Art of Problem Solving
2009 Iran MO (3rd Round)

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1 Suppose $n > 2$ and let $A_1, \dots, A_n$ be points on the plane such that no three are collinear.
(a) Suppose $M_1, \dots, M_n$ be points on segments $A_1A_2, A_2A_3, \dots, A_nA_1$ respectively. Prove that if $B_1, \dots, B_n$ are points in triangles $M_2A_2M_1, M_3A_3M_2, \dots, M_1A_1M_n$ respectively then

$$|B_1B_2| + |B_2B_3| + \cdots + |B_nB_1| \le |A_1A_2| + |A_2A_3| + \cdots + |A_nA_1|$$

Where $|XY|$ means the length of line segment between X and Y.

(b) If X, Y and Z are three points on the plane then by H_{XYZ} we mean the half-plane that it's boundary is the exterior angle bisector of angle $X\hat{Y}Z$ and doesn't contain X and Z ,having Y crossed out.
Prove that if $C_1, \dots, C_n$ are points in $H_{A_nA_1A_2}, H_{A_1A_2A_3}, \dots, H_{A_{n-1}A_nA_1}$ then

$$|A_1A_2| + |A_2A_3| + \cdots + |A_nA_1| \le |C_1C_2| + |C_2C_3| + \cdots + |C_nC_1|$$

Time allowed for this problem was 2 hours.

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2 Permutation π of $\{1, \dots, n\}$ is called **stable** if the set $\{\pi(k) - k | k = 1, \dots, n\}$ is consisted of exactly two different elements.
Prove that the number of stable permutation of $\{1, \dots, n\}$ equals to $\sigma(n) - \tau(n)$ in which $\sigma(n)$ is the sum of positive divisors of n and $\tau(n)$ is the number of positive divisors of n.

Time allowed for this problem was 75 minutes.

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3 An arbitary triangle is partitioned to some triangles homothetic with itself. The ratio of homothety of the triangles can be positive or negative.
Prove that sum of all homothety ratios equals to 1.

Time allowed for this problem was 45 minutes.

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4 Does there exists two functions $f, g : \mathbb{R} \to \mathbb{R}$ such that: $\forall x \neq y : |f(x) - f(y)| + |g(x) - g(y)| > 1$

Time allowed for this problem was 75 minutes.

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5 A ball is placed on a plane and a point on the ball is marked.
Our goal is to roll the ball on a polygon in the plane in a way that it comes

back to where it started and the marked point comes to the top of it. Note that We are not allowed to rotate without moving, but only rolling.
Prove that it is possible.

Time allowed for this problem was 90 minutes.

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6 Let z be a complex non-zero number such that $Re(z), Im(z) \in \mathbb{Z}$.
Prove that z is uniquely representable as $a_0+a_1(1+i)+a_2(1+i)^2+\dots+a_n(1+i)^n$ where $n \geq 0$ and $a_j \in \{0,1\}$ and $a_n = 1$.

Time allowed for this problem was 1 hour.

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7 A sphere is inscribed in polyhedral P. The faces of P are coloured with black and white in a way that no two black faces share an edge.
Prove that the sum of surface of black faces is less than or equal to the sum of the surface of the white faces.

Time allowed for this problem was 1 hour.

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8 Sone of vertices of the infinite grid $\mathbb{Z}^2$ are missing. Let's take the remainder as a graph. Connect two edges of the graph if they are the same in one component and their other components have a difference equal to one. Call every connected component of this graph a **branch**.
Suppose that for every natural n the number of missing vertices in the $(2n+1) \times (2n+1)$ square centered by the origin is less than $\frac{n}{2}$.
Prove that among the branches of the graph, exactly one has an infinite number of vertices.

Time allowed for this problem was 90 minutes.