Canada National Olympiad 2011

1 Consider 70-digit numbers with the property that each of the digits $1, 2, 3, ..., 7$ appear 10 times in the decimal expansion of n (and $8, 9, 0$ do not appear). Show that no number of this form can divide another number of this form.

2 Let $ABCD$ be a cyclic quadrilateral with opposite sides not parallel. Let X and Y be the intersections of AB, CD and AD, BC respectively. Let the angle bisector of $\angle AXD$ intersect AD, BC at E, F respectively, and let the angle bisectors of $\angle AYB$ intersect AB, CD at G, H respectively. Prove that $EFGH$ is a parallelogram.

3 Amy has divided a square into finitely many white and red rectangles, each with sides parallel to the sides of the square. Within each white rectangle, she writes down its width divided by its height. Within each red rectangle, she writes down its height divided by its width. Finally, she calculates x, the sum of these numbers. If the total area of white equals the total area of red, determine the minimum of x.

4 Show that there exists a positive integer N such that for all integers $a > N$, there exists a contiguous substring of the decimal expansion of a, which is divisible by 2011.
Note. A contiguous substring of an integer a is an integer with a decimal expansion equivalent to a sequence of consecutive digits taken from the decimal expansion of a.

5 Let d be a positive integer. Show that for every integer S, there exists an integer $n > 0$ and a sequence of n integers $\epsilon_1, \epsilon_2, ..., \epsilon_n$, where $\epsilon_i = \pm 1$ (not necessarily dependent on each other) for all integers $1 \leq i \leq n$, such that $S = \sum_{i=1}^{n} \epsilon_i (1 + id)^2$.