PAGE NO.:

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Assignment	Parameter Estimation
Subject	UCS654

Ques1 Let $(X_1, X_2, \ldots,)$ be a random sample of size n taken from a Normal Population with parameters mean $= \theta_1$ and variance $= \theta_2$.
Find the Maximum Likelihood Estimates of these two parameters.

Sol^n: PDF (Probability ~~Mass~~ Density function) of Normal Distribution is given by

$$f(x) = \frac{1}{\sqrt{\theta_2}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\theta_1}{\sqrt{\theta_2}}\right)^2}$$

where $\theta_2 = \sigma^2$

$\theta_1 = \mu$

where x is value of random Variable X.

Acc. to question $X_1 \ldots X_n$ are random values from the distribution, which makes likelihood function as follows

$$L = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{1}{2}\frac{(x_i-\theta_1)^2}{\theta_2}}$$

where L is function of θ_1, θ_2 given x_i

PAGE NO:

Taking log on both sides

$$\log(\mathcal{L}) = \log\left(\sqrt{2\pi\theta_2}^{\,-n} \prod_{i=1}^{n} e^{-\frac{1}{2}\frac{(x_i-\theta_1)^2}{\theta_2}}\right)$$

$$\log(\mathcal{L}) = -\frac{n}{2}\log(2\pi\theta_2) + \left(\frac{-1}{2\theta_2}\right)\sum_{i=1}^{n}(x_i-\theta_1)^2$$

[using property $\log(AB) = \log A + \log B$]

Differentiationg on both side with respect to θ_1

$$\frac{1}{\mathcal{L}}\frac{\partial \mathcal{L}}{\partial \theta_1} = \frac{-1}{2\theta_2}\sum_{i=1}^{n} 2(x_i-\theta_1)(-1)$$

$$\frac{1}{\mathcal{L}}\frac{d\mathcal{L}}{d\theta_1} = \frac{1}{2\theta_2}\sum_{i=1}^{n} 2(x_i-\theta_1)$$

Equationg $\frac{\partial L}{d\theta_1} = 0$

$$\Rightarrow \quad (\mathcal{L}) \cdot \frac{1}{2\theta_2}\sum_{i=1}^{n} 2(x_i-\theta_1) = 0$$

Either $\mathcal{L} = 0$

or $\frac{1}{2\theta_2}\sum_{i=1}^{n} 2(x_i-\theta_1) = 0$

$\mathcal{L} = 0$ can't be possible as probability density of already happened can't be zero.

$$\Rightarrow -\frac{1}{2\theta_2} \sum_{i=1}^{n} 2(\theta_1 - x_i) = 0$$

$$\Rightarrow \sum_{i=1}^{n} 2\theta_1 = \sum_{i=1}^{n} 2x_i$$

$$\Rightarrow n\theta_1 = \sum_{i=1}^{n} x_i$$

$$\Rightarrow \theta_1 = \frac{\sum_{i=1}^{n} x_i}{n}$$

$\Rightarrow \theta_1 =$ Sample mean.

Now differentiating ~~θ_2~~ $\mathcal{L}$ wrt θ_2.

$$\frac{\partial \mathcal{L}}{\partial \theta_2}\left(\frac{1}{\mathcal{L}}\right) = -\frac{n}{2}\frac{2\pi}{2\pi\theta_2} + \sum_{i=1}^{n}(x_i - \theta_1)^2\left(\frac{1}{2\theta_2^2}\right)$$

Putting $\frac{\partial \mathcal{L}}{\partial \theta_2} = 0$

Again $\mathcal{L}$ can't be zero

so,

$$-\frac{n}{2\theta_2} + \sum_{i=1}^{n}(x_i - \theta_1)^2 \frac{1}{2\theta_2^2} = 0$$

$$\sum_{i=1}^{n}(x_i - \theta_1)^2 = n\theta_2$$

$$\theta_2 = \frac{\sum_{i=1}^{n}(x_i - \theta_1)^2}{n}$$

$\theta_2 =$ variance of sample (taken as population) where θ_1 is population mean. ($\theta_1 =$ sample mean only as found above).

PAGE NO.:

Ques2 Let $X_1, X_2 \dots X_n$ be a random sample from $B(m, \theta)$ distribution where $\theta \in (0,1)$ is unknown and m is a known positive integer. Compute value of θ using the MLE.

Solⁿ: PMF (Probability Mass function) of $B(m, \theta)$ i.e, Binomial distribution would be

$$P(X=k) = {}^{m}C_{k} \, \theta^{k} (1-\theta)^{m-k}$$

where k represents number of success out of m.

Let $X_1 \dots X_n$ be random sample from $B(m,\theta)$ dist. where for a X_i, it represents number of successes in i-th trial of experiment.

So likelihood function becomes

$$L(\theta) = \prod_{i=1}^{n} {}^{m}C_{x_i} \, \theta^{x_i} (1-\theta)^{m-x_i}$$

Taking Logarithm on both sides

$$\log L = \log\left(\prod_{i=1}^{n} {}^{m}C_{x_i} \, \theta^{x_i} (1-\theta)^{m-x_i} \right)$$

$$\log L = \sum_{i=1}^{n} \left[\log {}^{m}C_{x_i} + x_i \log\theta + (m-x_i) \log(1-\theta) \right]$$

Performing differentiation wrt θ and equating to zero

$$\frac{1}{L}\frac{dL}{d\theta} = \frac{1}{\theta}\sum_{i=1}^{n} x_i - \frac{1}{1-\theta}\sum_{i=1}^{n} (m-x_i)$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$\Rightarrow \mathcal{L} \cdot \left(\frac{1}{\theta} \sum_{i=1}^{n} x_i - \frac{1}{1-\theta} \sum_{i=1}^{n} m - x_i \right) = 0$$

$\mathcal{L}$ can't be zero. Probability of real happened events can't be zero.

$$\frac{1}{\theta} \sum_{i=1}^{n} x_i = \frac{1}{1-\theta} \sum_{i=1}^{n} m - x_i$$

$$(1-\theta) \sum_{i=1}^{n} x_i = \theta n m - \theta \sum_{i=1}^{n} x_i$$

$$\theta = \frac{\sum x_i}{nm}$$ where i goes from 1 to n

$$\theta = \frac{\text{Sample Mean}}{m}$$