1) (1)

$$\begin{array}{c|ccc} & a & b & 1-a-b \\ \hline \frac{1}{6} & 3,4 & 2,3 & 3,2 \\ \frac{2}{3} & 6,1 & 0,2 & 3,3 \\ \frac{1}{6} & 4,6 & 3,4 & 4,5 \\ \hline & A & B & C \end{array}$$

$4a+3b+2-2a-2b=2a+b+2$

$a+2b+3-3a-3b=3-2a-b$

$6a+4b+5-5a-5b=a+5-b$

best response function

$B_2(a_{-2}) = \{a_2 \text{ in } A_2 \quad u_2(a_2, a_{-2}) \geq u_2(a'_2, a_{-2}) \text{ for all } a'_2 \text{ in } A_2\}$

$(a,b) \to u_2((a,b),(\frac{1}{6},\frac{2}{3})) \geq u_2(\boxed{}, (\frac{1}{6},\frac{2}{3}))$

set of actions

any other probability dist.
for P1 strategy profile

$C = \frac{1}{3} + 2 + \frac{5}{6} = \frac{19}{6}$ ✓ best response

$B = \frac{1}{2} + \frac{4}{3} + \frac{2}{3} = \frac{5}{2}$

$C = \frac{2}{3} + \frac{2}{3} + 1 = \frac{7}{3}$

b)

Safety levels

$$\begin{array}{c|ccc|c} & A & B & C & \\ D & 3,4 & 2,3 & 3,2 & \rightarrow (2) \\ E & 6,1 & 0,2 & 3,3 & \rightarrow (0) \\ F & 4,6 & 3,4 & 4,5 & \rightarrow \boxed{(3)} \\ & 1 & \boxed{2} & \boxed{2} & \end{array}$$

Safety levels

(3,4) or (4,5)

$v_1 = \max_{a_1} \min_{a_{-1}} v_1(a_1, a_{-1}) \Rightarrow \{F\}$

$v_2 = \max_{a_2} \min_{a_{-2}} v_2(a_2, a_{-2}) \rightarrow \{B, C\} \Rightarrow \{C\}$ is better

Safety level and maximin strategy are necessarily the same

c)

$$\begin{array}{c|ccc|} & p_2 & q_2 & 1-p_2-q_2 \\ p_1 & \overset{*}{3},4 & 2,3 & \overset{*}{3},2 \\ q_1 & \overset{*}{6},1 & 0,2 & 3,3 \\ 1-p_1-q_1 & \overset{*}{4},6^* & 3,4^* & \overset{*}{4},5^* \\ & \text{NE} & & \text{NE} \end{array}$$

All of the non-marked entries are not NE since with changing one of the strategies of either P1 or P2, one can gain more reword.

2) a)

$$A = \begin{array}{c|ccc|c} & y_0 & y_1 & y_2 & \\ \hline x_0 & 0 & 1 & 2 & 0 \\ x_1 & 2 & -1 & -2 & -2 \\ x_2 & 3 & -3 & 0 & -3 \\ \hline & 3 & 1 & 2 & \end{array}$$

$* \quad \min_{y} \left\{ \max_{i} \sum_{j=0}^{2} A_{ij} y_j \right\}$ subject to $\sum_{j=0}^{2} y_j = 1, \; y \geq 0$

$*' \quad \max_{x} \left\{ \min_{i} \sum_{j=0}^{2} A_{ij} x_j \right\}$ subject to $\sum_{j=0}^{2} x_j = 1, \; x \geq 0$

$* \min \left\{ \max \left\{ y_1 + 2y_2, \; 2y_0 - y_1 - 2y_2, \; 3y_0 - 3y_1 \right\} \right\}$ s.t $\sum_{j=0}^{2} y_j = 1 \quad y \geq 0$

$*' \max \left\{ \min \left\{ 2x_1 + 3x_2, \; x_0 - x_1 - 3x_2, \; 2x_0 - 2x_1 \right\} \right\}$ s.t $\sum_{j=0}^{2} x_j = 1 \quad x \geq 0,$

b) → these programs can be further reduced to these formats

$\max_{x} \sum_{i=0}^{2} x_j$ s.t $Ax \leq 1, \; x \geq 0$, $\min_{y} \sum_{i=0}^{2} y_j$ S.t $A^T y \geq 1, \; y \geq 0$

website: SAS LP Solver

Game value = 0.5

I(0.75, 0.25, 0)

II(0.5, 0.5, 0)

2) A → payoff matrix → game value

$= \min_{1\le j\le n} (p^T A)_j$; $\max_p$, p is one solution to $\max_{\hat{p}} \min_{q} \hat{p}^T A q = v_1^*$

$\max_{1\le i\le m} (Aq)_i$; q is one solution to $\min_{\hat{q}} \max_{p} p^T A \hat{q} = v_2^*$

→ p, q are maxmin and minimax respectively

$\Rightarrow \min_{1\le j\le n} (p^T A)_j = \max_{1\le i\le m} (Aq)_i$

Since this is a zero sum game $\Rightarrow v_1^* \le v_2^*$

→ it is enough to show $v_2^* \le v_1^*$

proof by contradiction $\Rightarrow v_2^* > v_1^*$

→ Choosing q' yielded us a value $v_2' \geq v_2^*$

$\Rightarrow$ q' is the better strategy

$\Rightarrow$ q was the best

$\Rightarrow q = q'$

$\Rightarrow v_2^* = v_1^*$

3)b) $\left[6/37,\ 20/37,\ 0,\ 11/37\right]\begin{bmatrix} 5 & 8 & 3 & 1 & 6 \\ 4 & 2 & 6 & 3 & 5 \\ 2 & 4 & 6 & 4 & 1 \\ 1 & 3 & 2 & 5 & 3 \end{bmatrix}$

$= \left[\frac{121}{37}, \frac{121}{37}, \frac{160}{37}, \frac{121}{37}, \frac{169}{37}\right]$ $\quad \min = \frac{121}{37}$

$$\begin{bmatrix} 5 & 8 & 3 & 1 & 6 \\ 4 & 2 & 6 & 3 & 5 \\ 2 & 4 & 6 & 4 & 1 \\ 1 & 3 & 2 & 5 & 3 \end{bmatrix}\begin{bmatrix} 14/37 \\ 4/37 \\ 0 \\ 19/37 \\ 0 \end{bmatrix} = \begin{bmatrix} 121/37 \\ 121/37 \\ 112/37 \\ 121/37 \end{bmatrix}$$

$\max = \frac{121}{37}$

value of the game $= \frac{121}{37}$

ب)

	-1	0	2
-1	$(0^*, -1^*)$	$(-1, -2)$	$(-3, -4)$
0	$(1^*, -1^*)$	$(0, 0^*)$	$(-2, -2)$
2	$(2^*, -1^*)$	$(2, 0^*)$	$(0, 2^*)$

$[-1, -1]$ NE_1

$[0, -1]$ NE_2

$[2, -1]$ NE_3

ج)

	1	2	3	...	n	
1	^{*1}	-1^*	-1^*		-1	$q_1 - \sum_{i=2}^{n} q_i$
2	^{*0}	1^*	-1		-1	
3	0	0	1^*		-1	$q_2 - \sum_{i=3}^{n} q_i$
.	0	0	0			
.	0	.				
.						
n	0	0	0		1	q_n

P_1 $\quad -P_1 + P_2$ $\quad P_n - \sum_{i=2}^{n} P_i$

q_1

$$\Rightarrow q_1 - q_2 = q_2 \Rightarrow q_1 = 2q_2$$

$$q_2 - q_3 = q_3 \Rightarrow q_2 = 2q_3$$

$$\dots$$

$$q_{n-2} - q_{n-1} = q_{n-1} \Rightarrow q_{n-2} = 2q_{n-1}$$

$$q_{n-1} - q_n = q_n \Rightarrow q_{n-1} = 2q_n$$

$$q_1 + q_2 + \dots + q_n = 1 \Rightarrow 2^{n-1} q_n + 2^{n-2} q_n + \dots + 2 q_n + q_n = 1$$

$$\Rightarrow (2^n - 1) q_n = 1 \Rightarrow q_n = \frac{1}{2^n - 1} \qquad q_{n-1} = \frac{1 \times 2}{2^n - 1}$$

$$q_j = \frac{2^{n-j}}{2^n - 1}$$

rows:

$P_1 = -P_1 + P_2 \Rightarrow P_2 = 2P_1$

$P_2 = -P_2 + P_3 \Rightarrow P_3 = 2P_2$

$P_{n-1} = -P_{n-1} + P_n \Rightarrow P_n = 2P_{n-1}$

$P_1 + P_2 + \cdots + P_n = 1 \Rightarrow P_1 + 2P_1 + \cdots + 2^{n-1} P_1 = 1$

$\Rightarrow P_1 = \frac{1}{2^n - 1} \qquad P_2 = \frac{2}{2^n - 1} \qquad P_j = \frac{2^{j-1}}{2^n - 1}$

6.

Step #1: For each player i, traverse through each $i+1$ $i+2$ $i+3$ (mod n) players strategies and find the best strategy according to those. i.e. each player has the following table (first strategy is denoted by A and second by B)

$i+1$	$i+2$	$i+3$	i
A	A	A	A/B
A	A	B	A/B
A	B	A	A/B
A	B	B	A/B
B	A	A	A/B
B	A	B	A/B
B	B	A	A/B
B	B	B	A/B

if computing each row takes $O(1)$ this step takes $8n$

Step #2: Construct the following graph

- for each player i add V_i which consists of 8 nodes corresponding to each of the rows in player's #i respective table.

- make a node in V_i and V_{i+1} adjacent iff the corresponding row in V_i has xyz under $i+1$ $i+2$ $i+3$ and corresponding row in V_{i+1} has xyz under i $i+1$ $i+2$

ex.

$\overset{i+1}{A} \cdot \overset{i+2}{A}\ \overset{i+3}{A}\ \overset{i}{B}$ ———— $\overset{i+1}{A}\ \overset{i+2}{A}\ \overset{i+3}{B}\ \overset{i}{A}$

⋮

Construction of graph takes $8 \times 8n + 8n \in O(n)$

Step #3: for each node v in the first bag, find a path travelling through every bag once and back to v.

this can be done simply by BFS modifications

add $v \to$ mark 1

add $N(V) \to$ mark 2

add $N(N(V)) \to$ mark 3

⋮

add $\underbrace{N(N(\dots(V)))}_{S} \to$ mark $n-1$

is $v \in N(S)$?
- yes → return yes NE exists
- NO → go to next v of V_1
  - last v? → return NO

this step is linear

$\longrightarrow O(n)$

why would this work?

if NE exists $\longleftrightarrow$ Algorithm returns yes

$\rightarrow$: NE exists $\Rightarrow$ Same Strategy profile exists for n players

Assume the SP = $P_1 P_2 P_3 \dots P_n P_1 P_2 P_3$

for each 4 consecutive letter in this profile

$P_i P_{i+1} P_{i+2} P_{i+3}$ Since it is NASH,

player i best option with respect to players $i+1$ $i+2$ $i+3$ best options is P_i

$\Rightarrow$ $P_i P_{i+1} P_{i+2} P_{i+3}$

exists as a node in V_i

also for $P_{i+1} P_{i+2} P_{i+3} P_{i+4}$

exists as a node in V_{i+1}

(Since player's $i+1$ best option with respect to $P_{i+2} P_{i+3} P_{i+4}$ is P_{i+1}

$\rightarrow$ these two node are linked by definition $\Rightarrow$

there exists a cycle from P_1 through P_i $i=1 \dots n$ to P_1

$\Leftarrow$: we have a yes from our Alg $\Rightarrow$ there is a cycle passing through each bag back to first node

$\Rightarrow$ there exists a strategy for player i which is compatible for a strategy of plyer k $i+1 < k < i+3$

$\Rightarrow$ Since strategies in the tables were the best ones $\rightarrow$ this yields a nash.

7) a) if (i,j) is an NE $\Rightarrow$ each of the entries in the corresponding row #i has to be less equal x.

Since each of the values are chosen randomly and independently, the probability of one entry to be less than x is x (values are $\in [0,1]$)

$\Rightarrow$ there is $n-1$ entries in #i row and $n-1$ entries in #j column.

$\Rightarrow$ each probability of row's values of being $\leq x = x$

each probability of column's values of being $\leq y = y$

$\Rightarrow$ (i,j) is an NE with probability $x^{n-1} y^{n-1}$

b) Assume each of the columns maximum values in second entry is marked.

For the pure NE to not exist, in each of the n rows, maximum has to be any other $n-1$ values other than marked one.

For each row, assume pair (x_i, y_i) has a mark $\Rightarrow$ x_i has not to be among maximums of the current row's first values.

probability of choosing any other values in a row than the marked one.

Since all of the probabilities are uniform, the chance of marking same entry is $\frac{1}{n}$ $\Rightarrow$ not hitting $= \frac{n-1}{n}$

This value stands for every row →

$$P(\text{No NE}) = \left(1 - \frac{1}{n}\right)^n$$

$$\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e}$$

#1

8) a) By fixing strategy p and changing q, a set of $u(p,q)$ can allocate values in $[-1,1] \Rightarrow \inf_q u(p,q) = -1$

Among all strategies that yield a u with value of -1, one strategy with exactly value -1 is feasible

$\Rightarrow \sup_p \inf_q u(p,q) = -1$ $\qquad \min\{-1,-1,-1,\dots,0,1\} = -1$

$\max\{-1\} = -1$

#2 By fixing strategy q and varying p among possible values $u(p,q)$ can allocate values in $[-1,1] \Rightarrow \sup_p u(p,q) = 1$

Among all strategies that yield $u=1$, a strategy with value 1 is constructable $\Rightarrow$

$\inf_q \sup_p u(p,q) = 1$ $\qquad \max\{-1,0,1\} = 1$

$\min\{1\} = 1$

b)

	q_1 1	q_2 2	q_3 3	q_4 4	q_5 5	...
1	0	1	1	1	1	$\rightarrow q_2 + \sum_{i=3}^{\infty} q_i$
2	-1	0	1	1	1	$\rightarrow -q_1 + \sum_{i=3}^{\infty} q_i$
3	-1	-1	0	1	1	
4	-1	-1	-1	0	1	
5	-1	-1	-1	-1	0	
⋮						

using indifference principle:

$$q_2 + \sum_{i=3}^{\infty} q_i = -q_1 + \sum_{i=3}^{\infty} q_i \Rightarrow q_2 = -q_1$$

$\Rightarrow$ since $q_1, q_2 \geq 0 \Rightarrow q_2 = q_1 = 0$