Romania Team Selection Test 2008

Day 1 May 1st

1 Let n be an integer, $n \geq 2$. Find all sets A with n integer elements such that the sum of any nonempty subset of A is not divisible by $n+1$.

2 Let a_i, b_i be positive real numbers, $i = 1, 2, \ldots, n$, $n \geq 2$, such that $a_i < b_i$, for all i, and also

$$b_1 + b_2 + \cdots + b_n < 1 + a_1 + \cdots + a_n.$$

Prove that there exists a $c \in \mathbb{R}$ such that for all $i = 1, 2, \ldots, n$, and $k \in \mathbb{Z}$ we have

$$(a_i + c + k)(b_i + c + k) > 0.$$

3 Let $ABCDEF$ be a convex hexagon with all the sides of length 1. Prove that one of the radii of the circumcircles of triangles ACE or BDF is at least 1.

4 Prove that there exists a set S of $n-2$ points inside a convex polygon P with n sides, such that any triangle determined by 3 vertices of P contains exactly one point from S inside or on the boundaries.

5 Find the greatest common divisor of the numbers

$$2^{561} - 2, 3^{561} - 3, \ldots, 561^{561} - 561.$$

Day 2

1 Let $n \geq 3$ be an odd integer. Determine the maximum value of

$$\sqrt{|x_1 - x_2|} + \sqrt{|x_2 - x_3|} + \ldots + \sqrt{|x_{n-1} - x_n|} + \sqrt{|x_n - x_1|},$$

where x_i are positive real numbers from the interval $[0, 1]$.

2 Are there any sequences of positive integers $1 \leq a_1 < a_2 < a_3 < \ldots$ such that for each integer n, the set $\{a_k + n \mid k = 1, 2, 3, \ldots\}$ contains finitely many prime numbers?

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3 Show that each convex pentagon has a vertex from which the distance to the opposite side of the pentagon is strictly less than the sum of the distances from the two adjacent vertices to the same side.

Note. If the pentagon is labeled $ABCDE$, the adjacent vertices of A are B and E, the ones of B are A and C etc.

4 Let G be a connected graph with n vertices and m edges such that each edge is contained in at least one triangle. Find the minimum value of m.

Day 3

1 Let ABC be a triangle with $\measuredangle BAC < \measuredangle ACB$. Let D, E be points on the sides AC and AB, such that the angles ACB and BED are congruent. If F lies in the interior of the quadrilateral $BCDE$ such that the circumcircle of triangle BCF is tangent to the circumcircle of DEF and the circumcircle of BEF is tangent to the circumcircle of CDF, prove that the points A, C, E, F are concyclic.

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2 Let ABC be an acute triangle with orthocenter H and let X be an arbitrary point in its plane. The circle with diameter HX intersects the lines AH and AX at A_1 and A_2, respectively. Similarly, define B_1, B_2, C_1, C_2. Prove that the lines A_1A_2, B_1B_2, C_1C_2 are concurrent.

Remark. The triangle obviously doesn't need to be acute.

3 Let $m,\ n \geq 3$ be positive odd integers. Prove that $2^m - 1$ doesn't divide $3^n - 1$.

4 Let n be a nonzero positive integer. A set of persons is called a n-balanced set if in any subset of 3 persons there exists at least two which know each other and in each subset of n persons there are two which don't know each other. Prove that a n-balanced set has at most $(n-1)(n+2)/2$ persons.

Day 4 June 12th

1 Let $ABCD$ be a convex quadrilateral and let $O \in AC \cap BD$, $P \in AB \cap CD$, $Q \in BC \cap DA$. If R is the orthogonal projection of O on the line PQ prove that the orthogonal projections of R on the sidelines of $ABCD$ are concyclic.

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2 Let $m, n \geq 1$ be two coprime integers and let also s an arbitrary integer. Determine the number of subsets A of $\{1, 2, ..., m+n-1\}$ such that $|A| = m$ and $\sum_{x \in A} x \equiv s \pmod{n}$.

3 Let $n \geq 3$ be a positive integer and let $m \geq 2^{n-1}+1$. Prove that for each family of nonzero distinct subsets $(A_j)_{j \in \overline{1,m}}$ of $\{1, 2, ..., n\}$ there exist i, j, k such that $A_i \cup A_j = A_k$.

Day 5 June 13th

1 Let n be a nonzero positive integer. Find n such that there exists a permutation $\sigma \in S_n$ such that

$$\left| \{ |\sigma(k) - k| \ : \ k \in \overline{1,n} \} \right| = n.$$

2 Let ABC be a triangle and let $\mathcal{M}_a$, $\mathcal{M}_b$, $\mathcal{M}_c$ be the circles having as diameters the medians m_a, m_b, m_c of triangle ABC, respectively. If two of these three circles are tangent to the incircle of ABC, prove that the third is tangent as well.

3 Let $\mathcal{P}$ be a square and let n be a nonzero positive integer for which we denote by $f(n)$ the maximum number of elements of a partition of $\mathcal{P}$ into rectangles such that each line which is parallel to some side of $\mathcal{P}$ intersects at most n interiors (of rectangles). Prove that

$$3 \cdot 2^{n-1} - 2 \leq f(n) \leq 3^n - 2.$$