India

International Mathematical Olympiad Training Camp

2010

$\boxed{1}$ Let ABC be a triangle in which $BC < AC$. Let M be the mid-point of AB, AP be the altitude from A on BC, and BQ be the altitude from B on to AC. Suppose that QP produced meets AB (extended) at T. If H is the orthocenter of ABC, prove that TH is perpendicular to CM.

$\boxed{2}$ Two polynomials $P(x) = x^4 + ax^3 + bx^2 + cx + d$ and $Q(x) = x^2 + px + q$ have real coefficients, and I is an interval on the real line of length greater than 2. Suppose $P(x)$ and $Q(x)$ take negative values on I, and they take non-negative values outside I. Prove that there exists a real number x_0 such that $P(x_0) < Q(x_0)$.

$\boxed{3}$ For any integer $n \geq 2$, let $N(n)$ be the maximum number of triples (a_j, b_j, c_j), $j = 1, 2, 3, \cdots, N(n)$, consisting of non-negative integers a_j, b_j, c_j (not necessarily distinct) such that the following two conditions are satisfied:

(a) $a_j + b_j + c_j = n$, for all $j = 1, 2, 3, \cdots N(n)$; (b) $j \neq k$, then $a_j \neq a_k$, $b_j \neq b_k$ and $c_j \neq c_k$. Determine $N(n)$ for all $n \geq 2$.

$\boxed{4}$ Let a, b, c be positive real numbers such that $ab + bc + ca \leq 3abc$. Prove that

$$\sqrt{\frac{a^2+b^2}{a+b}} + \sqrt{\frac{b^2+c^2}{b+c}} + \sqrt{\frac{c^2+a^2}{c+a}} + 3 \leq \sqrt{2}(\sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a})$$

$\boxed{5}$ Given an integer $k > 1$, show that there exist an integer an $n > 1$ and distinct positive integers $a_1, a_2, \cdots a_n$, all greater than 1, such that the sums $\sum_{j=1}^n a_j$ and $\sum_{j=1}^n \phi(a_j)$ are both k-th powers of some integers. (Here $\phi(m)$ denotes the number of positive integers less than m and relatively prime to m.)

$\boxed{6}$ Let $n \geq 2$ be a given integer. Show that the number of strings of length n consisting of $0'$s and $1'$s such that there are equal number of 00 and 11 blocks in each string is equal to

$$2\binom{n-2}{\left\lfloor \frac{n-2}{2} \right\rfloor}$$

$\boxed{7}$ Let $ABCD$ be a cyclic quadrilaterla and let E be the point of intersection of its diagonals AC and BD. Suppose AD and BC meet in F. Let the midpoints of AB and CD be G and H respectively. If Γ is the circumcircle of triangle EGH, prove that FE is tangent to Γ.

$\boxed{8}$ Call a positive integer **good** if either $N = 1$ or N can be written as product of *even* number of prime numbers, not necessarily distinct. Let $P(x) = (x - a)(x - b)$, where a, b are positive integers.

(a) Show that there exist distinct positive integers a, b such that $P(1), P(2), \cdots, P(2010)$ are all good numbers. (b) Suppose a, b are such that $P(n)$ is a good number for all positive integers n. Prove that $a = b$.

9 Let $A = (a_{jk})$ be a 10×10 array of positive real numbers such that the sum of numbers in row as well as in each column is 1. Show that there exists $j < k$ and $l < m$ such that

$$a_{jl}a_{km} + a_{jm}a_{kl} \geq \frac{1}{50}$$

10 Let ABC be a triangle. Let Ω be the brocard point. Prove that $\left(\frac{A\Omega}{BC}\right)^2 + \left(\frac{B\Omega}{AC}\right)^2 + \left(\frac{C\Omega}{AB}\right)^2 \geq 1$

11 Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x+y) + xy = f(x)f(y)$ for all reals x, y

12 Prove that there are infinitely many positive integers m for which there exists consecutive odd positive integers $p_m < q_m$ such that $p_m^2 + p_m q_m + q_m^2$ and $p_m^2 + m \cdot p_m q_m + q_m^2$ are both perfect squares. If m_1, m_2 are two positive integers satisfying this condition, then we have $p_{m_1} \neq p_{m_2}$

