3-rd Czech–Polish–Slovak Match 2003

Žilina, June 15–18, 2003

1. Given an integer $n \geq 2$, solve in real numbers the system of equations

$$\begin{array}{rcl} \max\{1, x_1\} & = & x_2 \\ \max\{2, x_2\} & = & 2x_3 \\ \cdots\cdots\cdots & & \cdots \\ \max\{n, x_n\} & = & nx_1. \end{array}$$

2. In an acute-angled triangle ABC the angle at B is greater than 45°. Points D, E, F are the feet of the altitudes from A, B, C respectively, and K is the point on segment AF such that $\angle DKF = \angle KEF$.

   (a) Show that such a point K always exists.

   (b) Prove that $KD^2 = FD^2 + AF \cdot BF$.

3. Numbers p, q, r lie in the interval $(\frac{2}{5}, \frac{5}{2})$ and satisfy $pqr = 1$. Prove that there exist two triangles of the same area, one with the sides a, b, c and the other with the sides pa, qb, rc.

4. Point P lies on the median from vertex C of a triangle ABC. Line AP meets BC at X, and line BP meets AC at Y. Prove that if quadrilateral $ABXY$ is cyclic, then triangle ABC is isosceles.

5. Find all natural numbers $n \geq 2$ for which all binomial coefficients

$$\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}$$

   are even numbers.

6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the condition

$$f(f(x) + y) = 2x + f(f(y) - x) \quad \text{for all } x, y \in \mathbb{R}.$$