Iran Team Selection Test 2008

## Day 1

**1** Find all functions $f:\mathbb{R}\longrightarrow\mathbb{R}$ such that for each $x,y\in\mathbb{R}$:

$$f(xf(y))+y+f(x)=f(x+f(y))+yf(x)$$

**2** Suppose that $I$ is incenter of triangle $ABC$ and $l'$ is a line tangent to the incircle. Let $l$ be another line such that intersects $AB,AC,BC$ respectively at $C',B',A'$. We draw a tangent from $A'$ to the incircle other than $BC$, and this line intersects with $l'$ at $A_1$. $B_1,C_1$ are similarly defined. Prove that $AA_1,BB_1,CC_1$ are concurrent.

**3** Suppose that $T$ is a tree with $k$ edges. Prove that the $k$-dimensional cube can be partitioned to graphs isomorphic to $T$.

## Day 2

**4** Let $P_1,P_2,P_3,P_4$ be points on the unit sphere. Prove that $\sum_{i\neq j}\frac{1}{|P_i-P_j|}$ takes its minimum value if and only if these four points are vertices of a regular pyramid.

**5** Let $a,b,c>0$ and $ab+bc+ca=1$. Prove that:

$$\sqrt{a^3+a}+\sqrt{b^3+b}+\sqrt{c^3+c}\geq 2\sqrt{a+b+c}.$$

**6** Prove that in a tournament with 799 teams, there exist 14 teams, that can be partitioned into groups in a way that all of the teams in the first group have won all of the teams in the second group.

## Day 3

**7** Let $S$ be a set with $n$ elements, and $F$ be a family of subsets of $S$ with $2^{n-1}$ elements, such that for each $A,B,C\in F$, $A\cap B\cap C$ is not empty. Prove that the intersection of all of the elements of $F$ is not empty.

**8** Find all polynomials $p$ of one variable with integer coefficients such that if $a$ and $b$ are natural numbers such that $a+b$ is a perfect square, then $p(a)+p(b)$ is also a perfect square.

**9** $I_a$ is the excenter of the triangle $ABC$ with respect to $A$, and $AI_a$ intersects the circumcircle of $ABC$ at $T$. Let $X$ be a point on $TI_a$ such that $XI_a^2 = XA.XT$. Draw a perpendicular line from $X$ to $BC$ so that it intersects $BC$ in $A'$. Define $B'$ and $C'$ in the same way. Prove that $AA'$, $BB'$ and $CC'$ are concurrent.

**Day 4**

**10** In the triangle $ABC$, $\angle B$ is greater than $\angle C$. $T$ is the midpoint of the arc $BAC$ from the circumcircle of $ABC$ and $I$ is the incenter of $ABC$. $E$ is a point such that $\angle AEI = 90^\circ$ and $AE \parallel BC$. $TE$ intersects the circumcircle of $ABC$ for the second time in $P$. If $\angle B = \angle IPB$, find the angle $\angle A$.

**11** $k$ is a given natural number. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for each $m, n \in \mathbb{N}$ the following holds:

$$f(m) + f(n) \mid (m+n)^k$$

**12** In the acute-angled triangle $ABC$, $D$ is the intersection of the altitude passing through $A$ with $BC$ and $I_a$ is the excenter of the triangle with respect to $A$. $K$ is a point on the extension of $AB$ from $B$, for which $\angle AKI_a = 90^\circ + \frac{3}{4}\angle C$. $I_aK$ intersects the extension of $AD$ at $L$. Prove that $DI_a$ bisects the angle $\angle AI_aB$ iff $AL = 2R$. ($R$ is the circumradius of $ABC$)