CSI - 3105 Design & Analysis of Algorithms
Course 8

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Fall 2020

Algorithm *Dijkstra*(G, s)

for each vertex $v \in V$ **do**
    $d(v) = \infty$
end for
$d(s) = 0$
$S = \{\,\}$
$Q = V$
while $Q \neq \{\,\}$ **do**
    $u =$ vertex in Q for which $d(u)$ is minimum
    delete u from Q
    insert u into S
    for each edge (u, v) **do**
        $d(v) = \min\{d(v), d(u) + wt(u, v)\}$
    end for
end while

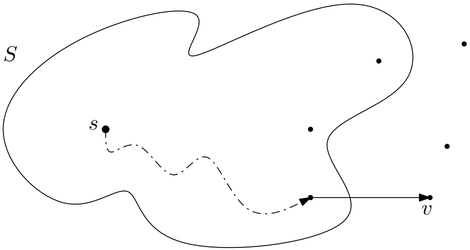
Dijkstra Algorithm is Correct

Theorem

Let $G = (V, E)$ be a weighted directed graph and $s \in V$ be a source vertex. Dijkstra algorithm finds the lengths of the shortest paths from s to all vertices in V.

Special Paths and Induction Hypotheses

PROOF: We say that a path from s to a vertex v is *special* if all vertices on that path belong to S, except maybe v.



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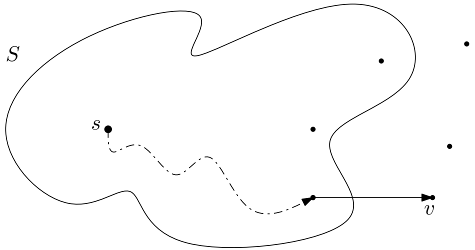

A *special path* from s to v.

We prove by induction that

(a) if a vertex u is in S, then $d(u)$ gives the length of a shortest path from s to u and

(b) if a vertex u is not in S, then $d(u)$ gives the length of a shortest *special path* from s to u.

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We address induction steps (a) and (b) separately.

Induction Step (a)

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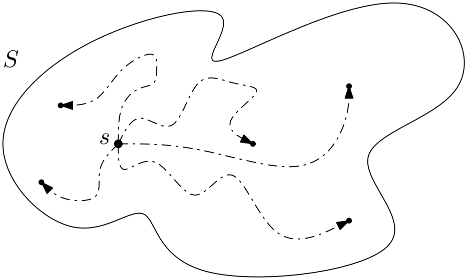

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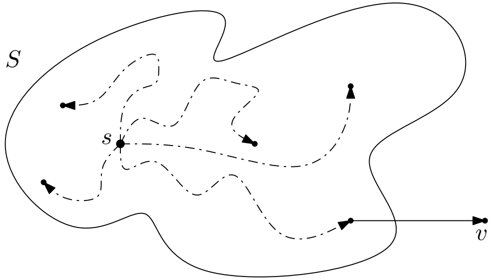
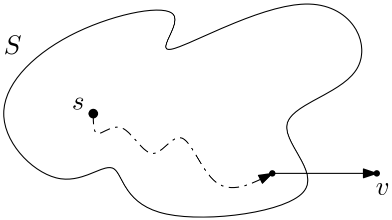


As for node v, it is about to be inserted in S. Before adding it to S, we must check that $d(v)$ gives the length of a shortest path from s to v.

Induction Step (a)



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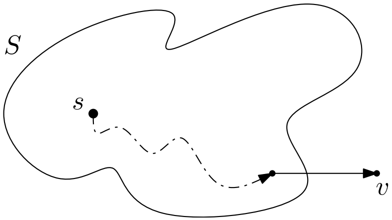

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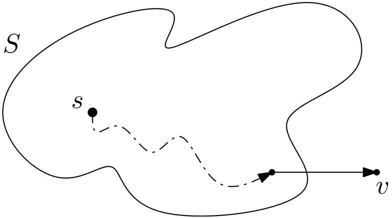

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(1) By the induction hypothesis (b), we already know that $d(v)$ is less than or equal to the length of any special path from s to v.

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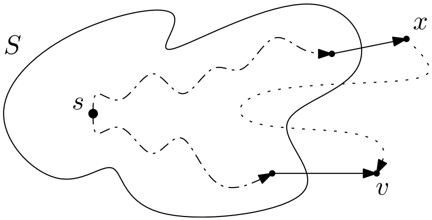

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(1) By the induction hypothesis (b), we already know that $d(v)$ is less than or equal to the length of any special path from s to v.

(2) A non-special path from s to v is one which contains at least one vertex $x \neq v$ that is not in S.

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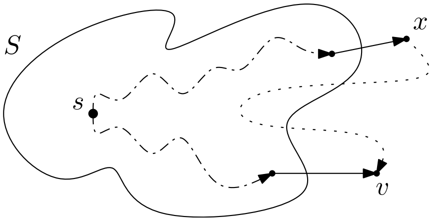

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$$d(x)$$

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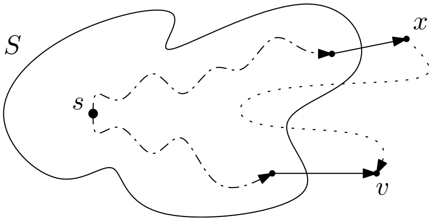

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$$d(x) \geq d(v). \qquad \text{(do you see why?)}$$

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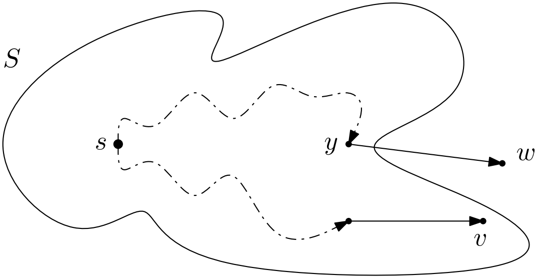
So the induction step is complete for (a).

Induction Step (b)

Induction step (b): Consider a node $w \neq v$ which is not in S.



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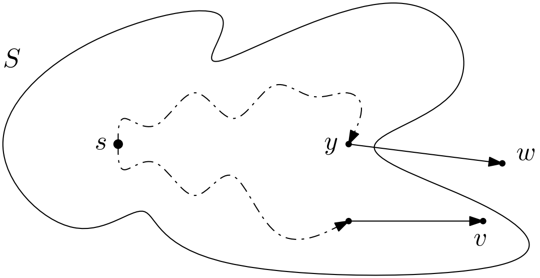

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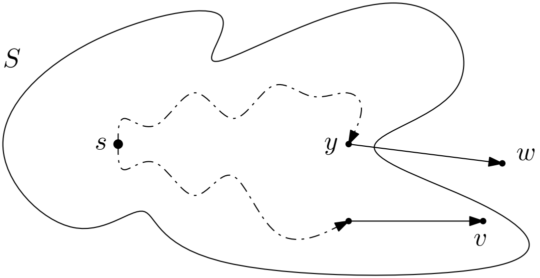

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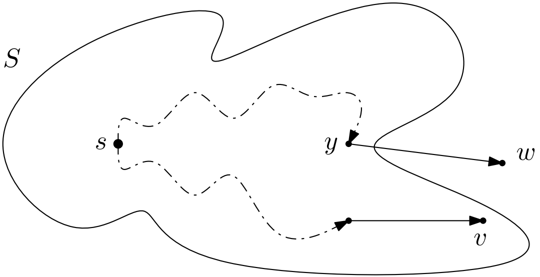

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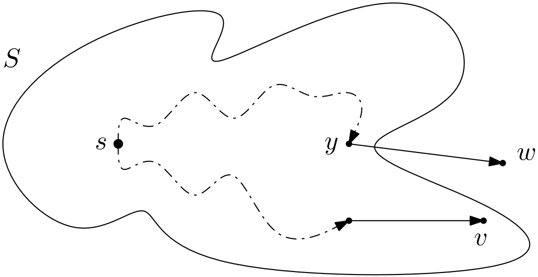

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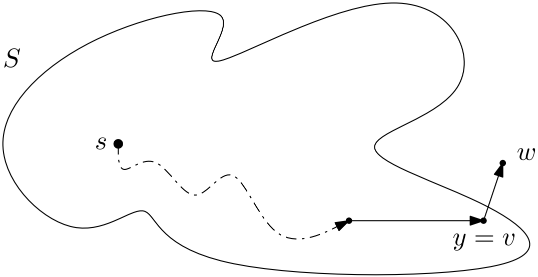

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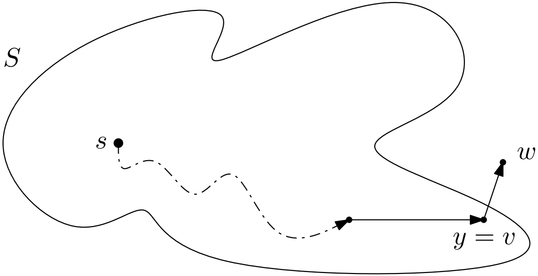

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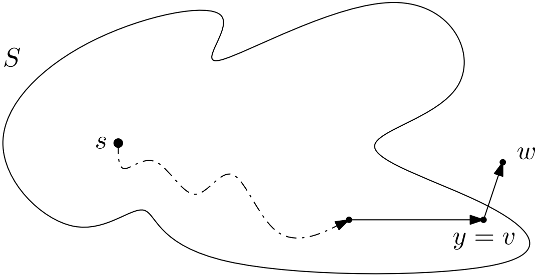

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