A Project Report

On

SIGN PERMUTATION

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Certificate

This is to certify that the project report entitled "**SIGN PERMUTATION"** submitted by Ms. MAHIMA NANDANA K (ID No. 2023B4A70660H) and Ms.AISHWINA DUBEY (ID No. 2023B4AA0672H) in partial fulfillment of the requirements of the course MATH F266, Study Project Course, embodies the work done by them under my supervision and guidance.

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Abstract

This report provides a comprehensive analysis of the sign function in permutation theory. We explore multiple perspectives on the parity of permutations, beginning with the fundamental decomposition of permutations into transpositions and establishing that the parity of the number of transpositions in such decompositions is invariant. We then develop the theory of the alternating group, examine the sign function through determinants and linear transformations, and characterize the sign homomorphism algebraically. The report concludes with an analysis of the minimal number of transpositions required to represent a permutation.We emphasize both algebraic structure and geometric intuition, presenting alternative proofs to highlight different mathematical perspectives.Through a blend of theoretical proofs and illustrative examples, this study highlights the elegance and utility of the sign function in mathematical structures and applications. We analyze permutation matrices and their relationship to the sign function, demonstrating that the determinant of a permutation matrix equals precisely the sign of the corresponding permutation. The report thoroughly examines the alternating group of even permutations as a normal subgroup of the symmetric group, studying its distinctive properties including simplicity for $n \geq 5$. Our investigation extends to secular equations and the Cayley-Hamilton theorem, establishing deep connections between permutation signs, characteristic polynomials, and matrix eigenvalues. The report addresses computational aspects of determinants, including efficient calculation methods via cofactor matrices and the adjoint, block matrix techniques, and applications in inner product spaces

Contents

1 Introduction

Throughout our discussion, we assume $n \geq 2$. The symmetric group S_n consists of all permutations of the set $\{1, 2, \ldots, n\}$.A permutation is a rearrangement or ordering of elements within a set. A fundamental observation in the theory of permutations is that every permutation in S_n can be expressed as a product of cycles, and each cycle can be further written as a product of transpositions (2-cycles).(A transposition swaps two elements).This leads to significant structural insights, particularly regarding the classification of permutations into even and odd types.

1.1 Breaking a cycle into Transpositions

Any cycle in S_n can be written as a product of transpositions.

The identity permutation (1) can be expressed as (12)(12), and more generally, a k-cycle with $k \geq 2$ can be written as:

$$(i_1 i_2 \ldots i_k) = (i_1 i_2)(i_2 i_3) \cdots (i_{k-1} i_k) \tag{1}$$

For instance, a 3-cycle (abc) (where a, b, c are distinct) is a product of two transpositions:

$$(abc) = (ab)(bc) \tag{2}$$

There are alternative ways to express this, including $(abc) = (bc)(ac) = (ac)(ab)$.

Since each permutation in S_n can be expressed as a product of disjoint cycles, and each cycle can be written as a product of transpositions, we can conclude that every permutation in S_n can be expressed as a product of transpositions. Although the disjoint cycle decomposition of a permutation is unique up to the order of the cycles, the representation as a product of transpositions is generally not unique. This non-uniqueness raises an important question: is there any invariant property in the various ways of expressing a permutation as a product of transpositions?

1.2 Examples

Example 1.1. *Consider the permutation $\sigma = (15342)$. Two different expressions for σ as a product of transpositions are:*

$$\sigma = (15)(53)(34)(42) \tag{3}$$

$$\sigma = (13)(24)(32)(13)(32)(24)(45)(24)(32)(13) \tag{4}$$

The second representation is a non-minimal representation of σ.

Example 1.2. *Consider $\tau = (12)(243)(35)$. Note that these cycles are not disjoint. Two different ways to express τ as a product of transpositions are:*

$$\tau = (12)(12)(15)(13)(14)(12) \tag{5}$$
$$\tau = (12)(24)(43)(35) \tag{6}$$

When we write a general permutation $\sigma \in S_n$ as

$$\sigma = \tau_1 \tau_2 \cdots \tau_r \tag{7}$$

where each τ_i is a transposition, the specific transpositions are not uniquely determined. However, there exists a fundamental parity constraint: the value of r mod 2 is unique. For example, the two expressions for (15342) in the first example involve 4 and 10 transpositions, which are even numbers. Similarly, in the second example, the permutation (12)(243)(35) is expressed using 6 and 4 transpositions, both even numbers.It is not possible to represent this permutation as a product of odd number of transpositions.

A permutation can only be expressed as a product of even or odd number of transpositions, and parity is a constant for a given permutation.This uniqueness of parity is the basis for classifying permutations as either even or odd.

2 Definition of the Sign

This section establishes this fundamental theorem and defines the sign function.

2.1 Definition

The sign of a permutation σ, denoted $\text{sgn}(\sigma)$, is defined as $(-1)^r$, where r is the number of transpositions in a decomposition of σ.
The sign is 1 if it is even and -1 if it is odd.Permutations with the sign $+1$ are called **even permutations**, and those with sign -1 are called **odd permutations**. This classification is also referred to as the **parity** of the permutation.

Theorem 2.1 (Invariance of Parity)**.** *Let $\sigma \in S_n$ be expressed as a product of transpositions in two different ways:*

$$\sigma = \tau_1\tau_2\cdots\tau_r = \tau_1'\tau_2'\cdots\tau_{r'}' \tag{8}$$

Then $r \equiv r' \pmod 2$.
(i.e)parity of a permutation remains constant.

The invariance theorem guarantees that the sign is well-defined, as the parity of r is the same regardless of which factorization into transpositions we choose.
The Levi-Civita symbol $\epsilon_{i_1,i_2,\ldots,i_n}$ is defined based on the sign of a permutation:

$$\epsilon_{i_1,i_2,\ldots,i_n} = \operatorname{sgn}(\sigma)$$

2.2 Examples

Example 2.2. *The permutation $\sigma = (15342)$ from our earlier example can be expressed using 4 or 10 transpositions, both even numbers.Therefore, $\operatorname{sgn}(\sigma) = (-1)^4 = 1$, making it an even permutation.*

Example 2.3. *Every transposition in S_n has sign -1 and is therefore odd.*

Example 2.4. *The identity permutation is $(12)(12)$, a product of 2 transpositions, so it has sign $+1$ and is even.*

Example 2.5. *The permutation $(146)(25)$ is $(14)(46)(25)$, a product of three transpositions, so it has sign -1 and is odd.*

Example 2.6. *The 5-cycle (12345) is $(12)(23)(34)(45)$, a product of 4 transpositions, so $\operatorname{sgn}(12345) = 1$, making it even.*

Example 2.7. *What is the sign of a k-cycle? Since any k-cycle can be expressed as*

$$(i_1i_2\cdots i_k) = (i_1i_2)(i_2i_3)\cdots(i_{k-1}i_k) \tag{9}$$

which involves $k-1$ transpositions,

$$\operatorname{sgn}(i_1i_2\cdots i_k) = (-1)^{k-1} \tag{10}$$

*In other words, a cycle with even length has sign -1 (is odd) and a cycle with odd length has sign $+1$ (is even).Note that the parity of a cycle is **opposite** to the parity of its length. Ex:a 2-cycle (transposition) is odd.*

2.3 Other Properties

An important property of the sign function is that it is multiplicative, as stated in the following theorem.

Theorem 2.8 (Multiplicativity of Sign)**.** *For any permutations $\sigma, \sigma' \in S_n$,*

$$\operatorname{sgn}(\sigma\sigma') = \operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma') \tag{11}$$

Proof. If σ is a product of k transpositions and σ' is a product of k' transpositions, then $\sigma\sigma'$ can be written as a product of $k + k'$ transpositions. Therefore,

$$\operatorname{sgn}(\sigma\sigma') = (-1)^{k+k'} = (-1)^k(-1)^{k'} = \operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma') \tag{12}$$

□

Corollary 2.9. *Inverting and conjugating a permutation does not change its sign.*

Proof. Since $\operatorname{sgn}(\sigma\sigma^{-1}) = \operatorname{sgn}(1) = 1$, we have $\operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma^{-1}) = 1$, which implies $\operatorname{sgn}(\sigma^{-1}) = \operatorname{sgn}(\sigma)^{-1} = \operatorname{sgn}(\sigma)$ since the sign values are ± 1.

Similarly, if $\sigma' = \pi\sigma\pi^{-1}$ for some permutation π, then

$$\operatorname{sgn}(\sigma') = \operatorname{sgn}(\pi)\operatorname{sgn}(\sigma)\operatorname{sgn}(\pi^{-1}) = \operatorname{sgn}(\pi)\operatorname{sgn}(\sigma)\operatorname{sgn}(\pi) = \operatorname{sgn}(\sigma) \tag{13}$$

since $\operatorname{sgn}(\pi)^2 = 1$ for any permutation π. □

The multiplicative property allows us to compute the sign of a permutation using any decomposition into cycles, not necessarily disjoint ones. For a permutation expressed as a product of cycles, we compute the sign of each cycle based on its length and then multiply these signs.

In our earlier example:

$$\operatorname{sgn}((15342)) = \operatorname{sgn}(15342) = (-1)^{5-1} = (-1)^4 = 1 \tag{14}$$

Remark 2.10. *Identity has sign 1.This can be proved by considering identity as a product of zero transpositions.*

The concept of even and odd permutations originates from Évariste Galois, who laid the groundwork for group theory.

3 Determinants and Sign of a Permutation

The concept of the sign of a permutation is not just some abstract property—it plays a crucial role in other areas of mathematics, particularly in linear algebra. One of the most fundamental places it appears is in the **determinant** of a square matrix.

Think of a determinant as a sum, made up of carefully selected products of elements from a matrix, multiplied by either $+1$ or -1. The way these signs are assigned is not random,they come directly from the **sign of a permutation**.

3.1 How Permutations Appear in Determinants:

The determinant of an $n \times n$ matrix (a_{ij}) is given by:

$$\det(a_{ij}) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)} \tag{15}$$

This sum runs over all possible permutations of n elements, and each term picks one entry from each row and column. The sign of each term is determined by whether the permutation involved is **even** or **odd**.

For example, for a 2×2 matrix:

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \operatorname{sgn}(1) a_{11} a_{22} + \operatorname{sgn}(12) a_{12} a_{21} = a_{11} a_{22} - a_{12} a_{21} \tag{16}$$

This connection between permutations and determinants leads to an elegant interpretation of the sign function through linear transformations.

3.2 Permutation Matrices

Now, let's look at permutations as actual transformations.

For any permutation $\sigma \in S_n$, we can define a function $T_\sigma : \mathbb{R}^n \to \mathbb{R}^n$ that rearranges the **standard basis vectors** according to σ. This means:

$$T_\sigma(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n) = x_1 e_{\sigma(1)} + x_2 e_{\sigma(2)} + \cdots + x_n e_{\sigma(n)}. \tag{17}$$

In other words, T_σ sends the standard basis vector e_i to $e_{\sigma(i)}$ and extends by linearity to all of $\mathbb{R}^n$. This transformation permutes the standard basis according to σ. When represented as a matrix, T_σ becomes a permutation matrix. A permutation matrix is a square matrix with exactly one 1 in each

row and column and all other entries are 0.

For example, the permutation $\sigma = (231)$, which swaps positions so that 1 moves to 2, 2 moves to 3, and 3 moves to 1, corresponds to the permutation matrix:

$$P_{(231)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Also,

$$P_{(15342)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure 1: Permutation matrix for $\sigma = (15342)$

Every permutation can be thought of as one of these matrices, and the **determinant of a permutation matrix** is exactly equal to the **sign of the permutation**.

Example 3.1. *Let $\sigma = (142)$ in S_4. Then $T_\sigma(e_1) = e_4$, $T_\sigma(e_2) = e_2$, $T_\sigma(e_3) = e_3$, and $T_\sigma(e_4) = e_1$. As a matrix:*

$$[T_\sigma] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{18}$$

Example 3.2. *Let $\sigma = (14)(23)$ in S_4. Then:*

$$[T_\sigma] = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{19}$$

The mapping $\sigma \mapsto T_\sigma$ is a group homomorphism from S_n to the group of $n \times n$ invertible matrices. That is, $T_{\sigma_1\sigma_2} = T_{\sigma_1} T_{\sigma_2}$. Taking determinants, we get:

$$\det(T_{\sigma_1}) \det(T_{\sigma_2}) = \det(T_{\sigma_1\sigma_2}) \tag{20}$$

What is $\det(T_\sigma)$? Since each row and column of T_σ has exactly one 1 and the rest 0s, the determinant formula reduces to a single term:

$$\det(T_\sigma) = \operatorname{sgn}(\sigma) \tag{21}$$

This gives us a profound geometric interpretation: the sign of a permutation is the determinant of its associated permutation matrix. Since determinants measure orientation-preserving or orientation-reversing properties of linear transformations, even permutations preserve orientation, while odd permutations reverse it.

This connection also provides another proof of the multiplicative property of the sign function. Since matrix multiplication corresponds to composition of linear transformations and determinants are multiplicative for matrices, we have:

$$\operatorname{sgn}(\sigma_1\sigma_2) = \det(T_{\sigma_1\sigma_2}) = \det(T_{\sigma_1}T_{\sigma_2}) = \det(T_{\sigma_1})\det(T_{\sigma_2}) = \operatorname{sgn}(\sigma_1)\operatorname{sgn}(\sigma_2) \tag{22}$$

4 Algebraic Characterization of the Sign Function

4.1 Algebraic Characterization

The sign function on S_n has a remarkable characterization: it is a unique nontrivial group homomorphism from S_n to the multiplicative group $\{+1, -1\}$. This characterization highlights the fundamental nature of the sign function in permutation theory.

Theorem 4.1 (Characterization of Sign)**.** *For $n \geq 2$, let $h : S_n \to \{+1, -1\}$ be a function that satisfies $h(\sigma\sigma') = h(\sigma)h(\sigma')$ for all $\sigma, \sigma' \in S_n$. Then $h(\sigma) = 1$ for all σ (the trivial homomorphism) or $h(\sigma) = \operatorname{sgn}(\sigma)$ for all σ. Thus, if h is multiplicative and not identically 1, then $h = \operatorname{sgn}$.*

4.2 Applications

This theorem has a fascinating application in quantum mechanics.
In physics, particularly quantum mechanics, the states of a system of n identical particles are represented in such a way that swapping two particles should not affect physical observables.

This means the function describing how the system changes under swaps must be a multiplicative function on S_n. By Theorem 4.1, it must be either be constantly 1 or exactly the sign function.

- **Bosons** (e.g., photons) correspond to the case where $h(\sigma) = 1$ for all σ, meaning that swapping particles has no effect.
- **Fermions** (e.g., electrons) correspond to $h(\sigma) = \text{sgn}(\sigma)$, meaning that the substitution of two particles introduces a negative sign in the wave-function.

This distinction between bosons and fermions is fundamental to modern physics and follows directly from the algebraic properties of the symmetric group!

5 The Alternating Group

5.1 Structure and Examples of A_n

The set of even permutations in S_n forms a subgroup known as the alternating group, denoted A_n. This section explores the structure and properties of this important subgroup.

$$A_n = \{\sigma \in S_n : \text{sgn}(\sigma) = 1\} \tag{23}$$

To verify that A_n is a subgroup of S_n, we check the properties of the subgroup:

1. The identity permutation is even, so $(1) \in A_n$.
2. If $\sigma, \tau \in A_n$, then $\text{sgn}(\sigma\tau) = \text{sgn}(\sigma)\,\text{sgn}(\tau) = (1)(1) = 1$, so $\sigma\tau \in A_n$.
3. If $\sigma \in A_n$, then $\text{sgn}(\sigma^{-1}) = \text{sgn}(\sigma) = 1$, so $\sigma^{-1} \in A_n$.

In addition, A_n is a **normal** subgroup of S_n. To verify that A_n is a subgroup of S_n,

A subgroup H of a group G is normal if for all $g \in G$ and $h \in H$, we have:

$$ghg^{-1} \in H$$

for all $g \in G$ and $h \in H$.

Taking any $\sigma \in A_n$, so σ is an even permutation. Consider conjugation by some $\tau \in S_n$, that is, we compute $\tau\sigma\tau^{-1}$.
The sign function satisfies:

$$\text{sgn}(\tau\sigma\tau^{-1}) = \text{sgn}(\tau) \cdot \text{sgn}(\sigma) \cdot \text{sgn}(\tau^{-1}).$$

Since $\text{sgn}(\tau^{-1}) = \text{sgn}(\tau)$, we get:

$$\text{sgn}(\tau\sigma\tau^{-1}) = \text{sgn}(\tau) \cdot 1 \cdot \text{sgn}(\tau) = 1.$$

So, $\tau\sigma\tau^{-1}$ is also an even permutation, which means that it belongs to A_n.

For $n \geq 5$, A_n is simple (i.e., has no nontrivial normal subgroups).

Example 5.1. *For $n = 2$, we have $S_2 = \{(1), (12)\}$ and $A_2 = \{(1)\}$.*

Example 5.2. *For $n = 3$, $A_3 = \{(1), (123), (132)\}$, which is a cyclic group of order 3 (either of the non-identity element is a generator).*

Example 5.3. *The group A_4 consists of 12 permutations:*

$$(1), (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23)$$

Example 5.4. *Every 3-cycle is even, so A_n contains all the 3-cycles when $n \geq 3$. In general, A_n is non-abelian for $n \geq 4$ because for example,* (123) *and* (124) *do not commute.*

$$(123)(124) = (13)(24) \tag{24}$$

$$(124)(123) = (14)(23) \tag{25}$$

$$(13)(24) \neq (14)(23) \tag{26}$$

Remark: A_1=S_1

5.2 Properties of A_n

Key properties:

- The product of two even permutations is even.
- The inverse of an even permutation is even.
- $|A_n| = n!/2$

The name "alternating" in the alternating group relates to the behavior of the multi-variable polynomial:

$$\prod_{1 \leq i < j \leq n} (X_j - X_i) \tag{27}$$

This polynomial is alternating in the sense that permuting its variables by $\sigma \in S_n$ changes the polynomial by exactly the sign of σ:

$$\prod_{i<j}(X_{\sigma(j)} - X_{\sigma(i)}) = \operatorname{sgn}(\sigma) \prod_{i<j}(X_j - X_i) \tag{28}$$

A polynomial whose overall value changes by ± 1 when permutation of each pair is done is called alternating polynomial.

The alternating group A_n is precisely the group of permutations that leave this polynomial unchanged.

Theorem 5.5. *For $n \geq 2$, $|A_n| = n!/2$.*

Proof. Choose any transposition, say $\tau = (12)$. Then $\tau \notin A_n$ since transpositions are odd. For any $\sigma \notin A_n$, we have $\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) = (-1)(-1) = 1$, so $\sigma\tau \in A_n$.

This means $\sigma \in A_n\tau$, where $A_n\tau$ denotes the set of permutations of the form $\pi\tau$ for $\pi \in A_n$. We obtain a decomposition of S_n into two disjoint parts:

$$S_n = A_n \cup A_n\tau \tag{29}$$

This union is disjoint since elements of A_n have sign 1 and elements of $A_n\tau$ have sign -1. Moreover, $A_n\tau$ has the same size as A_n (multiplication on the right by τ gives a bijection between the sets), so $n! = |S_n| = |A_n| + |A_n\tau| = 2|A_n|$, which implies $|A_n| = n!/2$. □

The sizes of the smallest symmetric and alternating groups are:

n	1	2	3	4	5	6	7
$\lvert S_n\rvert$	1	2	6	24	120	720	5040
$\lvert A_n\rvert$	1	1	3	12	60	360	2520

Just as all elements of S_n are products of transpositions, the elements of A_n can be expressed as products of 3-cycles:

Theorem 5.6. *For $n \geq 3$, each element of A_n is a product of 3-cycles.*

Proof. The identity (1) can be written as (123)(132), which is a product of 3-cycles.

For a non-identity element $\sigma \in A_n$, we express it as a product of transpositions in S_n:

$$\sigma = \tau_1\tau_2\cdots\tau_r \tag{30}$$

We know that $\sigma \in A_n$, so it has sign 1 and hence r must be even. Therefore, we can group the transpositions into successive pairs $\tau_i\tau_{i+1}$ where $i = 1, 3, \ldots$ is odd.

We'll show that each product of two transpositions can be expressed as a product of 3-cycles, which will in turn establish that σ is a product of 3-cycles.

Case 1: If $\tau_i = \tau_{i+1}$, then $\tau_i\tau_{i+1} = (1) = (123)(132)$, so we can replace $\tau_i\tau_{i+1}$ with a product of two 3-cycles.

Case 2: If τ_i and τ_{i+1} share exactly one element, we can write $\tau_i = (ab)$ and $\tau_{i+1} = (ac)$ where $b \neq c$, which gives

$$\tau_i\tau_{i+1} = (ab)(ac) = (abc) \tag{31}$$

which is a 3-cycle, so we can replace $\tau_i\tau_{i+1}$ with a single 3-cycle.

Case 3: If τ_i and τ_{i+1} have no elements in common, we can write $\tau_i = (ab)$ and $\tau_{i+1} = (cd)$ where a, b, c, d are all distinct (so $n \geq 4$), which gives

$$\tau_i\tau_{i+1} = (ab)(cd) = (ab)(bc)(bc)(cd) = (abc)(bcd) \tag{32}$$

so we can replace $\tau_i\tau_{i+1}$ with a product of two 3-cycles.

In all cases, we express each pair of transpositions as either one or two 3-cycles, showing that σ is a product of 3-cycles. □

Remark 5.7. *Unlike the parity constraint on expressing permutations as products of transpositions, there is no similar restriction on the number of 3-cycles needed to express an element of A_n. In fact, the identity permutation can be written as a product of m 3-cycles for any $m \geq 2$. This follows from:*

$$(1) = (123)(132) \tag{33}$$
$$(1) = (123)(123)(123) \tag{34}$$
$$(1) = (123)(132)(123)(132) \tag{35}$$

This shows that (1) can be written as a product of 2, 3, or 4 3-cycles. By multiplying any of these expressions by $(123)^{3k}$ for $k \geq 1$ (which equals the identity), we can express (1) as a product of $2 + 3k$, $3 + 3k$, or $4 + 3k$ 3-cycles. This covers all integers $m \geq 2$ except $m = 1$, confirming that the identity cannot be expressed as a single 3-cycle.

6 Minimal Number of Transpositions for a Permutation

Given a permutation $\sigma \in S_n$, a natural question arises: what is the minimum number of transpositions needed to express σ? This section provides a formula for this minimum in terms of the cycle structure of the permutation.

Theorem 6.1 (Minimal Representation of a Transposition)**.** *Let $\sigma \in S_n$ be expressed as a product of m disjoint cycles, including 1-cycles. If we write $\sigma = \tau_1\tau_2\cdots\tau_r$ where each τ_i is a transposition, then the smallest possible value of r is $n - m$.*

To illustrate this theorem with some examples:

Example 6.2. *Consider $\sigma = (1234765)$ in S_7. Here $n = 7$, $m = 1$ (a single 7-cycle), and $n - m = 6$. An expression of σ as a product of 6 transpositions is:*

$$(1234765) = (15)(16)(17)(14)(13)(12) \tag{36}$$

Alternatively, we could write:

$$(1234765) = (12)(23)(34)(47)(76)(65) \tag{37}$$

If we view σ in S_{10} as $(1234765)(8)(9)(10)$ (explicitly writing the fixed points as 1-cycles), then $n = 10$, $m = 4$, and $n - m = 6$ again.

Example 6.3. *Let $\sigma = (156)(2847)$ in S_8. Then $n = 8$, $m = 2$, and $n-m = 6$. An expression of σ as a product of 6 transpositions is:*

$$(156)(2847) = (15)(56)(28)(84)(47) \tag{38}$$

Example 6.4. *It's important to note that the theorem applies to the canonical decomposition of permutations into disjoint cycles. For instance, $\sigma = (12)(23)(34)$ is a product of 3 transpositions that are not disjoint, and if we were to incorrectly use $n = 4$ and $m = 3$, we'd get $n - m = 1$, suggesting σ is a transposition, which is false. The correct computation is to first multiply out the cycles to get $\sigma = (1234)$, a 4-cycle, so $n = 4$, $m = 1$, and $n - m = 3$, which correctly gives the minimum number of transpositions needed.*

This theorem not only provides a formula for the minimum number of transpositions but also connects the cycle structure of a permutation to its representation complexity, offering another perspective on the structure of the symmetric group.

6.1 Minimal Factorization Problems

Given a permutation $\sigma \in S_n$, finding a minimal-length decomposition into transpositions is straightforward using the cycle structure, as we've seen: the minimum number of transpositions needed is $n - c$ where c is the number of cycles in σ (including fixed points).

However, several related problems have interesting complexity aspects:

- **Sorting by Transpositions**: What is the minimum number of transpositions (not necessarily adjacent) needed to sort a permutation? This problem has polynomial-time algorithms.
- **Sorting by Adjacent Transpositions**: What is the minimum number of adjacent transpositions needed to sort a permutation? This equals the number of inversions and can be computed in $O(n \log n)$ time.
- **Sorting by Reversals**: What is the minimum number of reversals (operations that reverse a contiguous segment) needed to transform one permutation into another? This problem is NP-hard in general but has polynomial-time approximation algorithms.

7 Determinants and their Properties

7.1 Traces and Determinants

Trace is an important and useful function from the matrix ring F_n (ring of $n \times n$ matrices over F) into F (an arbitrary field). The trace of a square matrix is the sum of its diagonal elements.

Example 7.1. *Let*

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

The trace of A is given by:

$$tr(A) = 1 + 4 = 5$$

The trace function primarily relates to the additive properties of matrices. Some of the key properties of traces are:

1. $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$ for any two $n \times n$ matrices A and B.
2. $\text{tr}(cA) = c\text{tr}(A)$ for any scalar c.
3. For any two $n \times n$ matrices A and B,

$$\text{tr}(AB) = \text{tr}(BA).$$

Proof:
The elements of the matrix product AB are given by:

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}.$$

The trace of AB is the sum of its diagonal elements:

$$\text{tr}(AB) = \sum_{i=1}^{n} (AB)_{ii}.$$

Substituting the formula for $(AB)_{ii}$:

$$\text{tr}(AB) = \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} B_{ki}.$$

Similarly, for the matrix product BA:

$$(BA)_{ij} = \sum_{k=1}^{n} B_{ik} A_{kj}.$$

Thus, the trace of BA is:

$$\text{tr}(BA) = \sum_{i=1}^{n} (BA)_{ii}.$$

Substituting the formula for $(BA)_{ii}$:

$$\text{tr}(BA) = \sum_{i=1}^{n} \sum_{k=1}^{n} B_{ik} A_{ki}.$$

Since addition is commutative, i.e., $A_{ik}B_{ki} = B_{ik}A_{ki}$, we conclude:

$$\text{tr}(AB) = \text{tr}(BA).$$

4. It is invariant for similar matrices (i.e., if $B = P^{-1}AP$ for some invertible matrix P), then:

$$\text{tr}(A) = \text{tr}(B).$$

Proof:
Given $B = P^{-1}AP$,

$$\text{tr}(B) = \text{tr}(P^{-1}AP).$$

Using the cyclic property of trace:

$$\text{tr}(B) = \text{tr}(P^{-1}PA).$$

Since $P^{-1}P = I$, we get:

$$\text{tr}(B) = \text{tr}(A).$$

On the other hand,The determinant is a more powerful function that maps an $n \times n$ matrix to the field F and is closely tied to the multiplicative properties of matrices.
In the earlier sections, we have already looked at how permutations and determinants are related. In this section, we look into some of the key properties of determinants and their various applications.

The determinant serves several crucial purposes:

- It provides a criterion for matrix invertibility
- It enables the construction of polynomials whose roots are the characteristic roots of a matrix
- It plays a key role in solving systems of linear equations
- It helps define the determinant of a linear transformation, independent of the chosen basis

7.2 Definition and Basic Properties

Definition 7.2. *If $A = (\alpha_{ij})$ is an $n \times n$ matrix over a field F, then the determinant of A, written* $\det A$*, is the element*

$$\sum_{\sigma \in S_n} (-1)^\sigma \alpha_{1\sigma(1)} \alpha_{2\sigma(2)} \cdots \alpha_{n\sigma(n)}$$

in F.

We sometimes use the notation

$$\begin{vmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{vmatrix}$$

for the determinant of the matrix of the form

$$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{pmatrix}.$$

Note that the determinant of a matrix A is the sum (modulo signs) of all possible products of entries of A, where exactly one entry is taken from each row and column of A.
Some key properties:

- $\det(AB) = \det(A)\det(B)$
- $\det(A^T) = \det(A)$
- A matrix A is invertible if and only if $\det(A) \neq 0$
- For a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det = ad - bc$

One of the most fundamental applications of determinants is their connection to matrix invertibility.

Theorem 7.3. *A matrix A is invertible if and only if* $\det A \neq 0$.

7.3 Determinants of Special Matrices

Lemma 7.4. *The determinant of a triangular matrix is the product of its entries on the main diagonal.*

From this lemma, we can deduce important special cases.

1. If $A = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a diagonal matrix, then $\det A = \lambda_1\lambda_2 \cdots \lambda_n$.
2. If $A = I$ is the identity matrix, then $\det A = 1$.
3. If $A = \lambda I$ is a scalar matrix, then $\det A = \lambda^n$.

An important observation: if a row (or column) of a matrix consists entirely of zeros, then the determinant is zero, as each term in the expansion would include at least one zero factor.

Given a matrix $A = (\alpha_{ij})$ in F_n, we can consider its rows $v_1, v_2, \ldots, v_n$ as vectors in $F^{(n)}$. This perspective allows us to view the determinant as a function of these n vectors, denoted $\det A = d(v_1, v_2, \ldots, v_n)$.

Lemma 7.5 (Scalar Multiplication Property)**.** *If $A \in F_n$ and $\gamma \in F$, then*

$$d(v_1, \ldots, v_{i-1}, \gamma v_i, v_{i+1}, \ldots, v_n) = \gamma \cdot d(v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_n).$$

This lemma states that if all elements in one row of A are multiplied by a fixed scalar γ, then the determinant of the resulting matrix is the original determinant multiplied by γ.

Lemma 7.6 (Additivity Property)**.**

$$d(v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_n) + d(v_1, \ldots, v_{i-1}, u_i, v_{i+1}, \ldots, v_n) = d(v_1, \ldots, v_{i-1}, v_i + u_i, v_{i+1}, \ldots, v_n).$$

Example 7.7. *Consider the matrices:*

$$A = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 5 & 3 \\ 4 & 2 \end{pmatrix}$$

These matrices differ only in their first row. Let's calculate:

$$\det A = 3 \cdot 2 - 1 \cdot 4 = 6 - 4 = 2$$
$$\det B = 5 \cdot 2 - 3 \cdot 4 = 10 - 12 = -2$$

Now, let's create a matrix C where the first row is the sum of the first rows of A and B:

$$C = \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}$$

Its determinant is:

$$\det C = 8 \cdot 2 - 4 \cdot 4 = 16 - 16 = 0$$

We observe that $\det A + \det B = 2 + (-2) = 0 = \det C$*, confirming the additivity property.*

Lemma 7.8. *If two rows of A are equal (that is, $v_r = v_s$ for $r \neq s$), then* $\det A = 0$.

Let

$$A = \begin{bmatrix} a & b & c \\ a & b & c \\ d & e & f \end{bmatrix}$$

$\det(A) = \operatorname{sgn}(1) \cdot a \cdot b \cdot f + \operatorname{sgn}(12) \cdot b \cdot a \cdot f + \operatorname{sgn}(13) \cdot c \cdot b \cdot d + \operatorname{sgn}(23) \cdot a \cdot c \cdot e + \operatorname{sgn}(123) \cdot b \cdot c \cdot d + \operatorname{sgn}(132) \cdot c \cdot a \cdot e = 0$

Lemma 7.9. *Interchanging two rows of a matrix A , changes the sign of its determinant.*

Example 7.10. *Let*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

Now interchange the first and second rows to get matrix B:

$$B = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & 3 \\ 5 & 6 & 0 \end{bmatrix}$$

Then,
$\det(A) = 1(1\cdot 0 - 4\cdot 6) - 2(0\cdot 0 - 4\cdot 5) + 3(0\cdot 6 - 1\cdot 5) = -24 + 40 - 15 = 1$
$\det(B) = 0(2\cdot 0 - 3\cdot 6) - 1(1\cdot 0 - 3\cdot 5) + 4(1\cdot 6 - 2\cdot 5) = 0 + 15 - 16 = -1$

$$\det(B) = -\det(A) = -1$$

Hence, interchanging two rows reverses the sign of the determinant.

Corollary 7.11. *If the matrix B is obtained from A by a permutation of the rows of A, then $\det A = \pm \det B$, with the sign being $+1$ if the permutation is even, and -1 if the permutation is odd.*

Importantly, all the properties we've established for row operations apply equally to column operations, as $\det A = \det A^T$.

7.4 Multiplicative Property and Applications

Theorem 7.12. *For any matrices $A, B \in F_n$, $\det(AB) = (\det A)(\det B)$.*

Example 7.13. *Let*

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now compute the product AB:

$$AB = \begin{bmatrix} 1\cdot 2 + 2\cdot 1 + 0\cdot 0 & 1\cdot 0 + 2\cdot 1 + 0\cdot 0 & 0 \\ 0\cdot 2 + 1\cdot 1 + 0\cdot 0 & 0\cdot 0 + 1\cdot 1 + 0\cdot 0 & 0 \\ 3\cdot 2 + 4\cdot 1 + 1\cdot 0 & 3\cdot 0 + 4\cdot 1 + 1\cdot 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 0 \\ 1 & 1 & 0 \\ 10 & 4 & 1 \end{bmatrix}$$

Now computing the determinants:

$$\det(A) = 1(1\cdot 1 - 0\cdot 4) - 2(0\cdot 1 - 0\cdot 3) + 0(0\cdot 4 - 1\cdot 3) = 1$$

$$\det(B) = 2(1\cdot 1 - 0\cdot 0) - 0(1\cdot 1 - 0\cdot 0) + 0(1\cdot 0 - 1\cdot 0) = 2$$

$\det(AB) = 4(1\cdot 1 - 0\cdot 4) - 2(1\cdot 1 - 0\cdot 10) + 0(1\cdot 4 - 1\cdot 10) = 4 - 2 + 0 = 2$

Therefore,
$\det(AB) = \det(A)\det(B) = 1\cdot 2 = 2$

Corollary 7.14. *If A is invertible, then $\det A \neq 0$ and $\det(A^{-1}) = (\det A)^{-1}$.*

Example 7.15. *Let*

$$A = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$$

First, compute $\det(A)$*:*

$$\det(A) = (4)(6) - (7)(2) = 24 - 14 = 10$$

Now compute the inverse A^{-1}*:*

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

Now compute $\det(A^{-1})$*:*

$$\det(A^{-1}) = \left(\frac{1}{10}\right)^2 \cdot (6 \cdot 4 - (-7)(-2)) = \frac{1}{100}(24 - 14) = \frac{10}{100} = \frac{1}{10}$$

Thus,

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Corollary 7.16. *If* A *is invertible, then for any matrix* B*,* $\det(ABA^{-1}) = \det B$*.*

Example 7.17. *Let*

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$$

First, compute $\det(A)$*:*

$$\det(A) = 2 \cdot 1 - 1 \cdot 1 = 1$$

So A *is invertible.*

Now compute A^{-1}*:*

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \quad \textit{(since } \det(A) = 1\textit{, no need to divide)}$$

Now compute ABA^{-1}*:*

$$AB = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 3 & 5 \end{bmatrix}$$

$$ABA^{-1} = \begin{bmatrix} 6 & 5 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 \cdot 1 + 5 \cdot (-1) & 6 \cdot (-1) + 5 \cdot 2 \\ 3 \cdot 1 + 5 \cdot (-1) & 3 \cdot (-1) + 5 \cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -2 & 7 \end{bmatrix}$$

Now compute the determinant of ABA^{-1}*:*

$$\det(ABA^{-1}) = 1 \cdot 7 - 4 \cdot (-2) = 7 + 8 = 15$$

Now compute $\det(B)$*:*

$$\det(B) = 3 \cdot 5 = 15$$

$$\therefore \det(ABA^{-1}) = \det(B)$$

Lemma 7.18. $\det A = \det(A^T)$.

7.5 Cramer's Rule for Solving Linear Equations

Determinants provide an elegant method for solving systems of linear equations, known as Cramer's rule.

Consider the system of linear equations:

$$\begin{aligned} \alpha_{11}x_1 + \alpha_{12}x_2 + \cdots + \alpha_{1n}x_n &= \beta_1 \\ \alpha_{21}x_1 + \alpha_{22}x_2 + \cdots + \alpha_{2n}x_n &= \beta_2 \\ &\vdots \\ \alpha_{n1}x_1 + \alpha_{n2}x_2 + \cdots + \alpha_{nn}x_n &= \beta_n \end{aligned}$$

Theorem 7.19 (Cramer's Rule)**.** *If the determinant* Δ *of the system of linear equations is non-zero, then the solution of the system is given by* $x_i = \Delta_i/\Delta$*, where* Δ_i *is the determinant obtained from* Δ *by replacing in* Δ *the* i*-th column by* $\beta_1, \beta_2, \ldots, \beta_n$*.*

Example 7.20. *Consider the system of equations:*

$$\begin{aligned} 2x + y + z &= 8 \\ -3x - y + 2z &= -11 \\ -2x + y + 2z &= -3 \end{aligned}$$

This can be written in matrix form as $AX = B$*, where:*

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 8 \\ -11 \\ -3 \end{bmatrix}$$

Now, compute the determinant of A:

$$\det(A) = 2\begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} -3 & 2 \\ -2 & 2 \end{vmatrix} + 1\begin{vmatrix} -3 & -1 \\ -2 & 1 \end{vmatrix}$$

$$= 2((-1)(2) - (2)(1)) - 1((-3)(2) - (-2)(2)) + 1((-3)(1) - (-1)(-2))$$

$$= 2(-2-2) - 1(-6+4) + 1(-3-2) = 2(-4) - 1(-2) + 1(-5) = -8+2-5 = -11$$

Now, construct matrices A_1, A_2, and A_3 by replacing the 1st, 2nd, and 3rd columns of A with B, respectively:

$$A_1 = \begin{bmatrix} 8 & 1 & 1 \\ -11 & -1 & 2 \\ -3 & 1 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 8 & 1 \\ -3 & -11 & 2 \\ -2 & -3 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2 & 1 & 8 \\ -3 & -1 & -11 \\ -2 & 1 & -3 \end{bmatrix}$$

Now compute:

$$x = \frac{\det(A_1)}{\det(A)}, \quad y = \frac{\det(A_2)}{\det(A)}, \quad z = \frac{\det(A_3)}{\det(A)}$$

$$x = \frac{\det(A_1)}{\det(A)} = \frac{-30}{-11} = \frac{30}{11}, \quad y = \frac{\det(A_2)}{\det(A)} = \frac{-29}{-11} = \frac{29}{11}, \quad z = \frac{\det(A_3)}{\det(A)} = \frac{1}{-11} = -\frac{1}{11}$$

Final Answer

$$\boxed{x = \frac{30}{11}, \quad y = \frac{29}{11}, \quad z = -\frac{1}{11}}$$

7.6 Secular Equations and Cayley-Hamilton Theorem

Definition 7.21. *Given $A \in F_n$, the secular equation of A is the polynomial* $\det(xI - A)$ *in* $F[x]$.

The secular equation is often called the characteristic polynomial.

Example 7.22. *Given the matrix:*

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$$

The secular (characteristic) equation is found by solving:

$$\det(xI - A) = 0$$

First compute $xI - A$:

$$xI - A = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} x-2 & 0 \\ -1 & x-3 \end{bmatrix}$$

Now compute the determinant:

$$\det(xI - A) = (x-2)(x-3) - (0)(-1) = (x-2)(x-3)$$

The Secular Equation is

$$(x-2)(x-3) = 0$$

The Eigenvalues are

$$x = 2, \quad x = 3$$

Example 7.23. *For the matrix $A = \begin{pmatrix} 4 & 5 \\ 3 & 2 \end{pmatrix}$, we have:*

$$xI - A = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} - \begin{pmatrix} 4 & 5 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} x-4 & -5 \\ -3 & x-2 \end{pmatrix}$$

Taking the determinant:

$$\det(xI - A) = (x-4)(x-2) - (-5)(-3) = x^2 - 6x + 8 - 15 = x^2 - 6x - 7$$

So the secular equation of A is $x^2 - 6x - 7$.

The secular equation has several important properties:

1. If λ is a root of $\det(xI - A)$, then $\det(\lambda I - A) = 0$, which by our earlier theorem means $\lambda I - A$ is not invertible. Therefore, λ is a characteristic root of A.

2. Conversely, if λ is a characteristic root of A, then $\lambda I - A$ is not invertible, so $\det(\lambda I - A) = 0$, making λ a root of the secular equation.

3. Most importantly, the multiplicity of a root λ in the secular equation equals its multiplicity as a characteristic root of A.

Theorem 7.24. *The characteristic roots of A are the roots, with the correct multiplicities, of the secular equation* $\det(xI - A)$ *of A.*

This theorem can be proved using the properties of similar matrices and the schur triangularization theorem.

Theorem 7.25. *For any square matrix $A \in \mathbb{C}^{n\times n}$, there exists a unitary matrix U (i.e., $U^{-1} = U^*$, where U^* is the conjugate transpose of U) such that:*

$$UAU^{-1} = T$$

where T is an upper triangular matrix, and the diagonal entries of T are the eigenvalues of A.

Theorem 7.26 (Cayley-Hamilton)**.** *Every matrix $A \in F_n$ satisfies its secular equation.*

7.7 Additional Properties

7.7.1 Cofactor Matrix

Definition 7.27. *Given a matrix $A = (\alpha_{ij})$, let A_{ij} be the matrix obtained from A by removing the i-th row and j-th column. The cofactor of α_{ij} is defined as $M_{ij} = (-1)^{i+j} \det A_{ij}$.*

Theorem 7.28 (Cofactor Expansion)**.** *For any matrix $A = (\alpha_{ij})$:*

$$\det A = \sum_{j=1}^{n} \alpha_{ij} M_{ij} \quad \textit{for any fixed row } i$$

or

$$\det A = \sum_{i=1}^{n} \alpha_{ij} M_{ij} \quad \textit{for any fixed column } j$$

Proof. This follows from the multilinearity properties of determinants and the sign changes associated with row/column exchanges. □

Example 7.29. *Let's compute the determinant of* $A = \begin{pmatrix} 2 & 0 & 1 \\ 3 & -1 & 2 \\ 4 & 1 & 0 \end{pmatrix}$ *using cofactor expansion along the first row:*

$$M_{11} = (-1)^{1+1} \begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} = (-1)^2 \cdot ((-1) \cdot 0 - 2 \cdot 1) = -2$$

$$M_{12} = (-1)^{1+2}\begin{vmatrix} 3 & 2 \\ 4 & 0 \end{vmatrix} = (-1)^3 \cdot (3 \cdot 0 - 2 \cdot 4) = -(-8) = 8$$
$$M_{13} = (-1)^{1+3}\begin{vmatrix} 3 & -1 \\ 4 & 1 \end{vmatrix} = (-1)^4 \cdot (3 \cdot 1 - (-1) \cdot 4) = 7$$

Therefore:

$$\det A = 2 \cdot (-2) + 0 \cdot 8 + 1 \cdot 7 = -4 + 7 = 3$$

7.7.2 Block Matrices

For matrices with a special block structure, we can simplify determinant calculations.

Theorem 7.30. *If A and B are square submatrices, then:*

$$\det \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = (\det A)(\det B)$$

This result generalizes to block triangular matrices with multiple diagonal blocks.

7.7.3 Adjoint and Matrix Inversion

The cofactors of a matrix can be used to construct its inverse when the determinant is non-zero.

Definition 7.31. *The adjoint (or classical adjoint) of a matrix A is the transpose of its cofactor matrix:*

$$adj(A)_{ij} = M_{ji}$$

Theorem 7.32. *For any square matrix A:*

$$A \cdot adj(A) = adj(A) \cdot A = (\det A) \cdot I$$

Therefore, if $\det A \neq 0$, then:

$$A^{-1} = \frac{1}{\det A} \cdot adj(A)$$

This provides an explicit formula for the inverse of a matrix, though it's generally not the most computationally efficient method for large matrices.

7.7.4 Determinants in Inner Product Spaces

When working with inner product spaces, particularly over the complex field, determinants interact with important concepts like Hermitian matrices and unitary transformations.

Theorem 7.33. *If A is a Hermitian matrix (i.e., $A = A^*$, where A^* is the conjugate transpose), then all its eigenvalues are real.*

Theorem 7.34. *If A is a unitary matrix (i.e., $A^*A = AA^* = I$), then all its eigenvalues have absolute value 1.*

These properties have important consequences for the determinants of such matrices:

Corollary 7.35. *The determinant of a Hermitian matrix is real.*

Corollary 7.36. *The determinant of a unitary matrix has absolute value 1.*

8 Some Examples

8.1 Question 1: Number of Even Permutations Sending i to j

*Fix integers $i, j \in \{1, 2, \ldots, n\}$. How many **even permutations** $\sigma \in S_n$ satisfy $\sigma(i) = j$?*

Solution

We begin by recalling a few key facts:

- *The total number of permutations in S_n is $n!$.*
- *Exactly half of these are even, so the number of even permutations is $\frac{n!}{2}$.*
- *The number of permutations (regardless of parity) that send a fixed i to a fixed j is $(n-1)!$. This is because once we fix $\sigma(i) = j$, the remaining $n-1$ elements can be mapped arbitrarily to the remaining $n-1$ values.*

Now we want to count how many of these $(n-1)!$ permutations are even. *Observe:*

- *The group S_n acts transitively on the set of positions, and the sign function* $\operatorname{sgn} : S_n \to \{\pm 1\}$ *is a group homomorphism.*
- *For any fixed i and j, the set $\{\sigma \in S_n : \sigma(i) = j\}$ is in bijection with S_{n-1}, the permutations of the remaining $n-1$ elements.*
- *Since S_{n-1} also has half even and half odd permutations, exactly half of the $(n-1)!$ permutations that send i to j will be even.*

Answer:

$$\boxed{\frac{(n-1)!}{2}}$$

even permutations in S_n send i to j.

Question 2: Even Permutations Centralizing a 3-Cycle

Let $\tau = (1\ 2\ 3) \in S_n$ for $n \geq 3$. How many even permutations $\sigma \in A_n$ satisfy:

$$\sigma\tau\sigma^{-1} = \tau \ ?$$

Solution

We want the number of ***even*** *permutations in S_n that commute with $\tau = (1\ 2\ 3)$, i.e., the number of even elements in the centralizer $C_{S_n}(\tau)$.*

Step 1: Describe the Centralizer in S_n

- τ acts nontrivially only on the subset $\{1, 2, 3\}$. - The centralizer consists of permutations that:

- *act as a power of τ on $\{1, 2, 3\}$, and*
- *can act arbitrarily on $\{4, \ldots, n\}$.*

- There are exactly 3 such powers of τ: $id, (1\ 2\ 3), (1\ 3\ 2)$, all of which are ***even permutations****.*

- On $\{4, \ldots, n\}$, we can permute arbitrarily: $(n-3)!$ ways.

So total number of permutations commuting with τ in S_n is:

$$|C_{S_n}(\tau)| = 3 \cdot (n-3)!$$

Step 2: Count the Even Ones

- For each power of τ, which is even, the sign of the whole permutation depends only on the sign of the permutation of $\{4, \ldots, n\}$. - Half of the permutations of $(n-3)!$ are even.

So for each power of τ, there are:

$$\frac{(n-3)!}{2} \text{ even permutations}$$

Hence, total number of even permutations that centralize τ is:

$$\boxed{3 \cdot \frac{(n-3)!}{2} = \frac{3(n-3)!}{2}}$$

Question 3: Number of Even Derangements

*Let D_n denote the number of derangements of n elements (i.e., permutations $\sigma \in S_n$ such that $\sigma(i) \neq i$ for all i). How many of these are **even permutations**?*

Solution

Step 1: Total Number of Derangements

- The number of derangements of n elements is:

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} = \left\lfloor \frac{n!}{e} + \frac{1}{2} \right\rfloor$$

Step 2: Parity Argument

- The set of derangements is closed under taking inverses: if σ is a derangement, so is σ^{-1}. - The sign of a permutation equals the sign of its inverse:

$$\operatorname{sgn}(\sigma^{-1}) = \operatorname{sgn}(\sigma)$$

- Hence, derangements split evenly into even and odd permutations (provided D_n is even, which it always is for $n \geq 2$).

Final Answer:

$$\boxed{\frac{D_n}{2}}$$

even derangements in S_n for $n \geq 2$.

9 Applications

The sign of a permutation has far-reaching applications across mathematics and physics. In this section, we briefly outline some of these applications.

9.1 Determinants and Matrix Theory

The connection between permutation signs and determinants is fundamental to linear algebra. The determinant formula:

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)} \tag{39}$$

provides a connection between combinatorial properties of permutations and algebraic properties of matrices.

This connection leads to important results such as:

- *The determinant of a product of matrices is the product of their determinants.*
- *A matrix is invertible if and only if the determinant is non-zero.*
- *The determinant of a matrix is equal to the the product of its eigenvalues (counting multiplicity).*

9.2 Alternating Polynomials and Schur Functions

The alternating polynomial $\prod_{i<j}(X_j - X_i)$ we looked into earlier is the Vandermonde determinant, a fundamental object in the theory of symmetric functions. More generally, alternating polynomials (polynomials that change sign under odd permutations) play an important role in representation theory and algebraic combinatorics.

Schur functions, defined as ratios of alternating polynomials, form an important basis for the ring of symmetric functions and have various applications in representation theory, algebraic geometry, and mathematical physics.

9.3 Quantum Mechanics

In quantum mechanics, the wave function of a system of identical particles must transform in specific ways under particle exchange. For bosons (like photons), the wave function remains unchanged, and for fermions (like electrons), it acquires a factor of -1.

This corresponds mathematically to the wave function that transforms it according to either the trivial representation or the sign representation of the symmetric group. The Pauli exclusion principle, states that no two fermions can occupy the same quantum state, and is a direct consequence of this anti-symmetry requirement.

9.4 Galois Theory

In Galois theory, the solvability of a polynomial equation is related to the structure of permutation groups. The fact that A_n is a simple group for $n \geq 5$ leads to the non-solvability of the general quintic equation. The difference between even and odd permutations plays a crucial role in understanding which polynomials have solutions that can be expressed in radicals.

9.5 Cryptography and Coding Theory

Permutations play a significant role in modern cryptography:

- *Permutation-based block ciphers use permutations as a core component of their design*
- *The security of certain cryptographic schemes relies on the computational difficulty of specific problems in permutation groups*
- *Error-correcting codes often employ permutation groups to analyze code properties*

Example 9.1 (S-Boxes in Cryptography)**.** *In block ciphers such as AES (Advanced Encryption Standard), substitution boxes (S-boxes) implement non-linear transformations that can be described using permutations. The design of these S-boxes often considers algebraic properties related to permutation groups to resist cryptanalysis.*

9.6 Quantum Computing

The theory of permutation groups has connections to quantum information theory:

- *Quantum error-correcting codes can be constructed using properties of permutation groups*
- *Certain quantum algorithms exploit group-theoretic properties, including those of permutation groups*

- *The representation theory of the symmetric group provides tools for analyzing quantum systems with identical particles*

These modern applications demonstrate the continuing relevance and vitality of permutation theory and the sign function in contemporary mathematics and its applications.

9.7 Surprising Number-Theoretic Connections

The sign function reveals unexpected links to number theory through various counting problems and congruences.

Wilson's theorem states that for any prime p:

$$(p-1)! \equiv -1 \pmod{p} \tag{40}$$

This can be reinterpreted in terms of permutation signs: the product of all permutations in S_{p-1} has sign $(-1)^{p-1}$ when viewed in a certain way, providing a group-theoretic perspective on this classical number-theoretic result.

9.8 Other Concepts

Several other interesting concepts include the following:

Computational aspects: *Algorithms for efficiently computing the sign of a permutation and finding the minimal transposition factorizations have practical applications in computational group theory.*

Generalized sign characters: *The sign function is a group character of S_n. The representation theory of the symmetric group includes studying more general characters and their properties.*

Sign functions on other groups: *The concept of parity can be extended to other groups, such as Coxeter groups, where elements can be expressed as products of reflections.*

10 Conclusion

The sign of a permutation stands as a fundamental concept in group theory, with rich connections to various areas of mathematics and physics. This report has explored its definition, properties, and different ways of understanding it, including its relation to transpositions, determinants, and permutation matrices. The sign function provides a simple yet powerful classification of permutations into even and odd, leading to the formation of the alternating

group,a subgroup of the symmetric group. We have explored multiple perspectives on this concept:

- *As the parity of the number of transpositions in a factorization*
- *Through the alternating group, which comprises all even permutations*
- *Via determinants of permutation matrices*
- *As the unique non-trivial homomorphism from S_n to $\{+1, -1\}$*

The fact that such a simple function on permutations exhibits so many different characterizations and applications demonstrates its profound nature. The sign function reveals structural properties of the symmetric group and provides a bridge between combinatorial group theory and linear algebra, representation theory, and mathematical physics.

Through these diverse perspectives, we gain a deeper understanding of both the sign function itself and structure of the symmetric group, one of the most fundamental objects in all of mathematics.

The determinant function stands out for its elegant mathematical properties:

- *It preserves multiplication:* $\det(AB) = (\det A)(\det B)$
- *It characterizes invertibility: A matrix is invertible if and only if its determinant is non-zero*
- *It connects to eigenvalues: The determinant equals the product of all eigenvalues (counting multiplicities)*
- *It has geometric meaning as a volume scaling factor*

These properties make determinants indispensable in various mathematical and applied fields, from solving systems of linear equations to understanding transformations in geometry, analyzing stability in differential equations, and characterizing important classes of matrices in physics and engineering applications.

11 References

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