Sharygin Geometry Olympiad 2016

– Grade 9

– Day 1

1 The diagonals of a parallelogram $ABCD$ meet at point O. The tangent to the circumcircle of triangle BOC at O meets ray CB at point F. The circumcircle of triangle FOD meets BC for the second time at point G. Prove that $AG = AB$.

2 Let H be the orthocenter of an acute-angled triangle ABC. Point X_A lying on the tangent at H to the circumcircle of triangle BHC is such that $AH = AX_A$ and $X_A \neq H$. Points X_B, X_C are defined similarly. Prove that the triangle $X_AX_BX_C$ and the orthotriangle of ABC are similar.

3 Let O and I be the circumcenter and incenter of triangle ABC. The perpendicular from I to OI meets AB and the external bisector of angle C at points X and Y respectively. In what ratio does I divide the segment XY?

4 One hundred and one beetles are crawling in the plane. Some of the beetles are friends. Every one hundred beetles can position themselves so that two of them are friends if and only if they are at unit distance from each other. Is it always true that all one hundred and one beetles can do the same?

– Day 2

5 The center of a circle ω_2 lies on a circle ω_1. Tangents XP and XQ to ω_2 from an arbitrary point X of ω_1 (P and Q are the touching points) meet ω_1 for the second time at points R and S. Prove that the line PQ bisects the segment RS.

6 The sidelines AB and CD of a trapezoid meet at point P, and the diagonals of this trapezoid meet at point Q. Point M on the smallest base BC is such that $AM = MD$. Prove that $\angle PMB = \angle QMB$.

7 From the altitudes of an acute-angled triangle, a triangle can be composed. Prove that a triangle can be composed from the bisectors of this triangle.

8 The diagonals of a cyclic quadrilateral meet at point M. A circle ω touches segments MA and MD at points P, Q respectively and touches the circumcircle of $ABCD$ at point X. Prove that X lies on the radical axis of circles ACQ and BDP.

(Proposed by Ivan Frolov)

– Grade 10

– Day 1

1 A line parallel to the side BC of a triangle ABC meets the sides AB and AC at points P and Q, respectively. A point M is chosen inside the triangle APQ. The segments MB and MC meet the segment PQ at points E and F, respectively. Let N be the second intersection point of the circumcircles of the triangles PMF and QME. Prove that the points A, M, N are collinear.

2 Let I and I_a be the incenter and excenter (opposite vertex A) of a triangle ABC, respectively. Let A' be the point on its circumcircle opposite to A, and A_1 be the foot of the altitude from A. Prove that $\angle IA_1I_a = \angle IA'I_a$.

(Proposed by Pavel Kozhevnikov)

3 Assume that the two triangles ABC and $A'B'C'$ have the common incircle and the common circumcircle. Let a point P lie inside both the triangles. Prove that the sum of the distances from P to the sidelines of triangle ABC is equal to the sum of distances from P to the sidelines of triangle $A'B'C'$.

4 The Devil and the Man play a game. Initially, the Man pays some cash s to the Devil. Then he lists some 97 triples $\{i, j, k\}$ consisting of positive integers not exceeding 100. After that, the Devil draws some convex polygon $A_1A_2...A_{100}$ with area 100 and pays to the Man, the sum of areas of all triangles $A_iA_jA_k$. Determine the maximal value of s which guarantees that the Man receives at least as much cash as he paid.

Proposed by Nikolai Beluhov, Bulgaria

– Day 2

5 Does there exist a convex polyhedron having equal number of edges and diagonals?

(A diagonal of a polyhedron is a segment through two vertices not lying on the same face)

---

6 A triangle ABC is given. The point K is the base of the external bisector of angle A. The point M is the midpoint of the arc AC of the circumcircle. The point N on the bisector of angle C is such that $AN \parallel BM$. Prove that the points M, N, K are collinear.

(Proposed by Ilya Bogdanov)

---

7 Restore a triangle by one of its vertices, the circumcenter and the Lemoine's point.

(The Lemoine's point is the intersection point of the reflections of the medians in the correspondent angle bisectors)

---

8 Let ABC be a non-isosceles triangle, let AA_1 be its angle bisector and A_2 be the touching point of the incircle with side BC. The points B_1, B_2, C_1, C_2 are defined similarly. Let O and I be the circumcenter and the incenter of triangle ABC. Prove that the radical center of the circumcircle of the triangles $AA_1A_2, BB_1B_2, CC_1C_2$ lies on the line OI.
