Sharygin Geometry Olympiad 2008

– Grade level 8

1 (B.Frenkin) Does a convex quadrilateral without parallel sidelines exist such that it can be divided into four congruent triangles?

2 (F.Nilov) Given right triangle ABC with hypothenuse AC and $\angle A = 50^\circ$. Points K and L on the cathetus BC are such that $\angle KAC = \angle LAB = 10^\circ$. Determine the ratio CK/LB.

3 (D.Shnol) Two opposite angles of a convex quadrilateral with perpendicular diagonals are equal. Prove that a circle can be inscribed in this quadrilateral.

4 (F.Nilov, A.Zaslavsky) Let CC_0 be a median of triangle ABC; the perpendicular bisectors to AC and BC intersect CC_0 in points A', B'; C_1 is the meet of lines AA' and BB'. Prove that $\angle C_1CA = \angle C_0CB$.

5 (A.Zaslavsky) Given two triangles ABC, $A'B'C'$. Denote by α the angle between the altitude and the median from vertex A of triangle ABC. Angles β, γ, α', β', γ' are defined similarly. It is known that $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$. Can we conclude that the triangles are similar?

6 (B.Frenkin) Consider the triangles such that all their vertices are vertices of a given regular 2008-gon. What triangles are more numerous among them: acute-angled or obtuse-angled?

7 (F.Nilov) Given isosceles triangle ABC with base AC and $\angle B = \alpha$. The arc AC constructed outside the triangle has angular measure equal to β. Two lines passing through B divide the segment and the arc AC into three equal parts. Find the ratio α/β.

8 (B.Frenkin, A.Zaslavsky) A convex quadrilateral was drawn on the blackboard. Boris marked the centers of four excircles each touching one side of the quadrilateral and the extensions of two adjacent sides. After this, Alexey erased the quadrilateral. Can Boris define its perimeter?

– Grade level 9

1 (A.Zaslavsky) A convex polygon can be divided into 2008 congruent quadrilaterals. Is it true that this polygon has a center or an axis of symmetry?

2 (F.Nilov) Given quadrilateral $ABCD$. Find the locus of points such that their projections to the lines AB, BC, CD, DA form a quadrilateral with perpendicular diagonals.

3 (R.Pirkuliev) Prove the inequality

$$\frac{1}{\sqrt{2\sin A}}+\frac{1}{\sqrt{2\sin B}}+\frac{1}{\sqrt{2\sin C}}\leq\sqrt{\frac{p}{r}},$$

where p and r are the semiperimeter and the inradius of triangle ABC.

4 (F.Nilov, A.Zaslavsky) Let CC_0 be a median of triangle ABC; the perpendicular bisectors to AC and BC intersect CC_0 in points A_c, B_c; C_1 is the common point of AA_c and BB_c. Points A_1, B_1 are defined similarly. Prove that circle $A_1B_1C_1$ passes through the circumcenter of triangle ABC.

5 (N.Avilov) Can the surface of a regular tetrahedron be glued over with equal regular hexagons?

6 (B.Frenkin) Construct the triangle, given its centroid and the feet of an altitude and a bisector from the same vertex.

7 (A.Zaslavsky) The circumradius of triangle ABC is equal to R. Another circle with the same radius passes through the orthocenter H of this triangle and intersect its circumcirle in points X, Y. Point Z is the fourth vertex of parallelogram $CXZY$. Find the circumradius of triangle ABZ.

8 (J.-L.Ayme, France) Points P, Q lie on the circumcircle ω of triangle ABC. The perpendicular bisector l to PQ intersects BC, CA, AB in points A', B', C'. Let A", B", C" be the second common points of l with the circles $A'PQ$, $B'PQ$, $C'PQ$. Prove that AA", BB", CC" concur.

– Grade level 10

1 (B.Frenkin) An inscribed and circumscribed n-gon is divided by some line into two inscribed and circumscribed polygons with different numbers of sides. Find n.

2	(A.Myakishev) Let triangle $A_1B_1C_1$ be symmetric to ABC wrt the incenter of its medial triangle. Prove that the orthocenter of $A_1B_1C_1$ coincides with the circumcenter of the triangle formed by the excenters of ABC.
3	(V.Yasinsky, Ukraine) Suppose X and Y are the common points of two circles ω_1 and ω_2. The third circle ω is internally tangent to ω_1 and ω_2 in P and Q respectively. Segment XY intersects ω in points M and N. Rays PM and PN intersect ω_1 in points A and D; rays QM and QN intersect ω_2 in points B and C respectively. Prove that $AB = CD$.
4	(A.Zaslavsky) Given three points C_0, C_1, C_2 on the line l. Find the locus of incenters of triangles ABC such that points A, B lie on l and the feet of the median, the bisector and the altitude from C coincide with C_0, C_1, C_2.
5	(I.Bogdanov) A section of a regular tetragonal pyramid is a regular pentagon. Find the ratio of its side to the side of the base of the pyramid.
6	(B.Frenkin) The product of two sides in a triangle is equal to $8Rr$, where R and r are the circumradius and the inradius of the triangle. Prove that the angle between these sides is less than 60°.
7	(F.Nilov) Two arcs with equal angular measure are constructed on the medians AA' and BB' of triangle ABC towards vertex C. Prove that the common chord of the respective circles passes through C.
8	(A.Akopyan, V.Dolnikov) Given a set of points inn the plane. It is known that among any three of its points there are two such that the distance between them doesn't exceed 1. Prove that this set can be divided into three parts such that the diameter of each part does not exceed 1.
–	First Round
1	(B.Frenkin, 8) Does a regular polygon exist such that just half of its diagonals are parallel to its sides?
2	(V.Protasov, 8) For a given pair of circles, construct two concentric circles such that both are tangent to the given two. What is the number of solutions, depending on location of the circles?
3	(A.Zaslavsky, 8) A triangle can be dissected into three equal triangles. Prove that some its angle is equal to 60°.

4 (D.Shnol, 8–9) The bisectors of two angles in a cyclic quadrilateral are parallel. Prove that the sum of squares of some two sides in the quadrilateral equals the sum of squares of two remaining sides.

5 (Kiev olympiad, 8–9) Reconstruct the square $ABCD$, given its vertex A and distances of vertices B and D from a fixed point O in the plane.

6 (A. Myakishev, 8–9) In the plane, given two concentric circles with the center A. Let B be an arbitrary point on some of these circles, and C on the other one. For every triangle ABC, consider two equal circles mutually tangent at the point K, such that one of these circles is tangent to the line AB at point B and the other one is tangent to the line AC at point C. Determine the locus of points K.

7 (A.Zaslavsky, 8–9) Given a circle and a point O on it. Another circle with center O meets the first one at points P and Q. The point C lies on the first circle, and the lines CP, CQ meet the second circle for the second time at points A and B. Prove that $AB = PQ$.

8 (T.Golenishcheva-Kutuzova, B.Frenkin, 8–11) a) Prove that for $n > 4$, any convex n-gon
can be dissected into n obtuse triangles.

9 (A.Zaslavsky, 9–10) The reflections of diagonal BD of a quadrilateral $ABCD$ in the bisectors of angles B and D pass
through the midpoint of diagonal AC. Prove that the reflections of diagonal AC in the bisectors of angles A and C pass
through the midpoint of diagonal BD (There was an error in published condition of this problem).

10 (A.Zaslavsky, 9–10) Quadrilateral $ABCD$ is circumscribed arounda circle with center I. Prove that the projections of points B and D to the lines IA and IC lie on a single circle.

11 (A.Zaslavsky, 9–10) Given four points A, B, C, D. Any two circles such that one of them contains A and B, and the other one contains C and D, meet. Prove that common chords of all these pairs of circles pass through a fixed point.

12 (A.Myakishev, 9–10) Given a triangle ABC. Point A_1 is chosen on the ray BA so that segments BA_1 and BC are equal. Point A_2 is chosen on the ray CA so that segments CA_2 and BC are equal. Points B_1, B_2 and C_1, C_2 are chosen similarly. Prove that lines A_1A_2, B_1B_2, C_1C_2 are parallel.

13 (A.Myakishev, 9–10) Given triangle ABC. One of its excircles is tangent to the side BC at point A_1 and to the extensions of two other sides. Another excircle is tangent to side AC at point B_1. Segments AA_1 and BB_1 meet at point N. Point P is chosen on the ray AA_1 so that $AP = NA_1$. Prove that P lies on the incircle.

14 (V.Protasov, 9–10) The Euler line of a non-isosceles triangle is parallel to the bisector of one of its angles. Determine this
angle (There was an error in published condition of this problem).

15 (M.Volchkevich, 9–11) Given two circles and point P not lying on them. Draw a line through P which cuts chords of equal length from these circles.

16 (A.Zaslavsky, 9–11) Given two circles. Their common external tangent is tangent to them at points A and B. Points X, Y on these circles are such that some circle is tangent to the given two circles at these points, and in similar way (external or internal). Determine the locus of intersections of lines AX and BY.

17 (A.Myakishev, 9–11) Given triangle ABC and a ruler with two marked intervals equal to AC and BC. By this ruler only, find the incenter of the triangle formed by medial lines of triangle ABC.

18 (A.Abdullayev, 9–11) Prove that the triangle having sides a, b, c and area S satisfies the inequality

$$a^2 + b^2 + c^2 - \frac{1}{2}(|a-b| + |b-c| + |c-a|)^2 \geq 4\sqrt{3}S.$$

19 (V.Protasov, 10-11) Given parallelogram $ABCD$ such that $AB = a$, $AD = b$. The first circle has its center at vertex A and passes through D, and the second circle has its center at C and passes through D. A circle with center B meets the first circle at points M_1, N_1, and the second circle at points M_2, N_2. Determine the ratio M_1N_1/M_2N_2.

20 (A.Zaslavsky, 10–11) a) Some polygon has the following property:
if a line passes through two points which bisect its perimeter then this line

bisects the area of the polygon. Is it true that the polygon is central symmetric?
b) Is it true that any figure with the property from part a) is central symmetric?

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21 (A.Zaslavsky, B.Frenkin, 10–11) In a triangle, one has drawn perpendicular bisectors to its sides and has measured their segments lying inside the triangle.

a) All three segments are equal. Is it true that the triangle is equilateral?

b) Two segments are equal. Is it true that the triangle is isosceles?

c) Can the segments have length 4, 4 and 3?

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22 (A.Khachaturyan, 10–11) a) All vertices of a pyramid lie on the facets of a cube but not on its edges, and each facet contains at least one vertex. What is the maximum possible number of the vertices of the pyramid?

b) All vertices of a pyramid lie in the facet planes of a cube but not on the lines
including its edges, and each facet plane contains at least one vertex. What is the
maximum possible number of the vertices of the pyramid?

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23 (V.Protasov, 10–11) In the space, given two intersecting spheres of different radii and a point A belonging to both spheres. Prove that there is a point B in the space with the following property:
if an arbitrary circle passes through points A and B then the second points of its meet with the given spheres are equidistant from B.

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24 (I.Bogdanov, 11) Let h be the least altitude of a tetrahedron, and d the least distance between its opposite edges. For what values of t the inequality $d > th$ is possible?
