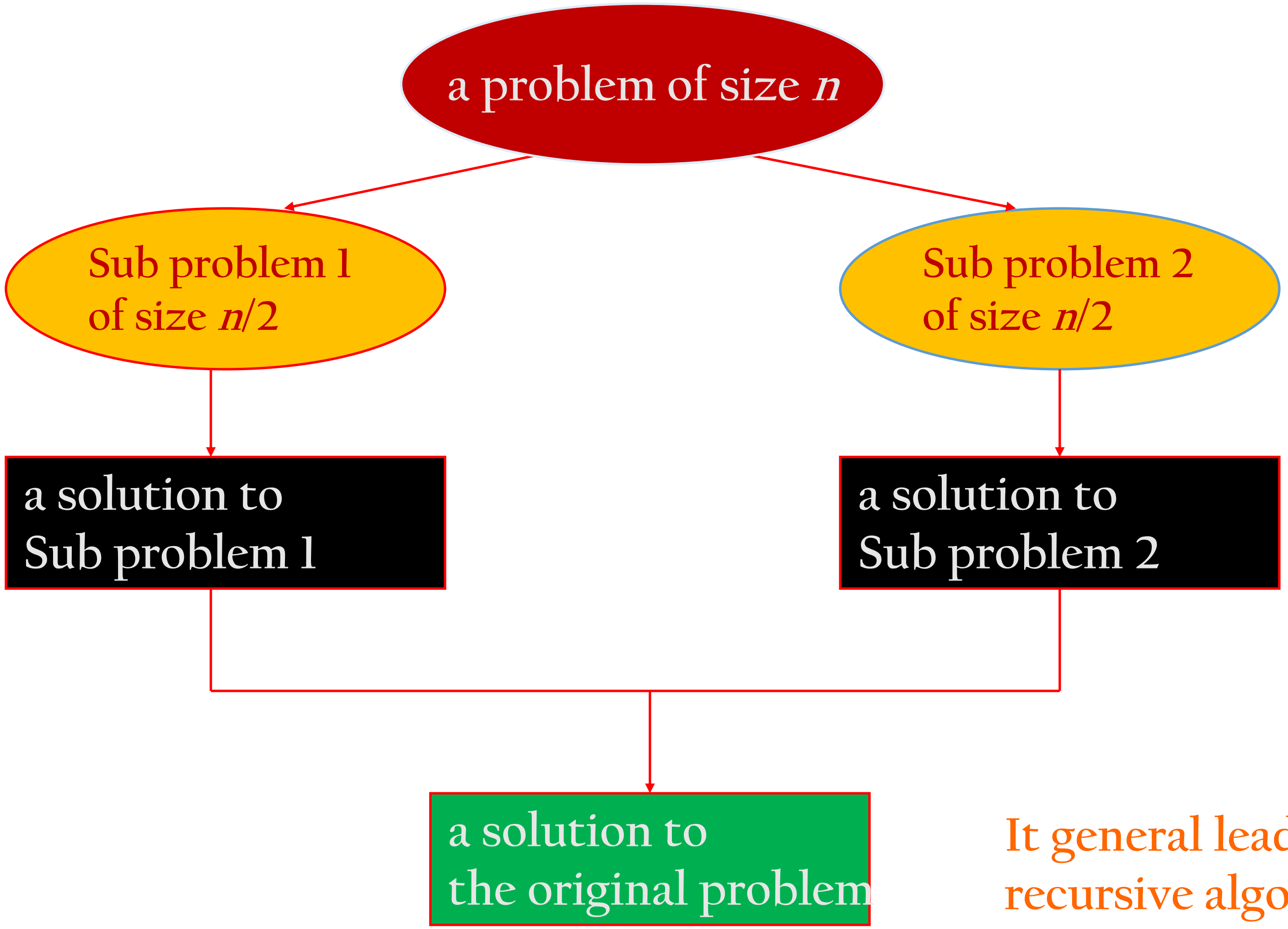
Chapter - 2
Divide and Conquer Algorithms

Essence of Divide and Conquer

- In divide and conquer approach, the problem in hand, is divided into smaller sub-problems and then each problem is solved independently.
- When we keep on dividing the sub problems into even smaller sub-problems, we may eventually reach a stage where no more division is possible.
- Those "atomic" smallest possible sub-problem (fractions) are solved. The solution of all sub-problems is finally merged in order to obtain the solution of an original problem.

Divide-and-Conquer Technique



It general leads to a recursive algorithm!

divide-and-conquer steps

- Broadly, we can understand *divide-and-conquer* approach in a three-step process.

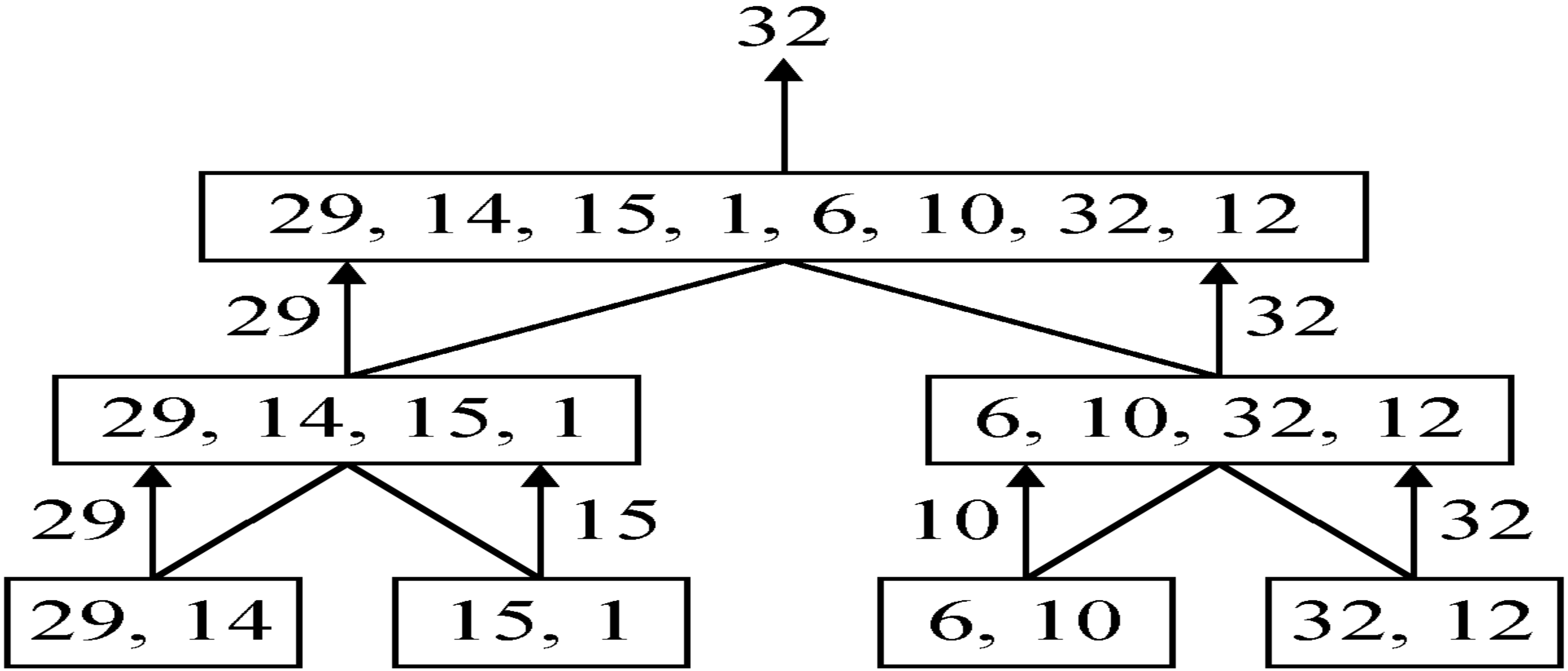
1) *Divide:* a problem to be solved is broken into a number of sub problems. Normally, the sub problems are similar to the original

2) *Conquer:* the sub problems are then solved independently, usually recursively.
3) *Combine:* finally, the solutions to the sub problems are combined to provide the answer to the original problem. The solutions to get a solution to the sub problems And finally a solution to the original problem.

- *Divide and Conquer* algorithms are normally recursive.

A simple example

- finding the maximum of a set S of n numbers



A general divide-and-conquer algorithm

Step 1: If the problem size is small, solve this problem directly; otherwise, *split* the original problem into 2 sub-problems with equal sizes.

Step 2: Recursively solve these 2 sub-problems by applying this algorithm.

Step 3: Merge the solutions of the 2 sub-problems into a solution of the original problem.

```
DAC(problem P)
{
        if Small(P) return S(P);
        else {

          divide P into smaller instances P1, P2, …, Pk, k≥1;

          Apply DAC to each of these subproblems;

          return Combine(DAC(P1), DAC(P2),…,DAC (Pk));
        }
}
```

Application of **divide-and-conquer**

- The following computer algorithms are based on **divide-and-conquer** programming approach −

  1. Finding Max and Min
  2. binary Search
  3. Merge Sort
  4. Quick Sort
  5. Strassen's Matrix Multiplication

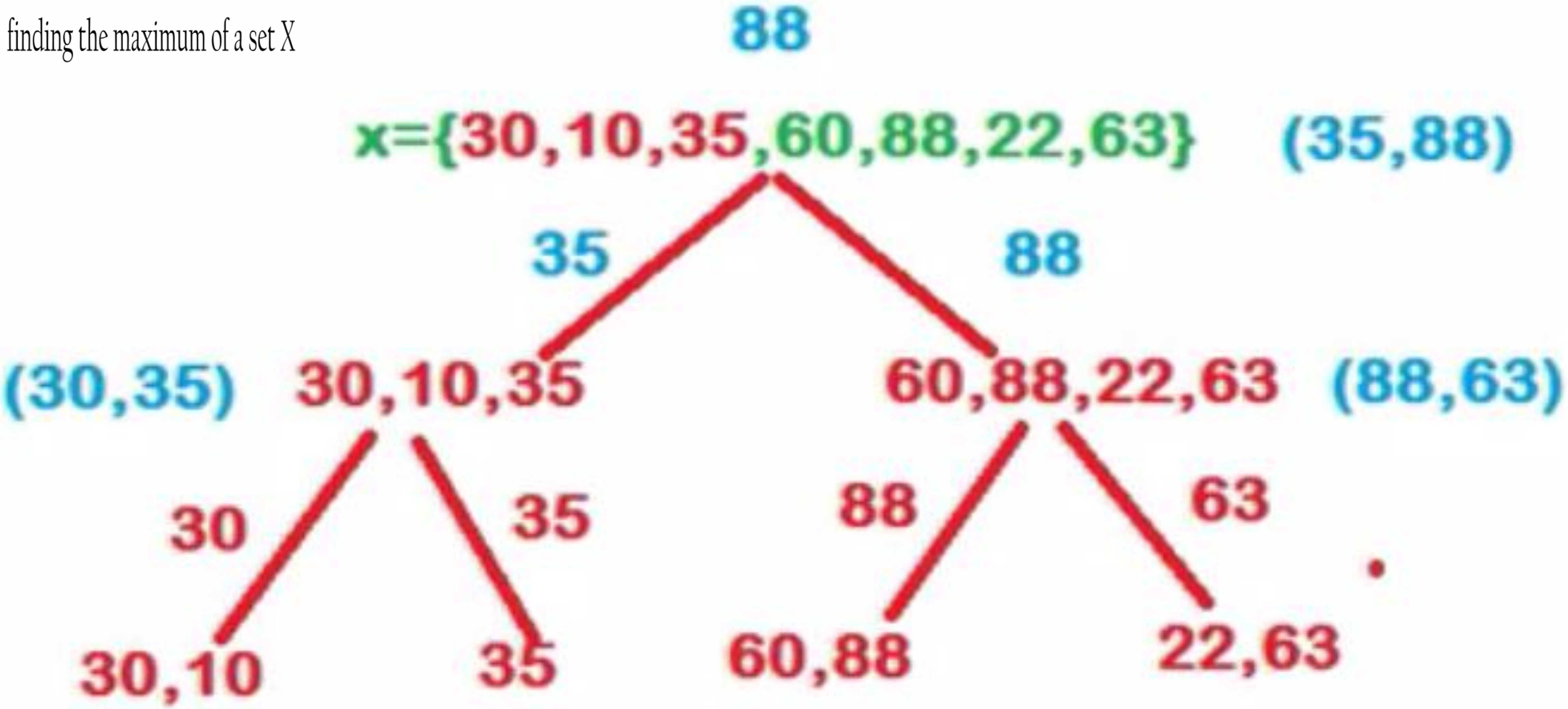
Note: There are various ways available to solve any computer problem, but the mentioned are a good example of divide and conquer approach.

Cont….

- *sorting*: ordering a list of values
- *searching*: finding the position of a value within a list
- Algorithm analysis should begin with a clear statement of the task to be performed.
- This allows us both to check that the algorithm is correct and to ensure that the algorithms we are comparing perform the same task.
- Although there are many ways that algorithms can be compared, we will focus on two that are of primary importance to many data processing algorithms:
- *time complexity*: how the number of steps required depends on the size of the input
- *space complexity*: how the amount of extra memory or storage required depends on the size of the input

1. Finding Max and Min

finding the maximum of a set X



Algorithm for maximum and minimum using divide-and-conquer

```
MaxMin(i, j, max, min)
// a[1:n] is a global array. Parameters i and j are integers,   // 1≤i≤j≤n. The effect is to set max and min to the largest and  // smallest values in a[i:j].
{
    if (i=j) then max := min := a[i]; //Small(P)
    else if (i=j-1) then // Another case of Small(P)
        {
            if (a[i] < a[j]) then max := a[j]; min := a[i];
            else max := a[i]; min := a[j];
        }
    else
    {
        // if P is not small, divide P into sub-problems.
        // Find where to split the set.
        mid := ( i + j )/2;
        // Solve the sub-problems.
        MaxMin( i, mid, max, min );
        MaxMin( mid+1, j, max1, min1 );
        // Combine the solutions.
        if (max < max1) then max := max1;
        if (min > min1) then min := min1;
    }}
```

Complexity:

Now what is the number of element comparisons needed for MaxMin? If T(n) represents this number, then the resulting recurrence relation is

$$T(n) = \begin{cases} 0 & n=1 \\ 1 & n=2 \\ T(n/2) + T(n/2) + 2 & n>2 \end{cases}$$

Note that 3n/2 – 2 is the best, average, worst case number of comparison when n is a power of two.

2. *binary search*

- divide sequence into two halves by comparing search key to midpoint
- recursively search in one of the two halves
- combine step is empty

Algorithm binary-search

Input: A sorted sequence of n elements stored in an array.

Output: The position of x (to be searched).

Step 1: If only one element remains in the array, solve it directly.

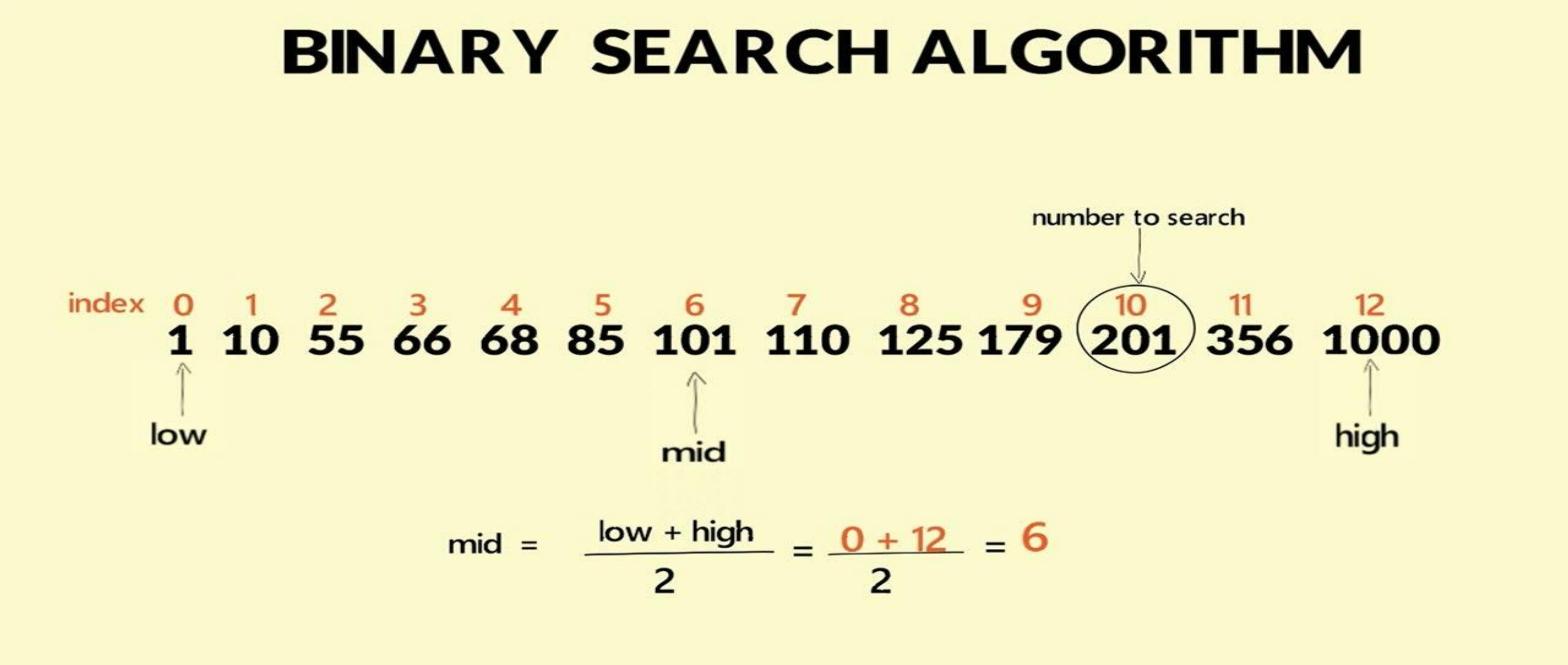
Step 2: Compare x with the middle element of the array.

Step 2.1: If x = middle element, then output it and stop.

Step 2.2: If x < middle element, then recursively solve the problem with x and the left half array.

Step 2.3: If x > middle element, then recursively solve the problem with x and the right half array.

Example of BinSearch



Algorithm BinSearch(a, low, high, x)

```
// a[]: sorted sequence in nondecreasing order
// low, high: the bounds for searching in a []
// x: the element to be searched
// If x = a[j], for some j, then return j else return –1
  if (low > high) then return –1          // invalid range
  if (low = high) then                    // if small P
    if (x == a[i]) then return i
    else return -1
  else            // divide P into two smaller subproblems
    mid = (low + high) / 2
    if (x == a[mid]) then return mid
    else if (x < a[mid]) then
        return BinSearch(a, low, mid-1, x)
    else return BinSearch(a, mid+1, high, x)
```

Binary Search

Precondition: S is a sorted list

```
index binsearch(number n, index low, index high,
          const keytype S[], keytype x)
      if low ≤ high then
          mid = (low + high) / 2
          if x = S[mid] then
              return mid
          elsif x < s[mid] then
              return binsearch(n, low, mid-1, S, x)
          else
              return binsearch(n, mid+1, high, S, x)
      else
          return 0
   end binsearch
```

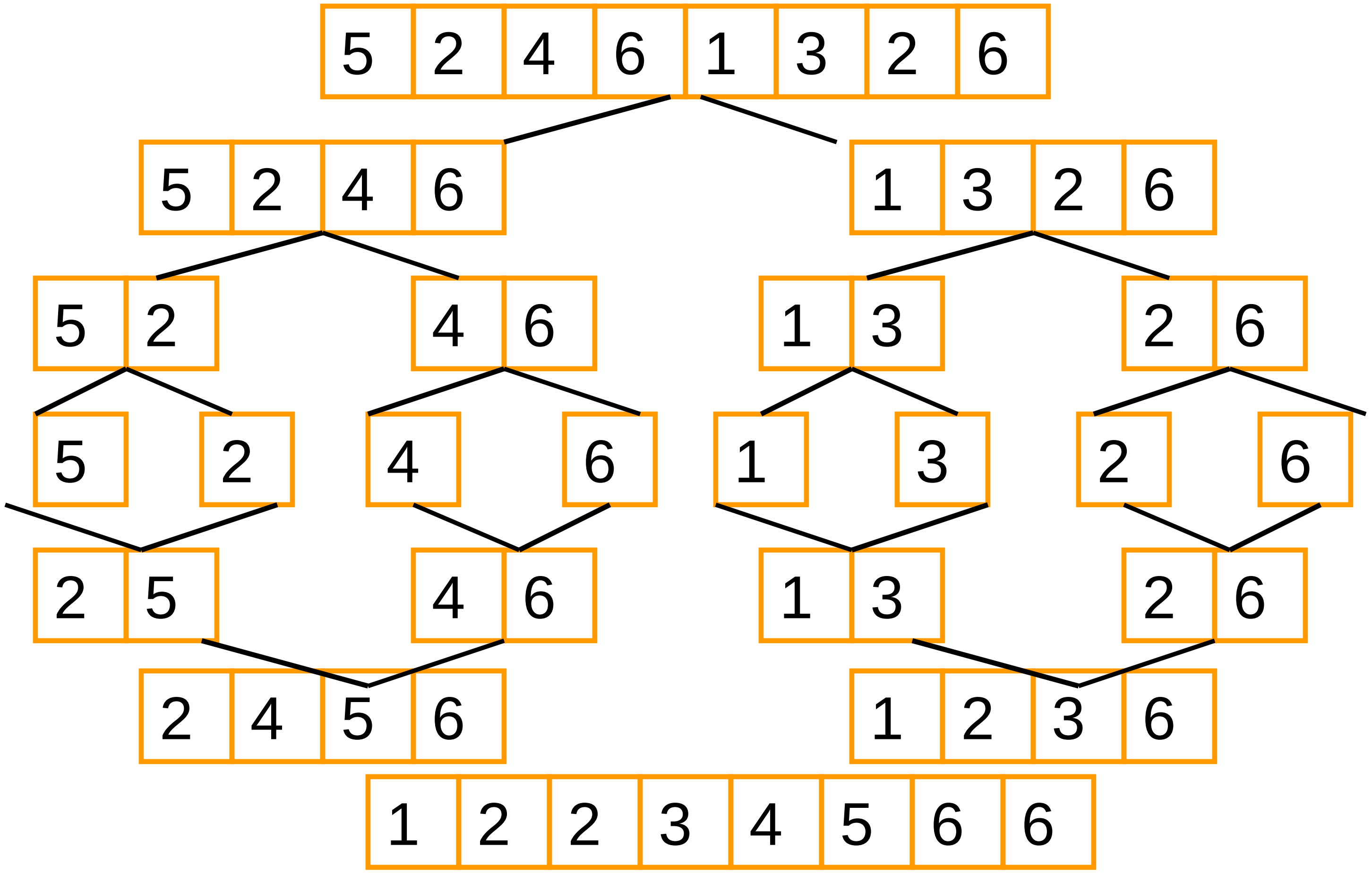
Analysis of Binary Search

- **Time efficiency T(N) = T(N/2) + 1**
  - Worst case: *O(log(n))*
  - Best case: O(1)
- **Optimal for searching a sorted array**
- **Limitations: must be a sorted array (not linked list)**

3. Merge sort

- DIVIDE the input sequence in half
- RECURSIVELY sort the two halves. Basis of the recursion is sequence with 1 key
- COMBINE the two sorted subsequences by merging them

Merge sort Example



Merge sort

- Split array A[0..n-1] in two about equal halves and make copies of each half in arrays B and C
- Sort arrays B and C recursively
- Merge sorted arrays B and C into array A as follows:
  - Repeat the following until no elements remain in one of the arrays:
    - compare the first elements in the remaining unprocessed portions of the arrays
    - copy the smaller of the two into A, while incrementing the index indicating the unprocessed portion of that array
  - Once all elements in one of the arrays are processed, copy the remaining unprocessed elements from the other array into A.

Merge Sort algorithm

- The Merge Sort function keeps on splitting an array into two halves until a condition is met where we try to perform Merge Sort on a subarray of size 1, i.e., **p == r**.
- And then, it combines the individually sorted subarrays into larger arrays until the whole array is merged.

1.ALGORITHM-MERGE SORT

2.1. If p**<r**

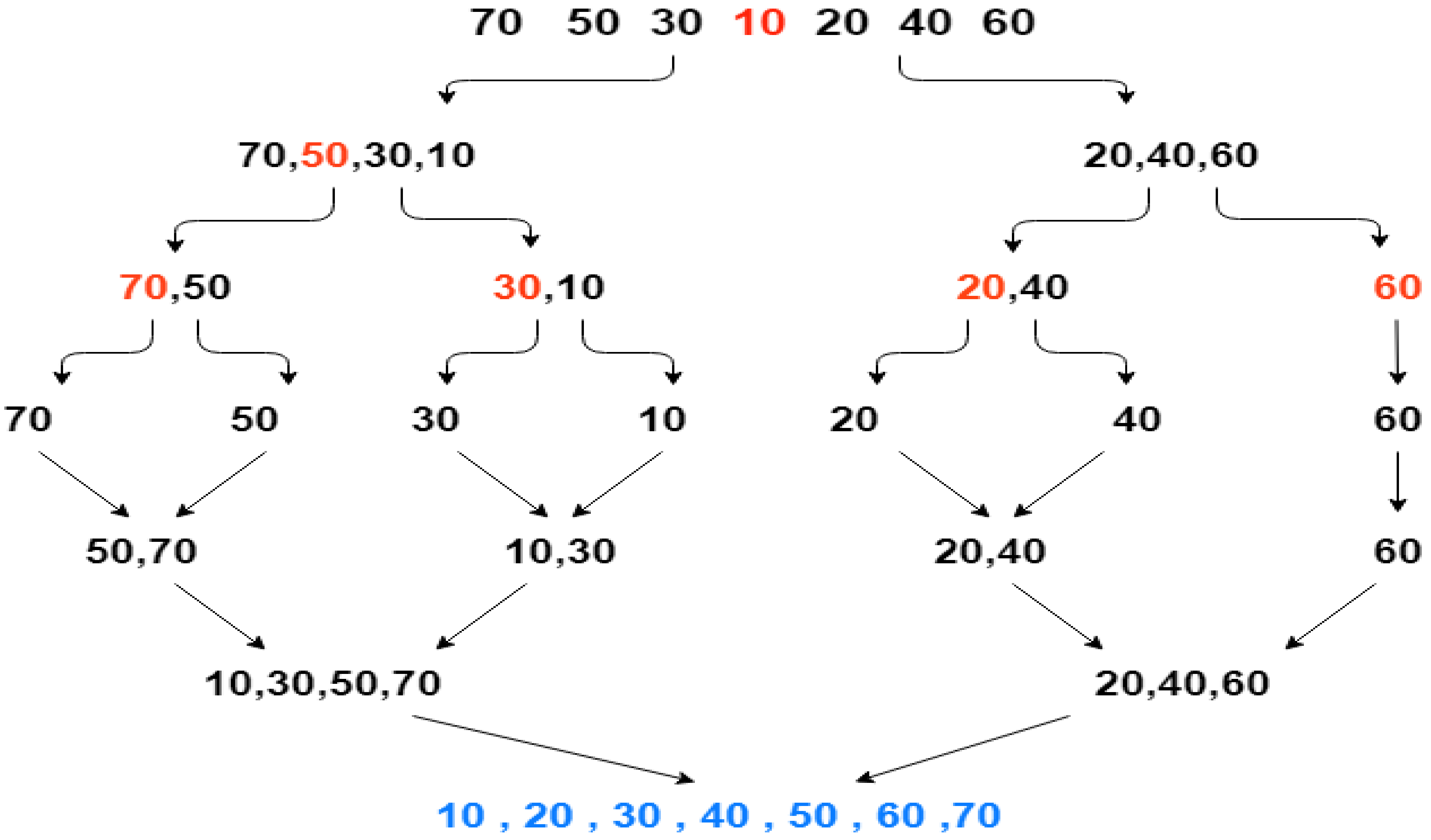
3.2. Then q → (p+ r)/2

4.3. MERGE-SORT (A, p, q)

5.4. MERGE-SORT (A, q+1,r)

6.5. MERGE (A, p, q, r)

- Here we called **MergeSort(A, 0, length(A)-1)** to sort the complete array.
- As you can see in the image given below, the merge sort algorithm recursively divides the array into halves until the base condition is met, where we are left with only 1 element in the array. And then, the merge function picks up the sorted sub-arrays and merge them back to sort the entire array.
- The following figure illustrates the dividing (splitting) procedure.


70 50 30 10 20 40 60
70,50,30,10
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10,30,50,70
20,40,60
10 , 20 , 30 , 40 , 50 , 60 ,70

Algorithm Merge-Sort

Input: A set S of n elements.

Output: The sorted sequence of the inputs in non decreasing order.

Step 1: If $|S| \le 2$, solve it directly.

Step 2: Recursively apply this algorithm to solve the left half part and right half part of S, and the results are stored in S_1 and S_2, respectively.

Step 3: Perform the two-way merge scheme on S_1 and S_2.

Method of Solving Recurrences

1. Substitution method
2. Iteration method
3. Master method

3. Master Theorem (Simplified)

Theorem: (Simplified Master Theorem) Let $a \geq 1$, $b > 1$ be constants and let $T(n)$ be the recurrence

$$T(n) = aT(n/b) + n^k,$$

defined for $n \geq 0$. (As usual let us assume that n is a power of b. The basis case, $T(1)$ can be any constant value.) Then

Case 1: if $a > b^k$ then $T(n) \in \Theta(n^{\log_b a})$.

Case 2: if $a = b^k$ then $T(n) \in \Theta(n^k \log n)$.

Case 3: if $a < b^k$ then $T(n) \in \Theta(n^k)$.

master method:

- a is the number of subproblems that are solved recursively; i.e. the number of recursive calls.
- b is the size of each subproblem relative to n; n/b is the size of the input to the recursive call.
- f(n) is the cost of dividing and recombining the subproblems.

Master Theorem

- Let T(n) be a monotonically increasing function that satisfies

$$T(n) = a\,T(n/b) + f(n)$$

$$T(1) = c$$

where $a \geq 1$, $b \geq 2$, $c>0$. If $f(n)$ is $\Theta(n^d)$ where $d \geq 0$ then

$$T(n) = \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log n) & \text{If } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

Master Theorem: Pitfalls

- You cannot use the Master Theorem if
  - T(n) is not monotone, e.g. T(n) = sin(x)
  - f(n) is not a polynomial, e.g., $T(n)=2T(n/2)+2^n$
  - b cannot be expressed as a constant, e.g. $T(n) = T(\sqrt{n})$
- The three cases of the master theorem do not cover all possibilities.
- The master theorem provides a general method for solving recurrences

Master Theorem: Example 1

- Let $T(n) = T(n/2) + ½\, n^2 + n$. What are the parameters?

  $a =$ 1

  $b =$ 2

  $d =$ 2

  Therefore, which condition applies?

  $1 < 2^2$, case 1 applies

- We conclude that

$$T(n) \in \Theta(n^d) = \Theta\ (n^2)$$

Master Theorem: Example 2

- Let $T(n) = 2\,T(n/4) + \sqrt{n} + 42$. What are the parameters?

  a = 2

  b = 4

  d = 1/2

  Therefore, which condition applies?

  $2 = 4^{1/2}$, case 2 applies

- We conclude that

$$T(n) \in \Theta(n^d \log n) = \Theta(\log n \sqrt{n})$$

Master Theorem: Example 3

- Let $T(n)= 3\,T(n/2) + 3/4n + 1$. What are the parameters?

  a = 3

  b = 2

  d = 1

  Therefore, which condition applies?

  $3 > 2^1$, case 3 applies

- We conclude that

$$T(n) \in \Theta(n^{\log_b a}) = \Theta(n^{\log_2 3})$$

- Note that $\log_2 3 \approx 1.584\ldots$, can we say that $T(n) \in \Theta(n^{1.584})$

  No, because $\log_2 3 \approx 1.5849\ldots$ and $n^{1.584} \notin \Theta(n^{1.5849})$

Recurrence Relation for Merge sort

- Let $T(n)$ be worst case time on a sequence of n keys
- If $n = 1$, then $T(n) = \Theta(1)$ (constant)
- If $n > 1$, then $T(n) = 2\,T(n/2) + \Theta(n)$
  - two sub problems of size $n/2$ each that are solved recursively
  - $\Theta(n)$ time to do the merge

Merge sort

- Sort into nondecreasing order

```
        [25] [57] [48] [37] [12] [92] [86] [33]
pass 1
        [25   57] [37   48] [12   92] [33   86]
pass 2
        [25   37   48   57] [12   33   86   92]
pass 3
        [12   25   33   37   48   57   86   92]
```

- $\log_2 n$ passes are required.
- time complexity: O(nlogn)

Time complexity

- Time complexity:

  T(n): # of comparisons

$$T(n)=\begin{cases} 2T(n/2)+1 & , n>2 \\ 1 & , n\leq 2 \end{cases}$$

- Calculation of T(n):

  Assume $n = 2^k$,

$$\begin{aligned} T(n) &= 2T(n/2)+1 \\ &= 2(2T(n/4)+1)+1 \\ &= 4T(n/4)+2+1 \\ &\quad\vdots \\ &=2^{k-1}T(2)+2^{k-2}+\ldots+4+2+1 \\ &=2^{k-1}+2^{k-2}+\ldots+4+2+1 \\ &=2^k-1 = n-1 \end{aligned}$$

Time complexity of the general algorithm

- Time complexity:

$$T(n)=\begin{cases} 2T(n/2)+S(n)+M(n) & , n \geq c \\ b & , n < c \end{cases}$$

where S(n) : time for splitting

M(n) : time for merging

b : a constant

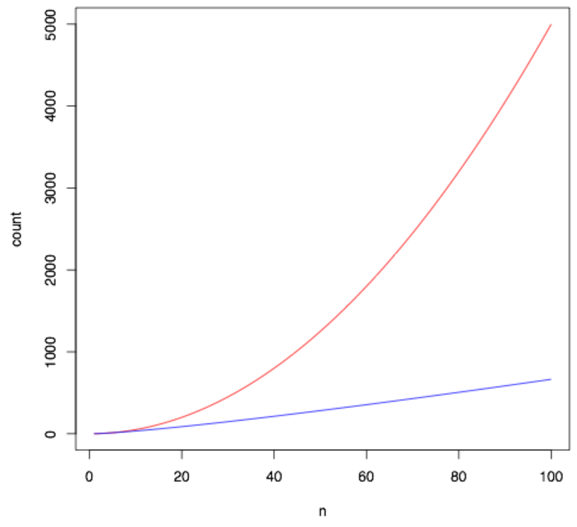
c : a constant

Best case: split in the middle — $\Theta(n \log n)$
Worst case: sorted array! — $\Theta(n^2)$
Average case: random arrays — $\Theta(n \log n)$

Note: Considered the method of choice for internal sorting of large files ($n \geq 10000$)

Recap: Divide and Conquer Algorithms

- The divide and conquer strategy often reduces the number of iterations of the main loop from *n* to $\log_2 n$
  - binary search: $\mathcal{O}(\log_2 n)$
  - merge sort: $\mathcal{O}(n \times \log_2 n)$
  - Quick Sort: $\mathcal{O}(n \times \log_2 n)$
- It may not look like much, but the reduction in the number of iterations is significant for larger problems




Quick
Sort
Merge
Sort

5. Strassen's Matrix Multiplication

- First we will discuss the general method of matrix multiplication and later we will discuss Strassen's matrix multiplication algorithm

Problem Statement

- Let us consider two matrices A and B. We want to calculate the resultant matrix C by multiplying A and B.

Method

- First, we will discuss naïve method and its complexity. Here, we are calculating $\boldsymbol{C = AXB}$. Using Naïve method, two matrices ($\boldsymbol{A}$ and $\boldsymbol{B}$) can be multiplied if the order of these matrices are $\boldsymbol{p} \times \boldsymbol{q}$ and $\boldsymbol{q} \times \boldsymbol{r}$. Following is the algorithm

Algorithm: Matrix-Multiplication (X, Y, Z)

```
for i = 1 to p do
   for j = 1 to r do
      Z[i,j] := 0
      for k = 1 to q do
         Z[i,j] := Z[i,j] + X[i,k] × Y[k,j]
```

Complexity

- Here, we assume that integer operations take ***O(1)*** time. There are three **for** loops in this algorithm and one is nested in other. Hence, the algorithm takes ***O(n^3)*** time to execute.

Strassen's Matrix Multiplication Algorithm

- In this context, using Strassen's Matrix multiplication algorithm, the time consumption can be improved a little bit.

Matrix Multiplication

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \times \underbrace{\begin{bmatrix} e & f \\ g & h \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}}_{C}$$

Multiplications: 8

Additions: 4

Addition of 2 matrices takes $O(N^2)$ time.

$$T(N) = 8T(N/2) + O(N^2)$$

Time Complexity of the above method is $O(N^3)$.

Faster Matrix Multiplication

- Clever idea due to Strassen
- Start with a recursive version of the straight forward algorithm
  - divide *A*, *B* and *C* into 4 quadrants
  - compute the 4 quadrants of *C* by doing 8 matrix multiplications on the quadrants of *A* and *B*, plus $\Theta(n^2)$ scalar operations

Strassen's Matrix Multiplication

- There is a way to get all the required information with only 7 matrix multiplications, instead of 8.
- Recurrence for new algorithm is
  - $T(\mathrm{n}) = 7T(n/2) + \Theta(n^2)$

You just need to remember 4 Rules :

- AHED (Learn it as 'Ahead')
- Diagonal
- Last CR
- First CR

Also, consider X as (Row +) and Y as (Column -) matrix

$$p1 = a(f - h) \qquad p2 = (a + b)h$$
$$p3 = (c + d)e \qquad p4 = d(g - e)$$
$$p5 = (a + d)(e + h) \qquad p6 = (b - d)(g + h)$$
$$p7 = (a - c)(e + f)$$

The A x B can be calculated using above seven multiplications.
Following are values of four sub-matrices of result C

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{X} \times \underbrace{\begin{bmatrix} e & f \\ g & h \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} p5 + p4 - p2 + p6 & p1 + p2 \\ p3 + p4 & p1 + p5 - p3 - p7 \end{bmatrix}}_{C}$$

X , Y and **C** are square metrices of size N x N
a, b, c and d are submatrices of A, of size N/2 x N/2
e, f, g and h are submatrices of B, of size N/2 x N/2
p1, p2, p3, p4, p5, p6 and p7 are submatrices of size N/2 x N/2

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \qquad Y = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

-AHED
-Diagonals
-Last CR
-First CR

Check for Row (+)

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Check for Column (-)

$$Y = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

-AHED
-Diagonals
-Last CR
-First CR

Check for Row (+)

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Check for Column (-)

$$Y = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

-AHED
-Diagonals
-Last CR
-First CR

Idea: Reduce the number of recursive calls to 7.

Formula:

p1 = a(f-h)
p2 = (a+b)h
p3 = (c+d)e
p4 = d(g-e)

p5 = (a+d)(e+h)
p6 = (b-d)(g+h)
p7 = (a-c)(e+f)

$$\underset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \times \underset{B}{\begin{bmatrix} e & f \\ g & h \end{bmatrix}} = \underset{C}{\begin{bmatrix} p5 + p4 - p2 + p6 & p1 + p2 \\ p3 + p4 & p1 + p5 - p3 - p7 \end{bmatrix}}$$

A, B and C are square metrices of size N x N
a, b, c and d are submatrices of A, of size N/2 x N/2
e, f, g and h are submatrices of B, of size N/2 x N/2
p1, p2, p3, p4, p5, p6 and p7 are submatrices of size N/2 x N/2

Complexity in Strassen matrix

$$\underset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \times \underset{B}{\begin{bmatrix} e & f \\ g & h \end{bmatrix}} = \underset{C}{\begin{bmatrix} p5 + p4 - p2 + p6 & p1 + p2 \\ p3 + p4 & p1 + p5 - p3 - p7 \end{bmatrix}}$$

Multiplications: 7

Addition & Subtraction of 2 matrices takes **$O(N^2)$** time.

$$T(N) = 7T(N/2) + O(N^2)$$

Time Complexity of the above method is **$O(N^{\log 7})$**

example

$$\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 & 4 \\ 3 & 5 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 & 4 \\ 3 & 5 \end{bmatrix}$$

$$\begin{array}{c} \text{1st Row} \\ \text{Times} \\ \text{1st Column} \\ \\ \\ \\ \text{2nd Row} \\ \text{Times} \\ \text{1st Column} \end{array} \left[\begin{array}{c|c} (1)(-1)+(0)(3) & (1)(4)+(0)(5) \\ -1+0 & 4+0 \\ -1 & 4 \\ \hline (-3)(-1)+(2)(3) & (-3)(4)+(2)(5) \\ 3+6 & -12+10 \\ 9 & -2 \end{array}\right] \begin{array}{c} \text{1st Row} \\ \text{Times} \\ \text{2nd Column} \\ \\ \\ \\ \text{2nd Row} \\ \text{Times} \\ \text{2nd Column} \end{array}$$

$$\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 & 4 \\ 3 & 5 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 & 4 \\ 3 & 5 \end{bmatrix}$$

Final Answer: $\begin{bmatrix} -1 & 4 \\ 9 & -2 \end{bmatrix}$

General Divide-and-Conquer Recurrence

$T(n) = aT(n/b) + f(n)$ where $f(n) \in \Theta(n^d)$, $d \geq 0$

Master Theorem:
If $a < b^d$, $T(n) \in \Theta(n^d)$
If $a = b^d$, $T(n) \in \Theta(n^d \log n)$
If $a > b^d$, $T(n) \in \Theta(n^{\log_b a})$

Note: The same results hold with O instead of Θ.

Examples: $T(n) = 4T(n/2) + n \Rightarrow T(n) \in ?$ $\Theta(n^2)$

$T(n) = 4T(n/2) + n^2 \Rightarrow T(n) \in ?$ $\Theta(n^2 \log n)$

$T(n) = 4T(n/2) + n^3 \Rightarrow T(n) \in ?$ $\Theta(n^3)$

$T(n) = 7T(n/2) + n^2 \Rightarrow T(n) \in ?$ $\Theta(n^{\log_2 7}) = \Theta(n^{2.1})$

(Strassen's Algorithm for Matrix Multiplication)


Learning is key to success