Computational Complexity

Exam #1-Take Home
Start Date: 8 Farvardin 1398
Due Date: 24:00 p.m. 10 Farvardin 1398

Some Rules.
1. Please hand in your worksheet prior to 24:00 p.m. 10 Farvardin 1398. Overdue papers are not acceptable at all.
2. You are allowed to use the Internet and the textbooks. You are NOT allowed to talk with your classmates about the problems and the solutions.
3. Partial answers are appreciated, so write down your thinking even though you may think it is insufficient or imperfect.

Problem 1.
(a) Prove that if $\mathbf{P} = \mathbf{NP}$ then $\mathbf{P} = \mathbf{NP} = \mathbf{CoNP}$.
(b) A Boolean formula ϕ with $n + m$ variable $x_1, \ldots, x_n, y_1, \ldots, y_m$ is said to be good if

$$\exists\, x \in \{0,1\}^n\ \forall\, y \in \{0,1\}^m,\ \phi(x, y) = 1.$$

Consider the following problem.

GBF:
Instance: A Boolean formula ϕ.
Query: Is ϕ good?

Prove that if $\mathbf{P} = \mathbf{NP}$, then GBF$\in \mathbf{P}$.

(Hint: First prove that the language

$$L = \{(\phi, x) : \phi \text{ is a Boolean formula and } x \in \{0,1\}^n \text{ and } \forall y \in \{0,1\}^m,\ \phi(x, y) = 1\}$$

is in $\mathbf{CoNP}$.)

Problem 2. A search problem can be written as a subset $S \subseteq \{0,1\}^* \times \{0,1\}^*$, where for every $(x, y) \in S$, x is the instance (input) and y is a solution (output) for x. Note that y is not unique in terms of x. For example, if S is the Hamiltonian Cycle Problem, then $(G, C) \in S$ if G is a simple graph and C is a Hamiltonian cycle for G.
A search problem $S \subseteq \{0,1\}^* \times \{0,1\}^*$ is said to be in the class $\mathcal{P}^*$ if there is a polynomial time (deterministic) Turing machine M which gets an $x \in \{0,1\}^*$ and returns a $y \in \{0,1\}^*$ such that $(x, y) \in S$ if such y exists and returns NO otherwise.
A search problem $S \subseteq \{0,1\}^* \times \{0,1\}^*$ is said to be in the class $\mathcal{NP}^*$ if (1) there is a polynomial $p(n)$ such that for every $(x, y) \in S$ we have $|y| \leq p(|x|)$ and (2) there is a polynomial time (deterministic) Turing machine M which gets an $(x, y) \in \{0,1\}^* \times \{0,1\}^*$ and returns YES if $(x, y) \in S$ and returns NO otherwise.
(a) Prove that $\mathcal{P}^* \subseteq \mathcal{NP}^*$.
(b) Prove that $\mathbf{P} = \mathbf{NP}$ if and only if $\mathcal{P}^* = \mathcal{NP}^*$.

Problem 3. Define the classes Γ_1 and Γ_2 as follows.

Γ_1: The class of all languages L for which there exists a polynomial-time non-deterministic Turing machine N such that $L = L(N)$, and for every input x, the computation tree of $N(x)$ admits at most one accepting path.

Γ_2: The class of all languages L for which there exists a polynomial-time non-deterministic Turing machine N such that for every input x, we have

$$x \in L \Longleftrightarrow \text{The computation tree of } N(x) \text{ admits at most one accepting path.}$$

Answer the following questions. Justify your answers as best as you can.

(a) Is it true that $\Gamma_1 = \Gamma_2$?
(b) Is it possible to define "complete" problems in Γ_1 or Γ_2?!
(c) Let USAT be the language of all Boolean formulas admitting *exactly one* satisfying assignment. Does USAT belong to any of the classes Γ_1 or Γ_2? Is it complete in any of them under polynomial time (Karp) reduction?
(d) Which of the following inclusions are true, and which are (possibly) false?

$$\mathbf{P} \subseteq \Gamma_1 \subseteq \mathbf{NP} \cap \Gamma_2 \subseteq \mathbf{NP} \subseteq \mathbf{Co} - \Gamma_1$$

(e) Are Γ_1 and Γ_2 closed under intersection?

Problem 4. Consider the following decision problem.

Instance: A finite set S and some of its subsets $A_1, \ldots, A_m$.
Query: Does there exist a partition (S_1, S_2) of S, such that for all $i = 1, \ldots, m$, A_i intersects both S_1 and S_2?

(a) Show that the above problem is **NP**-complete.
(Hint: Recall the reduction from 3-SAT to Hitting Set Problem you have observed in the first Exam, and try to modify it!)

(b) Explain why this problem belongs to **P**, provided that $|A_1| = \cdots |A_m| = 2$.

Good Luck and Happy New Year!