Computational Complexity and Game Theory

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Outline

- Preliminary
- Function Complexity classes
- **MNE** is **PPAD**-Complete
- Conclusion

Preliminary

- Decision Problems
- P and NP
- Completeness
- NP-Complete Problems
- Co-NP

Decision Complexity

A **decision problem** is a problem that can be posed as a yes-no question of the input values.

- Is *m*, a number, prime?
- Does graph *G* has a Hamiltonian path?
- Does the TM *M* halt on the input *w*?

Decision Complexity

A **decision problem** is a problem that can be posed as a yes-no question of the input values.

P	NP

Decision Complexity

A language ***L*** is in **P** if and only if there exists a deterministic Turing machine ***M***, such that

- ***M*** runs for polynomial time on all inputs
- For all ***x*** in ***L***, ***M*** outputs **1**
- For all ***x*** not in ***L***, ***M*** outputs **0**

A language ***L*** is in **NP** if and only if there exist polynomials ***p*** and ***q***, and a deterministic TM ***M***, such that

- For all ***x*** and ***y***, the ***M*** runs in time ***p*(|*x*|)** on input
- For all ***x*** in ***L***, there exists a string ***y*** of length ***q*(|*x*|)** such that **M(x,y)=1**
- For all ***x*** not in ***L*** and all strings ***y*** of length ***q*(|*x*|)**, **M(x,y) = 0**

Decision Complexity

P

$\stackrel{?}{=}$

NP

Completeness

A problem ***p*** is called **hard** for a complexity class ***C*** under a given type of reduction if there exists a reduction (of the given type) from any problem in ***C*** to ***p***. If a problem is both **hard** for the class and a member of the class, it is **complete** for that class (for that type of reduction).

NP-Complete problems

- SAT
- Clique
- TSP
- Hamiltonian Path
- Independent Set

Co-NP

- **NP** is not known to be closed under complement
  - Known as **NP vs Co-NP** problem
  - **TAUT**

Function Complexity

- Functions and Computation
- FP and FNP
- TFNP
- PPA, PPAD, PPP, PLS

Function Complexity

A function problem is a problem that given input x, has to return y such that some relation R(x, y) holds, and reject O.W.

- The actual solution to SAT: FSAT
- Return the IS of graph G, if exists.
- Factorize integer I

Function Complexity

FP	FNP

Decision Complexity

A binary relation **P(*x*,*y*)** is in **FP** if and only if there is a deterministic polynomial time algorithm that, given ***x***, can find some ***y*** such that **P(*x*,*y*)** holds.

A binary relation **P(*x*,*y*)**, where ***y*** is at most polynomially longer than ***x***, is in FNP if and only if there is a deterministic polynomial time algorithm that can determine whether **P(*x*,*y*)** holds given both ***x*** and ***y***.

Function Complexity

FP	$\stackrel{?}{=}$	FNP

Some remarks

- **FSAT** is **FNP-Complete**
- **SAT** is self-reducible
- Any **NP-C** problem is self-reducible
- Conjecture: IntFact is not self-reducible
- ***NP* = *P*** if and only if ***FP* = *FNP***
- *If **A** is in **NPC** and in **TFNP*** Hence ***NP*** = ***CoNP***

Total functions

A function defined for all possible input values.

TFNP

Total functions

A function defined for all possible input values.

A binary relation **P(*x*,*y*)** is in **TFNP** if and only if there is a deterministic polynomial time algorithm that can determine whether **P(*x*,*y*)** holds given both ***x*** and ***y***, and for every ***x***, there exists a ***y*** which is at most polynomially longer than ***x*** such that **P(*x*,*y*)** holds.

The Class PLS

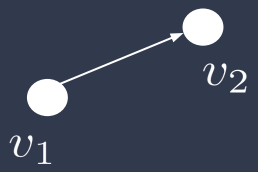
"Every DAG has a sink."

Suppose that a DAG with vertex set $\{0,1\}^n$ is defined by two circuits:

node id $\longrightarrow$ $\boxed{C}$ $\longrightarrow$ {node id_1, …, node id_k}

node id $\longrightarrow$ $\boxed{F}$ $\longrightarrow$ $\mathbb{R}$ [redacted]

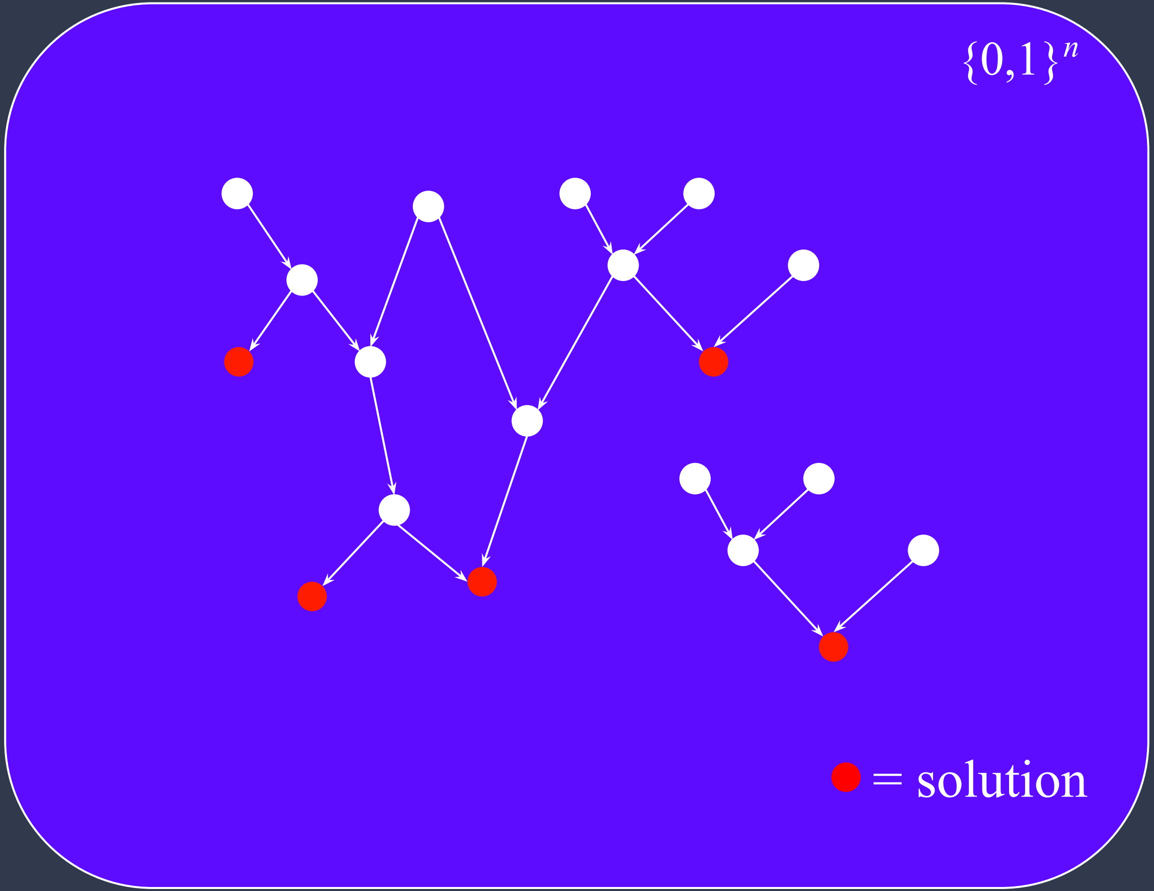
$$v_2 \in C(v_1) \quad \wedge \quad F(v_2) > F_{(}v_1)$$



FIND SINK: Given C, F: Find x s.t. F(x) ≥ F(y), for all $y \in C(x)$.

PLS = { *Function problems in FNP reducible to* FIND SINK }

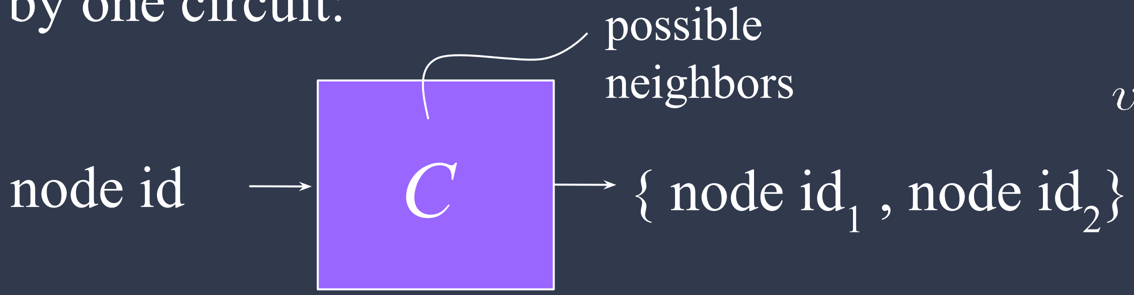
The DAG



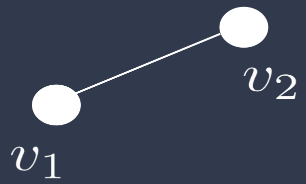
The Class PPA

“If a graph has a node of odd degree, then it must have another.”

Suppose that an exponentially large graph with vertex set $\{0,1\}^n$ is defined by one circuit:



$$v_1 \in C(v_2) \quad \wedge \quad v_2 \in C(v_1)$$



ODD DEGREE NODE: Given C: If 0^n has odd degree, find another node with odd degree. Otherwise say “yes”.

PPA = $\{$ *Search problems in FNP reducible to* ODD DEGREE NODE$\}$

The Class PPP

"If a function maps n elements to n-1 elements, then there is a collision."

Suppose that an exponentially large graph with vertex set $\{0,1\}^n$ is defined by one circuit:

$$\text{node id} \longrightarrow \boxed{C} \longrightarrow \text{node id}$$

COLLISION: Given C: Find x s.t. $C(x) = 0^n$; or find $x \neq y$ s.t. $C(x)=C(y)$.

PPP = { *Search problems in FNP reducible to* COLLISION }

The PPAD Class

Suppose that an exponentially large graph with vertex set $\{0,1\}^n$ is defined by two circuits:

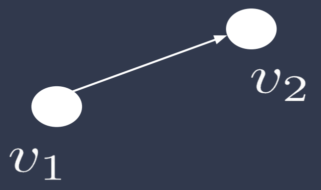
possible previous

node id $\longrightarrow$ $\boxed{P}$ $\longrightarrow$ node id

node id $\longrightarrow$ $\boxed{N}$ $\longrightarrow$ node id

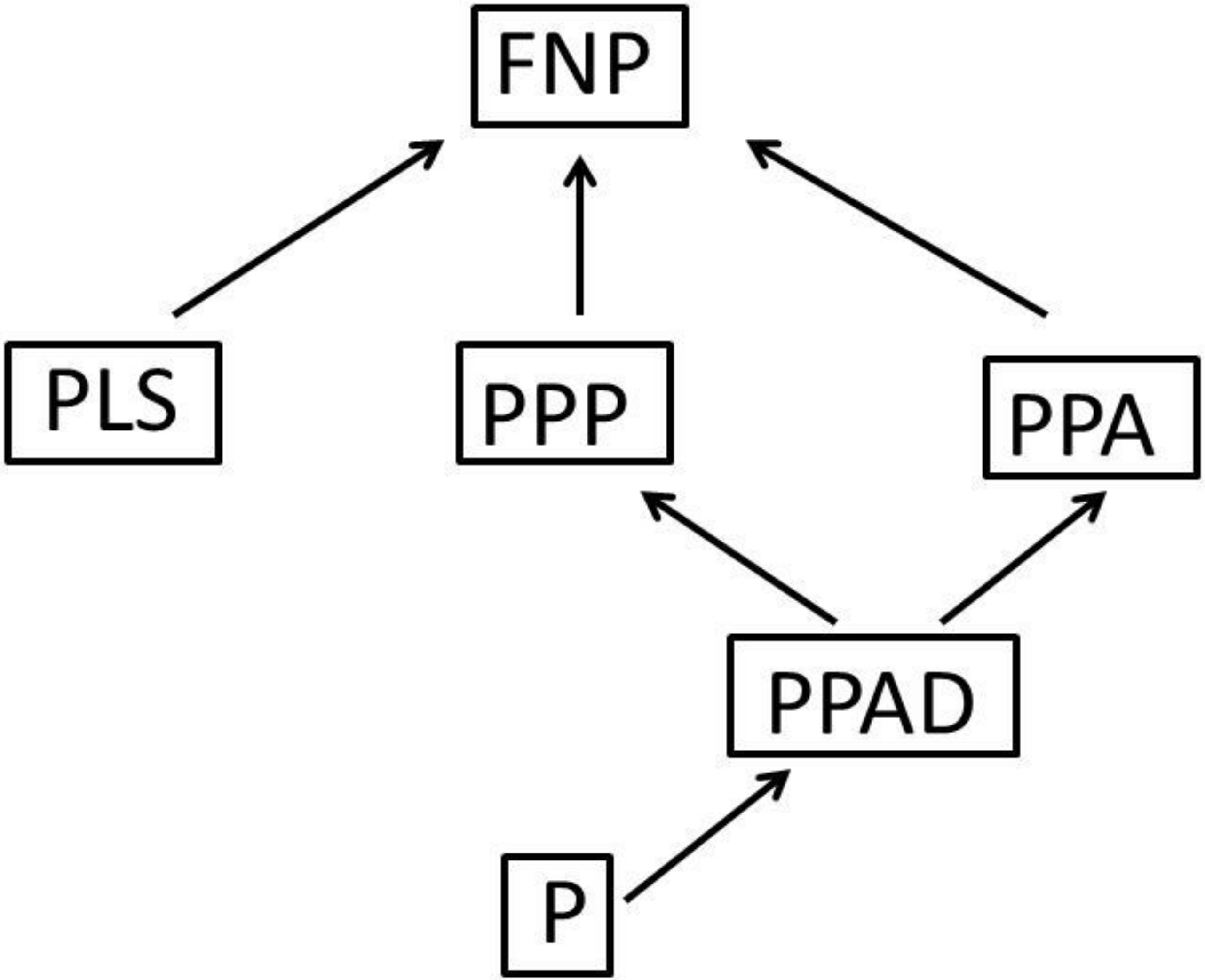
possible next

$$P(v_2) = v_1 \wedge N(v_1) = v_2$$



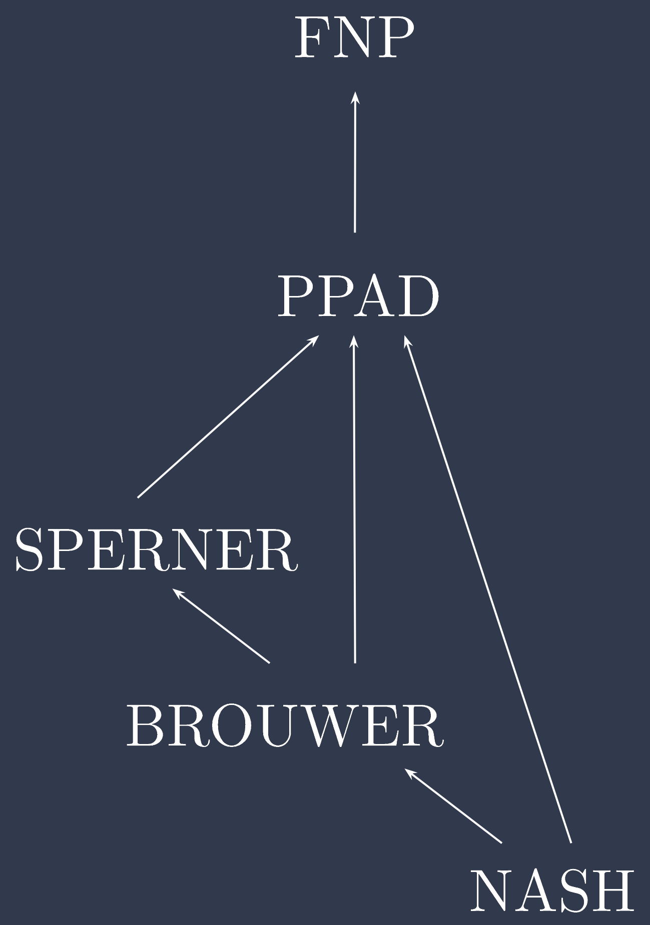
END OF THE LINE: {Given P and N: If 0^n is an unbalanced node, find another unbalanced node. Otherwise say "yes".}

PPAD = *{ Search problems in FNP reducible to END OF THE LINE}*


FNP
PLS
PPP
PPA
PPAD
P

NASH is PPAD-Complete

- Nash to Brouwer Theorem
- Brouwer Theorem to Sperner
- Sperner to END-OF-THE-LINE
- The other direction


FNP
PPAD
SPERNER
BROUWER
NASH

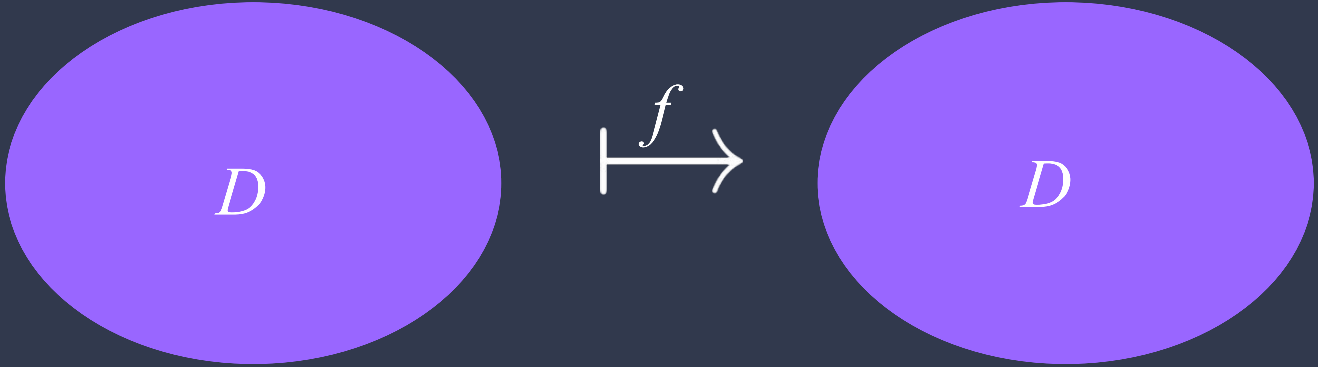
Brouwer's fixed point theorem

Theorem: Let $f : D \longrightarrow D$ be a continuous function from a convex and compact subset D of the Euclidean space to itself.

Then there exists an $x \in D$ s.t. $x = f(x)$.

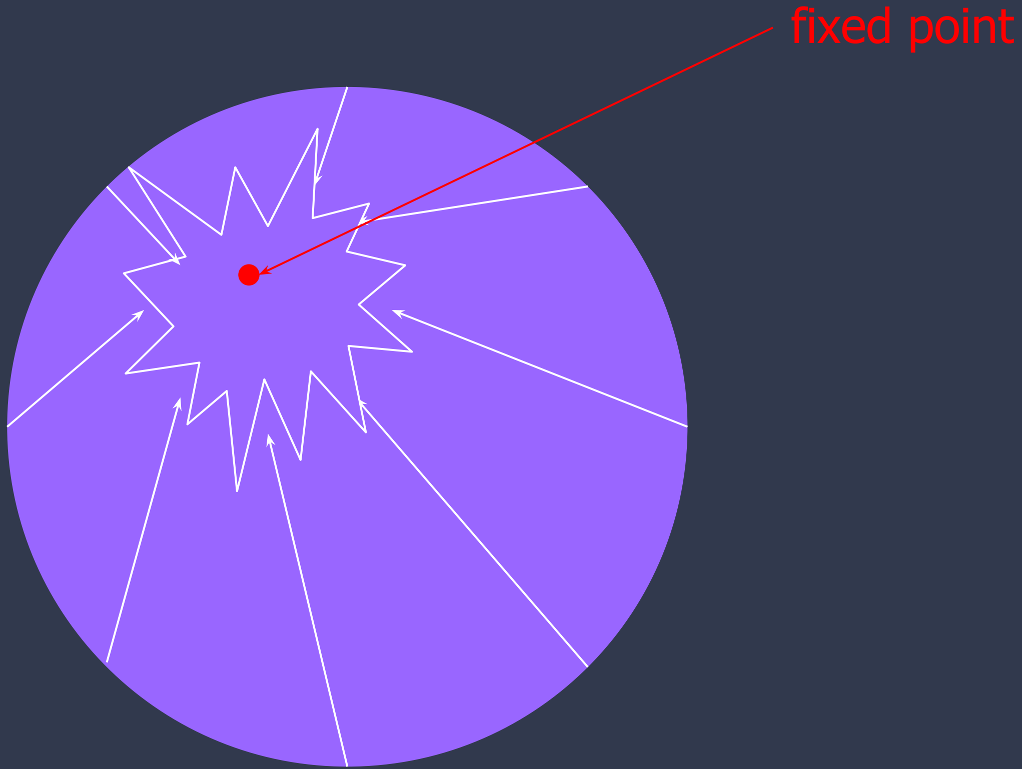
closed and bounded

Below we show a few examples, when D is the 2-dimensional disk.

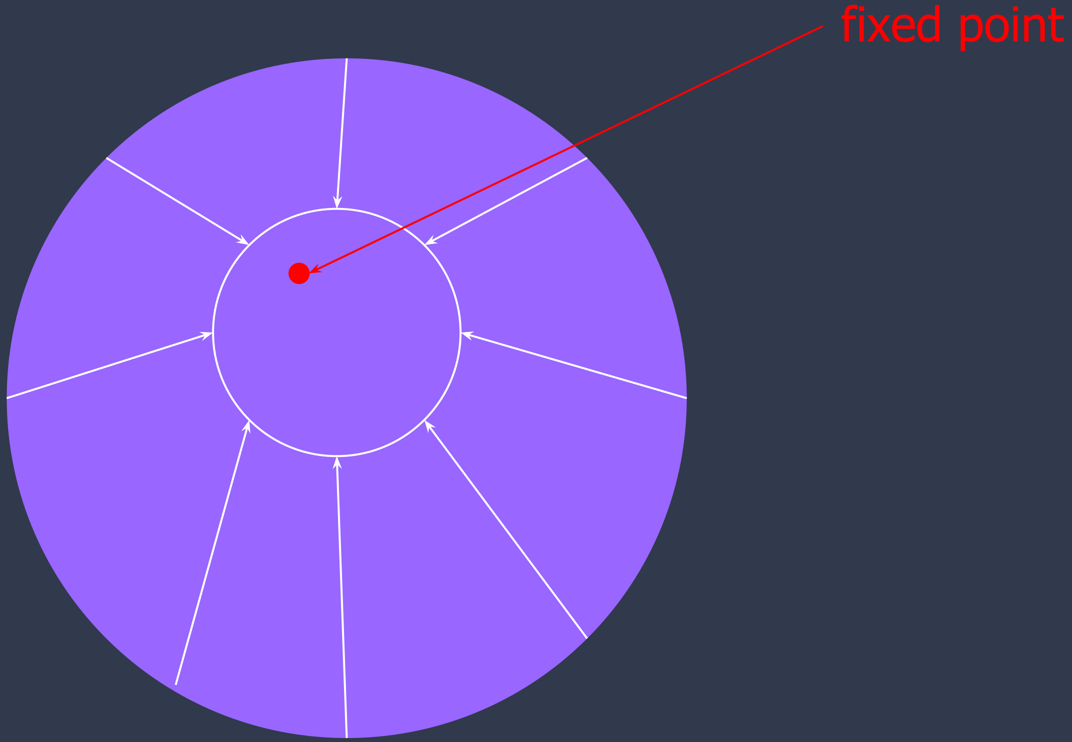


N.B. All conditions in the statement of the theorem are necessary.

Brouwer's fixed point theorem



Brouwer's fixed point theorem



Nash's Function

$$\Delta \ni x \overset{f}{\mapsto} y \in \Delta :$$

$$y_p(s_p) := \frac{x_p(s_p) + \mathrm{Gain}_{p;s_p}(x)}{1 + \sum_{s'_p \in S_p} \mathrm{Gain}_{p;s'_p} x)}$$

where: $\mathrm{Gain}_{p;s_p}(x) = \max\{u_p(s_p; x_{-p}) - u_p(x), 0\}$

Visualizing Nash's Construction

Kick / Dive	Left	Right
Left	1 , -1	-1 , 1
Right	-1 , 1	1, -1

Penalty Shot Game

$\mapsto$

$f: [0,1]^2 \rightarrow [0,1]^2$, continuous
such that
fixed points $\equiv$ Nash eq.

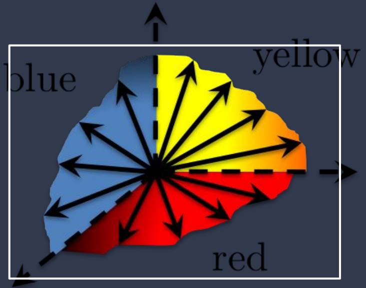
Visualizing Nash's Construction

Kick / Dive	Left	Right
Left	1 , -1	-1 , 1
Right	-1 , 1	1, -1

Penalty Shot Game

$\mapsto$

Visualizing Nash's Construction



Kick / Dive	Left	Right
Left	1 , -1	-1 , 1
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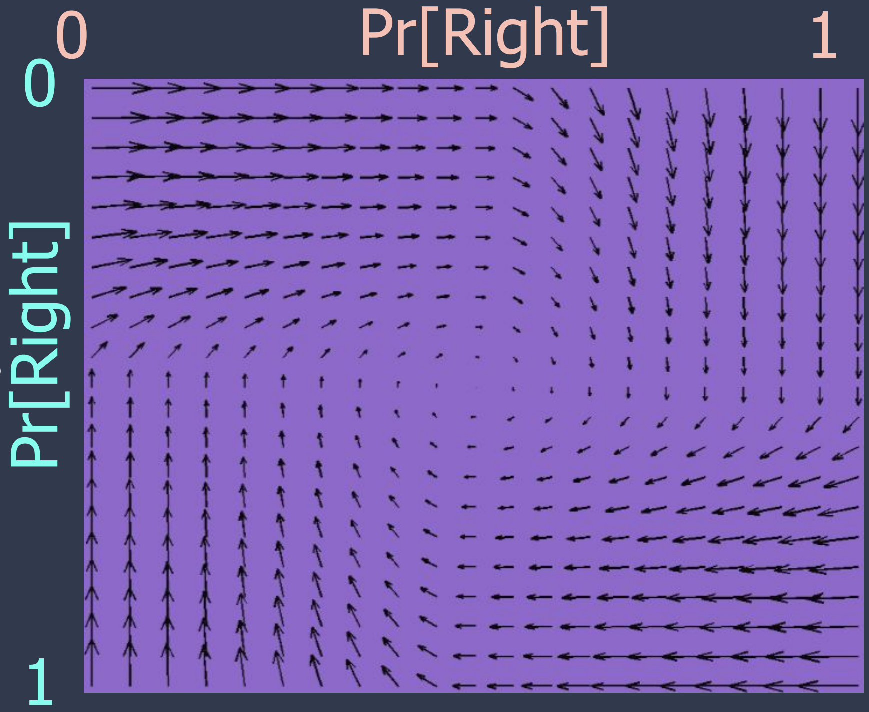
Penalty Shot Game

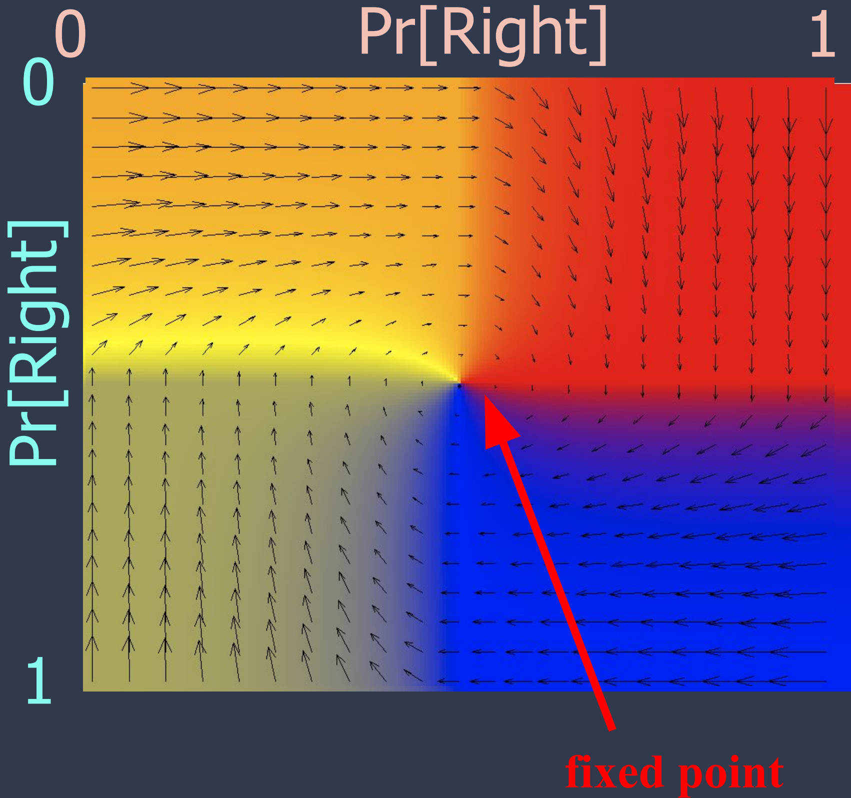


Visualizing Nash's Construction

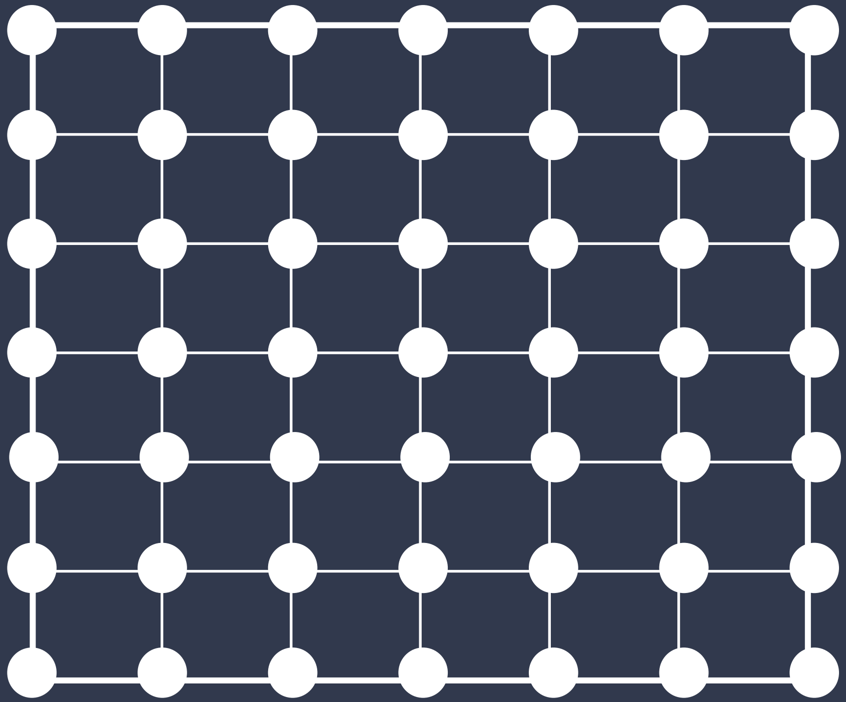
		½	½
	Kick / Dive	Left	Right
½	Left	1 , -1	-1 , 1
½	Right	-1 , 1	1, -1

Penalty Shot Game

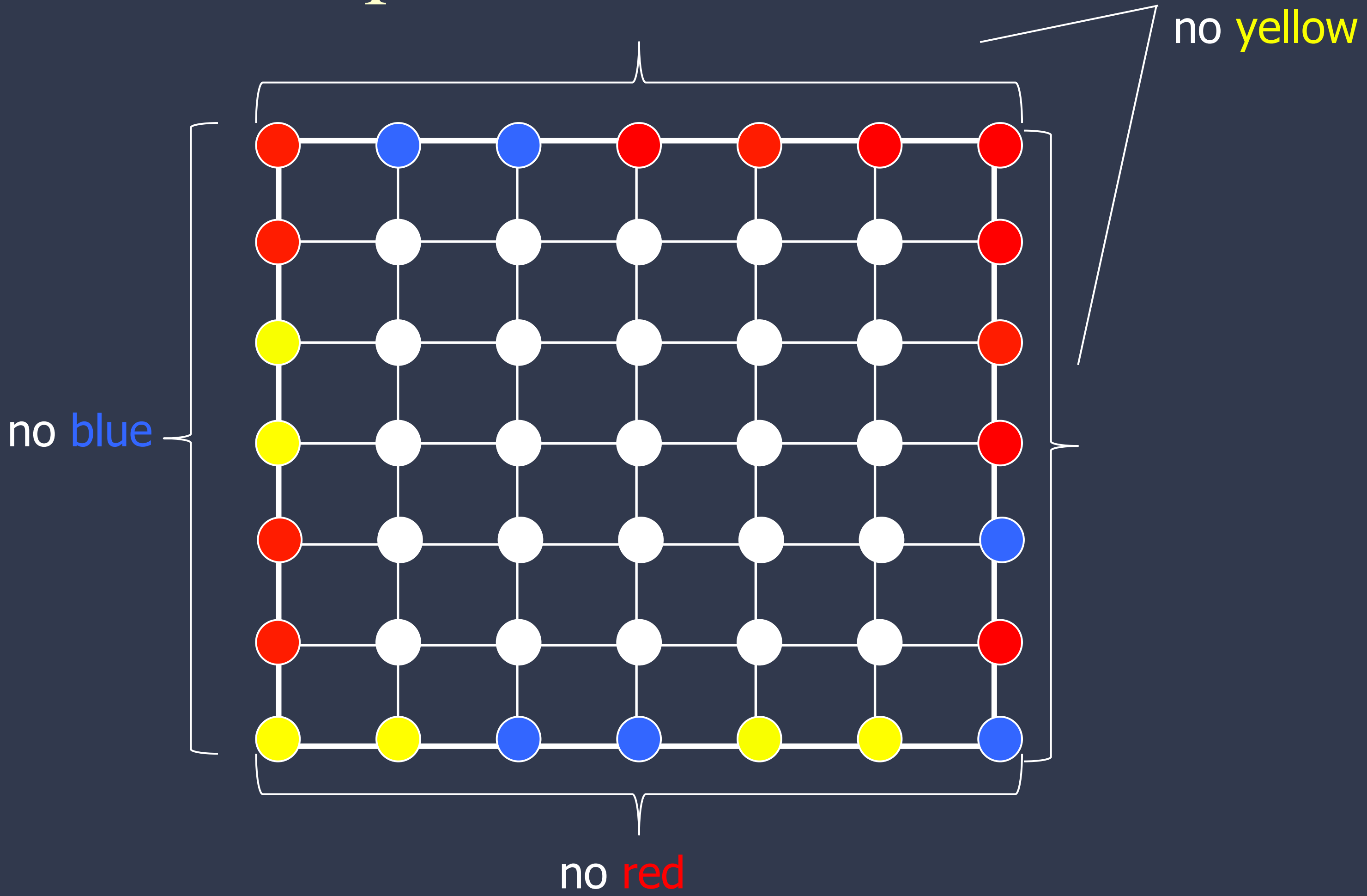
$\mapsto$



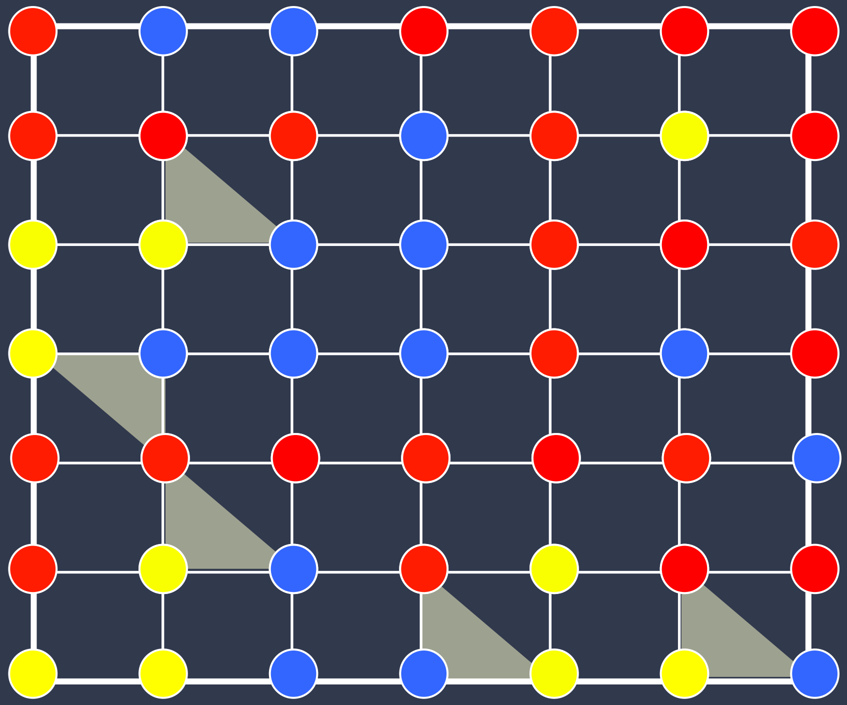
Sperner's Lemma



Sperner's Lemma



Sperner's Lemma



2D-Brouwer on the Square

Suppose f: $[0,1]^2 \rightarrow [0,1]^2$, continuous

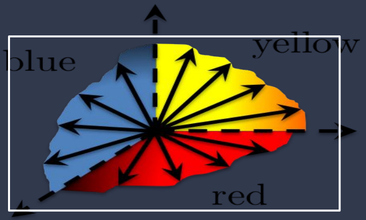
$$\forall \epsilon > 0,\ \exists \delta(\epsilon) > 0, s.t.$$
$$d(z,w) < \delta(\epsilon) \implies d(f(z), f(w)) < \epsilon$$

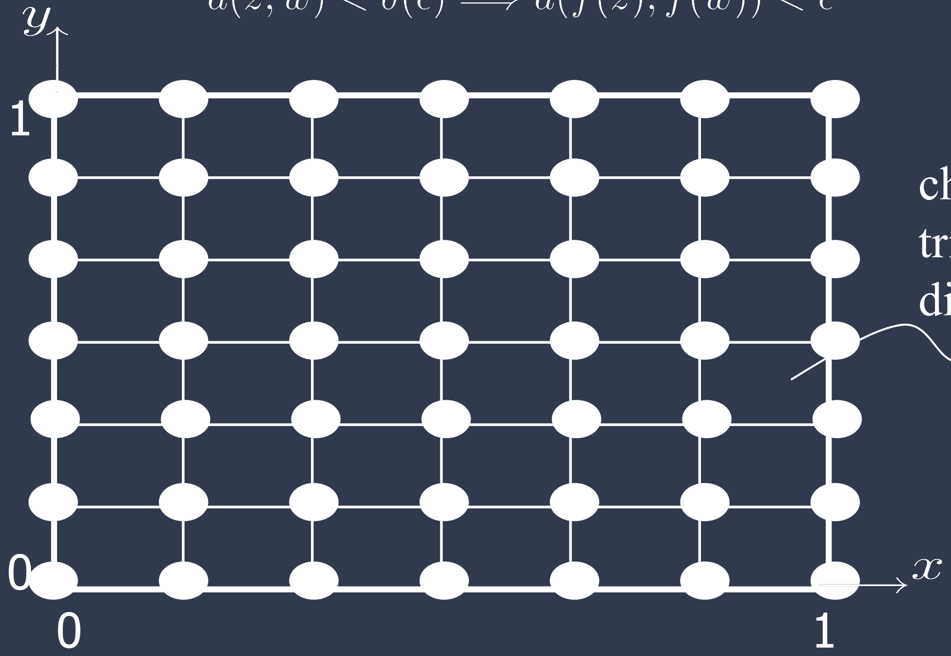
2D-Brouwer on the Square

Suppose f: $[0,1]^2 \rightarrow [0,1]^2$, continuous

$$\forall \epsilon > 0, \ \exists \delta(\epsilon) > 0, s.t.$$
$$d(z,w) < \delta(\epsilon) \implies d(f(z), f(w)) < \epsilon$$

color the nodes of the triangulation according to the direction of
$f(x) - x$





choose some ϵ and triangulate so that the diameter of cells is $\delta < \delta(\epsilon)$

2D-Brouwer on the Square

say d is the ℓ_∞ norm

Suppose f: $[0,1]^2 \rightarrow [0,1]^2$, continuous

$$\forall \epsilon > 0, \;\; \exists \delta(\epsilon) > 0, s.t.$$

$$d(z, w) < \delta(\epsilon) \implies d(f(z), f(w)) < \epsilon$$

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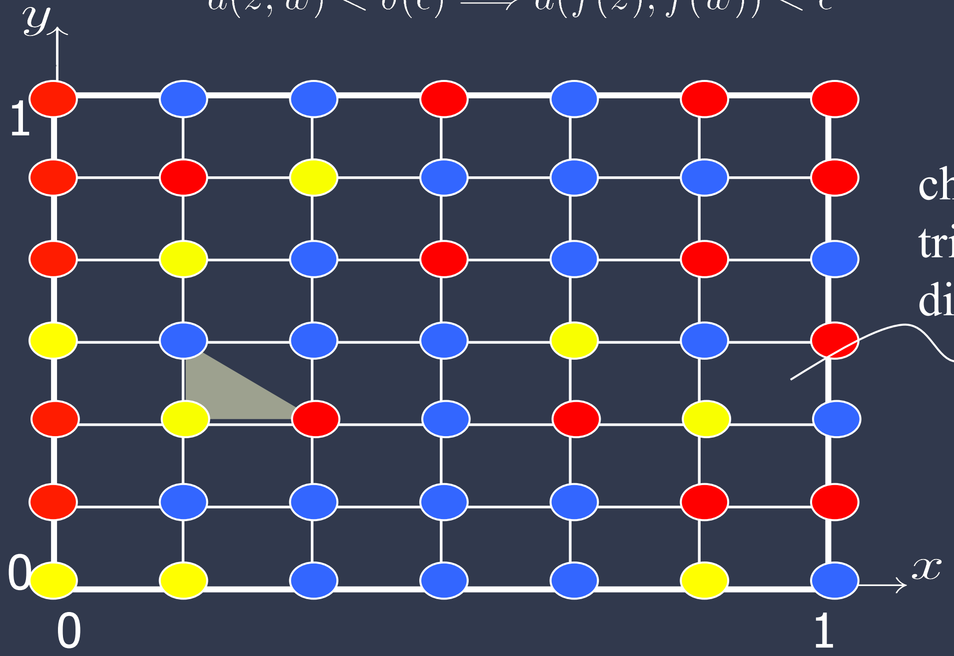


choose some ϵ and triangulate so that the diameter of cells is $\delta < \delta(\epsilon)$

find a trichromatic triangle, guaranteed by Sperner

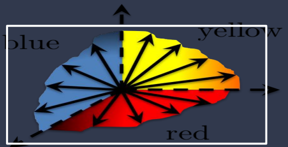
2D-Brouwer on the Square

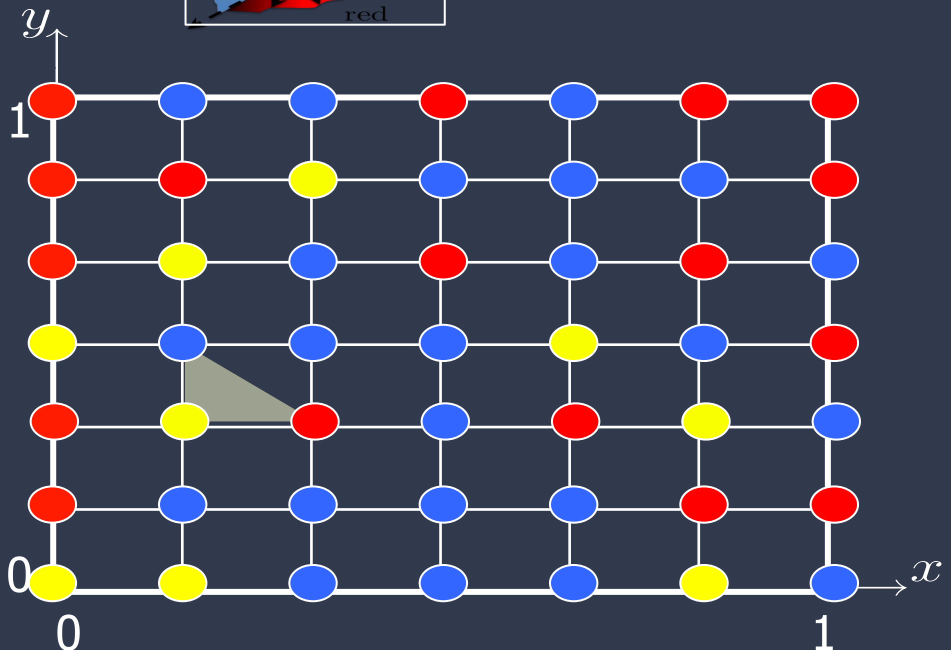
Suppose f: $[0,1]^2 \rightarrow [0,1]^2$, continuous

must be uniformly continuous (by the Heine-Cantor theorem)

$$\forall \epsilon > 0, \ \exists \delta(\epsilon) > 0, s.t.$$

$$d(z,w) < \delta(\epsilon) \implies d(f(z), f(w)) < \epsilon$$





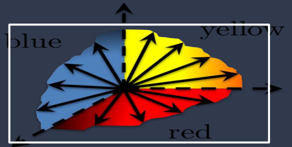
Claim: If z^{Y} is the yellow corner of a trichromatic triangle, then

$$|f(z^{\mathrm{Y}}) - z^{\mathrm{Y}}|_\infty < \epsilon + \delta.$$

Proof of Claim

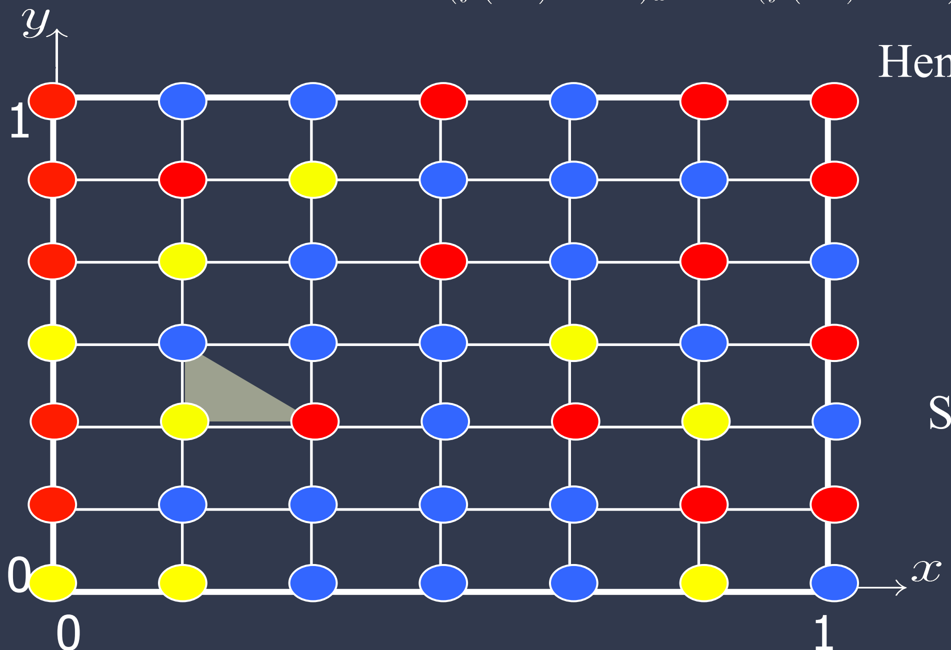
Claim: If z^{Y} is the yellow corner of a trichromatic triangle, then $|f(z^{\mathrm{Y}}) - z^{\mathrm{Y}}|_\infty < \epsilon + \delta.$

Proof: Let $z^{\mathrm{Y}}, z^{\mathrm{R}}, z^{\mathrm{B}}$ be the yellow/red/blue corners of a trichromatic triangle.



By the definition of the coloring, observe that the product of

$$(f(z^Y) - z^Y)_x \text{ and } (f(z^B) - z^B)_x \text{ is } \leq 0.$$



Hence:

$$\begin{aligned}
&|(f(z^Y) - z^Y)_x| \\
&\quad \leq |(f(z^Y) - z^Y)_x - (f(z^B) - z^B)_x| \\
&\quad \leq |(f(z^Y) - f(z^B))_x| + |(z^Y - z^B)_x| \\
&\quad \leq d(f(z^Y), f(z^B)) + d(z^Y, z^B) \\
&\quad \leq \epsilon + \delta.
\end{aligned}$$

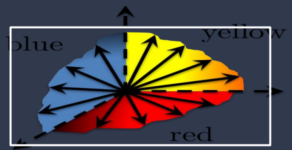
Similarly, we can show:

$$|(f(z^Y) - z^Y)_y| \leq \epsilon + \delta.$$

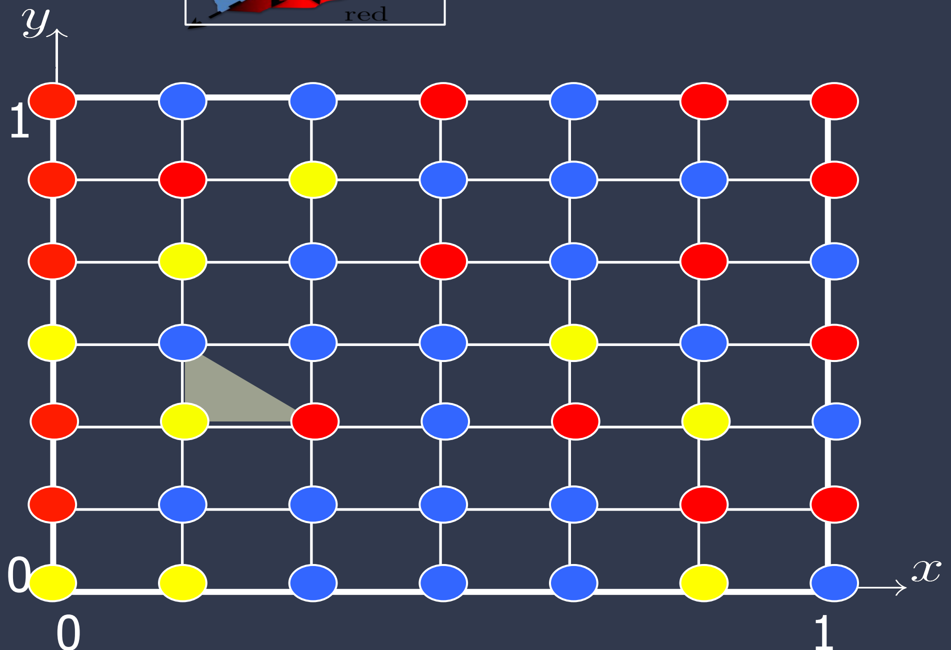
$\square$

2D-Brouwer on the Square

Suppose f: $[0,1]^2 \rightarrow [0,1]^2$, continuous



$$\forall \epsilon > 0, \ \exists \delta(\epsilon) > 0, s.t.$$
$$d(z, w) < \delta(\epsilon) \implies d(f(z), f(w)) < \epsilon$$



Claim: If z^{Y} is the yellow corner of a trichromatic triangle, then

$$|f(z^{\mathrm{Y}}) - z^{\mathrm{Y}}|_\infty < \epsilon + \delta.$$

Choosing $\delta = \min(\delta(\epsilon), \epsilon)$

$$|f(z^Y) - z^Y|_\infty < 2\epsilon.$$

Conclusion

- NASH is hard then, what to do?
- Alternatives, tractability of alternatives
- Open problems

NASH is important

- Universality
- But is NASH credible now?

If finding this Nash equilibrium is intractable (i.e. there exist some specialized instances when one cannot find it in any reasonable amount of time), then it loses universality, and therefore loses credibility as a predictor of behavior.

Approximate Nash equilibria

$$\sum_{s \in S_{-p}} u_p(js)x_s > \varepsilon + \sum_{s \in S_{-p}} u_p(j' s)x_s \Longrightarrow x^p_{j'} = 0.$$

The players have low incentive to change their strategy!

Epsilon-equilibria is PPAD-Complete on games with any number of players, yet more! tractable on games with constant number of player.

Remarks

- There is a FPTAS scheme for computing NASH!

(Fully Polynomial Time Approximation Scheme)

- There is sub-exponential algorithm that computes epsilon-equilibria on games with constant number of players.

$$O\left(n^{\log n/\varepsilon^2}\right)$$

Open Problem

- Whether or not there exists a PTAS for approximate nash equilibria?

- Better than $O\left(n^{\log n/\varepsilon^2}\right)$ algorithm for constant number of players ep-NASH?

Thank you

Questions?

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