Baltic Way 2004

–	November 7th
1	Given a sequence $a_1, a_2, \ldots$ of non-negative real numbers satisfying the conditions: 1. $a_n + a_{2n} \geq 3n$; 2. $a_{n+1} + n \leq 2\sqrt{a_n\,(n+1)}$ for all $n \in \mathbb{N}$ (where $\mathbb{N} = \{1, 2, 3, ...\}$). (1) Prove that the inequality $a_n \geq n$ holds for every $n \in \mathbb{N}$. (2) Give an example of such a sequence.
2	Let $P(x)$ be a polynomial with a non-negative coefficients. Prove that if the inequality $P\left(\frac{1}{x}\right)P(x) \geq 1$ holds for $x = 1$, then this inequality holds for each positive x.
3	Let p, q, r be positive real numbers and n a natural number. Show that if $pqr = 1$, then $$\frac{1}{p^n + q^n + 1} + \frac{1}{q^n + r^n + 1} + \frac{1}{r^n + p^n + 1} \leq 1.$$
4	Let x_1, x_2, ..., x_n be real numbers with arithmetic mean X. Prove that there is a positive integer K such that for any integer i satisfying $0 \leq i < K$, we have $\frac{1}{K-i}\sum_{j=i+1}^{K} x_j \leq X$. (In other words, prove that there is a positive integer K such that the arithmetic mean of each of the lists $\{x_1, x_2, ..., x_K\}$, $\{x_2, x_3, ..., x_K\}$, $\{x_3, ..., x_K\}$, ..., $\{x_{K-1}, x_K\}$, $\{x_K\}$ is not greater than X.)
5	Determine the range of the following function defined for integer k, $$f(k) = (k)_3 + (2k)_5 + (3k)_7 - 6k$$ where $(k)_{2n+1}$ denotes the multiple of $2n+1$ closest to k
6	A positive integer is written on each of the six faces of a cube. For each vertex of the cube we compute the product of the numbers on the three adjacent faces. The sum of these products is 1001. What is the sum of the six numbers on the faces?

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7 Find all sets X consisting of at least two positive integers such that for every two elements $m, n \in X$, where $n > m$, there exists an element $k \in X$ such that $n = mk^2$.

8 Let $f(x)$ be a non-constant polynomial with integer coefficients, and let u be an arbitrary positive integer. Prove that there is an integer n such that $f(n)$ has at least u distinct prime factors and $f(n) \neq 0$.

9 A set S of $n-1$ natural numbers is given ($n \geq 3$). There exist at least at least two elements in this set whose difference is not divisible by n. Prove that it is possible to choose a non-empty subset of S so that the sum of its elements is divisible by n.

10 Is there an infinite sequence of prime numbers p_1, p_2, $\ldots$, p_n, p_{n+1}, $\ldots$ such that $|p_{n+1} - 2p_n| = 1$ for each $n \in \mathbb{N}$?

11 Given a table $m \times n$, in each cell of which a number $+1$ or -1 is written. It is known that initially exactly one -1 is in the table, all the other numbers being $+1$. During a move, it is allowed to chose any cell containing -1, replace this -1 by 0, and simultaneously multiply all the numbers in the neighbouring cells by -1 (we say that two cells are neighbouring if they have a common side). Find all (m, n) for which using such moves one can obtain the table containing zeros only, regardless of the cell in which the initial -1 stands.

12 There are $2n$ different numbers in a row. By one move we can interchange any two numbers or interchange any 3 numbers cyclically (choose a, b, c and place a instead of b, b instead of c, c instead of a). What is the minimal number of moves that is always sufficient to arrange the numbers in increasing order ?

13 The 25 member states of the European Union set up a committee with the following rules:
1) the committee should meet daily;
2) at each meeting, at least one member should be represented;
3) at any two different meetings, a different set of member states should be represented;
4) at n^{th} meeting, for every $k < n$, the set of states represented should include at least one state that was represented at the k^{th} meeting.

For how many days can the committee have its meetings?

14 We say that a pile is a set of four or more nuts. Two persons play the following game. They start with one pile of $n \geq 4$ nuts. During a move a player takes one of the piles that they have and split it into two nonempty sets (these sets are not necessarily piles, they can contain arbitrary number of nuts). If the player cannot move, he loses. For which values of n does the first player have a winning strategy?

15 A circle is divided into 13 segments, numbered consecutively from 1 to 13. Five fleas called A, B, C, D and E are sitting in the segments $1, 2, 3, 4$ and 5. A flea is allowed to jump to an empty segment five positions away in either direction around the circle. Only one flea jumps at the same time, and two fleas cannot be in the same segment. After some jumps, the fleas are back in the segments $1, 2, 3, 4, 5$, but possibly in some other order than they started. Which orders are possible ?

16 Through a point P exterior to a given circle pass a secant and a tangent to the circle. The secant intersects the circle at A and B, and the tangent touches the circle at C on the same side of the diameter through P as the points A and B. The projection of the point C on the diameter is called Q. Prove that the line QC bisects the angle $\angle AQB$.

17 Consider a rectangle with sidelengths 3 and 4, pick an arbitrary inner point on each side of this rectangle. Let x, y, z and u denote the side lengths of the quadrilateral spanned by these four points. Prove that $25 \leq x^2 + y^2 + z^2 + u^2 \leq 50$.

18 A ray emanating from the vertex A of the triangle ABC intersects the side BC at X and the circumcircle of triangle ABC at Y. Prove that $\frac{1}{AX} + \frac{1}{XY} \geq \frac{4}{BC}$.

19 Let D be the midpoint of the side BC of a triangle ABC. Let M be a point on the side BC such that $\angle BAM = \angle DAC$. Further, let L be the second intersection point of the circumcircle of the triangle CAM with the side AB, and let K be the second intersection point of the circumcircle of the triangle BAM with the side AC. Prove that $KL \parallel BC$.

20 Three fixed circles pass through the points A and B. Let X be a variable point on the first circle different from A and B. The line AX intersects the other two circles at Y and Z (with Y between X and Z). Show that the ratio $\frac{XY}{YZ}$ is independent of the position of X.

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