USA Team Selection Test 2012

– December TST

1 In acute triangle ABC, $\angle A < \angle B$ and $\angle A < \angle C$. Let P be a variable point on side BC. Points D and E lie on sides AB and AC, respectively, such that $BP = PD$ and $CP = PE$. Prove that as P moves along side BC, the circumcircle of triangle ADE passes through a fixed point other than A.

2 Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that for every pair of real numbers x and y,

$$f(x + y^2) = f(x) + |yf(y)|.$$

3 Determine, with proof, whether or not there exist integers $a, b, c > 2010$ satisfying the equation

$$a^3 + 2b^3 + 4c^3 = 6abc + 1.$$

4 There are 2010 students and 100 classrooms in the Olympiad High School. At the beginning, each of the students is in one of the classrooms. Each minute, as long as not everyone is in the same classroom, somebody walks from one classroom into a different classroom with at least as many students in it (prior to his move). This process will terminate in M minutes. Determine the maximum value of M.

– January TST

1 Consider (3-variable) polynomials

$$P_n(x, y, z) = (x - y)^{2n}(y - z)^{2n} + (y - z)^{2n}(z - x)^{2n} + (z - x)^{2n}(x - y)^{2n}$$

and

$$Q_n(x, y, z) = [(x - y)^{2n} + (y - z)^{2n} + (z - x)^{2n}]^{2n}.$$

Determine all positive integers n such that the quotient $Q_n(x, y, z)/P_n(x, y, z)$ is a (3-variable) polynomial with rational coefficients.

2 In cyclic quadrilateral $ABCD$, diagonals AC and BD intersect at P. Let E and F be the respective feet of the perpendiculars from P to lines AB and CD. Segments BF and CE meet at Q. Prove that lines PQ and EF are perpendicular to each other.

3 Determine all positive integers n, $n \geq 2$, such that the following statement is true:
If $(a_1, a_2, ..., a_n)$ is a sequence of positive integers with $a_1 + a_2 + \cdots + a_n = 2n - 1$, then there is block of (at least two) consecutive terms in the sequence with their (arithmetic) mean being an integer.

4 Find all positive integers $a, n \geq 1$ such that for all primes p dividing $a^n - 1$, there exists a positive integer $m < n$ such that $p \mid a^m - 1$.