CSI - 3105 Design & Analysis of Algorithms Course 14

Jean-Lou De Carufel

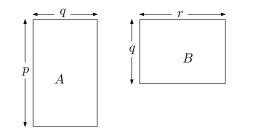
Fall 2020

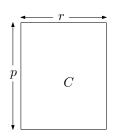
Matrix Chain Multiplication

 $A: p \times q \text{ matrix}$

 $B: q \times r \text{ matrix}$

 $C = A B : p \times r \text{ matrix}$





C has pr entries, each of which can be computed in O(q) time. So C can be computed in O(pqr) time. We define the cost of multiplying A and B to be pqr.

 $\textit{A}_1:10\times100$

 $A_2: 100 \times 5$

 $A_3:5\times50$

How to compute $A_1A_2A_3$:

 $A_1:10 \times 100$

 $A_2:100\times 5$

 $A_3: 5 \times 50$

How to compute $A_1A_2A_3$:

• Compute A_1A_2 , $cost = 10 \times 100 \times 5 = 5000$

 $A_1:10 \times 100$

 $A_2: 100 \times 5$

 $A_3:5\times50$

How to compute $A_1A_2A_3$:

•

- Compute A_1A_2 , $cost = 10 \times 100 \times 5 = 5000$
- Compute $(A_1A_2)A_3$, $cost = 10 \times 5 \times 50 = 2500$.

 $A_1:10 \times 100$

 $A_2: 100 \times 5$

 $A_3:5\times50$

How to compute $A_1A_2A_3$:

•

- Compute A_1A_2 , $cost = 10 \times 100 \times 5 = 5000$
- Compute $(A_1A_2)A_3$, $cost = 10 \times 5 \times 50 = 2500$.

For a total cost of 5000 + 2500 = 7500.

 $A_1:10 \times 100$

 $A_2: 100 \times 5$

 $A_3:5\times50$

How to compute $A_1A_2A_3$:

0

- Compute A_1A_2 , $cost = 10 \times 100 \times 5 = 5000$
- Compute $(A_1A_2)A_3$, $cost = 10 \times 5 \times 50 = 2500$.

For a total cost of 5000 + 2500 = 7500.

•

• Compute A_2A_3 , $cost = 100 \times 5 \times 50 = 25000$

 $A_1:10 \times 100$

 $A_2: 100 \times 5$

 $A_3: 5 \times 50$

How to compute $A_1A_2A_3$:

0

- Compute A_1A_2 , $cost = 10 \times 100 \times 5 = 5000$
- Compute $(A_1A_2)A_3$, $cost = 10 \times 5 \times 50 = 2500$.

For a total cost of 5000 + 2500 = 7500.

•

- Compute A_2A_3 , $cost = 100 \times 5 \times 50 = 25000$
- Compute $A_1(A_2A_3)$, $cost = 10 \times 100 \times 50 = 50000$.

 $A_1:10 \times 100$

 $A_2: 100 \times 5$

 $A_3: 5 \times 50$

How to compute $A_1A_2A_3$:

•

- Compute A_1A_2 , $cost = 10 \times 100 \times 5 = 5000$
- Compute $(A_1A_2)A_3$, $cost = 10 \times 5 \times 50 = 2500$.

For a total cost of 5000 + 2500 = 7500.

•

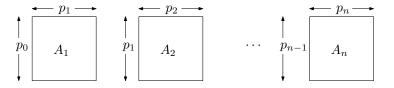
- Compute A_2A_3 , $cost = 100 \times 5 \times 50 = 25000$
- Compute $A_1(A_2A_3)$, $cost = 10 \times 100 \times 50 = 50000$.

For a total cost of 25000 + 50000 = 75000.

Which one is better?

In general,

- $p_0, p_1, ..., p_n$: positive integers
- $A_1A_2,...,A_n$: matrices such that A_i has p_{i-1} rows and p_i columns.



Compute the best order to compute $A_1A_2 \cdot ... \cdot A_n$, i.e., minimize the total cost.

Consider the best order to compute $A_iA_{i+1} \cdot ... \cdot A_j$. In the **last** step, we multiply

$$\underbrace{(A_i \cdot \ldots \cdot A_k)}_{\text{already computed}} \underbrace{(A_{k+1} \cdot \ldots \cdot A_j)}_{\text{already computed}}$$

for some k such that $i \le k \le j-1$.

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Consider the best order to compute $A_iA_{i+1} \cdot ... \cdot A_j$. In the **last** step, we multiply

$$\underbrace{(A_i \cdot \ldots \cdot A_k)}_{\text{already computed}} \underbrace{(A_{k+1} \cdot \ldots \cdot A_j)}_{\text{already computed}}$$

for some k such that $i \le k \le j-1$.

How did we compute $A_i \cdot ... \cdot A_k$?

Consider the best order to compute $A_iA_{i+1} \cdot ... \cdot A_j$. In the **last** step, we multiply

$$\underbrace{(A_i \cdot \ldots \cdot A_k)}_{\text{already computed}} \underbrace{(A_{k+1} \cdot \ldots \cdot A_j)}_{\text{already computed}}$$

for some k such that $i \le k \le j-1$.

How did we compute $A_i \cdot ... \cdot A_k$? In the best order.

Consider the best order to compute $A_iA_{i+1} \cdot ... \cdot A_j$. In the **last** step, we multiply

$$\underbrace{(A_i \cdot \ldots \cdot A_k)}_{\text{already computed}} \underbrace{(A_{k+1} \cdot \ldots \cdot A_j)}_{\text{already computed}}$$

for some k such that $i \le k \le j-1$.

How did we compute $A_i \cdot ... \cdot A_k$? In the best order. How did we compute $A_{k+1} \cdot ... \cdot A_i$?

Consider the best order to compute $A_iA_{i+1} \cdot ... \cdot A_j$. In the **last** step, we multiply

$$\underbrace{(A_i \cdot \ldots \cdot A_k)}_{\text{already computed}} \underbrace{(A_{k+1} \cdot \ldots \cdot A_j)}_{\text{already computed}}$$

for some k such that $i \le k \le j-1$.

How did we compute $A_i \cdot ... \cdot A_k$? In the best order. How did we compute $A_{k+1} \cdot ... \cdot A_i$? In the best order!

Consider the best order to compute $A_i A_{i+1} \cdot ... \cdot A_i$. In the **last** step, we multiply

$$\underbrace{(A_i \cdot \ldots \cdot A_k)}_{\text{already computed}} \underbrace{(A_{k+1} \cdot \ldots \cdot A_j)}_{\text{already computed}}$$

for some k such that $i \leq k \leq j-1$.

How did we compute $A_i \cdot ... \cdot A_k$? In the best order. How did we compute $A_{k+1} \cdot ... \cdot A_i$? In the best order!

minimum cost to compute $A_i \cdot ... \cdot A_i$

minimum cost to compute $A_i \cdot ... \cdot A_k$

minimum cost to compute $A_{k+1} \cdot ... \cdot A_i$

$$p_{i-1}p_kp_j$$

For $1 \le i \le j \le n$, define

$$m(i,j) = \text{minimum cost to compute } A_i \cdot ... \cdot A_j$$
.

We want to compute m(1, n).

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We want to compute m(1, n).

If we know k, then

$$m(i,j) = m(i,k) + m(k+1,j) + p_{i-1}p_kp_j.$$

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If we know k, then

$$m(i,j) = m(i,k) + m(k+1,j) + p_{i-1}p_kp_j.$$

But we do not know k, so try all values of k, $i \le k \le j-1$ and take the best one.

For $1 \le i \le j \le n$, define

$$m(i,j) = \text{minimum cost to compute } A_i \cdot ... \cdot A_j$$
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We want to compute m(1, n).

If we know k, then

$$m(i,j) = m(i,k) + m(k+1,j) + p_{i-1}p_kp_j.$$

But we do not know k, so try all values of k, $i \le k \le j-1$ and take the best one.

Recurrence:

• For $1 \le i \le n$: m(i, i) = 0.

For $1 \le i \le j \le n$, define

$$m(i,j) = \text{minimum cost to compute } A_i \cdot ... \cdot A_j$$
.

We want to compute m(1, n).

If we know k, then

$$m(i,j) = m(i,k) + m(k+1,j) + p_{i-1}p_kp_j.$$

But we do not know k, so try all values of k, $i \le k \le j-1$ and take the best one.

Recurrence:

- For $1 \le i \le n$: m(i, i) = 0.
- For $1 \le i < j \le n$:

$$m(i,j) = \min_{i \le k \le j-1} \{m(i,k) + m(k+1,j) + p_{i-1}p_kp_j\}$$



Step 3: Solve the Recurrence Bottom-Up

Compute, in this order,

$$m(1,1), m(2,2), ..., m(n, n)$$

 $m(1,2), m(2,3), ..., m(n-1, n)$
 $m(1,3), m(2,4), ..., m(n-2, n)$
 $m(1,4), m(2,5), ..., m(n-3, n)$
 \vdots
 $m(1, n-1), m(2, n)$
 $m(1, n)$

Algorithm

Algorithm Matrix Chain Multiplication

```
1: for i = 1 to n do
      m(i, i) = 0
 3: end for
 4: for \ell = 2 to n do
    // Compute m(1, \ell), m(2, \ell + 1), ..., m(n - \ell + 1, n)
 6:
     for i = 1 to n - \ell + 1 do
 7:
       // Compute m(i, i + \ell - 1)
 8:
      i = i + \ell - 1
         // Compute m(i,j) using the recurrence
10:
    m(i, j) = \infty
11:
          for k = i to i - 1 do
12:
             m(i, j) = \min \{ m(i, j), m(i, k) + m(k + 1, j) + p_{i-1} p_k p_i \}
13:
          end for
14:
       end for
15: end for
16: return m(1, n)
```

Matrices		s	A_1	A_2		A_3	A_4	A_5	A_6	
Dimensions		ns 3	0 × 35	35 ×	15	15×5	5×10	10×20	20×2	
			p	$p_0 \times p_1$	$p_1 \times$	p_2	$p_2 \times p_3$	$p_3 \times p_4$	$p_4 \times p_5$	$p_5 \times p_6$
$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$										
								$-p_{i-1}p_kp_j$	$\}$ $1 \leq i < i$	$\leq j \leq 6$
	$i \setminus j$	1	2	3 2625 0	4	5	6			
	1	0	15750					_		
	2		0	2625						
	3			0	750					
	4				0	1000				
	5					0	5000			
	6						0			

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Matrices							A_4	A_5	A_6	
Dimensions		ns 3	0 × 35	35 ×	15	15 × 5	5 × 10	10 × 20	20 × 2!	
			p	$p_0 \times p_1$	$p_1 \times$	p_2	$p_2 \times p_3$	$p_3 \times p_4$	$p_4 \times p_5$	$p_5 \times p_6$
$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$										
	$i \setminus j$	1	2	3 7875 2625 0	4	5	6			
	1	0	15750	7875				-		
	2		0	2625	4375					
	3			0	750					
	4				0	1000				
	5					0	5000			
	6						0			

Matrices		I .	A_2			A_5	A_6		
Dimensions									
		$p_0 \times p_1$	$p_1 \times p_2$	$p_2 \times p_3$	$p_3 \times p_4$	$p_4 \times p_5$	$p_5 \times p_6$		
$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1} p_k p_j \} & 1 \le i < j \le 6 \end{cases}$									
$i \setminus j$	1	2 3	4	5 6					
1	0 15	750 7875	9375						
2		0 2625	4375	7125					
3		0	750 2	2500					
$\frac{1}{1}$	0 15	2 3 750 7875 0 2625 0	9375	<u> </u>					

Matrices			A_1		!	A_3	A_4	A_5	A_6	
Dimensions										
		p_0	$\times p_1$	$p_1 \times$	p_2	$p_2 \times p_3$	$p_3 \times p_4$	$p_4 \times p_5$	$p_5 \times p_6$	
$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$										
$i \setminus j$	1	2	3	4	5	6				
1 2 3 4 5 6	0 :	15750	7875	9375			_			
2		0	2625	4375	7125	;				
3			0	750	2500	5375				
4				0	1000	3500				
5					0	5000				
6						0				

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	Mat	Matrices			A_2	A_3	A_4	A_5	A_6
Ī	Dime	nsions					5×10		
			$p_0 \times$	p_1 p_1	$\times p_2$	$p_2 \times p_3$	$p_3 \times p_4$	$p_4 \times p_5$	$p_5 \times p_6$
) + m(k	+1, j) +	$-p_{i-1}p_kp_j$	$i = j$ $1 \le i < j$	< <i>j</i> ≤ 6
	$i \setminus j$	1	2 750 78 0 26	3 4	5	6			
	1	0 15	750 78	375 937	5 1187	5	_		
	2		0 26	525 437	5 7125				
	3		(0 750	2500	5375			

$$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$$

$i \setminus j$	1	2	3	4	5	6
1	0	15750	7875	9375	11875	
2		0	2625	4375	7125	10500
3			0	750	2500	5375
4				0	1000	3500
5					0	5000
6						0

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	Matrices			_	A_3	A_4	A_5	A_6
Dime	nsions					5 × 10		
		$p_0 \times$	p_1 p_1	$\times p_2$	$p_2 \times p_3$	$p_3 \times p_4$	$p_4 \times p_5$	$p_5 \times p_6$
m(i	$,j)=\langle$	$ \begin{cases} 0 \\ \min_{i \le k < j} \end{cases} $	{ m(i, k)+m(k	+1, j) +	$-p_{i-1}p_kp_j$	$i = j$ $1 \le i <$	$j \le 6$
$i \setminus j$	1	2 750 78 0 20	3 4	5	6			
1	0 15	750 78	875 937	5 1187	5 15125	-		
2		0 20	625 437	5 7125	10500			
3			0 750	2500	5375			

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Mat	rices	/			A_2		A_3		A_4		A_5		A_6	
Dime	nsions	30	× 35	35 ×	15	15 ×	5	5 × 1	0	10×20		0	20	× 25
		p_0	\times p_1	$p_1 \times$	<i>p</i> ₂	$p_2 \times p_2$) 3	$p_3 \times p$) 4	<i>p</i> ₄	$\times p$	5	<i>p</i> ₅	$\times p_6$
$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{m(i,k) + m(k+1,j) + p_{i-1}p_kp_j\} \end{cases} 1 \le i < j \le 6$														
$i \setminus j$	1	2	3	4	5	6		$i \setminus j$	1	2	3	4	5	6
1 2 3	0 1	L5750						1	0	1				
2		0						2		0				
3			0					3			0			
4				0				4				0		
5					0			5					0	
6						0		6						0

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Mat	rices	5	A_1	A_2	!	A_3 A_4		•			A_6		
Dime	nsio	ns 30) × 35	35×15		15 × 5	5×10		10	× 2	0	20	× 25
		p_0	$0 \times p_1$	$p_1 \times$	p_2	$p_2 \times p_3$	$p_3 \times p_3$	04	<i>p</i> ₄	$\times p$	5	<i>p</i> ₅	$\times p_6$
$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1} p_k p_j \} & 1 \le i < j \le 6 \end{cases}$													
$i \setminus j$	1	2	3	4	5	6	$i \setminus j$	1	2	3	4	5	6
1 2 3	0	15750					1	0	1				
2		0	2625				2		0	2			
3			0	750			3			2 0	3		
4				0			4				0		
5					0		5					0	
6						0	6						0

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Mat	rices	5	A_1	A_2		A_3	A ₄		A ₅			A_6	
Dime	nsio					15 × 5			-	× 2	-	20	× 25
		p_0	$p_1 \times p_1$	$p_1 \times$	p_2	$p_2 \times p_3$	$p_3 \times p$	2 4	<i>p</i> ₄	× p	5	<i>p</i> ₅	\times p_6
$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$													
$i \setminus j$	1	2	3	4	5	6	$i \setminus j$	1	2	3	4	5	6
1	0	15750	7875				1	0	1	1			
1 2 3		0	2625	4375			2		0	2	3		
3			0	750			3			0	3		
4				0	1000	0	4				0	4	
5					0	5000	5					0	5
6						0	6						0

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Mat	rices		A_1	A_2		A_3		A_4		A ₅		A_6	
Dime	nsior	ıs 30	× 35	35 ×	15	15×5	5×10		0 10 imes 2		0	20	× 25
		p_0	$\times p_1$	$p_1 \times p_2$		$p_2 \times p_3$	$p_3 \times p_4$		<i>p</i> ₄	× p	5	<i>p</i> ₅	$\times p_6$
$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$													
$i \setminus j$	1	2	3	4	5	6	$i \setminus j$	1	2	3	4	5	6
1	0	15750	7875				1	0	1	1			
1 2 3		0	2625	4375			2		0	2	3		
3			0	750	2500	1	3			0	3	3	
4				0	1000	1	4				0	3	
5					0	5000	5					0	5
6						0	6						0

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Mat	rices	5	A_1	A_2		A_3	A_4		A_5			A_6	
Dime	nsio	ns 30) × 35	35 ×	15	15 × 5	$.5 \times 5$ 5×10		$0 10 \times 20$			20	× 25
		$ p_0 $	$p_1 \times p_1$	$p_1 \times$	$p_1 \times p_2$ $p_2 \times p_3$ $p_3 \times p_4$ $p_4 \times p_5$					5	$p_5 \times p_6$		
$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$													
$i \setminus j$	1	2	3	4	5	6	$i \setminus j$	1	2	3	4	5	6
1	0	15750	7875	9375			1	0	1	1	3		
1 2 3		0	2625	4375	7125		2		0	2	3	3	
3			0	750	2500)	3			0	3	3	
4				0	1000	3500	4				0	4	5
5			0	5000	5					0	5		
6						0	6						0

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Mat	rices	5	A_1	A_2		A_3	A_4		A_5			A_6	
Dime	nsio	ns 30) × 35	35 ×	15	15×5	15×5 5×10		$0 10 \times 20$		0	20	× 25
		$ p_0 $	$0 \times p_1$	$p_1 \times p_2$ $p_2 \times p_3$ $p_3 \times p_4$ $p_4 \times p_4$					$\times p$	5	<i>p</i> ₅	$\times p_6$	
$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$													
$i \setminus j$	1	2	3	4	5	6	$i \setminus j$	1	2	3	4	5	6
1	0	15750	7875	9375			1	0	1	1	3		
1 2 3		0	2625	4375	7125	5	2		0	2	3	3 3	
3			0	750	2500	5375	3			0	3	3	3
4				0	1000	3500	4				0	4	5
5				0	5000	5					0	5	
6						0	6						0

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Matrices			A_1	A_2	2	A_3	A_4		A ₅			A_6	
Dimensions						15×5			10×20		0	20	× 25
		p_0	$\times p_1$	$p_1 \times$	p_2	$p_2 \times p_3$	$p_3 \times p$) 4	<i>p</i> ₄	$\times p$	5	<i>p</i> ₅ :	$\times p_6$
m(i	(,j) =	$\begin{cases} 0 \\ m \\ i \leq k \end{cases}$	in { <i>m</i> (<< <i>j</i>	(i,k) $+$	- m(k	$+\ 1,j)\ +$	$-p_{i-1}p_k$, p j }	<i>i</i> 1	$= j$ $\leq j$; i <	$j \le$	6
$i \setminus j$	1	2	3	4	5	6	$i \setminus j$	1	2	3	4	5	6
1	0	15750	7875	9375	11875	5	1	0	1	1	3	3	
2		0	2625	4375	7125	10500	2		0	2	3	3	3
3			0	750	2500	5375	3			0	3	3	3
4				0	1000	3500	4				0	4	5
5 6					0	5000	5					0	5
6						Λ	6	1					Λ

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Matrices			A_1	A_2		A_3 A_4		A_5			A_6			
Dime	nsions			35×15					0 10		0×20		20 :	× 25
		p_0	\times p_1	$p_1 \times$	$p_1 \times p_2 p_2 \times p_3 p_3 \times p_4$					<i>p</i> ₄	$\times p$	5	$p_5 \times p_6$	
$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$													6	
$i \setminus j$	1	2	3	4	5	6	i 🔪	$j \mid$	1	2	3	4	5	6
1	0 1	5750	7875	9375	1187	5 15125	1		0	1	1	3	3	3
1 2 3		0	2625	4375	7125	10500	2			0	2	3	3	3
3			0	750	2500	5375	3				0	3	3	3
4				0	1000	3500	4					0	4	5
5					0	5000	5						0	5
6						0	6							0

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There are 3 nested loops, so this algorithm takes $O(n^3)$ time.

More careful counting:

- *l*: 2 to *n*
 - For each ℓ , we have i: 1 to $n \ell + 1$
 - For each i, we have k: i to $i + \ell 2$

Algorithm

Algorithm Matrix Chain Multiplication

```
1: for i = 1 to n do
      m(i, i) = 0
 3: end for
 4: for \ell = 2 to n do
    // Compute m(1, \ell), m(2, \ell + 1), ..., m(n - \ell + 1, n)
 6:
     for i = 1 to n - \ell + 1 do
 7:
      // Compute m(i, i + \ell - 1)
 8:
      i = i + \ell - 1
       // Compute m(i,j) using the recurrence
10:
    m(i, j) = \infty
11:
          for k = i to i - 1 do
12:
             m(i, j) = \min \{ m(i, j), m(i, k) + m(k + 1, j) + p_{i-1} p_k p_i \}
13:
          end for
14:
       end for
15: end for
16: return m(1, n)
```

$$\sum_{\ell=2}^{n} \sum_{i=1}^{n-\ell+1} \sum_{k=i}^{i+\ell-2} 1$$

$$\sum_{\ell=2}^{n} \sum_{i=1}^{n-\ell+1} \sum_{k=i}^{i+\ell-2} 1$$

$$= \sum_{\ell=2}^{n} \sum_{i=1}^{n-\ell+1} (\ell-1)$$

$$\sum_{\ell=2}^{n} \sum_{i=1}^{n-\ell+1} \sum_{k=i}^{i+\ell-2} 1$$

$$= \sum_{\ell=2}^{n} \sum_{i=1}^{n-\ell+1} (\ell-1)$$

$$= \sum_{\ell=2}^{n} (n-\ell+1)(\ell-1)$$

$$\begin{split} &\sum_{\ell=2}^{n} \sum_{i=1}^{n-\ell+1} \sum_{k=i}^{i+\ell-2} 1 \\ &= \sum_{\ell=2}^{n} \sum_{i=1}^{n-\ell+1} (\ell-1) \\ &= \sum_{\ell=2}^{n} (n-\ell+1)(\ell-1) \\ &= \sum_{n=0}^{n} (n-\ell+1)(\ell-1) \quad \text{since the summand is 0 when } \ell=0 \end{split}$$

$$\sum_{\ell=2}^{n} \sum_{i=1}^{n-\ell+1} \sum_{k=i}^{i+\ell-2} 1$$

$$= \sum_{\ell=2}^{n} \sum_{i=1}^{n-\ell+1} (\ell-1)$$

$$= \sum_{\ell=2}^{n} (n-\ell+1)(\ell-1)$$

$$= \sum_{\ell=1}^{n} (n-\ell+1)(\ell-1)$$
 since the summand is 0 when $\ell=0$

$$= \frac{n^3-n}{6}$$

$$= \Theta(n^3).$$