IMO 1973

Day 1

1. Prove that the sum of an odd number of vectors of length 1, of common origin O and all situated in the same semi-plane determined by a straight line which goes through O, is at least 1.

2. Establish if there exists a finite set M of points in space, not all situated in the same plane, so that for any straight line d which contains at least two points from M there exists another straight line d', parallel with d, but distinct from d, which also contains at least two points from M.

3. Determine the minimum value of $a^2 + b^2$ when (a, b) traverses all the pairs of real numbers for which the equation
$$x^4 + ax^3 + bx^2 + ax + 1 = 0$$
has at least one real root.

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Day 2

1. A soldier needs to check if there are any mines in the interior or on the sides of an equilateral triangle ABC. His detector can detect a mine at a maximum distance equal to half the height of the triangle. The soldier leaves from one of the vertices of the triangle. Which is the minimum distance that he needs to traverse so that at the end of it he is sure that he completed successfully his mission?

2. G is a set of non-constant functions f. Each f is defined on the real line and has the form $f(x) = ax + b$ for some real a, b. If f and g are in G, then so is fg, where fg is defined by $fg(x) = f(g(x))$. If f is in G, then so is the inverse f^{-1}. If $f(x) = ax+b$, then $f^{-1}(x) = \frac{x-b}{a}$. Every f in G has a fixed point (in other words we can find x_f such that $f(x_f) = x_f$. Prove that all the functions in G have a common fixed point.

3. Let $a_1, \ldots, a_n$ be n positive numbers and $0 < q < 1$. Determine n positive numbers $b_1, \ldots, b_n$ so that:

   a.) $k < b_k$ for all $k = 1, \ldots, n$, *b.)* $q < \frac{b_{k+1}}{b_k} < \frac{1}{q}$ for all $k = 1, \ldots, n-1$, *c.)* $\sum\limits_{k=1}^{n} b_k < \frac{1+q}{1-q} \cdot \sum\limits_{k=1}^{n} a_k$.