National Math Olympiad (3rd Round) 2003

1   suppose this equation: $x^2 + y^2 + z^2 = w^2$. show that the solution of this equation (if w,z have same parity) are in this form:
x=2d(XZ-YW), y=2d(XW+YZ),z=d(X^2 +Y^2 -Z^2 -W^2),w=d(X^2 +Y^2 +Z^2 +W^2)

2   assume ABCD a convex quadrilatral. P and Q are on BC and DC respectively such that angle BAP= angle DAQ .prove that [ADQ]=[ABP]
([ABC] means its area) iff the line which crosses through the orthocenters of these traingles , is perpendicular to AC.

3   assume that A is a finite subset of prime numbers, and a is an positive integer. prove that there are only finitely many positive integers m s.t: prime divisors of aˆm-1 are contained in A.

4   XOY is angle in the plane.A,B are variable point on OX,OY such that 1/OA+1/OB=1/K (k is constant).draw two circles with diameter OA and OB.prove that common external tangent to these circles is tangent to the constant circle(ditermine the radius and the locus of its center).

5   Let p be an odd prime number. Let S be the sum of all primitive roots modulo p. Show that if $p-1$ isn't squarefree (i. e., if there exist integers k and m with $k > 1$ and $p-1 = k^2m$), then $S \equiv 0 \mod p$.

If not, then what is S congruent to   mod p ?

6   let the incircle of a triangle ABC touch BC,AC,AB at A1,B1,C1 respectively. M and N are the midpoints of AB1 and AC1 respectively. MN meets A1C1 at T . draw two tangents TP and TQ through T to incircle. PQ meets MN at L and B1C1 meets PQ at K . assume I is the center of the incircle .

prove IK is parallel to AL

7 $f_1, f_2, \ldots, f_n$ are polynomials with integer coefficients. Prove there exist a reducible $g(x)$ with integer coefficients that $f_1+g, f_2+g, \ldots, f_n+g$ are irreducible.

8   A positive integer n is said to be a *perfect power* if $n = a^b$ for some integers a, b with $b > 1$. (a) Find 2004 perfect powers in arithmetic progression. (b) Prove that perfect powers cannot form an infinite arithmetic progression.

9   Does there exist an infinite set S such that for every $a, b \in S$ we have $a^2 + b^2 - ab \mid (ab)^2$.

10   let p be a prime and a and n be natural numbers such that $(p\char`\^a -1 )/ (p-1) = 2 \char`\^n$

find the number of natural divisors of na. :)

11   assume that X is a set of n number.and $0 \le k \le n$.the maximum number of permutation which acting on X st every two of them have at least k component in common,is $a_{n,k}$.and the maximum nuber of permutation st every two of them have at most k component in common,is $b_{n,k}$.

a)proeve that :$a_{n,k} \cdot b_{n,k-1} \le n!$

b)assume that p is prime number,determine the exact value of $a_{p,2}$.

12   There is a lamp in space.(Consider lamp a point)

Do there exist finite number of equal sphers in space that the light of the lamp can not go to the infinite?(If a ray crash in a sphere it stops)

13   here is the most difficult and the most beautiful problem occurs in 21th iranian (2003) olympiad

assume that P is n-gon ,lying on the plane ,we name its edge 1,2,..,n.

if S=s1,s2,s3,.... be a finite or infinite sequence such that for each i, si is in {1,2,...,n},

we move P on the plane according to the S in this form: at first we reflect P through the s1

(s1 means the edge which iys number is s1)then through s2 and so on like the figure below.

a)show that there exist the infinite sequence S sucth that if we move P according to S we cover all the plane

b)prove that the sequence in a) isn't periodic.

c)assume that P is regular pentagon ,which the radius of its circumcircle is 1,and D is circle ,with radius 1.00001 ,arbitrarily in the plane .does exist a sequence S such that we move P according to S then P reside in D completely?

14   n \geq 6 is an integer. evaluate the minimum of f(n) s.t: any graph with n vertices and f(n) edge contains two cycle which are distinct(also they have no comon vertice)?

15 Assume $m \times n$ matrix which is filled with just 0, 1 and any two row differ in at least $n/2$ members, show that $m \leq 2n$.
(for example the diffrence of this two row is only in one index
110
100)

Edited by Myth

16 Segment AB is fixed in plane. Find the largest n, such that there are n points $P_1, P_2, \ldots, P_n$ in plane that triangles ABP_i are similar for $1 \leq i \leq n$. Prove that all of P_i's lie on a circle.

17 A simple calculator is given to you. (It contains 8 digits and only does the operations $+,-,*,/,\sqrt{}$) How can you find $3^{\sqrt{2}}$ with accuracy of 6 digits.

18 In tetrahedron $ABCD$, radius four circumcircles of four faces are equal. Prove that $AB = CD$, $AC = BD$ and $AD = BC$.

19 An integer n is called a good number if and only if $|n|$ is not square of another intger. Find all integers m such that they can be written in infinitely many ways as sum of three different good numbers and product of these three numbers is square of an odd number.

20 Suppose that M is an arbitrary point on side BC of triangle ABC. B_1, C_1 are points on AB, AC such that $MB = MB_1$ and $MC = MC_1$. Suppose that H, I are orthocenter of triangle ABC and incenter of triangle MB_1C_1. Prove that A, B_1, H, I, C_1 lie on a circle.

21 Let ABC be a triangle. W_a is a circle with center on BC passing through A and perpendicular to circumcircle of ABC. W_b, W_c are defined similarly. Prove that center of W_a, W_b, W_c are collinear.

22 Let $a_1 = a_2 = 1$ and
$$a_{n+2} = \frac{n(n+1)a_{n+1} + n^2 a_n + 5}{n+2} - 2$$
for each $n \in \mathbb{N}$. Find all n such that $a_n \in \mathbb{N}$.

23 Find all homogeneous linear recursive sequences such that there is a T such that $a_n = a_{n+T}$ for each n.

24 A, B are fixed points. Variable line l passes through the fixed point C. There are two circles passing through A, B and tangent to l at M, N. Prove that circumcircle of AMN passes through a fixed point.

25 Let A, B, C, Q be fixed points on plane. M, N, P are intersection points of AQ, BQ, CQ with BC, CA, AB. D', E', F' are tangency points of incircle of ABC with BC, CA, AB. Tangents drawn from M, N, P (not triangle sides) to incircle of ABC make triangle DEF. Prove that DD', EE', FF' intersect at Q.

26 Circles C_1, C_2 intersect at P. A line Δ is drawn arbitrarily from P and intersects with C_1, C_2 at B, C. What is locus of A such that the median of AM of triangle ABC has fixed length k.

27 $S \subset \mathbb{N}$ is called a square set, iff for each $x, y \in S$, $xy + 1$ is square of an integer.
a) Is S finite?
b) Find maximum number of elements of S.

28 There are n points in $\mathbb{R}^3$ such that every three form an acute angled triangle. Find maximum of n.

29 Let $c \in \mathbb{C}$ and $A_c = \{p \in \mathbb{C}[z] | p(z^2 + c) = p(z)^2 + c\}$.

a) Prove that for each $c \in C$, A_c is infinite.

b) Prove that if $p \in A_1$, and $p(z_0) = 0$, then $|z_0| < 1.7$.

c) Prove that each element of A_c is odd or even.

Let $f_c = z^2 + c \in \mathbb{C}[z]$. We see easily that $B_c := \{z, f_c(z), f_c(f_c(z)), \dots\}$ is a subset of A_c. Prove that in the following cases $A_c = B_c$.

d) $|c| > 2$.

e) $c \in \mathbb{Q} \backslash \mathbb{Z}$.

f) c is a non-algebraic number

g) c is a real number and $c \notin [-2, \frac{1}{4}]$.