The Fibonacci Numbers

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Algorithm *fib*(n)

Input: An integer $n \geq 0$.

Output: F_n.

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if n ≤ 1 then
    return n
else
    return fib(n − 1) + fib(n − 2)
end if
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If $n = 0$ or $n = 1$, we do

$$\left.\begin{array}{l}\text{comparison “}n \leq 1\text{”} \\ \text{return value}\end{array}\right\} \quad 2 \text{ steps}$$

The Running Time of $fib(n)$

If $n \geq 2$, we do

comparison "$n \leq 1$"	:	1 step
compute $n - 1$	:	1 step
call $fib(n - 1)$	:	?
compute $n - 2$	:	1 step
call $fib(n - 2)$	:	?
compute sum of two results	:	1 step
return output	:	1 step

The Running Time of $fib(n)$

If $n \geq 2$, we do

comparison "$n \leq 1$"	:	1 step
compute $n - 1$	:	1 step
call $fib(n-1)$	:	$T(n-1)$ steps
compute $n - 2$	:	1 step
call $fib(n-2)$	:	?
compute sum of two results	:	1 step
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$$T(n) = \begin{cases} 2 & \text{if } n = 0, \\ 2 & \text{if } n = 1, \\ T(n-1) + T(n-2) + 5 & \text{if } n \geq 2. \end{cases}$$

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What do we do with this?

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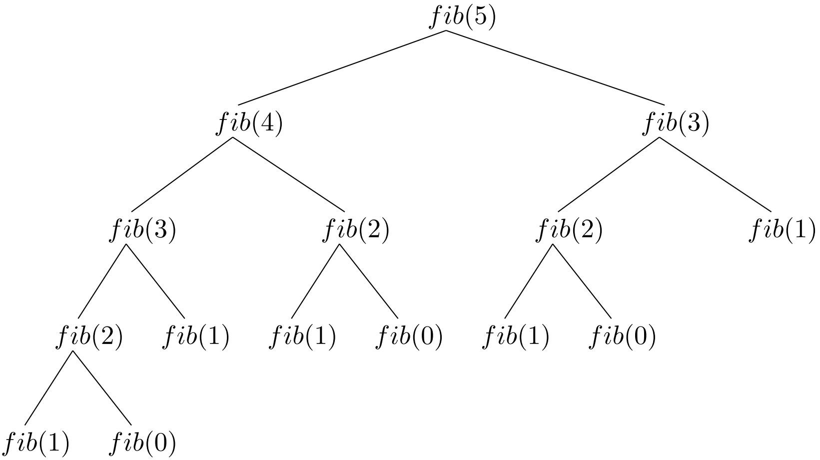
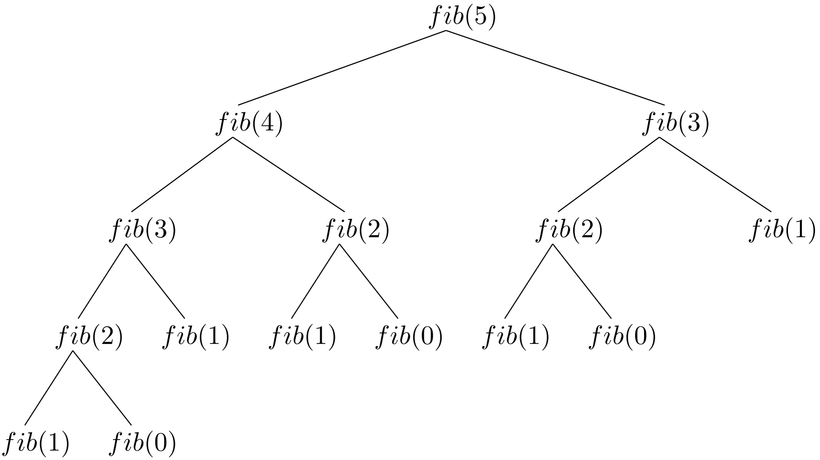
Why is *fib*(n) so slow?

The Running Time of *fib*(n)



The Running Time of *fib*(n)



Too many things are called multiple times.

A Better Algorithm

Algorithm *fib2*(n)

Input: An integer $n \geq 0$.

Output: F_n.

```
if n ≤ 1 then
    return n
else
    initialize array f[0..n]
    f[0] = 0
    f[1] = 1
    for i = 2 to n do
        f[i] = f[i − 1] + f[i − 2]
    end for
    return f[n]
end if
```

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But! Is it realistic to say that $fib2(n)$ takes a linear number of steps?

In our analysis, one step corresponds to

$$\left.\begin{array}{l}\text{comparison}\\ \text{addition}\\ \text{subtraction}\end{array}\right\}\quad \text{involving very large numbers}$$

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When running *fib*2(n), we do roughly n additions of numbers, each of these numbers is at most F_n, each of these numbers has roughly n bits.

Therefore, *fib*2(n) makes a *quadratic number* of bit-operations, i.e. $O(n^2)$ bit-operations.

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Therefore, the running time, in terms of bit-operations:

Running time of $fib(n)$:	exponential
Running time of $fib2(n)$:	quadratic