China Girls Math Olympiad 2011

Day 1

1 Find all positive integers n such that the equation $\frac{1}{x}+\frac{1}{y}=\frac{1}{n}$ has exactly 2011 positive integer solutions (x,y) where $x \leq y$.

2 The diagonals AC, BD of the quadrilateral $ABCD$ intersect at E. Let M, N be the midpoints of AB, CD respectively. Let the perpendicular bisectors of the segments AB, CD meet at F. Suppose that EF meets BC, AD at P, Q respectively. If $MF \cdot CD = NF \cdot AB$ and $DQ \cdot BP = AQ \cdot CP$, prove that $PQ \perp BC$.

3 The positive reals a,b,c,d satisfy $abcd=1$. Prove that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{9}{a+b+c+d} \geqslant \frac{25}{4}$.

4 A tennis tournament has $n>2$ players and any two players play one game against each other (ties are not allowed). After the game these players can be arranged in a circle, such that for any three players A, B, C, if A, B are adjacent on the circle, then at least one of A, B won against C. Find all possible values for n.

Day 2

5 A real number $\alpha \geq 0$ is given. Find the smallest $\lambda = \lambda(\alpha) > 0$, such that for any complex numbers z_1, z_2 and $0 \leq x \leq 1$, if $|z_1| \leq \alpha |z_1 - z_2|$, then $|z_1 - xz_2| \leq \lambda |z_1 - z_2|$.

6 Do there exist positive integers m, n, such that $m^{20}+11^n$ is a square number?

7 There are n boxes $B_1, B_2, \ldots, B_n$ from left to right, and there are n balls in these boxes. If there is at least 1 ball in B_1, we can move one to B_2. If there is at least 1 ball in B_n, we can move one to B_{n-1}. If there are at least 2 balls in B_k, $2 \leq k \leq n-1$ we can move one to B_{k-1}, and one to B_{k+1}. Prove that, for any arrangement of the n balls, we can achieve that each box has one ball in it.

8 The A-excircle (O) of $\triangle ABC$ touches BC at M. The points D, E lie on the sides AB, AC respectively such that $DE \parallel BC$. The incircle (O_1) of $\triangle ADE$

touches DE at N. If $BO_1 \cap DO = F$ and $CO_1 \cap EO = G$, prove that the midpoint of FG lies on MN.