Chapter 6: P vs NP

§6.1 Introduction

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Intuitively speaking,

polynomial means good, fast, efficient, easy, ...

exponential means bad, slow, "try all possible solutions", difficult, ...

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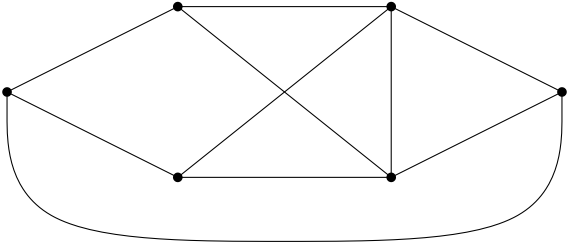
- Is a given input graph connected?
- Is a given input graph bipartite?
- Is a given input sequence sorted?
- Does a given input graph contain an Euler cycle? An *Euler cycle* is a cycle that traverses each edge exactly once.

Other Problems

HAM-CYCLE

input: An undirected graph $G = (V, E)$ stored using adjacency lists.

question: Does G contain a Hamiltonian cycle? A *Hamiltonian cycle* is a cycle that traverses each vertex exactly once.



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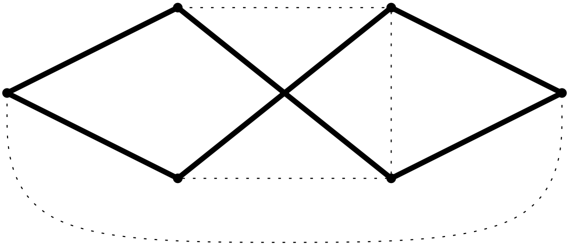

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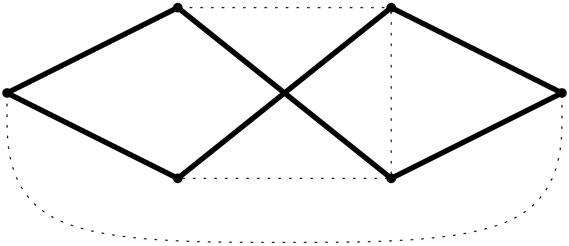

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Traveling Salesman Problem (TSP)

input:

- A complete directed graph $G = (V, E)$, where each edge $(u, v) \in E$ has a weight $wt(u, v) > 0$.
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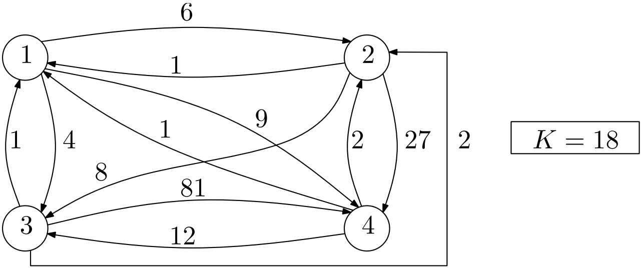

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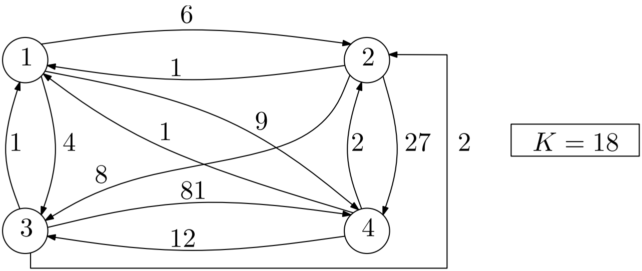

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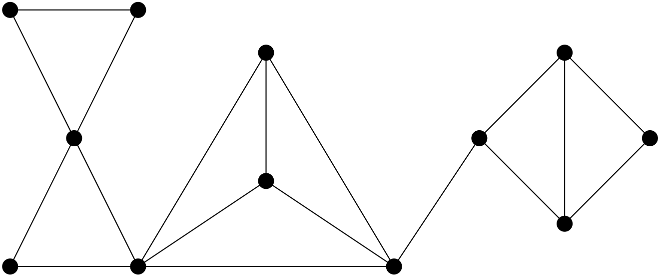

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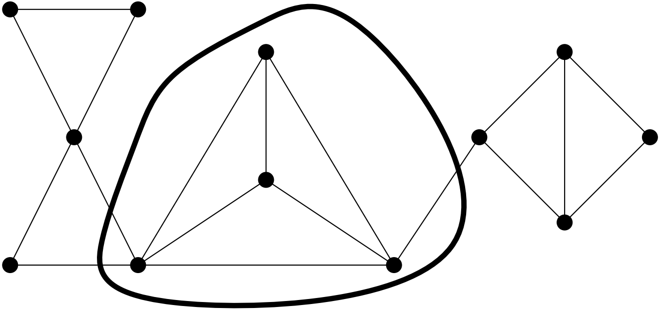

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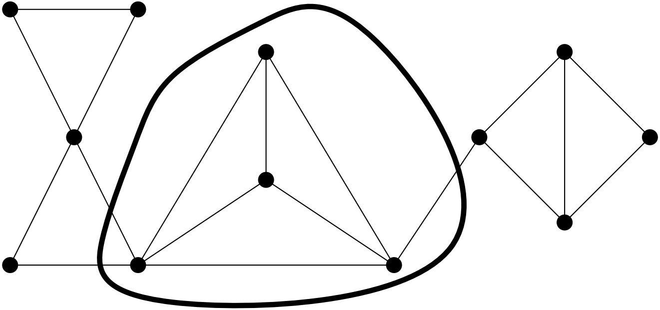

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- Do we have $|V'| = K$?
- For each $u, v \in V'$ such that $u \neq v$, is $\{u, v\}$ an edge in E?

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A decision problem A is in NP if

- If for a given input I, the answer to the question $A(I)$ is YES, then there exists a proof/solution/certificate C such that
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The following problems are in NP:

HAM-CYCLE, TSP, SUBSET-SUM, CLIQUE