# MAXIMUM FLOW

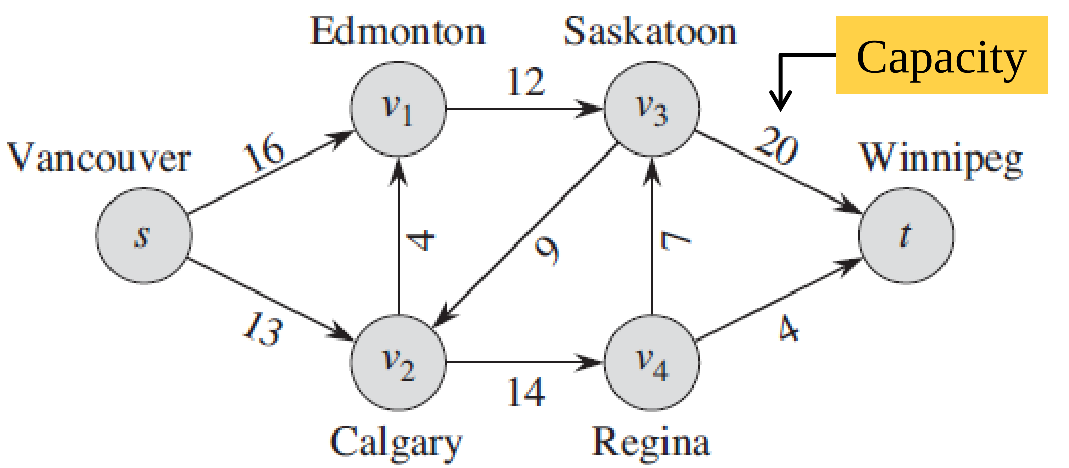
**Max-Flow Min-Cut Theorem (Ford Fukerson's Algorithm)**

# What is Flow Network?

- Flow network is a directed graph $G = (V,E)$ such that each edge has a non-negative capacity $c(u,v) \geq 0$.
- Two distinguished vertices exist in $G$ namely:
  - Source (produces flow and denoted by $s$): In-degree of this vertex is 0.
  - Sink (consumes flow and denoted by $t$): Out-degree of this vertex is 0.
- Flow in a network is an integer-valued function $f$ defined on the edges of $G$ satisfying $0 \leq f(u,v) \leq c(u,v)$, for every edge $(u,v)$ in $E$.

# Contd…

- Each edge $(u,v) \in E$ has a non-negative capacity $c(u,v)$.
- If $(u,v) \notin E$ assume $c(u,v)=0$.
- Two special vertices, source $s$ and sink $t$.
- Every vertex $v \in V$ is on some path from $s$ to $t$.




$c(s,v_1) = 16$
$c(v_1,s) = 0$
$c(v_2,s) = 0$
$c(v_4,t) = 4$ …

# Conditions for Flow Network

- For each edge $(u,v)$ in $E$, the flow $f(u,v)$ in $G$ is a real valued function $f : V \times V \rightarrow R$ that satisfies the following properties:

- Capacity constraints – The flow along an edge can not exceed its capacity. $\forall (u,v) \in E\ f(u,v) \leq c(u,v)$

- Skew symmetry – The net flow from $u$ to $v$ must be the opposite of the net flow from $v$ to $u$.
$$\forall (u,v) \in E\ f(u,v) = -f(v,u)$$

- Flow conservation – Unless $u$ is $s$ or $t$, the net flow to a node is zero. $\forall u \in V : u \neq s \text{ and } u \neq t \Rightarrow \sum_{w \in V} f(u,w) = 0$

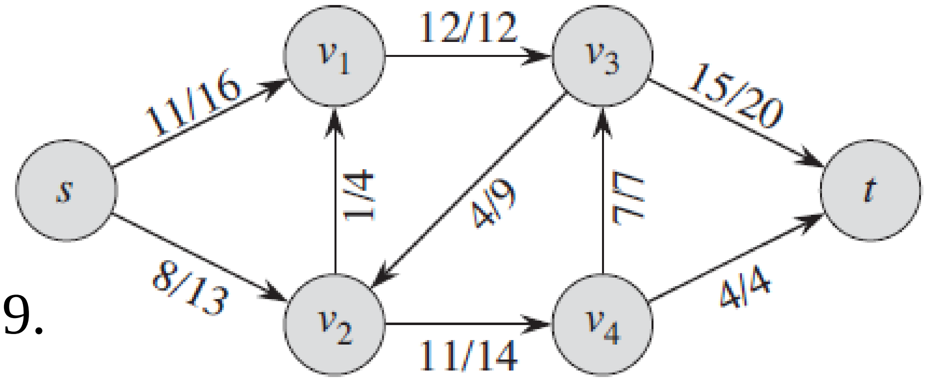
# The Value of a Flow

- The value of a flow is given by:

$$|f| = \sum_{v \in V} f(s,v) = \sum_{v \in V} f(v,t)$$

- The flow into the sink node ($t$) is same as flow going out from the source node ($s$) and thus the flow is conserved.
- Total amount of flow from source $s$ is equal to total amount of flow into the sink $t$.

# Example




- A flow $f$ in $G$ with value $|f| = 19$.
- Each edge $(u, v)$ is labeled by $f(u,v)/c(u,v)$. Slash notation separates the flow and capacity; it does not indicate division.

$$|f| = \sum_{v \in V} f(s,v) = \sum_{v \in V} f(v,t) = 19$$

- The flow across nodes $v_1$, $v_2$, $v_3$, and $v_4$ are also conserved as flow into them = flow out of them.
- $v_1$: $11 + 1 = 12$
- $v_2$ : $8 + 4 = 1 + 11$
- $v_3$: $12 + 7 = 4 + 15$, and
- $v_4$: $11 = 7 + 4$.

| **Edge** | **Capacity** | **Flow** | **Comment** |
|---|---|---|---|
| $(s,v_1)$ | 16 | 11 | $11 \leq 16$ Valid |
| $(s,v_2)$ | 13 | 8 | $8 \leq 13$ Valid |
| $(v_2,v_1)$ | 4 | 1 | $1 \leq 4$ Valid |
| $(v_1,v_3)$ | 12 | 12 | $12 \leq 12$ Valid |
| $(v_3,v_2)$ | 9 | 4 | $4 \leq 9$ Valid |
| $(v_2,v_4)$ | 14 | 11 | $11 \leq 14$ Valid |
| $(v_4,v_3)$ | 7 | 7 | $7 \leq 7$ Valid |
| $(v_3,t)$ | 20 | 15 | $15 \leq 20$ Valid |
| $(v_4,t)$ | 4 | 4 | $4 \leq 4$ Valid |

# The Maximum Flow Problem

- Given:
  - Graph $G(V,E)$,
  - $f(u,v)$ = flow on edge $(u,v)$,
  - $c(u,v)$ = capacity of edge $(u,v)$,
  - $s$ = source node, $t$ = sink node.
- Maximize: $|f|$
- Subject to:

$$\sum_{v \in V} f(u,v) - \sum_{v \in V} f(v,u) = 0, \forall u \neq s,t$$

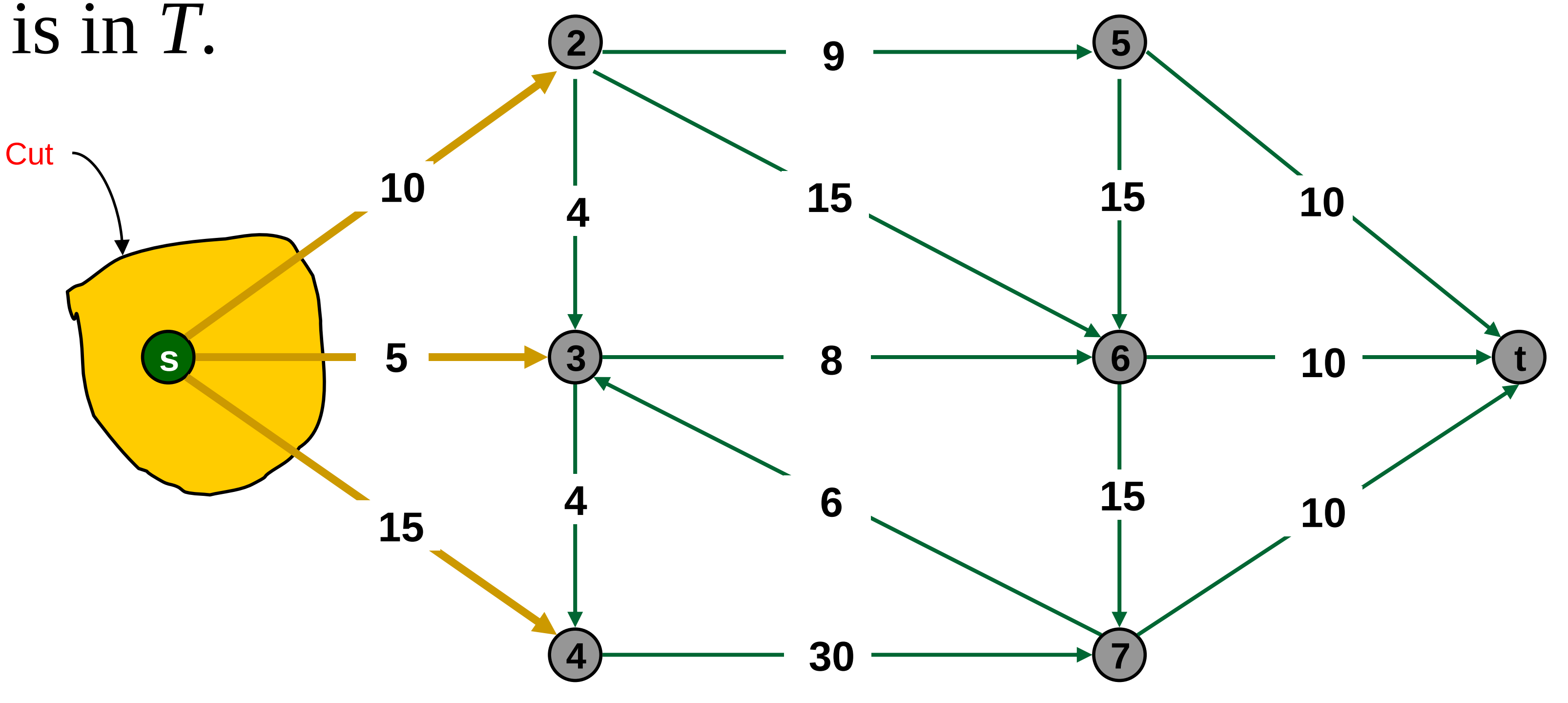
$$\sum_{v \in V} f(s,v) = \sum_{v \in V} f(v,t) = |f|$$

$$0 \leq f(u,v) \leq c(u,v), \forall (u,v) \in E$$

In simple terms maximize the $s$ to $t$ flow, while ensuring that the flow is feasible.

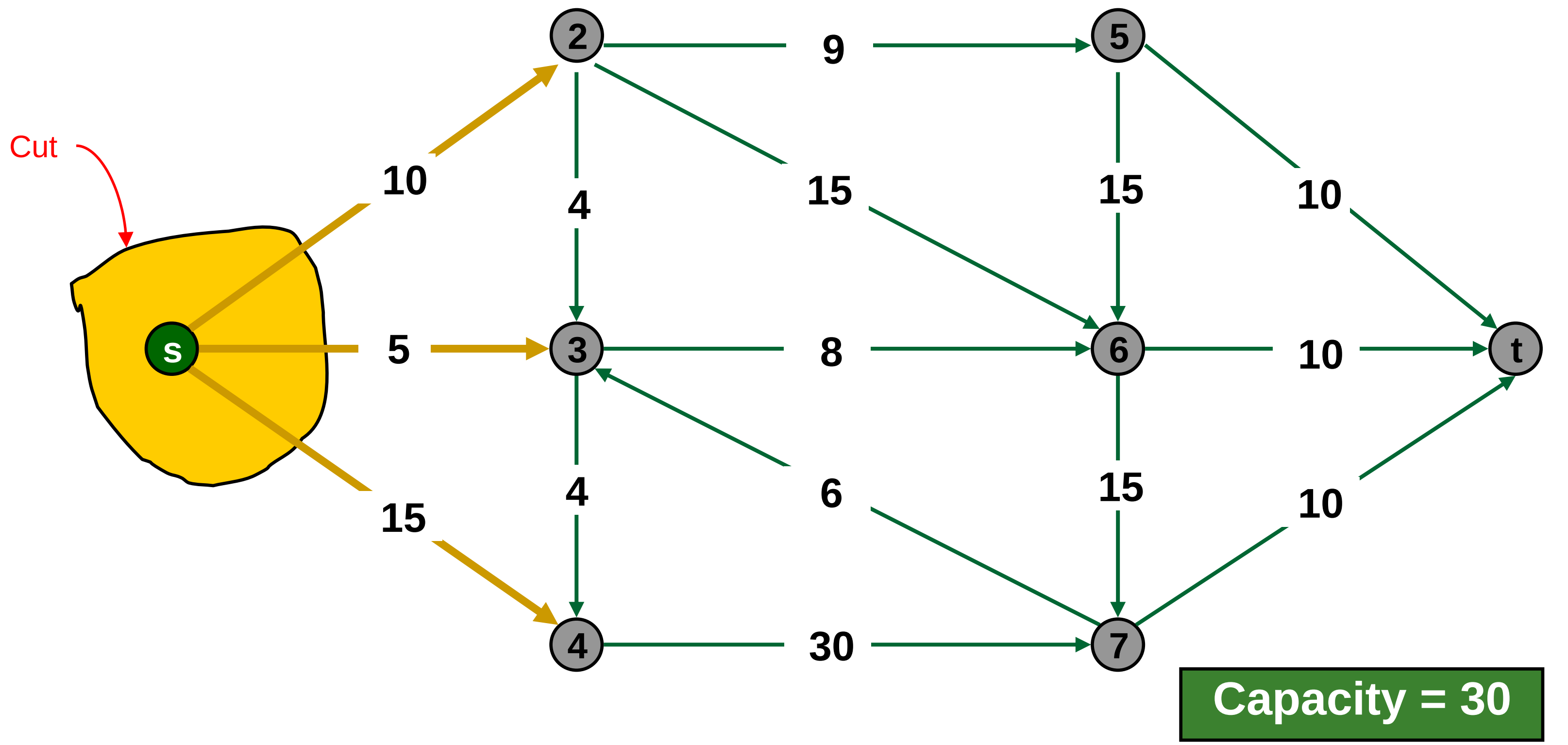
# Cuts of Flow Networks

- A Cut in a network is a partition of $V$ into $S$ and $T$ ($T = V - S$) such that $s$ (source) is in $S$ and $t$ (target) is in $T$.



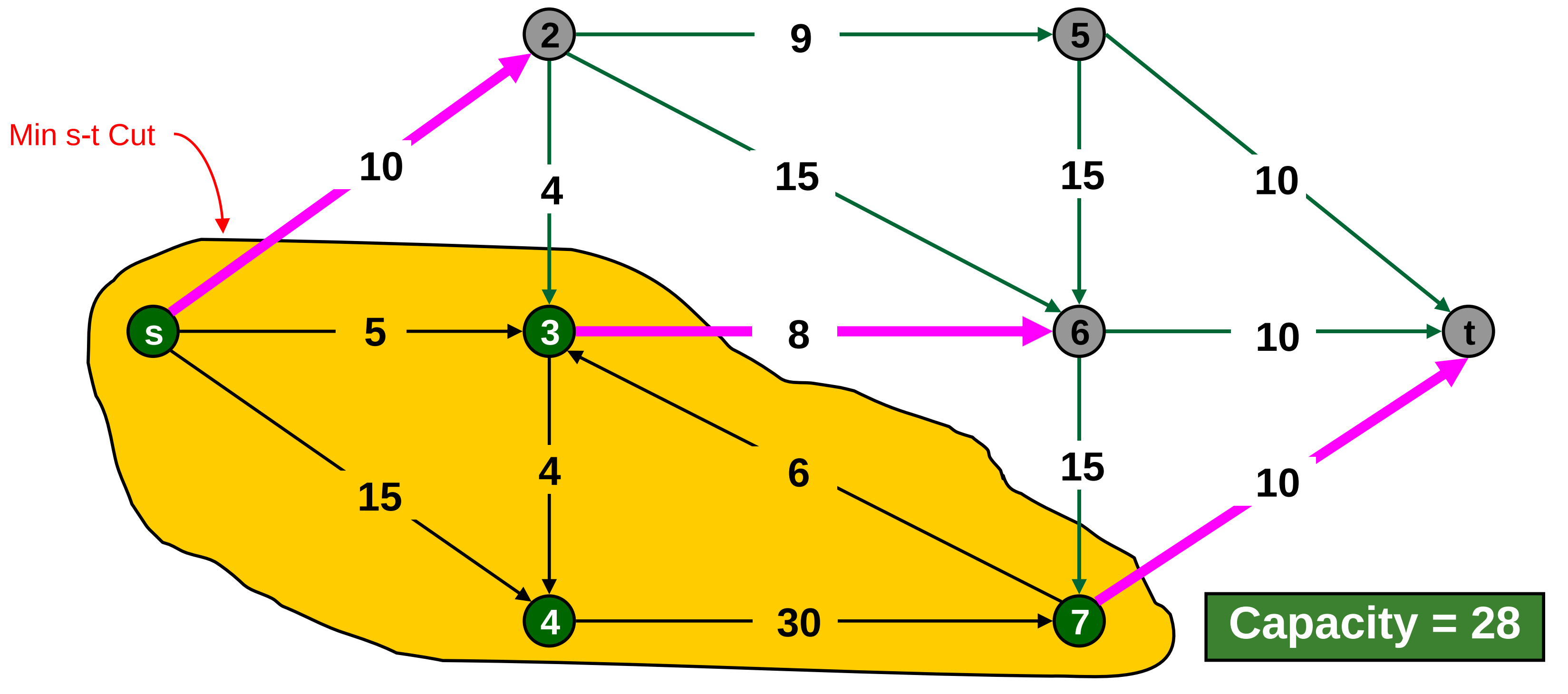
# Capacity of Cut (S,T)

$$c(S,T) = \sum_{u \in S, v \in T} c(u,v)$$



# Min Cut

- Min s-t cut (also called as a Min Cut) is a cut of minimum capacity

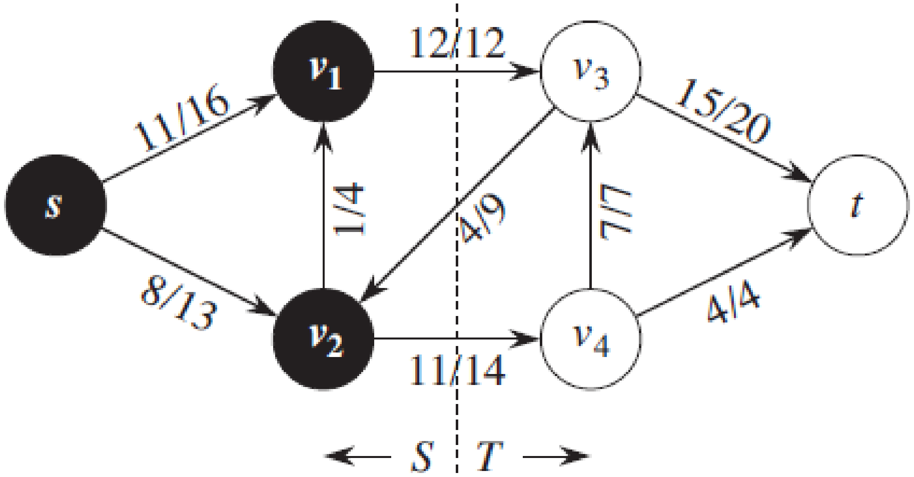


# Flow of Min Cut

- Let $f$ be the flow and let $(S,T)$ be a cut. Then $|f| \le$ capacity$(S,T)$.

$$
\begin{aligned}
|f| &= f(S,T) \\
&= \sum_{u\in S}\sum_{v\in T} f(u,v) - \sum_{u\in S}\sum_{v\in T} f(v,u) \\
&\le \sum_{u\in S}\sum_{v\in T} f(u,v) \\
&\le \sum_{u\in S}\sum_{v\in T} c(u,v) \\
&= c(S,T)\,.
\end{aligned}
$$

Capacity = $c(v_1,v_3) + c(v_2,v_4)$
= 12 + 14 = 26.
Flow = $f(v_1,v_3) + f(v_2,v_4) - f(v_3,v_2)$
= 12 + 11 – 4 = 19.



# Ford-Fulkerson Method

- FORD-FULKERSON-METHOD($G$,$s$,$t$)

1. initialize flow $f$ to 0
2. **while** there exists an augmenting path $p$ in the residual network $G_f$
3. augment flow $f$ along $p$
4. **return** $f$

# Residual Network

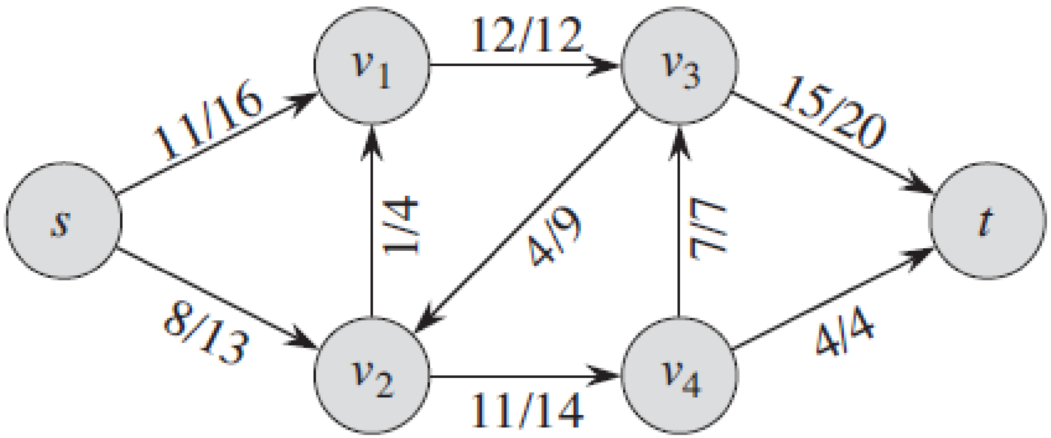
- Given a flow network $G = (V,E)$ and a flow $f$, the ***residual network*** of $G$ induced by $f$ is $G_f = (V,E_f)$, where

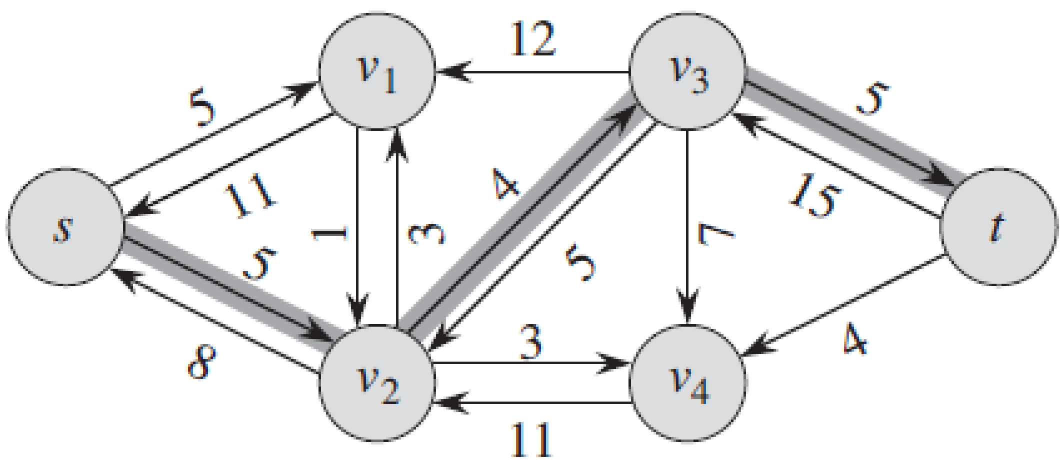
  $E_f = \{(u,v) \in V \times V : c_f(u,v) > 0\}$ with residual capacity $c_f$

$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E , \\ f(v,u) & \text{if } (v,u) \in E , \\ 0 & \text{otherwise} . \end{cases}$$

# Contd…

$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E , \\ f(v,u) & \text{if } (v,u) \in E , \\ 0 & \text{otherwise} . \end{cases}$$



# Augmenting Path

- An augmenting path $p$ is a simple path from $s$ to $t$ on a residual network $G_f$.
- The maximum capacity by which we can increase the flow on each edge in an augmenting path $p$ is known as the residual capacity of $p$, given by

$$\boxed{c_f(p) = \min\{c_f(u,v) : (u,v) \text{ is on } p\}}$$

# Max-flow min-cut theorem

- If $f$ is a flow in a flow network $G = (V,E)$ with source $s$ and sink $t$, then the following conditions are equivalent:

1. $f$ is a maximum flow in $G$.
2. The residual network $G_f$ contains no augmenting paths.
3. $|f| = c(S,T)$ for some cut $(S,T)$ of $G$.

Note:

If $|f| = c(S,T)$, then $c(S,T)$ is the required min-cut.

# Basic Ford-Fulkerson Algorithm

FORD-FULKERSON$(G, s, t)$

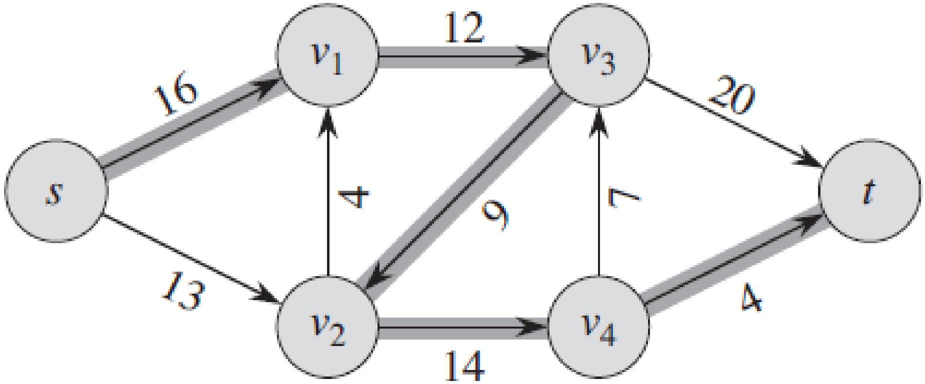
1 **for** each edge $(u, v) \in G.E$
2 $\quad (u, v).f = 0$
3 **while** there exists a path $p$ from $s$ to $t$ in the residual network $G_f$
4 $\quad c_f(p) = \min\{c_f(u, v) : (u, v) \text{ is in } p\}$
5 $\quad$ **for** each edge $(u, v)$ in $p$
6 $\quad\quad$ **if** $(u, v) \in E$
7 $\quad\quad\quad (u, v).f = (u, v).f + c_f(p)$
8 $\quad\quad$ **else** $(v, u).f = (v, u).f - c_f(p)$

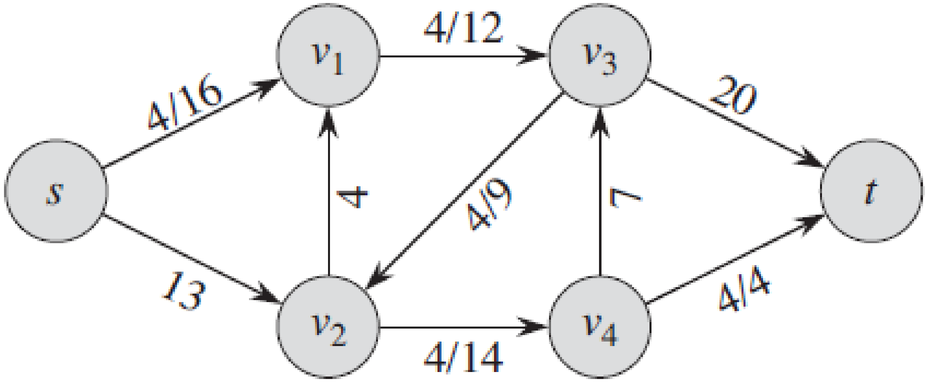
- Line 7. Add residual capacity to the flow over an existing edge $(u,v)$ in $E$.
- Line 8. Subtract residual capacity from the flow over an existing edge $(v,u)$ in $E$.

# Example

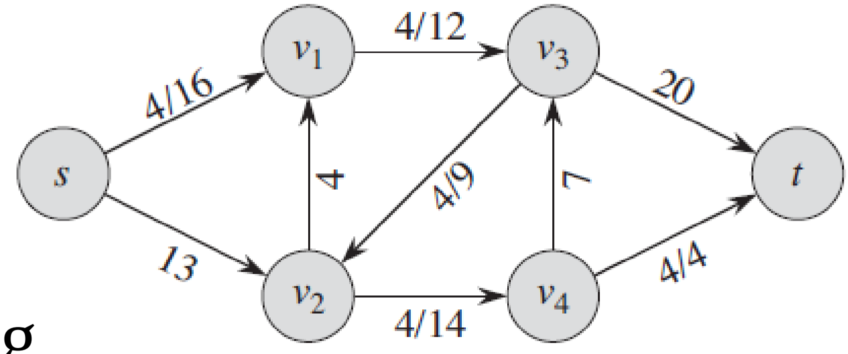
Flow = 4

- Initial flow = 0. Thus original network and initial residual network is same.
- Path: $s \to v_1 \to v_3 \to v_2 \to v_4 \to t$.
- Residual capacity: min(16, 12, 9, 14, 4) = 4.
- All edges in path exists in $E$, so add 4 to the initial flow for all the edges in the path (0 + 4 = 4).



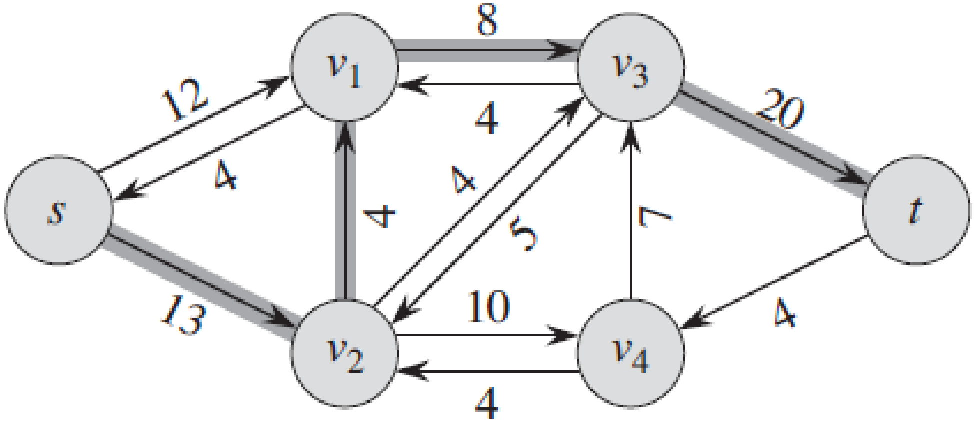



# Contd…




- Update residual network using,

$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E\,, \\ f(v,u) & \text{if } (v,u) \in E\,, \\ 0 & \text{otherwise}\,. \end{cases}$$



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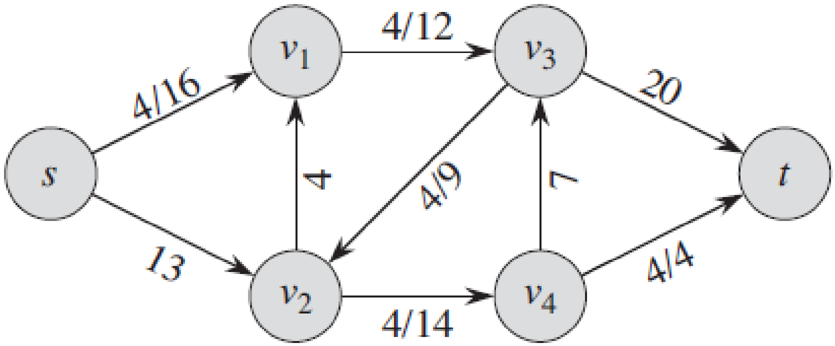
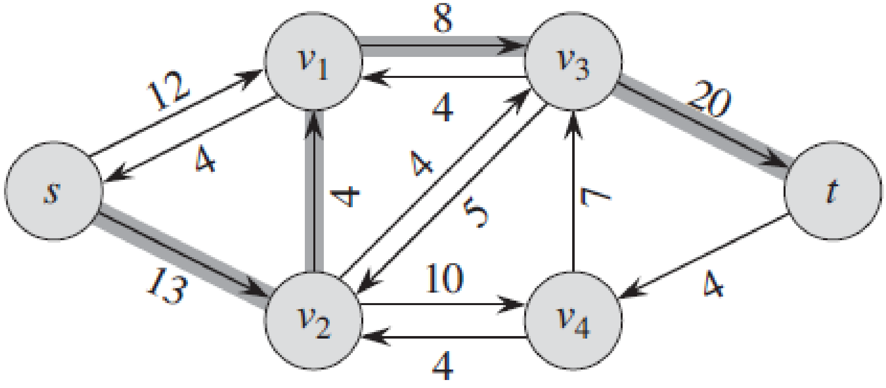
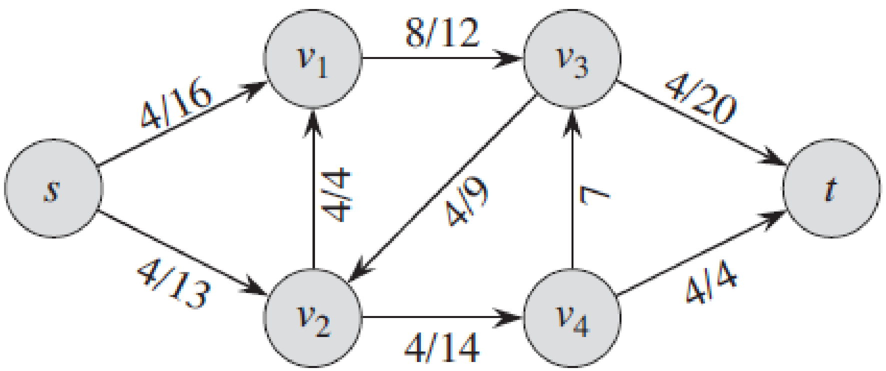



- Path: $s \longrightarrow v_2 \longrightarrow v_1 \longrightarrow v_3 \longrightarrow t$.
- Residual capacity: min(13, 4, 8, 20) = 4.
- All edges in path exists in $E$, so add 4 to the flow for all the edges in the path.

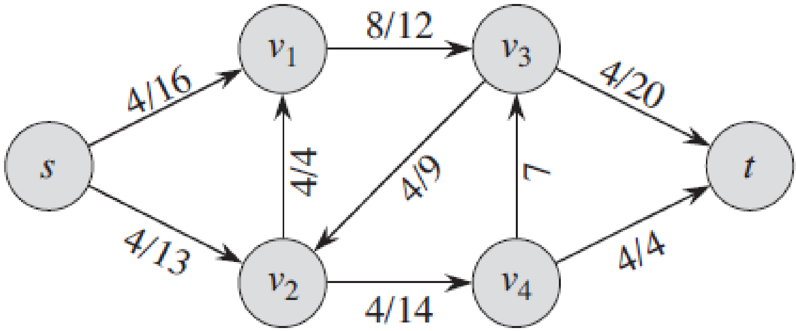
**Flow = 4 + 4**



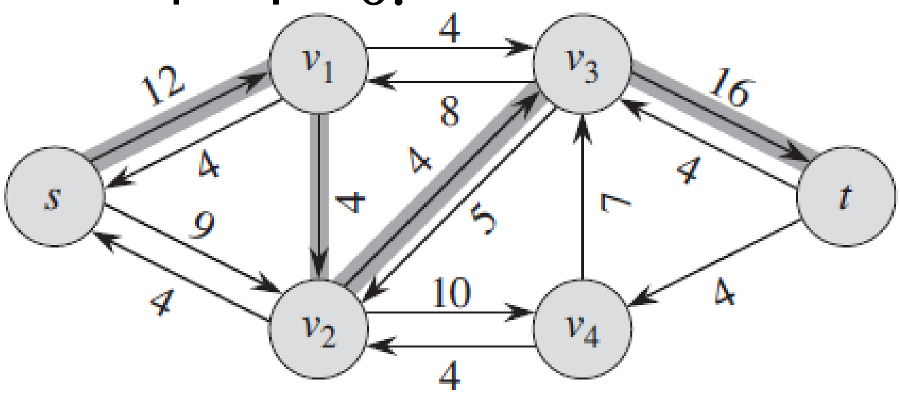
# Contd…

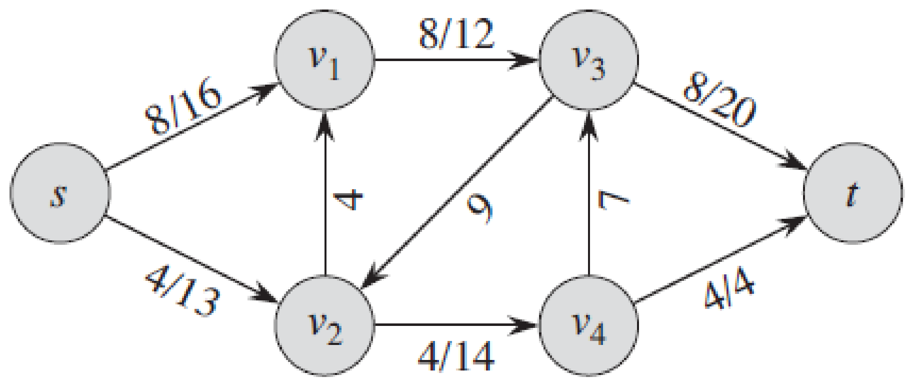
```
for each edge (u, v) in p
    if (u, v) ∈ E
        (u, v).f = (u, v).f + c_f(p)
    else (v, u).f = (v, u).f − c_f(p)
```




- Update residual network.
- Path: $s \rightarrow v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow t$.
- Residual capacity: min(12, 4, 4, 16) = 4.
- Edges ($v_1$,$v_2$) and ($v_2$,$v_3$) in path doesn't exist in $E$, so subtract residual capacity (4) from the previous flow for the edges ($v_2$,$v_1$) and ($v_3$,$v_2$). For both the edges updated flow is 4 – 4 = 0.

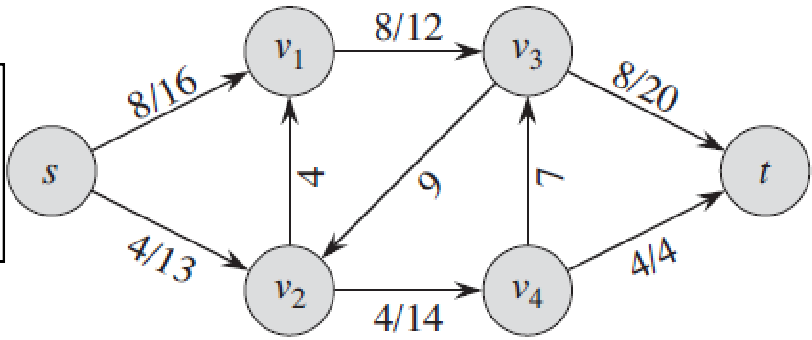
Flow = 4 + 4 + 4



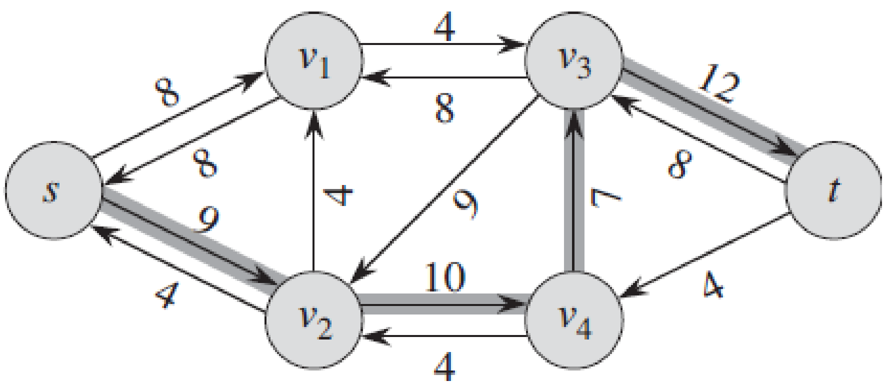
# Contd…

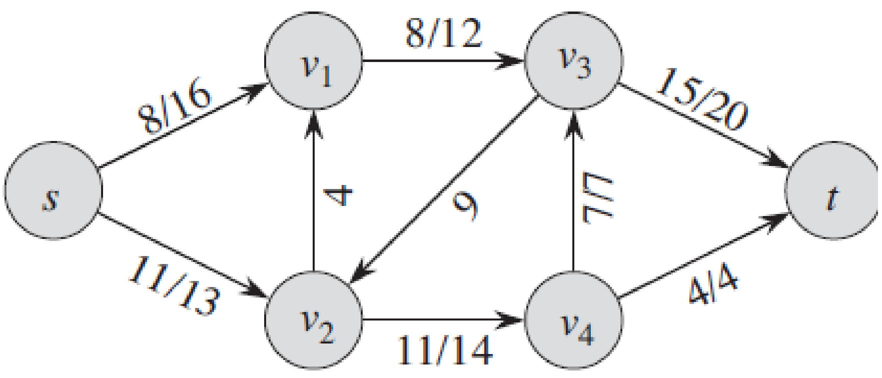
```
for each edge (u, v) in p
    if (u, v) ∈ E
        (u, v).f = (u, v).f + c_f(p)
    else (v, u).f = (v, u).f − c_f(p)
```




- Update residual network.
- Path: $s \rightarrow v_2 \rightarrow v_4 \rightarrow v_3 \rightarrow t$.
- Residual capacity: min(9, 10, 7, 12) = 7.
- All edges in path exists in $E$, so add 7 to the flow for all the edges in the path.

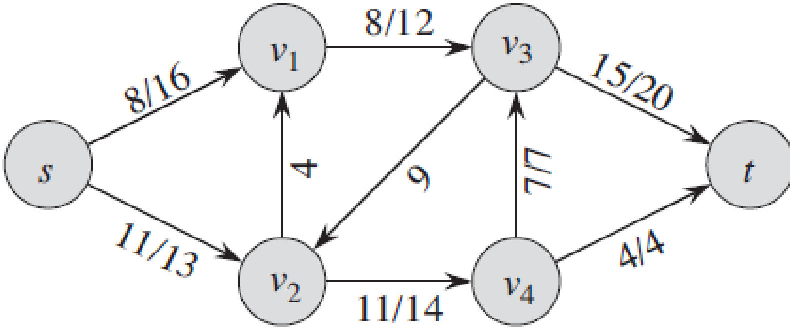
Flow = 4 + 4 + 4 + 7



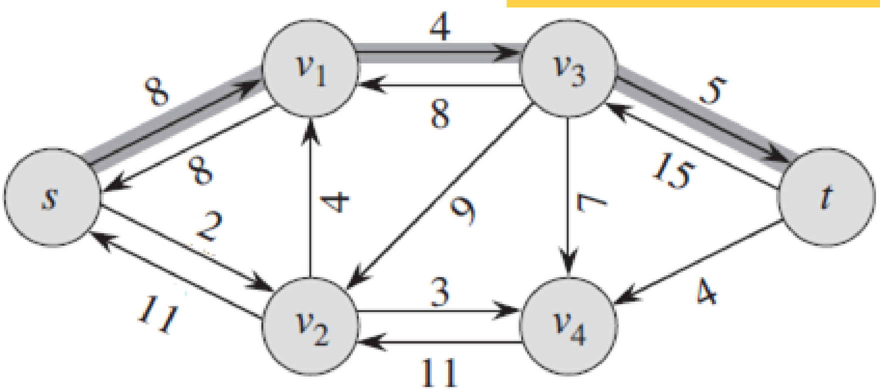
# Contd…

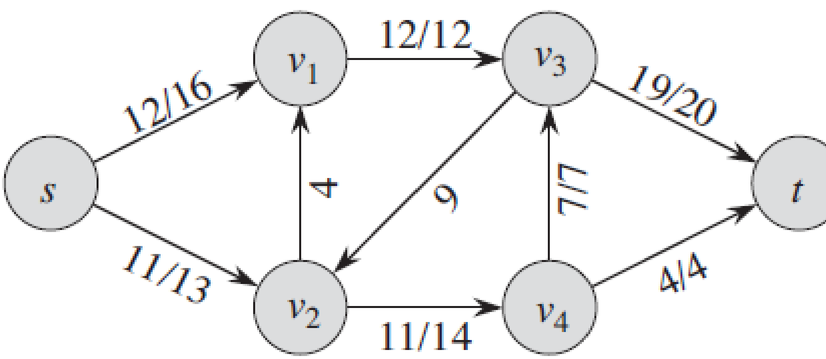
**for** each edge $(u, v)$ in $p$
  **if** $(u, v) \in E$
    $(u, v).f = (u, v).f + c_f(p)$
  **else** $(v, u).f = (v, u).f - c_f(p)$




- Update residual network.
- Path: $s \rightarrow v_1 \rightarrow v_3 \rightarrow t$.
- Residual capacity: min(8, 4, 5) = 4.
- All edges in path exists in $E$, so add 4 to the flow for all the edges in the path.

**Flow = 4 + 4 + 4 + 7 + 4**



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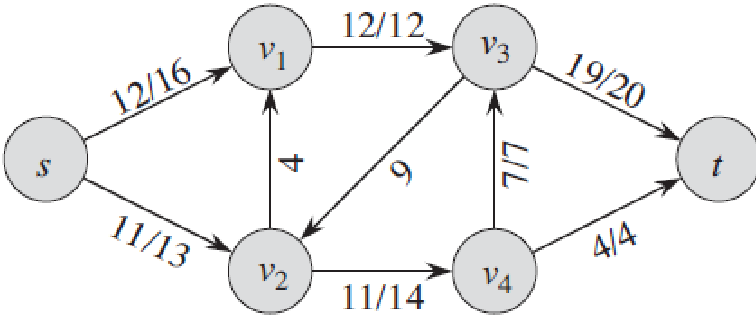
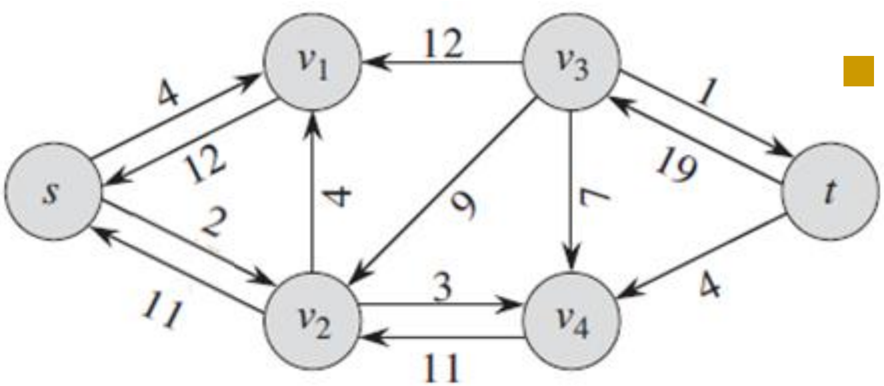


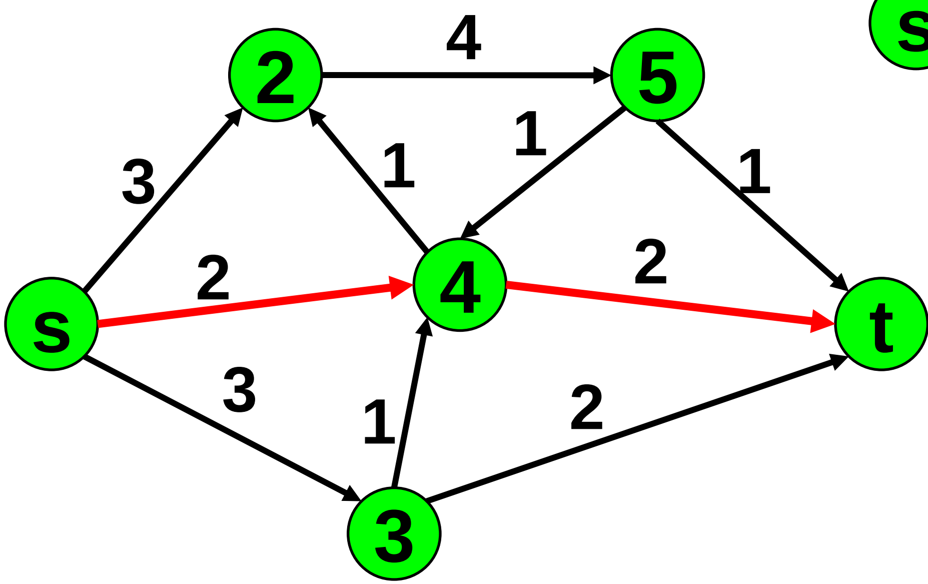

- Update residual network.
- No path exists in the residual network from *s* to *t*.
- Loop terminates and the final flow is

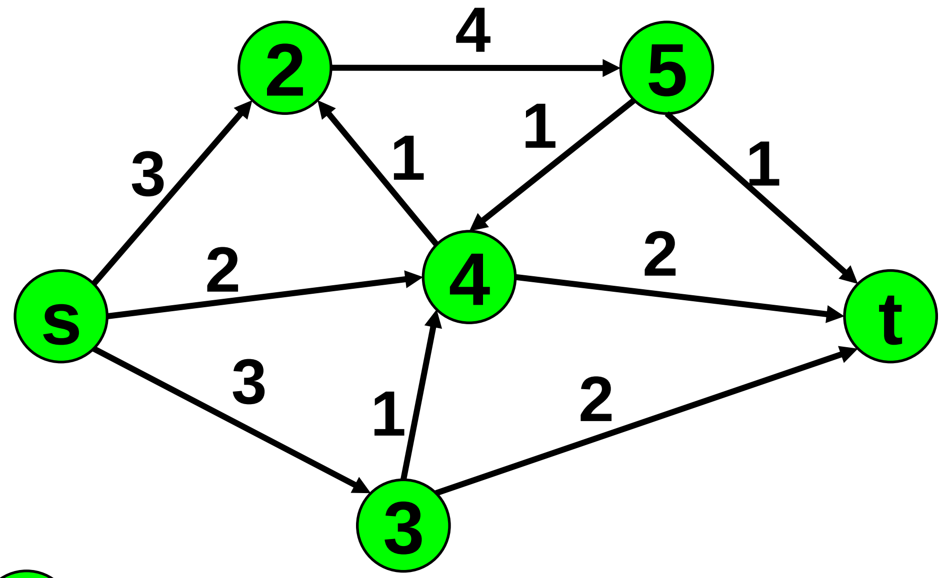
**Flow = 4 + 4 + 4 + 7 + 4 = 23**

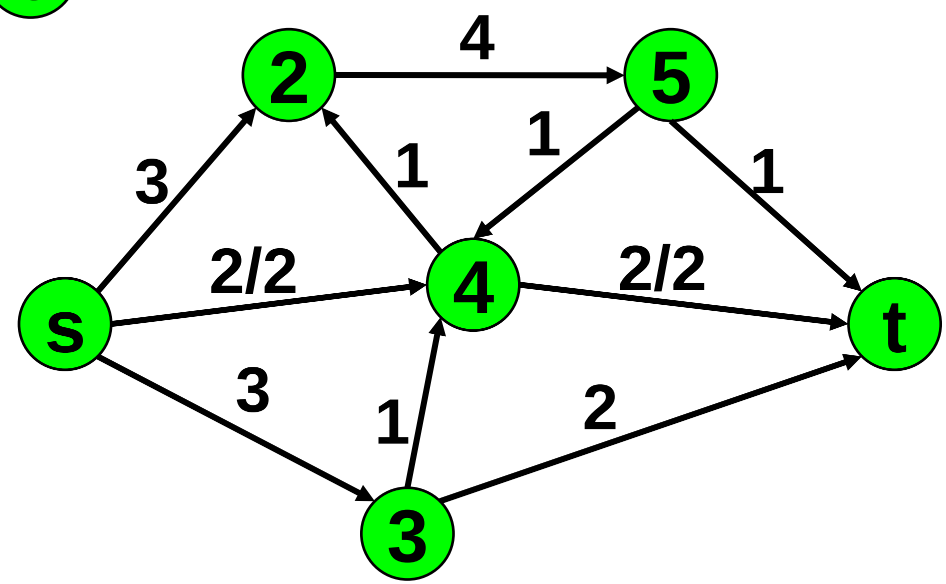



- $|f| = 23$.
- $c(\{s, v_1, v_2, v_4\}, \{v_3, t\}) = 23$.

# Example 2

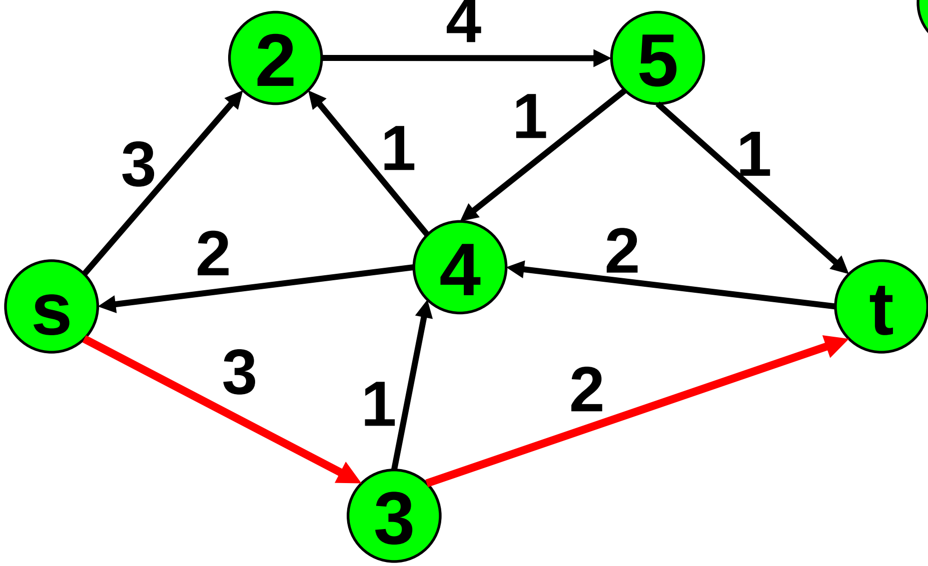


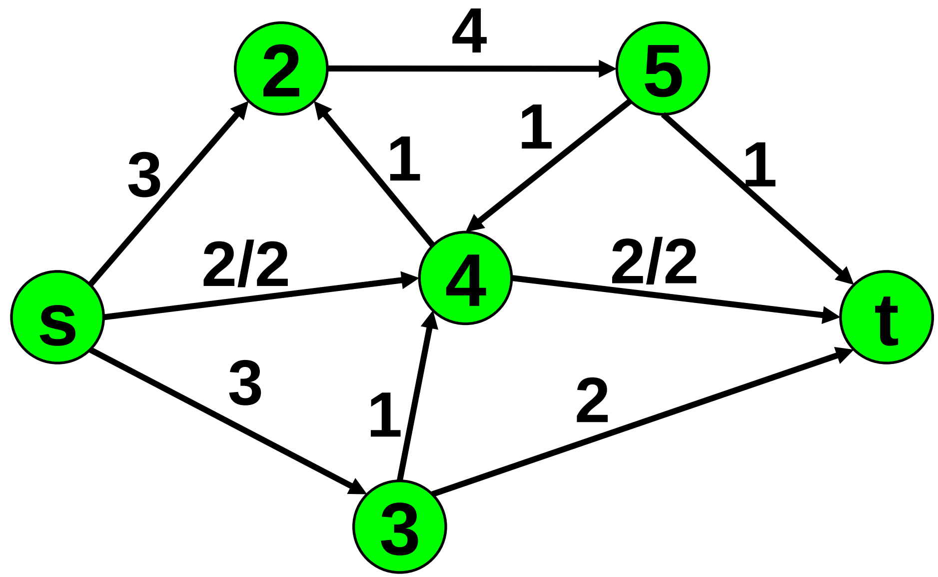



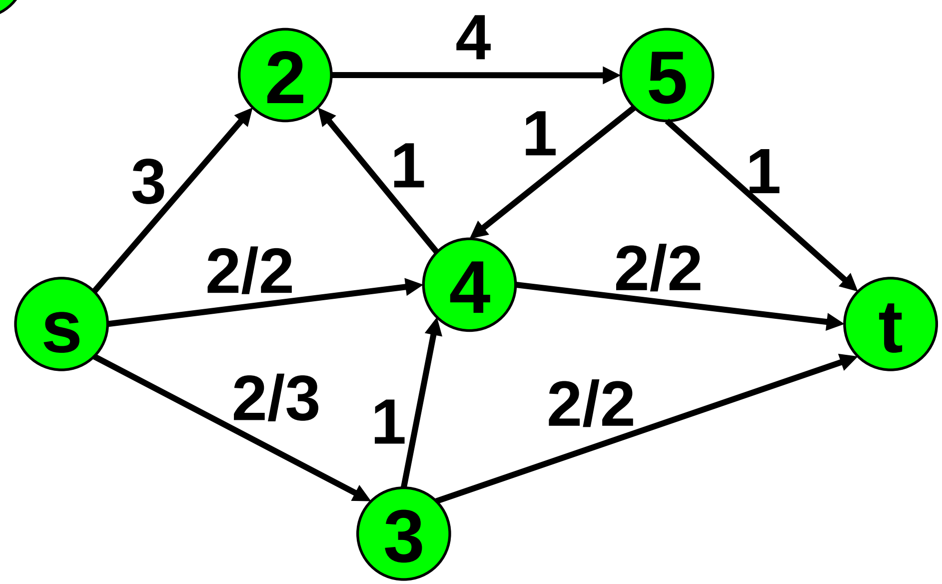




Flow = 2

# Contd…

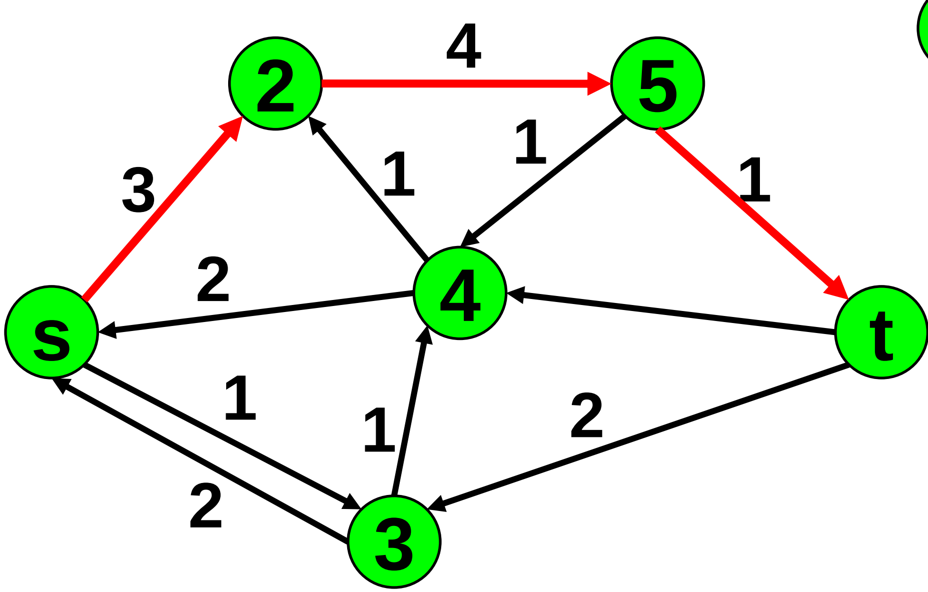


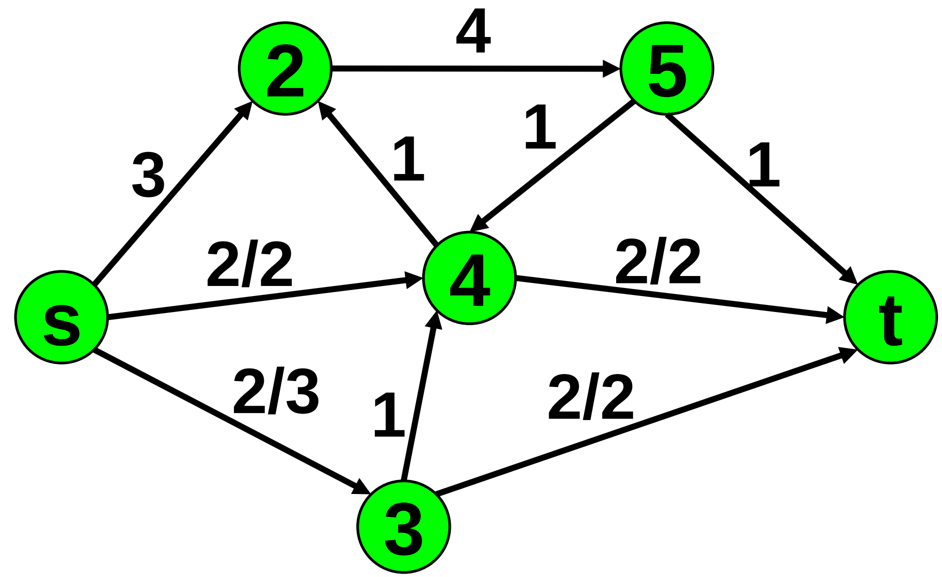



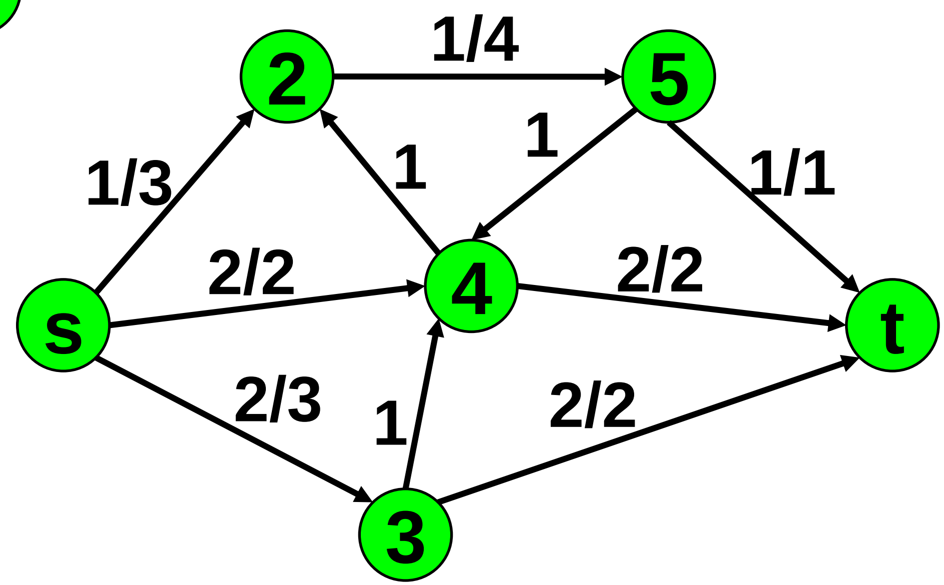

Flow = 2 + 2

# Contd…




Flow = 2 + 2 + 1



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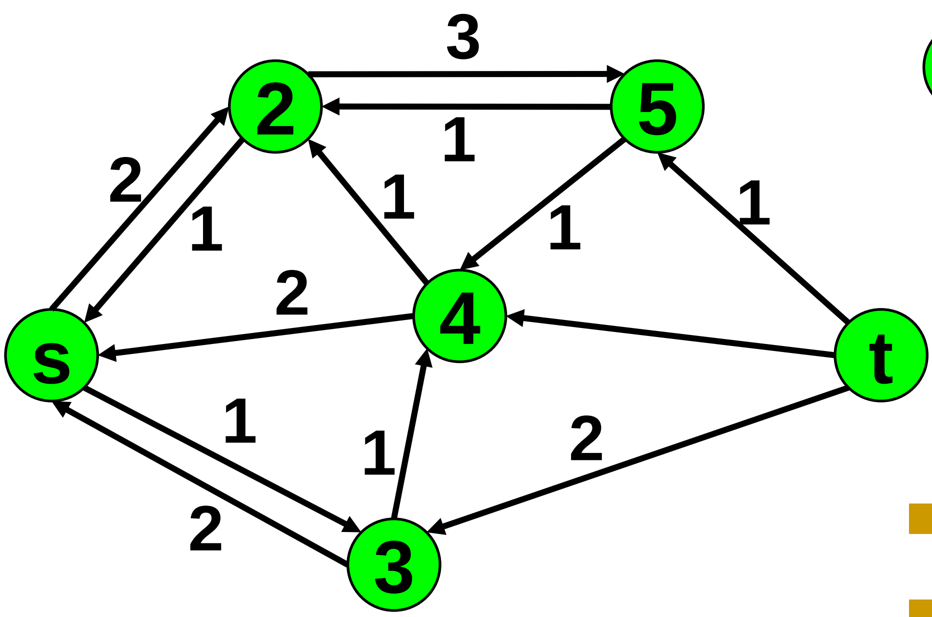


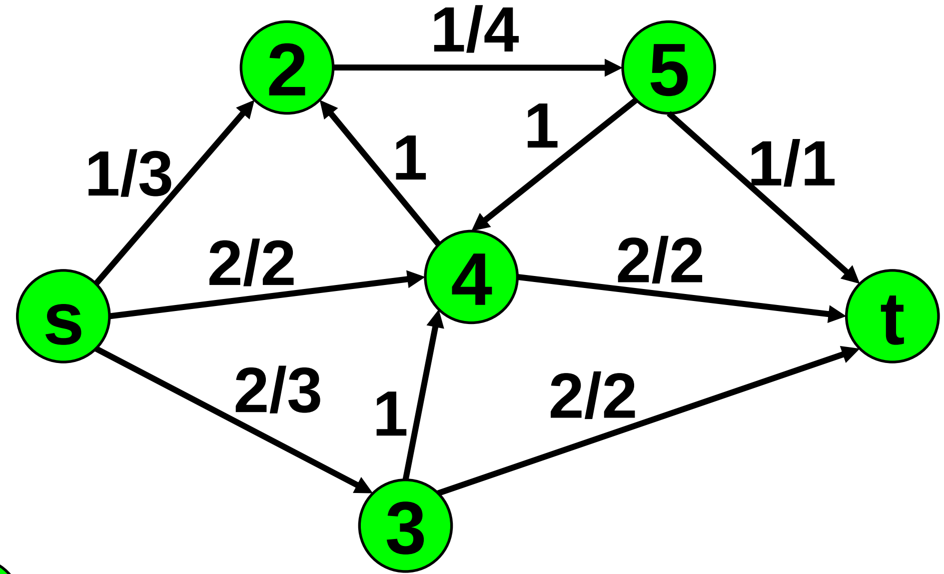


- $|f| = 5$.
- $c(\{s,2,3,4,5\},\{t\}) = 5$.

Flow = 2 + 2 + 1 = 5

# Analysis

- If $f^*$ is the maximum flow, then algorithm executes while loop of lines 3–8 at most $|f^*|$ times, assuming flow increases by at least one unit in each iteration.
- The time to find a path in a residual network is O(E).
- The time to update capacity and flow values is O(1).
- Each iteration of while loop thus takes O(E) time, as does the initialization in lines 1–2.
- Thus, the total running time of the FORD-FULKERSON algorithm is $O(E\,|f^*|)$.

# Edmonds-Karp algorithm

- It’s similar to a Ford-Fulkerson method.
- It chooses the augmenting path as a shortest path from $s$ to $t$ in the residual network.
- Edmonds-Karp algorithm runs in $O(VE^2)$ time.

# Applications

- Application area of max flow min cut is very vast.

- Interested students may refer the document available at http://www.cs.princeton.edu/~wayne/cs423/lectures/max-flow-applications