China Girls Math Olympiad 2016

Day 1

1 Let $n \geq 3$ be an integer. Put n^2 cards, each labelled $1, 2, \ldots, n^2$ respectively, in any order into n empty boxes such that there are exactly n cards in each box. One can perform the following operation: one first selects 2 boxes, takes out any 2 cards from each of the selected boxes, and then return the cards to the other selected box. Prove that, for any initial order of the n^2 cards in the boxes, one can perform the operation finitely many times such that the labelled numbers in each box are consecutive integers.

2 In $\triangle ABC, BC = a, CA = b, AB = c$, and Γ is its circumcircle. (1) Determine a necessary and sufficient condition on a, b and c if there exists a unique point $P(P \neq B, P \neq C)$ on the arc BC of Γ not passing through point A such that $PA = PB + PC$. (2) Let P be the unique point stated in (1). If AP bisects BC, prove that $\angle BAC < 60^\circ$.

3 Let m and n are relatively prime integers and $m > 1, n > 1$. Show that:There are positive integers a, b, c such that $m^a = 1 + n^b c$, and n and c are relatively prime.

4 Let n is a positive integers ,$a_1, a_2, \cdots, a_n \in \{0, 1, \cdots, n\}$. For the integer j $(1 \leq j \leq n)$,define b_j is the number of elements in the set $\{i|i \in \{1, \cdots, n\}, a_i \geq j\}$ .For example When $n = 3$,if $a_1 = 1, a_2 = 2, a_3 = 1$,then $b_1 = 3, b_2 = 1, b_3 = 0$. (1) Prove that

$$\sum_{i=1}^{n}(i + a_i)^2 \geq \sum_{i=1}^{n}(i + b_i)^2.$$

(2) Prove that

$$\sum_{i=1}^{n}(i + a_i)^k \geq \sum_{i=1}^{n}(i + b_i)^k,$$

for the integer $k \geq 3$.

Day 2

5 Define a sequence $\{a_n\}$ by

$$S_1 = 1,\ S_{n+1} = \frac{(2 + S_n)^2}{4 + S_n}(n = 1,\ 2,\ 3,\ \cdots).$$

Where S_n the sum of first n terms of sequence $\{a_n\}$.
For any positive integer n ,prove that

$$a_n \geq \frac{4}{\sqrt{9n+7}}.$$

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6 Find the greatest positive integer m, such that one of the 4 letters C, G, M, O can be placed in each cell of a table with m rows and 8 columns, and has the following property: For any two distinct rows in the table, there exists at most one column, such that the entries of these two rows in such a column are the same letter.

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7 In acute triangle $ABC, AB < AC$, I is its incenter, D is the foot of perpendicular from I to BC, altitude AH meets BI, CI at P, Q respectively. Let O be the circumcenter of $\triangle IPQ$, extend AO to meet BC at L. Circumcircle of $\triangle AIL$ meets BC again at N. Prove that $\frac{BD}{CD} = \frac{BN}{CN}$.

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8 Let $\mathbb{Q}$ be the set of rational numbers, $\mathbb{Z}$ be the set of integers. On the coordinate plane, given positive integer m, define

$$A_m = \left\{(x, y) \mid x, y \in \mathbb{Q}, xy \neq 0, \frac{xy}{m} \in \mathbb{Z}\right\}.$$

For segment MN, define $f_m(MN)$ as the number of points on segment MN belonging to set A_m.

Find the smallest real number λ, such that for any line l on the coordinate plane, there exists a constant $\beta(l)$ related to l, satisfying: for any two points M, N on l,

$$f_{2016}(MN) \leq \lambda f_{2015}(MN) + \beta(l)$$
