Balkan MO 2013

– June 30th

1 In a triangle ABC, the excircle ω_a opposite A touches AB at P and AC at Q, while the excircle ω_b opposite B touches BA at M and BC at N. Let K be the projection of C onto MN and let L be the projection of C onto PQ. Show that the quadrilateral $MKLP$ is cyclic.

(*Bulgaria*)

2 Determine all positive integers x, y and z such that $x^5 + 4^y = 2013^z$.

(*Serbia*)

3 Let S be the set of positive real numbers. Find all functions $f\colon S^3 \to S$ such that, for all positive real numbers x, y, z and k, the following three conditions are satisfied:

(a) $xf(x, y, z) = zf(z, y, x)$,

(b) $f(x, ky, k^2z) = kf(x, y, z)$,

(c) $f(1, k, k + 1) = k + 1$.

(*United Kingdom*)

4 In a mathematical competition, some competitors are friends; friendship is mutual, that is, when A is a friend of B, then B is also a friend of A.
We say that $n \geq 3$ different competitors $A_1, A_2, \ldots, A_n$ form a *weakly-friendly cycle* if A_i is not a friend of A_{i+1} for $1 \leq i \leq n$ (where $A_{n+1} = A_1$), and there are no other pairs of non-friends among the components of the cycle.

The following property is satisfied:

"for every competitor C and every weakly-friendly cycle $\mathcal{S}$ of competitors not including C, the set of competitors D in $\mathcal{S}$ which are not friends of C has at most one element"

Prove that all competitors of this mathematical competition can be arranged into three rooms, such that every two competitors in the same room are friends.

(*Serbia*)