CSE 5824: Advanced Machine Learning
Self-Evaluation Exam Solution

Instructor: Prof. Dawei Zhou
January 18, 2022

First Name:			
Last Name:			
Email:			
VT ID:			
Q	Topic	Max Score	Score
1	Probability	24	
2	Iterative Algorithms	21	
3	MLE	20	
4	Quadratic Optimization	17	
5	Distance Metric	18	
Total:		100	

- This exam book has **6** pages, including this cover page and a blank page at the end.
- You have 45 minutes in total.
- Good luck!

1 Probability (24 points)

(7 points.) If A and B are **DISJOINT** events, and $P(B) > 0$, what is the value of $P(A|B)$?

A and B are **DISJOINT** events

$\therefore P(AB) = 0$

$P(A|B) = \frac{P(AB)}{P(B)} = 0$

(7 points.) Suppose that the PDF of a random variable X is as follows:

$$f(x) = \begin{cases} \frac{4}{3}(1 - x^3), & \text{for } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Then what is the value of $P(X < 0)$?

$P(X < 0) = \int_{-\infty}^{0} f(x)\, dx = 0$

(10 points.) Suppose that X is a random variable for which $E(X) = \mu$ and $Var(X) = \sigma^2$, and let c be an arbitrary constant. What is the value of $E[(X-c)^2]$? **Hint: What is the definition of variance?**

$E[(X - c)^2] = E[X^2] - 2cE[X] + c^2 = \sigma^2 + \mu^2 - 2c\mu + c^2 = (\mu - c)^2 + \sigma^2$

2 Iterative Algorithms and Big-O (21 points)

[7 points.] Write a program/function (**Not Pseudocode**) to calculate the following sum (c++ or any of your favorite programming language). Your program takes an integer n $(n \geq 1)$ as the input; and returns r defined as $r = \sum_{i=1}^{n} \sum_{j=1}^{i} j$.

the c++ or c code for the following iterative algorithm (Algorithm 1)

Algorithm 1 An Iterative Algorithm

Require: An integer $n \geq 0$;
Ensure: r.

```
1: r ← 0;
2: for i ← 1 to n do
3:     for j ← 1 to i do
4:         r ← r + j
5:     end for
6: end for
```

(7 points.) What is the value r returned by your program? Express your answer as a function of n and use *closed-form* solution.

$$
\begin{aligned}
r &= \sum_{i=1}^{n} \sum_{j=1}^{i} j \\
&= \sum_{i=1}^{n} 1/2 * i(i+1) = \frac{1}{2} \sum_{i=1}^{n} (i^2 + i) \\
&= \frac{1}{2} \left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \right)
\end{aligned}
$$

(7 points.) Using $O()$ notation, give the worst-case running time of your program.

$$O(n^3)$$

3 Maximum Likelihood Estimation (20 points)

[10 points.] Suppose we flip a coin, and observe either a head or a tail. The probability of observing a head in each trial is p $(0 \leq p \leq 1)$. If we flip the coin five times, and observe *(head, head, tail, tail, head)*, what is the maximum likelihood estimate of p? Justify your answer.

$P(head, head, tail, tail, head) = p^3(1-p)^2$
We use calculus to find the MLE of p by taking the derivative of the likelihood function and setting it to 0.

$$\frac{dP}{dp} = 3p^2(1-p)^2 - 2p^3(1-p) = 0$$

$$2p = 3(1-p)$$
$$p = 0.6$$

[10 points.] Suppose we flip a coin, and observe either a head or a tail. The probability of observing a head in the first trial is p $(0 \leq p \leq 1)$. The probability of observing a head in the second trial is $2p$. The probability of observing a head in the third and fourth trials is $3p$, $4p$ respectively. The probability of observing a head in the fifth trial is $5p$. If we flip the coin five times, and observe *(head, head, tail, tail, head)*, what is the maximum likelihood estimate of p? Justify your answer.
$P(head, head, tail, tail, head) = p * 2p * (1-3p) * (1-4p) * 5p = 120p^5 - 70p^4 + 10p^3$

$$\frac{dP}{dp} = 600p^4 - 280p^3 + 30p^2 = 0$$
$$p = \frac{1}{6} \; or \; p = \frac{3}{10}$$
$$0 \leq 5p \leq 1 \;\; \therefore p = \frac{1}{6}$$

4 Quadratic Optimization (17 points)

[7 points.] Solve the following optimization problem

$$\text{argmin}_x \; 3x^2 - 12x + 8$$

$f(x) = 3(x-2)^2 - 4$. therefore $x_* = 2, and\ f_* = -4$

$f'(x) = 6x - 12$ set $f'(x) = 0$. therefore $x_* = 2, and\ f_* = -4$

[10 points.] Solve the following optimization problem

$$\begin{aligned} &\text{argmin}_x \; 3x^2 - 12x + 8 \\ &\text{subject to } 6 \leq x \leq 10 \end{aligned}$$

$f'(x) = 6x - 12 > 0\ when\ 6 \leq x \leq 10.$

therefore $x_* = 6, and\ f_* = 44$

5 Distance Metric (18 points)

[18 points]. Given two data points $x_1 = (0, 0)'$ and $x_2 = (1, 2)'$ in two-dimensional space, what is the L_2 distance between them? What is the L_1 distance between them? What is the L_∞ distance between them? Justify your answer.

$L_2 = \sqrt{(0-1)^2 + (0-2)^2} = \sqrt{5}$

$L_1 = |0-1| + |0-2| = 3$

$L_\infty = max(|0-1|, |0-2|) = 2$