India
National Olympiad
2002

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1. For a convex hexagon $ABCDEF$ in which each pair of opposite sides is unequal, consider the following statements.

   (a_1) AB is parallel to DE. $(a_2) AE = BD$.

   (b_1) BC is parallel to EF. $(b_2) BF = CE$.

   (c_1) CD is parallel to FA. (c_2) $CA = DF$.

   (a) Show that if all six of these statements are true then the hexagon is cyclic.

   (b) Prove that, in fact, five of the six statements suffice.

2. Find the smallest positive value taken by $a^3 + b^3 + c^3 - 3abc$ for positive integers a, b, c . Find all a, b, c which give the smallest value

3. If x, y are positive reals such that $x + y = 2$ show that $x^3y^3(x^3 + y^3) \leq 2$.

4. Is it true that there exist 100 lines in the plane, no three concurrent, such that they intersect in exactly 2002 points?

5. Do there exist distinct positive integers a, b, c such that a, b, c, $-a+b+c$, $a-b+c$, $a+b-c$, $a+b+c$ form an arithmetic progression (in some order).

6. The numbers $1, 2, 3, \ldots, n^2$ are arranged in an $n \times n$ array, so that the numbers in each row increase from left to right, and the numbers in each column increase from top to bottom. Let a_{ij} be the number in position i, j. Let b_j be the number of possible values for a_{jj}. Show that

$$b_1 + b_2 + \cdots + b_n = \frac{n(n^2 - 3n + 5)}{3}.$$