Numerical Methods

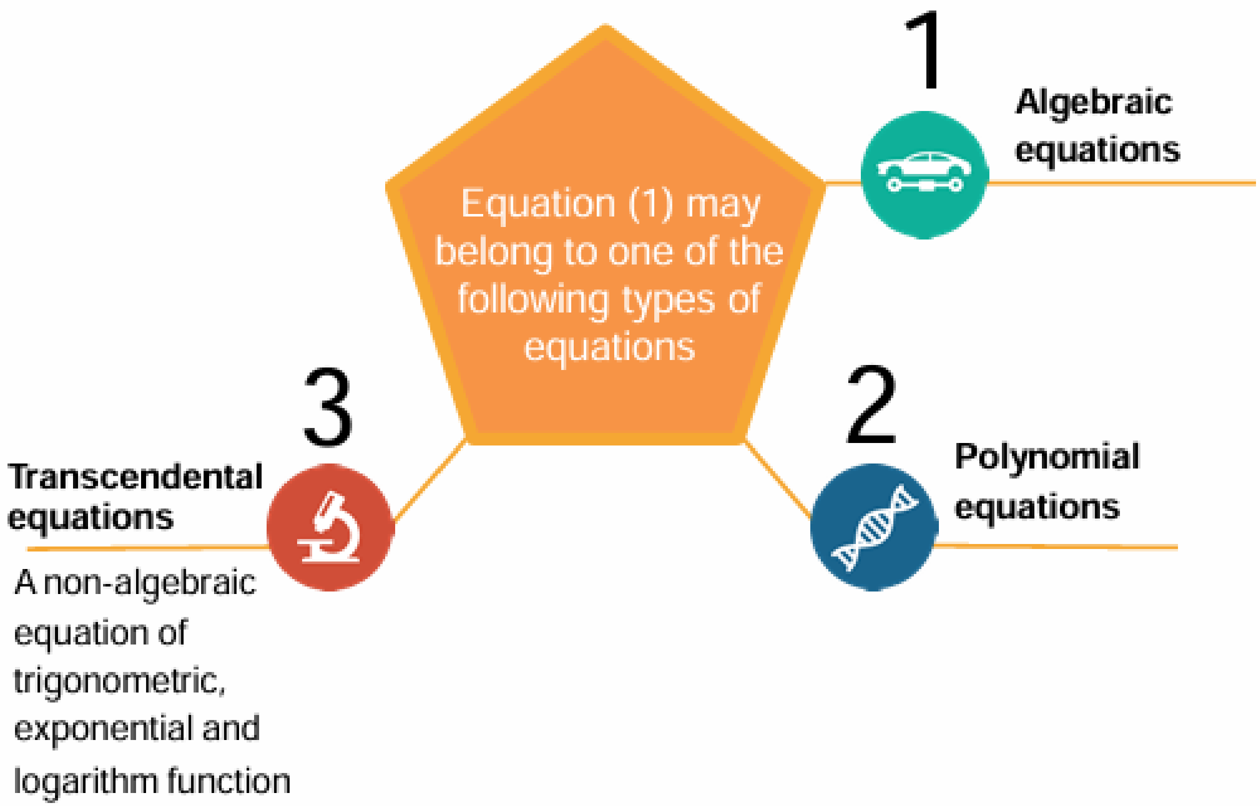
Adrita Alam
Lecturer,Dept of CSE
Varendra University, Rajshahi
Email: adritaalam.prima@gmail.com

Outline

- Introduction
- Approximations and Errors
- Roots of Equations
- Systems of linear algebraic equations
- Curve Fitting
- Numerical Differentiation and Integration

Roots of Equations

INTRODUCTION (Cont.)



INTRODUCTION (Cont.)

Example 1: Algebraic Equation

$$4x - 3x^2y - 15 = 0$$

Example 2: Polynomial Equation

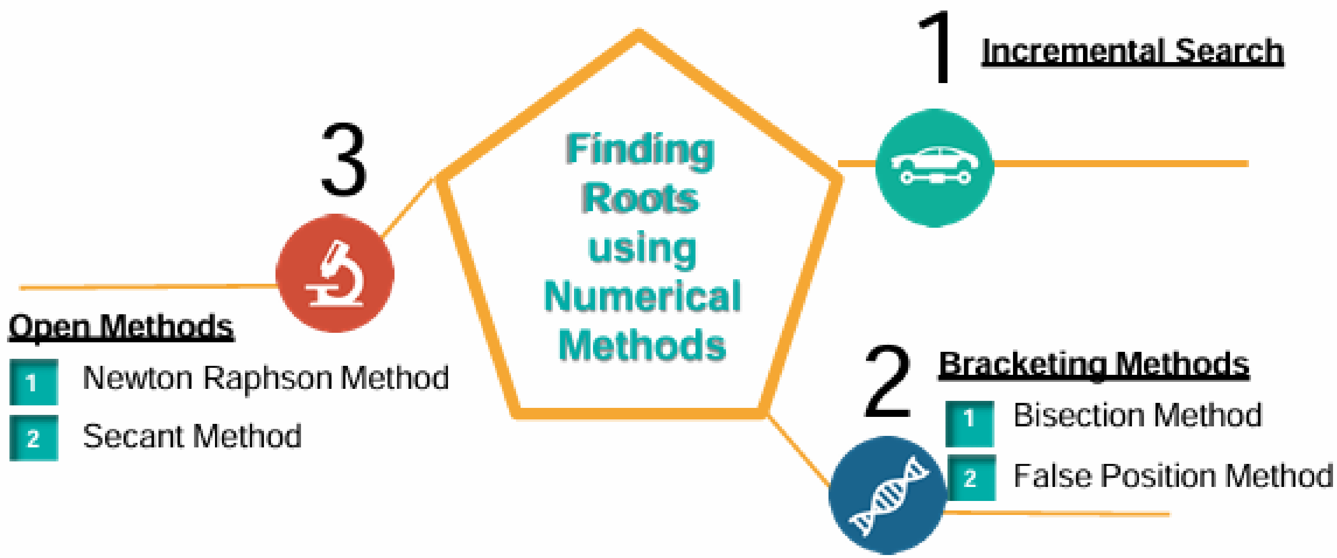
$$x^2 + 2x - 4 = 0$$

Example 3: Transcendental Equation

$$\sin(2x) - 3x = 0$$

INTRODUCTION (Cont.)

Three types of Numerical Methods shall be considered to find the roots of the equations:



Prior to the numerical methods, a **graphical method** of finding roots of the equations are presented.

Bracketing Methods

- Figure 1 illustrates the basic idea of bracketing method– that is guessing an interval containing the root(s) of a function.
- Starting point of the interval is a lower bound, xl. End point of the interval is an upper bound, xu.
- By using bracketing methods, the interval will split into two subintervals and the size of the interval is successively reduced to a smaller interval.
- The subintervals will reduce the range of intervals until its distance is less than the desired accuracy of the solution

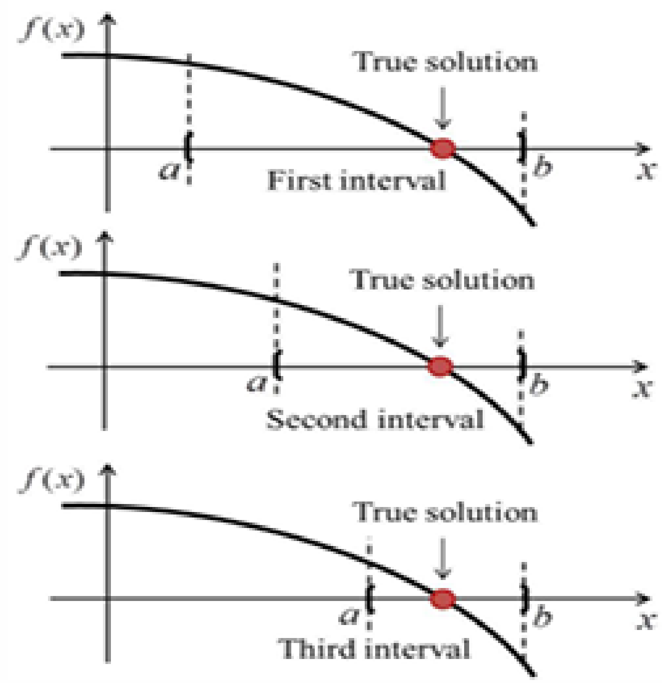


Figure 4: Graphical Illustration of Bracketing Method

Bracketing Methods

- Bracketing methods always converge to the true solution.
- There are two types bracketing methods; bisection method and false position method.

BISECTION METHOD

- Bisection method is the simplest bracketing method.
- The lower value, x_l and the upper value, x_u which bracket the root(s) are required.
- The procedure starts by finding the interval $[x_l, x_u]$ where the solution exist.
- As shown in **Figure 5**, at least one root exist in the interval $[x_l, x_u]$ if $f(x_l) \cdot f(x_u) < 0$

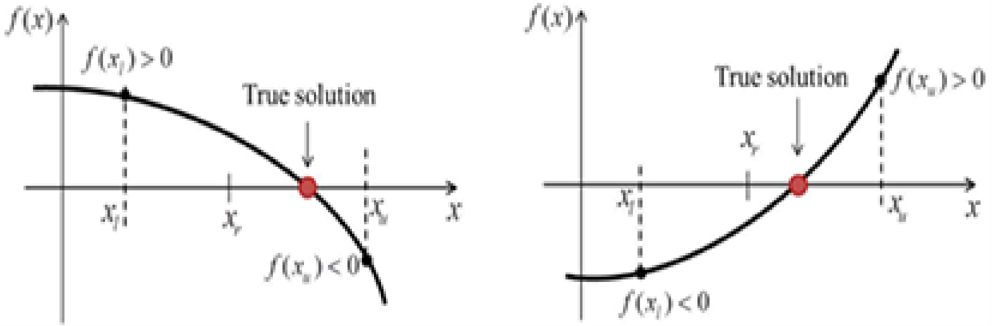


Figure 5: Solution of $f(x) = 0$

BISECTION METHOD (Cont.)

Algorithm

For the continuous equation of one variable, $f(x) = 0$,

Step 1: Choose the lower guess, x_l and the upper guess, x_u that bracket the root such that the function has opposite sign over the interval, $x_l \leq x \leq x_u$.

Step 2: The estimation root, x_r is computed by using

$$x_r = \frac{x_l + x_u}{2}$$

Step 3: Use the following evaluations to identify the subinterval that the root lies

- ✓ If $f(x_l) \cdot f(x_r) < 0$, then the root lies in the lower subinterval. Therefore, set $x_u = x_r$ and repeat **Step 2.**
- ✓ If $f(x_l) \cdot f(x_r) > 0$, then the root lies in the upper subinterval. Therefore set $x_l = x_r$ and repeat **Step 2.**
- ✓ If $f(x_l) \cdot f(x_r) = 0$, then the root is equal to x_r. Terminate the computation.

Step 4: Calculate the approximate percent relative error,

$$\varepsilon_a = \left| \frac{x_r^{present} - x_r^{previous}}{x_r^{present}} \right| \times 100\%$$

Step 5: Compare with. If $\varepsilon_a < \varepsilon_s$, then stop the computation. Otherwise go to **Step 2** and repeat the process by using the new interval.

Bisection Method: Example 1

Find the root of the equation $x^3 + 4x^2 - 1 = 0$.

Solution

Let, $a = 0$ and $b = 1$.

Now, $f(0) = (0)^3 + 4(0)^2 - 1 = -1 < 0$ and
$f(1) = (1)^3 + 4(1)^2 - 1 = 4 > 0$.
i.e., $f(a)$ and $f(b)$ has opposite signs.

Therefore, $f(x)$ has a root in the interval $[a, b] = [0, 1]$

$x_c = (0 + 1)/2 = 0.5$,
$f(0.5) = 0.125$. Now $f(a)$ and $f(x_c)$ has opposite signs

So, the next interval is $[0, 0.5]$

Bisection Method: Example 1

Find the root of the equation $x^3 + 4x^2 - 1 = 0$.

Solution

a	b	$x_c = (a+b)/2$	$f(a)$	$f(b)$	$f(x_c)$
0	1	0.5	-1	4	0.125
0	0.5	0.25	-1	0.125	- 0.73438
0.25	0.5	0.375	-0.73438	0.125	- 0.38477
0.375	0.5	0.4375	-0.38477	0.125	- 0.15063
0.4375	0.5	0.46875	-0.15063	0.125	- 0.0181
0.46875	0.5	0.484375	-0.0181	0.125	0.05212
0.46875	0.484375	0.476563	- 0.0181	0.05212	0.01668

... and so we approach the root 0.472834.

Can you use Bisection method to find a zero of :

$f(x) = x^3 - 3x + 1$ in the interval $[0,2]$?

Answer:

$f(x)$ is continuous on $[0,2]$

and $f(0) * f(2) = (1)(3) = 3 > 0$

$\Rightarrow$ Assumptions are not satisfied

$\Rightarrow$ Bisection method can not be used

Advantages:

- Simple and easy to implement
- One function evaluation per iteration
- The size of the interval containing the zero is reduced by 50% after each iteration
- The number of iterations can be determined a priori
- No knowledge of the derivative is needed
- The function does not have to be differentiable

Disadvantages:

- Slow to converge
- Good intermediate approximations may be discarded
- We need two initial guesses a and b which bracket the root.
- It is among the slowest methods to find the root.
- When an interval contains more than one root, the bisection method can find only one of them.

Bisection Methods: Class work

- Find the real root of the equation f(x)=x3 – x – 1= 0 correct to 2 decimal places.(ε=0.01).

Answer: 1.328125

- Find the positive root, between 0 and 1, of the equation x = e^{-x} to a tolerance of 0.05%.

  Hints:
  f(x) = xe^{-x} - 1
  initial guesses: 0,1
  Answer: 0.567 (up to three decimal places)

False Position Method or Regular Falsi Method

- It is an improvement of the Bisection method.
- The bisection method converges slowly due to its behavior in redefined the size of interval that containing the root.
- The procedure begins by finding an initial interval $[x_l, x_u]$ that bracket the root.
- $f(x_l)$ and $f(x_u)$ are then connected using a straight line.
- The estimated root, x_r is the x-value where the straight line crosses x-axis.
- **Figure 6** indicates the graphical illustration of False Position method.

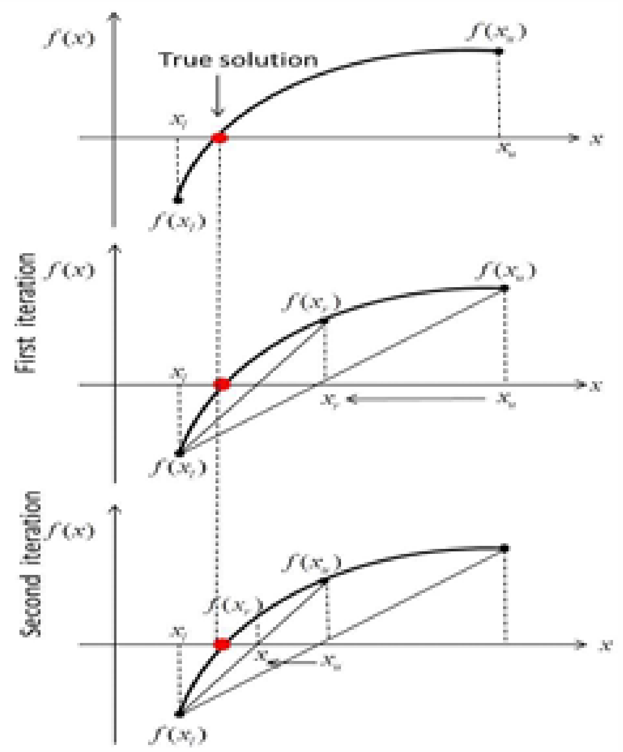


Figure 6: Graphical Illustration of False Position Method

False Position Method or Regular Falsi Method

The equation of the chord joining the two points [a, f(a)] and [b, f(b)] if given by –

$$y - f(a) = \frac{f(b)-f(a)}{(b-a)} . (x - a)$$

Let y = 0 be the point of intersection of the chord equation (given above) with the x-axis.

$$x = \frac{af(b)-bf(a)}{f(b)-f(a)}$$

False Position Method or Regular Falsi Method

For the continuous equation of one variable, $f(x) = 0$,

Step 1: Choose the lower guess, x_l and the upper guess, x_u that bracket the root such that the function has opposite sign over the interval, $x_l \leq x \leq x_u$.

Step 2: The estimation root, x_r is computed by using

$$x_r = x_u - \left[\frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)} \right]$$

Step 3: Use the following evaluations to identify the subinterval that the root lies

- ✓ If $f(x_l) \cdot f(x_r) < 0$, then the root lies in the lower subinterval. Therefore, set $x_u = x_r$ and repeat **Step 2.**
- ✓ If $f(x_l) \cdot f(x_r) > 0$, then the root lies in the upper subinterval. Therefore set $x_l = x_r$ and repeat **Step 2.**
- ✓ If $f(x_l) \cdot f(x_r) = 0$, then the root is equal to x_r. Terminate the computation.

Step 4: Calculate the approximate percent relative error,

$$\varepsilon_a = \left| \frac{x_r^{present} - x_r^{previous}}{x_r^{present}} \right| \times 100\%$$

Step 5: Compare with. If $\varepsilon_a < \varepsilon_s$, then stop the computation. Otherwise go to **Step 2** and repeat the process by using the new interval.

The Method of False Position: Example

Find the real root of the equation till 2 decimal place

$$f(x) = x^3 - 2x - 5 = 0$$

We observe that $f(2) = -1$ and $f(3) = 16$

And hence a root lies between 2 and 3. Then

x0	x1	x2	f(x0)	f(x1)	f(x2)
2	3	2.058824	-1	16	-0.3908
2.058824	3	2.081264	-0.3908	16	-0.1472
2.081264	3	2.089639	-0.1472	16	-0.05468
2.089639	3	2.09274	-0.05468	16	-0.0202
2.09274	3	2.093884	-0.0202	16	-0.00745

x1	2.059
x2	2.081
x3	2.090
x4	2.093

x_4 is correct to 2 decimal places.

False Position Method or Regular Falsi Method

Advantages:

- Simple and easy to implement
- Brackets the root

Disadvantages:

- Can be very slow to converge
- Like Bisection, need an initial interval around the root