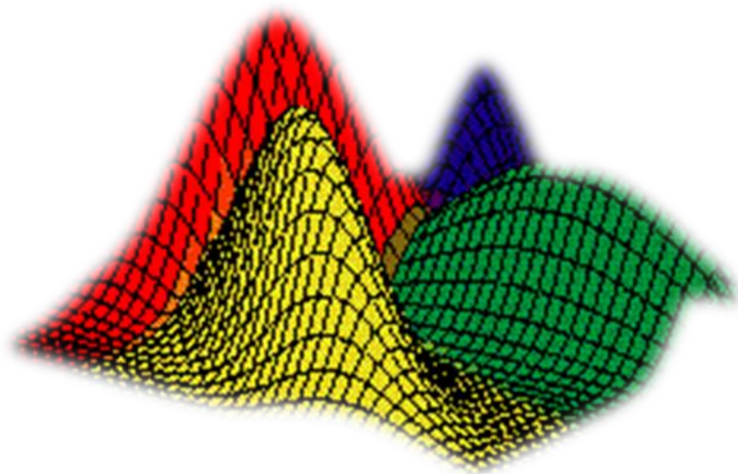
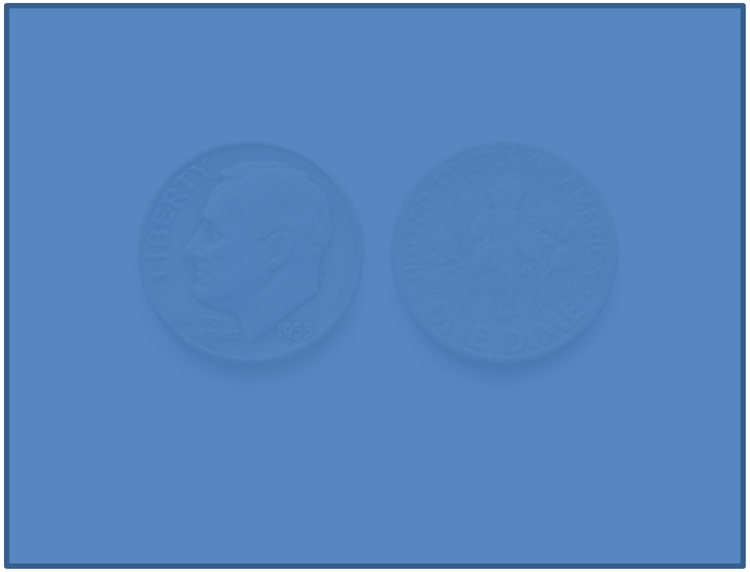


CSE 473: Pattern Recognition

Hidden Markov Model



Main issues using HMMs

- **Evaluation problem.**

Given the HMM $M=(A, B, \pi)$ and the observation sequence $O=o_1 o_2 \ldots o_K$, calculate the probability that model M has generated sequence O.

$O=o_1 \ldots o_K$ denotes a sequence of observations $o_k \in \{v_1, \ldots, v_M\}$.

Main issues using HMMs

•**Decoding problem.**

Given the HMM $M=(A, B, \pi)$ and the observation sequence $O=o_1\, o_2 \ldots o_K$, calculate the most likely sequence of hidden states s_i that produced this observation sequence O.

$O=o_1 \ldots o_K$ denotes a sequence of observations $o_k \in \{v_1, \ldots, v_M\}$.

Main issues using HMMs

•**Learning problem.**

Given some training observation sequences $O=o_1\, o_2 \ldots o_K$ and general structure of HMM (numbers of hidden and visible states), determine HMM parameters $M=(A, B, \pi)$ that best fit training data.

$O=o_1 \ldots o_K$ denotes a sequence of observations $o_k \in \{v_1, \ldots, v_M\}$.

The evaluation Problem

Given the HMM $M=(A, B, \pi)$ and the observation sequence $O=o_1\ o_2\ ...\ o_K$, calculate the probability that model M has generated sequence O.

where, $O=o_1...o_K$ denotes a sequence of observations $o_k \in \{v_1, \ldots, v_M\}$.

The evaluation Problem

Given the HMM $M=(A, B, \pi)$ and the observation sequence $O=o_1\, o_2 \ldots o_K$, calculate the probability that model M has generated sequence O.

where, $O=o_1 \ldots o_K$ *denotes a sequence of observations* $o_k \in \{v_1, \ldots, v_M\}$.

Alternately, find $P(O|M)$ or $P(o_1\, o_2 \ldots o_K|M)$

The evaluation Problem

Given the HMM $M=(A, B, \pi)$ and the observation sequence $O=o_1\, o_2 \ldots o_K$, calculate the probability that model M has generated sequence O.

where, $O=o_1 \ldots o_K$ *denotes a sequence of observations* $o_k \in \{v_1, \ldots, v_M\}$.

Alternately, find $P(O|M)$ or $P(o_1\, o_2 \ldots o_K|M)$

For simplicity we write it as $P(O)$ or $P(o_1\, o_2 \ldots o_K)$

The evaluation Problem

Objective:

- find P(O) or P(o_1 o_2 ... o_K)

$$P(O) = \sum_i p(O, \Omega_i)$$

where, Ω_i is a possible state sequence

$$S_{i_1}, S_{i_2}, \ldots, S_{i_m}, \ldots, S_{i_K}$$

The evaluation Problem

Objective:

- find $P(O)$ or $P(o_1\ o_2 \ldots o_K)$

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There are N^K possible state sequences!!

The evaluation Problem

Objective:

- find P(O) or P(o_1 o_2 ... o_K)

$$P(O) = \sum_i p(O, \Omega_i)$$

where, Ω_i is a possible state sequence

$$S_{i_1}, S_{i_2}, \ldots, S_{i_m}, \ldots, S_{i_K}$$

Complexity is O(N^K)

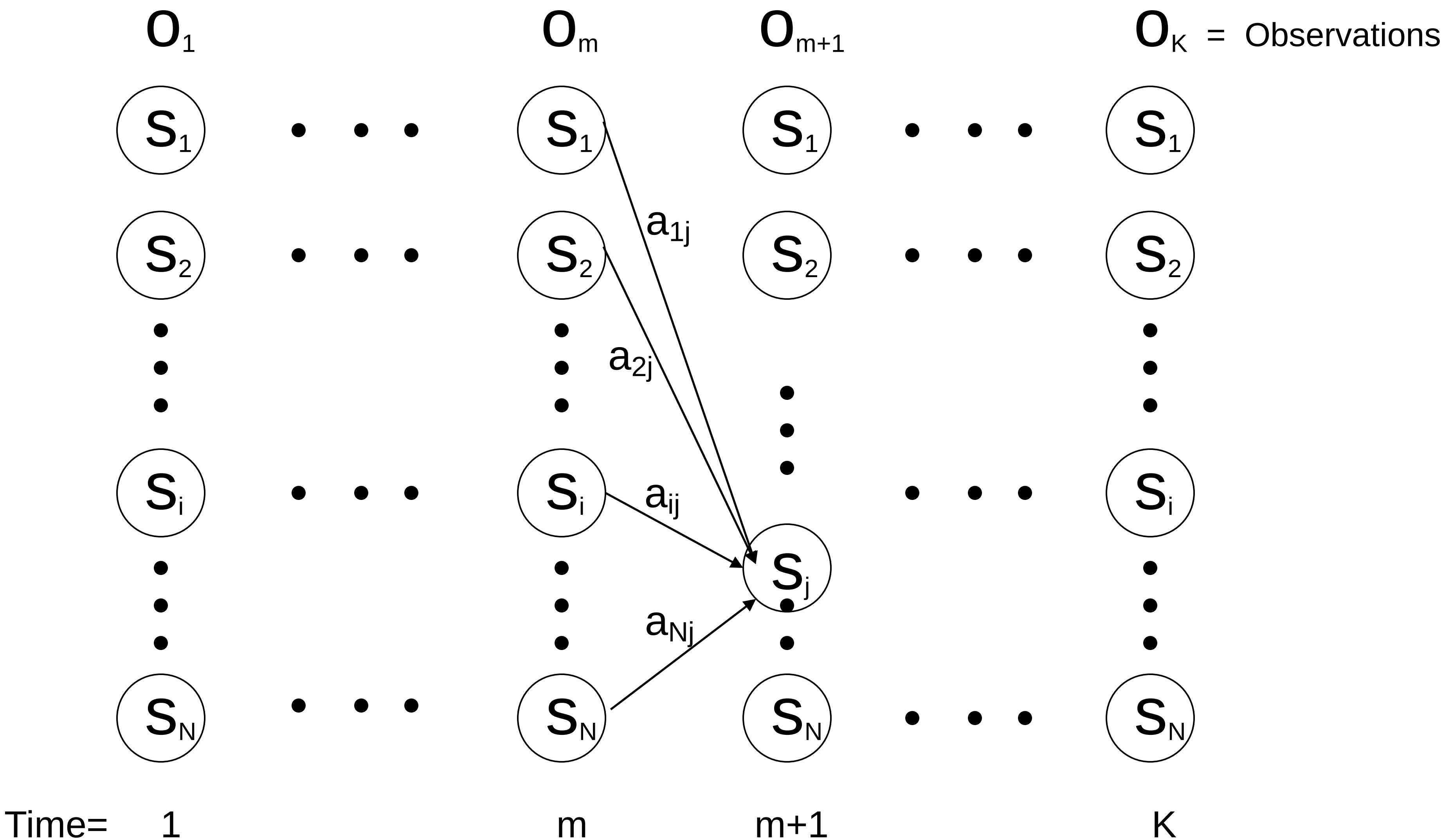
Alternate Solution to The evaluation Problem

- Use **Forward-Backward HMM algorithms** for efficient calculations.

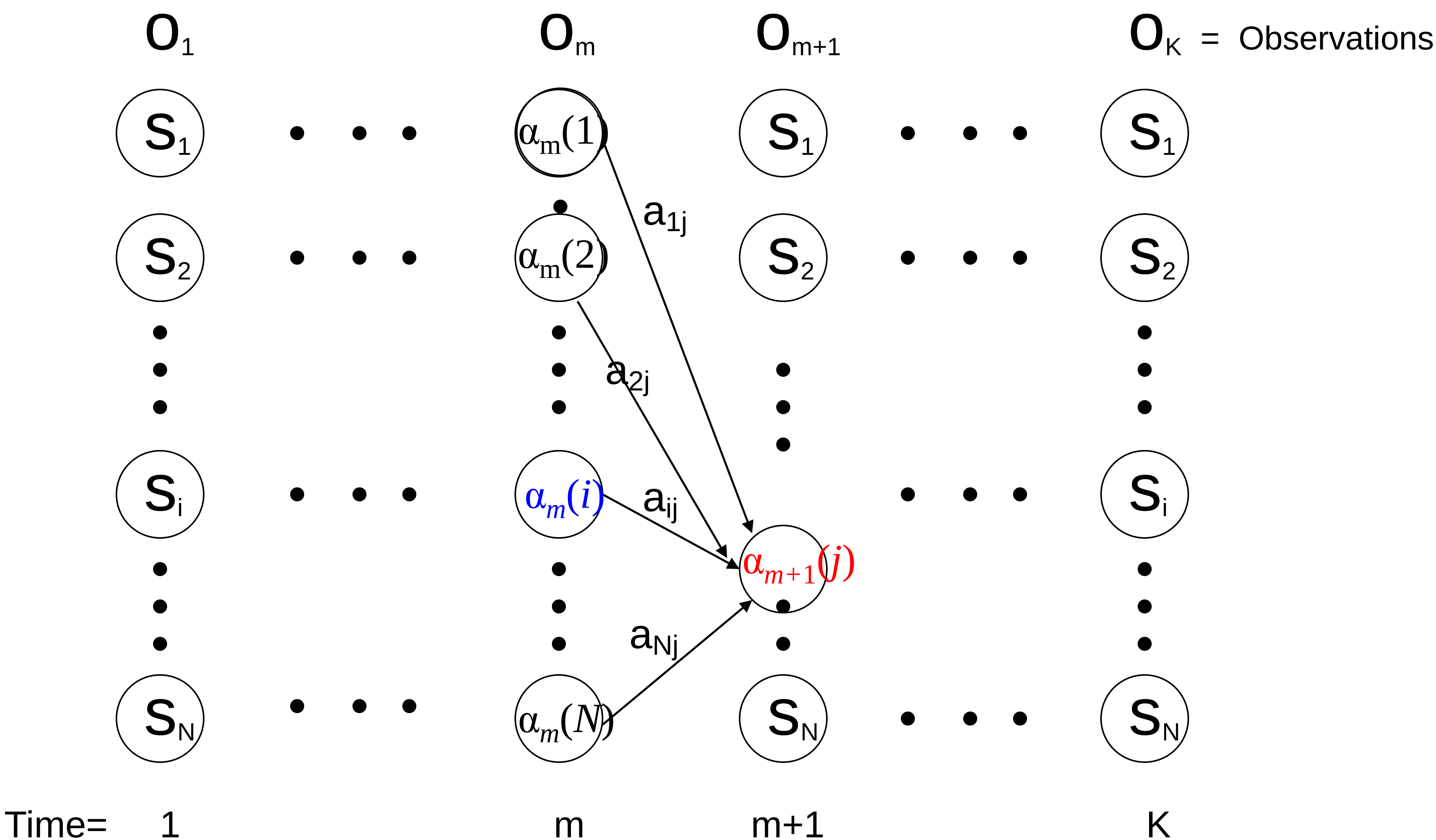
- Define the forward variable $\alpha_m(i)$ as the joint probability of
  - the partial observation sequence $o_1\, o_2 \ldots o_m$ and
  - the hidden state at time m is s_i :

$$\alpha_m(i) = P(o_1, o_2 \ldots o_m, q_m = s_i)$$

Trellis representation of an HMM



Trellis representation of an HMM



Trellis representation of an HMM

Therefore, we can write,

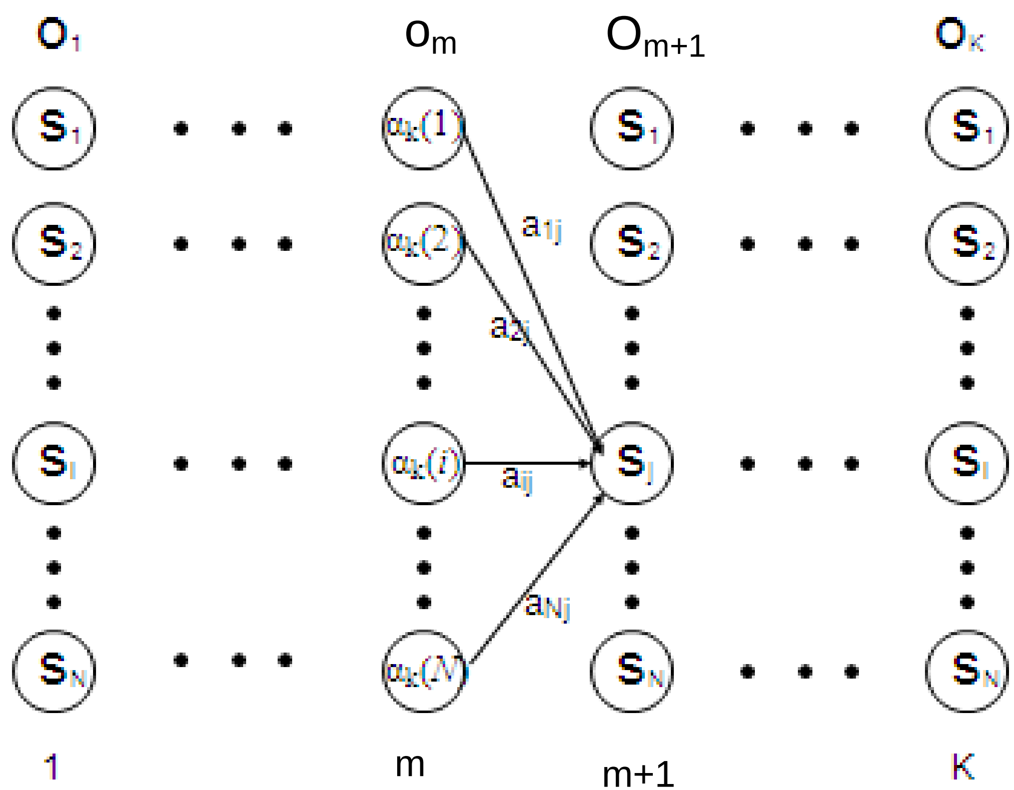
$\alpha_{m+1}(j) = P(o_1\, o_2 \ldots\, o_{m+1}, q_{m+1} = s_j)$

$= \Sigma_i\, P(o_1\, o_2 \ldots\, o_{m+1}, q_m = s_i, q_{m+1} = s_j)$

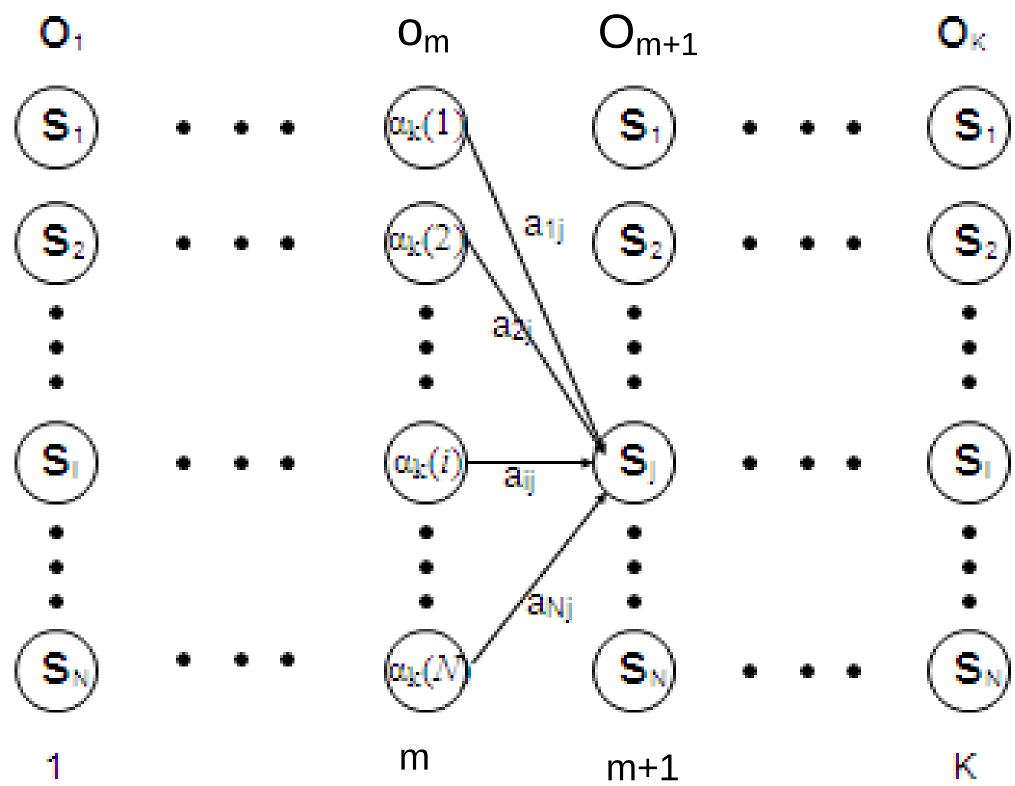
$= \Sigma_i\, P(o_1\, o_2 \ldots o_m, q_m = s_i)\, a_{ij}\, b_j(o_{m+1})$

$= [\Sigma_i\, \alpha_m(i)\, a_{ij}]\, b_j(o_{m+1})$,

for $1 <= j <= N$, $1 <= m <= K-1$.



Trellis representation of an HMM



Now $P(o_1\, o_2 \ldots o_K)$

can be written as $\Sigma_i\, P(o_1\, o_2 \ldots o_K,\, q_K = s_i) = \Sigma_i\, \alpha_K(i)$

Forward recursion for HMM

- Initialization:

$$\alpha_1(i) = P(o_1, q_1 = s_i) = \pi_i \, b_i(o_1), \quad 1 <= i <= N.$$

- Forward recursion:

$$\alpha_{m+1}(j) = [\Sigma_i \, \alpha_m(i) \, a_{ij}] \, b_j(o_{m+1}), \quad 1 <= j <= N, \; 1 <= m <= K-1.$$

- Termination:

$$P(o_1 o_2 \ldots o_K) = \Sigma_i \, \alpha_K(i)$$

- Complexity :

N^2K operations.

Backward recursion for HMM

- Define the backward variable $\beta_m(j)$ as the conditional probability of
  - the partial observation sequence $o_{m+1}\, o_{m+2} \ldots o_K$
  - given that the hidden state at time m is s_j :

$$\beta_m(j) = P(o_{m+1}\, o_{m+2} \ldots o_K \mid q_m = s_j)$$

Backward recursion for HMM

- Define $\beta_m(j)$ in terms of $\beta_{m+1}(i)$'s:

- $\beta_{m+1}(i)$ is the conditional probability of

    - the partial observation sequence $o_{m+2}\, o_{m+3} \ldots o_K$
    - given that the hidden state at time m+1 is s_i :

$$\beta_{m+1}(i) = P(o_{m+2}\, o_{m+3} \ldots o_K \mid q_{m+1} = s_i)$$

Backward recursion for HMM

- Define $\beta_m(j)$ in terms of $\beta_{m+1}(i)$'s: the probability of

where $\beta_{m+1}(i) = P(o_{m+2}\, o_{m+3} \ldots o_K \mid q_{m+1} = s_i)$

Now,

$\beta_m(j) = P(o_{m+1}\, o_{m+2} \ldots o_K \mid q_m = s_j)$

$= \Sigma_i\, P(o_{m+1}\, o_{m+2} \ldots o_K,\, q_{m+1} = s_i \mid q_m = s_j)$

$= \Sigma_i\, P(o_{m+2}\, o_{m+3} \ldots o_K \mid q_{m+1} = s_i)\, a_{ji}\, b_i(o_{m+1})$

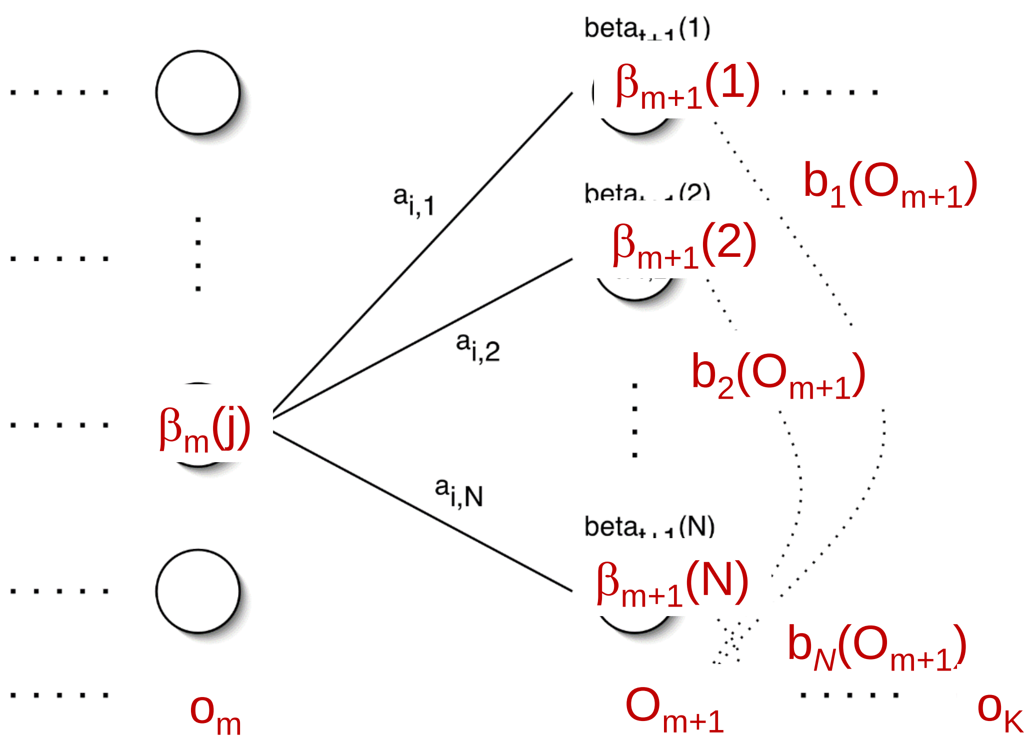
$= \Sigma_i\, \beta_{m+1}(i)\, a_{ji}\, b_i(o_{m+1})$, $\quad 1 \le j \le N,\ 1 \le m \le K-1.$

Backward recursion for HMM

$= \Sigma_i \, P(o_{m+1}\, o_{m+2} \ldots o_K \,,\, q_{m+1} = s_i \mid q_m = s_j\,)$

$= \Sigma_i \, P(o_{m+2}\, o_{m+3} \ldots o_K \mid q_{m+1} = s_i)\; a_{ji}\; b_i\,(o_{m+1}\,)$

$= \Sigma_i \, \beta_{m+1}(i)\; a_{ji}\; b_i\,(o_{m+1}\,)$, $1 <= j <= N,\ 1 <= m <= K-1.$



Backward recursion for HMM

- <u>Initialization:</u>

  $\beta_K(i) = 1 \quad , 1 <= i <= N.$

- <u>Backward recursion:</u>

  $\beta_m(j) = \Sigma_i \, \beta_{m+1}(i) \, a_{ji} \, b_i \, (o_{m+1}) \, , \qquad 1 <= j <= N, \ 1 <= m <= K-1.$

- <u>Termination:</u>

  $P(o_1 \, o_2 \, ... \, o_K) = \Sigma_i \, P(o_1 \, o_2 \, ... \, o_K \, , q_1 = s_i) =$

  $\Sigma_i \, P(o_1 \, o_2 \, ... \, o_K \, | q_1 = s_i) \, P(q_1 = s_i) = \Sigma_i \, \beta_1(i) \, b_i \, (o_1) \, \pi_I$

Main issues using HMMs (2)

- **Decoding problem.**

Given the HMM $M=(A, B, \pi)$ and the observation sequence $O=o_1 o_2 ... o_K$, calculate the most likely sequence of hidden states s_i that produces this observation sequence O.

$O=o_1...o_K$ denotes a sequence of observations $o_k \in \{v_1, \ldots, v_M\}$.

Decoding problem

Given the HMM $M=(A, B, \pi)$ and the observation sequence $O=o_1\, o_2 \ldots o_K$, calculate the most likely sequence of hidden states s_i that produces this observation sequence O.

We want to find:
the state sequence $Q= q_1 \ldots q_K$ maximizing

$$P(Q \mid o_1\, o_2 \ldots o_K)$$

Decoding problem

Given the HMM $M=(A, B, \pi)$ and the observation sequence $O=o_1\,o_2 \ldots o_K$, calculate the most likely sequence of hidden states s_i that produces this observation sequence O.

We want to find:
the state sequence $Q= q_1 \ldots q_K$ maximizing

$$P(Q \mid o_1\,o_2 \ldots o_K)$$

or, equivalently

$$P(Q , o_1\,o_2 \ldots o_K)$$

Decoding problem

- Find max value of $P(Q , o_1 \, o_2 \ldots o_K)$

Brute Force Method:

Try for all possible sequences of states

N^K possible sequences for Q

Decoding problem

- Use efficient **Viterbi algorithm** instead

- Define variable $\delta_m(i)$ as the maximum probability of
  - producing observation sequence $o_1 \, o_2 \dots o_m$
  - when moving along any hidden state sequence $q_1 \dots q_{m-1}$ and
  - getting into $q_m = s_i$

Decoding problem

- Use efficient **Viterbi algorithm** instead

- Define variable $\delta_m(i)$ as the maximum probability of
  - producing observation sequence $o_1\, o_2 \ldots o_m$
  - when moving along any hidden state sequence $q_1 \ldots q_{m-1}$ and
  - getting into $q_m = s_i$

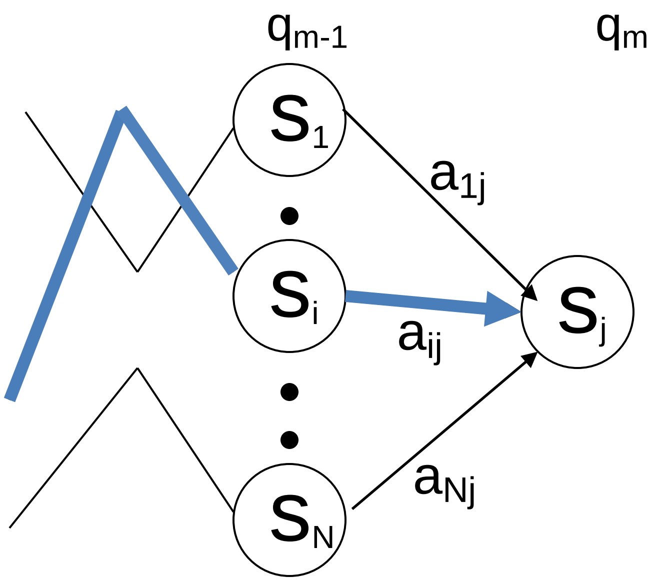
Therefore, $\delta_m(i) = \max P(q_1 \ldots q_{m-1}, q_m = s_i, o_1\, o_2 \ldots o_m)$

where max is taken over all possible paths $q_1 \ldots q_{m-1}$.

Decoding problem

- General idea:

  if best path ending in $q_m = s_j$ goes through $q_{m-1} = s_i$ then it should coincide with best path ending in $q_{m-1} = s_i$.



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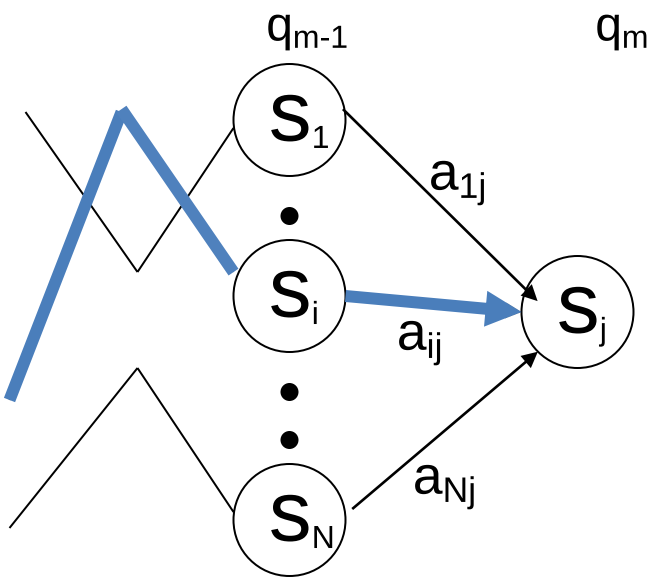

- $\delta_m(j) = \max P(q_1 \ldots q_{m-1}, q_m = s_j, o_1 o_2 \ldots o_m)$

Decoding problem

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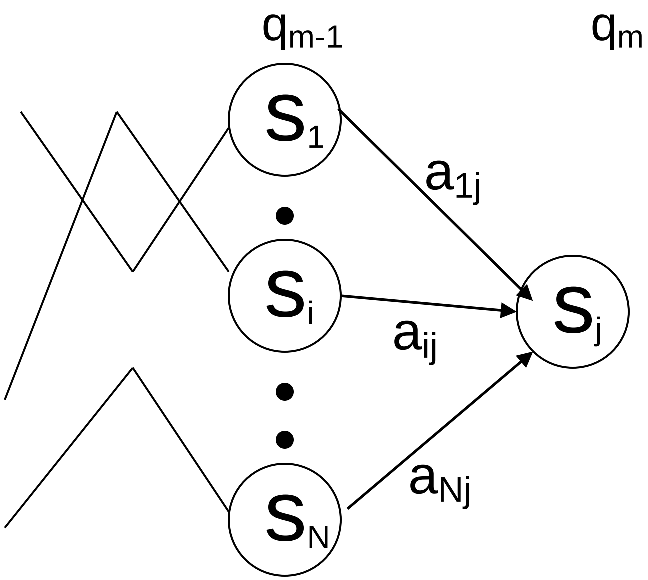


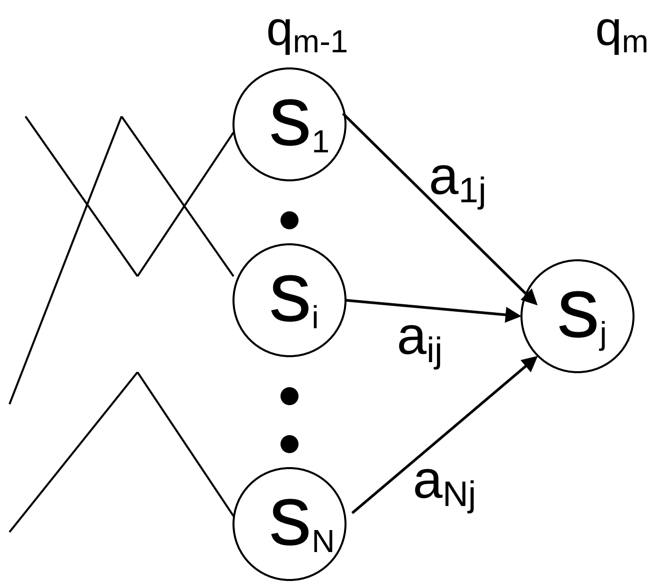
- $\delta_m(j) = \max P(q_1 \ldots q_{m-1}, q_m = s_j, o_1 o_2 \ldots o_m)$

$$= \max_i [\, a_{ij}\, b_j(o_m) \; \max P(q_1 \ldots q_{m-1} = s_i, o_1 o_2 \ldots o_{m-1}) \,]$$

Decoding problem

- General idea:

 if best path ending in $q_m = s_j$ goes through $q_{m-1} = s_i$ then it should coincide with best path ending in $q_{m-1} = s_i$.



- $\delta_m(j) = \max P(q_1 \ldots q_{m-1}, q_m = s_j, o_1 o_2 \ldots o_m)$

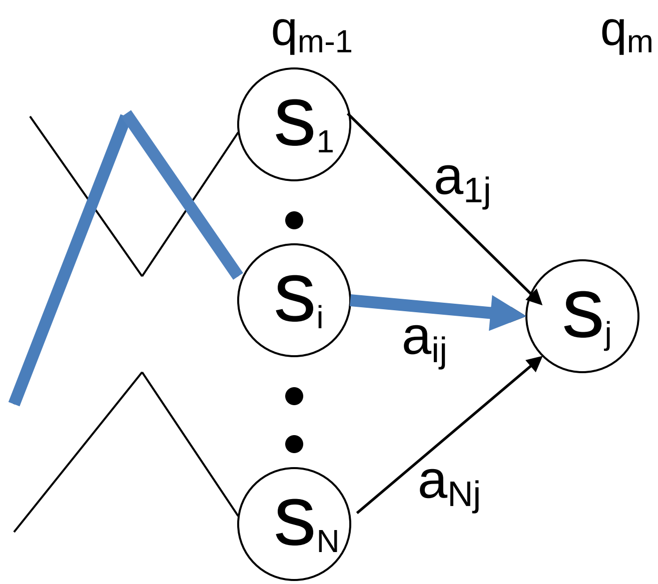
$$= \max_i [\, a_{ij}\, b_j(o_m) \ \max P(q_1 \ldots q_{m-1} = s_i, o_1 o_2 \ldots o_{m-1})\,]$$

$$= \max_i [\, a_{ij}\, b_j(o_m)\, \delta_{m-1}(i)]$$

Decoding problem

- General idea:

  if best path ending in $q_m = s_j$ goes through $q_{m-1} = s_i$ then it should coincide with best path ending in $q_{m-1} = s_i$.



- To backtrack best path, keep info that predecessor of s_j was s_i.

Decoding problem

- Initialization:

$$\delta_1(i) = \max P(q_1 = s_i\ ,\ o_1) = \pi_i\, b_i\,(o_1)\ ,\ 1 <= i <= N$$

- Forward recursion:

$$\delta_m(j) = \max_i [\ a_{ij}\, b_j\,(o_m)\ \delta_{m-1}(i)\ ]\ , \qquad 1<=j<=N,\ 2<=m<=K.$$

- Termination: choose best path ending at time K

$$\max_i [\ \delta_K(i)\ ]$$

- Backtrack best path.

Issues in HMMs (3)

•**Learning/Training problem.**

Given some training observation sequences $O=o_1\,o_2 \ldots o_K$ and general structure of HMM (numbers of hidden and visible states), determine HMM parameters $M=(A, B, \pi)$ that best fit training data.

$O=o_1,\ldots, o_m,\ldots,o_K$ *denotes a sequence of observations* where, $o_m \in \{v_1,\ldots,v_M\}$.

Learning/Training Problem

Given some training observation sequences $O=o_1\, o_2 \ldots o_K$ and general structure of HMM (numbers of hidden and visible states), determine HMM parameters $M=(A, B, \pi)$ that best fit training data.

•There is no algorithm producing optimal parameter values.

Learning/Training Problem

Given some training observation sequences $O=o_1\, o_2 \ldots o_K$ and general structure of HMM (numbers of hidden and visible states), determine HMM parameters $M=(A, B, \pi)$ that best fit training data.

- There is no algorithm producing optimal parameter values.

- Use iterative Expectation-Maximization (EM) algorithm to find local maximum of $P(O \mid M)$ - **Baum-Welch algorithm.**

Learning/Training Problem

Idea of EM :

Initialization: Assume initial value of A, B, π

and calculate $P(O \mid M)$

E-step:

Estimate new values of the model parameters: A, B, π

from the previous values of A, B, π

M-step:

Find $P(O \mid M)$ with the new values of A, B, π

Repeat these steps until $P(O \mid M)$ declines

Learning/Training Problem

- we need to calculate the following parameters:

$$a_{ij} = P(s_j \mid s_i) = \frac{\text{No. of transitions from state } s_i \text{ to state } s_j}{\text{No. of transitions out of state } s_i}$$

$$b_i(v_n) = P(v_n \mid s_i) = \frac{\text{No. of times observation } v_n \text{ occurs in state } s_i}{\text{No. of times in state } s_i}$$

$$\pi(i) = P(s_i) = \frac{\text{No. of times in state } s_i \text{ at time } m=1}{\text{No. of times in any state at time } m=1}$$

Baum-Welch algorithm

- the algorithm estimates the expected value::

$$a_{ij} = P(s_j \mid s_i) = \frac{\text{Expected No. of transitions from state } s_i \text{ to state } s_j}{\text{Expected No. of transitions out of state } s_i}$$

$$b_i(v_n) = P(v_n \mid s_i) = \frac{\text{Expected No. of times observation } v_n \text{ occurs in state } s_i}{\text{Expected No. of times in state } s_i}$$

$$\pi(i) = P(s_i) = \frac{\text{Expected No. of times in state } s_i \text{ at time } m = 1}{\text{Expected No. of times in any state at time } m = 1}$$

Baum-Welch algorithm

- Define variable $\xi_m(i,j)$ as the probability of being in state s_i at time m and in state s_j at time m+1, given the observation sequence $o_1\, o_2 \dots o_K$.

$$\xi_m(i,j) = P(q_m = s_i\ ,\ q_{m+1} = s_j \mid o_1\, o_2 \dots o_K)$$

Baum-Welch algorithm

- Define variable $\xi_m(i,j)$ as the probability of being in state s_i at time m and in state s_j at time m+1, given the observation sequence $o_1\, o_2 \ldots o_K$.

$$\xi_m(i,j) = P(q_m = s_i\,,\, q_{m+1} = s_j \mid o_1\, o_2 \ldots o_K)$$

$$\xi_m(i,j) = \frac{P(q_m = s_i\,,\, q_{m+1} = s_j\,,\, o_1\, o_2 \ldots o_K)}{P(o_1\, o_2 \ldots o_K)}$$

Baum-Welch algorithm

- Define variable $\xi_m(i,j)$ as the probability of being in state s_i at time m and in state s_j at time m+1, given the observation sequence $o_1\, o_2 \ldots o_K$.

$$\xi_m(i,j) = P(q_m = s_i\ , q_{m+1} = s_j \mid o_1\, o_2 \ldots o_K)$$

$$\xi_m(i,j) = \frac{P(q_m = s_i\ , q_{m+1} = s_j\ , o_1\, o_2 \ldots o_K)}{P(o_1\, o_2 \ldots o_K)}$$

$$= \frac{P(q_m = s_i\ , o_1\, o_2 \ldots o_m)\ a_{ij}\ b_j\,(o_{m+1})\ P(o_{m+2} \ldots o_K \mid q_{m+1} = s_j)}{P(o_1\, o_2 \ldots o_K)}$$

Baum-Welch algorithm

- Define variable $\xi_m(i,j)$ as the probability of being in state s_i at time m and in state s_j at time $m+1$, given the observation sequence $o_1\, o_2 \ldots o_K$.

$$\xi_m(i,j) = P(q_m = s_i\ ,\ q_{m+1} = s_j \mid o_1\, o_2 \ldots o_K)$$

$$= \frac{P(q_m = s_i\ ,\ o_1\, o_2 \ldots o_m)\ a_{ij}\ b_j(o_{m+1})\ P(o_{m+2} \ldots o_K \mid q_{m+1} = s_j)}{P(o_1\, o_2 \ldots o_K)}$$

$$= \frac{\alpha_m(i)\ a_{ij}\ b_j(o_{m+1})\ \beta_{m+1}(j)}{P(o_1\, o_2 \ldots o_K)}$$

Baum-Welch algorithm

- Define variable $\gamma_m(i)$ as the probability of being in state s_i at time m, given the observation sequence $o_1\, o_2 \ldots o_K$.

$$\gamma_m(i) = P(q_m = s_i \mid o_1\, o_2 \ldots o_K)$$

$$\gamma_m(i) = \frac{P(q_m = s_i\, ,\, o_1\, o_2 \ldots o_k)}{P(o_1\, o_2 \ldots o_k)} = \frac{\alpha_m(i)\, \beta_m(i)}{P(o_1\, o_2 \ldots o_k)}$$

Baum-Welch algorithm

- We calculated $\xi_m(i,j) = P(q_m = s_i , q_{m+1} = s_j \mid o_1\, o_2 \dots o_K)$
  and $\gamma_m(i) = P(q_m = s_i \mid o_1\, o_2 \dots o_K)$

- Expected number of transitions from state s_i to state s_j
  $= \Sigma_m \xi_m(i,j)$

- Expected number of transitions out of state $s_i = \Sigma_m \gamma_m(i)$

- Expected number of times observation v_n occurs in state s_i =
  $= \Sigma_m \gamma_m(i)$, m is such that $o_m = v_n$

Baum-Welch algorithm: E-Step

Estimate the expected values as

$$a_{ij} = \frac{\text{Expected No. of transitions from state } s_i \text{ to state } s_j}{\text{Expected No. of transitions out of state } s_i} = \frac{\sum_m \xi_m(i,j)}{\sum_m \gamma_m(i)}$$

$$b_i(v_m) = \frac{\text{Expected No. of times observation } v_n \text{ occurs in state } s_i}{\text{Expected No. of times in state } s_i} = \frac{\sum_{m, O_m = v_n} \gamma_m(i)}{\sum_m \gamma_m(i)}$$

$$\pi(i) = P(s_i) = \frac{\text{Expected No. of times in state } s_i \text{ at time } m = 1}{\text{Expected No. of times in any state at time } m = 1} = \frac{\gamma_1(i)}{\sum_i \gamma_1(i)}$$

Learning/Training Algorithm

Initialization: Assume initial value of A, B, π

and calculate $P(O \mid M)$

E-step:

Estimate new values of the model parameters: A, B, π

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Repeat these steps until $P(O \mid M)$ declines