Computational Complexity

Problem set #3
Due date: 12 Tir 1398 (via lms.iut.ac.ir)

Solutions to the problems on this set might partially be found on the Internet. It is not acceptable to copy such solutions. Write with your own notation and formulation. For the randomized FPT algorithms, **you don't need to present your derandomization**.

Solve at least 6 problems to get a complete mark.

Problem 1. Suppose that $\mathrm{tw}(G) = k$. Prove that there is a tree decomposition for G of width k, where $|V(T)| \leq |V(G)| - k$. Also, prove that $|E(G)| \leq k|V(G)| - \binom{k+1}{2}$.

Problem 2. In the slides of Session 27, elaborate completely that why the recursion given in page 6 for Independent Set Problem and page 11 for Dominating Set Problem work?

Problem 3. Here we are going to see the parameterized problems with structural parameters of the graph. Recall that a *regular* graph with all vertices of degree d is called a *d-regular* graph.

(a) In the Feedback Vertex Set (FVS) problem, we are given an undirected graph G and a nonnegative integer k, and the objective is to determine whether there exists a set $S \subseteq V(G)$ such that $|S| \leq k$ and $G - S$ is an acyclic graph (a forest). Show that FVS admits a kernel with $\mathcal{O}(k)$ vertices on undirected regular graphs.

(b) Show that Clique, parameterized by the solution size k, is **FPT** on d-regular graphs for every fixed integer d. Also show that the problem is **FPT** with parameter $k + d$.

Problem 4. In the Cluster Vertex Deletion problem, we are given a graph G and an integer k, and the task is to delete at most k vertices from G to obtain a cluster graph (a disjoint union of cliques). Obtain a $3^k n^{\mathcal{O}(1)}$-time algorithm for Cluster Vertex Deletion.

Problem 5. Give a randomized **FPT** algorithm for the problem of deciding whether a given directed graph contains a cycle of length at least k. Your algorithm should use color coding with k^2 colors, and have running time $2^{\mathcal{O}(k^2)} n^{\mathcal{O}(1)}$. Then improve the running time to $k^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$ by only using colors $\{1, \ldots, k+1\}$ and assigning color $k+1$ with probability $1 - \frac{1}{k^2}$.

Problem 6. For given a Feedback Vertex Set instance (G, k), obtain a randomized **FPT** algorithm in time $4^k n^{\mathcal{O}(1)}$ that either reports a failure or finds a feedback vertex set in G of size at most k. Moreover, if the algorithm is given a yes-instance, it returns a solution with a constant probability. (Note that you don't have to use the color coding techniques)

Problem 7. A graph G is called *d-degenerate* if every subgraph of G contains a vertex of degree at most d.

(a) Prove that graphs of treewidth k are k-degenerate.
(b) For $K_{m,n}$ (complete bipartite graph), show that $tw(K_{m,n}) = \min\{m, n\}$.

Problem 8. Recall the problems Feedback Vertex Set (FVS) and Vertex Cover (VC). Let G be an n-vertex graph given together with its tree decomposition of width at most k. Show that FVS and VC can be solved in G in time $k^{\mathcal{O}(k)} \cdot n$ and $2^k \cdot k^{\mathcal{O}(1)} \cdot n$, respectively.

Good Luck and Have Fun!