

# Tournament Of Towns 2003

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## Fall Junior A-Level

- [1] An increasing arithmetic progression consists of one hundred positive integers. Is it possible that every two of them are relatively prime?
- [2] Smallville is populated by unmarried men and women, some of them are acquainted. Two city matchmakers are aware of all acquaintances. Once, one of matchmakers claimed: I could arrange that every brunette man would marry a woman he was acquainted with. The other matchmaker claimed I could arrange that every blonde woman would marry a man she was acquainted with. An amateur mathematician overheard their conversation and said Then both arrangements could be done at the same time! Is he right?
- [3] Find all positive integers  $k$  such that there exist two positive integers  $m$  and  $n$  satisfying

$$m(m+k) = n(n+1).$$

- [4] Several squares on a  $15 \times 15$  chessboard are marked so that a bishop placed on any square of the board attacks at least two of marked squares. Find the minimal number of marked squares.
- [5] A point  $O$  lies inside of the square  $ABCD$ . Prove that the difference between the sum of angles  $OAB, OBC, OCD, ODA$  and  $180^\circ$  does not exceed  $45^\circ$ .
- [6] An ant crawls on the outer surface of the box in a shape of rectangular parallelepiped. From ants point of view, the distance between two points on a surface is defined by the length of the shortest path ant need to crawl to reach one point from the other. Is it true that if ant is at vertex then from ants point of view the opposite vertex be the most distant point on the surface?
- [7] Two players in turn play a game. First Player has cards with numbers  $2, 4, \dots, 2000$  while Second Player has cards with numbers  $1, 3, \dots, 2001$ . In each his turn, a player chooses one of his cards and puts it on a table; the opponent sees it and puts his card next to the first one. Player, who put the card with a larger number, scores 1 point. Then both cards are discarded. First Player starts. After 1000 turns the game is over; First Player has used all his cards and Second Player used all but one. What are the maximal scores, that players could guarantee for themselves, no matter how the opponent would play?

## Fall Junior O-Level

- 1 There is  $3 \times 4 \times 5$  - box with its faces divided into  $1 \times 1$  - squares. Is it possible to place numbers in these squares so that the sum of numbers in every stripe of squares (one square wide) circling the box, equals 120?
- 2 In 7-gon  $A_1A_2A_3A_4A_5A_6A_7$  diagonals  $A_1A_3, A_2A_4, A_3A_5, A_4A_6, A_5A_7, A_6A_1$  and  $A_7A_2$  are congruent to each other and diagonals  $A_1A_4, A_2A_5, A_3A_6, A_4A_7, A_5A_1, A_6A_2$  and  $A_7A_3$  are also congruent to each other. Is the polygon necessarily regular?
- 3 For any integer  $n + 1, \dots, 2n$  ( $n$  is a natural number) consider its greatest odd divisor. Prove that the sum of all these divisors equals  $n^2$ .
- 4 There are  $N$  points on the plane; no three of them belong to the same straight line. Every pair of points is connected by a segment. Some of these segments are colored in red and the rest of them in blue. The red segments form a closed broken line without self-intersections (each red segment having only common endpoints with its two neighbors and no other common points with the other segments), and so do the blue segments. Find all possible values of  $N$  for which such a disposition of  $N$  points and such a choice of red and blue segments are possible.
- 5 25 checkers are placed on 25 leftmost squares of  $1 \times N$  board. Checker can either move to the empty adjacent square to its right or jump over adjacent right checker to the next square if it is empty. Moves to the left are not allowed. Find minimal  $N$  such that all the checkers could be placed in the row of 25 successive squares but in the reverse order.

## Fall Senior A-Level

- [1] Smallville is populated by unmarried men and women, some of them are acquainted. Two city matchmakers are aware of all acquaintances. Once, one of matchmakers claimed: I could arrange that every brunette man would marry a woman he was acquainted with. The other matchmaker claimed I could arrange that every blonde woman would marry a man she was acquainted with. An amateur mathematician overheard their conversation and said Then both arrangements could be done at the same time! Is he right?

- [2] Prove that every positive integer can be represented in the form

$$3^{u_1}2^{v_1} + 3^{u_2}2^{v_2} + \dots + 3^{u_k}2^{v_k}$$

with integers  $u_1, u_2, \dots, u_k, v_1, \dots, v_k$  such that  $u_1 > u_2 > \dots > u_k \geq 0$  and  $0 \leq v_1 < v_2 < \dots < v_k$ .

- [3] An ant crawls on the outer surface of the box in a shape of rectangular parallelepiped. From ants point of view, the distance between two points on a surface is defined by the length of the shortest path ant need to crawl to reach one point from the other. Is it true that if ant is at vertex then from ants point of view the opposite vertex be the most distant point on the surface?
- [4] In a triangle  $ABC$ , let  $H$  be the point of intersection of altitudes,  $I$  the center of incircle,  $O$  the center of circumcircle,  $K$  the point where the incircle touches  $BC$ . Given that  $IO$  is parallel to  $BC$ , prove that  $AO$  is parallel to  $HK$ .
- [5] Two players in turn play a game. First Player has cards with numbers  $2, 4, \dots, 2000$  while Second Player has cards with numbers  $1, 3, \dots, 2001$ . In each his turn, a player chooses one of his cards and puts it on a table; the opponent sees it and puts his card next to the first one. Player, who put the card with a larger number, scores 1 point. Then both cards are discarded. First Player starts. After 1000 turns the game is over; First Player has used all his cards and Second Player used all but one. What are the maximal scores, that players could guarantee for themselves, no matter how the opponent would play?
- [6] Let  $O$  be the center of insphere of a tetrahedron  $ABCD$ . The sum of areas of faces  $ABC$  and  $ABD$  equals the sum of areas of faces  $CDA$  and  $CDB$ . Prove that  $O$  and midpoints of  $BC, AD, AC$  and  $BD$  belong to the same plane.
- [7] A  $m \times n$  table is filled with signs "+" and "-". A table is called irreducible if one cannot reduce it to the table filled with "+", applying the following operations (as many times as one wishes). a) It is allowed to flip all the signs in a row or in a column. Prove that an

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irreducible table contains an irreducible  $2 \times 2$  sub table. *b)* It is allowed to flip all the signs in a row or in a column or on a diagonal (corner cells are diagonals of length 1). Prove that an irreducible table contains an irreducible  $4 \times 4$  sub table.

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## Fall Senior O-Level

- 1 For any integer  $n + 1, \dots, 2n$  ( $n$  is a natural number) consider its greatest odd divisor. Prove that the sum of all these divisors equals  $n^2$ .
- 2 What least possible number of unit squares ( $1 \times 1$ ) must be drawn in order to get a picture of  $25 \times 25$ -square divided into 625 of unit squares?
- 3 A salesman and a customer altogether have 1999 rubles in coins and bills of 1, 5, 10, 50, 100, 500, 1000 rubles. The customer has enough money to buy a Cat in the Bag which costs the integer number of rubles. Prove that the customer can buy the Cat and get the correct change.
- 4 Each side of  $1 \times 1$  square is a hypotenuse of an exterior right triangle. Let  $A, B, C, D$  be the vertices of the right angles and  $O_1, O_2, O_3, O_4$  be the centers of the incircles of these triangles. Prove that a) The area of quadrilateral  $ABCD$  does not exceed 2; b) The area of quadrilateral  $O_1O_2O_3O_4$  does not exceed 1.
- 5 A paper tetrahedron is cut along some of so that it can be developed onto the plane. Could it happen that this development cannot be placed on the plane in one layer?

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## Spring Junior A-Level Paper

- 1 Johnny writes down quadratic equation

$$ax^2 + bx + c = 0.$$

with positive integer coefficients  $a, b, c$ . Then Pete changes one, two, or none  $+$  signs to  $-$ . Johnny wins, if both roots of the (changed) equation are integers. Otherwise (if there are no real roots or at least one of them is not an integer), Pete wins. Can Johnny choose the coefficients in such a way that he will always win?

- 2 Triangle  $ABC$  is given. Prove that  $\frac{R}{r} > \frac{a}{h}$ , where  $R$  is the radius of the circumscribed circle,  $r$  is the radius of the inscribed circle,  $a$  is the length of the longest side,  $h$  is the length of the shortest altitude.
- 3 In a tournament, each of 15 teams played with each other exactly once. Let us call the game *odd* if the total number of games previously played by both competing teams was odd.
- (a) Prove that there was at least one *odd* game.
- (b) Could it happen that there was exactly one *odd* game?
- 4 A chocolate bar in the shape of an equilateral triangle with side of the length  $n$ , consists of triangular chips with sides of the length 1, parallel to sides of the bar. Two players take turns eating up the chocolate. Each player breaks off a triangular piece (along one of the lines), eats it up and passes leftovers to the other player (as long as bar contains more than one chip, the player is not allowed to eat it completely).
- A player who has no move or leaves exactly one chip to the opponent, loses. For each  $n$ , find who has a winning strategy.
- 5 What is the largest number of squares on  $9 \times 9$  square board that can be cut along their both diagonals so that the board does not fall apart into several pieces?
- 6 A trapezoid with bases  $AD$  and  $BC$  is circumscribed about a circle,  $E$  is the intersection point of the diagonals. Prove that  $\angle AED$  is not acute.

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## Spring Junior O-Level Paper

- [1] 2003 dollars are placed into  $N$  purses, and the purses are placed into  $M$  pockets. It is known that  $N$  is greater than the number of dollars in any pocket. Is it true that there is a purse with less than  $M$  dollars in it?
- [2] Two players in turns color the sides of an  $n$ -gon. The first player colors any side that has 0 or 2 common vertices with already colored sides. The second player colors any side that has exactly 1 common vertex with already colored sides. The player who cannot move, loses. For which  $n$  the second player has a winning strategy?
- [3] Points  $K$  and  $L$  are chosen on the sides  $AB$  and  $BC$  of the isosceles  $\triangle ABC$  ( $AB = BC$ ) so that  $AK + LC = KL$ . A line parallel to  $BC$  is drawn through midpoint  $M$  of the segment  $KL$ , intersecting side  $AC$  at point  $N$ . Find the value of  $\angle KNL$ .
- [4] Each term of a sequence of positive integers is obtained from the previous term by adding to it its largest digit. What is the maximal number of successive odd terms in such a sequence?
- [5] Is it possible to tile  $2003 \times 2003$  board by  $1 \times 2$  dominoes placed horizontally and  $1 \times 3$  rectangles placed vertically?

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## Spring Senior A-Level Paper

- 1 A triangular pyramid  $ABCD$  is given. Prove that  $\frac{R}{r} > \frac{a}{h}$ , where  $R$  is the radius of the circumscribed sphere,  $r$  is the radius of the inscribed sphere,  $a$  is the length of the longest edge,  $h$  is the length of the shortest altitude (from a vertex to the opposite face).
- 2  $P(x)$  is a polynomial with real coefficients such that  $P(a_1) = 0, P(a_{i+1}) = a_i$  ( $i = 1, 2, \dots$ ) where  $\{a_i\}_{i=1,2,\dots}$  is an infinite sequence of distinct natural numbers. Determine the possible values of degree of  $P(x)$ .
- 3 Can one cover a cube by three paper triangles (without overlapping)?
- 4 A right triangle  $ABC$  with hypotenuse  $AB$  is inscribed in a circle. Let  $K$  be the midpoint of the arc  $BC$  not containing  $A$ ,  $N$  the midpoint of side  $AC$ , and  $M$  a point of intersection of ray  $KN$  with the circle. Let  $E$  be a point of intersection of tangents to the circle at points  $A$  and  $C$ . Prove that  $\angle EMK = 90^\circ$ .
- 5 Prior to the game John selects an integer greater than 100.  
Then Mary calls out an integer  $d$  greater than 1. If John's integer is divisible by  $d$ , then Mary wins. Otherwise, John subtracts  $d$  from his number and the game continues (with the new number). Mary is not allowed to call out any number twice. When John's number becomes negative, Mary loses. Does Mary have a winning strategy?
- 6 The signs "+" or "-" are placed in all cells of a  $4 \times 4$  square table. It is allowed to change a sign of any cell altogether with signs of all its adjacent cells (i.e. cells having a common side with it). Find the number of different tables that could be obtained by iterating this procedure.
- 7 A square is triangulated in such way that no three vertices are collinear. For every vertex (including vertices of the square) the number of sides issuing from it is counted. Can it happen that all these numbers are even?



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## Spring Senior O-Level Paper

- [1] Two players in turns color the sides of an  $n$ -gon. The first player colors any side that has 0 or 2 common vertices with already colored sides. The second player colors any side that has exactly 1 common vertex with already colored sides. The player who cannot move, loses. For which  $n$  the second player has a winning strategy?
- [2] 100-gon made of 100 sticks. Could it happen that it is not possible to construct a polygon from any lesser number of these sticks?
- [3] Point  $M$  is chosen in triangle  $ABC$  so that the radii of the circumcircles of triangles  $AMC$ ,  $BMC$ , and  $BMA$  are no smaller than the radius of the circumcircle of  $ABC$ . Prove that all four radii are equal.
- [4] In the sequence  $00, 01, 02, 03, \dots, 99$  the terms are rearranged so that each term is obtained from the previous one by increasing or decreasing one of its digits by 1 (for example, 29 can be followed by 19, 39, or 28, but not by 30 or 20). What is the maximal number of terms that could remain on their places?
- [5] Prove that one can cut  $a \times b$  rectangle,  $\frac{b}{2} < a < b$ , into three pieces and rearrange them into a square (without overlaps and holes).