Research Article

Open Access

International Annals of Science
Vol. 1 Issue 1, pp: 15-24, October 2016, Available at: journals.aijr.in

International Annals of Science
Multidisciplinary Science Journal

Existence and Stability of the Libration Points in the Circular Restricted Three Body Problem with Variable Masses

Abdullah*[1], Mehtab Alam[2]

[1] Department of Mathematics, College of Science, Al-Zulfi, Majmaah University, Kingdom of Saudi Arabia.
[2] HPS, Talimabad Campus, Sangam Vihar, New Delhi, India

Received: 09 October 2016; Revised: 19 October 2016; Accepted: 20 October 2016; Published: 21 October 2016

ABSTRACT

We have investigated the existence and stability of the libration points in the circular restricted three body problem with the variation of all the masses (primaries and infinitesimal body) with time. We have used the Meshcherskii transformation for finding the autonomized equations of motion and found at most nine libration points. We have drawn the zero velocity curves and Poincare surface of sections for the different values of parameter k. Finally, we have checked the stability and found that all the libration points are unstable.

Keywords: Variable mass, Autonomized system, Zero velocity curves, Libration points, Poincare surface of sections, Stability

1. Introduction

In the present age, the restricted problem is the common model in the celestial mechanics with different perturbations as different shapes of the primaries, radiation factor, Pointing-Robretson Drag, Variable masses, resonance, relativistic effects etc. Many mathematicians and astronomers have studied these models and investigated about their stability, chaos, fractals, periodic orbits, non-linear oscillations etc. Jeans [1] studied the two body problem with variable mass. Mescherskii [2-3] investigated on the mechanics of bodies with variable mass. Szebehely [4] given the theory about the stability and periodic orbits around the libration points in his book "Theory of Orbits". Bhatnagar [5] studied the periodic orbits of collision in the restricted problem of three bodies in a three-dimensional coordinate system. Simmons [6] has studied about the stability of the restricted three body problem with radiation pressure. Singh [7-10] has studied about the stability of the restricted three body problem with the perturbations as variable mass, radiation pressure and Coriolis and centrifugal forces. Zhang [11] studied about the photo gravitational restricted three body problem with variable mass. Abouelmagd [12, 13] studied the effect of oblateness in the perturbed restricted three body problem and also with variable mass. Shalini [14] investigated the Existence and Stability of the libration point L_4 in the R3BP, when the smaller primary is a heterogeneous axis symmetric body with N layers. Abdullah [15] investigated the stability of the Lagrangian solutions in the photo gravitational circular restricted four-body

* Corresponding author:
email: a.ansari@mu.edu.sa
DOI : http://dx.doi.org/10.21467/ias.1.1.15-24

problem with the effect of oblateness and variable mass. Mittal [16] investigated the stability of the Lagrangian solutions for the restricted four-body problem with variable mass. Taking inspiration from all these models, we have decided to study about the existence and stability of the libration points in the circular restricted three body problem in which the masses of the primaries as well as the mass of the infinitesimal body vary with time. We have studied our problem in various sections. In the first section, we have introduced the problem. In the second section, we have evaluated the equations of motion of the infinitesimal variable mass in the cartesian form and in the autonomized form. In the third section, we have shown the libration points (collinear, triangular and coplanar points), zero velocity curves and Poincare surface of sections for the different values of the parameters. In the fourth section, we have shown the stability of the libration points. And finally in the fifth section, we have concluded the problem. Our problem has many application in this space age particularly in the field of Astrodynamics, Astronomy and Astrophysics.

2. Equations of Motion

Let the three masses m1, m2 and m varies with time be the masses of the primaries and infinitesimal body respectively. The primaries are revolving in the circular orbits around their center of mass which is considered as origin. The line joining these primaries is taken as x-axis and the perpendicular line of x-axis and passing through the origin is taken as y-axis. The line through the origin and perpendicular to the plane of motion of the primaries is taken as z-axis. Let us consider the synodic coordinate system, initially coincident with the inertial coordinate system, with angular velocity w about z-axis. Using the procedure of Abdullah [15], we can write the equations of motion of the infinitesimal variable mass m in the circular restricted three body problem when the variation of mass is non-isotropic and originates from one point as

$$\frac{\dot{m}}{m}(\dot{x}-\omega y)+(\ddot{x}-2\omega\dot{y})=\Omega_x,$$

$$\frac{\dot{m}}{m}(\dot{y}+\omega x)+(\ddot{y}+2\omega\dot{x})=\Omega_y,$$

$$\frac{\dot{m}}{m}\dot{z}+\ddot{z}=\Omega_z, \qquad (1)$$

where, $\Omega=\frac{1}{2}\omega^2(x^2+y^2+z^2)+\frac{\mu_1}{r_1}+\frac{\mu_2}{r_2}$,

$r_i^2=(x-x_i)^2+y^2+z^2, (i=1,2)$, are the distance from the primaries to the infinitesimal body.

$\mu_i=\frac{m_i}{m_1+m_2}, (i=1,2)$, are the masses of the primaries.

Using Meshcherskii [3] transformation

$$x=\xi R(t),\ y=\eta R(t),\ z=\zeta R(t),$$

$$\frac{dt}{d\tau}=R^2(t),\ r_i=\rho_i R(t),\ (i=1,2),$$

the particular solutions of the Gylden-Meshcherskii problem

$$\omega(t)=\frac{\omega_0}{R^2(t)},\ x_1=\xi_1 R(t),\ x_2=\xi_2 R(t),$$

and the unified Meshcherskii law

$$\mu(t)=\frac{\mu_0}{R(t)},\ \mu_1(t)=\frac{\mu_{10}}{R(t)},$$

$$\mu_2(t)=\frac{\mu_{20}}{R(t)},\ \mu(t)=\mu_1(t)+\mu_2(t),$$

$$m=\frac{m_0}{R(t)},\ R(t)=\sqrt{at^2+2bt+c},$$

where $a,b,c,\mu_0,\mu_{10},\mu_{20},m_0$ are constants. We transform the system (1) to the autonomous form

$$\xi''-2\omega_0\eta'-(a\,t+b)\xi'=\frac{\partial\Omega}{\partial\xi},$$

$$\eta''+2\omega_0\xi'-(a\,t+b)\eta'=\frac{\partial\Omega}{\partial\eta},$$

$$\zeta''-(a\,t+b)\zeta'=\frac{\partial\Omega}{\partial\zeta}. \qquad (2)$$

where,

$$\Omega=\frac{1}{2}((a\,t+b)^2+\omega_0^2-\Delta)(\xi^2+\eta^2+\zeta^2)$$

$$-(a\,t+b)\xi\eta-\frac{\zeta^2}{2}+\frac{\mu_{10}}{\rho_1}+\frac{\mu_{20}}{\rho_2},$$

Abdullah *et al.*, *Int. Annals Sci.;* Vol. 1 Issue 1, pp: 15-24, October 2016

$$\rho_i^2 = (\xi - \xi_i)^2 + \eta^2 + \zeta^2,\ \Delta = ac - b^2,$$

$$\xi_1 = \frac{-\mu_{20}}{\mu_0}\rho_{12}, \xi_2 = \frac{\mu_{10}}{\mu_0}\rho_{12}.$$

Dash (′) is the differentiation w.r.to τ. Taking unit of mass, distance and time at initial time t_0 such that

$$\mu_0 = G,\ \rho_{12} = 1,\ \omega_0 = 1, \text{a } t_0 + b = \alpha_1 (\text{constant})$$

So that $\Delta = 1 - k$, where k is constant of a particular integral of Gylden-Meshcherskii problem and consequently G = *k*. Introducing the mass parameter υ expressed as

$$\frac{\mu_{10}}{\mu_0} = 1 - \upsilon, \frac{\mu_{20}}{\mu_0} = \upsilon, 0 < \upsilon \le \frac{1}{2},$$

where υ is the ratio of the mass of the primaries to the total mass of the primaries. Finally, the autonomized system (2) becomes

$$\xi'' - 2\eta' - \alpha_1 \xi' = \frac{\partial \Omega}{\partial \xi},$$

$$\eta'' + 2\xi' - \alpha_1 \eta' = \frac{\partial \Omega}{\partial \eta},$$

$$\zeta'' - \alpha_1 \zeta' = \frac{\partial \Omega}{\partial \zeta}. \quad (3)$$

where,

$$\Omega = \frac{1}{2}(\alpha_1^2 + k)(\xi^2 + \eta^2 + \zeta^2)$$

$$-\alpha_1 \xi \eta - \frac{\zeta^2}{2} + \frac{k(1-\upsilon)}{\rho_1} + \frac{k\upsilon}{\rho_2},$$

$$\rho_i^2 = (\xi - \xi_i)^2 + \eta^2 + \zeta^2,$$

$$\xi_1 = -\upsilon, \xi_2 = 1 - \upsilon.$$

3. Locations of the Equilibrium Points of the Autonomized System

The equilibrium points with the variable masses are obtained from the solution of the equations

$$\Omega_\xi = 0, \Omega_\eta = 0, \Omega_\zeta = 0,$$

i.e.

$$(\alpha_1^2 + k)\xi - \alpha_1 \eta - \frac{k(1-\upsilon)(\xi+\upsilon)}{\rho_1^3} - \frac{k\upsilon(\xi+\upsilon-1)}{\rho_2^3} = 0 \quad (4)$$

$$(\alpha_1^2 + k)\eta - \alpha_1 \xi - \frac{k(1-\upsilon)\eta}{\rho_1^3} - \frac{k\upsilon\eta}{\rho_2^3} = 0, \quad (5)$$

$$(\alpha_1^2 + k)\zeta - \zeta - \frac{k(1-\upsilon)\zeta}{\rho_1^3} - \frac{k\upsilon\zeta}{\rho_2^3} = 0, \quad (6)$$

3.1. Tables for Equilibrium Points

We have given all the Lagrangian points for the values of ν = 0.019, α_1 = 0.2, 0.4, 0.9 and *k* = 0.4, 0.7, 1, 10, 100 in the Table 1- 6 and also in the Figures 1-10. From Table 1-3, we have given the values in $\xi\eta-$ plane and from Table 4-6, we have given the values in the $\xi\zeta-$plane. Correspondingly from Figure 1-5, we have shown the libration points for the different five values of k in the $\xi\eta-$plane and from Figure 6-10, we have shown the libration points for the different five values of *k* in the $\xi\zeta-$plane.

Table 1: *Values of equilibrium points at ν = 0.019, α_1 = 0.2.*

k	$((\xi, \eta)$	Number of points
0.4	(−0.019,0),(0.81989109, 0.84389109), (−0.83389109, −0.84389109), (0.98554574, 0), (1.078554574, −0.13), (−0.615389109, 0.6001389109), (0.612389109, −0.579109)	7
0.7	(−0.019,0),(0.74989109,0.7789109), (−0.76389109, −0.78389109), (0.98554574, 0), (1.12394, −0.112), (−0.66389109,0.62389109), (0.662389109, −0.599109), (0.8079985, −0.059956), (0.874574, −0.27432)	9
1	(−0.019,0), (0.71989109,0.7689109), (−0.73389109, −0.77389109), (0.98554574,0), (1.14394, −0.091), (−0.68389109,0.633389109), (0.695389109, −0.599109), (0.8079985, −0.039956), (0.867574, −0.3432)	9
10	(−0.019,0),(0.60689109,0.791689), (−0.4789109, −0.88138), (0.98554574,0), (1.175394, −0.016), (−0.8615109,0.5129), (0.8079985, −0.0091)	7
100	(−0.019,0), (0.539109,0.828689), (0.4689109, −0.87138), (0.98554574,0), (1.175394,0), (−0.9615109,0.2976), (0.8079985,0)	7

Table 2: *Values of equilibrium points at ν = 0.019, α_1 = 0.4.*

k	(ξ, η)	Number of points
0.4	(−0.019,0), (0.94989109,0.96389109), (−0.96389109, −0.96389109), (0.98554574,0), (1.048554574, −0.101), (−0.54389109,0.5189109), (0.52389109, −0.519109)	7
0.7	(−0.019,0), (0.7999109,0.816109), (−0.81589109, −0.8169109), (0.98554574,0), (1.0794, −0.112), (−0.59389,0.5789109), (0.582389109, −0.569109)	7
1	(−0.019,0).(0.7599109,0.7816109), (−0.7689109, −0.7879109), (0.98554574,0), (1.099794, −0.1112), (−0.626783,0.5989109), (0.6199109, −0.579109), (0.8189794, −0.12), (0.837794, −0.182)	9
10	(−0.019,0), (0.62689109,0.771689), (−0.599109, −0.818918), (0.98554574,0), (1.175394, −0.0219), (−0.7715109,0.6129), (0.8079985, −0.0091)	7
100	(−0.019,0), (0.539109,0.828689), (0.4689109, −0.87138), (0.98554574,0), (1.175394,0), (−0.9615109,0.2976), (0.8079985,0)	7

Table 3: *Values of equilibrium points at ν = 0.019, α_1 = 0.9.*

k	(ξ, η)	Number of points
0.4	(−0.019,0), (0.75989109,0.76389109), (−0.77389109, −0.77389109), (−0.4189109,0.40189109), (0.399109, −0.4019109)	5
0.7	(−0.019,0), (0.7299109,0.736109), (−0.73589109, −0.7469109), (0.98554574,0), (1.0594, −0.063), (−0.48389,0.4589109), (0.4589109, −0.459109)	7
1	(−0.019,0), (0.71599109,0.726109), (−0.72689109, −0.7379109), (0.98554574,0), (1.07, −0.071), (−0.524783,0.4989109), (0.500991, −0.5009109)	7
10	(−0.019,0), (0.65689109,0.74689), (−0.659109, −0.75918), (0.98554574,0), (1.156394, −0.0419), (−0.735109,0.6129), (0.8079985, −0.0091)	7
100	(−0.019,0), (0.539109,0.828689), (0.4689109, −0.87138), (0.98554574,0), (1.175394,0), (−0.9615109,0.2976), (0.8079985,0)	7

Table 4: *Values of equilibrium points at ν = 0.019, α_1 = 0.2.*

k	(ξ, ζ)	Number of points
0.4	(−0.019,0), (0.8049109,0), (−0.979109,0), (0.979109,0), (1.165109,0)	5
0.7	(−0.019,0), (0.807709,0), (−0.987809,0), (0.98594,0), (1.1689109,0)	5
1	(−0.019,0), (0.79809,0), (−0.989109,0), (0.98554574,0), (1.1809,0)	5
10	(−0.019,0), (0.809109,0), (−1.00859,0), (0.98554574,0), (1.176394,0), (0.09109,1.0268), (0.09109, −1.0268)	7
100	(−0.019,0), (0.39109,0.91586), (0.39109, −0.91586), (0.98554574,0), (1.175394,0), (−1.00615109,0), (0.8079985,0)	7

Table 5: *Values of equilibrium points at ν = 0.019, α_1 = 0.4.*

k	(ξ, ζ)	Number of points
0.4	(−0.019,0), (0.781109,0), (−0.905109,0), (0.986109,0), (1.135109,0)	5
0.7	(−0.019,0), (0.788709,0), (−0.937809,0), (0.98594,0), (1.1489109,0)	5
1	(−0.019,0), (0.79809,0), (−0.959109,0), (0.98554574,0), (1.1609,0), (0,1.8377), (0, −1.8377)	7
10	(−0.019,0), (0.809109,0), (−1.001359,0), (0.98554574,0), (1.176394,0), (0.09109,1.0268), (0.09109, −1.0268)	7
100	(−0.019,0), (0.39109,0.91586), (0.39109, −0.91586), (0.98554574,0), (1.175394,0), (−1.00615109,0), (0.8079985,0)	7

Table 6: *Values of equilibrium points at* $\nu = 0.019$, $\alpha_1 = 0.9$.

k	(ξ, ζ)	Number of points
0.4	$(-0.019,0),(0.6549109,0),(-0.699109,0),(0.989109,0),(1.0715109,0),(0,1.24),(0,-1.24)$	7
0.7	$(-0.019,0),(0.717709,0),(-0.77809,0),(0.98594,0),(1.089109,0),(0,1.11),(0,-1.11)$	7
1	$(-0.019,0),(0.74809,0),(-0.829109,0),(0.98554574,0),(1.109,0),(0.01,1.07),(0.01,-1.07)$	7
10	$(-0.019,0),(0.804109,0),(-0.9815,0),(0.98554574,0),(1.166394,0),(0.104,0.9971),$ $(0.104,-0.9971)$	7
100	$(-0.019,0),(0.39109,0.91586),(0.39109,-0.91586),(0.98554574,0),(1.175394,0),$ $(-1.00615109,0),(0.8079985,0)$	7

According to the above tables, we found at-most 9 libration points and at-least 5 libration points.

3.2. Equilibrium Points

i- $(\xi,\eta)-$plane $\qquad \alpha_1 = 0.2, \quad \nu = 0.019$

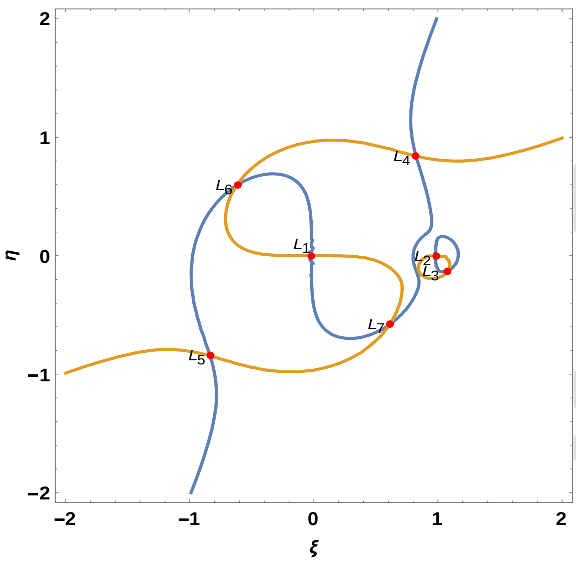

Figure 1: *Locations of equilibrium points at k = 0.4*

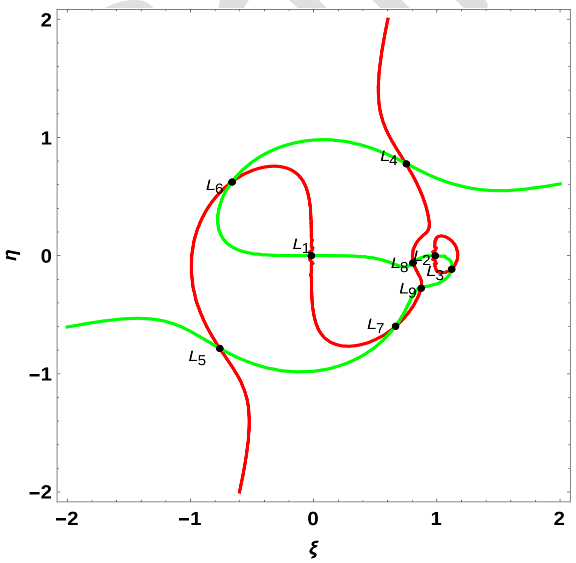

Figure 2: *Locations of equilibrium points at k = 0.7*

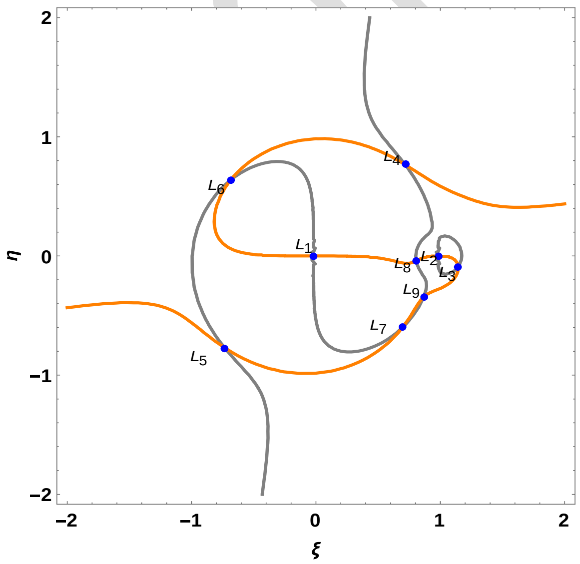

Figure 3: *Locations of equilibrium points at k = 1*

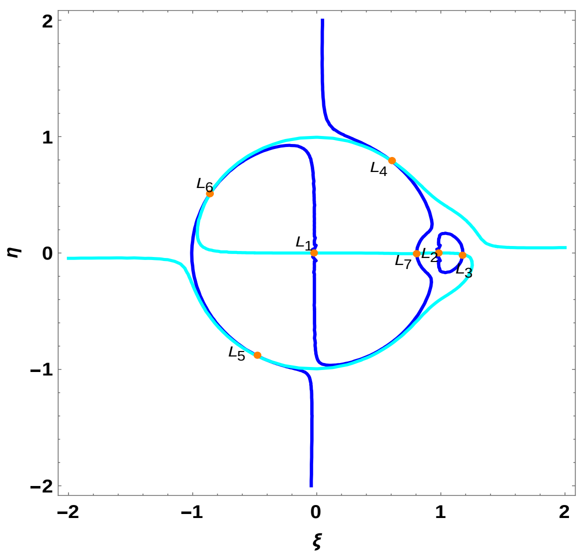

Figure 4: *Locations of equilibrium points at k = 10*

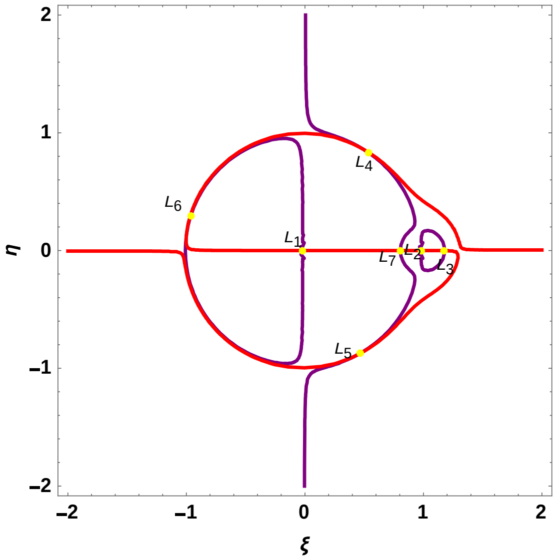

Figure 5: *Locations of equilibrium points at k = 100*

ii- $(\xi,\zeta)-\text{plane}$ $\alpha_1 = 0.2,\quad \nu = 0.019$

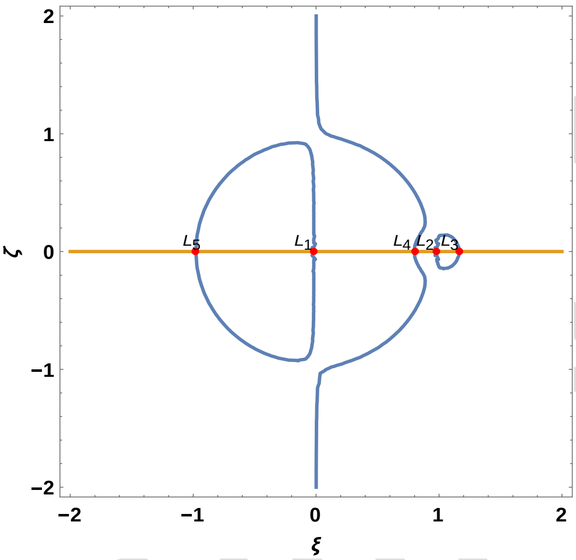

Figure 6: *Locations of equilibrium points at k = 0.4*

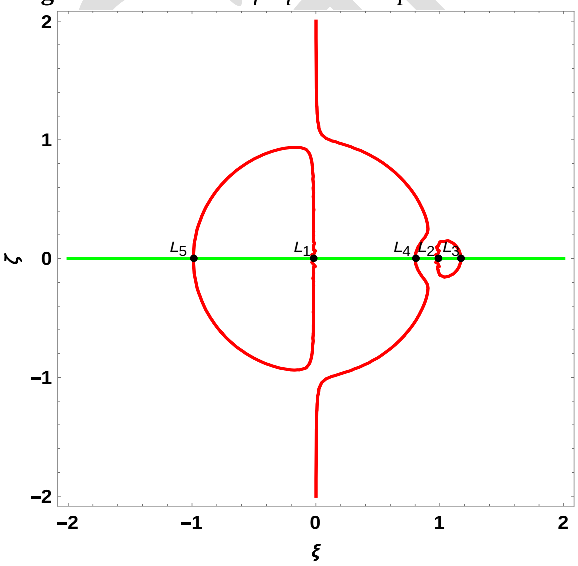

Figure 7: *Locations of equilibrium points at k = 0.7*

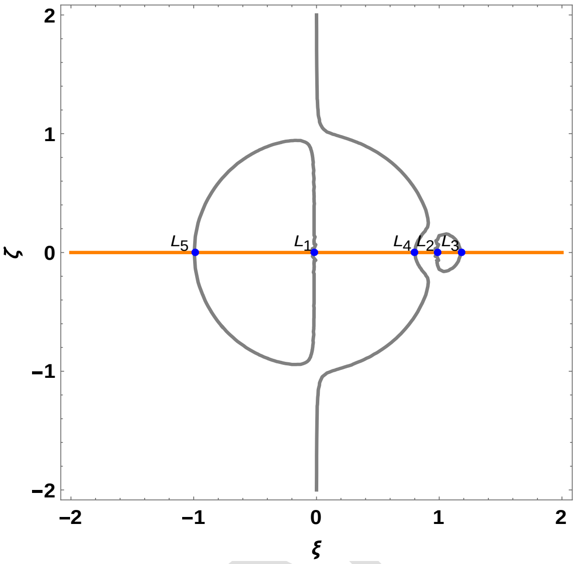

Figure 8: *Locations of equilibrium points at k = 1*

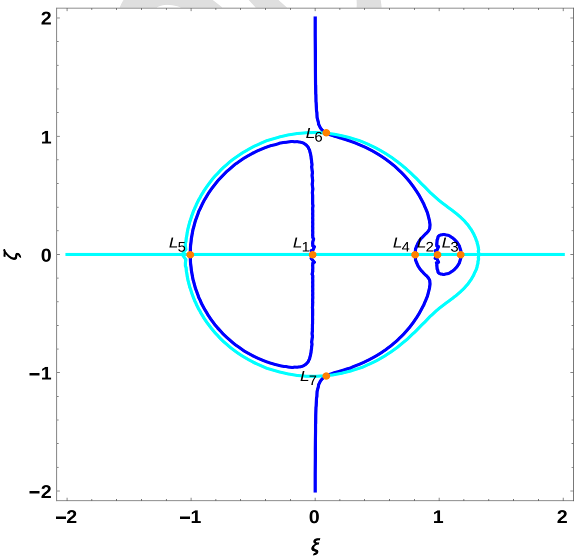

Figure 9: *Locations of equilibrium points at k = 10*

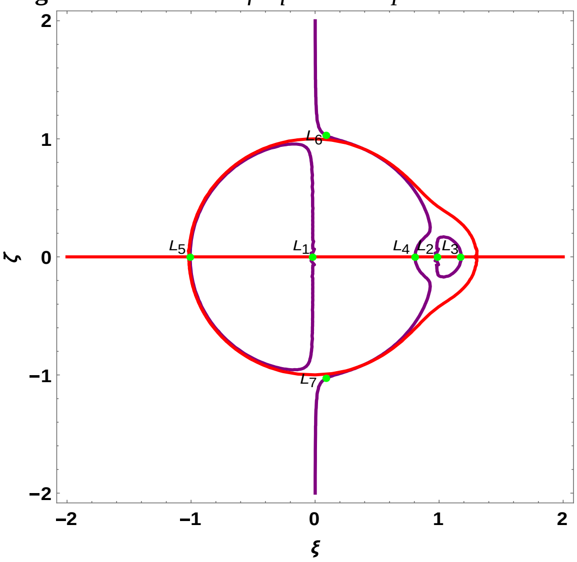

Figure 10: *Locations of equilibrium points at k = 100*

3.3. Zero-Velocity Curves

We have drawn the zero velocity curves for the different five values of k and found that as we increase the values of k the regions of motion are expanding in both $\xi\eta-$plane (Figure 11-15) and $\xi\zeta-$plane (Figure 16-20).

i. $(\xi,\eta)-$plane $\alpha_1 = 0.2,\quad \nu = 0.019$

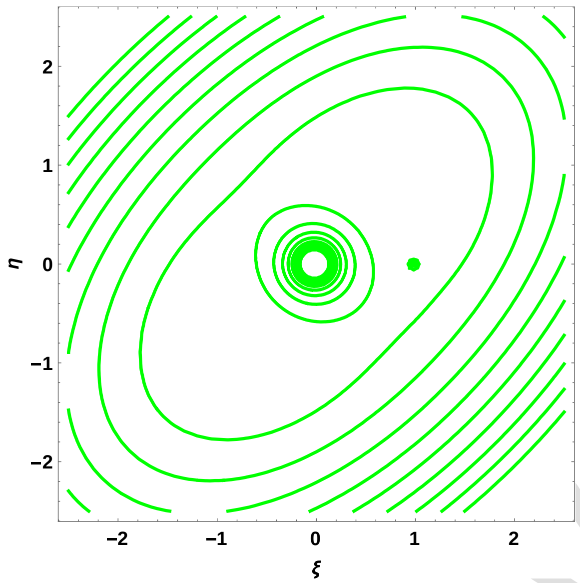

Figure 11: *Zero-velocity curves for C = 1.0096, k = 0.4*

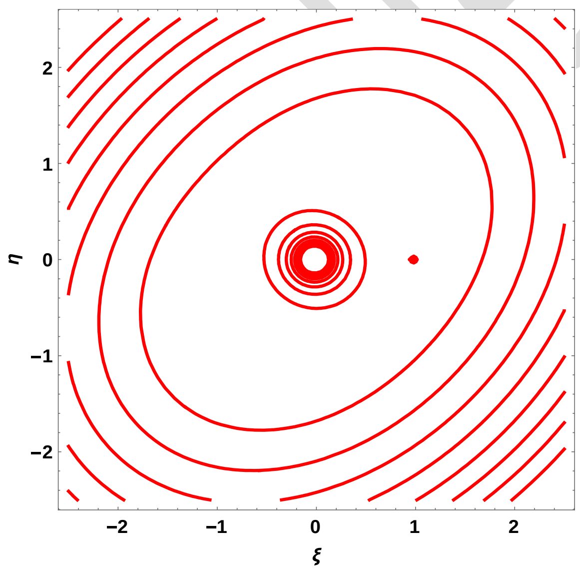

Figure 12: *Zero-velocity curves for C = 1.91903, k = 0.7*

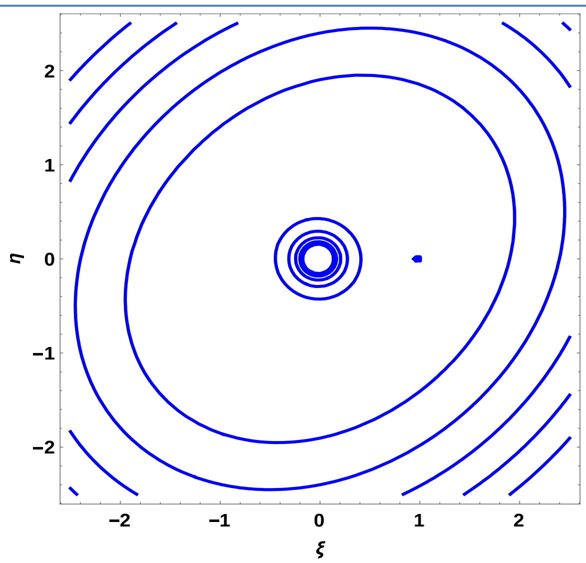

Figure 13: *Zero-velocity curves for C = 2.81908, k = 1*

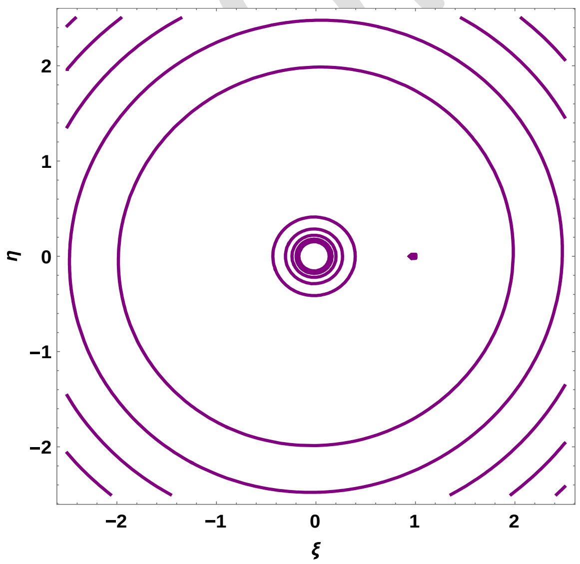

Figure 14: *Zero-velocity curves for C = 29.6734, k = 10*

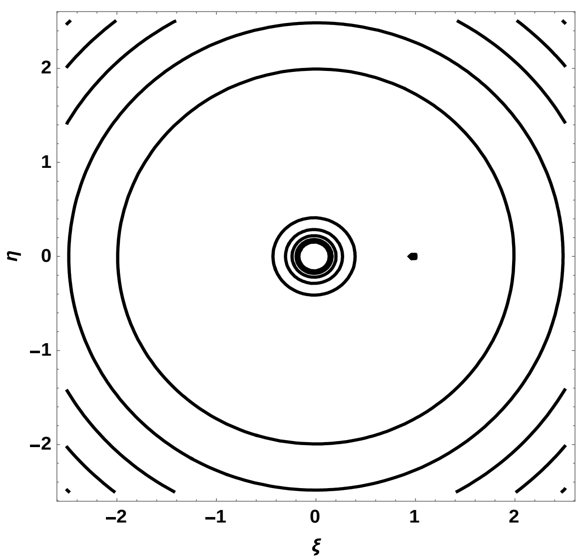

Figure 15: *Zero-velocity curves for C = 298.019, k = 100*

ii. (ξ,ζ) – plane $\alpha_1 = 0.2,\quad \nu = 0.019$

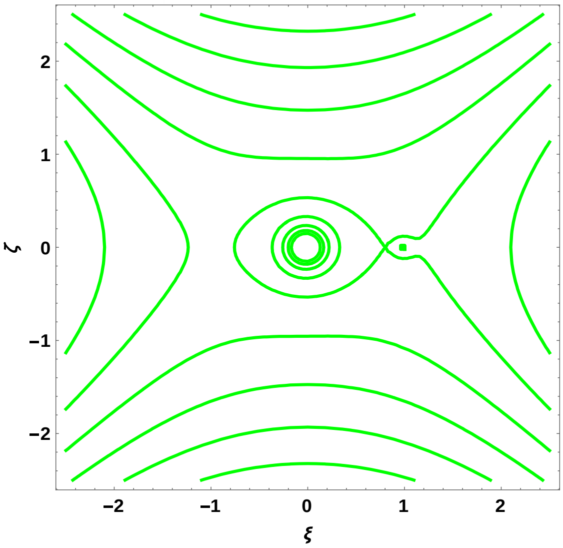

Figure 16: *Zero-velocity curves for C = 1.32392, k = 0.4*

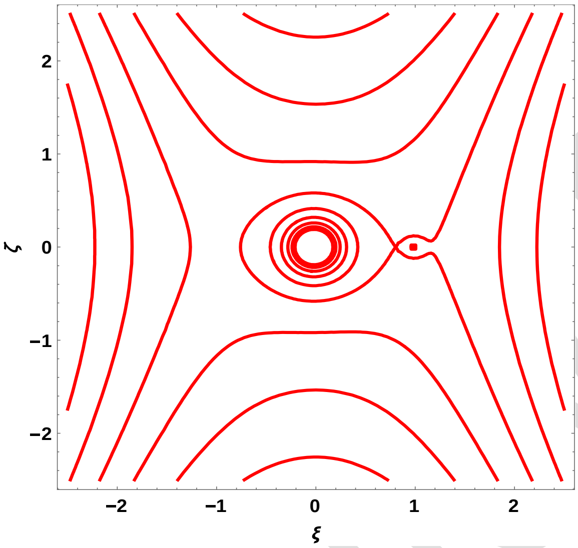

Figure 17: *Zero-velocity curves for C = 2.29756, k = 0.7*

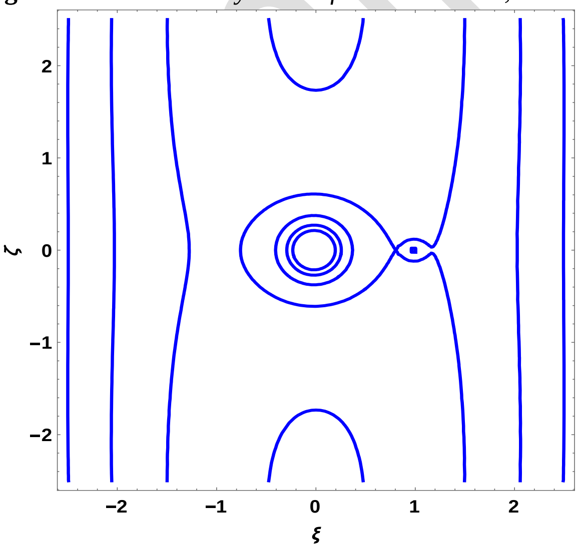

Figure 18: *Zero-velocity curves for C = 3.27138, k = 1*

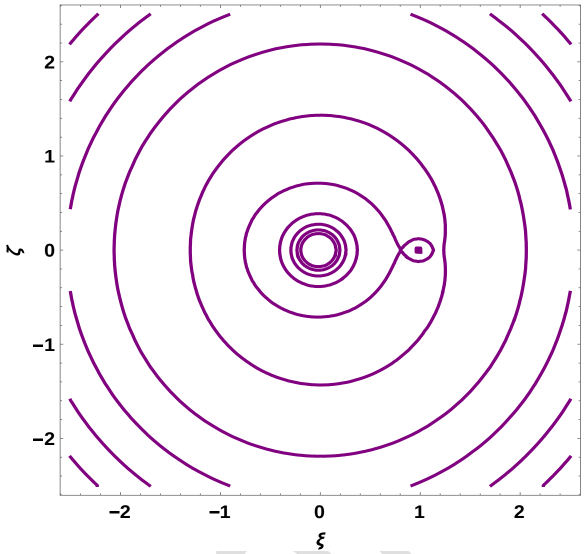

Figure 19: *Zero-velocity curves for C = 32.476, k = 10*

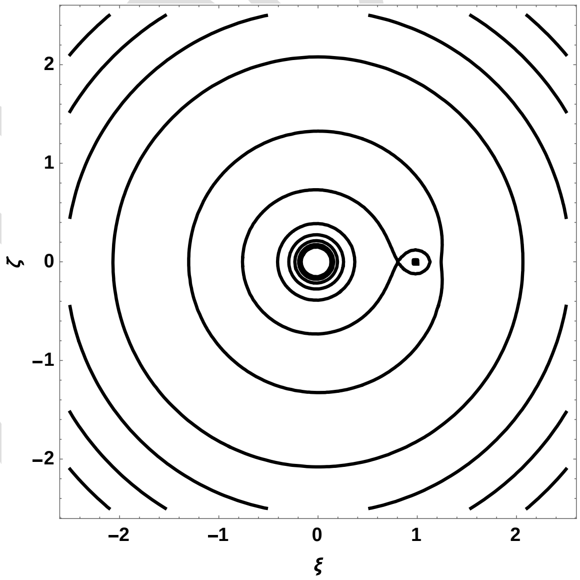

Figure 20: *Zero-velocity curves for C = 324.521, k = 100*

3.4 Poincare Surface of Section

We also have drawn the poincare surface of sections for the different five values of k and found that as we are increasing the values of k, the surfaces are expanding. Poincare surface of sections for the different values of k is represented in Figure 21. We can also draw the poincare surface of sections for the other values of α_1.

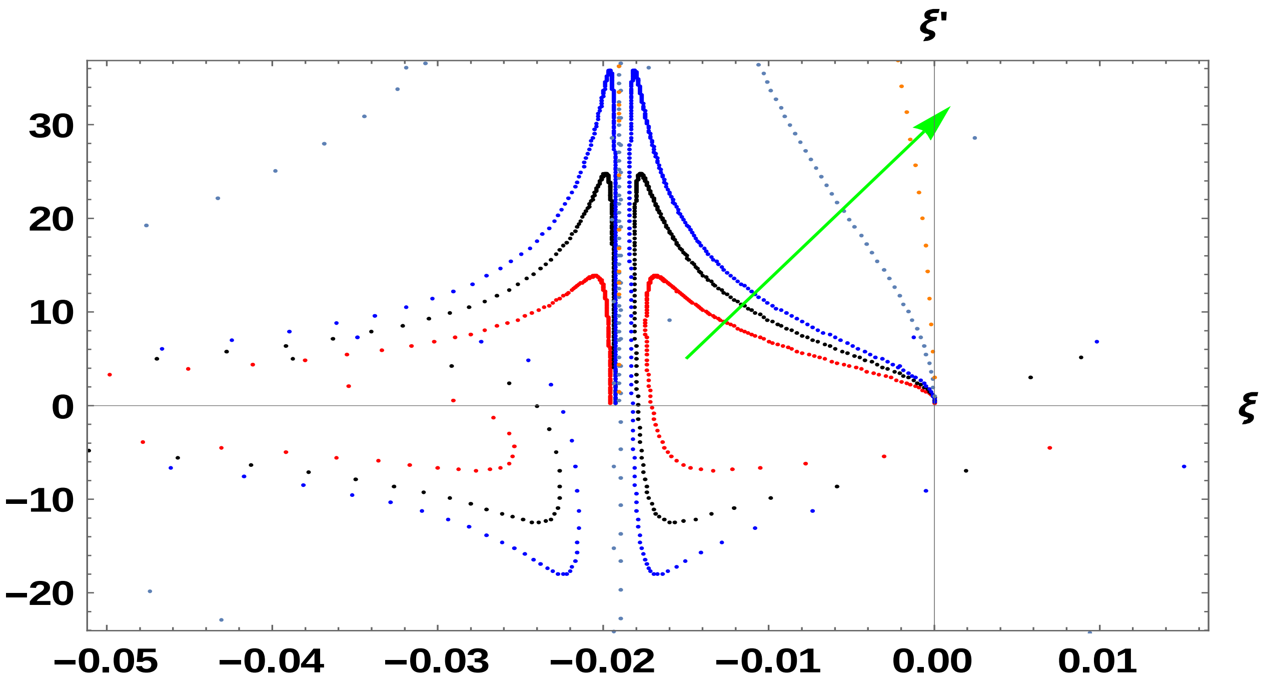

Figure 21: *Poincare Surface of Sections k -> {0.4(Red), 0.7 (Black), 1(Blue), 10 (Magenta), 100(Orange)}*

4. Stability of the Equilibrium Points of the Autonomized Equations

Following the procedure of the stability of the equilibrium points given by Mccuskey [17], we can write the autonomized equation when $\xi=\xi_0+\alpha, \eta=\eta_0+\beta, \zeta=\zeta_0+\gamma,$ as

$$\alpha''-2\beta'-\alpha_1\alpha'=\alpha(\Omega_{\xi\xi})_0+\beta(\Omega_{\xi\eta})_0+\gamma(\Omega_{\xi\zeta})_0,$$
$$\beta''+2\alpha'-\alpha_1\beta'=\alpha(\Omega_{\eta\xi})_0+\beta(\Omega_{\eta\eta})_0+\gamma(\Omega_{\eta\zeta})_0,$$
$$\gamma''-\alpha_1\gamma'=\alpha(\Omega_{\zeta\xi})_0+\beta(\Omega_{\zeta\eta})_0+\gamma(\Omega_{\zeta\zeta})_0, \qquad (7)$$

Where α, β and γ are the small displacements of the infinitesimal body from the libration point. Suffix zero denotes the value at the libration point.

To solve equation (7), let $\alpha=Ae^{\lambda\tau}, \beta=Be^{\lambda\tau}, \gamma=Ce^{\lambda\tau}$, where A, B and C are parameters. Substituting these values in equation (7) and rearranging, we get

$$A(\lambda^2-\alpha_1\lambda-(\Omega_{\xi\xi})_0)-B(2\lambda+(\Omega_{\xi\eta})_0)-C(\Omega_{\xi\zeta})_0=0,$$
$$A(2\lambda+(\Omega_{\eta\xi})_0)+B(\lambda^2-\alpha_1\lambda-(\Omega_{\eta\eta})_0)-C(\Omega_{\eta\zeta})_0=0,$$
$$-A(\Omega_{\zeta\xi})_0-B(\Omega_{\zeta\eta})_0+C(\lambda^2-\alpha_1\lambda-(\Omega_{\zeta\zeta})_0)=0, \qquad (8)$$

The equation (8) will have a non-trivial solution for A, B and C if

$$\begin{vmatrix} \lambda^2-\alpha_1\lambda-(\Omega_{\xi\xi})_0 & -(2\lambda+(\Omega_{\xi\eta})_0) & -(\Omega_{\xi\zeta})_0 \\ 2\lambda+(\Omega_{\eta\xi})_0 & \lambda^2-\alpha_1\lambda-(\Omega_{\eta\eta})_0 & -(\Omega_{\eta\zeta})_0 \\ -(\Omega_{\zeta\xi})_0 & -(\Omega_{\zeta\eta})_0 & \lambda^2-\alpha_1\lambda-(\Omega_{\zeta\zeta})_0 \end{vmatrix}=0,$$

$$\begin{aligned}
&\lambda^6-3\alpha_1\lambda^5+\lambda^4(4+3\alpha_1^2-(\Omega_{\xi\xi})_0-(\Omega_{\eta\eta})_0\\
&-(\Omega_{\zeta\zeta})_0)+\alpha_1\lambda^3(-4-\alpha_1^2+2(\Omega_{\xi\xi})_0\\
&+2(\Omega_{\eta\eta})_0+2(\Omega_{\zeta\zeta})_0)+\lambda^2(-(\Omega_{\xi\eta})_0^2\\
&-(\Omega_{\xi\zeta})_0^2+(\Omega_{\xi\xi})_0(\Omega_{\eta\eta})_0-(\Omega_{\eta\zeta})_0^2\\
&-4(\Omega_{\zeta\zeta})_0+(\Omega_{\xi\xi})_0(\Omega_{\zeta\zeta})_0+(\Omega_{\zeta\zeta})_0(\Omega_{\eta\eta})_0\\
&-\alpha_1^2(\Omega_{\xi\xi})_0-\alpha_1^2(\Omega_{\eta\eta})_0-\alpha_1^2(\Omega_{\zeta\zeta})_0)\\
&+\alpha_1\lambda((\Omega_{\xi\eta})_0^2+(\Omega_{\xi\zeta})_0^2-(\Omega_{\xi\xi})_0(\Omega_{\eta\eta})_0\\
&+(\Omega_{\eta\zeta})_0^2-(\Omega_{\xi\xi})_0(\Omega_{\zeta\zeta})_0-(\Omega_{\eta\eta})_0(\Omega_{\zeta\zeta})_0)\\
&+((\Omega_{\xi\zeta})_0^2(\Omega_{\eta\eta})_0-2(\Omega_{\xi\eta})_0(\Omega_{\xi\zeta})_0(\Omega_{\eta\zeta})_0\\
&+(\Omega_{\xi\xi})_0(\Omega_{\xi\eta})_0^2+(\Omega_{\xi\eta})_0^2(\Omega_{\zeta\zeta})_0\\
&-(\Omega_{\xi\xi})_0(\Omega_{\eta\eta})_0(\Omega_{\zeta\zeta})_0),
\end{aligned}$$

We have solved this polynomial for the different values of the libration points given in the tables and found that in all the cases, λ has mixed values, i.e. some values are real and some values are complex. Hence all the libration points are unstable.

5. Conclusion

We have investigated the existence and stability of libration points in the circular restricted three body problem. We have evaluated the equations of motions when the mass of the primaries as well as the infinitesimal bodies varies with time, which are different from the classical case by the factor α_1 and k. We found at most 9 libration points and at least 5 libration points given in the tables. We have determined the zero velocity curves for the different values of k and found that the regions of motion are expanding after increasing the values of k. We also have drawn the Poincare surface of sections for the different values of k and observed that the surfaces of sections are expanding when we are increasing the values of k. Finally, we have checked the stability for each libration points given in the tables and found that the libration points are unstable.

Acknowledgement

We are thankful to the Deanship of Scientific Research, College of Science in Zulfi, Majmaah University, Kingdom of Saudi Arabia, for providing all the research facilities in the completion of this research work.

References

[1] J. H. Jeans," Astronomy and Cosmogony," *Cambridge University Press, Cambridge*, 1928.

[2] Meshcherskii, I.V.: Studies on the Mechanics of Bodies of Variable Mass. *GITTL, Moscow*, 1949.

[3] I. V. Meshcherskii, "Works on the Mechanics of Bodies of Variable Mass," *GITTL, Moscow,* 1952.

[4] V. Szebehely, "Theory of Orbits," *Academic Press, New York,* 1967.

[5] K. B. Bhatnagar, "Periodic orbits of collision in the restricted problem of three bodies in a Three-dimensional coordinate system," *Indian Journal of Pure and Applied Mathematics, India*, 3(1), 101 -117, 1972.

[6] J. F. L. Simmons, et.al., "The restricted three body problem with radiation pressure,"*Celestial Mechanics,* 35, 145, 1985.

[7] J. Singh, A. J. C. Macdonald, J. C. Brown, "Effect of perturbations on the location of equilibrium points in the restricted problem of three bodies with variable mass," *Celest. Mech.* 32 (4), 297-305, 1984.

[8] J. Singh, B. Ishwar, "Effect of perturbations on the stability of triangular points in the restricted problem of three bodies with variable mass," *Celest. Mech.* 35, 201-207, 1985.

[9] J. Singh, "Photogravitational restricted three body problems with variable mass," *Indian J. of Pure and Applied Math,* 32 (2), 335-341, 2003.

[10] J. Singh, "Stability of photogravitational restricted three body problem with variable mass, *Astrophysics and Space Sci.* 326 (2), 305-314, 2010.

[11] M. J. Zhang, C. Y. Zhao, Y. Q. Xiong, "On the triangular libration points in photo-gravitational restricted three body problem with variable mass," *Astrophysics Space Sci.* 337, 107-113, 2012, DOI 10.1007/s10509-011-0821-8.

[12] E. I. Abouelmagd, A. Mostafa, "Out of plane equilibrium points locations and the forbidden movement regions in the restricted three-body problem with variable mass," *Astrophysics Space Sci.* 357, 58, 2015, DOI 10.1007/s10509-015-2294-7.

[13] E. I. Abouelmagd, et.al., "The effect of oblateness in the perturbed restricted three body problem," *Meccanica* 48, 2479-2490, 2013.

[14] K. Shalini, et.al., "Existence and Stability of the libration point *L4* in the R3BP, when the smaller primary is a heterogeneous axis symmetric body with *N* layers," *J. Appl. Environ. Biol. Sci., 6(1), 249-257, 2016.*

[15] Abdullah, "Stability of the equilibrium points in the circular restricted four body problem with oblate primary and variable mass," *International Journal of Advanced Astronomy, 4(1), 14-19, 2016.*

[16] A. Mittal, et. al., "Stability of libration points in the restricted four-body problem with variable mass," *Astrophysics and space science,* 361, 329, 2016, DOI 10.1007/s10509-016-2901-2.

[17] S. W. Mccuskey, "Introduction to Celestial Mechanics," *Addison–Wesley, USA,* 1963.