Matrix Multiplication

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

$$\begin{aligned} C &= AB \\ c_{ij} &= \sum_{k=1}^{n} a_{ik} b_{kj} \end{aligned}$$

Matrix Multiplication

Example

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 3 \\ 1 & 5 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 & 7 \\ -2 & 1 & 0 \\ 6 & 1 & -1 \end{pmatrix} = \begin{pmatrix} -2 & -1 & 8 \\ 14 & 5 & -3 \\ -6 & 5 & 7 \end{pmatrix}$$

Matrix Multiplication

Input: Two square matrices $A_{n \times n}$ and $B_{n \times n}$.

Output: AB.

```
Initialize matrix C_{n×n}
for i = 1 to n do
    for j = 1 to n do
        c_{ij} = 0
        for k = 1 to n do
            c_{ij} = c_{ij} + a_{ik} b_{kj}
        end for
    end for
end for
return C
```

Matrix Multiplication

Input: Two square matrices $A_{n\times n}$ and $B_{n\times n}$.
Output: AB.

```
Initialize matrix C_{n×n}
for i = 1 to n do
    for j = 1 to n do
        c_{ij} = 0
        for k = 1 to n do
            c_{ij} = c_{ij} + a_{ik} b_{kj}
        end for
    end for
end for
return C
```

$$T(n) = \Theta\left(n^3\right)$$

How can we Divide and Conquer?

Assume $n = 2^k$.

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 4 & 7 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 \\ -2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 5 & 1 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 3 & -3 & 1 \\ 4 & 5 & 4 & 9 \\ 5 & 1 & -2 & -1 \\ 4 & 7 & 5 & 9 \end{pmatrix}$$

How can we Divide and Conquer?

Assume $n = 2^k$.

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 4 & 7 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 \\ -2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 5 & 1 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 3 & -3 & 1 \\ 4 & 5 & 4 & 9 \\ 5 & 1 & -2 & -1 \\ 4 & 7 & 5 & 9 \end{pmatrix}$$

$$\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} & \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 2 & 4 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 7 & 1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 9 & 3 \\ 4 & 5 \end{pmatrix} & \begin{pmatrix} -3 & 1 \\ 4 & 9 \end{pmatrix} \\ \begin{pmatrix} 5 & 1 \\ 4 & 7 \end{pmatrix} & \begin{pmatrix} -2 & -1 \\ 5 & 9 \end{pmatrix} \end{pmatrix}$$

How can we Divide and Conquer?

Assume $n = 2^k$.

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 4 & 7 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 \\ -2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 5 & 1 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 3 & -3 & 1 \\ 4 & 5 & 4 & 9 \\ 5 & 1 & -2 & -1 \\ 4 & 7 & 5 & 9 \end{pmatrix}$$

$$\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} & \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 2 & 4 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 7 & 1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 9 & 3 \\ 4 & 5 \end{pmatrix} & \begin{pmatrix} -3 & 1 \\ 4 & 9 \end{pmatrix} \\ \begin{pmatrix} 5 & 1 \\ 4 & 7 \end{pmatrix} & \begin{pmatrix} -2 & -1 \\ 5 & 9 \end{pmatrix} \end{pmatrix}$$

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

How can we Divide and Conquer?

Assume $n = 2^k$.

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 4 & 7 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 \\ -2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 5 & 1 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 3 & -3 & 1 \\ 4 & 5 & 4 & 9 \\ 5 & 1 & -2 & -1 \\ 4 & 7 & 5 & 9 \end{pmatrix}$$

$$\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} & \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 2 & 4 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 7 & 1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 9 & 3 \\ 4 & 5 \end{pmatrix} & \begin{pmatrix} -3 & 1 \\ 4 & 9 \end{pmatrix} \\ \begin{pmatrix} 5 & 1 \\ 4 & 7 \end{pmatrix} & \begin{pmatrix} -2 & -1 \\ 5 & 9 \end{pmatrix} \end{pmatrix}$$

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$\begin{aligned} C_{11} &= A_{11}B_{11} + A_{12}B_{21} \\ C_{12} &= A_{11}B_{12} + A_{12}B_{22} \\ C_{21} &= A_{21}B_{11} + A_{22}B_{21} \\ C_{22} &= A_{21}B_{12} + A_{22}B_{22} \end{aligned}$$

How can we Divide and Conquer?

Assume $n = 2^k$.

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 4 & 7 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 \\ -2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 5 & 1 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 3 & -3 & 1 \\ 4 & 5 & 4 & 9 \\ 5 & 1 & -2 & -1 \\ 4 & 7 & 5 & 9 \end{pmatrix}$$

$$\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} & \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 2 & 4 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 7 & 1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 9 & 3 \\ 4 & 5 \end{pmatrix} & \begin{pmatrix} -3 & 1 \\ 4 & 9 \end{pmatrix} \\ \begin{pmatrix} 5 & 1 \\ 4 & 7 \end{pmatrix} & \begin{pmatrix} -2 & -1 \\ 5 & 9 \end{pmatrix} \end{pmatrix}$$

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$\begin{aligned} C_{11} &= A_{11}B_{11} + A_{12}B_{21} \\ C_{12} &= A_{11}B_{12} + A_{12}B_{22} \\ C_{21} &= A_{21}B_{11} + A_{22}B_{21} \\ C_{22} &= A_{21}B_{12} + A_{22}B_{22} \\ T(n) = 8\,T(n/2) + 4\,(n/2)^2 &= 8\,T(n/2) + n^2 \end{aligned}$$

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 8T(n/2) + n^2 & \text{if } n \geq 2 \end{cases}$$

Assume $n = 2^k$.

$$T(n) = 8T(n/2) + n^2$$

$$= 8\left(8T(n/2^2) + \left(\tfrac{n}{2}\right)^2\right) + n^2$$

$$= 8^2 T(n/2^2) + 2n^2 + n^2$$

$$= 8^2\left(8 \cdot T(n/2^3) + \left(\tfrac{n}{2^2}\right)^2\right) + 2n^2 + n^2$$

$$= 8^3 T(n/2^3) + 4n^2 + 2n^2 + n^2$$

$$= 8^3\left(8 \cdot T(n/2^4) + \left(\tfrac{n}{2^3}\right)^2\right) + 4n^2 + 2n^2 + n^2$$

$$= 8^4 \cdot T(n/2^4) + 8n^2 + 4n^2 + 2n^2 + n^2$$

$$= 8^4 \cdot T(n/2^4) + (2^0 + 2^1 + 2^2 + 2^3)n^2$$

$$\vdots$$

$$= \underbrace{8^k}_{(2^3)^k = (2^k)^3 = n^3} \cdot \underbrace{T(n/2^k)}_{T(1) = 1} + \underbrace{(2^0 + 2^1 + \dots + 2^{k-1})}_{2^k - 1 = n - 1} n^2$$

$$= n^3 + (n-1)n^2$$

$$= O(n^3)$$, no improvement !

Strassen Algorithm (1969)

$$
\begin{array}{lllll}
S_1 = B_{12} - B_{22} & S_2 = A_{11} + A_{12} & S_3 = A_{21} + A_{22} & S_4 = B_{21} - B_{11} & S_5 = A_{11} + A_{22} \\
S_6 = B_{11} + B_{22} & S_7 = A_{12} - A_{22} & S_8 = B_{21} + B_{22} & S_9 = A_{11} - A_{21} & S_{10} = B_{11} + B_{12}
\end{array}
$$

Strassen Algorithm (1969)

$$\begin{array}{lllll} S_1 = B_{12} - B_{22} & S_2 = A_{11} + A_{12} & S_3 = A_{21} + A_{22} & S_4 = B_{21} - B_{11} & S_5 = A_{11} + A_{22} \\ S_6 = B_{11} + B_{22} & S_7 = A_{12} - A_{22} & S_8 = B_{21} + B_{22} & S_9 = A_{11} - A_{21} & S_{10} = B_{11} + B_{12} \end{array}$$

$$\begin{aligned}
P_1 &= A_{11} S_1 = A_{11} B_{12} - A_{11} B_{22} \\
P_2 &= S_2 B_{22} = A_{11} B_{22} + A_{12} B_{22} \\
P_3 &= S_3 B_{11} = A_{21} B_{11} + A_{22} B_{11} \\
P_4 &= A_{22} S_4 = A_{22} B_{21} - A_{22} B_{11} \\
P_5 &= S_5 S_6 = A_{11} B_{11} + A_{11} B_{22} + A_{22} B_{11} + A_{22} B_{22} \\
P_6 &= S_7 S_8 = A_{12} B_{21} + A_{12} B_{22} - A_{22} B_{21} - A_{22} B_{22} \\
P_7 &= S_9 S_{10} = A_{11} B_{11} + A_{11} B_{12} - A_{21} B_{11} - A_{21} B_{12}
\end{aligned}$$

Strassen Algorithm (1969)

$$
\begin{array}{lllll}
S_1 = B_{12} - B_{22} & S_2 = A_{11} + A_{12} & S_3 = A_{21} + A_{22} & S_4 = B_{21} - B_{11} & S_5 = A_{11} + A_{22} \\
S_6 = B_{11} + B_{22} & S_7 = A_{12} - A_{22} & S_8 = B_{21} + B_{22} & S_9 = A_{11} - A_{21} & S_{10} = B_{11} + B_{12}
\end{array}
$$

$$
\begin{aligned}
P_1 &= A_{11} S_1 = A_{11} B_{12} - A_{11} B_{22} \\
P_2 &= S_2 B_{22} = A_{11} B_{22} + A_{12} B_{22} \\
P_3 &= S_3 B_{11} = A_{21} B_{11} + A_{22} B_{11} \\
P_4 &= A_{22} S_4 = A_{22} B_{21} - A_{22} B_{11} \\
P_5 &= S_5 S_6 = A_{11} B_{11} + A_{11} B_{22} + A_{22} B_{11} + A_{22} B_{22} \\
P_6 &= S_7 S_8 = A_{12} B_{21} + A_{12} B_{22} - A_{22} B_{21} - A_{22} B_{22} \\
P_7 &= S_9 S_{10} = A_{11} B_{11} + A_{11} B_{12} - A_{21} B_{11} - A_{21} B_{12}
\end{aligned}
$$

$$P_5 + P_4 - P_2 + P_6$$

Strassen Algorithm (1969)

$$
\begin{array}{lllll}
S_1 = B_{12} - B_{22} & S_2 = A_{11} + A_{12} & S_3 = A_{21} + A_{22} & S_4 = B_{21} - B_{11} & S_5 = A_{11} + A_{22} \\
S_6 = B_{11} + B_{22} & S_7 = A_{12} - A_{22} & S_8 = B_{21} + B_{22} & S_9 = A_{11} - A_{21} & S_{10} = B_{11} + B_{12}
\end{array}
$$

$$
\begin{aligned}
P_1 &= A_{11} S_1 = A_{11} B_{12} - A_{11} B_{22} \\
P_2 &= S_2 B_{22} = A_{11} B_{22} + A_{12} B_{22} \\
P_3 &= S_3 B_{11} = A_{21} B_{11} + A_{22} B_{11} \\
P_4 &= A_{22} S_4 = A_{22} B_{21} - A_{22} B_{11} \\
P_5 &= S_5 S_6 = A_{11} B_{11} + A_{11} B_{22} + A_{22} B_{11} + A_{22} B_{22} \\
P_6 &= S_7 S_8 = A_{12} B_{21} + A_{12} B_{22} - A_{22} B_{21} - A_{22} B_{22} \\
P_7 &= S_9 S_{10} = A_{11} B_{11} + A_{11} B_{12} - A_{21} B_{11} - A_{21} B_{12}
\end{aligned}
$$

$$
\begin{array}{cccccccc}
 & \multicolumn{7}{l}{P_5 + P_4 - P_2 + P_6} \\
= & A_{11} B_{11} & + A_{11} B_{22} & + A_{22} B_{11} & + A_{22} B_{22} & & & \\
 & & & - A_{22} B_{11} & & + A_{22} B_{21} & & \\
 & & - A_{11} B_{22} & & & & - A_{12} B_{22} & \\
 & & & & - A_{22} B_{22} & - A_{22} B_{21} & + A_{12} B_{22} & + A_{12} B_{21} \\
\hline
\end{array}
$$

Strassen Algorithm (1969)

$$
\begin{array}{lllll}
S_1 = B_{12} - B_{22} & S_2 = A_{11} + A_{12} & S_3 = A_{21} + A_{22} & S_4 = B_{21} - B_{11} & S_5 = A_{11} + A_{22} \\
S_6 = B_{11} + B_{22} & S_7 = A_{12} - A_{22} & S_8 = B_{21} + B_{22} & S_9 = A_{11} - A_{21} & S_{10} = B_{11} + B_{12}
\end{array}
$$

$$
\begin{aligned}
P_1 &= A_{11} S_1 = A_{11} B_{12} - A_{11} B_{22} \\
P_2 &= S_2 B_{22} = A_{11} B_{22} + A_{12} B_{22} \\
P_3 &= S_3 B_{11} = A_{21} B_{11} + A_{22} B_{11} \\
P_4 &= A_{22} S_4 = A_{22} B_{21} - A_{22} B_{11} \\
P_5 &= S_5 S_6 = A_{11} B_{11} + A_{11} B_{22} + A_{22} B_{11} + A_{22} B_{22} \\
P_6 &= S_7 S_8 = A_{12} B_{21} + A_{12} B_{22} - A_{22} B_{21} - A_{22} B_{22} \\
P_7 &= S_9 S_{10} = A_{11} B_{11} + A_{11} B_{12} - A_{21} B_{11} - A_{21} B_{12}
\end{aligned}
$$

$$
\begin{array}{rccccccccccccc}
 & \multicolumn{13}{l}{P_5 + P_4 - P_2 + P_6} \\
= & A_{11}B_{11} & + & A_{11}B_{22} & + & A_{22}B_{11} & + & A_{22}B_{22} & & & & & & \\
 & & & & - & A_{22}B_{11} & & & + & A_{22}B_{21} & & & & \\
 & & - & A_{11}B_{22} & & & & & & & - & A_{12}B_{22} & & \\
 & & & & & & - & A_{22}B_{22} & - & A_{22}B_{21} & + & A_{12}B_{22} & + & A_{12}B_{21} \\
\hline
= & A_{11}B_{11} & & & & & & & & & & & + & A_{12}B_{21}
\end{array}
$$

Strassen Algorithm (1969)

$$
\begin{array}{lllll}
S_1 = B_{12} - B_{22} & S_2 = A_{11} + A_{12} & S_3 = A_{21} + A_{22} & S_4 = B_{21} - B_{11} & S_5 = A_{11} + A_{22} \\
S_6 = B_{11} + B_{22} & S_7 = A_{12} - A_{22} & S_8 = B_{21} + B_{22} & S_9 = A_{11} - A_{21} & S_{10} = B_{11} + B_{12}
\end{array}
$$

$$
\begin{aligned}
P_1 &= A_{11} S_1 &&= A_{11} B_{12} - A_{11} B_{22} \\
P_2 &= S_2 B_{22} &&= A_{11} B_{22} + A_{12} B_{22} \\
P_3 &= S_3 B_{11} &&= A_{21} B_{11} + A_{22} B_{11} \\
P_4 &= A_{22} S_4 &&= A_{22} B_{21} - A_{22} B_{11} \\
P_5 &= S_5 S_6 &&= A_{11} B_{11} + A_{11} B_{22} + A_{22} B_{11} + A_{22} B_{22} \\
P_6 &= S_7 S_8 &&= A_{12} B_{21} + A_{12} B_{22} - A_{22} B_{21} - A_{22} B_{22} \\
P_7 &= S_9 S_{10} &&= A_{11} B_{11} + A_{11} B_{12} - A_{21} B_{11} - A_{21} B_{12}
\end{aligned}
$$

$$
\begin{array}{cccccccccccccccc}
 & \multicolumn{15}{l}{P_5 + P_4 - P_2 + P_6} \\
= & A_{11}B_{11} & + & A_{11}B_{22} & + & A_{22}B_{11} & + & A_{22}B_{22} & & & & & & \\
 & & & & - & A_{22}B_{11} & & & + & A_{22}B_{21} & & & & \\
 & & - & A_{11}B_{22} & & & & & & & - & A_{12}B_{22} & & \\
 & & & & & & - & A_{22}B_{22} & - & A_{22}B_{21} & + & A_{12}B_{22} & + & A_{12}B_{21} \\
\hline
= & A_{11}B_{11} & & & & & & & & & & & + & A_{12}B_{21} \\
= & C_{11} & & & & & & & & & & & &
\end{array}
$$

Strassen Algorithm (1969)

$$
\begin{array}{lllll}
S_1 = B_{12} - B_{22} & S_2 = A_{11} + A_{12} & S_3 = A_{21} + A_{22} & S_4 = B_{21} - B_{11} & S_5 = A_{11} + A_{22} \\
S_6 = B_{11} + B_{22} & S_7 = A_{12} - A_{22} & S_8 = B_{21} + B_{22} & S_9 = A_{11} - A_{21} & S_{10} = B_{11} + B_{12}
\end{array}
$$

$$
\begin{aligned}
P_1 &= A_{11} S_1 = A_{11} B_{12} - A_{11} B_{22} \\
P_2 &= S_2 B_{22} = A_{11} B_{22} + A_{12} B_{22} \\
P_3 &= S_3 B_{11} = A_{21} B_{11} + A_{22} B_{11} \\
P_4 &= A_{22} S_4 = A_{22} B_{21} - A_{22} B_{11} \\
P_5 &= S_5 S_6 = A_{11} B_{11} + A_{11} B_{22} + A_{22} B_{11} + A_{22} B_{22} \\
P_6 &= S_7 S_8 = A_{12} B_{21} + A_{12} B_{22} - A_{22} B_{21} - A_{22} B_{22} \\
P_7 &= S_9 S_{10} = A_{11} B_{11} + A_{11} B_{12} - A_{21} B_{11} - A_{21} B_{12}
\end{aligned}
$$

$$
\begin{array}{rlllllllllllllll}
 & \multicolumn{15}{l}{P_5 + P_4 - P_2 + P_6} \\
= & A_{11}B_{11} & + & A_{11}B_{22} & + & A_{22}B_{11} & + & A_{22}B_{22} & & & & & & & \\
 & & & & - & A_{22}B_{11} & & & + & A_{22}B_{21} & & & & & \\
 & & - & A_{11}B_{22} & & & & & & & - & A_{12}B_{22} & & & \\
 & & & & & & - & A_{22}B_{22} & - & A_{22}B_{21} & + & A_{12}B_{22} & + & A_{12}B_{21} \\
\hline
= & A_{11}B_{11} & & & & & & & & & & & + & A_{12}B_{21} \\
= & C_{11} & & & & & & & & & & & &
\end{array}
$$

$$C_{11} = P_5 + P_4 - P_2 + P_6$$

Strassen Algorithm (1969)

$$
\begin{array}{lllll}
S_1 = B_{12} - B_{22} & S_2 = A_{11} + A_{12} & S_3 = A_{21} + A_{22} & S_4 = B_{21} - B_{11} & S_5 = A_{11} + A_{22} \\
S_6 = B_{11} + B_{22} & S_7 = A_{12} - A_{22} & S_8 = B_{21} + B_{22} & S_9 = A_{11} - A_{21} & S_{10} = B_{11} + B_{12}
\end{array}
$$

$$
\begin{aligned}
P_1 &= A_{11} S_1 = A_{11} B_{12} - A_{11} B_{22} \\
P_2 &= S_2 B_{22} = A_{11} B_{22} + A_{12} B_{22} \\
P_3 &= S_3 B_{11} = A_{21} B_{11} + A_{22} B_{11} \\
P_4 &= A_{22} S_4 = A_{22} B_{21} - A_{22} B_{11} \\
P_5 &= S_5 S_6 \;= A_{11} B_{11} + A_{11} B_{22} + A_{22} B_{11} + A_{22} B_{22} \\
P_6 &= S_7 S_8 \;= A_{12} B_{21} + A_{12} B_{22} - A_{22} B_{21} - A_{22} B_{22} \\
P_7 &= S_9 S_{10} = A_{11} B_{11} + A_{11} B_{12} - A_{21} B_{11} - A_{21} B_{12}
\end{aligned}
$$

$$
\begin{array}{rlllllll}
 & P_5 + P_4 - P_2 + P_6 & & & & & & \\
= & A_{11}B_{11} & + A_{11}B_{22} & + A_{22}B_{11} & + A_{22}B_{22} & & & \\
 & & & - A_{22}B_{11} & & + A_{22}B_{21} & & \\
 & & - A_{11}B_{22} & & & & - A_{12}B_{22} & \\
 & & & & - A_{22}B_{22} & - A_{22}B_{21} & + A_{12}B_{22} & + A_{12}B_{21} \\
\hline
= & A_{11}B_{11} & & & & & & + A_{12}B_{21} \\
= & C_{11} & & & & & &
\end{array}
$$

$$
\begin{aligned}
C_{11} &= P_5 + P_4 - P_2 + P_6 \\
C_{12} &= P_1 + P_2 \\
C_{21} &= P_3 + P_4 \\
C_{22} &= P_5 + P_1 - P_3 - P_7
\end{aligned}
$$

Strassen Algorithm (1969)

$$
\begin{array}{lllll}
S_1 = B_{12} - B_{22} & S_2 = A_{11} + A_{12} & S_3 = A_{21} + A_{22} & S_4 = B_{21} - B_{11} & S_5 = A_{11} + A_{22} \\
S_6 = B_{11} + B_{22} & S_7 = A_{12} - A_{22} & S_8 = B_{21} + B_{22} & S_9 = A_{11} - A_{21} & S_{10} = B_{11} + B_{12}
\end{array}
$$

$$
\begin{aligned}
P_1 &= A_{11} S_1 = A_{11} B_{12} - A_{11} B_{22} \\
P_2 &= S_2 B_{22} = A_{11} B_{22} + A_{12} B_{22} \\
P_3 &= S_3 B_{11} = A_{21} B_{11} + A_{22} B_{11} \\
P_4 &= A_{22} S_4 = A_{22} B_{21} - A_{22} B_{11} \\
P_5 &= S_5 S_6 = A_{11} B_{11} + A_{11} B_{22} + A_{22} B_{11} + A_{22} B_{22} \\
P_6 &= S_7 S_8 = A_{12} B_{21} + A_{12} B_{22} - A_{22} B_{21} - A_{22} B_{22} \\
P_7 &= S_9 S_{10} = A_{11} B_{11} + A_{11} B_{12} - A_{21} B_{11} - A_{21} B_{12}
\end{aligned}
$$

$$
\begin{array}{llllllllllllllll}
 & P_5 + P_4 - P_2 + P_6 \\
= & A_{11}B_{11} & + & A_{11}B_{22} & + & A_{22}B_{11} & + & A_{22}B_{22} \\
 & & & & - & A_{22}B_{11} & & & + & A_{22}B_{21} \\
 & & - & A_{11}B_{22} & & & & & & & - & A_{12}B_{22} \\
 & & & & & & - & A_{22}B_{22} & - & A_{22}B_{21} & + & A_{12}B_{22} & + & A_{12}B_{21} \\
\hline
= & A_{11}B_{11} & & & & & & & & & & & + & A_{12}B_{21} \\
= & C_{11}
\end{array}
$$

$$
\begin{aligned}
C_{11} &= P_5 + P_4 - P_2 + P_6 \\
C_{12} &= P_1 + P_2 \\
C_{21} &= P_3 + P_4 \\
C_{22} &= P_5 + P_1 - P_3 - P_7 \\
T(n) &= 7\,T(n/2) + 18(n/2)^2 = 7\,T(n/2) + (9/2)n^2
\end{aligned}
$$

$$T(n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ 7\,T(n/2) + \frac{9}{2} n^2 & \text{if} \quad n \geq 2 \end{cases}$$

Assume $n = 2^k$.

$$T(n) = 7\,T(n/2) + \frac{9}{2} n^2$$

$$= 7\left(7\,T(n/2^2) + \frac{9}{2}(n/2)^2\right) + \frac{9}{2} n^2$$

$$= 7^2\,T(n/2^2) + \frac{7}{4}\cdot\frac{9}{2}\cdot n^2 + \frac{9}{2} n^2$$

$$= 7^2\left(7\,T(n/2^3) + \frac{9}{2}\left(\frac{n}{2^2}\right)^2\right) + \frac{7}{4}\cdot\frac{9}{2} n^2 + \frac{9}{2} n^2$$

$$= 7^3\,T\left(\frac{n}{2^3}\right) + \left(\frac{7}{4}\right)^2\cdot\frac{9}{2} n^2 + \frac{7}{4}\cdot\frac{9}{2}\cdot n^2 + \frac{9}{2} n^2$$

$$= 7^3\left(7\,T\left(\frac{n}{2^4}\right) + \frac{9}{2}\left(\frac{n}{2^3}\right)^2\right) + \left(\frac{7}{4}\right)^2\frac{9}{2} n^2 + \frac{7}{4}\cdot\frac{9}{2} n^2 + \frac{9}{2} n^2$$

$$= 7^4\,T\left(\frac{n}{2^4}\right) + \left(\frac{7}{4}\right)^3\frac{9}{2} n^2 + \left(\frac{7}{4}\right)^2\frac{9}{2} n^2 + \frac{7}{4}\cdot\frac{9}{2} n^2 + \frac{9}{2} n^2$$

$$= 7^4\,T\left(\frac{n}{2^k}\right) + \frac{9}{2}\left(\left(\frac{7}{4}\right)^0 + \left(\frac{7}{4}\right)^1 + \left(\frac{7}{4}\right)^2 + \left(\frac{7}{4}\right)^3\right) n^2$$

$\vdots$

$T(1) = 1$

$$= 7^k\,T\left(\frac{n}{2^k}\right) + \frac{9}{2}\left(\left(\frac{7}{4}\right)^0 + \left(\frac{7}{4}\right)^1 + \dots + \left(\frac{7}{4}\right)^{k-1}\right) n^2$$

Recall:

$$\sum_{i=0}^{k-1} x^i = 1 + x + x^2 + \dots + x^{k-1} = \frac{x^k - 1}{x - 1} \quad (x \neq 1)$$

$$= 7^k + \frac{9}{2} \cdot \frac{(7/4)^k - 1}{\frac{7}{4} - 1} n^2$$

$$= 7^k + 6\left((7/4)^k - 1\right) n^2$$

$$= \left(2^{\log_2(7)}\right)^k + 6\left(\left(2^{\log_2(7/4)}\right)^k - 1\right) n^2$$

$$= \left(2^k\right)^{\log_2(7)} + 6\left(\left(2^k\right)^{\log_2(7/4)} - 1\right) n^2$$

$$= n^{\log_2(7)} + 6\left(n^{\log_2(7/4)} - 1\right) n^2$$

$$= 7\, n^{\log_2(7)} - 6\, n^2$$

$$= O\left(n^{\log_2(7)}\right),$$

where $\log_2(7) \approx 2.81$

Strassen Algorithm (1969)

$$T(n) = 7\,T(n/2) + (9/2)n^2$$
$$T(n) = O\left(n^{\log_2(7)}\right),$$

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Coppersmith & Winograd (1990): $O\left(n^{2.38}\right)$