Art. 63. Definition :

A circle is the locus of a point in a plane which moves so that its distance from a given point is always equal to a given distance. The given point is called the centre and the given distance is the radius of the circle.

Art. 63. (a) The equation of a circle with centre at (h, k) and radius 'a' is

$$\boxed{(x-h)^2 + (y-k)^2 = a^2} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Cor. *If the centre of the circle is the origin, the equation becomes* $x^2 + y^2 = a^2$ (2)

Art. 64. Equation of the circle in Polar Co-ordinates.

Let the centre C of the circle be at (r_1, θ_1) and a be the radius of the circle. Let P be any point (r, θ) on the circle. Then the angle $\angle COP = \theta - \theta_1$ and from the Δ OCP

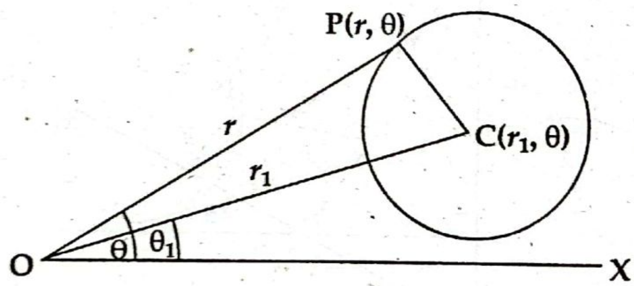

Fig : 27

$CP^2 = OC^2 + OP^2 - 2OC.\ OP \cos COP$

or, $a^2 = r_1^2 + r^2 - 2rr_1 \cos(\theta - \theta_1)$. The equation of the circle in terms of polar co-ordinates is

$$r^2 + r_1^2 - 2r_1 r \cos(\theta - \theta_1) = a^2 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where (r_1, θ_1) is the contric of the circle.

Cor. 1. If the pole lies on the circle, then $r_1 = a$. The equation of the circle now is

$$r = 2a \cos(\theta_1 - \theta) \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Cor. 2. If $\theta_1 =$ zero, we see that initial line is a diameter and the pole is an extremity of the diameter. The equation of the circle of radius a is $r = 2a \cos\theta$ ... (3)

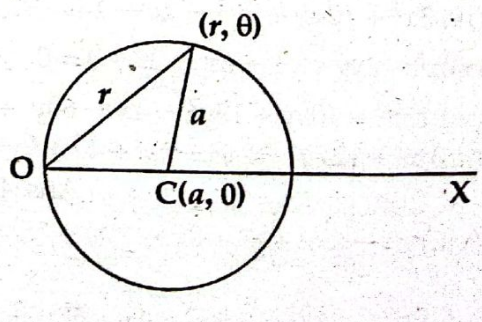

Fig : 28

Cor. 3. If the centre is the origin (i.e. the pole), the equation of the circle becomes

$$r = a \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Art. 65. (a) General equation to a circle is

$x^2 + y^2 + 2gx + 2fy + c = 0$

or, $(x+g)^2 + (y+f)^2 = g^2 + f^2 - c$ ... ... (1)

The centre of the circle is at $(-g, -f)$ ... ... (2)

The radius of the circle is positive

$a^2 = g^2 + f^2 - c$ or, $a = \sqrt{(g^2 + f^2 - c)}$ ... ... (3)

Note : 1. If $g^2 + f^2 - c = 0$, the rdius of the circle is zero and the equation (1) represents a real circle.

Note. 2. If $g^2 + f^2 - c = 0$, the radius of the circle is zero and the equation (1) represents a circle of zero radius. This is known as **point circle.**

Note. 3. If $g^2 + f^2 - c < 0$. i.e. $g^2 + f^2 - c$ is negative. The radius $\sqrt{(g^2 + f^2 - c)}$ becomes imaginary. The equation ... (1) represents an imaginary circle.

(b) General equation of a circle in Polar Co -ordinates.

The equation, (1) Art. 64 is

$r^2 + r_1^2 - 2rr_1 \cos(\theta - \theta_1) = a^2$

or, $r^2 - 2rr_1 \cos\theta_1 \cos\theta - 2r_1 r \sin\theta_1 \sin\theta = a^2 - r_1^2$

or, $r^2 + 2r(g\cos\theta + f\sin\theta) + c = 0$ ... ... (1)

which is the general equation of a circle

where $g = -r_1 \cos\theta$

$f = r_1 \sin\theta$

$c = r_1^2 - a^2$ ... ... (2)

The co-ordinates of centre $= (\sqrt{(g^2 + f^2)}, \tan^{-1}(f/g))$ ... ... (3)

The radius of the circle $= \sqrt{(g^2 + f^2 - c)}$ ... ... (4)

Art. 66. The equation of the circle whose diameter is the line joining the points (x_1, y_1) and (x_2, y_2) is ...

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

Art. 67. Position of a points with respect to a circle.

Any point (x_1, y_1) will be outside, on or inside the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ ... ... (1)

according as its distance from the centre $(-g, -f)$ is greater, equal or less than the radius $\sqrt{(g^2 + f^2 - c)}$ of the circle (1) i.e. according as $(x_1 + g)^2 + (y_1 + f)^2 > = < g^2 + f^2 - c$

or, $x_1^2 + y_1^2 + 2gx_1 + 2f_1 + c > = < 0$.

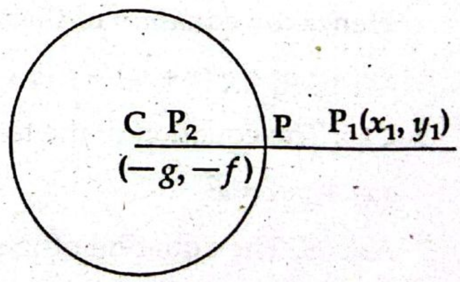

Fig : 29

Rule. *Substitue the given point in the equation of the circle and the point will be outside, on or inside the circle according as the results is + ve, zero or — ve*

Art. 68. The equation of the tangent to the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ (See Art. 52 *a*) VVI

(Alternative Method).

Let P (x_1, y_1) and Q (x_2, y_2) be the two points on the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

The equation of the line PQ is

$$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2} \text{ or, } \frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Since the points (x_1, y_1) and (x_2, y_2) are on the circle (1) they will satisfy Eq. (1).

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

subtracting (4) from (3), we have

$$x_1^2 - x_2^2 + y_1^2 - y_2^2 + 2g(x_1 - x_2) + 2f(y_1 - y_2) = 0$$

or, $\dfrac{y_1 - y_2}{x_1 - x_2} = -\dfrac{x_1 + x_2 + 2g}{y_1 + y_2 + 2f}$

or, $\dfrac{y - y_1}{x - x_1} = -\dfrac{x_1 + x_2 + 2g}{y_1 + y_2 + 2f}$ by (2) $\quad \ldots \quad \ldots \quad \ldots \quad (5)$

When Q coincides with P, the line PQ becomes the tangent at P of the circle (1). Then in limit $x_1 = x_2$ and $y_1 = y_2$ in eq. (5) we have,

or, $\dfrac{y - y_1}{x - x_1} = -\dfrac{2x_1 + 2g}{2y_1 + 2f} = -\dfrac{x_1 + g}{y_1 + f}$

or, $xx_1 + yy_1 + xg + yf = x_1^2 + y_1^2 + x_1g + y_1f \quad \ldots \quad \ldots \quad (6)$

Adding $x_1g + y_1f$ to both sides of Eq. (6) we have

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) = x_1^2 + y_1^2 + 2gx_1 + 2fy_1$

or, $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) = -c$ ……… by (3)

or, $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

Hence the equation of the tangent to the circle (1) is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

Cor. The equation of the tangent at (x, y) to the circle $x^2 + y^2 = a^2$ is

$$xx_1 + yy_1 = a^2$$

Art. 69. The equation of the normal (x_1, y_1) to the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ is

$y(x_1 + g) - x(y_1 + f) + fx_1 - gy_1 = 0$ (See Art. 53 (a))

Cor. The equation of the nomal

at (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $xy_1 - x_1y = 0$

Note : *In the case of a circle the equation of normal at any point on it can be easily found since it is a straight line joining the point and the centre of circle.*

Art. 70. Length and mid point of the chord intercepted by the circle $x^2 + y^2 = a^2$ on the line $y = mx + c$.

The line joining centre O to the middle point C of a chord of a circle is perpendicular to the chord whose equation is

$y = mx + c$ ... ... (1)

Hence the equation of OC is $y - 0 = -1/m\, x$ or, $my + x = 0$ ... ... (2)

Solve (1) and (2), the co-ordinates of the middle point C of the chord AB is $-mc/(1 + m^2)$, $c/(1 + m^2)$ ... ... (3)

Again OC = perpendicular distance of C (0,0) from AB

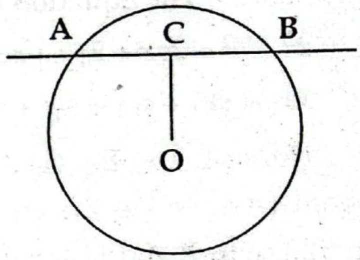

Fig : 30

$OC = \dfrac{c}{\sqrt{(1+m^2)}}$ and $CA^2 = a^2 - c^2/(1 + m^2)$

Length of chord AB = 2AC.

$$= 2\sqrt{\left\{\frac{a^2(1+m^2) - c^2}{(1+m^2)}\right\}}$$

Art. 70. (a) Length of the tangent T from a external point (x_1, y_1) to the circle. $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$T^2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

Cor. The length of the tangent T from an external point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is given by $T^2 = x_1^2 + y_1^2 - a^2$

Note **:** The equation of the circle is to be arranged in such a way that right hand side is zero and the co-efficients of x^2 and y^2 must be unity.

Condition of Tangency

Art. 71. Show that the straight lines $y = mx \pm a\sqrt{(1 + m^2)}$ are always tangents to the circle $x^2 + y^2 = a^2$

Let $y = mx + c$ ... (1) meet the circle $x^2 + y^2 = a^2$... ... ... (2)

$x^2 + (mx + c)^2 = a^2$ [put value of y from (1) in (2)]

or, $(1 + m^2)x^2 + 2mcx + c^2 - a^2 = 0$ ... ... (3)

If the line (1) always touches the circle (2) then x has equal roots in (2)

$\therefore\ 4m^2c^2 = 4(1 + m^2)(c^2 - a^2)$ or $c^2 = a^2(1 + m^2)$

or, $c = \pm a\sqrt{(1 + m^2)}$ ... ... (4)

which is, therefore, the condition of tangency of the lines (1) to the circle (2)

Put the values of c in (1), then the straight lines

$y = mx \pm a\sqrt{(1 + m^2)}$

are always tangents to the circle $x^2 + y^2 = a^2$ whatever be the value of m.

The general equation of the tangent to the circle $x^2 + y^2 = a^2$ is represented by the form
$y = mx \pm a\sqrt{(1 + m^2)}$ ... ... (5)

where m is a variable parameter.

Cor. $y = mx \pm a\sqrt{(1 + m^2)}$ are the equation of the tangents to the circle which are parallel to $y = mx$.

Art. 72. Chord of contact : If two tangents be drawn to a circle from an external point the straight line joning the points of contact is called the Chord of contact of tangent drawn from it. From the fig (31) T and T′ are the points of contact of the tangents PT and PT′ and TT′ is the chord of contact.

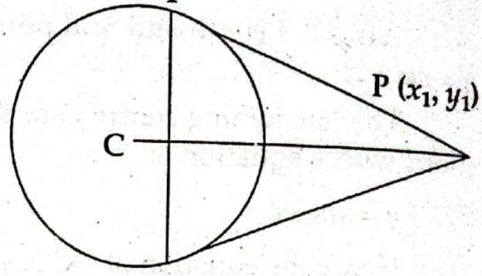

Fig : 31

Art. 73. The equation of the chord of contact of the tangents drawn to a circle.

$x^2 + y^2 + 2gx + 2fy + c = 0$ from a point (x_1, y_1) is

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ ... ... (1)

Note : **1.** The Eq. (1) Art. 73. indicates that it is of the same form as that of equation of the tangent (Art. 68) to the circle at (x_1, y_1). The reason is obvious. If P is on the circle. T and T′ coincide with P and in this limiting position of chord of contact becomes the tangent to the circle at the point P (x_1, y_1)

Hence the identical form.

Note : **2.** If the point P (x_1, y_1) be within the circle, the two tangents will be imaginary; yet the equation of the chord of contact obtained by above method is the same as above and is real. We thus see, since this line is alway real that you may have a real straight line joining the imaginary points of contact of two imaginary tangents. There is nothing surprising in this for real curves can pass throjugh imaginary points.

Ex. Find the points of intersection of the circle $x^2 + y^2 = 1$ with the straight line $x = 2$.

The figure shows that the straight line does not meet the circle at all ... (1)

Put the value of x in the circle $x^2 + y^2 = 1$

$y^2 = -3$

or, $y = \pm\sqrt{(-3)}$

$= \pm\sqrt{3}i$

The straight line $x = 2$, cuts the circle $x^2 + y^2 = 1$ in the two imaginary points, $(2, i\sqrt{3})$ and $(2, -i\sqrt{3})$ ... ... (2)

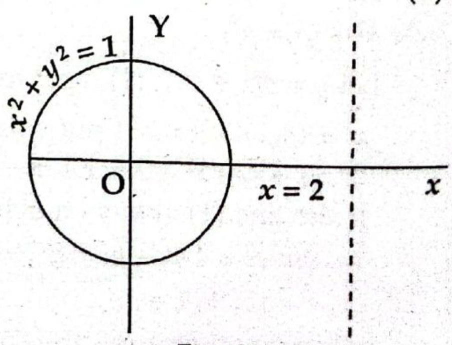

Fig : 32

Of the two statements (1) and (2) the mathematician prefers statement (2). One great advantage in this that he can give his proposition in a form which has no exceptions. Thus he can say. "Every straight line meets a circle in two points."

Note : 3. The equation (7) can be written in the form

$y(y_1+f) = -(x_1+g)\,x - (gx_1+fy_1+c) = y = \dfrac{(x_1+g)}{y_1+f}x - \dfrac{gx_1+fy_1+c}{y_1+f}$

'm' of the chord is $\dfrac{-(x_1+g)}{y_1+f}$ ... ... (i)

Again the equation to the line passing through P (x_1, y_1) and C $(-g, -f)$ is $\dfrac{(y-y_1)}{y_1+f} = \dfrac{x-x_1}{x_1+g}$ or, $y = \dfrac{y_1+f}{x_1+g}(x-x_1)+y_1$

'm' of the chord is $\dfrac{(y_1+f)}{x_1+g}$ ... ... (ii)

From (i) and (ii), it is seen that PC is at right angles to TT'

Art. 74. Find the polar of the point (x_1, y_1) w.r., to the circle $x^2+y^2+2gx+2fy+c=0$

If the points of contact $T(x_2, y_2)$; $T'(x_3, y_3)$; the tangents at these points are $xx_2+yy_2+g(x+x_2)+f(y+y_2)+c=0$ ... ... (1)

and $xx_3+yy_3+g(x+x_3)+f(y+y_3)+c=0$ (see fig 31) ... ... (2)

Both of these tangents (1) and (2) will pass through P (x_1, y_1); we have, $x_1x_2+y_1y_2+g(x_1+x_2)+f(y_1+y_2)+c=0$ ... ... (3)

$x_1x_3+y_1y_3+g(x_1+x_3)+f(y_1+y_3)+c=0$ ... ... (4)

relations (3) and (4) show that the two points

$T(x_2,y_2)$; and $T'(x_3,y_3)$; are on the line

$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$ $\therefore$ ... ... (5)

which is, therefore, the required equation of the polar of the point (x_1,y_1); (see Art. 57)

Note : 1. It will be noticed that the equation of the polar is the same as that of the chord of contact of the tangents real and imaginary from that point to the conic.

Note : 2. The pole is different from the pole of polar co-ordinates.

Cor. If $g=0$, $f=0$ in (5); then $xx_1+yy_1=a^2$ which is the polar of the point $P(x_1,y_1)$ w.r. to the circle $x^2+y^2=a^2$

Art. 75. Pole of line.

Find the pole of a line with respect to any circle $x^2+y^2=a^2$

Let the equation to the line and circle respectively be

$Ax+By+C=0$ .... (1) and $x^2+y^2=a^2$ ... ... (2)

If (x_1, y_1) be the pole then the equations to the polar is $xx_1+yy_1=a^2$

Compare equations (1) and (3), then $\dfrac{x_1}{A} = \dfrac{y_1}{B} = -\dfrac{a^2}{C}$ or $x_1 = -\dfrac{A}{C}a^2$, $y_1 = \dfrac{-B}{C}a^2$

The required pole $\left(-\dfrac{A}{C}a^2, \dfrac{-B}{C}a^2\right)$

Similarly from the $\Delta\, T_1 O_1 P_1$ and $\Delta\, T_2 O_2 P_2$

$$\frac{T_1 O_2}{T_2 O_2} = \frac{O_1 P_1}{O_2 P_2} = \frac{r_1}{r_2} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Therefore, the points T_1 and T_2 divide $O_1 O_2$ internally and externally in the radii of the circles.

These points are known as the **centre of similitude** and the circle on $T_2 T_1$ as diameter is called **the circle of similitude of the two circles.**

If the co-ordinates of T_2 be (x_1, y_1) then any line through it is

$$y - y_1 = m\,(x - x_1) \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Now the condition of tangency is applied. The length of perpendicular from O_1 on this line (3) is equal to radius of the circle r_1. This will give the value of m, hence the equations of tangents are obtained. So for the other pair which passes through T_1.

Art. 82. The power of a point.

If any line drawn through a fixed point P meets the circle at Q and R. the product **PQ. PR** is constant and is called the **Power of P. *w. r.* to the circle.**

Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Let P (x_1, y_1) be any point. The equation PR is of the form (See Art. 14).

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Now putting the value of x and y from (2) in (1) we have $(x_1 + r\cos\theta)^2 + (y_1 + r\sin\theta)^2 + 2g(x_1 + r\cos\theta) + 2(fy_1 + r\sin\theta) + c = 0 \quad \ldots \quad \ldots \quad \ldots \quad (3)$

The roots r_1, r_2 of the equation (3) are the measure PQ and PR Hence

$$r_1 . r_2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

or, PQ. PR $= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c =$ (length of tangent)^2

The power of P is (+ ve) or (— ve) according as PQ and P R are drawn in the same or in opposite senses i. e. according as PQ and PR are drawn in the same or in opposite senses i. e. according as P is outside, or inside the circle. The power of P is zero when P lies on the circle.

Art. 83. A. R. Khalifas Solution of a class of problems on circles in Analytic Geometry (Published by Dhaka University studies Vol. XI June 1963).

Proposition.

To find the general equation of all circles passing through two given points (x_1, y_1) and (x_2, y_2)

In the consideration of the fact that the equation of a circle must be of the second degree, in which the co-efficients of x^2 and y^2 must be equal and the term containing xy must be absent (for rectangular co-ordinates) and in order that the circle may pass through the two points, the co-ordinates of the points must satisfy it, the equation of a circle passing through the given points (x_1, y_1) and (x_2, y_2) may be written as

$$S \equiv (x - x_1)\,(x - y_2) + (y - y_1)\,(y - y_2) = 0 \quad \ldots \quad \ldots \quad (1)$$

It may be seen in passing that this equation is obtained in Analytical conics in another context, namely as the equation of the circle drawn upon the straight line joining the two given points of its diameter. We do not require that properly. This may only be noted how easily the equation can be written down at once from above considerations.

The equation of a st. line through the two points (x_1, y_1) and (x_2, y_2) is written very easily from similar considerations in the same way as

$L = (x - x_1)(y - y_2) - (x - x_2)(y - y_1) = 0$ ... ... (2)

Therefore, the general equation of all circles passing through the two points (x_1, y_1) and (x_2, y_2) is $S = AL$.

Or, $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) =$

$A\{(x - x_1)(y - y_2) - (x - x_2) + (y - y_1)\} = 0$ ... ... (3)

where A is a constant

Now we work out some problems with the help of this result.

EXAMPLES

VVS

Ex. 1. Find the equation of the circle passing through the three points (—3, 2), (1, 7) and (5, — 3)

The general equation of all circles passing through the first two points $(-3, 2), (1, 7)$ is

$(x + 3)(x - 1) + (y - 2)(y - 7) = A\{(x + 3)(y - 7) - (x - 1)(y - 2)\}$ ... (1)

Since the circle passes through the third point $(5, -3)$, we have $8.4 + (-5)(-10) = A\{(8(-10) - 4(-5)\}$ or, $A = -30/41$

Thus from (4), we have

$30(x^2 + y^2 + 2x - 9x + 11) = -41(-5x + 4y - 23)$

or, $30(x^2 + y^2) - 145x - 106y - 613 = 0$

which is the rquired equation.

Ex. 2. Find the pole of the straight line $2x - y = 6$ w. r. to the circle $5x^2 + 5y^2 = 9$

Let $P(x_1, y_1)$ be the co-ordinates of the pole.

The equation to the polar of $P(x_1, y_1)$ with respect to the circle

$5x^2 + 5y^2 = 9$ is $5xx_1 + 5yy_1 = 9$ ... ... (1)

This polar is identical with the line $2x - y = 6$ ... ... (2)

Now comparing the co-efficients of x, y from (1) and (2) we have. $5x_1/2 = -5y_1/1 = 9/6$, $x_1 = 3/5, y_1 = -3/10$ $\therefore$ $(3/5, -3/10)$ Ans.

Ex. 3. ABCD is a square whose side is a, taking AB and AD as axes, prove that the equation to the circle circumscribing the square is $x^2 + y^2 = a\ (x + y)$ [D. U. 1975]

If AB and AD are axes, clearly the centre of the circle is at $(a/2, a/2)$ and radius is $\frac{1}{2}\sqrt{(a^2 + a^2)} = a/\sqrt{2}$

The required equation of the circle is $(x - a/2)^2 + (y - a/2)^2 = (a/\sqrt{2})^2$

or, $x^2 + y^2 - ax - ay = 0$ or $x^2 + y^2 = a(x + y)$ proved.

Ex. 4. Find the locus of the middle points of a system of parallel chords of a circle.

$x^2 + y^2 + 2gx + 2fy + c = 0$ ... ... (i)

The equation to the chord in terms of its middle points (x_1, y_1) is $T = S_1$

or, $(x - x_1)(x_1 + g) + (y - y_1)(y_1 + f) = 0$ ... ... (ii)

It is evident that if the chords be parallel to the given line $lx + my = 0$ and (ii) be one of such chord, then comparing the equation $lx + my = 0$ with the equation (ii) we have

$(x_1 + g) : l = (y_1 + f) : m$

Thus the locus of (x_1, y_1) is the line $x + g : l = y + f : m$ which is satisfied by the equations $x + g = 0$ and $y + f = 0$ i. e ; the co-ordinates of the centre $(-g, -f)$ of circle.

Hence the locus of the middle points of a system of parallel chord is a straight line passing through the centre, which is the diameter of the system.

Ex. 5. Show that the locus of the poles of tangents to the circle $x^2 + y^2 = a^2$ *w. r.* to the circle $x^2 + y^2 = 2bx$ is the conic.

$(a^2 - b^2)x + a^2y^2 - 2a^2bx + a^2b^2 = 0$

The polar of any point (x_1, y_1) w. r. to the circle

$x^2 + y^2 - 2bx = 0$ is

$xx_1 + yy_1 - b(x + x_1) = 0$ or, $x(x_1 - b) + yy_1 - bx_1 = 0$ ... ... (i)

(i) is tangent to the circle $x^2 + y^2 = a^2$

$$\frac{-bx_1}{\sqrt{\{(x_1 - b)^2 + y_1^2\}}} = \pm a$$

or. $b^2x_1^2 = a^2(x_1 - b)^2 + y_1^2a^2$ or $b^2x_1^2 = a^2x^2_1 - 2a^2x_1b + a^2b^2 + y_1^2a^2$

or, $x_1^2(a^2 - b^2) - 2a^2bx_1 + a^2b^2 + y_1^2a^2 = 0$

Hence the locus of (x_1, y_1) is $x^2(a^2 - b^2) + a^2y^2 - 2a^2bx + a^2b^2 = 0$

Ex. 6. Show that the equation to the pair of tangents drawn from the origin to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

is $(gx + fy)^2 + c(x^2 + y^2)$

The equation to the pair tangents is $S S_1 = T^2$

or, $(x^2 + y^2 + 2gx + 2fy + c)c = x\ [x.0 + y.0 + g(x + 0) + f(y + 0) + c]^2$

or, $c(x^2 + y^2 + 2gx + 2fy + c) = (gx + fy + c)^2$

or, $c(x^2 + y^2) + c(2gx + 2fy + c) = (gx + fy)^2 + 2(gx + fy)c + c^2$

or, $c(x^2 + y^2) = (gx + fy)^2$

Ex. 7. prove that, if the polar of a point P w. r. to the circle $x^2 + y^2 = 37$ touches the circle. $(x - 3)^2 + (y + 2)^2 = 25$, the locus of P must be a conic, whose equation you are required to obtain.

Let the co-ordinates of P be (x_1, y_1)

Polar of (x_1, y_1) w. r. to $x^2 + y^2 = 37$ is $xx_1 + yy_1 = 37$

If (1) touches $(x-3)^2+(y+2)^2=25$

$$\frac{3x_1-2y_1-37}{\sqrt{(x_1^2+y_1^2)}}=5$$

Length of the perpendicular from the centre (3, − 2) to the line(1) is equal to the radius of the circle. 5

or, $(3x_1-2y_1-37)^2=25(x_1^2+y_1^2)$

or, $16x_1^2+21y_1^2+12x_1y_1+222x_1-148y_1-1369=0$

The locus of (x_1, y_1) is $16x^2+21y^2+12xy+222x-148y-1369=0$

Ex. 8. Find the equation of the circle inscribed in the triangle determined by the lines.

$2x-3y+21=0$ ... ... (i)

$3x-2y+20=0$ ... ... (ii)

$2x+3y+9=0$ ... ... (iii)

The centre of the circles lies at the point of intersection of the bisectors of the interior angles of the triangle, Let (h, k) be the co-ordinates of the centre of the incircle. Bisector of angle (i) and (ii) is

$$\frac{2h-3k+21}{\sqrt{(4+9)}}=+\frac{3h-2k-6}{\sqrt{(9+4)}} \text{ or, } h+k+3=0 \quad \ldots \quad \ldots \quad \text{(iv)}$$

Biector of angle (i) and (iii) is

$$\frac{2h-3k+21}{\sqrt{(4+9)}}=+\frac{2h+3k+9}{\sqrt{(4+9)}} \text{ or, } 6k-12=0 \text{ or, } k=2 \quad \ldots \quad \ldots \quad \text{(v)}$$

From (iv), $h=-1$. Co. ordinates of centre $(-1, 2)$

Radius of the circle, $=\dfrac{2(-1)+3(2)+9}{\sqrt{13}}=\sqrt{13}$

Hence equation to circle is $(x+1)^2+(y-2)^2=(\sqrt{13})^2$, or, $x^2+y^2+2x-4y-8=0$

Ex. 9. Find the equation of the direct common tangents to the circles $x^2+y^2=16$ and $x^2+y^2+6x-8y=0$

The centres of the two circles are (0 0) and (−3, 4) and their radii are 4 and 5 respectively. So the point dividing externally the line of centres in the ratio 4 : 5 is given by

$$x_1=\frac{5.0-4.(-3)}{5-4}=12,\ y_1=\frac{5.0-4.4}{5-4}=-16$$

The direct tangents pass through (12, − 16)

Hence their equation are given by $y+16=m(x-12)$ ... ... (1)

In order that this line may touch the circle if the perpendicular from the centre (0, 0) of the circle on the line (i) is equal to the radius 4.

$$\frac{12m+16}{\sqrt{(1+m^2)}}=\pm 4 \text{ or, } 8m^2+24m+15=0 \qquad m=\frac{-24\pm\sqrt{(576-480)}}{16}=\frac{-6\pm\sqrt{6}}{4}$$

The equation (i) becomes $y+16=\pm\dfrac{-6\pm\sqrt{6}}{4}(x-12)$ or, $6x+4y-8=\pm 16(x-12)$

which are the required equations.

Ex. 10. Find the equation of the transverse common tangents of the circle $x^2+y^2-x-2y+1=0$ and $2x^2+2y^2-3x+4y=0$

The centres of the circles are

$$x^2+y^2-x-2y+1=0 \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

$$x^2+y^2-\frac{3}{2}x+2y=0 \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

are $(\frac{1}{2}, 1)$ and $(\frac{3}{4}, -1)$ and their radii are $\frac{1}{2}$ and $\frac{5}{4}$ (See fig 34) respectively.

The point dividing O_1 and O_2 internally in the ratio of $\frac{1}{2} : \frac{5}{4}$, i. e.,

co-ordinates of T_1 are $x_1 = \dfrac{5/4\frac{1}{2}+\frac{1}{2}.3/4}{\frac{5}{4}+\frac{1}{2}} = \frac{4}{7}$, $y_1 = \dfrac{5/4.1+\frac{1}{2}(-1)}{\frac{5}{4}+\frac{1}{2}} = \frac{3}{7}$

The transverce tangents are the lines through T_1 and distance from the centre O_1 (1/2, 1) so they are given by

$$y-3/7 = m(x-4/7) \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

$$\frac{1-3/7-m\ (1/2-4/7)}{\sqrt{(1+m^2)}} = \pm\frac{1}{2}$$

or, $\left(\frac{4}{7}+m/14\right)^2 = \frac{1}{4}(1+m^2)$ or, $(8+m)^2 = \frac{196}{4}(1+m^2)$

or, $64+16m+m^2 = 49+49m^2$

or, $48m^2-16m-15=0$ or, $m=\frac{3}{4}, -\frac{5}{12}$

Thus the transverse tangents are $y-3/7 = \frac{3}{4}\left(x-\frac{4}{7}\right)$ and $y-\frac{3}{7} = -\left(\frac{5}{12}\right)\left(x-\frac{4}{7}\right)$

or, $4y=3x$ and $12y+5x=8$

Ex. 11. The circle $x^2+y^2-2x+4y-20=0$ meets the straight line $2x-y=2$ at A and B. Find the equation of the straight line joining A and B with the origin and obtain the angle between them.

The equation of the straight lines OA and OB is obtained by making the equation of the circle $x^2+y^2-2x+4y-20=0$

homogeneous with the line $2x-y=2$ or, $\frac{1}{2}(2x-y)=1$

The joint equation is $x^2+y^2-2x\frac{1}{2}(2x-y)+4y\frac{1}{2}(2x-y)-20\{\frac{1}{2}(2x-y)\}^2=0$

or, $x^2+y^2-2x^2+xy+4xy-2y^2-5(4x^2+y^2-4xy)=0$

or, $21x^2+6y^2-25xy=0$

And the angle between the straight lines is $\tan 2\varphi = \dfrac{2\sqrt{(h^2-ab)}}{a+b}$

or, $\tan 2\varphi = \dfrac{2\sqrt{(625/4-21.6)}}{21+6}$ $\therefore$ $2\varphi = \tan^{-1}\frac{11}{27}$

EXERCISE VII

1. Find the equation to the circle whose radius is 8 and the centre is (—4, 2). vvI

Ans. $x^2 + y^2 + 8x + 4y + 4 = 0$

2. Find the equation to the circle which is tangent to both axes, its centre being in the first quadrant and radius is 8.

Ans. $x^2 + y^2 - 16x - 16y + 64 = 0$

2 (a) Find the equation of a circle which touches both the axes and passes through the pt. $(-2, -1)$. [D. U. 1984] Ans. $x^2 + y^2 + 10x + 10y + 25 = 0$, $x^2 + y^2 + 2x + 2y + 1 = 0$

3. Find the equation of the circle which touches the axes of co-ordinates and passes through (3, 4).

Ans. $x^2 + y^2 - 2ax - 2ay + a^2 = 0$. where $a = 7 \pm \sqrt{24}$

4. Find the co-ordinates and centre of the circle. vvI [R. U. 1958]

$x^2 + y^2 - 6x + 14y + 33 = 0$. Ans. $(3, -7), 5$

5. Find the equation of the circle passing through the points (use Khalifa's method).

(a) $(1, 3), (2, -1), (-1, 1)$ Ans : $5x^2 + 5y^2 - 11x - 9y - 12 = 0$

(b) $(-4, -3), (-1, -7), (0, 0)$ Ans : $x^2 + y^2 + x + 7y = 0$

6. Show that following points are concyclic

(a) $(3, 5), (3, -5), (2, 4), (2, -4)$

(b) $(am_1, a/m_1), (am_2, a/m_2), (am_3, a/m_3), (am_4, a/m_4)$ if $m_1 m_2 m_3 m_4 = 1$.

[R. U. 1963, D. U. 1964]

7. Find the equation of the circle which passes through (3, 5) and (5, −3) and has its centre on the line $2x + y = 27$ (use Khalifa's method) vvI

Ans. $x^2 + y^2 - 24x - 6y - 68 = 0$ [D. U. 1965]

8. Determine the centres and radii of the circles $x^2 + y^2 - 2x + 2y - 7 = 0$, $x^2 + y^2 - 6x - 2y - 6 = 0$ and $x^2 + y^2 - 8x - 4y - 5 = 0$ and show that their centres are collinear. vvI [D. U. 1962]

9. Find the locus of the middle points of the chords of the circles $x^2 + y^2 = a^2$ which subtends a right angle at the origin. Ans. $x^2 + y^2 = a^2/2$ [D. U. 1959, '77, '84 ; R. U. 1980]

10. Find the equation of the circle circumscribing the triangle formed by the lines.

(a) $2x - y + 7 = 0, 3x + 5y - 9 = 0, x - 7y - 13 = 0$

Ans. $169x^2 + 169y^2 - 8x + 498y - 3707 = 0$

(b) $x + 2y - 5 = 0, 2x + y - 7 = 0, x - y + 1 = 0$ Ans. $3x^2 + 3y^2 - 13x - 11y + 20 = 0$

11. Find the equation of the circle inscribed in the triangle formed by the lines $2x - 3y + 21 = 0, 3x - 2y - 6 = 0, 2x + 3y + 9 = 0$

Ans. $x^2 + y^2 + 2x - 4y - 8 = 0$

12. Find the area of an equilateral triangle inscribed in the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ Ans. $(g^2 + f^2 - c)\, 3\sqrt{3}/4$

13. Show that the circles $x^2 + y^2 - 4x + 6y + 8 = 0$ and $x^2 + y^2 - 10x - 6y + 14 = 0$ touch at the points (3, −1)

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14. Find the equation of the circle through the points of intersection of the circles. $x^2 + y^2 + 2x + 3y - 7 = 0$ and $x^2 + y^2 + 8x - 2y - 1 = 0$ and through the point (1, 2). Ans $x^2 + y^2 + 4x - 7y - 5 = 0$ [R. U. 1963]

15. Show that the equation $(3x - 3y + 1)^2 + (6x - 1)(3y + 2) = 0$ represents a circle and that the straight lines $x = \frac{1}{6}$ and $y = -2/3$ touch it. Assig (Assignment)

16. Find the equation to the circle which is concentric with the circle $x^2 + y^2 - 8x + 12y + 15 = 0$ and passes through (5, 4) Ans. $x^2 + y^2 - 8x + 12y - 49 = 0$

17. Show that the equation $(y - x + 3)^2 + 2(x - 2)(y + 2) = 0$ represents a circle of which $x = 2$ and $y + 2 = 0$ are two tangents. What is represented by $x - y = 3$.

Ans. $x - y = 3$ is the equation of the polar of the point (2, − 2) w.r. to the given circle.

18. Find the equation to the circle through the point (1, 2) (3, 4) and tangents to the lines $3x + y - 3 = 0$ [একটি বৃত্ত (1, 2) (3, 4) বিন্দুগামী এবং $3x + y - 3 = 0$ সরলরেখাকে স্পর্শ করিলে বৃত্তটির সমীকরণ নির্ণয় কর।] [C. U. 1982] Ans. $x^2 + y^2 - 8x - 2y + 7 = 0$, $x^2 + y^2 - 3x - 7y + 9/2 = 0$.

19. Find the equation of the circle through the points of intersection of $x^2 + y^2 - 4 = 0$ and $x^2 + y^2 - 2x - 4y + 4 = 0$ and touching the line $x + 2y = 0$ [একটি বৃত্ত $x^2 + y^2 - 4 = 0$ এবং $x^2 + y^2 - 2x - 4y + 4 = 0$ বৃত্তদ্বয়ের ছেদবিন্দুগামী এবং $x + 2y = 0$ সরলরেখাকে স্পর্শ করলে, বৃত্তটির সমীকরণ নির্ণয় কর।]

Ans. $x^2 + y^2 - x - 2y = 0$ [D. U. 1964 ; C. U. 1980]

20. Prove that the locus of a point whose distance from a fixed point is in a constant ratio to the tangent drawn from it to a given circle is a circle.

21. Find the polar of the point $(-16/3, -1/3)$ *w. r.* to the circle $x^2 + y^2 + 4x - 6y + 3 = 0$ [$x^2 + y^2 + 4x - 6y + 3 = 0$ বৃত্তটি অনুসারে $(-16/3, -1/3)$ বিন্দুটির পোলার নির্ণয় কর।]

Ans. $x + y + + 2 = 0$ [D. U. 1980]

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22. Prove that the polar of the point (p, q) with respect to the circle $x^2 + y^2 = a^2$ touches $(x - c)^2 + (y - d)^2 = b^2$ if $b^2 (p^2 + q^2) = (a^2 - cp - dq)^2$

23. Find the equation of the circle which has for its diameter the chord cut off on the $3x + y + 5 = 0$ by the circle $x^2 + y^2 = 16$ Ans. $x^2 + y^2 + 3x + y - 11 = 0$

24. Find the pole of the lines $2x - y + 10 = 0$ *w. r.* to $x^2 + y^2 - 7x - 5y - 1 = 0$

Ans. (3/2, − 3/2) [D. U. 1961)

25. Show that the polar of the point (h, k) w. r. to the circle $x^2 + y^2 - 2\lambda x + c = 0$ where λ is a variable parameter, always passes through a fixed point.

26. Find the equations of the tangents to the circle $x^2 + y^2 - 4x + 6y - 3 = 0$ which are parallel to the straight line $3x - 4y + 1 = 0$

Ans. $3x - 4y + 2 = 0$, $3x - 4y - 38 = 0$ [D. U. 1952, '57, R. U. 1985]

27. Show that the equation of the circle which has for its diameter the chord cut off on straight line $ax + by + c = 0$ by the circle $(a^2 + b^2)(x^2 + y^2) = 2c^2$ is $(a^2 + b^2)(x^2 + y^2) + 2c(ax + by) = 0$