Romania Team Selection Test 2014

Day 1

1 Let ABC be a triangle, let A', B', C' be the orthogonal projections of the vertices A ,B ,C on the lines BC, CA and AB, respectively, and let X be a point on the line AA'.Let γ_B be the circle through B and X, centred on the line BC, and let γ_C be the circle through C and X, centred on the line BC.The circle γ_B meets the lines AB and BB' again at M and M', respectively, and the circle γ_C meets the lines AC and CC' again at N and N', respectively.Show that the points M, M', N and N' are collinear.

2 Let $n \geq 2$ be an integer. Show that there exist $n+1$ numbers $x_1, x_2, \ldots, x_{n+1} \in \mathbb{Q} \setminus \mathbb{Z}$, so that $\{x_1^3\} + \{x_2^3\} + \cdots + \{x_n^3\} = \{x_{n+1}^3\}$, where $\{x\}$ is the fractionary part of x.

3 Let $A_0A_1A_2$ be a scalene triangle. Find the locus of the centres of the equilateral triangles $X_0X_1X_2$, such that A_k lies on the line $X_{k+1}X_{k+2}$ for each $k = 0, 1, 2$ (with indices taken modulo 3).

4 Let k be a nonzero natural number and m an odd natural number . Prove that there exist a natural number n such that the number $m^n + n^m$ has at least k distinct prime factors.

5 Let n be an integer greater than 1 and let S be a finite set containing more than $n + 1$ elements.Consider the collection of all sets A of subsets of S satisfying the following two conditions :
(a) Each member of A contains at least n elements of S.
(b) Each element of S is contained in at least n members of A.

Determine $\max_A \min_B |B|$, as B runs through all subsets of A whose members cover S , and A runs through the above collection.

Day 2

1 Let ABC be a triangle and let X,Y ,Z be interior points on the sides BC, CA, AB, respectively. Show that the magnified image of the triangle XYZ under a homothety of factor 4 from its centroid covers at least one of the vertices A, B, C.

2 Let a be a real number in the open interval $(0,1)$. Let $n \geq 2$ be a positive integer and let $f_n \colon \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = x + \frac{x^2}{n}$. Show that

$$\frac{a(1-a)n^2 + 2a^2n + a^3}{(1-a)^2n^2 + a(2-a)n + a^2} < (f_n \circ \ \cdots \ \circ f_n)(a) < \frac{an + a^2}{(1-a)n + a}$$

where there are n functions in the composition.

3 Determine all positive integers n such that all positive integers less than n and coprime to n are powers of primes.

4 Let f be the function of the set of positive integers into itself, defi ned by $f(1) = 1$, $f(2n) = f(n)$ and $f(2n+1) = f(n) + f(n+1)$. Show that, for any positive integer n, the
number of positive odd integers m such that $f(m) = n$ is equal to the number of positive
integers **less or equal to** n and coprime to n.

[mod: the initial statement said less than n, which is wrong.]

Day 3

1 Let ABC be an isosceles triangle, $AB = AC$, and let M and N be points on the sides BC and CA, respectively, such that $\angle BAM = \angle CNM$. The lines AB and MN meet at P. Show that the internal angle bisectors of the angles BAM and BPM meet at a point on the line BC.

2 For every positive integer n, let $\sigma(n)$ denote the sum of all positive divisors of n (1 and n, inclusive). Show that a positive integer n, which has at most two distinct prime factors, satisfies the condition $\sigma(n) = 2n - 2$ if and only if $n = 2^k(2^{k+1} + 1)$, where k is a non-negative integer and $2^{k+1} + 1$ is prime.

3 Determine the smallest real constant c such that

$$\sum_{k=1}^{n} \left(\frac{1}{k} \sum_{j=1}^{k} x_j \right)^2 \leq c \sum_{k=1}^{n} x_k^2$$

for all positive integers n and all positive real numbers $x_1, \cdots, x_n$.

4 Let n be a positive integer and let A_n respectively B_n be the set of nonnegative integers $k < n$ such that the number of distinct prime factors of $\gcd(n,k)$ is even (respectively odd). Show that $|A_n| = |B_n|$ if n is even and $|A_n| > |B_n|$ if n is odd.

Example: $A_{10} = \{0, 1, 3, 7, 9\}$, $B_{10} = \{2, 4, 5, 6, 8\}$.

Day 4

1 Let $\triangle ABC$ be an acute triangle of circumcentre O. Let the tangents to the circumcircle of $\triangle ABC$ in points B and C meet at point P. The circle of centre P and radius $PB = PC$ meets the internal angle bisector of $\angle BAC$ inside $\triangle ABC$ at point S, and $OS \cap BC = D$. The projections of S on AC and AB respectively are E and F. Prove that AD, BE and CF are concurrent.

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2 Let p be an odd prime number. Determine all pairs of polynomials f and g from $\mathbb{Z}[X]$ such that

$$f(g(X)) = \sum_{k=0}^{p-1} X^k = \Phi_p(X).$$

3 Let $n \in \mathbb{N}$ and S_n the set of all permutations of $\{1, 2, 3, ..., n\}$. For every permutation $\sigma \in S_n$ denote $I(\sigma) := \{i : \sigma(i) \leq i\}$.
Compute the sum $\sum_{\sigma \in S_n} \frac{1}{|I(\sigma)|} \sum_{i \in I(\sigma)} (i + \sigma(i))$.

Day 5

1 Let ABC a triangle and O his circumcentre.The lines OA and BC intersect each other at M ; the points N and P are defined in an analogous way.The tangent line in A at the circumcircle of triangle ABC intersect NP in the point X ; the points Y and Z are defined in an analogous way.Prove that the points X , Y and Z are collinear.

2 Let m be a positive integer and let A, respectively B, be two alphabets with m, respectively $2m$ letters. Let also n be an even integer which is at least $2m$. Let a_n be the number of words of length n, formed with letters from A, in which appear all the letters from A, each an even number of times. Let b_n be the number of words of length n, formed with letters from B, in which appear all the letters from B, each an odd number of times. Compute $\frac{b_n}{a_n}$.

3 Let n a positive integer and let $f\colon [0,1] \to \mathbb{R}$ an increasing function. Find the value of :

$$\max_{0\leq x_1\leq\cdots\leq x_n\leq 1} \sum_{k=1}^{n} f\left(\left|x_k - \frac{2k-1}{2n}\right|\right)$$