# Art of Problem Solving
## 2001 USA Team Selection Test

USA Team Selection Test 2001

**Day 1** June 9th

**1** Let $\{a_n\}_{n\geq 0}$ be a sequence of real numbers such that $a_{n+1} \geq a_n^2 + \frac{1}{5}$ for all $n \geq 0$. Prove that $\sqrt{a_{n+5}} \geq a_{n-5}^2$ for all $n \geq 5$.

**2** Express

$$\sum_{k=0}^{n}(-1)^k(n-k)!(n+k)!$$

in closed form.

**3** For a set $S$, let $|S|$ denote the number of elements in $S$. Let $A$ be a set of positive integers with $|A| = 2001$. Prove that there exists a set $B$ such that

(i) $B \subseteq A$;
(ii) $|B| \geq 668$;
(iii) for any $u, v \in B$ (not necessarily distinct), $u + v \notin B$.

**Day 2** June 10th

**4** There are 51 senators in a senate. The senate needs to be divided into $n$ committees so that each senator is on one committee. Each senator hates exactly three other senators. (If senator A hates senator B, then senator B does *not* necessarily hate senator A.) Find the smallest $n$ such that it is always possible to arrange the committees so that no senator hates another senator on his or her committee.

**5** In triangle $ABC$, $\angle B = 2\angle C$. Let $P$ and $Q$ be points on the perpendicular bisector of segment $BC$ such that rays $AP$ and $AQ$ trisect $\angle A$. Prove that $PQ < AB$ if and only if $\angle B$ is obtuse.

**6** Let $a, b, c$ be positive real numbers such that

$$a + b + c \geq abc.$$

Prove that at least two of the inequalities

$$\frac{2}{a} + \frac{3}{b} + \frac{6}{c} \geq 6, \quad \frac{2}{b} + \frac{3}{c} + \frac{6}{a} \geq 6, \quad \frac{2}{c} + \frac{3}{a} + \frac{6}{b} \geq 6$$

are true.

**Day 3**

**7** Let $ABCD$ be a convex quadrilateral such that $\angle ABC = \angle ADC = 135^\circ$ and

$$AC^2 \cdot BD^2 = 2 \cdot AB \cdot BC \cdot CD \cdot DA.$$

Prove that the diagonals of the quadrilateral $ABCD$ are perpendicular.

**8** Find all pairs of nonnegative integers $(m, n)$ such that

$$(m+n-5)^2 = 9mn.$$

**9** Let $A$ be a finite set of positive integers. Prove that there exists a finite set $B$ of positive integers such that $A \subseteq B$ and

$$\prod_{x \in B} x = \sum_{x \in B} x^2.$$

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