APMO 2011

1 Let a, b, c be positive integers. Prove that it is impossible to have all of the three numbers $a^2+b+c, b^2+c+a, c^2+a+b$ to be perfect squares.

2 Five points A_1, A_2, A_3, A_4, A_5 lie on a plane in such a way that no three among them lie on a same straight line. Determine the maximum possible value that the minimum value for the angles $\angle A_iA_jA_k$ can take where i, j, k are distinct integers between 1 and 5.

3 Let ABC be an acute triangle with $\angle BAC = 30^\circ$. The internal and external angle bisectors of $\angle ABC$ meet the line AC at B_1 and B_2, respectively, and the internal and external angle bisectors of $\angle ACB$ meet the line AB at C_1 and C_2, respectively. Suppose that the circles with diameters B_1B_2 and C_1C_2 meet inside the triangle ABC at point P. Prove that $\angle BPC = 90^\circ$.

4 Let n be a fixed positive odd integer. Take $m+2$ **distinct** points $P_0, P_1, \ldots, P_{m+1}$ (where m is a non-negative integer) on the coordinate plane in such a way that the following three conditions are satisfied:
1) $P_0 = (0, 1), P_{m+1} = (n+1, n)$, and for each integer $i, 1 \leq i \leq m$, both x- and y- coordinates of P_i are integers lying in between 1 and n (1 and n inclusive).
2) For each integer $i, 0 \leq i \leq m$, P_iP_{i+1} is parallel to the x-axis if i is even, and is parallel to the y-axis if i is odd.
3) For each pair i, j with $0 \leq i < j \leq m$, line segments P_iP_{i+1} and P_jP_{j+1} share at most 1 point.
Determine the maximum possible value that m can take.

5 Determine all functions $f : \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers, satisfying the following two conditions:
1) There exists a real number M such that for every real number $x, f(x) < M$ is satisfied.
2) For every pair of real numbers x and y,

$$f(xf(y)) + yf(x) = xf(y) + f(xy)$$

is satisfied.