

Problems from the 2016 IMO Shortlist

— Algebra (A)

**A1** Let  $a, b, c$  be positive real numbers such that  $\min(ab, bc, ca) \geq 1$ . Prove that

$$\sqrt[3]{(a^2 + 1)(b^2 + 1)(c^2 + 1)} \leq \left( \frac{a + b + c}{3} \right)^2 + 1.$$

**A2** Find the smallest constant  $C > 0$  for which the following statement holds: among any five positive real numbers  $a_1, a_2, a_3, a_4, a_5$  (not necessarily distinct), one can always choose distinct subscripts  $i, j, k, l$  such that

$$\left| \frac{a_i}{a_j} - \frac{a_k}{a_l} \right| \leq C.$$

**A3** Find all positive integers  $n$  such that the following statement holds: Suppose real numbers  $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$  satisfy  $|a_k| + |b_k| = 1$  for all  $k = 1, \dots, n$ . Then there exists  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ , each of which is either  $-1$  or  $1$ , such that

$$\left| \sum_{i=1}^n \varepsilon_i a_i \right| + \left| \sum_{i=1}^n \varepsilon_i b_i \right| \leq 1.$$

**A4** Find all functions  $f : (0, \infty) \rightarrow (0, \infty)$  such that for any  $x, y \in (0, \infty)$ ,

$$xf(x^2)f(f(y)) + f(yf(x)) = f(xy) (f(f(x^2)) + f(f(y^2))).$$

**A5** Consider fractions  $\frac{a}{b}$  where  $a$  and  $b$  are positive integers.  
 (a) Prove that for every positive integer  $n$ , there exists such a fraction  $\frac{a}{b}$  such that  $\sqrt{n} \leq \frac{a}{b} \leq \sqrt{n+1}$  and  $b \leq \sqrt{n+1}$ .  
 (b) Show that there are infinitely many positive integers  $n$  such that no such fraction  $\frac{a}{b}$  satisfies  $\sqrt{n} \leq \frac{a}{b} \leq \sqrt{n+1}$  and  $b \leq \sqrt{n}$ .

**A6**

The equation

$$(x-1)(x-2)\cdots(x-2016) = (x-1)(x-2)\cdots(x-2016)$$

is written on the board, with 2016 linear factors on each side. What is the least possible value of  $k$  for which it is possible to erase exactly  $k$  of these 4032 linear factors so that at least one factor remains on each side and the resulting equation has no real solutions?

**A7**

Find all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f(0) \neq 0$  and for all  $x, y \in \mathbb{R}$ ,

$$f(x+y)^2 = 2f(x)f(y) + \max\{f(x^2+y^2), f(x^2) + f(y^2)\}.$$

**A8**

Find the largest real constant  $a$  such that for all  $n \geq 1$  and for all real numbers  $x_0, x_1, \dots, x_n$  satisfying  $0 = x_0 < x_1 < x_2 < \cdots < x_n$  we have

$$\frac{1}{x_1 - x_0} + \frac{1}{x_2 - x_1} + \cdots + \frac{1}{x_n - x_{n-1}} \geq a \left( \frac{2}{x_1} + \frac{3}{x_2} + \cdots + \frac{n+1}{x_n} \right)$$

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Combinatorics (C)

**C1**

The leader of an IMO team chooses positive integers  $n$  and  $k$  with  $n > k$ , and announces them to the deputy leader and a contestant. The leader then secretly tells the deputy leader an  $n$ -digit binary string, and the deputy leader writes down all  $n$ -digit binary strings which differ from the leaders in exactly  $k$  positions. (For example, if  $n = 3$  and  $k = 1$ , and if the leader chooses 101, the deputy leader would write down 001, 111 and 100.) The contestant is allowed to look at the strings written by the deputy leader and guess the leaders string. What is the minimum number of guesses (in terms of  $n$  and  $k$ ) needed to guarantee the correct answer?

**C2**

Find all positive integers  $n$  for which all positive divisors of  $n$  can be put into the cells of a rectangular table under the following constraints:

- each cell contains a distinct divisor;
- the sums of all rows are equal; and
- the sums of all columns are equal.

- C3** Let  $n$  be a positive integer relatively prime to 6. We paint the vertices of a regular  $n$ -gon with three colours so that there is an odd number of vertices of each colour. Show that there exists an isosceles triangle whose three vertices are of different colours.
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- C4** Find all integers  $n$  for which each cell of  $n \times n$  table can be filled with one of the letters  $I, M$  and  $O$  in such a way that:
- in each row and each column, one third of the entries are  $I$ , one third are  $M$  and one third are  $O$ ; and
  - in any diagonal, if the number of entries on the diagonal is a multiple of three, then one third of the entries are  $I$ , one third are  $M$  and one third are  $O$ .
- Note.** The rows and columns of an  $n \times n$  table are each labelled 1 to  $n$  in a natural order. Thus each cell corresponds to a pair of positive integer  $(i, j)$  with  $1 \leq i, j \leq n$ . For  $n > 1$ , the table has  $4n - 2$  diagonals of two types. A diagonal of first type consists all cells  $(i, j)$  for which  $i + j$  is a constant, and the diagonal of this second type consists all cells  $(i, j)$  for which  $i - j$  is constant.
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- C5** Let  $n \geq 3$  be a positive integer. Find the maximum number of diagonals in a regular  $n$ -gon one can select, so that any two of them do not intersect in the interior or they are perpendicular to each other.
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- C6** There are  $n \geq 3$  islands in a city. Initially, the ferry company offers some routes between some pairs of islands so that it is impossible to divide the islands into two groups such that no two islands in different groups are connected by a ferry route.
- After each year, the ferry company will close a ferry route between some two islands  $X$  and  $Y$ . At the same time, in order to maintain its service, the company will open new routes according to the following rule: for any island which is connected to a ferry route to exactly one of  $X$  and  $Y$ , a new route between this island and the other of  $X$  and  $Y$  is added.
- Suppose at any moment, if we partition all islands into two nonempty groups in any way, then it is known that the ferry company will close a certain route connecting two islands from the two groups after some years. Prove that after some years there will be an island which is connected to all other islands by ferry routes.
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- C7** There are  $n \geq 2$  line segments in the plane such that every two segments cross and no three segments meet at a point. Geoff has to choose an endpoint of each segment and place a frog on it facing the other endpoint. Then he will clap his hands  $n - 1$  times. Every time he claps, each frog will immediately jump

forward to the next intersection point on its segment. Frogs never change the direction of their jumps. Geoff wishes to place the frogs in such a way that no two of them will ever occupy the same intersection point at the same time.

- (a) Prove that Geoff can always fulfill his wish if  $n$  is odd.
- (b) Prove that Geoff can never fulfill his wish if  $n$  is even.

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**C8** Let  $n$  be a positive integer. Determine the smallest positive integer  $k$  with the following property: it is possible to mark  $k$  cells on a  $2n \times 2n$  board so that there exists a unique partition of the board into  $1 \times 2$  and  $2 \times 1$  dominoes, none of which contain two marked cells.

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— Geometry (G)

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**G1** Triangle  $BCF$  has a right angle at  $B$ . Let  $A$  be the point on line  $CF$  such that  $FA = FB$  and  $F$  lies between  $A$  and  $C$ . Point  $D$  is chosen so that  $DA = DC$  and  $AC$  is the bisector of  $\angle DAB$ . Point  $E$  is chosen so that  $EA = ED$  and  $AD$  is the bisector of  $\angle EAC$ . Let  $M$  be the midpoint of  $CF$ . Let  $X$  be the point such that  $AMXE$  is a parallelogram. Prove that  $BD, FX$  and  $ME$  are concurrent.

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**G2** Let  $ABC$  be a triangle with circumcircle  $\Gamma$  and incenter  $I$  and let  $M$  be the midpoint of  $\overline{BC}$ . The points  $D, E, F$  are selected on sides  $\overline{BC}, \overline{CA}, \overline{AB}$  such that  $\overline{ID} \perp \overline{BC}$ ,  $\overline{IE} \perp \overline{AI}$ , and  $\overline{IF} \perp \overline{AI}$ . Suppose that the circumcircle of  $\triangle AEF$  intersects  $\Gamma$  at a point  $X$  other than  $A$ . Prove that lines  $XD$  and  $AM$  meet on  $\Gamma$ .

*Proposed by Evan Chen, Taiwan*

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**G3** Let  $B = (-1, 0)$  and  $C = (1, 0)$  be fixed points on the coordinate plane. A nonempty, bounded subset  $S$  of the plane is said to be *nice* if

- (i) there is a point  $T$  in  $S$  such that for every point  $Q$  in  $S$ , the segment  $TQ$  lies entirely in  $S$ ; and
- (ii) for any triangle  $P_1P_2P_3$ , there exists a unique point  $A$  in  $S$  and a permutation  $\sigma$  of the indices  $\{1, 2, 3\}$  for which triangles  $ABC$  and  $P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$  are similar.

Prove that there exist two distinct nice subsets  $S$  and  $S'$  of the set  $\{(x, y) : x \geq 0, y \geq 0\}$  such that if  $A \in S$  and  $A' \in S'$  are the unique choices of points in (ii), then the product  $BA \cdot BA'$  is a constant independent of the triangle  $P_1P_2P_3$ .

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**G4** Let  $ABC$  be a triangle with  $AB = AC \neq BC$  and let  $I$  be its incentre. The line  $BI$  meets  $AC$  at  $D$ , and the line through  $D$  perpendicular to  $AC$  meets  $AI$  at  $E$ . Prove that the reflection of  $I$  in  $AC$  lies on the circumcircle of triangle  $BDE$ .

**G5** Let  $D$  be the foot of perpendicular from  $A$  to the Euler line (the line passing through the circumcentre and the orthocentre) of an acute scalene triangle  $ABC$ . A circle  $\omega$  with centre  $S$  passes through  $A$  and  $D$ , and it intersects sides  $AB$  and  $AC$  at  $X$  and  $Y$  respectively. Let  $P$  be the foot of altitude from  $A$  to  $BC$ , and let  $M$  be the midpoint of  $BC$ . Prove that the circumcentre of triangle  $XS Y$  is equidistant from  $P$  and  $M$ .

**G6** Let  $ABCD$  be a convex quadrilateral with  $\angle ABC = \angle ADC < 90^\circ$ . The internal angle bisectors of  $\angle ABC$  and  $\angle ADC$  meet  $AC$  at  $E$  and  $F$  respectively, and meet each other at point  $P$ . Let  $M$  be the midpoint of  $AC$  and let  $\omega$  be the circumcircle of triangle  $BPD$ . Segments  $BM$  and  $DM$  intersect  $\omega$  again at  $X$  and  $Y$  respectively. Denote by  $Q$  the intersection point of lines  $XE$  and  $YF$ . Prove that  $PQ \perp AC$ .

**G7** Let  $I$  be the incentre of a non-equilateral triangle  $ABC$ ,  $I_A$  be the  $A$ -excentre,  $I'_A$  be the reflection of  $I_A$  in  $BC$ , and  $l_A$  be the reflection of line  $AI'_A$  in  $AI$ . Define points  $I_B$ ,  $I'_B$  and line  $l_B$  analogously. Let  $P$  be the intersection point of  $l_A$  and  $l_B$ .

- Prove that  $P$  lies on line  $OI$  where  $O$  is the circumcentre of triangle  $ABC$ .
- Let one of the tangents from  $P$  to the incircle of triangle  $ABC$  meet the circumcircle at points  $X$  and  $Y$ . Show that  $\angle XIY = 120^\circ$ .

**G8** Let  $A_1, B_1$  and  $C_1$  be points on sides  $BC, CA$  and  $AB$  of an acute triangle  $ABC$  respectively, such that  $AA_1, BB_1$  and  $CC_1$  are the internal angle bisectors of triangle  $ABC$ . Let  $I$  be the incentre of triangle  $ABC$ , and  $H$  be the orthocentre of triangle  $A_1B_1C_1$ . Show that

$$AH + BH + CH \geq AI + BI + CI.$$

– Number Theory (N)

**N1** For any positive integer  $k$ , denote the sum of digits of  $k$  in its decimal representation by  $S(k)$ . Find all polynomials  $P(x)$  with integer coefficients such that for any positive integer  $n \geq 2016$ , the integer  $P(n)$  is positive and

$$S(P(n)) = P(S(n)).$$

*Proposed by Warut Suksompong, Thailand*

**N2** Let  $\tau(n)$  be the number of positive divisors of  $n$ . Let  $\tau_1(n)$  be the number of positive divisors of  $n$  which have remainders 1 when divided by 3. Find all positive integral values of the fraction  $\frac{\tau(10n)}{\tau_1(10n)}$ .

**N3** A set of positive integers is called *fragrant* if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let  $P(n) = n^2 + n + 1$ . What is the least possible positive integer value of  $b$  such that there exists a non-negative integer  $a$  for which the set

$$\{P(a+1), P(a+2), \dots, P(a+b)\}$$

is fragrant?

**N4** Let  $n, m, k$  and  $l$  be positive integers with  $n \neq 1$  such that  $n^k + mn^l + 1$  divides  $n^{k+l} - 1$ . Prove that

$-m = 1$  and  $l = 2k$ ; or  
 $-l|k$  and  $m = \frac{n^{k-l}-1}{n^l-1}$ .

**N5** Let  $a$  be a positive integer which is not a perfect square, and consider the equation

$$k = \frac{x^2 - a}{x^2 - y^2}.$$

Let  $A$  be the set of positive integers  $k$  for which the equation admits a solution in  $\mathbb{Z}^2$  with  $x > \sqrt{a}$ , and let  $B$  be the set of positive integers for which the equation admits a solution in  $\mathbb{Z}^2$  with  $0 \leq x < \sqrt{a}$ . Show that  $A = B$ .

**N6** Denote by  $\mathbb{N}$  the set of all positive integers. Find all functions  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that for all positive integers  $m$  and  $n$ , the integer  $f(m) + f(n) - mn$  is nonzero and divides  $mf(m) + nf(n)$ .

*Proposed by Dorlir Ahmeti, Albania*

**N7** Let  $P = A_1A_2 \dots A_k$  be a convex polygon in the plane. The vertices  $A_1, A_2, \dots, A_k$  have integral coordinates and lie on a circle. Let  $S$  be the area of  $P$ . An odd positive integer  $n$  is given such that the squares of the side lengths of  $P$  are integers divisible by  $n$ . Prove that  $2S$  is an integer divisible by  $n$ .



# Art of Problem Solving

## 2016 IMO Shortlist

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N8

Find all polynomials  $P(x)$  of odd degree  $d$  and with integer coefficients satisfying the following property: for each positive integer  $n$ , there exists  $n$  positive integers  $x_1, x_2, \dots, x_n$  such that  $\frac{1}{2} < \frac{P(x_i)}{P(x_j)} < 2$  and  $\frac{P(x_i)}{P(x_j)}$  is the  $d$ -th power of a rational number for every pair of indices  $i$  and  $j$  with  $1 \leq i, j \leq n$ .

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