India
Regional Mathematical Olympiad
2011

Advanced RMO

1. Let ABC be an acute angled scalene triangle with circumcentre O and orthocentre H. If M is the midpoint of BC, then show that AO and HM intersect on the circumcircle of ABC.

2. Let n be a positive integer such that $2n+1$ and $3n+1$ are both perfect squares. Show that $5n+3$ is a composite number.

3. Let $a, b, c > 0$. If $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in arithmetic progression, and if $a^2+b^2, b^2+c^2, c^2+a^2$ are in geometric progression, show that $a = b = c$.

4. Find the number of 4-digit numbers with distinct digits chosen from the set $\{0, 1, 2, 3, 4, 5\}$ in which no two adjacent digits are even.

5. Let $ABCD$ be a convex quadrilateral. Let E, F, G, H be the midpoints of AB, BC, CD, DA respectively. If AC, BD, EG, FH concur at a point O, prove that $ABCD$ is a parallelogram.

6. Find the largest real constant λ such that

$$\frac{\lambda abc}{a+b+c} \leq (a+b)^2 + (a+b+4c)^2$$

For all positive real numbers a, b, c.

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1. Let ABC be a triangle. Let D, E, F be points respectively on the segments BC, CA, AB such that AD, BE, CF concur at the point K. Suppose $\frac{BD}{DC} = \frac{BF}{FA}$ and $\angle ADB = \angle AFC$. Prove that $\angle ABE = \angle CAD$.

2. Let $(a_1, a_2, a_3, ..., a_{2011})$ be a permutation of the numbers $1, 2, 3, ..., 2011$. Show that there exist two numbers j, k such that $1 \leq j < k \leq 2011$ and $|a_j - j| = |a_k - k|$

3. A natural number n is chosen strictly between two consecutive perfect squares. The smaller of these two squares is obtained by subtracting k from n and the larger by adding l to n. Prove that $n - kl$ is a perfect square.

4. Consider a 20-sided convex polygon K, with vertices $A_1, A_2, ..., A_{20}$ in that order. Find the number of ways in which three sides of K can be chosen so that every pair among them has at least two sides of K between them. (For example $(A_1A_2, A_4A_5, A_{11}A_{12})$ is an admissible triple while $(A_1A_2, A_4A_5, A_{19}A_{20})$ is not.

5. Let ABC be a triangle and let BB_1, CC_1 be respectively the bisectors of $\angle B, \angle C$ with B_1 on AC and C_1 on AB, Let E, F be the feet of perpendiculars drawn from A onto BB_1, CC_1 respectively. Suppose D is the point at which the incircle of ABC touches AB. Prove that $AD = EF$

6. Find all pairs (x, y) of real numbers such that

$$16^{x^2+y} + 16^{x+y^2} = 1$$