India
International Mathematical Olympiad Training Camp
2012

Practice Tests

Day 1

1. Let ABC be an isosceles triangle with $AB = AC$. Let D be a point on the segment BC such that $BD = 2DC$. Let P be a point on the segment AD such that $\angle BAC = \angle BPD$. Prove that $\angle BAC = 2\angle DPC$.

2. Let $a \geq b$ and $c \geq d$ be real numbers. Prove that the equation
$$(x + a)(x + d) + (x + b)(x + c) = 0$$
has real roots.

3. How many 6-tuples (a, b, c, d, e, f) of natural numbers are there for which $a > b > c > d > e > f$ and $a + f = b + e = c + d = 30$ are simultaneously true?

Day 2

1. Let $ABCD$ be a trapezium with $AB \parallel CD$. Let P be a point on AC such that C is between A and P; and let X, Y be the midpoints of AB, CD respectively. Let PX intersect BC in N and PY intersect AD in M. Prove that $MN \parallel AB$.

2. Let $0 < x < y < z < p$ be integers where p is a prime. Prove that the following statements are equivalent: $(a) x^3 \equiv y^3 \pmod{p}$ and $x^3 \equiv z^3 \pmod{p}$ $(b) y^2 \equiv zx \pmod{p}$ and $z^2 \equiv xy \pmod{p}$

3. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function such that $f(x + y + xy) = f(x) + f(y) + f(xy)$ for all $x, y \in \mathbb{R}$. Prove that f satisfies $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

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Team Selection Tests

Day 1

1 The cirumcentre of the cyclic quadrilateral $ABCD$ is O. The second intersection point of the circles ABO and CDO, other than O, is P, which lies in the interior of the triangle DAO. Choose a point Q on the extension of OP beyond P, and a point R on the extension of OP beyond O. Prove that $\angle QAP = \angle OBR$ if and only if $\angle PDQ = \angle RCO$.

2 Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_m z^m$ be a polynomial with complex coefficients such that $a_m \neq 0, a_n \neq 0$ and $n > m$. Prove that

$$\max_{|z|=1}\{|P(z)|\} \geq \sqrt{2|a_m a_n| + \sum_{k=m}^{n} |a_k|^2}$$

3 Determine the greatest positive integer k that satisfies the following property: The set of positive integers can be partitioned into k subsets $A_1, A_2, \ldots, A_k$ such that for all integers $n \geq 15$ and all $i \in \{1, 2, \ldots, k\}$ there exist two distinct elements of A_i whose sum is n.

Proposed by Igor Voronovich, Belarus

Day 2

1 Determine all sequences $(x_1, x_2, \ldots, x_{2011})$ of positive integers, such that for every positive integer n there exists an integer a with

$$\sum_{j=1}^{2011} j x_j^n = a^{n+1} + 1$$

Proposed by Warut Suksompong, Thailand

2 Show that there exist infinitely many pairs (a, b) of positive integers with the property that $a + b$ divides $ab + 1$, $a - b$ divides $ab - 1$, $b > 1$ and $a > b\sqrt{3} - 1$

3 Suppose that 1000 students are standing in a circle. Prove that there exists an integer k with $100 \leq k \leq 300$ such that in this circle there exists a contiguous group of $2k$ students, for which the first half contains the same number of girls as the second half.

Proposed by Gerhard Wginger, Austria

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Day 3

1. Let ABC be a triangle with $AB = AC$ and let D be the midpoint of AC. The angle bisector of $\angle BAC$ intersects the circle through D, B and C at the point E inside the triangle ABC. The line BD intersects the circle through A, E and B in two points B and F. The lines AF and BE meet at a point I, and the lines CI and BD meet at a point K. Show that I is the incentre of triangle KAB.

   Proposed by Jan Vonk, Belgium and Hojoo Lee, South Korea

2. Let S be a nonempty set of primes satisfying the property that for each proper subset P of S, all the prime factors of the number $\left(\prod_{p \in P} p\right) - 1$ are also in S. Determine all possible such sets S.

3. In a $2 \times n$ array we have positive reals s.t. the sum of the numbers in each of the n columns is 1. Show that we can select a number in each column s.t. the sum of the selected numbers in each row is at most $\frac{n+1}{4}$.

Day 4

1. A quadrilateral $ABCD$ without parallel sides is circumscribed around a circle with centre O. Prove that O is a point of intersection of middle lines of quadrilateral $ABCD$ (i.e. barycentre of points A, B, C, D) iff $OA \cdot OC = OB \cdot OD$.

2. Find the least positive integer that cannot be represented as $\frac{2^a - 2^b}{2^c - 2^d}$ for some positive integers a, b, c, d.

3. Let $\mathbb{R}^+$ denote the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ satisfying

$$f(x) + f(y) \leq \frac{f(x+y)}{2}, \frac{f(x)}{x} + \frac{f(y)}{y} \geq \frac{f(x+y)}{x+y},$$

   for all $x, y \in \mathbb{R}^+$.