Romanian Masters In Mathematics 2013

Day 1 March 1st

1 For a positive integer a, define a sequence of integers $x_1, x_2, \dots$ by letting $x_1 = a$ and $x_{n+1} = 2x_n + 1$ for $n \geq 1$. Let $y_n = 2^{x_n} - 1$. Determine the largest possible k such that, for some positive integer a, the numbers $y_1, \dots, y_k$ are all prime.

2 Does there exist a pair (g, h) of functions $g, h : \mathbb{R} \to \mathbb{R}$ such that the only function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(g(x)) = g(f(x))$ and $f(h(x)) = h(f(x))$ for all $x \in \mathbb{R}$ is identity function $f(x) \equiv x$?

3 Let $ABCD$ be a quadrilateral inscribed in a circle ω. The lines AB and CD meet at P, the lines AD and BC meet at Q, and the diagonals AC and BD meet at R. Let M be the midpoint of the segment PQ, and let K be the common point of the segment MR and the circle ω. Prove that the circumcircle of the triangle KPQ and ω are tangent to one another.

Day 2 March 2nd

1 Suppose two convex quadrangles in the plane P and P', share a point O such that, for every line l trough O, the segment along which l and P meet is longer then the segment along which l and P' meet. Is it possible that the ratio of the area of P' to the area of P is greater then 1.9?

2 Given a positive integer $k \geq 2$, set $a_1 = 1$ and, for every integer $n \geq 2$, let a_n be the smallest solution of equation

$$x = 1 + \sum_{i=1}^{n-1} \left\lfloor \sqrt[k]{\frac{x}{a_i}} \right\rfloor$$

that exceeds a_{n-1}. Prove that all primes are among the terms of the sequence $a_1, a_2, \dots$

3 A token is placed at each vertex of a regular $2n$-gon. A *move* consists in choosing an edge of the $2n$-gon and swapping the two tokens placed at the endpoints of that edge. After a finite number of moves have been performed, it turns out that every two tokens have been swapped exactly once. Prove that some edge has never been chosen.