$n, r \in \mathbb{Z}^+ \quad n \geq r \geq 0$

$${}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$n \in \mathbb{R}$

$${}^nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

$n \in \mathbb{Z}^+$

$${}^{-n}C_r = \frac{(-n)(-n-1)(-n-2)\cdots(-n-r+1)}{r!}$$

$$= \frac{(-1)^r\, n(n+1)(n+2)\cdots(n+r-1)}{r!}$$

$$= \frac{(-1)^r\,(n+r-1)!}{(n-1)!\,r!} = (-1)^r\, {}^{n+r-1}C_r.$$

$$\frac{1}{1-x} = (1-x)^{-1}$$

$$(1+x)^{-n} = \sum_{r=0}^{\infty} (-1)^r \cdot {}^{n+r-1}C_r\, x^r.$$

$${}^nC_r = \frac{n!}{(n-r)!\,r!}$$

<u>Date - 14.10.23</u>

Example - 9.7

$${}^nC_r = {}^{n+r-1}C_r$$

For $n \in \mathbb{Z}^+$

$$(1+x)^{-n} = \sum_{r=0}^{\infty} (-1)^r\, {}^{n+r-1}C_r\, x^r$$

$$(1+x)^{-n} = {}^{-n}C_0 + {}^{-n}C_1 x + {}^{-n}C_2 x^2 + \dots {}^{-n}C_n x^n$$

$$= \sum_{r=0}^{\infty} {}^{-n}C_r x^r$$

Example - 9.8 -
Find the coeff. of x^5 in $(1-2x)^{-7}$

Solu^n - $y = -2x$

$$(1+y)^{-7} = \sum_{r=0}^{\infty} (-1)^r \, {}^{7+r-1}C_r \, y^r$$

$$= \sum_{r=0}^{\infty} (-1)^r \, {}^{6+r}C_r \, y^r$$

$$= \sum_{r=0}^{\infty} (-1)^5 \, {}^{6+5}C_5 \, (-2x)^5$$

$$= -\sum_{r=0}^{\infty} {}^{11}C_5 \, (-32) x^5$$

Coeff of x^5 will be $(-1)^5 \cdot {}^{7+5-1}C_5 (-2)^5$

Example - 9.9 - For each real number n, the Maclauren series expansion to r $(1+x)^n$ is

$$1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \dots$$

Solu^n - $$1 + \sum_{r=1}^{\infty} \frac{n(n-1)(n-2) \dots (n-r+1)}{r!} x^r$$

Example - 9.10 - Determine the coefficient of x^{15} in $(x^2 + x^3 + x^4 + \dots)^4$

Solu^n - $\{x^2 (1 + x + x^3 + \dots \infty)\}^4$

$$\Rightarrow x^{2 \times 4} \left(\frac{1}{1-x}\right)^4 = \frac{x^8}{(1-x)^4}$$

$\Rightarrow \quad x^8 (1-x)^{-4}$

$\Rightarrow \quad x^8 \sum_{r=0}^{\infty} (-1)^r \cdot {}^{n+r-1}C_r \, x^r$

$= \quad x^8 (-1)^{7} \cdot {}^{4+7-1}C_7 \, x^7$

$\Rightarrow \quad - x^8 \cdot {}^{10}C_7 \, x^7$

$\therefore \Rightarrow \quad -{}^{10}C_7 \, x^{15}$

Example – 9.14 – In how many ways can a police captain distribute 24 rifle shells to four police officer so that each officer get at least 3 shells but not more than 8.

Solu^n –
$$c_1 + c_2 + c_3 + c_4 = 24$$
$$3 \le c_1, c_2, c_3, c_4 \le 8$$
$$3 \le c_i \le 8.$$

$f(x) = f_1(x) \cdot f_2(x) \cdot f_3(x) \cdot f_4(x)$

$= (x^3 + x^4 + x^5 + x^6 + x^7 + x^8)^4$

The solu^n will be the coeff. of x^{24}.

$\Rightarrow (x^3)^4 (1 + x + x^2 + x^3 + x^4 + x^5)^4$

$\Rightarrow x^{12} \times \dfrac{(1-x^6)^4}{(1-x)^4}$

$\Rightarrow x^{12}(1-x)^{-4}$.

$\Rightarrow x^{12}\sum_{r=0}^{\infty}(-1)^r\,{}^{n+r-1}C_r\,x^r$.

$r = 12$.

$\Rightarrow x^{12}\cdot(-1)^{12}\cdot{}^{4+12-1}C_{12}\,x^{12}$.

$\Rightarrow x^{12}\cdot{}^{15}C_{12}\,x^{12}$

$\Rightarrow {}^{15}C_{12}\,x^{24} = {}^{15}C_3\,x^{24}$

$\Rightarrow$ coeff of x^{12} $(1-x^6)^4\cdot(1-x)^{-4}$

$\Rightarrow (1-{}^4C_1x^6+{}^4C_2x^{12}-{}^4C_3x^{18}+{}^4C_4x^{24})\times$

$({}^{-4}C_0 + {}^{-4}C_1(-x) + \dots {}^{-4}C_2(-x)^2 + {}^{-4}C_3(-x)^3$

$+ \dots)$

$\Rightarrow$ coeff of x^{12} —

$${}^{-4}C_{12}(-1)^{12} + [(-{}^4C_1)\times\underbrace{({}^{-4}C_6)}_{}{}^{(-1)^6} + {}^4C_2\times({}^{-4}C_0)]$$

$$(1+x)^{-n} = \sum_{k=0}^{\infty}{}^{-n}C_k(-1)^k x^k = \sum_{k=0}^{\infty}{}^{n+k-1}C_k(-x)^k$$

Ex-9.2

1a. Find the generating funcn for ${}^8C_0, {}^8C_1, {}^8C_2, \dots {}^8C_8$

Solun — $f(x) = {}^8C_0x^0 + {}^8C_1x^1 + {}^8C_2x^2 + \dots {}^8C_8x^8$

$= (1+x)^8$

Date - 18.10.23

1 (b) Find the generating for:
$^8C_1 \cdot 2x\, ^8C_2 \cdot 3x\, ^8C_3 \ldots \ldots 8x\, ^8C_8$

$\Rightarrow f(x) = {}^8C_1 x^0 + 2 \cdot {}^8C_2 x^1 + 3 \cdot {}^8C_3 x^2 + \ldots 8 \cdot {}^8C_8 x^7$

$(1+x)^8 = {}^8C_0 + {}^8C_1 x + {}^8C_2 + \ldots + {}^8C_8 x^8$

Diff. w.r.t. x :-

$8(1+x)^7 = {}^8C_1 + 2 \cdot {}^8C_2 x + 3 \cdot {}^8C_3 x^2 + \ldots + 8 \cdot {}^8C_8 x^7$

Q2. Find the generating func^n for $1, -1, 1, -1, 1, -1, \ldots$

$f(x) = 1 - x + x^2 - x^3 + \ldots$

$= \dfrac{1}{1+x}$

Q3 $0, 0, 0, +6, -6, 6, -6, \ldots$

$f(x) = 6x^3 - 6x^4 + 6x^5 - 6x^6 \ldots$

$= 6x^3(1 - x + x^2 - x^3 \ldots)$

$= \dfrac{6x^3}{1+x}$

Date - 25.10.23

① Find the generating funcn for -

(e) 1, 0, 1, 0, 1, 0, ...

(f) 0, 0, 1, a, a^2, a^3, ...

Sum of ∞ G.P

$$\frac{a}{1-r}$$

Solun - (e) $x^0 + x^2 + x^4 + x^6 + \ldots$

$$f(x) = 1 + x^2 + x^4 + x^6 + \ldots$$

$$= \frac{1}{-x^2+1} = \frac{1}{(-x+1)(x+1)}$$

(f) $f(x) = 1x^2 + ax^3 + a^2x^4 + a^3x^5 + \ldots$

$$= \frac{x^2}{1-ax}$$

② Determine the sequence generated by each of the following generating funcn

(a) $f(x) = (2x-3)^3$

$$(2x-3)^3 = \sum_{k=0}^{3} {}^3C_k (2x)^k (-3)^{3-k}$$

$$= {}^3C_0(-3)^3 + {}^3C_1(2x)(-3)^2 + {}^3C_2(2x)^2(-3)^1 + {}^3C_3(2x)^3$$

$$= -27, 54x, -36x^2, \ldots$$

(b) $f(x) = \frac{x^4}{1-x}$

$= x^4(1 + x + x^2 + \ldots)$

$= x^4 + x^5 + x^6 + \ldots$

$= 0, 0, 0, 0, 0, 1, 1, 1, 1, \ldots$

(c) $f(x) = \dfrac{x^3}{1-x^2}$

$= x^3\left(1 + x^2 + x^4 + x^6 + \cdots\right)$

$= x^3 + x^5 + x^7 + x^9 + \cdots$

$= 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, \cdots$

(d) $f(x) = \dfrac{1}{1+3x}$

$= \dfrac{1}{1-(-3x)}$

$= \left((-3x)^0 + (-3x)^1 + (-3x)^2 + \cdots\right)$

$= \left(1 + (-3x) + (9x^2) + (-27x^3) + \cdots\right)$

$= (1, -3, 3^2, -3^3, 3^4, \cdots)$

$= 1, -3, 9, -27, \cdots$

(e) $f(x) = \dfrac{1}{3-x}$

$= \dfrac{1}{3(1-x/3)}$

$= \dfrac{1}{3}\left\{(x/3)^0 + \left(\dfrac{x}{3}\right)^1 + \left(\dfrac{x}{3}\right)^2 + \left(\dfrac{x}{3}\right)^3 + \cdots\right\}$

$= \dfrac{1}{3}\left\{1, \dfrac{1}{3}, \left(\dfrac{1}{3}\right)^2, \cdots\right\}$

$= 1/3, (1/3)^2, (1/3)^3, (1/3)^4, \cdots$

(f) $f(x) = \frac{1}{1-x} + 3x^7 - 11$

$= 1 + x + x^2 + \cdots + 3x^7 - 11$

$= -10 + x + x^2 + x^3 + x^4 + x^5 + x^6 + 4x^7 + \cdots$

$= -10, 1, 1, 1, 1, 1, 1, 4, 1, 1, 1 \ldots$

(3) In each of the following, the functions $f(x)$ is the generating function for the sequence $a_0, a_1, a_2, \ldots$ where the sequence $b_0, b_1, b_2, \ldots$ is generated by the function $g(x)$. Express $g(x)$ in terms of $f(x)$

(a) $b^3 = 3$

$b_n = a_n$

$n \in N,\ n \neq 3$

$f(x) = a_0 + a_1x + a_2x^2 + \cdots$

$g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots$

$= b_0 + b_1x + b_2x^2 + 3x^3 + \cdots$

$= a_0 + a_1x + a_2x^2 + 3x^3 + a_4x^4 + \cdots$

$= f(x) - a_3x^3 + 3x^3$

$= f(x) + x^3(3 - a_3)$

(b) $b_3 = 3$

$b_7 = 7$

$b_n = a_n,\ n \in N$

$f(x) = a_0 + a_1x + a_2x^2 + \cdots$

$g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots$

$= a_0 + a_1x + a_2x^2 + 3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + 7x^7 + \cdots$

$$g(x) = f(x) - ax^3 - ax^7 + 3x^3 + 7x^7$$
$$= f(x) + x^3(3-a) + x^7(7-a)$$

(c) $b_1 = 1$, $b_3 = 3$
$b_n = 2a_n$

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$
$$g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \dots$$
$$= 2a_0 + x + 2a_2x^2 + 3x^3 + 2a_4x^4 + \dots$$
$$= 2(f(x) - a_1x - a_3x^3) + x + 3x^3$$
$$= 2f(x) + x(1-2a_1) + x^3(3-2a_3)$$

(d) $b_1 = 1$, $b_3 = 3$, $b_7 = 7$, $b_n = 2a_n + 5$

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$
$$g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \dots$$
$$= 2a_0 + 5 + x + (2a_2+5)x^2 + 3x^3 + (2a_4+5)x^4 + (2a_5+5)x^5 + (2a_6+5)x^6 + (7x^7) + \dots$$

$$= 2a_0 + x + 2a_2x^2 + 3x^3 + 2a_4x^4 + 2a_5x^5 + 2a_6x^6 + 7x^7 + 5 + 5x^2 + 5x^3 + \dots$$

$$= 2\{f(x) - a_1x - a_3x^3 - a_7x^7\} + x + 3x^3 + 7x^7 + \frac{5}{1-x} - 5(x + x^3 + x^7)$$

$$= 2f(x) + x(1-2a_1) + x^3(3-2a_3) + x^7(7-2a_7) + \frac{5}{1-x} - 5(x + x^3 + x^7)$$

Date - 28.10.23

Exercise - 9.2

Q4 Determine the constant in $\left(3x^2 - \frac{2}{x}\right)^{15}$

Solu^n - $\left(3x^2 - \frac{2}{x}\right)^{15} = \frac{1}{x^{15}}\left(3x^3 - 2\right)^{15}$

$$\boxed{(x+y)^n = \sum_{k=0}^{n} {}^nC_k\, x^k\, y^{n-k}}$$

$\boxed{k=5} \rightarrow$

$$= \frac{1}{x^{15}} \times {}^{15}C_k\, (3x^3)^k\, (-2)^{15-k}$$

The constant term is ${}^{15}C_5 \cdot (3)^5 (-2)^{10}$

Q5 (a) Find the coefficient of x^7 in $(1+x+x^2+x^3+\ldots)^{15}$

Solu^n - $\left(\frac{1}{1-x}\right)^{15} = (1-x)^{-15} = (1+(-x))^{-15}$

$$\sum_{k=0}^{\infty} (-1)^k\, {}^{15+k-1}C_k (-x)^k$$

$$k=7$$

$$\Rightarrow\ -\ {}^{15+7-1}C_7 (-x)^7 = +\ {}^{21}C_7$$

Chapter - 10

Recurrence Relation

- First - order linear recurrence relation :-

$5, 15, 45, 135, \ldots$

$a_{n+1} = 3a_n$ $\quad a_0 = 5$

first order linear R.R → $a_{n+1} = 3a_n, \; n \geq 0$ → homogenous.

- When the successor depends on its immediate predecessor then the relation is said to be of first order.

→ if $a_{n+1} = 3a_n + \text{something}$ → inhomogenous

eg- $a_{n+1} = 3a_n, \; n \geq 0 \quad a_0 = 5$

$a_0 = 5$

$a_1 = 3a_0 = 15 = 3 \cdot 5$

$a_2 = 3a_1 = 45 = 3^2 \cdot 5$

$a_3 = 3a_2 = 3^3 \cdot 5$

$\vdots$

$a_n = 3^n \cdot 5$

- The unique solution of the recurrence relations $a_{n+1} = da_n$, where $n \geq 0$, d is a constant and $a_0 = A$ is given by $a_n = Ad^n, \; n \geq 0$

10) Solve the recurrence relation $a_{n+1} = 7a_n, \; n \geq 0$ and $a_2 = 98$

Solu^n - $a_2 = 7a_1$

$98 = 7a_1$

$a_1 = 14$

$a_1 = 7a_0 \quad = \quad 14 = 7a_0$

$a_0 = 2.$

Also, $a_2 = 7^2 a_0$

$98 = 49a_0$

$a_0 = 2.$

$a_1 = 7a_0$

$a_2 = 7^2 a_0$

$a_3 = 7^3 a_0$

$\vdots$

$a_n = 7^n \cdot 2$

Example - 10.2

If A bank pays 6% (annual) interest on savings, compounding the interest monthly. If Bonnie deposits ₹1000 on the first day of May, how much will this deposit be worth a year later.

Solu^n - Annual interest rate is 6%. So monthly interest rate is $(6\%)/12 = 0.5\% = 0.005$. For $0 \leq n \leq 12$, let P_n denotes the value of Bonnie's deposit at the end of a month.

$P_0 = 1000$

$P_{n+1} = P_n + 0.005 P_n$

$P_{n+1} = 1.005 P_n$

So, $P_n = (1000)(1.005)^n$

So, $P_{12} = (1000)(1.005)^{12} = 1000 \times 1.06 = ₹1060$

Example 10.4 -

Q Find a_{12} if $a_{n+1}^2 = 5a_n^2$ where $a_n > 0$ for $n \geq 0$, $a_0 = 2$.

Solun - Let us consider $b_n = a_n^2$, $b_{n+1} = 5b_n$

$b_0 = 4$

$b_n = 4 \cdot 5^n$

$a_n = 2 \cdot (\sqrt{5})^n$

$a_{12} = 2 \cdot (\sqrt{5})^{12}$