All-Russian Olympiad 2002

– Grade level 9

Day 1

1 Can the cells of a 2002×2002 table be filled with the numbers from 1 to 2002^2 (one per cell) so that for any cell we can find three numbers a, b, c in the same row or column (or the cell itself) with $a = bc$?

2 Point A lies on one ray and points B, C lie on the other ray of an angle with the vertex at O such that B lies between O and C. Let O_1 be the incenter of $\triangle OAB$ and O_2 be the center of the excircle of $\triangle OAC$ touching side AC. Prove that if $O_1A = O_2A$, then the triangle ABC is isosceles.

3 On a plane are given 6 red, 6 blue, and 6 green points, such that no three of the given points lie on a line. Prove that the sum of the areas of the triangles whose vertices are of the same color does not exceed quarter the sum of the areas of all triangles with vertices in the given points.

4 A hydra consists of several heads and several necks, where each neck joins two heads. When a hydra's head A is hit by a sword, all the necks from head A disappear, but new necks grow up to connect head A to all the heads which weren't connected to A. Heracle defeats a hydra by cutting it into two parts which are no joined. Find the minimum N for which Heracle can defeat any hydra with 100 necks by no more than N hits.

Day 2

1 There are eight rooks on a chessboard, no two attacking each other. Prove that some two of the pairwise distances between the rooks are equal. (The distance between two rooks is the distance between the centers of their cell.)

2 We are given one red and $k > 1$ blue cells, and a pack of $2n$ cards, enumerated by the numbers from 1 to $2n$. Initially, the pack is situated on the red cell and arranged in an arbitrary order. In each move, we are allowed to take the top card from one of the cells and place it either onto the top of another cell on which the number on the top card is greater by 1, or onto an empty cell. Given k, what is the maximal n for which it is always possible to move all the cards onto a blue cell?

3 Let O be the circumcenter of a triangle ABC. Points M and N are choosen on the sides AB and BC respectively so that the angle AOC is two times greater than angle MON. Prove that the perimeter of triangle MBN is not less than the lenght of side AC

4 From the interval $(2^{2n}, 2^{3n})$ are selected $2^{2n-1}+1$ odd numbers. Prove that there are two among the selected numbers, none of which divides the square of the other.

– Grade level 10

Day 1

1 The polynomials P, Q, R with real coefficients, one of which is degree 2 and two of degree 3, satisfy the equality $P^2+Q^2=R^2$. Prove that one of the polynomials of degree 3 has three real roots.

2 A quadrilateral $ABCD$ is inscribed in a circle ω. The tangent to ω at A intersects the ray CB at K, and the tangent to ω at B intersects the ray DA at M. Prove that if $AM=AD$ and $BK=BC$, then $ABCD$ is a trapezoid.

3 Prove that for every integer $n>10000$ there exists an integer m such that it can be written as the sum of two squares, and $0<m-n<3\sqrt[4]{n}$.

4 There are 2002 towns in a kingdom. Some of the towns are connected by roads in such a manner that, if all roads from one city closed, one can still travel between any two cities. Every year, the kingdom chooses a non-self-intersecting cycle of roads, founds a new town, connects it by roads with each city from the chosen cycle, and closes all the roads from the original cycle. After several years, no non-self-intersecting cycles remained. Prove that at that moment there are at least 2002 towns, exactly one road going out from each of them.

Day 2

1 For positive real numbers a,b,c such that $a+b+c=3$, show that:

$$\sqrt{a}+\sqrt{b}+\sqrt{c}\geq ab+bc+ca.$$

2 We are given one red and $k>1$ blue cells, and a pack of $2n$ cards, enumerated by the numbers from 1 to $2n$. Initially, the pack is situated on the red cell and arranged in an arbitrary order. In each move, we are allowed to take the top card from one of the cells and place it either onto the top of another cell on which the number on the top card is greater by 1, or onto an empty cell. Given k, what is the maximal n for which it is always possible to move all the cards onto a blue cell?

3 Let A' be the point of tangency of the excircle of a triangle ABC (corrsponding to A) with the side BC. The line a through A' is parallel to the bisector of $\angle BAC$. Lines b and c are analogously dened. Prove that a,b,c have a common point.

4 On a plane are given finitely many red and blue lines, no two parallel, such that any intersection point of two lines of the same color also lies on another line of the other color. Prove that all the lines pass through a single point.

– Grade level 11

Day 1

1 The polynomials P, Q, R with real coefficients, one of which is degree 2 and two of degree 3, satisfy the equality $P^2+Q^2=R^2$. Prove that one of the polynomials of degree 3 has three real roots.

2 Several points are given in the plane. Suppose that for any three of them, there exists an orthogonal coordinate system (determined by the two axes and the unit length) in which these three points have integer coordinates. Prove that there exists an orthogonal coordinate system in which all the given points have integer coordinates.

3 Prove that if $0<x<\frac{\pi}{2}$ and $n>m$, where n,m are natural numbers,

$$2\left|\sin^n x-\cos^n x\right|\le 3\left|\sin^m x-\cos^m x\right|.$$

4 There are some markets in a city. Some of them are joined by one-way streets, such that for any market there are exactly two streets to leave it. Prove that the city may be partitioned into 1014 districts such that streets join only markets from different districts, and by the same one-way for any two districts (either only from first to second, or vice-versa).

Day 2

1 Determine the smallest natural number which can be represented both as the sum of 2002 positive integers with the same sum of decimal digits, and as the sum of 2003 integers with the same sum of decimal digits.

2 The diagonals AC and BD of a cyclic quadrilateral $ABCD$ meet at O. The circumcircles of triangles AOB and COD intersect again at K. Point L is such that the triangles BLC and AKD are similar and equally oriented. Prove that if the quadrilateral $BLCK$ is convex, then it is tangent [has an incircle].

3 On a plane are given finitely many red and blue lines, no two parallel, such that any intersection point of two lines of the same color also lies on another line of the other color. Prove that all the lines pass through a single point.

4 Prove that there exist infinitely many natural numbers n such that the numerator of $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...+\frac{1}{n}$ in the lowest terms is not a power of a prime number.