APMO 2010

1 Let ABC be a triangle with $\angle BAC \neq 90^\circ$. Let O be the circumcenter of the triangle ABC and Γ be the circumcircle of the triangle BOC. Suppose that Γ intersects the line segment AB at P different from B, and the line segment AC at Q different from C. Let ON be the diameter of the circle Γ. Prove that the quadrilateral $APNQ$ is a parallelogram.

2 For a positive integer k, call an integer a *pure* $k-th$ *power* if it can be represented as m^k for some integer m. Show that for every positive integer n, there exists n distinct positive integers such that their sum is a pure $2009-$th power and their product is a pure $2010-$th power.

3 Let n be a positive integer. n people take part in a certain party. For any pair of the participants, either the two are acquainted with each other or they are not. What is the maximum possible number of the pairs for which the two are not acquainted but have a common acquaintance among the participants?

4 Let ABC be an acute angled triangle satisfying the conditions $AB > BC$ and $AC > BC$. Denote by O and H the circumcentre and orthocentre, respectively, of the triangle ABC. Suppose that the circumcircle of the triangle AHC intersects the line AB at M different from A, and the circumcircle of the triangle AHB intersects the line AC at N different from A. Prove that the circumcentre of the triangle MNH lies on the line OH.

5 Find all functions f from the set $\mathbb{R}$ of real numbers into $\mathbb{R}$ which satisfy for all $x, y, z \in \mathbb{R}$ the identity $f(f(x) + f(y) + f(z)) = f(f(x) - f(y)) + f(2xy + f(z)) + 2f(xz - yz)$