Art of Problem Solving

2016 ISI Entrance Examination

ISI Entrance Examination 2016

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1 In a sports tournament of n players, each pair of players plays against each other exactly one match and there are no draws.Show that the players can be arranged in an order $P_1, P_2, ..., P_n$ such that P_i defeats P_{i+1} for all $1 \le i \le n-1$.

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2 Consider the polynomial $ax^3 + bx^2 + cx + d$ where a, b, c, d are integers such that ad is odd and bc is even.Prove that not all of its roots are rational..

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3 If $P(x) = x^n + a_1 x^{n-1} + ... + a_{n-1}$ be a polynomial with real coefficients and $a_1^2 < a_2$ then prove that not all roots of $P(x)$ are real.

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4 Given a square $ABCD$ with two consecutive vertices, say A and B on the positive x-axis and positive y-axis respectively. Suppose the other vertice C lying in the first quadrant has coordinates (u, v). Then find the area of the square $ABCD$ in terms of u and v.

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5 Prove that there exists a right angle triangle with rational sides and area d if and only if x^2, y^2 and z^2 are squares of rational numbers and are in Arithmetic Progression

Here d is an integer.

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6 Suppose in a triangle $\triangle ABC$, A , B , C are the three angles and a , b , c are the lengths of the sides opposite to the angles respectively. Then prove that if $sin(A - B) = \frac{a}{a+b} \sin A \cos B - \frac{b}{a+b} \sin B \cos A$ then the triangle $\triangle ABC$ is isoscelos.

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7 f is a differentiable function such that $f(f(x)) = x$ where $x \in [0, 1]$.Also $f(0) = 1$.Find the value of

$$\int_0^1 (x - f(x))^{2016} dx$$

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8 Suppose that $(a_n)_{n \ge 1}$ is a sequence of real numbers satisfying $a_{n+1} = \frac{3a_n}{2+a_n}$.
(i) Suppose $0 < a_1 < 1$, then prove that the sequence a_n is increasing and hence show that $\lim_{n\to\infty} a_n = 1$.

(ii) Suppose $a_1 > 1$, then prove that the sequence a_n is decreasing and hence show that $\lim_{n\to\infty} a_n = 1$.
