IMO 2005

Merida, Yutacan, Mexico

Day 1 - *13 July 2005*

$\boxed{1}$ Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC, B_1, B_2 on CA and C_1, C_2 on AB, such that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths.

Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.

Bogdan Enescu, Romania

$\boxed{2}$ Let $a_1, a_2, \ldots$ be a sequence of integers with infinitely many positive and negative terms. Suppose that for every positive integer n the numbers $a_1, a_2, \ldots, a_n$ leave n different remainders upon division by n.

Prove that every integer occurs exactly once in the sequence $a_1, a_2, \ldots$.

$\boxed{3}$ Let x, y, z be three positive reals such that $xyz \geq 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0.$$

Hojoo Lee, Korea

IMO 2005

Merida, Yutacan, Mexico

Day 2 - *14 July 2005*

$\boxed{4}$ Determine all positive integers relatively prime to all the terms of the infinite sequence

$$a_n = 2^n + 3^n + 6^n - 1,\ n \geq 1.$$

$\boxed{5}$ Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and BC not parallel with DA. Let two variable points E and F lie of the sides BC and DA, respectively and satisfy $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R.

Prove that the circumcircles of the triangles PQR, as E and F vary, have a common point other than P.

$\boxed{6}$ In a mathematical competition, in which 6 problems were posed to the participants, every two of these problems were solved by more than $\frac{2}{5}$ of the contestants. Moreover, no contestant solved all the 6 problems. Show that there are at least 2 contestants who solved exactly 5 problems each.

Radu Gologan and Dan Schwartz