

# Tournament Of Towns 2001

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## Fall - Junior A-Level

- 1 Do there exist positive integers  $a_1 < a_2 < \cdots < a_{100}$  such that for  $2 \leq k \leq 100$  the greatest common divisor of  $a_{k-1}$  and  $a_k$  is greater than the greatest common divisor of  $a_k$  and  $a_{k+1}$ ?
- 2 Let  $n \geq 3$  be an integer. A circle is divided into  $2n$  arcs by  $2n$  points. Each arc has one of three possible lengths, and no two adjacent arcs have the same lengths. The  $2n$  points are colored alternately red and blue. Prove that the  $n$ -gon with red vertices and the  $n$ -gon with blue vertices have the same perimeter and the same area.
- 3 Let  $n \geq 3$  be an integer. Each row in an  $(n-2) \times n$  array consists of the numbers  $1, 2, \dots, n$  in some order, and the numbers in each column are all different. Prove that this array can be expanded into an  $n \times n$  array such that each row and each column consists of the numbers  $1, 2, \dots, n$ .
- 4 Let  $n \geq 2$  be an integer. A regular  $(2n+1)$ -gon is divided into  $2n-1$  triangles by diagonals which do not meet except at the vertices. Prove that at least three of these triangles are isosceles.
- 5 Alex places a rook on any square of an empty  $8 \times 8$  chessboard. Then he places additional rooks one rook at a time, each attacking an odd number of rooks which are already on the board. A rook attacks to the left, to the right, above and below, and only the first rook in each direction. What is the maximum number of rooks Alex can place on the chessboard?
- 6 Several numbers are written in a row. In each move, Robert chooses any two adjacent numbers in which the one on the left is greater than the one on the right, doubles each of them and then switches them around. Prove that Robert can make only a finite number of moves.
- 7 It is given that  $2^{333}$  is a 101-digit number whose first digit is 1. How many of the numbers  $2^k$ ,  $1 \leq k \leq 332$  have first digit 4?

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## Fall - Junior O-Level

- 1 In the quadrilateral  $ABCD$ ,  $AD$  is parallel to  $BC$ .  $K$  is a point on  $AB$ . Draw the line through  $A$  parallel to  $KC$  and the line through  $B$  parallel to  $KD$ . Prove that these two lines intersect at some point on  $CD$ .
- 2 Clara computed the product of the first  $n$  positive integers, and Valerie computed the product of the first  $m$  even positive integers, where  $m \geq 2$ . They got the same answer. Prove that one of them had made a mistake.
- 3 Kolya is told that two of his four coins are fake. He knows that all real coins have the same weight, all fake coins have the same weight, and the weight of a real coin is greater than that of a fake coin. Can Kolya decide whether he indeed has exactly two fake coins by using a balance twice?
- 4 On an east-west shipping lane are ten ships sailing individually. The first five from the west are sailing eastwards while the other five ships are sailing westwards. They sail at the same constant speed at all times. Whenever two ships meet, each turns around and sails in the opposite direction. When all ships have returned to port, how many meetings of two ships have taken place?
- 5 On the plane is a set of at least four points. If any one point from this set is removed, the resulting set has an axis of symmetry. Is it necessarily true that the whole set has an axis of symmetry?

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## Fall - Senior A-Level

- 1 On the plane is a triangle with red vertices and a triangle with blue vertices.  $O$  is a point inside both triangles such that the distance from  $O$  to any red vertex is less than the distance from  $O$  to any blue vertex. Can the three red vertices and the three blue vertices all lie on the same circle?
- 2 Do there exist positive integers  $a_1 < a_2 < \dots < a_{100}$  such that for  $2 \leq k \leq 100$ , the least common multiple of  $a_{k-1}$  and  $a_k$  is greater than the least common multiple of  $a_k$  and  $a_{k+1}$ ?
- 3 An  $8 \times 8$  array consists of the numbers  $1, 2, \dots, 64$ . Consecutive numbers are adjacent along a row or a column. What is the minimum value of the sum of the numbers along the diagonal?
- 4 Let  $F_1$  be an arbitrary convex quadrilateral. For  $k \geq 2$ ,  $F_k$  is obtained by cutting  $F_{k-1}$  into two pieces along one of its diagonals, flipping one piece over, and the glueing them back together along the same diagonal. What is the maximum number of non-congruent quadrilaterals in the sequence  $\{F_k\}$ ?
- 5 Let  $a$  and  $d$  be positive integers. For any positive integer  $n$ , the number  $a + nd$  contains a block of consecutive digits which constitute the number  $n$ . Prove that  $d$  is a power of 10.
- 6 In a row are 23 boxes such that for  $1 \leq k \leq 23$ , there is a box containing exactly  $k$  balls. In one move, we can double the number of balls in any box by taking balls from another box which has more. Is it always possible to end up with exactly  $k$  balls in the  $k$ -th box for  $1 \leq k \leq 23$ ?
- 7 The vertices of a triangle have coordinates  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(x_3, y_3)$ . For any integers  $h$  and  $k$ , not both 0, both triangles whose vertices have coordinates  $(x_1+h, y_1+k)$ ,  $(x_2+h, y_2+k)$  and  $(x_3+h, y_3+k)$  has no common interior points with the original triangle.
  - (a) Is it possible for the area of this triangle to be greater than  $\frac{1}{2}$ ?
  - (b) What is the maximum area of this triangle?

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## Fall - Senior O-Level

- [1] An altitude of a pentagon is the perpendicular drop from a vertex to the opposite side. A median of a pentagon is the line joining a vertex to the midpoint of the opposite side. If the five altitudes and the five medians all have the same length, prove that the pentagon is regular.
- [2] There exists a block of 1000 consecutive positive integers containing no prime numbers, namely,  $1001! + 2, 1001! + 3, \dots, 1001! + 1001$ . Does there exist a block of 1000 consecutive positive integers containing exactly five prime numbers?
- [3] On an east-west shipping lane are ten ships sailing individually. The first five from the west are sailing eastwards while the other five ships are sailing westwards. They sail at the same constant speed at all times. Whenever two ships meet, each turns around and sails in the opposite direction. When all ships have returned to port, how many meetings of two ships have taken place?
- [4] On top of a thin square cake are triangular chocolate chips which are mutually disjoint. Is it possible to cut the cake into convex polygonal pieces each containing exactly one chip?
- [5] The only pieces on an  $8 \times 8$  chessboard are three rooks. Each moves along a row or a column without running to or jumping over another rook. The white rook starts at the bottom left corner, the black rook starts in the square directly above the white rook, and the red rook starts in the square directly to the right of the white rook. The white rook is to finish at the top right corner, the black rook in the square directly to the left of the white rook, and the red rook in the square directly below the white rook. At all times, each rook must be either in the same row or the same column as another rook. Is it possible to get the rooks to their destinations?

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## Spring - Junior A-Level

- 1 In a certain country 10% of the employees get 90% of the total salary paid in this country. Supposing that the country is divided in several regions, is it possible that in every region the total salary of any 10
- 2 In three piles there are 51, 49, and 5 stones, respectively. You can combine any two piles into one pile or divide a pile consisting of an even number of stones into two equal piles. Is it possible to get 105 piles with one stone in each?
- 3 Point  $A$  lies inside an angle with vertex  $M$ . A ray issuing from point  $A$  is reflected in one side of the angle at point  $B$ , then in the other side at point  $C$  and then returns back to point  $A$  (the ordinary rule of reflection holds). Prove that the center of the circle circumscribed about triangle  $\triangle BCM$  lies on line  $AM$ .
- 4 Several non-intersecting diagonals divide a convex polygon into triangles. At each vertex of the polygon the number of triangles adjacent to it is written. Is it possible to reconstruct all the diagonals using these numbers if the diagonals are erased?
- 5 (a) One black and one white pawn are placed on a chessboard. You may move the pawns in turn to the neighbouring empty squares of the chessboard using vertical and horizontal moves. Can you arrange the moves so that every possible position of the two pawns will appear on the chessboard exactly once? (b) Same question, but you don't have to move the pawns in turn.
- 6 Let  $AH_A$ ,  $BH_B$  and  $CH_C$  be the altitudes of triangle  $\triangle ABC$ . Prove that the triangle whose vertices are the intersection points of the altitudes of triangles  $\triangle AH_BH_C$ ,  $\triangle BH_AH_C$  and  $\triangle CH_AH_B$  is equal to triangle  $\triangle H_AH_BH_C$ .
- 7 Alex thinks of a two-digit integer (any integer between 10 and 99). Greg is trying to guess it. If the number Greg names is correct, or if one of its digits is equal to the corresponding digit of Alex's number and the other digit differs by one from the corresponding digit of Alex's number, then Alex says hot; otherwise, he says cold. (For example, if Alex's number was 65, then by naming any of 64, 65, 66, 55 or 75 Greg will be answered hot, otherwise he will be answered cold.)  
(a) Prove that there is no strategy which guarantees that Greg will guess Alex's number in no more than 18 attempts. (b) Find a strategy for Greg to find out Alex's number (regardless of what the chosen number was) using no more than 24 attempts. (c) Is there a 22 attempt winning strategy for Greg?

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## Spring - Junior O-Level

- 1 The natural number  $n$  can be replaced by  $ab$  if  $a + b = n$ , where  $a$  and  $b$  are natural numbers. Can the number 2001 be obtained from 22 after a sequence of such replacements?
- 2 One of the midlines of a triangle is longer than one of its medians. Prove that the triangle has an obtuse angle.
- 3 Twenty kilograms of cheese are on sale in a grocery store. Several customers are lined up to buy this cheese. After a while, having sold the demanded portion of cheese to the next customer, the salesgirl calculates the average weight of the portions of cheese already sold and declares the number of customers for whom there is exactly enough cheese if each customer will buy a portion of cheese of weight exactly equal to the average weight of the previous purchases. Could it happen that the salesgirl can declare, after each of the first 10 customers has made their purchase, that there just enough cheese for the next 10 customers? If so, how much cheese will be left in the store after the first 10 customers have made their purchases? (The average weight of a series of purchases is the total weight of the cheese sold divided by the number of purchases.)
- 4
  - a. There are 5 identical paper triangles on the table. Each can be moved in any direction parallel to itself (i.e., without rotating it). Is it true that then any one of them can be covered by the 4 others?
  - b. There are 5 identical equilateral paper triangles on the table. Each can be moved in any direction parallel to itself. Prove that any one of them can be covered by the 4 others in this way.
- 5 On a square board divided into  $15 \times 15$  little squares there are 15 rooks that do not attack each other. Then each rook makes one move like that of a knight. Prove that after this is done a pair of rooks will necessarily attack each other.

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## Spring - Senior A-Level

- [1] Find at least one polynomial  $P(x)$  of degree 2001 such that  $P(x) + P(1 - x) = 1$  holds for all real numbers  $x$ .
- [2] At the end of the school year it became clear that for any arbitrarily chosen group of no less than 5 students, 80
- [3] Let  $AH_A$ ,  $BH_B$  and  $CH_C$  be the altitudes of triangle  $\triangle ABC$ . Prove that the triangle whose vertices are the intersection points of the altitudes of  $\triangle AH_BH_C$ ,  $\triangle BH_AH_C$  and  $\triangle CH_AH_B$  is congruent to  $\triangle H_AH_BH_C$ .
- [4] There are two matrices  $A$  and  $B$  of size  $m \times n$  each filled only by 0s and 1s. It is given that along any row or column its elements do not decrease (from left to right and from top to bottom). It is also given that the numbers of 1s in both matrices are equal and for any  $k = 1, \dots, m$  the sum of the elements in the top  $k$  rows of the matrix  $A$  is no less than that of the matrix  $B$ . Prove for any  $l = 1, \dots, n$  the sum of the elements in left  $l$  columns of the matrix  $A$  is no greater than that of the matrix  $B$ .
- [5] In a chess tournament, every participant played with each other exactly once, receiving 1 point for a win,  $1/2$  for a draw and 0 for a loss. (a) Is it possible that for every player  $P$ , the sum of points of the players who were beaten by  $P$  is greater than the sum of points of the players who beat  $P$ ? (b) Is it possible that for every player  $P$ , the first sum is less than the second one?
- [6] Prove that there exist 2001 convex polyhedra such that any three of them do not have any common points but any two of them touch each other (i.e., have at least one common boundary point but no common inner points).
- [7] Several boxes are arranged in a circle. Each box may be empty or may contain one or several chips. A move consists of taking all the chips from some box and distributing them one by one into subsequent boxes clockwise starting from the next box in the clockwise direction.  
(a) Suppose that on each move (except for the first one) one must take the chips from the box where the last chip was placed on the previous move. Prove that after several moves the initial distribution of the chips among the boxes will reappear. (b) Now, suppose that in each move one can take the chips from any box. Is it true that for every initial distribution of the chips you can get any possible distribution?

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## Spring - Senior O-Level

- [1] A bus that moves along a 100 km route is equipped with a computer, which predicts how much more time is needed to arrive at its final destination. This prediction is made on the assumption that the average speed of the bus in the remaining part of the route is the same as that in the part already covered. Forty minutes after the departure of the bus, the computer predicts that the remaining travelling time will be 1 hour. And this predicted time remains the same for the next 5 hours. Could this possibly occur? If so, how many kilometers did the bus cover when these 5 hours passed? (Average speed is the number of kilometers covered divided by the time it took to cover them.)
- [2] The decimal expression of the natural number  $a$  consists of  $n$  digits, while that of  $a^3$  consists of  $m$  digits. Can  $n + m$  be equal to 2001?
- [3] Points  $X$  and  $Y$  are chosen on the sides  $AB$  and  $BC$  of the triangle  $\triangle ABC$ . The segments  $AY$  and  $CX$  intersect at the point  $Z$ . Given that  $AY = YC$  and  $AB = ZC$ , prove that the points  $B$ ,  $X$ ,  $Z$ , and  $Y$  lie on the same circle.
- [4] Two persons play a game on a board divided into  $3 \times 100$  squares. They move in turn: the first places tiles of size  $1 \times 2$  lengthwise (along the long axis of the board), the second, in the perpendicular direction. The loser is the one who cannot make a move. Which of the players can always win (no matter how his opponent plays), and what is the winning strategy?
- [5] Nine points are drawn on the surface of a regular tetrahedron with an edge of 1 cm. Prove that among these points there are two located at a distance (in space) no greater than 0.5 cm.