MAT 1241

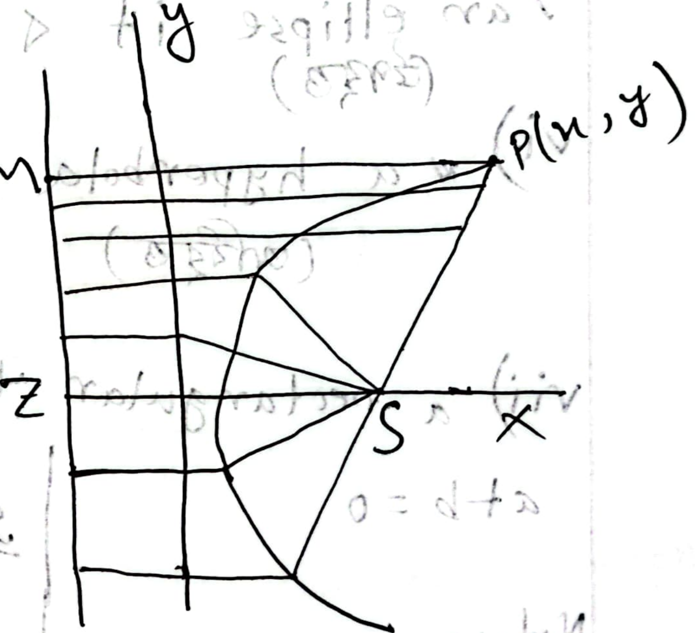
Lecture-1
21.10.2024

General Eqn of Second Degree

Q.1 Define conic section / conic.

⇒ If a point P moves in a plane such a way that the ratio of its distance PS from a fixed point S in the plane to its perpendicular distance PM from a fixed st. line ZM is always constant then the locus of P is called conic
(সঞ্চারপথ বিন্দু)

the ratio $= \frac{SP}{PM}$



General eqn of conic:

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \cdots (i)$$

here, $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

$= abc + 2fgh - af^2 - bg^2 - ch^2 \quad \text{(i)}$

Eq^n(i) represent.

i) Two parallel lines if $\Delta = 0$ and $ab - h^2 = 0$

ii) Two ~~para~~ perpendicular lines if $\Delta = 0$ and $a + b = 0$

iii) a circle if $\Delta \neq 0$, $a = b$, $h = 0$

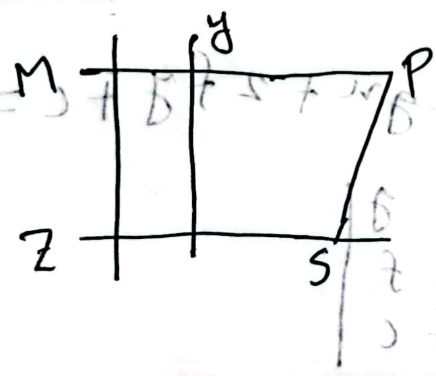
iv) a parabola if $\Delta \neq 0$, $ab - h^2 = 0$
(পরাবৃত্ত)

v) an ellipse if $\Delta \neq 0$ and $ab - h^2 > 0$
(উপবৃত্ত)

vi) a hyperbola if $\Delta \neq 0$, $ab - h^2 < 0$
(অধিবৃত্ত)

vii) a rectangular hyperbola if $\Delta \neq 0$, $ab - h^2 < 0$
$a + b = 0$

Note:



$\frac{SP}{PM} = e$ (ecentricity → উৎকেন্দ্রিকতা)

i) $e = 0$ implies that circle

ii) $e < 1$ implies that ellipse

iii) $e > 1$ implies that hyperbola

standard form: of gen eq^n(i) $\rightarrow ax^2 + 2hxy + by^2 = c$ $\rightarrow -(gx_1 + fy_1 + c)$

$\frac{SP}{SPM} = 0 = e$; $SP = 0$; $\boxed{SP = \sqrt{(x-a)^2 + (y-b)^2}}$

Page-86, Example-3) Test the nature of the conic given by

$9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0$

Here,

$a = 9$, $g = -9$

$h = -12$,

$b = 16$, $f = \frac{-101}{2} =$

$c = 19$

$\therefore \Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$\neq 0$ and; $ab - h^2 = 9 \times 16 - (12)^2$

$= 0$

the given conic is a parabola.

Example-4 Find the centre of the conic represen-ted by $2x^2 + y^2 - 3xy - 5x + 4y + 6 = 0$

Sol^n: Given eq^n is $2x^2 + y^2 - 3xy - 5x + 4y + 6 = 0$

Let $f(x,y) = 2x^2 + y^2 - 3xy - 5x + 4y + 6 = 0$

$\therefore \dfrac{\partial f}{\partial x} = 4x - 3y - 5 = 0 \quad \ldots (i)$

and, $\dfrac{\partial f}{\partial y} = 2y - 3x + 4 = 0 \quad \ldots (ii)$

solving (i) and (ii) we get,

$x = 2,$

$y = 1$

$\therefore$ centre is $(2,1)$

Example-6 Reduce the equation $6x^2 + 5xy - 6y^2 - 4x + 7y + 11 = 0$ to the standard form. Find also its lengths, equations and direction of the axes.

Solve: Let $f(x,y) = 6x^2 + 5xy - 6y^2 - 4x + 7y + 11 = 0 \quad \ldots (i)$

$\therefore \dfrac{\partial f}{\partial x} = 12x + 5y - 4 = 0 \quad \ldots (ii)$

and $\dfrac{\partial f}{\partial y} = 5x - 12y + 7 = 0 \quad \ldots (iii)$

Solving (ii) and (iii)

$x = \frac{1}{13}, \; y = 8/3$

Note:

i) calculate the centre

ii) Find $c' = -(gx_1 + fy_1 + c)$ from the give conic

the centre of conic(i) is $\left(\frac{1}{13}, \frac{8}{3}\right)$
$x_1, \; y_1$

Assume,

$$C := C_1 = -(gx_1 + fy_1 + c)$$

$$= (-2 \times \frac{1}{13} + \frac{7}{2} \times \frac{8}{13} + 11)$$

from ...(i)

$g = -2$; $c = 11$

$f = \frac{7}{2}$

$$= -13$$

therefore, the standard eq^n of the conic(i) is $6x^2 + 5xy - 6y^2 = c$

or, $6x^2 + 5xy - 6y^2 = -13$ ... which is the required standard form

Lecture - 2

24.10.2024

ii) page-83 Example | Reduce the eq^n $16x^2 - 24xy + 9y^2 - 104x - 172y - 44 = 0$ to standard form.

Sol^n: Here, $a = 16$, $h = -12$, $b = 9$ | $g = -57$, $f = -86$, $c = -44$

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$ and $ab - h^2$

$\therefore ab - h^2 = 16 \times 9 - (-12)^2$

$= 0$

Hence the given conic is a parabola

From the given conic $(4x - 3y)^2 = 104x + 172y + 44$ ...(i)

here,

$4x - 3y = 0$ and $104x + 172y + 44 = 0$ are not intersect at right angle since $m_1 m_2 \neq -1$

Let us, introduce a constant λ in eq^n(i).

$(4x - 3y + \lambda)^2 = \lambda^2 + 2\lambda(4x - 3y) + 104x + 172y + 44$

$\Rightarrow (4x - 3y + \lambda)^2 = (104 + 8\lambda)x + (172 - 6\lambda)y + \lambda^2 + 44$ ...(ii)

Now,

$4x - 3y + \lambda = 0$ | $(104 + 8\lambda)x + (172 - 6\lambda)y + \lambda^2 + 44 = 0$

since these two are $\perp^r$ to each other then,

$$a_1a_2 + b_1b_2 = 0$$

or, $4(104+8\lambda) - 3(172-6\lambda) = 0$

$\therefore \lambda = 2$

$$\frac{|a\alpha + b\beta + c|}{\sqrt{a^2+b^2}}$$

putting $\lambda = 2$ in eq^n(ii), we get

$$(4x - 3y + 2)^2 = 40(3x+4y-1)$$

or,

$$25\left(\frac{4x-3y+2}{\sqrt{4^2+3^2}}\right)^2 = 40\times 5\left(\frac{3x+4y-1}{\sqrt{3^2+4^2}}\right)$$

or, $Y^2 = 8X$, where, $Y = \dfrac{4x-3y+2}{\sqrt{4^2+3^2}}$

$$X = \frac{3x+4y-1}{\sqrt{3^2+4^2}}$$

which is the standard form of parabola.

H.W. section

p-65(example) Reduce the eq^n $32x^2+52xy-7y^2 - 64x - 5y - 148 = 0$ to standard form.

Soln:

Let $f(x,y) \equiv 32x^2+52xy-7y^2-64x-5y-148=0$ ... (i)

Now, through eq^n(i),

$$\frac{\partial f}{\partial x} = 64x+52y-64=0 \quad \text{...(ii)}$$

and $$\frac{\partial f}{\partial y} = 52x-14y-52=0 \quad \text{...(iii)}$$

solving eq(ii) and eq^n(iii), we have,

$$x_1 = x = 1, \quad y_1 = y = 0$$

$$\therefore (x_1, y_1) = (1, 0)$$

$\therefore$ the centre of conic (i) is $(1, 0)$

and, here,

$$g = -32$$
$$h = -26$$
$$c = -148$$

Assume,

$$C = c_1 = -(gx_1 + fy_1 + c)$$
$$= -(-32\times 1 + (-26).0 - 148)$$
$$= -180$$

therefore, the standard form of eq^n conic (i) is,

$$32x^2 + 52xy - 7y^2 = -180 \text{ (Ans)}$$

or, $32x^2 + 52xy - 7y^2 - 180 = 0$ (Ans)

When xy term is removed, let the equation be;

$a_1x^2 + b_1y^2 = 180$

$a_1 + b_1 = 32 - 7 = 25$

and, $a = 32$, $b = -7$, $h = 26$

$a_1 b_1 = -32 \cdot 7 - (26)^2 = -900$

$\therefore a_1 = 45, \quad b_1 = -20$

$\therefore$ eq^n is $45x^2 - 20y^2 = 180$

or, $x^2/4 - y^2/\frac{9}{2} = 1$, which is a hyperbola.

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Example-08

Reduce the eq^n $4x^2 - 4xy + y^2 - 8x - y + 6 = 0$ to standard from. Find all its properties.

Soln:

Given, $4x^2 - 4xy + y^2 - 8x - y + 6 = 0$ ...(i)

From the eq^n (i)

$a = 4$, $b = 1$, $h = -2$, $g = -4$, $f = -3$, $c = 5$

$\therefore \Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$70 - 64$

and, $ab - h^2 = 4 - 4 = 0$

$\therefore \Delta \neq 0$ and $ab - h^2 = 0$

Hence eq^n (i) reprensts a parabola.

(i) $\therefore (2x-y)^2 = 8x+6y+5$ ---- (ii)

Lets introduce a new constant λ in eqn (ii)

$(2x-y+\lambda)^2 = 8x+6y-5+\lambda^2+2\lambda(2x-y)$

or, $(2x-y+\lambda)^2 = x(4\lambda+8)+y(-2\lambda+6)+(\lambda^2-5)$ ---- (iii)

$\therefore$ $(2x-y+\lambda) = 0$ | or, $x(4\lambda+8)+y(-2\lambda+6)+(\lambda^2-5)=0$

since these two are perpendicular to each other then,

$\therefore a_1a_2+b_1b_2=0$

$\therefore$ ~~$x(4\lambda+8)+y(-2\lambda+3)+\lambda$~~

$2(4\lambda+8)(-1)+(-2\lambda+6)=0$, or $\lambda=-1$

Now, put $\lambda=-1$ in eqn (iii),

$(2x-y-1)^2 = 4(x+2y-1)$

or, $\left(\dfrac{2x-y-1}{\sqrt{2^2+1^2}}\right)^2 5 = 4\sqrt{5}\left(\dfrac{x+2y-1}{\sqrt{1^2+2^2}}\right)$

or, $Y^2 = 4/\sqrt{5}\,X$ | where, $Y = \dfrac{2x-y-1}{\sqrt{2^2+1^2}}$

$X = \left(\dfrac{x+2y-1}{\sqrt{1^2+2^2}}\right)$

$\therefore Y^2 = 4/\sqrt{5}\,X$ is the standard eqn of parabola.

Chapter-7 (The Circle) Lecture-3

28.10.2024

#1: Standard eq^n of circle $(x-h)^2 + (y-k)^2 = a^2$

where $(h,k) \to$ centre, $a \to$ radius

#2: General eq^n of circle: $x^2+y^2+2gx+2fy+c=0$; its $\boxed{\text{centre } (-g, -f)}$, $\boxed{\text{radius} = \sqrt{g^2+f^2-c}}$

#3: Eq^n of a circle passes through the end points of diameter (x_1, y_1) and (x_2, y_2) is:

$$\boxed{(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0}$$

#4: Eq^n of a circle passes through two points (x_1, y_1) and (x_2, y_2) is:

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2) = A\left\{(x-x_1)(y_1-y_2) - (y-y_1)(x_1-x_2)\right\}$$

where A is any real constant.

page(111) Example - 1 | Find the eqn of a circle passing through the three points $(-3,2)$, $(1,7)$, $(5,-3)$

Soln: The general eqn of a circle passing through $(-3,2)$ and $(1,7)$ is:

$$\cancel{(x+3)}\ (x+3)(x-1)+(y-2)(y-7) = A\left\{(x+3)(2-7) - (y-2)(-3-1)\right\}$$

or, $x^2+3x-x-3+y^2-2y-7y+14 = A(-5x-15+4y-8)$ ...(i)

since, $(5,-3)$ lies on eqn (i)....

$$25+9+10+27+14 = A(-25-12-23)$$

$$\therefore A = -\frac{47}{30}$$

Substituting the value of A in eqⁿ(i), we get,

$$x^2+y^2+2x-9y+11 = -\frac{41}{30}(-5x+4y-23)$$

or, $30x^2+30y^2+60x-270y+330 = -205x-164y+943$

$\therefore 30x^2+30y^2-145x-106y-613=0$ (Ans)

Page-(115) Exer-7 Q.7 | Find the eqⁿ of a circle which passes through (3, 5) and (5, -3) and has it's centre on the line $2x+y=27$

Solⁿ: The general eqⁿ of circle passing through (3,5) and (5,-3) is:

$$(x-3)(x-5)+(y-5)(y+3) = A\left\{(x-3)(5+3)-(y-5)(3-5)\right\} \quad \text{...(i)}$$

or, $x^2-8x+15+y^2-2y-15 = A(8x-24+2y-10)$

or, $x^2+y^2-8x-2y-8Ax-2Ay+34A=0$

or, $x^2+y^2-8(1+A)x-2(1+A)y+34A=0$ ...(ii)

whose centre is

$$\{\underset{-g}{4(1+A)}, \underset{-f}{(1+4)}\}$$

which lie on,

$$2x+y-27=0$$

So, $8(1+A)+1+A-27=0$

or, $A=2$

Now, putting $A=2$ in eq^n (ii) we get,

$$x^2+y^2-24x-6y+68=0 \quad \text{(Ans)}$$

Page (176) (Ex.-7) Q.18 Find the ~~the~~ eq^n of circle through the points $(1,2)$, $(3,4)$ and tangents to the line $3x+y-3=0$

Sol^n: The general eq^n of circle passing $(1,2)$ and $(3,4)$ is,

$$(x-1)(x-3)+(y-2)(y-4)=A\left\{(x-1)(2-4)-(y-2)(1-3)\right\}$$

or, $x^2 - 3x - x + 3 + y^2 - 4y - 2y + 8 = A(-2x+2+2y-4)$

or, $x^2+y^2-4x-6y+11 = A(-2x+2y-2)$

or, $x^2+y^2-2(2-A)x-2(3+A)y+11+2A=0$ --- (i) whose centre is

$$(-g, -f) = \{(2-A), (3+A)\}$$

$\therefore$ radius, $= \sqrt{g^2 + f^2 + c}$

$$= \sqrt{(2-A)^2 + (3+A)^2 - 11 - 2A}$$

$$= \sqrt{4 - 4A + A^2 + 9 + 6A + A^2 - 11 - 2A}$$

$$= \sqrt{2A^2 + 2}$$

We know,

$\perp^r$ distance of $\underline{3x + y - 3 = 0}$ from $(2-A, 3+A)$ radius is

$$\frac{|3(2-A) + (3+A) - 3|}{\sqrt{3^2 + 1^2}} = \sqrt{2A^2 + 2}$$

or, $(6 - 3A + 3 + A - 3)^2 = 10(2A^2 + 2)$

or, $(2A - 6)^2 - 20A^2 - 20 = 0$

or, $4A^2 - 24A + 36 - 20A^2 - 20 = 0$

or, $\cancel{4}\,{-4}(-A^2 + 6A \cancel{+9} - 9 + 5A^2 + 5) = 0$

or, $4A^2 + 6A - 4 = 0 \quad \ldots (ii)$

$\cancel{\therefore A = \left(1/2, -2\right)}$

or, $2A(A+2) - 1(A+2) = 0$

or, $(A+2)(2A-1) = 0$

$\therefore A = -2$ and, $A = \frac{1}{2}$

Now, putting $A = -2$ and $A = \frac{1}{2}$ in eq^n(i)

we get,

$A = -2$,

$x^2 + y^2 - 8x - 2y + 7 = 0$ (Ans)

again, $A = \frac{1}{2}$

$x^2 + y^2 - 3x - 7y + 12 = 0$ (Ans)

H.W. (Exercise- VII)

page - 115 ((1-6), 8)

page - 116 (17, 19)

H.W.

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Exercise - VII

Q.1 Find the equation to the circle whose radius is 8 and centre is $(-4, 2)$

Soln: Here,

centre $(h, k) = (-4, 2)$

radius, $a = 8$

$\therefore$ eqn of circle is:

$$(x+4)^2 + (y-2)^2 = 8^2$$

or, $x^2 + y^2 + 8x - 4y - 44 = 0$

$\therefore$ Eqn is $x^2 + y^2 + 8x - 4y - 44 = 0$ (Ans)

Q.2 Find the eqn to the circle which is tangent to both axes, its centre being in the first quadrant and radius is 8.

Soln: Since, the circle is tangent to both axes then $h = k = a$

here, radius, $a = 8$

$\therefore h = k = 8$

$\therefore$ Circle eqn is : $(x-8)^2+(y-8)^2=8^2$

or, $x^2+y^2-16x+64-16y+64=64$

or, $x^2+y^2-16x-16y+64=0$ (Ans)

3.9(a) Find the eqn of a circle which touches both the axes and passes through the pt. $(-2,-1)$

Soln: Let, (h,k) be the centre of a circle with radius a.

$\therefore$ required eqn of circle :

$$(x-h)^2+(y-k)^2=a^2 \;\cdots (i)$$

since, circle touches both the axes then,

$$h=k=a$$

from eqn (i),

$$(x-a)^2+(y-a)^2=a^2$$

or, $x^2+y^2-2ax-2ay+a^2=0 \cdots (ii)$

the pt. $(-2,-1)$ passes through eqn (ii).

we get,

$4+1+4a+2a+a^2=0$

or, $a^2+6a+5=0 \quad \ldots (iii)$

Solving eq^n (iii), we have,

$$a=-1,-5$$

Now putting $a=-1$ and $a=-5$ into eq^n(ii), we have,

$a=-1$,

$x^2+y^2+2x+2y+1=0$ (~~Ans~~)

and $a=-5$

$x^2+y^2+10x+10y+25=0$ (~~Ans~~)

$\therefore$ The eq^n are:

$x^2+y^2+2x+2y+1=0$ and $x^2+y^2+10x+10y+25=0$ (Ans)

Q.3 Find the eq^n of the circle which touches the axes of co-ordinates and passes through $(3,4)$

Soln: Let the centre be (h,k) and radius a.

Since it touches both axes, then

$$h=k=a$$

General eqn of circle:

$(x-a)^2+(y-a)^2=a^2$ | $\because h=k=a$

or, $x^2+y^2-2ax-2ay+a^2=0 \cdots$ (i)

since (3, 4) passes through eqn (i) we get,

$9+16-6a-8a+a^2=0$

or, $a^2-14a+25=0 \cdots$ (ii)

from eqn (ii), solving it we have,

$a=7\pm2\sqrt{6}$

since, radius cannot be negative, $a=7+2\sqrt{6}$

$\therefore$ required eqn is $x^2+y^2-2ax-2ay+a^2=0$

where $a=7+2\sqrt{6}$ (Ans)

Q.4 Find the co-ordinates and centre of the circle. $x^2+y^2-6x+14y+33=0$

Soln. Given, $x^2+y^2-6x+14y+33=0$

or, $x^2-6x+y^2+14y=-33$

or, $(x-3)^2-9+(y+7)^2-49=-33$

or, $(x-3)^2+(y+7)^2=5^2$ ...(i)

comparing eqn (i) with $(x-h)^2+(y-k)^2=a^2$ we have,

$$\text{centre}(h,k)=(3,-7)$$

$$\text{radius}, a=5$$

Q.5 Find the eqn of the circle passing through the points.

a) $(1,3), (2,-1), (-1,1)$

b) $(-4,-3), (-1,-7), (0,0)$

Soln.

5(a) Given that,

$$(1,3), (2,-1), (-1,1)$$

General eqn of circle passing through $(1,3)$ $(2,-1)$

$$(x-1)(x-2)+(y-3)(y+1)=A\begin{Bmatrix}(x-1)(3+1)-\\(y-3)(1-2)\end{Bmatrix}$$

or, $x^2-2x-x+2+y^2+y-3y-3=A\{4x-4-(-y+3)\}$

$A\{4x-4+y-3\}$

or, $x^2+y^2-3x-2y-1=4xA+yA-7A$ ...(i)

~~or~~ since, $(-1,1)$ lies on eq^n (i), we have,

$$(-1)^2 + 1^2 + 3 - 2 - 1 = A(4.(-1) + 1 - 7)$$

$$\therefore A = -\frac{1}{\cancel{2}} \frac{1}{5}$$

putting $A = \frac{1}{5}$ in eq^n we have,

$$x^2 + y^2 - 3x - 2y - 1 = \frac{1}{5}(4x + y - 7)$$

or, $5x^2 + 5y^2 - 15x - 4x - 10y - y - 5 + 7 = 0$

or, $5x^2 + y^2 - 19x - 11y + 2 = 0$ (Ans)

5 b) Given,

$(-4,3)$, $(-1,-7)$, $(0,0)$

General eq^n of circle passing through $(-4,-3)$, $(-1,-7)$

$$(x+4)(x+1) + (y+3)(y+7) = A\left\{ (x+4)(-3+7) - (y+3)(-4+1) \right\}$$

or, $x^2 + x + 4x + 4 + y^2 + 7y + 3y + 21 = A\{4x + 16 + 3y + 9\}$

or, $x^2 + y^2 + 5x + 10y + 25 = A\{4x + 3y + 25\}$ ...... (i)

Since, (0,0) lies on eqⁿ(i), whe have

$0+0+0+0+25 = 25A$

or, $\therefore A = 1$

putting $A=1$ into eqⁿ(i) we have,

$x^2+y^2+5x+10y+25 = 4x+3y+25$

or $x^2+y^2+x+7y+0 = 0$

Q.6 Show that following points are concyclic

a) $(3,5)$, $(3,-5)$, $(2,4)$, $(2,-4)$

b) $(am_1, a/m_1)$, $(am_2, a/m_2)$, $(am_3, a/m_3)$, ~~and~~ $(am_4, a/m_4)$.

Solⁿ:

a) General Eqⁿ of circle is:

$x^2+y^2+2gx+2fy+c=0 \quad \ldots (i)$

~~eqⁿ(i) lies o~~ Here, $(3,5)$, $(3,-5)$, $(2,4)$, $(2,-4)$ lies on the circle.

$9+25+6g+10f+c=0 \quad \ldots (i)$

$9+25+6g-10f+c=0 \quad \ldots (ii)$

$4+16+4g+8f+c=0 \quad \ldots (iii)$

$4+16+4g-8f+c= \quad \ldots (iv)$

by (1) - (2), $\Rightarrow$ $f = 0$

$20f = 0$

by (1) + (2),

$68 + 12g + 2c = 0$

$\Rightarrow 6g + c = -34 \quad \cdots (v)$

by (3) + (4),

$40 + 8g + 2c = 0$

$\Rightarrow 4g + c = -20 \quad \cdots (vi)$

by (5) - (6),

$2g = -14$

$\Rightarrow g = -7$

or,

$20 + 4g - 8f + c = 0$

$\Rightarrow 20 - 28 + c = 0$

$\therefore c = 0$

Radius of circle $= \sqrt{g^2 + f^2 - c}$

$= \sqrt{49 + 0 - 8}$

$= \sqrt{41}$

$\therefore$ Eqn of circle:

$(x+7)^2 + y^2 = 41$

All the points lie on the circle. Hence the points are concyclic.

3.8 Determine the centres and radius of the circles $x^2+y^2-2x+2y-7=0$, $x^2+y^2-6x-2y-6=0$ and

$x^2+y^2-8x-4y-5=0$ and show that their centres are collinear.

Soln: We know that,

General eqn of circle is:

$$(x-h)^2+(y-k)^2=a^2$$

where (h,k) is the centre and a is radius

Given,

$$x^2+y^2-2x+2y-7=0 \quad \cdots (i)$$

$$x^2+y^2-6x-2y-6=0 \quad \cdots (ii)$$

$$x^2+y^2-8x-4y-5=0 \quad \cdots (iii)$$

~~for eqn(i)~~ By rewritting eqn(i), we have:

$$(x-1)^2-1+(y+1)^2-1=7$$

or, $(x-1)^2+(y+1)^2=(3)^2$

$\therefore$ centre $(h,k)=(1,-1)$ and radius, $a=3$

again, rewritting eqn(ii), we have:

$$(x^2-6x)+(y^2-2y)=6$$

or, $(x-3)^2-9+(y-1)^2-1=6$

or, $(x-3)^2+(y-1)^2=4^2$

here, centre $(h,k) = (3,1)$ and radius, $a=4$

and, Rewritting the eq^n(iii) we have:

$$(x^2-8x)+(y^2-4y)=5$$

or, $(x-4)^2-16+(y-2)^2-4=5$

or, $(x-4)^2+(y-2)^2=5^2$

here, centre $(h,k) = \cancel{(3,1)}$ $(4,2)$ and radius, $a=5$

for eq^n(i),

centre $= (1,-1)$

for eq^n(ii), centre $= (3,1)$

for eq^n(iii), centre $= (4,2)$

Slope between $(1,-1)$ and $(3,1)$:

$$\text{slope}_1 = \frac{1-(-1)}{3-1} = 1$$

slope between $(3,1)$ and $(4,2)$:

$$\text{slope}_2 = \frac{2-1}{4-3} = 1$$

Since, slope1 = slope2, these points are collinear

page-116

Q.17

[illegible]

[illegible]

Page-116

Q.10

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(01.2)

Vector Analysis

Lecture-4
31.10.2024

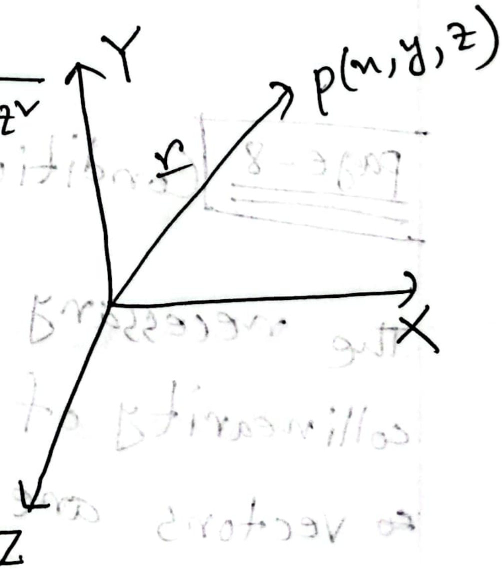
Define: Vectors, Positive vectors, Unit Vectors, null vectors.

Quantity

scalar quantity — only magnitude

vector quantity — both magnitude and direction

Position Vectors: Any vector of the form $\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is said to be position vector, Radius vector of a point $p(x, y, z)$ in the scope.



It's magnitude $r = |\underline{r}| = \sqrt{x^2+y^2+z^2}$

$\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

$\underline{r} = (x_1-x_2)\hat{i} + (y_1-y_2)\hat{j}$

Unit Vector: For any non empty vector $\vec{A}$, there ~~are~~ is a vector which is denoted by $\hat{a}$ and defined by $\hat{a} = \frac{\vec{A}}{|\vec{A}|}$

here direction of unit vector (^) is in the direction of $\vec{A}$ or is along to the $\vec{A}$.

null vector: শূন্য vector

sign → •

initial → terminal

can be defined:

$\boxed{\vec{AB}}$, $\boxed{\vec{A}}$, $\boxed{\mathbf{A}}$, $\boxed{\underline{a}}$, $\boxed{\hat{i}}$, $\boxed{\underline{r}}$, $\boxed{\vec{r}}$

$\boxed{\text{page-8}}$ **Condition for collinearity of vector:**

The necessary and sufficient condition for the collinearity of three points A, B, C whose position ~~co~~ vectors are, $\underline{a}, \underline{b}, \underline{c}$ is that for any three scalar x, y, z then $x\underline{a} + y\underline{b} + z\underline{c} = 0$ and

$x+y+z=0$

$\boxed{\text{Note:}}$ here, x, y, z all are not zero

$\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$A \longrightarrow \underline{a} = x\hat{i} + y\hat{j} + z\hat{k}$

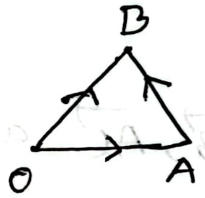
page-12 **Example-4:** Show that $\underline{a} - 2\underline{b} + 3\underline{c}$, $2\underline{a} + 3\underline{b} - 4\underline{c}$, $-7\underline{b} + 10\underline{c}$ are collinear where $\underline{a}, \underline{b}, \underline{c}$ are none coplanar.

Sol^n: Let,

$\vec{OA} = \underline{a} - 2\underline{b} + 3\underline{c}$

$\vec{OB} = 2\underline{a} + 3\underline{b} - 4\underline{c}$

$\vec{OC} = -7\underline{b} + 10\underline{c}$



$\vec{OA} + \vec{AB} = \vec{OB}$

$\vec{AB} = \vec{OB} - \vec{OA}$

Now,

$\vec{AB} = \vec{OB} - \vec{OA}$

$= a + 5b - 7c$

$\vec{BC} = \vec{OC} - \vec{OB}$

$= -2a - 10b - 14c$

$= -2(a + 5b - 7c) = -2\vec{AB}$

$\vec{AC} = \vec{OC} - \vec{OA}$

$= -a - (5b + 7c)$

$= -(a + 5b - 7c)$

$\therefore \vec{AC} = -\vec{AB}$

$\therefore$ These three vectors are collinear.

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Example: 5 | Show that the four points $-\underline{a}+4\underline{b}-3\underline{c}$, $3\underline{a}+2\underline{b}-5\underline{c}$; $3\underline{a}+8\underline{b}-5\underline{c}$, $-3\underline{a}+2\underline{b}+\underline{c}$ are coplanar (সমতলীয়).

proof: $\vec{OA} = -\underline{a}+4\underline{b}-3\underline{c}$

$\vec{OB} = 3\underline{a}+2\underline{b}-5\underline{c}$

$\vec{OC} = -3\underline{a}+8\underline{b}-5\underline{c}$

$\vec{OD} = -3\underline{a}+2\underline{b}+\underline{c}$

$\vec{AB} = \vec{OB}-\vec{OA} = 4\underline{a}-2\underline{b}-2\underline{c}$

$\vec{AC} = \vec{OC}-\vec{OA} = -2\underline{a}+4\underline{b}-2\underline{c}$

$\vec{AD} = \vec{OD}-\vec{OA} = -2\underline{a}-2\underline{b}+4\underline{c}$

Here, $\vec{AB}$, $\vec{AC}$ and $\vec{AD}$ will be collinear if

$l\vec{AB}+m\vec{AC} = \vec{AD}$ --- (i) where l, m are scalar

or, $l(4\underline{a}-2\underline{b}-2\underline{c}) + m(-2\underline{a}+4\underline{b}-2\underline{c}) = -2\underline{a}-2\underline{b}+4\underline{c}$

or, $(4l-2m)\underline{a} + (-2l+4m)\underline{b} + (2l-2m)\underline{c} = -2\underline{a}-2\underline{b}+4\underline{c}$

Equating both sides,

$4l-2m = -2$ --- (ii)

$-2l+4m = -2$ --- (iii)

$-2l-2m = 4$ --- (iv)

Solving (ii) and (iii) we get $l = -1$

$m = -1$

---

page-16

Q.10 If $\underline{a}_1 = 2\hat{i} - \hat{j} + \hat{k}$, $\underline{a}_2 = \hat{i} + 3\hat{j} - 2\hat{k}$, $\underline{a}_3 = -2\hat{i} + \hat{j} - 13\hat{k}$ and $\underline{a}_4 = 3\hat{i} + 2\hat{j} + 5\hat{k}$;

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find ~~the~~ the scalars x, y and z such that

$\underline{a}_4 = x\underline{a}_1 + y\underline{a}_2 + z\underline{a}_3$

Soln:

Given that,

vectors are,

$\underline{a}_1 = 2\hat{i} - \hat{j} + \hat{k}$ $\qquad$ $\underline{a}_3 = -2\hat{i} + \hat{j} - 3\hat{k}$

$\underline{a}_2 = \hat{i} + 3\hat{j} - 2\hat{k}$ $\qquad$ $\underline{a}_4 = 3\hat{i} + 2\hat{j} + 5\hat{k}$

and, $\underline{a}_4 = x\underline{a}_1 + y\underline{a}_2 + z\underline{a}_3$

$$3\hat{i} + 2\hat{j} + 5\hat{k} = x(2\hat{i} - \hat{j} + \hat{k}) + y(\hat{i} + 3\hat{j} - 2\hat{k}) + z(-2\hat{i} + \hat{j} - 3\hat{k})$$

or, $$3\hat{i} + 2\hat{j} + 5\hat{k} = (2x + y - 2z)\hat{i} + (-x + 3y + z)\hat{j} + (x - 2y - 3z)\hat{k}$$

Equatting the like terms,

$2x + y - 2z = 3 \quad \cdots (i)$

$-x + 3y + z = 2 \quad \cdots (ii)$

$x - 2y - 3z = 5 \quad \cdots (iii)$

এখানে সহগ বসবে

Dy মানে y column বরাবর সহগ replace হবে

let,

$$D = \begin{vmatrix} 2 & 1 & -2 \\ -1 & 3 & 1 \\ 1 & -2 & -3 \end{vmatrix} = -14 \qquad D_y = \begin{vmatrix} 2 & 3 & -2 \\ -1 & 2 & 1 \\ 1 & 5 & -3 \end{vmatrix} = -14$$

$$D_x = \begin{vmatrix} 3 & 1 & -2 \\ 2 & 3 & 1 \\ 5 & -2 & -3 \end{vmatrix} = 28 \qquad D_z = \begin{vmatrix} 2 & 1 & 3 \\ -1 & 3 & 2 \\ 1 & -2 & 5 \end{vmatrix} = 42$$

Apply Cramer's Rule

$$x = \frac{D_x}{D} = \frac{28}{-14} = -2 \qquad z = \frac{D_z}{D} = \frac{42}{-14} = -3$$

$$y = \frac{D_y}{D} = \frac{-14}{-14} = 1$$

$$\therefore \underline{a}_4 = -2\underline{a}_1 + \underline{a}_2 - 3\underline{a}_3$$

H.W. (page-15 (1, 2, 3)) (page-16 (8, 9)) } Exercise - I (vector) page-15 Q.1 If $a = 3i - j - 4k$, $b = -2i + 4j - 3k$, $c = i + 2j - k$ find the value of (i) $a+b+c$, (ii) $2a - b + 3c$,

(iii) $|3a - 2b + 4c|$

Solⁿ: (i)
Given,

$a = 3i - j - 4k$

$b = -2i + 4j - 3k$

$c = i + 2j - k$

$\therefore a+b+c = (3i - j - 4k) + (-2i + 4j - 3k) + (i + 2j - k)$

$= 2i + 5j - 8k$

$\therefore a+b+c = 2i + 5j - 8k$ (Ans)

Solⁿ: (ii)

Given,

$a = 3i - j - 4k$

$b = -2i + 4j - 3k$

$c = i + 2j - k$

Now,

$2a = 6i - 2j - 8k$

$-b = 2i - 4j + 3k$

$3c = 3i + 6j - 3k$

$2a - b + 3c = (6i - 2j - 8k) + (2i - 4j + 3k) + (3i + 6j - 3k)$

$= 11i - 8k$

$\therefore 2a - b + 3c = 11i - 8k$ (Ans)

Soln: (iii) Given,

$$a = 3i - j - 4K$$
$$b = -2i + 4j - 3K$$
$$c = i + 2j - K$$

Now,

$$3a = 9i - 3j - 12K$$
$$-2b = 4i - 8j + 6K$$
$$4c = 4i + 8j - 4K$$

Now,
$$3a - 2b + 4c = (9i - 3j - 12K) + (4i - 8j + 6k) + (4i + 8j - 4k)$$
$$= 17i - 3j - 10K \quad \cancel{\text{Ans}}$$

Finding the magnitude:

$$|3a - 2b + 4c| = \sqrt{(17)^2 + (-3)^2 + (-10)^2}$$
$$= \sqrt{398}$$

Q.2 Find the unit vector parallel to $3a - 2b + 4c$ where $a = 3i - j - 4K$, $b = -2i + 4j - 3K$, $c = i + 2j - K$.

Soln: Given,

$$a = 3i - j - 4K$$
$$b = -2i + 4j - 3K$$
$$c = i + 2j - K$$

~~Now, 3a =~~

Now,
$$3a - 2b + 4c = 3(3i - j - 4k) - 2(-2i + 4j - 3k) + 4(i + 2j - k)$$

$= 7i - 3j - 12k + 4i - 8j + 6k + 4i + 8j - 4k$

$= (7+4+4)i + (-3-8+8)j + (-12+6-4)k$

$= 17i - 3j - 10k$ | $\therefore 3a - 2b + 4c = 17i - 3j - 10k$

The magnitude of $3a - 2b + 4c$ is:

$$|3a - 2b + 4c| = \sqrt{(17)^2 + (-3)^2 + (-10)^2}$$

$$= \sqrt{398}$$

Unit vector of $3a - 2b + 4c = \dfrac{3a - 2b + 4c}{|3a - 2b + 4c|} = \dfrac{17i - 3j - 10k}{\sqrt{398}}$

$\therefore$ Unit vector is $\dfrac{17}{\sqrt{398}}i - \dfrac{3}{\sqrt{398}}j - \dfrac{10}{\sqrt{398}}k$.

Q.8 Show that the three points $-2a + 3b + 5c$, $a + 2b + 3c$, $7a - c$ are collinear. Where a, b, c are three noncoplanar vectors.

Soln: Let, $P = -2a + 3b + 5c$

$Q = a + 2b + 3c$

$R = 7a - c$

Now, ~~Q-P~~

$PQ =$ ~~PQ~~ $= \cancel{(a + 2b + 3c) - (-2a + 3b + 5c)}$

$= \cancel{3a - b - 2c}$

If the three points P, Q, R form an triangle, then area should be zero to be collinear of the three points.

$$\therefore \text{Area} = \frac{1}{2}\begin{vmatrix} -2 & 3 & 5 \\ 1 & 2 & 3 \\ 7 & 0 & -1 \end{vmatrix}$$

$$= 0$$

Since, area $= 0$ the three points are collinear. (showed).

Q.9 Show that the following vectors are coplanar.

i) $-6a+3b+2c$, $3a-2b+4c$, $5a+7b-3c$, $-13a+17b-c$.

ii) $5a+6b+7c$; $7a-8b+9c$, $3a+20b+5c$

Soln: i) Given,

$$v_1 = -6a+3b+2c$$

$$v_2 = 3a-2b+4c$$

$$v_3 = 5a+7b-3c$$

$$v_4 = -13a+17b-c$$

If the scalar tripple product of any three vectors ~~coplanar~~ is zero, the vectors are coplanar.

$$\therefore [v_1, v_2, v_3] = \det\begin{vmatrix} -6 & 3 & 2 \\ 3 & -2 & 4 \\ 5 & 7 & -3 \end{vmatrix}$$

$= 281$

since the determinant is not zero. The vectors are not coplanar.

Soln: ii) Given,

$$v_1 = 5a + 6b + 7c$$

$$v_2 = 7a - 8b + 9c$$

$$v_3 = 3a + 20b + 5c$$

If the three vectors are coplanar, then the triple product of the three vectors will be zero.

$$[v_1, v_2, v_3] = \det\begin{vmatrix} 5 & 6 & 7 \\ 7 & -8 & 9 \\ 3 & 20 & 5 \end{vmatrix}$$

$$= 0$$

since the determinant is not zero. the vectors are coplanar.

Chapter-2 (Scalar and Cross Product)

① $\underline{a} \cdot \underline{b} = ab\cos\theta$

② $\underline{a} \times \underline{b} = ab\sin\theta\,\hat{n} = |\underline{a} \times \underline{b}|\,\hat{n}$

③ Perpendicular vector formed with two vectors a and b $\hat{n} = \dfrac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|}$

④ Angle between two vectors $\underline{a}$ and $\underline{b}$

(i) $\cos\theta = \dfrac{\underline{a} \cdot \underline{b}}{ab}$ (ii) $\sin\theta = \dfrac{|\underline{a} \times \underline{b}|}{ab}$

⑤ Projection of $\underline{a}$ along $\underline{b}$,

$$a\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{b}|}$$

⑥ Projection of $\underline{b}$ along $\underline{a}$,

$$b\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}|}$$

⑦ Scalar tripple product $\underline{a} \cdot (\underline{b} \times \underline{c})$

⑧ ~~Sc~~ Vector tripple product $\underline{a} \times (\underline{b} \times \underline{c})$

⑨ Volume of a parallelopiped, $V = \underline{a} \cdot (\underline{b} \times \underline{c})$

(10) Area of triangle with sides $\underline{a}$ and $\underline{b}$ is : $\boxed{\frac{1}{2}|\underline{a} \times \underline{b}|}$

(11) Area of ~~triangle~~ rectangle with sides $\underline{a}$ and $\underline{b}$ is : $|\underline{a} \times \underline{b}|$

(12) Area of parallelogram with two diagonals d_1 and d_2 is : $\frac{1}{2}|\underline{d_1} \times \underline{d_2}|$

Page-30 Example-5 | Find the unit vector perpendicular to each other vectors $2\hat{i} + \hat{j} + \hat{k}$ and $\hat{i} - \hat{j} + 2\hat{k}$

Soln: Let $\underline{a} = 2\hat{i} + \hat{j} + \hat{k}$, $\underline{b} = \hat{i} - \hat{j} + 2\hat{k}$

$$\therefore \underline{a} \times \underline{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix}$$

$$= \hat{i}(2+1) - \hat{j}(4-1) + \hat{k}(-2-1)$$

$$= 3\hat{i} - 3\hat{j} - 3\hat{k}$$

$$\therefore |\underline{a} \times \underline{b}| = \sqrt{3^2 + (-3)^2 + (-3)^2} = 3\sqrt{3}$$

$$\therefore \text{unit vector, } \hat{n} = \frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|} = \frac{3\hat{i} - 3\hat{j} - 3\hat{k}}{3\sqrt{3}}$$

H.W. Section (pdf)

Q.1 Find the angle between the vectors $\underline{a} = 3\hat{i} + 2\hat{j} + 6\hat{k}$ and $\underline{b} = 2\hat{i} + 4\hat{j} - 4\hat{k}$

Soln: Let, θ be the angle betⁿ a ~~& b~~ and b,

we know that,

$$a = \sqrt{3^2 + 2^2 + 6^2} = \sqrt{49} = 7$$

$$b = \sqrt{4 + 16 + 16} = \sqrt{36} = 6$$

$$\underline{a} \cdot \underline{b} = ab\cos\theta$$

$$\text{or, } \cos\theta = \frac{a \cdot b}{ab}$$

$$= \frac{(3\hat{i} + 2\hat{j} + 6\hat{k}) \cdot (2\hat{i} + 4\hat{j} - 4\hat{k})}{6 \cdot 7}$$

$$= \frac{6 + 8 - 24}{47}$$

$$\therefore \theta = \cos^{-1}\left(\frac{-5}{21}\right)$$

Q.2 Find the scalar product of the vectors $(2, 3, 1)$ and $(3, 1, -2)$ and also find the angle betⁿ them.

Soln: Let, $\underline{a} = 2\hat{i} + 3\hat{j} + \hat{k}$

$\underline{b} = 3\hat{i} + \hat{j} - 2\hat{k}$

$$\therefore \underline{a} \cdot \underline{b} = (2\hat{i} + 3\hat{j} + \hat{k}) \cdot (3\hat{i} + \hat{j} - 2\hat{k})$$

$$= 6 + 3 - 2$$

$$= 7$$

$$a = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$$

$$b = \sqrt{3^2 + 1^2 + 2^2} = \sqrt{14}$$

again, $\underline{a} \cdot \underline{b} = ab\cos\theta$

$$\text{or, } \cos\theta = \frac{a \cdot b}{ab} = \frac{7}{\sqrt{14} \cdot \sqrt{14}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$$

Q.3 Find the cross product of the of the two vectors $\hat{i}+2\hat{j}+3\hat{k}$ and $3\hat{i}-4\hat{j}+2\hat{k}$.

Soln.

Let, $\underline{a} = \hat{i}+2\hat{j}+3\hat{k}$

$\underline{b} = 3\hat{i}-4\hat{j}+2\hat{k}$

Now,

$$\underline{a}\times\underline{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 3 & -4 & 2 \end{vmatrix}$$

$$= i(4+12) - j(2-9) + \hat{k}(-4-6)$$

$$= 16\hat{i} + 7\hat{j} - 10\hat{k}$$

$\therefore\ \underline{a}\times\underline{b} = 16\hat{i}+7\hat{j}-10\hat{k}$ (Ans)

Q.4 Find the sine of the angle between the vectors $\hat{i}+2\hat{j}+3\hat{k}$ and $3\hat{i}-4\hat{j}+2\hat{k}$

Soln.

Let, $\underline{a} = \hat{i}+2\hat{j}+3\hat{k}$

$\underline{b} = 3\hat{i}-4\hat{j}+2\hat{k}$

Now,

$$\underline{a}\times\underline{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 3 & -4 & 2 \end{vmatrix}$$

$$= i(4+12) - j(2-9) + \hat{k}(-4-6)$$

$$= 16\hat{i} + 7\hat{j} - 10\hat{k}$$

We know,

$$|\underline{a}\times\underline{b}| = \sin\theta\, ab$$

or, $$\sin\theta = \frac{|\underline{a}\times\underline{b}|}{ab}$$

$$= \frac{9\sqrt{5}}{\sqrt{14}\cdot\sqrt{29}}$$

$a = \sqrt{1^2+2^2+3^2}$

$= \sqrt{14}$

$b = \sqrt{3^2+(-4)^2+2^2} = \sqrt{29}$

Q.5 (Example-5)

$\therefore \theta = \sin^{-1}\left(\frac{9\sqrt{5}}{\sqrt{29}.\sqrt{14}}\right)$ (Ans)

$|a \times b| = \sqrt{16^2+7^2+10^2}$
$= 9\sqrt{5}$

Q.6 Find the angles which the vector $3\hat{i}-6\hat{j}+2\hat{k}$ makers, with the co-ordinate axes.

Solⁿ: Let,
$a = 3\hat{i}-6\hat{j}+2\hat{k}$ made angles α, β, γ with the positive direction of x, y and respectively,

$|a| = \sqrt{3^2+6^2+2^2} = 7$

$\therefore a.\hat{i} = |a|.\cos\alpha$

or, $(3\hat{i}-6\hat{j}+2\hat{k}).\hat{i} = 7\cos\alpha$

or, $\cos\alpha = 7/3$

$\therefore \cos\alpha = 7/3$

$\therefore \alpha = \cos^{-1}(7/3)$

$a.\hat{j} = |a|.\cos\beta$

$(3\hat{i}-6\hat{j}+2\hat{k}).\hat{j} = 7.\cos\beta$

or, $\cos\beta = -6/7$

$\therefore \beta = \cos^{-1}(-6/7)$

$a.\hat{k} = |a|.\cos\gamma$

$(3\hat{i}-6\hat{j}+2\hat{k}).\hat{k} = 7.\cos\gamma$

or, $2 = 7\cos\gamma$

or, $\cos = 2/7$

$\therefore \gamma = \cos^{-1}(2/7)$

$\therefore$ angles are: $\alpha = \cos^{-1}(7/3)$, $\beta = \cos^{-1}(-6/7)$, $\gamma = \cos^{-1}(2/7)$

~~Q.6~~ Q.7 Find the area of triangle whose vertices are A(1,3,2), B(2,-1,1), C(-1,2,3).

Sol^n: $\vec{AB} = (1,-4,-1)$ $\vec{AC} = (-2,-1,1)$

$\therefore$ Area of the triangle is $\frac{1}{2}|\vec{AB} \times \vec{AC}|$

$$\therefore \text{Area} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -4 & -1 \\ -2 & -1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left|\{\hat{i}(-5) - \hat{j}(-1) + \hat{k}(-1-8)\}\right|$$

$$= \frac{1}{2}\left|\sqrt{(-5)^2 + 1^2 + (-9)^2}\right|$$

$$= \frac{1}{2} \times \sqrt{107}$$

$\therefore$ Area $= \frac{1}{2}\sqrt{107}$ sq unit (Ans)

Q.8 Find the volume of the parallelopiped where the ~~eg~~ edges are represented by $a = 2\hat{i} - 3\hat{j} + 4\hat{k}$, $b = \hat{i} + 2\hat{j} - \hat{k}$ and $c = 3\hat{i} - \hat{j} + 2\hat{k}$

Sol^n: Volume of abc is $\underline{a} \cdot (\underline{b} \times \underline{c})$

$$= (2\hat{i} - 3\hat{j} + 4\hat{k}) \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$$

$$= (2\hat{i} - 3\hat{j} \cdot 4\hat{k}) \cdot (3\hat{i} - 5\hat{j} - 7\hat{k})$$

$$= 6 + 15 - 28$$

$$= -7$$

Volume can not be negative. $\therefore$ Volume $= 7$ cubic unit

(Ans)

Q.9 What is the unit vector perpendicular to each of the vector $U = 2\hat{i} + \hat{j} - \hat{k}$ and $V = -6\hat{i} + 3\hat{j} + 5\hat{i}$. Calculate the sine of the angle between these vectors.

Soln. Perpendicular unit vector of u and and v is

$$= \frac{\underline{u} \times \underline{v}}{|\underline{u} \times \underline{v}|} \quad \therefore \quad \underline{u} \times \underline{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ -6 & 3 & 5 \end{vmatrix}$$

$$= \hat{i}8 - \hat{j}(4) + \hat{k}(12)$$

$$= 8\hat{i} - 4\hat{j} + 12\hat{k}$$

$$\therefore |\underline{u} \times \underline{v}| = \sqrt{8^2 + 4^2 + 12^2}$$

$$= \sqrt{224}$$

$\therefore$ The unit vector is $= \dfrac{8\hat{i} - 4\hat{j} + 12\hat{k}}{\sqrt{224}}$ (Ans)

$\therefore$ We know,

$$\sin\theta = \frac{|\underline{u} \times \underline{v}|}{|\underline{u}| \times |\underline{v}|}$$

$$|\underline{u}| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$$

$$|\underline{v}| = \sqrt{6^2 + 3^2 + 5^2} = \sqrt{70}$$

$$= \frac{\sqrt{224}}{\sqrt{6} \cdot \sqrt{70}}$$

$$\therefore \theta = \cancel{\cos}\sin^{-1}\left(\frac{2\sqrt{30}}{15}\right) \text{ (Ans)}$$

Q.10 Find the unit vectors perpendicular to each of the

(i) $\hat{i} + \hat{j} + 2\hat{k}$, $2\hat{i} + 3\hat{j} + \hat{k}$

(ii) $4\hat{i} - \hat{j} + 3\hat{k}$, $-2\hat{i} + \hat{j} - 2\hat{k}$

(iii) $2\hat{i} + 3\hat{j} - \hat{k}$, $3\hat{i} - 2\hat{j} + \hat{k}$

(i) Let, $\underline{a} = \hat{i} + \hat{j} + 2\hat{k}$, $\underline{b} = 2\hat{i} + 3\hat{j} + \hat{k}$

$$\therefore \underline{a} \times \underline{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix} = |i(-5) - j(-3) + \hat{k}(1)|$$

$$= -5\hat{i} + 3\hat{j} + \hat{k}$$

$$\therefore |\underline{a} \times \underline{b}| = \sqrt{(-5)^2 + 3^2 + 1^2} = \sqrt{35}$$

$$\therefore \text{unit vector} = \frac{-5\hat{i} + 3\hat{j} + \hat{k}}{\sqrt{35}}$$

(ii) Let,

$\underline{a} = 4\hat{i} - \hat{j} + 3\hat{k}$, $\underline{b} = -2\hat{i} + \hat{j} - 2\hat{k}$

$$\therefore \underline{a} \times \underline{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix} = i(-1) - \hat{j}(-2) + \hat{k}(2)$$

$$= -i + 2\hat{j} + 2\hat{k}$$

$$\therefore |\underline{a} \times \underline{b}| = \sqrt{1^2 + 2^2 + 2^2} = 3$$

$$\therefore \text{unit vector} = \frac{-\hat{i} + 2\hat{j} + 2\hat{k}}{3}$$

(iii) Let,

$\underline{a} = 2\hat{i} + 3\hat{j} - \hat{k}$, $\underline{b} = 3\hat{i} - 2\hat{j} + \hat{k}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ 3 & -2 & 1 \end{vmatrix} = \left| i(1) - \hat{j}(5) + \hat{k}(-10) \right|$$

$$= \hat{i} - 5\hat{j} - 20\hat{k}$$

$$\therefore |\underline{a} \times \underline{b}| = \sqrt{1+5^2+10^2} = \sqrt{126}$$

$$\therefore \text{unit vector} = \frac{\hat{i} - 5\hat{j} - 10\hat{k}}{\sqrt{126}}$$

Q.11 Find the vector product of the following pairs of vectors and angle between them.

(i) $4\hat{i} - 5\hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + 3\hat{k}$

(ii) $3\hat{i} + \hat{j} + 2\hat{k}$ and $2\hat{i} + 3\hat{j} - \hat{k}$

Soln: (i)
Let $\underline{a} = 4\hat{i} - 5\hat{j} + \hat{k}$ and $b = \hat{i} + 2\hat{j} + 3\hat{k}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -5 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \hat{i}(-17) - \hat{j}(11) + \hat{k}(13)$$

$$= -17\hat{i} - 11\hat{j} + 13\hat{k}$$

$$\therefore \sin\theta = \frac{|a \times \underline{b}|}{ab} = \frac{\sqrt{(-17)^2 + (-11)^2 + 13^2}}{\sqrt{4^2+(-5)^2+1^2} \cdot \sqrt{1^2+2^2+3^2}}$$

$\therefore \theta = \sin^{-1}\left(\frac{\sqrt{173}}{19}\right)$ (Ans)

(ii) Let,

$\underline{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\underline{b} = 2\hat{i} + 3\hat{j} - k$

$\therefore a \times \underline{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 2 & 3 & -1 \end{vmatrix} = \hat{i}(-7) - \hat{j}(-7) + \hat{k}(7)$

$= -7\hat{i} + 7\hat{j} + 7\hat{k}$

$\therefore |\underline{a} \times \underline{b}| = \sqrt{7^2 + 7^2 + 7^2} = 7\sqrt{3}$

$\therefore \cancel{\cos\theta}\ \sin\theta = \frac{|a \times b|}{ab} = \frac{7\sqrt{3}}{\sqrt{3^2+1^2+2^2} \times \sqrt{2^2+3^2+(-1)^2}}$

$= \frac{\sqrt{3}}{2}$

$\therefore \theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = 60$

$\therefore \theta = 60$ (Ans)

Q.12 ~~Find~~ (i) Show that the vectors $a = 9\hat{i} + \hat{j} - 6\hat{k}$ and $b = 4\hat{i} - 6\hat{j} + 5\hat{k}$ are at right angles to one another, where $\hat{i}, \hat{j}, \hat{k}$ are unit vector along x, y, z axes respectively.

(ii) Show that the two vectors $2\hat{i} + 3\hat{j} - 4\hat{k}$ and $8\hat{i} - 12\hat{j} - 5\hat{k}$ are perpendicular to each other.

(iii) Determine the value of a so that $2\hat{i} + a\hat{j} + 2\hat{k}$ and $4\hat{i} - 2\hat{j} - 2\hat{k}$ are perpendicular.

Sol^n:

i) Let, $\underline{a} = 9\hat{i} + \hat{j} - 6\hat{k}$ and $\underline{b} = 4\hat{i} - 6\hat{j} + 5\hat{k}$

$\therefore \underline{a} \cdot \underline{b} = (9\hat{i} + \hat{j} - 6\hat{k}) \cdot (4\hat{i} - 6\hat{j} + 5\hat{k})$

$= 36 - 6 - 30$

$= 0$

$\therefore$ $\underline{a}$ and $\underline{b}$ are perpendicular to each other.

ii) Let,

$\underline{a} = 2\hat{i} + 3\hat{j} - 4\hat{k}$ and $\underline{b} = 8\hat{i} - 12\hat{j} - 5\hat{k}$

$\therefore \underline{a} \cdot \underline{b} = (2\hat{i} + 3\hat{j} - 4\hat{k}) \cdot (8\hat{i} - 12\hat{j} - 5\hat{k})$

$= 16 - 36 + 20 = 0$

$\therefore$ $\underline{a}$ and $\underline{b}$ are perpendicular to each other

iii) Let, $\underline{a} = 2\hat{i} + a\hat{j} + 2\hat{k}$ and $\underline{b} = 4\hat{i} - 2\hat{j} - 2\hat{k}$

Here, $\underline{a}$ an$\underline{b}$ are perpendicular

$\therefore \underline{a} \cdot \underline{b} = 0$

$8 - 2a - 4 = 0$ $\qquad \therefore a = 2$ (Ans)

$\therefore a = 2$

Q.13 Find the unit vector parallel to xy plane and perpendicular to the unit vector $4\hat{i} - 3\hat{j} + \hat{k}$

Sol^n: Unit vector $4\hat{i} - 3\hat{j} + k$ parallel to xy plane and perpendicular to k that is:

$$K \times (4\hat{i} - 3\hat{j} + \hat{k}) = 4(\hat{i} \times \hat{k}) - 3(\hat{k} \times \hat{j}) + (\hat{k} \times \hat{k})$$

$$= 4\hat{j} + 3\hat{i}$$

$$= 3\hat{i} + 4\hat{j}$$

$$\therefore \text{unit vector} = \frac{3\hat{i} + 4\hat{j}}{5} = \frac{1}{5}(3\hat{i} + 4\hat{j})$$

(Ans)

Q.14

Determine a unit vector perpendicular to the plane of $A = 3\hat{i} - 5\hat{j} + k$ and $B = 2\hat{i} - 4\hat{j} - 7\hat{k}$

perpendicular vector,

$$A \times B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -5 & 1 \\ 2 & -4 & -7 \end{vmatrix} = \hat{i}(39) - \hat{j}(-23) + \hat{k}(22)$$

$$= 39\hat{i} + 23\hat{j} + 22\hat{k}$$

$$\therefore \text{unit vector} = \frac{39\hat{i} + 23\hat{j} + 22\hat{k}}{\sqrt{39^2 + 23^2 + 22^2}}$$

$$= \frac{1}{\sqrt{2534}}(39\hat{i} + 23\hat{j} + 22\hat{k})$$

Q.15 $a = \hat{i} + 2\hat{j} - 3k$, $b = 2\hat{i} + \hat{j} - \hat{k}$, $c = \hat{i} + 3\hat{j} - 2\hat{k}$ find...
(i) $a \cdot (\underline{b} \times \underline{c})$ (ii) $(\underline{a} \times \underline{b}) \times (\underline{b} \times \underline{c})$ and also magnitude

Soln: (i)

$$\underline{b} \times \underline{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 1 & -1 \end{vmatrix} = \hat{i}(-1) - \hat{j}(5) + \hat{k}(-3)$$

$$= \hat{i} - 5\hat{j} - 3\hat{k}$$

$$\underline{a} \cdot |\underline{b} \times \underline{c}| = (\hat{i} + 2\hat{i} - 3\hat{k}) \cdot (\hat{i} - 5\hat{j} - 3\hat{k})$$

$$= 1 - 10 + 9 = 0$$

~~(ii) $\underline{a} = 9\hat{i} + \hat{j} - 6\hat{k}$~~

~~$\underline{b} = 4\hat{i} - 6\hat{j} + 5\hat{k}$ $\therefore$ $\underline{a} \cdot \underline{b} = 36 - 6 - 30 = 0$~~

~~$\therefore$ $\underline{a}$ and $\underline{b}$ perpendicular to each other.~~

ii)

$$\underline{a} \times \underline{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 1 & -1 \end{vmatrix} = \hat{i}(-2+3) - \hat{j}(-1+6) + \hat{k}(-3)$$

$$= \hat{i} - 5\hat{j} - 3\hat{k}$$

$$\underline{b} \times \underline{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & 3 & -2 \end{vmatrix} = \hat{i}(-2+3) - \hat{j}(-4+1) + \hat{k}(5)$$

$$= \hat{i} + 3\hat{j} + 5\hat{k}$$

$$(\underline{a} \times \underline{b}) \times (\underline{b} \times \underline{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -5 & -3 \\ 1 & 3 & 5 \end{vmatrix} = \hat{i}(-25+9) - \hat{j}(5+3) + \hat{k}(5+5$$

$$= -14\hat{i} - 8\hat{j} + 20\hat{k}$$

$\therefore$ magnitude $= \sqrt{14^2+8^2+20^2} = 2\sqrt{165}$

Q.16 Find the area of the parallelogram whose diagonals or $3\hat{i}+\hat{j}-2\hat{k}$ and $\hat{i}-3\hat{j}+4\hat{k}$.

Soln:

$$\text{Area} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix}$$

$$= \frac{1}{2}\left| i(-2) - j(14) + \hat{k}(-10) \right|$$

$$= \frac{1}{2}\sqrt{2^2+14^2+10^2}$$

$$= 5\sqrt{3} \text{ unit square}$$

$\therefore$ Area $= 5\sqrt{3}$ unit square (Ans).

Q.17 If the position vectors of three points $A(2,4,-$ $B(1,2,-3)$ and $C(3,1,2)$ find a vector perpendicular to the plane. ABC

Soln: $\overrightarrow{AB} = (-1,-2,-2)$ $\qquad$ $\overrightarrow{BC} = (2,-1,5)$

$= -\hat{i}-2\hat{j}-2\hat{k}$ $\qquad$ $= 2\hat{i}-\hat{j}+5\hat{k}$

$\therefore$ perpendicular vector $= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -2 & -2 \\ 2 & -1 & 5 \end{vmatrix}$

$= \hat{i}(-10-2) - \hat{j}(-5+4) + \hat{k}(1+4)$

$= -12\hat{i} + \hat{j} + 5\hat{k}$

$\therefore$ Required vector is. $-12\hat{i} + \hat{j} + 5\hat{k}$

Q.18 Show that the points whose position vectors are given by $\hat{i}+2\hat{j}+3\hat{k}$, $2\hat{i}+3\hat{j}+4\hat{k}$, $3\hat{i}+4\hat{j}+5\hat{k}$ are collinear.

Soln: If they are collinear $= \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$

or, $1(15-16) - 2(10-12) + 3(8-9)$

or, $-1+4-3 = 0$

$\therefore$ Those position vectors are collinear. (Ans)

Vector Calculas (chap-4)

Lecture-6

11.11.2024

Vector calculas two tyes:

i) vector differentiation.

ii) vector integration.

Vector Differentiation

Equations:

① Vector differential operator / delta / nabla :-

$$\nabla = \hat{i}\,\partial/\partial x + \hat{j}\,\partial/\partial y + \hat{k}\,\partial/\partial z$$

② For any vector $\underline{r} = r_x\hat{i} + r_y\hat{j} + r_z\hat{k}$

(i) $$\frac{d\underline{r}}{dt} = d/dt\left(r_x\hat{i} + r_y\hat{j} + r_z\hat{k}\right) = \frac{dr_x}{dt}\hat{i} + \frac{dr_y}{dt} + \frac{dr_z}{dt}\hat{k}$$

(ii) $$\int r\,dt = \int (r_x\hat{i} + r_y\hat{j} + r_z\hat{k})\,dt$$
$$= \hat{i}\int r_x\,dt + \hat{j}\int r_y\,dt + \hat{k}\int r_z\,dt$$

③ gradient of a scalar function $\phi = \phi(x_1, y_1, z_1)$

$$\vec{\nabla}\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

④ Divergence of a vector function $\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$

is $$\vec{\nabla}\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

(5) Curl of a vector function $\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$ is

$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix}$$

(6) Directional derivative of a scalar function ϕ at a point (x, y, z) in the direction of $\underline{a}$ is:

$$\vec{\nabla}\phi \cdot \hat{a}$$

Example-3 (page-103) | If $\vec{V} = xy^2\hat{i} - 2xyz\hat{j} + z^2x^2\hat{k}$. Find the curl $\vec{V}$.

Soln: curl $\vec{V} = \vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy^2 & -2xyz & z^2x^2 \end{vmatrix}$

$$= \hat{i}\left[\partial/\partial y(z^2x^2) - \partial/\partial z(-2xyz)\right] - \hat{j}\left[\partial/\partial x(z^2x^2) - \partial/\partial z(xy^2)\right] + \hat{k}\left[\partial/\partial x(-2xyz) - \partial/\partial y(xy^2)\right]$$

$$= \hat{i}(2xy) - \hat{j}(2xz^2) + \hat{k}(-2yz - 2yx)$$

$$= 2xy\hat{i} - 2xz^2\hat{j} + (-2yz - 2yx)\hat{k}$$

Example-6 | Find the directional derivative of $\phi = 4xy - 3x^2z^2$ at $(2,-1,2)$ in the direction $2\hat{i} - 3\hat{j} + 6\hat{k}$

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Soln: Given that, $\phi = 4xy - 3x^2z^2$

$$\vec{\nabla}\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}\,\partial/\partial x\{4xy - 3x^2z^2\} + \hat{j}\,\partial/\partial y\{4xy - 3z^2x^2\}$$

$$+ \hat{k}\,\partial/\partial z\{4xy - 3x^2z^2\}$$

or, $\vec{\nabla}\phi = \hat{i}(4y - 6z^2x) + \hat{j}(4x - 0) + \hat{k}(0 - 6x^2z)$

At $(2,-1,2)$,

$$\vec{\nabla}\phi = -52\hat{j} + 8\hat{j} - 48\hat{k}$$

unit vector of a is $\hat{a} = \frac{a}{|a|} = \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{\sqrt{2^2 + (-3)^2 + 6^2}}$

$$= \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{7}$$

The, direction derivative $= \vec{\nabla}\phi \cdot \hat{a}$

$$= (-52\hat{i} + 8\hat{j} - 48\hat{k}) \cdot \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{7}$$

$$= \frac{-104-24-288}{7} = -\frac{416}{7}$$

$\therefore$ Direction derivative $= -\frac{416}{7}$ (Ans)

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Exercise-11(a)

Find the directional derivative of $p=4e^{2x^2-y+z}$ at the point $(1,1,-1)$ in a direction towards the point $(-3,5,6)$.

$\frac{d}{dx} e^x = e^x$

$\frac{d}{dx}(e^{mx}) = me^{mx}$

$\frac{d}{dx}(e^{mx^2+2x}) = e^{mx^2+2x} \cdot \frac{d}{dx}(mx^2+2x)$

$= e^{mx^2+2x} \cdot (2mx+2)$

Soln: Given that,

$$p = 4e^{2x^2-y+z}$$

$$\therefore \vec{\nabla} \cdot p = \hat{i}\, \partial p/\partial x + \hat{j}\, \frac{\partial p}{\partial y} + \hat{k}\, \frac{\partial p}{\partial z}$$

$$= \hat{i}\, \partial/\partial x\left[4e^{2x^2-y+z}\right] + \hat{j}\, \partial/\partial y\left[4e^{2x^2-y+z}\right] +$$

$$\hat{k}\, \partial/\partial z\left[4e^{2x^2-y+z}\right]$$

$$= \hat{i}\left(4e^{2x^2-y+z} \cdot 4x\right) + \hat{j}\left(4e^{2x^2-y+z} \cdot (-1)\right)$$

$$+ \hat{k}\left(4 \cdot e^{2x^2-y+z}(1)\right)$$

$$= 4e^{2x^2-y+z}\left(4x\hat{i} - \hat{j} + \hat{k}\right)$$

At $(1, 1, -1)$, $\vec{\nabla} p = 4e^{0}(4\hat{i} - \hat{j} + \hat{k})$
$= 4(4\hat{i} - \hat{j} + \hat{k})$

$\hat{a} = -3\hat{i} + 5\hat{j} + 6\hat{k}$

or, unit of $\hat{a} = \dfrac{-3\hat{i} + 5\hat{j} + 6\hat{k}}{\sqrt{(-3)^2 + 5^2 + 6^2}} = \dfrac{-3\hat{i} + 5\hat{j} + 6\hat{k}}{\sqrt{70}}$

$\therefore$ directional ~~dire~~ derivative,

$$\vec{\nabla} p \cdot \hat{a} = 4(4\hat{i} - \hat{j} + \hat{k}) \cdot \frac{(-3\hat{i} + 5\hat{j} + 6\hat{k})}{\sqrt{70}}$$

$$= \frac{4}{\sqrt{70}}(-12 - 5 + 6)$$

$$= \frac{-44}{\sqrt{70}} \text{ (Ans)}$$

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Ex-14

~~Find~~ Show that $\underline{r} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$ is irrotational or not solonoidal.

Soln: $\vec{\nabla} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial y \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix}$

$= \hat{i}\left[\partial/\partial y(3xz^2 - y) - \partial/\partial z(3x^2 - z)\right] - \hat{j}\left[\partial/\partial x(3xz^2 - y)\right.$
$\left. - \partial/\partial z(6xy + z^3)\right] + \hat{k}\left[\partial/\partial x(3x^2 - z)\right.$

$- \partial/\partial y(6xy + z^3)]$

$= \hat{i}(-1+1) - \hat{j}(3z^2 - 3z^2) + \hat{k}(6x - 6x)$

$= 0$

$\therefore \vec{\nabla} \times \vec{r} = 0$ | Here, $\vec{r}$ is irrotational vector.

2nd part:

$\vec{\nabla} \cdot \vec{r} = (\hat{i}\,\partial/\partial x + \hat{j}\,\partial/\partial y + \hat{k}\,\partial/\partial z) \cdot \{(6x + z^3)\hat{i} + (3x^2 - z)\hat{j} + \hat{k}(3xz^2 - y)\}$

$= \partial/\partial x(6xy + z^3) + \partial/\partial y(3x^2 - z) + \partial/\partial z(3xz^2 - y)$

$= 6y + 0 + 6xz \neq 0$

Hence, $\vec{r}$ is not solonoidal

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Q.24 | Determine the constant a so that $\vec{v} = (3x + y)\hat{i} + (2y - z)\hat{j} + (x + az)\hat{k}$ is solenoidal.

Soln: Given that,

$\vec{v} = (3x + y)\hat{i} + (2y - z)\hat{j} + (x + az)\hat{k}$

or, $\partial/\partial x(3x + y) + \partial/\partial y(2y - z) + \partial/\partial z(x + az) = 0$

or, $3 + 2 + a = 0$

$\therefore a = -5$

H.W. Section

page-102 Example-1 | If $\phi = xy + 2yz + 3xz$, find grad of ϕ.

$$\text{grad } \phi = \vec{\nabla}\phi = \vec{\nabla}(xy + 2yz + 3xz)$$

$$= \hat{i}\left[\partial/\partial x\,(xy + 2yz + 3xz)\right] + \hat{j}\left[\partial/\partial y\,(xy + 2yz + 3xz)\right] + K\left[\partial/\partial z\,(xy + 2yz + 3xz)\right]$$

or, $\hat{i}(y + 3z) + \hat{j}(x + 2z) + \hat{k}(2y + 3x)$.

$\therefore \text{grad } \phi = (y + 3z)\hat{i} + (x + 2z)\hat{j} + (2y + 3x)\hat{k}$ (Ans)

page-102 Example-2 | If $\vec{V} = (2x^2z)\hat{i} - (xy^2z)\hat{j} + (3yz^2)\hat{k}$. find $\text{div } \vec{V}$ at $(2, -1, 3)$

Solⁿ: $\text{div } V = \vec{\nabla}\cdot\vec{V} = (\hat{i}\,\partial/\partial x + \hat{j}\,\partial/\partial y + K\,\partial/\partial z) \cdot (2x^2z)\hat{i} - (xy^2z)\hat{j} + (3xz^2)\hat{k}$

$$= \overset{4xz}{\cancel{4xy}} - 2xyz + 6yz$$

$\therefore \text{div } V = 4xz - 2xyz + 6yz$ (Ans) page 103

Example-4 | $\vec{A} = (xy)\hat{i} - 3yz^2\hat{j} + y^2 z\hat{k}$, find the div $\vec{A}$ at the point $C(-1, -2, 1)$

Sol^n: $\text{div}\,\vec{A} = \vec{\nabla}\cdot\vec{A} = \left(\hat{i}\,\partial/\partial x + \partial/\partial y + \partial/\partial z\right)\left\{(xy)\hat{i} - (3yz^2)\hat{j} + (y^2 z)\hat{k}\right\}$

or, $\partial/\partial x(xy) + \partial/\partial y(-3yz^2) + \partial/\partial z(y^2 z)$

or, $y - 3z^2 + y^2$

At $(1, -2, 1)$, $\text{div}\,\vec{A} = 2 - 3\cdot 1 + 1 = -4$

$\therefore$ $\text{div}\,\vec{A}$ at $(1, -2, 1) = -4$ (Ans)

Exercise:

Q.11(i): Find the directional derivative of $\phi = x^2yz^2 + 4y^2z$ at $(-1, -2, -1)$ in the direction $5\hat{i} - 2\hat{j} - 3\hat{k}$.

Sol^n: Given that,

$$\phi = x^2yz^2 + 4y^2z$$

$$\vec{\nabla}\phi = \hat{i}\,\partial\phi/\partial x + \hat{j}\,\partial\phi/\partial y + \hat{k}\,\partial\phi/\partial z$$

$$= \hat{i}\,\partial/\partial x(x^2yz^2 + 4y^2z) + \hat{j}\,\partial/\partial y(x^2yz^2 + 4y^2z) + \hat{k}\,\partial/\partial z(x^2yz^2 + 4y^2z)$$

$$= \hat{i}(2xyz^2) + \hat{j}(x^2z^2 + 8yz) + \hat{k}(2x^2yz + 4y^2z)$$

At point $(-1, -2, -1)$

$$\vec{\nabla}\phi = \hat{i}\{2.(-1).(-2).(-1)^2\} + \hat{j}\{(-1)^2(-1)^2 + 8.(-2)(-1)\}$$
$$+ \hat{k}\{2(-1)^2(-2)(-1) + 4.(-2)^2\}$$

$$= 4\hat{i} + 17\hat{j} + 20\hat{k}$$

Now,

unit vector of $\underline{a} = \frac{a}{|\underline{a}|} = \frac{5\hat{i} - 2\hat{j} - 3\hat{k}}{\sqrt{5^2 + (-2)^2 + (-3)^2}}$

$$= \frac{5\hat{i} - 2\hat{j} - 3\hat{k}}{\sqrt{38}}$$

The directional derivative,

$$\vec{\nabla}\phi . \hat{a} = (4\hat{i} + 17\hat{j} + 20\hat{k}) . \frac{5\hat{i} - 2\hat{j} - 3\hat{k}}{\sqrt{38}}$$

$$= \frac{20 - 34 - 60}{\sqrt{38}} = \frac{-74}{\sqrt{38}}$$

$\therefore$ directional derivate $\vec{\nabla}\phi . \hat{a} = \frac{-74}{\sqrt{38}}$ (Ans)

11(ii) If $a = (x+y+1)\hat{i} + \hat{j} + (-x-y)\hat{k}$, show that, $\hat{a} . \text{curl } \hat{a} = 0$

Soln: The curl of $\hat{a}$ is:

$$\vec{\nabla} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x+y+1) & \cancel{(x-y)}\ 1 & -x-y \end{vmatrix}$$

$$= \hat{i}\left[\partial/\partial y(-x-y) - \partial/\partial z(1)\right] - \hat{j}\left[\partial/\partial x(-x-y) - \partial/\partial z(x+y+1)\right] + \hat{k}\left[\partial/\partial x(1) - \partial/\partial y(x+y+1)\right]$$

$$= -\hat{i} + \hat{j} - \hat{k}$$

Now, $\hat{a} . (\vec{\nabla} \times \vec{a}) = \left[(x+y+1)\hat{i} + \hat{j} + (x-y)\hat{k}\right] . (-\hat{i} + \hat{j} - \hat{k})$

$$= -(x+y+1) + 1 - (x-y)$$

$$= -x - y - 1 + 1 + x + y$$

$$= 0$$

$\therefore \vec{a} . \text{curl } \vec{a} = 0$ (showed)

Q.12 Find the divergence and curl of the vector field

a) $x^2 z\hat{i} - 2yz^3\hat{j} + xy^2 z\hat{k}$ at the point $(1, 2, -1)$ and $(1, -1, -1)$.

b) If $V = (x^2 y, zx, yz)$

Soln: a) Given,

$$\vec{F} = x^2 z\hat{i} - 2yz^3\hat{j} + xy^2 z\hat{k}$$

Now,
divergence of $\vec{F}$ is:

$$\vec{\nabla}.\vec{F} = \left[\partial/\partial x(x^2 z)\right]\hat{i} + \left[\partial/\partial y(-2yz^3)\right]\hat{j} + \left[\partial/\partial z(xy^2 z)\right]\hat{k}$$

$$= 2xz\hat{i} - 2z^3\hat{j} + xy^2\hat{k}$$

$$\therefore \vec{\nabla}.\vec{F} = 2xz\hat{i} - 2z^3\hat{j} + xy^2\hat{k}$$

At $(1, 2, -1)$,

$$\vec{\nabla}.\vec{F} = 2(1)(-1) - 2(-1)^3 + (1)(2^2)$$

$$= -2 + 2 + 4 = 4 \text{ (Ans)}$$

At $(1, -1, -1)$,

$$\vec{\nabla}.\vec{F} = 2(1)(-1) - 2(-1)^3 + (1)(-1^2)$$

$$= -2 + 2 + 1 = 1 \text{ (Ans)}$$

Curl of $\vec{F}$,

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 z & -2yz^3 & xy^2 z \end{vmatrix}$$

$$= \hat{i}\left[\partial/\partial y(xy^2z) - \partial/\partial z(-2yz^3)\right] - \hat{j}\left[\partial/\partial x(xy^2z) - \partial/\partial z(x^2z)\right]$$
$$+ \hat{k}\left[\partial/\partial x(-2yz^3) - \partial/\partial y(x^2z)\right]$$

$$= \hat{i}(2xyz + 6yz^2) - \hat{j}(y^2z - x^2) + \hat{k}(0-0)$$

$$\therefore \vec{\nabla}\times\vec{F} = (2xyz+6yz^2)\hat{i} - (y^2z - x^2)\hat{j}$$

Now,

At $(1, 2, -1)$,

$$\vec{\nabla}\times\vec{F} = \{2(1)(2)(-1) + 6(2)(-1)^2\}\hat{i} - \{(2^2)(-1) - 1^2\}\hat{j}$$

$$= 8\hat{i} + 5\hat{j} \text{ (Ans)}$$

Again, At $(1, -1, -1)$,

$$\vec{\nabla}\times\vec{F} = \{2(1)(-1)(-1) + 6(-1)(-1)^2\}\hat{i} - \{(-1)^2(-1) - 1^2\}\hat{j}$$

$$= 8\hat{i} + 2\hat{j} \text{ (Ans)}$$

(b) Given,

$$V = (x^2y,\ zx,\ yz)$$

$$\therefore \vec{V} = x^2y\hat{i} + zx\hat{j} + yz\hat{k}$$

divergence of $\vec{V}$

$$\vec{\nabla}\cdot\vec{V} = \{\partial/\partial x(x^2y)\} + \{\partial/\partial y(zx)\} + \{\partial/\partial z(yz)\}$$

$$= 2xy + y$$

$$\therefore \vec{\nabla}\cdot\vec{V} = 2xy + y$$

Curl of $\vec{V}$ is:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 y & zx & yz \end{vmatrix}$$

$$= \hat{i}[z - x] - \hat{j}[0 - 0] + \hat{k}[z - x^2]$$

$$= (z - x)\hat{i} + (z - x^2)\hat{k} \text{ (Ans)}$$

Q.20 Show that $F = (2x^2 + 8xy^2 z,\ 3x^2 y - 3xy,\ -4y^2 z^2 - 2x^3 z)$ is not solenodial but $v = xyz^2$, F is solenoidal.

Soln: Given,

$$F = 2x^2 + 8xy^2 z,\ 3x^2 y - 3xy,\ -4y^2 z^2 - 2x^3 z$$

$$\therefore \vec{F} = (2x^2 + 8xy^2 z)\hat{i} + (3x^2 y - 3xy)\hat{j} + (-4y^2 z^2 - 2x^3 z)\hat{k}$$

if $\vec{F}$ is solenoidal its divergence $= 0$.

$$\therefore \vec{\nabla} \cdot \vec{F} = \partial/\partial x(2x^2 + 8xy^2 z) + \partial/\partial y(3x^2 y - 3xy) + \partial/\partial z(-4y^2 z^2 - 2x^3 z)$$

$$= 4x + 8y^2 z + 3x^2 - 3x + (-8y^2 z - 2x^3)$$

$$= 4x + 3x^2 - 3x + 8y^2 z - 8y^2 z - 2x^3$$

$$= 3x^2 - 2x^3 + x$$

$\therefore$ since $\vec{\nabla} \cdot \vec{F} \neq 0$, $\vec{F}$ vector is not solenoidal.

Q.26 Given $v = (x+3y,\ y-3z,\ x-2z)$, determine whether or not v is solenoidal.

Soln. Given, $v = (x+3y,\ y-3z,\ x-2z)$

$$\therefore \vec{v} = (x+3y)\hat{i} + (y-3z)\hat{j} + (x-2z)\hat{k}$$

if vector field $\vec{v}$ is solenoidal then divergence must be zero.

$\therefore$ divergence of $\vec{v}$:

$$\vec{\nabla}\cdot\vec{v} = \partial/\partial x(x+3y) + \partial/\partial y(y-3z) + \partial/\partial z(x-2z)$$

$$= \cancel{3y}\ 1 + (-1) - 2$$

$$= 0$$

Lecture-7

14.11.2024

Vector Integration

(1) Vector Integration of $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$

$$\int \vec{v}\,dt = \int (v_x\hat{i} + v_y\hat{j} + v_z\hat{k})\,dt$$

$$= \hat{i}\int v_x dt + \hat{j}\int v_y dt + \hat{k}\int v_z dt$$

(2) The most important vector integration are:

i) line integral $= \int_c \vec{F}\cdot d\vec{r} = \int_c (F_x\hat{i} + F_y\hat{j} + F_z\hat{k})\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$

$$= \int_c (F_x dx + F_y dy + F_z dz)$$

ii) Surface integral $= \int_s \vec{F} \cdot d\vec{s}$

$$= \int_s (F_x \hat{i} + F_y \hat{j} + F_z \hat{k}) \cdot (ds_x \hat{i} + ds_y \hat{j} + ds_z \hat{k})$$

$$= \int_s (F_x ds_x + F_y ds_y + F_z ds_z)$$

iii) Volume integral $= \int_v \vec{F} \cdot d\vec{v}$

$$= \int_v (F_x \hat{i} + F_y \hat{j} + F_z \hat{k}) \cdot (dv_x \hat{i} + dv_y \hat{j} + dv_z \hat{k})$$

$$= \int_v (F_x dv_x + F_y dv_y + F_z dv_z)$$

Lecture-10

25.11.2024

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Example-2

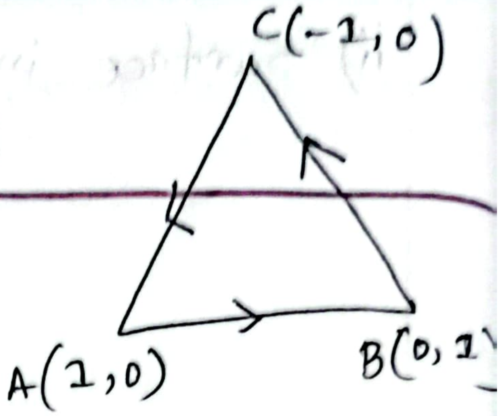
Find the integrals $\oint (y^2 dx - x^2 dy)$ about the triangle whose vertices are $(0,2), (0,1), (-1,0)$

Let,

$$I_{ABC} = \oint_{ABC} (y^2 dx - x^2 dy)$$

$$= \int_{AB} (y^2 dx - x^2 dy) + \int_{BC} (y^2 dx - x^2 dy) + \int_{CA} (y^2 dx - x^2 dy)$$

$\therefore I_{ABC} = I_{AB} + I_{BC} + I_{CA} \cdots (i)$



Now,

Eqn of AB:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$$

or, $\frac{x - 1}{0 - 1} = \frac{y - 0}{1 - 0}$

or, $-y = x - 1 \quad \therefore y = 1 - x \cdots (ii)$

Eqn of BC:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$$

or, $\frac{x - 0}{-1 - 0} = \frac{y - 1}{0 - 1}$

or, $-y + 1 = -x \quad \therefore y = 1 + x \cdots (iii)$

Eqn of CA:

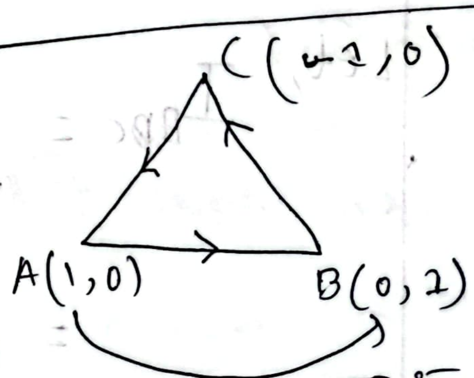
$y = 0 \cdots (iv)$

Along AB:

$y = 1 - x$

or, $\frac{dy}{dx} = \frac{d}{dx}(1 - x)$

$\therefore dy = -dx$



x এর সাপেক্ষে
তাই x এর দূরত্ব নির্ণয় হবে

$\therefore I_{AB} = \int (y^2 dx - x^2 dx) = \int_1^0 (1-x)^2 dx + x^2 dx$

$= \int_1^0 (1 - 2x + 2x^2)\, dx$

$= \left[x - \frac{2x^2}{2} + 2 - \frac{x^3}{3}\right]_1^0$

$= 0 - (1 - 1 + \frac{2}{3})$

$= -\frac{2}{3}$

Along BC:

$y = 1 + x$

or, $\frac{dy}{dx} = dx(1+x)$

$\therefore dy = dx$

$\therefore I_{BC} = \int_{BC} (y^2 dx - x^2 dy)$

$= \int_0^{-1} (1+x^2) dx - x^2 dx$

$= \int_0^{-1} (1 + 2x)\, dx$

$= \left[x + \frac{2x^2}{2}\right]_0^{-1} = -1 + 1$

$= 0$

Along CA:

$y = 0$

or, $dy = 0$

$\therefore I_{CA} = \int_{CA} (y^2 dx - x^2 dy)$

$= 0$

From eqn(i),

$I_{ABC} = I_{AB} + I_{BC} + I_{CA}$

$= -\frac{2}{3} + 0 + 0$

$= -\frac{2}{3}$ (Ans)

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ Lecture-8

$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$ 18.11.2024

work done (কৃতকাজ), $\omega = \int_a^b \vec{F} \cdot d\vec{r} = \int_a^b (F_x dx + F_y dy + F_z dz)$

$dx\hat{i} + dy\hat{j} + dz\hat{k}$

A vector $\vec{F}$ is said to irrotational if (i) $\vec{\nabla} \times \vec{F} = \underline{0}$, or $\vec{F} = \vec{\nabla}\phi$

or, $\oint \vec{F} \cdot d\vec{r} = 0$ for any closed curve

If $\vec{F}$ is conservative, then $\vec{\nabla} \times \vec{F} = 0$ that is $\vec{F}$ is irrotational.

Q. State the Green's theorem with example

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Statement: Let R be closed region bounded by a closed regular curve C whose boundary is ~~at~~ cut at most in two points by parallel to axes. Then if $M(x,y)$, $N(x,y)$, $\frac{\partial M}{\partial y}$, $\frac{\partial N}{\partial x}$ are continuous in R then,

$$\oint_C (M\,dx + N\,dy) = \oint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

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Here,
$$\vec{n} = \frac{d\vec{s}}{ds}$$
$$\therefore \vec{n}\,ds = d\vec{s}$$

Divergence / Gauss's theorem:

The surface integral of the normal component of a vector $\vec{F}$ taken over a closed surface is equal to the volume integral of the divergance of $\vec{F}$ taken throughout enclosed surface.

Mathematically,
$$\int_s \vec{F}\cdot d\vec{s} = \int_v \vec{\nabla}\cdot\vec{F}\,dv$$
$$\int_n \vec{F}\cdot\vec{n}\cdot ds = \int_v \vec{\nabla}\cdot\vec{F}\,dv$$

Stoke's theorem:

The tangential line integral of a vector function around any closed curve is equal to the normal surface integral of the curl of that function over any closed surface

Mathematically, we can express,
$$\int_c \vec{F}\cdot d\vec{r} = \int (\vec{\nabla}\times\vec{F})\cdot d\vec{s}$$
$$= \int (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,ds$$
$$\vec{n} = \frac{d\vec{s}}{ds}$$
or, $\vec{n}\,ds = d\vec{s}$

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Exampl-1

Find the area bounded by a simple closed curve C is given $\oint_c (xdy - ydx)$

Soln. Given close curve, $\oint_c (xdy - ydx)$

comparing with $\oint_c Mdx + Ndy$ we get,

$$M = -y, \quad N = x$$

$$\therefore \frac{\partial M}{\partial y} = -1, \quad \frac{\partial N}{\partial x} = 1$$

According to Green's theorem,

$$\oint Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$\text{or, } \int_c -ydx + xdy = 2\int_R dxdy$$

$$\text{or, } \int_c -ydx + xdy = 2A, \left[\text{where } A = \int_R dxdy \right.$$

$$\therefore A = \frac{1}{2}\int_c -ydx + xdy \text{ (Ans)}$$

Note: অঙ্ক দেখে বুঝে নেওয়া লাগবে কোন theorem দিয়ে solve করতে হবে।

page-250 | Example-11

Find the integral $\iint_S \phi \vec{n}\, ds$ if $\vec{F} = (x+2y)\hat{i} - 3z\hat{j} + x\hat{k}$, $\phi = 4x + 3y - 2z$ and S is the surface of the $2x + y + 2z = 0$ bounded by $x = 0$, $x = 1$, $y = 0$, $y = 2$

Soln: Integral $I = \iint_S \phi \vec{n}\, ds \quad \cdots (i)$

$$\vec{F} = (x+2y)\hat{i} - 3z\hat{j} + x\hat{k} \quad \cdots (ii)$$

$$\phi = 4x + 3y - 2z \quad \cdots (iii)$$

Eqn of surface $2x + y + 2z = 0 \quad \cdots (iv)$

$$\therefore \vec{\nabla}(2x + y + 2z) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x + y + 2z)$$

$$= 2\hat{i} + \hat{j} + 2\hat{k}$$

$$\therefore \vec{n} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{3}$$

[$\vec{n}$ unit normal vector] to (iv)

Now, $\vec{n} \cdot \hat{k} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{3} \cdot \hat{k} = \frac{2}{3}$

(i) We know, $$\iint_S \phi \vec{n}\, ds = \iint_R \phi \vec{n} \cdot \frac{dx\,dy}{|\vec{n} \cdot \hat{k}|}$$

$$= \int_{x=0}^{1}\int_{y=0}^{2} (4x+3y-2z)\left(\frac{2\hat{i}+\hat{j}+3\hat{k}}{3}\right)\frac{3}{2}\,dx\,dy$$

$$= \frac{1}{2}(2\hat{i}+\hat{j}+2\hat{k})\int_{x=0}^{1}\int_{y=0}^{2}(4x+3y-2z)\,dx\,dy$$

$$= \frac{1}{2}(2\hat{i}+\hat{j}+2\hat{k})\int_{x=0}^{1}\left[\frac{3y^2}{2}\right]_0^2 dx$$

$$= \frac{1}{2}(2\hat{i}+\hat{j}+2\hat{k})\int_{x=0}^{1}(6-0)\,dx$$

$$= \frac{1}{\cancel{2}}(2\hat{i}+\hat{j}+2\hat{k})\times \cancel{6}^{3}\times[x]_0^1$$

$$= (6\hat{i}+3\hat{j}+6\hat{k}) \text{ (Ans)}$$

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Example-10) Verify stoke's theorem for the vector $\vec{F} = 2y\hat{i} + 3x\hat{j} - z^2\hat{k}$ where S is the upper half surface of the sphere $x^2+y^2+z^2=9$ and C is its boundary.

Soln: We know,

the stoke's theorem is :

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,ds \quad \ldots (i)$$

Given that,

$$\vec{F} = 2y\hat{i} + 3x\hat{j} - z^2\hat{k}$$

---

surface of sphere : $x^2 + y^2 + z^2 - 9 = 0$

L.H.S.

$$= \oint_C \vec{F} \cdot d\vec{r}$$

$$= \oint_C (2y\hat{i} + 3x\hat{j} - z^2\hat{k}) \cdot (dx\hat{i} \cancel{+ dy\hat{i}} + dy\hat{j} + dz\hat{k})$$

$$= \oint_C (2y\,dx + 3x\,dy - z^2\,dz)$$

~~x =~~ In xy plane

$x^2 + y^2 = 9$, $\underline{z = 0}$

means xy plane

$$= \oint_0^{2\pi} \left(6\sin\theta \cdot (-3\sin\theta\, d\theta) + 3.3\cos\theta \cdot 3\cos\theta\, d\theta\right)$$

$\therefore x = 3\cos\theta$

$y = 3\sin\theta$

where $0 < \theta < 2\pi$

$$= \int_0^{2\pi} (-18\sin^2\theta + 27\cos^2\theta)\, d\theta$$

$x = a\cos\theta$

$y = b\sin\theta$

a ও b একই হলে বৃত্ত

$$= \int_0^{2\pi} \left\{-9(1 - \cos 2\theta) + \frac{27}{2}(1 + \cos 2\theta)\right\} d\theta$$

$1 + \cos 2\theta = 2\cos^2\theta$

$1 - \cos 2\theta = 2\sin^2\theta$

$$= \left[-9\left(\theta - \frac{\sin 2\theta}{2}\right) + \frac{27}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\right]_0^{2\pi}$$

$$= \left[-9(2\pi - 0) + \frac{27}{2}(2\pi + 0)\right]$$

$$= -18\pi + 27\pi$$

$$= 9\pi \quad \underline{\text{(Ans)}}$$