USA Team Selection Test 2013

– December TST

1 A social club has $2k+1$ members, each of whom is fluent in the same k languages. Any pair of members always talk to each other in only one language. Suppose that there were no three members such that they use only one language among them. Let A be the number of three-member subsets such that the three distinct pairs among them use different languages. Find the maximum possible value of A.

2 Find all triples (x, y, z) of positive integers such that $x \leq y \leq z$ and

$$x^3(y^3 + z^3) = 2012(xyz + 2).$$

3 Let ABC be a scalene triangle with $\angle BCA = 90^\circ$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. The circumcircle of triangle DKL intersects segment AB at a second point T (other than D). Prove that $\angle ACT = \angle BCT$.

4 Let $f : \mathbb{N} \to \mathbb{N}$ be a function, and let f^m be f applied m times. Suppose that for every $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $f^{2k}(n) = n + k$, and let k_n be the smallest such k. Prove that the sequence $k_1, k_2, \ldots$ is unbounded.

– January TST

1 Two incongruent triangles ABC and XYZ are called a pair of *pals* if they satisfy the following conditions:
(a) the two triangles have the same area;
(b) let M and W be the respective midpoints of sides BC and YZ. The two sets of lengths $\{AB, AM, AC\}$ and $\{XY, XW, XZ\}$ are identical 3-element sets of pairwise relatively prime integers.

Determine if there are infinitely many pairs of triangles that are pals of each other.

2 Let ABC be an acute triangle. Circle ω_1, with diameter AC, intersects side BC at F (other than C). Circle ω_2, with diameter BC, intersects side AC at

E (other than C). Ray AF intersects ω_2 at K and M with $AK < AM$. Ray BE intersects ω_1 at L and N with $BL < BN$. Prove that lines AB, ML, NK are concurrent.

3 In a table with n rows and $2n$ columns where n is a fixed positive integer, we write either zero or one into each cell so that each row has n zeros and n ones. For $1 \le k \le n$ and $1 \le i \le n$, we define $a_{k,i}$ so that the i^{th} zero in the k^{th} row is the $a_{k,i}^{\text{th}}$ column. Let $\mathcal{F}$ be the set of such tables with $a_{1,i} \ge a_{2,i} \ge \cdots \ge a_{n,i}$ for every i with $1 \le i \le n$. We associate another $n \times 2n$ table $f(C)$ from $C \in \mathcal{F}$ as follows: for the k^{th} row of $f(C)$, we write n ones in the columns $a_{n,k} - k + 1, a_{n-1,k} - k + 2, \ldots, a_{1,k} - k + n$ (and we write zeros in the other cells in the row).

(a) Show that $f(C) \in \mathcal{F}$.
(b) Show that $f(f(f(f(f(f(C)))))) = C$ for any $C \in \mathcal{F}$.

4 Determine if there exists a (three-variable) polynomial $P(x,y,z)$ with integer coefficients satisfying the following property: a positive integer n is *not* a perfect square if and only if there is a triple (x,y,z) of positive integers such that $P(x,y,z) = n$.

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