IMO Shortlist 2002

– Geometry

1 Let B be a point on a circle S_1, and let A be a point distinct from B on the tangent at B to S_1. Let C be a point not on S_1 such that the line segment AC meets S_1 at two distinct points. Let S_2 be the circle touching AC at C and touching S_1 at a point D on the opposite side of AC from B. Prove that the circumcentre of triangle BCD lies on the circumcircle of triangle ABC.

2 Let ABC be a triangle for which there exists an interior point F such that $\angle AFB = \angle BFC = \angle CFA$. Let the lines BF and CF meet the sides AC and AB at D and E respectively. Prove that

$$AB + AC \geq 4DE.$$

3 The circle S has centre O, and BC is a diameter of S. Let A be a point of S such that $\angle AOB < 120^\circ$. Let D be the midpoint of the arc AB which does not contain C. The line through O parallel to DA meets the line AC at I. The perpendicular bisector of OA meets S at E and at F. Prove that I is the incentre of the triangle CEF.

4 Circles S_1 and S_2 intersect at points P and Q. Distinct points A_1 and B_1 (not at P or Q) are selected on S_1. The lines A_1P and B_1P meet S_2 again at A_2 and B_2 respectively, and the lines A_1B_1 and A_2B_2 meet at C. Prove that, as A_1 and B_1 vary, the circumcentres of triangles A_1A_2C all lie on one fixed circle.

5 For any set S of five points in the plane, no three of which are collinear, let $M(S)$ and $m(S)$ denote the greatest and smallest areas, respectively, of triangles determined by three points from S. What is the minimum possible value of $M(S)/m(S)$?

6 Let $n \geq 3$ be a positive integer. Let $C_1, C_2, C_3, \ldots, C_n$ be unit circles in the plane, with centres $O_1, O_2, O_3, \ldots, O_n$ respectively. If no line meets more than two of the circles, prove that

$$\sum_{1 \leq i < j \leq n} \frac{1}{O_iO_j} \leq \frac{(n-1)\pi}{4}.$$

7 The incircle Ω of the acute-angled triangle ABC is tangent to its side BC at a point K. Let AD be an altitude of triangle ABC, and let M be the midpoint of the segment AD. If N is the common point of the circle Ω and the line KM (distinct from K), then prove that the incircle Ω and the circumcircle of triangle BCN are tangent to each other at the point N.

8 Let two circles S_1 and S_2 meet at the points A and B. A line through A meets S_1 again at C and S_2 again at D. Let M, N, K be three points on the line segments CD, BC, BD respectively, with MN parallel to BD and MK parallel to BC. Let E and F be points on those arcs BC of S_1 and BD of S_2 respectively that do not contain A. Given that EN is perpendicular to BC and FK is perpendicular to BD prove that $\angle EMF = 90^\circ$.

– Number Theory

1 What is the smallest positive integer t such that there exist integers $x_1, x_2, \ldots, x_t$ with

$$x_1^3 + x_2^3 + \ldots + x_t^3 = 2002^{2002}\,?$$

2 Let $n \geq 2$ be a positive integer, with divisors $1 = d_1 < d_2 < \ldots < d_k = n$. Prove that $d_1d_2 + d_2d_3 + \ldots + d_{k-1}d_k$ is always less than n^2, and determine when it is a divisor of n^2.

3 Let $p_1, p_2, \ldots, p_n$ be distinct primes greater than 3. Show that $2^{p_1p_2\cdots p_n} + 1$ has at least 4^n divisors.

4 Is there a positive integer m such that the equation

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} = \frac{m}{a+b+c}$$

has infinitely many solutions in positive integers a, b, c?

5 Let $m, n \geq 2$ be positive integers, and let $a_1, a_2, \ldots, a_n$ be integers, none of which is a multiple of m^{n-1}. Show that there exist integers $e_1, e_2, \ldots, e_n$, not all zero, with $|\,e_i\,| < m$ for all i, such that $e_1a_1 + e_2a_2 + \ldots + e_na_n$ is a multiple of m^n.

6 Find all pairs of positive integers $m, n \geq 3$ for which there exist infinitely many positive integers a such that

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is itself an integer.

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– Algebra

1 Find all functions f from the reals to the reals such that

$$f\left(f(x)+y\right)=2x+f\left(f(y)-x\right)$$

for all real x,y.

2 Let $a_1, a_2, \ldots$ be an infinite sequence of real numbers, for which there exists a real number c with $0 \leq a_i \leq c$ for all i, such that

$$\left|a_i-a_j\right| \geq \frac{1}{i+j} \quad \text{for all } i,\ j \text{ with } i \neq j.$$

Prove that $c \geq 1$.

3 Let P be a cubic polynomial given by $P(x)=ax^3+bx^2+cx+d$, where a,b,c,d are integers and $a\ne0$. Suppose that $xP(x)=yP(y)$ for infinitely many pairs x,y of integers with $x\ne y$. Prove that the equation $P(x)=0$ has an integer root.

4 Find all functions f from the reals to the reals such that

$$\left(f(x)+f(z)\right)\left(f(y)+f(t)\right)=f(xy-zt)+f(xt+yz)$$

for all real x,y,z,t.

5 Let n be a positive integer that is not a perfect cube. Define real numbers a,b,c by

$$a=\sqrt[3]{n}\,, \qquad b={1\over a-[a]}\,, \qquad c={1\over b-[b]}\,,$$

where $[x]$ denotes the integer part of x. Prove that there are infinitely many such integers n with the property that there exist integers r,s,t, not all zero, such that $ra+sb+tc=0$.

6 Let A be a non-empty set of positive integers. Suppose that there are positive integers $b_1, \dots b_n$ and $c_1, \dots, c_n$ such that

- for each i the set $b_iA + c_i = \{b_ia + c_i \colon a \in A\}$ is a subset of A, and

- the sets $b_iA + c_i$ and $b_jA + c_j$ are disjoint whenever $i \neq j$

Prove that
$$\frac{1}{b_1} + \dots + \frac{1}{b_n} \leq 1.$$

– Combinatorics

1 Let n be a positive integer. Each point (x, y) in the plane, where x and y are non-negative integers with $x + y < n$, is coloured red or blue, subject to the following condition: if a point (x, y) is red, then so are all points (x', y') with $x' \leq x$ and $y' \leq y$. Let A be the number of ways to choose n blue points with distinct x-coordinates, and let B be the number of ways to choose n blue points with distinct y-coordinates. Prove that $A = B$.

2 For n an odd positive integer, the unit squares of an $n \times n$ chessboard are coloured alternately black and white, with the four corners coloured black. A it tromino is an L-shape formed by three connected unit squares. For which values of n is it possible to cover all the black squares with non-overlapping trominos? When it is possible, what is the minimum number of trominos needed?

3 Let n be a positive integer. A sequence of n positive integers (not necessarily distinct) is called **full** if it satisfies the following condition: for each positive integer $k \geq 2$, if the number k appears in the sequence then so does the number $k - 1$, and moreover the first occurrence of $k - 1$ comes before the last occurrence of k. For each n, how many full sequences are there ?

4 Let T be the set of ordered triples (x, y, z), where x, y, z are integers with $0 \leq x, y, z \leq 9$. Players A and B play the following guessing game. Player A chooses a triple (x, y, z) in T, and Player B has to discover A*'s* triple in as few moves as possible. A *move* consists of the following: B gives A a triple (a, b, c) in T, and A replies by giving B the number $|x + y - a - b| + |y + z - b - c| + |z + x - c - a|$. Find the minimum number of moves that B needs to be sure of determining A*'s* triple.

5 Let $r \geq 2$ be a fixed positive integer, and let F be an infinite family of sets, each of size r, no two of which are disjoint. Prove that there exists a set of size $r-1$ that meets each set in F.

6 Let n be an even positive integer. Show that there is a permutation $(x_1, x_2, \ldots, x_n)$ of $(1,\ 2,\ \ldots, n)$ such that for every $i \in \{1,\ 2,\ ...,\ n\}$, the number x_{i+1} is one of the numbers $2x_i$, $2x_i - 1$, $2x_i - n$, $2x_i - n - 1$. Hereby, we use the cyclic subscript convention, so that x_{n+1} means x_1.

7 Among a group of 120 people, some pairs are friends. A *weak quartet* is a set of four people containing exactly one pair of friends. What is the maximum possible number of weak quartets ?