4-th Czech–Slovak Match 1998

Modra, June 4–7, 1998

1. Let P be an interior point of the parallelogram $ABCD$. Prove that $\angle APB + \angle CPD = 180^\circ$ if and only if $\angle PDC = \angle PBC$.

2. A polynomial $P(x)$ of degree $n \geq 5$ with integer coefficients has n distinct integer roots, one of which is 0. Find all integer roots of the polynomial $P(P(x))$.

3. Let $ABCDEF$ be a convex hexagon such that $AB = BC$, $CD = DE$, $EF = FA$. Prove that
$$\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \geq \frac{3}{2}.$$
When does equality occur?

4. Find all functions $f : \mathbb{N} \to \mathbb{N} \setminus \{1\}$ satisfying
$$f(n) + f(n+1) = f(n+2)f(n+3) - 168 \quad \text{for all } n \in \mathbb{N}.$$

5. In a triangle ABC, T is the centroid and $\angle TAB = \angle ACT$. Find the maximum possible value of $\sin\angle CAT + \sin\angle CBT$.

6. In a summer camp there are n girls $D_1, D_2, \ldots, D_n$ and $2n-1$ boys $C_1, C_2, \ldots, C_{2n-1}$. The girl D_i, $i = 1, 2, \ldots, n$, knows only the boys $C_1, C_2, \ldots, C_{2i-1}$. Let $A(n,r)$ be the number of different ways in which r girls can dance with r boys forming r pairs, each girl with a boy she knows. Prove that
$$A(n,r) = \binom{n}{r} \cdot \frac{n!}{(n-r)!}.$$

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