Art of Problem Solving
2008 USA Team Selection Test

USA Team Selection Test 2008

Day 1

1 There is a set of n coins with distinct integer weights $w_1, w_2, \ldots, w_n$. It is known that if any coin with weight w_k, where $1 \le k \le n$, is removed from the set, the remaining coins can be split into two groups of the same weight. (The number of coins in the two groups can be different.) Find all n for which such a set of coins exists.

2 Let P, Q, and R be the points on sides BC, CA, and AB of an acute triangle ABC such that triangle PQR is equilateral and has minimal area among all such equilateral triangles. Prove that the perpendiculars from A to line QR, from B to line RP, and from C to line PQ are concurrent.

3 For a pair $A = (x_1, y_1)$ and $B = (x_2, y_2)$ of points on the coordinate plane, let $d(A, B) = |x_1 - x_2| + |y_1 - y_2|$. We call a pair (A, B) of (unordered) points *harmonic* if $1 < d(A, B) \le 2$. Determine the maximum number of harmonic pairs among 100 points in the plane.

Day 2

4 Prove that for no integer n is $n^7 + 7$ a perfect square.

5 Two sequences of integers, $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$, satisfy the equation

$$(a_n - a_{n-1})(a_n - a_{n-2}) + (b_n - b_{n-1})(b_n - b_{n-2}) = 0$$

for each integer n greater than 2. Prove that there is a positive integer k such that $a_k = a_{k+2008}$.

6 Determine the smallest positive real number k with the following property. Let $ABCD$ be a convex quadrilateral, and let points A_1, B_1, C_1, and D_1 lie on sides AB, BC, CD, and DA, respectively. Consider the areas of triangles AA_1D_1, BB_1A_1, CC_1B_1 and DD_1C_1; let S be the sum of the two smallest ones, and let S_1 be the area of quadrilateral $A_1B_1C_1D_1$. Then we always have $kS_1 \ge S$.

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Day 3

7 Let ABC be a triangle with G as its centroid. Let P be a variable point on segment BC. Points Q and R lie on sides AC and AB respectively, such that $PQ \parallel AB$ and $PR \parallel AC$. Prove that, as P varies along segment BC, the circumcircle of triangle AQR passes through a fixed point X such that $\angle BAG = \angle CAX$.

8 Mr. Fat and Ms. Taf play a game. Mr. Fat chooses a sequence of positive integers $k_1, k_2, \dots, k_n$. Ms. Taf must guess this sequence of integers. She is allowed to give Mr. Fat a red card and a blue card, each with an integer written on it. Mr. Fat replaces the number on the red card with k_1 times the number on the red card plus the number on the blue card, and replaces the number on the blue card with the number originally on the red card. He repeats this process with number k_2. (That is, he replaces the number on the red card with k_2 times the number now on the red card plus the number now on the blue card, and replaces the number on the blue card with the number that was just placed on the red card.) He then repeats this process with each of the numbers $k_3, \dots k_n$, in this order. After has has gone through the sequence of integers, Mr. Fat then gives the cards back to Ms. Taf. How many times must Ms. Taf submit the red and blue cards in order to be able to determine the sequence of integers $k_1, k_2, \dots k_n$?

9 Let n be a positive integer. Given an integer coefficient polynomial $f(x)$, define its [i]signature modulo n[/i] to be the (ordered) sequence $f(1), \dots, f(n)$ modulo n. Of the n^n such n-term sequences of integers modulo n, how many are the signature of some polynomial $f(x)$ if
a) n is a positive integer not divisible by the square of a prime.
b) n is a positive integer not divisible by the cube of a prime.