Pan African 2003

Day 1

1. Let $N_0 = \{0, 1, 2 \cdots\}$. Find all functions: $N_0 \to N_0$ such that:

   (1) $f(n) < f(n+1)$, all $n \in N_0$;

   (2) $f(2) = 2$;

   (3) $f(mn) = f(m)f(n)$, all $m, n \in N_0$.

2. The circumference of a circle is arbitrarily divided into four arcs. The midpoints of the arcs are connected by segments. Show that two of these segments are perpendicular.

3. Does there exists a base in which the numbers of the form:

$$10101, 101010101, 1010101010101, \cdots$$

   are all prime numbers?

Day 2

1. Let $\mathbb{N}_0 = \{0, 1, 2 \cdots\}$. Does there exist a function $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that:

$$f^{2003}(n) = 5n, \forall n \in \mathbb{N}_0$$

where we define: $f^1(n) = f(n)$ and $f^{k+1}(n) = f(f^k(n))$, $\forall k \in \mathbb{N}_0$?

2. Find all positive integers n such that 21 divides $2^{2^n} + 2^n + 1$.

3. Find all functions $f : R \to R$ such that:

$$f(x^2) - f(y^2) = (x + y)(f(x) - f(y)), x, y \in R$$