IE579 Game Theory and Multi-Agent Reinforcement Learning

Homework 1

Due: 2019. 10. 4

Submit the pdf file in the KLMS

(15pts) Problem 1 (Sharing game)

Two people are trying to share a 100$. They each simultaneously announce a real number from 0 to 100$. If the sum of the numbers is less than or equal to 100$, each person receives the number he announced. If the sum is greater than 100$, then the player who announced the smaller number, say x, receives x while the other person receives $100 - x$. If the sum is greater than 100 and both numbers are the same, then each receives 50$.

(1) Formulate this situation a strategic game (i.e., define players, actions, payoff functions)

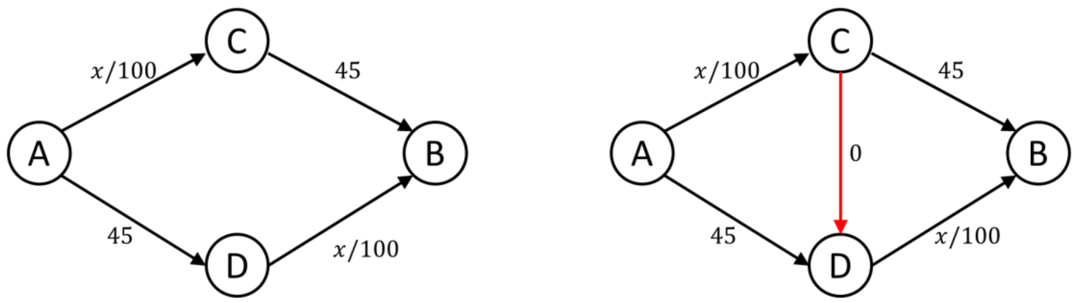
(2) Show that (50, 50) is a Nash equilibrium

(3) Can you find more Nash equilibria? Prove your answer

(15pts) Problem 2 (Nash equilibrium route for reaching a destination)

Consider the following traffic network. You need to reach destination B starting from A. The number on the edge represents the travel time required to move between two nodes. For example, the travel time from node C to B is 45 (minutes).

- The travel time for the routes A-C and D-B depends on the number of car x using that route.
- The travel time for the routes A-D and C-B are fixed as 45.



(a) Current (a) New road

Now, assume that 4000 cars want to move from A to B as part of the morning commute. The two possible routes that each car can choose are: the upper route through C and the lower route through D.

(1) Formulate this problem as a game by 4000 players

(2) What is a Nash equilibrium? Describe the conditions where each car is not willing to deviate from. (express this condition using x, the number of cars moving upper route A-C-B)

(3) What will happen if the city constructs a new really fast highway between C and D. Assume the travel time between C and D is 0. How will this new road affect the commuting time?

(15Pts) Problem 3 (Normal form game with three-players)

Consider the 3-player normal form game bellows:

	L	R
T	$(5,5,5)$	$(2,6,2)$
B	$(6,2,2)$	$(3,3,-1)$
N		

(when player 3 choose to play N)

	L	R
T	$(2,2,6)$	$(-1,3,3)$
B	$(3,-1,3)$	$(0,0,0)$
F		

(when player 3 choose to play F)

Here, each player has two strategies:

- (T, B) for player 1,
- (L, R) for player 2
- (N, F) for player 3.

In this notation, player 3 gets to select left or right table, player 2 selects the column, and player 1 selects the row. For example, if they play $(\mathrm{T}, \mathrm{L}, \mathrm{F})$ they each get (2,2,6) respectively.

(1) List all of the pure strategy Nash equilibrium profiles of this game.

(2) Compute the maxmin value (or security level) for player 1.

(3) Suppose it is common knowledge that player 3 will play N with probability 0.8 and F with probability 0.2. This induces a game just between player 1 and player 2. Show the Normal Form of this induced game (include payoffs for all three players).

(4) What are the pure strategy Nash Equilibria in the induced game?

(15pts) Problem 4 (Extension of Cournot Duopoly game to multiple firms)

In Cournot Duopoly game, assume there are n firms.

- Each firm can choose the production quantity $q_i \geq 0$ for $i = 1, \dots n$
- The market price decrease as the firms' total production $Q = \sum_{i=1}^{n} q_i$ increases as:

$$P(Q) = \begin{cases} a - Q & \text{if } Q \leq a \\ 0 & \text{if } Q > a \end{cases}$$

- The cost function of each firm i is $C_i(q_i) = cq_i$ with $c < a$
- The firm i's profit, equal to its revenue minus its cost, is

$$u_i(q_1, \dots, q_n) = q_i P(Q) - C_i(q_i)$$

(1) Find the best response function of each firm and set up the conditions for $(q_1^*, \dots, q_n^*)$ to be a Nash equilibrium

(2) How the price $P(Q)$ changes as the number of firms n increases

(20pts) Problem 5 (Nash and Correlated equilibria for a two player game)

Consider the following two player game:

	L	R
T	$6, 6$	$2, 8$
B	$8, 2$	$0, 0$

(1) Find all pure strategy Nash equilibria and the payoff vector for each one

(2) Find all mixed strategy Nash equilibria and the payoff vector for each one

(3) Identify all correlated equilibria by specifying the set of inequalities that they need to satisfy

(4) Among all the correlated equilibria, provide one with the largest sum of payoffs

(20pts) Problem 6 (Nash equilibria for a two player zero-sum game)

In a two player zero-sum game defined by an $(m \times n)$-dimensional matrix $A = \{a_{ij}\}$, assume that Player 1 wants to minimize the outcome and Player 2 wants to maximize the outcome.

If Player 1 adopts the i^*th row as his strategy, where i^* satisfies the inequalities

$$\bar{V}(A) \triangleq \max_j a_{i^*j} \leq \max_j a_{ij}, \quad i = 1,2,..m,$$

then his losses will be no greater than $\bar{V}$, which is the security level for his losses. The strategy "row i^*" that yields this security level will be called a security strategy for Player 1.

On the other hand, if Player 2 adopts the j^*th column as his strategy, where j^* satisfies the inequalities

$$\underline{V}(A) \triangleq \min_i a_{ij^*} \geq \min_i a_{ij}, \quad j = 1,2,..n,$$

then his gains will be no smaller than $\underline{V}$, which is the security level for his gains. The strategy "column j^*" that yields this security level will be called a security strategy for Player 2.

Now, prove or disprove the following statements:

(1) If $\bar{V}(A) = \underline{V}(A)$, an ordered pair of strategies provides a Nash equilibrium for A if, and only if, the first of these is a security strategy for Player 1 and the second one is a security strategy for Player 2.

(2) If {row i_1, column j_1} and {row i_2, column j_2} are two Nash equilibrium strategy pairs, then {row i_1, column j_2}, {row i_2, column j_1} are also Nash equilibrium strategy pairs.

(Bonus problem 20pts) Problem 7 (Algorithm of Nash equilibria for a two player game)

Consider the two player normal-form game.

(1) Prove or disprove that it is always possible to find a Nash equilibrium using **Algorithm 1** if there exist a pure Nash equilibrium.

Algorithm1

Initialize action pair $\left(a_1^{(0)}, a_2^{(0)}\right)$

(i) $a_1^{(1)} = \mathrm{BR}_1\left(a_2^{(0)}\right)$ where BR_1 is the best response function of player 1.

(ii) $a_2^{(1)} = \mathrm{BR}_2\left(a_2^{(1)}\right)$ where BR_2 is the best response function of player 2.

(iii) $\left(a_1^{(0)}, a_2^{(0)}\right) \leftarrow \left(a_1^{(1)}, a_2^{(1)}\right)$

Repeat (i) - (iii) until $\left(a_1^{(0)}, a_2^{(0)}\right) = \left(a_1^{(1)}, a_2^{(1)}\right)$.

Then, $\left(a_1^{(0)}, a_2^{(0)}\right)$ is the Nash equilibrium.

---

(2) Assume that there exist a pure Nash equilibrium for the given two player normal-form game. Suggest the algorithm to find a Nash equilibrium of two player normal form game. Prove that how it works and analyze the efficiency of the algorithm.