CSI - 3105 Design & Analysis of Algorithms
Course 13

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Fall 2020

Chapter 5: Dynamic Programming

Section 5.1: Shortest Paths in Acyclic Graphs

Let $G = (V, E)$ be a directed acyclic graph, where each edge (u, v) has a weight $wt(u, v) > 0$.

Topological sorting: vertices are numbered $v_1, v_2, ..., v_n$ such that for each edge (v_i, v_j), we have $i < j$.

Let $s = v_1$ and $t = v_n$.

How do we compute the shortest path from s to t?

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Can we do better than Dijkstra algorithm, using the topological ordering of G?

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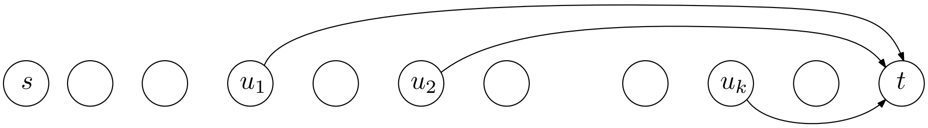

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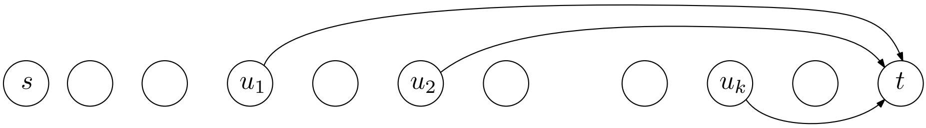

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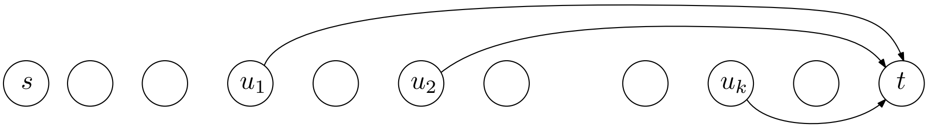

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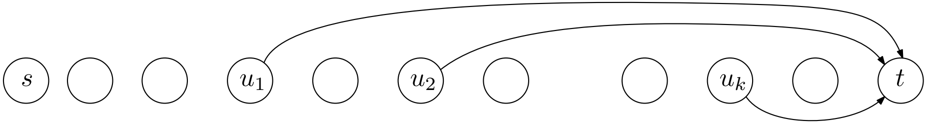

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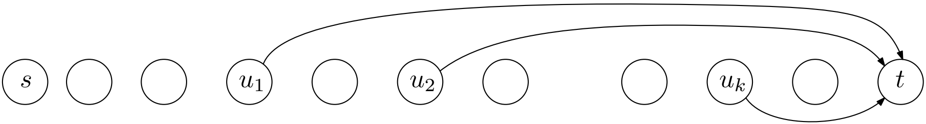

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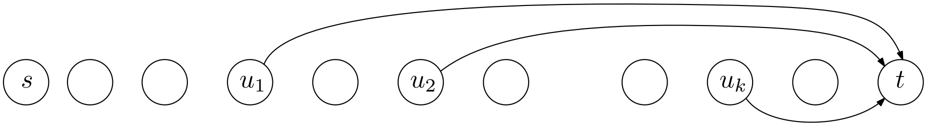

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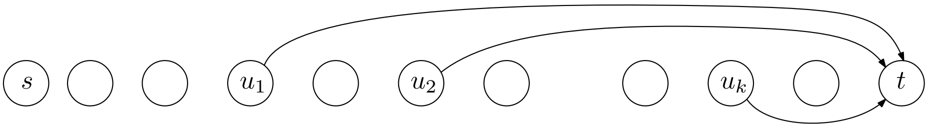

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In other words, the shortest path from s to t contains the shortest path from s to one of $u_1, u_2, \ldots, u_k$.

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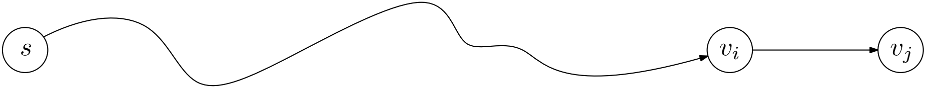
Recurrence:

- $d(v_1) = 0$
- For $2 \leq j \leq n$,

$$d(v_j) = \min_{(v_i, v_j) \in E} \{d(v_i) + wt(v_i, v_j)\}$$



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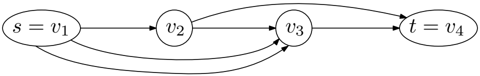

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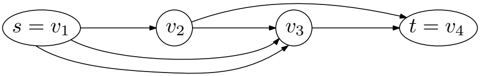
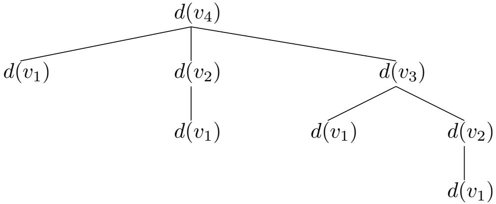


Récursion tree:



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Do you remember Fibonacci?

Algorithm:

- $d(v_1) = 0$
- For $j = 2$ to n:
  - $k = indegree(v_j)$
  - Let $u_1, u_2, ..., u_k$ be all the vertices that have an edge to v_j.
  - $d(v_j) = \infty$
  - For $i = 1$ to k
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Running time

$$O\left(\sum_{j=1}^{n} (1 + indegree(v_j))\right) = O(|V| + |E|)$$