Balkan MO 2003

– May 4th

1 Can one find 4004 positive integers such that the sum of any 2003 of them is not divisible by 2003?

2 Let ABC be a triangle, and let the tangent to the circumcircle of the triangle ABC at A meet the line BC at D. The perpendicular to BC at B meets the perpendicular bisector of AB at E. The perpendicular to BC at C meets the perpendicular bisector of AC at F. Prove that the points D, E and F are collinear.

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3 Find all functions $f:\mathbb{Q}\to\mathbb{R}$ which fulfill the following conditions:

a) $f(1)+1>0$;

b) $f(x+y)-xf(y)-yf(x)=f(x)f(y)-x-y+xy$, for all $x,y\in\mathbb{Q}$;

c) $f(x)=2f(x+1)+x+2$, for every $x\in\mathbb{Q}$.

4 A rectangle $ABCD$ has side lengths $AB=m$, $AD=n$, with m and n relatively prime and both odd. It is divided into unit squares and the diagonal AC intersects the sides of the unit squares at the points $A_1=A, A_2, A_3, \ldots, A_k=C$. Show that

$$A_1A_2-A_2A_3+A_3A_4-\cdots+A_{k-1}A_k=\frac{\sqrt{m^2+n^2}}{mn}.$$