Combinatorics and Combinatorial Geometry

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1. General Combinatorics

(Extremal Principle:) A problem can often be solved by considering a notion of a minimum (or a maximum) condition in the problem. We shall demonstrate this principle with a sample problem.

Problem: A set S of points on a plane has the property that if $A, B \in S$, then the midpoint of A and B is also in S. Prove that either $|S| = 1$ or S is infinite.

Proof: For X, Y in the plane, denote the distance between X and Y by $d(X, Y)$. Suppose S is finite. Choose $A, B \in S$ such that $d(A, B)$ is minimum. Let X be the midpoint of A and B. Hence, $X \in S$ and $d(A, X) = \frac{1}{2}d(A, B) < d(A, B)$, contradicting the minimality of $d(A, B)$. Hence, S is infinite. $\square$

Warning: Do not take the minimum value of an infinite set without care. For example, if S is an infinite set of real numbers in $(0, 1)$, the minimum value and the maximum value of S may not exist. The term you may be looking for are **infimum** and **supremum**. The infimum of S is the largest real number r such that $x \geq r$ for all $x \in S$. The supremum of S is the smallest real number R such that $x \leq R$ for all $x \in S$.

However, if S is an infinite set of positive integers, then the minimum of S exists. This is called the **well-ordering principle**.

(Intermediate Value Theorem:) Let f be a continuous function on the real numbers with $f(0) < 0$ and $f(1) > 0$. Then $f(r) = 0$ for some $r \in (0, 1)$.

(Discrete Version:) Let f be a function on the integers such that $f(0) < 0, f(m) > 0$ for some $m > 0$ and $f(n + 1) \leq f(n) + 1$ for all $n \in \mathbb{Z}$. Then $f(t) = 0$ for some $t \in \mathbb{Z}, 0 < t < m$.

This is a notion of an existence proof which is non-constructive, i.e. we have no idea what r is, we just know it exists. Let's demonstrate this with an example.

Problem: Alice begins at the bottom of a hill at 9am and begins running to the top of the hill, arriving there at noon. On the next day, Alice begins at the top of the hill at 9am and begins running to the bottom of the hill, arriving there at noon. Alice's speed can change and she may even change directions in the middle of her run. Prove that there exists a time between 9am and noon such that Alice was at the same position at this time on the two days.

Proof: Let f, g denote the function of her first run and her second run, with 0 in the range of the functions denoting the bottom of the hill and 1 the top of the hill, i.e. $f(9) = 0, f(12) = 1, g(9) = 1, g(12) = 0$. Note that f, g are continuous. You need to prove there exists $t \in (9, 12)$ such that $f(t) = g(t)$. Let $h(t) = f(t) - g(t)$. Note that h is also continuous. Then $h(9) = -1, h(12) = 1$. Then there exists $t \in (9, 12)$ such that $h(t) = 0$, i.e. $f(t) - g(t) = 0$. $\square$

In-Lecture Problems:

1. Given $2n$ points in a plane with no three collinear, with n red points and n blue points, prove that there exists a pairing of the red and blue points such that the n segments joining each pair are pairwise non-intersecting.

2. A finite set S of points on a plane satisfies the property the area of any triangle with vertices in S has area at most one. Prove that S can be covered by a triangle with area four.

3. In a $1 \times 6n$ grid, $4n$ squares are coloured red and $2n$ squares are coloured blue. Prove that there exists $3n$ consecutive squares in the grid that contains $2n$ red squares and n blue squares.

Colouring and Patterns

Problems often involved constructing a combinatorial object that involves patterns of colours and numbers. These problems often require a natural feel for the integers and for symmetry.

Problem: Each point on the plane is coloured one of two colours. Prove that there exists two points of distance 1 of the same colour. Solve the same problem with three colours instead of two colours. Prove that the same statement does not hold for seven colours.

Solution: Take an equilateral triangle of length 1. By Pigeonhole Principle, two of these points have the same colour. For the second part, let A, B, C be vertices of an equilateral triangle of length 1. Let D be the reflection of A through BC. Rotate B, C, D about A to images B', C', D' such that $DD' = 1$. Prove that you cannot colour A, B, C, D, B', C', D' with three colours. For the final part, tile the plane with regular hexagons of length $1/2$ and colour the hexagons using seven colours accordingly. $\square$

Problem: A mathematics competition has 5 problems and n contestants, where each contestant is assigned an integer score of at least 1 and at most 7 on each problem. It turns out every pair of contestants have at most one problem whose scores are common. Find the maximum possible value of n.

Solution: We'll let the reader prove that $n \leq 7^2 = 49$. Let $C_{i,j}$, $1 \leq i, j \leq 7$ be the 49 contestants. Assign contestant $C_{i,j}$ a score of $ki + j \mod 7$ on problem k (where the arithmetic is performed on $\{1, 2, \cdots, 7\}$) This works. Hence, the maximum possible value of n is 49. $\square$

In-Lecture Problems

1. The squares of a 8×8 board are tiled with 21 triminos (3×1 pieces) with one empty square. Find all possible locations of this empty square.

2. Let n be an odd positive integer. n computers are in a room and each pair of computers is joined by a cable. Each computer and cable are to be assigned a colour so that no two computers are assigned the same colour, no computer is assigned the same colour as a cable joining it, and no two cables joined to a common computer are assigned the same colour. Prove that this can be done in n colours.

3. Each point on the circumference of a circle is coloured either red or blue. Prove that there exist three distinct points on this circumference X, Y, Z all of the same colour such that $|XY| = |XZ|$.

Other Techniques:

This is a check-list of other topics in combinatorics you should know for math olympiads. This wouldn't be covered in the lecture due to lack of time. If you have any questions, or want a mini-lecture on one of these topics outside of lecture time, feel free to ask me and I will accomodate you.

Pigeonhole Principle:
Given a sequence of $n^2 + 1$ distinct real numbers, prove that the sequence contains a decreasing subsequence of length $n + 1$ or an increasing subsequence of length $n + 1$.

Binomial Theorem, Bijections, Luca's Theorem:
a.) For any non-negative integer n, prove that $\sum_{k=0}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$.

b.) Let n be a positive integer. Let $f(n)$ be the number of factors of 2 in $n!$ and $g(n)$ be the number of 1's in the binary expansion of n. Prove that $f(n) + g(n) = n$.

Double Counting - Counting Objects in Two Different Ways:
Given a permutation $(x_1, \cdots, x_n)$ of $(1, 2, \cdots, n)$, a fixed point is said to be an element x_i such that $x_i = i$. Let $f(n, k)$ be the number of permutations of $(1, 2, \cdots, n)$ with exactly k fixed points. Prove that

$$\sum_{k=0}^{n} kf(n, k) = n!.$$

Graph Theory Techniques: (Often applied to problems involving people and/or grids)
a.) Let n be positive integers. A group of $2n$ people are in a room where each pair is classified as friends or strangers. It turns out that no three people are all mutually friends. Find the maximum possible number of pairs of people that are friends.

b.) Let $n \geq 3$ be a positive integer. Let G be a grid whose entries are all $0, 1$ or -1 such that each row and each column contains exactly one 1 and one -1. Prove that the rows and the columns of the grid can be re-ordered such that the resulting grid is the negative of G.

Partitions and Generating Functions:
Given a positive integer n, a partition of n is defined to be an *unordered* tuple of positive integers $(x_1, \cdots, x_t)$ whose sum of entries is n. For example, the partitions of 5 are $(4, 1), (3, 2), (3, 1, 1), (2, 1, 1, 1)$, $(1, 1, 1, 1, 1)$. Let $f(n)$ be the number of partitions of n where each part is odd. Let $g(n)$ be the number of partitions of n where the parts are pairwise distinct. Prove that $f(n) = g(n)$.

Linear Recurrence Relations:
Let a, b, c be the roots of the polynomial $x^3 - x^2 + 4x + 7$. Prove that $a^n + b^n + c^n$ is an integer for all positive integers n and find all $n \in \mathbb{N}$ such that $a^n + b^n + c^n$ is divisible by 4.

Partially-Ordered Sets, Chains and Anti-Chains:
((LYM) Lubell-Yamamoto-Meshalkin Inequality:) Let S be a set of subsets of $\{1, 2, \cdots, n\}$ such that no subset in S contains another subset in S. Let a_k be the subsets in S of size k. Prove that

$$\sum_{k=0}^{n} \frac{a_k}{\binom{n}{k}} \leq 1.$$

2. Combinatorial Geometry

Combinatorial geometry problems are simply Euclidean geometry problems with combinatorial aspects to them. These notes will not cover the elementary Euclidean geometry facts. We first want to define convex sets. These sets become extremely insightful ways to view points on a plane to solve problems.

We will denote a plane by $\mathbb{R}^2$. A set $S \subseteq \mathbb{R}^2$ is said to be **convex** if for every $A, B \in S$, the line segment joining A, B is also in S. Formally, for every $\lambda \in [0, 1]$, $\lambda \cdot A + (1 - \lambda) \cdot B \in S$.

For example, triangles, rectangles, circles, ellipses are convex sets. Figures with "holes" in it or a horseshoe are not convex sets. Any polygon with an internal angle larger than 180^o is not convex.

Given a set of points S in a plane, the **convex hull** of S is the intersection of all convex sets containing S. It is seen as the "smallest" convex set containing S. We are interested in the case where S is **finite**. The convex hull of a finite set of points is a convex polygon, whose vertices are in S. The remaining vertices of S are on the boundary or interior of the polygon.

A finite set S in $\mathbb{R}^2$ is said to be in **general position** if no three points in S are collinear. S is said to be in **convex position** if the points in S are vertices of a convex polygon.

In-Lecture Problems

1. A set S of n points are in convex position if and only if every set of four points are in convex position.
2. Five points in $\mathbb{R}^2$ are in general position, Prove that four of these points that are in convex position.[1]
3. Let S be a set of n points on a plane in general position. Prove that the points in S can be labeled $P_1, P_2, \cdots, P_n$ such that the broken line $P_1P_2P_3, \cdots, P_n$ does not intersect itself.

As you notice, it is extremely useful to study the convex hull of a finite set of points in $\mathbb{R}^2$. It often trivializes the analysis required to solve a problem.

Other Tips:
• Let $\mathcal{P}$ be a convex n-gon in $\mathbb{C}$ (or $\mathbb{R}^2$) and $\alpha_1, \cdots, \alpha_n \in \mathbb{C}$ (or $\mathbb{R}^2$) be the complex number for the coordinates of P. Prove that the interior and boundary of $\mathcal{P}$ is defined by

$$\left\{ \sum_{i=1}^{n} \lambda_i \alpha_i \mid \lambda_i \in \mathbb{R}, \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

• If S is a finite set of points in $\mathbb{R}^2$, you can place the points onto the Cartesian plane such that no two points have the same x-coordinate or the same y-coordinate. (Why?) If there are $n^2 + 1$ points, then there exists a sequence of $n + 1$ points with increasing x-coordinates such that the y-coordinates are increasing or the y-coordinates are decreasing. (Why?)

• Recall **Pick's Theorem:** On the Cartesian plane, given a simple polygon with vertices on lattice points, let x be number of lattice points interior of the polygon and i be the number of lattice points on the boundary of the polygon. Then the area of the polygon is $x + i/2 - 1$. (Why?)

[1] Erdos-Szekeres Theorem states that for any $k \in \mathbb{N}$, there exists a positive integer N such that any N points in general position contains k points in convex position. An open problem is to find the smallest possible N, for each k. The Erdos-Szekeres Conjecture states the minimum is $N = 2^k + 1$.

General Combinatorics Problem

1. Given $2n$ points on the plane, prove that there exists a line in the plane such that there are n points on each side of the line.

2. A 5×5 grid of light bulbs contains a light switch for each row and each column. The light switch toggles every light in the corresponding row or a column. Prove that given any initial state of the light bulbs, you can hit a finite sequence of light switches such that in each row and each column, there are more lights on than lights off.

3. Given a set S of n points in the plane, prove there exists at least $\sqrt{n}$ points in S such that no three are vertices of an equilateral triangle.

4. Consider two system of points in the plane $\{A_1, A_2, \cdots, A_n\}$ and $\{B_1, B_2, \cdots, B_n\}$ having different centroids. Prove that there is a point P in the plane such that
$$\sum_{i=1}^{n} |PA_i| = \sum_{i=1}^{n} |PB_i|.$$

5. Let n, k be positive integers such that $n \geq 2k$. You are to colour each subset of $\{1, 2, \cdots, n\}$ of size k such that every pair of disjoint subsets of size k are assigned different colours. Prove that this can be done in $n - 2k + 2$ colours. [2]

6. There are computers are in a room where some pairs of computers are joined by a cable. Let m be the number of cables. The computers are to be coloured in a manner such that no two computers joined by a cable are assigned the same colour. Prove that this can be done with at most $\frac{1}{2} + \sqrt{2m + \frac{1}{4}}$ colours.

7. (Romania TST 1999) A is a set with n elements. Let $A_1, A_2, \cdots, A_m$ be subsets of A such that $|A_i| = 3$ and $|A_i \cap A_j| \leq 1$ for distinct $i, j \in \{1, 2, \cdots, m\}$. Prove that there exists a subset X of A such that $|X| \geq \lfloor \sqrt{2n} \rfloor$ and X does not contain A_i for each i.

8. The sets $A_1, A_2, \cdots, A_{35}$ are given with the property that $|A_i| = 27$ for $1 \leq i \leq 35$, such that the intersection of every three of them has exactly one element. Show that the intersection of $A_1, A_2, \cdots, A_{35}$ is non-empty.

9. Each of six people initially knows a piece of gossip that the other five does not know. When a telephone call takes place between two of these people, they tell each other every piece of gossip they know at the time. What is the minimum number of phone calls required so that in the end, all six people know all six pieces of gossip?

10. (**Sperner's Theorem:**) Let S be a set of subsets of $\{1, 2, \cdots, n\}$ such that no subset in S contains another subset in S. Prove that
$$|S| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

[2] This is in fact the minimum number of colours required. But this fact is much harder to prove.

Convexity and Collinearity Problems

1. (Romania 2003) Find all functions f from a plane onto itself so that for every 4-tuple of points A, B, C, D that are the vertices of a convex quadrilateral, $f(A), f(B), f(C), f(D)$ are the vertices of a concave quadrilateral.

2. Given a set S of 2005 points in the plane in general position, prove that there exists 401 pairwise disjoint convex quadrilaterals with vertices in S.

3. A set of points S in general position, with at least three points, such that if $A, B, C \in S$, then the circumcentre of ΔABC is in S. Prove that S is infinite.

4. (IMO 1999) Find all finite set of points S such that for every pair of points $A, B \in S$, the perpendicular bisector of A and B is an axis of symmetry for S.

5. (Romania TST 1991) A, B, C, D, E are points on the plane such that the area of any triangle formed by three of these vertices is greater than three. Prove that the area of some triangle formed by three of these vertices is greater than four.

6. **(Sylvester's Theorem:)** Let S be a finite set of points so that every line that passes through two points in S passes through a third point in S. Prove that the points in S are collinear.

7. Given a set S of n points in general position, prove that there exists $\binom{n-3}{2}$ convex quadrilaterals whose vertices are in S.

8. (IMO Shortlist 2001) Let $n \geq 4$ be a positive integer. Let $S = \{P_1, \cdots, P_n\}$ be a set of n points in the plane with no three collinear and no four concyclic. For each $1 \leq t \leq n$, let a_t be the number of unordered triples (i, j, k) such that P_t is in the interior of the circumcircle of $\Delta P_i P_j P_k$. Let

$$m(S) = a_1 + a_2 + \cdots + a_n.$$

   Prove that there exists a positive integer function $f(n)$, depending only on n such that S are the vertices of a convex polygon if and only if $m(S) = f(n)$.

9. (USA TST 2005) Find all finite set of points S, with no three collinear, such that if $A, B, C \in S$ then there exists $D \in S$ such that $A, B, C, D \in S$ are the vertices of a parallelogram.

10. Given a set of $n \geq 3$ in general position, a subset K of S is called good if points in K are vertices of a convex polygon with no other points of S in its interior. For each $3 \leq k \leq n$, let c_k be the number of good subsets of S of size k. Prove that the sum

$$\sum_{k=3}^{n} (-1)^k c_k$$

   is dependent only on n and not the configuration of S.

Convex Polygons and Diagonals Problems

In this section, convex hulls are not as useful as before. The convex hull is our given polygon. Time to rely on your puzzle solving skills and intuition from Euclidean geometry.

Also recall **Ptolemy's Inequality:** Let $ABCD$ be a convex quadrilateral. Then

$$AB \cdot CD + BC \cdot AD \geq AC \cdot BD.$$

1. (Romania TST 2007) Given a convex $2n$-gon $\mathcal{P}$ with vertices $P_1, P_2, \cdots, P_{2n}$ and an interior point Q that does not lie on any diagonals on the $2n$-gon, prove that there is a side so that none of the lines $QP_i, i = 1, 2, \cdots, 2n$ intersect that side at its interior.

2. (USA 2003) A convex polygon $\mathcal{P}$ in the plane is dissected into smaller convex polygons by drawing all of its diagonals. The lengths of all sides and all diagonals of the polygon $\mathcal{P}$ are rational numbers. Prove that the length of all sides of the polygons in the dissection are also rational numbers.

3. (Romania TST 2008) For any convex n-gon, there exists $n-2$ points in the interior such that every triangle formed by three vertices of the n-gon contains at exactly one of these points in its interior.

4. (Romania TST 2008) Let $ABCDEF$ be a convex hexagon with all side lengths 1. Prove that the circumradius of one of ΔACE, ΔBDF is at least 1.

5. (Russia 1994) Let $n, k \in \mathbb{N}$ with $2 \leq k \leq \lfloor n/2 \rfloor$. A set S of k points are in the interior of a given convex $2n$-gon $\mathcal{P}$. Prove that there exists $2k$ vertices on $\mathcal{P}$ whose corresponding $2k$-gon contains the points in S.

6. (IMO 1996 Shortlist) Let $ABCDEF$ be a convex hexagon with $AB = BC, CD = DE, EF = FA$. Prove that
$$\frac{AB}{BE} + \frac{CD}{DA} + \frac{EF}{FC} \geq \frac{3}{2}.$$

7. (Romania TST 2005) Let n be a positive integer. A convex $(4n+2)$-gon has area 1. Prove that there exists two consecutive sides of the $(4n+2)$-gon which form a triangle of area at most $1/6n$.

8. (IMO 2006) Let $\mathcal{P}$ be a regular 2006-gon. A diagonal is called good if its endpoints divide the boundary of $\mathcal{P}$ into two parts, each composed of an odd number of sides of $\mathcal{P}$. The sides of $\mathcal{P}$ are also called good. Suppose $\mathcal{P}$ has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of $\mathcal{P}$. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.

9. (Iran 2005) A simple polygon is one where the perimeter of the polygon does not intersect itself (but is not necessarily convex). Prove that a simple polygon $\mathcal{P}$ contains a diagonal which is completely inside $\mathcal{P}$ such that the diagonal divides the perimeter into two parts both containing at least $n/3-1$ vertices. (Do not count the vertices which are endpoints of the diagonal.)

10. (IMO 2006) Assign to each side b of a convex polygon of $\mathcal{P}$ the maximum area of a triangle that has b as a side and is contained in $\mathcal{P}$. Show that the sum of the areas assigned to the sides of $\mathcal{P}$ is at least twice the area of $\mathcal{P}$.

Covering Problems

Not much to say here. It really is Euclidean geometry with extra care and imagination.

1. Prove that a connected figure $P \subseteq \mathbb{R}^2$ with diameter 1 can be covered by a hexagon with side length $\frac{1}{\sqrt{3}}$. (The diameter of P is the smallest r such that $d(x, y) \leq r$ for all $x, y \in P$.)

2. A finite set of points S in $\mathbb{R}^2$ has the property that every three points can be contained in a disc of radius 1. Prove that every point in S can be contained in a disc of radius 1.

3. a.) Prove that a finite collection of squares of total area 3 can be arranged to cover a unit square.

   b.) Prove that a finite collection of squares of total area 1/2 can be placed inside a unit square without overlap.

4. (USA 2007) A square grid on the Euclidean plane consists of all points (m, n), where m and n are integers. Is it possible to cover all grid points by an infinite family of discs with non-overlapping interiors if each disc in the family has radius at least 5?

5. (Putnam 1998) Let $D_1, D_2, \cdots, D_n$ be a collection of open discs (circles without the boundary) in the plane whose union contains a given set E. For $i = 1, 2, \cdots, n$, let D_i' be the open disc concentric with D_i with three times the radius as D_i. Prove that there exists a subset S of $\{1, 2, \cdots, n\}$ such that D_i, D_j are disjoint whenever $i, j \in S$ and $E \subseteq \bigcup_{i \in S} D_i'$.

Miscellaneous (Yet Beautiful) Combinatorial Geometry Problems

1. (Romania TST 1996) Let $n \geq 3$ be a fixed positive integer. A function $f : \mathbb{R}^2 \to \mathbb{R}$ has the property that for any points $P_1, P_2, \cdots, P_n$ which are vertices of a regular n-gon, we have

$$\sum_{i=1}^{n} f(P_i) = 0.$$

   Prove that $f(P) = 0$ for all points $P \in \mathbb{R}^2$.

2. (Germany 2004) Let k be a positive integer. A finite number of chords are drawn on a circle of radius 1 such that every diameter of the circle intersects at most k of the chords. Prove that the sum of the lengths of the chords is less than $k\pi$.

3. (Iran 1998) Let $n \geq 2$ and $C_1, \cdots, C_n$ be circles of radius 1 such that no two circles are tangent, and the union of the n circles are connected. Let S be the set of pairwise intersection points of the circles. Prove that $|S| \geq n$.

4. (Crux) Given $n \geq 4$ points S on a plane such that no three points are collinear and no four points are concyclic, let $f(n)$ be the number of (unordered) pair of points in S such that there exists a circle containing these two points, whose interior or boundary does not contain any other point in S. Prove that $f(n) \leq 3n - 6$.

5. (IMO 1989) Let n, k be positive integers and let S be a set of n points in the plane such that no three points of S are collinear and for every point P of S there are at least k points in S equidistant from P. Prove that $k < \frac{1}{2} + \sqrt{2n}$.

Hints

General Combinatorics

1. Place the $2n$ points onto the Cartesian plane with no two points having the same x-coordinates.
2. Consider the maximum number of lights on in a configuration accessible from the initial configuration of lights.
3. Consider the largest set of points with no three points forming an equilateral triangle.
4. Place points on coordinates. Consider the line $y = 0$. What happens when P approaches $(-\infty, 0)$ and when P approaches $(\infty, 0)$?
5. Colour all sets containing 1 with colour C_1. How should you colour the remaining sets?
6. If l is the minimum number of colours required, how many cables are needed (in terms of l) ?
7. Consider the largest subset X of A satisfying the given condition.
8. There are $\binom{34}{2} = 561$ pairs of sets chosen from $A_2, \cdots, A_{35}$. Since $|A_1| = 27$, what can you conclude? (Thanks to Jarno Sun for the resulting solution.)
9. Keep track of how many calls involving a certain piece of gossip is required so that everyone knows that piece of gossip. Also keep track of the total of new pieces of gossip learned by the six people for each call.
10. Prove the LYM-Inequality. (See Page 3) Define a chain to be a sequence of $n+1$ subsets $S_0, S_1, \cdots, S_n$ of $\{1, 2, \cdots, n\}$ such that $S_0 = \emptyset$, $|S_{i+1}| = |S_i| + 1$. How many maximum chains are there? How many subsets in S can be in a chain?

Convexity and Collinearity Problems

1. Five points in general position satisfy what property ?
2. Same hint as (1).
3. Consider the convex hull and recall that the circumcentre of a triangle is external to the triangle for a certain kind of triangle.
4. Convex hull again. Solve the problem in the case where the points in S are in convex position first. Now can the convex hull contain interior points ?
5. Consider the triangle of maximum area and suppose it has area at most four. Where can the two other points lie?
6. Suppose the points are not collinear. Choose a point $P \in S$ and a line l (that passes through two points in S) such that the distance between P and l is positive and minimal.
7. By induction. Take two points A, B of maximum distance and place the points on the Cartesian plane with A, B having the same y-coordinate.
8. Consider every four-tuple of points in S.
9. Consider the three points in S which forms the triangle of maximum area.
10. Consider the sum $\sum_{T \subseteq S, |T| \geq 3} (-1)^{|T|}$ and compare this to the expression in question. Prove they are equal.

Convex Polygons and Diagonals Problem

1. Consider a diagonal that splits the vertices on either side of the diagonal in half. Consider the side that P is on. Let $P_1, \cdots, P_{n+1}$ be the points on the polygon not containing P. Consider PP_i for $i = 1, \cdots, n-1$.
2. Solve the case for $n = 4$. (This becomes more of an algebra problem.) Now extend to general n.
3. You see it or you don't. Triangulate the polygon in a "zig-zag" way. Place one point in each triangle to solve the problem.
4. Suppose $\angle A + \angle C + \angle E \geq 360^o$. Prove that ΔACE works.
5. Induction on k. For $k = 1$, draw the line through the two points and see what sides this line touches. For general k, consider the convex hull of the $2k$ points.
6. Use Ptolemy's Inequality.
7. For $n = 1$, join AD, BE, CF. Use these diagonals to form three interior-disjoint quadrilaterals. For general n, dissect the polygon into hexagons with no two diagonals incident.
8. Let D be a diagonal of length l. (i.e. joins vertices that are l vertices apart.) Prove that in the resulting $(l+1)$-gon, there are at most $l/2$ good triangles.
9. Triangulate the polygon in any way. Construct a graph with faces as vertices and incidency as edges.
10. Once I can solve this, I'll let you know.

Covering Problems

1. Let A, B be on the boundary or exterior of P such that $d(A, B) = 1$. Draw circles of radii 1 with centres at A, B, then P must be in its interior. Where does length 1 appear in a regular hexagon of side length $1/\sqrt{3}$?
2. Consider triangle of largest area and consider its circumcircle.
3. No hints given.
4. Suppose this can be done. Consider three circles such that the triangle whose vertices are the centres of these circles, contain no other circle.
5. Choose D_1 of maximum radius. Iteratively, choose D_i of maximum radius disjoint from $D_1, \cdots, D_{i-1}$.

Miscelleaneous (Yet Beautiful) Combinatorial Geometry Problems

1. Suppose the convex polygon is centred at 0 and inscribed in a circle of radii 1. Further suppose the vertices are at $e^{2\pi i/n}$ for $i = 0, \cdots, n-1$. Consider another polygon centred at 1 with radii 1 and vertices at $1 + e^{2\pi i/n}$.
2. No hints given.
3. For each intersection point P_i on circle C_j, prove that the number of circles containing P_i is at most the number of intersection points on C_j.
4. Set as a graph where each pair of points are joined by an edge if and only if there exists a circle containing these two points but no other points in S. Prove that this graph is planar.
5. Count pairs $(P_a, (P_b, P_c))$ such that $P_aP_b = P_aP_c$ and a, b, c are pairwise distinct in two different ways.