Canada National Olympiad 2006

1 Let $f(n,k)$ be the number of ways of distributing k candies to n children so that each child receives at most 2 candies. For example $f(3,7)=0, f(3,6)=1, f(3,4)=6$. Determine the value of $f(2006,1)+f(2006,4)+\ldots+f(2006,1000)+f(2006,1003)+\ldots+f(2006,4012)$.

2 Let ABC be acute triangle. Inscribe a rectangle $DEFG$ in this triangle such that $D\in AB, E\in AC, F\in BC, G\in BC$. Describe the locus of (i.e., the curve occupied by) the intersections of the diagonals of all possible rectangles $DEFG$.

3 In a rectangular array of nonnegative reals with m rows and n columns, each row and each column contains at least one positive element. Moreover, if a row and a column intersect in a positive element, then the sums of their elements are the same. Prove that $m=n$.

4 Consider a round-robin tournament with $2n+1$ teams, where each team plays each other team exactly one. We say that three teams X, Y and Z, form a *cycle triplet* if X beats Y, Y beats Z and Z beats X. There are no ties.
a)Determine the minimum number of cycle triplets possible.
b)Determine the maximum number of cycle triplets possible.

5 The vertices of a right triangle ABC inscribed in a circle divide the circumference into three arcs. The right angle is at A, so that the opposite arc BC is a semicircle while arc BC and arc AC are supplementary. To each of three arcs, we draw a tangent such that its point of tangency is the mid point of that portion of the tangent intercepted by the extended lines AB, AC. More precisely, the point D on arc BC is the midpoint of the segment joining the points D' and D'' where tangent at D intersects the extended lines AB, AC. Similarly for E on arc AC and F on arc AB. Prove that triangle DEF is equilateral.