1. Mathematical Modeling of Engineering Problems

A mathematical model can be broadly defined as a formulation or equation that expresses the essential features of a physical system or process in mathematical terms

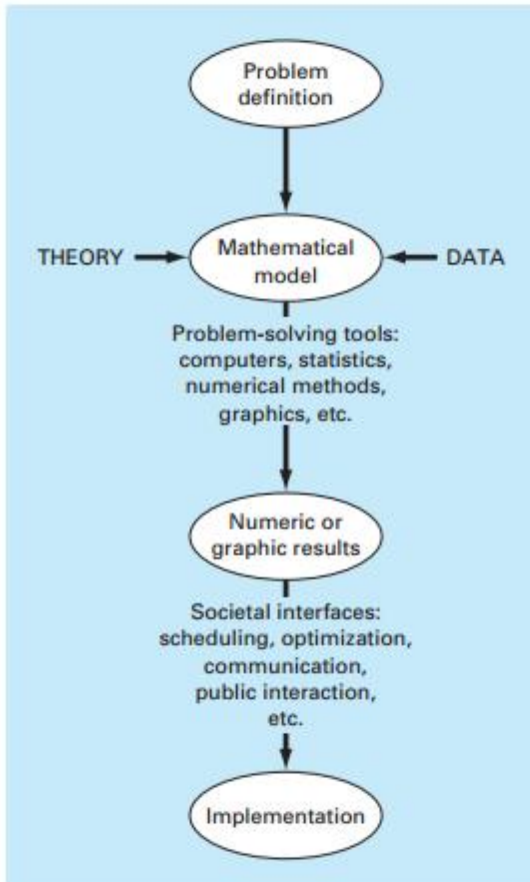


FIGURE 1.1
The engineering problem-solving process.

$$dependent\ variable = f(independent\ variables;\ parameters, forcing\ functions)$$

Where:

- where the dependent variable is a characteristic that usually reflects the behavior or state of the system
- the independent variables are usually dimensions, such as time and space
- the parameters are reflective of the system's properties or composition
- forcing functions are external influences acting upon the system.

The mathematical expression, or model, of the second law is the well-known equation

$$F = ma \tag{1.1}$$

Where:

- F is the net force acting on the body
- m is the mass of the object
- a is the acceleration

Eq 1.1 can be written as

$$a = \frac{F}{m}$$

Since we know

$$a = \frac{dv}{dt}$$

We can write eq 1.1 as

$$\frac{dv}{dt} = \frac{F}{m} \tag{1.2}$$

Next, we will express the net force in terms of measurable variables and parameters. For a body falling within the vicinity of the earth, the net force is composed of two opposing forces: the downward pull of gravity F_D and the upward force of air resistance F_U

$$F = F_D + F_U$$

If the downward force is assigned a positive sign, the second law can be used to formulate the force due to gravity, as

$$F_D = mg$$

Air resistance can be formulated in a variety of ways. A simple approach is to assume that it is linearly proportional to velocity and acts in an upward direction, as in

$$F_U = -cv$$

where c is a proportionality constant called the drag coefficient.

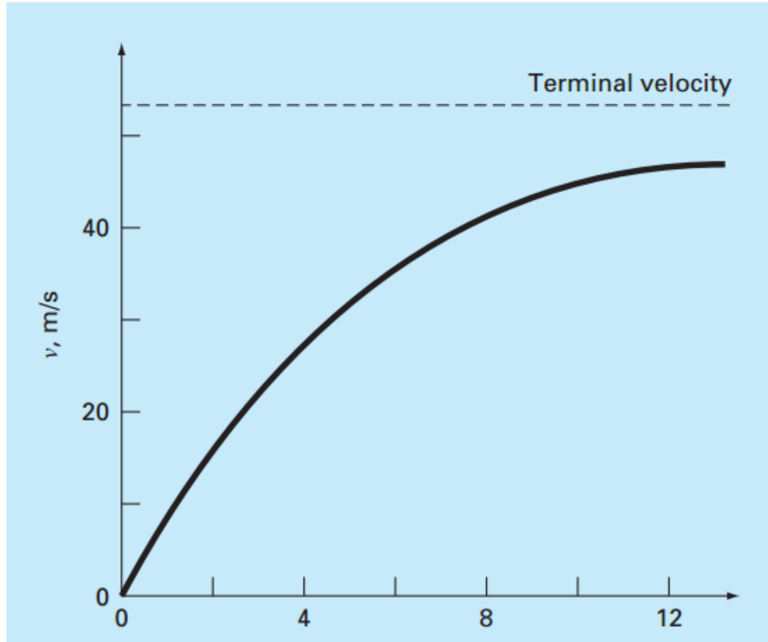
Thus, we can write eq 1.2 as

$$\frac{dv}{dt} = g - \frac{c}{m}v \tag{1.3}$$

Equation (1.3) is a model that relates the acceleration of a falling object to the forces acting on it. It is a differential equation because it is written in terms of the differential rate of change (dv/dt) of the variable that we are interested in predicting. If the parachutist is initially at rest, i.e., $v = 0$ at $t = 0$, then eq 1.3 can be solved for v as a function of time as

$$v(t) = \frac{gm}{c}\left(1 - \exp\left(\frac{c}{m}t\right)\right) \tag{1.4}$$

FIGURE 1.3
The analytical solution to the falling parachutist problem as computed in Example 1.1. Velocity increases with time and asymptotically approaches a terminal velocity.



Equation 1.10 is called the analytical or exact solution. Unfortunately, there are many mathematical models that cannot be solved exactly. In many of these cases, the only alternative is to develop a numerical solution that approximates the exact solution.

For newton's second law, by realizing that the time rate of change of velocity can be approximated by

$$\frac{dv}{dt} \approx \frac{\Delta v}{\Delta t} = \frac{v(t_{i+1}) - v(t)}{t_{i+1} - t_i} \tag{1.5}$$

Substituting eq 1.5 into eq 1.4 we get

$$\frac{v(t_{i+1}) - v(t)}{t_{i+1} - t_i} = g - \frac{c}{m} v(t_i) \tag{1.6}$$

The equation can then be rearranged to yield,

$$v(t_{i+1}) = v(t_i) + \left[g - \frac{c}{m} v(t_i)\right] (t_{i+1} - t_i) \tag{1.7}$$

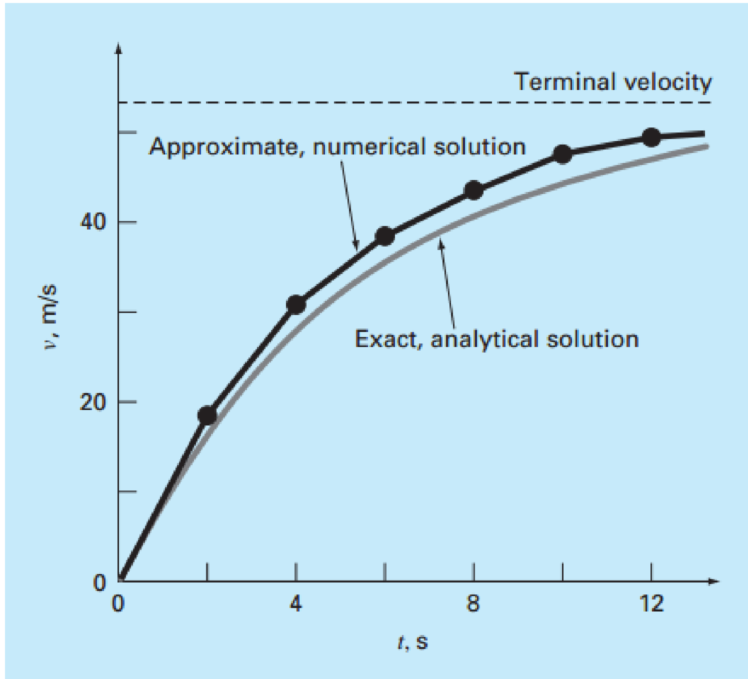


FIGURE 1.5
Comparison of the numerical and analytical solutions for the falling parachutist problem.

2. Approximations and Truncation Errors

Since numerical methods are all about approximating an exact solution by means iterative approaches as opposed to analytical approach, it is very important of analyze the error that it incurs. In this section we shall study such error analysis techniques.

2.1. Error Definitions

Numerical errors arise from the use of approximations to represent exact mathematical operations and quantities. These include *truncation errors*, which result when approximations are used to represent exact mathematical procedures, and *round-off errors*, which result when numbers having limited significant figures are used to represent exact numbers.

$$True\ value = Approximation + Error$$

$$E_t = true\ value - approximation$$

A shortcoming of this definition is that it takes no account of the order of magnitude of the value under examination. For example, an error of a centimeter is much more significant if we are measuring a rivet than a bridge. One way to get around is

$$True\ percent\ relative\ error = \frac{|true\ value - approximation|}{|true\ value|} 100\%$$

However, since eq 2.2 relies on the true value, which is not often known, for many iterative numerical algorithms we use the absolute percent relative error, which is defined as

$$\varepsilon_a = \frac{|current\ approximation - previous\ approximation|}{|current\ approximation|} 100\%$$

And we continue our iteration until

$$\varepsilon_a \geq \varepsilon_s$$

Where ε_s is called the *tolerance level*. Sometimes we also keep a counter of how many iterations we have performed thus far and we only allow a certain number of iterations due to limited computer CPU time. For example, we can continue an iterative algorithm until

$$iterations \leq maximum\ iterations$$

3. Truncation Errors and the Taylor Series

Truncation errors are those that result from using an approximation in place of an exact mathematical procedure. This often arises when use only a finite number of terms to approximate something which is analytically formulated in terms on infinite number of terms.

3.1. The Taylor Series

The Taylor's series is of great value to the analysis of numerical algorithms. In essence, the Taylor series provides a means to predict a function value at one point in terms of the function value and its derivatives at another point. In particular, the theorem states that any smooth function can be approximated as a polynomial. A useful way to gain insight into the Taylor series is to build it term by term. For example, the first term in the series is

$$f(x_{i+1}) \approx f(x_i)$$

This relationship, called the *zero-order approximation*. This is true if the function does not change much. However, we need more sophisticated techniques for many functions with dynamical behaviors. The first order approximation is

$$f(x_{i+1}) \approx f(x_i) + f'(x_i)(x_{i+1} - x_i)$$

In general, the Taylor series is written as

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \frac{f^{(3)}(x_i)}{3!}(x_{i+1} - x_i)^3 + \cdots + \frac{f^{(n)}}{n!}(x_{i+1} - x_i)^n + R_n$$

A remainder term R_n is included to account for all terms from $n + 1$ to ∞:

$$R_n = \frac{\bar{f}^{(n+1)}}{(n+1)!}(x_{i+1} - x_i)^{n+1}$$

where the subscript n connotes that this is the remainder for the nth-order approximation and the term $\bar{f}^{(n+1)}$ is defined as:

$$\bar{f}^{(n+1)} = \frac{1}{x_{i+1} - x_i}\int_{x_i}^{x_{i+1}} f^{(n+1)}(z)\, dz$$

It is often convenient to define

$$h = x_{i+1} - x_i$$

so our Taylor series becomes

$$f(x_{i+1}) = \sum_{k=0}^{n} \frac{f^{(k)}(x_i)}{k!} h^k + R_n$$

$$R_n = \frac{\bar{f}^{(n+1)}}{(n+1)!} h^{n+1}$$

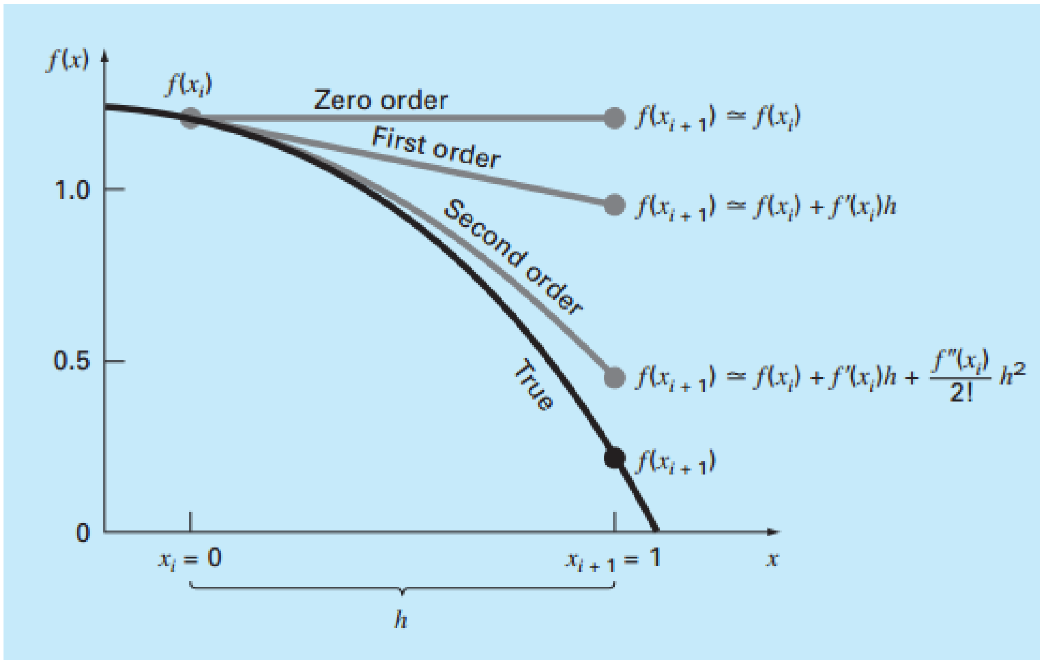


FIGURE 4.1
The approximation of $f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$ at $x = 1$ by zero-order, first-order, and second-order Taylor series expansions.

3.1.1. The Remainder of the Taylor Series Expansion

Before demonstrating how the Taylor series is actually used to estimate numerical errors, we must explain why we included the argument ξ. Suppose that we truncated the Taylor series expansion after the zero-order term to yield

$$f(x_{i+1}) \approx f(x_i)$$

3.1.2. Using the Taylor Series to Estimate Truncation Error

Recall from the falling parachutist problem, we were interested in estimating $v(t)$ which can be expanded as

$$v(t_{i+1}) = v(t_i) + v'(t_i)h + \frac{v''(t_i)}{2} h^2 + \cdots + R_n$$

Now let us truncate the series after the first derivative term

$$v(t_{i+1}) = v(t_i) + v'(t_i)h + R_1$$

Equation 3.8 can now be solved for $v'(t_i)$ as

$$v'(t_i) = \frac{v(t_{i+1}) - v(t_i)}{h} - \frac{R_1}{h}$$

In equation 3.9, the second term is called the *truncation error*

$$\frac{R_1}{h} = \frac{\bar{v}''}{2} h$$

Or

$$\frac{R_1}{h} = \mathcal{O}(h)$$

Which states that, the estimate of the derivative has a truncation error of order h.

In other words, the error of our derivative approximation should be proportional to the step size. ***Consequently, if we halve the step size, we would expect to halve the error of the derivative***.

3.2. Error Propagation

The purpose of this section is to study how errors in numbers can propagate through mathematical functions. For example, if we multiply two numbers that have errors, we would like to estimate the error in the product.

3.2.1. Stability and Condition

The condition of a mathematical problem relates to its sensitivity to changes in its input values. We say that a computation is numerically unstable if the uncertainty of the input values is grossly magnified by the numerical method.

Given a small change Δx in x, the relative change in x is $\frac{x+\Delta x - x}{x} = \frac{\Delta x}{x}$ while the relative change in $f(x)$ is $\frac{f(x+\Delta x)-f(x)}{f(x)}$. Taking the ratio yields

$$\frac{x}{f(x)} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Which is defined as the condition number of some numerical algorithm or function $f(x)$.

The condition number provides a measure of the extent to which an uncertainty in x is magnified by $f(x)$. Functions with large condition number is said to be ill conditioned.

- A value of 1 tells us that the function's relative error is identical to the relative error in x.
- A value greater than 1 tells us that the relative error is amplified.
- A value less than 1 tells us that it is attenuated

Readings for this chapter:

- Book: Numerical Methods for Engineers
- Chapter: 1
- Sections: 3.1-3.3 and 4.1-4.2