Canada National Olympiad 2005

1 An equilateral triangle of side length n is divided into unit triangles. Let $f(n)$ be the number of paths from the triangle in the top row to the middle triangle in the bottom row, such that adjacent triangles in a path share a common edge and the path never travels up (from a lower row to a higher row) or revisits a triangle. An example is shown on the picture for $n = 5$. Determine the value of $f(2005)$.

2 Let (a, b, c) be a Pythagorean triple, i.e. a triplet of positive integers with $a^2 + b^2 = c^2$.

a) Prove that $\left(\frac{c}{a} + \frac{c}{b}\right)^2 > 8$. b) Prove that there are no integer n and Pythagorean triple (a, b, c) satisfying $\left(\frac{c}{a} + \frac{c}{b}\right)^2 = n$.

3 Let S be a set of $n \geq 3$ points in the interior of a circle. a) Show that there are three distinct points $a, b, c \in S$ and three distinct points A, B, C on the circle such that a is (strictly) closer to A than any other point in S, b is closer to B than any other point in S and c is closer to C than any other point in S. b) Show that for no value of n can four such points in S (and corresponding points on the circle) be guaranteed.

4 Let ABC be a triangle with circumradius R, perimeter P and area K. Determine the maximum value of: $\frac{KP}{R^3}$.

5 Let's say that an ordered triple of positive integers (a, b, c) is [i]n-powerful[/i] if $a \leq b \leq c, \gcd(a, b, c) = 1$ and $a^n + b^n + c^n$ is divisible by $a + b + c$. For example, $(1, 2, 2)$ is 5-powerful. a) Determine all ordered triples (if any) which are n-powerful for all $n \geq 1$. b) Determine all ordered triples (if any) which are 2004-powerful and 2005-powerful, but not 2007-powerful.