Art of Problem Solving

2009 Sharygin Geometry Olympiad

Sharygin Geometry Olympiad 2009

1 Points B_1 and B_2 lie on ray AM, and points C_1 and C_2 lie on ray AK. The circle with center O is inscribed into triangles AB_1C_1 and AB_2C_2. Prove that the angles B_1OB_2 and C_1OC_2 are equal.

2 Given nonisosceles triangle ABC. Consider three segments passing through different vertices of this triangle and bisecting its perimeter. Are the lengths of these segments certainly different?

3 The bisectors of trapezoid's angles form a quadrilateral with perpendicular diagonals. Prove that this trapezoid is isosceles.

4 Let P and Q be the common points of two circles. The ray with origin Q reflects from the first circle in points A_1, A_2,... according to the rule "the angle of incidence is equal to the angle of reflection". Another ray with origin Q reflects from the second circle in the points B_1, B_2,... in the same manner. Points A_1, B_1 and P occurred to be collinear. Prove that all lines A_iB_i pass through P.

5 Given triangle ABC. Point O is the center of the excircle touching the side BC. Point O_1 is the reflection of O in BC. Determine angle A if O_1 lies on the circumcircle of ABC.

6 Find the locus of excenters of right triangles with given hypotenuse.

7 Given triangle ABC. Points M, N are the projections of B and C to the bisectors of angles C and B respectively. Prove that line MN intersects sides AC and AB in their points of contact with the incircle of ABC.

8 Some polygon can be divided into two equal parts by three different ways. Is it certainly valid that this polygon has an axis or a center of symmetry?

9 Given n points on the plane, which are the vertices of a convex polygon, $n > 3$. There exists k regular triangles with the side equal to 1 and the vertices at the given points.
- Prove that $k < \frac{2}{3}n$. - Construct the configuration with $k > 0.666n$.

10 Let ABC be an acute triangle, CC_1 its bisector, O its circumcenter. The perpendicular from C to AB meets line OC_1 in a point lying on the circumcircle of AOB. Determine angle C.

11 Given quadrilateral $ABCD$. The circumcircle of ABC is tangent to side CD, and the circumcircle of ACD is tangent to side AB. Prove that the length of diagonal AC is less than the distance between the midpoints of AB and CD.

12 Let CL be a bisector of triangle ABC. Points A_1 and B_1 are the reflections of A and B in CL, points A_2 and B_2 are the reflections of A and B in L. Let O_1 and O_2 be the circumcenters of triangles AB_1B_2 and BA_1A_2 respectively. Prove that angles O_1CA and O_2CB are equal.

13 In triangle ABC, one has marked the incenter, the foot of altitude from vertex C and the center of the excircle tangent to side AB. After this, the triangle was erased. Restore it.

14 Given triangle ABC of area 1. Let BM be the perpendicular from B to the bisector of angle C. Determine the area of triangle AMC.

15 Given a circle and a point C not lying on this circle. Consider all triangles ABC such that points A and B lie on the given circle. Prove that the triangle of maximal area is isosceles.

16 Three lines passing through point O form equal angles by pairs. Points A_1, A_2 on the first line and B_1, B_2 on the second line are such that the common point C_1 of A_1B_1 and A_2B_2 lies on the third line. Let C_2 be the common point of A_1B_2 and A_2B_1. Prove that angle C_1OC_2 is right.

17 Given triangle ABC and two points X, Y not lying on its circumcircle. Let A_1, B_1, C_1 be the projections of X to BC, CA, AB, and A_2, B_2, C_2 be the projections of Y. Prove that the perpendiculars from A_1, B_1, C_1 to B_2C_2, C_2A_2, A_2B_2, respectively, concur if and only if line XY passes through the circumcenter of ABC.

18 Given three parallel lines on the plane. Find the locus of incenters of triangles with vertices lying on these lines (a single vertex on each line).

19 Given convex n-gon $A_1 \dots A_n$. Let P_i $(i = 1, \dots, n)$ be such points on its boundary that A_iP_i bisects the area of polygon. All points P_i don't coincide with any vertex and lie on k sides of n-gon. What is the maximal and the minimal value of k for each given n?

20 Suppose H and O are the orthocenter and the circumcenter of acute triangle ABC; AA_1, BB_1 and CC_1 are the altitudes of the triangle. Point C_2 is the reflection of C in A_1B_1. Prove that H, O, C_1 and C_2 are concyclic.

21 The opposite sidelines of quadrilateral $ABCD$ intersect at points P and Q. Two lines passing through these points meet the side of $ABCD$ in four points which are the vertices of a parallelogram. Prove that the center of this parallelogram lies on the line passing through the midpoints of diagonals of $ABCD$.

22 Construct a quadrilateral which is inscribed and circumscribed, given the radii of the respective circles and the angle between the diagonals of quadrilateral.

23 Is it true that for each n, the regular $2n$-gon is a projection of some polyhedron having not greater than $n+2$ faces?

24 A sphere is inscribed into a quadrangular pyramid. The point of contact of the sphere with the base of the pyramid is projected to the edges of the base. Prove that these projections are concyclic.