1 Combinatorics

1. Wlog $a_1 < a_2 < \cdots < a_n$, and let $M = a_n - a_1 + 1$. For each integer t, let $I_t \subset \{0, 1, \ldots, M\}$ denote the set of integers $0 \le k \le M$ such that $t + k \in X$. By the pigeonhole principle, we can find distinct $s \neq t$ such that $I_s = I_t$. We claim that $I_{s+1} = I_{t+1}$. To prove this, it suffices to show that if $s + 1 + M \in X$ than $t + 1 + M \in X$. Therefore, assume $s + 1 + M \in X$.

   Note that by the condition of the problem, for each integer l, exactly one of $l - a_1, l - a_2, \ldots, l - a_n$ is in X. Thus, letting $l = s + 1 + M + a_1$, we conclude that exactly one of $s + 1 + M, s + 1 + M - a_2 + a_1, \ldots, s + 1 + M - a_n + a_1$ is in X. Thus, we conclude that the numbers $s + 1 + M + a_1 - a_2, \ldots, s + 1 + M + a_1 - a_n$ are not in X, and thus that

   $$M + 1 + (a_1 - a_2), \ldots, M + 1 + (a_1 - a_n)$$

   are not in $I_s = I_t$. Thus, we conclude that $t + 1 + M + a_2 - a_1, \ldots, t + 1 + M + a_n - a_1$ are not in X, and so letting $l = t + 1 + M + a_1$ we conclude that $t + 1 + M \in X$.

   We thus have that $I_{s+1} = I_{t+1}$, and thus by induction that $I_{s+k} = I_{t+k}$ for all positive k. Similarly, we conclude that $I_{s-k} = I_{t-k}$ for all positive integers k, and thus $I_k = I_{k+(s-t)}$ for all integers k. Thus, as $k \in X$ if and only if $0 \in I_k$, we conclude that $X = X + (s - t)$.

2. We first show that the people can win with probability $1 - \frac{1}{2^n}$. To do this, A person simply says the number $(-1)^r \cdot n^r$ where r is the number of red hats she sees. Now, if there are $n - r$ blue hats and r red hats, then $n - r$ people see r red hats and r people see $r - 1$ red hats. Thus, the sum of the numbers said will be

   $$(-1)^r \cdot \left((n - r) \cdot n^r - r \cdot n^{r-1}\right) = (-1)^r n^{r-1}(n^2 - r(n + 1)).$$

   Since $n^2 > r(n+1)$ if $r < n$ the people will win as long as the hats aren't all red, which is an event with probability $\frac{1}{2^n}$.
   Now, we must show that there is no strategy that does any better. Note that there are 2^n possible events (each person gets 1 of 2 possible hat colors) and each strategy works for some subset of them. Thus, all we must show is that there is no strategy that works in all cases. So suppose there is. For each event E, let $r(E)$ denote the number of red hats, E_i denote the number that the i'th person says under the winning strategy, and $S(E)$ denote the sum of the E_i. Since the strategy always works, we must have $(-1)^{r(E)}S(E)$ is always positive. Let us consider the sum $\sum_E (-1)^{r(E)}S(E)$. Note that this can be rewritten as

   $$\sum_{i=1}^{n} \sum_{E} (-1)^E r(E) E_i.$$

   Now, fix some i. Let E' denote the event E but with the i'th person receiving the opposite hat coloring. Then $r(E) = r(E') \pm 1$ and $E_i = E'_i$ since the colors that person i sees are the same for E and E'. Thus,

   $$(-1)^{r(E)} E_i + (-1)^{r(E')} E'_i = 0.$$

   Since the map $E \to E'$ pairs of all events, we must have that $\sum_E (-1)^E r(E) E_i = 0$, and thus $\sum_E (-1)^{r(E)} S(E) = 0$. This is a contradiction, which completes the proof.

3. Note that the sum of all elements in the grid is $18 \cdot 37 = 666$ and thus the sum in each row, column, and diagonal must be 111. Now, let R_i, C_j refer to the sum of the numbers in row i and column j respectively, and let D_k denote the sum of the numbers in diagonal k. Then

   $$333 = R_1 + R_3 + R_5 + C_1 + C_3 + C_5 - D_1 - D_3 - D_5$$

   and this letter sum is twice the sum of the numbers in those squares whose co-ordinates are both odd. Since 333 is an odd number, this is impossible.

2 Algebra

1. In fact jacob can make $f(2) = 0$. To begin with, Jacob starts lowering the x coefficient, making $f(2)$ decrease by 2. Since David can only increase $f(2)$ by 1 each term, Jacob can $f(2)$ decrease until $f(2) = 1$ or $f(2) = 0$. If the latter happens, Jacob has won. else, David can make $f(2) = 2$ or $f(2) = 0$. In the former case, jacob lowers the coefficient by 1 and wins. In the latter, Jacob has already won.

2. We claim that $f(x) = x$ or $f(x) = 0$. Assume $f(x)$ is not identically 0.

   Setting $x = y = 0$ gives $f(0) = 0$. Next, setting $y = 0$ gives

   $$f(x^2)x^2 = f(x^4) \tag{1}$$

and setting $x = 0$ gives

$$yf(y)f(y^2) = f(y^4). \tag{2}$$

Moreover, setting $y = x$ gives

$$f(x^2) = xf(x) \text{ or } f(x^2) + x^2 = 0. \tag{3}$$

Now, if $f(c^2) = -c^2$ then (1) gives $f(c^4) = -c^4$, and similarly $f(c^8) = -c^8$. Plugging $y = c^2$ into (2) now gives $c^8 = -c^8$, and thus $c = 0$. Hence, (3) implies that $f(x^2) = xf(x)$ for all x. Plutting in (1) and $f(y^2) = yf(y)$ into the original equation expanded out gives

$$f(y^2)\left(f(x^2) - x^2\right) = 0.$$

Thus, letting y be such that $f(y) \neq 0$ and thus $f(y^2) = yf(y) \neq 0$, we get that $f(x^2) = x^2$. Finally,

$$xf(x) = f(x^2) = x^2$$

and thus $f(x) = x$, as desired.

3. We proceed by induction on n, the case of $n = 1$ being obvious(as it is vacuous). So let the claim be prove for $n-1$. Given a sequence $a_1, \ldots, a_n$ with even sum create a sequence of length $n-1$, given by $a_1, a_2, \ldots, a_{n-2}, |a_{n-1} - a_n|$. This latter sequences also has even sum, and $|a_{n-1} - a_n| \leq \max(a_{n-1}, a_n) - 1 \leq n-1$. Hence, our induction hypothesis applies, and we can write

$$a_1 \pm a_2 \pm \cdots \pm |a_{n-1} - a_n| = 0$$

for some choice of signs. Expanding out the absolute value now gives the result.

3 Number Theory

1. Let k be such that $10^{k-2} > N$, and let $x = \lceil\sqrt{10^k N}\rceil$. Then

$$\sqrt{10^k N} \leq x < \sqrt{10^k N} + 1.$$

Squaring, we get

$$10^k N \leq x^2 < 10^k N + 2\sqrt{10^k N} + 1 < 10^k N + 2 \cdot 10^{k-1} + 1.$$

Hence the first k digit of x give N.

2. Assume that there are only finitely many primes p satisfying the conclusion of the statement, and let S be their product. Let $m = 9a^3S^4$. Then

$$m^3 - a = a(729a^2S^{12} - 1) = a(27a^4S^6 - 1)(27a^4S^6 + 1).$$

Now let p be a prime dividing $27a^4S^6 + 1$. Then letting $n = 9a^2S^3$ we see that p satisfies the conclusions of the statement, and is relatively prime to S. This is a contradiction, thus the statement must be true.

3. The answer is no. Note that A_n is equal to the number of residue classes $x \bmod n$ such that $n \mid x^n + 1$. Letting $n = \prod_{i=1}^m p_i^{e_i}$ this is equal to the product over all i of the number $t_{n,i}$ of residue classes $x \bmod p_i^{e_i}$.

Suppose first that n is even. Now, if $4 \mid n$ then $A_n = 0$ since $x^2 + 1$ is never divisible by 4. Thus n must be equal to $2\prod_{i=2}^m p_i^{e_i}$. We claim in this case that each $t_{n,i}$ is even. To see this, note that if $x^n + 1$ is divisible by $p_i^{e_i}$ then so is x^{-1}. Now $x = x^{-1} \mod p_i^{e_i}$ is equivalent to $x^2 = 1 \mod p_i^{e_i}$ and the only solutions are $x = \pm 1$. Thus, in order for $t_{n,i}$ to be odd it must be the case that $1^n + 1$ or $(-1)^n + 1$ is divisible by $p_i^{e_i}$ but not both. However, both are equal to 2, and so $t_{n,i}$ is odd as desired. Now, 130 is not divisible by 4, so we must have $n = 2p^e$ for some prime p.

Then A_n is equal to the number of residue classes $x \bmod p^e$ such that $x^{2p^e} = -1 \mod p^e$. Thus, p must be a prime that is $1 \mod 4$. If $e = 1$, then $x^{2p} = x^2 \mod p$ and so the number of solutions is 2. If $e > 1$, then as $(1+pr)^{p^e} = 1 \mod p^e$ the number of solutions is divisible by p^{e-1}, and as x modulo p can have at most 2 values the number of such values is at most $2p^{e-1}$. Thus, $e = 2$ and $p \mid 130$ and $2p > 130$. Since the only odd primes dividing 130 are 5 and 13 this is impossible.

Finally, suppose that n is odd. Then we claim that $t_{n,i}$ is odd for all i. To see this, note again that if $x^n + 1$ is divisible by $p_i^{e_i}$ then so is x^{-1}. Now $x = x^{-1} \mod p_i^{e_i}$ is equivalent to $x^2 = 1 \mod p_i^{e_i}$ and the only solutions are $x = \pm 1$. Thus, in order for $t_{n,i}$ to be odd it must be the case that $1^n + 1$ or $(-1)^n + 1$ is divisible by $p_i^{e_i}$ but not both. However, the former is 2 and the latter is 0 as n is odd, and so $t_{n,i}$ is odd as desired. Thus $|A_n|$ is odd, which completes the proof.

4 Geometry

1. The answer is al P on the circumcircle of BCD.

   Wlog P is at east as close to A as to C. Note that $\sqrt{2}BC = \sqrt{2}CD = BD$, and thus the equation is

   $$PC \cdot BD = PB \cdot CD + PD \cdot BC,$$

   and so by the converse to Ptolemy's inequality we deduce that $PBCD$ is cyclic, and thus P lies on the circumcircle of $ABCD$ on arc BAD. Conversely, if P lies on this arc then $PBCD$ is cyclic, and so the equality holds by Ptolemy. The other case is identical.

2. Let O be the circumcircle of $ABCD$. We claim that $DHOC$ is concyclic. To see this, let $y = \angle DAF$. Then by cyclicity of $ABCD$ we have $y = \angle DCB$. Since $AFHD$ and $BFHC$ are concyclic, we conclude that $\angle FHD = \angle FHC = \pi - y$ and thus $\angle DCH = 2y = \angle DCO$, where the latter is because O is the circumcenter of DBC. Thus, we concluden that $DCOH$ is cyclic and likewise, so is $ABOH$. Applying the radial axis theorem to the 3 circles $DCOH, ABOH, ABCD$ we conclude that AB, DC, OH are concurrent, and thus E, H, O are collinear. Thus, it suffices to show that EO i perpendicular to FH. Applying the radical axis theorem to the 3 circles $ABCD, ADHF, CBHF$ we likewise learn that H, F, K are collinear. Thus it suffices to prove that EO is perpendicular to FK. That this is true is Brokard's theorem.

3. The answer is no. We shall need the following claim: Let ABC be a triangle, and I be the incenter. Now let M, N be the midpoints of AB, BC respectively. Then I and A lie on opposite sides of MN. To prove this, simply note that since the incircle is inside ABC, the diameter of the incircle is less than the altitude from A to BC, and thus the altitude from I to BC is less than half the altitude from A to BC.

   Now, let $ABCD$ be our tetrahedron, and let I_{ABC} be the incenter of ABC, with $I_{ACD}, I_{ABD}, I_{BCD}$ be defined similarly. Suppose that these 4 points are coplanar. Then either they make a convex quadrilateral or one of them – wlog I_{ABC} lies inside the triangle formed by the other 3 points. But $I_{ACD}, I_{ABD}, I_{BCD}$ all lie on one side of the plane ABC, and so this latter possibility is impossible. So assume that the four incenters form a convex quadrilateral, so that the two diagonals intersect. Wlog, $I_{ABC}I_{BCD}$ intersects $I_{ACD}I_{ABD}$. Now let M, N, K, L be the midpoints of AB, AC, BD, CD. Then $MN||BC||KL$, and thus $MNKL$ lie on a plane P.

   Now, by our initial claim, I_{ABC} and I_{DBC} lie on the same side as BC of the plane P, and I_{ACD} and I_{DBC} lie on the same side as AD of the plane P. But evidently AD and BC lie on opposite sides of P. Thus the segments $I_{ABC}I_{BCD}$ $I_{ACD}I_{ABD}$ are on opposite sides of the plane P and therefore cannot intersect, which is a contradiction.