National Math Olympiad (Second Round) 2017

1 a) Prove that there doesn't exist sequence $a_1, a_2, a_3, ... \in \mathbb{N}$ such that: $\forall i < j : gcd(a_i + j, a_j + i) = 1$

b) Let p be an odd prime number. Prove that there exist sequence $a_1, a_2, a_3, ... \in \mathbb{N}$ such that: $\forall i < j : p \not| gcd(a_i + j, a_j + i)$

2 Let $ABCD$ be an isosceles trapezoid such that $AB||CD$. Suppose that there exists a point P in $ABCD$ such that: $\angle APB > \angle ADC, \angle DPC > \angle ABC$. Prove that: $AB + CD > DA + BC$

3 Let n be a natural number divisible by 3. We have a table $n \times n$ and each square is colored by black or white. Suppose that for all $m \times m$ subsquares from the table $(m \neq 1)$, the number of black squares member of that is not more than whites. Find the maximum number of black squares.

4 $x, y > 0$ are two real numbers such that $x^4 - y^4 = x - y$. prove that:

$$\frac{x-y}{x^6 - y^6} \leq \frac{4}{3}(x+y)$$

5 There are five smart kids sitting around a round table. Their teacher says: "I gave a few apples to some of you, and none of you have the same amount of apple. Also each of you will know the amount of apple that the person to your left and the person to your right has."
The teacher tells the total amount of apples, then asks the kids to guess the difference of the amount of apple that the two kids in front of them have.

a) If the total amount of apples is less than 16, prove that at least one of the kids will guess the difference correctly. b) Prove that the teacher can give the total of 16 apples such that no one can guess the difference correctly.

6 Let ABC be a triangle and X be a point on its circumcircle. Q, P lie on a line BC such that $XQ \perp AC, XP \perp AB$. Let Y be the circumcenter of $\triangle XQP$. Prove that ABC is equilateral triangle if and if only Y moves on a circle when X varies on the circumcircle of ABC.