Problems at least as hard as Clique

1905072 - Mahir Labib Dihan

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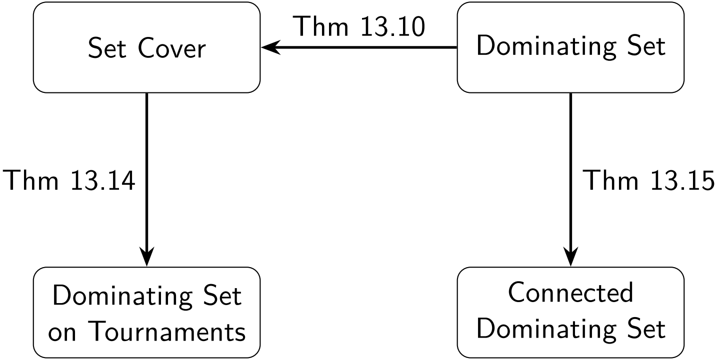
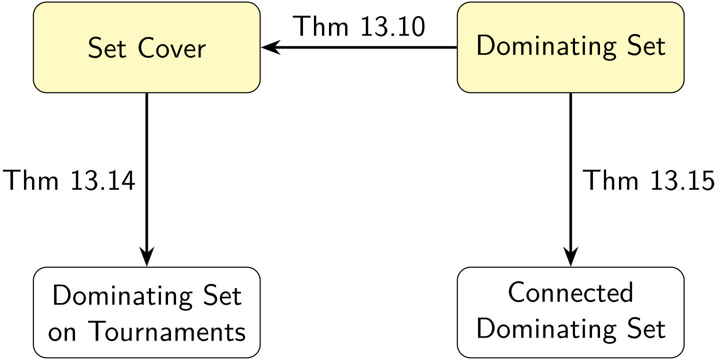
Outline

- We will see 3 Parameterized Reductions.



Outline

- From Dominating Set to Set Cover.



Theorem 13.10

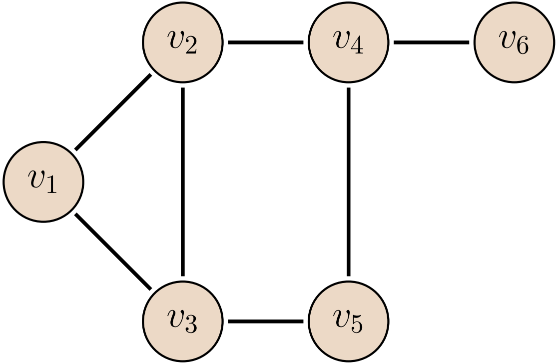
Theorem

There is a parameterized reduction from Dominating Set to Set Cover.

Theorem 13.10: Construction

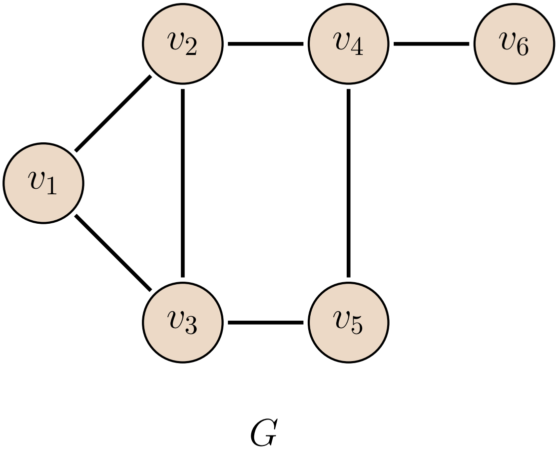
The reduction starts with an instance (G, k) of Dominating Set, and outputs an equivalent instance $(\mathcal{F}, U, k)$ of Set Cover.

■ Let G be an undirected graph. We create an instance $(\mathcal{F}, U, k)$ of Set Cover as follows.



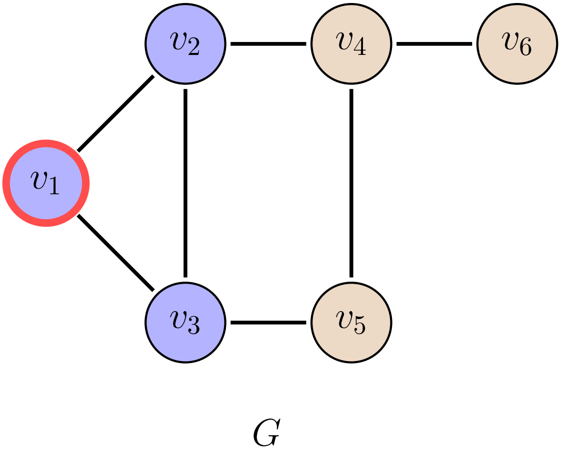
G

- We let $U := V(G)$.



$U = \{v_1, v_2, v_3, v_4, v_5, v_6\}$

■ For every $v \in V(G)$, we introduce the set $N_G[v]$ (the closed neighborhood of v) into $\mathcal{F}$.



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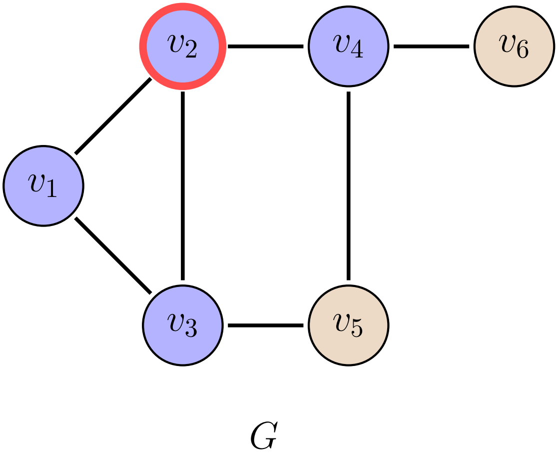

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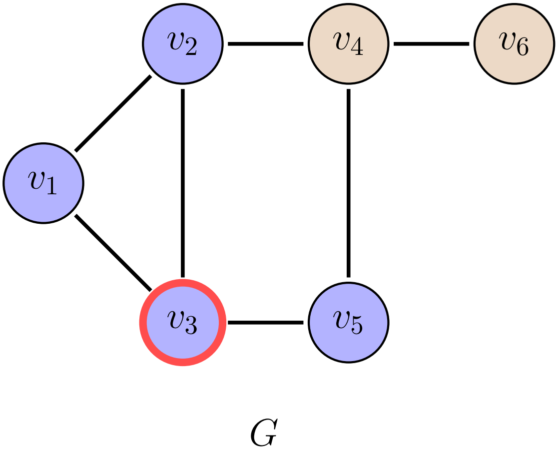

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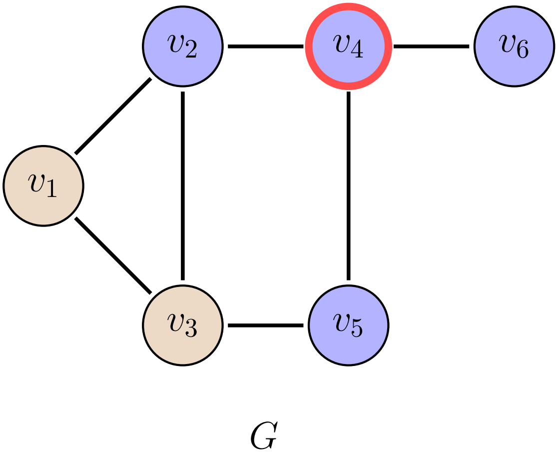

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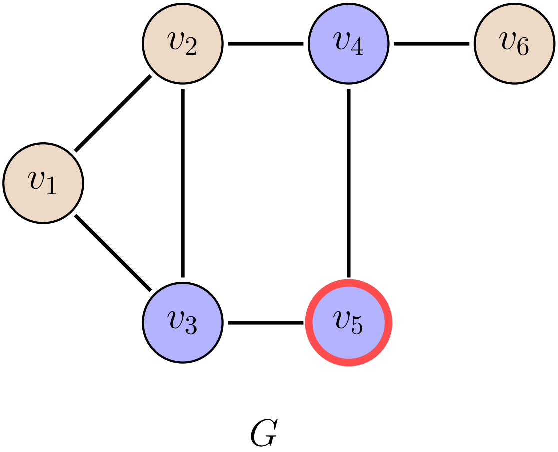

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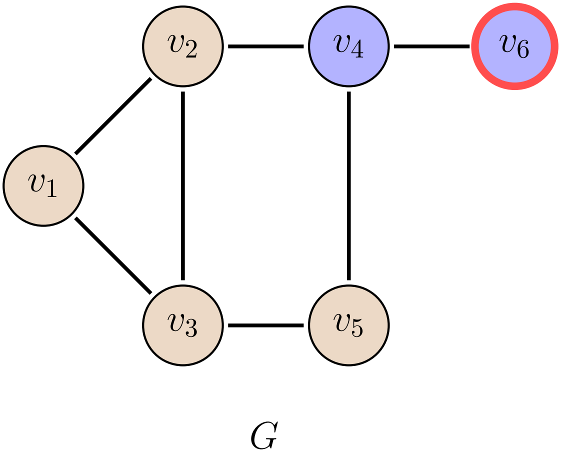

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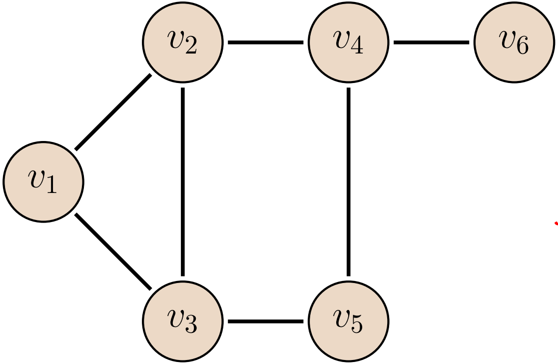

$$U = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$

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$$(G, k) \quad \Rightarrow \quad (\mathcal{F}, U, k)$$

Theorem 13.10

We now claim that G admits a dominating set of size k if and only if $(\mathcal{F}, U, k)$ is a yes-instance.

Theorem 13.10: Necessity

- Suppose that D is a dominating set of size k in G.
- Then the union of the corresponding k sets of F covers U.
- An uncovered element would correspond to a vertex of G not dominated by D.

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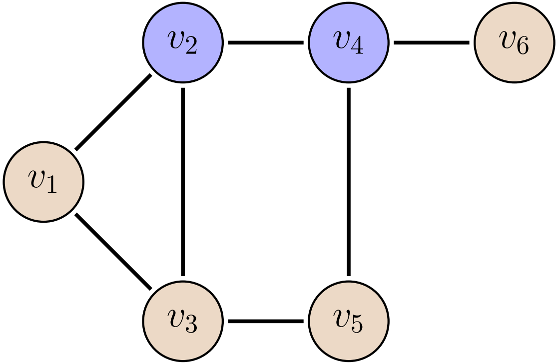

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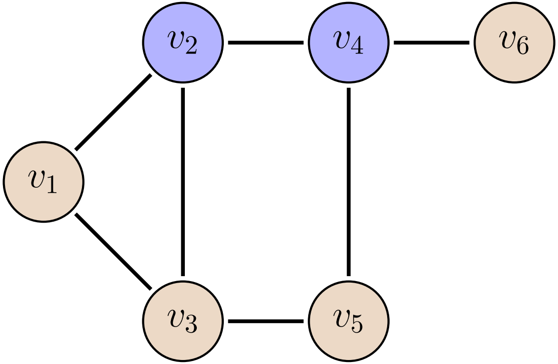
■ Then the union of the corresponding k sets of F covers U: an uncovered element would correspond to a vertex of G not dominated by D.



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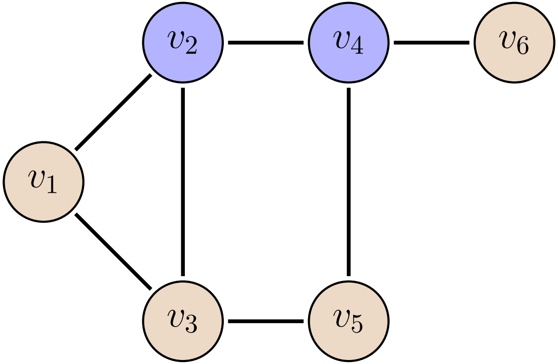

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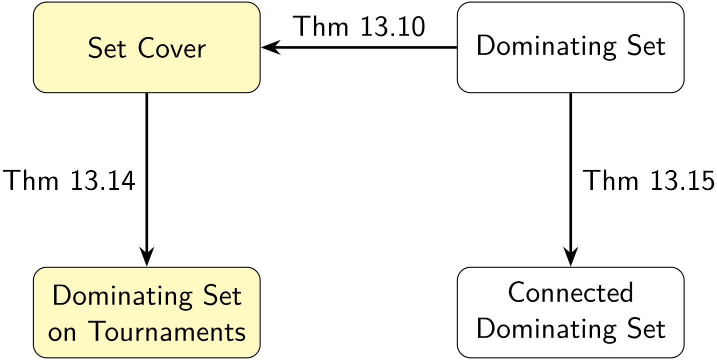
$$S_b \cup S_d = \{a, b, c, d\} \cup \{b, d, e, f\} = \{a, b, c, d, e, f\} = U$$

Theorem 13.10: Sufficiency

- Suppose that the union of k sets in F is U.
- Then the corresponding k vertices of G dominate every vertex.
- A vertex not dominated in G would correspond to an element of U not covered by the k sets.

Outline

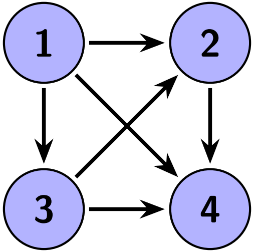
- From Set Cover to Dominating Set on Tournaments.



Tournament

Definition

A tournament is a directed graph T such that for every pair of vertices $u, v \in V(T)$, exactly one of (u, v) or (v, u) is a directed edge (also often called an arc) of T.



k-paradoxical tournament

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Sufficiently small tournaments that do not admit a dominating set of size k.

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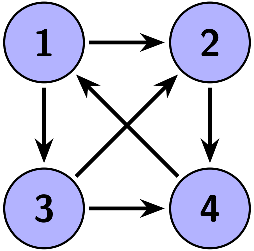

1-paradoxical

Theorem 13.14

Theorem

There is a parameterized reduction from Set Cover to Dominating Set on Tournaments.

Theorem 13.14: Construction

The reduction starts with an instance $(\mathcal{F}, U, k)$ of Set Cover, and outputs an equivalent instance $(T, k+1)$ of Dominating Set on Tournaments.

- Let $U = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and $\mathcal{F} = \{X_1, X_2, X_3, X_4, X_5\}$.

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■ The first step is a construction of a $(k+1)$-paradoxical tournament $S = T_{k+1}$ on r_{k+1} vertices.

$$U = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

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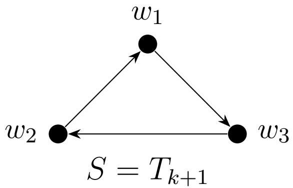

Theorem 13.14: Vertex Set

The vertex set of the constructed tournament T is defined as follows:

(i) For every $e \in U$, create a set of r_{k+1} vertices $V_e = \{v_{e,w} : w \in V(S)\}$, one for each vertex of S. Let $V_w = v_{e,w} : e \in U$, and let $V_U = \bigcup_{e \in U} V_e = \bigcup_{w \in V(S)} V_w$.

(ii) For every $X \in \mathcal{F}$, create one vertex v_X. Let $V_{\mathcal{F}} = v_X : X \in \mathcal{F}$.

(iii) Moreover, create one vertex v^*.

- The vertex set of the constructed tournament T is defined as follows:

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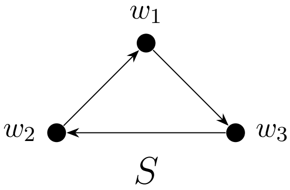

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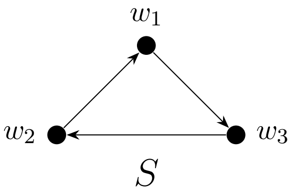

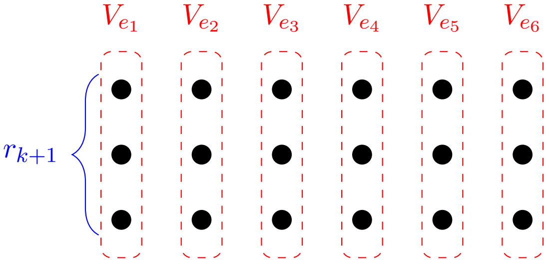

- Let $V_w = v_{e,w} : e \in U$,

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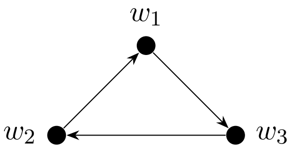

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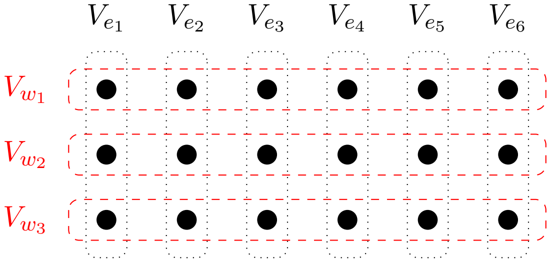

- Let $V_U = \bigcup_{e \in U} V_e = \bigcup_{w \in V(S)} V_w$.

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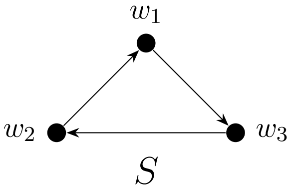

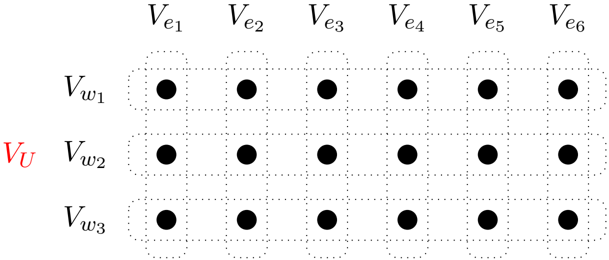

(ii) For every $X \in \mathcal{F}$, create one vertex v_X.

$$U = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

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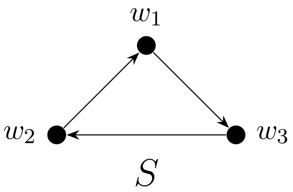

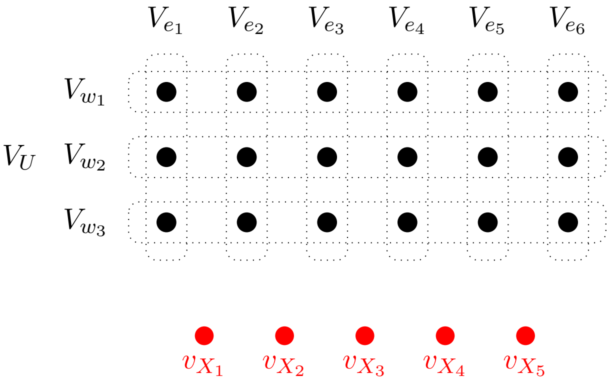

- Let $V_{\mathcal{F}} = v_X : X \in \mathcal{F}$.

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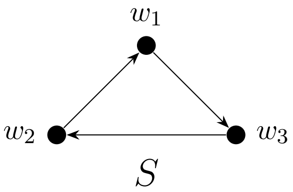

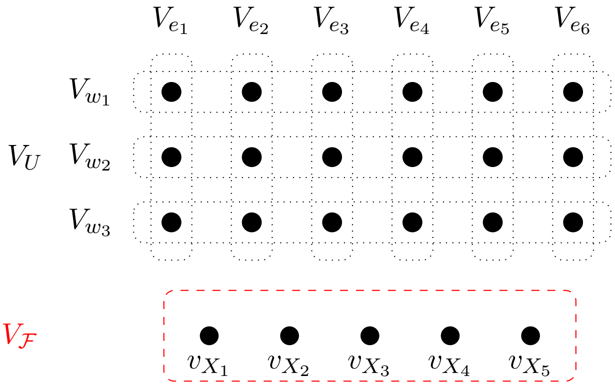

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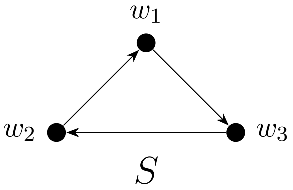

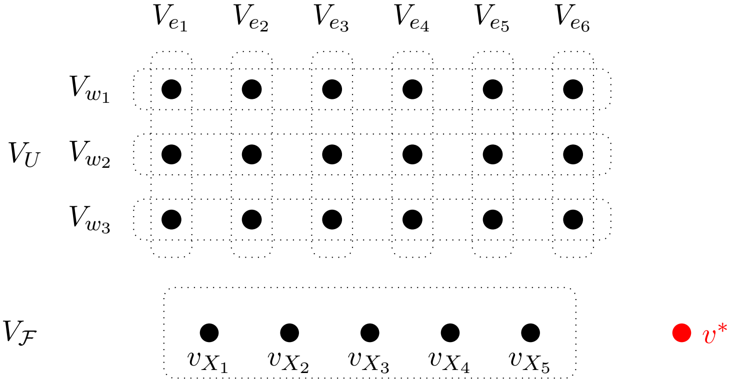

Theorem 13.14: Edge Set

We now create the edge set of T.

(i) For every set $X \in \mathcal{F}$ and every element $e \in U$, if $e \in X$ then introduce an edge from v_X to every vertex of V_e, and if $e \notin X$ then introduce an edge from every vertex of V_e to v_X.

(ii) For every set $X \in \mathcal{F}$, introduce an edge (v^*, v_X).

(iii) For every element $e \in X$ and $w \in V(S)$, introduce an edge $(v_{e,w}, v^*)$.

(iv) For every $w_1, w_2 \in V(S)$ with $w1 \neq w2$, introduce an edge from every vertex of V_{w1} to every vertex of V_{w2} if $(w_1, w_2) \in E(S)$, and introduce the reverse edges if $(w_2, w_1) \in E(S)$.

(v) For every $w \in V(S)$, put edges between vertices of V_w arbitrarily.

(vi) Finally, put the edges between vertices of $V_{\mathcal{F}}$ arbitrarily.

(i) For every set $X \in \mathcal{F}$ and every element $e \in U$, if $e \in X$ then introduce an edge from v_X to every vertex of V_e.

$$U = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

$$\mathcal{F} = \{X_1, X_2, X_3, X_4, X_5\}$$

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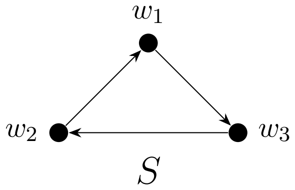

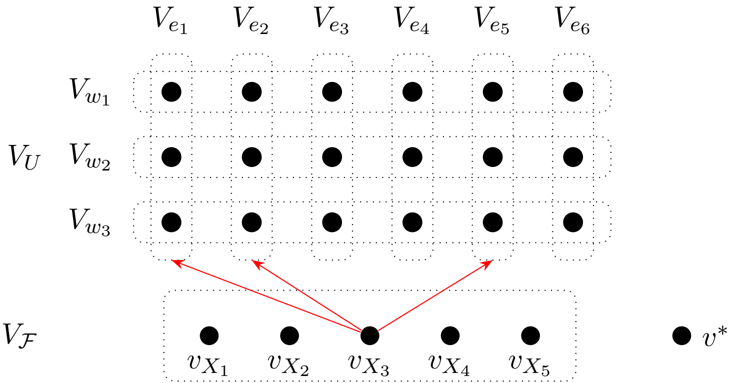

(i) ...And if $e \notin X$ then introduce an edge from every vertex of V_e to v_X.

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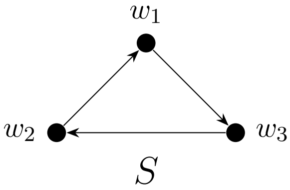

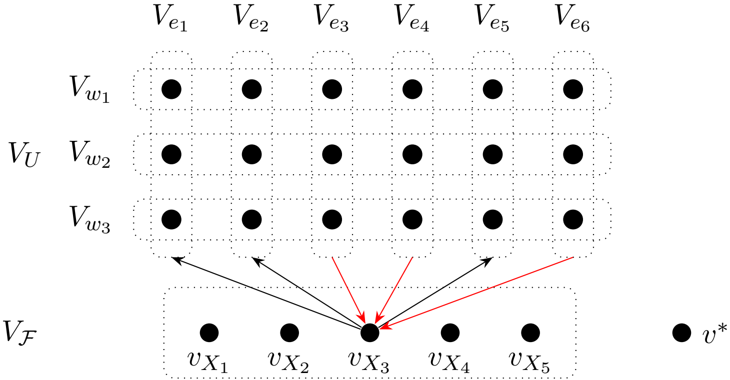

(ii) For every set $X \in \mathcal{F}$, introduce an edge (v^*, v_X).

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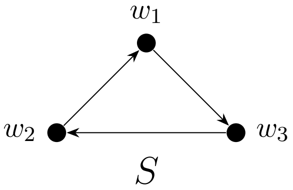

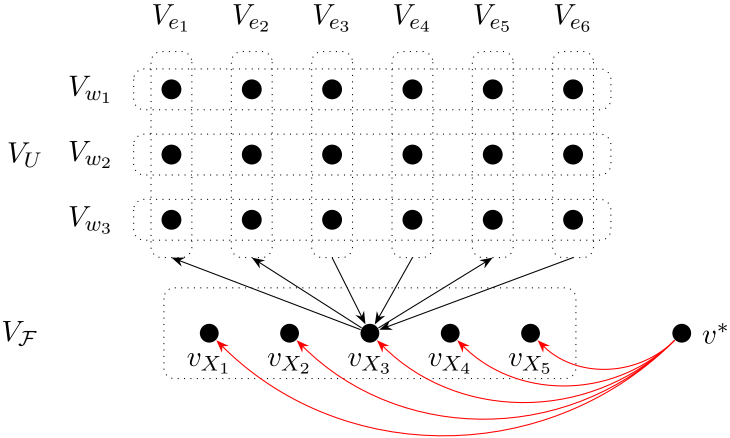

(iii) For every element $e \in X$ and $w \in V(S)$, introduce an edge $(v_{e,w}, v^*)$.

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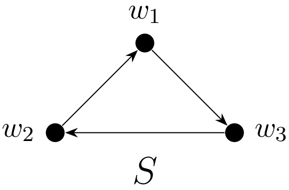
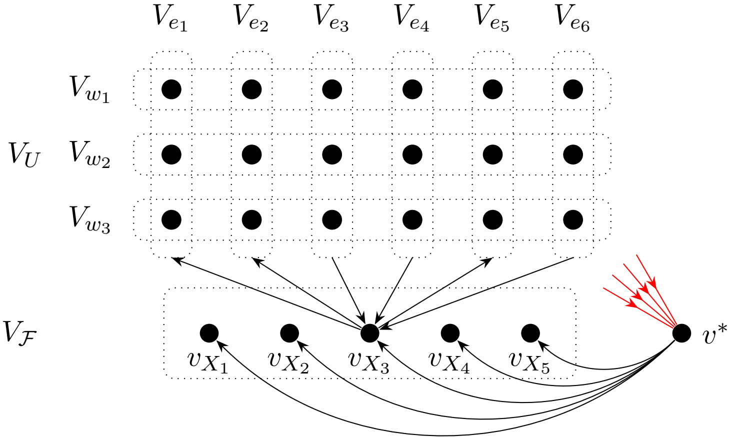
$$X_3 = \{e_1, e_2, e_5\}$$



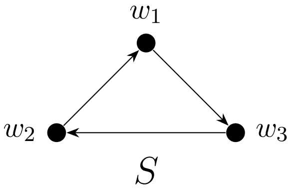


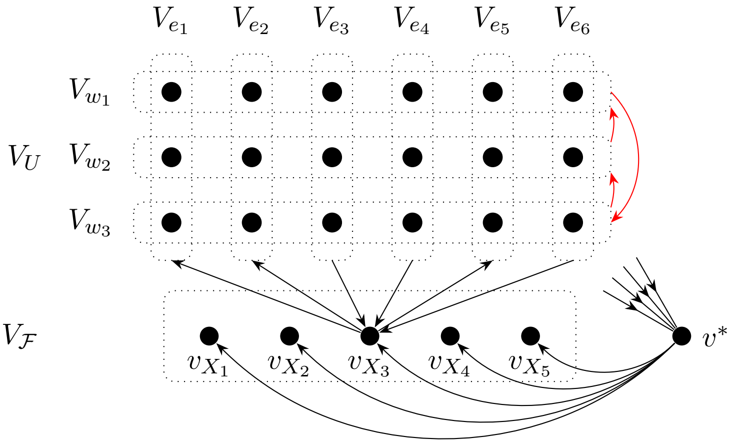
(iv) For every $w_1, w_2 \in V(S)$ with $w_1 \neq w_2$, introduce an edge from every vertex of V_{w_1} to every vertex of V_{w_2} if $(w_1, w_2) \in E(S)$.

$$U = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

$$\mathcal{F} = \{X_1, X_2, X_3, X_4, X_5\}$$

$$X_3 = \{e_1, e_2, e_5\}$$



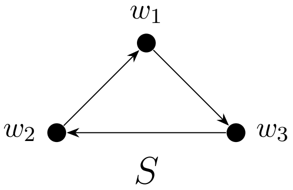


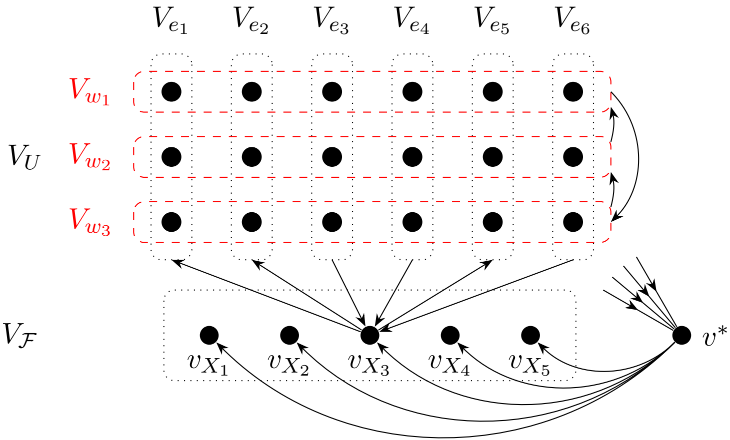
(v) For every $w \in V(S)$, put edges between vertices of V_w arbitrarily (To make tournament).

$$U = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

$$\mathcal{F} = \{X_1, X_2, X_3, X_4, X_5\}$$

$$X_3 = \{e_1, e_2, e_5\}$$



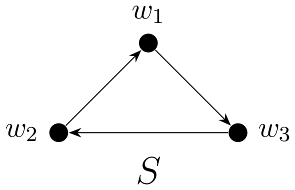


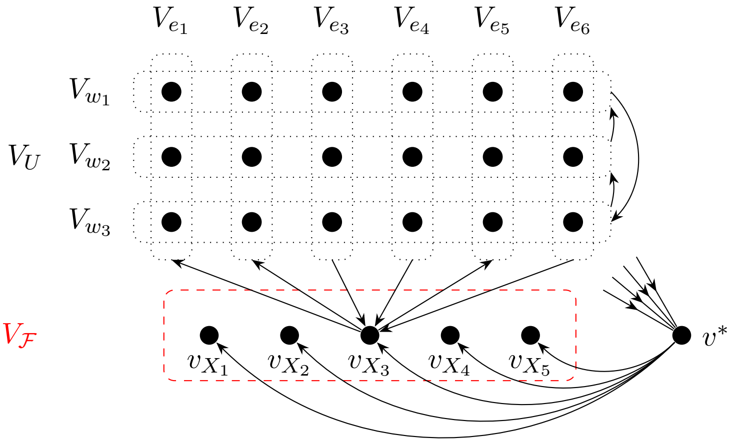
(vi) Finally, put the edges between vertices of $V_{\mathcal{F}}$ arbitrarily (To make tournament).

$$U = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

$$\mathcal{F} = \{X_1, X_2, X_3, X_4, X_5\}$$

$$X_3 = \{e_1, e_2, e_5\}$$



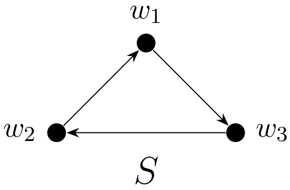


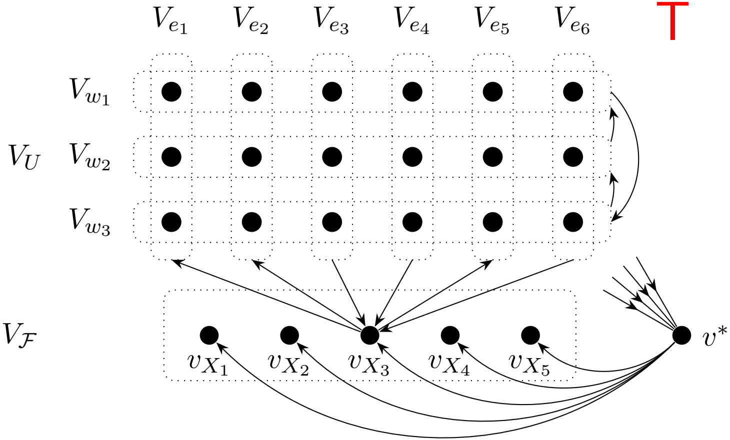
- It is easy to see that the constructed digraph T is indeed a tournament.

$$U = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

$$\mathcal{F} = \{X_1, X_2, X_3, X_4, X_5\}$$

$$X_3 = \{e_1, e_2, e_5\}$$





Theorem 13.14

We now claim that $(\mathcal{F}, U, k)$ is a yes-instance if and only if T admits a dominating set of size $k + 1$.

Theorem 13.14: Necessity

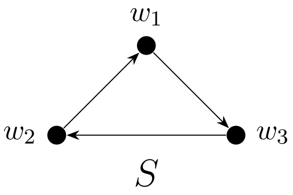
- Assume first that $\mathcal{G} \subseteq \mathcal{F}$ is a subfamily of size at most k such that $\bigcup \mathcal{G} = U$. Consider $D = \{v^*\} \cup \{v_X : X \in \mathcal{G}\}$.
- Clearly $|D| \leq k + 1$, and observe that D is a dominating set of T: each vertex of $V_{\mathcal{F}}$ is dominated by v^*, while each vertex $v_{e,w} \in V_U$ is dominated by a vertex $v_X \in D$ for $X \in G$ such that $e \in X$.

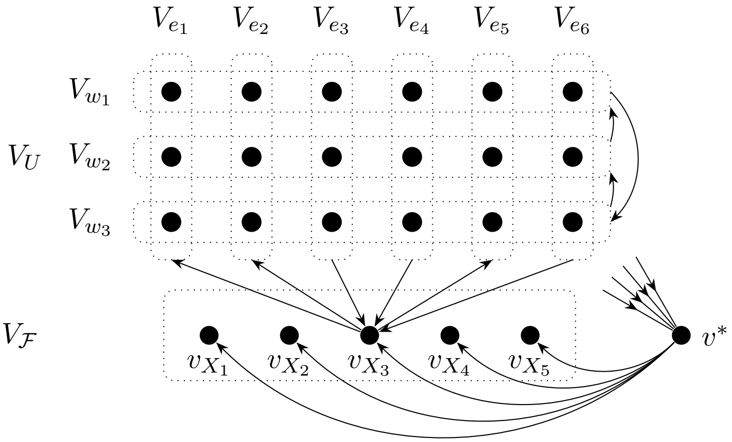
■ Assume first that $\mathcal{G} \subseteq \mathcal{F}$ is a subfamily of size at most k such that $\bigcup \mathcal{G} = U$ ($\mathcal{G}$ is a set-cover).

$$U = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

$$\mathcal{F} = \{X_1, X_2, X_3, X_4, X_5\}$$

$$X_3 = \{e_1, e_2, e_5\}$$



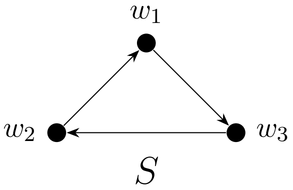


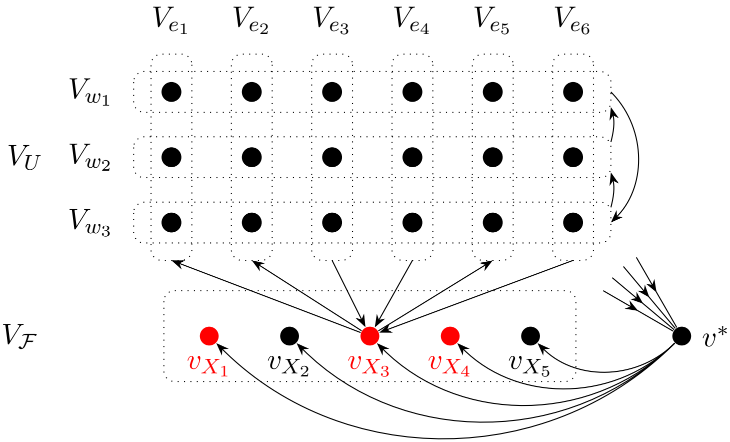
■ Consider $D = \{v^*\} \cup \boxed{\{v_X : X \in \mathcal{G}\}}$. Clearly $|D| \leq k+1$.

$$U = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

$$\mathcal{F} = \{X_1, X_2, X_3, X_4, X_5\}$$

$$X_3 = \{e_1, e_2, e_5\}$$



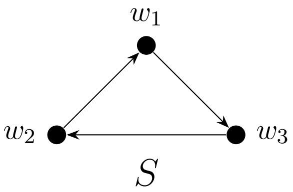


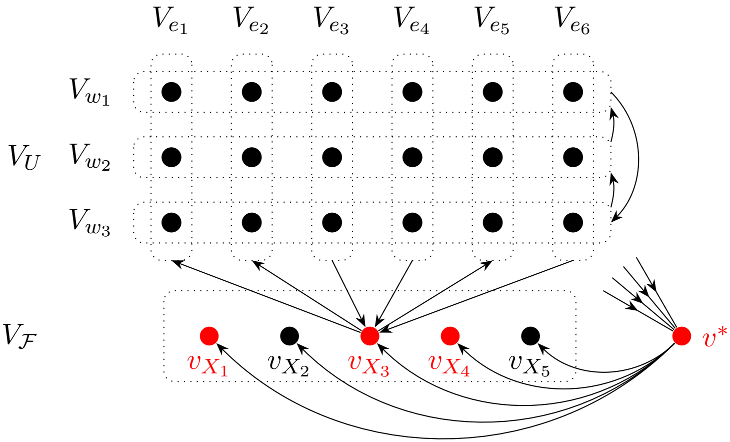
■ Consider $D = \boxed{\{v^*\}} \cup \{v_X : X \in \mathcal{G}\}$. Clearly $\boxed{|D| \leq k+1}$.

$$U = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

$$\mathcal{F} = \{X_1, X_2, X_3, X_4, X_5\}$$

$$X_3 = \{e_1, e_2, e_5\}$$



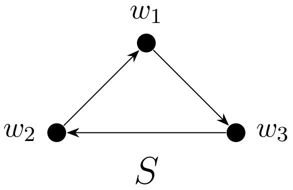


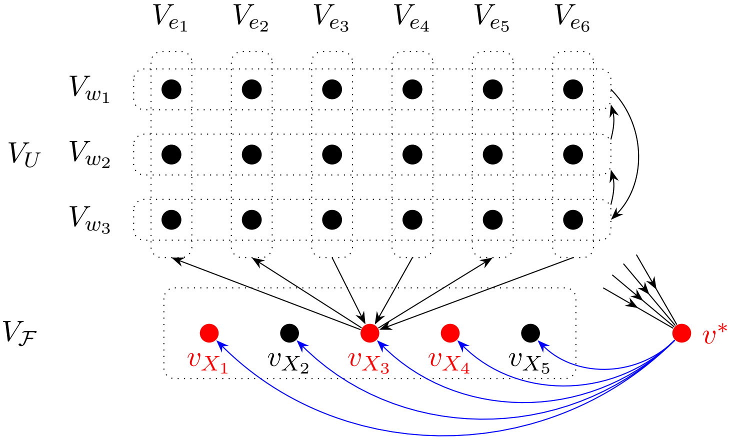
■ Observe that D is a dominating set of T: each vertex of $V_{\mathcal{F}}$ is dominated by v^*.

$$U = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

$$\mathcal{F} = \{X_1, X_2, X_3, X_4, X_5\}$$

$$X_3 = \{e_1, e_2, e_5\}$$



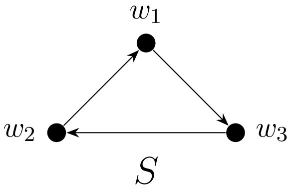


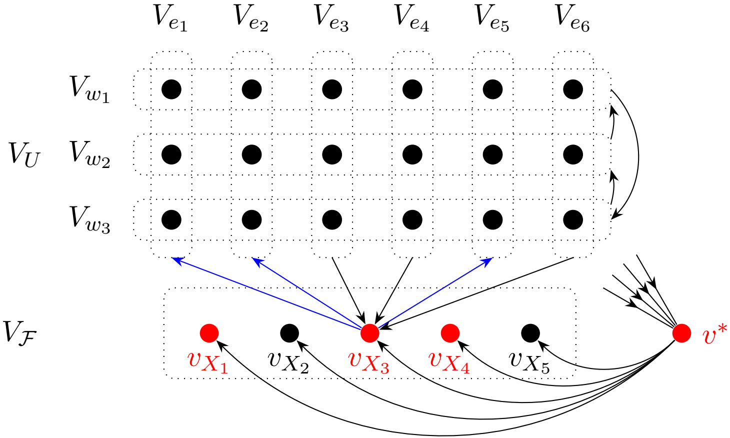
■ While each vertex $v_{e,w} \in V_U$ is dominated by a vertex $v_X \in D$ for $X \in G$ such that $e \in X$.

$$U = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

$$\mathcal{F} = \{X_1, X_2, X_3, X_4, X_5\}$$

$$X_3 = \{e_1, e_2, e_5\}$$





Theorem 13.14: Sufficiency

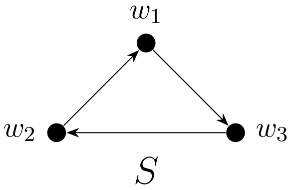
- Conversely, suppose that T admits a dominating set D such that $|D| \leq k+1$.
- Since D has to dominate v^*, either D contains v^* or at least one vertex of V_U.
- Consequently, $|D \cap V_{\mathcal{F}}| \leq k$. Let $\mathcal{G} = \{X \in \mathcal{F} : v_X \in D\}$. Clearly $|\mathcal{G}| \leq k$, so it suffices to prove that $\bigcup \mathcal{G} = U$.

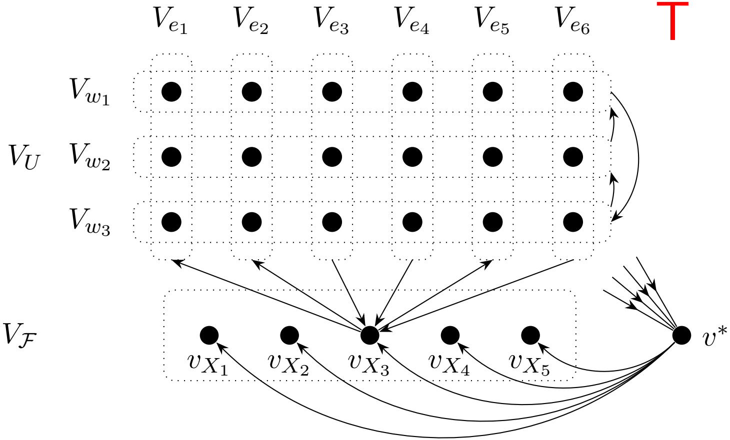
- Conversely, suppose that T admits a dominating set D such that $|D| \le k+1$.

$$U = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

$$\mathcal{F} = \{X_1, X_2, X_3, X_4, X_5\}$$

$$X_3 = \{e_1, e_2, e_5\}$$



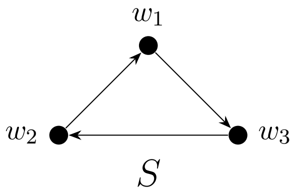


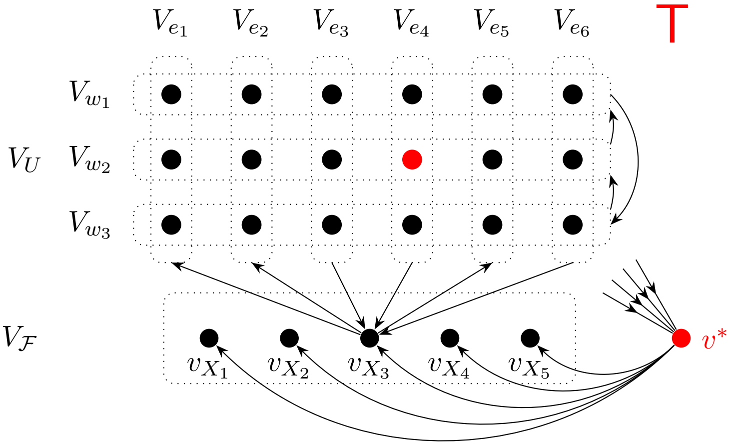
- Since D has to dominate v^*, either D contains v^* or at least one vertex of V_U.

$$U = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

$$\mathcal{F} = \{X_1, X_2, X_3, X_4, X_5\}$$

$$X_3 = \{e_1, e_2, e_5\}$$



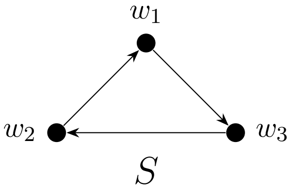


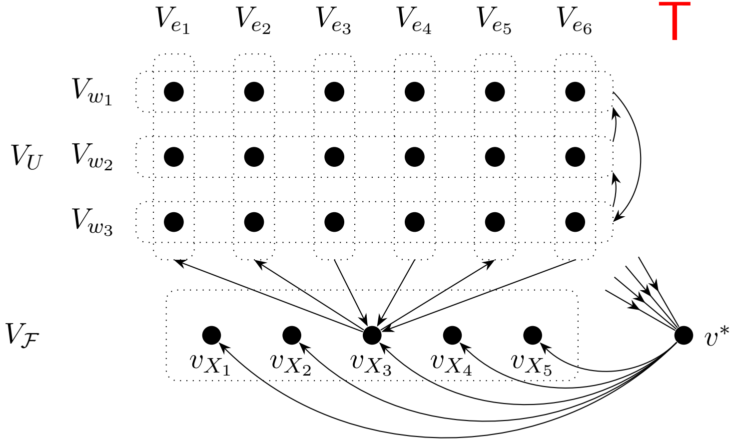
- Consequently, $|D \cap V_{\mathcal{F}}| \leq k$. Which means V_F contains at most k vertices of D.

$$U = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

$$\mathcal{F} = \{X_1, X_2, X_3, X_4, X_5\}$$

$$X_3 = \{e_1, e_2, e_5\}$$



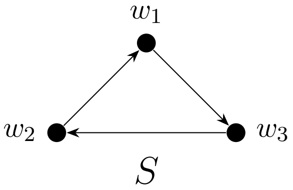


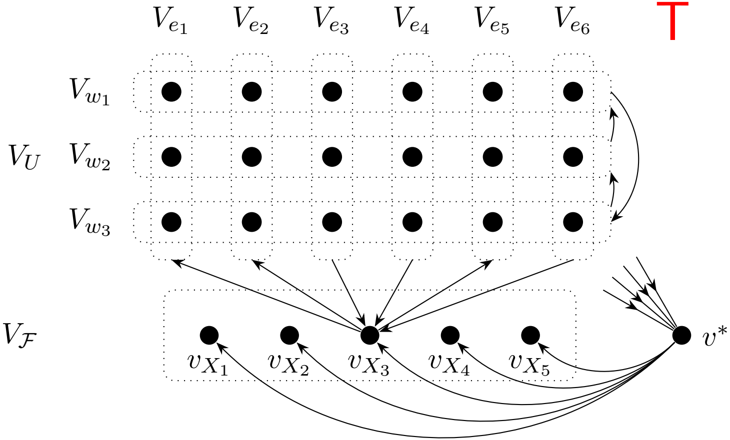
■ Let $\mathcal{G} = \{X \in \mathcal{F} : v_X \in D\}$. Clearly $|\mathcal{G}| \leq k$. So, it suffices to prove that $\bigcup \mathcal{G} = U$.

$$U = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

$$\mathcal{F} = \{X_1, X_2, X_3, X_4, X_5\}$$

$$X_3 = \{e_1, e_2, e_5\}$$





Theorem 13.14: Sufficiency

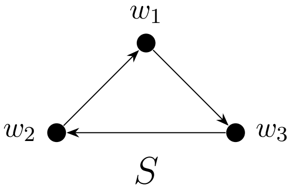
We will prove this by contradiction!!

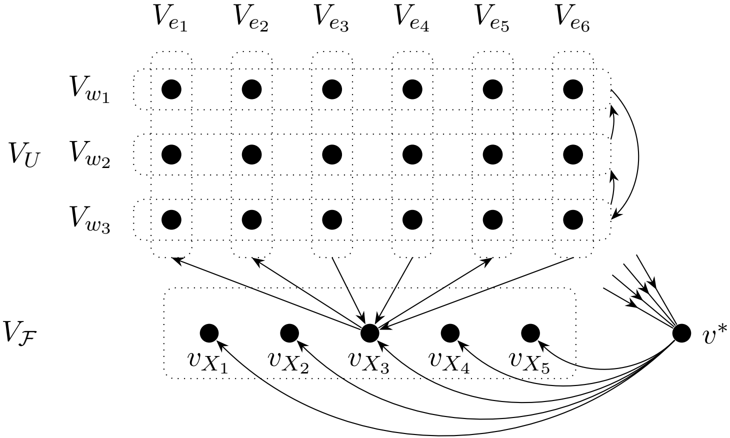
■ For the sake of contradiction assume that there exists some $e_1 \in U$ that does not belong to any set of $\mathcal{G}$.

$$U = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

$$\mathcal{F} = \{X_1, X_2, X_3, X_4, X_5\}$$

$$X_3 = \{e_1, e_2, e_5\}$$



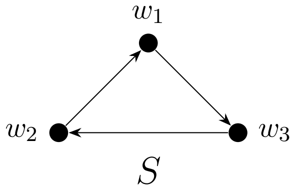


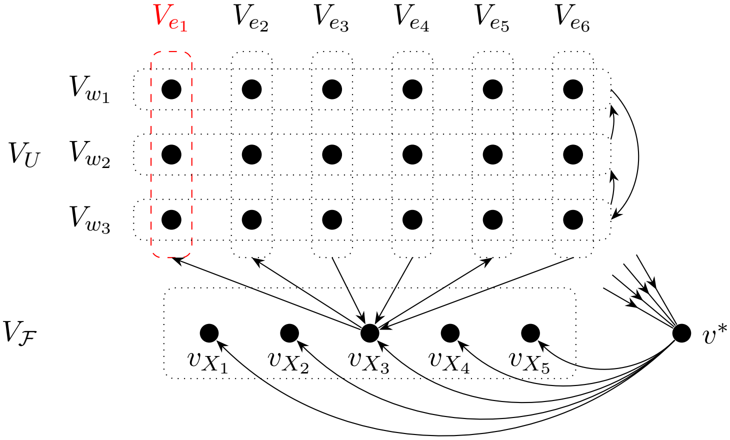
■ For the sake of contradiction assume that there exists some $e_1 \in U$ that does not belong to any set of $\mathcal{G}$.

$$U = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

$$\mathcal{F} = \{X_1, X_2, X_3, X_4, X_5\}$$

$$X_3 = \{e_1, e_2, e_5\}$$



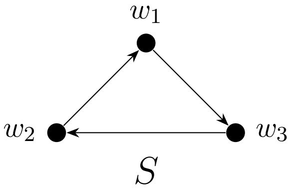


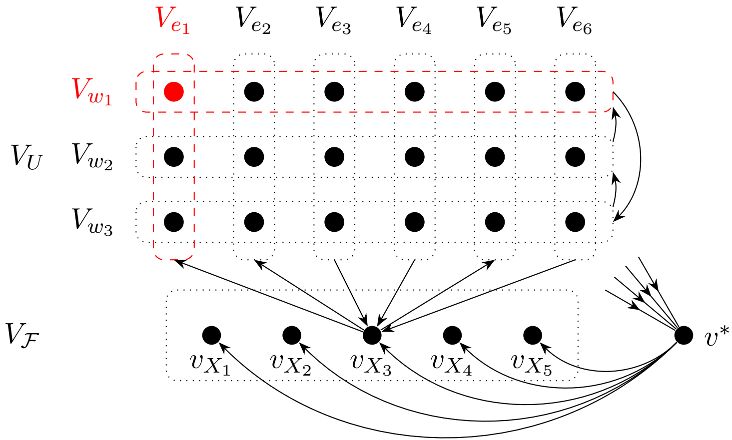
■ Since S is $(k+1)$-paradoxical, we have that there exists some vertex $w_1 \in V(S)$ that is not dominated by Z in S.

$$U = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

$$\mathcal{F} = \{X_1, X_2, X_3, X_4, X_5\}$$

$$X_3 = \{e_1, e_2, e_5\}$$



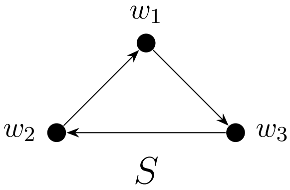


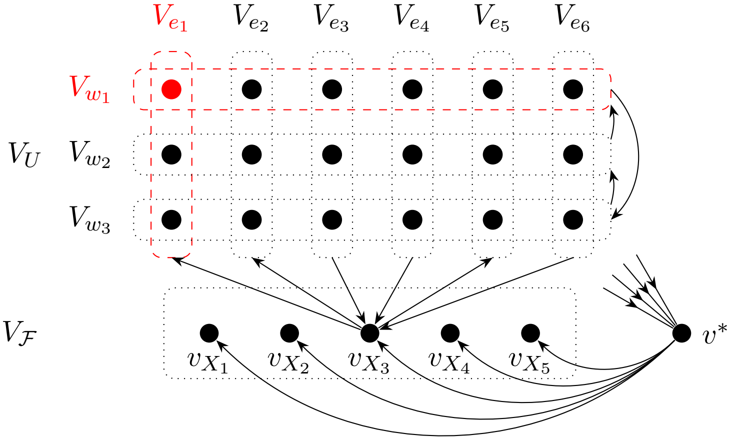
- V_U can't dominate V_{w_1} and $V_{\mathcal{F}}$ can't dominate V_{e_1}.

$$U = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

$$\mathcal{F} = \{X_1, X_2, X_3, X_4, X_5\}$$

$$X_3 = \{e_1, e_2, e_5\}$$



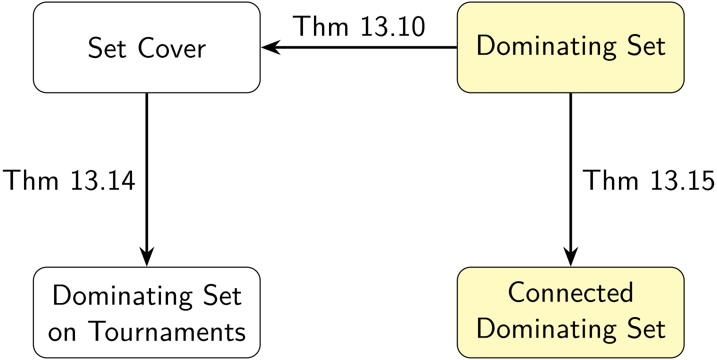


Theorem 13.14: Sufficiency

We infer that v_{e_1,w_1} is not dominated by D at all, which contradicts the assumption that D is a dominating set in T.

Outline

- From Dominating Set to Connected Dominating Set.



Connected Dominating Set

Definition

Connected Dominating Set is the variant of Dominating Set where we additionally require that the dominating set induce a connected graph.

Theorem 13.15

Theorem

There is a parameterized reduction from Dominating Set to Connected Dominating Set.

Theorem 13.15: Construction

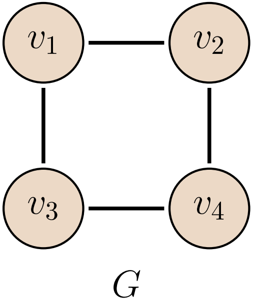
Let (G, k) be an instance of Dominating Set. We construct a graph G' the following way.

(i) For every vertex $v \in V(G)$, two adjacent vertices v^x, v^y are created in G'.

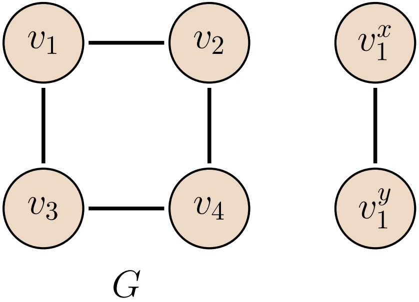
(ii) We make the set $\{v^x : v \in V(G)\}$ a clique K of size $|V(G)|$.

(iii) We make v^x and u^y adjacent if v and u are adjacent in G.

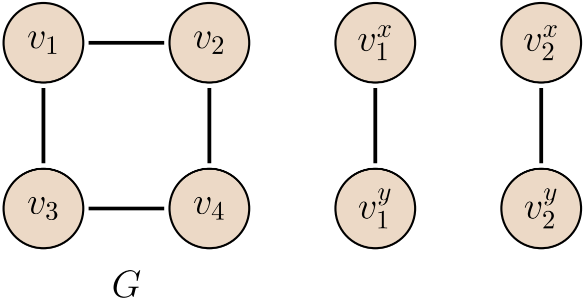
■ Let (G, k) be an instance of Dominating Set. We construct a graph G' the following way.



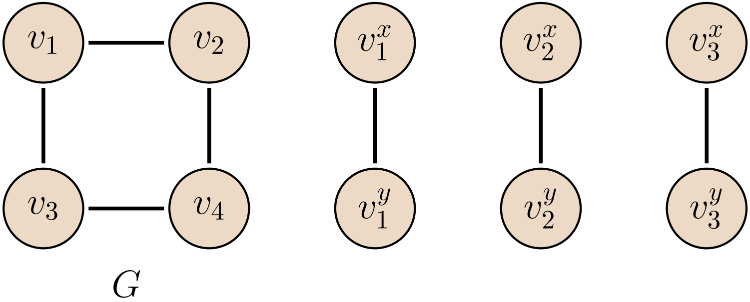
(i) For every vertex $v \in V(G)$, two adjacent vertices v^x, v^y are created in G'.



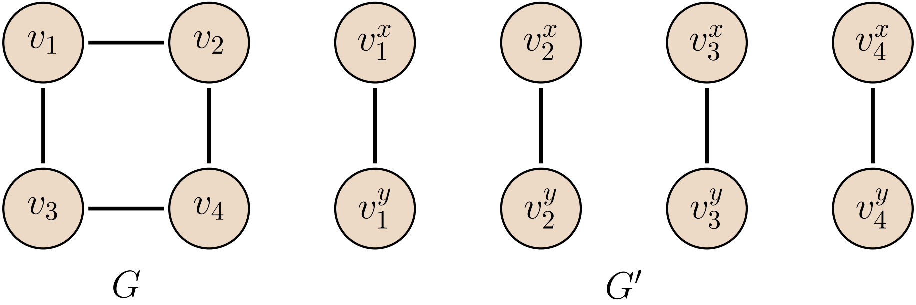
(i) For every vertex $v \in V(G)$, two adjacent vertices v^x, v^y are created in G'.



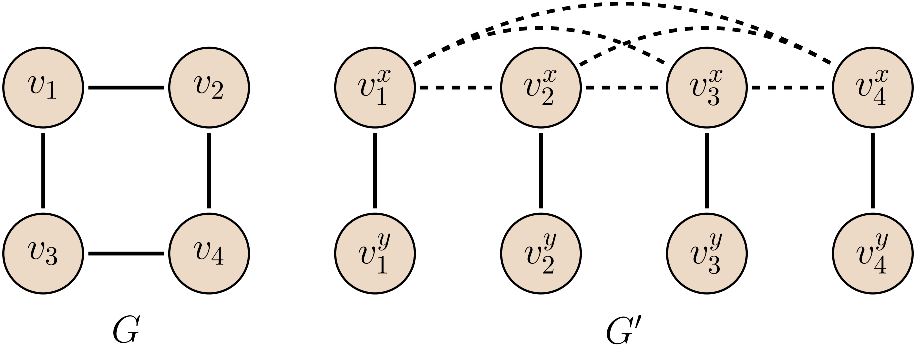
(i) For every vertex $v \in V(G)$, two adjacent vertices v^x, v^y are created in G'.



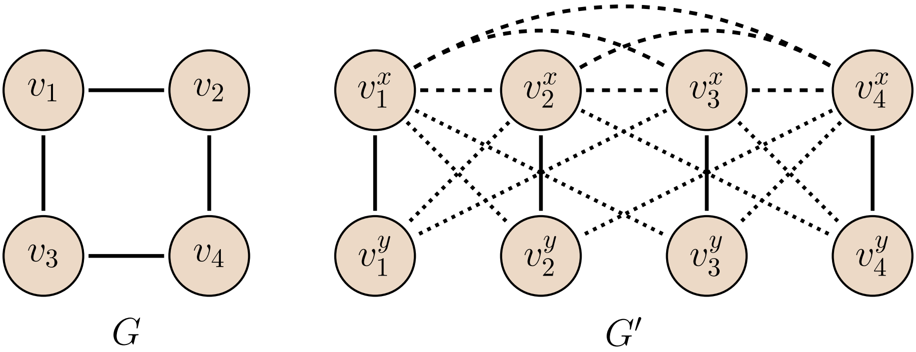
(i) For every vertex $v \in V(G)$, two adjacent vertices v^x, v^y are created in G'.



(ii) We make the set $\{v^x : v \in V(G)\}$ a clique K of size $|V(G)|$.



(iii) We make v^x and u^y adjacent if v and u are adjacent in G.



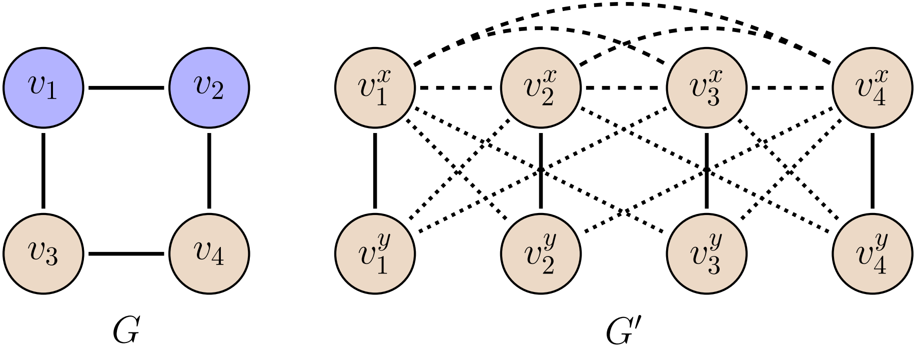
Theorem 13.15: Proof

We claim that (G, k) is a yes-instance of Dominating Set if and only if (G', k) is a yes-instance of Connected Dominating Set.

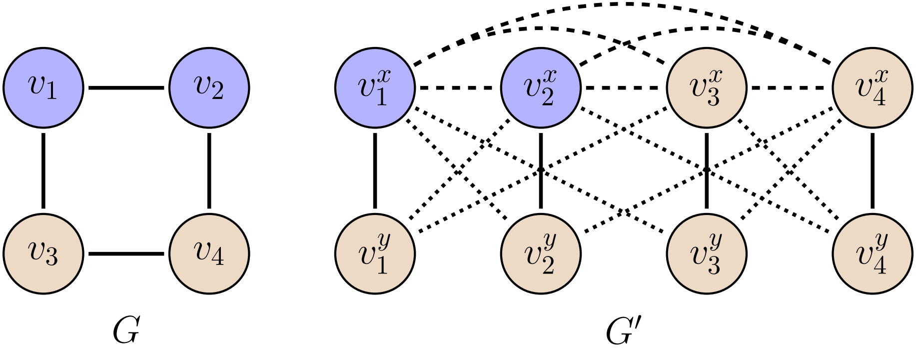
Theorem 13.15: Necessity

- Suppose first that $S = \{v_1, ..., v_k\}$ is a dominating set of size k in G.
- Then we claim that $S' = \{v_1^x, ..., v_k^x\}$ is a connected dominating set of size k in G'.
- Clearly, $G'[S']$ is a clique and hence it is connected.
- To see that S' is a dominating set in G', observe that v_1^x dominates K, and if u is dominated by v_i in G, then u^y is dominated by v_i^x in G'.

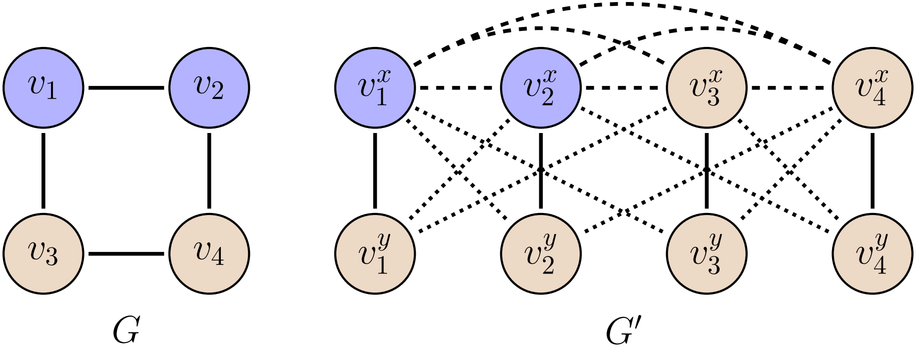
- Suppose first that $S = \{v_1, ..., v_k\}$ is a dominating set of size k in G.



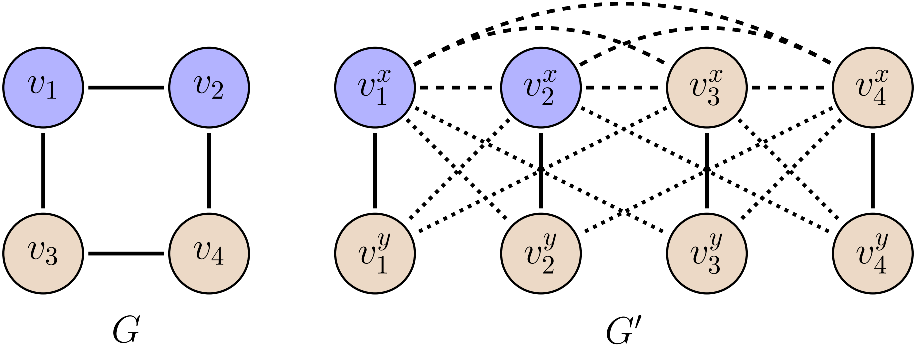
■ Then we claim that $S' = \{v_1^x, ..., v_k^x\}$ is a connected dominating set of size k in G'.



- Clearly, $G'[S']$ is a clique and hence it is connected.



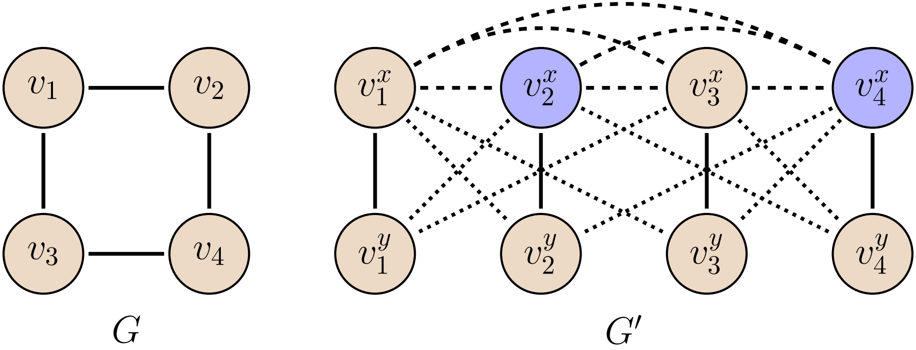
■ To see that S' is a dominating set in G', observe that v_1^x dominates K, and if u is dominated by v_i in G, then u^y is dominated by v_i^x in G'.



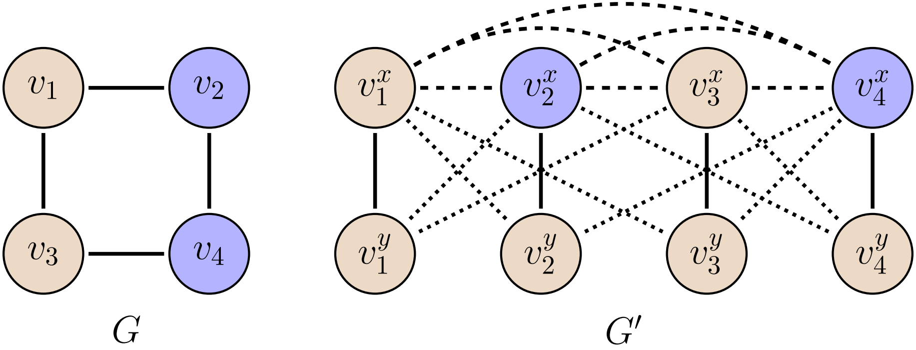
Theorem 13.15: Sufficiency

- Let S' be a connected dominating set of size k in G'.
- Let v be in S if at least one of v^x and v^y is in S'; clearly, $|S| \leq |S'| = k$.
- We claim that S is a dominating set of G.
- Consider any vertex $u \in V(G)$.
- Vertex u^y of G' is dominated by some vertex v^x or v^y that belongs to S'.
- Then v is in S and, by the construction of G', it dominates u in G, as required.

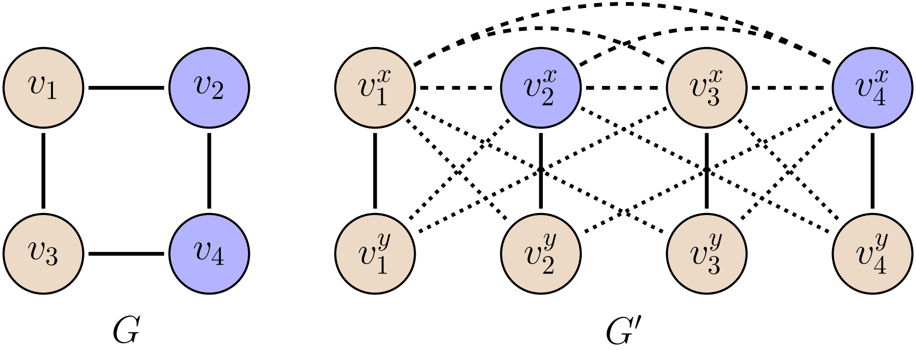
■ Let S' be a connected dominating set of size k in G'.



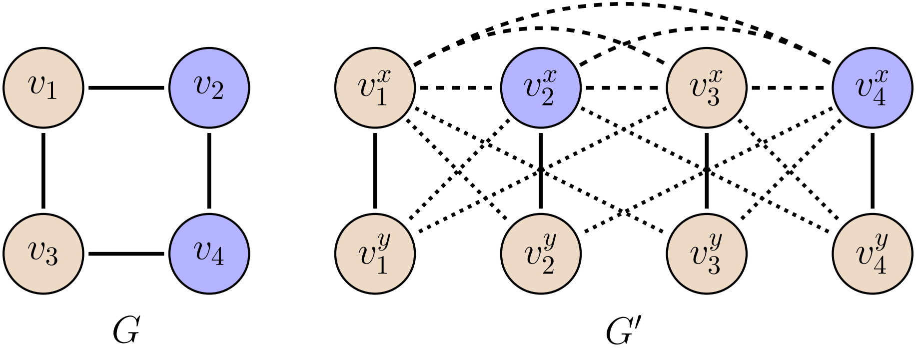
- Let v be in S if at least one of v_x and v_y is in S'; clearly, $|S| \leq |S'| = k$.



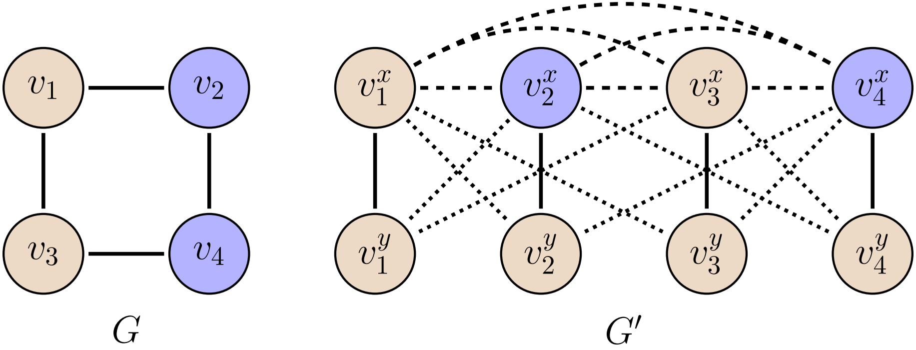
- We claim that S is a dominating set of G.



■ Consider any vertex $u \in V(G)$. Vertex u^y of G' is dominated by some vertex v^x or v^y that belongs to S'.



- Then v is in S and, by the construction of G', it dominates u in G, as required.



Thank You :)