JBMO ShortLists 2001

1. Find the positive integers n that are not divisible by 3 if the number $2^{n^2-10}+2133$ is a perfect cube.

   [hide="Note"] The wording of this problem is perhaps not the best English. As far as I am aware, just solve the diophantine equation $x^3 = 2^{n^2-10} + 2133$ where $x, n \in \mathbb{N}$ and $3 \nmid n$.

2. Let P_n ($n = 3, 4, 5, 6, 7$) be the set of positive integers $n^k + n^l + n^m$, where k, l, m are positive integers. Find n such that:

   i) In the set P_n there are infinitely many squares.

   ii) In the set P_n there are no squares.

3. Find all the three-digit numbers $\overline{abc}$ such that the 6003-digit number $\overline{abcabc\ldots abc}$ is divisible by 91.

4. The discriminant of the equation $x^2 - ax + b = 0$ is the square of a rational number and a and b are integers. Prove that the roots of the equation are integers.

5. Let $x_k = \frac{k(k+1)}{2}$ for all integers $k \geq 1$. Prove that for any integer $n \geq 10$, between the numbers $A = x_1 + x_2 + \ldots + x_{n-1}$ and $B = A + x_n$ there is at least one square.

6. Find all integers x and y such that $x^3 \pm y^3 = 2001p$, where p is prime.

7. Prove that there are are no positive integers x and y such that $x^5 + y^5 + 1 = (x+2)^5 + (y-3)^5$.

   [hide="Note"] The restriction x, y are positive isn't necessary.

8. Prove that no three points with integer coordinates can be the vertices of an equilateral triangle.

9. Consider a convex quadrilateral $ABCD$ with $AB = CD$ and $\angle BAC = 30°$. If $\angle ADC = 150°$, prove that $\angle BCA = \angle ACD$.

10. A triangle ABC is inscribed in the circle $\mathcal{C}(O, R)$. Let $\alpha < 1$ be the ratio of the radii of the circles tangent to $\mathcal{C}$, and both of the rays $(AB$ and $(AC$. The numbers $\beta < 1$ and $\gamma < 1$ are defined analogously. Prove that $\alpha + \beta + \gamma = 1$.

11. Consider a triangle ABC with $AB = AC$, and D the foot of the altitude from the vertex A. The point E lies on the side AB such that $\angle ACE = \angle ECB = 18°$.

    If $AD = 3$, find the length of the segment CE.

12. Consider the triangle ABC with $\angle A = 90°$ and $\angle B \neq \angle C$. A circle $\mathcal{C}(O, R)$ passes through B and C and intersects the sides AB and AC at D and E, respectively. Let S be the foot of the perpendicular from A to BC and let K be the intersection point of AS with the segment DE. If M is the midpoint of BC, prove that $AKOM$ is a parallelogram.

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13 At a conference there are n mathematicians. Each of them knows exactly k fellow mathematicians. Find the smallest value of k such that there are at least three mathematicians that are acquainted each with the other two.

Rewording of the last line for clarification: Find the smallest value of k such that there (always) exists 3 mathematicians X, Y, Z such that X and Y know each other, X and Z know each other and Y and Z know each other.