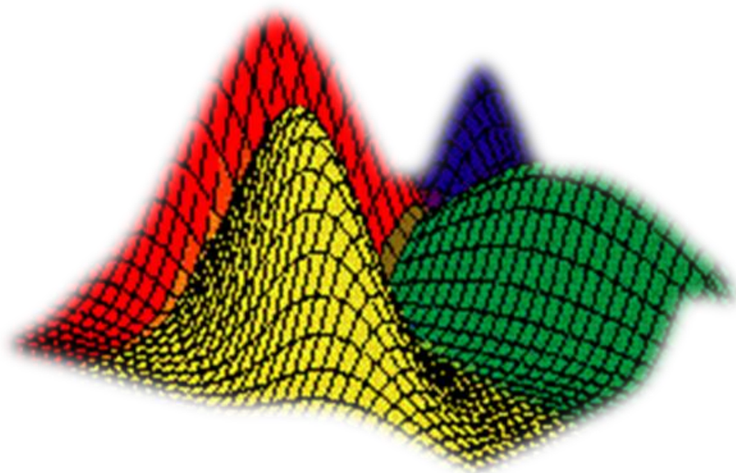
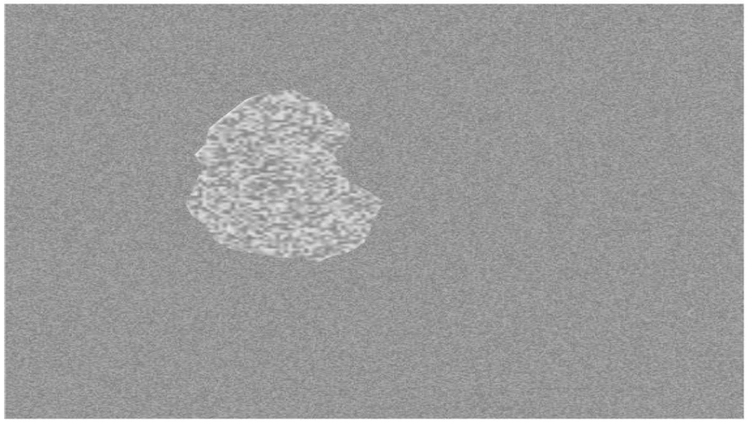


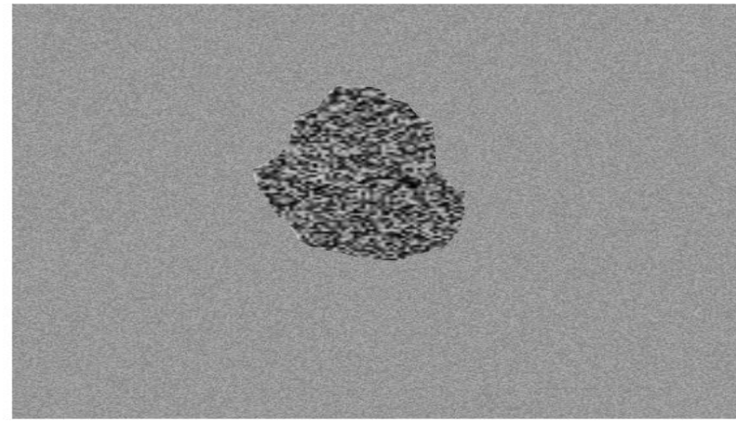
CSE 473
Pattern Recognition

Bayesian Classifier and its Variants

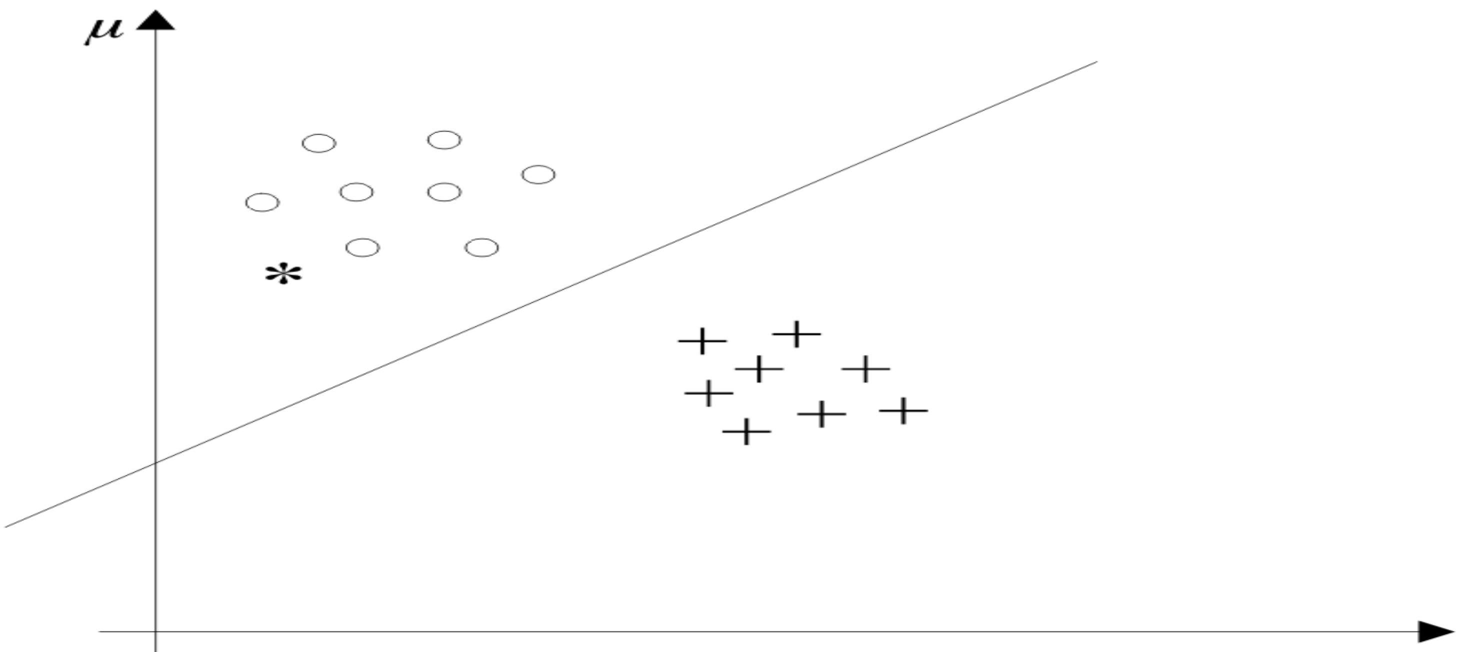
Recall from Lecture 1



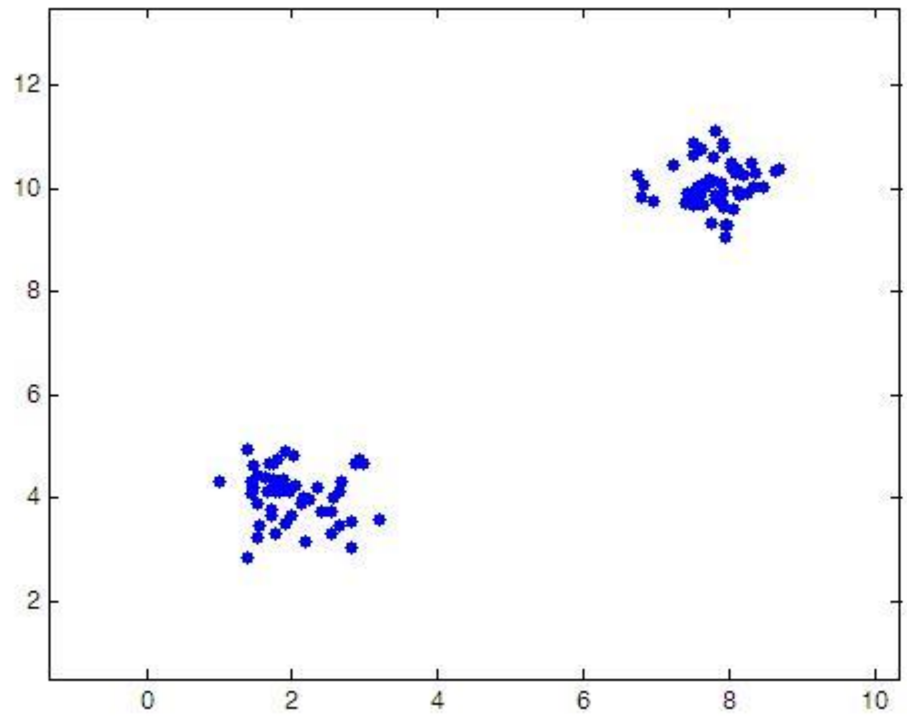
(a)



(b)



Sample Data for Sessional on Bayesian Classification



Sample Data for Bayesian Classification

Feature 1	Feature 2	Class
1.7044	3.6651	1
1.6726	4.6705	1
1.4597	4.194	1
1.9761	4.1965	1
1.9126	3.4987	1
1.5214	3.9072	1
2.6463	3.473	1
2.2205	3.9642	1
6.8104	10.0517	2
7.5809	9.8897	2
8.1287	9.8605	2
7.9081	9.6332	2
7.9162	9.9677	2
7.9415	9.278	2
8.0842	10.3062	2
7.7494	9.3382	2
8.1146	9.9617	2

Naïve Bayes (Summary)

- Simplify the probability expression
- Robust to
  - isolated noise points
  - missing values
  - irrelevant attributes

Naïve Bayes (Issues)

- Over simplification
  - Use other techniques such as Bayesian Belief Networks (BBN)

Bayesian Belief Networks

- Let we have *l* random variables
- The joint probability is given by,

$$p(x_1, x_2, \ldots, x_\ell) = p(x_\ell \mid x_{\ell-1}, \ldots, x_1) \cdot p(x_{\ell-1} \mid x_{\ell-2}, \ldots, x_1) \cdot \ldots$$

$$\ldots \cdot p(x_2 \mid x_1) \cdot p(x_1)$$

Bayesian Belief Networks

The formula

$$p(x_1, x_2, \ldots, x_\ell) = p(x_\ell \mid x_{\ell-1}, \ldots, x_1) \cdot p(x_{\ell-1} \mid x_{\ell-2}, \ldots, x_1) \cdot \ldots$$
$$\ldots \cdot p(x_2 \mid x_1) \cdot p(x_1)$$

can be written as

$$p(x_1, x_2, \ldots, x_\ell) = p(x_1) \cdot \prod_{i=2}^{\ell} p(x_i \mid A_i)$$

where

$$A_i \subseteq \{x_{i-1}, x_{i-2}, \ldots, x_1\}$$

– For example, if ℓ=6, then we could assume:

$$p(x_6 \mid x_5, \ldots, x_1) = p(x_6 \mid x_5, x_4)$$

Then:

$$A_6 = \{x_5, x_4\} \subseteq \{x_5, \ldots, x_1\}$$

– Simialrly, if we assume

$$p(x_5|x_4,\ldots,x_1) = p(x_5|x_4)$$

$$p(x_4|x_3,x_2,x_1) = p(x_4|x_2,x_1)$$

$$p(x_3|x_2,x_1) = p(x_3|x_2)$$

$$p(x_2|x_1) = p(x_2)$$

Then:

$$A_5 = \{x_4\},\ A_4 = \{x_2,x_1\},\ A_3 = \{x_2\}, A_2 = \emptyset$$

– Simialrly, if we assume

$$p(x_5|x_4, \ldots, x_1) = p(x_5|x_4)$$

$$p(x_4|x_3, x_2, x_1) = p(x_4|x_2, x_1)$$

$$p(x_3|x_2, x_1) = p(x_3|x_2)$$

$$p(x_2|x_1) = p(x_2)$$

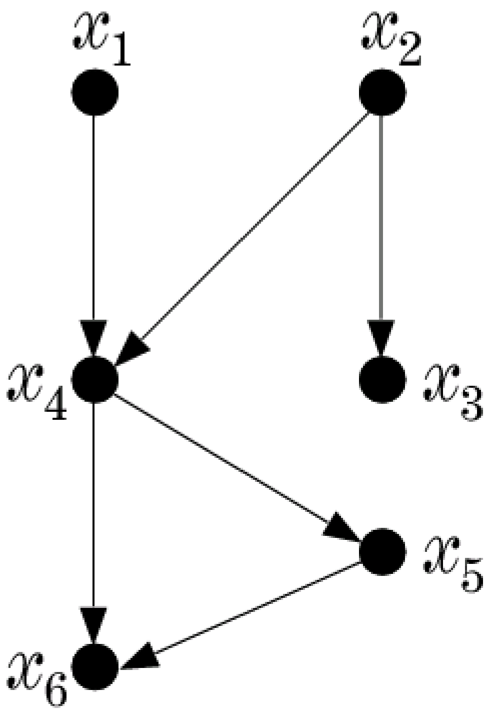
Then:

$$A_5 = \{x_4\},\ A_4 = \{x_2, x_1\},\ A_3 = \{x_2\}, A_2 = \emptyset$$

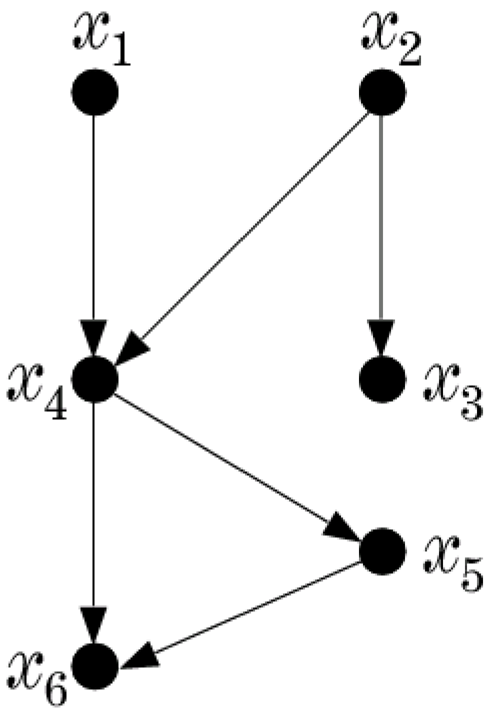
– The above is a generalization of the Naïve – Bayes. For the Naïve – Bayes the assumption is:

$$A_i = \emptyset, \text{ for } i=1, 2, \ldots, \ell$$

– A graphical way to portray conditional dependencies

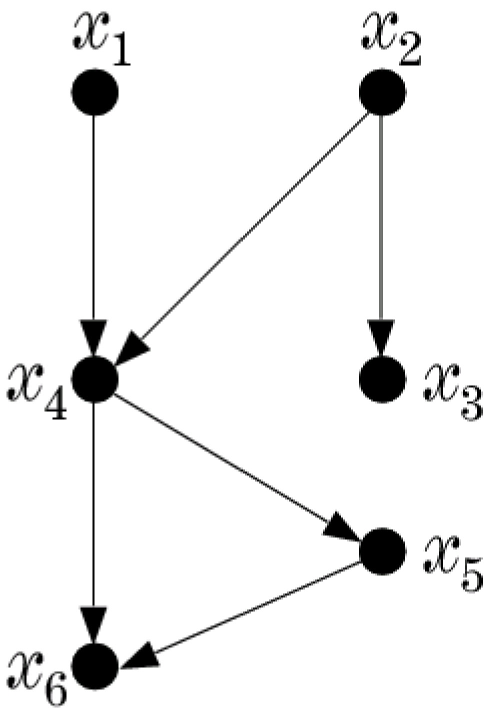


– A graphical way to portray conditional dependencies



- According to this figure, we have :
  - x_6 is conditionally dependent on x_4, x_5.
  - x_5 on x_4
  - x_4 on x_1, x_2
  - x_3 on x_2
  - x_1, x_2 are conditionally independent on other variables.

– A graphical way to portray conditional dependencies



- According to this figure, we have :
  - x_6 is conditionally dependent on x_4, x_5.
  - x_5 on x_4
  - x_4 on x_1, x_2
  - x_3 on x_2
  - x_1, x_2 are conditionally independent on other variables.

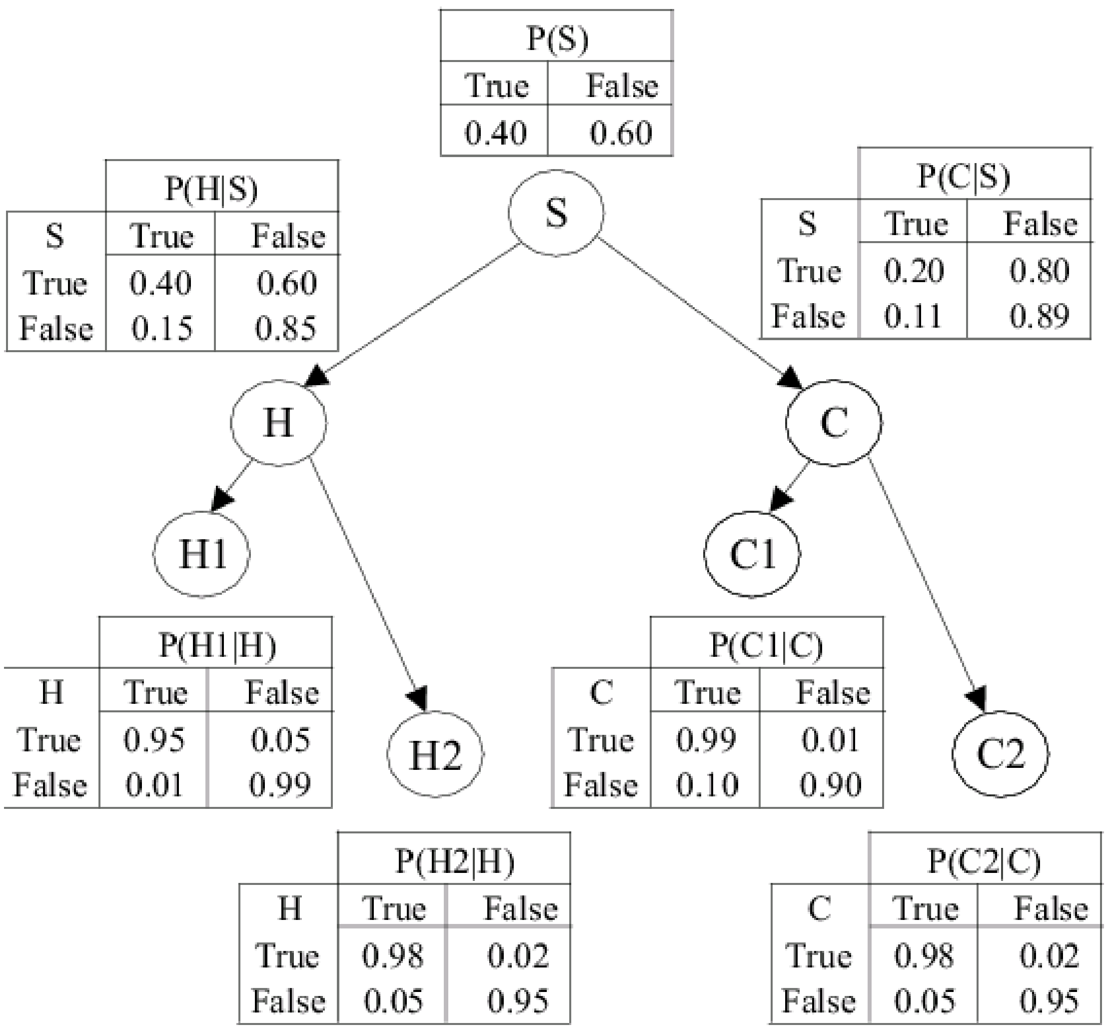
- For this case:

$$p(x_1, x_2, \ldots, x_6) = p(x_6 \mid x_5, x_4) \cdot p(x_5 \mid x_4) \cdot p(x_3 \mid x_2) \cdot p(x_2) \cdot p(x_1)$$

- Bayesian Networks
  - a directed acyclic graph (DAG)
  - the nodes correspond to random variables
  - arc represents parent-child (*dependence*) relationship

- A Bayesian Network is specified by:
  - The prior probabilities of its root nodes.
  - The conditional probabilities of the non-root nodes, given their parents, for ALL possible combinations.

– A Bayesian Network from a medical application



P(S)	
True	False
0.40	0.60

	P(H|S)	
S	True	False
True	0.40	0.60
False	0.15	0.85

	P(C|S)	
S	True	False
True	0.20	0.80
False	0.11	0.89

	P(H1|H)	
H	True	False
True	0.95	0.05
False	0.01	0.99

	P(C1|C)	
C	True	False
True	0.99	0.01
False	0.10	0.90

	P(H2|H)	
H	True	False
True	0.98	0.02
False	0.05	0.95

	P(C2|C)	
C	True	False
True	0.98	0.02
False	0.05	0.95

- This BBN models conditional dependencies concerning smokers' (S), tendencies to develop cancer (C) and heart disease (H), together with variables corresponding to heart (H1, H2) and cancer (C1, C2) medical tests

- Training: given a topology, probabilities are estimated from training data. There are also methods that learn the topology.

- any joint probability can be obtained by multiplying the prior (root nodes) and the conditional (non-root nodes) probabilities.

- Probability Inference: Given a pattern (evidence), the goal is to compute the conditional probabilities for some of the other variables (class)

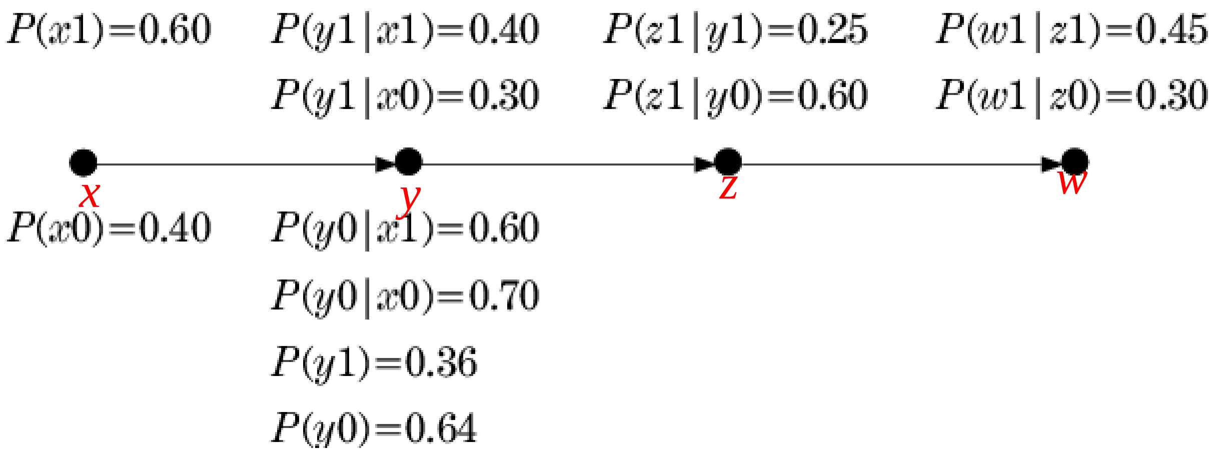
- Example: Consider the Bayesian network of the figure:

$P(x1) = 0.60$ $\quad P(y1|x1) = 0.40$ $\quad P(z1|y1) = 0.25$ $\quad P(w1|z1) = 0.45$

$P(y1|x0) = 0.30$ $\quad P(z1|y0) = 0.60$ $\quad P(w1|z0) = 0.30$

- Random variables: *x*, *y*, *z*, *w*
- *x*0 means *x* = 0
- *x*1 means *x* = 1

- We can calculate the other probabilities



Example: $p(y_1)$:

$$P(y1) = \sum_x P(y1, x) = P(y1, x1) + P(y1, x0)$$

$$P(y1) = P(y1|x1)P(x1) + P(y1|x0)P(x0) = (0.4)(0.6) + (0.3)(0.4) = 0.36$$

- We can calculate the other probabilities

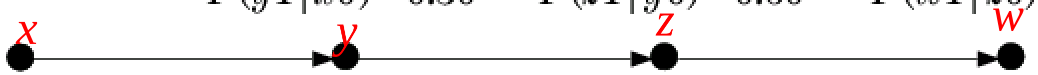
$P(x1)=0.60$	$P(y1\,|\,x1)=0.40$	$P(z1\,|\,y1)=0.25$	$P(w1\,|\,z1)=0.45$
	$P(y1\,|\,x0)=0.30$	$P(z1\,|\,y0)=0.60$	$P(w1\,|\,z0)=0.30$

$P(x0)=0.40$	$P(y0\,|\,x1)=0.60$	$P(z0\,|\,y1)=0.75$	$P(w0\,|\,z1)=0.55$
	$P(y0\,|\,x0)=0.70$	$P(z0\,|\,y0)=0.40$	$P(w0\,|\,z0)=0.70$
	$P(y1)=0.36$	$P(z1)=0.47$	$P(w1)=0.37$
	$P(y0)=0.64$	$P(z0)=0.53$	$P(w0)=0.63$

- Given this info, we can answer any probabilistic query:

$P(x1)=0.60$ $P(y1|x1)=0.40$ $P(z1|y1)=0.25$ $P(w1|z1)=0.45$

$P(y1|x0)=0.30$ $P(z1|y0)=0.60$ $P(w1|z0)=0.30$



$P(x0)=0.40$ $P(y0|x1)=0.60$ $P(z0|y1)=0.75$ $P(w0|z1)=0.55$

$P(y0|x0)=0.70$ $P(z0|y0)=0.40$ $P(w0|z0)=0.70$

$P(y1)=0.36$ $P(z1)=0.47$ $P(w1)=0.37$

$P(y0)=0.64$ $P(z0)=0.53$ $P(w0)=0.63$

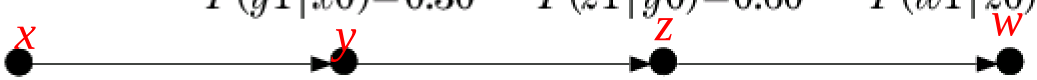
a) If x is measured to be $x=1$ ($x1$), compute $P(\text{z}1|\text{x}1)$ *and* $P(w0|x1)$.

b) If w is measured to be $w=1$ ($w1$) compute $P(\text{z}1|\text{w}1)$].

a) If x is measured to be $x=1$ ($x1$), compute P(z1|x1) and $P(w0|x1)$.

$P(x1)=0.60$ $\quad P(y1|x1)=0.40$ $\quad P(z1|y1)=0.25$ $\quad P(w1|z1)=0.45$

$P(y1|x0)=0.30$ $\quad P(z1|y0)=0.60$ $\quad P(w1|z0)=0.30$



$P(x0)=0.40$ $\quad P(y0|x1)=0.60$ $\quad P(z0|y1)=0.75$ $\quad P(w0|z1)=0.55$

$P(y0|x0)=0.70$ $\quad P(z0|y0)=0.40$ $\quad P(w0|z0)=0.70$

$P(y1)=0.36$ $\quad P(z1)=0.47$ $\quad P(w1)=0.37$

$P(y0)=0.64$ $\quad P(z0)=0.53$ $\quad P(w0)=0.63$

$$
\begin{aligned}
P(z1|x1) &= P(z1|y1, x1)P(y1|x1) + P(z1|y0, x1)P(y0|x1) \\
&= P(z1|y1)P(y1|x1) + P(z1|y0)P(y0|x1) \\
&= (0.25)(0.4) + (0.6)(0.6) = 0.46
\end{aligned}
$$

a) If x is measured to be $x=1$ ($x1$), compute P(z1|x1) and $P(w0|x1)$.

$P(x1)=0.60$ $P(y1|x1)=0.40$ $P(z1|y1)=0.25$ $P(w1|z1)=0.45$

$P(y1|x0)=0.30$ $P(z1|y0)=0.60$ $P(w1|z0)=0.30$

$P(x0)=0.40$ $P(y0|x1)=0.60$ $P(z0|y1)=0.75$ $P(w0|z1)=0.55$

$P(y0|x0)=0.70$ $P(z0|y0)=0.40$ $P(w0|z0)=0.70$

$P(y1)=0.36$ $P(z1)=0.47$ $P(w1)=0.37$

$P(y0)=0.64$ $P(z0)=0.53$ $P(w0)=0.63$

$$P(w0|x1) = P(w0|z1, x1)P(z1|x1) + P(w0|z0, x1)P(z0|x1)$$

$$= P(w0|z1)P(z1|x1) + P(w0|z0)P(z0|x1)$$

$$= (0.55)(0.46) + (0.7)(0.54) = 0.63$$

b) If w is measured to be w=1 (w1) compute P(z1|w1|)].

$P(x1)=0.60$ $\quad$ $P(y1|x1)=0.40$ $\quad$ $P(z1|y1)=0.25$ $\quad$ $P(w1|z1)=0.45$

$P(y1|x0)=0.30$ $\quad$ $P(z1|y0)=0.60$ $\quad$ $P(w1|z0)=0.30$

$P(x0)=0.40$ $\quad$ $P(y0|x1)=0.60$ $\quad$ $P(z0|y1)=0.75$ $\quad$ $P(w0|z1)=0.55$

$P(y0|x0)=0.70$ $\quad$ $P(z0|y0)=0.40$ $\quad$ $P(w0|z0)=0.70$

$P(y1)=0.36$ $\quad$ $P(z1)=0.47$ $\quad$ $P(w1)=0.37$

$P(y0)=0.64$ $\quad$ $P(z0)=0.53$ $\quad$ $P(w0)=0.63$

$$P(z1|w1) = \frac{P(w1|z1)P(z1)}{P(w1)} = \frac{(0.45)(0.47)}{0.37} = 0.57$$

c) CAN WE CALCULATE P(x0|w1|)]?

$P(x1)=0.60$ $P(y1|x1)=0.40$ $P(z1|y1)=0.25$ $P(w1|z1)=0.45$

$P(y1|x0)=0.30$ $P(z1|y0)=0.60$ $P(w1|z0)=0.30$

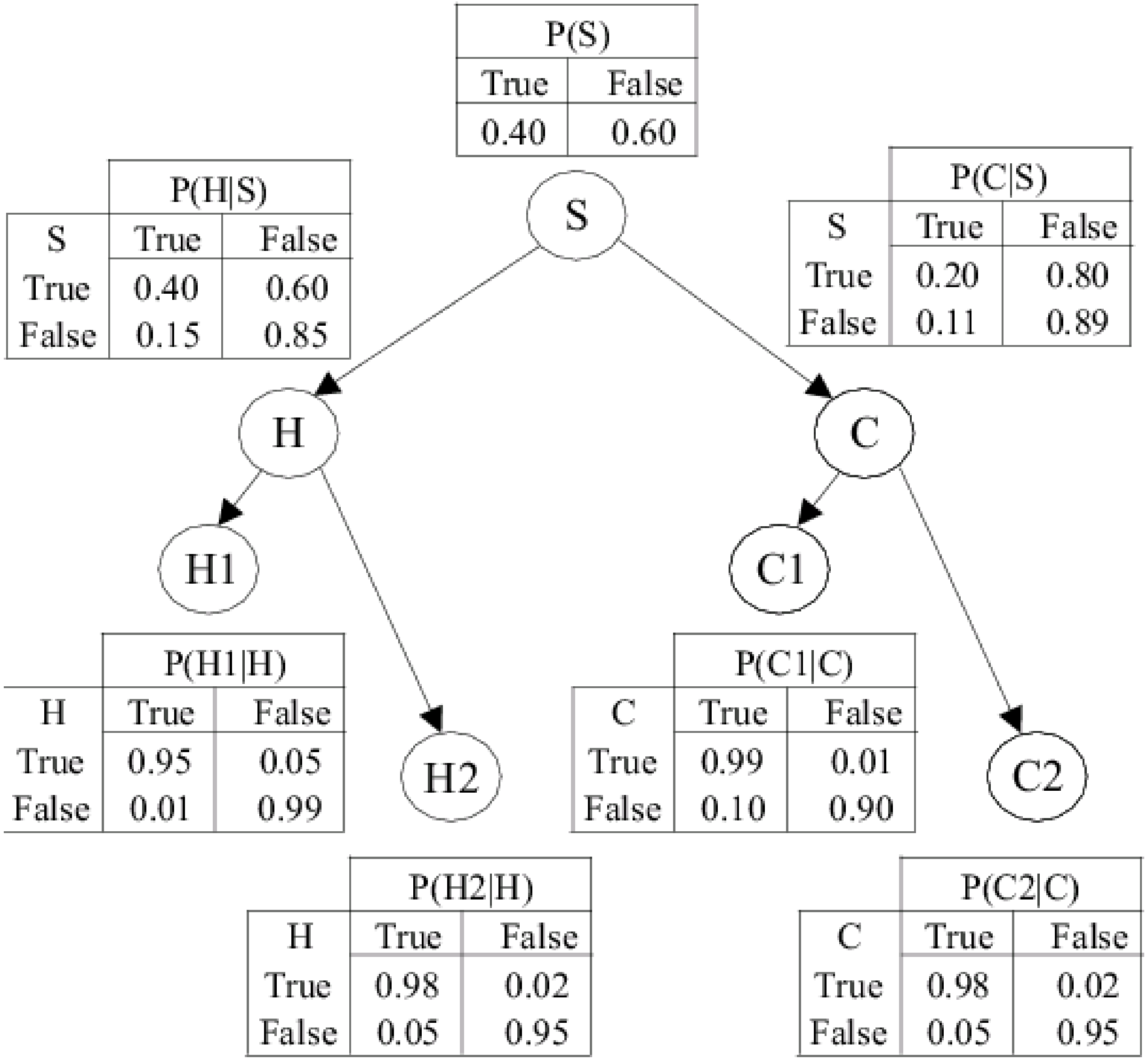
$P(x0)=0.40$ $P(y0|x1)=0.60$ $P(z0|y1)=0.75$ $P(w0|z1)=0.55$

$P(y0|x0)=0.70$ $P(z0|y0)=0.40$ $P(w0|z0)=0.70$

$P(y1)=0.36$ $P(z1)=0.47$ $P(w1)=0.37$

$P(y0)=0.64$ $P(z0)=0.53$ $P(w0)=0.63$

What’s about more complex networks?



P(S)	
True	False
0.40	0.60

	P(H|S)	
S	True	False
True	0.40	0.60
False	0.15	0.85

	P(C|S)	
S	True	False
True	0.20	0.80
False	0.11	0.89

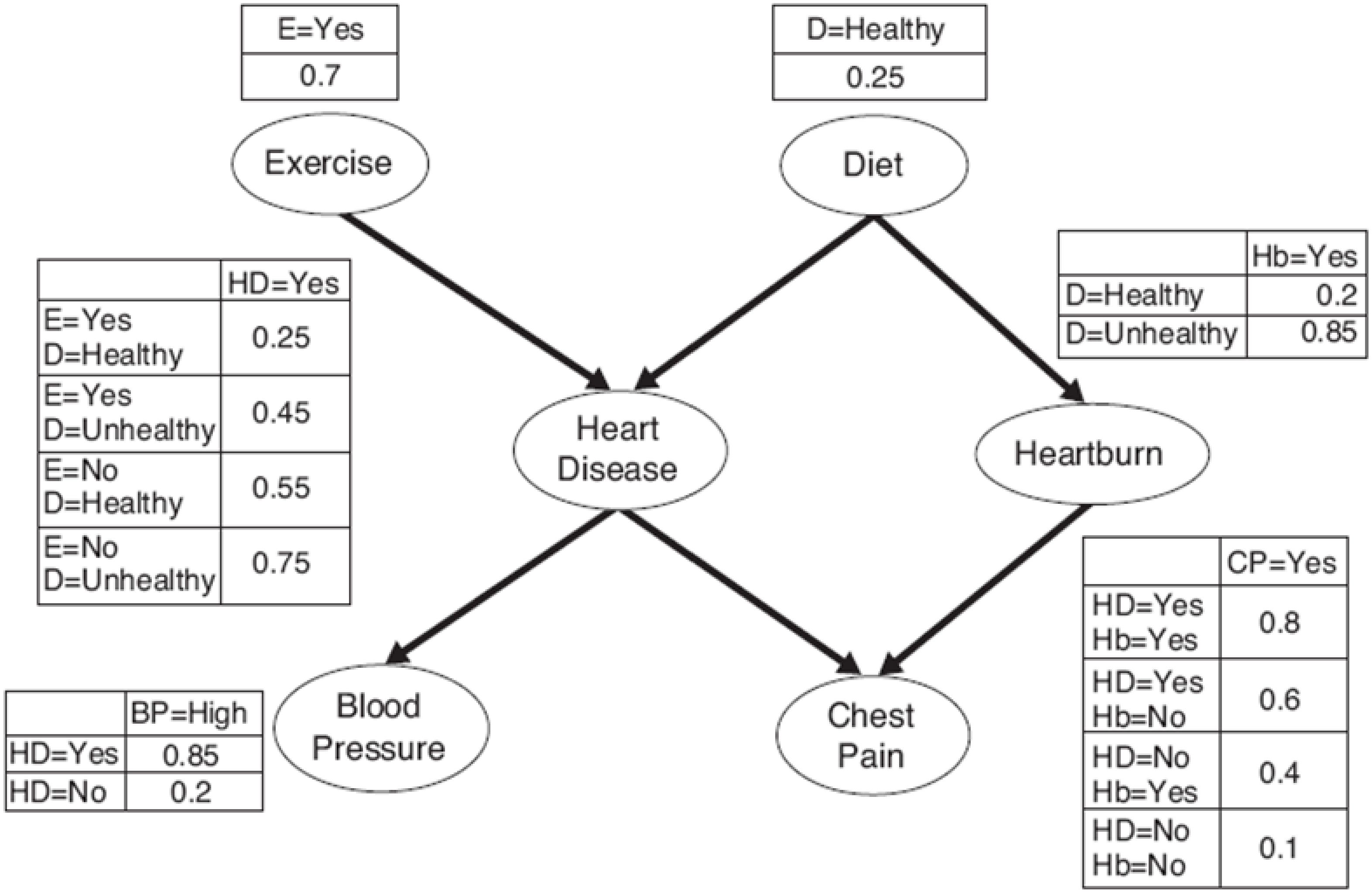
	P(H1|H)	
H	True	False
True	0.95	0.05
False	0.01	0.99

	P(C1|C)	
C	True	False
True	0.99	0.01
False	0.10	0.90

	P(H2|H)	
H	True	False
True	0.98	0.02
False	0.05	0.95

	P(C2|C)	
C	True	False
True	0.98	0.02
False	0.05	0.95

What's about more complex networks?



E=Yes
0.7

D=Healthy
0.25

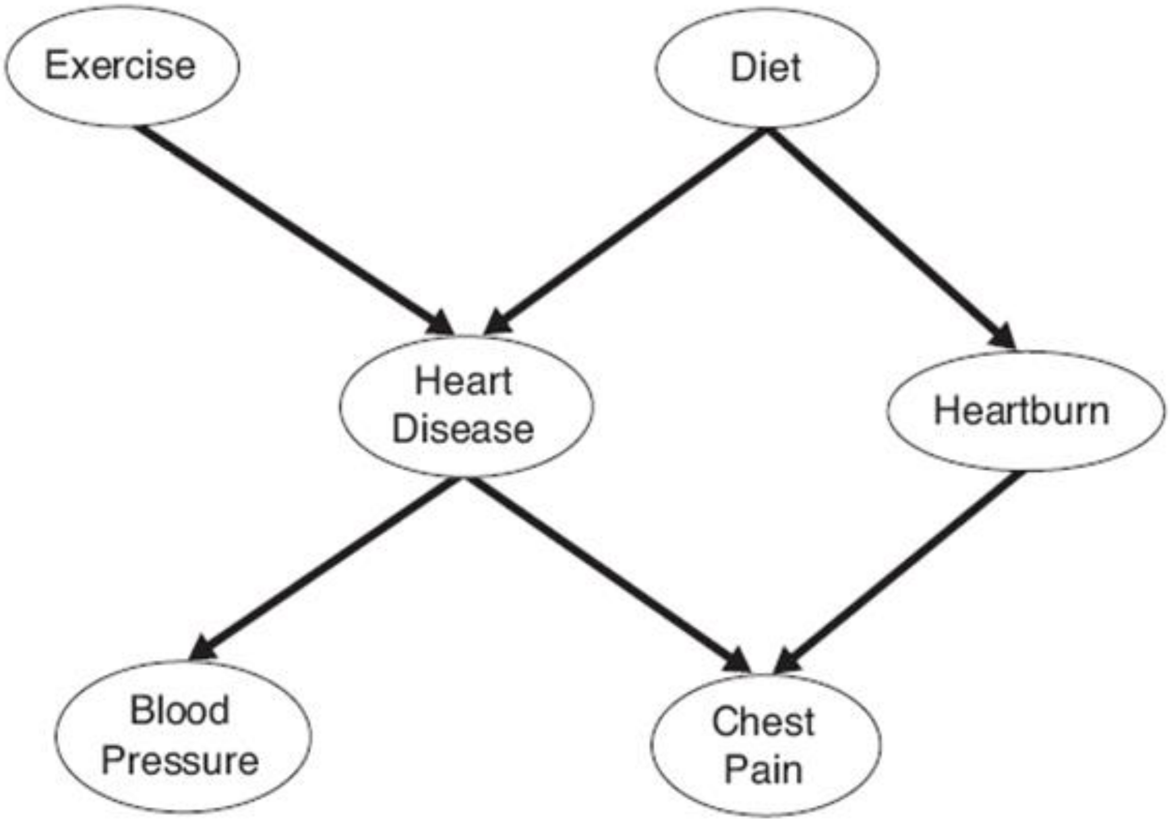
	HD=Yes
E=Yes D=Healthy	0.25
E=Yes D=Unhealthy	0.45
E=No D=Healthy	0.55
E=No D=Unhealthy	0.75

	Hb=Yes
D=Healthy	0.2
D=Unhealthy	0.85

	BP=High
HD=Yes	0.85
HD=No	0.2

	CP=Yes
HD=Yes Hb=Yes	0.8
HD=Yes Hb=No	0.6
HD=No Hb=Yes	0.4
HD=No Hb=No	0.1

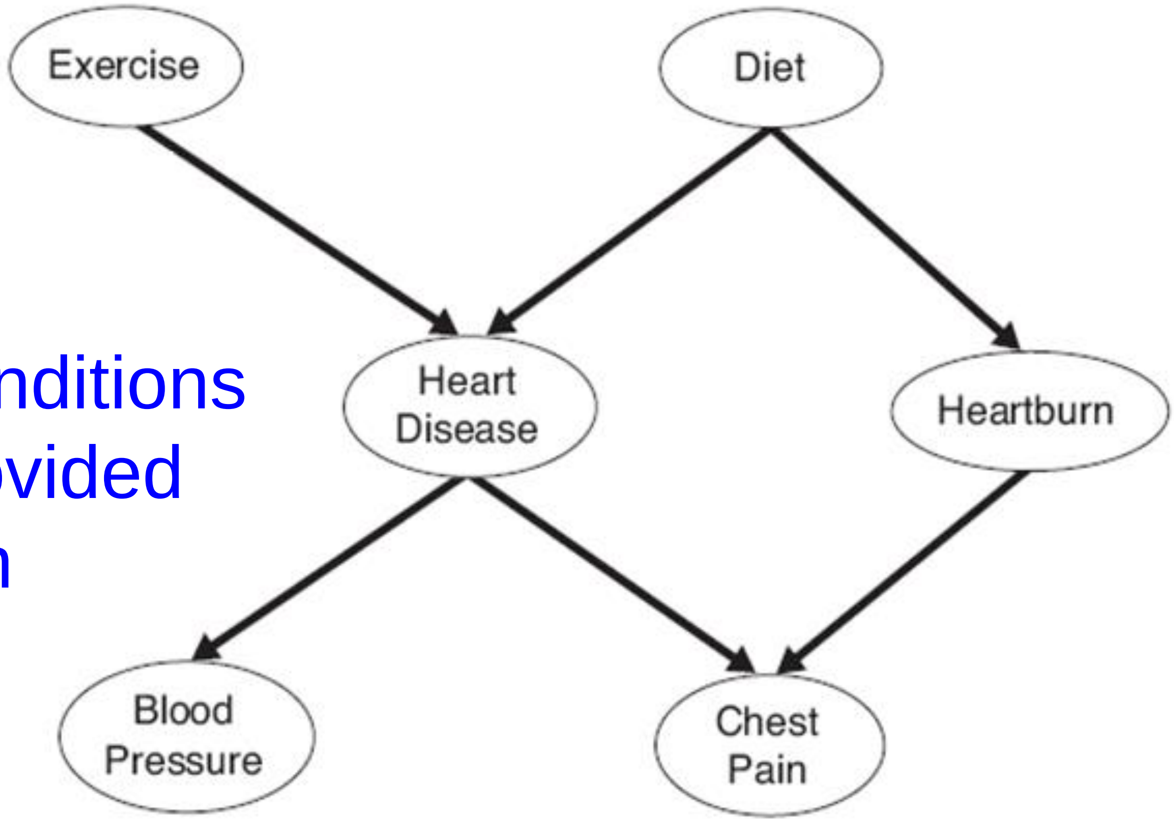
We will study this graph



We can show:

- $P(D|E)=P(D)$
- $P(Hb|HD, E, D)= P(Hb|D)$
- $P(CP|Hb, HD, E, D)= P(CP|Hb, HD)$
- $P(BP|CP, Hb, HD, E, D)= P(BP|HD)$
- However, $P(HD|E,D)$ cannot be simplified

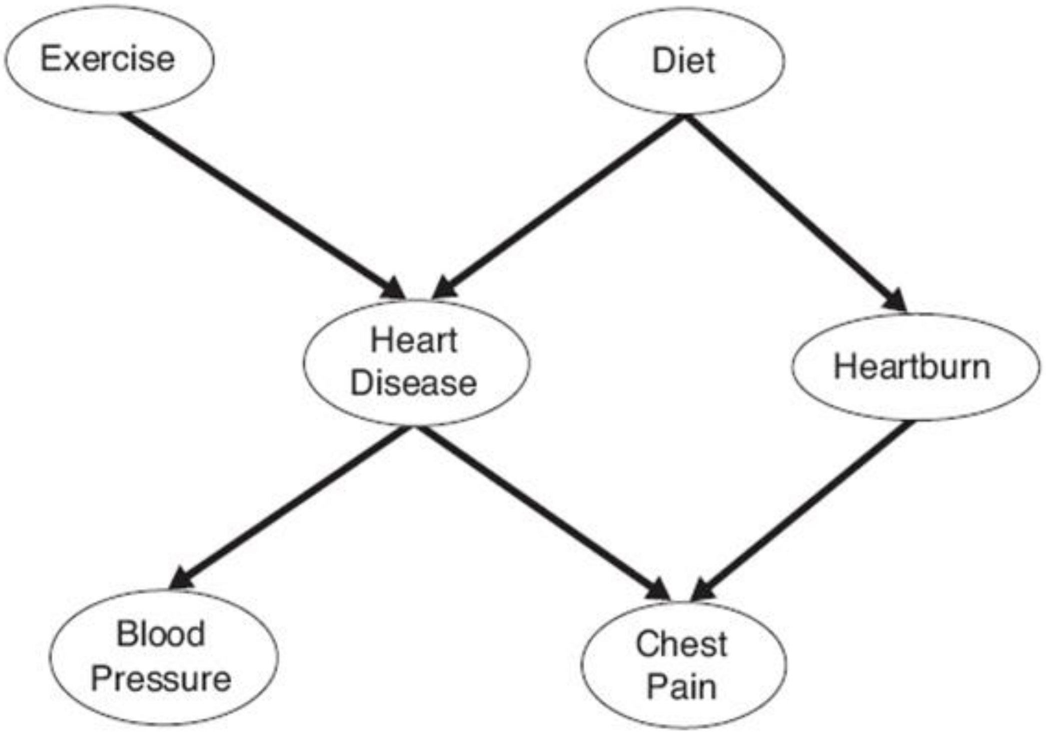
We will study this graph



Non-descendant conditions can be removed provided all parents are given

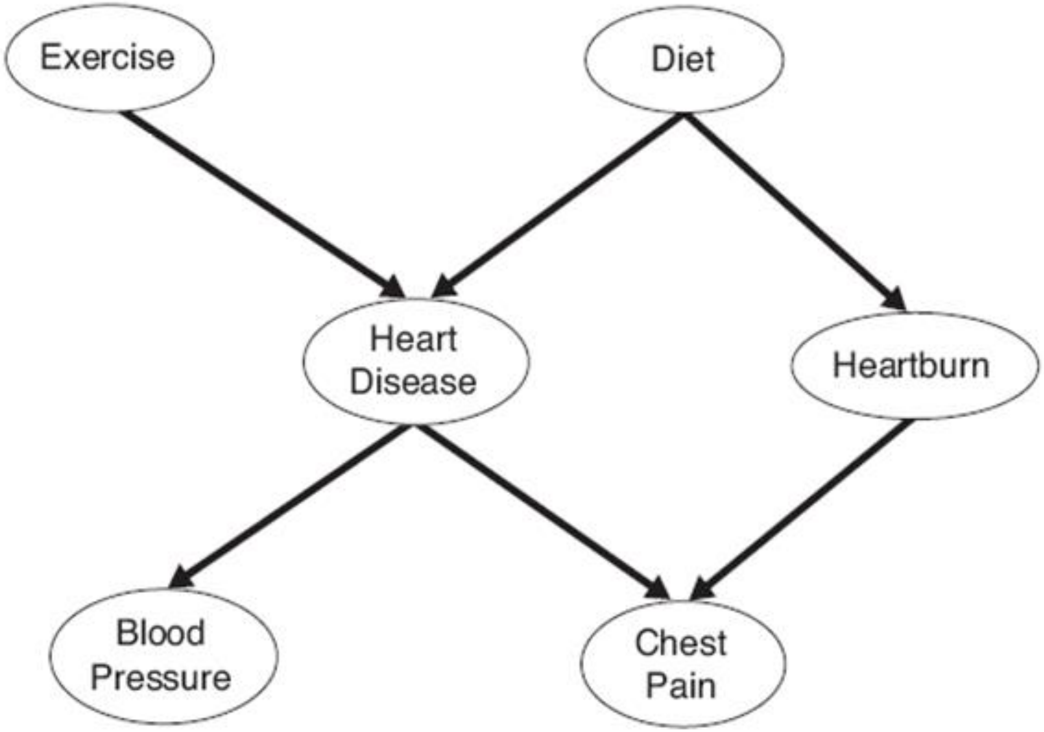
We can show:

- *P*(D|E)=P(D)
- *P (Hb|HD, E, D,CP)= P* (*Hb*|*D, CP*)
- *P (CP*|*Hb, HD, E, D)= P* (*CP*|Hb, HD)
- *P (BP*|*CP, Hb, HD, E, D)= P* (*BP*|HD)
- However, *P* (HD|E,D) cannot be simplified



Exercise:

- *$P(CP|HD, BP, E, D) = ?$*



Exercise:

- *P (CP|HD, BP, E, D)*= No simplification

- BBN Model Building

$T = \{X_1, X_2, X_3, \cdots, X_d\}$ Set of ordered variables

for $j = 1$ to d do

$X_{T(j)} =$ jth highest order variable

$\pi(X_{T(j)}) = \{X_{T(1)}, X_{T(2)}, X_{T(3)}, \cdots, X_{T(j-1)}\}$: preceding variables

remove non-dependent variables

create links between $X_{T(j)}$ and remaining $\pi(X_{T(j)})$

We will study this graph

$T = \{X_1, X_2, X_3, \cdots, X_d\}$ Set of ordered variables

for $j = 1$ to d do

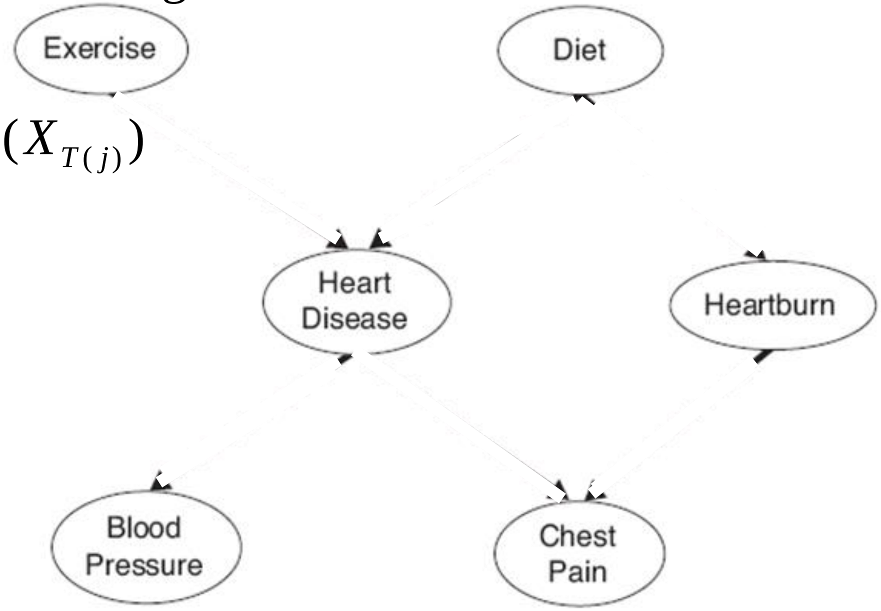
$X_{T(j)} =$ jth highest order variable

$\pi(X_{T(j)}) = \{X_{T(1)}, X_{T(2)}, X_{T(3)}, \cdots, X_{T(j-1)}\}$: preceding variables

remove non-dependent variables

create links between $X_{T(j)}$ and remaining $\pi(X_{T(j)})$

Order: *E, D, HD, Hb, CP, BP*



We will study this graph

$T = \{X_1, X_2, X_3, \cdots, X_d\}$ Set of ordered variables

for $j = 1$ to d do

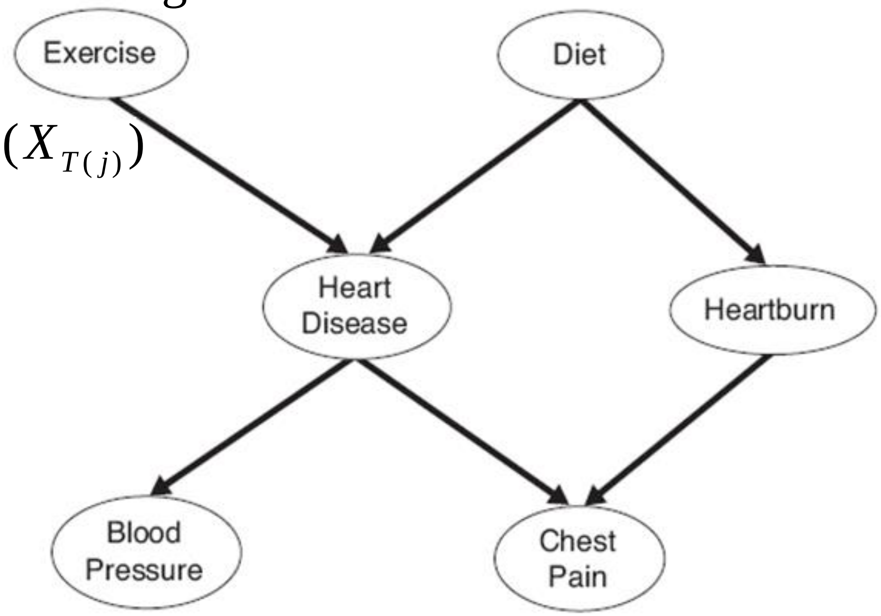
$X_{T(j)} = j$th highest order variable

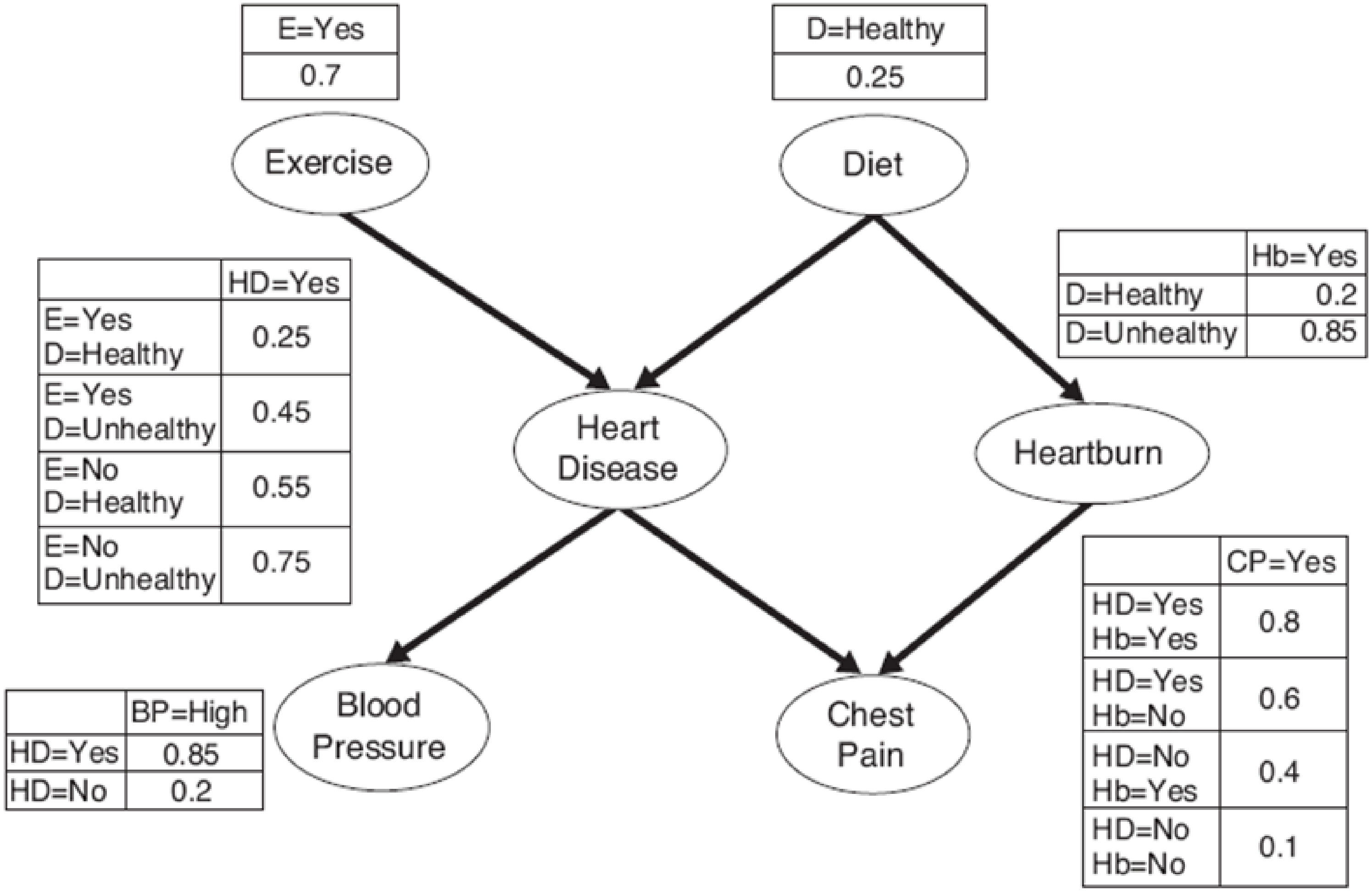
$\pi(X_{T(j)}) = \{X_{T(1)}, X_{T(2)}, X_{T(3)}, \cdots, X_{T(j-1)}\}$: preceding variables

remove non-dependent variables

create links between $X_{T(j)}$ and remaining $\pi(X_{T(j)})$

Order: *E, D, HD, Hb, CP, BP*





E=Yes
0.7

D=Healthy
0.25

	HD=Yes
E=Yes D=Healthy	0.25
E=Yes D=Unhealthy	0.45
E=No D=Healthy	0.55
E=No D=Unhealthy	0.75

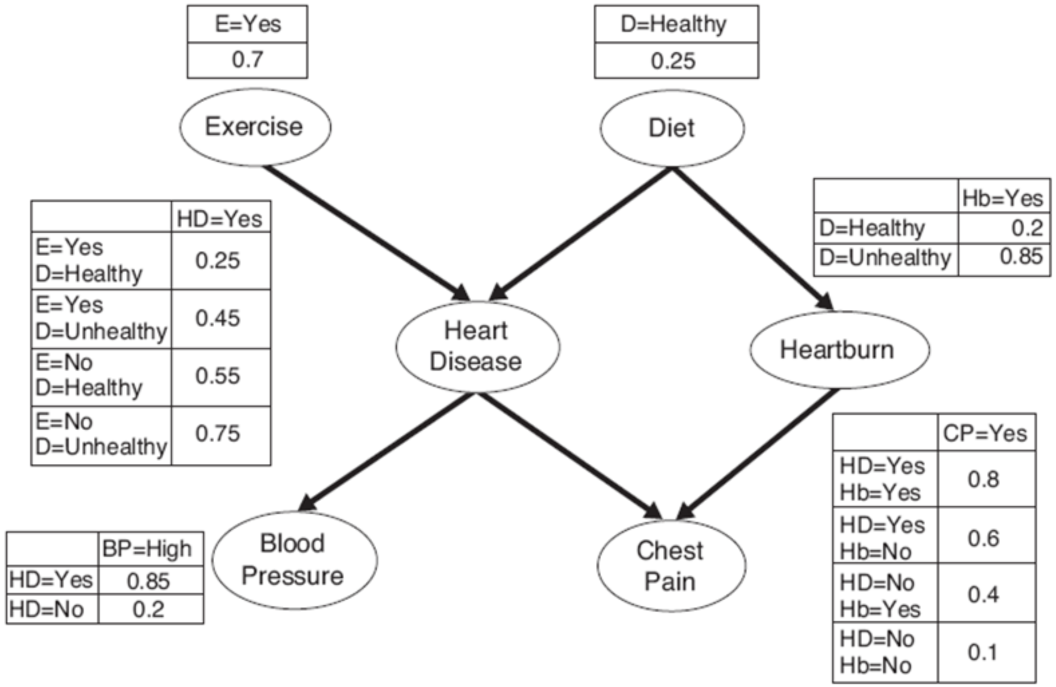
	Hb=Yes
D=Healthy	0.2
D=Unhealthy	0.85

	BP=High
HD=Yes	0.85
HD=No	0.2

	CP=Yes
HD=Yes Hb=Yes	0.8
HD=Yes Hb=No	0.6
HD=No Hb=Yes	0.4
HD=No Hb=No	0.1

Calculate P(HD=yes) ?

Calculate P(HD=yes) ?



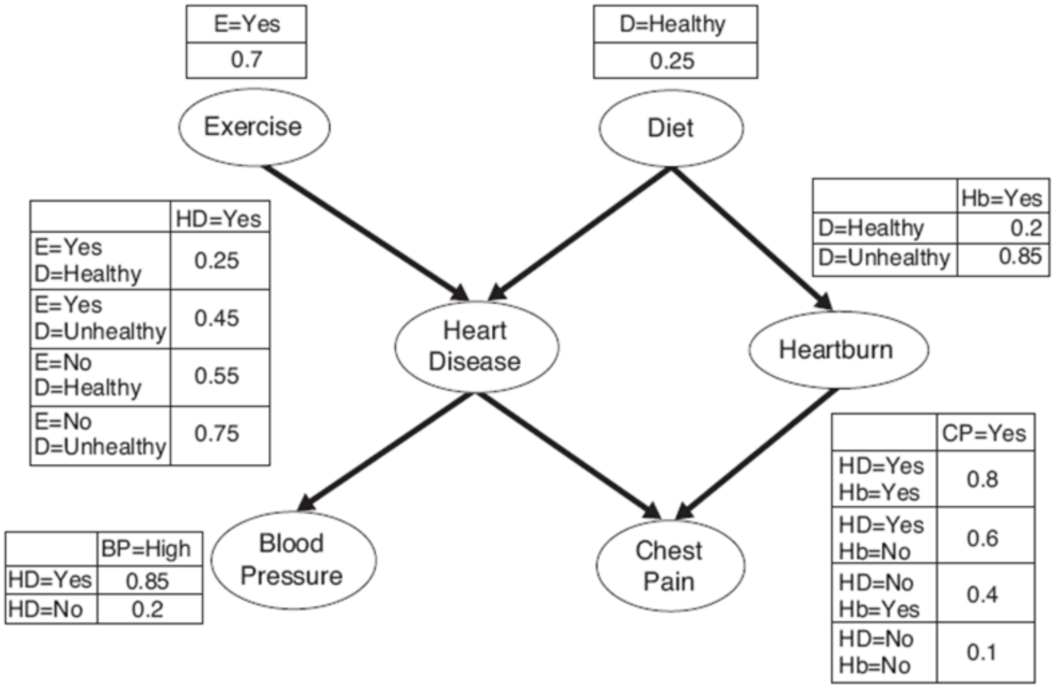
$$P(HD = Yes) = \sum_{\alpha} \sum_{\beta} P(HD = yes \mid E = \alpha, D = \beta) P(E = \alpha, D = \beta)$$

where,

α = Set of Values of Exercise(E) = {Yes, No}

β = Set of Values of Diet(D) = {Healthy, Not Healthy}

Calculate P(HD=yes) ?



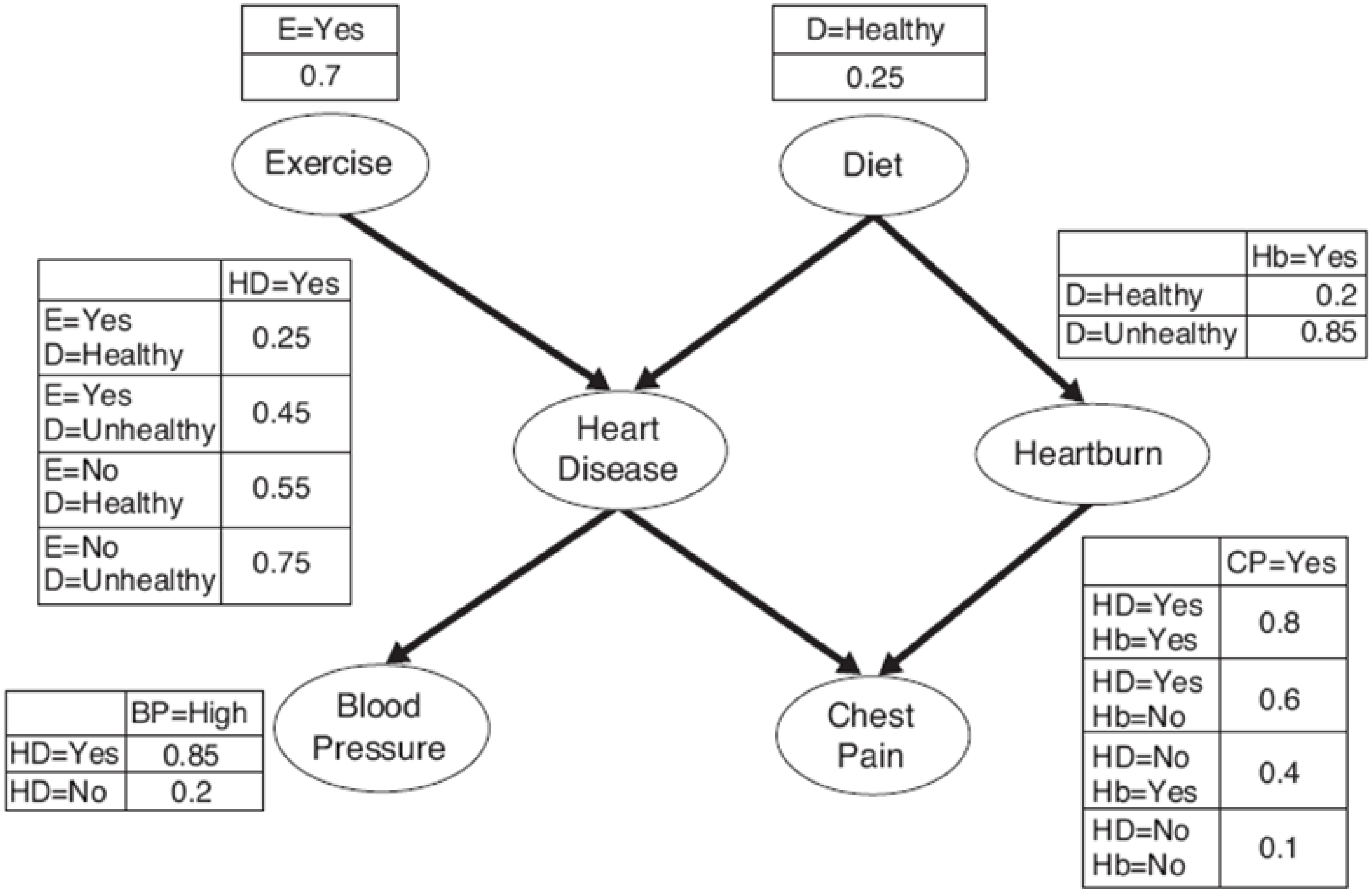
$$P(HD=Yes)=\sum_{\alpha}\sum_{\beta}P(HD=yes\,|\,E=\alpha,D=\beta)P(E=\alpha,D=\beta)$$

$$=\sum_{\alpha}\sum_{\beta}P(HD=yes\,|\,E=\alpha,D=\beta)P(E=\alpha)P(D=\beta)$$

$$=0.25\times0.7\times0.25+0.45\times0.7\times0.75+0.55\times0.3\times0.25$$

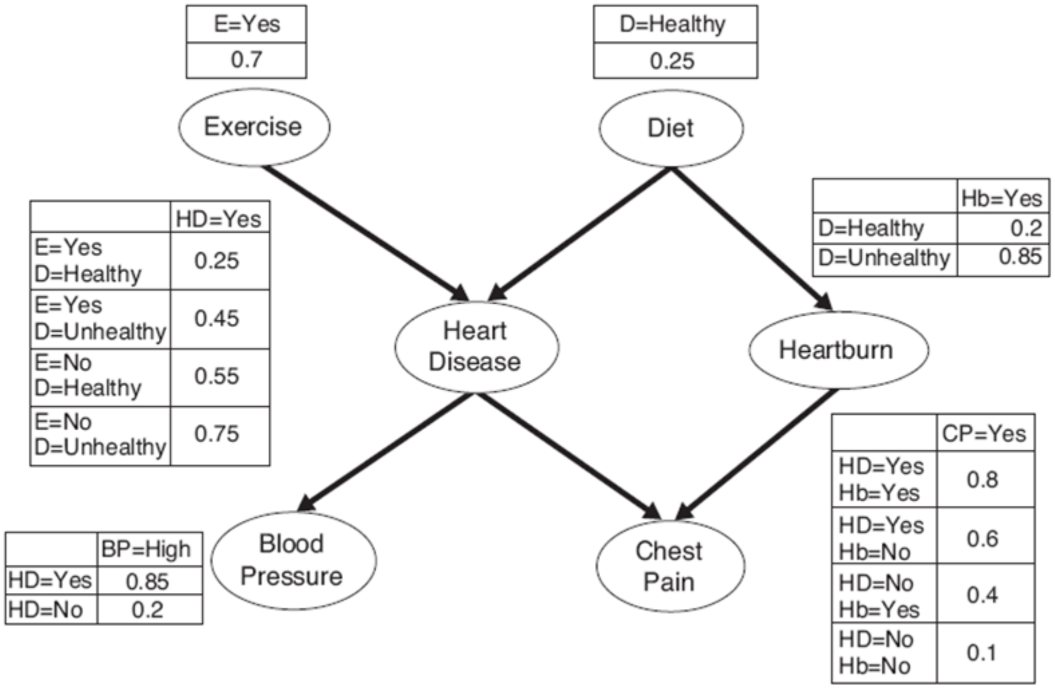
$$+0.75\times0.3\times0.75$$

$$=0.49$$



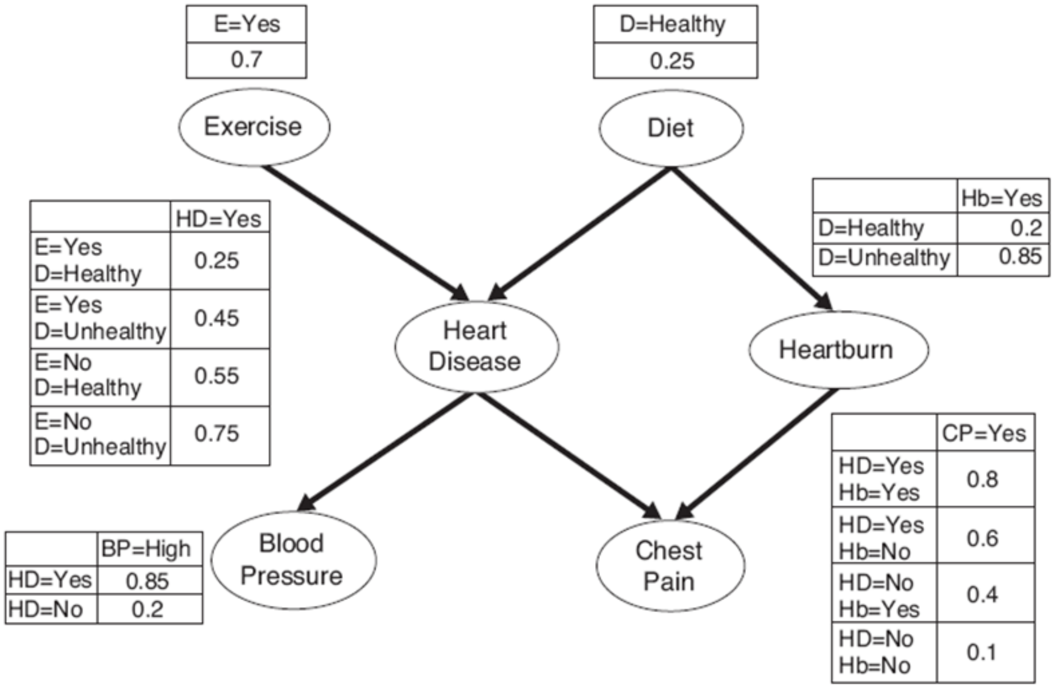
Calculate: P(HD=yes|BP=High)?

Calculate P(HD=yes|BP=High)



$P(HD = yes \mid BP = High)$ can be written as $\dfrac{P(BP = High \mid HD = yes)P(HD = yes)}{P(BP = High)}$

Calculate P(HD=yes|BP=High)

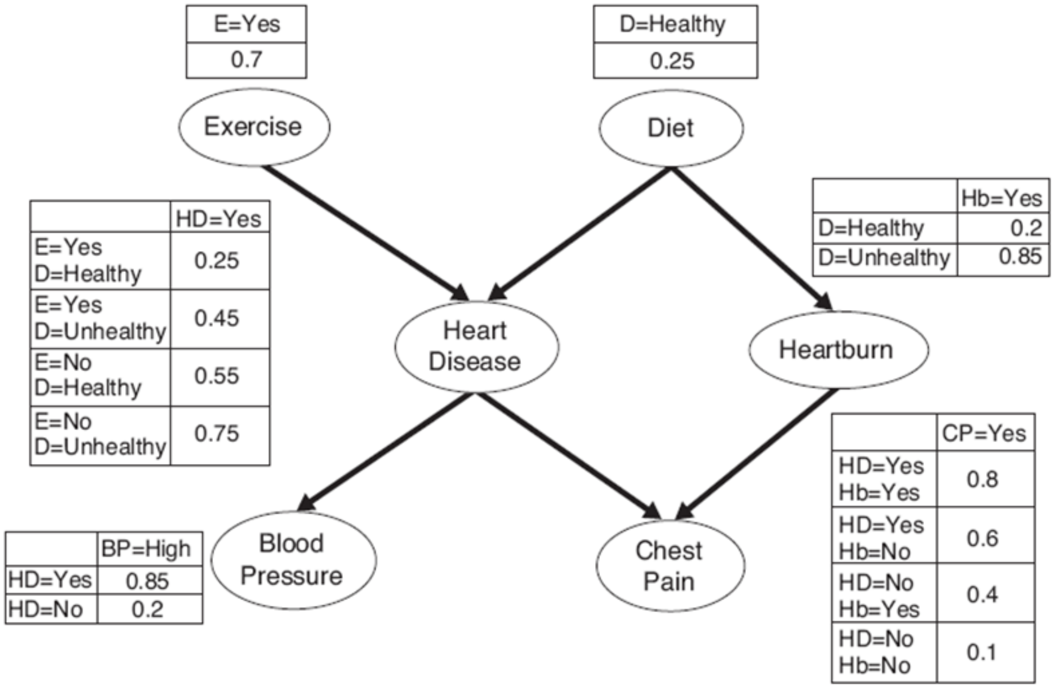


$$P(BP = High) = \sum_{\gamma} P(BP = high \mid HD = \gamma) P(HD = \gamma)$$

where,

γ = Set of Values of Heart Disease (HD) = {Yes, No}

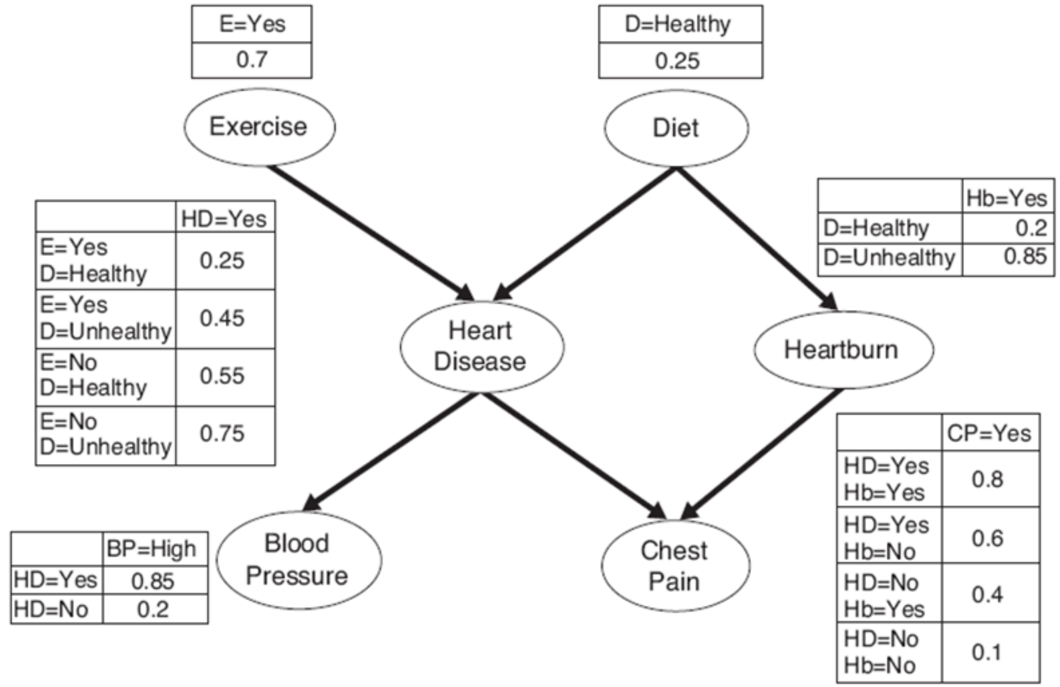
Calculate P(HD=yes|BP=High)



$$P(BP = High) = \sum_{\gamma} P(BP = high \mid HD = \gamma) P(HD = \gamma)$$

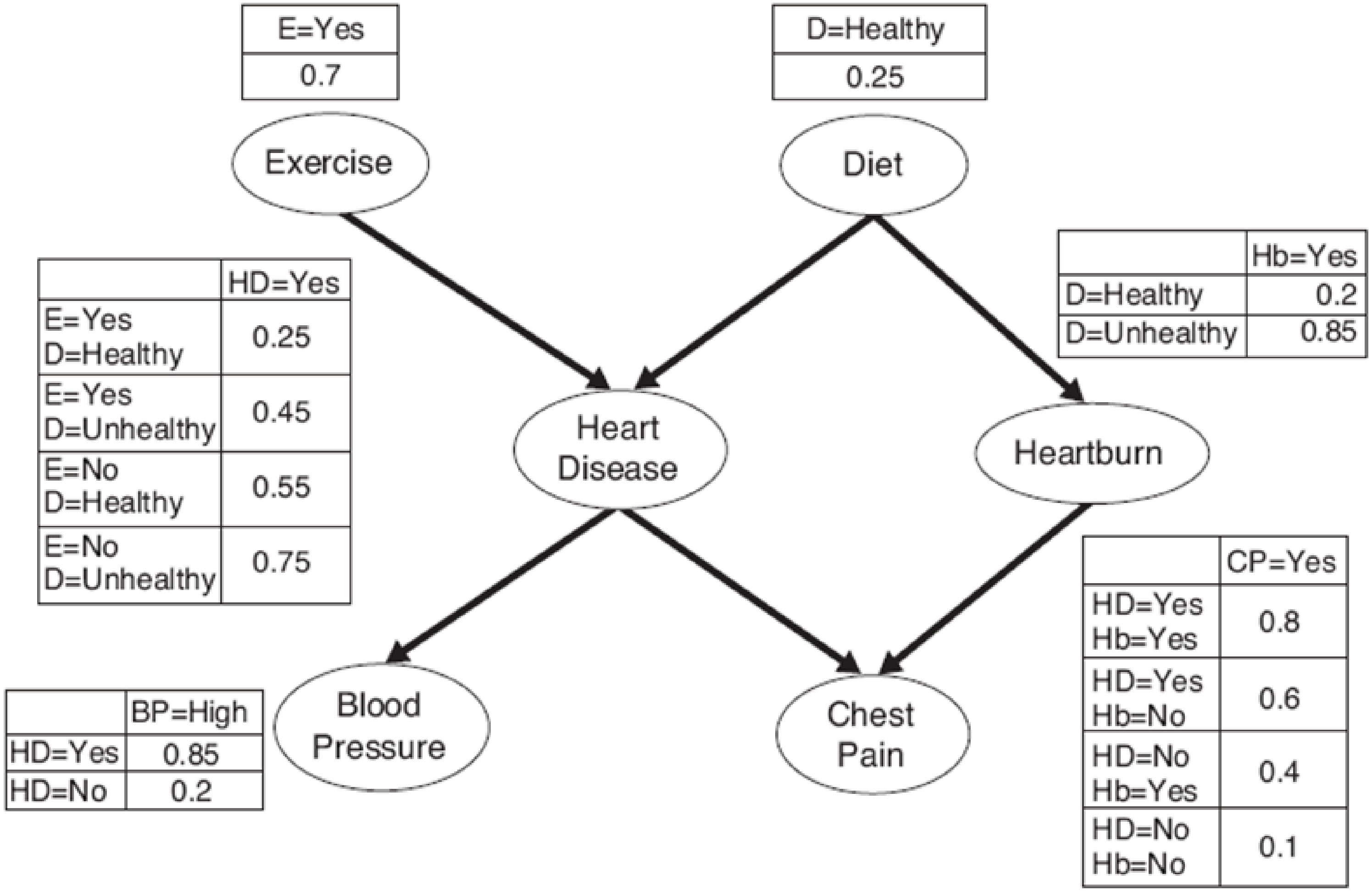
$$= 0.85 \times 0.49 + 0.2 \times 0.51 = 0.5185$$

Calculate P(HD=yes|BP=High)



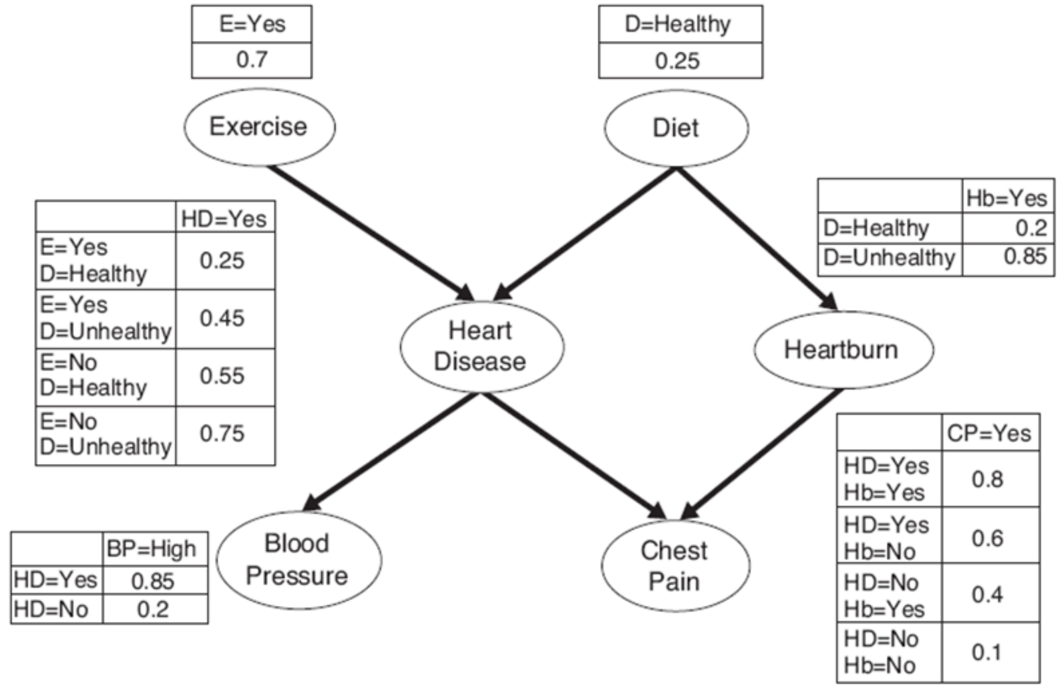
$$P(HD = yes \mid BP = High) = \frac{P(BP = High \mid HD = yes)P(HD = yes)}{P(BP = High)}$$

$$= \frac{0.85 \times 0.49}{0.5185} = 0.8033$$



Calculate P(HD=yes|BP=high, D=Healthy, E=yes)?

Calculate P(HD=yes|BP=high, D=Healthy, E=yes) ?



$$P(HD = yes \mid BP = high, D = Healthy, E = Yes)$$

$$= \frac{P(BP = high \mid HD = yes, D = Healthy, E = Yes)}{P(BP = high \mid D = Healthy, E = Yes)} \times P(HD = yes \mid D = Healthy, E = Yes)$$

How is this formula true?

$$P(HD = yes \mid BP = high, D = Healthy, E = Yes)$$

$$= \frac{P(BP = high \mid HD = yes, D = Healthy, E = Yes)}{P(BP = high \mid D = Healthy, E = Yes)} \times P(HD = yes \mid D = Healthy, E = Yes)$$

Let $$P(X \mid Y) = \frac{P(Y \mid X)}{P(Y)} \times P(X)$$

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Let $$P(X \mid Y) = \frac{P(Y \mid X)}{P(Y)} \times P(X)$$

Now add *Z* and *W* as condition $$P(X \mid Y, Z, W) = \frac{P(Y \mid X, Z, W)}{P(Y \mid Z, W)} \times P(X \mid Z, W)$$

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Similarly,

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Similarly,

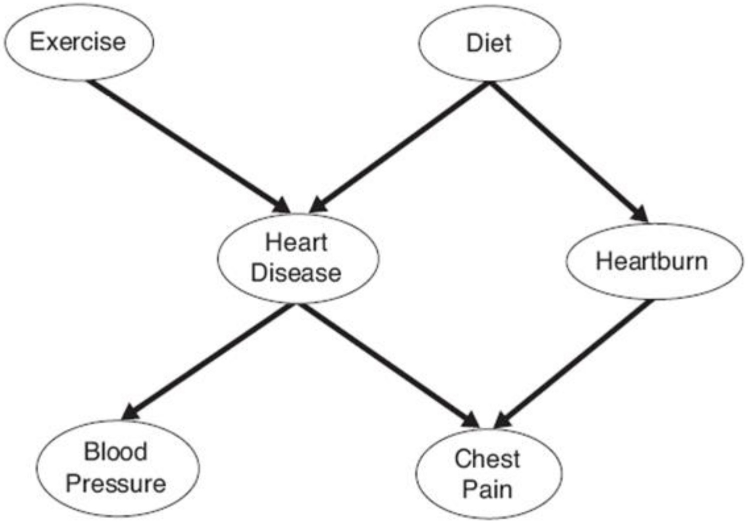
$$P(HD = yes \mid BP = high) = \frac{P(BP = high \mid HD = yes)}{P(BP = high)} \times P(HD = yes)$$

Now add conditions *D= Healthy* and *E= Yes* to above formula

Similarly,

$$P(BP = high \mid D = Healthy, E = Yes)$$

$$= \sum_{\gamma} P(BP = high \mid HD = \gamma, D = Healthy, E = Yes) \times P(HD = \gamma \mid D = Healthy, E = Yes)$$



Similarly,

$$P(BP = high \mid D = Healthy, E = Yes)$$
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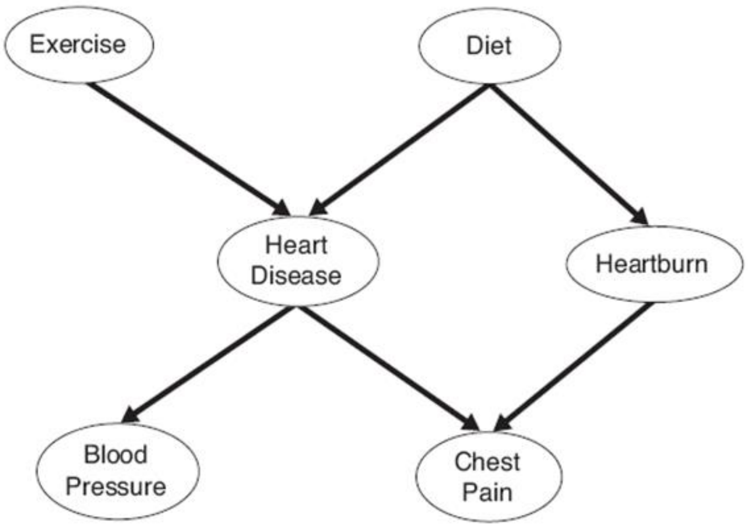
Proof:

$$P(BP = high) = \sum_{\gamma} P(BP = high \mid HD = \gamma) \times P(HD = \gamma)$$



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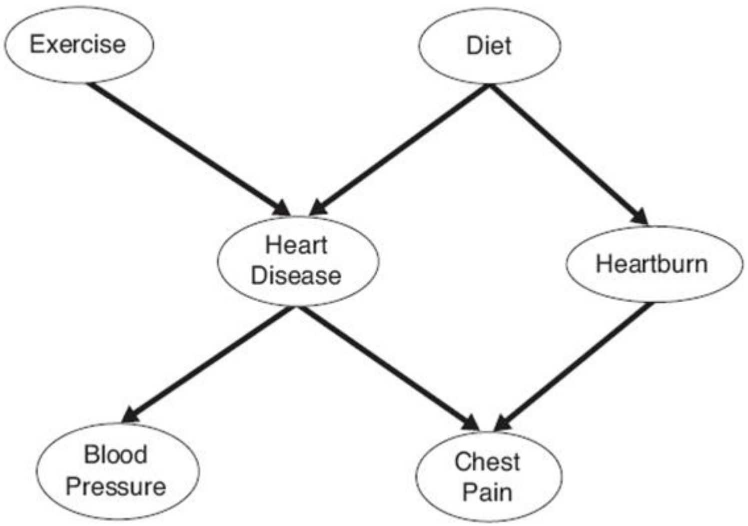
Proof:



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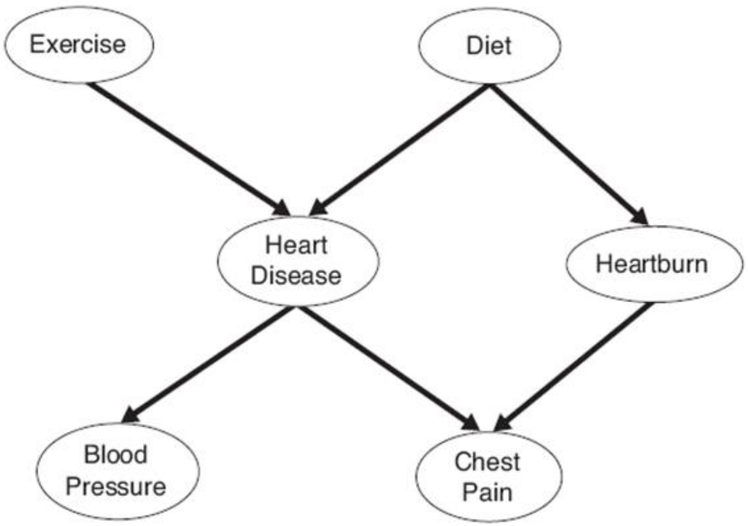

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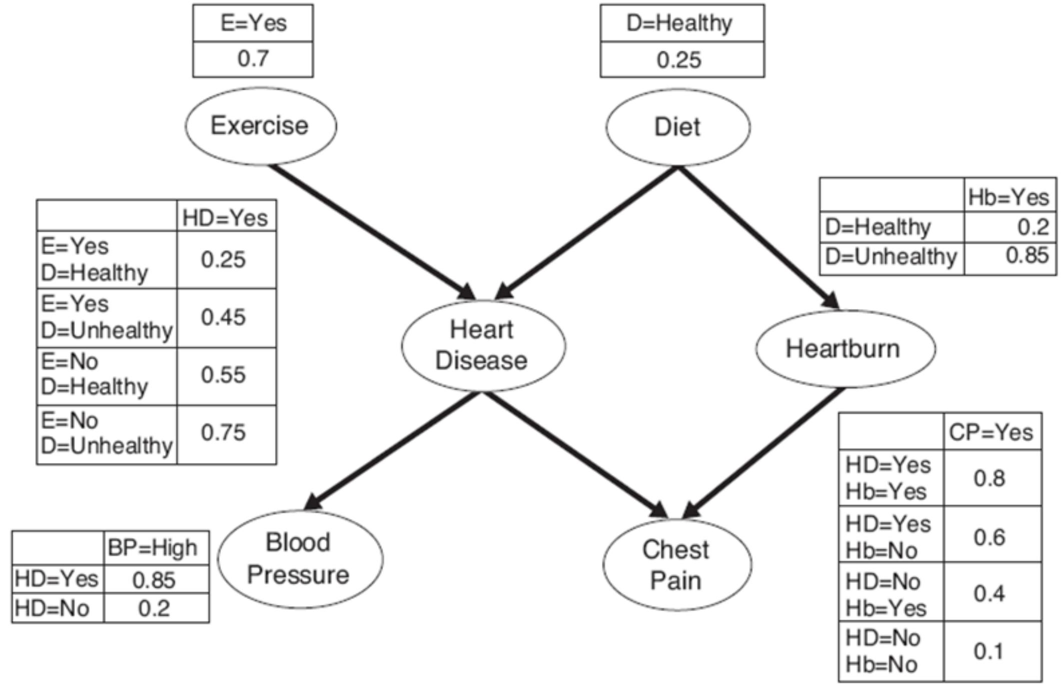

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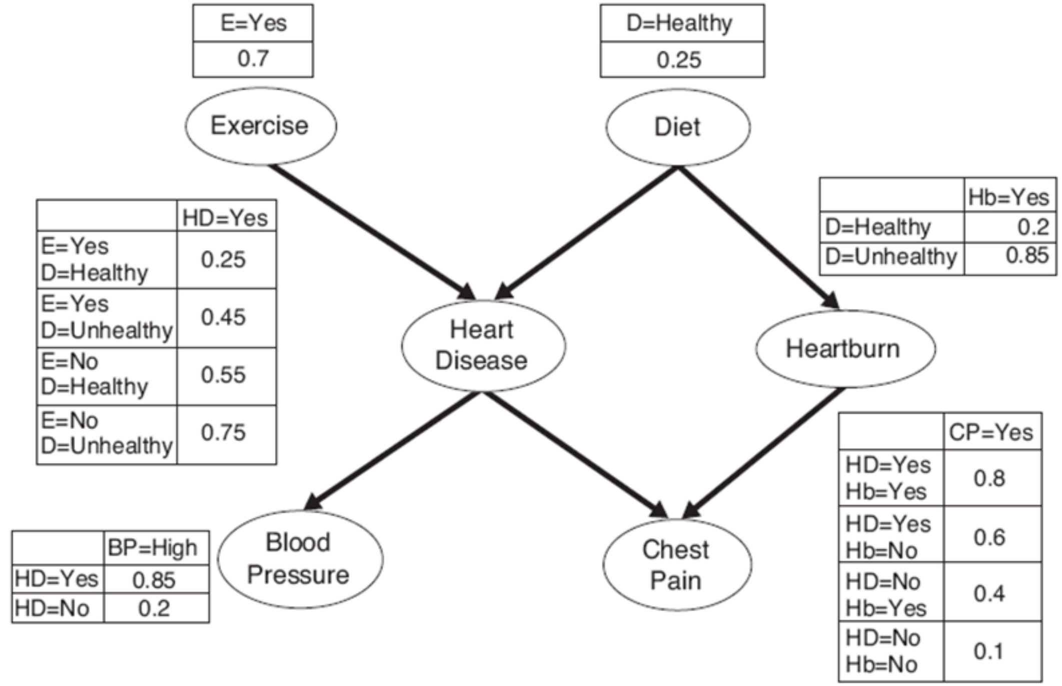

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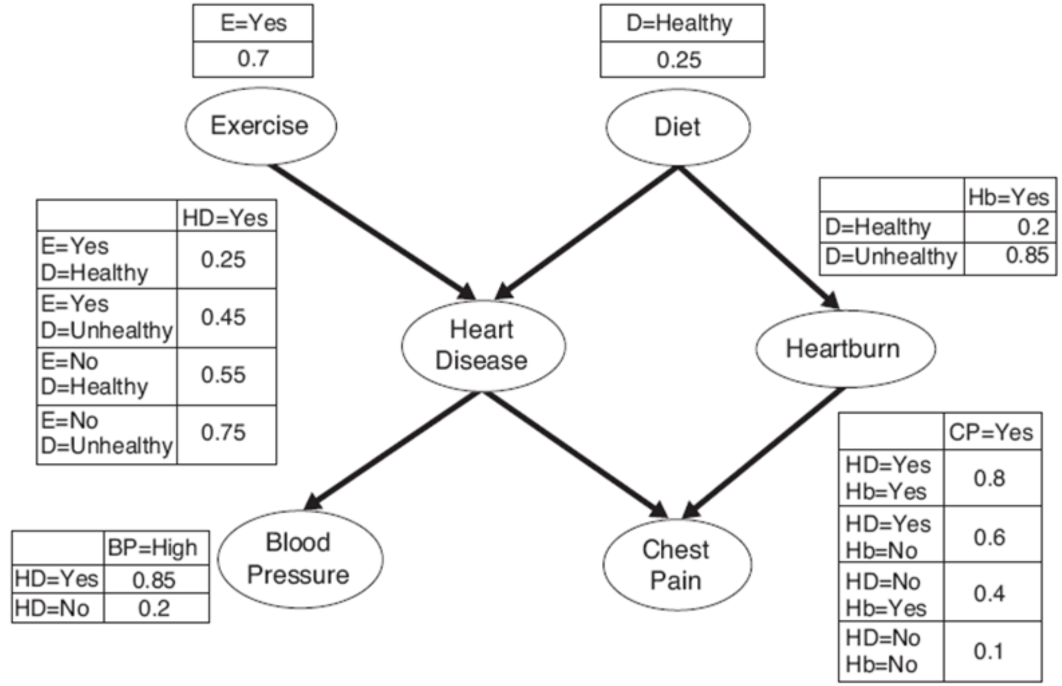

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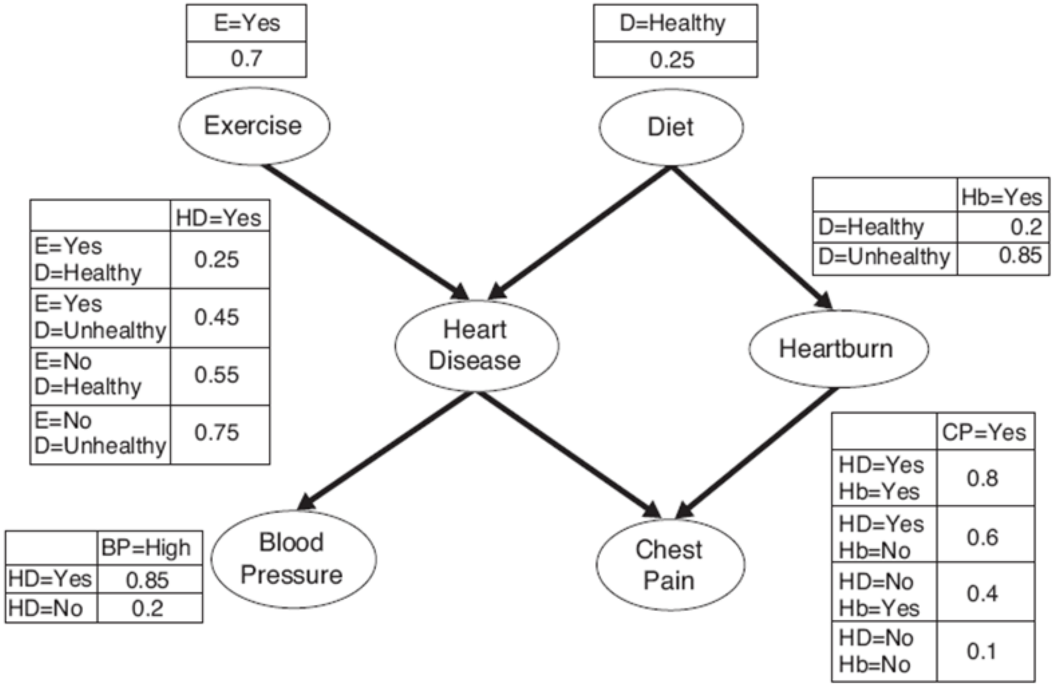

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$$= \frac{0.85 \times 0.25}{0.85 \times 0.25 + 0.2 \times 0.75} = 0.5862$$

Review of Bayesian Classifier and its variants

- underlying probability densities were known
- training sample are used to estimate the probabilities

Linear Classifier: Introduction

- Classifies linearly separable patterns
- Assume proper forms for the discriminant functions
- may not be optimal
- very simple to use

Linear discriminant functions and decisions surfaces

- Definition

Let a pattern vector $\mathbf{x} = \{x_1, x_2, x_3, \ldots,\}$

a weight vector $\mathbf{w} = \{w_1, w_2, w_3, \ldots,\}$

A discriminant function :

$$g(\mathbf{x})=x_1w_1+x_2w_2+x_3w_3+\ldots$$

OR

$$g(\mathbf{x}) = w^tx + w_0 \qquad (1)$$

where w is the weight vector and w_0 the bias

Linear discriminant functions and decisions surfaces

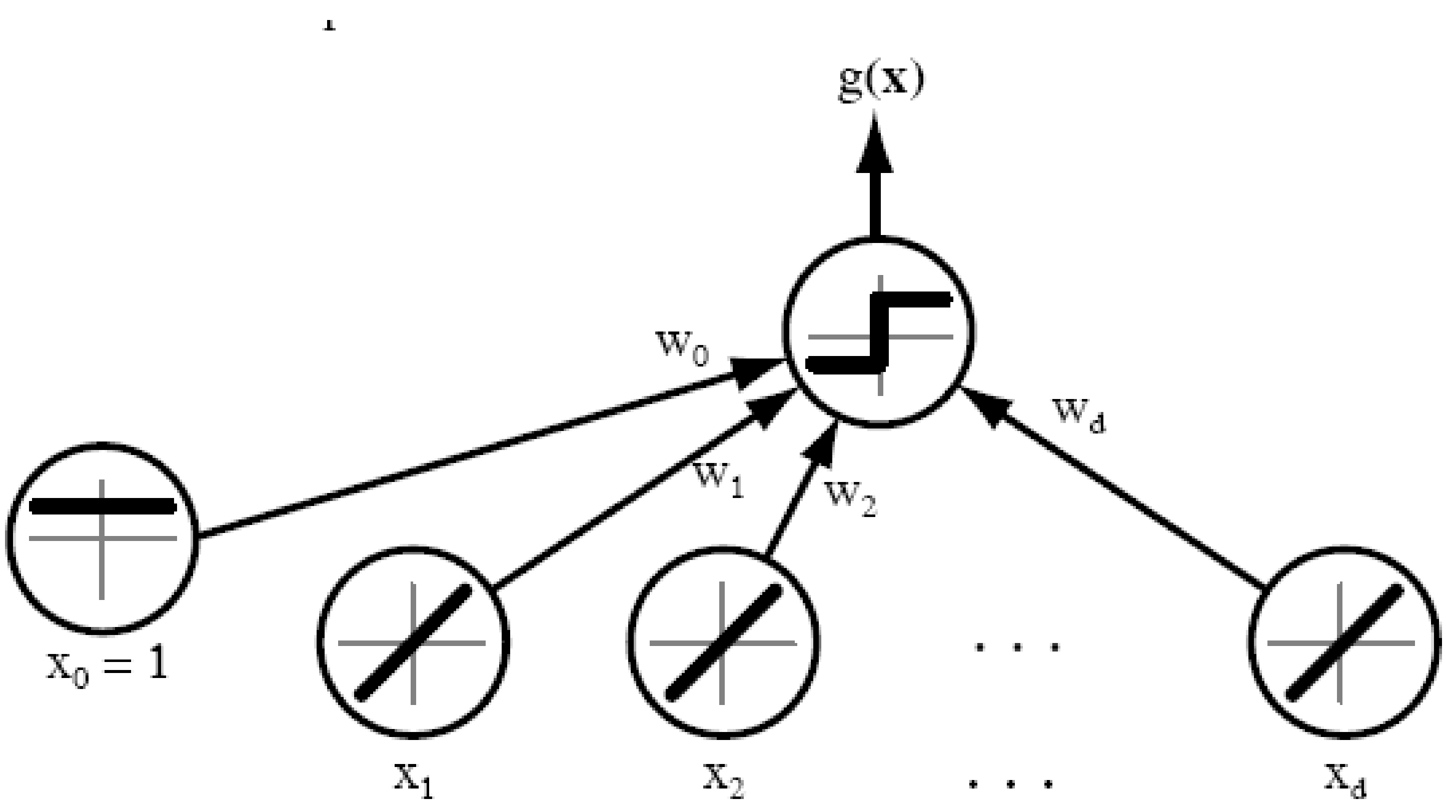
- Classify a new pattern **x** as follows

  Decide class ω_1 *if* $g(\mathrm{x}) > 0$

  and class ω_2 if $g(\mathrm{x}) < 0$

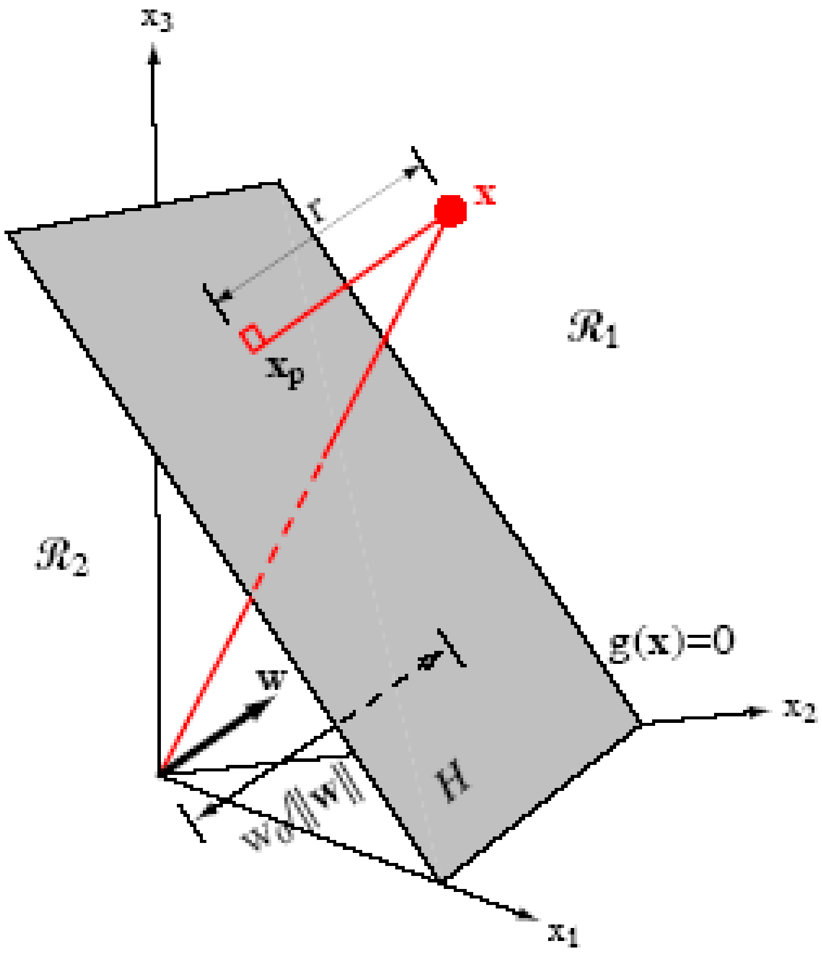
  If $g(x) = 0 \Rightarrow x$ is assigned to either class

Linear discriminant functions and decisions surfaces

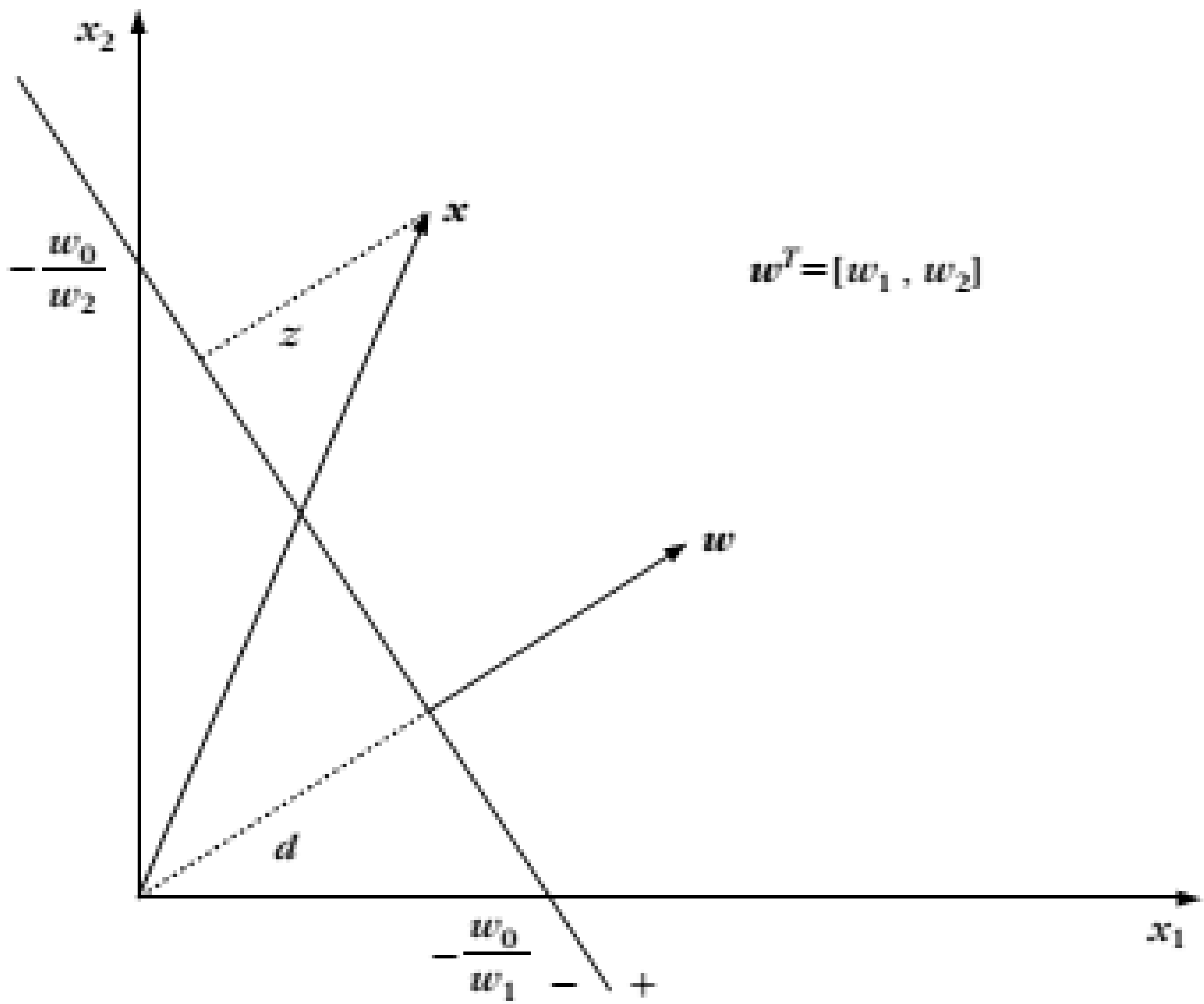


- The equation $g(x) = 0$ is the decision surface that separates patterns

- When $g(x)$ is linear, the decision surface is a hyperplane



x_3
r
x
$\mathcal{R}_1$
x_p
$\mathcal{R}_2$
g(x)=0
w
x_2
H
$w_0/\|w\|$
x_1



x_2
$-\frac{w_0}{w_2}$
x
$w^T=[w_1, w_2]$
z
w
d
x_1
$-\frac{w_0}{w_1}$
$-$
$+$

A little bit mathematics

- The Problem: Consider a two class task with ω_1, ω_2
  - $g(\underline{x}) = \underline{w}^T \underline{x} + w_0 = 0 =$
    $w_1 x_1 + w_2 x_2 + \ldots + w_l x_l + w_0$
  - Assume $\underline{x}_1, \underline{x}_2$ on the decision hyperplane:
    $0 = \underline{w}^T \underline{x}_1 + w_0 = \underline{w}^T \underline{x}_2 + w_0 \Rightarrow$
    $\underline{w}^T (\underline{x}_1 - \underline{x}_2) = 0 \ \forall \underline{x}_1, \underline{x}_2$

- Hence:

$$\boxed{\underline{w} \perp \text{on the hyperplane}}$$

$$g(\underline{x}) = \underline{w}^T \underline{x} + w_0 = 0$$

- Hence:

$$\boxed{\underline{w} \perp \text{on the hyperplane}}$$

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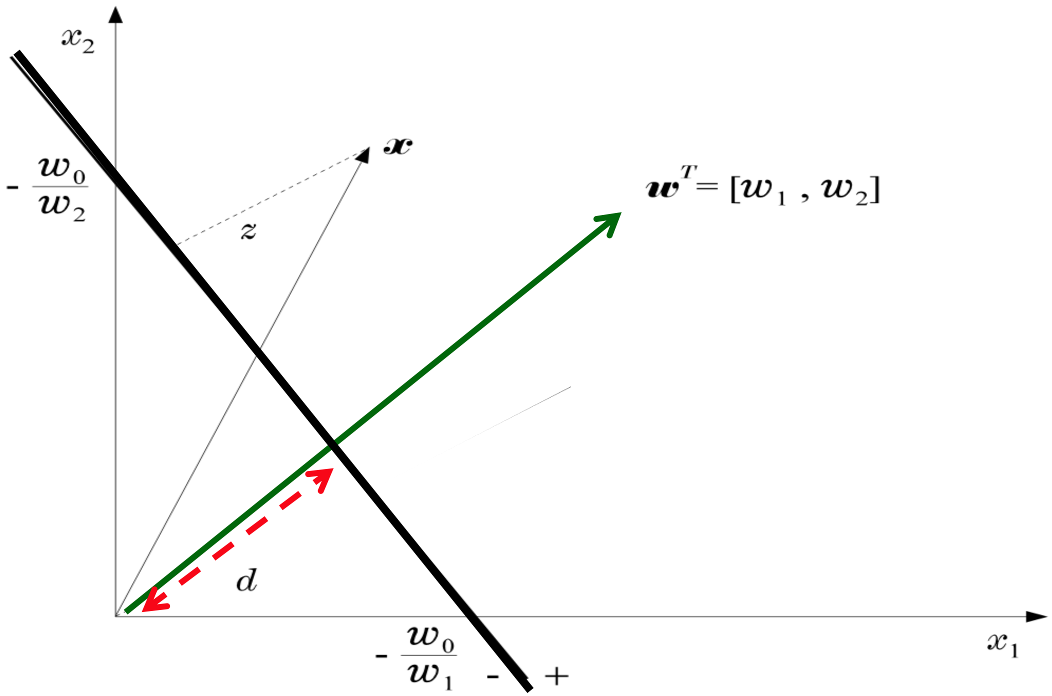


$$d = \frac{|w_0|}{\sqrt{w_1^2 + w_2^2}}, \quad z = \frac{|g(\underline{x})|}{\sqrt{w_1^2 + w_2^2}}$$

- The Perceptron Algorithm

  - Assume linearly separable classes, i.e.,

$$\exists \ \underline{w}^*: \ w^{*T} \ \underline{x} > 0 \ \ \forall \underline{x} \in \omega_1$$

$$\underline{w}^{*T} \ \underline{x} < 0 \ \ \forall \underline{x} \in \omega_2$$

- The Perceptron Algorithm

  - Assume linearly separable classes, i.e.,

$$\exists\ \underline{w}^*:\ \underline{w}^{*T}\,\underline{x} > 0\ \ \forall \underline{x} \in \omega_1$$
$$\underline{w}^{*T}\,\underline{x} < 0\ \ \forall \underline{x} \in \omega_2$$

  - The case $\underline{w}^{*T}\,\underline{x} + w_0^*$ falls under the above formulation, since

    - $\underline{w}' \equiv \begin{bmatrix} \underline{w}^* \\ w_0^* \end{bmatrix},\ \underline{x}' = \begin{bmatrix} \underline{x} \\ 1 \end{bmatrix}$

    - $\underline{w}^{*T}\,\underline{x} + w_0^* = \underline{w}'^T\,\underline{x}' = 0$

– Our goal: Compute a solution, i.e., a hyperplane $\underline{w}$, so that

$$\underline{w}^T \underline{x} (><) 0 \quad \underline{x} \in \begin{cases} \omega_1 \\ \omega_2 \end{cases}$$

- The steps
  - Define a cost function to be minimized
  - Choose an algorithm to minimize the cost function
  - The minimum corresponds to a solution

– The Cost Function

$$J(\underline{w}) = \sum_{\underline{x} \in Y} (\delta_x \underline{w}^T \underline{x})$$

- Where Y is the subset of the vectors wrongly classified by $\underline{w}$.

- $$\delta_x = -1 \text{ if } \underline{x} \in Y \text{ and } \underline{x} \in \omega_1$$
  $$\delta_x = +1 \text{ if } \underline{x} \in Y \text{ and } \underline{x} \in \omega_2$$

– The Cost Function

$$J(\underline{w}) = \sum_{\underline{x} \in Y} (\delta_x \underline{w}^T \underline{x})$$

- Where Y is the subset of the vectors wrongly classified by $\underline{w}$.
- when Y=(empty set) a solution is achieved and

$$J(\underline{w}) = 0$$

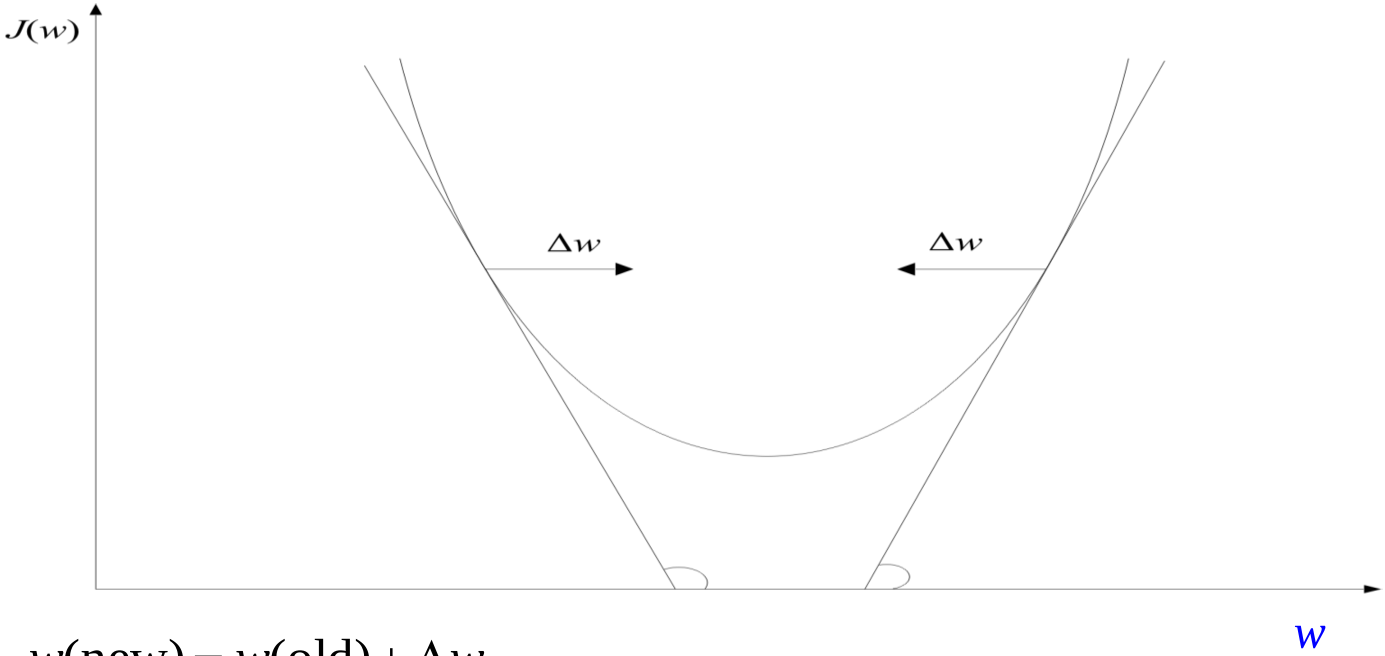
otherwise

$$J(\underline{w}) \geq 0$$

- $J(\underline{w})$ is piecewise linear (WHY?)

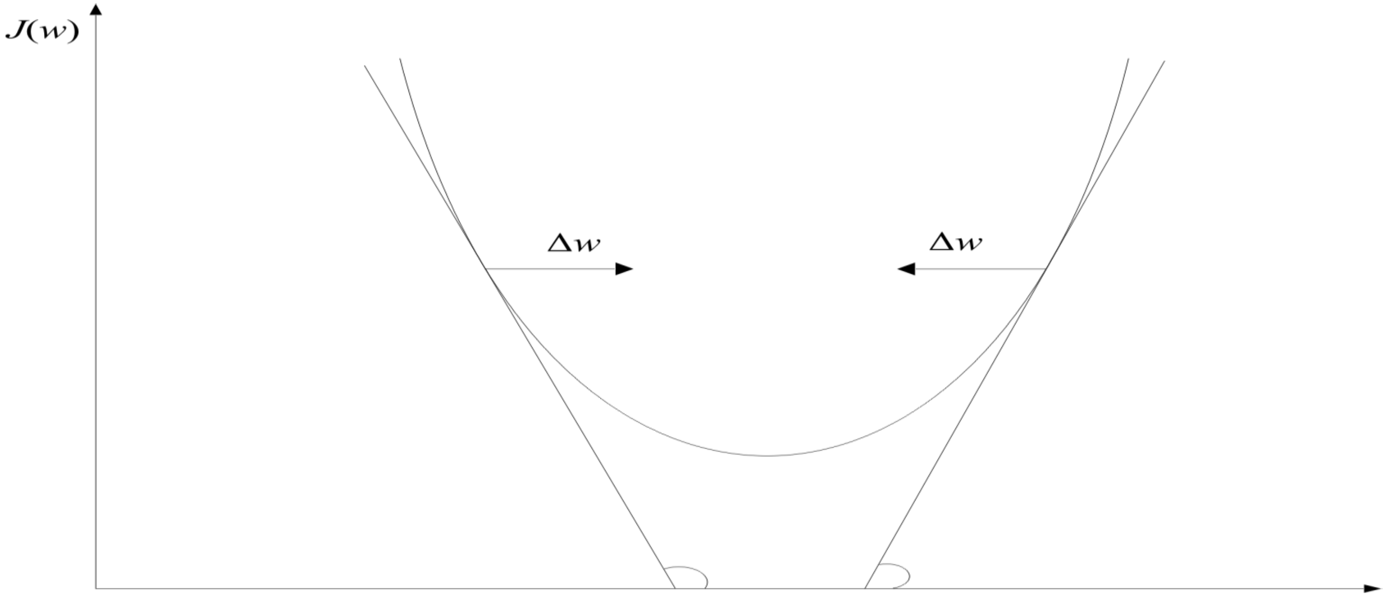
– The Algorithm

- The philosophy of the gradient descent is adopted.



$$\underline{w}(\text{new}) = \underline{w}(\text{old}) + \Delta\underline{w}$$

$$\Delta\underline{w} = -\mu \frac{\partial J(\underline{w})}{\partial \underline{w}} \bigg|\, \underline{w} = \underline{w}(\text{old})$$



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$$\Delta \underline{w} = -\mu \frac{\partial J(\underline{w})}{\partial \underline{w}} \Big|_{\underline{w} = \underline{w}(\text{old})}$$

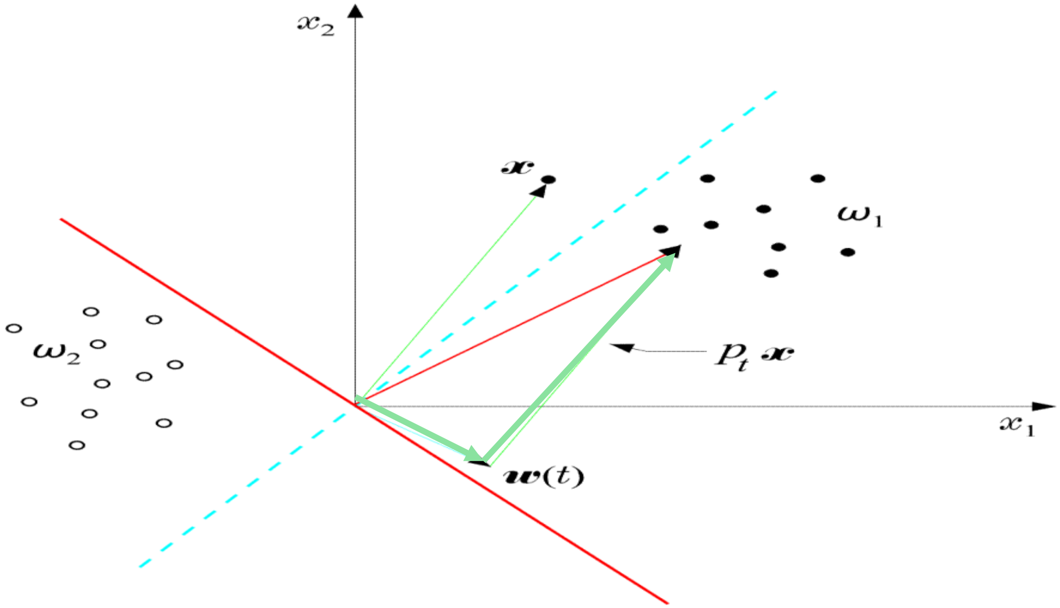
- Wherever valid

$$\frac{\partial J(\underline{w})}{\partial \underline{w}} = \frac{\partial}{\partial \underline{w}} \left(\sum_{\underline{x} \in Y} \delta_x \underline{w}^T \underline{x} \right) = \sum_{\underline{x} \in Y} \delta_x \underline{x}$$

- $$\boxed{\underline{w}(t+1) = \underline{w}(t) - \rho_t \sum_{\underline{x} \in Y} \delta_x \underline{x}}$$

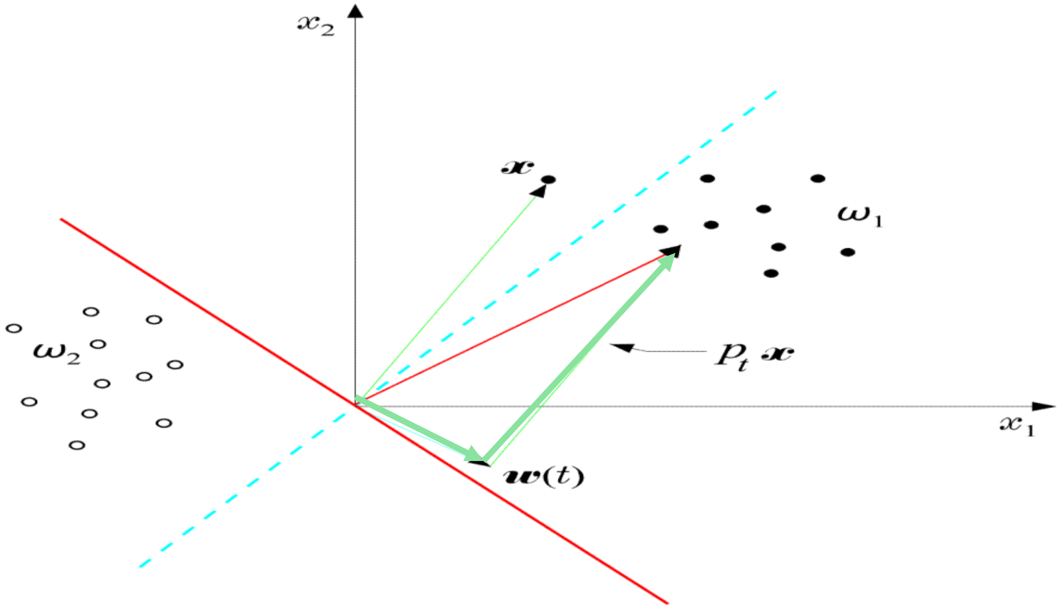
This is the celebrated Perceptron Algorithm

– An example:



$$\begin{aligned}\underline{w}(t+1) &= \underline{w}(t) - \rho_t \delta_x \underline{x} \\ &= \underline{w}(t) + \rho_t \underline{x} \quad (\delta_x = -1)\end{aligned}$$

– An example:



$$\underline{w}(t+1) = \underline{w}(t) - \rho_t \delta_x \underline{x}$$
$$= \underline{w}(t) + \rho_t \underline{x} \quad (\delta_x = -1)$$

– The perceptron algorithm **converges** in a **finite** number of iteration steps to a solution **if patterns are linearly separable**

– Example: At some stage t the perceptron algorithm results in

$$w_1 = 1,\ w_2 = 1,\ w_0 = -0.5$$

$$x_1 + x_2 - 0.5 = 0$$

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$$w_1 = 1, \;\; w_2 = 1, \;\; w_0 = -0.5$$

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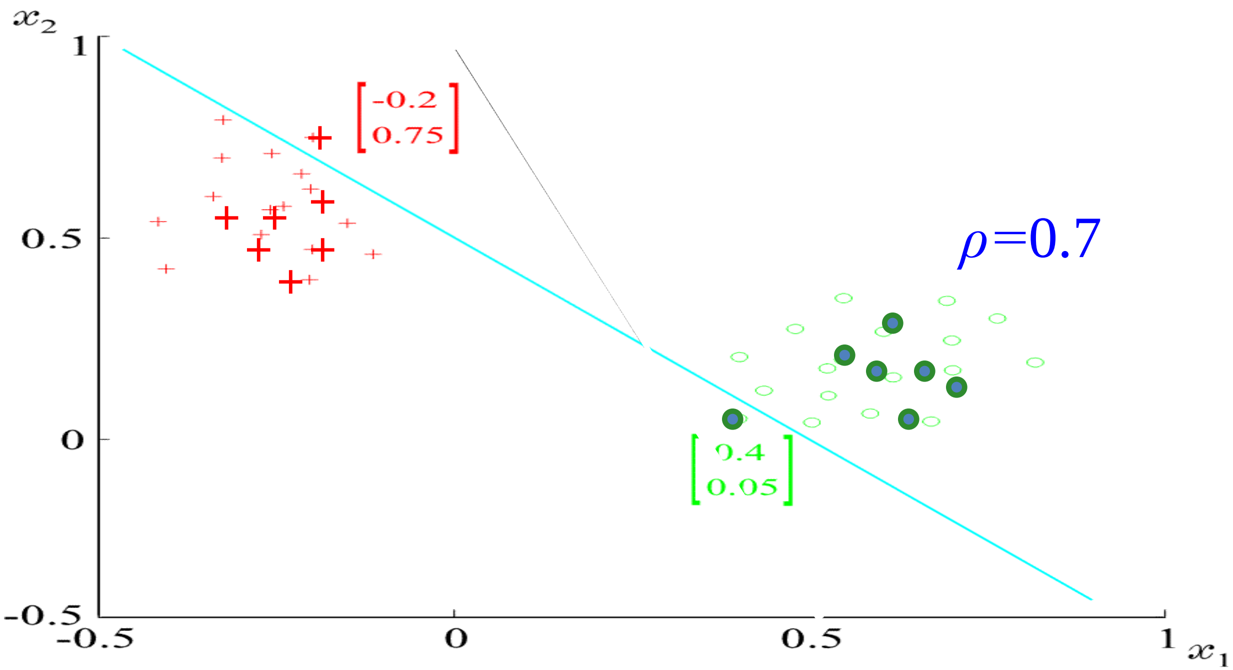
The corresponding hyperplane is



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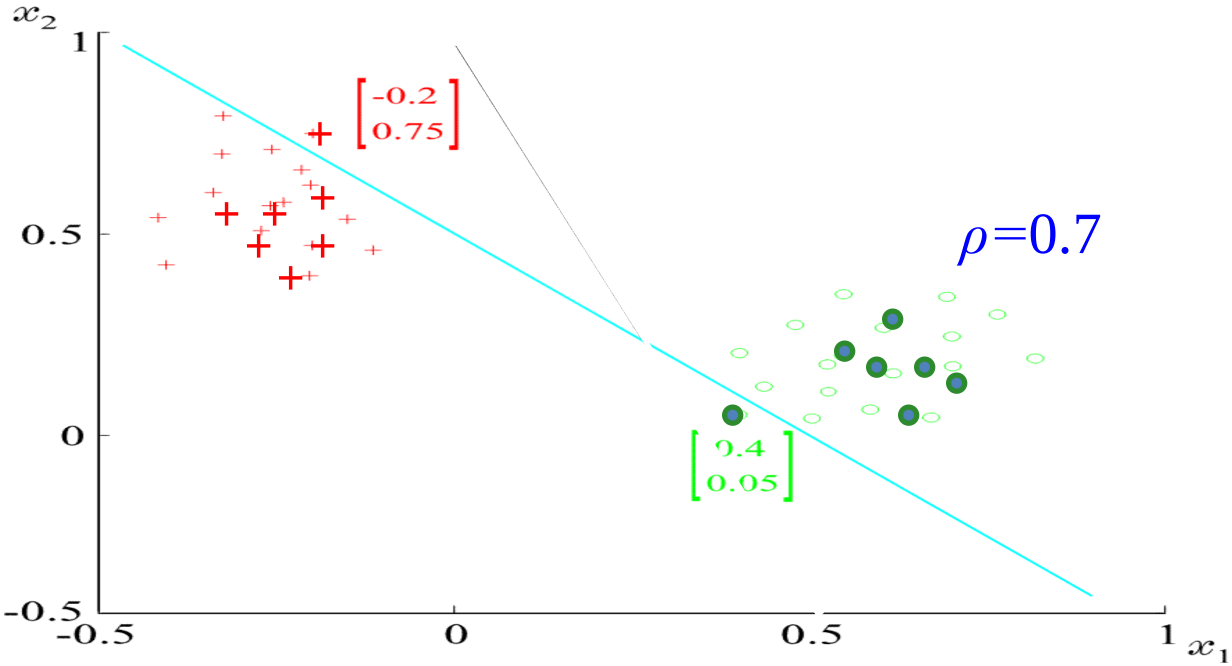

$$\underline{w}(t+1) = \begin{bmatrix} 1 \\ 1 \\ -0.5 \end{bmatrix} - 0.7(-1)\begin{bmatrix} 0.4 \\ 0.05 \\ 1 \end{bmatrix} - 0.7(+1)\begin{bmatrix} -0.2 \\ 0.75 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.42 \\ 0.51 \\ -0.5 \end{bmatrix}$$

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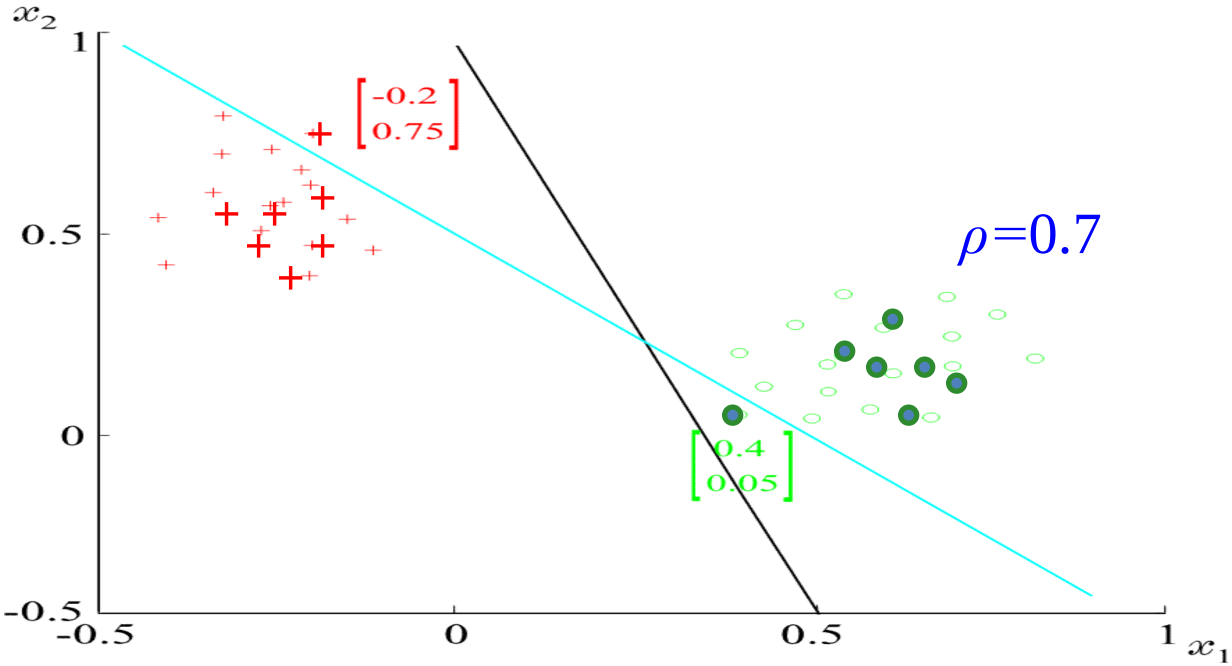

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