All-Russian Olympiad 2005

– Grade level 9

Day 1

1 Given a parallelogram $ABCD$ with $AB < BC$, show that the circumcircles of the triangles APQ share a second common point (apart from A) as P, Q move on the sides BC, CD respectively s.t. $CP = CQ$.

2 Lesha put numbers from 1 to 22^2 into cells of 22×22 board. Can Oleg always choose two cells, adjacent by the side or by vertex, the sum of numbers in which is divisible by 4?

3 Given three reals $a_1,\ a_2,\ a_3 > 1$, $S = a_1 + a_2 + a_3$. Provided $\frac{a_i^2}{a_i - 1} > S$ for every $i = 1,\ 2,\ 3$ prove that

$$\frac{1}{a_1 + a_2} + \frac{1}{a_2 + a_3} + \frac{1}{a_3 + a_1} > 1.$$

4 Given 365 cards, in which distinct numbers are written. We may ask for any three cards, the order of numbers written in them. Is it always possible to find out the order of all 365 cards by 2000 such questions?

Day 2

1 Ten mutually distinct non-zero reals are given such that for any two, either their sum or their product is rational. Prove that squares of all these numbers are rational.

2 Find the number of subsets $A \subset M = \{2^0,\ 2^1,\ 2^2, \dots, 2^{2005}\}$ such that equation $x^2 - S(A)x + S(B) = 0$ has integral roots, where $S(M)$ is the sum of all elements of M, and $B = M \setminus A$ (A and B are not empty).

3 We have an acute-angled triangle ABC, and AA', BB' are its altitudes. A point D is chosen on the arc ACB of the circumcircle of ABC. If $P = AA' \cap BD, Q = BB' \cap AD$, show that the midpoint of PQ lies on $A'B'$.

4 100 people from 50 countries, two from each countries, stay on a circle. Prove that one may partition them onto 2 groups in such way that neither no two countrymen, nor three consecutive people on a circle, are in the same group.

– Grade level 10

Day 1

1 Find the least positive integer, which may not be represented as $\frac{2^a-2^b}{2^c-2^d}$, where a, b, c, d are positive integers.

2 In a $2 \times n$ array we have positive reals s.t. the sum of the numbers in each of the n columns is 1. Show that we can select a number in each column s.t. the sum of the selected numbers in each row is at most $\frac{n+1}{4}$.

3 Given 2005 distinct numbers $a_1, a_2, \ldots, a_{2005}$. By one question, we may take three different indices $1 \le i < j < k \le 2005$ and find out the set of numbers $\{a_i, a_j, a_k\}$ (unordered, of course). Find the minimal number of questions, which are necessary to find out all numbers a_i.

4 w_B and w_C are excircles of a triangle ABC. The circle w_B' is symmetric to w_B with respect to the midpoint of AC, the circle w_C' is symmetric to w_C with respect to the midpoint of AB. Prove that the radical axis of w_B' and w_C' halves the perimeter of ABC.

Day 2

1 We select 16 cells on an 8×8 chessboard. What is the minimal number of pairs of selected cells in the same row or column?

2 We have an acute-angled triangle ABC, and AA', BB' are its altitudes. A point D is chosen on the arc ACB of the circumcircle of ABC. If $P = AA' \cap BD, Q = BB' \cap AD$, show that the midpoint of PQ lies on $A'B'$.

3 Positive integers $x > 1$ and y satisfy an equation $2x^2 - 1 = y^{15}$. Prove that 5 divides x.

4 A white plane is partitioned onto cells (in a usual way). A finite number of cells are coloured black. Each black cell has an even (0, 2 or 4) adjacent (by the side) white cells. Prove that one may colour each white cell in green or red such that every black cell will have equal number of red and green adjacent cells.

– Grade level 11

Day 1

1 Find the maximal possible finite number of roots of the equation $|x-a_1|+\cdots+|x-a_{50}|=|x-b_1|+\cdots+|x-b_{50}|$, where $a_1,\,a_2,\,\ldots,a_{50},\,b_1,\ldots,\,b_{50}$ are distinct reals.

2 Given 2005 distinct numbers $a_1,\,a_2,\ldots,a_{2005}$. By one question, we may take three different indices $1\le i<j<k\le 2005$ and find out the set of numbers $\{a_i,\,a_j,\,a_k\}$ (unordered, of course). Find the minimal number of questions, which are necessary to find out all numbers a_i.

3 Let A', B', C' be points, in which excircles touch corresponding sides of triangle ABC. Circumcircles of triangles $A'B'C$, $AB'C'$, $A'BC'$ intersect a circumcircle of ABC in points $C_1\ne C$, $A_1\ne A$, $B_1\ne B$ respectively. Prove that a triangle $A_1B_1C_1$ is similar to a triangle, formed by points, in which incircle of ABC touches its sides.

4 Integers $x>2$, $y>1$, $z>0$ satisfy an equation $x^y+1=z^2$. Let p be a number of different prime divisors of x, q be a number of different prime divisors of y. Prove that $p\ge q+2$.

Day 2

1 Do there exist a bounded function $f:\mathbb{R}\to\mathbb{R}$ such that $f(1)>0$ and $f(x)$ satisfies an inequality $f^2(x+y)\ge f^2(x)+2f(xy)+f^2(y)$?

2 Do there exist 12 rectangular parallelepipeds $P_1,\,P_2,\ldots,P_{12}$ with edges parallel to coordinate axes OX, OY, OZ such that P_i and P_j have a common point iff $i\ne j\pm 1$ modulo 12?

3 A quadrilateral $ABCD$ without parallel sides is circumscribed around a circle with centre O. Prove that O is a point of intersection of middle lines of quadrilateral $ABCD$ (i.e. barycentre of points A, B, C, D) iff $OA\cdot OC=OB\cdot OD$.

4 100 people from 25 countries, four from each countries, stay on a circle. Prove that one may partition them onto 4 groups in such way that neither no two countrymans, nor two neighbours will be in the same group.