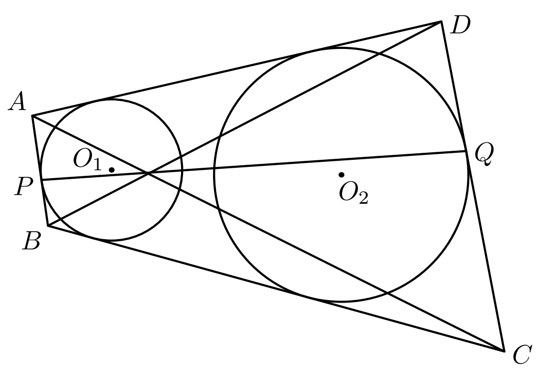
China Girls Math Olympiad 2013

Day 1

1 Let A be the closed region bounded by the following three lines in the xy plane: $x = 1, y = 0$ and $y = t(2x - t)$, where $0 < t < 1$. Prove that the area of any triangle inside the region A, with two vertices $P(t, t^2)$ and $Q(1, 0)$, does not exceed $\frac{1}{4}$.

2 As shown in the figure below, $ABCD$ is a trapezoid, $AB \parallel CD$. The sides DA, AB, BC are tangent to $\odot O_1$ and AB touches $\odot O_1$ at P. The sides BC, CD, DA are tangent to $\odot O_2$, and CD touches $\odot O_2$ at Q. Prove that the lines AC, BD, PQ meet at the same point.



3 In a group of m girls and n boys, any two persons either know each other or do not know each other. For any two boys and any two girls, there are at least one boy and one girl among them,who do not know each other. Prove that the number of unordered pairs of (boy, girl) who know each other does not exceed $m + \frac{n(n-1)}{2}$.

4 Find the number of polynomials $f(x) = ax^3 + bx$ satisfying both following conditions:
(i) $a, b \in \{1, 2, \ldots, 2013\}$;
(ii) the difference between any two of $f(1), f(2), \ldots, f(2013)$ is not a multiple of 2013.

Day 2

5 For any given positive numbers $a_1, a_2, \dots, a_n$, prove that there exist positive numbers $x_1, x_2, \dots, x_n$ satisfying $\sum_{i=1}^{n} x_i = 1$, such that for any positive numbers $y_1, y_2, \dots, y_n$ with $\sum_{i=1}^{n} y_i = 1$, the inequality $\sum_{i=1}^{n} \frac{a_i x_i}{x_i + y_i} \geq \frac{1}{2} \sum_{i=1}^{n} a_i$ holds.

6 Let S be a subset of $\{0, 1, 2, \dots, 98\}$ with exactly $m \geq 3$ (distinct) elements, such that for any $x, y \in S$ there exists $z \in S$ satisfying $x + y \equiv 2z \pmod{99}$. Determine all possible values of m.

7 As shown in the figure, $\odot O_1$ and $\odot O_2$ touches each other externally at a point T, quadrilateral $ABCD$ is inscribed in $\odot O_1$, and the lines DA, CB are tangent to $\odot O_2$ at points E and F respectively. Line BN bisects $\angle ABF$ and meets segment EF at N. Line FT meets the arc $\widehat{AT}$ (not passing through the point B) at another point M different from A. Prove that M is the circumcenter of $\triangle BCN$.

8 Let n (≥ 4) be an even integer. We label n pairwise distinct real numbers arbitrarily on the n vertices of a regular n-gon, and label the n sides clockwise as $e_1, e_2, \dots, e_n$. A side is called *positive* if the numbers on both endpoints are increasing in clockwise direction. An unordered pair of distinct sides $\{e_i, e_j\}$ is called *alternating* if it satisfies both conditions:

(i) $2 \mid (i + j)$; and

(ii) if one rearranges the four numbers on the vertices of these two sides e_i and e_j in increasing order $a < b < c < d$, then a and c are the numbers on the two endpoints of one of sides e_i or e_j.

Prove that the number of alternating pairs of sides and the number of positive sides are of different parity.