CHAPTER TWO

BETA AND GAMMA FUNCTIONS

2.1. Introduction

The **gamma function** first came into view in connection with the interpolation problem for **factorials (Davis,** 1959). This problem was posed by **Stirling** (1692 – 1764), **Goldbach** (1690–1764) and **Daniel Bernoulli** (1700–1784). It was solved by Euler (1707–1783) in two letters to Goldbach in 1729 and 1730, first by means of an infinite product and later as an integral. The modern notation is due to **Legendre**. He called the integral which **Euler** obtained for $|\overline{(n)}$ the **second Eulerian integral.** Euler's derivation of this integral began with another integral which Legendre called the **first Eulerian integral.** The first and second Eulerian intergals are generally known as the **Beta function** and **Gamma Function** respectively.

2.2 The Factorial Function

Let us calculate the values of some integrals . For $\alpha > 0$,

$$\int_0^\infty e^{-\alpha x}dx = \left[-\frac{1}{\alpha}e^{-\alpha x}\right]_0^\infty = -\frac{1}{\alpha}[0-1] = \frac{1}{\alpha} \quad (1)$$

Now we differentiate both sides of this equation repeatedly with respect to α . Then

$$\int_0^\infty -xe^{-\alpha x}dx = -\frac{1}{\alpha^2} \quad \text{or,} \int_0^\infty xe^{-\alpha x}dx = \frac{1}{\alpha^2}$$

$$\int_0^\infty x^2e^{-\alpha x}dx = -\frac{2}{\alpha^3} = \frac{|\underline{2}}{\alpha^{2+1}}$$

$$\int_0^\infty x^3e^{-\alpha x}dx = \frac{6}{\alpha^4} = \frac{|\underline{3}}{\alpha^{3+1}}$$

... ...

Or , in gereral

$$\int_0^\infty x^ne^{-\alpha x}dx = \frac{|\underline{n}}{\alpha^{n+1}} \quad (2)$$

Putting $\alpha = 1$ in (2), we get

$$\int_0^\infty x^ne^{-x}\,dx = |\underline{n} \quad (n = 1, 2, 3\ldots) \quad (3)$$

Thus we have a definite integral whose value is $|\underline{n}$ for positive intergal n.

We can use (3) to give a meaning to $|\underline{0}$

Putting n = O in (3) , we get .

$$|\underline{o} = \int_0^\infty e^{-x}\,dx = \left[-e^{-x}\right]_0^\infty = 1.$$

2.3 Definitions of Beta and Gamma Functions

The **Beta Function** denoted by β(m, n) is defined as

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \text{ where } m > 0, n > o \quad (1)$$

which is convergent for positive m and n. Evidently β(m, n) is symmetric in m and n for the transformation $x = 1 - y$, that is, β (m, n) = β(n, m)

Thus substituting $x = 1 - y$ in (1) we get

$$\left.\begin{matrix} x = o \\ y = 1 \end{matrix}\right\} \quad \left.\begin{matrix} x = 1 \\ y = 0 \end{matrix}\right\} \quad dx = -dy$$

$$\therefore \beta(m, n) = \int_1^0 (1-y)^{m-1}y^{n-1}.-dy$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\,dy = \beta(n,m)$$

The **Gamma Function** denoted by $|\overline{(n)}$ is defined as

$$|\overline{(n)} = \int_0^\infty e^{-x}x^{n-1}\,dx, \text{ where } n > o \quad (2)$$

This integral converges for n > o.. The restriction n > o is imposed because the integral diverges at the lower limit for other values of n.

In particular, $\Gamma(1) = \int_1^{\infty} e^{-x}\, dx = \left[-e^{-x}\right]_0^{\infty} = 1$

On the other hand, the **Gamma Function** was defined by **Weierstrass** by the equation

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1+\frac{z}{n}\right) e^{-\frac{z}{n}}\right]$$

where z is a complex number and

$$\gamma = \lim_{n\to\infty} \left\{\sum_{k=1}^{n} \frac{1}{k} - \log n\right\} = 0.5772157 \ldots\ldots\ldots$$

is Euler's constant. **Euler** showed that

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \left[\left(1+\frac{1}{n}\right)^{z} \left(1+\frac{z}{n}\right)^{-1}\right].$$

Also due to Euler $\Gamma(z) = \lim_{n\to\infty} \dfrac{\lfloor n \; n^z}{z(z+1)\ldots.(z+n)}$.

Definition. The **incomplete gamma function**

$\gamma(z, \alpha)$ and its **complement** $\Gamma(z, \alpha)$ are defined by the formulas $\gamma(z, \alpha) = \int_0^{\alpha} e^{-t} t^{z-1}\, dt$, $Re z > o$, $|\arg \alpha| < \pi$,

and $\Gamma(z, \alpha) = \int_{\alpha}^{\infty} e^{-t} t^{z-1}\, dt$, $|\arg \alpha| < \pi$ respectively so that

$\gamma(z, \alpha) + \Gamma(z, \alpha) = \Gamma(z)$.

2.4 Different forms of Beta Function

From definition we have

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx,\ m > 0,\ n > 0 \quad (1)$$

(i) Substituting $x = \sin^2\theta$ in (1) so that $dx = 2 \sin\theta \cos\theta\, d\theta$

Also when $x = 0, \theta = 0$ and $x = 1,\ \theta = \frac{\pi}{2}$.

Thus (1) becomes

$$\beta(m, n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} (1-\sin^2\theta)^{n-1}\, 2\sin\theta \cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta \quad (2)$$

Again let $2m - 1 = p$ and $2n - 1 = q$

then $m = \frac{p+1}{2}$ and $n = \frac{q+1}{2}$

Thus from (2) we have

$$\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta\, d\theta \quad (3)$$

(ii) If we put $x = \frac{y}{1+y}$ in (1) then.

$dx = \left\{\frac{1+y-y}{(1+y)^2}\right\} dy = \frac{1}{(1+y)^2}\, dy$. and

$1 - x = 1 - \frac{y}{1+y} = \frac{1}{1+y}$.

When $\left.\begin{matrix} x = 0 \\ y = 0 \end{matrix}\right\}$ $\left.\begin{matrix} x = 1 \\ y = \infty \end{matrix}\right\}$

$$\text{Therefore, } \beta(m, n) = \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m-1}} \cdot \frac{1}{(1+y)^{n-1}} \cdot \frac{dy}{(1+y)^2}$$

$$= \int_0^{\infty} \frac{y^{m-n}}{(1+y)^{m+n}}. \quad (4)$$

Also since $\beta(m,n) = \beta(n, m)$, we have

$$\beta(m, n) = \int_0^{\infty} \frac{y^{n-1}\, dy}{(1+y)^{n+m}} \quad (5)$$

(iii) If we put $x = \frac{y}{a}$ in (1), then $dx = \frac{1}{a} dy$

When $\left.\begin{matrix} x = 0 \\ y = 0 \end{matrix}\right\}$ $\left.\begin{matrix} x = 1 \\ y = a \end{matrix}\right\}$

Therefore, $\beta(m, n) = \int_0^a \frac{y^{m-1}}{a^{m-1}} \left(1 - \frac{y}{a}\right)^{n-1} \frac{1}{a} dy$

$$= \int_0^a \frac{y^{m-1}(a-y)^{n-1}}{a^{m+n-1}} dy \quad (6)$$

2.5 Reduction Formula for $\Gamma(n)$.

From definition we have $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$ (2)

Replacing n by (n + 1) in (2) we get

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^n dx$$

Integrating by parts taking e^{-x} as second function, we get

$$\Gamma(n+1) = \left[-x^n e^{-x}\right]_0^\infty + \int_0^\infty n x^{n-1} e^{-x} dx$$

$$= 0 + n \int_0^\infty e^{-x} x^{n-1} dx = n\Gamma(n).$$

$\therefore \Gamma(n+1) = n\Gamma(n)$ (3) which is the **reduction formula** for $\Gamma(n)$.

[**Note :** $x^n e^{-x}$ vanishes for both the limits as

$$\lim_{x\to 0} x^n e^{-x} = \lim_{x\to 0} \frac{x^n}{e^x} = 0 \text{ and}$$

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{x^n}{1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \dots \dots} = 0 \;]$$

Now replacing n by n −1 in (3) we get $\Gamma(n) = (n-1)\Gamma(n-1)$

Similarly .

$\Gamma(n-1) = (n-2)\Gamma(n-2)$

$\Gamma(n-2) = (n-3)\Gamma(n-3)$

....

$\Gamma(3) = 2\Gamma(2)$

$\Gamma 2 = 1\Gamma 1 = 1$. Since $\Gamma 1 = 1$.

Thus $\Gamma(n+1) = n(n-1)(n-2) \dots\dots 3.2.1 = \lfloor n$ (4)

provided n is a positive integer.

Replacing n by n−1 in (4), we can write $\Gamma(n) = \lfloor n-1$

This shows that Gamma function can be regarded as a generalization of the elementary factorial function. So the Gamma Function is also known as the **generalized factorial function.**

Also we have from reduction formula $\Gamma(n) = \frac{\Gamma(n+1)}{n}$ (5)

putting n = o in (5), we have $\Gamma 0 = \frac{\Gamma(o+1)}{0} = \frac{1}{0} = \infty.$

By repeated application of (5), we can prove that the gamma function becomes infinite when n is a negative integer *i. e.* $\Gamma(-n) = \infty$.

But it is to be noted that the gamma function has finite value for negative values of n which are not integers.

2.6 The value of $\Gamma\left(\frac{1}{2}\right)$ and graph of the gamma function

By definition we have $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx, n > o$ (2)

Substituting $n = \frac{1}{2}$ in (2), we get

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x} x^{-\frac{1}{2}} dx \quad (3)$$

Again putting $x = y^2$ in (3) so that $dx = 2ydy$.

Then $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-y^2} \frac{1}{y} . 2ydy = 2 \int_0^\infty e^{-y^2} dy$ (4)

Similarly, we can write

$$\Gamma\left(\frac{1}{2}\right) = 2 \int_0^\infty e^{-x^2} dx \quad (5)$$

Therefore, $\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = \Gamma\left(\frac{1}{2}\right) . \Gamma\left(\frac{1}{2}\right)$.

$$= 4 \int_0^\infty \int_0^\infty e^{-(x^2 + y^2)} dxdy \quad (6)$$

Changing to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ and $dxdy = rd\theta dr$, we get.

$$\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = 4 \int_0^\infty \int_0^{\frac{\pi}{2}} e^{-r^2} rd\theta dr$$

$$= 4 . \frac{\pi}{2} \int_2^\infty e^{-r^2} rdr$$

$$= 2\pi \left[-\frac{1}{2} . e^{-r^2}\right]_0^\infty = -\pi(0 - 1) = \pi .$$

$$\therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} = 1.722 .$$

Now we know that $\Gamma(n) = \frac{\Gamma(n+1)}{n}$

Putting $n = -\frac{1}{2}$ in (5) of article 2.5 we get

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\left(\Gamma -\frac{1}{2} + 1\right)}{-\frac{1}{2}} = \frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{2}} = -2\sqrt{\pi}$$

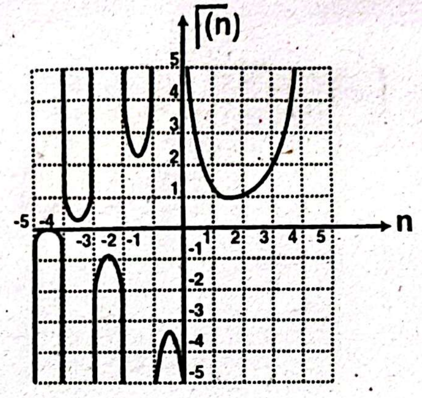
Similarly ,

$$\Gamma(-3/2) = \frac{\Gamma(-3/2+1)}{-3/2} = \frac{\Gamma -\left(\frac{1}{2}\right)}{-3/2}$$

$$= \frac{-2\sqrt{\pi}}{-3/2} = \frac{4\sqrt{\pi}}{3}$$

and $\Gamma\left(-\frac{5}{2}\right) = \frac{\Gamma(-5/2+1)}{-5/2} = \frac{\Gamma(-3/2)}{-5/2} = -\frac{8}{15}\sqrt{\pi}$

and so on. The graph of $\Gamma(n)$ may be shown as below under the definition that the function becomes continuous function of n except when n = 0 or any negative integer.



Graph of Γ(n)

Table of values

n	$\Gamma(n)$
1.00	1.0000
1.10	0.9514
1.20	0.9182
1.30	0.8975
1.40	0.8873
1.50	0.8862
1.60	0.8935
1.70	0.9086
1.80	0.9314
1.90	0.9618
2.00	1.0000

2.7 Transformations of gamma function

By definition $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$ (1)

(i) If we put $x = \lambda y$ in (1) so that $dx = \lambda dy$.
Then we have

$$\Gamma(n) = \int_0^\infty e^{-\lambda y} (\lambda y)^{n-1} \lambda dy$$

$$= \int_0^\infty e^{-\lambda y} \lambda^n . y^{\lambda-1} dy$$

or, $$\frac{\Gamma(n)}{\lambda^n} = \int_0^\infty e^{-\lambda y} . y^{\lambda-1} dy \quad (2)$$

(ii) If we put $e^{-x} = y$ i.e. $e^x = \frac{1}{y}$

or, $x = \log\frac{1}{y}$ in (1) so that $\left.\begin{array}{l} x = 0 \\ y = 1 \end{array}\right\}$ $\left.\begin{array}{l} x = \infty \\ y = 0 \end{array}\right\}$ and

$$dx = \frac{1}{\frac{1}{y}} \cdot -\frac{1}{y^2} dy = -\frac{1}{y} dy.$$

Then we have $$\Gamma(n) = \int_1^0 y. \left(\log\frac{1}{y}\right)^{n-1} . -\frac{1}{y} dy.$$

$$= \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy \quad (3)$$

(iii) If put $x^n = y$ in (1) then $nx^{n-1} dx = dy$

$$\therefore dx = \frac{dy}{nx^{n-1}}$$

So from (1) we have

$$\Gamma(n) = \int_0^\infty e^{-y^{\frac{1}{n}}} \left(y^{\frac{1}{n}}\right)^{n-1} . \frac{dy}{n.\, y^{\frac{n-1}{n}}}$$

$$= \frac{1}{n} \int_0^\infty e^{-y^{\frac{1}{n}}} dy \quad (4)$$

or, $$n\Gamma(n) = \int_0^\infty e^{-y^{\frac{1}{n}}} dy \text{ But } n\Gamma(n) = \Gamma(n+1)$$

Thus $$\Gamma(n+1) = \int_0^\infty e^{-y^{\frac{1}{n}}} dy \quad (5)$$

Also, if we replace n by $\frac{1}{2}$ in (4), we get

$$\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-y^2} dy \text{ But } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\therefore \int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2}$$

(iv) If we put $x = y^2$ in (I) so that $dx = 2ydy$

Then $$\Gamma(n) = \int_0^\infty e^{-y^2} (y^2)^{n-1} 2ydy$$

$$= 2\int_0^\infty e^{-y^2} y^{2n-1} dy \quad (6)$$

(v) If we put $x = -(m+1)\log y$ so that

$$dx = -(m+1). \frac{1}{y} dy.$$

Then $$\Gamma(n) = \int_1^0 e^{(m+1)\log y} \{-(m+1)\log y\}^{n-1}.$$

$$-(m+1)\frac{1}{y} dy$$

$$= \int_0^1 y^{m+1} (m+1)^n \{\log y^{-1}\}^{n-1} \frac{1}{y} dy$$

$$= (m+1)^n \int_0^1 y^m \left(\log\frac{1}{y}\right)^{n-1} dy \quad (7)$$

2.8 Relation between beta and gamma functions.

There is a very useful formula which expresses the beta function in terms of gamma function.

From definition, we have $\Gamma(m) = \int_0^\infty e^{-t} t^{m-1}\, dt$

Putting $t = x^2$ so that $dt = 2x\,dx$.

Then $\Gamma(m) = 2\int_0^\infty e^{-x^2} x^{2m-1}\, dx$ (8)

Similarly, putting $t = y^2$ in $\Gamma(n) = \int_0^\infty e^{-t} t^{n-1}\, dt$

we get $\Gamma(n) = 2\int_0^\infty e^{-y^2} y^{2n-1}\, dy$ (9)

$$\therefore \Gamma(m)\Gamma(n) = 4\int_0^\infty e^{-x^2} x^{2m-1}\, dx \int_0^\infty e^{-y^2} y^{2n-1}\, dy$$

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1}\, dx\,dy \quad (10)$$

Now changing to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ and $dx\,dy = r\,d\theta\,dr$. To cover the region in (10) which is the entire first quadrant r must vary from 0 to ∞ and θ from 0 to $\frac{\pi}{2}$.

Thus (10) becomes $\Gamma(m)\Gamma(n)$

$$= 4\int_0^\infty \int_0^{\pi/2} e^{-r^2} r^{2m-1} \cos^{2m-1}\theta\, r^{2n-1} \sin^{2n-1}\theta\, r\,d\theta\,dr.$$

$$= \left[2\int_0^\infty e^{-r^2} . r^{2(m+n)-1}\, dr\right]\left[2\int_0^{\frac{\pi}{2}} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta\right]$$

$$= \Gamma(m+n).\beta(m,n) \quad \therefore \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

which is the required relation between beta and gamma functions.

Corollary 1 : $\int_0^{\frac{\pi}{2}} \sin^p x \cos^q x\, dx = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

$$= \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}.$$

In particular, when q = 0 and p = n,

we have $\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} . \frac{\sqrt{\pi}}{2}$

Similarly, when q = n and p = o, we have

we have $\int_0^{\frac{\pi}{2}} \cos^n x\, dx = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} . \frac{\sqrt{\pi}}{2}$

or, equivalently, we can write

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \int_0^{\frac{\pi}{2}} \cos^n x\, dx$$

$= \frac{n-1}{n}, \frac{n-3}{n-2}, \frac{n-5}{n-4} \dots \frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$ when n is even

$= \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.\frac{4}{5}.\frac{2}{3}.1$ when n is odd.

These integrals are known as **Walli's formulae.**

Corollary 2 : From the relation between beta and gamma functions, we have

$\Gamma(m)\Gamma(n) = \Gamma(m+n)\,\beta(m,n)$

$$= \Gamma(m+n)\int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\, dy.$$

Now putting $m + n = 1$ so that $n = 1 - m$

Then $\Gamma(m)\Gamma(1-m) = \Gamma 1 . \int_0^\infty \frac{y^{m-1}}{1+y} dy.$

Now we know that $\Gamma 1 = 1$ and

$$\int_0^\infty \frac{y^{m-1}}{1+y} dy = \frac{\pi}{\sin m\pi}, 0<m<1$$

Then $\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin m\pi}$

This relation is known as **Euler's reflection formula**.

Also if we multiply both sides by m. then we have

$m\Gamma(m)\Gamma(1-m) = \frac{m\pi}{\sin m\pi}$

or, $\Gamma(m+1)\Gamma(1-m) = \frac{m\pi}{\sin m\pi}$.

[since $m\Gamma(m) = \Gamma(m+1)$.]

2.9 Reduction of definite integral to beta and gamma functions.

(i) Prove that

$$\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy = \int_0^1 \frac{(x^{m-1}+x^{n-1})}{(1+x)^{m+n}} dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Proof : By definition we have

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx \quad (1)$$

Putting $x = \frac{1}{1+y}$ i.e. $y = \frac{1}{x} - 1$ so that $dx = -\frac{1}{(1+y)^2} dy$

Limits

$\left.\begin{matrix} x=0 \\ y=\infty \end{matrix}\right\}$ $\quad$ $\left.\begin{matrix} x=1 \\ y=0 \end{matrix}\right\}$

Then $\beta(m,n) = \int_\infty^0 \frac{1}{(1+y)^{m-1}} \cdot \frac{y^{n-1}}{(1+y)^{n-1}} \cdot \frac{-1}{(1+y)^2} dy$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy \quad (2)$$

Now by using the properties of definite intergal, equation (2) can be written as

$$\beta(m,n) = \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}} dy + \int_1^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy \quad (3)$$

Now putting $y = \frac{1}{t}$ in the second integral of (3), we get

$$\int_1^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy = \int_1^0 \left(\frac{1}{t}\right)^{n-1} \frac{1}{\left(1+\frac{1}{t}\right)^{m+n}} \cdot \frac{-1}{t^2} dt$$

$$= \int_0^1 \frac{1}{t^{n-1}} \cdot \frac{t^{m+n}}{(1+t)^{m+n}} \cdot \frac{1}{t^2} dt$$

$$= \int_0^1 \frac{t^{m-1}}{(1+t)^{m+n}} dt = \int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}} dy.$$

[Since change of variables does not change the value of the integral when limits are same]

Then from (3) we have

$$\beta(m,n) = \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}} dy + \int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}} dy$$

$$= \int_0^1 \frac{(y^{m-1}+y^{n-1})}{(1+y)^{m+n}} dy = \int_0^1 \frac{(x^{m-1}+x^{n-1})}{(1+x)^{m+n}} dx$$

or, $\beta(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}} dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

(ii) Prove that

$$\int_0^\infty \frac{y^{m-1} dy}{(ay+b)^{m+n}} = \frac{\Gamma(m)\Gamma(n)}{a^m b^n \Gamma(m+n)}.$$

Proof : We know that

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad (4)$$

Putting $x = \frac{a}{b} y$ in (4) so that $dx = \frac{a}{b} dy$.

When $\left.\begin{array}{l} x = 0 \\ y = 0 \end{array}\right\}$ and $\left.\begin{array}{l} x = \infty \\ y = \infty \end{array}\right\}$

Then $$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^\infty \frac{\frac{a^{m-1}}{b^{m-1}} y^{n-1}}{\left(1 + \frac{a}{b} y\right)^{m+n}} \frac{a}{b} dy$$

$$= \int_0^\infty \frac{a^{m-1}}{b^{m-1}} y^{m-1} \cdot \frac{b^{m+n}}{(ay+b)^{m+n}} \cdot \frac{a}{b} dy$$

$$= a^m b^n \int_0^\infty \frac{y^{m-1}}{(ay+b)^{m+n}} dy.$$

$$\therefore \int_0^\infty \frac{y^{m-1}}{(ay+b)^{m+n}} dy. = \frac{1}{a^m b^n} \int_0^\infty \frac{y^{m-1}}{(1+x)^{m+n}} dy.$$

$$= \frac{1}{a^m b^n} \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

(iii) Prove that

$$\int_0^{\pi/2} \frac{\sin^{2m-1}\theta \cos^{2n-1}\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}} d\theta = \frac{\Gamma(m)\Gamma(n)}{2a^m b^n \Gamma(m+n)}$$

Proof : We know that

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad (4)$$

Putting $x = \frac{a}{b} \tan^2\theta$ in (4) so that

$dx = \frac{a}{b} 2\tan\theta \sec^2\theta \, d\theta$ and

$$1 + x = 1 + \frac{a}{b} \tan^2\theta = \frac{a\sin^2\theta + b\cos^2\theta}{b\cos^2\theta}$$

When $\left.\begin{array}{l} x = 0 \\ \theta = 0 \end{array}\right\}$ $\left.\begin{array}{l} x = \infty \\ \theta = \frac{\pi}{2} \end{array}\right\}$

$$\therefore \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^{\pi/2} \frac{\left(\frac{a}{b}\tan^2\theta\right)^{m-1} \frac{2a}{b} \tan\theta \sec^2\theta \, d\theta}{\left(\frac{a\sin^2\theta + b\cos^2\theta}{b\cos^2\theta}\right)^{m+n}}$$

$$= 2\int_0^\infty \frac{a^m}{b^m} \cdot b^{m+n} \frac{\sin^{2m-2}\theta}{\cos^{2m-2}\theta} \cdot \frac{\cos^{2m+2n}\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}} \cdot \frac{\sin\theta}{\cos^3\theta} d\theta$$

$$= 2a^m b^n \int_0^{\pi/2} \frac{\sin^{2m-1}\theta \cos^{2n-1}\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}} d\theta$$

$$\therefore \int_0^{\pi/2} \frac{\sin^{2m-1}\theta \cos^{2n-1}\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}} d\theta.$$

$$= \frac{1}{2a^m b^n} \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$$= \frac{1}{2a^m b^n} \cdot \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad (5)$$

(iv) Prove that

$$\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(m)}{\Gamma\left(m + \frac{1}{2}\right)} = \frac{2^{2m-1}\{\Gamma(m)\}^2}{\Gamma(2m)}.$$

Proof : *From (5), we have*

$$\int_0^{\pi/2} \frac{\sin^{2m-1}\theta \cos^{2n-1}\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}} = \frac{1}{2a^m b^n} \cdot \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad (5).$$

putting $a = b = 1$ in (5) we get

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)} \quad (6)$$

Now putting $2n = 1$ *i.e* $2n-1 = o$ *in (6), we get*

$$\int_0^{\pi/2} \sin^{2m-1}\theta d\theta = \frac{\Gamma(m).\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(m+\frac{1}{2}\right)} = \frac{\Gamma(m).\sqrt{\pi}}{2\Gamma\left(m+\frac{1}{2}\right)}$$

$$\therefore \int_0^{\pi/2} \sin^{2m-1}\theta d\theta = \frac{\sqrt{\pi}}{2}.\frac{\Gamma(m)}{\Gamma\left(m+\frac{1}{2}\right)} \quad (7)$$

Also putting $m = n$ *in (6), we get*

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta d\theta = \frac{\Gamma(m)\Gamma(m)}{2\Gamma(m+m)}$$

$$\text{or, } \int_0^{\pi/2} (\sin\theta\cos\theta)^{2m-1} d\theta = \frac{\{\Gamma(m)\}^2}{2\Gamma(2m)}$$

$$\int_0^{\pi/2} \frac{(2\sin\theta\cos\theta)^{2m-1}}{2^{2m-1}} d\theta = \frac{\{\Gamma(m)\}^2}{2\Gamma(2m)}$$

$$\text{or, } \int_0^{\pi/2} (\sin 2\theta)^{2m-1} d\theta = \frac{2^{2m-2}\{\Gamma(m)\}^2}{\Gamma(2m)}.$$

Now putting $2\theta = \varphi$ *so that* $d\theta = \frac{1}{2}d\varphi$.

$$\left.\begin{matrix}\theta = 0\\ \varphi = 0\end{matrix}\right\} \quad \left.\begin{matrix}\theta = \frac{\pi}{2}\\ \varphi = \pi\end{matrix}\right\}$$

$$\int_0^{\pi} (\sin\varphi)^{2m-1}.\frac{1}{2}d\varphi = \frac{2^{2m-2}\{\Gamma(m)\}^2}{\Gamma(2m)}$$

$$\text{or, } \int_0^{\pi/2} (\sin\varphi)^{2m-1} \varphi d\varphi = \frac{2^{2m-2}\{\Gamma(m)\}^2}{\Gamma(2m)}$$

Now replacing φ *by* θ, *we have*

$$\int_0^{\pi/2} \sin^{2m-1}\theta d\theta = \frac{2^{2m-2}\{\Gamma(m)\}^2}{\Gamma(2m)} \quad (8)$$

Now from (7) and (8) it is obvious that

$$\frac{\sqrt{\pi}}{2}.\frac{\Gamma(m)}{\Gamma\left(m+\frac{1}{2}\right)} = \frac{2^{2m-2}\{\Gamma(m)\}^2}{\Gamma(2m)}$$

$$\text{or, } \Gamma(2m) = \frac{2^{2m-1}}{\sqrt{\pi}}.\Gamma(m)\Gamma\left(m+\frac{1}{2}\right).$$

This relation is known as **duplication formula** for the gamma function.

Alternative Proof of $\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

By definition of gamma function, we have

$$\Gamma(m) = \int_0^{\infty} e^{-x} x^{m-1} dx,\ m > o \quad (1)$$

Putting $x = \lambda t$ in (1). so that $dx = \lambda dt$.

Limits $\left.\begin{matrix}x = o\\ t = o\end{matrix}\right\} \quad \left.\begin{matrix}x = \infty\\ t = \infty\end{matrix}\right\}$

Thus from (1), we have

$$\Gamma(m) = \int_0^{\infty} e^{-\lambda t}(\lambda t)^{m-1}\lambda dt$$

$$= \int_0^{\infty} e^{-\lambda t}.\lambda^m.t^{m-1} dt \quad (2)$$

$$\text{Or, } \frac{\Gamma(m)}{\lambda^m} = \int_0^{\infty} e^{-\lambda t} t^{m-1} dt \quad (3)$$

Multiplying both sides of (2) by $e^{-\lambda}\lambda^{n-1}$, we get

$$\Gamma(m)\, e^{-\lambda}\lambda^{n-1} = \int_0^{\infty} e^{-\lambda t}.e^{-\lambda}.\lambda^m.\lambda^{n-1}t^{m-1} dt$$

$$= \int_0^{\infty} e^{-\lambda(1+t)}.\lambda^{(m+n)-1}.t^{m-1} dt. \quad (4)$$

Integrating both sides of (4) with respct to λ from o to ∞, we get

$$\Gamma(m)\int_0^{\infty} e^{-\lambda}\lambda^{n-1}\,d\lambda = \int_0^{\infty}\left\{\int_0^{\infty} e^{-\lambda(1+t)}\lambda^{m+n-1}.\,d\lambda\right\} t^{m-1}\,dt.$$

Or, $$\Gamma(m)\,\Gamma(n) = \int_0^{\infty}\frac{\Gamma(m+n)}{(1+t)^{m+n}}\,t^{m-1}\,dt.$$

since $$\frac{\Gamma(m+n)}{(1+t)^{m+n}} = \int_0^{\infty} e^{-\lambda(1+t)}\lambda^{m+n-1}\,d\lambda \text{ using (3)}$$

Or, $$\Gamma(m)\,\Gamma(n) = \Gamma(m+n)\,.\int_0^{\infty}\frac{t^{m-1}dt}{(1+t)^{m+n}}$$

$$= \Gamma(m+n)\ \beta(m,n).$$

since $$\beta(m,n) = \int_0^{\infty}\frac{t^{m-1}}{(1+t)^{m+n}}\,dt$$

$$\therefore\quad \beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

WORKED OUT EXAMPLES

Example 1. *(i) Evaluate* $I = \int_0^1 x^7(1-x)^3 dx$

Solution : $$\int_0^1 x^7(1-x)^3 dx = \int_0^1 x^{8-1}(1-x)^{4-1}dx$$

$$= \beta(8,4) = \frac{\Gamma(8)\,\Gamma(4)}{\Gamma(8+4)}$$

$$= \frac{\Gamma(8)\,.\,3.\,2.\,1}{11.10.\,9.\,8\,\Gamma(8)}$$

$$= \frac{3.2.1}{11.10.9.8} = \frac{1}{11.10.3.4} = \frac{1}{1320}.$$

Example 1. *(ii) Prove that* $\int_0^{\infty} e^{-t^2}dt = \frac{\sqrt{\pi}}{2}.$

Proof : Let $I = \int_0^{\infty} e^{-t^2}dt \quad (1)$

Putting $t^2 = x$ so that $2tdt = dx$

$$\therefore dt = \frac{1}{2}\cdot\frac{dx}{t} = \frac{1}{2}\frac{dx}{\sqrt{x}}$$

$$\text{Or, } dt = \frac{1}{2}x^{-\frac{1}{2}}dx.$$

Limits $\left.\begin{matrix} t=o \\ x=o\end{matrix}\right\}\ \left.\begin{matrix} t=\infty \\ x=\infty\end{matrix}\right\}$

Thus from (1), we have

$$I = \int_0^{\infty} e^{-t^2}dt = \int_0^{\infty} e^{-x}.\frac{1}{2}\,x^{-\frac{1}{2}}dx$$

$$= \frac{1}{2}\int_0^{\infty} e^{-x}x^{\frac{1}{2}-1}\,dx$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \text{ since } \int_0^{\infty} e^{-x}x^{n-1}\,dx = \Gamma(n)$$

$$= \frac{1}{2}\sqrt{\pi} = \frac{\sqrt{\pi}}{2}$$

$$\therefore \int_0^{\infty} e^{-t^2}dt = \frac{\sqrt{\pi}}{2}.$$

Example 1. (iii) Express the integral $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$ in terms of beta function. **D.U. M.Sc. 1990**

Solution : $$I = \int_0^1 \frac{dx}{\sqrt{1-x^3}} \quad (1)$$

By definition of beta function, we have

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx,\ m>o,\ n>o.$$

Putting $x^3 = t$ in (1) so that $3x^2dx = dt$

$$x = t^{\frac{1}{3}} \therefore x^2 = t^{\frac{2}{3}}.$$

$$dx = \frac{1}{3t^{2/3}}\,dt.$$

Limits $\left.\begin{array}{l} x = 0 \\ t = 0 \end{array}\right\}$ $\left.\begin{array}{l} x = 1 \\ t = 1 \end{array}\right\}$ $dx = \dfrac{1}{3t^{2/3}}$

Thus from (1), we get

$$I = \int_0^1 \frac{1}{(1-t)^{\frac{1}{2}}} \frac{1}{3t^{2/3}}\,dt.$$

$$= \frac{1}{3}\int_0^1 t^{-2/3}(1-t)^{-\frac{1}{2}}\,dt$$

$$= \frac{1}{3}\int_0^1 t^{\frac{1}{3}-1}(1-t)^{\frac{1}{2}-1}\,dt.$$

$$= \frac{1}{3}\beta\left(\frac{1}{3}, \frac{1}{2}\right).$$

Therefore, $I = \displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^3}} = \frac{1}{3}\beta\left(\frac{1}{3}, \frac{1}{2}\right)$.

Example 2. Evaluate the integral

$$I = \int_0^1 (1-x)^{\frac{1}{2}} x^3 dx$$

Solution :

By definition $\beta(m, n) = \displaystyle\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$.

$$\therefore \quad I = \int_0^1 (1-x)^{\frac{1}{2}} x^3 dx = \int_0^1 x^{4-1}(1-x)^{\frac{3}{2}-1}\,dx$$

$$= \beta(4, 3/2) = \frac{\Gamma(4)\,\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(4+\frac{3}{2}\right)} = \frac{\Gamma(4)\,\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{11}{2}\right)}$$

$$= \frac{3.2.1.\frac{1}{2}.\,\Gamma\left(\frac{1}{2}\right)}{\frac{9}{2}.\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)} = \frac{2^5}{9\times 7\times 5} = \frac{32}{315}.$$

Example 3. Evaluate $\displaystyle\int_0^\infty x^5 e^{-4x} dx$.

Solution : Let $I = \displaystyle\int_0^\infty x^5 e^{-4x} dx$.

Putting $4x = y$ then $x = \frac{1}{4}y$ and $dx = \frac{1}{4}dy$.

Limits : $\left.\begin{array}{l} x = 0 \\ y = 0 \end{array}\right\}$ $\left.\begin{array}{l} x = \infty \\ y = \infty \end{array}\right\}$

$$\therefore \quad I = \int_0^\infty x^5 e^{-4x}\,dx = \int_0^\infty \left(\frac{1}{4}y\right)^5 e^{-y}\,\frac{1}{4}dy$$

$$= \frac{1}{4^6}\int_0^\infty y^5 e^{-y}\,dy = \frac{1}{4^6}\int_0^\infty e^{-y}\,y^{6-1}\,dy$$

$$= \frac{\Gamma(6)}{4^6} = \frac{|\underline{5}}{4^6} = \frac{5.\,4.\,3.\,2.\,1}{4.\,4.\,4^4} = \frac{5.3}{8.64} = \frac{15}{512}.$$

Example 4. Evaluate $\displaystyle\int_0^\infty e^{-y^2} y^5\,dy$.

Solution : Let $I = \displaystyle\int_0^\infty e^{-y^2} y^5 dy$.

Putting $y^2 = z$ i.e $y = z^{\frac{1}{2}}$

$$\therefore \quad 2ydy = dz, \text{ Or, } ydy = \frac{1}{2}dz.$$

Limits $\left.\begin{matrix} y = 0 \\ z = 0 \end{matrix}\right\} \left.\begin{matrix} y = \infty \\ z = \infty \end{matrix}\right\}$

$$I = \int_0^\infty e^{-y^2} y^4 . \, y\,dy = \int_0^\infty e^{-z} z^2 . \frac{1}{2} dz$$

$$= \frac{1}{2}\int_0^\infty e^{-z} z^{3-1}\, dz = \frac{1}{2}\Gamma(3)$$

$$= \frac{1}{2}\lfloor 2 . = \frac{1}{2} . 2 . 1 = 1$$

Thus $\int_0^\infty e^{-y^2} . y^5\, dy = 1.$

Formulae used

(i) $\Gamma(n) = \int_0^1 e^{-x} x^{n-1}\, dx$

(ii) $\Gamma n = (n-1)!$

Example 5. Evaluate the integral

$$I = \int_0^1 x^{5/2}(1-x)^{3/2}\, dx$$

Solution : $I = \int_0^1 x^{5/2}(1-x)^{3/2}\, dx$

$$= \int_0^1 x^{7/2-1}(1-x)^{5/2-1}\, dx.$$

$$= \beta\,(7/2, 5/2) = \frac{\Gamma\left(\frac{7}{2}\right).\Gamma\frac{5}{2}}{\Gamma\left(\frac{7}{2}+\frac{5}{2}\right)} = \frac{\Gamma\left(\frac{7}{2}\right).\Gamma\left(\frac{5}{2}\right)}{\Gamma(6)}$$

$$= \frac{\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{5.\,4.\,3.\,2.\,1} = \frac{5.3\,3\sqrt{\pi}.\sqrt{\pi}}{2^5.\,5.\,4.\,3.\,2}$$

$$= \frac{3\pi}{32 \times 8} = \frac{3\pi}{256}$$

Example 6. Evaluate the integral

$$I = \int_0^a y^7 \sqrt{a^4 - y^4}\, dy.$$

Solution : $I = \int_0^a y^7 \sqrt{a^4 - y^4}\, dy$

Putting $y^4 = a^4 x$

$$\therefore\ y = ax^{1/4},\ dy = \frac{1}{4} ax^{-3/4}\, dx.$$

Limits $\left.\begin{matrix} y = 0 \\ x = 0 \end{matrix}\right\}$ and $\left.\begin{matrix} y = a \\ x = 1 \end{matrix}\right\}$

Thus the integral becomes

$$I = \int_0^1 a^7 x^{7/4}.\, a^2(1-x)^{1/2}\, \frac{1}{4} a x^{-3/4}\, dx$$

$$= \frac{a^{10}}{4}\int_0^1 x(1-x)^{\frac{1}{2}}\, dx = \frac{a^{10}}{4}\int_0^1 x^{2-1}(1-x)^{\frac{3}{2}-1}\, dx$$

$$= \frac{a^{10}}{4}\,\beta(2, 3/2)$$

$$= \frac{a^{10}}{4}.\frac{\Gamma(2).\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{7}{2}\right)}$$

$$= \frac{a^{10}}{4}.\frac{1.\,\Gamma(3/2)}{\frac{5}{2}.\frac{3}{2}.\Gamma\left(\frac{3}{2}\right)} = \frac{a^{10}}{15}.$$

Example 7. Prove that $\int_0^4 y\sqrt[3]{(64-y^3)}\, dy = \frac{128\pi}{9\sqrt{3}}$

Proof : $I = \int_0^4 y\sqrt[3]{64 - y^3}\, dy$

Putting $y^3 = 64x$ then $y = 4x^{\frac{1}{3}}$ and $dy = \frac{4}{3}x^{-\frac{2}{3}}\, dx.$

Limits $\left.\begin{matrix} y=0 \\ x=0 \end{matrix}\right\}$ $\left.\begin{matrix} y=4 \\ x=1 \end{matrix}\right\}$

Thus the integral becomes

$$I = \int_0^1 4x^{1/3} \cdot 4(1-x)^{\frac{1}{3}} \cdot \frac{4}{3}x^{-\frac{2}{3}}\, dx$$

$$= \frac{64}{3}\int_0^1 x^{-\frac{1}{3}}(1-x)^{\frac{1}{3}}\, dx$$

$$= \frac{64}{3}\int_0^1 x^{\frac{2}{3}-1}(1-x)^{\frac{4}{3}-1}\, dx$$

$$= \frac{64}{3}\beta\left(\frac{2}{3}, \frac{4}{3}\right) = \frac{64}{3} \cdot \frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{2}{3}+\frac{4}{3}\right)}$$

$$= \frac{64}{3}\,\frac{\Gamma\left(\frac{2}{3}\right)\frac{1}{3}\Gamma\left(\frac{1}{3}\right)}{\Gamma(2)}$$

$$= \frac{64}{9}\Gamma\left(\frac{1}{3}\right) \cdot \Gamma\left(1-\frac{1}{3}\right)$$

$$= \frac{64}{9} \cdot \frac{\pi}{\sin\frac{\pi}{3}}$$

$$= \frac{64}{9} \cdot \frac{\pi}{\frac{\sqrt{3}}{2}} = \frac{128\pi}{9\sqrt{3}}.$$

Example 8. Find $\int_0^\infty \frac{x^3 dx}{(1+x)^5}$.

Solution : By definition of beta function we have

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx \quad m > 0, n > 0 \quad (1)$$

Putting $x = \frac{y}{1+y}$ in (1) so that $y = \frac{x}{1-x}$

$1 - x = \frac{1}{1+y}$ and $dx = \frac{1}{(1+y)^2}dy$.

Limits

$\left.\begin{matrix} x=0 \\ y=0 \end{matrix}\right\}$ $\left.\begin{matrix} x=1 \\ y=\infty \end{matrix}\right\}$

Thus from (1) we, have

$$\beta(m, n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m-1}} \cdot \frac{1}{(1+y)^{n-1}} \cdot \frac{1}{(1+y)^2}dy$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}dy \quad (2)$$

Now applying (2) in the given integral we have

$$I = \int_0^\infty \frac{x^3}{(1+x)^5}dx = \int_0^\infty \frac{x^{4-1}}{(1+x)^{4+1}}dx$$

$$= \beta(4, 1) = \frac{\Gamma 4\, \Gamma 1}{\Gamma(4+1)} = \frac{\Gamma 4}{4\Gamma 4} = \frac{1}{4}$$

Exmple 9. Evaluate the integral

$\int_0^1 x^2(1-x^3)^{\frac{3}{2}} dx$ using beta function.

Putting $z = x^3$ then $dz = 3x^2\, dx$.

$\therefore\ x^2\, dx = \frac{1}{3}dz$

Limits $\left.\begin{matrix} x=0 \\ z=0 \end{matrix}\right\}$ $\left.\begin{matrix} x=1 \\ z=1 \end{matrix}\right\}$

Thus the integral becomes

$$I = 3\int_0^1 (1-z)^{3/2} dz = 3\int_0^1 z^{1-1}(1-z)^{\frac{5}{2}-1} dz$$

$$= 3\beta\left(1, \frac{5}{2}\right)$$

$$= 3\frac{\Gamma 1\,\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{7}{2}\right)} = \frac{3\Gamma\left(\frac{5}{2}\right)}{5/2\,\Gamma(5/2)} = \frac{6}{5}.$$

Example 10. Prove that $\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}.$

Proof : $I = \int_0^\infty \frac{dx}{1+x^4}$ Putting $x = \sqrt{\tan\theta}$

$$1 + x^4 = 1 + \tan^2\theta = \sec^2\theta.$$

$$dx = \frac{1}{2\sqrt{\tan\theta}}\sec^2\theta d\theta$$

Limits $\left.\begin{array}{l} x = 0 \\ \theta = 0 \end{array}\right\}$ $\left.\begin{array}{l} x = \infty \\ \theta = \frac{\pi}{2} \end{array}\right\}$

$$\therefore I = \int_0^{\pi/2} \frac{1}{\sec^2\theta}\cdot\frac{1}{2\sqrt{\tan\theta}}\sec^2\theta d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\sqrt{\cot\theta}\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\sin^{-\frac{1}{2}}\theta\cos^{\frac{1}{2}}\theta d\theta$$

$$= \frac{1}{2}\cdot\frac{1}{2}\,\beta\left(\frac{-\frac{1}{2}+1}{2}, \frac{\frac{1}{2}+1}{2}\right)$$

$$= \frac{1}{4}\beta\left(\frac{1}{4}, \frac{3}{4}\right) = \frac{1}{4}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}+\frac{3}{4}\right)}.$$

$$= \frac{1}{4}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) = \frac{1}{4}\cdot\frac{\pi}{\sin\frac{\pi}{4}} = \frac{1}{4}\cdot\frac{\pi}{\frac{1}{\sqrt{2}}} = \frac{\pi}{2\sqrt{2}}.$$

Formulae used

(i) $\int_0^{\pi/2}\sin^p\theta\cos^q\theta d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

(ii) $\Gamma(m)\,\Gamma(1-m) = \frac{\pi}{\sin m\pi}$ where $0 < m < 1$.

Example 11 (a). Evaluate $\int_0^{\pi/2}\sin^4\theta\cos^2\theta d\theta$

Let $I = \int_0^{\pi/2}\sin^4\theta\cos^2\theta d\theta$

$$= \frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{4+2+2}{2}\right)} = \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\Gamma(4)} = \frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{2\lfloor 3}$$

$$= \frac{3.\sqrt{\pi}.\sqrt{\pi}}{8.2.3.2.1} = \frac{\pi}{32}.$$

Formulae used

(i) $\int_0^{\pi/2}\sin^p\theta\cos^q\theta d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$

(ii) $\Gamma(n) = (n-1)\Gamma(n-1)$.

(b) Evaluate $\int_0^{\pi/8}\sin^2 8x\cos^5 4x dx$

Let $I = \int_0^{\pi/8}\sin^2 8x\cos^5 4x dx.$

Putting $4x = \theta$ then $x = \frac{1}{4}\theta$ and $dx = \frac{1}{4}d\theta$

Limits $\left.\begin{matrix} x = 0 \\ \theta = 0 \end{matrix}\right\}\ \left.\begin{matrix} x = \frac{\pi}{8} \\ \theta = \frac{\pi}{2} \end{matrix}\right\}$

$$\therefore\ I = \int_0^{\pi/2} \sin^2 2\theta \cos^5\theta .\ \frac{1}{4}\, d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} (2\sin\theta \cos\theta)^2 \cos^5\theta\, d\theta .$$

$$= \int_0^{\pi/2} \sin^2\theta \cos^7\theta\, d\theta = \frac{\Gamma\left(\frac{2+1}{2}\right) . \Gamma\left(\frac{7+1}{2}\right)}{2\Gamma\left(\frac{2+7+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(4)}{2\Gamma\frac{11}{2}} = \frac{\Gamma\left(\frac{3}{2}\right) . \underline{|3}}{2 . \frac{9}{2} . \frac{7}{2} . \frac{5}{2} . \frac{3}{2}\Gamma\left(\frac{3}{2}\right)}$$

$$\frac{3 . 2 . 1 . 8 .}{9 . 7 . 5 . 3} = \frac{16}{315} .$$

Example 12. Show that

$$\int_0^1 x^{m-1} (1 - x^a)^n\, dx = \frac{1}{a} \frac{\underline{|n}\ \Gamma\left(\frac{m}{a}\right)}{\Gamma\left(\frac{m}{a} + n + 1\right)}$$

Proof : $L.H.S = \int_0^1 x^{m-1} (1 - x^a)^n\, dx.$

We substitute $x^a = \sin^2\theta$ i. e. $x = (\sin^2\theta)^{\frac{1}{a}} = \sin^{\frac{2}{a}} \theta.$

Then $dx = \frac{2}{a} \sin^{\frac{2}{a}-1} \theta . \cos\theta d\theta$

Limits $\left.\begin{matrix} x = 0 \\ \theta = 0 \end{matrix}\right\}\ \left.\begin{matrix} x = 1 \\ \theta = \frac{\pi}{2} \end{matrix}\right\}$

Thus $L.H.S = \int_0^{\pi/2} \sin^{\frac{2(m-1)}{a}} \theta \cos^{2n}\theta .\ \frac{2}{a} \sin^{\frac{2}{a}-1} \theta .\ \cos\theta d\theta.$

$$= \frac{2}{a}\int_0^{\pi/2} \sin^{\frac{2m}{a}-1} \theta \cos^{2n+1} \theta d\theta$$

$$= \frac{2}{a}\int_0^{\pi/2} \sin^{\frac{2m}{a}-1} \theta \cos^{2(n+1)-1}\theta d\theta$$

$$= \frac{2}{a} \frac{\Gamma\left(\frac{m}{a}\right)\Gamma(n+1)}{2\Gamma\left(\frac{m}{a} + n + 1\right)} = \frac{1}{a} . \frac{\Gamma\left(\frac{m}{a}\right) . \underline{|n}}{\Gamma\left(\frac{m}{a} + n + 1\right)}$$

Example 13. Prove that $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \frac{\pi}{\sqrt{2}}.$

Proof. $L. H. S = \int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \frac{\sqrt{\sin\theta}}{\sqrt{\cos\theta}} d\theta .$

$$= \int_0^{\pi/2} \sin^{\frac{1}{2}} \theta \cos^{-\frac{1}{2}} \theta d\theta$$

$$= \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right) \Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\frac{\frac{1}{2}-\frac{1}{2}+2}{2}\right)} = \frac{\Gamma\left(\frac{3}{4}\right) . \Gamma\left(\frac{1}{4}\right)}{2\Gamma 1} . = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(1 - \frac{1}{4}\right)$$

$$= \frac{1}{2} . \frac{\pi}{\sin\frac{\pi}{4}} = \frac{1}{2} . \frac{\pi}{\frac{1}{\sqrt{2}}} = \frac{\pi}{\sqrt{2}} .$$

Example 14. Prove that

$$\int_0^1 x^m (\log x)^n\, dx = \frac{(-1)^n \underline{|n}}{(m+1)^{n+1}};\ n > 0,\ m > -1.$$

Proof : L. H. S = $\int_0^1 x^m (\log x)^n dx$

Putting $x = e^{-y}$ i. e. $\log x = -y$

then $dx = -e^{-y}\, dy$. $\left.\begin{matrix} x = 0 \\ y = \infty \end{matrix}\right\}$ $\left.\begin{matrix} x = 1 \\ y = 0 \end{matrix}\right\}$

$$\therefore \quad \text{L. H. S} = \int_0^1 x^m (\log x)^n dx = -\int_\infty^0 e^{-my} (-y)^n e^{-y} dy.$$

$$= \int_0^\infty (-y)^n e^{-(m+1)y} dy.$$

Again Putting $(m+1)y = z$ then $dy = \frac{1}{m+1} dz$

$\left.\begin{matrix} y = 0 \\ z = 0 \end{matrix}\right\}$ $\left.\begin{matrix} y = \infty \\ z = \infty \end{matrix}\right\}$

$$\therefore \text{L. H. S.} = \int_0^\infty (-y)^n e^{-(m+1)y} dy$$

$$= \int_0^\infty \left(\frac{-z}{m+1}\right)^n . e^{-z} . \frac{1}{m+1} dz$$

$$= \frac{(-1)^n}{(m+1)^{n+1}} \int_0^\infty e^{-z} z^n dz = \frac{(-1)^n}{(m+1)^{n+1}} \Gamma(n+1)$$

$$= \frac{(-1)^n \lfloor n}{(m+1)^{n+1}}$$

since $\Gamma(n+1) = \lfloor n$ and $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx.$

Example 15. Prove that $\int_0^\infty x^p e^{-x^q} dx = \frac{1}{q} \Gamma\left(\frac{p+1}{q}\right).$

Proof : L. H. S = $\int_0^\infty x^p e^{-x^q} dx$

Putting $x^q = z$ then $x = z^{\frac{1}{q}}$

$$dx = \frac{1}{q} . z^{\frac{1}{q}-1} dz$$

Limits $\left.\begin{matrix} x = 0 \\ z = 0 \end{matrix}\right\}$ $\left.\begin{matrix} x = \infty \\ z = \infty \end{matrix}\right\}$

$$\text{L. H. S} = \int_0^\infty z^{p/q} e^{-z} . \frac{1}{q} . z^{1/q-1} dz$$

$$= \frac{1}{q} \int_0^\infty e^{-z} . z^{\frac{p+1}{q}-1} dz = \frac{1}{q} . \Gamma\left(\frac{p+1}{q}\right).$$

Example 16. Show that

$$\int_0^1 \frac{x^n dx}{\sqrt{1-x^2}} = \frac{1.\,3.\,5.\,\ldots.(n-1)}{2.\,4.\,6.\,\ldots..\,n} . \frac{\pi}{2}.$$

if n is an even positive integer.

Proof : L. H. S = $\int_0^1 \frac{x^n}{\sqrt{1-x^2}} dx.$

Putting $x = \sin\theta$ then $dx = \cos\theta d\theta$

Limits $\left.\begin{matrix} x = 0 \\ \theta = 0 \end{matrix}\right\}$ $\left.\begin{matrix} x = 1 \\ \theta = \frac{\pi}{2} \end{matrix}\right\}$ $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \cos\theta$.

$$\therefore \text{L. H. S} = \int_0^1 \frac{x^n}{\sqrt{1-x^2}} dx = \int_0^{\pi/2} \frac{\sin^n\theta \cos\theta}{\cos\theta} d\theta$$

$$= \int_0^{\pi/2} \sin^n \theta d\theta.$$

When n is an even integer, applying **Walli's formula,** we get

$$\int_0^{\pi/2} \sin^n \theta d\theta = \frac{n-1}{n} . \frac{n-3}{n-2} \cdots \frac{3}{4} . \frac{1}{2} . \frac{\pi}{2}.$$

$$= \frac{1.\,3.\,5.\,\ldots\ldots(n-3)\,(n-1)}{2.\,4.\,6.\,\ldots\,(n-2)n} . \frac{\pi}{2}.$$

$$\therefore \int_0^1 \frac{x^n dx}{\sqrt{1-x^2}} = \frac{1.\,3.\,5.\,\ldots\,(n-3)(n-1)}{2.\,4.\,6.\,\ldots\,(n-2)n} . \frac{\pi}{2}.$$

Example 17. Prove that $\Gamma(n)\Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}\,\Gamma(2n)}{2^{2n-1}}$

Proof : L. H. S $=\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right)$

$$=\{(n-1)(n-2)\ldots 3.\,2.\,1\}\left\{\left(n+\frac{1}{2}-1\right)\left(n+\frac{1}{2}-2\right)\ldots\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)\right\}$$

$$=\left\{\frac{(2n-2)(2n-4)\ldots 6.4.\,2}{2^{n-1}}\right\}\left\{\frac{(2n-1)(2n-3)\ldots 5.\,3.\,1.\sqrt{\pi}}{2^n}\right\}$$

$$=\frac{(2n-1)(2n-2)(2n-3)(2n-4)\ldots 6.5.\,4.\,3.\,2.\,1.\sqrt{\pi}}{2^{2n-1}}$$

$$=\frac{\sqrt{\pi}\,\Gamma(2n)}{2^{2n-1}} \text{ since } \Gamma(n)=\underline{|n-1}\,.$$

Example 18. Evaluate $\int_0^1 (1-x^{2m})^{\frac{1}{m}}\,x dx.$

Let $I=\int_0^1 (1-x^{2m})^{\frac{1}{m}}\,xdx=\int_0^1 (1-(x^2)^m)^{1/m}\,xdx.$

Putting $x^2=y$ then $2xdx=dy$, or, $xdx=\frac{1}{2}dy$

Limits $\left.\begin{matrix}x=0\\ y=0\end{matrix}\right\}\ \left.\begin{matrix}x=1\\ y=1\end{matrix}\right\}$

$$\therefore\quad I=\frac{1}{2}\int_0^1\left(1-y^m\right)^{\frac{1}{m}}dy.$$

Again Putting $y^m=\sin^2\theta$. i. e. $y=\sin^{\frac{2}{m}}\theta.$

$$\therefore\quad dy=\frac{2}{m}\cdot\sin^{\frac{2}{m}-1}\theta\cos\theta d\theta.$$

Limits $\left.\begin{matrix}y=0\\ \theta=0\end{matrix}\right\}\ \left.\begin{matrix}y=1\\ \theta=\frac{\pi}{2}\end{matrix}\right\}$

$$\therefore I=\frac{1}{2}\int_0^{\pi/2}(1-\sin^2\theta)^{\frac{1}{m}}\,\frac{2}{m}\sin^{\frac{2}{m}-1}\theta\cos\theta d\theta.$$

$$=\frac{1}{m}\int_0^{\pi/2}\cos^{\frac{2}{m}}\theta\,\sin^{\frac{2}{m}-1}\theta\cos\theta d\theta.$$

$$=\frac{1}{m}\int_0^{\pi/2}\cos^{\frac{2}{m}+1}\theta\sin^{\frac{2}{m}-1}\theta d\theta.$$

$$=\frac{1}{m}\cdot\frac{\Gamma\frac{1}{2}\left(\frac{2}{m}+1+1\right)\Gamma\frac{1}{2}\left(\frac{2}{m}-1+1\right)}{2\,\Gamma\frac{1}{2}\left(\frac{2}{m}+1+\frac{2}{m}-1+2\right)}$$

$$=\frac{1}{2m}\cdot\frac{\Gamma\left(\frac{1}{m}+1\right)\Gamma\left(\frac{1}{m}\right)}{\Gamma\left(\frac{2}{m}+1\right)}\cdot$$

$$=\frac{1}{2m}\cdot\frac{\frac{1}{m}\cdot\Gamma\left(\frac{1}{m}\right)\Gamma\left(\frac{1}{m}\right)}{\frac{2}{m}\Gamma\left(\frac{2}{m}\right)}=\frac{1}{4m}\cdot\frac{\left\{\Gamma\left(\frac{1}{m}\right)\right\}^2}{\Gamma\left(\frac{2}{m}\right)}.$$

Thus $\int_0^1 (1-x^{2m})^{\frac{1}{m}}\,xdx=\frac{1}{4m}\,\frac{\left\{\Gamma\left(\frac{1}{m}\right)\right\}^2}{\Gamma\left(\frac{2}{m}\right)}.$

Exmaple 19. Show that $\int_0^{\infty} e^{-x^4} x^2 dx \times \int_0^{\infty} e^{-x^4} dx = \frac{\pi}{8\sqrt{2}}$.

Proof : Let $I = \int_0^{\infty} e^{-x^4} x^2 dx \times \int_0^{\infty} e^{-x^4} dx$

Again let $I_1 = \int_0^{\infty} e^{-x^4} x^2 dx$ and $I_2 = \int_0^{\infty} e^{-x^4} dx$.

Putting $x^4 = z$ then $4x^3 dx = dz$.

Or, $x^2 dx = \frac{1}{4} z^{-\frac{1}{4}} dz$.

Limits $\left.\begin{matrix} x = 0 \\ z = 0 \end{matrix}\right\} \left.\begin{matrix} x = \infty \\ z = \infty \end{matrix}\right\}$

$$\therefore I_1 = \frac{1}{4}\int_0^{\infty} e^{-z} z^{-\frac{1}{4}} dz = \frac{1}{4}\int_0^{\infty} e^{-z}. z^{3/4-1} dz = \frac{1}{4}\Gamma\left(\frac{3}{4}\right).$$

$$I_2 = \int_0^{\infty} e^{-x^4} dx = \frac{1}{4}\int_0^{\infty} e^{-z}. z^{-3/4} dz$$

$$= \frac{1}{4}\int_0^{\infty} e^{-z}. z^{\frac{1}{4}-1} dz = \frac{1}{4}\Gamma\left(\frac{1}{4}\right).$$

Now $I = I_1 \times I_2 = \frac{1}{4}\Gamma\left(\frac{3}{4}\right) \cdot \frac{1}{4}\Gamma\left(\frac{1}{4}\right)$

$$= \frac{1}{16} \Gamma\left(\frac{1}{4}\right) \Gamma\left(1 - \frac{1}{4}\right) = \frac{1}{16} \cdot \frac{\pi}{\sin\frac{\pi}{4}}$$

$$= \frac{1}{16} \cdot \frac{\pi}{\sqrt{\frac{1}{2}}} = \frac{\pi}{8\sqrt{2}}.$$

Thus $\int_0^{\infty} e^{-x^4} x^2 dx \times \int_0^{\infty} e^{-x^4} dx = \frac{\pi}{8\sqrt{2}}$.

Example 20. Show that.

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}} dx = \frac{\beta(m,n)}{a^n (1+a)^m}$$

$$= \frac{\Gamma(m)\,\Gamma(n)}{a^n (1+a)^m\,\Gamma(m+n)}$$

Proof :

Let $I = \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}} dx$ (1)

Putting $\frac{x(1+a)}{(a+x)} = y$ **Limits** : $\left.\begin{matrix} x = o \\ y = o \end{matrix}\right\} \left.\begin{matrix} x = 1 \\ y = 1 \end{matrix}\right\}$

or, $x + ax = ay + xy$

or, $x + ax - xy = ay$

or, $x(1 + a - y) = ay$

or, $x = \frac{ay}{1 + a - y}$.

$$dx = \left\{\frac{(1 + a - y). a - ay (o - 1)}{(1 + a - y)^2}\right\} dy$$

$$\text{or, } dx = \left\{\frac{a(1 + a) - ay + ay}{(1 + a - y)^2}\right\} dy$$

$$\text{or, } dx = \frac{a(1 + a)}{(1 + a - y)^2} dy.$$

$$1 - x = 1 - \frac{ay}{1 + a - y} = \frac{1 + a - y - ay}{1 + a - y}$$

$$= \frac{(1 + a) - y(1 + a)}{1 + a - y} = \frac{(1 + a)(1 - y)}{1 + a - y}$$

$$a + x = a + \frac{ay}{1 + a - y} = \frac{a + a^2 - ay + ay}{1 + a - y} = \frac{a(1 + a)}{1 + a - y}$$

Thus from (1), we have

$$= \int_0^1 \frac{a^{m-1} y^{m-1} (1 + a)^{n-1} (1 - y)^{n-1} (1 + a - y)^{m+n} a(1 + a)}{(1 + a - y)^{m-1}. (1 + a - y)^{n-1}. a^{m+n} (1 + a)^{m+n} (1 + a - y)^2}$$

$$\text{or, } I = \int_0^1 \frac{y^{m-1}(1-y)^{n-1}\,dy}{a^n(1+a)^m}$$

$$= \frac{1}{a^n(1+a)^m}\int_0^1 y^{m-1}(1-y)^{n-1}\,dy.$$

$$= \frac{1}{a^n(1+a)^m}\,\beta(m,n) = \frac{\beta(m,n)}{a^n(1+a)^m}.$$

$$\text{since } \beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$$

$$\therefore\ I = \frac{\Gamma m\,\Gamma n}{a^n(1+a)^m\,\Gamma(m+n)}.$$

Hence

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}\,dx}{(a+x)^{m+n}} = \frac{\beta(m,n)}{a^n(1+a)^m} = \frac{\Gamma m\,\Gamma n}{a^n(1+a)^m\,\Gamma(m+n)}.$$

Example 21. Prove that

$$\int_0^\infty e^{-\alpha\lambda^2}\cos\beta\lambda\,d\lambda = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\,e^{-\frac{\beta^2}{4\alpha}}$$

$$\text{or, } \int_0^\infty e^{-ax^2}\cos bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{-\frac{b^2}{4a}}$$

Proof : Let $I = I(a\,.\,b) = \int_0^\infty e^{-ax^2}\cos bx\,dx.$

$$\text{Then } \frac{\partial I}{\partial b} = \int_0^\infty \left(-xe^{-ax^2}\right)\sin bx\,dx.$$

$$= \int_0^\infty \left(\frac{-2axe^{-ax^2}}{2a}\right)\sin bx\,dx.$$

$$= \left[\sin bx.\,\frac{e^{-ax^2}}{2a}\right]_0^\infty - \frac{b}{2a}\int_0^\infty \cos bx.\,e^{-ax^2}\,dx.$$

$$= 0 - \frac{b}{2a}\int_0^\infty e^{-ax^2}\cos bx\,dx.$$

$$= 0 - \frac{b}{2a}I = -\frac{b}{2a}I.$$

$$\therefore \frac{1}{I}\frac{\partial I}{\partial b} = -\frac{b}{2a} \quad (1)$$

$$\text{or, } \frac{\partial}{\partial b}(\log I) = -\frac{b}{2a}.$$

Integration both sides with respect to b, we get

$\log I = -\dfrac{b^2}{4a} + C_1$, C_1 being some constant of intergation.

$$\text{or, } \log I = \log e^{-\frac{b^2}{4a}} + \log C = \log C\,e^{-\frac{b^2}{4a}}$$

$$\text{or, } I = Ce^{-\frac{b^2}{4a}}. \qquad \text{where } C_1 = \log C$$

$$\text{or, } I = I(a\,.\,b) = Ce^{-\frac{b^2}{4a}} \quad (2)$$

When $b = 0$, we have $I(a,o) = Ce^o = C.$

$$\therefore \boxed{C = I(a,o)}$$

$$C = I(a\,,o) = \int_0^\infty e^{-ax^2}.\,1\,dx = \int_0^\infty e^{-ax^2}\,dx.$$

$$\text{or, } C = \int_0^\infty e^{-(\sqrt{a}x)^2}\,dx.$$

Putting $\sqrt{a}x = t.$

$$\sqrt{a}\,dx = dt$$

Limits : $\left.\begin{matrix} x = 0 \\ t = 0\end{matrix}\right\}\ \left.\begin{matrix} x = \infty \\ t = \infty\end{matrix}\right\}$ $\therefore\ dx = \dfrac{1}{\sqrt{a}}dt.$

$$\text{or, } C = \frac{1}{\sqrt{a}}\int_0^\infty e^{-t^2}\,dt.$$

$$= \frac{1}{\sqrt{a}} \cdot \frac{\sqrt{\pi}}{2} \quad \text{since} \int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

Putting the value of C in (2), we get

$$I = \frac{1}{2}\sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}}.$$

Example 22. Prove that $\int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}$, $0 < p < 1$

Proof : $I = \int_0^\infty \frac{x^{p-1}}{1+x} dx.$

$$= \int_0^1 \frac{x^{p-1}}{1+x} dx + \int_1^\infty \frac{x^{p-1}}{1+x} dx \quad (1)$$

Putting $x = \frac{1}{y}$ *in the 2nd integral so the* $dx = -\frac{1}{y^2} dy$.

$$1 + x = 1 + \frac{1}{y} = \frac{1+y}{y}$$

Limits $\left.\begin{matrix} x = 1 \\ y = 1 \end{matrix}\right\} \left.\begin{matrix} x = \infty \\ y = 0 \end{matrix}\right\}$

$$\int_1^\infty \frac{x^{p-1}}{1+x} dx = \int_1^0 \left(\frac{1}{y}\right)^{p-1} \frac{y}{1+y} \cdot -\frac{1}{y^2} dy.$$

$$= \int_0^1 y^{-p+1} \frac{1}{1+y} \cdot \frac{1}{y} dy.$$

$$= \int_0^1 \frac{y^{-p}}{1+y} dy = \int_0^1 \frac{x^{-p}}{1+x} dx$$

Thus from (1), we get

$$I = \int_0^1 \frac{x^{p-1}}{1+x} dx + \int_0^1 \frac{x^{-p}}{1+x} dx.$$

$$= \int_0^1 \frac{x^{p-1} + x^{-p}}{(1+x)} dx \quad (2).$$

Now using binomial theorem, we get

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots + (-1)^r x^r + \cdots$$

$$= \sum_{r=0}^{\infty} (-x)^r = \sum_{r=0}^{n} (-x)^r + \sum_{r=n+1}^{\infty} (-x)^r$$

$$= \sum_{r=0}^{n} (-x)^r + \frac{(-x)^{n+1}}{(1+x)}$$

Therefore, $I = \int_0^1 \frac{x^{p-1} + x^{-p}}{1+x} dx$

$$= \int_0^1 (x^{p-1} + x^{p}) \left[1 - x + x^2 - \ldots + (-x)^n + \frac{(-x)^{n+1}}{1+x}\right] dx.$$

$$= \sum_{k=0}^{n} (-1)^k \int_0^1 (x^{p-1+k} + x^{p+k})\, dx + \int_0^1 (x^{p-1} + x^{-p})(-1)^{n+1} \frac{x^{n+1}}{1+x} dx.$$

$$= \sum_{k=0}^{n} (-1)^k \left[\frac{x^{p+k}}{p+k} + \frac{x^{-p+1+k}}{-p+1+k}\right]_0^1 + (-1)^{n+1} \int_0^1 (x^{p-1} + x^{p}) \frac{x^{n+1}}{1+x} dx.$$

$$= \sum_{k=0}^{n} (-1)^k \left[\frac{1}{p+k} + \frac{1}{k+1-p}\right] + R_n$$

where $R_n = (-1)^{n+1} \int_0^1 (x^{p-1} + x^{p}) \frac{x^{n+1}}{1+x} dx$

$$|R_n| = \int_0^1 (x^{p-1} + x^{-p}) \frac{x^{n+1}}{1+x} dx < \int_0^1 (x^{p-1} + x^{-p})\, x^{n+1} dx.$$

$$= \int_0^1 (x^{n+p} + x^{n+1-p})\, dx.$$

$$= \left[\frac{x^{n+p+1}}{n+p+1} + \frac{x^{n-p+2}}{n-p+2}\right]_0^1$$

$$= \left[\frac{1}{n+p+1} + \frac{1}{n-p+2}\right] \to 0 \text{ as } n \to \infty$$

Therefore, $I = \int_0^1 \frac{x^{p-1} + x^{-p}}{1+x} dx$

$$= \sum_{k=0}^{n} (-1)^k \left[\frac{1}{k+p} + \frac{1}{k-p+1}\right]$$

which is a standard Fourier series.

and $\sum_{k=0}^{n} (-1)^k \left[\frac{1}{k+p} + \frac{1}{k-p+1}\right] = \frac{\pi}{\sin p\pi}$.

Hence, $I = \int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}$

Complement or reflection Formula.

Example 23. $\Gamma p \,\Gamma 1-p = \frac{\pi}{\sin p\pi}$, $0 \angle p \angle 1$.

proof : By definition of the beta integral of the second kind we have

$$\beta(p, q) = \int_0^1 \frac{x^{p-1} dx}{(1+x)^{p+q}}, p > o, q > 0 \quad (1)$$

putting $p + q = 1$. 1.e $q = 1 - p$. in (1)

we get $\beta(p, 1-p) = \int_0^\infty \frac{x^{p-1}}{1+x} dx$.

or, $\frac{\Gamma p \,\Gamma 1-p}{\Gamma p+1-p} = \int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}$

or, $\frac{\Gamma p \,\Gamma 1-p}{\Gamma 1} = \frac{\pi}{\sin p\pi}$

or, $\Gamma p \,\Gamma 1-p = \frac{\pi}{\sin p\pi}$.

Example 24. *Prove that*

$$I = \int_{-\infty}^{\infty} \frac{e^{at}}{1+e^t} dt = \Gamma a \;\Gamma(1-a.)$$

Proof : $I = \int_{-\infty}^{\infty} \frac{e^{at}}{1+e^t} dt$. Putting $e^t = x$.

$e^{at} = (e^t)^a = x^a$. Then $e^t dt = dx$.

$$\therefore dt = \frac{dx}{e^t} = \frac{dx}{x}$$

Limits : $\left.\begin{matrix} t = -\infty \\ x = 0 \end{matrix}\right\} \left.\begin{matrix} t = \infty \\ x = \infty \end{matrix}\right\}$

$$I = \int_0^\infty \frac{x^a}{1+x} \cdot \frac{dx}{x} = \int_0^\infty \frac{x^{a-1}\, dx}{1+x}$$

$$= \int_0^\infty \frac{x^{a-1}\, dx}{(1+x)^{(1-a)+a}} = \beta(a, 1-a)$$

$$= \Gamma a \;\Gamma(1-a.)$$

Hence $\int_{-\infty}^{\infty} \frac{e^{at}}{1+e^t} dt = \Gamma a \;\Gamma(1-a)$.

Example 25. *Establish the* ***Gauss integral***

$$\int_0^\infty e^{-\lambda x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}, \lambda > o.$$

Proof : This integral is convergent, since $e^{-\lambda x^2} < e^{-\lambda x}$

for $x > 1$. *and* $\int_0^\infty e^{-\lambda x} dx$ is finite. To evaluate the given integral, we reduce it to gamma integral (function) :

$I = \int_{-0}^{\infty} e^{-\lambda x^2} dx$ Putting $\lambda x^2 = z$ $x = \left(\frac{z}{\lambda}\right)^{\frac{1}{2}}$

$$2\lambda x dx = dz.$$

Limits $\left.\begin{matrix} x = 0 \\ z = 0 \end{matrix}\right\} \left.\begin{matrix} x = \infty \\ z = \infty \end{matrix}\right\}$

$$I = \int_0^\infty e^{-z} \cdot \frac{dz}{2\lambda x}.$$

$$= \frac{1}{2\lambda}\int_0^\infty e^{-z}x^{-1}dz = \frac{1}{2\lambda}\int_0^\infty e^{-z} \cdot \frac{z^{-\frac{1}{2}}}{\lambda^{-\frac{1}{2}}}\, dz.$$

$$= \frac{1}{2\sqrt{\lambda}}\int_0^\infty e^{-z} z^{\frac{1}{2}-1}\, dz.$$

$$= \frac{1}{2\sqrt{\lambda}}\cdot \Gamma\left(\frac{1}{2}\right) = \frac{1}{2\sqrt{\lambda}}.\sqrt{\pi} = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}.$$

Example 26. *Prove that* $\displaystyle\int_0^\infty \frac{\cos x}{x^p}\,dx = \frac{\pi}{2\,\Gamma(p)\cos\left(\frac{p\pi}{2}\right)}$

Proof : By definition of gamma function, we have

$$\Gamma(p) = \int_0^\infty e^{-\lambda}\lambda^{p-1}\,d\lambda \quad (1).\ 0<p<1.$$

Putting $\lambda = ux$ in (1). so that

$d\lambda = x\,du$

Limits $\left.\begin{matrix}\lambda = 0\\ u = 0\end{matrix}\right\}$ $\left.\begin{matrix}\lambda = \infty\\ u = \infty\end{matrix}\right\}$

$$\therefore \Gamma(p) = \int_0^\infty e^{-ux}(ux)^{p-1}\,x\,du$$

$$= \int_0^\infty e^{-ux}u^{p-1}.x^{p-1}.x\,du$$

$$= \int_0^\infty e^{-ux}u^{p-1}.x^{p}\,du.$$

or. $\displaystyle\frac{\Gamma(p)}{x^p} = \int_0^\infty u^{p-1}e^{-ux}\,du.$

or. $\displaystyle\frac{1}{x^p} = \frac{1}{\Gamma(p)}\int_0^\infty u^{p-1}e^{-ux}\,du \quad (2)$

$$\therefore \int_0^\infty \frac{\cos x}{x^p}\,dx = \frac{1}{\Gamma(p)}\int_0^\infty\int_0^\infty u^{p-1}e^{-ux}\cos x\,dx\,du.$$

$$= \frac{1}{\Gamma(p)}\int_0^\infty\left[\int_0^\infty e^{-ux}\cos x\,dx\right]u^{p-1}\,du$$

$$= \frac{1}{\Gamma(p)}\int_0^\infty \frac{u}{u^2+1}u^{p-1}\,du$$

since $\displaystyle\int_0^\infty e^{-ux}\cos x\,dx = \frac{u}{u^2+1}$

$$= \frac{1}{\Gamma(p)}\int_0^\infty \frac{u^p}{1+u^2}\,du \quad (3)$$

Putting $u^2 = v$ in (3) so that

Limits : $\left.\begin{matrix}u = 0\\ v = 0\end{matrix}\right\}$ $\left.\begin{matrix}u = \infty\\ v = \infty\end{matrix}\right\}$

$u = v^{\frac{1}{2}}$, $2u\,du = dv.$

Thus from (3), we have

$$\int_0^\infty \frac{\cos x}{x^p}\,dx = \frac{1}{\Gamma(p)}\int_0^\infty \frac{v^{\frac{p}{2}}}{1+v}\cdot\frac{dv}{2v^{\frac{1}{2}}}$$

$$= \frac{1}{2\,\Gamma(p)}\int_0^\infty \frac{v^{\frac{p-1}{2}}}{1+v}\,dv.$$

$$= \frac{1}{2\,\Gamma(p)}\int_0^\infty \frac{v^{\frac{p-1}{2}+1-1}}{1+v}\,dv.$$

$$= \frac{1}{2\,\Gamma(p)}\int_0^\infty \frac{v^{\frac{p+1}{2}-1}}{1+v}\,dv.$$

$$= \frac{1}{2\,\Gamma(p)}\cdot\frac{\pi}{\sin\left(\frac{p+1}{2}\right)\pi}$$

$$= \frac{1}{2\,\overline{|p}} \cdot \frac{\pi}{\sin\left(\frac{\pi}{2}+\frac{p\pi}{2}\right)}$$

$$= \frac{1}{2\,\overline{|p}} \cdot \frac{\pi}{\cos\frac{p\pi}{2}}$$

Hence $\displaystyle\int_0^\infty \frac{\cos x}{x^p}\,dx = \frac{\pi}{2\overline{|p}\,\cos\frac{p\pi}{2}}$.

Applications of gamma functions in Mechanics.

Example 27. The equation of motion of a particle moving from rest, towards a centre of attraction point, situated at a distance 'a' from it, is given by $\frac{d^2x}{dt^2}+\frac{k}{x}=0$, where k is a constant. Applying gamma function show that the particle will reach the centre in a time given by $T = a\sqrt{\frac{\pi}{2k}}$.

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Proof : The given equation of motion is $\frac{d^2x}{dt^2}+\frac{k}{x}=0$,

or, $\frac{d^2x}{dt^2} = -\frac{k}{x}$ (1)

Multiplying both sids of (1) by $\frac{2dx}{dt}$, we get

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} = -\frac{2k}{x}\frac{dx}{dt}$$

$$\frac{d}{dt}\left(\frac{dx}{dt}\right)^2 = -\frac{2k}{x}\frac{dx}{dt}$$

Integrating both sides, we get

$$\left(\frac{dx}{dt}\right)^2 = -2k\log x + c_1 \quad (2)$$

where c_1 is a constant. Initially when $x=a, \frac{dx}{dt}=0$

Thus from (2), we get $0 = -2k\log a + c_1$

$\therefore \boxed{c_1 = 2k\log a.}$

Therefore, $\left(\frac{dx}{dt}\right)^2 = 2k\log a - 2k\log x$

$$= 2k\log\frac{a}{x}$$

$$\therefore \frac{dx}{dt} = \sqrt{(2k)}\sqrt{\log\frac{a}{x}}$$

or, $dt = \frac{1}{\sqrt{2k}}\frac{1}{\sqrt{\log\frac{a}{x}}}dx$ (3)

Integrating both sides of (3), we get

$$\int_0^T dt = \frac{1}{\sqrt{2k}}\int_0^a \left(\log\frac{a}{x}\right)^{-\frac{1}{2}}dx.$$

or, $T = \frac{1}{\sqrt{2k}}\int_0^a\left(\log\frac{a}{x}\right)^{-\frac{1}{2}}dx$ (4)

Putting $\log\frac{a}{x} = p$. or, $\frac{a}{x} = e^p$

$$\therefore x = ae^{-p}.$$

Limits

When $\left.\begin{matrix} x=0 \\ p=\infty \end{matrix}\right\}$ when $\left.\begin{matrix} x=a \\ p=0 \end{matrix}\right\}$ $dx = -ae^{-p}\,dp$.

Thus from (4), we have

$$T = \frac{1}{\sqrt{2k}}\int_\infty^0 p^{-\frac{1}{2}}\cdot -ae^{-p}\,dp.$$

$$= \frac{a}{\sqrt{2k}}\int_0^\infty e^{-p}p^{-\frac{1}{2}}\,dp.$$

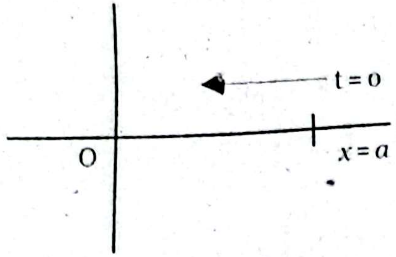
$$= \frac{a}{\sqrt{2k}}\int_0^\infty e^{-p}p^{\frac{1}{2}-1}\,dp.$$

since $\overline{|(n)} = \int_0^\infty e^{-x}x^{n-1}\,dx$.

$$= \frac{1}{\sqrt{2k}}\overline{\left|\left(\frac{1}{2}\right)\right.} = \frac{a}{\sqrt{2k}}\sqrt{\pi} = a\sqrt{\frac{\pi}{2k}}. \text{ Since } \overline{\left|\left(\frac{1}{2}\right)\right.} = \sqrt{\pi}$$

Hence $T = a\sqrt{\frac{\pi}{2k}}$.

Example 28. A particle moves with an acceleration which is always towards and equal to μ divided by the distance from a fixed point O. If it starts from rest at a distance a from O, show that it will arrive at O in time a $\sqrt{\dfrac{\pi}{2\mu}}$.



Proof : Let the distance be x.

Then the equation of motion is $\dfrac{d^2x}{dt^2} = -\dfrac{\mu}{x}$ (1)

[as accleration is always towards]

Multiplying both sids of (1) by $2\dfrac{dx}{dt}$, we get

$$2\frac{dx}{dt}\cdot\frac{d^2x}{dt^2} = -2\frac{\mu}{x}\frac{dx}{dt}.$$

$$\text{or, } \frac{d}{dt}\left(\frac{dx}{dt}\right)^2 = -2\mu.\frac{1}{x}\frac{dx}{dt}.$$

Integrating both sides, we get $\left(\dfrac{dx}{dt}\right)^2 = -2\mu \log x + c$ (2)

When $x = a$, $\dfrac{dx}{dt} = 0$. Thus (2) becomes

$$o = -2\mu\log a + c \quad \therefore \quad c = 2\mu\log a.$$

Putting the value of c in (2), we get

$$\left(\frac{dx}{dt}\right)^2 = -2\mu \log x + 2\mu\log a.$$

$$\text{or, } \left(\frac{dx}{dt}\right)^2 = 2\mu\,(\log a - \log x) = 2\mu \log\frac{a}{x}$$

$\therefore \dfrac{dx}{dt} = -\sqrt{2\mu}\sqrt{\log\dfrac{a}{x}}$ [since the particle moves towards the point O).

$$\text{or, } \frac{dx}{\sqrt{\log\frac{a}{x}}} = -\sqrt{2\mu}\,dt.$$

$$\text{or, } \sqrt{2\mu}\,dt = -\frac{dx}{\sqrt{\log\frac{a}{x}}}$$

Integrating both sides and taking the limits, we get

$$\sqrt{2\mu}\,t = -\int_a^o \frac{dx}{\sqrt{\log\frac{a}{x}}} = \int_o^a \frac{dx}{\sqrt{\log\frac{a}{x}}} \quad (3)$$

Putting $\log\dfrac{a}{x} = z^2 \Rightarrow \dfrac{a}{x} = e^{z^2}$

$$\therefore\ x = ae^{-z^2}$$

$$dx = ae^{-z^2}.\,-2zdz$$

Limits $\left.\begin{matrix} x = o \\ z = \infty \end{matrix}\right\}$ $\left.\begin{matrix} x = a \\ z = o \end{matrix}\right\}$

Thus from (3), we get

$$\sqrt{2\mu}\,t = \int_\infty^o \frac{-2a\,z\,e^{-z^2}\,dz}{z}$$

$$\text{or, } \sqrt{2\mu}\,t = 2a\int_0^\infty e^{-z^2}\,dz \quad (4)$$

Putting $z^2 = y$ so that $2z\,dz = dy$

Limits $\left.\begin{matrix} z = o \\ y = o \end{matrix}\right\}$ $\left.\begin{matrix} z = \infty \\ y = \infty \end{matrix}\right\}$

$$\text{Or, } dz = \frac{1}{2z}dy\,.\ \text{or, } dz = \frac{1}{2}y^{-\frac{1}{2}}\,dy.$$

Thus from (4), we get

$$\sqrt{2\mu}t = 2a\int_0^\infty e^{-y}.\frac{1}{2}y^{-\frac{1}{2}}\,dy.$$

$$= a\int_0^\infty e^{-y}\,y^{\frac{1}{2}-1}\,dy$$

$$= a\,\Gamma\left(\frac{1}{2}\right) \text{ since } \Gamma(n) = \int_0^\infty e^{-t}\, t^{n-1}\, dt.$$

$$= a\sqrt{\pi} \text{ since } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

$$\therefore \quad t = \frac{a\sqrt{\pi}}{\sqrt{2\mu}}$$

$$\text{or, } t = a\sqrt{\frac{\pi}{2\mu}}.$$

EXERCISES 2 (A)

1. Evaluate the following integrals :

(i) $\int_0^1 \frac{x^2}{\sqrt{1-x}}\,dx$ (ii) $\int_0^4 x^{3/2}(4-x)^{\frac{5}{2}}\,dx$

(iii) $\int_0^b y^5\sqrt{b^2-y^2}\,dy$ (iv) $\int_0^a x^3(a-x)^2\,dx.$

Answer : (i) $\frac{16}{15}$ (ii) 12π (iii) $\frac{8b^7}{105}$ (iv) $\frac{1}{60}a^6.$

2. Evaluate the following integrals :

(i) $\int_0^\infty e^{-x^2}\,dx$ (ii) $\int_0^\infty x^6 e^{-3x}\,dx$

(iii) $\int_0^\infty e^{-x^2} x^9\,dx$ (iv) $\int_0^\infty \sqrt{x}\, e^{-x^2}\,dx.$

Answer : (i) $\frac{\sqrt{\pi}}{2}$ (ii) $\frac{80}{243}$ (iii) 12 (iv) $\frac{1}{2}\Gamma\left(\frac{3}{4}\right).$

3. Evaluate the following integrals :

(i) $\int_0^\pi \sin^5\theta\, \cos^4\theta\, d\theta$ (ii) $\int_0^\pi \sin^6\theta\, \cos\theta\, d\theta.$

(iii) $\int_0^{\pi/6} \cos^4 3x.\ \sin^2 6x\, dx.$ (iv) $\int_0^{\pi/4} \cos^3 2\theta.\ \sin^2 4\theta\, d\theta.$

Answer : (i) $\frac{16}{315}$ (ii) $\frac{32}{3003}$ (iii) $\frac{5\pi}{192}$ (iv) $\frac{16}{105}$

4. Prove that $\int_0^{\frac{\pi}{2}} \sin^8\theta\, d\theta = \int_0^{\frac{\pi}{2}} \cos^8\theta\, d\theta = \frac{35\pi}{256}$

5. Show that

(i) $\int_0^1 \frac{dx}{(1-x)^{\frac{1}{4}}} = \frac{4}{3}$ (ii) $\int_0^1 \left(1-x^{\frac{2}{3}}\right)^{\frac{3}{2}} dx = \frac{3\pi}{32}.$

6. In terms of gamma function evaluate the the following integrals :

(i) $\int_0^1 \frac{x^3}{\sqrt{1-x^3}}$ (ii) $\int_0^1 \frac{dx}{\sqrt{x\log\left(\frac{1}{x}\right)}}$

(iii) $\int_0^1 \left(1-\frac{1}{x}\right)^{\frac{1}{3}} dx$

Answer. (i) $\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{1}{2}\right) / 3\,\Gamma\left(\frac{11}{6}\right).$

(ii) $\sqrt{(2\pi)}$

(iii) $-\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{2}{3}\right).$

7. (i) Show that $\Gamma(x)\,\Gamma(1-x) = \int_0^\infty \frac{u^{x-1}}{1+u}\,du.$

(ii) prove that $\int_0^1 x^2\left(\log\frac{1}{x}\right)^3 dx = \frac{2}{27}.$

8. Applying beta and gamma functions evaluate the followings :

(i) $\int_0^1 \frac{dx}{(1-x^4)^{\frac{1}{2}}}$ (ii) $\int_0^{\pi/2} r(\sin\theta)\,d\theta$

Answer : (i) $\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{4\,\Gamma\left(\frac{3}{4}\right)}$ (ii) $\frac{2\,\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}\right)}$

9. Evaluate each of the following integrals :

(i) $\displaystyle\int_0^1 \frac{dx}{\sqrt{\log \frac{1}{x}}}$

(ii) $\displaystyle\int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx$ (iii) $\displaystyle\int_0^\infty \frac{x^t}{t^x} dx.$

Answer : (i) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

(ii) $\dfrac{\Gamma(n+1)}{(m+1)^{n+1}}.$

(iii) $\dfrac{\Gamma(t+1)}{(\log t)^{t+1}}.$

10. (i) Prove that $\displaystyle\int_0^1 \frac{dx}{x^n} = \sum_{n=1}^{\infty} \frac{1}{n^n}.$ **[D. U. S. 1982]**

(ii) Evaluate in terms of gamma function :

$\displaystyle\int_0^1 \left(\frac{x}{\log x}\right)^{\frac{1}{3}} dx.$ **[D. U. P. 1985]**

Answer : $(-1)^{\frac{1}{3}} \left(\frac{3}{4}\right)^{\frac{1}{3}} \Gamma\left(\frac{2}{3}\right)$

11. Show that $\Gamma(p)\,\Gamma(1-p) = \dfrac{\pi}{\sin p\pi}$ if

$\dfrac{\pi}{\sin p\pi} = \dfrac{1}{p} + 2p \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{p^2 - n^2}$ where $0 < p < 1.$

[D. U. S. 1983]

12. Prove that

(i) $\displaystyle\int_0^\infty \frac{e^{-5x}}{\sqrt{x}} dx = \sqrt{\frac{\pi}{5}}.$

(ii) $\displaystyle\int_0^\infty \left(\frac{1}{x} \log \frac{1}{x}\right)^{\frac{1}{2}} dx = \sqrt{2\pi}$

13. Prove that (i) $\displaystyle\int_0^\infty \frac{x\,dx}{1 + x^6} = \frac{\pi}{3\sqrt{3}};$

(ii) $\displaystyle\int_0^\infty \frac{y^2 dy}{1 + y^4} = \frac{\pi}{2\sqrt{2}}.$

14. Show that

$\displaystyle\int_0^\infty \frac{x^{p-1} \log x}{1 + x} dx = -\pi^2 \operatorname{cosec} p\pi \cot p\pi.$

Hence show that $\displaystyle\int_0^\infty \frac{\log x}{1 + x^4} dx = -\frac{\pi^2 \sqrt{2}}{16}$ **[D. U. S. 1983]**

15. Prove that

(i) $\beta(m, n) = \displaystyle\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1 + x)^{m+n}} dx$ where $m, n > 0.$

(ii) $\dfrac{\beta(m,n)}{r^n(1 + r)^m} = \displaystyle\int_0^1 \frac{x^{m-1}(1 - x)^{n-1}}{(x + r)^{m+n}} dx.$ **[D. U. S. 1982]**

16. Show that

$2\displaystyle\int_0^{\pi/2} (\cos\theta)^r\, d\theta = 2\int_0^{\pi/2} (\sin\theta)^r\, d\theta$

$= \beta\left(\dfrac{r+1}{2}, \dfrac{1}{2}\right), r > -1.$

where β denotes the beta function. **[D. U. P. 1984]**

17. Show that $\displaystyle\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - \frac{1}{2}\sin^2\varphi}} = \frac{\{\Gamma(\frac{1}{4})\}^2}{4\sqrt{\pi}}$ **[D.U.S.1986]**

18. Evaluate in terms of the gamma **[D. U. P. 1985]**

function : $\displaystyle\int_0^{\pi/2} \sqrt{(\sin 2x)}\,dx.$ **(Supplementary)**

Answer : $\dfrac{\sqrt{2}}{\sqrt{\pi}} \left\{\Gamma\left(\frac{3}{4}\right)\right\}^2.$

19. Prove that $\int_0^\infty x^m e^{-ax^n}\,dx = \frac{1}{na^{\frac{m+1}{n}}}\,\Gamma\left(\frac{m+1}{n}\right)$

20. Evaluate $\int_0^1 x^r \left[\log \frac{1}{x}\right]^s dx$

Answer : $\frac{s+1}{(r+1)^{s+1}}$

21. Show that $\beta(p,q) = \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}}\,du$

Hence show that $I = \int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^4} = \frac{5\pi}{16}$.

22. Show that $\int_0^1 \frac{x^n dx}{\sqrt{1-x^2}} = \frac{2.4.6.\ldots(n-1)}{1.3.5.\ldots n}$

if n is an odd positive integer.

23. Show that $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}}{n}\cdot\frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$

24. Prove that

(i) $\Gamma\left(\frac{2k+1}{2}\right) = \frac{1.3.5.\ldots(2k-1)}{2^k}\sqrt{\pi} = \frac{(2k)!}{4^k k!}\sqrt{\pi}$.

(ii) $\Gamma\left(\frac{1}{n}\right)\cdot\Gamma\left(\frac{2}{n}\right)\cdot\Gamma\left(\frac{3}{n}\right)\cdots\Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}$

(iii) $\Gamma\left(\frac{1}{2}-n\right) = (-1)^n\pi$ if n is a positive integer.

25. Prove that

$\int_0^\infty \frac{x^a}{(1+x)^b}\,dx = \beta(a+1, b-a-1)$, where $a>-1$, $b>a+1$.

26. Prove that

$\int_0^1 x^a(1-x^b)^{c-1}\,dx = \frac{1}{b}\,\beta\left(\frac{a+1}{b}, c\right)$ where $\frac{a+1}{b}>0$ and $c>0$.

27. Establish the following formulae for the specified values of the variables :

(i) $\Gamma(\alpha) = \int_0^1 \left(\log\frac{1}{x}\right)^{\alpha-1} dx$, $\alpha>0$.

(ii) $\Gamma(\alpha) = p^\alpha \int_0^1 x^{p-1}\left(\log\frac{1}{x}\right)^{\alpha-1} dx$; $\alpha>0$, $p>0$.

28. If n is an integer and $p>-1$ then show that

$\int_{-1}^1 x^n(1-x^2)^p\,dx = 0$ if n is odd

$= \beta\left(\frac{n-1}{2}, P+1\right)$ if n is even.

29. Show that $\int_0^1 x^m(1-x^n)^p\,dx = \frac{1}{n}\,\beta\left(\frac{m+1}{n}, P+1\right)$

where $\frac{m+1}{n}>0$, $p>-1$.

30. (i) Prove that

$\beta(p+1, q) = \frac{p}{p+q}\,\beta(p,q)$, $p>0$, $q>0$.

(ii) Prove that $\beta(n, n) = \beta\left(n, \frac{1}{2}\right)/2^{2n-1}$.

31. Prove that

$$\int_0^1 x^{p-1}(1-x^r)^{q-1}\,dx = \frac{1}{r}\beta\left(\frac{p}{r}, q\right)$$

where $p > o, q > o, r > o$.

32. Show that

(i) $$\int_{-1}^{+1}(1+x)^m(1-x)^n\,dx = \frac{2^{m+n+1}\,\Gamma(m+1)\,\Gamma(n+1)}{\Gamma(m+n+2)}$$

(ii) $$\int_a^b (x-a)^m(b-x)^n\,dx = \frac{(b-a)^{m+n+1}\,\Gamma(m+1)\,\Gamma(n+1)}{\Gamma(m+n+2)}$$

33. (i) Show that $$\int_0^1 \frac{dx}{(1-x^6)^{\frac{1}{6}}} = \frac{\pi}{3}.$$

(ii) Prove that

(ii) $$\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\ldots\ldots\Gamma\left(\frac{8}{9}\right) = \frac{16}{3}\pi^4$$

34. Prove that $$\int_0^\pi \frac{\sqrt{\sin x}\,dx}{(5+3\cos x)^{\frac{3}{2}}} = \frac{\left\{\Gamma\left(\frac{3}{4}\right)\right\}^2}{2\sqrt{\pi}}.$$

35. Prove that

$$\int_0^\infty x e^{-x^2}\,dx \times \int_0^\infty x^2 e^{-x^4}\,dx = \frac{\pi}{16\sqrt{2}}.$$

36. Prove that

$$\int_0^{\frac{\pi}{2}} (x^2)^p(1-x^2)^q\,dx = \frac{\Gamma\left(p+\frac{1}{2}\right)\Gamma(q-1)}{\Gamma\left(p+q+\frac{3}{2}\right)}.$$

37. If $a > b$, then show that

$$\int_0^\pi \frac{\sin^{n-1}x}{(a+b\cos x)^n}\,dx = \frac{2^{n-1}}{(a^2-b^2)^{\frac{n}{2}}}\,\beta\left(\frac{n}{2}, \frac{n}{2}\right)$$

Note : $\left(\frac{a-b}{a+b}\right)\tan^2\frac{x}{2} = \theta$(say) where $n > o$ **[J. U. H. 1995]**

2. 10 Asymptotic representation of f(z) for large | z |

To describe the behaviour of a given function $f(z)$ as $|z| \to \infty$ within a sector $\alpha \le \arg z \le \beta$, it is in many cases sufficient to derive an expression of the form.

$$f(z) = \varphi(z)\,[\,1 + r(z)\,] \qquad (1)$$

where $\varphi(z)$ is a function of a simpler structure than $f(z)$ and $r(z)$ converges uniformly to zero as $|z| \to \infty$ within the given sector. Formulas of this type are called **asymptotic representations** of $f(z)$ for large $|z|$. It follows from (1) that the ratio $\frac{f(z)}{\varphi(z)}$ converge to unity as $|z| \to \infty$ that is, the two functions $f(z)$ and $\varphi(z)$ are **"asymptotically equal"** a fact we indicate by writing $f(z) \approx \varphi(z)$, $|z| \to \infty$, $\alpha \le \arg z \le \beta$ .... (2)

An estimate of $|r(z)|$ gives the size of the error committed when $f(z)$ is replaced by $\varphi(z)$ for large but finite $|z|$.

Now we look for a description of the behaviour of the function $f(z)$ as $|z| \to \infty$ which is more exact than that given by (1). Suppose we succeed in deriving the formula.

$$f(z) = \varphi(z)\left[\sum_{n=0}^{N} a_n z^{-n} + r_N(z)\right],\ a_0 = 1,\ N = 1, 2, 3, \ldots \qquad (3)$$

where $z^N r_N(z)$ converges uniformly to zero as $|z| \to \infty$, $\alpha \le \arg z \le \beta$.[We have to note that (3) reduces to (1) for N = o.] Then we write $f(z) \approx \varphi(z) \sum_{n=o}^{\infty} a_n z^{-n}, |z| \to \infty$, $\alpha \le \arg z \le \beta$ (4) and the right hand side is called an **asymptotic series** or **asymptotic expansion** of f(z) for large | z |.

2. 11 Stirling's asymptotic formula

For large n, $\lfloor n \approx \sqrt{2\pi n}\, n^n e^{-n}$ approximately which is the **asymptotic expansion** of $\lfloor n$ or $\lceil n+1$.

By definion of gamma function, we have

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x} dx = \int_0^\infty e^{\log x^n} e^{-x} dx$$

$$= \int_0^\infty e^{n\log x} . e^{-x} dx$$

$$= \int_0^\infty e^{n\log x - x} dx \quad (1)$$

The function nlog$x$− x has a **relative maximum** for x = n.

Therefore, putting x = n + y in (1),

we get $$\Gamma(n+1) = \int_{-n}^\infty e^{n\log(n+y)-(n+y)} dy$$

$$= \int_{-n}^\infty e^{n\log n\left(1+\frac{y}{n}\right) - y} . e^{-n} dy$$

$$= e^{-n}\int_{-n}^\infty e^{n\log n + n\log\left(1+\frac{y}{n}\right) - y} dy$$

$$= e^{-n}\int_{-n}^\infty n^n\, e^{n\log\left(1+\frac{y}{n}\right) - y} dy.$$

$$= e^{-n}\, n^n \int_{-n}^\infty e^{n\log\left(1+\frac{y}{n}\right) - y} dy \quad (2)$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\dots\dots$$

$$\log\left(1+\frac{y}{n}\right) = \frac{y}{n} - \frac{y^2}{2n^2} + \frac{y^3}{3n^3} - \dots\dots$$

$$\therefore\ n\log\left(1+\frac{y}{n}\right) = y - \frac{y^2}{2n} + \frac{y^3}{3n^2} - \dots\dots\dots\dots$$

$$\text{or, } n\log\left(1+\frac{y}{n}\right) - y = -\frac{y^2}{2n} + \frac{y^3}{3n^2} - \dots\dots\dots \quad (3)$$

Combining (2) & (3), we get

$$\lceil n+1 = e^{-n} n^n \int_{-n}^\infty e^{-\frac{y^2}{2n} + \frac{y^3}{2n^2} + \dots\dots} dy \quad (4)$$

Putting y = $\sqrt{n}$ v . so that

$dy = \sqrt{n}\, dv$.

$$\Gamma(n+1) = n^n e^{-n}\sqrt{n}\int_{-\sqrt{n}}^\infty e^{\frac{-v^2}{2} + \frac{v^3}{3\sqrt{n}} - } dv \quad (5)$$

When n is large, a close approximation is

$$\Gamma(n+1) \sim n^n e^{-n}\sqrt{n}\int_{-\infty}^\infty e^{-\frac{v^2}{2}} dv.$$

$$\sim n^n . e^{-n}\sqrt{n}\, 2\int_0^\infty e^{-\frac{v^2}{2}} dv \quad (6)$$

Putting $\frac{v}{\sqrt{2}} = t$ $\left\{\begin{array}{l}\text{Limits of } t \text{ are} \\ o \text{ to } \infty.\end{array}\right.$

$dv = \sqrt{2}\, dt.$

$$\sim n^n e^{-n}\sqrt{n}\, 2\sqrt{2}\int_0^\infty e^{-t^2} dt$$

$$\sim n^n e^{-n}\sqrt{n}\, 2\sqrt{2}.\frac{\sqrt{\pi}}{2}$$

Since $\int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}$

$$\sim n^n e^{-n}\sqrt{2n\pi}$$

$\therefore \Gamma(n+1) \sim \sqrt{2n\pi}\, n^n e^{-n}$

Hence $\lfloor n \sim \sqrt{2n\pi}\, n^n e^{-n}$.

Since $\Gamma(n+1) = \lfloor n$.

which is known as the **S tirling's asymptotic expansion** of gamma function $\Gamma(n+1)$ or $\lfloor n$.

Remarks '$\sim$' means "is approximately equal to when n is large"

2. 12 Error functions.

The **error function** is defined by

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \quad (1)$$

and some times the **Complementary error function** denoted by erfc(x) is defined by

$$erfc\,(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt \quad (2)$$

Adding (1) & (2), we get

$$erf(x) + erfc(x) = \frac{2}{\sqrt{\pi}}\left[\int_0^x e^{-t^2}dt + \int_x^\infty e^{-t^2}dt\right]$$

$$= \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}dt = \frac{2}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2} = 1$$

erf(x) + erfc (x) = 1

$\therefore$ *erf c(x) = 1 – erf(x).*

Hence erfc (x) is also defined by

erfc(x) = 1 – erf(x).

Note : (1) *Prove that* $\int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}$.

$I = \int_0^\infty e^{-t^2}dt$. *Putting* $t^2 = x$. *i.e* $t = \sqrt{x}$

Limits $\quad 2tdt = dx$.

$\left.\begin{matrix} t = 0 \\ x = 0\end{matrix}\right\}$ $\left.\begin{matrix} t = \infty \\ x = \infty\end{matrix}\right\}$ $\quad dt = \frac{1}{2t}dx = \frac{1}{2}x^{-\frac{1}{2}}dx$

Thus $I = \int_0^\infty e^{-x}\cdot\frac{1}{2}x^{-\frac{1}{2}}dx$

$$= \frac{1}{2}\int_0^\infty e^{-x}x^{-\frac{1}{2}}dx$$

$$= \frac{1}{2}\int_0^\infty e^{-x}x^{\frac{1}{2}-1}dx$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \qquad \text{Since } \Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx$$

$$= \frac{1}{2}\cdot\sqrt{\pi} \qquad \text{Since } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$= \frac{\sqrt{\pi}}{2}. \qquad \text{Hence } \int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}.$$

2. 13 Generalization of the error function.

$$E_n(x) = \frac{1}{\Gamma\left(\frac{n+1}{n}\right)}\int_0^x e^{-t^n}dt$$

$$E_2(x) = \frac{1}{\Gamma\left(\frac{3}{2}\right)}\int_0^x e^{-t^2}dt$$

$$= \frac{1}{\frac{1}{2}\Gamma\frac{1}{2}} \int_0^x e^{-t^2}\, dt$$

$$= \frac{1}{\frac{1}{2}\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

$$= \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt = erf(x)$$

$\therefore\ erf(x) = E_2(x).$

2.14 The asymptotic expansion of the error function erf(x).

We know that

$$erf(x) = 1 - erfc(x) = 1 - \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt \quad (1)$$

$$\text{Now} \int_x^\infty e^{-t^2}\, dt = -\frac{1}{2}\int_x^\infty \frac{1}{t}\frac{d}{dt}\left(e^{-t^2}\right) dt$$

(on integration by parts)

$$= -\frac{1}{2}\left[\frac{1}{t}e^{-t^2}\right]_x^\infty + \frac{1}{2}\int_x^\infty -\frac{1}{t^2}e^{-t^2}\, dt$$

$$= -\frac{1}{2}\left[0 - \frac{1}{x}e^{-x^2}\right] - \frac{1}{2}\int_x^\infty \frac{1}{t^2}e^{-t^2}\, dt.$$

$$= \frac{e^{-x^2}}{2x} - \frac{1}{2}\int_x^\infty \frac{1}{t^2}e^{-t^2}\, dt.$$

$$= \frac{e^{-x^2}}{2x} - \frac{1}{2}\int_x^\infty \frac{1}{t^2}\cdot\frac{\frac{d}{dt}\left(e^{-t^2}\right)}{-2t}\, dt$$

$$= \frac{e^{-x^2}}{2x} + \frac{1}{2^2}\int_x^\infty \frac{1}{t^3}\frac{d}{dt}\left(e^{-t^2}\right) dt.$$

(again on integration by parts)

$$= \frac{e^{-x^2}}{2x} + \frac{1}{2^2}\left[\frac{1}{t^3}e^{-t^2}\right]_x^\infty + \frac{1}{2^2}\int_x^\infty \frac{3}{t^4}e^{-t^2}\, dt$$

$$= \frac{e^{-x^2}}{2x} + \frac{1}{2^2}\left\{0 - \frac{e^{-x^2}}{x^3}\right\} + \frac{1.3}{2^2}\int_x^\infty \frac{e^{-t^2}}{t^4}\, dt.$$

$$= \frac{e^{-x^2}}{2x} - \frac{e^{-x^2}}{2^2x^3} + \frac{1.3}{2^2}\int_x^\infty \frac{e^{-t^2}}{t^4}\, dt.$$

$$= \frac{e^{-x^2}}{2x} - \frac{e^{-x^2}}{2^2x^3} + \frac{1.3}{2^2}\int_x^\infty \frac{1}{t^4}\frac{\frac{d}{dt}\left(e^{-t^2}\right)}{-2t}\, dt.$$

$$= \frac{e^{-x^2}}{2x} - \frac{e^{-x^2}}{2^2x^3} - \frac{1.3}{2^3}\int_x^\infty \frac{1}{t^5}\frac{d}{dt}\left(e^{-t^2}\right) dt$$

(again on integration by parts.)

$$= \frac{e^{-x^2}}{2x} - \frac{e^{-x^2}}{2^2x^3} - \frac{1.3}{2^3}\left[\frac{1}{t^5}e^{-t^2}\right]_x^\infty$$

$$- \frac{1.3.5}{2^3}\int_x^\infty \frac{1}{t^6}e^{-t^2}\, dt.$$

$$= \frac{e^{-x^2}}{2x} - \frac{e^{-x^2}}{2^2x^3} + \frac{1.3}{2^3x^5}e^{-x^2} - \frac{1.3.5}{2^3}\int_x^\infty \frac{e^{-t^2}}{t^6}\, dt.$$

Proceeding in this way, we finally obtain

$$\int_x^\infty e^{-t^2}\, dt$$

$$= \frac{e^{-x^2}}{2x}\left[1 - \frac{1}{2x^2} + \frac{1.3}{2^2x^4} - \frac{1.3.5}{2^3x^6} + \cdots + (-)^n \frac{1.3.5\cdots(2n-1)}{2^n x^{2n}}\right]$$

$$+ \frac{(-1)^{n+1}}{2^{n+1}}\, 1.\,3.\,5\cdots(2n+1)\int_x^\infty \frac{e^{-t^2}}{t^{2n+2}}\, dt.$$

$$= \frac{e^{-x^2}}{2x}\left[1 - \frac{1}{2x^2} + \frac{1.3}{2^2x^4} - \frac{1.3.5}{2^3x^6} + \cdots\right.$$

$$+ (-1)^n . \frac{1.3.5 \ldots\ldots (2n-1)}{2^n x^{2n}}$$

$$\left. + (-1)^{n+1} \frac{1.3.5 \ldots (2n+1)}{2^n} x e^{x^2} \int_x^\infty \frac{e^{-t^2}}{t^{2n+2}} dt \right].$$

$$\text{erfc}(x) \quad = 1 - \text{erf}(x) = 1 - \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\, dt.$$

$$= \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} - \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\, dt.$$

$$= \frac{2}{\sqrt{\pi}}\left\{\int_0^\infty e^{-t^2}\, dt - \int_0^x e^{-t^2} dt\right\}$$

$$= \frac{2}{\sqrt{\pi}}\left[\int_0^\infty e^{-t^2}\, dt + \int_x^0 e^{-t^2} dt\right]$$

$$= \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\, dt.$$

$$1 - \text{erf}(x) = \frac{2}{\sqrt{\pi}} \cdot \frac{e^{-x^2}}{2x}$$

$$\left[1 - \frac{1}{2x^2} + \frac{1.3}{2^2x^4} - \frac{1.3.5}{2^3x^6} + \cdots + (-1)^n \frac{1.3.5 \cdots (2n-1)}{2^n . x^{2n}} + r_n(x)\right]$$

$$\text{where } r_n(x) = (-1)^{n+1} . \frac{1.3.5. \cdots (2n+1)}{2^n} x e^{x^2} \int_x^\infty \frac{e^{-t^2}}{t^{2n+2}} dt$$

and $r_n(x)$ converges uniformly to zero for large value of n.

Hence

$$\text{erf}(x) \approx 1 - \frac{e^{-x^2}}{\sqrt{\pi}x}\left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{1.3.5 \cdots (2n-1)}{(2x^2)^n}\right]$$

which is the **asymptotic representation or asymptotic expansion** of the error function.

2.15. Important deductions from error functions

If erf(x) represents the error function then

(i) $\text{erf}(-x) = -\text{erf}(x)$

(ii) $\text{erf}(0) = 0$

(iii) $\text{erf}(\infty) = 1$

(iv) $\text{erf}(x) = \dfrac{2}{\sqrt{\pi}}\left(x - \dfrac{x^3}{3\lfloor 1} + \dfrac{x^5}{5\lfloor 2} - \dfrac{x^7}{7\lfloor 3} + \cdots\right)$

Proof : (i) By definition of error function, we have

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} dt \quad (1)$$

$$\therefore \text{erf}(-x) = \frac{2}{\sqrt{\pi}}\int_0^{-x} e^{-t^2} dt \quad (2)$$

Putting $t = -r$ *so that* $dt = -dr$.

Limits $\left.\begin{matrix} t = o \\ r = o \end{matrix}\right\}$ $\left.\begin{matrix} t = -x \\ r = x \end{matrix}\right\}$

Thus from (2), we get

$$\text{erf}(-x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-r^2}(-dr)$$

$$= -\frac{2}{\sqrt{\pi}}\int_0^x e^{-r^2} dr$$

$$= -\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} dt.$$

$= -\text{erf}(x)$ *by (1)*

$\therefore$ $\text{erf}(-x) = -\text{erf}(x)$.

(ii) $\text{erf}(o) = O$

Proof : By defn of error function, we have

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} dt \quad (1)$$

When $x = 0$, from (1), we have $\text{erf}(o) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^0 e^{-t^2} dt$

$$= \frac{2}{\sqrt{\pi}} . 0 = 0.$$

Hence $\text{erf}(O) = 0$.

(iii) $\text{erf}(\infty) = 1$.

Proof : By def ^{n} of error function, we have

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \quad (1)$$

When $x = \infty$, from (1), we have

$$\operatorname{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2}\, dt.$$

$$= \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = 1. \text{ since} \int_0^\infty e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2}.$$

$$\text{(iv) } \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3\lfloor 1} + \frac{x^5}{5\lfloor 2} - \frac{x^7}{7\lfloor 3} + \cdots \right)$$

Proof : By definition of error function, we have

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x {}^{-t^2}\, dt.$$

$$= \frac{2}{\sqrt{\pi}} \int_0^x \left(1 - \frac{t^2}{\lfloor 1} + \frac{t^4}{\lfloor 2} - \frac{t^6}{\lfloor 3} + \frac{t^8}{\lfloor 4} - \cdots \right) dt$$

$$= \frac{2}{\sqrt{\pi}} \left[t - \frac{t^3}{3\lfloor 1} + \frac{t^5}{5\lfloor 2} - \frac{t^7}{7\lfloor 3} + \frac{t^9}{9\lfloor 4} - \cdots \right]_0^x$$

$$= \frac{2}{\sqrt{\pi}} \left[x - \frac{x^3}{3\lfloor 1} + \frac{x^5}{5\lfloor 2} - \frac{x^7}{7\lfloor 3} + \frac{x^9}{9\lfloor 4} - \cdots \right]$$

2.16. Definition of gamma function in the complex domain.

The gamma function is defined by the formula

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt, \ Rez > 0 \quad (1)$$

whenever the complex variable z has a positive real part Rez.

We can write the integral (1) as a sum of two integrals i.e.

$$\Gamma(z) = \int_0^1 e^{-t}\, t^{z-1}\, dt + \int_1^\infty e^{-t}\, t^{z-1}\, dt \quad (2)$$

where it can be easily shown that the first integral defines a function P(z) which is analytic in the half-plane Rez>0, while the second integral defines an entire function. It follows that the function $\Gamma(z) = P(z)+Q(z)$ is analytic in the half plane Rez >0.

Note 1. Analytic functions

A single valued function which is defined and differentiable at each point of a domain D is said to be **analytic** in that domain.

[A function is said to be **analytic at a point** if its derivative exists not only at that point but in some neighbourhood of that point.]

Note 2. Analytic continuation

If we have two domains D_1 and D_2 with points in common and a function f_1 that is analytic in D_1, there may exist a function f_2, which is analytic in D_2, such that $f_2(z) = f_1(z)$ for each z in the intersection $D_1 \cap D_2$. If so, we call f_2 an **analytic continuation** of f_1 into the domain D_2.

The values of $\Gamma(z)$ in the rest of the complex plane can be found by the analytic continuation of the function defined by

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt, \ Rez > 0.$$

First we replace the exponential in the integral for P(z) by its power series expansion and then we integrate term by term, obtaining

$$P(z) = \int_0^1 t^{z-1}\, dt. \sum_{k=o}^{\infty} \frac{(-1)^k}{\lfloor k}\, t^k$$

$$= \sum_{k=o}^{\infty} \frac{(-1)^k}{\lfloor k} \int_0^1 t^{k+z-1}\, dt = \sum_{k=o}^{\infty} \frac{(-1)^k}{\lfloor k}\, \frac{1}{z+k} \quad (3)$$

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where it is permissible to reverse the order of integration and summation since

$$\int_0^1 \left|t^{z-1}\right| dt. \sum_{k=0}^{\infty}\left|\frac{(-1)^k}{\underline{|k}}\, t^k\right| = \int_0^1 t^{x-1}\, dt. \sum_{k=0}^{\infty}\frac{t^k}{\underline{|k}}$$

$$= \int_0^1 e^t\, t^{x-1}\, dt < \infty .$$

The last integral converges for $x = \text{Re}z > 0$. The terms of the series (3) are analytic functions of z if $z \neq 0, -1, -2, \ldots\ldots\ldots\ldots$ Moreover, in the region $|z+k| \geq \delta > 0,\ k = 0, 1, 2, \ldots\ldots\ldots\ldots$ the series (3) is majorized by the convergent series $\sum_{k=0}^{\infty}\frac{1}{\underline{|k}\,\delta}$ and hence is uniformly convergent in this region. Using Weierstrass' theorem and the arbitrariness of δ, we conclude that the sum of the series (3) is a **meromorphic function** with simple poles at the points $z = 0, -1, -2, \ldots\ldots\ldots$ For $\text{Re}z>0$ this function coincides with the integral $P(z)$ and hence is the analytic continuation of $P(z)$.

The function $\Gamma(z)$ differs from $P(z)$ by the term $Q(z)$, which is an entire function. Therefore, $\Gamma(z)$ is a meromorphic function of the complex variable z with simple poles at the points $z = 0, -1, -2 \ldots\ldots\ldots\ldots$ An analytic expression for Γz, suitable for defining $\Gamma(z)$ in the whole complex plane, is given by

$$\Gamma(z) = \sum_{k=0}^{\infty}\frac{(-1)^k}{\underline{|k}}\cdot\frac{1}{z+k} + \int_1^{\infty} e^{-t}\, t^{z-1}\, dt \quad (4)$$

$$z \neq 0, -1, -2, \ldots\ldots$$

2.17. Some basic relations satisfied by the gamma function

(i) $\Gamma(z+1) = z\,\Gamma(z)$

(ii) $\Gamma(z)\,\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$

(iii) $2^{2z-1}\,\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2z)$.

Proof : (i) Let us asume that $\text{Re}z > 0$ and use the integral representation $\Gamma(z) = \int_0^{\infty} e^{-t}\, t^{z-1}\, dt$ (1)

An integration by parts gives

$$\Gamma(z+1) = \int_0^{\infty} e^{-t}\, t^{z}\, dt = \left[t^z . -e^{-t}\right]_0^{\infty}$$

$$+ \int_0^{\infty} z.\, t^{z-1}\, e^{-t}\, dt.$$

$$= 0 + z\int_0^{\infty} e^{-t}\, t^{z-1}\, dt = z\,\Gamma(z).$$

$$\therefore \Gamma(z+1) = z\,\Gamma(z).$$

The validity of this result for arbitrary complex $z \neq 0, -1, -2, \ldots\ldots\ldots$ is an immediate consequence of the principle of analytic continuation, since both sides of the formula are analytic everywhere except at the points $z = 0, -1, -2, \ldots\ldots\ldots\ldots$

(ii) Let us temporarily assume that $0 < \text{Re}z < 1$ and use

$$\Gamma(z) = \int_0^{\infty} e^{-t}\, t^{z-1}\, dt \quad (1)$$

Now $\Gamma(1-z) = \int_0^\infty e^{-s} s^{1-z-1} ds$

$$= \int_0^\infty e^{-s} s^{-z} ds$$

$$\therefore \Gamma(z)\,\Gamma(1-z) = \int_0^\infty\int_0^\infty e^{-s} \cdot e^{-t} \cdot s^{-z} \cdot t^{z-1}\, dsdt.$$

$$= \int_0^\infty\int_0^\infty e^{-(s+t)} s^{-z} t^{z-1}\, ds\, dt \quad (2)$$

Introducing the new variables $u = s + t,\ v = \frac{t}{s}$

$$u = s + t = s\left(1 + \frac{t}{s}\right) = s(1+v)$$

$$\because\ s = \frac{u}{1+v} = f(u,v)$$

$$t = u - s = u - \frac{u}{1+v} = \frac{u + uv - u}{1+v} = \frac{uv}{1+v}$$

$$dsdt = \left|\frac{\partial(s,t)}{\partial(u,v)}\right| dudv.$$

$$\text{where } \frac{\partial(s,t)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{1+v} & -\frac{u}{(1+v)^2} \\ \frac{v}{1+v} & \frac{u}{(1+v)^2} \end{vmatrix}$$

$$= \frac{u}{(1+v)^3} + \frac{uv}{(1+v)^3}$$

$$= \frac{u(1+v)}{(1+v)^3} = \frac{u}{(1+v)^2}$$

$$\therefore\ dsdt = \frac{u}{(1+v)^2} \cdot dudv$$

$$e^{-(s+t)} s^{-z} t^{z-1} dsdt$$

$$= e^{-(s+t)} \frac{1}{s}\left(\frac{t}{s}\right)^{z-1} . dsdt$$

$$= e^{-u} \cdot \frac{1+v}{u} \cdot v^{z-1} \cdot \frac{u}{(1+v)^2} \cdot dudv.$$

$$= e^{-u} v^{z-1} \frac{dudv}{1+v}.$$

The limits of u & v will be o to ∞.

Thus from (2), we have

$$\Gamma(z)\,\Gamma(1-z) = \int_0^\infty\int_0^\infty e^{-u} v^{z-1} \frac{dudv}{1+v}$$

$$= \int_0^\infty \left[-e^{-u}\right]_0^\infty \frac{v^{z-1}}{1+v}\, dv.$$

$$= \int_0^\infty \frac{v^{z-1}}{1+v}\, dv = \frac{\pi}{\sin \pi z}$$

$$\therefore \Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

The Legendre-Duplication formula

(iii) $2^{2z-1}\,\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2z)$

Proof : Let us asume that Rez > 0, then by definition of gamma function, we have

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt \quad (1)$$

$$\Gamma\left(z+\frac{1}{2}\right)=\int_0^\infty e^{-s}s^{z+\frac{1}{2}-1}\,ds$$

$$=\int_0^\infty e^{-s}s^{2z-1}s^{\frac{1}{2}}\,ds.$$

$$=\int_0^\infty e^{-s}s^{\frac{2z-1}{2}}\,ds.$$

$$2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)=\int_0^\infty\!\!\int_0^\infty 2^{\frac{2z-1}{2}}\cdot e^{-s}\cdot e^{-t}\cdot t^{z-1}\cdot s^{\frac{2z-1}{2}}\,ds\,dt$$

$$=\int_0^\infty\!\!\int_0^\infty e^{-(s+t)}\cdot 2^{2z-1}\cdot t^{\frac{1}{2}}\cdot t^{z-1}\cdot s^{\frac{2z-1}{2}}\,t^{-\frac{1}{2}}\,ds\,dt$$

$$=\int_0^\infty\!\!\int_0^\infty e^{-(s+t)}\cdot 2^{2z-1}\,t^{\frac{2z-1}{2}}\,s^{\frac{2z-1}{2}}\,t^{-\frac{1}{2}}\,ds\,dt$$

$$=\int_0^\infty\!\!\int_0^\infty e^{-(s+t)}\left(2\sqrt{st}\right)^{2z-1}t^{-\frac{1}{2}}\,ds\,dt$$

Putting $\alpha^2=s,\ \beta^2=t$ *i.e.* $\alpha=\sqrt{s},\ \beta=\sqrt{t}$

$2\alpha d\alpha=ds,\ 2\beta d\beta=dt.$

Limits of α *and* β *are o to* ∞

$$=\int_0^\infty\!\!\int_0^\infty e^{-(\alpha^2+\beta^2)}(2\alpha\beta)^{2z-1}\cdot\frac{1}{\beta}\cdot 4\alpha\beta\,d\alpha\,d\beta.$$

$$=4\int_0^\infty\!\!\int_0^\infty e^{-(\alpha^2+\beta^2)}(2\alpha\beta)^{2z-1}\alpha\,d\alpha\,d\beta\quad(2)$$

Similarly, $2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)=$

$$=4\int_0^\infty\!\!\int_0^\infty e^{-(\alpha^2+\beta^2)}(2\alpha\beta)^{2z-1}\beta\,d\alpha\,d\beta\quad(3)$$

Adding (2) & (3), we get

$$2.\,2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$$

$$=4\int_0^\infty\!\!\int_0^\infty e^{-(\alpha^2+\beta^2)}(2\alpha\beta)^{2z-1}(\alpha+\beta)\,d\alpha\,d\beta.$$

Or, $2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$

$$=2\int_0^\infty\!\!\int_0^\infty e^{-(\alpha^2+\beta^2)}(2\alpha\beta)^{2z-1}(\alpha+\beta)\,d\alpha\,d\beta\quad(4)$$

Or, $2^{2z-1}\Gamma z\,\Gamma\left(z+\frac{1}{2}\right)$

$$=4\iint_\sigma e^{-(\alpha^2+\beta^2)}(2\alpha\beta)^{2z-1}(\alpha+\beta)\,d\alpha\,d\beta\quad(5)$$

where the last integral is over the sector

$\sigma:0\le\alpha\le\infty,\ 0\le\beta<\infty.$

Again putting $\alpha^2+\beta^2=u,\ 2\alpha\beta=v$

we get $2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)=\int_0^\infty v^{2z-1}dv\int_0^\infty\frac{e^{-u}}{\sqrt{u-v}}du$

$$=\int_0^\infty e^{-v}v^{2z-1}dv\int_0^\infty\frac{e^{-(u-v)}}{\sqrt{u-v}}du$$

$$=2\int_0^\infty e^{-v}v^{2z-1}dv\int_0^\infty e^{-w^2}dw.\ \text{where } u-v=w^2$$

$$=2\Gamma(2z)\cdot\frac{\sqrt{\pi}}{2}=\sqrt{\pi}\,\Gamma(2z)$$

Hence $2^{2z-1}\Gamma(z)\cdot\Gamma\left(z+\frac{1}{2}\right)=\sqrt{\pi}\,\Gamma(2z)$.

2.18. Weierstrass formula

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-\frac{z}{n}}$$

where γ is the **Euler-Mascheroni constant.**

Proof : By definition of gamma function in complex domain, we have $\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt, \ Re\, z > 0$ (1)

Also from the relation between beta and gamma functions, we have $\beta(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$ (2)

Putting $u = z$ and $v = 1 - z$ in (2), we get

$$\Gamma(z)\Gamma(1-z) = \Gamma(1)\, \beta(z, 1-z) = \int_0^{\infty} \frac{y^{z-1}}{1+y} dy = \frac{\pi}{\sin \pi z}$$

$$\therefore \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \quad (3)$$

This result (3) is valid only for $Re\,z>0$, $Re\,(1-z)>0$ i.e. $0<Re\,z<1$. The result can be extended throughout the entire complex plane by analytic continuation.

The formula (2) can also be used to find the partial fraction expansion of $\frac{\Gamma'(z)}{\Gamma(z)}$. Set $u = h, v = z - h$ in (2) and use

$\beta(u, v) = \int_0^1 t^{u-1} (1-t)^{v-1} dt$. Then we have

$$\frac{\Gamma(z-h)\Gamma(h)}{\Gamma(z)} = \int_0^1 t^{h-1} (1-t)^{z-h-1} dt.$$

$$\text{Or, } \frac{\Gamma(z-h)\Gamma(h)}{\Gamma(z)} = \int_0^1 t^{h-1} (1-t)^{z-h-1} dt + \int_0^1 t^{h-1} dt - \int_0^1 t^{h-1} dt.$$

$$= \frac{1}{h} + \int_0^1 [(1-t)^{z-h-1} - 1]\, t^{h-1} dt, \quad 0<h<Re\,z \quad (4)$$

$\because$ Since $\int_0^1 t^{h-1} dt = \frac{1}{h}$;

Now consider the limit $h \to 0$. In this limit the **recursion relation** $\Gamma(z+1) = z\Gamma(z)$ and the **Taylor series** for $\Gamma(h+1)$ give

$$\Gamma(h) = \frac{1}{h}\Gamma(h+1) = \frac{1}{h}[\Gamma(1) + h\Gamma'(1) + \ldots\ldots]$$

$$= \frac{1}{h}[1 - h\gamma + \ldots\ldots]$$

$$= \frac{1}{h} - \gamma + \ldots\ldots \text{ where } \gamma = -\Gamma'(1)$$

$$\therefore \Gamma(h) = \frac{1}{h} - \gamma + \ldots \quad (5)$$

where $\gamma = -\Gamma'(1)$ (6) is **Euler's constant.**

Similarly, the Taylor series for $\Gamma(z-h)$ gives

$$\frac{\Gamma(z-h)}{\Gamma(z)} = 1 - h\frac{\Gamma'(z)}{\Gamma(z)} + \ldots\ldots (7)$$

Putting (5) & (7) in (4), we get

$$\left[1 - h\frac{\Gamma'(z)}{\Gamma(z)} + \ldots\ldots\right]\left\{\frac{1}{h} - \gamma + \ldots\ldots\right\}$$

$$= \frac{1}{h} + \int_0^1 t^{h-1} [(1-t)^{z-h-1} - 1]\, dt \quad (8)$$

Comparing the coefficients of powers of h, in particular the coefficient of h^0 from both sides of (8), we get

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \int_0^1 \frac{1}{t}[1-(1-t)^{z-1}]\, dt \quad (9)$$

We can use the expansion

$$\frac{1}{t} = \frac{1}{1-(1-t)} = \{1-(1-t)\}^{-1}$$

$$= \sum_{n=0}^{\infty} (1-t)^n \quad (10)$$

and do the integral (9) term by term to obtain

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \sum_{n=0}^{\infty}\left[\frac{1}{n+1}-\frac{1}{n+z}\right]$$

Or
$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+z}\right)$$

or.
$$\frac{\Gamma'(z)}{\Gamma(z)} + \frac{1}{z} = -\gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+z}\right) \qquad (11)$$

which is a partial fraction expansion of $\frac{\Gamma'(z)}{\Gamma(z)}$. Rewriting it

the form $\frac{d}{dz}\log\left[z\Gamma(z)\right] = \frac{d}{dz}\left[(-\gamma z) + \sum_{n=1}^{\infty}\left\{\frac{z}{n} - \log\left(1+\frac{z}{n}\right)\right\}\right]$

Integrating from o to z, we get

$$\log z\Gamma(z) = -\gamma z + \sum_{n=1}^{\infty}\left\{\frac{z}{n} - \log\left(1+\frac{z}{n}\right)\right\}$$

Now exponentiating both sides, we get

$$z\Gamma(z) = e^{-\gamma z + \sum_{n=1}^{\infty}\left\{\frac{z}{n} - \log\left(1+\frac{z}{n}\right)\right\}}$$

Or.
$$z e^{\gamma z}. \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}} = \frac{1}{\Gamma(z)} \qquad (12)$$

which is an infinite product function for the gamm function

Setting $z = 1$ in (12) and then taking logarithm on bo sides, we get another expression for **Euler's constant** γ :

$$\gamma = -\Gamma'(1) = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\dots\dots+\frac{1}{n}-\log n\right)$$

2.19 Logarithmic derivative of gamma functions

The logarithmic derivative of gamma function can be defined as $\frac{d}{dz}\{\log\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} = \psi(z)$

Three important relations for $\psi(z)$

(i) show that $\psi(z) + \psi\left(z+\frac{1}{2}\right) + 2\log 2 = 2\psi(2z)$

Proof : From the **Duplication formula of gamma function** we have

$$2^{2z-1}\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2z) \quad (1)$$

Taking log on both sides of (1), we get

$$(2z-1)\log 2 + \log\Gamma(z) + \log\Gamma\left(z+\frac{1}{2}\right) = \log\sqrt{\pi} + \log\Gamma(2z) \quad (2)$$

Differentiating both sides of (2) with respect to z, we get

$$2\log 2 - 0 + \frac{\Gamma'(z)}{\Gamma(z)} + \frac{\Gamma'\left(z+\frac{1}{2}\right)}{\Gamma\left(z+\frac{1}{2}\right)} = 0 + \frac{\Gamma'(2z)}{\Gamma(2z)}.2$$

Or, $2\log 2 + \psi(z) + \psi\left(z+\frac{1}{2}\right) = 2\psi(2z)$

hence $\psi(z) + \psi\left(z+\frac{1}{2}\right) + 2\log 2 = 2\psi(2z)$.

Note : $\boxed{\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin\pi z}}$

(ii) Verify the formula

$$\Gamma(3z) = \frac{3^{3z-\frac{1}{2}}}{2\pi}\,\Gamma(z)\,\Gamma\left(z+\frac{1}{3}\right)\Gamma\left(z+\frac{2}{3}\right)$$

Proof : By reduction formula of gamma function, we have

$$\Gamma(3z) = \lfloor 3z-1$$

$$= (3z-1)(3z-2)(3z-3)(3z-4) \ldots\ldots\ldots 4.3.2.1$$

$$= 3\left(z-\frac{1}{3}\right) 3\left(z-\frac{2}{3}\right) 3(z-1)\, 3\left(z-\frac{4}{3}\right) \ldots\ldots \left(3.\frac{4}{3}\right) 3\frac{3}{3}\, 3.\frac{2}{3}.\, 3.\frac{1}{3}$$

$$= 3^{3z-1}\left\{\left(z-\frac{1}{3}\right)\left(z-\frac{2}{3}\right)(z-1)\left(z-\frac{4}{3}\right) \ldots\ldots \frac{4}{3}\cdot\frac{3}{3}\cdot\frac{2}{3}\cdot\frac{1}{3}\right\}$$

$$= 3^{3z-1}\left\{\left(z-\frac{1}{3}\right)\left(z-\frac{4}{3}\right) \ldots\ldots \frac{4}{3}\cdot\frac{1}{3}\right\} \left\{\left(z-\frac{2}{3}\right)\left(z-\frac{5}{3}\right) \ldots\ldots \frac{5}{3}\cdot\frac{2}{3}\right\} \left\{(z-1)(z-2) \ldots\ldots 3.2.1\right\}$$

$$= 3^{3z-1}\left\{\left(z-\frac{1}{3}\right)\left(z-\frac{4}{3}\right) \ldots\ldots \frac{4}{3}\cdot\frac{1}{3}\right\} \times \left\{\left(z-\frac{2}{3}\right)\left(z-\frac{5}{3}\right) \ldots\ldots \frac{5}{3}\cdot\frac{2}{3}\right\} \lfloor z-1$$

$$= 3^{3z-1} \frac{\left\{\left(z-\frac{1}{3}\right)\left(z-\frac{4}{3}\right) \ldots \frac{4}{3}\frac{1}{3}\Gamma\left(\frac{1}{3}\right)\right\}}{\Gamma\left(\frac{1}{3}\right)} \times \frac{\left\{\left(z-\frac{2}{3}\right)\left(z-\frac{5}{3}\right) \ldots \frac{5}{3}\frac{2}{3}\Gamma\left(\frac{2}{3}\right)\right\}}{\Gamma\left(\frac{2}{3}\right)} \lfloor z-1$$

$$= 3^{3z-1} \frac{\Gamma\left(z-\frac{1}{3}+1\right) . \Gamma\left(z-\frac{2}{3}+1\right) . \Gamma(z-1+1)}{\Gamma\left(\frac{1}{3}\right) . \Gamma\left(\frac{2}{3}\right)}$$

$$= 3^{3z-1} \frac{\Gamma\left(z+\frac{2}{3}\right) \Gamma\left(z+\frac{1}{3}\right) . \Gamma(z)}{\Gamma\left(\frac{2}{3}\right) . \Gamma\left(1-\frac{2}{3}\right)}$$

$$= 3^{3z-1} \frac{\Gamma(z)\, \Gamma\left(z+\frac{1}{3}\right) \Gamma\left(z+\frac{2}{3}\right)}{\dfrac{\pi}{\sin\frac{2\pi}{3}}} \qquad \text{Since } \Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

$$= 3^{3z-1} \frac{\Gamma(z)\, \Gamma\left(z+\frac{1}{3}\right) \Gamma\left(z+\frac{2}{3}\right)}{\dfrac{2\pi}{\sqrt{3}}} \qquad \text{Since } \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$

$$= 3^{3z-1+\frac{1}{2}} \frac{\Gamma(z)\, \Gamma\left(z+\frac{1}{3}\right) \Gamma\left(z+\frac{2}{3}\right)}{2\pi}$$

$$\therefore \Gamma(3z) = \frac{3^{3z-\frac{1}{2}}}{2\pi} . \Gamma(z) . \Gamma\left(z+\frac{1}{3}\right) . \Gamma\left(z+\frac{2}{3}\right).$$

(iii) Derive the formula

$$3\psi(3z) = \psi(z) + \psi\left(z+\frac{1}{3}\right) + \psi\left(z+\frac{2}{3}\right) + 3\log 3.$$

Proof : We know that

$$\Gamma(3z) = \frac{3^{3z-\frac{1}{2}}}{2\pi} \Gamma(z)\, \Gamma\left(z+\frac{1}{3}\right) \Gamma\left(z+\frac{2}{3}\right) \qquad (1)$$

Taking log on both sides of (1), we get

$$\log \Gamma(3z) = \log 3^{3z-\frac{1}{2}} - \log 2\pi + \log \Gamma(z) + \log \Gamma\left(z+\frac{1}{3}\right) + \log \Gamma\left(z+\frac{2}{3}\right)$$

$$\text{Or, } \log \Gamma(3z) = \left(3z-\frac{1}{2}\right) \log 3 - \log 2\pi + \log \Gamma(z) + \log \Gamma\left(z+\frac{1}{3}\right) + \log \Gamma\left(z+\frac{2}{3}\right) \qquad (2)$$

Differentiating both sides of (2), with respect to z, we get

$$\frac{\Gamma'(3z)}{\Gamma(3z)}\cdot 3 = 3\log 3 - 0 + \frac{\Gamma'(z)}{\Gamma(z)} + \frac{\Gamma'\left(z+\frac{1}{3}\right)}{\Gamma\left(z+\frac{1}{3}\right)} + \frac{\Gamma'\left(z+\frac{2}{3}\right)}{\Gamma\left(z+\frac{2}{3}\right)}$$

Or, $3\,\Psi(3z) = 3\log 3 + \Psi(z) + \Psi\left(z+\frac{1}{3}\right) + \Psi\left(z+\frac{2}{3}\right)$

Hence $3\,\Psi(3z) = \Psi(z) + \Psi\left(z+\frac{1}{3}\right) + \Psi\left(z+\frac{2}{3}\right) + 3\log 3.$

Note : $\frac{d}{dz}\left\{\log\Gamma(z)\right\} = \frac{\Gamma'(z)}{\Gamma(z)} = \Psi'(z).$

WORKED OUT EXAMPLES

Example 1.

Derive the following representation of $\gamma(z,\alpha)$:

$$\gamma(z,\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k\,\alpha^{k+z}}{k!\,(k+z)}$$

Proof : $\gamma(z,\alpha) = \int_0^{\alpha} e^{-t}\, t^{z-1}\, dz \quad (1)$

$$= \int_0^{\alpha} t^{z-1}\left(1 - \frac{t}{1!} + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \dots\dots\right) dz$$

$$= \int_0^{\alpha}\left(t^{z-1} - \frac{t^z}{1!} + \frac{t^{z+1}}{2!} - \frac{t^{z+2}}{3!} + \frac{t^{z+3}}{4!} - \dots\dots\right) dz$$

$$= \left[\frac{t^z}{z} + \frac{t^{z+1}}{1!\,(z+1)} + \frac{t^{z+2}}{2!\,(z+2)} - \frac{t^{z+3}}{3!\,(z+3)} + \frac{t^{z+4}}{4!\,(z+4)} - \dots\dots\right]_0^{\alpha}$$

$$= \frac{\alpha^z}{z} - \frac{\alpha^{z+1}}{1!\,(z+1)} + \frac{\alpha^{z+2}}{2!\,(z+2)} - \frac{\alpha^{z+3}}{3!\,(z+3)} + \frac{\alpha^{z+4}}{4!\,(z+4)} - \dots\dots$$

$$= \frac{\alpha^{0+z}}{0!\,(0+z)} - \frac{\alpha^{1+z}}{1!\,(1+z)} + \frac{\alpha^{2+z}}{2!\,(2+z)} - \frac{\alpha^{3+z}}{3!\,(3+z)} + \frac{\alpha^{4+z}}{4!\,(4+z)} - \dots\dots\dots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k\,\alpha^{k+z}}{k!\,(k+z)}, \quad z \neq 0, -1, -2, \dots\dots\dots$$

Hence $\gamma(z,\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k\,\alpha^{k+z}}{k!\,(k+z)}$), $z \neq 0, -1, -2, \dots$

Example 2. show that

$\Psi(1) = \Gamma'(1) = -\gamma = \int_0^{\infty} e^{-t} \log t\, dt.$ **D. U. M. Sc. 1989.**

Proof : By definition of gamma function, we have

$$\Gamma(z) = \int_0^{\infty} e^{-t}\, t^{z-1}\, dt. \; Re\, z > 0 \quad (1)$$

whenever the complex variable z has a positive real part Re z.

Note : $\frac{d}{dx}(a^x) = a^x \log_e a$

Differentiating both sides of (1) with respect to z, we get

$$\Gamma'(z) = \int_0^{\infty} e^{-t}\, t^{z-1} \log_e t \; dt \quad (2)$$

Putting z = 1 in (2) we get

$$\Gamma'(1) = \int_0^{\infty} e^{-t}\,.\,1.\log_e t\; dt$$

Or, $\Gamma'(1) = \int_0^{\infty} e^{-t} \log t\; dt \quad (3)$

Now the theory of gamma function is related to the theory of another special function i.e. the logarithmic derivative of

$$\Gamma(z) : \Psi(Z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad (4)$$

since $\Gamma(z)$ is a meromorphic function with no zeros, $\Psi(z)$ can have no singular points other than the poles $z = -n$ $(n = 0, 1, 2....)$ of $\Gamma(z)$.

Putting $z = 1$ in (4) we get

$$\Psi(1) = \frac{\Gamma'(1)}{\Gamma(1)} = \Gamma'(1) \quad \text{Since } \Gamma(1) = 1.$$

$$\therefore \Psi(1) = \Gamma'(1) \quad (5)$$

Combining (3) & (5), we get

$$\Psi(1) = \Gamma'(1) = \int_0^\infty e^{-t} \log t \, dt.$$

Also we have $\Psi(1) = \Gamma'(1) = -\gamma$, where

$\gamma = 0.57721566$ --- is **Euler's constant.**

Thus we have $\Psi(1) = \Gamma'(1) = -\gamma = \int_0^\infty e^{-t} \log t \, dt.$

Note : The function $\Psi(z)$ satisfies the following relations:

(i) $\Psi(z + 1) = \frac{1}{z} + \Psi(z)$

(ii) $\Psi(1 - z) - \Psi(z) = \pi \cot \pi z$

(iii) $\Psi(z) + \Psi\left(z + \frac{1}{2}\right) + 2 \log 2 = 2\Psi(2z).$

Theorem 1. If $\Gamma(a + ib) = p + iq$ then $\Gamma(a - ib) = p - iq$ and conversely.

Proof : By definition of gamma function in complex domain, we have

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \quad (1)$$

Putting $z = a + ib$ in (1), we get

$$\Gamma(a + ib) = \int_0^\infty e^{-t} t^{a+ib-1} dt$$

$$= \int_0^\infty e^{-t} t^{a-1} . t^{ib} dt$$

$$= \int_0^\infty e^{-t} t^{a-1} e^{\log t^{ib}} dt$$

$$= \int_0^\infty e^{-t} t^{a-1} e^{ib \log t} dt$$

$$= \int_0^\infty e^{-t} t^{a-1} [\cos(b \log t) + i \sin(b \log t)] dt.$$

$$= \int_0^\infty e^{-t} t^{a-1} \cos(b \log t) \, dt$$

$$+ i \int_0^\infty e^{-t} t^{a-1} \sin(b \log t) \, dt$$

$$\therefore P = \int_0^\infty e^{-t} t^{a-1} \cos(b \log t) \, dt$$

$$\text{and } q = \int_0^\infty e^{-t} t^{a-1} \sin(b \log t) \, dt.$$

Again let $\Gamma(a-ib) = r - is$. putting $z = a - ib$

in (1), we get $\Gamma(a-ib) = \int_0^\infty e^{-t}\, t^{a-ib-1}\, dt$

$$= \int_0^\infty e^{-t}\, t^{a-1}\, t^{-ib}\, dt$$

$$= \int_0^\infty e^{-t}\, t^{a-1}\, e^{\log t^{-ib}}\, dt$$

$$= \int_0^\infty e^{-t}\, t^{a-1}\, e^{-ib\log t}\, dt$$

$$= \int_0^\infty e^{-t}\, t^{a-1}\, \{\cos(b\log t) - i\sin(b\log t)\}\, dt.$$

$$= \int_0^\infty e^{-t}\, t^{a-1}\cos(b\log t) - i\int_0^\infty e^{-t}\, t^{a-1}\sin(b\log t)\, dt$$

$$\therefore\ r = \int_0^\infty e^{-t}\, t^{a-1}\cos(b\log t)\, dt = p$$

$$s = \int_0^\infty e^{-t}\, t^{a-1}\sin(b\log t)\, dt = q$$

Therefore, $\Gamma(a-ib) = p - iq$

By similar procedure conversely, we can easily show that if $\Gamma(a-ib) = p - iq$ then $\Gamma(a+ib) = p + iq$.

Corollary : When a = o and p = o, we have if $\Gamma(ib) = iq$ then $\Gamma(-ib) = -iq$ and conversely.

Theorem 2.

(i) $|\Gamma(a+ib)|^2 = \Gamma(a+ib)\,\Gamma(a-ib) = |\Gamma(a-ib)|^2$

(ii) $|\Gamma(ib)|^2 = \Gamma(ib)\,\Gamma(-ib) = |\Gamma(-ib)|^2$

Proof : (i) We know that if (i) $\Gamma(a+ib) = p + iq$

then $\Gamma(a-ib) = p - iq$ and conversely.

Here $|\Gamma(a+ib)|^2 = |p+iq|^2 = p^2 + q^2 = (p+iq)(p-iq)$

$$= \Gamma(a+ib)\,.\,\Gamma(a-ib)\,.$$

Also $|\Gamma(a-ib)|^2 = |p-iq|^2 = p^2 + q^2 = \Gamma(a+ib)\,\Gamma(a-ib)$.

Hence $|\Gamma(a+ib)|^2 = \Gamma(a+ib)\,\Gamma(a-ib) = |\Gamma(a-ib)|^2$

Proof : (ii) Taking a = o in (i), we get

$|\Gamma(ib)|^2 = \Gamma(ib)\,\Gamma(-ib) = |\Gamma(-ib)|^2$.

Remarks $|\Gamma(z_1)\,\Gamma(z_2)\ldots\ldots\Gamma(z_n)| =$

$$|\Gamma(z_1)|\,|\Gamma(z_2)|\ldots\ldots|\Gamma(z_n)|.$$

where $z_1, z_2, \ldots\ldots, z_n$ are complex numbers.

Example 3. Prove the followings for real y :

(i) $|\Gamma(iy)|^2 = \dfrac{\pi}{y\sinh \pi y}$

(ii) $\Gamma\left(\frac{1}{2} + iy\right)\Gamma\left(\frac{1}{2} - iy\right) = \dfrac{\pi}{\cosh \pi y}$

(iii) $\left|\Gamma\left(\frac{1}{2} + iy\right)\right|^2 = \dfrac{\pi}{\cosh \pi y}$

Proof : (i) By theorem we know that

$\Gamma(z)\,\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$ where z is a complex number.

Or, $\Gamma(z)\,.\,\{(-z)\,\Gamma(-z)\} = \dfrac{\pi}{\sin \pi z}$

Or, $\Gamma(z)\,\Gamma(-z) = \dfrac{\pi}{-z\sin \pi z}$ ...(1)

Putting z = iy in (1), we get

$$\Gamma(iy)\,\Gamma(-iy) = \frac{\pi}{-iy\sin \pi iy} = \frac{\pi}{-i^2 y \sinh \pi y}$$

Or, $$\Gamma(iy)\,\Gamma(-iy) = \frac{\pi}{y\sinh \pi y} \quad (2)$$

Or, $$|\Gamma(iy)|^2 = \frac{\pi}{y\sinh \pi y}$$

Since $\Gamma(iy)\,\Gamma(-iy) = |\Gamma(iy)|^2$.

Proof (ii) By theorem we have

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sinh \pi z} \quad (1)$$

Putting $z = \frac{1}{2} + iy$ in (1), we get

$$\Gamma\left(\frac{1}{2}+iy\right)\Gamma\left(\frac{1}{2}-iy\right) = \frac{\pi}{\sin\left(\frac{\pi}{2}+i\pi y\right)}$$

$$= \frac{\pi}{\cos i\pi y} = \frac{\pi}{\cosh \pi y}$$

Proof (iii) since $\Gamma\left(\frac{1}{2}+iy\right)\Gamma\left(\frac{1}{2}-iy\right) = \left|\Gamma\left(\frac{1}{2}+iy\right)\right|^2$

$$\therefore \left|\Gamma\left(\frac{1}{2}+iy\right)\right|^2 = \frac{\pi}{\cosh \pi y}.$$

EXERCISES 2(B)

1. Prove that $\displaystyle\int_0^\infty \frac{x^{2m}}{1+x^{2n}}\,dx = \frac{\pi}{2n\sin\left(\frac{2m+1}{2n}\right)\pi}$.

2. Prove that

$$\int_0^\infty \frac{\cosh 2yt}{(\cosh t)^{2x}}\,dt = 2^{2x-2}\frac{\Gamma(x+y)\,\Gamma(x-y)}{\Gamma(2x)}$$

where Re x>0, Re x>| Re y |

3. Prove that

$$\frac{\lfloor n}{z^{n+1}} = (-1)^n \int_0^1 t^{z-1}(\log t)^n\,dt$$

$$= \int_0^1 e^{-tz}\, t^n\,dt,\ Re(z) > 0.$$

4. Prove that

(i) $\gamma(z+1,\alpha) = z\,\gamma(z,\alpha) - e^{-\alpha}\alpha^z$

(ii) $\Gamma(z+1,\alpha) = z\,\Gamma(z,\alpha) + e^{-\alpha}\alpha^z$

5. If $0 < Re(z) < 1$, then prove that

$$\int_0^\infty t^{z-1}\cos t\,dt = \Gamma(z)\cos\frac{\pi z}{2}.$$

6. If $-1 < Re(z) < 1$, then prove that

$$\int_0^\infty t^{z-1}\sin t\,dt = \Gamma(z)\sin\frac{\pi z}{2}.$$

7. If $0 < Re(z)$ 1, then show that

$$\int_0^\infty \frac{\cos t}{t^z}\,dt = \frac{\pi}{2\,\Gamma(z)\cos\frac{\pi z}{2}}.$$

8. If $0 < Re(z) < 2$ then prove that

$$\int_0^\infty \frac{\sin t}{t^z}\,dt = \frac{\pi}{2\,\Gamma(z)\sin\frac{\pi z}{2}}.$$

9. Prove that $\Gamma(z) = \displaystyle\int_{-\infty}^{\infty} e^{-e^t} e^{zt}\,dt,\ Re(z) > 0.$

10. Prove the followings :

(i) $\psi(z+1) - \psi(z) = \dfrac{1}{z}$

(ii) $\psi(1-z) - \psi(z) = \pi \cot \pi z$

11. Prove the folowings :

(i) $\psi(n+1) = \sum\limits_{k=1}^{n} \dfrac{1}{k} - \gamma$

(ii) $\psi\left(n+\dfrac{1}{2}\right) = 2\sum\limits_{k=1}^{n} \dfrac{1}{2k-1} - \gamma - 2\log 2.$

12. Prove that

$$\psi(z) = \int_0^\infty \left\{e^{-t} - \frac{1}{(1+t)^z}\right\}\frac{dt}{t}, \ Re(z) > o.$$

13. Prove that

$$\psi(z) = \frac{1}{\Gamma(z+1)} \int_0^1 e^{-t}\, t^z \log t\, dt \ Re(z) > -1.$$

14. Prove that

$$\psi(z) = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+z}\right) - \gamma.$$

CHAPTER THREE

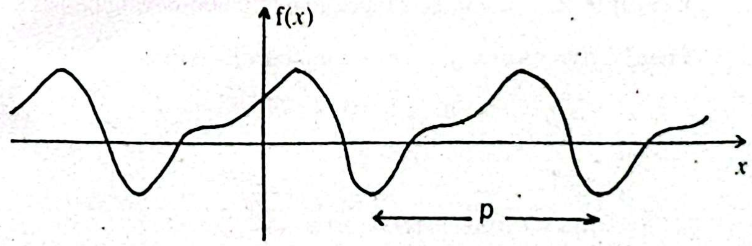
FOURIER SERIES AND FOURIER INTEGRALS

3.1 Introduction

Periodic functions occur frequently in engineering problems. The representation of these engineering problems in terms of simple periodic functions, such as sine and cosine' is a matter of great practical importance, which leads to **Fourier series.** These series, named after the French physicist **JOSEPH FOURIER** (1768—1830), are a very powerful tool in connection with various problems involving ordinary and partial differential equations. Here we shall discuss basic concepts, facts and techniques in connection with Fourier series. Some illustrative examples and also some important engineering applications of these series will be included.

3.2 Periodic functions and Trigonometric series

Definition : The function $f(x)$ of a real variable x is said to be **periodic** if there exists a non-zero number p, independent of x, such that the equation $f(x+p) = f(x)$ holds for all values of x. The least value of p>0 is called the **least period** or simply the **period** of f(x).



Graph of periodic function.