Romania Team Selection Test 2010

– TST 1

1 Given an integer number $n \geq 3$, consider n distinct points on a circle, labelled 1 through n.
Determine the maximum number of closed chords $[ij]$, $i \neq j$, having pairwise non-empty intersections.

Jnos Pach

2 Let n be a positive integer number and let $a_1, a_2, \ldots, a_n$ be n positive real numbers. Prove that $f : [0, \infty) \to \mathbb{R}$, defined by

$$f(x) = \frac{a_1 + x}{a_2 + x} + \frac{a_2 + x}{a_3 + x} + \cdots + \frac{a_{n-1} + x}{a_n + x} + \frac{a_n + x}{a_1 + x},$$

is a decreasing function.

Dan Marinescu et al.

3 Two rectangles of unit area overlap to form a convex octagon. Show that the area of the octagon is at least $\frac{1}{2}$.

Kvant Magazine

4 Two circles in the plane, γ_1 and γ_2, meet at points M and N. Let A be a point on γ_1, and let D be a point on γ_2. The lines AM and AN meet again γ_2 at points B and C, respectively, and the lines DM and DN meet again γ_1 at points E and F, respectively. Assume the order M, N, F, A, E is circular around γ_1, and the segments AB and DE are congruent. Prove that the points A, F, C and D lie on a circle whose centre does not depend on the position of the points A and D on the respective circles, subject to the assumptions above.

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5 Let a and n be two positive integer numbers such that the (positive) prime factors of a be all greater than n.
Prove that $n!$ divides $(a-1)(a^2-1)\cdots(a^{n-1}-1)$.

AMM Magazine

– TST 2

1 Given a positive integer number n, determine the minimum of

$$\max\left\{\frac{x_1}{1+x_1}, \frac{x_2}{1+x_1+x_2}, \cdots, \frac{x_n}{1+x_1+x_2+\cdots+x_n}\right\},$$

as $x_1, x_2, \ldots, x_n$ run through all non-negative real numbers which add up to 1.

Kvant Magazine

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2 (a) Given a positive integer k, prove that there do not exist two distinct integers in the open interval $(k^2, (k+1)^2)$ whose product is a perfect square.
(b) Given an integer $n > 2$, prove that there exist n distinct integers in the open interval $(k^n, (k+1)^n)$ whose product is the n-th power of an integer, for all but a finite number of positive integers k.

AMM Magazine

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3 Let γ_1 and γ_2 be two circles tangent at point T, and let ℓ_1 and ℓ_2 be two lines through T. The lines ℓ_1 and ℓ_2 meet again γ_1 at points A and B, respectively, and γ_2 at points A_1 and B_1, respectively. Let further X be a point in the complement of $\gamma_1 \cup \gamma_2 \cup \ell_1 \cup \ell_2$. The circles ATX and BTX meet again γ_2 at points A_2 and B_2, respectively. Prove that the lines TX, A_1B_2 and A_2B_1 are concurrent.

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4 Let n be an integer number greater than or equal to 2, and let K be a closed convex set of area greater than or equal to n, contained in the open square $(0,n) \times (0,n)$. Prove that K contains some point of the integral lattice $\mathbb{Z} \times \mathbb{Z}$.

Marius Cavachi

– TST 3

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1 Let n be a positive integer and let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $x_1 x_2 \cdots x_n = 1$. Prove that

$$\sum_{i=1}^{n} x_i^n (1+x_i) \geq \frac{n}{2^{n-1}} \prod_{i=1}^{n} (1+x_i).$$

IMO Shortlist

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2 Let ABC be a triangle such that $AB \neq AC$. The internal bisector lines of the angles ABC and ACB meet the opposite sides of the triangle at points B_0 and

C_0, respectively, and the circumcircle ABC at points B_1 and C_1, respectively. Further, let I be the incentre of the triangle ABC. Prove that the lines B_0C_0 and B_1C_1 meet at some point lying on the parallel through I to the line BC.

Radu Gologan

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3 Given a positive integer a, prove that $\sigma(am) < \sigma(am+1)$ for infinitely many positive integers m. (Here $\sigma(n)$ is the sum of all positive divisors of the positive integer number n.)

Vlad Matei

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4 Let X and Y be two finite subsets of the half-open interval $[0,1)$ such that $0 \in X \cap Y$ and $x+y=1$ for no $x \in X$ and no $y \in Y$. Prove that the set $\{x+y-\lfloor x+y \rfloor : x \in X \text{ and } y \in Y\}$ has at least $|X|+|Y|-1$ elements.

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– TST 4 (All Geometry)

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1 Let P be a point in the plane and let γ be a circle which does not contain P. Two distinct variable lines ℓ and ℓ' through P meet the circle γ at points X and Y, and X' and Y', respectively. Let M and N be the antipodes of P in the circles PXX' and PYY', respectively. Prove that the line MN passes through a fixed point.

Mihai Chis

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2 Let ABC be a scalene triangle. The tangents at the perpendicular foot dropped from A on the line BC and the midpoint of the side BC to the nine-point circle meet at the point A'\,; the points B' and C' are defined similarly. Prove that the lines AA', BB' and CC' are concurrent.

Gazeta Matematica

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3 Let $\mathcal{L}$ be a finite collection of lines in the plane in general position (no two lines in $\mathcal{L}$ are parallel and no three are concurrent). Consider the open circular discs inscribed in the triangles enclosed by each triple of lines in $\mathcal{L}$. Determine the number of such discs intersected by no line in $\mathcal{L}$, in terms of $|\mathcal{L}|$.

B. Aronov et al.

– TST 5

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1 Each point of the plane is coloured in one of two colours. Given an odd integer number $n \geq 3$, prove that there exist (at least) two similar triangles whose similitude ratio is n, each of which has a monochromatic vertex-set.

Vasile Pop

2 Let ℓ be a line, and let γ and γ' be two circles. The line ℓ meets γ at points A and B, and γ' at points A' and B'. The tangents to γ at A and B meet at point C, and the tangents to γ' at A' and B' meet at point C'. The lines ℓ and CC' meet at point P. Let λ be a variable line through P and let X be one of the points where λ meets γ, and X' be one of the points where λ meets γ'. Prove that the point of intersection of the lines CX and $C'X'$ lies on a fixed circle.

Gazeta Matematica

3 Let p be a prime number,let $n_1, n_2, \ldots, n_p$ be positive integer numbers, and let d be the greatest common divisor of the numbers $n_1, n_2, \ldots, n_p$. Prove that the polynomial

$$\frac{X^{n_1} + X^{n_2} + \cdots + X^{n_p} - p}{X^d - 1}$$

is irreducible in $\mathbb{Q}[X]$.

Beniamin Bogosel

– TST 6

1 A nonconstant polynomial f with integral coefficients has the property that, for each prime p, there exist a prime q and a positive integer m such that $f(p) = q^m$. Prove that $f = X^n$ for some positive integer n.

AMM Magazine

2 Let ABC be a scalene triangle, let I be its incentre, and let A_1, B_1 and C_1 be the points of contact of the excircles with the sides BC, CA and AB, respectively. Prove that the circumcircles of the triangles AIA_1, BIB_1 and CIC_1 have a common point different from I.

Cezar Lupu & Vlad Matei

3 Let n be a positive integer number. If S is a finite set of vectors in the plane, let $N(S)$ denote the number of two-element subsets $\{\mathbf{v}, \mathbf{v'}\}$ of S such that

$$4\,(\mathbf{v} \cdot \mathbf{v}') + (|\mathbf{v}|^2 - 1)(|\mathbf{v}'|^2 - 1) < 0.$$

Determine the maximum of $N(S)$ when S runs through all n-element sets of vectors in the plane.

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