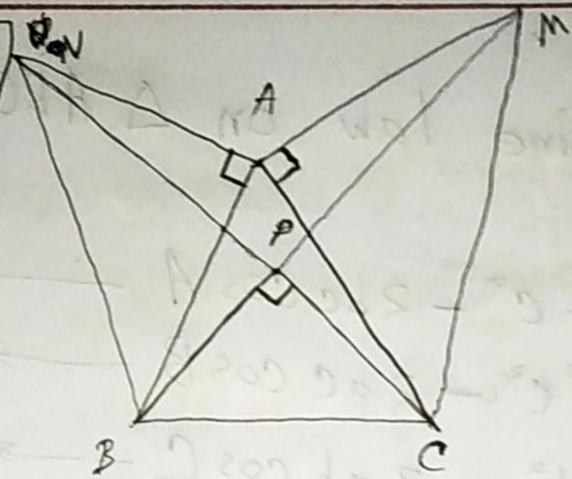
$\boxed{P-2}$



PM = PN ?

~~By~~ Here

$$\angle PBC = \angle PCB = \angle ABN = \angle ACM = 45^\circ$$

So

$$\angle PBN = \angle B \quad \text{and} \quad \angle PCN = \angle C$$

Let,

$BC = a$; $CA = b$; $AB = c$

By using trigonometry we get,

$$BP = PC = \frac{a}{\sqrt{2}} \; ; \; AM = b \; ; \; AN = c$$

Now,

applying cosine law on $\triangle ABC$ we get,

$$a^2 = b^2 + c^2 - 2bc\cos A \longrightarrow \text{(i)}$$

$$b^2 = a^2 + c^2 - 2ac\cos B \longrightarrow \text{(ii)}$$

$$c^2 = a^2 + b^2 - 2ab\cos C \longrightarrow \text{(iii)}$$

By using it again on $\triangle PCM$, $\triangle PBN$ and $\triangle AMN$ we get

$$PM^2 = \frac{a^2}{2} + 2b^2 - 2ab\cos C$$

$$PN^2 = \frac{a^2}{2} + 2c^2 - 2ac\cos B$$

$$MN^2 = b^2 + c^2 - 2bc\cos MAN$$

$$= b^2 + c^2 + 2bc\cos A \quad \left[\text{As, } \angle MAN = 180° - \angle A \text{ and } \cos(180-\theta) = -\cos\theta\right]$$

So,

$$PM^2 + PN^2 = a^2 + 2b^2 + 2c^2 - 2ab\cos C - 2ac\cos B$$

$$= a^2 + 2b^2 + 2c^2 + (c^2 - a^2 - b^2) + (b^2 - a^2 - c^2) \quad [\text{By (ii) and (iii)}]$$

$$= a^2 + 2b^2 + 2c^2 + c^2 - a^2 - b^2 + b^2 - a^2 - c^2$$

$$= 2b^2 + 2c^2 - a^2$$

$$= 2b^2 + 2c^2 - b^2 - c^2 + 2bc\cos A \quad [\text{By (i)}]$$

$$= b^2 + c^2 + 2bc\cos A$$

$$= MN^2$$

Therefore by pythagorean theorem

$\angle MPN = 90^\circ$ as needed

$\square$

$\boxed{P-3}$

<u>Answer:</u> The ~~only~~ solutions for this functional equation ~~is~~, are

$$f(x) = x^2 \text{ and } f(x) = 0$$

<u>Proof:</u> Let,

$$P(x, y) : f(x^2+y) = f(f(x)-y) + 4f(x)y$$

So,

$$P(0,0) : f(0) = f(f(0)) \longrightarrow (i)$$

Now,

$$P(0, f(0)) : f(f(0)) = f(f(0)-f(0)) + 4f(0)f(0)$$

$$\Rightarrow f(0) = f(0) + 4f(0)^2$$

$$\Rightarrow 4f(0)^2 = 0$$

$$\Rightarrow f(0) = 0$$

Now,

$P(x,0):\ f(x^2) = f(f(x))$

And,

$P(0,x):\ f(x) = f(-x)$

So, function f is a even function.

Now,

$P(x, f(x)):\ f(x^2+f(x)) = f(f(x) - f(x)) + 4f(x)f(x) \longrightarrow$ (ii)

$P(x, -x^2):\ f(x^2-x^2) = f(f(x)+x^2) + 4f(x)(-x^2) \longrightarrow$ (iii)

From, (ii) + (iii) we get,

$$f(x^2+f(x)) + f(0) = f(0) + 4f(x)^2 + f(x^2+f(x)) - 4f(x)x^2$$

$$\Rightarrow 4f(x)^2 = 4f(x)x^2$$

$$\Rightarrow 4f(x)^2 - 4f(x)x^2 = 0$$

$$\Rightarrow f(x)^2 - f(x)x^2 = 0$$

$$\Rightarrow f(x)(f(x) - x^2) = 0$$

So the solution for the function $f(x)=x^2$ and $f(x)=0$. which both satisfy the equation.

$\square$

$\boxed{P-1}$

Let the $\gcd(a_1, a_2, a_3, \cdots a_n) = d$ and

$x_i = \frac{a_i}{d}$

So, $\gcd(x_1, x_2, x_3, \cdots x_n) = 1$

Let the product $\prod_{i=1}^{n} x_i = Q$

Now, let p be a ~~prim~~ odd prime number which divides $x_i^n + Q$ for all $i \in \{1, 2, 3, \ldots, n\}$

Therefore,

$x_i^n \equiv -Q \pmod{p}$ for all $i \in \{1, 2, \ldots, n\}$

By multiplying all of these equations we get

$$x_1^n \cdot x_2^n \cdot x_3^n \cdot \ldots \cdot x_n^n \equiv (-Q)^n \pmod{p}$$

$$\Rightarrow Q^n \equiv -Q^n \pmod{p} \quad \left[(-Q)^n = -Q^n \text{ as } n \text{ is odd}\right]$$

$\longrightarrow$ (i)

As,
$$\gcd(x_1, x_2, x_3, x_4 \dots x_n) = 1$$
there is at least one x_i such that

$$p \nmid x_i^n ;\ p \mid x_i^n + Q$$

So, $P \nmid Q$ but $Q^n \equiv -Q^n \pmod{P}$

As $P \nmid Q^n$ it makes a contradiction to the statement "p is odd"

So,
$x_i^n + Q$ is not divisible to any odd prime for all $i \in \{1, 2, 3, 4 \dots, n\}$

So the,
$\gcd(x_1^n + Q, x_2^n + Q \dots x_n^n + Q)$ must be 1

or 2

So we can write,

$$\gcd\left(x_1^n+Q, x_2^n+Q, \ldots, x_n^n+Q\right) \leq 2\gcd\left(x_1, x_2, \ldots, x_n\right)^n$$

Now,

$$2\gcd\left(a_1, a_2, a_3 \ldots a_n\right)^n = d^n \times \left\{2\gcd\left(x_1, x_2, x_3 \ldots x_n\right)^n\right\}$$

$$\geq d^n\left\{\gcd\left(x_1^n+Q, x_2^n+Q, \ldots, x_n^n+Q\right)\right\}$$

$$= \gcd\left(x_1^n d^n + Qd^n, \ldots x_n^n d^n + Qd^n\right)$$

$$= \gcd\left(a_1^n+P, a_2^n+P, \ldots, a_n+P\right)$$

$\square$