Tournament Of Towns 2009

Day 1

$\boxed{1}$ Each of 10 identical jars contains some milk, up to 10 percent of its capacity. At any time, we can tell the precise amount of milk in each jar. In a move, we may pour out an exact amount of milk from one jar into each of the other 9 jars, the same amount in each case. Prove that we can have the same amount of milk in each jar after at most 10 moves.

(4 points)

$\boxed{2}$ Mike has 1000 unit cubes. Each has 2 opposite red faces, 2 opposite blue faces and 2 opposite white faces. Mike assembles them into a $10 \times 10 \times 10$ cube. Whenever two unit cubes meet face to face, these two faces have the same colour. Prove that an entire face of the $10 \times 10 \times 10$ cube has the same colour.

(6 points)

$\boxed{3}$ Find all positive integers a and b such that $(a + b^2)(b + a^2) = 2^m$ for some integer m.

(6 points)

$\boxed{4}$ Let $ABCD$ be a rhombus. P is a point on side BC and Q is a point on side CD such that $BP = CQ$. Prove that centroid of triangle APQ lies on the segment BD.

(6 points)

$\boxed{5}$ We have N objects with weights $1, 2, \cdots, N$ grams. We wish to choose two or more of these objects so that the total weight of the chosen objects is equal to average weight of the remaining objects. Prove that

(a) (2 points) if $N + 1$ is a perfect square, then the task is possible;

(b) (6 points) if the task is possible, then $N + 1$ is a perfect square.

$\boxed{6}$ On an in

finite chessboard are placed 2009 $n \times n$ cardboard pieces such that each of them covers exactly n^2 cells of the chessboard. Prove that the number of cells of the chessboard which are covered by odd numbers of cardboard pieces is at least n^2.

(9 points)

$\boxed{7}$ Anna and Ben decided to visit Archipelago with 2009 islands. Some pairs of islands are connected by boats which run both ways. Anna and Ben are playing during the trip:

Anna chooses the

first island on which they arrive by plane. Then Ben chooses the next island which they could visit. Thereafter, the two take turns choosing an island which they have not yet visited. When they arrive at an island which is connected only to islands they had already visited, whoever's turn to choose next would be the loser. Prove that Anna could always win, regardless of the way Ben played and regardless of the way the islands were connected.

(12 points for Juniors and 10 points for Seniors)

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Day 2

$\boxed{1}$ Is it possible to cut a square into nine squares and colour one of them white, three of them grey and

ve of them black, such that squares of the same colour have the same size and squares of different colours will have different sizes?

(3 points)

$\boxed{2}$ There are forty weights: $1, 2, \cdots, 40$ grams. Ten weights with even masses were put on the left pan of a balance. Ten weights with odd masses were put on the right pan of the balance. The left and the right pans are balanced. Prove that one pan contains two weights whose masses differ by exactly 20 grams.

(4 points)

$\boxed{3}$ A cardboard circular disk of radius 5 centimeters is placed on the table. While it is possible, Peter puts cardboard squares with side 5 centimeters outside the disk so that:

(1) one vertex of each square lies on the boundary of the disk; *(2)* the squares do not overlap; *(3)* each square has a common vertex with the preceding one.

Find how many squares Peter can put on the table, and prove that the fi

rst and the last of them must also have a common vertex.

(4 points)

$\boxed{4}$ We only know that the password of a safe consists of 7 different digits. The safe will open if we enter 7 different digits, and one of them matches the corresponding digit of the password. Can we open this safe in less than 7 attempts?

(5 points for Juniors and 4 points for Seniors)

$\boxed{5}$ A new website registered 2000 people. Each of them invited 1000 other registered people to be their friends. Two people are considered to be friends if and only if they have invited each other. What is the minimum number of pairs of friends on this website?

(5 points)

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Day 3

1 One hundred pirates played cards. When the game was over, each pirate calculated the amount he won or lost. The pirates have a gold sand as a currency; each has enough to pay his debt.

Gold could only change hands in the following way. Either one pirate pays an equal amount to every other pirate, or one pirate receives the same amount from every other pirate.

Prove that after several such steps, it is possible for each winner to receive exactly what he has won and for each loser to pay exactly what he has lost.

(4 points)

2 A non-square rectangle is cut into N rectangles of various shapes and sizes. Prove that one can always cut each of these rectangles into two rectangles so that one can construct a square and rectangle, each

figure consisting of N pieces.

(6 points)

3 Every edge of a tetrahedron is tangent to a given sphere. Prove that the three line segments joining the points of tangency of the three pairs of opposite edges of the tetrahedron are concurrent.

(7 points)

4 Denote by $[n]!$ the product $1 \cdot 11 \cdot 111 \cdot ... \cdot \underbrace{111...1}_{\text{n ones}}$.($n$ factors in total). Prove that $[n+m]!$ is divisible by $[n]! \times [m]!$

(8 points)

5 Let XYZ be a triangle. The convex hexagon $ABCDEF$ is such that $AB; CD$ and EF are parallel and equal to $XY; YZ$ and ZX, respectively. Prove that area of triangle with vertices at the midpoints of $BC; DE$ and FA is no less than area of triangle XYZ.

(8 points)

6 Anna and Ben decided to visit Archipelago with 2009 islands. Some pairs of islands are connected by boats which run both ways. Anna and Ben are playing during the trip:

Anna chooses the

first island on which they arrive by plane. Then Ben chooses the next island which they could visit. Thereafter, the two take turns choosing an island which they have not yet visited. When they arrive at an island which is connected only to islands they had already visited, whoever's turn to choose next would be the loser. Prove that Anna could always win, regardless of the way Ben played and regardless of the way the islands were connected.

(12 points for Juniors and 10 points for Seniors)

7 At the entrance to a cave is a rotating round table. On top of the table are n identical barrels, evenly spaced along its circumference. Inside each barrel is a herring either with its head up or its head down. In a move, Ali Baba chooses from 1 to n of the barrels and turns them upside down. Then the table spins around. When it stops, it is impossible to tell which barrels have been turned over. The cave will open if the heads of the herrings in all n barrels are up or are all down. Determine all values of n for which Ali Baba can open the cave in a finite number of moves.

(11 points)

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Day 4

$\boxed{1}$ We only know that the password of a safe consists of 7 different digits. The safe will open if we enter 7 different digits, and one of them matches the corresponding digit of the password. Can we open this safe in less than 7 attempts?

(5 points for Juniors and 4 points for Seniors)

$\boxed{2}$ $A; B; C; D; E$ and F are points in space such that AB is parallel to DE, BC is parallel to EF, CD is parallel to FA, but $AB \neq DE$. Prove that all six points lie in the same plane.

(4 points)

$\boxed{3}$ Are there positive integers $a; b; c$ and d such that $a^3 + b^3 + c^3 + d^3 = 100^{100}$?

(4 points)

$\boxed{4}$ A point is chosen on each side of a regular 2009-gon. Let S be the area of the 2009-gon with vertices at these points. For each of the chosen points, reflect it across the midpoint of its side. Prove that the 2009-gon with vertices at the images of these reflections also has area S.

(4 points)

$\boxed{5}$ A country has two capitals and several towns. Some of them are connected by roads. Some of the roads are toll roads where a fee is charged for driving along them. It is known that any route from the south capital to the north capital contains at least ten toll roads. Prove that all toll roads can be distributed among ten companies so that anybody driving from the south capital to the north capital must pay each of these companies.

(5 points)