CSI - 3105 Design & Analysis of Algorithms
Course 4

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Fall 2020

Selection Problem

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Output : The k-th smallest element in S.

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What is the bottleneck?

Can we solve this problem without sorting?

First Attempt for a Faster Algorithm

Algorithm *Select*(S, k)

Input: Sequence S of n numbers and an integer k with $1 \leq k \leq n$.

Output: The k-th smallest element of S.

if $|S| = 1$ **then**
    return the only element in S
else
    Choose an element p in S (called the *pivot*)
    Split S into $S_<$, $S_=$ and $S_>$
    if $k \leq |S_<|$ **then**
        Run *Select*$(S_<, k)$
    else if $k > |S_<| + |S_=|$ **then**
        Run *Select*$(S_>, k - |S_<| - |S_=|)$
    else
        return p
    end if
end if

The running time of $Select(S, k)$ depends on the pivot p.
In the worst case,

- S is sorted
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Using the Master Theorem with $a = 1$, $b = 2$ and $d = 1$, we find $T(n) = O(n)$.

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How *close* to a good case do we need to be to get a linear-time algorithm?

General Approach

Assume that all numbers are different. (The purpose of this assumption is only to simplify the discussion.)

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Assume that there is a constant $0 < \alpha < 1$ such that in $\mathit{Select}(S, p)$, it is always possible to choose a pivot p satisfying $|S_<| \le \alpha\, n$ and $|S_>| \le \alpha\, n$

Then the running time satisfies

$$\begin{aligned} T(n) &= T(\alpha\, n) + n \\ &= T(\alpha^2\, n) + \alpha\, n + n \\ &= T(\alpha^3\, n) + \alpha^2\, n + \alpha\, n + n \\ &\vdots \\ &= O(n) \end{aligned}$$

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Blum, Floyd, Pratt, Rivest and Tarjan (1973) discovered the following technique.

The Algorithm

Step 1 : Divide the input sequence into $\frac{n}{5}$ groups, each of size 5.

Step 2 : For $i = 1, 2, ..., n/5$, compute the median of the i-th group, call this median m_i.

Step 3 : Compute the median p of $m_1, m_2, ..., m_{n/5}$.

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Is p a good pivot? Why?

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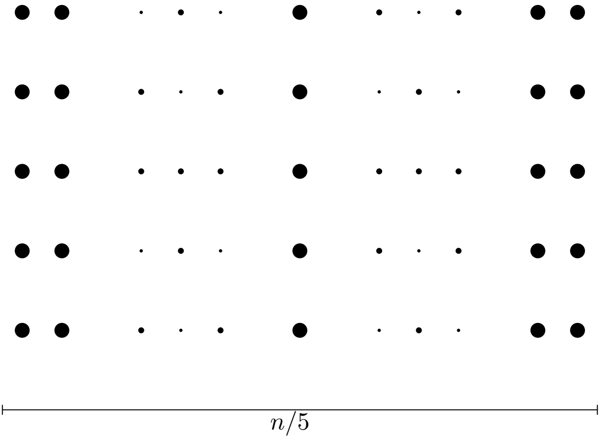
Is p a good pivot? Why?

We have to figure out how many numbers in S are larger than p.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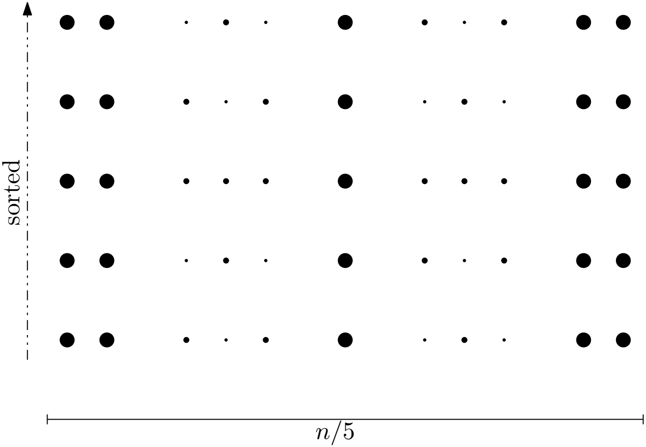
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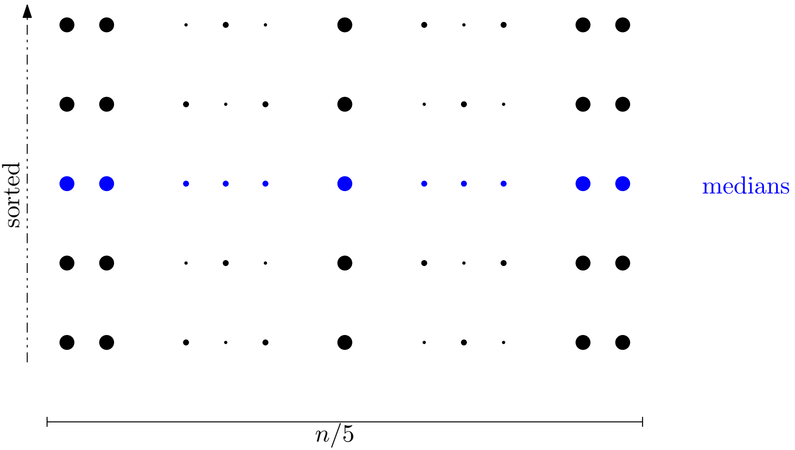
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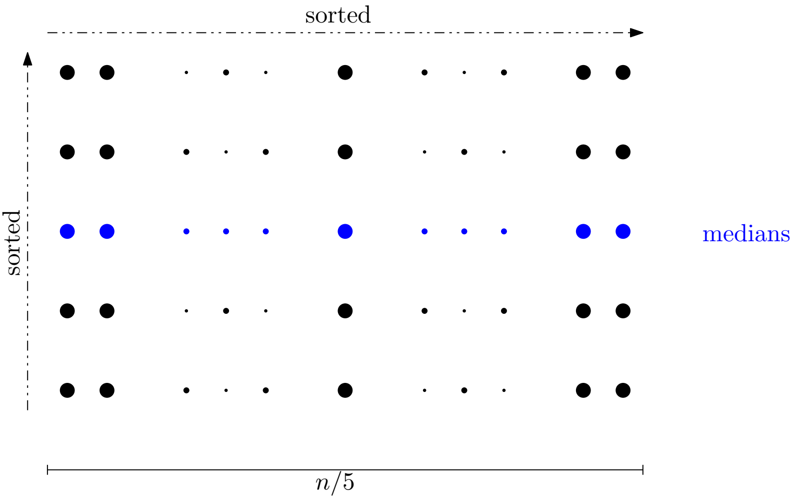
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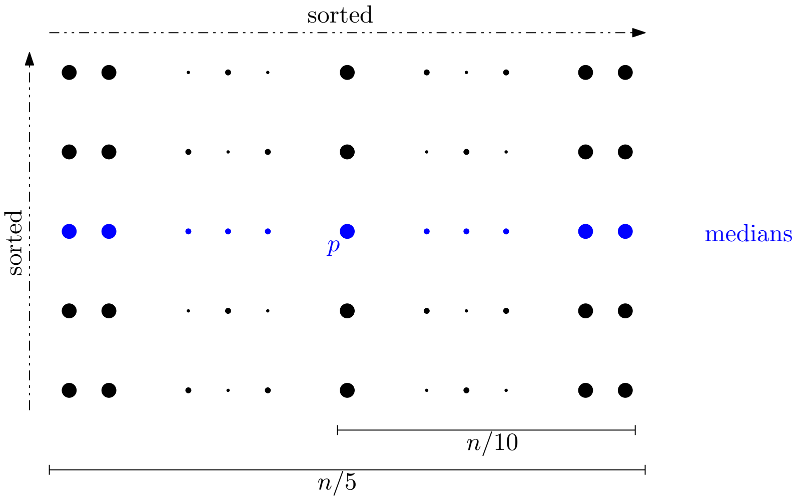
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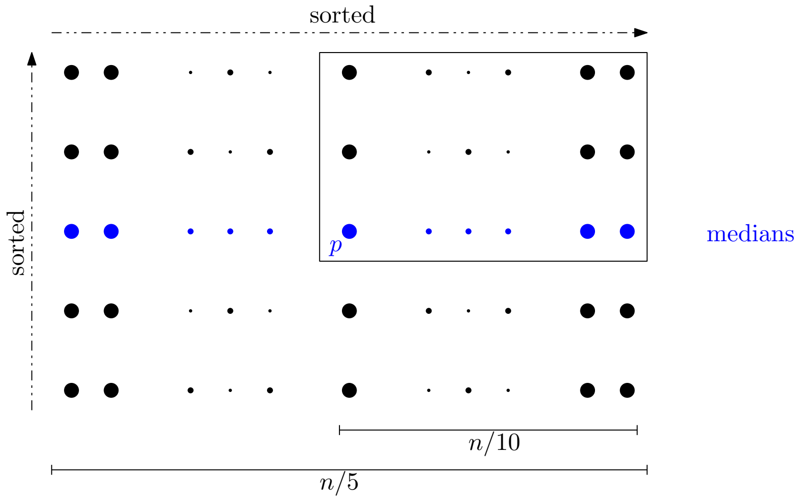
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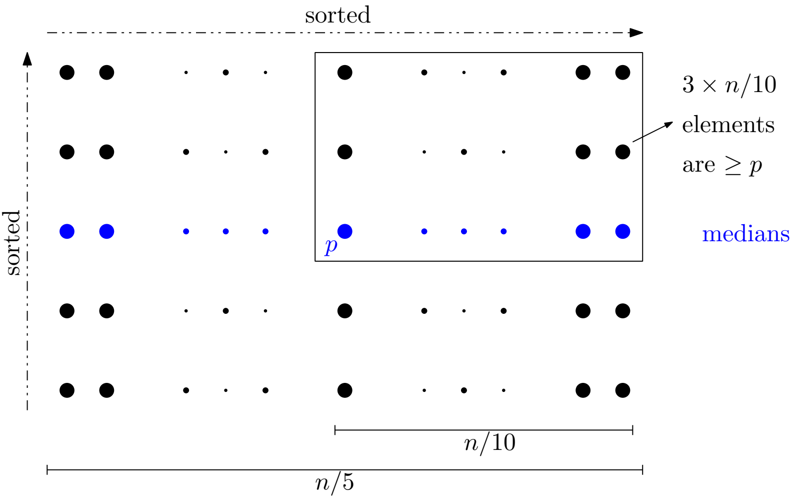
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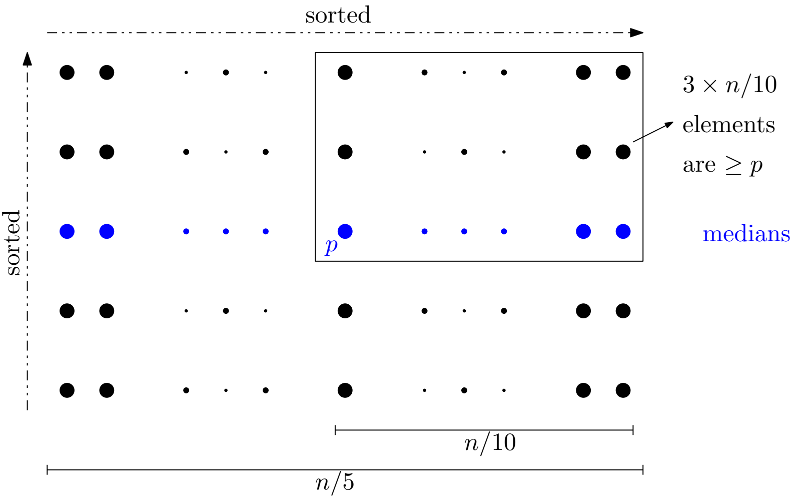
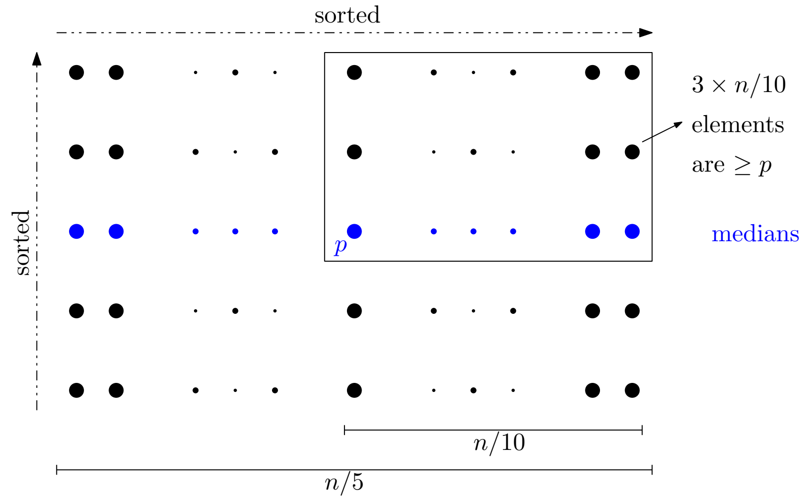


Hence, at least $\frac{3}{10}n$ elements are $\geq p$.



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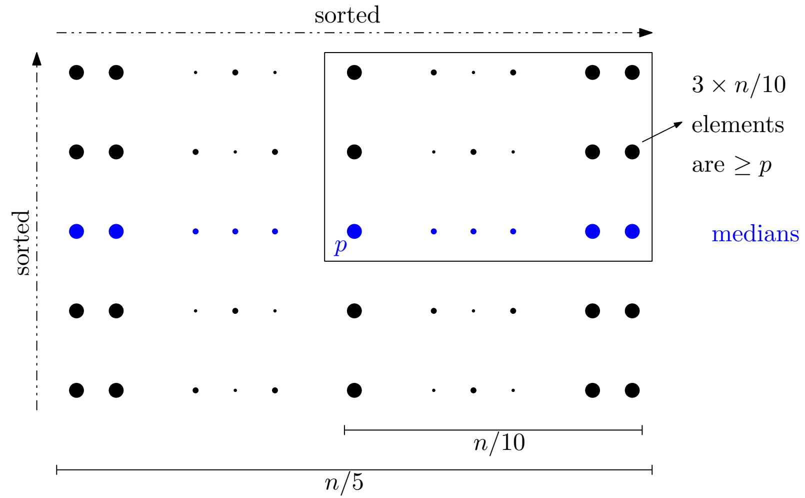
Thus, at most $\frac{7}{10}n$ elements are $\leq p$.



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In other words, $|S_<| \leq \frac{7}{10}n$.

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Hence, with this choice of pivot, we have $\alpha = \frac{7}{10}$.

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Hence: $T(n) = T\left(\frac{1}{5}n\right) + T\left(\frac{7}{10}n\right) + O(n)$

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The Master theorem does not apply.

We use induction to show that $T(n) = O(n)$.

$$\begin{aligned} T(n) = \; & O(n) && \text{Step 1} \\ & + O(n) && \text{Step 2} \\ & + \;\; ? && \text{Step 3} \\ & + O(n) + T\left(\tfrac{7}{10}n\right) && \text{Step 4} \end{aligned}$$

How do we do Step 3: Recursively compute the $\frac{n}{10}$-th smallest element of the sequence $m_1, m_2, \ldots, m_{n/5}$. This takes $T(n/5)$ time.

We obtain the recurrence

$$T(n) = n + T\left(\frac{n}{5}\right) + T\left(\frac{7}{10}n\right)$$

→ Unfolding get messy

→ Master theorem does not apply!

Let us use induction to show that $T(n) = O(n)$.

<u>Claim:</u> $T(n) \le c \cdot n$ for some constant c

<u>Proof:</u> By choosing c sufficiently large, the claim is true for "small" n (this is the base case of the induction).

Let n be "large" and assume $T(m) \le c \cdot m$ for all $1 \le m < n$. Then

$$T(n) = n + T\left(\tfrac{n}{5}\right) + T\left(\tfrac{7}{10}n\right)$$

$$\le n + c \cdot \tfrac{n}{5} + c \cdot \tfrac{7}{10} n \quad \text{by the induction hypothesis}$$

$$= n + \tfrac{9}{10} c \cdot n$$

Is $n + \frac{9}{10} c \cdot n \le cn$?

Yes, provided that $c \geq 10$. $\square$

<u>Conclusion:</u> The k-th smallest element in a sequence of n numbers can be computed in $O(n)$ time.