Mathematics-II

Question Bank – 1

1 Find the work done by the force field $F(x, y) = (y, -x)$ along the path from (1,0) to (0,1) along the parabola $y = x^2$.

Sol Given $F(x, y) = (y, -x)$ along path from parabola $y = x^2$ from $(1,0)$ to $(0,1)$.

We use line integral, $W = \int_C F\, dr$.

Given path is $y = x^2$.

Set $x = t$ and $y = t^2$. ; t varies from 1 to 0

The differential displacement is

$\frac{dx}{dt} = 1$; $\frac{dy}{dt} = 2t$ and $dr = (dx, dy)$.

$= (1, 2t)\, dt$.

we have $F(t) = (t^2, -t)$.

From Dot product ;

$F \cdot dr = (t^2, -t) \cdot (1, 2t) = t^2(1) + (-t)(2t)$.

$= -t^2$.

Now $W = \int_1^0 F\, dt = \int_0^1 -t^2\, dt = -\left[\frac{t^3}{3}\right]_1^0 = \frac{1}{3}$.

$\therefore W = \frac{1}{3}$

functions $\sin(\frac{1}{x})$

2 Find the divergence of a vector valued function $x^2 y\, \vec{i} - (z^3 - 3x)\vec{j} + 4y^2\vec{k}$.

e. Find the divergence of a vector valued function $xy^2\vec{i} - (z^2 - 3x^2)\vec{j} + 4y^3\vec{k}$ at the point $(1,2,3)$

Sol The divergence of a vector field F is

$F = Pi + Qj + Rk$ is given by:

$\nabla F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

Given $F = xy^2 i - (z^2 - 3x^2) j + 4y^3 k$

Now, Computing partial derivatives in ∇F.

$\frac{\partial P}{\partial x} = y^2$; $\frac{\partial Q}{\partial y} = 0$; $\frac{\partial R}{\partial z} = 0$

So, $\nabla F = y^2 + 0 + 0 = y^2$.

Now ∇F at $(1,2,3)$ is.

$\nabla F(1,2,3) = 2^2 = 4$.

$\therefore \nabla F = 4$.

3 Find the gradient of scalar valued function $f(x, y, z) = 2x^3y^2 + 3y^3x^2 + z^3$ at the point $(1, 0, 0)$.

7.

d. Find the gradient of scalar valued function $f(x,y,z) = 2x^3y^2+3y^3x^2+z^3$ at the point $(1,0,0)$.

Sol: Given, $f(x,y,z) = 2x^3y^2+3y^3x^2+z^3$.

$\because$ The gradient of a scalar-valued function $f(x,y,z)$ is

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

Now $\frac{\partial f}{\partial x} = 6x^2y^2+6y^3x$.

Now $\frac{\partial f}{\partial y} = 4x^3y+9y^2x^2$.

also $\frac{\partial f}{\partial z} = 3z^2$.

Now ∇f at $(1,0,0)$ is,

$$\nabla f(1,0,0) = [6(1)(0)+0, 4(0)+9(0), 3(0)]$$

$$= (0,0,0).$$

$\therefore \nabla f(1,0,0) = (0,0,0)$

4

Find $\int_C xdy - ydx$, where C is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ traversed clockwise.

(b) Find $\int_C xdy-ydx$, where C is ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ traverse clockwise.

Sol: Given line integral $I = \oint_C (xdy-ydx)$.

By Green's theorem,

$$\oint_C Pdx+Qdy = \iint_R \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA.$$

where $P=-y$ & $Q=x$.

Now $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y} = 1-(-1) = 2$.

Now area of ellipse is $\frac{x^2}{4}+\frac{y^2}{9}=1$.

$$A = \pi \times a \times b = 6\pi.$$

Thus $I = \iint_R 2\,dA = 2\times 6\pi = 12\pi$.

$\because$ C is traversed clockwise,

$$I = -12\pi.$$

5

Verify whether the vector valued function $\vec{F} = 3x^2y\,\vec{i} + (z^2 - 3x^2)\vec{j} + 4y^2\vec{k}$ is solenoidal or not?

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f. Verify whether the vector valued function $\vec{F} = 3x^2y\,\vec{i} + (z^2-3x^2)\vec{j} + 4y^2\vec{k}$ is solenoidal or not?
Sol: If $\nabla F = 0$, i.e., its divergence is Zero,
Then F is solenoidal.
Given $F = (3x^2y)\hat{i} + (z^2-3x^2)\hat{j} + 4y^2\hat{k}$
Divergence of F $\Rightarrow \nabla F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$
$\Rightarrow \nabla F = 6xy + 0 + 0$.
$\Rightarrow \nabla F = 6xy$
Thus $\nabla F \neq 0 = 6xy$.
So, the vector field F is not solenoidal.
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Verify Stoke’s theorem to compute the work done by force field given by $\overline{F} = y^2\overline{i} + x^2\overline{j} - (x + 2)\overline{k}$ and C is the boundary of the triangle with vertices (0, 0, 0), (1, 0, 0) and (1, 1, 0).

7(c). Verify stoke's theorem to compute the work done by force field given by $F = y^2\vec{i} + x^2\vec{j} - (x+2)\bar{k}$ and C is the boundary of triangle with vertices (0,0,0), (1,0,0) and (1,1,0).
Sol: By stoke's theorem states for vector field F, along a closed curve C is equal to surface integral of curl of $\bar{F}$ over a surface S bounded by C.
$\oint_C F\cdot d\gamma = \iint_S (\nabla \times F)\cdot n\, ds$.
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	Sol: given force field, $F = y^2 i + x^2 j - (x+2)k$. $= (y^2, x^2, -(x+2))$ The curl of a vector field $F = (P, Q, R)$ is. $\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+2) \end{vmatrix}$ $\Rightarrow \nabla \times F = i\left[\frac{\partial}{\partial y}(-(x+2)) - \frac{\partial}{\partial z}(x^2)\right] - j\left[\frac{\partial}{\partial x}(-(x+2)) - \frac{\partial}{\partial z}(y^2)\right] + k\left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(y^2)\right]$ $= (0, 1, 2x - 2y)$ Given curve C with vertices $(0,0,0), (1,0,0)$ and $(1,1,0)$. in xy-plane $(z=0)$. $\because z = 0$, normal vector to the surface in $n = k = (0,0,1)$ By stoke's theorem, $\iint_S (\nabla \times F) \cdot n \, ds$, $\because n = (0,0,1)$ $\Rightarrow \iint_S (0, 1, 2x-2y) \cdot (0,0,1) = \iint_S (2x - 2y)\, ds$ $\because$ By given condition, x varies from 0 to 1. for a fixed x, y varies from 0 to x (eq'n line $y = x$) $\int_0^1 \int_0^x (2x - 2y)\, dy\, dx = \int_0^1 (2xy - y^2)_0^x \, dx = \int_0^1 x^2 dx$. $= \frac{1}{3}$ $\because$ we have surface integral is $\frac{1}{3}$ & line integral is $\frac{1}{3}$. Thus, stoke's theorem verified. $\oint_C F \cdot dx = \frac{1}{3}$.
7	Find the value of a, b, c so that the vector $\vec{F} = \left(axy + bz^3\right)\vec{i} + \left(3x^2 - cz\right)\vec{j} + (3xz^2 - y)\vec{k}$ may be irrational. 5(a). Find the value of a, b, c, so that vector $\vec{F}$ is $\vec{F} = (axy + bz^3)\hat{i} + (3x^2 - cz)\hat{j} + (3xz^2 - y)\hat{k}$ may be irrational. Sol: Given $F = (axy + bz^3)\hat{i} + (3x^2 - cz)\hat{j} + (3xz^2 - y)\hat{k}$ Now $\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy + bz^3 & 3x^2 - cz & 3xz^2 - y \end{vmatrix} = 0$ $= \hat{i}(c-1) - \hat{j}(3(b-1)z^2) + \hat{k}(6x - ax) = 0$ [$\because$ To ensure that F is irrotational, curl must be zero.] i.e., $\nabla \times F = 0$ Now for irrotationality $\hat{i} \Rightarrow c - 1 = 0 \Rightarrow c = 1$. " " $\hat{j} \Rightarrow b - 1 = 0 \Rightarrow b = 1$ " " $\hat{k} \Rightarrow 6x - ax = 0 \Rightarrow 6 - a = 0$ $\Rightarrow a = 6$ $\therefore a = 6, b = 1, c = 1$
8	Find the directional derivative of function $f(x, y, z) = x^2 y^3 - 4xz$ at the point $(-1, 4, 2)$ in the direction $-\vec{i} + 2\vec{j}$.

	4(c). Find the directional derivative of function $f(x,y,z) = x^2y^3 - 4xz$ at the point $(-1, 4, 2)$ in the direction $-\vec{i} + 2\vec{j}$ Sol: Consider $f(x,y,z) = x^2y^3 - 4xz$. Now gradient of $f(x,y,z)$ is, $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ $= (2xy^3 - 4z, 3x^2y^2, -4x)$. Now ∇f at $(-1, 4, 2)$. $\Rightarrow \nabla f = (-136, 48, 4)$ Now given direction vector is $v = -\vec{i} + 2\vec{j} = (-1, 2, 0)$. Now $\lvert v \rvert = \sqrt{1+4+1} = \sqrt{5}$. The unit vector in the given direction is. $u = \left(\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, \frac{0}{\sqrt{5}}\right) = \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right)$ Now the direction derivative is $D_u f = \nabla f \cdot u$. $\Rightarrow D_u f = (-136, 48, 4) \cdot \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right)$. $= \frac{136}{\sqrt{5}} + \frac{96}{\sqrt{5}} + 0$. $= \frac{232}{\sqrt{5}}$ $\because$ By rationalizing denominator, $D_u f = \frac{232\sqrt{5}}{\sqrt{5}\,\sqrt{5}}$ $\therefore D_u f = \frac{232\sqrt{5}}{5}$, is required directional derivative
9	Calculate the work done by the force field $F(x, y) = (2x + y)\vec{i} + (x - y)\vec{j}$ along the path from $(0, 0)$ to $(2, 1)$ following the parabola $y = \frac{x^2}{4}$. 6(b). Calculate the work done by the force field $F(x,y) = (2x+y)\hat{i} + (x-y)\hat{j}$ along the path from $(0,0)$ to $(2,1)$ following the parabola $y = \frac{x^2}{4}$. Sol: Consider given $F(x,y)$ along path $(0,0)$ to $(2,1)$. Now line integral of the vector field: $W = \int_C F \cdot dr$. given path is $y = \frac{x^2}{4}$, Let $x = t$ then $y = \frac{t^2}{4}$: so, position vector, $r(t) = t\hat{i} + \frac{t^2}{4}\hat{j}$ limits of t are from $t=0$ to $t=2$: so, $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = \frac{t}{2}$. Then $dr = dt\,\hat{i} + \frac{t}{2}\,dt\,\hat{j}$

	Now $F_x = 2t + \frac{t^2}{4}$ and $F_y = t - \frac{t^4}{4}$ Now dot product as, $F \cdot dr = (2t + \frac{t^2}{4})dt + (t - \frac{t^2}{4}) \cdot \frac{t}{2} dt$ $= (2t + \frac{t^2}{4} + \frac{t^2}{2} - \frac{t^3}{8})dt$ $= (2t + t^2 - \frac{t^3}{8})dt$ Now $W = \int_0^2 (2t + t^2 - \frac{t^3}{8})dt$. $= (t^2)_0^2 + (\frac{t^3}{3})_0^2 - \frac{1}{8}(\frac{t^4}{4})_0^2$ $= 4 + \frac{8}{3} - \frac{1}{2}$ $= \frac{40}{6} - \frac{3}{6} = \frac{37}{6}$. $\therefore W \cong 6.17$ units. So, work done by the force field is, 6.17 units.
10	Find the directional derivative of function $f(x, y, z) = x^3y - x^2y - z$ at the point $(1, 1, 2)$ in the direction $2\vec{i} - 3\vec{j} + 6\vec{k}$. 5(c). Find the directional derivative of function is $f(x,y,z) = x^3y - x^2y - z$ at point $(1,1,2)$ in the direction $2\hat{i} - 3\hat{j} + 6\vec{k}$ Sol^n Given $f(x,y,z) = x^3y - x^2y - z$. at $(1,1,2)$. Now gradient vector of $f(x,y,z)$ is given by: $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$. $= (3x^2y - 2xy, x^3 - x^2, -1)$ Now ∇f at $(1,1,2)$ is. So, $\nabla f(1,1,2) = (1,0,-1)$ The given direction vector is $v = (2,-3,6)$ The magnitude of v is, $\|v\| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = \sqrt{49} = 7$. So, the unit vector u in direction of v is. $u = (\frac{2}{7}, \frac{-3}{7}, \frac{6}{7})$ Now directional derivative is $D_uf = \nabla f \cdot u$. $\Rightarrow D_uf = (1,0,-1) \cdot (\frac{2}{7}, \frac{-3}{7}, \frac{6}{7})$ $= \frac{2}{7} + 0 - \frac{6}{7}$ $\therefore D_uf = \frac{-4}{7}$ This is our required directional derivative of f.
11	A Fluid motion is given by $v = (y + z)i + (z + x)j + (x + y)k$, Is the motion is irrotational and solenoidal ?

4(b). A fluid motion is given by
$v = (y+z)i + (z+x)j + (x+y)k$, is the motion is irrotational and solenoidal.

Sol: Given $v = (y+z)i + (z+x)j + (x+y)k$.

Irrotationality: curl of v.

$$\nabla \times v = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y+z) & (z+x) & (x+y) \end{vmatrix}$$

$$\Rightarrow \nabla \times v = i\left[\frac{\partial}{\partial y}(x+y) - \frac{\partial}{\partial z}(z+x)\right] + j\left[\frac{\partial}{\partial z}(y+z) - \frac{\partial}{\partial x}(x+y)\right] + k\left[\frac{\partial}{\partial x}(z+x) - \frac{\partial}{\partial y}(y+z)\right]$$

$$\Rightarrow \quad = i(1-1) + j(1-1) + k(1-1)$$

$$\Rightarrow \nabla \times v = 0$$

$\therefore$ curl $v = 0$, i.e., $\nabla \times v = 0$, the motion is irrotational.

Solenoidal property: Divergence of v.

$$\nabla \cdot v = \frac{\partial}{\partial x}(y+z) + \frac{\partial}{\partial y}(z+x) + \frac{\partial}{\partial z}(x+y).$$

$$= 0 + 1 + 1 = 2$$

$\because \nabla \cdot v \neq 0$, the motion is not solenoidal.

12

Find the unit normal vector to the surface $z = x^2 + y^2$ at the point $(-1, -2, -5)$.

5(b). Find the unit normal vector to the surface $z = x^2 + y^2$ at the point $(-1, -2, -5)$.

Sol. Given $z = x^2 + y^2$ at $(-1, -2, -5)$.

Now level surface $F(x, y, z) = x^2 + y^2 - z = 0$

The normal vector to the surface is given by gradient ∇F,

$$\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$$

$$= (2x, 2y, -1)$$

Now ∇F at $(-1, -2, -5) = (-2, -4, -1)$.

The unit normal vector is obtained by dividing by the magnitude of ∇F:

$$|\nabla F| = \sqrt{(-2)^2 + (-4)^2 + (-1)^2}$$

$$= \sqrt{21}$$

so, $n = \frac{1}{\sqrt{21}}(-2, -4, -1)$.

$$\therefore n = \left(\frac{-2}{\sqrt{21}}, \frac{-4}{\sqrt{21}}, \frac{-1}{\sqrt{21}}\right)$$

which is our required unit normal vector to surface at point $(-1, -2, 5)$.

13

Find the circulation of $F(x, y) = y\vec{i} + x\vec{j}$ around the square with vertices at (0, 0), (1, 0), (1, 1), and (0, 1), traversed counter clockwise.

	7(b). Find the circulation of $F(x,y) = y\bar{i} + x\bar{j}$ around the square with vertices at $(0,0), (1,0), (1,1)$ and $(0,1)$, traversed counter clockwise. Sol: Given $F(x,y) = y\bar{i} + x\bar{j}$ around vertices. Consider Green's theorem states that for a simple closed curve C, $\oint_C F \cdot dr = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$, where $dr = dx\bar{i} + dy\bar{j}$ where $F = (P, Q) \implies P = y$ and $Q = x$, $\frac{dQ}{dx} = 1$ & $\frac{\partial P}{\partial y} = 1$ Thus $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 1 = 0$ ∵ curl is zero, the integral over the closed region is also zero. $\oint_C F \cdot dr = \iint_R 0 \, dA = 0$. Thus the circulation of F around the given ~~region~~ square is 0. $\therefore \oint_C F \cdot dr = 0$.
14	A particle moves along a curve whose parametric equations are $x = e^{-t}$, $y = 2\cos cos\, 3t$, $z = \sin sin\, 3t$. Find the velocity and acceleration at $t = 0$. 4(a). A particle moves along a curve whose parametric eq'ns are $x = e^{-t}$, $y = 2\cos 3t$, $z = \sin 3t$. Find velocity and acceleration at $t = 0$ Sol: Given parametric eq'n of a particle's is $x = e^{-t}$, $y = 2\cos(3t)$, $z = \sin(3t)$ Now, the velocity of V is, $V = \frac{dr}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$ $\implies V = (-e^{-t}, -6\sin(3t), 3\cos(3t))$ Now velocity at $t = 0$. $\implies V(0) = (-e^0, -6\sin(0), 3\cos(0))$. $= (-1, 0, 3)$. Now the acceleration vector a is, $a = \frac{dv}{dt} = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right)$ $\implies a = (e^{-t}, -18\cos(3t), -9\sin(3t))$ Now, acceleration vector a at $t = 0$. $\implies a(0) = (e^0, -18\cos(0), -9\sin(0))$. $= (1, -18, 0)$.