IMO Winter Camp 2010 Warm Up Problems

Algebra

A1 Let $x, y, z \in \mathbb{R}$ be non-negative real numbers such that $0 \leq x, y, z \leq 1$. Find the maximum possible value of

$$x + y + z - xy - yz - zx.$$

Determine all triples (x, y, z) for which this maximum is attained.

A2 For any positive integer k, prove that

$$2k^2 + k < \sum_{j=1}^{2k} \sqrt{k^2 + j} < 2k^2 + k + \frac{1}{2}.$$

A3 Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{1}{1+ab} + \frac{1}{1+bc} + \frac{1}{1+ca} \geq \frac{3}{2}.$$

A4 $P(x)$ is a polynomial of odd degree with real coefficients. Show that the equation $P(P(x)) = 0$ has at least as many distinct real roots, as the equation $P(x) = 0$.

A5 Find all functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that $\forall x, y, z \in \mathbb{R}$, we have that if $x^3 + f(y) \cdot x + f(z) = 0$, then $f(x)^3 + y \cdot f(x) + z = 0$.

Number Theory

N1 Let n be a positive integer. The sum of the positive divisors of n is s. Prove that the sum of the reciprocal of these divisors is $\frac{s}{n}$.

N2 A rational number x is written on a blackboard. In each step, you erase x and replace it with either $x+1$ or $-\frac{1}{x}$. (If $x = 0$, you must choose $x+1$). Prove that for any rational number p, if p currently appears on the blackboard, then you can make 0 appear after a finite number of steps.

N3 Find all triples of positive integers (a, b, c) such that $a^2 + 2^{b+1} = 3^c$.

N4 Find all positive integers that can be written in the form

$$\frac{m^2 + n^2 + 1}{mn},$$

for all positive integers m, n.

N5 Find all prime numbers p such that the following statement is true: there are exactly p ordered pairs of integers (x, y) such that $0 \leq x, y < p$ and $y^2 \equiv x^3 - x \bmod p$.

Geometry

G1 A circle $\mathcal{C}$ and a point P are given on the same plane. Given any point Q on the circumference of $\mathcal{C}$, let M be the midpoint of PQ. Find the locus of point M, i.e. find all possible locations for point M.

G2 Let A be a point outside of a circle $\mathcal{C}$. Two lines pass through A, one intersecting $\mathcal{C}$ at B, C, with B closer to A than C, and the other intersecting $\mathcal{C}$ at D, E, with D closer to A than E. The line passing through D parallel to AC intersects $\mathcal{C}$ a second time at F and the line AF intersects $\mathcal{C}$ a second time at G. Let $M = EG \cap AC$. Prove that

$$\frac{1}{|AM|} = \frac{1}{|AB|} + \frac{1}{|AC|}.$$

G3 Let ABC be a triangle and D be a point on side BC. The internal angle bisector of $\angle ADB$ and that of $\angle ACB$ intersect at P. The internal angle bisector of $\angle ADC$ and that of $\angle ABC$ intersect at Q. Let M be the midpoint of PQ. Prove that $|MA| < |MD|$.

G4 A convex quadrilateral $ABCD$ has $|AD| = |CD|$ and $\angle DAB = \angle ABC < 90°$. The line through D and the midpoint of BC intersects line AB at point E. Prove that $\angle BEC = \angle DAC$.

G5 Let ABC be a triangle with $|AB| > |AC|$. Let its incircle touch side BC at E. Let AE intersect this incircle again at D. Let F be the second point on AE such that $|CE| = |CF|$. Let CF intersect BD at G. Prove that $|CF| = |FG|$.

Combinatorics

C1 Find the number of subsets of $\{1, 2, \cdots, 10\}$ that contain its own size. For example, the set $(1, 3, 6)$ has 3 elements and contains 3.

C2 A sequence of non-negative integers is defined by $G_0 = 0, G_1 = 0$ and $G_n = G_{n-1} + G_{n-2} + 1$ for every $n \geq 2$. Prove that for every positive integer m, there exists a positive integer a such that G_a, G_{a+1} are both divisible by m.

C3 A set S of ≥ 3 points in a plane has the property that no three points are collinear, and if A, B, C are three distinct points in S, then the circumcentre of ΔABC is also in S. Prove that S is infinite.

C4 Let n, k be positive even integers. A survey was done on n people where on each of k days, each person was asked whether he/she was happy on that day and answered either "yes" or "no". It turned out that on any two distinct days, exactly half of the people gave different answers on the two days. Prove that there were at most $n - \frac{n}{k}$ people who answered "yes" the same number of times he/she answered "no" over the k days.

C5 There are $n \geq 5$ people in a room, where each pair is classified as friends or strangers. No three people are mutually friends. There also exist an odd number of people $P_1, \cdots, P_m$ such that P_i is friends with P_{i+1} for all $i \in \{1, \cdots, m\}$, where the indices are taken modulo m. Prove that there exists one person who is friends with at most $2n/5$ people.

IMO Winter Camp 2010 Warmup Solutions

Algebra

A1 Let $x, y, z \in \mathbb{R}$ be non-negative real numbers such that $0 \leq x, y, z \leq 1$. Find the maximum possible value of

$$x + y + z - xy - yz - zx.$$

Determine all triples (x, y, z) for which this maximum is attained.

Solution: The maximum possible value is 1 and is attained when $(x, y, z) = (1, t, 0)$ for any $0 \leq t \leq 1$, or any of its permutation solutions.

Note that

$$x + y + z - xy - yz - zx = -(1 - x)(1 - y)(1 - z) + 1 - xyz \leq 1.$$

Equality holds when at least one of x, y, z is equal to 1 and at least one of x, y, z is equal to 0. Clearly, this is possible. Therefore, the maximum possible value of $x + y + z - xy - yz - zx$ is 1 and is attained when $(x, y, z) = (1, t, 0)$ for any $0 \leq t \leq 1$, and any of its permutation solutions. $\square$

Source: Diamonds by Tran Phuong

Comments: When the expressions $x + y + z, xy + yz + zx, xyz$ appear, you should always consider polynomials of degree three and/or terms of the form $(c - x)(c - y)(c - z)$ for some constant c. This may even help you factor terms in other similar problems.

A2 For any positive integer k, prove that

$$2k^2 + k < \sum_{j=1}^{2k} \sqrt{k^2 + j} < 2k^2 + k + \frac{1}{2}.$$

Solutions: By subtracting $2k^2$ from all three terms, it suffices to show that

$$k < \sum_{j=1}^{2k} (\sqrt{k^2 + j} - k) < k + \frac{1}{2}.$$

Note that

$$\sqrt{k^2 + j} - k = \frac{j}{\sqrt{k^2 + j} + k},$$

and

$$\frac{j}{2k + 1} < \frac{j}{\sqrt{k^2 + j} + k} < \frac{j}{2k}.$$

Therefore,

$$\sum_{j=1}^{2k} \frac{j}{2k + 1} < \sum_{j=1}^{2k} \frac{j}{\sqrt{k^2 + j} + k} < \sum_{j=1}^{2k} \frac{j}{2k}.$$

Hence,

$$\frac{2k(2k+1)}{2(2k+1)} < \sum_{j=1}^{2k} \frac{j}{\sqrt{k^2+j}+k} < \frac{2k(2k+1)}{2(2k)}.$$

Since $\frac{j}{\sqrt{k^2+j}+k} = \sqrt{k^2+j} - k$, we conclude that

$$k < \sum_{j=1}^{2k} \left(\sqrt{k^2+j} - k\right) < k + \frac{1}{2}.$$

This solves the problem. $\square$

Source: Mongolian Mathematical Olympiad 2009

Comments: The appearance of the square root suggests that you rationalize the numerator of some expression. It should be natural for you to consider the term $\sqrt{k^2+j} - k$, since $k < \sqrt{k^2+j} < k+1$ for $1 \le j \le 2k$. Bounding the term $j/(\sqrt{k^2+j}+k)$ between $\frac{j}{2k+1}$ and $\frac{j}{2k}$ should also be natural since you are looking for rational lower and upper bounds of $\frac{j}{\sqrt{k^2+j}+k}$. $\square$

A3 Let a, b, c be positive real numbers such that $a^2+b^2+c^2 = 3$. Prove that

$$\frac{1}{1+ab} + \frac{1}{1+bc} + \frac{1}{1+ca} \ge \frac{3}{2}.$$

Solution 1: Note that $ab \le \frac{a^2+b^2}{2}$. Therefore, $\frac{1}{1+ab} \ge \frac{1}{1+\frac{a^2+b^2}{2}}$ for all $a, b \ge 0$. Therefore, by AM-HM inequality, we have

$$\sum_{cyc} \frac{1}{1+ab} \ge \sum_{cyc} \frac{1}{1+\frac{a^2+b^2}{2}} \ge \frac{9}{\sum_{cyc}(1+\frac{a^2+b^2}{2})} = \frac{9}{3+a^2+b^2+c^2} = \frac{3}{2},$$

as desired. $\square$

Solution 2: By adding the terms on the left-hand side and cross multiplying, the inequality becomes equivalent to

$$3 + (ab+bc+ca) \ge abc(a+b+c) + 3a^2b^2c^2.$$

By homogenizing (i.e. using the condition $a^2+b^2+c^2 = 3$ to make every term have the same degree), we have

$$3 \cdot \frac{(a^2+b^2+c^2)^3}{27} + \frac{(a^2+b^2+c^2)^2(ab+bc+ca)}{9} \ge \frac{(a^2+b^2+c^2)abc(a+b+c)}{3} + 3a^2b^2c^2.$$

By clearing denominators, this inequality becomes equivalent to

$$(a^2+b^2+c^2)^3 + (a^2+b^2+c^2)^2(ab+bc+ca) \ge 3(a^2+b^2+c^2)abc(a+b+c) + 27a^2b^2c^2.$$

For any non-negative real numbers x, y, z, let

$$[x, y, z] = \sum_{sym} a^x b^y c^z.$$

We recall Muirhead's Majorization inequality, which states that if x, y, z, u, v, w are non-negative real numbers such that $x \ge y \ge z, u \ge v \ge w, x + y + z = u + v + w$, $x \ge u, x + y \ge u + v$, then $[x, y, z] \ge [u, v, w]$. $[x, y, z]$ is said to *majorize* $[u, v, w]$. Note that

$$(a^2 + b^2 + c^2)^3 = \frac{1}{2}[6, 0, 0] + 3[4, 2, 0] + [2, 2, 2],$$

$$(a^2 + b^2 + c^2)^2(ab + bc + ca) = [5, 1, 0] + \frac{1}{2}[4, 1, 1] + [3, 3, 0] + 2[3, 2, 1],$$

$$3(a^2 + b^2 + c^2)abc(a + b + c) = \frac{3}{2}[4, 1, 1] + 3[3, 2, 1],$$

$$27a^2b^2c^2 = \frac{9}{2}[2, 2, 2].$$

Hence, after cancellation of terms, the previous inequality becomes equivalent to

$$\frac{1}{2}[6, 0, 0] + 3[4, 2, 0] + [5, 1, 0] + [3, 3, 0] \ge [4, 1, 1] + [3, 2, 1] + \frac{7}{2}[2, 2, 2].$$

By Muirhead's inequality, $[5, 1, 0] \ge [4, 1, 1], [3, 3, 0] \ge [3, 2, 1], \frac{1}{2}[6, 0, 0] + 3[4, 2, 0] \ge \frac{1}{2}[2, 2, 2] + 3[2, 2, 2] = \frac{7}{2}[2, 2, 2]$. Hence, the inequality holds. $\square$

Source: Belarus Mathematical Olympiad 1999

Comments: You should always be able to solve any three-variable symmetric inequality, that can be homogenized, has equality case $a = b = c$ and/or $a = b, c = 0$ and its permutation solutions, and whose variables are non-negative real numbers, using Muirhead's and Schur's inequality. Muirhead's inequality states that $[x, y, z] \ge [u, v, w]$ whenever $[x, y, z]$ majorizes $[u, v, w]$. An equivalent version of Schur's inequality is $[x + 2, 0, 0] + [x, 2, 0] \ge 2[x + 1, 1, 1]$. For proofs of these inequalities, see Chapter 3.2 of [2].

You may be thinking that solving problems in this manner is not elegant and can be considered ugly. But the truth of the matter is that a lot of mathematics is initially done by getting your hands dirty and working with cumbersome expressions. Muirhead's and Schur's inequality is in the arsenal of every top math olympian in the world and should always be used as a last resort to solve inequalities of this type. To test your ability to use these inequalities, do try the following problem.

Exercise: Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \ge a^2 + b^2 + c^2.$$

A4 $P(x)$ is a polynomial of odd degree with real coefficients. Show that the equation $P(P(x)) = 0$ has at least as many distinct real roots, as the equation $P(x) = 0$.

Solution: Let $x_1, \cdots, x_n$ be all of the distinct roots of the equation $P(x) = 0$. We want to show that $P(P(x)) = 0$ has at least n distinct real roots.

For each $i = 1, \cdots, n$, consider the equation $P(x) = x_i$. It has at least one real root a_i since P is a polynomial of odd degree. Now, for $i, j \in \{1, 2, \cdots, n\}, i \ne j$, if $a_i = a_j$, then $P(a_i) = P(a_j)$. Therefore,

$x_i = x_j$, which is impossible since $x_1, \cdots, x_n$ are pairwise distinct. Therefore, the a_i's are also pairwise distinct. But each a_i is a solution to $P(P(x)) = 0$, since $P(P(a_i)) = P(x_i) = 0$. Hence the equation $P(P(x)) = 0$ has at least n distinct real roots $a_1, a_2, ..., a_n$. The result follows. $\square$

Source: Russian Mathematical Olympiad 2000

Comments: When polynomials in question have certain properties such as having odd degree, a question you should ask yourself is what are the differences between polynomials of odd degree and those of even degree? The answer will often give you clues as to how to approach a problem. If you have not used the fact that the polynomial has odd degree, you may not be approaching the problem in a correct way.

A5 Find all functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that $\forall x, y, z \in \mathbb{R}$, we have if $x^3 + f(y) \cdot x + f(z) = 0$, then $f(x)^3 + y \cdot f(x) + z = 0$.

Solution: We will call $x^3 + f(y) \cdot x + f(z) = 0$ relation (A) and $f(x)^3 + y \cdot f(x) + z = 0$ relation (B).

We first prove that f is surjective. Let m be an arbitrary real number. We will show that $f(x) = m$ for some $x \in \mathbb{R}$. Since (A) is a cubic polynomial in x, given any fixed $y, z \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that (x, y, z) satisfies (A). We choose $y = 1, z = -m^3 - m$. We can choose $x \in \mathbb{R}$ such that (x, y, z) satisfies (A). Therefore, (x, y, z) satisfies (B), i.e. $f(x)^3 + f(x) - (m^3 + m) = 0$. This implies that $(f(x) - m)(f(x)^2 + m \cdot f(x) + m^2 + 1) = 0$. The latter factor, as a quadratic equation in $f(x)$, has negative discriminant, hence cannot be zero. Therefore, $f(x) = m$. This proves that f is surjective.

We now show that $f(x) = 0$ if and only if $x = 0$. Since f is surjective, $f(r) = 0$ for some $r \in \mathbb{R}$. Note that $(x, y, z) = (0, a, r)$ satisfies (A) for any choice of $a \in \mathbb{R}$. Hence, $(0, a, r)$ satisfies (B), i.e. $f(0)^3 + af(0) + r = 0$ holds for all $a \in \mathbb{R}$. Therefore, $af(0) = r - f(0)^3$ for all $a \in \mathbb{R}$. The right hand side is a constant. Therefore, $f(0) = 0$. This implies $r = 0$. Hence, $f(x) = 0$ if and only if $x = 0$.

We now show that f is injective. Assume $f(a) = f(b)$ for some $a, b \in \mathbb{R}$. Take any $z \neq 0$. Then $f(z) \neq 0$. There exists x for which $x^3 + f(a)x + f(z) = 0$. Note that $x \neq 0$, since $f(z) \neq 0$. Then (x, a, z) and (x, b, z) satisfy (A), since $f(a) = f(b)$ and $x^3 + f(a)x + f(z) = 0$. Therefore, $(x, a, z), (x, b, z)$ satisfy (B), i.e. $x^3 + af(x) + z = x^3 + bf(x) + z = 0$, hence $(a - b)f(x) = 0$. But $x \neq 0$, implying $f(x) \neq 0$. Hence $a = b$. Therefore, f is injective.

We now show that (A) and (B) are equivalent, i.e. (x, y, z) satisfies (A) if and only if (x, y, z) satisfies (B). From the condition given in the problem, (x, y, z) satisfies (A) implies (x, y, z) satisfies (B). To prove the converse, let $x, y, z \in \mathbb{R}$ be a triple such that (x, y, z) satisfies (B), i.e. we have have $f(x)^3 + y \cdot f(x) + z = 0$. Let $z' \in \mathbb{R}$ such that (x, y, z') satisfies (A). This is possible since f is surjective. Therefore, (x, y, z') satisfies (B), i.e. $z' = -f(x)^3 - yf(x)$. But this term is also equal to z. Therefore, $z' = z$, implying (x, y, z) satisfies (A). Therefore, (A) and (B) are equivalent.

Take (x, y, z) such that $x = 1, f(y) = -1, z = 0$. Choosing such a y is possible since f is surjective. Then (x, y, z) satisfies (A). Hence, $f(1)^3 + yf(1) = 0$. Therefore, $y = -f(1)^2$, which implies $f(-f(1)^2) = -1$, by the definition of y. Take (x, y, z) such that $x = 1, y = 0, f(z) = -1$. Then (x, y, z) satisfies (A). Hence, $f(1)^3 + z = 0$, implying $z = -(f(1))^3$. Therefore, $f(-f(1)^3) = -1$. But f is injective and $f(z) = 1$.

Therefore, $-f(1)^2 = -f(1)^3$, or equivalently, $f(1)^2(f(1)-1) = 0$. Since f is injective, $f(1) \neq f(0) = 0$. Therefore, $f(1) = 1$.

Take $x = 1, z = -y-1$. Since $f(1) = 1$, (x, y, z) satisfies (B). Since (A) and (B) are equivalent, (x, y, z) satisfies (A), i.e.

$$f(-y-1) = -f(y) - 1, \forall y \in \mathbb{R}. \tag{1}$$

Substitute $y = 0$ into (1) yields

$$f(-1) = -1. \tag{2}$$

Take $x = -1, z = y+1$. Since $f(-1) = -1$, (x, y, z) satisfies (B), which implies (x, y, z) satisfies (A), i.e. $f(y+1) = f(y) + 1, \forall y \in \mathbb{R}$. Inductively, we can show that

$$f(y+k) = f(y) + k, \forall y \in \mathbb{R}, k \in \mathbb{Z}. \tag{3}$$

Because $f(0) = 0$, from (3), we have that

$$f(x) = x, \forall x \in \mathbb{Z}. \tag{4}$$

From (1) and (3), we get that $f(-y-1) = -f(y) - 1 = -(f(y+1)-1) - 1 = -f(y+1)$. Therefore,

$$f(y) = f(-y), \forall y \in \mathbb{R}. \tag{5}$$

Let x, y be any fixed real numbers and let $z = -f(x)^3 - yf(x)$. Then (x, y, z) satisfies (A). Hence, (x, y, z) satisfies (B), i.e. $x^3 + f(y)x + f(z) = 0$. Therefore, $f(z) = -x^3 - f(y)x$. By the definition of z, we now have

$$f(-f(x)^3 - yf(x)) = -x^3 - f(y)x, \forall x, y \in \mathbb{R}. \tag{6}$$

Using (5), we get that

$$f(f(x)^3 + yf(x)) = x^3 + f(y)x, \forall x, y \in \mathbb{R}. \tag{7}$$

If x is an integer, then by (4), $f(x)$ and x^3 are also integers. Therefore, by (3) and (7), we have $x^3 + xf(y) = f(f(x)^3 + yf(x)) = f(x)^3 + f(yf(x)) = x^3 + f(yx)$. (We substitute $y \leftarrow yf(x)$ and $k \leftarrow f(x)^3$ into (3) for the second last assertion and the fact that $f(x) = x$ for all $x \in \mathbb{Z}$, for the last assertion.) Therefore,

$$xf(y) = f(yx), \forall x \in \mathbb{Z}, y \in \mathbb{R}. \tag{8}$$

Now, if m, n are integers with $n \neq 0$, then by (8), we have

$$f\left(x + \frac{m}{n}\right) = f\left(\frac{nx+m}{n}\right) = \frac{f(nx+m)}{n} = \frac{nf(x)+m}{n} = f(x) + \frac{m}{n}.$$

Hence,

$$f(x+q) = f(x) + q, \forall x \in \mathbb{R}, q \in \mathbb{Q}. \tag{9}$$

Substituting $x = 0$ into (9) gives us

$$f(q) = q, \forall q \in \mathbb{Q}. \tag{10}$$

Substituting $y = -f(x)^2$ into (7) gives us $0 = f(0) = -x^3 - f(-f(x)^2)x$. Therefore, $f(-f(x)^2) = -x^2$. By (5), we have

$$f(f(x)^2) = x^2, \forall x \in \mathbb{R}. \tag{11}$$

Let $c > 0$ be an arbitrary real number. Then since f is surjective, there exists $d \in \mathbb{R}$ such that $c = f(d)^2$. Since $c \neq 0$, $f(0) = 0$ and f is injective, $d \neq 0$. Therefore, by (11), we have $f(c) = f(f(d)^2) = d^2 > 0$. Hence,

$$c > 0 \Rightarrow f(c) > 0, \forall c \in \mathbb{R} \tag{12}$$

Assume for some x, we have $f(x) < x$. Then there exists a rational number r for which $f(x) < r < x$. Then by (12), we have $f(x - r) > 0$ and by (9), we have $r > f(x) = f(r + (x - r)) = r + f(x - r) > r$. This is a contradiction. So we cannot have $f(x) < x$. Similarly, we cannot have $f(x) > x$. So the only possibility for f is $f(x) = x$ for all $x \in \mathbb{R}$. It is easy to check $f(x) = x$ satisfies the conditions of the problem and so is the only solution.

Source: German Team Selection Test 2009

Comments: This problem is difficult in the sense that the solution requires many steps. However, every step demonstrates an important technique that you should know to solve functional equations.

The first thing you should almost always do is to find solutions to the functional equation by inspection. The most common ones are $f(x) = c$ for some constant c, $f(x) = x + c$ for some constant c, quadratic equations and $f(x) = \frac{1}{x^d}$ for some integer d.

You next want to establish properties of the functional equation, i.e. what is $f(0)$? If $f(x) = 0$, does it mean $x = 0$? Is f one-to-one? Is f surjective? (Don't do the latter two questions if one of the solutions to the functional equation is, say, $f(x) = x^2$. You already know that this function is neither one-to-one or surjective over the reals.)

Another technique is solving the problem over the rationals first, and then use detailed (and careful) continuity arguments to solve the problem over all of the reals.

Number Theory

N1 Let n be a positive integer. The sum of the positive divisors of n is s. Prove that the sum of the reciprocal of these divisors is $\frac{s}{n}$.

Solution: Let $d_1, \cdots, d_t$ be the positive divisors of n with $1 = d_1 < d_2 < \cdots < d_t = n$. Note that $n = d_i d_{t+1-i}$ for each i. Therefore, $\frac{1}{d_i} = \frac{d_{t+1-i}}{n}$. Hence,

$$\sum_{i=1}^{t} \frac{1}{d_i} = \sum_{i=1}^{t} \frac{d_{t+1-i}}{n} = \frac{1}{n} \sum_{i=1}^{t} d_{t+1-i} = \frac{1}{n} \sum_{i=1}^{t} d_i = \frac{s}{n}. \square$$

Source: Original

Comments: Always remember that the positive divisors of n come in pairs as shown in this solution. The exception is when n is a perfect square. In this case, $\sqrt{n}$ is paired with itself.

N2 A rational number x is written on a blackboard. In each step, you erase x and replace it with either $x+1$ or $-\frac{1}{x}$. (If $x = 0$, you must replace x with $x+1$). Prove that for any rational number p, if p currently appears on the blackboard, then you can make 0 appear in a finite number of steps.

Solution: By adding 1 repeatedly to p (which is a legal step), we may assume that $p \geq 0$. If $p = 0$, we are already done. Let $p = \frac{m}{n}$, where $m, n > 0$. I claim that after a finite number of steps, you can obtain a rational number with a non-negative numerator non-negative which is strictly smaller than m. Let $n = qm - r$ where $0 \leq r < m$. We erase $\frac{m}{n}$ and replace it with $\frac{-n}{m}$. By adding 1 to this q times, we get a fraction $\frac{r}{m}$ and $0 \leq r < m$. We obtain a number with a non-negative numerator which is smaller than m. Repeating this procedure gives us fractions with strictly smaller non-negative numerators. Hence, we can eventually obtain a rational number with numerator 0, which is the number zero. $\square$

Source: John Conway's Talk on Knot Theory

Comments: This is motivated by the Euclidean algorithm and using a general technique in number theory called descent. A strategy for this type of problem generally involves solving this problem for certain values of p with small numerator and denominator. It should be pretty natural to try to lower the numerator and/or denominator of the number on the board. I will leave as an exercise that starting at 0, you can make p appear after a finite number of steps.

Another note is that we used a variant of the division algorithm. Instead of writing $n = qm + r$ with $0 \leq r < m$, which is the more traditional method, we wrote it as $n = qm - r$ with $0 \leq r < m$ to make the problem easier. The other useful variant is writing n as $n = qm + r$ where $-m/2 < r \leq m/2$.

N3 Find all triples of positive integers (a, b, c) such that $a^2 + 2^{b+1} = 3^c$.

Solution: The answers are $(1, 2, 2)$ and $(7, 4, 4)$.

Since 2^{b+1} is even and 3^c is odd, a^2 is odd. Therefore, $a^2 \equiv 1 \mod 4$. Since $2^{b+1} \equiv 0 \mod 4$, $3^c \equiv 1 \mod 4$. This implies $3^c \equiv 1 \mod 4$. Hence, $(-1)^c \equiv 1 \mod 4$. We conclude that c is even. Therefore,

$$2^{b+1} = (3^{\frac{c}{2}} - a)(3^{\frac{c}{2}} + a).$$

Hence, $3^{\frac{c}{2}} - a$ and $3^{\frac{c}{2}} + a$ are both powers of 2. Since a is odd, both of these terms are even, so none of these terms are 1. Hence, we can let s, t be positive integers such that $3^{\frac{c}{2}} - a = 2^s$ and $3^{\frac{c}{2}} + a = 2^t$. Subtracting the former from the latter yields $2a = 2^t - 2^s$. Hence, $a = 2^{t-1} - 2^{s-1}$. Since a is odd, $s = 1$. Therefore, $a = 2^{t-1} - 1$. Substituting this into $3^{\frac{c}{2}} + a = 2^t$ yields $3^{\frac{c}{2}} + 2^{t-1} - 1 = 2^t$, or equivalently,

$$3^{\frac{c}{2}} - 1 = 2^{t-1}.$$

If $t = 1$, then $3^{\frac{c}{2}} = 2$, which has no integer solutions. If $t = 2$, then $3^{\frac{c}{2}} - 1 = 2$. Hence, $c = 2$, $a = 2^{t-1} - 1 = 2^{2-1} - 1 = 1$. Therefore, $1 + 2^{b+1} = 3^2$, implying $b = 2$. Hence, $(1, 2, 2)$ is a solution to the equation and is easily verified to be a solution. Otherwise, $t > 2$, implying $4 \mid 2^{t-1}$. Hence, $3^{\frac{c}{2}} \equiv 1 \mod 4$, which implies $(-1)^{\frac{c}{2}} \equiv 1 \mod 4$. Hence, $\frac{c}{2}$ is even. Therefore,

$$(3^{\frac{c}{4}} - 1)(3^{\frac{c}{4}} + 1) = 2^{t-1}.$$

From this, we have that both $3^{\frac{c}{4}} - 1, 3^{\frac{c}{4}} + 1$ are powers of 2 and are consecutive even integers. This is only possible when the smaller integer is 2 and the larger integer is 4. Therefore, $3^{\frac{c}{4}} - 1 = 2$, implying $c = 4$. Therefore, $3^2 - 1 = 2^{t-1} = a + 1$, or equivalently, $a = 7$. Finally, $7^2 + 2^{b+1} = 3^4$, implying $b = 4$. Therefore, $(7, 4, 4)$ is a solution to the equation and is easily verified to be a solution. $\square$

Source: Italian Team Selection Test 2008

Comments: One technique to solve Diophantine equations is consider reduce everything modulo a certain positive integer. The existence of a square and a power of 2 suggests that mod 4 and mod 8 are possibilities. Problems like this test your comfort level with divisibility problems. You should always try to find as many solutions by inspection as you can as a first step. (However, I do not expect you to initially find the solution $(7, 4, 4)$.)

Try to use modular arithmetic to solve the following problem.

Exercise: Let m, n be positive integers. Find the minimum possible value of $|12^m - 5^n|$.

N4 Find all positive integers that can be written in the form

$$\frac{m^2 + n^2 + 1}{mn},$$

for all positive integers m, n.

Solution: The answer is 3.

Let $S = \{(m, n) \mid mn \text{ divides } m^2 + n^2 + 1, m, n \in \mathbb{N}\}$ and

$$f(m, n) = \frac{m^2 + n^2 + 1}{mn}.$$

Note that $(m, n) \in S$ if and only if $(n, m) \in S$.

If $m = 1$, then n divides $n^2 + 2$. This implies $n \mid 2$. Therefore, $n = 1$ or $n = 2$. Both yield the property that $\frac{m^2+n^2+1}{mn} = 3$. This holds similar for $n = 1$. Therefore, if $m = 1$ or $n = 1$, then $f(m, n) = 3$.

We next consider the case $m = n$. If $(m, n) \in S$ and $m = n$, then m^2 divides $2m^2 + 1$. Therefore, $m^2 \mid 1$. Therefore, $m = 1$. Hence, if $(m, m) \in S$, then $m = 1$. This also yields that $f(m, n) = 3$.

Now, suppose $(m, n) \in S$ with $1 < m < n$. Suppose $k = \frac{m^2+n^2+1}{mn}$. I claim there exists $0 < n' < m$ such that $(m, n') \in S$. Rewriting the equation gives us

$$n^2 - kmn + (m^2 + 1) = 0.$$

Hence, n is an integer root of the equation $x^2 - kmx + (m^2 + 1) = 0$. Let n' be the other root. Since $n + n' = km$, which is an integer, n' is an integer. Since $nn' = m^2 + 1 > 0$ and $n > 0$, $n' > 0$. Note that $f(m, n) = f(m, n')$, by rewriting the quadratic equation back to the form $f(m, n')$. Therefore, $(m, n') \in S$. I claim that $n' < m$. Suppose on the contrary that $n' \geq m$. Since $(m, n') \in S$ and $n, m > 1$, $n' \neq m$ by the argument in the case $m = n$. Suppose $n' > m$. Let $n = m + a, n' = m + b$ for some positive integers a, b. Then $nn' = (m+a)(m+b) = m^2+1$, since the product of the roots of $x^2 - kmx + (m^2+1)$ is m^2+1. Hence, $m(a + b) + ab = 1$, which is impossible since $a, b \geq 1$. Therefore, $n' < m$. This proves the claim.

Hence, $(m, n') \in S$, which implies $(n', m) \in S$. Note that $k = f(m, n) = f(m, n') = f(n', m)$. If $n' = 1$, then from an earlier argument, $k = 3$. Otherwise, we repeat this process for $1 < n' < m$. Every time we perform the previous paragraph, $\min\{m, n\}$ strictly decreases and remains positive and $f(m, n)$ remains constant. Hence, eventually, at least one of $m, n = 1$. By an earlier argument, this results in $k = 3$. Since $f(m, n)$ remains constant at every step, we conclude that $f(m, n) = 3$. This completes the problem. □

Source: Arthur Engel's Problem-Solving Strategies

Comments: The technique used in this problem is called *Vieta-jumping* or *descent*. It is extremely helpful in solving Diophantine equations with quadratic terms. By letting S be a solution set, you can claim statements of the form if $(a, b) \in S$, then $(c, d) \in S$ for certain choices of a, b, c, d. By doing so, you can create a series of solutions for which $\min\{m, n\}$ is decreasing. Try to solve the following problem from the 1988 International Mathematical Olympiad using *Vieta-jumping*.

Exercise: Suppose m, n are positive integers such that $mn + 1$ divides $m^2 + n^2$. Prove that

$$\frac{m^2 + n^2}{mn + 1}$$

is a perfect square.

N5 Find all prime numbers p such that the following statement is true: there are exactly p ordered pairs of integers (x, y) such that $0 \leq x, y < p$ and $y^2 \equiv x^3 - x \bmod p$.

Solution: The answers are $p = 2$ and all primes p such that $p \equiv 3 \bmod 4$.

If $p = 2$, then we can check that the only pairs that work are $(0,0)$ and $(1,0)$. There are exactly two solutions. Hence $p = 2$ is an answer.

We may assume now that p is odd. For $a \in \{0, 1, \cdots, p-1\}$, we call a a quadratic residue modulo p if and only if $n^2 \equiv a \bmod p$ for some integer n. Otherwise, we call a a quadratic non-residue. We recall that the product of two quadratic residues is a quadratic residue, and the product of two quadratic non-residues is a quadratic residue. Also, the product of a quadratic residue and a quadratic non-residue, is a quadratic non-residue. Also recall that -1 is a quadratic residue of p if and only if $p \equiv 1 \bmod 4$ or $p = 2$. (*)

Note that $(-1,0), (0,0), (1,0)$ are the only solutions involving $y = 0$ and $x \in \{-1, 0, 1\}$. There are three such solutions. We will henceforth only consider the cases when $x \in \{2, \cdots, p-2\}$ and $y \neq 0$.

We will call $x \in \{2, \cdots, p-2\}$ solvable if there exists y such that $y^2 \equiv x^3 - x \bmod p$.

If $p \equiv 1 \bmod 4$, then note that $x^3 - x = -1 \cdot ((-x)^3 - (-x))$. By the arguments made in (*), since -1 is a quadratic residue mod 4, x is solvable if and only if $-x$ is solvable. For each $x \in \{2, \cdots, p-2\}$ which is solvable, there are exactly two values of y such that (x, y) is a solution to the original equation. This is because (x, y) is a solution if and only if $(x, -y)$ is a solution. Note that $x \not\equiv -x$ and $y \not\equiv -y \bmod p$ since p is odd and $y \neq 0$. Hence the number of values of $x \in \{2, \cdots, p-2\}$ which is solvable is even and each solvable x yields two values of y. Hence, the number of ordered pair solutions with $x \in \{2, \cdots, p-2\}$ is divisible by 4. Including the three aforementioned solutions, the number of solutions (x, y) is congruent to 3 mod 4. But $p \equiv 1 \bmod 4$. Therefore, the number of solutions (x, y) cannot equal to p.

If $p \equiv 3 \bmod 4$, again, note that $x^3 - x = -1 \cdot ((-x)^3 - (-x))$. Since -1 is a quadratic non-residue mod p and the product of a quadratic residue and a quadratic non-residue is a quadratic non-residue, x is solvable if and only if $-x$ is not solvable. Hence, exactly half of the value of $x \in \{2, \cdots, p-2\}$ are solvable and using a similar argument as in the case $p \equiv 1 \bmod 4$, each solvable x yields two solutions. Therefore, there are $\frac{p-3}{2} \cdot 2 = p - 3$ solutions. Adding the aforementioned three solutions, there are p solutions total. Hence, $p \equiv 3 \bmod 4$ is a solution to the problem. $\square$

Source: Turkish Team Selection Test 2005

Comments: The facts in the paragraph marked (*) are important topics to learn for number theory, especially the fact that $n^2 \equiv -1 \bmod p$ has a solution if and only if $p = 2$ or $p \equiv 1 \bmod 4$. Please read Theorem 4.9 and Chapter 7 of [3] for more details.

Another important point is that the very least you should be able to do for this problem, even without knowledge of quadratic residues and of fact (*), is to conjecture your answer for p by solving this problem for small primes p, say up to $p = 17$. The motivation behind this given solution arises from the work you do for small values of p. Only then would you be able to consider the differences between primes 1 and 3 mod 4. The least that you should be able to write, is something along the lines of "I conjecture that the solution to this problem are $p = 2$ and all primes $p \equiv 3 \bmod 4$". Of course, now that you have read and understood this solution, I will expect much more than this. :) Using fact (*), you can try the following exercise.

Exercise: Prove that there are no pairs of positive integers a, b such that $4ab - a - b$ is a perfect square.

Geometry

G1 A circle $\mathcal{C}$ and a point P are given on the same plane. Given any point Q on the circumference of $\mathcal{C}$, let M be the midpoint of PQ. Find the locus of point M, i.e. find all possible locations for point M.

Solution 1: The locus of M is exactly the dilation of $\mathcal{C}$ about point P with ratio $\frac{1}{2}$. The image is therefore the circle with radius half of that of $\mathcal{C}$ whose centre is the midpoint of P and O, where O is the centre of $\mathcal{C}$. $\square$

Solution 2: Let O be the centre of $\mathcal{C}$, r the radius of $\mathcal{C}$ and N the midpoint of PO. Since $\frac{|PM|}{|MQ|} = \frac{|PN|}{|NO|} = 1$, we have $\frac{|MN|}{|QO|} = \frac{1}{2}$. Hence, $|MN| = \frac{r}{2}$. Therefore, M lies on the circle with radius $\frac{r}{2}$ centered at N.

Conversely, let M be any point on the circle centered at N with radius $\frac{r}{2}$. Let Q be on the circumference of $\mathcal{C}$ such that QO is parallel to MN. Then $\frac{|MN|}{|QO|} = \frac{|PN|}{|PO|} = \frac{1}{2}$, which implies P, M, Q are collinear and $\frac{|PM|}{|PQ|} = \frac{1}{2}$. Therefore, M is the midpoint of PQ. $\square$

Source: Original

G2 Let A be a point outside of a circle $\mathcal{C}$. Two lines pass through A, one intersecting $\mathcal{C}$ at B, C, with B closer to A than C, and the other intersecting $\mathcal{C}$ at D, E, with D closer to A than E. The line passing through D parallel to AC intersects $\mathcal{C}$ a second time at F and the line AF intersects $\mathcal{C}$ a second time at G. Let $M = EG \cap AC$. Prove that

$$\frac{1}{|AM|} = \frac{1}{|AB|} + \frac{1}{|AC|}.$$

Solution: Note that $\angle GAM = \angle FAM = \angle AFD = \angle GFD = \angle GED$, the latter assertion following from that $DEFG$ is cyclic. Therefore, $\Delta GAM \sim \Delta AEM$. Hence, we have $\frac{|MA|}{|MG|} = \frac{|ME|}{|MA|}$, which implies $|MA|^2 = |MG| \cdot |ME| = |MB| \cdot |MC|$. The assertion follows from that $DEFG$ is cyclic. We shall call this property (*).

Note that M lies on the segment AB. Therefore, $|AB| = |MA| + |MB|$ and $|AC| = |MA| + |MC|$. Hence,

$$\frac{1}{|AM|} = \frac{1}{|AB|} + \frac{1}{|AC|} = \frac{|AB| + |AC|}{|AB| \cdot |AC|} = \frac{2|MA| + |MB| + |MC|}{(|MA| + |MB|)(|MA| + |MC|)}.$$

This equality holds if and only if $(|MA| + |MB|)(|MA| + |MC|) = 2|MA|^2 + |MA||MB| + |MA||MC| \Leftrightarrow |MB| \cdot |MC| = |MA|^2$, which is true by property (*). $\square$

Source: Romanian Team Selection Test 2006

Comments: If you are not familiar with Power of a Point, please read Chapter 6 of [1]. It is an important and powerful technique to solve certain geometry problems, especially those involving circles and tangent lines. Power of a Point is the motivation for this solution.

G3 Let ABC be a triangle and D be a point on side BC. The internal angle bisector of $\angle ADB$ and that of $\angle ACB$ intersect at P. The internal angle bisector of $\angle ADC$ and that of $\angle ABC$ intersect at Q. Let M be the midpoint of PQ. Prove that $|MA| < |MD|$.

Solution: Note that P is the excentre of $\triangle ACD$ opposite C, and Q is the excentre of $\triangle ABD$ opposite B. Therefore, $\angle PAD = \frac{180-\angle CAD}{2}$ and $\angle QAD = \frac{180-\angle BAD}{2}$.

Note also that $\angle QDP = \angle QDA + \angle ADP = \frac{1}{2}(\angle BDA + \angle CDA) = 90^o$. Therefore, $|MP| = |MQ| = |MD|$. Consider the circle centred at M passing through P, Q, D. To prove that $|MA| < |MD|$, it suffices to show that A is in the interior of the circle, i.e. $\angle QAP$ is obtuse. But

$$\angle QAP = \angle QAD + \angle PAD = \frac{180 - \angle BAD}{2} + \frac{180 - \angle CAD}{2} = 180 - \frac{\angle BAD + \angle CAD}{2} = 180 - \angle BAC$$

and this angle is greater than 90^o (and less than 180^o). Therefore, $|MA| < |MD|$. $\square$

Source: Original and similar to one on Irish Mathematical Olympiad 2005

Comments: It is important to observe excentres in a diagram. Their existence, along with incentres, often simplifies problems involving angle bisectors.

G4 A convex quadrilateral $ABCD$ has $|AD| = |CD|$ and $\angle DAB = \angle ABC < 90°$. The line through D and the midpoint of BC intersects line AB in point E. Prove that $\angle BEC = \angle DAC$.

Solution: Let F be on side AB such that $|DA| = |DF|$, which is also equal to $|DC|$. Since $\angle AFD = \angle DAF = \angle DAB = \angle ABC$, $DF||BC$. Extend DF to meet EC at G. Since $DF||BC$ and ED passes through the midpoint of BC, D is the midpoint of FG. Hence, $|DA| = |DC| = |DF| = |DG|$. Hence, $ACGF$ is cyclic. Then $\angle FAC = \angle FGC = \angle DGC = \angle BCE$. Since $\angle DAB = \angle ABC$, $\angle DAC = \angle DAB - \angle FAC = \angle ABC - \angle BCE = \angle BEC$, as desired. $\square$

Source: Bulgarian Mathematical Olympiad 1998

Comments: Construction problems in geometry are fun. Of course, there are solutions that does not involve extra constructions given in this solution. However, the construction of point F on AB is quite natural given that $|AD| = |CD|$ and $\angle DAB = \angle ABC$. You will be given three equal lengths in AD, CD, FD and that FD is parallel to AB. Many new properties of the diagram arise from this one point. Wow! :)

G5 Let ABC be a triangle with $|AB| > |AC|$. Let its incircle touch side BC at E. Let AE intersect this incircle again at D. Let F be the second point on AE such that $|CE| = |CF|$. Let CF intersect BD at G. Prove that $|CF| = |FG|$.

Solution: Let the line tangent to the incircle at D intersect BC (extended) at P. Since $|PE| = |PD|$ and $|CE| = |CF|$, $CF||PD$.

By Menelaus' Theorem on ΔBCG, we have that

$$\frac{|GD|}{|DB|} \cdot \frac{|BE|}{|EC|} \cdot \frac{|CF|}{|FG|} = 1.$$

Since $CF||PD$, $\frac{|GD|}{|DB|} = \frac{|CP|}{|PB|}$. Therefore, to prove that $|CF| = |FG|$, it suffices to prove that

$$\frac{|CP|}{|PB|} = \frac{|CE|}{|EB|}.$$

Note that the polar of P with respect to the incircle is line DE, which also contains A. Therefore, the polar of A contains P. Hence, the polar of A is XY, where X, Y are points where the incircle touches AC, AB, respectively. Hence, P lies on XY. Since $|AX| = |AY|, |BY| = |BE|, |CE| = |CX|$, we have that

$$\frac{AY}{YB} \cdot \frac{BE}{EC} \cdot \frac{CX}{XA} = 1. \tag{13}$$

Hence, AE, BX, CY are concurrent by Ceva's Theorem. Since X, Y, P are collinear, by Menelaus' Theorem on ΔABC, we have

$$\frac{AY}{YB} \cdot \frac{BP}{PC} \cdot \frac{CX}{XA} = -1. \tag{14}$$

Comparing equations (13) and (14), yields

$$\frac{|CP|}{|PB|} = \frac{|CE|}{|EB|},$$

as desired. □

Source: China Team Selection Test 2008

Comments: Ceva's and Menealus' Theorem are essential pieces of knowledge for proving concurrency and collinearity involving triangles. Please learn these theorems if you haven't done so already.

Poles and polars have come to the rescue. Those familiar with poles and polars and harmonic conjugates will realize that (E, P) divides (B, C) harmonically. To learn more about these powerful techniques, please consult Chapter 10 and 11 of [1].

Combinatorics

C1 Find the number of subsets of $\{1, 2, \cdots, 10\}$ that contain its own size. For example, the set $(1, 3, 6)$ has 3 elements and contains 3.

Solution: The answer is 512.

For each $k \in \{0, 1, \cdots, 10\}$, we want to find how many elements there are of size k. Note that $k \neq 0$ since $0 \notin \{1, \cdots, 10\}$. Every subset of size k must contain k. The other $k - 1$ elements can be anything. Therefore, there are $\binom{9}{k-1}$ subsets of size k that contains k. Therefore, the number of subsets that contain its own size is

$$\sum_{k=1}^{10} \binom{9}{k-1} = \sum_{k=0}^{9} \binom{9}{k} = 2^9 = 512.$$

$\square$

Source: Original

C2 A sequence of non-negative integers is defined by $G_0 = 0, G_1 = 0$ and $G_n = G_{n-1} + G_{n-2} + 1$ for every $n \geq 2$. Prove that for every positive integer m, there exists a positive integer a such that G_a, G_{a+1} are both divisible by m.

Solution: Consider each pair (G_t, G_{t+1}) modulo m. Since there are at m^2 pairs of integers (x, y) modulo m, the sequence (G_t, G_{t+1}) modulo m eventually repeats, by Pigeonhole Principle. It suffices to prove that $(G_a, G_{a+1}) = (0, 0) = (G_0, G_1)$ modulo m, for some positive integer a. Suppose the sequence $\{(G_t, G_{t+1}\}$ repeats the first time at index $t = a$. Then $(G_a, G_{a+1}) = (G_j, G_{j+1})$ modulo m for some $0 \leq j < t$. If $j = 0$, then $(G_a, G_{a+1}) = (0, 0)$, implying G_a, G_{a+1} are both divisible by m and we are done. Otherwise, note that the pair that appears before (x, y) is $(y, x + y - 1)$. In other words, given a pair of consecutive terms in the sequence, the previous pair of consecutive terms is determined. Hence, $(G_a, G_{a+1}) = (G_j, G_{j+1})$ implies $(G_{j-1}, G_j) = (G_{a-1}, G_a)$, contradicting that the sequence repeats the first time at index a. Therefore, G_a, G_{a+1} are both divisible by m. $\square$

Source: Estonian Mathematical Olympiad 2007

Comments: Pigeonhole Principle is a powerful tool to solve certain existence type of combinatorics problems. It is an art to determine what are the pigeons and what are the holes.

C3 A set S of ≥ 3 points in a plane has the property that no three points are collinear, and if A, B, C are three distinct points in S, then the circumcentre of ΔABC is also in S. Prove that S is infinite.

Solution: Suppose on the contrary that S is finite. Consider the convex hull $\mathcal{P}$ of S, i.e. the smallest polygon containing all of S (on the polygon's boundary and interior). Since $|S| \geq 3$, $\mathcal{P}$ contains at least three vertices. Let AB be a side of $\mathcal{P}$ and C be any other vertex of $\mathcal{P}$. Note that $\mathcal{P}$ is contained in the halfplane of AB containing C and there are no points in S in the halfplane not containing C. If $\angle ACB > 90°$, then the circumcircle of ΔACB lies in the halfplane of AB not containing C, which does not contain any point in S. Hence, the circumcircle of ΔACB is not in S. This contradicts the assumption given in the problem. Otherwise, $\angle ACB$ is acute. We let O_1 be the circumcentre of ΔACB and recursively, define O_{i+1} to be the circumcentre of ΔAO_iB. Hence, $O_i \in S$ for all $i \geq 1$. Hence, $\Delta AO_1B = 2\angle ACB$. We choose the smallest index k such that $\angle AO_kB \geq 90°$. Therefore, $\angle AO_kB = 2\angle AO_{k-1}B$. Since $\angle AO_{k-1}B \leq 90°$ by our choice of k, $\angle AO_{k-1}B < 180°$. If $\angle AO_kB = 90°$, then O_{k+1} lies on AB, contradicting the fact that no three points in S are collinear. (Note that $O_i \neq A, B$ for any i, since the circumcentre of a triangle cannot be any of the vertices of the triangle.) Otherwise, $90 < \angle AO_kB < 180°$, which implies O_{k+1} lies on the halfplane of AB not containing O_k. Hence, O_{k+1} cannot lie in S, but this contradicts the fact that $O_i \in S$ for all $i \geq 1$. Hence, S cannot be finite, implying S is infinite. □

Source: Unknown

Comments: When you are given a finite set of points in the plane in a problem, you should always consider the convex hull of the set. It is likely to simplify the problem. Try the following problem from the 1999 International Mathematical Olympiad by considering the convex hull.

Exercise: Find all finite set of points S such that for every pair of points $A, B \in S$, the perpendicular bisector of A and B is an axis of symmetry for S.

C4 Let n, k be positive even integers. A survey was done on n people where on each of k days, each person was asked whether he/she was happy on that day and answered either "yes" or "no". It turned out that on any two distinct days, exactly half of the people gave different answers on the two days. Prove that there were at most $n - \frac{n}{k}$ people who answered "yes" the same number of times he/she answered "no" over the k days.

Solution: Consider a grid with k rows and n columns. Label the days from 1 to k and the people from 1 to n. We place a 1 in the i^{th} row and j^{th} column if person j is happy on day i and a 0 otherwise. Since on any two days, exactly half of the people gave different answers on the two days, we have that for any two rows, exactly half of the columns differ. Hence, the number of pairs of squares in the same column that differ, is $\binom{k}{2} \cdot \frac{n}{2}$. For each person that answered yes the same number of times as he/she answered no, the number of pairs of squares in the person's column that differ is $(\frac{k}{2})^2$. Hence, the maximum number of people that could have answered yes the same number of times as he/she answered no, is

$$\frac{\binom{k}{2}\frac{n}{2}}{(\frac{k}{2})^2} = \frac{\frac{nk(k-1)}{4}}{\frac{k^2}{4}} = \frac{n(k-1)}{k} = n - \frac{n}{k},$$

as desired. □

Source: Iran Mathematical Olympiad 2006

Comments: An important key to solving combinatorics problem is find out what you are suppose to be counting. If you can count a certain object in two different ways, you can yield some extremely neat and powerful identities and relationships. If you take the time to solve enough combinatorics problems, it will become more natural as to what objects you should be counting. Use this technique to solve the following problem from the 1998 International Mathematical Olympiad

Exercise: In a contest, there are m candidates and n judges, where $n \geq 3$ is an odd integer. Each candidate is evaluated by each judge as either pass or fail. Suppose that each pair of judges agrees on at most k candidates. Prove that

$$\frac{k}{m} \geq \frac{n-1}{2n}.$$

C5 There are $n \geq 5$ people in a room, where each pair is classified as friends or strangers. No three people are mutually friends. There also exist an odd number of people $P_1, \cdots, P_m$ such that P_i is friends with P_{i+1} for all $i \in \{1, \cdots, m\}$, where the indices are taken modulo m. Prove that there exists one person who is friends with at most $2n/5$ people.

Solution: Given a positive integer $m \geq 3$, we will call an ordered set of m people $P_1, \cdots, P_m$ a cycle if P_i is friends with P_{i+1} for all $i \in \{1, \cdots, m\}$, where the indices are taken modulo m. If m is odd, we call a cycle an odd cycle. Given the condition in the problem, an odd cycle exists. Consider the odd cycle in the room $P_1, \cdots, P_g$ with the smallest number of people. Since no three people who are mutually friends, the odd cycle must contain at least five people, i.e. $g \geq 5$

I claim that P_i is not friends with P_j if $|i-j| \neq 1 \bmod g$. Without loss of generality, suppose P_1 is friends with P_j for some $j \neq 2, j \neq g$. Consider the set of vertices $(P_1, \cdots, P_j)$ and $(P_1, P_j, P_{j+1}, \cdots, P_g)$. These are both cycles and the sum of the the two sets is $j + (g - j + 2) = g + 2$, which is odd. Therefore, one of these cycles is odd, and smaller than g. This contradicts the minimality of g.

I claim that every person outside of the cycle is friends with at most two people inside the cycle. Suppose there exists a person A outside of the cycle who is friends with three people in the cycle, say P_i, P_j, P_k with $i < j < k$. Note that no two of P_i, P_j, P_k are friends, since otherwise, A and these two people are three mutual friends, which is not allowed. Consider the cycle drawn as a graph with P_i, P_j, P_k appearing in clockwise order. Let x be the distance of the path between P_i and P_j on the cycle not containing P_k. Define y, z analogously. Therefore, $x + y + z = g$. Since no two of P_i, P_j, P_k are friends, $x, y, z > 1$. Therefore, $\max\{x, y, z\} < g-2$. Since g is odd, at least one of x, y, z is odd. Without loss of generality, suppose x is odd. Then $AP_iP_{i+1} \cdots P_jA$ is an odd cycle of length $x+2 < g$, contradicting the minimality of g.

Therefore, every person outside of the cycle is friends with at most two people inside the cycle. Hence, the number of pairs of friends with one person in the pair in the cycle and the other person in the pair outside of the cycle, is at most $2(n-g)$. Therefore, there exists a person inside the cycle who is friends with at most $\frac{2(n-g)}{g}$ friends outside of the cycle. Since this person is friends with exactly two people inside the cycle, this person is friends with at most $\frac{2(n-g)}{g} + 2 = \frac{2n}{g} \leq \frac{2n}{5}$, since $g \geq 5$, as desired. $\square$

Source: Unknown

Comments: Knowledge in graph theory is of utmost importance to have to solve this problem. Please read [4] to learn what graphs are and their important properties.

References

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http://math.ucalgary.ca/∼atang/mathcircles/AT-GraphTheory.pdf