APMO 2004

1 Determine all finite nonempty sets S of positive integers satisfying

$$\frac{i+j}{(i,j)} \qquad \text{is an element of S for all i,j in S,}$$

where (i,j) is the greatest common divisor of i and j.

2 Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Prove that the area of one of the triangles AOH, BOH and COH is equal to the sum of the areas of the other two.

3 Let a set S of 2004 points in the plane be given, no three of which are collinear. Let $\mathcal{L}$ denote the set of all lines (extended indefinitely in both directions) determined by pairs of points from the set. Show that it is possible to colour the points of S with at most two colours, such that for any points p, q of S, the number of lines in $\mathcal{L}$ which separate p from q is odd if and only if p and q have the same colour.

Note: A line ℓ separates two points p and q if p and q lie on opposite sides of ℓ with neither point on ℓ.

4 For a real number x, let $\lfloor x \rfloor$ stand for the largest integer that is less than or equal to x. Prove that

$$\left\lfloor \frac{(n-1)!}{n(n+1)} \right\rfloor$$

is even for every positive integer n.

5 Prove that the inequality

$$\left(a^2+2\right)\left(b^2+2\right)\left(c^2+2\right) \geq 9\left(ab+bc+ca\right)$$

holds for all positive reals a, b, c.