{1}{2}\left[(k+2)(k+1)\, z^k\right]_{z=1}$$

$$= \frac{1}{2}(k+2)(k+1)$$

$$\therefore \quad \mathbf{Z^{-1}\left[\left(\frac{z}{z-1}\right)^3\right] = \frac{(k+2)(k+1)}{2}}$$

Self-Assessment Exercises 2.3

Ex.1. $Z^{-1}\left[\frac{z^2}{z^2+1}\right]$ **Ans. :** $\cos\left(\frac{k\pi}{2}\right)$, $k \geq 0$

Ex.2. $Z^{-1}\left[\frac{10z}{(z-1)(z-2)}\right]$ **Ans. :** $10(2^k - 1)$, $k \geq 0$

Ex.3. $Z^{-1}\left[\frac{4z^3}{(z-1)(2z-1)^2}\right]$ **Ans. :** $4 - (k+3)\left(\frac{1}{2}\right)^k$, $k \geq 0$

Ex.4. $Z^{-1}\left[\frac{1}{(z-2)(z-3)}\right]$ **Ans. :** $3^{k-1} - 2^{k-1}$, $k \geq 1$, $|z| > 3$

Ex.5. $Z^{-1}\left[\frac{z^3}{\left(z-\frac{1}{4}\right)^2(z-1)}\right]$ **Ans. :** $\frac{16}{9} - \frac{4}{9}\left(\frac{1}{4}\right)^k - \frac{1}{3}(k+1)\left(\frac{1}{4}\right)^k$ **[May 2012]**

Ex.6. $Z^{-1}\left[\dfrac{z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}\right]$ **Ans. :** $20\left[\left(\frac{1}{4}\right)^k - \left(\frac{1}{5}\right)^k\right]$

Ex.7. $Z^{-1}\left[\dfrac{z}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{2}\right)}\right]$ **Ans. :** $\left(\frac{1}{2}\right)^k + \left(-\frac{1}{2}\right)^k$

Ex.8. $Z^{-1}\left[\dfrac{36z}{(4z-1)\,(3z-1)}\right]$ **Ans. :** $36\left[\left(\frac{1}{3}\right)^k - \left(\frac{1}{4}\right)^k\right]$

Ex.9. $Z^{-1}\left[\dfrac{z^3}{\left(z-\frac{1}{2}\right)(z-2)\,(z+2)}\right]$ **Ans. :** $\frac{2}{3}(2^k) + \frac{2}{5}(-2)^k - \frac{1}{15}\left(\frac{1}{2}\right)^k$

Ex.10. $Z^{-1}\left[\dfrac{z^3}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)^2}\right]$ **Ans. :** $4\left(\frac{1}{3}\right)^k - 3\left(\frac{1}{2}\right)^k + 3k\left(\frac{1}{2}\right)^k$

Ex.11. $Z^{-1}\left[\dfrac{3z}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}\right]$ **Ans. :** $18\left[\left(\frac{1}{2}\right)^k - \left(\frac{1}{3}\right)^k\right]$

Ex.12. $Z^{-1}\left[\dfrac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}\right]$ **Ans. :** $\frac{1}{6}\left[\left(\frac{1}{2}\right)^{k+1} - \left(\frac{1}{3}\right)^{k+1}\right]$

Ex.13. $Z^{-1}\left[\dfrac{1}{(z-3)\,(z-4)}\right]$ **Ans. :** $4^{k-1} - 3^{k-1}$

Ex.14. $Z^{-1}\left[\dfrac{1}{(z-4)^3}\right]$ **Ans. :** $\frac{1}{2}(k-1)\,(k-2)\,4^{k-3}$

Ex.15. $Z^{-1}\left[\dfrac{z+2}{(z-1)^2}\right]$ **Ans. :** $(3k-2)$

Ex.16. $Z^{-1}\left[\dfrac{1}{(z-4)\,(z-5)}\right]$, $4 < |z| < 5$ **Ans. :** $\begin{cases} -5^{k-1}, & k \le 0 \\ -4^{k-1}, & k \ge 0 \end{cases}$

Ex.17. $Z^{-1}\left[\dfrac{1}{(z-3)\,(z-4)}\right]$, $|z| < 4$ **Ans. :** $3^{k-1} - 4^{k-1}$, $k < 0$

Ex.18. $Z^{-1}\left[\dfrac{1}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}\right]$, $|z| < \frac{1}{5}$ **Ans. :** $100\left(\frac{1}{5}\right)^k - 80\left(\frac{1}{4}\right)^k$

Ex.19. $Z^{-1}\left[\dfrac{z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}\right], \frac{1}{5} < |z| < \frac{1}{4}$ **Ans. :** $\left\{\begin{matrix} -20\left(\frac{1}{4}\right)^k \\ -20\left(\frac{1}{5}\right)^k \end{matrix}\right\} \begin{matrix}(k < 0)\\(k \geq 0)\end{matrix}$

Ex.20. $Z^{-1}\left[\dfrac{2z^2+3z}{(z+2)\,(z-4)}\right]$ **Ans. :** $\frac{1}{6}(-2)^k + \frac{11}{6}\,4^k$

Ex.21. $Z^{-1}\left[\dfrac{z^3-20z}{(z-3)^2\,(z-4)}\right]$ **Ans. :** $\frac{1}{2}\,[\,2^k + 2k^2\,2^k\,] - 4^k$

Ex.22. $Z^{-1}\left[\dfrac{z}{z^2+11z+24}\right]$ **Ans. :** $\frac{1}{5}\,[\,(-3)^k - (-8)^k\,]$

Ex.23. $Z^{-1}\left[\dfrac{Z}{(z+3)^2\,(z-2)}\right]$ **Ans. :** $-\frac{1}{25}\,(-3)^k - \frac{k}{5}(-3)^k + \frac{1}{25}2^k$

Ex.24. $Z^{-1}\left[\dfrac{z^3-20z}{(z-3)^2(z-4)}\right]$ **Ans. :** $\frac{1}{2}\;2^k + k^2\,2^k - 4^k$

Ex.25. $Z^{-1}\left[\dfrac{2z}{(z-1)(z^2+1)}\right]$ **Ans. :** $1 - \frac{(i)^k}{i+1} - \frac{(-i)^k}{1-i}$

Ex.26. $Z^{-1}\left[\dfrac{z^2}{(z-1)\,(z-2)}\right]$ **Ans. :** $\frac{1}{2}\,[\,3^{k+1} - 1\,]$

Ex.27. $Z^{-1}\left[\dfrac{5z}{(2-z)\,(3z-1)}\right]$ **Ans. :** $\left(\frac{1}{3}\right)^k - 2^k$

Ex.28. $Z^{-1}\left[\dfrac{3z^2-18z+26}{(z-2)\,(z-3)\,(z-4)}\right]$ **Ans. :** $2^{k-1} + 3^{k-1} + 4^{k-1}\;,\; k > 0$

Ex.29. $Z^{-1}\left[\dfrac{3z^2+2}{(5z-1)\,(5z+2)}\right]$ **Ans. :** $\frac{13}{75}\left(\frac{1}{5}\right)^k + \frac{4}{5}\left(-\frac{2}{5}\right)^k$

Ex.30. $Z^{-1}\left[\dfrac{z}{(z-1)^2}\right]$ **Ans. :** k

Ex.31. $Z^{-1}\left[\dfrac{z^2+z}{(z-1)\,(z^2+1)}\right]$ **Ans. :** $1 + \frac{1}{2}\,[(i)^{k-2} + (-i)^{k-2}\,]$

Ex.32. $Z^{-1}\left[\dfrac{z+3}{(z+1)\,(z-2)}\right]$ **Ans. :** $2\,(-1)^{k-1} - (-2)^{k-1}$

2.11 Difference Equations :

Let {f (k) } be a sequence. The equation in terms of f (k), f (k+1), f (k+2) .. is called as difference equation.

The performance of discrete systems is expressed in terms of a difference equation. Z Transforms and Inverse Z Transforms are useful for solving difference equations.

- **Shifting theorem :**

1. $Z[f(k-n)] = z^{-n} z[f(k)]$
2. $Z[f(k+1)] = z F(z) - z f(0)$
3. $Z[f(k+2)] = z^2 F(z) - z^2 f(0) - z f(1)$
4. $Z[f(k+3)] = z^3 F(z) - z^3 f(0) - z^2 f(1) - z f(2)$

Illustrative Examples

Example : 1

Solve the difference equation

$$6 f(k+2) - f(k+1) - f(k) = 0 \qquad \ldots(1)$$

where, $f(0) = 0, f(1) = 1, k \geq 0$

Solution : Take Z. T. of equation (1)

$$6Z\{ f(k+2) - Z\{f(k+1) - Z\{ f(k) \} = Z\{0\}$$

$$6[z^2 F(z) - z^2 f(0) - z f(1)] - [z F(z) - z f(0)] - F(z) = 0$$

$$(6z^2 - z - 1) F(z) = 6z$$

$$F(z) = \frac{6z}{6z^2 - z - 1} \qquad \ldots(2)$$

Use of Inverse z Transform :

$$Z^{-1}[F(z)] = Z^{-1}\left[\frac{6z}{6z^2 - z - 1}\right]$$

$$\{f(k)\} = Z^{-1}\left[\frac{6z}{(3z+1)(2z-1)}\right] = Z^{-1}\left[\frac{z}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}\right]$$

$$= Z^{-1}\left[z\left(\frac{\frac{6}{5}}{z-\frac{1}{2}} - \frac{\frac{6}{5}}{z+\frac{1}{3}}\right)\right] \quad \text{Use method of partial fraction}$$

$$= \frac{6}{5} Z^{-1}\left[\frac{z}{z-\frac{1}{2}} - \frac{z}{z+\frac{1}{3}}\right]$$

$$\mathbf{f(k)} = \frac{\mathbf{6}}{\mathbf{5}}\left[\left(\frac{\mathbf{1}}{\mathbf{2}}\right)^{\mathbf{k}} - \left(-\frac{\mathbf{1}}{\mathbf{3}}\right)^{\mathbf{k}}\right]$$

Example : 2

Solve the difference equation **Dec. 2010, May 2016**

$$y\,(k+1)+\frac{1}{2}\,y\,(k) = \left(\frac{1}{2}\right)^k \qquad \ldots(1)$$

where, $y\,(0) = 0,\ k \geq 0$

Solution :

Take Z.T. of both sides of equation (1)

$$Z\,\{\,y\,(k+1)\,\}+\frac{1}{2}Z\,\{\,y\,(k)\,\} = Z\left\{\left(\frac{1}{2}\right)^k\right\}$$

$$zY\,(z)\ -zy\,(0)+\frac{1}{2}\,Y(z) = \frac{z}{z-\frac{1}{2}}$$

$$\therefore \qquad \left(z+\frac{1}{2}\right)Y\,(z) = \frac{z}{\left(z-\frac{1}{2}\right)} \qquad \ldots(2)$$

Take Z^{-1} on both sides of equation (2)

$$Z^{-1}\,[y(z)] = Z^{-1}\left[\frac{z}{\left(z+\frac{1}{2}\right)\left(z-\frac{1}{2}\right)}\right]$$

$$y\,(k) = Z^{-1}\left[\,z\left(\frac{1}{\left(z+\frac{1}{2}\right)\left(z-\frac{1}{2}\right)}\right)\right]$$

$$= Z^{-1}\left[\,z\left(\frac{1}{z-\frac{1}{2}}-\frac{1}{z+\frac{1}{2}}\right)\right] \qquad \text{Use method of Partial fractions}$$

$$= Z^{-1}\left[\frac{z}{\left(z-\frac{1}{2}\right)}-\frac{z}{\left(z+\frac{1}{2}\right)}\right]$$

$$\mathbf{y\,(k)} = \left\{\left(\frac{1}{2}\right)^k\right\}-\left\{\left(-\frac{1}{2}\right)^k\right\}$$

Example : 3

Solve the difference equation

$$6x(k+2)-5x\,(k+1)+x(k) = 0 \qquad \ldots\,(1)$$

$x(0) = 0\ ,\ x(1) = 2,\ k \geq 0$ **Dec. 2006**

Solution :

Take Z.T. of both sides of equation (1)

$$6Z[\,x\,(k+2)]-5Z[x\,(k+1)]+Z\,\{x\,(k) = 0$$

$\therefore \quad 6\,[z^2x\,(z) - z^2x(0) - zx(1)] - 5[zx\,(z) - zx(0)] + x(z) = 0$

$$(6z^2 - 5z + 1)\,x(z) - 12z = 0$$

$$(6z^2 - 5z + 1)\,x(z) = 12z$$

$$\therefore \quad x(z) = \frac{12z}{6z^2 - 5z + 1} \qquad \dots (2)$$

Take Z^{-1} of both sides of equation (2)

$$Z^{-1}\,[x(z)\,] = Z^{-1}\left[\frac{12z}{6z^2 - 5z + 1}\right]$$

$$x(k) = Z^{-1}\left[\frac{12z}{(3z-1)\,(2z-1)}\right]$$

$$= Z^{-1}\left[\frac{12z}{3\left(z - \frac{1}{3}\right)2\left(z - \frac{1}{2}\right)}\right] = 2\,Z^{-1}\left[\frac{z}{\left(z - \frac{1}{3}\right)\left(z - \frac{1}{2}\right)}\right]$$

$$= 2\,Z^{-1}\left[z\left(\frac{1}{\left(z - \frac{1}{3}\right)\left(z - \frac{1}{2}\right)}\right)\right] = 2\,Z^{-1}\left[z\left(\frac{6}{z - \frac{1}{2}} - \frac{6}{z - \frac{1}{3}}\right)\right]$$

$$= 2\,Z^{-1}\left[\frac{6z}{z - \frac{1}{2}} - \frac{6z}{z - \frac{1}{3}}\right] = 12\left[Z^{-1}\left(\frac{z}{z - \frac{1}{2}}\right) - Z^{-1}\left(\frac{z}{z - \frac{1}{3}}\right)\right]$$

$$= \mathbf{12}\left\{\left(\frac{\mathbf{1}}{\mathbf{2}}\right)^{\mathbf{k}} - \left(\frac{\mathbf{1}}{\mathbf{3}}\right)^{\mathbf{k}}\right\}$$

Example : 4

Solve $\quad 12\,f\,(k+2) - 7\,f\,(k+1) + f\,(k) = 0$, where …(1)
$f\,(0) = 0$, $f\,(1) = 3$, $k \geq 0$.

Solution :

Take Z.T. of equation (1)

$$12\,Z\,[f_{(k+2)}\,] - 7\,Z\,[f_{(k+1)}\,] + Z\,[\,f_{(k)}\,] = 0$$

$$12\,[\,z^2\,F(z) - z^2\,f_{(0)} - z\,f_{(1)}\,] - 7\,[\,z\,\;F(z) - z\,f_{(0)}\,] + F(z) = 0$$

$$[\,12z^2 - 7z + 1\,]\,F(z) = 36z$$

$$\therefore \quad F(z) = \frac{36z}{12z^2 - 7z + 1}$$

$$= \frac{36z}{(4z-1)\,(3z-1)} = \frac{36z}{4\left[\,z - \frac{1}{4}\right]3\left[z - \frac{1}{3}\right]}$$

$$= \frac{3z}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{3}\right)} \qquad \dots (2)$$

Taking Z^{-1} of equation (2)

$$Z^{-1}[F(z)] = Z^{-1}\left[\frac{3z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{3}\right)}\right]$$

$$f(k) = 3Z^{-1}\left[z\left(\frac{1}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{3}\right)}\right)\right]$$

$$= 3Z^{-1}\left[z\left(\frac{12}{\left(z-\frac{1}{3}\right)}-\frac{12}{\left(z-\frac{1}{4}\right)}\right)\right]$$

$$= 36\,Z^{-1}\left[\frac{z}{z-\frac{1}{3}}-\frac{z}{z-\frac{1}{4}}\right]$$

$$= 36\left[Z^{-1}\left[\frac{z}{z-\frac{1}{3}}\right]-z^{-1}\left[\frac{z}{z-\frac{1}{4}}\right]\right] = \mathbf{36}\left[\left(\frac{\mathbf{1}}{\mathbf{3}}\right)^{\mathbf{k}}-\left(\frac{\mathbf{1}}{\mathbf{4}}\right)^{\mathbf{k}}\right]$$

Example : 5

Solve the difference equation

$u_{k+2} + 4\,u_{k+1} + 3u_k = 3^k$ … (1)

with $u_0 = 0$, $u_1 = 1$, $k \geq 0$

Solution :

Take Z.T. of equation (1)

$$Z\{u_{k+2}\} + 4Z\{u_{k+1}\} + 3Z\{u_k\} = Z\{3^k\}$$

$$\therefore \quad [z^2 U_{(z)} - z^2 u_{(0)} - z\,u_{(1)}] + 4[z\,U_{(z)} - z\,u_{(0)}] + 3U_{(z)} = \frac{z}{z-3}$$

$$U_{(z)}[z^2 + 4z + 3] - z = \frac{z}{z-3}$$

$$U_{(z)}[(z+1)(z+3)] = z + \frac{z}{z-3}$$

$$U_{(z)} = \frac{z}{(z+1)(z+3)} + \frac{z}{(z+1)(z+3)(z-3)}$$

$$= \frac{z^2 - 3z + z}{(z+1)(z+3)(z-3)}$$

$$U_{(z)} = \frac{z^2 - 2z}{(z+1)(z+3)(z-3)} \quad \ldots(2)$$

Take Z^{-1} of both sides of equation (2)

$$Z^{-1}U_{(z)} = Z^{-1}\left[\frac{z^2 - 2z}{(z+1)(z+3)(z-3)}\right]$$

$$U_{(k)} = Z^{-1}\left[z\left(\frac{z-2}{(z+1)(z+3)(z-3)}\right)\right]$$

$$= Z^{-1}\left[z\left(\frac{A}{(z+1)} + \frac{B}{(z+3)} + \frac{C}{(z-3)}\right)\right]$$

$$= Z^{-1}\left[z\left(\frac{\frac{3}{8}}{z+1} + \frac{\frac{-5}{12}}{z+3} + \frac{\frac{1}{24}}{z-3}\right)\right]$$

(use method of partial fraction)

$$= Z^{-1}\left[\frac{3}{8}\left(\frac{z}{(z+1)}\right) - \frac{5}{12}\left(\frac{z}{(z+3)}\right) + \frac{1}{24}\left(\frac{z}{(z-3)}\right)\right]$$

$$\mathbf{U_{(k)} = \frac{3}{8}\{(-1)^k\} - \frac{5}{12}\{(-3)^k\} + \frac{1}{24}\{(3)^k\} \ , |z|3}$$

Example : 6

Solve the difference equation

$y_{k+2} + 6\, y_{k+1} + 9y_k = 2^k$ ….(1)

where $y_0 = 0$, $y_1 = 0$, $k \geq 0$

Solution :

Taking Z.T. of equation (1)

$$Z\{y_{k+2}\} + 6Z\{y_{k+1}\} + Z\{9y_k\} = z\{2^k\}$$

$$[z^2\, Y(z) - z^2\, Y(0) - z\, Y(1)] + 6\,[zY(z) - zY(0)] + 9y(z) = \frac{z}{z-2}$$

$$\therefore \quad [z^2 + 6z + 9]\, Y(z) = \frac{z}{z-2}$$

$$Y(z) = \frac{z}{(z-2)(z^2+6z+9)}$$

$$Y(z) = \frac{z}{(z-2)(z+3)^2} \quad \ldots(2)$$

Take Z^{-1} on both sides of equation (2)

$$Z^{-1}[\,y(z)] = Z^{-1}\left[\frac{z}{(z-2)(z+3)^2}\right] = Z^{-1}\left[z\left(\frac{1}{(z-2)(z+3)^2}\right)\right]$$

$$= Z^{-1}\left[z\left(\frac{A}{z-2} + \frac{B}{z+3} + \frac{C}{(z+3)^2}\right)\right]$$

$$= Z^{-1}\left[z\left(\frac{\frac{1}{25}}{z-2}\right) - \left(\frac{\frac{1}{25}}{z+3}\right) + \left(\frac{\frac{1}{5}}{(z+3)^2}\right)\right]$$

$$= \frac{1}{25} Z^{-1}\left[\frac{z}{z-2} - \frac{z}{z+3} - \frac{5z}{(z+3)^2}\right]$$

$$\mathbf{y_{(k)} = \frac{1}{25}[(2)^k - (-3)^k + 5k\,(-3)^k]}$$

Self-Assessment Exercises 2.4

Solve the following difference equations :

Ex.1 $y_{k+1} + \frac{y_k}{4} = \left(\frac{1}{4}\right)^k$ $y_0 = 0,\ k \geq 0$ **Ans. :** $y_k = 2\left[\left(\frac{1}{4}\right)^k \left(-\frac{1}{4}\right)^k\right]$

Ex.2 $f_{k+2} - f_{k+1} + f_k = 2^k$ $f_0 = 2, f_1 = 1, k \geq 0$ **Ans. :** $f_k = 1 - 2k + 2^k$

Ex.3 $x_{k+2} - 3x_{k+1} + 2x_k = 0$ $x_0 = 0, x_1 = 1, k \geq 0$ **Ans. :** $x_k = 1 - 2^k$

Ex.4 $6y_{k+2} - y_{k+1} - y_k = 0$ $y_0 = 1, y_1 = 1, k \geq 0$ **Ans.:** $y_k = \frac{8}{5}\left(\frac{1}{2}\right)^k - \frac{3}{5}\left(-\frac{1}{3}\right)^k$

Ex.5 $f_k + 3f_{k-1} - 4f_{k-2} = 0, k \geq 2$ $f_0 = 3, f_1 = -2,$ **Ans. :** $f_k = 2 + (-4)^k$

Ex.6 $y_{k+2} - 4y_{k+1} + 3y_k = 5^k, k \geq 0$ $y_0 = 0, y_1 = 1, k \geq 0$

Ex.7 $f(k+1) - f(k) = 1$ $f(0) = 0, k \geq 0$ **Ans. :** $f(k) = k$

Ex.8 $f(k+1) + \frac{1}{2} f(k) = \left(\frac{1}{2}\right)^k$ **(May 2016)** $f(0) = 0, k \geq 0$ **Ans.** : $\left(\frac{1}{2}\right)^k - \left(-\frac{1}{2}\right)^k$

Ex.9 $f(k+2) + 3f(k+1) + 2f(k) = 0$ $f(0) = 0, f(1) = 1, k \geq 0$ **Ans. :** $(-1)^k - (-2)^k$

(May 2014)

Ex.10 $f(k+2) + 3f(k+1) + 2f(k) = 0$ $f(0) = 0, f(1) = 2, k \geq 0$

Ans. : $2[(-1)^k - (-2)^k]$

Ex.11 $12f(k+2) - 7f(k+1) + f(k) = 0$ $f(0) = 0, f(1) = 3, k \geq 0$ **(Dec. 2016)**

Ans. : $36\left[\left(\frac{1}{3}\right)^k - \left(\frac{1}{4}\right)^k\right]$

Ex.12 $6y(k+2) - 5y(k+1) + y(k) = 0$ $y(0) = 0, y(1) = 3, k \geq 0$

Ans. : $18\left[\left(\frac{1}{2}\right)^k - \left(\frac{1}{3}\right)^k\right]$

Ex.13 $x(k) - 4x(k-2) = \left(\frac{1}{2}\right)^k k \geq 0$ **Ans. :** $-\frac{1}{15}\left(\frac{1}{2}\right)^k + \frac{2}{3}(2)^k + \frac{2^k}{5}(-2)^k$

Ex.14 $y(k+2) - 3y(k+1) + 2y(k)\ U(k), f(0) = 0$ **Ans. :** $2^{k-1} - 1$

Ex.15 $4x(k) + x(k-2) = 4\left(\frac{1}{2}\right)^k \sin\left(\frac{k\pi}{2}\right), k \geq 0$ **Ans. :** $(k+1)\left(\frac{1}{2}\right)^k \sin\left(\frac{k\pi}{2}\right), |z| > \frac{1}{2}$

Descriptive Questions

Q. 1 Find the Fourier sine and cosine transform of $x\,e^{-ax}$

Q. 2 Solve the integral equation $\int_0^\infty f(x)\sin\lambda u\,dx = e^{-\lambda}$

Q. 3 Find the Fourier transform of

$$F(x) = \begin{cases} 0 & 0 \le x < a \\ x & , a \le x \le b \\ 0 & x > b \end{cases}$$

Q. 4 Find the Fourier sine and cosine transform of

$$f(x) = \begin{cases} x_2 & 0 \le x < 1 \\ 0 & , \quad x > 0 \end{cases}$$

Q. 5 Obtain the Z Transform : $\cosh\left[\frac{k\pi}{2} + \alpha\right]$, $k \ge 0$

Q. 6 Find the Z transforms of the following for $k \ge 0$

1. $Z\{\ k^2 a^k\ \}$
2. $Z\{\ (1 + k)\,8^k\ \}$
3. $Z\{\ k^2 e^{-ak}\ \}$
4. $Z^{-1}\left[\frac{1}{(z-2)(z-3)}\right]$
5. $Z^{-1}\left[\frac{3z^2 + 2}{(5z-1)(5z+2)}\right]$
6. $f(k+2) + 3f(k+1) + 2f(k) = 0 \quad f(0) = 0, f(1) = 2, k \ge 0$
7. $x(k) - 4x(k-2) = \left(\frac{1}{2}\right)^k \ k \ge 0$

Q. 7 Find the Fourier cosine transform of $e^{-x} + e^{-2x}$, $x > 0$

Q. 8 Using Fourier cosine integral of e^{-mx} $(m > 0)$ prove that

$$\int_0^\infty \frac{m\cos\lambda x\,d\lambda}{m^2 + \lambda^2} = \frac{\pi}{2} e^{-mx}, (m > 0, x > 0)$$

Q. 9 By considering Fourier cosine transform of e^{-mx}, $(m > 0)$ prove that.

$$\int_0^\infty \frac{\cos\lambda x}{x^2+m^2}\,d\lambda = \frac{\pi}{2m} e^{-mx}\ m > 0, x > 0,$$

Q. 10 Find the Fourier sine transform of the following function.

$$f(x) = \begin{cases} x & , 0 \le x \le 1 \\ 2 - x & , 1 < x \le 2 \\ 0 & , x > 2 \end{cases}$$

Q. 11 Using Fourier integral representation show that.

$$\int_0^\infty \frac{\lambda^3 \sin \lambda x}{\lambda^4+4}\, d\lambda = \frac{\pi}{2} e^{-x} \cos x,\ x > 0$$

Q. 12 Obtain Z Transform of $\left\{\left(\frac{1}{3}\right)^{|k|}\right\}$

Q. 13 Obtain Z Transform of $\left\{\left(\frac{2}{3}\right)^{|k|}\right\}$

Q. 14 Obtain $z^{-1}\left[\frac{3z^2+2z}{z^2-3z+2}\right]$, $|z| > 2$, $k \geq 0$

Q. 15 Obtain $Z^{-1}\left[\frac{z^2}{z^2+1}\right]$

Q. 16 Obtain $Z^{-1}\left[\frac{1}{(z-3)(z-2)}\right]$, $|z| > 3$

Q. 17 Solve the difference equation

$$y(k+1) + \frac{1}{2} y(k) = \left(\frac{1}{2}\right)^k \qquad \ldots(1)$$

where, $y(0) = 0$, $k \geq 0$

Q. 18 Solve the difference equation

$$6x(k+2) - 5x(k+1) + x(k) = 0 \qquad \ldots(1)$$

$x(0) = 0$, $x(1) = 2$, $k \geq 0$

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Unit 3

Statistics

Syllabus :

Measures of central tendency, Standard deviation, Coefficient of variation, Moments, Skewness and Kurtosis, Curve fitting : fitting of straight line, parabola and related curves, Correlation and Regression, Reliability of Regression Estimates.

- **Definition :**

Statistics is the science which deals with methods of collecting, classifying, Presenting, comparing numerical data collected to throw light on any sphere of enquiry.

- **Variable (or Variate) :**

A quantity which can vary from one individual to another is called a variable or variate.

e.g. Heights, weights, ages, wages of persons, rain fall records of cities, Income. Quantities which can take any numerical value within a certain range are called **continuous variables.**

e.g. Height, weight, temperature, time, As the child grows, his/her height takes all possible values from 50cm to 100cm. No. of rooms in a house.

Quantities which are incapable of taking all possible values are called **discrete or discontinuous variables.**

[**Discrete:** The variable which can assume only particular values are called as discrete variables. e.g. No. of children in a family, No. of defective in a lot.]

e.g. No. of workers in a factory, No. of defective products, the no. of telephone calls on different dates.

Ungrouped data: The data does not give any useful information, it is rather confusing to mind, these are called raw data or ungrouped data.

Grouped data: If we express the data in ascending or descending order of magnitude, this does not reduce the bulk of the data we condense the data into classes or groups.

Range: Difference between largest and smallest numbers occurring in the data.

Frequency distribution: It is a tabular arrangement by which large mass of raw data is summarized by forming number of groups or categories.

Exclusive and Inclusive class- intervals: Class- intervals of the type$\{x: a \le x < b\} = [a, b]$ are called 'exclusive' since they exclude the upper limit of the class.

The following data are classified on this basis.

Income (Rs.)	50 - 100	100-150	150-200	200-250	250-300
No. of persons	88	70	52	30	23

In this method, the upper limit of one class is the lower limit of the next class.

In this example there are 88 persons whose income is from Rs.50 to Rs.99.99.

A person whose income is Rs.100 is included in the class Rs.100 to Rs.150.

Class- intervals of the type $\{x: a \le x \le b\} = [a, b]$ are called 'Inclusive' since they include the upper limit of the class. The following data are classified on this basis.

Income (Rs.)	50 – 99	100 – 149	150 – 199	200 – 249	250 – 299
No. of persons.	60	38	22	16	7

However, to ensure continuity and to get correct class-limits exclusive method of classification should be adopted. To convert inclusive class- interval into exclusive, we have to make an adjustment.

3.1 Measure of Central Tendency :

A figure which is used to represent a whole series should neither have the lowest value nor the highest in the series, but somewhere between these two limits, possibly in the center, where most of the items of the series cluster, such figures are called Measures of central tendency (or Averages).

- **There are five types of averages in common use:**

1. Arithmetic (Average or) Mean.	2. Median
3. Mode	4. Geometric Mean
5. Harmonic mean	

1. Arithmetic Mean:

a) In case of individual observations: (i.e. where frequency is not given)

i) Direct Method: If the variable 'x' takes the values $x_1, x_2, x_3, \ldots, x_n$ then A.M.

$\bar{x}$ is given by $\bar{x} = \frac{1}{n}(x_1 + x_2 + x_3 + \cdots + x_n) = \frac{1}{n}\sum x$

ii) Short cut method (or shift of origin) : Shifting the origin to an arbitrary point 'A' then A.M. $\overline{x}$ is given by $\overline{x} = A + \frac{1}{n}\sum d$

where deviation d = x-A.

n = no. of observations, A = assumed mean.

b) In the case of discrete series: (i.e. where frequency is given)

i) Direct method: The frequency distribution is

x	x_1	x_2	x_3	…	x_n
f	f_1	f_2	f_3	…	f_n

then $\overline{x} = \frac{f_1 x_1 + f_2 x_2 + \cdots + f_n x_n}{f_1 + f_2 + f_3 + \cdots + f_n} = \frac{1}{N}\sum fx$ where $N = f_1 + f_2 + f_3 + \cdots + f_n$

ii) Short cut method (or shift of origin): Shifting the origin to an arbitrary point 'A' then A.M. $\overline{x}$ is given by $\overline{x} = A + \frac{1}{N}\sum fd$ where deviation d = x-A.

$N = f_1 + f_2 + f_3 + \cdots + f_n$, A = assumed mean.

Note : If the frequencies are given in terms of class- intervals the mid values of class-intervals are considered as 'x'.

iii) In the case of continuous series having equal class- intervals say of width 'h' we use a different formula.

(i.e. shift of origin and change of scale or step deviation method).

Let $u = \frac{x - A}{h}$ then A.M. $\overline{x}$ is given by $\overline{x} = A + \frac{h}{N}\sum fu$ where

$N = f_1 + f_2 + f_3 + \cdots + f_n$, A = assumed mean.

Weighted Arithmetic mean: If the variate values are not of equal importance, we may attach to them 'weight' $w_1, w_2, w_3, \ldots w_n$ as measures of their importance.

The weighted mean $\overline{x}_W$ is defined as

$$\overline{x}_W = \frac{w_1 x_1 + w_2 x_2 + w_3 x_3 + \cdots + w_n x_n}{w_1 + w_2 + w_3 + \cdots + w_n}$$

$$= \frac{\sum wx}{\sum w}.$$

Property: (Mean of composite series) : If $\overline{x}_i$ (i = 1, 2,…k) be the Arithmetic mean of 'k' distributions with respective frequencies n_i (i = 1, 2, 3, …k) then mean $\overline{x}$ of the whole distribution is given by

$$\overline{x} = \frac{n_1 \overline{x}_1 + n_2 \overline{x}_2 + \cdots + n_k \overline{x}_k}{n_1 + n_2 + \cdots + n_k} = \frac{\sum n\overline{x}}{\sum n}$$

2. Median:

i) Median is the measure of central value of the variable when the values are arranged in ascending or descending order of magnitude.

(Median divides the distribution into two equal parts)

e.g. 3, 4, 4, **5**, 6, 7, 8, 3, 4, 4, 5, **7, 9**, 11, 13, 15, 17.

Median = 5, $\frac{7 + 9}{2} = 8$

ii) For an ungrouped frequency distribution if the 'n' values of the variate are arranged in ascending or descending order of magnitude.

a) when n = odd the middle value i.e. $\left(\frac{n + 1}{2}\right)^{th}$ value gives the median.

b) when n = even there are two middle values $\left(\frac{n}{2}\right)^{th}$ and $\left(\frac{n}{2} + 1\right)^{th}$. The arithmetic mean of these two values gives the median.

iii) For a grouped frequency distribution the median is given by the formula:

Median = $L + \frac{h}{f}\left(\frac{N}{2} - C\right)$ where L = lower limit of median class, where median class is the class corresponding to cumulative frequency just greater than $\frac{N}{2}$.

h = the width of median class. f = the frequency of the median class.

C = cumulative frequency of the class preceding the median class. N = $\sum f$.

iv) For discrete frequency distribution: median is obtained by considering cumulative frequencies.

Find $\frac{N + 1}{2}$ where N = $\sum f$ find cumulative frequency just greater than $\frac{N + 1}{2}$ the corresponding value of 'x' is the median.

3. Mode:

i) Mode is the value which occurs most frequently in a set of observations. [The mode or modal value of the distribution is that value of the variate for which frequency is maximum].

e.g. i) 2, 3, 3, 3, 5, 7, 7, 9 mode = 3

ii) 2, 3, 2, 4, 2, 5, 7, 5, 6, 8, 9 mode = 3.

iii) 1, 3, 5, 7, 8, 10 no mode.

iv) 7, 4, 3, 5, 6, 3, 3, 2, 4, 3, 4, 3, 3, 4, 4, 3, 2, 2, 4, 3, 5, 4, 3, 4, 3, 4, 3, 1, 2, 3 mode = 3.

x	1	2	3	4	5	6	7
frequency	1	4	12	9	2	1	1

ii) In case of continuous frequency distribution mode is given by the formula:

Mode = $L + h\dfrac{f_m - f_1}{2f_m - f_1 - f_2}$ Where f_m = frequency of the modal class.

f_1 and f_2 are the frequencies of the classes preceding and succeeding the modal class respectively. L = lower limit, h = length of the interval.

iii) where mode is ill-defined i.e. where the method of grouping also fails, its value can be ascertained by the formula Mode = 3median – 2 mean.

This measure is called the empirical mode.

Mean – Mode = 3[mean - median]

Harmonic Mean: Harmonic mean of a number of observations is the reciprocal of the arithmetic mean of the reciprocals of the given values. Thus the harmonic mean H of 'n' observations $x_1, x_2, x_3, \ldots x_n$ is

$$H = \frac{1}{\frac{1}{n}\sum\frac{1}{x}} = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_n}}$$

if $x_1, x_2, x_3, \ldots x_n$ (none of them being zero) have the frequencies $f_1, f_2, f_3, \ldots f_n$ respectively the harmonic mean is given by

$$H = \frac{1}{\frac{1}{N}\sum\frac{f}{x}} = \frac{N}{\frac{f_1}{x_1} + \frac{f_2}{x_2} + \frac{f_3}{x_3} + \cdots + \frac{f_n}{x_n}}.$$

3.2 Measures of Dispersion :

- **Dispersion:**

The variation or scattering or deviation of the different values of a variable from their average is known as dispersion. Dispersion indicates the extent to which the values vary among themselves.

Distribution A	75	85	95	105	115	125
Distribution B	10	20	30	70	180	290

Arithmetic Mean of each distribution is $\frac{600}{6}$ = 100.

In distribution A, the values of the variate differ from 100 but the difference is small, In distribution B the values(or items) are widely scattered and lie far from the mean. Although the A.M. is the same, yet the two distribution widely differ from each other in their formation.

The following are the Measures of Dispersion:

i) Range

ii) Quartile deviation or semi inter quartile deviation

iii) Average (or Mean) deviation

iv) Standard deviation

i) Range : Range is the difference between the extreme values of the variate.

Range = L – S where L = largest S = smallest.

Co-efficient of Range = $\frac{L - S}{L + S}$.

ii) Average deviation or Mean deviation: if $x_1, x_2, x_3, \ldots x_n$ occurs $f_1, f_2, f_3, \ldots f_n$ times respectively and $N = \sum f$ the mean deviation from the average A(usually Mean or Median) is given by

Mean Deviation = $\frac{1}{N} \sum f \mid x - A \mid$ where $\mid x - A \mid$ represents the modulus or the absolute value of the deviation (x - A).

Co-efficient of mean deviation = $\frac{\text{Mean Deviation}}{\text{Average from which it is calculated}}$

iii) Standard deviation (S.D.):

Root Mean Square Deviation (R.M.S.): The r.m.s. deviation denoted by S is defined as the positive square root of the mean of the squares of the deviations from an arbitrary origin A.

thus $S = + \sqrt{\frac{1}{N} \sum f (x - A)^2}$

when the deviations are taken from the mean $\bar{x}$ the r.m.s. deviation is called the standard deviation and is denoted by σ.

Thus S. D. = $\sigma \sqrt{\frac{1}{N} \sum f (x - \bar{x})^2}$

Note: The Square of the S.D. (i.e. σ^2) is called Variance.

Variance = $(S.D.)^2 = \sigma^2$.

- **Short- cut methods for calculating standard deviation:**

i) Direct Method: $\sigma = \sqrt{\frac{1}{N} \sum fx^2 - \left(\frac{1}{N} \Sigma fx\right)^2}$

ii) Change of origin:

Let the origin be shifted to an arbitrary point 'A' and d = x – A then

$$\sigma = \sqrt{\frac{1}{N} \sum fd^2 - \left(\frac{1}{N} \Sigma fd\right)^2}$$

iii) Shift of origin and change of scale (or step deviation method):

Let the origin be shifted to an arbitrary point 'A' and the new scale be $\frac{1}{h}$ times the original scale let $u = \frac{x - A}{h}$ then $\sigma = h \sqrt{\frac{1}{N} \sum fu^2 - \left(\frac{1}{N} \Sigma fu\right)^2}$

- **Relation between measures of dispersion:**

  i) Quartile deviation = $\frac{2}{3}$ S.D.

  ii) Mean deviation = $\frac{4}{5}$ S.D.

- **Coefficient of Variation:**

  The ratio of the S.D. to the mean i.e. $\frac{\sigma}{\bar{x}}$ is as the coefficient of variation. As this is ratio having no dimension. It is used for comparing the variations between two groups with different means.

  C.V. = $\frac{\sigma}{\bar{x}} \times 100$

Illustrative Examples

Example : 1

The mean yearly salary of employees of a company was Rs.20,000 the mean yearly salaries of male and female employees were Rs.20,800 and Rs.16, 800 respectively. Find out the percentage of males and females employed by the company.

Solution :

Let P_1 and P_2 represent percentage of males and females respectively.

then $P_1 + P_2 = 100$. Mean annual salary of all employees $\bar{x}$ = Rs.20,000

Mean annual salary of male employees $\bar{x}_1$ = Rs. 20,800

Mean annual salary of Female employees $\bar{x}_2$ = Rs.16,800

Now

$$\bar{x} = \frac{\bar{x}_1 P_1 + \bar{x}_2 P_2}{100}$$

$$20{,}000 = \frac{20.800\ P_1 + 16.800\ P_2}{100}$$

$$208\ P_1 + 168\ P_2 = 20.000$$

$$26\ P_1 + 21\ P_2 = 2500$$

$$26P_1 + 21(100 - P_1) = 2500$$

$$P_! = 80,\ P_2 = 20$$

Hence % of males and females is 80 and 20 respectively.

Example : 2

Find the mean of the following data:

Marks below	10	20	30	40	50	60	70	80	90	100
No. of students	5	9	17	29	45	60	70	78	83	86

The frequency distribution table can be written as:

Marks	**Mid value (x)**	**f**	$u = \frac{x - A}{h}$	**fu**
0 – 10	5	5	-5	-25
10 – 20	15	4	-4	-16
20 – 30	25	8	-3	-24
30 – 40	35	12	-2	-24
40 – 50	45	16	-1	-16
50 – 60	55	15	00	0
60 – 70	65	10	1	10
70 – 80	75	8	2	16
80 – 90	85	5	3	15
90 – 100	95	3	4	12
		$\Sigma f = 86$		$\Sigma fu = - 52$

Mean $\bar{x} = A + \frac{h}{N} \Sigma fu = 55 + \frac{10}{86} (- 52) = 48.95$ marks.

Example : 3

Obtain the median of the following frequency distribution:

x	1	2	3	4	5	6	7	8	9
f	8	10	11	16	20	25	15	9	6
c.f.	8	18	29	45	65	90	105	114	120

Here N = 120 $\therefore \frac{N + 1}{2} = 60.5$

cumulative frequency just greater than $\frac{N + 1}{2}$ is 65 is 5 hence median is 55.

Example : 4

Find the median from the following data:

Marks	10-20	20-30	30-40	40-50	50-60	60-70	70-80
No. of students (f)	24	18	14	10	42	22	24
c.f.	24	42	56	66	108	130	154

$N = \sum f = 154, \ \frac{N}{2} = \frac{154}{2} = 77$

Median class is 50 – 60 and L = 50 h = 10, f = 42, c = 66

Here Median $= L + \frac{h}{f}\left(\frac{N}{2} - c\right) = 50 + \frac{10}{42}(77 - 66) = 52.62.$

Example : 5

Find the median from the following data:

Marks	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80
No. of students (f)	15	20	25	24	10	33	71	51
c.f.	15	35	60	84	94	127	198	249

$N = \sum f = 249, \frac{N}{2} = \frac{249}{2} = 124.5$

Median class is 50 – 60 and L = 50, h = 10 f = 33, c = 94

Here Median $= L + \frac{h}{f}\left(\frac{N}{2} - c\right)$

$= 50 + \frac{10}{33}(124.5 - 94) = 59.24$ marks.

Example : 6

Find the mode of the following data:

Marks	1- 5	6-10	11-15	16-20	21-25	26-30	31-35	36-40	41-45
No. of candidates (f)	7	10	16	32	24	18	10	5	1

Here the greatest frequency is 32 lies in the class interval 16 – 20. Hence the modal class is 16 – 20. But the actual limits of this class are 15.5 – 20.5

$L = 15.5, f_m = 32, f_1 = 16, f_2 = 24, h = 5$

$$\text{Mode} = L + \frac{f_m - f_1}{2f_m - f_1 - f_2} \times h = 15.5 + \frac{32 - 16}{64 - 16 - 24} \times 5 = 18.83$$

Example : 7

Find the Harmonic Mean of the following data:

Marks x	f	$\frac{1}{x}$	$\frac{f}{x}$
10	2	0.100	0.200
20	3	0.050	0.150.
40	6	0.025	0.150
60	5	0.017	0.085
120	4	0.0008	0.032

$\sum \frac{f}{x} = 0.617$

$$\text{Harmonic Mean} = \frac{N}{\sum \frac{f}{x}} = \frac{20}{0.617} = 32.4$$

Example : 8

Find the Mean deviation from the median of the following frequency Distribution,

Marks	Mid value (x)	f	c.f.	$\lvert x - M_d \rvert$	$f \lvert x - M_d \rvert$
0-10	5	5	5	23	115
10-20	15	8	13	13	104
20-30	25	15	28	3	45
30-40	35	16	44	7	112
40-50	45	6	50	17	102
		$\Sigma f = 50$			$\Sigma f \lvert x - M_d \rvert = 478$

Here

$\frac{N}{2} = \frac{50}{2} = 25$ Median class corresponds to c.f. 28 i.e.

Median class is 20 – 30.

$$\text{Median } M_d = L + \frac{h}{f}\left(\frac{N}{2} - C\right) = 20 + \frac{10}{15}(25 - 13) = 28.$$

$$\text{Mean Deviation from Median} = \frac{1}{N}\sum f \lvert x - M_d \rvert = \frac{478}{50} = 9.56.$$

Example : 9

Find the Mean and Standard Deviation of the following series.

Marks obtained	No. of candidates (f)	Mid-values	$u = \frac{x - 47.5}{5}$	fu	fu^2
15 – 20	2	17.5	– 6	– 12	72
20 – 25	5	22.5	– 5	– 25	125
25 – 30	8	27.5	– 4	– 32	128
30 – 35	11	32.5	– 3	– 33	99
35 – 40	15	37.5	– 2	– 30	60
40 – 45	20	42.5	– 1	– 20	20
45 – 50	20	47.5	0	0	0
50 – 55	17	52.5	1	17	17
55 – 60	16	57.5	2	32	64
60 – 65	13	62.5	3	39	117
65 – 70	11	67.5	4	44	176
70 – 75	5	72.5	5	25	125
	N = 143			5	1003

$$\text{Mean } \bar{x} = A + \frac{h}{N}\sum fu = 47.5 + \frac{5}{143} \times 5 = 47.7$$

$$\text{S. D.} = \sigma_x = h\sigma_u = h\sqrt{\frac{1}{N}\sum fu^2 - \left(\frac{1}{N}\sum fu\right)^2}$$

$$= 5\sqrt{\frac{1003}{143} - \left(\frac{5}{143}\right)^2} = 13.2$$

Example : 10

The following data related to the ages of a group of government employees.

Calculate the Mean and Standard Deviation of the following series.

Age	No. of Employees (f)	Mid-values	$u = \frac{x - 37.5}{5}$	fu	fu^2
20 – 25	170	22.5	-3	– 510	1530

25 – 30	110	27.5	-2	– 220	440
30 – 35	80	32.5	-1	– 80	80
35 – 40	45	37.5	0	0	0
40 – 45	40	42.5	1	40	40
45 – 50	30	47.5	2	60	120
50 – 55	25	52.5	3	75	225
	N = 500			– 635	2435

$$\text{Mean } \bar{x} = A + \frac{h}{N}\sum fu = 37.5 + \frac{5}{500} \times (-635) = 31.15$$

$$\text{S. D.} = \sigma_x = h\sigma_u = h\sqrt{\frac{1}{N}\sum fu^2 - \left(\frac{1}{N}\sum fu\right)^2}$$

$$= 5\sqrt{\frac{2435}{500} - \left(-\frac{635}{500}\right)^2} = 9.0237187$$

Example : 11

Calculate the Mean and Standard Deviation of the following data.

Marks obtained	**No. of students(f)**	**Mid-values**	$u = \frac{x - 25}{10}$	**f u**	fu^2
0 – 10	5	5	– 2	– 10	20
10 – 20	8	15	–1	– 8	8
20 – 30	15	25	0	0	0
30 – 40	16	35	1	16	16
40 – 50	6	45	2	6	24
	N = 50			4	68

$$\text{Mean } \bar{x} = A + \frac{h}{N}\sum fu = 25 + \frac{10}{50} \times 4 = 25.8$$

$$\text{S. D.} = \sigma_x = h\sigma_u = h\sqrt{\frac{1}{N}\sum fu^2 - \left(\frac{1}{N}\sum fu\right)^2}$$

$$= 10\sqrt{\frac{68}{50} - \left(\frac{4}{50}\right)^2}$$

$$= 11.634432$$

Example : 12

Calculate the Mean and Standard Deviation of the following data.

Marks obtained	No. of students(f)	Mid-values	$u = \frac{x - 25}{10}$	fu	fu^2
0 – 10	12	5	– 2	– 24	48
10 – 20	15	15	– 1	– 15	15
20 – 30	40	25	0	0	0
30 – 40	22	35	1	22	22
40 – 50	11	45	2	22	44
	N = 100			5	129

$$\text{Mean } \bar{x} = A + \frac{h}{N}\sum fu = 25 + \frac{10}{100} \times 5 = 25.5$$

$$\text{S. D.} = \sigma_x = h\sigma_u = h\sqrt{\frac{1}{N}\sum fu^2 - \left(\frac{1}{N}\sum fu\right)^2}$$

$$= 10\sqrt{\frac{129}{100} - \left(\frac{5}{100}\right)^2} = 11.346806$$

Example : 13

Calculate the Mean and Standard Deviation of the following data giving the age distribution of 542 members.

Age in years	No. of members(f)	Mid-values	$u = \frac{x - 55}{10}$	fu	fu^2
20 – 30	3	25	– 3	– 9	27
30 – 40	61	35	– 2	– 122	244
40 – 50	132	45	– 1	– 132	132
50 – 60	153	55	0	0	0
60 – 70	140	65	1	140	140
70 – 80	51	75	2	102	204
80 – 90	2	85	3	6	18
	N = 542			– 15	765

$$\text{Mean } \bar{x} = A + \frac{h}{N}\sum fu = 55 + \frac{10}{542} \times (-15) = 54.723247$$

$$\text{S. D.} = \sigma_x = h\sigma_u = h\sqrt{\frac{1}{N}\sum fu^2 - \left(\frac{1}{N}\sum fu\right)^2}$$

$$= 10\sqrt{\frac{765}{542} - \left(\frac{-15}{542}\right)^2} = 11.877176$$

Example : 14

Calculate standard deviation for the following frequency distribution.

Decide whether arithmetic mean is good average.

Wages in Rs. per day	No. of Labourers (f)	Mid-values	$u = \frac{x - 25}{10}$	fu	fu^2
0 – 10	5	5	– 2	– 10	20
10 – 20	9	15	– 1	– 9	9
20 – 30	15	25	0	0	0
30 – 40	12	35	1	12	12
40 – 50	10	45	2	20	40
50 – 60	3	55	3	9	27
	N = 54			22	108

$$\text{Mean } \bar{x} = A + \frac{h}{N}\sum fu = 25 + \frac{10}{54} \times (22) = 29.074$$

$$\text{S. D.} = \sigma_x = h\sigma_u = h\sqrt{\frac{1}{N}\sum fu^2 - \left(\frac{1}{N}\sum fu\right)^2}$$

$$= 10\sqrt{\frac{108}{54} - \left(\frac{22}{54}\right)^2} = 13.54$$

S.D. 13.54 is quite a large value and A.M. 29.074 is not a good average.

(The mean = $\frac{54}{6}$ = 9 it is distorted by the usually high labourers compared to other labourers.) or S.D. is very much deviated from arithmetic mean therefore A.M. is not good average.

Example : 15

Goals scored by two teams A and B in football season are:

No. of goals in a match		0	1	2	3	4
No. of Matches.	A	27	9	8	5	4
	B	17	9	6	5	3

Find out which team is more consistent scorer.

Solution : Frequency distribution table for Team- A and Team-B

Team-A					Team-B				
No. of goals	**Matches f**	**d = x – 2**	**f d**	**fd^2**	**No. of goals**	**Matches f**	**d = x-2**	**fd**	**fd^2**
0	27	– 2	– 54	108	0	17	– 2	– 34	68
1	9	– 1	– 9	9	1	9	– 1	– 9	9
2	8	0	0	0	2	6	0	0	0
3	5	1	5	5	3	5	1	5	5
4	4	2	8	16	4	3	2	6	12
	53		– 50	138		40		– 32	94

For Team-A

$$\bar{x} = A + \frac{1}{N}\sum fd = 2 + \frac{-50}{53} = 1.06$$

$$\sigma_A = \sqrt{\frac{1}{N}\sum fd^2 - \left(\frac{1}{N}\sum fd\right)^2}$$

$$\sigma_A = \sqrt{\frac{138}{53} - \left(-\frac{50}{53}\right)^2} = 1.31$$

$$\text{c. v.} = \frac{\sigma_A}{\bar{x}} \times 100 = \frac{1.31}{1.06} \times 100$$

$$= 123.6\%$$

For Team-A

$$\bar{x} = A + \frac{1}{N}\sum fd = 2 + \frac{-32}{40} = 1.2$$

$$\sigma_B = \sqrt{\frac{1}{N}\sum fd^2 - \left(\frac{1}{N}\sum fd\right)^2}$$

$$\sigma_B = \sqrt{\frac{94}{40} - \left(-\frac{32}{40}\right)^2} = 1.3$$

$$\text{c. v.} = \frac{\sigma_B}{\bar{x}} \times 100 = \frac{1.3}{1.2} \times 100$$

$$= 108.3\%$$

Since $(c.v.)_B < (c.v.)_A$ Therefore Team-B is more consistent.

Example : 16

The following are scores of two batsmen A and B in a series of innings:

A(x)	**B(y)**	**d_A = x - 51**	**d_B = y-51**	**d_A^2**	**d_B^2**
12	47	-39	-4	1521	16
115	12	64	-39	4096	1521
6	16	-45	-35	2025	1225
73	42	22	-9	484	81

7	4	-44	-47	1936	2209
19	51	-32	0	1024	0
119	37	68	-14	4624	196
36	48	-15	-3	225	9
84	13	33	-38	1089	1444
29	0	-22	-51	484	2601
=500	=270	= - 10	= - 240	= 17508	= 9302

Find out which batsman is more consistent.

$$\bar{x} = A + \frac{1}{n}\sum d \quad = \quad 51 - \frac{10}{10} = 50$$

$$\bar{y} = A + \frac{1}{n}\sum d \quad = \quad 51 - \frac{240}{10} = 27$$

$$\sigma_A \quad = \quad \sqrt{\left(\frac{1}{n}\sum d^2\right) - \left(\frac{1}{n}\sum d\right)^2}$$

$$= \quad \sqrt{\frac{1}{10}(17508) - \left(-\frac{10}{10}\right)^2}$$

$$= \quad 41.83,\ \text{C.V.} = 83.6\%$$

$$\sigma_B \quad = \quad \sqrt{\left(\frac{1}{n}\sum d^2\right) - \left(\frac{1}{n}\sum d\right)^2}$$

$$= \quad \sqrt{\frac{1}{10}(9302) - \left(-\frac{240}{10}\right)^2}$$

$$= \quad 18.82,\ \text{C.V.} = 69.6\%$$

A.M. of A > A.M. of B, $(\text{c.v.})_B < (\text{c.v.})_A$, B is more consistent.

3.3 Moments, Skewness, Kurtosis :

- **Moments:**

The r^{th} moment about any point A is denoted by μ'_r and is defined as

$\mu'_r = \frac{1}{N}\sum f(x - A)^r$ where $N = \sum f$. It can be seen that putting r = 0, 1, 2, 3, 4…..etc. we get.

$$\mu'_0 \quad = \quad 1,\ \mu'_1 = \frac{1}{N}\sum f(x - A) = \frac{1}{N}\sum fx - A \text{ or } \mu'_1 = \bar{x} - A$$

$$\mu'_2 \quad = \quad \frac{1}{N}\sum f(x - A)^2 = S^2, \text{ the mean square deviation.}$$

$$\mu_3' = \frac{1}{N}\sum f(x-A)^3 \text{ and so on.}$$

The r^{th} moment about the mean $\bar{x}$ of a distribution is denoted by μ_r and is given by $\mu_r = \frac{1}{N}\sum f(x-\bar{x})^r$ where $\bar{x}$ is arithmetic mean of the distribution.

Putting r = 0, 1, 2, 3, 4…..etc. we get.

$$\mu_0 = \frac{1}{N}\sum f = 1,\ \mu_1 = \frac{1}{N}\sum f(x-\bar{x}) = 0,$$

$$\mu_2 = \frac{1}{N}\sum f(x-\bar{x})^2, \text{ this gives the variance of the distribution.}$$

$$\mu_3 = \frac{1}{N}\sum f(x-\bar{x})^3, \text{ this gives the third moment of the distribution about the mean and so on.}$$

- **Relation between μ_r and μ_r' :**

The value of μ_r' can be calculated with much less calculations as compared to μ_r by selecting appropriate A, we have seen this in calculation of Arithmetic Mean (A.M.) and standard Deviation (S.D.) hence we express μ_r in terms of μ_r'.

By definition,

$$\mu_r' = \frac{1}{N}\sum f(x-\bar{x})^r = \frac{1}{N}\sum f(x - A + A - \bar{x})^r,$$

let d = x – A

$$\therefore \quad \frac{1}{N}\sum fd = \frac{1}{N}\sum fx - \frac{A}{N}\sum f \text{ or } \bar{d} = \bar{x} - A = \mu_1'$$

Thus $\mu_r = \frac{1}{N}\sum f(d-\bar{d})^r$ Expand $(d-\bar{d})^r$ binomially we get

$$\mu_r = \frac{1}{N}\sum f\left(d^r - c_1^r\ d^{r-1}\,\bar{d} + c_2^r\, d^{r-2}(\bar{d})^2 + \cdots + (-1)^r(\bar{d})^r\right)$$

$$\mu_r = \mu_r' - c_1^r\,\mu_1'\,\mu_r' + c_2^r\,\mu_r'\,(\mu_r')^2 + \ldots + (-1)^r(\mu_1')^r$$

where $\frac{1}{N}\sum fd^r = \mu_r,\ \bar{d} = \mu_1'$, we seen that, $\mu_0 = 1,\ \mu_1 = 0$

by putting r = 2, 3, 4,……etc. we get

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$\mu_3 = \mu_3' - 3\mu_2'\,\mu_1' + 2(\mu_1')^3$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4.$$

- **Skewness:**

Skewness signifies departure from symmetry. We study skewness to have an idea about the shape of the curve which we draw with the given data.

If the frequency curve stretches to the right as in fig.(a) i.e. the mean is to the right of the mode then the distribution is right skewed or is said to have positive skewness.

If the curve stretches to left of mode is to the right of the mean then the distribution is said to have negative skewness.

The different measures of skewness are:

i) Pearson's coefficient of skewness . = $\dfrac{3\text{ (mean-median)}}{\text{standard deviation}}$

ii) Coefficient of skewness $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3}$

- **Kurtosis:**

To get complete idea of the distribution in addition to the knowledge of mean dispersion and skewness, we should have an idea of the flatness or Peakedness of the curve. It is measured by the coefficient β_2 is given by $\beta_2 = \dfrac{\mu_4}{\mu_2^2}$.

The curve of fig.(a) which is neither flat nor peaked is called the normal curve or Mesokurtic curve $\gamma = \beta_2 - 3$. Gives the excess of kurtosis. For a normal distribution $\beta_2 = 3$ and the excess is zero. The curve of fig.(c) which is flatter than the normal curve is called Platykurtic and that of fig.(b) which is more peaked is called Leptokurtic. For Platykurtic curves $\beta_2 < 3$. For Leptokurtic curves $\beta_2 > 3$.

(Skewness: Measures the degree of asymmetric or the departure from symmetry.

Kurtosis: Measures the degree of Peakedness of a distribution.)

Illustrative Examples

Example : 1

If the first four moments of a distribution about the value 5, are equal to -4, 22, -117 and 560, determine the central moments (β_1) and (β_2).

Solution :

The first four moments about the arbitrary origin 5 are

$$\mu_1' = -4,\ \mu_2' = 22,\ \mu_3' = -117,\ \mu_4' = 560$$

$$\mu_1' = \frac{1}{N}\sum f\,(x-5) = \frac{1}{N}\sum fx - 5 = \overline{x} - 5$$

$$\therefore \quad \text{Mean} = \overline{x} = \mu_1' + 5 = -4 + 5 = 1$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = 22 - (-4)^2 = 6$$

$$\mu_3 = \mu_3' - 3\mu_2'\,\mu_1' + 2(\mu_1')^3$$

$$= -117 - 3(22)\,(-4) + 2(-4)^3 = 19.$$

$$\mu_4 = \mu_4' - 4\mu_3'\,\mu_1' + 6\mu_2'\,(\mu_1')^2 - 3(\mu_1')^4$$

$$= 560 - 4(-117)(-4) + 6(22)(-4)^2 - 3(-4)^4$$

$$\mu_4 = 32$$

$$\text{Coefficient of skewness} = \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(19)^2}{(6)^3} = 1.6773$$

$$\text{Kurtosis} = \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{32}{(6)^2} = 0.8889.$$

Example : 2

Calculate the first four moments about the mean of the given distribution. Also find skewness β_1 and kurtosis (β_2).

x	2.0	2.5	3.0	3.5	4.0	4.5	5.0
f	4	36	60	90	70	40	10

Solution :

Taking A = 3.5 and $u = \frac{x - A}{h}$

$$= \frac{x-3.5}{0.5}$$

we prepare the table for calculating μ_1', μ_2', μ_3', μ_4', β_1, β_2.

x	**f**	$u = \frac{x - 3.5}{0.5}$	**fu**	**fu^2**	**fu^3**	**fu^4**
2.0	4	– 3	– 12	36	– 108	324
2.5	36	– 2	– 72	144	– 288	576
3.0	60	– 1	– 60	60	-60	60
3.5	90	0	0	0	0	0
4.0	70	1	70	70	70	70
4.5	40	2	80	160	320	640
5.0	10	3	30	90	270	810
	$N = \Sigma f = 310$		$\Sigma fu = 36$	$\Sigma fu^2 = 560$	$\Sigma fu^3 = 204$	$\Sigma fu^4 = 2480$

Here $\mu_r' = \frac{h^r \sum fu^r}{\sum f} = \frac{h^r}{N} \sum fu^r$

Now, $\mu_1' = \frac{h}{N} \sum fu = \frac{0.5}{310} \times 36 = 0.058064$

$$\mu_2' = \frac{h^2}{N}\sum fu^2 = \frac{(0.5)^2}{310} \times 560 = 0.451612$$

$$\mu_3' = \frac{h^3}{N}\sum fu^3 = \frac{(0.5)^3}{310} \times 204 = 0.08225$$

$$\mu_4' = \frac{h^4}{N}\sum fu^4 = \frac{(0.5)^4}{310} \times 2480 = 0.5$$

$$\therefore \quad \mu_1 = 0,$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = 0.451612 - (0.058064)^2 = 0.44824$$

$$\mu_3 = \mu_3' - 3\mu_2'\,\mu_1' + 2(\mu_1')^3 = 0.08225 - 3(0.451612)\,(0.058064) + 2(0.058064)^3$$

$$= 0.0039826$$

$$\mu_4 = \mu_4' - 4\mu_3'\,\mu_1' + 6\mu_2'\,(\mu_1')^2 - 3(\mu_1')^4$$

$$= 0.5 - 4(0.08225)(0.058064) + 6(0.451612)(0.058064)^2 - 3(0.058064)^4$$

$$\mu_4 = 0.48999.$$

Coefficient of skewness $= \beta_1 = \dfrac{\mu_3^2}{\mu_2^3} = \dfrac{(0.0039826)^2}{(0.44824)^3} = 1.76549 \times 10^{-4}$.

Kurtosis $= \beta_2 = \dfrac{\mu_4}{\mu_2^2} = \dfrac{0.48999}{(0.44824)^2} = 2.43874$.

Example : 3

Calculate the first four moments of the following distribution about the mean and hence find skewness β_1 and kurtosis (β_2).

Solution : First we calculate the moments about assumed mean x = 4.

x	f	d = x - 4	f d	fd^2	fd^3	fd^4
0	1	-4	-4	16	-64	256
1	8	-3	-24	72	-216	648
2	28	-2	-56	112	-224	448
3	56	-1	-56	56	-56	56
4	70	0	0	0	0	0
5	56	1	56	56	56	56
6	28	2	56	112	224	448
7	8	3	24	72	216	648
8	1	4	4	16	64	256
	N = 256		0	512	0	2816

We know that, $\mu'_r = \frac{1}{N}\sum f(x-A)^r = \frac{1}{N}\sum fd^r.$

$$\mu'_1 = \frac{1}{N}\sum fd = 0$$

$$\mu'_2 = \frac{1}{N}\sum fd^2 = \frac{512}{256} = 2$$

$$\mu'_3 = \frac{1}{N}\sum fd^3 = 0$$

$$\mu'_4 = \frac{1}{N}\sum fd^4 = \frac{2816}{256} = 11.$$

By using the relation between μ_r and μ'_r we find four moments about the mean are.

$$\mu_r = 0,$$

$$\mu_2 = \mu'_2 - (\mu'_1)^2 = 2 - 0 = 2$$

$$\mu_3 = \mu'_3 - 3\mu'_1\,\mu'_2 + 2(\mu'_1)^3 = 0 - (3)(2)(0) + 2(0) = 0$$

$$\mu_4 = \mu'_4 - 4\mu'_3\,\mu'_1 + 6\mu'_2\,(\mu'_1)^2 - 3(\mu'_1)^4$$

$$= 11 - (4)(0)(0) + (6)(2)(0) - (3)(0) = 0$$

Coefficient of skewness $= \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(0)^2}{(2)^3} = 0$

Kurtosis $= \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{11}{(2)^2} = 2.75$

Example : 4

The first three moments of a distribution about the value 2 of a distribution are 1, 16 and -40. Find the mean, standard deviation and skewness of the distribution.

Solution : The first three moments about the arbitrary origin 2 are

$$\mu'_1 = 1,\ \mu'_2 = 16,\ \mu'_3 = -40.$$

$$\mu'_1 = \frac{1}{N}\sum f(x-2) = \frac{1}{N}\sum fx - 2 = \bar{x} - 2$$

$\therefore$ Mean $= \bar{x} = \mu'_1 + 2 = 1 + 2 = 3$

$$\mu_2 = \mu'_2 - (\mu'_1)^2 = 16 - (1)^2 = 15$$

$\therefore$ S.D. $= \sigma = \sqrt{15} = 3.873$

$$\mu_3 = \mu'_3 - 3\mu'_2\,\mu'_1 + 2(\mu'_1)^3$$

$$= -40 - 3(16)(1) + 2(1)^3 = -86$$

Coefficient of skewness $= \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(-86)^2}{(15)^3} = 2.19.$

Example : 5

The first four moments about the working mean 30.2 of a distribution are 0.255, 6.222, 30.211 and 440.25. Calculate the moments about the mean.

Also evaluate skewness (β_1) and kurtosis (β_2), and comment upon the skewness and kurtosis of the distribution.

Solution : The first four moments about the arbitrary origin 30.2 are

$\mu_1' = 0.255$, $\mu_2' = 6.222$, $\mu_3' = 30.211$, $\mu_4' = 440.25$

$$\mu_1' = \frac{1}{N}\sum f(x - 30.2) = \frac{1}{N}\sum fx - 30.2 = \bar{x} - 30.2$$

$$\therefore \quad \text{Mean} = \bar{x} = \mu_1' + 30.2 = 0.255 + 30.2 = 30.455$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = 6.222 - (0.255)^2 = 6.15698$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$

$$= 30.211 - 3(6.222)(0.255) + 2(0.255)^3 = 25.48433$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$

$$= 440.25 - 4(30.211)(0.255) + 6(6.222)(0.255)^2 - 3(0.255)^4$$

$$\mu_4 = 411.8496.$$

$$\text{Coefficient of skewness} = \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(25.48433)^2}{(6.15698)^3} = 2.78255$$

$$\text{Kurtosis} = \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{411.8496}{(6.15698)^2} = 10.86434.$$

$$\gamma_1 = \sqrt{\beta_1} = 1.6681$$

This indicates considerable skewness of the distribution $\gamma_2 = \beta_2 - 3 = 7.86434$

This shows that the distribution is leptokurtic. (because $\beta_2 > 3$).

3.4 Correlation :

If the change in one variable affects a change in the other variable the variables are said to be correlated and the relation between them is called correlation.

If the two variables deviate in the same direction i.e. If the increase (or decrease) in one results in a corresponding increase (or decrease) in the other, correlation is said to be direct or positive.

e.g. The correlation between income and expenditure is positive.

If the two variable in opposite directions i.e. If the increase (or decrease) in one results in a corresponding decrease (or increase) in the other, correlation is said to be inverse or negative.

e.g. i) the correlation between price and demand is negative

ii) The correlation between volume and the pressure of a perfect gas is negative. If the deviation in one variable is followed by a corresponding proportional deviation in the other is said to be perfect correlation.

3.5 Karl Pearson's Coefficient of Correlation:

(Or Product Moment Correlation Coefficient):

Correlation coefficient between two variables x and y, usually denoted by r(x, y) or r_{xy} is a numerical measure of relationship between them and is defined as:

$$r_{xy} = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum(x_i - \bar{x}) \sum(y_i - \bar{y})}}$$

$$= \frac{\sum XY}{\sqrt{\sum X^2 \ \sum Y^2}}$$

$$r_{xy} = \frac{\text{cov}(x,y)}{\sigma_x \ \sigma_y}.$$

where
$$\text{cov}(x, y) = \frac{1}{n}\sum x\, y - \bar{x}\,\bar{y}$$

$$\sigma_x = \sqrt{\frac{1}{n}\sum x^2 - (\bar{x})^2},$$

$$\sigma_y = \sqrt{\frac{1}{n}\sum y^2 - (\bar{y})^2}$$

$$\bar{x} = \frac{1}{n}\sum x,\ \bar{y} = \frac{1}{n}\sum y.\ n = \text{no. of values (or entries)}$$

- **Correlation formulae:**

i) **Frequency is not given:** put

$u = x - a,\ v = y - b,\ \bar{u} = \frac{1}{n}\sum u,\ \bar{v} = \frac{1}{n}\sum v$

n = no. of values (or entries) $\sigma_u^2 = \frac{1}{n}\sum u^2 - (\bar{u})^2,\ \sigma_v^2 = \frac{1}{n}\sum v^2 - (\bar{v})^2,$

$\text{cov}(u, v) = \frac{1}{n}\sum uv - \bar{u}\,\bar{v}\ ,\ r = \frac{\text{cov}(u,v)}{\sigma_u\ \sigma_v}.$

ii) Frequency is given: sum of the frequency = N = $\sum f$ and

Put $u = x - a,\ v = y - b,\ \bar{u} = \frac{1}{n}\sum fu,\ \bar{v} = \frac{1}{n}\sum fv\ \ \sigma_u^2 = \frac{1}{n}\sum fu^2 - (\bar{u})^2,$

$\sigma_v^2 = \frac{1}{n}\sum fv^2 - (\bar{v})^2,\ \text{cov}(u, v) = \frac{1}{n}\sum fuv - \bar{u}\,\bar{v}\ ,\ r = \frac{\text{cov}(u,v)}{\sigma_u\ \sigma_v}.$

Illustrative Examples

Example : 1

Find the coefficient of correlation for the following data:

x	10	14	18	22	26	30
y	18	12	24	6	30	36

Solution : We construct a table as follows :

x	y	$X = x - \bar{x}$	$Y = y - \bar{y}$	X^2	Y^2	X Y
10	18	-10	-3	100	9	30
14	12	-6	-9	36	81	54
18	24	-2	3	4	9	-6
22	6	2	-15	4	225	-30
16	30	6	9	36	81	54
30	36	10	15	100	225	150
120	126	0	0	280	630	252

where
$$\bar{x} = \frac{1}{n}\sum x = \frac{120}{6} = 20$$

$$\bar{y} = \frac{1}{n}\sum y = \frac{120}{6} = 21$$

$$r = \frac{\sum XY}{\sqrt{\sum X^2 \ \sum Y^2}} = \frac{252}{\sqrt{280 \times 630}} = 0.6$$

Example : 2

Calculate karl-pearson's coefficient of correlation between price and supply of commodity from the following data.

Price (Rs.)(x)	17	18	19	20	21	22	23	24	25	26
Supply(kg) (y)	38	37	38	33	32	33	34	29	26	23

Solution : We construct a table as follows:

x	y	$X = x - \bar{x}$	$Y = y - \bar{y}$	X^2	Y^2	X Y
17	38	-4.5	5.7	20.25	32.49	-25.65
18	37	-3.5	4.7	12.25	22.09	-16.45

19	38	-2.5	5.7	6.25	32.49	-14.25
20	33	-1.5	0.7	2.25	0.49	-1.05
21	32	-0.5	-0.3	0.25	0.09	0.15
22	33	0.5	0.7	0.25	0.49	0.35
23	34	1.5	1.7	2.25	2.89	2.55
24	29	2.5	-3.3	6.25	10.89	-8.25
25	26	3.5	-6.3	12.25	39.69	-22.05
26	23	4.5	-9.3	20.25	86.49	-41.85
215	323	0	0	82.5	228.1	126.5

where $\overline{x} = \frac{1}{n}\sum x = \frac{215}{10} = 21.5$

$$\overline{y} = \frac{1}{n}\sum y = \frac{323}{10} = 32.3$$

$$r = \frac{\sum XY}{\sqrt{\sum X^2 \ \sum Y^2}} = \frac{-126.5}{\sqrt{82.5 \times 228.1}} = 0.9221485$$

Example : 3

Calculate karl pearson's coefficient of correlation from the following data, taking 100 and 50 as assumed averages of x and y respectively.

x	104	111	104	114	118	117	105	108	106	100	104	105
y	57	55	47	45	45	50	64	63	66	62	69	61

Solution :

We construct a table as follows:

x	**y**	**X = x - 100**	**Y = y - 50**	X^2	Y^2	**X Y**
104	57	4	7	16	49	28
111	55	11	5	121	25	55
104	47	4	-3	16	9	-12
114	45	14	-5	196	25	-70
118	45	18	-5	324	25	-90
117	50	17	0	289	0	0
105	64	5	14	25	196	70

108	63	8	13	64	169	104
106	66	6	16	36	256	96
100	62	0	12	0	144	0
104	69	4	19	16	361	76
105	61	5	11	25	121	55
1296	684	96	84	1128	1380	312

Karl-pearson's coefficient of correlation is given by

$$\bar{x} = \frac{1}{n}\sum x = \frac{96}{12} = 8$$

$$\bar{y} = \frac{1}{n}\sum y = \frac{84}{12} = 7.$$

$$\text{cov}(x, y) = \frac{1}{n}\sum x\,y - \bar{x}\,\bar{y}$$

$$= \frac{1}{12}(312) - (8)(7) = 26 - 56 = -30$$

$$\sigma_x = \sqrt{\frac{1}{n}\sum x^2 - (\bar{x})^2}$$

$$= \sqrt{\frac{1}{12}(1128) - (8)^2} = \sqrt{94 - 64} = \sqrt{30} = 5.48$$

$$\sigma_y = \sqrt{\frac{1}{n}\sum y^2 - (\bar{y})^2}$$

$$= \sqrt{\frac{1}{12}(1380) - (7)^2} = \sqrt{115 - 49} = \sqrt{66} = 8.124$$

$$r = \frac{\text{cov}(u,v)}{\sigma_u\,\sigma_v} = -\frac{30}{(5.48)\,(8.124)} = -\frac{30}{44.52} = -0.674.$$

Example : 4

Given $n = 6$, $\sum(x - 18.5) = -3$, $\sum(y - 50) = 20$, $\sum (x - 18.5)^2 = 19$,
$\sum (y - 50)^2 = 850$, $\sum (x - 18.5)(y - 50) = -120$.
Calculate the coefficient of correlation.

Solution : We have $u = x = 18.5$, $v = y - 50$, $\bar{u} = -0.5$, $\bar{v} = 3.33$ then,

$$r = \frac{\frac{1}{n}\Sigma uv = \overline{uv}}{\sqrt{\left(\frac{1}{n}\Sigma u^2 - (\bar{u})^2\right)\left(\frac{1}{n}\Sigma v^2 - (\bar{v})^2\right)}} = -0.9395$$

Example : 5

From a group of 10 students marks obtained by each in papers of mathematics and applied mechanics are given as:

Marks in Mathematics (x)	23	28	42	17	26	35	29	37	16	46
Marks in Applied mechanics (y)	25	22	38	21	27	39	24	32	18	44

Solution : We construct a table as follows:

x	y	$X = x - \bar{x}$	$Y = y - \bar{y}$	X^2	Y^2	XY
23	25	-6.9	-4	47.61	16	27.6
28	22	-1.9	-7	3.61	49	13.3
42	38	12.1	9	146.41	81	108.9
17	21	-12.9	-8	166.41	64	103.2
26	27	-3.9	-2	15.21	4	7.8
35	39	5.1	10	26.01	100	51.0
29	24	-0.9	-5	0.81	25	4.5
37	32	7.1	3	50.41	9	21.3
16	18	-13.9	-11	193.21	121	152.9
46	44	16.1	15	259.21	225	241.5
299	290	00	00	908.9	694	732

Karl-pearson's coefficient of correlation is given by

$$\bar{x} = \frac{1}{n}\sum x = \frac{299}{10} = 29.9,\ \bar{y} = \frac{1}{n}\sum y = \frac{290}{10} = 29.$$

$$r = \frac{\sum XY}{\sqrt{\sum X^2\ \sum Y^2}} = \frac{732}{\sqrt{(908.9) \times (694)}} = \frac{732}{794.2} = 0.9217.$$

Example : 6

Calculate the coefficient of correlation from the following data:

x	5	9	15	19	24	28	32
y	7	9	14	21	23	29	30
f	6	9	13	20	16	11	7

Solution :

We construct a table as follows:

x	y	f	u = x - 19	v = y - 21	fu	fv	fu^2	fv^2	fuv
5	7	6	-14	-14	-84	-84	1176	1176	1176
9	9	9	-10	-12	-90	-108	900	1296	1080
15	14	13	-4	-7	-52	-91	208	637	364
19	21	20	0	0	0	0	0	0	0
24	23	16	5	2	80	32	400	64	160
28	29	11	9	8	99	88	891	704	792
32	30	7	13	9	91	63	1183	567	819
		82			44	-100	4758	4444	4391

Put u = x - 19, v = y – 21, $\sum f = N = 82$, $\bar{u} = \frac{1}{N}\sum fu = \frac{44}{82} = 0.5366$,

$$\bar{v} = \frac{1}{N}\sum f\,v = -\frac{100}{82} = -1.2196$$

$$r = \frac{cov(u,v)}{\sigma_u\,\sigma_v} = -\frac{54.20}{(7.598)\,(7.26)} = 0.9825.$$

$$\sigma_u^2 = \frac{1}{N}\sum fu^2 - (\bar{u})^2 = \frac{1}{82}(4758) - (0.5366)^2 = 57.7364$$

$$\therefore \quad \sigma_u = 7.598$$

$$\sigma_v^2 = \frac{1}{N}\sum fv^2 - (\bar{v})^2 = \frac{1}{82}(4444) - (-1.2196)^2 = 52.708$$

$$\therefore \quad \sigma_v = 7.26$$

$$cov(u, v) = \frac{1}{N}\sum fuv - \bar{u}\bar{v} = \frac{1}{82}(4391) - (0.5366)(-1.2096) = 54.20$$

3.6 Regression :

Regression is the estimation or prediction of unknown values of one variable from known values of another variable. i.e. One is interested to know the nature of relationship between the two variables.

- **Lines of Regression:**

Let the equation of line of regression of y on x by

$y = a + bx$……..(i) then $\bar{y} = a + b\bar{x}$ …….(ii) $\therefore$ $y - \bar{y} = b(x - \bar{x})$……..(iii)

The normal equations are $\sum y = na + b\sum x$, $\sum xy = a\sum x + b\sum x^2$, ……..(iv)

Shifting the origin to $(\bar{x}\ \bar{y})$ then eqn.(iv) becomes

$\sum(x - \bar{x})\ (y - \bar{y}) = a\sum(x - \bar{x}) + b\sum(x - \bar{x})^2$……….(v)

Since

$$\frac{\sum(x - \bar{x})\ (y - \bar{y})}{n\sigma_x\ \sigma_y} = r$$

$$\sum (x - \bar{x}) = 0, \frac{1}{N}\sum (x - \bar{x})^2 = \sigma_x^2, \text{ from equation (v)we get}$$

$$nr\sigma_x\ \sigma_y = a(0) + bn\ \sigma_x^2\ , \therefore b = r\frac{\sigma_y}{\sigma_x}.$$

Hence from (iii) the line of regression of y on x is $y - \bar{y} = b_{yx}\ (x - \bar{x})$

Where $b_{yx} = r\frac{\sigma_y}{\sigma_x}$. is called the regression coefficient of y on x.

Similarly the line of regression of x on y is $x - \bar{x} = b_{yx}\ (y - \bar{y})$

Where $b_{xy} = r\frac{\sigma_x}{\sigma_y}$. is called the regression coefficient of x on y.

Now $b_{yx}.\ b_{xy} = r\frac{\sigma_y}{\sigma_x}.\ r\frac{\sigma_x}{\sigma_y}$ then $r = \sqrt{b_{yx}\cdot\ b_{xy}\cdot}$.

Note :

i) If r = 0 the two lines of regression becomes $x = \bar{x}$ and $y = \bar{y}$ these are two straight lines parallel to X and Y axes respectively, and passing through their means $\bar{x}$ and $\bar{y}$ they are mutually perpendicular.

ii) If $r = \pm 1$ the two lines of regression will coincide.

Illustrative Examples

Example : 1

If θ be the acute angle between two regression lines in the case of two variables x and y show that $\tan\theta = \frac{1 - r^2}{r}\ \frac{\sigma_x\ \sigma_y}{\sigma_x^2 + \sigma_y^2}$. where r, σ_x, σ_y have their usual meaning, Explain the significance when r = 0 and r = ± 1.

Solution :

The lines of regression are $(y - \bar{y}) = b_{yx}\ (x - \bar{x})$……….(1)

$(x - \bar{x}) = b_{xy}(x - \bar{x}) \quad \therefore\ (y - \bar{y}) = \frac{1}{b_{xy}}\ (x - \bar{x})$………..(2)

Here $m_1 = r\frac{\sigma_y}{\sigma_x}$ and $m_2 = \frac{1}{r}\frac{\sigma_y}{\sigma_x}$, now, $\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$

$$\tan\theta = \frac{\frac{1}{r}\frac{\sigma_y}{\sigma_x} - r\frac{\sigma_y}{\sigma_x}}{1 + r\frac{\sigma_y}{\sigma_x}.\frac{1}{r}\frac{\sigma_y}{\sigma_x}} = \frac{1-r^2}{r}.\frac{\frac{\sigma_y}{\sigma_x}.\sigma_x^2}{\sigma_x^2+\sigma_y^2} = \frac{1-r^2}{r}\frac{\sigma_x\,\sigma_y}{\sigma_x^2+\sigma_y^2} \ \ldots\ldots\ldots..(3)$$

i) If $r = 0$, then there is no relationship between the two variables and they are independent. on putting the value $r = 0$ in(3) we get $\tan\theta = 0$, $\theta = \frac{\pi}{2}$ so the lines (1) and (2) are perpendicular.

ii) If $r = 1$ or -1, on putting these values of r in (3) we get $\tan\theta = 0$ or $\theta = 0$ i.e. lines (1) and (2) coincide. The correlation between the two variables is perfect.

Example : 2

Calculate the coefficient of correlation, obtain the least square regression line y on x for the following data.

x	1	2	3	4	5	6	7	8	9
y	9	8	10	12	11	13	14	16	15

Also obtain an estimate of y which should correspond on the average to x = 6.2.

Solution : We construct a table as follows:

x	y	$u = x - \bar{x}$	$v = y - \bar{y}$	u^2	v^2	uv
1	9	-4	-3	16	9	12
2	8	-3	-4	9	16	12
3	10	-2	-2	4	4	4
4	12	-1	0	1	0	0
5	11	0	-1	0	1	0
6	13	1	1	1	1	1
7	14	2	2	4	4	4
8	16	3	4	9	16	12
9	15	4	3	16	9	12
45	108	0	0	60	60	57

The coefficient of correlation is given by

$\bar{x} = \frac{1}{n}\sum x = \frac{45}{9} = 5$, $\bar{y} = \frac{1}{n}\sum y = \frac{108}{9} = 12.$

$$r = \frac{\sum uv}{\sqrt{\sum u^2 \ \sum v^2}} = \frac{57}{\sqrt{(5) \times (12)}} = 0.95.$$

$$\sigma_u = \sqrt{\frac{1}{n}\sum u^2 - (\bar{u})^2} = \sqrt{\frac{1}{9}(60) - (0)^2}.$$

$$\sigma_u = 2.582.\ \sigma_v = \sqrt{\frac{1}{n}\sum v^2 - (\bar{v})^2} = \sqrt{\frac{1}{9}(60) - (0)^2} = 2.582.$$

$$b_{vu} = r\frac{\sigma_v}{\sigma_u} = \frac{(0.95)(2.582)}{2.582} = 0.95$$

The line of regression v on u is $v - \bar{v} = b_{vu}(u - \bar{u})$

$\therefore \quad [(y - \bar{y}) - 0] = 0.95\,[(x - \bar{x}) - 0]$

$y - 12 = (0.95)(x - 5) \ \therefore \ y = 0.95x + 7.25$

when $x = 6.2$ then $y = 0.95(6.2) + 7.25 = 13.14$. then.

Example : 3

Given $8x - 10y + 66 = 0$, $40x - 18y = 214$, and variance of $x = 9$.

Find i) Average value of x and y.

ii) The correlation coefficient between two variables.

iii) The standard deviation of y

Solution :

Since both lines of regression pass through the point $(\bar{x}, \bar{y})$ we have

$8\bar{x} - 10\bar{y} = -66$

$40\bar{x} - 18\bar{y} = 214$ Solving these two equations we get mean of x and y are

$\bar{x} = 13$, $\bar{y} = 17$.

The equations of lines of regression can be written as $= \frac{8}{10}x + \frac{66}{10}$, $b_{yx} = \frac{8}{10}$.

$x = \frac{18}{40}y + \frac{214}{40}$, $b_{xy} = \frac{18}{40}$ then $r = \sqrt{b_{yx} \times b_{xy}} = \sqrt{\frac{8}{10} \times \frac{18}{40}} = 0.6$

Given variance of x is $\sigma_x^2 = 9$, S.D. $= \sigma_x = 3$. To find, S.D. of y is

$\sigma_y = b_{yx}\frac{\sigma_x}{r} = \frac{8}{10} \times \frac{3}{0.6} = 4.$

Example : 4

Given the following data: The regression equations are $4x - 5y + 33 = 0$.

$20x - 9y + 107 = 0$, and variance of $x = 9$. Find

i) Average value of x and y.

ii) The correlation coefficient between two variables.

iii) The standard deviation of y.

Solution :

$$\bar{x} = 30, \bar{y} = 40,$$

$$b_{yx} = \frac{5}{6}$$

$$b_{xy} = \frac{8}{15}, r = 0.6$$

$$\sigma_y = \frac{5}{6} \times \frac{3}{0.6} = 4.17$$

Example : 5

Given the following data:

	Variable (x)	Variable (y)
A.M.	8.2	12.4
S.D.	6.2	20

and coefficient of correlation between x and y is 0.9. Find the linear regression, estimate the value of x given y = 10.

Solution :

Since $\bar{x} = 8.2$, $\bar{y} = 12.4$, $\sigma_x = 6.2$, $\sigma_y = 20$, $r = 0.9$.

$$b_{xy} = r\frac{\sigma_x}{\sigma_y} = (0.9) \times \frac{6.2}{20} = 0.279.$$

The line of regression x on y is $x - \bar{x} = b_{xy}(y - \bar{y})$

$(x - 8.2) = (0.279)(y - 12.4)$

$\therefore \quad x = 0.279y + 4.7404.$

when y = 10 then x = 7.5304.

Example : 6

Following are the marks of students in a particular subject. Find the average marks.

Marks	10 - 12	12 - 14	14 - 16	16 - 18	18 - 20	20 - 22	22 - 24	24 - 26
Students	3	6	10	15	24	42	75	90

26 - 28	28 - 30	30 - 32	32 - 34	34 - 36	36 - 38	38 - 40	40 - 42
79	55	36	26	19	13	9	7

Solution :

First prepare the table, where a = 25, h = 2

Class	Students	Mid value (x)	f_x	$d = \begin{Bmatrix} x - a \\ x - 25 \end{Bmatrix}$	f_d	$f_u = \frac{x - 25}{2}$	f_u
10 - 12	3	11	33	- 14	- 42	- 7	- 21
12 - 14	6	13	78	- 12	- 72	- 6	- 36
14 - 16	10	15	150	- 10	- 100	- 5	- 50
16 - 18	15	17	255	- 8	120	- 4	- 60
18 - 20	24	19	456	- 6	- 144	- 3	- 72
20 - 22	42	21	882	- 4	- 168	- 2	- 84
22 - 24	75	23	1725	- 2	- 150	- 1	- 75
24 - 26	90	25	2250	0	0	0	0
26 - 28	79	27	2133	2	158	1	79
28 - 30	55	29	1595	4	220	2	110
30 - 32	36	31	1116	6	216	3	108
32 - 34	21	33	858	8	208	4	104
34 - 36	19	35	665	10	190	5	95
36 - 38	13	37	481	12	156	6	78
38 - 40	9	39	351	14	126	7	63
40 - 42	7	41	287	16	112	8	295
Σ	509	-	13315	-	590	-	295

i) Direct method, mean $= \bar{x} = \frac{\Sigma f_x}{\Sigma f}$

$$= \frac{13315}{509} = 26.16$$

ii) Short wt method, mean $= \bar{x} = a + \frac{\Sigma f_d}{\Sigma f} = 25 + \frac{590}{509} = 26.16$

iii) Step deviation method, mean $= \bar{x} = a + h\left(\frac{\Sigma f_u}{\Sigma f}\right)$

$$= 25 + 2\left[\frac{295}{509}\right] = 26.16$$

Example : 7

Following are the monthly wages of workers. Find the average monthly wages.

Wages	0 - 10	10 - 20	20 - 30	30 -40	40 - 50	50 - 60	60 – 70
Workers	5	12	30	45	50	37	21

Solution :

Wages	**Work-ers (f)**	**Midpo int (x)**	f_x	**d = x - 35**	f_d	$u = \frac{x - 35}{10}$	f_u
0 - 10	5	5	25	- 30	- 150	- 3	- 15
10 - 20	12	15	180	- 20	- 240	- 2	- 24
20 - 30	30	25	750	- 10	- 300	- 1	- 30
30 - 40	45	35	1575	0	0	0	0
40 - 50	50	45	2250	10	500	1	50
50 - 60	37	55	2035	20	740	2	74
60 - 70	21	65	1365	30	630	3	63
Σ	200	-	8180	-	1180	-	118

i) Direct method: mean $= \bar{x} = \frac{\Sigma f_x}{\Sigma f} = \frac{8180}{200} = 40.9$

ii) Short wt method: mean $= \bar{x} = a + \frac{\Sigma f_d}{\Sigma f} = 35 + \frac{1180}{200} = 40.9$

iii) Step deviation method: mean $= \bar{x} = a + h\left(\frac{\Sigma f_u}{\Sigma f}\right)$

$$= 35 + 10\left[\frac{118}{200}\right] = 40.9$$

Example : 8

Calculate the mean for following data by

i) Direct method

ii) Shortcut method

iii) Step deviation method

Weights of students (x)	15	20	25	30	35	40	45	50	55
Students (f)	2	22	19	14	3	4	6	1	1

Solution :

$$\Sigma f = 72, \Sigma f_x = 2185, \Sigma f_d = -515, \Sigma f_u = -101$$

Where $h = 5, d = x - a = x - 35$

i) $\bar{x} = \frac{\Sigma f_x}{\Sigma f} = \frac{2185}{72} = 30.34$

ii) $\bar{x} = a + \frac{\Sigma f_d}{\Sigma f} = 35 + \frac{-515}{72} = 30.34$

iii) $\bar{x} = \bar{a} + h\left(\frac{\Sigma f_u}{\Sigma f}\right) = 35 + \left[\frac{-101}{72}\right] 5 = 30.34$

Example : 9

Calculate the mean for the following data.

Weight of articles	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60
No. of articles	14	17	22	26	23	16

Solution :

$$\Sigma f = 120, \Sigma f_x = , \Sigma f_d = 210, \Sigma f_u = 162$$

$\therefore \quad \bar{x} =$ mean by all three methods $= 31.75$

3.7 Example of Median, Quartiles :

- **Median :**

If the given values of a distribution are arranged in ascending or descending order of its magnitude then

i) Medium = Middle item, if no. of distributions are odd

= Mean value of middle two values, if distributions are even.

Illustrative Examples

Example : 1

Find the median value for the following distribution

2, 9, 17, 6, 5, 27, 8, 35, 20, 22, 40.

Solution :

Arrange the numbers in ascending order

2, 5, 6, 8, 9, 17, 20, 22, 27, 35, 40.

$\therefore$ Median = 17

Example : 2

Find the median value for the following distribution

2, 9, 17, 6, 5, 27, 8, 38, 20, 35, 20, 22, 40, 24.

Solution :

Arrange the numbers in ascending order

2, 5, 6, 8, 9, 17, 20, 20, 22, 24, 27, 35, 38, 40.

This distribution has 14 observations, even no of observations

$$\therefore \quad \text{Median} = \frac{20 + 20}{2}$$

$$= 20$$

3.8 Quartiles :

Quartiles divides the frequency distribution into four equal parts after arranging values/ observations in ascending order.

Q_1 = Lower quartile (between lower value and Q_2)

Q_2 = Median

Q_3 = Upper quartile (between Q_2 and upper extreme value)

Formulae

1. $N = \Sigma f$

2. $$Q_1 = L + \left[\frac{\left(\frac{N}{4}\right) - C}{f}\right]h$$

3. $$Q_2 = L + \left[\frac{\left(\frac{N}{2}\right) - C}{f}\right]h$$

4. $$Q_3 = L + \left[\frac{\left(\frac{3N}{4}\right) - C}{f}\right]h$$

Where l = lower limit of class interval in which Q_1| Q_2 | Q_3 lies

f = frequency of that respective class internal.

h = width of that respective class interval

c = cumulative frequency of the class preceding the class in which Q_1|Q_2|Q_3 lies.

Illustrative Examples

Example : 1

Calculate the median, lower and upper quartiles from the following data.

Class	10 - 20	20 - 30	30 - 40	40 - 50	50 – 60
Frequency	7	6	9	2	6

Solution :

Given frequency distribution table is as below.

Class	**Frequency (f)**	**Cumulative frequency (c)**
10 - 20	7	7
20 - 30	6	13
30 - 40	9	22
40 - 50	2	24
50 - 60	6	30
	$\Sigma f = N = 30$	

To find Q_1 :

$$N = 30$$

$$\therefore \quad \frac{N}{4} = \frac{30}{4} = 7.5 \text{ which lies in } 20 – 30$$

$$\therefore \quad L = 20,\ f = 6,\ c = cf = \text{preceding frequency} = 7$$

$$\therefore \quad Q_1 = L + \left[\frac{\left(\frac{N}{4}\right) - C}{f}\right] h = 20 + \left[\frac{7.5 - 7}{6}\right](10) = 14.16$$

$$\therefore \quad Q_1 = 14.16$$

To find Q_2 :

$$\frac{N}{2} = \frac{30}{2} = 15 \text{ which lies between } 30 – 40$$

$$\therefore \quad L = 30,\ f = 9,\ c = 13$$

$$\therefore \quad Q_2 = L + \left[\frac{\left(\frac{N}{2}\right) - C}{f}\right] h$$

$$= 30 + \left[\frac{15 - 13}{9}\right](10) = 32.22$$

$$\therefore \quad Q_2 = 32.22$$

To find Q_3 :

$$\frac{3N}{4} = \frac{90}{4} = 22.5, \text{ which lies in } 40 – 50$$

$$\therefore \quad L = 40, f = 2, c = 22$$

$$\therefore \quad Q_3 = L + \left[\frac{\left(\frac{3N}{4}\right) - C}{f}\right] h = 40 + \left[\frac{22.5 - 22}{2}\right] = 42.5$$

$$\therefore \quad Q_3 = 42.5$$

Example : 2

Calculate the median, lower and upper quartiles from the following distribution

Class	5 - 10	10 - 15	15 - 20	20 - 25	25 - 30	30 - 35	35 - 40	40 – 45
Frequency	5	6	15	10	5	4	2	2

Solution :

Given distribution table can be written as

Class	5 - 10	10 - 15	15 - 20	20 - 25	25 - 30	30 - 35	35 - 40	40 – 45
Frequency	5	6	15	10	5	4	2	2
Cumulative Frequency	5	11	26	36	41	45	47	49

$$N = \Sigma f = 49$$

i) $$\frac{N}{4} = \frac{49}{4} = 12.25, \text{ which lies in } 15 – 20$$

$$\therefore \quad L = 15, f = 15, c = 11$$

$$\therefore \quad Q_1 = L + \left[\frac{\left(\frac{N}{4}\right) - C}{f}\right] h = 15 + \left[\frac{\left(\frac{49}{4}\right) - 11}{15}\right] (5) = 15.4$$

$$\therefore \quad Q_1 = 15.4$$

ii) $$\frac{N}{2} = \frac{49}{2} = 24.5, \text{ which lies in } 15 - 20$$

$$\therefore \quad L = 15, f = 15, c = 11$$

$$\therefore \quad Q_2 = L + \left[\frac{\left(\frac{N}{2}\right) - C}{f}\right] h$$

$$= 15 + \left[\frac{24.5 - 11}{15}\right] (5) = 19.5$$

$$\therefore \quad Q_2 = 19.5$$

iii) $\frac{3N}{4} = \frac{(3)(49)}{4} = 36.75$, which lies in 25 – 30

$\therefore \quad L = 25, f = 5, c = 36$

$$\therefore \quad Q_3 = L + \left[\frac{\left(\frac{3N}{4}\right) - C}{f}\right] h = 25 + \left[\frac{36.75 - 36}{5}\right] (5) = 25.75$$

$\therefore \quad Q_3 = 25.75$

Exercise No. 1

Ex. 1 Calculate the median, lower and upper quartiles from the following data

Class	10 - 20	20 - 30	30 - 40	40 - 50	50 – 60
Frequency	8	7	10	3	7

Ex. 2 Calculate the median, lower and upper quartiles from the following table

Class	3 - 8	8 - 13	13 - 18	18 - 23	23 - 27	27 - 32	32 - 37	37 - 42
Frequency	12	88	58	17	23	29	18	5

Ex. 3 Calculate the median, lower and upper quartiles from the following table

Weight	70-80	80-90	90-100	100-110	110-120	120-130	130-140	140-150
No. of Persons	12	18	35	42	50	45	20	08

3.9 Examples on Standard Devitation :

Example : 1

Calculate the mean and standard deviation for the following

Size of item	6	7	8	9	10	11	12
Frequency	3	6	9	13	8	5	4

Solution :

First we prepare the table let a = 9

Size (x)	**Frequency (f)**	**d = x - 9**	f_d	d^2	$f_d^{\,2}$	f_x
6	3	- 3	- 9	9	27	18
7	6	- 2	- 12	4	24	42
8	9	- 1	- 9	1	9	72
9	13	0	0	0	0	117

10	8	1	8	1	8	80
11	5	2	10	4	20	55
12	4	3	12	9	36	48
Σ	48	-	00	-	124	432

i) Mean $= \bar{x} = a = \frac{\Sigma f_d}{\Sigma f} = 9 + \frac{0}{48} = 9$

ii) Mean $= \bar{x} = \frac{\Sigma f_x}{\Sigma f} = \frac{432}{48} = 9$

iii) S. D. $= \sigma = \sqrt{\frac{\Sigma f_d^2}{\Sigma f} - \frac{(f_d)^2}{(\Sigma f)^2}} = \sqrt{\left(\frac{124}{48}\right) - \frac{(0)^2}{(48)^2}} = 1.607$

Example : 2

Calculate the standard deviation for the following data

Marks	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60
Students	3	16	26	31	16	8

Solution : First prepare the table with a = 35, h = 10, d = x – a, $u = \frac{x - a}{h}$

Class	**Frequencies(f)**	**Midpoint (x)**	$u = \frac{x - 35}{10}$	f_u	u^2	f_u^2
0 - 10	3	5	- 3	- 9	9	27
10 - 20	16	15	- 2	- 32	4	64
20 - 30	26	25	- 1	- 26	1	26
30 - 40	31	35	0	0	0	0
40 - 50	16	45	1	16	1	16
50 - 60	8	55	2	16	4	32
Σ	100	-	-	- 35	-	165

We have

$$\text{S. D.} = \sigma = h\sqrt{\frac{\Sigma f_u^2}{\Sigma f} - \left(\frac{\Sigma f_u}{\Sigma f}\right)^2}$$

$$= 10\sqrt{\frac{165}{100} - \left(\frac{-35}{100}\right)^2} = 12.35$$

Example : 3

Find the standard deviation for the following data

Age under	10	20	30	40	50	60	70	80
No. of boys	15	30	53	75	100	110	115	125

Solution :

First we should prepare table by grouping no. of boys in the interval 10 – 20, 20 – 30, ---

Age	**Boys (f)**	**Midpoint (x)**	$u = \frac{x - 35}{10}$	f_u	u^2	f_u^2
0 - 10	15	5	- 3	- 45	9	135
10 - 20	15	15	- 2	- 30	4	60
20 - 30	23	25	- 1	- 23	1	23
30 - 40	22	35	0	0	0	0
40 - 50	25	45	1	25	1	25
50 - 60	10	55	2	20	4	40
60 - 70	5	65	3	15	9	45
70 - 80	10	75	4	40	16	160
Σ	125	-	-	2	-	488

$$\therefore \quad \sigma = h\sqrt{\frac{\Sigma f_u^2}{\Sigma f} - \left(\frac{\Sigma f_u}{\Sigma f}\right)^2} = 10\sqrt{\frac{488}{125} - \left(\frac{2}{125}\right)^2} = 19.76$$

Example : 4

Calculate the standard deviation of scores of a player in 10 matches played by him.

47, 12, 16, 42, 4, 51, 37, 48, 13, 0

Solution : In this example, neither frequencies nor intervals are given. This is an ungrouped or raw data

$$\text{Here } \bar{x} = \frac{47 + 12 + 16 + 42 + 4 + 51 + 37 + 48 + 13 + 0}{10} = \frac{270}{10} = 27$$

Prepare a table

x	47	12	16	42	4	51	37	48	13	0	Σ 270
d = x - 27	20	- 15	- 11	15	- 23	24	10	21	- 14	- 27	0
d^2	400	225	121	225	529	576	100	441	196	729	3542

$$\therefore \quad \sigma = \text{S. D.} = \sqrt{\frac{\Sigma\, d^2}{n} - \left(\frac{\Sigma d}{n}\right)^2} = \sqrt{\left(\frac{3542}{10}\right) - \left(\frac{0}{10}\right)^2} = 18.82$$

Example : 5

Calculate the standard deviation of the following distribution

x	5	6	7	8	9	10	11	12	13	14	15
f	18	15	34	47	68	90	80	62	35	27	11

Solution :

Let d = x – 10, $\Sigma\, f = 487$, $\Sigma\, f_d = 36$, $\Sigma\, f_d^2 = 2602$

$$\therefore \quad \sigma = \sqrt{\frac{\Sigma\, f_d^2}{\Sigma\, f} - \left(\frac{\Sigma\, f_d}{\Sigma\, f}\right)^2} = \sqrt{\frac{2602}{487} - \left(\frac{36}{487}\right)^2} = 2.311$$

Example : 6

Calculate the standard deviation of scores made by a batsman in 10 matches as below.

12, 115, 6, 73, 7, 19, 119, 36, 84, 29.

Solution :

$\bar{x} = 50$, $d = x - 51$, $\Sigma\, d = -10$, $\Sigma\, d^2 = 17508$, $n = 10$

$$\therefore \quad \sigma = \sqrt{\frac{\Sigma\, d^2}{n} - \left(\frac{\Sigma\, d}{n}\right)^2} = \sqrt{\frac{17508}{10} - \left(\frac{-10}{10}\right)^2} = 41.8$$

Example : 7

Calculate the standard deviation of scores made by a batsmen in 10 matches as below

47, 12, 16, 42, 4, 51, 37, 48, 13, 0.

Solution :

In this example, neither frequencies nor intervals are given. This is a ungrouped or raw data

$$\text{Here } \bar{x} = \frac{47 + 12 + 16 + 42 + 4 + 51 + 37 + 48 + 13 + 0}{10} = \frac{270}{10} = 27$$

Prepare a table

x	47	12	16	42	4	51	37	48	13	0	Σ 270
d = x - 27	20	- 15	- 11	15	- 23	24	10	21	- 14	- 27	0
d^2	400	225	121	225	529	576	100	441	196	729	3542

$$\therefore \quad \sigma = \text{S. D.} = \sqrt{\frac{\Sigma\, d^2}{n} - \left(\frac{\Sigma d}{n}\right)^2} = \sqrt{\left(\frac{3542}{10}\right) - \left(\frac{0}{10}\right)^2} = 18.82$$

Exercise No. 2

Ex. 1 Calculate the standard deviation and mean for following data.

Marks	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50	50 60	60 – 70
Students	5	12	30	45	50	37	21

[Hint : $N = \Sigma f = 200$, $a = 35$, $u = \dfrac{x - 35}{10}$, $\Sigma f_u = 118$, $\Sigma f_u^2 = 510$

$\therefore \quad \bar{x} = 40.9$, σ = S. D = 10.839]

Ex. 2 The annual salaries of a group of employees are given below.

Salaries (1000)	45	50	55	60	65	70	75	80
Persons	3	5	8	7	9	7	4	7

[Hint: $N = \Sigma f = 50$, $a = 60$, $h = 5$, $u = \dfrac{x - 60}{5}$, $\Sigma f_u = 36$, $\Sigma f_u^2 = 240$

$$\therefore \qquad \sigma = h\sqrt{\frac{\Sigma f_u^2}{\Sigma f} - \left(\frac{\Sigma f_u}{\Sigma f}\right)^2} = 10.35]$$

Ex. 3 Calculate the standard deviation for the following data

Class	100-109	110-119	120-129	130-139	140-149	150-159	160-169	170-179
Frequency	15	44	133	150	125	82	35	16

[Hint: Take $a = 145$, $h = 10$, $\Sigma f = 600$, $\Sigma f_u = -408$, $\Sigma f_u^2 = 1684$

$$\therefore \quad \text{S. D} = \sigma = h\sqrt{\frac{\Sigma f_x^2}{\Sigma f} - \left(\frac{f_u}{\Sigma f}\right)^2} = 15.28]$$

Ex. 4 Calculate the standard deviation for the following data.

Weights	0 - 10	11 - 20	21 - 30	31 - 40	41 - 50	51 – 60
Students	3	16	26	31	16	8

[Hint: Take $a = 35$, $h = 10$, $u = \dfrac{x - 35}{10}$, $\Sigma f_u = -35$, $\Sigma f_u^2 = 165$, $\Sigma f = 100$

$\therefore \qquad \sigma = 12.35$]

Ex. 5 Calculate the standard deviation for the following data

Heights	20 - 25	25 - 30	30 - 35	35 - 40	40 - 45	45 – 50
No. of Girls	170	110	80	45	40	35

[Hint: $N = \Sigma f = 480$, $a = 35.5$, $h = 5$, $\Sigma f_u = -220$, $\Sigma f_u^2 = 1310$ $\therefore \quad \sigma = 7.936$]

Example : 9

Annual salaries of a group of employees are given below.

Salaries (1000)	45	50	55	60	65	70	75	80
Employees	2	2	1	30	12	1	1	1

[Hint: a = 60, h = 5, $u = \frac{x - 60}{5}$, $\Sigma f = N = 50$, $\Sigma f_u = 10$, $\Sigma f_u^2 = 68$

$$\therefore \quad \sigma = h\sqrt{\frac{\Sigma f_x^2}{\Sigma f} - \left(\frac{\Sigma f_u}{\Sigma f}\right)^2} = 5\sqrt{\frac{68}{50} - \left(\frac{10}{50}\right)^2} = 5\sqrt{1.32} = 5.74$$

3.10 Examples on Coefficient of Variation :

If σ and $\bar{x}$ respectively denotes the standard deviation and mean of data A then

coefficient of variation of A = cov (A) = $\frac{\sigma}{\bar{x}} \times 100$

Note :

1. If. (mean of data A) > (mean of data B)
   Then team A is more run taker
2. If. cov (A) > cov (B)
   Then i) Group B is more consistent
   ii) Group A has more variability

Illustrative Examples

Example : 1

Two brands of types are tested with following results of their life in kilometers

Life in kms (000)	20 - 25	25 - 30	30 - 35	35 - 40	40 – 45
Brand A	1	22	64	10	3
Brand B	0	24	76	0	0

Determine:

(i) Which brand tyres have greater average life ?

(ii) Which brand tyres will be preferred for use?

Solution :

First we should find mean and S. D. fir both brands

Brand A :

Life	Frequency (f)	Midpoint (x)	$u = \frac{x - 32.5}{5}$	f_u	u^2	f_u^2
20 - 25	1	22.5	- 2	- 2	4	4
25 - 30	22	27.5	- 1	- 22	1	22
30 - 35	64	32.5	0	0	0	0
35 - 40	10	37.5	1	10	1	10
40 - 45	3	42.5	2	06	4	12
Σ -	100	-	-	-8	-	48

$$\bar{x}_A = a + \left(\frac{\Sigma f_u}{\Sigma f}\right) h = 32.5 + \left(\frac{-8}{100}\right) 5 = 32.1$$

$$\sigma_A = h\sqrt{\frac{\Sigma f_u^2}{\Sigma f} - \left(\frac{\Sigma f_u}{\Sigma f}\right)^2}$$

$$= 5\sqrt{\left(\frac{48}{100}\right) - \left(\frac{-8}{100}\right)^2} = 3.441$$

$$\therefore \quad \text{cov}(A) = \frac{\sigma_A}{\bar{x}_A} \times 100 = \frac{3.441}{32.1} \times 100 = 10.72$$

For Brand B :

Life	Frequency (f)	Midpoint (x)	$u = \frac{x - 32.5}{5}$	f_u	u^2	f_u^2
20 - 25	0	22.5	- 2	0	4	0
25 - 30	24	27.5	- 1	- 24	1	24
30 - 35	76	32.5	0	0	0	0
35 - 40	0	37.5	1	0	1	0
40 - 45	0	42.5	2	0	4	0
Σ -	100	-	-	- 24	0	24

$$\therefore \quad \bar{x}_B = a + \left(\frac{\Sigma f_u}{\Sigma f}\right) h = 32.5 + \left(\frac{-24}{100}\right) 5 = 31.3$$

$$\sigma_B = h\sqrt{\frac{\Sigma f_u^2}{\Sigma f} - \left(\frac{\Sigma f_u}{\Sigma f}\right)^2}$$

$$= 5\sqrt{\left(\frac{24}{100}\right) - \left(\frac{-24}{100}\right)^2} = 2.136$$

$\therefore \quad cov\ (B) \quad = \quad \frac{\sigma_B}{\bar{x}_B} \times 100 = \frac{2.136}{31.3} \times 100 = 6.824$

Conclusion :

i) $\bar{x}_A > \bar{x}_B \Rightarrow$ Brand A tyres have more life

ii) cov (A) > cov (B) $\Rightarrow$ Brand B tyres will be preferred to a

Example : 2

Goals scored by two teams in a hockey match session are as below:

Calculate the coefficient of variation and hence state, which team is more consistent.

Goals	0	1	2	3	4	5
Played by A	15	12	07	06	05	03
Played by B	18	12	06	03	02	01

Solution :

For team A, $\bar{x}_A \;=\; \frac{15+12+7+6+5+3}{6} = \frac{48}{6} = 8$

For team B, $\bar{x}_B \;=\; \frac{18+12+6+3+2+1}{6} = \frac{42}{6} = 7$

Table

A (x)	d = x - 8	d^2	B (y)	D = y - 7	D^2
15	7	49	18	11	121
12	4	16	12	5	25
7	-1	1	6	-1	1
6	-2	4	3	-4	16
5	-3	9	2	-5	25
3	-5	25	1	-6	36
Σ48	0	104	42	0	224

Team A : $\bar{x}_A \;=\; 8$

$$\sigma_A \;=\; \sqrt{\frac{\Sigma D^2}{N} - \left(\frac{\Sigma D}{N}\right)^2} = \sqrt{\left(\frac{104}{48}\right) - 0} = \sqrt{\frac{13}{6}} = \sqrt{2.1666} = 1.471$$

$\therefore \quad cov\ (A) \quad = \quad \frac{\sigma_A}{x_A} \times 100 = \frac{1.471}{8} \times 100 = 18.39$

Team B :

$$\bar{x}_B = 7$$

$$\sigma_B = \sqrt{\frac{\Sigma D^2}{N} - \left(\frac{\Sigma D}{N}\right)^2} = \sqrt{\left(\frac{224}{42}\right) - 0} = \sqrt{\frac{112}{21}} = \sqrt{5.333} = 2.309$$

$$\therefore \quad \text{cov (B)} = \frac{\sigma_B}{\bar{x}_B} \times 100 = \frac{2.309}{7} \times 100 = 32.99$$

$$\therefore \quad \text{cov (A)} < \text{cov (B)} \Rightarrow \text{team A is more consistent.}$$

Example : 3

Goals scored by two teams in a football match session were as follows.

Goals scored	0	1	2	3	4	5
Team A Played	15	10	07	05	03	02
Team B Played	20	10	05	04	02	01

Calculate the coefficient of variation and state which team is more consistent ?

Solution :

Hint : $\bar{x}_A = 7$, $\bar{x}_B = 7$, $\sigma_A = 4.43$, $\sigma_B = 6.48$

$\therefore$ cov (A)= 63.29, cov (B) = 92.57

$\therefore$ Team B has more variability.

$\therefore$ Team A is more consistent.

Example : 4

Following are scores of two batsmen A and B in a series of innings

A	12	115	6	73	19	119	29	84	36	7
B	47	12	16	42	51	37	0	13	48	4

State (i) Which player is more run taker?

(iii) Which player is more consistent?

Solution :

Hints:

$$\bar{x}_A = \frac{12 + 115 + 6 + 73 + 19 + 119 + 29 + 84 + 36 + 7}{10} = \frac{500}{10} = 50$$

$$\bar{x}_B = \frac{47 + 12 + 16 + 42 + 51 + 37 + 0 + 13 + 48 + 4}{10} = \frac{270}{10} = 27$$

$$\therefore \quad \sigma_A = \sqrt{\frac{\Sigma d^2}{N} - \left(\frac{\Sigma d}{N}\right)^2} = 41.8$$

$$\therefore \quad \sigma_B = \sqrt{\frac{\Sigma d^2}{N} - \left(\frac{\Sigma d}{N}\right)^2} = 18.82$$

$$\text{cov (A)} = 83.6, \text{cov (B)} = 69.6$$

$$\therefore \quad \text{cov (A)} < \text{cov (B)} \Rightarrow \text{B is more consistent.}$$

$$\bar{x}_B < \bar{x}_A \Rightarrow \text{A is more run taker.}$$

Example : 5

Find standard deviation and coefficient of variance for the following data.

Marks	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60	60 - 70	70 – 80
Students	12	18	35	42	50	45	20	08

Solution : $[N = \Sigma f = 230$, let $a = 40$, $u = \dfrac{x - 40}{10}$, $\bar{x} = 40.43$

$$\sigma = 17.26$$

$$\therefore \quad \text{cov (A)} = \frac{17.26}{40.43} \times 100 = \frac{\sigma}{\bar{x}} \times 100 = 42.69]$$

Example : 6

Goals scored by team A and B in a football season were as below

Goals	0	1	2	3	4
Team A	27	9	8	5	4
Team B	17	9	6	5	3

Which team was more consistent ?

Solution : Team A : $\bar{x}_A = 1.0566$, $\sigma_A = 1.309$, cov (A) = 123.90

Team B : $\bar{x}_B = 1.2$, $\sigma_B = 1.307$, cov (B) = 108.97

$\therefore$ Team B is more consistent.

Example : 7

Scores obtained by two batsmen A and B in 10 matches are given below. State, which batsman has good average of runs and which player is more consistent.

A	30	66	60	34	20	38	44	62	80	46
B	34	70	55	48	45	30	46	38	60	34

Solution :

Player A : $\bar{x}_A = 48$, $\sigma_A = 17.776$, cov (A) = 37.03

Player B : $\bar{x}_B = 46$, $\sigma_B = 12.107$, cov (B) = 26.32

(i) Batsman A is more run taker

(ii) Player B is more consistent.

Example : 8

Following table gives the marks obtained in a paper of mathematics out of 50 by the students of D_2 divisions A and B.

Class	0 - 5	5 - 10	10 - 15	15 - 20	20 - 25	25 - 30	30 - 35	30 - 40	40 - 45	45 - 50
A	2	6	8	8	15	18	12	11	9	4
B	3	5	7	9	12	16	11	5	6	2

State, which Batch has more variability

Solution :

Division A: $\bar{x}_A$ = 26.854, σ_A = 11.173, cov (A) = 41.60

Division B: $\bar{x}_B$ = 24.934, σ_B = 10.927, cov (B) = 43.824

$\therefore$ Team B has greater variability and team A is more consistent.

Example : 9

Find the mean, standard deviation and coefficient of variation for the following data.

Marks obtained up to	10	20	30	40	50	60	70	80
No. of Students	12	30	65	107	157	202	222	230

Solution :

First prepare the table by using class intervals such as marks obtained by students in 0 – 10, 10 – 20, 20 – 30, 30 – 40, --- as below

Marks	0 - 10	10 - 20	20 -30	30 - 40	40 - 50	50 – 60	60 - 70	70 – 80
Students	12	18	35	42	50	45	20	08

$\therefore$ mean = $\bar{x}$ = 40: 43, σ = 17.26, cov (A) = 42.69

3.11 Examples on Combined Mean :

Let $\bar{x}_1$ and σ_1 be the mean and standard deviation of a sample size of n_1 and $\bar{x}_2$, σ_2 be the mean and standard deviation of a sample of size n_2.

Then

i) $\bar{x}_{12}$ = combined mean of two samples = $\dfrac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$

ii) $\bar{x}_{13} = \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2 + n_3\bar{x}_2}{n_1 + n_2 + n_3}$

iii) σ_{12} = combined standard deviation = $\sqrt{\dfrac{n_1\sigma_1^2 + n_2\sigma_2^2 + n_1d_1^2 + n_2d_2^2}{n_1 + n_2}}$

Where $d_1 = |\bar{x}_1 - \bar{x}_{12}|$, $d_2 = |\bar{x}_2 - \bar{x}_{12}|$

Example : 1

An analysis of monthly wages paid to workers is two firms A and B, belonging to same domain of production, is as below.

	A	B
No. of Workers	500	600
Average salary	3000	3500
S. D. of distribution of wages	88	120

Determine :

i) Which firm pays larger on salaries?

ii) Which firm has more consistency?

iii) Which firm has greater variability?

iv) What is average of salary if A and B taken together ?

v) What is S. D. of individual worker if A and B taken together ?

Solution :

i) Total monthly payment of A = $500 \times 3000 = 15.00000$

Total monthly payment of B = $600 \times 3500 = 21,00,000$

Firm B pays larger amount on salaries.

ii) $x_A = 3000$, $x_B = 3500$, $\sigma_A = 88$, $\sigma_B = 120$

$$\text{cov}(A) = \frac{\sigma_A}{x_A} \times 100 = \frac{88}{3000} \times 100 = 2.93$$

$$\text{cov}(B) = \frac{\sigma_B}{x_B} \times 100 = \frac{120}{3500} \times 100 = 3.42$$

$$\text{cov}(A) < \text{cov}(B)$$

Firm B has greater variability and

Firm A is more consistent

iii) Average salary = combined mean of (A and B)

$$x_{12} = \frac{n_1\overline{x}_1 + n_2\overline{x}_2}{n_1 + n_2}$$

$$= \frac{500(3000) + 600(3500)}{500 + 600}$$

$$= \frac{1500000 + 2100000}{1100} = \frac{3600000}{1100} = 3272.70 \cong 3273$$

iv) $d_1 = |\,\overline{x}_1 - \overline{x}_{12}\,| = |\,3000 - 3273\,| = 273$

$d_2 = |\,\overline{x}_2 - \overline{x}_{12}\,| = |\,3500 - 3273\,| = 227$

$$\text{Combined S. D} = \sigma_{12} = \sqrt{\frac{n_1\sigma_1^2 + n_2\sigma_2^2 + n_1\sigma_1^2 + n_2\sigma_2^2}{n_1 + n_2}}$$

$$= \sqrt{\frac{(560)\,(88)^2 + (600)\,(120)^2 + (500)\,(273)^2 + 600(227)^2}{500 + 600}}$$

$$= \sqrt{\frac{(500 \times 7744) + (600 \times 14400) + (500)\,(74529) + (600)\,(51529)}{1100}}$$

$$= \sqrt{\frac{80693900}{1100}}$$

$$= \sqrt{73358.09}$$

$$= 270.84$$

Example : 2

The mean and variance of two villages are given below

	Village x	**Village y**
No. of persons	500	600
Average income	186	175
Variance of income	81	100

i) In which village the variation in income is greater?

ii) What is the combined S. D. it x, y are taken together?

Solution :

i) Given $\bar{x} = 186$, $N_1 = 500$, $\bar{y} = 175$, $N_2 = 600$

$\sigma_x^2 = 81$, $\sigma_x = ?$, $\sigma_y^2 = 100$, $\sigma_y = 10$

We know σ^2 = variance and $\sqrt{\text{variance}} = \sigma = \text{S. D}$

$$\therefore \quad \text{cov}(x) = \frac{\sigma_x}{\bar{x}} \times 100 = \frac{9}{186} \times 100 = 4.839$$

$$\text{cov}(y) = \frac{\sigma_y}{\bar{y}} \times 100$$

$$= \frac{10}{175} \times 100 = 5.714$$

$\therefore$ cov (x) < cov (y)

$\therefore$ Variation in income of village y is greater.

ii) $$\bar{x}_{12} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2} = \frac{(500)\,(186) + (600)\,(175)}{500 + 600} = 180$$

$d_1 = | \bar{x} - 180 | = | 175 - 180 | = 5$

$d_2 = | \bar{y} - 180 | = | 186 - 180 | = 6$

$$\text{Combined S. D} = \sigma_{12} = \sqrt{\frac{n_1\sigma_1^2 + n_2\sigma_2^2 + n_1\sigma_1^2 + n_2\sigma_2^2}{n_1 + n_2}}$$

$$= \sqrt{\frac{60000 + 40500 + 15000 + 18000}{1100}}$$

$$= 11.02$$

Example : 3

An analysis of wages paid to workers in 02 firms A and B is as below

	A	**B**
No. of workers	550	650
Average monthly salary	Rs 1450	Rs 1400
S. D. of distribution	100	140

Calculate the monthly average wages and combined S. D. by taking two firms A and B together.

Solution :

$$N_1 = 550, N_2 = 650, \bar{x}_A = 1450, \bar{x}_B = 1400$$

$$\text{Combined mean} = \bar{x}_{12} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

$$= 1422.92$$

$$d_1 = 27.\ 08, d_2 = 22.92$$

$$\therefore \quad \sigma_{12} = \sqrt{\frac{n_1d_1^2 + n_2d_2^2 + n_1\sigma_1^2 + n_2\sigma_2^2}{n_1 + n_2}}$$

$$= 12578$$

Example : 4

The number of employees wages per employee and variance of the wages per employee are given below for two factories A and B.

	A	**B**
No. of employees	100	150
Average wages of employee	3200	2800
Variance of wages	625	729

i) Which factory have greater variation in wages

ii) What is average salary/wages if A and B are taken together.

Solution :

$N_1 = 100$, $N_2 = 150$, $\overline{x}_1 = 3200$, $\overline{x}_2 = 2800$, $\sigma_1 = 25$, $\sigma_2 = 27$

i) cov (A) $= \frac{25}{3200} \times 100 = 0.781$, cov (B) $= \frac{27}{2800} \times 100 = 0.964$

$\therefore$ Factory B has greater variation in monthly wages

ii) combined mean $= \overline{x}_{12} = \frac{n_1\overline{x}_1 + n_2\overline{x}_2}{n_1 + n_2} = \frac{320000 + 420000}{250}$

$$= \frac{740000}{250} = 2960$$

3.12 Formulae on Moments and Examples :

The arithmetic mean of variance power of the deviations (xi - $\overline{x}$) is called the moment of the distribution.

I) r^{th} moment of variable x about mean $\overline{x}$

$$= \mu_r = \frac{1}{N} \Sigma (xi - \overline{x})^r \text{, for ungrouped data}$$

$$= \frac{1}{N} \ fi\ (xi - \overline{x})^r \text{, for frequency distribution}$$

II) r^{th} moment about any value "a"

$$= \mu_r' = \frac{1}{N} \ \Sigma f_i d_i^r,\ d_i = xi - a$$

$$= {}^{r}_{h} \frac{1}{N} \ \Sigma f_i u_i^r, \text{ where } u_i = \frac{xi - a}{h}$$

Note :

1. $\mu_r = \frac{\Sigma (xi - \overline{x})^r}{N} = \frac{\Sigma f_i (xi - \overline{x})^r}{N}$

2. $\mu_o = \frac{\Sigma f_i (xi - \overline{x})^o}{N} = \frac{\Sigma f_i}{N} = \frac{\Sigma f_i}{\Sigma f_i} = 1 \ \boxed{\mu_o = 1}$

3. $\mu_1 = \frac{\Sigma f_i (xi - \overline{x})^1}{N} = \frac{\Sigma f_i xi}{N} - \frac{\Sigma f_i \overline{x}}{N}$

$$= \frac{\Sigma f_i xi}{N} - \left(\frac{\Sigma f_i}{N}\right)\overline{x} = \overline{x} - \left(\frac{N}{N}\right)\overline{x} = \overline{x} - \overline{x} = 0$$

$\boxed{\mu_1 = 0}$

4. $\mu_o' = 1$

III) relations between μr and μ_r' :

a) $\mu_o = 1$

b) $\mu_1 = 0$

c) $\mu_2 = \mu_2' .. \left(\mu_1'\right)^2$

d) $\mu_3 = \mu_3' - 3\,\mu_2\,\mu_1' + 2\left(\mu_1'\right)^3$

e) $\mu_4 = \mu_4' - 4\,\mu_3'\,\mu_1' + 6\mu_2'\left(\mu_1'\right)^2 - 3\left(\mu_1'\right)^4$

IV) Relations between μ_1' and μ_r :

a) $\mu_o' = 1$

b) $\mu_1' = \bar{x} - A$

c) $\mu_2' = \mu_2 + \left(\mu_1'\right)^2$

d) $\mu_3' = \mu_3 + 3\mu_2\,\mu_1' + \mu_1'^3$

e) $\mu_4' = \mu_4 + \mu_3\,\mu_1' + 6\mu_2\left(\mu_1'\right)^2 + \left(\mu_1'\right)^4$

Illustrative Examples

Example : 1

The first 04 moments about "4" of variables are – 1.5, 17, - 30, 108. Find the first 04 moments about mean and hence $\beta_1, \beta_2, \bar{x}, \sigma$, variance

Solution :

Given

$$a = 4,\ \mu_1' = -1.5,\ \mu_2' = 17,\ \mu_3' = -30,\ \mu_4' = 108$$

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - \left(\mu_1'\right)^2 = 17 - (-1.5)^2 = 14.75$$

$$\mu_3 = \mu_3' - 3\,\mu_2'\,\mu_1' + 2\left(\mu_1'\right)^3 = 39.75$$

$$\mu_4 = \mu_4' - 4\,\mu_3'\,\mu_1' + 6\mu_2'\left(\mu_1'\right)^2 - 3\left(\mu_1'\right)^4 = 142.3125$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(39.75)^2}{(14.75)^3} = 0.4926$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{142.31}{(14.75)^2} = 0.6543$$

$$\bar{x} = a + \mu_1' = 4 + (-1.5) = 2.5$$

$$\sigma = \sqrt{\mu_2} = \sqrt{14.75} = 3.84$$

$$\text{Variance} = \sigma^2 = \mu_2 = 14.75$$

Example : 2

The first 04 moments about working mean "44.5" of a distribution are – 0.4, 2.99, - 0.08, 27.63. Calculate the moments about mean, β_1, β_2, mean, S. D. variance.

Solution :

$$\mu_1 = 0,\ \mu_2 = 2.83,\ \mu_3 = 3.38,\ \mu_4 = 30.295,\ \beta_1 = 0.504,\ \beta_2 = 3.782,$$

$$\text{Mean } \bar{x} = a + \mu_1' = 44.5 + (-0.4) = 44.1,\ \sigma^2 = \text{variance} = \mu_2 = 2.83,\ \sigma = 1.682$$

Example : 3

Find coefficient of skewness, kurtosis if the first 04 moment about, "5" are 2, 20, 40, 50. And hence find, $\bar{x}$, σ, variance.

Solution :

$$\mu_1 = 0,\ \mu_2 = 16,\ \mu_3 = 64,\ \mu_4 = 162,\ \bar{x} = 7,\ \sigma = \sqrt{\mu_2} = 4$$

$$\text{Variance} = \sigma^2 = 4,\ \beta_1 = 1,\ \beta_2 = 0.6328$$

Example : 4

Find μ_1, μ_2, μ_3, μ_4, β_1, β_2, σ, σ^2, $\bar{x}$ if

$a = 2$, $\mu_1' = 1$, $\mu_2' = 2.5$, $\mu_3' = 5.5$, $\mu_4' = 16$

Solution : $\bar{x} = a + \mu_1' = 3,\ \mu_1 = 0,\ \mu_2 = 1.5,\ \mu_3 = 0,\ \mu_4 = 6$

$$\sigma = \sqrt{\mu_2} = \sqrt{1.5} = 1.224,\ \sigma^2 = 1.5,\ \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{0}{(7.5)^3} = 0$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{6}{(1.5)^2} = \frac{6}{2.25} = 2.66$$

Example : 5

The first 03 moments about "2" of a distribution are 1, 16, - 40. Find first 03 moments about mean and hence find $\bar{x}$, σ, β_1, σ^2

Solution : $\mu_1 = 0,\ \mu_2 = 15,\ \mu_3 = -86,\ \bar{x} = 3,\ \sigma = \sqrt{15} = 3.873,\ \beta_1 = 2.192,\ \sigma^2 = 15.$

Example : 6

The first 04 moments about working mean "30.5" of a distribution are 0.0375, 0.4546, 0.0609, 0.5074. Calculate the moments about mean and hence , β_1, β_2, $\bar{x}$, σ.

Solution :

$$\mu_1 = 0,\ \mu_2 = 0.3139,\ \mu_3 = 0.0098,\ \mu_4 = 0.3033,\ \beta_1 = 1.1143 \times 10^{-4}$$

$$\beta_2 = 0.3349,\ \bar{x} = 30.5 + 0.0375 = 30.5375,\ \sigma = \sqrt{0.3139} = 0.5602.$$

Example : 7

The first 04 moments about the working mean (44.5) of a distribution are – 0.4, 2.99, -0.08 and 27.63. Calculate the moments about mean. β_1, β_2, $\bar{x}$.

Solution : $a = 44.5$, $\mu_1 = 0$, $\mu_2 = 2.83$, $\mu_3 = 3.38$, $\mu_4 = 30.295$, $\beta_1 = 0.504$

$\beta_2 = 3.782$, $\bar{x} = 44.5 + (-0.4) = 44.1$

Example : 8

Find the coefficient of skewness, kurtosis about a point "48" if $d = x - 48$, $\Sigma f = 100$, $\Sigma f_d = 50$, $\Sigma f_d^2 = 1970$, $\Sigma f_d^3 = 2948$ $\Sigma f_d^4 = 86752$.

Solution :

We know (i) $\mu_1' = \dfrac{\Sigma f_d}{\Sigma f} = 0.5$

(ii) $\mu_2' = \dfrac{\Sigma f_d^2}{\Sigma f} = 19.7$

(iii) $\mu_3' = \dfrac{\Sigma f_d^3}{\Sigma f} = 29.48$

(iv) $\mu_4' = \dfrac{\Sigma f_d^4}{\Sigma f} = 867.52$

$\mu_1 = 0$, $\mu_2 = 19.45$, $\mu_3 = 0.18$, $\mu_4 = 837.92$,

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(0.18)^2}{(19.45)^3} = \frac{0.0324}{7357.98} = 4.40 \times 10^{-6}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{837.92}{(19.45)^2} = 2.21$$

Example : 9

Calculate the first 04 moments about the mean and also find the values of β_1, β_2. From the following distribution.

Marks	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60	60 – 70
Students	8	12	20	30	15	10	5

Solution :

First we prepare the table with assumed mean a = 35

Marks	**Midpoint (x)**	**f**	**u**	f_u	u^2	f_u^2	u^3	f_u^3	u^4	f_u^4
0 - 10	5	8	- 3	- 24	9	72	- 27	- 216	81	648
10 - 20	15	12	- 2	- 24	4	48	- 8	- 96	16	192

20 - 30	25	20	- 1	- 20	1	20	- 1	- 20	1	20
30 - 40	35	30	0	0	0	0	0	0	0	0
40 - 50	45	15	1	15	1	15	1	15	1	15
50 - 60	55	10	2	20	4	40	8	80	16	160
60 - 70	65	05	3	15	9	45	27	135	81	405
Σ	-	100	-	- 18	-	240	-	- 102	-	1440

$$\therefore \quad \mu_1' = h\frac{\Sigma f_u}{\Sigma f} = \frac{-18}{100} \times 10 = -18$$

$$\mu_2' = h^2\left[\frac{\Sigma f_u^2}{\Sigma f}\right] = 240,$$

$$\mu_3' = h^3\left[\frac{\Sigma f_u^3}{\Sigma f}\right] = -1020,$$

$$\mu_4' = h^4\left[\frac{\Sigma f_u^4}{\Sigma f}\right] = 14400$$

$$\mu_1 = 0,\ \mu_2 = 236.76,\ \mu_3 = 264.336,\ \mu_4 = 141290.11,$$

$$\beta_1 = 0.005,\ \beta_2 = 2.52.$$

Example : 10

From the following data, calculate the moments about (i) assumed mean "25" (ii) actual mean.

Class	0 - 10	10 - 20	20 - 30	30 - 40
Frequency	1	3	4	2

Solution :

Prepare a table with h = 10, a = 25, $u = \frac{x - 25}{10}$, $\Sigma f = 10$,

$$\Sigma f_u = -3,\ \Sigma f_u^2 = 9,\ \Sigma f_u^3 = -9,\ \Sigma f_u^4 = 21$$

$$\mu_1' = \left(\frac{\Sigma f_u}{\Sigma f}\right)h = -3,\ \mu_2' = h^2\left[\frac{\Sigma f_u^2}{\Sigma f}\right] = 90,$$

$$\mu_3' = -900,\ \mu_4' = 21000$$

$$\mu_1 = 0,\ \mu_2 = 81,\ \mu_3 = -144,\ \mu_4 = 14817.$$

Example : 11

From the following frequency distribution table calculate

$\mu_1, \mu_2, \mu_3, \mu_4, \sigma, \beta_1, \beta_2$.

x	2	3	4	5	6

f	1	3	7	3	1

Solution :

We have $\bar{x} = \frac{2 + 9 + 28 + 25 + 6}{15} = \frac{\Sigma f_u}{\Sigma f} = \frac{60}{15} = 4$ = actual mean

Prepare a table

x	f	u = x - 4	f_u	u^2	$f_u{}^2$	u^3	$f_u{}^3$	u^4	$f_u{}^4$
2	1	-2	-2	4	4	-8	-8	16	16
3	3	-1	-3	1	3	-1	-3	1	3
4	7	0	0	0	0	0	0	0	0
5	3	1	3	1	3	1	3	1	3
6	1	2	2	4	4	8	8	16	16
Σ -	15	-	0	-	14	-	0	-	38

Solution :

We have $h = 1$

$$\mu_1 = h\frac{\Sigma f_u}{\Sigma f} = 0$$

$$\mu_2 = h^2\left[\frac{\Sigma f_u{}^2}{\Sigma f}\right] = 0.933,$$

$$\mu_3 = h^3\left[\frac{\Sigma f_u{}^3}{\Sigma f}\right] = 0,$$

$$\mu_4 = h^4\left[\frac{\Sigma f_u{}^4}{\Sigma f}\right] = 2.533$$

$$\sigma = \sqrt{\mu_2} = \sqrt{0.933} = 0.966,$$

$$\beta_1 = 0, \beta_2 = 2.91$$

3.13 Examples on Correlation :

1. The distribution for one variate x is called univariate distribution.
2. The distribution involving more than one variate is bivariate distribution.
3. If the change in one vanate x affects the change in other vaniate y then x, y are called correlated.
4. If increase in x increases y then correlation is positive.
5. If decrease in x decreases y then correlation is positive.
6. If increase (decrease) in x decreases (increases) y then correlation is negative.

7. If x and y are only two variates and if $\frac{y}{x}$ = constant then correlation is called as linear or perfect otherwise it is called as nonlinear correlation.
8. The formula for measure the intensity or the degree of linear relationship between two variates (variable) was developed by Karl Pearson and called as correlation coefficient.
9. Correlation coefficient of x, y = $r = r(x, y) = \frac{\text{cov}(x,y)}{\sigma x \, \sigma y}$

   Where cov (x, y) = covariance of x, y = $\frac{\Sigma xy}{n} - \left(\frac{\Sigma x}{n}\right)\left(\frac{\Sigma y}{n}\right)$

   Correlation coefficient = $r = r(x, y) = \frac{n\Sigma xy - (\Sigma_x)(\Sigma_y)}{\sqrt{n\Sigma x^2 - (\Sigma x)^2}\sqrt{n\Sigma y^2 - (\Sigma y)^2}}$
10. cov (x, y) = cov (y, x)
11. cov (x, x) = variance x = $\sigma_2^{\ 2}$
12. r always lies between – 1 + 1 ie. $-1 \le r \le 1$
13. If r = 0 then there is no relation
14. If r = $\pm$ 1 then relationship between x and y is very strong.

Illustrative Examples

Example : 1

Calculate the correlation coefficient between x and y from the following data.

x	78	89	99	60	59	79	68	61
y	125	137	156	112	107	136	123	108

Solution : Firstly we prepare the table

x	x = x - 69	x^2	y	y = y - 112	y^2	xy
78	9	81	125	13	169	117
89	20	400	137	25	625	500
99	30	900	156	44	1936	1320
60	- 9	81	112	0	0	0
59	- 10	100	107	- 5	25	50
79	10	100	136	24	576	240
68	- 1	1	123	11	121	- 11
61	- 8	64	108	- 4	16	32
Σ	41	1727	-	108	3468	2248

We have n = 8

$$\therefore \quad r = \frac{n\Sigma xy - (\Sigma x)(\Sigma y)}{\sqrt{n\Sigma x^2 - (\Sigma x)^2}\sqrt{n\Sigma y^2 - (\Sigma y)^2}}$$

$$= \frac{8(2248) - (41)(108)}{\sqrt{8(1727) - (41)^2}\sqrt{8(3468) - (108)^2}}$$

$$= \frac{13556}{13968.95} = 0.97$$

Note:

(Here 69 112 are selected randomly to reduce the calculations by reducing given variates in size)

Example : 2

Calculate the correlation coefficient of x, y from the following data.

x	9	8	7	6	5	4	3	2	1
y	15	16	14	13	11	12	10	8	9

Solution :

We have n = 9

Prepare the table

x	**y**	**x^2**	**y^2**	**xy**
9	15	81	225	135
8	16	64	256	128
7	14	49	196	98
6	13	36	169	78
5	11	25	121	55
4	12	16	144	48
3	10	9	100	30
2	8	4	64	16
1	9	1	81	9
45	108	285	1356	597

$$\therefore \quad r = \frac{n\Sigma xy - (\Sigma x)(\Sigma y)}{\sqrt{n\Sigma x^2 - (\Sigma x)^2}\sqrt{n\Sigma y^2 - (\Sigma y)^2}}$$

$$= \frac{9(597) - (45)(108)}{\sqrt{9(285) - (45)^2}\sqrt{9(1356) - (108)^2}}$$

$$= \frac{513}{540} = 0.95$$

Example : 3

Following data includes the production data of two items x and y year wise. Calculate the correlation coefficient of x and y.

Year	2002	2003	2004	2005	2006	2007	2008	2009
x	100	102	104	107	105	112	103	99
y	15	12	13	11	12	12	19	26

Solution :

We have $\bar{x} = \frac{100 + 102 + 104 + 107 + 105 + 112 + 103 + 99}{8} = \frac{832}{8} = 104$

$$\bar{y} = \frac{15 + 12 + 13 + 11 + 12 + 12 + 19 + 26}{8} = \frac{120}{8} = 15$$

We prepare the table firstly

x	**x = x - 104**	**x^2**	**y**	**y = y - 15**	**y^2**	**xy**
100	- 4	16	15	0	0	0
102	- 2	4	12	- 3	9	6
104	0	0	13	- 2	4	0
107	3	9	11	- 4	16	- 12
105	1	1	12	- 3	9	- 3
112	8	64	12	- 3	9	- 24
103	- 1	1	19	4	16	- 4
99	- 5	25	26	11	121	- 55
Σ	0	120	120	0	184	- 92

$\therefore$ $$r = \frac{n\Sigma xy - (\Sigma x)(\Sigma y)}{\sqrt{n\Sigma x^2 - (\Sigma x)^2}\sqrt{n\Sigma y^2 - (\Sigma y)^2}}$$

$$= \frac{8(-92) - 0}{\sqrt{8(120) - 0}\ \sqrt{8(184) - 0}}$$

$$= \frac{-92}{\sqrt{120}\sqrt{184}} = -0.619$$

Example : 4

Calculate the correlation coefficient from the following data.

x	48	35	17	23	47
y	45	20	40	25	45

Solution :

Hint - $\bar{x} = 34, \bar{y} = 35, x = x - 34, y = y - 35,$

$\Sigma x = 0, \Sigma y = 0, \Sigma x^2 = 776, \Sigma y^2 = 550, \Sigma xy = 280$

$\therefore \quad r = 0.429$

Example : 5

Two examiners A and B awarded the marks to seven students as below.

Marks by A	40	44	28	30	45	38	31
Marks by B	32	39	26	30	28	34	27

Solution :

$n = 7$, Let $x = x - 30, y = y - 30, \Sigma x = 46, \Sigma y = 6$

$\Sigma xy = 153, \Sigma x^2 = 590, \Sigma y^2 = 115, \Sigma xy = 153$

$$\therefore \quad r = \frac{n\Sigma xy - (\Sigma x)(\Sigma y)}{\sqrt{n\Sigma x^2 - (\Sigma x)^2}\sqrt{n\Sigma y^2 - (\Sigma y)^2}} = 0.638$$

Example : 6

Find the correlation coefficient of x and y if

$n = 50, \Sigma x = \Sigma(x - 40) = 30, \Sigma y = \Sigma(y - 20) = 70, \Sigma x^2 = 170, \Sigma y^2 = 165,$

$\Sigma xy = 140$

Solution :

$$r = \frac{50(140) - (70)(30)}{\sqrt{8500 - 900}\sqrt{8250 - 4900}}$$

$$= \frac{7000 - 2100}{\sqrt{7600}\sqrt{3350}} = 0.0167$$

Example : 7

Find the correlation coefficient of x and y if

$n = 25, \Sigma x = 100, \Sigma y = 125$

$\Sigma x^2 = 250, \Sigma y^2 = 500, \Sigma xy = 2$

Solution :

$$\therefore \quad r = \frac{n\Sigma xy - (\Sigma_x)(\Sigma_y)}{\sqrt{n\Sigma x^2 - (\Sigma x)^2}\sqrt{n\Sigma y^2 - (\Sigma y)^2}}$$

$$= \frac{25(212) - (100)(125)}{\sqrt{6250 - (100)^2}\ \sqrt{12500 - (125)^2}} = 0.9646$$

Example : 8

Compute the coefficient of correlation from following data

x	10	14	18	22	26	30
y	18	12	24	6	30	36

Solution :

$$\bar{x} = \frac{120}{6} = 20, \ \bar{y} = \frac{126}{6} = 21$$

$$x = x - 20, y = y - 21, r = 0.60$$

Exercise No. 3

Example : 1

Calculate the correlation coefficient for the following data

x	2	4	5	6	8	11
y	18	12	10	8	7	6

Solution : $r = -0.92$

Example : 2

Calculate the correlation coefficient from the following data.

Maths (x)	23	28	42	17	26	35	29	37	16	46
Stat (y)	25	22	38	21	27	39	24	32	18	44

Solution : $x = x - 35, y = y - 39, r = 0.927$

3.14 Examples on Lines of Regressions :

The line of best fit for the given distribution is called the line of regression

There are 02 such lines

(i) Line of regression of y on x :

$$y - \bar{y} = r\left(\frac{\sigma_y}{\sigma_x}\right)(x - \bar{x})$$

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$y = (b_{yx})\, x + C$$

Where $\bar{x}$ = mean value of x, $\bar{y}$ = mean value of y

$$b_{yx} = r\left(\frac{\sigma_y}{\sigma_x}\right) = \text{regression coefficient of y on x}$$

and r is the correlation coefficient of x and y

(ii) Line of regression of x on y :

$$x - \bar{x} = r\left(\frac{\sigma_x}{\sigma_y}\right)(y - \bar{y})$$

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

$$x = (b_{xy})\, y + k$$

Where $b_{xy} = \left(r\,\frac{\sigma_x}{\sigma_y}\right)$ = regression coefficient of y on x

Note :

1. $(b_{xy})(b_{yx}) = \left(r\,\frac{\sigma_x}{\sigma_y}\right)\left(r\,\frac{\sigma_y}{\sigma_x}\right) r^2$

   $\therefore \quad r^2 = (b_{xy})(b_{yx})$
2. The correlation coefficient r is geometric mean of two regression coefficients.
3. The point – $(\bar{x},\bar{y})$ always lies on lines of regression.
4. $b_{yx} = \left(r\,\frac{\sigma_x}{\sigma_y}\right) = (r)\left(\frac{\sigma_y}{\sigma_x}\right) = \left(\frac{\text{cov}(x, y)}{\sigma_x\,\sigma_y}\right)\left(\frac{\sigma_y}{\sigma_x}\right) = \frac{\text{cov}(x, y)}{(\sigma_x)^2}$
5. $b_{xy}\,\frac{r\,\sigma_x}{\sigma_y} = \frac{\text{cov}(x, y)}{\sigma_x\,\sigma_y} \cdot \frac{\sigma_x}{\sigma_y} = \frac{\text{cov}(x, y)}{(\sigma_y)^2}$

Illustrative Examples

Example : 1

Two regressions equations of the variables x and y are given by

$$x = 19.13 - 0.87y \text{ and}$$

$$y = 11.64 - 0.50\,x$$

Find $\bar{x},\bar{y}$ and correlation coefficient

Solution :

We know, $(\bar{x},\bar{y})$ satisfies the line of regression

$$\bar{x} = 19.13 - 0.87\,\bar{y}$$

$$\bar{y} = 11.64 - 0.50\,\bar{x}$$

Solving we get $\bar{x} = 15.79$, $\bar{y} = 3.74$

We have $b_{xy} = -0.87$, $b_{yx} = -0.50$

We know, $r^2 = (b_{xy})(b_{yx})$

$$r^2 = (-0.87)(-0.50) = 0.43$$

$$\boxed{r = \underline{+}\,0.66}$$

Example : 2

The equation of lines of regression are

$$2x = 8 - 3y,\ 2y = 5 - x$$

Hence find $\overline{x}, \overline{y}$ and r

Solution :

$$\overline{x} = 1,\ b_{xy} = -\frac{3}{2},\ \overline{y} = 2,\ b_{yx} = -\frac{1}{2},\ r = 0.866.$$

Example : 3

The equations of lines of regression are

$$x - 0.9075\,y + 41.1375 = 0$$

$$0.48\,x - y + 67.72 = 0$$

Then find $\overline{x}, \overline{y}$, r

Solution :

$$\overline{x} = 36,\ \overline{y} = 85,\ r = 0.66$$

Example : 4

Find the correlation coefficient between x and y if two lines of regression are

$$2x - 9y + 6 = 0,\ x - 2y + 1 = 0$$

Solution :

Let the line of regression of x on y is $2x - 9y + 6 = 0$

$$\therefore \quad x = \left(\frac{9}{2}\right) y - 3$$

$$\therefore \quad b_{xy} = \frac{9}{2}$$

Let the line of regression of y on x is $x - 2y + 1 = 0$

$$\therefore \quad y = \frac{1}{2}x + \frac{1}{2}$$

$$\therefore \quad b_{xy} = \frac{1}{2}$$

We know, $r^2 = (b_{xy})(b_{yx})$

$$= \left(\frac{9}{2}\right)\left(\frac{1}{2}\right) = \frac{9}{4}$$

$$r = \pm\frac{3}{2}, > 1, \text{ which is wrong as } -1 \le r \le 1$$

Hence the selection of regression lines was wrong.

Let line of regression of x on y is $x - 2y + 1 = 0$

$\therefore \quad x = 2y - 1$

$\therefore \quad b_{xy} = 2$

Let line of regression of y on x is $2x - 9y + 6 = 0$

$\therefore \quad y = \frac{2}{9}x + \frac{2}{3}$

$\therefore \quad \sigma_{yx} = \frac{2}{9}$

$\therefore \quad r^2 = (b_{xy})(b_{yx})$

$\quad = (2)\left(\frac{2}{9}\right) = \frac{4}{9}$

$\therefore \quad \boxed{r = \pm\frac{2}{3}}$

Example : 5

If $\bar{x} = 36$, $\bar{y} = 85$, $\sigma_x = 11$, $\sigma_y = 8$, $r = 0.66$

Then find the lines of regression of x on y and y on x.

Solution :

1. Line x on y is $x - \bar{x} = r\left(\frac{\sigma_x}{\sigma_y}\right)(y - \bar{y})$

$$x - 36 = (0.66)\left(\frac{11}{8}\right)(y - 85)$$

$\therefore \quad \boxed{x = -41.1375 + 0.9075\, y}$

2. Line of y on x is $y - \bar{y} = r\left(\frac{\sigma_y}{\sigma_x}\right)(x - \bar{x})$

$$y - 85 = (0.66)\left(\frac{8}{11}\right)(x - 36)$$

$\therefore \quad \boxed{y = 67.72 + 0.48\, x}$

Example : 6

From the following data, obtain

(i) two regression lines (ii) correlation coefficient (iii) value of y at x = 6.2

x	1	2	3	4	5	6	7	8	9

y	9	8	10	12	11	13	14	16	15

Solution :

$$\bar{x} = \frac{1+2+3+4+5+6+7+8+9}{9} = \frac{45}{9} = 5,$$

$$\bar{y} = \frac{9+8+10+12+11+13+14+16+15}{9} = \frac{108}{9} = 12$$

Construct a table

x	x = x - 5	x^2	y	y = y - 12	y^2	xy
1	- 4	16	9	- 3	09	12
2	- 3	9	8	- 4	16	12
3	- 2	4	10	- 2	4	4
4	- 1	1	12	0	0	0
5	0	0	11	- 1	1	0
6	1	1	13	1	1	1
7	2	4	14	2	4	8
8	3	9	16	4	16	12
9	4	16	15	3	9	12
Σ -	0	60	-	0	60	57

i) We know $r = \dfrac{n\Sigma xy - (\Sigma x)(\Sigma y)}{\sqrt{n\Sigma x^2 - (\Sigma x)^2}\sqrt{n\Sigma y^2 - (\Sigma y)^2}} = \dfrac{9(57)}{\sqrt{9(60)}\sqrt{9(60)}} = \dfrac{57}{60} = 0.9$

ii) $\sigma_x = \sqrt{\dfrac{\Sigma y^2}{n} - \left(\dfrac{\Sigma y_2}{n}\right)} = \sqrt{\dfrac{60}{9} - 0} = \sqrt{\dfrac{60}{9}}$

iii) $\sigma_y = \sqrt{\dfrac{\Sigma y^2}{n} - \left(\dfrac{\Sigma y}{n}\right)^2} = \sqrt{\dfrac{60}{9}}$

iv) Regression line of y on x is

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

$$y - 12 = (0.95)\left(\frac{\sqrt{6019}}{\sqrt{6019}}\right)(x - 5)$$

$$\therefore \quad y = (0.95)\,x + 7.25$$

v) Regression line of x on y is

$$x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y})$$

$$x - 5 = (0.95)\left(\frac{\sqrt{6019}}{\sqrt{6019}}\right)(y - 12)$$

$$\therefore \quad x = (0.95)\,y - 6.4$$

vi) $y(6.2) = (0.95)(6.2) + 7.25 = 13.14$

Example : 7

From the following data, obtain the regression lines

x	6	2	10	4	8
y	9	11	5	8	7

Hence find x, y at y = 10 and x = 20 respectively.

Solution :

$$\bar{x} = \frac{30}{5} = 6,\ \bar{y} = \frac{40}{5} = 8$$

Let, $x = x - 6$, $y = y - 8$, $\Sigma x = 0$, $\Sigma y = 0$, $\Sigma x^2 = 40$, $\Sigma y^2 = 20$, $\Sigma xy = -26$

$$r = \frac{n\Sigma xy - (\Sigma x)(\Sigma y)}{\sqrt{n\Sigma x^2 - (\Sigma x)^2}\sqrt{n\Sigma y^2 - (\Sigma y)^2}} = \frac{5(-26)}{\sqrt{5(40)}\sqrt{5(20)}}$$

$$= \frac{-26}{\sqrt{800}} = \frac{-26}{28.28} = -0.91$$

$$b_{xy} = r\frac{\sigma_x}{\sigma_y} = -1.3,\ b_{yx} = r\frac{\sigma_y}{\sigma_x} = -0.65$$

i) Regression line of y on x is $y - 8 = (-0.65)(x - 6)$

$$\boxed{y = 11.9 - 0.65x}$$

$$\therefore \boxed{y(20) = 1.1}$$

ii) Regression line of x on y is $x - 5 = (-1.3)(y - 8)$

$$\boxed{x = 16.4 - 1.3y}$$

$$\therefore \quad x(10) = 16.4 - 13$$

$$\therefore \boxed{x(10) = 3.4}$$

Example : 8

Calculate the regression lines of x on y and y on x if following table gives the scores in aptitude test and productivity of 10 workers selected at random.

Score	60	62	65	70	72	48	53	73	65	82
Productivity	68	60	62	80	85	40	52	62	60	81

Solution :

We have $\bar{x} = \frac{650}{10} = 65, \bar{y} = \frac{650}{10} = 65$

Let, x = x – 65, y = y – 65, $\Sigma x = 0$, $\Sigma y = 0$, $\Sigma x^2 = 894$, $\Sigma y^2 = 1752$, $\Sigma xy = 1044$

$$b_{xy} = r\frac{\sigma_x}{\sigma_y} = \frac{\Sigma xy}{\sigma_x\,\sigma_y}\,\frac{\sigma_x}{\sigma_y} = \frac{\Sigma xy}{\sigma_y^2} = \frac{1044}{1752} = 0.596$$

$$b_{yx} = r\frac{\sigma_y}{\sigma_y} = \frac{\Sigma xy}{\sigma_x^2} = \frac{1044}{894} = 1.168$$

∴ Regression of line x on y is x = 26.26 + 0.596 y

∴ Regression of line y on x is y = - 10.92 + 1.168

Example : 9

Obtain the regression lines y on x and y at x = 70 from the following table

x	40	50	38	60	65	50	35
y	38	60	55	70	60	48	30

Solution :

We have $\bar{x} = \frac{\Sigma x}{n} = \frac{338}{7} = 48.29,$

$$\bar{y} = \frac{\Sigma y}{n} = \frac{361}{7} = 51.57$$

Let, x = x – 48, y = y – 50, $\Sigma x^2 = 774$, $\Sigma y^2 = 1173$, $\Sigma xy = 732$, $\Sigma x = 2$, $\Sigma y = 11$,

$$b_{yx} = r\left(\frac{\sigma_y}{\sigma_x}\right) = \frac{n\Sigma xy - (\Sigma x)(\Sigma y)}{n\Sigma x^2 - (\Sigma x)^2} = \frac{7(732) - (2)(11)}{7(774) - (2)^2} = 0.942$$

Regression of y on x is $y - \bar{y} = b_{yx}(x - \bar{x})$

$$y - 51.57 = (0.942)(x - 48.29)$$

$$\therefore \quad \boxed{y = (0.942)x + 6.08}$$

$$\therefore \quad y(x = 70) = (0.942)(70) + 6.08 = 65.94 + 6.08 = 7202 = 72.02$$

Exercise No. 4

Ex. 1 Find $\bar{x}$, $\bar{y}$ and r from the following lines of regression.

i) y = 0.516x + 33.73, x = 0.512y + 32.52

[**Ans:** r = 0.514]

ii) y = 0.516x + 33.73, x = 0.512y + 32.52

[**Ans:** $\bar{x}$ = 67.6, $\bar{y}$ = 68.61]

iii) $x = 0.4y + 6.4$, $y = - 0.6x + 406$

[**Ans:** $\bar{x} = 6$, $\bar{y} = 1$, $r = - 0.49$]

iv) $x = 19.17 – 0.87y$, $y = 11.64 – 0.50x$

[**Ans:** $\bar{x} = 15.935$, $\bar{y} = 3.67$, $r = - 0.435$]

v) $3x + 2y = 26$, $6x + y = 31$

[**Ans:** $\bar{x}= 4$, $\bar{y}= 7$, $r = - 0.5$]

vi) $20x – 9y – 107 = 0$, $4x – 5y + 33 = 0$

[**Ans:** $\bar{x} = 13$, $\bar{y} = 17$, r]

vii) $7x – 16y + 9 = 0$, $5y – 4x – 3 = 0$

[**Ans:** $\bar{x} = - \frac{3}{29}$, $\bar{y} = \frac{15}{19}$, $r = \frac{3}{9}$]

Ex. 2 If two regression coefficient are 0.8, 0.4, what would be the correlation coefficient ?
[**Hint:** $r^2 = (b_{xy})\ (b_{yx}) = (0.8)\ (0.4) = 0.32 \Rightarrow r = \pm\ 0.56$]

3.15 Examples on least square method for curve fitting :

1. The normal equations for fitting the line $y = ax + b$ for the data having "n" no. of observations are

$$\Sigma y = a\,\Sigma x + nb$$
$$\Sigma xy = a\,\Sigma x^2 + b\Sigma x$$

2. The normal equations for fitting the parabolic curve
$y = ax^2 + bx + e$, for the data having "n" observations are

$$\Sigma y = a\,\Sigma x^2 + bx + ne$$
$$\Sigma xy = a\,\Sigma x^3 + b\Sigma x^2 + c\Sigma x$$
$$\Sigma x^2 y = a\Sigma x^4 + b\Sigma x^3 + c\Sigma x^2$$

Example : 1

Fit a line $y = ax + b$ to the data

x	- 1	0	1	2	3	4	6
y	1	3	5	7	9	11	15

Solution :

Firstly construct a table as per requirements of normal equation.

x	y	x^2	xy

0 - 1	1	1	- 1
0	3	0	0
1	5	1	5
2	7	4	14
3	9	9	27
4	11	16	44
6	15	36	90
15	51	67	179

We have $n = 7$

Normal equations are

$$\Sigma y = a\Sigma x + nb$$

$$\Sigma xy = a\Sigma x^2 + b\Sigma x$$

$\therefore$ $51 = 159 + 7b$

$179 = 679 + 15b$

$225\,a + 105\,b = 765$

$469\,a + 105\,b = 1253$

$\therefore$ $2449 = 488$

$\therefore$ $\boxed{a = 2}$

Secondly $7b = 51 - 15a$

$= 51 - 30$

$= 21$

$\therefore$ $\boxed{b = 3}$

$\therefore$ Required line is $\boxed{y = 2x + 3}$

Example : 2

Fit a parabolic curve $y = a + bx + cx^2$ to the given data.

x	- 2	- 1	0	1	2	3
y	- 1	2	3	2	- 1	- 6

Solution :

The normal equations to fit a parabola to the given data are

$$\Sigma y = na + b\Sigma x + c\Sigma x^2$$

$$\Sigma xy = a\,\Sigma x + b\Sigma x^2 + c\Sigma x^3$$
$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^3$$

Construct a table

x	**y**	x^2	x^3	x^4	**xy**	x^2y
- 2	- 1	4	- 8	16	2	- 4
- 1	2	1	- 1	1	- 2	2
0	3	0	0	0	0	0
1	2	1	1	1	2	2
2	- 1	4	8	16	- 2	- 4
3	- 6	9	27	81	- 18	- 54
$\Sigma = 3$	- 1	19	27	115	- 18	- 58

We get

$$6a + 3b + 19c = -1$$
$$3a + 19b + 27c = -18$$
$$19a + 27b + 115c = -58$$

Solving we get, $a = 3, b = 0, c = -1$

$\therefore$ Required parabola is $\boxed{y = 3 - x^2}$

Exercise No. 5

Example : 1

Fit a line to the following data

x	1	2	3	4
y	- 1	4	9	14

Ans : $y = 5x - 6$

Example : 2

Fit a line $p = mw + c$ to the following data

w	50	70	100	120
p	12	15	21	25

Ans. : $p = 2.2785 + (0.1879)\,w$

Example : 3

Fit a line to the given data

x	0	1	3	6	8
y	1	3	2	5	4

Ans. : $y = 1.6 + (0.38)x$

Example : 4

Fit a parabolic curve $y = a + bx + cx^2$ to the following data

x	- 3	- 2	- 1	0	1	2	3
y	1.1	1.3	1.6	2.0	2.7	3.4	4.1

Ans. : Parabolic curve is $y = 2.07 + (0.51)x + (0.06)x^2$

Example : 5

Fit a parabolic curve to the following data

x	- 2	- 1	0	1	2	3	4
y	7	1	- 1	1	7	17	31

Ans. : $y = 2x^2 – 1$

Example : 6

Fit parabolic curve to the following data

(i)

x	0	1	2	3	4	5
y	1	3	7	13	21	31

Ans. : $y = x^2 + x + 1$

(ii)

x	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y	1.1	1.3	1.6	2.0	2.7	3.4	4.1

Ans. : $y = 1.04 – 198x + 0.244x^2$

3.16 Curve Fitting : (Least Square Approximation):

Curve fitting is the process of constructing a curve, or mathematical function, that has the best fit to a series of data points, possibly subject to constraints.

Let (x_i, y_i), i = 1, 2, 3, …., n be a given set of n pairs of values. x being independent variable and y is dependent variable.

i.e. for equation $y = f(x)$

Fitting of curves to a set of numerical data is of ..,..,.., considerable importance i.e. theoretical as well as practical importance in the study of correlation and regression.

Fitting of a straight line:

Let us consider the fitting of a straight line

$y = a + bx$ to a set of n pts (x_i, y_i) for $i = 1, 2, 3, ..., n$

To determine a and b the normal equations are

$$\Sigma y_i = na + b\Sigma x_i$$

$$\Sigma x_i y_i = a\Sigma x_i + b\Sigma x_i^2$$

Fitting of second degree parabola :

Let $y = a + bx + c_x^2$ …(I)

be the II[nd] degree parabola of best fit to set of n pts.

$(x_i y_i), i = 1, 2, …., n.$

To determine a, b, and c the normal equations are

$$\Sigma y_i = na + b\Sigma x_i + c\Sigma x_i^2$$

$$\Sigma x_i y_i = a\Sigma x + b\Sigma x_i^2 + c\Sigma x_i^3$$

$$\Sigma x_i^2 y_i = a\Sigma x_i^2 + b\Sigma x_i^3 + c\Sigma x_i^4$$

Example : 1

Fit a straight line to the following data.

x	1	2	3	4	6	8
y	2.4	3	3.6	4	5	6

Solution :

Let the straight line be $y = a + bx$

Then the normal equations are $\Sigma y_i = 6a + b\Sigma x_i$

$$\Sigma x_i y_i = a\Sigma x_i + b\Sigma x_i^2 \quad ….(i)$$

The values of Σx_i, Σy_i etc. are calculated below :

x	**y**	x^2	**xy**
1	2.4	1	2.4
2	3	4	6
3	3.6	9	10.8
4	4	16	16
6	5	36	30
8	6	64	48
$\Sigma x_i = 24$	$\Sigma y_i = 24$	$\Sigma x_i^2 = 130$	$\Sigma x_i y_i = 113.2$

The equations (i) become $24 = 6a + 24b$ and

$$113.2 = 24a + 130b.$$

i.e. $$4 = a + 4b \quad \text{....(ii)}$$

$$113.2 = 24a + 130b \quad \text{....(iii)}$$

Multiplying (ii) by 24 and subtracting from (iii),

We get $a = 1.976$ and $b = 0.506$.

Hence the required line of best fit is

$$y = 1.976 + 0.506x$$

Example : 2

Fit a straight line to the following data:

x :	6	7	7	8	8	8	9	9	10
y :	5	5	4	5	4	3	4	3	3

Solution :

Let the straight line be $y = a + bx$

Then the normal equations are $\Sigma y_i = 9a + b\Sigma x_i$

$$\Sigma x_i y_i = a\Sigma x_i + b\Sigma x_i^2 \quad \text{....(i)}$$

The values are Σx_i, Σy_i etc. are calculated below.

x	**y**	**x^2**	**xy**
6	5	36	30
7	5	49	35
7	4	49	28
8	5	64	40
8	4	64	32
8	3	64	24
9	4	81	36
9	3	81	27
10	3	100	30
$\Sigma x_i = 72$	$\Sigma y_i = 36$	$\Sigma x_i^2 = 588$	$\Sigma x_i y_i = 282$

The equations (i) become $36 = 9a + 72b$ and

$$282 = 72a + 588b$$

i.e., $$4 = a + 8b \quad \text{....(ii)}$$

$$47 = 12a + 98b \quad \text{....(iii)}$$

Multiplying (ii) by 12 and subtracting from (iii),

we get $b = -0.5$ and $a = 8$

Hence the required line of best fit is

$$y = 8 - 0.5x.$$

Example : 3

Fit a straight line to the following data :

x :	12	15	21	25
y :	50	70	100	120

Solution :

Let the straight line be $y = a + bx$

Then the normal equations are $\Sigma y_i = 4a + b\Sigma x_i$

$$\Sigma x_i y_i = a\Sigma x_i + b\Sigma x_i^2 \quad \text{....(i)}$$

The values of Σx_i, Σy_i etc. are calculated below.

x	**y**	$\mathbf{x^2}$	**xy**
12	50	144	600
15	70	225	1050
21	100	441	2160
25	120	625	3000
$\Sigma x_i = 73$	$\Sigma y_i = 340$	$\Sigma x_i^2 = 1435$	$\Sigma x_i y_i = 6750$

The equations (i) become $340 = 40 + 73b$ and,

$$6750 = 73a + 1435b$$

i.e.,

$$340 = 40 + 73b \quad \text{....(ii)}$$

$$6750 = 73a + 1435b \quad \text{....(iii)}$$

Multiplying (ii) by $\frac{73}{4}$ and subtracting from (iii),

We get $a = -11.7998$ and $b = 5.3041$.

Hence the required line of best fit is

$$y = -11.7998 + 5.3041x.$$

Note :

For the sake of convenience it is sometime advisable to change the origin and scale with the substitutions $X = (x - A)/h$ and $Y = (y - B)/h$, where A and B are the assumed means (or middle values) of x and y respectively and h is the width of the interval.

Example : 4

Fit a second degree parabola to the following data

x :	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y :	1.1	1.3	1.6	2.0	2.7	3.4	4.1

Solution :

Let the second degree parabola of fit be

$$y = a + bx + cx^2$$

Here we take $X = \dfrac{x - \bar{x}}{h}$

Where $\bar{x} = 2.5, \; h = 0.5$

$\therefore$ $X = \dfrac{x - 2.5}{0.5}$

So the parabola of fit $y = a + bx + cx^2$ becomes

$$y = a + bx + cx^2 \qquad \ldots(i)$$

The normal equations are

$$\Sigma y_i = na + b\Sigma x_i + c\Sigma x_i^2$$

$$\Sigma x_i y_i = a\Sigma x_i + b\Sigma x_i^2 + c\Sigma x_i^3$$

$$\Sigma x_i^2 y_i = a\Sigma x_i^2 + b\Sigma x_i^3 + c\Sigma x_i^4 \qquad \ldots(ii)$$

The values of Σx_i, Σy_i etc. are calculated below :

x	**y**	$X = \dfrac{x - 2.5}{0.5}$	x^2	x^3	x^4	**xy**	x^2y
1.0	1.1	- 3	9	- 27	81	- 3.3	9.9
1.5	1.3	- 2	4	- 8	16	- 2.6	5.2
2.0	1.6	- 1	1	- 1	1	- 1.6	1.6
2.5	2.0	0	0	0	0	0	0
3.0	2.7	1	1	1	1	2.7	2.7
3.5	3.4	2	4	8	16	6.8	13.6
4.0	4.1	3	9	27	81	12.3	36.9
	$\Sigma y_i = 16.2$	$\Sigma x_i = 0$	$\Sigma x_i^2 = 28$	$\Sigma x_i^3 = 0$	$\Sigma x_i^4 = 196$	$\Sigma x_i y_i = 14.3$	69.9

The equations (iii) become

$$16.2 = 7a + 28c \qquad \ldots(iii)$$

$$14.3 = 28b \qquad \ldots(iv)$$

$$69.9 = 28a + 196 \qquad \ldots(v)$$

From (iv), $b = 0.511$

Multiplying (iii) by 4 and subtracting from (v)

we get $a = 2.07, c = 0.061$

Equation (i) becomes,

$$y = a + b\left(\frac{x - 2.5}{0.5}\right) + c\left(\frac{x - 2.5}{0.5}\right)^2$$

$$y = a + b(2x - 5) + c(2x - 5)^2$$

$$y = 2.07 + 0.511(2x - 5) + 0.061(2x - 5)^2$$

$$y = 2.07 + 1.022x - 2.555 + 0.244x^2 - 1.22x + 1.525$$

$$y = 1.04 - 0.198x + 0.244x^2$$

which is required equation of parabola.

Example : 5

Fit a second degree parabola to the following data

x :	0	1	2	3	4
y :	1	1.8	1.3	2.5	6.3

Solution :

Let the second degree parabola of fit be

$$y = a + bx + cx^2$$

we take $X = \frac{x - \bar{x}}{h}$

Where $\bar{x} = 2, \; h = 1$

$\therefore$ $X = x - 2$

So the parabola of fit $y = a + bx + cx^2$ becomes

$$y = a + bx + cx^2 \quad \ldots(i)$$

The normal equations are

$$\Sigma y_i = na + b\Sigma x_i + c\Sigma x_i^2$$

$$\Sigma x_i y_i = a\Sigma x_i + b\Sigma x_i^2 + c\Sigma x_i^3$$

$$\Sigma x_i^2 y_i = a\Sigma x_i^2 + b\Sigma x_i^3 + c\Sigma x_i^4 \quad \ldots(ii)$$

The values of Σx_i, Σy_i etc. are calculated below.

x	**y**	**X = x - 2**	**x^2**	**x^3**	**x^4**	**xy**	**x^2y**
0	1	- 2	4	- 8	16	- 2	4
1	1.8	- 1	1	- 1	1	- 1.8	1.8

2	1.3	0	0	0	0	0	0
3	2.5	1	1	1	1	2.5	2.5
4	6.3	2	4	8	16	12.6	25.2
	$\Sigma y_i = 12.9$	$\Sigma x_i = 0$	$\Sigma x_i^2 = 10$	$\Sigma x_i^3 = 0$	$\Sigma x_i^4 = 34$	$\Sigma x_i y_i = 11.3$	$\Sigma x_i^2 y_i = 33.5$

The equations (ii) become

$$12.9 = 5a + 10c \quad \ldots(iii)$$

$$11.3 = 10b \quad \ldots(iv)$$

$$33.5 = 10a + 34c \quad \ldots(v)$$

From (iv), $b = 1.13$

Multiplying (iii) by 2 and subtracting from (v)

we get, $a = 1.48, c = 0.55$

Equation (i) becomes,

$$y = a + b(x-2) + c(x-2)^2$$

$$y = 1.48 + 1.13(x-2) + 0.55(x-2)^2$$

$$y = 1.48 + 1.13x - 2.26 + 0.55x^2 - 2.2x + 2.2$$

$$y = 1.42 - 1.07x + 0.55x^2$$

which is required equation of parabola.

Example : 6

Following is the data given for values of x and y.

Fit a second degree polynomial of the type $ax^2 + bx + c$, where a, b, c are constants.

x :	-3	-2	-1	0	1	2	3
y :	12	4	1	2	7	15	30

Solution :

Let the second degree parabola of fit be

$$y = ax^2 + bx + c \quad \ldots(i)$$

The normal equations are

$$\Sigma y_i = cn + b\Sigma x_i + a\Sigma x_i^2$$

$$\Sigma x_i y_i = c\Sigma x_i + b\Sigma x_i^2 + a\Sigma x_i^3$$

$$\Sigma x_i^2 y_i = c\Sigma x_i^2 + b\Sigma x_i^3 + a\Sigma x_i^4 \quad \ldots(ii)$$

The values of Σx_i, Σy_i etc. are calculated below :

x	**y**	**x^2**	**x^3**	**x^4**	**xy**	**x^2y**
- 3	12	9	- 27	81	- 36	108

- 2	4	4	- 8	16	- 8	16
- 1	1	1	- 1	1	- 1	1
0	2	0	0	0	0	0
1	7	1	1	1	7	7
2	15	4	8	16	30	60
3	30	9	27	81	90	270
$\Sigma x_i = 0$	$\Sigma y_i = 71$	$\Sigma x_i^2 = 28$	$\Sigma x_i^3 = 0$	$\Sigma x_i^4 = 196$	$\Sigma x_i y_i = 82$	$\Sigma x_i^2 y_i = 462$

The equations (ii) become $71 = 7c + 28a$ …(iii)

$82 = 28b$ …(iv)

$462 = 28c + 196a$ …(v)

From (iv), $b = 2.92$

Multiplying (iii) by 4 and subtracting from (v)

we get, $a = 2.14, c = 1.52$

$\therefore$ $a = 2.14, b = 2.92, c = 1.52$

Equation (i) becomes,

$$y = 2.14x^2 + 2.92x + 1.52$$

which is required equation of parabola.

Example : 7

Fit a straight line to the following data.

x :	0	1	2	3
y :	2	5	8	11

Solution :

Let the straight line be $y = a + bx$

Then the normal equations are $\Sigma y_i = 4a + b\Sigma x_i$

$\Sigma x_i y_i = a\Sigma x_i + b\Sigma x_i^2$ …(i)

The values of Σx_i, Σy_i etc. are calculated below.

x	**y**	**x^2**	**xy**
0	2	0	0
1	5	1	5
2	8	4	16

3	11	9	33
$\Sigma x_i = 6$	$\Sigma y_i = 26$	$\Sigma x_i^2 = 14$	$\Sigma x_i y_i = 54$

The equations (i) become $26 = 4a + 6b$

and $54 = 6a + 14b$

i.e. $13 = 2a + 3b$ …(ii)

$27 = 3a + 7b$ …(iii)

Multiplying (ii) by 3 and (iii) by 2, then subtracting (ii) from (iii)

we get, $a = 2$, and $b = 3$

Hence the required line of best fit is

$$y = 2 + 3x$$

Example : 8

Fit a straight line of the type $y = a + bx$ to the following data. **Nov – 2017**

x :	0	5	10	15	20	25
y :	12	15	17	22	24	30

Solution :

Let the straight line be $y = a + bx$

Then the normal equations are $\Sigma y_i = 6a + b\Sigma x_i$

$\Sigma x_i y_i = a\Sigma x_i + b\Sigma x_i^2$ …(i)

The values of Σx_i, Σy_i etc. are calculated below :

x	**y**	**x^2**	**xy**
0	12	0	0
5	15	25	75
10	17	100	170
15	22	225	330
20	24	400	480
25	30	625	750
$\Sigma x_i = 75$	$\Sigma y_i = 120$	$\Sigma x_i^2 = 1375$	$\Sigma x_i y_i = 1805$

The equations (i) become $120 = 6a + 75b$

and $1805 = 75a + 1375b$

i.e. $40 = 2a + 25b$ …(ii)

$361 = 15a + 275b$ …(iii)

Multiplying (ii) by 11 and subtracting from (iii)

we get, $a = 11.2857$, and $b = 0.6971$

Hence the required line of best fit is

$$y = 11.2857 + 0.6971x$$

Example : 9

Fit a second degree parabola to the following data.

x :	0	1	2	3	4
y :	1	0	3	10	21

Solution :

We want to estimate an equation of type $y = a + bx + cx^2$

Here we take $x = \frac{x - \overline{x}}{h}$

Where $\overline{x} = 2, \; h = 1$

Example : 10

Fit a second degree parabola to the function 2^x at points 0, 1, 2, 3, 4.

Solution :

Here $y = 2^x$ at 0, 1, 2, 3, 4.

x :	0	1	2	3	4
y :	1	2	4	8	16

Let the parabola of fit be $y = a + bx + cx^2$

We take $x = \frac{x - \overline{x}}{h}$

Where $\overline{x} = 2, \; h = 1$

$\therefore$ $x = x - 2$

So the parabola of fit $y = a + bx + cx^2$ becomes

$$y = a + bx + cx^2 \quad \ldots(i)$$

The normal equations are

$$\Sigma y_i = na + b\Sigma x_i + c\Sigma x_i^2$$
$$\Sigma x_i\, y_i = a\Sigma x_i + b\Sigma x_i^2 + c\Sigma x_i^3$$
$$\Sigma x_i^2\, y_i = a\Sigma x_i^2 + b\Sigma x_i^3 + c\Sigma x_i^4 \quad \ldots(ii)$$

The values of Σx_i, Σy_i etc. are calculated below.

x	y	X = x - 2	x^2	x^3	x^4	xy	x^2y
0	1	- 2	4	- 8	16	- 2	4
1	2	- 1	1	- 1	1	- 2	2
2	4	0	0	0	0	0	0
3	8	1	1	1	1	8	8
4	16	2	4	8	16	32	64
	$\Sigma y_i = 31$	$\Sigma x_i = 0$	$\Sigma x_i^2 = 10$	$\Sigma x_i^3 = 0$	$\Sigma x_i^4 = 34$	$\Sigma x_i y_i = 36$	$\Sigma x_i^2 y_i = 78$

The equations (ii) become

$$31 = 5a + 10c \quad \text{...(iii)}$$

$$36 = 10b \quad \text{...(iv)}$$

$$78 = 10a + 34c \quad \text{...(v)}$$

From (iv), $b = 3.6$

$$x = x - 2, (h = 1)$$

Let, x = x- 2 so that the parabola of fit $y = a + bx + cx^2$

becomes $y = a + bx + cx^2$ ...(i)

The normal equations are

$$\Sigma y_i = na + b\Sigma x_i + c\Sigma x_i^2$$

$$\Sigma x_i y_i = a\Sigma x_i + b\Sigma x_i^2 + c\Sigma x_i^3$$

$$\Sigma x_i^2 y_i = a\Sigma x_i^2 + b\Sigma x_i^3 + c\Sigma x_i^4 \quad \text{...(ii)}$$

The values of Σx_i, Σy_i etc. are calculated below.

x	y	X = x - 2	x^2	x^3	x^4	xy	x^2y
0	1	- 2	4	- 8	16	- 2	4
1	0	- 1	1	- 1	1	0	0
2	3	0	0	0	0	0	0
3	10	1	1	1	1	10	10
4	21	2	4	8	16	42	84
	$\Sigma y_i = 35$	$\Sigma x_i = 0$	$\Sigma x_i^2 = 10$	$\Sigma x_i^3 = 0$	$\Sigma x_i^4 = 34$	$\Sigma x_i y_i = 50$	$\Sigma x_i^2 y_i = 98$

The equations (ii) become

$$35 = 5a + 10c \quad \text{...(iii)}$$

$$50 = 10b \quad \text{...(iv)}$$

$$98 = 10a + 34c \quad \text{...(v)}$$

From (iv), $b = 5$

Multiplying (iii) by 2 and subtracting from (v) we get, $a = 3, c = 2$

$\therefore$ Equation (i) becomes,

$$y = a + b(x - 2) + c(x - 2)^2$$
$$y = 3 + 5(x - 2) + 2(x - 2)^2$$
$$y = 3 + 5x - 10 + 2x^2 - 8x + 8$$
$$y = 1 - 3x + 2x^2$$

which is required equation of parabola.

Multiplying (iii) by 2 and subtracting from (v)

we get, $c = 1.142, a = 3.916$

$\therefore$ Equation (i) becomes,

$$y = a + b(x - 2) + c(x - 2)^2$$
$$y = 3.916 + 3.6(x - 2) + 1.142(x - 2)^2$$
$$y = 3.916 + 3.6x - 7.2 + 1.142x^2 - 4.568x + 4.568$$
$$y = 1.284 - 0.968x + 1.142x^2$$

which is required equation of parabola.

Exercise No. 6

1. Fit a straight line to the following data

x :	0	1	2	3	4
y :	1	1.8	3.3	4.5	6.3

Ans. : $y = 0.72 + 1.33x$

2. Find the best values of a, b assuming that the following values of x, y are connected by the relation $y = a + bx$

x :	0	1	2	3	4
y :	1	2.9	4.8	6.7	8.6

Ans. : $a = 1, b = 1.9$

3. S. T. the line of fit to the following data is given by $y = 3.9 + 1.5x$.

x :	1	2	3	4	5
y :	5	7	9	10	11

4. Obtain the least squares line fit to the following data.

x :	0.2	0.4	0.6	0.8	1
y :	0.447	0.632	0.775	0.894	1

Ans. : $y = 0.3392 + 0.684x$

5. Fit a second degree parabola for the following data.

x :	-2	-1	0	1	2
y :	15	1	1	3	19

Ans. : $y = - 1.057 + x + 4.4285x^2$

6. Fit a second degree parabola to the following data.

x :	1989	1990	1991	1992	1993	1994	1995	1996	1997
y :	352	356	357	358	360	361	361	360	359

Ans. : $y = - 1062526.37 + 1066.37x – 0.267x^2$

7. Fit a straight line to the following data by least square method. **May – 2018**

x :	0	6	8	10	14	16	18	20
y :	3	12	15	18	24	27	30	33

Ans. : $y = 1.5a + 3$

8. By the method of least square, find the straight line that best fits the following data.

May – 2017

x :	1	2	3	4	5
y :	14	27	40	55	68

Ans. : $y = 13.6x$

9. Fit a straight line to the following data by least square method.

x :	1	2	3	4	5	6
y :	6	4	3	5	4	2

Ans. : $y = 5.7999 – 0.514x$

10. Fit a second degree parabola for the following data.

x :	1	2	3	4
y :	1.7	1.8	2.3	3.2

Ans. : $y = 2 – 0.5x + 0.2x^2$

11. Fit a least square straight line to the following data.

x :	2	7	9	1	5	12
y :	13	21	23	14	15	21

Ans. : $y = 12.45 + 0.8977x$

12. Fit a least square straight line to the following data.

x :	2	4	6	8	10	12
y :	1.8	1.5	1.4	1.1	1.1	0.9

Ans. : $y = 1.9 - 0.086x$

13. Fit a least square straight line to the following data.

x :	20	60	100	140	180	220	260	300	340	380
y :	0.18	0.37	0.35	0.78	0.56	0.75	1.18	1.36	1.17	1.65

Ans. : $y = 0.069 + 0.0038x$

14. Fit a straight line to the following data by least square method.

x :	1	3	4	6	8	9	11	14
y :	1	2	4	4	5	7	8	9

Ans. : $y = 0.5454 + 0.6363x$

15. In a study between the amount of rainfall and the quantity of air pollution removed the following data were collected.

Daily Rainfall in 0.01 cm(x)	4.3	4.5	5.9	5.6	6.1	5.2	3.8	2.1
Pollution Removed (mg/m^3)(y)	12.6	12.1	11.6	11.8	11.4	11.8	13.2	14.1

Obtain by the method of least square, a relation of the form $y = a + bx$ which best fits to these observations.

Ans. : $y = 15.49 - 0.675x$

16. If a curve of the form $x = ay^2 + by + c$ satisfies the data.

x :	- 6	- 8	- 4	6	22	44	72
y :	0	1	2	3	4	5	6

Find the best values of a, b, c.

Ans. : $a = 3, b = -5, c = -6$

17. Fit a parabola of the form $y = ax^2 + by + c$ to the following data using least square criteria.

x :	1	2	3	4	5	6	7
y :	- 5	- 2	5	16	31	50	73

Ans. : $y = 2x^2 - 3x - 4$

18. Values of x and y are tabulated as under.

x :	1	1.5	2.0	2.5
y :	25	56.2	100	156

Find the law of the form $x = ay^n$ to satisfy the given by data.

[Hint : $x = ay^n$, Taking logarithm we get

$$\log x = \log a + n \log y$$

$$\therefore \quad x = c + nY, \text{ where } \log x = X, \log a = C, \log y = Y$$

which is a straight line equation].

Ans. : $n = 0.5$, $c = - 0.6988$, $a = 0.2$, $x = 0.2\, y^{0.5}$

19. Find the best values of a, b, c assuming that the following values of x, y are connected by the relation $y = ax^2 + bx + c$

x :	1	2	3	4	5
y :	3.38	8.25	16.6	28.5	44

Ans. : $a = 1.772$, $b = - 0.383$, $c = 2.103$

20. Fit a parabola $y = a + bx + cx^2$ to the following data.

x :	1	2	3	4	5	6	7	8	9
y :	2	6	7	8	10	11	11	10	9

Ans. : $y = 7.4 + 0.85x + 0.1232x^2$

Illustrative Examples

Example : 1

Fit a straight line to the following data.

x :	o	1	2	3	4
y :	1	1.8	3.3	4.5	6.3

Solution :

Let the straight line to be fitted to the data by

$$y = a + bx. \text{ then the normal equations are.}$$

$$\Sigma y = na + b\Sigma x. \; ; \quad \Sigma xy = a\Sigma x + b\Sigma x^2$$

x	y	xy	x^2
0	1	0	0
1	1.8	1.8	1
2	3.3	6.6	4
3	4.5	13.5	9
4	6.3	25.2	16
10	16.9	47.1	30

Here $n = 5$

$$16.9 = 5a + 10b \qquad \text{...(i)}$$

$$47.1 = 10a + 30b \qquad \text{...(ii)}$$

Solving (i) and (ii) we get a = 0.72 b = 1.33

Hence the equation of the line of best fit is

$$y = 0.72 + 1.33x.$$

Example : 2

Fit a straight line to the following data treating y as independent variable.

x	1	2	3	4	5
y	5	7	9	10	11

Solution :

Let the line of best fit be.

$$X = a + by,$$

Then normal equations are

$$\Sigma x = na + b\Sigma y \ ; \ \ \Sigma xy = a\Sigma y + b\Sigma y^2$$

x	y	xy	y^2
1	5	5	25
2	7	14	49
3	9	27	81
4	10	40	100
5	11	55	121
15	42	141	376

$$15 = 5a + 42b \ \Rightarrow 5a = 42b - 15$$

$$141 = 42\,a + 376\,b \qquad a = \frac{42}{5}b - 3$$

$$141 = 42\left(\frac{42}{5}b - 3\right) + 376\,b$$

$$141 = \frac{1764}{5}b - 126 + 376\,b.$$

$$267 = \frac{3644}{5}b.\ b = \frac{267 \times 5}{3644}$$

$$b = \frac{1335}{3644} = 0.3663556 = 0.37$$

$$a = (8.4)\,(0.37) - 3 = 0.108 = 0.11$$

$$\boxed{x = 0.11 + 0.37\,y.}$$

Example : 3

Fit a second degree parabola to the following data.

x	1	2	3	4	5	6	7	8	9
y	2	6	7	8	10	11	11	10	9

$$\Sigma x = 45,\ \bar{x} = 5,\ \Sigma y = 74,\ \bar{y} = 8.22$$

Solution :

Let x = x – 5 and y = y – 7. and let the curve of best fit be

$$y = a + bx + cx^2.\text{ The normal equations are}$$

$$\Sigma y = na + b\Sigma x + c\Sigma x^2$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$

$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4.$$

x	**y**	**x**	**y**	**xy**	**x^2**	**x^3**	**x^4**
1	2	- 4	- 5	20	16	- 64	256
2	6	- 3	- 1	3	9	- 27	81
3	7	- 2	0	0	4	- 8	16
4	8	- 1	1	- 1	1	- 1	1
5	10	0	3	0	0	0	0
6	11	1	4	4	1	1	1
7	11	2	4	8	4	8	16
8	10	-3	3	-9	9	27	81

9	9	4	2	8	16	64	256
Σ = 46	74	0	11	51	60	0	708

$$11 = 9a + 6c$$

$$51 = 60b$$

$$-9 = 60a + 708c$$

$$a = 3, \quad b = 0.85$$

$$c = -0.27$$

Hence the curve of best fit is

$$y = 3 + 0.85x - 0.27x^2$$

$$y = 3 + 0.85(x - 5) + 0.27(x - 5)^2$$

$$y = 3 + 0.85x - 4.25 - 0.27x^2 + 2.7x - 6.75$$

$$y = -1 + 3.55x - 0.77x^2$$

Example : 4

Use method of least squares to fit y = mx + c to

x	0	1	2	3	4	5
y	-8	-5	-2	1	4	7

Solution :

Let the equation of st. line of best fit y = mx + c.

Normal equation form 'm' is

$$\Sigma xy = m\Sigma x^2 + c\Sigma x \quad \ldots(i)$$

$$\Sigma y = m\Sigma x + 6c \quad \ldots(ii)$$

6 = No. of absent we prepare the table.

x	**y**	x^2	**xy**
0	- 8	0	0
1	- 5	1	- 5
2	- 2	4	- 4
3	1	9	3
4	4	16	16
5	7	25	35

$$\Sigma x = 15, \Sigma y = -3, \Sigma x^2 = 55, \Sigma xy = 45.$$

Equation (i) and (ii) become

$$m(55) + c(15) = 45$$

$$m(15) + c(6) = -3$$

Solving the above equations simultaneously.

We get $m = 3$ and $c = -8$

Example : 5

If $x = ay^2 + by + c$, find a, b, c by least square method.

Solution : Let $x = ay^2 + by + c \rightarrow$ (i) Normal equations for a, b, c are

$\Sigma y^2x = a\Sigma y^4 + b\Sigma y^3 + c\Sigma y^2 \rightarrow$ (ii) $\Sigma y = 15, \Sigma x = 85$

$\Sigma yx = a\Sigma y^3 + b\Sigma y^2 + c\Sigma y \rightarrow$ (iii) $\Sigma yx = 345, \Sigma y^2 = 155$

$\Sigma x = a\Sigma y^2 + b\Sigma y + 5c \rightarrow$ (iv) $\Sigma y^2x = 1503$ $\Sigma y^3 = 225$

$\Sigma y^4 = 979$

y	x	yx	y^2	y^2x	y^3	y^4
1	3	3	1	3	1	1
2	6	12	4	24	8	16
3	13	39	9	1.17	27	81
4	24	96	16	384	64	256
5	39	195	25	975	125	625

$\therefore$ (i) (iii) (iv) become.

$$a(979) + b(225) + c(55) = 1503$$

$$a(225) + b(55) + c(15) = 345$$

$$a(55) + b(15) + c(5) = 85$$

Solving by cramer's rule a = 2, b = - 3, c = 4

$\therefore$ $x = 2x^2 - 3y + 4.$

3.17 Non – polynomial Approximatim :

In certain problems of science and engineering often we have to fit an empirical law, a non-polynomial function to data, the most used empirical laws are

(i) $y = a\,e^{bx}$ (ii) $y = ax^b$ (iii) $y = a(b^x)$

These formulas can be reduced to linear equations by taking logarithms and introducing new variables.

1. $y = a\,e^{bx}$ yields

   $\text{Log } y = \text{Log } a + bx$

   i.e. $Y = c_0 + c_1 x$

   where $Y = \text{Log } y$, $c_0 = \text{Log } a$, $c_1 = b$.

2. Similarly $y = ax^b$ yields

$\text{Log } y = \text{Log } a + b \text{ Log } x$

i.e. $Y = c_0 + c_1X$

where $Y = \text{Log}y$, $c_0 = \text{Log } a$, $x = \text{Log } x$, $c_1 = b$.

3. $y = ab^x$ yields

$\text{Log } y = \text{Log } a + x \text{ Log } b$

$Y = c_0 + c_1$

where $Y = \text{Log } y$, $c_0 = \text{Log } a$, $c_1 = \text{Log } b$.

By least square approximation we determine c_0, c_1 and then a and b are found.

Illustrative Examples

Example : 1

Determine a and b so that $y = a\, e^{dx}$ fits the data.

X :	1	2	3	4
Y :	7	11	17	27

Solution :

$$Y = a\, e^{dx} \text{ yields on having logarithm}$$

$$\text{Log } y = \text{Log } a + bx$$

$$Y = c_0 + c_1 x$$

We have $Y \text{ Log } y\ c_0 \text{ Log } a - c_1 = b$

Hence we have the table.

X :	1	2	3	4
Y :	1.96	2.40	2.63	3.30

The second line in this table is obtained by taking the logarithm of the entries in the original table.

Now $\Sigma x = 4$, $\Sigma x_1 = 10$, $\Sigma x_i^2 = 30$, $\Sigma y_i = 10.45$, $\Sigma x_i y_i = 28.44$.

yielding the normal equations

$$4 c_0 + 10 c_1 = 10.48$$

Solving we get $c_0 = 1.5$, $c_1 = 0.45$

$$10 c_0 + 30 c_1 = 28.44$$

Hence $a = e^{c_o} = e^{1.5} = 4.48$, $b = c_1 = 0.45$

Example : 2

Fit an equation of the type $y = ab^x$ to the following data by the method of least square technique.

x :	2	3	4	5	6
y :	144	172.8	207.4	248.8	298.5

Solution : Taking logarithm to $y = ab^x$ this equation becomes

$$\text{Log } y = \text{Log } a + (\text{Log } b)\, x$$

where $Y = \text{Log } y,\ c_0 = \text{Log } a,\ c_1 = \text{Log } b.$

$$Y = c_0 + c_1 x$$

We have the following table

x	**Y = log y**	**x^2**	**xY**
2	4.97	4	9.94
3	5.15	9	15.45
4	5.33	16	21.32
5	5.52	25	27.60
6	5.70	36	34.20
20	26.67	90	108.51

The normal equations are

$$\Sigma Y = c_0 \Sigma_1 + c_1 \Sigma x$$

$$\Sigma xY = c_0 \Sigma x + c_1 \Sigma x^2$$

$$26.67 = 5\, c_0 + 20\, c_1$$

$$108.51 = 20 c_0 + 90\, c_1$$

By using cramer's rule we solve

The above equations we get

$$\frac{c_0}{230.1} = \frac{-\, c_1}{-\, 9.15} = \frac{1}{50}$$

$$\therefore \quad c_0 = 4.602,\ c_1 = 0.183$$

Now, $\text{Log } a = c_0 \Rightarrow a = e^{c_o} = e^{4.602} = 99.68$

$$\text{Log } b = c_1 \Rightarrow b = e^{c_1} = e^{0.183} = 1.2008$$

$$\therefore \quad y = (99.68)\, (1.2008)^x.$$

Exercise No. 6

Example : 1

Determine the constants in $y = ae^{bx}$ fitting the data.

x :	1	2	3	4
f :	60	30	20	15

Ans.: a = 84.8, b = - 0.456

Example : 2

Find the constants in $y = ae^{bx}$ fitting the data.

x :	77	100	185	239	285
f :	2.1	3.4	7.0	11.1	19.6

Ans.: a = 1.2, b = 0.0096

Example : 3

Determine the constants in $y = ax^b$ fitting the data.

x :	2.2	2.7	3.5	4.1
f :	65	60	53	50

Ans.: a = 91.8, b = - 0.434

Example : 4

By the method of least squares fit the following data.

$$y = ax + b$$

x :	- 1	0	2	3	4	5
y :	- 9	- 7	- 3	- 1	1	3

Ans.: a = 0.64, b = - 2.43

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Unit 4

Probability and Probability Distributions

Syllabus :

Probability, Theorems on Probability, Bayes Theorem, Random variables, Mathematical Expectation, Probability density function, Probability distributions: Binomial, Poisson, Normal and Hypergometric, Test of Hypothesis: Chi–Square test, t–distribution. .

4.1 Introduction :

The notion of the probability of a particular event of a trial is subject to diverse meanings or interpretations. The word "probability" or "chance" is a very commonly used in day to day conversation. For example: if a weather forecaster quoted as saying that "there is a 40 percent chance of rain today in a certain region," we all most likely have some intuitive thought as to what is being said. Indeed, the majority of us would most likely interpret this statement in one of two possible ways.

Probability is used throughout engineering. In electrical engineering, signals and noise are analyzed by means of probability theory. Civil, mechanical, and industrial engineers use statistics and probability to test and account for variations in materials and goods. Chemical engineers use probability and statistics to assess experimental data and control and improve chemical processes.

In this chapter, we present some well known rules, or axioms, that used in probability theory. In order to understand probability theory; we need to prerequisites set theory and to study the concept of the sample space and the events of an experiment.

4.2 Sets: Definition & Types :

Probability makes widespread use of set operations, so let's revisit relevant notation and terminology of set theory.

- **Set :**

A set is a well defined collection of distinct elements. If S is a set and x is an element of the set then we write $x \in S$ and if x is not an element of the set then we write $x \notin S$.

Elements of the set can be enumerated by roster method or by the property method. In roaster method, we list all the elements in braces { } separated by commas. For instance, the set of possible outcomes of a dice roll is {1, 2, 3, 4, 5, 6}.

Alternatively, we can enumerate elements of sets by a certain property, say P in braces {}. For example, the set of even integer can be written as {2k | k is an integer}. Similarly, set of all real numbers between [0, 2] can be written as $\{x \mid 0 \le x \le 2\}$

- **Empty Set:**

A set is said to be empty set if it has no element. It is denoted by ϕ. For example, let A = {all integers between 1 and 2}. Then $A = \phi$.

- **Universal set:**

A set which contains all objects that could conceivably be of interest in a particular context. It is denoted by U.

- **Subsets:**

If each element of a set B is also an element of set A then B is said to be subset of A, written as $B \subset A$. For every set A, $A \subset A$ and $\phi \subset A$.

- **Equal sets:**

Two sets A and B are said to be equal sets if every element of set A are in B and vice–versa. Also if $A \subset B$ and $B \subset A$ then $A = B$.

4.3 Operations on Sets :

Suppose A and B are two sets and U is the Universal set. Then

Union of sets : $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

Intersection of sets: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

Difference of sets : $A - B = \{x \mid x \in A \text{ but } x \notin B\}$

$B - A = \{x \mid x \in B \text{ but } x \notin A\}$

Compliment of sets: Compliment of set A with respect to universal set U is defined as

$A^C = \{x \mid x \in U \text{ but } x \notin A\}$

Disjoint sets : Two sets A and B are said to disjoint if $A \cap B = \phi$.

Suppose we have infinitely many sets $S_1, S_2, \ldots, S_n, \ldots$, then union and intersections all these sets are defined as:

$$\bigcup_{n=1}^{\infty} S_n = \{x \mid x \in S_n \text{ for some } n\}$$

$$\bigcap_{n=1}^{\infty} S_n = \{x \mid x \in S_n \text{ for all } n\}$$

Partition: A collection of sets is said to be a partition of a set S if the sets in collection are disjoint and their union is S**.** That is, sets $S_1, S_2, \ldots, S_n, \ldots$, are said to be partition of S if $\bigcap_{i=1}^{n} S_i = \phi$ and $\bigcup_{i=1}^{n} S_i = S$.

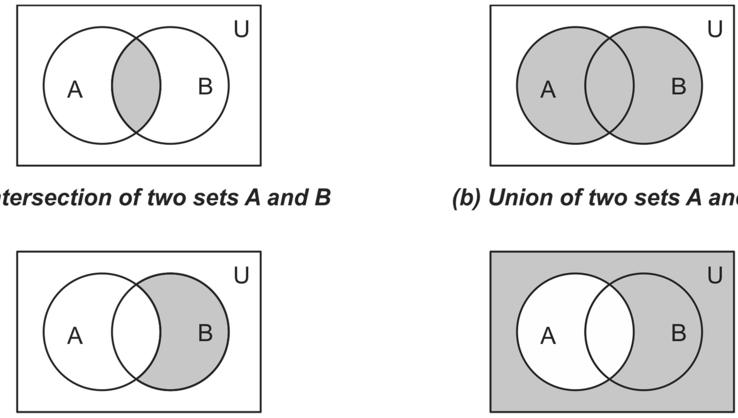
Ordered Pair: If x and y are two objects then ordered pair of x and y is given by (x, y).

4.4 Venn Diagrams and Algebra of Sets :

Sets and the associated operations are easy to visualize in terms of Venn diagrams.

- **Venn Diagrams:**

A graphical representation of points that is very useful for illustrating logical relations among them is known as Venn diagram. A universal set U can be represented geometrically by the set of all points inside a rectangle. Subsets of U are represented by the set of points inside a circle within the rectangle. Figure 4.1 represents the Venn diagrams of various logical relations of sets.



(a) Intersection of two sets A and B *(b) Union of two sets A and B*

(c) Difference of sets B and A (B - A) *(d) Compliment of Sets A (U - A)*

Fig. 4.1: Examples of Venn diagrams. (a) – The shaded region is $A \cap B$. (b) – The shaded region is $A \cup B$. (c) – The shaded region is $B – A$. (d) – The shaded region is A^c.

Algebra of Sets: Set operations have various properties. We list few of them as follows:

Commutative law for union	$A \cup B = B \cup A$
Commutative law for intersection	$A \cap B = B \cap A$
Associative law for union	$A \cup (B \cup C) = (A \cup B) \cup C$
Associative law for intersection	$A \cap (B \cap C) = (A \cap B) \cap C$
First Distributive law	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Second Distributive law	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Demorgan's first law	$(A \cap B)^c = A^c \cup B^c$
Demorgan's second law	$(A \cup B)^c = A^c \cap B^c$
For any sets A and B	$A = (A \cap B) \cup (A \cap B^c)$
For any set A	$A \cap U = A$; $A \cup U = U$

4.5 Sample Spaces and Events: Definition & Examples :

In context of probability, an ***experiment*** is any activity that produces outcomes. For example, when we rolled a dice, flip coin(s), dealing with cards, taking samples out of a warehouse, all these have some outcomes. An experiment may have finitely or infinitely many outcomes.

Suppose an experiment has outcomes $m_1, m_2, \ldots, m_n$. *Then the set of all these outcomes is called the* ***sample spaces (probability space)*** *of the experiment*. It is denoted by S. i.e. $S = \{m_1, m_2, \ldots, m_n\}$ is a sample space. Some examples are:

Example 1

If the outcome of an experiment consist only determination of the sex of a newborn child, then sample space S = {g, b}.

Example 2

If the experiment consists of the rolling of dice then sample space S = {1, 2, 3, 4, 5, 6}.

For an experiment, there may be more than one sample space. For example, consider a bag in which there are some kinds of balls. A certain number of the balls are made of glass, say, and rest of the plastic. Some of them are red and some are blue. If we now perform an experiment in which a ball is drawn from the bag, we might be interested in one of the following:

a) We might be concerned with whether a glass (g) or a plastic (p) ball is drawn. Then sample space is S = {g, p}.

b) We might be concerned with whether a red (r) or a blue (b) ball is drawn. Then sample space is S = {r, b}.

Each element of a sample space is called an outcome or sample point or simple event or an elementary event.

Any subset of a sample space is called an ***event*** *and is commonly denoted by E.* That is, an event is a set consisting of possible outcomes of the experiment. Some examples of events are the following:

Example 3

In Example 1, if E = {b}, then E is the event that the child is a boy

Example 4

In Example 2, if E = {2, 3}, then E is the event that the number on dice is either 2 or 3

Note : Total number of events associated with a random experiment is the total number of possible subsets of the sample space S. If n(S) = N, then there are 2^N subsets of S and thus 2^N events in the sample spaces.

We explain below some important events with the help of examples.

- **Sure Event:**

Suppose a dice is thrown. Then the sample space is S = {1, 2, 3, 4, 5, 6}. Let event E = {number less than 7} = {1, 2, 3, 4, 5, 6}. This is called sure event. Thus a sure event is the whole sample space itself.

- **Impossible Event:**

Suppose a dice is thrown. Then the sample space is S = {1, 2, 3, 4, 5, 6}. Let event E = {number greater than 6} = ϕ. This is an impossible event.

Example 5

Suppose a coin is tossed once. Then sample space S = {H, T}. Let event E = {both H and T} = ϕ. This is an impossible event. But if we let Event E = {not both H and T} = {H, T}, whole space itself. This is a sure event.

- **Equally Likely Events:**

The outcomes of an experiment are said to be equally likely if any one of them cannot be expected to occur in preference to another. For Instance, in rolling a dice, the outcomes 1 or 2 or 3 or 4 or 5 or 6 are equally likely.

- **Mutually Exclusive Events:**

The outcomes of an experiment are said to be mutually exclusive if they cannot occur together. Suppose we have two events E_1 and E_2. Then both events are said to be mutually exclusive if both events are disjoints, i.e.. $E_1 \cap E_2 = \phi$.

Example 6

Suppose a dice is thrown. Then the sample space is S = {1, 2, 3, 4, 5, 6}. Let event E_1 = {odd numbers} = {1, 3, 5} and event E_2 = {even numbers} = {2, 4, 6}. Then. $E_1 \cap E_2 = \phi$. Therefore both events are mutually exclusive.

Example 7

In tossing a coin, events E_1 = {H} and E_2 = {T} are mutually exclusive.

- **Independent Events:**

Two events are said to be independent if the occurrence or non–occurrence of the one does not affect the probability of the occurrence of the other. For instance, event of getting a head on the first coin and the event of getting a tail on the second coin in simultaneous throws of two coins are independent.

4.6 Probability: Definition & Properties :

An assignment of real numbers to the events defined in a sample space S is known as the probability measure. Consider a random experiment with a sample space S, and let E be a particular event defined in S.

A. Relative frequency definition:

If an experiment has *n* equally likely simple events and if the number of ways that an event *E* can occur is *m*, then the probability of *E*, *P*(*E*), is

$$P(E) = \frac{\text{Number of way that E can occur}}{\text{Number of Possible Outcomes}} = \frac{m}{n}$$

So, if *S* is the sample space of the experiment, then

$$P(E) = \frac{N(E)}{N(S)}$$

Otherwise we can define: Suppose that random experiment is repeated n times. If event E occurs n(E) times, then the probability of event E, denoted by P(E), is defined as

$$P(E) = \lim_{n\to\infty} \frac{n(E)}{n}, \text{ for every } E \subseteq S,$$

where $\frac{n(E)}{n}$ is known as the relative frequency of the event E. It is clear that for any event E, the relative frequency of E will have the following properties:

R$_1$. $0 \leq \frac{n(E)}{n} \leq 1$, where $\frac{n(E)}{n} = 0$ if E occurs in none of the n repeated experiments and $\frac{n(E)}{n} = 1$ if E occurs in all of the n repeated experiments.

R$_2$. If E_1 and E_2 are mutually exclusive events, then $n(E_1 \cup E_2) = n(E_1) + n(E_2)$ and $\frac{n(E_1 \cup E_2)}{n} = \frac{n(E_1)}{n} + \frac{n(E_2)}{n}$.

B. Axiomatic definition:

Let S be a finite sample space and E be an event in S. Then in the axiomatic definition, the probability P(E) of the event E is real number assigned to E which satisfies the following three axioms:

A$_1$: $P(E) \geq 0$.

A$_2$: $P(S) = 1$.

A$_3$: $P(E_1 \cup E_2) = P(E_1) + P(E_2)$, if $E_1 \cap E_2 = \phi$

Suppose sample space is not finite, then axiom **A$_3$** can be modified as:

A$_4$: If $\{E_i\}_{1\leq i\leq\infty}$ is a sequence of mutually exclusive events in S, where $E_i \cap E_j = \phi$ for $i \neq j$, then

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

C. Elementary properties of Probability:

On using the above axioms, some useful properties of probability can be obtained, listed as follows:

C_1: $P(\phi) = 0$;

C_2: $P(E_1) \leq P(E_2)$ if $E_1 \subset E_2$;

C_3: $P(E_1) \leq 1$;

C_4: $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$.

C_5: If an event E is sure event then $P(E^c) = 0$ and $P(E) = 1$.

C_6: For any event E, $0 \leq P(E) \leq 1$

4.7 Equally Likely Events :

A) Finite Sample Space: Consider a finite sample space S with n finite elements

$$S = \{E_1, E_2, \ldots, E_n\}$$

Where $E_i^{'s}$ are elementary events. Let $P(E_i) = p_i$. Then

F1. $0 \leq p_i \leq 1 \quad i = 1, 2, \ldots, n$

F2. $\sum_{i=1}^{n} p_i = p_1 + p_2 + \cdots + p_n = 1$

F3. If $E = \bigcup_{i \in I} E_i$, where I is a collection of subscript, then

$$P(E) = \sum_{E_i \in E} P(E_i) = \sum_{i \in I} p_i$$

B) Equally Likely Events:

When all elementary events E_i ($i = 1, 2, \ldots, n$) are equally likely, that is, $E_1 = E_2 = \ldots = E_n$ then from (F1), we get, $n\, p_i = 1$ and hence from (F3)

$$P(E) = \frac{n(E)}{n}$$

where n(E) is the number of outcomes belongings to event E and n is the number of sample points in S.

4.8 Complimentary Events :

Suppose an experiment is performed and E is an event. Compliment of E is the event E^c consisting of all outcomes not in E. Since E and E^c have no common outcomes, then $P(E) + P(E^c)$ is the sum of the probabilities of all the outcomes in the sample spaces, and thus

$P(E) + P(E^c) = 1$.

4.9 Conditional Probability :

The conditional probability of an event E_1 given event E_2, denoted by $P\left(\frac{E_1}{E_2}\right)$, is defined as

$$P\left(\frac{E_1}{E_2}\right) = \frac{P(E_1 \cap E_2)}{P(E_2)} \quad P(E_2) > 0 \qquad \ldots(1)$$

where $P(E_1 \cap E_2)$ is the joint probability of E_1 and E_2.

Similarly,

$$P\left(\frac{E_2}{E_1}\right) = \frac{P(E_1 \cap E_2)}{P(E_1)} \quad P(E_1) > 0 \qquad \text{.... (2)}$$

is the conditional probability of an event E_2 given event E_1.

From Eq. (1) and Eq. (2), we have

$$P(E_1 \cap E_2) = P\left(\frac{E_1}{E_2}\right) P(E_2) = P\left(\frac{E_2}{E_1}\right) P(E_1)$$

Equating last two equalities, we obtain

$$P\left(\frac{E_1}{E_2}\right) = \frac{P\frac{E_2}{E_1} P(E_1)}{P(E_2)} \qquad \text{.... (3)}$$

Eq. (3) known as ***Bayes' rule***.

4.10 Total Probability Theorem :

A. **Bayes' Theorem:** If $E_1, E_2, \ldots, E_n$ are mutually exclusive events whose union is the sample space and E is any event, then

$$P\frac{E_i}{E} = \frac{P\left(\frac{E}{E_i}\right) P(E_i)}{\sum_{i=1}^{n} P\left(\frac{E}{E_i}\right) P(E_i)}$$

B. **Multiplication Theorem**: If two events E_1 and E_2 are *independent*, then probability that they will both occur is equal to the product of their individual probabilities.

i.e., $P(E_1 \cap E_2) = P(E_1) P(E_2)$

In general, if events E_1 and E_2 are independent then from Eq. (1) and Eq. (2), it follows immediately that

$$P\left(\frac{E_1}{E_2}\right) = P(E_1) \text{ and } P\left(\frac{E_2}{E_1}\right) = P(E_2)$$

If two events E_1 and E_2 are independent, then E_1 and E_2^c are also independent. That is,

$P(E_1 \cap E_2^c) = P(E_1) P(E_2^c)$

Therefore from Eq. (1), we have

$$P\left(\frac{E_1}{E_2^c}\right) = \frac{P(E_1 \cap E_2^c)}{P(E_2^c)} = \frac{P(E_1)P(E_2^c)}{P(E_2^c)} = P(E_1)$$

Similarly, if there are three events, E_1, E_2 and E_3 are independent, then

$$P(E_1 \cap E_2 \cap E_3) = P(E_1) P(E_2) P(E_3)$$

To distinguish between the mutual exclusiveness and independence of a collection of events, we summarize as follows:

I_1. If $\{E_i\}$, where, i = 1, 2, …, n is a sequence of mutually exclusive events, then

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$$

I_2. If $\{E_i\}$, where, i = 1, 2, …, n is a sequence of independent events, then

$$P\left(\bigcap_{i=1}^{n} E_i\right) = \prod_{i=1}^{n} P(E_i)$$

Illustrative Examples

Example : 1

A dice is thrown one. Find the probability of getting a number greater than 4.

Solution:

Here sample space $S = \{1, 2, 3, 4, 5, 6\}$

Therefore, $n(S) = 6$

Numbers greater than 4 = {5, 6}, hence n (E) = 2

$$\therefore \quad P(E) = \frac{n(E)}{n(s)} = \frac{2}{6} = \frac{1}{3}$$

Example : 2

If there are three children in a family, find the probability that there is one girl in the family.

Solution:

Let us denote b for boy and g for girl, then the sample space is:

S = {(b, b, b), (b, b, g), (b, g, g), (b, g, b), (g, b, b,), (g, b, g), (g, g, b), (g, g, g)}

Therefore, n(S) = 8

E = Number of favorable events that there is one girl in the family =

E = {(b, b, g), (b, g, b), (g, b, b,)}

Therefore, n (E) = 3 , hence

$$\therefore \quad P(E) = \frac{n(E)}{n(s)} = \frac{3}{8}$$

Example : 3

Find the probability of getting a red colored king from a well shuffled pack of 52 card when a card is drawn.

Solution: As there are 2 red colored kings in a pack of cards.

So the number of favorable events n(E) = 2.

Here, n(S) = 52

∴ Probability of getting one red king is

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{52} = \frac{1}{26}$$

Example : 4

From a well shuffled pack of cards, three cards are drawn at random. Find the probability that they from a King, Queen, Jack combination.

Solution:

3 cards can be drawn in $^{52}C_3$ ways.

i.e. $n(S) = {}^{52}C_3 = \frac{52 \cdot 51 \cdot 50}{1 \cdot 2 \cdot 3} = 22100$

King, Queen and Jack each can be chosen in $^{4}C_1$ ways.

The combination can be chosen in $^{4}C_1 \times {}^{4}C_1 \times {}^{4}C_1 = 64$

i.e. $n(E) = 64$

∴ Probability $P(E) = \frac{n(E)}{n(S)} = \frac{64}{22100} = \frac{16}{5525}$

Example : 5

Three coins are tossed once. Find the probability of getting

(a) at least 2 heads

(b) at most 2 heads

(c) exactly 2 heads

Solution:

Sample space is

S = {(H,H,H), (H,H,T), (H,T,T), (H,T,H), (T,H,H), (T,H,T), (T,T,H), (T,T,T)}

∴ n (S) = 8

Let E be the set of favorable event, then

(a) favorable event with at least 2 heads is

E_1 = {(H, H, H), (H, H, T), (H, T, H), (T, H, H)}

∴ $n(E_1) = 4$

and so

Probability $P(E_1) = \frac{n(E)}{n(S)} = \frac{4}{8} = \frac{1}{2}$

(b) favorable event with at most 2 heads is

E_2 = {(H, H, T), (H, T, T), (H, T, H), (T, H, H), (T, H, T), (T, T, H), (T, T, T)}

∴ $n(E_2) = 7$

and so

Probability $P\,(E_2) = \frac{n(E)}{n(s)} = \frac{7}{8}$

(c) favorable event with at exactly 2 heads is

$E_3 = \{(H, H, T), (H, T, H), (T, H, H)\}$

$\therefore\ n\,(E_1) = 3$

and so

Probability $P\,(E) = \frac{n(E)}{n(s)} = \frac{3}{8}$

Example : 6

A class contains 10 men and 20 women of which half the men and half the women have brown eyes. Find the probability that a person chosen at random is a man or has brown eyes.

Solution:

Let A = person is a man

B = person has brown eyes

We calculate $P\,(A \cup B)$

Then, $P(A) = \frac{10}{30} = \frac{1}{3}, P(B) = \frac{15}{30} = \frac{1}{2}, P(A \cap B) = \frac{5}{30} = \frac{1}{6}$

$$\therefore \quad P\,(A \cup B) = P\,(A) + P\,(B) - P\,(A \cap B)$$

$$= \frac{1}{3} + \frac{1}{2} - \frac{1}{6} = \frac{2}{3}$$

Example : 7

Find the probability of drawing a heart on each of the two consecutive draws from a well shuffled pack of cards if the card is not replaced after the first draw.

Solution:

Let us define the event A to be heart on the first draw and event B to be a heart on the second draw. Then

$$P(A \cap B) = P\,(A).\,P(B/A)$$

Now $P(A) = \frac{13}{52} = \frac{1}{4}$

When we get a heart on the first draw, the second draw has 51 outcomes and 12 are favourable.

$$\therefore \quad P\,(B/A) = \frac{12}{51} = \frac{4}{17}$$

$$\therefore \quad P\,(A \cap B) = P\,(A).\,P\,(B/A) = \frac{1}{4} \times \frac{4}{17} = \frac{1}{17}$$

Example : 8

Two cards are drawn from a pack of 52 cards, one after the other without replacement. Find the chance that one of these cards is an ace and the other is queen of possible shade.

Solution:

Here we find two possibilities:

(i) The first card drawn in an ace and the second is a queen shade. The probability of this compound event is

$$= \frac{4}{52} \times \frac{2}{51} = \frac{2}{663}$$

(ii) The first card drawn is a queen and the second is an ace of opposite color. The probability of this compound event is

$$= \frac{4}{52} \times \frac{2}{51} = \frac{2}{663}$$

Hence the required probability $= \frac{2}{663} + \frac{2}{663} = \frac{4}{663}$

Example : 9

Mohan speak truth in 60 % and ram in 70% cases. In what percentage of cases are they likely to contradict each other in stating the same fact?

Solution:

They will contradict each other if one of them speak truth and the other tells a lie. Therefore,

Probability that's Mohan tells a lie and Ram speak the truth is

$$= \left(1 - \frac{60}{100}\right) \times \frac{70}{100} = \frac{7}{25}$$

and

Probability that's Mohan speaks truth and Ram tells a lie

$$= \left(1 - \frac{70}{100}\right) \times \frac{60}{100} = \frac{9}{50}$$

Therefore, probability that they contradict each other

$$= \frac{7}{25} + \frac{9}{50} = \frac{23}{50}$$

and,

so percentage of cases in which they contradict each other

$$= \frac{23}{50} \times 100 = 46\%$$

Example : 10

A bag contains 2 white and 3 red balls and a bag contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and is found to be red. Find the probability that it was drawn from the bag B.

Solution:

Let E_1 be the event that the ball is drawn from the bag A, E_2 be the event that it is drawn from the bag B and E is the event that the drawn ball is red.

As the drawn ball is found to be red so we have to find $P\left(\frac{E_2}{E}\right)$.

Since both the bags are equally likely to be selected to draw the ball, so we have

$P(E_1) = \frac{1}{2} = P(E_2)$

Also,

$P\left(\frac{E}{E_1}\right)$ = the probability that the ball drawn is red if it is drawn from the bag A $= \frac{3}{5}$

Similarly, $P\left(\frac{E}{E_2}\right) = \frac{5}{9}$

Hence by Bays Theorem, we have

$$P\left(\frac{E_2}{E}\right) = \frac{P(E_2)\,P\left(\frac{E}{E_2}\right)}{P(E_1)P\left(\frac{E}{E_1}\right) + P(E_2)\,P\left(\frac{E}{E_2}\right)}$$

$$= \frac{\frac{1}{2}\times\frac{5}{9}}{\frac{1}{2}\times\frac{3}{5}+\frac{1}{2}\times\frac{5}{9}} = \frac{5}{9}\times\frac{45}{52} = \frac{25}{52}$$

Example : 11

There are three shooters A, B and C. If shooter A hit the target 1 out of 4 times, shooter B hit the target 2 out of 3 times; shooter C hit the target 3 out of 4 times. Find the probability of at least two shooters hit the target.

Solution:

Let E_1 be the event of A hitting the target, E_2 be the event of B hitting the target and E_3 be the event of C hitting the target. Therefore,

$$P(E_1) = \frac{1}{4}; \quad P(E_2) = \frac{2}{3}; \quad P(E_3) = \frac{3}{4}$$

and so

$$P(E_1^C) = \frac{3}{4}; \quad P(E_2^C) = \frac{1}{3}; \quad P(E_3^C) = \frac{1}{4}$$

Therefore, probability of the event that "at least two shooter hitting the target" is equal to P (at least two shooter hitting the target)

$$= P(E_1 \cap E_2 \cap E_3^C) + P(E_1 \cap E_2^C \cap E_3) + P(E_1^C \cap E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3) \quad \ldots(1)$$

Now, $P(E_1 \cap E_2 \cap E_3^C) = P(E_1)\,P(E_2)P(E_3^C) = \frac{1}{4} \times \frac{2}{3} \times \frac{1}{4} = \frac{1}{24}$

$P(E_1 \cap E_2^C \cap E_3) = P(E_1)\,P(E_2^C)P(E_3) = \frac{1}{4} \times \frac{1}{3} \times \frac{3}{4} = \frac{1}{16}$

$P(E_1^C \cap E_2 \cap E_3) = P(E_1^C)\,P(E_2)P(E_3) = \frac{3}{4} \times \frac{2}{3} \times \frac{3}{4} = \frac{3}{8}$

$P(E_1 \cap E_2 \cap E_3) = P(E_1)\,P(E_2)\,P(E_3) = \frac{1}{4} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{8}$

Hence from (1), we have

$$\text{P (at least two shooter hitting the target)} = \frac{1}{24} + \frac{1}{16} + \frac{3}{8} + \frac{1}{8} = \frac{29}{48}$$

Example : 12

First bag contains 6 white and 8 red balls. Second bag contains 9 white and 10 red balls. One ball is drawn at random from the first bag and put into the second bag without noticing its color. A ball is drawn at random from the second bag. What is the probability that it is red?

Solution:

As there are two types of balls, red and white, in first bag. One ball is drawn at random from the first bag and put into the second bag without noticing its color. Let E_1 be the event of drawing a white ball from the first bag, E_2 be the event of the transferring a red ball from the first bag and E_3 be the event of the transferring a red ball from second bag.

Now,

Probability of drawing white ball from first bag = $P(E_1) = \frac{6}{14} = \frac{3}{7}$

Probability of drawing red ball from first bag = $P(E_2) = \frac{8}{14} = \frac{4}{7}$

Therefore, $P\left(\frac{E_3}{E_1}\right) = \frac{10}{20} = \frac{1}{2}$

Also, $P\left(\frac{E_3}{E_2}\right) = \frac{11}{20}$

Case I : If white ball is transferred from first bag to the second bag and then red ball is drawn from it. Therefore probability of this event is

$$P(E_1 \cap E_3) = P(E_1)P\left(\frac{E_3}{E_1}\right) = \frac{3}{7} \times \frac{1}{2} = \frac{3}{14}$$

Case II : If red ball is transferred from first bag to the second bag and then red ball is drawn from it. Therefore probability of this event is

$$P(E_2 \cap E_3) = P(E_2)\, P\left(\frac{E_3}{E_2}\right) = \frac{4}{7} \times \frac{11}{20} = \frac{44}{140}$$

Hence the total probability of getting red ball is

$$P(E) = P(E_1) + P(E_2) = \frac{3}{14} + \frac{44}{140} = \frac{74}{140}$$

Example : 13

A committee of 5 persons is to be selected randomly from a group of 5 men and 10 women.

(a) Find the probability that the committee consists of 2 men and 3 women

(b) Find the probability that the committee consists of all women.

Solution :

The number of total outcomes is $n(s) = {}^{15}C_5$

(a) Let E_1 be the event that the committee consists of 2 men and 3 women. Then the number of outcomes belonging to E_1 is given by

$$n(E_1) = {}^{5}C_2 \times {}^{10}C_3$$

Thus,

$$P(E_1) = \frac{n(E_1)}{n(S)} = \frac{{}^{5}C_2 \times {}^{10}C_3}{{}^{15}C_5} = \frac{400}{1001} \approx 0.4$$

(b) Let E_2 be the event that the committee consists of all women. Then the number of outcomes belonging to E_2 is given by

$$n(E_2) = {}^{5}C_0 \times {}^{10}C_5$$

Thus,

$$P(E_1) = \frac{n(E_1)}{n(S)} = \frac{{}^{5}C_0 \times {}^{10}C_5}{{}^{15}C_5} = \frac{36}{429} \approx 0.084$$

Example : 14

A number is selected at random from {1, 2, 3,…, 100 }. Given that the numbered selected is divisible by 2, find the probability that it is divisible by 3 or 5.

Solution: Let

E_2 = Event that the number is divisible by 2;

E_3 = Event that the number is divisible by 3;

E_5 = Event that the number is divisible by 5;

Then the desired probability is

$$P(E_3 \cup E_5 \backslash E_2) = \frac{P[(E_3 \cup E_5) \cap E_2]}{P(E_2)}$$

$$= \frac{P[(E_3 \cap E_2) \cup (E_5 \cap E_2)]}{P(E_2)}$$

$$= \frac{P(E_3 \cap E_2) + P(E_5 \cap E_2) - P(E_3 \cap E_5 \cap E_2)}{P(E_2)}$$

where

$P(E_3 \cap E_2)$ = event that the number is divisible by 6

$P(E_5 \cap E_2)$ = event that the number is divisible by 10

$P(E_3 \cap E_5 \cap E_2)$ = event that the number is divisible by 30

Now,

$$P(E_3 \cap E_2) = \frac{16}{100}; \quad P(E_5 \cap E_2) = \frac{10}{100}; \quad P(E_3 \cap E_5 \cap E_2) = \frac{3}{100}$$

Thus,

$$P(E_3 \cup E_5 \backslash E_2) = \frac{\frac{16}{100} + \frac{10}{100} - \frac{3}{100}}{\frac{50}{100}} = \frac{23}{50} = 0.46$$

Self–Assessment Exercise 4.1

1. If two dice are thrown, what is probability that the sum of the digits showing on the top face of the dice is less than 6? **Ans.: 5 / 18**

2. A bag contains 4 white, 3 red and 2 black balls. If a ball is drawn at random, find the probability that it is white. **Ans.: 4 / 9**

3. From a set of 20 card numbered as 1, 2, 3, …, 19,20, one is drawn at random. Find the probability of having a number divisible by 3 or 7. **Ans.: 2 / 5**

4. Imagine we are rolling a fair dice. There are six equally likely outcomes: 1, 2, 3, 4, 5 and 6. What is the probability of getting a five? **Ans.: 1 / 6**

5. Two unbiased dice are thrown. Find the probability that

(a) Both the dice show the same number **Ans.: 1 / 6**

(b) The total of the numbers on the dice is 8 **Ans.: 5 / 36**

(c) The total of the numbers on the dice is 13 **Ans.: 0**

6. A card is drawn from a well shuffled pack of 52 cards. Find the probability of getting neither a heart nor a red king. **Ans.: 19/26**

7. A card is drawn from a pack of 52 cards. What is the probability of getting a queen of club or a king of heart? **Ans.: 1/26**

8. A bag contains 4 white, 5 red and 6 blue balls. Three balls are drawn at random from the bag. The probability that all of them are red is? **Ans.: 2/91**

9. A box contains two white, three red and four black balls. If at least one red ball is to be included in the draw, the number of ways of drawing 3 balls is given by (all balls are considered to be different) **Ans.: 64**

10. Two events A and B have probabilities 0.25 and 0.50 respectively. The probability that both A and B occur simultaneously is 0.12. Then the probability that neither A nor B occur is. **Ans.: 0.37**

11. A bag contains 2 white and 3 back balls. Four persons A, B, C and D in the order named each draw one ball and do not replace it. The person to draw a white ball receives Rs. 200. Determine their expectations. **Ans.: Rs. 40**

12. Find the probability of drawing one rupee coin from a purse with two compartments, one of which contains 3 fifty paise coins and 2 one rupee coins and the other 2 fifty paise coins 3 one rupee coins. **Ans. : 1/2**

13. A bag contains 5 red, 4 black and 3 white balls, 4 balls are drawn from the bag. Find the probability of having at least 3 red balls if the balls are not replaced at the drawn.

Ans.: 5/33

14. Let $P(A) = 0.9$ and $P(B) = 0.8$. Show that $P(A \cap B) \geq 0.7$.

15. Given that $P(A) = 0.9$, $P(B) = 0.8$ and $P(A \cap B) = 0.75$. Find

(a) $P(A \cup B)$ **Ans.: 0.95**

(b) $P(A \cap B^C)$ **Ans.: 0.15**

(c) $P(A^C \cap B^C)$ **Ans.: 0.05**

16. Prove that : $P\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} P(E_i)$

17. Two numbers are chosen at random from among the numbers 1 to 10 without replacement. Find the probability that the second number chosen is 5. **Ans.: 1 / 10**

18. Two boys and two girls enter a music hall and take four seats at random in a row. What is the probability that the girls take the two end seats? **Ans.: 1 / 6**

19. A box contains 6 red balls, 4 white balls and 5 blue balls. Three balls are drawn successively from the box. Find the probability that they are drawn in the order red, white and blue if each ball is not replaced. **Ans.: 4 / 91**

20. A six faced die is so biased that it is twice as likely to show an even number as an odd number when thrown. It is thrown twice. What is the probability that the sum of the two numbers thrown is even? **Ans.: 5 / 9**

4.11 Measures of Mean and Standard Deviation :

In day to day life, many process or studies generates tables of numbers called *data*. A statistic is measure that is used in the analysis of data to enable us to draw conclusion from the data. Two measure elementary tools of statistics are measure of mean and variation.

A. **Measure of Mean:** Given a table x of real numbers which takes n value $A_1, A_2, \ldots, A_n$ need not be all distinct, then the mean or centre or average of these data sets is denoted by $\bar{x}$ and is given by

$$\bar{x} = \frac{A_1 + A_2 + \cdots + A_n}{n} = \sum_{i=1}^{n} \frac{A_i}{n}$$

If the data is given in the form of frequency distribution, then

$$\bar{x} = \frac{f_1 A_1 + f_2 A_2 + \cdots + f_n A_n}{f_1 + f_2 + \cdots + f_n} = \sum_{i=1}^{n} \frac{f_i A_i}{f_i},$$

where each f_i are frequencies of corresponding data values A_i

B. **Measure of Standard Deviation:** For the standard deviation of, $A_1, A_2, \ldots, A_n$ suppose $\bar{x}$ is the mean. Then we define the following two definitions of standard deviations that are routinely used in statistics.

1. If $A_1, A_2, \ldots, A_n$ is a sample of data within a much larger population, then the standard deviation of this sample is

$$S = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (A_i - \bar{x})^2}$$

2. If $A_1, A_2, \ldots, A_n$ are all of the data points, then the standard deviation is

$$\sigma = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (A_i - \bar{x})^2}$$

Example : 1

Find the mean and standard deviation of the data set:

1.1, 1.3, 1.6, 2.0, 2.1, 2.2, 2.4, 2.5

Solution:

Let $\bar{x}$ be the mean of the data set then

$$\bar{x} = \frac{1.1 + 1.3 + 1.6 + 2.0 + 2.1 + 2.2 + 2.4 + 2.5}{8} = 2.025$$

From Table 1,

$$\sum_{i=1}^{n} (A_i - \bar{x})^2 = 1.9650$$

Therefore,

$$S = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (A_i - \bar{x})^2} = \sqrt{\frac{1.9650}{7}} = 0.5298$$

$(A_i - \bar{x})^2$
$(1.1 - 2.025)^2$
$(1.3 - 2.025)^2$
$(1.6 - 2.025)^2$
$(2.0 - 2.025)^2$
$(2.1 - 2.025)^2$
$(2.2 - 2.025)^2$
$(2.4 - 2.025)^2$
$(2.4 - 2.025)^2$

Table – 1

4.12 Random Variable and Probability Distribution :

A random variable is often defined as a variable whose values are determined by chance. In other words, we can say the value taken by the variable is known only when the outcome of a random experiment is known. Mathematically we defined as follows:

- **Definition (Random Variable):**

Let S be a sample space. Then a random variable X(s) is a single valued real function that assigns a real number called the value of X(s) to each sample point s of S.

The sample space S is called domain of the random variable X(s) and the collection of all values of X(s) is known as range of the random variable X(s). If the sample space is either finite or countably infinite, the random variable is said to be discrete.

Example: In an experiment of tossing a coin once, we can defined random random variable as X(H) = 1 and X(T) = 0. We could also define another random variable say Y and Z such that Y(H) = 0 and Y(T) = 1 or Z(H) = 0 and Z(T) = 0.

- **Events defined by random variable:**

If X(s) is a random variable and x is a fixed real number, we can define the event X(s) = x, and is written as {X(s) = x}. Similarly we can define other events $X(s) \leq x$ and $X(s) \geq x$.

- **Types of Random Variable :**

(i) **Discrete** random variables have a countable number of outcomes

Examples: dice, cards, coin, counts, etc.

(ii) **Continuous** random variables have an infinite continuum of possible values.

Examples: blood pressure, weight, the speed of a car, the real numbers between 0 to 1.

- **Definition (Distribution Function):**

The probability distribution on a random variable X(s) is function P which assigns a probability P(x) to each value x that the random variable can assume and such that their sum is equal to 1.

Mathematically,

A distribution function for X(s) is a real valued function P whose domain is sample space S and satisfies following property:

(i) $P(x) \geq 0$ for all $x \in S$, and

(ii) $\sum_{x \in S} P(x) = 1$

For any subset E of S, we define the probability of E to be the number P(E) given by

$P(E) = \sum_{x \in E} P(x)$

4.13 Mean and Variance of Random Variables:

Mean of Random variable : Let $x_1, x_2, x_3, \ldots, x_n$ are values of random variable and $P_1, P_2, P_3, \ldots, P_n$ are corresponding probabilities then mean or expected value of a random variable X, denoted by μ or E(X), is defined as

$$\mu = \sum_{i=1}^{n} \frac{p_i x_i}{p_i} = \sum p_i x_i$$

Variance of Random variable: The formula for Variance of a probability distribution is

$\sigma^2 = \sum (x_i - \mu)^2 p_i$

If μ is a fraction, then we have another shape of this formula which is

$\sigma^2 = \sum (x_i - \mu)^2 p_i$

and hence standard deviation is

$\sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$

or

$\sigma = \sqrt{\sum p_i x_i^2 - \mu^2}$

4.14 Binomial Probability Distribution :

There are many possible discrete probability distributions we are now going to discuss one of them.

Suppose there are n independent trials, each of which results are performed in a "success" with probability p ($0 \leq p \leq 1$) and in a "failure" with probability $q = 1 - p$. If $X = r$ represents the number of successes that occur in the n trials, then X is said to be a binomial random variable with parameters (n, p). Therefore probability of r success in n trails is

$P(X = r) = {}^{n}_{r}C\, p^r q^{(n-r)}$

where r = 0,1,2,…, n. Such distributions are called binomial probability distribution.

If we repeat the experiment N times, then the frequency function of the binomial distribution is given by

$F(r) = N[P(X = r)] = N\, {}^{n}_{r}C\, p^r q^{(n-r)}$

The mean and variance of the binomial random variable X are

$\mu = E(X) = np$

$\sigma^2 = \text{Var}(X) = npq$

The probability mass functions of two binomial random variables with respective parameters (10, .5) and (10, .6) are presented in Figure 4.2.

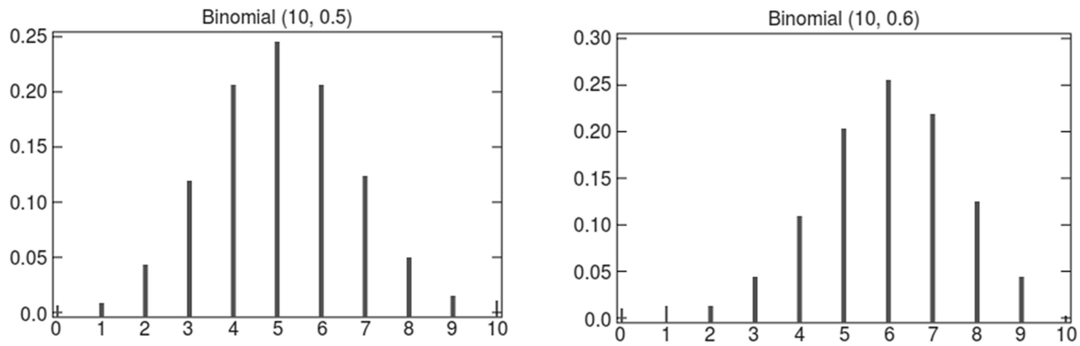


Fig. 4.2 : Binomial probability mass function

4.15 Poisson Probability Distribution :

The Poisson distribution is also a discrete probability distribution. This was discovered by French mathematician S. D. Poisson (1781 – 1840) in year 1837.

Suppose there are n numbers of trails in an experiment, which is indefinitely large. Let p denotes the probability of successes in any trail. We assume that p is indefinitely small i.e. we are dealing with a rare event. Let X = r denotes the Poisson variable corresponding to this random experiment. Therefore possible value of X are 0, 1, 2,… , n.

The Poisson distribution is obtained as a limiting case of the corresponding binomial distribution of the experiment under the conditions:

(a) n, the number of trails is indefinitely large, i.e. $n \to \infty$

(b) p, the probability of success in a trail is indefinitely small, i.e. $p \to 0$

(c) The product np of n and p is constant; denote it by $\lambda = np$, known as parameter of the Poisson distribution.

Then Poisson probability function (Poisson distribution) is given by

$$P(X = r) = \frac{e^{-\lambda}\lambda^r}{r!} \quad r = 0,1,\ldots$$

A graph of this mass function when $\lambda = 4$ is given in Figure 4.3.

The mean and variance of Poisson random variable X are

$$\mu = E(X) = \lambda$$

$$\sigma^2 = Var(X) = \lambda$$

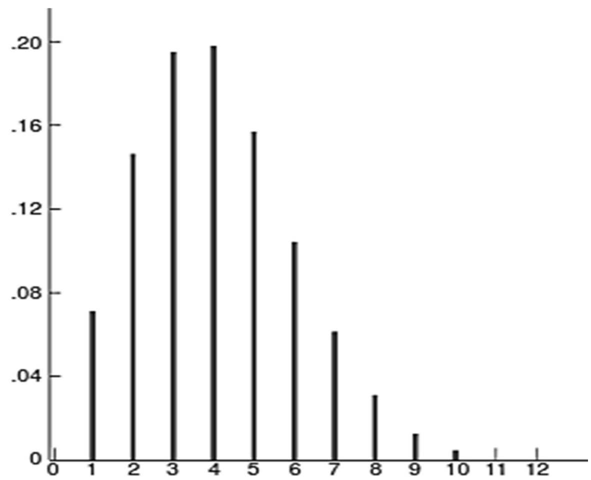


Fig. 4.3 : The Poisson probability mass function with $\lambda = 4$.

4.16 Poisson Frequency Distribution :

If a random experiment, satisfying the requirement of Poisson distribution is repeated N times, then the expected frequency of getting r ($0 \le r \le n$) successes is given by

$$F(r) = N[P(X = r)] = N\frac{e^{-\lambda}\lambda^r}{r!}$$

where r = 0,1,2, …

Some examples of Poisson random variables are:

(i) The number of telephone calls arriving at a switching center during various intervals of time.

(ii) The number of people in a community living to 100 years of age.

(iii) The number of misprints on the page of a book.

(iv) The number of transistors that fail on their first day of use.

(v) The number of customers entering a bank during various intervals of time.

Remark:

The distribution to be used in solving a problem is generally given in the problem, if it is not given, then the student should make use of Poisson distribution only when the event in the problem is of rare nature, i.e. the probability of happening of event is quite near to zero.

4.17 Hypergeometric Probability Distribution :

We known that binomial distribution is applied whenever we draw a random sample with replacement. Suppose a bin contains N + M batteries, of which N are acceptable quality and the others M are defective. A sample of size n is to be randomly chosen (without replacements) in the sense that the set of sampled batteries is equally likely to be any of the ${}^{N+M}_{n}C$ subject of size n. If we assume X = r denote the number of acceptable batteries in the sample, then

$$P(x = r) = \begin{cases} \dfrac{\left({}^{N}_{r}C\right)\left({}^{m}_{n-r}C\right)}{N + {}^{M}_{n}C} & ; \ r = 0, 1, 2, \dots, n \\ \theta & ; \quad \text{otherwise} \end{cases}$$

Any random variable X = r, whose probability mass function is given by P(X = r) is said to be a Hypergeometric random variable with parameters N, M, n.

4.18 Normal Probability Distribution :

The normal distribution was introduced by the French mathematician De Moivre in 1733 and was used by him to approximate probabilities associated with binomial random variables when the binomial parameter n is large. It is also known as Gaussian distribution.

Normal distribution is a limiting case of the Binomial distribution under the following conditions.

(a) when n, the number of trials is very large and

(b) p, the probability of successes, is close to $\frac{1}{2}$.

- **Normal Distribution Equation (P.D.F.):**

  The general equation of the normal distribution is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1(x-\mu)^2}{2\sigma^2}}$$

where x, $\mu \in (-\infty, \infty)$ and $\sigma > 0$. The parameters μ and σ are respectively mean and standard deviation of the distribution. Random variable X is called a normal vitiate if it follows a normal distribution. Normal *pdf*s is denoted by X = N(**μ , σ**).

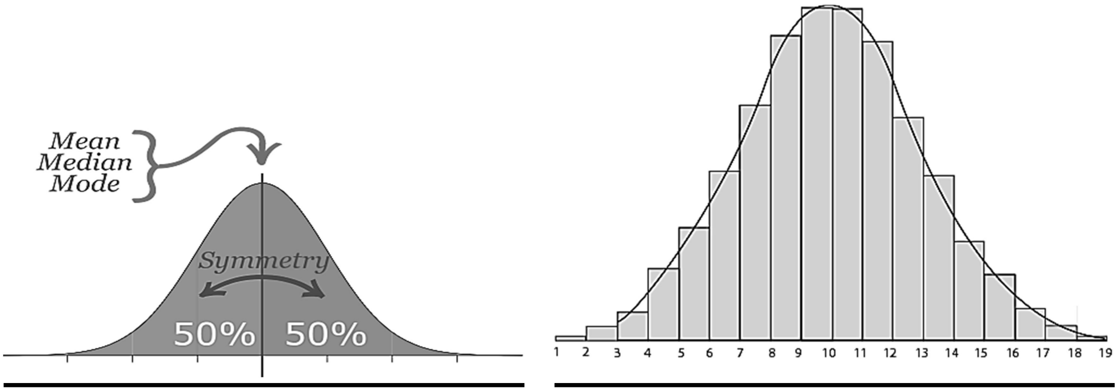


Fig. 4.4 : Graph showing normal probabilities

- **Basic properties of the Normal Distribution:**

  (a) The normal distribution is symmetrical about the line $x = \mu$.

  (b) It is unimodal

  (c) For a normal distribution, mean = median = mode.

  (d) Area under the curve = 1

  (e) The point of inflection of the Normal curve are given by $x = \mu \pm \sigma$.

- **Applications of Normal distribution**

  (a) Calculation of hit probability of a shot.

  (b) Statistical inference in most branches of science.

  (c) Calculation of errors made by chance in experimental measurement.

- **Standard form of Normal distribution**

If X is a normal random variable with mean μ and S.D. σ, then the random variable defined by

$$z = \frac{x - \mu}{\sigma}$$

is said to be standard normal variate with mean 0 and S.D. 1 i.e. $z = N(0, 1)$.

The probability density function for the standard normal variate is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\ e^{\frac{-z^2}{2}}$$

where. $z \in (-\infty, \infty)$. The integral $\int_0^z f(z)\, dz$ cannot be evaluated analytically. The value of this integral for various positive values of z has been given in the Table 2.

Illustrative Examples

Example : 1

Find the probability distribution of X, getting a head, in an experiment when a coin is tossed twice.

Solution:

Let getting a head is random variable X. Let X(s) denotes the number heads obtained. In this case we have X(s) = {0, 1, 2}. Here sample space S = {HH, HT, TH, TT}.

Now,

$P(x = 0)$ = Probability of getting no head = $\frac{1}{4}$

$P(x = 1)$ = Probability of getting one head = $\frac{2}{4}$

$P(x = 2)$ = Probability of getting two heads = $\frac{1}{4}$

Therefore, we have

X	0	1	2
P(X)	1/4	2/4	1/4

Example : 2

Find the probability distribution of X(s), getting the number of aces, in an experiment when 2 cards are drawn from a well shuffled deck of 52 cards with replacement.

Solution:

Let getting the number of aces is random variable X. Let X(s) denotes the number aces obtained. In this case we have X(s) = {0, 1, 2}.

Now , $P(x = 0)$ = Probability of getting no ace card = 1 – P (getting ace cards)

$$= \quad 1 - \frac{25}{169} = \frac{144}{169}$$

$P(x = 1)$ = Probability of getting one ace card = $\frac{24}{169}$

$P(x = 2)$ = Probability of getting two heads = $\frac{1}{169}$

Therefore, we have

X	0	1	2
P(X)	144/169	24/169	1/169

Example : 3

Find the probability distribution of Y(s), getting the number of sixes, in an experiment when a dice is tossed twice.

Solution:

Here, we have Y(s) = {0, 1, 2}.

Now

$P(y = 0)$ = Probability of getting no six $= \frac{25}{36}$

$P(y = 1)$ = Probability of getting one ace card $= \frac{5}{18}$

$P(y = 2)$ = Probability of getting two heads $= \frac{1}{36}$

Therefore, we have

X	0	1	2
P(X)	25/36	5/18	1/36

Example : 4

Find the mean and variance of the number of tails in a single throw of two coins.

Solution:

Let X be the number of tails. Then X can take values 0, 1 and 2.

Now,

$P(x = 0)$ = Probability of getting no tail$= \frac{1}{4}$

$P(x = 1)$ = Probability of getting one tail $= \frac{1}{2}$

$P(x = 2)$ = Probability of getting two tails $= \frac{1}{4}$

Therefore, we have

X	0	1	2
P(X)	1/4	1/2	¼

x_i	p_i	$p_i x_i$	$p_i x_i^2$
0	1/4	0	0

1	1/2	1/2	1/2
2	1/4	1/2	1

Table: 1

To find mean and variance, we construct a table

From Table we observed that

$\sum p_i x_i = 1$ and $\sum p_i x_i^2 = \frac{3}{2}$

Now mean

$\mu = \sum p_i x_i = 1$

and

Variance

$\sigma = \sum p_i x_i^2 - \mu^2 = \frac{3}{2} - 1 = \frac{1}{2}.$

Example : 5

It is known that disks produced by a certain company will be defective with probability .01 independently of each other. The company sells the disks in packages of 10 and offers a money–back guarantee that at most 1 of the 10 disks is defective. What proportion of packages is returned? If someone buys three packages, what is the probability that exactly one of them will be returned?

Solution:

If X is the number of defective disks in a package, then assuming that customers always take advantage of the guarantee, it follows that X is a binomial random variable with parameters (10, .01). Hence the probability that a package will have to be replaced is

$$\begin{aligned} P(X > 1) &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - {}^{10}_{0}C(.01)^0 (0.99)^{10} - {}^{10}_{1}C(.01)^1 (0.99)^9 \\ &\approx 0.05 \end{aligned}$$

Example : 6

A coin is tossed six times. What is the probability of obtaining 4 or more heads.

Solution:

When a coin is tossed, the probability of having a head is $p = \frac{1}{2}$.

Therefore, $q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$.

Here $n = 6$, $r = 4, 5, 6$

Therefore,

Probability of having 4 heads i.e. $P(4) = {}^6_4C(1/2)^4(1/2)^2 = 15\left(\frac{1}{2}\right)^6$

Probability of having 5 heads i.e. $P(5) = {}^6_5C(1/2)^5\ (1/2)^1 = 6\left(\frac{1}{2}\right)^6$

Probability of having 6 heads i.e. $P(6) = {}^6_6C(1/2)^6\ (1/2)^0 = \left(\frac{1}{2}\right)^6$

Hence, probability of having 4 or more heads

$15\left(\frac{1}{2}\right)^6 + 6\left(\frac{1}{2}\right)^6 + \left(\frac{1}{2}\right)^6 = \frac{22}{64} = \frac{11}{32}.$

Example : 7

Five cards are drawn at random with replacement from a deck of 52 cards. What is the probability of getting 4 spades?

Solution:

Probability of having a spade $= p = \frac{13}{52} = \frac{1}{4}$

Therefore, $q = 1 - p = 1 - \frac{1}{4} = \frac{3}{4}$

Here $n = 5$, $r = 4$

Therefore,

Probability of getting 4 heads i.e. $P(4) = {}^5_4C(1/4)^4\ (3/4)^1 = \frac{15}{1024}$

Example : 8

A and B are exactly two equally strong chess players, which of the following two events is more probable?

(a) A beats B exactly in 4 games out of 6 , or

(b) A beats B exactly in 7 games out of 9.

Solution: Here, $p = q = \frac{1}{2}$

(a) Probability of A beating B in exactly 4 games out of 6.

$= {}^6_4C\ (1/2)^4\ (1/2)^2 = \frac{15}{64}$

(b) Probability of A beating B in exactly 7 games out of 9

$= {}^9_7C\ (1/2)^7\ (1/2)^2 = \frac{9}{128}$

Since, $\frac{15}{64} > \frac{9}{128}$, therefore first event is more probable.

Example : 9

An unbiased coin is thrown 10 times. Find the probability of getting exactly 6 heads, at least 6 heads.

Solution: Here, $p = q = \frac{1}{2}$ and n = 10. Therefore

Probability of getting exactly 6 heads is = $P(6) = {}^{10}_{6}C(1/2)^6\,(1/2)^4$

Events of at least 6 heads occur when coins shows up head 6, 7, 8, 9, or 10 times.

Hence, probabilities of these events are:

$$P(7) = {}^{10}_{7}C(1/2)^7\,(1/2)^3$$

$$P(8) = {}^{10}_{8}C(1/2)^8\,(1/2)^2$$

$$P(9) = {}^{10}_{9}C(1/2)^9(1/2)^1$$

$$P(10) = {}^{10}_{10}C(1/2)^{10}$$

Therefore,

$$P(\text{at least 6 heads}) = P(6) + P(7) + P(8) + P(9) + P(10).$$

Example : 10

For a random variable X, if binomial variable is B (n = 6, p), then find p if 9P(r = 4) = P(r = 2).

Solution: Given that n = 6, we have to find p, then clearly q = 1 – p and 9 P(4) = P(2).

As we know that

$$P(r) = {}^{n}_{r}C\,p^r\,q^{n-r}$$

Therefore, by definition

$$9\,P(4) = P(2)$$

$$9.\,{}^{6}_{4}C\,p^4\,(1-p)^2 = {}^{6}_{2}C\,p^2(1-p)^4$$

implies, $9p^2 = (1-p)^2$

therefore, $8p^2 + 2p - 1 = 0$

implies, $p = \frac{1}{4}$ or $p = -\frac{1}{2}$

Since $p = -\frac{1}{2}$ is not possible, therefore $P = \frac{1}{4}$.

Example : 11

Mean and variance of Binomial distribution are 6 and 2 respectively. Find $P(r \geq 1)$.

Solution: Here r denotes the number of successes in n trails. Given that

Mean = np = 6 and npq = 2

Then, $q = \frac{npq}{np} = \frac{1}{3}$.

and $p = 1 - \frac{1}{3} = \frac{2}{3}$

Also, $np = 6$ $\therefore$ $n = 9$

Hence, $P(r \geq 1) = 1 - P(r = 0) = 1 - q^n = 1 - \frac{1}{3^9} = 0.999949$

Example : 12

The components of a 6–component system are to be randomly chosen from a bin of 20 used components. The resulting system will be functional if at least 4 of its 6 components are in working condition. If 15 of the 20 components in the bin are in working condition, what is the probability that the resulting system will be functional?

Solution :

If X is the number of working components chosen, then X is Hyper geometric with parameters 15, 5, 6. The probability that the system will be functional is

$$P(X \geq 4) = \Sigma_{r=4}^{6} P(X = r)$$

$$= \frac{\left({}^{15}_{4}C\right)\left({}^{5}_{2}C\right) + \left({}^{15}_{5}C\right)\left({}^{5}_{1}C\right) + \left({}^{15}_{6}C\right)\left({}^{5}_{0}C\right)}{\left({}^{20}_{6}C\right)}$$

$$\approx 0.8687$$

Example : 13

A room has 4 sockets. from a collection of 12 bulbs of which only 5 are good. A person selects 4 bulbs at random (without replacement) and puts them in the sockets. Find the probability that

(i) The room is lighted

(ii) Exactly one bulb in the selected bulbs is good.

Solution: Given that N = 12, M = 5, n = 4, X = r = no of good bulbs in the sample.

Therefore,

$$P(X = r) = \frac{\left({}^{5}_{r}C\right)\left({}^{7}_{4-r}C\right)}{{}^{12}_{4}C}, \ r = 0, 1, 2, 3, 4$$

(i) The room is lighted even if a single bulb is good. Therefore, the required probability is

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \frac{\left({}^{5}_{0}C\right)\left({}^{7}_{4}C\right)}{{}^{12}_{4}C} = 0.9292$$

(ii) $$P(X = 1) = \frac{\left({}^{5}_{1}C\right)\left({}^{7}_{3}C\right)}{{}^{12}_{4}C} = 0.707$$

Example : 14

Out of 100 bulbs sample, the probability of a bulb to be defective is 0.03. Using Poisson distribution, obtain the probability that in a sample of 100 bulbs none is defective.

Solution:

Let X = r be the Poisson variable i.e. no. of defective bulbs in a sample of 100 bulbs.

Therefore by Poisson distribution,

$$P(X = r) = \frac{e^{-\lambda}\lambda^r}{r!},\ r = 0,1,2,\ldots$$

Here, n = 100, p= 0.03

$$\therefore \quad \lambda = np = 100 \times 0.03 = 3$$

Hence,

$$P(X = r) = \frac{e^{-\lambda}\lambda^r}{r!} = \frac{e^{-3}3^r}{r!}$$

$$\therefore \quad P\text{ (none is defective)} = P(X = 0) = \frac{e^{-3}3^0}{0!} = \frac{0.0479\times 1}{1} = 0.0479.$$

Example : 15

2% of bolts manufactured by a factory are found to be defective. Find the probability that in a packet of 200 bolts not more than 3 bolts will come out to be defective. (Take e^{-4} = 0.0183)

Solution:

Let X = r be the passion variable i.e. no. of defective bolts in a packet of 200 bolts.

Therefore by Poisson distribution

$$P(X= r) = \frac{e^{-\lambda}\lambda^r}{r!},\ r = 0,1,2,\ldots$$

Here, n = 200, p = 2%= 0.02

$$\therefore \quad \lambda = np = 200 \times 0.02 = 4$$

Hence,

$$P(X= r) = \frac{e^{-\lambda}\lambda^r}{r!} = \frac{e^{-4}4^r}{r!}$$

$$\therefore \quad P\text{ (not more than 3 defective)} = P(X \le 3)$$

$$= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$

$$= \frac{e^{-4}4^0}{0!} + \frac{e^{-4}4^1}{1!} + \frac{e^{-4}4^2}{2!} + \frac{e^{-4}4^3}{3!}$$

$$= e^{-4}\left[1 + 4 + \frac{16}{2} + \frac{64}{6}\right]$$

$$= 0.4331$$

Example : 16

Suppose that the average number of accidents occurring weekly on a particular stretch of a highway equals 3. Calculate the probability that there is at least one accident this week.

Solution:

Let X denote the number of accidents occurring on the stretch of highway in question during this week. Because it is reasonable to suppose that there are a large number of cars passing along that stretch, each having a small probability of being involved in an accident, the number of such accidents should be approximately Poisson distributed.

Hence,

$$P(X \geq 1) = 1 - P[X = 0]$$
$$= 1 - \frac{e^{-3}\,3^0}{0!} = 1 - e^{-3} \approx 0.9502$$

Example : 17

If the average number of claims handled daily by an insurance company is 5, what proportion of days has less than 3 claims? What is the probability that there will be 4 claims in exactly 3 of the next 5 days? Assume that the number of claims on different days is independent.

Solution:

Because the company probably insures a large number of clients, each having a small probability of making a claim on any given day, it is reasonable to suppose that the number of claims handled daily, call it X, is a Poisson random variable. Since E(X) = 5, the probability that there will be fewer than 3 claims on any given day is

$$P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2)$$
$$= \frac{e^{-5}\,5^0}{0!} + \frac{e^{-5}\,5^1}{1!} + \frac{e^{-5}\,5^2}{2!}$$
$$= e^{-5}\left[1 + 5 + \frac{25}{2}\right] \approx 0.1247$$

Example : 18

Assuming that the diameters of 1000 plugs taken consecutively from machine from a normal distribution with mean 0.7515 cm and SD 0.0020 cm. How many of the plugs are likely to be approved if the acceptable diameter is 0.752 ± 0.004 cm?

Solution:

Let x be the number of plugs likely to be approved.

Here $\sigma = 0.0020, \mu = 0.7515,$

$$x_1 = 0.756\ ,\ x_2 = 0.748$$

This implies,

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{0.756 - 0.7515}{0.0020} = 2.25$$

and

$$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{0.748 - 0.7515}{0.0020} = -1.75$$

A_1 is the area for the $z_1 = 2.25 = 0.4878$

A_2 is the area for the $z_2 = 1.75 = 0.4599$

$\therefore$ $P(x) = 0.4878 + 0.4599 = 0.9477.$

Therefore, numbers of plugs likely to be approved = 1000 × 0.9477 = 948 approx.

Example : 19

The mean weight of 500 students is 63 kgs and the SD is 8 kgs. Assuming that the weights are normally distributed, find how many students weight 52kgs? The weights are recorded to the nearest kg.

Solution:

Since the weights are recorded to the nearest kg, the students weighting 52 kg have their actual weights between x = 51.5 and 52.5 kg. So the area under the curve from x = 51.5 and x = 52.5 kg is to be obtained.

We know that $z = \dfrac{x - \mu}{\sigma}$

This implies, $z_1 = \dfrac{51.5 - 63}{8} = -1.4375 = -1.44$ (appx)

and $z_2 = \dfrac{52.5 - 63}{8} = -1.3125 = -1.31$ (appx)

The number of students weight 52 kg = $500 \int_{51.5}^{52.5} f(x)\, dx$

The frequency curve for the given curve is $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1(x-\mu)^2}{2\sigma^2}}$

$\therefore$ The number of students weighting 52 kg $= \dfrac{500}{\sqrt{2\pi}} \int_{-1.44}^{-1.31} e^{-z^2/2}\, dz$

$$= 500\,(A_1 - A_2)$$

Where,

A_1 is the area for the $z_1 = -1.44 = 0.4251$.

A_2 is the area for the $z_2 = -1.31 = 0.4049$.

$\therefore$ The number of students weighting 52 kg = 500 (0.4251 – 0.4049)

= 10 students approximately.

Self–Assessment Exercise 4.2

1. Two cards are drawn simultaneously. Find the probability distribution of number of queens. **[Ans. 188/221, 32/221, 1/221]**

2. Find the mean and variance of number of sixes in two tosses of a die. **[Ans. 1/3, 5/18]**

3. A die is thrown 12 times. Getting an even number is a success. What is the probability of at least six successes? **[Ans. 193/512]**

4. A coin is tossed five times. Find the probability of getting 3 heads. **[Ans. 5/16]**

5. A coin is tossed 10 times. Find the probability of having at least 7 tails. **[Ans. 175/1024]**

6. In a certain factory tuning out razor blades, there is a small chance of 0.002 for any black to be defective. The Blades are supplied in packets of 10, use Poisson distribution to calculate approx, number of packets containing no defective, one defective and two defective blades respectively in a consignment of 10000 packets.

[Ans. 9802, 196, 2 packets]

7. Let z be a standard normal variate, then find

(i) $P(0 \le z \le 1.42)$ **[Ans. 0.4222]**

(ii) $P(z \ge -1.28)$ **[Ans. 0.8997]**

(iii) $P(0.81 \le z \le 1.94)$ **[Ans. 0.1828]**

8. If X is a normal variate with mean 30 and S.D. is 5 then find

(i) $P(26 \le X \le 40)$ **[Ans. 0.7563]**

(ii) $P(X \ge 45)$ **[Ans. 0.00135]**

9. In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find the mean and S.D. of the distribution. **[Ans. 50, 10]**

10. The wages of the workers are normally distributed with a standard deviation of 100. The probability that workers get less than ₹ 300 is 33%. What is the mean of the distribution? **[Ans. 344]**

4.19 Sampling Distribution :

Suppose that we draw all possible samples of size n from a given population. Suppose further that we compute a statistic (e.g., a mean, proportion, standard deviation) for each sample. The probability distribution of this statistic is called a sampling distribution.

- **Sampling Distribution of Mean :**

If $\bar{x}$ is the mean of random sample of size n, drawn from a population with mean μ and standard deviation σ, then the sampling distribution of $\bar{x}$ is approximately a normal distribution.

- **Standard Error of the mean:**

It is the standard deviation of a sampling distribution and is given by

Standard error = $\bar{\sigma} = \frac{\sigma}{\sqrt{N}}$

where, σ is the standard deviation of given population.

4.20 Concepts of Null Hypothesis and Alternative Hypothesis :

- **Hypothesis :**

A Hypothesis is the statement or an assumption about relationships between variables. It is also called significance testing.

- **Criteria for Hypothesis Construction:**
  a) It should be correctly testable, whether it is right or wrong.
  b) It should be specific and accurate.
  c) The statements in the hypothesis should not be contradictory.
  d) It should specify variables between which the relationship is to be established.
  e) It should describe one issue only.
- **Hypothesis Testing Steps:**
  a) Null and alternative hypotheses
  b) Level of Significance
  c) Test statistic
  d) Critical region
  e) Errors

A. Null hypotheses :

The null hypothesis means "no difference in the population" i.e both is same. It is denoted by H_0. The acceptance of null hypothesis implies we have no evidence to believe otherwise and indicates that the difference is not so significant and it's only due to sampling fluctuation.

B. Alternative hypotheses :

The alternative hypothesis (H_1) claims "H_0 is false". Its acceptance depends on the rejection of the null hypothesis.

C. Test statistic:

This is an example of a one–sample test of a mean when σ (variance) is known. Use this statistic to test the problem:

$$z_{stat} = \frac{\bar{x} - \mu_0}{SE_{\bar{x}}}$$

where μ_0 = population mean assuming H_0 is true

$$SE_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$

D. Level of Significance :

Sometimes, the experimenter will fix some value also known as the significance level. It is the maximum probability of rejecting the null hypothesis when it is true. It is usually expressed in %. Most significant levels are 1%, 5%.

E. Critical or Rejection region :

Critical Region is the region which corresponds to a predetermined level of significance. The set of values of the test statistics which leads to rejection or acceptance of the null hypothesis is called region of rejection or acceptance. Critical value is that value of statistics which separates the critical region from the acceptance. It lies at the boundary of the regions of acceptance and rejection.

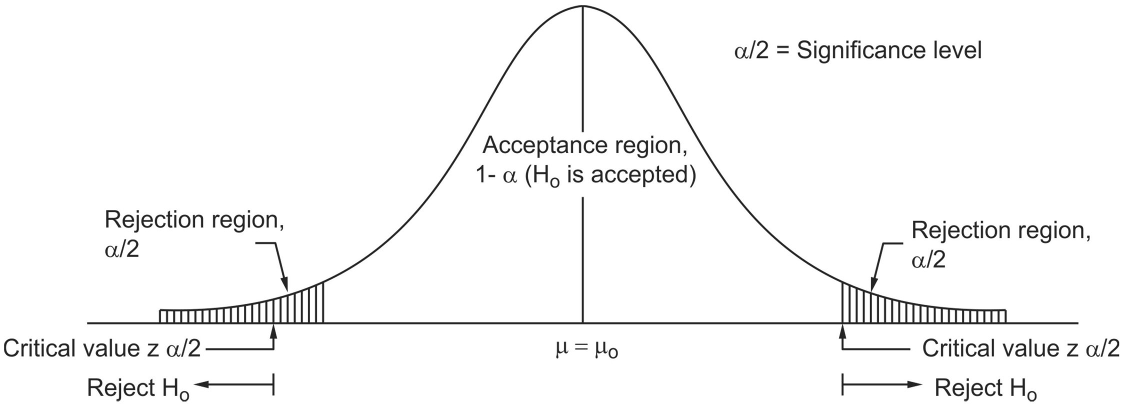


Fig. 4.1 : Areas of Accepted and Rejection of Ho (Two – Tailed tes)

Critical value of test statistics at 1% and 5% level of significance are given in following table:

Type of test	**Level of Significance**	
	1%	5%
Two tailed	2.58	1.96
One tailed	2.33	1.645

F. Errors in Hypothesis Testing:

Since decision of acceptance or rejection of H_0 is based on sampling, it is subjected to two kinds of error.

Type I error: A type I error consists of rejecting the null hypothesis H_0, when it was true.

Type II error: A type II error consists of not rejecting H_0, when H_0 is false.

4.21 Hypothesis Statements :

Case I: When H_0: $\mu = \mu_0$ vs. H_1: $\mu < \mu_0$ (left–sided)

This is a one–tailed test with the critical region in the left–tail of the test statistic.

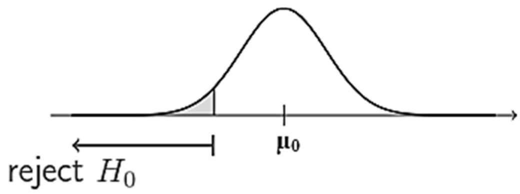


Fig. 4.2

Case II: When H_0: $\mu = \mu_0$ vs. H_1: $\mu > \mu_0$ (right–sided)

This is a one–tailed test with the critical region in the right–tail of the test statistic.

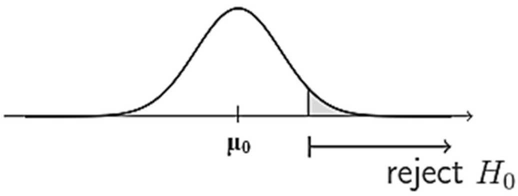


Fig. 4.3

Case III: When H_0: $\mu = \mu_0$ vs. H_1: $\mu \neq \mu_0$ (two–sided)

This is a two–tailed test with the critical region in the both side of the test statistic.

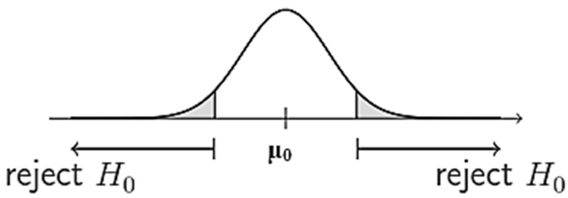


Fig. 4.4

4.22 Degree of Freedom :

The degrees of freedom in a statistical calculation represent how many values involved in our calculations have the freedom to vary. We can calculate degrees of freedom to help ensure the statistical validity of chi–square tests, t–tests and even the more advanced F–tests.

The statistical formula to determine degrees of freedom is quite simple. It states that degrees of freedom equal the number of values in a data set minus 1, and looks like this:

$$df = N - 1,$$

where N is the number of values in the data set (sample size).

Example: Let's say we have a data set of 4 (N=4).

We will call the data set **x** and create a list with the values for each data.

X: 15, 30, 25, 10

This data set has a mean (also called average) of 20, which can be calculated by adding the values and dividing by N i.e.

Mean = (15 + 30 + 25 + 10)/4 = 20.

Using the formula, the degrees of freedom would be calculated as df = N – 1

df = 4 – 1 = 3

This indicates that, in this data set, three numbers have the freedom to vary as long as the mean remains 20.

4.23 Chi–Square Distribution (χ^2) :

It is a continuous probability distribution. It has the zero value at its lower limit and extends to infinity in the positive direction. It is mainly applied in testing of hypothesis for testing the independence of attributes, testing the goodness of fit of a model. It is denoted as x^2.

- **Assumptions for χ^2 test:**
  - **a)** Independent observations
  - **b)** A sample size of at least 50 observations.
  - **c)** Random sampling
  - **d)** All observations must be used
  - **e)** For the test to be accurate, the expected frequency should be at least 5.
- **Properties of Chi–square distribution:**
  - i) The value of χ^2 is never negative since the differences between the observed and expected frequencies are always squared.
  - ii) It has only one parameter *i.e.* degree of freedom (df).
  - iii) Its shaped is depends upon the number of degree of freedoms.
  - iv) In general
    - a) If df is small, the shape of the distribution is skewed to the right.
    - b) If df is large, the shape of the distribution becomes more and more symmetrical
  - v) Its mean = degree of freedom (df)
  - vi) Its variance = twice degree of freedom (2 df)

4.24 Conducting Chi–Square Analysis :

A. Test for goodness of fit of χ^2 distribution:

Step 1. Make a hypothesis based on your basic biological question

Step 2. Let $O_1, O_2, \ldots, O_i, \ldots, O_k$ be set of observed frequencies given and $E_1, E_2, \ldots, E_i, \ldots, e_k$ be corresponding expected frequencies obtained under hypothesis Ho such that

$$\sum_{i=1}^{k} Oi = N = \sum_{i=1}^{k} E_i$$

Step 3. Create a table with observed frequencies, expected frequencies, and then calculate chi–square values using the formula:

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} = \sum_{i=1}^{k} \left(\frac{O_i}{E_i}\right)^2 - N$$

Step 4. Assume p = number of parameters estimated for fitting the probability distribution and then calculate degrees of freedom with = k – p – 1.

Step 5. Find the chi–square statistic in the Chi–Square Distribution table.

Step 6. If chi–square statistic > your calculated chi–square value, you *do not reject* your null hypothesis H_0 and vice versa.

B. Chi–Square as a test in 2 x 2 Contingency tables:

χ^2 is used to test the association between the attributes when the sample data is presented in the form of contingency table.

Step 1. Make a hypothesis based on your basic biological question

Step 2. Calculate observed frequencies and make 2 x 2 Contingency tables. Determine the expected frequencies using the formula

$$E_{ij} = \frac{R_i \times C_j}{n}.$$

Step 3. Create a table with observed frequencies, expected frequencies, and then calculate chi–square values using the formula:

$$\chi^2 = \sum \frac{(O - E)^2}{E}$$

Step 4. Calculate degrees of freedom with (n –1) (m –1).

Step 5. Find the chi–square statistic in the Chi–Square Distribution table.

Step 6. If chi–square statistic > your calculated chi–square value, you *do not reject* your null hypothesis H_0 and vice versa.

Illustrative Examples

Example : 1

Consider a population {2, 3, 4, 5, 6}. Write all possible random sample of size two which can be drawn with replacement from this population.

Solution: In a random sampling with replacement, these numbers are arranging as following ways:

(2,2) (2,3) (2,4) (2,5) (2,6); (3,2) (3,3) (3,4) (3,5) (6,6); (4,2) (4,3) (4,4) (4,5) (4,6);

(5,2) (5,3) (5,4) (5,5) (5,6); (6,2) (6,3) (6,4) (6,5) (6,6);

Mean of random sample size are:

2, 2.5, 3.0, 3.5, 4.0; 2.5, 3.0, 3.5, 4.0, 4.5; 3.0, 3.5, 4.0, 4.5, 5.0;

3.5, 4.0, 4.5, 5.0, 5.5; 4.0, 4.5, 5.0, 5.5, 6.0.

Hence the sampling distribution is:

Sampling mean	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0
frequency	1	2	3	4	4	4	3	2	1

Example : 2

Consider a population {2, 3, 4, 5, 6}. Write all possible random sample of size two which can be drawn with replacement from this population. Find the standard error of the population.

Solution: From above **Example** we find the sampling distribution table as follows.

Sampling mean	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0
frequency	1	2	3	4	4	4	3	2	1

From above table,

Mean of S.D. = 6.

and hence $\bar{\sigma} = 2.32$

Example : 3

The average number of defective articles per day in a certain factory is claimed to be less than the average of all the factories. The average for all the factories is 30.5. A random sample of 100 days showed the following distribution

Class Limit	16–20	21–25	26–30	31–35	36–40	Total
No. of Days	12	22	20	30	16	100

Test the claim that the average is less than or equal to the figure of all factories, at 5% level of significance.

Solution: We can easily calculate the mean and S.D. of the above samples.

Mean = 28.8

Standard Deviation: 6.35

Here define Hypothesis as:

$H_0 : \mu < 30.5 \quad H_1 : \mu \geq 30.5$

Now, $SE_x = \dfrac{SD}{\sqrt{n}} = \dfrac{6.35}{\sqrt{100}} = .0635$

$Z_{stat} = \dfrac{28.8 - 30.5}{0.635} = -2.68$

$| Z_{stat} | > Z_{cric} = 2.33$

Since the computed value is greater than tabular value, then H_0 is rejected and hence our claim is not significant i.e. average is less than or equal to the figure of all factories.

Example : 4

State bank of India utilizes four teller windows to render fast service to the customers. On a particular day 800 customers were observed. They were given service at the different windows as follows.

Window Number	Expected number of customer
1	150
2	250
3	170
4	230

Test at 5% level of significance, whether the customers are uniformly distributed over the window.

Solution:

Set up the hypothesis as follows:

H_0: customers on all windows are uniformly distributed

H_1: customers are not equal on all windows.

Under the null hypothesis H_0, the expected frequencies are

Window Number	Expected number of customer (E_i)
1	200
2	200
3	200
4	200

Therefore chi–square distribution is :

Window Number	Expected number of customer O_i	Expected number of customer E_i	$O_i - E_i$	$(O_i - E_i)^2$	$\frac{(O_i - E_i)^2}{E_i}$
1	150	200	–50	2500	12.5
2	250	200	50	2500	12.5
3	170	200	–30	900	4.5
4	230	200	30	900	4.5
x^2=					**34**

Also at 5% level of significance and d.f. = 3, $\chi^2_{(stat)} = 7.81$

Here $\chi^2(\text{cal}) = 34 > \chi^2(\text{stat}) = 7.81$, so we reject H_0, and hence we conclude that difference is significant. This implies that customers are not uniformly distributed over all windows.

Example : 5

Following table gives the number of clothes sold per day on a retail shop

Days	No of sold clothes	$(O_i - E_i)^2$
Monday	120	0
Tuesday	130	100
Wednesday	110	100
Thursday	115	25
Friday	135	225
Saturday	110	100

Test at 5% level of significance, whether selling of clothes are day dependent.

Solution:

Set up the hypothesis as follows:

H_0: Selling of clothes not dependent on days.

H_1: Selling of clothes are days dependents.

Now, $$\chi^2 = \sum_{i=1}^{5} \frac{(O_i - E_i)^2}{E_i} = \frac{550}{120} = 4.5833$$

Also at 5% level of significance and d.f. = 5, $\chi^2_{(stat)} = 11.07$

Here $\chi^2_{(cal)} = 4.583 < \chi^2_{(stat)} = 11.07 = 7.81$, so we accept H_0. Hence selling of clothes are days independent.

Example : 6

In an experiment of pea breading, the following frequencies of seeds were obtained.

Round and Green	Wrinkled and green	Round and yellow	Wrinkled and yellow	Total
222	120	32	150	524

Theory predicts that the frequencies should be in proportion 8:2:2:1. Examine the correspondence between theory and experiment.

Solution:

Set up the hypothesis as follows:

H_0: There is very high degree of correspondence between theory and experiment.

H_1: There is very low degree of correspondence between theory and experiment.

The chi–square distribution is:

O_i	E_i	$O_i - E_i$	$(O_i - E_i)^2$	$\frac{(O_i - E_i)^2}{E_i}$
222	$\frac{8}{13} \times 524 = 323$	–101	10201	31.58
120	$\frac{2}{13} \times 524 = 81$	39	1521	18.78
32	$\frac{8}{13} \times 524 = 81$	–49	2401	29.64
150	$\frac{1}{13} \times 524 = 40$	110	12100	302.5
			χ^2	382.50

Also at 5% level of significance and d.f. =3, $\chi^2_{(stat)} = 7.81$

Here $\chi^2_{(cal)} = 382.50 > \chi^2_{(stat)} = 7.81$, so we reject H_0, and hence we conclude that difference is significant. This implies that there is very low degree of correspondence between theory and experiment.

Example : 7

In a survey of 200 girls of which 40% were intelligent, 30% had uneducated fathers, while 20% of the unintelligent girls had educated fathers. Do these figures support the hypothesis that educated fathers have intelligent girls? Test at 5% level of significance.

Solution:

Set up the hypothesis as follows:

H_0: NO association exists between the educated fathers and intelligent girls.

H_1: An association exists between the educated fathers and intelligent girls

Calculate observed frequencies:

	Intelligent girls	**Unintelligent girls**	
Educated fathers	56	24	80
Uneducated fathers	24	96	120
Total	80	120	200

Calculation of expected frequencies

	Intelligent girls	**Unintelligent girls**	
Educated fathers	80*80/200 = 32	80*120/200 = 48	80
Uneducated fathers	120*80/200 = 48	120 * 120/ 200 = 72	120
Total	80	120	200

Calculation of χ^2

O	**E**	**(O – E)**	**$(O - E)^2$**	**$(O - E)^2/E$**
56	32	24	576	18
24	48	–24	576	12
24	48	–24	576	12
96	72	24	576	8
χ^2 =				50

Also at 5% level of significance and d.f. = 1, $\chi^2_{(stat)} = 3.84$

Since the commuted value is greater than the tabular value, so we reject H0. Hence an association exists between the educated fathers and intelligent girls.

Self–Assessment Exercise 4.3

1. In a survey of 2000 students of which 55% were undergraduate, 20% favored the autonomous colleges., while 40% of the post graduates opposed. Test at 5% level of significance that opinions of undergraduates and post graduates students on autonomous status of colleges are independent.
2. Out of 8000 graduates in town 800 are females. Out of 1600 graduate's employees 120are females. Use Chi –Square test to determine if any distinction is made in appointment on the basis of sex. Value of Chi –Square for 5% level for one degree of freedom is 3.84
3. In a survey of 200 boys of which 75 were intelligent, 40 had skilled fathers, while 85 of the unintelligent boys had unskilled fathers. Do these figures support the hypothesis that skilled fathers have intelligent boys? Test at 5% level of significance.
4. In a telephone exchange, the probability that any one call is wrongly connected is 0.02. What is the minimum number of calls required to ensure a probability 0.1 that at least one call is wrongly connected. **[Ans. 6 calls approx]**
5. In a certain examination, the percentage of passes and distinction were 48 and 10 respectively. Estimate the average marks obtained by the candidates, the minimum passes and distinction marks being 40 and 75 respectively. **[Ans. 38.57]**
6. In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find the mean and standard deviation of distribution. **[Ans. 49.92, 10.05]**
7. Fit a Binomial distribution to the following data.

x	0	1	2	3	4	5
f	2	22	63	76	96	56

8. The following is a 2 x 2 contingency table:

Eye color in father	**Eye color in son**	
	Not light	**Light**
Not light	23	15
Light	15	47

Test whether the eye color in son is associated with the eye color in father.

◆◆◆

Unit 5

Vector Calculus

Syllabus :

Vector differentiation, Gradient, Divergence and Curl, Directional derivative, Solenoid and Irrigational fields, Vector identities. Line, Surface and Volume integrals, Green's Lemma, Gauss's Divergence theorem and Stoke's theorem.

- **Scalar :**
  A quantity which has got magnitude alone but not direction is known as scalar quantity.
  e.g. : Mass, Length, Time, Temperature, Work, Energy, Statistical data.
- **Vector :**
  A quantity which has got both magnitude and direction is called vector quantity.
  e.g. : Velocity, Acceleration, Force, Displacement, Momentum.
- **Position Vector :**
  A point P in space can be associated with a vector by jointing the point P with some point O in space called origin of reference. Thus $\overrightarrow{OP} = \vec{a}$ is associated with point P and is called position vector of the point P.

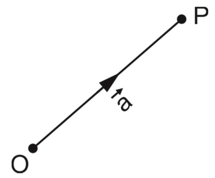


Fig.: 5.1

5.1 Physical Interpretation of Vector Differentiations :

- **Product of Vectors :**

1. **Scalar product (or) Dot product :**

If $\vec{a}$ and $\vec{b}$ are two vectors inclined at an angle θ with each other, then the dot product of vectors $\vec{a}$ and $\vec{b}$ are defined as

$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta = ab\cos\theta$, where $a = |\vec{a}|$ and $b = |\vec{b}|$

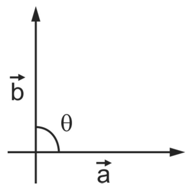


Fig.: 5.2

Note :

i) If vectors $\vec{a}$ and $\vec{b}$ are in the same direction (i.e. parallel) then $\theta = 0$ $\cos\theta = 1$, $\vec{a}\cdot\vec{b} = ab$

ii) If vectors $\vec{a}$ and $\vec{b}$ are in the opposite directions $\theta = \pi$, $\cos\theta = -1$ $\vec{a}\cdot\vec{b} = -ab$.

iii) If vectors $\vec{a}$ and $\vec{b}$ are perpendicular (or orthogonal) $\theta = \frac{\pi}{2}$, $\cos\theta = 0$, $\vec{a}\cdot\vec{b} = 0$

iv) If two vectors are equal $\vec{a} = \vec{b}$ then

$\vec{a}\cdot\vec{a} = a^2$

v) $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

2. Vector (or cross) product :

If $\vec{a}$ and $\vec{b}$ are two vectors inclined at an angle θ with each other then the cross product of vectors $\vec{a}$ and $\vec{b}$ is defined as $\vec{a}\times\vec{b} = ab\sin\theta\hat{n}$, where $\hat{n}$ is a unit vector normal to the plane containing the vectors $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ form a right handed system.

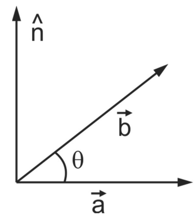


Fig.: 5.3

$$\therefore \qquad |\vec{a}\times\vec{b}| = ab\sin\theta, \qquad \hat{n} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$$

Note :

i) If vectors $\vec{a}$ and $\vec{b}$ are in the same direction (i.e. parallel) $\theta = 0$, $\sin\theta = 0$, $\vec{a}\times\vec{b} = 0$

ii) If vectors $\vec{a}$ and $\vec{b}$ are in the opposite direction $\theta = \pi$, $\sin\theta = 0$, $\vec{a}\times\vec{b} = 0$

iii) If vectors $\vec{a}$ and $\vec{b}$ are perpendicular (or orthogonal), $\theta = \frac{\pi}{2}$, $\sin\theta = 1$, $\vec{a}\times\vec{b} = ab\hat{n}$

iv) If two vectors are equal $\vec{a} = \vec{b}$ $\quad \therefore \vec{a}\times\vec{a} = 0$

v) $(\vec{a}\times\vec{b}) = -(\vec{b}\times\vec{a})$

vi) Let $\vec{a} = a_1i + a_2j + a_3k$, $\vec{b} = b_1i + b_2j + b_3k$

then $$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

- **Unit vector :**

A vector whose magnitude is unity (or one) is called unit vector.

If $\vec{a}$ is any vector and $\hat{a}$ a unit vector in the direction of $\vec{a}$ then we have

$$\vec{a} = |\vec{a}|\hat{a} \qquad \therefore \hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

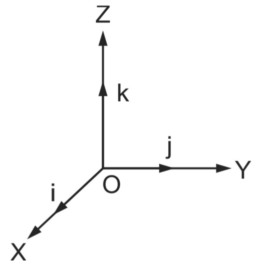


Fig.: 5.4

Note : Let i, j, k are unit vectors along X–axis, Y–axis and Z–axis. Then the dot product of unit vectors are ;

$i \cdot i = 1, \quad j \cdot j = 1, \quad k \cdot k = 1, \quad i \cdot j = 0, \quad j \cdot k = 0, \quad k \cdot i = 0$

The cross product of unit vectors are

$i \times j = k, \; j \times k = i, \; k \times i = j, \; i \times i = 0, \; j \times j = 0, \; k \times k = 0$

- **Scalar Triple Product :**

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors then their dot product is expressed as

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

where,

$$\vec{a} = a_1 i + a_2 j + a_3 k$$
$$\vec{b} = b_1 i + b_2 i + b_3 k$$
$$\vec{c} = c_1 i + c_2 j + c_3 k$$

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}\ , \ |\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}\ , \ |\vec{c}| = \sqrt{c_1^2 + c_2^2 + c_3^2}$$

if any two vectors are equal then $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

- **Vector Triple Product :**

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ are there vectors then their vector (or cross) product is expressed as $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\, \vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

- **Angle between two vectors :**

The angle between two vectors is given by $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$

where,

$$\vec{a} = a_1 i + a_2 j + a_3 k, \quad \vec{b} = b_1 i + b_2 j + b_3 k$$
$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} \qquad |\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$$

- **Distance between two points :**

The distance between two points is given by,

$$\overrightarrow{PQ} = (x_2 - x_1)\, i + (y_2 - y_1)\, j + (z_2 - z_1)\, k$$

Fig.: 5.5

- **Position Vector :**

The position vector

$$\vec{r} = xi + yj + zk \text{ and}$$
$$|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$$
$$\therefore x^2 + y^2 + z^2 = r^2 \qquad \text{…. Equation of sphere}$$

Coordinates of centre are C (0, 0, 0) and radius = r.

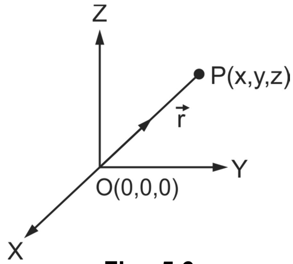


Fig.: 5.6

The dot product, cross product and unit vector are :

$$\vec{r}\cdot\vec{r} = r^2,\quad \vec{r}\times\vec{r} = 0,\quad \vec{r} = |\vec{r}|\,\hat{r} \qquad \therefore \quad \hat{r} = \frac{\vec{r}}{|\vec{r}|}$$

or $$\vec{r} = r\hat{r} \quad \text{i.e. } \hat{r} = \frac{\vec{r}}{r} \quad (\text{i.e. } r = |\vec{r}|)$$

Perpendicular distance of the point (x_1, y_1, z_1) from the plane $ax + by + cz + d = 0$ is

$$p = \left| \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}} \right|$$

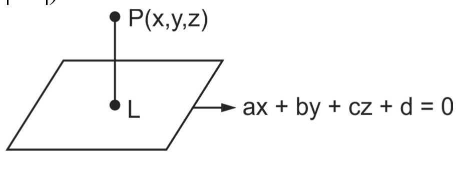


Fig.: 5.7

- **Direction Cosines :**

Let any line OP make angles α, β, γ with x, y, z – axes respectively, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the direction cosines of this line which are usually denoted by l, m, n.

$l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$ and $l^2 + m^2 + n^2 = 1$

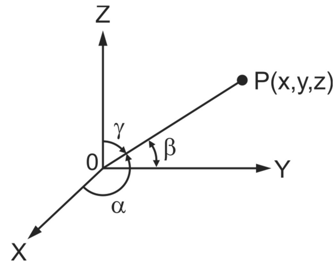


Fig.: 5.8

- **Direction Ratios :**

If the direction cosines of a line be proportional to a, b, c then these are called proportional direction cosines or direction ratios of the line, then $\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{a^2 + b^2 + c^2}} = \frac{1}{\sqrt{a^2 + b^2 + c^2}}$

$$\therefore \quad l = \frac{a}{\sqrt{a^2 + b^2 + c^2}},\quad m = \frac{b}{\sqrt{a^2 + b^2 + c^2}},\quad n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

Note : Any unit vector can be expressed as,

$\hat{r} = \cos\alpha i + \cos\beta j + \cos\gamma k = li + mj + nk$

$|\hat{r}| = \sqrt{\cos^2\alpha + \cos^2\beta + \cos^2\gamma} = \sqrt{l^2 + m^2 + n^2} = 1$

- **Equation of a Straight Line :**

i) Equations of the line through the point (x_1, y_1, z_1) and having direction cosines l, m, n are

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

ii) The equations of the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \Rightarrow \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

where a, b, c are direction ratios of the line.

1. Differentiation of Vectors :

If a vector $\vec{r}$ varies continuously as a scalar variable t changes, then $\vec{r}$ is said to be a function of t and is written as $\vec{r} = \vec{F}$ (t). Here we define derivative of a vector function $\vec{r}$ = F (t) as

$$\frac{d\vec{r}}{dt} = \frac{d\vec{F}}{dt} = \lim_{\delta t \to 0} \frac{\vec{F}(t+\delta t) - \vec{F}(t)}{\delta t}$$

2. General rules of differentiation :

If ϕ $\vec{u}$, $\vec{v}$, $\vec{w}$ are scalar and vector functions of a scalar variable t, we have.

i) $\frac{d}{dt}\left[\vec{u} - \vec{v} - \vec{w}\right] = \frac{d\vec{u}}{dt} + \frac{d\vec{v}}{dt} - \frac{d\vec{w}}{dt}$

ii) $\frac{d}{dt}(\phi\,\vec{u}) = \vec{u}\frac{d\phi}{dt} + \frac{d\vec{u}}{dt}\phi$

iii) $\frac{d}{dt}\left(\vec{u}\cdot\vec{v}\right) = \vec{u}\cdot\frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt}\cdot\vec{v}$

iv) $\frac{d}{dt}\left(\vec{u}\times\vec{v}\right) = \vec{u}\times\frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt}\times\vec{v}$

v) $\frac{d}{dt}\left[\vec{u}\cdot\left(\vec{v}\times\vec{w}\right)\right] = \frac{d\vec{u}}{dt}\cdot\left(\vec{v}\times\vec{w}\right) + \vec{u}\cdot\left(\frac{d\vec{v}}{dt}\times\vec{w}\right) + \vec{u}\cdot\left(\vec{v}\times\frac{d\vec{w}}{dt}\right)$

vi) $\frac{d}{dt}\left[\vec{u}\times\vec{v}\times\vec{w}\right] = \frac{d\vec{u}}{dt}\times\left(\vec{v}\times\vec{w}\right) + \vec{u}\times\left(\frac{d\vec{v}}{dt}\times\vec{w}\right) + \vec{u}\times\left(\vec{v}\times\frac{d\vec{w}}{dt}\right)$

Obs. 1. If $\vec{r}$ (t) has a constant magnitude, then $\vec{r}\cdot\frac{d\vec{r}}{dt} = 0$. For $\vec{r}(t)\cdot\vec{r}(t) = |\vec{r}(t)|^2$ = constant.

$$\therefore \quad \vec{r}\cdot\frac{d\vec{r}}{dt} = 0 \qquad \text{i.e. either } \frac{d\vec{r}}{dt} = 0 \text{ or } \frac{d\vec{r}}{dt} \perp \vec{r}(t).$$

Obs. 2. If $\vec{r}$ (t) has constant (fixed) direction then

$\vec{r}\times\frac{d\vec{r}}{dt} = 0$, let $\hat{r}$ be a unit vector in the direction of $\vec{r}$ (t) so that $\vec{r}(t) = r\,\hat{r}$ $r = |\vec{r}|$.

$$\frac{d\vec{r}}{dt} = r\frac{d\hat{r}}{dt} + \frac{dr}{dt}\hat{r} \text{ and } \vec{r}\times\frac{d\vec{r}}{dt}$$

$$= r\,\hat{r}\times\left[r\frac{d\hat{r}}{dt} + \frac{dr}{dt}\hat{r}\right]$$

$$\vec{r}\times\frac{d\vec{r}}{dt} = r^2\,\hat{r}\times\frac{d\hat{r}}{dt} = 0$$

$\left(\text{since of } \hat{r} \text{ is constant}, \frac{d\hat{r}}{dt} = 0\right)$

- **Geometrical Interpretation of $\frac{d\vec{r}}{dt}$.**

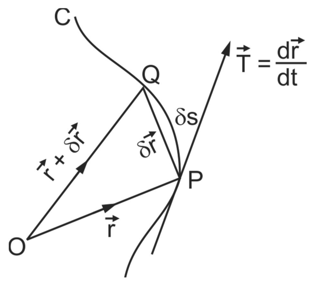


Fig.: 5.9

Let $\overrightarrow{OP} = \vec{r}$ and $\overrightarrow{OQ} = \vec{r} + \delta\vec{r}$ then

$$\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ} \quad \text{or} \quad \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

Since $\frac{d\vec{r}}{dt} = \lim_{\delta t \to 0} \frac{\delta\vec{r}}{\delta t}$ as $Q \to P$, $\delta t \to 0$ the direction of $\frac{d\vec{r}}{dt}$ approaches the direction of the tangent to the curve at P.

i.e. : The vector $\frac{d\vec{r}}{dt}$ is along the tangent to the space curve traced out by P.

Let $\overrightarrow{OP}$ makes a position vector $\vec{r}$ and t is time then $\frac{d\vec{r}}{dt}$, $\frac{d^2\vec{r}}{dt^2}$ are velocity and acceleration. $\left(\lim_{\theta \to P} \frac{\text{chord } PQ}{\text{Arc}PQ} = 1\right)$

Illustrative Examples

Example : 1

If $\vec{F} = \vec{a}\cos wt + \vec{b}\sin wt$, where $\vec{a}$, $\vec{b}$, w are constant show that,

i) $\frac{d^2\vec{F}}{dt^2} + w^2\vec{F} = 0$

ii) $\vec{F} \times \frac{d\vec{F}}{dt} = w(\vec{a} \times \vec{b}) = \text{constant}$.

Solution : Differentiating $\vec{F}$ with respect t we get

$$\frac{d\vec{F}}{dt} = \vec{a}(-w \sin t\, wt) + \vec{b}(w \cos wt)$$

$$\frac{d^2\vec{F}}{dt^2} = \vec{a}(-w^2\cos wt) + \vec{b}(-w^2\sin wt)$$

i) $$\frac{d^2\vec{F}}{dt^2} = -w^2(\vec{a}\cos wt + \vec{b}\sin wt) = -w^2\vec{F}$$

$$\frac{d^2\vec{F}}{dt^2} + w^2\vec{F} = 0$$

ii)
$$\vec{F} \times \frac{d\vec{F}}{dt} = (\vec{a}\cos wt + \vec{b}\sin wt) \times (\vec{a}(-w\sin wt) + \vec{b}(w\cos wt))$$

$$\vec{F} \times \frac{d\vec{F}}{dt} = (\vec{a} \times \vec{a})(-w\sin wt\cos wt) + (\vec{a} \times \vec{b})(w\cos^2 wt)$$
$$+ (\vec{b} \times \vec{a})(-w\sin^2 wt) + (\vec{b} \times \vec{b})(w\sin wt\cos wt)$$
$$= 0 + w(\vec{a} \times \vec{b})\cos^2 wt + w(\vec{a} \times \vec{b})\sin^2 wt + 0$$

$$\vec{F} \times \frac{d\vec{F}}{dt} = w(\vec{a} \times \vec{b}) \qquad \left[\because \vec{a} \times \vec{a} = 0, \vec{b} \times \vec{b} = 0, \vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})\right]$$

Example : 2

If $\frac{d\vec{a}}{dt} = \vec{c} \times \vec{a}$, $\frac{d\vec{b}}{dt} = \vec{c} \times \vec{b}$ then prove that $\frac{d}{dt}(\vec{a} \times \vec{b}) = \vec{c} \times (\vec{a} \times \vec{b})$

Solution: By product formula of differentiation

The derivative of $\frac{d}{dt}(\vec{a} \times \vec{b}) = \frac{d\vec{a}}{dt} \times \vec{b} + \vec{a} \times \frac{d\vec{b}}{dt}$

$$\therefore \frac{d}{dt}(\vec{a} \times \vec{b}) = (\vec{c} \times \vec{a}) \times \vec{b} + \vec{a} \times (\vec{c} \times \vec{b})$$
$$= -\vec{b} \times (\vec{c} \times \vec{a}) + \vec{a} \times (\vec{c} \times \vec{b})$$
$$\frac{d}{dt}(\vec{a}\times\vec{b}) = -[(\vec{b} \cdot \vec{a})\,\vec{c} - (\vec{b} \cdot \vec{c})\vec{a}] + [(\vec{a} \cdot \vec{b})\vec{c} - (\vec{a} \cdot \vec{c})\,\vec{b}]$$
$$= -(\vec{a} \cdot \vec{b})\vec{c} + (\vec{c} \cdot \vec{b})\vec{a} + ((\vec{a} \cdot \vec{b})\vec{c} - (\vec{c} \cdot \vec{a})\,\vec{b}$$
$$= (\vec{c} \cdot \vec{b})\vec{a} - (\vec{c} \cdot \vec{a})\vec{b}$$
$$= \boldsymbol{\vec{c} \times (\vec{a} \times \vec{b})}$$

Example : 3

Prove that $\frac{d}{dt}\left[\vec{v}\cdot\frac{d\vec{v}}{dt}\times\frac{d^2\vec{v}}{dt^2}\right] = \vec{v}\cdot\frac{d\vec{v}}{dt} \times \frac{d^3\vec{v}}{dt^3}$

Solution : The derivative of scalar triple product is

$$\frac{d}{dt}\left[\vec{v}\cdot\frac{d\vec{v}}{dt}\times\frac{d^2\vec{v}}{dt^2}\right] = \frac{d\vec{v}}{dt} \cdot \frac{d\vec{v}}{dt} \times \frac{d^2\vec{v}}{dt^2} + \vec{v} \cdot \frac{d^2\vec{v}}{dt^2} \times \frac{d^2\vec{v}}{dt^2} + \vec{v} \cdot \frac{d\vec{v}}{dt} \times \frac{d^3\vec{v}}{dt^3}$$
$$= 0 + 0 + \vec{v} \cdot \frac{d\vec{v}}{dt} \times \frac{d^3\vec{v}}{dt^3}$$
$$= \boldsymbol{\vec{v} \cdot \frac{d\vec{v}}{dt} \times \frac{d^3\vec{v}}{dt^3}}$$

(If any two vectors are equal then $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$]

Example : 4

If $\vec{F} = \vec{a}\sinh t + \vec{b}\cosh t$ then prove that

i) $\vec{F} \cdot \dfrac{d\vec{F}}{dt} \times \dfrac{d^2\vec{F}}{dt^2} = 0$

ii) $\dfrac{d\vec{F}}{dt} \times \dfrac{d^2\vec{F}}{dt^2} = \vec{a} \times \vec{b} = \text{constant}$

Solution : Differentiating $\vec{F}$ w.r.t. t we get,

$$\frac{d\vec{F}}{dt} = \vec{a}\,(\cosh t) + \vec{b}\,(\sinh t)$$

$$\frac{d^2\vec{F}}{dt^2} = \vec{a}\,(\sinh t) + \vec{b}\,(\cosh t)$$

i) $\vec{F} \cdot \dfrac{d\vec{F}}{dt} \times \dfrac{d^2\vec{F}}{dt^2} = 0 \quad \because \vec{F} = \dfrac{d^2\vec{F}}{dt^2}$

[By scalar triple product if any two vectors are equal then $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$]

ii) $$\frac{d\vec{F}}{dt} \times \frac{d^2\vec{F}}{dt^2} = [\vec{a}\,(\cosh t) + \vec{b}(\sinh t)] \times [\vec{a}(\sinh t) + \vec{b}(\cosh t)]$$

$$\frac{d\vec{F}}{dt} \times \frac{d^2\vec{F}}{dt^2} = (\vec{a} \times \vec{a})\,(\cosh t \sinh t) + (\vec{a} \times \vec{b})\,(\cosh^2 t)$$

$$+ (\vec{b} \times \vec{a})\,(\sinh^2 t) + (\vec{b} \times \vec{b})\,(\sinh t \cosh t)$$

$$= 0 + (\vec{a} \times \vec{b}) \cosh^2 t - (\vec{a} \times \vec{b}) \sinh^2 t + 0$$

$$\boldsymbol{\frac{d\vec{F}}{dt} \times \frac{d^2\vec{F}}{dt^2} = (\vec{a} \times \vec{b}) = \textbf{constant.}}$$

$[\because (\vec{a} \times \vec{a}) = 0,\ \vec{b} \times \vec{b} = 0,\ \ \cosh^2 t - \sin^2 h\, t = 1 \quad \vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})]$

Example : 5

If $\vec{r} = t\,(\cos t\ \vec{a} + \sin t\ \vec{b})$ where $\vec{a}, \vec{b}$ are constant vectors. Prove that $\dfrac{d\vec{r}}{dt} \times \left[\dfrac{d^2\vec{r}}{dt^2} + \vec{r}\right] = 2\,(\vec{a} \times \vec{b})$

Solution : Differentiating $\vec{r}$ w.r.t. t by using product formula we get,

$$\frac{d\vec{r}}{dt} = (\cos t\ \vec{a} + \sin t\vec{b}) + t(-\sin t\vec{a} + \cos t\vec{b})$$

$$\frac{d^2\vec{r}}{dt^2} = (-\sin t\vec{a} + \cos t\ \vec{b}) + (-\sin t\vec{a} + \cos t\vec{b}) + t(-\cos t\vec{a} - \sin t\vec{b})$$

$$\frac{d^2\vec{r}}{dt^2} = (-2\sin t - t\cos t)\,\vec{a} + (2\cos t - t\sin t)\vec{b}$$

$$\vec{r} + \frac{d^2\vec{r}}{dt^2} = t\,(\cos t\,\vec{a} + \sin t\,\vec{b}) + (-2\sin t - t\cos t)\,\vec{a} + (2\cos t - t\sin t)\,\vec{b}$$

$$\vec{r} + \frac{d^2\vec{r}}{dt^2} = -2\sin t\,\vec{a} + 2\cos t\,\vec{b}$$

$$\frac{d\vec{r}}{dt} \times \left(\vec{r} + \frac{d^2\vec{r}}{dt^2}\right) = [(\cos t - t\sin t)\,\vec{a} + (\sin t + t\cos t)\,\vec{b}] \times [(-2\sin t)\,\vec{a} + (2\cos t)\,\vec{b}]$$

$$\frac{d\vec{r}}{dt} \times \left(\vec{r} + \frac{d^2\vec{r}}{dt^2}\right) = (\cos t - t\sin t)\,(-2\sin t)\,(\vec{a} \times \vec{a}) + (\cos t - t\sin t)\,(2\cos t)\,(\vec{a} \times \vec{b})$$

$$+ (\sin t + t\cos t)\,(-2\sin t)\,(\vec{b} \times \vec{a}) + (\sin t + t\cos t)\,(2\cos t)\,(\vec{b} \times \vec{b})$$

$$\frac{d\vec{r}}{dt} \times \left(\vec{r} + \frac{d^2\vec{r}}{dt^2}\right) = [2\cos^2 t - 2t\sin t\cos t + 2\sin^2 t + 2t\sin t\cos t]\,(\vec{a} \times \vec{b})$$

$$= 2\,(\cos^2 t + \sin^2 t)\,(\vec{a} \times \vec{b})$$

$$= \mathbf{2\,(\vec{a} \times \vec{b}) = constant.}$$

$[\because (\vec{a} \times \vec{a}) = 0,\ \vec{b} \times \vec{b} = 0,\ (\vec{b} \times \vec{a}) = -(\vec{a} \times \vec{b}),\ \cos^2 t + \sin^2 t = 1]$

Example : 6

If $\vec{a} = \sin\theta i + \cos\theta j + \theta k,\ \vec{b} = \cos\theta i - \sin\theta j - 3k$

$\vec{c} = i + 2j + 3k.$

Find $\frac{d}{d\theta}[\vec{a} \times (\vec{b} \times \vec{c})]$ at $\theta = \frac{\pi}{2}$

Solution : By vector triple product $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\,\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \left(\frac{1}{2}\sin 2\theta + 2\cos^2\theta - 3\theta\cos\theta + 3\theta\right) i + (-\sin^2\theta - \sin 2\theta$$

$$+ 3\theta\sin\theta + 6\theta)j + (-3\sin\theta - 6\cos\theta)\,k.$$

$$\frac{d}{d\theta}[\vec{a} \times (\vec{b} \times \vec{c})] = [\cos 2\theta - 4\cos\theta\sin\theta - 3\cos\theta + 3\theta\sin\theta + 3)i + [-2\sin\theta\cos\theta$$

$$-2\cos 2\theta + 3\sin\theta + 3\theta\cos\theta + 6)j + [-3\cos\theta + 6\sin\theta]k.$$

At $\theta = \frac{\pi}{2}$ we get,

$$\frac{d}{d\theta}[\vec{a} \times (\vec{b} \times \vec{c})] = \left(-1 - 0 - 0 + 3\frac{\pi}{2} + 3\right) i + (-0 + 2 + 3 + 0 + 6)j + (0 + 6)k$$

$$= \mathbf{\left(2 + \frac{3\pi}{2}\right) i + 11j + 6\,k}$$

Example : 7

The position vector of a point at any time 't' is given by $\vec{r} = e^t(\cos t\, i + \sin t\, j)$ show that $\vec{a} = 2\vec{v} - 2\vec{r}$ where $\vec{a}$ and $\vec{v}$ are acceleration and velocity of a particle. Also find the angle between the radius vector and the acceleration.

Solution : Differentiating $\vec{r}$ w.r.t. t we get,

$$\vec{v} = \frac{d\vec{r}}{dt} = e^t[(\cos t - \sin t)\, i + (\sin t + \cos t)\, j]$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = 2e^t(-\sin t\, i + \cos t\, j) \qquad \text{.... (i)}$$

$$2\vec{v} - 2\vec{r} = 2e^t[(\cos t - \sin t - \cos t)\, i + (\cos t + \sin t - \sin t)\, j]$$

$$\vec{a} = 2e^t(-\sin t\, i + \cos t\, j) \qquad \text{.... (ii)}$$

From (i) and (ii) we get $\vec{a} = 2\vec{v} - 2\vec{r}$

The angle between the radius vector and acceleration is given by

$$\mathbf{\cos\theta = \frac{\vec{a}\cdot\vec{r}}{|\vec{a}|\,|\vec{r}|} = 0 \Rightarrow \theta = \frac{\pi}{2}} \qquad \left(\because \vec{a}\cdot\vec{r} = 0\right)$$

Example : 8

A particle is moving along the curve $x = t^3 + 1$, $y = t^2$, $z = t$. Find the velocity and acceleration at t = 1

Solution : The position vector $\vec{r} = xi + yj + zk$

$$\vec{r} = (t^3 + 1)i + t^2 j + tk$$

Velocity $\quad \vec{v} = \frac{d\vec{r}}{dt} = 3t^2 i + 2tj + k$

At t = 1 $\quad \vec{v} = 3i + 2j + k$

Acceleration $\quad \vec{a} = \frac{d^2\vec{r}}{dt^2} = 6ti + 2j$

At t = 1 $\quad \mathbf{\vec{a} = 6i + 2j}$

Example : 9

Find the angle between the tangents to the curve $\vec{r} = (t^3 + 2)\, i + (4t - 5)\, j + (2t^2 - 6t)\, k$ at t = 0 and t = 2.

Solution : The tangent vector $\vec{T} = \frac{d\vec{r}}{dt} = 3t^2\, i + 4\, j + (4t - 6)\, k$

at t = 0 $\vec{T}_1 = 4j - 6k$, at t = 2.

$$\vec{T}_2 = 12i + 4j + 2k \qquad |\vec{T}_1| = \sqrt{52}\ ,\ |\vec{T}_2| = \sqrt{164}$$

$\therefore$ The angle between tangents is

$$\cos\theta = \frac{\vec{T}_1 \cdot \vec{T}_2}{|\vec{T}_1|\,|\vec{T}_2|} = \frac{0 + 16 - 12}{\sqrt{52}\sqrt{164}} = \frac{4}{\sqrt{52}\sqrt{164}}.$$

5.2 The Vector Differential Operator Del. (∇) :

The vector differential operator ∇ is defined as $\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$ it involves the operators of partial differentiation such as $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ and the component of unit vectors along the axes i, j, k with the help of ∇ we will define the gradient the divergence and the curl.

5.3 Gradient :

- **The gradient of a scalar function :**

  Let the scalar function ϕ (x, y, z) be continuous and differentiable, then the gradient of the function $\phi(x, y, z)$ is defined as

$$\text{Grad}\,\phi = \nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} = \text{vector quantity}$$

- **Geometrical Interpretation of Gradient :**

  If $d\vec{r}$ is differential of position vector $\vec{r}$, then $d\vec{r} = dx\,i + dy\,j + dz\,k$

$$\therefore \nabla\phi\cdot d\vec{r} = \left(i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}\right)\cdot(i\,dx + j\,dy + k\,dz)$$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi$$

- **Since $d\phi$ is total differential of ϕ :**

  If $d\vec{r}$ is on a level surface $\phi(x, y, z) = c$ then $d\phi = 0$ so that $\nabla\phi\cdot d\vec{r} = 0$ and therefore $\nabla\phi$ has a direction normal to the level surface.

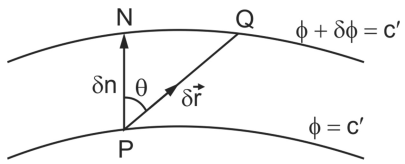


Fig.: 5.10

Let $\nabla\phi = |\nabla\phi|\,\hat{n}$ where $\hat{n}$ is a unit vector normal to the surface $\phi(x, y, z) = c$ and Q is a point on the surface $\phi + \delta\phi = c'$. Let PN = δn be the perpendicular distance between the two surfaces at P. Then the rate of change of ϕ in the direction of normal to the surface $= \frac{\partial\phi}{\partial n} = \lim_{\delta n \to 0}$

$$\frac{\delta\phi}{\delta n} = \lim_{\delta n \to 0}\frac{\nabla\phi\cdot\delta\vec{r}}{\delta n} \quad \text{since } \delta\phi = \nabla\phi\cdot\delta\vec{r}$$

$$\therefore \quad \frac{\partial\phi}{\partial n} = \lim_{\partial n \to 0} |\nabla\phi|\frac{\hat{n}\cdot d\vec{r}}{\delta n} = |\nabla\phi|$$

Since $\delta n = PN = PQ\cos\theta = \delta r\cos\theta = \hat{n}\cdot\delta\vec{r}$

Hence the gradient of a scalar field ϕ is a vector normal to the surface ϕ (x, y, z) = c and having a magnitude equal to the rate of change of ϕ along this normal.

- **Level surface :**

  If $\phi(x, y, z)$ be a single valued continuous scalar point function through any point p of the region considered, we can draw a surface such that at each point on it the function has the same value at P. Such a surface is called a level surface of the function. Thus the surface $\phi(x, y, z)$ = constant, is a level surface. No two level surfaces (corresponding to two different values of the constant) can intersect, otherwise the single valued function ϕ will have two different values at the point of intersection. If $\phi(x, y, z)$ represents potential at the point (x, y, z) the equipotential surface $\phi(x, y, z) = c$, is an example of a level surface. Similarly, if ϕ denotes temperature, the surface of constant temperature is called isothermal surface.

- **Some Properties of Gradient :**

i) If ϕ be constant, then grad $\phi = 0$

ii) If ϕ_1 and ϕ_2 be any two differentiable scalar functions, then.

(a) $\nabla(\phi_1 \pm \phi_2) = \nabla\phi_1 \pm \nabla\phi_2$

(b) $\phi(c_1\phi_1 + c_2\phi_2) = c_1\nabla\phi_1 + c_2\nabla\phi_2$

(c) $\nabla(\phi_1\phi_2 = \phi_1\nabla\phi_2 + \phi_2\nabla\phi_1$

(d) $\nabla\left(\frac{\phi_1}{\phi_2}\right) = \frac{\phi_2\nabla\phi_1 - \phi_1\nabla\phi_2}{\phi_2^2}$, $\phi_2 \neq 0$

(e) if $\phi_2 = f(\phi_1)$ then $\nabla\phi_2 = \nabla f(\phi_1) = f'(\phi_1)\,\nabla\phi_1$

(f) If $\vec{r} = xi + yj + zk$ and $r = |\vec{r}|$ then $\nabla f(r) = f'(r)\nabla r = \frac{f'(r)}{r}\vec{r}$

5.4 Directional Derivative :

Let $|\delta\vec{r}| = \delta r = PQ$ and $\hat{u}$ be the unit vector in the direction of PQ. Then $\delta\vec{r} = \delta r\,\hat{u}$. The directional derivative $\frac{\partial\phi}{\partial r}$ of ϕ at P in the direction of $\hat{u}$ is defined as $\frac{\partial\phi}{\partial r} = \lim_{\delta r - \infty} \frac{\delta\phi}{\delta r}$

where Q $\to$ P in direction of $\hat{u}$.

Since $\hat{u} \cdot \hat{n} = \cos\theta$ and $\cos\theta = \frac{\delta n}{\delta r}$, we have

$$\delta r = \frac{\delta n}{\cos\theta} = \frac{\delta n}{\hat{u} \cdot \hat{n}}$$

$$\therefore \quad \frac{\partial\phi}{\partial r} = \lim_{\delta r \to 0} \frac{\delta n}{\delta n/\hat{u} \cdot \hat{n}} = \lim_{\delta r \to 0} \hat{u} \cdot \hat{n}\frac{\delta\phi}{\delta n}$$

$$\frac{\partial\phi}{\partial r} = \hat{u} \cdot \hat{n}\frac{\partial\phi}{\partial n} = \hat{u} \cdot \hat{n}\,|\nabla\phi| = \hat{u} \cdot \nabla\phi$$

$\therefore \quad \text{D.D.} = \nabla\phi \cdot \hat{u} \qquad$ where $\hat{u} = \dfrac{\vec{u}}{|\vec{u}|}$

Since $\quad |\nabla\phi| = \dfrac{\partial\phi}{\partial n} \quad$ and $\hat{n}\,|\nabla\phi| = \nabla\phi$

Hence the directional derivative $\dfrac{\partial\phi}{\partial r}$ is the resolved part of $\nabla\phi$ in the direction $\hat{u}$. Clearly $|\nabla\phi|$ gives the magnitude of maximum rate of change of ϕ.

Illustrative Examples

Example : 1

If $\vec{a} = a_1i + a_2j + a_3k$ and $\vec{r} = xi + yj + zk$, $r = |\vec{r}|$ then prove that :

i) $\nabla(\vec{a}\cdot\vec{r}) = \vec{a}$ ii) $(\vec{a}\cdot\nabla)\vec{r} = \vec{a}$ iii) $\nabla f(r) = f'(r)\,\nabla r = \dfrac{f'(r)}{r}\vec{r}$

iv) $\nabla \int r^n\,dr = r^{n-1}\,\vec{r}$ v) $\nabla r^m = mr^{m-2}\vec{r}$ vi) $\nabla\dfrac{1}{r^m} = -\dfrac{m}{r^{m+2}}\vec{r}$

vii) $\nabla \log r = \dfrac{\vec{r}}{r^2}$ viii) $\nabla|\vec{r}|^2 = 2\vec{r}$ ix) $\nabla\left(\dfrac{\vec{a}\cdot\vec{r}}{r^n}\right) = \dfrac{\vec{a}}{r^n} - \dfrac{n(\vec{a}\cdot\vec{r})}{r^{n+2}}$

x) $\vec{b} \times \nabla\,[\vec{a}\cdot\nabla \log r] = \dfrac{\vec{b}\times\vec{a}}{r^2} - 2\,\dfrac{(\vec{a}\cdot\vec{r})\,(\vec{b}\times\vec{r})}{r^4}$

Solution :

i) The dot product of $\vec{a}\cdot\vec{r} = a_1x + a_2y + a_3z$

$$\nabla(\vec{a}\cdot\vec{r}) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(a_1x + a_2y + a_3z)$$

$$\nabla(\vec{a}\cdot\vec{r}) = i\,a_1 + j\,a_2 + k\,a_3 = \vec{a}$$

$$\therefore \quad \nabla(\vec{a}\cdot\vec{r}) = \vec{a}$$

ii) The dot product of $\vec{a}$ and ∇ is

$$\vec{a}\cdot\nabla = (a_1i + a_2j + a_3k)\cdot\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)$$

$$= a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}$$

$$\therefore \quad (\vec{a}\cdot\nabla)\vec{r} = \left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\right)(xi + yj + zk)$$

$$= a_1i + a_2j + a_3k = \vec{a}$$

$$(\vec{a}\cdot\nabla)\vec{r} = \vec{a}$$

iii) $\nabla f(r) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) f(r)$

$= i\, f'(r)\frac{\partial r}{\partial x} + j\, f'(r)\frac{\partial r}{\partial y} + k\, f'(r)\frac{\partial r}{\partial z}$

$\nabla f(r) = f'(r)\left(i\frac{\partial r}{\partial x} + j\frac{\partial r}{\partial y} + k\frac{\partial r}{\partial z} \right)$

$= f'(r)\, \nabla r = \frac{f'(r)}{r}\vec{r}$

Since $r^2 = x^2 + y^2 + z^2$

$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$

$\therefore \quad \nabla r = i\frac{\partial r}{\partial x} + j\frac{\partial r}{\partial y} + k\frac{\partial r}{\partial z}$

$\nabla r = i\frac{x}{r} + j\frac{y}{r} + k\frac{z}{r} = \frac{1}{r}\vec{r}$

iv) Let $f(r) = \int r^n\, dr$ then $f'(r) = r^n$

$\therefore \quad \nabla f(r) = \frac{f'(r)}{r}\vec{r} \quad \Rightarrow \quad \nabla\int r^n\, dr = \frac{r^n}{r}\vec{r} = r^{n-1}\vec{r}$

v) Let $f(r) = r^m$ and $f'(r) = mr^{m-1}$

$\nabla r^m = \frac{mr^{m-1}}{r}\vec{r} = mr^{m-2}\vec{r}$

vi) let $f(r) = \frac{1}{r^m} = r^{-m}$ and $f'(r) = -mr^{-m-1}$

$\therefore \quad \nabla\frac{1}{r^m} = -\frac{mr^{m-1}}{r}\vec{r} = -\frac{m}{r^{m+2}}\vec{r}$

vii) Let $f(r) = \log r$ and $f'(r) = \frac{1}{r}$

$\nabla \log r = \frac{1/r}{r}\vec{r} = \frac{1}{r^2}\vec{r}$

viii) We know that $|\vec{r}| = r$

$\therefore \quad |\vec{r}|^2 = r^2$ and $f(r) = r^2,\ f'(r) = 2r$

$\nabla|\vec{r}|^2 = \frac{2r}{r}\vec{r} = 2\vec{r}$

ix) $\nabla\left(\frac{\vec{a}\cdot\vec{r}}{r^n}\right) = \frac{1}{r^n}\nabla(\vec{a}\cdot\vec{r}) + (\vec{a}\cdot\vec{r})\,\nabla\left(\frac{1}{r^n}\right)$

$= \frac{1}{r^n}\vec{a} + (\vec{a}\cdot\vec{r})\left(-\frac{n}{r^{n+2}}\vec{r}\right)$

$\nabla\left(\frac{\vec{a}\cdot\vec{r}}{r^n}\right) = \frac{\vec{a}}{r^n} - \frac{n}{r^{n+2}}(\vec{a}\cdot\vec{r})\,\vec{r}$

x) Consider $\vec{a} \cdot \nabla \log r = \vec{a} \cdot \frac{\vec{r}}{r^2} = \frac{\vec{a} \cdot \vec{r}}{r^2}$

$$\nabla[\vec{a} \cdot \nabla \log r] = \nabla\left(\frac{\vec{a}\cdot\vec{r}}{r^2}\right) = \frac{\vec{a}}{r^2} - \frac{2}{r^4}(\vec{a} \cdot \vec{r})\,\vec{r} \text{ (by previous example n = 2)}$$

$$\therefore \quad \mathbf{\vec{b} \times \nabla[\vec{a} \cdot \nabla \log r] = \frac{\vec{b} \times \vec{a}}{r^2} - \frac{2}{r^4}(\vec{a} \cdot \vec{r})\,(\vec{b} \times \vec{r})}$$

Example : 2

If θ is the acute angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point $(1, -2, 1)$, show that $\cos\theta = \frac{3}{7\sqrt{6}}$

Solution :

Angle between two surfaces at a point is the angle between the normals to the surfaces at that point. Now let

$$\phi_1 = xy^2z - 3x - z^2 = 0 \text{ and } \phi_2 = 3x^2 - y^2 + 2z - 1 = 0$$

Then
$$\vec{N}_1 = \nabla\phi_1 = (y^2z - 3)\, i + (2xyz)\, j + (xy^2 - 2z)k$$
$$= i - 4j + 2k \quad \text{at} \quad (1, -2, 1)$$
$$\vec{N}_2 = \nabla\phi_2 = 6xi - 2yj + 2k = 6i + 4j + 2k \quad \text{at} \quad (1, -2, 1)$$
$$\therefore \quad \cos\theta = \frac{\vec{N}_1 \cdot \vec{N}_2}{|\vec{N}_1|\,|\vec{N}_2|} = \left|\frac{6 - 16 + 4}{\sqrt{21}\sqrt{56}}\right| = \left|\frac{-6}{14\sqrt{6}}\right| = \mathbf{\frac{3}{7\sqrt{6}}}$$

Example : 3

Find the value of constants λ and μ so that the surfaces $\lambda x^2 - \mu yz = (\lambda + 2)\, x$ and $4x^2y + z^3 = 4$ intersect orthogonally at the point $(1, -1, 2)$.

Solution :

Let
$$\phi_1 = \lambda x^2 - \mu yz - (\lambda + 2)\, x = 0 \qquad \text{.... (i)}$$
$$\phi_2 = 4x^2y + z^3 - 4 = 0 \qquad \text{.... (ii)}$$

The given point $(1, -1, 2)$ must lie on both the surfaces.

Substituting in (i) we get $\lambda + 2\mu = \lambda + 2 \Rightarrow \mu = 1$

The surfaces (i) and (ii) will intersect orthogonally if the normal to them at $(1, -1, 2)$ are perpendicular to each other.

$\therefore$ at $(1, -1, 2)$
$$\nabla\phi_1 \cdot \nabla\phi_2 = 0 \qquad \text{.... (iii)}$$

Now
$$\nabla\phi_1 = \{(2\lambda x - (\lambda + 2)\}i - \mu zj - \mu yk$$
$$= (\lambda - 2)\, i - 2j + k \text{ at } (1, -1, 2) \text{ as } \mu = 1$$

and
$$\nabla\phi_2 = 8xyi + 4xj + 3z^2k$$

$$= -8i + 4j + 12k \quad \text{at } (1, -1, 2)$$

Substituting the values of $\nabla\phi_1$ and $\nabla\phi_2$ in (iii) we get

$$\{(\lambda - 2)i - 2j + k\} \cdot (-8i + 4j + 12k) = 0$$

$$-8(\lambda - 2) - 8 + 12 = 0 \quad \Rightarrow \quad \lambda = \frac{5}{2} = 2.5$$

$$\therefore \ \lambda = \mathbf{2.5} \quad \textbf{and } \mu = \mathbf{1}$$

Example : 4

Find the directional derivative of the function $\phi = x^2 - y^2 + 2z^2$ at the point P (1, 2, 3) in the direction of the line PQ where Q has co–ordinates (5, 0, 4). In what direction it will be maximum and what is its value?

Solution :

We have $\overrightarrow{PQ} = (5i + 4k) - (i + 2j + 3k) = 4i - 2j + k$

Let $\hat{u}$ be a unit vector along $\overrightarrow{PQ}$ then $\hat{u} = \dfrac{4i - 2j + k}{\sqrt{16 + 4 + 1}} = \dfrac{1}{\sqrt{21}}(4i - 2j + k)$

Now $\quad \nabla\phi = 2xi - 2yj + 4zk$

$\quad = 2i - 4j + 12k \quad$ at the point (1, 2, 3)

$\therefore$ Directional derivative in the direction of $\overrightarrow{PQ} = \nabla\phi \cdot \hat{u}$

$$= (2i - 4j + 12k) \cdot \frac{1}{\sqrt{21}}(4i - 2j + k)$$

$$= \frac{1}{\sqrt{21}}(8 + 8 + 12) = \frac{28}{\sqrt{21}} = \frac{4}{3}\sqrt{21}$$

The directional derivative is maximum in the direction of the normal to the given surface this is the direction of $\nabla\phi$. $\therefore$ the maximum value of directional derivative at P(1, 2, 3)

$$= |\nabla\phi|_P = |2i - 4j + 12k|$$

$$= \mathbf{\sqrt{4 + 16 + 144} = 2\sqrt{41}}$$

Example : 5

Find the directional derivative of $\phi = xy^2 + yz^2$ at (2, –1, 1) in the direction normal to the surface $xy = z^3$ at the point (1, 4, 2)

Solution :

$$\nabla\phi = i(y^2) + j(2xy + z^2) + k(2yz)$$

$$\nabla\phi = i - 3j - 2k \quad \text{at the point } (2, -1, 1)$$

let $\quad \phi_1 = xy - z^3 = 0$

then $\quad \nabla\phi_1 = yi + xj - 4z^2k$

$$\therefore \quad \hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{4i + j - 12k}{\sqrt{16 + 1 + 144}}$$

$$\hat{u} = \frac{1}{\sqrt{161}}(4i + j - 12k)$$

$\therefore$ Directional derivative in the direction normal to the surface is $= \nabla\phi \cdot \hat{u}$

$$\text{D.D.} = \nabla\phi \cdot \hat{u} = (i - 3j - 2k) \cdot \frac{1}{\sqrt{161}}(4i + j - 12k)$$

$$= \frac{1}{\sqrt{161}}(4 - 3 + 24)$$

$$\textbf{D.D.} = \frac{\mathbf{25}}{\sqrt{\mathbf{161}}}$$

Example : 6

Find the directional derivative of $\phi = x\log z - y^2 + 4$ at $(-1, 2, 1)$ in the direction of the tangent to the curve $x = e^{-t}$, $y = 2\sin t + 1$, $z = t - \cos t$ at $t = 0$

Solution :

Now $\nabla\phi = \log z\, i - 2y\, j + \frac{x}{z} k = (0)\, i - 4j - k$ at the point $(-1, 2, 1)$

let $\vec{r} = xi + yj + zk = e^{-t}i + (2\sin t + 1)j + (t - \cos t)\, k$

$$\vec{T} = \frac{d\vec{r}}{dt} = -e^{-t}i + 2\cos t j + (1 + \sin t)\, k = -i + 2j + k \quad \text{at } t = 0$$

$$\therefore \text{D.D.} = \nabla\phi \cdot \hat{T} = [(0)i - 4j - k] \cdot \frac{1}{\sqrt{6}}(-i + 2j + k) = \frac{1}{\sqrt{6}}(-8 - 1)$$

$$\textbf{D.D.} = -\frac{\mathbf{9}}{\sqrt{\mathbf{6}}}$$

Example : 7

If the directional derivations of $\phi = a(x + y) + b(y + z) + c(x + z)$ has a maximum value 12 in the direction parallel to the line $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-1}{3}$ Find the values of a, b, c.

Solution :

Here the direction vector $\vec{u} = i + 2j + 3k$ and

$\nabla\phi = i(a + c) + j(a + b) + k(b + c)$ has maximum magnitude $|\nabla\phi| =$ 12 in the direction parallel to the given line.

$$\therefore \quad \hat{u} = \frac{\nabla\phi}{|\nabla\phi|} \quad \Rightarrow \quad \nabla\phi = |\nabla\phi|\, \hat{u}$$

$\Rightarrow$ $i(a + c) + j\,(a + b) + k(b + c) = 12\,(i + 2j + 3k)$ equating co–efficient of unit vectors we get,

$$a + c = 12 \quad \text{.... (i)}$$

$$a + b = 24 \quad \text{.... (ii)}$$

$$b + c = 36 \quad \text{.... (iii)}$$

(m) – (i) we get,

$$b - a = 24 \quad \text{.... (iv)}$$

From (ii) & (iv) we get $b = 0$, $a = 24$, $c = 36$

Example : 8

A scalar field f has at the point (1, 2) directional derivative +2 in the direction towards (2, 2) and –2 in the direction towards (1, 1) determine gradf. at (1, 2) also find the direction towards (4, 6).

Solution :

Given : $\nabla f \cdot \hat{a} = +2$

$$\therefore \quad \nabla f \cdot \hat{a} = \left(i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} \right) \cdot \hat{a}$$

$$+2 = \left(i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} \right) \cdot i$$

$$\therefore \quad \frac{\partial f}{\partial x} = 2$$

ii) $$\nabla f \cdot \hat{b} = \left(i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} \right) \cdot \hat{b}$$

$$-2 = \left(i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} \right) \cdot (-j)$$

$$\therefore \quad \frac{\partial f}{\partial y} = 2$$

$$\therefore \quad \nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} = 2i + 2j$$

iii) $$\text{D.D.} = \nabla f \cdot \hat{c}$$

$$= (2i + 2j) \cdot \frac{1}{5}(3i + 4j) = \mathbf{\frac{1}{5}(6 + 8) = \frac{14}{5}}$$

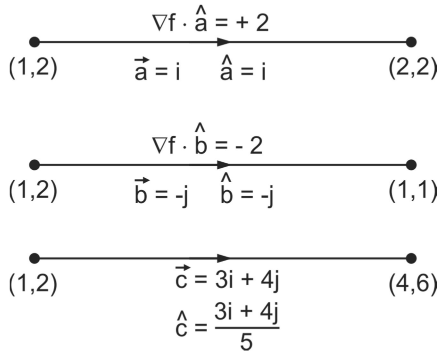


Fig.: 5.11

5.5 Divergence and Curl :

5.5.1 Divergence of a Vector Field :

When the vector operator ∇ operates scalarly on vector point function the result is a scalar quantity, this is known as the divergence of vector function. Thus if $\vec{F} = F_1 i + F_2 j + F_3 k$ is a vector point function, then

$$\text{div}\vec{F} = \nabla \cdot \vec{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \cdot (F_1 i + F_2 j + F_3 k)$$

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \text{Scalar quantity}$$

5.5.2 Physical Interpretation of Divergence :

Consider a small parallelopiped of dimensions δx, δy, δz with its edges parallel to the co–ordinate axes.

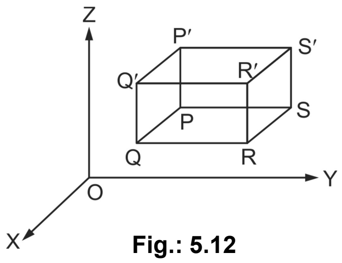


Fig.: 5.12

Let the co–ordinates of P be (x, y, z) the mass of fluid flowing through the face PQ Q′P′ per unit time is

$$\rho(V)_y\, \delta x\delta z \quad = \quad V_y\delta x\delta z = F(y) \text{ say} \qquad \text{.... (i)}$$

and the mass of the fluid flowing out through the opposite face SRR′S′ is

$$F(y + \delta y) \quad = \quad F(y) + \frac{\partial F}{\partial y}\delta y = \left(V_y + \frac{\partial Vy}{\partial y} dy\right)\delta x\delta z \qquad \text{.... (ii)}$$

by using Taylor's series.

Taking the difference of (i) and (ii) we obtain the decrease of mass inside the parallelopiped per unit time due to one pair of face as $\frac{\partial Vy}{\partial y}\delta x\delta y\delta z$.

Now taking into account the other two pairs of faces, we get the total decrease in the mass of the fluid per unit time in the parallelopiped as $\left(\frac{\partial Vx}{\partial x} + \frac{\partial Vy}{\partial y} + \frac{\partial Vz}{\partial z}\right)\delta x\delta y\delta z$ ...(iii)

Hence the rate of loss of fluid per unit volume is

$$\frac{\partial Vx}{\partial x} + \frac{\partial Vy}{\partial y} + \frac{\partial Vz}{\partial z} \quad = \quad \nabla \cdot \vec{v} = \text{div } \vec{v}$$

This is true exactly in the limit as the parallelopiped shrinks top as δx, δy, δz approach zero.

5.5.3 Curl of a Vector Field :

When the vector operator ∇ operates vectorially on a vector point function the result is a vector quantity, this is known as the curl of vector function. Thus if $\vec{F} = F_1 i + F_2 j + F_3 k$ is a vector point function, then

$$\text{curl } \vec{F} = \nabla \times \vec{F} \quad = \quad \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \quad i\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - j\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + k\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= \quad \text{Vector quantity}$$

5.5.4 Physical Interpretation of Curl :

Consider the rotation of a rigid body about a fixed axis. Referring to figure ω is the angular velocity of the body $\vec{v}$ is the linear velocity of the point P and $\overrightarrow{OP} = \vec{r}$ taking 'O' as origin of co–ordinates.

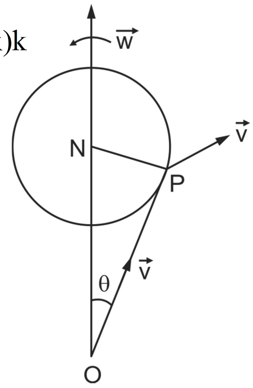


Fig.: 5.13

Let $\vec{r} = xi + yj + zk$ and $\vec{\omega} = \omega_1 i + \omega_2 j + \omega_3 k$

We have
$$\vec{v} = \vec{\omega} \times \vec{r}$$
$$= (\omega_2 z - \omega_3 y)i + (\omega_3 x - \omega_1 z)j + (\omega_1 y - \omega_2 x)k$$

$\therefore$
$$\text{Curl } \vec{v} = \nabla \times \vec{v}$$
$$= 2(\omega_1 i + \omega_2 j + \omega_3 k) = 2\vec{\omega}$$

$\therefore$
$$\vec{\omega} = \frac{1}{2}\nabla \times \vec{v} = \frac{1}{2} \text{ curl } \vec{v}$$

- **Vector Identities :**

i) $\hat{r} = li + mj + nk$, $|\hat{r}| = \sqrt{l^2 + m^2 + n^2} = \sqrt{1} = 1$

ii) $(\vec{u} \cdot \nabla)\vec{r} = \vec{u}$

iii) $(\vec{u} \cdot \nabla)\phi = \vec{u} \cdot \nabla\phi$

5.5.5 Properties of Divergence and Curl :

1. For a constant vector $\vec{A}$, div $\vec{A} = \nabla \cdot \vec{A} = 0$, curl $\vec{A} = \nabla \times \vec{A} = 0$
2. If $\vec{u}$ and $\vec{v}$ are vector functions and ϕ is scalar function then

i) $\nabla \cdot (\vec{u} \pm \vec{v}) = \nabla \cdot \vec{u} \pm \nabla \cdot \vec{v}$

$\nabla \times (\vec{u} \pm \vec{v}) = \nabla \times \vec{u} \pm \nabla \times \vec{v}$

ii) $\nabla \cdot (\phi\vec{u}) = \nabla\phi \cdot \vec{u} + \phi(\nabla \cdot \vec{u})$

$\nabla \times (\phi\vec{u}) = \nabla\phi \times \vec{u} + \phi(\nabla \times \vec{u})$

iii) $\nabla \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot (\nabla \times \vec{u}) - \vec{u} \cdot (\nabla \times \vec{v})$

$\nabla \times (\vec{u} \times \vec{v}) = (\nabla \cdot \vec{v})\,\vec{u} - (\nabla \cdot \vec{u})\,\vec{v} + (\vec{v} \cdot \nabla)\vec{u} - (\vec{u} \cdot \nabla)\,\vec{v}$

iv) $\nabla(\vec{u} \cdot \vec{v}) = (\vec{u} \cdot \nabla)\vec{v} + (\vec{v} \cdot \nabla)\,\vec{u} + \vec{u} \times (\nabla \times \vec{v}) + \vec{v} \times (\nabla \times \vec{u})$

v) If $\vec{u}$ = constant, $(\vec{u} \times \nabla) \times \vec{v} = \nabla(\vec{u} \cdot \vec{v}) - (\nabla \cdot \vec{v})\,\vec{u}$

e.g. : Show that $(\vec{a} \times \nabla) \times \vec{r} = -2\vec{a}$

$$(\vec{a} \times \nabla) \times \vec{r} = \nabla(\vec{a} \cdot \vec{r}) - (\nabla \cdot \vec{r})\vec{a} = \vec{a} - 3\vec{a} = -2\vec{a}$$

Illustrative Examples

Example : 1

If $\vec{r} = xi + yj + zk$ and $r = |\vec{r}|$ then show that

i) $\nabla \cdot \vec{r} = 3$

ii) $\nabla \times \vec{r} = 0$

iii) $\nabla r = \frac{1}{r}\vec{r}$

Solution :

i) $$\nabla \cdot \vec{r} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \cdot (xi + yj + zk)$$

$$\nabla \cdot \vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$$

$$\therefore \quad \nabla \cdot \vec{r} = 3$$

ii) $$\nabla \times \vec{r} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = i(0-0) - j(0-0) + k(0-0)$$

$$\therefore \quad \nabla \times \vec{r} = 0$$

iii) Let $f(r) = r$ then $f'(r) = 1$

$$\mathbf{\nabla f(r) = \frac{f'(r)}{r}\vec{r} \Rightarrow \quad \nabla r = \frac{1}{r}\vec{r}}$$

Example : 2

Find the divergence and curl of the vector $\vec{F} = (xyz)\, i + (3x^2y)j + (xz^2 - y^2z)k$ at the point (2, –1, 1).

Solution :

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z)$$

$$\nabla \cdot \vec{F} = yz + 3x^2 + 2xz - y^2$$

$$\nabla \cdot \vec{F} = 14 \text{ at the point } (2, -1, 1)$$

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & 3x^2y & zx^2 - y^2z \end{vmatrix}$$

$$\nabla \times \vec{F} = i(-y^2 - 0) - j(z^2 - xy) + k(6xy - xz)$$

$$\mathbf{\nabla \times \vec{F} = -i - 3j - 14k = -(i + 3j + 14k)}$$

Example : 3

Find div $\vec{F}$ and curl $\vec{F}$, where $\vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$

Solution :

Since $$\vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$$

$$\vec{F} = i(3x^2 - 3yz) + j(3y^2 - 3xz) + k(3z^2 - 3xy)$$

Now, $$\text{div}\,\vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy)$$

$$\nabla \cdot \vec{F} = 6x + 6y + 6z = 6(x + y + z)$$

and $$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = i(-3x + 3x) - j(-3y + 3y) + k(-3z + 3z) = 0$$

$$\therefore \quad \nabla \times \vec{\mathbf{F}} = \mathbf{0}$$

Example : 4

If $\vec{r}_1$ and $\vec{r}_2$ are vectors joining the fixed points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ to a variable point $P(x, y, z)$ using vector identities prove that :

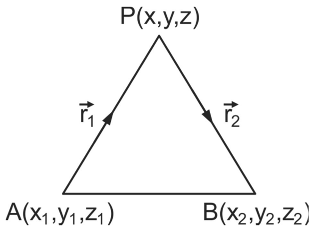


Fig.: 5.14

i) $\nabla (\vec{r}_1 \cdot \vec{r}_2) = \vec{r}_1 + \vec{r}_2$

ii) $\nabla \cdot (\vec{r}_1 \times \vec{r}_2) = 0$

iii) $\nabla \times (\vec{r}_1 \times \vec{r}_2) = 2 (\vec{r}_1 - \vec{r}_2)$

Solution :

Let $$\vec{r}_1 = (x - x_1)i + (y - y_1)j + (z - z_1)k$$

and $$\vec{r}_2 = (x - x_2)i + (y - y_2)j + (z - z_2)k$$

$$\nabla \cdot \vec{r}_1 = 3, \quad \nabla \times \vec{r}_1 = 0, \quad \nabla \cdot \vec{r}_2 = 3, \quad \nabla \times \vec{r}_2 = 0$$

i)
$$\nabla(\vec{u} \cdot \vec{v}) = (\vec{u} \cdot \nabla)\vec{v} + (\vec{v} \cdot \nabla)\vec{u} + \vec{u} \times (\nabla \times \vec{v}) + \vec{v} \times (\nabla \times \vec{u})$$
$$\nabla(\vec{r}_1 \cdot \vec{r}_2) = (\vec{r}_1 \cdot \nabla)\vec{r}_2 + (\vec{r}_2 \cdot \nabla)\vec{r}_1 + \vec{r}_1 \times (\nabla \times \vec{r}_2) + \vec{r}_2 \times (\nabla \times \vec{r}_1)$$
$$\nabla(\vec{r}_1 \cdot \vec{r}_2) = \vec{r}_1 + \vec{r}_2 + 0 + 0 = \vec{\mathbf{r}}_1 + \vec{\mathbf{r}}_2$$

ii)
$$\nabla \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot (\nabla \times \vec{u}) - \vec{u} \cdot (\nabla \times \vec{v})$$
$$\nabla \cdot (\vec{r}_1 \times \vec{r}_2) = \vec{r}_2 \cdot (\nabla \times \vec{r}_1) - \vec{r}_1 \cdot (\nabla \times \vec{r}_2) = 0$$
$$\boldsymbol{\nabla \cdot (\vec{r}_1 \times \vec{r}_2) = 0}$$

iii)
$$\nabla \times (\vec{u} \times \vec{v}) = (\nabla \cdot \vec{v})\vec{u} - (\nabla \cdot \vec{u})\vec{v} + (\vec{v} \cdot \nabla)\vec{u} - (\vec{u} \cdot \nabla)\vec{v}$$
$$\nabla \times (\vec{r}_1 \times \vec{r}_2) = (\nabla \cdot \vec{r}_2)\vec{r}_1 - (\nabla \cdot \vec{r}_1)\vec{r}_2 + (\vec{r}_2 \cdot \nabla)\vec{r}_1 - (\vec{r}_1 \cdot \nabla)\vec{r}_2$$
$$\nabla \times (\vec{r}_1 \times \vec{r}_2) = 3\vec{r}_1 - 3\vec{r}_2 + \vec{r}_2 - \vec{r}_1$$
$$= \mathbf{2\vec{r}_1 + 2\vec{r}_2}$$

Example : 5

If $\vec{F} = \phi\nabla\psi$ where ϕ, ψ are scalar point functions then show that $\vec{F} \cdot (\nabla \times \vec{F}) = 0$

Solution :

We have $$\text{curl } \vec{F} = \nabla \times \vec{F} = \nabla \times (\phi\nabla\psi)$$
$$\nabla \times \vec{F} = \nabla\phi \times \nabla\psi + \phi (\nabla \times \nabla\psi)$$
$$\therefore \quad \vec{F} \cdot (\nabla \times \vec{F}) = (\phi\nabla\psi) \cdot [\nabla\phi \times \nabla\psi + \phi (\nabla \times \nabla\psi)]$$

$$\vec{F}\cdot(\nabla\times\vec{F}) = \phi[\nabla\psi\cdot(\nabla\phi\times\nabla\psi)] + (\phi)^2\,[\nabla\psi\cdot(\nabla\times\nabla\psi)]$$

$$\mathbf{\vec{F}\cdot(\nabla\times\vec{F}) = 0}$$

in scalar triple product $\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$ if any two vectors are equal.

Example : 6

If r and $\vec{r}$ have their usual meaning and $\vec{a}$, $\vec{b}$ are constant vectors, prove that :

i) $\nabla\cdot(r^n\,\vec{r}) = (3+n)r^n$

ii) $\nabla\times(r^n\,\vec{r}) = 0$

iii) $\nabla\cdot\left[r\,\nabla\left(\frac{1}{r^3}\right)\right] = \frac{3}{r^4}$

iv) curl $[(\vec{r}\times\vec{a})\times\vec{b}] = \vec{b}\times\vec{a}$

v) $\nabla\times\left(\frac{\vec{a}\times\vec{r}}{r^n}\right) = \frac{2-n}{r^n}\vec{a} + \frac{n}{r^{n+2}}(\vec{a}\cdot\vec{r})\vec{r}$

Solution :

i) Using vector identities we have $\nabla\cdot(\phi\vec{u}) = \nabla\phi\cdot\vec{u} + \phi\,(\nabla\cdot\vec{u})$

$$\therefore\quad \nabla\cdot r^n\vec{r} = \nabla r^n\cdot\vec{r} + r^n\,\nabla\cdot\vec{r} = \frac{nr^{n-1}}{r}\vec{r}\cdot\vec{r} + r^n\,(3)$$

$$\nabla\cdot r^n\vec{r} = \frac{nr^n}{r^2}r^2 + 3r^n = \mathbf{(3+n)r^n} \qquad (\because \vec{r}\cdot\vec{r} = r^2 \;;\; \nabla\cdot\vec{r} = 3)$$

put $n = -3$ we get

$$\nabla\left(\frac{1}{r^3}\right) = (3-3)r^{-3} = 0$$

ii)
$$\nabla\times(\phi\vec{u}) = \nabla\phi\times\vec{u} + \phi\,(\nabla\times\vec{u})$$

$$\nabla\times(r^n\vec{r}) = \nabla r^n\times\vec{r} + r^n(\nabla\times\vec{r})$$

$$= \frac{nr^{n-1}}{r}(\vec{r}\times\vec{r}) + r^n\,(\nabla\times\vec{r})$$

$$= \mathbf{0} \qquad [\because (\vec{r}\times\vec{r} = 0, \nabla\times\vec{r} = 0]$$

iii)
$$\nabla\left(\frac{1}{r^3}\right) = \nabla r^{-3} = \frac{-3r^{-4}}{r}\vec{r} = -\frac{3}{r^5}\vec{r}$$

$$r\,\nabla\frac{1}{r^3} = r\left(-\frac{3}{r^5}\vec{r}\right) = -\frac{3}{r^4}\vec{r}$$

$$\nabla\cdot\left[r\,\nabla\frac{1}{r^3}\right] = -3\,\nabla\cdot\left(\frac{1}{r^4}\vec{r}\right) = -3[\nabla r^{-4}\cdot\vec{r} + r^{-4}\,(\nabla\cdot\vec{r})]$$

$$= -3\left[\frac{-4r^{-5}}{r}\vec{r}\cdot\vec{r} + r^{-4}(3)\right]$$

$$\nabla\cdot\left[r\,\nabla\frac{1}{r^3}\right] = (-3)\left[-\frac{4}{r^4} + \frac{3}{r^4}\right] = (-3)\left(\frac{-1}{r^4}\right) = \mathbf{\frac{3}{r^4}}$$

iv) $\text{curl}\,[(\vec{r}\times\vec{a})\times\vec{b}] = -\nabla\times[\vec{b}\times(\vec{r}\times\vec{a})]$

$$= -\nabla\times[(\vec{b}\cdot\vec{a})\vec{r}-(\vec{b}\cdot\vec{r})\,\vec{a}]$$

$$= -(\vec{b}\cdot\vec{a})\,(\nabla\times\vec{r})+\nabla\times[(\vec{b}\cdot\vec{r})\,\vec{a}]$$

$$= 0+\nabla\,(\vec{b}\cdot\vec{r})\times\vec{a}+(\vec{b}\cdot\vec{r})\,(\nabla\times\vec{a})$$

$$\boldsymbol{\nabla\times[\vec{b}\times(\vec{r}\times\vec{a})] = \vec{b}\times\vec{a}} \qquad (\because \nabla\times\vec{r}=0\ ;\ \nabla\times\vec{a}=0)$$

v) $\nabla\times\left(\dfrac{\vec{a}\times\vec{r}}{r^n}\right) = \nabla\times[\vec{a}\times r^{-n}\vec{r}]$

$$= (\nabla\cdot r^{-n}\vec{r})\,\vec{a}-(\nabla\cdot\vec{a})\,r^{-n}\vec{r}+(r^{-n}\vec{r}\cdot\nabla)\,\vec{a}-(\vec{a}\cdot\nabla)r^{-n}\vec{r}$$

$$= [\nabla r^{-n}\cdot\vec{r}+r^{-n}\,(\nabla\cdot\vec{r})]\vec{a}-0+0-\nabla'\,(r^{-n}\vec{r})\ (\nabla'=\vec{a}\cdot\nabla)$$

$$= \left[\frac{-nr^{-n-1}}{r}\vec{r}\cdot\vec{r}+r^{-n}(3)\right]\vec{a}-r^{-n}\,\nabla'\,\vec{r}-\vec{r}\,\nabla' r^{-n}$$

$$= \left[-\frac{n}{r^{n+2}}\,r^2+\frac{3}{r^n}\right]\vec{a}-r^{-n}\,(\vec{a}\cdot\nabla)\vec{r}-\vec{r}\,(\vec{a}\cdot\nabla)r^{-n}$$

$$= \left[-\frac{n}{r^n}+\frac{3}{r^n}\right]\vec{a}-r^{-n}\,\vec{a}-\vec{r}\,(\vec{a}\cdot\nabla r^{-n})$$

$$= \left[\frac{-n+3}{r^n}\right]\vec{a}-\frac{\vec{a}}{r^n}-\vec{r}\left(\vec{a}\cdot\frac{-nr^{-n-1}}{r}\vec{r}\right)$$

$$= \left[\frac{-n+3-1}{r^n}\right]\vec{a}-\vec{r}(-n)\left(\frac{\vec{a}\cdot\vec{r}}{r^{n+2}}\right)$$

$$= \boldsymbol{\frac{2-n}{r^n}\,\vec{a}+\frac{n}{r^{n+2}}\,(\vec{a}\cdot\vec{r})\vec{r}}$$

- **Repeated operations by ∇ :**

1. Div (grad ϕ) = $\nabla\cdot\nabla\phi=\nabla^2\phi$ ($\nabla^2\phi=0$, Laplace equation)
2. Curl (grad ϕ) = $\nabla\times\nabla\phi=0$ (Irrotational vector field)
3. Div (curl $\vec{F}$) = $\nabla\cdot(\nabla\times\vec{F})=0$ (Solenoidal vector field)
4. curl $\left(\text{curl}\,\vec{F}\right) = \nabla\times(\nabla\times\vec{F}) = (\nabla\cdot\vec{F})\,\nabla-\nabla^2\vec{F}$

Illustrative Examples

Example : 1

With usual notations prove that $\nabla^2 f(r) = f''(r)+\dfrac{2}{r}f'(r)$ also determine f(r) such that $\nabla^2 f(r)=0$

Solution :

We know that $\nabla f(r) = \dfrac{f'(r)}{r}\vec{r}$, then

$$\nabla^2 f(r) = \nabla \cdot \nabla f(r) = \nabla \cdot \left(\frac{f'(r)}{r}\vec{r}\right)$$

$$= \nabla \frac{f'(r)}{r} \cdot \vec{r} + \frac{f'(r)}{r}(\nabla \cdot \vec{r})$$

$$\nabla^2 f(r) = \left[\frac{1}{r}\nabla f'(r) + f'(r)\,\nabla\frac{1}{r}\right] \cdot \vec{r} + \frac{f'(r)}{r}\ (3)$$

$$\nabla^2 f(r) = \left[\frac{1}{r}\frac{f''(r)}{r}\vec{r} + f'(r)\left(-\frac{1}{r^3}\vec{r}\right)\right] \cdot \vec{r} + \frac{3}{r}f'(r)$$

$$= \frac{f''(r)}{r^2}r^2 - \frac{f'(r)}{r^3}r^2 + \frac{3}{r}f'(r)$$

$$\nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r), \quad \text{since } \nabla^2 f(r) = 0 \text{ then}$$

$$f''(r) + \frac{2}{r}f'(r) = 0 = 1\,\frac{f''(r)}{f'(r)} = -\frac{2}{r} \qquad \text{on integration}$$

we get, $\log f'(r) = -2\log r + \log c$

$$\therefore \quad \mathbf{f'(r) = \frac{C}{r^2} \quad \Rightarrow \quad f(r) = -\frac{C}{r} + C_1}$$

$$\therefore \quad \mathbf{f(r) = A + \frac{B}{r} \qquad (A = C_1\ ,\ \ B = -C)}$$

Example : 2

Prove the following :

i) $\nabla^2 r^n = n(n+1)\,r^{n-2}$

ii) $\nabla^2 \dfrac{1}{r^n} = \dfrac{n(n-1)}{r^{n+2}}$

iii) $\nabla^2\left(\nabla \cdot \dfrac{\vec{r}}{r^2}\right) = \dfrac{2}{r^4}$

iv) $\nabla^2(\phi\psi) = \phi\nabla^2\psi + 2\nabla\phi \cdot \nabla\psi + \psi\nabla^2\phi$

v) $\nabla^4(r^2 \log r) = \dfrac{6}{r^2}$

Solution :

i) $f(r) = r^n,\ \ f'(r) = nr^{-n-1}\ f''(r) = n(n-1)\,r^{n-2}$

by formula

$$\nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)$$

$$\nabla^2 r^n = n(n-1)\,r^{n-2} + \frac{2}{r}nr^{n-1} = n(n+1)r^{n-2}$$

ii) $$f(r) = \frac{1}{r^n} = r^{-n},\ \ f'(r) = -nr^{-n-1},\ f''(r) = -n(-n-1)\,r^{-n-2}$$

by using formula

$$\nabla^2 \frac{1}{r^n} = n(n+1)\, r^{-n-2} + \frac{2}{r}(-n\, r^{-n-1})$$

$$\therefore \quad \nabla^2 \frac{1}{r^n} = \frac{n(n-1)}{r^{n+2}}$$

iii) $$\nabla \cdot \frac{\vec{r}}{r^2} = \nabla \cdot r^{-2}\vec{r} = \nabla r^{-2} \cdot \vec{r} + r^{-2}(\nabla \cdot \vec{r})$$

$$= -2\frac{r^{-3}}{r}\vec{r} \cdot \vec{r} + 3r^{-2} = -\frac{2}{r^2} + \frac{3}{r^2} = \frac{1}{r^2}$$

$$\nabla^2\left(\nabla \cdot \frac{\vec{r}}{r^2}\right) = \nabla^2\left(\frac{1}{r^2}\right) = \frac{2.1}{r^4} = \frac{2}{r^4}$$

put n = 2

iv) $$\nabla^2(\phi\psi) = \nabla \cdot \nabla(\phi\psi) = \nabla \cdot [\phi\nabla\psi + \psi\nabla\phi]$$

$$= \nabla\phi \cdot \nabla\psi + \phi(\nabla \cdot \nabla\psi) + \nabla\psi \cdot \nabla\phi \cdot \nabla\phi + \psi(\nabla \cdot \nabla\phi)$$

$$\nabla^2(\phi\psi) = \phi\nabla^2\psi + 2\nabla\phi \cdot \nabla\psi + \psi\nabla^2\phi$$

v) $$\nabla^4(r^2 \log r) = \nabla^2[\nabla^2(r^2 \log r)] = \nabla^2 F(r) = f''(r) + \frac{2}{r} f'(r)$$

$$\nabla^4(r^2 \log r) = -\frac{6}{r^2} + \frac{2}{r} \cdot \frac{6}{r} = \frac{6}{r^2}$$

$$F(r) = \nabla^4(r^2 \log r) \qquad = \nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$$

$$f(r) = r^2 \log r \qquad = 3 + 2\log r + \frac{2}{r}\, r\,(1 + 2\log r)$$

$$f'(r) = r^2 \cdot \frac{1}{r} + 2r \cdot \log r \qquad = 5 + 6 \log r$$

$$f''(r) = 1 + 2\left(\log r + r \cdot \frac{1}{r}\right) \quad F(r) = 5 + 6\log r$$

$$\mathbf{f''(r) = 3 + 2\log r} \qquad \mathbf{F'(r) = \frac{6}{r}, \quad F''(r) = -\frac{6}{r^2}}$$

Example : 3

If $\nabla \cdot \vec{F} = 0$ Show that curl curl curl curl $\vec{F} = \nabla^4\vec{F}$.

Solution :

$$\text{Consider curl curl } \vec{F} = \nabla \times (\nabla \times \vec{F})$$

$$\text{curl curl } \vec{F} = (\nabla \cdot \vec{F})\nabla - \nabla^2\vec{F} = 0 - \nabla^2\vec{F} = \vec{F}_1$$

$$\text{curl curl curl curl } \vec{F} = \nabla \times \nabla \times \nabla \times \nabla \times \vec{F} = \nabla \times \nabla \times \vec{F}_1$$

$$= (\nabla \cdot \vec{F}_1)\nabla - \nabla^2\vec{F}_1 = 0 - \nabla^2\vec{F}_1 = -\nabla^2(-\nabla^2\vec{F})$$

$$\therefore \quad \text{curl curl curl curl } \vec{F} = \nabla^4\vec{F}$$

5.6 Irrotational and Solenoidal Fields :

1. The divergence of a vector is zero everywhere in a field, that field is called solenoidal i.e. div $\vec{V} = 0$, $\nabla \cdot \vec{V} = 0$. Suppose $\nabla \cdot \vec{V} = 0$ then $\vec{V}$ determines a solenoidal field and we know that from vector identities $\nabla \cdot (\nabla \times \vec{F}) = 0$ from the above equations we have $\vec{V} = \nabla \times \vec{F}$.

   i.e. The solenoidal field $\vec{V}$ can be expressed as the curl of the another vector $\vec{F}$.

2. The curl of a vector is zero everywhere in a filed, that field is called irrotational. curl $\vec{F} = 0$ i.e. $\nabla \times \vec{F} = 0$. Suppose $\nabla \times \vec{F} = 0$ then $\vec{F}$ determines an irrotational field and we know that from vector identities $\nabla \times (\nabla\phi) = 0$. From the above equations we have $\vec{F} = \nabla\phi$.

   i.e. The irrotational field $\vec{F}$ can be expressed as the gradient of scalar function (i.e. $\vec{F} = \nabla\phi$)

   To determine ϕ :

   i) $\phi = \int (F_1 dx + F_2 dy + F_3 dz) + C$. Integrate F_1 with respect to x keeping y, z as constants.

      Integrate F_2 with respect to y keeping z as constant, excluding 'x' terms.

      Integrate F_3 with respect to z, excluding x, y, terms.

   ii) If the vector function contains 'r' terms then use $\int \vec{F} \cdot d\vec{r} + C$.

Illustrative Examples

Example : 1

Verify whether the following vector field $\vec{F} = (y \sin z - \sin x)\, i + (x \sin z + 2yz)\, j + (xy \cos z + y^2)\, k$ is irrotational. If so find the corresponding scalar potential function ϕ such that $\vec{F} = \nabla\phi$

Solution :

For irrotational vector field $\nabla \times \vec{F} = 0$

$$\therefore \quad \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial x \\ y\sin z - \sin x & x\sin z + 2yz & xy\cos z + y^2 \end{vmatrix}$$

$$\nabla \times \vec{F} = i[(x \cos z + 2y) - (x \cos z + 2y)] - j[y \cos z - y \cos z] + k\,[\sin z - \sin z] = 0$$

$\therefore \quad \nabla \times \vec{F} \quad 0$ i.e. $\vec{F}$ is irrotational field.

To find scalar potential function ϕ.

$$\phi = \int [(y\sin z - \sin x)\, dx + 2yzdy + 0] + C$$

$$\phi = xy\sin z + \cos x + y^2 z + C$$

$$\nabla\phi = i(y \sin z - \sin x) + j\,(x \sin z + 2yz) + k\,(xy \cos z + y^2)$$

$$\nabla\phi = \vec{\mathbf{F}}$$

Example : 2

If $\vec{F}_1 = (y + z)i + (z + x)j + (x + y)k$ and $\vec{F}_2 = (x^2 - yz)\, i + (y^2 - xz)\, j + (z^2 - xy)\, k$ then show that $\vec{F}_1 \times \vec{F}_2$ is solenoidal.

Solution :

For Solenoidal vector field div $\vec{F} = \nabla \cdot \vec{F} = 0$

let $\vec{F} = \vec{F}_1 \times \vec{F}_2$ then

$$\nabla \cdot \vec{F} = \nabla \cdot (\vec{F}_1 \times \vec{F}_2) = \vec{F}_2 \cdot (\nabla \times \vec{F}_1) - \vec{F}_1 \cdot (\nabla \times \vec{F}_2)$$

$$= \vec{F}_2 \cdot (0) - \vec{F}_1 \cdot (0) = 0$$

Now,
$$\nabla \times \vec{F}_1 = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y+z & z+x & x+y \end{vmatrix}$$

$$= i(1-1) - j\,(1-1) - k\,(1-1) = 0$$

$$\nabla \times \vec{F}_2 = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 - yz & y^2 - xz & z^2 - xy \end{vmatrix}$$

$$= i(-x + x) - j\,(-y + y) + k\,(-z + z)$$

$$= \mathbf{0}$$

$\therefore$ **$\vec{F}_1 \times \vec{F}_2$ is solenoidal field.**

Example : 3

Prove that the vector field $\vec{F} = \dfrac{xi + yj}{x^2 + y^2}$ is solenoidal as well as irrotational.

Solution :

For solenoidal field $\nabla \cdot \vec{F} = 0$

$$\therefore \quad \nabla \cdot \vec{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} \right)\left(\frac{xi + yj}{x^2 + y^2} \right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2} \right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2 + y^2} \right)$$

$$\nabla \cdot \vec{F} = \frac{1}{x^2 + y^2}(1) + x\left(\frac{-2x}{(x^2 + y^2)^2} \right) + \frac{1}{(x^2 + y^2)}(1) + \left(-\frac{2y}{(x^2 + y^2)^2} \right) y$$

$$= \frac{1}{(x^2 + y^2)^2}\,[2x^2 + 2y^2 - 2x^2 - 2y^2] = 0$$

$\therefore$ $\vec{F}$ is solenoidal field.

For irrotational vector field $\nabla \times \vec{F} = 0$

$$\therefore \quad \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \dfrac{x}{x^2+y^2} & \dfrac{y}{x^2+y^2} & 0 \end{vmatrix}$$

$$\nabla \times \vec{F} = i(0-0) - j(0-0) + k\left(-\frac{2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2}\right)$$

$$\boldsymbol{\nabla \times \vec{F} = 0}$$

$\therefore$ **$\vec{F}$ is irrotational field.**

Example : 4

Show that the vector field $f(r)\,\vec{r}$ is irrotational. Determine f(r) such that the field is solenoidal.

Solution :

For irrotational field $\nabla \times \vec{F} = 0$

$$\therefore \quad \nabla \times f(r)\,\vec{r} = \nabla f(r) \times \vec{r} + f(r)\,(\nabla \times \vec{r})$$

$$= \frac{f'(r)}{r}(\vec{r} \times \vec{r}) + f(r)\,(\nabla \times \vec{r})$$

$$\nabla \times f(r)\,\vec{r} = 0 \qquad (\because \vec{r} \times \vec{r} = 0, \ \ \nabla \times \vec{r} = 0)$$

For solenoidal field $\nabla \cdot \vec{F} = 0$

$$\therefore \quad \nabla \cdot f(r)\,\vec{r} = \nabla f(r) \cdot \vec{r} + f(r)\,(\nabla \cdot \vec{r})$$

$$0 = \frac{f'(r)}{r}\,\vec{r} \cdot \vec{r} + f(r)\,(3) \qquad (\nabla \cdot \vec{r} = 3 \ ; \ \vec{r} \cdot \vec{r} = r^2)$$

$$\therefore \quad r\,f'(r) + 3\,f(r) = 0 \quad \Rightarrow \quad \frac{f'(r)}{f(r)} = -\frac{3}{r}$$

On integration we get

$$\log f(r) = -3 \log r + \log c$$

$$\therefore \quad \boldsymbol{f(r) = \frac{c}{r^3}}$$

Example : 5

If ϕ, ψ satisfy Laplace equation then prove that the vector $(\phi\nabla\psi - \psi\nabla\phi)$ is solenoidal.

Solution : For solenoidal field $\nabla \cdot \vec{F} = 0$

$$\therefore \quad \nabla \cdot \vec{F} = \nabla \cdot (\phi\nabla\psi - \psi\nabla\phi)$$

$$= \nabla \cdot (\phi\nabla\psi) - \nabla \cdot (\psi\nabla\phi)$$

$$= \nabla\phi \cdot \nabla\psi + \phi\nabla^2\psi - \nabla\psi \cdot \nabla\phi - \psi\nabla^2\phi$$

$$\boldsymbol{= 0 \qquad (\because \nabla^2\psi = 0, \ \ \nabla^2\phi = 0 \ \ \text{Laplace equation})}$$

Example : 6

Show that $\vec{F} = \frac{1}{r}[r^2\vec{a} + (\vec{a}\cdot\vec{r})\vec{r}]$ is irrotational. Hence find ϕ such $\vec{F} = \nabla\phi$.

Solution :

For irrotational field $\nabla \times \vec{F} = 0$

$$\therefore \quad \nabla \times \vec{F} = \nabla \times (r\vec{a}) + \nabla \times \left[\left(\frac{\vec{a}\cdot\vec{r}}{r}\right)\vec{r}\right]$$

$$\nabla \times \vec{F} = \nabla r \times \vec{a} + r(\nabla \times \vec{a}) + \nabla\left(\frac{\vec{a}\cdot\vec{r}}{r}\right) \times \vec{r} + \left(\frac{\vec{a}\cdot\vec{r}}{r}\right)(\nabla \times \vec{r})$$

$$= \frac{1}{r}(\vec{r} \times \vec{a}) + 0 + \left[\frac{1}{r}\nabla(\vec{a}\cdot\vec{r}) + (\vec{a}\cdot\vec{r})\nabla\frac{1}{r}\right] \times \vec{r} + 0$$

$$= \frac{1}{r}(\vec{r} \times \vec{a}) + \frac{1}{r}(\vec{a} \times \vec{r}) - \frac{(\vec{a}\cdot\vec{r})}{r^3}(\vec{r} \times \vec{r}) \qquad \left(\begin{array}{c}\because \nabla \times \vec{a} = 0 \\ \nabla \times \vec{r} = 0 \\ \vec{r} \times \vec{r} = 0\end{array}\right)$$

$$= -\frac{\vec{a} \times \vec{r}}{r} + \frac{\vec{a} \times \vec{r}}{r} - 0$$

$$= \mathbf{0}$$

$\therefore$ **$\vec{F}$ is irrotational field.**

- **To determine scalar potential function ϕ :**

$$\vec{F}\cdot d\vec{r} = \left[r\vec{a} + \frac{(\vec{a}\cdot\vec{r})}{r}\vec{r}\right]\cdot d\vec{r}$$

$$= r(\vec{a}\cdot d\vec{r}) + \frac{(\vec{a}\cdot\vec{r})}{r}\vec{r}\cdot d\vec{r}$$

$$\vec{F}\cdot d\vec{r} = r(\vec{a}\cdot d\vec{r}) + \frac{\vec{a}\cdot\vec{r}}{r}r\,dr$$

$$= d[r(\vec{a}\cdot\vec{r})]$$

$$\therefore \quad \phi = \int \vec{F}\cdot d\vec{r} + C$$

$$= \int d[r(\vec{a}\cdot\vec{r})] + C \qquad (\because \vec{r}\cdot d\vec{r} = r\,dr)$$

$$\phi = r(\vec{a}\cdot\vec{r}) + C$$

$$\nabla\phi = \nabla[r(\vec{a}\cdot\vec{r}) + C] = r\nabla(\vec{a}\cdot\vec{r}) + (\vec{a}\cdot\vec{r})\nabla r + 0$$

$$\nabla\phi = r\vec{a} + \frac{\vec{a}\cdot\vec{r}}{r}\vec{r} = \vec{F}$$

$$\therefore \quad \vec{\mathbf{F}} = \nabla\phi$$

Self–Assessment Exercise : 5.1

Ex.1: A particle moves on the curve $x = 2t^2$, $y^2 = t^2 - 4t$, $z = 3t - 5$, where t is time. Find the components of velocity and acceleration at time t = 1 in the direction $\hat{i} - 3\hat{j} + 2\hat{k}$.

$$\left[\textbf{Ans. } \frac{8\sqrt{14}}{7}\ \frac{-2}{\sqrt{14}}\right]$$

Ex.2: A particle moves along the curve $\bar{r} = (t^3 - 4t)\hat{i} + (t^2 + 4t)\hat{j} + (8t^2 - 3t^3)\hat{k}$ where t denotes time. Find the magnitudes of acceleration along the tangent and normal at time t = 2.

[Ans. 16, $2\sqrt{73}$]

Ex.3: Find the angle between tangent to the curve $x = t$, $y = t^2$, $z = t^3$ at $t = \pm 1$.

$$\left[\textbf{Ans. } \cos^{-1}\left(\frac{3}{7}\right)\right]$$

Ex.4: A particle moves so that its position vector is given be $\bar{r} = \cos wt\,\hat{i} + \sin wt\,\hat{j}$. Show that the velocity $\bar{V}$ of the particle is perpendicular to $\bar{r}$ and $\bar{r} \times \bar{v}$ is a constant vector.

Ex.5: Find the unit tangent vector at any point on the curve x = 3 cost, y = 3sint, z = 4t.

$$\left[\textbf{Ans. } \hat{T} = \frac{1}{5}(-3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 4\hat{k})\right]$$

Ex.6: If $\bar{r} = \cos nt\,\hat{i} + \sin(nt)\,\hat{j}$ where n is a constant and t varies, show that $\bar{r} \times \frac{d\bar{r}}{dt} = n\hat{k}$.

Ex.7: If $\bar{r} = \bar{a}\sinh t + \bar{b}\cosh t$ then prove that

(i) $\frac{d^2\bar{r}}{dt^2} = \bar{r}$, (ii) $\frac{d\bar{r}}{dt} \times \frac{d^2\bar{r}}{dt^2}$ = constant, (iii) $\bar{r} \cdot \frac{d\bar{r}}{dt} \times \frac{d^2\bar{r}}{dt^2} = 0$.

Self–Assessment Exercise : 5.2

Ex.1: Find $\nabla\phi$ for

i) $\phi = 3x^2y - y^3z^2$ at the point (1, 2, –1) **Ans. :** $12\hat{i} - 9\hat{j} - 16\hat{k}$

ii) $\phi = x^3 + y^3 + 3xyz = 3$ at the point (1, 2, –1) **Ans. :** $-3\hat{i} + 9\hat{j} + 6\hat{k}$

iii) $\phi = x^2y + 2xz - 4 = 0$ at the point (2, –2, 3) **Ans. :** $-\hat{i} + 2\hat{j} + 2\hat{k}$

Ex.2: Find unit vector normal to the surface.

$xy^3z^2 = 4$ at the point (–1, –1, 2) **Ans. :** $\frac{-1}{\sqrt{11}}\hat{i} + 3\hat{j} - \hat{k}$

Ex.3: Find angle between normal to the surfaces $xy^2z = 3xz^2$ and $3x^2 - y^2 + xz = 1$ at the point (1, –2, 1) **Ans. :** $\cos\theta = \frac{3}{7\sqrt{6}}$

(**Hint :** Angle between two surfaces is same as angle between their normal)

Ex.4 The temperature at a point (x, y, z) in space is given by $T(x, y, z) = x^2 + y^2 - z$. A mosquito located at (1, 1, 2) desire to fly in such direction that it will get warm as soon as possible in what direction should it fly. **Ans. :** $2\hat{i} + 2\hat{j} - \hat{k}$

(**Hint :** Along normal to the surface)

Ex.5 In what direction from (3, 1, –2) is the directional derivative of $\phi = x^2y^2z^4$ maximum and what is its magnitude. **Ans. :** $96\hat{i} + 288\hat{j} - 288\hat{k}$ $960T_9$

Ex.6 Find directional derivative of $\phi = xy^2 + yz^3$ at the point $(2_1 - 1_1\ 1)$ in the direction of normal to the surface $x \log z - y^2 + 4 = 0$ at (–1, 2, 1) **Ans. :** $\frac{15}{\sqrt{17}}$

Ex.7 Find angle between the tangent planes to the surfaces $x \log z = y^2 - 1$ and $x^2y = 2 - z$ at the point (1, 1, 1) **Ans. :** $\cos{-1}\left(\frac{-1}{\sqrt{30}}\right)$

(**Hint :** Angle between two plane is same as angle between their normal)

Ex.8 Find direction derivative of $x^2y + y^3z$ at $(2_1 - 1_1\ 1)$ along the direction which make equal angle with co–ordinate axes. **Ans. :** $\frac{-2}{\sqrt{3}}$

Ex.9 If direction derivative of $\phi = a(x + y) + b(y + z) + c(x + z)$ has maximum magnitude 12 in the direction.

Parallel to the line $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-1}{3}$ Find a, b, c. **Ans. :** $0, \pm\frac{24}{\sqrt{14}}, \pm\frac{12}{\sqrt{14}}$

Ex.10 Find direction derivative of the function $F = x^2 - y^2 + 2z^2$ at the point P(1, 2, 3) in the direction of the line PQ where Q is the point (5, 0, 4). **Ans. :** $\pm\frac{28}{\sqrt{21}}$

Ex.11 Find the directional derivative of $\phi = x^2 + 2y^2 - 3z^2$ at (1, 2, 1) in the direction tangent to $x = t^2 + t$, $y = 2t$, $z = 2 - t$ at $t = 1$. **Ans. :** $2\sqrt{14}$

Ex.12 Show that the vector field $\overline{F} = (x^2 + xy^2)\ \hat{i} + 1y^2 + x^2y)\ \hat{j}$ is irrotational. Hence find the scalar potentialϕ such that $\overline{F} = \nabla\phi$. **Ans. :** $\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{x^2y^2}{2} + C$

Ex.13 Prove that the vector $F(r)\ \overline{r}$ is irrotational.

Ex.14 Show that the vector field $\overline{F} = (\sin y + z) + \hat{i}\ (x \cos y - z)\ \hat{j} + (x - y)\ \hat{k}$ is irrotational. Find scalar ϕ such that $\overline{F} = \text{grad}\ \phi$. **Ans. :** $\phi = x \sin y + xz - yz + C$

Ex.15 Show that the vector field $\overline{F} = 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$ is irrotational. Find the scalar potential ϕ. Such that $\overline{F} = \text{grad}\phi$. **Ans. :** $\phi = x^2yz^3 + C$

Ex.16 Show that $\overline{F} = \frac{\overline{r}}{r^2}$ is irrotational.

Ex.17 If $\overline{F} = (y + z)\,\hat{i} + (z + z)\,\hat{j} + (x + y)\,\hat{k}$ then show that curl curl curl cur $\overline{F} = \nabla^4\overline{F}$

(**Hint :** First show that $\nabla\cdot\overline{F} = 0$)

Ex.18 Show that the vector field $\overline{F} = (y^2 \cos x + z^2)\,\hat{i} + 2y\sin x\,\hat{j} + 2xz\hat{k}$ is irrotational. Hence find ϕ such that $\overline{F} = \nabla\phi$. **Ans. :** $\phi = y^2 \sin x + z^2x + C$

Ex.19 Show that the vector $\overline{V} = (x + 3y)\,\hat{i} + (y - 3z)\,\hat{j} + (x - 2z)\,\hat{k}$ is solenoidal.

Ex.20 Define F(r) such that $\overline{F} = F(r)\,\overline{r}$ is solenoidal. **Ans. :** $F(r) = \dfrac{C}{r^3}$

Ex.21 Show that the vector field $\overline{F} = (x^2 - y^2 + x)\,\hat{i} + (x + y)\,\hat{j}$ is irrotational. Find a scalar ϕ such that $\overline{F} = \text{grad}\,\phi$. **Ans. :** $\phi = \dfrac{1}{3}x^3 - xy^2 + \dfrac{1}{2}(x^2 - y^2) + C$

Ex.22 If $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$ Prove that :

i) $\nabla\cdot(r^n\overline{r}) = (n + 3)y^n$ ii) $\nabla\times(r^n\overline{r}) = 0$

iii) $\nabla\cdot\left(\dfrac{\overline{r}}{r^3}\right) = 0$ iv) $\nabla^2\left[\nabla\cdot\left(\dfrac{\overline{r}}{r^2}\right)\right] = \dfrac{2}{r^4}$

v) $\nabla\cdot\left\{r\nabla\left(\dfrac{1}{r^3}\right)\right\} = \dfrac{3}{r^4}$ vi) $\nabla^2(r^n) = n\,(n + 1)\,r^{n-2}$

Ex.23 If $\overline{a}$ is constant vector and $\overline{r} = x\hat{i} + y\,\hat{j} + z\hat{k}$ Prove that curl $(\overline{a}\times\overline{r}) = 2\overline{a}$

Ex.24 If $u\overline{F} = \nabla v$ where u, v are scalar field and $\overline{F}$ is a vector field show that $\overline{F}\,.\,\text{curl}\,\overline{F} = 0$.

Ex.25 Show that $\overline{F} = \dfrac{1}{r}\,[r^2\overline{a} + (\overline{a}\cdot\overline{r})\overline{r}\,]$ is irrotational. Hence find scalar potential ϕ.

$[\phi = r\,(\overline{a}\cdot\overline{r}) + c]$

Ex.26 If $\overline{F}_1 = yz\hat{i} + xz\hat{j} + xy\hat{k}$, $\overline{F}_2 = (\overline{a}\cdot\overline{r})\overline{a}$ then show that $\overline{F}_1\times\overline{F}_2$ is solenoidal.

Ex.27 Prove that $\overline{F} = \dfrac{1}{x^2 + y^2}(x\hat{i} + y\hat{j}\,)$ is solenoidal.

Ex.28 Find the function F(r) so that $F(r)\overline{r}$ is solenoidal.

Ex.29 If ϕ, ψ satisfy Laplace equation then prove that $(\phi\nabla\psi - \psi\nabla\phi)$ is solenoidal.

Ex.30 Evaluate : $\nabla\cdot(r^3\overline{r}\,)$

Descriptive Questions

Q.1 Find the directional derivative of $\phi = x^2yz^3$ at (2, 1, –1) along the vector $-4\hat{i} - 4\hat{j} + 12\hat{k}$.

Q.2 Find the directional derivative at $xy^2 + yz^3$ at 2, –1, 1
along the line $2(x - 2) = y + 1 = z - 1$

Q.3 Find the direactional derivative of $\phi = xy^2 + yz^3$ at (1, –1, 1) along the direction normal to the surface $x^2 + y^2 + z^2 = 9$ at (1, 2, 2)

Q.4 Show that vector field

$\overline{F} = (x^2 - yz)\hat{i} + (y^2 - xz)\hat{j} + (x^2 - xy)\hat{k}$ is irrotatioal hence find scalar potential ϕ such that $\overline{F} = \nabla\phi$

Q.5 Show that the vector field

$\overline{F} = (x + 2y + 4z)\hat{i} + (2x - 3y - z)\hat{j} + (4x - y + 2z)\hat{k}$ is irrotational and hence find scalar potential ϕ such that $\overline{F} = \nabla\phi$

Q.6 Show that vector field

$\overline{F} = (y^2\cos x + z^2)\hat{i} + (2y\sin x)\hat{j} + 2xz\hat{k}$ is irrotational and find scalar potential ϕ such that $\overline{F} = \nabla\phi$

Q.7 Prove the following :

i) $\nabla^4(r^2 \log r) = \dfrac{6}{r^2}$

ii) $\overline{a} \cdot\nabla\left[\overline{b} \cdot\nabla\left(\dfrac{1}{r}\right)\right] = \dfrac{3\,(\overline{a}\cdot\overline{r}\ \overline{b}\cdot\overline{r})}{r^5} - \dfrac{(\overline{a}\cdot\overline{b})}{r^3}$

iii) $\nabla^2F(r) = F''(r) + \dfrac{2}{r}F'(r)$

iv) $\nabla\cdot\left[\ r\,\nabla\left(\dfrac{1}{r^3}\right)\right] = \dfrac{3}{r^4}$

v) $\overline{b} \times\nabla\,(\overline{a} \cdot\nabla \log r) = \dfrac{\overline{b} \times\overline{a}}{r^2} - \dfrac{2\,(\overline{a}\cdot\overline{r})}{r^4}(\overline{b} \times\overline{r})$

vi) $\nabla^4\,(r^2 \log r) = \dfrac{6}{r^2}$

vii) $\nabla\times\left(\ \overline{a} \times\nabla\dfrac{1}{r}\right) = \dfrac{\overline{a}}{r^3} - \dfrac{3\,(\overline{a}\cdot\overline{r})}{r^3}\,\overline{r}$

viii) $\nabla^2\left(\nabla\cdot\dfrac{\overline{r}}{r^2}\right) = \dfrac{2}{r^4}$

Q.8 Find the contant a & b so that the surface $ax^2 - byz = (a + z)\,x$ will be orthogonal to surface $4x^2y + z^3 = 4$ at the point (1, –1, 2)

Q.9 Show that the vector field $F(r)\,\overline{r}$ is always irrotational and determine F(r) such that the field is solenoidal also.

Q.10 Find directional derivative of $\phi = xy + yz^2$ at (1, –1, 1) towards the point (2, 1, 2)

Q.11 Show that $\overline{F} = (2xz^3 + 6y)\hat{i} + (6x - 2yz)\hat{j} + (3x^2z^2 - y^2)\hat{k}$ is irrotational. Find the scalar potential ϕ sucha that $\overline{F} = \nabla\phi$

Q.12 Show that the vector field given by $\overline{A} = (x^2 + xy^2)\hat{i} + (y^2 + x^2y)\hat{j}$ is irrotational f find scalar potential such that $\overline{F} = \nabla\phi$.

Q.13 Find the directional derivative of $\phi = x^2 - y^2 + 2z^2$ at the point P(1, 2, 3) in the direction of line PQ, where Q is (5, 0, 4)

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Unit 6

Complex Variables

Syllabus :

Functions of complex variables, Analytic functions, Cauchy – Riemann equations, conformal mapping, Bilinear Transformation , Cauchy's Integral theorem, Cauchy Integral formula, Laurent's series and Residue theorem.

6.1 Introduction :

Theory of complex functions / variables are widely used in the study of fluid dynamics, conduction of heat flow, electrostatics, electromagnetic Engineering, two dimensional potential problems. thermodynamics, electrical fields etc.

6.1.1 Prerequisites :

1. The complex numbers are denoted by $z = x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$
2. $|z| = \sqrt{x^2 + y^2}$
3. $\bar{z} = x - iy$
4. The complex number $x + iy$ is geometrically denoted by a point (x, y) on XY plane
5. The X axis is called real axis & Y axis is called as imaginary axis
6. $|z| = |a|$ is $x^2 + y^2 = a^2$, a circle with centre $(0, 0)$ and radius a units in XOY plane or z– plane.
7. $z_0 = x_0 + iy_0 \equiv (x_0, y_0)$
8. $|z - z_0| = |a|$ means $|(x + iy) - (x_0 + iy_0)| = |a|$

   $\therefore\ |(x - x_0) + i(y - y_0)| = |a|$

   $\therefore\ \sqrt{(x - x_0)^2 + (y - y_0)^2} = |a|$

   $\therefore\ (x - x_0)^2 + (y - y_0)^2| = a^2$, which is a circle with centre (x_0, y_0) and radius a units
9. $|z - z_0| < a$, means $(x - x_0)^2 + (y - y_0)^2 < a^2$, which is a interior part of circle with centre (x_0, y_0) and radius a units.

10. $| z - z_0 | > a$, means $(x - x_0)^2 + (y - y_0)^2 > a^2$, which is a exterior part of circle, whose centre (x_0, y_0) and radius a units.

11. $a < | z - z_0 | < b$, gives a region lies between two concentric circles having centre (x_0, y_0) and having radii a, b and that region is called as an open circular ring or open annulus.

12. $| z + z_0 | = |a|$, means $(x + x_0)^2 + (y + y_0)^2 = a^2$, a circle with centre $(-x_0, - y_0)$ and radius a units.

13. If $f(x, y) = 0$ then $\frac{dy}{dx}$, gives the slope of tangent to the curve $f(x, y) = 0$

14. If m_1, m_2 are slopes of tangents to the two curves and if $m_1 \times m_2 = -1$, then two curves will be orthogonal to each other.

6.1.2 Complex Functions / Complex Variables / Function of Complex Variables :

Let u(x, y) and v(x, y) be real valued functions of real numbers x, y. Let $z = x + iy$, where $i = \sqrt{-1}$, then $[u(x, y) + i\, v(x, y)]$ is called as complex function and denoted by f (z)

$\therefore \quad f(z) = u(x, y) + i\, v(x, y) = u + iv$

$\therefore \quad w = f(z) = u(x, y) + i\, v(x, y) = u + iv$

Note that u(x, y) and v(x, y) are called real and imaginary parts of complex function w or f(z).

Example :

(i) If $w = z^3$

then $w = (x + iy)^3 = x^3 + 3x^2(iy) + 3x\,(iy)^2 + (iy)^3$

$= (x^3 - 3xy^2) + i\,(3x^2y - y^3)$

$\therefore \quad u = x^3 - 3xy^2, \; v = 3x^2y - y^3$

(ii) If $w = \overline{z}$

then $w = x - iy = x + i(-y)$

$\therefore \quad u = x, \; v = -y$

6.1.3 Single Valued Function / Multivalued Function :

If w = f (z) and if for each value of z, there exists only one value of w then w is called as a single valued function, otherwise it is called as a multivalued function.

Example : (1) If $w = z^2$ and if $z = 4$ then $w = 4^2 = 16$

$\therefore w = z^2$ is a single valued function

(2) If $w = \sqrt{z}$ and if $z = 4$ then $w = \sqrt{4} = \pm 2$

$\therefore w = \sqrt{z}$ is multivalued function

- **Continuity :**

  A complex function f (z) is said to be continuous at point $z = z_0$, if $\lim_{z \to z_0} f(z) = f(z_0)$

  (i) A function is said to be continuous in a domain R, if it is continuous at every point in domain R.

  (ii) A complex function w = u + iv is continuous, if both u and v are continuous functions.

- **Complex Differentiation :**

  Let $\quad w = f(z)$ be a single valued function of z

  then $\quad \frac{dw}{dz} = \lim_{h \to 0} \left[\frac{f(z+h) - f(z)}{h}\right]$

  (i) $\frac{dw}{dz} = f'(z)$

  (ii) $\frac{dw}{dz}$ exists only if limit on r.h.s. exists.

  (iii) The formula for differentiation of f (z) is as same to the formula for differentiation in real calculus and hence all formulae of differentiation of real calculus are valid in complex differentiation as below :

1) $\frac{d}{dz}$ (constant) = 0 $\qquad$ 2) $\frac{d}{dz}[f \pm g] = \frac{df}{dz} \pm \frac{dg}{dz}$

3) $\frac{d}{dz}[cf(z)] = c\frac{d}{dz}[f(z)]$ $\qquad$ 4) $\frac{d}{dz}[f.g] = f\frac{dg}{dz} \pm g\frac{df}{dz}$

5) $\frac{d}{dz}[\tan z] = \sec^2 z$ $\qquad$ 6) $\frac{d}{dz}[z^n] = n\,z^{n-1}$

7) $\frac{d}{dz}[\cos z] = -\sin z$ $\qquad$ 8) $\frac{d}{dz}\left(\frac{f}{g}\right) = \frac{gdf - fdg}{g^2}$

9) $\frac{d}{dz}[\sec z] = \sec z \tan z$ $\qquad$ 10) $\frac{d}{dz}(\log z) = \frac{1}{z}$, etc…..

6.2 Analytic Functions :

A function f(z) , which is a single valued function and possesses a unique derivative at all points of domain R, is called as an analytic function of z in region R.

An analytic function is also known as **entire / regular / holomorphic function.**

Example :

(1) If $f(z) = z^2$ then $f'(z) = 2z$, exists everywhere

(2) If $f(z) = z^3$ then $f'(z) = 3z^2$, exists everywhere

- **Harmonic function :**

  A real valued function u(x, y) is called as a harmonic function, if it satisfies a Laplace's equation $\quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

6.3 Cauchy Riemann (C–R) Equations : (Cartesian Form)

The complex function f(z) = u(x, y) + iv(x, y) will be an analytic function is region R if it satisfies the C – R equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

where the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x, y in the region R.

Note : Consider an analytic function f (z) = (u + iv), then

(i) u and v are always harmonic functions .

(ii) u and v satisfies C–R equations.

(iii) real part u and imaginary part v are called as conjugate harmonic functions of each other.

(iv) differentiation of an analytic function is also an analytic function

(v) polynomial function is always an analytic function

(vi) $u = c_1$, $v = c_2$ forms mutually orthogonal trajectories

(vii) if f & g are analytic functions then (f ± g), $\left(\frac{f}{g}\right)$,(f. g) are also analytic functions if g(z) ≠ 0

(viii) **Milne Thompson method :** Analytic functions f (z) and f ′(z) can be expressed in terms of only z by replacing x = z , y = 0

(ix) $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$

6.4 Cauchy Riemann (C–R) Equations : (Polar Form)

The complex function f (z) = u(r, θ) + iv(r, θ) will be analytic if

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Note :

The complex function f (z) = u(r, θ) + iv(r, θ), can be expressed in terms of only z by replacing r = z, θ = 0, if f (z) is an analytic function.

Illustrative Examples

Type : (A)

Example : 1

Find an analytic function f (z) = u + iv,

where $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 6$ …. (1)

and hence express f (z) in terms of only z

Solution :

We know, f(z) will be an analytic function

if $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \ldots (2)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \ldots (3)$$

Diff (1) partially w.r.to x (y constant)

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$$

$$\therefore \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2 + 6x \qquad \because (2) \text{ above}$$

Integrate partially w.r. to y (x constant)

$$v = 3x^2y - 3\left(\frac{y^3}{3}\right) + 6xy + A(x)$$

$$v = 3x^2y - y^3 + 6xy + A(x) \quad \ldots (4)$$

Differentiate (4) partially w.r.to x (y constant)

$$\frac{\partial v}{\partial x} = 3y(2x) - 0 + 6y + A'(x)$$

$$\frac{\partial v}{\partial x} = 6xy+6y+A'(x)$$

$$\therefore \quad -\frac{\partial u}{\partial y} = 6xy+6y+A'(x) \qquad \because (3) \text{ above}$$

$$\therefore \quad \frac{\partial u}{\partial y} = -6xy-6y-A'(x) \quad \ldots (5)$$

Diff (1) partially w.r. to y (x constant)

$$\frac{\partial u}{\partial y} = 0 - 6xy + 0 - 6y$$

$$= -6xy - 6y \quad \ldots (6)$$

From (5) and (6) we get

$$-6xy - 6y = -6xy - 6y - A'(x)$$

$$\therefore \quad -A'(x) = 0$$

$$\therefore \quad A'(x) = 0$$

$$\therefore \quad A(x) = c$$

$\therefore$ From (4), $$v = 3x^2y - y^3 + 6xy+c \quad \ldots (7)$$

$\therefore$ Required analytic function f(z) is

$$f(z) = u+iv$$

$$f(z) = (x^3-3xy^2+3x^2-3y^2+6) + i(3x^2y - y^3 + 6xy+c) \quad \ldots (8)$$

Milne–Thompson Method :

Put x = z, y = 0 in (8)

$\therefore$ f(z) in terms of only z is

$$f(z) = (z^3 + 3z^2 + 6) + i\,(c)$$

$$= \mathbf{z^3 + 3z^2 + 6 + ic}$$

Example : 2

Find a regular function f(z) = u+iv

where $v = -\sin x \sinh y$ …. (1)

Also express f(z) only in terms of z.

Solution :

Diff (1) partially w.r. to y (x constant)

$$\frac{\partial v}{\partial y} = -\sin x \cosh y \quad \text{…. (2)}$$

$$\frac{\partial u}{\partial x} = -\sin x \cosh y \quad \text{…. (3)}$$

$\because$ C–R equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

Integrate (3) partially w.r. to x (y constant)

$$\therefore \quad u = (-\cosh y)(-\cos x) + A(y)$$

$$u = \cos x \cosh y + A(y) \quad \text{…. (4)}$$

Differentiate (4) partially w.r.to y (x constant)

$$\therefore \quad \frac{\partial u}{\partial y} = \cos x \sinh y + A'(y)$$

$$-\frac{\partial v}{\partial x} = \cos x \sinh y + A'(y) \qquad \because \text{ by C – R equation } \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$$

$$\therefore \quad \frac{\partial v}{\partial x} = -\cos x \sinh y - A'(y) \quad \text{…. (5)}$$

Diff (1) partially w.r. to x (y constant)

$$\frac{\partial v}{\partial x} = -\cos x \sinh y \quad \text{…. (6)}$$

From (5) and (6), we get

$$-\cos x \sinh y - A'(y) = -\cos x \sinh y$$

$$\therefore \quad -A'(y) = 0$$

$$\therefore \quad A'(y) = 0$$

$$\therefore \quad A(y) = c$$

$\therefore$ Required, $u = \cos x \cosh y + c$

$\therefore$ Required analytic function f (z) is

$$f(z) = (\cos x \cosh y + c) + i(-\sin x \sinh y) \qquad \ldots.(7)$$

Milnes Thompson Method :

Put $x = z,\ y = 0$ in (7)

$$\mathbf{f(z) = \cos z + c}$$

Example : 3

If $v = y + e^x \cos y$, show that

(i) v is a harmonic function

(ii) find u, harmonic conjugate of v, such that f(z) = u +iv be an analytic function

(iii) Express f(z) in terms of only z

(iv) Show that $u(x, y) = c_1$, $v(x, y) = c_2$, where c_1, c_2 are constants, cuts each other orthogonally.

Solution :

i) Let

$$v = y + e^x \cos y \qquad \ldots.(1)$$

$$\therefore \quad \frac{\partial v}{\partial x} = 0 + e^x \cos y$$

$$\frac{\partial^2 v}{\partial x^2} = e^x \cos y$$

$$\frac{\partial v}{\partial y} = 1 + e^x(-\sin y)$$

$$\frac{\partial^2 v}{\partial y^2} = 0 + e^x(-\cos y) = -e^x \cos y$$

$$\therefore \quad \frac{\partial^2 \mathbf{v}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{v}}{\partial \mathbf{y}^2} = \mathbf{e^x \cos y - e^x \cos y = 0}$$

$\therefore$ v is a harmonic function.

(ii) To find u, harmonic conjugate of v

Diff. (1) partially w.r. to x (y constant)

$$\therefore \quad \frac{\partial v}{\partial x} = e^x \cos y$$

$$-\frac{\partial u}{\partial y} = e^x \cos y \qquad \because \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\therefore \quad \frac{\partial u}{\partial y} = -e^x \cos y \qquad \ldots.(2)$$

Integrate (2) partially w.r. to y (x constant)

$$\therefore \quad u = -e^x(\sin y) + A'(x) \qquad \ldots.(3)$$

Differentiate (3) partially w.r. to x (y constant)

$$\frac{\partial u}{\partial x} = -e^x \sin y + A'(x)$$

$$\therefore \quad \frac{\partial v}{\partial y} = -e^x \sin y + A'(x) \qquad \because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \ldots (4)$$

Differentiate (1) partially w.r. to y (x constant)

$$\therefore \quad \frac{\partial v}{\partial y} = 1 + e^x (-\sin y)$$

$$= 1 - e^x \sin y \qquad \ldots (5)$$

From (4) and (5), $\dfrac{\partial v}{\partial y} = -e^x \sin y + A'(x) = 1 - e^x \sin y$

$$\therefore \quad -e^x \sin y + A'(x) = 1 - e^x \sin y$$

$$\therefore \quad A'(x) = 1$$

$$\therefore \quad A(x) = x$$

From (3) $\quad u = -e^x \sin y + x$

$$u = x - e^x \sin y \qquad \ldots (6)$$

Required Analytic function f (z) is

$$f(z) = u + iv$$

$$= \mathbf{(x - e^x \sin y) + i\,(y + e^x \cos y)} \qquad \ldots \mathbf{(7)}$$

(iii) To express f (z) in terms of z by using Milne–Thompson method :

put x = z and y = 0 in (7)

$$f(z) = [\,z - e^z (0)\,] + i\,(0 + e^z (1))$$

$$= \mathbf{z + i\, e^z}$$

(iv) Orthogonal trajectories :

We know that $\dfrac{dy}{dx}$ gives the slope of tangent to curve $f(x, y) = 0$

we have $\quad u = x - e^x \sin y \quad v = y + e^x \cos y$

Let $\quad u = x - e^x \sin y = c_1 \qquad \ldots (8)$

$$\therefore \quad \frac{dy}{dx} = -\frac{\left(\dfrac{\partial u}{\partial x}\right)}{\left(\dfrac{\partial u}{\partial y}\right)}$$

$$= -\left[\frac{(1 - e^x \sin y)}{-e^x \cos y}\right] = m_1 \qquad \ldots (9)$$

Let $\quad v = y + e^x \cos y = c_2 \qquad \ldots (10)$

$$\therefore \qquad \frac{dy}{dx} = -\frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)}$$

$$= -\left[\frac{0 + e^x \cos y}{1 - e^x \sin y}\right]$$

$$= -\left[\frac{e^x \cos y}{1 - e^x \sin y}\right] = m_2 \qquad \ldots. (11)$$

$$\therefore \qquad (m_1)(m_2) = \left(\frac{1 - e^x \sin y}{-e^x \cos y}\right)\left(\frac{-e^x \cos y}{1 - e^x \sin y}\right)$$

$$= \mathbf{-1}$$

$\therefore$ Product of slopes of tangents to curves (8) & (10) is (–1)

$\therefore$ Curves (8) & (10) forms orthogonal system/trajectory.

Type : B

Example : 1

Determine an analytic function f (z) = u + iv in terms of z if $u = x^2 - y^2 - y$

Solution :

Let $u = u = x^2 - y^2 - y \qquad \ldots. (1)$

Differentiate (1) partially w.r. to x & y , we get

$$\frac{\partial u}{\partial x} = 2x \qquad \ldots. (2)$$

$$\frac{\partial u}{\partial y} = -2y - 1 \qquad \ldots. (3)$$

we know , if f (z) = u + iv, is an analytic function

$$\text{then, } f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} + i\left(-\frac{\partial u}{\partial y}\right) \qquad (\because \text{C – R equations})$$

$$f'(z) = 2x + i(2y + 1) \qquad (\because 2 \ \& \ 3)$$

Put x = z , y = 0 Milne–Thompson method

$$= 2z + i(0 + 1)$$

$$f'(z) = 2z + i$$

Integrate w.r. to z , $f(z) = 2\left(\frac{z^2}{2}\right) + iz + c$

$$\mathbf{f(z) = z^2 + iz + c}$$

Example : 2

If $v = 4x^3y - 4xy^3$, find harmonic conjugate of v such that f(z) = u +iv becomes an analytic function.

Solution : Let $v = 4x^3y - 4xy^3$ …. (1)

Differentiate (1) partially w.r. to x & y

$$\therefore \quad \frac{\partial v}{\partial x} = 12x^2y - 4y^3 \quad \text{…. (2)}$$

$$\therefore \quad \frac{\partial v}{\partial y} = 4x^3 - 12xy^2 \quad \text{…. (3)}$$

We know, $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \quad \left(\because \text{C–R equations } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right)$$

$$f'(z) = (4x^3 - 12xy^2) + i\,(12x^2y - 4y^3)$$

$$= (4z^3 - 0) + i\,(0 - 0) \quad (\because x = z, y = 0 \text{ Milnes–Thompson method})$$

$$f'(z) = 4z^3 + 0$$

Integrate w.r. to z we get

$$f(z) = 4\left(\frac{z^4}{4}\right) + c$$

$$f(z) = z^4 + c \quad \text{…. (4)}$$

$$= (x + iy)^4 + c$$

$$= x^4 + 4(x^3)(iy) + 6(x^2)(iy)^2 + 4(x)(iy)^3 + (iy)^4 + c$$

$$= x^4 + 4iyx^3 + 6x^2(-y^2) + 4x(-iy^3) + i^4y^4 + c$$

$$= x^4 + 4iyx^3 - 6x^2y^2 - 4ixy^3 + y^4 + c$$

$$= (x^4 - 6x^2y^2 + y^4) + i\,(4yx^3 - 4xy^3) + c$$

$$= (x^4 - 6x^2y^2 + y^4 + c) + i\,(4yx^3 - 4xy^3)$$

$$f(z) = u + iv$$

$$\therefore \quad \mathbf{u = x^4 - 6x^2y^2 + y^4 + c}$$

Type : C

Example : 1

Find an analytic function f(z) = u + iv

if $u + v = e^{-x}(\cos y - \sin y)$ …. (1)

Solution :

Differentiate (1) partially w.r.to x (y constant)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = -e^{-x}(\cos y - \sin y) \quad \ldots.(2)$$

Differentiate (1) partially w.r.to y (x constant)

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = e^{-x}(-\sin y - \cos y) = -e^{-x}(\sin y + \cos y)$$

$$-\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} = -e^{-x}(\sin y + \cos y) \quad \ldots.(3) \qquad (\because \text{C–R equations})$$

Adding (2) & (3) we get

$$2\frac{\partial u}{\partial x} = -e^{-x}(\cos y - \sin y) - e^{-x}(\sin y + \cos y)$$

$$= -e^{-x}(\cos y - \sin y + \sin y + \cos y)$$

$$2\frac{\partial u}{\partial x} = -2e^{-x}\cos y$$

$$\therefore \quad \frac{\partial u}{\partial x} = -e^{-x}\cos y \quad \ldots.(4)$$

Subtract (3) from (2), we get

$$2\frac{\partial v}{\partial x} = -e^{-x}(\cos y - \sin y) + e^{-x}(\sin y + \cos y)$$

$$= -e^{-x}(\cos y - \sin y - \sin y - \cos y)$$

$$= 2e^{-x}\sin y$$

$$\therefore \quad \frac{\partial v}{\partial x} = e^{-x}\sin y \quad \ldots.(5)$$

We know, $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$= (-e^{-x}\cos y) + i(e^{-x}\sin y)$$

$$f'(z) = e^{-x}[-\cos y + i\sin y]$$

$$\therefore \quad f'(z) = e^{-z}[-1 - 0] \qquad (\because x = z, y = 0)$$

$$f'(z) = -e^{-z} = -e^{-z}$$

Integrate w.r.to z, $f(z) = e^{-z} + c$

$$\therefore \quad \mathbf{f(z) = e^{-z} + c}$$

Example : 2

Find an analytic function f(z) = u +iv

if $2u + v = e^{x}(\cos y - \sin y) \quad \ldots.(1)$

Solution : Differentiate (1) Partially w.r.to x (y constant)

$$2\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = e^{x}(\cos y - \sin y) \quad \ldots.(2)$$

Differentiate (1) partially w.r.to y (x constant)

$$2\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y} = e^{x}(-\sin y-\cos y)$$

$$-2\frac{\partial v}{\partial x}+\frac{\partial u}{\partial x} = e^{-x}(-\sin y-\cos y) \quad \ldots (3) \qquad (\because \text{C–R equations})$$

Multiply equation (2) by 2 & add with (3),

$$4\frac{\partial u}{\partial x}+2\frac{\partial v}{\partial x}-2\frac{\partial u}{\partial x}+\frac{\partial u}{\partial x} = 2(e^{x})(\cos y-\sin y)+e^{x}(-\sin y-\cos y)$$

$$5\frac{\partial u}{\partial x} = e^{x}[2\cos y-2\sin y-\sin y-\cos y]$$

$$5\frac{\partial u}{\partial x} = e^{x}[\cos y-3\sin y]$$

$$\therefore \frac{\partial u}{\partial x} = \frac{e^{x}}{5}[\cos y-3\sin y] \quad \ldots (4)$$

Multiply equation (3) by 2 and subtract from (2),

$$2\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}+4\frac{\partial v}{\partial x}-2\frac{\partial u}{\partial x} = e^{x}(\cos y-\sin y)-2e^{x}(-\sin y-\cos y)$$

$$5\frac{\partial v}{\partial x} = e^{x}[\cos y-\sin y+2\sin y+2\cos y]$$

$$5\frac{\partial v}{\partial x} = e^{x}[3\cos y+\sin y]$$

$$\therefore \frac{\partial v}{\partial x} = \frac{e^{x}}{5}[3\cos y+\sin y] \quad \ldots (5)$$

We know,

$$f'(z) = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$= \frac{e^{x}}{5}[\cos y-3\sin y]+i\frac{e^{x}}{5}[3\cos y+\sin y] \quad \because \text{from (4) \& (5)}$$

$$f'(z) = \frac{e^{x}}{5}[\cos y-3\sin y+3i\cos y+i\sin y]$$

$$= \frac{e^{z}}{5}[1-0+3i+0] \quad \because x=z,\ y=0$$

$$f'(z) = \frac{e^{z}}{5}(1+3i)$$

Integrate w.r.to z,

$$\mathbf{f(z) = \frac{1+3i}{5}\,e^{z}+c}$$

Type : D

Example : 1

Determine the value of p such that

$$f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{px}{y}\right) \quad \text{.... (1)}$$

becomes an analytic function.

Solution :

We know $\quad f(z) = u + iv$

$$\therefore \quad u = \frac{1}{2}\log(x^2+y^2)$$

and $\quad v = i\tan^{-1}\left(\frac{px}{y}\right)$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{1}{2}\left(\frac{1}{x^2+y^2}\right)2x$$

$$= \frac{x}{(x^2+y^2)} \quad \text{.... (2)}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}\left(\frac{1}{x^2+y^2}\right)2y$$

$$= \frac{y}{(x^2+y^2)} \quad \text{.... (3)}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+\left(\frac{px}{y}\right)^2}\left(\frac{p}{y}\right) = \frac{y^2}{(y^2+p^2x^2)}\left(\frac{p}{y}\right)$$

$$= \frac{py}{y^2+p^2x^2} \quad \text{.... (4)}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+\left(\frac{px}{y}\right)^2}\left(-\frac{px}{y^2}\right) = \frac{y^2}{(y^2+p^2x^2)}\left(-\frac{px}{y^2}\right)$$

$$= -\frac{px}{(y^2+p^2x^2)} \quad \text{... (5)}$$

We know , f (z) = u + iv , will be analytic if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$$\therefore \quad \frac{x}{y^2+x^2} = -\frac{-px}{y^2+p^2x^2}$$

Comparing, we get , $-p = 1$, $p^2 = 1$

$$\therefore \quad \therefore \quad p = -1,\ p^2 = 1$$

$$\therefore \quad \therefore \quad p = -1,\ p = \pm 1$$

$\therefore \quad \boxed{p = -1}$

Secondly we know $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

$$\therefore \frac{y}{x^2+y^2} = -\left(\frac{py}{y^2+p^2x^2}\right) = -\frac{\mathbf{-py}}{\mathbf{y^2+p^2x^2}}$$

$\therefore \quad p = -1, p^2 = 1 \Rightarrow p = -1, p = \pm 1$

$\therefore \quad \boxed{p = -1}$

Example : 2

Determine value of p for which the function

$f(z) = r^n \cos n\theta + i\, r^n \sin p\theta$ becomes an analytic function …(1)

Solution :

We know, $f(z) = u + iv$

$\therefore \quad u = r^n \cos n\theta, \; v = r^n \sin p\theta$

$$\therefore \quad \frac{\partial u}{\partial r} = nr^{n-1} \cos n\theta$$

$$\frac{\partial u}{\partial \theta} = -\; nr^n \sin n\theta$$

$$\frac{\partial v}{\partial r} = nr^{n-1} \sin p\theta$$

$$\frac{\partial v}{\partial \theta} = pr^n \cos p\theta$$

We know, f (z) will be analytic function if

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$nr^{n-1} \cos n\theta = \frac{1}{r}(\, pr^n \cos p\theta)$$

$$\therefore \quad nr^{n-1} \cos n\theta = pr^{n-1} \cos p\theta$$

$\therefore \quad \boxed{p = n}$

Also $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$

$$\therefore \quad nr^{n-1} \sin p\theta = -\frac{1}{r}\;(-\,nr^{n-1} \sin n\theta)$$

$$\mathbf{nr^{n-1} \sin p\theta = nr^{n-1} \sin n\,\theta}$$

$\therefore \quad \boxed{n = p}$

$\therefore \quad \boxed{p = n}$

Type : E

Example : 1

Determine the analytic function f(z) such that

$$R(f'(z)) = 3x^2 - 4y - 3y^2$$

Solution : We know , if f(z) is an analytic function then f '(z) is also an analytic function

Let $f'(z) = u + iv$

$$u = 3x^2 - 4y - 3y^2 \quad \text{.... (1)}$$

$$\frac{\partial u}{\partial x} = 6x \quad \text{.... (2)}$$

$$\frac{\partial u}{\partial y} = -4 - 6y \quad \text{.... (3)}$$

But, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 6x,$ (By C – R equations)

$$\therefore \quad \frac{\partial v}{\partial y} = 6x$$

Integrate w.r.to y partially (x constant)

$$\therefore \quad v = 6xy + A(x) \quad \text{.... (4)}$$

Differentiate w.r.to x partially (y constant)

$$\therefore \quad \frac{\partial v}{\partial x} = 6y + A'(x)$$

$$-\frac{\partial u}{\partial y} = 6y + A'(x) \quad (\because \text{ C – R equations})$$

$$\frac{\partial u}{\partial y} = -6y - A'(x) \quad \text{.... (5)}$$

From (3) & (5) , $-4 - 6y = -6y - A'(x)$

$$\therefore \quad -4 = -A'(x)$$

$$\therefore \quad A'(x) = 4$$

$$\therefore \quad A(x) = 4x + c$$

From (4), $v = 6xy + 4x + c$

$$\therefore \quad f'(z) = u + iv = (3x^2 - 4y - 3y^2) + i\,(6xy + 4x + c)$$

$$\therefore \quad f'(z) = 3z^2 + 4\,iz + ic \quad (\because x = z, y = 0)$$

Integrate w.r. to z, we get

$$f(z) = 3\left(\frac{z^3}{3}\right) + 4i\left(\frac{z^2}{2}\right) + icz + c_1$$

$$\mathbf{f(z) = z^3 + 2iz^2 + c_1 z + c_2}$$

Example : 2

Prove that the analytic function with constant modulus is constant

Solution :

Let $f(z) = u + iv$ .... (1)

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{.... (2)}$$

Given that, f (z) has constant modulus

$$\therefore \quad |f(z)| = C$$

$$\therefore \quad |u + iv| = C$$

$$\sqrt{u^2 + v^2} = C$$

$$u^2 + v^2 = C^2 \qquad \text{.... (3)}$$

Differentiate (3) w.r.to x partially, we get

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

$$\therefore \quad u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \qquad \text{.... (4)}$$

Differentiate (3) w.r.to y partially we get

$$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0 \qquad \text{.... (5)}$$

$$-u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = 0 \qquad \text{.... (6)} \ (\because (2))$$

Multiply (4) by u and (6) by v and add

$$u^2\frac{\partial u}{\partial x} + uv\frac{\partial v}{\partial x} - uv\frac{\partial v}{\partial x} + v^2\frac{\partial u}{\partial x} = 0$$

$$\therefore \quad (u^2 + v^2)\frac{\partial u}{\partial x} = 0$$

$$\therefore \quad \frac{\partial u}{\partial x} = 0 \qquad \because u^2 + v^2 \neq 0$$

Similarly we can show $\frac{\partial v}{\partial y} = 0$

We know , $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ $=$ $\mathbf{0 + i0 = 0}$

$\therefore$ f (z) is a constant function

Example : 3

Find the condition, under which the function

$u = px^3 + q\,x^2 y + r\,xy^2 + sy^3$ becomes a harmonic function

Solution :

$$\therefore \quad \frac{\partial u}{\partial x} = 3px^2 + 2qxy + ry^2 + 0$$

$$\frac{\partial^2 u}{\partial x^2} = 6px + 2qy + 0$$

$$\frac{\partial u}{\partial y} = 0 + qx^2 + 2rxy + 3sy^2$$

$$\frac{\partial^2 u}{\partial y^2} = 0 + 2rx + 6sy$$

We know that u will be harmonic if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\therefore \quad (6px + 2\,qy) + (2rx + 6sy) = 0$$

$$(6px + 2\,rx) + (2qy + 6sy) = 0$$

$$(3p + r)\,2x + (q + 3s)\;2y = 0$$

$$\therefore \quad \mathbf{3p + r = 0, \quad q + 3s = 0}$$

$$\therefore \quad \mathbf{r = -3p, \quad q = -3s}$$

Example : 4

If f(z) and $f(\overline{z})$ are analytic functions, then prove that f(z) is constant

Solution : Let

$$f(z) = u + iv \qquad \text{.... (1)}$$

$$\therefore \quad f(\overline{z}) = u - iv \qquad \text{.... (2)}$$

As f(z) is an analytic function

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{.... (3)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{.... (4)}$$

As $f(\overline{z})$ is an analytic function

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \qquad \text{.... (5)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \qquad \text{.... (6)}$$

Adding (3) & (5) , we get

$$2\frac{\partial u}{\partial x} = 0$$

$$\therefore \quad \frac{\partial u}{\partial x} = 0$$

Adding (4) & (6) , we get

$$2\frac{\partial u}{\partial y} = 0$$

$$\therefore \quad \frac{\partial u}{\partial y} = 0$$

But $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$f'(z) = 0 - i\,0 = 0$$

$$\mathbf{f'(z) = 0}$$

$\therefore$ f(z) is a constant function.

Example : 5

If f(z) is an analytic function of z, and f(z) = u +iv prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |\,[\,R\ f(z)\,]\,|^2 = 2\,|f'(z)|^2$$

Solution : $$f(z) = u + iv \qquad \text{.... (1)}$$

$$\therefore \quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad \text{.... (2)}$$

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \qquad \text{.... (3)}$$

As f(z) is an analytic function, u & v must be harmonic functions

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \qquad \text{.... (4)}$$

$$\therefore \quad \text{L.H.S.} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) (\,|R\ f(z)\,|^2\,)$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u^2 = \frac{\partial^2}{\partial x^2}(\,u^2\,) + \frac{\partial^2}{\partial y^2}(u^2)$$

$$= \frac{\partial}{\partial x}\left(2u\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(2u\frac{\partial u}{\partial y}\right)$$

$$= 2u\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}(2)\frac{\partial u}{\partial x} + 2u\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}(2)\frac{\partial u}{\partial y}$$

$$= 2u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial u}{\partial y}\right)^2$$

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] \qquad (\because \text{ equation (4)})$$

$$\mathbf{= 2\,|f'(z)|^2 = R.H.S.} \qquad (\because \textbf{ equation (3)})$$

Example : 6

If f(z) is an analytic function show that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\ |f'(z)|^2$$

where f (z) = u+ iv

Solution :

We know

$$|f(z)| = \sqrt{u^2+v^2}$$

$$|f(z)|^2 = u^2+v^2 \qquad \ldots.(1)$$

$$f'(z) = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}$$

$$|f'(z)| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2} = \sqrt{\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2}$$

$$\therefore \quad |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2 \qquad \ldots.(2)$$

$$= \left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2 \qquad \ldots.(3)$$

As u, v are harmonic functions, we know

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 \qquad \ldots.(4)$$

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2} = 0 \qquad \ldots.(5)$$

$$\therefore \quad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f_{(z)}|^2 = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^2+v^2)$$

$$= \frac{\partial^2}{\partial x^2}(u^2+v^2)+\frac{\partial^2}{\partial y^2}(u^2+v^2)$$

$$= \frac{\partial}{\partial x}\left[2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}\right]+\frac{\partial}{\partial y}\left[2u\frac{\partial u}{\partial y}+2v\frac{\partial v}{\partial y}\right]$$

$$= 2u\frac{\partial^2 u}{\partial x^2}+2\left(\frac{\partial u}{\partial x}\right)^2+2v\frac{\partial^2 v}{\partial x^2}+2\left(\frac{\partial v}{\partial x}\right)^2+2u\frac{\partial^2 u}{\partial y^2}+2\left(\frac{\partial u}{\partial y}\right)^2+2v\frac{\partial^2 v}{\partial y^2}+2\left(\frac{\partial v}{\partial y}\right)^2$$

$$= 2u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial^2 y}\right)+2v\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial^2 y}\right)+2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial u}{\partial y}\right)^2+2\left(\frac{\partial v}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2$$

$$= 2u(0)+2v(0)+2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right]+2\left[\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]$$

$$= 2\,[\,|f'(z)|^2+|f'(z)|^2\,] \qquad \because \text{Refer (4), (5)}$$

$$= 4\,|f'(z)|^2 \qquad \because \text{Refer (2), (3)}$$

= **R.H.S.**

Self–Assessment Exercise 6.1

Ex.1 Determine the harmonic conjugates of following functions u or v such that f(z) = u +iv becomes an analytic function.

1. $u = x^2 - y^2 - y$ **Ans.:** $v = 2xy + x + c,\ f(z) = z^2 + iz + c$
2. $v = y^2 - x^2$ **Ans. :** $u = 2xy + c,\ f(z) = -iz^2 + c$
3. $u = 3xy^2 - x^3$ **Ans. :** $v = y^3 - 3x^2y + c,\ f(z) = -z^3 + ic$
4. $v = x^2 - y^2 + \frac{x}{x^2 + y^2}$ **Ans. :** $u = -2xy + \frac{y}{x^2+y^2} + c,\ f(z) = iz^2 + \frac{i}{z} + c$
5. $u = e^{-x}(x \sin y - y \cos y)$ **Ans. :** $v = e^{-x}(y \sin y + x \cos y) + c,\ f(z) = iz\, e^{-z}$
6. $u = \log\sqrt{x^2+y^2}$ **Ans. :** $v = \tan^{-1}\left(\frac{y}{x}\right) + c,\ f(z) = \log z + c$
7. $u = e^{-2xy} \sin(x^2 - y^2)$ **Ans. :** $v = -e^{-2xy}\cos(x^2 - y^2) + c,\ f(z) = -i\, e^{iz^2} + ci$
8. $u = x^3y - xy^3$ **Ans. :** $v = x^4 + y^4 - 6x^2y^2 + c\ ,\ f(z) = z^4 + e$
9. $u = \frac{y}{y^2+x^2}$ **Ans. :** $f(z) = \frac{i}{z}$
10. $u = e^x[(x^2-y^2) \cos y - 2xy \sin y]$ **Ans. :** $f(z) = z^2e^z + ic$
11. $u = x \sin x \cosh y - y \cos x \sinh y$ **Ans. :** $f(z) = z \sin z + c$
12. $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ **Ans.:** $f(z) = z^3 + 3z^2 + 1 + ic$
13. $u = e^{2x}(x \cos 2y - y \sin 2y)$ **Ans. :** $f(z) = ze^{2z} + ic$
14. $u = \cos x \cosh y$ **Ans. :** $f(z) = \cos z + c$
15. $u = y + e^x \cos y$ **Ans. :** $f(z) = e^z + ic - iz$
16. $v = e^x \sin y$ **Ans. :** $f(z) = e^z + c$
17. $v = e^{-x}(x \cos y + y \sin y)$ **Ans. :** $f(z) = 1 + ize^{-z}$
18. $v = -\sin x \sinh y$ **Ans. :** $f(z) = \cos z + c$
19. $v = \frac{x - y}{x^2 + y^2}$ **Ans. :** $f(z) = \frac{1}{z} + \frac{i}{z} + ic$
20. $v = e^{-x}(x \sin y - y \cos y)$ **Ans. :** $f(z) = \bar{z}\ e^{-\bar{z}} + c$
21. $u = r^2 \cos 2\theta - r \cos\theta + 2$ **Ans. :** $u = -r^2\sin 2\theta + r\sin\theta + c$
22. $u = e^x(x\cos y - y\sin y) + 2\sin x \sinh y + x^3 - 3xy^2 + y$ **Ans. :** $f(z) = ze^z + z^3 + 2i\cos z - iz + e$
23. $u = -r^3\sin 3\theta$ **Ans. :** $v = r^3\cos 3\theta + c$
24. $u = r^2 \cos 2\theta$ **Ans. :** $v = r^2\sin 2\theta + c$
25. $u = r^3\cos 3\theta + r\sin\theta$ **Ans. :** $v = r^3\sin 3\theta - r\cos\theta + c$
26. $u = r^3\cos 3\theta$ **Ans. :** $v = r^3\sin 3\theta + c$

27. $v = r\sin\theta - \frac{1}{r}\sin\theta$ **Ans. :** $f(z) = z + \frac{1}{z} = \frac{z^2+1}{z}$

28. $v = r^n \sin n\theta$ **Ans. :** $f(z) = r^n\cos n\theta = z^n$

29. $u = x^2 - y^2 - 2xy - 2x + 3y$ **Ans. :** $f(z) = z^2 + iz^2 - 2z - 3iz + 6i + c$

30. $u = r\cos\theta + \frac{1}{r}\cos\theta$ **Ans. :** $f(z) = r\sin\theta - \frac{1}{r}\sin\theta + c$

31. $u = e^{-x}\cos y + xy$ **Ans. :** $v = y^2 - x^2 - 2e^{-x}\sin y + c$

Ex.2 Find the analytic function f(z) = u+iv if

32. $u - v = \frac{x-y}{x^2+4xy+y^2}$ **Ans. :** $f(z) = c - iz^3$

33. $2u + v = e^x(\cos y - \sin y)$ **Ans. :** $f(z) = \left(\frac{1+3i}{5}\right)e^z + c$

34. $u - v = \frac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$ **Ans. :** $f(z) = -\frac{1}{2}\cot\left(\frac{z}{2}\right) + c$

35. $u + v = \frac{2\sin x}{e^{2y} - e^{-2y} - 2\cos 2x}$

36. $u - v = x^3 + 3x^2y - 3xy^2 - y^3$ **Ans. :** $f(z) = -iz^3 + c$

37. $u + v = r^2(\cos 2\theta + \sin 2\theta)$ **Ans. :** $f(z) = \frac{z^2}{2} + c$

38. $u - v = e^x[(x-y)\cos y - (x+y)\sin y]$ **Ans. :** $f(z) = z\,e^z + 1$

39. $u + v = e^x[\cos y + \sin y] + \frac{x-y}{x^2+y^2}$ **Ans. :** $f(z) = e^z + \frac{1}{z} + c$

40. $u + v = \sin x\cosh y + \cos x\sinh y$ **Ans. :** $f(z) = \sin z + c$

41. $u + v = 3x + 3y + \frac{x-y}{x^2+y^2}$ **Ans. :** $f(z) = 6z + \frac{1}{z} + c$

Ex.3 Find the values of a and b so that f(z)=u +iv is an analytic function

42. $f(z) = (x + ay) + i(bx + cy)$ **Ans. :** $a = -b,\ c = 1$

43. $f(z) = \cos x(\cosh y + a\sinh y) + i\sin x(\cosh y + b\sinh y)$ **Ans. :** $a = b = -1$

44. $f(z) = (r^2\cos 2\theta) + i(r^2\sin a\theta)$ **Ans. :** $a = 2$

45. $f(z) = (x^2 + axy + by^2) + i(px^2 + qxy + y^2)$ **Ans. :** $a = 2,\ b = -1,\ p = -1,\ q = 2$

46. $f(z) = \log\sqrt{x^2+y^2} + i\tan^{-1}\left(\frac{ay}{x}\right)$ **Ans. :** $a = 1$

Ex.4 Various examples (miscellaneous examples)

47. Determine the analytic function f(z)= u + iv
if Im $[f'(z)] = 12xy - 6y$

48. Show that the analytic function with constant amplitude is constant

49. Find the orthogonal trajectories of the family of curves $x^3y - yx^3 = c_1$

Ans.: $x^4 + y^4 - 6x^2y^2 = c_2$

50. Determine the constant 'a' such that

    $u = e^{ax} \cos 5y$ becomes harmonic function. Hence find harmonic conjugate of u

    Ans. : $a = 5$, $v = e^{5x} \sin 5y + c$

51. Find the value of 'a' for which

    $f(z) = (e^x \cos y) + i (e^x \sin a\, y)$ is an analytic function

52. Find the orthogonal trajectories of the family of curves $r^2 \cos 2\theta = c_1$

    Ans. : $r^2 \sin 2\theta = c_2$

6.5 Argands Diagram :

The geometrical representation of a complex number (x + iy) by a point (x, y) on XOY plane is called as Argands diagram.

Ex. : 1. The complex number (2 + 3i) is represented by (2, 3) in first quadrant.

2. The complex number $4 + 5i \equiv (4, 5)$ is in first quadrant

3. The complex number $-1 + i \equiv (-1, 1)$ is in second quadrant

6.6 Transformation :

We know w = f(z) = u + iv is called as a complex variable function

$$\therefore \quad f(z) = u + iv$$

$$\therefore \quad f(x + iy) = u + iv$$

This equation involves 4 variables x, y, u, v and hence f(x + iy) = u + iv cannot be plotted on a single frame. Hence the points (x, y) are plotted on XOY plane and points (u, v) are plotted on another UOV plane with the help of w = f (z). This process is called as a transformation of points (x, y) on XOY plane to points (u, v) on UOV plane.

Example :

Let $w = z^3$ and $z = x + iy$

$$\therefore \quad w = (x + iy)^3$$

$$= x^3 + 3x^2 (iy) + 3x (iy)^2 + (iy)^3$$

$$w = (x^3 - 3xy^2) + i (3x^2y - y^3)$$

$$\therefore \quad u = x^3 - 3xy^2$$

$$v = 3x^2y - y^3.$$

If x = 2, y = 3 then $z = x + iy = 2 + 3i \equiv (2, 3)$ lies in first quadrant of xy plane XOY or Z plane.

and $f(z) = w = u + iv$

$$= [8 - 3(2)(9)] + i [3(4)(3) - 27]$$

$$= [-46] + i [9] \equiv (-46, 9)$$ lies in second quadrant of UOV Plane or W plane

∴ The point (2, 3) on XOY plane is transformed to the point (–46, 9) on UOV/W plane by transformation $w = z^3$.

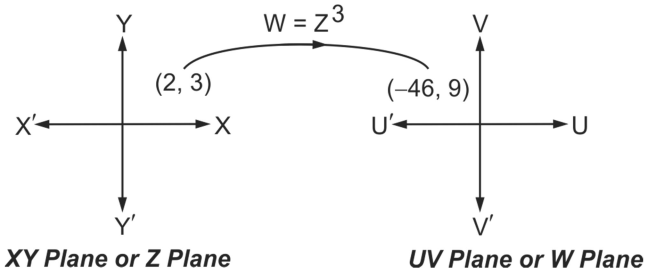

Fig.: 6.1

6.6.1 Conformal Transformation :

Let c_1, c_2 be any two curves in XY Plane, which cuts each other at point P and makes an angle of intersection as α.

Let the transformation $w = f(z)$ transforms the curves c_1, c_2 in XY plane and suppose it takes new shapes as curves c_1', c_2' in UV Plane and let the angle between c_1' and c_2' be β.

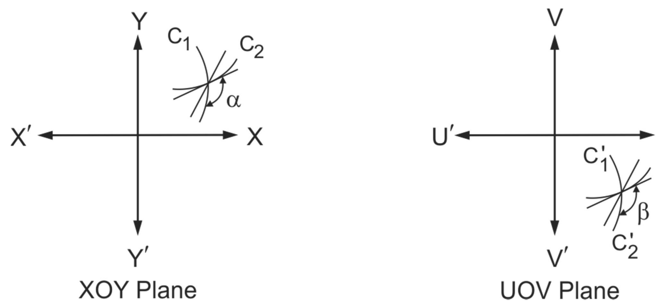

Fig.: 6.2

If $\alpha = \beta$, then the transformation $w = f(z)$ is called as a conformal transformation / mapping.

6.6.2 Some Standard Transformations :

1. Translation : $w = z + c$, where c is complex constant.
2. Magnification : $w = cz$, where c is real constant.
3. Inversion and Reflection : $w = \frac{1}{z}$.
4. Isogonal Transformation : $w = \bar{z}$

6.7 Bilinear Transformation /Mobius Transformation :

(i) The transformation $w = \frac{az + b}{cz + d}$, where a, b, c, d are complex/real constants, is called as a bilinear Transformation.

(ii) The bilinear transformation can also be found by using the formula.

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

(iii) The mapping $z = \frac{-dw + b}{cw - a}$ is inverse mapping of $w = \frac{az + b}{cz + d}$ and also called as bilinear transformation.

Illustrative Examples

Example : 1

Find a bilinear transformation w = f (z) which maps the points z = ∞, 1, i on Z plane to the points w = 0, 1 , – i on W plane respectively.

Solution :

Let the required bilinear transformation is,

$$w = \frac{az + b}{cz + d}$$

$$w = \frac{\left(\frac{a}{b}\right) z + 1}{\left(\frac{c}{b}\right) z + \left(\frac{d}{b}\right)} \qquad \text{…(1)}$$

$$w = \frac{\left(\frac{a}{b}\right) + \frac{1}{z}}{\left(\frac{c}{b}\right) + \left(\frac{d}{b}\right)\frac{1}{z}} \qquad \text{…(2)}$$

Put z = ∞, w = 0 in (2), we get,

$$0 = \frac{\left(\frac{a}{b}\right) + 0}{\left(\frac{c}{b}\right) + 0}$$

$$\therefore \quad \frac{a}{b} = 0 \qquad \text{…(3)}$$

Put z = 1, w = 1 in (1), we get,

$$1 = \frac{\left(\frac{a}{b}\right) + 1}{\left(\frac{c}{b}\right) + \left(\frac{d}{b}\right)} = \frac{0 + 1}{\left(\frac{c}{b}\right) + \left(\frac{d}{b}\right)} \qquad \because \frac{a}{b} = 0$$

$$= \frac{1}{\left(\frac{c}{b}\right) + \left(\frac{d}{b}\right)}$$

$$\therefore \qquad \frac{c}{b} + \frac{d}{b} = 1 \qquad \ldots(4)$$

Lastly put z = i, w = – i in (1), we get

$$-i = \frac{\left(\frac{a}{b}\right)(i) + 1}{\left(\frac{c}{b}\right) i + \left(\frac{d}{b}\right)}$$

$$\left(\frac{c}{b}\right) - i\left(\frac{d}{b}\right) = 1 \qquad \because \frac{a}{b} = 0 \qquad \ldots(5)$$

Solving (4), (5), we get $\left(\frac{d}{b}\right) = 0, \frac{c}{b} = 1$

$\therefore$ Required bilinear transformation is

$$w = \frac{0+1}{(1)\,z + 0} = \frac{1}{z}$$

$$\therefore \qquad \boxed{\mathbf{w = \frac{1}{z}}}$$

Example : 2

Find the bilinear transformation w = f(z) which maps the points z = – i, 0, 2 + i on Z – plane to the points w = 0, – 2i, 4 on W – plane respectively.

Solution : Let the required bilinear transformation be

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)\,(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)\,(z_2 - z_1)}$$

Given : $w_1 = 0,\ w_2 = -2i,\ w_3 = 4$

$z_1 = -i,\ z_2 = 0,\ z_3 = 2 + i$

$$\therefore \qquad \frac{(w-0)(-2i-4)}{(w-4)\,(-2i-0)} = \frac{(z+i)(0-2-i)}{(z-2-i)\,(0+i)}$$

$$\frac{-2w\,(i+2)}{-2i\,(w-4)} = \frac{-(z+i)\,(2+i)}{i(z-2-i)}$$

$$\frac{w}{w-4} = \frac{-(z+i)}{(z-2-i)}$$

$$wz - 2w - wi = -[wz + wi - 4z - 4i]$$

$$wz - 2w - wi = -wz - wi + 4z + 4i$$

$$wz - 2w + wz = 4z + 4i$$

$$2wz - 2w = 4\,(z+i)$$

$$2w\,(z-1) = 4\,(z+i)$$

$$\boxed{\mathbf{w = \frac{2\,(z+i)}{(z-1)}}}$$

Example : 3

Find the bilinear transformation which maps z = 1, i, –1 on Z plane into the points w = i, o, – i on W plane respectively.

Solution : Let, $z_1 = 1, z_2 = i, z_3 = -1$

$w_1 = i, w_2 = 0, w_3 = -i$

We know, the bilinear transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\therefore \quad \frac{(w-i)(0+i)}{(w+i)(0-i)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\therefore \quad \frac{w-i}{w+i} = -\frac{(z-1)}{(z+1)}\cdot\frac{(i+1)}{(i-1)}\frac{(i+1)}{(i+1)}$$

$$= -\left[\frac{(z-1)}{(z+1)}\cdot\frac{(1+2i+i^2)}{(i^2-1)}\right]$$

$$= -\left[\frac{(z-1)}{(z+1)}\cdot\frac{2i}{(-1-1)}\right] = \frac{i(z-1)}{(z+1)}$$

$$\therefore \quad \frac{w-i}{w+i} = \frac{i(z-1)}{(z+1)}$$

By components and Dividends rule $\frac{N+D}{N-D}$

$$\frac{w-i+w+i}{w-i-w-i} = \frac{iz-i+z+1}{iz-i-z-1}$$

$$\frac{2w}{-2i} = \frac{iz-i+z+1}{iz-i-z-1}$$

$$w = \frac{i^2z+1+iz+i}{-iz+i+z+1}$$

$$\therefore \quad \mathbf{w} = \frac{\mathbf{iz+i+i^2z+1}}{\mathbf{-iz+i+z+1}} = \frac{\mathbf{1+iz}}{\mathbf{1-iz}}$$

Example : 4

Find a bilinear transformation w = f (z) which maps z = 1, i, – 1 on z–plane to the points w = 0, 1, ∞ on W plane respectively.

Solution : Let the required bilinear transformation is

$$w = \frac{az+b}{cz+d} \qquad \ldots(1)$$

$$w = \frac{\left(\frac{a}{b}\right)z+1}{\left(\frac{c}{b}\right)z+\left(\frac{d}{b}\right)} \qquad \ldots(2)$$

We have, $w = \infty, z = -1$

$$\therefore \qquad \infty = \frac{\left(\frac{a}{b}\right)(-1)+1}{\left(\frac{c}{b}\right)(-1)+\left(\frac{d}{b}\right)}$$

$$\therefore \qquad \frac{1}{\infty} = \frac{-\left(\frac{c}{b}\right)+\left(\frac{d}{b}\right)}{\left(-\frac{a}{b}\right)+1}$$

$$0 = \frac{-\left(\frac{c}{d}\right)+\left(\frac{d}{b}\right)}{\left(-\frac{a}{b}\right)+1}$$

$$\therefore \qquad \frac{c}{b} = \frac{d}{b} \qquad \ldots(3)$$

Put $z = 1, w = 0$ in (2)
$$0 = \frac{\left(\frac{a}{b}\right)(1)+1}{\left(\frac{c}{b}\right)(1)+\left(\frac{d}{b}\right)}$$

$$\therefore \qquad \frac{a}{b} = -1 \qquad \ldots(4)$$

Lastly put $z = i, w = 1$ in (2)

$$1 = \frac{\left(\frac{a}{b}\right)i+1}{\left(\frac{c}{b}\right)i+\left(\frac{d}{b}\right)} = \frac{-i+1}{\left(\frac{c}{b}\right)i+\left(\frac{c}{b}\right)} \qquad \because \frac{c}{b} = \frac{d}{b}$$

$$= \frac{1-i}{\left(\frac{c}{b}\right)(1+i)}$$

$$\therefore \qquad \frac{c}{b} = \frac{1-i}{1+i} = \frac{(1-i)(1-i)}{(1+i)(1-i)}$$

$$= \frac{1-2i+i^2}{1-i^2} = \frac{-2i}{2} = -i$$

$$\therefore \qquad \frac{c}{b} = -i = \frac{d}{b}$$

$\therefore$ Required Bilinear transformation is

$$w = \frac{-z+1}{-iz-i}$$

$$= \frac{1-z}{-i(1+z)} = \mathbf{i\left(\frac{1-z}{1+z}\right)}$$

Example : 5

Find a bilinear transformation w = f(z) which transforms the points z = ∞, i, 0 on z plane to the points w = –1, – i, 1 on w plane respectively.

Solution :

Required Bilinear Transformation is

$$w = \frac{az+b}{cz+d} \qquad \ldots(1)$$

$$w = \frac{\left(\frac{a}{b}\right)z+1}{\left(\frac{c}{b}\right)z+\left(\frac{d}{b}\right)} \qquad \ldots(2)$$

$$w = \frac{\left(\frac{a}{b}\right)+\frac{1}{z}}{\left(\frac{c}{b}\right)+\left(\frac{d}{b}\right)\frac{1}{z}} \qquad \ldots(3)$$

Put z = ∞, w = –1 in (3),

$$-1 = \frac{\left(\frac{a}{b}\right)+0}{\left(\frac{c}{b}\right)+0}$$

$$\left(\frac{a}{b}\right) = -\left(\frac{c}{b}\right) \qquad \ldots(4)$$

Put z = 0, w = 1 in (2),

$$1 = \frac{0+1}{0+\left(\frac{d}{b}\right)}$$

$$\therefore \quad \frac{d}{b} = 1 \qquad \ldots(5)$$

Lastly put z = i, w = – i in (2)

$$-i = \frac{\left(\frac{a}{b}\right)i+1}{\left(\frac{c}{b}\right)i+\left(\frac{d}{b}\right)}$$

$$-i = \frac{\left(\frac{a}{b}\right)i+1}{-\left(\frac{a}{b}\right)i+1} \qquad \because \frac{d}{b} = 1, \ \frac{c}{b} = -\frac{a}{b}$$

$$\left(\frac{a}{b}\right)i^2 - i = \left(\frac{a}{b}\right)i+1$$

$$-\frac{a}{b}-\frac{a}{b}i = i+1$$

$$\therefore \quad \left(\frac{a}{b}\right) = -1 \qquad \ldots(6)$$

$\therefore$ Required bilinear transformation is,

$$w = \frac{-z+1}{z+1} = \mathbf{\frac{1-z}{1+z}}$$

Example : 6

Find a bilinear transformation w = f(z) which maps the points z = 0, – 1, ∞ on z plane to the points w = – 1, – (2 + i), i on w plane respectively.

Solution :

Required Bilinear Transformation is,

$$w = \frac{az+b}{cz+d} \qquad \ldots(1)$$

$$w = \frac{\left(\frac{a}{b}\right)z+1}{\left(\frac{c}{b}\right)z+\left(\frac{d}{b}\right)} \qquad \ldots(2)$$

$$w = \frac{\left(\frac{a}{b}\right)+\frac{1}{z}}{\left(\frac{c}{b}\right)+\left(\frac{d}{b}\right)\frac{1}{z}} \qquad \ldots(3)$$

Put z = ∞, w = i in (3),
$$i = \frac{\left(\frac{a}{b}\right)+0}{\left(\frac{c}{b}\right)+0}$$

$$\therefore \quad \frac{a}{b} = i\left(\frac{c}{b}\right) \qquad \ldots(4)$$

Put z = 0, w = – 1 in (2),

$$-1 = \frac{0+1}{0+\left(\frac{d}{b}\right)}$$

$$\therefore \quad \frac{d}{b} = -1 \qquad \ldots(5)$$

Lastly put, z = – 1, w = – (2 + i) in (2),

$$-(2+i) = \frac{\left(\frac{a}{b}\right)(-1)+1}{\left(\frac{c}{b}\right)(-1)+\left(\frac{d}{b}\right)}$$

$$-(2+i) = \frac{i\left(\frac{c}{b}\right)(-1)+1}{-\left(\frac{c}{b}\right)-1} \qquad \because \frac{a}{b} = i\left(\frac{c}{b}\right) \ \& \ \frac{d}{b} = -1$$

$$-(2+i) = \frac{i\left(\frac{c}{b}\right)-1}{\left(\frac{c}{b}\right)+1}$$

$$-(2+i)\left(\frac{c}{b}+1\right) = i\left(\frac{c}{b}\right)-1$$

$$-(2+i)\left(\frac{c}{b}\right)-i\left(\frac{c}{b}\right) = -1+2+i$$

$$-\frac{c}{b}(2+i+i) = (1+i)$$

$$-\frac{c}{b}(2+2i) = (1+i)$$

$$-2\frac{c}{b}(1+i) = (1+i)$$

$$\therefore \quad \left(\frac{c}{b}\right) = -\frac{1}{2} \qquad \ldots(6)$$

$$\frac{a}{b} = i\left(-\frac{1}{2}\right) \qquad \ldots(7)$$

$\therefore$ Required Bilinear Transformation is,

$$w = \frac{\left(\frac{a}{b}\right)z+1}{\left(\frac{c}{b}\right)z+\left(\frac{d}{b}\right)} = \frac{zi\left(-\frac{1}{2}\right)+1}{-\left(\frac{1}{2}\right)z+(-1)} = \frac{iz-2}{-z-2}$$

$$\mathbf{w} = \left(\frac{\mathbf{iz-2}}{\mathbf{z+2}}\right)$$

Example : 7

Find a bilinear transformation w = f(z) which maps points z = – 1, i , 1 on z plane to the points w = 1, i , – 1 on w plane respectively.

Solution :

Let, $z_1 = -1, z_2 = i, z_3 = 1$

$w_1 = 1, w_2 = i, z_3 = -1$

We know bilinear transformation is,

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_3)(z_2-z_1)}$$

$$\therefore \quad \frac{(w-1)(i+1)}{(w+1)(i-1)} = \frac{(z+1)(i-1)}{(z-1)(i+1)}$$

$$\therefore \quad \frac{w-1}{w+1} = \frac{(z+1)}{(z-1)} \cdot \frac{(i-1)^2}{(i+1)^2}$$

$$= \left(\frac{z+1}{z-1}\right) \cdot \left(\frac{i^2-2i+1}{i^2+2i+1}\right)$$

$$= \left(\frac{z+1}{z-1}\right)(-1) \quad = -\left(\frac{z+1}{z-1}\right)$$

$$\therefore \quad (w-1)(z-1) = -(z+1)(w+1) = -[zw+z+w+1]$$

$$wz - w - z + 1 = -zw - z - w - 1$$

$$2zw = -2$$

$$\boxed{\mathbf{w = -\frac{1}{z}}}$$

Example : 8

Find a bilinear transformation which maps the points z = 0, 1, ∞ on z plane to the points w = –1, –i, 1 on w plane respectively.

Solution : We know, the B.T. is

$$w = \frac{az+b}{cz+d} \qquad \ldots(1)$$

$$w = \frac{\left(\frac{a}{b}\right)z+1}{\left(\frac{c}{b}\right)z+\left(\frac{d}{b}\right)} \qquad \ldots(2)$$

$$w = \frac{\left(\frac{a}{b}\right)+\frac{1}{z}}{\left(\frac{c}{b}\right)+\left(\frac{d}{b}\right)\frac{1}{z}} \qquad \ldots(3)$$

Put z = ∞, w = 1 in (3)

$$1 = \frac{\left(\frac{a}{b}\right)+0}{\left(\frac{c}{b}\right)+0}$$

$$\therefore \quad \boxed{\left(\frac{a}{b}\right) = \left(\frac{c}{b}\right)} \qquad \ldots(4)$$

Put z = 0, w = – 1 in (2),

$$-1 = \frac{0+1}{0+\left(\frac{d}{b}\right)}$$

$$\therefore \qquad \boxed{\frac{d}{b} = -1} \qquad \ldots(5)$$

Put z = 1, w = – i in (2),

$$-i = \frac{\left(\frac{a}{b}\right) + 1}{\left(\frac{c}{b}\right)(1) + \left(\frac{d}{b}\right)}$$

$$-i = \frac{\left(\frac{a}{b}\right) + 1}{\left(\frac{a}{b}\right) - 1} \qquad \left\{\because \frac{a}{b} = \frac{c}{b} \text{ and } \frac{d}{b} = -1\right\}$$

$$-\left(\frac{a}{b}\right) i + i - \frac{a}{b} = 1$$

$$-\frac{a}{b}(i+1) = 1 - i$$

$$\therefore \quad -\frac{a}{b} = \frac{1-i}{1+i}$$

$$\frac{a}{b} = \frac{(i-1)}{(i+1)} \times \frac{(i-1)}{(i-1)}$$

$$= \frac{i^2 - 2i + 1}{i^2 - 1} = \frac{-2i}{-2} = i$$

$$\left(\frac{a}{b}\right) = i$$

∴ Required B.T. is

$$\mathbf{w = \frac{iz + 1}{iz - 1}}$$

Example : 9

Find a Bilinear Transformation w = f(z) which transforms the points z = – 2, 0, 2 on XY plane to the points w = 0, i, – i on UV plane respectively.

Solution :

Let, $z_1 = -2, z_2 = 0, z_3 = 2$

$w_1 = 0, w_2 = i, w_3 = -i$

Required B. T. is,

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

$$\therefore \quad \frac{(w - 0)(i + i)}{(w + i)(i - 0)} = \frac{(z + 2)(0 - 2)}{(z - 2)(0 + 2)}$$

$$\frac{w}{w + i}(2) = \frac{(z + 2)}{(z - 2)}(-1)$$

$$2w\,(z-2) = -(z+2)\,(w+i)$$
$$2wz - 4w = -zw - zi - 2w - 2i$$
$$2zw - 4w + zw + 2w = -zi - 2i$$
$$3zw - 2w = -zi - 2i$$
$$w\,(3z-2) = -i\,(z+2)$$
$$\therefore \quad \mathbf{w = \frac{i\,(z+2)}{(2-3z)}}$$

Example : 10

Show that the transformation W = coshz transforms the lines parallel to x–axis on XOY plane to the hyperbola in UOV plane.

Solution :

Given transformation is

$$W = \cosh z$$
$$u + iv = \cosh (x+iy) = \cos [i\,(x+iy)] \qquad \because \cos (ix) = \cosh x$$
$$= \cos (ix - y)$$
$$= \cos (ix) \cos y + \sin (ix) \sin y$$
$$= \cosh x \cos y + i \sinh x \sin y$$
$$\therefore \quad u = \cosh x \cos y \qquad \ldots(1)$$
$$v = \sinh x \sin y \qquad \ldots(2)$$

Equation of a line parallel to x–axis is

$$y = b \qquad \ldots(3)$$

Equations (1) & (2) changes to

$$u = \cosh x \cos b$$
$$v = \sinh x \sin b$$
$$\therefore \quad \cosh x = \frac{u}{\cos b} \qquad \ldots(4)$$
$$\sinh x = \frac{v}{\sin b} \qquad \ldots(5)$$

Squaring and subtracting, we get

$$\cosh^2 x - \sinh^2 x = \frac{u^2}{\cos^2 b} - \frac{v^2}{\sin^2 b}$$
$$\therefore \quad 1 = \frac{u^2}{\cos^2 b} - \frac{v^2}{\sin^2 b}$$
$$\therefore \quad \mathbf{\frac{u^2}{\cos^2 b} - \frac{v^2}{\sin^2 b} = 1}$$

which is an equation of hyperbola in UOV plane.

Example : 11

Find the map of the circle |z| = 1 on XOY plane to W plane by the transformation W = (1 + i) z.

Solution :

We know, the circle |z| = 1 is

$$x^2 + y^2 = 1 \quad \ldots(1)$$

The given transformation is

$$W = (1 + i) z$$
$$u + iv = (1 + i)(x + iy)$$
$$= x + iy + ix + i^2y$$
$$= x + iy + ix - y$$
$$u + iv = (x - y) + i(x + y)$$
$$\therefore \quad u = x - y \quad \ldots(2)$$
$$v = x + y \quad \ldots(3)$$

Solving (2) and (3), we get

$$x = \frac{u+v}{2}, \; y = \frac{v-u}{2}$$

∴ Equation (1) changes to,

$$\left(\frac{u+v}{2}\right)^2 + \left(\frac{v-u}{2}\right)^2 = 1$$
$$u^2 + 2uv + v^2 + v^2 - 2uv + u^2 = 4$$
$$2u^2 + 2v^2 = 4$$
$$\therefore \quad \mathbf{u^2 + v^2 = 2}$$

which is a circle with centre (0, 0), rad = $\sqrt{2}$ units.

Example : 12

Show that the transformation $w = \frac{1}{z}$ transform the circle |z – 2| = 2 in z–plane to the line in w – plane.

Solution :

Given equation of circle on XOY plane is

$$|z - 2| = 2$$
$$|x + iy - 2| = 2$$
$$|(x - 2) + iy| = 2$$
$$\therefore \quad (x - 2)^2 + y^2 = 4$$
$$\therefore \quad x^2 + y^2 = 4x \quad \ldots(1)$$

Given transformation is $w = \frac{1}{z}$

$$\therefore \quad z = \frac{1}{w}$$

$$x + iy = \frac{1}{u + iv} = \frac{u - iv}{(u + iv)(u - iv)}$$

$$= \frac{u - iv}{u^2 + v^2}$$

$$= \left[\frac{u}{(u^2 + v^2)}\right] + i\left[\frac{-v}{u^2 + v^2}\right]$$

$$\therefore \quad x = \frac{u}{u^2 + v^2}$$

$$y = -\frac{v}{(u^2 + v^2)}$$

$\therefore$ Equation (1) changes/transforms to

$$\frac{u^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2} = 4\left(\frac{u}{u^2 + v^2}\right)$$

$$\frac{u^2 + v^2}{(u^2 + v^2)^2} = \frac{4u}{(u^2 + v^2)}$$

$$\therefore \quad 4u = 1$$

$$\therefore \quad \mathbf{u = \frac{1}{4}}$$ **, a line on UV plane.**

Example : 13

Find the image of $|z| = 4$ on XY plane to UV plane by the transformation. $w = z + 1 - 4i$

Solution : The given curve on xy plane is

$$|z| = 4$$

$$\therefore \quad x^2 + y^2 = 16 \qquad \ldots(1)$$

Given transformation is

$$w = z + 1 - 4i$$

$$u + iv = (x + iy) + 1 - 4i$$

$$= (x + 1) + i(y - 4)$$

$$\therefore \quad u = x + 1, \; v = y - 4$$

$$\therefore \quad x = u - 1, \; y = v + 4$$

$\therefore$ Equation (1) changes/transforms to

$$(u - 1)^2 + (v + 4)^2 = 16 \qquad \ldots(2)$$

which is a circle on W plane with centre (1, –4) and radius 4 units.

$\therefore$ Transformation of circle on XY plane is a circle on UV plane.

Example : 14

Find the image of hyperbola $4x^2 - 9y^2 = 36$ in XY plane on UV plane by using transformation $w = (1 - i) z$.

Solution :

Given parabola is

$$4x^2 - 9y^2 = 36 \qquad \ldots(1)$$

Given transform is

$$\begin{aligned} w &= (1 - i) z \\ u + iv &= (1 - i)(x + iy) \\ &= x + iy - ix - i^2 y \\ u + iv &= (x + y) + i(y - x) \\ \therefore \quad u &= x + y \ \& \ v = y - x \\ y &= \frac{u + v}{2}, \ x = \frac{u - v}{2} \end{aligned}$$

Equation (1) changes to

$$\begin{aligned} 4\left(\frac{u - v}{2}\right)^2 - 9\left(\frac{u + v}{2}\right)^2 &= 36 \\ 4(u^2 - 2uv + v^2) - 9(u^2 + 2uv + v^2) &= 144 \\ 4u^2 - 8uv + 4v^2 - 9u^2 - 18uv - 9v^2 &= 144 \\ -5u^2 - 5v^2 - 26uv &= 144 \\ \mathbf{5u^2 + 5v^2 + 26uv + 144} &= \mathbf{0} \end{aligned}$$

Example : 15

Show that $w = \frac{z - i}{1 - iz}$ maps upper half of Z plane onto interior of unit circle in W plane.

Solution :

We know that upper half of Z plane is 1st and 2nd quadrant.

$$\therefore \quad y > 0 \qquad \ldots(1)$$

The given transformation is

$$\begin{aligned} w &= \frac{z - i}{1 - iz} \\ w - wiz - z + i &= 0 \\ -z(wi + 1) &= -w - i = -(w + i) \\ \therefore \quad z &= \frac{w + i}{1 + wi} \\ x + iy &= \frac{u + iv + i}{1 + i(u + iv)} \end{aligned}$$

$$= \frac{u + i\,(v+1)}{1 + iu - v}$$

$$= \frac{[u + i(v+1)]}{(1-v) + iu} \times \frac{[(1-v) - iu]}{[(1-v) - iu]}$$

$$= \frac{u\,(1-v) - iu^2 + i(v+1)\,(1-v) - i^2\,u(v+1)}{(1-v)^2 + u^2}$$

$$= \frac{u\,(1-v) - iu^2 + i\,(1-v^2) + u(v+1)}{(1-v)^2 + u^2}$$

$$= \frac{(u - uv + uv + u) + i\,(1 - v^2 - u^2)}{(1-v)^2 + u^2}$$

$$x + iy = \left[\frac{2u}{(1-v)^2 + u^2}\right] + i\left[\frac{1 - v^2 - u^2}{(1-v)^2 + u^2}\right]$$

$$\therefore \quad x = \frac{2u}{(1-u)^2 + v^2},\ y$$

$$= \frac{1 - v^2 - u^2}{(1-u)^2 + v^2} \qquad \ldots(2)$$

But $y > 0$

$$\therefore \quad \frac{1 - v^2 - u^2}{(1-u)^2 + v^2} > 0$$

$$\therefore \quad 1 - u^2 - v^2 > 0$$

$$\therefore \quad 1 > u^2 + v^2$$

$$\therefore \quad \mathbf{u^2 + v^2 < 1}$$

Which is a interior of circle with centre (0, 0) and radius 01 unit.

Example : 16

Show that the transformation $w = \frac{i - z}{i + z}$ transforms the area covered under $|z| < 1$ to the area which is right half of UV plane or W plane.

Solution :

The given curve is $|z| < 1$

$$\therefore \quad x^2 + y^2 < 1 \qquad \ldots(1)$$

Given transformation is

$$w = \frac{i - z}{i + z}$$

$$\therefore \quad w\,(i + z) = i - z$$

$$\therefore \quad z = \frac{i\,(1 - w)}{(1 + w)}$$

$$\therefore \quad x + iy = \frac{i\,(1 - u - iv)}{1 + u + iv}$$

$$= \frac{v + i(1-u)}{(1+u) + iv}$$

$$= \frac{[v + i\,(1-u)]\,[(1+u) - iv]}{[(1+u) + iv]\,[(1+u) - iv]}$$

$$= \frac{2v + i\,(1 - u^2 - v^2)}{(1+u)^2 + v^2}$$

$$= \left[\frac{2v}{(1+u)^2 + v^2}\right] + i\left[\frac{1 - u^2 - v^2}{(1+u)^2 + v^2}\right]$$

$$\therefore \quad x = \frac{2v}{(1+u)^2 + v^2}, \; y = \frac{1 - u^2 - v^2}{(1+u)^2 + v^2} \qquad \ldots(2)$$

$\therefore$ Equation (1) changes to

$$\frac{4v^2}{[(1+u)^2 + v^2]^2} + \frac{(1 - u^2 - v^2)^2}{[(1+u)^2 + v^2]^2} < 1$$

$$\therefore \quad 4v^2 + (1 - u^2 - v^2)^2 < [(1+u)^2 + v^2]^2$$

$$\therefore \quad 4v^2 + 1 + u^4 + v^4 - 2u^2 + 2u^2v^2 - 2v^2 < 1 + 4u + 6u^2 + 4u^3 + u^4 + v^4 + 2v^2 + 4v^2u + 2u^2v^2$$

$$\therefore \quad 0 < 4u + 8u^2 + 4u^3 + 4v^2u$$

$$0 < 4u\,[1 + 2u + u^2 + v^2]$$

$$0 < (4u)\,[(1+u)^2 + v^2]$$

$$\therefore \quad 4u > 0 \qquad \because \; (1 + u^2) + v^2 > 0$$

$$\therefore \quad \mathbf{u > 0}$$

Which is a right half of UV/ W plane.

Example : 17

Consider a region bounded by $x = 1$, $y = 1$ and $y = 1 - x$ is XY plane. Determine the transformation of this bounded region is UV plane by using the transformation $w = z^2$

Solution :

The region bounded by $x = 1$, $y = 1$ and $y = 1 - x$ is a triangle as shown in following figure.

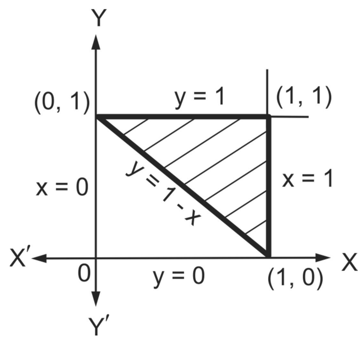

Fig.: 6.3

Given transformation is $\quad w = z^2$

$$\therefore \quad u + iv = (x + iy)^2$$

$$= x^2 + 2ixy + i^2y^2$$

$$= (x^2 - y^2) + i\,(2xy)$$

$$\therefore \quad u = x^2 - y^2, \quad v = 2xy$$

(i) Boundary x = 1 :

we get, $\quad u = 1 - y^2, \quad v = 2y$

$$\therefore \quad u = 1 - \left(\frac{v}{2}\right)^2$$

$$= 1 - \frac{v^2}{4}$$

$$v^2 = -4(u-1), \text{ a parabola symmetric on U axis.}$$

which cuts U axis at (0, 2), (0, – 2) and V axis at (1, 0).

(ii) Boundary y = 1 :

we get, $u = x^2 - 1, \quad v = 2x$

$$\therefore \quad u = \left(\frac{v}{2}\right)^2 - 1$$

$$= \frac{v^2}{4} - 1$$

$$\therefore \quad v^2 = 4(u+1), \text{ a parabola symmetric on U axis.}$$

which cuts U axis at (0, 2), (0, –2) and V axis at (–1, 0).

(iii) Boundary y = 1 – x :

we get, $u = x^2 - (1-x)^2 = 2x - 1$

$$v = 2xy$$

$$= 2x(1-x)$$

$$\therefore \quad v = 2\left(\frac{u+1}{2}\right)\left[1 - \frac{(u+1)}{2}\right]$$

$$= (u+1)\left(\frac{1-u}{2}\right)$$

$$= \mathbf{\frac{1-u^2}{2}}$$

$$\therefore \quad u^2 = 1 - 2v, \text{ which is a parabola symmetric on V axis and}$$

cuts U axis at (–1, 0) and (1, 0).

$\therefore$ New transformed region would be as shown below.

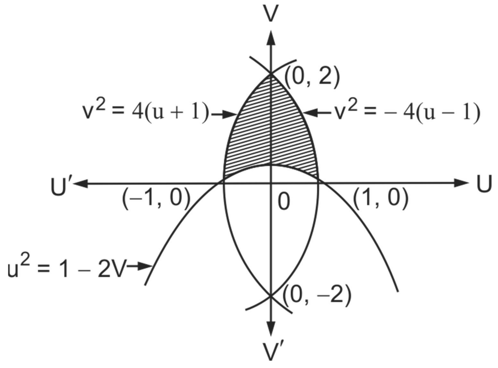

Fig.: 6.4

Self–Assessment Exercise 6.2

Ex. 1 Find the bilinear transformations w = f(z) which transforms/maps the following points on z plane to w–plane respectively.

Sr. No.	Points on z	Points on w	Answer
1.	$\infty, i, 0$	$0, i, \infty$	$w = -\frac{1}{z}$
2.	$1, i, -1$	$0, 1, \infty$	$w = \frac{i(1-z)}{1+z}$
3.	$0, 1, 2$	$-1, i, 1$	$w = \frac{z+(i-1)}{z-(i-1)}$
4.	$0, 1, \infty$	$i, -1, -i$	$w = -\frac{(1+iz)}{(z+i)}$
5.	$2, i, -2$	$1, i, -1$	$w = \frac{3z+2i}{6+iz}$
6.	$0, -1, i$	$i, 0, \infty$	$w = \frac{z+1}{z-i}$
7.	$0, -i, 2i$	$5i, \infty, \frac{-i}{3}$	$w = \frac{5i-3z}{1-iz}$
8.	$-1, 0, 1$	$0, i, 3i$	$w = \frac{-3i(z+1)}{(z-3)}$
9.	$0, -i, -1$	$i, 1, 0$	$w = -i\left(\frac{z+1}{z-1}\right)$
10.	$1, i, -1$	$2, i, -2$	$w = \frac{2i-6z}{iz-3}$
11.	$\infty, i, 0$	$0, i, \infty$	$w = \frac{-1}{z}$
12.	$1, 0, -1$	$i, 1, \infty$	$w = \frac{1+(2i-1)z}{z+1}$
13.	$0, 1, \infty$	$-1, -i, 1$	$w = \frac{z-i}{z+i}$
14.	$0, i, \infty$	$0, \frac{1}{2}, \infty$	$w = \frac{-iz}{2}$
15.	$-1, i, i+1$	$0, 2i, 1-i$	$w = \frac{-2i(z+1)}{4z-1-5i}$
16.	$-1, 0, 1$	$1, i, -1$	$w = \frac{z-i}{iz-1}$
17.	$1, i, -1$	$i, 0, -i$	$w = \frac{1+iz}{1-iz}$

18.	$-1, \infty, i$	$\infty, i, 1$	$w = \dfrac{iz + z + 1}{z + 1}$
19.	$1, i, -1$	$2, i, -2$	$w = \dfrac{2i - 6z}{iz - 3}$
20.	$-1, i, 1$	$1, i, -1$	$w = -\dfrac{1}{z}$
21.	$0, -1, i$	$2, \infty, \left(\dfrac{5+i}{2}\right)$	$w = \dfrac{3z+2}{z+1}$
22.	$-1, 0, 1$	$0, i, 3i$	$w = \dfrac{3i\,(1+z)}{(3-z)}$
23.	$-i, 0, 2+i$	$0, -2i, 4$	$w = \dfrac{2\,(z+i)}{z-1}$
24.	$1, 0, i$	$\infty, -2, -\left(\dfrac{1+i}{2}\right)$	$w = \dfrac{2+zi}{z-1}$
25.	$1, i, -1$	$i, 0, -i$	$w = \dfrac{i-z}{i+z}$
26.	$2, i, -2$	$1, i, -1$	$w = \dfrac{3z+2i}{6+iz}$
27.	$0, 1, 2$	$1, \dfrac{1}{2}, \dfrac{1}{3}$	$w = \dfrac{1}{z+1}$
28.	$-2, 0, 2$	$0, i, -i$	$w = \dfrac{i\,(z+2)}{2-3z}$

29. Find the image of circle $|z| = 2$ on XY plane onto W plane by transformation $w = z+3+4i$.

Ans. : a circle $(u-3)^2 + (v-4)^2 = 1^2$ on UV plane.

30. Find the image of circle $|z| = 3$ on XY plane onto the plane W by transformation $w = 4z$.

Ans. : a circle with centre (0, 0) and rad = 12 units on UV plane.

31. Find the image of half plane $x > 0$ under the transformation $w = i\,(z + 1)$.

Ans. : $w = u + iv$, $u = -y$, $v = x + 1 \Rightarrow x > 0 \Rightarrow v > 1$

32. Find the map of the square whose vertices are $-1 - i$, $1 - i$, $1 + i$, $-1 + i$ by the transformation $w = (1 + i)\,z$.

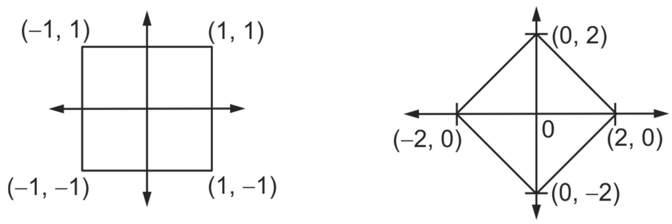

Fig.: 6.5

Ans. : It is a square with vertices $-2i, 2, -2, 2i$

33. Find the image of vertices of a square whose vertices are 0, 1, 1+i, i by using $w = (1-i)z$.

Ans. : Now vertices are 0, $1 - i$, 2, $1 + i$

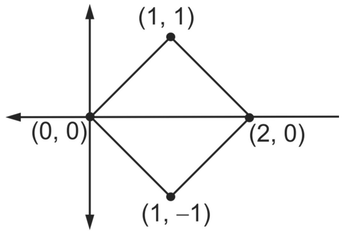

Fig.: 6.6

34. Find the image of the region $y > 1$ under the transformation $w = (1 - i)z$

Ans. : $u + v > 2$

35. Show that the straight line $y = x$ on XY plane maps to a circle on UV plane through the transformation $w = \dfrac{z-1}{z+1}$. **Ans. :** $u^2 + v^2 + 2v + 1 = 0 \Rightarrow (u-0)^2 + (v+1)^2 = 2$

36. Show that the transformation $w = \dfrac{2z+3}{z-4}$ transforms the circle $x^2 + y^2 = 4x$ on XY plane to the line $4u + 3 = 0$ on UV plane.

37. Find the map of a circle $|z - i| = 1$ on XY plane onto the w–plane by using transformation $w = \dfrac{1}{z}$. **Ans. :** $v = -\dfrac{1}{2}$, a straight line in UV plane.

6.8 Complex Integration :

The Cauchy's Integral theorem is one of the very important tool in evaluation of complex integrals and plays very important role in complex function theory. The complicated complex integrals can be easily evaluated with the help of Cauchy's Integral formula and theory of residues.

Simple Curve : A curve having no self–intersections is called as a simple curve.

Closed Curve : A curve whose end points coincides is called as a closed curve.

Simple Closed Curve : A curve having no self–intersections and whose end points coincides is called as a simple closed curve and also called as contour.

Examples : Ellipse, circle, square, rectangle, triangle are examples of closed curves/contours.

6.8.1 Line Integral or Complex Integration :

Let f(z) be a continuous function of z defined at all points of a curve c from point A to B, where $z = x + iy$ and $dz = dx + i\,dy$.

The integral $\underset{A}{\overset{B}{\int}}_{C} f(z)\,dz$ is called as a line integral or complex integral of f(z) along path C from point A to point B.

Note :

1. We know, $f(z) = u + iv$, $dz = dx + i\,dy$.
2. $\int_C F(z)\,dz = \int_C (u+iv)(dx + i\,dy) = \int_C (udx - vdy) + i\int(vdx + udy)$
3. The evaluation of complex integral depends upon the path of curve C from point A to point B.
4. $x + iy \equiv (x, y)$
5. The curve c is called as a path of integration.
6. If c is a closed curve then line integral $= \oint_C F(z)\,dz$

Illustrative Examples

Example : 1

Evaluate $\int_C F(z)\,dz$, where $f(z) = x + iy^2$ along (i) a line $y = x$ from (0, 0) to (1, 1), (ii) a parabola $y = x^2$ from (0, 0) to (1, 1).

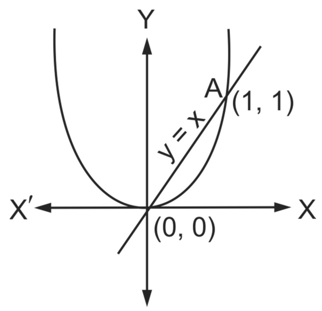

Fig.: 6.7

Solution :

Given $f(z) = x + iy^2$

We know, $z = x + iy$

$\therefore \quad dz = dx + idy$

$$\therefore \quad \int_C f(z)\,dz = \int_C (x + iy^2)(dx + idy)$$

$$= \int_C xdx + xi\,dy + i\,y^2dx + i^2y^2dy$$

$$= \int_C (xdx - y^2dy) + i\int_C (xdy + y^2dx) \qquad \ldots(1)$$

(i) along a line y = x from (0, 0) to (1, 1) :

$$\int_C f(z)\,dz = \int_{(0,0)}^{(1,1)} (xdx - y^2dy) + i\int_{(0,0)}^{(1,1)} (xdy + y^2dx) \qquad \text{from (1)}$$

$$= \int_{(0,0)}^{(1,1)} (xdx - y^2dy) + i\int_{(0,0)}^{(1,1)} (xdx + y^2dy) \qquad (\because y = x,\ dy = dx)$$

$$= \left(\frac{x^2}{2}\right)_0^1 - \left(\frac{y^3}{3}\right)_0^1 + i\left(\frac{x^2}{2}\right)_0^1 + i\left(\frac{y^3}{3}\right)_0^1$$

$$= \frac{1}{2}(1-0) - \frac{1}{3}(1-0) + \frac{i}{2}(1-0) + \frac{i}{3}(1-0)$$

$$= \frac{1}{2} - \frac{1}{3} + \frac{i}{2} + \frac{i}{3} = \frac{1}{6} + i\left(\frac{5}{6}\right)$$

(ii) along a parabola $y = x^2$ from (0, 0) to (1, 1) :

$$y = x^2$$

$$\therefore \quad dy = 2x\,dx$$

$$\therefore \quad \int_C f(z)\,dz = \int_C (xdx - y^2dy) + i\int_C (xdy + y^2dx), \qquad \text{from (1)}$$

$$= \int_C (xdx - y^2dy) + i\int_C (x\,2x\,dx + x^4dx)$$

$$= \left(\frac{x^2}{2}\right)_0^1 - \left(\frac{y^3}{3}\right)_0^1 + i\left(2\frac{x^3}{3}\right)_0^1 + i\left(\frac{x^5}{5}\right)_0^1$$

$$= \frac{1}{2}(1-0) - \frac{1}{3}(1-0) + \frac{2i}{3}(1-0) + \frac{i}{5}(1-0)$$

$$= \frac{1}{2} - \frac{1}{3} + \frac{2i}{3} + \frac{i}{5}$$

$$= \frac{1}{6} + i\left(\frac{2}{3} + \frac{1}{5}\right)$$

$$= \boldsymbol{\frac{1}{6} + \left(\frac{13}{15}\right) i}$$

Note :
1. In above example, though f(z) was same but the paths of integration in (i) and (ii) were different and hence the answers of $\int$ f(z) dz are also different.
2. The evaluation of integral depends upon the path, if f(z) is not analytic function of z.

Example : 2

Evaluate $\int_C (x^2 - y^2 + i2xy)\,dz$ along

(i) a line y = x from (0, 0) to (1, 1)

(ii) a parabola $y^2 = x$ from (0, 0) to (1, 1).

Solution :

$$\int_C f(z)\,dz = \int_C (x^2 - y^2 + i2xy)(dx + i\,dy)$$

$$= \int_C [(x^2 - y^2)\,dx - 2xy\,dy] + i\int_C [2xydx + (x^2 - y^2)\,dy] \qquad \ldots(1)$$

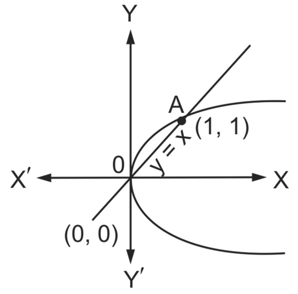

Fig.: 6.8

(i) along the line y = x from (0, 0) to (1, 1) :

$y = x, \therefore dy\ dx$

$\therefore$ from (1),

$$\int_C f(z)\,dz = \int_{(0,0)}^{(1,1)} [(x^2 - x^2)\,dx - 2xxdx] + i\int_{(0,0)}^{(1,1)} [2yydy + (y^2 - y^2)\,dy]$$

$$= 0 - 2\left(\frac{x^3}{3}\right)_0^1 + i\left[2\left(\frac{y^3}{3}\right)_0^1 + 0\right]$$

$$= -\frac{2}{3}(1-0) + i\,\frac{2}{3}(1-0) = \frac{2}{3}(i-1)$$

(ii) along the parabola $y^2 = x$ from (0, 0) to (1, 1) :

$$x = y^2$$

$$\therefore \quad dx = 2y\,dy$$

From (1),

$$\therefore \quad \int_C f(z)\,dz = \int_C [(x^2 - x)dx - 2y^2ydy)] + i\int_C [(2y^2y2ydy + (y^4 - y^2)\,dy]$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} - 2\left(\frac{y^4}{4}\right) + i\left(4\,\frac{y^5}{5}\right) + i\left(\frac{y^5}{5}\right) - i\left(\frac{y^3}{3}\right)\right]_{(0,0)}^{(1,1)}$$

$$= \frac{1}{3}(1-0) - \frac{1}{2}(1-0) - \frac{1}{2}(1-0) + \frac{4i}{5}(1-0)$$

$$+ \frac{i}{5}(1-0) - \frac{i}{3}(1-0)$$

$$= \frac{1}{3} - \frac{1}{2} - \frac{1}{2} + \frac{4i}{5} + \frac{i}{5} - \frac{i}{3}$$

$$= \left(\frac{2-3-3}{6}\right) + i\left(\frac{12+3-5}{15}\right)$$

$$= -\frac{4}{6} + i\left(\frac{10}{15}\right)$$

$$= -\frac{2}{3} + \frac{2i}{3} = \mathbf{\frac{2}{3}(-1+i) = \frac{2}{3}(i-1)}$$

Note: 1. In above example

(i) f(z) was same.

(ii) paths (i) and (ii) were different.

(iii) evaluations along both paths gives same answer.

2. Evaluations on both paths (i) and (ii) gave same answers because $f(z) = (x^2 - y^2) + i2xy$ is an analytic function.

3. Evaluation of $\int_C F(z)\, dz$ is independent of path if f (z) is an analytic function.

Example : 3

Evaluate $\int_C \bar{z}^2 dz$, where c is a curve along a line y = 2x from (1, 2) to (3, 6)

Solution :

We know,

$$\bar{z} = x - iy$$

$$\bar{z}^2 = (x - iy)^2 = x^2 + (iy)^2 - 2ixy$$

$$= (x^2 - y^2) + i\,(-2xy)$$

$$\therefore \int_C \bar{z}^2 dz = \int_C [(x^2 - y^2) + i\,(-2xy)]\,[dx + idy]$$

$$= \int_{(1,2)}^{(3,6)} [(x^2 - 4x^2) + i\,(-2x2x)]\,[dx + i2dx] \qquad \because y = 2x, \quad \therefore dy = 2dx$$

$$= \int_1^3 [-3x^2 - 4ix^2]\,[1 + 2i]\,dx$$

$$= (1 + 2i)\int_1^3 (-3x^2 - 4ix^2)\,dx$$

$$= -(1 + 2i)\left[3\,\frac{x^3}{3} + 4i\left(\frac{x^3}{3}\right)\right]_1^3$$

$$= -(1 + 2i)\left[x^3 + \frac{4i}{3}x^3\right]_1^3 = -(1 + 2i)\left[1 + \frac{4i}{3}\right][x^3]_1^3$$

$$= -(1 + 2i)\left[1 + \frac{4i}{3}\right](27 - 1)$$

$$= \mathbf{-26\,(1 + 2i)\left(1 + \frac{4i}{3}\right)}$$

$$= \mathbf{\frac{130}{3}\,(2i - 1)}$$

Example : 4

Evaluate $\int_{1-i}^{2+i} [2x + iy + 1]\, dz$ along the path $x = t + 1$, $y = 2t^2 - 1$

Solution :

Given path is $x = t + 1$ and $y = 2t^2 - 1$

$\therefore \quad dx = dt$ and $dy = 4tdt$

Limits : Lower Limit : At $(1 - i) \equiv (0, -1) \equiv (x, y)$

$$x = t + 1 \text{ and } x = 1 \quad \therefore\ t = 0$$

also, $$y = 2t^2 - 1 \text{ and } y = -1 \quad \therefore\ t = 0$$

Upper Limit : At $(2 + i) \equiv (2,1) = (x, y)$

$$x = t + 1 \text{ and } x = 2$$

$$\therefore \quad t + 1 = 2 \quad \therefore\ t = 1$$

Also, $$y = 2t^2 - 1,\ y = 1$$

$$\therefore \quad 2t^2 - 1 = 1 \quad \therefore\ t = 1$$

$$\begin{aligned}
\therefore \quad \int_{1-i}^{2+i} [2x + iy + 1]\, dz &= \int_{1-i}^{2+i} (2x + iy + 1)(dx + idy) \\
&= \int_{t=0}^{1} [2(t + 1) + i(2t^2 - 1) + 1][dt + i(4tdt)] \\
&= \int_0^1 [2t + 2 + 2it^2 - i + 1][1 + 4it]\, dt \\
&= \int_0^1 [2t + 8it^2 + 3 + 12it + 2it^2 - 8t^3 - i + 4t]\, dt \\
&= \int_0^1 [6t + 10it^2 + 3 + 12it - 8t^3 - i]\, dt \\
&= \left[6\frac{t^2}{2} + 10i\left(\frac{t^3}{3}\right) + 3t + 12i\left(\frac{t^2}{2}\right) - 8\left(\frac{t^4}{4}\right) - it\right]_0^1 \\
&= \left[3t^2 + \frac{10i}{3}t^3 + 3t + 6it^2 - 2t^4 - it\right]_0^1 \\
&= \left[3 + \frac{10i}{3} + 3 + 6i - 2 - i\right] \\
&= 4 + i\left(\frac{10}{3} + 5\right) \\
&= \mathbf{4 + \left(\frac{25}{3}\right) i}
\end{aligned}$$

Example : 5

Evaluate $\int_{1-i}^{2+3i} (z^2 + z)\, dz$ along a straight line.

Solution :

The integral path is a line from point (1, – 1) to (2, 3).

$\therefore$ Equation of a line through these points is

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

$$y + 1 = \left(\frac{3+1}{2-1}\right)(x - 1)$$

$$y + 1 = 4(x - 1) = 4x - 4$$

$$y = 4x - 5$$

$$\therefore \quad dy = 4\, dx$$

$$\therefore \quad \int_{1-i}^{2+3i} (z^2 + z)\, dz = \int_{(1,-1)}^{(2,3)} [(x + iy)^2 + (x + iy)]\,(dx + i\, dy)$$

$$= \int_{(1,-1)}^{(2,3)} [(x^2 - y^2 + 2ixy) + (x + iy)]\,(dx + i\, dy)$$

$$= \int_{(1,-1)}^{(2,3)} (x^2 - y^2 + 2xiy + x + iy)]\,(dx + i\, dy)$$

$$= \int_{(1,-1)}^{(2,3)} [x^2 - (4x-5)^2 + 2ix\,(4x-5) + i\,(4x-5) + x)]\,(dx + 4i\, dx)$$

$$= (1+4i)\int_{(1,-i)}^{(2,3)} [x^2 - (4x-5)^2 + 8ix^2 - 10ix + 4ix - 5i + x]\, dx$$

$$= (1+4i)\int_{1}^{2} [x^2\,(1+8i) - 6ix - 5i + x - (4x-5)^2]\, dx$$

$$= (1+4i)\left[(1+8i)\left(\frac{x^3}{3}\right) - 6i\left(\frac{x^2}{2}\right) - 5ix + \frac{x^2}{2} - \frac{(4x-5)^3}{12}\right]_1^2$$

$$= (1+4i)\left[\left(\frac{1+8i}{3}\right)(8-1) - 3i\,(4-1) - 5i\,(2-1) + \frac{1}{2}(4-1) - \frac{1}{12}(27-(-1))\right]$$

$$= (1+4i)\left[\left(\frac{1+8i}{3}\right)(7) - 9i - 5i + \frac{3}{2} - \frac{28}{12}\right]$$

$$= (1+4i)\left[\frac{7}{3} + \frac{3}{2} - \frac{28}{12} + \frac{56i}{3} - 14i\right] = (1+4i)\left[\left[\frac{28+18-28}{12}\right] + \left[\frac{56i - 42i}{3}\right]\right]$$

$$= (1+4i)\left[\frac{3}{2} + \frac{14i}{3}\right] \quad = \mathbf{\frac{1}{6}\,[-103 + 64i]}$$

Example : 6

Evaluate $\int_C \bar{z}\, dz$, where c is a straight line from (0, 0) to (2, 0) to (3, 3)

Solution :

We know,

$$z = x + iy$$
$$\bar{z} = x - iy$$
$$dz = dx + i\, dy$$

Let O ≡ (0, 0), A ≡ (2, 0), B ≡ (3, 3)

i) Equation of line OA is y = 0, ∴ dy = 0.

ii) Equation of line AB is y = 3 (x – 2) = 3x – 6,

$$dy = 3dx.$$

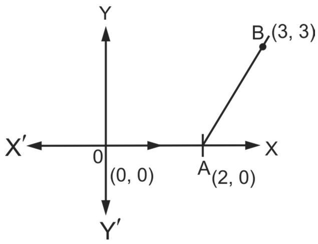

Fig.: 6.9

$$\therefore \quad \int_C \bar{z}\, dz = \int_{O-A-B} \bar{z}\, dz$$

$$= \int_{OA} \bar{z}\, dz + \int_{AB} \bar{z}\, dz$$

$$= \int_{(0,0)}^{(2,0)} (x - iy)(dx + idy) + \int_{(2,0)}^{(3,3)} (x - iy)(dx + idy)$$

$$= \int_0^2 (x - 0)(dx + 0) + \int_{(2,0)}^{(3,3)} [x - 3xi - 6i][dx + i\, 3dx]$$

$$= \int_0^2 x\, dx + \int_2^3 (1 + 3i)[x - 3xi - 6i]\, dx$$

$$= \left(\frac{x^2}{2}\right)_0^2 + (1 + 3i)\left[\frac{x^2}{2} - 3i\left(\frac{x^2}{2}\right) + 6ix\right]_2^3$$

$$= \frac{1}{2}[4 - 0] + [1 + 3i]\left[\left(\frac{1 - 3i}{2}\right)(9 - 4) + 6i\,(3 - 2)\right]$$

$$= \frac{1}{2}(4) + (1 + 3i)\left[\frac{5}{2} - \frac{15i}{2} + 6i\right]$$

$$= 2 + (1 + 3i)\left(\frac{5}{2} - \frac{3i}{2}\right) = 2 + (1 + 3i)\left(\frac{5 - 3i}{2}\right) = \mathbf{9 + 6i}$$

Example : 7

Evaluate the integral $\int_C \frac{dz}{z - a}$, where c is the circle with centre at (a, 0) and radius r units.

Solution :

We know the circle with centre at (a, 0) and radius r units is,

$$(x-a)^2 + (y-0)^2 = r^2$$

$\therefore \quad |z-a| = r$ and let θ be amplitude of $(z-a)$

$\therefore \quad (z-a) = re^{i\theta}$, where $\quad 0 \le \theta \le 2\pi$

$\therefore \quad z = a + re^{i\theta,} \quad \because$ a, r are constants

$$dz = rie^{i\theta}\, d\theta$$

$$\therefore \quad \int_C \frac{dz}{z-a} = \int_C \frac{ri\, e^{i\theta}\, d\theta}{r\, e^{i\theta}} = \int_0^{2\pi} i\, d\theta$$

$$= i\,(2\pi - 0) = \mathbf{2\pi i}$$

Example : 8

Evaluate $\int_C (z-z^2)\, dz$, where c is the upper half of a circle $|z-2| = 3$.

Solution :

$$|z-2| = 3$$

Let θ be amplitude of $(z-2)$, $0 \le \theta \le \pi$ modulus of $(z-2)$ is 3

$\therefore \quad z-2 = 3e^{i\theta}, 0 \le \theta \le \pi \quad (\because$ upper half of circle)

This is a circle having centre (2, 0), rad = 3

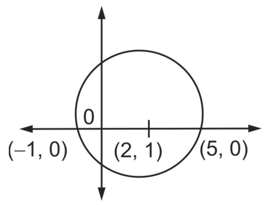

Fig.: 6.10

$\therefore \quad z = 2 + 3e^{i\theta}, z^2 = (2+3e^{i\theta})^2$

$\therefore \quad z - z^2 = (2+3e^{i\theta}) - (2+3e^{i\theta})^2$

$$= -2 - 9e^{i\theta} - 9e^{2i\theta}$$

and $\quad dz = 3i\, e^{i\theta}\, d\theta$

$$\therefore \quad \int_C (z-z^2)\, dz = -\int_0^{\pi} [2 + 9e^{i\theta} + 9e^{2i\theta}]\, d\theta\, (3i\, e^{i\theta})$$

$$= -3i \int_0^{\pi} (2e^{i\theta} + 9e^{2i\theta} + 9e^{3i\theta})\, d\theta \qquad \because e^{i\theta} = \cos\theta + i\sin\theta$$

$$= -3i \left(\frac{2e^{i\theta}}{2} + \frac{9e2^{i\theta}}{2i} + \frac{9e3^{i\theta}}{3i}\right)_0^{\pi} \qquad \because \sin n\pi = 0$$

$$= \mathbf{30} \qquad \because \cos 2n\pi = 1$$

6.9 Cauchy's Integral Theorem :

If f(z) is an analytic function and if f′(z) is a continuous function at each point on and within a closed curve c, then $\oint_C f(z)\, dz = 0$.

Illustrative Examples

Example : 1

Verify Cauchy's Integral theorem for $f(z) = z^2$, where c is a closed curve along the sides of an rectangle whose vertices are (0, 0), (2, 0), (2, 2), (0, 2).

Solution :

Given $f(z) = z^2$, which is analytic function on and within a closed curve.

To prove that, $\oint_C f(z)\, dz = \oint_C z^2\, dz = 0$

The closed curve C is along the sides OA – AB – BD – DO as shown in figure, whose equations are respectively y = 0, x = 2, y = 2, x = 0.

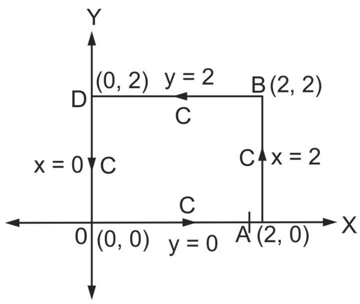

Fig.: 6.11

$$\therefore \quad \oint_C f(z)\, dz = \oint_C z^2\, dz$$

$$= \oint_C (x+iy)^2 (dx + i dy)$$

$$= \oint_C (x^2 - y^2 + 2ixy)(dx + i dy)$$

$$= \int_{OA} + \int_{AB} + \int_{BD} + \int_{DO}$$

$$= \int_0^2 x^2 dx + \int_0^2 (4 - y^2 + 4iy)\, i dy + \int_2^0 (x^2 - 4 + 4ix)\, dx + \int_2^0 -y^2 i dy$$

$$= \frac{1}{3}(x^3)_0^2 + i\left[4y - \frac{y^3}{3} + 2iy^2\right]_0^2 + \left[\frac{x^3}{3} - 4x + 2ix^2\right]_2^0 - i\left(\frac{y^3}{3}\right)_2^0$$

$$= \frac{1}{3}(8-0) + i\left[8 - \frac{8}{3} + 8i\right] + \left[-\frac{8}{3} + 8 - 8i\right] - \frac{i}{3}(-8)$$

$$= \frac{8}{3} + 8i - \frac{8i}{3} + 8i^2 - \frac{8}{3} + 8 - 8i + \frac{8i}{3}$$

$$\boldsymbol{\int f(z)\, dz = 0}$$

$\therefore$ Cauchy's Integral theorem is verified.

Example : 2

Verify Cauchy's Integral theorem for f(z) = z, where closed curve is along the sides of a triangle whose vertices are (0, 0), (4, 0), (4, 4).

Solution :

Given f(z) = z is an analytic function on and within a closed curve C, as drawn in following figure as O – A – B – O.

The equations of sides of triangle OA,

OB, BO are respectively y = 0, x = 4 and x = y.

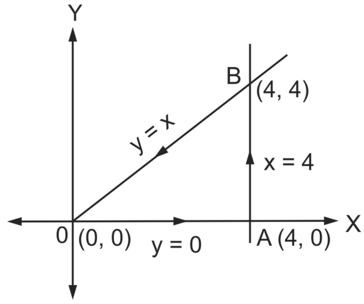

Fig.: 6.12

$$\therefore \quad \oint_C f(z)\,dz = \oint_C z\,dz$$

$$= \oint_C (x + iy)(dx + idy)$$

$$= \int_{\substack{OA \\ \left(\substack{y=0 \\ dy=0}\right)}} + \int_{\substack{AB \\ \left(\substack{x=4 \\ dx=0}\right)}} + \int_{\substack{BO \\ \left(\substack{x=y \\ dx=dy}\right)}}$$

$$= \int x\,dx + \int (4 + iy)\,idy + \int (y + iy)(dy + idy)$$

$$= \int_0^4 x\,dx + i\int_0^4 (4 + iy)\,dy + (1 + i)\int_4^0 (y + iy)\,dy$$

$$= \left(\frac{x^2}{2}\right)_0^4 + i\left[4y + i\frac{y^2}{2}\right]_0^4 + (1 + i)\left[\frac{y^2}{2} + i\frac{y^2}{2}\right]_4^0$$

$$= \frac{1}{2}(16 - 0) + i[16 + 8i] + (1 + i)[(0) - (8 + 8i)]$$

$$= 8 + 16i - 8 - [8 + 8i + 8i - 8]$$

$$= 8 + 16i - 8 - 8 - 8i - 8i + 8$$

$$\boldsymbol{\oint_C f(z)\,dz = 0}$$

Cauchy Integral theorem is verified.

Example : 3

Verify Cauchy′s Integral Theorem for f(z) = z – 1, over the closed curve c along |z – 2| = 4.

Solution : As $|z - 2| = 4$n and θ be the amplitude

$$\therefore \quad z - 2 = 4e^{i\theta}$$

$$\therefore \quad z = 2 + 4e^{i\theta}$$

$$dz = 4i\,e^{i\theta}\,d\theta$$

$$\therefore \quad \oint_C f(z)\,dz = \int_0^{2\pi} (z - 1)\,dz = \int_0^{2\pi} (2 + 4\,e^{i\theta} - 1)\,4ie^{i\theta}\,d\theta$$

$$= 4i\int_0^{2\pi} [e^{i\theta} + 4e^{2i\theta}]\,d\theta$$

$$= 4i\left[\frac{e^{i\theta}}{i} + 4\,\frac{e^{2i\theta}}{2i}\right]_0^{2\pi}$$

$$= 4i\left[\left(\frac{e^{2\pi i} - 1}{i}\right) + \frac{2}{i}[e^{4\pi i} - 1]\right]$$

$$= 4\,[(\cos 2\pi + i\sin 2\pi - 1) + 2(\cos 4\pi + i\sin 4\pi - 1)]$$

$$= 4\,[(1 + 0 - 1) + 2\,(1 + 0 - 1)]$$
$$= 0$$
$$\therefore \quad \boldsymbol{\oint_C f(z)\,dz = 0}$$

$\therefore$ Cauchy Integral theorem is verified.

6.10 Cauchy's Integral Formula :

If f(x) is an analytic function inside and on a closed curve c enclosing a simply connected region R and if point 'a' lies in the interior of R then,

(i) $$\oint_C \frac{f(z)}{(z-a)}\,dz = 2\pi i\, f(a)$$

(ii) $$\oint_C \frac{f(z)}{(z-a)^2}\,dz = \frac{2\pi i}{\lfloor 1}\, f'(a)$$

(iii) $$\oint_C \frac{f(z)}{(z-a)^3}\,dz = \frac{2\pi i}{\lfloor 2}\, f''(a)$$

(iv) $$\oint_C \frac{f(z)}{(z-a)^n}\,dz = \frac{2\pi i}{\lfloor n-1}\, f^{(n-1)}(a)$$, where $n \in N$, a natural number.

where $f'(a) = [f'(z)]_{z=a}$, $f''(a) = [f''(z)]_{z=a}$, ….

Illustrative Examples

Example : 1

Evaluate the integral $\oint_C \left[\frac{z^3+1}{(z-2)^3}\right] dz$, where c is the closed curve $|z| = 3$.

Solution :

The function $f(z) = z^3 + 1$ is an analytic on and within closed curve $|z| = 3$

The point $a = 2 = (2, 0)$ lies inside the circle $|z| = 3$ ie $x^2 + y^2 = 9$, a circle whose centre is (0, 0) and radius 3 units.

$\therefore$ By Cauchy's Integral formula

$$\oint_C \frac{f(z)}{(z-a)^3}\,dz = \frac{2\pi i}{\lfloor 2}\, f''(a)$$

$$\therefore \quad \oint_C \left[\frac{z^3+1}{(z-2)^3}\right] dz = \frac{2\pi i}{\lfloor 2}\,[6z]_{z=2}$$

$$= \frac{2\pi i}{2}\,(12) = \mathbf{12\,\pi\, i}$$

Example : 2

Evaluate the integral $\oint_C \left[\frac{e^{4z}}{(z+1)^4}\right] dz$, where c is the closed curve $|z| = 2$.

Solution :

Let $f(z) = e^{4z}$, which is an analytic function on and within a closed curve $|z| = 2$ i.e. a circle $x^2 + y^2 = 4$, having centre at (0, 0) and radius 2 units.

$\therefore$ Here point $a = -1 \equiv (-1, 0)$ lies within closed curve $|z| = 2$.

$\therefore$ By Cauchy′s Integral formula

$$\oint_C \frac{f(z)}{(z-a)^4} dz = \frac{2\pi i}{\lfloor 3} f'''(a) \quad , \quad \text{where} \quad f(z) = e^{4z}, \; f'(z) = 4e^{4z}$$

$$\oint_C \left[\frac{e^{4z}}{(z+1)^4}\right] dz = \frac{2\pi i}{\lfloor 3} (64\, e^{4z})_{z=-1} \qquad f''(z) = 16e^{4z},\; f'''(z) = 64e^{4z}$$

$$= \frac{2\pi i}{6} (64\, e^{-4})$$

$$= \frac{64\pi i}{3} e^{-4} = \mathbf{\frac{64\pi i}{3e^4}}$$

Example : 3

Evaluate the integral $\oint_C \frac{(z+1)}{(z^3 - 4z)} dz$, where c is a closed curve $|z + 2| = 1.5$

Solution :

The function $\frac{z+1}{z^3 - 4z} = \frac{z+1}{z(z^2-4)} = \frac{z+1}{z(z-2)(z+2)}$ has simple poles at $z = 0$, $z = 2$, $z = -2$. Also we can say this function is not analytic at $z = 0$, $z = 2$, $z = -2$

The given closed curve, is

$$|z + 2| = 1.5$$

$$|x + iy + 2| = 1.5$$

$$|(x+2) + iy| = 1.5$$

$$\therefore \quad (x+2)^2 + y^2 = (1.5)^2$$

which is a circle whose centre (–2, 0) and radius 1.5 units.

$\therefore$ $z = 0 \equiv (0, 0)$ lies outside circle

$z = 2 \equiv (2, 0)$ lies outside circle

$z = -2 \equiv (-2, 0)$ lies inside circle

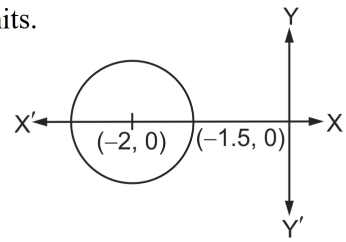

Fig.: 6.13

$$\therefore \quad \oint_C \frac{(z+1)}{(z^3 - 4z)} dz = \oint_C \left[\frac{z+1}{z(z-2)(z+2)}\right] dz$$

$$= \oint_C \frac{\left[\dfrac{z+1}{z(z-2)}\right]}{(z+2)} dz \qquad , \quad \text{(Note this step carefully)}$$

$$= \oint \frac{f(z)}{(z+2)} dz, \qquad , \quad \text{where } f(z) = \frac{z+1}{z^2-2z}$$

$$= 2\pi i\, f(a) \quad = 2\pi i\, f(-2)$$

$$= 2\pi i \left[\frac{-2+1}{4+4}\right] = 2\pi i\left(-\frac{1}{8}\right) = -\boldsymbol{\frac{\pi i}{4}}$$

Example : 4

Evaluate the integral $\oint_C \left[\dfrac{z^3}{(z^2-9)}\right] dz$, where c is a closed curve $|z| = 2$.

Solution :

The function $\dfrac{z^3}{z^2-9} = \dfrac{z^3}{(z+3)(z-3)}$, which has simple poles at $z = 3, -3$.

The closed curve $|z| = 2$ is a circle with centre (0, 0) and radius 02 units.

The poles $z = 3 \equiv (3, 0)$ and $z = -3 \equiv (-3, 0)$ lies outside the circle $|z| = 2$

$\therefore$ $f'(z)$ exists for all points on and inside $|z| = 2$

where $\qquad f(z) = \dfrac{z^3}{(z-3)(z+3)}$

$$\oint_C \mathbf{f(z)\, dz} = \oint_C \left[\frac{\mathbf{z^3}}{\mathbf{(z-3)(z+3)}}\right] \mathbf{dz} = \mathbf{0}, \textbf{ by Cauchy Integral theorem.}$$

Example : 5

Evaluate $\oint_C \left[\dfrac{e^z}{(z+1)(z-1)}\right] dz$, where c is a closed curve $|z+1| = 1$.

Solution :

The closed curve is $|z+1| = 1$ and $(x+1)^2 + (y-0)^2 = 1^2$, which is a closed curve (circle) with centre at (–1,0) and radius 01 unit.

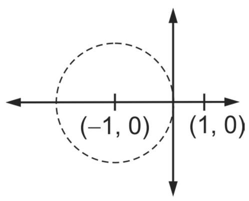

Fig.: 6.14

The function $\dfrac{e^z}{(z+1)(z-1)}$ is not analytic at $z = -1, z = 1$.

But $z = -1 \equiv (-1, 0)$, which lies inside $|z+1| = 1$

$$\therefore \quad \oint_C \left[\frac{e^z}{(z+1)(z-1)}\right] = \oint_C \frac{(e^z/z-1)}{(z+1)} dz$$

$$= \oint \frac{f(z)}{z+1} dz, \qquad \text{where } f(z) = \frac{e^z}{z-1}$$

$$= 2\pi i\,[f(-1)] = 2\pi i\left[\frac{e^{-1}}{-1-1}\right] = -\boldsymbol{\frac{\pi i}{e}}$$

Example : 6

Evaluate $\int_C \left[\frac{z^2}{(z-2)(z-3)}\right] dz$, where c is a circle $|z| = 4$.

Solution :

$f(z) = z^2$ is an analytic function within and on a close curve $|z| = 4$.

$\therefore$ $z = 2 \equiv (2, 0)$, $z \equiv (3, 0)$ both lies in side $|z| = 4$.

$$\therefore \quad \oint_C \frac{z^2}{(z-2)(z-3)} dz = \oint_C z^2 \left[\frac{1}{z-3} - \frac{1}{z-2}\right] dz$$

$$= \oint_C \left(\frac{z^2}{z-3}\right) dz - \oint_C \left(\frac{z^2}{z-2}\right) dz$$

$$= \frac{2\pi i}{1} (z^2)_{z=3} - \frac{2\pi i}{1} (z^2)_{z=2}$$

$$= 18\pi i - 8\pi i$$

$$= \mathbf{10\pi i}$$

Example : 7

Evaluate $\oint_C \frac{\log z}{(z-2)^3} dz$, where c is $|z-2| = \frac{1}{2}$.

Solution :

The closed curve $|z-2| = \frac{1}{2}$ is a circle $(x-2)^2 + (y-0)^2 = \left(\frac{1}{2}\right)^2$, whose centre is (2, 0) and radius $\frac{1}{2}$ units and $a = 2 \equiv (2, 0)$ which lies inside closed above.

$$\therefore \quad \oint \frac{f(z)}{(z-a)^3} dz = \frac{2\pi i}{\lfloor 2} f''(a), \text{ where } f(z) = \log z, f'(z) = \frac{1}{2} \log z, f'''(z) = -\frac{1}{z^2}$$

$$\therefore \quad = \oint \frac{\log z}{(z-2)^3} dz$$

$$= \frac{2\pi i}{2} \left[-\frac{1}{z^2}\right]_{z=2}$$

$$= \pi i \left(-\frac{1}{4}\right) = -\frac{\boldsymbol{\pi i}}{\mathbf{4}}$$

Example : 8

Evaluate $\oint_C \frac{\cos^2 z}{\left(z - \frac{\pi}{4}\right)^3} dz$, where c is $|z| = 1$.

Solution :

The closed curve $|z| = 1$ is a circle $x^2 + y^2 = 1$ with centre at (0, 0) and radius = 1 unit.

and $a = \frac{\pi}{4} = \frac{3.14}{4} = 0.78 \equiv (0.78,0)$, which lies in $|z| = 1$.

$$\therefore \quad \oint_C \frac{f(z)}{(z-a)^3}\,dz = \frac{2\pi i}{\lfloor 2} f''(a) \text{, where } f(z) = \cos^2 z,\ f'(z) = -\sin 2z,\ f'' = -2\cos 2z$$

$$\therefore \quad \oint_C \frac{\cos^2 z}{\left(z-\frac{\pi}{4}\right)^3}\,dz = \frac{2\pi i}{2}[-2\cos 2z]_{z=\frac{\pi}{4}}$$

$$= \pi i\left[-2\cos\left(\frac{\pi}{2}\right)\right] = \mathbf{0}$$

Example : 9

Evaluate $\oint_C \left[\frac{12z^2+7z+1}{z+1}\right] dz$, where c is $|z| = \frac{1}{2}$

Solution :

The closed curve c is $|z| = \frac{1}{2}$, a circle whose centre is (0, 0) and radius $\left(\frac{1}{2}\right)$ unit.

The point $z = -1$ lies outside $|z| = \frac{1}{2}$

$\therefore \quad \left[\frac{12z^2+7z+1}{z+1}\right]$ is analytic at all points on and inside $|z| = \frac{1}{2}$.

$$\therefore \quad \oint_C \left[\frac{12z^2+7z+1}{z+1}\right] dz = \mathbf{0} \quad \text{, by Cauchy's Integral theorem}$$

Example : 10

Evaluate $\int_C \frac{z+4}{(z^2+2z+5)}\,dz$, where c is a closed curve $|z+1-i| = 2$.

Solution :

The closed curve is $|z+1-i| = 2$

$$|x+iy+1-i| = 2$$

$$|(x+1)+i(y-1)| = 2$$

$$\therefore \quad (x+1)^2+(y-1)^2 = 2^2$$

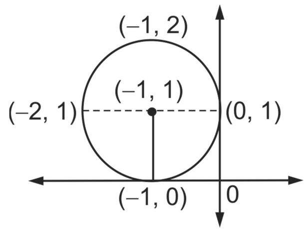

Fig.: 6.15

This is a circle whose centre (–1, 1) and radius 2 units

$$\begin{aligned} z^2+2z+5 &= z^2+2z+1+4 \\ &= (z+1)^2+(-4i^2) \\ &= (z+1)^2-(2i^2) \\ &= (z+1+2i)(z+1-2i) \\ &= [z-(-1-2i)][z-(-1+2i)] \end{aligned}$$

$\therefore \quad z = -1-2i \equiv (-1, -2)$ lies outside of $|z+1-i| = 2$

& $z = -1 + 2i \equiv (-1, 2)$ lies on the closed curve $|z + 1 - i| = 2$

$$\therefore \quad \oint_C \frac{z+4}{z^2+2z+5}\,dz = \oint_C \left[\frac{(z+4)}{(z+1+2i)\,(z+1-2i)}\right] dz$$

$$= \oint_C \left[\frac{(z+4)/(2+1+2i)}{[z-(-1+2i)]}\right] dz$$

$$= \oint \frac{f(z)}{z-(-1+2i)}\,dz, \text{ where } f(z) = \frac{z+4}{z+1+2i}$$

$$= 2\pi i \left[\frac{z+4}{z+1+2i}\right]_{z \to -1+2i}$$

$$= 2\pi i \left[\frac{-1+2i+4}{-1+2i+1+2i}\right]$$

$$= 2\pi i \left[\frac{3+2i}{4i}\right]$$

$$= \boldsymbol{\frac{\pi\,(3+2i)}{2}}$$

Example : 11

Evaluate $\oint_C \left(\frac{2z^2+z}{z^2-1}\right) dz$, where c is $|z-1| = 1.5$,

Solution :

The closed curve is $|z-1| = 1.5$

$$|(x-1)+iy| = 1.5$$

$(x-1)^2 + (y-0)^2 = (1.5)^2$, which is a circle with centre (1, 0) and rad 1.5 units.

$z = 1 \equiv (1, 0)$, lies inside the closed curve.

$z = -1 \equiv (-1, 0)$, lies outside the closed curve.

$$\oint_C \frac{2z^2+z}{z^2-1}\,dz = \oint_C \frac{2z^2+z}{(z+1)\,(z-1)}\,dz$$

$$= \oint \frac{(2z^2+z/z+1)}{(z-1)}\,dz$$

$$= \oint \frac{f(z)}{z-1}\,dz, \text{ where } f(z) = \left(\frac{2z^2+z}{z+1}\right)$$

$$= 2\pi i\,[f(1)] \quad , \quad \text{by Cauchy Integral Formula}$$

$$= 2\pi i \left[\frac{2+1}{1+1}\right]$$

$$= \mathbf{3\pi i}$$

Example : 12

Evaluate $\oint_C \frac{6z-1}{(z+2)(z-4)} dz$, where c is $|z| = 3$.

Solution :

The closed curve $|z| = 3$ is $x^2 + y^2 = 9$, a circle curve centre (0, 0) and radius 03 units.

$z = -2 \equiv (-2, 0)$, which lies inside the circle $x^2 + y^2 = 9$

$z = 4 \equiv (4, 0)$, which lies outside the circle $x^2 + y^2 = 9$

$$\therefore \quad \oint \frac{6z-1}{(z+2)(z-4)} dz = \oint_C \frac{[(6z-1)/z-4]}{z+2} dz$$

$$= \oint \frac{f(z)}{z+2} dz, \quad \text{where } f(z) = \frac{6z-1}{z-4}$$

$$= 2\pi i\,[f(-2)], \text{ by Cauchy Integral formula}$$

$$= 2\pi i \left[\frac{-12-1}{-2-2}\right]$$

$$= 2\pi i \left(\frac{13}{4}\right)$$

$$= \boldsymbol{\frac{13\pi i}{2}}$$

Example : 13

Evaluate the integral $\oint_C \left[\frac{4z+1}{z(z+2)(z-3)}\right] dz$, where c is a closed curve $|z| = 1$.

Solution :

The closed curve c is $|z| = 1$, a circle $x^2 + y^2 = 1$, with centre (0, 0) and radius 01 unit.

$z = 0 \equiv (0, 0)$, lies inside closed curve c

$z = -2 \equiv (-2, 0)$, lies outside closed curve c

$z = 3 \equiv (3, 0)$, lies outside closed curve c

$$\therefore \quad \oint_C \left[\frac{4z+1}{z(z+2)(z-3)}\right] dz = \oint_C \frac{[4z+1/(z+2)(z-3)]}{z} dz$$

$$= \oint \frac{f(z)}{z} dz, \text{ where } f(z) = \frac{4z+1}{z^2-z-6}$$

$$= 2\pi i\,[f(0)]$$

$$= 2\pi i \left[\frac{0+1}{0-0-6}\right] = -\boldsymbol{\frac{\pi i}{3}}$$

Example : 14

Evaluate $\oint_C \frac{e^{4z}}{\left(z-\frac{\pi}{4}i\right)} dz$, where c is $|z-2|+|z+2|=6$.

Solution :

The closed curve c is

$$\begin{aligned}
|z-2|+|z+2| &= 6 \\
|(x-2)+iy|+|(x+2)+iy| &= 6 \\
\sqrt{(x-2)^2+y^2}+\sqrt{(x+2)^2+y^2} &= 6 \\
\therefore \quad \sqrt{(x-2)^2+y^2} &= 6-\sqrt{(x+2)^2+y^2} \\
\text{Squaring,} \quad (x-2)^2+y^2 &= 36+[(x+2)^2+y^2]-12\sqrt{(x+2)^2+y^2} \\
x^2-4x+4+y^2 &= 36+x^2+4x+4+y^2-12\sqrt{(x+2)^2+y^2} \\
-8x-36 &= -12\sqrt{(x+2)^2+y^2} \\
2x+9 &= 3\sqrt{(x+2)^2+y^2} \\
\text{Squaring,} \quad 4x^2+36x+81 &= 9[x^2+4x+4+y^2] \\
&= 9x^2+36x+36+9y^2 \\
45 &= 5x^2+9y^2 \\
\therefore \quad \frac{x^2}{9}+\frac{y^2}{5} &= 1, \text{ an ellipse which cuts x – axis at (3, 0),}
\end{aligned}$$

(–3, 0) and y – axis at $\left(0,-\sqrt{5}\right)$, $\left(0,+\sqrt{5}\right)$

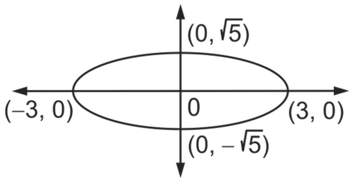

Fig.: 6.16

$z=\frac{\pi}{4}i = \frac{3.14}{4}i = 0.78\,i \equiv (0, 0.78)$, which lies inside ellipse.

$$\begin{aligned}
\therefore \quad \oint_C \frac{e^{4z}}{\left(z-\frac{\pi}{4}i\right)} dz &= \oint_C \frac{f(z)}{\left(z-\frac{\pi}{4}i\right)} dz, \text{ where } f(z)=e^{4z} \\
&= 2\pi i\left[f\left(\frac{\pi}{4}\right)\right], \text{ by Cauchy Integral formula} \\
&= 2\pi i\left[e^{4\left(\frac{\pi}{4}\right)}\right] \\
&= \mathbf{2\pi i\, e^{\pi}}
\end{aligned}$$

Example : 15

Evaluate $\oint_C \dfrac{\sin(\pi z)^2 + \cos(\pi z)^2}{(z+2)^2 (z-1)} dz$, where c is $|z-1| = 1$

Solution :

The closed curve is

$$|z-1| = 1$$
$$|(x-1) + iy| = 1$$
$$(x-1)^2 + y^2 = 1,$$

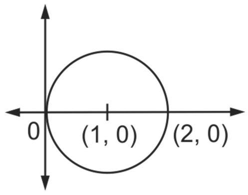

Fig.: 6.17

which is a circle having centre at (1, 0) and rad = 1 unit

$z = -2 \equiv (-2, 0)$ lies outside c

$z = 1 \equiv (1, 0)$ lies inside c

$$\therefore \quad \oint_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z+2)^2 (z-1)} dz = \oint_C \frac{[\sin\pi z^2 + \cos\pi z^2/(z+2)^2]}{(z-1)} dz$$

$$= \oint \frac{f(z)}{(z-1)} dz, \text{ Where, } f(z) = \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z+2)^2}$$

$$= 2\pi i\, f(1), \text{ by cauchy integral formula}$$

$$= 2\pi i \left[\frac{\sin\pi + \cos\pi}{(1+2)^2}\right]$$

$$= 2\pi i \left(\frac{-1}{9}\right)$$

$$= -\frac{2\pi i}{9}$$

Example : 16

Evaluate $\oint_C \dfrac{z^2}{(z-4)} dz$, where c is a unit circle $|z| = 1$.

Solution :

The function $f(z) = \dfrac{z^2}{z-4}$ is an analytic function everywhere inside and on $|z| = 1$ but its pole/singular point $z \equiv (4, 0)$ lies outside of $|z| = 1$.

$$\therefore \quad \oint_C f(z) = 0, \text{ by Cauchy's Integral Theorem}$$

$$\therefore \quad \oint_C \left(\frac{z^2}{z-4}\right) dz = 0$$

Example : 17

Evaluate $\oint_C \frac{5z-2}{(z+1)} dz$, where c is a closed curve $|z-1| = 1$.

Solution :

The function $f(z) = \frac{5z-2}{z+1}$ is analytic everywhere within and closed curve $(x-1)^2 + (y-0)^2 = 1^2$, a circle with centre (1, 0) and radius 01 unit but its singular point $z = -1 \equiv (-1, 0)$ lies outside the closed curve. By Cauchy′s Integral Theorem.

$$\oint_C \frac{5z-2}{(z+1)} dz = 0$$

Example : 18

Evaluate $\oint_C \frac{z^3}{(z+1)(z-4i)^2} dz$, where c is an ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Solution : We have

$z = -1$, $z = 4i$ are singular points of given function and $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is an ellipse as shown in figure.

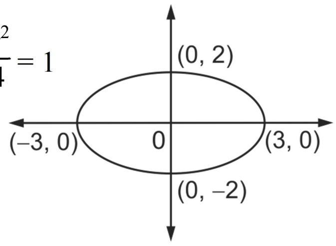

Fig.: 6.18

The point $z = -1 \equiv (-1, 0)$ lies inside c and

$z = 4i \equiv (0, 4)$ lies outside c

$$\therefore \quad \oint_C \frac{z^3}{(z+1)(z-4i)^2} dz = \oint_C \frac{[z^3/(z-4i)^2]}{(z+1)} dz$$

$$= \oint \frac{f(z)}{z+1} dz, \quad \text{where} \quad f(z) = \frac{z^3}{(z-4i)^2}$$

$$= 2\pi i\,(f(-1))$$

$$= 2\pi i \left[\frac{(-1)^3}{(1-4i)^2}\right] = \frac{-2\pi i}{-15-8i} = \mathbf{\frac{2\pi i}{15+8i}}$$

Example : 19

Evaluate $\oint_C \frac{z^4}{(z+4)(z-i)^2} dz$, where c is an ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$

Solution :

The closed curve is an ellipse as drawn in figure.

the points $z = -4 \equiv (-4, 0)$, lies outside c and

$z = i \equiv (0, 1)$, lies inside c

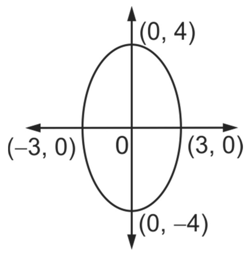

Fig.: 6.19

$$\therefore \quad \oint_C \frac{z^4}{(z+4)(z-i)^2} dz = \oint_C \frac{(z^2/z+4)}{(z-i)^2} dz$$

$$= \oint \frac{f(z)}{(z-i)^2} dz$$

$$= \frac{2\pi i}{\lfloor 1} f'(i) \text{ by Cauchy Integral Formula}$$

$$\boxed{\text{Where } f(z) = \frac{z^4}{z+4}, \ \therefore \ f'(z) = \frac{(z+4)4z^3 - z^4(1)}{(z+4)^2}}$$

$$= 2\pi i \left[\frac{(i+4)\,4i^3 - i^4}{(i+4)^2}\right]$$

$$= 2\pi i \left[\frac{4 - 16i - 1}{i^2 + 8i + 16}\right] = \mathbf{2\pi i \left[\frac{3 - 16i}{15 + 8i}\right]}$$

Example : 20

Evaluate $\oint_C \left[\frac{z^2 + \cos^2 z}{\left(z - \frac{\pi}{4}\right)^3}\right] dz$, where c is a circle $|z| = 1$.

Solution :

The curve $|z| = 1$ is a circle with centre (0, 0), radius 01 unit and $z = \frac{\pi}{4} = \frac{3.14}{4} = 0.78 \equiv$ (0.78, 0) lies within c.

$$\therefore \quad \oint_C \frac{z^2 + \cos^2 z}{\left(z - \frac{\pi}{4}\right)^3} dz = \oint \frac{f(z)}{\left(z - \frac{\pi}{4}\right)^3} dz$$

Where $f(z) = z^2 + \cos^2 z$
$f'(z) = 2z - \sin 2z$
$f''(z) = 2 - 2\cos 2z$

$$= \frac{2\pi i}{\lfloor 2} [f''(z)]_{z = \frac{\pi}{4}}$$

$$= \frac{2\pi i}{2} [2 - 2\cos 2z]_{z = \frac{\pi}{4}}$$

$$= \mathbf{\pi i\, [2 - 0] = 2\pi i}$$

Example : 21

Evaluate $\oint \frac{\sin 2z}{\left(z + \frac{\pi}{3}\right)^4} dz$, where c is $|z| = 2$.

Solution : The curve $|z| = 2$ is a circle with centre (0, 0), rad = 2 units.

$z = -\frac{\pi}{3} = -\frac{3.14}{3} = -1.04 \equiv (-1.04, 0)$ lies within $|z| = 2$.

$$\therefore \quad \oint \frac{\sin 2z}{\left(z + \frac{\pi}{3}\right)^4} dz = \int \frac{f(z)}{\left(z + \frac{\pi}{3}\right)^4} dz = \frac{2\pi i}{\lfloor 3} [f'''(z)]_{z = -\frac{\pi}{3}}, \text{ where } f(z) = \sin 2z$$

$$= \frac{2\pi i}{6} [-8\cos 2z]_{z = -\frac{\pi}{3}} = \mathbf{\frac{4\pi i}{3}}$$

Example : 22

Evaluate $\oint \frac{z+3}{(z-2)(z+1)^2} dz$, where c is boundary of a square whose vertices are (0, 1.5), (0, –1.5), (1.5, 0), (– 1.5, 0)

Solution :

The closed curve c is as drawn in figure.

z = 2 ≡ (2, 0) lies outside c and z = – 1 ≡ (– 1, 0) lies inside C.

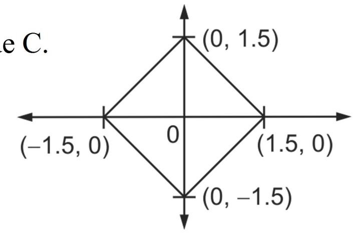

Fig.: 6.20

$$
\begin{aligned}
\therefore \quad \oint \frac{z+3}{(z-2)(z+1)^2} dz &= \oint_C \frac{(z+3/z-2)}{(z+1)^2} dz \\
&= \oint_C \frac{f(z)}{(z+1)^2} dz \\
&= \frac{2\pi i}{\lfloor 1} f'(-1) \\
&= 2\pi i \left[\frac{(z-2)(1)-(z+3)(1)}{(z-2)^2}\right]_{z=-1}, \quad \text{where } f(z) = \frac{z+3}{z-2} \\
&= 2\pi i \left[\frac{-5}{9}\right] \\
&= -\frac{\mathbf{10\pi i}}{\mathbf{9}}
\end{aligned}
$$

6.11 Residue Theorem :

Let f(z) be some function of z.

(i) Zero : If f(z) = 0 at z = a then z = a is called as zero of f(z).

(ii) Pole : If f(z) = ∞ at z = b then z = b is called as pole of f(z).

(iii) Multiple/Repeated Pole : If a pole is repeated for more than once then it is called as multiple pole.

(iv) $\left.\begin{matrix}\text{Residue of } f(z) \\ \text{at simple pole} \\ z = b\end{matrix}\right\} = [(z-b)F(z)]_{z=b}$

(v) $\left.\begin{matrix}\text{Residue of } f(z) \\ \text{at multiple pole} \\ z = c \text{ repeated for} \\ n \text{ times}\end{matrix}\right\} = \frac{1}{\lfloor n-1} \left[\frac{d^{n-1}}{dz^{n-1}} (z-c)^n F(z)\right]_{z=C}$

- **Residue Theorem :**

If f(z) is an analytic function within and on a simple closed curve c, except at finite number of isolated singular points within (inside) closed curve c, then,

$$
\begin{aligned}
\oint F(z)\, dz &= 2\pi i\, [\text{sum of residues of } f(z) \text{ at all singular points within c}] \\
&= 2\pi i\, [R_1 + R_2 + R_3 + \;\; + R_n]
\end{aligned}
$$

Illustrative Examples

Example : 1

Evaluate $\oint \frac{dz}{z^2(z+8)}$ dz, where c is a closed curve $|z| = 3$.

Solution :

The closed curve $|z| = 3$ is a circle $x^2 + y^2 = 9$.

Let, $$f(z) = \frac{1}{z^2(z+8)}$$

$\therefore$ Poles/singular points of f(z) are z = 0, z = – 8, out of which only z = 0 ≡ (0, 0) lies inside/within c and z = – 8 ≡ (– 8, 0) lies outside c.

Hence we should find residue of f(z) only at z = 0 which is repeated 02 times.

$$R = \left.\begin{matrix}\text{Residue of } f(z)\\ \text{at } z = 0\end{matrix}\right\} = \frac{1}{\lfloor 2-1}\left(\frac{d^{2-1}}{dz^{2-1}}\left[z^2\frac{1}{z^2(z+8)}\right]\right)_{z=0}$$

$$= \frac{1}{\lfloor 1}\left[\frac{d}{dz}\left(\frac{1}{z+8}\right)\right]$$

$$= \left[-\frac{1}{(z+8)^2}\right]_{z=0} = -\frac{1}{64}$$

$\therefore$ By Residue theorem,

$$\oint_C \frac{dz}{z^2(z+8)} dz = 2\pi i(R)$$

$$\therefore \oint_C \frac{dz}{z^2(z+8)} dz = 2\pi i\left(-\frac{1}{64}\right) = -\boldsymbol{\frac{\pi i}{32}}$$

Example : 2

Evaluate $\oint_C \left[\frac{\sin\pi z^2 + \cos\pi z^2}{(z+2)^2(z-1)}\right]$ dz, where c is a closed curve $|z - 1| = 1$.

Solution :

The closed curve is $|z - 1| = 1$, which is a circle whose centre is (1, 0) and radius 01 unit.

Let, $$f(z) = \frac{\sin\pi z^2 + \cos\pi z^2}{(z+2)^2(z-1)}$$

$\therefore$ z = – 2 and z = 1 are poles/singular points of f(z) out of which z = – 2 ≡ (– 2, 0) lies outside c and z = 1 ≡ (1, 0) lies within c. Hence we, should find Residue of f(z) only at z = 1.

$$\therefore \left.\begin{matrix}\text{Residue of } f(z)\\ \text{at } z = 1\end{matrix}\right\} = \left((z-1)\left[\frac{\sin\pi z^2 + \cos\pi z^2}{(z+2)^2(z-1)}\right]\right)_{z=1}$$

$$= \frac{\sin\pi + \cos\pi}{(1+2)^2} = -\frac{1}{9}$$

$\therefore$ By Residue theorem,

$$\therefore \quad \oint_C \left[\frac{\sin\pi z^2 + \cos\pi z^2}{(z+2)^2 (z-1)}\right] dz = 2\pi i \left(-\frac{1}{9}\right)$$

$$= -\frac{\mathbf{2\pi i}}{\mathbf{9}}$$

Example : 3

Evaluate $\oint_C \left[\frac{1-8z}{z(z-2)(z+3)}\right] dz$, where c is a closed curve $|z| = 4$.

Solution :

Let $$f(z) = \frac{1-8z}{z(z-2)(z+3)}$$

$\therefore$ $z = 0, z = 2, z = -3$ are singular points/poles of f(z).

The closed curve is a circle $|z| = 4$, i.e. $x^2 + y^2 = 16$.

The poles $z = 0 \equiv (0, 0)$, $z = 2 \equiv (2, 0)$, $z = -3 \equiv (-3, 0)$ lies within/inside closed curve c. Hence we should find residues at all three simple poles.

$$R_1 = \{\text{Residue at } z = 0\} = \left[z \frac{1-8z}{z(z-2)(z+3)}\right]_{z=0}$$

$$= \frac{1}{(0-2)(0+3)} = -\frac{1}{6}$$

$$R_2 = \{\text{Residue at } z = 2\} = \left[(z-2) \frac{1-8z}{z(z-2)(z+3)}\right]_{z=2}$$

$$= \frac{1-16}{2(5)} = -\frac{15}{10} = -\frac{3}{2}$$

$$R_3 = \{\text{Residue at } z = -3\} = \left[(z+3) \frac{1-8z}{z(z-2)(z+3)}\right]_{z=-3}$$

$$= \frac{1+24}{(-3)(-3-2)}$$

$$= \frac{25}{15} = \frac{5}{3}$$

$\therefore$ By Residue theorem,

$$\oint_C \left[\frac{1-8z}{z(z-2)(z+3)}\right] dz = 2\pi i \, [R_1 + R_2 + R_3]$$

$$= 2\pi i \left[-\frac{1}{6} - \frac{3}{2} + \frac{5}{3}\right]$$

$$= 2\pi i \left[\frac{-1-9+10}{6}\right] = \mathbf{0}$$

Example : 4

Evaluate the integral $\oint \frac{4z+1}{z(z+2)(z-3)} dz$, where c is $|z| = 1$.

Solution :

The closed curve c is $|z| = 1$ which is $x^2 + y^2 = 1$, a circle with centre (0, 0) and radius 01 unit.

Let, $$f(z) = \frac{4z+1}{z(z+2)(z-3)}$$

$\therefore$ z = 0, z = –2, z = 3 are poles of f(z), out of which z = 0 lies inside the c and z = –2, z = 3 lies outside c.

$\therefore$ We should find Residue of f(z) only at z = 0.

$$\therefore \quad \{\text{Residue at } z = 0\} = \left[z \frac{1+4z}{z(z+2)(z-3)}\right]_{z=0} = -\frac{1}{6}$$

$\therefore$ By Residue theorem,

$$\oint_C F(z)\, dz = \oint_C \frac{4z+1}{z(z+2)(z-3)} dz$$

$$= 2\pi i\left(-\frac{1}{6}\right) = -\boldsymbol{\frac{\pi i}{3}}$$

Example : 5

Evaluate : $\oint_C \left[\frac{4z+11}{z^3-16z}\right] dz$, where c is $|z-1| = 2$.

Solution :

The closed curve c is $(x-1)^2 + (y-0)^2 = 2^2$, a circle whose centre is (1, 0) and radius 2 units.

Let, $$f(z) = \frac{4z+11}{z(z+4)(z-4)}$$

z = 0, z = – 4, z = 4 are poles of f(z), out of which only z = 0 lies inside/within given closed curve.

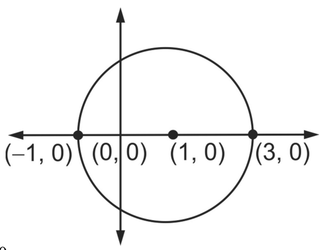

Fig.: 6.21

$$\therefore \quad \text{Residue of } f(z) \text{ at } z = 0\} = \left[z\left[\frac{4z+11}{z(z-4)(z+4)}\right]\right]_{z=0}$$

$$= \frac{11}{(-4)(4)} = -\frac{11}{16}$$

$\therefore$ By Residue theorem,

$$\oint_C \left[\frac{4z+11}{z^3-16z}\right] dz = 2\pi i\left(-\frac{11}{16}\right)$$

$$= -\left(\boldsymbol{\frac{11\pi i}{6}}\right)$$

Example : 6

Evaluate $\oint_C \frac{dz}{(z^2+4)^3}$, where c is the circle $|z - i| = 2$.

Solution :

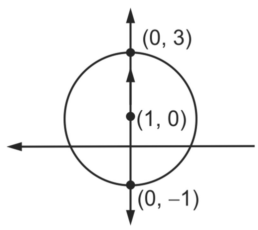

Fig.: 6.22

The closed curve c is, $\quad |z - i| = 2$

$$\therefore \quad |x + iy - i| = 2$$

$$\therefore \quad |x + i(y-1)| = 2$$

$$\therefore \quad (x-0)^2 + (y-1)^2 = 2^2,$$

which is a circle with centre (0, 1) and radius 02 units.

We know, $\quad z^2 + 4 = 0$

$$\therefore \quad z^2 = -4$$

$$\therefore \quad z = \pm 2i$$

$\therefore \quad (z^2+4)^3 = 0$, gives us six poles $z = -2i, -2i, -2i, 2i, 2i, 2i$ out of which $z = -2i \equiv (0, -2)$ lies outside c and $z = 2i \equiv (0, 2)$ lies inside closed curve c.

$$\therefore \left.\begin{array}{c}\text{Residue of } f(z) \text{ at } z = +2i \\ \text{which is repeated 03 times}\end{array}\right\} = \frac{1}{\lfloor 2}\left[\frac{d^2}{dz^2}(z-2i)^3 \frac{1}{(z+2i)^3 (z-2i)^3}\right]_{z=2i}$$

$$= \frac{1}{2}\left[\frac{d^2}{dz^2}\left(\frac{1}{(z+2i)^3}\right)\right]_{z=2i}$$

$$= \frac{1}{2}\left[\frac{d}{dz}[-3(z+2i)^{-4}]\right]_{z=2i}$$

$$= \frac{1}{2}[12(z+2i)^{-5}]_{z=2i} \quad = 6\left[\frac{1}{(z+2i)^5}\right]_{z=2i}$$

$$= 6\left[\frac{1}{(4i)^5}\right] = \frac{-3i}{512}$$

$\therefore \quad$ By Cauchy′s Residue theorem.

$$\oint \frac{dz}{(z^2+4)^3} = 2\pi i \text{ [sum of residues]}$$

$$= 2\pi i\left[\frac{-3i}{512}\right] = \mathbf{\frac{3\pi}{256}}$$

Example : 7

Evaluate $\oint_C \frac{e^{2z}}{(z^2+\pi^2)} dz$, where c is a closed curve $|z| = 4$.

Solution :

The closed curve c is $|z| = 4$, which is a circle with centre (0, 0), radius 04 units.

We know, $\quad [z^2 + \pi^2]^2 = [(z+\pi i)(z-\pi i)]^2$

$$= (z+\pi i)^2 (z-\pi i)^2$$

$\therefore \quad z = -\pi i \equiv (0, -\pi) = (0, -3.14)$, lies inside c

$z = \pi i \equiv (0, \pi) = (0, 3.14)$, lies inside c

$$\therefore \quad R_1 = R\,[z = \pi i]$$

$$= \frac{1}{\lfloor 1}\left[\frac{d}{dz}(z-\pi i)^2\cdot\frac{e^{2z}}{(z+\pi i)^2\,(z-\pi i)^2}\right]_{z=\pi i}$$

$$= \frac{d}{dz}\left[\frac{e^{2z}}{(z+\pi i)^2}\right]_{z=\pi i}$$

$$= \left[\frac{(z+\pi i)^2\,2e^{2z} - e^{2z}\,(2)\,(z+\pi i)}{(z+\pi i)^4}\right]_{z=\pi i}$$

$$= \left[\frac{2(z+\pi i)\,e^{2z} - 2\,e^{2z}}{(z+\pi i)^3}\right]_{z=\pi i}$$

$$= \left[\frac{2\,e^{2z}\,[z+\pi i-1]}{(z+\pi i)^3}\right]_{z=\pi i}$$

$$= \frac{2e^{2\pi i}\,[\pi i+\pi i\ -1]}{(\ \pi i+\pi i)^3} = \frac{2e^{2\pi i}\,[2\pi i-1]}{(\ 2\pi i)^3}$$

$$= \frac{2e^{2\pi i}\,[2\pi i-1]}{8\pi^3 i^3}$$

$$= \frac{2\,(\cos 2\pi + i\sin 2\pi)\,(2\pi i-1)}{8\pi^3\,(-i)},\ \cos 2\pi = 1,\ \sin 2\pi = 0, -\frac{1}{i} = i$$

$$\left[i^3 = -i, -\frac{1}{i} = \frac{i^2}{q} = i\right]$$

$$= \frac{2\,[1+0]}{8\pi^3}\,[i(2\pi i-1)]$$

$$= \frac{1}{4\pi^3}\,[2\pi i^2 - i] = \frac{1}{4\pi^3}\,(-2\pi - i)$$

$$R_1 = -\frac{(2\pi + i)}{4\pi^3}$$

$$R_2 = R\,[z = -\pi i]$$

$$= \frac{1}{\lfloor 1}\left[\frac{d}{dz}(z+\pi i)^2\,\frac{e^{2z}}{(z+\pi i)^2\,(z-\pi i)^2}\right]_{z=-\pi i}$$

$$= \frac{d}{dz}\left[\frac{e^{2z}}{(z-\pi i)^2}\right]_{z=-\pi i}$$

$$= \left[\frac{(z-\pi i)^2\,2e^{2z} - e^{2z}\,2\,(z-\pi i)}{(z-\pi i)^4}\right]_{z=-\pi i}$$

$$= \left[\frac{2e^{2z}\,(z-\pi i) - 2e^{2z}}{(z-\pi i)^3}\right]_{z=-\pi i}$$

$$= \left[\frac{2e^{2z}(z-\pi i-1)}{(z-\pi i)^3}\right]_{z=-\pi i}$$

$$= \frac{2e^{-2\pi i}(-\pi i-\pi i-1)}{(-\pi i-\pi i)^3}$$

$$= \frac{2[\cos 2\pi - i\sin 2\pi][-2\pi i-1]}{(-2\pi i)^3}$$

$$= \frac{2[1-0](-2\pi i-1)}{-8\pi^3 i^3} = \frac{(2\pi i+1)}{4\pi^3(-i)}$$

$$= \frac{i(2\pi i+1)}{4\pi^3}$$

$$R_2 = \frac{(-2\pi+i)}{4\pi^3} = \frac{(i-2\pi)}{4\pi^3}$$

$$\therefore \oint_C \frac{e^{2z}}{(z^2+\pi^2)^2}\,dz = 2\pi i\,[R_1+R_2]$$

$$= 2\pi i\left[\frac{-(2\pi+i)}{4\pi^3}+\frac{i-2\pi}{4\pi^3}\right] = \frac{2\pi i}{4\pi^3}[-2\pi-i+i-2\pi]$$

$$= \frac{2\pi i}{4\pi^3}(-4\pi) = -\frac{\mathbf{2i}}{\mathbf{\pi}}$$

Self–Assessment Exercise 6.3

E.1 : Evaluate $\int_C z^2\,dz$, where c is a curve on straight, line joining.

the points origin and (2 + i). **Ans. :** $\left(\frac{2+11i}{3}\right)$.

E.2 : Evaluate $\int_C z^2\,dz$, where c is a straight line joining the points (1, 1) and (2, 4).

Ans. : $-\frac{1}{3}[86+18i]$ **.**

E.3 : Evaluate $\int_C z^2\,dz$, where $z = 2x$, $0 \le x \le 1$, $z = 2 + i(x+1)$, $1 \le x \le 2$.

[Hint : Ans. : $\int_C z^2\,dz = \int_0^1 z^2\,dz + \int_1^2 z^2\,dz = \int_0^1 (4x^2)\,2dx + \int_1^2 [2+i(x-1)]^2\,i dx$

$$= \mathbf{\frac{1}{3}(2+11i)}$$

E.4 : Evaluate $\int_C z^2\,dz$, where $z = x + ix^2$, $0 \le x \le 1$ **Ans. :** $\frac{2}{3}(-1+i)$.

E.5 : Evaluate $\int_C (z - z^2)\,dz$, where c is the lower half of a circle having its centre at (2, 0) and radius 03 units. **Ans. :** –30.

E.6 : Evaluate $\int_C \bar{z}^2\,dz$, along a line $y = \frac{x}{2}$ from (0, 0) to (2, 1). **Ans. :** $\frac{5}{3}(2 - i)$.

E.7 : Evaluate $\int_C \frac{dz}{(z-a)^m}$, $m \neq -1$ along a circle $|z - a| = r$ **Ans. :** 0.

E.8 : Evaluate $\int_C (x^2 - iy)\,(dx + idy)$ from point (0, 0) to (1, 1) along a line, (i) y = x and (ii) along a parabola $y = x^2$. **Ans. :** $\frac{1}{6}(5 - i), \frac{1}{6}(5 + i)$.

E.9 : Evaluate $\int_C (x - y + ix^2)\,(dx + idy)$ from point (0, 0) to (1, 0) to (1, 1) along a straight line.

Ans. : $\left[\left(\frac{1}{2} + \frac{i}{3}\right)\right]$ and $\left[\frac{1}{2}(-2 + i)\right]$

E.10 : Verify Cauchy's Integral Theorem for f(z) = z + 8 over the curve c along sides of a rectangle whose vertices are (–1, 0), (1, 0), (1, 1), (– 1, 1).

E.11 : Verify Cauchy's Integral Theorem for $f(z) = z^2 + 1$ over the curve c along the sides of a triangle whose equations are y = 0, x = 0, y = 1 – x.

E.13 : Verify Cauchy's Integral Theorem for f(z) = z over the curve c along |z| = 6.

E.14 : Evaluate $\int_C (z^2 + 3z)\,dz$ along a circle |z| = 2 from point (2, 0) to (0, 2).

Ans. : $-\frac{1}{3}(44 + 8i)$ **.**

E.15 : Evaluate the integral, $\oint_C \left[\frac{e^{5z}}{(z+i)^4}\right] dz$, where c is the closed curve |z| = 3.

Ans. : $\frac{125\pi i e^{-5i}}{3}$ **.**

E.16 : Evaluate the integral, $\oint_C \frac{z+6}{(z^3 - 4z)}\,dz$, where c is the closed curve |z –2| = 1.5.

Ans. : $\left(\frac{\pi i}{2}\right)$

E.17 : Evaluate the integral, $\oint_C \left[\frac{z-1}{z(z^2-4)}\right] dz$, where c is the closed curve |z| = 1.5.

Ans. : $2\pi i\left(\frac{1}{4}\right)$

E.18 : Evaluate $\oint_C \left[\frac{z^2 + z + 1}{z^2 - 7z + 12}\right] dz$, where c is a closed curve $\frac{x^2}{25} + \frac{x^2}{9} = 1$

E.19 : Evaluate $\oint_C \frac{z^2+1}{z(2z+1)}\,dz$, where c is $|z| = 1$. **Ans. :** $\frac{-\pi i}{2}$

E.20 : Evaluate $\oint_C \left[\frac{z}{z^2-3z+2}\right] dz$, where c is $|z-2| = \frac{1}{2}$. **Ans. :** $4\pi i$

E.21 : Evaluate $\oint_C \frac{e^{2z}}{(z-1)(z-2)}\,dz$, where c is $|z| = 3$. **Ans. :** $2\pi i\,[e^4 - e^2]$

E.22 : Evaluate $\oint_C \frac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^2}\,dz$, where c is the circle $|z| = 1$. **Ans. :** πi

E.23 : Evaluate $\oint_C \left(\frac{z+2}{z^2+1}\right) dz$, where c is $|z-i| = \frac{1}{2}$. **Ans. :** $\pi\,(i+2)$

E.24 : Evaluate $\oint_C \frac{e^{-2z}}{(z-i)^2}\,dz$, where c is the curve along the sides of vertices at (–2, 2), (–2, –2), (2, –2), (2, 2) . **Ans. :** $-4\pi i\,[\cos 2 - i\sin 2]$

E.25 : Evaluate $\oint_C \frac{\cos\pi z^2}{(z-1)(z-2)}\,dz$, where c is $|z| = 3$. **Ans. :** $4\pi i$

E.26 : Evaluate $\oint_C \frac{z^4}{(z-3i)}\,dz$, where c is $|z-2| < 5$ **Ans. :** $162\,\pi i$

E.27 : Evaluate $\oint_C \left(\frac{z^2+1}{z^2-1}\right) dz$, where c is $|z-1| = 1$. **Ans. :** $2\pi i$

E.28 : Evaluate $\oint_C \left(\frac{2z^2+z}{z^2-1}\right) dz$, where c is $|z-1| = 1$. **Ans. :** $3\pi i$

E.29 : Evaluate $\oint_C \frac{z\,dz}{(z-1)(z-2)^2}$ where c is $|z-2| = \frac{1}{2}$. **Ans. :** $-2\pi i$

E.30 : Evaluate $\oint_C \left(\frac{4z^2+z}{z^2-1}\right) dz$, where c is $|z-1| = 3$. **Ans. :** $\frac{\pi i}{3}$

E.31 : Evaluate $\oint_C \frac{\sin 2z}{\left(z+\frac{\pi}{3}\right)^4}\,dz$, where c is $|z| = 2$. **Ans. :** $\frac{4\pi i}{3}$

E.32 : Evaluate $\oint_C \frac{z+2}{(z+1)^2(z-2)}\,dz$, where c is $|z-i| = 2$. **Ans. :** $-\frac{8\pi i}{9}$

E.33 : Evaluate $\oint_C \frac{\cos\pi z}{z(z+2)}\,dz$, where c is $|z+3-i| = 3$. **Ans. :** $-\pi i$

E.34 : Evaluate $\oint_C \frac{z-3}{z^2+2z+5}\,dz$, where c is $|z+1+i| = 2$. **Ans. :** $\pi(2+i)$

E.35 : Evaluate $\oint_C \frac{2z^2+z+5}{\left(z-\frac{3}{2}\right)^2}\,dz$, where c is $\frac{x^2}{4}+\frac{y^2}{9} = 1$. **Ans. :** $14\pi i$

E.36 : Evaluate $\oint_C \frac{e^z}{(z+1)(z+2)}\,dz$, where c is $|z+1| = \frac{1}{2}$ **Ans. :** $\frac{2\pi i}{e}$

E.37 : Evaluate $\oint_C \frac{1-2z}{z(z-1)(z-2)}\,dz$, where c is $|z| = \frac{3}{2}$ **Ans. :** $3\pi i$

E.38 : Evaluate $\oint_C \cot z\,dz$, where c is the circle $|z| = 5$. **Ans. :** $6\pi i$

E.39 : Evaluate $\oint \log z\,dz$, where c is the circle $|z| = 1$. **Ans. :** $2\pi i$

E.40 : Evaluate $\oint_C \frac{dz}{z^2}$, where c is $|z| = 1$. **Ans. :** 0

E.41 : Evaluate $\oint \frac{z^4-1}{z^2(2z+1)(z+2)}\,dz$, where c is $|z| = 1$. **Ans. :** 0

E.42 : Evaluate $\oint \left[\frac{2z+3}{z^2+z+1}\right] dz$, where c is $\left|z+\frac{\sqrt{3}\,i}{2}\right| = 1$. **Ans. :** $\frac{2\pi}{\sqrt{3}}(i\sqrt{3}-2)$

E.43 : Evaluate $\oint \frac{\sinh z}{\left(z-\frac{\pi}{2}i\right)}\,dz$, where c is $|z-3| = 2$. **Ans. :** -2π

E.44 : Evaluate $\oint \left(\frac{z+2}{z^2+4}\right) dz$, where c is $|z-1| = 1$. **Ans. :** 0

E.45 : Evaluate $\oint \frac{e^{3iz}}{(z+\pi)^3}\,dz$, where c is $|z-\pi| = 3$. **Ans. :** 0

E.46 : Evaluate $\oint \frac{e^{2z}}{(z-\pi i)}\,dz$, where c is $|z-1| + |z+1| = 1$.

6.12 Application of Cauchy's Residue Theorem, to Evaluate the Integrals, Over the Circles, of the type $\int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta$:

Method : We consider, in these cases, the closed curve c is $|z| = 1$, a unit circle having centre at (0, 0) and radius 01 unit.

(i) Let $z = e^{i\theta}$, where θ varies from 0 to 2π. $\therefore\ dz = ie^{i\theta}d\theta$.

(ii) $\therefore\ d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$

(iii) We know, $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right) = \frac{1}{2z}(z^2+1)$

(iv) We know, $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{1}{2i}\left[z-\frac{1}{z}\right] = \frac{1}{2iz}(z^2-1)$

Hence we get $\int_0^{2\pi} f(\sin\theta,\cos\theta)\,d\theta = \oint_C f(z)\,dz$ and can be evaluated by using, Cauchy's Residue Theorem.

Illustrative Examples

Example : 1

Evaluate $\int_0^{2\pi} \left[\frac{d\theta}{13 + 5\cos\theta}\right]$

Solution : Let the closed curve be |z| = 1, a unit circle.

Put, $z = e^{i\theta}, \therefore d\theta = \frac{dz}{iz}, \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$$\therefore \int_0^{2\pi} \frac{d\theta}{13 + 5\cos\theta} = \oint_C \frac{\frac{dz}{iz}}{13 + \frac{5}{2}\left(z + \frac{1}{z}\right)}$$

$$= \frac{1}{i}\oint_C \frac{dz}{z\left[13 + \frac{5}{2}\left(z + \frac{1}{z}\right)\right]} = \frac{1}{i}\oint_C \frac{dz}{z\left[\frac{13z + \frac{5}{2}z^2 + \frac{5}{2}}{z}\right]}$$

$$= \frac{1}{i}\oint_C \frac{dz}{\frac{5}{2}\left[z^2 + \frac{26}{5}z + 1\right]} = \frac{2}{5i}\oint_C \frac{dz}{\left(z^2 + \frac{26}{5}z + 1\right)}$$

$$= \frac{2}{5i}\oint_C \frac{dz}{\left(z + \frac{1}{5}\right)(z + 5)} \qquad \ldots(1)$$

The integrand in (2) has simple poles at $z = -5, z = -\frac{1}{5}$

$\therefore$ z = (–5, 0) which lies outside |z| = 1 and $z \equiv (-\frac{1}{5}, 0)$ lies inside |z| = 1. Hence we should find Residue of integrand only at $z = -\frac{1}{5}$

$$\therefore \text{Residue}\left[\text{at } z = -\frac{1}{5}\right] = \left(\left(z + \frac{1}{5}\right)\left[\frac{1}{\left(z + \frac{1}{5}\right)(z + 5)}\right]\right)_{z = -\frac{1}{5}}$$

$$= \left(\frac{1}{z + 5}\right)_{z = -\frac{1}{5}} = \frac{1}{-\frac{1}{5} + 5} = \frac{5}{24}$$

By Cauchy′s Residue theorem, from (1)

$$\int_0^{2\pi} \left[\frac{d\theta}{13 + 5\cos\theta}\right] = \frac{2}{5i}\oint_C \frac{dz}{\left(z + \frac{1}{5}\right)(z + 5)} = 2\pi i\ [\text{Summation of Residues}]$$

$$= 2\pi i\left[\frac{2}{5i}\left(\frac{5}{24}\right)\right] = \boldsymbol{\frac{\pi}{6}}$$

Example : 2

Evaluate $\int_0^{2\pi} \left(\frac{d\theta}{5 + 4\sin\theta}\right)$

Solution : Let c be a closed curve $|z| = 1$

Let $z = e^{i\theta}$, $\therefore$ $dz = ie^{i\theta}\, d\theta$, $d\theta = \frac{dz}{iz}$

$$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\int_0^{2\pi} \frac{d\theta}{(5 + 4\sin\theta)} = \oint_C \frac{\left(\frac{dz}{iz}\right)}{5 + 4\frac{1}{2i}\left(z - \frac{1}{z}\right)} = \frac{1}{i}\oint_C \frac{dz}{z\left[5 + \frac{2}{i}\left(z - \frac{1}{z}\right)\right]}$$

$$= \frac{1}{i}\oint_C \frac{dz}{z\left[\frac{5zi + 2z^2 - 2}{zi}\right]} = \oint_C \frac{dz}{2z^2 + 5iz - 2}$$

$$= \oint_C \frac{dz}{2\left(z^2 + \frac{5iz}{2} z - 1\right)}$$

$$= \frac{1}{2}\oint_C \frac{dz}{(z + 2i)\left(z + \frac{i}{2}\right)} \qquad \ldots(1)$$

The integrand in (2) has poles at $z = -\frac{i}{2}$, $z = -2i$, out of which $z = -\frac{i}{2}$ lies within $|z| = 1$ and $z = -2i$ lies outside of $|z| = 1$. Hence we should find Residue of integrand only at $z = -\frac{i}{2}$.

$$\text{Residue}\left[\text{at } z = -\frac{i}{2}\right] = \left(\left(z + \frac{i}{2}\right)\left[\frac{1}{(z + 2i)\left(z + \frac{i}{2}\right)}\right]\right)_{z = -\frac{i}{2}}$$

$$= \left(\frac{1}{z + 2i}\right)_{z = -\frac{i}{2}}$$

$$= \frac{1}{-\frac{i}{2} + 2i} = \frac{1}{\frac{3i}{2}} = \frac{2}{3i}$$

$\therefore$ By Cauchy′s Residue theorem, from (1)

$$\int_0^{2\pi} \left(\frac{d\theta}{5 + 4\sin\theta}\right) = \frac{1}{2}\oint \frac{dz}{(z + 2i)\left(z + \frac{i}{2}\right)}$$

$$= \frac{2\pi i}{2}\left[\frac{2}{3i}\right] = \mathbf{\frac{2\pi}{3}}$$

Example : 3

Show that $\int_0^{2\pi} \frac{d\theta}{5 - 4\cos\theta} = \frac{2\pi}{3}$

Solution :

Let $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$$\therefore \int_0^{2\pi} \frac{d\theta}{5 - 4\cos\theta} = \int_0^{2\pi} \frac{\frac{dz}{iz}}{5 - \frac{4}{2}\left(z + \frac{1}{z}\right)} = \int_0^{2\pi} \frac{dz}{iz\left[\frac{5z - 2z^2 - 2}{z}\right]}$$

$$= \frac{1}{i}\oint_C \frac{dz}{-2z^2 + 5z - 2} = -\frac{1}{2i}\oint_C \frac{dz}{\left(z^2 - \frac{5}{2}z + 1\right)}$$

$$= -\frac{1}{2i}\oint \frac{dz}{(z-2)\left(z - \frac{1}{2}\right)} \qquad \ldots(1)$$

$\therefore$ The integral of (1) has poles at $z = 2$, $z = \frac{1}{2}$, out of which $z = 2$ lies outside $z = \frac{1}{2}$ lies inside $|z| = 1$

$$\text{Residue}\left[\text{at } z = \frac{1}{2}\right] = \left(\left(z - \frac{1}{2}\right)\left[\frac{1}{(z-2)\left(z - \frac{1}{2}\right)}\right]\right)_{z = \frac{1}{2}}$$

$$= \left(\frac{1}{z-2}\right)_{z = \frac{1}{2}} = \frac{1}{\frac{1}{2} - 2} = -\frac{2}{3}$$

By Cauchy Residue theorem, from (1).

$$\int_0^{2\pi} \frac{d\theta}{5 - 4\cos\theta} = -\frac{1}{2i}\oint_C \frac{dz}{(z-2)\left(z - \frac{1}{2}\right)}$$

$$= -\frac{1}{2i}[2\pi i \text{ (sum of residues)}]$$

$$= -\frac{1}{2i}(2\pi i)\left(-\frac{2}{3}\right) = \mathbf{\frac{2\pi}{3}}$$

Example : 4

Evaluate $\int_0^{2\pi} \frac{d\theta}{17 - 8\cos\theta}$ …(1)

Solution : Let $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$$= \int_0^{2\pi} \frac{\frac{dz}{iz}}{17 - \frac{8}{2}\left(z + \frac{1}{z}\right)} = \oint_C \frac{dz}{iz\left[17 - 4\left(\frac{z^2+1}{z}\right)\right]}$$

$$= \oint_C \frac{dz}{iz\left[\frac{17z - 4z^2 - 4}{z}\right]} = \frac{1}{i}\oint_C \frac{dz}{-4z^2 + 17z - 4}$$

$$= \frac{1}{-4i}\oint_C \frac{dz}{z^2 - \frac{17}{4}z + 1}$$

$$= \frac{1}{-4i}\oint \frac{dz}{(z-4)\left(z - \frac{1}{4}\right)} \quad \ldots(2)$$

This integrand has poles at $z = 4$, $z = \frac{1}{4}$, out of which $z = 4$ lies outside $|z| = 1$ and $z = \frac{1}{4}$ lies inside $|z| = 1$ and hence we should find residue only at $z = \frac{1}{4}$

$$\text{Residue}\left[\text{at } z = \frac{1}{4}\right] = \left(\left(z - \frac{1}{4}\right)\left[\frac{1}{(z-4)\left(z-\frac{1}{4}\right)}\right]\right)_{z=\frac{1}{4}}$$

$$= \frac{1}{\left(\frac{1}{4} - 4\right)} = -\frac{4}{15}$$

$\therefore$ From equation (2) and equation (2)

$$\int_0^{2\pi}\left[\frac{d\theta}{17 - 8\cos\theta}\right] = \frac{1}{-4i}\oint_C \frac{dz}{(z-4)\left(z-\frac{1}{4}\right)}$$

$$= \frac{1}{-4i}\left[2\pi i\left(\frac{-4}{15}\right)\right] = \mathbf{\frac{2\pi}{15}}$$

Example : 5

Evaluate $\int_0^{2\pi} \frac{d\theta}{(5 - 3\cos\theta)^2}$ ...(1)

Solution :

Let c be a closed curve $|z| = 1$.

$\therefore$ Let $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{1}{2z}(z^2 + 1)$

$$\therefore \quad \int_0^{2\pi} \frac{d\theta}{(5 - 3\cos\theta)^2} = \oint_C \frac{\frac{dz}{iz}}{\left[5 - 3\cdot\frac{1}{2z}(z^2+1)\right]^2} = \frac{1}{i}\oint_C \frac{dz}{z\left[\frac{10z - 3z^2 - 3}{2z}\right]^2}$$

$$= \frac{1}{i}\oint_C \frac{4z\,dz}{[3z^2 - 10z + 3]^2} = \frac{1}{i}\oint_C \frac{4z\,dz}{[(3z-1)(z-3)]^2}$$

$$= \frac{1}{i}\oint_C \frac{4z\,dz}{(3z-1)^2 (z-3)^2} \qquad \ldots(1)$$

$\therefore$ Integral in (1) has double pole at $z = \frac{1}{3}$ which lies inside $|z| = 1$ and double pole $z = 3$, which lies outside $|z| = 1$.

$$\text{Residue}\left[\text{at } z = \frac{1}{3}\right] = \left\{\frac{d}{dz}\left[\left(z - \frac{1}{3}\right)^2 \frac{4z}{9\left(z - \frac{1}{3}\right)^2 (z-3)^2}\right]_{z=\frac{1}{3}}\right.$$

$$= \frac{4}{9}\left\{\frac{d}{dz}\left(\frac{z}{(z-3)^2}\right)\right\}_{z=\frac{1}{3}}$$

$$= \frac{4}{9}\left[\frac{(z-3)^2(1) - 2(z-3)(z)}{(z-3)^4}\right]_{z=\frac{1}{3}}$$

$$= \frac{4}{9}\left[\frac{(z-3) - 2z}{(z-3)^3}\right]_{z=\frac{1}{3}} = \frac{4}{9}\left[\frac{-3-z}{(z-3)^3}\right]_{z=\frac{1}{3}}$$

$$= \frac{4}{9}\left[\frac{-3-\frac{1}{3}}{\left(\frac{1}{3}-3\right)^3}\right] = \frac{4}{9}\left[\frac{\frac{-10}{3}}{\frac{-512}{27}}\right]$$

$$= \frac{5}{64}$$

$\therefore$ From (1) and Residue theorem

$$\int_0^{2\pi} \frac{d\theta}{(5 - 3\cos\theta)^2} = \left[\frac{1}{i}\right] 2\pi i \left[\frac{+5}{64}\right] = \mathbf{\frac{5\pi}{32}}$$

Example : 6

Evaluate $\displaystyle\int_0^{2\pi} \frac{\sin 2\theta}{5 + 4\cos\theta}\, d\theta$

Solution : Let $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $\therefore\ z^2 = e^{2i\theta}$, $\frac{1}{z^2} = e^{-2i\theta}$

$$\therefore \quad \sin 2\theta = \frac{e^{2i\theta} - e^{-2i\theta}}{2i} = \frac{1}{2i}\left[z^2 - \frac{1}{z^2}\right]$$

$$= \frac{1}{2iz^2}[z^4 - 1] \text{ and } \cos\theta = \frac{1}{2z}(z^2 + 1)$$

$$\therefore \int_0^{2\pi}\left[\frac{\sin 2\theta}{5 + 4\cos\theta}\right] d\theta = \oint_C \frac{\frac{1}{2iz^2}(z^4 - 1)}{5 + 4\frac{1}{2z}(z^2+1)} \cdot \frac{dz}{iz}$$

$$= \frac{1}{2i^2} \oint_C \frac{z^4 - 1}{z^3 \left[5 + \frac{2}{z}(z^2+1)\right]} dz$$

$$= \frac{1}{2i^2} \oint_C \frac{z^4 - 1}{z^3 \left[\frac{5z + 2z^2 + 2}{z}\right]} dz$$

$$= \frac{1}{2i^2} \oint_C \frac{z^4 - 1}{z^2\,[2z^2 + 5z + 2]} dz$$

$$= \frac{-1}{2} \oint_C \frac{z^4 - 1}{z^2\,(2z+1)\,(z+2)} dz \qquad \ldots(1)$$

$\therefore$ The integrand in (1) has poles at $z = 0$, $z = -2$, $z = -\frac{1}{2}$, out of which $z = 0$ is double pole and $z = -\frac{1}{2}$ simple pole which lies inside $|z| = 1$ and $z = -2$ lies outside $|z| = 1$

R_1 = Residue (at $z = 0$) $= \frac{d}{dz}\left\{z^2 \frac{z^4 - 1}{z^2(2z+1)\,(z+2)}\right\}_{z=0}$

$$= \frac{d}{dz}\left[\frac{z^4 - 1}{(2z^2 + 5z + 2)}\right]_{z=0}$$

$$= \left[\frac{(2z^2 + 5z + 2)\,4z^3 - (z^4 - 1)\,(4z + 5)}{(2z^2 + 5z + 2)^2}\right]_{z=0}$$

$$= \frac{5}{4}$$

R_2 = Residue $\left(\text{at } z = -\frac{1}{2}\right)$ $= \left(\left(z + \frac{1}{2}\right)\left[\frac{z^4 - 1}{z^2 2\left(z + \frac{1}{2}\right)(z+2)}\right]\right)_{z = \frac{-1}{2}}$

$$= \left[\frac{z^4 - 1}{2z^2\,(z+2)}\right]_{z = \frac{-1}{2}}$$

$$= \frac{\frac{1}{16} - 1}{2\left(\frac{1}{4}\right)\left(-\frac{1}{2} + 2\right)} = \frac{-\frac{15}{16}}{\frac{3}{4}} = -\frac{5}{4}$$

From (1) and by Cauchy's Residue Theorem

$$\oint_0^{2\pi} \left(\frac{\sin 2\theta}{5 + 4\cos\theta}\right) d\theta = -\frac{1}{2} \oint \frac{z^4 - 1}{z^2\,(2z+1)\,(z+2)} dz$$

$$= -\frac{1}{2}\,[2\pi i]\left[\frac{5}{4} - \frac{5}{4}\right]$$

$$= \mathbf{0}$$

Example : 7

Evaluate $\int_0^{2\pi} \frac{\sin^2\theta}{5 + 4\sin\theta}\, d\theta$

Solution : Let $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right) = \frac{1}{2iz}(z^2 - 1)$

$$\therefore \quad \int_0^{2\pi} \frac{\sin^2\theta}{5 + 4\sin\theta}\, d\theta = \oint_C \frac{\frac{-1}{4}\left(z - \frac{1}{z}\right)^2}{5 + 4\frac{1}{2iz}(z^2 - 1)} \frac{dz}{iz}$$

$$= -\frac{1}{4i} \oint_C \frac{(z^2 - 1)^2}{z^3\left[\frac{10iz + 4z^2 - 4}{2iz}\right]} dz$$

$$= -\frac{1}{4i} \oint_C \frac{(z^2 - 1)^2}{\frac{z^2\,[5iz + 2z^2 - 2]}{i}} dz$$

$$= -\frac{1}{4} \oint_C \frac{(z^2 - 1)^2}{z^2\,[2z^2 + 5iz - 2]} dz$$

$$= -\frac{1}{8} \oint_C \frac{(z^2 - 1)^2}{z^2\,(z + 2i)\left(z + \frac{i}{2}\right)} dz \qquad \ldots(1)$$

The integrand in (1) has poles at $z = 0$, $z = -2i$, $z = -\frac{i}{2}$, out of which $z = 0$ a double pole and $z = -\frac{i}{2}$ simple pole which lies inside $|z| = 1$ and $z = -2i$ lies outside $|z| = 1$.

$$\therefore\ R_1 = \text{Residue (at } z = 0) = \left(\frac{d}{dz} z^2\left[\frac{(z^2 - 1)^2}{z^2\,(z + 2i)\left(z + \frac{i}{z}\right)}\right]\right)_{z=0}$$

$$= \left(\frac{d}{dz}\left[\frac{(z^2 - 1)^2}{(z + 2i)\left(z + \frac{i}{z}\right)}\right]\right)_{z=0}$$

$$= \left(\frac{d}{dz}\left[\frac{(z^2 - 1)^2}{\left(z^2 + \frac{5}{2}\,iz - 1\right)}\right]\right)_{z=0}$$

$$= \left[\frac{\left(z^2 + \frac{5iz}{2}\,z - 1\right) 2(z^2 - 1)\,(2z) - (z^2 - 1)^2\left(2z + \frac{5i}{2}\right)}{\left(z^2 + \frac{5}{2}\,zi - 1\right)^2}\right]_{z=0}$$

$$= \frac{0 - (+1)\left(\frac{5i}{2}\right)}{(-1)^2} = -\frac{5i}{2}$$

$$R_2 = \text{Residue}\left(\text{at } z = -\frac{i}{2}\right) = \left[\left(z+\frac{i}{2}\right)\left[\frac{(z^2-1)^2}{z^2\,(z+2i)\left(z+\frac{i}{2}\right)}\right]\right]_{z=\frac{-i}{2}}$$

$$= \left[\frac{(z^2-1)^2}{z^2\,[z+2i]}\right]_{z=\frac{-i}{2}} = \frac{\left(\frac{i^2}{4}-1\right)^2}{\left(\frac{i^2}{4}\right)\left(-\frac{i}{2}+2i\right)}$$

$$= \frac{\left(-\frac{1}{4}-1\right)^2}{-\frac{1}{4}\left(\frac{3i}{2}\right)} = \frac{\frac{25}{16}}{\frac{-3i}{8}} = \left(\frac{25}{16}\right)\left(-\frac{8}{3i}\right) = -\frac{25}{6i}$$

$$= \frac{25}{6}\,i$$

From (1) and by Cauchy's Residue Theorem

$$\oint_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}\,d\theta = -\frac{1}{8}(2\pi i)\left(-\frac{5i}{2}+\frac{25i}{6}\right) = -\frac{1}{8}(2\pi i)\left(\frac{-15i+25i}{6}\right)$$

$$= -\frac{\pi i}{24}(25i-15i) = -\frac{\pi i}{24}\,(10i)$$

$$= \boldsymbol{\frac{5\pi}{12}}$$

Self–Assessment Exercises 6.4

Evaluate the following integrals :

E.1 : $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$ **Ans. :** $-\pi$

E.2 : $\int_0^{2\pi} \frac{d\theta}{5-4\cos\theta}$ **Ans. :** $\frac{2\pi}{3}$

E.3 : $\int_0^{2\pi} \left(\frac{d\theta}{3+2\cos\theta}\right)$ **Ans. :** $\frac{\pi}{\sqrt{5}}$

E.4 : $\int_0^{2\pi} \frac{d\theta}{(5-3\cos\theta)^2}$ **Ans. :** $\frac{5\pi}{32}$

E.5 : $\frac{d\theta}{(2+\cos\theta)^2}$ **Ans. :** $\frac{2\pi}{3\sqrt{3}}$

E.6 : $\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}$, $0 < a < 1$ **Ans. :** $\frac{2\pi}{1-a^2}$

E.7 : $\int_0^{2\pi} \frac{d\theta}{1+\sin^2\theta}$ **Ans. :** $\pi\sqrt{2}$

E.8 : $\int_0^{2\pi} \frac{\cos\theta}{5+4\cos\theta}\, d\theta$ **Ans. :** $\frac{\pi}{6}$

E.9 : $\int_0^{2\pi} \left[\frac{\cos 2\theta}{1-2a\cos\theta+a^2}\right] d\theta,\quad 0 < a < 1$ **Ans. :** $\frac{2\pi a^2}{(1-a^2)}$

E.10 : $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta},\quad a > b > 0$ **Ans. :** $\frac{2\pi}{\sqrt{a^2-b^2}}$

E.11 : $\int_0^{2\pi} \frac{d\theta}{97-72\cos\theta}$ **Ans. :** $\frac{2\pi}{65}$

E.12 : $\int_0^{2\pi} \frac{\cos\theta}{13-12\cos\theta}\, d\theta,$ **Ans. :** 0

E.13 : $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta},\quad a > b > 0$ **Ans. :** $\frac{2\pi}{\sqrt{a^2-b^2}}$

E.14 : $\int_0^{2\pi} \frac{\cos\theta}{5-4\cos\theta}\, d\theta$ **Ans. :** $\frac{13\pi}{8}$

E.15 : $\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\, d\theta$ **Ans. :** $\frac{2\pi}{b^2}(a-\sqrt{a^2-b^2})$

E.16 : $\int_0^{2\pi} \frac{d\theta}{13+5\cos\theta}$ **Ans. :** $\frac{\pi}{6}$

E.17 : $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta$ **Ans. :** $\frac{\pi}{12}$

E.18 : $\int_0^{2\pi} \frac{\cos\theta}{3+\sin\theta}\, d\theta$ **Ans. :** $\mathbf{0}$

E.19 : $\int_0^{2\pi} \frac{d\theta}{\pi+\cos\theta}$ **Ans. :** $\frac{\pi}{\sqrt{\pi^2-1}}$

E.20 : $\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta}$ **Ans. :** $\frac{\pi}{2}$

E.21 : $\int_0^{2\pi} \frac{\sin^2\theta}{5-4\cos\theta}\, d\theta$ **Ans. :** $\frac{\pi}{4}$

E.22 : $\int_0^{2\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta}\, d\theta$ **Ans. :** $\frac{3\pi}{8}$

E.23 : $\int_0^{2\pi} \left[\frac{\sin^2\theta}{3+2\cos\theta}\right] d\theta$ **Ans. :** $\mathbf{0}$

E.24 : $\int_0^{2\pi} \frac{d\theta}{[1 - 2a\sin\theta + a^2]}$, $0 < a < 1$ **Ans. :** $\frac{2\pi}{1 - a^2}$

E.25 : $\int_0^{2\pi} \frac{\cos 2\theta \, d\theta}{16 - 8\cos\theta + 1}$ **Ans. :** $\frac{\pi}{120}$

E.26 : $\int_0^{2\pi} \left[\frac{\sin 2\theta}{5 + 4\cos\theta}\right] d\theta$ **Ans. :** 0

E.27 : $\int_0^{2\pi} \left[\frac{\sin^2\theta}{5 + 4\sin\theta}\right] d\theta$ **Ans. :** $\frac{5\pi}{12}$

E.28 : $\int_0^{2\pi} \frac{d\theta}{(5 - 3\sin\theta)^2}$ **Ans. :** $\frac{5\pi}{32}$

E. 29 : $\int_0^{2\pi} \left(\frac{1 + 2\cos\theta}{5 + 4\cos\theta}\right) d\theta$ **Ans. :** 0

E. 30 : $\int_0^{2\pi} \left(\frac{\sin^2\theta - 2\cos\theta}{2 + \cos\theta}\right) d\theta$ **Ans. :** $\frac{2\pi}{\sqrt{3}}$

6.13 Laurent's Series :

Let f(z) be an analytic function of z on two concentric circles c_1, c_2 with radius r_1, r_2 respectively with centre at a and also analytic in the annular region R bounded by c_1, c_2 then at each point z on R, f(z) can be expressed as a convergent series including positive and negative powers of (z – a) as below and called as Laurents Series.

$$\begin{aligned} f(z) &= a_0 + a_1(z - z_0) + a_2 (z - z_0)^2 + a_3 (z - z_0)^3 + + + \\ &\quad + b_1 (z - z_0)^{-1} + b_2 (z - z_0)^{-2} + b_3 (z - z_0)^{-3} + + + \\ &= \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n} \\ &= A + B \end{aligned}$$

where, $$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)\,dz}{(z - z_0)^{n+1}}, \quad n = 0, 1, 2, 3, \ldots\ldots.$$

$$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)\,dz}{(z - z_0)^{n+1}}, \quad n = 1, 2, 3, 4 \ldots\ldots.$$

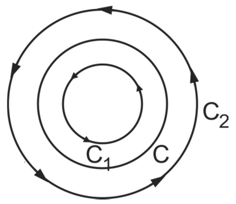

Fig.: 6.23

Where,

1. A is called as analytic part and B is called Principal Part of Laurents series.
2. If Principal Part B contains terms $b_1 (z - z_0)^{-1}$ then $x = x_0$ is called simple pole of f(z) and if B contains two terms $b_1 (z - z_0)^{-1} + b_2 (z - z_0)^{-2}$ then $z = z_0$ is called as double pole of f(z).

Illustrative Examples

Example : 1

Express $f(z) = \dfrac{1}{z(z-1)^2}$ as a Laurent's series at the point z = 1.

Solution :

To express the given f(z) as Laurent's series at z = 1,

We put, $z_1 = z - 1$

$z = z_1 + 1$

$$\therefore \quad f(z) = \frac{1}{z(z-1)^2} = \frac{1}{(z_1+1)\,z_1^2} = \frac{1}{z_1^2}(1+z_1)^{-1}$$

$$= \frac{1}{z_1^2}[1 - z_1 + z_1^2 - z_1^3 + \ldots.+]$$

$$= \frac{1}{z_1^2} - \frac{1}{z_1} + 1 - z_1 + z_1^2 \ldots\ldots\ldots$$

$$= \mathbf{\frac{1}{(z-1)^2} - \frac{1}{(z-1)} + 1 - (z-1) + (z-1)^2 \ldots.} \qquad \because z_1 = z - 1$$

which is valid for $|z-1| < 1$

Example : 2

Obtain the Laurent's series for $f(z) = \dfrac{1}{(1-z)(z+2)}$ for $1 < |z| < 2$.

Solution : We can write ,

$$\frac{1}{(1-z)(z+2)} = \frac{1}{3(1-z)} + \frac{1}{3(z+2)}, \text{ by partial fractions} \qquad \ldots(1)$$

Firstly,
$$\frac{1}{1-z} = -\frac{1}{(z-1)} \qquad \text{for } |z| > 1$$

$$= -\frac{1}{z\left(1-\frac{1}{z}\right)} = -\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} +++\right] = -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} \ldots\ldots.$$

$$= -\sum_{n=1}^{\infty} \frac{1}{z^n} \qquad \ldots(2)$$

Secondly for $|z| < 2$,

$$\frac{1}{z+2} = \frac{1}{2\left(1+\frac{z}{2}\right)} = \frac{1}{2}\left[1 + \frac{z}{2}\right]^{-1}$$

$$= \frac{1}{2}\left[1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^3 + \ldots\ldots\right]$$

$$= \frac{1}{2} - \frac{z}{2^2} + \frac{z^2}{2^3} - \frac{z^3}{2^4} + - + -$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{2^{n+1}} \qquad \ldots(3)$$

Put (2) and (3) in (1), we get

$$f(z) = \frac{1}{(1-z)(z+2)} = \frac{1}{3(1-z)} + \frac{1}{3(z+2)}$$

$$= -\frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{z^n} + \frac{1}{3}\sum_{n=0}^{\infty} \frac{(-1)^n z^n}{2^{n+1}}$$ **, which is valid for $1 < |z| < 2$.**

Example : 3

Find the Laurent's expression of $f(z) = \dfrac{7z-2}{(z+1)(z)(z-2)}$, $1 < |z+1| < 3$.

Solution :

Let $z + 1 = t$, $z = t - 1$, and $z - 2 = t - 1 - 2 = t - 3$

$$\therefore \quad f(z) = \frac{7z-2}{(z+1)(z)(z-2)} = \frac{7(t-1)-2}{t(t-1)(z-2)}$$

$$= \frac{7t-9}{t(t-1)(t-3)}$$

$$= \frac{A}{t} + \frac{B}{t-1} + \frac{C}{t-3} = -\frac{3}{t} + \frac{1}{t-1} + \frac{2}{t-3}, \qquad \text{(by Partial Fraction)}$$

$$= -\frac{3}{t} + \frac{1}{t\left(1-\frac{1}{t}\right)} - \frac{2}{3\left(1-\frac{t}{3}\right)}$$

$$= -\frac{3}{t} + \frac{1}{t}\left(1-\frac{1}{t}\right)^{-1} - \frac{2}{3}\left(1-\frac{t}{3}\right)^{-1}$$

$$= -\frac{3}{t} + \frac{1}{t}\left(1 + \frac{1}{t} + \frac{1}{t^2} + \frac{1}{t^3} + +\right) - \frac{2}{3}\left(1 + \frac{t}{3} + \frac{t^2}{9} + \frac{t^3}{27} + +\right)$$

$$= \left(-\frac{3}{t} + \frac{1}{t} + \frac{1}{t^2} + \frac{1}{t^3} + +\right) - \frac{2}{3}\left(1 + \frac{t}{3} + \frac{t^2}{9} + \frac{t^3}{27} + +\right)$$

$$= \left(-\frac{2}{t} + \frac{1}{t^2} + \frac{1}{t^3} + +\right) - \frac{2}{3}\left(1 + \frac{t}{3} + \frac{t^2}{9} + \frac{t^3}{27} + +\right)$$

$$= -\frac{2}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + + -\frac{2}{3}\left(1 + \frac{1}{3}(z+1) + \frac{1}{9}(z+1)^2 + +\right) \qquad 1 < |z+1| < 3$$

Example : 4

Express $f(z) = \dfrac{z^2 - 6z - 1}{(z-1)(z-3)(z+2)}$ as a Laurent's series on the region $3 < (z+2)| < 5$.

Solution :

First resolve f(z) into partial fractions.

$$f(z) = \frac{z^2 - 6z - 1}{(z-1)(z-3)(z+2)} = \frac{A}{z-1} + \frac{B}{z-3} + \frac{C}{z+2}$$

$$= \frac{1}{z-1} + \frac{1}{z-3} + \frac{1}{z+2}$$

$$= \frac{1}{t-3} + \frac{1}{t-5} + \frac{1}{t} \qquad \text{Put, } z+2 = t \Rightarrow z = t-2\}$$

$$= \frac{1}{t\left(1-\frac{3}{t}\right)} + \frac{1}{-5\left(1-\frac{t}{5}\right)} + \frac{1}{t} = \frac{1}{t} + \frac{1}{t}\left(1-\frac{3}{t}\right)^{-1} - \frac{1}{5}\left(1-\frac{t}{5}\right)^{-1} \ldots\ldots$$

$$= \frac{1}{t} + \frac{1}{t}\left[1 + \frac{3}{t} + \frac{9}{t^2} + +\right] - \frac{1}{5}\left[1 + \frac{t}{5} + \frac{t^2}{25} + +\right]$$

$$= \frac{1}{t} + \frac{1}{t}\sum_{n=0}^{\infty}\left(\frac{3}{t}\right)^n - \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{t}{5}\right)^n = \frac{1}{t} + \sum_{n=0}^{\infty}\frac{3^n}{t^{n+1}} - \sum_{n=0}^{\infty}\frac{t^n}{5^{n+1}}$$

$$= \mathbf{\frac{1}{z+2} + \sum_{n=0}^{\infty}\frac{3^n}{(z+2)^{n+1}} - \sum_{n=0}^{\infty}\frac{(z+2)^n}{5^{n+1}}}, \qquad \mathbf{3 < |z+2| < 5}$$

Example : 5

Find Laurent's series of $f(z) = (3 - z)\sin\dfrac{1}{z+2}$ about $z = -2$.

Solution :

Let $z + 2 = x$, $\quad \therefore z = x - 2$

$$f(z) = (3-z)\sin\left(\frac{1}{z+2}\right) = (3-x+2)\sin\left(\frac{1}{x}\right)$$

$$= (5-x)\left(\frac{1}{x} - \frac{(1/x)^3}{3!} + \frac{(1/x)^5}{5!} + \right)$$

$$= (5-x)\left(\frac{1}{x} - \frac{1}{6x^3} + \frac{1}{120x^5} - + - +\right)$$

$$= \frac{5-x}{x} - \frac{5-x}{6x^3} + \frac{5-x}{120x^5} \ldots.$$

$$= \frac{5}{x} - 1 - \frac{5}{6x^3} + \frac{1}{6x^2} + \frac{1}{24x^5} - \frac{1}{120x^4} \ldots.$$

$$= -1 + \frac{5}{x} + \frac{1}{6x^2} - \frac{5}{6x^3} - \frac{1}{120x^4} + \frac{1}{24x^5} \ldots.$$

$$= \; -(1) + \frac{5}{z+2} + \frac{1}{6\,(z+2)^2} - \frac{5}{6\,(z+2)^3} - \frac{1}{120\,(z+2)^4} \; \ldots.$$

$$= \; \mathbf{-1 + 5\,(z+2)^{-1} + \frac{1}{6}(z+2)^{-2} - \frac{5}{6}(z+2)^{-3} \ldots}\text{, **a Laurent's series.**}$$

Descriptive Questions

Q. 1 **Determine the harmonic conjugates of following functions u or v such that f(z) = u +iv becomes an analytic function.**

1. $u = \log\sqrt{x^2+y^2}$
2. $u = x \sin x \cosh y - y \cos x \sinh y$
3. $v = e^{-x}(x \cos y + y \sin y)$
4. $u = r^3 \cos 3\theta$

Q. 2 **Find the analytic function f(z) = u+iv if**

1. $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$
2. $u + v = e^x[\cos y + \sin y] + \dfrac{x-y}{x^2+y^2}$
3. Find the image of circle $|z| = 3$ on XY plane onto the plane W by transformation $w = 4z$.

Q. 3 Evaluate $\int_C z^2\, dz$, where c is a straight line joining the points (1, 1) and (2, 4).

Q. 4 Evaluate $\int_C \bar{z}^2\, dz$, along a line $y = \frac{x}{2}$ from (0, 0) to (2, 1).

Q. 5 Verify Cauchy's Integral Theorem for $f(z) = z + 8$ over the curve c along sides of a rectangle whose vertices are (–1, 0), (1, 0), (1, 1), (– 1, 1).

Q. 6 Evaluate $\oint_C \left[\dfrac{z}{z^2 - 3z + 2}\right] dz$, where c is $|z - 2| = \frac{1}{2}$.

Q. 7 Evaluate $\oint_C \dfrac{\sin 2z}{\left(z + \frac{\pi}{3}\right)^4}\, dz$, where c is $|z| = 2$.

Q. 8 Evaluate $\oint \dfrac{e^{2z}}{(z - \pi i)}\, dz$, where c is $|z-1| + |z+1| = 1$.

Q. 9 $\displaystyle\int_0^{2\pi} \left(\frac{d\theta}{3 + 2\cos\theta}\right)$

Q. 10 $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta}\, d\theta$

Q. 11 $\displaystyle\int_0^{2\pi} \left(\frac{1 + 2\cos\theta}{5 + 4\cos\theta}\right) d\theta$

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MCQ’s

(Multiple Choice Questions)

MCQ's

Multiple Choice Question's

1 Unit — Linear Differential Equations (LDE) and Applications

1. For linear differential equation with constant coefficient of type $\phi(D)\,y = F(x)$ with $D \equiv \dfrac{d}{dx}$, the auxiliary equation is given by

(a) $F(x) = 0$ (b) $\phi(D) = 0$

(c) $\phi(D)\,y = 0$ (d) $F(D) = 0$

Ans. : (b)

Explanation : By definition, the auxiliary equation of $\phi(D)\,y = F(x)$ is $\phi(D) = 0$

2. For $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y = e^{-4x}$, $D \equiv \dfrac{d}{dx}$, the auxiliary equation is

(a) $D^2 + 5D - 6 = 0$ (b) $D^2 - 5D + 6 = 0$

(c) $D^2 + 5D + 6 = 0$ (d) $D^2 - 5D - 6 = 0$

Ans. :(c)

Explanation :

Use $D = \dfrac{d}{dx}$ in given differential equation

$$D^2 y + 5Dy + 6y = e^{-4}$$

$$(D^2 + 5D + 6)y = e^{-4}$$

which is of type $\phi(D) = F(x)$

A.E. is $\phi(D) = 0$

$$D^2 + 5D + 6 = 0$$

3. The auxiliary equation of $\frac{d^2y}{dx^2} - 4y = 0$ is

(a) $m^2 - 4 = 0$ (b) $m^2 + 4 = 0$

(c) $m^2 + 1 = 0$ (d) $m^2 - 1 = 0$

Ans. : (a)

Explanation :

$$\text{Use} \quad D \equiv \frac{d}{dx}$$

$$(D^2 - 4)\, y = 0$$

$$\text{A.E. is} \quad D^2 - 4 = 0$$

Or by taking D as m,

$$\text{A.E. is} \quad m^2 - 4 = 0$$

4. The roots of the auxiliary equation of $2D^2 y + Dy - 6y = 0$ are

(a) $2, -\frac{3}{2}$ (b) $-2, \frac{3}{2}$ (c) $-2, \frac{3}{2}$ (d) $-2, -\frac{3}{2}$

Ans. : (b)

Explanation :

$$\text{Use} \quad D \equiv \frac{d}{dx}$$

$$(2D^2 + D - 6)\, y = 0$$

$$\text{A.E.is} \quad 2D^2 + D - 6 = 0$$

$$2D^2 + 4D - 3D - 6 = 0$$

$$2D\,(D + 2) - 3\,(D + 2) = 0$$

$$D + 2 = 0 \text{ and } 2D - 3 = 0$$

$D = -2, \frac{3}{2}$ are roots of the auxiliary equation.

5. The complimentary function of $(D^3 + D^2 - 2D)\, y = 0$ with $D \equiv \frac{d}{dx}$ is

(a) $c_1 e^x + c_2 e^{-x}$ (b) $c_1 + c_2 e^{-2x}$

(c) $c_1 + c_2 e^x + c_3 e^{-x}$ (d) $c_1 + c_2 e^x + c_3 - e^{-2x}$

Ans. : (d)

Explanation : The auxiliary equation is

$$D^2 + D^2 - 2D = 0$$

$$D\,(D^2 + D - 2) = 0$$

$$D\,(D - 1)\,(D + 2) = 0$$

$$D = 0, 1, -2 \quad \text{(Real and distinct)}$$

Complimentary function $= c_1 e^{0x} + c_2 e^x + c_3 e^{-2x}$

$$= c_1 + c_2 e^x + c_3 e^{-2x}$$

6. The complimentary function of $D^2 y + y\ 6Dy + 9y = 0$ with $D \equiv \dfrac{d}{dx}$ is

(a) $c_1 e^{3x} + c_2 e^{3x}$ (b) $c_1 e^{-3x} + c_2 e^{3x}$

(c) $(c_1 + c_2 x) e^{-3x}$ (d) $c_1 \cos 3x + c_2 \sin 3x$

Ans .: (c)

Explanation :

A.E. is
$$D^2 + 6D + 9 = 0$$
$$(D + 3)^2 = 0$$
$$D = -3, -3 \text{ (real \& repeated)}$$
$$CF = (c_1 + c_1 x) e^{-3x}$$

7. The solution of the ordinary differential equation $\dfrac{d^2 y}{dx^2} + \dfrac{dy}{dx} - 6y = 0$ is

(a) $y = c_1 e^{3x} + c_2 e^{-2x}$ (b) $y = c_1 e^{3x} + c_2 e^{2x}$

(c) $y = c_1 e^{-3x} + c_2 e^{2x}$ (d) $y = c_1 e^{-3x} + c_2 e^{-2x}$

Ans. : (c)

Explanation :

$$(D^2 + D - 6)\, y = 0$$

A.E. is
$$D^2 + D - 6 = 0$$
$$(D + 3)(D - 2) = 0,$$
$$D = -3, 2$$

As differential equation is homogeneous, solution is

$$y = CF$$
$$y = c_1 e^{-3x} + c_2 e^{2x}$$

8. The complimentary function of $\dfrac{d^2 y}{dx^2} + 4y = \cos 2x$ is

(a) $c_1 \cos 2x + c_2 \sin 2x$ (b) $c_1 e^{2x} + c_2 e^{-2x}$

(c) $(c_1 + c_2 x) e^{2x}$ (d) $(c_1 + c_2 x) e^{-2x}$

Ans. : (a)

Explanation :

$$(D^2 + 4)\, y = \cos 2x$$

A.E. is
$$D^2 + 4 = 0$$
$$(D - 2i)(D + 2i) = 0$$
$$D = \pm 2i \text{ (complex roots)}$$

$$CF = e^{0x}[c_1 \cos 2x + c_2 \sin 2x]$$
$$= c_1 \cos 2x + c_2 \sin 2x$$

9. The solution of linear differential equation $(D^4 + 4D^2)\, y = 0$ with $D \equiv \frac{d}{dx}$ is

(a) $y = c_1 \cos 2x + c_2 \sin 2x$
(b) $y = c_1 + c_2 \cos 2x + c_3 \sin 2x$
(c) $y = c_1 + c_2 x + c_3 \cos 2x + c_4 \sin 2x$
(d) $y = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x$

Ans . : (c)

Explanation :

A.E. is
$$D^4 + 4D^2 = 0$$
$$D^2 (D^2 + 4) = 0$$
$$D^2 (D - 2i)(D + 2i) = 0$$
$$D = 0, 0 \pm 2i$$
Solution is
$$y = (c_1 + c_2 x) e^{0x} + e^{0x} [c_3 \cos 2x + c_4 \sin 2x]$$

10. The solution of $2\frac{d^2 y}{dx^2} - 2\frac{dy}{dx} - y = 0$ is

(a) $y = c_1 \cos\left(\frac{\sqrt{3}}{2} x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2} x\right)$

(b) $y = e^{x/2}\left[c_1 \cos\left(\frac{\sqrt{3}}{2} x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2} x\right)\right]$

(c) $y = c_1 e^{(1+\sqrt{3})\,x} + c_2 e^{(1-\sqrt{3})\,x}$

(d) $y = c_1 e^{i\left(\frac{1+\sqrt{3}}{2}\right) x} + c_2 e^{-i\left(\frac{1-\sqrt{3}}{2}\right) x}$

Ans. : (c)

Explanation:

$$(2D^2 - 2D - 1)\, y = 0$$
A. E. is
$$2D^2 - 2D - 1 = 0$$
$$D = \frac{2 \pm \sqrt{(-2)^2 - 4(2)(-1)}}{2(1)}$$
$$D = \frac{2 \pm \sqrt{12}}{2}$$
$$D = 1 \pm \sqrt{3}$$
$$D = 1 + \sqrt{3} \text{ and } 1 - \sqrt{3}$$
$$CF = c_1\, e^{(1+\sqrt{3})x} + c_2\, e^{(1-\sqrt{3})x}$$
Solution is
$$y = c_1 e^{(1+\sqrt{3})x} + c_2 e^{(1-\sqrt{3})x}$$

11. The complimentary function of $(D^4 + 8D^2 + 16)\, y = 0$ with $D \equiv \frac{d}{dx}$ is

(a) $(c_1 + c_2\, x)\, e^{2x} + (c_3 + c_4\, x)\, e^{-2x}$

(b) $(c_1 + c_2\, x) \cos 2x + (c_3 + c_4\, x) \sin 2x$

(c) $(c_1 + c_2\, x)\, e^{2x} + c_3 \cos 2x + c_4 \sin 2x$

(d) $(c_1 \cos 2x + c_2 \sin 2x)^2$

Ans. : (b)

Explanation :

$$D^2 + 8D^2 + 16 = 0$$

$$(D^2 + 4)^2 = 0D^2 + 4=0$$

And $$D^2 + 4 = 0$$

$$D = \pm 2i\,,\, \pm 2i \text{ (repeated imaginary roots)}$$

$$CF = (c_1 + c_2\, x) \cos 2x + (c_3 + c_4\, x) \sin 2x$$

12. The solution of $\frac{d^2 y}{dx^2} + \frac{dy}{dx} + y = 0$ is

(a) $y = c_1 \cos\left(\frac{\sqrt{3}}{2}\, x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}\, x\right)$

(b) $y = e^{-x} \left[c_1 \cos\left(\sqrt{3}\, x\right) + c_2 \sin\left(\sqrt{3}\, x\right)\right]$

(c) $y = e^{\frac{-x}{2}} \left[c_1 \cos\left(\frac{\sqrt{3}}{2}\, x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}\, x\right)\right]$

(d) $y = c_1\, e^{\frac{\sqrt{3}}{2} x} + c_2\, e^{-\frac{\sqrt{3}}{2} x}$

Ans. : (c)

Explanation :

$$D^2 + D + 1 = 0$$

$$D = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$D = \frac{1 - \pm i\sqrt{3}}{2}$$

$$D = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$CF = e^{\frac{-1}{2} x} \left[c_1 \cos\left(\frac{\sqrt{3}}{2}\, x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}\, x\right)\right]$$

So solution is $$y = e^{\frac{-1}{2} x} \left[c_1 \cos\left(\frac{\sqrt{3}}{2}\, x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}\, x\right)\right]$$

13. A solution of differential equation $\frac{d^2 y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$ is

(a) $y = e^{2x} + e^{-3x}$ (b) $y = e^{2x} + e^{3x}$

(c) $y = e^{-2x} + e^{3x}$ (d) $y = e^{-2x} + e^{-3x}$

Ans : (b)

Explanation :

A.E. is
$$D^2 - 5D + 6 = 0$$
$$(D - 3)(D - 2) = 0$$
$$D = 3, 2$$
So solution is
$$y = c_1 e^{3x} - c_2 e^{3x}$$

14. For $\phi(D)\, y = F(x)$; $D \equiv \frac{d}{dx}$ particular Integral is given by

(a) $\frac{1}{f(D)} F(x)$ (b) $\phi(D) F(x)$ (c) $\frac{1}{F(x)} \phi(D)$ (d) $F(x) \frac{1}{f(D)}$

Ans . : (a)

Explanation : By definition $\quad$ $\text{P.I.} = \frac{1}{f(D)} F(x)$

15. For $\frac{d^2 y}{dx^2} + 4\frac{dy}{dx} + 3y = 3e^{2x}$, particular integral is

(a) $\frac{1}{15} e^{2x}$ (b) $\frac{1}{5} e^{2x}$ (c) $3e^{2x}$ (d) $c_1 e^{-x} + c_2 e^{-3x}$

Ans. : (b)

Explanation :

Use $\quad D \equiv \frac{d}{dx}$ in given D.E.

$$(D^2 + 4D + 3)\, y = 3e^{2x}$$

$$\text{Particular Integral} = \frac{1}{f(D)} F(x)$$
$$= \frac{1}{D^2 + 4D + 3} 3e^{2x}$$
$$= 3\frac{1}{D^2 + 4D + 3} e^{2x}$$
$$= 3\frac{1}{(3)^2 + 4(2) + 3} e^{2x}$$
$$= \frac{1}{5} e^{2x}$$

16. For $\frac{d^2 y}{dx^2} - 6\frac{dy}{dx} + 8y = 5e^{4x}$, particular integral is

(a) 0 (b) $\frac{5}{2}e^{4x}$ (c) $\frac{5}{2}xe^{4x}$ (d) $\frac{5}{2}$

Ans. : (c)

Explanation :

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2 - 6D + 8} 5e^{4x} \\
&= 5\frac{1}{(4)^2 - 6(4) + 8} e^{4x} \\
&= 5\frac{1}{0} e^{4x} \\
&= 5x\frac{1}{2D - 6} e^{4x} \\
&= 5x\frac{1}{2(4) - 6} e^{4x} = \frac{5}{2} x e^{4x}
\end{aligned}$$

17. For $(D^2 + 9) y = \sin 2x$, particular integral is

(a) $\frac{1}{13}\sin 2x$ (b) $\frac{1}{13}\cos 2x$

(c) $\frac{1}{5}\sin 2x$ (d) $-4 \sin 2x$

Ans. : (c)

Explanation :

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2 + 9} \sin 2x \\
&= \frac{1}{-4 + 9} \sin 2x \\
&= \frac{1}{5} \sin 2x
\end{aligned}$$

18. The differential equation $\frac{d^2 y}{dx^2} + \frac{dy}{dx} + \sin y = 0$ is

(a) linear (b) non–linear

(c) homogeneous (d) of degree 2

Ans. : (b)

Explanation : D.E. is nonlinear because of sin y

19. Which one of the following does not satisfy the differential equation $\frac{d^3 y}{dx^3} - y = 0$?

(a) $y = e^{x}$ (b) $y = e^{-x}$

(c) $y = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2}x\right)$ (d) $y = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x\right)$

Ans. : (b)

Explanation : $y = e^{-x}$ satisfies the given D.E. while remaining does not satisfies the D.E.

20. Which of the following is solution of the differential equation $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + (q-1)y = 0$ for $p = 4$, $q = 5$

(a) $y = e^{-3x}$ (b) $y = x e^{-x}$

(c) $y = x e^{-2x}$ (d) $y = x^2 e^{-2x}$

Ans. : (c)

Explanation :

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$$

$$(D^2 + 4D + 4)\,y = 0$$

A.E. is $(D+2)^2 = 0$

$D = -2, -2$

Solution is $y = (c_1 + c_2 x)\,e^{-2x}$

Taking $c_1 = 0$ and $c_2 = 1$

We get $y = x e^{-2x}$

21. Which of the following is a solution of the differential equation $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + (q-1)y = 0$ for $p = 4$, $q = 4$

(a) $y = e^{3x} + e^{x}$ (b) $y = e^{-3x} + e^{-x}$

(c) $y = e^{3x}$ (d) $y = e^{x}$

Ans. : (b)

Explanation :

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 0$$

A.E. is $(D^2 + 4D + 3) = 0$

$(D+3)(D+1) = 0$

$D = -3, -1$

Solution is $y = c_1 e^{-3x} + c_2 e^{-x}$

By taking $c_1 = c_2 = 1$

We get $y = e^{-3x} + e^{-x}$

22. For $(D^2 + D + 1)\ y = \cos 2x$, the particular integral is

(a) $\frac{1}{13}\ (3 \cos 2x - 2 \sin 2x)$ (b) $\frac{-1}{13}(3 \cos 2x + 2 \sin 2x)$

(c) $-\frac{1}{13}\ (3 \cos 2x - 2 \sin 2x)$ (d) $\frac{1}{13}\ (3 \cos 2x + 2 \sin 2x)$

Ans. : (c)

Explanation :

$$\begin{aligned} \text{P.I.} &= \frac{1}{f(D)}\ F(x) \\ &= \frac{1}{D^2 + D + 1} \cos 2x \\ &= \frac{1}{-4 + D + 1} \cos 2x \\ &= \frac{1}{D - 3} \cos 2x \\ &= \frac{(D + 3)}{(D - 3)(D + 3)} \cos 2x \\ &= \frac{(D + 3)}{(D^2 - 9)} \cos 2x \\ &= \frac{(D + 3)}{-4 - 9} \cos 2x \\ &= \frac{1}{-13}\ (-2 \sin 2x + 3 \cos 2x) \end{aligned}$$

23. Particular Integral of $(D^2 - 2D + 1)\ y = e^x$ is

(a) $\frac{x^2}{2}\ e^x$ (b) $\frac{x}{2}\ e^x$

(c) $\frac{x}{4} e^{x^2}$ (d) $\frac{x^2}{5}\ e^{2x}$

Ans. : (a)

Explanation :

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 - 2D + 1}\ e^x \\ &= \frac{1}{1 - 2 + 1}\ e^x \\ &= \frac{1}{0}\ e^x \\ &= x \frac{1}{2D - 2}\ e^x \end{aligned}$$

$$= x\frac{1}{0}e^x$$

$$= x^2\frac{1}{2}\ e^x$$

24. Particular Integral of $(D^2 + 4)\, y = \sin 3x$ is

(a) $-\frac{1}{10}\sin 3x$ (b) $\frac{1}{2}\sin 3x$

(c) $\frac{1}{3}\sin 2x$ (d) $-\frac{1}{5}\sin 5x$

Ans. : (d)

Explanation :

$$\text{P.I.} = \frac{1}{D^2 + 4}\sin 3x$$

$$= \frac{1}{-9+4}\sin 3x$$

$$= \frac{-1}{5}\sin 3x$$

25. The particular Integral of $\frac{d^2 y}{dx^2} + \frac{dy}{dx} + y = \sin^2 x$ is

(a) $\frac{1}{2} - \frac{2}{13}(2\sin 2x - 2\cos 2x)$ (b) $\frac{1}{2} + \frac{2}{13}(2\sin 2x - 3\cos 2x)$

(c) $\frac{1}{2} + \frac{1}{26}(-2\sin 2x + 3\cos 2x)$ (d) $\frac{1}{2} + \frac{2}{13}(2\cos 2x - 3\sin 2x)$

Ans. :(c)

Explation :

$$\text{P.I.} = \frac{1}{D^2 + D + 1}\sin^2 x$$

$$= \frac{1}{D^2 + D + 1}\left[\frac{1}{2} - \frac{1}{2}\cos 2x\right]$$

$$= \frac{1}{2}\frac{1}{D^2 + D + 1}e^{0x} - \frac{1}{2}\frac{1}{D^2 + D + 1}\cos 2x$$

$$= \frac{1}{2}\frac{1}{0+0+1}\ e^{0x} - \frac{1}{2}\frac{1}{-4+D+1}\cos 2x$$

$$= \frac{1}{2} - \frac{1}{2}\frac{1}{D-3}\cos 2x$$

$$= \frac{1}{2} - \frac{1}{2}\frac{(D+3)}{(D-3)\,(D+3)}\cos 2x$$

$$= \frac{1}{2} - \frac{1}{2}\frac{D+3}{D^2 - 9}\cos 2x$$

$$= \frac{1}{2} - \frac{1}{2}\frac{D+3}{-4-9}\cos 2x$$

$$= \frac{1}{2} + \frac{1}{26}\,(-2\sin 2x + 3\cos 2x)$$

26. Particular Integral of $(D^2 + a^2)\,y = \cos ax$ is

(a) $\frac{-x}{2a}\cos ax$ (b) $\frac{1}{2a}\sin ax$

(c) $\frac{1}{2a}\sin ax$ (d) $\frac{x}{2a}\sin ax$

Ans. : (d)

Explanation :

$$\text{P.I.} = \frac{1}{D^2 + a^2}\cos ax$$

$$= \frac{1}{-a^2 + a^2}\cos ax$$

$$= x\frac{1}{2D}\cos ax$$

$$= x\frac{1}{2}\int \cos ax\,dx$$

$$= x\frac{1}{2}\frac{\sin ax}{a}$$

27. $\frac{1}{f(D)}\,x^n \sin ax$ is equal to

(a) $e^{iax}\frac{1}{f(D+ia)}\,x^n$ (b) Imaginary part of $e^{iax}\frac{1}{f(D+ia)}\,x^n$

(c) Real part of $e^{iax}\frac{1}{f(D+ia)}\,x^n$ (d) None of these

Ans. : (b)

Explanation :

$$e^{iax} = \cos ax + i\sin ax$$

$$x^n\,e^{i\,ax} = x^n\cos ax + i\,x^n\sin ax$$

$$\text{Im}\,(x^n e^{i\,ax}) = x^n\sin ax$$

28. $\frac{1}{f(D)}\,x^n\cos ax$ is equal to

(a) real part of $e^{iax}\frac{1}{f(D+ia)}\,x^n$ (b) imaginary part of $e^{iax}\frac{1}{f(D+ia)}\,x^n$

(c) $e^{iax}\frac{1}{f(D+ia)}\,x^n$ (d) None of these

Ans. : (a)

Explanation : From above Real part of (x^h $e^{i\,ax}$) = x^n cos ax

29. $\frac{1}{D-a}\phi(x)$ is equal to

(a) $e^{ax}\int \phi(x)\, e^{x}\, dx$ (b) $\int e^{-ax}\phi(x)\, dx$

(c) $e^{ax}\int e^{-ax}\phi(x)\, dx$ (d) $e^{ax}\int \phi(x)\, dx$

Ans. : (c)

Explanation :

By formula

$$\frac{1}{D-a}\phi(x) = e^{ax}\int a^{-ax}\phi(x)\, dx$$

30. $\frac{1}{D-3i}$ $\sec^{3x}$ is equal to

(a) $e^{3ix}\int e^{-3ix}\sec 3x\, dx$ (b) $\int e^{-3ix}\sec 3x\, dx$

(c) $e^{3ix}\int e^{-3ix}\, dx$ (d) $e^{-3ix}\int e^{3ix}\sec 3x\, dx$

Ans ; (a)

Explanation : Apply the above formula

31. Solution of $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$ is

(a) $y = c_1 e^{x} + c_2 e^{2x}$ (b) $y = c_1 + (c_2 + c_3 x)\, e^{x}$

(c) $y = (c_1 + c_2 x + c_3 x^2)\, e^{x}$ (d) $y = c_1 + c_2 e^{x}$

Ans. (b)

Explanation :

$$(D^3 + 2D^2 + D)y = 0$$

A.E. is

$$D(D^2 + 2D + 1) = 0$$
$$D(D+1)^2 = 0$$
$$D = 0, -1, -1$$
$$y = c_1 e^{ox} + (c_2 + c_3 x)\, e^{-x}$$
$$y = c_1 + (c_2 + c_3\, x)\, e^{-x}$$

32. For $(D^2 - 4D + 4)\, y = x^4 e^{2x}$ the particular integral is

(a) $\frac{x^6}{120}\, e^{2x}$ (b) $\frac{x^5}{20}\, e^{2x}$

(c) $\frac{x^6}{30}\, e^{2x}$ (d) $\frac{x^6 6}{60}\, e^{2x}$

Ans. (c)

Explanation :

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2 - 4D + 4}\, x^4 e^{2x} \\
&= e^{2x} \frac{1}{(D + 2 - 2)^2}\, x^4 \\
&= e^{2x} \frac{1}{D^2}\, x^4 \\
&= e^{2x} \frac{1}{D} \frac{1}{D} x^4 \\
&= e^{2x} \frac{1}{D} \frac{x^5}{5} \\
&= e^{2x} \frac{x^6}{30}
\end{aligned}$$

33. For $(D^3 - 3D + 2)\, y = 0$, the complimentary function is

(a) $(c_1 + c_2\, x)\, e^{x} + c_3\, e^{-2x}$ (b) $(c_1 + c_2\, x)\, e^{-2x} + c_3\, e^{x}$

(c) $(c_1 + c_2\, x)\, e^{x} + c_3\, e^{-2x}$ (d) $(c_1 + c_2)\, e^{x} - c_3\, e^{-2x}$

Ans. : (c)

Explanation :

A.E is $D^2 - 3D + 2 = 0$

$D = 1$ satisfies the above

By synthetic division method

$$\begin{array}{c|cccc}
1 & 1 & 0 & -3 & 2 \\
 & & 1 & 1 & -2 \\
\hline
 & 1 & 1 & -2 & \boxed{0}
\end{array}$$

$$\begin{aligned}
(D - 1)(D^2 + D - 2) &= 0 \\
(D - 1)(D + 2)(D - 1) &= 0 \\
(D - 1)^2 (D + 2) &= 0 \\
D &= 1, 1, -2
\end{aligned}$$

34. The complimentary function for the solution of differential equation $y'' + 5y' + 6y = x^4$ is

(a) $(c_1 + c_2\, x)\, e^{3x}$ (b) $c_1\, e^{-3x} + c_2\, e^{-2x}$

(c) $c_1\, e^{3x} + c_2\, e^{2x}$ (d) $c_1\, e^{-3x} + c_2\, e^{2x}$

Ans. : (b)

Explanation :

$$(D^2 + 5D + 6)\, y = x^4$$

AE is $D^2 + 5D + 6 = 0$

$$(D+3)(D+2) = 0$$
$$D = -3, -2$$
$$CF = c_1 e^{-3x} + c_2 e^{-2x}$$

35. On putting $x = e^z$, the transformed differential equation of $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = x$ is

(a) $\frac{d^2y}{dz^2} - y = e^z$ (b) $\frac{d^2y}{dz^2} + y = e^z\ z$

(c) $\frac{dy}{dz} + y = e^z$ (d) $\frac{dy}{dz} - y = e^{z^2}$

Ans. : (b)

Explanation :

Use $D \equiv \frac{d}{dz}$, given D. E. $\Rightarrow$

$$D(D-1)y + Dy + y = e^z$$
$$(D^2+1)y = e^z$$

36. The complimentary function for the solution of differential equation $x^2\ y'' + xy' + y = \log x \sin \log x$, is

(a) $c_1 + c_2 \sin(\log x)$ (b) $c_1 \cos(\log x) + c_2 \sin(\log x)$

(c) $c_1\ {}_1\cos(\log x) + c_2$ (d) $(c_1 + c_2 x)\sin(\log x)$

Ans. : (b)

Explanation :

Put $x = e^z$ and use $D \equiv \frac{d}{dz}$, given D.E. $\Rightarrow$

$$D(D-1)y + Dy + y = z \sin z$$
$$(D^2+1)y = z \sin z$$

AE is $$(D-i)(D+i) = 0$$
$$D = \pm i$$
$$CF = c_1 \cos z + c_2 \sin z$$
$$CF = c_1 \cos \log x + c_2 \sin \log x$$

37. The solution of the differential equation $y'' + 2y' + y = 0$, $y(0) = 1$, $y'(0) = -1$, $y = y(x)$ is

(a) $y = x e^{-x}$ (b) $y = -x e^{-x}$

(c) $y = -e^{-x}$ (d) $y = e^{-x}$

Ans. : (d)

Explanation :

$$(D^2 + 2D + 1)\,y = 0$$

A.E. is $D^2 + 2D + 1 = 0$

$$(D + 1)^2 = 0$$

$$D = -1, -1$$

$$y = (c_1 + c_2 x)\, e^{-x} \quad \text{...(1)}$$

$y(0) = 1 \Rightarrow 1 = (c_1 + 0)\, e^{-0}$

$\Rightarrow \quad c_1 = 1$

$$\frac{dy}{dx} = c_2\, e^{-x} - (c_1 + c_2\, x)\, e^{-x}$$

$$y'(0) = -1 \Rightarrow$$

$$-1 = c_2 - (1 + 0)\, e^{-0}$$

$\Rightarrow \quad c_2 = 0$

Hence (1) $\Rightarrow \quad y = e^{-x}$

38. The solution of the differential equation $\frac{d^2y}{dx^2} + a^2 y = 0,\ y(0) = 0,\ y' = a$ is

(a) $y = \cos x$ (b) $y = \sin x$

(c) $y = \cos ax$ (d) $y = \sin ax$

Ans. : (d)

Explanation :

$$(D^2 + a^2)\,y = 0$$

$\Rightarrow \quad y = c_1 \cos ax + c_2 \sin ax$

$$y(0) = 0$$

$\Rightarrow \quad 0 = c_1 \cos 0 + c_2 \sin 0$

$$C_1 = 0$$

$$y' = -c_1\, a \sin ax + c_2\, a \cos ax$$

$$y'(0) = a \Rightarrow$$

$$a = -c_1\, a \sin 0 + c_2\, a \cos 0$$

$$a = c_2\, a \Rightarrow c_2 = 1$$

Hence solution is $y = \sin ax$

39. The complementary function for the solution of the differential equation

$2x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - 3y = x^3$ is

(a) $c_1 x + c_2 x^{-\frac{3}{2}}$ (b) $c_1 x^2 + c_2 x$

(c) $c_1 x^{-\frac{3}{2}} + c_2\, x^{\frac{3}{2}}$ (d) $c_1 x + c_2 x^{\frac{3}{2}}$

Ans. : (a)

Explanation :

Given is C anch y D.E.

Put $x = e^z$ and use $D \equiv \frac{d}{dz}$

Given D.E. ⇒ $2D(D-1)y + 3Dy - 3y = 3z$

$$(2D^2 + D - 3)y = e^{3z}$$

A.E. is

$$2D^2 + D - 3 = 0$$

$$(2D+3)(D-1) = 0$$

$$D = \frac{-3}{2}, 1$$

$$CF = c_1 e^z + c_2 e^{\frac{3z}{2}}$$

$$= c_1 x + c_2 x^{-\frac{3}{2}}$$

40. The solution of $2x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} - 3y = 0$ is

(a) $y = c_1 x + c_2 x^{\frac{-3}{2}}$ (b) $y = c_1 x^2 + c_2 x$

(c) $y = c_1 x^{-\frac{3}{2}} + c_2 x^{\frac{3}{2}}$ (d) $y = c_1 x + c_2 x^{\frac{3}{2}}$

Ans. : (a)

Explanation :

Put $x = e^z$ and use $D \equiv \frac{d}{dz}$

Given D. E.⇒

$$2D(D-1) + 3Dy - 3y = 0$$

$$(2D^2 + D - 3)y = 0$$

$$D = \frac{-3}{2}, 1$$

$$y = c_1 e^z + c_2 e^{-\frac{3}{2}z}$$

$$y = c_1 x + c_2 x^{-\frac{3}{2}}$$

41. For the differential equation $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8x = 0$ with initial condition $x(0) = 1$ and $\left.\frac{dx}{dt}\right|_{t=0} = 0$ the solution is

(a) $x(t) = 2e^{-6t} - e^{-2t}$ (b) $x(t) = 2e^{-2t} - e^{-4t}$

(c) $x(t) = -e^{-6t} + 2e^{-4t}$ (d) $x(t) = e^{-2t} + 2e^{-4t}$

Ans. : (b)

Explanation : The solution of given differential equation is

$$x = c_1 e^{-4t} + c_2 e^{-2t}$$

$$\frac{dx}{dt} = -4c_1 e^{-4t} - c_2 e^{-2t}$$

$$x(0) = 1$$

$$\Rightarrow \quad 1 = c_1 + c_2$$

$$\left.\frac{dx}{dt}\right|_{t=0} = 0$$

$$\Rightarrow \quad 0 = -4c_1 - 2c_2$$

Solving we get

$$c_1 = -1$$

and $c_2 = 2$

Hence solution is

$$x = -e^{-4t} + 2e^{-2t}$$

42. The general solution of $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$ is

(a) $y = Ax + Bx^2$ (b) $y = Ax + B \log x$

(c) $y = Ax + Bx^2 \log x$ (d) $y = Ax + B x \log x$

Ans. : (d)

Explanation :

Put $x = e^2$ and

Using $D = \frac{d}{dz}$, we get

$$(D^2 - 2D + 1)\, y = 0$$

$$y = (c_1 + c_2 z)\, e^z$$

$$y = c_1 x + c_2 x \log x$$

or $y = Ax + Bx \log x$

43. If e^{-x} and xe^{-x} are fundamental solutions of $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + y = 0$ is, the value of a is

(a) 1 (b) 3

(c) 2 (d) 4

Ans. : (c)

Explanation :

$y = e^{-x}$ satisfies the

D.E. $e^{-x} - a e^{-x} + e^{-x} = 0$

$$\Rightarrow \quad 2e^{-x} - a e^{-x} = 0$$

$$\Rightarrow \quad e^{-x}(2-a) = 0$$

$$\Rightarrow \quad 2-a = 0$$

$$\Rightarrow \quad a = 2$$

44. If $D \equiv \frac{d}{dz}$ and $z = \log x$ then differential equation $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 6x$ becomes

(a) $D(D-1)y = 6e^z$ (b) $D(D-1)y = 6e^{2z}$

(c) $D(D+1)y = 6e^{2z}$ (d) $D(D+1)y = 6e^z$

Ans. : (c)

Explanation :

$$D(D-1)y + 2Dy = 6e^z$$

$$(D^2 + D)y = 6e^z$$

$$D(D+1)y = 6e^z$$

45. The solution of the equation $\frac{d^2y}{dx^2} - y = k$, (k is nonzero constant) which vanishes when $x = 0$ and which tends to a finite limit as x tends to infinity, is

(a) $y = k(1 + e^{-x})$ (b) $y = k(e^{-x} - 1)$

(c) $y = k(e^x + e^{-x} - 2)$ (d) $y = k(e^x - 1)$

Ans. : (b)

46. 2, 2 and – 1 are roots of auxiliary equation of a homogeneous differential equation with constant coefficient and y and x are variables. Then differential equation is

(a) $(D^3 + 3D^2 + 4)y = 0$ (b) $(D^2 + 3D^2 - 4)y = 0$

(c) $(D^3 - 3D^2 + 4)y = 0$ (d) $(D^3 - 3D^2 - 4)y = 0$

Ans. : (c)

Explanation :

$$D = 2, 2, -1$$

$$(D-2)^z(D+1) = 0$$

$$D^3 - 3D^2 + 4 = 0$$

Respective differential equation is $(D^3 - 3D^2 + 4)y = 0$

47. –3, 4 are roots of auxiliary equation of a homogeneous differential equation with constant coefficients. Then differential equation is

(a) $(D^2 + D - 12)y = 0$ (b) $(D^2 - D - 12)y = 0$

(c) $(D^2 + D + 12)y = 0$ (d) $(D^2 + 7D + 12)y = 0$

Ans. : (b)

Explanation :

$$D = -3, 4$$

A.E. is $$(D + 3)(D - 4) = 0$$

$$D^2 - D - 12 = 0$$

The respective differential equation is

$$(D^2 - D - 12)\, y = 0$$

48. Which one of the following is not a solution of $\frac{d^2y}{dx^2} + y = 1$?

(a) $y = 1$ (b) $y = 1 + \cos x$

(c) $y = 1 + \sin x$ (d) $y = 2 + \sin x + \cos x$

Ans. : (d)

Explanation :

$$(D^2 + 1)\, y = 1$$

$$D = \pm i$$

$$CF = c_1 \cos x + c_2 \sin x$$

$$PI = \frac{1}{D^2 + 1} e^{0x}$$

$$PI = 1$$

The complete solution is

$$y = CF + PI$$

$$y = c_1 \cos x + c_2 \sin x + 1$$

Hence (a), (b), (c) are solution but (d) can not be solution.

49. The differential equation $\frac{d^2y}{dx^2} + \sin x \frac{dy}{dx} + ye^x = \sin hx$ is

(a) First order & linear (b) First order and nonlinear

(c) Second order and linear (d) Second order and nonlinear

Ans. : (c)

Explanation : Follow from definition of linear D.E.

50. If the function 1 and $-e^{-3x}$ form a fundamental system of solution of differential equation $y'' + ay' + by = 0$ then

(a) $a = 3, b = 0$ (b) $a = 0, b = -3$

(c) $a = -3, b = 0$ (d) $a = 0, b = -3$

Ans. : (a)

Explanation :

From functions 1 and $-e^{-3x}$

$$D = 0, -3$$
$$D(D+3) = 0$$
$$(D^2 + 3D)\,y = 0$$
$$D^2 y + 3Dy = 0$$
$$\Rightarrow \quad a = 3 \text{ and } b = 0$$

Test Your Knowledge

1. Which of the following is not a solution of the equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$

(a) $y = e^{2x}$ (b) $y = e^{-3x}$

(c) $y = 3e^{2x} + 5e^{-3x}$ (d) $y = c_1 e^{2x} + c_2 e^{-3x} + 1$

2. The solution of $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$ is

(a) $y = (c_1 + c_2 x)\, e^{3x}$ (b) $y = (c_1 + c_2)\, e^{3x}$

(c) $y = (c_1 + c_2 x)\, e^{-3x}$ (d) $y = (c_1 + c_2)\, e^{-3x}$

3. The solution of $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$ has the solution

(a) $c_1 e^{x} + c_2 e^{2x}$ (b) $c_1 e^{-x} + c_2 e^{2x}$

(c) $c_1 e^{x} \cos(2x + c_2)$ (d) $c_1 e^{-x} \sin(2x + c_2)$

4. Complimentary solution of $(D^4 + 2D^3 - 3D^2)\, y = x^4$ is

(a) $c_1 x + c_2 x^2 + c_3 e^{x} + c_4 e^{-3x}$ (b) $(c_1 + c_2 x) + (c_3 e^{x} + c_4 e^{-x})$

(c) $(c_1 + c_2 x) + (c_3 e^{x} + c_4 e^{-3x})$ (d) $(c_1 + c_2 x)\, e^{x} + c_3 e^{2x} + c_4 e^{-3x}$

5. Complimentary function of $(D^2 - 4)\, y = \sin 2x$ is

(a) $c_1 x^{2x} + c_2 e^{2x}$ (b) $c_1 + c_2 x)e^{2x}$

(c) $c_1 e^{2x} + c_2 e^{-2x}$ (d) $c_1 \cos 2x + c_2 \sin 2x$

6. The solution of $\frac{d^2y}{dx^2} + y = 0$ satisfying the initial conditions $y(0) = 1$, $y = \left(\frac{p}{2}\right) = 2$ is

(a) $y = 2\cos x + \sin x$ (b) $y = \cos x + 2\sin x$

(c) $y = \cos x + \sin x$ (d) $y = 2\cos x + 2\sin x$

7. If general solution of $(D^2 + 2D + 2)\, y = 0$ is $y = e^{-x}(c_1 \cos x + c_2 \sin x)$ and $y(0) = 0$; $y'(0) = 1$ then particular solution is

(a) $y = e^{x} \sin x$ (b) $y = e^{-x} \cos x$

(c) $y = e^{-x} \sin x$ (d) $y = e^{x} \cos x$

8. The solution of $(D^4 - 6D^3 + 12D^2 - 8D)y = 0$ is given by

(a) $y = a + (b + cx + dx^2)e^{2x}$ (b) $y = (b + cx + dx^2)e^{2x}$

(c) $y = a + bx + cx^2 + dx^3$ (d) $y = a + bx + cx^2 + d\,e^{2x}$

9. The particular integral of $\frac{d^2y}{dx^2} + a^2y = \sin ax$ is

(a) $-\frac{x}{2a}\cos ax$ (b) $\frac{x}{2a}\cos ax$

(c) $\frac{-ax}{2}\cos ax$ (d) $\frac{ax}{2}\cos ax$

10. The solution of the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{3x}$ is

(a) $y = ae^{x} + be^{2x} + \frac{1}{2}e^{3x}$ (b) $y = ae^{-x} + be^{-2x} + \frac{1}{2}e^{3x}$

(c) $y = ae^{x} + be^{-2x} + \frac{1}{2}e^{3x}$ (d) $y = ae^{-x} + be^{2x} + \frac{1}{2}e^{3x}$

11. The particular integral of $(D^2 + 4)y = \cos 2x$ is

(a) $\frac{x}{4}\cos 2x$ (b) $\frac{x}{4}\sin 2x$

(c) $x\cos 2x$ (d) $\frac{1}{4}\sin 2x$

12. The particular integral of $\frac{d^2y}{dx^2} + y = \sin x$ is

(a) $-\frac{x}{2}\cos x$ (b) $\frac{x}{2}\cos x$

(c) $-\frac{x}{2}\sin x$ (d) $\frac{1}{2}\sin x$

13. The particular integral of $\frac{d^2y}{dx^2} - y = \cos x$ is

(a) $-\frac{1}{2}\cos x$ (b) $\frac{1}{2}\cos x$

(c) $-\frac{1}{2}\sin x$ (d) $\frac{1}{2}\sin x$

14. The particular integral of $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x$ is

(a) $3\sin 2x + 2\cos 2x$ (b) $\frac{1}{13}(3\sin 2x + 2\cos 2x)$

(c) $13(3\sin 2x + 2\cos 2x)$ (d) $-\frac{1}{13}(3\sin 2x + 2\cos 2x)$

15. The particular integral of $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 5e^{3x}$ is

(a) $\frac{5\,e^{3x}}{18}$ (b) $\frac{e^{3x}}{36}$

(c) $5e^{3x}$ (d) $\frac{5\,e^{3x}}{36}$

16. The particular integral of $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = e^{x}$ is

(a) e^{x} (b) $\frac{e^{x}}{3}$

(c) e^{x^2} (d) $\frac{e^{x}}{2}$

17. The particular integral of $2\frac{d^2y}{dx^2} + \frac{dy}{dx} + 3y = e^{2x}$ is

(a) $\frac{e^{2}}{13}$ (b) $\frac{e^{2x}}{13}$

(c) $e^{\frac{2x}{27}}$ (d) $13\,e^{2x}$

18. The particular integral of $4\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{3x}$ is

(a) $\frac{e^{3x}}{7}$ (b) $\frac{e^{2x}}{47}$

(c) $\frac{e^{3x}}{27}$ (d) $\frac{e^{3x}}{47}$

19. The particular integral of $2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = e^{-x}$ is

(a) $-x\,e^{-x}$ (b) $x\,e^{-x}$

(c) $-x\,e^{x}$ (d) $x\,e^{x^2}$

20. The particular integral of $\frac{d^2y}{dx^2} - 4y = e^{2x}$ is

(a) $x\,e^{2x}$ (b) $2\,xe^{2x}$

(c) $\frac{x\,e^{2x}}{2}$ (d) $\frac{x\,e^{2x}}{4}$

21. The particular integral of $\frac{d^2y}{dx^2} + y = x^2$ is

(a) x^2 (b) $x^2 + 2$

(c) $x^2 - 2$ (d) None of these

22. The particular integral of $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=x^2$ is

(a) x^2-2x (b) x^2+2x

(c) x^2+2 (d) x^2-2

23. The particular integral of $2\frac{d^2y}{dx^2}-y=x^3$ is

(a) x^3+12x (b) $-(x^3+12x)$

(c) x^3+12 (d) $-x^3-12$

24. The particular integral of $\frac{d^2y}{dx^2}+y=1$ is

(a) Zero (b) x

(c) 1 (d) x^2

25. The solution of $(D^3-3D^2+4)y=1$ is

(a) $y=c_1e^x+c_2e^{2x}+c_3e^{2x}$ (b) $y=c_1e^x+(c_2+c_3x)e^{2x}$

(c) $y=c_1e^{2x}+c_2e^x+c_3e^{3x}$ (d) $y=c_1e^x+c_2e^{-2x}+c_3e^{-3x}$

26. The particular integral of $(D^2+9)y=\sin 3x$ is

(a) $-\frac{1}{3}\cos 3x$ (b) $\frac{1}{3}\cos 3x$

(c) $-\frac{x}{9}\cos 3x$ (d) $-\frac{x}{6}\cos 3x$

27. The particular integral of $(D^2+6D+5)y=4e^x$ is

(a) $\frac{1}{3}e^x$ (b) e^x

(b) $\frac{1}{2}e^x$ (d) $\frac{x}{2}e^x$

28. The solution of $y''-5y'-6y=0$ is

(a) $y=c_1e^{3x}+c_2e^{2x}$ (b) $y=e^{-3x}+e^{-2x}$

(c) $y=c_1e^{-x}+c_2e^{6x}$ (d) $y=c_1e^{3x}+c_2e^{2x}$

29. The solution of $2y''-y'-10y=0$ is

(a) $y=c_1e^{-2x}+c_2e^{\frac{5x}{2}}$ (b) $y=c_1e^{5x}+c_2e^{-2x}$

(c) $y=c_1e^{5x}+c_2e^{-x}$ (d) $y=c_1e^{-4x}+c_2e^{5x}$

30. The solution of $\frac{dx}{y^2}=\frac{dy}{x^2}=\frac{dz}{x^2+y^2}$ contains

(a) $2x=2y+c_1$ (b) $x^2=y^2+c_1$

(c) $x^3-y^3=c_1$ (d) $x-y=c_1$

31. The solution of $4\frac{d^2x}{dt^2} = -9x$ is

(a) $x = c_1 \cos 3t + c_2 \sin 3t$

(b) $x = c_1 \cos\left(\frac{3t}{2}\right) + c_2 \sin\left(\frac{3t}{2}\right)$

(c) $x = c_1 e^{3t} + c_2 e^{2t}$

(d) $x = c_1 e^{\frac{3}{2}t} + c_2 e^{-\frac{3}{2}t}$

32. If the set of multipliers for $\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y}$ is 1, 1, 1 then corresponding solution is

(a) $x^2 + y^2 + z^2 = c_1$

(b) $x + y + z = c_1$

(c) $x^3 + y^3 + z^3 = c_1$

(d) $xyz = c_1$

33. If the set of multipliers for $\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y}$ is x, y, z then respective solution is

(a) $x^2 + y^2 + z^2 = c_1$

(b) $x + y + z = c_1$

(c) $x^3 + y^3 + z^3 = c_1$

(d) $xyz = c_1$

34. If the set of multipliers for $\frac{dx}{x(2y^4 - z^4)} = \frac{dy}{y(z^4 - 2x^4)} = \frac{dz}{z(x^4 - y^4)}$ is x^3, y^3, z^3 then solution is

(a) $x^2 + y^2 + z^2 = c_1$

(b) $x + y + z = c_1$

(c) $x^3 + y^3 + z^3 = c_1$

(d) $x^4 + y^4 + z^4 = c_1$

35. If the set of multipliers for $\frac{dx}{x(2y^4 - z^4)} = \frac{dy}{y(z^4 - 2x^4)} = \frac{dz}{z(x^4 - y^4)}$ is $\frac{1}{x}, \frac{1}{y}, \frac{2}{z}$ then solution contains.

(a) $x^2 + y^2 + 2z^2 = c_1$

(b) $2x^2 + 2y^2 + z^2 = c_1$

(c) $xy^2z = c_1$

(d) $xyz^2 = c_1$

36. For the system $\frac{dx}{dt} = 3x + 8y, \frac{dy}{dt} = -x - 3y$, on using $D \equiv \frac{d}{dt}$ and eliminating x we get

(a) $(D^2 + 1)\, y = 0$

(b) $(D^2 - 1)\, y = 0$

(c) $(D^2 + 9)\, y = 0$

(d) $(D^2 - 9)\, y = 0$

37. The solution of $\frac{dx}{dt} = 3x + 8y, \frac{dy}{dt} = -x - 3y$ contains $x = -4c_1 e^t - 2c_2 e^{-t}$,

$y = c_1 e^t - c_2 e^{-t}$ particular solution subjected to the conditions $x(0) = 6y(0) = -2$ is

(a) $x = -e^t - e^{-t}$
$y = 4e^t + 2e^{-t}$

(b) $x = 4e^t + 2e^{-t}$
$y = -e^t + e^{-t}$

(c) $x = -4e^t - 2e^{-t}$
$y = e^t + e^{-t}$

(d) $x = e^t + e^{-t}$
$y = -4e^t - 2e^{-t}$

38. The particular integral of $\frac{d^2y}{dx^2} + y = 2 \cosh 3x$ is

(a) $\frac{1}{4} \cosh 2x$ (b) $\frac{1}{5} \cosh 3x$

(c) $\frac{1}{4} \sinh 3x$ (d) $-\frac{1}{4} \sinh 3x$

39. The particular integral of $(D^2 + 1)\, y = \cosh x$ is

(a) $x\sinh x$ (b) $\frac{1}{2} \cosh x$

(c) $x\cosh x$ (d) $-\frac{1}{2} \cosh x$

40. The particular integral of $(D^2 - 4)\, y = \cosh 2x$ is

(a) $\frac{1}{2}\, x \cosh 2x$ (b) $\frac{1}{2}\, x \sinh 2x$

(c) $\frac{1}{4}\, x \sinh 2x$ (d) $-\frac{1}{8} \cosh 2x$

41. The solution of $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{3x}$ is

(a) $y = c_1 e^{x} + c_2 e^{-x} + \frac{1}{2} e^{3x}$ (b) $y = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{2} e^{3x}$

(c) $y = c_1 e^{x} + c_2 e^{-x} + \frac{1}{2} e^{2x}$ (d) $y = c_1 e^{-x} + c_2 e^{2x} + \frac{1}{2} e^{-3x}$

42. The particular integral of $(D^3 - D)\, y = e^{x} + e^{-x}$ is

(a) $\frac{1}{2} (e^{x} + e^{-x})$ (b) $\frac{1}{2} x^2 (e^{x} + e^{-x})$

(c) $\frac{1}{2}\, x\, (e^{x} + e^{-x})$ (d) $\frac{1}{2} e^{x} + \frac{1}{2}\, x e^{-x}$

43. If general solution of $\frac{d^2y}{dx^2} + y = 0$ is $y = c_1 \cos x + c_2 \sin x$ and $y(0) = 1$, $y\left(\frac{p}{2}\right) = 2$ then particular solution is

(a) $\cos x + 2 \sin x$ (b) $\cos x + \sin x$

(c) $\cos x - \sin x$ (d) $2 (\cos x + \sin x)$

44. The solution of $(D^4 + 8D^2 + 16)\, y = 0$ is

(a) $y = (c_1 + c_2 x) \cos 2x + (c_3 + c_4 x) \sin 2x$

(b) $y = c_1 e^{2x} + c_2 e^{-x} + c_3 e^{x} + c_4 e^{-2x}$

(c) $y = (c_1 + c_2 x) e^{2x} + (c_3 + c_4 x) e^{-2x}$

(d) $y = (c_1 + c_2 x) \cosh 2x + (c_3 + c_4 x) \sinh 2x$

45. The particular integral of $(D^2 + 4)\, y = x$ is

(a) $x\, e^{2x}$ (b) $x \cos 2x$

(c) $x \sin 2x$ (d) $\frac{x}{4}$

46. Consider $\frac{dy}{dx} + y = e^x; y(0) = 1$. The value of y (1) is

(a) $e + e^{-1}$ (b) $\frac{1}{2}(e - e^{-1})$

(c) $\frac{1}{2}(e + e^{-1})$ (d) $2(e - e^{-1})$

47. The general solution of $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 0$ is

$y = e^{-t}(c_1 \cos t + c_2 \sin t)$ and $y(0) = 0$, $\left.\frac{dy}{dt}\right|_{t=0} = -1$ then particular solution is

(a) $-t \sin t$ (b) $-e^{-t}(1 - \cos t)$

(c) $-\left(\frac{t + \sin t}{2}\right)$ (d) $-e^{-t} \sin t$

48. For the equation $x''(t) + 3x'(t) + 2x(t) = 5$ the solution x (t) approaches which of the following values as $t \to \infty$

(a) zero (b) $\frac{5}{2}$

(c) 5 (d) 10

49. Transforming $x^2 \frac{d^3 y}{dx^3} + 2x \frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$ into linear differential equation with constant coefficients by using $x = e^z$ and $D \equiv \frac{d}{dz}$ we get

(a) $(D^3 + D^2 - 2D)\, y = 0$ (b) $(D^3 - D^2 - 2D)\, y = 0$

(c) $(D^2 - 3D - 2)\, y = 0$ (d) $(D^2 - 3D + 2)\, y = 0$

50. $y = c_1 e^{2x} + c_2 e^{3x}$ is the general solution of the differential equation

(a) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$ (b) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

(c) $\frac{d^2y}{dx^2} + 6y = 0$ (d) $\frac{d^2y}{dx^2} - 6y = 0$

51. To transform $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{1}{x}$ into a linear differential equation with constant coefficient, which of the following substitution is useful?

(a) $z = e^x$ (b) $\log z = x$

(c) $x = e^z$ (d) $\log x = z$

52. Transforming $x\frac{d^2y}{dx^2}+\frac{dy}{dx}=\frac{1}{x}$ into linear differential equation with constant coefficient by using $x = e^z$ and $D\equiv\frac{d}{dz}$ we get

(a) $(D^2 + 1)\,y = 0$ (b) $(D^2 - 1)\,y = 0$

(c) $(D^2 - 2)\,y = 1$ (d) $D^2\,y = 1$

53. The particular integral of $(D^2 - 3D + 2)\,y = 12$

(a) 12 (b) 6 (c) 3 (d) 4

54. The particular integral of $(D^2 + 6D + 9)\,y = 5^x$

(a) $\frac{1}{64}5^x$ (b) $\frac{1}{(\log 5+3)^2}5^x$

(c) $\frac{1}{(\log 5+3)^2}e^x$ (d) $\frac{1}{64}e^x$

55. The particular integral of $(D^2 + 4D + 4)y = \log 3$

(a) $\frac{1}{25}\log 3$ (b) $\frac{1}{25}e^x$

(c) $\frac{1}{4}\log 3$ (d) $\frac{1}{4}e^x$

56. Transforming legendre's differential equation $(1 + 4x)^2 y'' + (1 + 4x)\,y' + 4y = 8(1 + 4x)^2$ into linear Differential equation with constant coefficient by using $1 + 4x = e^z$ and $D\equiv\frac{d}{dz}$, we get

(a) $(16D^2 + 12D - 4)\,y = 8e^{2z}$ (b) $(16D^2 - 12D + 4)\,y = 8e^{2z}$

(c) $(16D^2 + 12D + 4)\,y = 8e^{2z}$ (c) $(4D^2 - 3D - 1)\,y = 2e^{2z}$

57. Transforming $(1+x)^3\frac{d^2y}{dx^2}+3(x+1)^2\frac{dy}{dx}+(1+x)y=6\log(x+1)$ into linear differential equation with constant coefficient by using $1 + x = e^z$ and $D\equiv\frac{d}{dz}$ we get

(a) $(D^2 + 2D - 1)\,y = 6z\,e^{-z}$ (b) $(D^2 + 2D + 1)\,y = 6z\,e^z$

(c) $(D^2 + 2D + 1)\,y = 6z\,e^{-z}$ (d) $(D^2 - 2D + 1)\,y = 6z\,e^z$

58. Transforming $(x - 2)^2 y'' - 3(x - 2)\,y' + 4y = x$ into linear differential equation with constant coefficient by using $x - 2 = e^z$ and $D\equiv\frac{d}{dz}$ we get

(a) $(D^2 - 4D + 4)\,y = e^z + 2$ (b) $(D^2 + 4D + 4)\,y = e^z - 2$

(c) $(D^2 - 4D + 4)\,y = e^z - 2$ (d) $(D^2 + 4D + 4)y = e^z$

59. Complimentary function of $(D - 1)^2 (D^2 + 1)\,y = e^x + \sin 2\frac{x}{2}$ is

(a) $(c_1 + c_2 x)\,e^x + c_3\cos x + c_4\sin x$ (b) $c_1 e^x + c_2\cos x + c_3\sin x$

(c) $c_1 e^x + c_2 e^x + c_3 e^{-x}$ (d) $(c_1 + c_2 x)\,e^{-x} + c_3\cos x + c_4\sin x$

60. The particular solution of $(D^2 + 2D + 1)\, y = x$ is

(a) $x + 2$ (b) $x - 2$ (c) x (d) $-x$

61. Transforming $(4x^2 D^2 + 16x D + 9)\, y = 0$ into linear D.E. with constant coefficients, we get the differential equation

(a) $\frac{d^2y}{dz^2} + 9y = 0$ (b) $\frac{d^2y}{dz^2} - 9y = 0$

(c) $4\frac{d^2y}{dz^2} + 12\frac{dy}{dz} + 9y = 0$ (d) $\frac{d^2y}{dz^2} + 3\frac{dy}{dz} + 3y = 0$ where $z = \log x$

62. Transforming Legendre's differential equation $(2+3x)^2 \frac{d^2y}{dx^2} + 3\,(2 + 3x)\frac{dy}{dx} - 36y = 0$ into linear D.E. with constant coefficients by using $2 + 3x = e^z$ and $D \equiv \frac{d}{dz}$ we get

(a) $(D^2 - 4)\, y = 0$ (b) $(D^2 + 2D - 36)\, y = 0$

(c) $(D^2 + 4D - 36)\, y = 0$ (d) $(9D^2 + 3D - 36)\, y = 0$

63. Wronskian of e^x and e^{-x} is

(a) 0 (b) 2

(c) -2 (d) e^{2x}

64. The complimentary function of $x^2 y'' - x y' + 2y = x \log x$ is

(a) $c_1 \cos \log x + c_2 \sin x$ (b) $x\,(c_1 \cos \log x + c_2 \sin \log x$

(c) $c_1 \cos \log x + c_2 \sin \log x$ (d) $(c_1 \cos \log x + c_3 \sin x)/x$

65. The complimentary function of $(D^4 - a^4)\, y = F(x)$ is

(a) $(c_1 + c_2 x)\, e^{ax} + (c_3 + c_4 x)\, e^{-ax}$

(b) $c_1 e^{a^2x} + c_2 e^{-a^2x}$

(c) $c_1 e^{ax} + c_2 e^{-ax} + c_3 \cos ax + c_4 \sin ax$

(d) $(c_1 + c_2 x) \cos ax + (c_3 + c_4 x) \sin ax$

66. The homogeneous linear differential equation whose auxiliary equation has roots 1, – 1 is

(a) $(D^2 - 2D + 1)\, y = 0$ (b) $(D^2 + 2D + 1)\, y = 0$

(c) $(D^2 + 1)\, y = 0$ (d) $(D^2 - 1)\, y = 0$

67. The homogeneous linear differential equation whose auxiliary equation has roots $-1 \pm i$ is

(a) $(D^2 - 2D + 1)\, y = 0$ (b) $(D^2 - 2D - 2)\, y = 0$

(c) $(D^2 + 2D + 2)\, y = 0$ (d) $(D^2 + 2D + 1)\, y = 0$

68. The homogenous linear differential equation whose auxiliary equation has roots 0, –1, – 1 is

(a) $(D^3 - 2D^2 + D)\, y = 0$ (b) $(D^3 + 2D^2 + D)\, y = 0$

(c) $(D^2 - 2D + 1)\, y = 0$ (d) $D\,(D - 1)\, y = 0$

69. The particular integral of $(D + 1)^2 y = e^{-x}$ is

(a) e^{-x} (b) $\frac{1}{2} e^{-x}$

(c) $-\frac{1}{2} xe^{-x}$ (d) $\frac{1}{2} x^2 e^{-x}$

70. The solution of $(D^2 + 1)^2 y = 0$ with $D \equiv \frac{d}{dx}$ is

(a) $y = c_1 \cos x + c_2 \sin x$

(b) $y = c_1 e^x + c_2 e^{-x}$

(c) $y = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x$

(d) $y = (c_1 + c_2\ x) e^x + (c_3 + c_4 x) e^{-x}$

71. If $f(D) = D^2 + 1$ then $\frac{1}{f(D)} \cosh 3x$ is equal to

(a) $\frac{1}{4} \cosh 3x$ (b) $\frac{1}{10} \cosh 3x$

(c) $\frac{-1}{8} \cosh 3x$ (d) $\frac{1}{5} \cosh 3x$

72. If $f(D) = D^2 + 5$, then $\frac{1}{f(D)} \sin 2x$ is equal to

(a) $\cos 2x$ (b) $\frac{1}{2} x \sin 2x$

(c) $\sin 2x$ (d) $\frac{1}{9} \sin 2x$

73. The particular integral of $(D^2 + 36)\, x = 4 \cos 6t$ is

(a) $\frac{\sin 6t}{3}$ (b) $\frac{-\cos 6t}{3}$

(c) $\frac{t \sin 6t}{3}$ (d) $\frac{4t \cos 6t}{3}$

74. $(x^2 D^2 + x D + 7)\, y = \frac{2}{x}$ converted form to a linear differential equation with constant coefficient is

(a) $\frac{d^2y}{dz^2} + 7y = 2e^{-z}$ where $x = e^z$ (b) $\frac{d^2y}{dz^2} + 7y = 2e^z$ where $x = e^z$

(c) $\frac{d^2y}{dz^2} + 2\frac{dy}{dz} + 7y = 2e^{-z}$ where $x = e^z$ (d) $\frac{d^2y}{dz^2} + 2\frac{dy}{dz} + 7y = 2e^z$ where $x = e^z$

75. If particular integral of $(D^2 - 4)\, y = \sin 2x$ is $k \sin 2x$ then value of k is

(a) $\frac{-1}{8}$ (b) $\frac{x}{4}$

(c) $\frac{-x}{4}$ (d) $\frac{-1}{4}$

76. The Wronskian of y_1 and y_2 is defined as

(a) $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ (b) $\begin{vmatrix} y_1 & y_2 \\ y_2 & y_1 \end{vmatrix}$

(c) $\begin{vmatrix} y_1 & y_2 \\ y_2' & y_1' \end{vmatrix}$ (d) $\begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix}$

77. The Wronskian W of x and e^x is

(a) $x\, e^2 + e^x$ (b) $e^x\, x - x\, e^x$

(c) $x\, e^x - e^x$ (d) $x - e^{2x}$

78. If particular integral of $(D^3 + 4D)\, y = \sin 2\, x$ is $k\, x \sin 2x$, then value of k is

(a) $-\frac{1}{8}$ (b) $-\frac{1}{4}$

(c) $-\frac{1}{4x}$ (d) $\frac{x}{4}$

79. With usual notations, the value of $\frac{1}{D^2 - 2D + 4}\, e^x \cos x$ is

(a) $\frac{1}{4} e^x \cos x$ (b) $\frac{1}{2} e^x \cos x$

(c) $\frac{1}{2} x\cos x$ (d) $\frac{1}{4} x\, e^x$

80. The Wronskian of e^x and $e^x x$ is

(a) e^x (b) $e^{2x}\, (2x + 1)$

(c) e^{2x} (d) $e^x\, (2x + 1)$

Answers Key

1 (d)	2 (a)	3 (c)	4 (c)	5 (c)	6 (b)	7 (c)	8 (a)	9 (a)	10 (a)
11 (b)	12 (a)	13 (a)	14 (d)	15 (d)	16 (b)	17 (b)	18 (d)	19 (b)	20 (d)
21 (c)	22 (a)	23 (b)	24 (c)	25 (b)	26 (d)	27 (a)	28 (c)	29 (a)	30 (c)
31 (b)	32 (b)	33 (a)	34 (d)	35 (d)	36 (b)	37 (b)	38 (b)	39 (b)	40 (c)
41 (a)	42 (c)	43 (a)	44 (a)	45 (d)	46 (c)	47 (d)	48 (b)	49 (b)	50 (b)
51 (c)	52 (d)	53 (b)	54 (b)	55 (c)	56 (b)	57 (c)	58 (a)	59 (a)	60 (b)
61 (c)	62 (a)	63 (c)	64 (b)	65 (c)	66 (d)	67 (c)	68 (b)	69 (d)	70 (c)
71 (b	72 (c)	73 (c)	74 (a)	75 (a)	76 (a)	77 (c)	78 (a)	79 (b)	80 (c)

◆◆◆

Unit 2 Transforms

Fourier Transform

1. If f(x) is defined in the interval $-\infty < x < \infty$ and $F(\lambda)$ is Fourier transform of f(x) then.

(a) $F(\lambda) = \int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x}\, dx$

(b) $F(\lambda) = \int_{-\infty}^{\infty} f(x)\, e^{i\lambda x}\, dx$

(c) $F(\lambda) = \int_{0}^{\infty} e^{-i\lambda x}\,{}^{-iyx}\, dx$

(d) $F(\lambda) = \int_{0}^{\infty} f(x)\, e^{i\lambda x} dx$

Ans.: (a)

Explanation : Standard definition

2. If $F(\lambda)$ is Fourier transform of f(x) then.

(a) $f(x) = \frac{1}{2\pi}\int_{0}^{\infty} F(\lambda)\, e^{i\lambda x}\, d\lambda$

(b) $f(x) = \frac{1}{2\pi}\int_{0}^{\infty} F(\lambda) e^{-i\lambda x}\, d\lambda$

(c) $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x}\,{}^{-i\lambda x}\, d\lambda$

(d) $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda) e^{-i\lambda x}\, d\lambda$

Ans.: (c)

Explanation : Standard definition

3. If Fc (λ) is Fourier cosine transform of function f(x) then.

(a) $F_c(\lambda) = \int_{0}^{\infty} f(x) \cos\lambda x\, d\lambda$

(b) $F_c(\lambda) = \int_{0}^{\infty} f(x) \cos\lambda x\, d\lambda$

(c) $F_c(\lambda) = \int_{0}^{\infty} f(x) \sin \lambda x\, d\lambda$

(d) $F_c(\lambda) = \int_{0}^{\infty} f(x) \sin \lambda x\, dx$

Ans. : (b)

Explanation : Standard definition

4. If $F_s(\lambda)$ is Fourier sine transform of function f(x) then.

(a) $F_s(\lambda) = \int_0^{\infty} f(x) \cos\lambda x \, d\lambda$ (b) $F_s(\lambda) = \int_0^{\infty} f(x) \cos\lambda x \, dx$

(c) $F_s(\lambda) = \int_0^{\infty} f(x) \sin \lambda x \, d\lambda$ (d) $F_c(\lambda) = \int_0^{\infty} f(x) \sin \lambda x \, dx$

Ans. : (d)

Explanation : Standard definition

5. If $F_c(\lambda)$ is Fourier consine transform of function f(x) then.

(a) $f(x) = \frac{2}{\pi}\int_0^{\infty} F_c(\lambda) \cos\lambda x \, dx$ (b) $f(x) = \frac{2}{\pi}\int_0^{\infty} F_c(\lambda) \cos\lambda x \, d\lambda$

(c) $f(x) = \frac{2}{\pi}\int_0^{\infty} F_s(\lambda) \cos\lambda x \, d\lambda$ (d) $f(x) = \frac{2}{\pi}\int_0^{\infty} F_s(\lambda) \sin \lambda x \, d\lambda$

Ans. : (b)

Explanation : Standard definition

6. If $F_s(\lambda)$ is Fourier sine transform of function f(x) then.

(a) $f(x) = \frac{2}{\pi}\int_0^{\infty} F_c(\lambda) \cos\lambda x \, dx$ (b) $f(x) = \frac{2}{\pi}\int_0^{\infty} F_c(\lambda) \cos\lambda x \, d\lambda$

(c) $f(x) = \frac{2}{\pi}\int_0^{\infty} F_s(\lambda) \sin \lambda x \, dx$ (d) $f(x) = \frac{2}{\pi}\int_0^{\infty} F_s(\lambda) \sin \lambda x \, d\lambda$

Ans. : (d)

Explanation : Standard definition

7. The Fourier transform of $f(x) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$

(a) $\frac{2 \sin \lambda}{\lambda}$ (b) $\frac{\sin \lambda}{\lambda}$ (c) $-\frac{2 \sin \lambda}{\lambda}$ d) $-\frac{\sin \lambda}{\lambda}$

Ans. : (a)

Explanation : f(x) is even function

$$F(\lambda) = 2\int_0^{\infty} f(x) \cos\lambda \, x \, dx$$

$$= 2\int_0^{1} \cos\lambda \, x \, dx + 2\int_1^{\infty} 0 \cos\lambda x \, dn$$

$$= 2\left[\frac{\sin\lambda x}{\lambda}\right]_0^1 + 0$$

$$= \mathbf{\frac{2\sin\lambda}{\lambda}}$$

8. The Fourier cosine transform of

$f(x) = \begin{cases} 1 & \text{for } 0 \le x < 1 \\ 0 & \text{for } x > 1 \end{cases}$ is

(a) $\frac{2\sin\lambda}{\lambda}$ (b) $\frac{\sin\lambda}{\lambda}$ (c) $-\frac{2\sin\lambda}{\lambda}$ (d) $-\frac{\sin\lambda}{\lambda}$

Ans. : (b)

Explanation :

$$Fc(\lambda) = \int_0^\infty f(x)\cos\lambda x\,dx$$

$$= \int_0^1 \cos\lambda x\,dx + \int_1^\infty 0\,dx$$

$$= \left[\frac{\sin\lambda x}{\lambda}\right]_0^1 + 0 = \mathbf{\frac{\sin\lambda}{\lambda}}$$

9. For

$f(x) = \begin{cases} 1 & \text{for } 0 \le x < 1 \\ 0 & \text{for } x > 1 \end{cases}$, if $F_c(\lambda) = \frac{\sin\lambda}{\lambda}$ their value of integral $\int_0^\infty \frac{\sin\lambda}{\lambda}\,d\lambda$ is

(a) π (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) 2π

Ans. : (c)

Explanation : By definition of inverse cosine transform

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\lambda)\cos\lambda x\,d\lambda$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda}{\lambda}\cos\lambda x\,d\lambda$$

Put $x = 0$ in the above

$$f(0) = \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda}{\lambda}\cos 0\,d\lambda$$

$$1 = \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda}{\lambda}\,d\lambda$$

$$\mathbf{\int_0^\infty \frac{\sin\lambda}{\lambda}\,d\lambda = \frac{\pi}{2}}$$

10. The Fourier sine transform of

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x < 1 \\ 0 & \text{for } x > 1 \end{cases} \text{ is}$$

(a) $\dfrac{\cos\lambda}{\lambda}$ (b) $\dfrac{2\cos\lambda}{\lambda}$ (c) $\dfrac{1+\cos\lambda}{\lambda}$ (d) $\dfrac{1-\cos\lambda}{\lambda}$

Ans. : (d)

Explanation :

$$\begin{aligned}
F_s(\lambda) &= \int_0^\infty f(x)\sin\lambda x\,dx \\
&= \int_0^1 \sin\lambda x\,dx + \int_1^\infty 0\,dx \\
&= \left[\frac{-\cos\lambda x}{\lambda}\right]_0^1 \\
&= \left[\frac{-\cos\lambda}{\lambda} + \frac{\cos 0}{\lambda}\right] \\
&= \frac{-\cos\lambda + 1}{\lambda}
\end{aligned}$$

11. The Fourier cosine transform of $f(x) = e^{-2x}, x \ge 0$ is

(a) $\dfrac{1}{\lambda^2+4}$ (b) $\dfrac{2}{\lambda^2+4}$ (c) $\dfrac{\lambda}{\lambda^2+4}$ (d) $\dfrac{2\lambda}{\lambda^2+4}$

Ans. :(b)

Explanation :

$$\begin{aligned}
F_c(\lambda) &= \int_0^\infty f(x)\cos\lambda x\,dx \\
&= \int_0^\infty e^{-2x}\cos\lambda\,x\,dx \\
&= \left\{\frac{e^{-2x}}{(-2)^2+\lambda^2}\,[-2\cos\lambda x + \lambda\sin\lambda x]\right\}_{x=0}^{x=\infty} \\
&= \left\{0 - \frac{e^{-0}}{4+\lambda^2}[-2\cos 0 + \lambda\sin 0]\right\} \\
&= \mathbf{\frac{2}{4+\lambda^2}}
\end{aligned}$$

12. The Fourier cosine transform of $f(x) = e^{-4x}, x \ge 0$ is.

(a) $\dfrac{1}{\lambda^2+16}$ (b) $\dfrac{\lambda}{\lambda^2+16}$ (c) $\dfrac{4}{\lambda^2+16}$ (d) $\dfrac{4\lambda}{\lambda^2+16}$

Ans. : (c)

Explanation :

$$F_c(\lambda) = \int_0^{\infty} f(x) \cos\lambda x \, dx$$

$$= \int_0^{\infty} e^{-4x} \cos\lambda x \, dx$$

$$= \left\{ \frac{e^{-4x}}{(-4)^2 + \lambda^2} [-4 \cos \lambda x + \lambda \sin \lambda x] \right\}_{x=0}^{x=\infty}$$

$$= \left\{ 0 - \frac{e^{-0}}{16 + \lambda^2} [-4 \cos 0 + \lambda \sin 0] \right\}$$

$$= \frac{\mathbf{4}}{\boldsymbol{\lambda 2 + 16}}$$

13. The Fourier sine transform of $f(x) = e^{-kx}$ (where $x \geq 0$ and k is any real positive real number is

(a) $\frac{1}{\lambda^2 + k^2}$ (b) $\frac{1}{\lambda^2 + k^2}$ (c) $\frac{\lambda}{\lambda^2 + k^2}$ (d) $\frac{\lambda k}{\lambda^2 + k^2}$

Ans. : (c)

Explanation :

$$F_s(\lambda) = \int_0^{\infty} f(x) \sin \lambda x \, dx$$

$$= \int_0^{\infty} e^{-kx} \sin \lambda x \, dx$$

$$= \left\{ \frac{e^{-kx}}{k^2 + \lambda^2} [-k \sin \lambda x - \lambda \cos \lambda x] \right\}_{x=0}^{x=8}$$

$$= \frac{\boldsymbol{\lambda}}{\boldsymbol{\lambda^2 + 4^2}}$$

14. The Fourier sine transform of $\frac{1}{x}$ is

(a) π (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) 2π

Ans. : (b)

Explanation :

$$F_s(\lambda) = \int_0^{\infty} f(x) \sin \lambda x \, dx$$

$$= \int_0^\infty \frac{1}{x} \sin \lambda x \, dx \quad \text{.... (1)}$$

$$\text{Put} \quad \lambda x = t \ \therefore \ x = \frac{1}{\lambda} t$$

$$dx = \frac{1}{\lambda} dt$$

$$(1) \Rightarrow \quad F_s(\lambda) = \int_0^\infty \frac{\lambda}{t} \sin t \frac{dt}{\lambda}$$

$$= \int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$$

15. If Fourier sine transform of $\frac{e^{-ax}}{x}$ (a> 0) is $\tan^{-1}\frac{\lambda}{a}$ then the value of integral $\int_0^\infty \tan^{-1}\frac{x}{a}$ sin x dx is

(a) $\frac{\pi}{2} e^{-a}$ (b) $\frac{2}{\pi} e^{-a}$ (c) $\frac{\pi}{2} e^{a}$ d) $\frac{2}{\pi} e^{a}$

Ans. :(a)

Explanation :

$$f(x) = \frac{e^{-ax}}{x}$$

$$F_s(\lambda) = \tan^{-1}\frac{\lambda}{a}$$

By definition

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(\lambda) \sin \lambda x \, dx$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \tan^{-1}\frac{\lambda}{a} \sin \lambda x \, d\lambda$$

$$\int_0^\infty \tan^{-1}\frac{\lambda}{a} \sin \lambda x \, d\lambda = \frac{\pi}{2} f(x)$$

Put x = 1 in above

$$\int_0^\infty \tan^{-1}\frac{\lambda}{a} \sin \lambda \, d\lambda = \frac{\pi}{2} f(1)$$

$$\int_0^\infty \tan^{-1}\frac{x}{a} \sin x \, dx = \frac{\pi}{2} e^{-a}$$

16. If $\int_0^\infty F_s(\lambda)\sin\lambda x\, d\lambda = \frac{\pi}{2} e^{-x}\cos x,\ x > 0$, then the inverse sine transformation of $F_s(\lambda)$ is

(a) $\frac{\pi}{2} e^{-x}\cos x$ (b) $\frac{\pi}{2} e^{-x}\sin x$ (c) $e^{-x}\cos x$ (d) $e^{-x}\sin x$

Ans .: (c)

Explanation :

The inverse sine transform of $F_s(\lambda)$ is f(x) and it is given by $f(x) = \frac{2}{\pi}\int_0^\infty F_s(\lambda)\sin\lambda x\, d\lambda$ comparing given equation with the above, we get $f(x) = e^{-x}\cos x$.

17. If $\int_0^\infty F_s(\lambda)\sin\lambda x\, d\lambda = \frac{\pi}{2}$ for $0 < x < \pi = 0$ for $x > \pi$ then the inverse sine transformation of $F_s(\lambda)$ is

(a) $f(x) = \frac{\pi}{2}\sin x$ for $0 < x < \pi$

$= 0$ for $x > \pi$

(b) $f(x) = \frac{\pi}{2}\cos x$ for $0 < x < \pi$

$= 0$ for $x > \pi$

(c) $f(x) = \frac{\pi}{2} e^{ix}$ for $0 < x < \pi$

$= 0$ for $x > \pi$

(d) $f(x) = \frac{\pi}{2}$ for $0 < x < \pi$

$= 0$ for $x > \pi$

Ans. : (d)

Explanation :

Just like solution of above we get required f (x)

18. If $\int_0^\infty f(x)\cos\lambda x\, dx = e^{-\lambda},\ \lambda > 0$ then the fourier cosine transform of f(x) is

(a) $F_c(\lambda) = \frac{2}{\pi} e^{-\lambda}$ (b) $F_c(\lambda) = \frac{\pi}{2} e^{-\lambda}$ (c) $F_s(\lambda) = e^{-\lambda}$ (d) $F_c(\lambda) = e^{-\lambda}\cos\lambda$

Ans. : (c)

Explanation :

By definition

$$F_c(\lambda) = \int_0^\infty f(x) \cos\lambda x \, dx$$

$$\Rightarrow \quad F_c(\lambda) = e^{-\lambda} \text{ by using given equation}$$

19. If $\int_0^\infty f(x) \sin \lambda x \, dx = \frac{\lambda}{y^2+4}$, $\lambda > 0$ then the Fourier sine transform of f(x) is

(a) $F_s(\lambda) = \frac{2 \quad \lambda}{\pi\lambda^2 + 4}$ (b) $F_s(\lambda) = \frac{2 \quad \lambda}{\pi\lambda^2 + 4}$

(c) $F_s(\lambda) = \frac{2\lambda \sin \lambda}{\pi\lambda^2 + 4}$ (d) $F_s(\lambda) = \frac{\lambda}{\lambda^2 + 4}$

Ans. : (d)

Explanation : By definition

$$F_s(\lambda) = \int_0^\infty f(x) \sin \lambda x \, dx$$

By using given equation

$$F_s(\lambda) = \frac{\lambda}{y^2 + 4}$$

20. If $\int_0^\infty f(x) \cos\lambda x \, dx = \begin{cases} 1-\lambda, & 0 \le \lambda \le 1 \\ 0, & \lambda \ge 1 \end{cases}$ then Fourier cosine transform of f(x) is

(a) $F_c(\lambda) = 1 - \lambda$, (b) $Fc(\lambda) = (1-\lambda) \sin \lambda$

(c) $F_c(\lambda) = \begin{cases} (1-\lambda) \sin \lambda, & 0 \le \lambda \le 1 \\ 0, & \lambda \ge 1 \end{cases}$ (d) $F_c(\lambda) \begin{cases} 1-\lambda, & 0 \le \lambda \le 1 \\ 0, & \lambda \ge 1 \end{cases}$

Ans. : (d)

Explanation :

Fourier cosine transform of f(x)

$$F_c(\lambda) = \int_0^\infty f(x) \cos\lambda x \, dx$$

$$\Rightarrow \quad F_c(\lambda) = \begin{cases} (1-\lambda), & 0 \le \lambda \le 1 \\ 0, & \lambda \ge 1 \end{cases}$$

21. $f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(u) \cos\lambda u \cos\lambda x \, du \, d\lambda$ represents

(a) Fourier inverse transform (b) Fourier cosine integral of f (x)

(c) Fourier sine integral of f (x) (d) Complex exponential form of Fourier series of f(x)

Ans. : (b)

Explanation : Follow from standard definition

22. $f(x) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(u) \sin \lambda u \sin \lambda x \, du \, d\lambda$ represents

(a) Fourier inverse transform
(b) Fourier cosine integral of f(x)
(c) Fourier sine integral of f(x)
(d) Complex exponential form of fourier series of f(x)

Ans.: (c)

Explanation : Because of standard definition

23. $f(x) = \int_0^\infty [A(\lambda) \cos\lambda x + B(\lambda) \sin \lambda x] \, d\lambda$

where $A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u) \cos\lambda u \, du$

$B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u) \sin\lambda u \, du$ represents

(a) Fourier integral representation of f(x)
(b) Fourier cosine integral of f(x)
(c) Fourier sine integral of f(x)
(d) Complex exponential form of Fourier series of f(x)

Ans. : (a)

Explanation : Standard form

24. The convolution theore for fourier transform states that..

(a) F[f (x)* g (x)] = F[f(x) * F[g (x)]
(b) F[f(x) * g(x)] = F[f(x)] F[g(x)]

Ans. : (b)

Explanation : It is statement of convolntion theorem for Fourier Transform.

25. In the Fourier integral representation

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1-i\lambda}{1+\lambda^2}\right) e^{i\lambda} \, d\lambda = \begin{cases} 0 & , x < 0 \\ e^{-x} & , x > 0 \end{cases} \quad F(\lambda) \text{ is}$$

(a) $\frac{\sin \lambda}{1+\lambda^2}$ (b) $\frac{\cos \lambda}{1+\lambda^2}$ (c) $\frac{1-i\lambda}{1+\lambda^2}$ (d) $\frac{1+\lambda^2}{1-i\lambda}$

Ans. : (c)

Explanation : Compare given equation with $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x} F(\lambda)\, d\lambda$

26. The Fourier transform $F(\lambda)$ of $f(x) = e^{-|x|}$ is given by

(a) $\frac{3}{1+\lambda^2}$ (b) $\frac{2}{1+\lambda^2}$ (c) $\frac{1}{1-\lambda^2}$ (d) $\frac{2}{1-\lambda^2}$

Ans. : (b)

Explanation : By straightforward calculation

27. The Fourier transform $F(\lambda)$ of $f(x) = \begin{cases} \sin x & , x > 0 \\ 0 & , x < 0 \end{cases}$ is

(a) $\frac{1}{1-\lambda^2}$ (b) $\frac{1}{1-\lambda^2}$ (c) $\frac{i\lambda}{1-\lambda^2}$ (d) $\frac{i\lambda}{1+\lambda^2}$

Ans. : (a)

Explanation : By straightforward calculation

28. The fourier transform $F(\lambda)$ of $f(x) = \begin{cases} 2x + x & , x > 0 \\ 0 & , x < 0 \end{cases}$ is

(a) $-\frac{1}{\lambda^2} - \frac{2i}{\lambda}$ (b) $\frac{-1}{\lambda^2} + \frac{2i}{\lambda}$ (c) $\frac{1}{\lambda^2} - \frac{2i}{\lambda}$ (d) $\frac{1}{\lambda^2} + \frac{2i}{\lambda}$

Explanation : By straightforward calculation

29. The inverse Fourier transform f(x) defined in $-\infty < x < \infty$ of $F(\lambda) = \pi\left[\frac{1 - i\lambda}{1+\lambda^2}\right]$ is

(a) $\frac{1}{2}\int_{0}^{\infty}\left[\frac{\cos \lambda x + \lambda \sin \lambda x}{1+\lambda^2} + i\,\frac{-\lambda \cos \lambda x + \sin \lambda x}{1+\lambda^2}\right] d\lambda$

(b) $\frac{1}{2}\int_{-\infty}^{\infty}\left[\frac{\cos \lambda x + \lambda \sin \lambda x}{1+\lambda^2} + i\,\frac{-\lambda \cos \lambda x + \sin \lambda x}{1+\lambda^2}\right] d\lambda$

(c) $\frac{1}{2}\int_{-\infty}^{\infty}\left[i\,\frac{-\lambda \cos \lambda x + \sin \lambda x}{1+\lambda^2}\right] d\lambda$

(d) $\frac{1}{2}\int_{-\infty}^{\infty}\left[\frac{\cos \lambda x + \lambda \sin \lambda x}{1-\lambda^2} + i\,\frac{-\lambda \cos \lambda x + \sin \lambda x}{1-\lambda^2}\right] d\lambda$

Ans. : (b)

Explanation : Formula used is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x} F(\lambda)\, d\lambda$$

Test Your Knowledge

1. $z\left\{\cos\left(\frac{\pi}{2}\right)k\right\}$

(a) $\frac{z^2}{z^2+1}$ (b) $\frac{z^2}{z^2-1}$ (c) $\frac{z}{z^2+1}$ (d) $\frac{z^2}{(z^2+1)^2}$

2. $z\left\{\sin\left(\frac{\pi}{2}\right)k\right\}$

(a) $\frac{z^2}{(z^2+1)}$ (b) $\frac{z}{z^2-1}$ (c) $\frac{z}{z^2+1}$ (d) $\frac{z}{(z^2-1)^2}$

3. $z\{k\}, k \geq 0$

(a) 1 (b) $\frac{z}{(z-1)^2}$ (c) $\frac{z}{z-1}$ (d) $\frac{z}{(z-1)^2}$

4. $z\{a^k\}, k < 0, |z| < |a|$

(a) $-\left(\frac{a}{z-a}\right)$ (b) $-\left[\frac{z}{z-a}\right]$ (c) $\frac{z}{z-a}$ (d) $\frac{z}{z+a}$

5. $Z\{(-4)^k\}, k > 0, |z| > 4$

(a) $\frac{1}{z+4}$ (b) $\frac{z}{4-z}$ (c) $\frac{z}{z-4}$ (d) $\frac{z}{z+4}$

6. $z\left\{\frac{a^k}{k!}\right\}, k \geq 0$

(a) $\frac{z}{(z-a)\,k!}$ (b) $\frac{kz}{z-a}$ (c) $e^{\frac{a}{z}}$ (d) $\frac{kz}{k\,(z-a)}$

7. $z^{-1}[4] =$

(a) $-\left[\frac{4}{z-4}\right]$ (b) $4\,\delta(k)$ (c) $4\,U(k)$ (d) 4

8. $z\{2^k + (2)^k\}, k \geq 0, |z| > 2$

(a) $\frac{z^2}{z^2-2}$ (b) $\frac{z^2}{4-z^2}$ (c) $\frac{z^2}{z^2+4}$ (d) $\frac{z^2}{(z^2-6)}$

9. $z\{(3)^k - (-2)^k\}, k \geq 0, |z| > 3$

(a) $\frac{5z}{z^2-z-6}$ (b) $\frac{5z}{z^2+z-6}$ (c) $\frac{5z}{z^2-z+6}$ (d) $\frac{5z}{z^2+z+6}$

10. $z\{(-1)^k + 1\}, k \geq 0, |z| > 1$

(a) $\left(\frac{z}{z+1}\right)\left(\frac{z}{z-1}\right)$ (b) $\frac{2z^2}{z^2-1}$ (c) $\frac{z^2}{z^2-1}$ (d) None of these

11. $z\{(-2)^k - (2)^k\}, k \geq 0, |z| > 2$

(a) $-\left(\frac{4z}{z^2-2}\right)$ (b) $\left(\frac{4z}{z^2-4}\right)$ (c) $-\left(\frac{4z}{(z^2-4)}\right)$ (d) $\left(\frac{4z}{z^2+4}\right)$

12. $z\{k\,a^k\}$, $k \geq 0$, $|z| > |a|$

(a) $\dfrac{z}{(z-a)^2}$ (b) $\dfrac{az}{(z-a)^2}$ (c) $\dfrac{a}{(z-a)^2}$ (d) None of these

13. $z^{-1}\left[\dfrac{z}{z-1}\right]$, $|z| > 1$ For all k

(a) 1 (b) U(k) (c) U(–k) (d) $\delta(k)$

14. $z^{-1}\left[\dfrac{z}{z-1}\right]$, $k \geq 0$, $|z| > 1$

(a) $\{1^k\}$ (b) U(k) (c) $\delta(k)$ (d) $(-1)^k$

15. $z^{-1}\left[\dfrac{z}{2z+1}\right]$, $|z| > \dfrac{1}{2}$, $k \geq 0$

(a) $\left(-\dfrac{1}{2}\right)^k$ (b) $\dfrac{1}{2}\left(-\dfrac{1}{2}\right)^k$ (c) $\dfrac{1}{2}\left(\dfrac{1}{2}\right)^k$ (d) $\dfrac{1}{2}\left[(-1)^k + 1\right]$

16. $z{-}1\left[\dfrac{z}{4z-1}\right]$

(a) $\left(\dfrac{1}{4}\right)^k$ (b) $\dfrac{1}{4}\left(\dfrac{1}{4}\right)^k$ (c) $\dfrac{1}{4}\left(-\dfrac{1}{4}\right)^k$ (d) None of these

17. $z^{-1}\left[\dfrac{1}{z-4}\right]$

(a) $\dfrac{1}{4}(4^k - 1)$ (b) 4k (c) 4k–1 (d) None of these

18. $z^{-1}\left[\dfrac{z}{z^2-3z+2}\right]$

(a) $2^k + 3^k$ (b) $2^k - 1$ (c) $2^k + 1$ (d) $2^k - 3^k$

19. $z\left\{\left(\dfrac{1}{4}\right)^{|k|}\right\}$

(a) $\dfrac{z}{z-\frac{1}{4}} - \dfrac{\frac{1}{4}z}{1-\frac{1}{4}z}$ (b) $\dfrac{4z}{4z-1} + \dfrac{z}{1-4z}$

(c) $\dfrac{4z}{4z-1} + \dfrac{z}{4-z}$ (d) None of these

20. $z\{2, -1, 2\}$

(a) $2 + (-1)\,z^{-1} + 2z^{-2}$ (b) $2z - \dfrac{1}{z} + \dfrac{2}{z^2}$

(c) $\dfrac{2}{z} - \dfrac{1}{z^2} + \dfrac{2}{z^3}$ (d) $2z - z^2 + 2z^3$

21. z {4, 2, 1} is

(a) $4z + 2 + \frac{1}{z}$ (b) $4z^{-1} + 2z^{-1} + z^{-2}$

(c) $4 + 2z + z^2$ (d) $\frac{4}{z^2} + \frac{2}{z} + 1$

22. z {– 4 , 0, 4} is

(a) $-4z + \frac{4}{z}$ (b) $-\frac{4}{z} + 4z$ (c) $-4z^2 + 4z$ (d) $-4 + \frac{4}{z^2}$

23. z $\{4^k U(k)\}$

(a) $\frac{z}{z-4}$ (b) $\frac{z}{z+4}$ (c) $\frac{z}{z-U(k)}$ (d) $U\left(\frac{k}{4}\right)$

24. $z^{-1} - \left[\frac{z(z+1)}{(z-1)^3}\right] =$

(a) k (b) k^2 (c) $\{(-1)^k + (3)^k$ (d) None of these

25. z $\{k\, 5^{-k}\}$, $k \geq 0$

(a) $\frac{(z-5)^2}{5z^2}$ (b) $\frac{5z}{(z-5)^2}$ (c) $\frac{5z}{(z+5)^2}$ (d) $\frac{5\,(z-5)}{z^2}$

26. z $\{e^{-2k}\, 3^k\}$, $k \geq 0$

(a) $\frac{z}{z-2e^3}$ (b) $\frac{z}{z+2e^2}$ (c) $\frac{z}{z-3e^2}$ (d) $\frac{z}{(z-3e)^2}$

27. $z\left\{\frac{2^k}{k!}\right\}$, $k \geq 0$

(a) $e^{\frac{z}{2}}$ (b) e^{2z} (c) e^{kz} (d) $e^{\frac{2}{z}}$

28. z {cos(πk) } , $k \geq 0$, $|z| = 1$

(a) $\frac{z}{z+1}$ (b) $\frac{z-1}{z+2}$ (c) $\frac{z\,(z+1)}{(z+1)^2}$ (d) $\frac{z\,(z-1)}{(z+1)^2}$

29. If z {(f(k)}= F(z) then z { k f(k) }

(a) $\left(z\frac{d}{dz}\right)F(z)$ (b) $-z\frac{d}{dz}F(z)$ (c) $-\frac{d}{dz}[\,F(z)\,]$ (d) $\left(-z\frac{d}{dz}\right)^2 F(z)$

30. z {sin 2k}

(a) $\frac{z\sin 2}{z^2 - 2z\cos 2 + 1}$ (b) $\frac{z\,(z-\cos 2)}{z^2 - 2z\cos 2 + 4}$ (c) $\frac{z\sin 2}{z^2 - 2z\cos 2 + 4}$ (d) $\frac{(z\sin 2)}{z^2 - 2z\cos 2 + 4}$

31. z {cos2k}

(a) $\frac{z\sin^2}{z2 - 2z\cos 2 + 1}$ (b) $\frac{z\,(z-\cos 2)}{z^2 - 2z\cos 2 + 1}$

(c) $\frac{z\sin 2}{z^2 - 2z\cos 2 + 4}$ (d) $\frac{z\,(z-\cos 2)}{z^2 - 2z\cos 2 + 4}$

32. $z\{z\ 6^k\}$, $k < 0$, $|z| = |a|$

(a) $\left\{-\left(\frac{z}{z-6}\right)\right\}$ (b) $\frac{z}{z-6}$ (c) $\frac{6z}{z-6}$ (d) $\frac{6z}{6-z}$

33. $z\{8\ \delta(k)\}$

(a) 4 (b) $8\left(\frac{z}{z-1}\right)$ (c) 8 (d) 8k

34. $z\{2^{3k+1}\}$, $k \geq 0$

(a) $\frac{z}{z-8}$ (b) $\frac{2z}{z-8}$ (c) $\frac{z}{z-3k}$ (d) $\frac{2z}{z+8}$

35. $z\{2^k - 3^k\}$, $k \geq 0$

(a) $\frac{z}{z^2+5z+6}$ (b) $\frac{z}{z^2-5z+6}$ (c) $\frac{-z}{z^2-5z+6}$ (d) $\frac{6z}{z^2-5z+6}$

36. $z\{e^{-n-pk}\}$

(a) $e^{-n}\left[\frac{z}{z-e^{-p}}\right]$ (b) $e^{n}\left(\frac{z}{z-e^{-p}}\right)$ (c) $e^{-n}\left[\frac{z}{z+e^{-p}}\right]$ (d) $e^{n}\left[\frac{z}{z+e^{-p}}\right]$

37. $z\{4^{3k-1}\}$, $k \geq 0$

(a) $\frac{4z}{z-256}$ (b) $\frac{z}{z-64}$ (c) $\frac{z}{z+64}$ (d) $\frac{z}{256-4z}$

38. $z\left\{\frac{1}{k!}\right\}$

(a) $e^{\frac{1}{z}}$ (b) $e^{\frac{2}{z}}$ (c) $e^{\frac{k}{z}}$ (d) $e^{-\frac{1}{z}}$

39. $z\left[\frac{1}{k}\right]$

(a) $e^{\frac{1}{z}}$ (b) $e^{\frac{-1}{z}}$ (c) $-\log(1 - z^{-1})$ (d) $e^{\frac{1}{k}}$

40. $z\{e^{2k} + 4\}$

(a) $\frac{z}{z-e} + \frac{4}{z-1}$ (b) $\frac{z}{z-e^2} + \frac{4z}{z-1}$ (c) $\frac{z}{z-e^2} + \frac{4}{z-1}$ (d) $\frac{z}{z-e^2} + \frac{4z}{z+1}$

Answers Key

1. (a)	2. (c)	3. (b)	4. (c)	5. (d)	6. (c)	7. (d)	8. (d)	9. (a)	10. (b)
11. (c)	12. (b)	13. (b)	14. (a)	15. (b)	16. (b)	17. (c)	18. (b)	19. (c)	20. (a)
21. (a)	22. (d)	23. (a)	24. (b)	25. (b)	26. (b)	27. (d)	28. (d)	29. (b)	30. (a)
31. (b)	32. (a)	33. (c)	34. (b)	35. (c)	36. (a)	37. (d)	38. (a)	39. (c)	40. (b)

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3 Unit — Statistics

1. The arithmetic mean ($\bar{x}$) of following distribution

x	0	1	2	4
f	1	4	3	2

Where f is frequency of corresponding variable, is given by

(a) 1 (b) 1.2 (c) 1.8 (d) 2

Ans.: (b)

Explanation : for frequency distribution e

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{(0)(1) + (1)(4) + (2)(3) + (4)(2)}{1 + 4 + 3 + 2}$$

$$= \frac{0 + 4 + 6 + 8}{10} = \frac{18}{10} = 1.8$$

2. The arithmetic mean ($\bar{x}$) of following distribution

x	1	2	3	4	5
f	2	4	5	3	1

Where f is frequency of corresponding variable, is given by

(a) 2 (b) 3 (c) 2.8 (d) 2.5

Ans.: (c)

Explanation :

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{(1)(2) + (2)(4) + (3)(5) + (4)(3) + (5)(1)}{2 + 4 + 5 + 3 + 1}$$

$$= \frac{2 + 8 + 15 + 12 + 5}{15} = \frac{42}{15} = 2.8$$

3. The standard deviation (σ) of a distribution 2, 3, 5 is given by

(a) 2 (b) 3.88 (c) 1.25 (d) 4.12

Ans.: (c)

Explanation :

$$\sigma = \sqrt{\frac{\sum x_1^2}{n}(\bar{x})^2} = \sqrt{\frac{38}{3} - (3.33)^2} = 1.25$$

4. For a given distribution if value of $\sum fx^2 = 188$, $N = \sum f = 10$ $\bar{x} = 3.5$ then value of standard deviation (σ) is given by

(a) 2.56 (b) 4.12 (c) 4.88 (d) 5.13

Ans.: (a)

Explanation :

$$\sigma = \sqrt{\frac{\sum x^2}{n}(\bar{x})^2} = \sqrt{18.8 - (3.5)^2} = 2.56$$

5. For a given distribution if value of $\sum fx^2 = 122$, $N = \sum f = 5$, $\bar{x} = 4$ then value of standard deviation (σ) is given by

(a) 2 (b) 3.88 (c) 5.13 (d) 2.90

Ans.: (a)

Explanation :

$$\sigma = \sqrt{\frac{\sum x^2}{n}(\bar{x})^2} = \sqrt{24.4 - 16} = 2.90$$

6. The standard deviation of the following frequency distribution is

Wages in rupees earned per day	0 – 10	10 – 20	20 – 30
No. of labours	5	9	15

(a) 9.32 (b) 7.55 (c) 8.30 (d) 12.70

Ans.: (b)

Explanation : Let A = 15 Here h = 10

Wages in rupees earned per day	Middle value x	No. of labours f	$u = \frac{x - A}{h}$ A = 15	$f_1 u_1$	$f_1 u_1^2$
0 – 10	5	5	– 1	– 5	5
10 – 20	15	9	0	0	0
20 – 30	25	15	1	15	15
$\sum$	–	29	–	10	20

$$\text{S.D.} \quad = \quad \sigma = h\sqrt{\frac{\sum f_1 u_1^2}{\sum f_1} - \left(\frac{\sum f_1 u_1}{\sum f_1}\right)^2}$$

$$= \quad 10\sqrt{\frac{20}{29} - \left(\frac{10}{29}\right)^2} = 7.55$$

7. The following table gives the marks obtained in a paper of statistics out of 25

Class interval	0 – 5	5 – 10	10 – 15	15 – 20	20 – 25
No. of students	2	6	8	8	15

The standard deviation (S.D) is _______

(a) 7.29 (b) 7.35 (c) 6.30 (d) 5.75

Ans.: (c)

Explanation :

$$\text{S.D} \quad = \quad \sigma = h\sqrt{\frac{\sum f_1 u_1^2}{\sum f_1} - \left(\frac{\sum f_1 u_1}{\sum f_1}\right)^2}$$

Class interval	Middle value x	No. of students f	$u = \frac{x - A}{h}$ A = 12.5	$f_1 u_1$	$f_1 u_1^2$
0 – 5	2.5	2	–1	–2	2
5 – 10	7.5	6	– 0.5	– 3	1.5
10 – 15	12.5	8	0	0	0
15 – 20	17.5	8	0.5	4	2
20 – 25	22.5	15	1	15	15
$\sum$	–	39	–	14	20.5

$$\therefore \quad \text{S.D.} \quad = \quad \sigma = 10\sqrt{\left(\frac{20.5}{39}\right) - \left(\frac{14}{39}\right)^2} = 6.30$$

8. From the following data

Team	S.D (σ)	A.M$\bar{x}$
A	2	5
B	2.5	4

The more consistent team is

(a) Team A (b) Team B

(c) Team equally consistent (d) can't say

Ans.: (a)

Explanation :

We have coefficient of variation

$$(\text{C.V.}) = \frac{\sigma}{x} \times 100$$

$$\text{For team A, } (\text{C.V})_A = \frac{2}{5} \times 100 = 40$$

$$\text{For team B, } (\text{C.V})_B = \frac{2.5}{4} \times 100 = 62.5$$

As $(\text{C.V.})_A < (\text{C.V})_B$, team A is more consistent

9. From the following data which term is more consistent

Team	S.D (σ)	A.M $\bar{x}$
A	2.5	10
B	4	16

(a) A (b) B (c) Equally consistent (d)Can't say

Ans.: (c)

Explanation :

$$\text{For team A, } (\text{C.V.})_A = \frac{\sigma}{x} \times 100 = \frac{2.5}{10} \times 100$$

$$(\text{C.V.})_A = 25$$

$$\text{For team B, } (\text{C.V.})_B = \frac{4}{16} \times 100 = 25$$

$\therefore$ $(\text{C.V.})_A = (\text{C.V.})_B$, Both teams are equally consistent

10. From the following data : the most consistent player is

Player	S.D (σ)	A.M ($\bar{x}$)
Sachin	10	45
Rahul	13	38
Sourav	11	42

(a) Sachin (b) Rahul (c) Sourav (d) EquallyConsistent

Ans.: (a)

Explanation :

$$\text{For Sachin, } (\text{C.V.})_A = \frac{\sigma}{x} \times 100 = \frac{10}{45} \times 100 = 22.22$$

$$\text{For Rahul, } (\text{C.V.})_B = \frac{13}{38} \times 100 = 34.21$$

For Saurav, $(C.V.)_C = \frac{11}{42} \times 100 = 26.19$

Since $(C.V)_A < (C.V.)_C < (C.V.)_B$

Sachin is most consistent player

11. If arithmetic mean of four numbers is 15, one item 19 is replaced by 23, then new arithmetic mean is,

(a) 16 (b) 17 (c) 18 (d) 20

Ans.: (a)

Explanation :

Given, $\bar{x} = \frac{\sum x_i}{n}$

$\therefore \quad 15 = \frac{x_1 + x_2 + x_2 + x_4}{4}$

$60 = x_1 + x_2 + x_3 + x_4$

Let $x_1 = 19$

$\therefore \quad x_2 + x_3 + x_4 = 60 - 19 = 41$ is mean of three numbers

$\therefore$ New mean Now $x_1 = 23$

$\therefore \quad x_1 + x_2 + x_3 + x_4 = 41 + 23 = 64$

$\text{Mean} = \frac{64}{4} = 16$

New arithmetic mean $= 16$

12. The first moment of the distribution about the value 4 is 6, Arithmetic mean of the distribution is,

(a) 11 (b) 15 (c) 10 (d) 12

Ans.: (c)

Explanation :

Given : $a = 4, \ \mu_1 = 6$

$\therefore \quad \bar{x} = \mu_1 + a = 6 + 4 = 10$

13. The first and second moments of the distribution about the value 2 are 3 and 17, second moment about the mean is,

(a) – 6 (b) 7 (c) – 8 (d) 8

Ans.: (d)

Explanation :

Given $\mu_1' = 3 \quad \mu_2' = 17$

and $a = 2$

Second moment about mean is,

$$\mu_2 = \mu_2' - (\mu_1')^2 = 17 - (3)^2 = 17 - 9$$

$$\mu_2 = 8$$

14. The first four moments about the working mean A = 5 are – 1, 10, 11, 16 then value of second moment μ_2 about mean is given by

(a) 10 (b) 9 (c) 8 (d) 7

Ans.: (b)

Explanation :

$$\mu_2 = \mu_2 - (\mu_2')^2 = 10 - (-1)^2 = 9$$

15. If the first four moments of a distribution about the working mean A = 2 are 1, 5, – 20, 10 then the second moment μ_2 about mean of distribution is given by

(a) 1 (b) 2 (c) 3 (d) 4

Ans.: (d)

Explanation :

$$\mu_2 = \mu_2' (\mu_2')^2 = 5 - (-1)^2 = 4$$

16. If the first four moments of a distribution about the working mean A = 2 are 1, 5, – 20, 10 then the third moment μ_3 about mean of distribution is given by

(a) – 10 (b) 30 (c) – 33 (d) – 20

Ans.: (c)

Explanation :

$$\mu_3 = \mu_3' - 3\,\mu_2'\mu_1' + 2\,(\mu_1')^2$$

$$= -20 - 3\,(5)\,(1) + 2\,(1)^2$$

$$= -20 - 15 + 2 = -33$$

17. If the first four moments of a distribution about the working mean A = 2 are 1, 5, – 20, 19 then the variance of distribution is given by

(a) 1 (b) 2 (c) 3 (d) 4

Ans.: (d)

Explanation :

$$\text{Variance} = (\text{S.D})^2$$

Now $$\mu_2 = \mu_2' (\mu_1')^2 = 5 - (1)^2 = 4$$

$$\text{S.D} = \sqrt{\mu_2} = \sqrt{4} = +2$$

$\therefore$ $$\text{Variance} = = (2)^2 = 4$$

18. The value of second and third moments about mean of distribution are 2.83 and 2.38 respectively. The coefficient of skewnessβ_1 is equal to

(a) 0.10 (b) 0.50 (c) 0.20 (d) 0.30

Ans.: (b)

Explanation :

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{1(3.38)^2}{(2.83)^3} = 0.5040$$

19. The first four moments about the working mean 44.5 of a distribution are – 0.4, 2.99, – 0.08 and 27.63 then value of moments about mean μ_2 is _______

(a) 2.83 (b) 3.99 (c) 2.2 (d) 5.9

Ans.: (a)

Explanation :

Given $A = 44.5$

$\mu_1' = -0.4,\ \mu_2' = 2.99,\ \mu_3' = -0.008,\ \mu_4' = 27.63$

$\mu_2 = \mu_2' - \mu_1'^2 = 2.83$

20. The first three moments about the working mean 44.5 of a distribution are – 0.4, 2.99, – 0.08 then value of moment about mean μ_3 is

(a) 2.83 (b) 4.30 (c) 3.38 (d) 30.3

Ans.: (c)

Explanation : Given $A = 44.5$

$\mu_1' = -0.4,\ \mu_2' = 2.99,\ \mu_3' = -0.008$

$\mu_3 = \mu_3' = 3\mu_2'\mu_1' + 2\mu_1'^3 = 3.38$

21. The first four moments about the working mean 44.5 of a distribution are – 0.4, 2.99, 0.08 and 27.63 then value of moment about mean μ_4 is _______

(a) 15.19 (b) 30.30 (c) –29.20 (d) 37.99

Ans.: (b)

Explanation : Given $A = 44.5$

$\mu_1' = -0.4,\ \mu_2' = 2.99,\ \mu_3' = -0.008,\ \mu_4' = 27.63$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_2'^2 - 3\mu_1'^4 = 30.30$

22. The first four moments about the working mean 44.5 of a distribution are – 0.4, 2.99, – 0.04 and 27.63 then distribution is _______

(a) platykurtic (b) Mesokurtic (c) Leptokurtic (d) Equal distribution

Ans.: (c)

Explanation : Given

$$A = 44.5$$

$$\mu_1' = -0.4, \mu_2' = 2.99, \mu_3' = -0.008, \mu_4' = 27.63$$

$$\mu_2 = \mu_2' - \mu_1'^2 = 2.83$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4 = 30.30$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = 3.78$$

∴ Distribution is Leptokurtic.

23. The first three moments about the value 2 of a distribution are 1, 16 and – 40 then value of A.M. is ______

(a) 2 (b) 4 (c) 3 (d) – 2

Ans.: (c)

Explanation : Given

$$A = 2$$

$$\mu_1' = 1, \mu_2' = 16, \mu_3' = -40$$

$$A.M = A + \mu_1' = 3$$

24. The first three moments about the value 2 of a distribution are 1, 16 and – 40 then value of standard deviation (σ) is ______

(a) 3.87 (b) 4.03 (c) 13.31 (d) – 4.09

Ans.: (a)

Explanation : Given :

$$A = 2$$

$$\mu_1' = 1, \mu_2' = 16, \mu_3' = -40$$

$$\mu_2 = \mu_2' - \mu_1'^2 = 15$$

∴ $$S.D = \sigma$$

$$= \sqrt{\mu_2} = 3.87$$

25. The first three moments about the value of 2 of a distribution are 1, 16 and – 40 then the value of coefficient of skewness β_1 of distribution is ______

(a) 0.05 (b) 2.19 (c) – 0.05 (d) 1.28

Ans.: (b)

Explanation :Given :

$$A = 2$$

$$\mu_1' = 1, \mu_2' = 16, \mu_3' = -40$$

and $$\mu_2 = \mu_2' - \mu_1'^2 = 15$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + \mu_1'^3 = -86$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}$$

$$= 2.19$$

26. For the data of distribution :

$n = 100, \sum fd = 50, \sum fd^2 = 1970, \sum fd^3 = 2948$ where $d = X - 48$ then value of third moment about mean $\mu_3 =$ ______

(a) 0.18 (b) 1.20 (c) – 0.37 (d) – 1.49

Ans.: (a)

Explanation : Given

$$n = 100 \text{ and } A = 48$$

$$\mu_1' = \frac{\sum fd}{n} = \frac{50}{100} = 0.5$$

$$\mu_2' = \frac{1970}{100} = 19.7$$

$$\mu_3' = \frac{2948}{100} = 29.48$$

$$\mu_3 = \mu_3' - 3\,\mu_2'\mu_1' + 2\,\mu_1'^3 = 0.18$$

27. For the data of distribution : $n = 100, \sum fd = 50, \sum fd^2 = 1970, \sum fd^3 = 2948$ $\sum fd^4 = 86752$ where $d = X - 48$ then value of moment about mean $\mu_4 =$ ______

(a) 187.32 (b) 837.92 (c) 83.31 (d) 29.29

Ans.: (b)

Explanation :

Given :

$$n = 100 \text{ and } \quad A = 48$$

$$\mu_1' = \frac{\sum fd}{n} = \frac{50}{100} = 0.5 \; ; \quad \mu_2' = \frac{1970}{100} = 19.7$$

$$\mu_3' = \frac{2948}{100} = 29.48 \; ; \quad \mu_4' = \frac{86752}{100} = 867.52$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\,\mu_2'\mu_1'^2 - 3\,\mu_1'^4 = 837.92$$

28. For the data of distribution : $n = 100, \sum fd = 50, \sum fd^2 = 1970, d = X - 48$ then value of moment about mean $\mu_2 =$ ______

(a) 19.45 (b) 15.32 (c) 12.39 (d) 35.32

Ans.: (a)

Explanation :

Given :

$$n = 100 \text{ and } \quad A = 48$$

$$\mu_1' = \frac{\sum fd}{n} = \frac{50}{100} = 0.5 \; ;$$

$$\mu_2' = \frac{1970}{100} = 19.7$$

$$\mu_2 = \mu_2' - \mu_1'^2 = 19.45$$

29. For the data of distribution : $n = 100$, $\sum fd = 50$, $\sum fd^2 = 1970$, $\sum fd^3 = 2948$, $\sum fd^4 = 86752$ where $d = X - 48$ then value of coefficient of skewness β_1 is _______

(a) 0.23 (b) 0.0000044 (c) – 0.20 (d) 0.35

Ans.: (b)

Explanation :

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 \text{ and } A = 48$$

$$\mu_1' = \frac{\sum fd}{n} = \frac{50}{100} = 0.5\ ; \quad \mu_2' = \frac{1970}{100} = 19.7$$

$$\mu_3' = \frac{2948}{100} = 29.48\ ; \quad \mu_4' = \frac{86752}{100} = 867.52$$

$$\mu_2 = \mu_2' - 3\mu_1'^2 = 19.45$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1^3 = 0.18$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0.0000044 \text{ or } \beta_1 = 4.40 \times 10^{-6}$$

30. For the data of distribution : $n = 100$, $\sum fd = 50$, $\sum fd^2 = 1970$, $\sum fd^3 = 2948$, $\sum fd^4 = 86752$ where $d = X - 48$ then value of coefficient of Kurtosis β_2 is _______

(a) 3.31 (b) 2.21 (c) 2.71 (d) 3.94

Ans.: (b)

Explanation :

Given : $n = 100$ and $A = 48$

$$\mu_1' = \frac{\sum fd}{n} = \frac{50}{100} = 0.5\ ; \quad \mu_2' = \frac{1970}{100} = 19.7$$

$$\mu_3' = \frac{2948}{100} = 29.48\ ; \quad \mu_4' = \frac{86752}{100} = 867.52$$

$$\mu_2 = \mu_2' - \mu_2'^2 = 19.45$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4 = 837.92$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = 2.21$$

31. The first four moments about working mean 30.2 of a distribution are 0.255, 6.222, 30.211 and 400.25 then central moment μ_2 is _______

(a) 7.32 (b) 1.92 (c) 6.16 (d) 3.62

Ans.: (c)

Explanation :

Given : $A = 30.2$

$\mu'_1 = 0.255$; $\mu'_2 = 6.222$

$\mu'_3 = 30.211$; $\mu'_4 = 400.25$

$\mu_2 = \mu'_2 - \mu'^2_1 = 6.16$

32. The first four moments about working mean 30.2 of a distribution are 0.255, 6.222, 30.211 and 400.25 the central moment μ_3 is _______

(a) 25.48 (b) 11.35 (c) 32.29 (d) 17.32

Ans.: (a)

Explanation:

Given : $A = 30.2$

$\mu'_1 = 0.255$; $\mu'_2 = 6.222$

$\mu'_3 = 30.211$; $\mu'_4 = 400.25$

$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu'^3_1 = 25.48$

33. The first four moments about working mean 30.2 of a distribution are 0.255, 6.222, 30.211 and 400.25 the central moment μ_4 is ___

(a) 371.85 (b) 341.57 (c) 270.71 (d) 291.53

Ans.: (a)

Explanation:

Given : $A = 30.2$

$\mu'_1 = 0.255$; $\mu'_2 = 6.222$

$\mu'_3 = 30.211$; $\mu'_4 = 400.25$

$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu'^2_1 - 3\mu'^4_1 = 371.85$

34. The first four moments about working mean 30.2 of a distribution are 0.255, 6.222, 30.211 and 400.25 then value of coefficient of skewnessβ_1 is____

(a) 3.31 (b) 0.07 (c) 2.78 (d) 0.0

Ans.: (c)

Explanation :

Given : $A = 30.2$

$\mu'_1 = 0.255$; $\mu'_2 = 6.222$

$\mu'_3 = 30.211$; $\mu'_4 = 400.25$

$\mu_2 = \mu'_2 - \mu'^2_1 = 6.16$;

$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1\ 2\ \mu_1'^{3} = 25.48$$

$$\beta_1 = \frac{\mu_3^{2}}{\mu_2^{3}} = 2.78$$

35. The first four moments about working mean 3.02 of a distribution are 0.255, 6.222, 30.211 and 400.25 then value of coefficient of skewness β_2 is_______

(a) 2.99 (b) 6.20 (c) 3.51 (d) 9.80

Ans.: (d)

Explanation : Given : A = 30.2

$\mu'_1 = 0.255$; $\mu'_2 = 6.222$

$\mu'_3 = 30.211$; $\mu'_4 = 400.25$

$\mu_2 = \mu'_2 - \mu'^2_1 = 6.16$;

$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\,\mu'_2\mu'^2_1 - 3\mu'^4_1 = 371.85$

$$\beta_2 = \frac{\mu_4}{\mu_2^{3}} = 9.80$$

36. The first four moments about working mean of the distribution are 0, 2.5, 0.7 and 18.75 then moment about mean μ_2 is _______

(a) 11.5 (b) 12.5 (c) 2.5 (d) 7.5

Ans.: (c)

Explanation :

$\mu'_1 = 0$; $\mu'_2 = 2.5$

$\mu'_3 = 0.7$; $\mu'_4 = 18.75$

$\mu_2 = \mu'_2 - \mu'^2_1 = 2.5$

37. The first four moments about working mean of the distribution are 0.25, 0.7 and 18.75 then coefficient of skewness β_2 is _______

(a) 2.87 (b) 3 (c) 0.37 (d) 3.87

Ans.: (b)

Explanation :

$\mu'_1 = 0$; $\mu'_2 = 2.5$

$\mu'_3 = 0.7$; $\mu'_4 = 18.75$

$\mu_2 = \mu'_2 - \mu'^2_1 = 2.5$;

$\mu_2 = \mu'_4 - 4\mu'_3\mu'_1 + 6\,\mu_2'^{3}\mu_1^{2} - 3\mu_1'^{3} = 18.75$

$$\beta_2 = \frac{\mu_4}{\mu_2^{2}} = 3$$

38. The first four moment of a distribution about the value 5 are 2, 20, 40 and 50 then the value of central moment μ_3 is _______

(a) 57 (b) 25 (c) – 64 (d) 40

Ans. (c)

Explanation :

$$\mu'_1 = 2 ; \qquad \mu'_2 = 20$$

$$\mu'_3 = 40 ; \qquad \mu'_4 = 50$$

$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\,\mu'^3_1 = -64$$

39. The first four moment of a distribution about the value 5 are 2, 20, 40 and 50 then the value of central moment μ_4 is _______

(a) 50 (b) 157 (c) 22.39 (d) 162

Ans.: (d)

Explanation :

$$A = 5$$

$$\mu'_1 = 2 ; \qquad \mu'_2 = 20$$

$$\mu'_3 = 40 ; \qquad \mu'_4 = 50$$

$$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu'^2_1 - 3\,\mu'^4_1 = 162$$

40. The first four moments about working mean of the distribution are 0, 2.5, 0.7 and 18.75 then moment about mean μ_3 is _______

(a) 0.7 (b) 5.7 (c) 0.32 (d) 1.32

Ans.: (a)

Explanation :

$$\mu'_4 = 0 ; \qquad \mu'_2 = 2.5$$

$$\mu'_3 = 0.7 ; \qquad \mu'_4 = 18.75$$

$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\,\mu'^3_1 = 0.7$$

41. The first four moments about working mean of the distribution are 0, 2.5, 0.7 and18.75 then moment about mean μ_4 is _______

(a) 18.75 (b) 22.35 (c) 17.32 (d) 91.40

Ans.: (a)

Explanation :

$$\mu'_4 = 0 ; \qquad \mu'_2 = 2.5$$

$$\mu'_3 = 0.7 ; \qquad \mu'_4 = 18.75$$

$$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu'^2_1 - 3\mu'^4_1 = 18.75$$

42. The first four moments about working mean of the distribution are 0, 2.5, 0.7, and 18.75 then coefficient of skewness β_1 is _______

(a) 0.0 (b) 0.51 (c) 0.03 (d) – 0.5

Ans.: (c)

Explanation

$$\mu'_4 = 0\ ; \qquad \mu'_2 = 2.5$$

$$\mu'_3 = 0.7\ ; \qquad \mu'_4 = 18.75$$

$$\mu_2 = \mu'_2 - \mu'^2_1 = 2.5\ ;$$

$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu_1^3 = 0.7$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0.03$$

43. The first two moments about the working mean 30.2 of a distribution are0.255, 6.222 then value of S.D. (σ) is _______

(a) 3 (b) 3.22 (c) 2.48 (d) 5.7

Ans.: (c)

Explanation :

$$A = 30.2$$

$$\mu'_1 = 0.255 \qquad \mu'_2 = 6.222$$

$$\text{S.D.}(\sigma) = \sqrt{\mu_2} \quad \text{and} \qquad \mu_2 = \mu'_2 - \mu'^2_2 = 6.16$$

$$\text{S.D} = 2.48$$

44. The first two moments about working mean 44.5 of a distribution are– 0.4, 2.99 then the value of the arithmetic mean (A.M) is _______

(a) 44.1 (b) 42.5 (c) 1.35 (d) 44.5

Ans.: (a)

Explanation :

$$A = 44.5$$

$$\mu'_1 = -0.4 \qquad \mu'_2 = 2.99$$

$$\text{A.M.} = A + \mu'_1 = 44.1$$

45. The first two moments about the working mean 44.5 of a distribution are – 0.4, 2.99 then the value of standard deviation (S.D) (σ) is _______

(a) 2.32 (b) 2.87 (c) – 0.32 (d) 1.68

Ans.: (d)

Explanation :

$$A = 44.5$$

$$\mu_1' = -0.4 \qquad \mu_2' = 2.99$$

$$\text{S.D. } (\sigma) = \sqrt{\mu_2} \text{ and } \qquad \mu_2 = \mu_2' - \mu_1'^2 = 2.83$$

$$\text{S.D.} = 1.68$$

46. The first two moments of a distribution about the value 5 are 2, 20 then A.M. is ____

(a) 16 (b) 7 (c) 4 (d) 13

Ans.: (b)

Explanation :

$$A = 5$$

$$\mu_1' = 2\,; \qquad \mu_2' = 20$$

$$\text{A.M.} = A + \mu_1' = 7$$

47. The first two moments about the working mean 5 of a distribution are 2, 20 then standard deviation (σ) is _______

(a) 7 (b) 9 (c) 4 (d) 16

Ans.: (c)

Explanation :

$$A = 5;$$

$$\mu_1' = 2 \qquad \mu_2' = 20$$

$$\mu_2 = \mu_2' - \mu_2'^2 = 16$$

$$\text{S.D. } (\sigma) = \sqrt{\mu_2} = 4$$

48. The value of central moment μ_2 of the following distribution is

x	1	2	3	4	5
f	6	15	23	42	62

(a) 2.75 (b) 3.01 (c) – 1.72 (d) 1.34

Ans.: (d)

Explanation : $\mu_2 = \mu_2' - \mu_1'^2$

And $\mu_1' = \dfrac{\sum f_1 d_1}{\sum f_1}$ And $\mu_2' = \dfrac{\sum f_1 d_1^2}{\sum f_1}$

x	f	d = x – A A = 3	$f_1 d_1$	$f_1 d_1^2$
1	6	– 2	– 12	24
2	15	– 1	– 15	15

3	23	0	0	0
4	42	1	42	42
5	62	2	124	248
$\sum$	148	–	139	329

$$\mu_1' = 0.939 \qquad \mu_2' = 2.22$$

$$\mu_2 = \mu_2' - \mu_2'^2 = 1.34$$

49. For the data : n = 10; $\bar{u}$ = – 5.1, $\bar{v}$ = –10, $\sum u_i v_i$ = = 1242, $\sum u_i^2$ = 1169, $\sum v_i^2$ = 1694 the value of coefficient of correlation r is ____

(a) 0.74 (b) 0.92 (c) 0.65 (d) 0.89

Ans.: (b)

Explanation :

$$n = 10, \bar{u} = -5.1, \bar{v} = -10, \sum u_i v_i = 1242$$

$$\sum u_i^2 = 1169, \sum v_i^2 = 1694$$

$$r = \frac{\text{cor}(u,v)}{\sigma_u \sigma_v} \text{ and cov}(u, v) = \frac{1}{n}\sum u_i v_i - \bar{u}\,\bar{v} = 73.2$$

$$\sigma_u^2 = \frac{1}{n}\sum u_i^2 - \bar{u}^2 = 90.89 \quad \Rightarrow \sigma_u = 9.53$$

$$\sigma_v^2 = \frac{1}{n}\sum v_i^2 - \bar{u}^2 = 69.4 \quad \Rightarrow \sigma_v = 8.33$$

$$\therefore \quad r = \frac{73.2}{(9.53)(8.33)} = 0.92$$

50. For a given distribution, if value of cov (x, y) = – 5.2, σ_x = 2.82 the value of regression coefficient b_{yx} is _______

(a) – 0.55 (b) – 0.85 (c) – 0.75 (d) – 0.65

Ans.: (d)

Explanation :

We have $$b_{yx} = \frac{\text{cor}(x, y)}{\sigma_x^2} = \frac{-5.2}{(2.85)^2} = -0.65$$

51. For a given distribution cov (x, y) = 35.25, σ_y = 5.82 the value of regression coefficient b_{xy} is ____

(a) 1.80 (b) 2.8 (c) 1.04 (d) 2

Ans.: (c)

Explanation :

We have $$b_{yx} = \frac{\text{cor}(x, y)}{\sigma_y^2} = \frac{35.25}{(5.82)^2} = 1.04$$

52. Given n = 25, $\sum x = 75$, $\sum y = 100$, $\sum x^2 = 250$, $\sum y^2 = 500$, $\sum xy = 325$ then value of coefficient of correlation r is ___

(a) – 0.7 (b) 0.8 (c) 0.5 (d) 0.9

Ans.: (c)

Explanation :

Given

$$n = 25, \sum x = 75, \sum y = 100, \sum x^2 = 250, \sum y^2 = 500, \sum xy = 325$$

$$r = \frac{\text{cor}(x, y)}{\sigma_x \sigma_y};$$

$$\bar{x} = \frac{\sum x}{n} = 3;$$

$$\bar{y} = \frac{\sum y}{n} = 4$$

$$\text{Cor}(x, y) = \frac{1}{n}\sum xy - \bar{x}\,\bar{y} = 1$$

$$\sigma_x^2 = \frac{1}{n}\sum x^2 - \bar{x}^2 = 1 \quad \Rightarrow \sigma_x = 1$$

$$\sigma_y^2 = \frac{1}{n}\sum y^2 - \bar{y}^2 = 1 \quad \Rightarrow \sigma_y = 4$$

$\therefore$ $r = 0.5$

53. The two regression equation of the variables x and y are x = 19.3 – 0.87 y, y = 11.64 – 0.50 x then values of $\bar{x}$ and $\bar{y}$ are _______

(a) $\bar{x} = 11.64$; $\bar{y} = 19.3$ (b) $\bar{x} = 0.50$; $\bar{y} = 0.87$

(c) $\bar{x} = 16.23$; $\bar{y} = 3.52$ (d) $\bar{x} = 17.3$; $\bar{y} = 4.50$

Ans.: (c)

Explanation : Two regression lines are

$$x = 19.3 - 0.87\,y$$

$$y = 11.64 - 0.50\,x$$

$\therefore$ $x + 0.87y = 19.3\ ;\ 0.50x + y = 11.64$

Sincce $\bar{x}$ and $\bar{y}$ satisfies these equations

$$\bar{x} + 0.87\,\bar{y} = 19.3$$

$$0.50\,\bar{x} + \bar{y} = 11.64$$

$$\bar{y} = 16.23,\ \bar{y} = 3.52$$

54. The two regression equation of the variables x and y are x = 19.3 – 0.87 ; y = 11.64 – 0.50 x then correlation coefficient between x and y is_______

(a) 0.8 (b) 0.7 (c) – 0.66 (d) + 0.66

Ans.: (c)

Explanation : Two regression lines are

$$x = 19.3 - 0.87\,y\,; \qquad y = 11.64 - 0.50\,x$$

$$\therefore \quad b_{xy} = -0.87 \qquad b_{yx} = -0.50$$

$$r = \sqrt{b_{xy}\,b_{yx}} = 0.66$$

Since both b_{xy} and b_{yx} are negative

$$\therefore \quad r = -0.66$$

55. Two regression lines are $5y - 8x + 17 = 0$ and $2y - 5x + 14 = 0$ then mean values of x and y are _______

(a) $\bar{x} = 3\,; \bar{y} = 4$ (b) $\bar{x} = 8\,; \bar{y} = 2$ (c) $\bar{x} = 5\,; \bar{y} = 5$ (d) $\bar{x} = 4\,; \bar{y} = 3$

Ans.: (d)

Explanation :

Let $(\bar{x}, \bar{y})$ be the point of intersection of lines (1) and (2),

Therefore, $(\bar{x}, \bar{y})$ satisfies the Equations (1) and (2)

$\therefore$ put $x = \bar{x}$ and $y = \bar{y}$ in equations (1) and (2)

$$\therefore \quad 8\bar{x} - 5\bar{y} = 17 \text{ and } 5\bar{x} - 2\bar{y} = 14$$

Solving simultaneously, we get

$$\bar{x} = 4\,; \qquad \bar{y} = 3$$

$$\text{Mean} \quad \bar{x} = 4 \qquad \bar{y} = 3$$

56. Two regression lines are $5y - 8x + 17 = 0$ and $2y - 5x + 14 = 0$ also $\sigma_y^2 = 16$ then variance of x is _______

(a) 8 (b) 2 (c) 16 (d) 4

Ans.: (d)

Explanation : To find σ_x

$$\sigma_x = \frac{b_{xy} \times b_y}{r\,(x, y)} = \frac{(0.4) \times (4)}{0.8} = 2$$

(Take r from above example)

$$\sigma_x^2 = 4$$

57. Two regression lines are $8x - 10y + 66 = 0\,;\; 40x - 18y = 214$ then the mean values of x and y are _______

(a) $\bar{x} = 8, \bar{y} = 18$ (b) $\bar{x} = 40, \bar{y} = 10$ (c) $\bar{x} = 13, \bar{y} = 17$ (d) $\bar{x} = 9, \bar{y} = 11$

Ans.: (c)

Explanation : The regression lines are

$$8x - 10y + 66 = 0 \text{ and } 40x - 18y = 214$$

Since mean values of x and y are $\bar{x}$, $\bar{y}$ satisfies these equations.

$$8\bar{x} - 10\bar{y} = -66 \Rightarrow 40\bar{x} - 18\bar{y} = 214$$

$$\Rightarrow \therefore \quad \bar{x} = 13 ; \bar{y} = 17$$

58. The equation of two regression lines obtained in correlation analysis are 4x – 5y + 33 = 0;

20x – 9y – 107 = 0, then the correlation coefficient between x and y is _______

(a) 0.7 (b) 0.6 (c) – 0.8 (d) 9

Ans.: (b)

Explanation : Let us consider the lines of regression of y on x is from Equation (1),

$$y = \frac{4}{5}x + \frac{33}{5} = 0.8x + \frac{33}{5}$$

$$\therefore \quad b_{yx} = 0.8$$

And the lines of regression of x on y is from equation (2),

$$x = \frac{9}{20}y + \frac{107}{20} = 0.45y + \frac{107}{20}$$

$$\therefore \quad b_{xy} = 0.45$$

$\therefore$ Correlation co–efficient r (x, y)

$$r(x, y) = \sqrt{b_{yx} \cdot b_{xy}} = \sqrt{0.8 \cdot (0.45)}$$

$$r = 0.6$$

59. The equation of two regression line obtained in a correlation analysis are 4x – 5y + 33 = 0; 20x – 9y – 107 = 0 and variance of y is 16 then variance of x is _______

(a) 9 (b) 8 (c) 3 (d) 4

Ans.: (a)

Explanation:

Since $\sigma_y = 4$

$$\sigma_x = \frac{b_{xy} \times \sigma_y}{r(x, y)} = \frac{(0.45)(4)}{0.6} = 2$$

(Take r from above example)

$$\sigma_x = 3 \text{ variance in x is } \sigma_x^2$$

$$\sigma_x^2 = 9$$

60. If two lines of regression are 9x + y –λ = 0 and 4x + y = μ and mean of x and y are respectively $\bar{x}$ – 2, $\bar{y}$ = – 3 then values of λ and μ are

(a) λ = 15; μ = 5 (b) λ = 2 ; μ = – 3 (c) λ = 19 ; μ = 17 (d) λ = 12 ; μ = 15

Ans.: (a)

Explanation : Let $(\bar{x}, \bar{y})$ be the point of intersection of given two regression lines

Put $x = \bar{x}$ and $y = \bar{y}$

$$\therefore \quad 9\bar{x} + \bar{y} = \lambda \quad \Rightarrow \quad 9(2) - 3 = \lambda \Rightarrow \lambda = 15 \text{ and}$$

$$4\bar{x} + \bar{y} = \mu \quad \Rightarrow \quad 4(2) - 3 = \mu \Rightarrow \mu = 5$$

61. If two lines of regression are $9x + y - 15 = 0$ and $4x + y = 5$ and $\bar{x} = 2$, $\bar{y} = -3$ then coefficient of correlation r is _______

(a) – 0.8 (b) – 0.67 (c) 0.67 (d) 0.8

Ans.: (b)

Explanation : Let the regression line of y on x is form equation (4),

$$y = -4x + 15$$

$$\Rightarrow \quad b_{yx} = \text{regression coefficient of y on x}$$

$$b_{yx} = -4$$

and the regression line of x on y is from equation (3)

$$x = -\frac{y}{9} - \frac{15}{9} \therefore \quad b_{xy} = -\frac{1}{9}$$

$$= -0.1111 \text{ is the coefficient of regression of x on y}$$

$$r = \sqrt{b_{xy} \cdot b_{yx}} = \sqrt{4 \times 0.1111}$$

$$r(x, y) = 0.6666$$

62. If regression coefficient y on x is – 0.50 and x on y is −0.87 then coefficient of correlations is _______

(a) 0.8 (b) −0.66 (c) −0.3 (d) 0.9

Ans.: (b)

Explanation :

$$b_{yx} = -0.50 \text{ and} \quad b_{xy} = -0.87$$

$$r = \sqrt{b_{xy} \cdot b_{yx}} = \sqrt{(-0.50) \times (-0.87)} = 0.66$$

Since both b_{yx} and b_{xy} are negative

$$\therefore \quad r = -0.66$$

63. If the two regression coefficient are 0.8 and0.45 then coefficient of correlation r is ____

(a) 0.3 (b) 0.8 (c) 0.9 (d) 0.6

Ans.: (d)

Explanation : $b_{yx} = 0.8$ and $b_{xy} = 0.45$

$$r = \sqrt{b_{xy} \cdot b_{yx}} = \sqrt{(0.8 \times 0.45)} = 0.6$$

Test Your Knowledge

1. The standard deviation (σ) of a distribution 2, 3, 5 is given by
   (a) 2 (b) 3.88 (c) 1.25 (d) 4.12

2. For a given distribution if value of $\sum fx^2 = 122$, $N = \sum f = 5$, $\bar{x} = 4$ then value of standard deviation (σ) is given by
   (a) 2 (b) 3.88 (c) 5.13 (d) 2.90

3. From the following data

Team	S.D (σ)	A.M $\bar{x}$
A	2	5
B	2.5	4

   The more consistent team is
   (a) Team A (b) Team B
   (c) Team equally consistent (d) can't say

4. If arithmetic mean of four numbers is 15, one item 19 is replaced by 23, then new arithmetic mean is,
   (a) 16 (b) 17 (c) 18 (d) 20

5. If the first four moments of a distribution about the working mean A = 2 are 1, 5, – 20, 10 then the third moment μ_3 about mean of distribution is given by
   (a) – 10 (b) 30 (c) – 33 (d) – 20

6. The first four moments about the working mean 44.5 of a distribution are – 0.4, 2.99, – 0.08 and 27.63 then value of moments about mean μ_2 is _______
   (a) 2.83 (b) 3.99 (c) 2.2 (d) 5.9

7. The first three moments about the value 2 of a distribution are 1, 16 and – 40 then value of standard deviation (σ) is ____
   (a) 3.87 (b) 4.03 (c) 13.31 (d) – 4.09

8. For the data of distribution : $n = 100$, $\sum fd = 50$, $\sum fd^2 = 1970$, $\sum fd^3 = 2948$, $\sum fd^4 = 86752$ where $d = X - 48$ then value of coefficient of skewnessβ_1 is _______
   (a) 0.23 (b) 0.0000044 (c) – 0.20 (d) 0.35

9. The first four moments about working mean 30.2 of a distribution are 0.255, 6.222, 30.211 and 400.25 the central moment μ_4 is _______
   (a) 371.85 (b) 341.57 (c) 270.71 (d) 291.53

10. The first four moments about working mean of the distribution are 0.25, 0.7 and 18.75 then coefficient of skewnessβ_2 is _______
   (a) 2.87 (b) 3 (c) 0.37 (d) 3.87

11. The first two moments about the working mean 30.2 of a distribution are 0.255, 6.222 then value of S.D. (σ) is _______

(a) 3 (b) 3.22 (c) 2.48 (d) 5.7

12. The value of central moment μ_2 of the following distribution is

x	1	2	3	4	5
f	6	15	23	42	62

(a) 2.75 (b) 3.01 (c) – 1.72 (d) 1.34

13. Given n = 25, $\sum x = 75$, $\sum y = 100$, $\sum x^2 = 250$, $\sum y^2 = 500$, $\sum xy = 325$ then value of coefficient of correlation r is _______

(a) – 0.7 (b) 0.8 (c) 0.5 (d) 0.9

14. Two regression lines are $5y - 8x + 17 = 0$ and $2y - 5x + 14 = 0$ also $\sigma_y^2 = 16$ then variance of x is _______

(a) 8 (b) 2 (c) 16 (d) 4

15. If two lines of regression are $9x + y - \lambda = 0$ and $4x + y = \mu$ and mean of x and y are respectively $\bar{x} - 2$, $\bar{y} = -3$ then values of λ and μ are

(a) $\lambda = 15; \mu = 5$ (b) $\lambda = 2 ; \mu = -3$ (c) $\lambda = 19 ; \mu = 17$(d) $\lambda = 12 ; \mu = 15$

16. The probability of drawing Ace from a well shuttled pack of cards is _______

(a) $\frac{1}{52}$ (b) $\frac{3}{13}$ (d) $\frac{3}{52}$ (d) $\frac{1}{13}$

17. Two dice are thrown. The probability of getting double is

(a) $\frac{1}{6}$ (b) $\frac{1}{36}$ (c) $\frac{4}{36}$ (d) $\frac{1}{3}$

18. If probability of success p = 0.7 then probability of failure q = _______

(a) 0.7 (b) 1.7 (c) – 0.7 (d) 0.3

19. Two dice are thrown at a time. What is the probability of getting 10 points.

(a) $\frac{2}{3}$ (b) $\frac{1}{4}$ (c) $\frac{1}{6}$ (d) $\frac{1}{12}$

20. 20% of bolts produced by machine are defective. The mean and standard deviation of defective bolts in total of 900 bolts are respectively.

(a) 180 and 12 (b) 12 and 100 (c) 12 and 180 (d) 9 and 0.8

21. Slope of regression line of x on y is

(a) $r\frac{\sigma_x}{\sigma_y}$ (b) r(x, y) (c) $\frac{\sigma_x}{\sigma_y}$ (d) $\frac{\sigma_y}{\sigma_x}$

22. In regression line y on x, b_{yx} is given by

(a) cov (x, y) (b) r(x, y) (c) $\frac{\text{cov}(x, y)}{\sigma_x^2}$ (d) $\frac{\text{cov}(x, y)}{\sigma_y^2}$

23. If the two regression coefficient are 0.16 and 4 then the correlation coefficient is

(a) 0.08 (b) –0.8 (c) 0.8 (d) 0.64

24. If covariance between x and y is 10 and the variance of x and y are 16 and 9 respectively then coefficient of correlation r(x, y) is

(a) 0.833 (b) 0.633 (c) 0.527 (d) 0.745

25. Given the following data $r = 0.5$, $\Sigma xy = 350$, $\sigma x = 1$, $\sigma y = 4$, $\bar{x} = 68$, $\bar{y} = 62.125$. The value of n (number of observation) is

(a) 5 (b) 7 (c) 8 (d) 10

Answers Key

1. (c)	2. (a)	3. (a)	4. (a)	5. (c)	6. (a)	7. (a)	8. (b)	9. (a)	10. (b)
11. (c)	12. (d)	13. (c)	14. (d)	15. (a)	16. (d)	17. (a)	18. (d)	19. (d)	20. (a)
21. (a)	22. (c)	23. (c)	24. (a)	25. (a)					

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Unit 4 Probability and Probability Distribution

1. A bag contains 5 white 6 red balls. Two balls are drawn at random. Then the probability of both will be white is _______

(a) $\frac{5}{7}$ (b) $\frac{2}{11}$ (c) $\frac{6}{11}$ (d) $\frac{5}{11}$

Ans.: (b)

Explanation : Total balls $= 5 + 6 = 11$

Total No. of ways $= {}^{11}C_2 = \frac{11 \times 10}{2 \times 1} = 55$

Out of 5 white balls 2 drawn at random $= 5C_2 = 10$

$\therefore$ Probability of both will be while $= \frac{10}{55} = \frac{2}{11}$

2. The probability of drawing Ace from a well shuttled pack of cards is _______

(a) $\frac{1}{52}$ (b) $\frac{3}{13}$ (d) $\frac{3}{52}$ (d) $\frac{1}{13}$

Ans.: (d)

Explanation : Total cards $=$ 52, Total Ace $= 4$

Required probability $= \frac{4}{52} = \frac{1}{13}$

3. A box contains 10 white 5 red 7 black balls. Three balls are drawn at random then probability of all will be black is _______

(a) $\frac{3}{22}$ (b) $\frac{3}{7}$ (c) $\frac{15}{7}$ (d) $\frac{1}{44}$

Ans.: (d)

Explanation : Total $= 10 + 5 + 7 = 22$

Total No. of ways $= {}^{22}C_3 = 1540$

Out of 7 black ball 3 drawn at random $= {}^{7}C_3 = 35$

$\therefore$ Probability of all three will be black $= \frac{35}{1540} = \frac{1}{44}$

4. From a well shuffled pack of cards two cards are drawn at random. Find the probability that they from a King, Queen combination.

(a) $\frac{8}{663}$ (b) $\frac{1}{26}$ (c) $\frac{2}{13}$ (d) $\frac{4}{51}$

Ans.: (a)

Explanation : From 52 cards two cards at random can be drawn in ${}^{53}C_2$ ways

$$\therefore \quad n = {}^{53}C_2 = \frac{52!}{2!\,50!} = \frac{52 \times 51 \times 50!}{2 \times 50!} = 1326$$

There are 4 king and 4 Queen cards. One king card from 4 cards can be drawn in ${}^4C_1$ways.

One Queen card from 4 cards can be drawn in ${}^4C_1$ways.

Combination is ${}^4C_1 \times {}^4C_1 = \frac{4!}{1!\,3!} \times \frac{4!}{1!\,3!} \times \frac{4 \times 3!}{3!} \times \frac{4 \times 3!}{3!} = 16$

$$P\text{ (King Queen)} = \frac{16}{1326} = \frac{8}{663}$$

5. Two dice are thrown. The probability of getting double is

(a) $\frac{1}{6}$ (b) $\frac{1}{36}$ (c) $\frac{4}{36}$ (d) $\frac{1}{3}$

Ans. (a)

Explanation : S = [(1, 1), (1, 2), (1,3) (1, 4) (1, 5), (1, 6)
(2, 1), (2, 2), (2,3) (2, 4) (2, 5), (2, 6)
(3, 1), (3, 2), (3,3) (3, 4) (3, 5), (3, 6)
(4, 1), (4, 2), (4,3) (4, 4) (4, 5), (4, 6)
(5, 1), (5, 2), (5,3) (5, 4) (5, 5), (5, 6)
(6, 1), (6, 2), (6,3) (6, 4) (6, 5), (6, 6)]

S = 36

There are six doublets

$$\therefore \quad P\text{ (getting doublet)} = \frac{6}{36} = \frac{1}{6}$$

6. If a throw is made with two dice. Find the probability of getting score 4

(a) $\frac{1}{4}$ (b) $\frac{1}{6}$ (c) $\frac{1}{12}$ (d) $\frac{1}{3}$

Ans.: (c)

Explanation : From above example,

S = 36 ; n = 36

Pairs with sum 4 are (1, 3) (2, 2), (3, 1)

$\therefore \quad m = 3;$

$$\therefore \quad P\text{ (getting score 4)} = \frac{3}{36} = \frac{1}{12}$$

7. Four cards are drawn from a pack of cards then probability of all are diamonds is __

(a) $\frac{11}{4165}$ (b) $\frac{4}{13}$ (c) $\frac{9}{52}$ (d) $\frac{17}{7250}$

Ans.: (a)

Explanation : Total cards = 52

Total No. of ways = $^{52}C_4 \frac{52 \times 51 \times 50 \times 49}{1 \times 2 \times 3 \times 4} = 270725$

Total diamonds cards in a pack = 13

∴ out of 13 card 4 can be drawn

No. of ways = $^{13}C_4 = 715$

Required probability $= \frac{715}{270725} = \frac{11}{4165}$

8. If probability of success p = 0.7 then probability of failure q = _______

(a) 0.7 (b) 1.7 (c) – 0.7 (d) 0.3

Ans.: (d)

Explanation: $p + q = 1 \therefore q = 0.3$

9. The probability of strike a large is $\frac{1}{5}$. If six bombs are there then probability that exactly two will strike the target is _______

(a) 0.5 (b) 0.2 (c) 0.738 (d) 0.2458

Ans.: (d)

Explanation : Given

Let x = Probability of a bomb dropped from a plane will strike the target,

i.e. $p = \frac{1}{5} = 0.2$

∴ $q = 1 - p = 0.8 \; (\because p + q = 1)$

We have, n = 6, r = 2 Binomial distribution, probability of r success.

$p\,(x = r) = {}^nC_r p^r q^{n-r}$

Here, n = 6

∴ Probability of exactly two bombs will strike the target is,

$P\,(2) = {}^5C_2\, p^2\, q^4 = {}^6C_2\, (0.2)^2\, (0.8)^4 = 0.2458$

10. Two dice are thrown at a time. What is the probability of getting 10 points.

(a) $\frac{2}{3}$ (b) $\frac{1}{4}$ (c) $\frac{1}{6}$ (d) $\frac{1}{12}$

Ans.: (d)

Explanation :

n = 36

pairs of sum 10 are : (4, 6), (5, 5), (6, 4)

∴ m = 3

P (getting 10 points) $= \frac{m}{n} = \frac{3}{36} = \frac{1}{12}$

11. If a throw is made with two dice, find the probability of getting score at least 10.

(a) $\frac{1}{4}$ (b) $\frac{1}{6}$ (c) $\frac{2}{3}$ (d) $\frac{1}{12}$

Ans.: (b)

Explanation:

$$n = 36$$

Pairs of sum at least 10 are :

(4, 6) (5, 5) (5, 6) (6, 4) (6, 6)

i.e. $m = 6$

$$P \text{ (getting at least 10 sum)} = \frac{m}{n} = \frac{6}{36} = \frac{1}{6}$$

12. What is the probability of getting two heads, if an unbiased coin is thrown four tones.

(a) $\frac{5}{16}$ (b) $\frac{16}{5}$ (c) 16 (d) $\frac{3}{8}$

Ans.: (d)

Explanation :

Given : $r = 2 ; p = \frac{1}{2}$

$q = \frac{1}{2}$; $n = 4$

By Binomial distribution :

$$P (r = 2) = {}^nC_r p^r q^{n-r} = {}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = 6 \times \frac{1}{4} \times \frac{1}{4} = \frac{3}{8}$$

13. Out of 800 families with 4 children each, then find the number of families having no girls.

(a) 100 (b) 125 (c) 50 (d) 9

Ans.: (c)

Explanation :

Given : $N = 800, n = 4; p = \frac{1}{2}, q = \frac{1}{2}, r = 0$

$$P (r = 0) = {}^nC_r p^r q^{n-r} {}^4C_0\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^4 = 1 \times 1 \times \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

$\therefore$ Number of families having no girls $= 800 \times \frac{1}{16} = 50$

14. Given, the probability of a man now aged 60 years will live 70 years, is 0.65. What is the probability that out of 10 men, (60 years old) 2 men will live upto 70 years.

(a) 0.004281 (b) 0.003281 (c) 0.23 (d) 0.4281

Ans.: (a)

Explanation :

Here $P = 0.65$, $q = 1 - p = 0.35$; $r = 2$; $n = 10$

P (only two men will live upto 70 years)

$$= {}^nC_r p^r q^{n-r}$$

$$= {}^{10}C_2\,(0.65)^2\,(0.35)^8$$

$$= 0.004281$$

15. Given the probability that a present hit target in shooting practice is 0.4. If the shoots 10 times then the probability that the man hits the target is,

(a) $1 - (0.6)^{10}$ (b) $(0.6)^{10}$ (c) $1 - (0.4)^{10}$ (d) (0.4) 10.4

Ans.: (a)

Explanation :

Here $p = 0.4$; $q = 1 - p = 0.6$; $n = 10$; $r = 0$

(No hits the target)

P (r = 0) : (No hits the target)

$$= {}^nC_r p^r q^{n-r} = {}^{10}C_3\,(0.4)^{10}\,(0.6)^{10}$$

$$= 1 \times 1 \times (0.6)^{10} = (0.6)^{10}$$

Hence, the probability that the man hits the target,

$$= 1 - p\,(r = 0) = 1 - (0.6)^{10}$$

16. A box contains 100 bulbs, out of which 10 are defective. A sample of 5 bulbs are drawn. What is the probability of none is defective.

(a) $(10)^5$ (b) $(9)^5$ (c) $\left(\frac{9}{10}\right)^5$ (d) $\left(\frac{10}{9}\right)^5$

Ans.: (c)

Explanation :

Given, $p = \frac{10}{100} = 0.1$; $q = 1 - p = 0.9$

$n = 5$; $r = 0$ (none of defective)

By Binomial distribution,

$$P\,(r = 0) = {}^nC_r p^r q^{n-r} = {}^5C_0\,(0.1)\,0\,(0.9)^5$$

$$= 1 \times 1 \times (0.9)^5 = (0.9)^5 = \left(\frac{9}{10}\right)^5$$

17. In 100 set of tosses of a coin, the number of cases you expect 7 head and 4 tail are,

(a) 10 (b) 20 (c) 9 (d) 12

Ans.: (d)

Explanation :

Given $N = 100$;

$n = 10$; $p = \frac{1}{2}$; $q = \frac{1}{2}$; $r = 7$;

$$P(r = 7) = N\,[{}^nC_r p^r q^{n-r}] = 100\left[{}^{10}C_2\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^3\right]$$

$$= 100\left[120 \times \frac{1}{2^7} \times \frac{1}{2^3}\right] = 100\left[120 \times \frac{1}{2^{10}}\right] = 11.72 = 12$$

18. 20% of bolts produced by machine are defective. The mean and standard deviation of defective bolts in total of 900 bolts are respectively.

(a) 180 and 12 (b) 12 and 100 (c) 12 and 180 (d) 9 and 0.8

Ans.: (a)

Explanation :

Given : $p = 20\% = \frac{20}{100} = 0.2\ ;\ q = 0.8;\ \ n = 900.$

By Binomial distribution

$$\text{Mean} = np = 900 \times 0.2 = 180$$

and Standard deviation $= \sqrt{npq} = \sqrt{(900) \times (0.2) \times (0.8)} = 12$

19. The mean and standard deviation of a binomial distribution are $\frac{3}{4}$ and $\frac{7}{8}$ respectively then the probability of success in a single trials is,

(a) 0.12 (b) 0.8 (c) 0.2 (d) 0.75

Ans.: (c)

Explanation :

Given : Mean (np) $= \frac{3}{4}$;

Standard deviation $= \sqrt{npq} = \frac{7}{9}$

$\therefore$ $np = \frac{3}{4}$ and $npq = \frac{49}{81}$

$\therefore$ $\left(\frac{3}{4}\right) q = \frac{49}{81} \Rightarrow q = \frac{49}{81} \times \frac{3}{4} = \frac{196}{243}$

$q = 0.8$

$\therefore$ $p = 1 - q = 0.2$

20. The mean and variance of binomial probability distribution are 6 and 4 respectively then number of trails 'n' is,

(a) 10 (b) 18 (c) 10 (d) 9

Ans.: (b)

Explanation :

Given : Mean $= np = 6$;

and variance $= npq = 4$

$\therefore \quad npq = 4 \Rightarrow \quad q = \frac{4}{6} = \frac{2}{3}$

$p = 1 - q = 1 - \frac{4}{6} = \frac{2}{6} = \frac{1}{3}$

$\therefore \quad np = 6$

$\Rightarrow \quad n\left(\frac{1}{3}\right) = 6 \Rightarrow n = 18$

Number of trials = 18

21. The mean and standard deviation of a Binomial distribution are 25 and 5 respectively. Number of trials 'n' is,

(a) 42 (b) 40 (c) 9 (d) 44

Ans.: (a)

Explanation :

Given : Mean $= np = 25$;

and standard deviation $= \sqrt{npq} = 3$

$\Rightarrow \quad npq = 9$

$\therefore \quad q = \frac{9}{25} = 0.4$

$\Rightarrow \quad p = 0.6$; $np = 25$

$n = \frac{25}{p} = \frac{25}{0.6} = 42$

22. The mean and standard deviation of Binomial probability distribution are 6 and $\sqrt{2}$ respectively then what is the value of p ($r \geq 2$).

(a) 0.88 (b) 0.99 (c) 0.1 (d) 0.55

Ans.: (b)

Explanation :

Given : Mean (np) = 6 ;

Standard derivation $\left(\sqrt{npq}\right) = \sqrt{2}$

$(npq) = 2$

Also given $r \geq 2$

$\therefore \quad np = 6$ and $npq = 2$

$\Rightarrow \quad q = \frac{1}{3}$ and $p = 1 - q = 1 - \frac{1}{3} = \frac{2}{3}$

and $\quad n = \frac{6}{p} = \frac{6}{(2/3)} = 9$

$p\,(r = 0, 1) = {}^nC_0 p^r q^{n-r}$

$$= {}^{n}C_{10}\left(\frac{2}{3}\right)^{0}\left(\frac{1}{3}\right)^{9} + {}^{9}C_{1}\left(\frac{2}{3}\right)^{1}\left(\frac{1}{3}\right)^{8}$$

$$= 1 \times 2 \times \frac{1}{3^9} + 9 \times \frac{2}{3} \times \frac{1}{3^8} = \frac{2}{3^9} + 9\frac{2}{3^9} = 10\left(\frac{2}{3^9}\right)$$

$$p\,(r \geq 2) = 1 - p\,(r = 0, 1)$$

$$= 1 - 10\left(\frac{2}{3^9}\right)$$

$$= 0.999$$

23. On an average the box containing 10 articles is likely to have 2 defectives. In a consignment of 100 boxes, the total number of boxes expected to have exactly one defective are _______

(a) 72 (b) 89 (c) 35 (d) 29

Ans.: (d)

Explanation :

$n = 10, \quad p = 0.2$

$\therefore \quad q = 1 - p = 0.8$

By Binomial distribution

The Probabilityof one defective $= {}^{10}C_1\, p^1\, q^9 = 0.2684$

$\therefore$ Total no. of boxes $= 100 \times 0.268 = 26.8 \approx 29.$

24. On an average the box containing 10 articles is likely to have 2 defectives. In a consignment of 100 boxes, the total number of boxes expected to have exactly at least one defective are _______

(a) 29 (b) 72 (c) 89 (d) 35

Ans.: (c)

Explanation :

$n = 10, \quad p = 0.2$

$\therefore \quad q = 1 - p = 0.8$

By Binomial Distribution

The Probability of at least one defective $= p\,(x \geq 1)$

$= P\,(1) + P\,(2) + \ldots.\, P\,(10)$

$= 1 - [P\,(0)] = 1 - [{}^{10}C_0\, p^0\, q^{10}] = 0.8926$

$\therefore$ Total no. of boxes $= 100 \times 0.8926 = 89.26 \approx 89.$

25. In 10 tosses of a coin the probability to get at least 8 heads is _______

(a) 0.231 (b) 0.5 (c) 0.39 (d) 0.055

Ans.: (d)

Explanation : $n = 10$

Probability to get head $= p = 0.5$

$\therefore \quad q = 0.5$

By Binomial Distribution

$$P(r \geq 8) = P(8) + P(9) + P(10)$$
$$= {}^{10}C_8\, p^8 q^2 + {}^{10}C_9\, p^9 q + {}^{10}C_{10}\, p^{10} q^0 = 0.055$$

26. In a Binomial distribution mean = 2 and mean + variance = 3 then $P(r \geq 2)$ is _____

(a) 0.531 (b) 0.688 (c) 0.259 (d) 0.317

Ans.: (b)

Explanation: Mean = np = 2 andvariance = npq

$\therefore$ mean + variance = 3 $\Rightarrow$ variance = 1

$\therefore$ $\frac{npq}{np} = \frac{1}{2} \Rightarrow$ q = 0.5

$\therefore$ p = 0.5 and n = 4

By Binomial Distribution,

$$P(r \geq 2) = P(2) + P(3) + P(4) = 1 - [P(0) + P(1)]$$
$$= 1 - \left[{}^{4}C_0\, p^0 q^4 + {}^{4}C_1\, p^1 q^3\right]$$
$$P(r \geq 2) = 0.688$$

27. In a Binomial distribution mean = 2 and mean + variance = 3 then $P(r \leq 2)$ is ___

(a) 0.531 (b) 0.317 (c) 0.688 (d) 0.259

Ans.: (c)

Explanation : Mean = np = 2 andvariance = npq

$\therefore$ mean + variance = 3 $\Rightarrow$ variance = 1

$\therefore$ $\frac{npq}{np} = \frac{1}{2} \Rightarrow$ q = 0.5

$\therefore$ p = 0.5 and n = 4

By Binomial Distribution,

$$P(r \leq 2) = P(0) + P(1) + P(2)$$
$$= {}^{4}C_0\, p^0 q^4 + {}^{4}C_1\, p q^3 + {}^{4}C_2\, p^2 q^2 = 0.688$$

28. One percent of articles from a certain machine are defective then the probability of no defective in a sample of 100 is_______

(a) 0.421 (b) 0.723 (c) 0.368 (d) 0.579

Ans.: (c)

Explanation :

Step I : Given, One percent of articles are deflective,

$\therefore$ p = 0.01 andhere n = 100

Mean z = np = 1

Step II : By Poisson's Distribution,

$$P(r) = \frac{z^r e^{-z}}{r!}$$

Probability of one defective is,

$$P\,(r = 1) = \frac{z\,e^{-z}}{1!} = 0.3679$$

$$P\,(0) = \frac{z\,e^{-z}}{0!} = 0.2231$$

29. The average number of misprints per page of a book is 1.5 then number of pages containing more than one misprints if book contains 900 pages using Possion's distribution are,

(a) 802 (b) 705 (c) 699 (d) 578

Ans.: (c)

Explanation:

$$\text{Mean (m)} = 15$$

By Poission's Distribution,

The probability of a book containing more than one misprints = P (1) + P (2) + …

$$= 1 - P\,(0) = 1 - \left[\frac{z^0\,e^{-m}}{0!}\right] = 0.7768$$

$$\text{No. of pages} = 900 \times 0.78 = 699.12 \approx 699$$

30. Number of road accidents follows a Poission's distribution with mean 5, In a certain month on the highway the probability of more than 2 accidents is ….

(a) 0.875 (b) 0.357 (c) 0.523 (d) 0.125

Ans.: (a)

Explanation :

$$\text{Mean} = z = 5$$

Probability of more than 2 accidents

$$= P\,(3) + P\,(4) + \ldots = 1 - [P\,(0) + P\,(1) + P\,(2)]$$

$$= 1 - \left[\frac{z^0\,e^{-z}}{0!} + \frac{z\,e^{-z}}{1!} + \frac{z^2\,e^{-z}}{2!}\right] = 0.875$$

31. In a certain factory manufacturing condensers there is a chance that 1% of the condenses are defective. The condensers are packed in a boxes of 100 then using poisson distribution number of boxes contains more than 2 faulty condensers = ____

(a) 80 (b) 8 (c) 4 (d) 3

Ans.: (b)

Explanation :

$$P = 1\% = 0.01 \text{ and } n = 100$$

$$\text{Mean} = m = np = 1$$

By Poission's Distribution,

$$P\,(r > 2) = P\,(3) + P\,(4) + \ldots = 1 - [P\,(0) + P(1) + P(2)]$$

$$= 1 - \left[\frac{z^0\,e^{-z}}{0!} + \frac{z\,e^{-z}}{1!} + \frac{z^2\,e^{-z}}{2!}\right] = 1 - e^{-z}$$

$$\left[1 + z + \frac{z^2}{2} + ...\right] = 0.08030$$

Number if boxes containing more than 2 faulty condensers

$$= 0.08030 \times 100 = 8.03 \approx 8$$

32. If the probability of bad reaction from a certain injection is 0.001 then the chance that out of 2000 individuals more than 2 will get a bad reaction is_______

(a) 0.999 (b) 0.231 (c) 0.677 (d) 0.3233

Ans.: (d)

Explanation :

Step I : Given $P = 0.001$, $n = 2000$

More than2, means $r = 3$ to 2000

$\therefore$ $P(r > 2) = 1 - P(r = 0, 1, 2)$

Step II : By Poisson Distribution (as $n \to \infty$, $P \to 0$)

$$P(r) = \frac{z^r e^{-z}}{r!}$$

33. If x is a normal variable with mean 30 and S.D. 5, the probability of $(26 \le x \le 40)$ is _______

[Given : $z = 0.8, A = 0.2881, z = 2, A = 0.4772$]

(a) 0.7653 (b) – 0.1891 (c) 0.7653 (d) 0.1891

Ans.: (c)

Explanation :

Given mean $= \mu = 30$ and S.D $= \sigma = 5$

$$\therefore \quad P(26 \le x \le 40) = P\left(\frac{26-\mu}{\sigma} \le z = \frac{x-\mu}{\sigma} \le \frac{40-\mu}{\sigma}\right)$$

$$= P(0.8 \le z \ge 2)$$

$$= P(-0.8 \le z \le 0) + P(0 \le z \le 2)$$

$$= 0.2881 + 0.4772 = 0.7653$$

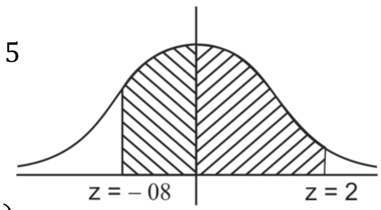

Fig. : 1

34. Suppose the height of American men are normally distributed with mean $\mu = 68$ inches and S.D, $\sigma = 2.5$. The probability of American men who are atleast 72 inches tall is _______

[**Given :** Area corresponding 1.6 is 0.4452]

(a) 0.0123 (b 0.234 (c) 0.312 (d) 0.0548

Ans.: (d)

Explanation :

$$\mu = 68, \quad \sigma = 2.5, \quad x = 72$$

$$z = \frac{x-\mu}{\sigma} = 1.6 \; (+\text{ ve})$$

$$\therefore \quad P(x \ge 72) = P(z \ge 1.6)$$

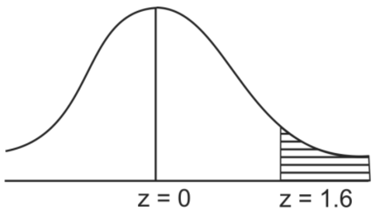

Fig. : 2

$$= \frac{1}{2} - A\,(1.6)$$

$$= \frac{1}{2} - 0.4452$$

$$= 0.0548$$

35. Suppose to x is normal distribution with mean $\mu = 70$ and S.D, $\sigma = 2$, then P $(68 \le x \le 74)$ is _______

[**Given :** For z = 1, A = 0.314 and for z = 2, A = 0.4772]

(a) 0.8184 (b 0.1359 (c) 0.9772 (d) 0.8413

Ans.: (a)

Explanation:

at $x_1 = 68, \quad x_2 = 74$

$\mu = 70, \quad \sigma = 2,$

$$z_1 = \frac{x_1 - \mu}{\sigma} = -1 \ (-ve)$$

$$z_2 = \frac{x_2 - \mu}{\sigma} = 2 \ (+ ve)$$

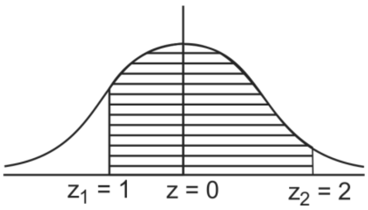

Fig. : 3

$\therefore$ P $= (68 \le x \le 74)$

P $= (-1 \le z \le 2)$

$= A(-1) + A(2)$

$= 0.3143 + 0.4772$ **= 0.8184**

36. Suppose x is normal distribution with mean $\mu = 70$ and S.D. $\sigma = 2$ then P $(x \le 73)$ is _______

[Given : A = 0.4332 for z = 1.5]

(a) 0.0668 (b) 0.9332 (c) 0.4332 (d) 0.0658

Ans.: (b)

Explanation :

$x = 73, \quad \mu = 70, \sigma = 2$

$$z = \frac{x - \mu}{\sigma} = \frac{73 - 70}{2} = \frac{3}{2} = 1.5 \ (+ ve)$$

$\therefore$ P $(x \le 73)$ = P $(z \le 1.5)$

$$= \frac{1}{2} + A\,(1.5)$$

$= 0.5 + 0.4332$

$= 0.9332$

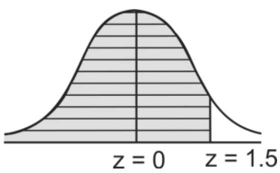

Fig. : 4

37. Suppose x is normal distribution with mean $\mu = 50$ and S.D. $\sigma = 2$ then $P(x \le 45)$ is ______

[Given : Area corresponding to 2.5 is 0.4938]

(a) 0.62 (b) 0.9938 (c) 0.0062 (d) 0.4938

Ans.: (c)

Explanation :

$$\mu = 50, \sigma = 2$$

$$x = 45$$

$$z = \frac{x-\mu}{\sigma} = \frac{45-50}{2} = \frac{-5}{2} = -2.5 \text{ (–ve)}$$

$$\therefore \quad P(x \le 45) = P(z \le -2.5)$$

$$= 0.5 - A(2.5)$$

$$= 0.5 - 0.4938$$

$$= 0.0062$$

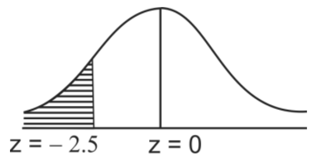

Fig. : 5

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May 2018

Solved University Questions Paper

Q. 1 (a) Solve any two:

i) $(D^2 + 4)\, y = e^x + x^2$

Solution : The auxiliary equation is $m^2 + 4 = 0$ and its roots are $m = \pm 2i$ imaginary and distinct

$\therefore$

$$C.F = C_1 \cos 2x + C_2 \sin 2x$$

$$P.I = \frac{1}{D^2+4}(e^x + x^2) = \frac{1}{D^2+4} e^x + \frac{1}{D^2+4} x^2$$

$$= \frac{1}{(1)^2+4} e^x + \frac{1}{4}\left[1+\frac{D^2}{4}\right]^{-1} x^2$$

$$= \frac{1}{5} e^x + \frac{1}{4}\left[1 - \frac{D^2}{4} + \frac{D^4}{16} \ldots\right] x^2 \qquad \begin{bmatrix} Dx^2 = 2x \\ D^2 x^2 = 2 \\ D^3 x^2 = 0 \end{bmatrix}$$

$$= \frac{1}{5} e^x + \frac{1}{4}\left[x^2 - \frac{1}{4}(2)\right]$$

The general solution is y = C.F. and P.I

$$\boxed{y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{5} e^x + \frac{1}{4}\left(x^2 - \frac{1}{2}\right)}$$

---

ii) $(D^2 + 6D + 9)\, y = x^{-4} e^{-3x}$

Solution : The auxiliary equation is $m^2 + 6m + 9 = 0$ its roots are $m = -3, -3$ real and equal

$$C.F = (C_1 + C_2 x)\, e^{-3x}$$

$$P.I = \frac{1}{D^2+6D+9} e^{-3x} x^{-4} = e^{-3x} \frac{1}{(D-3)^2 + 6(D-3) + 9} x^{-4}$$

$$= e^{-3x} \frac{1}{D^2 - 6D + 9 + 6D - 18 + 9} x^{-4} = e^{-3x} \frac{1}{D^2} x^{-4}$$

$$P.I = e^{-3x} \cdot \frac{1}{D} \int x^{-4}\, dx = e^{-3x} \int \frac{x^{-3}}{-3}\, dx = e^{-3x} \frac{x^{-2}}{6}$$

The general solution is y = C.F. + P.I $\quad \boxed{y = (C_1 + C_2 x)\, e^{-3x} + \frac{e^{-3x}}{6x^2}}$

iii) $x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + y = \frac{1}{x} \sin(\log x)$

Solution :

Put $x = e^z$

$\therefore$ $z = \log x , \frac{1}{x} = e^{-z}$

Now $x \frac{dy}{dx} = Dy$ where $D = \frac{d}{dz}$

$$x^2 \frac{d^2y}{dx^2} = D(D-1)y$$

Substituting these values in the given differential equations.

We get , $D(D-1)y + 3Dy + y = e^{-z} \sin z$

$$(D^2 - D + 3D + 1)y = e^{-z} \sin z$$

$$(D^2 + 2D + 1)y = e^{-z} \sin z$$

The auxiliary equation is $m^2 + 2m + 1 = 0$

Its roots are $m = -1, -1$ real and equal.

$$\text{C.F.} = (C_1 + C_2 z)e^{-z} = (C_1 + C_2 \log x)\frac{1}{x}$$

$$\text{P.I} = \frac{1}{D^2 + 2D + 1} e^{-z} \sin z = e^{-z} \frac{1}{(D-1)^2 + 2(D-1) + 1} \sin z$$

$$= e^{-z} \frac{1}{D^2 - 2D + 1 + 2D - 2 + 1} \sin z = e^{-z} \frac{1}{D^2} \sin z$$

$$= e^{-z} \frac{1}{(-1^2)} \sin z \qquad [\text{Put } D^2 = -(1^2)]$$

$$\text{P.I} = -e^{-z} \sin z = -\frac{1}{x} \sin(\log x)$$

The complete solution is y = C.F. + P.I.

$$\boxed{y = (C_1 + C_2 \log x)\frac{1}{x} - \frac{1}{x}\sin(\log x)}$$

(b) Solve the integral equation :

$$\int_0^\infty f(x) \sin \lambda x \, dx = \begin{cases} 1 - \lambda & 0 \le \lambda \le 1 \\ 0 & \lambda \ge 1 \end{cases}$$

Solution :

The inverse Fourier sine transform is

$$f(x) = \frac{2}{\pi}\int_0^\infty f_s(\lambda) \sin \lambda x \, d\lambda$$

$$f(x) = \frac{2}{\pi}\int_0^1 (1-\lambda) \sin \lambda x \, d\lambda + \int_1^\infty (0) \sin \lambda x \, d\lambda$$

$$f(x) = \frac{2}{\pi}\left[(1-\lambda)\left(\frac{-\cos\lambda x}{x}\right) - (0-1)\left(\frac{-\sin\lambda x}{x^2}\right) + 0\right]_0^1$$

$$\therefore \quad f(x) = \frac{2}{\pi}\left[0 - \frac{\sin x}{x^2} - (1)\left(-\frac{1}{x}\right) + 0\right]$$

$$\mathbf{f(x) = \frac{2}{\pi}\left[-\frac{\sin x}{x^2} + \frac{1}{x}\right] = \frac{2}{\pi}\left(\frac{x - \sin x}{x^2}\right)}$$

Q. 2 (a) An uncharged condenser of capacity C charged by applying an e.m.f. of value E sin $\frac{t}{\sqrt{LC}}$ through leads of inductance L and of negligible resistance. The charge θ on the plate of condenser satisfies the differential equation $\frac{d^2\theta}{dt^2} + \frac{\theta}{LC} = \frac{E}{L} \sin \frac{t}{\sqrt{LC}}$. Find the charge θ at any time t.

Solution : We take $w^2 = \frac{1}{LC}$ then the differential equation becomes,

$$\frac{d^2\theta}{dt^2} + w^2 \theta = \frac{E}{L} \sin wt$$

Its auxiliary equation is $m^2 + w^2 = 0$ and roots are $m = \pm$ iw imaginary and distinct.

$$C.F. = C_1 + \cos wt + C_2 \sin wt$$

$$P.I = \frac{1}{D^2 + w^2}\frac{E}{L} \sin wt. \qquad \text{[if we put } D^2 = -w \text{ this is a case of failure]}$$

$$= \frac{E}{L}\, t. \frac{1}{2D} \sin wt = \frac{E}{2L}\, t \int \sin wt \, dt = \frac{-E}{2LW} + \cos wt$$

$$P.I = \frac{-E}{2LW} t \cos wt.$$

The complete solution is θ = C.F. + P.I.

$$\theta = C_1 \cos wt + C_2 \sin wt + \frac{E}{2LW} t \cos wt \qquad \ldots(1)$$

$$I = \frac{d\theta}{dt} = -C_1 w \sin wt + C_2 w \cos wt + \frac{E}{2LW}(\cos wt - tw \sin wt) \quad \ldots(2)$$

The initial conditions are $\theta = 0$, $\frac{d\theta}{dt} = 0$ when $t = 0$

$\therefore$ From (i) we get $0 = C_1 + 0 + 0 \Rightarrow C_1 = 0$

From (ii) we get $0 = 0 + C_2\, w + \frac{E}{2LW}\,(1-0) = C_2\, w + \frac{E}{2LW}$

$$\therefore \quad C_2 = \frac{E}{2LW^2}$$

$$\therefore \quad \boxed{\theta = \frac{-E}{2LW^2}\sin wt + \frac{E}{2LW}\, t\cos wt}$$

(b) Solve any one of the following

i) Find $\mathbf{z\left\{\cos\left(k\frac{\pi}{2}+\frac{\pi}{4}\right)\right.,\ k \geq 0}$

Solution :

$$f(k) = \cos\left(k\frac{\pi}{2}+\frac{\pi}{4}\right) \operatorname{cosk}\frac{\pi}{2}\cos\frac{\pi}{4} - \operatorname{sink}\frac{\pi}{2}\cos\frac{\pi}{4}$$

$$\therefore \quad z\{f(k)\} = \cos\frac{\pi}{4}\, z\left\{\cos\frac{\pi}{2}\right\} - \sin\frac{\pi}{4}\, z\left\{\operatorname{sink}\frac{\pi}{2}\right\}$$

$$f(z) = \cos\frac{\pi}{4}\,\frac{z\left(z-\cos\frac{\pi}{2}\right)}{z^2-2z\cos\frac{\pi}{2}+1} - \sin\frac{\pi}{4}\cdot\frac{z\sin\frac{\pi}{2}}{z^2-2z\cos\frac{\pi}{2}+1}$$

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ii) Using inversion integral method find :

$$\mathbf{z^{-1}\left\{\frac{z}{(z-1)(z-2)}\right\}}$$

Solution :

$$f(z) = \frac{z}{(z-1)(z-2)}\ ,\ z = 1, 2, \text{ are simple poles.}$$

i) the residue of f (z) at z = 1 is

$$r_1 = \lim_{z\to 1}\,(z-1)\,z^{k-1}\frac{z}{(z-1)(z-2)}\ \lim_{z\to 1}\frac{z^k}{z-2}$$

$$r_1 = \frac{(+1)^k}{(1-2)} = -(1)^k = -1$$

ii) The residue of f (z) at z = 2 is

$$r_2 = \lim_{z\to 2}\,(z-2)\,z^{k-1}\frac{z}{(z-1)(z-2)} = \lim_{z\to 2}\frac{z^k}{z-1}$$

$$r_2 = \frac{(2)^k}{(2-1)} = (2)^k$$

The inverse z transform of $\frac{z}{(z-1)(z-2)}$ is $z^{-1}\left\{\frac{z}{(z-1)(z-2)}\right\} = r_1 + r_2 = 2^k - 1$, $k \geq 0$.

(c) Solve the following difference equation $y_{k+2} - 5\, y_{k+1} + 6\, y_k = u_k$ with $y_0 = 0$, $y_1 = 1$ and $u_k = 1$ for $k \geq 0$.

Solution : Taking z – Transform of the given equation

$$z\,\{y\,(k+2)\} - 5z\,\{y\,(k+1)\} + 6z\,\{y\,(k) \quad = \quad z\,\{u\,(k)\}$$

$$[z^2\, y\,(z) - z^2\, y\,(0) - z\, y\,(1)] - 5\,[z\, y\,(z) - z\, y\,(0)] + 6y\,(z) \quad = \quad \frac{z}{z-1}$$

$$(z^2 - 5z + 6)\, y\,(z) \quad = \quad \frac{z}{z-1} + z\frac{z_2}{z-1}$$

$$\therefore \qquad y\,(z) \quad = \quad \frac{z^2}{(z-1)\,(z^2-5z+6)} = \frac{z^2}{(z-1)\,(z-2)\,(z-3)}$$

$$y\,(z) \quad = \quad \frac{1}{2}\,\frac{z}{z-1} - 2\frac{z}{z-2} + \frac{3}{2}\,\frac{z}{z-3}$$

$$z^{-1}\,\{y\,(z)\} \quad = \quad \frac{1}{2}\,z^{-1}\left\{\frac{z}{z-1}\right\} - 2\,z^{-1}\left\{\frac{z}{z-2}\right\} + \frac{3}{2}\,z^{-1}\left\{\frac{z}{z-3}\right\}$$

$$\boxed{y\,(k) \quad = \quad \frac{1}{2}\,(1)^k - 2\,(2)^k + \frac{3}{2}\,(3^k)}$$

Q. 3 (a) The first four moments of a distribution about the value 5 are – 4, 22, –117 and 560 respectively. Find the moments about the mean. Also calculate β_1 and β_2.

Solution : Given : $a = 5$, $\mu_1' = -4$, $\mu_2' = 22$, $\mu_3' = -117$, $\mu_4' = 560$

The central moments are :

$$\mu_1 \quad = \quad 0$$

$$\mu_2 \quad = \quad \mu_2' - (\mu_1')^2 = 22 - (-4)^2 = 6$$

$$\mu_3 \quad = \quad \mu_3' - 3\mu_2'\,\mu_1' + 2(\mu_1')3$$

$$= \quad (-117) - 3(22)\,(-4) + 2(-4)^3$$

$$\mu_3 \quad = \quad 19$$

$$\mu_4 \quad = \quad \mu_4' - 4\mu_3'\,\mu_1' + 6\,\mu_2'\,(\mu_1')^2 - 3(\mu_1')^4$$

$$\mu_4 \quad = \quad 560 - 4\,(-117)\,(-4) + 6\,(22)\,(-4)^2 - 3\,(-4)^4$$

$$\mu_4 \quad = \quad 32$$

$$\beta_1 \quad = \quad \frac{\mu_3^2}{\mu_2^3} = \frac{(19)^2}{(6)^3} = 1.67,$$

$$\beta_2 \quad = \quad \frac{\mu_4}{\mu_2^2} = \frac{32}{(6)^2} = 0.89.$$

(b) Fit a straight line of the form y = ax + b to the following data by the least square method:

x	0	6	8	10	14	16	18	20
y	3	12	15	18	24	27	30	33

Solution : The table is as follows :

x	y	x^2	xy
0	3	0	0
6	12	36	72
8	15	64	120
10	18	100	180
14	24	196	336
16	27	256	432
18	30	324	540
20	33	400	660
92	162	1376	2340

The normal equations are

$$\Sigma y = a\,\Sigma x + nb$$

$$\Sigma xy = a\,\Sigma x^2 + b\,\Sigma x$$

$$162 = 92\,a + 8\,b$$

$$2340 = 1376\,a + 92\,b$$

Solving above equations we get

$$a = 1.5 \quad b = 3$$

$$\therefore \quad \boxed{y = 1.5x + 3}$$

(c) There is a small probability of $\frac{1}{1000}$ for any computer produced to be defective. Determine in a sample of 2000 computers the probability that there are :

i) no defectives and ii) 2 defectives

Solution : Step : I

Given : $p = \frac{1}{1000}$,

$n = 2000$

$\therefore \quad \lambda = np = 2$

By Poisson′s distribution we have

$$p(r) = e^{-\lambda}\frac{\lambda^r}{r!} \quad r = 0, 1, 2, \ldots.$$

$$\therefore \quad p(r) = e^{-2}\frac{(2)^r}{r!} \quad (\text{since } \lambda = 2)$$

i) no defectives : $p(r = 0) = e^{-2}\frac{(2)^o}{(0)!} = 0.1353$

ii) Two defectives : $p(r = 2) = e^{-2}\frac{(2)^2}{2!} = 0.2707$

Q. 4 (a) The life time of an article has a normal distribution with mean 400 hours and standard deviation 50 hours. Assuming normal distribution. Find the expected number of articles out of 2000 whose life time lies between 335 hours to 465 hours. [Given z = 1.3, A = 0.4032]

Solution :

Given mean $\mu = 400$

and S.D. $= \sigma = 50$

Then $z = \frac{x-\mu}{\sigma} = \frac{x-400}{50}$

When $x = 335$

$z_1 = \frac{335-400}{50} = -1.3$

When $x = 465$

$z_2 = \frac{465-400}{50} = 1.3$

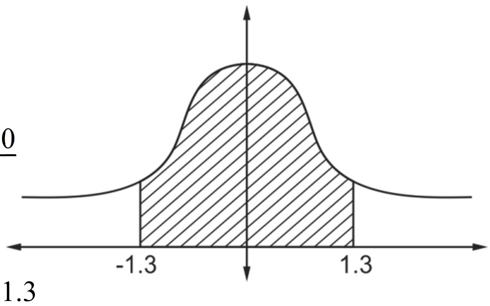

Fig.: 1

By normal distribution we have

$$p(335 < x < 465) = P(-1.3 < z < 1.3) = 2\,p(z = 1.3)$$

$$= 2(0.4032) = 0.8064$$

$\therefore$ Expected number of articles $= 2000 \times 0.8064$

$= 1612.8 = 1613$

(b) On an average 20% of the workers in an industry suffer with a certain diseases. If 12 workers are chosen from the industry find the probability that :

i) Exactly 2 workers suffer from the diseases.

ii) At least one worker suffers from the disease.

Solution :

Given: $p = \frac{20}{100} = 0.2$ and $n = 12$

$q = 1 - p = 1 - 0.2 = 0.8$

By using binomial distribution we have.

$$p(r) = {}^{n}C_r\, p^r\, q^{n-r} = {}^{12}C_r\,(0.2)^r\,(0.8)^{12-r}$$

i) Exactly two workers suffer from the diseases.

$$p(r = 2) = {}^{12}C_r\ (0.2)^2\,(0.8)^{10} = 0.2825$$

ii) At least one worker suffers from the disease.

$$p(r \geq 1) = 1 - p(r = 0) = 1 - {}^{12}C_0\,(0.2)^0\,(0.8)^{12} = 0.9313$$

(c) Obtain the line of regression of y on x for the following data. Also estimate the value of y for x = 1.

x	2	4	5	6	8	11
y	18	12	10	8	7	5

Solution : The table is as follows :

x	**y**	**x^2**	**xy**
2	18	4	36
4	12	16	48
5	10	25	50
6	8	36	48
8	7	64	56
11	5	121	55
36	60	266	293

Here n = 6 $\Sigma x = 36,$ $\Sigma y = 0$

$\Sigma x^2 = 266$ $\Sigma xy = 293$

Now $\bar{x} = \frac{1}{n} \quad \Sigma x = \frac{36}{6} = 6$

$\bar{y} = \frac{1}{n} \quad \Sigma y = \frac{60}{6} = 10$

Variance of $x = \sigma_x^2 = \frac{1}{n} \Sigma x^2 - (\bar{x})^2$

$= \frac{256}{6} - (6)^2 = 8.3233$

$\text{cov}(x, y) = \frac{1}{n} \Sigma xy - \bar{x}\,\bar{y} = \frac{293}{6} - (6)(10)$

$= -11.1667.$

$B_{yx} = \frac{\text{cov}(x, y)}{\text{Variance}(x)} = \frac{-11.1667}{8.3233}$

$B_{yx} = -1.34$

∴ The line regression y on x is $y - \bar{y}\ b_{yx}(x - \bar{x})$

∴ $y - 10 = -1.34(x - 6)$

$y = -1.34x + 18.04$

For x = 10 , y = 4.64

Q. 5 (a) Find the directional derivative of the function $\phi = x^2 y z^3$ at (2, 1, – 1) towards the point (1, –1, 1).

Solution : $\nabla\phi = i(2xyz^3) + j(x^2 z^3) + k(3x^2 y z^2)$

$(\nabla\phi)_{(2,1,-1)} = -4i - 4j + 12k$

$\bar{u} = (1-2)i + (-1, -1)j + (1+1)k = -i - 2j + 2k,$

$\hat{u} = \frac{\bar{u}}{|\bar{u}|}$

$\hat{u} = \frac{1}{3}(-i - 2j + 2k)$

∴ The directional derivative $= \nabla\phi \cdot \hat{u}$

$= (-4i \_ 4j + 2k) \cdot \frac{1}{3}(-i - 2j + 2k)$

$= \frac{1}{3}(4 + 8 + 24$

$= \frac{36}{3} = 12$

(b) Show that $\bar{F}$ = (y cos z) i + (x cos z) j – (xy sinz) k is irrotational. Find scalar of ϕ such $\bar{F} = \nabla\phi$.

Solution : For irrotational field $\nabla \times \bar{F} = 0$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y\cos z & x\cos z & -xy\sin z \end{vmatrix}$$

$$\nabla \times \bar{F} = i\,(-x\sin z + x\sin z) - j\,(-y\sin z + y\cos z) + k\,(\cos z - \cos z)$$

$$\nabla \times \bar{F} = 0$$

$\therefore$ $\bar{F}$ is irrotational.

Scalar potential function $\phi = \int (f_1\,dx + f_2\,dy + f_3\,dz) + c$

$$\phi = \int [y\cos z\,dx + 0 + 0]\;c = xy\cos z + c$$

$$\bar{F} = \nabla\phi = i\,(y\cos z) + j\,(x\cos z) - k\,(xy\sin z)$$

$\therefore$ $\bar{F} = \nabla\phi.$

(c) Evaluate : **(4)**

$\int_C \bar{F} \cdot d\bar{r}$ for $\bar{F}$ $3x^2$ i + (2xz – y) j + zk along the curve $x = t^2$, y = t, $z = t^3$ from t = 0, t = 1.

Solution :

$$\bar{F}\cdot d\bar{r} = 3x^2\,dx + (2xz - y)\,dy + z\,dz.$$

$$x = t^2 \quad dx = 2t\,dt$$

$$y = t \quad dy = dt$$

$$z = t^3 \quad dz = 3\,t^2\,dt$$

$$\therefore \quad \bar{F}\cdot d\bar{r} = 3t^4 \cdot 2t\,dt + (2t^2\,t^3 - t)\,dt + t^3 \cdot 3t^2\,dt$$

$$\bar{F}\cdot d\bar{r} = (6\,t^5 + 2t^5 - t + 3t^5)\,dt = (11t^5 - t)\,dt$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_{t=0}^{1} (11t^5 - t)\,dt = \left(\frac{11t^6}{6} - \frac{t^2}{2}\right)_0^1$$

$$= \frac{11}{6} - \frac{1}{2} = \frac{8}{6} = \frac{4}{3}$$

Q. 6 (a) Show that (any one)

i) $\nabla \times (\bar{r}\, r^3) = 0$

Solution :

$$\nabla \times (\phi\, \bar{u}) = \nabla\phi \times \bar{u} + \phi\,(\nabla \times \bar{u})$$

$$\nabla \times (\bar{r}\, r^3) = \nabla r^3 \times \bar{r} + r^3\,(\nabla \times \bar{r})$$

$$= \frac{3r^2}{r}(\bar{r} \times \bar{r}) + r^3\,(\nabla \times \bar{r})$$

$$= 0 + 0 = 0 \qquad (\because \bar{r} \times \bar{r} = 0, \nabla \times \bar{r} = 0)$$

ii) $\nabla^2 (r^2 \log r) = 5 + 6 \log r$

Solution :

$$f(r) = r^2 \log r$$

$$f'(r) = 2r \log r + r^2 \frac{1}{r} = 2r \log r + r$$

$$f''(r) = 2\left(\log r + r\frac{1}{r}\right) + 1 = 2 \log r + 2 + 1$$

$$f''(r) = 3 + 2 \log r.$$

$$\therefore \quad \nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$$

$$\nabla^2 (r^2 \log r) = 3 + 2 \log r + \frac{2}{r}\, r\,(2 \log r + 1)$$

$$= 3 + 2 \log r + 4 \log r + 2$$

$$\nabla^2 (r^2 \log r) = 5 + 6 \log r$$

(b) Find the directional derivative of the function $\phi = x^2 y + xyz + z^3$ at (1, 2, –1) along the direction 8i + 8j.

Solution :

$$\nabla\phi = i\,(2xy + yz) + j\,(x^2 + xz) + k\,(xy + 3z^2)$$

$$(\nabla\phi)_{(1,2,-1)} = i\,(4 - 2) + j\,(1 - 1) + k\,(2 + 3)$$

$$(\nabla\phi)_{(1,2,-1)} = 2i + (0)\,j + 5k.$$

$$\bar{u} = 8i + 8j$$

$$\hat{u} = \frac{\bar{u}}{|\bar{u}|} = \frac{8i + 8j}{8\sqrt{1+1}} = \frac{i + j}{\sqrt{2}}$$

$$\therefore \quad \text{Directional derivative} = \nabla\phi \cdot \hat{u} = (2i + 5k) \cdot \frac{1}{\sqrt{2}}\,(I + j)$$

$$\textbf{D.D} = \frac{1}{\sqrt{2}}\,(2 + 0) = \sqrt{2}\,.$$

(c) Find the work done in moving a particle along the circle $x^2 + y^2 = 4$ under field of force $\overline{F} = xi + y^2 j$.

Solution :

The parametric co–ordinates of circle are $x = 2\cos\theta$ $y = 2\sin\theta$

$dx = -2\sin\theta\, d\theta$, $dy = 2\cos\theta\, d\theta$ and θ is varying from 0, to 2π,

$$\overline{F}\cdot d\overline{r} = x\,dx + y^2\,dy$$

$$\therefore \quad \overline{F}\cdot d\overline{r} = 2\cos\theta(-2\sin\theta\, d\theta) + 4\sin^2\theta\,(2\cos\theta\, d\theta)$$

$$\therefore \quad \text{work done} = \int_{\theta=0}^{2\pi} \overline{F}\cdot d\overline{r} \int_{\theta=0}^{2\pi} (-4\sin\theta\cos\theta + 8\sin^2\theta\cos\theta)\,d\theta$$

$$= -4\left(\frac{\sin^2\theta}{2}\right)_0^{2\pi} + 8\left(\frac{\sin^3\theta}{3}\right)_0^{2\pi} = 0 + 0 = 0$$

Work done = 0

Q. 7 (a) Find an analytic function $f(z) = u + iv$ where $v = 4\pi^3 y - 4x y^3$.

Solution :

$$\frac{\partial v}{\partial x} = 12x^2 y - 4y^3,$$

$$\frac{\partial v}{\partial y} = 4x^3 - 12xy^2$$

By C – R equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

$$\therefore \quad -\frac{\partial u}{\partial y} = 12x^2 y - 4y^3 \text{ integrate w.r.t. y keeping x at constant.}$$

$$u = -6x^2 y^2 + y^4 + f(x)$$

$$\frac{\partial u}{\partial x} = -12xy^2 + f'(x) \Rightarrow \left(\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\ \text{C – R equation.}\right)$$

$$4x^3 - 12xy^2 = -12xy^2 + f'(x) \Rightarrow f'(x) = 4x^3$$

$$\therefore \quad f(x) = \int 4x^3\,dx + C = x^4 + C$$

$$\therefore \quad u = -6x^2 y^2 + y^4 + x^4 + C$$

The analytic function $f(x) = u + iv$.

$$f(z) = (x^4 - 6x^2 y^2 + y^4) + i(4x^3 y - 4xy^3) + C$$

(b) Find a bilinear transformation w = f (z) which transforms the points z = ∞, i, 0 on z–plane to the points w = 0, i, ∞ on w–plane respectively.

Solution : The bilinear transformation $w = \dfrac{az+b}{cz+d}$

$$w = 0,\quad z = \infty \Rightarrow w = \frac{a+\frac{b}{z}}{c+\frac{d}{z}} \Rightarrow 0 = \frac{a+0}{c+0} \Rightarrow a = 0 \qquad \ldots(1)$$

$$w = i,\quad z = i \Rightarrow w = \frac{az+b}{cz+d} \Rightarrow i = \frac{ai+b}{ci+d} \Rightarrow ai + b = -c + id \ldots(2)$$

$$w = \infty,\quad z = 0 \Rightarrow \frac{1}{w} = \frac{cz+d}{az+b} \Rightarrow 0 = \frac{0+d}{0+b} \Rightarrow d = 0 \qquad \ldots(3)$$

From (1), (2), (3) we get $b = -c$, $\Rightarrow c = -b$

$$\therefore \quad \mathbf{w} = \frac{\mathbf{az+b}}{\mathbf{cz+d}} = \frac{\mathbf{0+b}}{\mathbf{-bz+0}} = -\frac{\mathbf{1}}{\mathbf{z}}$$

(c) Use residne theorem to evaluate the integral $\oint_C \left[\dfrac{4z-1}{z^2-z-6}\right] dz$ where C is a closed curve |z| = 4.

Solution : $z = -2, 3$ are simple pole lies inside the circle $|z| = 4$

i) The residue of f (z) at $z = -2$ is

$$f(z) = \frac{4z-1}{z^2-z-6}$$

$$r_1 = \lim_{z \to -2} (z+2)\frac{4z-1}{(z+2)(z-3)} = \frac{4(-2)-1}{(-2-3)} = \frac{-9}{-5} = \frac{9}{5}$$

ii) The residue of f (z) at z = 3 is

$$r_2 = \lim_{z \to 3} (z-3)\frac{4z-1}{(z+2)(z-3)} = \frac{12-1}{5} = \frac{11}{5}$$

∴ by residuce theorem we have $f(z)\int_c dz = 2\pi i\,(r_i + r_2)$

$$\therefore \quad \oint_C \frac{4z-1}{z^2-z-6}\,dz = 2\pi i\left(\frac{9}{5}+\frac{11}{5}\right) = 2\pi I\,(4) = 8\pi i$$

Q. 8 (a) Find an analytic function f(z) = u (r, θ) + iv (r, θ) where v = r^n sin θ.

Solution :

$$\frac{\partial v}{\partial r} = n\,r^{n-1}\sin n\theta$$

$$\frac{\partial v}{\partial \theta} = n\,r^{n}\cos n\theta$$

By C – R equations
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta},$$
$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

$$\therefore \quad -\frac{1}{r}\frac{\partial u}{\partial \theta} = n\,r^{n-1}\sin n\theta$$

$$\Rightarrow \quad \frac{\partial u}{\partial \theta} = -n\,r^{n}\sin n\theta.$$

Integrating w.r.t. θ keeping r as constant.

$$\therefore \quad u = -n\int r^n \sin n\,\theta\, d\theta + f(r)$$
$$u = r^n \cos n\theta + f(r)$$
$$\frac{\partial u}{\partial r} = n\,r^{n-1}\cos n\theta + f'(r) \Rightarrow \frac{1}{r}\frac{\partial v}{\partial \theta} = n\,r^{n-1}\cos n\,\theta + f'(r)$$

$$\therefore \quad \frac{1}{r}\,n\,r^n\cos n\theta = n\,r^{n-1}\cos n\theta + f'(r) = f'(r) = 0$$

$$\therefore \quad f(r) = C.$$
$$u = r^n \cos n\theta + C.$$

The analytic function f (z) = u + iv.

$$\therefore \quad \mathbf{f(z) = r^n \cos n\theta + i\,r^n \sin n\theta + C}$$
$$\mathbf{= r^n e^{i n \theta} + C.}$$

(b) Find the mapping of $y^2 = 2y - x^2$ on z–plane through the transformation $w = \frac{2}{z}$ on w–plane.

Solution :

The transformation $w = \frac{2}{z}$ can be written as.

$$z = \frac{2}{w} \quad \Rightarrow \quad x + iy = \frac{2}{u+iv} \times \frac{u-iv}{u-iv} = \frac{2\,(u-iv)}{u^2+v^2}$$

$$\therefore \quad x = \frac{2u}{u^2+v^2} \qquad y = -\frac{2v}{u^2+v^2}$$

Now $\quad y^2 = 2y - x^2 \quad \Rightarrow \quad x^2 + y^2 = 2y$

$$\therefore \quad \frac{4u^2}{(u^2+v^2)^2} + \frac{4v^2}{(u^2+v^2)^2} = -\frac{4v}{u^2+v^2}$$

$$4\,\frac{(u^2+v^2)}{(u^2+v^2)^2} = -\frac{4v}{(u^2+v^2)} \quad \Rightarrow -v = 1 \text{ or } v + 1 = 0$$

(c) Evaluate the integral $\oint_C \left[\frac{\sin \pi z^2 + 6z}{(z-1)(z+2)}\right] dz$ where C is a closed curve $|z| = 3$.

Solution :

Given : $$f(z) = \frac{\sin \pi z^2 + 6z}{(z-1)(z+2)}$$

$\therefore$ $z = 1, -2$ are simple poles lies inside the circle $|z| = 3$.

i) The residue of f (z) at z = 1 is

$$r_1 = \lim_{z \to 1} (z-1) f(x)$$

$$= \lim_{z \to 1} (z-1) \frac{\sin \pi z^2 + 6z}{(z-1)(z+2)} = \frac{0+6}{3} = 2$$

ii) The residue of f (z) at z = – 2 is

$$r_2 = \lim_{z \to -2} (z+2) f(z)$$

$$= \lim_{z \to -2} (z+2) \frac{\sin \pi z^2 + 6z}{(z-1)(z+2)} = \frac{0-12}{-3} = 4$$

By residue theorem we have $\oint_C f(x)\, dz = 2\pi i\, (r_1 + r_2)$

$$\therefore \quad \oint_C \left[\frac{\sin \pi z^2 + 6z}{(z-1)(z+2)}\right] dz = 2\pi i\,(2+4) = 12\pi i$$

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Solved University Questions Paper

Q. 1 (a) Solve any two the following differential equations **(4)**

i) $$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{-x}\, x \cos x$$

Solution : To find complementary function (C.F.) :

Auxiliary equation is,

$$D^2 + 2D + 1 = 0$$

$$\therefore \quad D = -1, -1 \;;$$

Real and repeated roots.

$$\therefore \quad \text{C.F.} = y\,(c_1 + c_2\, x)\, e^{-x}$$

To find particular integral (P.I) :

$$\begin{aligned}
\text{P.I.} &= \frac{1}{f(D)}\, x \\
&= \frac{1}{D^2 + 2D + 1}\, e^{-x}\, x \cos x \\
&= e^{-x}\left[\frac{1}{(D-1)^2 + 2\,(D-1) + 1}\, x \cos x\right] \\
&= e^{-x}\left[\frac{1}{D^2}\, x \cos x\right] \\
&= e^{-x}\left[x\left(\frac{1}{D^2} \cos x\right) - \frac{20}{D^4} \cos x\right] \\
&= e^{-x}\left[x\, \frac{1}{-1} \cos x - \frac{2D}{(-1)^2} \cos x\right] \\
&= e^{-x}\,[-x \cos x - 2D\,(\cos x)] \\
&= e^{-x}\,[-x \cos x + 2 \sin x]
\end{aligned}$$

$$\begin{aligned}
\therefore \quad \text{G.S.} &= \text{C.F.} + \text{P.I} \\
&= (c_1 + c_2\, x)\, e^{-x} + e^{-x}\,[2 \sin x - x \cos x]
\end{aligned}$$

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ii) $$(2x+1)^2 \frac{d^2y}{dx^2} + 2\,(2x+1)\frac{dy}{dx} + 4y = 4 \sin\,[2 \log (2x+1)] \qquad \ldots(1)$$

Solution : This is a Legendre′s linear differential equation :

Put $\quad 2x + 1 = e^z$

$\therefore \quad z = \log(2x+1)$

$$(2x+1)\frac{dy}{dx} = 2\,Dy \text{ and } (2x+1)^2\frac{d^2y}{dx^2} = 2^2\,D\,(D-1)\,y,\ D = \frac{d}{dz}$$

$\therefore$ Equation (1) changes to

$$4D\,(D-1)\,y + 2\ \ 2Dy + + 4y = 4\sin(2z)$$

$$\therefore \quad 4\,[D^2 - D + D + 1]\,y = 4\sin(2z)$$

$$\therefore \quad (D^2+1)\,y = \sin(2z) \qquad \ldots(2)$$

This is a *l*.d.e with constant coefficients.

$\therefore$ Auxiliary equation is $D^2 + 1 = 0$

$$\therefore \quad D = 0 \pm i$$

$$\therefore \quad \text{C.F.} = y = e^{0x}\,[c_1\cos z + c_2\sin z]$$

$$y = c_1\cos z + c_2\sin z,$$

$$\text{P.I.} = \frac{1}{f(D)}\ x$$

$$= \frac{1}{D^2+1}\sin 2z$$

$$= \frac{1}{-4+1}\ \sin 2z$$

$$= -\frac{1}{3}\ \sin 2z$$

$$\therefore \quad \text{G.S.} = y = c_1\cos z + c_2\sin z - \frac{1}{3}\ \sin 2z, \text{ where } z = \log(2x+1)$$

iii) $\mathbf{\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}}$ **, by method of variation of parameters.**

Solution : Auxiliary equation is

$$D^2 + 3D + 2 = 0$$

$$\therefore \quad D = -1, -2$$

$$\therefore \quad \text{C.F.} = y = c_1\,e^{-x} + c_2\,e^{-2x}$$

To find P.I. :

Let, $y_1 = e^{-z}$, $y_2 = e^{-2x}$, $c_1 = u$, $c_2 = v$

$$\therefore \quad \text{P.I.} = uy_1 + vy_2$$

We know, $w = y_1\,y_2^1 - y_2\,y_1^1$

$$= e^{-x}\,(-2e^{-2x}) - e^{-2x}\,(-\,e^{-x}) = -\,e^{-3x}$$

$$u = \int \frac{-y_2 - x}{w}\,dx$$

$$= \int -\frac{e^{-2x}\,e^{e^x}}{-e^{-3x}}\ dx$$

$$= \int e^x e^{e^x} dx$$

$$= \int e^t dt, \text{ where } t = e^x$$

$$= e^t = e^{e^x}$$

Secondly, $$v = \int \frac{y_1 X}{W} dx$$

$$= \int \frac{e^{-x} e^{e^x}}{-e^{-3x}} dx = -\int e^{2x} e^{e^x} dx$$

$$= -\int e^x e^x e^{e^x} dx = -\int t e^t dt, \text{ where } e^x = t$$

$$= -\left[(t)(e^t) - \int (1) e^t dt\right]$$

$$= -[t e^t - e^t]$$

$$= e^t [1 - t] = e^{e^x} [1 - e^x]$$

$\therefore$ $$\text{P.I.} = y = uy_1 + vy_2$$

$$= e e^{e^x} (e^{-x}) + e^{e^x} (1 - e^x)(e^{-2x})$$

$$= e^{e^x} [e^{-x} + e^{-2x} - e^{-x}] = e^{-2x} e^{e^x}$$

$\therefore$ $$\text{G.S.} = \text{C.F} + \text{P.I}$$

$$= c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} e^{e^x}$$

(b) Solve the integral equation : $\int_0^\infty f(x) \sin\lambda x \, dx = e^{-\lambda}, \lambda > 0$ **(4)**

Solution :

The integral in the left side of given equation is a Fourier sine transform of f(x). To solve the given equation means to find the unknown in equation i.e. f (x).

Let $$\phi_s(\lambda) = e^{-\lambda}$$

$\therefore$ By using, Inverse Fourier sine Transform, we get

$$f(x) = F_s^{-1} [\phi_s(\lambda)]$$

$$= \frac{2}{\pi} \int_0^\infty \phi_s(\lambda) \sin x\lambda \, d\lambda$$

$\therefore$ $$f(x) = \frac{2}{\pi} \int_0^\infty e^{-\lambda} \sin x\lambda \, d\lambda$$

$$= \frac{2}{\pi} \left[\frac{e^{-\lambda}}{1 + x^2} (-\sin x\lambda - x \cos x\lambda)\right]_0^\infty$$

$$f(x) = \frac{2}{\pi} \left[\frac{x}{1 + x^2}\right]$$

Q. 2 (a) An electrical circuit consisting of inductance L, condenser of capacity c is connected in series with applied alternating emf (sin nt) at t = 0. Find the current i and charge q at any time t by assuming $w^2 = \frac{1}{LC}$ and $w \neq n$. (4)

Solution : The differential equation for given electrical circuit is

$$L\frac{di}{dt} + \frac{q}{c} = \sin(nt)$$

$$\therefore \quad L\frac{d^2i}{dt^2} + \frac{q}{c} = \sin(nt)$$

$$\therefore \quad \frac{d^2i}{dt^2} + \left(\frac{1}{LC}\right) q = \frac{1}{L}\sin(nt)$$

$$\therefore \quad (D^2 + w^2)\, q = \frac{1}{L}\sin(nt)$$

$\therefore$ A.E. is $D^2 + w^2 = 0$

$$\therefore \quad D = 0 \pm iw$$

$$\therefore \quad \text{C.F.} = y = e^{0t}[c_1 \cos wt + c_2 \sin wt] = c_1 \cos wt + c_2 \sin wt$$

$$\therefore \quad \text{P.I} = \frac{1}{f(D)}\,\frac{1}{L}\sin(nt) = \frac{1}{D^2 + w^2}\left[\frac{1}{L}\sin(nt)\right]$$

$$= \frac{1}{-n^2 + w^2}\left(\frac{\sin nt}{L}\right) = \frac{\sin nt}{L(w^2 - n^2)},\quad w \neq n$$

$$\therefore \quad \text{G.S.} = q = \text{C.F.} + \text{P.I.}$$

$$q = c_1 \cos wt + c_2 \sin wt + \frac{\sin nt}{L(w^2 - n^2)}$$

and
$$i = \frac{di}{dt} = -c_1 w \sin wt + c_2 w \cos wt + \frac{n \cos nt}{L(w^2 - n^2)}$$

(b) Solve any one of the following (4)

i) Obtain $z\{k^2\}$, $k \geq 0$.

Solution :

We know $z\{1\} = \frac{z}{z-1}$

We know, $z\{k^2\} = z\{k^2(1)\}$

$$= \left(-z\frac{d}{dz}\right)^2 [z\{1\}] = \left(z\frac{d}{dz}\right)^2\left[\frac{z}{z-1}\right]$$

$$= z\frac{d}{dz}\left[z\frac{d}{dz}\left(\frac{z}{z-1}\right)\right]$$

$$= z\frac{d}{dz}\left[z\left(\frac{(z-1)(1)-z}{(z-1)^2}\right)\right] = z\frac{d}{dz}\left[z\left(\frac{-1}{(z-1)^2}\right)\right]$$

$$= z\frac{d}{dz}\left[\frac{-z}{(z-1)^2}\right]$$

$$= -z\left[\frac{(z-1)^2(1)-z\,2\,(z-1)}{(z-1)^4}\right]$$

$$= \frac{z(z+1)}{(z-1)^3}$$

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ii) Obtain $z^{-1}\left[\frac{6z}{(z+2)(z-4)}\right]$ **(4)**

Solution : Resolve $\frac{1}{(z+2)(z-4)}$ into partial fractions

$$\therefore \quad \frac{1}{(z+2)(z-4)} = \frac{A}{z+2}+\frac{B}{z-4} = \frac{\left(-\frac{1}{6}\right)}{z+2}+\frac{\left(\frac{1}{6}\right)}{z-4}$$

$$= \frac{1}{6}\left[\frac{1}{z-4}-\frac{1}{z+2}\right]$$

$$\therefore \quad \frac{6z}{(z+2)(z-4)} = \frac{z}{z-4}-\frac{z}{z+2}$$

$$\therefore \quad z^{-1}\left[\frac{6z}{(z+2)(z-4)}\right] = z^{-1}\left[\frac{z}{z-4}\right]-z^{-1}\left[\frac{z}{z+2}\right]$$

$$= \{4^k\}-\{(-2)^k\},\ k \geq 0.$$

(c) Solve the difference equation **(4)**

$x_{k+2}-3x_{k+1}+2x_k = 0$ **…(1)**

where $x_0 = 0$, $x_1 = 1$ for $k \geq 0$.

Solution : Take Z.T. of equation (1)

$$\therefore \quad z\{x_{k+2}\}-3z\{x_{k+1}\}+2z\{x_k\} = z\{0\}$$

$$[z^2x(2)-z^2x(0)-zx(1)]-3[zx(2)-zx(0)]+2x(z) = 0$$

$$\therefore \quad (z^2-3z+2)\,x(z) = z \qquad \because x(0)=0,\ x(1)=1$$

$$\therefore \quad x(z) = \frac{z}{z^2-3z+2} \qquad \ldots(2)$$

Take inverse Z.T of equation (2)

$$z^{-1}[x(z)] = z^{-1}\left[\frac{z}{(z-1)(z-2)}\right]$$

$$x(k) = x_k = z^{-1}\left[\frac{z}{z-2}-\frac{z}{z-1}\right]$$

$$= \{2^k\}-\{1^k\}$$

$$x_k = \{2^k\}-1$$

Q. 3 (a) The first 04 moments of a distribution about the value "4" are – 1.5, 17, – 30, 108. Find the first 04 central moments and β_1, β_2.

Solution :

Given : $a = 4$, $\mu'_1 = 1.5$, $\mu'_2 = 17$, $\mu'_3 = -30$, $\mu'_4 = 108$

We know

$$\mu_1 = 0$$

$$\mu_2 = \mu'_2 - (\mu'_1)^2 = 17 - (-1.5)^2$$

$$= 17 - 2.25 = 14.75$$

$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^3$$

$$= -30 - 3(17)(-1.5) + 2(-1.5)^3 = 39.75$$

$$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2(\mu'_1)^2 - 3(\mu'_1)^4 = 142.31$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(39.75)^2}{(14.75)^3} = 0.4924$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{142.31}{(14.75)^2} = 0.6541$$

(b) Fit a straight line of the form y = ax + b to the following data by least square method: (4)

x	–2	1	3	6	8	9
y	17	14	12	9	7	6

Solution : Firstly we prepare the table

	x	y	x^2	xy
	–2	17	4	–34
	1	14	1	14
	3	12	9	36
	6	9	36	54
	8	7	64	56
	9	6	81	54
Σ	25	65	195	180

We have $n = 6$

We know, normal equations are

$$\Sigma y = a\,\Sigma x + nb$$

$$\Sigma xy = a\,\Sigma x^2 + b\,\Sigma x$$

$$\therefore \quad 25a + 6b = 65$$

$$195a + 25b = 180$$

$$\therefore \quad a = -1, \; b = 15$$

$\therefore$ Required line is $\boxed{y = -x + 15}$

(c) A series of 05 one day matches is to be played between India and Srilanka. Assuming that the probability of India's win in each match is 0.6 and results of all the 05 matches independent of each other, find the probability that India wins the series.

Solution : The India will win the 05 one day matches series, if India wins 03, 04, 05 matches.

We have $n = 5$, $p = 0.6$, $q + p = 1$, $q = 1 - p = 0.4$

Let x denotes the number of matches those will be won by India of 05 matches.

$\therefore$ The probability distribution of $x = P(x = r)$

$$= nC_r \, p^r \, q^{n-r}$$

$$= 5C_r \, (0.6)^r \, (0.4)^{5-r}$$

$\therefore$ The probability that India wins the series

$$= P(x = 3) + P(x = 4) + P(x = 5)$$

$$= 5C_3 \, (0.6)^3 \, (0.4)^2 + 5C_4 \, (0.6)^4 \, (0.4)^1 + 5C_5 \, (0.6)^5 \, (0.4)^0$$

$$= 0.6826$$

Q. 4 (a) The height of a student in a school follows a normal distribution with mean 190 cm and variance 80 cm^2. Among the 1000 students from the school, how many are expected to have height above 200 cm? (4)

(Given : z = 1.118 , A = 0.3686)

Solution : Let x denotes the height of a student in the school.

$$\therefore \quad x \sim N(190, 80)$$

$$\therefore \quad z = \frac{x - 190}{\sqrt{80}} \sim N(0, 1)$$

The probability that a student has height above 200 cm

$$= P(x > 200)$$

$$= P(z > 1.118)$$

$$= 0.5 - 0.3686$$

$$= 0.1314$$

Expected no. of students among 1000 having height above 200 cms.

$$1000 \, P(x > 200) = 131.4$$

$$= 131.$$

(b) In a factory manufacturing razor blades, there is a small chance of $\left(\frac{1}{500}\right)$ for any blade to be defective. The blades are supplied in a packet of 10. Use Poisson distribution to calculate the approximate number of packets containing at least one defective blade in a consignment of 10000 packets. (4)

Solution : Let x denotes the number of defective blades in a packet of 10

$$\therefore \quad x \sim \text{Poisson}\left(m = np,\ m = (10)\left(\frac{1}{500}\right) = 0.02\right)$$

$\therefore$ The probability that a packet contains at least one defective blade

$$= P(x \geq 1)$$
$$= 1 - p(x = 0)$$
$$= 1 - e^{-0.02} = 0.0198$$

The probability of function x $= P(x = r) = \dfrac{e^{-0.02}(0.02)^r}{r!}$, r = 0, 1, 2,…

Expected no. of packets out of 1000 containing at least one defective blade

$$= [P((x \geq 1)] \times 10000$$
$$= 198.$$

(c) For a bivariate data, the regression equation of y on x is 4x + y = μ and the regression equation of x on y is 9x + y = λ. Find the values of μ and λ. Also find the correlation coefficient between x and y, if the means of x and y are 2 and – 3 respectively. (4)

Solution : Given that $\bar{x} = 2,\ \bar{y} = -3$

We know the point $(\bar{x}, \bar{y})$ always line on lines of regression.

$$\therefore \quad 4\bar{x} + \bar{y} = \mu$$
$$9\bar{x} + \bar{y} = \lambda$$
$$\therefore \quad \mu = 5, \lambda = 15$$

$\therefore$ Regression equation of y and x is

$$4x + y = 5$$
$$\therefore \quad y = -4x + 5$$
$$\therefore \quad byx = -4$$

Regression line of x on y is

$$9x + y = 15$$
$$\therefore \quad x = -\frac{1}{9}y + \frac{15}{9}$$
$$\therefore \quad bxy = -\frac{1}{9}$$

The correlation coefficient = $r = \sqrt{(byx)\,(bxy)} = \sqrt{(-4)\left(\frac{-1}{9}\right)}$

$= \sqrt{\frac{4}{9}} = \pm\frac{2}{3}$

Q. 5 (a) Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point (1, –2, 1) in the direction of a vector (2i – j – 2k). **(4)**

Solution : Given : $\phi = x^2yz + 4xz^2$

$$\therefore \quad \nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$= i\,(2xyz + 4z^2) + j\,(x^2z) + k\,(x^2y + 8xz)$$

$$= 0i + j + 6k \text{ at } (1, -2, 1)$$

Let $\bar{a} = 2i - j - 2k$

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{2i - j - 2k}{\sqrt{4+1+4}} = \frac{2i - j - 2k}{3}$$

$$\therefore \quad \text{D.D. } [\phi] = \nabla\phi \cdot \hat{a} = (0i + j + 6k) = \frac{2i - j - 2k}{3} = -\frac{13}{3}$$

(b) Show that the vector $\bar{F} = (y + z)\,i + (z + x)\,j + (x + y)\,k$ is irrotational. Hence find the scalar potential function ϕ such that $\bar{F} = \nabla\phi$.

Solution : We know, $\bar{F}$ will be irrotational if, $\nabla \times \bar{F} = 0$

$$\therefore \quad \nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix}$$

$$= i\,(1-1) - j\,(1-1) + k\,(1-1)$$

$$= 0$$

$\therefore \quad \bar{F}$ = is a irrotational vector field.

Secondly, we know

$$d\phi = \nabla\phi \cdot d\bar{r} = \bar{F} \cdot d\bar{r}$$

$$= [(y+z)\,i + (z+x)\,j + (x+y)\,k]\,[i\,dx + j\,dy + k\,dz]$$

$$= (y+z)\,dx + (z+x)\,dy + (x+y)\,dz$$

$$\therefore \quad \text{Integrating,} \quad \int d\phi = \int_{y,\,z\,=\,\text{constant}} (y+z)\,dx + \int_{\substack{x=0\\ z=\text{constant}}})\,(z+x)\,dy + \int_{\substack{x=0\\ y=0}} (x+y)\,dz$$

$$\boxed{\phi = (y+z)\,x + zy + c}$$

(c) Find the work done is moving a particle by the force field $\overline{F} = 3x^2\ i + (2xz - y)\ j + zk$ along the straight line from (0, 0, 0) to (2, 1, 3) **(5)**

Solution : We know the equation of a line passing through (0, 0, 0) and (2, 1, 3) is

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0}$$

$$\therefore \quad \frac{x}{2} = y = \frac{z}{3}$$

$$\therefore \quad x = 2y,\ z = 3y \Rightarrow dx = 2dy = 3dy$$

We know work done $W = \int_C \overline{F}\cdot d\overline{r}$

$$= \int_C [3x^2\ i\ (2xz - y)\ j + zk]\cdot[idx + jdy + kdz]\backslash$$

$$= \int_C [3x^2\ dx + (2xz - y)\ dy + z\ dz]$$

$$= \int_C 3\ (4^2 y)\ 2\ dy + [12y^2 - y]\ dy + qy\ dy$$

$$= \int_1 [24y^2 + 12y^2 - y + 9y]\ dy$$

$$= \int_0 (36y^2 + 8y)\ dy = \left[36\left(\frac{y^3}{3} + \frac{8y^2}{2}\right)\right]_0^1$$

$$= 12\ (1 - 0 + 4\ (1 - 0)$$

$$\text{W.D.} = 16$$

Q. 6 (a) Show that (any one) **(4)**

i) $\nabla\cdot[\phi\ \nabla\psi - \psi\ \nabla\phi]\ \phi\ \nabla^2\ \psi - \psi\nabla\phi$

Solution : Given $\phi.\psi$ are scalar point functions

We know, $\nabla\times(\phi\overline{u}) = [\nabla\phi\cdot\overline{u}] + \phi\ [\nabla\cdot\overline{u}]$

$$\text{L.H.S.} = \nabla\cdot[\phi\nabla\psi - \psi\nabla\phi] - \nabla\cdot(\phi\nabla\psi) - \nabla\cdot(\psi\ \nabla\phi)$$

$$= [\nabla\phi\cdot\nabla\psi + \phi\ (\nabla\cdot\nabla\psi)] - [\nabla\psi\cdot\nabla\phi + \psi\ (\nabla\cdot\nabla\psi)]$$

$$= \nabla\phi\cdot\nabla\psi + \phi\ \nabla^2\psi - \nabla\psi\cdot\nabla\phi - \psi\nabla^2\phi$$

$$= \phi\ \nabla^2\ \psi - \psi\ \nabla^2\ \phi \quad \because \overline{u}.\overline{v} = \overline{v}.\ \overline{u}$$

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ii) Show that $\nabla\cdot r\ \nabla\left(\frac{1}{r^5}\right) = \frac{15}{r^6}$

Solution : $\text{L.H.S.} = \nabla\cdot r\ \nabla\left(\frac{1}{r^5}\right)$

$$= \nabla\cdot[r\ \nabla\ \overline{r}^5] = \nabla\cdot[r - 5\overline{r}^7\ \overline{r}]$$

$$= -5\ \nabla\cdot\left(\overline{r}^6\ \overline{r}\right) \qquad \because \nabla\ r^n = n\ r^{n-2}\ \overline{r}$$

$$= -5\,(-6+3)\,\overline{r}^{6} \qquad \because \nabla\cdot r^{n}\,\overline{r} = (3+n)\,r^{n}$$

$$= 15\,\overline{r}^{6}$$

$$= \frac{15}{r^{6}}$$

(b) Find the directional derivative of $\phi = 4xz^3 - 3x^2 y^2$ at (2, –1, 3) along a line equally inclined with co–ordinate axes.

Solution : Given :

$$\phi = 4xz^3 - 3x^2 y^2$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$= i\,(4z^3 - 6xy^2) + j\,(-6x^2 y) + k\,(12xz^2)$$

$$= 20\,i + 24j + 36k \quad \text{at } (2, -1, 3)$$

$\overline{a}$ = vector which is equally inclined with co–ordinate axes.

$$= i + j + k$$

$$\hat{a} = \frac{\overline{a}}{|\overline{a}|} = \frac{i+j+k}{\sqrt{3}}$$

$$\text{D.D. } [\phi] = \nabla\phi\cdot\hat{a}$$

$$= (20i + 24j + 96k).\left(\frac{i+j+k}{\sqrt{3}}\right)$$

$$= \frac{140}{\sqrt{3}}$$

(c) Evaluate $\int_C \overline{F}\cdot d\overline{r}$

Where $\overline{F} = (2y+3)\,i + xzj + (yz - x)\,k$ along the curve $x = 2t^2$, $y = t$, $z = t^3$ from $t = 0$ to $t = 1$.

Solution : Given curve c is

$$x = 2t^2,, \; y = t \; z = t^3$$

$$dx = 4t\,dt. \; y = dt, \; z = 3t^2\,dt$$

$$\therefore \quad \int_C \overline{F}\cdot d\overline{r} = \int_C (2y+3)\,dx + xz\,dy + (y\,z - x)\,dz$$

$$= \int_0^1 (2t+3)\,4tdt + (2t^2)(t^3)\,dt + (t^4 - 2t^2)\,3t^2\,dt$$

$$= \int_0^1 (8t^2 + 18t + 2t^5 + 3t^6 - 6t^4)\,dt$$

$$= \left[8\frac{t^3}{3} + 18\frac{t^2}{2} + 2\frac{t^6}{6} + 3\frac{t^7}{7} - 6\frac{t^5}{5}\right]_{t=0}^{1}$$

$$= \frac{8}{3} + \frac{12}{2} + \frac{2}{6} + \frac{3}{7} - \frac{6}{5}$$

$$= \frac{560 + 1260 + 70 + 90 - 252}{210}$$

$$= \frac{1728}{210}$$

$$= \frac{864}{210}$$

Q. 7 (a) Determine the analytic function f (z) = u + iv if u = $3x^2y - y^3$ **(4)**

Solution : Let $u = 3x^2y - y^3$ …(1)

Differential (1) partially w.r.t x, (y constant)

$$\therefore \quad \frac{\partial u}{\partial x} = 6xy$$

$$\therefore \quad \frac{\partial v}{\partial y} = 6xy \qquad \because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

Integrate w.r.t. y (by keeping x constant partially

$$v = 6x\left(\frac{y^2}{2}\right) + f(x)$$

$$v = 3xy^2 + f(x) \qquad \text{…(2)}$$

$$\therefore \quad \frac{\partial v}{\partial x} = 3y^2 + f' + f'(x) \qquad \because \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

$$\therefore \quad \frac{\partial u}{\partial y} = -3y^2 - f'(x) \qquad \text{…(3)}$$

Differential (1) partially w.r.to y (x constant)

$$\therefore \quad \frac{\partial u}{\partial y} = 3x^2 - 3y^2 \qquad \text{…(4)}$$

From (3) and (4),

$$-f'(x) = 3x^2$$

$$f'(x) = -3x^2$$

$$\therefore \quad f(x) = -3\left(\frac{x^3}{3}\right) + c$$

$$\boxed{f(x) = -x^3 + c}$$

$$\therefore \quad \text{from (2)} \quad v = 3xy^2 - x^3 + c$$

$$\therefore \quad f(x) = u + iv$$

$$= (3x^2 y - y^3) + i(3xy^2 - x + c)$$

(b) Find a bilinear transformation which maps the points z = 1, i – 1 into the points w = i, 0, –i.

Solution : Let $z_1 = 1,\ z_2 = i,\ z_3 = -1$

and $w_1 = i,\ w_2 = 0,\ w_3 = -i$

We know the bilinear transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-1)}$$

$$\frac{(w-i)(0+i)}{(w+i)(0-i)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\therefore \quad \frac{w-i}{w+i} = -\left[\frac{(z-1)(i+1)(i+1)}{(z+1)(i-1)(i+1)}\right]$$

$$\therefore \quad \frac{w-i}{w+i} = \left[\frac{(z-1)(i^2+2i+1)}{(z+1)(i^2-1)}\right] = -\left[\frac{(z-1)(2i)}{(z+1)(-2)}\right]$$

$$\frac{w-i}{w+i} = \frac{i(z-1)}{(z+1)}$$

Cross multiplication and simplification gives as

$$\boxed{w = \frac{1+iz}{1-iz}}$$

(c) Evaluate $\int_C \left[\frac{4-3z}{z(z-1)(z-2)}\right] dz$ where c is the circle $|z| = \frac{3}{2}$ **(5)**

Solution :

Let $f(z) = \dfrac{4-3z}{z(z-1)(z-2)}$

$\therefore$ Poles of f (z) are z = 0, 1, 2

The given closed curve c is $|z| = \frac{3}{2}$

which is a circle with centre (0, 0) and radius $\frac{3}{2}$.

$\therefore$ out of three poles of f (z), z = 0, z = 1 lies inside and z = 2 lies out side closed curve c.

$\therefore$ we should only find residues of f (z) t z = 0, z = 1.

$$\therefore \quad R_1 = \left[\begin{matrix}\text{Residue of } f(z) \\ \text{at } z=0\end{matrix}\right] = \left[\frac{z(4-3z)}{z(z-1)(z-2)}\right]_{z=0} = \left[\frac{4-3z}{(z-1)(z-2)}\right]_{z=0} = 2$$

$$R_2 = \left[\begin{matrix}\text{Residue of } f(z) \\ \text{at } z=1\end{matrix}\right] = \left[\frac{(z-1)(4-3z)}{z(z-1)(z-2)}\right]_{z=1} = \left[\frac{4-3z}{z(z-2)}\right]_{z=1} = -1$$

$$\therefore \quad \int_C \left[\frac{4-3z}{z(z-1)(z-2)}\right] dz = 2\pi i\,[\text{sum of residues}]$$

$$= 2\pi i\,[R_1 + R_2]$$

$$= 2\pi i\,(2-1) = 2\pi i$$

Q. 8 (a) Determine the analytic function f (x) = u + iv if u = 2x – 2xy **…(1)**

Solution :
$$\frac{\partial u}{\partial x} = 2 - 2y \qquad \because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial y} = 2 - 2y$$

Integrate w.r.to y (x constant)

$$V = 2y - 2\left(\frac{y^2}{2}\right) + f(x)$$

$$V = 2y - y^2 + f'(x) \qquad \ldots(*)$$

$$\therefore \quad \frac{\partial v}{\partial x} = 0 - 0 + f'(x)$$

$$\frac{\partial v}{\partial x} = f'(x) \qquad \because \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

$$\therefore \quad -\frac{\partial u}{\partial y} = -f'(x) \qquad \ldots(2)$$

Differential (1) partially w.r.to y (x constant)

$$\therefore \quad \frac{\partial u}{\partial y} = -2x\,(1)$$

$$= -2x \qquad \ldots(3)$$

From (2) and (3), $-f'(x) = -2x$

$\therefore$ $f'(x) = 2x$

$\therefore$ Integrate, $f(x) = x^2 + c$

$\therefore$ From (*) $V = 2y - y^2 + x^2 + c$

$\therefore$ $f(z) = u + iv = (2x - 2xy) + i\,(x^2 - y^2 + 2y + c)$

(b) Determine /Find the image of the circle |z – 1| = 1 in the plane under the mapping $w = \frac{1}{z}$.

Solution : We know

$$|z - 1| = 1$$

$$|x + iy - 1| = 1$$

$$\therefore \quad (x-1)^2 + y^2 = 1$$

$$\therefore \quad x^2 + y^2 = 2x \qquad \ldots(*)$$

The transformation is $w = \frac{1}{z}$

$$\therefore \quad u + iv = \frac{1}{x + iy}$$

$$x + iy = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$$

$$\therefore \quad x = \frac{u}{u^2+v^2}, \; y = \frac{-v}{x^2+v^2} \quad \ldots(**)$$

From (*) and (**)

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} = 2\left(\frac{u}{u^2+v^2}\right)$$

$$\therefore \quad \boxed{2u - 1 = 0}$$

(c) Evaluate $\int_C \left[\frac{3z+4}{z(2z+1)}\right]$ **dz, where c is |z| = 1.**

Solution :

Given : Let $\quad f(z) = \frac{3z+4}{z(2z+1)}$

$\therefore$ Poles of f (z) are $z = 0,\ z = -\frac{1}{2}$, which lies inside $|z| = 1$

$$R_1 = \begin{bmatrix}\text{Residue of } f(z) \\ \text{at } z = 0\end{bmatrix} = \left[\frac{z(3z+4)}{z(2z+1)}\right]_{z=0} = \left[\frac{3z+4}{2z+1}\right]_{z=0} = 4$$

$$R_2 = \begin{bmatrix}\text{Residue of } f(z) \\ \text{at } z = -\frac{1}{2}\end{bmatrix} = \left[\frac{(2z+1)(3z+4)}{z(2z+1)}\right]_{z=-\frac{1}{2}} = \left[\frac{3z+4}{(2z+1)}\right]_{z=-\frac{1}{2}} = -5$$

$$\therefore \quad \int_C \left[\frac{3z+4}{z(2z+1)}\right] dz = 2\pi i\,[\text{sum of Residues}]$$

$$= 2\pi i\,[R_1 + R_2]$$

$$= 2\pi i\,[4 - 5] = -2\pi i$$

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