China Girls Math Olympiad 2009

Day 1

1 Show that there are only finitely many triples (x, y, z) of positive integers satisfying the equation $abc = 2009(a + b + c)$.

2 Right triangle ABC, with $\angle A = 90^\circ$, is inscribed in circle Γ. Point E lies on the interior of arc BC (not containing A) with $EA > EC$. Point F lies on ray EC with $\angle EAC = \angle CAF$. Segment BF meets Γ again at D (other than B). Let O denote the circumcenter of triangle DEF. Prove that A, C, O are collinear.

3 Let n be a given positive integer. In the coordinate set, consider the set of points $\{P_1, P_2, ..., P_{4n+1}\} = \{(x, y)|x, y \in \mathbb{Z}, xy = 0, |x| \leq n, |y| \leq n\}$.

Determine the minimum of $(P_1P_2)^2 + (P_2P_3)^2 + ... + (P_{4n}P_{4n+1})^2 + (P_{4n+1}P_1)^2$.

4 Let n be an integer greater than 3. Points $V_1, V_2, ..., V_n$, with no three collinear, lie on a plane. Some of the segments V_iV_j, with $1 \leq i < j \leq n$, are constructed. Points V_i and V_j are *neighbors* if V_iV_j is constructed. Initially, chess pieces $C_1, C_2, ..., C_n$ are placed at points $V_1, V_2, ..., V_n$ (not necessarily in that order) with exactly one piece at each point. In a move, one can choose some of the n chess pieces, and simultaneously relocate each of the chosen piece from its current position to one of its neighboring positions such that after the move, exactly one chess piece is at each point and no two chess pieces have exchanged their positions. A set of constructed segments is called *harmonic* if for any initial positions of the chess pieces, each chess piece $C_i (1 \leq i \leq n)$ is at the point V_i after a finite number of moves. Determine the minimum number of segments in a harmonic set.

Day 2

5 Let x, y, z be real numbers greater than or equal to 1. Prove that

$$\prod(x^2 - 2x + 2) \leq (xyz)^2 - 2xyz + 2.$$

6 Circle Γ_1, with radius r, is internally tangent to circle Γ_2 at S. Chord AB of Γ_2 is tangent to Γ_1 at C. Let M be the midpoint of arc AB (not containing S), and let N be the foot of the perpendicular from M to line AB. Prove that $AC \cdot CB = 2r \cdot MN$.

7 On a 10×10 chessboard, some $4n$ unit squares are chosen to form a region $\mathcal{R}$. This region $\mathcal{R}$ can be tiled by n 2×2 squares. This region $\mathcal{R}$ can also be tiled by a combination of n pieces of the following types of shapes (*see below*, with rotations allowed).

Determine the value of n.

8 For a positive integer n, $a_n = n\sqrt{5} - \lfloor n\sqrt{5} \rfloor$. Compute the maximum value and the minimum value of $a_1, a_2, \ldots, a_{2009}$.