Sharygin Geometry Olympiad 2016

– Grade 9

– Day 1

1 The diagonals of a parallelogram $ABCD$ meet at point O. The tangent to the circumcircle of triangle BOC at O meets ray CB at point F. The circumcircle of triangle FOD meets BC for the second time at point G. Prove that $AG = AB$.

2 Let H be the orthocenter of an acute-angled triangle ABC. Point X_A lying on the tangent at H to the circumcircle of triangle BHC is such that $AH = AX_A$ and $X_A \neq H$. Points X_B, X_C are defined similarly. Prove that the triangle $X_AX_BX_C$ and the orthotriangle of ABC are similar.

3 Let O and I be the circumcenter and incenter of triangle ABC. The perpendicular from I to OI meets AB and the external bisector of angle C at points X and Y respectively. In what ratio does I divide the segment XY?

4 One hundred and one beetles are crawling in the plane. Some of the beetles are friends. Every one hundred beetles can position themselves so that two of them are friends if and only if they are at unit distance from each other. Is it always true that all one hundred and one beetles can do the same?

– Day 2

5 The center of a circle ω_2 lies on a circle ω_1. Tangents XP and XQ to ω_2 from an arbitrary point X of ω_1 (P and Q are the touching points) meet ω_1 for the second time at points R and S. Prove that the line PQ bisects the segment RS.

6 The sidelines AB and CD of a trapezoid meet at point P, and the diagonals of this trapezoid meet at point Q. Point M on the smallest base BC is such that $AM = MD$. Prove that $\angle PMB = \angle QMB$.

7 From the altitudes of an acute-angled triangle, a triangle can be composed. Prove that a triangle can be composed from the bisectors of this triangle.

8 The diagonals of a cyclic quadrilateral meet at point M. A circle ω touches segments MA and MD at points P, Q respectively and touches the circumcircle of $ABCD$ at point X. Prove that X lies on the radical axis of circles ACQ and BDP.

(Proposed by Ivan Frolov)

– Grade 10

– Day 1

1 A line parallel to the side BC of a triangle ABC meets the sides AB and AC at points P and Q, respectively. A point M is chosen inside the triangle APQ. The segments MB and MC meet the segment PQ at points E and F, respectively. Let N be the second intersection point of the circumcircles of the triangles PMF and QME. Prove that the points A, M, N are collinear.

2 Let I and I_a be the incenter and excenter (opposite vertex A) of a triangle ABC, respectively. Let A' be the point on its circumcircle opposite to A, and A_1 be the foot of the altitude from A. Prove that $\angle IA_1I_a = \angle IA'I_a$.

(Proposed by Pavel Kozhevnikov)

3 Assume that the two triangles ABC and $A'B'C'$ have the common incircle and the common circumcircle. Let a point P lie inside both the triangles. Prove that the sum of the distances from P to the sidelines of triangle ABC is equal to the sum of distances from P to the sidelines of triangle $A'B'C'$.

4 The Devil and the Man play a game. Initially, the Man pays some cash s to the Devil. Then he lists some 97 triples $\{i, j, k\}$ consisting of positive integers not exceeding 100. After that, the Devil draws some convex polygon $A_1A_2...A_{100}$ with area 100 and pays to the Man, the sum of areas of all triangles $A_iA_jA_k$. Determine the maximal value of s which guarantees that the Man receives at least as much cash as he paid.

Proposed by Nikolai Beluhov, Bulgaria

– Day 2

5 Does there exist a convex polyhedron having equal number of edges and diagonals?

(A diagonal of a polyhedron is a segment through two vertices not lying on the same face)

6 A triangle ABC is given. The point K is the base of the external bisector of angle A. The point M is the midpoint of the arc AC of the circumcircle. The point N on the bisector of angle C is such that $AN \parallel BM$. Prove that the points M, N, K are collinear.

(Proposed by Ilya Bogdanov)

7 Restore a triangle by one of its vertices, the circumcenter and the Lemoine's point.

(The Lemoine's point is the intersection point of the reflections of the medians in the correspondent angle bisectors)

8 Let ABC be a non-isosceles triangle, let AA_1 be its angle bisector and A_2 be the touching point of the incircle with side BC. The points B_1, B_2, C_1, C_2 are defined similarly. Let O and I be the circumcenter and the incenter of triangle ABC. Prove that the radical center of the circumcircle of the triangles $AA_1A_2, BB_1B_2, CC_1C_2$ lies on the line OI.

Sharygin Geometry Olympiad 2013

–	First Round
1	Let ABC be an isosceles triangle with $AB = BC$. Point E lies on the side AB, and ED is the perpendicular from E to BC. It is known that $AE = DE$. Find $\angle DAC$.
2	Let ABC be an isosceles triangle ($AC = BC$) with $\angle C = 20^\circ$. The bisectors of angles A and B meet the opposite sides at points A_1 and B_1 respectively. Prove that the triangle A_1OB_1 (where O is the circumcenter of ABC) is regular.
3	Let ABC be a right-angled triangle ($\angle B = 90^\circ$). The excircle inscribed into the angle A touches the extensions of the sides AB, AC at points A_1, A_2 respectively; points C_1, C_2 are defined similarly. Prove that the perpendiculars from A, B, C to C_1C_2, A_1C_1, A_1A_2 respectively, concur.
4	*This post is removed.*
5	Four segments drawn from a given point inside a convex quadrilateral to its vertices, split the quadrilateral into four equal triangles. Can we assert that this quadrilateral is a rhombus?
6	Diagonals AC and BD of a trapezoid $ABCD$ meet at P. The circumcircles of triangles ABP and CDP intersect the line AD for the second time at points X and Y respectively. Let M be the midpoint of segment XY. Prove that $BM = CM$.
7	Let BD be a bisector of triangle ABC. Points I_a, I_c are the incenters of triangles ABD, CBD respectively. The line I_aI_c meets AC in point Q. Prove that $\angle DBQ = 90^\circ$.
8	Let X be an arbitrary point inside the circumcircle of a triangle ABC. The lines BX and CX meet the circumcircle in points K and L respectively. The line LK intersects BA and AC at points E and F respectively. Find the locus of points X such that the circumcircles of triangles AFK and AEL touch.
9	Let T_1 and T_2 be the points of tangency of the excircles of a triangle ABC with its sides BC and AC respectively. It is known that the reflection of the incenter of ABC across the midpoint of AB lies on the circumcircle of triangle CT_1T_2. Find $\angle BCA$.

10 The incircle of triangle ABC touches the side AB at point C'; the incircle of triangle ACC' touches the sides AB and AC at points C_1, B_1; the incircle of triangle BCC' touches the sides AB and BC at points C_2, A_2. Prove that the lines B_1C_1, A_2C_2, and CC' concur.

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11 a) Let $ABCD$ be a convex quadrilateral and $r_1 \le r_2 \le r_3 \le r_4$ be the radii of the incircles of triangles ABC, BCD, CDA, DAB. Can the inequality $r_4 > 2r_3$ hold?

b) The diagonals of a convex quadrilateral $ABCD$ meet in point E. Let $r_1 \le r_2 \le r_3 \le r_4$ be the radii of the incircles of triangles ABE, BCE, CDE, DAE. Can the inequality $r_2 > 2r_1$ hold?

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12 On each side of triangle ABC, two distinct points are marked. It is known that these points are the feet of the altitudes and of the bisectors.

a) Using only a ruler determine which points are the feet of the altitudes and which points are the feet of the bisectors.

b) Solve p.a) drawing only three lines.

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13 Let A_1 and C_1 be the tangency points of the incircle of triangle ABC with BC and AB respectively, A' and C' be the tangency points of the excircle inscribed into the angle B with the extensions of BC and AB respectively. Prove that the orthocenter H of triangle ABC lies on A_1C_1 if and only if the lines $A'C_1$ and BA are orthogonal.

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14 Let M, N be the midpoints of diagonals AC, BD of a right-angled trapezoid $ABCD$ ($\measuredangle A = \measuredangle D = 90^\circ$).
The circumcircles of triangles ABN, CDM meet the line BC in the points Q, R.
Prove that the distances from Q, R to the midpoint of MN are equal.

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15 (a) Triangles $A_1B_1C_1$ and $A_2B_2C_2$ are inscribed into triangle ABC so that $C_1A_1 \perp BC$, $A_1B_1 \perp CA$, $B_1C_1 \perp AB$, $B_2A_2 \perp BC$, $C_2B_2 \perp CA$, $A_2C_2 \perp AB$. Prove that these triangles are equal.

(b) Points A_1, B_1, C_1, A_2, B_2, C_2 lie inside a triangle ABC so that A_1 is on segment AB_1, B_1 is on segment BC_1, C_1 is on segment CA_1, A_2 is on segment AC_2, B_2 is on segment BA_2, C_2 is on segment CB_2, and the angles BAA_1, CBB_2, ACC_1, CAA_2, ABB_2, BCC_2 are equal. Prove that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are equal.

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16 The incircle of triangle ABC touches BC, CA, AB at points A_1, B_1, C_1, respectively. The perpendicular from the incenter I to the median from vertex C meets the line A_1B_1 in point K. Prove that CK is parallel to AB.

17 An acute angle between the diagonals of a cyclic quadrilateral is equal to ϕ. Prove that an acute angle between the diagonals of any other quadrilateral having the same sidelengths is smaller than ϕ.

18 Let AD be a bisector of triangle ABC. Points M and N are projections of B and C respectively to AD. The circle with diameter MN intersects BC at points X and Y. Prove that $\angle BAX = \angle CAY$.

19 a) The incircle of a triangle ABC touches AC and AB at points B_0 and C_0 respectively. The bisectors of angles B and C meet the perpendicular bisector to the bisector AL in points Q and P respectively. Prove that the lines PC_0, QB_0 and BC concur.

b) Let AL be the bisector of a triangle ABC. Points O_1 and O_2 are the circumcenters of triangles ABL and ACL respectively. Points B_1 and C_1 are the projections of C and B to the bisectors of angles B and C respectively. Prove that the lines O_1C_1, O_2B_1, and BC concur.

c) Prove that the two points obtained in pp. a) and b) coincide.

20 Let C_1 be an arbitrary point on the side AB of triangle ABC. Points A_1 and B_1 on the rays BC and AC are such that $\angle AC_1B_1 = \angle BC_1A_1 = \angle ACB$. The lines AA_1 and BB_1 meet in point C_2. Prove that all the lines C_1C_2 have a common point.

21 Chords BC and DE of circle ω meet at point A. The line through D parallel to BC meets ω again at F, and FA meets ω again at T. Let $M = ET \cap BC$ and let N be the reflection of A over M. Show that (DEN) passes through the midpoint of BC.

22 The common perpendiculars to the opposite sidelines of a nonplanar quadrilateral are mutually orthogonal. Prove that they intersect.

23 Two convex polytopes A and B do not intersect. The polytope A has exactly 2012 planes of symmetry. What is the maximal number of symmetry planes of the union of A and B, if B has a) 2012, b) 2013 symmetry planes?

c) What is the answer to the question of p.b), if the symmetry planes are replaced by the symmetry axes?

–	Grade level 8
–	Grade level 9
–	Grade level 10
1	A circle k passes through the vertices B, C of a scalene triangle ABC. k meets the extensions of AB, AC beyond B, C at P, Q respectively. Let A_1 is the foot the altitude drop from A to BC. Suppose $A_1P = A_1Q$. Prove that $\widehat{PA_1Q} = 2\widehat{BAC}$.
2	Let $ABCD$ is a tangential quadrilateral such that $AB = CD > BC$. AC meets BD at L. Prove that $\widehat{ALB}$ is acute. According to the jury, they want to propose a more generalized problem is to prove $(AB - CD)^2 < (AD - BC)^2$, but this problem has appeared some time ago
3	Let X be a point inside triangle ABC such that $XA.BC = XB.AC = XC.AC$. Let I_1, I_2, I_3 be the incenters of XBC, XCA, XAB. Prove that AI_1, BI_2, CI_3 are concurrent. Of course, the most natural way to solve this is the Ceva sin theorem, but there is an another approach that may surprise you;), try not to use the Ceva theorem :))
4	Given a square cardboard of area $\frac{1}{4}$, and a paper triangle of area $\frac{1}{2}$ such that the square of its sidelength is a positive integer. Prove that the triangle can be folded in some ways such that the squace can be placed inside the folded figure so that both of its faces are completely covered with paper. *Proposed by N.Beluhov, Bulgaria*
5	Let ABCD is a cyclic quadrilateral inscribed in (O). E, F are the midpoints of arcs AB and CD not containing the other vertices of the quadrilateral. The line passing through E, F and parallel to the diagonals of $ABCD$ meet at E, F, K, L. Prove that KL passes through O.
6	The altitudes AA_1, BB_1, CC_1 of an acute triangle ABC concur at H. The perpendicular lines from H to B_1C_1, A_1C_1 meet rays CA, CB at P, Q respectively. Prove that the line from C perpendicular to A_1B_1 passes through the midpoint of PQ.

7 Given five fixed points in the space. It is known that these points are centers of five spheres, four of which are pairwise externally tangent, and all these point are internally tangent to the fifth one. It turns out that it is impossible to determine which of the marked points is the center of the largest sphere. Find the ratio of the greatest and the smallest radii of the spheres.

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Sharygin Geometry Olympiad 2012

1 In triangle ABC point M is the midpoint of side AB, and point D is the foot of altitude CD. Prove that $\angle A = 2\angle B$ if and only if $AC = 2MD$.

2 A cyclic n-gon is divided by non-intersecting (inside the n-gon) diagonals to $n-2$ triangles. Each of these triangles is similar to at least one of the remaining ones. For what n this is possible?

3 A circle with center I touches sides AB, BC, CA of triangle ABC in points C_1, A_1, B_1. Lines AI, CI, B_1I meet A_1C_1 in points X, Y, Z respectively. Prove that $\angle YB_1Z = \angle XB_1Z$.

4 Given triangle ABC. Point M is the midpoint of side BC, and point P is the projection of B to the perpendicular bisector of segment AC. Line PM meets AB in point Q. Prove that triangle QPB is isosceles.

5 On side AC of triangle ABC an arbitrary point is selected D. The tangent in D to the circumcircle of triangle BDC meets AB in point C_1; point A_1 is defined similarly. Prove that $A_1C_1 \parallel AC$.

6 Point C_1 of hypothenuse AC of a right-angled triangle ABC is such that $BC = CC_1$. Point C_2 on cathetus AB is such that $AC_2 = AC_1$; point A_2 is defined similarly. Find angle AMC, where M is the midpoint of A_2C_2.

7 In a non-isosceles triangle ABC the bisectors of angles A and B are inversely proportional to the respective sidelengths. Find angle C.

8 Let BM be the median of right-angled triangle $ABC(\angle B = 90^\circ)$. The incircle of triangle ABM touches sides AB, AM in points A_1, A_2; points C_1, C_2 are defined similarly. Prove that lines A_1A_2 and C_1C_2 meet on the bisector of angle ABC.

9 In triangle ABC, given lines l_b and l_c containing the bisectors of angles B and C, and the foot L_1 of the bisector of angle A. Restore triangle ABC.

10 In a convex quadrilateral all sidelengths and all angles are pairwise different.
a) Can the greatest angle be adjacent to the greatest side and at the same time the smallest angle be adjacent to the smallest side?

b) Can the greatest angle be non-adjacent to the smallest side and at the same time the smallest angle be non-adjacent to the greatest side?

11 Given triangle ABC and point P. Points A', B', C' are the projections of P to BC, CA, AB. A line passing through P and parallel to AB meets the circumcircle of triangle $PA'B'$ for the second time in point C_1. Points A_1, B_1 are defined similarly. Prove that
a) lines AA_1, BB_1, CC_1 concur;
b) triangles ABC and $A_1B_1C_1$ are similar.

12 Let O be the circumcenter of an acute-angled triangle ABC. A line passing through O and parallel to BC meets AB and AC in points P and Q respectively. The sum of distances from O to AB and AC is equal to OA. Prove that $PB + QC = PQ$.

13 Points A, B are given. Find the locus of points C such that C, the midpoints of AC, BC and the centroid of triangle ABC are concyclic.

14 In a convex quadrilateral $ABCD$ suppose $AC \cap BD = O$ and M is the midpoint of BC. Let $MO \cap AD = E$. Prove that $\frac{AE}{ED} = \frac{S_{\triangle ABO}}{S_{\triangle CDO}}$.

15 Given triangle ABC. Consider lines l with the next property: the reflections of l in the sidelines of the triangle concur. Prove that all these lines have a common point.

16 Given right-angled triangle ABC with hypothenuse AB. Let M be the midpoint of AB and O be the center of circumcircle ω of triangle CMB. Line AC meets ω for the second time in point K. Segment KO meets the circumcircle of triangle ABC in point L. Prove that segments AL and KM meet on the circumcircle of triangle ACM.

17 A square $ABCD$ is inscribed into a circle. Point M lies on arc BC, AM meets BD in point P, DM meets AC in point Q. Prove that the area of quadrilateral $APQD$ is equal to the half of the area of the square.

18 A triangle and two points inside it are marked. It is known that one of the triangles angles is equal to $58°$, one of two remaining angles is equal to $59°$, one of two given points is the incenter of the triangle and the second one is its circumcenter. Using only the ruler without partitions determine where is each of the angles and where is each of the centers.

19 Two circles with radii 1 meet in points X, Y, and the distance between these points also is equal to 1. Point C lies on the first circle, and lines CA, CB are tangents to the second one. These tangents meet the first circle for the second time in points B', A'. Lines AA' and BB' meet in point Z. Find angle XZY.

20 Point D lies on side AB of triangle ABC. Let ω_1 and Ω_1, ω_2 and Ω_2 be the incircles and the excircles (touching segment AB) of triangles ACD and BCD. Prove that the common external tangents to ω_1 and ω_2, Ω_1 and Ω_2 meet on AB.

21 Two perpendicular lines pass through the orthocenter of an acute-angled triangle. The sidelines of the triangle cut on each of these lines two segments: one lying inside the triangle and another one lying outside it. Prove that the product of two internal segments is equal to the product of two external segments.

22 A circle ω with center I is inscribed into a segment of the disk, formed by an arc and a chord AB. Point M is the midpoint of this arc AB, and point N is the midpoint of the complementary arc. The tangents from N touch ω in points C and D. The opposite sidelines AC and BD of quadrilateral $ABCD$ meet in point X, and the diagonals of $ABCD$ meet in point Y. Prove that points X, Y, I and M are collinear.

23 An arbitrary point is selected on each of twelve diagonals of the faces of a cube.The centroid of these twelve points is determined. Find the locus of all these centroids.

24 Given are n $(n > 2)$ points on the plane such that no three of them are collinear. In how many ways this set of points can be divided into two non-empty subsets with non-intersecting convex envelops?

Sharygin Geometry Olympiad 2011

1 Does a convex heptagon exist which can be divided into 2011 equal triangles?

2 Let ABC be a triangle with sides $AB = 4$ and $AC = 6$. Point H is the projection of vertex B to the bisector of angle A. Find MH, where M is the midpoint of BC.

3 Let ABC be a triangle with $\angle A = 60^\circ$. The midperpendicular of segment AB meets line AC at point C_1. The midperpendicular of segment AC meets line AB at point B_1. Prove that line B_1C_1 touches the incircle of triangle ABC.

4 Segments AA', BB', and CC' are the bisectrices of triangle ABC. It is known that these lines are also the bisectrices of triangle $A'B'C'$. Is it true that triangle ABC is regular?

5 Given triangle ABC. The midperpendicular of side AB meets one of the remaining sides at point C'. Points A' and B' are defined similarly. Find all triangles ABC such that triangle $A'B'C'$ is regular.

6 Two unit circles ω_1 and ω_2 intersect at points A and B. M is an arbitrary point of ω_1, N is an arbitrary point of ω_2. Two unit circles ω_3 and ω_4 pass through both points M and N. Let C be the second common point of ω_1 and ω_3, and D be the second common point of ω_2 and ω_4. Prove that $ACBD$ is a parallelogram.

2010 Sharygin Geometry Olympiad

Sharygin Geometry Olympiad 2010

1 Does there exist a triangle, whose side is equal to some of its altitudes, another side is equal to some of its bisectors, and the third is equal to some of its medians?

Amir Hossein view topic

2 Bisectors AA_1 and BB_1 of a right triangle ABC $(\angle C = 90^\circ)$ meet at a point $I.$ Let O be the circumcenter of triangle $CA_1B_1.$ Prove that $OI \perp AB.$

Amir Hossein view topic

3 Points A', B', C' lie on sides BC, CA, AB of triangle $ABC.$ for a point X one has $\angle AXB = \angle A'C'B' + \angle ACB$ and $\angle BXC = \angle B'A'C' + \angle BAC.$ Prove that the quadrilateral $XA'BC'$ is cyclic.

Amir Hossein view topic

4 The diagonals of a cyclic quadrilateral $ABCD$ meet in a point $N.$ The circumcircles of triangles ANB and CND intersect the sidelines BC and AD for the second time in points $A_1, B_1, C_1, D_1.$ Prove that the quadrilateral $A_1B_1C_1D_1$ is inscribed in a circle centered at $N.$

Amir Hossein view topic

5 A point E lies on the altitude BD of triangle ABC, and $\angle AEC = 90^\circ.$ Points O_1 and O_2 are the circumcenters of triangles AEB and CEB; points F, L are the midpoints of the segments AC and $O_1O_2.$ Prove that the points L, E, F are collinear.

Amir Hossein view topic

6 Points M and N lie on the side BC of the regular triangle ABC (M is between B and N), and $\angle MAN = 30^\circ.$ The circumcircles of triangles AMC and ANB meet at a point $K.$ Prove that the line AK passes through the circumcenter of triangle $AMN.$

Amir Hossein view topic

7 The line passing through the vertex B of a triangle ABC and perpendicular to its median BM intersects the altitudes dropped from A and C (or their extensions) in points K and $N.$ Points O_1 and O_2 are the circumcenters of the triangles ABK and CBN respectively. Prove that $O_1M = O_2M.$

Amir Hossein view topic

8 Let AH be the altitude of a given triangle $ABC.$ The points I_b and I_c are the incenters of the triangles ABH and ACH respectively. BC touches the incircle of the triangle ABC at a point $L.$ Find $\angle LI_bI_c.$

Amir Hossein view topic

9 A point inside a triangle is called "*good*" if three cevians passing through it are equal. Assume for an isosceles triangle ABC $(AB = BC)$ the total number of "*good*" points is odd. Find all possible values of this number.

Amir Hossein view topic

10 Let three lines forming a triangle ABC be given. Using a two-sided ruler and drawing at most eight lines construct a point D on the side AB such that $\frac{AD}{BD} = \frac{BC}{AC}.$

Amir Hossein view topic

11 A convex $n-$gon is split into three convex polygons. One of them has n sides, the second one has more than n sides, the third one has less than n sides. Find all possible values of $n.$

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12 Let AC be the greatest leg of a right triangle $ABC,$ and CH be the altitude to its hypotenuse. The circle of radius CH centered at H intersects AC in point $M.$ Let a point B' be the reflection of B with respect to the point $H.$ The perpendicular to AB erected at B' meets the circle in a point K. Prove that

a) $B'M \parallel BC$

b) AK is tangent to the circle.

Amir Hossein view topic

13 Let us have a convex quadrilateral $ABCD$ such that $AB = BC.$ A point K lies on the diagonal $BD,$ and $\angle AKB + \angle BKC = \angle A + \angle C.$ Prove that $AK \cdot CD = KC \cdot AD.$

Amir Hossein view topic

14 We have a convex quadrilateral $ABCD$ and a point M on its side AD such that CM and BM are parallel to AB and CD respectively. Prove that $S_{ABCD} \geq 3S_{BCM}.$

Remark. S denotes the area function.

Amir Hossein view topic

15 Let AA_1, BB_1 and CC_1 be the altitudes of an acute-angled triangle $ABC.$ AA_1 meets $B_1C_$ in a point $K.$ The circumcircles of triangles A_1KC_1 and A_1KB_1 intersect the lines AB and AC for the second time at points N and L

Amir Hossein

respectively. Prove that

a) The sum of diameters of these two circles is equal to $BC,$

b) $\dfrac{A_1N}{BB_1}+\dfrac{A_1L}{CC_1}=1.$

view topic

16

A circle touches the sides of an angle with vertex A at points B and $C.$ A line passing through A intersects this circle in points D and $E.$ A chord BX is parallel to $DE.$ Prove that XC passes through the midpoint of the segment $DE.$

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17

Construct a triangle, if the lengths of the bisectrix and of the altitude from one vertex, and of the median from another vertex are given.

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18

A point B lies on a chord AC of circle $\omega.$ Segments AB and BC are diameters of circles ω_1 and ω_2 centered at O_1 and O_2 respectively. These circles intersect ω for the second time in points D and E respectively. The rays O_1D and O_2E meet in a point $F,$ and the rays AD and CE do in a point $G.$ Prove that the line FG passes through the midpoint of the segment $AC.$

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19

A quadrilateral $ABCD$ is inscribed into a circle with center $O.$ Points P and Q are opposite to C and D respectively. Two tangents drawn to that circle at these points meet the line AB in points E and $F.$ (A is between E and B, B is between A and F). The line EO meets AC and BC in points X and Y respectively, and the line FO meets AD and BD in points U and V respectively. Prove that $XV=YU.$

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20

The incircle of an acute-angled triangle ABC touches AB, BC, CA at points C_1, A_1, B_1 respectively. Points A_2, B_2 are the midpoints of the segments B_1C_1, A_1C_1 respectively. Let P be a common point of the incircle and the line CO, where O is the circumcenter of triangle $ABC.$ Let also A' and B' be the second common points of PA_2 and PB_2 with the incircle. Prove that a common point of AA' and BB' lies on the altitude of the triangle dropped from the vertex $C.$

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21

A given convex quadrilateral $ABCD$ is such that $\angle ABD+\angle ACD>\angle BAC+\angle BDC.$ Prove that

$$S_{ABD}+S_{ACD}>S_{BAC}+S_{BDC}.$$

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22

A circle centered at a point F and a parabola with focus F have two common points. Prove that there exist four points A, B, C, D on the circle such that the lines AB, BC, CD and DA touch the parabola.

Amir Hossein view topic

23

A cyclic hexagon $ABCDEF$ is such that $AB\cdot CF=2BC\cdot FA, CD\cdot EB=2DE\cdot BC$ and $EF\cdot AD=2FA\cdot DE.$ Prove that the lines AD, BE and CF are concurrent.

Amir Hossein view topic

24

Let us have a line ℓ in the space and a point A not lying on $\ell.$ For an arbitrary line ℓ' passing through A, XY (Y is on ℓ') is a common perpendicular to the lines ℓ and $\ell'.$ Find the locus of points $Y.$

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25

For two different regular icosahedrons it is known that some six of their vertices are vertices of a regular octahedron. Find the ratio of the edges of these icosahedrons.

Amir Hossein view topic

Sharygin Geometry Olympiad 2009

1 Points B_1 and B_2 lie on ray AM, and points C_1 and C_2 lie on ray AK. The circle with center O is inscribed into triangles AB_1C_1 and AB_2C_2. Prove that the angles B_1OB_2 and C_1OC_2 are equal.

2 Given nonisosceles triangle ABC. Consider three segments passing through different vertices of this triangle and bisecting its perimeter. Are the lengths of these segments certainly different?

3 The bisectors of trapezoid's angles form a quadrilateral with perpendicular diagonals. Prove that this trapezoid is isosceles.

4 Let P and Q be the common points of two circles. The ray with origin Q reflects from the first circle in points A_1, A_2,... according to the rule "the angle of incidence is equal to the angle of reflection". Another ray with origin Q reflects from the second circle in the points B_1, B_2,... in the same manner. Points A_1, B_1 and P occurred to be collinear. Prove that all lines A_iB_i pass through P.

5 Given triangle ABC. Point O is the center of the excircle touching the side BC. Point O_1 is the reflection of O in BC. Determine angle A if O_1 lies on the circumcircle of ABC.

6 Find the locus of excenters of right triangles with given hypotenuse.

7 Given triangle ABC. Points M, N are the projections of B and C to the bisectors of angles C and B respectively. Prove that line MN intersects sides AC and AB in their points of contact with the incircle of ABC.

8 Some polygon can be divided into two equal parts by three different ways. Is it certainly valid that this polygon has an axis or a center of symmetry?

9 Given n points on the plane, which are the vertices of a convex polygon, $n > 3$. There exists k regular triangles with the side equal to 1 and the vertices at the given points.
- Prove that $k < \frac{2}{3}n$. - Construct the configuration with $k > 0.666n$.

10 Let ABC be an acute triangle, CC_1 its bisector, O its circumcenter. The perpendicular from C to AB meets line OC_1 in a point lying on the circumcircle of AOB. Determine angle C.

11 Given quadrilateral $ABCD$. The circumcircle of ABC is tangent to side CD, and the circumcircle of ACD is tangent to side AB. Prove that the length of diagonal AC is less than the distance between the midpoints of AB and CD.

12 Let CL be a bisector of triangle ABC. Points A_1 and B_1 are the reflections of A and B in CL, points A_2 and B_2 are the reflections of A and B in L. Let O_1 and O_2 be the circumcenters of triangles AB_1B_2 and BA_1A_2 respectively. Prove that angles O_1CA and O_2CB are equal.

13 In triangle ABC, one has marked the incenter, the foot of altitude from vertex C and the center of the excircle tangent to side AB. After this, the triangle was erased. Restore it.

14 Given triangle ABC of area 1. Let BM be the perpendicular from B to the bisector of angle C. Determine the area of triangle AMC.

15 Given a circle and a point C not lying on this circle. Consider all triangles ABC such that points A and B lie on the given circle. Prove that the triangle of maximal area is isosceles.

16 Three lines passing through point O form equal angles by pairs. Points A_1, A_2 on the first line and B_1, B_2 on the second line are such that the common point C_1 of A_1B_1 and A_2B_2 lies on the third line. Let C_2 be the common point of A_1B_2 and A_2B_1. Prove that angle C_1OC_2 is right.

17 Given triangle ABC and two points X, Y not lying on its circumcircle. Let A_1, B_1, C_1 be the projections of X to BC, CA, AB, and A_2, B_2, C_2 be the projections of Y. Prove that the perpendiculars from A_1, B_1, C_1 to B_2C_2, C_2A_2, A_2B_2, respectively, concur if and only if line XY passes through the circumcenter of ABC.

18 Given three parallel lines on the plane. Find the locus of incenters of triangles with vertices lying on these lines (a single vertex on each line).

19 Given convex n-gon $A_1 \dots A_n$. Let P_i $(i = 1, \dots, n)$ be such points on its boundary that A_iP_i bisects the area of polygon. All points P_i don't coincide with any vertex and lie on k sides of n-gon. What is the maximal and the minimal value of k for each given n?

20 Suppose H and O are the orthocenter and the circumcenter of acute triangle ABC; AA_1, BB_1 and CC_1 are the altitudes of the triangle. Point C_2 is the reflection of C in A_1B_1. Prove that H, O, C_1 and C_2 are concyclic.

21 The opposite sidelines of quadrilateral $ABCD$ intersect at points P and Q. Two lines passing through these points meet the side of $ABCD$ in four points which are the vertices of a parallelogram. Prove that the center of this parallelogram lies on the line passing through the midpoints of diagonals of $ABCD$.

22 Construct a quadrilateral which is inscribed and circumscribed, given the radii of the respective circles and the angle between the diagonals of quadrilateral.

23 Is it true that for each n, the regular $2n$-gon is a projection of some polyhedron having not greater than $n+2$ faces?

24 A sphere is inscribed into a quadrangular pyramid. The point of contact of the sphere with the base of the pyramid is projected to the edges of the base. Prove that these projections are concyclic.

Sharygin Geometry Olympiad 2008

– Grade level 8

1 (B.Frenkin) Does a convex quadrilateral without parallel sidelines exist such that it can be divided into four congruent triangles?

2 (F.Nilov) Given right triangle ABC with hypothenuse AC and $\angle A = 50^\circ$. Points K and L on the cathetus BC are such that $\angle KAC = \angle LAB = 10^\circ$. Determine the ratio CK/LB.

3 (D.Shnol) Two opposite angles of a convex quadrilateral with perpendicular diagonals are equal. Prove that a circle can be inscribed in this quadrilateral.

4 (F.Nilov, A.Zaslavsky) Let CC_0 be a median of triangle ABC; the perpendicular bisectors to AC and BC intersect CC_0 in points A', B'; C_1 is the meet of lines AA' and BB'. Prove that $\angle C_1CA = \angle C_0CB$.

5 (A.Zaslavsky) Given two triangles ABC, $A'B'C'$. Denote by α the angle between the altitude and the median from vertex A of triangle ABC. Angles β, γ, α', β', γ' are defined similarly. It is known that $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$. Can we conclude that the triangles are similar?

6 (B.Frenkin) Consider the triangles such that all their vertices are vertices of a given regular 2008-gon. What triangles are more numerous among them: acute-angled or obtuse-angled?

7 (F.Nilov) Given isosceles triangle ABC with base AC and $\angle B = \alpha$. The arc AC constructed outside the triangle has angular measure equal to β. Two lines passing through B divide the segment and the arc AC into three equal parts. Find the ratio α/β.

8 (B.Frenkin, A.Zaslavsky) A convex quadrilateral was drawn on the blackboard. Boris marked the centers of four excircles each touching one side of the quadrilateral and the extensions of two adjacent sides. After this, Alexey erased the quadrilateral. Can Boris define its perimeter?

– Grade level 9

1 (A.Zaslavsky) A convex polygon can be divided into 2008 congruent quadrilaterals. Is it true that this polygon has a center or an axis of symmetry?

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2 (F.Nilov) Given quadrilateral $ABCD$. Find the locus of points such that their projections to the lines AB, BC, CD, DA form a quadrilateral with perpendicular diagonals.

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3 (R.Pirkuliev) Prove the inequality

$$\frac{1}{\sqrt{2\sin A}} + \frac{1}{\sqrt{2\sin B}} + \frac{1}{\sqrt{2\sin C}} \leq \sqrt{\frac{p}{r}},$$

where p and r are the semiperimeter and the inradius of triangle ABC.

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4 (F.Nilov, A.Zaslavsky) Let CC_0 be a median of triangle ABC; the perpendicular bisectors to AC and BC intersect CC_0 in points A_c, B_c; C_1 is the common point of AA_c and BB_c. Points A_1, B_1 are defined similarly. Prove that circle $A_1B_1C_1$ passes through the circumcenter of triangle ABC.

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5 (N.Avilov) Can the surface of a regular tetrahedron be glued over with equal regular hexagons?

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6 (B.Frenkin) Construct the triangle, given its centroid and the feet of an altitude and a bisector from the same vertex.

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7 (A.Zaslavsky) The circumradius of triangle ABC is equal to R. Another circle with the same radius passes through the orthocenter H of this triangle and intersect its circumcirle in points X, Y. Point Z is the fourth vertex of parallelogram $CXZY$. Find the circumradius of triangle ABZ.

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8 (J.-L.Ayme, France) Points P, Q lie on the circumcircle ω of triangle ABC. The perpendicular bisector l to PQ intersects BC, CA, AB in points A', B', C'. Let A", B", C" be the second common points of l with the circles $A'PQ$, $B'PQ$, $C'PQ$. Prove that AA", BB", CC" concur.

– Grade level 10

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1 (B.Frenkin) An inscribed and circumscribed n-gon is divided by some line into two inscribed and circumscribed polygons with different numbers of sides. Find n.

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2 (A.Myakishev) Let triangle $A_1B_1C_1$ be symmetric to ABC wrt the incenter of its medial triangle. Prove that the orthocenter of $A_1B_1C_1$ coincides with the circumcenter of the triangle formed by the excenters of ABC.

3 (V.Yasinsky, Ukraine) Suppose X and Y are the common points of two circles ω_1 and ω_2. The third circle ω is internally tangent to ω_1 and ω_2 in P and Q respectively. Segment XY intersects ω in points M and N. Rays PM and PN intersect ω_1 in points A and D; rays QM and QN intersect ω_2 in points B and C respectively. Prove that $AB = CD$.

4 (A.Zaslavsky) Given three points C_0, C_1, C_2 on the line l. Find the locus of incenters of triangles ABC such that points A, B lie on l and the feet of the median, the bisector and the altitude from C coincide with C_0, C_1, C_2.

5 (I.Bogdanov) A section of a regular tetragonal pyramid is a regular pentagon. Find the ratio of its side to the side of the base of the pyramid.

6 (B.Frenkin) The product of two sides in a triangle is equal to $8Rr$, where R and r are the circumradius and the inradius of the triangle. Prove that the angle between these sides is less than 60°.

7 (F.Nilov) Two arcs with equal angular measure are constructed on the medians AA' and BB' of triangle ABC towards vertex C. Prove that the common chord of the respective circles passes through C.

8 (A.Akopyan, V.Dolnikov) Given a set of points inn the plane. It is known that among any three of its points there are two such that the distance between them doesn't exceed 1. Prove that this set can be divided into three parts such that the diameter of each part does not exceed 1.

– First Round

1 (B.Frenkin, 8) Does a regular polygon exist such that just half of its diagonals are parallel to its sides?

2 (V.Protasov, 8) For a given pair of circles, construct two concentric circles such that both are tangent to the given two. What is the number of solutions, depending on location of the circles?

3 (A.Zaslavsky, 8) A triangle can be dissected into three equal triangles. Prove that some its angle is equal to 60°.

4 (D.Shnol, 8–9) The bisectors of two angles in a cyclic quadrilateral are parallel. Prove that the sum of squares of some two sides in the quadrilateral equals the sum of squares of two remaining sides.

5 (Kiev olympiad, 8–9) Reconstruct the square $ABCD$, given its vertex A and distances of vertices B and D from a fixed point O in the plane.

6 (A. Myakishev, 8–9) In the plane, given two concentric circles with the center A. Let B be an arbitrary point on some of these circles, and C on the other one. For every triangle ABC, consider two equal circles mutually tangent at the point K, such that one of these circles is tangent to the line AB at point B and the other one is tangent to the line AC at point C. Determine the locus of points K.

7 (A.Zaslavsky, 8–9) Given a circle and a point O on it. Another circle with center O meets the first one at points P and Q. The point C lies on the first circle, and the lines CP, CQ meet the second circle for the second time at points A and B. Prove that $AB = PQ$.

8 (T.Golenishcheva-Kutuzova, B.Frenkin, 8–11) a) Prove that for $n > 4$, any convex n-gon
can be dissected into n obtuse triangles.

9 (A.Zaslavsky, 9–10) The reflections of diagonal BD of a quadrilateral $ABCD$ in the bisectors of angles B and D pass
through the midpoint of diagonal AC. Prove that the reflections of diagonal AC in the bisectors of angles A and C pass
through the midpoint of diagonal BD (There was an error in published condition of this problem).

10 (A.Zaslavsky, 9–10) Quadrilateral $ABCD$ is circumscribed arounda circle with center I. Prove that the projections of points B and D to the lines IA and IC lie on a single circle.

11 (A.Zaslavsky, 9–10) Given four points A, B, C, D. Any two circles such that one of them contains A and B, and the other one contains C and D, meet. Prove that common chords of all these pairs of circles pass through a fixed point.

12 (A.Myakishev, 9–10) Given a triangle ABC. Point A_1 is chosen on the ray BA so that segments BA_1 and BC are equal. Point A_2 is chosen on the ray CA so that segments CA_2 and BC are equal. Points B_1, B_2 and C_1, C_2 are chosen similarly. Prove that lines A_1A_2, B_1B_2, C_1C_2 are parallel.

13 (A.Myakishev, 9–10) Given triangle ABC. One of its excircles is tangent to the side BC at point A_1 and to the extensions of two other sides. Another excircle is tangent to side AC at point B_1. Segments AA_1 and BB_1 meet at point N. Point P is chosen on the ray AA_1 so that $AP = NA_1$. Prove that P lies on the incircle.

14 (V.Protasov, 9–10) The Euler line of a non-isosceles triangle is parallel to the bisector of one of its angles. Determine this
angle (There was an error in published condition of this problem).

15 (M.Volchkevich, 9–11) Given two circles and point P not lying on them. Draw a line through P which cuts chords of equal length from these circles.

16 (A.Zaslavsky, 9–11) Given two circles. Their common external tangent is tangent to them at points A and B. Points X, Y on these circles are such that some circle is tangent to the given two circles at these points, and in similar way (external or internal). Determine the locus of intersections of lines AX and BY.

17 (A.Myakishev, 9–11) Given triangle ABC and a ruler with two marked intervals equal to AC and BC. By this ruler only, find the incenter of the triangle formed by medial lines of triangle ABC.

18 (A.Abdullayev, 9–11) Prove that the triangle having sides a, b, c and area S satisfies the inequality

$$a^2 + b^2 + c^2 - \frac{1}{2}(|a - b| + |b - c| + |c - a|)^2 \geq 4\sqrt{3}S.$$

19 (V.Protasov, 10-11) Given parallelogram $ABCD$ such that $AB = a$, $AD = b$. The first circle has its center at vertex A and passes through D, and the second circle has its center at C and passes through D. A circle with center B meets the first circle at points M_1, N_1, and the second circle at points M_2, N_2. Determine the ratio M_1N_1/M_2N_2.

20 (A.Zaslavsky, 10–11) a) Some polygon has the following property:
if a line passes through two points which bisect its perimeter then this line

bisects the area of the polygon. Is it true that the polygon is central symmetric?
b) Is it true that any figure with the property from part a) is central symmetric?

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21 (A.Zaslavsky, B.Frenkin, 10–11) In a triangle, one has drawn perpendicular bisectors to its sides and has measured their segments lying inside the triangle.

a) All three segments are equal. Is it true that the triangle is equilateral?

b) Two segments are equal. Is it true that the triangle is isosceles?

c) Can the segments have length 4, 4 and 3?

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22 (A.Khachaturyan, 10–11) a) All vertices of a pyramid lie on the facets of a cube but not on its edges, and each facet contains at least one vertex. What is the maximum possible number of the vertices of the pyramid?

b) All vertices of a pyramid lie in the facet planes of a cube but not on the lines
including its edges, and each facet plane contains at least one vertex. What is the
maximum possible number of the vertices of the pyramid?

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23 (V.Protasov, 10–11) In the space, given two intersecting spheres of different radii and a point A belonging to both spheres. Prove that there is a point B in the space with the following property:
if an arbitrary circle passes through points A and B then the second points of its meet with the given spheres are equidistant from B.

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24 (I.Bogdanov, 11) Let h be the least altitude of a tetrahedron, and d the least distance between its opposite edges. For what values of t the inequality $d > th$ is possible?
