Romanian Masters In Mathematics 2011

Day 1

1 Prove that there exist two functions $f, g\colon \mathbb{R} \to \mathbb{R}$, such that $f \circ g$ is strictly decreasing and $g \circ f$ is strictly increasing.

(Poland) Andrzej Komisarski and Marcin Kuczma

2 Determine all positive integers n for which there exists a polynomial $f(x)$ with real coefficients, with the following properties:

(1) for each integer k, the number $f(k)$ is an integer if and only if k is not divisible by n;
(2) the degree of f is less than n.

(Hungary) Gza Ks

3 A triangle ABC is inscribed in a circle ω.
A variable line ℓ chosen parallel to BC meets segments AB, AC at points D, E respectively, and meets ω at points K, L (where D lies between K and E). Circle γ_1 is tangent to the segments KD and BD and also tangent to ω, while circle γ_2 is tangent to the segments LE and CE and also tangent to ω.
Determine the locus, as ℓ varies, of the meeting point of the common inner tangents to γ_1 and γ_2.

(Russia) Vasily Mokin and Fedor Ivlev

Day 2

1 Given a positive integer $n = \prod_{i=1}^{s} p_i^{\alpha_i}$, we write $\Omega(n)$ for the total number $\sum_{i=1}^{s} \alpha_i$ of prime factors of n, counted with multiplicity. Let $\lambda(n) = (-1)^{\Omega(n)}$ (so, for example, $\lambda(12) = \lambda(2^2 \cdot 3^1) = (-1)^{2+1} = -1$).
Prove the following two claims:

i) There are infinitely many positive integers n such that $\lambda(n) = \lambda(n+1) = +1$;
ii) There are infinitely many positive integers n such that $\lambda(n) = \lambda(n+1) = -1$.

(Romania) Dan Schwarz

2 For every $n \geq 3$, determine all the configurations of n distinct points $X_1, X_2, \ldots, X_n$ in the plane, with the property that for any pair of distinct points X_i, X_j

there exists a permutation σ of the integers $\{1, \ldots, n\}$, such that $\mathrm{d}(X_i, X_k) = \mathrm{d}(X_j, X_{\sigma(k)})$ for all $1 \leq k \leq n$.
(We write $\mathrm{d}(X, Y)$ to denote the distance between points X and Y.)

(United Kingdom) Luke Betts

3 The cells of a square 2011×2011 array are labelled with the integers $1, 2, \ldots, 2011^2$, in such a way that every label is used exactly once. We then identify the left-hand and right-hand edges, and then the top and bottom, in the normal way to form a torus (the surface of a doughnut).
Determine the largest positive integer M such that, no matter which labelling we choose, there exist two neighbouring cells with the difference of their labels at least M.
(Cells with coordinates (x, y) and (x', y') are considered to be neighbours if $x = x'$ and $y - y' \equiv \pm 1 \pmod{2011}$, or if $y = y'$ and $x - x' \equiv \pm 1 \pmod{2011}$.)

(Romania) Dan Schwarz