APMO 2013

1 Let ABC be an acute triangle with altitudes AD, BE, and CF, and let O be the center of its circumcircle. Show that the segments OA, OF, OB, OD, OC, OE dissect the triangle ABC into three pairs of triangles that have equal areas.

2 Determine all positive integers n for which $\dfrac{n^2+1}{[\sqrt{n}]^2+2}$ is an integer. Here $[r]$ denotes the greatest integer less than or equal to r.

3 For $2k$ real numbers $a_1, a_2, ..., a_k$, $b_1, b_2, ..., b_k$ define a sequence of numbers X_n by

$$X_n = \sum_{i=1}^{k} [a_i n + b_i] \quad (n = 1, 2, ...).$$

If the sequence X_N forms an arithmetic progression, show that $\sum_{i=1}^{k} a_i$ must be an integer. Here $[r]$ denotes the greatest integer less than or equal to r.

4 Let a and b be positive integers, and let A and B be finite sets of integers satisfying
(i) A and B are disjoint;
(ii) if an integer i belongs to either to A or to B, then either $i+a$ belongs to A or $i-b$ belongs to B.
Prove that $a\,|A| = b\,|B|$. (Here $|X|$ denotes the number of elements in the set X.)

5 Let $ABCD$ be a quadrilateral inscribed in a circle ω, and let P be a point on the extension of AC such that PB and PD are tangent to ω. The tangent at C intersects PD at Q and the line AD at R. Let E be the second point of intersection between AQ and ω. Prove that B, E, R are collinear.