Angle Chasing Problems Showcase

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Let's assume that you have learnt all the theorems and lemmas needed for chasing angles. All of these problems can be solved using angle chasing alone (ya literally), but you are free to solve them as you like. But it is highly encouraged that you try them with angle chasing first, as it will toughen up your angle chasing skills :D

1 Required Theorems

You are really gonna need at least these theorems to continue:

Theorem 1.1. *The sum of the three angles of a triangle equals to* $180°$

Theorem 1.2. *In a triangle* ABC *with cicrcumcenter* O*, we have* $\angle BOC = 2 \times \angle BAC$

Theorem 1.3. *Let* ABC *be a triangle inscribed in a circle* ω*. Show that* $AC \perp CB$ *if and only if* AB *is a diameter of* ω*.*

Theorem 1.4. *Let* O *and* H *denote the circumcenter and orthocenter of an acute* $\triangle ABC$*, respectively. Show that* $\angle BAH = \angle CAO$*.*

Theorem 1.5. *Let* $ABCD$ *be a convex cyclic quadrilateral. Then* $\angle ABC + \angle CDA = 180°$ *and* $\angle ABD = \angle ACD$

Theorem 1.6. *Let* $ABCD$ *be a convex quadrilateral. Then the following are equivalent:*

1. $ABCD$ *is cyclic.*
2. $\angle ABC + \angle CDA = 180°$*.*
3. $\angle ABD = \angle ACD$*.*

Theorem 1.7. *Suppose* $\triangle ABC$ *is inscribed in a circle with center* O*. Let* P *be a point in the plane. Then the following are equivalent:*

1. PA *is tangent to* (ABC)*.*
2. $OA \perp AP$ *.*
3. $\angle PAB = \angle ACB$*.*

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*Thanks to Euclidean Geometry in Mathematical Olympiads (EGMO) - Evan Chen, Yufei Zhao Handouts for being wonderful sources for many of the problems here. They are great resources for further reading.

2 Easy

Problem 2.1. *Let ABC be an acute triangle with circumcenter O, and let K be a point such that KA is tangent to (ABC) and $\angle KCB = 90^\circ$. Point D lies on BC such that $KD \parallel AB$. Show that line DO passes through A.*

Problem 2.2. ***Extremely Useful:*** *Angles around the centers of a triangle ABC:*

1. *If I is the incenter of ABC then $\angle BIC = 90^\circ + \frac{a}{2}, \angle IBC = \frac{b}{2} and \angle ICB = \frac{c}{2}$.*
2. *If H is the orthocenter of ABC then $\angle BHC = 180^\circ - a, \angle HBC = 90^\circ - c$ and $\angle HCB = 90^\circ - b$.*
3. *If O is the circumcenter of ABC then $\angle BOC = 2a$ and $\angle OBC = \angle OCB = 90^\circ - a$.*
4. *If I_a is the A-excenter of ABC then $\angle AI_aB = \frac{c}{2}, \angle AI_aC = \frac{b}{2}$ and $\angle BI_aC = 90^\circ - \frac{a}{2}$*

Problem 2.3. ***Extremely Useful:*** *Pedal triangles of the centers of a triangle ABC:*

1. *If DEF is the triangle formed by projecting the incenter I onto sides BC, AC and AB, then I is the circumcenter of DEF and $\angle EDF = 90^\circ - \frac{a}{2}$.*
2. *If DEF is the triangle formed by projecting the orthocenter H onto sides BC, AC and AB, then H is the incenter of DEF and $\angle EDF = 180^\circ - 2a$.*
3. *The medial triangle of ABC is the pedal triangle of the circumcenter O of ABC and O is its orthocenter.*

Problem 2.4. *In scalene triangle ABC, let K be the intersection of the angle bisector of $\angle A$ and the perpendicular bisector of BC. Prove that the points A, B, C, K are concyclic.*

Problem 2.5. *Let AB and CD be two segments, and let lines AC and BD meet at X. Let the circumcircles of ABX and CDX meet again at O. Prove that triangles OAB and OCD are similar.*

Problem 2.6. *Let L, M, N are the midpoints of BC, CA, AB and AD, BE, CF are altitudes of $\triangle ABC$. Prove that*

- *O is the orthocenter of $\triangle LMN$.*
- *H is the incenter of $\triangle DEF$.*
- *D, E, F, L, M, N all lie on a circle.*
- *Let $BO \cap \odot ABC = Q$. Prove that $AQCH$ is a parallelogram*
- *Prove that the reflection of H on BC lies on the circumcenter.*
- *Prove that the reflection of the **Euler Line**[1] on the sides of $\triangle ABC$ concur at the circumcirle.*

[1]It is the line joining the orthocenter and the circumcenter

Problem 2.7. ***(Miquels theorem)*** *Let ABC be a triangle. Points X, Y and Z lie on sides BC, CA and AB, respectively. Prove that the circumcircles of triangles AYZ, BXZ, CXY meet at a common point.*

Problem 2.8. ***(Simson line)*** *Let ABC be a triangle, and let P be another point on its circumcircle. Let $X; Y; Z$ be the feet of perpendiculars from P to lines $BC; CA; AB$ respectively. Prove that $X; Y; Z$ are collinear.*

Problem 2.9. *Let $\angle AOB$ be a right angle, M and N points on rays OA and OB, respectively. Let $MNPQ$ be a square such that MN separates the points O and P. Find the locus of the center of the square when M and N vary.*

Problem 2.10. *An interior point P is chosen in the rectangle $ABCD$ such that $\angle APD + \angle BPC = 180$. Find $\angle DAP + \angle BCP$.*

Problem 2.11. *Let ABC be an acute triangle inscribed in circle ω. Let X be the midpoint of the arc BC of ω not containing A and define Y, Z similarly. Show that the orthocenter of XYZ is the incenter I of ABC.*

Problem 2.12. *Let ABC be an acute triangle. Let BE and CF be altitudes of $\triangle ABC$, and denote by M the midpoint of BC. Prove that ME, MF, and the line through A parallel to BC are all tangents to (AEF).*

Problem 2.13. *The incircle of $\triangle ABC$ is tangent to BC, CA, AB at D, E, F, respectively. Let M and N be the midpoints of BC and AC, respectively. Ray BI meets line EF at K. Show that $BK \perp CK$. Then show K lies on line MN.*

Problem 2.14. *Prove that if the orthocentre lie on the circumcircle then the triangle is a right angled triangle.*

Problem 2.15. *Prove that the isogonal conjugate of a point is a point at infinity if and only if it lies on the circumcircle.*

Problem 2.16. *Let ABC be a triangle with orthocenter H. If P is a point on (ABC) then its Simson line bisects PH .*

Problem 2.17. *Given a triangle ABC, let P lie on the circumcircle of the triangle and be the midpoint of the arc BC which does not contain A. Draw a straight line l through P so that l is parallel to AB. Denote by k the circle which passes through B, and is tangent to l at the point P. Let Q be the second point of intersection of k and the line AB (if there is no second point of intersection, choose $Q = B$). Prove that $AQ = AC$.*

3 Medium

Problem 3.1. ***IMO 2004 P4:*** ***E*** *Let ABC be an acute triangle with orthocenter H, and let W be a point on the side BC, between B and C. The points M and N are the feet of the altitudes drawn from B and C, respectively. ω_1 is the circumcircle of triangle BWN and X is a point such that WX is a diameter of ω_1. Similarly, ω_2 is the circumcircle of triangle CWM and Y is a point such that WY is a diameter of ω_2. Show that the points X, Y, and H are collinear.*

Problem 3.2. ***IMO Shortlist G1:*** ***E*** *Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.*

Problem 3.3. ***IOM 2017 P1:*** ***E*** *Let $ABCD$ be a parallelogram in which angle at B is obtuse and $AD > AB$. Points K and L on AC such that $\angle ADL = \angle KBA$(the points A, K, C, L are all different, with K between A and L). The line BK intersects the circumcircle ω of ABC at points B and E, and the line EL intersects ω at points E and F. Prove that $BF \parallel AC$.*

Problem 3.4. ***All Russian 2014 Grade 10 Day 1 P4:*** ***E*** *Given a triangle ABC with $AB > BC$, let Ω be the circumcircle. Let M, N lie on the sides AB, BC respectively, such that $AM = CN$. Let K be the intersection of MN and AC. Let P be the incentre of the triangle AMK and Q be the K-excentre of the triangle CNK. If R is midpoint of the arc ABC of Ω then prove that $RP = RQ$.*

Problem 3.5. ***USA TST 2000 P2:*** ***E*** *Let $ABCD$ be a cyclic quadrilateral and let E and F be the feet of perpendiculars from the intersection of diagonals AC and BD to AB and CD, respectively. Prove that EF is perpendicular to the line through the midpoints of AD and BC.*

Problem 3.6. ***IRAN 3rd Round 2016 P1:*** ***E*** *Let ABC be an arbitrary triangle, P is the intersection point of the altitude from C and the tangent line from A to the circumcircle. The bisector of angle A intersects BC at D . PD intersects AB at K, if H is the orthocenter then prove : $HK \perp AD$*

Problem 3.7. ***AoPS:*** ***E*** *I is the incenter of ABC, $PI, QI \perp BC$, PA, QA intersect BC at DE. Prove: $IADE$ is on a circle.*

Problem 3.8. ***IRAN 2nd Round 2016 P6:*** ***E*** *Let ABC be a triangle and X be a point on its circumcircle. Q, P lie on a line BC such that $XQ \perp AC, XP \perp AB$. Let Y be the circumcenter of $\triangle XQP$. Prove that ABC is equilateral triangle if and if only Y moves on a circle when X varies on the circumcircle of ABC*

Problem 3.9. ***BAMO 1999, P2:*** ***E*** *Let $O = (0, 0), A = (0, a)$, and $B = (0, b)$, where $0 < a < b$ are reals. Let γ be a circle with diameter AB and let P be any other point on γ. Line PA meets the x-axis again at Q. Prove that $\angle BQP = \angle BOP$*

Problem 3.10. ***CGMO 2012 P5:*** ***M*** *Let ABC be a triangle. The incircle of $\triangle ABC$ is tangent to AB and AC at D and E respectively. Let O denote the circumcenter of $\triangle BCI$. Prove that $\angle ODB = \angle OEC$*

Problem 3.11. ***Canada 1991 P3:*** ***M*** *Let P be a point inside circle ω. Consider the set of chords of ω that contain P. Prove that their midpoints all lie on a circle.*

Problem 3.12. ***Russia 1996:*** ***M*** *Points E and F are on side BC of convex quadrilateral $ABCD$ (with E closer than F to B). It is known that $\angle BAE = \angle CDF$ and $\angle EAF = \angle FDE$. Prove that $\angle FAC = \angle EDB$.*

Problem 3.13. ***JMO 2011 P5:*** ***M*** *Points A, B, C, D, E lie on a circle ω and point P lies outside the circle. The given points are such that:*

- *lines PB and PD are tangent to ω*
- *P, A, C are collinear*
- *$DE \parallel AC$*

Prove that BE bisects AC.

Problem 3.14. ***Canda 1997 P4:*** ***M*** *The point O is situated inside the parallelogram $ABCD$ such that $\angle AOB + \angle COD = 180°$. Prove that $\angle OBC = \angle ODC$.*

Problem 3.15. ***USAMO 2010 P1:*** ***M*** *Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto linesAX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.*

Problem 3.16. ***Romanian Masters in Mathematics 2018 P1:*** ***M*** *Let $ABCD$ be a cyclic quadrilateral an let P be a point on the side AB. The diagonals AC meets the segments DP at Q. The line through P parallel to CD mmets the extension of the side CB beyond B at K. The line through Q parallel to BD meets the extension of the side CB beyond B at L. Prove that the circumcircles of the triangles BKP and CLQ are tangent .*

Problem 3.17. *D be a point such that $ABDC$ is a parallelogram and E be the intersection of the B, C tangents. Prove that D, E are isogonal conjugates.*

Problem 3.18. ***China 2012 P1:*** ***M*** *In the triangle ABC, $\angle A$ is biggest. On the circumcircle of $\triangle ABC$, let D be the midpoint of $\widehat{ABC}$ and E be the midpoint of $\widehat{ACB}$. The circle c_1 passes through A, B and is tangent to AC at A, the circle c_2 passes through A, E and is tangent AD at A. c_1 and c_2 intersect at A and P. Prove that AP bisects $\angle BAC$.*

Problem 3.19. *Consider a circle C_1 and a point O on it. Circle C_2 with center O, intersects C_1 in two points P and Q. C_3 is a circle which is externally tangent to C_2 at R and internally tangent to C_1 at S and suppose that RS passes through Q. Suppose X and Y are second intersection points of PR and OR with C_1. Prove that QX is parallel with SY.*

4 Not So Easy...

That doesn't mean you shouldn't try them...

Problem 4.1. ***APMO 2014 P5: H*** *Circles ω and Ω meet at points A and B. Let M be the midpoint of the arc AB of circle ω (M lies inside Ω). A chord MP of circle ω intersects Ω at Q (Q lies inside ω). Let ℓ_P be the tangent line to ω at P, and let ℓ_Q be the tangent line to Ω at Q. Prove that the circumcircle of the triangle formed by the lines ℓ_P, ℓ_Q and AB is tangent to Ω.*

Problem 4.2. ***(Poncelet Point): H***

1. *Prove that for a quadruple W, X, Y, Z of points, the nine point circles of the triangles formed by points in this set, the pedal cicles of the points with respect to the triangle formed by the other three are concurrent. We call this the **Poncelet Point** of the quadruple.*

2. *Prove that the **Feurbach Point** is the Poncelet point of the quadruple A, B, C, I where I is the incenter of $\triangle ABC$.*

Problem 4.3. ***Canada 2007 P5: H*** *Let the incircle of triangle ABC touch sides BC, CA and AB at D, E and F, respectively. Let ω, ω_1, ω_2 and ω_3 denote the circumcircles of triangle ABC, AEF, BDF and CDE respectively. Let ω and ω_1 intersect at A and P, ω and ω_2 intersect at B and Q, ω and ω_3 intersect at C and R.*

- *Prove that ω_1, ω_2 and ω_3 intersect in a common point.*
- *Show that PD, QE and RF are concurrent.*

Problem 4.4. ***IMO 2011 P6: H*** *Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b and ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b and ℓ_c is tangent to the circle Γ.*

Problem 4.5. ***Canda 2013 P5: H*** *Let O denote the circumcentre of an acute-angled triangle ABC. Let point P on side AB be such that $\angle BOP = \angle ABC$, and let point Q on side AC be such that $\angle COQ = \angle ACB$. Prove that the reflection of BC in the line PQ is tangent to the circumcircle of triangle APQ.*

Problem 4.6. *Let ABC be an acute-angled triangle, and let P and Q be two points on its side BC. Construct a point C_1 in such a way that the convex quadrilateral $APBC_1$ is cyclic, $QC_1 \parallel CA$, and the points C_1 and Q lie on opposite sides of the line AB. Construct a point B_1 in such a way that the convex quadrilateral $APCB_1$ is cyclic, $QB1 \parallel BA$, and the points B_1 and Q lie on opposite sides of the line AC.*

Prove that the points $B_1, C_1, P,$ and Q lie on a circle