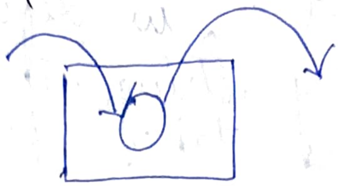
Date – 09.09.23 Ch-6 (Ex-6.1)

* Finite State Machine :-



Rs1 Rs5

Σ – A non-empty finite set
$\downarrow$
Alphabet

In any alphabet, we do not list elements that can be formed from other elements of Σ by juxtaposition.

i.e if $\Sigma = \{0, 1, 2, 3\}$
$\neq \{0, 1, 2, 01, 02, 12\}$

Def 6.1 - If Σ is an alphabet and $n \in Z^+$, we define the powers of Σ, recursively as follows :-

① $\Sigma^1 = \Sigma$

② $\Sigma^{n+1} = \{xy, x \in \Sigma \text{ and } y \in \Sigma^n\}$

Ex - 6.1 - Let Σ be an alphabet

$\Sigma^2 = \{xy, x \in \Sigma \text{ and } y \in \Sigma\}$

$\Sigma = \{0, 1\}$ & $\Sigma^2 = \{00, 01, 10, 11\}$

$\Sigma^3 = \{uv, u \in \Sigma \text{ and } x \in \Sigma, y \in \Sigma\}$

$\Sigma^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$

If $\Sigma = \{a, b, c\}$

$\Sigma^2 = \{aa, ab, ac, ba, bb, bc, ca, cb, cc\}$

For all n, $|\Sigma^n| = |\Sigma|^n$

Def 6.2 - For an alphabet Σ, we define $\Sigma^0 = \{\lambda\}$ where λ denotes the empty string, i.e. the string consisting of no symbols taken from $\Sigma = \{0, 1\}$

$\lambda = \{-\}$

⊙ λ is never an element in our alphabet Σ.

$\Rightarrow \lambda \notin \Sigma$

$\rightarrow \phi \subset \Sigma$

We observe that

① $\{\lambda\} \not\subset \Sigma$ because $\lambda \notin \Sigma$

② $\{\lambda\} \neq \phi$

$|\{\lambda\}| = 1$ (empty string)

$|\phi| = 0$

$|\{\lambda\}| = 1 \neq 0 = |\phi|$

Def 6.3 - If Σ is an alphabet then,

① $\Sigma^+ = \bigcup_{n=1}^{\infty} \Sigma^n = \bigcup_{n \in \mathbb{Z}^+} \Sigma^n$

② $\Sigma^* = \bigcup_{n=0}^{\infty} \Sigma^n = \lambda + \bigcup_{n \in \mathbb{Z}^+} \Sigma^n$.

$\Sigma^* = \lambda + \Sigma^+$

Def 6.4 - If $w_1, w_2 \in \Sigma^+$, then we may write

$w_1 = x_1 x_2 \text{ ----- } x_m$

$w_2 = y_1 y_2 \text{ ----- } y_n$ for $m, n \in \mathbb{Z}^+$

and $x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n \in \Sigma$
We can say that the strings $w1$ and $w2$ are equal and we write $w1 = w2$ if $m=n$ and $x_i = y_i \ \forall\ 1 \le i \le m$

It follows from this defn that two strings in Σ^+ are equal only when each is formed from the same no. of symbols from Σ and the corresponding symbols in the two strings match identically.

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Def-6.5 – Let $w = x_1 x_2 \dots x_n \in \Sigma^+$, where $x_i \in \Sigma$, for each $1 \le i \le n$. We define the length of w which is denoted by $\|w\|$ so the value is n.

For the case of λ,

$\|\{\lambda\}\| = 0$ (length)

$|\{\lambda\}| = 1$ (no. of element, an empty string)

$\Sigma = \{0, 1, 2\}$

$x = 01, \ y = 212, \ z = 01212 \in \Sigma^+$

$\|x\| = 2 \quad \|y\| = 3 \quad \|z\| = 5$

$\|z\| = \|x\| + \|y\|$

Def-6.6 – Let $x, y \in \Sigma^+$ with $x = x_1 x_2 \dots x_m$ and $y = y_1 y_2 \dots y_n$ so that $x_i, y_j \in \Sigma \ \forall\ 1 \le i \le m$ and $1 \le j \le n$. The concatenation of x and y which is xy is the string $x_1 x_2 \dots x_m y_1 y_2 \dots y_n$

eg – $\Sigma = \{0, 1, 2\}$, $x = 01$, $y = 210$, $xy = 01210$
$\|xy\| = 5$

$||xy|| = ||x|| + ||y||$ $\qquad x\lambda = \lambda x$

Q. Prove that $||x|| = ||\lambda x||$

$$||x|| = ||x|| + 0$$
$$= ||x|| + ||\lambda||$$
$$= ||x\lambda||$$
$$= ||\lambda x||$$

Def 6.7 - For each $x \in \Sigma^+$, we define the power of x by $x^0 = \lambda$, $x^1 = x$, $x^2 = xx$, $x^{n+1} = xx^2 \ldots x^{n-1} = xx^n$, where $n \in N$

(6.8 — X not in syllabus)

Def 6.9 - If $x, y, z \in \Sigma^+$ and $w = xyz$, then y is called a substring of w. If atleast one of x and z will be not equal to λ, then y is called a proper substring of w.

Example for 6.7 -

$\Sigma = \{0, 1\}$

$x = 01$, $\quad x^0 = \lambda$

$x^1 = 01$

$x^2 = 0101$

$x^3 = 01\,01\,01$

$||x^3|| = 6 = 3 \times 2 \Rightarrow ||x^n|| = n||x||$

Ex - 6.7 - For $\Sigma = \{0, 1\}$, let $w = 00101110 \in \Sigma^*$. Find out two substring of w

$\Rightarrow$ $w = xyz = 0010\,1110 = xyz$

① $x = 00$ $\quad y = 1011$ $\quad z = 10$

② $x = 00$ $\quad y = 10$ $\quad z = 1110$

Def 6.10 - For a given alphabet Σ, any subset of Σ^* is called a languge over Σ. This includes the subset ϕ, which we call the empty language.

Ex 6.2 With $\Sigma = \{0, 1\}$ the sets $A = \{0, 01, 001\}$ is languge

$B = \{0, 01, 001, 0001, \ldots\}$ language.

* Exercise 6.1 -

Q1. Let $\Sigma = \{a, b, c, d, e\}$ (a) What is $|\Sigma^2|$ & $|\Sigma^3|$?

$|\Sigma^2| = 25 = 5^2$, $|\Sigma^3| = 125 = 5^3$

(b) How many string in the Σ^* have length at most 5.

$5^0 + 5^1 + 5^2 + 5^3 + 5^4 + 5^5$

Q2. $\Sigma = \{0, 1\}$

$\Sigma^2 = \{00, 01, 10, 11\}$

$\Sigma^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$

Q3. If $x \in \Sigma^*$ and $||x^3|| = 36$ what is $||x||$?

$||x^3|| = 3||x|| = 36$

$||x|| = 12$

H/W - 8 and 10 (6.1)

Q4. Let $\Sigma = \{\beta, x, y, z\}$ where β denotes a blank, $x\beta \neq x$; $\beta\beta \neq \beta$ and $x\beta y \neq xy$ but $x \lambda y = xy$. Compute each of the following

a) $||\alpha\alpha|| = 0$ $\quad ||\beta|| = 1$ $\quad ||x\beta\beta y|| = 4$

$||\beta\alpha|| = 1$

Q8. $A = \{10, 11\}$, $B = \{00, 1\}$ be languages for alphabet $\Sigma = \{0, 1\}$.

(a) AB ; (b) BA ; (c) A^3 ; (d) B^2.

Solu^n – (a) $AB = \{10, 11\}\{00, 1\}$

$= \{1000, 1100, 101, 111\}$

(b) $BA = \{00, 1\}\{10, 11\}$

$= \{0010, 110, 0011, 111\}$

(c) $A^3 = \{10, 11\}\{10, 11\}\{10, 11\}$

$= \{1010, 1110, 1011, 1111\}\{10, 11\}$

$= \{101010, 111010, 101110, 111110, 101011, 111011, 101111, 111111\}$

(d) $B^2 = \{00, 1\}\{00, 1\}$

$= \{0000, 100, 001, 11\}$

Q10 $\Sigma = \{x, y, z\}$ $\quad A, B \subseteq \Sigma^*$ $\quad A = \{xy\}$

$B = \{d, x\}$. (a) AB ; (b) BA ; (c) B^3 ;

(d) B^+ ; (e) A^*

Solu^n – $AB = \{xy\}\{d, x\} = \{xyd, xyx\}$

$BA = \{d, x\}\{xy\} = \{dxy, xxy\}$

$B^3 = \{d, x\}\{d, x\}\{d, x\} = \{d^2, xd, dx, xx\}\{d, x\}$

$= \{d^3, xd^2, dxd, xxd, xxx, xdx, dxx, d^2x\}$

$B^{+} = B^{1} + B^{2} + \ldots\ldots B^{\infty}$.

=

$A^{*} = A^{0} + A^{1} + A^{2} + \ldots A^{\infty}$.

Date - 16.09.23

Ex - 6.2

Table 6.1

	t_0	t_1	t_2	t_3	t_4
State	S_0	S_1(5/-)	S_2(10/-)	S_3(20/-)	S_0
Input	5/-	5/-	10/- (20/-)	W	
Output	No	No	No (10/-)	Pepper-mint	

vending machine

20rs

Peppermint flavour - W

Sparemint flavour - B

5rs, 10rs, 20rs or 50rs.

Def 6.13 - A finite state machine is a five tuple

$M = \{S, I, O, V, W\}$ where

$S \to$ the set of internal states inside the machine

$I \to$ input alphabet for M.

$O \to$ output alphabet for M

$V: S \times I \to S$ is the next state function

$W: S \times I \to O$ is the output function.

example 6.17 :-

$M = (S, I, O, V, W)$

$S = \{S_0, S_1, S_2\}$

$I = O = \{0, 1\}$ and

V and W are given by the Table.

	v		w	
	0	1	0	1
S_0	S_0	S_1	0	0
S_1	S_2	S_1	0	0
S_2	S_0	S_1	0	1

Solu^n –

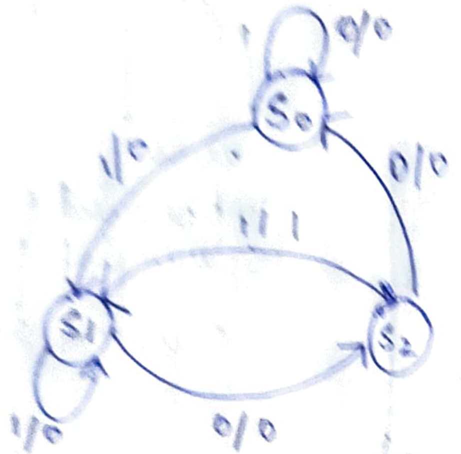


fig- state diagram

$v(S_1, 1) = S_1$

With S_0 designated as its starting ~~[illegible]~~ state, if input provided to M is the string 1010, then what will be output string?

$(S_1, 0)$ $(S_1, 0)$

1010.

$(S_0, 1)$ $(S_2, 1)$

Table 6.4

State	S_0	S_1	S_2	S_1
Input	1	0	1	0
Output	0	0	1	0

Output → 0010

Input - 110010

State	S_0	S_1	S_1	S_2	S_0	S_1
Input	1	1	0	0	1	0
Output	0	0	0	0	0	0

Output string = 000000

Ex-6.2

① Using the ~~[illegible]~~ previous finite state machine table, find out the output for the following & also determine the last internal state for the transition process.

(a) $x = 1010101$

(b) $x = 1001001$

(c) $x = 101001000$

Solu^n -

(a)

S_0	S_1	S_2	S_1	S_2	S_1	S_2
1	0	1	0	1	0	1
0	0	1	0	1	0	1

Output = 0010101

Final internal state = S_2

(b)

S_0	S_1	S_2	S_0	S_1	S_2	S_0
1	0	0	1	0	0	1
0	0	0	0	0	0	0

Output = 0000000

Final internal state = S_0

Example 6.18

① $S = \{s_0, s_1, s_2, s_3, s_4\}$

② $I = \{5¢, 10¢, 25¢, B, W\}$

③ $O = \{n, P, S, 5¢, 10¢, 15¢, 20¢, 25¢\}$

Table 6.5

	ν					w				
	5¢	10¢	25¢	B	W	5¢	10¢	25¢	B	W
s_0	s_1	s_2	s_4	s_0	s_0	n	n	5¢	n	n
s_1	s_2	s_3	s_4	s_1	s_1	n	n	10¢	n	n
s_2										
s_3										
s_4										

6.2 H/W

Exercise – 6.2(4)

Example – 6.10

Date - 20.09.23

Ch-7

7.1 Relations Revisited : Properties Of Relation.

Definition 7.1 - For two sets A, B any subset of $A \times B$ is called a relation from A to B.

Any subset of $A \times A$ is called a relation in A.

Example 7.1 (a) Define the relation R on the set Z by aRb or $(a, b) \in R$ if $a \leq b$.

$Z = \{1, 2, 3, 4\}$.

$\{(1,1), (1,2), (1,3), (1,4), (1,5)\} \rightarrow$ this will be a relation.

Definition 7.2 - A relation R on a set A is called reflexive if $\forall x \in A$, $(x, x) \in R$

Example 7.4 $A = \{1, 2, 3, 4\}$

$R_1 = \{(1,1), (2,2), (3,3), (3,4)\}$ ✗.

$R_2 = \{(x, y) : x, y \in A \text{ and } x \leq y\}$ ✓

Example 7.5 Given any set A with $|A| = n$. How many reflexive relation will be there?

Soln - ~~[illegible]~~ No. of relation will be $2^{n^2 - n}$

$|A| = n$

$|A \times A| = n^2$

$A = \{1,2,3\}$

$A \times A = \{(1,1), (1,2), (1,3)$
$(2,1), (2,2), (2,3)$
$(3,1), (3,2), (3,3)\}$.

Definition 7.3 - Relation R on a set A is called symmetric if $(x,y) \in R \Rightarrow (y,x) \in R$; $\forall x, y \in A$

Example 7.6 $A = \{1,2,3\}$

$R_1 = \{(1,2), (2,1), (1,3), (3,1)\}$ ✓
$R_2 = \{(1,1), (2,2), (3,3)\}$ ✓

Definition 7.4 - For a set if a relation R is called transitive if $\forall x, y, z \in A$ $(x,y), (y,z) \in R \Rightarrow (x,z) \in R$.

See example 7.8 & 7.9

Definition 7.5 - Given a relation R on a set A, R is called antisymmetric if $\forall a, b \in A$, aRb and $bRa \Rightarrow a = b$.

See example 7.11 & 7.12.

Definition 7.6 - A relation R on a set is called a partial order, or partial ordering relation if R is reflexive, anti-symmetric and transitive.

example - $A = \{1, 2, 3, 4\}$
xRy if $x \le y$, $y \le z$, $x \le z$.

Definition 7.7 - A relation R on a set A is called a equivalence relation if R is reflexive, symmetric and transitive.

Example - $A = \{1, 2, 3\}$

$R_1 = \{(1,1), (2,2), (3,3)\}$ ✓

$R_2 = \{(1,1), (2,2), (2,3), (3,2), (1,2)\}$ X

Exercise 7.1

① If $A = \{1, 2, 3, 4\}$, given an example of a relation R on A, i.e

(a) reflexive & symmetric but not transitive

(b) reflexive & transitive but not symmetric

(c) symmetric & transitive but not reflexive

Solu^n - (a) $R = \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,1), (2,3), (3,2)\}$.

(b) $R = \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,3), (1,3)\}$.

(c) $R = \{(1,2), (2,1), (1,3), (3,1), (2,3), (3,2)\}$

H/w 5, 6.

Date - 23.09.23

⑤ For each of the following relations determine whether the relation is reflexive, symmetric, anti-symmetric, transitive

(a) $R \subseteq Z^+ \times Z^+$ where aRb if a/b

Solu^n - Reflexive, Antisymmetric, Transitive

(b) $R \subseteq Z \times Z \rightarrow$ Transitive

(c) For a given universe U and a fixed subset C of U. ~~where~~ Define R on $P(U)$ such that for $A, B \subseteq U$
aRb, $A \cap C = B \cap C$

Soln – Reflexive Symmetric Transitive

(d) On the set A of all lines in R^2, define the relation R for two lines l_1 and l_2 by $l_1 R l_2$ if it l_1 is $\perp$ar to l_2

Soln – Symmetric

$l_1 \perp l_1 \rightarrow$ not possible

$l_1 \perp l_2 \rightarrow l_2 \perp l_1 \rightarrow$ symmetric

$l_1 \perp l_2$
and $l_2 \perp l_3 \Rightarrow l_1 \parallel l_3 \rightarrow$ not possible

7.2

Definition 7.8 :- If A, B and C are sets with $R_1 \subseteq A \times B$ and $R \subseteq B \times C$ then the composite relation $R_1 \circ R_2$ is a relation from A to C defined by $R_1 \circ R_2 =$
$\{ (x, z) : x \in A$ and $z \in C$ and $\exists$ $y \in B$ with $(x, y) \in R_1$ and $(y, z) \in R_2 \}$.

$A = \{1, 2, 3, 4\}$ $B = \{w, x, y, z\}$ $C = \{5, 6, 7\}$

$R_1 = \{(1, x), (2, x), (3, y), (3, z)\}$

$R_2 = \{(w, 5), (x, 6)\}$

$R_3 = \{(w, 5), (w, 6)\}$

$R_1 \circ R_2 = \{(1, 6), (2, 6)\}$

$R_1 \circ R_3 = \phi$

same example

Definition - 7.9 - Given a set A and a relation R on A, we define the powers of R recursively by (a) $R^1 = R$

(b) for $n \in Z^+$, $R^{n+1} = R \circ R^n$

Q. If $A = \{1, 2, 3, 4\}$

$R = \{(1,2), (1,3), (2,4), (3,2)\}$

$R^2 = R \circ R = \{(1,4), (1,2), (3,4)\}$

$R^3 = R^2 \circ R = \{(1,4)\}$

Definition - 7.10 - An $m \times n$ zero-one matrix, $E = (e_{ij})_{m \times n}$ is a rectangular array of numbers arranged in m rows and n - columns where each e_{ij} for $1 \le i \le m$ and $1 \le j \le n$ denotes the entry in the ith row and jth column of E and each such entry is either 0 or 1.

$$E = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

columns (B)

Rows (A)

$$\begin{array}{c|c} & \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Date - 27/09/23

* R_1 and R_2

$M(R_1)$ and $M(R_2)$

$$M(R_1 \circ R_2) = M(R_1) \circ M(R_2)$$

$A = \{1, 2, 3, 4\}$, $B = \{w, x, y, z\}$, $C = \{5, 6, 7\}$

$R_1 = \{(1, w), (1, x), (1, y), (2, z), (3, w), (4, z)\}$

$R_2 = \{(w, 5), (x, 5), (y, 5), (z, 6), (x, 6), (y, 6)\}$

$R_1 \circ R_2 = \{(1,5), (2,6), (3,5), (4,6), (1,6)\}$

$$M(R_1) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \qquad M(R_2) = \begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccc} 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$M(R_1 \circ R_2) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$M(R_1) M(R_2) = \begin{bmatrix} 1+1+1+0 & 0+1+1+0 & 0+0+0+0 \\ 0+0+0+0 & 0+0+0+1 & 0+0+0+0 \\ 1+0+0+0 & 0+0+0+0 & 0+0+0+0 \\ 0+0+0+0 & 0+0+0+1 & 0+0+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(Boolean multiplication)

Q. $A = \{1,2,3,4\}$

$R = \{(1,2), (1,3), (2,4), (3,2)\}$

Find $M(R^2)$, $M(R^4)$

Solnⁿ-

$R^2 = R \circ R = \{(1,2), (1,3), (2,4), (3,2)\} \circ \{(1,2), (1,3), (2,4), (3,2)\}$

$= \{(1,2), (1,4), (3,4)\}$.

$M(R^2) = M(R \circ R) = M(R) \cdot M(R)$

$= (M(R))^2$

$$\boxed{M(R^n) = (M(R))^n}$$

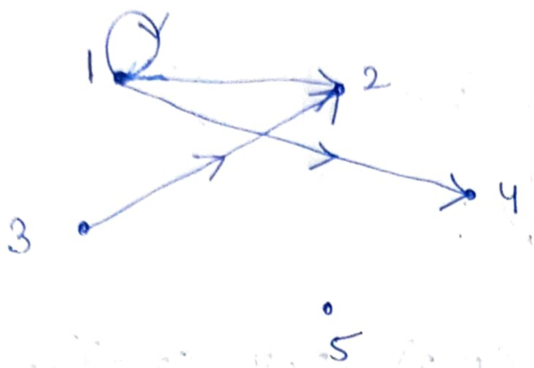
$M(R^2)$:-

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Definition:- 7.14- Let V be a finite non-empty set. A directed graph G on V is made up of elements of V called the ~~vertic~~ vertices or nodes of G and a subset E of $V \times V$ that contains the edges of G. ~~The~~ ~~that~~ V is called the vertex set.

Q. $V = \{1,2,3,4,5\}$

$R = \{(1,1), (1,2), (1,4), (3,2)\}$

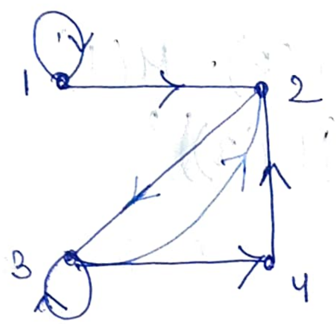


If $a \rightleftarrows b$, then

a ——— b, give no direction.

* Adjacent vertex : Connected edge.

⊙ Ex - 7.27 - for $A = \{1, 2, 3, 4\}$, let $R = \{(1,1), (1,2)$
$(2,3), (3,2), (3,3), (3,4), (4,2)\}$



Strongly connected - directed graph is connected.
Weakly connected - undirected graph correspon-ding to the directed graph is connected

only defined for directed graphs.

Definition 7.15 :- A directed graph G on V is called strongly connected if $x, y \in V$ where $x \neq y$, there is a path of directed edges from x to y.

Ex - 7.2 - ① H/W

Ex - 7.2 - ④ Let $A = \{1, 2, 3\}$, $B = \{w, x, y, z\}$, $C = \{4, 5, 6\}$

$R_1 \subseteq A \times B$; $R_2 \subseteq B \times C$ & $R_3 \subseteq B \times C$, where

$R_1 = \{(1, w), (3, w), (2, x), (1, y)\}$

$R_2 = \{(w, 5), (x, 6), (y, 4), (y, 6)\}$

$R_3 = \{(w, 4), (w, 5), (y, 5)\}$

(a) Determine $R_1 \circ (R_2 \cup R_3)$ and $(R_1 \circ R_2) \cup (R_1 \circ R_3)$

(b) $R_1 \circ (R_2 \cap R_3)$ and $(R_1 \circ R_2) \cap (R_1 \circ R_3)$

(a) $R_2 \cup R_3 = \{(w, 5), (w, 4), (x, 6), (y, 4),$
$(y, 5), (y, 6)\}$

$R_1 \circ (R_2 \cup R_3) = \{(1, 5), (1, 4), (2, 6), (1, 6),$
$(3, 4), (3, 5)\}$

$R_1 \circ R_2 = \{(1, 5), (3, 5), (2, 6), (1, 4), (1, 6)\}$

$R_1 \circ R_3 = \{(1, 4), (1, 5), (3, 4)\}$

$(R_1 \circ R_2) \cup (R_1 \circ R_3) = \{(1, 5), (3, 5), (2, 6), (1, 4), (1, 6),$
$(3, 4)\}$

Date - 30.09.23

Chapter - 8 : The Principle Of Inclusion And Exclusion

$C_1 \rightarrow$ the students have taken freshmen composition
$C_2 \rightarrow$ the students have taken Introduction to economics

$N(S) = 100 \quad N(C_1) = 35 \quad N(C_2) = 30$

$N(C_1 C_2) = 9$

$N(\bar{C_1} \bar{C_2}) = N(\overline{C_1 C_2}) + N(C_1 C_2)$

$\rightarrow$ No. of students taking freshman composition but no introduction to economics $= N(C_1 \bar{C_2})$

$= N(C_1) - N(C_1 C_2)$

$= 35 - 9 = 26$

$\rightarrow$ No. of students taking introduction to economics but not freshmen composition $= N(\bar{C_1} C_2)$

$= N(C_2) - N(C_1 C_2)$

$= 30 - 9$

$= 21$

$\rightarrow$ Neither of two $= N(\bar{C_1} \bar{C_2}) = N(\bar{C_1}) - N(\bar{C_1} C_2)$

$= N(S) - N(C_1) - (N(C_2) - N(C_1 C_2))$

$= N(S) - N(C_1) - N(C_2) + N(C_1 C_2)$

$= 44.$

Example - 8.2

Theorem - 8.1 (The principle of Inclusion & Exclusion)

Consider a set S with $|S| = N$ and conditions $C(i)$; $1 \le i \le t$, each of which may be satisfied by some of the elements of S. The no. of elements of S that satisfy none of the condition C_i is denoted by $\overline{N} = N(\overline{C_1}, \overline{C_2} \ldots \overline{C_t})$

$$\overline{N} = N - \sum_{1 \le i \le t} N(C_i) + \sum_{1 \le i < j \le t} N(C_i C_j) - \sum_{1 \le i < j < k \le t} N(C_i C_j C_k)$$

$$+ (-1)^t N(C_1 C_2 \ldots\ldots C_t)$$

Ex-8.4 Determine the number of +ve integers n where $1 \le n \le 100$ and is not divisible by 2, 3 or 5

Solu^n -
$N(C_1) = 50 = \lfloor 100/2 \rfloor$

$N(C_2) = 33 = \lfloor 100/3 \rfloor$

$N(C_3) = 20 = \lfloor 100/5 \rfloor$

$N(\overline{C_1}) = 50 \quad N(\overline{C_2}) = 67 \quad N(\overline{C_3}) = 20.$

$N(C_1 C_2) = \left\lfloor \frac{100}{6} \right\rfloor = 16$

$N(C_2 C_3) = \left\lfloor \frac{100}{15} \right\rfloor = 6.$

$N(C_1 C_3) = \left\lfloor \frac{100}{10} \right\rfloor = 10.$

$N(C_1 C_2 C_3) = \left\lfloor \frac{100}{30} \right\rfloor = 3$

$N(\overline{C_1}\,\overline{C_2}\,\overline{C_3}) = N - (N(C_1) + N(C_2) + N(C_3)) +$
$(N(C_1 C_2) + N(C_2 C_3) + N(C_1 C_3)) - N(C_1 C_2 C_3)$
$= 100 - (50 + 33 + 20) + (16 + 6 + 10) - 3 = 26.$

Ex-8.5 Find the no. of non-negative integer solutions to the equation $x_1+x_2+x_3+x_4=18$ where $x_i \le 7$ for $1 \le i \le 4$.

$$x_1+x_2+\dots x_n = k, \; x_i \geq 0$$

$$^{n+k-1}C_k.$$

Solu^n- Here S is the set of solu^n for $x_1+x_2+x_3+x_4=18$, where $x_i \geq 0$.

$$N = |S| = {}^{18+4-1}C_{18} = {}^{21}C_{18}$$

$N(\bar{C_1}\bar{C_2}\bar{C_3}\bar{C_4})$ $\quad x_i > 7$

$\Rightarrow x_i \geq 8$.

$C_1 \to x_1 \geq 8$

$C_2 \to x_2 \geq 8$

$C_3 \to x_3 \geq 8$

$C_4 \to x_4 \geq 8$

$N(C_1) \to y_1 = x_1 - 8 \Rightarrow y_1 + 8 + x_2 + x_3 + x_4 = 18$

$\Rightarrow y_1 + x_2 + x_3 + x_4 = 10.$

$\Rightarrow {}^{10+4-1}C_{10} = {}^{13}C_{10}$

$N(C_2) \to y_2 = x_2 - 8 = {}^{13}C_{10} = C_3 = C_4$

$z_1 = x_1 - 8, \; z_2 = x_2 - 8.$

$N(C_1C_2) \to z_1 + 8 + z_2 + 8 + x_3 + x_4 = 18.$

$z_1 + z_2 + x_3 + x_4 = 2.$

$\Rightarrow {}^{5}C_2 = N(C_2C_3) = N(C_3C_4)$

$= N(C_2C_4)$

$= N(C_1C_3)$

$= N(C_1C_4)$

Date - 04.10.23

Q. Find the number of non-ve integer solution to the equation $x_1 + x_2 + x_3 + x_4 = 18$ where $x_i \leq 7 \ \forall$ $1 \leq i \leq 4$

Solu^n - $C_1 : x_1 \geq 8$ $C_2 : x_2 \geq 8$ $C_3 : x_3 \geq 8$ $C_4 : x_4 \geq 8$

$x_1 \geq 8 \Rightarrow y_1 = x_1 - 8$.

$N = {}^{n+k-1}C_n$

$x_1 + x_2 + \dots x_k = n$.

$N(C_1) = N(C_2) = N(C_3) = N(C_4) = {}^{13}C_{10}$

$N(C_1C_2) = N(C_2C_3) = N(C_3C_4) = N(C_1C_3) = N(C_1C_4)$

$= N(C_2C_4)$

$= {}^5C_2$.

$N(\bar{C_1}\bar{C_2}\bar{C_3}\bar{C_4}) = {}^{21}C_{18} - {}^4C_1 \times {}^{13}C_{10} + {}^4C_2 \times {}^5C_2$

Q. Find the no. of non-ve integer solu^n to the equation $x_1 + x_2 + x_3 + x_4 = 19$ if $0 \leq x_i < 8 \ \forall \ 1 \leq i \leq 4$

Solu^n -

~~N(C1) = N(C2) = N(C3)~~

$C_1 : x_1 \geq 8$, $C_2 : x_2 \geq 8$ $C_3 : x_3 \geq 8$ $C_4 : x_4 \geq 8$.

$N(C_1 C_2 C_3 C_4) = {}^{19+4-1}C_{19} = {}^{22}C_{19}$.

$N(C_1) = N(C_2) = N(C_3) = N(C_4) = {}^{11+4-1}C_{11}$

$y_1 + 8 + x_2 + x_3 + x_4 = 19$ $= {}^{14}C_{11}$

$y_1 + x_2 + x_3 + x_4 = 11$

$x_1 = y_1 - 8$

$z_1 = x_1 - 8 \qquad z_2 = x_2 - 8$

$z_1 + 8 + z_2 + 8 + x_3 + x_4 = 19$

$z_1 + z_2 + x_3 + x_4 = 19 - 16 = 3$

$$N(C_1C_2) = N(C_1C_3) = N(C_1C_4) = N(C_2C_3)$$
$$= N(C_2C_4)$$
$$= N(C_3C_4)$$
$$= {}^{3+4-1}C_3$$
$$= {}^{6}C_3$$

$$N(\bar{C_1}\bar{C_2}\bar{C_3}\bar{C_4}) = {}^{22}C_{19} - {}^{4}C_1\,{}^{14}C_{11} + {}^{4}C_2\,{}^{6}C_3$$

Exercise 5, 6 - H/W

Exercise 8.1

Q1. Let S be a finite set with $|S| = N$ and let C_1, C_2, C_3, C_4 be four conditions each of which may be satisfied by $\cancel{N(\bar{C_2}\bar{C_3}\bar{C_4}) = N(C_1\bar{C_2}\bar{C_3}\bar{C_4}) + N(\bar{C_1}\bar{C_2}\bar{C_3}\bar{C_4})}$

~~Solu~~ one or more elements of S. Prove that

$$N(\bar{C_2}\bar{C_3}\bar{C_4}) = N(C_1\bar{C_2}\bar{C_3}\bar{C_4}) + N(\bar{C_1}\bar{C_2}\bar{C_3}\bar{C_4})$$

Solu^n - $N(C_1\bar{C_2}\bar{C_3}\bar{C_4}) = N(\bar{C_2}\bar{C_3}\bar{C_4}) - N(\bar{C_1}\bar{C_2}\bar{C_3}\bar{C_4})$

LHS -

$$N(C_1\bar{C_2}\bar{C_3}\bar{C_4}) = N(C_1) - \{N(C_1C_2) + N(C_1C_3) + N(C_1C_4)\} + \{N(C_1C_2C_3) + N(C_1C_2C_4) + N(C_1C_3C_4)\} - N(C_1C_2C_3C_4)$$

RHS -

$$N(\bar{C}_2\bar{C}_3\bar{C}_4) - N(\bar{C}_1\bar{C}_2\bar{C}_3\bar{C}_4)$$

$$\Rightarrow \Big[N - \{N(C_2) + N(C_3) + N(C_4)\} + \{N(C_2C_3) + N(C_2C_4)$$

$$+ N(C_3C_4)\} - N(C_2C_3C_4)\Big] - \Big[N - \{N(C_1) + N(C_2) +$$

$$+ N(C_3) + N(C_4)\} + \{N(C_1C_2) + N(C_1C_3) + N(C_1C_4) +$$

$$N(C_2C_3) + N(C_2C_4) + N(C_3C_4)\} - \{N(C_1C_2C_3) +$$

$$N(C_1C_2C_4) + N(C_2C_3C_4)\} + N(C_1C_2C_3C_4)\Big]$$

Date - 07.10.23

Ch-9 Generating functions

⊙ 9.1

⊙ $c_1 + c_2 + c_3 + c_4 = 25 \qquad 1 \le c_i \le 10$

Example 9.1 :

No. of oranges is 12
Grace, Mary & Frank

G	M	F
4	3	5
4	4	4
4	5	3
4	6	2
5	2	5
5	3	4
5	4	3
5	5	2
6	2	4
6	3	3
6	4	2
7	3	2
8	2	2

Grace at least 4
Mary at least 2
Frank at least 2

$c_1 + c_2 + c_3 = 12$

$4 \le c_1 \le 8 \,;\; 2 \le c_2 \,;\; 2 \le c_3 \le 5$

$$f(x) = (x^4 + x^5 + x^6 + x^7 + x^8) \times (x^2 + x^3 + x^4 + x^5 + x^6) \times (x^2 + x^3 + x^4 + x^5)$$

$\Rightarrow$ The coefficient of x^{12} in the above eqn will give the no. of non-neg integer soln for the original equation.

Example - 9.2 - If there is an unlimited number (or at least 24 of each color) of red, green, white and black jelly beans, in how many ways can Douglas select 24 of these candies so that he has an even no. of white beans & at least 6 black beans.

For red & green beans –

$f_1(x) = x^0 + x^1 + x^2 + \dots + x^{24}$.

For white beans –

$f_2(x) = x^0 + x^2 + x^4 + x^6 + \dots + x^{24}$.

For black beans –

$f_3(x) = x^6 + x^7 + \dots + x^{24}$.

$f(x) = f_1(x) * f_2(x) * f_3(x)$

$= (1 + x + x^2 + \dots + x^{24}) \times$
$(1 + x^2 + x^4 + \dots + x^{24}) \times$
$(x^6 + x^7 + \dots + x^{24})$.

The solu^n will be the coefficient of x^{24}.

Exercise 9.1

① For each of the following determine a generating function and indicate the coefficient in the function that is needed to solve the problem.

Find the no. of non-ve integer solutions.

(a) $c_1 + c_2 + c_3 + c_4 = 20$; $0 \le c_i \le 7$; $1 \le i \le 4$

Solu^n – $f_1(x) = x^0 + x^1 + \dots x^7 = 1 + x + x^2 + \dots + x^7$

$= f_2(x) = f_3(x) = f_4(x)$

$f(x) = (1 + x + x^2 + \dots + x^7)^4$.

The no. of non-ve integral solutions will be the coeff. of x^{20}.

(b) $c_1 + c_2 + c_3 + c_4 = 20$; $0 \le c_i$ $1 \le i \le 4$

c_2 and c_3 is even

$f_1(x) = 1 + x + x^2 + \ldots = \dfrac{1}{1-x}$ (from Binomial Theorem)

$f_2(x) = 1 + x^2 + x^4 + \ldots = \dfrac{1}{1-x^2}$ (")

$f_3(x) = f_2(x)$

$f_4(x) = f_1(x)$

$f(x) = (1 + x + x^2 + \ldots)^2 * (1 + x^2 + x^4 \ldots)^2$

$= \left(\dfrac{1}{1-x}\right)^2 * \left(\dfrac{1}{1-x^2}\right)^2$

Solu^n will be coeff. x^{20}.

(c) $C_1 + C_2 + C_3 + C_4 + C_5 = 30$; $2 \leq C_1 \leq 4$

$3 \leq C_i \leq 8$

for $2 \leq i \leq 5$

$f_1(x) = x^2 + x^3 + x^4$

$f_2(x) = f_3(x) = f_4(x) = f_5(x)$

$= x^3 + x^4 + x^5 + x^6 + x^7 + x^8$

$f(x) = (x^2 + x^3 + x^4)(x^3 + x^4 + x^5 + x^6 + x^7 + x^8)^4$

Solu^n will be coeff. of x^{30}.

Date – 11/10/23

Exercise 9.1

Q5. Find the generating function for the number of integer solution to the equation $c_1 + c_2 + c_3 + c_4 = 20$ where $-3 \leq c_1, -3 \leq c_2, -5 \leq c_3 \leq 5$ and $0 \leq c_4$.

Solu^n – $c_1 + c_2 + c_3 + c_4 = 20$

Let us consider,

$x_1 = c_1 + 3, \quad x_2 = c_2 + 3, \quad x_3 = c_3 + 5, \quad x_4 = c_4$

$x_1 - 3 + x_2 - 3 + x_3 - 5 + x_4 = 20$

$\Rightarrow \quad x_1 + x_2 + x_3 + x_4 = 31$

$x_1 \geq 0, x_2 \geq 0$

$-5 \leq x_3 - 5 \leq 5$

$\Rightarrow 0 \leq x_3 \leq 10$

$x_4 \geq 0$

$f_1(x) = 1 + x + x^2 + \dots + x^{31}$

$f_2(x) = 1 + x + x^2 + \dots + x^{31}$

$f_3(x) = 1 + x + x^2 + \dots + x^{10}$

$f_4(x) = 1 + x + x^2 + \dots + x^{31}$

$f(x) = f_1(x) \cdot f_2(x) \cdot f_3(x) \cdot f_4(x)$.

Solu^n will be the coefficient of x^{31}.

(0) 9.2

Def^n – Let $a_0, a_1, a_2, \dots$ be a sequence of real numbers. The function $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + \sum_{i=0}^{\infty} a x^i$ is called the generating for the given sequence.

$(1+x)^n = \sum_{k=0}^{n} {}^nC_k x^k = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_n x^n$.

(Binomial Theorem)

Generating funcⁿ - $({}^nC_0, {}^nC_1, {}^nC_2 \dots {}^nC_n)$

Example 9.5

(a) $(1-x^{n+1}) = (1-x)(1+x+x^2+\dots+x^n)$

$\Rightarrow \dfrac{1-x^{n+1}}{1-x} = 1+x+x^2+\dots+x^n$

(b) $1+x+x^2+\dots\infty = \dfrac{1}{1-x}$.

$(1, 1, 1 \dots$ upto $\infty)$

(c) $\dfrac{1}{1-x} = 1+x+x^2+\dots+\sum_{i=0}^{\infty} x^n$.

$\dfrac{d}{dx}\left(\dfrac{1}{1-x}\right) = \dfrac{d}{dx}(1+x+x^2+\dots)$

$\dfrac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+\dots$

(d) $\dfrac{x}{(1-x)^2} = x+2x^2+3x^3+4x^4+\dots$

Example 9.6

(a) $\dfrac{1}{1-y} = 1+y+y^2+\dots$

$y = 2x$

$\dfrac{1}{1-2x} = 1+4x^2+16x^4+\dots$

(b) $\{1, a, a^2, a^3, \ldots\} = \frac{1}{1-ax}$

$f(x) = \frac{1}{1-x}$

$\{1, 1, 1, \ldots 1, \ldots \infty\}$

$g(x) = f(x) - x^2$

$(1 + x + \cancel{x^2} + x^3 + \ldots \infty) - \cancel{x^2}$

$\{1, 1, 0, 1, 1, \ldots \infty\}$

$h(x) = f(x) + 2x^3$

$= 1 + x + x^2 + x^3 + \ldots \infty + 2x^3$

$\{1, 1, 1, 3, 1, 1, \ldots \infty\}$

(c) Find the generating function for the sequence
$0, 2, 6, 12, 20, 30, 42, \ldots$

$a_0 = 0^2 + 0 = 0$
$a_1 = 1^2 + 1 = 2$
$a_2 = 2^2 + 2 = 6$

$\vdots$

$a_n = n^2 + n$

$f(x) = (0 + 0^2)x^0 + (1 + 1^2)x^1 + (2 + 2^2)x^2 + \ldots$

$= (0x^0 + 1 \cdot x^1 + 2x^2 + \ldots) + (0^2 x^0 + 1^2 x^1 + 2^2 x^2 + \ldots)$

$= \frac{x}{(1-x)^2} + \frac{x(x+1)}{(1-x)^3}$

$0^2, 1^2, 2^2, 3^2, \ldots \ldots = \frac{x(x+1)}{(1-x)^3}$

$n, r \in \mathbb{Z}^+ \quad n \geq r > 0$

$${}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$n \in \mathbb{R}$

$${}^nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

$n \in \mathbb{Z}^+$

$${}^{-n}C_r = \frac{(-n)(-n-1)(-n-2)\cdots(-n-r+1)}{r!}$$

$$= \frac{(-1)^r\, n(n+1)(n+2)\cdots(n+r-1)}{r!}$$

$$= \frac{(-1)^r\,(n+r-1)!}{(n-1)!\,r!} = (-1)^r\, {}^{n+r-1}C_r$$

$$\frac{1}{1-x} = (1-x)^{-1}$$

$$(1+x)^{-n} = \sum_{r=0}^{\infty} (-1)^r \cdot {}^{n+r-1}C_r\, x^r$$