Balkan MO 2016

– May 7th

1 Find all injective functions $f:\mathbb{R}\to\mathbb{R}$ such that for every real number x and every positive integer n,

$$\left|\sum_{i=1}^{n} i\left(f(x+i+1)-f(f(x+i))\right)\right| < 2016$$

(Macedonia)

2 Let $ABCD$ be a cyclic quadrilateral with $AB<CD$. The diagonals intersect at the point F and lines AD and BC intersect at the point E. Let K and L be the orthogonal projections of F onto lines AD and BC respectively, and let M, S and T be the midpoints of EF, CF and DF respectively. Prove that the second intersection point of the circumcircles of triangles MKT and MLS lies on the segment CD.

(Greece - Silouanos Brazitikos)

3 Find all monic polynomials f with integer coefficients satisfying the following condition: there exists a positive integer N such that p divides $2(f(p)!)+1$ for every prime $p>N$ for which $f(p)$ is a positive integer.

Note: A monic polynomial has a leading coefficient equal to 1.

(Greece - Panagiotis Lolas and Silouanos Brazitikos)

4 The plane is divided into squares by two sets of parallel lines, forming an infinite grid. Each unit square is coloured with one of 1201 colours so that no rectangle with perimeter 100 contains two squares of the same colour. Show that no rectangle of size 1×1201 or 1201×1 contains two squares of the same colour.

Note: Any rectangle is assumed here to have sides contained in the lines of the grid.

(Bulgaria - Nikolay Beluhov)

Balkan MO 2015

– May 5th

1 If a, b and c are positive real numbers, prove that

$$a^3b^6 + b^3c^6 + c^3a^6 + 3a^3b^3c^3 \geq abc\left(a^3b^3 + b^3c^3 + c^3a^3\right) + a^2b^2c^2\left(a^3 + b^3 + c^3\right).$$

(Montenegro).

2 Let $\triangle ABC$ be a scalene triangle with incentre I and circumcircle ω. Lines AI, BI, CI intersect ω for the second time at points D, E, F, respectively. The parallel lines from I to the sides BC, AC, AB intersect EF, DF, DE at points K, L, M, respectively. Prove that the points K, L, M are collinear.
(Cyprus)

3 A committee of 3366 film critics are voting for the Oscars. Every critic voted just an actor and just one actress. After the voting, it was found that for every positive integer $n \in \{1, 2, \ldots, 100\}$, there is some actor or some actress who was voted exactly n times. Prove that there are two critics who voted the same actor and the same actress.
(Cyprus)

4 Prove that among 20 consecutive positive integers there is an integer d such that for every positive integer n the following inequality holds

$$n\sqrt{d}\left\{n\sqrt{d}\right\} > \frac{5}{2}$$

where by $\{x\}$ denotes the fractional part of the real number x. The fractional part of the real number x is defined as the difference between the largest integer that is less than or equal to x to the actual number x.

(Serbia)

Balkan MO 2014

–	May 4th
1	Let x, y and z be positive real numbers such that $xy + yz + xz = 3xyz$. Prove that $$x^2y + y^2z + z^2x \geq 2(x + y + z) - 3$$ and determine when equality holds. *UK - David Monk*
2	A *special number* is a positive integer n for which there exists positive integers a, b, c, and d with $$n = \frac{a^3 + 2b^3}{c^3 + 2d^3}.$$ Prove that i) there are infinitely many special numbers; ii) 2014 is not a special number. *Romania*
3	Let $ABCD$ be a trapezium inscribed in a circle Γ with diameter AB. Let E be the intersection point of the diagonals AC and BD . The circle with center B and radius BE meets Γ at the points K and L (where K is on the same side of AB as C). The line perpendicular to BD at E intersects CD at M. Prove that KM is perpendicular to DL. *Greece - Silouanos Brazitikos*
4	Let n be a positive integer. A regular hexagon with side length n is divided into equilateral triangles with side length 1 by lines parallel to its sides. Find the number of regular hexagons all of whose vertices are among the vertices of those equilateral triangles. *UK - Sahl Khan*

Balkan MO 2013

– June 30th

1 In a triangle ABC, the excircle ω_a opposite A touches AB at P and AC at Q, while the excircle ω_b opposite B touches BA at M and BC at N. Let K be the projection of C onto MN and let L be the projection of C onto PQ. Show that the quadrilateral $MKLP$ is cyclic.

(*Bulgaria*)

2 Determine all positive integers x, y and z such that $x^5 + 4^y = 2013^z$.

(*Serbia*)

3 Let S be the set of positive real numbers. Find all functions $f\colon S^3 \to S$ such that, for all positive real numbers x, y, z and k, the following three conditions are satisfied:

(a) $xf(x, y, z) = zf(z, y, x)$,

(b) $f(x, ky, k^2z) = kf(x, y, z)$,

(c) $f(1, k, k + 1) = k + 1$.

(*United Kingdom*)

4 In a mathematical competition, some competitors are friends; friendship is mutual, that is, when A is a friend of B, then B is also a friend of A.
We say that $n \geq 3$ different competitors $A_1, A_2, \ldots, A_n$ form a *weakly-friendly cycle* if A_i is not a friend of A_{i+1} for $1 \leq i \leq n$ (where $A_{n+1} = A_1$), and there are no other pairs of non-friends among the components of the cycle.

The following property is satisfied:

"for every competitor C and every weakly-friendly cycle $\mathcal{S}$ of competitors not including C, the set of competitors D in $\mathcal{S}$ which are not friends of C has at most one element"

Prove that all competitors of this mathematical competition can be arranged into three rooms, such that every two competitors in the same room are friends.

(*Serbia*)

Balkan MO 2012

– April 28th

1 Let A, B and C be points lying on a circle Γ with centre O. Assume that $\angle ABC > 90$. Let D be the point of intersection of the line AB with the line perpendicular to AC at C. Let l be the line through D which is perpendicular to AO. Let E be the point of intersection of l with the line AC, and let F be the point of intersection of Γ with l that lies between D and E.
Prove that the circumcircles of triangles BFE and CFD are tangent at F.

2 Prove that

$$\sum_{cyc}(x+y)\sqrt{(z+x)(z+y)} \geq 4(xy+yz+zx),$$

for all positive real numbers x, y and z.

3 Let n be a positive integer. Let $P_n = \{2^n, 2^{n-1}\cdot 3, 2^{n-2}\cdot 3^2, \dots, 3^n\}$. For each subset X of P_n, we write S_X for the sum of all elements of X, with the convention that $S_\emptyset = 0$ where $\emptyset$ is the empty set. Suppose that y is a real number with $0 \leq y \leq 3^{n+1} - 2^{n+1}$.
Prove that there is a subset Y of P_n such that $0 \leq y - S_Y < 2^n$

4 Let $\mathbb{Z}^+$ be the set of positive integers. Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that the following conditions both hold:
(i) $f(n!) = f(n)!$ for every positive integer n,
(ii) $m - n$ divides $f(m) - f(n)$ whenever m and n are different positive integers.

Balkan MO 2011

1 Let $ABCD$ be a cyclic quadrilateral which is not a trapezoid and whose diagonals meet at E. The midpoints of AB and CD are F and G respectively, and ℓ is the line through G parallel to AB. The feet of the perpendiculars from E onto the lines ℓ and CD are H and K, respectively. Prove that the lines EF and HK are perpendicular.

2 Given real numbers x, y, z such that $x + y + z = 0$, show that

$$\frac{x(x+2)}{2x^2+1} + \frac{y(y+2)}{2y^2+1} + \frac{z(z+2)}{2z^2+1} \geq 0$$

When does equality hold?

3 Let S be a finite set of positive integers which has the following property:if x is a member of S,then so are all positive divisors of x. A non-empty subset T of S is *good* if whenever $x, y \in T$ and $x < y$, the ratio y/x is a power of a prime number. A non-empty subset T of S is *bad* if whenever $x, y \in T$ and $x < y$, the ratio y/x is not a power of a prime number. A set of an element is considered both *good* and *bad*. Let k be the largest possible size of a *good* subset of S. Prove that k is also the smallest number of pairwise-disjoint *bad* subsets whose union is S.

4 Let $ABCDEF$ be a convex hexagon of area 1, whose opposite sides are parallel. The lines AB, CD and EF meet in pairs to determine the vertices of a triangle. Similarly, the lines BC, DE and FA meet in pairs to determine the vertices of another triangle. Show that the area of at least one of these two triangles is at least $3/2$.

Balkan MO 2009

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1 Solve the equation

$$3^x - 5^y = z^2.$$

in positive integers.

Greece

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2 Let MN be a line parallel to the side BC of a triangle ABC, with M on the side AB and N on the side AC. The lines BN and CM meet at point P. The circumcircles of triangles BMP and CNP meet at two distinct points P and Q. Prove that $\angle BAQ = \angle CAP$.

Liubomir Chiriac, Moldova

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3 A 9×12 rectangle is partitioned into unit squares. The centers of all the unit squares, except for the four corner squares and eight squares sharing a common side with one of them, are coloured red. Is it possible to label these red centres $C_1, C_2, \ldots, C_{96}$ in such way that the following to conditions are both fulfilled
i) the distances $C_1C_2, \ldots, C_{95}C_{96}, C_{96}C_1$ are all equal to $\sqrt{13}$,
ii) the closed broken line $C_1C_2 \ldots C_{96}C_1$ has a centre of symmetry?

Bulgaria

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4 Denote by S the set of all positive integers. Find all functions $f : S \to S$ such that

$$f(f^2(m) + 2f^2(n)) = m^2 + 2n^2$$

for all $m, n \in S$.

Bulgaria

Balkan MO 2008

–	May 6th
1	Given a scalene acute triangle ABC with $AC > BC$ let F be the foot of the altitude from C. Let P be a point on AB, different from A so that $AF = PF$. Let H, O, M be the orthocenter, circumcenter and midpoint of $[AC]$. Let X be the intersection point of BC and HP. Let Y be the intersection point of OM and FX and let OF intersect AC at Z. Prove that F, M, Y, Z are concyclic.
2	Is there a sequence $a_1, a_2, \ldots$ of positive reals satisfying simoultaneously the following inequalities for all positive integers n: a) $a_1 + a_2 + \ldots + a_n \le n^2$ b) $\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} \le 2008$?
3	Let n be a positive integer. Consider a rectangle $(90n+1) \times (90n+5)$ consisting of unit squares. Let S be the set of the vertices of these squares. Prove that the number of distinct lines passing through at least two points of S is divisible by 4.
4	Let c be a positive integer. The sequence $a_1, a_2, \ldots$ is defined as follows $a_1 = c$, $a_{n+1} = a_n^2 + a_n + c^3$ for all positive integers n. Find all c so that there are integers $k \ge 1$ and $m \ge 2$ so that $a_k^2 + c^3$ is the mth power of some integer.

Balkan MO 2007

– April 27th

1 Let $ABCD$ a convex quadrilateral with $AB = BC = CD$, with AC not equal to BD and E be the intersection point of it's diagonals. Prove that $AE = DE$ if and only if $\angle BAD + \angle ADC = 120$.

2 Find all real functions f defined on IR, such that

$$f(f(x) + y) = f(f(x) - y) + 4f(x)y,$$

for all real numbers x, y.

3 Find all positive integers n such that there exist a permutation σ on the set $\{1, 2, 3, \ldots, n\}$ for which

$$\sqrt{\sigma(1) + \sqrt{\sigma(2) + \sqrt{\ldots + \sqrt{\sigma(n-1) + \sqrt{\sigma(n)}}}}}$$

is a rational number.

4 For a given positive integer $n > 2$, let C_1, C_2, C_3 be the boundaries of three convex $n-$ gons in the plane , such that
$C_1 \cap C_2, C_2 \cap C_3, C_1 \cap C_3$ are finite. Find the maximum number of points of the sets $C_1 \cap C_2 \cap C_3$.

Balkan MO 2006

– April 29th

1 Let a, b, c be positive real numbers. Prove the inequality

$$\frac{1}{a\left(b+1\right)}+\frac{1}{b\left(c+1\right)}+\frac{1}{c\left(a+1\right)}\geq\frac{3}{1+abc}.$$

2 Let ABC be a triangle and m a line which intersects the sides AB and AC at interior points D and F, respectively, and intersects the line BC at a point E such that C lies between B and E. The parallel lines from the points A, B, C to the line m intersect the circumcircle of triangle ABC at the points A_1, B_1 and C_1, respectively (apart from A, B, C). Prove that the lines A_1E , B_1F and C_1D pass through the same point.

Greece

3 Find all triplets of positive rational numbers (m,n,p) such that the numbers $m+\frac{1}{np}$, $n+\frac{1}{pm}$, $p+\frac{1}{mn}$ are integers.

Valentin Vornicu, Romania

4 Let m be a positive integer and $\{a_n\}_{n\geq 0}$ be a sequence given by $a_0=a\in\mathbb{N}$, and

$$a_{n+1}=\begin{cases}\dfrac{a_n}{2} & \text{if } a_n\equiv 0 \pmod 2,\\ a_n+m & \text{otherwise.}\end{cases}$$

Find all values of a such that the sequence is periodical (starting from the beginning).

Balkan MO 2005

–	May 6th
1	Let ABC be an acute-angled triangle whose inscribed circle touches AB and AC at D and E respectively. Let X and Y be the points of intersection of the bisectors of the angles $\angle ACB$ and $\angle ABC$ with the line DE and let Z be the midpoint of BC. Prove that the triangle XYZ is equilateral if and only if $\angle A = 60^\circ$.
2	Find all primes p such that $p^2 - p + 1$ is a perfect cube.
3	Let a, b, c be positive real numbers. Prove the inequality $$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c + \frac{4(a-b)^2}{a+b+c}.$$ When does equality occur?
4	Let $n \geq 2$ be an integer. Let S be a subset of $\{1, 2, \ldots, n\}$ such that S neither contains two elements one of which divides the other, nor contains two elements which are coprime. What is the maximal possible number of elements of such a set S?

Balkan MO 2004

– May 7th

1 The sequence $\{a_n\}_{n\geq 0}$ of real numbers satisfies the relation:

$$a_{m+n} + a_{m-n} - m + n - 1 = \frac{1}{2}(a_{2m} + a_{2n})$$

for all non-negative integers m and n, $m \geq n$. If $a_1 = 3$ find a_{2004}.

2 Solve in prime numbers the equation $x^y - y^x = xy^2 - 19$.

3 Let O be an interior point of an acute triangle ABC. The circles with centers the midpoints of its sides and passing through O mutually intersect the second time at the points K, L and M different from O. Prove that O is the incenter of the triangle KLM if and only if O is the circumcenter of the triangle ABC.

4 The plane is partitioned into regions by a finite number of lines no three of which are concurrent. Two regions are called "neighbors" if the intersection of their boundaries is a segment, or half-line or a line (a point is not a segment). An integer is to be assigned to each region in such a way that:

i) the product of the integers assigned to any two neighbors is less than their sum;
ii) for each of the given lines, and each of the half-planes determined by it, the sum of the integers, assigned to all of the regions lying on this half-plane equal to zero.

Prove that this is possible if and only if not all of the lines are parallel.

Balkan MO 2003

–	May 4th
1	Can one find 4004 positive integers such that the sum of any 2003 of them is not divisible by 2003?
2	Let ABC be a triangle, and let the tangent to the circumcircle of the triangle ABC at A meet the line BC at D. The perpendicular to BC at B meets the perpendicular bisector of AB at E. The perpendicular to BC at C meets the perpendicular bisector of AC at F. Prove that the points D, E and F are collinear. *Valentin Vornicu*
3	Find all functions $f:\mathbb{Q}\to\mathbb{R}$ which fulfill the following conditions: a) $f(1)+1>0$; b) $f(x+y)-xf(y)-yf(x)=f(x)f(y)-x-y+xy$, for all $x,y\in\mathbb{Q}$; c) $f(x)=2f(x+1)+x+2$, for every $x\in\mathbb{Q}$.
4	A rectangle $ABCD$ has side lengths $AB=m$, $AD=n$, with m and n relatively prime and both odd. It is divided into unit squares and the diagonal AC intersects the sides of the unit squares at the points $A_1=A, A_2, A_3, \ldots, A_k=C$. Show that $$A_1A_2-A_2A_3+A_3A_4-\cdots+A_{k-1}A_k=\frac{\sqrt{m^2+n^2}}{mn}.$$

Balkan MO 2002

– April 27th

1 Consider n points $A_1, A_2, A_3, \dots, A_n$ $(n \geq 4)$ in the plane, such that any three are not collinear. Some pairs of distinct points among $A_1, A_2, A_3, \dots, A_n$ are connected by segments, such that every point is connected with at least three different points. Prove that there exists $k > 1$ and the distinct points $X_1, X_2, \dots, X_{2k}$ in the set $\{A_1, A_2, A_3, \dots, A_n\}$, such that for every $i \in \overline{1, 2k-1}$ the point X_i is connected with X_{i+1}, and X_{2k} is connected with X_1.

2 Let the sequence $\{a_n\}_{n\geq 1}$ be defined by $a_1 = 20$, $a_2 = 30$ and $a_{n+2} = 3a_{n+1} - a_n$ for all $n \geq 1$. Find all positive integers n such that $1 + 5a_n a_{n+1}$ is a perfect square.

3 Two circles with different radii intersect in two points A and B. Let the common tangents of the two circles be MN and ST such that M, S lie on the first circle, and N, T on the second. Prove that the orthocenters of the triangles AMN, AST, BMN and BST are the four vertices of a rectangle.

4 Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that for every positive integer n we have:
$$2n + 2001 \leq f(f(n)) + f(n) \leq 2n + 2002.$$

Balkan MO 2001

1 Let a, b, n be positive integers such that $2^n - 1 = ab$. Let $k \in \mathbb{N}$ such that $ab + a - b - 1 \equiv 0 \pmod{2^k}$ and $ab + a - b - 1 \neq 0 \pmod{2^{k+1}}$. Prove that k is even.

2 A convex pentagon $ABCDE$ has rational sides and equal angles. Show that it is regular.

3 Let a, b, c be positive real numbers with $abc \leq a + b + c$. Show that

$$a^2 + b^2 + c^2 \geq \sqrt{3}abc.$$

Cristinel Mortici, Romania

4 A cube side 3 is divided into 27 unit cubes. The unit cubes are arbitrarily labeled 1 to 27 (each cube is given a different number). A move consists of swapping the cube labeled 27 with one of its 6 neighbours. Is it possible to find a finite sequence of moves at the end of which cube 27 is in its original position, but cube n has moved to the position originally occupied by $27 - n$ (for each $n = 1, 2, \ldots, 26$)?

Balkan MO 2000

1 Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(xf(x)+f(y))=f^2(x)+y$$

for all $x,y\in\mathbb{R}$.

2 Let ABC be an acute-angled triangle and D the midpoint of BC. Let E be a point on segment AD and M its projection on BC. If N and P are the projections of M on AB and AC then the interior angule bisectors of $\angle NMP$ and $\angle NEP$ are parallel.

3 How many $1\times 10\sqrt{2}$ rectangles can be cut from a 50×90 rectangle using cuts parallel to its edges?

4 Show that for any n we can find a set X of n distinct integers greater than 1, such that the average of the elements of any subset of X is a square, cube or higher power.