
USAMO 1995

- April 27th
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- 1** Let p be an odd prime. The sequence $(a_n)_{n \geq 0}$ is defined as follows: $a_0 = 0$, $a_1 = 1, \dots, a_{p-2} = p - 2$ and, for all $n \geq p - 1$, a_n is the least positive integer that does not form an arithmetic sequence of length p with any of the preceding terms. Prove that, for all n , a_n is the number obtained by writing n in base $p - 1$ and reading the result in base p .
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- 2** A calculator is broken so that the only keys that still work are the \sin , \cos , and \tan buttons, and their inverses (the \arcsin , \arccos , and \arctan buttons). The display initially shows 0. Given any positive rational number q , show that pressing some finite sequence of buttons will yield the number q on the display. Assume that the calculator does real number calculations with infinite precision. All functions are in terms of radians.
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- 3** Given a nonisosceles, nonright triangle ABC , let O denote the center of its circumscribed circle, and let A_1 , B_1 , and C_1 be the midpoints of sides BC , CA , and AB , respectively. Point A_2 is located on the ray OA_1 so that OAA_1 is similar to OA_2A . Points B_2 and C_2 on rays OB_1 and OC_1 , respectively, are defined similarly. Prove that lines AA_2 , BB_2 , and CC_2 are concurrent, i.e. these three lines intersect at a point.
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- 4** Suppose q_0, q_1, q_2, \dots is an infinite sequence of integers satisfying the following two conditions:
- (i) $m - n$ divides $q_m - q_n$ for $m > n \geq 0$,
 - (ii) there is a polynomial P such that $|q_n| < P(n)$ for all n
- Prove that there is a polynomial Q such that $q_n = Q(n)$ for all n .
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- 5** Suppose that in a certain society, each pair of persons can be classified as either *amicable* or *hostile*. We shall say that each member of an amicable pair is a *friend* of the other, and each member of a hostile pair is a *foe* of the other. Suppose that the society has n persons and q amicable pairs, and that for every set of three persons, at least one pair is hostile. Prove that there is at least one member of the society whose foes include $q(1 - 4q/n^2)$ or fewer amicable pairs.
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Art of Problem Solving

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