EGMO 2014

Day 1 April 12th

1 Determine all real constants t such that whenever a, b and c are the lengths of sides of a triangle, then so are $a^2 + bct$, $b^2 + cat$, $c^2 + abt$.

2 Let D and E be points in the interiors of sides AB and AC, respectively, of a triangle ABC, such that $DB = BC = CE$. Let the lines CD and BE meet at F. Prove that the incentre I of triangle ABC, the orthocentre H of triangle DEF and the midpoint M of the arc BAC of the circumcircle of triangle ABC are collinear.

3 We denote the number of positive divisors of a positive integer m by $d(m)$ and the number of distinct prime divisors of m by $\omega(m)$. Let k be a positive integer. Prove that there exist infinitely many positive integers n such that $\omega(n) = k$ and $d(n)$ does not divide $d(a^2+b^2)$ for any positive integers a, b satisfying $a+b = n$.

Day 2 April 13th

4 Determine all positive integers $n \geq 2$ for which there exist integers $x_1, x_2, \ldots, x_{n-1}$ satisfying the condition that if $0 < i < n, 0 < j < n, i \neq j$ and n divides $2i + j$, then $x_i < x_j$.

5 Let n be a positive integer. We have n boxes where each box contains a non-negative number of pebbles. In each move we are allowed to take two pebbles from a box we choose, throw away one of the pebbles and put the other pebble in another box we choose. An initial configuration of pebbles is called *solvable* if it is possible to reach a configuration with no empty box, in a finite (possibly zero) number of moves. Determine all initial configurations of pebbles which are not solvable, but become solvable when an additional pebble is added to a box, no matter which box is chosen.

6 Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$f(y^2 + 2xf(y) + f(x)^2) = (y + f(x))(x + f(y))$$

for all real numbers x and y.