IMO 2008

Madrid, Spain

Day 1 - *16 July 2008*

$\boxed{1}$ Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of BC and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1 and C_2.

Prove that six points A_1 , A_2, B_1, B_2, C_1 and C_2 are concyclic.

Author: Andrey Gavrilyuk, Russia

$\boxed{2}$ **(i)** If x, y and z are three real numbers, all different from 1, such that $xyz = 1$, then prove that $\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$. (With the $\sum$ sign for cyclic summation, this inequality could be rewritten as $\sum \frac{x^2}{(x-1)^2} \geq 1$.)

(ii) Prove that equality is achieved for infinitely many triples of rational numbers x, y and z.

Author: Walther Janous, Austria

$\boxed{3}$ Prove that there are infinitely many positive integers n such that $n^2 + 1$ has a prime divisor greater than $2n + \sqrt{2n}$.

Author: Kestutis Cesnavicius, Lithuania

Day 2 - *17 July 2008*

4 Find all functions $f : (0, \infty) \mapsto (0, \infty)$ (so f is a function from the positive real numbers) such that

$$\frac{(f(w))^2 + (f(x))^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbes w, x, y, z, satisfying $wx = yz$.

Author: Hojoo Lee, South Korea

5 Let n and k be positive integers with $k \geq n$ and $k - n$ an even number. Let $2n$ lamps labelled 1, 2, ..., $2n$ be given, each of which can be either *on* or *off*. Initially all the lamps are off. We consider sequences of steps: at each step one of the lamps is switched (from on to off or from off to on).

Let N be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps $n + 1$ through $2n$ are all off.

Let M be number of such sequences consisting of k steps, resulting in the state where lamps 1 through n are all on, and lamps $n + 1$ through $2n$ are all off, but where none of the lamps $n + 1$ through $2n$ is ever switched on.

Determine $\frac{N}{M}$.

Author: Bruno Le Floch and Ilia Smilga, France

6 Let $ABCD$ be a convex quadrilateral with BA different from BC. Denote the incircles of triangles ABC and ADC by k_1 and k_2 respectively. Suppose that there exists a circle k tangent to ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD.

Prove that the common external tangents to k_1 and k_2 intersects on k.

Author: Vladimir Shmarov, Russia