USAMO 2013

**Day 1** April 30th

**1** In triangle $ABC$, points $P$, $Q$, $R$ lie on sides $BC$, $CA$, $AB$ respectively. Let $\omega_A$, $\omega_B$, $\omega_C$ denote the circumcircles of triangles $AQR$, $BRP$, $CPQ$, respectively. Given the fact that segment $AP$ intersects $\omega_A$, $\omega_B$, $\omega_C$ again at $X$, $Y$, $Z$, respectively, prove that $YX/XZ = BP/PC$.

**2** For a positive integer $n \geq 3$ plot $n$ equally spaced points around a circle. Label one of them $A$, and place a marker at $A$. One may move the marker forward in a clockwise direction to either the next point or the point after that. Hence there are a total of $2n$ distinct moves available; two from each point. Let $a_n$ count the number of ways to advance around the circle exactly twice, beginning and ending at $A$, without repeating a move. Prove that $a_{n-1} + a_n = 2^n$ for all $n \geq 4$.

**3** Let $n$ be a positive integer. There are $\frac{n(n+1)}{2}$ marks, each with a black side and a white side, arranged into an equilateral triangle, with the biggest row containing $n$ marks. Initially, each mark has the black side up. An *operation* is to choose a line parallel to the sides of the triangle, and flipping all the marks on that line. A configuration is called *admissible* if it can be obtained from the initial configuration by performing a finite number of operations. For each admissible configuration $C$, let $f(C)$ denote the smallest number of operations required to obtain $C$ from the initial configuration. Find the maximum value of $f(C)$, where $C$ varies over all admissible configurations.

**Day 2** May 1st

**4** Find all real numbers $x, y, z \geq 1$ satisfying

$$\min(\sqrt{x + xyz}, \sqrt{y + xyz}, \sqrt{z + xyz}) = \sqrt{x - 1} + \sqrt{y - 1} + \sqrt{z - 1}.$$

**5** Given positive integers $m$ and $n$, prove that there is a positive integer $c$ such that the numbers $cm$ and $cn$ have the same number of occurrences of each non-zero digit when written in base ten.

**6** Let $ABC$ be a triangle. Find all points $P$ on segment $BC$ satisfying the following property: If $X$ and $Y$ are the intersections of line $PA$ with the common external tangent lines of the circumcircles of triangles $PAB$ and $PAC$, then

$$\left(\frac{PA}{XY}\right)^2+\frac{PB\cdot PC}{AB\cdot AC}=1.$$

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