III GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
THE CORRESPONDENCE ROUND

1. (8) A triangle is cut into several (not less than two) triangles. One of them is isosceles (not equilateral), and all others are equilateral. Determine the angles of the original triangle.

2. (8) Each diagonal of a quadrangle divides it into two isosceles triangles. Is it true that the quadrangle is a diamond?

3. (8-9) Segments connecting an inner point of a convex non-equilateral n-gon to its vertices divide the n-gon into n equal triangles. What is the least possible n?

4. (8) Does a parallelogram exist such that all pairwise meets of bissectors of its angles are situated outside it?

5. A non-convex n-gon is cut into three parts by a straight line, and two parts are put together so that the resulting polygon is equal to the third part. Can n be equal to:

   a) (8) five?

   b) (8-10) four?

6. a) (8-9) What can be the number of symmetry axes of a checked polygon, that is, of a polygon whose sides lie on lines of a list of checked paper? (Indicate all possible values.)

   b) (10-11) What can be the number of symmetry axes of a checked polyhedron, that is, of a polyhedron consisting of equal cubes which border one to another by plane facets?

7. (8-9) A convex polygon is circumscribed around a circle. Points of contact of its sides with the circle form a polygon with the same set of angles (the order of angles may differ). Is it true that the polygon is regular?

8. (8-9) Three circles pass through a point P, and the second points of their intersection A, B, C lie on a straight line. Let A_1, B_1, C_1 be the second meets of lines AP, BP, CP with the corresponding circles. Let C_2 be the meet of lines AB_1 and BA_1. Let A_2, B_2 be defined similarly. Prove that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are equal.

9. (8-9) Suppose two convex quadrangles are such that the sides of each of them lie on the middle perpendiculars to the sides of the other one. Determine their angles.

10. (8-9) Find the locus of centers of regular triangles such that three given points A, B, C lie respectively on three lines containing sides of the triangle.

11. (8-10) A boy and his father are standing on a seashore. If the boy stands on his tiptoes, his eyes are at a height of 1 m above sea-level, and if he seats on father's shoulders, they are at a height of 2 m. What is the ratio of distances visible for him in two cases? (Find the answer to 0.1, assuming that the radius of Earth equals 6000 km.)

12. (9-10) A rectangle $ABCD$ and a point P are given. Lines passing through A and B and perpendicular to PC and PD respectively, meet at a point Q. Prove that $PQ \perp AB$.

13. (9-10) On the side AB of a triangle ABC, two points X, Y are chosen so that $AX = BY$. Lines CX and CY meet the circumcircle of the triangle, for the second time, at points U and V. Prove that all lines UV (for all X, Y, given A, B, C) have a common point.

14. (9-11) In a trapezium with bases AD and BC, let P and Q be the middles of diagonals AC and BD respectively. Prove that if $\angle DAQ = \angle CAB$ then $\angle PBA = \angle DBC$.

15. (9-11) In a triangle ABC, let AA', BB' and CC' be the bissectors. Suppose $A'B' \cap CC' = P$ and $A'C' \cap BB' = Q$. Prove that $\angle PAC = \angle QAB$.

16. (9-11) On two sides of an angle, points A, B are chosen. The middle M of the segment AB belongs to two lines such that one of them meets the sides of the angle at points A_1, B_1, and the other at points A_2, B_2. The lines A_1B_2 and A_2B_1 meet AB at points P and Q. Prove that M is the middle of PQ.

17. (9-11) What triangles can be cut into three triangles having equal radii of circumcircles?

18. (9-11) Determine the locus of vertices of triangles which have prescribed orthocenter and center of circumcircle.

19. (10-11) Into an angle A of size α, a circle is inscribed tangent to its sides at points B and C. A line tangent to this circle at a point M meets the segments AB and AC at points P and Q respectively. What is the minimum α such that the inequality $S_{PAQ} < S_{BMC}$ is possible?

20. (11) The base of a pyramid is a regular triangle having side of size 1. Two of three angles at the vertex of the pyramid are right. Find the maximum value of the volume of the pyramid.

21. (11) There are two pipes on the plane (the pipes are circular cylinders of equal size, 4 m around). Two of them are parallel and, being tangent one to another in the common generatrix, form a tunnel over the plane. The third pipe is perpendicular to two others and cuts out a chamber in the tunnel. Determine the area of the surface of this chamber.

IV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
THE CORRESPONDENCE ROUND

Below are the conditions of tasks constituting the correspondence round of the IV geometrical olympiad in honour of I.F.Sharygin.

Participation in the olympiad is possible for pupils of 8–11 forms (these are four elder forms in Russian school). In the list of tasks presented below, each task is indicated by the numbers of forms for which it is intended. However the participants may solve the tasks for elder forms as well.

Your work in a school copy-book containing the solutions for the tasks must be sent, at the latest, April 1, 2008 to the following address: Russia, 119002, Moscow, Bolshoy Vlasyevsky per., 11. Olympiad in honour of I.F.Sharygin.

The work must be written in Russian or in English. On the cover of the copy-book, the following information is obedient: all names (underline the surname); post address including post index, phone number and/or E-mail; the number of your present form; the number and address of your school; full names of your teachers in maths and/or of instructors of your math circle.

Please write down each task from a new page: first write down the condition, and then the solution. Present your solution in detail, including all essential arguments and calculations, with exact figures. The solution of a computational task must be completed by a distinctly presented answer. Please be accurate: you are interested in good understanding and true estimating of your work!

If your solution depends on some well-known theorem or a fact from a standard textbook, you may simply refer to it (explaining which theorem or fact is used). But if you use a fact not contained in the curriculum, you have to prove it (or to indicate the source).

Your work will be thoroughly examined, and you will receive the answer, at the latest, in the middle of May 2008. The winners of the correspondence round will be invited to the final round which will be held in summer 2008 in Dubna (near Moscow).

1. (8) Does a regular polygon exist such that just half of its diagonals are parallel to its sides?

2. (8) For a given pair of circles, construct two concentric circles such that both are tangent to the given two. What is the number of solutions, depending on location of the circles?

3. (8) A triangle can be dissected into three equal triangles. Prove that some its angle is equal to 60°.

4. (8–9) The bisectors of two angles in an inscribed quadrangle are parallel. Prove that the sum of squares of some two sides in the quadrangle equals the sum of squares of two remaining sides.

5. (8–9) Reconstruct the square $ABCD$, given its vertex A and distances of vertices B and D from a fixed point O in the plane.

6. (8-9) In the plane, given two concentric circles with the center A. Let B be an arbitrary point on some of these circles, and C on the other one. For every triangle ABC, consider two equal circles mutually tangent at the point K, such that one of these circles is tangent to the line AB at point B and the other one is tangent to the line AC at point C. Determine the locus of points K.

7. (8–9) Given a circle and a point O on it. Another circle with center O meets the first one at points P and Q. The point C lies on the first circle, and the lines CP, CQ meet the second circle for the second time at points A and B. Prove that $AB = PQ$.

8. (8–11) a) Prove that for $n > 4$, any convex n-gon can be dissected into n obtuse triangles.

   b) Prove that for any n, there exists a convex n-gon which cannot be dissected into less than n obtuse triangles.

   c) In a dissection of a rectangle into obtuse triangles, what is the least possible number of triangles?

9. (9–10) The lines symmetrical to diagonal BD of a rectangle $ABCD$ relative to bisectors of angles B and D pass through the midpoint of diagonal AC. Prove that the lines symmetrical to diagonal AC relative to bisectors of angles A and C pass through the midpoint of diagonal BD.

10. (9–10) Quadrangle $ABCD$ is circumscribed around a circle with center I. Prove that the projections of points B and D to the lines IA and IC lie on a single circle.

11. (9–10) Given four points A, B, C, D. Any two circles such that one of them contains A and B, and the other one contains C and D, meet. Prove that common chords of all these pairs of circles pass through a common point.

12. (9–10) Given a triangle ABC. Point A_1 is chosen on the ray BA so that segments BA_1 and BC are equal. Point A_2 is chosen on the ray CA so that segments CA_2 and BC are equal. Points B_1, B_2 and C_1, C_2 are chosen similarly. Prove that lines A_1A_2, B_1B_2, C_1C_2 are parallel.

13. (9–10) Given triangle ABC. One of its excircles is tangent to the side BC at point A_1 and to the extensions of two other sides. Another excircle is tangent to side AC at point B_1. Segments AA_1 and BB_1 meet at point N. Point P is chosen on the ray AA_1 so that $AP = NA_1$. Prove that P lies on the incircle.

14. (9–10) The line connecting the incenter and the orthocenter of a non-isosceles triangle is parallel to the bisector of one of its angles. Determine this angle.

15. (9–11) Given two circles and point P not lying on them. Draw a line through P which cuts chords of equal length from these circles.

16. (9–11) Given two circles. Their common external tangent is tangent to them at points A and B. Points X, Y on these circles are such that some circle is tangent to the given two circles at these points, and in similar way (external or internal). Determine the locus of intersections of lines AX and BY.

17. (9–11) Given triangle ABC and a ruler with two marked intervals equal to AC and BC. By this ruler only, find the incenter of the triangle formed by midlines of triangle ABC.

18. (9–11) Prove that the triangle having sides a, b, c and area S satisfies the inequality

$$a^2 + b^2 + c^2 - \frac{1}{2}(|a-b| + |b-c| + |c-a|)^2 \geq 4\sqrt{3}S.$$

19. (10-11) Given parallelogram $ABCD$ such that $AB = a$, $AD = b$. The first circle has its center at vertex A and passes through D, and the second circle has its center at C and passes through D. A circle with center B meets the first circle at points M_1, N_1, and the second circle at points M_2, N_2. Determine the ratio M_1N_1/M_2N_2.

20. (10–11) a) Some polygon has the following property: if a line passes through any two points which bisect its perimeter then this line bisects the area of the polygon. Is it true that the polygon is central symmetrical?

    b) Is it true that any figure with the property from part a) is central symmetrical?

21. (10–11) In a triangle, one has drawn middle perpendiculars to its sides and has measured their segments lying inside the triangle.

    a) All three segments are equal. Is it true that the triangle is equilateral?

    b) Two segments are equal. Is it true that the triangle is isosceles?

    c) Can the segments have length 4, 4 and 3?

22. (10–11) a) All vertices of a pyramid lie on the facets of a cube but not on its edges, and each facet contains at least one vertex. What is the maximum possible number of the vertices of the pyramid?

    b) All vertices of a pyramid lie in the facet planes of a cube but not on the lines including its edges, and each facet plane contains at least one vertex. What is the maximum possible number of the vertices of the pyramid?

23. (10–11) In the space, given two intersecting spheres of different radii and a point A belonging to both spheres. Prove that there is a point B in the space with the following property: if an arbitrary circle passes through points A and B then the second points of its meet with the given spheres are equidistant from B.

24. (11) Let h be the least altitude of a tetrahedron, and d the least distance between its opposite edges. For what values of t the inequality $d > th$ is possible?

Feel free to ask any questions via Organizing committee official e-mail *geomolymp@mccme.ru*

V GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
THE CORRESPONDENCE ROUND

Below are the conditions for the correspondence round of the V Geometrical Olympiad in honour of I.F.Sharygin.

Participation in the olympiad is possible for pupils of 8–11 forms (these are four elder forms in Russian school). In the list of tasks presented below, each task is indicated by the numbers of forms for which it is intended. However the participants may solve the tasks for elder forms as well (the solutions of problems for younger forms will not be considered).

Your work containing the solutions for the tasks (in Russian or English) must be sent, at the latest, April 1, 2009. We recommend to send the work by e-mail to geomolymp@mccme.ru in pdf, doc or jpg files. To avoid the loss of your work, please maintain the following rules.

1. Each work must be sent by a separate message.

2. If your work is contained in several files send it as an archive.

3. In the subject of the message, write "The work for Sharygin olympiad", and present the following information in the text:

- last name, first name;

- post address, phone number, E-mail;

- the current number of your form;

- the number and the address of your school;

- full names of your teachers in maths and/or of instructors of your math circle.

If you can't send your work by e-mail, please send it through the post to the following address: *Russia, 119002, Moscow, Bolshoy Vlasyevsky per., 11. Olympiad in honour of I.F.Sharygin.* In the title page present the information indicated in the item 3 above.

Please write down each task from a new page: first write down the condition, and next, the solution. Present your solution in detail, including all essential arguments and calculations, with exact figures. The solution of a computational task must be completed by a distinctly presented answer. Please be accurate: you are interested in good understanding and correct estimating of your work!

If your solution depends on some well-known theorem or a fact from a standard textbook, you may simply refer to it (to explain which theorem or fact is used). But if you use a fact not contained in the curriculum, you have to prove it (or to indicate the source).

Your work will be thoroughly examined, and you will receive the answer, at the latest, in the middle of May 2009. The winners of the correspondence round will be invited to the final round which will be held in summer 2009 in Dubna (near Moscow).

1. (8) Points B_1 and B_2 lie on ray AM, and points C_1 and C_2 lie on ray AK. The circle with center O is inscribed into triangles AB_1C_1 and AB_2C_2. Prove that the angles B_1OB_2 and C_1OC_2 are equal.

2. (8) Given nonisosceles triangle ABC. Consider three segments passing through different vertices of this triangle and bisecting its perimeter. Are the lengths of these segments certainly different?

3. (8) The bisectors of trapezoid's angles form a quadrilateral with perpendicular diagonals. Prove that this trapezoid is isosceles.

4. (8–9) Let P and Q be the common points of two circles. The ray with origin Q reflects from the first circle in points $A_1, A_2, \dots$ according to the rule "the angle of incidence is equal to the angle of reflection". Another ray with origin Q reflects from the second circle in the points $B_1, B_2, \dots$ in the same manner. Points A_1, B_1 and P occurred to be collinear. Prove that all lines A_iB_i pass through P.

5. (8–9) Given triangle ABC. Point O is the center of the excircle touching the side BC. Point O_1 is the reflection of O in BC. Determine angle A if O_1 lies on the circumcircle of ABC.

6. (8–9) Find the locus of excenters of right triangles with given hypotenuse.

7. (8–9) Given triangle ABC. Points M, N are the projections of B and C to the bisectors of angles C and B respectively. Prove that line MN intersects sides AC and AB in their points of contact with the incircle of ABC.

8. (8–10) Some polygon can be divided into two equal parts by three different ways. Is it certainly valid that this polygon has an axis or a center of symmetry?

9. (8–11) Given n points on the plane, which are the vertices of a convex polygon, $n > 3$. There exists k regular triangles with the side equal to 1 and the vertices at the given points.

   a) Prove that $k < \frac{2}{3}n$.

   b) Construct the configuration with $k > 0,666n$.

10. (9) Let ABC be an acute triangle, CC_1 its bisector, O its circumcenter. The perpendicular from C to AB meets line OC_1 in a point lying on the circumcircle of AOB. Determine angle C.

11. (9) Given quadrilateral $ABCD$. The circumcircle of ABC is tangent to side CD, and the circumcircle of ACD is tangent to side AB. Prove that the length of diagonal AC is less than the distance between the midpoints of AB and CD.

12. (9–10) Let CL be a bisector of triangle ABC. Points A_1 and B_1 are the reflections of A and B in CL; points A_2 and B_2 are the reflections of A and B in L. Let O_1 and O_2 be the circumcenters of triangles AB_1B_2 and BA_1A_2 respectively. Prove that angles O_1CA and O_2CB are equal.

13. (9–10) In triangle ABC, one has marked the incenter, the foot of altitude from vertex C and the center of the excircle tangent to side AB. After this, the triangle was erased. Restore it.

14. (9–10) Given triangle ABC of area 1. Let BM be the perpendicular from B to the bisecor of angle C. Determine the area of triangle AMC.

15. (9–10) Given a circle and a point C not lying on this circle. Consider all triangles ABC such that points A and B lie on the given circle. Prove that the triangle of maximal area is isosceles.

16. (9–11) Three lines passing through point O form equal angles by pairs. Points A_1, A_2 on the first line and B_1, B_2 on the second line are such that the common point C_1 of A_1B_1 and A_2B_2 lies on the third line. Let C_2 be the common point of A_1B_2 and A_2B_1. Prove that angle C_1OC_2 is right.

17. (9–11) Given triangle ABC and two points X, Y not lying on its circumcircle. Let A_1, B_1, C_1 be the projections of X to BC, CA, AB, and A_2, B_2, C_2 be the projections of Y. Prove that the perpendiculars from A_1, B_1, C_1 to B_2C_2, C_2A_2, A_2B_2, respectively, concur iff line XY passes through the circumcenter of ABC.

18. (9–11) Given three parallel lines on the plane. Find the locus of incenters of triangles with vertices lying on these lines (a single vertex on each line).

19. (10–11) Given convex n-gon $A_1 \dots A_n$. Let P_i $(i = 1, \dots, n)$ be such points on its boundary that A_iP_i bisects the area of polygon. All points P_i don't coincide with any vertex and lie on k sides of n-gon. What is the maximal and the minimal value of k for each given n?

20. (10–11) Suppose H and O are the orthocenter and the circumcenter of acute triangle ABC; AA_1, BB_1 and CC_1 are the altitudes of the triangle. Point C_2 is the reflection of C in A_1B_1. Prove that H, O, C_1 and C_2 are concyclic.

21. (10–11) The opposite sidelines of quadrilateral $ABCD$ intersect at points P and Q. Two lines passing through these points meet the side of $ABCD$ in four points which are the vertices of a parallelogram. Prove that the center of this parallelogram lies on the line passing through the midpoints of diagonals of $ABCD$.

22. (10–11) Construct a quadrilateral which is inscribed and circumscribed, given the radii of the respective circles and the angle between the diagonals of quadrilateral.

23. (10–11) Is it true that for each n, the regular $2n$-gon is a projection of some polyhedron having not greater than $n + 2$ faces?

24. (11) A sphere is inscribed into a quadrangular pyramid. The point of contact of the sphere with the base of the pyramid is projected to the edges of the base. Prove that these projections are concyclic.

V GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
THE CORRESPONDENCE ROUND. SOLUTIONS

1. (D.Prokopenko) (8) Points B_1 and B_2 lie on ray AM, and points C_1 and C_2 lie on ray AK. The circle with center O is inscribed into triangles AB_1C_1 and AB_2C_2. Prove that the angles B_1OB_2 and C_1OC_2 are equal.

   Solution. Let D be the common point of segments B_1C_1 and B_2C_2 (Fig.1). Then by theorem on exterior angles of a triangle, we have $\angle B_1OB_2 = \angle AOB_2 - \angle AOB_1 = \angle AB_1O - \angle AB_2O = (\angle AB_1C_1 - \angle AB_2C_2)/2 = \angle B_1DB_2/2$. Similarly $\angle C_1OC_2 = \angle C_1DC_2/2$, thus these angles are equal.

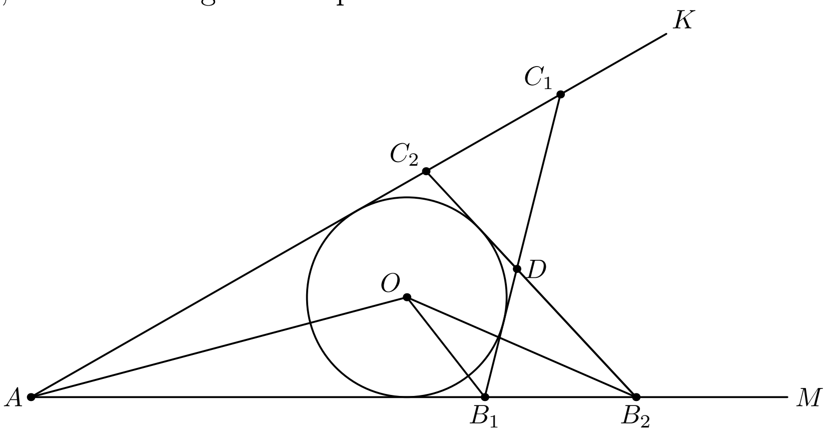


Fig.1

2. (B.Frenkin) (8) Given non-isosceles triangle ABC. Consider three segments passing through different vertices of this triangle and bisecting its perimeter. Are the lengths of these segments certainly different?

   Answer. Yes.

   Solution. Suppose for example that segments AA' and BB' are equal. Since the perimeters of triangles $AA'B$ and $AA'C$ are equal, we have $BA' = (AB + BC + CA)/2 - AB$. Similarly $AB' = (AB + BC + CA)/2 - AB$, and so triangles ABA' и BAB' are equal. Thus $\angle A = \angle B$, but this is impossible because triangle ABC is non-isosceles.

3. (D.Shnol) (8) The bisectors of trapezoid's angles form a quadrilateral with perpendicular diagonals. Prove that this trapezoid is isosceles.

   Solution. Let $KLMN$ be the quadrilateral formed by the bisectors (Fig. 3). Since AK, BK are the bisectors of adjacent trapezoid's angles, we have $\angle LKN = 90°$. Similarly $\angle LMN = 90°$. So $LK^2 + KN^2 = LM^2 + MN^2$. But by perpendicularity of the diagonals, $KL^2 + MN^2 = KN^2 + LM^2$. These two equalities yield that $KL = LM$ and $MN = NK$, thus $\angle NKM = \angle NMK$. But points K, M as common points of bisectors of adjacent angles, lie on the midline of the trapezoid, i.e. $KM \parallel AD$. So $\angle CAD = \angle BDA$ and the trapezoid is isosceles.

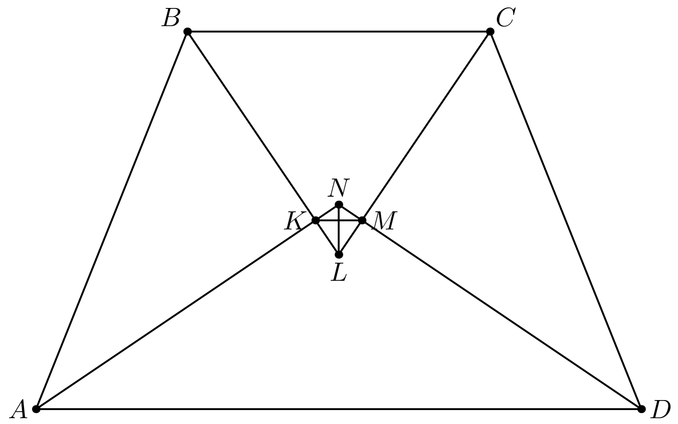


Fig.3

4. (D.Prokopenko) (8–9) Let P and Q be the common points of two circles. The ray with origin Q reflects from the first circle in points $A_1, A_2, \dots$ according to the rule "the angle of incidence is equal to the angle of reflection". Another ray with origin Q reflects from the second circle in the points $B_1, B_2, \dots$ in the same manner. Points A_1, B_1 and P occurred to be collinear. Prove that all lines A_iB_i pass through P.

   Solution. When the rays reflect from the circles, we have $QA_1 = A_1A_2 = A_2A_3 = \cdots$ and $QB_1 = B_1B_2 = B_2B_3 = \cdots$. So $\angle(PQ, PA_1) = \angle(PA_1, PA_2) = \angle(PA_2, PA_3) = \cdots$ and $\angle(PQ, PB_1) = \angle(PB_1, PB_2) = \angle(PB_2, PB_3) = \cdots$ (for oriented angles). Also, since points A_1, B_1, P are collinear, we have $\angle(PQ, PA_1) = \angle(PQ, PB_1)$. Thus for any i we have $\angle(PA_{i-1}, PA_i) = \angle(PB_{i-1}, PB_i)$, and by induction A_i, B_i, P are collinear.

5. (D.Shnol) (8–9) Given triangle ABC. Point O is the center of the excircle touching the side BC. Point O_1 is the reflection of O in BC. Determine angle A if O_1 lies on the circumcircle of ABC.

   Solution. The condition yields that $\angle BOC = \angle BO_1C = \angle A$. On the other hand, $\angle BOC = 180^\circ - (180^\circ - \angle B)/2 - (180^\circ - \angle C)/2 = (180^\circ - \angle A)/2$. So $\angle A = 60^\circ$.

6. (B.Frenkin) (8–9) Find the locus of excenters of right triangles with given hypotenuse.

   Solution. Let ABC be a right triangle with hypotenuse AB, and I_a, I_b, I_c be its excenters (Fig. 6). Then $\angle AI_cB = \angle AI_aB = \angle AI_bB = 45^\circ$, and points I_a, I_b lie on the same side from line AB, I_c on the other side. So these three points lie on two circles c_1, c_2 passing through A, B, such that their arc AB is equal to 90°. When C runs a semicircle with diameter AB then each excenter runs a quarter of the circle.

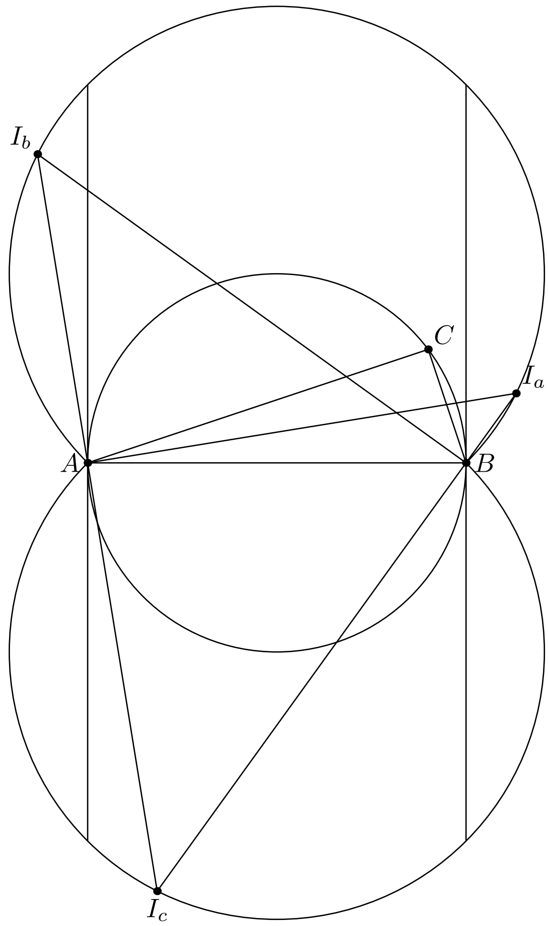


Fig.6

Namely, I_a runs the arc between B and the meet of the circle with l; I_b runs the arc between A and the meet of the circle with k; I_c runs the arc between the meets of the circle with k and l. When C runs the whole circle with diameter AB except points A, B, the excentres run the required locus, namely the arcs of c_1, c_2, lying outside the circle with diameter AB, except their ends A, B and their meets with k, l.

7. (V.Protasov) (8–9) Given triangle ABC. Points M, N are the projections of B and C to the bisectors of angles C and B respectively. Prove that line MN intersects sides AC and AB in their points of contact with the incircle of ABC.

   Solution. Let I be the incenter of ABC, P be the common point of MN and AC (Fig. 7). Points M, N lie on the circle with diameter BC, so $\angle MNB = \angle MCB = \angle ACI$. Hence C, I, P, N are concyclic and $\angle CPI = \angle CNI = 90°$. Thus, P is the touching point of AC with the incircle. For side AB the proof is similar.

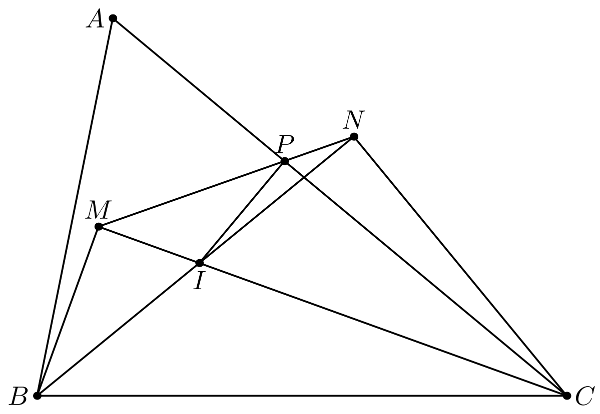


Fig.7

8. (S.Markelov) (8–10) Some polygon can be divided into two equal parts by three different ways. Is it certainly valid that this polygon has an axis or a center of symmetry?

   Answer. No, see Fig.8.

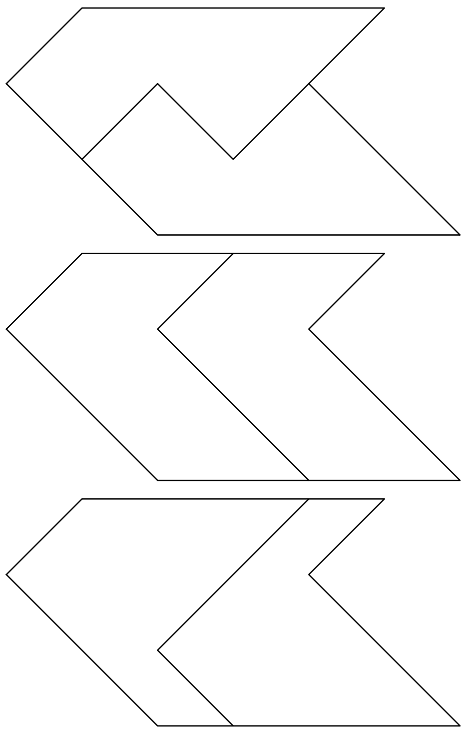


Fig.8

9. (V.A.Yasinsky) (8–11) Given n points on the plane, which are the vertices of a convex polygon, $n > 3$. There exist k regular triangles with the side equal to 1 and the vertices at the given points.

   a) Prove that $k < \frac{2}{3}n$.

   b) Construct the configuration with $k > 0,666n$.

   Solution. a) For any given point there exists a line passing through this point, such that all other given points lie on the same side from this line. This enables us to choose among all triangles having this point as a vertex, two "extreme"triangles (maybe coinciding), "left"and 'right'. We will call these two triangles "attached"to the given vertex.

   Lemma. Each triangle is attached at least three times.

Proof. Suppose that triangle ABC isn't "extremely left"for vertex C and isn't "extremely right"for vertex B (Fig.9).

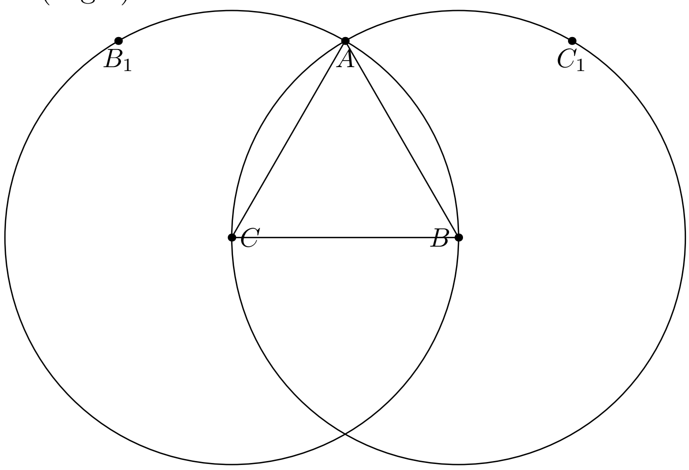


Fig.9

Then arcs AB_1 and AC_1 contain some of given points. But these points and A, B, C can't be vertices of a convex polygon. So ABC is attached by one of two indicated ways, i.e., it is "extremely left"for C or "extremely right"for B. Similarly it is "extremely left"for A or "extremely right"for C. Also it is "extremely left"for B or "extremely right"for A. Hence ABC is attached at least three times as required.

Suppose now that for n given points there exist k unit triangles. Since for any point there exist at most two attached triangles, $2n$ is the maximum number of attachments. Since each unit triangle is attached at least three times, $3k$ is the minimum number of attachments. Thus $3k \leq 2n$ and $k \leq \frac{2}{3}n$.

b) Consider the rhombus formed by two triangles. Rotating it around its obtuse-angled vertex we obtain m rhombuses.

If all rotation angles are less than $\pi/3$, then all vertices of obtained rhombuses form a convex polygon. Also we have $n = 3m + 1$, $k = 2m$, and for m sufficiently great we have $k > 0,666n$.

10. (F.Ivlev) (9) Let ABC be an acute triangle, CC_1 its bisector, O its circumcenter. The perpendicular from C to AB meets line OC_1 in a point lying on the circumcircle of AOB. Determine angle C.

    Solution. Let D be the common point of OC_1 and the perpendicular from C to AB. Since D lies on circle AOB and $AO = OB$, we have $\angle ADC_1 = \angle BDC_1$. So $AD/BD = AC_1/BC_1 = AC/BC$. On the other hand, $CD \perp AB$ implies $AC^2 + BD^2 = AD^2 + BC^2$. From these two relations $AC = AD$, i.e., D is the reflection of C in AB. But then CC_1 intersects the medial perpendicular of AB in the point symmetric to O (Fig.10). Since the medial perpendicular and the bisector meet on the circumcircle, AB bisects perpendicular radius. So $\angle C = 60°$.

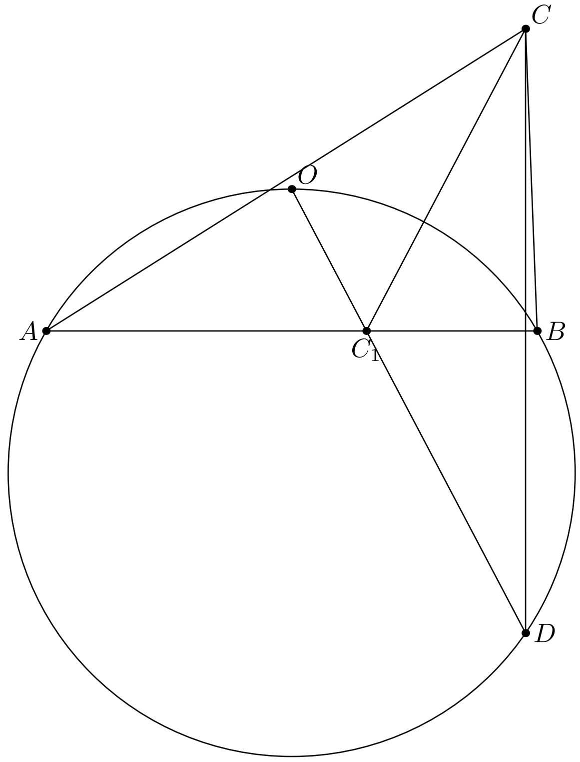


Fig.10

11. (A.Blinkov) (9) Given quadrilateral $ABCD$. The circumcircle of ABC is tangent to side CD, and the circumcircle of ACD is tangent to side AB. Prove that the length of diagonal AC is less than the distance between the midpoints of AB and CD.

   Solution. The condition implies $\angle BAC + \angle BCD = \angle ACD + \angle BAD = 180°$. Thus $\angle BCA = \angle CAD$, i.e. $AD \parallel BC$ and the segment between the midpoints of AB and CD is the medial line of the trapezoid and equals $(AD + BC)/2$. Also $\angle ACD = \angle ABC$ and $\angle BAC = \angle CDA$, so that triangles ABC and DCA are similar. Thus $AC^2 = AD \cdot BC$ and the assertion of the problem follows from Cauchi inequality.

12. (D.Prokopenko) (9–10) Let CL be a bisector of triangle ABC. Points A_1 and B_1 are the reflections of A and B in CL; points A_2 and B_2 are the reflections of A and B in L. Let O_1 and O_2 be the circumcenters of triangles AB_1B_2 and BA_1A_2 respectively. Prove that angles O_1CA and O_2CB are equal.

   Solution. The condition implies $CB_1/CA = CB/CA = BL/LA = B_2L/AL$, i.e., $B_1B_2 \parallel CL$. Similarly $A_1A_2 \parallel CL$. So $\angle AB_1B_2 = \angle BA_1A_2 = \angle C/2$. The reflection in CL transforms points B and A_1 to B_1 and A. Also it transforms A_2 to some point A'. We obtain $\angle A'AB_2 + \angle A'B_1B_2 = \angle A + \angle B + 2\angle C/2 = 180°$. Thus quadrilateral $AA'B_1B_2$ is cyclic and points O_1, O_2 are symmetric wrt CL (Fig.12).

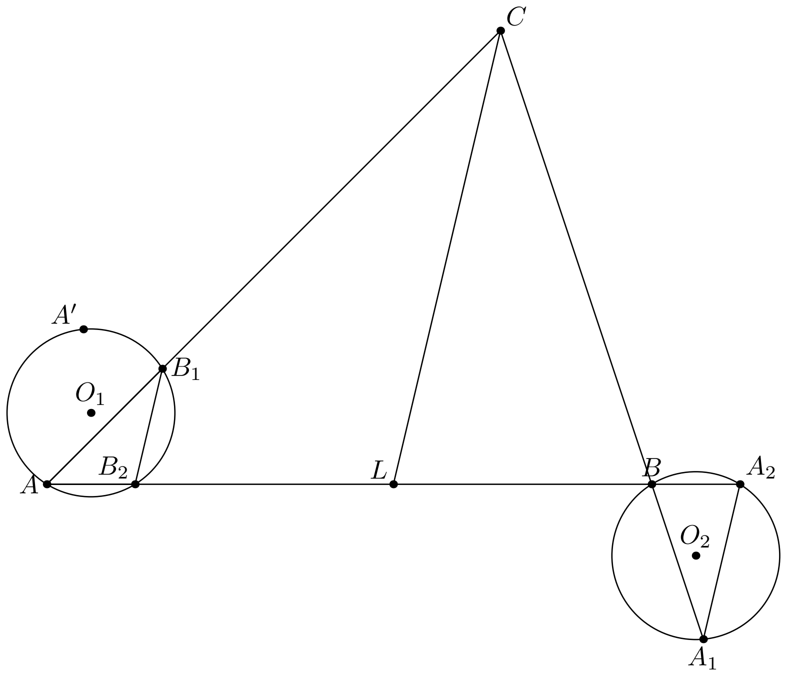


Fig.12

13. (A.Zaslavsky) (9–10) In triangle ABC, one has marked the incenter, the foot of altitude from vertex C and the center of the excircle tangent to side AB. After this, the triangle was erased. Restore it.

    Solution. Incenter I and excenter I_c lie on the bisector CC' of angle C. If r and r_c are the inradius and the exradius then $CI/CI_c = C'I/C'I_c = r/r_c$. So for any point X lying on the circle with diameter CC', the ratio XI/XI_c is the same. As the foot H of altitude to AB lies on this circle, we have $HI/HI_c = CI/C/I_c = C'I/C'I_c$, i.e., HC' and HC are the bisectors of angle IHI_c (Fig.13). Constructing these bisectors, we restore point C and line AB. As $\angle IAI_c = \angle IBI_c = 90°$, points A, B lie on the circle with diameter II_c. Constructing this circle and its common points with line AB, we restore the triangle.

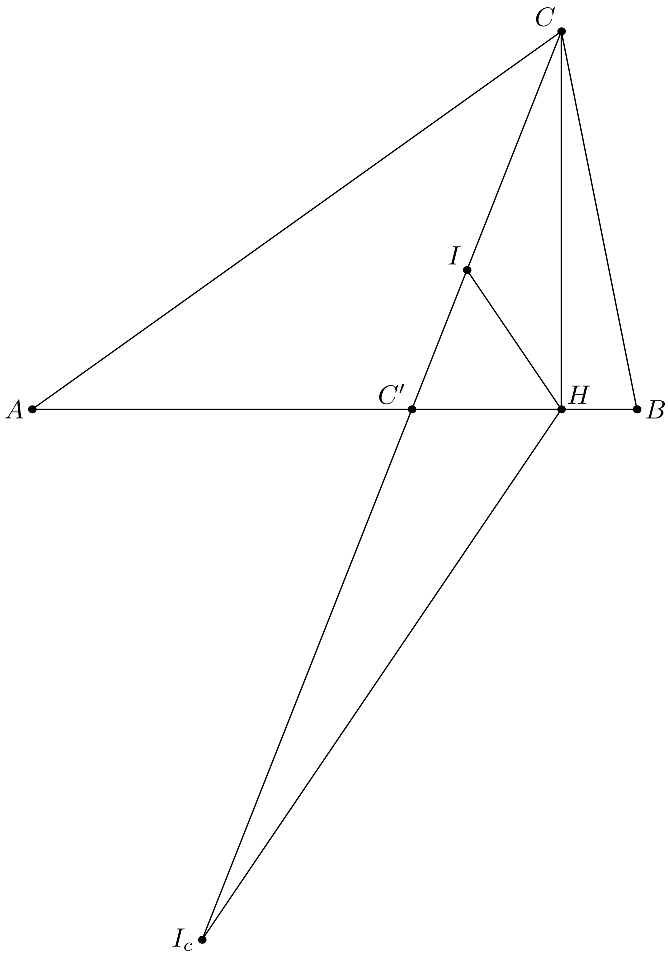


Fig.13

14. (V.Protasov) (9–10) Given triangle ABC of area 1. Let BM be the perpendicular from B to the bisector of angle C. Determine the area of triangle AMC.

First solution. Let the line passing through B and parallel to AC meet the bisector of angle C in point N (Fig.14). Since $\angle BNC = \angle ACN = \angle BCN$, triangle BCN is isosceles and BM is its median. Thus $S_{AMC} = \frac{1}{2}S_{ANC} = \frac{1}{2}S_{ABC} = \frac{1}{2}$.

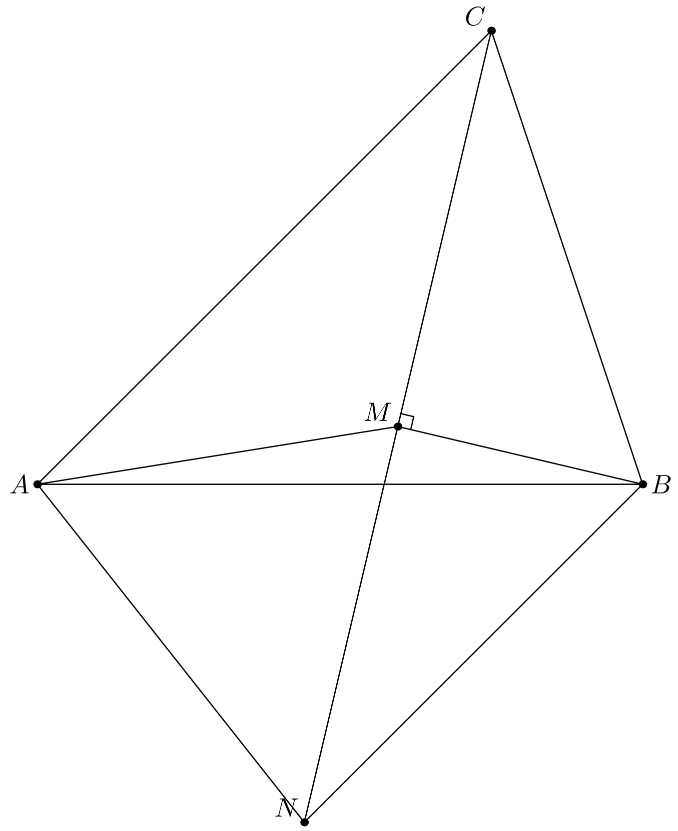


Fig.14

Second solution. Since $S_{AMC} = \frac{1}{2}AC \cdot CM \sin\frac{C}{2}$ and $CM = BC\cos\frac{C}{2}$, we have $S_{AMC} = \frac{1}{4}AC \cdot BC \sin C = \frac{S_{ABC}}{2} = \frac{1}{2}$.

15. (B.Frenkin) (9–10) Given a circle and a point C not lying on this circle. Consider all triangles ABC such that points A and B lie on the given circle. Prove that the triangle of maximal area is isosceles.

    Solution. Let C be the given point and A, B lie on the circle. If the tangent to the circle in A isn't parallel to CB, then moving point A, we can increase the distance from A to BC and the area of the triangle. Similarly the tangent at B is parallel to CA. So lines AC and BC are symmetric wrt the medial perpendicular to AB, and $AC = BC$ (Fig.15).

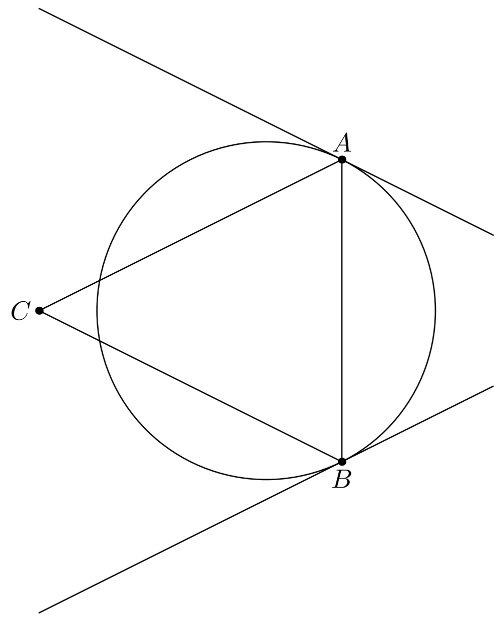


Fig.15

Note that the above argument does not depend on the location of the given point inside or outside the circle.

16. (A.Zaslavsky) (9–11) Three lines passing through point O form equal angles by pairs. Points A_1, A_2 on the first line and B_1, B_2 on the second line are such that the common point C_1 of A_1B_1 and A_2B_2 lies on the third line. Let C_2 be the common point of A_1B_2 and A_2B_1. Prove that angle C_1OC_2 is right.

    Solution. Let C_3 be the common point of lines OC_1 and A_2B_1 (Fig.16). Applying the Ceva and Menelaes theorems to triangle OA_2B_1 we obtain $C_2A_2/C_2B_1 = C_3A_2/C_3B_1 = OA_2OB_1$. So OC_2 is the external bisector of angle A_2OB_1, and $OC_2 \perp OC_1$.

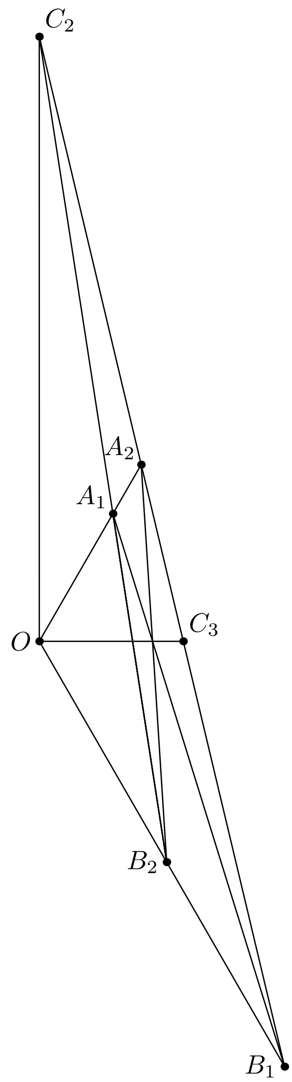


Fig.16

17. (A.Zaslavsky) (9–11) Given triangle ABC and two points X, Y not lying on its circumcircle. Let A_1, B_1, C_1 be the projections of X to BC, CA, AB, and A_2, B_2, C_2 be the projections of Y. Prove that the perpendiculars from A_1, B_1, C_1 to B_2C_2, C_2A_2, A_2B_2, respectively, concur iff line XY passes through the circumcenter of ABC.

    Solution. Let line XY pass through the circumcenter O. Fix point Y and move X along this line. The perpendiculars from A_1, B_1, C_1 to sidelines of $A_2B_2C_2$ move uniformly and remain self-parallel, so their common points move along some lines. When X coincides with O or Y, the three perpendiculars concur. Hence this is correct for all X.

    The above argument yields that for point Y fixed, the locus of points X such that the perpendiculars concur is line OY or the whole plane. Supposing the second case, take point C for X. Then A_1, B_1 coincide with C, and C_1 is the foot of the altitude from C. Since the three perpendiculars concur, we have $A_2B_2 \parallel AB$, and so Y lies on OC. Taking now another vertex for X, we obtain that Y coincides with O.

18. (B.Frenkin) (9–11) Given three parallel lines on the plane. Find the locus of incenters of triangles with vertices lying on these lines (a single vertex on each line).

Answer. The stripe whose bounds do not belong to the locus, are parallel to the given lines and are equidistant from the medium line and one of the extreme lines.

Solution. If we shift an arbitrary triangle whose vertices lie on the given lines by a vector parallel to these lines, its incenter is shifted by the same vector. So the desired locus is a stripe. Let us find its bounds.

Let b be the medium line and a, c the extreme lines. Let the vertex A, B, C lie on a, b, c respectively. Consider the diameter of the incircle, perpendicular to the given lines (Fig.18), and its endpoint nearest to a. This point lies nearer to a than the touching point of the incircle with AB, and thus it is nearer to a than b. Since another extreme point of the diameter lies nearer to a than c, the midpoint I of the diameter lies nearer to a than the line equidistant from b and c. Interchanging a and c in this argument, we obtain that I lies inside the stripe indicated in the answer.

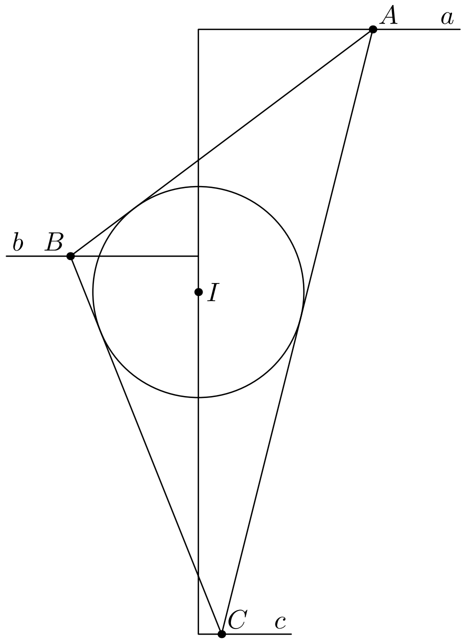


Fig.18

Now consider some triangle ABC with vertices on given lines. Move vertex B to make side AB perpendicular to the given lines. Now let point C tend to infinity. Then angles A and B of the triangle tend to right angles and point I tends to a vertex of the right isosceles triangle with hypotenuse AB. So I tends to the line equidistant from a and b. Similarly, starting from the same triangle, we can tend I to the line equidistant from b and c. Thus the locus of points I is the whole indicated stripe.

19. (B.Frenkin) (10–11) Given convex n-gon $A_1 \dots A_n$. Let $P_i\,(i = 1, \dots, n)$ be points on its boundary such that A_iP_i bisects the area of the polygon. All points P_i don't coincide with any vertex and lie on k sides of the n-gon. What is the maximal and the minimal value of k for each given n?

Answer. The minimal value is 3, the maximal value is equal to $n - 1$ for n even and equal to n for n odd.

Solution. Since segments A_iP_i bisect the area of the polygon, any two of them intersect. Let point P_i lie on side A_jA_{j+1}. Then points P_j and P_{j+1} lie on the boundary of the polygon in opposite directions from A_i, i.e., some of given points lie on three distinct sides of the polygon. Now let two vertices of the polygon be the vertices of a regular triangle and all other vertices lie near the third vertex of this triangle. Then all points P_i lie on three sides of the polygon.

If n is odd then it is evident that for a regular n-gon all P_i lie on different sides. Let $n = 2m$. Since segments A_mP_m and $A_{2m}P_{2m}$ intersect, points P_m and P_{2m} lie on the same side from diagonal A_mA_{2m}. There exist m sides of polygon lying on the other side from this diagonal, and point P_i can lie on one of these sides only if the corresponding vertex A_i lies between P_m and P_{2m}. But there exist at most $m-1$ such vertices, and so some side doesn't contain points P_i.

Consider now the n-gon such that vertices $A_1, \dots, A_{n-2}$ coincide with the vertices of a regular $(n-1)$-gon, and vertices A_{n-1}, A_n lie near the remaining vertex of this $(n-1)$-gon. Points P_i lie on all sides of this polygon except $A_{n-1}A_n$.

20. (D.Prokopenko) (10–11) Suppose H and O are the orthocenter and the circumcenter of acute triangle ABC; AA_1, BB_1 and CC_1 are the altitudes of the triangle. Point C_2 is the reflection of C in A_1B_1. Prove that H, O, C_1 and C_2 are concyclic.

    Solution. As $CA_1/CA = CB_1/CB = \cos C$, triangles ABC and A_1B_1C are similar. So, since $\angle ACO = \angle BCC_1 = 90^\circ - \angle B$, line CO contains an altitude of A_1B_1C, thus points C, O, C_2 are collinear (Fig.20). From similarity of ABC and A_1B_1C it follows as well that $CC_2/CC_1 = 2\cos C$. But it is known that $CH = 2CO\cos C$. Thus $CO \cdot CC_2 = CH \cdot CC_1$, which is equivalent to the assertion of the problem.

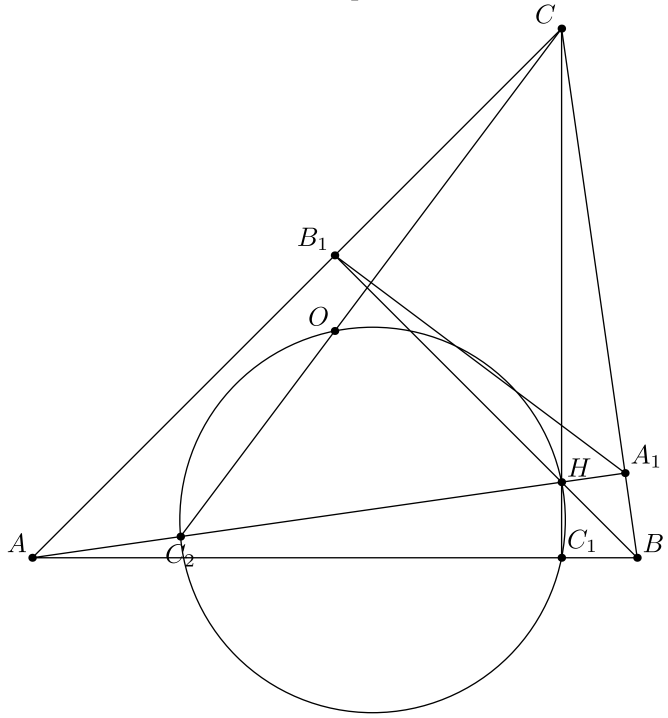


Fig.20

21. (F.Nilov) (10–11) The opposite sidelines of quadrilateral $ABCD$ intersect at points P and Q. Two lines passing through these points meet sides of $ABCD$ in four points which are vertices of a parallelogram. Prove that the center of this parallelogram lies on the line passing through the midpoints of diagonals of $ABCD$.

Solution. Using affine map, transform the parallelogram to a square and consider the coordinate system having the diagonals of this square for axes. Let the sides of the given quadrilateral intersect the coordinate axes in points $(\pm 1, 0)$, $(0, \pm 1)$, and the coordinates of P, Q be $(p, 0)$ and $(0, q)$ respectively. Then the equations of sidelines are $\frac{x}{p} \pm y = 1$, $\pm x + \frac{y}{q} = 1$; the vertices have coordinates $(\frac{p(q-1)}{pq-1}, \frac{q(p-1)}{pq-1})$, $(-\frac{p(q-1)}{pq+1}, \frac{q(p+1)}{pq+1})$, $(-\frac{p(q+1)}{pq-1}, -\frac{q(p+1)}{pq-1})$, $(\frac{p(q+1)}{pq+1}, -\frac{q(p-1)}{pq+1})$, and the midpoints of diagonals are collinear with the origin of the coordinate system.

22. (A.Zaslavsky) (10–11) Construct a quadrilateral which is inscribed and circumscribed, given the radii of the respective circles and the angle between the diagonals of the quadrilateral.

    Solution. If R, r are the radii of the circumcircle and the incircle and d is the distance between their centers O and I, then it is known that

$$\frac{1}{r^2} = \frac{1}{(R+d)^2} + \frac{1}{(R-d)^2}.$$

    So we can define d and construct these circles. The diagonals of all quadrilaterals with the given circumcircle and incircle intersect at the same point L lying on line OI, and their midpoints lie on the circle with diameter OL. Furthermore the segment between midpoints of the diagonals passes through I, and its length is equal to $OL \sin\phi$, where ϕ is the given angle. Constructing the chord of this length passing through I, we find the midpoints of the diagonals and so the vertices of the quadrilateral.

23. (V.Protasov) (10–11) Is it true that for each n, the regular $2n$-gon is a projection of some polyhedron having not greater than $n+2$ faces?

    Answer. Yes.

    Solution. Apply to regular $2n$-gon $A_1 \dots A_{2n}$ the dilation wrt A_nA_{2n} with coefficient $k > 1$ (Fig.23). Now bend the obtained polygon along line A_nA_{2n} to project its vertices $B_1, \dots, B_{n-1}, B_{n+1}, \dots, B_{2n-1}$ to the vertices of the original regular polygon. Then all lines B_iB_{2n-i} are parallel, and the polyhedron bounded by triangles $B_{n-1}B_nB_{n+1}$, $B_{2n-1}B_{2n}B_1$, trapezoids $B_iB_{i+1}B_{2n-i-1}B_{2n-i}$ and two halves of the $2n$-gon is the desired one.

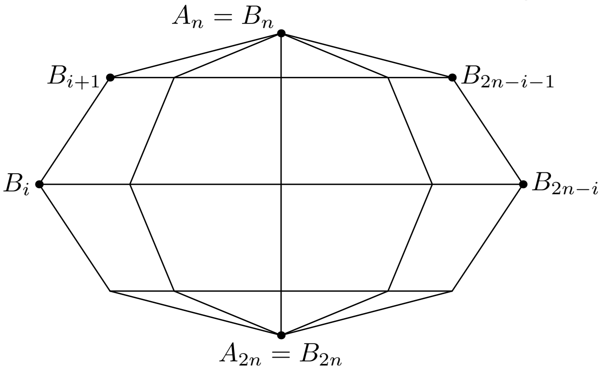


Fig.23

24. (F.Nilov) (11) A sphere is inscribed into a quadrangular pyramid. The point of contact of the sphere with the base of the pyramid is projected to the edges of the base. Prove that these projections are concyclic.

Solution. Let $ABCD$ be the base of the pyramid, P its touching point with the insphere, P' its touching point with the exsphere touching the base and the extensions of lateral faces. Then the ratio of distances from P to sidelines of the base is the same as for cotangents of halves of dihedral angles at the corresponding edges, and the ratio of distances from P' to sidelines is the same as for tangents of these half angles. Thus the lines joining each vertex of $ABCD$ with P and P' are symmetric wrt the bisector of the respective angle.

Now let K, L, M, N be the reflections of P wrt AB, BC, CD, DA. Since, for example, $BK = BP = BL$, the medial perpendicular to KL coincides with the bisector of angle KBL, i.e., line BP' (Fig.24). So P' is the circumcenter of $KLMN$. Using the homothety with center P and coefficient $1/2$ we obtain that the midpoint of PP' is the center of the circle passing through the projections of P, to the edges of the base.

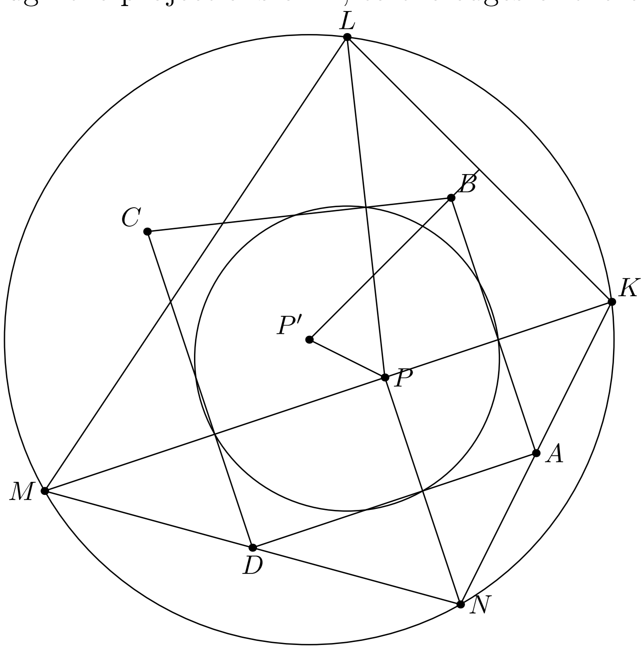


Fig.24

VI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
THE CORRESPONDENCE ROUND

Below is the list of problems for the first (correspondence) round of the VI Sharygin Geometrical Olympiad.

The olympiad is for high-school students of 8–11 grades (these are four elder grades in Russian school). In the list below each problem is indicated by the numbers of school grades, for which it is intended. However, the participants may solve problems for elder grades as well (solutions for younger forms will not be considered).

Your work containing the solutions for the problems (in Russian or in English) should be sent not later than April 1, 2010, by e-mail to geomolymp@mccme.ru in pdf, doc or jpg files. Please, follow several simple rules:

1. Each student sends his work in a separate message.

2. If your work consists of several files, send it as an archive.

3. In the subject of the message write "The work for Sharygin olympiad", and present the following personal information in the letter:

- last name, first name;

- post address, phone number, E-mail;

- the current number of your grade at school;

- the number and the mail address of your school;

- full names of your teachers in mathematics at school and/or of instructors of your extra math classes (if you attend additional math classes after school).

If you do not have an e-mail access, please, send your work by regular mail to the following address: *Russia, 119002, Moscow, Bolshoy Vlasyevsky per., 11. Olympiad in honour of Sharygin.* In the title page write your personal information indicated in the item 3 above.

In your work you should start writing the solution to each problem in a new page: first write down the statement of the problem, and then the solution. Present your solutions in detail, including all significant arguments and calculations. Provide all necessary figures. Solutions of computational problems have to be completed with a sharp answer. Please, be accurate to get a good understanding and correct estimating of your work !

If your solution depends on some well-known theorems from standard textbooks, you may simply refer to them instead of providing their proofs. However, any fact not from the standard curriculum should be either proved or properly referred (with an indication of the source).

Your work will be examined thoroughly, and your marks will be sent to you by the middle of May 2010. Winners of the correspondence round will be invited to take part in the final round in Summer 2010 in Dubna town (near Moscow).

1. (8) Does there exist a triangle, whose side is equal to some of its altitudes, another side is equal to some of its bisectrices, and the third side is equal to some of its medians ?

2. (8) Bisectors AA_1 and BB_1 of a right triangle ABC ($\angle C = 90^\circ$) meet at a point I. Let O be the circumcenter of the triangle CA_1B_1. Prove that $OI \perp AB$.

3. (8) Points A', B', C' lie on sides BC, CA, AB of a triangle ABC. For a point X one has $\angle AXB = \angle A'C'B' + \angle ACB$ and $\angle BXC = \angle B'A'C' + \angle BAC$. Prove that the quadrilateral $XA'BC'$ is cyclic.

4. (8) The diagonals of a cyclic quadrilateral $ABCD$ meet in a point N. The circumcircles of the triangles ANB and CND intersect the sidelines BC and AD for the second time in points A_1, B_1, C_1, D_1. Prove that the quadrilateral $A_1B_1C_1D_1$ is inscribed into a circle centered at N.

5. (8–9) A point E lies on the altitude BD of a triangle ABC, and $\angle AEC = 90°$. Points O_1 and O_2 are the circumcenters of the triangles AEB and CEB; points F, L are the midpoints of the segments AC and O_1O_2. Prove that the points L, E, F are collinear.

6. (8–9) Points M and N lie on the side BC of a regular triangle ABC (M is between B and N), and $\angle MAN = 30°$. The circumcircles of the triangles AMC and ANB meet at a point K. Prove that the line AK passes through the circumcenter of the triangle AMN.

7. (8–9) The line passing through the vertex B of a triangle ABC and perpendicular to its median BM intersects the altitudes dropped from A and C (or their extensions) in points K and N. Points O_1 and O_2 are the circumcenters of the triangles ABK and CBN respectively. Prove that $O_1M = O_2M$.

8. (8–10) Let AH be the altitude of a given triangle ABC. Points I_b and I_c are the incenters of the triangles ABH and ACH respectively; BC touches the incircle of ABC at a point L. Find $\angle LI_bI_c$.

9. (8–10) A point inside a triangle is called "good" if three cevians passing through it are equal. Assume for a isosceles triangle ABC ($AB = BC$) the total number of good points is odd. Find all possible values of this number.

10. (8–11) Let three lines forming a triangle ABC be given. Using a two-sided ruler and drawing at most eight lines construct a point D on the side AB such that $AD/BD = BC/AC$.

11. (8–11) A convex n-gon is split into three convex polygons. One of them has n sides, the second one has more than n sides, the third one has less than n sides. Find all possible values of n.

12. (9) Let AC be the greatest leg of a right triangle ABC, and CH be the altitude to its hypothenuse. The circle of radius CH centered at H intersects AC in point M. Let a point B' be the reflection of B with respect to the point H. The perpendicular to AB erected at B' meets the circle in a point K. Prove that:

    a) $B'M \parallel BC$;

    b) AK is tangent to the circle.

13. (9) Let us have a convex quadrilateral $ABCD$ such that $AB = BC$. A point K lies on the diagonal BD, and $\angle AKB + \angle BKC = \angle A + \angle C$. Prove that $AK \cdot CD = KC \cdot AD$.

14. (9–10) We have a convex quadrilateral $ABCD$ and a point M on its side AD such that CM and BM are parallel to AB and CD respectively. Prove that $S_{ABCD} \geq 3S_{BCM}$.

15. (9–11) Let AA_1, BB_1 and CC_1 be the altitudes of an acute-angled triangle ABC, AA_1 meets B_1C_1 in a point K. The circumcircles of the triangles A_1KC_1 and A_1KB_1 intersect the lines AB and AC for the second time at points N and L respectively. Prove that

a) the sum of diameters of these two circles is equal to BC;

b) $A_1N/BB_1 + A_1L/CC_1 = 1$.

16. (9–11) A circle touches the sides of an angle with vertex A at points B and C. A line passing through A intersects this circle in points D and E. A chord BX is parallel to DE. Prove that XC passes through the midpoint of the segment DE.

17. (9–11) Construct a triangle, if the lengths of the bisectrix and of the altitude from one vertex, and of the median from another vertex are given.

18. (9–11) A point B lies on a chord AC of a circle ω. Segments AB and BC are diameters of circles ω_1 and ω_2 centered at O_1 and O_2 respectively. These circles intersect ω for the second time in points D and E respectively. The rays O_1D and O_2E meet in a point F, and the rays AD and CE do in a point G. Prove that the line FG passes through the midpoint of the segment AC.

19. (9–11) A quadrilateral $ABCD$ is inscribed into a circle with center O. Points P and Q are opposite to C and D respectively. Two tangents drown to that circle at these points meet the line AB in points E and F (A is between E and B, B is between A and F). The line EO meets AC and BC in points X and Y respectively, and the line FO meets AD and BD in points U and V. Prove that $XV = YU$.

20. (10) The incircle of an acute-angled triangle ABC touches AB, BC, CA at points C_1, A_1, B_1 respectively. Points A_2, B_2 are the midpoints of the segments B_1C_1, A_1C_1 respectively. Let P be a common point of the incircle and the line CO, where O is the circumcenter of ABC. Let also A' and B' be the second common points of PA_2 and PB_2 with the incircle. Prove that a common point of AA' and BB' lies on the altitude of the triangle dropped from the vertex C.

21. (10–11) A given convex quadrilateral $ABCD$ is such that $\angle ABD + \angle ACD > \angle BAC + \angle BDC$. Prove that $S_{ABD} + S_{ACD} > S_{BAC} + S_{BDC}$.

22. (10–11) A circle centered at a point F and a parabola with focus F have two common points. Prove that there exist four points A, B, C, D on the circle such that the lines AB, BC, CD and DA touch the parabola.

23. (10–11) A cyclic hexagon $ABCDEF$ is such that $AB \cdot CF = 2BC \cdot FA$, $CD \cdot EB = 2DE \cdot BC$, and $EF \cdot AD = 2FA \cdot DE$. Prove that the lines AD, BE and CF concur.

24. (10–11) Let us have a line l in the space and a point A not lying on l. For an arbitrary line l' passing through A, XY (Y is on l') is a common perpendicular to the lines l and l'. Find the locus of points Y.

25. (11) For two different regular icosahedrons it is known that some six of their vertices are vertices of a regular octahedron. Find the ratio of the edges of these icosahedrons.

VI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN THE CORRESPONDENCE ROUND. SOLUTIONS

1. (B.Frenkin) (8) Does there exist a triangle, whose side is equal to some its altitude, another side is equal to some its bisectrix, and the third side is equal to some its median ?

   Solution. No, because the greatest side of a triangle is longer than any its median, bisector or altitude. Indeed, a segment joining a vertex of a triangle with an arbitrary point of the opposite side is shorter than one of two remaining sides. Thus each median or bisector is shorter than one of sides and so is shorter than the greatest side. This is correct also for the altitudes.

2. (D.Shvetsov) (8) Bisectors AA_1 and BB_1 of a right triangle ABC ($\angle C = 90°$) meet at a point I. Let O be the circumcenter of the triangle CA_1B_1. Prove that $OI \perp AB$.

   Solution. Let A_2, B_2, C_2 be the projections of A_1, B_1, I to AB (fig.2). Since AA_1 is a bisector, we have $AA_2 = AC$. On the other hand, AC_2 touches the incircle, thus the segment $A_2C_2 = AA_2 - AC_2$ is equal to the tangent to this circle from C. Similarly B_2C_2 is equal to the same tangent, i.e. C_2 is the midpoint of A_2B_2. By Phales theorem, C_2I meets segment A_1B_1 in its midpoint, which coincides with the circumcenter of triangle CA_1B_1 because this triangle is right-angled.

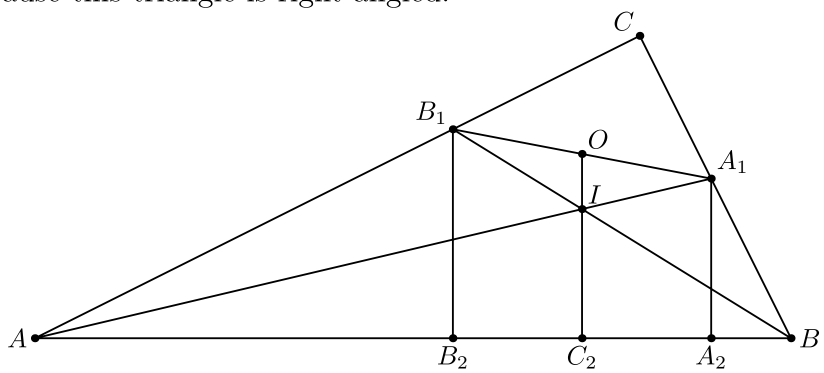


Fig. 2

3. (F.Nilov) (8) Points A', B', C' lie on sides BC, CA, AB of a triangle ABC. For a point X one has $\angle AXB = \angle A'C'B' + \angle ACB$ and $\angle BXC = \angle B'A'C' + \angle BAC$. Prove that the quadrilateral $XA'BC'$ is cyclic.

   Solution. Let Y be the common point of circles $AB'C'$ и $BC'A'$ distinct from C'. Then since $\angle B'YC' = \pi - \angle BAC$ and $\angle C'YA' = \pi - \angle CBA$, we obtain that $\angle A'YB' = \pi - \angle ACB$, i.e. Y lies also on circle $CA'B'$. Now note that $\angle AYB = \angle AYC' + \angle C'YB = \angle AB'C' + \angle C'A'B = 2\pi - \angle C'B'C - \angle CA'C' = \angle ACB + \angle A'C'B' = \angle AXB$ (fig.3). Similarly $\angle BYC = \angle BXC$, i.e. X and Y coincide.

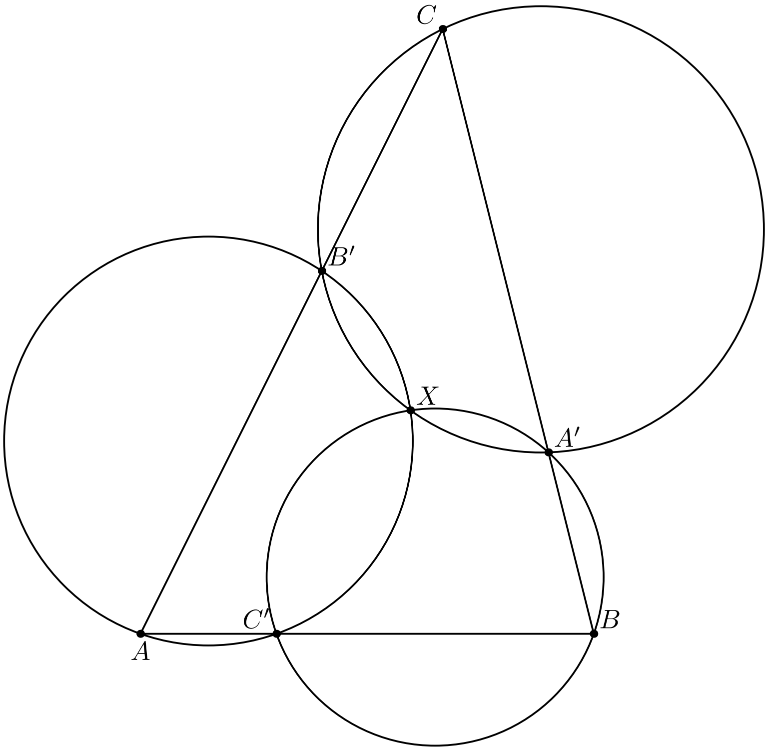


Fig. 3

4. (D.Shvetsov) (8) The diagonals of a cyclic quadrilateral $ABCD$ meet in a point N. The circumcircles of the triangles ANB and CND intersect the sidelines BC and AD for the second time in points A_1, B_1, C_1, D_1. Prove that the quadrilateral $A_1B_1C_1D_1$ is inscribed into a circle centered at N.

   Solution. Since pentagon A_1NB_1CD is cyclic, we obtain that $A_1N = B_1N$, because respective angles BDA and BCA are equal. Similarly $NC_1 = ND_1$. Also $\angle NA_1A = \angle ACD = \angle ABD = \angle DD_1N$ (fig.4). Thus $ND_1 = NA_1$, q.e.d.

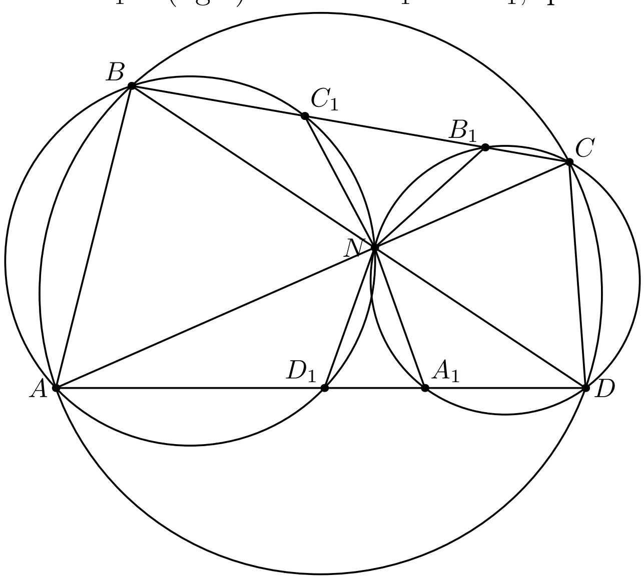


Fig. 4

5. (D.Shvetsov) (8–9) A point E lies on altitude BD of triangle ABC, and $\angle AEC = 90°$. Points O_1 and O_2 are the circumcenters of triangles AEB and CEB; points F, L are the

midpoints of segments AC and O_1O_2. Prove that points L, E, F are collinear.

Solution. Note that the medial perpendiculars to segments AE and EC are the medial lines of triangle AEC, thus they pass through F. So we must prove that FE is the median of triangle FO_1O_2. But $O_1O_2 \parallel AC$ because these two segments are perpendicular to BD. Let the line passing through E and parallel to AC meet FO_1 and FO_2 in points X and Y (fig.5). Since $FCEX$ and $FAEY$ are parallelograms, then $XE = FC = FA = EY$. Thus FE is the median of triangles FXY and FO_1O_2.

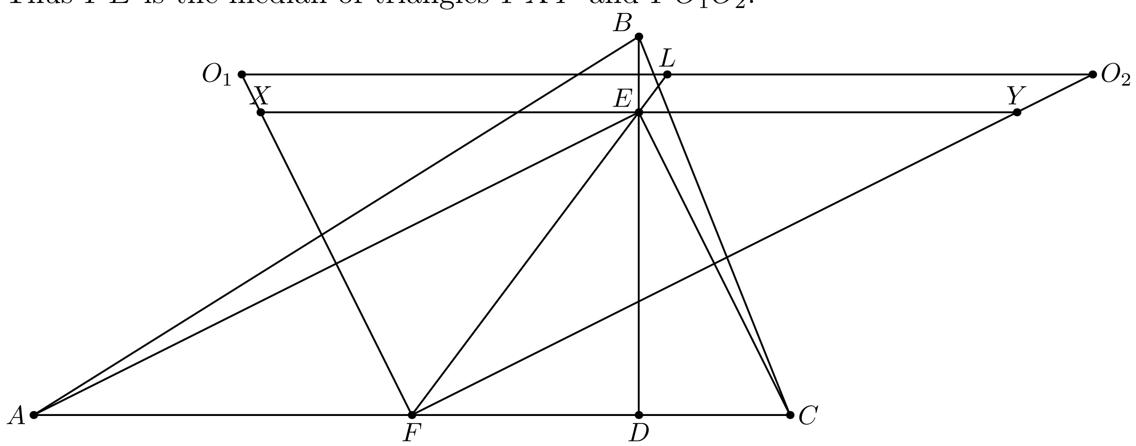


Fig. 5

6. (D.Shvetsov) (8–9) Points M and N lie on side BC of a regular triangle ABC (M is between B and N), and $\angle MAN = 30°$. The circumcircles of triangles AMC and ANB meet at a point K. Prove that line AK passes through the circumcenter of triangle AMN.

Solution. Since $\angle BAM + \angle NAC = \angle MAN$ and $AB = AC$, the reflection of B in AM coincides with the reflection of C in AN. Mark this point by L. Now $\angle ALM = \angle ABM = \angle ACM$, i.e. L lies on circle ACM. Similarly L lies on circle ABN and thus coincdes with K (fig.6). So $\angle KAN = \angle NAC = 30° - \angle BAM = 90° - \angle NMA$. But the theorem on inscribed angle implies that we have the same angle between line AN and the line connecting A with the circumcenter of triangle AMN.

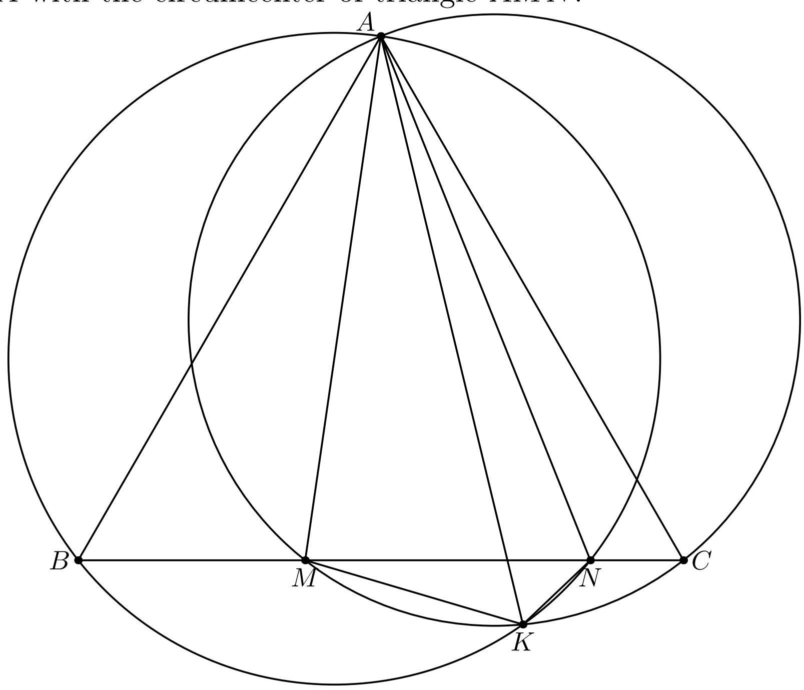


Fig. 6

7. (D.Shvetsov) (8–9) The line passing through vertex B of triangle ABC and perpendicular to its median BM intersects the altitudes dropped from A and C (or their extensions) in points K and N. Points O_1 and O_2 are the circumcenters of triangles ABK and CBN respectively. Prove that $O_1M = O_2M$.

   Solution. Consider parallelogram $ABCD$ (fig.7). Since $\angle BKA = \angle DKC = \angle BDA$, points A, B, K, D lie on a same circle and $O_1M \perp BD$. Similarly $O_2M \perp BD$. Also since triangles ABD and BCD are equal, the distances from their circumcenters to point M also are equal.

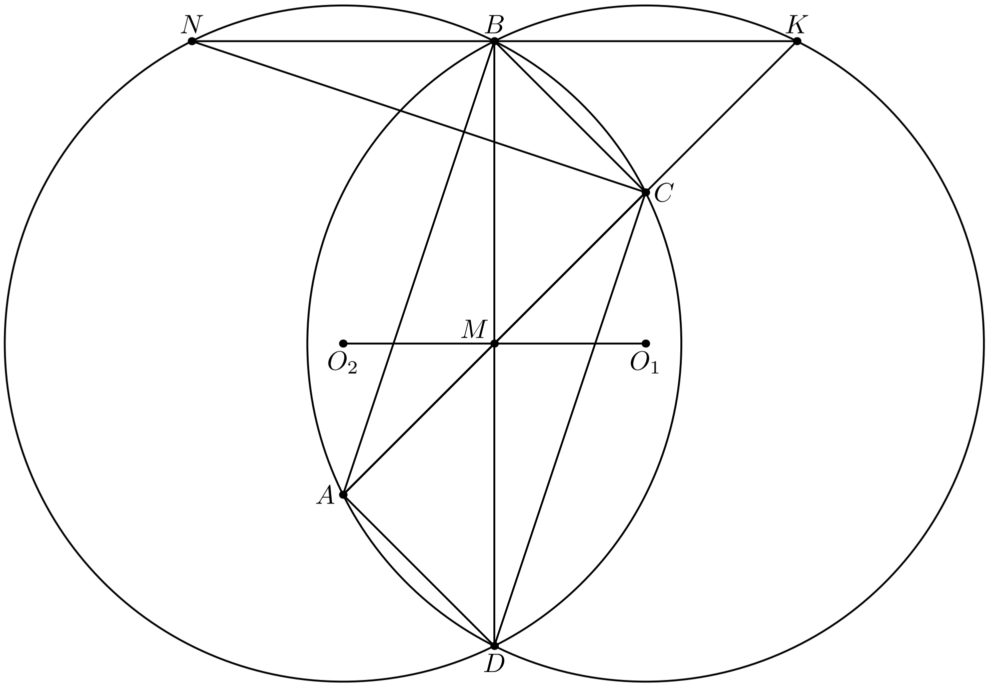


Fig. 7

8. (D.Shvetsov) (8–10) Let AH be an altitude of a given triangle ABC. Points I_b and I_c are the incenters of triangles ABH and CAH respectively; BC touches the incircle of triangle ABC at a point L. Find $\angle LI_bI_c$.

   Solution. We will prove that triangle LI_bI_c is right-angled and isosceles. Let L_b, L_c be the projections of I_b, I_c to BC, and r_b, r_c be the inradii of triangles AHB, AHC (fig.8). Since these triangles are right-angled, we have $r_b = (AH+BH-AB)/2$, $r_c = (AH+CH-AC)/2$ and $r_b-r_c = (BH-CH)/2-(AB-AC)/2 = (BH-CH)/2-(BL-CL)/2 = LH$. Thus $I_bL_b = LI_c = r_b$, $I_cL_c = LI_b = r_c$, i.e. triangles LI_bL_b and I_cL_cL are equal, $LI_b = LI_c$ and $\angle I_bLI_c = 90°$. So $\angle LI_bI_c = 45°$.

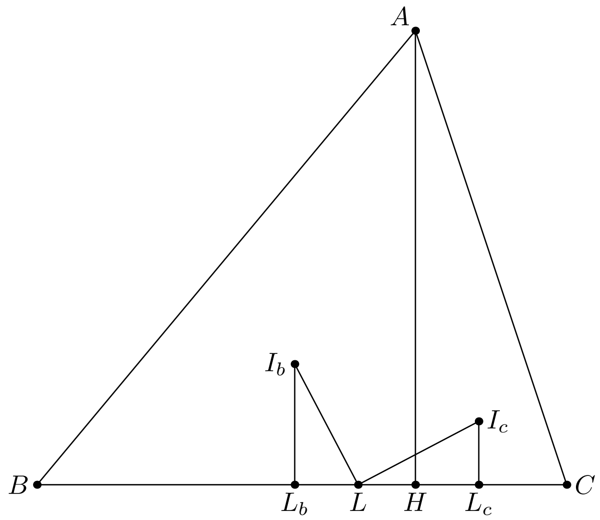


Fig. 8

9. (B.Frenkin) (8–10) A point inside a triangle is called "good" if three cevians passing through it are equal. Suppose the total number of good points is odd for an isosceles triangle ABC ($AB = BC$). Find all possible values of this number.

   Solution. Since the reflection of any good point in the altitude from B also is a good point and the total number of good points is odd, there exists a good point lying on this altitude. The cevian through this point from A is not shorter than the altitude from A. Hence the altitude from A is not shorter than the altitude from B and $AC \leq AB$. Also AC can't be longer than the altitude from B because in that case there exist two good points on this altitude. Suppose now that some good point doesn't lie on this altitude. Let AA', BB', CC' be respective cevians, and AA_1, CC_1 be the altitudes. Then $A_1A' = C_1C'$ and exactly one of points A', C' lies between the foot of respective altitude and vertex B. But this implies that respective cevians are shorter than AC. So they are shorter than BB_1 and we obtain a contradiction. Thus there exists exactly one good point.

10. (I.Bogdanov) (8–11) Let three lines forming a triangle ABC be given. Using a two-sided ruler and drawing at most eight lines, construct a point D on the side AB such that $AD/BD = BC/AC$.

    Solution. Construct lines a, b, c, parallel to BC, CA, AB and lying at the distance from them equal to the width of the ruler. Lines a, b, BC, AC form a rhombus, and its diagonal is the bisector of angle C. Let E be the common point of this bisector with c, and F be the common point of diagonals of trapezoid formed by lines c, AB, AC and BC (fig.10). Then EF meets AB in the sought point D.

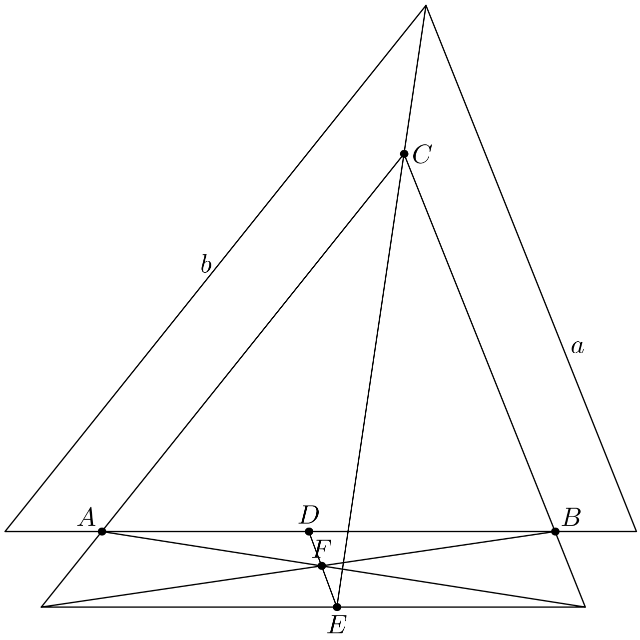


Fig. 10

11. (B.Frenkin) (8–11) A convex n-gon is split into three convex polygons. One of them has n sides, the second one has more than n sides, the third one has less than n sides. Find all possible values of n.

    Answer. $n = 4$ или $n = 5$.

    Solution. It is clear that $n > 3$. Suppose that $n > 5$. Then one of three parts of n-gon has at least $n+1$ sides, the second parts has at least n sides, the third part has at least three sides. If three pairs of the sides of these parts join inside the given polygon then at most three pairs can form its sides. If two pairs of sides join inside the polygon then at most four pairs can form its sides. In all cases the total number of sides of parts is not greater than $n+9$. If $n > 5$ this isn't possible. The examples for $n = 4, 5$ are given on fig. 11.

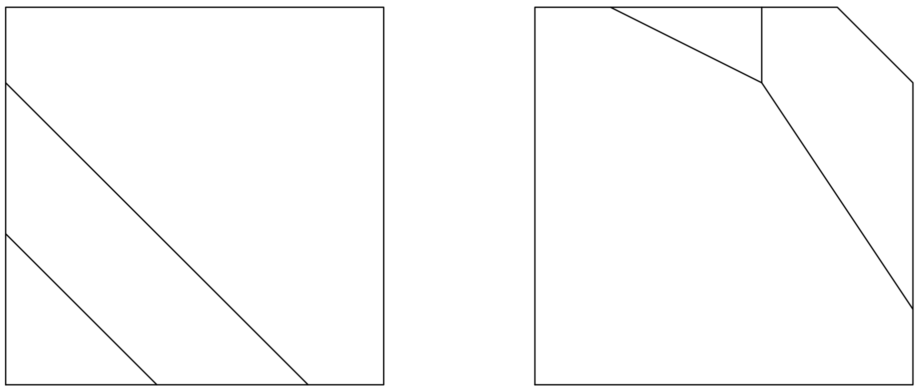


Fig. 11

12. (A.Blinkov, Y.Blinkov, M.Sandrikova) (9) Let AC be the greatest leg of a right triangle ABC, and CH be the altitude to its hypothenuse. The circle of radius CH centered at H intersects AC in point M. Let a point B' be the reflection of B with respect to the point H. The perpendicular to AB erected at B' meets the circle in a point K. Prove that:

    a) $B'M \parallel BC$;

b) AK is tangent to the circle.

Solution. a) Let N be an altitude of isosceles triangle CHM. Then $CN = NM$. Since $BH = B'H$ and $NH \parallel BC$, thus the line passing through B' and parallel to HN meets AC in point M (Phales theorem).

Second solution. Since $\angle CMH = \angle MCH = \angle CBB' = \angle CB'B = a$, points C, H, B' and M are concyclic. Thus $\angle CB'M = \angle CHM = 180^\circ - 2a$ and $\angle AB'M = a$ q.e.d.

b) From the right-angled triangle ABC we have: $CH^2 = AH \cdot BH$. Since $B'H = BH$ and $KH = CH$ then $KH^2 = AH \cdot B'H$, i.e. triangles AHK and KHB' are similar. This yields the assertion of the problem..

13. (S.Berlov) (9) Given a convex quadrilateral $ABCD$ such that $AB = BC$. A point K lies on the diagonal BD, and $\angle AKB + \angle BKC = \angle A + \angle C$. Prove that $AK \cdot CD = KC \cdot AD$.

    Solution. Let L be a point on BD such that $\angle ALB = \angle A$. Since triangles ABL and DBA are similar we have $BL \cdot BD = AB^2 = BC^2$. Thus triangles CBL and DBC are also similar, i.e. $\angle BLC = \angle C$ and L coincides with K. The sought equality clearly follows from these two similarities.

14. (S.Berlov) (9–10) Given a convex quadrilateral $ABCD$ and a point M on its side AD such that CM and BM are parallel to AB and CD respectively. Prove that $S_{ABCD} \geq 3S_{BCM}$.

    Solution. Since $\angle ABM = \angle BMC = \angle MCD$ we have $S_{ABM}/S_{BMC} = AB/MC$ and $S_{BMC}/S_{CMD} = BM/CD$. But triangles ABM and MCD are similar, so these two ratios are equal and $S_{BMC}^2 = S_{ABM} \cdot S_{MCD}$. By Cauchi inequality $S_{BMC} \leq (S_{ABM} + S_{MCD})/2$ which is equivalent to the assertion of the problem.

15. (D.Prokopenko, A.Blinkov) (9–11) Suppose AA_1, BB_1 and CC_1 are the altitudes of an acute-angled triangle ABC, AA_1 meets B_1C_1 in a point K. The circumcircles of triangles A_1KC_1 and A_1KB_1 intersect the lines AB and AC for the second time at points N and L respectively. Prove that

    a) the sum of diameters of these two circles is equal to BC;

    b) $A_1N/BB_1 + A_1L/CC_1 = 1$.

    Solution. a) Triangles AB_1C_1, A_1BC_1 and A_1B_1C are similar to triangle ABC with coefficients $\cos A$, $\cos B$, $\cos C$ respectively. Thus $\angle KA_1C_1 = \angle KA_1B_1 = 90^\circ - \angle A$, and by sinuses theorem the diameters of circumcircles of triangles AKB_1 and A_1KC_1 are equal to $B_1K/\cos A$ and $C_1K/\cos A$ respectively. So their sum is $B_1C_1/\cos A = BC$.

    b) An equality proved in p.a) can be written as

    $$\frac{A_1N}{\sin B} + \frac{A_1L}{\sin C} = BC.$$

    Dividing by BC we obtain sought relation.

16. (F.Nilov) (9–11) A circle touches the sides of an angle with vertex A at points B and C. A line passing through A intersects this circle in points D and E. A chord BX is parallel to DE. Prove that XC passes through the midpoint of segment DE.

    Solution. Note that $\angle BCD = \angle ECX$ because the respective arcs lie between parallel chords. Furthermore since $\angle ABD = \angle AEB$, triangles ABD and AEB are similar and so

$BD/BE = AD/AB$. Similarly $CD/CE = AD/AB$, i.e. $BD{\cdot}CE = CD{\cdot}BE = BC{\cdot}DE/2$ (the last equality follows from Ptolomeus theorem).

Now let CX meet DE at point M (fig. 16). Then triangles CBD and CME are similar, thus $BD \cdot CE = CB \cdot EM$. From this and previous equalities we obtain $EM = ED/2$.

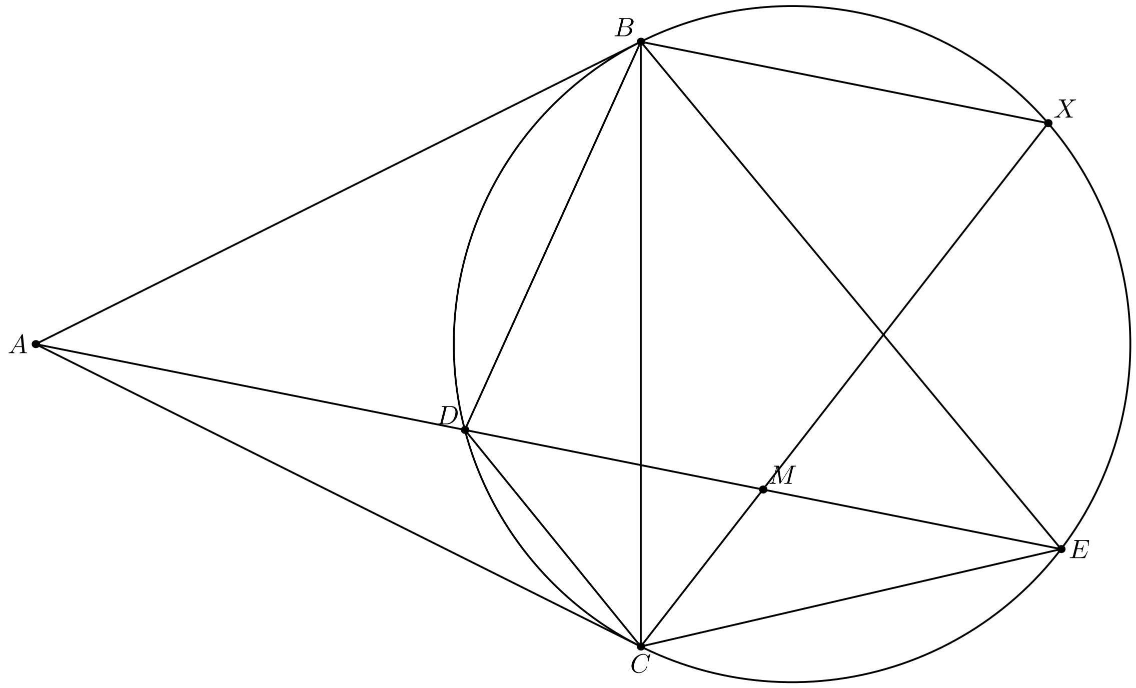


Fig. 16

17. (S.Tokarev) (9–11) Construct a triangle, if the lengths of the bisectrix and of the altitude from one vertex, and of the median from another vertex are given.

First solution. Let $l = CL$, $h = CH$ be the bisector and the altitude from vertex C, $m = BM$ be the median from vertex B and ϕ be the angle of right-angled triangle with hypothenuse l opposite to cathetus with length h. Let p be the line passing through C and parallel to AB, and B' be the reflection of point B in p.

Suppose that triangle ABC is constructed. Then $\angle CLB = \phi$ or $\angle CLB = 180^\circ - \phi$, and in both cases $\angle B'CM = 2\phi$. In fact, if $\angle CLB = \phi$, then $\angle B'CM = 360^\circ - 2\angle CBA - \angle BCA = 2(180^\circ - \angle CBA - \angle BCL) = 2\phi$ (Fig.17.1). The second case is similar.

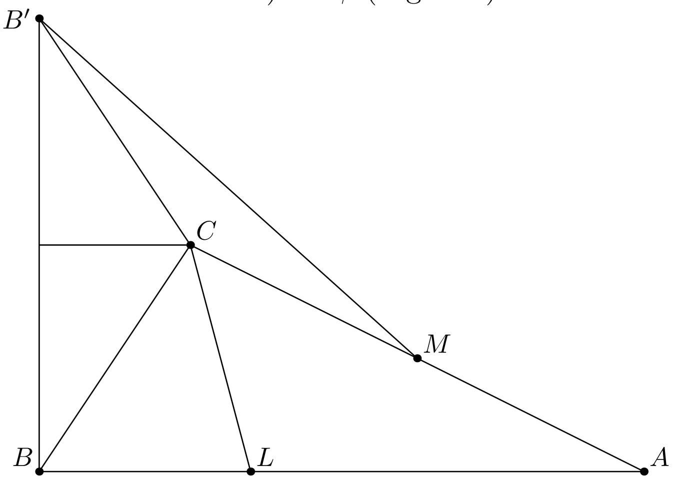


Fig. 17.1

Since $\angle B'CM = 2\phi$ we obtain the following construction.

Construct two parallel lines with distance h between them. Let B be a point lying on one of these lines and p be the second line. Now construct point B' and point M equidistant from two lines and such that $BM = m$. Construct angle ϕ and two arcs with endpoints B' and M equal to $360° - 4\phi$.

If C_1 and C_2 are the common points of these arcs with line p, and A_i $(i = 1, 2)$ is the reflection of C_i in M, then each of triangles A_1BC_1 and A_2BC_2 is sought.

Indeed, the altitudes of these triangles from C_1, C_2 are equal to h, and segment $BM = m$ is their common median. Furthermore if L_1, L_2 are the feet of respective bisectors then our construction yields that one of angles C_1L_1B and C_2L_2B is equal to ϕ, and the second one is equal to $180° - \phi$. Thus by definition of ϕ we obtain $C_1L_1 = C_2L_2 = l$.

Note. It is evident that if $l < h$ or $m < h/2$, then the solution doesn't exist. If $l = h$ and $m \geq h/2$, then the sought triangle is unique and isosceles (if $m = h/2$ it degenerates to a segment). When $l > h$ and $m = h/2$, we obtain two equal triangles symmetric wrt line BB'.

If $l > h$ and $m = l/2$ then one of two triangles is degenerated. In all other cases the problem has two solutions.

Second solution. Having an altitude and a bisector from vertex C, we can construct this vertex and line AB. Consider now the following map of this line to itself. For an arbitrary point X find a point Y such that its distances from X and AB are equal to the given median from B and to the half of the given altitude (fig. 17.2). Now find a common point X' for AB and for the reflection of CY in the bisector. Obviously this map is projective and B is its fixed point. So we obtain the well-known problem of constructing the fixed point of a projective map.

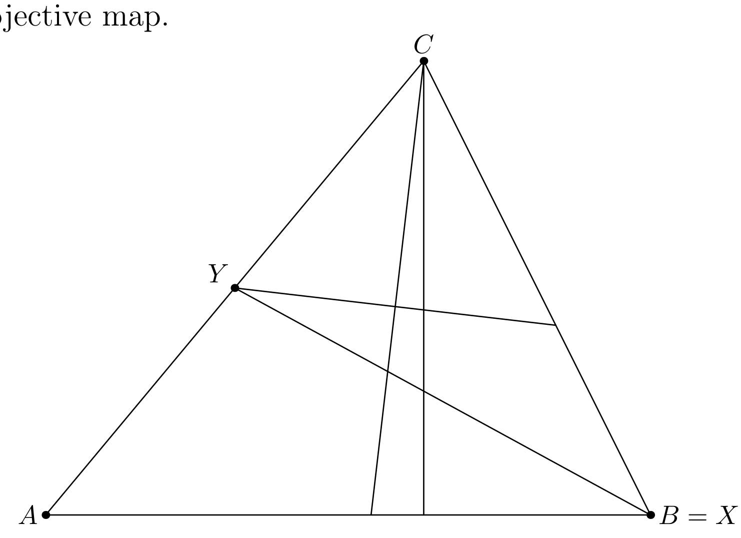


Fig. 17.2

18. (D.Prokopenko) (9–11) A point B lies on a chord AC of a circle ω. Segments AB and BC are diameters of circles ω_1 and ω_2 centered at O_1 and O_2 respectively. These circles intersect ω for the second time in points D and E respectively. The rays O_1D and O_2E meet in a point F, and the rays AD and CE meet in a point G. Prove that line FG passes through the midpoint of segment AC.

Solution. Since $\angle ADB = \angle BEC = 90°$, points D and E lie on the circle with diameter BG. Also $\angle FDG = \angle ADO_1 = \angle DAC = \angle GED$. Thus FD (and similarly FE) touches this circle (fig. 18). So GF is the symmedian of triangle GED. Since triangle GDE is similar to triangle GCA, this line is the median of the last triangle.

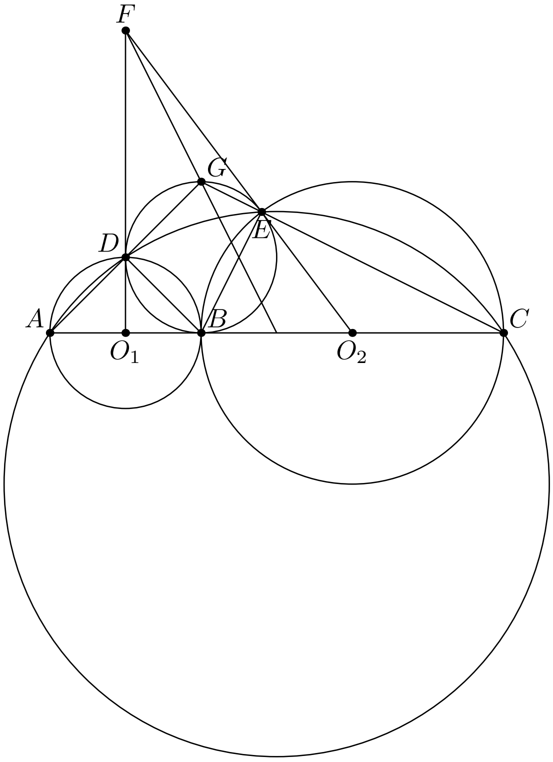


Fig. 18

19. (V.Yasinsky, Ukraine) (9–11) A quadrilateral $ABCD$ is inscribed into a circle with center O. Points P and Q are opposite to C and D respectively. Two tangents drawn to that circle at these points meet the line AB in points E and F (A is between E and B, B is between A and F). Line EO meets AC and BC in points X and Y respectively, and line FO meets AD and BD in points U and V. Prove that $XV = YU$.

Solution. It is sufficient to prove that $XO = OY$. Indeed, we then similarly have $UO = OV$ and so $XUYV$ is a parallelogram.

Let EO meet the circle in points P and Q (Fig. 19). The sought equality is equivalent to $(PX; OY) = (QY; OX)$. Projecting line EO to the circle from point C we obtain an equivalent equality $(PA; C'B) = (QB; C'A)$. It is correct because PQ, AB and the tangent in C' concur.

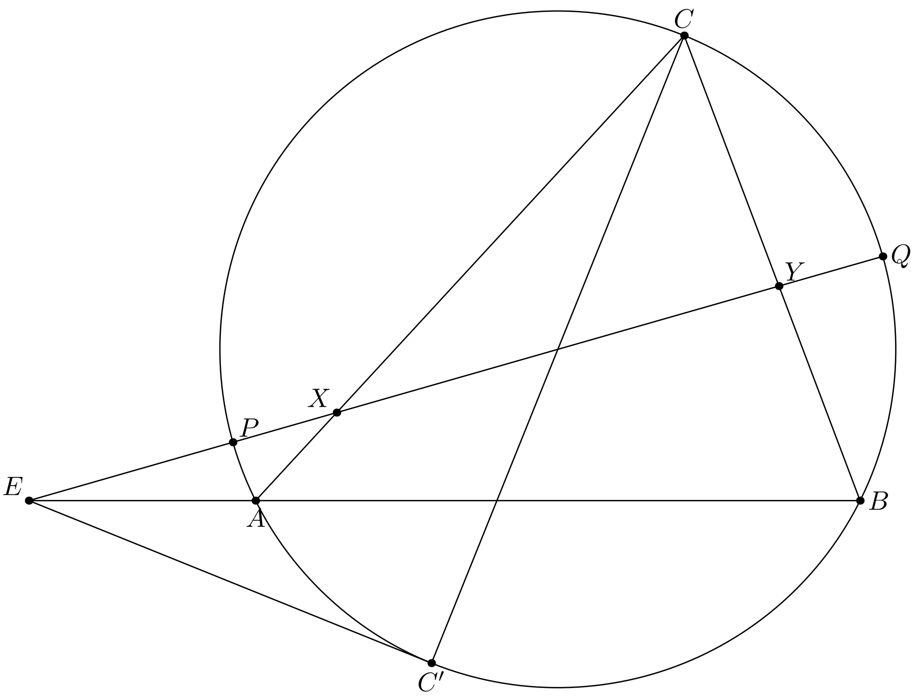


Fig. 19

20. (F.Ivlev) (10) The incircle of an acute-angled triangle ABC touches AB, BC, CA at points C_1, A_1, B_1 respectively. Points A_2, B_2 are the midpoints of segments B_1C_1, A_1C_1 respectively. Let P be a common point of the incircle and the line CO, where O is the circumcenter of ABC. Let also A' and B' be the second common points of PA_2 and PB_2 with the incircle. Prove that a common point of AA' and BB' lies on the altitude of the triangle dropped from the vertex C.

Solution. It is sufficient to prove that $\angle CAP = \angle A'AB$. Indeed, from this we obtain that line AA' is the reflection of AP in the bisector of angle A. Similarly line BB' is the reflection of BP in the bisector of angle B, and so the common point of these two lines lies on the reflection of line CP in the bisector of angle C, i.e. on the altitude.

Let Q be the common point of line AP with the incircle and S be the midpoint of arc B_1C_1 (fig. 20). Consider the composition f of the projections of incircle to itself from A and A_2. We have $f(B_1) = C_1$, $f(C_1) = B_1$, $f(Q) = A'$ and $f(S) = S$. Thus $(B_1Q; SC_1) = (C_1A'; SB_1)$, i.e. A' is the reflection of Q in AA_2, which proves the sought equality.

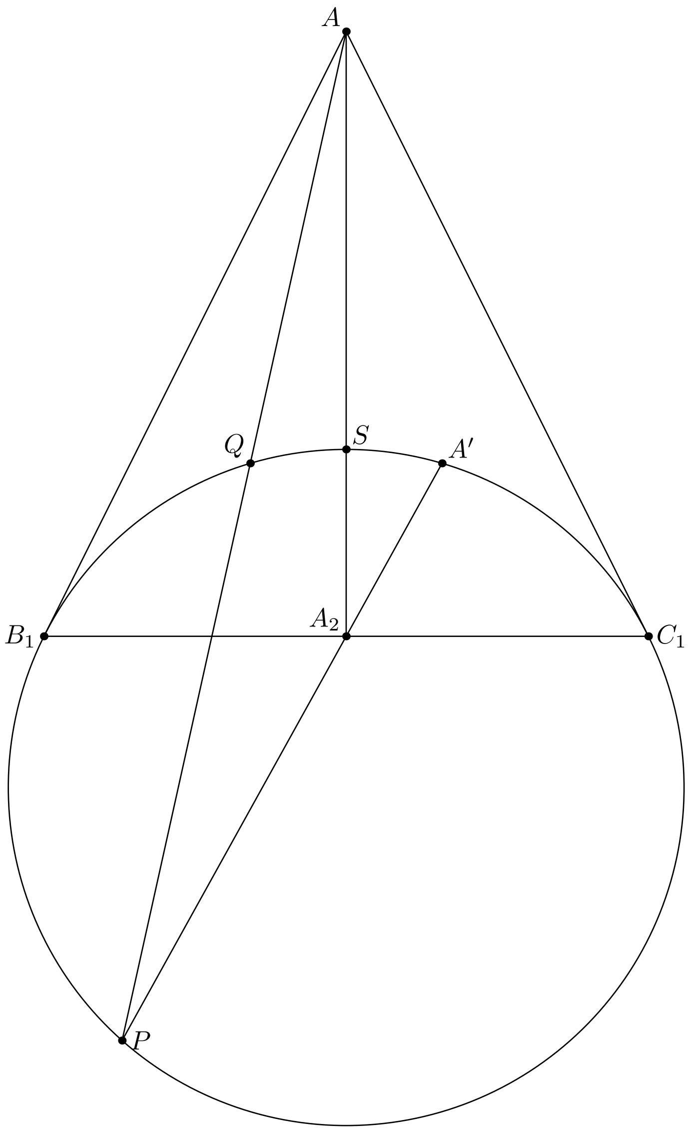


Fig. 20

21. (A.Akopjan) (10–11) A given convex quadrilateral $ABCD$ is such that $\angle ABD + \angle ACD > \angle BAC + \angle BDC$. Prove that $S_{ABD} + S_{ACD} > S_{BAC} + S_{BDC}$.

    Solution. If $AB \parallel CD$ then $\angle ABD = \angle BDC$ and $\angle ACD = \angle BAC$. Thus the given equality is equivalent to the fact that rays AB and DC intersect, i.e. the distance from C to line AB is less than the distance from D, and the distance from B to line CD is less than the distance from A. So $S_{ABD} > S_{ABC}$ and $S_{ACD} > S_{BCD}$.

22. (A.Zaslavsky) (10–11) A circle centered at a point F and a parabola with focus F have two common points. Prove that there exist four points A, B, C, D on the circle such that lines AB, BC, CD and DA touch the parabola.

    Solution. Take an arbitrary point A lying on the circle and outside the parabola. Line AF and the line passing through A and parallel to the axis of parabola intersect the circle in points symmetric wrt the axis. Thus the tangents from A to the parabola also intersect the circle in symmetric points B and D. Similarly the second tangents from B and D intersect the circle in point C symmetric to A. Thus A, B, C, D are the sought points.

23. (N.Beluhov, Bulgaria) (10–11) A cyclic hexagon $ABCDEF$ is such that $AB \cdot CF = 2BC \cdot FA$, $CD \cdot EB = 2DE \cdot BC$, and $EF \cdot AD = 2FA \cdot DE$. Prove that lines AD, BE and CF concur.

    Solution. Let P be the common point of FC and AD and G be the second common point of BP with the circumcircle of the hexagon. Then $2 = (AC; BF) = (DF; GB) = (DF; EB)$, and so points G and F coincide.

24. (A.Akopjan) (10–11) Given a line l in the space and a point A not lying on l. For an arbitrary line l' passing through A, XY (Y is on l') is the common perpendicular to the lines l and l'. Find the locus of points Y.

    Solution. Let the plane passing through A and perpendicular to l intersect l in point B. Let C be the projection of Y to this plane. Hence $BC \parallel XY$, thus $BC \perp AY$ and by three perpendiculars theorem $BC \perp AC$. So C lies on the circle with diameter AB, and Y lies on the cylinder constructed over this circle. It is clear that all points of the cylinder are on the sought locus.

25. (N.Beluhov, Bulgaria) (11) It is known for two different regular icosahedrons that some six of their vertices are vertices of a regular octahedron. Find the ratio of the edges of these icosahedrons.

    Solution. Note that no icosahedron can contain four vertices of octahedron. Indeed, between four vertices of octahedron there exist two opposite vertices that together with each of remaining vertices form an isosceles right-angled triangle . But no three vertices of icosahedron form such triangle.

    Thus one of given icosahedrons contains three vertices lying on one face of octahedron, and the other icosahedron contains three vertices lying on the opposite face. Now note that there exist only three different distances between the vertices of icosahedron: one is equal to the edge, the second is equal to the diagonal of a regular pentagon with the side equal to the edge, and the third is the distance between two opposite vertices. If three vertices form a regular triangle then the distance between them is one of two first types. Since two icosahedrons aren't equal, then some face of the octahedron is a face for one of them and a triangle formed by diagonals for the other. Thus the ratio of edges is the ratio of the diagonal and the side of a regular pentagon, i.e. $\dfrac{\sqrt{5}+1}{2}$.

VII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
THE CORRESPONDENCE ROUND

Below is the list of problems for the first (correspondence) round of the VII Sharygin Geometrical Olympiad.

The olympiad is intended for high-school students of 8–11 grades (these are four elder grades in Russian school). In the list below each problem is indicated by the numbers of school grades, for which it is intended. However, the participants may solve problems for elder grades as well (solutions for younger grades will not be considered).

Your work containing the solutions for the problems (in Russian or in English) should be sent not later than April 1, 2011, by e-mail to geomolymp@mccme.ru in pdf, doc or jpg files. Please, follow several simple rules:

1. Each student sends his work in a separate message (with delivery notification). The size of the message must not exceed 10 Mb.

2. If your work consists of several files, send it as an archive. If the size of your work exceeds 10 Mb cut it to several archives and send each of them by a separate message.

3. In the subject of the message write "The work for Sharygin olympiad", and present the following personal information in the body of your message:

- last name, first name;

- E-mail, post address, phone number;

- the current number of your grade at school;

- the number and the mail address of your school;

- full names of your teachers in mathematics at school and/or of instructors of your extra math classes (if you attend additional math classes after school).

If you have no e-mail access, please, send your work by regular mail to the following address: *Russia, 119002, Moscow, Bolshoy Vlasyevsky per., 11. Olympiad in honour of Sharygin.* In the title page write your personal information indicated in the item 3 above.

In your work you should start writing the solution to each problem in a new page. First write down the statement of the problem, and then the solution. Present your solutions in detail, including all significant arguments and calculations. Provide all necessary figures. Solutions of computational problems have to be completed with a distinctly presented answer. Please, be accurate to provide good understanding and correct estimating of your work !

If your solution depends on some well-known theorems from standard textbooks, you may simply refer to them instead of providing their proofs. However, any fact not from the standard curriculum should be either proved or properly referred (with an indication of the source).

You may (this isn't necessary) note the problems which you liked. Your opinion is interesting for the Jury.

Your work will be examined thoroughly, and your marks will be sent to you by the end of May 2011. Winners of the correspondence round will be invited to take part in the final round in Summer 2011 in Dubna town (near Moscow).

1. (8) Does a convex heptagon exist which can be divided into 2011 equal triangles?

2. (8) Let ABC be a triangle with sides $AB = 4$, $AC = 6$. Point H is the projection of vertex B to the bisector of angle A. Find MH, where M is the midpoint of BC.

3. (8) Let ABC be a triangle with $\angle A = 60°$. The midperpendicular of segment AB meets line AC at point C_1. The midperpendicular of segment AC meets line AB at point B_1. Prove that line B_1C_1 touches the incircle of triangle ABC.

4. (8) Segments AA', BB', CC' are the bisectrices of triangle ABC. It is known that these lines are also the bisectrices of triangle $A'B'C'$. Is it true that triangle ABC is regular?

5. (8) Given triangle ABC. The midperpendicular of side AB meets one of the remaining sides at point C'. Points A' and B' are defined similarly. For which original triangles triangle $A'B'C'$ is regular?

6. (8) Two unit circles ω_1 and ω_2 intersect at points A and B. M is an arbitrary point of ω_1, N is an arbitrary point of ω_2. Two unit circles ω_3 and ω_4 pass through both points M and N. Let C be the second common point of ω_1 and ω_3, and D be the second common point of ω_2 and ω_4. Prove that $ACBD$ is a parallelogram.

7. (8–9) Points P and Q on sides AB and AC of triangle ABC are such that $PB = QC$. Prove that $PQ < BC$.

8. (8–9) The incircle of right-angled triangle ABC ($\angle B = 90°$) touches AB, BC, CA at points C_1, A_1, B_1 respectively. Points A_2, C_2 are the reflections of B_1 in lines BC, AB respectively. Prove that lines $A_1A_2 and C_1C_2$ meet on the median of triangle ABC.

9. (8–9) Let H be the orthocenter of triangle ABC. The tangents to the circumcircles of triangles CHB and AHB at point H meet AC at points A_1 and C_1 respectively. Prove that $A_1H = C_1H$.

10. (8–9) The diagonals of trapezoid $ABCD$ meet at point O. Point M of lateral side CD and points P, Q of bases BC and AD are such that segments MP and MQ are parallel to the diagonals of the trapezoid. Prove that line PQ passes through point O.

11. (8–10) The excircle of right-angled triangle ABC ($\angle B = 90°$) touches side BC at point A_1 and touches line AC in point A_2. Line A_1A_2 meets the incircle of ABC for the first time at point A'; point C' is defined similarly. Prove that $AC || A'C'$.

12. (8–10) Let AP and BQ be the altitudes of acute-angled triangle ABC. Using a compass and a ruler, construct a point M on side AB such that $\angle AQM = \angle BPM$.

13. a) (8–10) Find the locus of centroids for triangles whose vertices lie on the sides of a given triangle (each side contains a single vertex).

    b) (11) Find the locus of centroids for tetrahedrons whose vertices lie on the faces of a given tetrahedron (each face contains a single vertex).

14. (9) In triangle ABC, the altitude and the median from vertex A form (together with line BC) a triangle such that the bisectrix of angle A is the median; the altitude and the median from vertex B form (together with line AC) a triangle such that the bisectrix of angle B is the bisectrix. Find the ratio of sides for triangle ABC.

15. (9–10) Given a circle with center O and radius equal to 1. AB and AC are the tangents to this circle from point A. Point M on the circle is such that the areas of quadrilaterals $OBMC$ and $ABMC$ are equal. Find MA.

16. (9–10) Given are triangle ABC and line l. The reflections of l in AB and AC meet at point A_1. Points B_1, C_1 are defined similarly. Prove that

    a) lines AA_1, BB_1, CC_1 concur;

    b) their common point lies on the circumcircle of ABC;

    c) two points constructed in this way for two perpendicular lines are opposite.

17. (9–11) a) Does there exist a triangle in which the shortest median is longer that the longest bisectrix?

    b) Does there exist a triangle in which the shortest bisectrix is longer that the longest altitude?

18. (9–11) On the plane, given are n lines in general position, i.e. any two of them aren't parallel and any three of them don't concur. These lines divide the plane into several parts. What is

    a) the minimal;

    b) the maximal

    number of these parts that can be angles?

19. (9–11) Does there exist a nonisosceles triangle such that the altitude from one vertex, the bisectrix from the second one and the median from the third one are equal?

20. (9–11) Quadrilateral $ABCD$ is circumscribed around a circle with center I. Points M and N are the midpoints of diagonals AC and BD. Prove that $ABCD$ is cyclic quadrilateral if and only if $IM : AC = IN : BD$.

21. (10–11) On a circle with diameter AC, let B be an arbitrary point distinct from A and C. Points M, N are the midpoints of chords AB, BC, and points P, Q are the midpoints of smaller arcs restricted by these chords. Lines AQ and BC meet at point K, and lines CP and AB meet at point L. Prove that lines MQ, NP and KL concur.

22. (10–11) Let CX, CY be the tangents from vertex C of triangle ABC to the circle passing through the midpoints of its sides. Prove that lines XY, AB and the tangent to the circumcircle of ABC at point C concur.

23. (10–11) Given are triangle ABC and line l intersecting BC, CA and AB at points A_1, B_1 and C_1 respectively. Point A' is the midpoint of the segment between the projections of A_1 to AB and AC. Points B' and C' are defined similarly.

    (a) Prove that A', B' and C' lie on some line l'.

    (b) Suppose l passes through the circumcenter of $\triangle ABC$. Prove that in this case l' passes through the center of its nine-points circle.

24. (10–11) Given is an acute-angled triangle ABC. On sides BC, CA, AB, find points A', B', C' such that the longest side of triangle $A'B'C'$ is minimal.

25. (10–11) Three equal regular tetrahedrons have the common center. Is it possible that all faces of the polyhedron that forms their intersection are equal?

VII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN THE CORRESPONDENCE ROUND. SOLUTIONS

1. (A.Zaslavsky) (8) Does a convex heptagon exist which can be divided into 2011 equal triangles?

   Solution. Yes, it does. For example, let T be a right-angled triangle such that its hypothenuse is equal to 1003 and one of its legs is equal to 1. Consider rectangles formed by two such triangles and compose a rectangle with side equal to 1003 from such rectangles. To one of these sides, attach an isosceles triangle composed from two triangles equal to T, and to the second side attach a quadrilateral composed from three such triangles (fig.1).

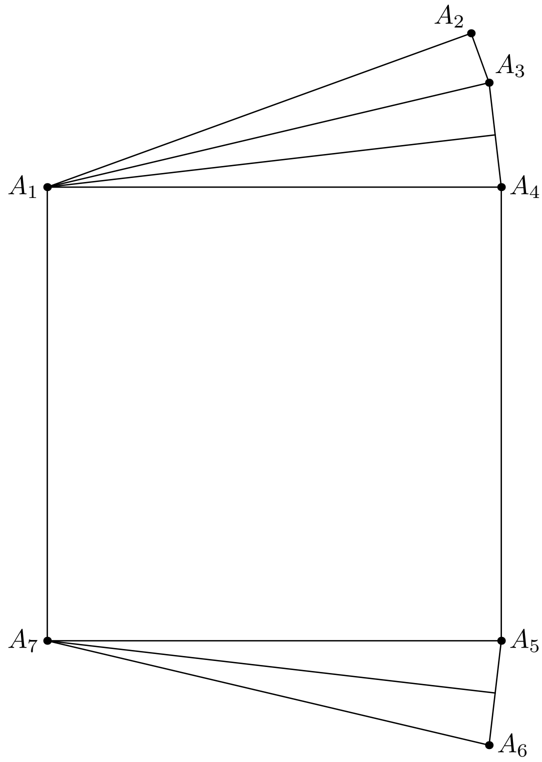


Fig. 1

   Second solution. Take a square with side equal to 34 and from three its angles cut isosceles right-angled triangles with legs equal to 3, 6 and 16. The obtained heptagon can be divided into isosceles right-angled triangles with legs equal to 1, and the number of these triangles is equal to $2 \cdot 34^2 - 9 - 36 - 256 = 2011$.

   Also we can take a rectangle $m \times n$ and cut from it isosceles right-angled triangles with legs x, y, z such that $2mn - x^2 - y^2 - z^2 = 2011$. The participants of olympiad found several such solutions.

   Third solution. (M.Amanzholov, Kazahstan) Let T be an isosceles triangle with the base equal to 1 and with the angle at the opposite vertex equal to $120°$. Then the regular triangle with side equal to 1 can be divided into three such triangles. On the other hand, from 335 regular triangles we can compose an isosceles trapezoid with bases equal to 168 and 167 and lateral sides equal to 1. Two such trapezoids with common greatest base form a convex hexagon which can be divided into 2010 triangles equal to T. Joining such triangle to its smallest side, we construct the sought heptagon.

2. (From Singapore olympiads) (8) Let ABC be a triangle with sides $AB = 4$, $AC = 6$. Point H

is the projection of vertex B to the bisector of angle A. Find MH, where M is the midpoint of BC.

Solution. Let D be the common point of BH and AC (fig.2). Then AH is the bisector and the altitude of triangle ABD. Thus this triangle is isosceles, i.e. $AD = BD$ and $BH = HD$. So MH ic the medial line of triangle BCD and $MH = CD/2 = (AC - AB)/2 = 1$.

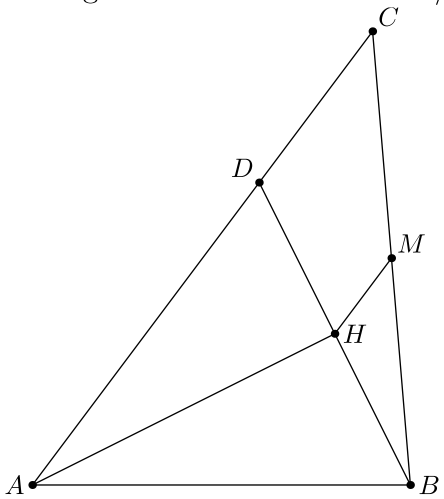


Fig. 2

3. (D.Shvetsov) (8) Let ABC be a triangle with $\angle A = 60°$. The midperpendicular of segment AB meets line AC at point C_1. The midperpendicular of segment AC meets line AB at point B_1. Prove that line B_1C_1 touches the incircle of triangle ABC.

   Solution. Let B_0, C_0 be the midpoints of AC, AB respectively. Since triangles AB_0B_1, AC_0C_1 are right-angled with $\angle A = 60°$, we have $AB_1 = 2AB_0 = AC$ and $AC_1 = 2AC_0 = AB$. Thus line B_1C_1 is the reflection of BC in the bisector of angle A. Since this bisector passes through the incenter and BC touches the incircle, B_1C_1 also touches the incircle.

4. (B.Frenkin) (8) Segments AA', BB', CC' are the bisectrices of triangle ABC. It is known that these lines are also the bisectrices of triangle $A'B'C'$. Is it true that triangle ABC is regular?

   Solution. Yes, the diagonal CC' of quadrilateral $A'C'B'C$ is the bisector of its angles C and C', thus it is the symmetry axis. Then $A'C = B'C$, $A'C' = B'C'$, $\angle CB'A' = \angle CA'B'$ and $\angle AB'C' = \angle BA'C'$. Similarly $\angle BC'A' = \angle BA'C' = \angle AB'C' = \angle AC'B'$. Thus triangles $AB'C'$ and $BA'C'$ are equal, i.e. $AB' = BA'$ and $AC = BC$. Similarly we obtain that $AB = BC$.

5. (B.Frenkin) (8) Given triangle ABC. The midperpendicular of side AB meets one of the remaining sides at point C'. Points A' and B' are defined similarly. For which original triangles triangle $A'B'C'$ is regular?

   Answer. For regular triangles and triangles with angles equal to 30, 30 and 120 grades.

   Solution. Consider a non-regular triangle ABC. Let AB be its greatest side. Then points A', B' lie on segment AB. From the condition of the problem we obtain that $C'C_0$, where C_0 is the midpoint of AB, is the bisector of segment $A'B'$, thus $CA' = A'B = AB' = CB'$, i.e. C' coincides with C and triangle ABC is isosceles. Also we have $2\angle A = \angle A + \angle CAB' = \angle CB'B = 60°$, so $\angle A = \angle B = 30°$.

6. (A.Akopjan) (8) Two unit circles ω_1 and ω_2 intersect at points A and B. M is an arbitrary point of ω_1, N is an arbitrary point of ω_2. Two unit circles ω_3 and ω_4 pass through both points M and N. Let C be the second common point of ω_1 and ω_3, and D be the second common point of ω_2 and ω_4. Prove that $ACBD$ is a parallelogram.

Solution. Let O_i be the center of circle ω_i. From the condition of the problem we obtain that O_1AO_2B, O_1CO_3M, O_3MO_4N, O_4NO_2D are rhombuses with sides equal to 1. Then $\overrightarrow{O_1C} = \overrightarrow{MO_3} = \overrightarrow{O_4N} = \overrightarrow{DO_2}$ and $\overrightarrow{O_1A} = \overrightarrow{BO_2}$. Thus $\overrightarrow{AC} = \overrightarrow{DB}$, q.e.d.

7. (A.Akopjan) (8–9) Points P and Q on sides AB and AC of triangle ABC are such that $PB = QC$. Prove that $PQ < BC$.

Solution. Let T be the fourth vertex of parallelogram $CBPT$. Then $PT = BC$ and $CT = BP = CQ$ (fig.7). Hence $\angle PQT > \angle TQC = \angle QTC > \angle QTP$, i.e. $PT > PQ$.

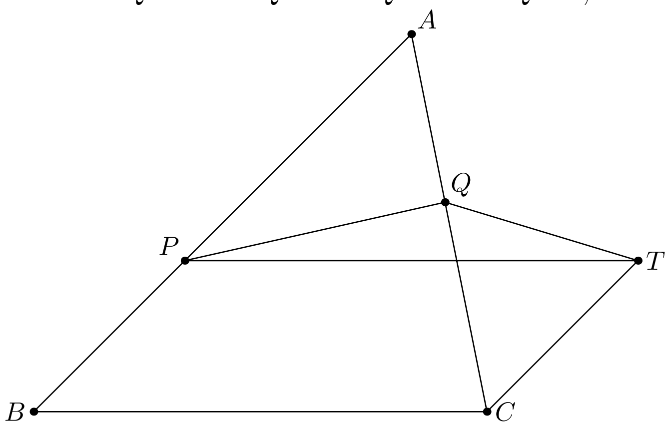


Рис. 7

8. (D.Shvetsov) (8–9) The incircle of right-angled triangle ABC ($\angle B = 90°$) touches AB, BC, CA at points C_1, A_1, B_1 respectively. Points A_2, C_2 are the reflections of B_1 in lines BC, AB respectively. Prove that lines A_1A_2 and C_1C_2 meet on the median of triangle ABC.

Solution. Let I be the incenter and P be the common point of line A_1A_2 with median BB_0 (fig.8). Since $\angle IA_1P = \angle IA_1B_1 = \angle C/2 = \angle PBA_1/2$, then $\angle BA_1P = \angle BPA_1$, i.e. $BP = BA_1$. Since $BA_1 = BC_1$, line C_1C_2 also passes through P.

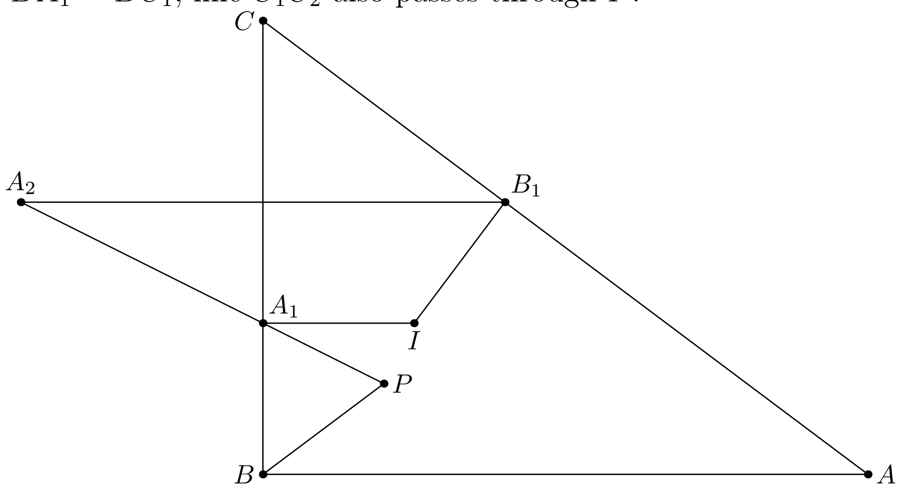


Fig. 8

9. (D.Shvetsov) (8–9) Let H be the orthocenter of triangle ABC. The tangents to the circumcircles

of triangles CHB and AHB at point H meet AC at points A_1 and C_1 respectively. Prove that $A_1H = C_1H$.

Solution. We have from the condition of the problem that $\angle AHC_1 = \angle ABH$. Thus $\angle C_1HB' = \angle AHB' - \angle ABH = \angle HAB = \pi/2 - \angle ABC$ (B' is the base of the altitude). Similarly $\angle B'HA_1 = \pi/2 - \angle ABC$, i.e triangle A_1HC_1 is isosceles.

10. (M.Volchkevich) (8–9) The diagonals of trapezoid $ABCD$ meet at point O. Point M of lateral side CD and points P, Q of bases BC and AD are such that segments MP and MQ are parallel to the diagonals of the trapezoid. Prove that line PQ passes through point O.

    Solution. By Thales theorem $AQ/QD = AM/MB = CP/PB$. Thus $AQ/PC = AD/BC = AO/CO$. Then triangles AOQ and COP are similar and $\angle AOQ = \angle COP$.

11. (D.Shvetsov) (8–10) The excircle of right-angled triangle ABC ($\angle B = 90°$) touches side BC at point A_1 and touches line AC at point A_2. Line A_1A_2 meets the incircle of ABC for the first time at point A'; point C' is defined similarly. Prove that $AC||A'C'$.

    Solution. Let I be the incenter and PQ be the diameter of the incircle parallel to AC (fig.11). Since $\angle PIC = \angle ACI = \angle BCI$ and $CA_1 = (AB + BC - AC)/2 = r = IP$, quadrilateral IPA_1C is an isosceles trapezoid. Then line A_1P is parallel to IC, i.e it coincides with A_1A_2. So P coincides with A', and similarly Q coincides with C'.

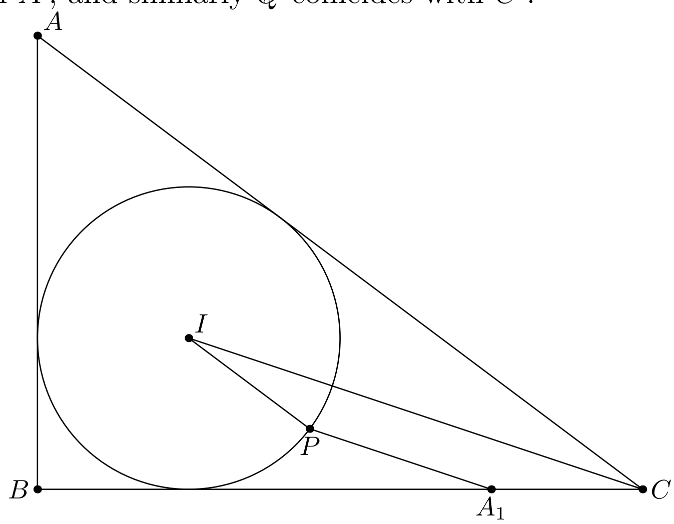


Fig.11

12. (V.Yasinsky) (8–10) Let AP and BQ be altitudes of acute-angled triangle ABC. Using a compass and a ruler, construct a point M on side AB such that $\angle AQM = \angle BPM$.

    Solution. Since points P, Q lie on the circle with diameter AB, $\angle BPQ = 180° - \angle A$. Then $\angle MPQ = \angle BPQ - \angle BPM = 180° - \angle A - \angle AQM = \angle AMQ$. Thus the circle passing through points P, Q, M touches AB (fig.12).

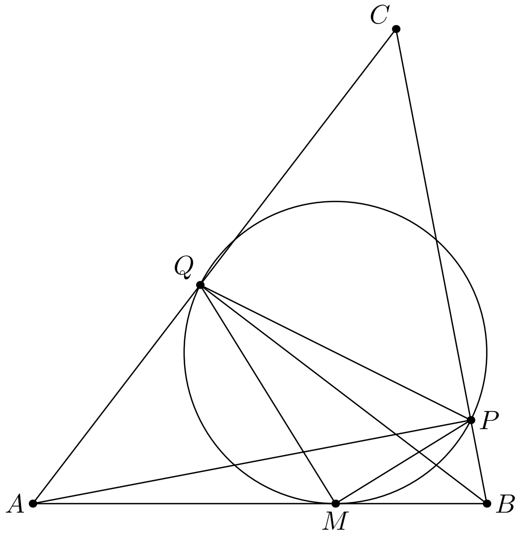


Fig.12

13. (B.Frenkin) a) (8–10) Find the locus of centroids for triangles whose vertices lie on the sides of a given triangle (each side contains a single vertex).

    b) (11) Find the locus of centroids for tetrahedrons whose vertices lie on the faces of a given tetrahedron (each face contains a single vertex).

    Solution. a) Let points A', B', C' lie on sides BC, CA, AB of triangle ABC. Since the midpoint C_0 of segment $A'B'$ lies inside triangle ABC, the distance from C_0 to AB is shorter than the respective altitude of ABC. Since the centroid M of $A'B'C'$ divides segment $C'C_0$ as $2:1$, the distance from M to AB is less than $2/3$ of this altitude. Similarly the distances from M to two remaining sidelines are less than $2/3$ of respective altitudes, i.e. M lies inside a hexagon formed by the sides of triangle and its reflections in the centroid of ABC. When two vertices of $A'B'C'$ tend to the same vertex of ABC, M tends to the boundary of this hexagon. Thus all its inner points belong to the sought locus.

    b) Similarly to a) we obtain that the sought locus is the polyhedron bounded by the faces of the given tetrahedron and by the planes parallel to the faces and dividing the respective altitudes in ratio $1:3$ starting from the vertex. Four of eight faces of this polyhedron are triangles, and four remaining faces are hexagons.

14. (B.Frenkin) (9) In triangle ABC, the altitude and the median from vertex A form (together with line BC) a triangle such that the bisectrix of angle A is the median; the altitude and the median from vertex B form (together with line AC) a triangle such that the bisectrix of angle B is the bisectrix. Find the ratio of sides for triangle ABC.

    Solution. Since the bisector of angle B bisects the angle between the altitude and the median, angle B is right. Thus the altitude from A coincides with AB, i.e. the bisector of angle A divides BC in ratio $1:3$. Then ratio $AB:AC$ also is equal to $1:3$ and by Pythagoras theorem $BC:AB=2\sqrt{2}$.

15. (V.Protasov) (9–10) Given a circle with center O and radius equal to 1. AB and AC are the tangents to this circle from point A. Point M on the circle is such that the areas of quadrilaterals $OBMC$ and $ABMC$ are equal. Find MA.

    Solution. Since $S_{OBMC}-S_{ABMC}=S_{ABC}-S_{OBC}+2S_{MBC}$, the locus of points for which $S_{OMBC}=S_{AMBC}$, is the bisector of segment OA. Thus $AM=OM=1$.

16. (P.Dolgirev) (9–10) Given are triangle ABC and line l. The reflections of l in AB and AC meet at point A_1. Points B_1, C_1 are defined similarly. Prove that

a) lines AA_1, BB_1, CC_1 concur;

b) their common point lies on the circumcircle of ABC;

c) two points constructed in this way for two perpendicular lines are opposite.

Solution. Firstly note that when l moves remaining parallel to itself with constant velocity, the reflections of l in AC and BC also move with constant velocities. Then C_1 moves along the line passing through C, i.e the common point of CC_1 with the circumcircle depends only on the direction of l. Now let A', B' be the common points of l with BC and AC (fig.16). Then $\angle C_1B'C = \angle CB'A'$, $\angle C'AC = \angle BA'C_1$. Thus C is the incenter or the excenter of triangle $A'B'C_1$, i.e. C_1C bisects angle $A'C_1B'$ or the adjacent angle. But the angle between lines $A'C_1$ and $B'C_1$ doesn't depend on l, thus the angle between CC_1 and C_1A' also doesn't depend on l. So when l rotates, lines AA_1, BB_1, CC_1 rotate with the same velocity. This yields all assertions of the problem.

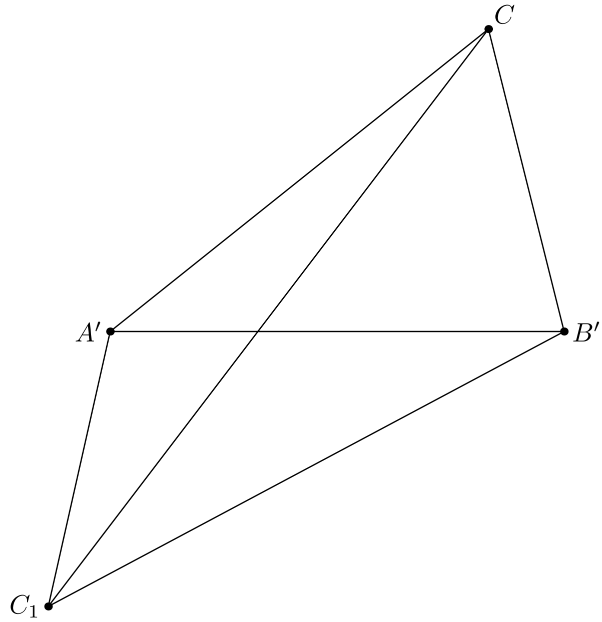


Fig.16

17. (B.Frenkin) (9–11) a) Does there exist a triangle in which the shortest median is longer that the longest bisectrix?

b) Does there exist a triangle in which the shortest bisectrix is longer that the longest altitude?

Solution. a) No, it doesn't. Let the lengths of sides BC, AC, AB be equal to a, b, c respectively and $a \le b \le c$. Now let CM be the median, AL be the bisector. If angle C isn't acute then $AL > AC$. Since $BC \le AC$, $\angle CMA$ isn' acute, thus $CM \le AC$ and $CM < AL$.

Now let $\angle C$ be acute. Since AB is the greatest side, $\angle C \ge 60^\circ$ and angles A, B are acute. Then the base H of altitude AH lies on segment BC. Thus AH (and AL) isn't less than $AC \cos\ 60^\circ = b\sqrt{3}/2$. But the square of CM is equal to $\frac{2a^2+2b^2-c^2}{4} \le \frac{2a^2+b^2}{4} \le \frac{3b^2}{4}$. So CM isn't greater than $b\sqrt{3}/2$ and can't exceed the bisector of angle A.

b) No, it doesn't. Let $a \le b \le c$ and l be the bisector of angle C. Then $(al+bl)\sin\frac{C}{2} = 2S_{ABC} = ab\sin C$, i.e. $l = \frac{2ab\cos\frac{C}{2}}{a+b}$. On the other hand, the altitude from A is equal to $h = b\sin C$. Since

$a + b \geq 2a$ and $C \geq 60°$, $h/l = (a+b)\sin\frac{C}{2}/a \geq 1$.

Note. It is easy to construct a triangle such that its shortest median is longer than its longest altitude.

18. (A.Zaslavsky) (9–11) On the plane, given are n lines in general position, i.e. any two of them aren't parallel and any three of them don't concur. These lines divide the plane into several parts. What is

    a) the minimal;

    b) the maximal

    number of these parts that can be angles?

    Solution. a) **Answer.** 3. Consider the convex envelope of all common points of given lines. Two lines passing through some vertex of this envelope divide the plane into four angles, and one of them contains all the remaining points. Thus the remaining lines don't intersect the vertical angle and the number of angles can't be less than three. The example with three angles can be constructed by induction: the line in turn must intersect all previous lines inside the triangle which is the convex envelope of common points.

    b) **Answer.** n if n is odd, $n-1$ if $n > 2$ is even. Construct a circle containing all common points. The given lines divide it into $2n$ arcs. Let AB, BC be two adjacent arcs, X, Y be the common points of the line passing through B, with the lines passing through A and C respectively. If X lies on segment BY then the part containing arc BC can't be an angle. Thus only one of two parts containing adjacent arcs can be an angle. Thus the number of angles isn't greater than n, and an equality is possible only when the part containing each second arc is an angle. But if n is even, this yields that there exist two angles containing the opposite arcs. Since these two angles are formed by the same lines, this isn't possible if $n > 2$. If n is odd then n parts formed by the sidelines of a regular n-gon are angles. It is evident that we can add one line without reduction of the number of angles.

    Second solution of a). (A.Goncharuk, Kharkov) Let polygon T be the union of all bounded parts. Then all angles are vertical to the angles of T, which are less than 180°. From the formula for the sum of angles we obtain that there exist three such angles. The polygon with three angles can be constructed in the following way. Take point D inside triangle ABC, inscribe a circle with sufficiently small radius in angle ADB and take $n-4$ points on the smaller arc formed by touching points. The tangents in these points and lines AC, BC, AD, BD form the sought polygon.

19. (A.Zaslavsky) (9–11) Does there exist a nonisosceles triangle such that the altitude from one vertex, the bisectrix from the second one and the median from the third one are equal?

    Solution. Yes, it does. Fix vertices A, B, construct point D which is the reflection of A in B, and consider an arbitrary point C such that $\angle BCD = 150°$. The altitude of triangle ABC from A is equal to the distance DH from D to BC, i.e. $CD/2$. The median BM from B as the medial line in triangle ACD also is equal to $CD/2$ (fig.19). Now move point C along arc BD, containing angle 150°. When C tends to B, the bisector from C tends to zero and the median from B tends to $AB/2$. When C tends to D, the median from B tends to zero and the bisector isn't less than BC. Thus there exists a point C for which the bisector is equal to two remaining segments.

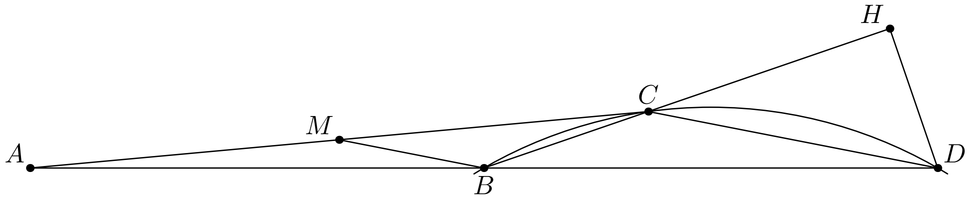


Fig.19

Note. When C moves from B to D, the bisector increases and the altitude decreases. Thus the angles of the sought triangle can be determined uniquely.

20. (N.Beluhov, A.Zaslavsky) (9–11) Quadrilateral $ABCD$ is circumscribed around a circle with center I. Points M and N are the midpoints of diagonals AC and BD. Prove that $ABCD$ is a cyclic quadrilateral if and only if $IM : AC = IN : BD$.

    Solution. Consider the case when $ABCD$ isn't a trapezoid. The other case needs no essential changes.

    By Newton theorem I lies on MN. Let $\lambda = MI : IN$. On the sides, construct points P, Q, R and S such that $AP : PB = CQ : QB = CR : RD = DS : SA = \lambda$. Let us prove that I is the midpoint of segments PR and QS.

    Use the masses method: place the unit masses in points A, C, and the masses equal to λ in points B, D. Replacing two first masses by the mass 2 in point M, and two remaining masses by the mass 2λ in point N, we obtain that I is the masscenter of four masses. On the other hand, we can replace the masses in A and B by the mass $1 + \lambda$ in point P and two remaining masses by the same mass in point R.

    Now since I is the midpoint of PR, and lines AB and CD are non-parallel tangents to the circle with center I, the angles formed by these lines with PR are equal i.e. PR is parallel to the bisector of some angle formed by these lines. Similarly QS is parallel to the bisector of some angle formed by AD and BC. Thus $ABCD$ is cyclic iff $PR \perp QS$. Since $PQRS$ is a parallelogram (with side parallel to AC and BD), this yields that $PQRS$ is a rhombus. But $PQ = QR \Leftrightarrow \frac{1}{1+\lambda}AC = \frac{\lambda}{1+\lambda}BD \Leftrightarrow \lambda = AC : BD$, q.e.d.

21. (V.Yasinsky) (10–11) On a circle with diameter AC, let B be an arbitrary point distinct from A and C. Points M, N are the midpoints of chords AB, BC, and points P, Q are the midpoints of smaller arcs related to these chords. Lines AQ and BC meet at point K, and lines CP and AB meet at point L. Prove that lines MQ, NP and KL concur.

    Solution. Lines PM and QN meet in the center O of the circle. Thus it suffices to apply the Desargues theorem to triangles PML and NQK.

22. (G.Feldman) (10–11) Let CX, CY be the tangents from vertex C of triangle ABC to the circle passing through the midpoints of its sides. Prove that lines XY, AB and the tangent at point C to the circumcircle of ABC concur.

    Solution. The homothety with center C and factor $1/2$ transforms line XY to the radical axis of point C and the circle passing through the midpoints A', B', C' of BC, CA, AB. On the other hand, the tangent at C to the circumcircle touches also the circle $A'B'C'$, i.e. it is the radical axis of this circle and point C. The common point of these radical axes lies on $A'B'$. Using the inverse homothety we obtain the assertion of the problem.

23. (N.Beluhov, M.Marinov) (10–11) Given are triangle ABC and line l intersecting BC, CA and AB at points A_1, B_1 and C_1 respectively. Point A' is the midpoint of the segment between the projections of A_1 to AB and AC. Points B' and C' are defined similarly.

(a) Prove that A', B' and C' lie on some line l'.

(b) Suppose l passes through the circumcenter of $\triangle ABC$. Prove that in this case l' passes through the center of its nine-points circle.

Solution. Let P_a, P_b, P_c be the midpoints of altitudes AH_a, BH_b, CH_c. It is evident that A', B', C' lie on the sidelines of triangle $P_aP_bP_c$ and divide its sides in the same ratios as points A_1, B_1, C_1 divide the sides of ABC. Thus a) immediately follows from Menelaos theorem. In addition, if l passes through some fixed point, then l' passes also through some fixed point, so it suffices to prove b) for two lines passing through the circumcenter O. Let us consider for example the lines passing through a vertex of ABC.

Let C_1 be the common point of CO and AB; X, Y be the projections of C_1 to AC and BC; A_0, B_0, C_0 be the midpoints of BC, CA, AB; U, V be the midpoints of XY and A_0B_0; Q be the common point of the bisector to segment A_0B_0 and of line UP_c (fig.23). Since $XY \parallel AB$, points C, V, U are collinear. Then $VQ/CP_c = UV/UC = C_1O/CC_1$, i.e. $VQ = OC_0/2$ and Q is the circumcenter of $A_0B_0C_0$.

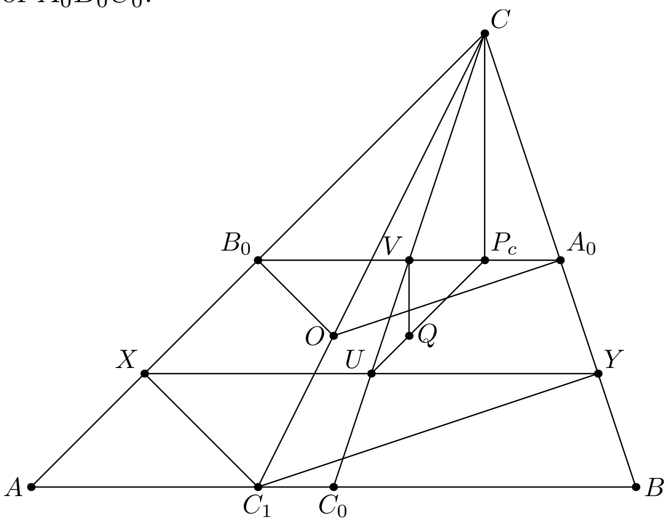


Fig.23

24. (A.Zaslavsky) (10–11) Given is an acute-angled triangle ABC. On sides BC, CA, AB, find points A', B', C' such that the longest side of triangle $A'B'C'$ is minimal.

Solution. Firstly let us prove that the sought triangle is pedal, i.e three perpendiculars from its vertices to respective sides of ABC concur. Indeed for an arbitrary triangle $A'B'C'$ circles $AB'C'$, $BC'A'$ and $CA'B'$ have some common point P. Let A'', B'', C'' be the projections of P to BC, CA, AB. Since $\angle A'PB' = \angle A''PB'' = \pi - \angle C$ etc., $\angle A''PA' = \angle B''PB' = \angle C''PC'$, and triangle $A''B''C''$ can be obtained from $A'B'C'$ by spiral similarity with center P and the factor less than 1.

Consider now point T with regular pedal triangle and prove that the pedal triangle of any point P has at least one greater side. Let A', B' be the projections of P to BC and AC. Then $A'B' = PC \sin C$, i.e. $A'B'$ isn't greater than the side of pedal triangle of T iff $PC \leq TC$. Similarly we obtain $PB \leq TB$, $PA \leq TA$. It is clear that these three inequalities can be correct only for point T.

Finally construct point T. From its definition we have $TA \cdot BC = TB \cdot AC = TC \cdot AB$. First equality defines the Apollonius circle passing through C and the bases of internal and external bisectors of angle C. Similarly the second equality defines the second circle. T is the common point of these circles lying inside triangle ABC.

25. (N.Beluhov) (10–11) Three equal regular tetrahedrons have the common center. Is it possible that all faces of the polyhedron that forms their intersection are equal?

Solution. Yes, it is possible. Let the first tetrahedron touch their common inscribed sphere at points A, B, C, D. Rotate these points around the common perpendicular (and bisector) of the segments AB and CD by $120°$ to obtain A', B', C', D' and by $240°$ to obtain A'', B'', C'', D'' (the twelve points form two regular hexagons). The tangential planes to the sphere in these twelve points form the three tetrahedrons needed. Indeed, for any two of these points there exists an isometry that maps this set of twelve points onto itself and maps one of these two points to another one. These isometries enable us to map any facet of the obtained polygon onto any other one.

VIII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
THE CORRESPONDENCE ROUND

Below is the list of problems for the first (correspondence) round of the VIII Sharygin Geometrical Olympiad.

The olympiad is intended for high-school students of 8–11 grades (these are four elder grades in Russian school). In the list below each problem is indicated by the numbers of school grades, for which it is intended. However, the participants may solve problems for elder grades as well (solutions for younger grades will not be considered).

Your work containing the solutions for the problems (in Russian or in English) should be sent not later than April 1, 2012, by e-mail to geomolymp@mccme.ru in pdf, doc or jpg files. Please, follow several simple rules:

1. Each student sends his work in a separate message (with delivery notification). The size of the message must not exceed 10 Mb.

2. If your work consists of several files, send it as an archive.

3. If the size of your message exceeds 10 Mb divide it into several messages.

4. In the subject of the message write "The work for Sharygin olympiad", and present the following personal information in the body of your message:

- last name;

- all other names;

- E-mail, post address, phone number;

- the current number of your grade at school;

- the number and/or the name and the mail address of your school;

- full names of your teachers in mathematics at school and/or of instructors of your extra math classes (if you attend additional math classes after school).

If you have no e-mail access, please, send your work by regular mail to the following address: *Russia, 119002, Moscow, Bolshoy Vlasyevsky per., 11. Olympiad in honour of Sharygin.* In the title page write your personal information indicated in the item 4 above.

In your work you should start writing the solution to each problem in a new page. First write down the statement of the problem, and then the solution. Present your solutions in detail, including all significant arguments and calculations. Provide all necessary figures. Solutions of computational problems have to be completed with a distinctly presented answer. Please, be accurate to provide good understanding and correct estimating of your work !

If your solution depends on some well-known theorems from standard textbooks, you may simply refer to them instead of providing their proofs. However, any fact not from the standard curriculum should be either proved or properly referred (with an indication of the source).

You may (this isn't necessary) note the problems which you liked. Your opinion is interesting for the Jury.

Winners of the correspondence round will be invited to take part in the final round in Summer 2012 in Dubna town (near Moscow). The list of the winners will be published on www.geometry.ru by the end of May 2012. If you want to know your detailed results, please use e-mail.

1. (8) In triangle ABC point M is the midpoint of side AB, and point D is the foot of altitude CD. Prove that $\angle A = 2\angle B$ if and only if $AC = 2MD$.

2. (8) A cyclic n-gon is divided by non-intersecting (inside the n-gon) diagonals to $n-2$ triangles. Each of these triangles is similar to at least one of the remaining ones.

   For what n this is possible?

3. (8) A circle with center I touches sides AB, BC, CA of triangle ABC in points C_1, A_1, B_1. Lines AI, CI, B_1I meet A_1C_1 in points X, Y, Z respectively. Prove that $\angle YB_1Z = \angle XB_1Z$

4. (8) Given triangle ABC. Point M is the midpoint of side BC, and point P is the projection of B to the perpendicular bisector of segment AC. Line PM meets AB in point Q. Prove that triangle QPB is isosceles.

5. (8) On side AC of triangle ABC an arbitrary point is selected D. The tangent in D to the circumcircle of triangle BDC meets AB in point C_1; point A_1 is defined similarly. Prove that $A_1C_1||AC$.

6. (8–9) Point C_1 of hypothenuse AC of a right-angled triangle ABC is such that $BC = CC_1$. Point C_2 on cathetus AB is such that $AC_2 = AC_1$; point A_2 is defined similarly. Find angle AMC, where M is the midpoint of A_2C_2.

7. (8–9) In a non-isosceles triangle ABC the bisectors of angles A and B are inversely proportional to the respective sidelengths. Find angle C.

8. (8–9) Let BM be the median of right-angled triangle ABC ($\angle B = 90°$). The incircle of triangle ABM touches sides AB, AM in points A_1, A_2; points C_1, C_2 are defined similarly. Prove that lines A_1A_2 and C_1C_2 meet on the bisector of angle ABC.

9. (8–9) In triangle ABC, given lines l_b and l_c containing the bisectors of angles B and C, and the foot L_1 of the bisector of angle A. Restore triangle ABC.

10. In a convex quadrilateral all sidelengths and all angles are pairwise different.

    a)(8–9) Can the greatest angle be adjacent to the greatest side and at the same time the smallest angle be adjacent to the smallest side?

    b)(9–11) Can the greatest angle be non-adjacent to the smallest side and at the same time the smallest angle be non-adjacent to the greatest side?

11. Given triangle ABC and point P. Points A', B', C' are the projections of P to BC, CA, AB. A line passing through P and parallel to AB meets the circumcircle of triangle $PA'B'$ for the second time in point C_1. Points A_1, B_1 are defined similarly. Prove that

    a) (8-10) lines AA_1, BB_1, CC_1 concur;

    b) (9–11) triangles ABC and $A_1B_1C_1$ are similar.

12. (9–10) Let O be the circumcenter of an acute-angled triangle ABC. A line passing through O and parallel to BC meets AB and AC in points P and Q respectively. The sum of distances from O to AB and AC is equal to OA. Prove that $PB + QC = PQ$.

13. (9–10) Points A, B are given. Find the locus of points C such that C, the midpoints of AC, BC and the centroid of triangle ABC are concyclic.

14. (9–10) In a convex quadrilateral $ABCD$ suppose $AC \cap BD = O$ and M is the midpoint of BC. Let $MO \cap AD = E$. Prove that $\frac{AE}{ED} = \frac{S_{\triangle ABO}}{S_{\triangle CDO}}$.

15. (9–11) Given triangle ABC. Consider lines l with the next property: the reflections of l in the sidelines of the triangle concur. Prove that all these lines have a common point.

16. (9–11) Given right-angled triangle ABC with hypothenuse AB. Let M be the midpoint of AB and O be the center of circumcircle ω of triangle CMB. Line AC meets ω for the second time in point K. Segment KO meets the circumcircle of triangle ABC in point L. Prove that segments AL and KM meet on the circumcircle of triangle ACM.

17. (9–11) A square $ABCD$ is inscribed into a circle. Point M lies on arc BC, AM meets BD in point P, DM meets AC in point Q. Prove that the area of quadrilateral $APQD$ is equal to the half of the area of the square.

18. (9–11) A triangle and two points inside it are marked. It is known that one of the triangle's angles is equal to 58°, one of two remaining angles is equal to 59°, one of two given points is the incenter of the triangle and the second one is its circumcenter. Using only the ruler without partitions determine where is each of the angles and where is each of the centers.

19. (10–11) Two circles with radii 1 meet in points X, Y, and the distance between these points also is equal to 1. Point C lies on the first circle, and lines CA, CB are tangents to the second one. These tangents meet the first circle for the second time in points B', A'. Lines AA' and BB' meet in point Z. Find angle XZY.

20. (10–11) Point D lies on side AB of triangle ABC. Let ω_1 and Ω_1, ω_2 and Ω_2 be the incirles and the excircles (touching segment AB) of triangles ACD and BCD. Prove that the common external tangents to ω_1 and ω_2, Ω_1 and Ω_2 meet on AB.

21. (10–11) Two perpendicular lines pass through the orthocenter of an acute-angled triangle. The sidelines of the triangle cut on each of these lines two segments: one lying inside the triangle and another one lying outside it. Prove that the product of two internal segments is equal to the product of two external segments.

22. (10–11) A circle ω with center I is inscribed into a segment of the disk, formed by an arc and a chord AB. Point M is the midpoint of this arc AB, and point N is the midpoint of the complementary arc. The tangents from N touch ω in points C and D. The opposite sidelines AC and BD of quadrilateral $ABCD$ meet in point X, and the diagonals of $ABCD$ meet in point Y. Prove that points X, Y, I and M are collinear.

23. (10–11) An arbitrary point is selected on each of twelve diagonals of the faces of a cube. The centroid of these twelve points is determined. Find the locus of all these centroids.

24. (10–11) Given are n ($n > 2$) points on the plane such that no three of them aren't collinear. In how many ways this set of points can be divided into two non-empty subsets with non-intersecting convex envelops?

VIII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN THE CORRESPONDENCE ROUND. SOLUTIONS

1. (M.Rozhkova) (8) In triangle ABC point M is the midpoint of side AB, and point D is the foot of altitude CD. Prove that $\angle A = 2\angle B$ if and only if $AC = 2MD$.

   Solution. Let K be the midpoint of AC (fig.1). Since DK is the median of a right-angled triangle ADC, we obtain that $AK = KD$ and $\angle ADK = \angle A$. On the other hand, MK is a medial line of ABC, therefore, $\angle DMK = \angle B$. Applying the external angle theorem to triangle DMK we obtain that the equalities $KD = DM$ and $\angle KDA = 2\angle KMD$ are equivalent.

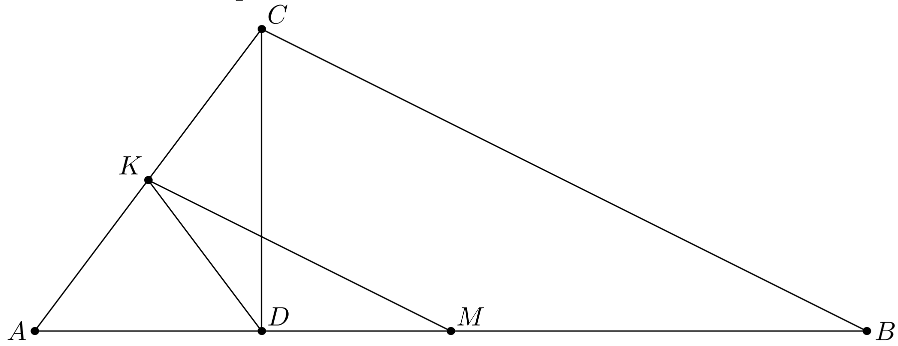


Fig.1

2. B.Frenkin) (8) A cyclic n-gon is divided by non-intersecting (inside the n-gon) diagonals to $n-2$ triangles. Each of these triangles is similar to at least one of the remaining ones.

   For what n this is possible?

   Answer. For $n = 4$ and for $n > 5$.

   Solution. It is clear that $n > 3$. Now if n is even then we can bisect a regular n-gon to two equal polygons by a diagonal passing through its center and divide these two polygons by the same way. Also we can construct on three sides of a regular $2k$-gon equal triangles with vertices on the circumcircle. Thus for odd $n > 5$ such situation is also possible. Prove that it isn't possible for $n = 5$.

   If the circumcenter of a pentagon doesn't lie on dividing diagonals then the triangle containing it is acute-angled and two remaining triangles are obtuse-angled, i.e. the condition of the problem can't be true. If the circumcenter lies on the diagonal then two triangles adjacent with this diagonal are right-angled and the third triangle is obtuse-angled. Thus the condition also isn't true.

3. (D.Shvetsov) (8) A circle with center I touches sides AB, BC, CA of triangle ABC in points C_1, A_1, B_1. Lines AI, CI, B_1I meet A_1C_1 in points X, Y, Z respectively. Prove that $\angle YB_1Z = \angle XB_1Z$

   Solution. Since $B_1I \perp AC$, it is sufficient to prove that $\angle YB_1A = \angle XB_1C$. Since CI is the medial bisector to A_1B_1, therefore $\angle YB_1A_1 = \angle C_1A_1B_1$, and since $\angle A_1B_1C = \angle B_1A_1C$, therefore $\angle YB_1A = \angle C_1A_1B$ (fig.3). Similarly $\angle XB_1C = \angle A_1C_1B = \angle C_1A_1B$.

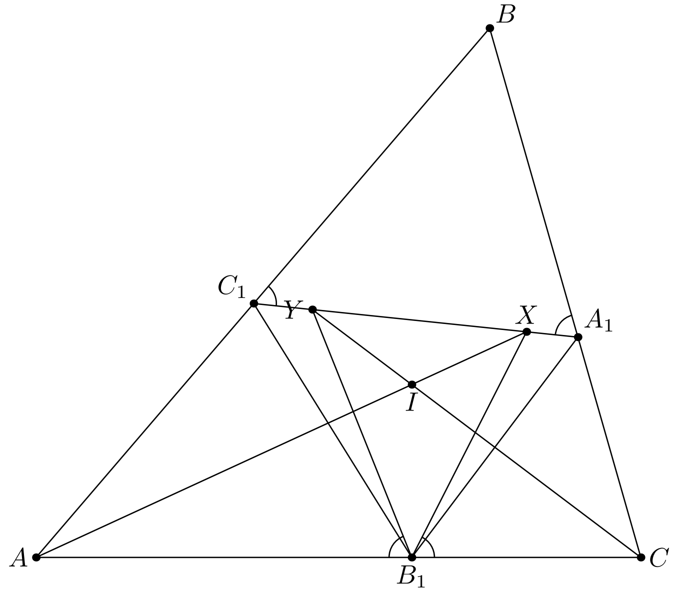


Fig.3

4. (A.Akopyan) (8) Given triangle ABC. Point M is the midpoint of side BC, and point P is the projection of B to the perpendicular bisector of segment AC. Line PM meets AB in point Q. Prove that triangle QPB is isosceles.

   Solution. Let D be the reflection of B in the medial bisector to AC, and T be the common point of AB and CD. Then $ACBD$ is an isosceles trapezoid, thus BDT is an isosceles triangle (fig.4). The line PM contains the medial line of this triangle, Therefore triangle QPB is also isosceles.

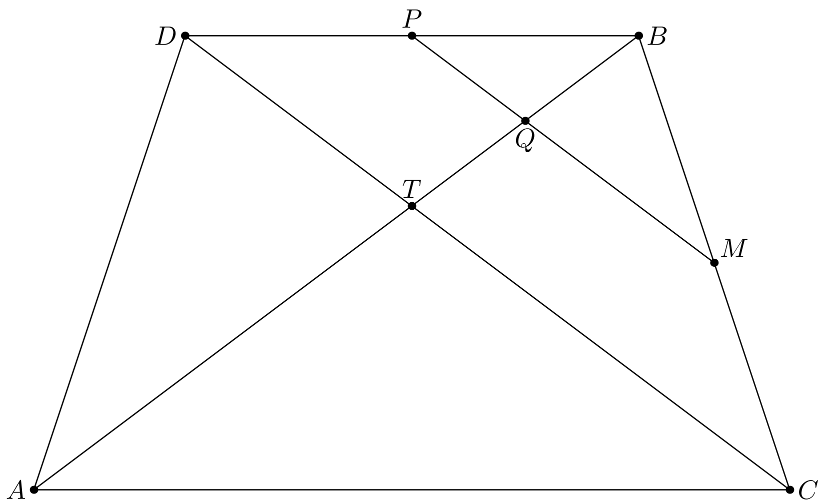


Fig.4

5. (D.Shvetsov) (8) On side AC of triangle ABC an arbitrary point is selected D. The tangent in D to the circumcircle of triangle BDC meets AB in point C_1; point A_1 is defined similarly. Prove that $A_1C_1||AC$.

   Solution. The condition yields that $\angle C_1DA = \angle DBC$ and $\angle A_1DC = \angle DBA$ (fig.5). Therefore A_1BC_1D is a cyclic quadrilateral, i.e. $\angle C_1A_1D = \angle C_1BD = \angle CDA_1$.

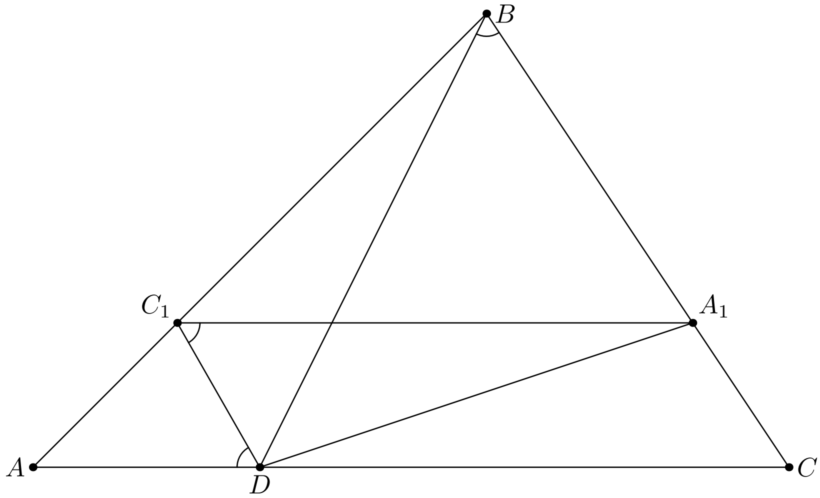


Fig.5

6. (D.Shvetsov) (8–9) Point C_1 of hypothenuse AC of a right-angled triangle ABC is such that $BC = CC_1$. Point C_2 on cathetus AB is such that $AC_2 = AC_1$; point A_2 is defined similarly. Find angle AMC, where M is the midpoint of A_2C_2.

   Answer. $135°$.

   Solution. Let I be the incenter of ABC. Since C_1 is the reflection of B in CI, and C_2 is the reflection of C_1 in AI, we obtain that $BI = IC_2$ and $\angle BIC_2 = 90°$. Similarly $BI = IA_2$ and $\angle BIA_2 = 90°$ (fig.6). Therefore, I is the midpoint of A_2C_2, and $\angle AIC = 135°$.

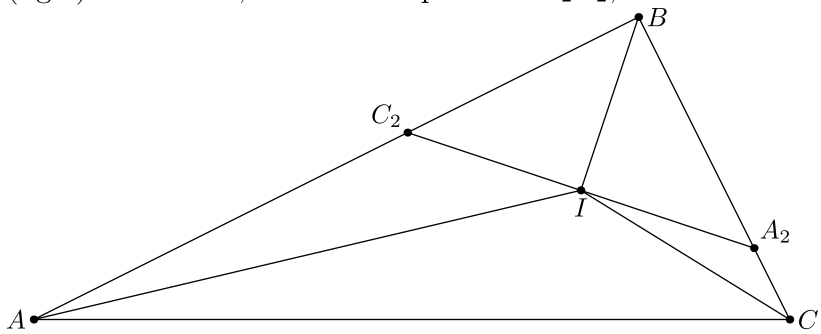


Fig.6

7. (B.Frenkin) (8–9) In a non-isosceles triangle ABC the bisectors of angles A and B are inversely proportional to the respective sidelengths. Find angle C.

   Answer. $60°$.

   Solution. Let AA_1, BB_1 be the bisectors of the given triangle, and AA_2, BB_2 be its altitudes. The condition yields that $AA_1/AA_2 = BB_1/BB_2$, therefore, $\angle A_1AA_2 = \angle B_1BB_2$. But $\angle A_1AA_2 = |\angle B - \angle C|$, $\angle B_1BB_2 = |\angle A - \angle C|$. Since the triangle isn't isosceles, an equality $\angle A - \angle C = \angle B - \angle C$ is impossible. Therefore, $\angle C = (\angle A + \angle B)/2 = 60°$.

8. (D.Shvecov) (8–9) Let BM be the median of right-angled triangle ABC ($\angle B = 90°$). The incircle of triangle ABM touches sides AB, AM in points A_1, A_2; points C_1, C_2 are defined similarly. Prove that lines A_1A_2 and C_1C_2 meet on the bisector of angle ABC.

   Solution. Since ABM, CBM are isosceles triangles, points A_1, C_1 are the midpoints of correspondent cathetus. Also the line A_1A_2 is perpendicular to the bisector of angle A, therefore it is the bisector of angle AA_1C_1 (fig.8). Similarly C_1C_2 is the bisector of angle CC_1A_1. Thus its common point is the excenter of triangle A_1BC_1 and lies on the bisector of angle B.

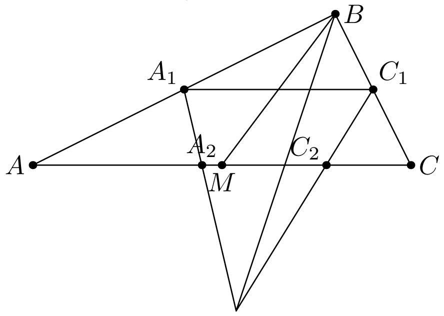


Fig.8

9. (A.Karluchenko) (8–9) In triangle ABC, given lines l_b and l_c containing the bisectors of angles B and C, and the foot L_1 of the bisector of angle A. Restore triangle ABC.

   Solution. Let I be the common point of l_b and l_c. Then IL_1 is the bisector of angle A. Thus we know the angles between the bisectors of the triangle and therefore we know the angles of the triangle. Construct an arbitrary triangle $A'B'C'$ with the same angles, find its incenter I', construct on the lines l_b, l_c the segments $IB'' = I'B'$, $IC'' = I'C'$ and pass the line through L_1 parallel to $B''C''$. This line meets l_b, l_c at the vertices B, C of the sought triangle. The construction of the vertex A is now evident.

10. (B.Frenkin, A.Zaslavsky) In a convex quadrilateral all sidelengths and all angles are pairwise different.

    a)(8–9) Can the greatest angle be adjacent to the greatest side and at the same time the smallest angle be adjacent to the smallest side?

    b)(9–11) Can the greatest angle be non-adjacent to the smallest side and at the same time the smallest angle be non-adjacent to the greatest side?

    Answer. a) Yes. b) No.

    Solution. a) Consider a triangle ABC with $AC > BC > AB$. Take on the segment AC a point P, such that $AP = BC$, construct the perpendicular from P to AC and take on this perpendicular a point D, lying outside the triangle and sufficiently near to P. Then AD is the greatest side of quadrilateral $ABCD$, CD is its smallest side, D is the greatest angle, and C is the smallest angle (fig.10).

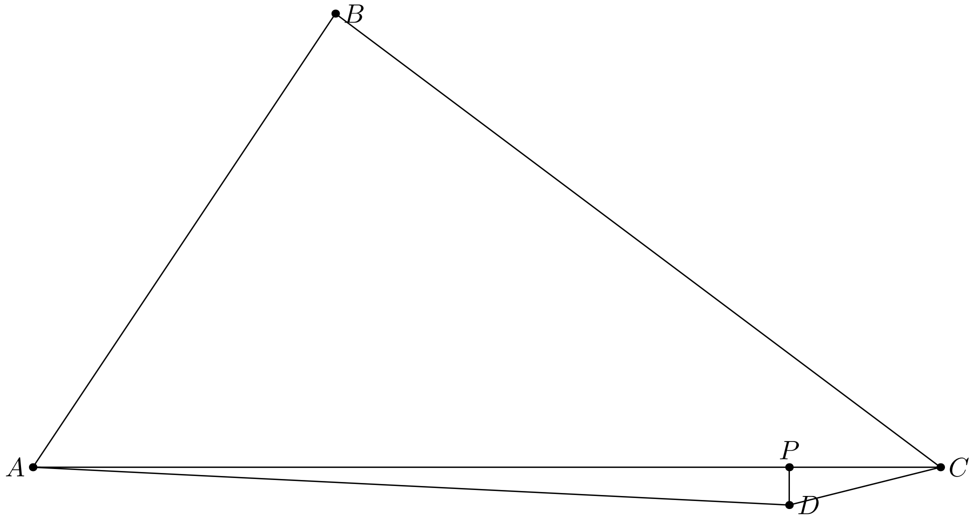


Fig.10

b) Suppose, that $ABCD$ is a quadrilateral satisfying to the condition. We can think that B is the greatest angle, and CD is the smallest side. Then the equality $AC^2 = AB^2 + BC^2 - 2AB \cdot BC \cos B = AD^2 + CD^2 - 2AD \cdot CD \cos D$ yields that AD is the greatest side, therefore, C is the smallest angle. Since $\angle C + \angle D < \pi$, the rays CB and DA meet at some point P. Since angle C is acute and $\angle C + \angle A < \pi$, we obtain that $\sin A > \sin C$. Since $PB/\sin A = AB/\sin P > CD/\sin P = PD/\sin C$, this yields that $PB > PD$. But $PB = PC - BC < PC - CD < PD$ — contradiction.

11. (Tran Q.H.) Given triangle ABC and point P. Points A', B', C' are the projections of P to BC, CA, AB. A line passing through P and parallel to AB meets the circumcircle of triangle $PA'B'$ for the second time in point C_1. Points A_1, B_1 are defined similarly. Prove that

    a) (8-10) lines AA_1, BB_1, CC_1 concur;

    b) (9–11) triangles ABC and $A_1B_1C_1$ are similar.

    Solution. Since PC is the diameter of the circumcircle of $PA'B'$, therefore the angle PC_1C is right, i.e. C_1 lies on the altitude of ABC. Similarly A_1, B_1 lie on the two remaining altitudes. Thus the lines AA_1, BB_1, CC_1 meet on the orthocenter H and the assertion a) is proved. Also A_1, B_1, C_1 lie on the circle with diameter PH, because the angles PA_1H, PB_1H, PC_1H are right. Therefore, the angle between the lines A_1C_1 and B_1C_1 is equal to the angle between the lines HA_1 and HB_1, which as the angle between two altitude of the triangle ABC is equal two the angle between its sidelines AC and BC. Thus the angles of the triangles ABC and $A_1B_1C_1$ are equal, i.e. these triangles are similar.

12. (M.Zhanbulatuly) (9–10) Let O be the circumcenter of an acute-angled triangle ABC. A line passing through O and parallel to BC meets AB and AC in points P and Q respectively. The sum of distances from O to AB and AC is equal to OA. Prove that $PB + QC = PQ$.

    Solution. An equality $\cos A + \cos B + \cos C = 1 + r/R$ yields, that in an acute-angled triangle the sum of distances from O to the sides is equal to the sum of the circumradius and the inradius. Thus we obtain that PQ passes through the incenter I. Then $\angle PIB = \angle IBA = \angle IBP$ and $PB = IP$. Similarly $QC = IQ$.

13. (A.Zaslavsky) (9–10) Points A, B are given. Find the locus of points C such that C, the midpoints of AC, BC and the centroid of triangle ABC are concyclic.

Answer. A circle having the center at the midpoint of AB and the radius equal to $AB\sqrt{3}/2$ without its common points with line AB.

Solution. Let the medians AA_0 and BB_0 of the triangle meet at the point M. From the condition we have that $AM \cdot AA_0 = AB_0 \cdot AC$, i.e. $AA_0^2 = \frac{3}{4}AC^2$. Similarly, $BB_0^2 = \frac{3}{4}BC^2$. Since for an arbitrary triangle the ratio of the sums of the squares of its medians and its sides is equal to $3/4$, these equalities yield that the median from C is equal to $AB\sqrt{3}/2$. It is clear that all points of the circle distinct from its common points with line AB lie on the sought locus.

14. (M.Volchkevich) (9–10) In a convex quadrilateral $ABCD$ suppose $AC \cap BD = O$ and M is the midpoint of BC. Let $MO \cap AD = E$. Prove that $\frac{AE}{ED} = \frac{S_{\triangle ABO}}{S_{\triangle CDO}}$.

    Solution. Let P be the common point of AB and MO. Applying the Menelaos theorem to triangles ABC and ABD, we obtain that $\frac{AP}{PB} \cdot \frac{BO}{OD} \cdot \frac{DE}{AE} = \frac{AP}{PB} \cdot \frac{BM}{MC} \cdot \frac{CO}{OA} = 1$. Therefore, $\frac{AE}{ED} = \frac{OA \cdot OB}{OC \cdot OD} = \frac{S_{\triangle ABO}}{S_{\triangle CDO}}$.

15. (A.Zaslavsky) (9–11) Given triangle ABC. Consider lines l with the next property: the reflections of l in the sidelines of the triangle concur. Prove that all these lines have a common point.

    Solution. Let the reflections of l concur at the point P. Then the reflections of P lie on l, therefore, the projections of P to the sidelines are collinear. By Simson theorem P lies on the circumcircle of ABC. Since the Simson's line of P bisects the segment between P and the orthocenter H of ABC, we obtain that l passes through H.

16. (F.Ivlev) (9–11) Given right-angled triangle ABC with hypothenuse AB. Let M be the midpoint of AB and O be the center of circumcircle ω of triangle CMB. Line AC meets ω for the second time in point K. Segment KO meets the circumcircle of triangle ABC in point L. Prove that segments AL and KM meet on the circumcircle of triangle ACM.

    First solution. Since $BMKC$ is a cyclic quadrilateral, therefore $\angle BMK = 90°$ and O lies on BK. Thus $\angle ABL = \angle MBK = \angle MCK = \angle A$. Therefore, $\angle MAL = \angle B$, and the angles between AL and KM is equal to angle A, i.e. angle ACM (fig.16).

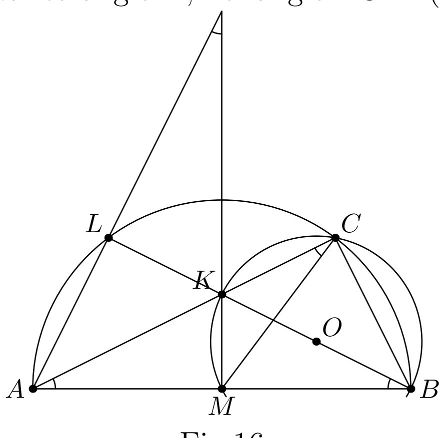


Fig.16

Second solution. Since KCB is a right angle, therefore O lies on KB. Since AB is a diameter of the circumcircle of ABC, therefore ALB is also a right angle. The angle KMB is right, because KCB is a right angle. Thus $K$is the orthocenter of the triangle formed by A, B and the common point of AL and MK. Then two right angles with vertices C and M leans on the same diameter.

17. (M.Rozhkova) (9–11) A square $ABCD$ is inscribed into a circle. Point M lies on arc BC, AM meets BD in point P, DM meets AC in point Q. Prove that the area of quadrilateral $APQD$ is equal to the half of the area of the square.

Solution. Since $\angle AMD = 45° = \angle OAD = \angle ODA$, therefore $\angle AQD = \angle AMD + \angle MAQ = \angle PAD$. Similarly, $\angle APD = \angle ADQ$ (fig.17). Thus the triangles APD and QDA are similar, i.e. $AQ \cdot PD = AD^2$, which yields the assertion of the problem.

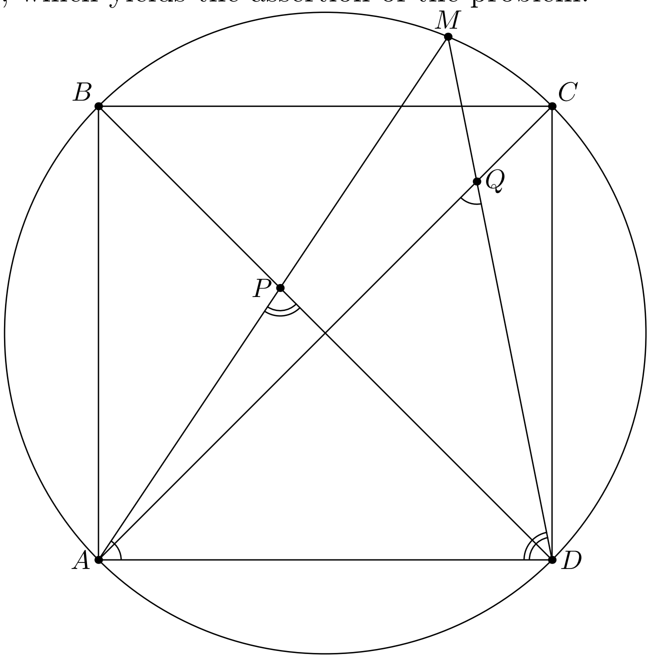


Fig.17

18. (B.Frenkin) (9–11) A triangle and two points inside it are marked. It is known that one of the triangle's angles is equal to 58°, one of two remaining angles is equal to 59°, one of two given points is the incenter of the triangle and the second one is its circumcenter. Using only the ruler without partitions determine where is each of the angles and where is each of the centers.

Solution. Construct the line passing through the marked points. It meets two sides of the triangle (for example AB and AC) and the prolongation of the third side (for example beyond the vertex C). Then AB is the greatest side of the triangle, BC is the smallest side and the marked poind nearest to BC is the incenter.

Prove these assertions. Let I be the incenter of given triangle and O be its circumcenter. Joining them with the vertices of the triangle and calculating the angles we obtain that O lies inside the triangle formed by I and the greatest side , and I lies inside the triangle formed by the smallest side and O. Thus the line OI meets the greatest and the smallest sides, therefore this line meet the prolongation of the third side. Also we obtain that O lies nearer to the greatest side, and I lies neare to the smallest side.

Now we have to examine wyich prolongation of side AC does OI meet. For this compare the lengths of the perpendiculars from O and I to AC. If r is the inradius, and R is the circumradius, then the distance from I to AC is equal to r, and the distance from O to AC is equal to $R\cos\ 59° > R/2 > r$, which yields the answer.

19. (A.Zaslavsky) (10–11) Two circles with radii 1 meet in points X, Y, and the distance between these points also is equal to 1. Point C lies on the first circle, and lines CA, CB are tangents

to the second one. These tangents meet the first circle for the second time in points B', A'. Lines AA' and BB' meet in point Z. Find angle XZY.

Answer. $150°$.

Solution. The condition yields that the distance between the centers of the circles is equal to $\sqrt{3}$, therefore by Euler formula these circles are the circumcircle and the excircle of the triangle $A'B'C$, i.e. $A'B'$ touches the second circle in a point C', lying on the line CZ (fig.19).

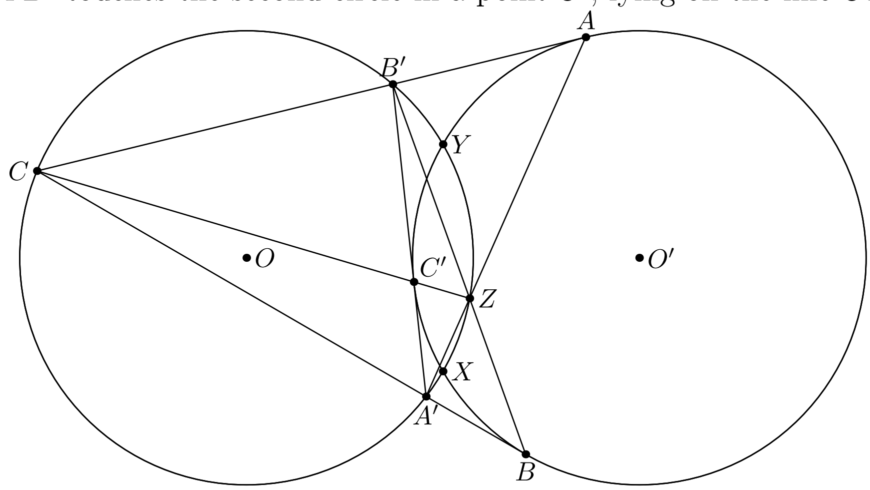


Fig.19

Let O and O' be the centers of the circles. Then $\angle A'O'A = \angle AO'C' + \frac{1}{2}\angle C'O'B = 2\angle ABC' + \angle C'AB = \angle CB'A' + \frac{1}{2}\angle CA'B'$, $\angle O'A'O = \angle O'A'B' + \angle B'A'O = \frac{\pi}{2} - \angle C'O'A' + \frac{\pi}{2} - \angle BCA = \pi - \angle BCA - \frac{1}{2}\angle CA'B' = \angle CB'A' + \frac{1}{2}\angle CA'B'$, and, since $O'A = OA'$, therefore $AO'A'O$ is an isosceles trapezoid. Thus $\angle O'AA' = \angle A'OO'$ and, similarly, $\angle O'BB' = \angle B'OO'$. Therefore, $\angle A'ZB' = 2\pi - \angle AO'B - \angle A'OB' = \pi - \angle C$, i.e. Z lies on the circumcircle and $\angle XZY = 150°$.

Note. We can prove that Z lies on the circumcircle on the other way. The point isogonally conjugated to Z wrt $A'B'C$ is the homothety center of the circles, which is an infinite point because the radii are equal.

20. (G.Feldman) (10–11) Point D lies on side AB of triangle ABC. Let ω_1 and Ω_1, ω_2 and Ω_2 be the incirles and the excircles (touching segment AB) of triangles ACD and BCD. Prove that the common external tangents to ω_1 and ω_2, Ω_1 and Ω_2 meet on AB.

    First solution. Let I_1, J_1, I_2, J_2 be the centers of ω_1, Ω_1, ω_2, Ω_2, and K_1, K_2 be the intersection points of I_1J_1, I_2J_2 with AB (fig.20). Then $I_1K_1/I_1C = J_1K_1/J_1C$, $I_2K_2/I_2C = J_2K_2/J_2C$ and, applying the Menelaos theorem to the triangle CK_1K_2, we obtain that I_1I_2 and J_1J_2 meet AB at the same point. The common external tangents also pass through this point.

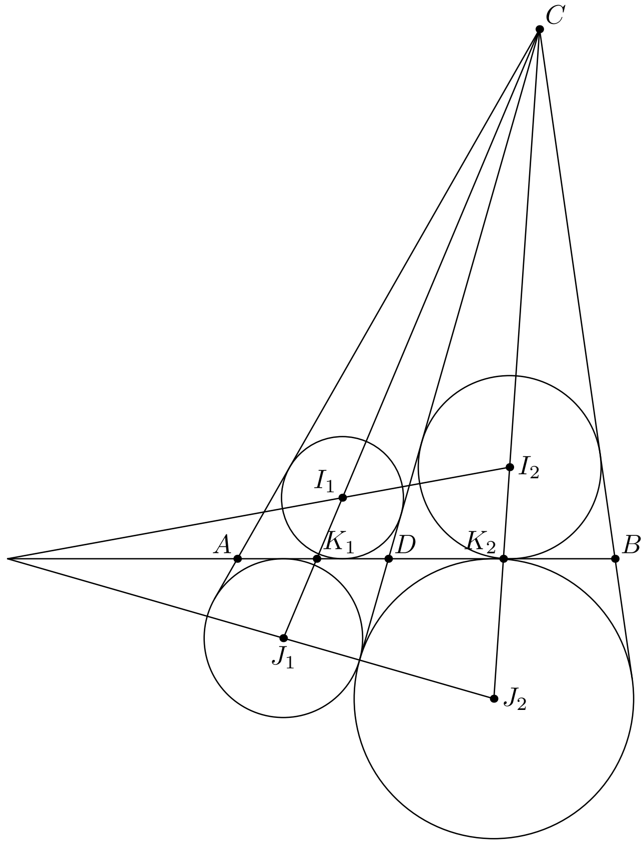


Fig.20

Second solution. Let the common external tangents to ω_1 and Ω_2 meet at a point P. Then applying the three caps theorem to ω_1, Ω_1, Ω_2 and to ω_1, ω_2, Ω_2, we obtain, that the intersection points of the common external tangents to Ω_1, Ω_2 and to ω_1, ω_2 coincide with the common point of the lines PC and AB. Thus these point coincide and lie on AB.

21. (N.Beluhov, E.Colev) (10–11) Two perpendicular lines pass through the orthocenter of an acute-angled triangle. The sidelines of the triangle cut on each of these lines two segments: one lying inside the triangle and another one lying outside it. Prove that the product of two internal segments is equal to the product of two external segments.

Solution. Let one of two lines meets BC, CA, AB at the points X_a, X_b, X_c, and the remaining line meets them at the points Y_a, Y_b, Y_c (fig.21). Then $\angle HY_aB = \angle X_bHA$ and $\angle HX_bA = \angle Y_aHB$, because the sidelines of these angles are perpendicular. Thus the triangles HBY_a and X_bAH are similar. The triangles HX_aB and Y_bAH are also similar. Therefore, $AX_b \cdot BY_a = AH \cdot BH = AY_b \cdot BX_a$. On the other hand applying the Menelaos theorem to the triangles CX_aX_b, CY_aY_b and the line AB, we obtain that $\frac{CA}{AX_b} \cdot \frac{X_bX_c}{X_cX_a} \cdot \frac{X_aB}{BC} = \frac{CA}{AY_b} \cdot \frac{Y_bY_c}{Y_cY_a} \cdot \frac{Y_aB}{BC} = 1$. These three equalities yield the assertion of the problem.

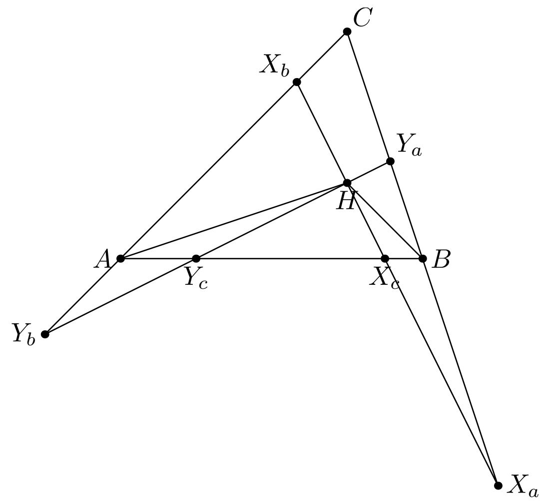


Fig.21

22. (F.Nilov) (10–11) A circle ω with center I is inscribed into a segment of the disk, formed by an arc and a chord AB. Point M is the midpoint of this arc AB, and point N is the midpoint of the complementary arc. The tangents from N touch ω in points C and D. The opposite sidelines AC and BD of quadrilateral $ABCD$ meet in point X, and the diagonals of $ABCD$ meet in point Y. Prove that points X, Y, I and M are collinear.

Solution. Let K, L — be the touching points of ω with AB and the great circle. Since L is the homothety center of the circles, and the tangents at the points K and N are parallel, therefore the points L, K, N are collinear. Also we have $\angle KAN = \angle NLA$, because the correspondent arcs are equal. Thus the triangles KAN and ALN are similar and $AN^2 = NK \cdot NL = NC^2$, i.e. quadrilateral $ABCD$ is inscribed into a circle with center N (fig.22). The line XY is the polar of the common point of AB and CD wrt this circle. And since $\angle NAM = \angle NBM = \angle NCI = \angle NDI = 90°$, therefore the points M and I are the poles of AB and CD. Thus they lie on XY.

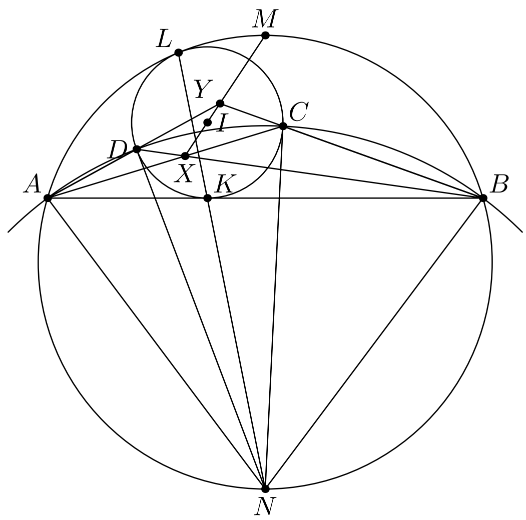


Fig.22

23. (A.Kanel) (10–11) An arbitrary point is selected on each of twelve diagonals of the faces of a cube. The centroid of these twelve points is determined. Find the locus of all these centroids.

Solution. Firstly note, that the locus of the midpoints of the segments with endpoints lying on two diagonals of a square is the square with the vertices coinciding with the midpoints of the sides of the original square. Thus the locus of the centroids of four points lying on the diagonals of two opposite faces of a cube is the square with the vertices coinciding with the centers of four remaining faces. Therefore we have to find the locus of centroids of three points each of them lies inside one of three such squares. It is clear that all such centroids lie inside an octahedron formed by the centers of the faces of the cube. Also, if one of three points lies on the central plane of the octahedron, and the distances from two remaining points to this plane don't exceed a half of the edge of the cube, then the distance from the centroid to this plane isn't greater than one third of the edge. Therefore all centroids lie inside the polyhedron obtained by the cutting off the octahedron six pyramids with the edges equal to one third of the edge of the octahedron. On the other hand all vertices of this polyhedron and therefore all its points lie on the sought locus.

24. (V.Yassinsky) (10–11) Given are n ($n > 2$) points on the plane such that no three of them aren't collinear. In how many ways this set of points can be divided into two non-empty subsets with non-intersecting convex envelops?

Answer. $n(n-1)/2$.

Solution. Since the convex envelops don't intersect, the two subsets lit on different sides from some line. Thus we have to examine in how many ways the given set of the points can be divided into two subsets by a line. Take a point O of the plane, which don't lie on any line joining the given points, and consider the polar correspondence with center O. The given points correspond to n lines, such that no two of them aren't parallel and no three don't concur. It is easy to prove by induction that these lines divide the plane into $n(n+1)/2+1$ parts, and $2n$ from these parts aren't limited.

Lemma. Let the polars a, b of the points A, B divide the plane into 4 angles. Then the poles of the lines, intersecting the segment AB, lie inside two vertical angles, and the poles of the lines which don't intersecting the segment AB lie inside two remaining angles.

In fact let the lines l and AB meet at the point X. Then the polar of X passes through the common point of a and b. When l rotates around X, its pole moves on this line, i.e. inside some pair of vertical angles formed by a and b. When X moves on AB its polar rotates around the common point of a and b, passing from one pair of vertical angles into the other when X passes through A, B. Lemma is proved.

Return to the problem. The lemma yields that two lines divide the given set of the points by the same way iff their poles lie inside the same part formed by the polars of the given points, or these poles lie on the different sides from all n polars. But the second case is possible iff the two points lie inside the not limited parts. In fact if two points P, Q lie on the different sides from all lines, then each of these lines intersect the segment PQ. Thus each of two rays prolongating this segment lies entirely inside one of the parts. Inversely, if the part containing the point P isn't limined, then ther exists a ray with endpoint in P, lying entirely inside this part and not parallel to any of n lines. The opposite ray intersect all lines and therefore contains a points lying on the different sides than P from these lines.

Thus, $2n$ not limited parts forms n pairs, each of them correspond to one way of dividing of the given set of the points. Each of limited parts also correspond to one way of dividing. Therefore we have $n(n-1)/2+1$ ways, but for one of them all n points belong to the same subset.

IX GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
THE CORRESPONDENCE ROUND

Below is the list of problems for the first (correspondence) round of the IX Sharygin Geometrical Olympiad.

The olympiad is intended for high-school students of 8–11 grades (these are four elder grades in Russian school). In the list below, each problem is indicated by the numbers of school grades, for which it is intended. However, the participants may solve problems for elder grades as well (solutions of problems for younger grades will not be considered).

The solutions for the problems (in Russian or in English) must be contained in pdf, doc or jpg files. Your work must be sent not later than on April 1, 2013, by e-mail to geomolymp@mccme.ru. Please, follow a few simple rules:

1. Each student sends his work in a separate message (with delivery notification). The size of the message must not exceed 10 Mb.

2. If your work consists of several files, send it as an archive.

3. If the size of your message exceeds 10 Mb divide it into several messages.

4. In the subject of the message write "The work for Sharygin olympiad", and present the following personal data in the body of your message:

- last name;

- all other names;

- E-mail, phone number, post address;

- the current number of your grade at school;

- the number of the last grade at your school;

- the number and/or the name and the mail address of your school;

- full names of your teachers in mathematics at school and/or of instructors of your extra math classes (if you attend additional math classes after school).

If you have no e-mail access, please send your work by ordinary mail to the following address: *Russia, 119002, Moscow, Bolshoy Vlasyevsky per., 11. Olympiad in honour of Sharygin.* In the title page, write down your personal information indicated in the item 4 above.

We recommend you to write your work on the special blanks that can be found on www.blank.geomolymp.mccme.ru. This will provide quick and qualitative examination of your work. If you type the work using a computer then please receive a blank on the site above and copy its bar code (a square with a pattern in the right upper corner of the blank) into your work as a picture. If your work consists of several files, copy this bar code into all files, this enables to identify the author of the work.

In your work, please start the solution for each problem in a new page. First write down the statement of the problem, and then the solution. Present your solutions in detail, including all significant arguments and calculations. Provide all necessary figures of sufficient size. If a problem has an explicit answer, this answer must be presented distinctly. Please, be accurate to provide good understanding and correct estimating of your work !

If your solution depends on some well-known theorems from standard textbooks, you may simply refer to them instead of providing their proofs. However, any fact not from the standard curriculum should be either proved or properly referred (with an indication of the source).

You may note the problems which you liked most (this is not obligatory). Your opinion is interesting for the Jury.

Winners of the correspondence round, the students of three grades before the last grade, will be invited to the final round in Summer 2013 in the city of Dubna, in Moscow region. (

For instance, if the last grade is 12, then we invite winners from 9, 10, and 11 grade.) Winners of the correspondence round, the students of the last grade, will be awarded by diplomas of the Olympiad. The list of the winners will be published on www.geometry.ru at the end of May 2013. If you want to know your detailed results, please use e-mail.

1. (8) Let ABC be an isosceles triangle with $AB = BC$. Point E lies on the side AB, and ED is the perpendicular from E to BC. It is known that $AE = DE$. Find $\angle DAC$.

2. (8) Let ABC be an isosceles triangle ($AC = BC$) with $\angle C = 20°$. The bisectors of angles A and B meet the opposite sides at points A_1 and B_1 respectively. Prove that the triangle A_1OB_1 (where O is the circumcenter of ABC) is regular.

3. (8) Let ABC be a right-angled triangle ($\angle B = 90°$). The excircle inscribed into the angle A touches the extensions of the sides AB, AC at points A_1, A_2 respectively; points C_1, C_2 are defined similarly. Prove that the perpendiculars from A, B, C to C_1C_2, A_1C_1, A_1A_2 respectively, concur.

4. (8) Let ABC be a nonisosceles triangle. Point O is its circumcenter, and point K is the center of the circumcircle w of triangle BCO. The altitude of ABC from A meets w at a point P. The line PK intersects the circumcircle of ABC at points E and F. Prove that one of the segments EP and FP is equal to the segment PA.

5. (8) Four segments drawn from a given point inside a convex quadrilateral to its vertices, split the quadrilateral into four equal triangles. Can we assert that this quadrilateral is a rhombus?

6. (8–9) Diagonals AC and BD of a trapezoid $ABCD$ meet at point P. The circumcircles of triangles ABP and CDP intersect the line AD for the second time at points X and Y respectively. Let M be the midpoint of segment XY. Prove that $BM = CM$.

7. (8–9) Let BD be a bisector of triangle ABC. Points I_a, I_c are the incenters of triangles ABD, CBD respectively. The line I_aI_c meets AC in point Q. Prove that $\angle DBQ = 90°$.

8. (8–9) Let X be an arbitrary point inside the circumcircle of a triangle ABC. The lines BX and CX meet that circumcircle in points K and L respectively. The line LK intersects BA and AC at points E and F respectively. Find the locus of points X such that the circumcircles of triangles AFK and AEL touch.

9. (8–9) Let T_1 and T_2 be the points of tangency of the excircles of a triangle ABC with its sides BC and AC respectively. It is known that the reflection of the incenter of ABC across the midpoint of AB lies on the circumcircle of triangle CT_1T_2. Find $\angle BCA$.

10. (8–9) The incircle of triangle ABC touches the side AB at point C'; the incircle of triangle ACC' touches the sides AB and AC at points C_1, B_1; the incircle of triangle BCC' touches the sides AB and BC at points C_2, A_2. Prove that the lines B_1C_1, A_2C_2, and CC' concur.

11. (8–9) a) Let $ABCD$ be a convex quadrilateral and $r_1 \le r_2 \le r_3 \le r_4$ be the radii of the incircles of triagles ABC, BCD, CDA, DAB. Can the inequality $r_4 > 2r_3$ hold?

    b)The diagonals of a convex quadrilateral $ABCD$ meet in point E. Let $r_1 \le r_2 \le r_3 \le r_4$ be the radii of the incircles of triangles ABE, BCE, CDE, DAE. Can the inequality $r_2 > 2r_1$ hold?

12. (8–11) On each side of a triangle ABC, two distinct points are marked. It is known that these points are the feet of the altitudes and of the bisectors.

    a) Using only a ruler determine which points are the feet of the altitudes and which points are the feet of the bisectors.

    b) Solve p.a) drawing only three lines.

13. (9–10) Let A_1 and C_1 be the tangency points of the incircle of triangle ABC with BC and AB respectively, A' and C' be the tangency points of the excircle inscribed into the angle B with the extensions of BC and AB respectively. Prove that the orthocenter H of triangle ABC lies on A_1C_1 if and only if the lines $A'C_1$ and BA are orthogonal.

14. (9–11) Let M, N be the midpoints of diagonals AC, BD of a right-angled trapezoid $ABCD$ ($\angle A = \angle D = 90°$). The circumcircles of triangles ABN, CDM meet the line BC in points Q, R. Prove that the distances from Q, R to the midpoint of MN are equal.

15. (9–11) a) Triangles $A_1B_1C_1$ and $A_2B_2C_2$ are inscribed into triangle ABC so that $C_1A_1 \perp BC$, $A_1B_1 \perp CA$, $B_1C_1 \perp AB$, $B_2A_2 \perp BC$, $C_2B_2 \perp CA$, $A_2C_2 \perp AB$. Prove that these triangles are equal.

    b) Points $A_1, B_1, C_1, A_2, B_2, C_2$ lie inside a triangle ABC so that A_1 is on segment AB_1, B_1 is on segment BC_1, C_1 is on segment CA_1, A_2 is on segment AC_2, B_2 is on segment BA_2, C_2 is on segment CB_2, and the angles $BAA_1, CBB_1, ACC_1, CAA_2, ABB_2, BCC_2$ are equal. Prove that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are equal.

16. (9–11) The incircle of triangle ABC touches BC, CA, AB at points A', B', C' respectively. The perpendicular from the incenter I to the median from vertex C meets the line $A'B'$ in point K. Prove that $CK \parallel AB$.

17. (9–11) An acute angle between the diagonals of a cyclic quadrilateral is equal to ϕ. Prove that an acute angle between the diagonals of any other quadrilateral having the same sidelengths is smaller than ϕ.

18. (9–11) Let AD be a bisector of triangle ABC. Points M and N are the projections of B and C respectively to AD. The circle with diameter MN intersects BC at points X and Y. Prove that $\angle BAX = \angle CAY$.

19. (10–11) a) The incircle of a triangle ABC touches AC and AB at points B_0 and C_0 respectively. The bisectors of angles B and C meet the perpendicular bisector to the bisector AL in points Q and P respectively. Prove that the lines PC_0, QB_0, and BC concur.

    b) Let AL be the bisector of a triangle ABC. Points O_1 and O_2 are the circumcenters of triangles ABL and ACL respectively. Points B_1 and C_1 are the projections of C and B to the bisectors of angles B and C respectively. Prove that the lines O_1C_1, O_2B_1, and BC concur.

    c) Prove that two points obtained in pp. a) and b) coincide.

20. (10–11) Let C_1 be an arbitrary point on the side AB of triangle ABC. Points A_1 and B_1 on the rays BC and AC are such that $\angle AC_1B_1 = \angle BC_1A_1 = \angle ACB$. The lines AA_1 and BB_1 meet in point C_2. Prove that all the lines C_1C_2 have a common point.

21. (10–11) Let A be a point inside a circle ω. One of two lines drawn through A intersects ω at points B and C, the second one intersects it at points D and E (D lies between A and E). The line passing through D and parallel to BC meets ω for the second time at point F, and the line AF meets ω at point T. Let M be the common point of the lines ET and BC, and N be the reflection of A across M. Prove that the circumcircle of triangle DEN passes through the midpoint of segment BC.

22. (10–11) The common perpendiculars to the opposite sidelines of a nonplanar quadrilateral are mutually orthogonal. Prove that they intersect.

23. (10–11) Two convex polytopes A and B do not intersect. The polytope A has exactly 2012 planes of symmetry. What is the maximal number of symmetry planes of the union of A and B, if B has a) 2012, b) 2013 symmetry planes?

    c) What is the answer to the question of p.b), if the symmetry planes are replaced by the symmetry axes?

IX GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN.
THE CORRESPONDENCE ROUND. SOLUTIONS.

1. (N.Moskvitin) Let ABC be an isosceles triangle with $AB = BC$. Point E lies on side AB, and ED is the perpendicular from E to BC. It is known that $AE = DE$. Find $\angle DAC$.

   Answer. $45°$.

   Solution. By the external angle theorem $\angle AED = 90° + \angle B = 270° - 2\angle A$ (fig.1). Therefore, $\angle EAD = (180° - \angle AED)/2 = \angle A - 45°$.

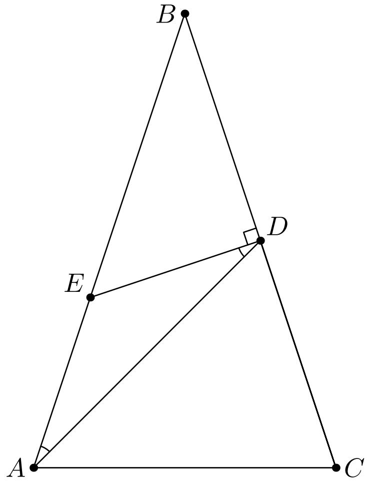


Fig.1

2. (L.Steingarts) Let ABC be an isosceles triangle ($AC = BC$) with $\angle C = 20°$. The bisectors of angles A and B meet the opposite sides in points A_1 and B_1 respectively. Prove that triangle A_1OB_1 (where O is the circumcenter of ABC) is regular.

   Solution. On sides BC and AC take points A' and B' such that $AB' = B'O = OA' = A'B$. It is clear that $A'B' \parallel AB$, i.e. $\angle CA'B' = \angle CBA = 80°$. Also $\angle A'OB = \angle A'BO = \angle BCO = 10°$. Thus $\angle CA'O = 20°$ and $\angle OA'B' = 60°$, i.e triangle $OA'B'$ is regular. Then $A'B' = A'B$ and $\angle A'BB' = \angle A'B'B = \angle ABB'$ (fig.2). Therefore B' coincides with B_1. Similarly A' coincides with A_1, q.e.d.

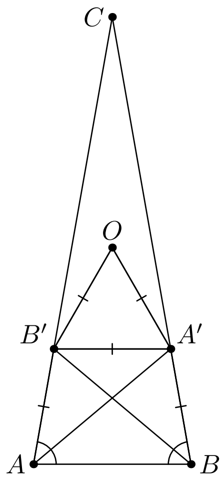


Fig.2

3. (D.Shvetsov) Let ABC be a right-angled triangle ($\angle B = 90°$). The excircle inscribed into the angle A touches the extensions of the sides AB, AC at points A_1, A_2 respectively; points C_1,

C_2 are defined similarly. Prove that the perpendiculars from A, B, C to C_1C_2, A_1C_1, A_1A_2 respectively, concur.

Solution.

Let I be the incenter of ABC, and D be the fourth vertex of rectangle $ABCD$. Since $AI \perp A_1A_2$, $CI \perp C_1C_2$, the perpendiculars from A to CC_1 and from C to AA_1 meet in the incenter J of triangle ACD. Then it is sufficient to prove that $DI \perp A_1C_1$. Let X, Y, Z be the projections of I to AB, BC, CD respectively. Then $BC_1 = XC_2 = ZD$ and $A_1B = CY = IZ$, thus triangles A_1BC_1 and IZD are equal, i.e. $\angle IDZ = \angle A_1C_1B$ (fig.3), that proves the required assertion.

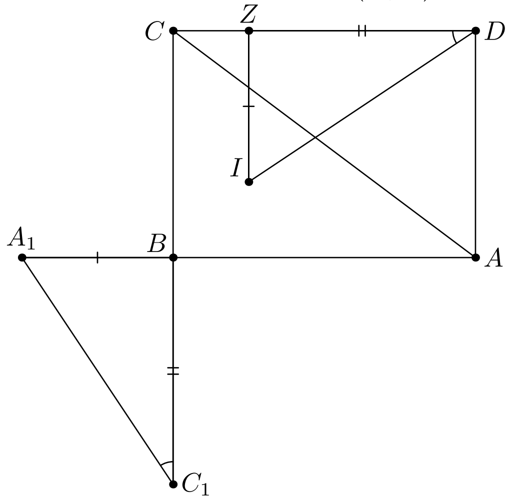


Fig.3

4. (F.Ivlev) Let ABC be a nonisosceles triangle. Point O is its circumcenter, and point K is the center of the circumcircle w of triangle BCO. The altitude of ABC from A meets w at a point P. The line PK intersects the circumcircle of ABC at points E and F. Prove that one of the segments EP and FP is equal to the segment PA.

   Solution. Points O and K lie on the bisector of segment BC, thus $OK \parallel AP$ and $\angle OPK = \angle POK = \angle OPA$. Therefore the reflection A' of A in OP lies on PK. Also $OA' = OA$, i.e. A' lies on the circumcircle of ABC (fig.4). Thus A' coincides with one of points E, F.

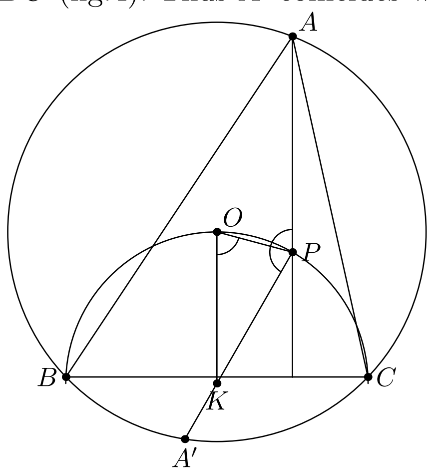


Fig.4

5. (B.Frenkin) Four segments drawn from a given point inside a convex quadrilateral to its vertices, split the quadrilateral into four equal triangles. Can we assert that this quadrilateral is a rhombus?

Answer. Yes.

Solution. Let $ABCD$ and O be the given quadrilateral and point. In equal triangles the angles opposite to equal sides are equal. Since $\triangle ABO = \triangle CBO$, angles BAO and BCO opposite to BO are equal. Similarly $\angle DAO = \angle DCO$, thus $\angle BAD = \angle BCD$. Two remaining angles of the quadrilateral are similarly equal, therefore $ABCD$ is a parallelogram.

There exist two adjacent angles with vertex O such that their sum is not less than π, suppose that these angles are $\angle AOB$ and $\angle COB$. The second angle is equal to some angle of triangle AOB. This can be only $\angle AOB$, because its sum with any of two remaining angles of AOB is less than π. The sides of equal triangles AOB and COB opposite to these angles are equal. Then $AB = BC$ and $ABCD$ is a rhombus.

6. (D.Shvetsov) Diagonals AC and BD of a trapezoid $ABCD$ meet at point P. The circumcircles of triangles ABP and CDP intersect the line AD for the second time at points X and Y respectively. Let M be the midpoint of segment XY. Prove that $BM = CM$.

   Solution.

   By condition, $\angle BXA = \angle BPA = \angle CPD = \angle CYD$ (fig.6). Thus $BXYC$ is an isosceles trapezoid, which proves the required assertion.

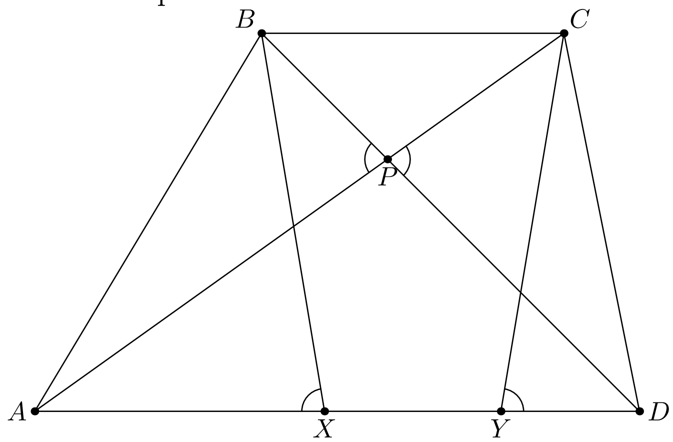


Fig.6

7. (D.Shvetsov) Let BD be a bisector of triangle ABC. Points I_a, I_c are the incenters of triangles ABD, CBD respectively. The line I_aI_c meets AC in point Q. Prove that $\angle DBQ = 90°$.

   Solution.

   Lines AI_a and CI_c meet in the incenter I of ABC. By the bisectrix theorem $AI_a/I_aI = AD/ID$, $CI_c/I_cI = CD/ID$. By the Menelaos theorem $QA/QC = AD/CD = AB/BC$. Therefore BQ is the external bisectrix of angle B, q.e.d.

8. (M.Plotnikov) Let X be an arbitrary point inside the circumcircle of a triangle ABC. The lines BX and CX meet the circumcircle for the second time at points K and L respectively. The line LK intersects BA and AC at points E and F respectively. Find the locus of points X such that the circumcircles of triangles AFK and AEL touch.

   Answer. The arc of the circle passing through B, C and the circumcenter O of ABC.

   Solution. Let the circles touche. Then the angles between their common tangent and lines AC and AB are equal to angles ALE and AKF respectively. Since these two angles are equal to angles ABX and ACX, their sum is equal to angle A and $\angle BXC = 2\angle A = \angle BOC$. Similarly we obtain that for any point of the arc the correspondent circles touche.

9. (M.Plotnikov) Let T_1 and T_2 be the points of tangency of the excircles of a triangle ABC with its sides BC and AC respectively. It is known that the reflection of the incenter of ABC across the midpoint of AB lies on the circumcircle of triangle CT_1T_2. Find $\angle BCA$.

   Answer. $90°$.

   Solution. Let D be the fourth vertex of parallelogram $ACBD$, J be the incenter of ABD, S_1, S_2 be the points of tangency of the incircle of ABC with AD and BD. Then $S_1T_1 \parallel AC$, $S_2T_2 \parallel BC$ and $\angle T_1JT_2 = \angle S_1JS_2 = \pi - \angle C$. Also $DS_1 = DS_2$, i.e. lines S_1T_1, S_2T_2 and DJ concur. Therefore J coincides with the common point of lines S_1T_1 and S_2T_2, i.e. $\angle C = 90°$ (fig.9).

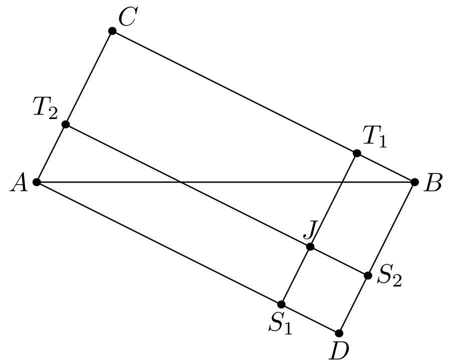


Fig.9

10. (D.Shvetsov) The incircle of triangle ABC touches the side AB at point C'; the incircle of triangle ACC' touches the sides AB and AC at points C_1, B_1; the incircle of triangle BCC' touches the sides AB and BC at points C_2, A_2. Prove that the lines B_1C_1, A_2C_2, and CC' concur.

    Solution.

    Since $AC' - BC' = AC - BC$, the incircles of triangles ACC' and BCC' touche CC' at the same point. Therefore $CB_1 = CA_2$. Also $AB_1 = AC_1$, $BA_2 = BC_2$, and if we find the angles of quadrilateral $A_2B_1C_1C_2$, we obtain that it is cyclic. Thus B_1C_1, A_2C_2 and CC' concur in the radical center of three circles: the circumcircle of $A_2B_1C_1C_2$ and the incircles of triangles ACC', BCC' (fig.10).

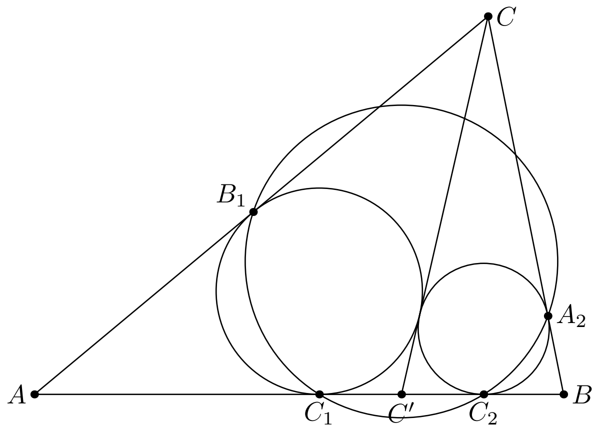


Fig.10

11. (P.Kozhevnikov) a) Let $ABCD$ be a convex quadrilateral and $r_1 \leq r_2 \leq r_3 \leq r_4$ be the radii of the incircles of triangles ABC, BCD, CDA, DAB. Can the inequality $r_4 > 2r_3$ hold?

    b)The diagonals of a convex quadrilateral $ABCD$ meet in point E. Let $r_1 \leq r_2 \leq r_3 \leq r_4$ be the radii of the incircles of triangles ABE, BCE, CDE, DAE. Can the inequality $r_2 > 2r_1$ hold?

    Answer. a) No. b) No.

    Solution. a) Suppose that $r_4 = r(ABC)$. It is sufficient to prove that $r(ABC)/2 < \max\{r(ABD), r(CB$ The midpoint K of AC lies inside one of triangles ABD, CBD, for example inside ABD. Then triangle AKL, where L is the midpoint of AB, lies inside triangle ABD, therefore $r(ABC)/2 = r(AKL) < r(ABD)$.

    b) Let $r = r_1$ be the inradius of triangle ABE. The diameters of the incircles of triangles BCE, ADE, are less than the altitudes of these triangles coinciding with altitudes h_a, h_b of ABE. Thus it is sufficient to prove that one of these altitudes is less than $4r$. Suppose that $AE \geq BE$. Then the semiperimeter $p < AE + BE \leq 2AE$ and $h_b = 2S/AE = 2pr/AE < 4r$.

    Comment. Note that the answer to both questions will be positive if we replace 2 to any smaller number.

12. (B.Frenkin) (8–11) On each side of triangle ABC, two distinct points are marked. It is known that these points are the feet of the altitudes and the bisectors.

    a) Using only a ruler determine which points are the feet of the altitudes and which points are the feet of the bisectors.

    b) Solve p.a) drawing only three lines.

    Solution. Preliminary hints. Since all points are distinct the triangle isn't isosceles. For each side, the foot of the altitude lies between the foot of the bisector and the smaller of two remaining sides. Thus it is sufficient to define the smallest and the greatest of the sides. We will denote the feet of the bisector and the altitude from vertex X as L_X and H_X respectively.

    Lemma. If $|AC| > |BC|$ then lines L_BL_A and H_BH_A meet the extension of side AB beyond B.

    Proof. Let L_BD be the perpendicular from L_B to AB, and CH be the altitude. By the bisector theorem $|L_BD| : |CH| = |AB| : (|BC| + |AB|)$. Similarly if L_AE is the perpendicular from L_A to AB, then $|L_AE| : |CH| = |AB| : (|AC| + |AB|)$. Furthermore $|AC| > |BC|$, $|L_BD > |L_AE|$, thus L_BL_A meets the extension of AB beyond B.

    Points H_B, H_A lie on the semicircle with diameter AB. Since $\angle H_AAB < \angle H_BBA$, the distance from H_A to AB is less than the distance from H_B to AB. The lemma is proved.

    Simple solution of p.a). Joining the given points with the opposite vertices we obtain two families of concurrent lines. Take two points of the same family on two sides and draw the line through them. By the lemma this line meets the extension of the third side beyond the vertex lying on the smaller of two sides. Therefore we can define the smaller of any two sides.

    Solution of p.b). Take for each vertex the nearest marked points on two adjacent sides and join these points. We will prove that *these lines meet the prolongation of the greatest side beyond the vertex of the medial angle and the extensions of two remaining sides beyond the vertex of the greatest angle.* From this we can define the greatest and the smallest side.

Let us prove the above assertion. Suppose that $|AB| > |AC| > |BC|$. The marked points nearest to the vertex of the smallest angle are the feet of the bisectors, and the points nearest to the vertex of the greatest angle are the feet of the altitudes. By the lemma, the lines joining these points meet the extension of BC beyond C and the extension of AB beyond B. The marked points nearest to B are H_C and L_A. By the lemma, line L_CL_A meets the extension of AC beyond C in some point P. Ray H_CL_A passes inside triangle H_CCP and thus intersects segment CP, q.e.d.

13. (F.Ivlev) Let A_1 and C_1 be the tangency points of the incircle of triangle ABC with BC and AB respectively, A' and C' be the tangency points of the excircle inscribed into the angle B with the extensions of BC and AB respectively. Prove that the orthocenter H of triangle ABC lies on A_1C_1 if and only if the lines $A'C_1$ and BA are orthogonal.

    Solution. Suppose that $A'C_1 \perp BA$. Then by Thales theorem the altitude from C divides segment A_1C_1 in ratio $A_1C : CA' = p-c : p-a$. The altitude from A passes through the same point. The inverse assertion is obtained similarly.

14. (D.Shvetsov) Let M, N be the midpoints of diagonals AC, BD of right-angled trapezoid $ABCD$ ($\angle A = \angle D = 90°$). The circumcircles of triangles ABN, CDM meet line BC in points Q, R. Prove that the distances from Q, R to the midpoint of MN are equal.

    Solution. Let X, Y be the projections of N and M to BC. Than we have to prove that $RY = XQ$. Since $\angle NQX = \angle NAB = \angle DBA$, triangles XQN and ABD are similar (fig.14). Thus $XQ = AB \cdot NX/AD$. But $NX = CD \sin \angle BCD/2 = CD \cdot AD/2BC$, therefore $XQ = AB \cdot CD/2BC = RY$.

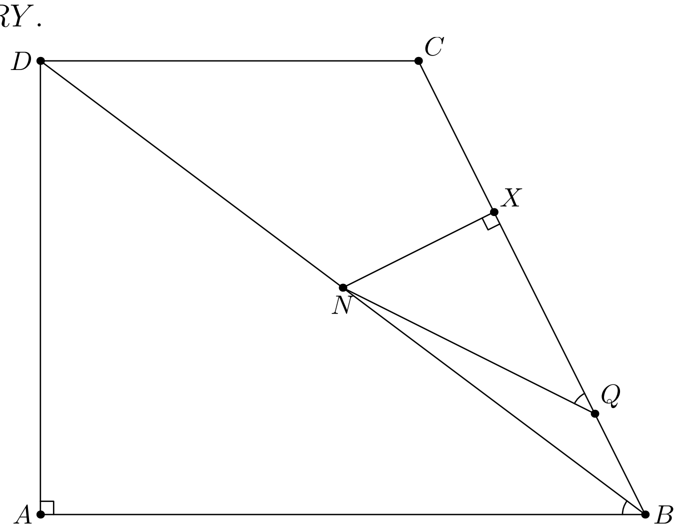


Fig.14

15. a) (V.Rastorguev) Triangles $A_1B_1C_1$ and $A_2B_2C_2$ are inscribed into triangle ABC so that $C_1A_1 \perp BC$, $A_1B_1 \perp CA$, $B_1C_1 \perp AB$, $B_2A_2 \perp BC$, $C_2B_2 \perp CA$, $A_2C_2 \perp AB$. Prove that these triangles are equal.

    b) (P.Kozhevnikov) Points $A_1, B_1, C_1, A_2, B_2, C_2$ lie inside triangle ABC so that A_1 is on segment AB_1, B_1 is on segment BC_1, C_1 is on segment CA_1, A_2 is on segment AC_2, B_2 is on segment BA_2, C_2 is on segment CB_2 and angles $BAA_1, CBB_1, ACC_1, CAA_2, ABB_2, BCC_2$ are equal. Prove that triangles $A_1B_1C_1$ and $A_2B_2C_2$ are equal.

    Solution. a) Inscribe triangle $A_2B_2C_2$ into triangle $A'B'C'$ in such a way that $C_2A_2 \perp B'C'$, $A_2B_2 \perp C'A'$, $B_2C_2 \perp A'B'$. It is clear that the corresponding sidelines of triangles ABC

and $B'C'A'$ are symmetric wrt the circumcenter of $A_2B_2C_2$. This symmetry maps $A_2B_2C_2$ to $B_1C_1A_1$. Therefore these triangles are equal and their circumcenters coincide.

b) Consider the chords AA', BB', CC', AA'', BB'', CC'' of the circumcircle of ABC lying on the lines A_1B_1, B_1C_1, C_1A_1, A_2C_2, B_2A_2, C_2B_2. By condition, arcs AC', BA', CB', AB'', CA'', BC'' are equal. Let their size be φ. The rotation around the circumcenter to φ maps AA', BB', CC' to BB'', CC'', AA'' respectively, thus it maps $A_1B_1C_1$ to $A_2B_2C_2$ (fig.15).

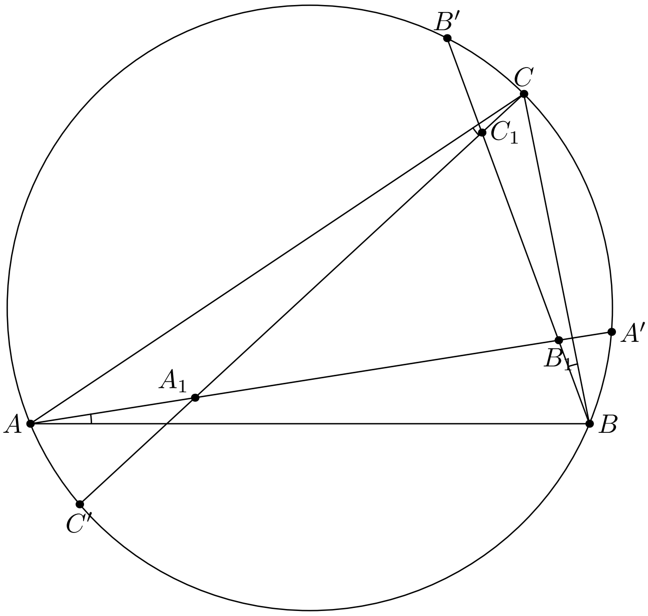


Fig.15

Comment. In a special case when triangle $A_1B_1C_1$ degenerates to a point, $A_2B_2C_2$ also degenerates to a point, and the distances from these two points to the circumcenter are equal. These points are *the Brockard points* of the triangle.

16. (F.Ivlev) The incircle of triangle ABC touches BC, CA, AB at points A', B', C' respectively. The perpendicular from the incenter I to the median from vertex C meets the line $A'B'$ in point K. Prove that $CK \parallel AB$.

Solution. The polar transformation wrt the incircle maps the perpendicular from I to the median into the infinite point of this median, the image of line $A'B'$ is point C, and the image of the line passing through C and parallel to AB is the common point P of $A'B'$ and IC'. Thus we have to prove that P lies on the median.

Since $IA' = IB'$, $\angle PIB' = \angle A$, $\angle PIA' = \angle B$, we have $B'P : A'P = BC : AC$. Since $CA' = CB'$, we have $\sin\angle ACP : \sin\angle BCP = BC : AC$, i.e. CP bisects AB (fig.16).

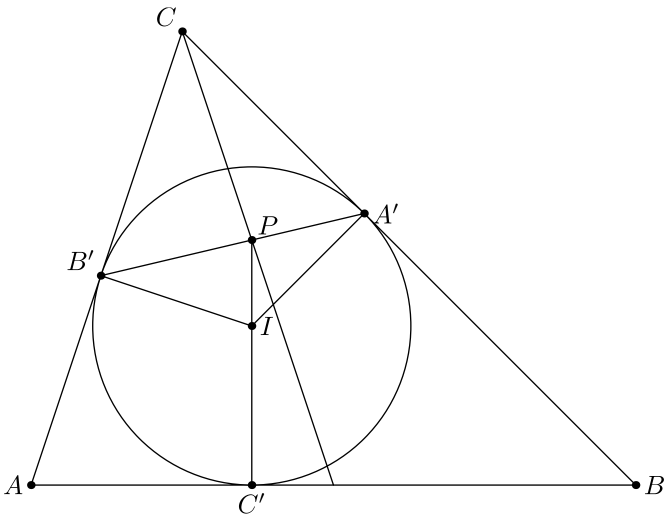


Fig.16

17. (A.Zaslavsky) An acute angle between the diagonals of a cyclic quadrilateral is equal to ϕ. Prove that an acute angle between the diagonals of any other quadrilateral having the same sidelengths is smaller than ϕ.

    Solution. Let the diagonals of quadrilateral $ABCD$ meet in point P. Put $PA = a$, $PB = b$, $PC = c$, $PD = d$ and express the sidelengths of $ABCD$ through a, b, c, d and $\cos\phi$. Then

    $$|AB^2 - BC^2 + CD^2 - CA^2| = 2\cos\phi(ab + bc + cd + da) = 2AC \cdot BD\cos\phi.$$

    By Phtolomeos theorem $AC \cdot BD \le AB \cdot CD + BC \cdot AD$, and the equality holds only for a cyclic quadrilateral.

18. (A.Ivanov) Let AD be a bisector of triangle ABC. Points M and N are the projections of B and C to AD. The circle with diameter MN intersects BC in points X and Y. Prove that $\angle BAX = \angle CAY$.

    Solution. Let B', C', X', Y' be the reflections of B, C, X, Y in MN. Then the diagonals of isosceles trapezoid $BB'CC'$ meet at point L, which is the reflection of A in the circle with diameter MN. The diagonals of isosceles trapezoid $XX'YY'$ inscribed into this circle also meet at L. The lateral sidelines of this trapezoid meet on the polar of L, passing through A and parallel to the bases of the trapezoid. By symmetry A is the common point of the sidelines, which implies the assertion of the problem (fig.18).

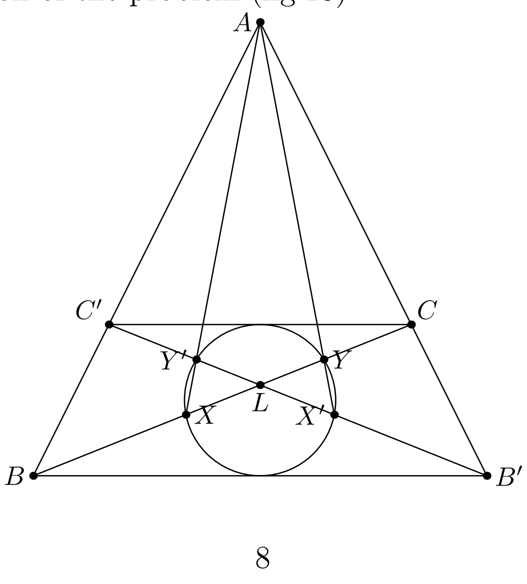


Fig.18

19. (D.Prokopenko) a) The incircle of a triangle ABC touches AC and AB at points B_0 and C_0 respectively. The bisectors of angles B and C meet the perpendicular bisector to the bisector AL in points Q and P respectively. Prove that the lines PC_0, QB_0, and BC concur.

b) Let AL be the bisector of a triangle ABC. Points O_1 and O_2 are the circumcenters of triangles ABL and ACL respectively. Points B_1 and C_1 are the projections of C and B to the bisectors of angles B and C respectively. Prove that the lines O_1C_1, O_1B_1, and BC concur.

c) Prove that two points obtained in pp. a) and b) coincide.

Solution. a) It is clear that $PQ \parallel B_0C_0$. Also P lies on the circumcircle of ACL. Thus $\angle PLA = \angle C/2$ and $\angle PLB = 90^\circ - \angle B/2 = \angle C_0A_0B$, where A_0 is the touching point of the incircle with BC. Therefore the corresponding sidelines of triangles PQL and $C_0B_0A_0$ are parallel i.e., these triangles are homothetic (fig.19a). The homothety center S lies on line LA_0. Thus lines P_0 and QB_0 meet in S, i.e. on line BC.

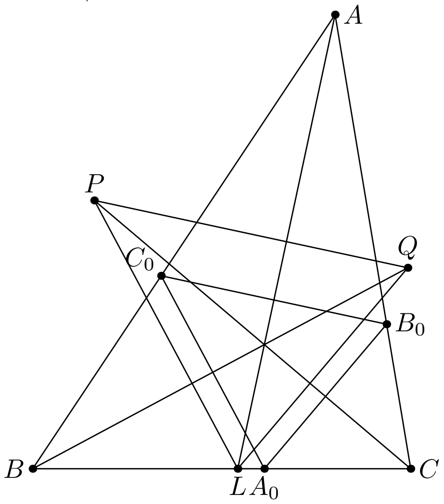


Fig.19a

b) First prove that points C_0, B_0, C_1 and B_1 are collinear. In fact, since the reflection of B in the bisector of angle C lies on AC, point C_1 lies on the medial line $A'C'$. Also we have $A'C_1 = BC/2$, and therefore $C'C_1 = |AC - BC|/2 = C'B_0$. This property defines the common point of $A'C'$ and B_0C_0. Thus lines O_1O_2 and C_1B_1 are parallel. Now quadrilateral BC_1IA_0 is cyclic, therefore $\angle C_1A_0B = 90^\circ - \angle A/2 = \angle O_1LB$ and $A_0C_1 \parallel LO_1$. Similarly $A_0B_1 \parallel LO_2$ (fig.19b). Thus triangles O_1O_2L and $C_1B_1A_0$ are homothetic.

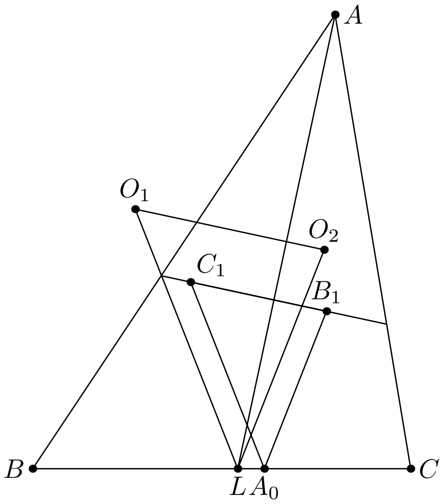


Fig.19b

c) Both homotheties of pp. a) and b) transform A_0 to L, and line B_0C_0 to the medial perpendicular to AL. Therefore their centers coincide.

20. (V.Yassinsky) Let C_1 be an arbitrary point on the side AB of triangle ABC. Points A_1 and B_1 on the rays BC and AC are such that $\angle AC_1B_1 = \angle BC_1A_1 = \angle ACB$. The lines AA_1 and BB_1 meet in point C_2. Prove that all the lines C_1C_2 have a common point.

Solution. By condition, quadrilaterals ACA_1C_1 and BCB_1C_1 are cyclic. Thus $\angle B_1BC_1 = \angle ACC_1$, $\angle A_1AC_1 = \angle BCC_1$, and therefore $\angle AC_2B = \pi - \angle C$, i.e. C_2 lies on the circle passing through A, B and the reflection C' of C wrt AB. Also $\angle BC'C_1 = \angle BAC_2$, thus $C'C_1$ passes through C_2 (fig.20).

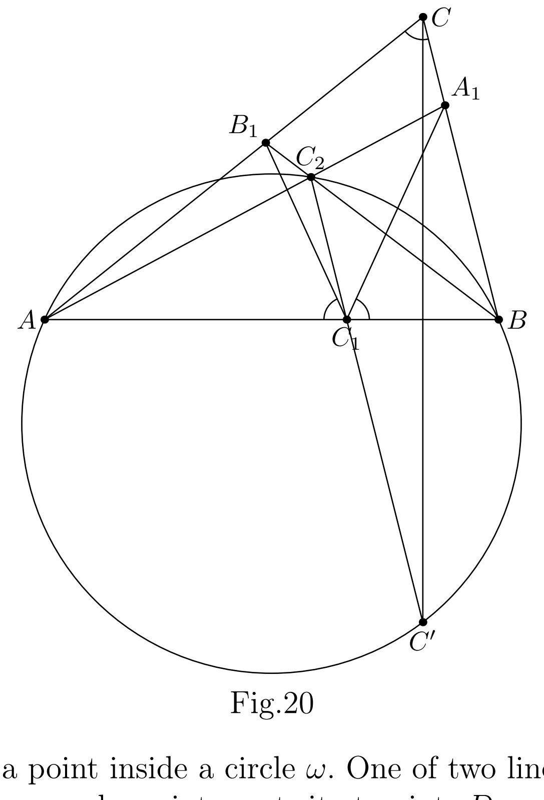


Fig.20

21. (D.Yassinsky) Let A be a point inside a circle ω. One of two lines drawn through A intersects ω at points B and C, the second one intersects it at points D and E (D lies between A and E). The line passing through D and parallel to BC meets ω for the second time at point F, and

the line AF meets ω at point T. Let M be the common point of the lines ET and BC, and N be the reflection of A across M. Prove that the circumcircle of triangle DEN passes through the midpoint of segment BC.

Solution. Firstly, project line AB to the circle from point D, and then project the circle to AB from point T. As a result we obtain that the image of A is M, the image of infinite point is A, and points B and C are fixed. From the equality of cross-ratios we obtain that $MB/MC = (AB/AC)^2$. Hence $AM = AB \cdot AC/(AB + AC)$. Now let K be the midpoint of BC. Then $AN \cdot AK = 2AM(AB + AC)/2 = AB \cdot AC = AD \cdot AE$, i.e. points D, E, K, N are concyclic.

22. (A.Zaslavsky) The common perpendiculars to the opposite sidelines of a nonplanar quadrilateral are mutually orthogonal. Prove that they intersect.

    Solution. Let K, L, M, N be the feet of common perpendiculars lying on the sides AB, BC, CD, DA of quadrilateral $ABCD$. The projection to the plane parallel to KM and LN transforms these lines to perpendicular lines $K'M'$ and $L'N'$. By three perpendiculars theorem the projections of AB and CD are perpendicular to $K'M'$, and the projections of BC and AD are perpendicular to $L'N'$. Therefore the projection of $ABCD$ is a rectangle $A'B'C'D'$, and $A'K' = D'M'$, $B'L' = A'N'$. Thus $AK/KB = DM/MC$, $BL/LC = AN/ND$ and by Menelaos theorem K, L, M, N are complanar.

23. (B.Frenkin) Two convex polytopes A and B do not intersect. The polytope A has exactly 2012 planes of symmetry. What is the maximal number of symmetry planes of the union of A and B, if B has a) 2012, b) 2013 symmetry planes?

    c) What is the answer to the question of p.b), if the symmetry planes are replaced by the symmetry axes?

    Answer. a) 2013. b) 2012. c) 1.

    Solution. a) *Estimation.* The symmetry transposes polyhedrons A and B or fixes each of them. In the first case it transposes the centroids of polyhedrons, thus the symmetry plane is the perpendicular bisector of the segment between the centroids. In the second case this plane is a symmetry plane of both polyhedrons A and B. Thus we have at most 1+2012=2013 planes. *Example.* Let A be regular 2012-gonal pyramid. Take a point outside A on its axis and construct a plane P passing through this point and perpendicular to the axis. Let B the reflection of A in P. Then all conditions are valid, and P and 2012 symmetry planes of A are the symmetry planes of the union.

    b) *Estimation.* Since A and B have a distinct number of symmetry planes, they aren't equal and can't be transposed by a symmetry. Thus each symmetry is a symmetry of polyhedron A, which has only 2012 symmetry planes. *Example.* Let A be a regular 2012-gonal pyramid. Take a point outside A on its axis, a plane passing through this point and perpendicular to the axis, and construct the reflection of the pyramid's base in this plane. Let B be a prism with this reflection as the base, disjoint from A. It is clear that B has 2013 symmetry planes: one of them is parallel to the bases of the prism and equidistant from them and 2012 remaining planes coincide with the symmetry planes of A.

    c) *Estimation.* Since A and B have a distinct number of symmetry axes they can't be transposed. Thus the sought symmetry fixes the centroid of each polyhedron. These centroids don't coincide because the polyhedrons are convex. Therefore the symmetry axis coincides with the line joining two centroids. *Example.* Let A be a regular 2011-gonal prism with horizontal bases. Then A has one vertical and 2011 horizontal symmetry axes. Now let B be a regular 2012-gonal prism with the same axis, disjoint from A. Then B has 2013 symmetry axes and the union of A and B has a vertical symmetry axis.

X GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
THE CORRESPONDENCE ROUND

Below is the list of problems for the first (correspondence) round of the X Sharygin Geometrical Olympiad.

The olympiad is intended for high-school students of four elder grades. In Russian school, these are 8-11. In the list below, each problem is indicated by the numbers of Russian school grades, for which it is intended. Foreign students of the last grade have to solve the problems for 11th grade, students of the preceding grade solve the problems for 10th grade etc. However, the participants may solve problems for elder grades as well (solutions of problems for younger grades will not be considered).

A complete solution of each problem or of each its item if there are any, costs 7 points. An incomplete solution costs from 1 to 6 points according to the extent of advancement. If no significant advancement was achieved, the mark is 0. The result of a participant is the total sum of marks for all problems.

In your work, please start the solution for each problem in a new page. First write down the statement of the problem, and then the solution. Present your solutions in detail, including all significant arguments and calculations. Provide all necessary figures of sufficient size. If a problem has an explicit answer, this answer must be presented distinctly. Please, be accurate to provide good understanding and correct estimating of your work !

If your solution depends on some well-known theorems from standard textbooks, you may simply refer to them instead of providing their proofs. However, any fact not from the standard curriculum should be either proved or properly referred (with an indication of the source).

You may note the problems which you liked most (this is not obligatory). Your opinion is interesting for the Jury.

The solutions for the problems (in Russian or in English) must be delivered up to April 1, 2014. For this, please apply **since January 2, 2014** to **http://olimpsharygin.olimpiada.ru** and follow the instructions given there. **Attention:** the solution of each problem must be contained in a separate pdf, doc or jpg file. We recommend to prepare the paper using a computer or to scan it rather than to photograph it. In the last two cases, please check readability of the file obtained.

If you have some technical problem, please contact us by e-mail **geomolymp@mccme.ru**.

It is also possible to send the solutions by e-mail to **geompapers@yandex.ru**. In this case, please follow a few simple rules:

1. Each student sends his work in a separate message (with delivery notification). The size of the message must not exceed 10 Mb.

2. If your work consists of several files, send it as an archive.

3. If the size of your message exceeds 10 Mb, divide it into several messages.

4. In the subject of the message write "The work for Sharygin olympiad", and present the following personal data in the body of your message:

- last name;
- all other names;
- E-mail, phone number, post address;
- the current number of your grade at school;
- the last grade at your high school;
- the number of the last grade in your school system;
- the number and/or the name and the mail address of your school;

- full names of your teachers in mathematics at school and/or of instructors of your extra math classes (if you attend additional math classes after school).

If you have no possibility to deliver the work in electronic form, please apply to the Organizing Committee to find a specific solution for this case.

Winners of the correspondence round, the students of three grades before the last grade, will be invited to the final round in Summer 2014 in the city of Dubna, in Moscow region. (For instance, if the last grade is 12, then we invite winners from 9, 10, and 11 grade.) Winners of the correspondence round, the students of the last grade, will be awarded with diplomas of the Olympiad. The list of the winners will be published on **www.geometry.ru** at the end of May 2014. If you want to know your detailed results, please contact us by e-mail **geomolymp@mccme.ru**.

1. (8) A right-angled triangle ABC is given. Its cathetus AB is the base of a regular triangle ADB lying in the exterior of ABC, and its hypothenuse AC is the base of a regular triangle AEC lying in the interior of ABC. Lines DE and AB meet at point M. The whole configuration except points A and B was erased. Restore the point M.

2. (8) A paper square with sidelength 2 is given. From this square, can we cut out a 12-gon having all sidelengths equal to 1, and all angles divisible by $45°$?

3. (8) Let ABC be an isosceles triangle with base AB. Line ℓ touches its circumcircle at point B. Let CD be a perpendicular from C to ℓ, and AE, BF be the altitudes of ABC. Prove that D, E, F are collinear.

4. (8) A square is inscribed into a triangle (one side of the triangle contains two vertices and each of two remaining sides contains one vertex). Prove that the incenter of the triangle lies inside the square.

5. (8) In an acute-angled triangle ABC, AM is a median, AL is a bisector and AH is an altitude (H lies between L and B). It is known that $ML = LH = HB$. Find the ratios of the sidelengths of ABC.

6. (8–9) Given a circle with center O and a point P not lying on it. Let X be an arbitrary point of this circle, and Y be a common point of the bisector of angle POX and the perpendicular bisector to segment PX. Find the locus of points Y.

7. (8–9) A parallelogram $ABCD$ is given. The perpendicular from C to CD meets the perpendicular from A to BD at point F, and the perpendicular from B to AB meets the perpendicular bisector to AC at point E. Find the ratio in which side BC divides segment EF.

8. (8–9) Given a rectangle $ABCD$. Two perpendicular lines pass through point B. One of them meets segment AD at point K, and the second one meets the extension of side CD at point L. Let F be the common point of KL and AC. Prove that $BF \perp KL$.

9. (8–9) Two circles ω_1 and ω_2 touching externally at point L are inscribed into angle BAC. Circle ω_1 touches ray AB at point E, and circle ω_2 touches ray AC at point M. Line EL meets ω_2 for the second time at point Q. Prove that $MQ \parallel AL$.

10. (8–9) Two disjoint circles ω_1 and ω_2 are inscribed into an angle. Consider all pairs of parallel lines l_1 and l_2 such that l_1 touches ω_1, and l_2 touches ω_2 (ω_1, ω_2 lie between l_1 and l_2). Prove that the medial lines of all trapezoids formed by l_1, l_2 and the sides of the angle touch some fixed circle.

11. (8–9) Points K, L, M and N lying on the sides AB, BC, CD and DA of a square $ABCD$ are vertices of another square. Lines DK and NM meet at point E, and lines KC and LM meet at point F. Prove that $EF \parallel AB$.

12. (9–10) Circles ω_1 and ω_2 meet at points A and B. Let points K_1 and K_2 of ω_1 and ω_2 respectively be such that K_1A touches ω_2, and K_2A touches ω_1. The circumcircle of triangle K_1BK_2 meets lines AK_1 and AK_2 for the second time at points L_1 and L_2 respectively. Prove that L_1 and L_2 are equidistant from line AB.

13. (9–10) Let AC be a fixed chord of a circle ω with center O. Point B moves along the arc AC. A fixed point P lies on AC. The line passing through P and parallel to AO meets BA at point A_1; the line passing through P and parallel to CO meets BC at point C_1. Prove that the circumcenter of triangle A_1BC_1 moves along a straight line.

14. (9–11) In a given disc, construct a subset such that its area equals the half of the disc area and its intersection with its reflection over an arbitrary diameter has the area equal to the quarter of the disc area.

15. (9–11) Let ABC be a non-isosceles triangle. The altitude from A, the bisector from B and the median from C concur at point K.

    а) Which of the sidelengths of the triangle is medial (intermediate in length)?

    б) Which of the lengths of segments AK, BK, CK is medial (intermediate in length)?

16. (9–11)Given a triangle ABC and an arbitrary point D.The lines passing through D and perpendicular to segments DA, DB, DC meet lines BC, AC, AB at points A_1, B_1, C_1 respectively. Prove that the midpoints of segments AA_1, BB_1, CC_1 are collinear.

17. (10–11) Let AC be the hypothenuse of a right-angled triangle ABC. The bisector BD is given, and the midpoints E and F of the arcs BD of the circumcircles of triangles ADB and CDB respectively are marked (the circles are erased). Construct the centers of these circles using only a ruler.

18. (10–11) Let I be the incenter of a circumscribed quadrilateral ABCD. The tangents to circle AIC at points A, C meet at point X. The tangents to circle BID at points B, D meet at point Y. Prove that X, I, Y are collinear.

19. (10–11) Two circles ω_1 and ω_2 touch externally at point P. Let A be a point of ω_2 not lying on the line through the centers of the circles, and AB, AC be the tangents to ω_1. Lines BP, CP meet ω_2 for the second time at points E and F. Prove that line EF, the tangent to ω_2 at point A and the common tangent at P concur.

20. (10–11) A quadrilateral $KLMN$ is given. A circle with center O meets its side KL at points A and A_1, side LM at points B and B_1, etc. Prove that if the circumcircles of triangles KDA, LAB, MBC and NCD concur at point P, then

a) the circumcircles of triangles KD_1A_1, LA_1B_1, MB_1C_1 and NC_1D_1 also concur at some point Q;

b) point O lies on the perpendicular bisector to PQ.

21. (10–11) Let $ABCD$ be a circumscribed quadrilateral. Its incircle ω touches sides BC and DA at points E and F respectively. It is known that lines AB, FE and CD concur. The circumcircles of triangles AED and BFC meet ω for the second time at points E_1 and F_1. Prove that $EF \parallel E_1F_1$.

22. (10–11) Does there exist a convex polyhedron such that it has diagonals and each of them is shorter than each of its edges?

23. (11) Let A, B, C and D be a triharmonic quadruple of points, i.e

$$AB \cdot CD = AC \cdot BD = AD \cdot BC.$$

Let A_1 be a point distinct from A such that the quadruple A_1, B, C and D is triharmonic. Points B_1, C_1 and D_1 are defined similarly. Prove that

a) A, B, C_1, D_1 are concyclic;

b) the quadruple A_1, B_1, C_1, D_1 is triharmonic.

24. (11) A circumscribed pyramid $ABCDS$ is given. The opposite sidelines of its base meet at points P and Q in such a way that A and B lie on segments PD and PC respectively. The inscribed sphere touches faces ABS and BCS at points K and L. Prove that if PK and QL are complanar then the touching point of the sphere with the base lies on BD.

X GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
THE CORRESPONDENCE ROUND. SOLUTIONS

1. (N.Moskvitin, V.Protasov) (8) A right-angled triangle ABC is given. Its cathetus AB is the base of a regular triangle ADB lying in the exterior of ABC, and its hypothenuse AC is the base of a regular triangle AEC lying in the interior of ABC. Lines DE and AB meet at point M. The whole configuration except points A and B was erased. Restore the point M.

   Solution. Since $\angle DAB = \angle EAC = 60°$, we have $\angle DAE = \angle BAC$, therefore triangles ADE and ABC are equal and $\angle ADE = 90°$. Thus triangleADM is right-angled with $\angle A = 60°$. Hence $AD = AB = AM/2$ (fig.1), i.e. M is the reflection of A with respect to B.

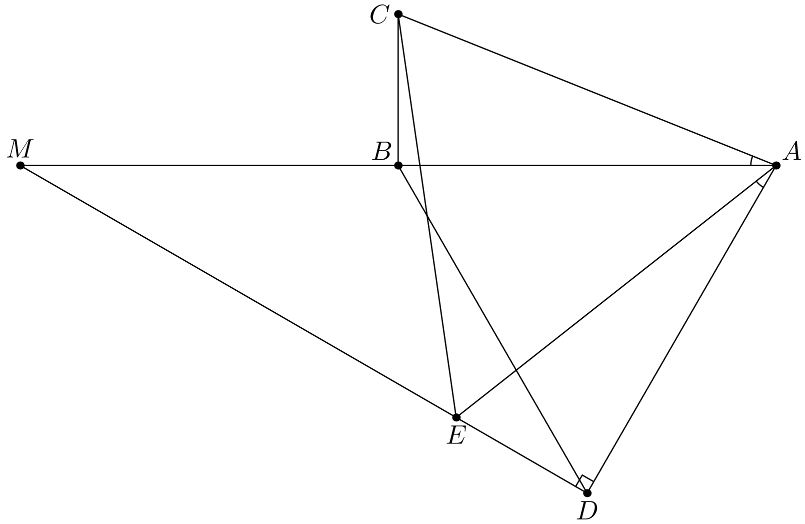


Fig.1

2. (K.Knop) (8) A paper square with sidelength 2 is given. From this square, can we cut out a 12-gon having all sidelengths equal to 1, and all angles divisible by 45°?

   Solution. Yes, see. fig.2. Points A, B, C, D lying on the medial lines of the given square are the vertices of the square with the side equal to $\sqrt{2} - 1$.

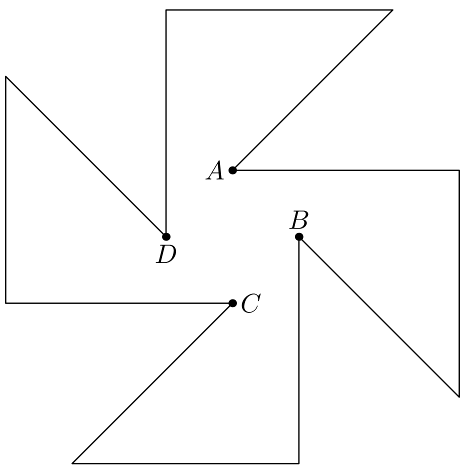


Fig.2

3. (N.Moskvitin) (8) Let ABC be an isosceles triangle with base AB. Line ℓ touches its

circumcircle at point B. Let CD be a perpendicular from C to ℓ, and AE, BF be the altitudes of ABC. Prove that D, E, F are collinear.

Solution. Let CH be the third altitude of the triangle. Since $\angle CBD = \angle CAB = \angle CBH$, the triangles CBD and CBH are equal, i.e. $BD = BH$. Also EH is the median of right-angled triangle AEB, thus $EH = HB = BD$ and $\angle BEH = \angle EBH = \angle EBD$. Therefore $EDBH$ is a parallelogram (fig.3) and $DE \parallel AB$. Since EF also is parallel to AB, lines DE and EF coincide.

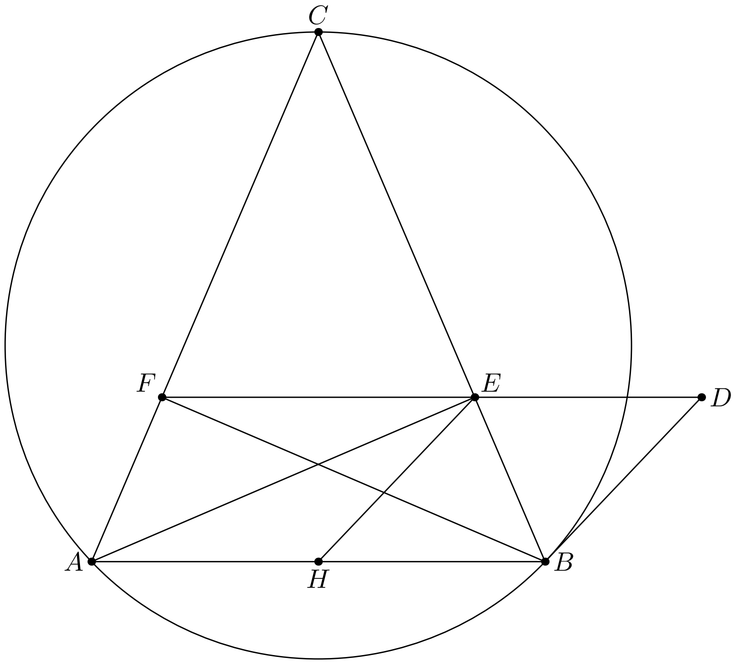


Fig.3

4. (B.Frenkin) (8) A square is inscribed into a triangle (one side of the triangle contains two vertices and each of two remaining sides contains one vertex). Prove that the incenter of the triangle lies inside the square.

   Solution. Let ABC be a triangle with incenter I, let vertices K and L of inscribed square lie on side AB, vertex M lie on AC and vertex N lie on BC (obviously angles A and B are equal). Take a perpendicular IH from I to AB and a segment DE passing through I parallel to AB with endpoints D and E lying on AC and BC respectively. We have to prove that $DE > IH$ and $H \in KL$. The first assertion is true because $IH = r$ and $DE > 2r$, where r is the radius of the incircle. Now let the extension of IH beyond I meet one of the sides of ABC at point F. We can suppose that $F \in AC$. Then H and K lie on the same side of L. Take a line passing through F, parallel to AB and intersecting BC at point G. It is sufficient to prove that $FG < FH$: then I and L lie on the same side of K and $I \in KL$.

   Note that FH contains a diameter of the incircle, thus F lies outside the incircle and $FH > 2r$. The perpendicular through F doesn't intersect the incircle. Therefore the touching points of AC and BC with the incircle lie between FG and AB. Hence the corresponding chord is greater than FG. Since it is less than $2r$, we have $FG < 2r < FH$, q.e.d.

   Comment. We see from the solution that a square can be replaced by a rectangle such that its greater side lies on the base of the triangle and is not greater than doubled smaller side.

5. (B.Frenkin) (8) In an acute-angled triangle ABC, AM is a median, AL is a bisector and AH is an altitude (H lies between L and B). It is known that $ML = LH = HB$. Find the ratios of the sidelengths of ABC.

   Answer. $AB : AC : BC = 1 : 2 : \frac{3\sqrt{2}}{2}$.

   Solution. By the property of the bisector $AC : AB = LA : LB = 2 : 1$. This follows also from the property of the median: take on the extension of AB beyond B segment $BD = AB$. Then BC is a median of the triangle ADC, and since $AL : LB = 2 : 1$, we obtain that AL also lies on a median. But AL is the bisector, therefore $AC = AD = 2AB$. Now by the Pythagor theorem we have: $AC^2 - AH^2 = AB^2 - BH^2$, or $4AB^2 - 25BH^2 = AB^2 - BH^2$, thus $AB = 2\sqrt{2}BH$ and $BC : AB = 6BH : AB = \frac{3\sqrt{2}}{2}$.

6. (A.Zaslavsky) (8–9) Given a circle with center O and a point P not lying on it. Let X be an arbitrary point of this circle, and Y be a common point of the bisector of angle POX and the perpendicular bisector to segment PX. Find the locus of points Y.

   Answer. The line perpendicular to ray OP and meeting it at the point on the distance from O equal to $(OP + OX)/2$.

   Solution. Let K, L be the projections of Y to OP and OX. By the definition of Y we have $YP = YX$ and $YK = YL$. Thus triangles YKP and YLX are equal i.e. $XL = PK$. Also $OL = OK$. Since the lengths of segments OP and OX are not equal, one of them is equal to the sum of OK and KP, and the second one is equal to their difference. Therefore $OK = (OP + OX)/2$ (fig.6). It is evident that the sought locus contains all points of the line.

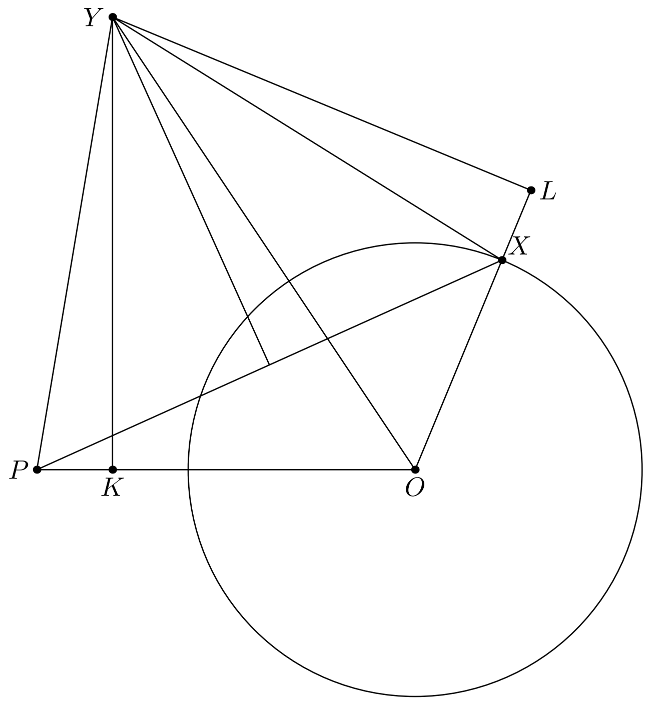


Fig.6

7. (V.Rumyantsev) (8–9) A parallelogram $ABCD$ is given. The perpendicular from C to CD meets the perpendicular from A to BD at point F, and the perpendicular from B

to AB meets the perpendicular bisector to AC at point E. Find the ratio in which side BC divides segment EF.

Answer. 1:2.

Solution. Let K be the reflection of A wrt B. Then E is the circumcenter of triangle ACK. On the other hand, since $BKCD$ is a parallelogram, we have $AF \perp CK$ and F is the orthocenter of triangle ACK. Therefore the median CB divides EF in ratio 1:2 (fig.7).

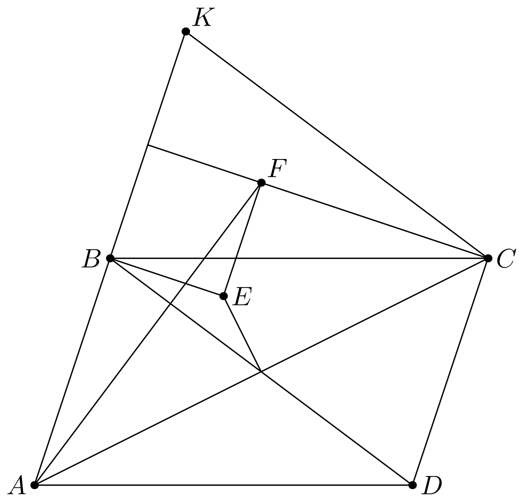


Fig.7

8. (R.Sadykov) (8–9) Given a rectangle $ABCD$. Two perpendicular lines pass through point B. One of them meets segment AD at point K, and the second one meets the extension of side CD at point L. Let F be the common point of KL and AC. Prove that $BF \perp KL$.

First solution. Since $\angle ABK = \angle CBL$, triangles ABK and CBL are similar. Thus triangles ABC and KBL are also similar and $\angle BKF = \angle BAF$. Therefore quadrilateral $ABFK$ is cyclic and $\angle BFK = 90^\circ$ (fig.8).

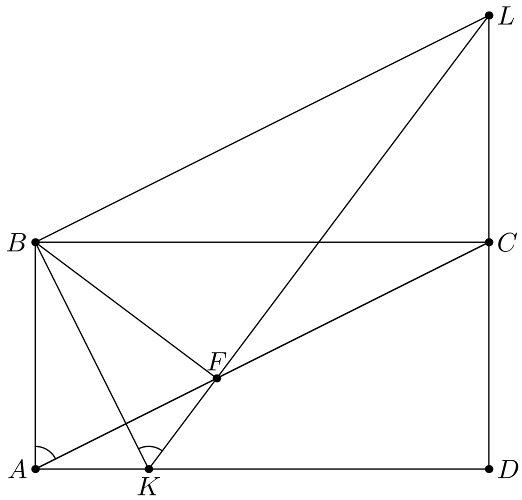


Fig.8

Second solution. Note that point B lies on the circumcircle of triangle KLD. Points A and C are the projections of B to lines KD and DL. Thus by the Simson theorem the projection of B to KL lies on AC, i.e, coincide with F q.e.d.

9. (D.Shvetsov) (8–9) Two circles ω_1 and ω_2 touching externally at point L are inscribed into angle BAC. Circle ω_1 touches ray AB at point E, and circle ω_2 touches ray AC at point M. Line EL meets ω_2 for the second time at point Q. Prove that $MQ \parallel AL$.

Solution. Let N be the second common point of ω_1 and AL (fig.9). Then the composition of the reflection in AL and the homothety with center A transforms arc NE to arc LM. Therefore angles ALE and MQE are equal, which yields the assertion of the problem.

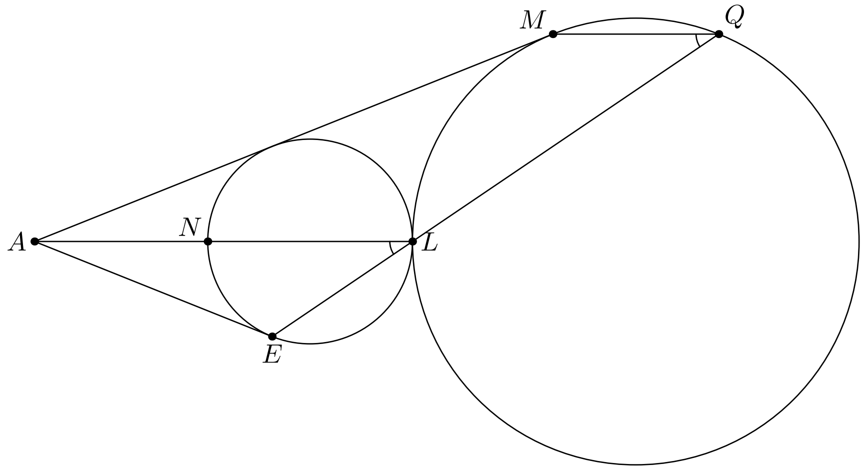


Fig.9

10. (M.Kungozhin) (8–9) Two disjoint circles ω_1 and ω_2 are inscribed into an angle. Consider all pairs of parallel lines l_1 and l_2 such that l_1 touches ω_1, and l_2 touches ω_2 (ω_1, ω_2 lie between l_1 and l_2). Prove that the medial lines of all trapezoids formed by l_1, l_2 and the sides of the angle touch some fixed circle.

Solution. Let O_1, O_2 be the centers of the given circles, r_1, r_2 be their radii, O be the midpoint of O_1O_2, l_1' be the line parallel to l_1 and passing through O_1, l_2' be the reflection of l_1' in the medial line (fig.10). Then the distance from O_2 to l_2' is equal to $|r_2 - r_1|$. Using the homothety with center O_1 and coefficient $1/2$, we obtain that the distance d from O to the medial line is equal to $|r_2 - r_1|/2$, i.e. all medial lines touch the circle with center O and radius d.

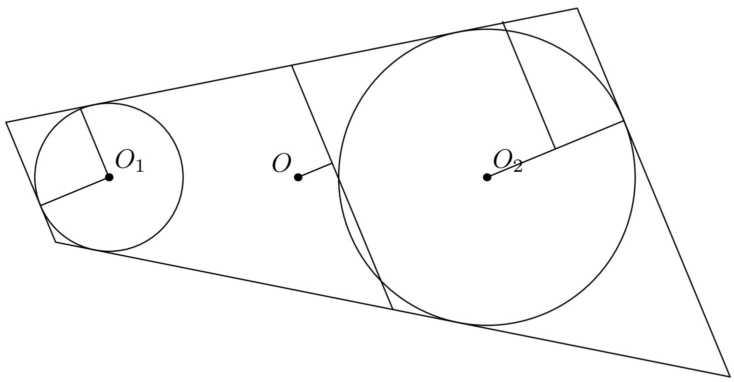


Fig.10

11. (M.Plotnikov) (8–9) Points K, L, M and N lying on the sides AB, BC, CD and DA of a square $ABCD$ are vertices of another square. Lines DK and NM meet at point E, and lines KC and LM meet at point F. Prove that $EF \parallel AB$.

Solution. Denote the common points of lines MN and LM with AB as P and Q respectively. Triangles AKN, BLK, CML and DMN are equal by the hypothenuse and the acute angle. Let $AK = a$ and $BK = b$, then $BL = CM = DN = a$, $CL = MD = NA = b$. Since triangles PKN and QLK are right-angled, we have $PA \cdot a = b^2$ and $BK \cdot b = a^2$. The similarity of triangles PEK and DEM implies that $KE/DE = (a + b^2/a)/b = (a^2 + b^2)/ab$, but the similarity of QFK и CFM implies that $FK/CF = (b + a^2/b)/a = (a^2 + b^2)/ab$. Thus $KE/DE = FK/CF$ and $EF \parallel AB$, q.e.d.

12. (I.Makarov) (9–10) Circles ω_1 and ω_2 meet at points A and B. Let points K_1 and K_2 of ω_1 and ω_2 respectively be such that K_1A touches ω_2, and K_2A touches ω_1. The circumcircle of triangle K_1BK_2 meets lines AK_1 and AK_2 for the second time at points L_1 and L_2 respectively. Prove that L_1 and L_2 are equidistant from line AB.

Solution. Since $\angle K_1AB = \angle AK_2B$, $\angle K_2AB = \angle AK_1B$, triangles AK_1B and K_2AB are similar (fig.12). Using the sinus theorem we obtain:

$$\frac{\sin \angle K_1AB}{\sin \angle K_2AB} = \frac{AK_1}{AK_2} = \frac{AL_2}{AL_1},$$

which is equivalent to the assertion of the problem.

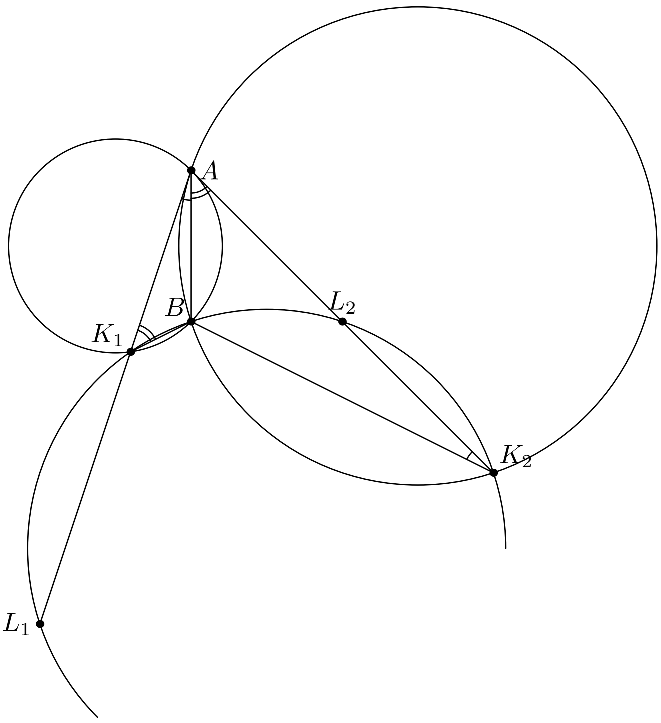


Fig.12

13. (D.Prokopenko, D.Shvetsov) (9–10) Let AC be a fixed chord of a circle ω with center O. Point B moves along the arc AC. A fixed point P lies on AC. The line passing through P and parallel to AO meets BA at point A_1; the line passing through P and parallel to CO meets BC at point C_1. Prove that the circumcenter of triangle A_1BC_1 moves along a straight line.

Solution. Let Q be the second common point of line AC and circle A_1PC_1. Then $\angle QA_1C_1 = \angle QPC_1 = \angle QCO = \angle QAO = \angle APA_1 = \angle QCA_1$. Therefore $QA_1 = QC_1$ and $\angle A_1QC_1 = \angle AOC = 2\angle A_1BC_1$, i.e. Q is the circumcenter of triangle A_1BC_1 (fig.13). Thus this circumcenter moves along line AC.

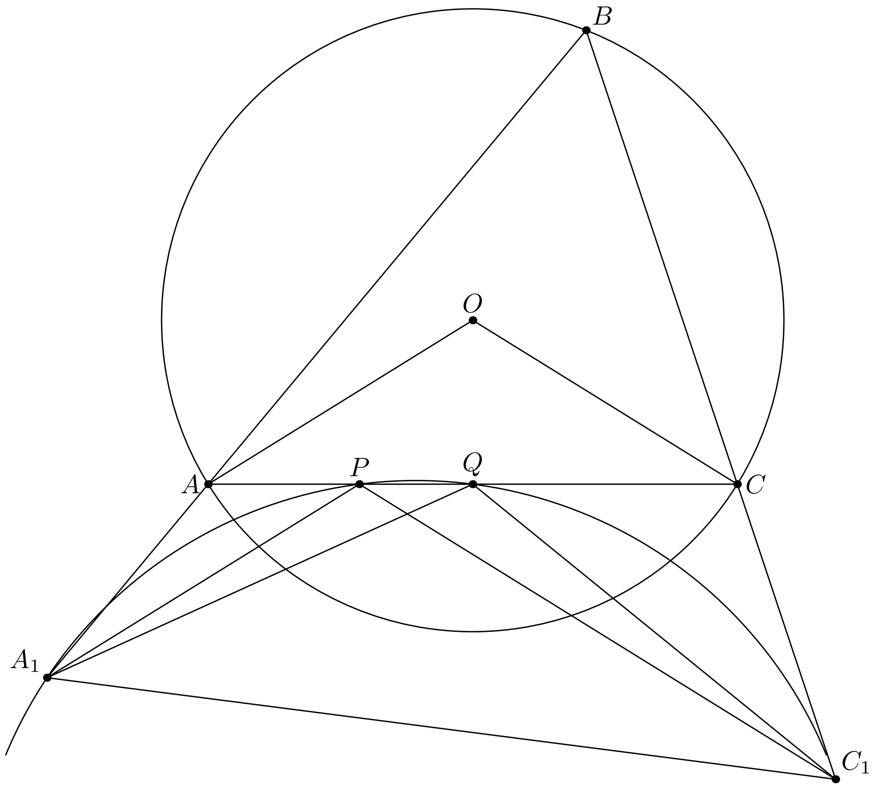


Fig.13

14. (Folklore) (9–11) In a given disc, construct a subset such that its area equals the half of the disc area and its intersection with its reflection over an arbitrary diameter has the area equal to the quarter of the disc area.

    Solution. Construct a disc, concentric to the given disc with the area equal to the half of the area of the given disc. Bisect the inner disc by an arbitrary diameter and bisect the external ring by the perpendicular diameter. Joining the half of the inner disc with the half of the ring we obtain the sought subset.

15. (9–11) Let ABC be a non-isosceles triangle. The altitude from A, the bisector from B and the median from C concur at point K.

    a) (B.Frenkin) Which of the sidelengths of the triangle is medial?

    b) (A.Zaslavsky) Which of the lengths of segments AK, BK, CK is medial?

    Answer. a) AC. b) BK.

    Solution. a) A bisector of a non-isosceles triangle lies between the corresponding altitude and median, and its altitude lies between the bisector and the smaller of two adjacent sides. Suppose that $AB < AC$. Then the bisector of angle A meets the bisector of angle B at a point, lying between K and AC. The bisector of angle C also passes through this point. Since it lies between the median and the smaller of two adjacent sides, we obtain that $AC < BC$. Thus AC is the medial side.

    Let now $AB > AC$. Then similarly the incenter lies between AK and AB, the bisector of angle C lies between CK and BC, thus $AC > BC$. Again AC is the medial side.

    b) By the condition the altitude from A lies inside the triangle, i.e. angles B and C are acute. Using the Ceva theorem we obtain that $\sin A = \cos C \mathrm{tgB}$ or $\mathrm{tgC} = \mathrm{tgB}\frac{1-\cos \mathrm{B}}{\cos \mathrm{B}}$.

Thus if $B < 60°$ then $C < B < 60° < A$, and if $B > 60°$ then $C > B > 60° > A$.

In the first case $\angle KBA < 30° < \angle KAB$ and $\angle KCB < C/2 < B/2 = \angle KBC$, therefore $KA < KB < KC$. Similarly in the second case we have $KA > KB > KC$.

Comment. Using the condition of the problem we can't define which of the sides is the greatest (the smallest), and which of segments KA, KC is the greatest (the smallest).

16. (D.Prokopenko) (9–11)Given a triangle ABC and an arbitrary point D.The lines passing through D and perpendicular to segments DA, DB, DC meet lines BC, AC, AB at points A_1, B_1, C_1 respectively. Prove that the midpoints of segments AA_1, BB_1, CC_1 are collinear.

    Solution. The circles with diameters AA_1, BB_1, CC_1 pass through the bases of the correspondent altitudes, thus the degrees of orthocenter H wrt these three circles are equal. Therefore line DH is their common radical axis and their centers are collinear.

    Comment. Applying the Menelaos theorem to triangle ABC and its medial triangle we can obtain that A_1, B_1, C_1 are also collinear.

17. (N.Moskvitin) (10–11) Let AC be the hypothenuse of a right-angled triangle ABC. The bisector BD is given, and the midpoints E and F of the arcs BD of the circumcircles of triangles ADB and CDB respectively are marked (the circles are erased). Construct the centers of these circles using only a ruler.

    Solution. We will use following well-known facts.

    1.) If two parallel lines are given then we can bisect a segment lying on one of them, using only a ruler.

    2.) If two parallel lines are given then we can construct a line parallel to them and passing through a fixed point not lying on these lines, using only a ruler.

    Note now that EF is the perpendicular bisector to BD. Thus its common points K, L with AB and BC are the vertices of a square $BKDL$. Using parallel lines BC and KD bisect segment BC. Using parallel lines AB and DL construct the line parallel to them trough the midpoint of BC. This line is the perpendicular bisector of BC, therefore it meets EF at the circumcenter of triangle BCD. The circumcenter of triangle ABD can be constructed similarly.

18. (A.Zaslavsky) (10–11) Let I be the incenter of a circumscribed quadrilateral ABCD. The tangents to circle AIC at points A, C meet at point X. The tangents to circle BID at points B, D meet at point Y. Prove that X, I, Y are collinear.

    Solution. Let J be the second common point of circles AIC and BID. The inversion wrt the incircle of $ABCD$ transforms A, B, C, D to the vertices of a parallelogram, also it transforms J to the center of this parallelogram. Therefore $AJ/CJ = AI/CI$, i.e line IJ is the symedian of triangle AIC, thus this line passes through X. Similarly it passes through Y.

19. (V.Yassinsky) (10–11) Two circles ω_1 and ω_2 touch externally at point P. Let A be a point of ω_2 not lying on the line through the centers of the circles, and AB, AC be the tangents to ω_1. Lines BP, CP meet ω_2 for the second time at points E and F. Prove that line EF, the tangent to ω_2 at point A and the common tangent at P concur.

Solution. The homothety with center P transforms B, C to E, F. Thus it transforms A to the pole of line EF wrt ω_2, i.e. the pole of EF lies on AP, which is equivalent to the assertion of the problem.

20. (N.Beluhov) (10–11) A quadrilateral $KLMN$ is given. A circle with center O meets its side KL at points A and A_1, side LM at points B and B_1, etc. Prove that if the circumcircles of triangles KDA, LAB, MBC and NCD concur at point P, then

 a) the circumcircles of triangles KD_1A_1, LA_1B_1, MB_1C_1 and NC_1D_1 also concur at some point Q;

 b) point O lies on the perpendicular bisector to PQ.

 Solution. Let $A_1'B_1'$ be a variable chord in the circle, equal to A_1B_1, i.e. obtained from A_1B_1 by rotation with center O. Easy computation of angles shows that the circle (LAB) is in fact the locus of the intersection $K' = AA_1' \cap BB_1'$ as $A_1'B_1'$ moves around the circle. Thus, since P is the intersection of four such loci, the lines AP, BP, CP and DP must intersect the circle in four points A', B', C', D', forming a quadrilateral equal to $A_1B_1C_1D_1$. Consider the rotation with center O, sending $A'B'C'D'$ to $A_1B_1C_1D_1$, and let it send P to some point Q. Then the lines A_1Q, B_1Q, C_1Q and D_1Q will intersect the circle in four points, forming a quadrilateral, equal to $ABCD$. The same loci argument, applied to the circumcircles of $\triangle KD_1A_1, \triangle LA_1B_1, \triangle MB_1C_1$ and $\triangle NC_1D_1$, shows that they are concurrent in Q. Also, since OQ is the image of OP under the rotation, we have $OP = OQ$, and (b) also follows.

21. (N.Poljansky, D.Skrobot) (10–11) Let $ABCD$ be a circumscribed quadrilateral. Its incircle ω touches sides BC and DA at points E and F respectively. It is known that lines AB, FE and CD concur. The circumcircles of triangles AED and BFC meet ω for the second time at points E_1 and F_1. Prove that $EF \parallel E_1F_1$.

 Solution. Let R be the common point of BC and AD. Then R and the touching points P и Q of the incircle with two remaining sides are collinear.

 Let EE_1 meet AD at point M. Consider three circles: the incircle of $ABCD$, AED and AID, where I is the incenter of $ABCD$. It is clear that the radical axis of AID and the incircle is the medial line of triangle FPQ. Since two remaining radical axes meet at M we obtain that $RM = MF$ (fig.21).

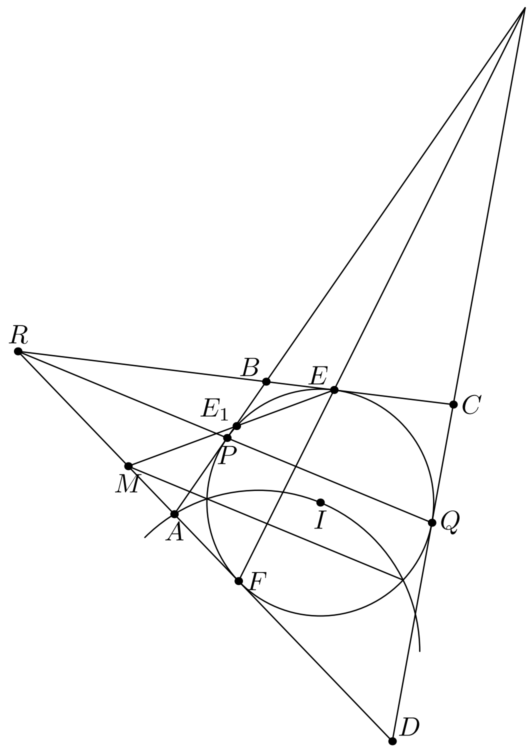


Fig.21

Similarly, FF_1 meets BC at point N, such that $RN = NE$. Therefore lines EE_1 and FF_1 are symmetric wrt the bisector of angle ERF. Thus points E_1 and F_1 are also symmetric and EFF_1E_1 is an isosceles trapezoid.

22. (A.Blinkov) (10–11) Does there exist a convex polyhedron such that it has diagonals and each of them is shorter than each of its edges?

    Solution. Yes, take a regular triangle ABC with side equal to 1 and two points S_1, S_2, symmetric wrt its plane and such that $S_1S_2 < S_1A = S_1B = S_1C < 1$. It is evident that the unique diagonal S_1S_2 of the obtained polyhedron is shorter than each of its edges.

23. (A.Akopyan) (11) Let A, B, C and D be a triharmonic quadruple of points, i.e

$$AB \cdot CD = AC \cdot BD = AD \cdot BC.$$

Let A_1 be a point distinct from A such that the quadruple A_1, B, C and D is triharmonic. Points B_1, C_1 and D_1 are defined similarly. Prove that

a) A, B, C_1, D_1 are concyclic;

b) the quadruple A_1, B_1, C_1, D_1 is triharmonic.

Solution. a) Consider three spheres touching the given plane at points A, B, C and externally touching each other. If the radii of these spheres are equal to x, y, z, then $AB = 2\sqrt{xy}$ etc. Thus there exist two spheres touching the plane at points D and D_1 and touching three given spheres. Therefore we can construct eight spheres a, b, c, d, a_1, b_1, c_1, d_1, touching the plane at A, B, C, D, A_1, B_1, C_1, D_1, and such that a and a_1 touch b, c, d etc.

Take an inversion of the space with the center at the touching point of c and d. It transforms these two spheres to two parallel planes, and the given plane, a and b will be transformed to three equal mutually touching spheres lying between these two planes. The images of c_1 and d_1 have to touch these three spheres, also each of these two spheres touches one of the planes, therefore they are symmetric wrt the plane containing the centers of three remaining spheres. Thus the images of A, B, C_1, D_1 are complanar and these points are concyclic.

b) Consider now an inversion with center D. It transforms d to the plane parallel to ABC, and the images of a, b, c are three equal mutually touching spheres. Therefore their touching points with the plane are the vertices of a regular triangle, and the image of D_1 is the center of this triangle. The images of A_1, B_1, C_1 are the vertices of a regular triangle with the same center, i.e. quadruple A_1, B_1, C_1, D_1 is triharmonic.

24. (F.Nilov) (11) A circumscribed pyramid $ABCDS$ is given. The opposite sidelines of its base meet at points P and Q in such a way that A and B lie on segments PD and PC respectively. The inscribed sphere touches faces ABS and BCS at points K and L. Prove that if PK and QL are complanar then the touching point of the sphere with the base lies on BD.

    First solution. Since P, Q, K and L are complanar, segments PL and QP meet at point R lying on BS. Let T be the touching point of the insphere with the base of the pyramid. Note that triangles QBK and QB are equal and triangles PBL and PBT are equal (by the equality of the correspondent tangents). Similarly triangles RKB and RLB are equal. Thus $\angle QTB = \angle QKB = \angle PLB = \angle PTB$. But in the circumscribed pyramid $\angle CTQ = \angle PTA$ and $\angle CTD + \angle ATB = 180^{circ}$, therefore $\angle PTB = 180°$.

    Second solution. Consider a projective map saving the insphere and transforming PQS to the infinite plane. It transforms the pyramid to the infinite prism, and by the complanarity of PK and QL we obtain that the facets of this prism passing through AB and BC form equal angles with plane $ABCD$. Thus the prism is symmetric wrt the plane passing through BD and perpendicular to $ABCD$. It is clear that the touching point of the insphere with the base lies on the plane of symmetry.

XI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
THE CORRESPONDENCE ROUND

Below is the list of problems for the first (correspondence) round of the X Sharygin Geometrical Olympiad.

The olympiad is intended for high-school students of four elder grades. In Russian school, these are 8-11. In the list below, each problem is indicated by the numbers of Russian school grades, for which it is intended. Foreign students of the last grade have to solve the problems for 11th grade, students of the preceding grade solve the problems for 10th grade etc. However, the participants may solve problems for elder grades as well (solutions of problems for younger grades will not be considered).

The full solution of each problem costs 7 points. The partial solution costs from 1 to 6 points. The solution without significant advancement costs 0 points. The result of the participant is the sum of all obtained marks.

In your work, please start the solution for each problem in a new page. First write down the statement of the problem, and then the solution. Present your solutions in detail, including all necessary arguments and calculations. Provide all necessary figures of sufficient size. If a problem has an explicit answer, this answer must be presented distinctly. Please, be accurate to provide good understanding and correct estimating of your work !

If your solution depends on some well-known theorems from standard textbooks, you may simply refer to them instead of providing their proofs. However, any fact not from the standard curriculum should be either proved or properly referred (with an indication of the source).

You may note the problems which you liked most (this is not obligatory). Your opinion is interesting for the Jury.

The solutions for the problems (in Russian or in English) must be sent not earlier than on January 8, 2015 and not later than on April 1, 2015. For sending your work, enter the site **http://geom.informatics.msk.ru** and follow the instructions.

Attention: The solutions must be contained in pdf, doc or jpg files. We recommend to prepare the paper using computer or to scan it rather than to photograph it. *In the last two cases, please check readability of the file before sending.*

If you have any technical problems with uploading of the work, write to **geomolymp@mccme.ru**.

The solutions can also be sent by e-mail to the special address **geompapers@yandex.ru** (*If you send the work to another address the Organizing Committee can't guarantee that it will be received*). In this case the work also will be loaded to the server. We recommend the authors to do this themselves. If you send your work by e-mail, please follow a few simple rules:

1. Each student sends his work in a separate message (with delivery notification).

2. If your work consists of several files, send it as an archive.

3. In the subject of the message write "The work for Sharygin olympiad", and present the following personal data in the body of your message:

- last name;

- all other names;

- E-mail, phone number, post address;

- the current number of your grade at school;

- the number of the last grade at your school;

- the number and/or the name and the mail address of your school;

- full names of your teachers in mathematics at school and/or of instructors of your extra math classes (if you attend additional math classes after school).

If you have no possibility to send the work by e-mail, please inform the Organizing Committee to find a specific solution for this case.

Winners of the correspondence round, the students of three grades before the last grade, will be invited to the final round in Summer 2015 in Moscow region. (For instance, if the last grade is 12, then we invite winners from 9, 10, and 11 grade.) The students of the last grade, winners of the correspondence round, will be awarded by diplomas of the Olympiad. The list of the winners will be published on www.geometry.ru at the end of May 2015. If you want to know your detailed results, please use e-mail.

1. (8) Tanya cut out a convex polygon from the paper, fold it several times and obtained a two-layers quadrilateral. Can the cutted polygon be a heptagon?

2. (8) Let O and H be the circumcenter and the orthocenter of a triangle ABC. The line passing through the midpoint of OH and parallel to BC meets AB and AC at points D and E. It is known that O is the incenter of triangle ADE. Find the angles of ABC.

3. (8) The side AD of a square $ABCD$ is the base of an obtuse-angled isosceles triangle AED with vertex E lying inside the square. Let AF be a diameter of the circumcircle of this triangle, and G be a point on CD such that $CG = DF$. Prove that angle BGE is less than half of angle AED.

4. (8) In a parallelogram $ABCD$ the trisectors of angles A and B are drawn. Let O be the common points of the trisectors nearest to AB. Let AO meet the second trisector of angle B at point A_1, and let BO meet the second trisector of angle A at point B_1. Let M be the midpoint of A_1B_1. Line MO meets AB at point N. Prove that triangle A_1B_1N is equilateral.

5. (8–9) Let a triangle ABC be given. Two circles passing through A touch BC at points B and C respectively. Let D be the second common point of these circles (A is closer to BC than D). It is known that $BC = 2BD$. Prove that $\angle DAB = 2\angle ADB$.

6. (8–9) Let AA', BB' and CC' be the altitudes of an acute-angled triangle ABC. Points C_a, C_b are symmetric to C' wrt AA' and BB'. Points A_b, A_c, B_c, B_a are defined similarly. Prove that lines A_bB_a, B_cC_b and C_aA_c are parallel.

7. (8–9) The altitudes AA_1 and CC_1 of a triangle ABC meet at point H. Point H_A is symmetric to H about A. Line H_AC_1 meets BC at point C'; point A' is defined similarly. Prove that $A'C'||AC$.

8. (8–9) Diagonals of an isosceles trapezoid $ABCD$ with bases BC and AD are perpendicular. Let DE be the perpendicular from D to AB, and let CF be the perpendicular from C to DE. Prove that angle DBF is equal to half of angle FCD.

9. (8–9) Let ABC be an acute-angled triangle. Construct points A', B', C' on its sides BC, CA, AB such that:
   - $A'B' \parallel AB$;
   - $C'C$ is the bisector of angle $A'C'B'$;
   - $A'C' + B'C' = AB$.

10. (8–9) The diagonals of a convex quadrilateral divide it into four similar triangles. Prove that is possible to inscribe a circle into this quadrilateral.

11. (8–10) Let H be the orthocenter of an acute-angled triangle ABC. The perpendicular bisector to segment BH meets BA and BC at points A_0, C_0 respectively. Prove that the perimeter of triangle A_0OC_0 (O is the circumcenter of $\triangle ABC$) is equal to AC.

12. (8–11) Find the maximal number of discs which can be disposed on the plane so that each two of them have a common point and no three have it.

13. (9–10) Let AH_1, BH_2 and CH_3 be the altitudes of a triangle ABC. Point M is the midpoint of H_2H_3. Line AM meets H_2H_1 at point K. Prove that K lies on the medial line of ABC parallel to AC.

14. (9–11) Let ABC be an acute-angled, nonisosceles triangle. Point A_1, A_2 are symmetric to the feet of the internal and the external bisectors of angle A wrt the midpoint of BC. Segment A_1A_2 is a diameter of a circle α. Circles β and γ are defined similarly. Prove that these three circles have two common points.

15. (9–11) The sidelengths of a triangle ABC are not greater than 1. Prove that $p(1-2Rr)$ is not greater than 1, where p is the semiperimeter, R and r are the circumradius and the inradius of ABC.

16. (9–11) The diagonals of a convex quadrilateral divide it into four triangles. Restore the quadrilateral by the circumcenters of two adjacent triangles and the incenters of two mutually opposite triangles.

17. (10–11) Let O be the circumcenter of a triangle ABC. The projections of points D and X to the sidelines of the triangle lie on lines l and L such that $l \parallel XO$. Prove that the angles formed by L and by the diagonals of quadrilateral $ABCD$ are equal.

18. (10–11) Let $ABCDEF$ be a cyclic hexagon, points K, L, M, N be the common points of lines AB and CD, AC and BD, AF and DE, AE and DF respectively. Prove that if three of these points are collinear then the fourth point lies on the same line.

19. (10–11) Let L and K be the feet of the internal and the external bisector of angle A of a triangle ABC. Let P be the common point of the tangents to the circumcircle of the triangle at B and C. The perpendicular from L to BC meets AP at point Q. Prove that Q lies on the medial line of triangle LKP.

20. (10–11) Given are a circle and an ellipse lying inside it with focus C. Find the locus of the circumcenters of triangles ABC, where AB is a chord of the circle touching the ellipse.

21. (10–11) A quadrilateral $ABCD$ is inscribed into a circle ω with center O. Let M_1 and M_2 be the midpoints of segments AB and CD respectively. Let Ω be the circumcircle of triangle OM_1M_2. Let X_1 and X_2 be the common points of ω and Ω, and Y_1 and Y_2 the second common points of Ω with the circumcircles of triangles CDM_1 and ABM_2. Prove that $X_1X_2 || Y_1Y_2$.

22. (10–11) The faces of an icosahedron are painted into 5 colors in such a way that two faces painted into the same color have no common points, even a vertices. Prove that for any point lying inside the icosahedron the sums of the distances from this point to the red faces and the blue faces are equal.

23. (11) A tetrahedron $ABCD$ is given. The incircles of triangles ABC and ABD with centers O_1, O_2, touch AB at points T_1, T_2. The plane π_{AB} passing through the midpoint of T_1T_2 is perpendicular to O_1O_2. The planes π_{AC}, π_{BC}, π_{AD}, π_{BD}, π_{CD} are defined similarly. Prove that these six planes have a common point.

24. (11) The insphere of a tetrahedron $ABCD$ with center O touches its faces at points A_1, B_1, C_1 и D_1.

    a) Let P_a be a point such that its reflections in lines OB, OC and OD lie on plane BCD. Points P_b, P_c and P_d are defined similarly. Prove that lines A_1P_a, B_1P_b, C_1P_c and D_1P_d concur at some point P.

    b) Let I be the incenter of $A_1B_1C_1D_1$ and A_2 the common point of line A_1I with plane $B_1C_1D_1$. Points B_2, C_2, D_2 are defined similarly. Prove that P lies inside $A_2B_2C_2D_2$.

XI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
THE CORRESPONDENCE ROUND. SOLUTIONS

1. (T.Kazitzyna) Tanya cut out a convex polygon from the paper, folded it several times and obtained a two-layers quadrilateral. Can the cut polygon be a heptagon?

   Solution. Yes, for example let angle B of a quadrilateral $ABCD$ be obtuse, and three remaining angles be acute. Take a point K on side CD such that $\angle CBK < 180° - \angle B$. Let points B_1, K_1 be symmetric to B, K about AD, and point K_2 be symmetric to K about BC. Then a heptagon $ABK_2CDK_1B_1$ is convex, and folding it by lines BC and AD, we obtain two-layers quadrilateral $ABCD$ (fig.1).

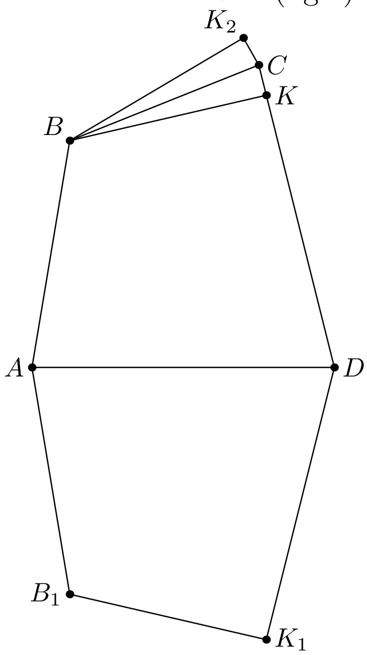


Fig.1

2. (M.Rozhkova) Let O and H be the circumcenter and the orthocenter of a triangle ABC respectively. The line passing through the midpoint of OH and parallel to BC meets AB and AC at points D and E respectively. It is known that O is the incenter of triangle ADE. Find the angles of ABC.

   Answer. $\angle A = 36°$, $\angle B = \angle C = 72°$.

   Solution. By the condition we obtain that AO is the bisector of angle A, i.e. $AB = AC$. Then $ODHE$ is a rhombus, $\angle ODH = 2\angle ODE = \angle B$, $\angle DOH = \angle DHO = 90° - \frac{\angle B}{2} = \angle BHD$.

   Let the line passing through H and parallel to AC meet AB at point K. Since $\angle HKB = \angle A = \angle HOB$, points H, O, K, B are concyclic. Since angle KHB is right, the center of the corresponding circle lies on AB, thus it coincides with D (fig.2). Therefore, $\angle HBD = \angle BHD = 90° - \frac{\angle B}{2}$. On the other hand this angle is equal to $\angle B - \frac{\angle A}{2}$, from this we obtain the answer.

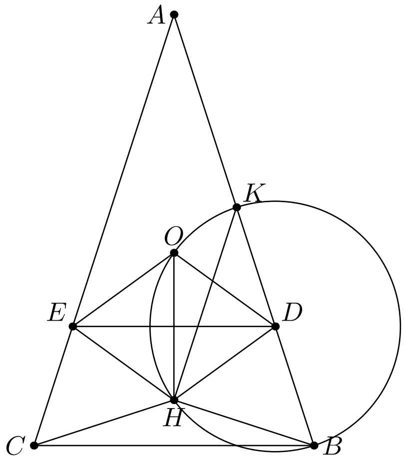


Fig.2

3. (N.Moskvitin) The side AD of a square $ABCD$ is the base of an obtuse-angled isosceles triangle AED with vertex E lying inside the square. Let AF be a diameter of the circumcircle of this triangle, and G be a point on CD such that $CG = DF$. Prove that angle BGE is less than half of angle AED.

   Solution. It is clear that F lies on sideline CD. Since $CG = DF$, we have $FG = CD = AB$, i.e. $ABGF$ is a parallelogram, and $\angle BGD = 180° - \angle AFD = \angle AED$. Thus we have to prove that $\angle BGE < \angle EGD$ or the distance from E to BG is less than its distance to CD. But the distances from E to CD and AF are equal, because FE bisects angle DFA, thus it is sufficient to prove that E is closer to BG, than to AF.

   A line through E parallel to AB meets AF at the center O of circle AED (fig.3). Therefore, $EO > AD/2 = AB/2$, which is equivalent to the desired inequality.

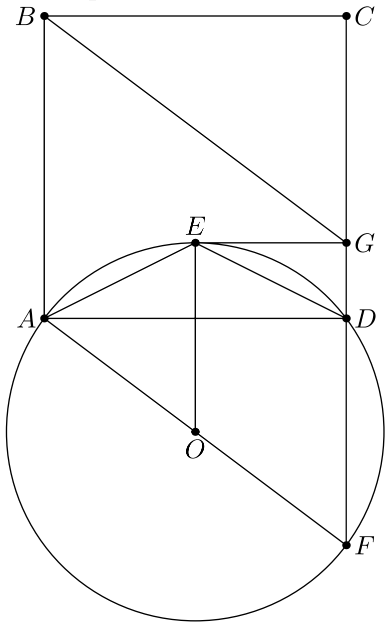


Fig.3

4. (L.Shteyngarts) In a parallelogram $ABCD$ the trisectors of angles A and B are drawn. Let O be the common points of the trisectors nearest to AB. Let AO meet the second trisector of angle B at point A_1, and let BO meet the second trisector of angle A at point

B_1. Let M be the midpoint of A_1B_1. Line MO meets AB at point N. Prove that triangle A_1B_1N is equilateral.

Solution. Let K be a common point of two remote trisectors. Then in triangle ABK $\angle K = 60°$, and AA_1 and BB_1 are its bisectors. Since $\angle A_1OB_1 = 120°$, quadrilateral A_1KB_1O is cyclic, and since KO bisects angle K, we obtain that $OA_1 = OB_1$. Therefore, $\angle MOA_1 = 60° = \angle A_1OB = \angle BON$. This yields that $ON = OA_1$ and $A_1N = A_1B_1 = B_1N$ (fig.4).

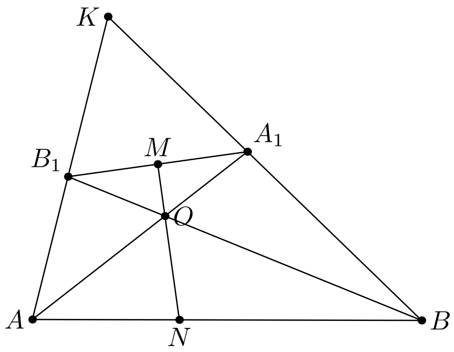


Fig.4

5. (V.Yassinsky) Let a triangle ABC be given. Two circles passing through A touch BC at points B and C respectively. Let D be the second common point of these circles (A is closer to BC than D). It is known that $BC = 2BD$. Prove that $\angle DAB = 2\angle ADB$.

   Solution. Since AD is a radical axis of two circles it meets segment BC at its midpoint M. Then $BM = BD$ and $\angle ADB = \angle DMB$. But $\angle ABM = \angle ADB$ as the angle between the chord and the tangent. By the exterior angle theorem $\angle DAB = \angle ABM + \angle AMB = 2\angle ADB$ (fig.5).

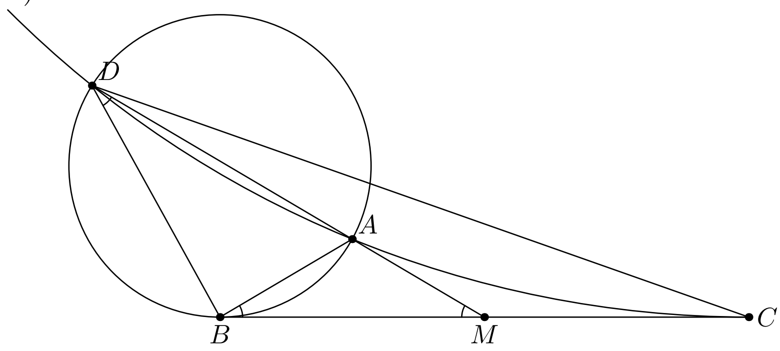


Fig.5

6. (A.Zaslavsky) Let AA', BB' and CC' be the altitudes of an acute-angled triangle ABC. Points C_a, C_b are symmetric to C' about AA' and BB' respectively. Points A_b, A_c, B_c, B_a are defined similarly. Prove that lines A_bB_a, B_cC_b and C_aA_c are parallel.

   First solution. Firstly prove next lemma.

   Let points Y', X' on sides XZ, YZ of triangle XYZ be such that $XY' = XY = X'Y$. Then $X'Y' \perp OI$, where O and I are the circumcenter and the incenter of the triangle.

   To prove the lemma it is sufficient to see that $X'O^2 - Y'O^2 = X'I^2 - Y'I^2$. Let x, y, z be the sidelengths of YZ, ZX, XY; X_0 be the the midpoint of YZ. Then $X'O^2 - OY^2 = X'X_0^2 - YX_0^2 = (z - x/2)^2 - (x/2)^2 = z(z - x)$. Similarly $Y'O^2 - OX^2 = z(z - y)$.

Also, $X'I^2 = r^2 + (z - (p - y))^2 = r^2 + (p - x)^2$, $Y'I^2 = r^2 + (p - y)^2$. Therefore, $X'O^2 - Y'O^2 = X'I^2 - Y'I^2 = z(y - x)$.

Now note that $A'A$, $B'B$, $C'C$ are the bisectors of triangle $A'B'C'$. Thus, for example, points A_b, B_a lie on $B'C'$, $A'C'$ respectively and $B'A_b = A'B_a = A'B'$. By the lemma A_bB_a is perpendicular to the line passing through the circumcenter and the incenter of triangle $A'B'C'$. Lines B_cC_b and A_cC_a are also perpendicular to this line, therefore these three lines are parallel.

Second solution. By previous solution B_a lies on $A'C'$, C_a lies on $A'B'$, A_b and A_c lie on $B'C'$. Since $A'B_a = A'B'$ and $A'C_a = A'C'$, we obtain that $B'B_a \parallel C'C_a$, thus $B'B_a/C'C_a = A'B'/A'C' = B'A_b/C'A_c$. Therefore triangles $B'A_bB_a$ and A_cC_aC' are similar, $\angle B_aA_bB' = \angle C_aA_cC'$ and $A_bB_a \parallel A_cC_a$. Similarly we prove that B_cC_b is parallel to these lines.

7. (D.Shvetsov) The altitudes AA_1 and CC_1 of a triangle ABC meet at point H. Point H_A is symmetric to H about A. Line H_AC_1 meets BC at point C'; point A' is defined similarly. Prove that $A'C'||AC$.

Solution. Since triangles AHC_1 and CHA_1 are similar, triangles AH_AC_1 and CH_CA_1 are also similar i.e. $\angle A'C_1B = \angle C'A_1B$. Therefore points A_1, C_1, A', C' are concyclic and lines A_1C_1 and $A'C'$ are antiparallel wrt angle B. Since A_1C_1 and AC are also antiparallel, $A'C' \parallel AC$ (fig.7).

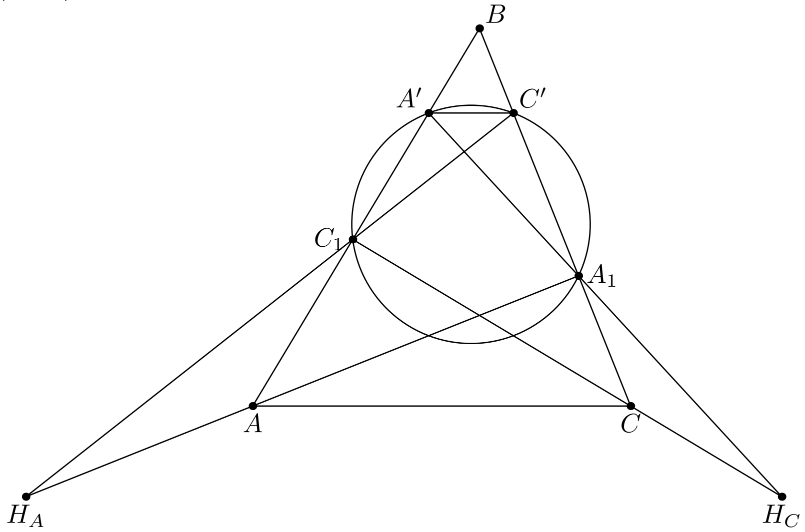


Fig.7

8. (N.Moskvitin) Diagonals of an isosceles trapezoid $ABCD$ with bases BC and AD are perpendicular. Let DE be the perpendicular from D to AB, and let CF be the perpendicular from C to DE. Prove that angle DBF is equal to half of angle FCD.

Solution. By condition $\angle EDB = 45° - (90° - \angle A) = \angle A - 45° = \angle BDC$. Thus the distances from B to lines DE and DC are equal. Since the trapezoid is isosceles, the distance from B to DC is equal to the distance from C to AB, which is equal to the distance from B to line AB parallel to CF. Therefore, BF bisects angle CFE and $\angle BFC = 45°$. Let the perpendicular to BF from F meet BD at point K. Then $\angle CFK = \angle CBK = 45°$, thus $BFKC$ is a cyclic quadrilateral and $CK \perp BC$. Since $CF \parallel AB$, altitude CK bisects angle FCD, and from cyclic quadrilateral $BFKC$ we obtain that $\angle DBF = \angle KCF = \angle FCD/2$ (fig.8).

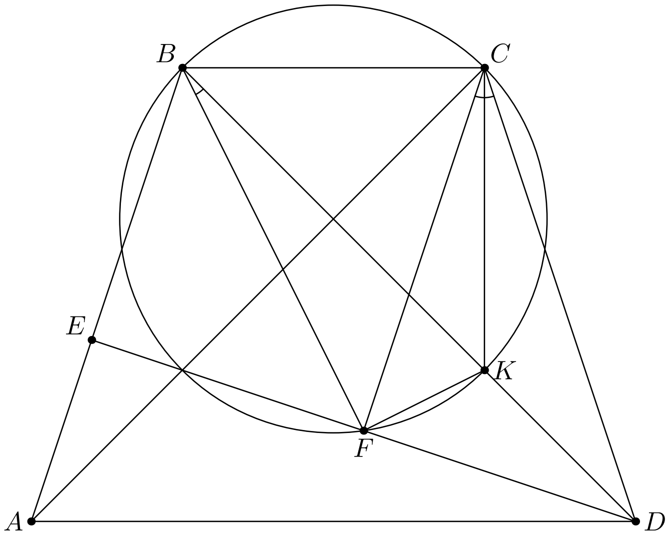


Fig.8

9. (a.Zaslavsky) Let ABC be an acute-angled triangle. Construct points A', B', C' on its sides BC, CA, AB such that:
- $A'B' \parallel AB$;
- $C'C$ is the bisector of angle $A'C'B'$;
- $A'C' + B'C' = AB$.

Solution. Let L be a common point of CC' and $A'B'$. Then $BC'/AC' = A'L/B'L = A'C'/B'C'$ and since $A'C' + B'C' = AB$ we obtain that $BC' = C'A'$, $AC' = C'B'$. Thus the reflections of C' in AC and BC lie on $A'B'$ and line CC' is symmetric to the altitude from C about the correspondent bisector i.e. CC' passes through the orthocenter of the given triangle (fig.9). The further construction is evident.

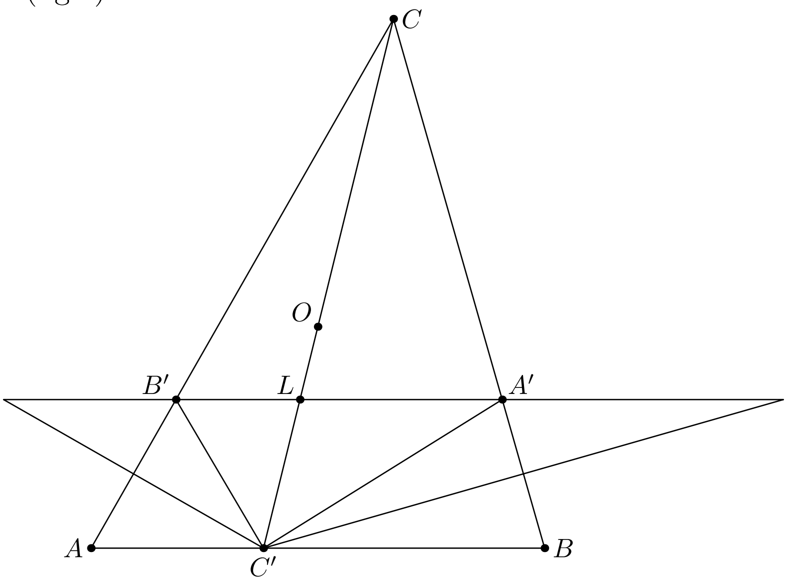


Fig.9

10. (B.Frenkin) The diagonals of a convex quadrilateral divide it into four similar triangles. Prove that it is possible to inscribe a circle into this quadrilateral.

Solution. Let the diagonals of a quadrilateral $ABCD$ meet at point L. If for example angle ALB is obtuse, then it is greater than any angle of triangle BLC and two adjacent triangles can not be similar. Therefore the diagonals are perpendicular. Now if $\angle ABL = \angle CBL$ then BL is an altitude and a bisector of triangle ABC, thus it is also a median and $AB = BC$. Then DL is an altitude and a median of triangle ADC, therefore $AD = DC$ and the quadrilateral id circumscribed.

If angles ABL and CBL are not equal then their sum is equal to 90°. If $\angle BCL = \angle DCL$ then reason as above. Else $ABCD$ is a rectangle with perpendicular diagonals, i.e a square. Therefore a circle can be inscribed into it.

11. (A.Sokolov) Let H be the orthocenter of an acute-angled triangle ABC. The perpendicular bisector to segment BH meets BA and BC at points A_0, C_0 respectively. Prove that the perimeter of triangle A_0OC_0 (O is the circumcenter of $\triangle ABC$) is equal to AC.

    Solution. It is known that the reflections of H in the sidelines of a triangle lie on its circumcircle, i.e. the distances from them to O are equal to the circumradius R. Therefore the distances from H to points O_a, O_c, symmetric to O about BC and BA, are also equal to R. Since $BO_a = BO_c = R$, points O_a, O_c lie on A_0C_0. Also $BOCO_a$ and $BOAO_c$ are rhombus, thus $CO_a \parallel OB \parallel AO_c$, i.e. ACO_aO_c is a parallelogram and $O_aO_c = AC$. But by construction O_aO_c is equal to the perimeter of A_0OC_0 (fig.11).

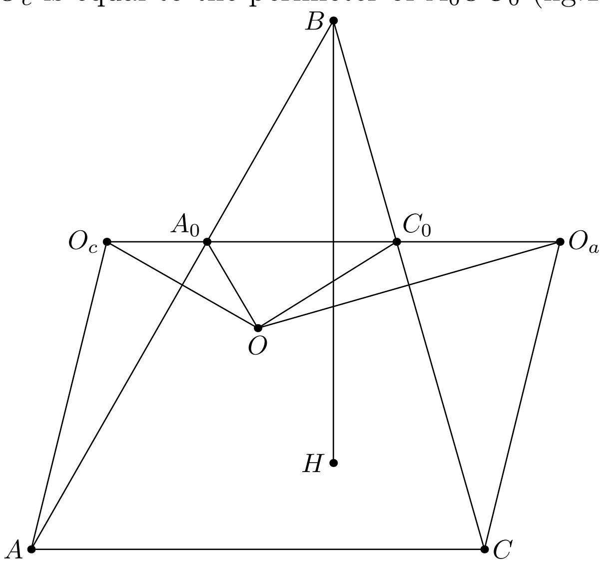


Fig.11

12. (A.Zaslavxky) Find the maximal number of discs which can be disposed on the plane so that each two of them have a common point and no three have it.

    Answer. 4.

    Solution. Consider one from n discs. Let A_iB_i be its common chords with the remaining discs. Since three discs do not intersect we obtain that for all i one of arcs A_iB_i does not contain the endpoints of the remaining chords. Cutting from each disc the segments limited by such arcs, we obtain n convex figures, each two of them have a common boundary. It is known that at most four such figures can exist on the plane. It is clear that four discs can satisfy the condition.

13. (A.Rudenko, D.Khilko) Let AH_1, BH_2 and CH_3 be the altitudes of a triangle ABC. Point M is the midpoint of H_2H_3. Line AM meets H_2H_1 at point K. Prove that K lies on the medial line of ABC parallel to AC.

Solution. Let P be the projection of H_3 to AC. Triangle H_3PH_2 is right-angled, and M is the midpoint of its hypothenuse, thus $MP = MH_2$ and $\angle MPH_2 = \angle MH_2A$. It is known that $\angle ABC = \angle H_1H_2P = \angle H_3H_2A$, therefore $MP \parallel KH_2$. From this we obtain that $\frac{AM}{AK} = \frac{AP}{AH_2}$. Triangles AH_2H_3 and ABC are similar, thus $\frac{AP}{AH_2} = \frac{AH_3}{AB}$. Then $\frac{AM}{AK} = \frac{AP}{AH_2} = \frac{AH_3}{AB}$, and $H_3M \parallel BK$ (fig.13). Also $\angle H_3H_2B = 90° - H_3H_2A = 90° - H_1H_2C = \angle BH_2K$. Therefore $\angle H_2BK = \angle H_3H_2B = \angle BH_2K$, and triangle BH_2K is isosceles. It is clear that the medial line parallel to AC is the perpendicular bisector to BH_2. Thus it passes through K.

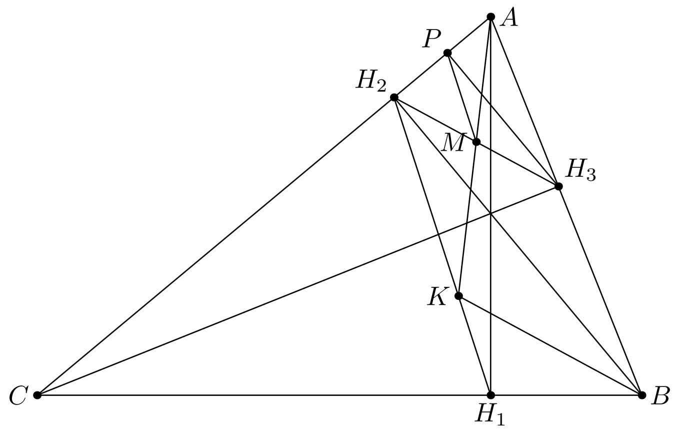


Fig.13

14. (A.Myakishev) Let ABC be an acute-angled, nonisosceles triangle. Point A_1, A_2 are symmetric to the feet of the internal and the external bisectors of angle A wrt the midpoint of BC. Segment A_1A_2 is a diameter of a circle α. Circles β and γ are defined similarly. Prove that these three circles have two common points.

    Solution. It is known that the circles having the feet of internal and external bisectors as opposite points are perpendicular to the circumcircle. Thus circles α, β, γ symmetric to them about the diameters of the circumcircles are also perpendicular to it, i.e. the degrees of the circumcenter O wrt these three circles are equal. Since the midpoints of the segments between the feet of the bisectors are concurrent, the centers of three circles are also concurrent by the Menelaos theorem. The perpendicular from O to the correspondent line is the common radical axis of three circles, therefore they have two common points.

15. (V.Yassinsky) The sidelengths of a triangle ABC are not greater than 1. Prove that $p(1-2Rr)$ is not greater than 1, where p is the semiperimeter, R and r are the circumradius and the inradius of ABC.

    Solution. Since the area of a triangle with sidelengths a, b, c is equal to $abc/4R = pr$, the desired inequality is equivalent to $a + b + c - abc \leq 2$. But

    $$a+b+c-abc = a+b+c(1-ab) \leq a+b+1-ab = 1+a+b(1-a) \leq 1+a+1-a = 2.$$

16. (B.Frenkin) The diagonals of a convex quadrilateral divide it into four triangles. Restore the quadrilateral by the circumcenters of two adjacent triangles and the incenters of two mutually opposite triangles.

    First solution. Let L be a common point of the diagonals of quadrilateral $ABCD$; O, I be the circumcenter and the incenter of triangle LAB; O' be the circumcenter of triangle LAD; I' be the incenter of triangle LCD. Then OO' is the perpendicular bisector to LA, and II' contains the bisector of angle LAB. Thus we can define the directions of lines LA, LB and construct the perpendicular bisector to LB.

Let X, Y, Z be the midpoints of arcs LA, LB, AB of circle LAB. Then I is the orthocenter of triangle XYZ and since we know angle ALB we can find angle XIY. Denote this angle as φ. Now we have to solve next problem.

An angle with vertex O and a point I are given. Construct on the sides of the angle such points X, Y that $OX = OY$ and $\angle XIY = \varphi$.

Take on the sides of the angles two arbitrary points X_1, Y_1 such that $OX_1 = OY_1$ and find such point I_1 on ray OI that $\angle X_1I_1Y_1 = \varphi$. The homothety with center O, transforming I_1 to I, transforms X_1, Y_1 to the desired points. The further construction is evident.

Second solution. In the notations of previous solution it is sufficient to find point L. In fact OO' is the perpendicular bisector to AL, and II' is the bisector of angle ALB. Constructing the perpendicular from L to OO' we find line AL. Reflecting it about II' we obtain line BL. Constructing a circle passing through L with center O we find A and B as its common points with AL and BL. The circle through L with center O' meets BL at D. Now construct the circle with center I', touching AL and BL, the tangent to this circle from D meets AL at C.

To find L use the trident theorem: a common point of the perpendicular bisector to a side of a triangle with its circumcircle lies on equal distances from the incenter and the endpoints of the side. Take an arbitrary circle ω_1 with center O. Let it meet OO' at point K. Constructing the perpendicular from K to II' and reflecting ω_1 about it, we obtain circle ω_2. Let OI meet ω_2 at point I_1. Reflecting I_1 about this perpendicular, we obtain point L' on ω_1. The homothety with center O, transforming I_1 to I, transforms L' to L.

17. (F.Nilov) Let O be the circumcenter of a triangle ABC. The projections of points D and X to the sidelines of the triangle lie on lines l and L such that $l \parallel XO$. Prove that the angles formed by L and by the diagonals of quadrilateral $ABCD$ are equal.

    Solution. By condition D and X lie on the circumcircle of ABC, and l and L are its Simson lines. Let chords CC', DD' and XX' be parallel to AB. By Simson lines properties l and OX are perpendicular to CD', and $L \perp CX'$. Thus we have to prove that arcs $X'D$ and $X'C'$ are equal. But these arcs are equal to $D'X$ and CX respectively, and the equality of these two arcs ic evident (fig.17).

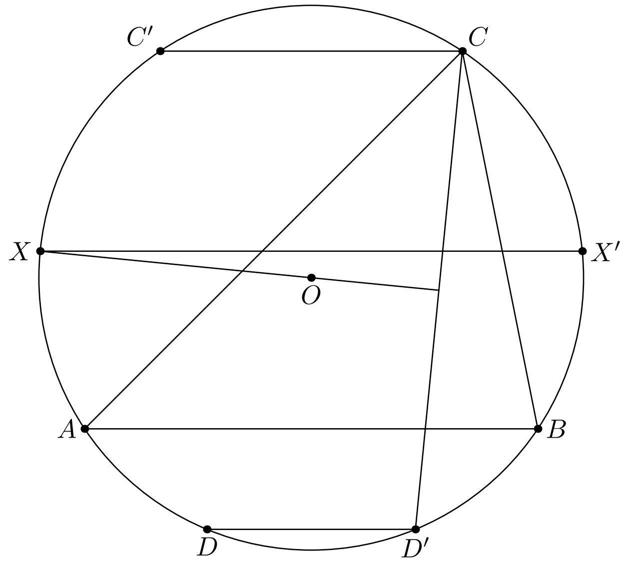


Fig.17

18. (V.Yassinsky) Let $ABCDEF$ be a cyclic hexagon, points K, L, M, N be the common points of lines AB and CD, AC and BD, AF and DE, AE and DF respectively. Prove that if three of these points are collinear then the fourth point lies on the same line.

    Solution. Consider a projective map saving the circumcircle and transforming L to its center. It transforms $ABCD$ and KL to a rectangle and its symmetry axis respectively. If one of points M, N lies on this axis then E and F are symmetric about it, therefore the remaining point also lies on KL.

19. (F.Ivlev) Let L and K be the feet of the internal and the external bisector of angle A of a triangle ABC. Let P be the common point of the tangents to the circumcircle of the triangle at B and C. The perpendicular from L to BC meets AP at point Q. Prove that Q lies on the medial line of triangle LKP.

    Solution. Since BC is the polar of P wrt the circumcircle ω of triangle ABC we obtain that P lies on the polar of L. Since the quadruple B, C, L, K is harmonic, K also lies on the polar of L. Therefore KP is the polar of L wrt ω, and the medial line of triangle KLP is the radical axis of ω and L. Prove that Q also lies on this axis.

    Let M be the midpoint of KL. Since M is the center of circle AKL perpendicular to ω, M lies on the polar of A. But M also lies on the polar of P, thus AP is the polar of M wrt ω and the common chord of ω and circle AKL. But LQ is the radical axis of circle AKL and L, therefore, Q is the common point of three radical axes.

20. (A.Zaslavsky) A circle and an ellipse lying inside it with a focus C are given. Find the locus of the circumcenters of triangles ABC, where AB is a chord of the circle touching the ellipse.

    Solution. Let CH be an altitude of triangle ABC. Then H lies on the circle having the greatest axis of the ellipse as diameter. Let O and R be the center and the radius of the given circle, and O' be the circumcenter of ABC. Using the cosine law to triangles $AO'O$ and $AO'C$, we have $R^2 = O'A^2 + O'O^2 - 2O'A \cdot O'O \cos\angle AO'O$, $OC^2 = O'C^2 + O'O^2 - 2O'C \cdot O'O \cos\angle CO'O$. Since $O'O \parallel CH$ and $O'A = O'C$, we obtain subtracting the second equality from the first one that $R^2 - OC^2 = 2O'O \cdot CH$.

Let the translation to vector CO transform H to H'. Then O, H' and O' are collinear and $OH' \cdot OO' = (R^2 - OC^2)/2$ do not depend on AB. Therefore O' and H' are symmetric about some circle concentric with the given one. Since the locus of points H' is a circle, The desired locus is also a circle.

21. (A.Yakubov) A quadrilateral $ABCD$ is inscribed into a circle ω with center O. Let M_1 and M_2 be the midpoints of segments AB and CD respectively. Let Ω be the circumcircle of triangle OM_1M_2. Let X_1 and X_2 be the common points of ω and Ω, and Y_1 and Y_2 the second common points of Ω with the circumcircles of triangles CDM_1 and ABM_2. Prove that $X_1X_2||Y_1Y_2$.

    Solution. Let K be a common point of AB and CD. Since angles OM_1K and OM_2K are right, OK is a diameter of Ω. Since arcs OX_1 and OX_2 of this circle are equal it is sufficient to prove that arcs KY_1 and KY_2 are also equal, or $\angle KM_1Y_1 = \angle KM_2Y_2$.

    Let N_1, N_2 be the second common points of circles CDM_1 and ABM_2 with AB and CD respectively. Then $KM_1 \cdot KN_1 = KC \cdot KD = KA \cdot KB$, therefore, $N_1K \cdot N_1M_1 = N_1A \cdot N_1B$. Thus the powers of N_1 wrt circles Ω and ABM_2 are equal, i.e. N_1 lies on M_2Y_2. Similarly N_2 lies on M_1Y_1 (fig.21). But it is clear that quadrilateral $M_1M_2N_2N_1$ is cyclic, which yields the desired equality.

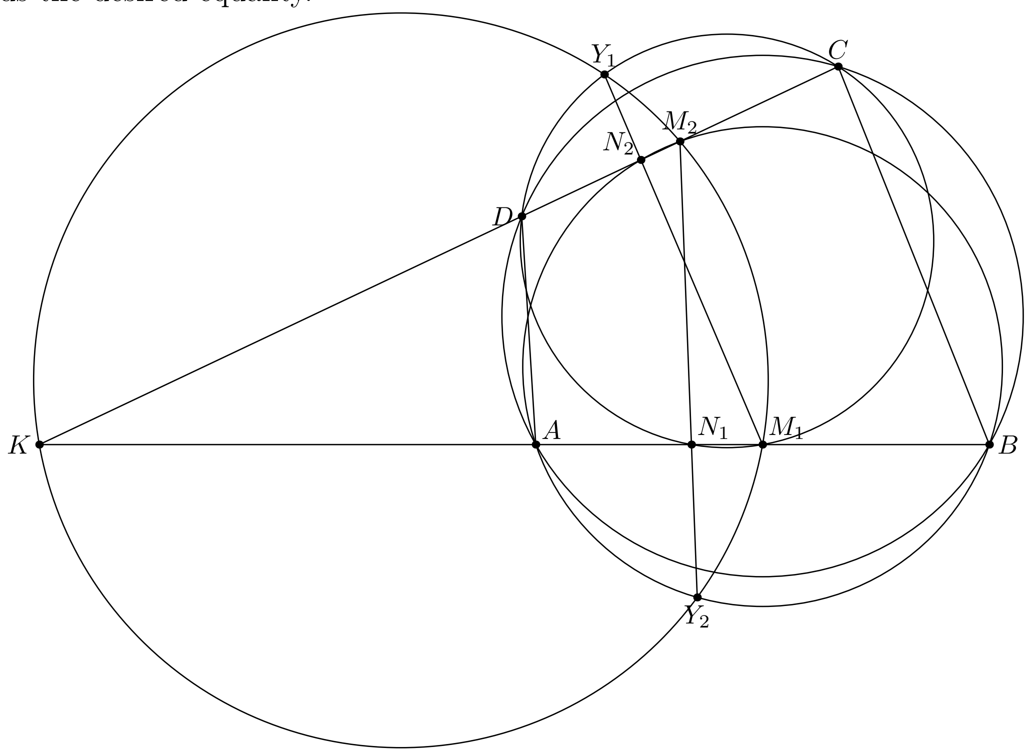


Fig.21

22. (A.Belov-Kanel) The faces of an icosahedron are painted into 5 colors in such a way that two faces painted into the same color have no common points, even vertices. Prove that for any point lying inside the icosahedron the sums of the distances from this point to the red faces and to the blue faces are equal.

    Solution. Prove that there exists a unique coloring satisfying the condition. Call the distance between two faces the minimal number of edges intersecting in the path from one face to the second one. Then the distance between two opposite faces is equal to

5. Also there exist 3 faces with distances 1 and 4 from any fixed face, and 6 faces with distances 2 and 3 from it.

Consider one of red faces. The faces with distances 1 or 2 from it can not be red. If the opposite face is red, then all remaining faces can not be red. If there exists a red face with distance 4 from the initial one, then there are only two faces without common vertices with two red faces. Since these two faces are adjacent only one from them can be red. Finally only three faces with distance 3 from the considered one can be red simultaneously. Thus there exists at most four red faces. This is also correct for all remaining colors, therefore there are exactly four faces of each color. The planes of four monochromatic faces form a regular tetrahedron. But for any point inside a tetrahedron the sum of the distances from it to the faces is equal to the altitude of the tetrahedron. This evidently yields the assertion of the problem.

23. (M.Yagudin) A tetrahedron $ABCD$ is given. The incircles of triangles ABC and ABD with centers O_1, O_2, touch AB at points T_1, T_2. The plane π_{AB} passing through the midpoint of T_1T_2 is perpendicular to O_1O_2. The planes π_{AC}, π_{BC}, π_{AD}, π_{BD}, π_{CD} are defined similarly. Prove that these six planes have a common point.

Solution. Consider four spheres having those circles as diametral sections. Then for example π_{AB} is the radical plane of two spheres touching AB, therefore it contains the radical center of four spheres. The remaining planes also pass through this point.

24. (N.Beluhov) The insphere of a tetrahedron $ABCD$ with center O touches its faces at points A_1, B_1, C_1 and D_1.

a) Let P_a be a point such that its reflections in lines OB, OC and OD lie on plane BCD. Points P_b, P_c and P_d are defined similarly. Prove that lines A_1P_a, B_1P_b, C_1P_c and D_1P_d concur at some point P.

b) Let I be the incenter of $A_1B_1C_1D_1$ and A_2 the common point of line A_1I with plane $B_1C_1D_1$. Points B_2, C_2, D_2 are defined similarly. Prove that P lies inside $A_2B_2C_2D_2$.

Solution. a) Let B_a be such a point that A_1B_a is a diameter in the circumcircle of $\triangle A_1C_1D_1$ with center O_b and radius R_B. Define C_a, D_a, O_b... and so on similarly. Let also the inscribed sphere of $ABCD$ be ω, and its inradius be r. Finally, denote by $d_a(X)$ the distance from a point X to the plane $(B_1C_1D_1)$, and similarly for $d_b(X)$ and so on.

By symmetry, B_a is the reflection of A_1 in BO. So, since the plane (BCD) touches ω, P_aB_a also touches ω. Let Q be the projection of P_a in the plane $(A_1C_1D_1)$. We see that $\angle P_aB_aO = 90 \Rightarrow \triangle P_aQB_a \sim \triangle B_aO_aO \Rightarrow d_b(P_a) : R_B = P_aB_a : r$. Analogously, $d_c(P_a) : R_C = P_aC_a : r$ and $d_d(P_a) : R_D = P_aD_a : r$. Since $P_aB_a = P_aC_a = P_aD_a$ (as tangents to a sphere), this means that the distances from P_a to the faces of the tetrahedron $A_1B_1C_1D_1$ are in ratios $d_b(P_a) : d_c(P_a) : d_d(P_a) = R_B : R_C : R_D$. Analogous reasoning shows that the distances from P_b to the corresponding faces of the same tetrahedron are in ratios $R_A : R_C : R_D$, and so on for P_c and P_d.

But the locus of the points whose distances to three given planes are in given ratios is a line trough the intersection of these planes, and the locus of the points whose distances to two given planes are in given ratio is a plane trough the intersection of these planes. Thus, the lines A_1P_a and B_1P_b lie in the same plane and intersect in some point P. By the loci argument, this point also lies in the lines C_1P_c and D_1P_d.

b) Notice that the interior of the tetrahedron $A_2B_2C_2D_2$ is the locus of the points X such that the four inequalities hold: $d_a(X)+d_b(X)+d_c(X) \geq 2d_d(X)$, $d_b(X)+d_c(X)+d_d(X) \geq 2d_a(X)$, and so on. This is easy to see using baricentric coordinates with respect to $A_1B_1C_1D_1$. Indeed, if α, β, γ and δ are the coordinates of some point X, and d_A and so on denote the equal distances from A_2 to the three corresponding faces of $A_1B_1C_1D_1$, then $d_a(X) = \beta d_B + \gamma d_C + \delta d_D$ and so on, yielding $3\alpha d_A = d_b(X)+d_c(X)+d_d(X)-2d_a(X)$ and so on. Thus, the inequalities hold exactly when α, β, γ and δ are positive, and this happens exactly when X lies inside $A_2B_2C_2D_2$ (more elementary, but not as simple arguments can also be applied).

Thus, it suffices to show that $R_A + R_B + R_C > 2R_D$ (and so on, symmetrically).

Notice that all faces of the tetrahedron $A_1B_1C_1D_1$ are acute-angled triangles, and the points O_a, O_b and so on are interior to them (this follows easily from the fact that its vertices are the tangency points of the inshere with the faces of $ABCD$). Obviously, $2R_A + 2R_B + 2R_C \geq B_1C_1 + C_1A_1 + A_1B_1$ (as diameters are greater than chords). Let K, L and M be the midpoints of the sides of $\triangle A_1B_1C_1$. The point O_d lies inside the quadrilateral, say, A_1LKB_1 (as it lies inside $\triangle KLM$), thus $A1L + LK + KB_1 > A_1O_d + O_dB_1$. But $A_1L + LK + KB_1 = \frac{1}{2}B_1C_1 + \frac{1}{2}C_1A_1 + \frac{1}{2}A_1B_1$, and the inequality desired follows.

XII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
THE CORRESPONDENCE ROUND

Below is the list of problems for the first (correspondence) round of the XII Sharygin Geometrical Olympiad.

The olympiad is intended for high-school students of four elder grades. In Russian school, these are 8-11. In the list below, each problem is indicated by the numbers of Russian school grades, for which it is intended. Foreign students of the last grade have to solve the problems for 11th grade, students of the preceding grade solve the problems for 10th grade etc. However, the participants may solve problems for elder grades as well (solutions of problems for younger grades will not be considered).

A complete solution of each problem costs 7 points. A partial solution costs from 1 to 6 points. A solution without significant advancement costs 0 points. The result of a participant is the sum of all obtained marks.

In your work, please start the solution for each problem in a new page. First write down the statement of the problem, and then the solution. Present your solutions in detail, including all necessary arguments and calculations. Provide all necessary figures of sufficient size. If a problem has an explicit answer, this answer must be presented distinctly. Please, be accurate to provide good understanding and correct estimating of your work !

If your solution depends on some well-known theorems from standard textbooks, you may simply refer to them instead of providing their proofs. However, any fact not from the standard curriculum should be either proved or properly referred (with an indication of the source).

You may note the problems which you liked most (this is not obligatory). Your opinion is interesting for the Jury.

The solutions for the problems (in Russian or in English) must be delivered not earlier than on January 8, 2016 and not later than on April 1, 2016. To upload your work, enter the site **http://geom.informatics.msk.ru** and follow the instructions.

Attention: The solutions must be contained in pdf, doc or jpg files. We recommend to prepare the paper using computer or to scan it rather than to photograph it. *In the last two cases, please check readability of the file before uploading.*

If you have any technical problems with uploading of the work, apply to **geomolymp@mccme.ru**.

The solutions can also be sent by e-mail to the special address **geompapers@yandex.ru**. (*If you send the work to another address the Organizing Committee can't guarantee that it will be received*). In this case the work also will be uploaded to the server. We recommend the authors to do this by their own. If you send your work by e-mail, please follow a few simple rules:

1. Each student sends his work in a separate message (with delivery notification).

2. If your work consists of several files, send it as an archive.

3. In the subject of the message write "The work for Sharygin olympiad", and present the following personal data in the body of your message:

- last name;

- all other names;

- E-mail, phone number, post address;

- the current number of your grade at school;

- the number of the last grade at your school;

- the number and/or the name and the mail address of your school;

- full names of your teachers in mathematics at school and/or of instructors of your extra math classes (if you attend additional math classes after school).

If you have no possibility to deliver the work by Internet, please inform the Organizing Committee to find a specific solution for this case.

Winners of the correspondence round, the students of three grades before the last grade, will be invited to the final round held in Summer 2016 in Moscow region. (For instance, if the last grade is 12 then we invite winners from 9, 10, and 11 grade.) The students of the last grade, winners of the correspondence round, will be awarded by diplomas of the Olympiad. The list of the winners will be published on **www.geometry.ru** at the end of May 2016 at latest. If you want to know your detailed results, please use e-mail **geomolymp@mccme.ru**.

1. (8) A trapezoid $ABCD$ with bases AD and BC is such that $AB = BD$. Let M be the midpoint of DC. Prove that $\angle MBC = \angle BCA$.

2. (8) Mark three nodes on a cellular paper so that the semiperimeter of the obtained triangle would be equal to the sum of its two smallest medians.

3. (8) Let AH_1, BH_2 be two altitudes of an acute-angled triangle ABC, D be the projection of H_1 to AC, E be the projection of D to AB, F be the common point of ED and AH_1. Prove that $H_2F \parallel BC$.

4. (8) In quadrilateral $ABCD$ $\angle B = \angle D = 90^\circ$ and $AC = BC + DC$. Point P of ray BD is such that $BP = AD$. Prove that line CP is parallel to the bisector of angle ABD.

5. (8) In quadrilateral $ABCD$ $AB = CD$, M and K are the midpoints of BC and AD. Prove that the angle between MK and AC is equal to the half-sum of angles BAC and DCA.

6. (8) Let M be the midpoint of side AC of triangle ABC, MD and ME be the perpendiculars from M to AB and BC respectively. Prove that the distance between the circumcenters of triangles ABE and BCD is equal to $AC/4$.

7. (8–9) Let all distances between the vertices of a convex n-gon ($n > 3$) be different.

   a) A vertex is called uninteresting if the closest vertex is adjacent to it. What is the minimal possible number of uninteresting vertices (for a given n)?

   b) A vertex is called unusual if the farthest vertex is adjacent to it. What is the maximal possible number of unusual vertices (for a given n)?

8. (8–9) Let $ABCDE$ be an inscribed pentagon such that $\angle B + \angle E = \angle C + \angle D$. Prove that $\angle CAD < \pi/3 < \angle A$.

9. (8–9) Let ABC be a right-angled triangle and CH be the altitude from its right angle C. Points O_1 and O_2 are the incenters of triangles ACH and BCH respectively; P_1 and P_2 are the touching points of their incircles with AC and BC. Prove that lines O_1P_1 and O_2P_2 meet on AB.

10. (8–9) Point X moves along side AB of triangle ABC, and point Y moves along its circumcircle in such a way that line XY passes through the midpoint of arc AB. Find the locus of the circumcenters of triangles IXY, where I is the incenter of ABC.

11. (8–10) Restore a triangle ABC by vertex B, the centroid and the common point of the symmedian from B with the circumcircle.

12. (9–10) Let BB_1 be the symmedian of a nonisosceles acute-angled triangle ABC. Ray BB_1 meets the circumcircle of ABC for the second time at point L. Let AH_A, BH_B, CH_C be the altitudes of triangle ABC. Ray BH_B meets the circumcircle of ABC for the second time at point T. Prove that H_A, H_C, T, L are concyclic.

13. (9–10) Given are a triangle ABC and a line ℓ meeting BC, AC, AB at points L_a, L_b, L_c respectively. The perpendicular from L_a to BC meets AB and AC at points A_B and A_C respectively. Point O_a is the circumcenter of triangle AA_bA_c. Points O_b and O_c are defined similarly. Prove that O_a, O_b and O_c are collinear.

14. Let a triangle ABC be given. Consider the circle touching its circumcircle at A and touching externally its incircle at some point A_1. Points B_1 and C_1 are defined similarly.

    a) (9–10) Prove that lines AA_1, BB_1 и CC_1 concur.

    b) (10–11) Let A_2 be the touching point of the incircle with BC. Prove that lines AA_1 and AA_2 are symmetric about the bisector of angle A.

15. (9–11) Let O, M, N be the circumcenter, the centroid and the Nagel point of a triangle. Prove that angle MON is right if and only if one of the triangle's angles is equal to $60°$.

16. (9–11) Let BB_1 and CC_1 be altitudes of triangle ABC. The tangents to the circumcircle of AB_1C_1 at B_1 and C_1 meet AB and AC at points M and N respectively. Prove that the common point of circles AMN and AB_1C_1 distinct from A lies on the Euler line of ABC.

17. (9–11) Let D be an arbitrary point on side BC of triangle ABC. Circles ω_1 and ω_2 pass through A and D in such a way that BA touches ω_1 and CA touches ω_2. Let BX be the second tangent from B to ω_1, and CY be the second tangent from C to ω_2. Prove that the circumcircle of triangle XDY touches BC.

18. (9–11) Let ABC be a triangle with $\angle C = 90°$, and K, L be the midpoints of the minor arcs AC and BC of its circumcircle. Segment KL meets AC at point N. Find angle NIC where I is the incenter of ABC.

19. (9–11) Let $ABCDEF$ be a regular hexagon. Points P and Q on tangents to its circumcircle at A and D respectively are such that PQ touches the minor arc EF of this circle. Find the angle between PB and QC.

20. (10–11) The incircle ω of a triangle ABC touches BC, AC and AB at points A_0, B_0 and C_0 respectively. The bisectors of angles B and C meet the perpendicular bisector to segment AA_0 at points Q and P respectively. Prove that PC_0 and QB_0 meet on ω.

21. (10–11) The areas of rectangles P and Q are equal, but the diagonal of P is greater. Rectangle Q can be covered by two copies of P. Prove that P can be covered by two copies of Q.

22. (10–11) Let M_A, M_B, M_C be the midpoints of the sides of a nonisosceles triangle ABC. Points H_A, H_B, H_C lying on the correspondent sides and distinct from M_A, M_B, M_C are such that $M_AH_B = M_AH_C$, $M_BH_A = M_BH_C$, $M_CH_A = M_CH_B$. Prove that H_A, H_B, H_C are the bases of the altitudes of ABC.

23. (10–11) A sphere touches all edges of a tetrahedron. Let a, b, c and d be the segments of the tangents to the sphere from the vertices of the tetrahedron. Is it true that that some of these segments necessarily form a triangle? (It is not obligatory to use all segments. The side of the triangle can be formed by two segments)

24. (11) A sphere is inscribed into a prism $ABCA'B'C'$ and touches its lateral faces $BCC'B'$, $CAA'C'$, $ABB'A'$ at points A_0, B_0, C_0 respectively. It is known that $\angle A_0BB' = \angle B_0CC' = \angle C_0AA'$.

    a) Find all possible values of these angles.

    b) Prove that segments AA_0, BB_0, CC_0 concur.

    c) Prove that the projections of the incenter to $A'B'$, $B'C'$, $C'A'$ are the vertices of a regular triangle.

XII Geometrical Olympiad in honour of I.F.Sharygin
The correspondence round. Solutions

1. (A.Trigub, 8) A trapezoid $ABCD$ with bases AD and BC is such that $AB = BD$. Let M be the midpoint of DC. Prove that $\angle MBC = \angle BCA$.

   Solution. Let the line BM meet AD at point K. Then $BCKD$ is a parallelogram, therefore $CK = BD = AB$. Thus we obtain, since $ABCK$ is an equilateral trapezoid, that $\angle BCA = \angle CBK = \angle MBC$ (fig.1).

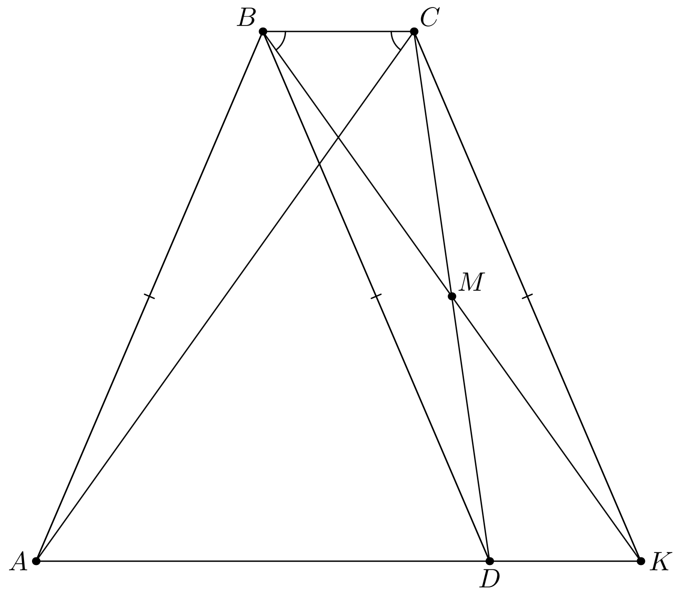


Fig.1

2. (L.Emelyanov, 8) Mark three nodes on a cellular paper so that the semiperimeter of the obtained triangle would be equal to the sum of its two smallest medians.

   Solution. Mark three vertices A, B, C of a right-angled triangle with legs $AC = 6$, $BC = 4$. Its median from C is equal to a half of hypothenuse AB, and its median from B by the Pythagorean theorem is equal to $\sqrt{BC^2 + (AC/2)^2} = \sqrt{4^2 + 3^2} = 5 = (AC + BC)/2$, hence ABC is the required triangle.

3. (E.Diomidov, 8) Let AH_1, BH_2 be two altitudes of an acute-angled triangle ABC, D be the projection of H_1 to AC, E be the projection of D to AB, F be a common point of ED and AH_1. Prove that $H_2F \parallel BC$.

   Solution. Let H be the orthocenter of triangle ABC. Using the Thales theorem we obtain (fig.3)
$$\frac{AF}{AH_1} = \frac{AF}{AH} \cdot \frac{AH}{AH_1} = \frac{AD}{AC} \cdot \frac{AH_2}{AD} = \frac{AH_2}{AC}.$$
   From this, also by the Thales theorem we obtain the required assertion.

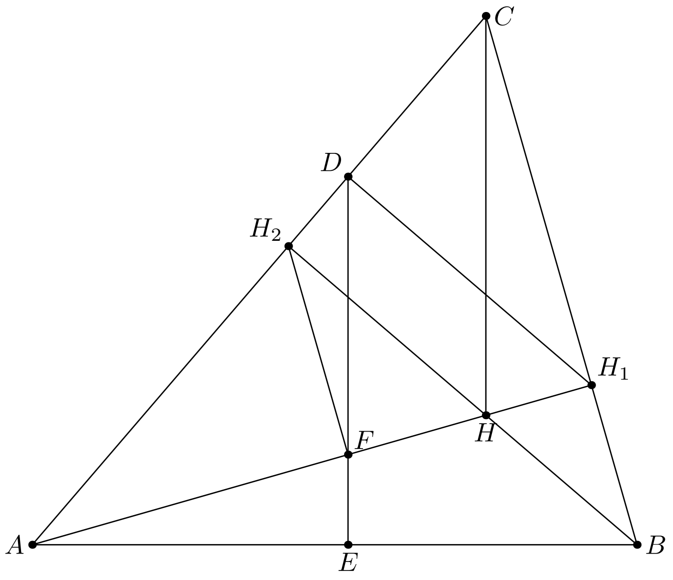


Fig.3

4. (A.Trigub, 8) In a quadrilateral $ABCD$ $\angle B = \angle D = 90^\circ$ and $AC = BC + DC$. The point P of ray BD is such that $BP = AD$. Prove that the line CP is parallel to the bisector of angle ABD.

   Solution. The assumption yields that the quadrilateral $ABCD$ is inscribed into the circle with diameter AC. Let K be a point of segment AC such that $AK{=}BC$ (fig.4). Then $CK = CD$, i.e. $\angle CKD = \angle CDK$. Now the triangles BCP and AKD are congruent because $AK = BC$, $AC = BP$ and $\angle KAD = \angle CAD = \angle CBD = \angle CBP$. Therefore $\angle BCP = \angle AKP = 180^\circ - \angle CKD = 90^\circ + \frac{\angle ACD}{2} = 90^\circ + \frac{\angle ABD}{2}$. On the other hand, $\angle CBP = 90^\circ - \angle ABD$, thus $\angle CPB = 180^\circ - \angle BCP - \angle CBP = \frac{\angle ABD}{2}$, and this yields the required assertion.

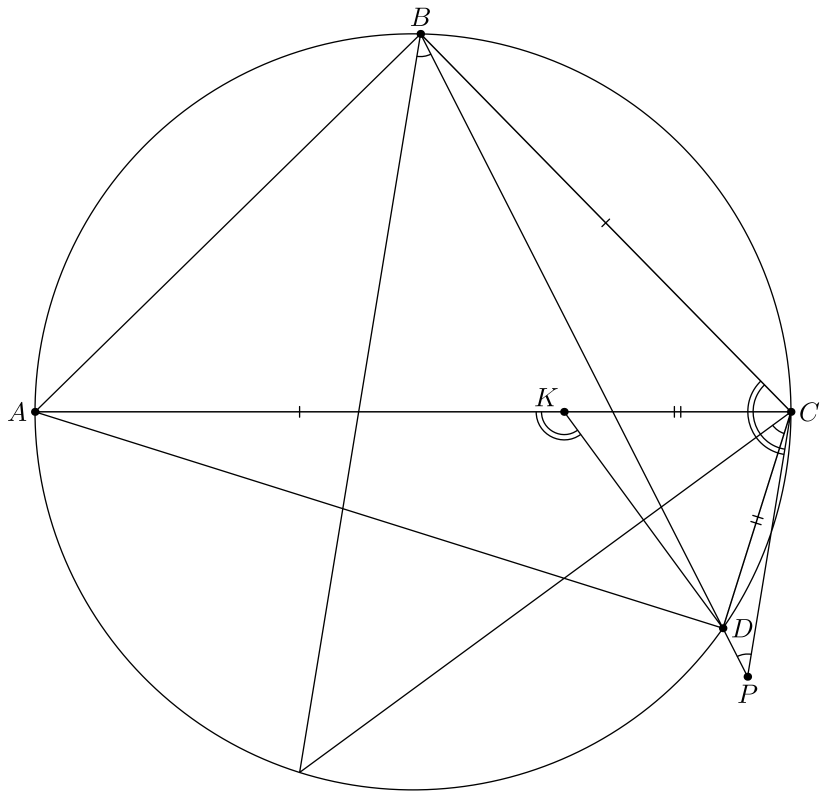


Fig.4

5. (M.Volchkevich, 8) In quadrilateral $ABCD$ $AB = CD$, M and K are the midpoints of BC and AD. Prove that the angle between MK and AC is equal to the half-sum of angles BAC and DCA.

   Solution. Construct parallelograms $ABMX$ and $DCMY$ (fig.5). Since $AX = BM = MC = DY$ and $AX \parallel BC \parallel DY$, triangles AXK and DYK are congruent. Hence $XK = KY$ and $\angle AKX = \angle DKY$, i.e. K is the midpoint of segment XY. Also we have $MX = AB = CD = MY$, therefore MK is the bisector of angle XMY, and this is equivalent to the required assertion.

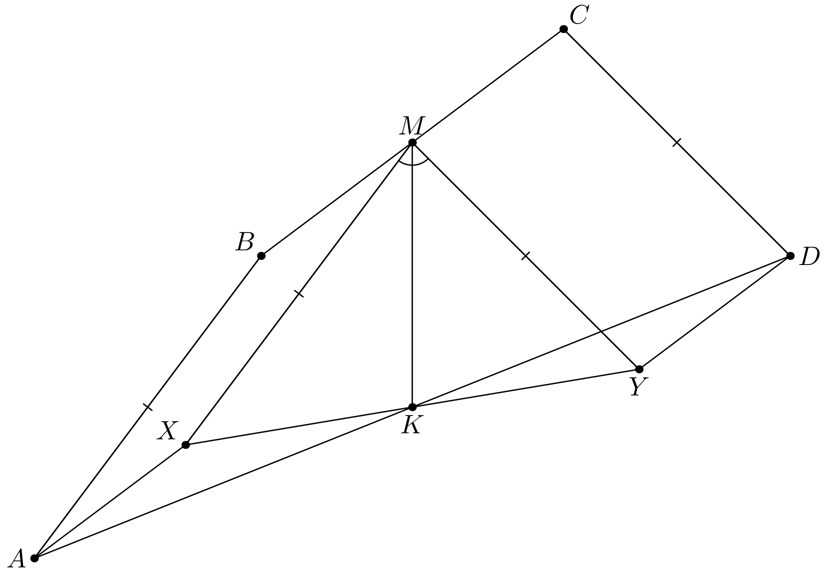


Fig.5

6. (M.Volchkevich, 8) Let M be the midpoint of side AC of triangle ABC, MD and ME be the perpendiculars from M to AB and to BC respectively. Prove that the distance between the circumcenters of triangles ABE and BCD is equal to $AC/4$.

   Solution. The segment between two circumcenters is a diagonal of the parallelogram formed by the perpendicular bisectors to segments AB, BD, BE and BC. Hence the projections of this segment to the lines AB and BC are equal to $AD/2$ and $CE/2$ respectively, i.e. they are equal to halves of the projections of segment $AM = MC$. Therefore the segment between the circumcenters is also equal to $AM/2 = AC/4$.

   Remark. From the solution we also obtain that this segment is parallel to AC.

7. (B.Frenkin, 8–9) Let all distances between the vertices of a convex n-gon ($n > 3$) be different.

   a) A vertex is called uninteresting if the closest vertex is adjacent to it. What is the minimal possible number of uninteresting vertices (for a given n)?

   b) A vertex is called unusual if the farthest vertex is adjacent to it. What is the maximal possible number of unusual vertices (for a given n)?

   Solution. a) **Answer**. 2.

   Example. Take a segment AB and a convex broken line ℓ close to it and having the same endpoints and the edges of equal length. Then ℓ and its reflection about AB form a convex polygon such that only vertices A and B are uninteresting in it. In such a way we obtain the desired n-gon for an arbitrary even $n > 2$. Now replace one of two copies of ℓ in the n-gon by an analogous broken line with the number of edges greater by 1. In this way we obtain a convex n-gon with an arbitrary odd $n > 3$, such that only the vertices A and B are uninteresting. In both cases a small shift of the vertices makes all distances between them different.

   Estimation. Let A be an interesting vertex of a convex n-gon, and B be the vertex closest to A. The diagonal AB divides the polygon into "right" and "left" parts. Let C be some vertex or right part distinct from A and B. Suppose that C is interesting and let D be the closest vertex. If D lies on the left part then in convex quadrilateral $ACBD$ we have $AB + CD < AD + CB$, i.e. the sum of the diagonals is less than the sum of two opposite sides, a contradiction. Thus D lies on the right part or on the boundary of two parts. Replacing vertices A, B to C, D we decrease the number of vertices in the right part. Since this process can not be infinite there exists an uninteresting vertex in the right part. Similarly there exists an uninteresting vertex in the left part therefore the number of uninteresting vertices is not less than two.

   b) **Answer**. 3.

   Example. Take a triangle ABC with $AB > BC > AC$. “Break” side AC a little to obtain a convex n-gon. Its unusual vertices are A, B, C only.

   Estimation. Let X be an unusual vertex, Y be the farthest vertex and Z be the vertex adjacent to Y and distinct from X. Then $XZ < XY$, hence angle XYZ is not the maximal angle of triangle XYZ and hence is acute.

Suppose that there exist more than three unusual vertices. A convex polygon has at most three acute angles. Thus there are two unusual vertices A and C for which the same vertex B is the farthest (and adjacent). Let D be an unusual vertex distinct from A, B, C and E be the farthest from it (and adjacent) vertex. Without loss of generality we can suppose that $ABED$ is a convex quadrilateral. In this quadrilateral $AB > AE, DE > BD$, i.e. the sum of the diagonals is less than the sum of two opposite sides, a contradiction.

8. (B.Frenkin, 8–9) Let $ABCDE$ be an inscribed pentagon such that $\angle B+\angle E = \angle C+\angle D$. Prove that $\angle CAD < \pi/3 < \angle A$.

   Solution. From the assumption we have $\smile AEDC+ \smile ABCD =\smile BAED+ \smile CBAE$, i.e. $\smile BAE = 2 \smile CD$. Since the sum of these two arcs is less than 2π, we obtain that $\smile CD < 2\pi/3$ and $\angle CAD > \pi/3$. On the other hand, since $\smile BAE < 4\pi/3$ we obtain $\smile BCDE > 2\pi/3$ and $\angle A > \pi/3$.

9. (M.Panov, 8–9) Let ABC be a right-angled triangle and CH be the altitude from its right angle C. The points O_1 and O_2 are the incenters of triangles ACH and BCH respectively; P_1 and P_2 are the touching points of their incircles with AC and BC. Prove that the lines O_1P_1 and O_2P_2 meet on AB.

   Solution. Let O_1P_1 and O_2P_2 meet AB at points K_1 and K_2. Then by Thales theorem $AK_1/K_1B = AP_1/P_1C$, $AK_2/K_2B = CP_2P_2B$. But these ratios are equal because triangles AHC and CHB are similar.

   Remark. From the solution we also obtain that the common point coincides with the touching point of the incircle of ABC with AB (fig.9).

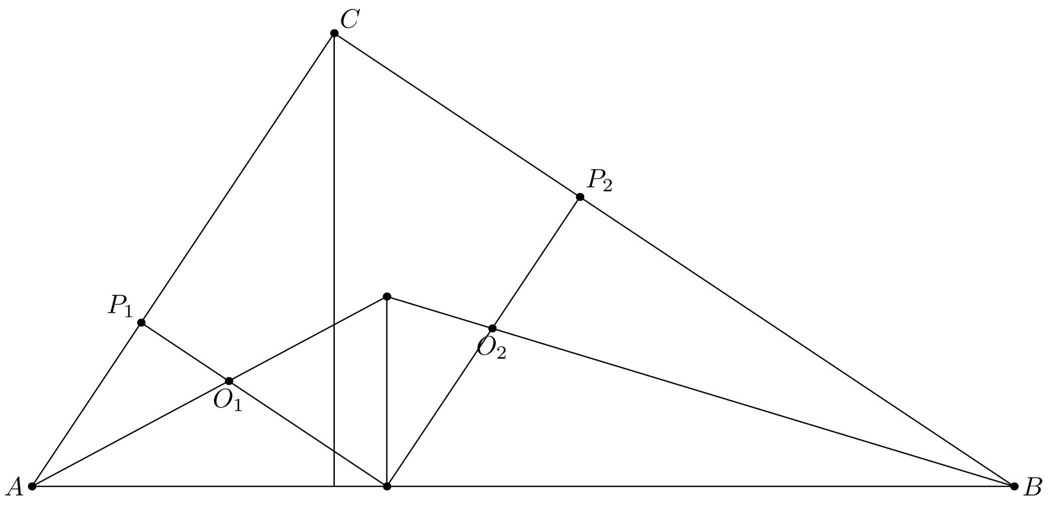


Fig.9

10. (D.Shvetsov, 8–9) The point X moves along the side AB of triangle ABC, and the point Y moves along its circumcircle in such a way that line XY passes through the midpoint of arc AB. Find the locus of the circumcenters of triangles IXY, where I is the incenter of ABC.

    Solution. Let U be the midpoint of arc AB. Since $\angle AYU = \angle ABU = \angle UAB$, triangles AUX and YUA are similar, i.e. $UX \cdot UY = UA^2$. It is known that U is the circumcenter of triangle IAB, therefore UI is a tangent to circle IXY (fig.10). Hence the center of this circle lies on the perpendicular from I to CI. Since the circle IXY cannot lie inside the circle ABC, the desired locus consists of two rays. The origins of these rays are the centers of two circles touching circle ABC internally and touching the side AB, i.e. the common points of the indicated line and the bisectors of the angles between AB and CU.

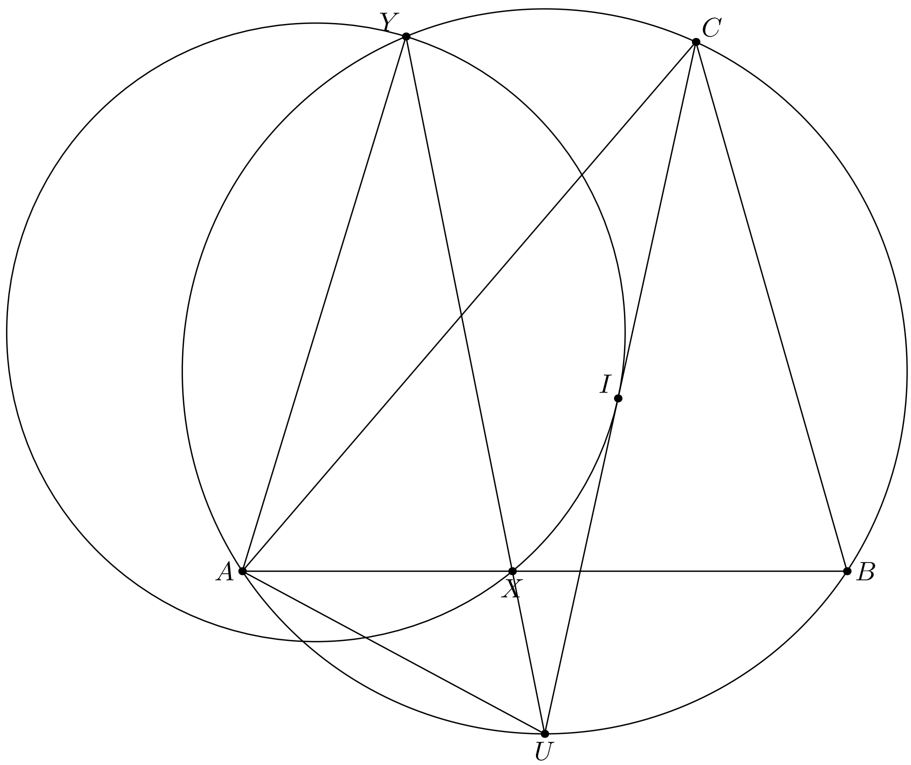


Fig.10

11. (A.Blinkov, 8–10) Restore a triangle ABC by vertex B, the centroid and the common point of the circumcircle and the symmedian going from B.

Solution. Let the median and the symmedian from B meet the circumcircle at points K and L respectively. Since $\angle ABK = \angle CBL$, the points K and L are equidistant from the midpoint M of AC (fig.11). From this we obtain the following construction.

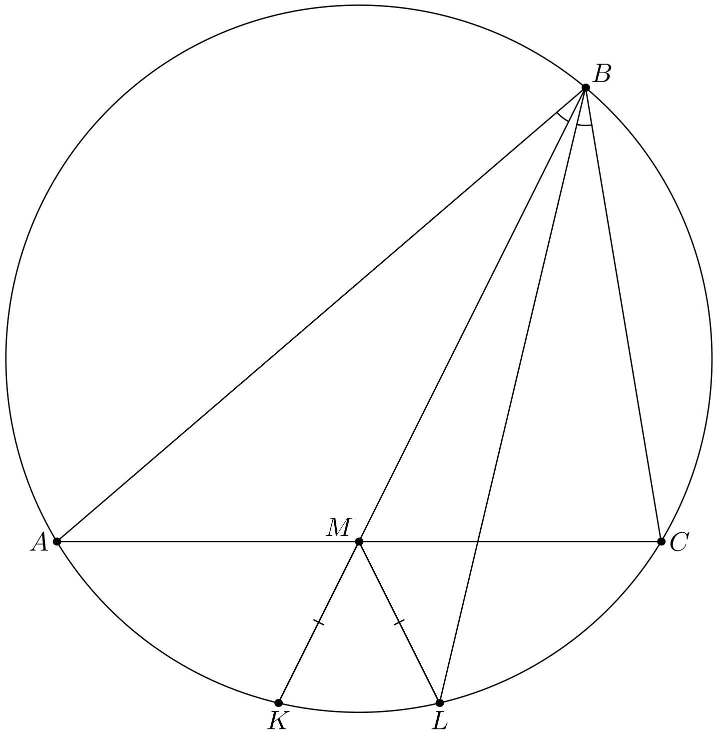


Fig.11

Extending the segment between B and the centroid by the half of its length we obtain point M. Construct the circle through L centered at M and find its common point K with BM, lying outside ray MB. Construct the circle BKL and find its common points A, C with the line passing through M and parallel to KL. The triangle ABC is the required one.

12. (S.Novikov, 9–10) Let BB_1 be the symmedian of a nonisosceles acute-angled triangle ABC. The ray BB_1 meets the circumcircle of ABC for the second time at point L. Let AH_A, BH_B, CH_C be the altitudes of triangle ABC. The ray BH_B meets the circumcircle of ABC for the second time at point T. Prove that H_A, H_C, T, L are concyclic.

First solution. Since the points A, C, H_A, H_C are concyclic it is sufficient to prove that the lines AC, H_AH_C and TL concur. Projecting the vertices of the harmonic quadrilateral $ABCL$ from T to the line AC we obtain that the common point of TL and AC forms a harmonic quadruple with A, C, H_B. The line H_AH_C meets AC at the same point.

Second solution. Let M be the midpoint of AC. Denote the circumcircles of triangles ABC, AHC, BH_AH_C and the circumcircle of quadrilateral AH_CH_AC by ω, ω_1, ω_2, ω_3 respectively. By the orthocenter's property the points H and T are symmetric about AC. Therefore the circles ω_1 and ω are also symmetric. Let ω_2 and ω meet for the second time at a point P, and let ω_2 and ω_1 meet for the second time at a point N.

It is known (see. for example the paper of Y.Blinkov "The orthocenter, the midpoint of the side, the common point of the tangents and one point more", Kvant, №1, 2014) that the points M, H and P are collinear, and $\angle BPH = 90°$.

Let the lines BP and AC meet at point S. Note that H is the orthocenter of triangle BMS. Therefore $SH \perp BM$. Since $SH \perp BN$ (because $\angle BNH = 90°$), we obtain that N lies on BM.

Let BM meet ω at a point D, and let the points N and N' be symmetric about AC. Since M is the midpoint of AC, and the arcs ANC and $AN'C$ are symmetric we obtain that the arcs AD and CN' of ω are equal. Line BD contains the median from B. Therefore BN' the symmedian of triangle ABC, i.e. the points N' and L coincide.

The lines NH and LT are symmetric about AC therefore they meet at S. Since S is the radical center of ω, ω_1, ω_2, ω_3 we obtain that S lies on H_CH_A (fig.12). The degrees of S wrt ω and ω_2 are equal, i.e. $SH_A \cdot SH_C = ST \cdot SL$. Therefore H_A, H_C, T, L are concyclic.

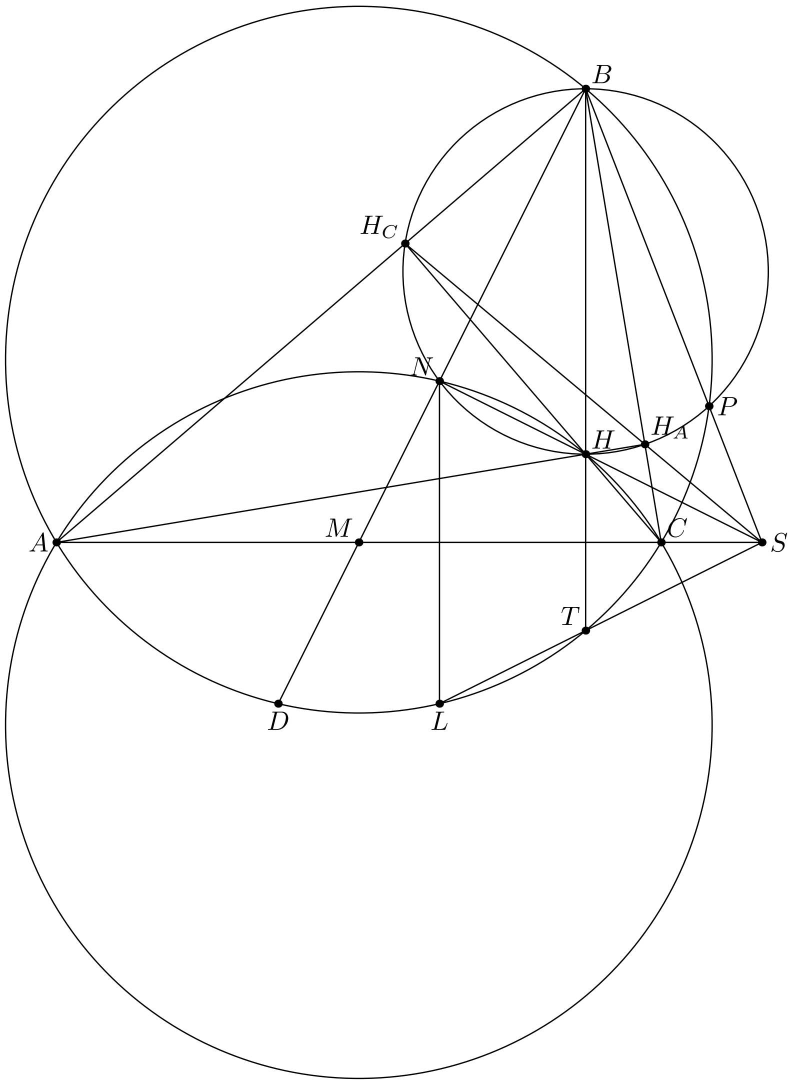


Fig.12

13. (R.Krytovsky, I.Frolov, 9–10) Given are a triangle ABC and a line ℓ meeting BC, AC, AB at points L_a, L_b, L_c respectively. The perpendicular from L_a to BC meets AB and AC at points A_B and A_C respectively. Point O_a is the circumcenter of triangle AA_bA_c. Points O_b and O_c are defined similarly. Prove that O_a, O_b and O_c are collinear.

Solution. Let Z be an arbitrary point of line AB; X, Y be the common points of the perpendicular from Z to AB with BC and CA respectively; and Z' be the circumcenter of triangle CXY. Then $\angle Z'CA = \pi/2 - \angle CXY = \angle B$, i.e. $Z'C$ touches the circumcircle of triangle ABC. If Z moves uniformly along AB then Z' also moves uniformly, and when Z coincides with A or B then Z' lies on the tangent to the circumcirle at this point. Thus if $A'B'C'$ is the triangle formed by three tangents then Z' divides segment $A'B'$ in the same ratio as Z divides AB. Applying this to points O_a, O_b, O_c and using the Menelaus theorem we obtain the required asssertion.

14. (A.Myakishev) Let a triangle ABC be given. Consider the circle touching its circumcircle at A and touching externally its incircle at some point A_1. Points B_1 and C_1 are defined similarly.

    a) (9–10) Prove that lines AA_1, BB_1 и CC_1 concur.

    b) (10–11) Let A_2 be the touching point of the incircle with BC. Prove that the lines AA_1 and AA_2 are symmetric about the bisector of angle A.

    Solution. a) Denote the first of the indicated circles by α. The point A is the center of the positive homothehy of α and the circumcircle of triangle ABC, and the point A_1 is the center of the negative homothety of α and this incircle. Therefore the line AA_1 passes through the center of the negative homothety between the incircle and the circumcircle. Two remaining lines also pass through this point.

    b) It is known that the center of the negative homothety between the incircle and the circumcircle is isogonally conjugated to the Gergonne point lying on the lines AA_2, BB_2 and CC_2. The desired assertion immediately follows from this.

15. (L.Emelyanov, 9–11) Let O, M, N be the circumcenter, the centroid and the Nagel point of a triangle. Prove that angle MON is right if and only if one of the triangle's angles is equal to 60°.

    Solution. Let I, H be the incenter and the orthocenter respectively of the triangle. The homothety with center M and coefficient $-1/2$ maps N and H to I and O respectively. Thus $\angle MON = \pi/2$ if and only if $IO = IH$. Let the line OH intersect the segments AC and BC. Then since AI and BI are the bisectors of angles HAO and HBO, we obtain that the points A, B, O, I, H are concyclic. Therefore $\angle AOB = 2\angle C = \angle AHB = \pi - \angle C$ and $\angle C = 60°$. The inverse assertion can be proved similarly.

16. (A.Doledenok, 9–11) Let BB_1 and CC_1 be the altitudes of triangle ABC. The tangents to the circumcircle of AB_1C_1 at B_1 and C_1 meet AB and AC at points M and N respectively. Prove that the common point of circles AMN and AB_1C_1 distinct from A lies on the Euler line of ABC.

    Solution. Let A_0, B_0, C_0 be the midpoints of BC, CA, AB; O, H be the circumcenter and the orthocenter of triangle ABC. The projection Z of A to line OH lies on circles AB_1HC_1 and AB_0OC_0, i.e., Z is the center of the spiral similarity mapping C_0 to B_0, and C_1 to B_1. Thus if we prove that this similarity maps M to N we obtain that circle AMN passes through Z.

    Note that point A_0 and the center of circle AB_1HC_1 are opposite on the nine points circle of triangle ABC. Hence lines A_0B_1 and A_0C_1 are tangents to circle AB_1HC_1, i.e. they coincide with lines B_1M and C_1N (fig.16). Projecting line AC to AB from point A_0 we obtain that $(N, B_1, B_0, \infty) = (C_1, M, \infty, C_0)$ or $NB_0/NB_1 = MC_0/MC_1$, q.e.d.

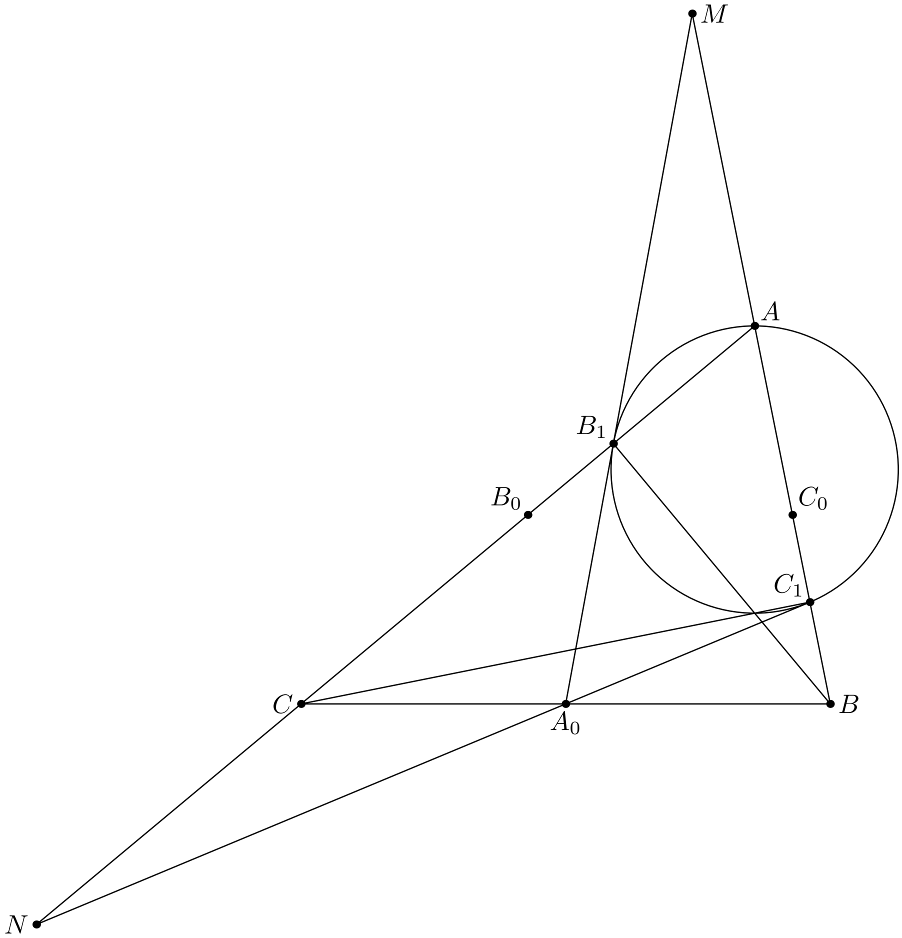


Fig.16

17. (D.Hilko, 9–11) Let D be an arbitrary point on the side BC of triangle ABC. The circles ω_1 and ω_2 pass through A and D in such a way that BA touches ω_1 and CA touches ω_2. Let BX be the second tangent from B to ω_1, and CY be the second tangent from C to ω_2. Prove that the circumcircle of triangle XDY touches BC.

    Solution. Take an inversion with center D and an arbitrary radius. Denote the images of all points by primes (fig.17).

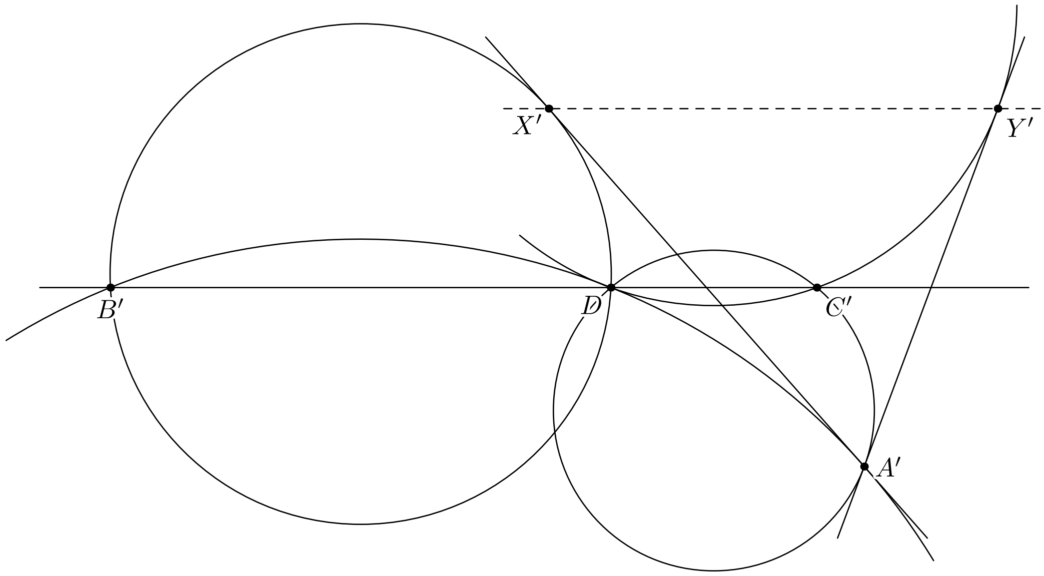


Fig.17

The circumcircle of triangle XDA touches BA and BX. Therefore the circumcircles of triangles $B'DA'$ and $B'DX'$ touch line $X'A'$. Then the radical axis $B'D$ of these circles bisects segment $X'A'$. Similarly circles $DC'Y'$ and $DC'A'$ touch line $Y'A'$. Then their radical axis DC' bisects segment $A'Y'$. Hence $B'C'$ is the medial line of triangle $X'A'Y'$ and $X'Y' \parallel B'D'$. Observe now that $X'Y'$ is the image of the circle passing through X, Y, D. Since $X'Y' \parallel B'C'$ this circle touches BC at point E.

18. (N.Moskvitin, 9–11) Let ABC be a triangle with $\angle C = 90°$, and K, L be the midpoints of the minor arcs AC and BC of its circumcircle. The segment KL meets AC at point N. Find angle NIC where I is the incenter of ABC.

    Answer. $45°$.

    Solution. It is known that points K and P are the circumcenters of triangles IAC and IBC respectively. Thus KP is the perpendicular bisector for segment CI. Then N is the touching point of AC with the incircle and $\angle NIC = 45°$ (fig.18).

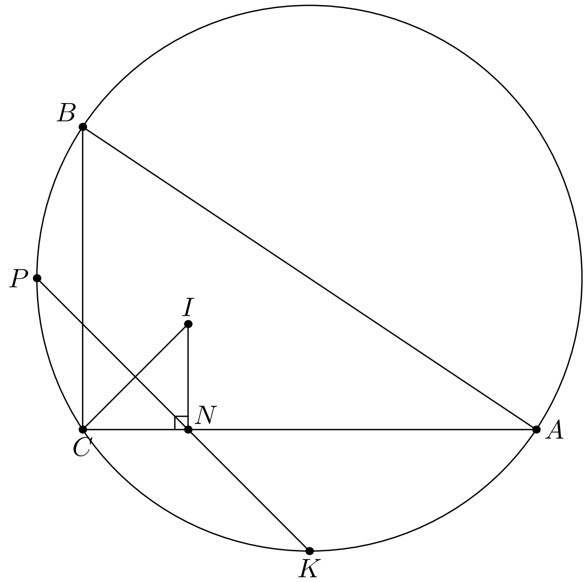


Fig.18

19. (A.Skutin, 9–11) Let $ABCDEF$ be a regular hexagon. The points P and Q on tangents to its circumcircle at A and D respectively are such that PQ touches the minor arc EF of this circle. Find the angle between PB and QC.

    Answer. $30°$.

    Solution. Let T be the touching point of PQ with the circle and M, N be the midpoints of segments AT, DT. Since PB and CQ are the symmedians of the triangles ABT, CDT respectively, we have $\angle ABP = \angle MBT$, $\angle DCQ = \angle NCT$. Since MN is the medial line of triangle ADT, we have $MN = AD/2 = BC$ and $MN \parallel BC$ (fig.19). Thus the angle between PB and QC is equal to $\angle PBM + \angle NCQ = \angle ABM + \angle NCD - \angle MBT - \angle TCN = 30°$.

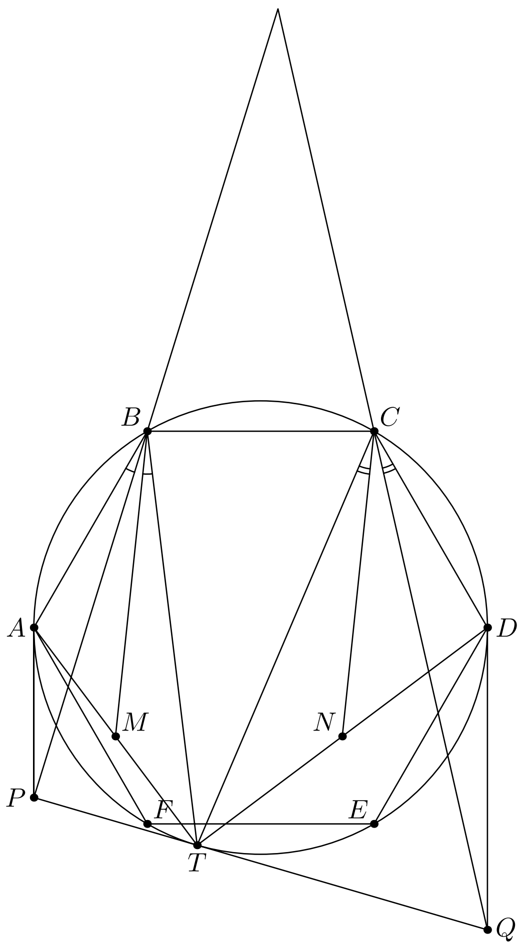


Fig.19

20. (D.Prokopenko, 10–11) The incircle ω of a triangle ABC touches BC, AC and AB at points A_0, B_0 and C_0 respectively. The bisectors of angles B and C meet the perpendicular bisector to segment AA_0 at points Q and P respectively. Prove that PC_0 and QB_0 meet on ω.

    Solution. The definition of points P, Q implies that they lie on the circumcircles of triangles ABA_0 and ACA_0 respectively. Therefore triangle AA_0Q is similar to triangle B_0A_0I, and triangle AA_0P is similar to triangle C_0A_0I (by three angles). Thus $A_0Q \cdot A_0B_0 = A_0I \cdot A_0A = A_0P \cdot A_0C_0$. Furthermore $\angle PA_0Q = (\angle B + \angle C)/2 = \angle B_0A_0C_0$, hence triangles A_0PQ and $A_0B_0C_0$ are similar (fig.20). Then triangles A_0B_0P and A_0C_0Q also are similar, i.e. the angle between B_0P and C_0Q is equal to angle $B_0A_0C_0$, and this yields the required assertion.

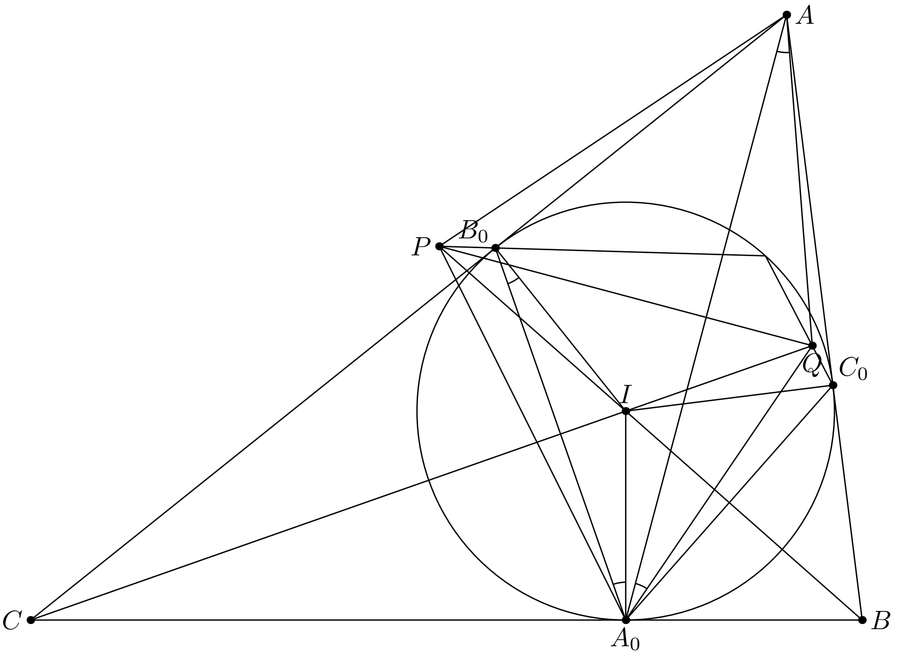


Fig.20

21. (A.Shapovalov, 10–11) The areas of rectangles P and Q are equal, but the diagonal of P is greater. Rectangle Q can be covered by two copies of P. Prove that P can be covered by two copies of Q.

    Solution. Let the width and the length of a rectangle mean its smaller and greater side respectively. From the assumption we have that the width of P is less than the width of Q, and the length of P is greater than the length of Q. If two copies of P cover Q, then they cover the disc with the diameter equal to the width of Q, therefore this disc can be covered by two bars with the width equal to the width of P. But the disc cannot be covered by bars if the sum of their widths is less than the diameter. Thus the width of P is at least the half of the width of Q. Then the length of Q is at least the half of the length of P, and clearly P can be covered by two copies of Q.

22. (A.Yakubov, 10–11) Let M_A, M_B, M_C be the midpoints of the sides of a nonisosceles triangle ABC. The points H_A, H_B, H_C lying on the corresponding sides and distinct from M_A, M_B, M_C are such that $M_AH_B = M_AH_C$, $M_BH_A = M_BH_C$, $M_CH_A = M_CH_B$. Prove that H_A, H_B, H_C are the bases of the altitudes of ABC.

    Solution. Consider a point X in the space such that $XM_A = M_AH_B$, $XM_B = M_BH_A$, $XM_C = M_CH_A$. Consider tetrahedron $XM_AM_BM_C$. The areas of all its faces are equal because triangles XM_AM_B and $H_CM_AM_B$ are congruent. Hence all faces are congruent and points H_C, M_A, M_B, M_C are concyclic. Therefore H_C is the base of the altitude.

23. (F.Ivlev, 10–11) A sphere touches all edges of a tetrahedron. Let a, b, c and d be the segments of the tangents to the sphere from the vertices of the tetrahedron. Is it true that some of these segments necessarily form a triangle? (Not all those segments must be used. Two segments may form one side of the triangle.)

    Answer. No.

Solution. Let β and γ be circles of radii 2 and 1 respectively that lie in the plane and touch externally. Construct their common external tangent and inscribe circle δ into the curvilinear triangle formed by two circles and this tangent. Clearly the radius of δ is less than 1, so the radii of three circles do not form a triangle. Now replace the common tangent of β and γ by circle α touching them externally with radius greater than 4. Construct three spheres with the same centers and radii as α, β, γ. Finally construct the sphere having the same radius as δ and touching three remaining spheres. The centers of these four spheres form a tetrahedron, and their touching points lie on the sphere touching all edges of this tetrahedron. The segments a, b, c, d are equal to the radii of α, β, γ, δ, therefore they don't form a triangle.

24. (I.I.Bogdanov, 11) A sphere is inscribed into a prism $ABCA'B'C'$ and touches its lateral faces $BCC'B'$, $CAA'C'$, $ABB'A'$ at points A_0, B_0, C_0 respectively. It is known that $\angle A_0BB' = \angle B_0CC' = \angle C_0AA'$.

a) Find all possible values of these angles.

b) Prove that segments AA_0, BB_0, CC_0 concur.

c) Prove that the projections of the incenter to $A'B'$, $B'C'$, $C'A'$ are the vertices of a regular triangle.

Solution. a) **Answer.** 60°.

Denote the value of these angles by θ. Since the triangles $CC'A_0$ and $CC'B_0$ are congruent we obtain that the angle A_0CC' is also equal to θ. Similarly $\angle B_0AA' = \angle C_0BB' = \theta$. Then $6\theta = 3\pi - (\angle C_0AB + \angle C_0AC + \angle A_0BC + \angle A_0CB + \angle B_0CA + \angle B_0AC)$. But for example $\angle C_0AB = \angle TAB$, where T is the touching point of the sphere with face ABC. From this and five similar equalities we obtain that the sum in the parentheses is equal to the sum of the angles of triangle ABC, i.e. $\theta = 60^\circ$.

b) By the previous part, $\angle AB_0C = \angle BA_0C = 2\pi/3$. Thus the lines AB_0 and BA_0 meet CC' at the same point K such that $CK = CB_0 = CA_0$ (the triangles CB_0K and CA_0K are regular because each of them has two angles equal to $\pi/3$). Therefore the points A, B, A_0, B_0 are complanar, i.e. the lines AA_0 and BB_0 intersect. Similarly each of these lines intersects CC_0. Since these three lines are not complanar the points of intersection coincide.

c) By the previous part, $\angle ATB = \angle BTC = \angle CTA = 2\pi/3$, i.e. T is the Torricelli point of triangle ABC. Consider another sphere touching the plane ABC from the opposite side at point T' and touching the planes of lateral faces. The ratios of distances from T and T' to the sidelines of ABC are equal to the ratios of the cotangents and tangents of the halves of the corresponding dihedral angles, therefore these points are isogonally conjugate in triangle ABC. Hence the insphere touches the face $A'B'C'$ at its Apollonius point. The projections of this point to $A'B'$, $B'C'$, $C'A'$ coincide with the projection of the center of the sphere and form a regular triangle.

XIII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
The correspondence round

Below is the list of problems for the first (correspondence) round of the XIII Sharygin Geometrical Olympiad.

The olympiad is intended for high-school students of four eldest grades. In Russian school, these are 8-11. In the list below, each problem is indicated by the numbers of Russian school grades, for which it is intended. Foreign students of the last grade have to solve the problems for 11th grade, students of the preceding grade solve the problems for 10th grade etc. However, the participants may solve problems for elder grades as well (solutions of problems for younger grades will not be considered).

A complete solution of each problem costs 7 points. A partial solution costs from 1 to 6 points. A text without significant advancement costs 0 points. The result of a participant is the sum of all obtained marks.

In your work, please start the solution for each problem in a new page. First write down the statement of the problem, and then the solution. Present your solutions in detail, including all necessary arguments and calculations. Provide all necessary figures of sufficient size. If a problem has an explicit answer, this answer must be presented distinctly. Please, be accurate to provide good understanding and correct estimating of your work !

If your solution depends on some well-known theorems from standard textbooks, you may simply refer to them instead of providing their proofs. However, any fact not from the standard curriculum should be either proved or properly referred (with an indication of the source).

You may note the problems which you liked most (this is not obligatory). Your opinion is interesting for the Jury.

The solutions for the problems (in Russian or in English) must be delivered not earlier than on January 8, 2017 and not later than on April 1, 2017. To upload your work, enter the site **http://geometry.ru/olimp/olimpsharygin.php** and follow the instructions.
Attention: The solutions must be contained in pdf, doc or jpg files. We recommend to prepare the paper using computer or to scan it rather than to photograph it. *In the last two cases, please check readability of the file before uploading.*

If you have any technical problems with uploading of the work, apply to **geomolymp@mccme.ru (DON'T SEND your work to this address)**.

Winners of the correspondence round, the students of three grades before the last grade, will be invited to the final round held in Summer 2017 in Moscow region. (For instance, if the last grade is 12 then we invite winners from 9, 10, and 11 grade.) The students of the last grade, winners of the correspondence round, will be awarded by diplomas of the Olympiad. The list of the winners will be published on **www.geometry.ru** at the end of May 2017 at latest. If you want to know your detailed results, please use e-mail **geomolymp@mccme.ru**.

1. (8) Mark on a cellular paper four nodes forming a convex quadrilateral with the sidelengths equal to four different primes.

2. (8) A circle cuts off four right-angled triangles from rectangle $ABCD$. Let A_0, B_0, C_0 and D_0 be the midpoints of the correspondent hypotenuses. Prove that $A_0C_0 = B_0D_0$.

3. (8) Let I be the incenter of triangle ABC; H_B, H_C the orthocenters of triangles ACI and ABI respectively; K the touching point of the incircle with the side BC. Prove that H_B, H_C and K are collinear.

4. (8) A triangle ABC is given. Let C' be the vertex of an isosceles triangle ABC' with $\angle C' = 120^\circ$ constructed on the other side of AB than C, and B' be the vertex of an equilateral triangle ACB' constructed on the same side of AC as ABC. Let K be the midpoint of BB'. Find the angles of triangle KCC'.

5. A segment AB is fixed on the plane. Consider all acute-angled triangles with side AB. Find the locus of

   а) (8) the vertices of their greatest angles;

   б) (8–9) their incenters.

6. (8–9) Let $ABCD$ be a convex quadrilateral with $AC = BD = AD$; E and F the midpoints of AB and CD respectively; O the common point of the diagonals. Prove that EF passes through the touching points of the incircle of triangle AOD with AO and OD.

7. (8–9) The circumcenter of a triangle lies on its incircle. Prove that the ratio of its greatest and smallest sides is less than two.

8. (8–9) Let AD be the base of trapezoid $ABCD$. It is known that the circumcenter of triangle ABC lies on BD. Prove that the circumcenter of triangle ABD lies on AC.

9. (8–9) Let C_0 be the midpoint of hypotenuse AB of triangle ABC; AA_1, BB_1 the bisectors of this triangle; I its incenter. Prove that the lines C_0I and A_1B_1 meet on the altitude from C.

10. (8–10) Points K and L on the sides AB and BC of parallelogram $ABCD$ are such that $\angle AKD = \angle CLD$. Prove that the circumcenter of triangle BKL is equidistant from A and C.

11. (8–11) A finite number of points is marked on the plane. Each three of them are not collinear. A circle is circumscribed around each triangle with marked vertices. Is it possible that all centers of these circles are also marked?

12. (9–10) Let AA_1, CC_1 be the altitudes of triangle ABC, B_0 the common point of the altitude from B and the circumcircle of ABC; and Q the common point of the circumcircles of ABC and $A_1C_1B_0$, distinct from B_0. Prove that BQ is the symmedian of ABC.

13. (9–11) Two circles pass through points A and B. A third circle touches both these circles and meets AB at points C and D. Prove that the tangents to this circle at these points are parallel to the common tangents of two given circles.

14. (9–11) Let points B and C lie on the circle with diameter AD and center O on the same side of AD. The circumcircles of triangles ABO and CDO meet BC at points F and E respectively. Prove that $R^2 = AF \cdot DE$, where R is the radius of the given circle.

15. (9–11) Let ABC be an acute-angled triangle with incircle ω and incenter I. Let ω touch AB, BC and CA at points D, E, F respectively. The circles ω_1 and ω_2 centered at J_1 and J_2 respectively are inscribed into $ADIF$ and $BDIE$. Let J_1J_2 intersect AB at point M. Prove that CD is perpendicular to IM.

16. (9–11) The tangents to the circumcircle of triangle ABC at A and B meet at point D. The circle passing through the projections of D to BC, CA, AB, meet AB for the second time at point C'. Points A', B' are defined similarly. Prove that AA', BB', CC' concur.

17. (9–11) Using a compass and a ruler, construct a point K inside an acute-angled triangle ABC so that $\angle KBA = 2\angle KAB$ and $\angle KBC = 2\angle KCB$.

18. (9–11) Let L be the common point of the symmedians of triangle ABC, and BH be its altitude. It is known that $\angle ALH = 180° - 2\angle A$. Prove that $\angle CLH = 180° - 2\angle C$.

19. (10–11) Let cevians AA', BB' and CC' of triangle ABC concur at point P. The circumcircle of triangle $PA'B'$ meets AC and BC at points M and N respectively, and the circumcircles of triangles $PC'B'$ and $PA'C'$ meet AC and BC for the second time respectively at points K and L. The line c passes through the midpoints of segments MN and KL. The lines a and b are defined similarly. Prove that a, b and c concur.

20. (10–11) Given a right-angled triangle ABC and two perpendicular lines x and y passing through the vertex A of its right angle. For an arbitrary point X on x define y_B and y_C as the reflections of y about XB and XC respectively. Let Y be the common point of y_b and y_c. Find the locus of Y (when y_b and y_c do not coincide).

21. (10–11) A convex hexagon is circumscribed about a circle of radius 1. Consider the three segments joining the midpoints of its opposite sides. Find the greatest real number r such that the length of at least one segment is at least r.

22. (10–11) Let P be an arbitrary point on the diagonal AC of cyclic quadrilateral $ABCD$, and PK, PL, PM, PN, PO be the perpendiculars from P to AB, BC, CD, DA, BD respectively. Prove that the distance from P to KN is equal to the distance from O to ML.

23. (10–11) Let a line m touch the incircle of triangle ABC. The lines passing through the incenter I and perpendicular to AI, BI, CI meet m at points A', B', C' respectively. Prove that AA', BB' and CC' concur.

24. (11) Two tetrahedrons are given. Each two faces of the same tetrahedron are not similar, but each face of the first tetrahedron is similar to some face of the second one. Does this yield that these tetrahedrons are similar?

XIII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
The correspondence round. Solutions

1. (A.Zaslavsky) (8) Mark on a cellular paper four nodes forming a convex quadrilateral with the sidelengths equal to four different primes.

   Solution. Take for example a quadrilateral with vertices $A(-3,0)$, $B(0,4)$, $C(12,-1)$, $D(12,-8)$. Its sidelengths are $AB=5$, $BC=13$, $CD=7$, $DA=17$.

2. (L.Shteingarts) (8) A circle cuts off four right-angled triangles from rectangle $ABCD$. Let A_0, B_0, C_0 and D_0 be the midpoints of the correspondent hypotenuses. Prove that $A_0C_0=B_0D_0$.

   Solution. Let the circle meet AB, BC, CD, DA at points K_1, K_2, L_1, L_2, M_1, M_2, N_1, N_2. Then $K_1K_2M_2M_1$ is an isosceles trapezoid, i.e. $AK_1-DM_1=BK_2-CM_2$, or $AK_1+CM_2=BK_2+DM_1$. Hence the projections of segments A_0C_0 and B_0D_0 to AB, equal to $AB-(AK_1+CM_2)/2$ and $AB-(BK_2+DM_1)/2$ respectively, are congruent. Similarly their projections to BC are congruent, therefore the lengths of these segments are equal.

3. (M.Plotnikov) (8) Let I be the incenter of triangle ABC; H_B, H_C the orthocenters of triangles ACI and ABI respectively; K the touching point of the incircle with the side BC. Prove that H_B, H_C and K are collinear.

   Solution. Since BH_B and CH_C are perpendicular to AI, the quadrilateral BH_BCH_C is a trapezoid and its diagonals divide each other as $BH_B:CH_C$. Since the projections M, N of H_B, H_C to AB and AC respectively coincide with the projections of I to these lines, we obtain that $BM=BK$ and $CN=CK$. Also since $\angle H_BBM=\angle H_CCN=90^\circ-\angle A/2$, the right-angled triangles H_BBM and H_CCN are similar. Therefore $BH_B:CH_C=BK:CK$, and the diagonals of the trapezoid meet at K (fig. 3).

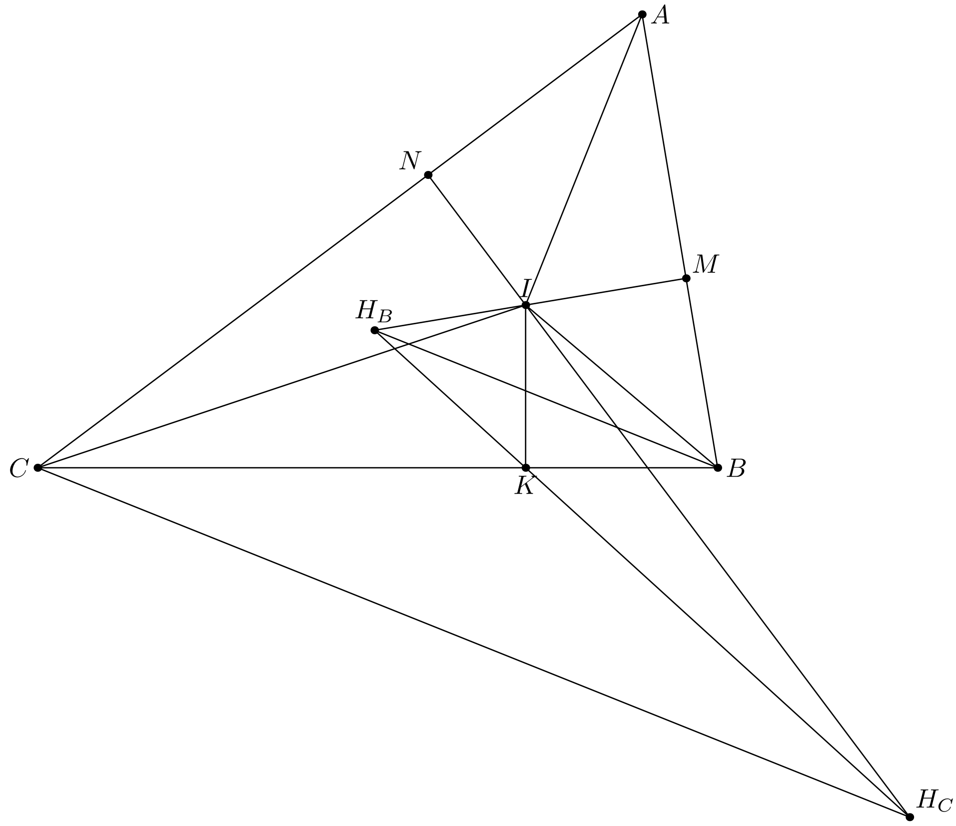


Fig. 3

4. (A.Zaslavsky) (8) A triangle ABC is given. Let C' be the vertex of an isosceles triangle ABC' with $\angle C' = 120°$ constructed on the other side of AB than C, and B' be the vertex of an equilateral triangle ACB' constructed on the same side of AC as ABC. Let K be the midpoint of BB'. Find the angles of triangle KCC'.

   Answer. $90°$, $30°$, $60°$.

   Solution. Let C'' be a vertex of parallelogram $B'C'BC''$. Then $B'C'' = BC' = AC'$, $B'C = AC$ and $\angle CB'C'' = \angle CAC'$ because the angle between $C''B'$ and AC' is equal to $\angle B'CA = 60°$. Therefore the triangles $C''B'C$ and $C'AC$ are congruent, and the angle between their corresponding sidelines $C''C$ and $C'C$ is equal to $60°$ (fig. 4). Thus the triangle $CC'C''$ is regular, and since K is the midpoint of $C'C''$, we obtain that $CK \perp C'K$ and $\angle C'CK = 30°$.

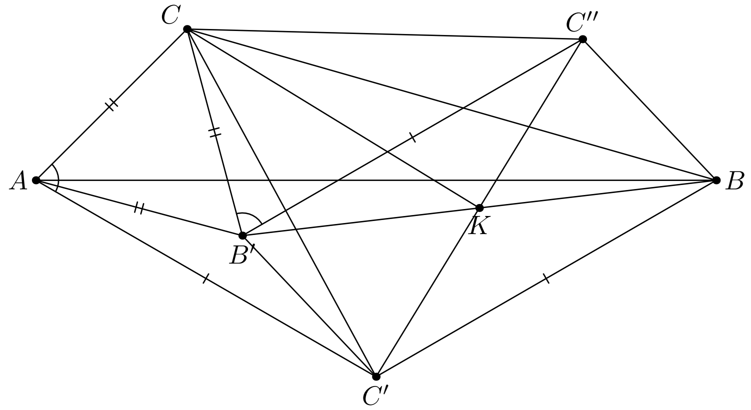


Fig. 4

This reasoning can be also formulated in the following way. Consider the rotations around C' by 120° and around C by 60°. Their composition maps B to B', hence it is the reflection about K. Since it maps C to C'', we obtain the indicated answer.

5. (B.Frenkin) A segment AB is fixed on the plane. Consider all acute-angled triangles with side AB. Find the locus of

   a) (8) the vertices of their greatest angles;

   b) (8–9) their incenters.

   Answer. a) The points A, B and the set of points lying inside or on the boundary of the intersection of two discs centered at A and B with radii AB, but outside the disc with diameter AB. b) The set of points lying inside the square $AKBL$, but outside the intersection of two discs centered at K and L with radii KA.

   Solution. a) If the vertex of the greatest angle does not coincide with A or B then AB is the greatest side of triangle ABC, i.e. $CA \leq AB$ and $CB \leq AB$. On the other hand, since angle C is acute, we obtain that C lies outside the circle with diameter AB.

   b) Let I be the incenter of ABC. Since angles A and B are acute, we have $\angle IAB < 45^\circ$ and $\angle IBA < 45^\circ$, i.e. I lies inside the square $AKBL$. On the other hand, since angle C is acute, we obtain that $\angle AIB < 135^\circ$ and I lies outside the intersection of the discs centered at K, L with radii KA.

6. (N.Moskvitin) (8–9) Let $ABCD$ be a convex quadrilateral with $AC = BD = AD$; E and F the midpoints of AB and CD respectively; O the common point of the diagonals. Prove that EF passes through the touching points of the incircle of triangle AOD with AO and OD.

   Solution. Let X, Y, Z be the touching points of the incircle with AO, OD, AD respectively. Then $DY = DZ$ and therefore $BY = AZ = AX$. Furthermore $OX = OY$. Applying the Menelaus theorem to the triangle AOB and the line XY, we obtain that this line passes through E. Similarly it passes through F (fig. 6).

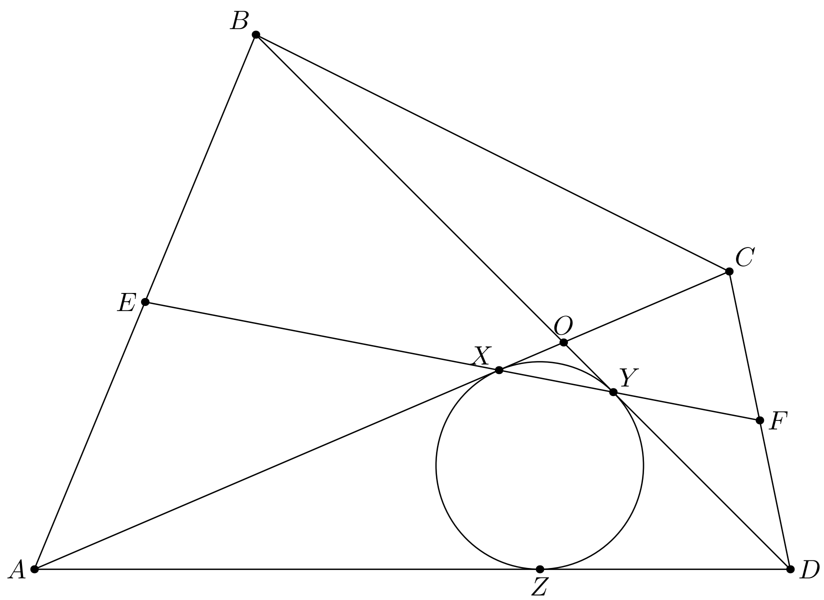


Fig. 6

7. (B.Frenkin) (8–9) The circumcenter of a triangle lies on its incircle. Prove that the ratio of its greatest and smallest sides is less than two.

   First solution. Since the circumcenter belongs to the given triangle ABC, this triangle is not obtuse-angled. If it is right-angled then the circumcenter O is the midpoint of its hypothenuse and coincides with the touching point of the incircle. Therefore the triangle is isosceles and right-angled and the assertion of the problem is valid. Suppose that the triangle is acute-angled and O lies on one of three arcs between the touching points. Let this arc is faced to the vertex A. Construct the perpendiculars from O to AB and AC. The foot of each of them (the midpoint of the corresponding side) lies between A and the touching point of the incircle ω with the side. Therefore, $AB > BC$ and $AC > BC$.

   Now we have to prove that the ratio of each of sides AB, AC to BC is less than 2. For example let D be the midpoint of AB. Let us prove that $AD < BC$. Let K and L be the touching points of ω with AB and BC. Then $BK = BL$, and we have to prove that $DK < CL$. But the perpendicular from D to AB passes through the point O on ω, hence DK is not greater than its radius. On the other hand CL is greater than the radius, because the perpendicular from C to BC does not intersect ω (the angle between BC and the tangent CA is acute). Q.e.d.

   Second solution. Use the Euler formula: $OI^2 = R^2 - 2Rr$, where I is the incenter, R, r are the radii of the circumcircle and the incircle. Since $OI = r$ we obtain that $r/R = \sqrt{2} - 1$. Each side of the triangle is a chord of the circumcircle tangent to the incircle. The greatest of these chords is equal to $2R$, and the shortest one touching the incircle at the point opposite to O is $2\sqrt{R^2 - 4r^2} > R$.

8. (Ye.Bakayev) (8–9) Let AD be the base of trapezoid $ABCD$. It is known that the circumcenter of triangle ABC lies on BD. Prove that the circumcenter of triangle ABD lies on AC.

   Solution. Let the perpendicular bisector to AB meet BD and AC at points K and L respectively. Then by the assumption $\angle BLK = \angle ACB = \angle CAD$. Hence $\angle CKL = \angle BDA$ which yields the required assertion (fig. 8).

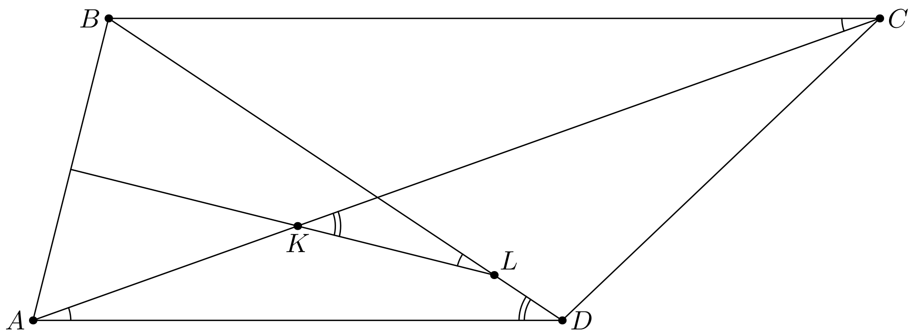


Fig. 8

9. (A.Zaslavsky) (8–9) Let C_0 be the midpoint of hypotenuse AB of triangle ABC; AA_1, BB_1 the bisectors of this triangle; I its incenter. Prove that the lines C_0I and A_1B_1 meet on the altitude from C.

   Solution. Use the following property of an arbitrary triangle.

Lemma. The line C_0I meets the altitude CH at the point lying at the distance r from C.

In fact, let C', C'' be the touching points of side AB with the incircle and the excircle respectively, and C_2 the point of the incircle opposite to C'. Point C is the homothety center of the incircle and the excircle, and C_2 and C'' are the corresponding points of these circles, therefore C, C_2, C'' are collinear. Furthermore $C'C_0 = C''C_0$, i.e. C_0I is the medial line of triangle $C'C''C_2$, and $C_0I \parallel CC_2$. Hence the lines CC_2, C_2I, C_0I and CH are the sidelines of a parallelogram, and we obtain the assertion of the lemma (fig. 9.1).

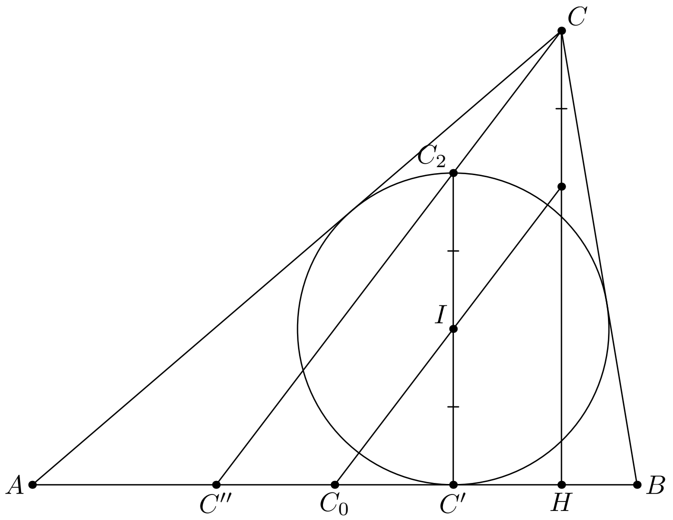


Fig. 9.1

Return to the problem. Denote the common point of C_0I and CH by H' (fig. 9.2). Since $CH' = r$, the distances from H' to CA, BC and AB are $d_b = r\cos\angle HCB = r\cos\angle BAC = r \cdot AC/AB$, $d_a = r \cdot BC/AB$ and $d_c = CH - r$ respectively. Since $(AB + BC + CA)r = AB \cdot CH = 2S_{ABC}$, we obtain that $d_c = d_a + d_b$. It is clear that the distances from A_1, B_1 to BC, CA andAB also have the similar property. By the Thales theorem all such points are collinear.

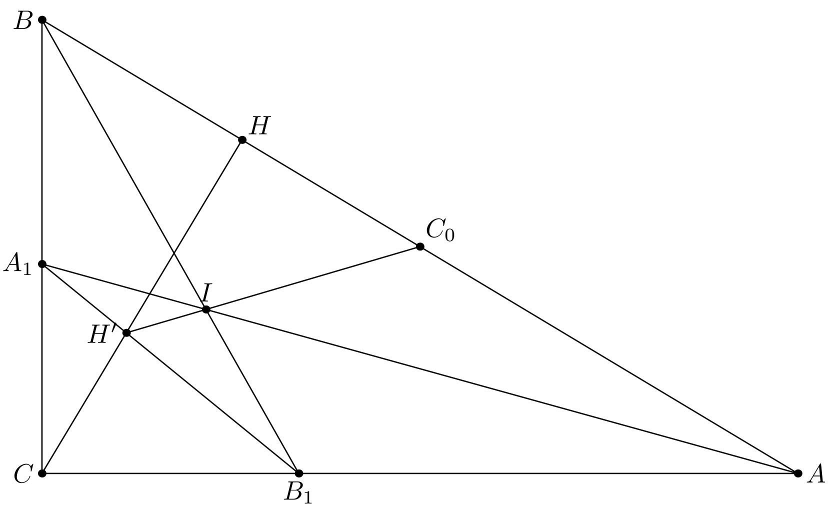


Fig. 9.2

10. (I.I.Bogdanov) (8–10) Points K and L on the sides AB and BC of parallelogram $ABCD$ are such that $\angle AKD = \angle CLD$. Prove that the circumcenter of triangle BKL is equidistant from A and C.

    Solution. The triangles AKD and CLD are similar by two angles, therefore $AK : CL = AD : CD$. Hence, when K moves along AB with constant velocity, L also moves along BC uniformly, and therefore the circumcenter of BKL moves along some line. If K, L are the projections of D to AB and BC respectively, the circumcenter of BKL coincides with the center of the parallelogram, and when K and L coincide with A and C respectively, the circumcenter lies on the perpendicular bisector to AC. Thus this perpendicular bisector is the locus of circumcenters.

11. (A.Tolesnikov) (8–11) A finite number of points is marked on the plane. Each three of them are not collinear. A circle is circumscribed around each triangle with marked vertices. Is it possible that all centers of these circles are also marked?

    Answer. No.

    Solution. Consider the circle having the minimal radius. Let it be the circumcircle of triangle ABC, and O be its center. If ABC is not a regular triangle, then some of its angles, for example C, is less than $60°$. But in this case $60° < \angle AOB < 120°$, i.e. $\sin\angle AOB > \sin\angle ACB$, and by the sinus theorem the circumradius of AOB is less than the radius of circle ABC, which contradicts to the definition of this circle. If ABC is regular then the centers A', B', C' of circles BOC, COA, AOB are also marked. But for example the triangle AOB' is regular, and its circumradius is less than the radius of circle ABC.

12. (D.Shvetsov) (9–10) Let AA_1, CC_1 be the altitudes of triangle ABC, B_0 the common point of the altitude from B and the circumcircle of ABC; and Q the common point of the circumcircles of ABC and $A_1C_1B_0$, distinct from B_0. Prove that BQ is the symmedian of ABC.

    Solution. Since A, C, A_1, C_1 are concyclic we obtain that the lines AC, A_1C_1 and B_0Q concur at the radical center N of circles ACA_1C_1, ABC and $A_1C_1B_0$. Let BQ meet AC and A_1C_1 at points P and M respectively (fig. 12). Projecting the circumcircle of triangle ABC from Q to AC, and projecting this line from B to A_1C_1 we obtain te equality of cross-ratios $(A_1C_1MN) = (CAPN) = (CABB_0) = \frac{BC}{BA} : \frac{B_0C}{B_0A}$. Since B_0 is the reflection of the orthocenter H of triangle ABC about AC, the second fraction is equal to $HC/HA = CA_1/AC_1$. Now applying the Menelaus theorem to the triangle A_1BC_1 and the line ACN we obtain that $A_1C_1MN = C_1N/A_1N$, i.e. $A_1M = C_1M$. Therefore BM is the median of triangle A_1BC_1 and the symmedian of ABC.

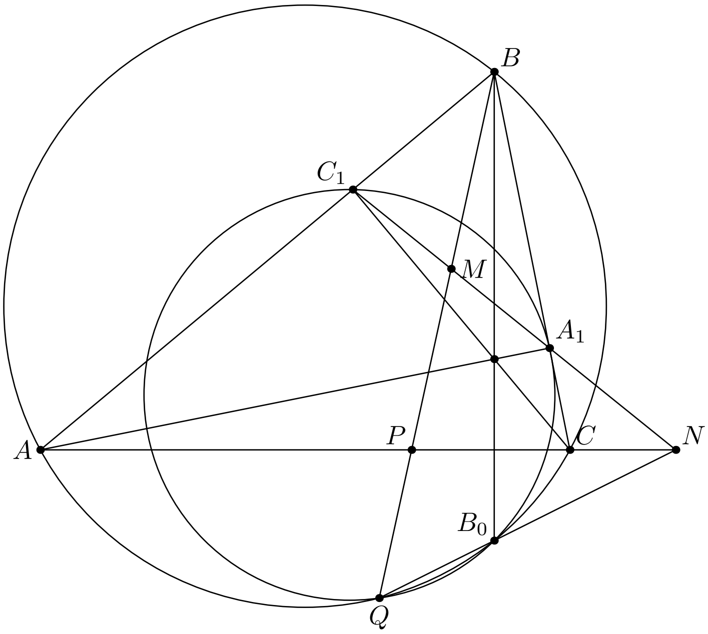


Fig. 12

13. (A.Zaslavsky) (9–11) Two circles pass through points A and B. A third circle touches both these circles and meets AB at points C and D. Prove that the tangents to this circle at these points are parallel to the common tangents of two given circles.

    Solution. Let the third circle touch two given circles at points X, Y, and their common tangent touch them at U, V (points X and U lie on the same circle). Since X is the homothety center of touching circles, the line XU meets the third circle at point P such that the tangent at this point is parallel to UV. Similarly YV passes through P. Also X, Y, U, V are collinear, therefore $PX \cdot PU = PY \cdot PV$. Hence P lies on AB and thus coincides with one of points C, D (fig. 13). The proof for the second point is similar.

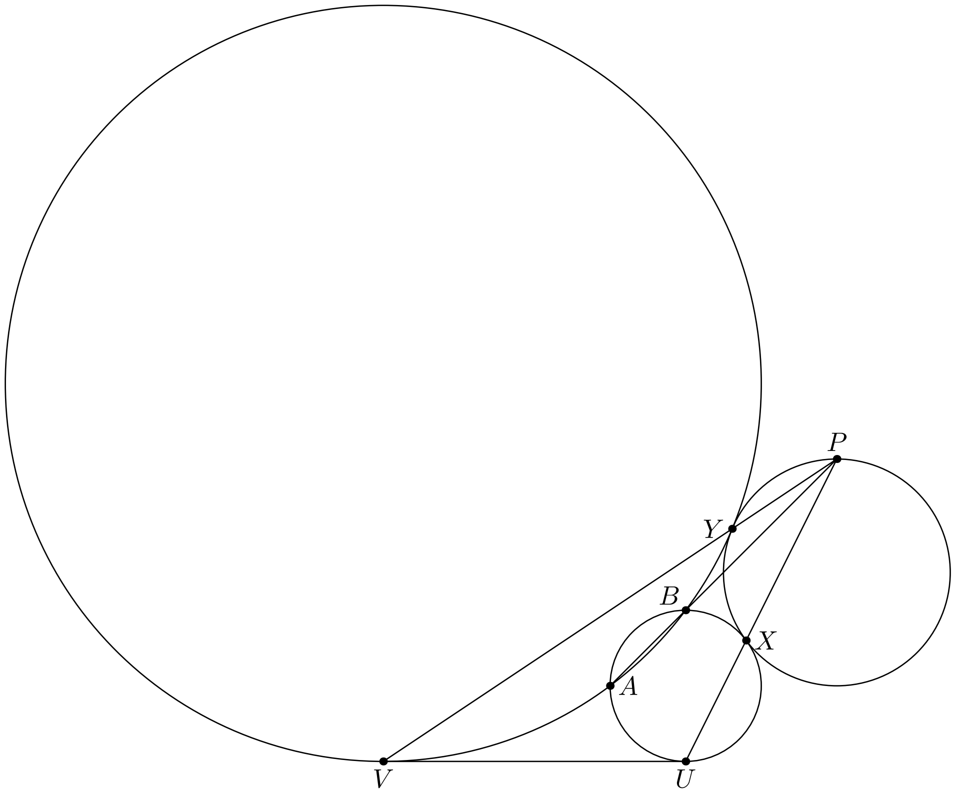


Fig. 13

14. (N.Moskvitin) (9–11) Let points B and C lie on the circle with diameter AD and center O on the same side of AD. The circumcircles of triangles ABO and CDO meet BC at points F and E respectively. Prove that $R^2 = AF \cdot DE$, where R is the radius of the given circle.

Solution. Since $ABFO$ is cyclic and $AO = OB$, we have (fig.14)

$$\frac{AF}{AO} = \frac{\sin\angle AOF}{\sin\angle ABO} = \frac{\sin\angle ABF}{\sin\angle ABO} = \frac{\sin\angle ABC}{\sin\angle BAD}.$$

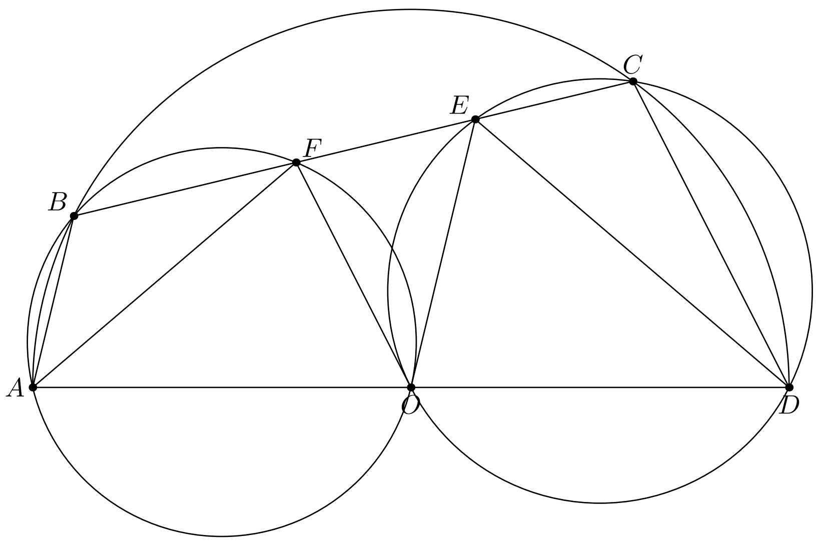


Fig. 14

Similarly, $DE/OD = \sin\angle BCD/\sin\angle CDA$. Since $ABCD$ is cyclic, the product of these ratios is equal to 1.

15. (K.Aleksiev) (9–11) Let ABC be an acute-angled triangle with incircle ω and incenter I. Let ω touch AB, BC and CA at points D, E, F respectively. The circles ω_1 and ω_2 centered at J_1 and J_2 respectively are inscribed into $ADIF$ and $BDIE$. Let J_1J_2 intersect AB at point M. Prove that CD is perpendicular to IM.

    Solution. Since DJ_1, DJ_2 are the bisectors of triangles DIA, DIB respectively, we have $AJ_1/J_1I = AD/ID$, $IJ_2/J_2B = CI/CB$. By the Menelaus theorem we obtain that the quadruple A, B, C, M is harmonic, i.e. M lies on FE (fig.15). Since C and D are the poles of lines EF and AB wrt the incircle we obtain that M is the pole of CD, therefore $CD \perp IM$.

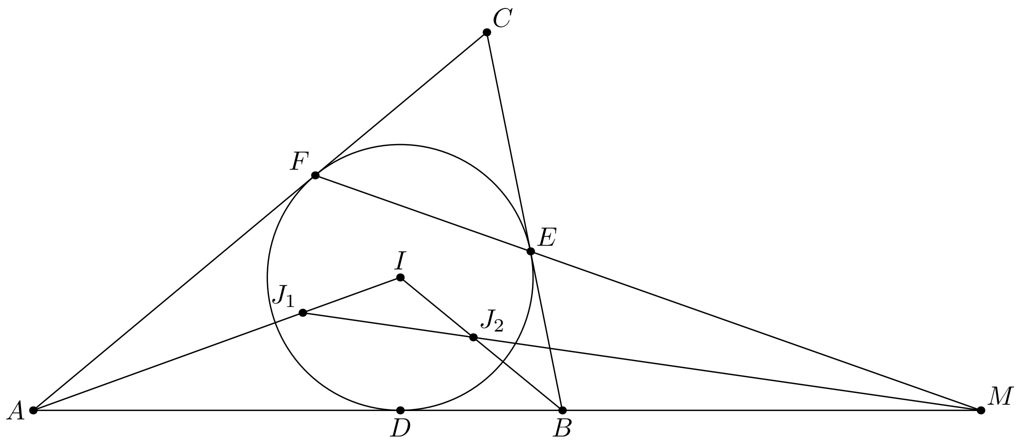


Fig. 15

16. (P.Ryabov) (9–11) The tangents to the circumcircle of triangle ABC at A and B meet at point D. The circle passing through the projections of D to BC, CA, AB, meet AB for the second time at point C'. Points A', B' are defined similarly. Prove that AA', BB', CC' concur.

    Solution. The pedal circle of point D coincides with the pedal circle of isogonally conjugated point D' which is the vertex of parallelogram $ACBD'$. Hence C' is the projection of D' to AB, i.e. the reflection of the foot of the altitude from C about the midpoint of AB. Similarly A', B' are the reflections of the feet of the altitudes from A and B about the midpoints of the corresponding sides. Therefore AA', BB' and CC' concur at the point isotomically conjugated to the orthocenter of the triangle.

17. (A.Trigub) (9–11) Using a compass and a ruler, construct a point K inside an acute-angled triangle ABC so that $\angle KBA = 2\angle KAB$ and $\angle KBC = 2\angle KCB$.

    Solution. Let the circle centered at K and passing through B meet AB and BC at points P and Q respectively, and let T be the midpoint of arc ABC of the circumcircle. Then $\angle KPB = \angle KBP = 2\angle KAP$, therefore $\angle KAP = \angle PKA$ and $AP = PK = KB$. Similarly $CQ = QK = KB$. Since $AP = CQ$, $AT = CT$ and $\angle PAT = \angle QCT$, the triangles TAP and TCQ are congruent i.e. $\angle TPB = \angle TQB$ and T lies on the circle BPQ.

Hence the center K of this circle lies on the perpendicular bisector to BT. Furthermore by the assumption $\angle AKC = 3\angle B/2$, i.e. K lies on the corresponding arc (fig.17).

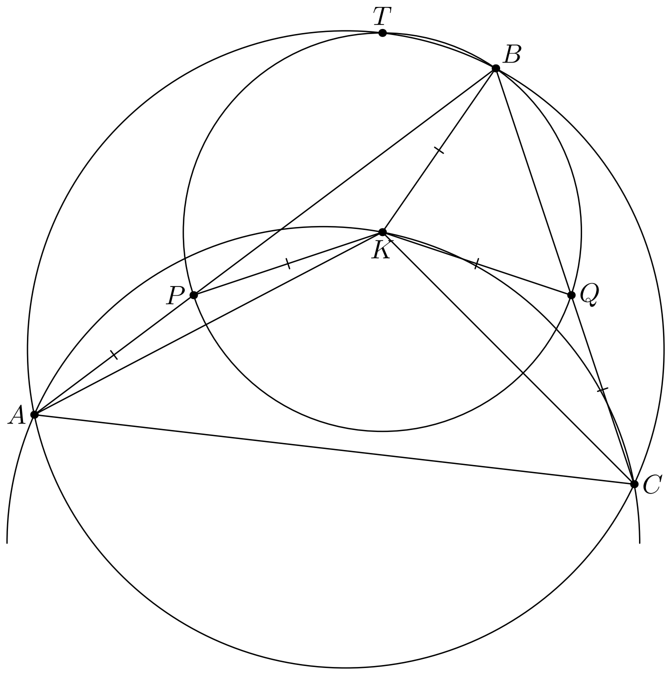


Fig. 17

Now let us prove that the constructed point K is in fact the required one. Denote again the common points of the sidelines with the circle centered at K and passing through B by P and Q. Since this circle passes through T, we obtain that $AP = CQ$. If $AP > PK = KB$ then $\angle PKA > \angle PAK$, $\angle KPB = \angle KBP > 2\angle BAK$, $\angle KBC > 2\angle KCB$ and $\angle AKC < 3\angle B/2$ which contradicts to the construction of K. Similarly if $AP < PK$ we have $\angle AKC > 3\angle B/2$.

18. (A.Trigub) (9–11) Let L be the common point of the symmedians of triangle ABC, and BH be its altitude. It is known that $\angle ALH = 180° - 2\angle A$. Prove that $\angle CLH = 180° - 2\angle C$.

    Solution. Let AA_1, CC_1 be the altitudes of the triangle. Then the symmedians AL, CL are the medians of triangles AC_1H, CA_1H, i.e. they pass through the midpoints M, N of segments HC_1, HA_1 respectively. But $\angle MNH = \angle C_1A_1H = 180° - 2\angle A$, therefore $\angle ALH = 180° - 2\angle A$ if and only if $HLMN$ is cyclic. Similarly this is equivalent to the condition $\angle CLH = 180° - 2\angle C$.

19. (D.Prokopenko) (10–11) Let cevians AA', BB' and CC' of triangle ABC concur at point P. The circumcircle of triangle $PA'B'$ meets AC and BC at points M and N respectively, and the circumcircles of triangles $PC'B'$ and $PA'C'$ meet AC and BC for the second time respectively at points K and L. The line c passes through the midpoints of segments MN and KL. The lines a and b are defined similarly. Prove that a, b and c concur.

    Solution. By the assumption $CM{\cdot}CB' = CN{\cdot}CA'$ and $CK{\cdot}CB' = CP{\cdot}CC' = CL{\cdot}CA'$. Hence $KL \parallel MN$ and c passes through C. Since MN and $A'B'$ are antiparallel, this line is the symmedian of triangle $CA'B'$ and so it divides C into two angles with the ratio of sinuses equal to $CB' : CA'$. The similar relations for two remaining angles and the Ceva theorem yield the required assertion.

20. (V.Luchkin, M.Fadin) (10–11) Given a right-angled triangle ABC and two perpendicular lines x and y passing through the vertex A of its right angle. For an arbitrary point X on x define y_B and y_C as the reflections of y about XB and XC respectively. Let Y be the common point of y_b and y_c. Find the locus of Y (when y_b and y_c do not coincide).

    Solution. Consider the point X' isogonally conjugated to X and its reflections U, V, W about AB, AC, BC respectively. Perpendicularity of x and y implies that U and V lie on y. Furthermore XB, XC are the perpendicular bisectors to UW, VW respectively. Therefore W lies on y_b, y_c, i.e. it coincides with Y (fig.20). Thus Y lies on the reflection of the isogonal image of x about BC. The required locus is this line without the points such that y_b and y_c coincide, i.e. the common point of this line with BC and the reflection of A about BC.

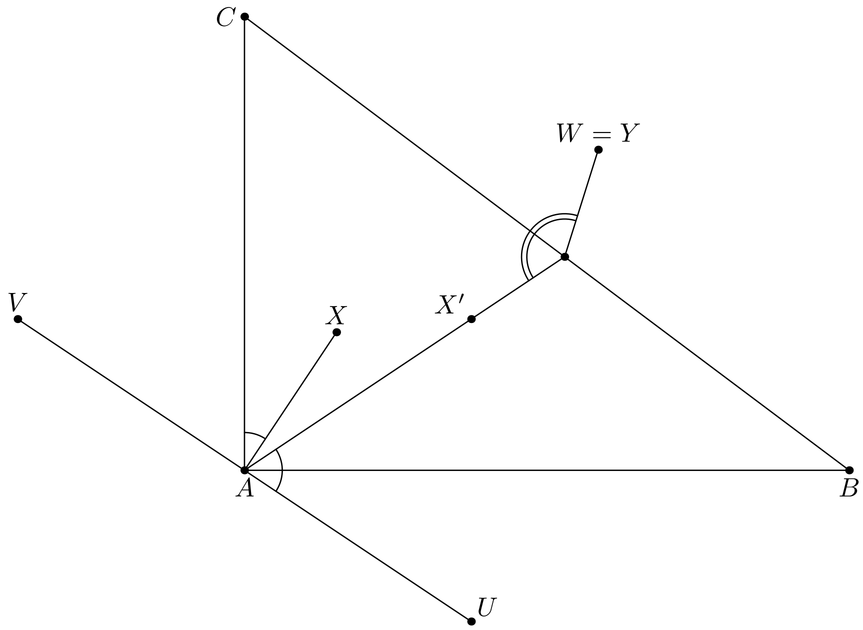


Fig. 20

21. (N.Beluhov) (10–11) A convex hexagon is circumscribed about a circle of radius 1. Consider the three segments joining the midpoints of its opposite sides. Find the greatest real number r such that the length of at least one segment is at least r.

    Solution. Let $A_1A_2 \dots A_6$ be the hexagon in question, circumscribed about a circle ω with center I, and let M_i be the midpoint of A_iA_{i+1} (indices run modulo 6, so that, say, $A_7 \equiv A_1$). If $A_1A_2A_3$ approaches an equilateral triangle and A_4, A_5, and A_6 all approach the midpoint of A_1A_3 then the lengths of M_1M_4, M_2M_5, and M_3M_6 all approach $\sqrt{3}$.

    We will show that $r = \sqrt{3}$ is indeed the answer to the problem. First we verify that I lies inside $M_1M_2 \dots M_6$. Suppose for example that it lies inside the triangle $M_1A_1M_6$. But then ω is contained inside $A_2A_1A_6$ and cannot touch all sides of $A_1A_2 \dots A_6$, a contradiction.

Let $\angle(ABCD)$ denote the angle such that rotation by it counterclockwise about A makes $\overrightarrow{AB}$ codirectional with $\overrightarrow{CD}$.

Since all M_i lie outside ω, we have $IM_i \geq 1$. Therefore, if $120° \leq \angle(IM_iIM_{i+3}) \leq 240°$ for some i then $M_iM_{i+3} \geq \sqrt{3}$ and we are done.

Suppose now that this does not happen for any i. Let j be such that $\angle(IM_jIM_{j+3}) \leq 120°$ and $\angle(IM_{j+3}IM_j) \geq 240°$. Then there is some k, $j \leq k \leq j+2$, such that $\angle(IM_kIM_{k+3}) \leq 120°$ and $\angle(IM_{k+1}IM_{k+4}) \geq 240°$. Without loss of generality, take $k = 4$. Then $120° \leq \angle IM_1IM_2 \leq 180°$ and consequently $M_1M_2 \geq \sqrt{3}$.

Consider the convex quadrilateral $M_1M_2M_4M_5$. If angle M_1 is right or obtuse then $M_2M_5 > M_1M_2 \geq \sqrt{3}$ and we are done. If angle M_2 is right or obtuse then $M_1M_4 > M_1M_2 \geq \sqrt{3}$ and we are done. It remains to consider the case when angles M_1 and M_2 are both acute.

In this case however $90° < \angle(M_1M_2M_4M_5) < 270°$. Since $\overrightarrow{M_3M_6} = -\overrightarrow{M_1M_2} + \overrightarrow{M_4M_5}$ (because $\overrightarrow{M_3M_6} = \overrightarrow{M_3M_4} + \overrightarrow{M_4M_5} + \overrightarrow{M_5M_6}$ and $\overrightarrow{M_1M_2} + \overrightarrow{M_3M_4} + \overrightarrow{M_5M_6} = \frac{1}{2}(\overrightarrow{A_1A_3} + \overrightarrow{A_3A_5} + \overrightarrow{A_5A_1}) = \mathbf{0}$), we have $M_3M_6 > M_1M_2 \geq \sqrt{3}$, and the proof is complete.

22. (M. Panov) (10–11) Let P be an arbitrary point on the diagonal AC of cyclic quadrilateral $ABCD$, and PK, PL, PM, PN, PO be the perpendiculars from P to AB, BC, CD, DA, BD respectively. Prove that the distance from P to KN is equal to the distance from O to ML.

    Solution. When P moves uniformly along AC, the lines KN and ML are translated uniformly and the point O moves uniformly as well. Thus $d(P, KN) - d(O, ML)$ is a linear function of the position of P. When $P = A$, this function equals 0 by the Simson theorem, and when P is the common point of AC and BD, it equals 0 because $KLMN$ is circumscribed about a circle centered at $P = O$ ($\angle NKP = \angle DAC = \angle DBC = \angle PKL$ because $AKPN$ and $BKPL$ are cyclic).

23. (I.Frolov) (10–11) Let a line m touch the incircle of triangle ABC. The lines passing through the incenter I and perpendicular to AI, BI, CI meet m at points A', B', C' respectively. Prove that AA', BB' and CC' concur.

    Solution. The polar transformation wrt the incircle maps BC, CA, AB, m to their touching points A_1, B_1, C_1 M with the incircle. Since IA' is the polar of the infinite point of perpendicular line IA, its common point with m is the pole of the line passing through M and parallel to IA. Since $IA \perp B_1C_1$, the line AA' is the polar of the projection of M to B_1C_1. Similarly the lines BB' and CC' are the polars of projections of M to A_1C_1 and A_1B_1 respectively. By the Simson theorem these projections are collinear, hence their polars concur.

24. (I.I.Bogdanov) (11) Two tetrahedrons are given. Each two faces of the same tetrahedron are not similar, but each face of the first tetrahedron is similar to some face of the second one. Does this yield that these tetrahedrons are similar?

    Answer. No.

    Solution. Let t be some number close to 1. Then there exist two tetrahedrons such that their bases are regular triangles with side equal to 1, the lateral edges of the first tetrahedron are equal to t, t^2, t^3, and the lateral edges of the second one are equal to $1/t$,

$1/t^2$, $1/t^3$. It is clear that the assumption is valid for these tetrahedrons but they are not similar.

XIV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
The correspondence round

Below is the list of problems for the first (correspondence) round of the XIV Sharygin Geometrical Olympiad.

The olympiad is intended for high-school students of four eldest grades. In Russian school, these are 8-11. In the list below, each problem is indicated by the numbers of Russian school grades, for which it is intended. Foreign students of the last grade have to solve the problems for 11th grade, students of the preceding grade solve the problems for 10th grade etc. However, the participants may solve problems for elder grades as well (solutions of problems for younger grades will not be considered).

A complete solution of each problem costs 7 points. A partial solution costs from 1 to 6 points. A text without significant advancement costs 0 points. The result of a participant is the sum of all obtained marks.

Please include the solution of each problem in a separate file. First write down the statement of the problem, and then the solution. Present your solutions in detail, including all necessary arguments and calculations. Provide all necessary figures of sufficient size. If a problem has an explicit answer, this answer must be presented distinctly. Please, be accurate to provide good understanding and correct estimating of your work !

If your solution depends on some well-known theorems from standard textbooks, you may simply refer to them instead of providing their proofs. However, any fact not from the standard curriculum should be either proved or properly referred (with an indication of the source).

You may note the problems which you liked most (this is not obligatory). Your opinion is interesting for the Jury.

The **solutions** for the problems (in Russian or in English) must be **delivered not before January 8, 2018 and not later than on April 1, 2018**. To upload your work, enter the site **https://contest.yandex.ru/geomshar/**, indicate the language (English) in the right upper part of the page, press "Registration"in the left upper part, and follow the instructions.

Attention:

1. The solution of each problem must be contained in a *separate* pdf, doc or jpg file. We recommend to prepare the paper using computer or to scan it rather than to photograph it. *In the last two cases, please check readability of the file before uploading.*

2. If you upload the solution of some problem more than once then only the last version is retained in the checking system. Thus if you need to change something in your solution then you have to upload the whole solution again.

If you have any technical problems with uploading of the work, apply to **geomshar@yandex.ru (DON'T SEND your work to this address)**.

The final round will be held in July–August 2018 in Moscow region. The winners of the correspondence round are invited to it if they don't graduate from school before. (For instance, if the last grade is 12 then we invite winners from 9–11 grades, and from 12 grade if they finish their education later.) The graduates, winners of the correspondence round, will be awarded by diplomas of the Olympiad. The list of the winners will be published on **www.geometry.ru** at the end of May 2018 at latest. If you want to know your detailed results, please use e-mail **geomshar@yandex.ru**.

1. (grade 8) Three circles lie inside a square. Each of them touches externally two remaining

circles. Also each circle touches two sides of the square. Prove that two of these circles are congruent.

2. (grade 8) A cyclic quadrilateral $ABCD$ is given. The lines AB and DC meet at point E, and the lines BC and AD meet at point F. Let I be the incenter of triangle AED, and a ray with origin F be perpendicular to the bisector of angle AID. In what ratio does this ray dissect the angle AFB?

3. (grade 8) Let AL be a bisector of triangle ABC, D be its midpoint, and E be the projection of D to AB. It is known that $AC = 3AE$. Prove that CEL is an isosceles triangle.

4. (grade 8) Let $ABCD$ be a cyclic quadrilateral. A point P moves along the arc AD which does not contain B and C. A fixed line l, perpendicular to BC, meets the rays BP, CP at points B_0, C_0 respectively. Prove that the tangent at P to the circumcircle of triangle PB_0C_0 passes through some fixed point.

5. (grades 8–9) The vertex C of equilateral triangles ABC and CDE lies on the segment AE, and the vertices B and D lie on the same side with respect to this segment. The circumcircles of these triangles centered at O_1 and O_2 meet for the second time at point F. The lines O_1O_2 and AD meet at point K. Prove that $AK = BF$.

6. (grades 8–9) Let CH be the altitude of a right-angled triangle ABC ($\angle C = 90°$) with $BC = 2AC$. Let O_1, O_2 and O be the incenters of triangles ACH, BCH and ABC respectively, and H_1, H_2, H_0 be the projections of O_1, O_2, O respectively to AB. Prove that $H_1H = HH_0 = H_0H_2$.

7. (grades 8–9) Let E be a common point of circles w_1 and w_2. Let AB be a common tangent to these circles, and CD be a line parallel to AB, such that A and C lie on w_1, B and D lie on w_2. The circles ABE and CDE meet for the second time at point F. Prove that F bisects one of arcs CD of circle CDE.

8. (grades 8–9) Restore a triangle ABC by the Nagel point, the vertex B and the foot of the altitude from this vertex.

9. (grades 8–9) A square is inscribed into an acute-angled triangle: two vertices of this square lie on the same side of the triangle and two remaining vertices lies on two remaining sides. Two similar squares are constructed for the remaining sides. Prove that three segments congruent to the sides of these squares can be the sides of an acute-angled triangle.

10. (grades 8–9) In the plane, 2018 points are given such that all distances between them are different. For each point, mark the closest one of the remaining points. What is the minimal number of marked points?

11. (grades 8–9) Let I be the incenter of a nonisosceles triangle ABC. Prove that there exists a unique pair of points M, N lying on the sides AC, BC respectively, such that $\angle AIM = \angle BIN$ and $MN \parallel AB$.

12. (grades 8–9) Let BD be the external bisector of a triangle ABC with $AB > BC$; K and K_1 be the touching points of side AC with the incicrle and the excircle centered at I and

I_1 respectively. The lines BK and DI_1 meet at point X, and the lines BK_1 and DI meet at point Y. Prove that $XY \perp AC$.

13. (grades 9–11) Let $ABCD$ be a cyclic quadrilateral, and M, N be the midpoints of arcs AB and CD respectively. Prove that MN bisects the segment between the incenters of triangles ABC and ADC.

14. (grades 9–11) Let ABC be a right-angled triangle with $\angle C = 90°$, K, L, M be the midpoints of sides AB, BC, CA respectively, and N be a point of side AB. The line CN meets KM and KL at points P and Q respectively. Points S, T lying on AC and BC respectively are such that $APQS$ and $BPQT$ are cyclic quadrilaterals. Prove that

    a) if CN is a bisector, then CN, ML and ST concur;

    b) if CN is an altitude, then ST bisects ML.

15. (grades 9–11) The altitudes AH_1, BH_2, CH_3 of an acute-angled triangle ABC meet at point H. Points P and Q are the reflections of H_2 and H_3 with respect to H. The circumcircle of triangle PH_1Q meets for the second time BH_2 and CH_3 at points R and S. Prove that RS is a medial line of triangle ABC.

16. (grades 9–11) Let ABC be a triangle with $AB < BC$. The bisector of angle C meets the line parallel to AC and passing through B, at point P. The tangent at B to the circumcircle of ABC meets this bisector at point R. Let R' be the reflection of R with respect to AB. Prove that $\angle R'PB = \angle RPA$.

17. (grades 10–11) Let each of circles α, β, γ touch two remaining circles externally, and all of them touch a circle Ω internally at points A_1, B_1, C_1 respectively. The common internal tangent to α and β meets the arc A_1B_1 not containing C_1, at point C_2. Points A_2, B_2 are defined similarly. Prove that the lines A_1A_2, B_1B_2, C_1C_2 concur.

18. (grades 10–11) Let C_1, A_1, B_1 be points on sides AB, BC, CA of triangle ABC, such that AA_1, BB_1, CC_1 concur. The rays B_1A_1 and B_1C_1 meet the circumcircle of the triangle at points A_2 and C_2 respectively. Prove that A, C, the common point of A_2C_2 and BB_1, and the midpoint of A_2C_2 are concyclic.

19. (grades 10–11) Let a triangle ABC be given. On a ruler, three segments congruent to the sides of this triangle are marked. Using this ruler construct the orthocenter of the triangle formed by the tangency points of the sides of ABC with its incircle.

20. (grades 10–11) Let the incircle of a nonisosceles triangle ABC touch AB, AC and BC at points D, E and F respectively. The corresponding excircle touches the side BC at point N. Let T be the common point of AN and the incircle, closest to N, and K be the common point of DE and FT. Prove that $AK||BC$.

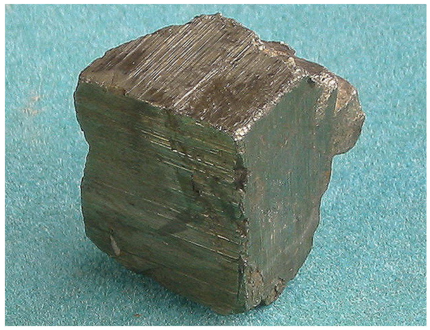
21. (grades 10–11) In the plane, a line l and a point A outside it are given. Find the locus of the incenters of acute-angled triangles having the vertex A and the opposite side lying on l.

22. (grades 10–11) Six circles of unit radius lie in the plane so that the distance between the centers of any two of them is greater than d. What is the least value of d such that there

always exists a straight line which does not intersect any of the circles and separates the circles into two groups of three?

23. (grades 10–11) The plane is divided into convex heptagons with diameters less than 1. Prove that an arbitrary disc with radius 200 intersects more than a billion of them.

24. (grades 10–11) A crystal of pyrite is a parallelepiped with dashed faces.



The dashes on any two adjacent faces are perpendicular. Does there exist a convex polytope with the number of faces not equal to 6, such that its faces can be dashed in such a manner?

XIV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
The correspondence round. Solutions

1. (L.Shteingarts, grade 8) Three circles lie inside a square. Each of them touches externally two remaining circles. Also each circle touches two sides of the square. Prove that two of these circles are congruent.

   Solution. If two circles are inscribed into the same angle of the square, then the third one can not touch them and two sides. Hence we can suppose that the circles are inscribed into the angles A, B and C of square $ABCD$. But then two circles inscribed into angles A and C are symmetric with respect to diagonal BD, therefore they are congruent.

2. (N.Moskvitin, grade 8) A cyclic quadrilateral $ABCD$ is given. The lines AB and DC meet at point E, and the lines BC and AD meet at point F. Let I be the incenter of triangle AED, and a ray with origin F be perpendicular to the bisector of angle AID. In which ratio this ray dissects the angle AFB?

   Answer. $1 : 3$.

   Solution. Note that the angle between the bisectors of angles AED and AFB is equal to the semisum of angles FAE and FCE, i.e. $90°$. Thus the angle between the bisector of angle AFB and the ray FK, where K is the projection of F to the bisector of angle AID, is equal to $180° - \angle EIK = 180° - (90° + \angle A/2) - (180° - \angle A/2 - \angle D/2)/2 = (\angle D - \angle A)/4 = \angle AFB/4$ (fig.2), therefore $\angle AFK = \angle AFB/4$.

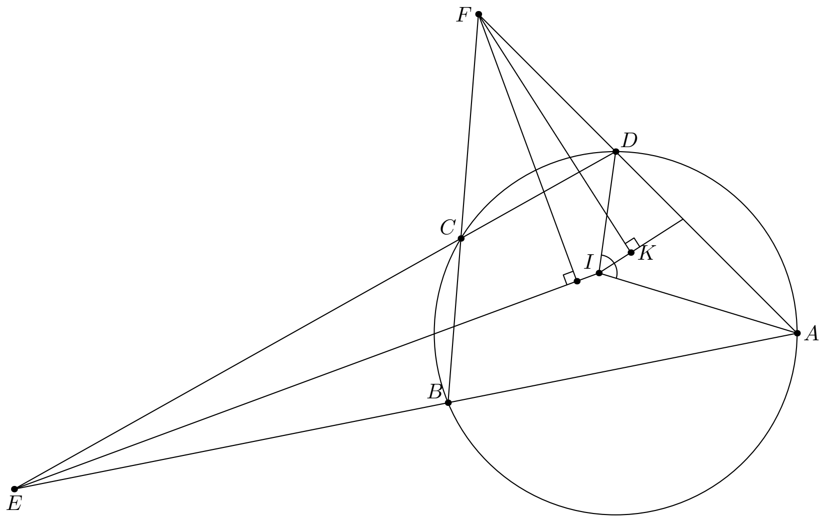


Fig. 2

3. (A.Zaslavsky, grade 8) Let AL be the bisector of triangle ABC, D be its midpoint, and E be the projection of D to AB. It is known that $AC = 3AE$. Prove that CEL is an isosceles triangle.

   Solution. Let F be the projection of L to AB, and G be the reflection of E about F. By the Thales theorem we have $AE = EF = FG$ and $AG = 3AE = AC$. Since AL is the bisector of angle A, and FL is the perpendicular bisector to segment EG, we obtain that $CL = LG = LE$ (fig.3).

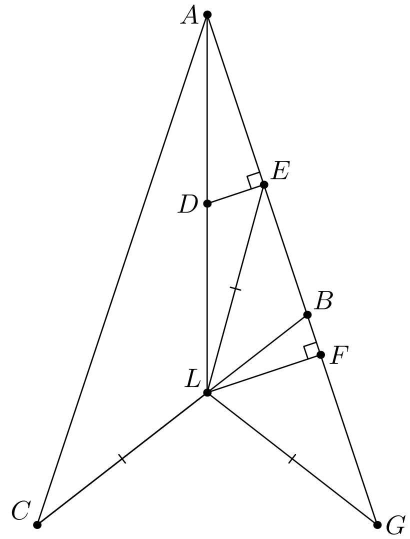


Fig. 3

4. (D.Shvetsov, grade 8) Let $ABCD$ be a cyclic quadrilateral. A point P moves along the arc AD which does not contain B and C. A fixed line l, perpendicular to BC, meets the rays BP, CP at points B_0, C_0 respectively. Prove that the tangent at P to the circumcircle of triangle PB_0C_0 passes through some fixed point.

   Solution. Let the tangent meet the circumcircle of $ABCD$ for the second time at point Q. Then $\angle BPQ = \angle B_0C_0P = 90^\circ - \angle BCP = 90^\circ - \angle BQP$ (fig.4). Thus $\angle PBQ = 90^\circ$, i.e. PQ is a diameter of circle $ABCD$. Hence all tangents pass through the center of this circle.

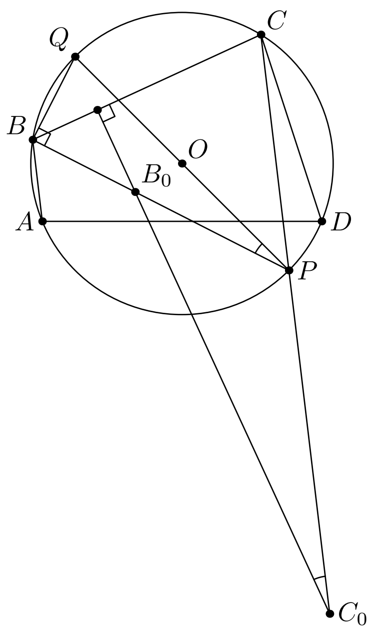


Fig. 4

5. (N.Moskvitin, grades 8–9) The vertex C of equilateral triangles ABC and CDE lies on the segment AE, and the vertices B and D lie on the same side with respect to this

segment. The circumcircles of these triangles centered at O_1 and O_2 meet for the second time at point F. The lines O_1O_2 and AD meet at point K. Prove that $AK = BF$.

Solution. Note that triangles ACD and BCE are congruent because $AC = BC$, $CD = CE$ and $\angle ACD = \angle BCE = 120°$. Also, since $\angle BFC = 120°$ and $\angle CFE = 60°$, we obtain that F lies on the segment BE. Finally triangles O_1CO_2 and ACD are similar, hence $\angle CO_1K = \angle CAK$, i.e the points A, O_1, K and C are concyclic (fig.5). Therefore $\angle ACK = 180° - \angle AO_1K = 60° - \angle CO_1K = 60° - \angle CBF = \angle BCF$, which yields the required equality.

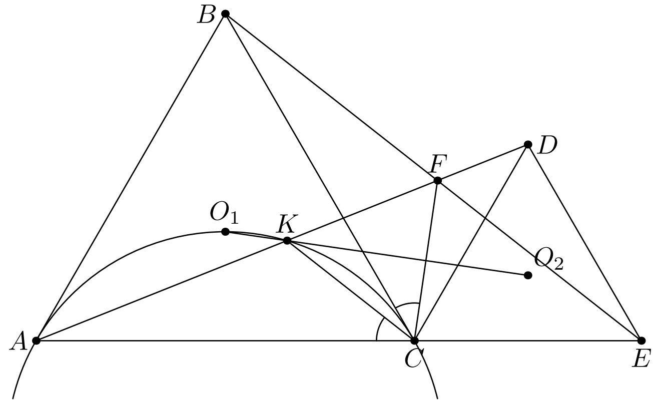


Fig. 5

6. (L.Shteingarts, grades 8–9) Let CH be the altitude of a right-angled triangle ABC ($\angle C = 90°$) with $BC = 2AC$. Let O_1, O_2 and O be the incenters of triangles ACH, BCH and ABC respectively, and H_1, H_2, H_0 be the projections of O_1, O_2, O respectively to AB. Prove that $H_1H = HH_0 = H_0H_2$.

   Solution. Similarity of triangles HAC and HCB implies $HO_2 = 2HO_1$, thus $HH_2 = HH_1$. So we have to prove that $H_1H_0 = 2H_0H_2$. But $H_1H_0 = AH_0 - AH_1 = AH_0(AB - AC)/AB = (AB + AC - BC)(AB - AC)/2AB = (BC^2 - BC(AB - AC))/2AB = BC(AC+BC-AB)/2AB$. Similarly $H_0H_2 = AC(AC+BC-AB)/2AB$, and we obtain the required equality.

7. (I.Spiridonov, grades 8–9) Let E be a common point of circles w_1 and w_2. Let AB be a common tangent to these circles, and CD be a line parallel to AB, such that A and C lie on w_1, B and D lie on w_2. The circles ABE and CDE meet for the second time at point F. Prove that F bisects one of arcs CD of circle CDE.

   Solution. Let the lines AC and BF meet at point H, and the lines BD and AF meet at point G.

   Since AB touches the circumcircle of triangle CAE, we have $(CA, CE) = (AB, AE)$. Since $ABEF$ is a cyclic quadrilateral, we have $(AB, AE) = (FB, FE)$. Then

$$(CH, CE) = (CA, CE) = (AB, AE) = (FB, FE) = (FH, FE)$$

$$(CH, CE) = (FH, FE)$$

   and $CHFE$ is a cyclic quadrilateral. Similarly we obtain that $DGFE$ is cyclic. Since $CFED$ is cyclic, we obtain that C, D, E, F, H, G are concyclic (fig.7).

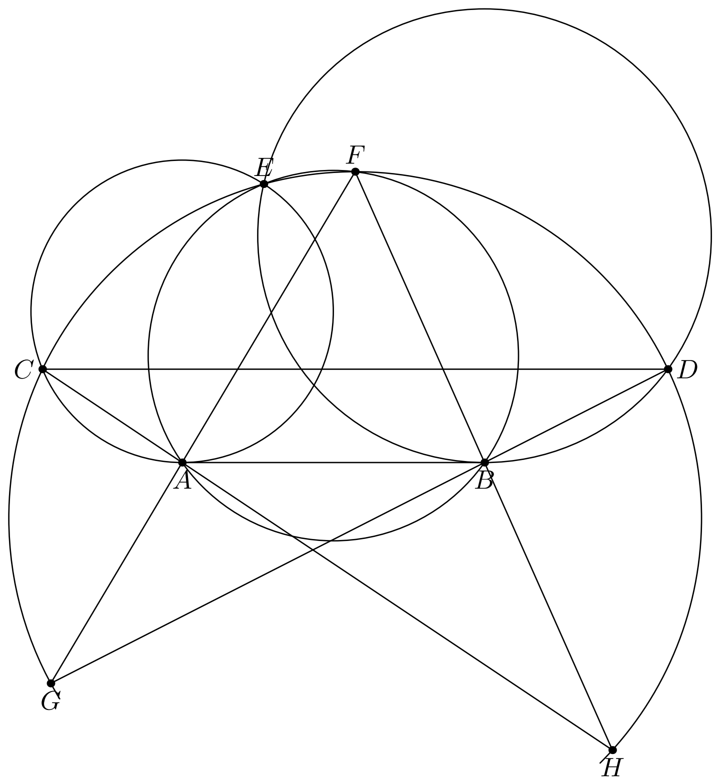


Fig. 7

Applying the Pascal theorem to cyclic hexagon $FFHCDG$ we obtain that the common points of FF and CD, FH and DG, HC and GF are collinear (line FF is the tangent at F to the circle CFD, denote this line by l). So A, B and the common point of l and CD are collinear. But $AB \parallel CD$, therefore $l \parallel CD$ and F is the midpoint of arc CD.

8. (K.Kadyrov, grades 8–9) Restore a triangle ABC by the Nagel point, the vertex B and the foot of the altitude from this vertex.

   Solution. Since the centroid of the triangle divides the segment between the Nagel point N and the incenter as $2:1$, we can find the radius of the incircle (we know the altitude and the distance from N to the base). Now, using the formulas for the area $S = bh_b/2 = pr = (p-b)r_b$, we can find the radius r_b of the excircle. Since the excircle touches the base at its common point with BN, we can construct this circle, draw the tangents to it from B, and restore the triangle.

9. (B.Frenkin, grades 8–9) A square is inscribed into an acute-angled triangle: two vertices of this square lie on the same side of the triangle and two remaining vertices lies on two remaining sides. Two similar squares are constructed for the remaining sides. Prove that three segments congruent to the sides of these squares can be the sides of an acute-angled triangle.

   Solution. Consider the greatest of three squares. Let its vertices K, L lie on AB, and the vertices M, N lie on BC, AC respectively. Draw the perpendiculars MX, NY to AC, BC respectively and the line passing through M, parallel to AC and meeting AB at point Z (fig.9). Since $MX < MN = ML < MZ$, the side of the inscribed square having the base on AC is greater than MX. Similarly the side of the square having the base on BC is greater than NY. Since $MN^2 - MX^2 = NX^2 < NY^2$, the triangle with the sidelengths MN, MX, NY is acute-angled. The more so, the sides of the three squares can form an acute-angled triangle.

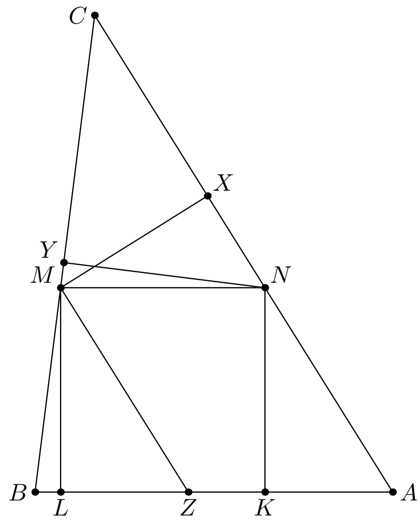


Fig. 9

10. (Folklore, grades 8–9) In the plane, 2018 points are given such that all distances between them are different. For each point, mark the closest one of the remaining points. What is the minimal number of marked points?

Answer. 449.

Solution. Divide the points into classes such that all points of the same class have the same closest point. Note that each class contains at most five points. In fact, if B is closest for $A_1, A_2, \dots, A_n$ then A_1A_2 is the greatest side of triangle A_1A_2B, thus $\angle A_1BA_2 > 60°$. The same is true for the angles $A_2BA_3, \dots, A_nBA_1$. Thus $n \leq 5$. A similar argument shows that if B is closest to $A_1, \dots, A_5$ then one of these points is closest to B and the class of B contains less than five points. Therefore at least $2n/9$ points of n points have to be marked, and for $n = 2018$ we have at least 449 marked points.

On the other hand, consider the configuration of 9 points on fig.10. Point A is closest for five points marked by a circ, and point B is closest for four points marked by a square. Take now 224 such groups, placed at a great distance one from another, and to one of them add two points such that C is closest to them. In this configuration $223 \cdot 2 + 3 = 449$ points are marked.

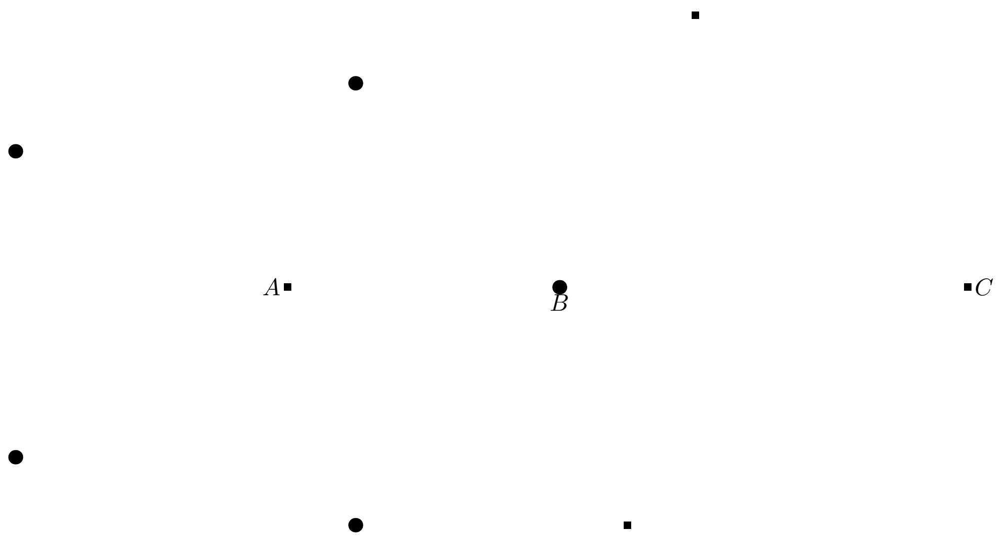


Fig. 10

11. (A.Zaslavsky, grades 8–9) Let I be the incenter of a nonisosceles triangle ABC. Prove that there exists a unique pair of points M, N lying on the sides AC, BC respectively, such that $\angle AIM = \angle BIN$ and $MN \parallel AB$.

    Solution. Consider the lines passing through A and B and parallel to IM, IN respectively. Since $MN \parallel AB$, their common point J lies on the ray CI and $\angle IAJ = \angle IBJ$. Thus the radii of circles AIJ and BIJ are equal, i.e. these circles are symmetric with respect to IJ. Hence the circle AIJ passes through the reflection B' of B about the bisector of angle C (fig.11). But A, B, I and B' are concyclic. Therefore the circles AIJ and BIJ coincide and J is the excenter of the triangle. Then $\angle AIM = \angle BIN = 90°$.

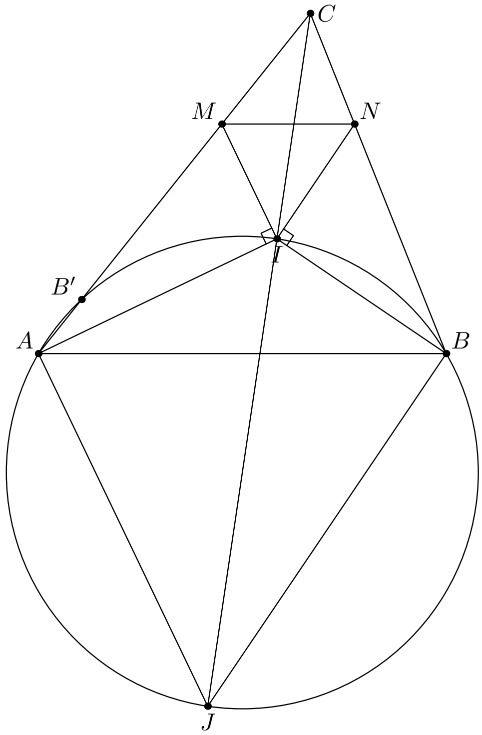


Fig. 11

12. (A.Didin, grades 8–9) Let BD be the external bisector of a triangle ABC with $AB > BC$; K and K_1 be the touching points of side AC with the incicrle and the excircle centered at I and I_1 respectively. The lines BK and DI_1 meet at point X, and the lines BK_1 and DI meet at point Y. Prove that $XY \perp AC$.

Solution. Since I and I_1 lie on the bisector of angle B, we have $BD \perp BI$. Hence B, K lie on the circle with diameter BI, and B, K_1 lie on the circle with diameter BI_1. Therefore $\angle YDK = \angle IBX$, $\angle YBI_1 = \angle KDX$, $\angle YBX = \angle YDX$ and points B, D, X, Y are concyclic (fig.12). Thus $\angle XYD = \angle XBD = 90° - \angle YDK$, i.e. $XY \perp AC$.

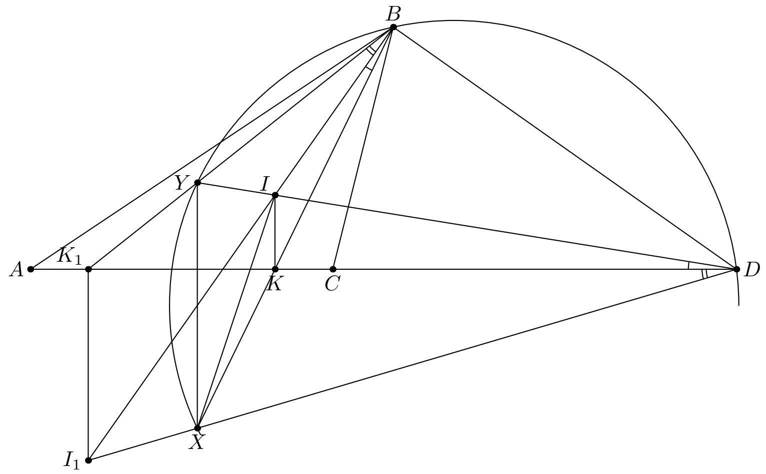


Fig. 12

13. (G.Feldman, grades 9–11) Let $ABCD$ be a cyclic quadrilateral, and M, N be the midpoints of arcs AB and CD respectively. Prove that MN bisects the segment between the incenters of triangles ABC and ADC.

Solution. Clearly the incenters I, J of triangles ABC and ADC lie on the segments CM and AN respectively. Also by the trident theorem $IM = AM = 2R \sin \angle ANM$. Thus the distance from I to MN is equal to $IM \sin \angle NMC = 2R \sin \angle ANM \sin \angle NMC$. We obtain the same expression for the distance from J to MN. Since I and J lie on the opposite sides from MN, this equality yields the assertion of the problem (fig.13).

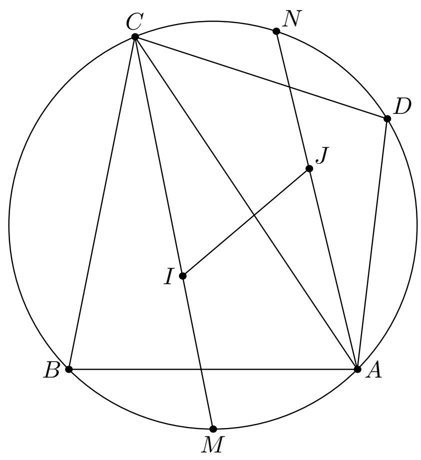


Fig. 13

14. (M.Kungozhin, grades 9–11) Let ABC be a right-angled triangle with $\angle C = 90°$, K, L, M be the midpoints of sides AB, BC, CA respectively, and N be a point of side AB. The line CN meets KM and KL at points P and Q respectively. Points S, T lying on AC and BC respectively are such that $APQS$ and $BPQT$ are cyclic quadrilaterals. Prove that

a) if CN is a bisector, then CN, ML and ST concur;

b) if CN is an altitude, then ST bisects ML.

Solution. a) By the assumption we have $CP = CM\sqrt{2} = AC/\sqrt{2}$, $CQ = BC/\sqrt{2}$. Hence $CS = CP \cdot CQ/AC = BC/2 = BL$. Similarly $CT = CM$. Therefore the segments ML and ST are symmetric with respect to CN and meet on this line.

b) From the similarity of triangles CMP, QLC and ACB we obtain that $CP = AC \cdot AB/2BC$, $CQ = BC \cdot AB/2AC$. Thus $CS = AB^2/4AC$, $CT = AB^2/4BC$ and the triangle CST is similar to CBA. Therefore ST is perpendicular to the median of ABC, and since the altitude of triangle CST is equal to $AB/4$, we obtain that its foot coincides with the midpoint of ML.

15. (D.Hilko, grades 9–11) The altitudes AH_1, BH_2, CH_3 of an acute-angled triangle ABC meet at point H. Points P and Q are the reflections of H_2 and H_3 with respect to H. The circumcircle of triangle PH_1Q meets for the second time BH_2 and CH_3 at points R and S. Prove that RS is a medial line of triangle ABC.

Solution. Consider the common point R' of the medial line M_2M_3 and the altitude BH_2. Prove that R' lies on the circumcircle of triangle PH_1Q.

Since BH_3H_2C is a cyclic quadrilateral, we have $H_3H \cdot HC = H_2H \cdot HB$. Then $HB \cdot HP = HC \cdot HQ$ and $PBQC$ is a cyclic quadrilateral. Therefore $\angle H_2PQ = \angle BCQ = \angle BAH_1$. Also, since R' lies on the medial line of ABC, we obtain that $\angle H_1AR' = \angle AH_1R'$. Now the triangles H_3HH_1 and BM_3R' are similar because $\angle M_3BR' = \angle H_3H_1H$, and $\angle M_3R'B = \angle HH_3H_1$. Thus

$$\frac{H_3H_1}{M_3R'} = \frac{HH_1}{BM_3}.$$

Since $H_3H = HQ$ and $BM_3 = M_3A$, we have

$$\frac{QH}{M_3R'} = \frac{HH_1}{AM_3}.$$

Clearly $\angle QHH_1 = \angle B = \angle AM_3R'$. This implies that triangles AM_3R' and HH_1Q are similar, hence $\angle HH_1Q = \angle M_3AR'$. Then $\angle QH_1R' = \angle HH_1Q - \angle HH_1R' = \angle M_3AR' - \angle R'AH_1 = \angle BAH_1 = \angle R'PQ$. Therefore PH_1QR' is a cyclic quadrilateral (fig.15). Hence $R = R'$, i.e R lies on the medial line of ABC. Similarly S lies on the medial line and we obtain the required assertion.

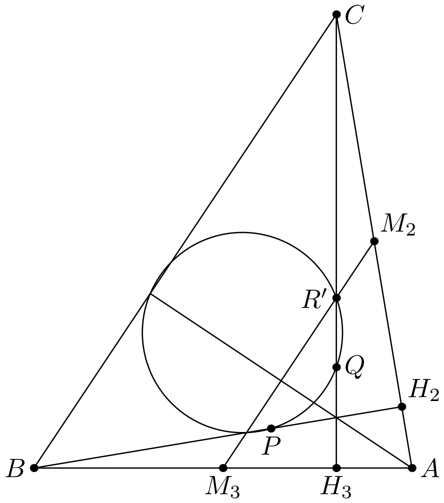


Fig. 15

16. (P.Ryabov, grades 9–11) Let ABC be a triangle with $AB < BC$. The bisector of angle C meets the line parallel to AC and passing through B, at point P. The tangent at B to the circumcircle of ABC meets this bisector at point R. Let R' be the reflection of R with respect to AB. Prove that $\angle R'PB = \angle RPA$.

    Solution. Since the lines BR and BP are symmetric with respect to the bisector of angle B, we obtain that P and R are isogonally conjugated with respect to ABC. Thus $\angle R'AB = \angle RAB = \pi - \angle CAP$, i.e. lines AR' and AC are symmetric with respect to the bisector of angle A. Similarly BR' and BC are symmetric with respect to the bisector of angle B. Therefore R' and C are isogonally conjugated with respect to triangle ABP, which yields the required assertion (fig.16).

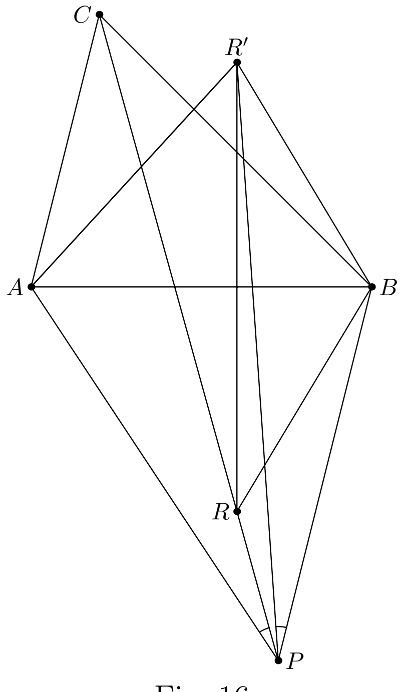


Fig. 16

17. (S.Takhaev, grades 10–11) Let each of circles α, β, γ touches two remaining circles externally, and all of them touch a circle Ω internally at points A_1, B_1, C_1 respectively. The common internal tangent to α and β meets the arc A_1B_1 not containing C_1 at point C_2. Points A_2, B_2 are defined similarly. Prove that the lines A_1A_2, B_1B_2, C_1C_2 concur.

   Solution. Let the tangents to Ω at A_1, B_1, C_1 form a triangle ABC. Without loss of generality suppose that Ω is the incircle (not the excircle) of this triangle. Note that, for example, C is the radical center of circles α, β and Ω, i.e. C lies on the common internal tangent to α and β. Also the common tangents to α, β, γ concur at their radical center, denote it by X. Hence we can reformulate the assertion of the problem as follows.

   A triangle ABC and a point X inside its incircle are given. The segments XA, XB, XC meet the incircle at A_2, B_2, C_2 respectively, and the sides BC, CA, AB touch it at A_1, B_1, C_1. Then A_1A_2, B_1B_2 and C_1C_2 concur.

   Applying the sinus theorem to triangles A_1CC_2 and B_1CC_2 we obtain

$$\frac{A_1C_2}{B_1C_2} = \frac{\sin\angle A_1CC_2}{\sin\angle B_1CC_2} \cdot \frac{\sin\angle CB_1C_2}{\sin\angle CA_1C_2} = \frac{\sin\angle A_1CC_2}{\sin\angle B_1CC_2} \cdot \frac{B_1C_2}{A_1C_2}.$$

   Now applying the Ceva theorem to triangles ABC and $A_1B_1C_1$ we obtain the required assertion.

18. (A.Polyanskii, N.Polyanskii, grades 10–11) Let C_1, A_1, B_1 be points on sides AB, BC, CA of triangle ABC, such that AA_1, BB_1, CC_1 concur. The rays B_1A_1 and B_1C_1 meet the circumcircle of the triangle at points A_2 and C_2 respectively. Prove that A, C, the common point of A_2C_2 and BB_1 and the midpoint of A_2C_2 are concyclic.

   Solution. Let K be the common point of A_2C_2 and AC, M be the midpoint of A_2C_2, and N be the second common point of circle ACM with A_2C_2. Then $KM\cdot KN = KA\cdot KC = KA_2 \cdot KC_2$, i.e the quadruple A_2, C_2, K, N is harmonic. Projecting A_2C_2 from B_1 to AA_1 we obtain that A_1, the common point of AA_1 with B_1C_1, A and the common point of BN with AA_1 also form a harmonic quadruple. Thus BN passes through the common point of AA_1, BB_1 and CC_1, i.e. coincides with BB_1.

19. (A.Myakishev, grades 10–11) Let a triangle ABC be given. On a ruler three segment congruent to the sides of this triangle are marked. Using this ruler construct the orthocenter of the triangle formed by the tangency points of the sides of ABC with its incircle.

   Solution. On the extension of AC beyond C, construct the segment $CX = BC$. We obtain the line BX parallel to the bisector of angle C. Similarly we can construct the line parallel to the bisector of C and passing through A, and having two parallel lines we can draw a line parallel to them and passing through an arbitrary point. Hence it is sufficient to construct the touching points A', B', C' of the incircle with BC, CA, AB (the bisectors of ABC are perpendicular to the sides of $A'B'C'$). On the extensions of AB beyond A and B, construct the segments $AU = BC$ and $BV = AC$ respectively. Since $AC' = s - BC$, where s is the semiperimeter of ABC, we obtain that C' is the midpoint of UV. Drawing the lines through U and V parallel to two bisectors, we obtain a parallelogram with diagonal UV, and drawing its second diagonal we find C'.

20. (A.Zimin, grades 10–11) Let the incircle of a nonisosceles triangle ABC touch AB, AC and BC at points D, E and F respectively. The corresponding excircle touches the side

BC at point N. Let T be the common point of AN and the incircle, closest to N, and K be the common point of DE and FT. Prove that $AK||BC$.

Solution. Let G be the point of the incircle opposite to F. Since the incircle and the excircle are homothetic with center A, we obtain that A, G and N are collinear, and $FT \perp AN$. The polar transformation with respect to the incircle maps ED into A, maps FT into the common point of tangents at F and T, i.e. the common midpoint of FN and BC, and maps the line through A parallel to BC into the common point L of ED and GF. Thus we have to prove that AL is a median of ABC (fig.20).

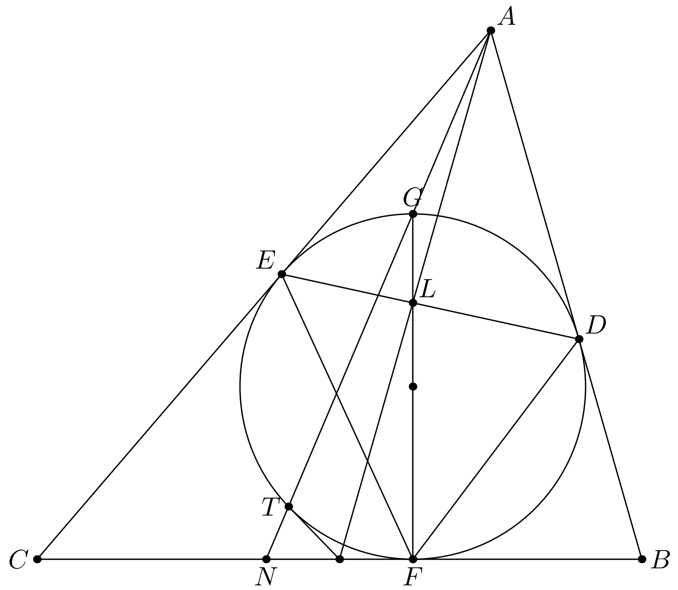


Fig. 20

Since $AE = AD$, we have $\sin\angle CAL : \sin\angle BAL = EL : DL$. Applying the sinus theorem to triangles EFL and EDL, we obtain $EL : DL = EF \sin\angle EFL : DF \sin\angle DFL$. But $\angle EFL = \angle C/2$, $\angle DFL = \angle B/2$, and $EF : DF = \cos\angle C/2 : \cos\angle B/2$. Therefore $\sin\angle CAL : \sin\angle BAL = AB : AC$, i.e. AL is a median.

21. (proposed by B.Frenkin, grades 10–11) In the plane a line l and a point A outside it are given. Find the locus of the incenters of acute-angled triangles having a vertex A and an opposite side lying on l.

    Solution. Let H be the projection of A to l. Since a triangle is acute-angled, we obtain that its incenter I and one of its vertices, for example B, lie on the opposite sides with respect to AH. Hence the distance from I to AH is less than the distance from I to AB which is equal to the inradius r, i.e. the distance from I to l. Therefore I lies inside the right angle formed by the bisectors of two angles between AH and l. Also it is clear that $r < AH/2$, i.e. I lies inside a strip bounded by l and the perpendicular bisector to AH. Finally, since angle A is acute, we have $AI = r/\sin\angle A/2 > r\sqrt{2}$, hence I lies between the branches of an equilateral hyperbola with focus A and directrix l. On the other hand, for an arbitrary point satisfying these conditions we can construct the circle centered at this point and touching l, draw the tangents to it from A and obtain an acute-angled triangle. So the required locus is bounded by the bisectors of angles between l and AH, the perpendicular bisector to AH and the corresponding branch of the hyperbola (the bounds are not included).

22. (N.Beluhov, grades 10–11) Six circles of unit radius lie in the plane so that the distance between the centers of any two of them is greater than d. What is the least value of d such that there always exists a straight line which does not intersect any of the circles and separates the circles into two groups of three?

    Solution. Let $O_1O_2O_3$ be an equilateral triangle of side d, O_4 be such that $O_1O_4 = d$ and $\angle O_2O_1O_4 = \angle O_4O_1O_3 = 150°$, and O_5 and O_6 be defined analogously so that the complete figure is rotationally symmetric about the center of $\triangle O_1O_2O_3$. The six circles centered at O_1, O_2, $\ldots$, O_6 show that $d \geq \frac{2}{\sin 15°} = 2(\sqrt{2}+\sqrt{6})$.

    Put $d = \frac{2}{\sin 15°}$. Let us show that a halving line always exists.

    Enumerate the circles' centers from 1 to 6, and let l be a straight line such that the six centers' projections onto l are distinct. The order of the projections from left to right gives us a permutation σ of the numbers 1 through 6.

    Rotate l counterclockwise until it makes a complete $360°$ turn. Each time that l becomes perpendicular to a line through two centers (it suffices to consider the case when no three centers are collinear), two neighbouring elements of σ switch their positions. Since there are $\binom{6}{2} = 15$ such lines, $2 \cdot 15 = 30$ such transpositions occur.

    We say that a transposition is *external* if, at the time when it takes place, the two centers involved are either the first two or the last two elements of σ (i.e., $AB\circ\circ\circ\circ \to BA\circ\circ\circ\circ$ or $\circ\circ\circ\circ AB \to \circ\circ\circ\circ BA$). Otherwise, we say that a transposition is *internal*.

    Since an external transposition corresponds to a side of the centers' convex hull, there are at least $2 \cdot 3 = 6$ external transpositions and at most $30 - 6 = 24$ internal ones.

    Since $\frac{360°}{24} = 15°$, there is some interval s of the rotation of l having length at least $15°$, containing no internal transpositions. This means that throughout s both the third and the fourth element of σ remain fixed. Let A and B be those elements.

    Consider the strip L bounded by the lines through A and B perpendicular to l. Throughout s, L does not contain any centers apart from A and B. Since the length of s is at least $15°$, there is some position of L during s such that the acute angle between AB and L is at least $15°$ and, consequently, the width of L is greater than two. The midline of this instance of L does the job.

23. (A.Kanel-Belov, 10–11) (grades 10–11) The plane is divided into convex heptagons with diameters less than 1. Prove that an arbitrary disc with radius 200 intersects most than a billion of them.

    Solution. Consider a disc K with radius R. Let k vertices of heptagons lie inside K. The average angle at these vertices is at most $2\pi/3$. (If a vertex is common for more than three heptagons then the average angle is less than $2\pi/3$, and if a vertex lies on a side then the average angle is at most $\pi/2$).

    On the other hand, consider the heptagons lying inside K or intersecting the bounding circle of K. Their average angle is $5\pi/7$. Let n of their vertices lie outside K, all of them lie at a distance not greater than 1 from K. Each angle in such vertex is less than π (of course this is true for any angle of a convex polygon).

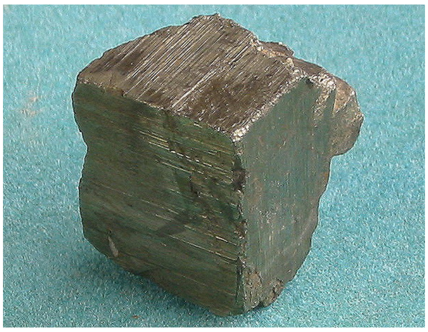
To satisfy the balance, the inequality $n\pi + k \cdot 2\pi/3 > (n + k)5\pi/7$ is needed. Hence $n > k/6$.

So the number of vertices lying at a distance not greater than 1 from K is greater than the number of vertices inside K divided by 6.

Now note that $(1 + 1/6)^6 > 2$, $(1 + 1/6)^{60} > 2^{10} > 1000$ and $(1 + 1/6)^{180} > 1000^3$.

Hence the number of heptagons (the number of angles divided by 7) intersecting a disc with radius 200 is at least 10^9.

24. (A.Solynin, grades 10–11) A crystal of pyrite is a parallelepiped with dashed faces.



The dashes on any two adjacent faces are perpendicular. Does there exist a convex polytope with the number of faces not equal to 6, such that its faces can be dashed in such a manner?

Answer. Yes.

Solution. Take a quadrilateral $ABCD$. Let the lines AB and CD meet at point X, and the lines AD and BC meet at point Y. Draw the plane passing through the line XY and perpendicular to the plane $ABCD$, and a point S on this plane such that $\angle XSY = 90°$. Now dash the faces SAB and SCD of pyramid $SABCD$ by lines parallel to SX, dash the faces SBC and SCD by lines parallel to SY, and dash the face $ABCD$ by perpendiculars to the plane SXY. Clearly the obtained dashes satisfy the condition.

XV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
The correspondence round

Below is the list of problems for the first (correspondence) round of the XV Sharygin Geometrical Olympiad.

The olympiad is intended for high-school students of four eldest grades. In Russian school, these are 8-11. In the list below, each problem is indicated by the numbers of Russian school grades, for which it is intended. Foreign students of the last grade have to solve the problems for 11th grade, students of the preceding grade solve the problems for 10th grade etc. However, the participants may solve problems for elder grades as well (solutions of problems for younger grades will not be considered).

A complete solution of each problem costs 7 points. A partial solution costs from 1 to 6 points. A text without significant advancement costs 0 points. The result of a participant is the sum of all obtained marks.

First write down the statement of the problem, and then the solution. Present your solutions in detail, including all necessary arguments and calculations. Provide all necessary figures of sufficient size. **If a problem has an explicit answer, this answer must be presented distinctly**. Please, be accurate to provide good understanding and correct estimating of your work !

If your solution depends on some well-known theorems from standard textbooks, you may simply refer to them instead of providing their proofs. However, any fact not from the standard curriculum should be either proved or properly referred (with an indication of the source).

You may note the problems which you liked most (this is not obligatory). Your opinion is interesting for the Jury.

The **solutions** for the problems (in Russian or in English) must be **delivered not before December 1, 2018 and not later than on March 1, 2019**. To upload your work, enter the site **https://contest.yandex.ru/geomshar/**, indicate the language (English) in the right upper part of the page, press "Registration"in the left upper part, and follow the instructions.

Attention:

1. The solution of each problem (and of each part of it if any) must be contained in a **separate** pdf, doc, docx or jpg file. If the solution is contained in several files then pack them to an archive (zip or rar) and load it.

2. We recommend to prepare the paper using computer or to scan it rather than to photograph it. **In all cases, please check readability of the file before uploading**.

3. If you upload the solution of some problem more than once then only the last version is retained in the checking system. **Thus if you need to change something in your solution then you have to upload the whole solution again.**

If you have any technical problems with uploading of the work, apply to **geomshar@yandex.ru (DON'T SEND your work to this address)**.

The final round will be held in July–August 2019 in Moscow region. The winners of the correspondence round are invited to it if they don't graduate from school before. (For instance, if the last grade is 12 then we invite winners from 9–11 grades, and from 12 grade if they finish their school education later.) The graduates, winners of the correspondence round, will be awarded by diplomas of the Olympiad. The list of the winners will be published on **www.geometry.ru** at the end of May 2019 at latest. If you want to know your detailed results, please use e-mail **geomshar@yandex.ru**.

1. (8) Let AA_1, CC_1 be the altitudes of triangle ABC, and P be an arbitrary point of side BC. Point Q on the line AB is such that $QP = PC_1$, and point R on the line AC is such that $RP = CP$. Prove that QA_1RA is a cyclic quadrilateral.

2. (8) The circle ω_1 passes through the center O of the circle ω_2 and meets it at points A and B. The circle ω_3 centered at A with radius AB meets ω_1 and ω_2 at points C and D (distinct from B). Prove that C, O, D are collinear.

3. (8) The rectangle $ABCD$ lies inside a circle. The rays BA and DA meet this circle at points A_1 and A_2. Let A_0 be the midpoint of A_1A_2. Points B_0, C_0, D_0 are defined similarly. Prove that $A_0C_0 = B_0D_0$.

4. (8) The side AB of triangle ABC touches the corresponding excircle at point T. Let J be the center of the excircle inscribed into angle A, and M be the midpoint of AJ. Prove that $MT = MC$.

5. (8–9) Let A, B, C and D be four points in general position, and ω be a circle passing through B and C. A point P moves along ω. Let Q be the common point of circles ABP and PCD distinct from P. Find the locus of points Q.

6. (8–9) Two quadrilaterals $ABCD$ and $A_1B_1C_1D_1$ are symmetric with respect to the point P. It is known that A_1BCD, AB_1CD and ABC_1D are cyclic quadrilaterals. Prove that the quadrilateral $ABCD_1$ is also cyclic.

7. (8–9) Let AH_A, BH_B, CH_C be the altitudes of the acute-angled triangle ABC. Let X be an arbitrary point of segment CH_C, and P be the common point of circles with diameters H_CX and BC, distinct from H_C. The lines CP and AH_A meet at point Q, and the lines XP and AB meet at point R. Prove that A, P, Q, R, H_B are concyclic.

8. (8–9) The circle ω_1 passes through the vertex A of the parallelogram $ABCD$ and touches the rays CB, CD. The circle ω_2 touches the rays AB, AD and touches ω_1 at point T. Prove that T lies on the diagonal AC.

9. (8–9) Let A_M be the midpoint of side BC of an acute-angled triangle ABC, and A_H be the foot of the altitude to this side. Points B_M, B_H, C_M, C_H are defined similarly. Prove that one of the ratios $A_MA_H : A_HA$, $B_MB_H : B_HB$, $C_MC_H : C_HC$ is equal to the sum of two remaining ratios.

10. (8–9) Let N be the midpoint of arc ABC of the circumcircle of triangle ABC, and NP, NT be the tangents to the incircle of this triangle. The lines BP and BT meet the circumcircle for the second time at points P_1 and T_1 respectively. Prove that $PP_1 = TT_1$.

11. (8–9) Morteza marks six points in the plane. He then calculates and writes down the area of every triangle with vertices in these points (20 numbers). Is it possible that all of these numbers are integers, and that they add up to 2019?

12. (8–11) Let $A_1A_2A_3$ be an acute-angled triangle inscribed into a unit circle centered at O. The cevians from A_i passing through O meet the opposite sides at points B_i $(i = 1, 2, 3)$ respectively.

    (a) Find the minimal possible length of the longest of three segments B_iO.

    (b) Find the maximal possible length of the shortest of three segments B_iO.

13. (9–10) Let ABC be an acute-angled triangle with altitude $AT = h$. The line passing through its circumcenter O and incenter I meets the sides AB and AC at points F and N respectively. It is known that $BFNC$ is a cyclic quadrilateral. Find the sum of the distances from the orthocenter of ABC to its vertices.

14. (9–11) Let the side AC of triangle ABC touch the incircle and the corresponding excircle at points K and L respectively. Let P be the projection of the incenter onto the perpendicular bisector of AC. It is known that the tangents to the circumcircle of triangle BKL at K and L meet on the circumcircle of ABC. Prove that the lines AB and BC touch the circumcircle of triangle PKL.

15. (9–11) The incircle ω of triangle ABC touches the sides BC, CA and AB at points D, E and F respectively. The perpendicular from E to DF meets BC at point X, and the perpendicular from F to DE meets BC at point Y. The segment AD meets ω for the second time at point Z. Prove that the circumcircle of the triangle XYZ touches ω.

16. (9–11) Let AH_1 and BH_2 be the altitudes of triangle ABC; let the tangent to the circumcircle of ABC at A meet BC at point S_1, and the tangent at B meet AC at point S_2; let T_1 and T_2 be the midpoints of AS_1 and BS_2 respectively. Prove that T_1T_2, AB and H_1H_2 concur.

17. (10–11) Three circles ω_1, ω_2, ω_3 are given. Let A_0 and A_1 be the common points of ω_1 and ω_2, B_0 and B_1 be the common points of ω_2 and ω_3, C_0 and C_1 be the common points of ω_3 and ω_1. Let $O_{i,j,k}$ be the circumcenter of triangle $A_iB_jC_k$. Prove that the four lines of the form $O_{ijk}O_{1-i,1-j,1-k}$ are concurrent or parallel.

18. (10–11) A quadrilateral $ABCD$ without parallel sidelines is circumscribed around a circle centered at I. Let K, L, M and N be the midpoints of AB, BC, CD and DA respectively. It is known that $AB \cdot CD = 4IK \cdot IM$. Prove that $BC \cdot AD = 4IL \cdot IN$.

19. (10–11) Let AL_a, BL_b, CL_c be the bisecors of triangle ABC. The tangents to the circumcircle of ABC at B and C meet at point K_a, points K_b, K_c are defined similarly. Prove that the lines K_aL_a, K_bL_b and K_cL_c concur.

20. (10–11) Let O be the circumcenter of triangle ABC, H be its orthocenter, and M be the midpoint of AB. The line MH meets the line passing through O and parallel to at point K lying on the circumcircle of ABC. Let P be the projection of K onto AC. Prove that $PH \parallel BC$.

21. (10–11) An ellipse Γ and its chord AB are given. Find the locus of orthocenters of triangles ABC inscribed into Γ.

22. (10–11) Let AA_0 be the altitude of the isosceles triangle ABC ($AB = AC$). A circle γ centered at the midpoint of AA_0 touches AB and AC. Let X be an arbitrary point of line BC. Prove that the tangents from X to γ cut congruent segments on lines AB and AC.

23. (10–11) In the plane, let a, b be two closed broken lines (possibly self-intersecting), and K, L, M, N be four points. The vertices of a, b and the points K L, M, N are in general position (i.e. no three of these points are collinear, and no three segments between them concur at an interior point). Each of segments KL and MN meets a at an even number of

points, and each of segments LM and NK meets a at an odd number of points. Conversely, each of segments KL and MN meets b at an odd number of points, and each of segments LM and NK meets b at an even number of points. Prove that a and b intersect.

24. (11) Two unit cubes have a common center. Is it always possible to number the vertices of each cube from 1 to 8 so that the distance between each pair of identically numbered vertices would be at most $4/5$? What about at most $13/16$?

XV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
The correspondence round. Solutions

1. (I.Kukharchuk, 8) Let AA_1, CC_1 be the altitudes of triangle ABC, and P be an arbitrary point of side BC. Point Q on the line AB is such that $QP = PC_1$, and point R on the line AC is such that $RP = CP$. Prove that QA_1RA is a cyclic quadrilateral.

   Solution. It is clear that A, C, A_1, C_1 are concyclic. Denote the corresponding circle by ω_1. Furthermore the midpoints X and Y of segments QC_1 and RC are the projections of P to AB and AC respectively, thus X, Y and A_1 lie on the circle ω_2 with diameter AP. Let O be symmetric to the center of ω_1 (the midpoint of AC) about the center of ω_2. By Thales theorem, the projections of O to AB and AC are the midpoints of segments AQ and AR respectively, i.e. O is the circumcenter of triangle AQR. Since O lies on the perpendicular bisector to AA_1, the point A_1 also lies on the circle ABC (fig.1).

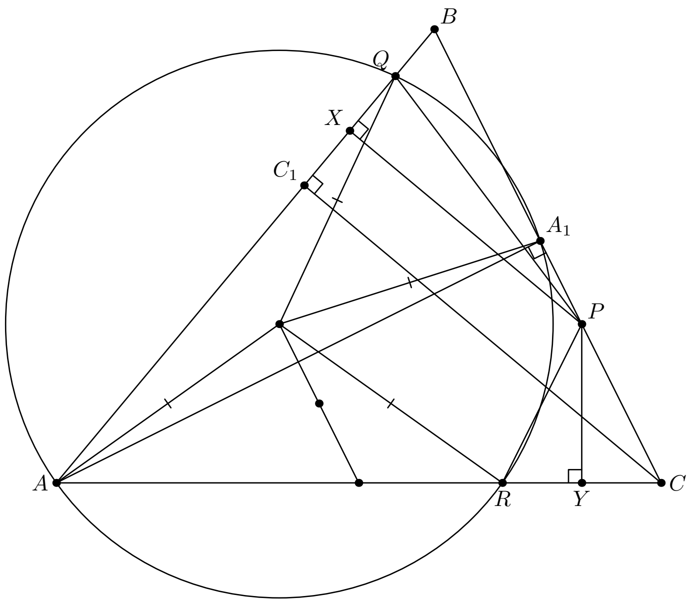


Fig. 1

2. (D.Shvetsov, 8) The circle ω_1 passes through the center O of the circle ω_2 and meets it at points A and B. The circle ω_3 centered at A with radius AB meets ω_1 and ω_2 at points C and D (distinct from B). Prove that C, O, D are collinear.

   Solution. Since the arcs AC and AB of ω_1 are congruent, we obtain that $\angle AOC = 180^\circ - \angle AOB$. But it is clear that $\angle AOD = \angle AOB$ (fig.2), so we obtain the required assertion.

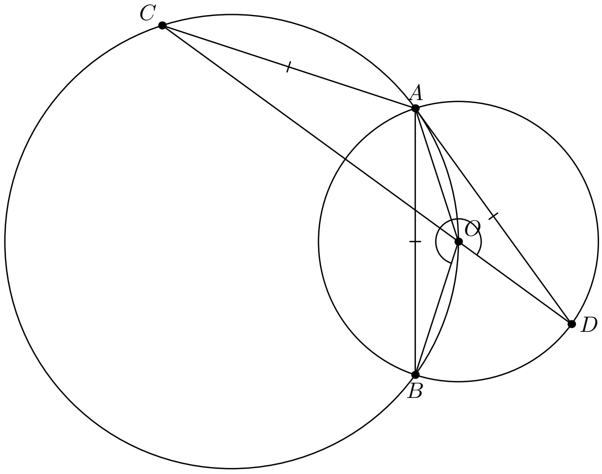


Fig. 2

3. (L.Shteingarts, 8) The rectangle $ABCD$ lies inside a circle. The rays BA and DA meet this circle at points A_1 and A_2. Let A_0 be the midpoint of A_1A_2. Points B_0, C_0, D_0 are defined similarly. Prove that $A_0C_0 = B_0D_0$.

   Solution. Let X, Y be the projections of the center of the circle to AB, CD respectively (fig.3). Then $BB_1 - AA_1 = (XB_1 - XB) - (XA_1 - XA) = AX - BX = DY - CY = CC_1 - DD_1$. Therefore the projection of segment A_0C_0 to the line AB, equal to $(A_1B_1 + C_1D_1 - AA_1 - CC_1)/2$, is congruent to the projection of segment B_0D_0 to the same line. Similarly the projections of these segments to the line AD are congruent, thus the segments are congruent too.

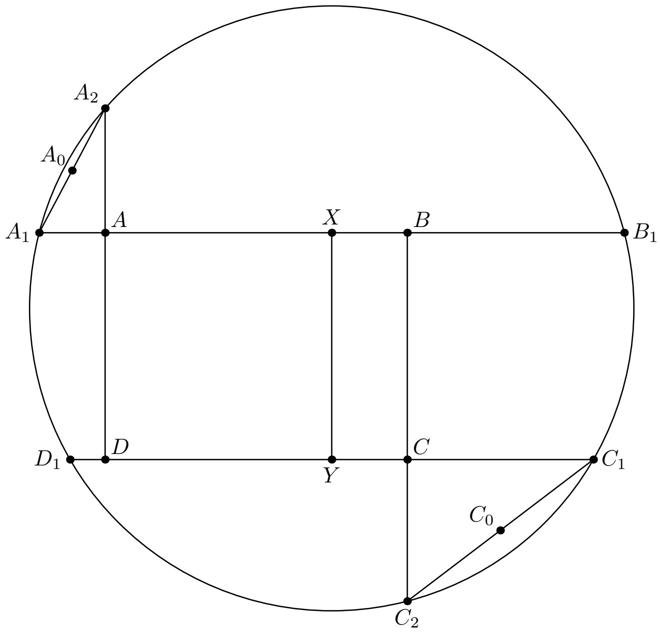


Fig. 3

4. (A.Trigub, 8) The side AB of triangle ABC touches the corresponding excircle at point T. Let J be the center of the excircle inscribed into angle A, and M be the midpoint of AJ. Prove that $MT = MC$.

Solution. Let R be the projection of J to AC. Then $CR = p - AC = AT$. Furthermore $MR = MA$ as a median of the right-angled triangle AJR, and $\angle MRA = \angle MAR = \angle MAT$ (fig.4). Hence the triangles MTA and MCR are congruent and $MT = MC$.

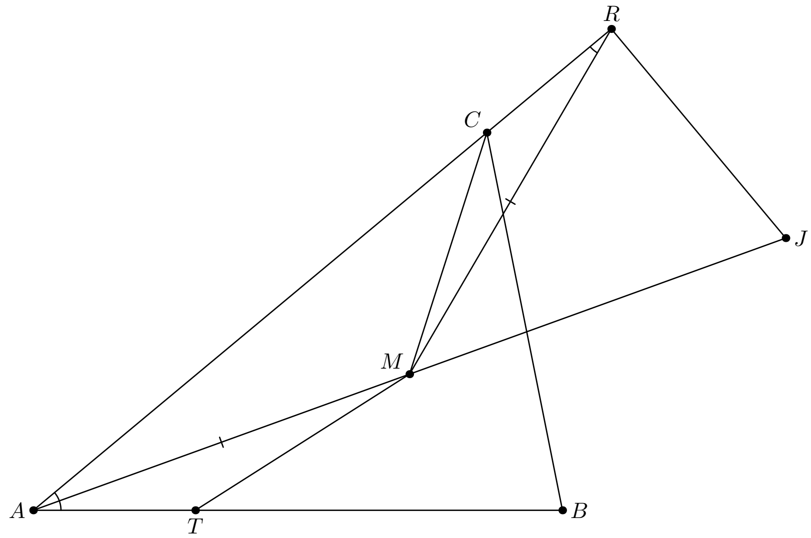


Fig. 4

5. (F.Ivlev, 8–9) Let A, B, C and D be four points in general position, and ω be a circle passing through B and C. A point P moves along ω. Let Q be the common point of circles ABP and PCD distinct from P. Find the locus of points Q.

   Solution. We have that $\angle(QA, QD) = \angle(QA, BA) + \angle(BA, DC) + \angle(DC, DQ) = \angle(QP, PB) + \angle(BA, DC) + \angle(PC, PQ) = \angle(PC, PB) + \angle(BA, DC)$ do not depend on P. Therefore the locus of Q is the circle passing through A and D.

6. (A.Akopyan, (8–9) Two quadrilaterals $ABCD$ and $A_1B_1C_1D_1$ are mutually symmetric with respect to the point P. It is known that A_1BCD, AB_1CD and ABC_1D are cyclic quadrilaterals. Prove that the quadrilateral $ABCD_1$ is also cyclic.

   Solution. We have $\angle(AD_1, D_1B) = \angle(AD_1, AB_1) + \angle(A_1B, D_1B) = \angle(A_1D, A_1B) + \angle(AB_1, B_1D) = \angle(AC, CD) + \angle(CD, BC) = \angle(AC, BC)$. Thus A, B, C, D_1 are concyclic.

7. (P.Bibikov, (8–9) Let AH_A, BH_B, CH_C be the altitudes of the acute-angled triangle ABC. Let X be an arbitrary point of segment CH_C, and P be the common point of circles with diameters H_CX and BC, distinct from H_C. The lines CP and AH_A meet at point Q, and the lines XP and AB meet at point R. Prove that A, P, Q, R, H_B are concyclic.

   Solution. Since $BCPH_C$ is a cyclic quadrilateral, we obtain $\angle CPH_C = 180° - \angle B = 180° - \angle AHH_C$, where H is the orthocenter of ABC. Hence $HQPH_C$ is cyclic, i.e. $\angle CQH = \angle HH_CP$. But $\angle HH_CP = \angle H_CRP$ because H_CP is an altitude of the right-angled triangle H_CRX. Thus A, R, P and Q are concyclic. Since H_CPH_BC is cyclic, we obtain $\angle PH_BA = \angle PH_CC = \angle PRB$, therefore H_B lies on the same circle (fig.7).

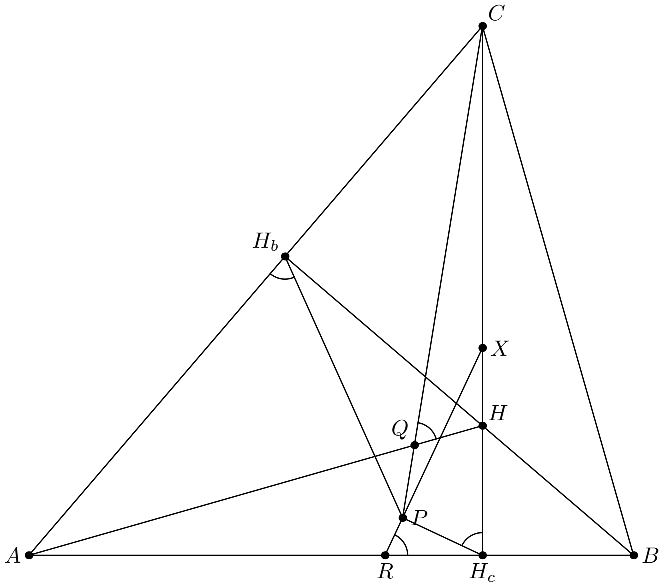


Fig. 7

8. (M.Etesamifard, 8–9) The circle ω_1 passes through the vertex A of the parallelogram $ABCD$ and touches the rays CB, CD. The circle ω_2 touches the rays AB, AD and touches ω_1 externally at point T. Prove that T lies on the diagonal AC.

   Solution. Let T' be the common point of ω_1 and the ray AC. The homothety with center T' mapping C to A maps the rays CB, CD to AD, AB respectively. Thus it maps ω_1 to circle ω' which touches these rays and is tangent to ω_1 at T'. Therefore ω' coincides with ω_2, and T' coincides with T.

9. (E.Bakaev, 8–9) Let A_M be the midpoint of side BC of an acute-angled triangle ABC, and A_H be the foot of the altitude to this side. Points B_M, B_H, C_M, C_H are defined similarly. Prove that one of the ratios $A_MA_H : A_HA$, $B_MB_H : B_HB$, $C_MC_H : C_HC$ is equal to the sum of two remaining ratios.

   Solution. Note that, for example, $C_MC_H = |AC_H - BC_H|/2$. Therefore $C_MC_H/C_HC = |\cot A - \cot B|/2$. The required assertion follows from this and two similar equalities.

10. (A.Trigub, 8–9) Let N be the midpoint of arc ABC of the circumcircle of triangle ABC, and NP, NT be the tangents to the incircle of this triangle. The lines BP and BT meet the circumcircle for the second time at points P_1 and T_1 respectively. Prove that $PP_1 = TT_1$.

    Solution. Let I be the incenter of ABC. Since BN is the external bisector of angle B, we have $\angle IBN = 90^\circ = \angle IPN = \angle ITN$. Thus B, I, N, T, P are concyclic, and since $IT = IP$, we obtain that BI bisects angle PBT. Hence P_1 and T_1 are symmetric with respect to the diameter of the circumcircle passing through N, i.e. $NP_1 = NT_1$. Furthermore $\angle NPB = \angle NTB$ and $NP = NT$, Therefore the triangles NPN_1 and NTT_1 are congruent and we obtain the required assertion (fig.10).

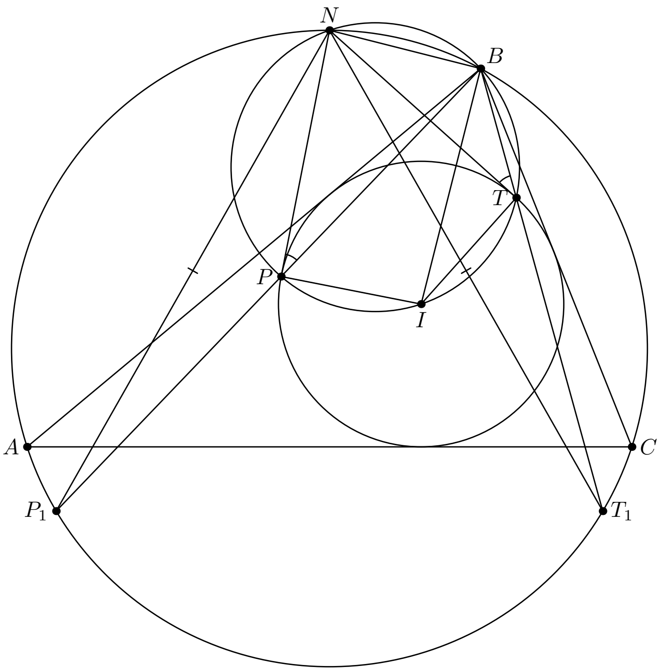


Fig. 10

11. (M.Saghafian, 8–9) Morteza marks six points in the plane. He then calculates and writes down the area of every triangle with vertices in these points (20 numbers). Is it possible that all of these numbers are integers, and that they add up to 2019?

Answer. No.

Solution. Consider any four of marked points. If they form a convex quadrilateral $ABCD$, then $S_{ABC} + S_{ACD} = S_{ABD} + S_{BCD}$. And if one point lies inside the triangle formed by three remaining ones, then the area of this triangle is equal to the sum of areas of three inner triangles. In both cases the sum of areas of four triangles formed by these points will be even. If we sum up all such sums then each triangle will be counted three times, therefore the sum of all 20 areas is also even.

12. (B.Frenkin, 8–11) Let $A_1A_2A_3$ be an acute-angled triangle inscribed into a unit circle centered at O. The cevians from A_i passing through O meet the opposite sides at points B_i $(i = 1, 2, 3)$ respectively.

(a) Find the minimal possible length of the longest of three segments B_iO.

(b) Find the maximal possible length of the shortest of three segments B_iO.

Answer. (a), (b) 1/2.

Solution. Firstly let us show that among two segments B_iO, the longer segment is directed to the shorter side (clearly, equality of sides implies equality of segments). Suppose, for example, that $A_1A_3 < A_2A_3$. Since $\angle OA_1A_2 = \angle OA_2A_1$, we have $\angle OA_2B_1 < \angle OA_1B_2$. For triangles A_1OB_2 and A_2OB_1, we have $A_1O = A_2O$, $\angle A_1OB_2 = \angle A_2OB_1$. Hence $BO_1 < B_2O$ as required.

(a) Suppose that in an acute-angled triangle $A_1A_2A_3$ with the circumcircle of radius 1 the side A_1A_2 is the shortest. Then the segment B_3O is the longest among B_iO. Since $\angle A_3 \leq 60°$, we have $\angle A_1OA_2 \leq 120°$ and $\angle OA_1A_2 \geq 30°$. From O, draw the perpendicular OP to A_1A_2. Then $1/2 \le OP \le B_3O$. The equality is attained for the equilateral triangle.

b) Suppose that in an acute-angled triangle $A_1A_2A_3$ with the circumcircle of radius 1 the side A_1A_2 is the shortest, and the side A_2A_3 is the longest, so that the segment B_1O is the shortest among B_iO. Let us move point A_1 along the circumcircle towards point A_2. Then the segment B_1O will increase because it will move away from the perpendicular from O to A_2A_3. When the angle $A_1A_3A_2$ will equal $180^\circ - 2\angle A_2A_1A_3$, we will obtain an isosceles triangle with $A_1A_3 = A_2A_3 \ge A_1A_2$.

In triangle $A_1B_1A_3$, the segment A_3O is a bisector, so $B_1O/A_1O = B_1A_3/A_1A_3 = B_1A_3/A_2A_3$. It is easily seen that the last ratio does not exceed $1/2$ for $A_1A_2 \le A_1A_3$. Hence $B_1O \le 1/2$. The equality is attained for the equilateral triangle.

13. (G.Filippovsky, 9–10) Let ABC be an acute-angled triangle with altitude $AT = h$. The line passing through its circumcenter O and incenter I meets the sides AB and AC at points F and N respectively. It is known that $BFNC$ is a cyclic quadrilateral. Find the sum of the distances from the orthocenter of ABC to its vertices.

Answer. $2h$.

Solution. Since $BNFC$ is cyclic, we have $\angle ONA = \angle B$. On the other hand, $\angle OAC = \pi/2 - \angle B$. Thus $AO \perp OI$. Draw the perpendicular IT to AH. Since AI bisects angle OAH, we obtain that the right-angled triangles AOI and ATI are congruent, i.e. $AT = AO = R$ and $h = AH = R + r$, where R and r are the circumradius and the inradius of triangle ABC (fig.13). It is known that the sum of distances from O to the sidelines of the triangle is equal to $R + r$, and the sum of distances from the orthocenter to the vertices is twice as large, which yields the answer.

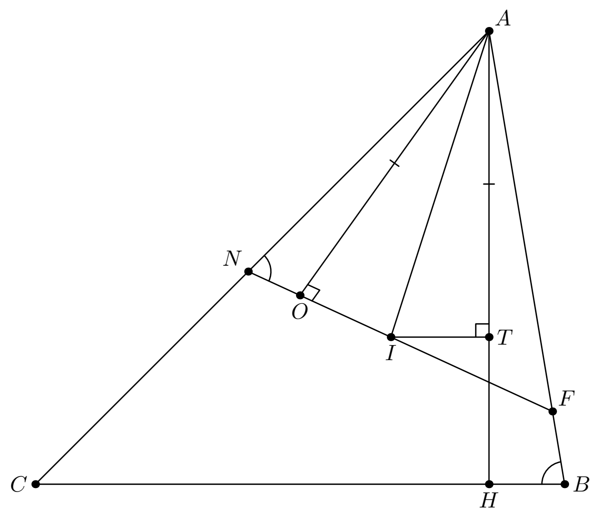


Fig. 13

14. (S.Arutyunyan, 9–11) Let the side AC of triangle ABC touch the incircle and the corresponding excircle at points K and L respectively. Let P be the projection of the incenter onto the perpendicular bisector of AC. It is known that the tangents to the circumcircle of triangle BKL at K and L meet on the circumcircle of ABC. Prove that the lines AB and BC touch the circumcircle of triangle PKL.

Solution. Suppose that $AB > BC$. Let M be the midpoint of AC, N be the midpoint of arc ABC, NW and KD be the diameters of the circumcircle and the incircle respectively. By the assumption, the tangents to the circle BKL at K and L meet at W, i.e. BW is the symedian of triangle BKL. Furthermore B, D, L are collinear and BW bisects segment KD. Hence triangles BKL and BDK are similar, i.e. $\angle BMC = \angle BID = (\angle C - \angle A)/2$. Then $\angle BMN = (\pi - \angle C + \angle A)/2 = \angle BNM$ and $BM = BN$. Let S be a point on the arc AWC such that $\angle SBC = \angle ABM$. Then $\angle SNB = \angle ABM + \angle BAC = \angle BMC = \angle NSB$, i.e. $BS = BN = BM$ (fig.14). By similarity of triangles ABM and SBC we have $AB \cdot BC = BM \cdot BS = BM^2 = (2AB^2 + 2BC^2 - AC^2)/4$. Therefore $AC^2 = 2(AB - BC^2)$, or $AC = \sqrt{2}KL$. Applying the Stewart theorem to triangle AWC and cevian WK we obtain that $WK^2 = WC^2 - AK \cdot KC = WI^2 - (AM^2 - MK^2) = WI^2 - MK^2 = WI^2 - PI^2 = WP^2$ (by the trident theorem, $WC = WI$). Thus P, K, L lie on the circle centered at W.

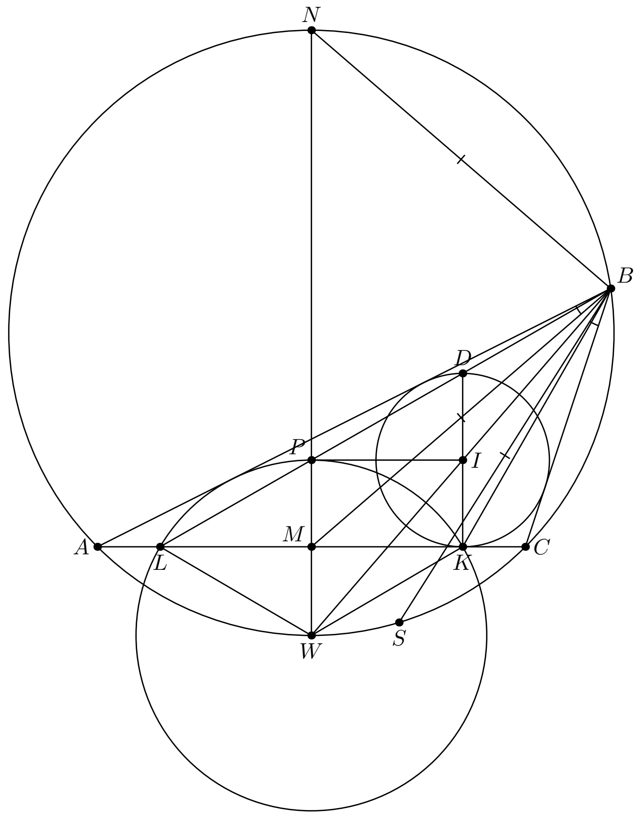


Fig. 14

Let R, r be the circumradius and the inradius of triangle ABC. Then the distance from W to line AB is equal to $BW \sin\frac{\angle B}{2} = 2R\cos\frac{\angle C - \angle A}{2}\sin\frac{\angle B}{2} = R(\sin\angle A + \sin\angle C)$. By the Carnot theorem $R + r = R(\sin\angle A + \sin\angle B + \sin\angle C)$, therefore this distance is equal to $R(1 - \cos\angle B) + r = WM + MP = WP$, which is equivalent to the required assertion.

15. (M.Etesamifard, 9–11) The incircle ω of triangle ABC touches the sides BC, CA and AB at points D, E and F respectively. The perpendicular from E to DF meets BC at point X, and the perpendicular from F to DE meets BC at point Y. The segment AD meets ω for the second time at point Z. Prove that the circumcircle of the triangle XYZ touches ω.

Solution. Let I be the center of ω. Note that $\angle FYX = \angle ICB = \angle FEX$, i.e. $XYEF$ is a cyclic quadrilateral. Now BC, EF and the tangent to ω at Z concur at the pole T of AD with respect to ω. Hence $TZ^2 = TF \cdot TE = TX \cdot TY$, i.e. TZ touches the circle XYZ (fig.15).

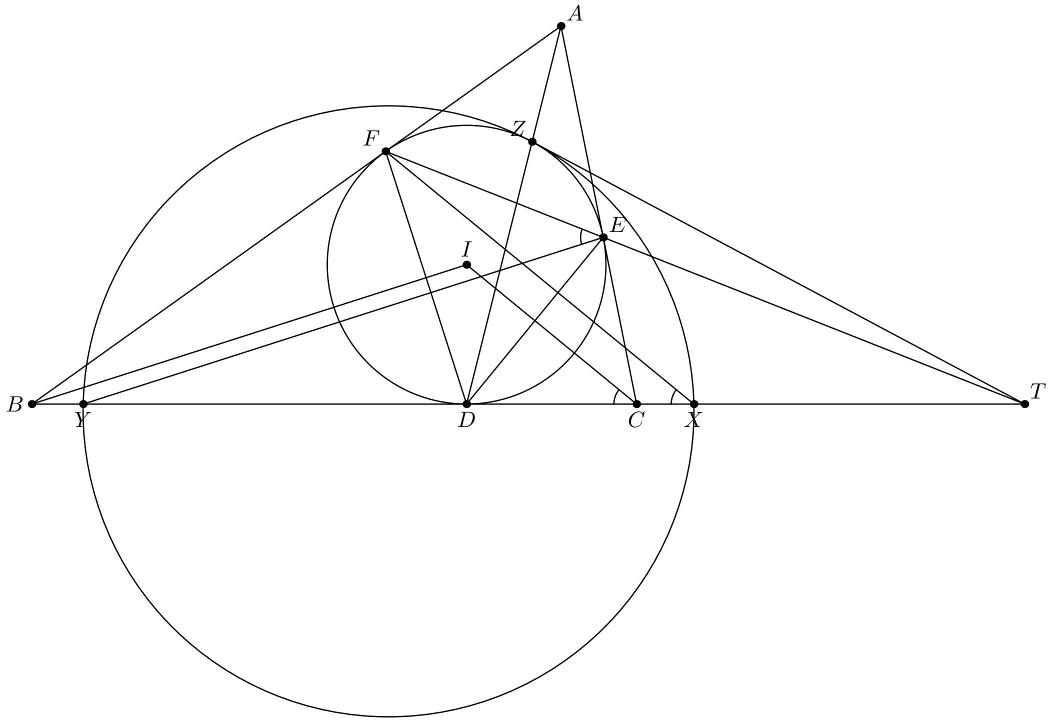


Fig. 15

16. (M.Plotnikov, 9–11) Let AH_1 and BH_2 be the altitudes of triangle ABC; let the tangent to the circumcircle of ABC at A meet BC at point S_1, and the tangent at B meet AC at point S_2; let T_1 and T_2 be the midpoints of AS_1 and BS_2 respectively. Prove that T_1T_2, AB and H_1H_2 concur.

    Solution. It is clear that T_1 lies on the medial line B_0C_0 of triangle ABC, and T_1A touches the circle AB_0C_0. Thus $T_1A^2 = T_1B_0 \cdot T_1C_0$. But B_0, C_0 lie on the nine-points circle (NPC) of triangle ABC, therefore T_1 lies on the radical axis of this circle and the circumcircle. By similar reasoning for T_2 we obtain that T_1T_2 is the radical axis of the circumcircle and the NPC. Since A, B, H_1, H_2 are concyclic, we obtain that the lines AB and H_1H_2 are the radical axes of the corresponding circle with the circumcircle and the NPC respectively. Clearly these three radical axes concur.

17. (E.Bakaev, 10–11) Three circles ω_1, ω_2, ω_3 are given. Let A_0 and A_1 be the common points of ω_1 and ω_2, B_0 and B_1 be the common points of ω_2 and ω_3, C_0 and C_1 be the common points of ω_3 and ω_1. Let $O_{i,j,k}$ be the circumcenter of triangle $A_iB_jC_k$. Prove that the four lines of the form $O_{ijk}O_{1-i,1-j,1-k}$ are concurrent or parallel.

    Solution. Let O be the radical center of the given circles. If O lies outside these circles then there exists a circle centered at O and perpendicular to three given circles. The inversion with respect to this circle saves all given circles. Therefore this inversion transposes A_0 and A_1, B_0 and B_1, C_0 and C_1, thus it transposes the circles $A_iB_jC_k$ and $A_{1-i}B_{1-j}C_{1-k}$. Hence the lines joining the centers of such pairs of circles pass through O.

If O lies inside the given circles then they are saved by the composition of the inversion and the central symmetry with center O. Therefore in this case four lines also pass through O.

18. (N.Beluhov, A.Zaslavsky, 10–11) A quadrilateral $ABCD$ without parallel and without equal sides is circumscribed around a circle centered at I. Let K, L, M and N be the midpoints of AB, BC, CD and DA respectively. It is known that $AB \cdot CD = 4IK \cdot IM$. Prove that $BC \cdot AD = 4IL \cdot IN$.

    Solution. Construct J such that $\triangle AJB \sim \triangle DIC$. Then $AJBI$ is cyclic. Let k be its circumcircle, and let IK meet k for the second time at J'. Then from $KJ : AB = IM : CD$, $IK \cdot KJ' = KA \cdot KB = AB^2/4$, and $4IK \cdot IM = AB \cdot CD$ it follows that $KJ = KJ'$ (fig. 18).

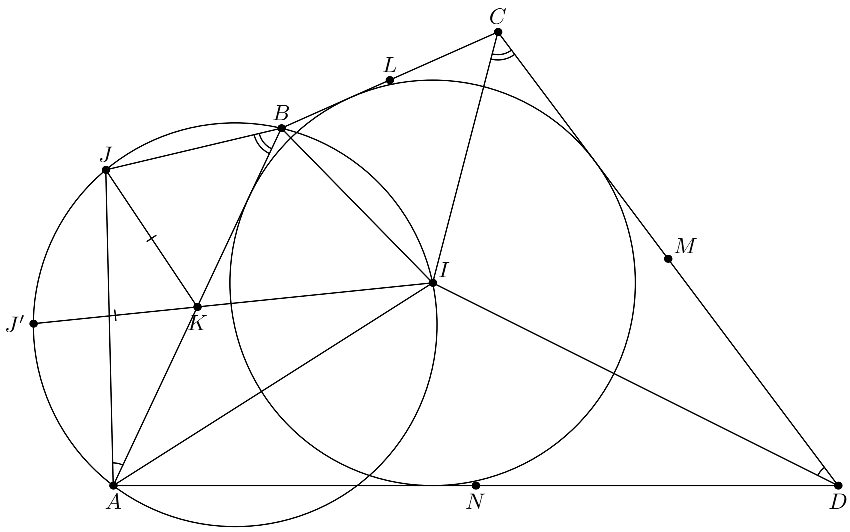


Fig. 18

If AB is a diameter of k, then $\angle AIB = 90^\circ$ and $AD \| BC$ — a contradiction. Therefore, AB is not a diameter of k. If $J = J'$, then $\angle ICB = \angle AIK$, $\angle IDA = \angle BIK$ and $BC = r(\cot \angle IBK + \cot \angle AIK) = r(\cot \angle IAK + \cot \angle BIK) = AD$ — a contradiction. From this and $KJ = KJ'$, it follows that J and J' are symmetric with respect to the perpendicular bisector of AB.

Therefore, $\triangle AIK \sim \triangle J'BK \simeq \triangle JAK \sim \triangle IDM$. From this, together with $\angle IAK = \angle IAD$ and $\angle IDM = \angle IDA$, it follows that $\triangle AIK \sim \triangle ADI \sim \triangle IDM$. Analogously, $\triangle BIK \sim \triangle BCI \sim \triangle ICM$.

Let P and Q be the midpoints of IA and IB. Then $\triangle IND \sim \triangle KPI \simeq \triangle IQK \sim \triangle CLI$. Therefore, $IN : ND = CL : LI$ and $4IL \cdot IN = AD \cdot BC$, as needed.

Note. A circumscribed quadrilateral $ABCD$ that has no parallel or equal sides satisfies the conditions of the problem if and only if its incenter I is the center of gravity of its four vertices A, B, C, and D.

19. (A.Utkin, 10–11) Let AL_a, BL_b, CL_c be the bisectors of triangle ABC. The tangents to the circumcircle of ABC at B and C meet at point K_a, points K_b, K_c are defined similarly. Prove that the lines K_aL_a, K_bL_b and K_cL_c concur.

Solution. Since ABK_c is an isosceles triangle, the sine law applied to triangles AL_cK_c and BL_aK_c implies that $\sin\angle AK_cL_c : \sin\angle BK_cL_c = AL_c : BL_c$. From this and two similar equalities we obtain the required assertion applying the Ceva theorem.

20. (A.Zaslavsky, 10–11) Let O be the circumcenter of triangle ABC, H be its orthocenter, and M be the midpoint of AB. The line MH meets the line passing through O and parallel to AB at point K lying on the circumcircle of ABC. Let P be the projection of K onto AC. Prove that $PH \parallel BC$.

Solution. Let Q be the projection of K to BC. Then PQ is the Simson line of K, therefore PQ bisects segment HK, and the angle between PQ and altitude CH (the Simson line of C) is equal to the half of angle COK. But OK is the perpendicular bisector for segment CL, where L is the second common point of CH with the circumcircle. Hence $\angle HCK = \angle CLK = \angle COK/2$, i.e. $PQ \parallel CK$. Thus PQ bisects segment CH. Also MH meets the circumcircle for the second time at point C' opposite to C, and $C'M = MH$. Therefore $CK \perp KC'$, i.e. the corresponding sides of triangles CPQ and BHC' are perpendicular. Then their medians are perpendicular too, therefore CH bisects segment PQ and $CPHQ$ is a parallelogram (fig.20).

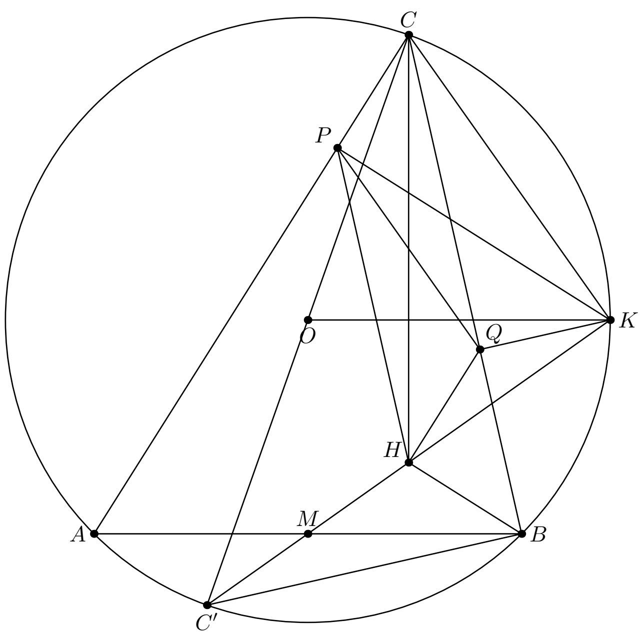


Fig. 20

21. (A.Sgibnev, A.Zaslavsky, 10–11) An ellipse Γ and its chord AB are given. Find the locus of orthocenters of triangles ABC inscribed into Γ.

Solution. Choose a coordinate system such that the line AB is the X-axis. Then the equation for Γ will be $(x-x_a)(x-x_B)+y(ax+by+c) = 0$, where $b > 0$. The coordinates of orthocenter H are (x_C, h), where h satisfies the condition of perpendicularity AH and BC:

$(x_C - x_A)(x_C - x_B) + hy_C = 0$, i.e. $h = -(x_C - x_A)(x_C - x_B)/y_C$. But by the Vieta theorem XH meets Γ for the second time at the point with ordinate $(x_C - x_A)(x_C - x_B)/by_C$. Thus the locus of orthocenters is an ellipse obtained by the contraction of Γ to AB with coefficient $-b$. Since this coefficient is equal to the ratio of squares of two diameters, perpendicular and parallel to AB, we obtain that this ellipse is similar to Γ and their major axes are perpendicular.

22. (P.Kozhevnikov, 10–11) Let AA_0 be the altitude of the isosceles triangle ABC ($AB = AC$). A circle γ centered at the midpoint of AA_0 touches AB and AC. Let X be an arbitrary point of line BC. Prove that the tangents from X to γ cut congruent segments on lines AB and AC.

First solution. For simplicity, we consider only the case when X lies inside segment BA_0. All other cases are similar.

Let B_0 and C_0 be the midpoints of segments AC and AB, respectively. Let one tangent meet segment AC_0 at P and let the other tangent meet segment CB_0 at Q.

By Newton's theorem for circumscribed quadrilateral $APXQ$, the midpoints of segments AA_0, AX, and PQ are collinear. Therefore, the midpoint R of segment PQ lies on the midline of triangle ABC opposite to vertex A.

Let S be the reflection of point A about point R. Then S lies on line BC, and quadrilateral $APSQ$ is a parallelogram (fig. 22.1). Therefore, $C_0P : A_0S = B_0Q : A_0S$ and $C_0P = B_0Q$.

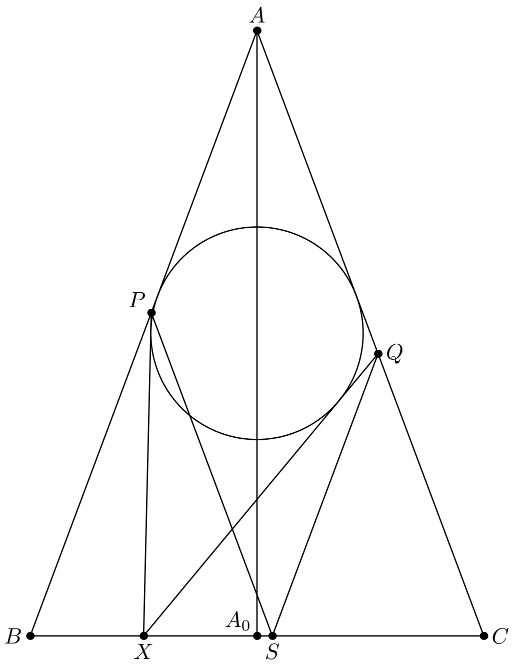


Fig. 22.1

Let one tangent meet ray C_0B at P', and let the other tangent meet ray B_0A at Q'. Similarly, $C_0P' = B_0Q'$. Therefore, $PP' = QQ'$, as needed.

Second solution. Let one of two tangents meet AB and AC at points Y_1 and Y_2, and the second one meet them at Z_1 and Z_2 respectively. Since the relation between these points is projective, it is sufficient to prove that $Y_1Z_1 = Y_2Z_2$ for three positions of X, i.e. by symmetry for some point distinct from the midpoint of BC. When X tends to B then one of points Z_1, Y_1 also tends to B, and the second one tends to the touching point P of γ with AB. Let Q be distinct from A point of AC such that BQ touches γ, and let B_0, C_0 be the midpoints of AC, AB respectively. Then we have $(B, C_0, P, \infty) = (Q, \infty, A, B_0)$, i.e. $AQ = BP$ (fig.22.2).

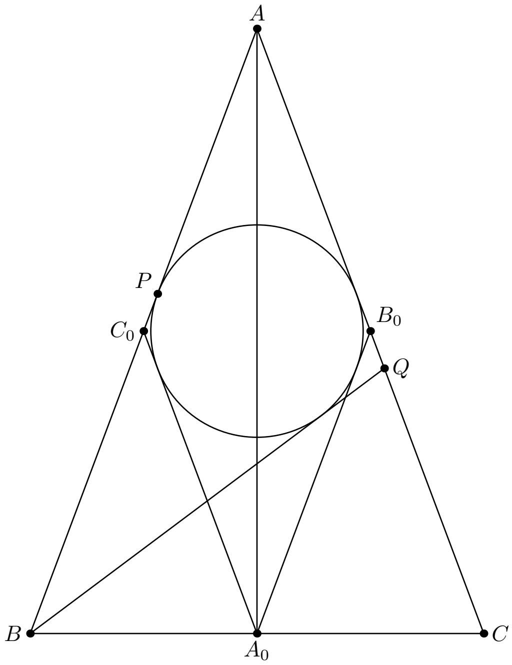


Fig. 22.2

23. (A.Skopenkov, (10–11) In the plane, let a, b be two closed broken lines (possibly self-intersecting), and K, L, M, N be four points. The vertices of a, b and the points K L, M, N are in general position (i.e. no three of these points are collinear, and no three segments between them concur at an interior point). Each of segments KL and MN meets a at an even number of points, and each of segments LM and NK meets a at an odd number of points. Conversely, each of segments KL and MN meets b at an odd number of points, and each of segments LM and NK meets b at an even number of points. Prove that a and b intersect.

First solution. Since the vertices of a are in general position, this broken line divides the plane into several parts which can be colored black and white regularly (i.e. in such a way that the colors of adjacent parts are different). See the proof, for example, in [Sk, §1.3, §2.2], [Sk18, §1.3, §2.2]. Let the "external" part be white. Consider an arbitrary point O of self-intersection of a and take segments $OA = OB = OC = OD = \varepsilon$ on its links passing through, such that $ABCD$ is a rectangle. If ε is sufficiently small then all common points of a with b and segments KL, LM, MN, NK lie outside this rectangle. If we construct such rectangles for all points of self-intersection and color them white, then the black part of the plane will be the union of several not intersecting polygons. Now recolor several rectangles to obtain a black polygon restricted by not self-intersecting broken line a'. Construct not self-intersecting broken line b' in the similar way. By the construction the broken lines a', b' intersect one another and meet segments KL, LM, MN, NK at the same points as a, b. Suppose that a' and b' do not intersect. Then they divide the plane into three parts, therefore two of points K, L, M, N lie in the same part. But this is impossible because a' separates K and L from M and N, and b' separates K and N from M and L. Thus a' and b' intersect, and hence the given broken lines intersect too.

Second solution. Let point C be in general position related to the vertices of a, b and points K, L, M, N. Denote the union of segments $CK \cup CL \cup CM \cup CN$ by γ.

As in the first solution color regularly black and white the parts into which a divides the plane. Denote the union of the black parts by α. Construct similarly the set β corresponding to b.

If a and b do not intersect then $a \cap \beta$ is a or $\emptyset$, and $\alpha \cap b$ is b or $\emptyset$. Then the following chain of comparisons modulo 2 yields a contradiction:

$$0 \underset{(1)}{=} |\partial(\gamma\cap\alpha\cap\beta)| \underset{(2)}{=} |\underbrace{\partial\gamma}_{=\{K,L,M,N\}}\cap\alpha\cap\beta| + |\gamma\cap\underbrace{\partial\alpha}_{=a}\cap\beta| + |\gamma\cap\alpha\cap\underbrace{\partial\beta}_{=b}| \underset{(3)}{=} 1+0+0 = 1.$$

Here (1) is true because $\gamma \cap \alpha \cap \beta$ is the union of a finite number of unclosed broken lines with even number of endpoints. The proof of (2) is not difficult (this is the «Leibnitz formula»).

Let us prove (3). We have

$$\{K, L, M, N\}\cap\alpha\cap\beta = (\{K, L, M, N\}\cap\alpha)\cap(\{K, L, M, N\}\cap\beta) = \{K, L\}\cap\{K, N\} = \{K\}.$$

If $a \cap \beta = \emptyset$ then $\gamma \cap a \cap \beta = \emptyset$. And if $a \cap \beta = a$ then

$$|\gamma \cap a \cap \beta| = |\gamma \cap a| = |KN \cap a| + |LM \cap a| = 1 + 1 = 0.$$

Thus in both cases $|\gamma \cap a \cap \beta| = 0$. Similarly $|\gamma \cap \alpha \cap b| = 0$.

Remarks. Similar reasoning about triple intersections demonstrates that the Borromeo rings cannot be uncoupled. See [Sk, §4].

The multidimensional version of the problem, the Borromeo rings lemma, can be proved similarly, see [AMS+]. This lemma is significant for the study of complexity of realizability of hypergraphs in multidimensional spaces, see [MTW11, ST17].

References

[AMS+] *S. Avvakumov, I. Mabillard, A. Skopenkov and U. Wagner.* Eliminating Higher-Multiplicity Intersections, III. Codimension 2, Israel J. Math., submitted, arxiv:1511.03501.

[MTW11] *J. Matoušek, M. Tancer, U. Wagner.* Hardness of embedding simplicial complexes in R^d, J. Eur. Math. Soc. 13:2 (2011), 259–295. arXiv:0807.0336.

[Sk] *A. Skopenkov.* Algebraic topology from the algorythmic point of view (Rus.), http://www.mccme.ru/circles/oim/algor.pdf.

[Sk18] *A. Skopenkov.* Invariants of graph drawings in the plane, Arnold J. Math., submitted, arXiv:1805.10237.

[ST17] *A. Skopenkov and M. Tancer,* Hardness of almost embedding simplicial complexes in R^d, Discr. Comp. Geom., to appear, arXiv:1703.06305.

24. (N.Beluhov, 11) Two unit cubes have a common center. Is it always possible to number the vertices of each cube from 1 to 8 so that the distance between each pair of identically numbered vertices would be at most 4/5? What about at most 13/16?

Solution. Let $\kappa = A_1A_2\dots A_8$ be one of the two cubes (with $A_1A_2A_3A_4$ a unit square and A_i adjacent to A_{i+4} for all i), d_1, d_2, d_3, and d_4 be the space diagonals of κ, λ be the second cube, and e_1, e_2, e_3, and e_4 be the space diagonals of λ. Let O be the common center of the two cubes, s be their common circumscribed sphere, μ be a positive real which does not exceed the diameter of s, and α be the central angle of a chord of length μ in s.

Let S_i be the set of all e_j such that the angle between d_i and e_j does not exceed α. Suppose that, for all $1 \le k \le 4$, the union of any k of the sets S_i contains at least k elements. Then, by Hall's representatives theorem, we can select a single representative e'_i from each S_i in such a way that all four representatives are distinct, and pair up the endpoints of each d_i with the endpoints of its corresponding e'_i in such a way that the distance between the two vertices in each pair is at most μ.

Let us then look at the possible values of k and the bounds on μ that they impose.

$k = 4$: Let P be the center of the spherical cap cut off from s by the plane $A_1A_2A_3A_4$. (That is, P lies on s, $PA_1 = PA_2 = PA_3 = PA_4$, and $A_1A_2A_3A_4$ separates O and P.)

We need to ensure that the union of the eight spherical caps with centers A_i and radii α contains all vertices of λ, i.e., that it covers s. This is true just if $\mu \ge PA_1$; denote the length of PA_1 by μ_4.

$k = 3$: Let Q be a point on the shorter great-circle arc $\smile A_1A_3$ of s such that $A_2Q = A_4Q = \mu$ and $A_1Q \le QA_3$. Choose R and S similarly on the shorter great-circle arcs $\smile A_1A_6$ and $\smile A_1A_8$.

Without loss of generality, we need to ensure that the union of the six spherical caps with centers A_2, A_4, A_5, A_3, A_6, and A_8 and radii α contains at least six vertices of λ. Equivalently, we need to ensure that the complement of this union to s contains at most two vertices of λ. This complement consists of two connected components which are symmetric with respect to O; therefore, it is necessary and sufficient to ensure that each component contains at most one vertex of λ. Since each component is contained within the equilateral spherical triangle QRS but contains points arbitrarily close to Q, R, and S, it is necessary and sufficient to have $QR \le 1$ – or, equivalently, $\mu \ge \mu_3$, where μ_3 is the value of μ for which equality is attained. It is easy to see that $\mu_3 > \mu_4$.

For $k = 2$ and $k = 1$, let T_i be the set of all d_j such that the angle between e_i and d_j does not exceed α.

$k = 2$: Without loss of generality, suppose that $S_1 \cup S_2$ does not contain e_1, e_2, and e_3. Then $T_1 \cup T_2 \cup T_3$ does not contain d_1 and d_2. From the case $k = 3$ we know that this is avoided just if $\mu \ge \mu_3$.

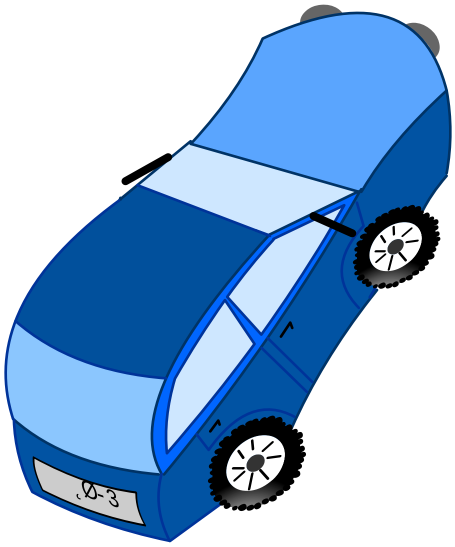
$k = 1$: Suppose that, without loss of generality, S_1 does not contain any e_i. Then the union of all T_i does not contain d_1. From the case $k = 4$ we know that this is avoided just if $\mu \geq \mu_4$.

Thus $\mu \geq \mu_3$ always works. In order to see that no $\mu < \mu_3$ works, let λ be the cube with center O and edge QR as in the case $k = 3$. At most one of the vertices Q and R of λ is paired with A_1; whatever vertex of κ we pair with the other one, the distance between them will be at least μ_3. Therefore, the shortest distance μ that satisfies the conditions of the problem is $\mu_3 = \sqrt{\frac{9-2\sqrt{2}-\sqrt{5}}{6}}$. Since $4/5 < \mu_3 < 13/16$, the answer to the first part of the problem is negative and the answer to the second part of the problem is positive.

III I.F.Sharygin olympiad in geometry

Final round. 8 form

1. Determine on which side is the steering wheel disposed in the car depicted in the figure.



2. By straightedge and compass, reconstruct a right triangle ABC ($\angle C = 90°$), given the vertices A, C and a point on the bisector of angle B.

3. The diagonals of a convex quadrilateral dissect it into four similar triangles. Prove that this quadrilateral can also be dissected into two congruent triangles.

4. Determine the locus of orthocenters of triangles, given the midpoint of a side and the bases of the altitudes drawn to two other sides.

5. Medians AA' and BB' of triangle ABC meet at point M, and $\angle AMB = 120°$. Prove that angles $AB'M$ and $BA'M$ are neither both acute nor both obtuse.

6. Two non-congruent triangles are called *analogous* if they can be denoted as ABC and $A'B'C'$ such that $AB = A'B'$, $AC = A'C'$ and $\angle B = \angle B'$. Do there exist three mutually analogous triangles?

III I.F.Sharygin olympiad in geometry

Final round. 9 form

1. Given a circumscribed quadrilateral $ABCD$. Prove that its inradius is smaller than the sum of the inradii of triangles ABC and ACD.

2. Points E and F are chosen on the base side AD and the lateral side AB of an isosceles trapezoid $ABCD$, respectively. Quadrilateral $CDEF$ is an isosceles trapezoid as well. Prove that $AE \cdot ED = AF \cdot FB$.

3. Given a hexagon $ABCDEF$ such that $AB = BC$, $CD = DE$, $EF = FA$, and $\angle A = \angle C = \angle E$. Prove that lines AD, BE, and CF are concurrent.

4. Given a triangle ABC. An arbitrary point P is chosen on the circumcircle of triangle ABH (H is the orthocenter of triangle ABC). Lines AP and BP meet the opposite sidelines of the triangle at points A' and B', respectively. Determine the locus of midpoints of segments $A'B'$.

5. Reconstruct a triangle, given the incenter, the midpoint of some side and the base of the altitude drawn to this side.

6. A cube with edge length $2n + 1$ is dissected into small cubes of size $1 \times 1 \times 1$ and bars of size $2 \times 2 \times 1$. Find the least possible number of cubes in such a dissection.

III I.F.Sharygin olympiad in geometry

Final round. 10 form

1. In an acute triangle ABC, altitudes at vertices A and B and bisector line at angle C intersect the circumcircle again at points A_1, B_1 and C_0. Using the straightedge and compass, reconstruct the triangle by points A_1, B_1 and C_0.

2. Points A', B', C' are the bases of the altitudes AA', BB' and CC' of an acute triangle ABC. A circle with center B and radius BB' meets line $A'C'$ at points K and L (points K and A are on the same side of line BB'). Prove that the intersection point of lines AK and CL belongs to line BO (O is the circumcenter of triangle ABC).

3. Given two circles intersecting at points P and Q. Let C be an arbitrary point distinct from P and Q on the former circle. Let lines CP and CQ intersect again the latter circle at points A and B, respectively. Determine the locus of the circumcenters of triangles ABC.

4. A quadrilateral $ABCD$ is inscribed into a circle with center O. Points C', D' are the reflections of the orthocenters of triangles ABD and ABC at point O. Lines BD and BD' are symmetric with respect to the bisector of angle ABC. Prove that lines AC and AC' are symmetric with respect to the bisector of angle DAB.

5. Each edge of a convex polyhedron is shifted such that the obtained edges form the frame of another convex polyhedron. Are these two polyhedra necessarily congruent?

6. Given are two concentric circles Ω and ω. Each of the circles b_1 and b_2 is externally tangent to ω and internally tangent to Ω, and each of the circles c_1 and c_2 is internally tangent to both Ω and ω. Mark each point where one of the circles b_1, b_2 intersects one of the circles c_1 and c_2. Prove that there exist two circles distinct from b_1, b_2, c_1, c_2 which contain all 8 marked points. (Some of these new circles may appear to be lines.))

IV RUSSIAN GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
The final round. Solutions. 8 form. First day

1. (B.Frenkin) Does a convex quadrilateral without parallel sidelines exist such that it can be divided into four equal triangles?

   Answer. Yes. See for example fig.8.1.

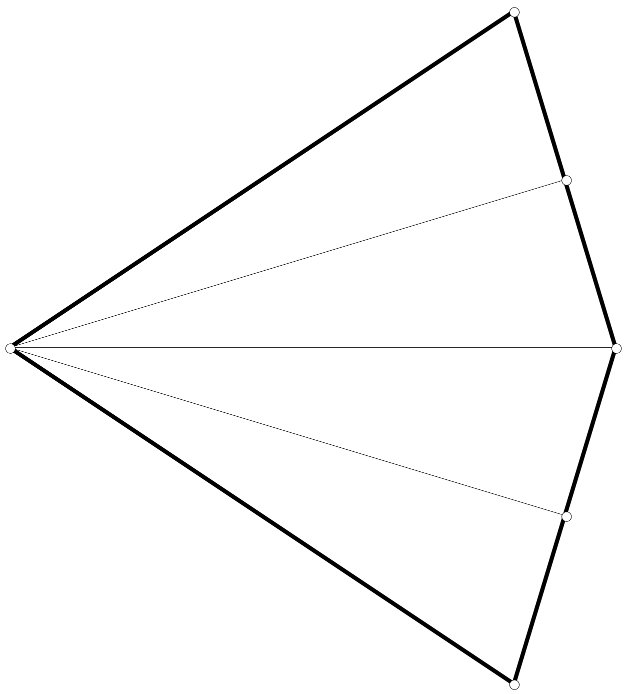


Fig.8.1.

2. (F.Nilov) Given right triangle ABC with hypothenuse AC and $\angle A = 50^\circ$. Points K and L on the cathetus BC are such that $\angle KAC = \angle LAB = 10^\circ$. Determine the ratio CK/LB.

   Answer. 2.

   Solution. Let L' is the reflection of L in AB (fig.8.2). As $\angle L'KA = 50^\circ = \angle KAL'$, we have $L'K = L'A = LA$. On the other hand, $\angle CAL = 40^\circ = \angle ACL$, i.e. $AL = CL$. So $CK = LL' = 2LB$.

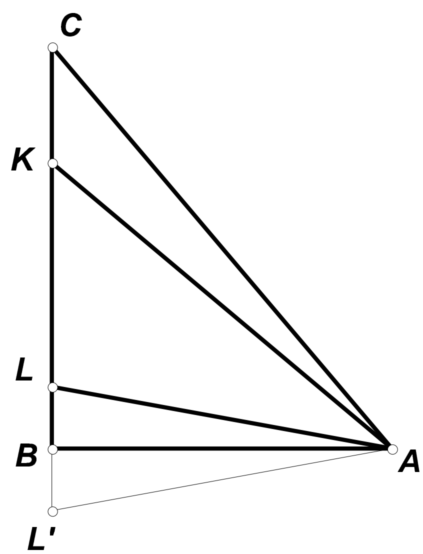


Fig.8.2.

3. (D.Shnol) Two opposite angles of a convex quadrilateral with perpendicular diagonals are equal. Prove that a circle can be inscribed in this quadrilateral.

   Solution. Let O be the common point of the diagonals in quadrilateral $ABCD$ with $\angle B = \angle D$. Suppose that $OB > OD$. Then point D' which is the reflection of D in AC lies on segment OB (fig.8.3). Thus by the property of external angle $\angle AD'O > \angle ABO$, $\angle CD'O > \angle CBO$. But then $\angle D = \angle AD'C > \angle B$ — a contradiction. So $OB = OD$ and AC is the symmetry axis of $ABCD$. Thus the bisectors of angles B, D and AC concur and their common point is the incenter of $ABCD$.

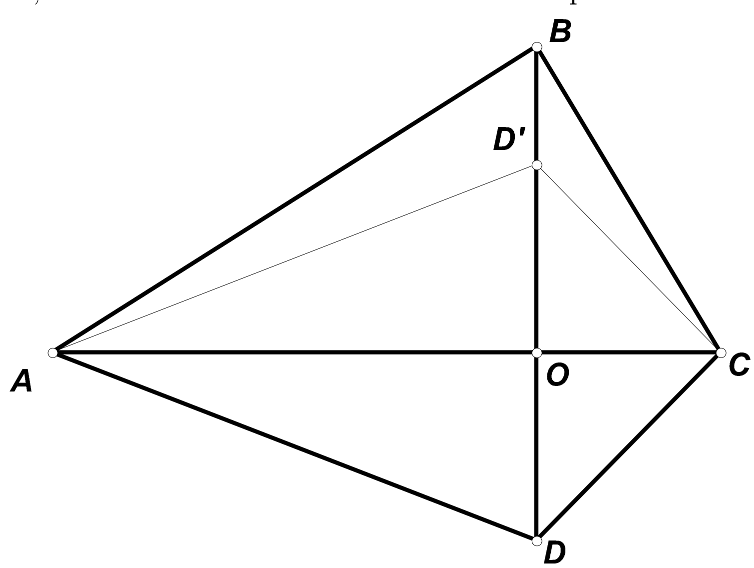


Fig.8.3.

4. (F.Nilov, A.Zaslavsky) Let CC_0 be a median of triangle ABC; the medial perpendiculars to AC and BC intersect CC_0 in points A', B'; C_1 is the meet of lines AA' and BB'. Prove that $\angle C_1CA = \angle C_0CB$.

Solution. Since triangles CAA', CBB' are isosceles, we have $\angle CAA' = \angle C_0CA$, $\angle CBB' = \angle C_0CB$. So the distances from C to the lines AC_1 and BC_1 are equal respectively to the distances from A and B to the line CC_0. But these distances are equal because CC_0 is a median. Thus C is equidistant from C_1A and C_1B. So $\angle CC_1A = \angle CC_1B$, and $\angle C_1CA - \angle C_1CB = \angle C_1BC - \angle C_1AC = \angle C_0CB - \angle C_0CA$ (fig.8.4). This is equivalent to the required assertion.

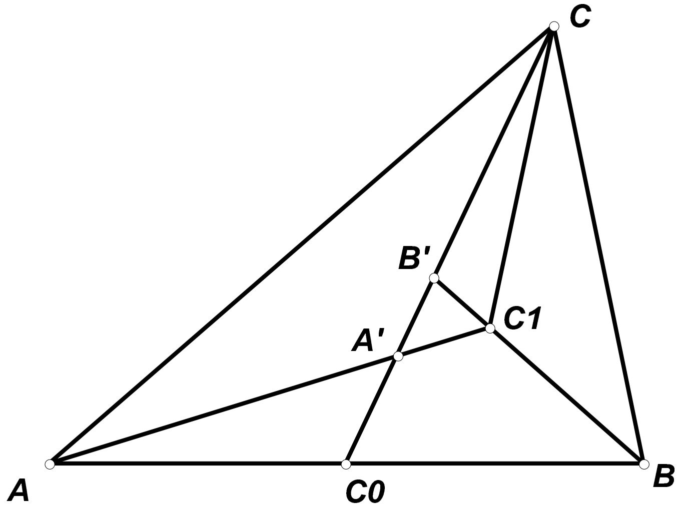


Fig.8.4.

5. (A.Zaslavsky) Given two triangles ABC, $A'B'C'$. Denote by α the angle between the altitude and the median from vertex A of triangle ABC. Angles β, γ, α', β', γ' are defined similarly. It is known that $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$. Can we conclude that the triangles are similar?

   Answer. No.

   Solution. Let the sidelines of $A'B'C'$ be parallel to the medians of ABC. Then the sidelines of ABC are parallel to the medians of $A'B'C'$ and the angles between the medians and the respective altitudes are the same for both triangles. But in general case these triangles aren't similar.

IV RUSSIAN GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

The final round. Solutions. 8 form. Second day

6. (B.Frenkin) Consider the triangles such that all their vertices are vertices of a given regular 2008-gon. What triangles are more numerous among them: acute-angled or obtuse-angled?

   Answer. Obtuse-angled.

   Solution. Fix two vertices A and B of one of given triangles. If they are the opposite vertices of the 2008-gon then for any third vertex C triangle ABC is right-angled. Otherwise denote by A', B' the vertices of the 2008-gon opposite to A, B respectively. Triangle ABC is acute-angled iff C lies on the smallest of two arcs of circumcircle, bounded by A', B'. So for any fixed A, B the number of acute-angled triangles having these two vertices is less than that of obtuse-angled. Thus the total number of obtuse-angled triangles is greater as well.

7. (F.Nilov) Given isosceles triangle ABC with base AC and $\angle B = \alpha$. The arc AC constructed outside the triangle has angular measure equal to β. Two lines passing through B divide the segment and the arc AC into three equal parts. Find the relation α/β.

   Answer. 1/3.

   Solution. Let points X, Y divide the segment AC into three equal parts ($AX = XY = YC$); U, V be the common points of rays BX, BY with arc AC; Z be the common point of BC and UV (fig.8.7). Since $UV \parallel AC$, we have $VZ = UV = VC$. So $\angle UCZ = 90°$. On the other hand, $\angle ACU = \angle UCV = \beta/6$, and $\angle BCA = 90° - \alpha/2$. Consequently $\beta = 3\alpha$.

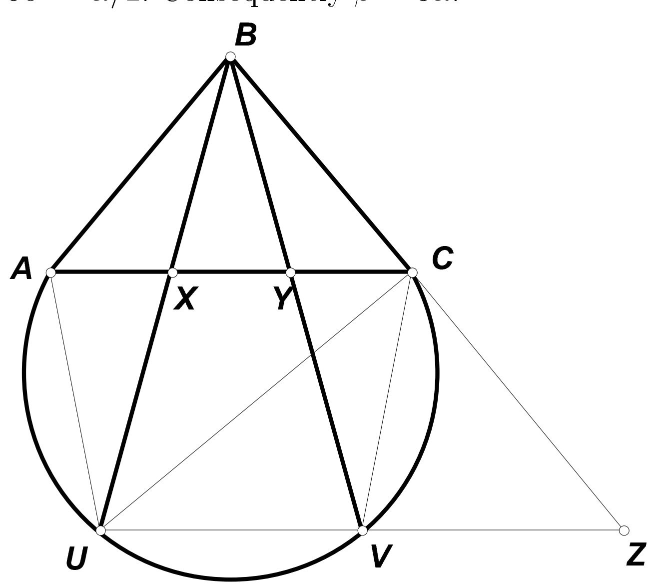


Fig.8.7.

8. (B.Frenkin, A.Zaslavsky) A convex quadrilateral was drawn on the blackboard. Boris marked the centers of four excircles each touching one side of the quadrilateral

and the extensions of two adjacent sides. After this, Alexey erased the quadrilateral. Can Boris define its perimeter?

Answer. Yes.

Solution. Let $ABCD$ be the quadrilateral formed by the excenters, and the vertex X of the original quadrilateral lies on AB. The sidelines of $ABCD$ are the external bisectors of the angles of original quadrilateral. So a billiard ball moving from X along a side of the original quadrilateral, will after the reflections in the sides of $ABCD$ continue to move along the sides. "Straighten"the trajectory of ball constructing quadrilaterals: A_1BCD_1 — the reflection of $ABCD$ in BC, $A_2B_1CD_1$ — the reflection of A_1BCD_1 in CD_1, and $A_2B_2C_1D_1$ — the reflection of $A_2B_1CD_1$ in D_1A_2. Then the trajectory of the ball transforms to segment XX', where X' lies on A_2B_2 and $A_2X' = AX$ (fig.8.8). Since $\angle X'XB = \angle XX'A_2$, we have $A_2B_2 \parallel AB$. Thus joining any other point of segment AB with the respective point of segment A_2B_2, after inverse reflections we obtain another quadrilateral satisfying the conditions of the problem. So there exists an infinite set of quadrilaterals having points A, B, C, D as excenters. But the perimeters of all these quadrilaterals are equal to $XX' = AA_2$ and so don't depend on X.

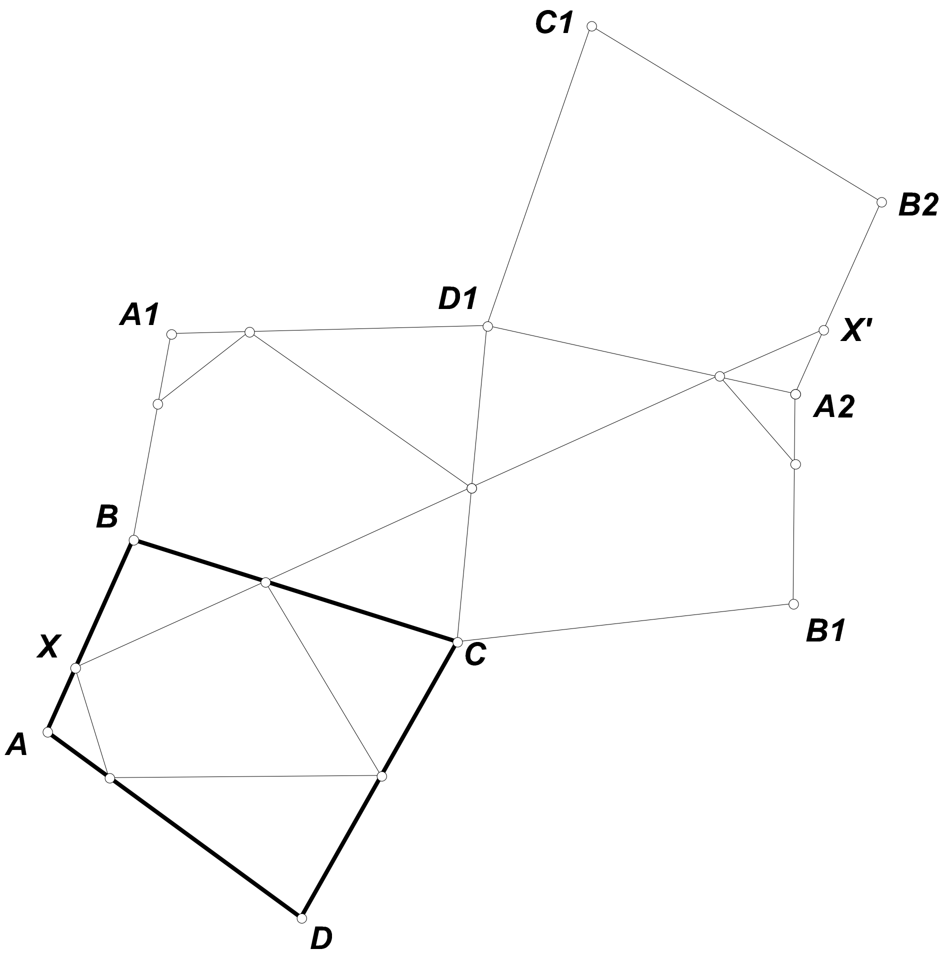


Fig.8.8.

IV RUSSIAN GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
The final round. Solutions. 9 form. First day

1. (A.Zaslavsky) A convex polygon can be divided into 2008 equal quadrilaterals. Is it true that this polygon has a center or an axis of symmetry?

   Answer. No. Take for example the trapezoid with bases equal to 1 and 2, and lateral sides equal to 1 and $\sqrt{2}$. Using the construction of fig.9.1, we can compose from such trapezoids a hexagon which isn't symmetric.

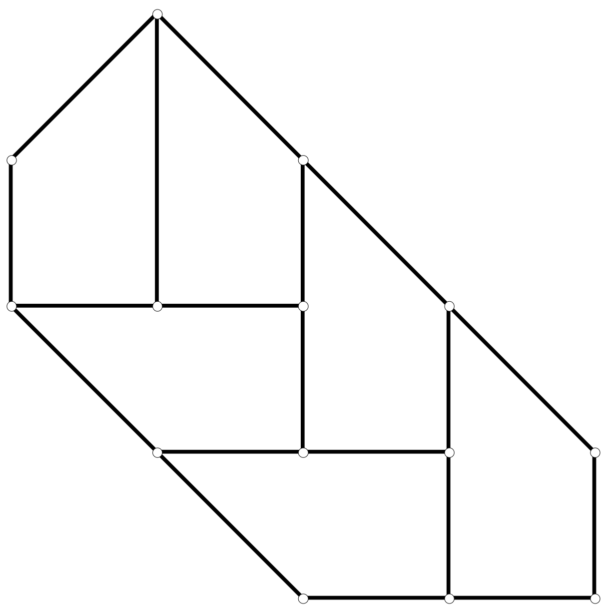


Fig.9.1.

2. (F.Nilov) Given quadrilateral $ABCD$. Find the locus of points such that their projections to the lines AB, BC, CD, DA form a quadrilateral with perpendicular diagonals.

   Solution. If the original quadrilateral is a trapezoid then the projections of a point in the locus to the lateral sidelines lie on a line parallel to the bases. It is evident that the set of such points is the line passing through the common point of lateral sidelines. Also it is clear that the required locus for a rectangle is the whole plane, and that for a parallelogram distinct from a rectangle such points don't exist.

   Let X be the common point of lines AB and CD, Y be a common point of lines BC и DA. Denote the projections of point P to AB, BC, CD, DA by K, L, M, N, and let O be the common point of KM and LN (fig.9.2). Since quadrilaterals $YLPN$ and $XKPM$ are cyclic, we have $\angle PLN = \angle PYA$ and $\angle PMK = \angle PXA$. So $\angle MOL = \pi - \angle C - \angle PLN - \angle PMK = \pi - \angle C - (\angle A - \angle XPY)$. Thus an equality $\angle MOL = \pi/2$ is equivalent to $\angle XPY = \angle A + \angle C - \pi/2$. So the required locus is a circle passing through X and Y.

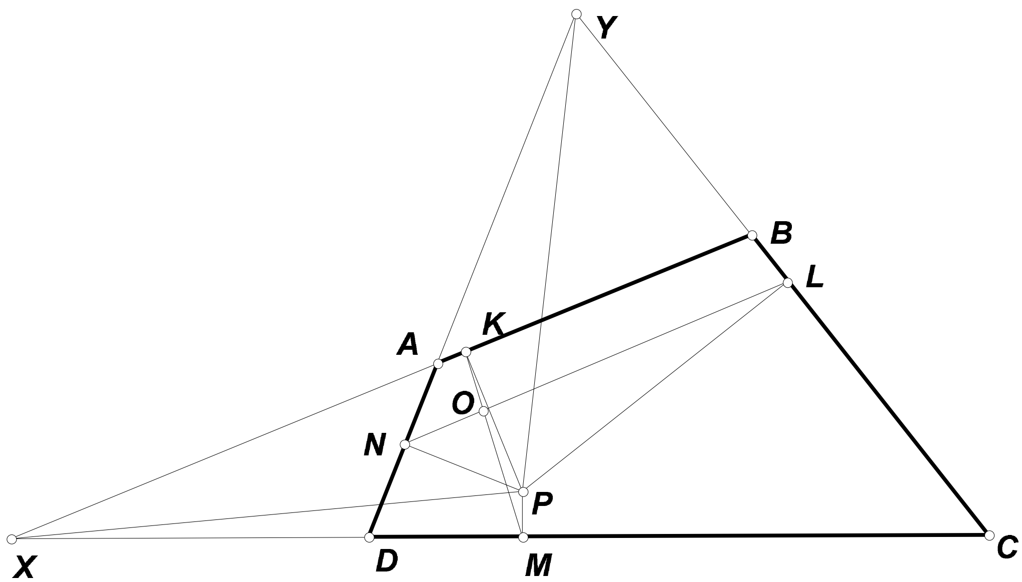


Fig.9.2.

3. (R.Pirkuliev) Prove the inequality

$$\frac{1}{\sqrt{2\sin A}}+\frac{1}{\sqrt{2\sin B}}+\frac{1}{\sqrt{2\sin C}}\le\sqrt{\frac{p}{r}},$$

where p and r are the semiperimeter and the inradius of triangle ABC.

Solution. Let R and S be the circumradius and the area of triangle ABC. Using the sinuses theorem and the formulaes $S=pr=abc/4R$, transform the right part of inequality:

$$\begin{aligned}\sqrt{\tfrac{p}{r}} &= \tfrac{p}{\sqrt{S}}=\tfrac{R(\sin A+\sin B+\sin C)}{\sqrt{2R^2\sin A\sin B\sin C}}\\ &= \sqrt{\tfrac{\sin A}{2\sin B\sin C}}+\sqrt{\tfrac{\sin B}{2\sin C\sin A}}+\sqrt{\tfrac{\sin C}{2\sin A\sin B}}.\end{aligned}$$

By Cauchi inequality:

$$\frac{2}{\sqrt{\sin A}}\le\sqrt{\frac{\sin B}{\sin C\sin A}}+\sqrt{\frac{\sin C}{\sin A\sin B}}.$$

Summing this inequality with two similar ones, we obtain the required assertion.

4. (F.Nilov, A.Zaslavsky) Let CC_0 be a median of triangle ABC; the medial perpendiculars to AC and BC intersect CC_0 in points A', B'; C_1 is the common point of AA' and BB'. Points A_1, B_1 are defined similarly. Prove that circle $A_1B_1C_1$ passes through the circumcenter of triangle ABC.

Solution. By the solution of problem 8.4, lines AA_1, BB_1, CC_1 concur in point L, and point C_1 lies on the circle passing through A, B and the circumcenter O of ABC. Hence $\angle OC_1L=\angle AC_1C-\angle AC_1O=\angle AC_1C-\angle ABO=(\pi-\angle C)-(\pi/2-\angle C)=\pi/2$, and C_1 lies on the circle with diameter OL. Similarly A_1 and B_1 lie on the same circle.

5. (N.Avilov) Can the surface of a regular tetrahedron be glued over with equal regular hexagons?

   Solution. Yes. For example, sticking together a tetrahedron from the development in fig.9.5 and cutting its surface by bold lines, we obtain two equal regular hexagons (obscure and light).

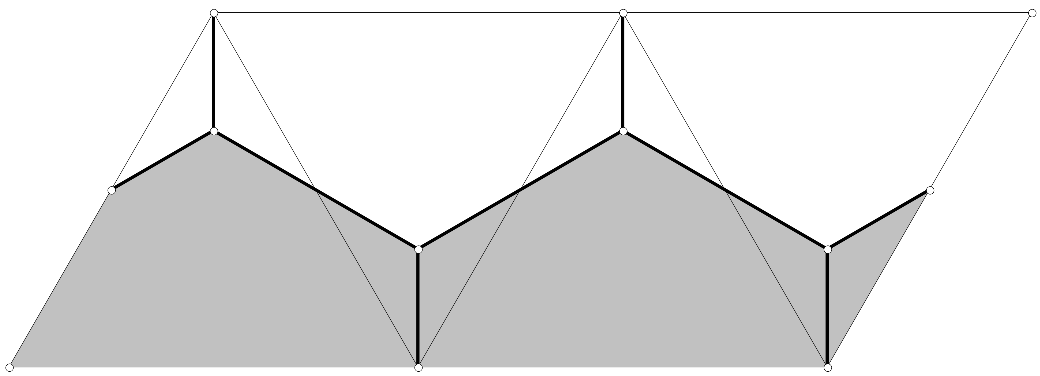


Fig.9.5.

IV RUSSIAN GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

The final round. Solutions. 9 form. Second day

6. (B.Frenkin) Construct the triangle, given its centroid and the feet of an altitude and a bisector from the same vertex.

   Solution. Let C_1, C_2 be the feet of the bisector and of the altitude from vertex C of triangle ABC, and M be its centroid. It is evident that C lies on the perpendicular from C_2 to line C_1C_2. The projection of M to this perpendicular divides the altitude in relation $2:1$. This enables to construct point C and the midpoint C_0 of side AB.

   Let C' be the common point of CC_1 and perpendicular l to C_1C_2 from C_0. Then C' lies on the circumcircle of ABC (fig.9.6). So the medial perpendicular to CC' intersect l in the circumcenter O. Constructing the circumcircle, we find A, B as its common points with C_1C_2.

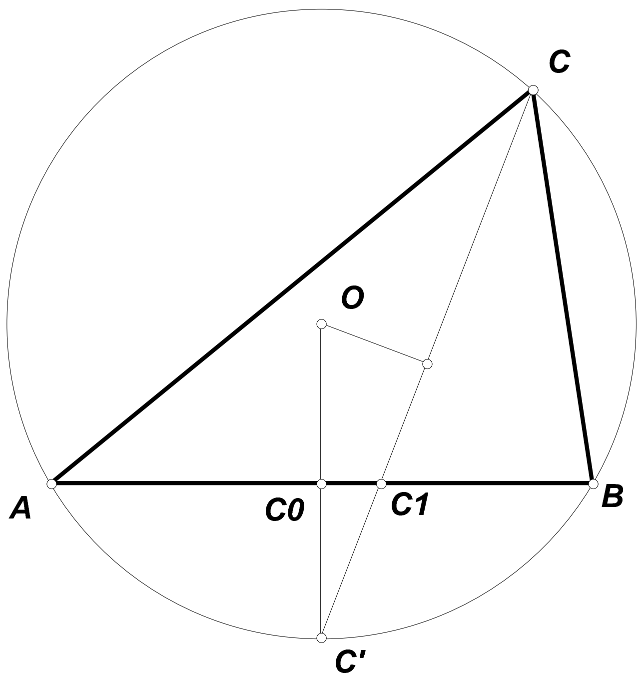


Fig.9.6.

7. (A.Zaslavsky) The circumradius of triangle ABC is equal to R. Another circle with the same radius passes through the orthocenter H of this triangle and intersect its circumcirle in points X, Y. Point Z is the fourth vertex of parallelogram $CXZY$. Find the circumradius of triangle ABZ.

   Answer. R.

   Solution. We will prove that Z lies on circle ABH with radius equal to R. Let H' be the second common point of circles XYH and ABH, C' be the orthocenter of triangle ABH' (fig.9.7). Then C' lies on the circle which is the reflection of ABH in AB, i.e. on the circumcircle of ABC. So $CH = C'H' = 2R|\cos C|$ and $CHH'C'$ is a parallelogram. Since CC' and HH' are the chords of equal circles ABC and

XHY, they are symmetric wrt the midpont of segment XY. Thus, $XCYH'$ is the parallelogram and H' coincides with Z.

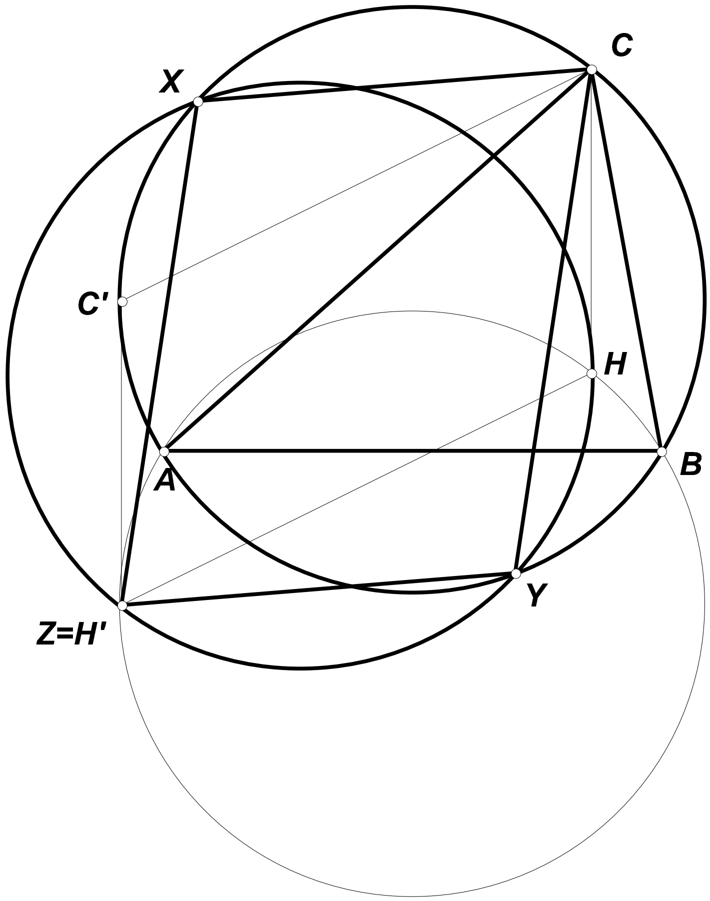


Fig.9.7.

8. (J.-L.Aime, France) Points P, Q lie on the circumcircle ω of triangle ABC. The medial perpendicular l to PQ intersects BC, CA, AB in points A', B', C'. Let A", B", C" be the second common points of l with the circles $A'PQ$, $B'PQ$, $C'PQ$. Prove that AA", BB", CC" concur.

   Solution. Let X, Y be the common points of ω and l. Consider the central projection from ω to l with center C. We obtain that $(AB; XY) = (B'A'; XY)$. Furthermore since $\angle A'PA - \angle B'PB - \angle XPY = \pi/2$, we have $(A'B'; XY) = (A"B"; YX)$. So $(AB; XY) = (A"B"; YX)$, and the common point of AA" and BB" lies on ω. Line CC" also passes through this point.

IV RUSSIAN GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
The final round. Solutions. 10 form. First day

1. (B.Frenkin) An inscribed and circumscribed n-gon is divided by some line into two inscribed and circumscribed polygons with different numbers of sides. Find n.

   Answer. 3.

   Solution. Suppose $n \neq 3$. If $n > 4$ then the boundary of at least one of obtained polygons contains the segments of three sides of the original polygon. So the incircle of this polygon coincides with the incircle of the original polygon. If $n = 4$ this also is correct because two obtained polygons have different number of sides and so the dividing line isn't a diagonal of original quadrilateral.

   Thus the dividing line is tangent to the incircle of the original n-gon, hence one of two parts obtained is a triangle. The vertices of the second part are $n-1$ vertices of the original polygon and two points lying on its sides. Since $n-1 \geq 3$, these vertices determine a unique circle which passes through the remain vertex of the n-gon and hence does not pass through the two points on the sides. Thus the cut polygon is not inscribed.

   Remark. It is possible to divide an arbitrary triangle by a line tangent to its incircle into a triangle and an inscribed-circumscribed quadrilateral.

2. (A.Myakishev) Let triangle $A_1B_1C_1$ be symmetric to ABC wrt the incenter of its medial triangle. Prove that the orthocenter of $A_1B_1C_1$ coincides with the circumcenter of the triangle formed by the excenters of ABC.

   Solution. Let H, I, O, M be the orthocenter, the incenter, the circumcenter, and the centroid of triangle ABC, I_0 be the incenter of its medial triangle. Obviously, the orthocenter H_1 of triangle $A_1B_1C_1$ is symmetric to H wrt I_0. On the other hand, in the triangle formed by the excenters of ABC, I is the orthocenter, ABC is the orthotriangle, and the circumcircle of ABC is the nine-points circle. So the circumcenter of excenter triangle is symmetric to I wrt O. Consider triangle IHH_1. Its median II_0 passes through M and is divided by this point in relation $2:1$. So M is the centroid of this triangle. But M also divides segment $HO0$ in relation $2:1$. Thus O is the midpoint of IH_1 (fig.10.2)

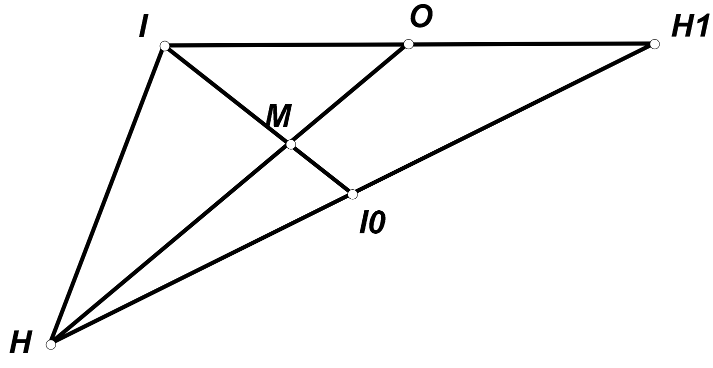


Fig.10.2.

3. (V.Yasinsky, Ukraine) Suppose X and Y are the common points of two circles ω_1 and ω_2. The third circle ω is internally tangent to ω_1 and ω_2 in P and Q respectively. Segment XY intersects ω in points M and N. Rays PM and PN intersect ω_1 in points A and D; rays QM and QN intersect ω_2 in points B and C respectively. Prove that $AB = CD$.

Solution. Point P is the homothety center of circles ω and ω_1. So $AD \parallel MN$ and segment AD is perpendicular to the center line of circles ω_1 and ω_2. Thus points A and D are symmetric wrt this line. Similarly B and C are symmetric wrt this line, and so $AB = CD$ (fig.10.3).

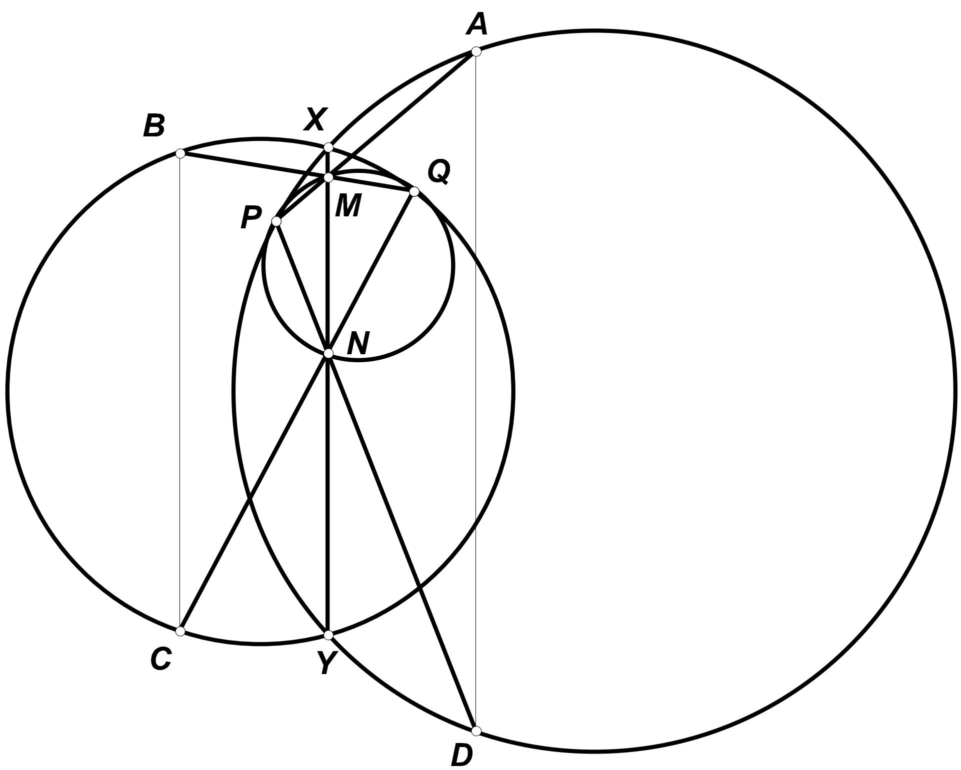


Fig.10.3.

4. (A.Zaslavsky) Given three points C_0, C_1, C_2 on the line l. Find the locus of incenters of triangles ABC such that points A, B lie on l and the feet of the median, the bisector and the altitude from C coincide with C_0, C_1, C_2.

Answer. The perpendicular to l passing through point C' on segment C_0C_2, such that $C_0C'^2 = C_0C_1 \cdot C_0C_2$.

Solution. Let C_3, C_4 be the points of contact between side AB and the incircle and the excircle of triangle ABC. Then C_0 is the midpoint of segment C_3C_4. On the other hand, points C_3, C_4 are the projections of incenter I and excenter I_c to AB (fig.10.4). Since these centers lie on CC_1, we have

$$\frac{C_2C_3}{C_2C_4} = \frac{CI}{CI_c} = \frac{r}{r_c} = \frac{C_1I}{C_1I_c} = \frac{C_1C_3}{C_1C_4}.$$

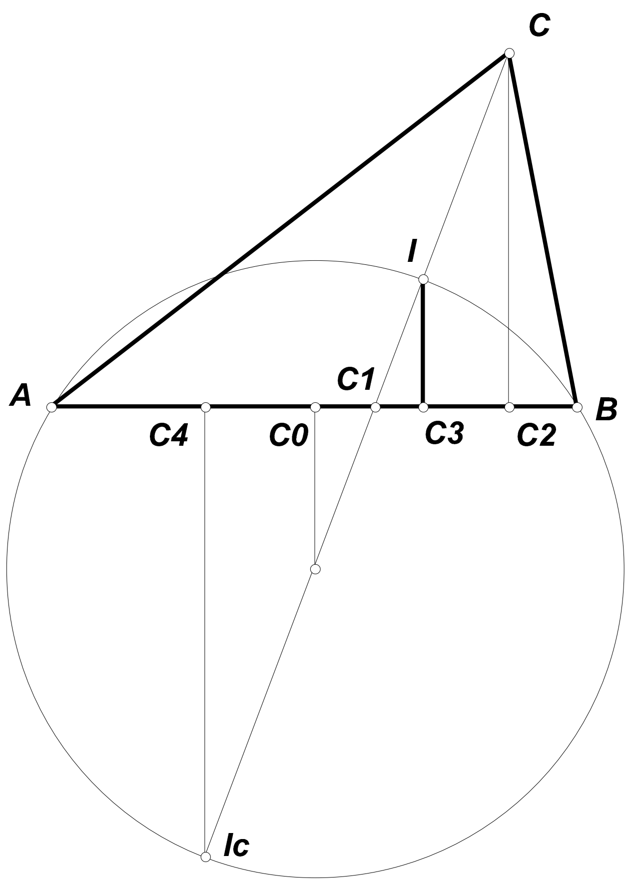


Fig.10.4.

This implies that C_3 coincides with C'. Now take an arbitrary point I, the projection of which to l coincides with C_3. The perpendiculars to l from C_2 and from C_0 intersect line C_1I in point C and in the circumcenter of triangle IAB. Finding the common points of the corresponding circle with l, we obtain the required triangle.

5. (I.Bogdanov) A section of a regular tetragonal pyramid is a regular pentagon. Find the ratio of its side to the side of the base of the pyramid.

   Answer. $\frac{3-\sqrt{5}}{\sqrt{2}}$.

   Solution. Let the plane of the section meet sides CD and DA of pyramid's base $ABCD$ in points X, Y. Then pentagon $ABCXY$ is the central projection of a regular pentagon. So the double relation of A, Y, D and the point at infinity of line AD is equal to double relation of four points in which one sideline of the regular pentagon intersects its remaining sidelines (fig.10.5). Thus:

$$\frac{DY}{AD} = \frac{3-\sqrt{5}}{2}.$$

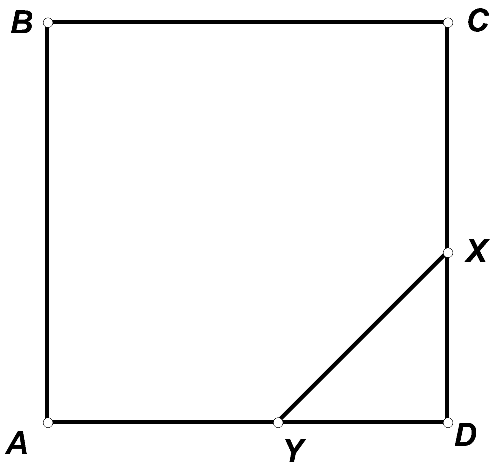


fig.10.5.

Point X divides segment CD in the same ratio. So the required ratio is

$$\frac{XY}{AB} = \frac{3-\sqrt{5}}{\sqrt{2}}.$$

Remark. Since the ratio of a side of the pentagon to a side of the base is determined unambiguously, the ratio of a lateral edge to a side of the base also is determined unambiguously. On the other hand, the planes of 8 faces of an icosahedron bound a regular octahedron. So the pyramid satisfying the conditions of the problem is a half of an octahedron, and its lateral edge is equal to the side of the base.

IV RUSSIAN GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

The final round. Solutions. 10 form. Second day

6. (B.Frenkin) The product of two sides in a triangle is equal to $8Rr$, where R and r are the circumradius and the inradius of the triangle. Prove that the angle between these sides is less than $60°$.

   Solution. Let the product of sides $AC = b$ and $BC = a$ of triangle ABC be equal to $8Rr$. Since the area of ABC is $S = pr = abc/4R$ where p is the semiperimeter, we have $4prR = abc = 8Rrc$. So $p = 2c$ or $a + b = 3c$. As $b < a + c$, this implies that $2a > 2c$ and $c < a$. Similarly $c < b$. Thus C as the strictly smallest angle of the triangle is less than $60°$.

7. (F.Nilov) Two arcs with equal angular measure are constructed on the medians AA' and BB' of triangle ABC towards vertex C. Prove that the common chord of the respective circles passes through C.

   Solution. Let the circle constructed on AA' intersect AC in point X, and the circle constructed on BB' intersect BC in point Y (fig.10.7). Since $\angle AXA' = \angle BYB'$, triangles CXA' and CYB' are similar. So $CX/CA' = CY/CB'$ and

$$CX \cdot CA = 2CX \cdot CB' = 2CY \cdot CA' = CY \cdot CB.$$

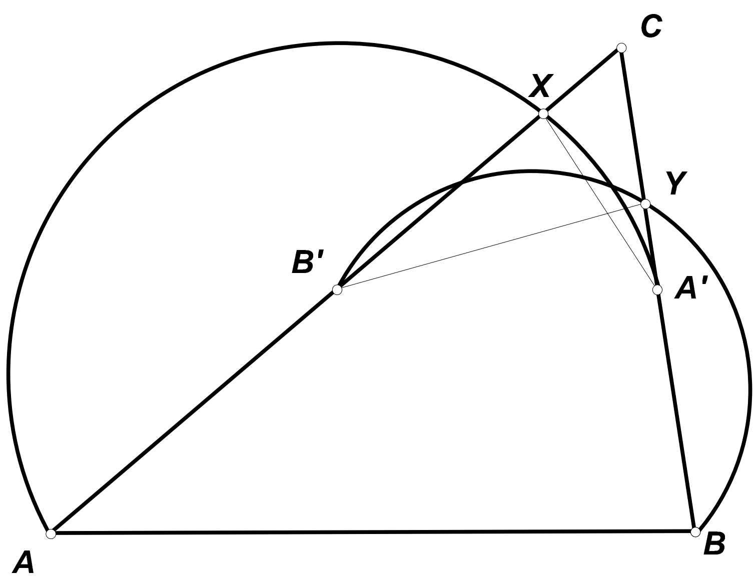


Рис.10.7.

   Thus the degrees of point C wrt both circles are equal and C lies on their radical axis.

8. (A.Akopyan, V.Dolnikov) Given a set of points inn the plane. It is known that among any three of its points there are two such that the distance between them doesn't exceed 1. Prove that this set can be divided into three parts such that the diameter of each part does not exceed 1.

Solution. Call two points 1-close if the distance between them does not exceed 1.

If the diameter of the given set V doesn't exceed $\sqrt{3}$ then V can be covered by a circle with radius 1. We can choose this circle so that it contains some points of V on its boundary. Denote the center of the circle and some point of V on its boundary by X and Y respectively.

Note that any two points in set $V \setminus B(Y,1)$ are 1-close. So the diameter of this set doesn't exceed 1. Furthermore segment $[X,Y]$ divides $V \bigcap B(Y,1)$ into two parts with diameters less or equal to 1. Thus we obtain the required dissection.

Let now two points $X, Y \in V$ exist such that $d(X,Y) > \sqrt{3}d$. Then the join of sets $V \setminus B(X,1)$, $V \setminus B(Y,1)$ and $V \bigcap B(X,1) \bigcap B(Y,1)$ contains V and each of these sets has the diameter less or equal to 1. In fact, any two points of $V \setminus B(X,1)$ or $V \setminus B(Y,1)$ are 1-close. Furthermore set $V \bigcap B(X,1) \bigcap B(Y,1)$ lies inside the set $B(X,1) \bigcap B(Y,1)$ with diameter less or equal to 1 (this diameter is the segment between common points of circles $S(X,1)$ and $S(Y,1)$).

V GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. First day. 8 form. Solutions.

1. (A.Blinkov, Y.Blinkov) Minor base BC of trapezoid $ABCD$ is equal to side AB, and diagonal AC is equal to base AD. The line passing through B and parallel to AC intersects line DC in point M. Prove that AM is the bisector of angle BAC.

   First solution. We have $\angle BMC = \angle ACD = \angle CDA = \angle BCM$ (first and third equality follow from parallelism of BM and AC, BC and AD; second equality follows from $AC = AD$). Thus, $BM = BC = AB$, and $\angle BAM = \angle BMA = \angle MAC$ (fig.8.1).

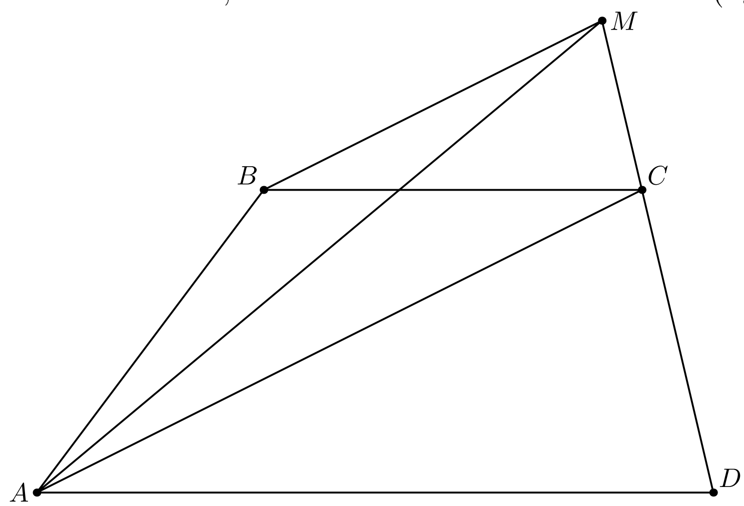


Fig.8.1

   Second solution. Let point P lie on the extension of side AB (beyond point B), and point K lie on the extension of diagonal AC (beyond point C). Then $\angle MCK = \angle ACD = \angle ADC = \angle BCM$, i.e CM is the bisector of angle BCK. Since AC bisects angle BAD and $BM \parallel AC$, then BM is the bisector of angle PBC. Thus M is the common point of two external bisectors of triangle ABC, therefore AM is the bisector of angle BAC.

2. (A.Blinkov) A cyclic quadrilateral is divided into four quadrilaterals by two lines passing through its inner point. Three of these quadrilaterals are cyclic with equal circumradii. Prove that the fourth part also is cyclic quadrilateral and its circumradius is the same.

   Solution. Let the parts adjacent to vertices A, B, C of cyclic quadrilateral $ABCD$ be cyclic quadrilaterals. Since angles A and C are opposite to equal angles in point of division L we have $\angle A = \angle C = 90°$. So two dividing lines are perpendicular. Thus angle B is also right and $ABCD$ is a rectangle. So the fourth quadrilateral is cyclic. Now the angles corresponding to arcs AL, BL, CL are equal, and since the radii of these circles also are equal, we have $AL = BL = CL$. So L is the center of the rectangle and the fourth circle has the same radius.

3. (A.Akopjan, K.Savenkov) Let AH_a and BH_b be the altitudes of triangle ABC. Points P and Q are the projections of H_a to AB and AC. Prove that line PQ bisects segment H_aH_b.

   Solution. Let CH_c be the third altitude of ABC. Then $\angle H_aH_cB = \angle H_bH_cA = \angle C$ because quadrilaterals CBH_cH_b and CAH_cH_a are cyclic. So the reflection of H_a in AB

lies on H_bH_c. Similarly the reflection of H_a in AC also lies on this line. Thus P and Q lie on the medial line of triangle $H_aH_bH_c$ (fig.8.3).

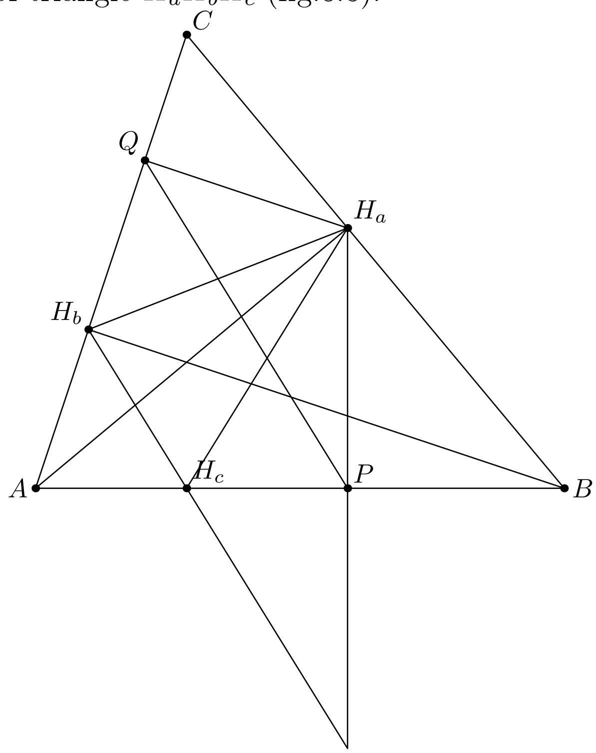


Fig.8.3

4. (N.Beluhov) Given is $\triangle ABC$ such that $\angle A = 57°$, $\angle B = 61°$ and $\angle C = 62°$. Which segment is longer: the angle bisector through A or the median through B?

   First solution. Let K be the midpoint of arc ABC in the circumcircle of ABC. Let also the circumcenter of the triangle be O, and AL and BM be the angle bisector and the median. Define $AL \cap CK = N$ and let AH be an altitude in $\triangle AKC$. Since $\angle A < \angle C$, B lies inside arc KC, therefore N lies inside segment AL and $AL > AN > AH$. Moreover $AH > KM$ as altitudes from a smaller and a greater angle in $\triangle AKC$. Finally, $KM = MO + OK = MO + OB > MB$, and the problem is solved: the angle bisector is longer (fig.8.4).

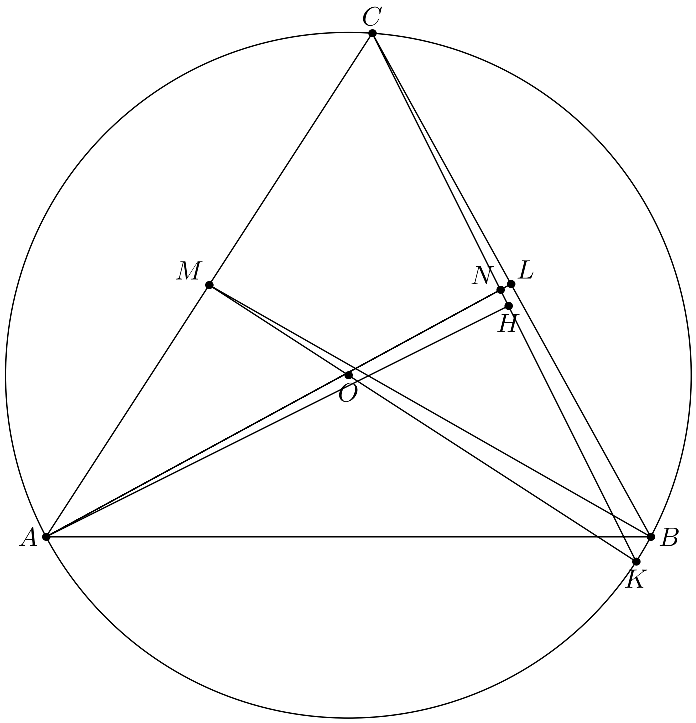


Fig.8.4

Second solution. Since $AB > BC$, we have $\angle MBC > 30°$. Construct the altitude AH and the perpendicular MK from M to BC. We have $AL > AH = 2MK > BM$, because $\sin\angle BMK = \frac{MK}{BM} > \frac{1}{2}$.

Third solution. (K.Ivanov, Moscow). Consider regular triangle ABC'. Since ray BC' lies inside angle ABC, we have that the bisector of angle A is longer that the altitude of regular triangle. In the other hand let M, N be the midpoints of AC and AC' respectively. Since ray AC lies inside angle $C'AB$, we have $\angle BMN > \angle BMA$. But $\angle BMA > 90°$ because $AB > BC$. Thus $BN > BM$ and the bisector of angle A is longer that the median from B.

V GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Second day. 8 form. Solutions.

5. (V.Protasov) Given triangle ABC. Point M is the projection of vertex B to bisector of angle C. K is the touching point of the incircle with side BC. Find angle MKB if $\angle BAC = \alpha$

   Solution. Let I be the incenter of ABC. Then quadrilateral $BMIK$ is cyclic because $\angle BMI = \angle BKI = 90^\circ$ (fig.8.5). Thus $\angle MKB = \angle MIB = \angle IBC + \angle ICB = \frac{\angle B+\angle C}{2} = 90^\circ - \frac{\alpha}{2}$.

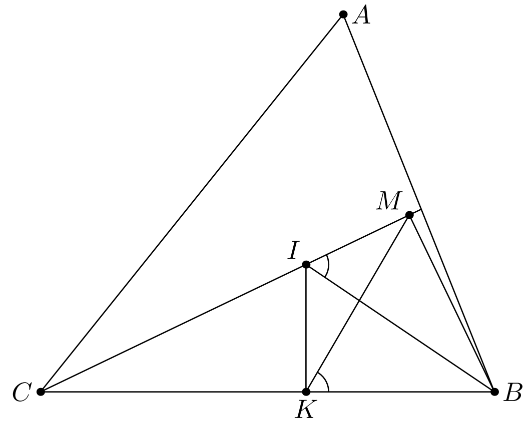


Fig.8.5

6. (S.Markelov) Can four equal polygons be placed on the plane in such a way that any two of them don't have common interior points, but have a common boundary segment?

   Solution. Yes, see fig.8.6.

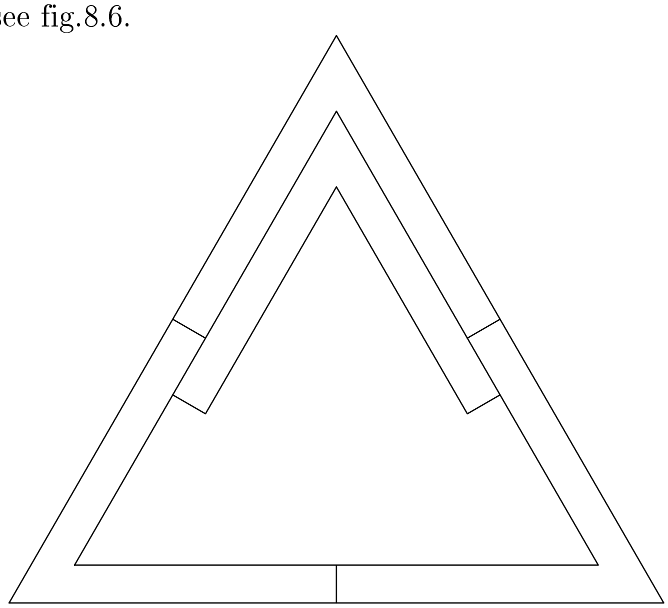


Fig.8.6

7. (D.Prokopenko) Let s be the circumcircle of triangle ABC, L and W be common points of angle's A bisector with side BC and s respectively, O be the circumcenter of triangle ACL. Restore triangle ABC, if circle s and points W and O are given.

Solution. Let O' be the circumcenter of ABC. Then lines $O'O$ and $O'W$ are perpendicular to sides AC and BC, so the directions of these sides are known. Also $\angle COL = 2\angle CAL = 2\angle LCW$, thus $\angle OCW = 90°$ (fig.8.7). Therefore C is the common point of s and the circle with diameter OW.

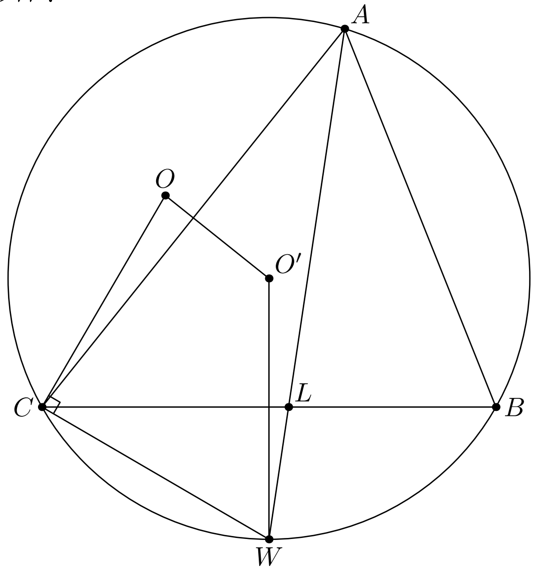


Fig.8.7

8. (N.Beluhov) A triangle ABC is given, in which the segment BC touches the incircle and the corresponding excircle in points M and N. If $\angle BAC = 2\angle MAN$, show that $BC = 2MN$.

Solution. We may assume that $AB > AC$, and therefore the points B, N, M, C lie on the line in this order. We will use the following well-known

Lemma. Let K be the midpoint of AB, and I and J be the incenter and the excenter opposite to A. Then $AN \parallel IK$ and $AM \parallel JK$.

Now the lemma shows that the original condition is equivalent to $\angle IKJ = 180 - \alpha/2$. We will show first that if $BC = 2MN$ then this is true. In this case, since the midpoints of BC and MN coincide, we have that M and N are midpoints of KC and KB, and therefore, IM and NJ are perpendicular bisectors of KC and KB. Thus triangles IKC and JKB are isosceles, and $\angle JKB = 90 - \beta/2, \angle IKC = \gamma/2$, yielding the claim (fig.8.8).

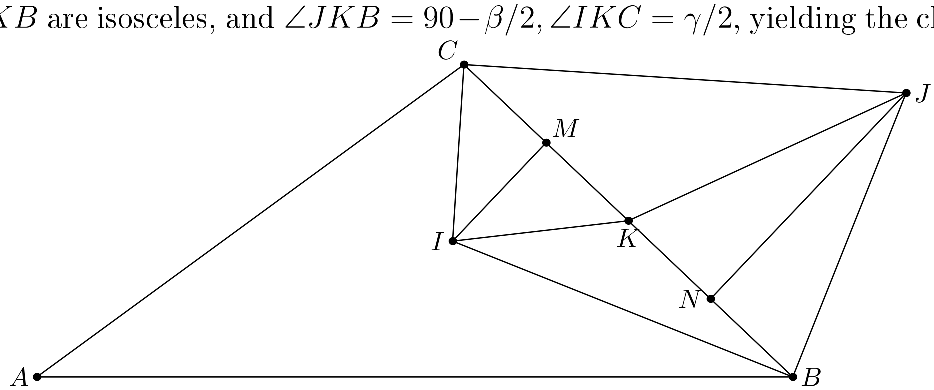


Fig.8.8

Now, consider the circle $(BICJ)$. Given α, we see that IJ is determined as a diameter, and BC as an arc constituting angle $90 + \alpha/2$. When the chord BC runs along the circle, its midpoint K runs along a smaller circle. In the same time the locus of the points K'

such that $\angle IKJ = 180 - \alpha/2$, consists of two arcs of circles with endpoints I and J. Obviously, these loci intersect in four points, symmetric to each other with respect to IJ and its perpendicular bisector, thus corresponding to four equal quadrilaterals $BICJ$. So this quadrilateral is completely determined by the condition $\angle IKJ = 180 - \alpha/2$. But the one obtained when $BC = 2MN$ satisfies this condition, hence the claim.

V GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. First day. 9 form. Solutions.

1. (A.Blinkov, Y.Blinkov) The midpoint of triangle's side and the base of the altitude to this side are symmetric wrt the touching point of this side with the incircle. Prove that this side equals one third of triangle's perimeter

   First solution. Let a, b be the lengths of two sides, and the altitude divide the third side into segments with lengths x, y (if the base of the altitude lies out of the side then one of these lengths is negative). By the Pythagorean theorem $x^2 - y^2 = a^2 - b^2$. But the touching point divides the side into segments with lengths $p - a$ and $p - b$. So the condition of the problem is equivalent to $x - y = 2(a - b)$. Dividing the first equality by the second one we obtain that $x + y = (a + b)/2 = 2p/3$.

   Second solution. Let c be the side in question, then $r/r_c = (p - c)/p$. Let K and P be the touching points of this side with the incircle and the excircle, I and Q be the centers of these circles. It is known that the midpoint of altitude CH lies on line IP. Using similarity of two pairs of triangles we obtain that $r = h/3$, $r_c = h$ (fig.9.1). From the first equality we obtain the assertion of the problem.

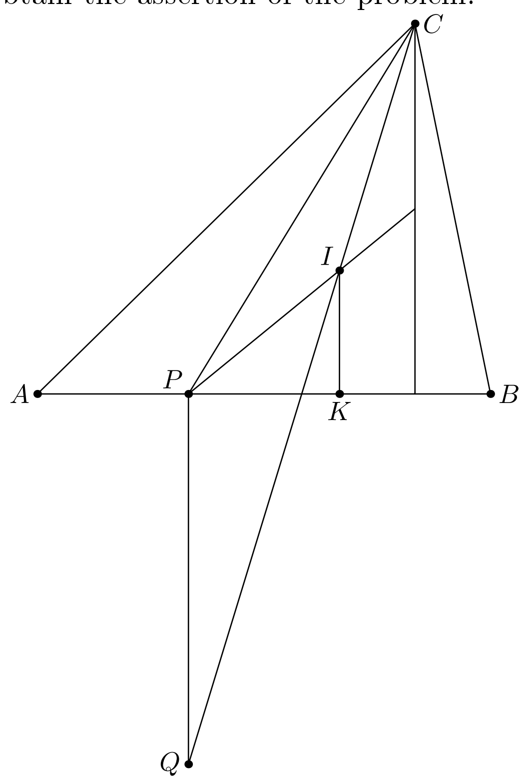


Fig.9.1

2. (O.Musin) Given a convex quadrilateral $ABCD$. Let R_a, R_b, R_c and R_d be the circumradii of triangles DAB, ABC, BCD, CDA. Prove that inequality $R_a < R_b < R_c < R_d$ is equivalent to

$$180^\circ - \angle CDB < \angle CAB < \angle CDB.$$

Solution. Let the angles of the quadrilateral satisfy the given inequality. Then $\sin\angle CAB > \sin\angle CDB$ and so $R_b < R_c$. Since angle CDB is obtuse, this implies that point A lies out of the circle CDB, thus $\angle CAD < \angle CBD$. As these angles are both acute, we have $\sin\angle CAD < \sin\angle CBD$ and $R_c < R_d$. Moreover $\angle ACB < \angle ADB < 90^\circ$, so $R_a < R_b$.

Conversely, from $R_b < R_c$ it follows that angle CAB lies between angles CDB and $180^\circ - \angle CDB$. If angle CDB is acute, we have $\angle ABD < \angle ACD$, and since $R_a < R_d$ then $\angle ABD > 180^\circ - \angle ACD$. But in this case we obtain by repeating previous argument that $R_b < R_a < R_d < R_c$.

3. (I.Bogdanov) Quadrilateral $ABCD$ is circumscribed, rays BA and CD intersect in point E, rays BC and AD intersect in point F. The incircle of the triangle formed by lines AB, CD and the bisector of angle B, touches AB in point K, and the incircle of the triangle formed by lines AD, BC and the bisector of angle B, touches BC in point L. Prove that lines KL, AC and EF concur.

   Solution. Let the incircle of $ABCD$ touch sides AB and BC in points U and V. Then we have
$$(EB;KU) = \frac{EK}{BK} : \frac{EU}{BU} = \frac{\operatorname{ctg}\frac{\angle BEC}{2}}{\operatorname{ctg}\frac{\angle B}{4}} : \frac{\operatorname{ctg}\frac{\angle BEC}{2}}{\operatorname{ctg}\frac{\angle B}{2}} = (FB;LV).$$
   This means that lines KL, EF, UV concur. Similarly lines AC, EF, UV concur (fig.9.3).

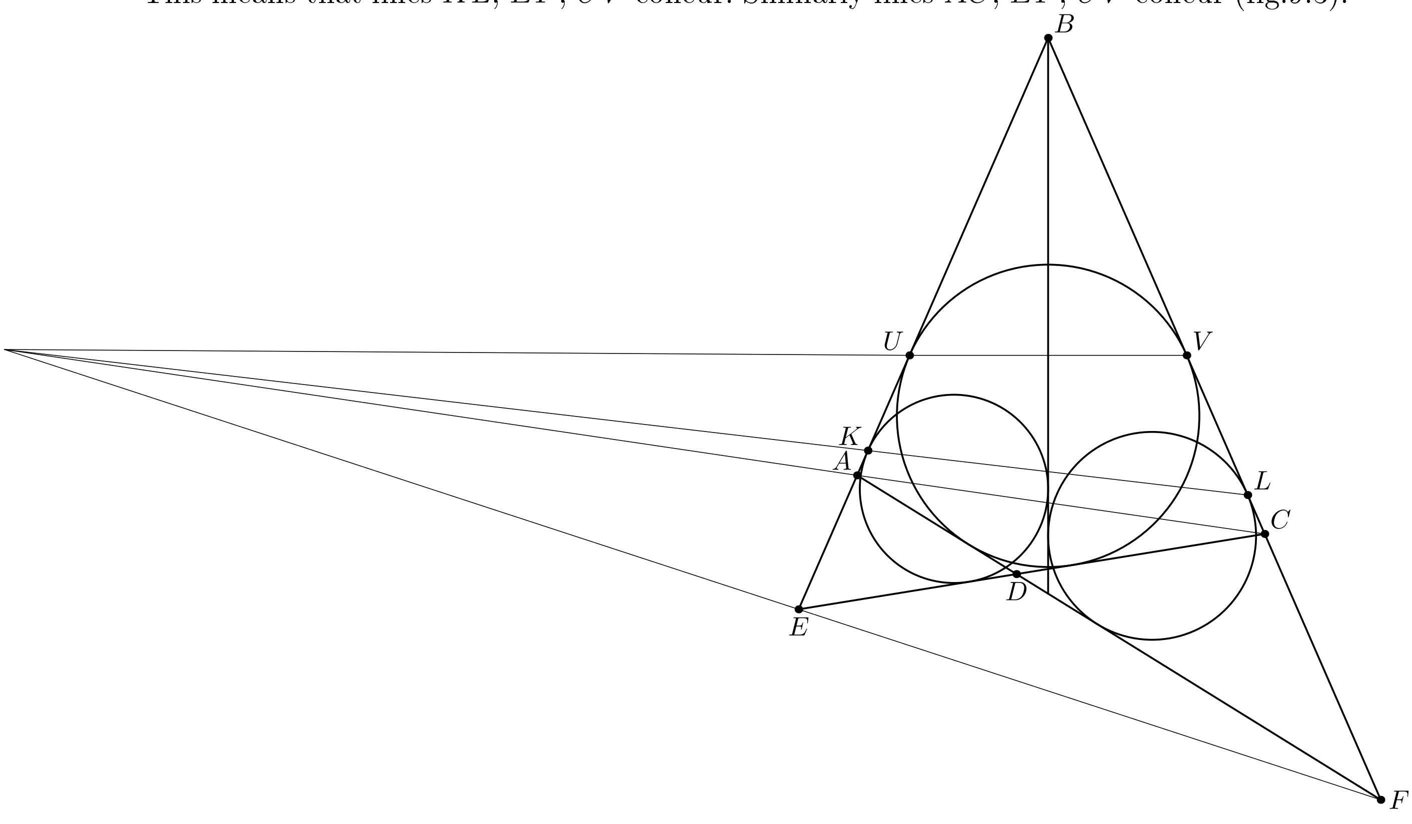


Fig.9.3

4. (N.Beluhov) Given regular 17-gon $A_1 \dots A_{17}$. Prove that two triangles formed by lines A_1A_4, A_2A_{10}, $A_{13}A_{14}$ and A_2A_3, A_4A_6 $A_{14}A_{15}$ are equal.

   Solution. Firstly note that $A_1A_4 \parallel A_2A_3$, $A_2A_{10} \parallel A_{14}A_{15}$, $A_{13}A_{14} \parallel A_4A_6$. So we have to prove that given triangles are central symmetric.

Let A, B, C, D, E, F be the midpoints of A_1A_2, A_3A_4, A_4A_{13}, A_6A_{14}, $A_{10}A_{14}$, $A_{15}A_2$ respectively. Lines BC, DE, FA as medial lines of three triangles are parallel to $A_3A_{13} \parallel A_6A_{10} \parallel A_1A_{15}$. Lines AD, BE, CF as axes of three isosceles trapezoids concur at the center of 17-gon. By dual Pappus theorem AB, CD, EF concur at some point P (fig.9.4). But these lines are the medial lines of three strips formed by parallel sidelines of given triangles. Therefore these triangles are symmetric wrt P.

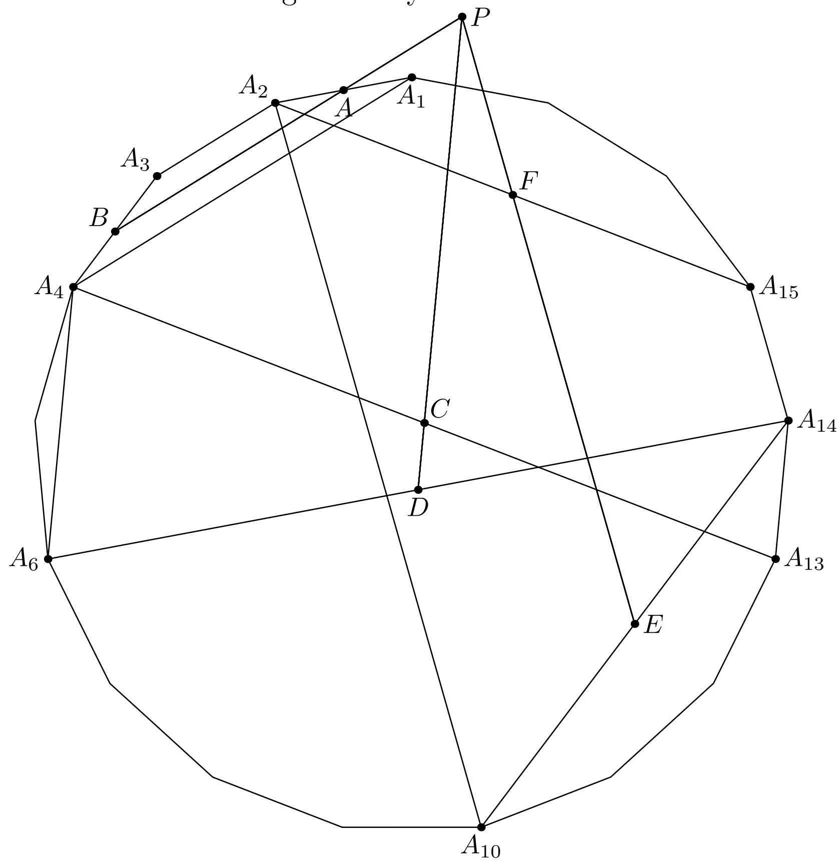


Fig.9.4

V GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Second day. 9 form. Solutions.

5. (B.Frenkin) Let n points lie on the circle. Exactly half of triangles formed by these points are acute-angled. Find all possible n.

   Answer. $n = 4$ or $n = 5$.

   Solution. It is evident that $n > 3$. Consider any quadrilateral formed by marked points. If the center of the circle lies inside this quadrilateral but not on its diagonal (call such quadrilateral "good"), then exactly two of four triangles formed by the vertices of the quadrilateral are acute-angled. In other cases less than two triangles are acute-angled. Therefore the condition of the problem is true only when all quadrilaterals are good. If $n = 4$ or $n = 5$ this is possible (consider for example the vertices of a regular pentagon).

   Now let $n > 5$. Consider one of marked points A and the diameter AA'. If point A' also is marked then the quadrilateral formed by A, A' and any two of remaining points isn't good. Otherwise there exist three marked points lying on the same side from AA'. The quadrilateral formed by these points and A isn't good.

6. (A.Akopjan) Given triangle ABC such that $AB - BC = \frac{AC}{\sqrt{2}}$. Let M be the midpoint of AC, and N be the base of the bisector from B. Prove that

$$\angle BMC + \angle BNC = 90^\circ.$$

   Solution. Let C' be the reflection of C in BN. Then $AC' = AB - BC$ and by condition $AM/AC' = AC'/AC$. Thus triangles $AC'M$ and ACC' are similar and $\angle AC'M = \angle C'CA = 90^\circ - \angle BNC$. Furthermore using the formula for a median we obtain that $BM^2 = AB \cdot BC$, so $BC'/BM = BM/BA$. Therefore triangles $BC'M$ and BMA are also similar and $\angle BMC' = \angle BAM$. Finally $\angle BMC = 180^\circ - \angle BMC - \angle C'MA = \angle MC'A$ q.e.d. (fig.9.6).

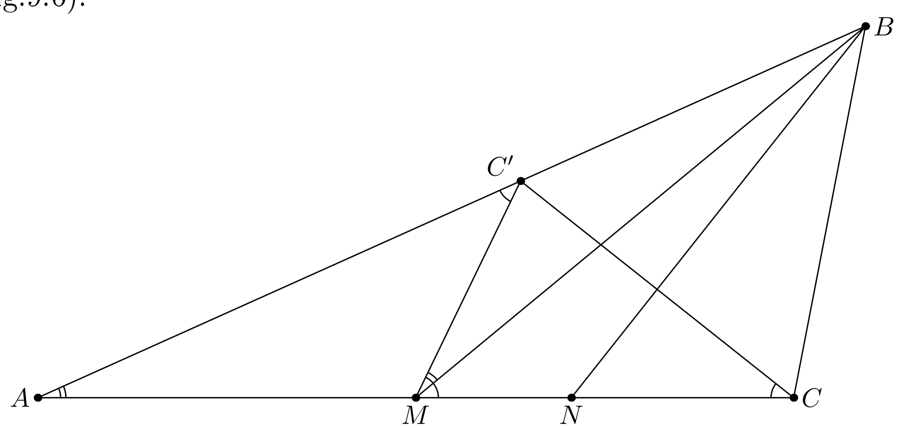


Fig.9.6

7. (M.Volchkevich) Given two intersecting circles with centers O_1, O_2. Construct the circle touching one of them externally and the second one internally such that the distance from its center to O_1O_2 is maximal.

   Solution. Let O, r be the center and the radius of some circle touching the two given; r_1, r_2 be the radii of the given circles. Then $OO_1 = r_1 - r$, $OO_2 = r_2 + r$, or $OO_1 = r_1 + r$,

$OO_2 = r_2 - r$, and in both cases $OO_1 + OO_2 = r_1 + r_2$. Therefore we must find the point satisfying this condition with maximal distance from line O_1O_2. It is known that the isosceles triangle has the minimal perimeter among all triangles with given base and altitude. Therefore the isosceles triangle has also the maximal altitude among all triangles with given side and the sum of two other sides. From this we obtain that the center of the required circle lies on equal distances $(r_1 + r_2)/2$ from points O_1 and O_2, and its radius is equal to $|r_1 - r_2|/2$.

8. (C.Pohoata, A.Zaslavsky) Given cyclic quadrilateral $ABCD$. Four circles each touching its diagonals and the circumcircle internally are equal. Is $ABCD$ a square?

Answer. Yes.

First solution. Let $AC \cap BD = P$, and let the incircles of the circular triangles ABP, BCP, CDP, DAP touch the circumcircle of $ABCD$ in K, L, M, N.

Consider the segment ABC. When a variable point X moves along the arc ABC from A to C, the radius of the circle inscribed in the segment and touching the arc in X changes as follows: it increases until X becomes the midpoint of the arc, and then decreases. Therefore, each value of radius is reached in exactly two, symmetrically situated positions of X.

Therefore $\smile AK = \smile LC$. Analogously $\smile AN = \smile MC$. So $\smile NK = \smile LM$. Analogously $\smile KL = \smile MN$. Now $\smile NL = \smile NK + \smile KL = 180^\circ$ i.e. NL is a diameter. Analogously KM is also a diameter.

Now symmetry with respect to O sends the pair of circles touching the circumcircle in M and N in the analogous pair touching it in K and L. So the same symmetry sends the common external tangent of the first pair in that of the second namely it sends AC in CA. Therefore AC is a diameter and similarly, BD is a diameter.

So $ABCD$ is a rectangle. Its diagonals divide the circumcircle into four sectors with equal radii of incircles. Therefore these sectors are also equal and $ABCD$ is square.

Second solution. Use **the Thebault theorem**: let point M lie on side AC of triangle ABC and two circles touch ray MB, line AC and internally the circumcircle of ABC. Then two centers of these circles and the incenter of ABC are collinear.

Applying the Thebault theorem to triangles ABC, BCD, CDA, DAB and the common point of diagonals we obtain that the inradii of these four triangles are equal. Calculating the areas of triangles as product of semiperimeter by inradius and finding the area of quadrilateral in two ways we obtain that $AC = BD$, so $ABCD$ is an isosceles trapezoid. Suppose that AD, BC are its bases and $AD > BC$. Then $S_{ABD}/S_{ABC} = AD/BC > (AD + BD + AB)/(BC + AB + AC)$, and the inradii of these triangles can't be equal. Thus $ABCD$ is a rectangle. As in the first solution $ABCD$ must be a square.

V GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. First day. 10 form. Solutions.

1. (D.Shvetsov) Let a, b, c be the lengths of some triangle's sides; p, r be the semiperimeter and the inradius of triangle. Prove an inequality

$$\sqrt{\frac{ab(p-c)}{p}}+\sqrt{\frac{ca(p-b)}{p}}+\sqrt{\frac{bc(p-a)}{p}}\geq 6r.$$

Solution. By Cauchi inequality the left part isn't less than

$$3\sqrt[3]{\frac{abc}{p}\sqrt{\frac{(p-a)(p-b)(p-c)}{p}}}=3\sqrt[3]{4r^2R}.$$

As $R \geq 2r$ we obtain the demanded inequality.

2. (F.Nilov) Given quadrilateral $ABCD$. Its sidelines AB and CD intersect in point K. Its diagonals intersect in point L. It is known that line KL pass through the centroid of $ABCD$. Prove that $ABCD$ is trapezoid.

Solution. Suppose that lines AD and BC intersect in point M. Let X, Y be the common points of these lines with line KL. Then $(AD; MX) = (BC; MY) = 1$. Therefore relations AX/XD and BY/YC are both greater or are both less than 1, and segment XY doesn't intersect the segment between the midpoints of AD and BC. As this last segment contains the centroid of $ABCD$, the condition of problem is true only when $AD \parallel BC$.

3. (A.Zaslavsky, A.Akopjan) The cirumradius and the inradius of triangle ABC are equal to R and r; O, I are the centers of respective circles. External bisector of angle C intersect AB in point P. Point Q is the projection of P to line OI. Find distance OQ.

Solution. Let A', B', C' be the excenters of ABC. Then I is the orthocenter of triangle $A'B'C'$, A, B, C are the bases of its altitudes and so the circumcircle of ABC is the Euler circle of $A'B'C'$. Thus the circumradius of $A'B'C'$ is $2R$, and its circumcenter O' is the reflection of I in O. Furthermore points A, B, A', B' lie on the circle. Line AB is the common chord of this circle and the circumcircle of ABC, and the external bisector of C is the common chord of this circle and the circumcircle of $A'B'C'$. So P is the radical center of three circles, and line PQ is the radical axis of circles ABC and $A'B'C'$ (fig.10.3). Therefore $OQ^2 - R^2 = (OQ + OO')^2 - 4R^2$. As $OO' = OI = \sqrt{R^2 - 2Rr}$, we have $OQ = R(R + r)/\sqrt{R^2 - 2Rr}$.

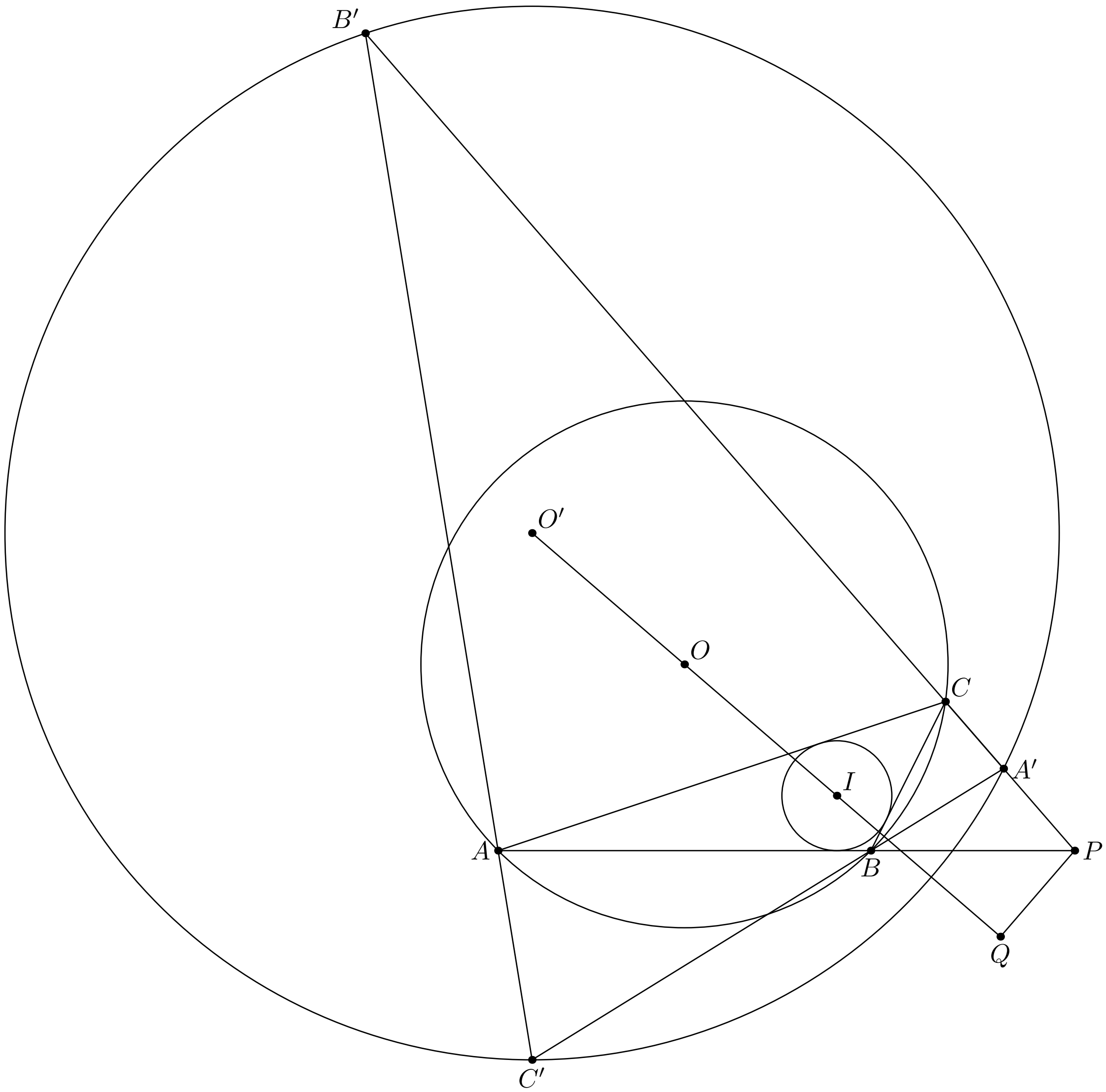


Fig.10.3

4. (C.Pohoata) Three parallel lines d_a, d_b, d_c pass through the vertex of triangle ABC. The reflections of d_a, d_b, d_c in BC, CA, AB respectively form triangle XYZ. Find the locus of incenters of such triangles.

   First solution. When d_a, d_b, d_c rotate around the vertices the symmetric lines rotate with the same velocity around the reflections of the vertices in opposite sidelines. Thus, firstly, the angles of XYZ don't depend on d_a, d_b, d_c, so all these triangles are similar, and secondly, points X, Y, Z move with equal angle velocity along three circles. Therefore the incenter also moves along some circles and it is sufficient to find three points of this circle.

   Take d_a, d_b coinciding with line AB. Let A', B' be the reflections of A, B in opposite sidelines. Then Z is the common point of lines AB' and BA', Y and X are the common points of these lines with the line parallel to AB and lying twice as far from C. Note that C and circumcenter O of ABC lie on equal distances from AB' and BA', so the bisector of angle XZY coincides with line CO. Also it is easy to see that the bisectors of angles ZXY and ZYX are perpendicular to AC and BC respectively.

   Consider the projections of O and of the incenter of XYZ to line AC. The projection of O is the midpoint of AC. Also it is the projection of the common point of AB' and d_c, because these two lines form equal angles with AC. Thus the projection of X and the incenter of XYZ is symmetric to the midpoint of AC wrt A (fig.10.4). Therefore the distance from the incenter to O is twice as large as the circumradius of ABC. When d_a, d_b, d_c are parallel to other sidelines of ABC, we obtain the same result. So the demanded locus is the circle with center O and radius twice as large as the circumradius of ABC .

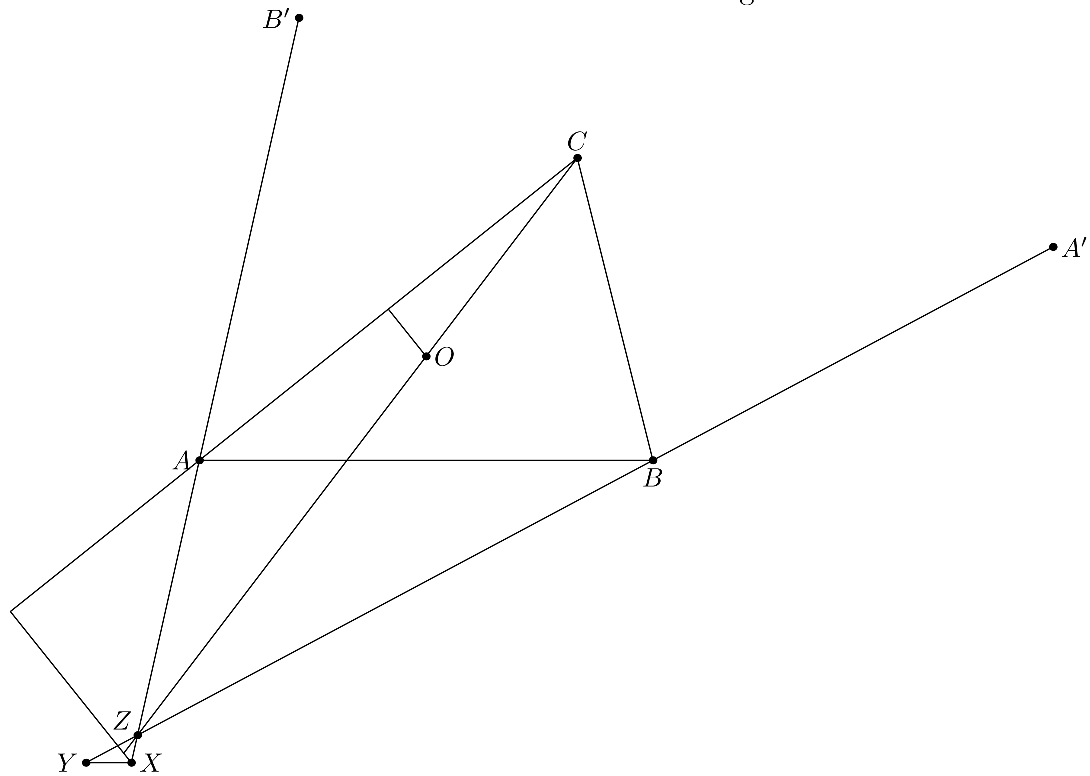


Fig.10.4

Second solution. As in the previous solution we obtain that when the direction of the lines d changes with a constant angular speed, so do the directions of XY, YZ, ZX. Therefore, the vertex X of triangle XYZ traces a circle with chord $B'C'$, and the angle bisector of $\angle YXZ$ rotates around the midpoint W_a of the arc $\smile B'C'$ with constant angular speed, too. So do the angle bisectors of $\angle Y$ and $\angle Z$ around the midpoints W_b, W_c of the corresponding arcs $\smile A'C'$ and $\smile A'B'$.

Therefore, their intersection I traces in the same time the circumcircles of triangles IW_aW_b, IW_bW_c and IW_cW_a. So, these three circumcircles do in fact coincide, and we are left to describe the circumcircle of triangle $W_aW_bW_c$.

We will show that all the points W_a, W_b, W_c are of distance $2R$ from O. Indeed, take W_a. Let BH_b, CH_c be the altitudes in triangle ABC, O_a be the circumcenter of triangle AH_bH_c, O' be the reflection of O in BC, and M_a be the midpoint of BC. The figures BCO', $H_bH_cO_a$ and $B'C'W_a$ are similar, and the figures BH_bB_1 and CH_cC_1 are also similar, therefore they are similar to $O'O_aW_a$, and M_aO_a is a mid-segment in triangle $O'OW_a$. Since M_aO_a is a diameter of the Euler circle, and thus equals R, the claim follows.

V GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Second day. 10 form. Solutions.

5. (D.Prokopenko) Rhombus $CKLN$ is inscribed into triangle ABC in such way that point L lies on side AB, point N lies on side AC, point K lies on side BC. O_1, O_2 and O are the circumcenters of triangles ACL, BCL and ABC respectively. Let P be the common point of circles ANL and BKL, distinct from L. Prove that points O_1, O_2, O and P are concyclic.

   Solution. It is evident that L is the base of the bisector of angle C, and lines LN, LK are parallel to sides BC, AC. Thus $\angle AO_1L = 2\angle ACL = \angle C = \angle ANL$, so point O_1 lies on the circumcircle of triangle ANL and coincides with the midpoint of arc ANL. Thus, $\angle O_1PL = \angle APL + \angle O_1PA = \angle C + \frac{\angle A + \angle B}{2} = \frac{\pi + \angle C}{2}$. Similarly $\angle O_2Pl = \frac{\pi+\angle C}{2}$. Therefore $\angle O_1PO_2 = \pi - \angle C$. But angle O_1OO_2 is also equal to $\pi - \angle C$, because lines OO_1, OO_2 are medial perpendiculars to AC and BC (fig.10.5).

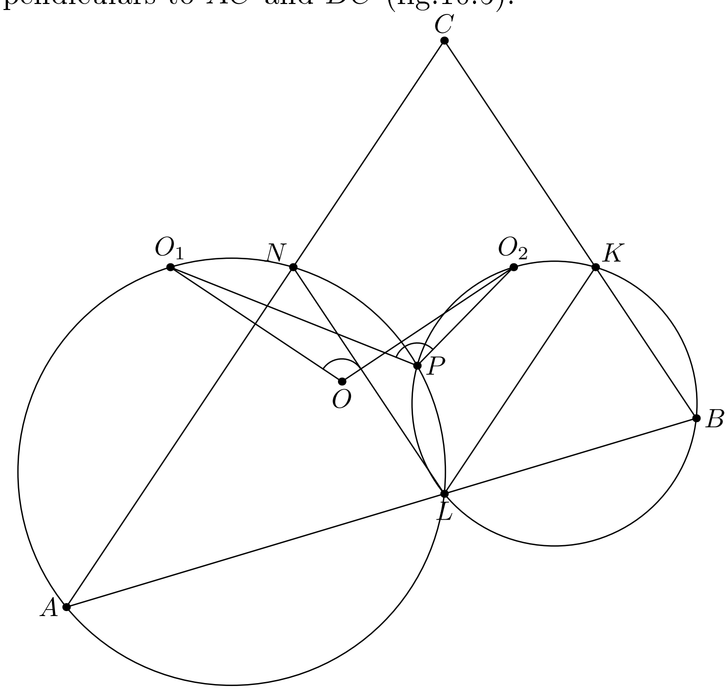


Fig.10.5

6. (A.Zaslavsky) Let M, I be the centroid and the incenter of triangle ABC, A_1 and B_1 be the touching points of the incircle with sides BC and AC, G be the common point of lines AA_1 and BB_1. Prove that angle CGI is right if and only if $GM \parallel AB$.

   First solution. Let C_1 be the touching point of incircle with side AB, C_2 be the second common point of incircle with CC_1. Then G lies on segment CC_1. As there exists central projection transforming the incircle to some circle and G to the center of this circle, then the cross-ratio $(CG; C_1C_2)$ is the same for any triangle and regular triangle. So this cross-ratio is equal to 3. Therefore we have the chain of equivalent assertions:

   - $\angle CGI = 90°$;

   - G is the midpoint of C_1C_2;

   - $CC_1 = 3CC_2$;

- $CC_1 = 3GC_1$;

- $GM \parallel AB$.

Second solution. Let $AC_1 = x, BA_1 = y, CB_1 = z$. By Menelaus' theorem,

$$\frac{y+x}{x} . \frac{GC_1}{GC} . \frac{z}{y} = 1 \Rightarrow \frac{GC_1}{GC} = k = \frac{xy}{z(x+y)} = \frac{m}{z},$$

where $m = \frac{xy}{x+y}$.

Now,

$$\angle IGC = 90^\circ \Leftrightarrow CI^2 - r^2 = GC^2 - GC_1^2 \Leftrightarrow z^2 =$$

$$= CC_1^2 (\frac{1}{(1+k)^2} - \frac{k^2}{(1+k)^2}) = CC_1^2 (\frac{1-k}{1+k}) = CC_1^2 (\frac{z-m}{z+m}).$$

But, by Stewart's theorem,

$$CC_1^2 = \frac{x}{x+y}(z+y)^2 + \frac{y}{x+y}(z+x)^2 - xy = z(z+4m).$$

Then, these two equations yield

$$z^2 = z(z+4m)(\frac{z-m}{z+m}) \Leftrightarrow z(z+m) = (z+4m)(z-m) \Leftrightarrow$$

$$\Leftrightarrow 2zm = 4m^2 \Leftrightarrow z = 2m \Leftrightarrow k = \frac{1}{2},$$

as needed.

7. (A.Glazyrin) Given points O, A_1, A_2 ... A_n on the plane. For any two of these points the square of distance between them is natural number. Prove that there exist two vectors $\vec{x}$ and $\vec{y}$, such that for any point A_i $\vec{OA_i} = k\vec{x} + l\vec{y}$, where k and l are some integer numbers.

   Solution. By condition we obtain that for all i, j the product $(\vec{OA_i}, \vec{OA_j})$ is a half of an integer number. Thus for any integer $m_1, \ldots, m_n$ the square of vector $m_1\vec{OA_1} + \cdots + m_n\vec{OA_n}$ is a natural number. Consider all points which are the ends of such vectors. Let X be the nearest to O of these points, Y be the nearest to O of considered points not lying on line OX. Divide the plane into parallelograms formed by vectors $\vec{x} = \vec{OX}$ and $\vec{y} = \vec{OY}$. By definition of points X, Y all marked points are vertices of parallelograms, therefore $\vec{x}$, $\vec{y}$ are demanded vectors.

8. (B.Frenkin) Can the regular octahedron be inscribed into regular dodecahedron in such way that all vertices of octahedron be the vertices of dodecahedron?

   Answer. No.

   Solution. If an octahedron is inscribed into a dodecahedron then their circumspheres coincide. Therefore two opposite vertices of the octahedron are opposite vertices of the dodecahedron, and all other vertices of the octahedrons are equidistant from these two vertices. But the dodecahedron has no vertices equidistant from two opposite vertices.

VI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. First day. 8 form. Solutions.

1. (M.Rozhkova, Ukraine) For a nonisosceles triangle ABC, consider the altitude from vertex A and two bisectrices from remaining vertices. Prove that the circumcircle of the triangle formed by these three lines touches the bisectrix from vertex A.

   Solution. Let I be the incenter of the triangle, B' be the foot of the bisectrix from vertex B and X be the common point of the bisectrix from C and the altitude from A (fig.8.1). Then $\angle AIB' = \angle A/2 + \angle C/2 = 90^\circ - \angle B/2 = \angle IXA$, and we obtain the required assertion. Other dispositions of points can be considered similarly.

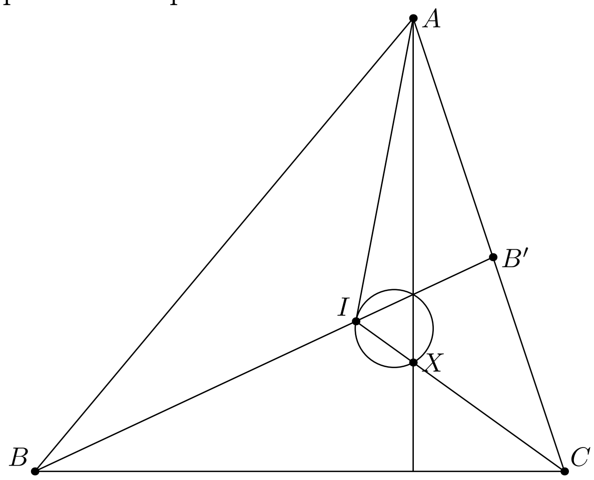


Fig.8.1

2. (A.Akopjan) Two points A and B are given. Find the locus of points C such that triangle ABC can be covered by a circle with radius 1.

   Solution. Evidently the locus in question is empty when $AB > 2$, and it is the circle with diameter AB when $AB = 2$. Let $AB < 2$, and P, Q be the common points of two circles with centers A, B and radii equal to 1. Then the locus in question is the union of unit circles with centers on the "lens"formed by arcs PQ of these circles. Let P_1, P_2, Q_1, Q_2 be points such that P is the midpoint of segments AP_1, BP_2, and Q is the midpoint of segments AQ_1, BQ_2. Construct four arcs: P_1Q_1 with center A and radius 2, P_2Q_2 with center B and 2, P_1P_2 with center P and radius 1, Q_1Q_2 with center Q and radius 1 (fig.8.2). These arcs bound the locus in question.

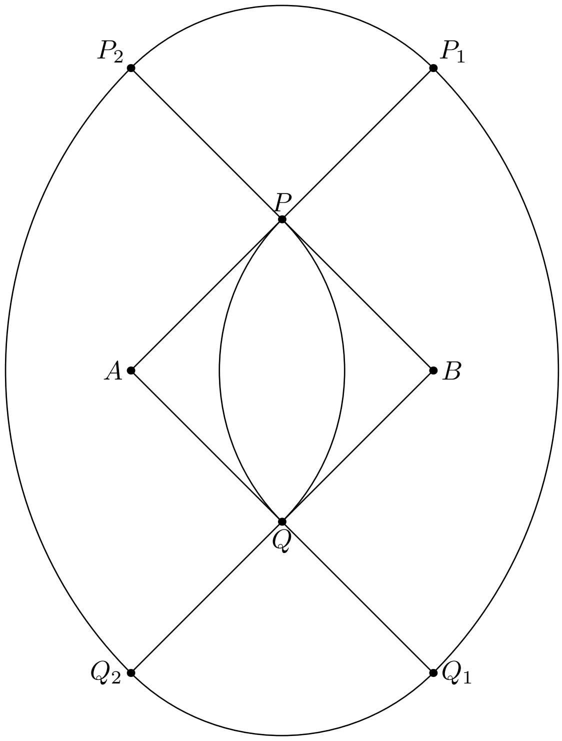


Fig.8.2

3. (S.Berlov, D.Prokopenko) Let $ABCD$ be a convex quadrilateral and K be the common point of rays AB and DC. There exists a point P on the bisectrix of angle AKD such that lines BP and CP bisect segments AC and BD respectively. Prove that $AB = CD$.

   Solution. Lines BP and CP contain the medians of triangles ABC and BCD, thus $S_{KAB} = S_{KBC} = S_{KCD}$. Since triangles KAB and KCD have equal altitudes, their bases are also equal.

4. (I.Bogdanov) Circles ω_1 and ω_2 inscribed into equal angles X_1OY and YOX_2 touch lines OX_1 and OX_2 at points A_1 and A_2 respectively. Also they touch OY at points B_1 and B_2. Let C_1 be the second common point of A_1B_2 and ω_1; C_2 be the second common point of A_2B_1 and ω_2. Prove that C_1C_2 is the common tangent of two circles.

   Solution. Triangles OA_1B_2 and OB_1A_2 are equal by two sides and the respective angle, thus $A_1B_2 = B_1A_2$. Moreover from $B_2C_1 \cdot B_2A_1 = B_2B_1^2 = B_1C_2 \cdot B_1A_2$ we obtain $B_2C_1 = B_1C_2$, and since $\angle A_1OB_1 = \angle B_2OA_2$, we have $\angle A_1C_1B_1 + \angle A_2C_2B_2 = 180°$. Therefore quadrilateral $B_2C_2B_1C_1$ is cyclic and so it is an isosceles trapezoid. Then $\angle B_2C_2C_1 = \angle C_2B_2B_1$ and C_2C_1 is the tangent (fig.8.4.).

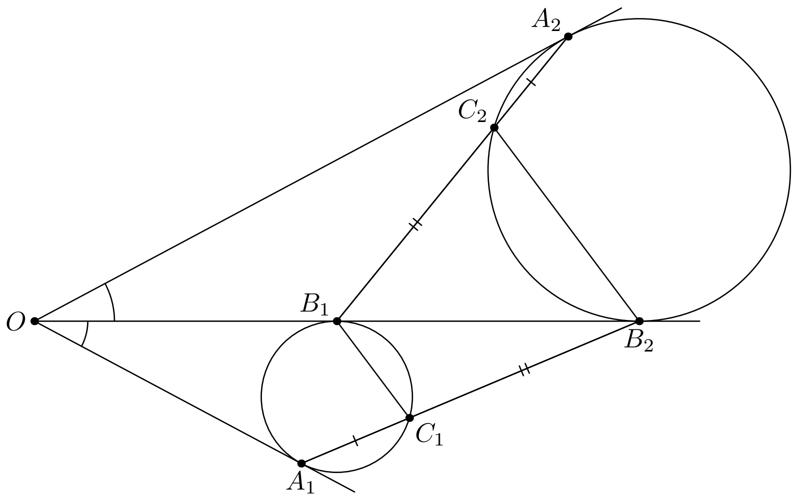


Fig.8.4

VI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Second day. 8 form. Solutions.

5. (B.Frenkin) Let AH, BL and CM be an altitude, a bisectrix and a median in triangle ABC. It is known that lines AH and BL are an altitude and a bisectrix of triangle HLM. Prove that line CM is a median of this triangle.

   Solution. Since $AH \perp LM$, we have $LM \parallel BC$, i.e. LM is the medial line of the triangle. Therefore BL is the bisectrix and the median, i.e. $AB = BC$. Now from equality of triangles BLM and BLH we obtain $BH = BM = AM = CH$. Thus $AB = AC$ and ABC is a regular triangle.

6. (D.Prokopenko) Let E, F be the midpoints of sides BC, CD of square $ABCD$. Lines AE and BF meet at point P. Prove that $\angle PDA = \angle AED$.

   First solution. Let the line passing through A and parallel to BF meet CD at point G. Since $ABFG$ is a parallelogram, we have $FG{=}AB$ and so $FD = DE$. By ?? Thales theorem, the line passing through D and parallel to BF is the median of triangle ADP. Evidently $AE \perp BF$, therefore this line is also the altitude (fig.8.6). Thus triangle ADP is isosceles as well as triangle AED. Angle EAD is their common angle at the base, therefore their angles at the opposite vertices are equal.

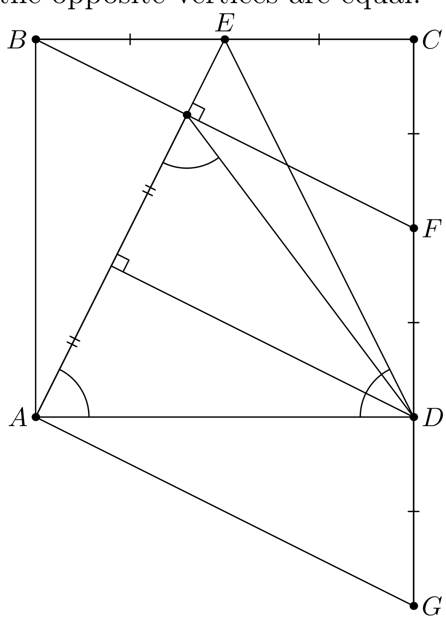


Fig.8.6

   Second solution. Let $AB = 1$. Since BP is the altitude of a right-angled triangle with legs 1 and $1/2$, we have $AP : PE = 4 : 1$. Then by Thales theorem the projection of segment DP to CD is equal to $4/5$. Similarly its projection to AD is equal to $3/5$. Therefore by Pythagorean theorem $DP = 1 = AD$ and we can proceed similarly to the previous solution.

7. (B.Frenkin) Each of two regular polygons P and Q was divided by a line into two parts. One part of P was attached to one part of Q along the dividing line so that the resulting polygon was regular and not congruent to P or Q. How many sides can it have?

Answer. 3 or 4.

Solution. Obviously the new polygon contains at least one vertex of each of given polygons. Both sides adjacent to any of these vertices are crossed by the dividing line because the new polygon can't contain two neighbouring vertices of any of the original polygons, otherwise it is equal to it. Thus the new polygon has two vertices which are the vertices of P or Q and one or two vertices on the dividing line, i.e. three or four vertices. Both cases are possible: we can cut off two small and equal right-angled triangles from two regular triangles or two small equal isosceles right-angled triangles from two squares, and then put together these two triangles.

8. (A.Zaslavsky) Bisectrices AA_1 and BB_1 of triangle ABC meet in I. Segments A_1I and B_1I are the bases of isosceles triangles with opposite vertices A_2 and B_2 lying on line AB. It is known that line CI bisects segment A_2B_2. Is it true that triangle ABC is isosceles?

Answer. No, the condition of the problem is true for any triangle with $\angle C = 120°$.

Solution. Let CC_1 be the bisectrix of angle C. Then CA_1 is the external bisectrix of angle ACC_1, i.e. point A_1 lies on equal distances from lines AC and CC_1. Also this point lies on equal distances from lines AC and AB, thus C_1A_1 is the bisectrix of angle CC_1B. Let J be the common point of lines C_1A_1 and BI. Since C_1A_1 and BI are the bisectrices of triangle BCC_1 with $\angle C = 60°$, we have $\angle IJA_1 = 120°$. Then quadrilateral $CIJA_1$ is cyclic and $IJ = JA_1$. Consider regular triangle IA_1K. Since $JK = JI$ and $\angle C_1JI = \angle C_1JK = 60°$, we have that K lies on C_1B, i.e. coincides with point A_2. Now we have $C_1A_2 = C_1I$ (fig.8.8). Similarly $C_1B_2 = C_1I$.

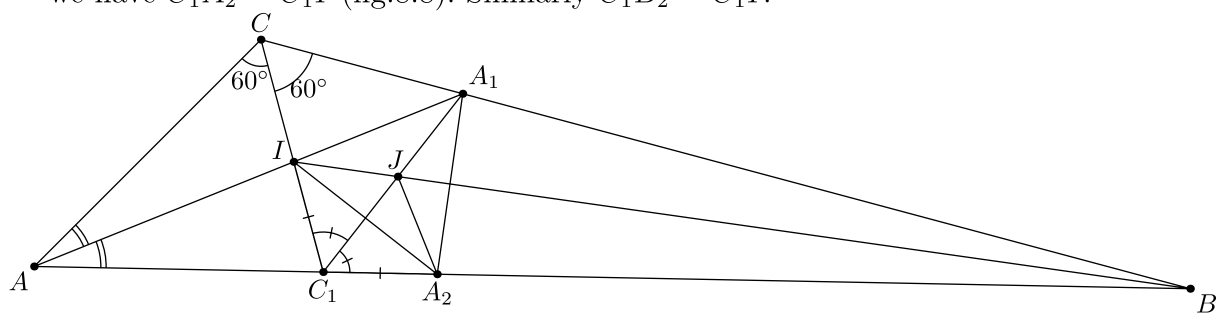


Fig.8.8

VI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. First day. 9 form. Solutions.

1. (B.Frenkin) For each vertex of triangle ABC, the angle between the altitude and the bisectrix from this vertex was found. It occurred that these angle in vertices A and B were equal. Furthermore the angle in vertex C is greater than two remaining angles. Find angle C of the triangle.

   Answer. 60°.

   Solution. The angle between the altitude and the bisectrix is equal to the half of absolute difference of angles adjacent to the opposite side of the triangle. Thus if these angles at vertices A and B are equal then $\angle A - \angle C = \angle B - \angle C$ or $\angle A - \angle C = \angle C - \angle B$. In the first case the triangle is isosceles i.e. the altitude and the bisectrix from C coincide which is impossible. In the second case $\angle C = (\angle A + \angle B)/2 = (180^\circ - \angle C)/2 = 60^\circ$.

2. (A.Akopjan) Two intersecting triangles are given. Prove that at least one of their vertices lies inside the circumcircle of the other triangle.

   Solution. If one of the circumcircles lies inside the other one, the assertion of the problem is evidently true. If each circumcircle lies outside the other, the triangles can't intersect. Let the circumcircles intersect at points P and Q, suppose that the assertion in question is false. Then all vertices of each triangle lie on the arc PQ of respective circle lying outside the second circle. But these arcs lie on the different semiplanes wrt line PQ. Thus the triangles also lie on the different semiplanes and can't intersect.

3. (V.Yasinsky, Ukraine) Points X, Y, Z lie on a line (in indicated order). Triangles XAB, YBC, ZCD are regular, the vertices of the first and the third triangle are oriented counterclockwise and the vertices of the second are oppositely oriented. Prove that AC, BD and XY concur.

   Solution. The rotation around B by 60° transforms A and C to X and Y respectively. Thus the angle between AC and XY is equal to 60°. Let P be the common point of these lines. Then since quadrilateral $AXPB$ is cyclic, we have $\angle APB = 60^\circ$ and quadrilateral $PYCB$ also is cyclic. Similarly we obtain that BD also passes through P (fig.9.3).

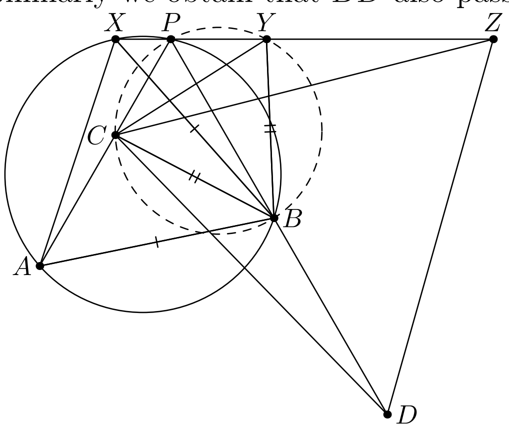


Fig.9.3

4. (A.Zaslavsky) In triangle ABC, touching points A', B' of the incircle with BC, AC and common point G of segments AA' and BB' were marked. After this the triangle was erased. Restore it by the ruler and the compass.

Solution. Let C' be the touching point of the incircle with AB; A_1, B_1, C_1 be the projections of G to the sidelines of $A'B'C'$. Then G as the Lemoine point of $A'B'C'$ is the centroid of triangle $A_1B_1C_1$. Therefore we have the following construction. Find point C_1 and its image C_2 under the homothety with center G and coefficient $-1/2$. Now construct the circles with diameters GA', GB' and find the common point of one of them with the reflection of the remaining one in C_2. This point lies on one side of $A'B'C'$, and the symmetric point on the other side. In general case the problem has two solutions.

VI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Second day. 9 form. Solutions.

5. (D.Shvetsov) The incircle of a right-angled triangle ABC ($\angle ABC = 90°$) touches AB, BC, AC in points C_1, A_1, B_1 respectively. One of the excircles touches the side BC in point A_2. Point A_0 is the circumcenter or triangle $A_1A_2B_1$; point C_0 is defined similarly. Find angle A_0BC_0.

   Solution. Points A_1 and A_2 are symmetric wrt the midpoint of BC, thus $A_0B = A_0C$. On the other hand, A_0 lies on the perpendicular bisector to segment A_1B_1, coinciding with the bisectrix of angle C. Therefore $\angle CBA_0 = \angle A_0CB = \angle C/2$. Similarly $ABC_0 = \angle A/2$, thus $\angle A_0BC_0 = 45°$.

6. (Y.Blinkov) An arbitrary line passing through vertex B of triangle ABC meets side AC at point K and the circumcircle in point M. Find the locus of circumcenters of triangles AMK.

   Solution. Let O be the circumcenter of AMK. Since $\angle AMK = \angle AMB = \angle C$, we have $\angle AOK = 2\angle C$ and $\angle OAC = 90° - \angle C$, i.e. this angle doesn't depend on K, M (fig.9.6). Therefore all circumcenters are collinear. When K moves from A to C they fill the lateral side of an isosceles triangle with base AC and angle at base equal to $90° - \angle C$. (If angle C is obtuse then the respective angle is equal to $\angle C - 90°$ and the locus in question lies on the same side wrt AC as point B).

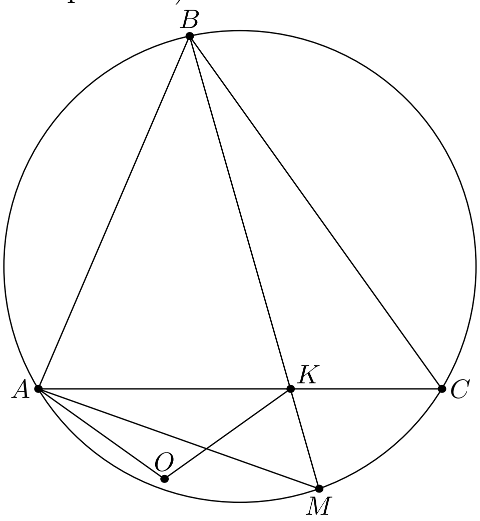


Fig.9.6

7. (N.Beluhov, Bulgaria) Given triangle ABC. Lines AL_a and AM_a are the internal and the external bisectrix of angle A. Let ω_a be the reflection of the circumcircle of $\triangle AL_aM_a$ in the midpoint of BC. Circle ω_b is defined similarly. Prove that ω_a and ω_b touch if and only if $\triangle ABC$ is right-angled.

   Solution. The circumcircle of triangle AL_aM_a is orthogonal to the circumcircle of triangle ABC. Also it is the locus of points X such that $BX : CX = AB : AC$. Therefore ω_a also is perpendicular to the circumcircle of ABC and is the locus of points X such that $BX : CX = AC : AB$. From similar conclusions for ω_b we obtain that when these circles touch, the touching point X lies on the circumcircle of ABC and $AX : BX : CX = BC :$

$CA : AB$. By Ptolomeus theorem one of products $AX \cdot BC$, $BX \cdot CA$, $CX \cdot AB$ is equal to the sum of two remaining ones. These products are proportional to the squares of sides of ABC, thus it is right-angled. The inverse assertion is obtained similarly.

8. (V.Gurovic) Given is a regular polygon. Volodya wants to mark k points on its perimeter so that any another regular polygon (maybe having a different number of sides) doesn't contain all marked points on its perimeter. Find the minimal k sufficient for any given polygon.

   Solution. First let us prove that five points do suffice. Let A, B, C, D be four successive vertices of the given polygon (the changes necessary for triangle are evident). Mark points A, B, point X on AB, point Y on BC near to B and point Z on CD near to C. Then line AB must contain a side of the polygon. he angle between this line and the side passing through Y is equal to some angle from a finite set. Clearly the respective common point can't lie on ray BA, and if it lies on the opposite ray then point Z is outside of polygon. Thus we restore line BC. Line CD and so the whole polygon are restored similarly.

   Now let us prove that four points don't suffice wnen the number of sides is great. Consider first the case when three of marked points lie on the line l. The regular triangle based on the respective side lies inside the polygon, thus the remaining marked point lies outside this triangle. Constructing two lines which pass through this point and form angles equal to 60° with l, we obtain a regular triangle containing all marked points on its perimeter.

   Now let marked points form a convex quadrilateral. Suppose that two of its opposite angles are less than 60°. Construct on diagonal AB passing through two remaining vertices, an arc which equals 240° and lies in the other semiplane than the center of the polygon. This arc meets the circumcircle of the polygon in two points near to A и B. The sides containing A and B form the angles lesser than 120° with AB, thus the respective part of the perimeter lies inside the arc, which contradicts our supposition. Therefore there exist two adjacent angles of the quadrilateral greater than 60°. Constructing two lines which form angles equal to 60° with the respective side of the quadrilateral and passes through two remaining vertices, we obtain the regular triangle containing all marked points on its perimeter.

VI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. First day. 10 form. Solutions.

1. (A.Zaslavsky) Let O, I be the circumcenter and the incenter of a right-angled triangle; R, r be the radii of respective circles; J be the reflection of the vertex of te right angle in I. Find OJ.

   Answer. $R - 2r$.

   Solution. Let ABC be the given triangle, $\angle C = 90^\circ$. Clearly the circle with center J and radius $2r$ touches AC, BC. Prove that it also touches the circumcircle of ABC.

   Consider the circle that touches AC and BC in points P, Q and touches the circumcircle from inside in point T. Since T is the homothety center of this circle and the circumcircle, lines TP, TQ secondly meet the circumcircle in midpoints B', A' of arcs AC, BC. Therefore lines AA' and BB' meet in point I, and applying Pascal theorem to hexagon $CAA'TB'B$ we obtain that P, I, Q are collinear. The respective line is perpendicular to the bisectrix of angle C, thus P, Q are the projections of J to AC and BC (fig.10.1).

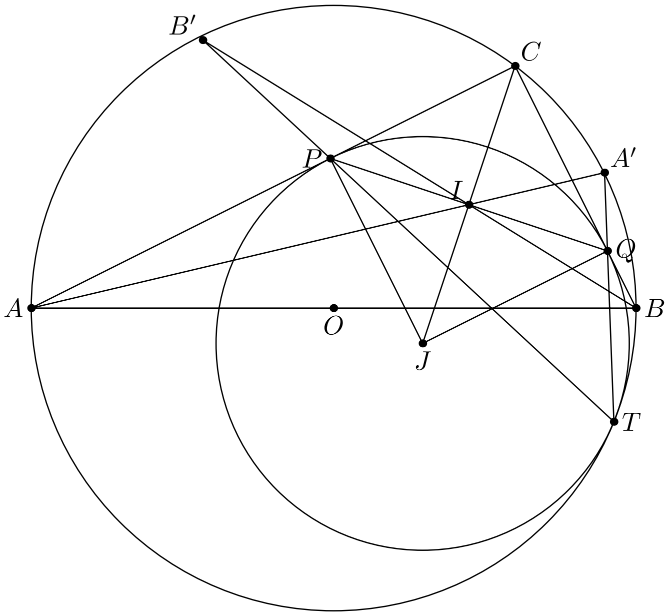


Fig.10.1

2. (P.Kozhevnikov) Each of two equal circles ω_1 and ω_2 passes through the center of the other one. Triangle ABC is inscribed into ω_1, and lines AC, BC touch ω_2. Prove that $\cos A + \cos B = 1$.

   Solution. Let O be the center of ω_2, P be the point of ω_1 opposite to O. Since CO is the bisectrix of angle ACB, points A and B are symmetric wrt line OP. Transform the sum of the cosinuses to the product: $\cos A + \cos B = 2\sin\frac{C}{2}\cos\frac{A-B}{2}$. From the indicated symmetry we obtain that $|A - B|/2 = \angle COP$, i.e. $OP\cos\frac{A-B}{2} = CO$, and since $C/2$ is the angle between CO and the tangent, we have that $CO\sin\frac{C}{2}$ is equal to the radius of ω_2, i.e. $OP/2$ (fig.10.2).

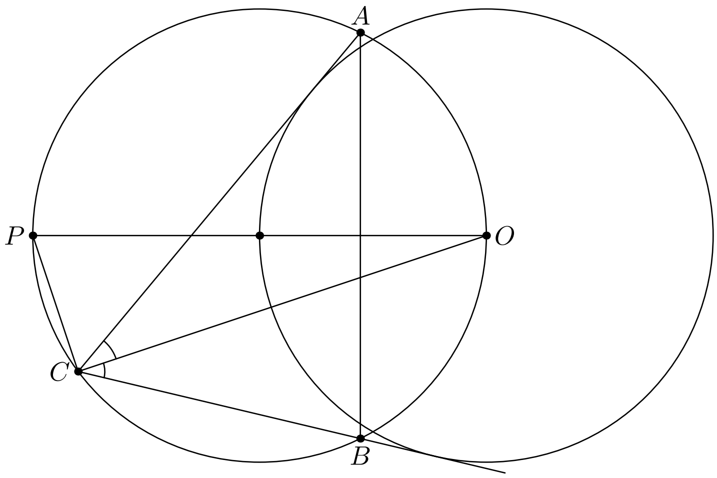


Fig.10.2

3. (A.Akopjan) All sides of a convex polygon were decreased in such a way that they formed a new convex polygon. Is it possible that all diagonals were increased?

   Answer. No.

   Solution. First prove the following lemma.

   Let ABC, ABC' be two triangles such that $AC > AC'$, $BC > BC'$. Then for any point K of segment AB we have $CK > C'K$.

   In fact, points A, B, C' lie on the same side of the bisector of segment CC'. Thus K also lies on this side, which is equivalent to the inequality in question.

   Let us prove now that the above situation is impossible for a quadrilateral. Indeed, otherwise we may suppose that one of the diagonals wasn't changed and the other was increased. Identifying equal diagonals, we obtain quadrilaterals $ABCD$, $AB'CD'$ with $AB > AB'$, $BC > B'C$, $CD > CD'$, $DA > D'A$. Let E be the common point of diagonals of $ABCD$. Then by lemma $BE > B'E$, $DE > D'E$ and $B'D' \leq B'E + D'E < BD$ — a contradiction.

   The impossibility for an arbitrary polygon can be proved by induction. As in the previous case, suppose that one diagonal wasn't changed and all remaining ones weren't decreased. Considering the parts to which the first diagonal divides the polygon, we reduce the problem to the polygon with a smaller number of sides.

4. (F.Nilov) Projections of two points to the sidelines of a quadrilateral lie on two concentric circles (projections of each point form a cyclic quadrilateral and the radii of circles are different). Prove that this quadrilateral is a parallelogram.

   Solution. Let the projections of point P to the sidelines lie on a circle with center O, and P' be the reflection of P in O. Then the projections of P' lie on the same circle and P, P' are the foci of some conic. Thus we obtain from the condition that the sidelines of the quadrilateral are the common tangents to two concentric conics. Therefore they form a parallelogram.

VI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Second day. 10 form. Solutions.

5. (D.Shvetsov) Let BH be an altitude of a right-angled triangle ABC ($\angle B = 90°$). The incircle of triangle ABH touches AB, AH in points H_1, B_1; the incircle of triangle CBH touches CB, CH in points H_2, B_2; point O is the circumcenter of triangle H_1BH_2. Prove that $OB_2 = OB_1$.

   Solution. Let I_1, I_2 be the incenters of triangles ABH, CBH. From similarity of these triangles $I_1H_1 : I_2H_2 = AB : BC$. Since segments I_1H_1 and I_2H_2 are perpendicular to AB and BC respectively, their projections to AC are equal. Then since O is the midpoint of H_1H_2, the projection of O to AC coincides with the midpoint of B_1B_2, which yields the assertion of the problem (fig.10.5).

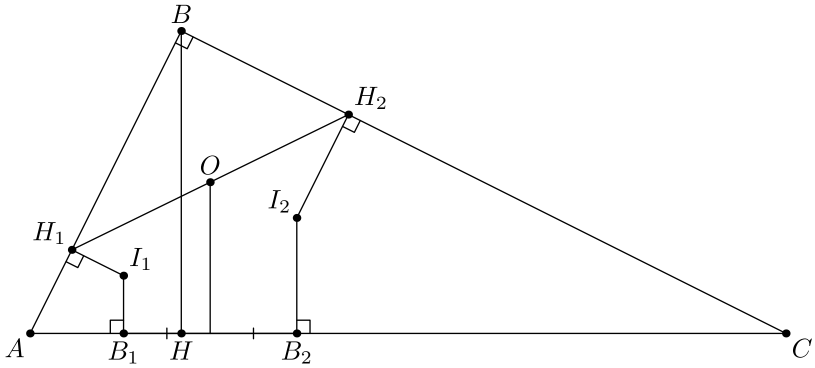


Fig.10.5

6. (F.Nilov) The incircle of triangle ABC touches its sides in points A', B', C'. It is known that the orthocenters of triangles ABC and $A'B'C'$ coincide. Is triangle ABC regular?

   Solution. Let O, I be the circumcenter and the incenter of ABC, H' be the orhocenter of $A'B'C'$, A'', B'', C'' be the second common point of lines $A'H'$, $B'H'$, $C'H'$ with the incircle. Then $\angle A''C''C' = \angle A''A'C' = \angle B''B'C' = \angle B''C''C'$, i.e. $A''B'' \parallel AB$. Therefore triangles ABC and $A''B''C''$ are homothetic. This homothety transforms O to I, and I to H'. Thus H' lies on OI. Then by condition, orthocenter H of triangle ABC lies on OI and $OI : IH = R : r$ (fig.10.6).

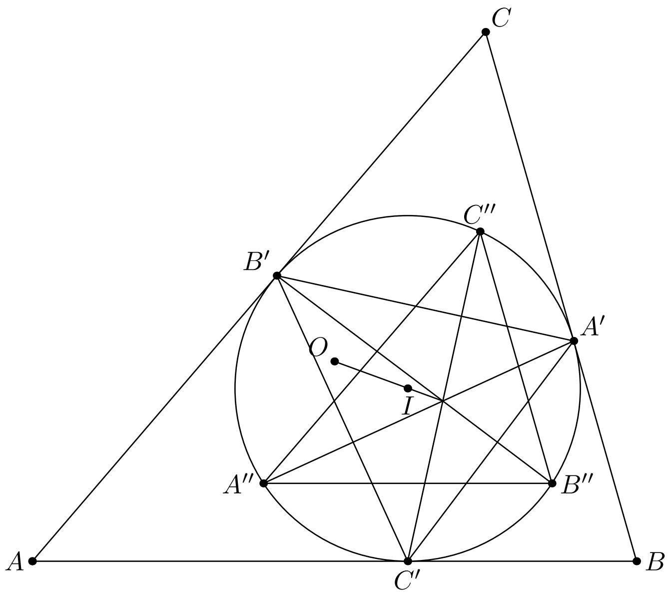


Fig.10.6

Suppose that triangle ABC isn't regular. Then two of its vertices, for example A, B, don't lie on OI. Since AI, BI are the bisectrices of angles OAH, OBH, we have $OI : IH = AO : OH = BO : BH$. Therefore $AH = BH = r$, which clearly isn't possible. Thus ABC is regular.

7. (B.Frenkin) Each of two regular polyhedrons P and Q was divided by the plane into two parts. One part of P was attached to one part of Q along the dividing plane and formed a regular polyhedron not equal to P and Q. How many faces can it have?

   Answer. 4 or 8.

   Solution. Obviously the new polyhedron R contains at least one vertex of each of given polyhedrons. Thus a polyhedral angle of R is equal to a polyhedral angle of P, and these polyhedrons have the same number of faces. Similarly Q has the same number of faces. All edges adjacent to any of the above vertices are crossed by the dividing plane because R can't contain two neighbouring vertices of any of the original polyhedrons, otherwise R is equal to it. Therefore not less than a half of faces of R is adjacent to the same vertex. This excludes dodecahedron and icosahedron. In the case of equality, the faces are triangles, and this excludes cube. The cases of tetrahedron and octahedron are possible if we cut off two equal pyramids from P and Q by the planes parallel to their symmetry planes.

8. (N.Beluhov, Bulgaria) Triangle ABC is inscribed into circle k. Points A_1, B_1, C_1 on its sides were marked, after this the triangle was erased. Prove that it can be restored uniquely if and only if AA_1, BB_1 and CC_1 concur.

   Solution. Fix triangle ABC and points A_1, B_1. Let A_2, B_2 be the second common points of AA_1, BB_1 with k; C' an arbitrary point of arc A_2CB_2; A', B' the second common points of $C'A_1$, $C'B_1$ with k; C_1 the common point of AB and $A'B'$. When C' moves from A_2 to B_2, point C_1 moves from A to B, thus there exist two triangles with the sides passing through A_1, B_1, C_1. The unique exclusion is the limiting position of C_1 when C' tends to C.

Let us prove now that if AA_1, BB_1, CC_1 concur then the triangle can be restored uniquely. Consider projective map that saves k and transforms the common point of these lines to the centroid of ABC, such that A_1, B_1, C_1 are transformed to the midpoints of its sides. Suppose that there exists another triangle $A'B'C'$ with sides passing through A_1, B_1, C_1. Let for example quadrilateral AB_1C_1A' be convex. Then quadrilaterals BC_1A_1B' and CA_1B_1C' are also convex. Since point A' lies outside the circumcircle of triangle AB_1C_1, we have $\angle B_1C'A_1 < \angle B_1AC_1$ and $\angle BC_1B' = \angle AC_1A' > \angle AB_1A'$. Similarly $\angle CA_1C' > \angle BC_1B'$ and $\angle AB_1A' > \angle CA_1C'$ — a contradiction.

VII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. First day. 8th grade. Solutions.

1. (A.Blinkov) The diagonals of a trapezoid are perpendicular, and its altitude is equal to the medial line. Prove that this trapezoid is isosceles.

   First solution. Consider the line passing through C and parallel to BD. Let E be the common point of this line and the extension of base AD. Then ACE is a right-angled triangle, thus its median from vertex C is equal to the half of the hypothenuse, i.e. to the medial line of the trapezoid. By condition, this median coincides with the corresponding altitude. Hence the diagonals of trapezoid are equal.

   Second solution. Let AD, BC be the bases of the trapezoid and O be the common point of its diagonals. Then the medians of right-angled triangles OAD, OBC are equal to halfs of their hypothenuses, i.e. the sum of these medians is equal to the medial line. On the other hand, the altitude of the trapezoid is equal to the sum of altitudes of these triangles. By the assumption the medians coincide with the altitudes, thus triangles OAD, OBC are isosceles, and this yields $AB = CD$.

2. (T.Golenishcheva-Kutuzova) Peter made a paper rectangle, put it on an identical rectangle and pasted both rectangles along their perimeters. Then he cut the upper rectangle along one of its diagonals and along the perpendiculars to this diagonal from two remaining vertices. After this he turned back the obtained triangles in such a way that they, along with the lower rectangle form a new rectangle.

   Let this new rectangle be given. Restore the original rectangle using compass and ruler.

   Solution. Let $ABCD$ be the obtained rectangle; O be its center; K, M be the midpoints of its shortest sides AB and CD; L, N be the meets of BC and AD respectively with the circle with diameter KM (fig.8.2). Then $KLMN$ is the desired rectangle. In fact, let P be the projection of M to LN. Since $\angle CLM = \angle OML = \angle MLO$, the triangles MCL and MPL are equal. Thus the bend along ML matches these triangles. Similarly the bend along MN matches triangles MDN and MPN. Finally, since the construction is symmetric wrt point O, the bend along KL and KN matches triangles BKL and AKN with triangle NKL.

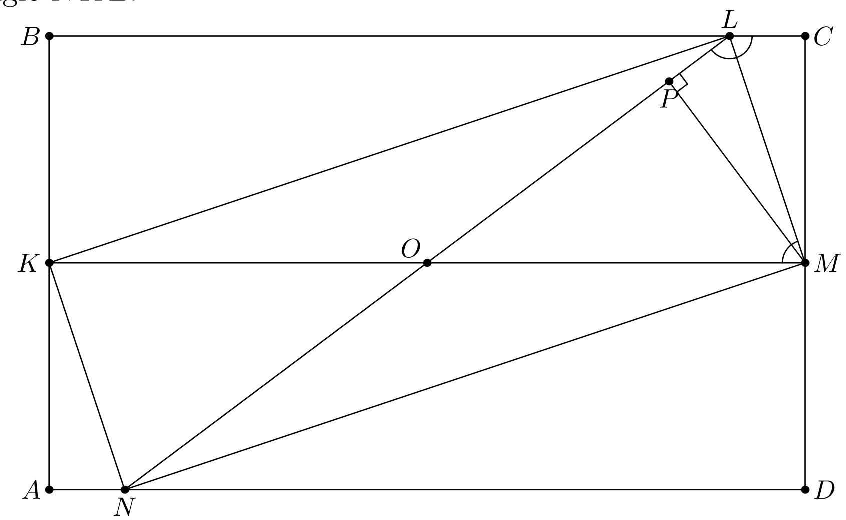


Fig.8.2

3. (A.Myakishev, D.Mavlo) The line passing through vertex A of triangle ABC and parallel to BC meets the circumcircle of ABC for the second time at point A_1. Points B_1 and C_1 are defined similarly. Prove that the perpendiculars from A_1, B_1, C_1 to BC, CA, AB respectively concur.

   First solution. Since A_1 is the reflection of A in the medial perpendicular to BC, the perpendicular from A_1 is the reflection of the altitude from A. Thus by the Thales theorem it passes through the reflection of the orthocenter in the circumcenter. The two remaining perpendiculars also pass through this point.

   Second solution. Let K, L and M be a common points of lines AA_1, BB_1 и CC_1 (fig.8.3). Prove that KC_1 is the altitude of triangle KLM. Since $KBCA$ is a parallelogram, and AC_1CB is an isosceles trapezoid, we have $KA = BC = AC_1$, $\angle KAB = \angle ABC = \angle BAC_1$. Therefore AB is the bisector and the altitude of isosceles triangle KAC_1. Thus $AB \perp KC_1$, and from this $CC_1 \perp KC_1$. Similarly LA_1 and MB_1 are also the altitudes of triangle KLM. Since the altitudes concur we obtain the assertion of the problem.

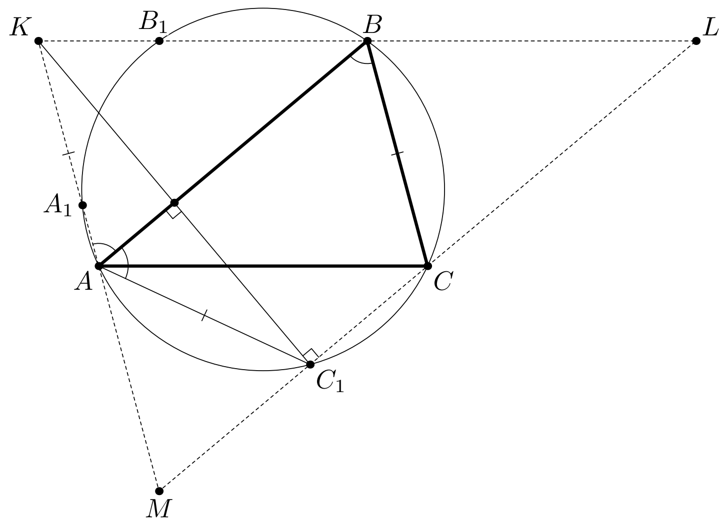


Fig.8.3.

4. (A.Shapovalov) Given the circle of radius 1 and several its chords with the sum of lengths 1. Prove that one can be inscribe a regular hexagon into that circle so that its sides don't intersect those chords.

   Solution. Paint the smallest arcs corresponding to given chords. If we rotate the painted arcs in such a way that the corresponding chords form a polygonal line, then the distance between the ends of it is less than 1, and since a chord with length 1 corresponds to an arc equal to 1/6 of the circle, the total length of painted arcs is less than 1/6 of the circle.

   Now inscribe a regular hexagon into the circle and mark one of its vertices. Rotate the hexagon, and when the marked vertex coincides with a painted point, paint the points corresponding to all remaining vertices. The total length of painted arcs increases at most 6 times, therefore there exists an inscribed regular hexagon with non-painted vertices. Obviously its sides don't intersect the given chords.

VII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Second day. 8th grade. Solutions.

5. (S.Markelov) A line passing through vertex A of regular triangle ABC doesn't intersect segment BC. Points M and N lie on this line, and $AM = AN = AB$ (point B lies inside angle MAC). Prove that the quadrilateral formed by lines AB, AC, BN, CM is cyclic.

   Solution. Since triangle BAN is isosceles, $\angle ANB = \frac{\angle MAB}{2}$ (fig.8.5). Similarly $\angle AMC = \frac{\angle NAC}{2}$. Thus the sum of these angles is equal to $60°$, and the angle between lines BN and CM is equal to $120°$, which yields the assertion of the problem.

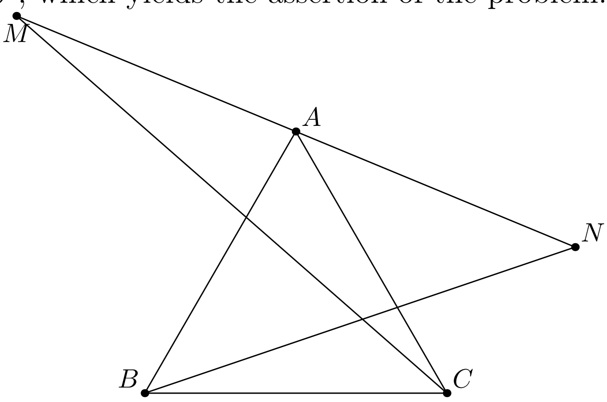


Fig.8.5

6. (D.Prokopenko) Let BB_1 and CC_1 be the altitudes of acute-angled triangle ABC, and A_0 is the midpoint of BC. Lines A_0B_1 and A_0C_1 meet the line passing through A and parallel to BC in points P and Q. Prove that the incenter of triangle PA_0Q lies on the altitude of triangle ABC.

   First solution. Since triangles BCB_1 and BCC_1 are right-angled, their medians B_1A_0, B_1C_0 are equal to the half of hypothenuse: $B_1A_0 = A_0C = A_0B = C_1A_0$. Now $\angle PB_1A = \angle CB_1A_0 = \angle B_1CA_0 = \angle PAC$, thus $PA = PB_1$ (fig.8.6.1). Similarly, $QA = QC_1$. Then the incircle of triangle A_0PQ touches its sides in points A, B_1, C_1, which yields the assertion of the problem.

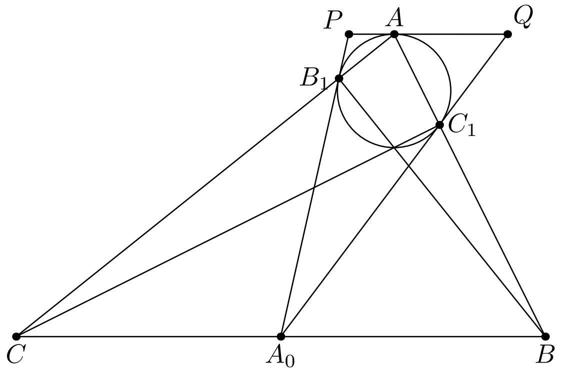


Fig.8.6

   Second solution. Let H be the orthocenter of ABC and O be the midpoint of AH. Then points A_0, B_1, C_1, O lie on the nine-points-circle of ABC, and A_0O is the diameter of this circle. On the other hand, points B_1, C_1 lie on the circle with diameter AH, thus this circle coincides with the incircle of APQ (fig.8.6.2).

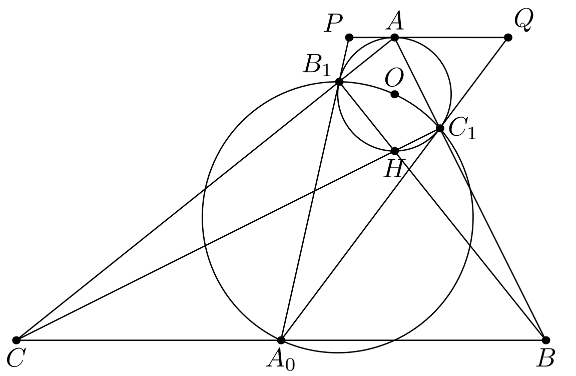


Fig.8.6.2

7. (A.Akopyan) Let a point M not lying on coordinates axes be given. Points Q and P move along Y- and X-axis respectively so that angle PMQ is always right. Find the locus of points symmetric to M wrt PQ.

   Solution. By condition, points P, Q, M and the origin O lie on the circle with diameter PQ. Thus point N symmetric to M wrt PQ also lies on this circle and $\angle PON = \angle PMN = \angle PNM = \angle POM$ (fig.8.7). Then N lies on the line symmetric to OM wrt the coordinates axes. On the other hand, if N is an arbitrary point of this line and P, Q are the common points of coordinates axes with circle OMN, then $\angle PMN = \angle PON = \angle POM = \angle PNM$ and $\angle PMQ = \angle POQ = \angle PNQ = 90°$, thus M and N are symmetric wrt PQ.

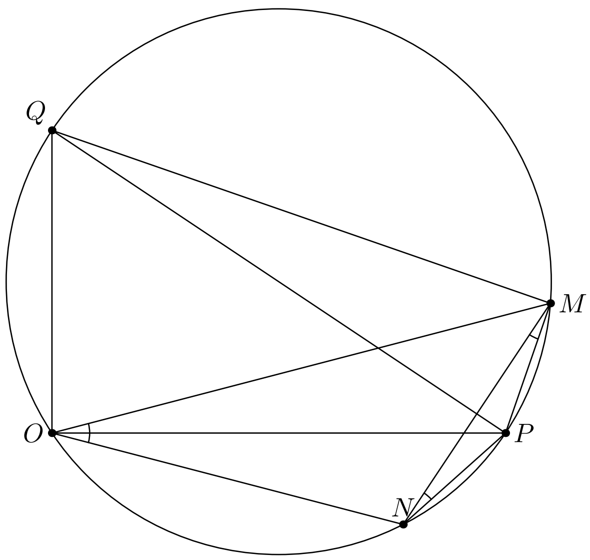


Fig.8.7

8. (A.Zaslavsky) Using only the ruler, divide the side of a square table into n equal parts. All lines drawn must lie on the surface of the table.

   Solution. Firstly bisect the side. Find center O of square $ABCD$ as a common point of its diagonals. Now let point X lie on side BC, Y be a common point of XO and AD, U be a common point of AX and BY, V be a common point of UC and XY (fig.8.8.1). Then line BV bisects the bases of trapezoid $CYUX$. The line passing through O and the midpoint of CY bisects sides AB and CD.

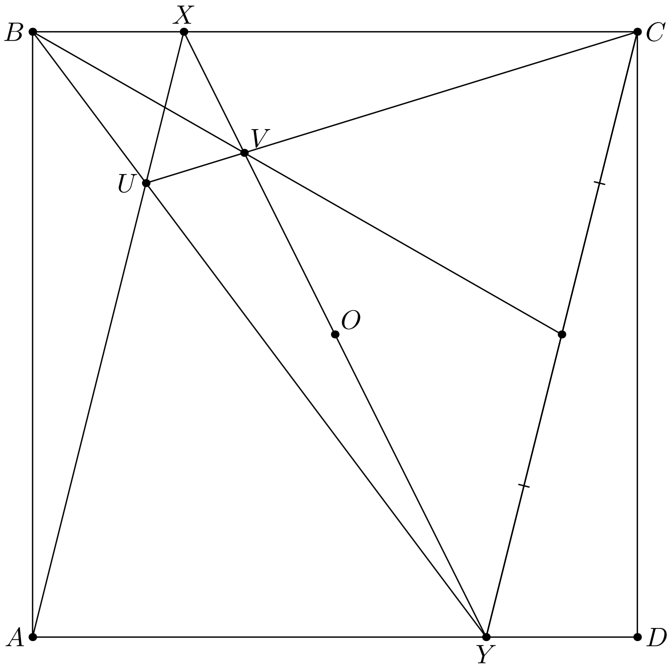


Fig.8.8.1

Now suppose that two opposite sides are divided into k equal parts. Let us demonstrate how to divide it into $k+1$ equal parts. Let $AX_1 = X_1X_2 = \cdots = X_{k-1}B$, $DY_1 = Y_1Y_2 = \cdots Y_{k-1}C$. Then by Thales theorem, lines $AY_1, X_1Y_2, \ldots, X_{k-1}C$ divide diagonal BD into $k+1$ equal parts (fig.8.8.2). Dividing similarly the second diagonal and joining the corresponding points by the lines parallel to BC we divide side AB into $k+1$ equal parts.

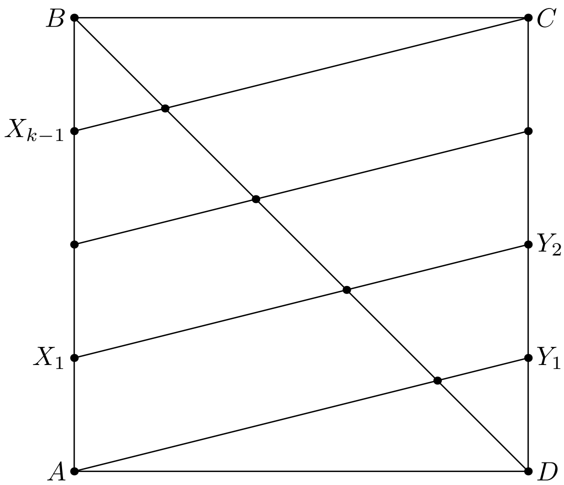


Fig.8.8.2

VII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. First day. 9th grade. Solutions.

1. (M.Kungozhin) Altitudes AA_1 and BB_1 of triangle ABC meet in point H. Line CH meets the semicircle with diameter AB, passing through A_1, B_1, in point D. Segments AD and BB_1 meet in point M, segments BD and AA_1 meet in point N. Prove that the circumcircles of triangles B_1DM and A_1DN touch.

   Solution. The angle between the tangent to circle B_1DM in point D and line AD is equal to angle MB_1D, which in its turn is equal to angle BAD (fig.9.1). Similarly the angle between the tangent to circle A_1DN and line BD is equal to angle ABD. Since $\angle BAD + \angle ADB = 90° = \angle ADB$, these tangents coincide.

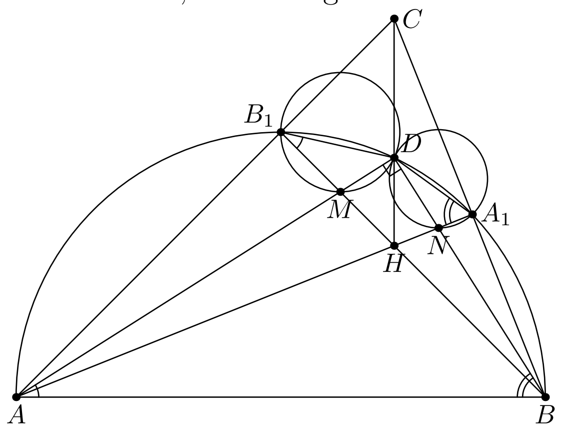


Fig.9.1

2. (D.Cheian) In triangle ABC, $\angle B = 2\angle C$. Points P and Q on the medial perpendicular to CB are such that $\angle CAP = \angle PAQ = \angle QAB = \frac{\angle A}{3}$. Prove that Q is the circumcenter of triangle CPB.

   Solution. Let D be the reflection of A in the medial perpendicular to BC. Then $ABCD$ is the isosceles trapezoid and its diagonal BD is the bisector of angle B. Thus $CD = DA = AB$. Now $\angle DAP = \angle C + \angle A/3 = (\angle A + \angle B + \angle C)/3 = 60°$. Thus triangle ADP is equilateral and $AP = AB$. Since AQ is the bisector of angle PAB, $QP = QB = QC$ (fig.9.2).

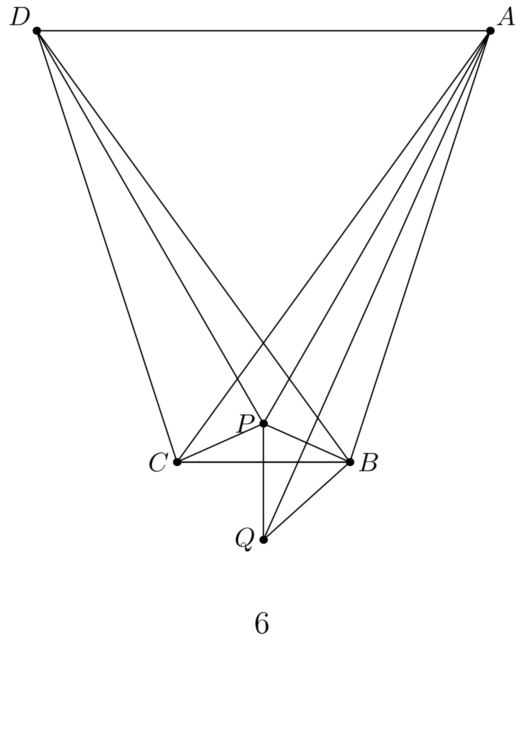


Fig.9.2

3. (A.Karluchenko) Restore the isosceles triangle ABC ($AB = AC$) if the common points I, M, H of bisectors, medians and altitudes respectively are given.

   Solution. Circumcenter O of the triangle lies on the extension of HM beyond point M, and $MO = HM/2$. Now BI, CI are the bisectors of angles OBH, OCH ($\angle CBH = \angle ABO = \pi/2 - \angle C$). Thus $BO/BH = CO/CH = IO/IH$, i.e. points B, C lie on the Apollonius circle of points O and H passing through I. But the circumcenter of triangle BIC lies on the circumcircle of ABC. So we obtain the following construction.

   Construct point O and the Apollonius circle. Now construct the circle with center O passing through the center of constructed circle. These two circles meet in points B, C, and line OH again meets the circle with center O in point A.

4. (A.Zaslavsky) Quadrilateral $ABCD$ is inscribed into a circle with center O. The bisectors of its angles form a cyclic quadrilateral with circumcenter I, and its external bisectors form a cyclic quadrilateral with circumcenter J. Prove that O is the midpoint of IJ.

   Solution. Let the bisectors of angles A and B, B and C, C and D, D and A meet in points K, L, M, N respectively (fig.9.4). Then line KM bisects the angle formed by lines AD and BC. If this angle is equal to ϕ, then by external angle theorem we obtain that $\angle LKM = \angle B/2 - \phi/2 = (\pi - \angle A)/2 = \angle C/2$ and thus $\angle LIM = \angle C$. On the other hand, the perpendiculars from L and M to BC and CD respectively form the angles with ML equal to $(\pi - \angle C)/2$, i.e. the triangle formed by these perpendiculars and ML is isosceles and the angle at its vertex is equal to C. Thus the vertex of this triangle coincides with I. So the perpendiculars from the vertices of $KLMN$ to the corresponding sidelines of $ABCD$ pass through I. Similarly the perpendiculars from the vertices of triangle formed by external bisectors pass through J.

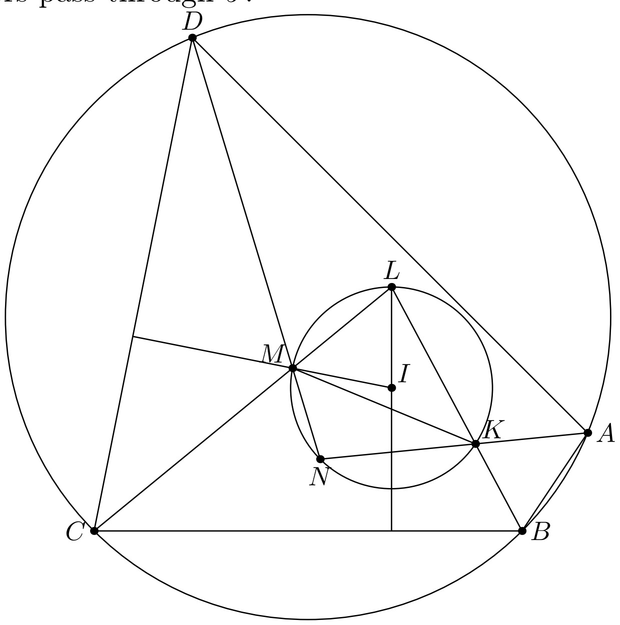


Fig.9.4

   Now let K' be the common point of external bisectors of angles A and B. Since quadrilateral $AKBK'$ is inscribed into the circle with diameter KK', the projections of K and K' to AB are symmetric wrt the midpoint of AB. From this and the above assertion, the projections of I and J to each side of $ABCD$ are symmetric wrt the midpoint of this side, and this is equivalent to the sought assertion.

Note. The similar property of a triangle is well-known: the circumcenter is the midpoint of the segment between the incenter and the circumcenter of the triangle formed by external bisectors.

VII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Second day. 9th grade. Solutions.

5. (B.Frenkin) It is possible to compose a triangle from the altitudes of a given triangle. Can we conclude that it is possible to compose a triangle from its bisectors?

   Solution. No. Consider a triangle with two sides equal to 2 and 3 and increase the angle between these sides. When the angle approaches to 180° the ratio of altitudes of triangle approaches to $1/2 : 1/3 : 1/5$, thus for any value of the angle it is possible to compose the triangle from the altitudes. On the other hand, the smallest bisector approaches to zero, and two remaining bisectors approach to different values. Thus for great values of the angle it is impossible to compose the triangle from the bisectors.

   Let us present the exact estimates. Firstly note that if a and b are two sides of a triangle, C is the angle between these sides, and l_c is the bisector of this angle, then the area of the triangle is equal to $S = ab\sin C/2 = (a+b)l_c \sin\frac{C}{2}/2$, thus $l_c = 2ab\cos\frac{C}{2}/(a+b)$. The lengths of bisectors l_a, l_b are determined similarly.

   Now let $a = 2$, $b = 3$. Then $\cos\frac{A}{2} > \cos\frac{B}{2}$. Thus

   $$l_a - l_b > 2c\cos\frac{A}{2}\left(\frac{b}{b+c} - \frac{a}{a+c}\right) = \frac{2c^2\cos\frac{A}{2}}{(c+2)(c+3)}.$$

   Take angle C sufficiently great such that $c > 4$, $\cos\frac{A}{2} > 0,9$, $\cos\frac{C}{2} < 0,1$. Then $l_a - l_b > l_c$ and it is impossible to compose a triangle from the bisectors. On the other hand, we have $h_b/h_a = 2/3$, $2/5 < h_c/h_a < 1/2$. Thus $h_b + h_c > h_a > h_b > h_c$ and it is possible to compose a triangle from the altitudes.

6. (P.Dolgirev) In triangle ABC AA_0 and BB_0 are medians, AA_1 and BB_1 are altitudes. The circumcircles of triangles CA_0B_0 and CA_1B_1 meet again in point M_c. Points M_a, M_b are defined similarly. Prove that points M_a, M_b, M_c are collinear and lines AM_a, BM_b, CM_c are parallel.

   Solution. Let O and H be the circumcenter and the orthocenter of triangle ABC. Since $\angle CA_0O = \angle CB_0O = \angle CA_1H = \angle CB_1H = 90°$, CO and CH are the diameters of circles CA_0B_0 and CA_1B_1 respectively. Thus the projection of point C to line OH lies on both circles, i.e. coincides with M_c (fig.9.6). This yields the assertion of the problem

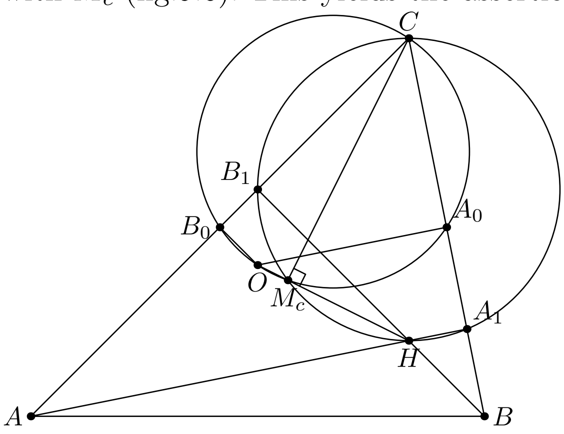


Fig.9.6

7. (I.Bogdanov) Circles ω and Ω are inscribed into the same angle. Line ℓ meets the sides of angles, ω and Ω in points A and F, B and C, D and E respectively (the order of points on the line is A, B, C, D, E, F). It is known that $BC = DE$. Prove that $AB = EF$.

   First solution. Let one side of the angle touch ω and Ω in points X_1, Y_1, and the second side touch them in points X_2, Y_2; U, V are the common points of X_1X_2 and Y_1Y_2 with AF. The midpoint of CD lies on the radical axis of the circles, i.e. the medial line of trapezoid $X_1Y_1Y_2X_2$, thus $BU = EV$ and $CU = DV$ (fig.9.7). This yields that $X_1U \cdot X_2U = Y_1V \cdot Y_2V$. Hence $FY_2/FX_2 = Y_2V/X_2U = X_1U/Y_1V = AX_1/AY_1$, i.e. $AX_1 = FY_2$. Now from $AB \cdot AC = AX_1^2 = FY_2^2 = FE \cdot FD$ we obtain the assertion of the problem.

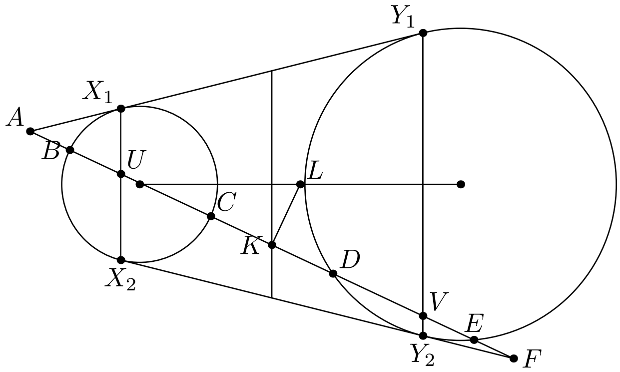


Fig.9.7

   Second solution. Prove that there exists exactly one line passing through a fixed point A on a side of the angle which satisfies the condition of the problem. In fact, the distances from the midpoint K of segment CD to the projections of the centers of the circles to the sought line are equal, thus K coincides with the projection of the midpoint L of the segment between the centers. Hence K is the common point of the circle with diameter AL and the radical axis, distinct from the midpoint of segment X_1Y_1. On the other hand, if F is a point such that $AX_1 = Y_2F$ then $AB \cdot AC = FE \cdot FD$ and $AD \cdot AE = FC \cdot FB$, thus AF is the sought line.

8. (B.Frenkin) A convex n-gon P, where $n > 3$, is dissected into equal triangles by diagonals non-intersecting inside it. Which values of n are possible, if P is circumscribed?

   Solution. Let us prove that $n = 4$.

   Lemma. Let a convex polygon be dissected into equal triangles by non-intersecting diagonals. Then each triangle of the dissection has at least one side which is a side (not a diagonal) of the polygon

   Proof. Let a triangle of the dissection have angles $\alpha \le \beta \le \gamma$; A, B, C be the corresponding vertices, AC and BC be the diagonals of the polygon. There are at least two another angles of dividing triangles adjacent to C. If one of them is greater than α, then the sum of angles adjacent to C is not less than $\gamma + \beta + \alpha = \pi$, but this sum can't be greater than an angle of convex polygon, a contradiction. Thus all angles in vertex C, except $\angle ACB$ are equal to α and $\alpha < \beta$.

   Consider the second triangle adjacent to BC. Since it is equal to $\triangle ABC$, its angle opposite to BC is equal to α. But angle C also is equal to α although it must be β or γ, a contradiction. Lemma is proved.

Now we turn to the solution of the problem.

Since the sum of angles of P is equal to $\pi(n-2)$ the number of dividing triangles is equal to $n-2$. By lemma, each of these triangles has at least one side coinciding with a side of P. Hence there are two triangles having two sides coinciding with sides of P.

Let KLM be one of these triangles, KL and LM be the sides of P. KM is the side of another dissection triangle KMN. One of its sides (for example KN) is a side of P. Since the triangles are equal, $\angle NKM$ is equal to $\angle LKM$, or to $\angle KML$. In the first case KM bisects the angle P and so passes through the incenter I. In the second case $KN \parallel LM$. Then I lies on the common perpendicular to these two segments, thus it lies inside parallelogram $KLMN$, i.e. belongs to at least one of triangles KLM, KMN.

Let $K'L'M'$ be the second triangle with two sides coinciding with sides of P. Similarly we obtain that I lies inside this or adjacent dissection triangle. If I lies inside one of triangles KLM, $K'L'M'$, then they are adjacent and $n = 4$. In the opposite case I lies inside triangle KMN which is adjacent to both these triangles. Then MN is a side of $\triangle K'L'M'$; let $M = M'$, $N = K'$. As above we obtain that $LM \parallel KN \parallel L'M$. But then sides LM and $L'M$ of the convex polygon lie on the same line, a contradiction.

$>$From the above argument we see that a convex quadrilateral satisfies the condition of the problem iff it is symmetric wrt one of its diagonals.

VII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. First day. 10th grade. Solutions.

1. (M.Rozhkova) In triangle ABC the midpoints of sides AC, BC, vertex C and the centroid lie on the same circle. Prove that this circle touches the circle passing through A, B and the orthocenter of triangle ABC.

   Solution. Let point C' be the reflection of C in the midpoint of AB. Then points A, B, C' and the orthocenter of ABC lie on the same circle. On the other hand, if A_0, B_0 are the midpoints of BC, AC, then triangle A_0B_0C is homothetic to triangle ABC' wrt cenroid M of ABC with coefficient $-1/2$. Thus the circumcircles of these triangles touch in M (fig.10.1).

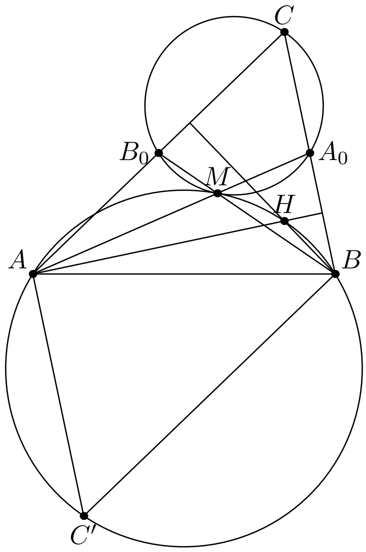
   

   Fig.10.1

2. (L.Emelyanov) Quadrilateral $ABCD$ is circumscribed. Its incircle touches sides AB, BC, CD, DA in points K, L, M, N respectively. Points A', B', C', D' are the midpoints of segments LM, MN, NK, KL. Prove that the quadrilateral formed by lines AA', BB', CC', DD' is cyclic.

   Solution. Let us begin with the assertion which follows from a simple calculation of angles.

   Lemma. Points A, B, C, D lie on the same circle iff the bisectors of angles formed by lines AB and CD are parallel to the bisectors of angles formed by lines AD and BC.

   In fact, consider the case when $ABCD$ is a convex quadrilateral, rays BA and DC meet in point E, rays DA and BC meet in point F. Then the angles between the bisectors of angles BED and BFD are equal to half-sums of opposite angles of the quadrilateral. This clearly yields the assertion of lemma. Another cases can be considered similarly.

   Now let us turn to the solution of the problem. Let I be the incenter of $ABCD$, r be the radius of its incircle. Then $IC' \cdot IA = r^2 = IA' \cdot IC$, i.e. points A, C, A', C' lie on the circle. By lemma, the bisectors of angles between AA' and CC' are parallel to the bisectors of angles between IA and IC, and hence to the bisectors of the angles between perpendicular lines KN and LM. Similarly the bisectors of the angles between BB' and

DD' are parallel to the bisectors of the angles between KL and MN. Using again the lemma we obtain the assertion of the problem.

3. (A.Akopyan) Given two tetrahedrons $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$. Consider six pairs of edges A_iA_j and B_kB_l, where (i, j, k, l) is a transposition of numbers $(1, 2, 3, 4)$ (for example A_1A_2 and B_3B_4). It is known that for all but one such pairs the edges are perpendicular. Prove that the edges in the remaining pair also are perpendicular.

   Solution. Let us prove firstly the following lemma.

   Lemma. Edges A_1A_2 and B_3B_4 are perpendicular iff the perpendiculars from points A_1, A_2 to planes $B_2B_3B_4$ and $B_1B_3B_4$ respectively intersect.

   Proof of Lemma. Let $A_1A_2 \perp B_3B_4$. Then there exists a plane passing through A_1A_2 and perpendicular to B_3B_4. The perpendiculars from the condition of Lemma lie on this plane and hence intersect. Conversely, if the perpendiculars intersect then the plane containing them is perpendicular to B_3B_4 and passes through A_1A_2.

   Now let $A_1A_2 \perp B_3B_4$, $A_1A_3 \perp B_2B_4$, $A_2A_3 \perp B_1B_4$. Then any two of three perpendiculars from A_1, A_2, A_3 to the corresponding faces of $B_1B_2B_3B_4$ intersect. Since these three perpendicular aren't complanar, this yields that they have a common point. Thus if the condition of the problem is true then all four perpendiculars from the vertices of one tetrahedron to the corresponding faces of the other have a common point and the edges in the sixth pair are perpendicular.

4. (V.Mokin) Point D lies on the side AB of triangle ABC. The circle inscribed in angle ADC touches internally the circumcircle of triangle ACD. Another circle inscribed in angle BDC touches internally the circumcircle of triangle BCD. These two circles touch segment CD in the same point X. Prove that the perpendicular from X to AB passes through the incenter of triangle ABC.

   Solution. Firstly prove next lemma.

   Lemma. Let a circle touch sides AC, BC of triangle ABC in points U, V and touch internally its circumcircle in point T. Then line UV passes through the incenter I of triangle ABC.

   Proof of Lemma. Let lines TU, TV intersect the circumcircle again in points X, Y. Since circles ABC and TUV are homothetic with center T, points X, Y are the midpoints of arcs AC, BC, i.e. lines AY and BX meet in point I (fig.10.4.1). Thus the assertion of the lemma follows from Pascal theorem applied to hexagon $AYTXBC$.

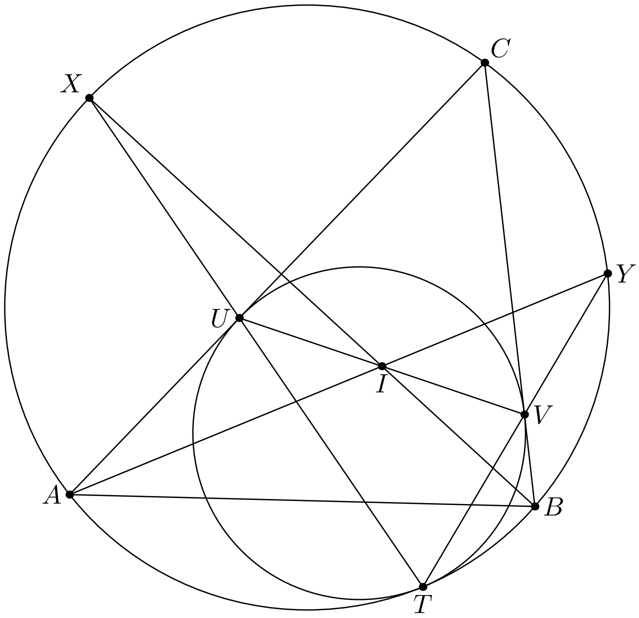


Fig.10.4.1

$>$From lemma and the condition of the problem we obtain that DI_1XI_2, where I_1, I_2 are the incenters of triangles ACD, BCD, is the rectangle (fig.10.4.2.). Let Y, C_1, C_2 be the projections of points X, I_1, I_2 to AB. Then $BY - AY = BC_2 + C_2Y - AC_1 - C_1Y = (BC_2 - DC_2) - (AC_1 - DC_1) = (BC - CD) - (AC - CD) = BC - AC$. Thus Y is the touching point of AB with the incircle.

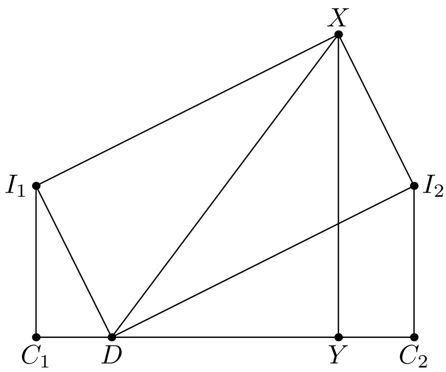


Fig.10.4.2

VII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Second day. 10th grade. Solutions.

5. (A.Blinkov) The touching point of the excircle with the side of a triangle and the base of the altitude to this side are symmetric wrt the base of the corresponding bisector. Prove that this side is equal to one third of the perimeter.

   Solution. By condition the radius r_c of the excircle touching side AB of triangle ABC is equal to altitude h_c. Since the square of the triangle equals $S = (p-c)r_c = ch_c/2$, we have $c = 2(p-c) = 2p/3$.

6. (M.Rozhkova) Prove that for any nonisosceles triangle $l_1^2 > \sqrt{3}S > l_2^2$, where l_1, l_2 are the greatest and the smallest bisectors of the triangle and S is its area.

   Solution. Let $a > b > c$ be the sidelengths of the triangle. Then l_2 is the bisector of angle A and $S = bc\sin A/2 = (b+c)l_2 \sin\frac{A}{2}/2$. Thus we can write down the right inequality as $\sqrt{3}(b+c)\sin\frac{A}{2}/2 > 2bc\cos\frac{A}{2}/(b+c)$ or $\sqrt{3}\operatorname{tg}\frac{A}{2} > 4bc/(b+c)^2$. But $\pi/6 < A/2 < \pi/2$, thus the left part is greater than 1, and the right part is less than 1 by Cauchy inequality.

   Since $C < \pi/3$, we have $\sqrt{3}S < 3ab/4$. On the other hand $l_1^2 = 4a^2b^2\cos^2\frac{C}{2}/(a+b)^2 = 2a^2b^2(1+\cos C)/(a+b)^2$. Since $b > c$, we have $\cos C > a/2b$ and $l_1^2 > a^2b(a+2b)/(a+b)^2$. Thus the left inequality follows from $a(a+2b)/(a+b)^2 = 1 - b^2/(a+b)^2 > 3/4$.

7. (G.Feldman) Point O is the circumcenter of acute-angled triangle ABC, points A_1, B_1, C_1 are the bases of its altitudes. Points A', B', C' lying on lines OA_1, OB_1, OC_1 respectively are such that quadrilaterals $AOBC'$, $BOCA'$, $COAB'$ are cyclic. Prove that the circumcircles of triangles AA_1A', BB_1B', CC_1C' have a common point.

   Solution. Let H be the orthocenter of ABC. Then $AH \cdot HA_1 = BH \cdot HB_1 = CH \cdot CH_1$, i.e the degrees of point H wrt circles AA_1A', BB_1B', CC_1C' are equal and H lies inside these circles. On the other hand $\angle BC'O = \angle BAO = \angle OBC_1$, i.e. triangles $OC'B$ and OBC_1 are similar and $OC_1 \cdot OC' = OB_0^2$ (fig.10.7). Thus the degrees of point O wrt all three circles are also equal, so these circles meet in two points lying on line OH.

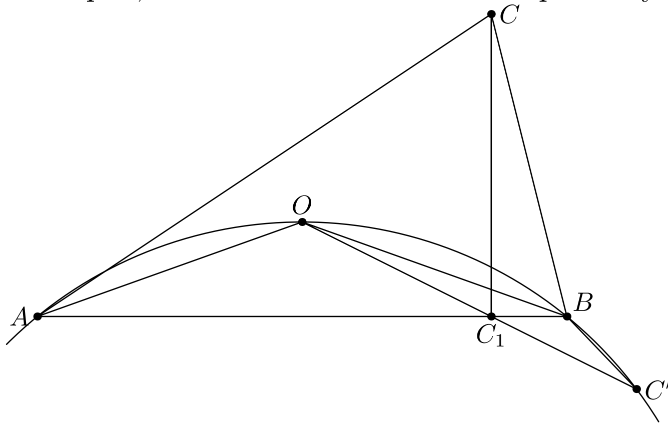


Fig.10.7

8. (S.Tokarev) Given a sheet of tin 6×6. It is allowed to bend it and to cut it but in such a way that it doesn't fall to pieces. How to make a cube with edge 2, divided by partitions into unit cubes?

Solution. The sought development is presented on Fig.10.8. Bold lines describe the cuts, thin and dotted lines describe the bends up and down. The central 2×2 square corresponds to the horizontal partition of the cube.

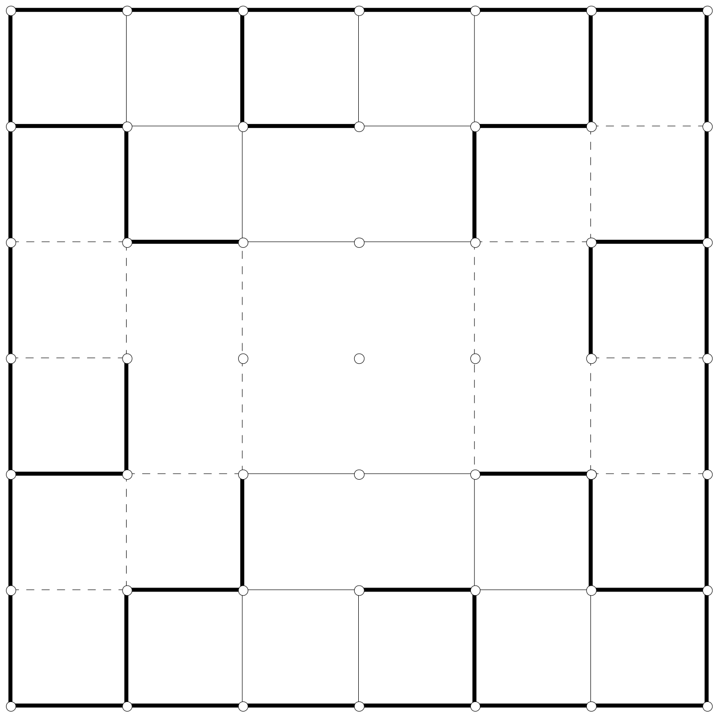


Fig.10.8.

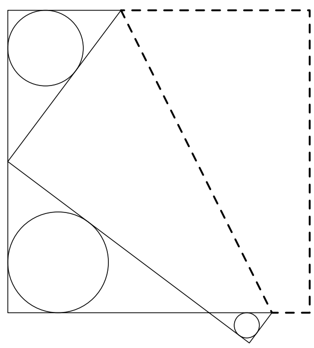
VIII Geometrical Olympiad in honour of I.F.Sharygin
Final round. First day. 8 form

Ratmino, 2012, July 31

1. Let M be the midpoint of the base AC of an acute-angled isosceles triangle ABC. Let N be the reflection of M in BC. The line parallel to AC and passing through N meets AB at point K. Determine the value of $\angle AKC$.

2. In a triangle ABC the bisectors BB' and CC' are drawn. After that, the whole picture except the points A, B', and C' is erased. Restore the triangle using a compass and a ruler.

3. A paper square was bent by a line in such way that one vertex came to a side not containing this vertex. Three circles are inscribed into three obtained triangles (see Figure). Prove that one of their radii is equal to the sum of the two remaining ones.



4. Let ABC be an isosceles triangle with $\angle B = 120°$. Points P and Q are chosen on the prolongations of segments AB and CB beyond point B so that the lines AQ and CP are perpendicular to each other. Prove that $\angle PQB = 2\angle PCQ$.

VIII Geometrical Olympiad in honour of I.F.Sharygin
Final round. First day. 9 form.

Ratmino, 2012, July 31

1. The altitudes AA_1 and BB_1 of an acute-angled triangle ABC meet at point O. Let A_1A_2 and B_1B_2 be the altitudes of triangles OBA_1 and OAB_1 respectively. Prove that A_2B_2 is parallel to AB.

2. Three parallel lines passing through the vertices A, B, and C of triangle ABC meet its circumcircle again at points A_1, B_1, and C_1 respectively. Points A_2, B_2, and C_2 are the reflections of points A_1, B_1, and C_1 in BC, CA, and AB respectively. Prove that the lines AA_2, BB_2, CC_2 are concurrent.

3. In triangle ABC, the bisector CL was drawn. The incircles of triangles CAL and CBL touch AB at points M and N respectively. Points M and N are marked on the picture, and then the whole picture except the points A, L, M, and N is erased. Restore the triangle using a compass and a ruler.

4. Determine all integer $n > 3$ for which a regular n-gon can be divided into equal triangles by several (possibly intersecting) diagonals.

VIII Geometrical Olympiad in honour of I.F.Sharygin
Final round. First day. 10 form.

Ratmino, 2012, July 31

1. Determine all integer n such that a surface of an $n \times n \times n$ grid cube can be pasted in one layer by paper 1×2 rectangles so that each rectangle has exactly five neighbors (by a line segment).

2. We say that a point inside a triangle is *good* if the lengths of the cevians passing through this point are inversely proportional to the respective side lengths. Find all the triangles for which the number of good points is maximal.

3. Let M and I be the centroid and the incenter of a scalene triangle ABC, and let r be its inradius. Prove that $MI = r/3$ if and only if MI is perpendicular to one of the sides of the triangle.

4. Consider a square. Find the locus of midpoints of the hypothenuses of right-angled triangles with the vertices lying on three different sides of the square and not coinciding with its vertices.

VIII Geometrical Olympiad in honour of I.F.Sharygin
Final round. Second day. 8 form.

Ratmino, 2012, August 1

5. Do there exist a convex quadrilateral and a point P inside it such that the sum of distances from P to the vertices of the quadrilateral is greater than its perimeter?

6. Let ω be the circumcircle of triangle ABC. A point B_1 is chosen on the prolongation of side AB beyond point B so that $AB_1 = AC$. The angle bisector of $\angle BAC$ meets ω again at point W. Prove that the orthocenter of triangle AWB_1 lies on ω.

7. The altitudes AA_1 and CC_1 of an acute-angled triangle ABC meet at point H. Point Q is the reflection of the midpoint of AC in line AA_1; point P is the midpoint of segment A_1C_1. Prove that $\angle QPH = 90°$.

8. A square is divided into several (greater than one) convex polygons with mutually different numbers of sides. Prove that one of these polygons is a triangle.

VIII Geometrical Olympiad in honour of I.F.Sharygin
Final round. Second day. 9 form.

Ratmino, 2012, August 1

5. Let ABC be an isosceles right-angled triangle. Point D is chosen on the prolongation of the hypothenuse AB beyond point A so that $AB = 2AD$. Points M and N on side AC satisfy the relation $AM = NC$. Point K is chosen on the prolongation of CB beyond point B so that $CN = BK$. Determine the angle between lines NK and DM.

6. Let ABC be an isosceles triangle with $BC = a$ and $AB = AC = b$. Segment AC is the base of an isosceles triangle ADC with $AD = DC = a$ such that points D and B share the opposite sides of AC. Let CM and CN be the bisectors in triangles ABC and ADC respectively. Determine the circumradius of triangle CMN.

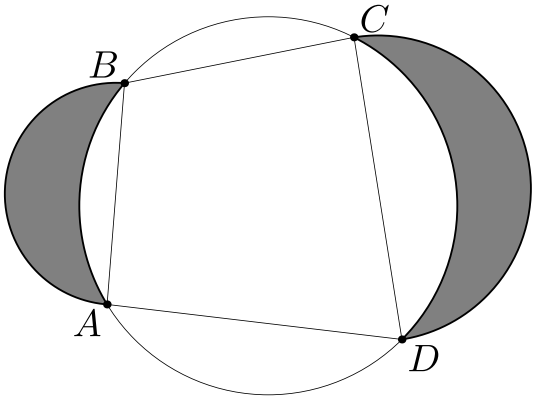
7. A convex pentagon P is divided by all its diagonals into ten triangles and one smaller pentagon P'. Let N be the sum of areas of five triangles adjacent to the sides of P decreased by the area of P'. The same operations are performed with the pentagon P'; let N' be the similar difference calculated for this pentagon. Prove that $N > N'$.

8. Let AH be an altitude of an acute-angled triangle ABC. Points K and L are the projections of H onto sides AB and AC. The circumcircle of ABC meets line KL at points P and Q, and meets line AH at points A and T. Prove that H is the incenter of triangle PQT.

VIII Geometrical Olympiad in honour of I.F.Sharygin
Final round. Second day. 10 form.

Ratmino, 2012, August 1

5. A quadrilateral $ABCD$ with perpendicular diagonals is inscribed into a circle ω. Two arcs α and β with diameters AB and CD lie outside ω. Consider two crescents formed by the circle ω and the arcs α and β (see Figure). Prove that the maximal radii of the circles inscribed into these crescents are equal.



6. Consider a tetrahedron $ABCD$. A point X is chosen outside the tetrahedron so that segment XD intersects face ABC in its interior point. Let A', B', and C' be the projections of D onto the planes XBC, XCA, and XAB respectively. Prove that $A'B' + B'C' + C'A' \leq DA + DB + DC$.

7. Consider a triangle ABC. The tangent line to its circumcircle at point C meets line AB at point D. The tangent lines to the circumcircle of triangle ACD at points A and C meet at point K. Prove that line DK bisects segment BC.

8. A point M lies on the side BC of square $ABCD$. Let X, Y, and Z be the incenters of triangles ABM, CMD, and AMD respectively. Let H_x, H_y, and H_z be the orthocenters of triangles AXB, CYD, and AZD. Prove that H_x, H_y, and H_z are collinear.

VIII Geometrical Olympiad in honour of I.F.Sharygin

Final round. First day. 8th form. Solutions

1. (A.Blinkov) Let M be the midpoint of the base AC of an acute-angled isosceles triangle ABC. Let N be the reflection of M in BC. The line parallel to AC and passing through N meets AB at point K. Determine the value of $\angle AKC$.

   Answer. $90°$.

   Solution. Let L be the common point of NK and BC (see Fig. 8.1). By means of the symmetry in BC we obtain $AM = MC = CN$ and $\angle MCB = \angle NCB$. Next, since $LN \parallel AC$, we have $\angle CNL = \angle LCM$, hence the triangle CNL is isosceles, and $LN = CN = AM$. Thus, the segments AM and LN are parallel and equal, hence the quadrilateral $ALNM$ is a parallelogram, and $AL \parallel MN \perp LC$. Finally, by the symmetry in BM we get $\angle AKC = \angle ALC = 90°$.

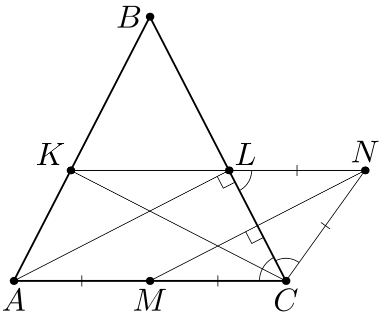


Figure 8.1

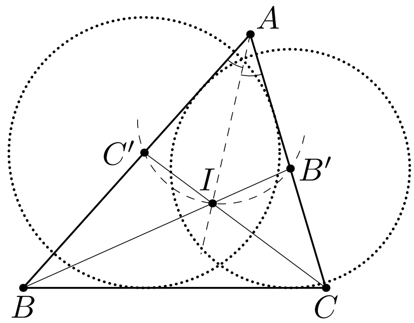


Figure 8.2

2. (A.Karlyuchenko) In a triangle ABC the bisectors BB' and CC' are drawn. After that, the whole picture except the points A, B', and C' is erased. Restore the triangle using a compass and a ruler.

   First solution. Let I be the incenter of triangle ABC. Then $\angle B'IC' = 180° - (\angle IBC + \angle ICB) = 90° + \frac{1}{2}\angle B'AC'$. So, denoting by O the center of the circumcircle ω of triangle BCI, we get $\angle B'OC' = 180° - \angle B'AC'$. Hence one can successively reconstruct points O and I (the latter is the meeting point of the smaller arc $B'C'$ of ω with the bisector of $\angle B'AC'$, see Fig. 8.2). Finally, the points B and C can be reconstructed and the meeting points of $B'I$, AC' and $C'I$, AB' respectively.

   Second solution. Since BB' is the bisector of $\angle B$, the point B' is equidistant from the lines BC and AB. Hence the circle with center B' tangent to AC' is also tangent to BC. Analogously, line BC is tangent to the circle with center C' touching AB' (see Fig. 8.2). So, to reconstruct the points B and C, it is sufficient to draw a common outer tangent to these two circles (sharing different sides of $B'C'$ with A) and to find its common points with AB' and AC'.

3. (L.Steingarts) A paper square was bent by a line in such way that one vertex came to a side not containing this vertex. Three circles are inscribed into three obtained triangles (see Figure). Prove that one of their radii is equal to the sum of the two remaining ones.

   Solution. Assume that square $ABCD$ is bent by line XY; denote the resulting points as in Fig. 8.3.2. Recall that in any right triangle, the incenter, the points of tangency of the incircle with the legs, and the vertex of the right angle form a square; hence the inradius is equal to the segment of the tangent line from that vertex. Hence the indiameters of triangles UDX, UAP, and PVY are $d_1 = UD + DX - XU$, $d_2 = UA + AP - UP$, and $d_3 = PV + VY - PY$ respectively. Denote $a = AB$ and notice that $UX = XC$ and $VY = YB$; therefore we obtain

$$d_1+d_3-d_2 = DU+(a-CX)-CX+PV+BY-PY-(a-DU)-(a-PY-BY)+(a-PV) = \\ = 2(DU + BY - CX).$$

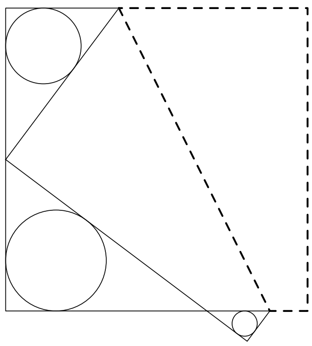


Figure 8.3.1

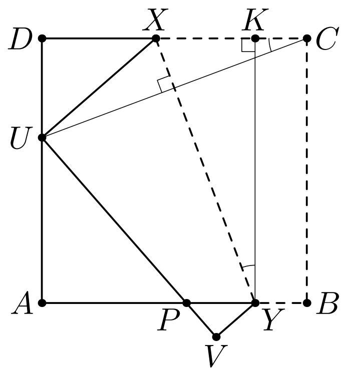


Figure 8.3.2

Let K be the projection of Y onto CD. The points C and U are symmetrical with respect to XY, hence $XY \perp CU$, and $\angle DCU = \angle KYX$. Moreover, $KY = CD = a$. Consequently, the right triangles CDU and YKX are congruent, hence $DU = KX = CX - CK = CX - BY$. This means exactly that $d_1 + d_3 - 2_d = 0$.

Remark. In the first part of the solution, one may also argue as follows. The right triangles DXY, VYP, and AUP are similar; hence the ratios of their inradii are the same as the ratios of their respective legs. Hence it suffices to prove the equality $DX + VY = AU$, or, equivalently, $DX + CK = a - DU$. The last relation follows from the relation $DU = KX$ which is proved in the second part of the Solution.

4. (A.Akopyan, D.Shvetsov) Let ABC be an isosceles triangle with $\angle B = 120°$. Points P and Q are chosen on the prolongations of segments AB and CB beyond point B so that the rays AQ and CP intersect and are perpendicular to each other. Prove that $\angle PQB = 2\angle PCQ$.

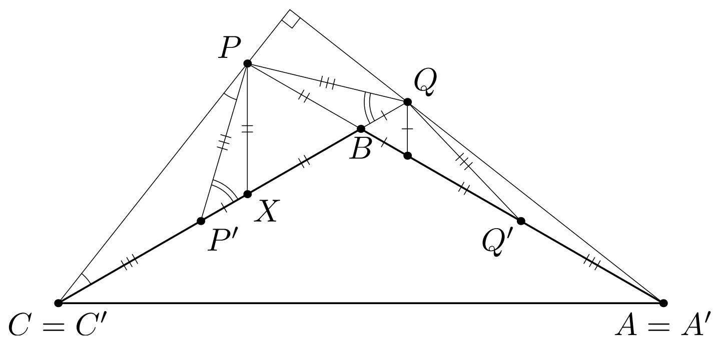


Figure 8.4

Solution. Let us choose points X and Q' on ray BC so that $BX = BP$ and $BQ' = BP + BQ$. Then triangle BPX is isosceles, and one of its angles is equal $60°$; hence it is equilateral, so $PX = BP$ and $\angle PXQ' = 120°$. Then triangles PBQ and PXQ' are congruent by SAS, hence $PQ' = PQ$ and $\angle PQ'B = \angle PQB$. Analogously, choosing point P' on ray BA so that $BP' = BP + BQ$, we get $QP' = QP$ and $\angle QP'B = \angle QPB$ (see Fig. 8.4).

Now let us prolongate segments BP' and BQ' beyond the points P' and Q' by the length $Q'A' = P'C' = PQ$. Then we have $BA' = BP' + P'A' = BP + BQ + PQ = BQ' + Q'C' = BC'$. Now, triangles $QP'A'$ and $PQ'C'$ are isosceles, so $\angle P'A'Q + \angle Q'C'P = \frac{1}{2}(\angle QP'B + \angle PQ'B) = \frac{1}{2}(\angle BPQ + \angle BQP) = 30°$. Hence the angle formed by the lines QA' and PC' is equal to $180° - (\angle P'A'Q + \angle Q'C'P + \angle BA'C' + \angle BC'A') = 90°$. But, if $BA' = BC' < BA$, then this angle should be less than $90°$, and if $BA' > BA$, then it should be greater than $90°$, So we obtain $A' = A$, $C' = C$, and $\angle PQB = \angle PQ'B = 2\angle PCQ'$, QED.

VIII Geometrical Olympiad in honour of I.F.Sharygin

Final round. Second day. 8th form. Solutions

5. (A.Akopyan) Do there exist a convex quadrilateral and a point P inside it such that the sum of distances from P to the vertices of the quadrilateral is greater than its perimeter?

   Answer. Yes.

   Solution. Consider a quadrilateral $ABCD$ such that $AD = BD = CD = x$, $AB = BC = y < x/4$, and a point P on the diagonal BD such that $PD = y$ (see Fig. 8.5). Then we have $PB + PD = BD = x$ and $PA = PC > AD - PD = x - y$, hence $PA + PB + PC + PD > 3x - 2y > 2x + 2y = AB + BC + CD + DA$.

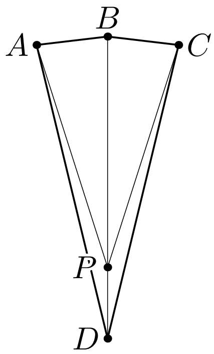


Figure 8.5

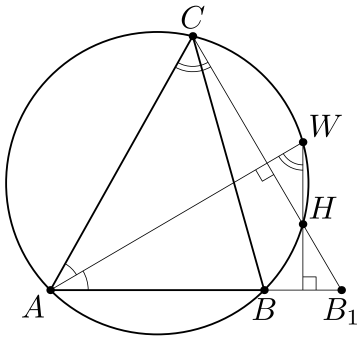


Figure 8.6

6. (A.Tumanyan) Let ω be the circumcircle of triangle ABC. A point B_1 is chosen on the prolongation of side AB beyond point B so that $AB_1 = AC$. The angle bisector of $\angle BAC$ meets ω again at point W. Prove that the orthocenter of triangle AWB_1 lies on ω.

   Solution. Let H be the second point of intersection of line CB_1 with ω. Since AW is an angle bisector in the isosceles triangle AB_1C, we have $B_1H \perp AW$. If the points C and W share a common side of AH, then $\angle AWH = \angle ACH = 90^\circ - \angle CAW = 90^\circ - \angle WAB$, which implies $WH \perp AB_1$ (see Fig. 8.6). If they share different sides, then $\angle AWH = 180^\circ - \angle ACH = 90^\circ + \angle WAB$, which again follows $WH \perp AB_1$. Finally, if these points coincide, then triangle AWB_1 is right-angled, and $H = W$ is its orthocenter.

   Thus, in any case point H lies on two altitudes of triangle AWB_1; hence H is its orthocenter.

7. (D.Shvetsov) The altitudes AA_1 and CC_1 of an acute-angled triangle ABC meet at point H. Point Q is the reflection of the midpoint of AC in line AA_1; point P is the midpoint of segment A_1C_1. Prove that $\angle QPH = 90^\circ$.

   First solution. Let K be the midpoint of AC. Since $KQ \parallel BC$, the line KQ bisects the altitude AA_1. So, the diagonals of quadrilateral AKA_1Q bisect each other and are perpendicular to each other, thus this quadrilateral is a rhombus. Moreover, from the symmetry we have $HQ = HK$.

   Analogously, let R be the reflection of K in CC_1; then CKC_1R is a rhombus, and $HQ = HR$ (see Fig. 8.7.1). So the segments A_1Q, AK, KC, and C_1R are parallel and equal, hence QA_1RC_1 is a parallelogram, and P is a midpoint of RQ. Consequently, HP is a median, and thus an altitude, in the isosceles triangle HQR, QED.

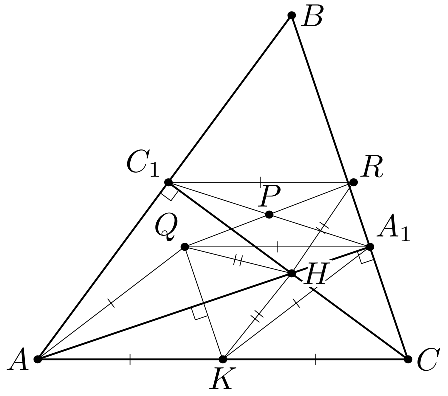


Figure 8.7.1

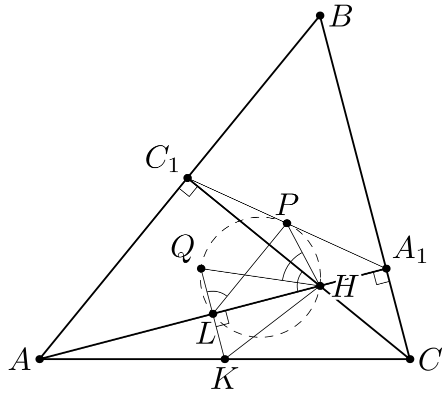


Figure 8.7.2

Second solution. Let L be the midpoint of AA_1. Then PL is a midline in the triangle AA_1C_1, so $\angle PLH = \angle BAA_1$, and hence $\angle PLQ = 90^\circ - \angle PLH = \angle C_1HA$. On the other hand, points A_1 and C_1 lie on the circle with diameter AC; therefore the triangles A_1C_1H and CAH are similar; hence the two angles $\angle PHC_1$ and $\angle KHA$ formed by their respective sides and medians are equal. Thus, $\angle QHA = \angle KHA = \angle PHC_1$, therefore $\angle PHQ = \angle C_1HA$ (see Fig. 8.7.2). So, $\angle PHQ = \angle C_1HA = \angle PLQ$, which implies that the points P, Q, L, and H are concyclic, and $\angle QPH = 180^\circ - \angle QLH = 90^\circ$.

8. (A.Zaslavsky) A square is divided into several (greater than one) convex polygons with mutually different numbers of sides. Prove that one of these polygons is a triangle.

   Solution. Suppose that a square is cut into n polygons. Then each of these polygons has at most one side lying on each side of the square; next, it shares at most one side with any of the other polygons. So the total number of its edges is at most $4 + (n-1) = n+3$.

   Thus, the number of sides of any polygon lies in the interval $[3, n+3]$. If none of them is a triangle, then the numbers of sides should be equal to $4, 5, \ldots, n+3$. So, there exists an $n+3$-gon, and it should share a segment with every side of the square. Therefore, any other polygon can share the segments with at most two sides of the square, and the number of its sides is at most $2 + (n-1) = n+1$. Hence there is no $(n+2)$-gon; a contradiction.

VIII Geometrical Olympiad in honour of I.F.Sharygin

Final round. First day. 9th form. Solutions

1. (L.Steingarts) The altitudes AA_1 and BB_1 of an acute-angled triangle ABC meet at point O. Let A_1A_2 and B_1B_2 be the altitudes of triangles OBA_1 and OAB_1 respectively. Prove that A_2B_2 is parallel to AB.

   Solution. By $\angle CAA_1 = 90^\circ - \angle ACB = \angle CBB_1$, the right triangles OA_1B and OB_1A are similar (see Fig. 9.1). So, their altitudes A_1A_2 and B_1B_2 divide the sides OB and OA in the same ratio. This exactly means that $A_2B_2 \parallel AB$.

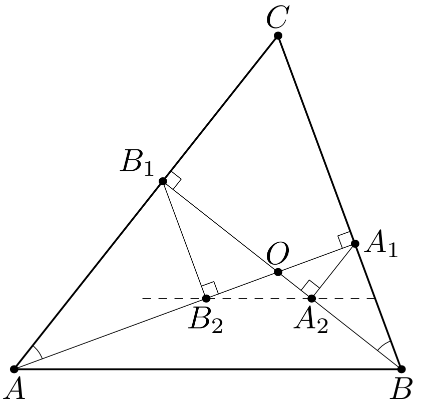


Figure 9.1

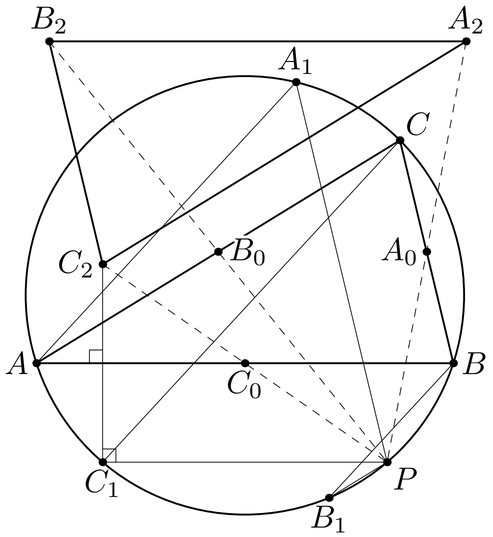


Figure 9.2

2. (D.Shvetsov, A.Zaslavsky) Three parallel lines passing through the vertices A, B, and C of triangle ABC meet its circumcircle again at points A_1, B_1, and C_1 respectively. Points A_2, B_2, and C_2 are the reflections of points A_1, B_1, and C_1 in BC, CA, and AB respectively. Prove that the lines AA_2, BB_2, CC_2 are concurrent.

   Solution. Let a, b, and c be the lines drawn through the points A_1, B_1 and C_1 and parallel to BC, CA, and AB respectively. We claim that these lines are concurrent, and their concurrency point lies on the circumcircle of ABC. Let c intersect the circumcircle at C_1 and P (if c is tangent to the circumcircle, then $P = C_1$). Then from $AB \parallel C_1P$ and $AA_1 \parallel CC_1$ we obtain $\smile BP = \smile C_1A = \smile A_1C$ (here, by $\smile XY$ we denote the measure of the arc passing from X to$Y$ clockwise). This means exactly that $A_1P \parallel BC$, hence a passes through P. Analogously, b also passes through P (see Fig. 9.2).

   Next, the points C_1 and P are symmetrical in the perpendicular bisector of AB, while the points C_1 and C_2 are symmetrical in AB; this implies that the points P and C_2 are symmetrical about the midpoint C_0 of the segment AB. Analogously, the points A_2 and B_2 are symmetrical to P about the midpoints A_0 and B_0 of the other two sides of ABC. Thus, $\overrightarrow{A_2B_2} = 2\overrightarrow{A_0B_0} = -\overrightarrow{AB}$ and analogously $\overrightarrow{A_2C_2} = -\overrightarrow{AC}$, $\overrightarrow{B_2C_2} = -\overrightarrow{BC}$. Therefore the triangles ABC and $A_2B_2C_2$ are centrally symmetric to each other, and the lines connecting their respective vertices are concurrent at the symmetry center.

3. (V.Protasov) In triangle ABC, the bisector CL was drawn. The incircles of triangles CAL and CBL touch AB at points M and N respectively. Points M and N are marked on the picture, and then the whole picture except the points A, L, M, and N is erased. Restore the triangle using a compass and a ruler.

First solution. Let K be the tangency point of the incircle of ABC with the side AB (clearly, point K lies on the segment MN). Notice that

$$MK = AK - AM = \frac{AB + AC - BC}{2} - \frac{AL + AC - LC}{2} = \frac{BL + LC - BC}{2} = LN.$$

Next, let I_a, I_b, and I be the centers of the incircles ω_a, ω_b, and ω of triangles ACL, BCL, and ABC respectively. By the angle bisector property, we get $\dfrac{AL}{IL} = \dfrac{AI_a}{I_aI} = \dfrac{AM}{MK}$, so $IL = \dfrac{AL \cdot MN}{AM}$.

Now we are ready to restore the triangle. It is easy to reconstruct successively points X, I (as the meeting point of the perpendicular to MN at K and a circle with center L), then I_a and I_b (as the meeting points of the bisectors of $\angle ALI$ and $\angle CLI$ with the perpendiculars to segment MN at its endpoints), then the circles ω_a and ω_b and, finally, the points C (the intersection of the tangents to ω_a drawn from A and L) and B (as the intersection of MN with a tangent to ω_b from C).

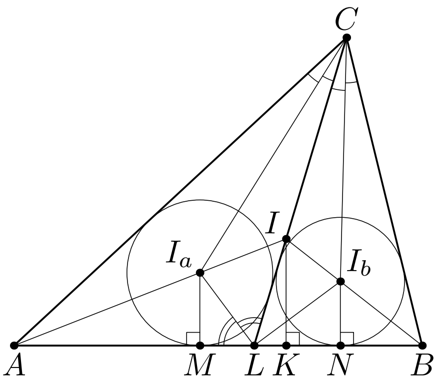


Figure 9.3

Second solution. First, let us prove the relation $1/AM + 1/ML = 1/LN + 1/NB$. Denote by $x = AC$, $y = CL$, $z = LA$ the side lengths of triangle ACL, by p, S, and r its semiperimeter, area, and inradius respectively, and by h the distance from C to line AB. Then we have $\dfrac{1}{AM} + \dfrac{1}{ML} = \dfrac{1}{p-y} + \dfrac{1}{p-x} = \dfrac{z}{(p-x)(p-y)}$. Next, by the Далее, $(p-x)(p-y) = \dfrac{S^2}{p(p-z)} = \dfrac{rp \cdot zh/2}{p(p-z)}$, which implies $\dfrac{1}{AM} + \dfrac{1}{ML} = \dfrac{2(p-z)}{rh} = \dfrac{2}{h\tan(\angle ACL/2)}$. Now notice that in the triangle BCL, the angle at C is the same, hence the value of $1/LN + 1/NB$ is the same.

Thus, knowing the lengths of segments AM, ML, and LN, we may reconstruct the length of NB and hence the point B. Further, from the relations $AC - CL = AM - LM$ and $BC - CL = BN - LN$ we get the difference $AC - BC = AM - LM - BN + LN = p$ of AC and BC, while the relation $AC/BC = AL/BL = q$ provides their ratio. Now it is easy to find the side lengths $AC = \dfrac{p}{q-1}$ and $BC = \dfrac{pq}{q-1}$, and to reconstruct the triangle.

4. (B.Frenkin) Determine all integer $n > 3$ for which a regular n-gon can be divided into equal triangles by several (possibly intersecting) diagonals.

Answer. All even n.

First solution. If $n = 2k$, then one may draw k main diagonals cutting the polygon into n congruent triangles.

Assume now that such cutting exists for some odd n. Consider the obtained triangles sharing a side with the initial n-gon P. All their angles opposite to these sides are equal; denote their value by α, and denote two other angles of an obtained triangle by β and γ. Two cases are possible.

Case 1. Assume that the angles β and γ are different, say, $\beta < \gamma$. We call a side of n-gon *β-side* or *γ-side* according to the angle of the triangle at its left endpoint (if observing from the center of P). Choose any β-side b, and consider the other side of angle β in the triangle adjacent to b. This side belongs to some diagonal of P, and the other endpoint of this diagonal also forms angle β with some side c of P (see Fig. 9.4.1). This angle β cannot be cut into some smaller angles, otherwise the angle adjacent to c is smaller than β; but it should be equal either to β or to $\gamma > \beta$, which is impossible.

Thus, this angle β belongs to a triangle with c as its side, and c is a γ-side; let us put b and c into correspondence. Conversely, considering angle β adjacent to any γ-side c we analogously find a β-side b corresponding to it. Thus all the sides are split into pairs, and their total number is even. A contradiction.

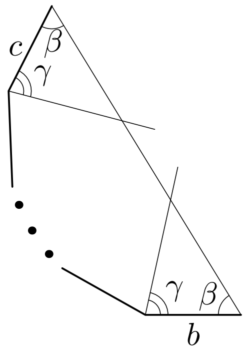


Figure 9.4.1

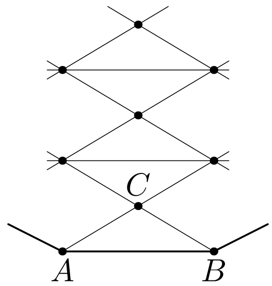


Figure 9.4.2

Case 2. Assume now that $\beta = \gamma$, and consider a triangle ABC containing the "bottom" side AB of P; its angle at C is α. The angle vertical to it is also α and belongs to some different triangle, whose side opposite to C is equal and parallel to AB. On the other side of it we find another triangle, and so on. Thus we obtain some chain of triangles (see Fig. 9.4.2); consider the last triangle UVW in this chain. If its orientation is different from that of ABC, then its top side (which is parallel to AB) is also a side of P, which is impossible for odd n (a regular n-gon has no parallel sides). Otherwise, the angle at the top vertex W is α, and W is a vertex of P. Then the sides UV and VW are (simultaneously) either sides of P or parts of its diagonals. In the first subcase we get $\alpha = \beta = \gamma = 60°$, so the angle of n-gon is $60°$, which is impossible since $n > 3$. In the second subcase, the angle of P contains (as minimum) the angle equal to α and two angles equal β (adjacent to the sides sharing the vertex W); thus $\alpha + 2\beta < 180°$. But $\alpha + 2\beta = 180°$ as the sum of three angles of a triangle; a contradiction.

Second solution. Here we present a different proof that the cutting is impossible for all odd $n > 3$.

Notice that no two diagonals are perpendicular to each other; hence for any internal meeting point of two drawn diagonals, there should exist the third diagonal passing through that point: otherwise these diagonals form two different angles which sum up to $180°$; but they should be equal to two angles of some triangle.

Next, we claim that at least two diagonals should pass through each vertex of P. Assume first that through some vertex, no diagonals are drawn. Then the triangle containing this vertex is a triangle formed by three consecutive vertices of P, and one of its angles is the angle of P. Then it is easy to see that every side of P belongs to some triangle which also contains one more side of P. Hence the sides are paired up, which is absurd.

Assume now that exactly one diagonal passes through some vertex A_i. This diagonal splits the angle at A into two different angles $\beta < \gamma$. Both these angles are adjacent to the sides of n-gon, therefore in any obtained triangle such angles are adjacent to a side equal to the side of P. Hence, the sum of all angles of triangles adjacent to the sides of P is $n(\beta + \gamma)$, which equals to the sum of all angles of P. Therefore, through each vertex passes exactly one diagonal, and the vertices are paired up, which is absurd again. The claim is proved.

Finally, assume that P is dissected into k triangles; the sum of all their angles is $180° \cdot k\pi$. The angles of P contribute $180°(n-2)$ to this sum, hence the sum of the angles at the internal points is $180°(k-n+2)$. Each such point contributes $360°$, so the number of internal points is $(k-n+2)/2$. On the other hand, each such point belongs to at least six triangles, while each vertex of P belongs to at least three of them. Hence the total number of triangles is at least $(3(k-n+2)+3n)/3 = k+2 > k$. A contradiction.

VIII Geometrical Olympiad in honour of I.F.Sharygin

Final round. Second day. 9th form. Solutions

5. (M.Kungozhin) Let ABC be an isosceles right-angled triangle. Point D is chosen on the prolongation of the hypothenuse AB beyond point A so that $AB = 2AD$. Points M and N on side AC satisfy the relation $AM = NC$. Point K is chosen on the prolongation of CB beyond point B so that $CN = BK$. Determine the angle between lines NK and DM.

   Answer. 45°.

   Solution. Let L be the projection of M onto AB. Notice that $\dfrac{ML}{CN} = \dfrac{AL}{BK} = \dfrac{AD}{BC} = \dfrac{1}{\sqrt{2}}$; hence we also have $\dfrac{LD}{CK} = \dfrac{AL + AD}{BK + BC} = \dfrac{1}{\sqrt{2}}$. Thus, the right triangles MLD and NCK are similar, and $\angle MDL = \angle NKC$ (see Fig. 9.5). Therefore the angle between the lines NK and MD is the same as the angle between KC and LD, which is equal to 45°.

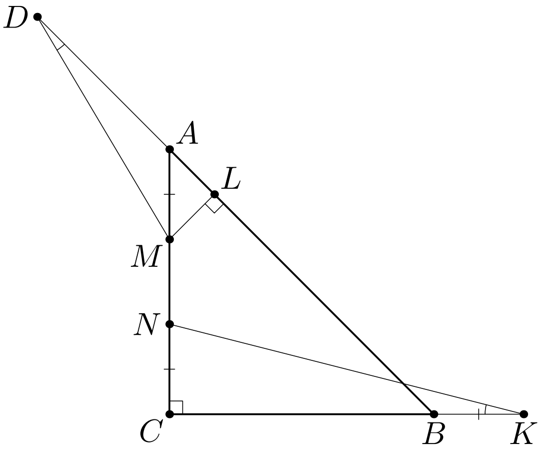


Figure 9.5

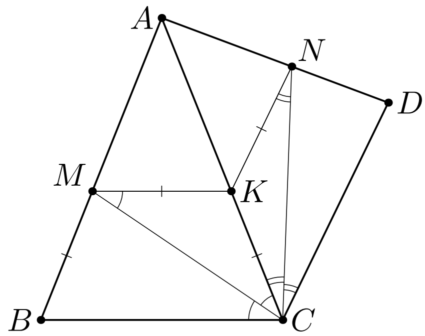


Figure 9.6

6. (M.Rozhkova) Let ABC be an isosceles triangle with $BC = a$ and $AB = AC = b$. Segment AC is the base of an isosceles triangle ADC with $AD = DC = a$ such that points D and B share the opposite sides of AC. Let CM and CN be the bisectors in triangles ABC and ADC respectively. Determine the circumradius of triangle CMN.

   Answer. $\dfrac{ab}{a+b}$.

   Solution. Choose a point K on segment AC so that $MK \parallel BC$. Then $\angle MCA = \angle MCB = \angle CMK$, so $MK = KC$. Moreover, by symmetry we get $KC = MB$. Next, by the bisector property we have $\dfrac{CK}{AK} = \dfrac{BM}{AM} = \dfrac{a}{b} = \dfrac{DN}{AN}$. Hence, $KN \parallel CD$. Now we may analogously obtain $KN = KC$. Thus, K is the circumcenter of triangle CMN, so its circumradius equals $KC = MB = ab/(a + b)$ (see Fig. 9.6).

7. (A.Belov) A convex pentagon P is divided by all its diagonals into ten triangles and one smaller pentagon P'. Let N be the sum of areas of five triangles adjacent to the sides of P decreased by the area of P'. The same operations are performed with the pentagon P'; let N' be the similar difference calculated for this pentagon. Prove that $N > N'$.

   Solution. Let $A_1A_2A_3A_4A_5$ be the initial pentagon, $B_1B_2B_3B_4B_5$ be the pentagon formed by its diagonals, and $C_1C_2C_3C_4C_5$ be the pentagon formed by the diagonals of $B_1B_2B_3B_4B_5$ (see Fig. 9.7). We will enumerate all the vertices cyclically, thus, for instance, $A_{i+5} = A_i$. For convenience, we will denote the area of polygon P by $[P]$.

Notice that $N' = \sum_i [B_iB_{i+1}B_{i+2}] - [B_1B_2B_3B_4B_5]$, since in the right-hand part the pentagon $C_1C_2C_3C_4C_5$ is counted with multiplicity -1, the triangles of the form $B_iB_{i+1}C_{i+3}$ — with multiplicity 1, and the triangles of the form $C_iC_{i+1}B_{i+3}$ with zero multiplicity. Thus the desired inequality is equivalent to

$$\sum_i [A_iA_{i+1}B_{i+3}] > \sum_i [B_iB_{i+1}B_{i+2}].$$

We will prove that $[A_iA_{i+1}B_{i+3}] > [B_{i+2}B_{i+3}B_{i+4}]$; adding up five such inequalities we will get the desired inequality.

Clearly, it is enough to deal with the case $i = 1$. Let us glue a triangle $A_1B_3B_4$ to each of the triangles $A_1A_2B_4$ and $B_3B_4B_5$; we get two triangles $A_1B_3A_2$ and $A_1B_3B_5$ with a common base A_1B_3. Finally, the distance from B_5 to the base is smaller than the distance from A_2; hence, $[A_1B_3A_2] > [A_1B_3B_5]$, QED.

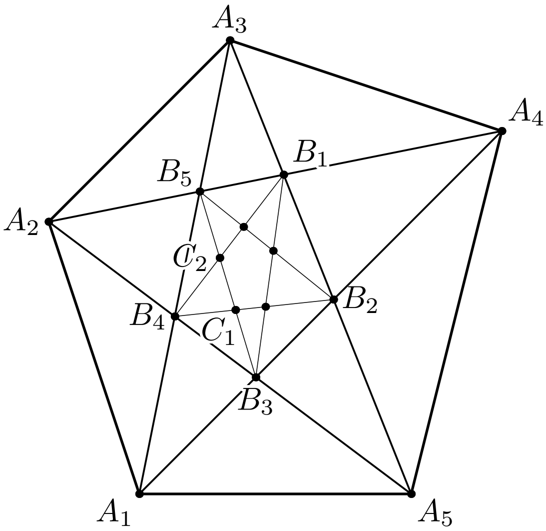


Figure 9.7

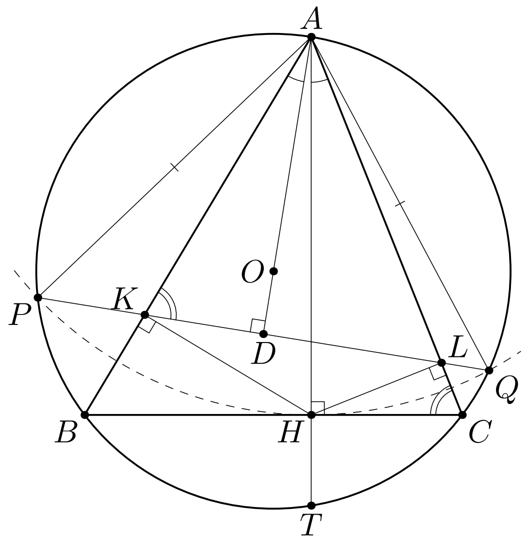


Figure 9.8

8. (M.Plotnikov) Let AH be an altitude of an acute-angled triangle ABC. Points K and L are the projections of H onto sides AB and AC. The circumcircle of ABC meets line KL at points P and Q, and meets line AH at points A and T. Prove that H is the incenter of triangle PQT.

   Solution. Let O be the center of the circumcircle Ω of triangle ABC. From right triangles ABH and ACH we get $AK \cdot AB = AH^2 = AL \cdot AC$, or $AK/AL = AC/AB$. Therefore, triangles ALK and ABC are similar, and $\angle AKL = \angle ACB$. Now, since $\angle OAB = \pi/2 - \angle ACB$, we get $OA \perp KL$, which means that OA is the perpendicular bisector to the chord PQ, so $AP = AQ$. This means that TA is the bisector of $\angle PTQ$ (see Fig. 9.8).

   Thus, the incenter I of triangle PQT lies on TA. Moreover, it is well-known that $AI = AP$. Thus, to prove that $I = H$ it suffices to show that $AH = AP$. Let D be the meeting point of the lines AO and KL, and let r be the radius of Ω. By the Pythagoras theorem, we have $AQ^2 - r^2 = AQ^2 - OQ^2 = (AD^2 + DQ^2) - (OD^2 + DQ^2) = AD^2 - (AD - r)^2$, which implies $AQ^2 = 2r \cdot AD$. On the other hand, notice that AH is a diameter of the circumcircle of AKL since $\angle AKH = \angle ALH = 90°$. Hence the similarity ratio of triangles AKL and ABC equals $AH/(2r)$. The segments AD and AH are the respective altitudes of these triangles, hence $AD/AH = AH/(2r)$, or $AH^2 = 2r \cdot AD = AQ^2$, QED.

Remark. The proof of the relation $AQ = AH$ may be shortened by means of the inversion with center A and radius AQ. Under this inversion, the line PQ and the circle Ω interchange, hence the points B and K also interchange, and $AQ^2 = AB \cdot AK = AH^2$.

VIII Geometrical Olympiad in honour of I.F.Sharygin

Final round. First day. 10th form. Solutions

1. (A.Shapovalov) Determine all integer n such that a surface of an $n \times n \times n$ grid cube can be pasted in one layer by paper 1×2 rectangles so that each rectangle has exactly five neighbors (by a line segment).

   Answer. All even n.

   Solution. Consider any even n. Divide each face into 2×2 squares, and paste each such square with two rectangles in such a way that the long sides of the rectangles in one square are adjacent to the short ones in a neighboring square. Let us show that such pasting is possible. It is easy to see that one may cover four side faces of the cube, leaving top and bottom faces uncovered. Next, one may paste one bordering row on the top face (the arrangement of the rectangles around the corner looks as on Fig. 10.1, or symmetrically to it). This row determines the arrangement of rectangles on the top face uniquely, and it is easy to see that all four bordering rows of squares will satisfy the conditions. The covering of the bottom face is analogous.

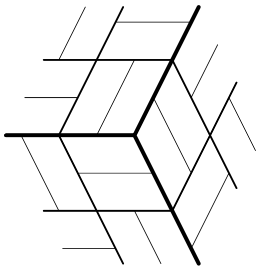
   

   Figure10.1

   Now assume that a pasting is possible for some odd n. The total number of rectangles is $6n^2/2 = 3n^2$; if each of them has five neighbors, then the total number of pairs of neighboring rectangles is $3n^2 \cdot 5/2$; but this number is not integer, which is absurd.

2. (A.Zaslavsky, B.Frenkin) We say that a point inside a triangle is *good* if the lengths of the cevians passing through this point are inversely proportional to the respective side lengths. Find all the triangles for which the number of good points is maximal.

   Answer. All acute-angled triangles.

   Solution. Let AA_1, BB_1, and CC_1 be the altitudes of a triangle ABC, and H be its orthocenter. Consider any good point P; let AA_P, BB_P, CC_P be the cevians passing through P. Then we have $AA_P/AA_1 = BB_P/BB_1 = CC_P/CC_1$; hence the right triangles AA_1A_P, BB_1B_P, and CC_1C_P are similar, so $\angle A_1AA_P = \angle B_1BB_P = \angle C_1CC_P$. There are two ways how these angles may be oriented. (Recall that an *oriented angle* $\angle(\ell, m)$ is the angle at which one needs to rotate ℓ clockwise to obtain a line parallel to m.)

   Case 1. Suppose that $\angle(A_1A, AA_P) = \angle(B_1B, BB_P) = \angle(C_1C, CC_P)$ (in particular, the triangle ABC is acute-angled; if, for instance, $\angle A \geq \pi/2$, then the angles $\angle B_1BB_P$ and $\angle C_1CC_P$ are acute, and their orientation s are opposite). The first equality yields that the points P, H. A, B are concyclic; analogously, P lies on the circumcircles of triangles ACH and BCH. But these three circles have exactly one common point H; hence $P = H$.

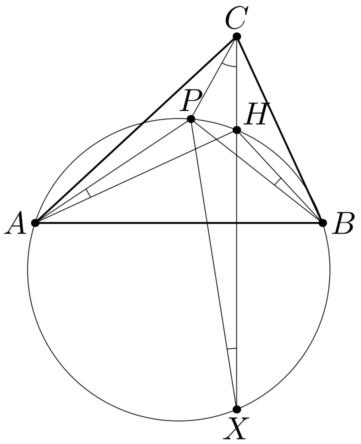


Figure 10.2.1

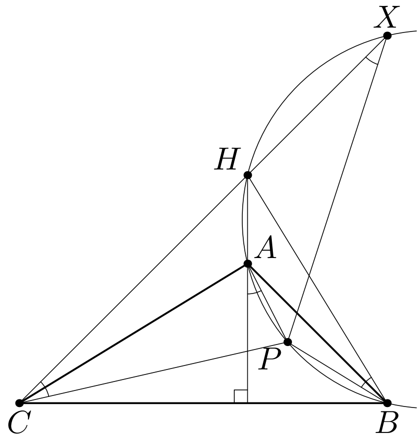


Figure 10.2.2

Case 2. Now suppose that two of the oriented angles are equal, while the third (say, $\angle(C_1C, CC_P)$) is opposite to them. Then, as in Case 1, the point P lies on the circumcircle Ω_C of triangle ABH (recall that $\angle(AH, HB) = -\angle(AC, CB)$, so Ω_C is symmetrical to the circumcircle Ω of ABC with respect to the line AB). Let X be the second meeting point of Ω_C with the line CH (then the points X and C are also symmetrical in AB; on Figs. 10.2.1 and 10.2.2 two possible configurations are shown). Then $\angle(PX, XC) = \angle(PB, BH) = -\angle(PC, CX)$; if these angles are nonzero, then this relation shows that the triangle PCX is isosceles, $PC = PX$. But then the point P lies on the perpendicular bisector AB of segment CX, which is impossible. Thus, $\angle(PB, BH) = 0$, and $P = H$.

Consequently, a point inside the triangle is good only if it is the orthocenter (and, obviously, the orthocenter of an acute-angled triangle is good). So, in an acute-angled triangle there exists exactly one good point, and there are no good point i other triangles.

Remark. In Case 2, one may apply a shorter (but less elementary) argument. The locus of points P satisfying the relation $\angle(B_1B, BB_P) = -\angle(C_1C, CC_P)$ is an equilateral hyperbola circumscribed about triangle ABC. Two such hyperbolas may have at most four common points, and these points are A, B, C, and H.

3. (A.Karlyuchenko) Let M and I be the centroid and the incenter of a scalene triangle ABC, and let r be its inradius. Prove that $MI = r/3$ if and only if MI is perpendicular to one of the sides of the triangle.

First solution. Let C_1 and C_2 be respectively the tangency points of side AB with the incircle ω and excircle ω_C of triangle ABC. Denote by C' the midpoint of AB. It is well known that $C_1C' = C_2C'$. Next, consider a homothety with center C mapping ω_C to *omega*; under this homothety, point C_2 maps to a point C_3 on ω opposite to C_1 (since the tangents in C_1 and C_3 to ω are parallel; see Fig. 10.3.1). Then IC' is a midline of triangle $C_1C_2C_3$, hence $C'I \parallel CC_2$. Thus, under a homothety with center M and coefficient -2, the point I maps to a point N lying on CC_2 (analogously, N lies on the segments connecting other vertices with the corresponding point of tangency of other excircles; N is called *the Nagel point* of triangle ABC). Therefore, N is obtained from M' by a homothety with center I and coefficient 3.

Now we turn to the problem. Suppose that $MI = r/3$. Then point N lies on ω. Without loss of generality we may assume that the tangent at N to ω intersects sides AC and BC; then ω and C share different sides of this tangent, and hence $N = C_3$. Since $IC_3 \perp AB$, we obtain $MI \perp AB$ (see Fig. 10.3.2).

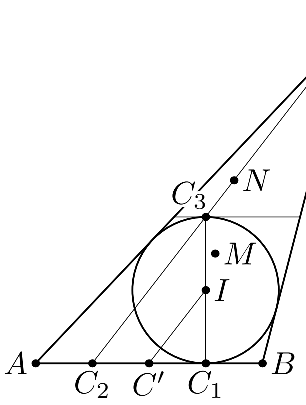


Figure 10.3.1

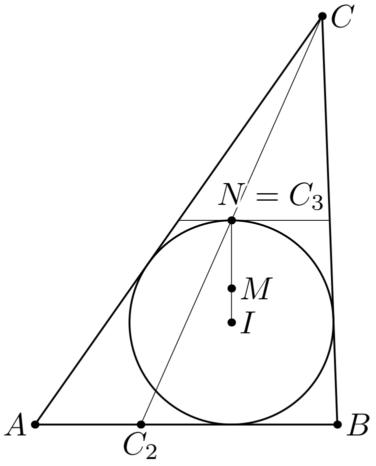


Figure 10.3.2

Conversely, if $AB \perp IM$, then N lies on the line IC_2; moreover, it also lies on the line CC_2. Since the triangle ABC is scalene, these lines are distinct, hence $N = C_2$, and thus $r = IN = 3IM$.

Second solution. Suppose that $AB \perp IM$. By the Pythagoras theorem, $AM^2 - BM^2 = (AC_1^2 + C_1M^2) - (BC_1^2 + C_1M^2) = (p-a)^2 - (p-b)^2 = c(b-a)$ (here, again, C_1 is the tangency point of AB with the incircle). Using the standard formula for the median length, we get $AM^2 = \frac{1}{9}(2b^2 + 2c^2 - a^2)$ and $BM^2 = \frac{1}{9}(2a^2 + 2c^2 - b^2)$, whence $c(b-a) = \frac{1}{3}(b-a)(a+b)$, or $a+b = 3c$, that is, $p = 2c$. It is easy to show that the converse is also true: namely, if $p = 2c$, then $AB \perp IM$. Finally, from $c(IM + r)/2 = S_{ABM} = S_{ABC}/3 = pr/3$ we obtain $IM + r = 4r/3$, or $IM = r/3$.

Conversely, assume that $MI = r/3$. Notice that $IA^2 + IB^2 + IC^2 = (\overrightarrow{IM} + \overrightarrow{MA})^2 + (\overrightarrow{IM} + \overrightarrow{MB})^2 + (\overrightarrow{IM} + \overrightarrow{MC})^2 = MA^2 + MB^2 + MC^2 + 2\overrightarrow{IM} \cdot (\overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC}) + 3MI^2 = MA^2 + MB^2 + MC^2 + 3MI^2$. So, if $MI = r/3$, then $IA^2 + IB^2 + IC^2 = MA^2 + MB^2 + MC^2 + \frac{1}{3}r^2 = \frac{1}{3}(a^2 + b^2 + c^2 + r^2)$. Next, by the Pythagoras theorem $IA^2 = r^2 + (p-a)^2$. Finally, using the relation $r^2 = S^2/p^2 = (p-a)(p-b)(p-c)/p$, we obtain

$$\frac{a^2 + b^2 + c^2 + r^2}{3} = (p-a)^2 + (p-b)^2 + (p-c)^2 + 3r^2,$$

or

$$\frac{a^2 + b^2 + c^2}{3} - (p-a)^2 - (p-b)^2 - (p-c)^2 = \frac{8r^2}{3} = \frac{8(p-a)(p-b)(p-c)}{3p},$$

which rewrites as $(p-2a)(p-2b)(p-2c) = 0$. It was mentioned above that if some expression in the brackets vanishes then IM is perpendicular to the corresponding side.

4. (B.Frenkin) Consider a square. Find the locus of midpoints of the hypothenuses of right-angled triangles with the vertices lying on three different sides of the square and not coinciding with its vertices.

Answer. All the points of a curvilinear octagon bounded by the arcs of eight parabolas with foci at the vertices of the square and directrices containing a side (non-adjacent to the focus); the midpoints of the sides of the square should be excluded (see Fig. 10.4.1).

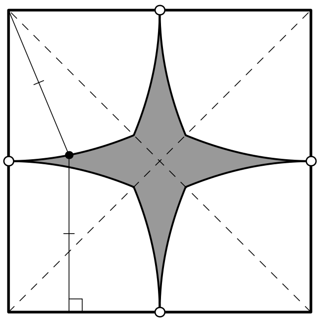


Figure 10.4.1

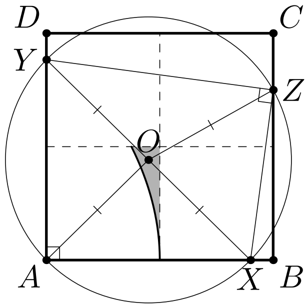


Figure 10.4.2

Solution. Notice first that the midpoint of the hypothenuse lies inside a square. If the endpoints of a hypothenuse lie on opposite sides of the square, then its midpoint lies on the midline of the square. Now assume that the endpoints X and Y of the hypothenuse of a triangle XYZ lie on sides AB and AD of a square $ABCD$ respectively, while the vertex Z lies on the side BC (see Fig. 10.4.2). Denote by O the midpoint of XY. The points A and Z belong to the circle with diameter XY; hence $OA = OX = OY = OZ$, and the distance from O to A is less that the distances to the other vertices of the square, but is not less than the distance from O to the line BC.

The locus of points equidistant from A and BC is the parabola with focus A and directrix BC (the vertex of this parabola is the midpoint of AB). So, the point O lies between this parabola and BC in the quarter of the square closest to A. Point O may lie on the parabola, but it cannot lie on the midline of the square (otherwise $Y = B$).

Analogously, one may consider other arrangements of points; taking the union of the obtained sets, we obtain the curvilinear octagon P bounded by the arcs of eight parabolas. The vertices of P are the midpoints of the sides of the square (they do not belong to the locus) and the points of intersection of parabolas with the diagonals of the square. Since the midlines of the square also lie in P, we obtain that the total locus lies in P. It remains to show that each point O in P (distinct from the midpoints of the sides) belongs to the locus.

If O lies on the midline parallel to AB, and it is not farther from AD than from BC, then one may take its projections onto AB and VD as the midpoints X and Y of a hypothenuse, and find Z as a meeting point of AD with the circle with diameter XY. If O lies in the quarter closest to A, between the parabola with focus A and directrix BC and the corresponding midline, then one chooses X and Y as the second intersection points of sides AB and AD with the circle with center O and radius OA, while Z may be chosen as any point of intersection of the same circle with side BC (such a point exists since the distance from O to BC is less than OA, but $OB > OA$).

VIII Geometrical Olympiad in honour of I.F.Sharygin

Final round. Second day. 10th form. Solutions

5. (F.Nilov) A quadrilateral $ABCD$ with perpendicular diagonals is inscribed into a circle ω. Two arcs α and β with diameters AB and CD lie outside ω. Consider two crescents formed by the circle ω and the arcs α and β (see Figure). Prove that the maximal radii of the circles inscribed into these crescents are equal.

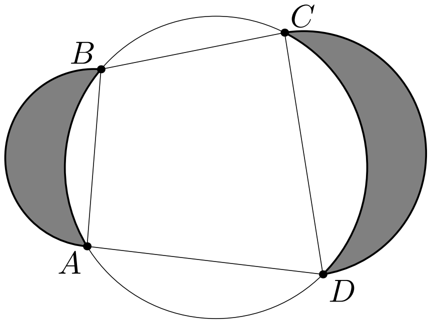


Figure 10.5.1

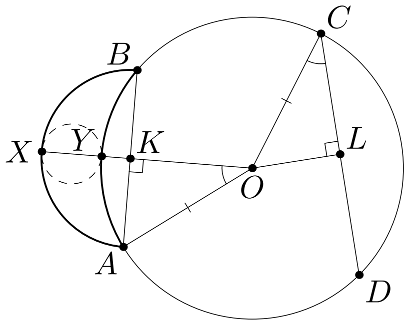


Figure 10.5.2

Solution. Let X and Y be respectively the midpoints of the arc α and the arc AB of the circle ω. Denote by O the center of ω. Then the crescent with vertices A and B is situated between the concentric circles centered at O with radii OY and OX (see Fig. 10.5.2). Hence the diameter of any circle inscribed into this crescent is at most XY; on the other hand, the circle with diameter XY lies inside the crescent. Thus the maximal diameter of a circle inside this crescent equals XY.

Since $AC \perp BD$, the sum of arcs AB and CD of the circle ω equals $180°$. Let K and L be the midpoints of segments AB and CD respectively; then $\angle AOK = 90° - \angle COL = \angle OCL$, hence the right triangles AOK and OCL are equal by hypothenuse and acute angle. So $OX = OK + KX = OK + KA = (AB + CD)/2$, and therefore $XY = (AB + CD)/2 - r$, where r is the radius of ω. Analogously we obtain that the maximal radius of a circle inscribed into the second crescent also equals $(AB+CD)/2-r$.

6. (V.Yassinsky) Consider a tetrahedron $ABCD$. A point X is chosen outside the tetrahedron so that segment XD intersects face ABC in its interior point. Let A', B', and C' be the projections of D onto the planes XBC, XCA, and XAB respectively. Prove that $A'B' + B'C' + C'A' \leq DA + DB + DC$.

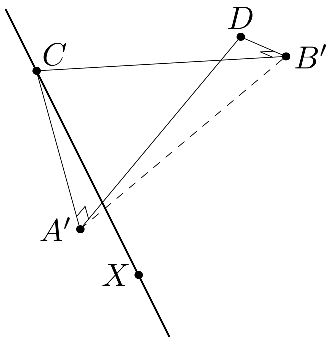


Figure 10.6

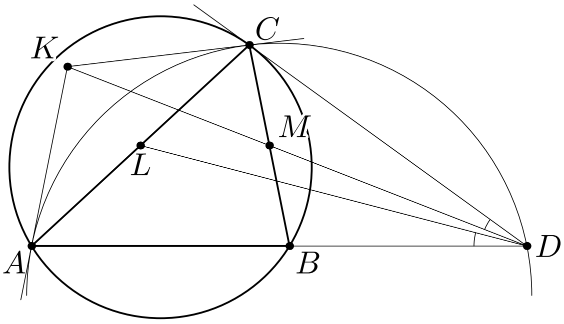


Figure 10.7

Solution. Since $DA' \perp (XBC)$, we get $\angle DA'C = 90°$; analogously, $\angle DB'C = 90°$ (see Fig. 10.6). Hence the points A' and B' lie on the sphere with diameter DC, and the distance between them is does not exceed the diameter: $A'B' \leq DK$. Analogously, we get $A'C' \leq DB$ and $B'C' \leq DA$. Adding up these inequalities we get the desired one.

7. (F.Ivlev) Consider a triangle ABC. The tangent line to its circumcircle at point C meets line AB at point D. The tangent lines to the circumcircle of triangle ACD at points A and C meet at point K. Prove that line DK bisects segment BC.

   Solution. It is well known that in any triangle XYZ, the symmedian XX' (that is, the line symmetrical to the median from the vertex X with respect to the angle bisector of angle X) passes through the common point of the tangents to the circumcircle of XYZ at Y and Z. Thus, the line DK is a symmedian in triangle ACD. Next, the triangles ACD and CBD are similar. So, denoting by DL and DM respectively their medians from D, we get $\angle CDK = \angle ADL = \angle CDM$, which implies that M lies on DK.

8. (D.Shvetsov) A point M lies on the side BC of square $ABCD$. Let X, Y, and Z be the incenters of triangles ABM, CMD, and AMD respectively. Let H_x, H_y, and H_z be the orthocenters of triangles AXB, CYD, and AZD. Prove that H_x, H_y, and H_z are collinear.

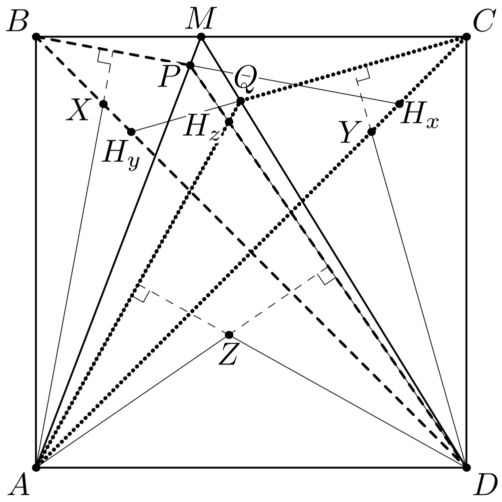


Figure 10.8

   Solution. Clearly, the points X and Y lie on the diagonals BD and AC respectively. Hence the lines AC and BD contain some altitudes of triangles AXB and CYD respectively. Let us choose points P and Q on the segments AM and DM respectively so that $AP = DQ = AD$. Then AX is an angle bisector (and hence an altitude) of an isosceles triangle ABP. Thus, the orthocenter H_x is the common point of the lines BP and AC. Analogously, H_y is the common point of the lines CQ and BD. Finally, from similar arguments we get that $AZ \perp DP$ and $BZ \perp AQ$, so H_z is the common point of the lines AQ and BP (see Fig. 10.8).

   Now let us apply the Desargues' theorem to the triangles BPD and CAQ. Since the lines BC, PA, and DQ which connect the corresponding points of these triangles are concurrent at M, we get that the common points of the lines containing the corresponding sides of these triangles are collinear. But these points are exactly H_x, H_y, and H_z.

IX Geometrical Olympiad in honour of I.F.Sharygin

Final round. Ratmino, 2013, August 1

Solutions

First day. 8 grade

8.1. (*N. Moskvitin*) Let $ABCDE$ be a pentagon with right angles at vertices B and E and such that $AB = AE$ and $BC = CD = DE$. The diagonals BD and CE meet at point F. Prove that $FA = AB$.

First solution. The problem condition implies that the right-angled triangles ABC and AED are equal, thus the triangle ACD is isosceles (see fig. 8.1a). Then $\angle BCD = \angle BCA + \angle ACD = \angle EDA + \angle ADC = \angle CDE$. Therefore, the isosceles triangles BCD and CDE are equal. Hence $\angle CBD = \angle CDB = \angle ECD = \angle DEC$.

Since the triangle CFD is isosceles and $BD = CE$, we obtain that $BF = FE$. Therefore $\triangle ABF = \triangle AEF$. Then $\angle AFB = \dfrac{\angle BFE}{2} = \dfrac{180^\circ - 2\angle FCD}{2} = 90^\circ - \angle ECD = 90^\circ - \angle DBC = \angle ABF$, hence $AB = AF$, QED.

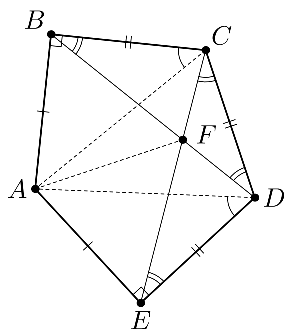


Fig. 8.1a

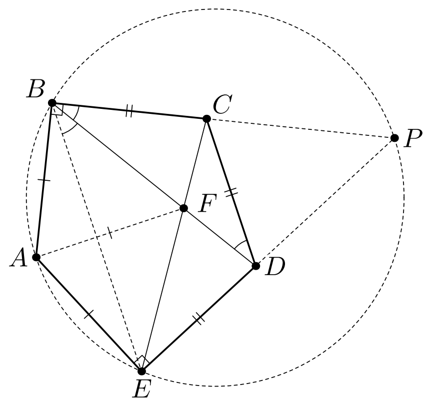


Fig. 8.1b

Second solution. Let BC meet DE at point P (see fig. 8.1b). Notice that $\angle CBD = \angle CDB = \angle DBE$, i.e., BD is the bisector of $\angle CBE$. Thus F is the incenter of $\triangle PBE$. Since the quadrilateral $PBAE$ is cyclic and symmetrical, we obtain that A is the midpoint of arc BE of the circle (PBE). Therefore, by the trefoil theorem we get $AF = AB$, QED.

Remark. The problem statement holds under the weakened condition of equality of side lengths. It is sufficient to say that $AB = AE$ and $BC = CD = DE$.

8.2. (*D. Shvetsov*) Two circles with centers O_1 and O_2 meet at points A and B. The bisector of angle O_1AO_2 meets the circles for the second time at points C and D. Prove that the distances from the circumcenter of triangle CBD to O_1 and to O_2 are equal.

First solution. Without loss of generality, suppose that C lies on the segment AD. Let P be the common point of the lines O_1C and O_2D (see fig. 8.2). The triangle AO_1C is isosceles, thus $\angle O_1CA = \angle O_1AC = \angle CAO_2$, therefore $O_1C \parallel AO_2$. Similarly, we obtain that $O_1A \parallel O_2D$. Hence O_1AO_2P is a parallelogram.

Let us prove that the quadrilateral $BCPD$ is cyclic, and O_1O_2PB is an isosceles trapezoid. Then the assertion of the problem follows. Indeed, then the circumcenter O of $\triangle BCD$ is equidistant from the points B and P, therefore O is equidistant from O_1 and O_2.

Notice that $O_1P = AO_2 = BO_2$ and $O_1B = O_1A = O_2P$, i.e., the triangles BO_1P and PO_2B are equal. Therefore $\angle BO_1P = \angle PO_2B$, and hence the quadrilateral O_1O_2PB is cyclic. Then $\angle O_1O_2B = \angle O_1PB$.

On the other hand, we have $\angle BDA = \frac{1}{2}\angle AO_2B = \angle AO_2O_1 = \angle O_1O_2B$ and $\angle O_2O_1P = \angle AO_2O_1$. Therefore $\angle BDA = \angle O_1PB = \angle O_2O_1P$, i.e., the quadrilateral $BCPD$ is cyclic, and $O_1O_2 \parallel BP$. From $O_1B = O_2P$ we obtain that O_1O_2PB is an isosceles trapezoid.

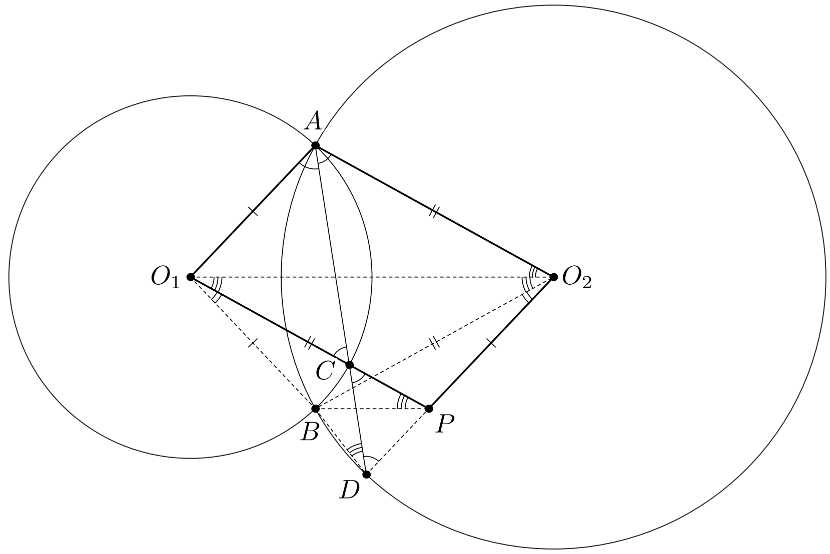


Fig. 8.2

Second solution. By $OO_1 \perp BC$ and $O_1O_2 \perp AB$, we get $\angle OO_1O_2 = \angle ABC = \dfrac{\angle AO_1C}{2}$. Similarly, we obtain $\angle OO_2O_1 = \dfrac{\angle AO_2D}{2}$. It remains to notice that $\angle AO_1C = \angle AO_2D$; it can be shown as in the previous solution.

8.3. (*B. Frenkin*) Each vertex of a convex polygon is projected to all nonadjacent sidelines. Can it happen that each of these projections lies outside the corresponding side?

Ответ: no.

Solution. Let AB be the longest side of the polygon (see fig. 8.3). Let us project all the vertices of the polygon different from A and B onto AB. Assume that none of the projections lies on the segment AB; then the projection of some side s different from AB strictly contains AB. However, this implies that $s > AB$, a contradiction.

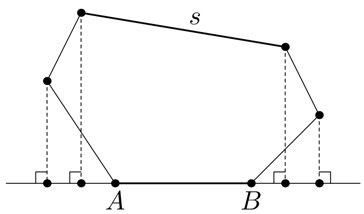


Fig. 8.3

8.4. (*A. Zaslavsky*) The diagonals of a convex quadrilateral $ABCD$ meet at point L. The orthocenter H of the triangle LAB and the circumcenters O_1, O_2, and O_3 of the triangles LBC, LCD, and LDA were marked. Then the whole configuration except for points H, O_1, O_2, and O_3 was erased. Restore it using a compass and a ruler.

Solution. Let O be the circumcenter of the triangle LAB (see fig. 8.4). Then the lines OO_1 and O_2O_3 are perpendicular to BD, while the lines O_1O_2 and O_3O are perpendicular to AC. Therefore, we can restore the perpendicular bisectors OO_1 and OO_3 to the sides LB and LA of the triangle LAB. The lines h_a and h_b passing through the orthocenter H of this triangle and parallel to OO_1 and OO_3 coincide with the altitudes of this triangle; i.e., they pass through A and B, respectively. Hence the reflections of h_a and h_b in OO_3 and OO_1, respectively, meet at point L. Now the construction is evident.

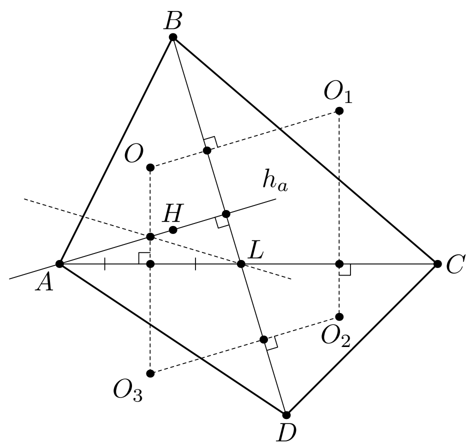


Рис. 8.4

IX Geometrical Olympiad in honour of I.F.Sharygin

Final round. Ratmino, 2013, August 2

Solutions

Second day. 8 grade

8.5. (*B. Frenkin*) The altitude AA', the median BB', and the angle bisector CC' of a triangle ABC are concurrent at point K. Given that $A'K = B'K$, prove that $C'K = A'K$.

Solution. Since the point K lies on the bisector of angle C, the distance from K to AC is the same as the distance to BC, i.e., this distance is equal to KA' (see fig. 8.5). Since $KA' = KB'$, this yields that $KB' \perp AC$. Thus the median BB' coincides with the altitude from B, and hence $AB = BC$. Then BK and CK are the angle bisectors in the triangle ABC, therefore AK is also an angle bisector; now, since AK is the altitude we have $AB = AC$. Therefore the triangle ABC is regular, and $A'K = B'K = C'K$.

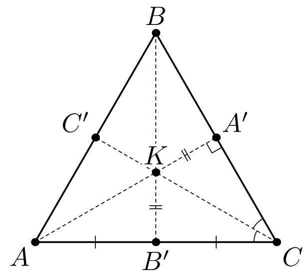
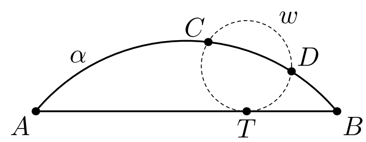


Fig. 8.5

8.6. (*F. Nilov*) Let α be an arc with endpoints A and B (see fig.). A circle ω is tangent to segment AB at point T and meets α at points C and D. The rays AC and TD meet at point E, while the rays BD and TC meet at point F. Prove that EF and AB are parallel.



Solution. Let us prove that the quadrilateral $CDEF$ is cyclic (see fig. 8.6); then the assertion of the problem follows. Indeed, then we have $\angle FEC = \angle FDC$ and $\angle FDC = 180^\circ - \angle BDC = \angle CAB$, i.e., $FE \parallel AB$.

Since AB is tangent to ω, we have $\angle TCD = \angle BTD$. Furthermore, we get $\angle FCE = \angle ACT = \angle ACD - \angle TCD = (180^\circ - \angle ABD) - \angle BTD = \angle TDB = \angle FDE$. Therefore the quadrilateral $CDEF$ is cyclic, QED.

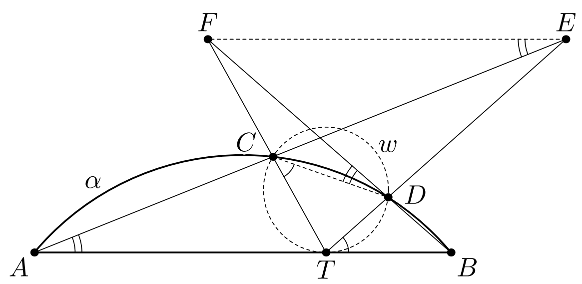


Fig. 8.6

8.7. (*B. Frenkin*) In the plane, four points are marked. It is known that these points are the centers of four circles, three of which are pairwise externally tangent, and all these three are internally tangent to the fourth one. It turns out, however, that it is impossible to determine which of the marked points is the center of the fourth (the largest) circle.

Prove that these four points are the vertices of a rectangle.

Solution. Let O_0 and R_0 be the center and the radius of the greatest circle, and let O_1, O_2, O_3 and R_1, R_2, R_3 be the centers and the radii of the remaining circles. Then $O_0O_i = R_0 - R_i$ ($i = 1$, 2, 3) and $O_iO_j = R_i + R_j$ (i, $j = 1$, 2, 3, $i \neq j$). Hence $O_0O_1 - O_2O_3 = O_0O_2 - O_3O_1 = O_0O_3 - O_1O_2 = R_0 - R_1 - R_2 - R_3 := d$.

If $d >$ (<)0, then the distance from O_0 to any of points O_1, O_2, O_3 is greater (less) than the distance between two remaining points. This enables us to determine O_0 which contradicts the condition. Indeed, if we colour the longer segments in each of the pairs (O_0O_1, O_2O_3), (O_0O_2, O_1O_3), and (O_0O_3, O_1O_2) in red and the shorter ones in blue then O_0 is the unique endpoint of three monochromatic segments.

If $d = 0$, then the marked points form a quadrilateral with equal opposite sides and equal diagonals. Such a quadrilateral has to be a rectangle.

8.8. (*I. Dmitriev*) Let P be an arbitrary point on the arc AC of the circumcircle of a fixed triangle ABC, not containing B. The bisector of angle APB meets the bisector of angle BAC at point P_a; the bisector of angle CPB meets the bisector of angle BCA at point P_c. Prove that for all points P, the circumcenters of triangles PP_aP_c are collinear.

Solution. Notice first that the lines PP_a and PP_c meet the circumcircle for the second time at the midpoints C' and A' of the arcs AB and AC, respectively (see fig. 8.8). Thus $\angle P_aPP_c = (\angle A + \angle C)/2 = 180^\circ - \angle AIC$, where I is the incenter of the triangle. Hence all circles PP_aP_c pass through I.

Now let us fix some point P and find the second common point J of circles PP_aP_c and ABC. For any other point P' we have $\angle JP'P'_c = \angle JP'A' = 180^\circ - \angle JPA' = 180^\circ - \angle JPP_c = \angle JIP_c = \angle JIP'_c$ (if P and P' lie on arcs CJ and AJ, respectively; the remaining cases can be considered similarly). Thus the circle $P'P'_aP'_c$ also passes through J.

Therefore the circumcenters of all triangles PP_aP_c lie on the perpendicular bisector of the segment IJ.

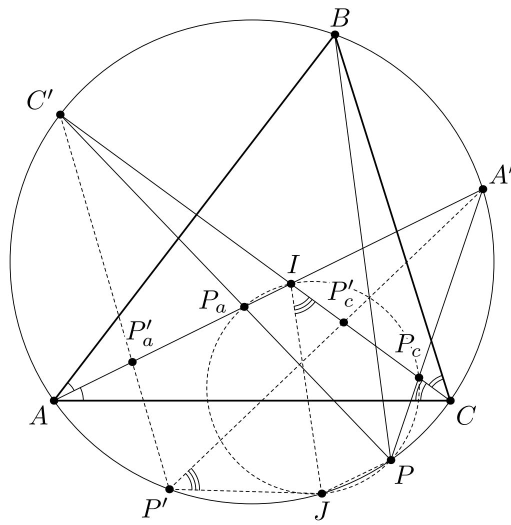


Рис. 8.8

Remark. Consider a “semiincircle” ω which is tangent to the segments BA, BC and to the arc APC). In a special case when P is the tangent point of ω and (ABC) we see that J coincides with P. Thus we can determine J as a touching point of the circumcircle and the semiincircle. It is known that J lies also on line IS, where S is the midpoint of arc ABC.

IX Geometrical Olympiad in honour of I.F.Sharygin

Final round. Ratmino, 2013, August 1

Solutions

First day. 9 grade

9.1. (*D. Shvetsov*) All angles of a cyclic pentagon $ABCDE$ are obtuse. The sidelines AB and CD meet at point E_1; the sidelines BC and DE meet at point A_1. The tangent at B to the circumcircle of the triangle BE_1C meets the circumcircle ω of the pentagon for the second time at point B_1. The tangent at D to the circumcircle of the triangle DA_1C meets ω for the second time at point D_1. Prove that $B_1D_1 \parallel AE$.

Solution. Let us take any points M and N lying outside ω on the rays B_1B and D_1D, respectively (see fig. 9.1). The angle MBE_1 is equal to the angle BCE_1 as an angle between a tangent line and a chord. Similarly, we get $\angle NDA_1 = \angle DCA_1$. Using the equality of vertical angles we obtain $\angle ABB_1 = \angle MBE_1 = \angle BCE_1 = \angle DCA_1 = \angle NDA_1 = \angle EDD_1$. Therefore, the arcs AD_1 and EB_1 are equal, and the claim follows.

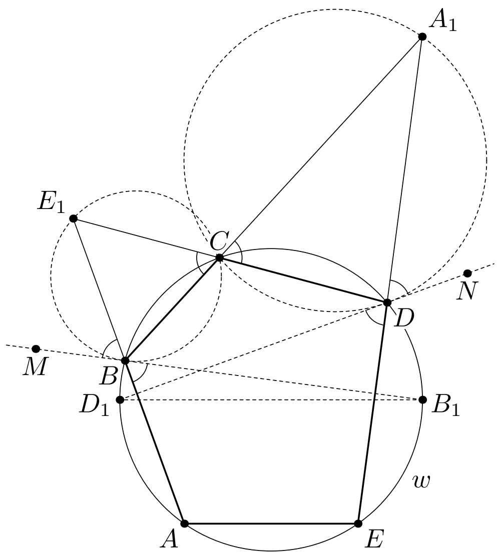


Fig. 9.1

9.2. (*F. Nilov*) Two circles ω_1 and ω_2 with centers O_1 and O_2 meet at points A and B. Points C and D on ω_1 and ω_2, respectively, lie on the opposite sides of the line AB and are equidistant from this line. Prove that C and D are equidistant from the midpoint of O_1O_2.

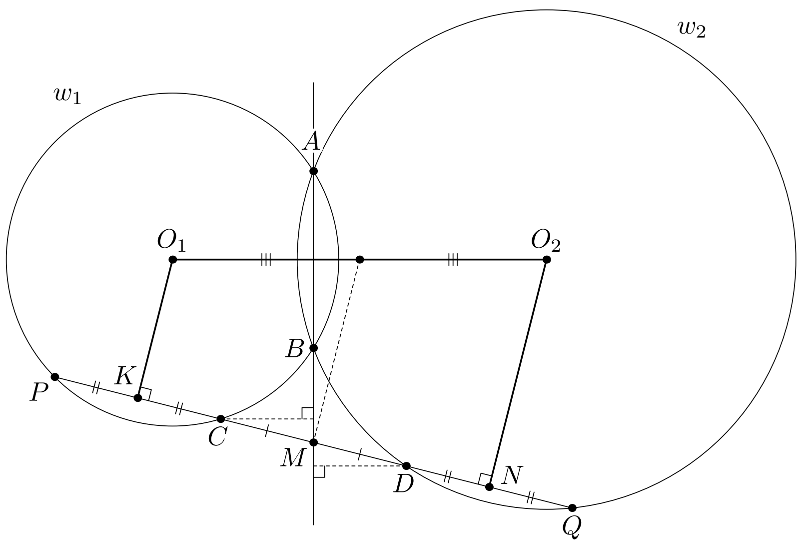


Fig. 9.2

Solution. Since the points C and D are equidistant from AB, the midpoint M of CD lies on AB (see fig. 9.2). Let P and Q be the second common points of the line CD with ω_1 and ω_2, respectively. Then $MC \cdot MP = MB \cdot MA = MD \cdot MQ$. Since $MC = MD$, we obtain that $MP = MQ$ and $PC = DQ$. Let K and N be the midpoints of PC and DQ, respectively. Then M is the midpoint of KN. Hence the midline of the right-angled trapezoid O_1KNO_2 is the perpendicular bisector of segment CD. Therefore the points C and D are equidistant from the midpoint of O_1O_2.

9.3. (*I. Bogdanov*) Each sidelength of a convex quadrilateral $ABCD$ is not less than 1 and not greater than 2. The diagonals of this quadrilateral meet at point O. Prove that $S_{AOB} + S_{COD} \leqslant 2(S_{AOD} + S_{BOC})$.

Solution. It suffices to prove that one of the ratios $\dfrac{AO}{OC}$ and $\dfrac{BO}{OD}$ is at most 2 and at least $\dfrac{1}{2}$. Indeed, assuming that $\dfrac{1}{2} \leqslant \dfrac{AO}{OC} \leqslant 2$ we get $S_{AOB} \leqslant 2S_{BOC}$ and $S_{COD} \leqslant 2S_{AOD}$; the claim follows. Thus, let us prove this fact.

Without loss of generality, we have $AO \leqslant OC$ and $BO \leqslant OD$. Assume, to the contrary, that $AO < \dfrac{OC}{2}$ and $BO < \dfrac{OD}{2}$. Let A' and B' be the points on segments OC and OD, respectively, such that $OA' = 2OA$ and $OB' = 2OB$ (see fig. 9.3). Then we have $A'B' = 2AB \geq 2$. Moreover, the points A' and B' lie on the sides of triangle COD and do not coincide with its vertices; hence the length of the segment $A'B'$ is less than one of the side lengths of this triangle. Let us now estimate the side lengths of COD.

The problem condition yields $CD \leqslant 2$. Since O lies between B and D, the length of the segment CO does not exceed the length of one of the sides CB and CD, therefore $CO \leqslant 2$. Similarly, $DO \leqslant 2$. Now, the length of $A'B'$ has to be less than one of these side lengths, which contradicts the fact that $A'B' \geq 2$.

Remark. The equality is achieved for the following degenerate quadrilateral. Consider a triangle ABC with $1 \leqslant AB, BC \leqslant 2$ and $AC = 3$, and take a point D on the segment AC such that $CD = 1$, $DA = 2$.

It is easy to see that the inequality is strict for any non-degenerate quadrilateral.

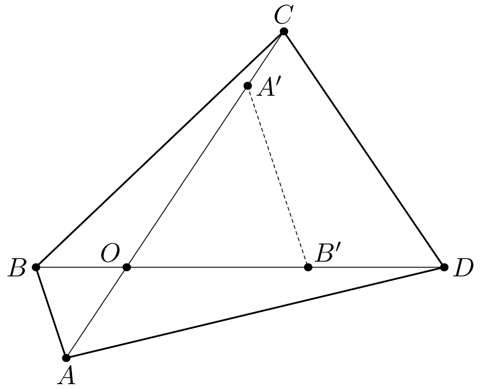


Fig. 9.3

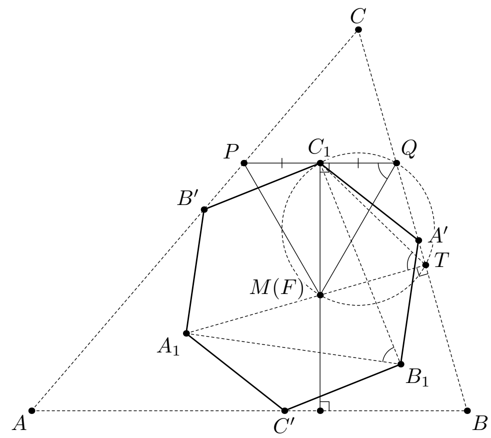


Fig. 9.4a

9.4. (*N. Beluhov*) A point F inside a triangle ABC is chosen so that $\angle AFB = \angle BFC = \angle CFA$. The line passing through F and perpendicular to BC meets the median from A at point A_1. Points B_1 and C_1 are defined similarly. Prove that the points A_1, B_1, and C_1 are three vertices of some regular hexagon, and that the three remaining vertices of that hexagon lie on the sidelines of ABC.

First solution. We will reconstruct the whole picture from the other end. Let us start with some regular hexagon $A_1B'C_1A'B_1C'$ (see fig. 9.4a). Next, let M be a point inside $\triangle A_1B_1C_1$ such that $\angle B_1MC_1 = 180^\circ - \alpha$, $\angle C_1MA_1 = 180^\circ - \beta$, and $\angle A_1MB_1 = 180^\circ - \gamma$ (this point lies inside the triangle $A_1B_1C_1$ since F lies inside the triangle ABC). Let us draw the lines through A', B', and C' perpendicular to A_1M, B_1M, and C_1M, respectively. Consider a triangle ABC formed by them. This triangle is similar to the initial triangle from the problem statement, so we may assume that it is exactly that triangle.

Thus we are only left to show that the lines AA_1, BB_1 and CC_1 are the medians of $\triangle ABC$, and M is its Fermat point (i.e., $M \equiv F$). Let the line parallel to AB through C_1 meet CA and CB at points P and Q, respectively. Construct $T = A_1M \cap CA'B$. Since $\angle A_1TA' = 90°$, point T belongs to the circumcircle of $A_1B'C_1A'B_1C'$, and the quadrilateral MC_1QT is cyclic. Therefore $\angle C_1QM = \angle C_1TM = \angle C_1TA_1 = \angle C_1B_1A_1 = 60°$. Similarly we get $\angle QPM = 60°$; thus $\triangle MPQ$ is equilateral, and C_1 the midpoint of PQ. Now, a homothety with center C shows that CC_1 is a median of $\triangle ABC$, and that CM passes through the third vertex of the equilateral triangle with base AB constructed outside ABC (this is a well-known construction for the Fermat point). By means of symmetry, the claim follows.

Second solution. Let A_p be a first Apollonius point (see fig. 9.4b). It is known that the pedal triangle $A_0B_0C_0$ of A_p is regular. Next, the Apollonius and the Torricelli point are isogonally conjugate. Therefore their pedal triangles have a common circumcircle ω.

Let us describe the point A_1 in a different way. Let E be the projection of F to BC. Then E lies on ω, and the line EF meets ω for the second time at point A_1. Notice that $\angle A_0EA_1 = 90°$; therefore A_0A_1 is a diameter of ω. Similarly we may define the points B_1 and C_1. Thus, the triangles $A_1B_1C_1$ and $A_0B_0C_0$ are symmetric with respect to the center of ω. Therefore, the hexagon $A_1B_0C_1A_0B_1C_0$ is regular. Now it remains to prove that the points A_1, B_1, and C_1 lie on the corresponding medians. This can be shown as in the previous solution.

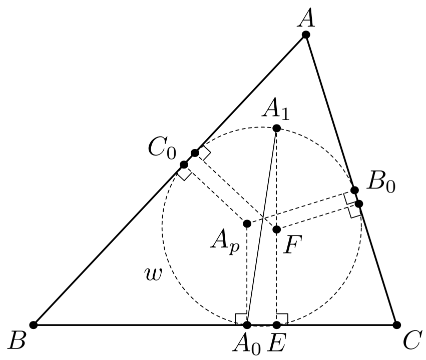


Fig. 9.4б

Solutions
Second day. 9 grade

9.5 (*V. Yassinsky*) Points E and F lie on the sides AB and AC of a triangle ABC. Lines EF and BC meet at point S. Let M and N be the midpoints of BC and EF, respectively. The line passing through A and parallel to MN meets BC at point K. Prove that $\frac{BK}{CK} = \frac{FS}{ES}$.

Solution. Let the lines passing through F and E and parallel to AK meet BC at points P and Q, respectively (see fig. 9.5). Since N is the midpoint of EF, we have $PM = MQ$, therefore $CP = BQ$ and

$$\frac{BK}{CK} = \frac{CP}{CK} \cdot \frac{BK}{BQ} = \frac{CF}{CA} \cdot \frac{BA}{BE}.$$

Applying now the Menelaus theorem to triangle AFE and line CB we obtain

$$\frac{CF}{CA} \cdot \frac{BA}{BE} \cdot \frac{ES}{FS} = 1,$$

QED.

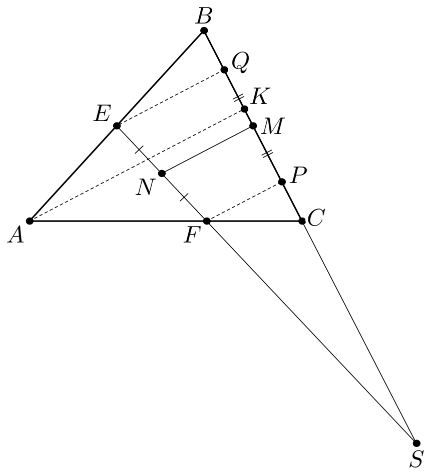


Fig. 9.5

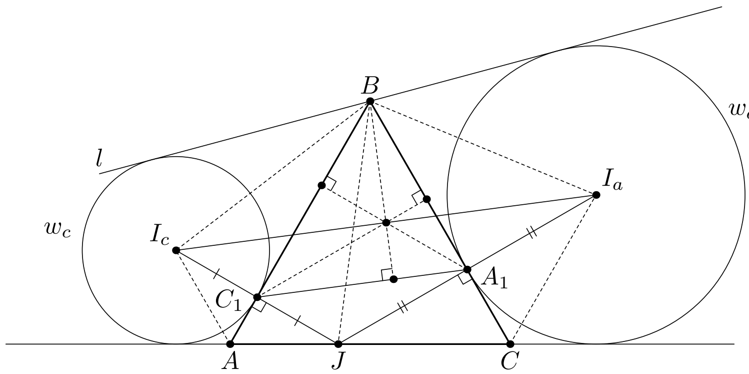


Fig. 9.6

9.6 (*D. Shvetsov, J. Zaytseva, A. Sokolov*) A line ℓ passes through the vertex B of a regular triangle ABC. A circle ω_a centered at I_a is tangent to BC at point A_1, and is also tangent to the lines ℓ and AC. A circle ω_c centered at I_c is tangent to BA at point C_1, and is also tangent to the lines ℓ and AC.

Prove that the orthocenter of triangle A_1BC_1 lies on the line I_aI_c.

Solution. By $\angle BAI_c = \angle BCI_a = 60^\circ$, the reflections of I_c and I_a in BA and BC respectively lie on AC. On the other hand, from $\angle ABI_c + \angle CBI_a = 60^\circ = \angle ABC$ we get that the reflections of BI_c and BI_a in AB and BC respectively meet AC at the same point J (see fig. 9.6). Hence A_1C_1 is the midline of triangle JI_aI_c. Then the altitudes of $\triangle A_1BC_1$ from A_1 and C_1 (which are parallel to the radii I_cC_1 and I_aA_1, respectively) are also the midlines of this triangle, thus meet at the midpoint of I_aI_c.

9.7 (*A. Karlyuchenko*) Two fixed circles ω_1 and ω_2 pass through point O. A circle of an arbitrary radius R centered at O meets ω_1 at points A and B, and meets ω_2 at points C and D. Let X be the common point of lines AC and BD. Prove that all the points X are collinear as R changes.

First solution. Let K be the second common point of ω_1 and ω_2 (see fig. 9.7). It suffices to prove that $\angle OKX = 90^\circ$.

We know that $OA = OB = OC = OD$. Therefore, the triangles AOB and COD are isosceles. Let α and β be the angles at their bases, respectively. Then we have $\angle BKC = \angle BKO + \angle CKO = \angle BAO + \angle CDO = \alpha + \beta$. Since the quadrilateral $ACBD$ is cyclic, we obtain that $\angle BXC = 180^\circ - \angle XBC - \angle XCB = 180^\circ - \angle CAD - \angle ADB = 180^\circ - \frac{1}{2}(\overset{\smile}{AB} + \overset{\smile}{CD})$, where $\overset{\smile}{AB}$ and $\overset{\smile}{CD}$ are the arcs of the circle with center O. We have $\overset{\smile}{AB} = 180^\circ - 2\alpha$ and $\overset{\smile}{CD} = 180^\circ - 2\beta$; thus $\angle BXC = \angle BKC$, i.e., the quadrilateral $BXKC$ is cyclic. Hence $\angle XKB = \angle XCB = 180^\circ - \angle ACB = 90^\circ - \alpha$. Therefore $\angle OKX = \angle BKX + \angle BKO = 90^\circ$, QED.

Second solution. Let OP and OQ be diameters of ω_1 and ω_2, respectively. Then $X \in PQ$; one may easily prove this by means of an inversion with center O. Indeed, let S be the common point of AB and CD, let M and N be the midpoints of AB and CD, respectively, and let Y be the second common point of the circles (ACS) and (BDS). Since the figures $AYBM$ and $CYDN$ are similar, we have $Y \in (OMSN)$, and the claim follows as Y and $(OMSN)$ are the images of X and PQ.

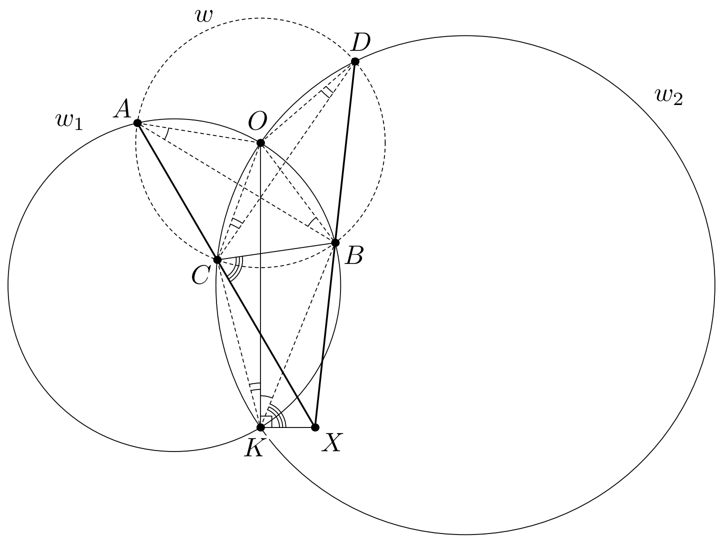


Fig. 9.7

9.8 (*V. Protasov*) Three cyclists ride along a circular road with radius 1 km counterclockwise. Their velocities are constant and different. Does there necessarily exist (in a sufficiently long time) a moment when all the three distances between cyclists are greater than 1 km?

Answer: no.

Solution. Solution. If one changes the velocities of cyclists by the same value, then the distances between them stay the same. Hence, it can be assumed that the first cyclist stays at a point A all the time.

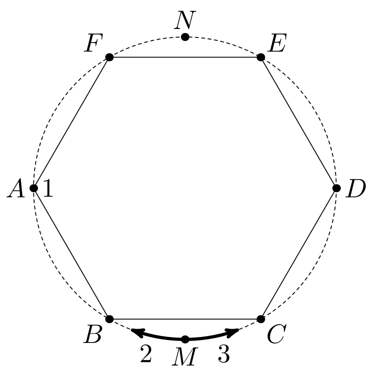


Fig. 9.8a

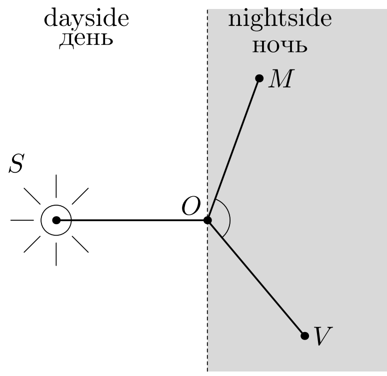


Fig. 9.8б

Let us inscribe a regular hexagon $ABCDEF$ in the circle. Let M and N be the midpoints of arcs BC and EF respectively. Suppose the second and the third cyclists start at the point M with equal velocities and go to opposite directions: the second does towards B, the third does towards C. The distance between them is less than 1 km, until they reach those points. Then the second one is located less than 1 km away from the first, i.e., from the point A, until he reaches the point F. Simultaneously, the third one reaches E, and the distance between the second and the third becomes 1 km. Then this distance is reduced monotone until they meet at the point N. We obtain a configuration symmetric to the initial one with respect to the axis AD, with the interchange of the second and the third cyclists. Then the process is repeated all over again.

Remark. It can be shown that this is the only possible example, up to a shift of velocities of cyclists. It corresponds to the case when the three velocities form an arithmetic progression. In all other cases there exists a moment when the distances between cyclists exceed not only 1 km, but $\sqrt{2}$ km! This is equivalent to the following theorem, whose proof is left to the reader:

Theorem. *If, under the assumptions of Problem 9.8, the velocities of cyclists do not form an arithmetic progression, then there exists a moment when the three radii to the cyclists form obtuse angles.*

By applying this fact, ancient astronomers could have rigorously shown the impossibility of geocentric model of the Universe. To this end, it suffices to consider the orbits of three objects: the Sun, Mercury, and Venus. Let us denote them by points S, V, M respectively and assume they move around the Earth (point O) along circular orbits. We suppose that they move on one plane (actually the planes of their orbits almost coincide). Their angular velocities are known to be different and not forming an arithmetic progression. Then there exists a moment when all the three angles between the rays OS, OM and OV are obtuse. Suppose an observer stands on the surface of the Earth at the point opposite to the direction of the ray OS. He is located on the nightside of the Earth and sees Mercury and Venus, since the angles SOM and SOV are both obtuse. The angular distance between those two planets, the angle MOV, is greater than 90°. However, the results of long-term observations available for ancient astronomers showed that the angular distance between Mercury and Venus never exceeds 76^o. This contradiction shows the impossibility of the geocentric model with circular orbits.

IX Geometrical Olympiad in honour of I.F.Sharygin

Final round. Ratmino, 2013, August 1

Solutions

First day. 10 grade

10.1 (*V. Yassinsky*) A circle k passes through the vertices B and C of a triangle ABC with $AB > AC$. This circle meets the extensions of sides AB and AC beyond B and C at points P and Q, respectively. Let AA_1 be the altitude of ABC. Given that $A_1P = A_1Q$, prove that $\angle PA_1Q = 2\angle BAC$.

Solution. Since $\angle A_1AP = 90^\circ - \angle ABC = 90^\circ - \angle AQP$, the ray AA_1 passes through the circumcenter O of the triangle APQ (see fig. 10.1). This circumcenter also lies on the perpendicular bisector ℓ of the segment PQ. Since $AB \neq AC$, the lines AO and ℓ are not parallel, so they have exactly one common point. But both O and A_1 are their common points, so $A_1 = O$. Therefore, the inscribed angle PAQ is the half of the central angle PA_1Q.

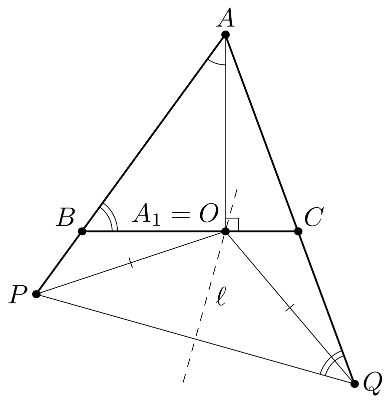


Рис. 10.1

10.2 (*A. Polyansky*) Let $ABCD$ be a circumscribed quadrilateral with $AB = CD \neq BC$. The diagonals of the quadrilateral meet at point L. Prove that the angle ALB is acute.

Solution. Assume to the contrary that $\angle ALB \geqslant 90^\circ$. Then we get $AB^2 \geqslant AL^2 + BL^2$ and $CD^2 \geqslant CL^2 + DL^2$; similarly, $AD^2 \leqslant AL^2 + DL^2$ and $BC^2 \leqslant BL^2 + CL^2$. Thus, $2AB^2 = AB^2 + CD^2 \geqslant AD^2 + BC^2$.

On the other hand, since the quadrilateral is circumscribed, we have $2AB = AB + CD = BC + AD$. This yields $AD \neq BC$ and

$$2(AD^2 + BC^2) = (AD + BC)^2 + (AD - BC)^2 > (2AB)^2 = 4AB^2.$$

A contradiction.

10.3 (*A. Karlyuchenko*) Let X be a point inside a triangle ABC such that $XA \cdot BC = XB \times AC = XC \cdot AB$. Let I_1, I_2, and I_3 be the incenters of the triangles XBC, XCA, and XAB, respectively. Prove that the lines AI_1, BI_2, and CI_3 are concurrent.

Solution 1. Consider a tetrahedron $ABCX'$ with

$$AB \cdot CX' = BC \cdot AX' = CA \cdot BX'. \qquad (*)$$

Denote by I'_a, I'_b and I'_c the incenters of the triangles BCX', ACX', and ABX'. Then $(*)$ implies that the bisectors AI'_b and BI'_a of the angles $X'AC$ and $X'BC$ meet the segment $X'C$ at the same point. This implies that the segments AI'_a and BI'_b have a common point. Similarly, each of them has a common point with the segment CI'_c. Since these three segments are not coplanar, all three of them have a common point.

Now, tending X' to X along the intersection circle of the three corresponding Apollonius spheres for the pairs (A, B), (B, C), and (A, C), we come to the problem statement.

Solution 2. Let I be the incenter of the triangle ABC, and let A_1, B_1, and C_1 be the feet of the respective bisectors in this triangle. Let T_c be the common point of the lines CI_3 and XI; define the points T_a and T_b similarly. We will prove that $T_a = T_b = T_c$.

Since $XB/XA = BC/AC$, the bisector XI_3 of the angle BXA passes through C_1. Applying the Menelaus theorem to the triangle $\triangle XIC_1$ and the line CI_3, and using the properties of the bisector AI_3 of the angle XAC_1, we obtain

$$\frac{XT_c}{T_cI} = \frac{XI_3}{I_cC_1} \cdot \frac{C_1C}{CI} = \frac{XA}{AC_1} \cdot \frac{C_1C}{CI} = \frac{XA}{CI} \cdot \frac{C_1C}{AC_1} = \frac{XA}{CI} \cdot \frac{\sin A}{\sin(C/2)}.$$

Similarly we get

$$\frac{XT_b}{T_bI} = \frac{XA}{BI} \cdot \frac{\sin A}{\sin(B/2)}.$$

But $\dfrac{BI}{CI} = \dfrac{\sin(C/2)}{\sin(B/2)}$, so $\dfrac{XT_c}{T_cI} = \dfrac{XT_b}{T_bI}$, as desired.

10.4 (*N. Beluhov*) We are given a cardboard square of area $1/4$ and a paper triangle of area $1/2$ such that all the squares of the side lengths of the triangle are integers. Prove that the square can be completely wrapped with the triangle. (In other words, prove that the triangle can be folded along several straight lines and the square can be placed inside the folded figure so that both faces of the square are completely covered with paper.)

Solution. 1. We say that a triangle is *elementary* if its area equals $\frac{1}{2}$, and the squares of its side lengths are all integral. Denote by Δ the elementary triangle with side lengths 1, 1, and $\sqrt{2}$.

Now we define the operation of *reshaping* as follows. Take a triangle ABC; let AM be one of its medians. Let us cut it along AM, and glue the pieces $\triangle ABM$ and $\triangle ACM$ along the equal segments BM and CM to obtain a new triangle with the side lengths AB, AC, and $2AM$.

2. We claim that for every elementary triangle δ, one may apply to it a series of reshapings resulting in Δ.

To this end, notice that a reshaping always turns an elementary triangle into an elementary triangle: indeed, reshaping preserves the area, and, by the median formula $4m_a^2 = 2b^2 + 2c^2 - a^2$, it also preserves the property that the side lengths are integral.

Now let us take an arbitrary elementary triangle δ. If its angle at some vertex is obtuse, then let us reshape it by cutting along the median from this vertex; the maximum side length of the new triangle will be strictly smaller than that of the initial one. Let us proceed on this way. Since all the squares of the side lengths are integral, we will eventually stop on some triangle δ' which is right- or acute-angled. The sine of the maximal angle of δ' is not less than $\sqrt{3}/2$, so the product of the lengths of the sides adjacent to this angle is at most $2/\sqrt{3}$. Hence both of them are unit, and the angle between them is right. Thus $\delta' = \Delta$, as desired.

3. Conversely, if δ' is obtained from δ by a series of reshapings, then δ can also be obtained from δ'. Therefore, each elementary triangle δ can be obtained from Δ.

4. Now, let us say that a triangle δ forms a *proper wrapping* if our cardboard square can be wrapped up completely with δ in such a way that each pair of points on the same side of δ equidistant from its midpoint comes to the same point on the same face of the folded figure. The triangle Δ forms a proper wrapping when folded along two its shorter midlines.

Suppose that a triangle $\delta = ABC$ forms a proper wrapping, and let AM be one of its medians. Consider the corresponding folding of this triangle. In it, let us glue together the segments BM and CM (it is possible by the definition of a proper wrapping), and cut our triangle along AM. We will obtain a folding of the reshaping of δ along AM; thus, this reshaping is also a proper wrapping. Together with the statement from part 3, this implies the problem statement.

Remark 1. From this solution, one may see that the following three conditions are equivalent:

(a) the triangle ABC is elementary;

(b) there exists a copy of $\triangle ABC$ such that all its vertices are integer points;

(c) there exist six integers p, q, r, s, t, u such that $p+q+r=s+t+u=0$ and $p^2+s^2=AB$, $q^2+t^2=BC$, $r^2+u^2=CA$.

Remark 2. The equivalence of the conditions (b) and (c) is obvious. The fact that (a) is also equivalent to them can be proved in different ways. E.g., one may start from Heron's formula; for the elementary triangle with side lengths $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ it asserts $2(ab+bc+ca)-(a^2+b^2+c^2)=1$. One may show — for instance, by the descent method — that all integral solutions of this equation satisfy (c).

Another approach is the following one. Consider an elementary triangle ABC and let it generate a lattice (that is, take all the endpoints X such that $\overrightarrow{AX}=k\overrightarrow{AB}+\ell\overrightarrow{AC}$ with integral k and ℓ). Using the cosine law, one easily gets that all the distances between the points of this lattice are roots of integers. Now, from the condition on the area, we have that the minimal area of a parallelogram with vertices in the lattice points is 1. Taking such a parallelogram with the minimal diameter, one may show that it is a unit square[1].

This lattice also helps in a different solution to our problem. For convenience, let us scale the whole picture with coefficient 2; after that, the vertices of the triangle have even coordinates, and its area is 2, and we need to wrap a unit square. Now, let us paint our checkered plane chess-like and draw on it the lattice of the triangles equal to ABC; their vertices are all the points with even coordinates. Notice that all the triangles are partitioned into two classes: the translations of ABC and the symmetric images of it.

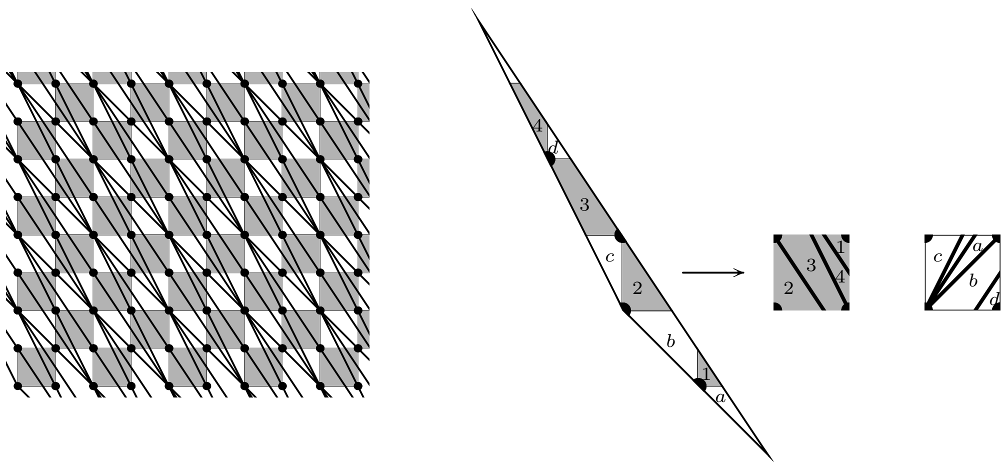


Рис. 10.4

Now, let us wrap a black square with vertex A with the triangle ABC, folding it by the sides of the cells. Then a black face of the square will get all black parts of the triangle; the parts from the black squares in even rows will be shifted, while those from the other black squares will be reflected at some points.

On the other hand, all the black squares in the even rows are partitioned by the triangles in the same manner; the partition of any other black square again can be obtained by a reflection of that first partition. Such a reflection interchanges the two classes of triangles. Finally, now it is easy to see that our black square will be completely covered: those its parts which are in the triangles of the first class — by the translations of the parts of ABC, and the others — by the reflections of the other black parts of ABC. The same applies to the other face of the square.

[1]Cf. problem 10.7 from the Final round of the 5th olympiad in honour of I.F.Sharygin, 2009.

IX Geometrical Olympiad in honour of I.F.Sharygin

Final round. Ratmino, 2013, August 2

Solutions

Second day. 10 grade

10.5 (*D. Shvetsov*) Let O be the circumcenter of a cyclic quadrilateral $ABCD$. Points E and F are the midpoints of arcs AB and CD not containing the other vertices of the quadrilateral. The lines passing through E and F and parallel to the diagonals of $ABCD$ meet at points E, F, K, and L. Prove that line KL passes through O.

Solution. For concreteness, let K lie on the line parallel to AC through E, as well as on the line parallel to BD through F (see fig. 10.5). Notice that

$$\measuredangle(KE, EF) = \measuredangle(AC, EF) = \frac{\smile{CF} + \smile{AE}}{2} = \frac{\smile{FD} + \smile{EB}}{2} = \measuredangle(BD, EF) = \measuredangle(KF, EF).$$

This means that the triangle KEF is isosceles, $KE = KF$. Hence the parallelogram $EKFL$ is in fact a rhombus, and KL is the perpendicular bisector of EF, thus it contains O.

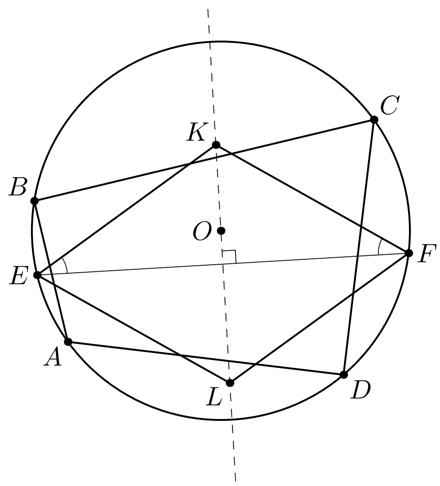


Рис. 10.5

10.6 (*D. Prokopenko*) The altitudes AA_1, BB_1, and CC_1 of an acute-angled triangle ABC meet at point H. The perpendiculars from H to B_1C_1 and A_1C_1 meet the rays CA and CB at points P and Q, respectively. Prove that the perpendicular from C to A_1B_1 passes through the midpoint of PQ.

Solution 1. Let N be the projection of C to A_1B_1. Consider a homothety h centered at C and mapping H to C_1; thus $h(P) = P_1$ and $h(Q(= Q_1$. We have $C_1P_1 \perp C_1B_1$ and $C_1Q_1 \perp C_1A_1$; it suffices to prove now that the line CN bisects P_1Q_1.

Let K and L be the projections of P_1 and Q_1, respectively, to the line A_1B_1. It is well known that $\angle CB_1A_1 = \angle AB_1C_1$; so, $\angle P_1B_1K = \angle P_1B_1C_1$, and the right-angled triangles P_1B_1K and $P_1B_1C_1$ are congruent due to equal hypothenuses and acute angles. Hence $B_1K = B_1C_1$. Similarly, $A_1L = A_1C_1$, so the length of KL equals the perimeter of $\triangle A_1B_1C_1$.

Since C is an excenter of the triangle $A_1B_1C_1$, the point N is the tangency point of the corresponding excircle with A_1B_1, so $B_1N = p - B_1C_1$. Then we have $KN = B_1C_1 + p - B_1C_1 = p$, thus N is the midpoint of KL. Finally, by the parallel lines P_1K, CN, and Q_1L we conclude that the line CN bisects P_1Q_1, as required.

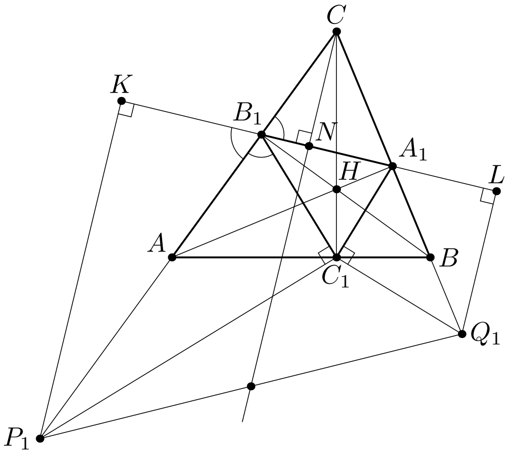


Рис. 10.6а

Solution 2. Denote $\angle BAC = \alpha$ and $\angle ABC = \beta$; then we also have $\angle ACC_1 = 90^\circ - \alpha$ and $\angle BCC_1 = 90^\circ - \beta$. By $\triangle AB_1C_1 \sim \triangle A_1BC_1 \sim \triangle ABC$ we get $\angle HPC = 90^\circ - \angle AB_1C_1 = 90^\circ - \beta$; similarly, $\angle HQC = 90^\circ - \alpha$. Next, let the perpendicular from C to A_1B_1 meet PQ at X. Then $\angle PCX = 90^\circ - \beta$ and $\angle QCX = 90^\circ - \alpha$.

We need to show that CX is a median in $\triangle CPQ$; since $\angle PCX = \angle QCH$, this is equivalent to the fact that CH is its symmedian. Therefore we have reduced the problem to the following known fact (see, for instance, A. Akopyan, "Geometry in figures", problem 4.4.6).

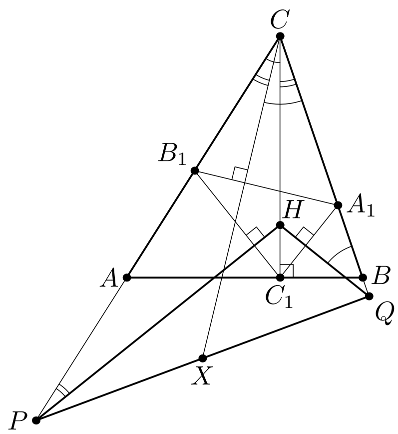


Рис. 10.6б

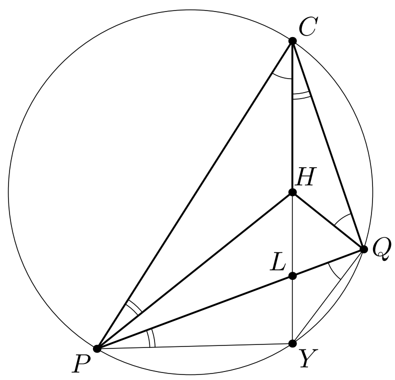


Рис. 10.6в

Lemma. *Assume that a point H inside a triangle CPQ is chosen so that $\angle CPH = \angle QCH$ and $\angle CQH = \angle PCH$. Then CH is a symmedian in this triangle.*

Proof. The triangles PHC and CHQ are similar due to two pairs of equal angles. Now, let Y be the second intersection point of the circumcircle of $\triangle CPQ$ with CH. Then $\angle YPH = \angle YPC - \angle CPH = (180^\circ - \angle YQC) - \angle YCQ = \angle HYQ$, and hence the triangles PHY and YHQ are also similar. From these similarities one gets

$$\left(\frac{PY}{YQ}\right)^2 = \frac{PH}{HY} \cdot \frac{HY}{HQ} = \frac{PH}{HQ} = \left(\frac{PC}{CQ}\right)^2,$$

so $CPYQ$ is a harmonic quadrilateral. This is equivalent to the statement of the Lemma.

Remark. One may easily obtain from the proof of the Lemma that H is a midpoint of CY.

Another proof of the Lemma (and even of the problem statement) may be obtained as follows. After noticing that the triangles PHC and CHQ are similar, it is easy to obtain the equality $\dfrac{CP}{CQ} = \dfrac{PH}{HC} = \dfrac{\sin\angle PCH}{\sin\angle QCH} = \dfrac{\sin\angle QCX}{\sin\angle PCX}$.

10.7 (*B. Frenkin*) In the space, five points are marked. It is known that these points are the centers of five spheres, four of which are pairwise externally tangent, and all these four are internally tangent to the fifth one. It turns out, however, that it is impossible to determine which of the marked points is the center of the fifth (the largest) sphere. Find the ratio of the greatest and the smallest radii of the spheres.

Answer. $\dfrac{\sqrt{7}+\sqrt{3}}{\sqrt{7}-\sqrt{3}} = \dfrac{5+\sqrt{21}}{2}$.

Solution. Denote by O and O' two possible positions of the center of the largest sphere (among the five marked points). and denote by A, B, and C the other three marked points.

Consider the points O, O', A, and B. In the configuration of spheres where O is the center of the largest sphere, denote by R, r', r_a, and r_b the radii of the spheres centered at O, O', A, and B, respectively. Then we have $OO' = R - r'$, $OA = R - r_a$, $OB = R - r_b$, $O'A = r' + r_a$, $O'B = r' + r_b$, and $AB = r_a + r_b$, which yields $OO' - AB = OA - O'B = OB - O'A$; denote this common difference by d. Similarly, from the configuration with O' being the center of the largest sphere we obtain $d = OO' - AB = O'A - OB = O'B - OA = -d$. Thus $d = 0$, and therefore $OO' = AB$, $OA = O'B$, and $OB = O'A$.

Applying similar arguments to the tuples (O, O', A, C) and (O, O', B, C) we learn $OO' = AB = AC = BC$ and $OA = O'B = OC = O'A = OB = O'C$. So, the triangle ABC is equilateral (let its side length be $2\sqrt{3}$), and the regular pyramids $OABC$ and $O'ABC$ are congruent. Thus the points O and O' are symmetrical to each other about (ABC). Moreover, we have $OO' = 2\sqrt{3}$, so the altitude of each pyramid has the length $\sqrt{3}$. Let H be the common foot of these altitudes, then $HO = HO' = \sqrt{3}$ and $HA = HB = HC = 2$, thus $OA = O'A = \sqrt{7}$. So the radii of the spheres centered at A, B, and C are equal to $\sqrt{3}$, while the radii of the other two spheres are equal to $\sqrt{7} - \sqrt{3}$ and $\sqrt{7} + \sqrt{3}$, whence the answer.

10.8 (*A. Zaslavsky*) In the plane, two fixed circles are given, one of them lies inside the other one. For an arbitrary point C of the external circle, let CA and CB be two chords of this circle which are tangent to the internal one. Find the locus of the incenters of triangles ABC.

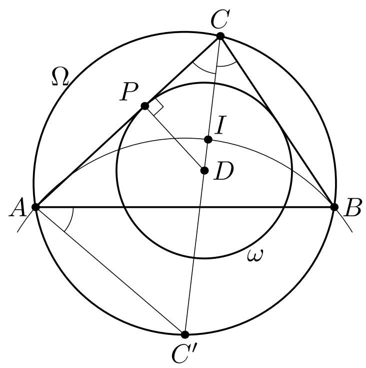


Рис. 10.8

Solution. Denote by Ω and ω the larger and the smaller circle, and by R and r their radii, respectively (see fig. 10.8). Denote by D the center of ω. Let C' be the midpoint of the arc AB of Ω not containing C, and let I be the incenter of $\triangle ABC$. Then the points I and D lie on CC'; next, it is well known that $C'I = C'A = 2R\sin\angle ACC'$.

On the other hand, denoting by P the tangency point of AC and ω, we have $\sin\angle ACC' = PD/CD = r/CD$. Next, the product $d = CD \cdot C'D$ is negated power of the point D with respect to Ω, thus it is constant. So we get $C'I = 2Rr/CD = C'D \cdot 2Rr/d$, whence

$$\overrightarrow{ID} = \overrightarrow{C'D} - \overrightarrow{C'I} = \overrightarrow{C'D} \cdot \left(1 - \frac{2Rr}{d}\right).$$

Thus, the point I lies on the circle obtained from Ω by scaling at D with the coefficient $\dfrac{2Rr}{d} - 1$.

Conversely, from every point I of this circle, one may find the points C and C' as the points of intersection of ID and Ω; the point C' is chosen as the image of I under the inverse scaling. For the obtained point C, the point I is the desired incenter; hence our locus is the whole obtained circle.

Remark. If $2Rr = d$, the obtained locus degenerates to the point D. In this case, one may obtain from our solution the *Euler formula* for the distance between the circumcenter and the incenter of a triangle.

X Geometrical Olympiad in honour of I.F.Sharygin

Final round. Ratmino, 2014, July 31

Solutions

First day. 8 grade

8.1. (*J. Zajtseva, D. Shvetsov*) The incircle of a right-angled triangle ABC touches its catheti AC and BC at points B_1 and A_1, the hypotenuse touches the incircle at point C_1. Lines C_1A_1 and C_1B_1 meet CA and CB respectively at points B_0 and A_0. Prove that $AB_0 = BA_0$.

First solution. Consider an excircle with center I_A touching side AC at point B_2 and the extension of side BC at point A_0'. Since $I_AB_2CA_0'$ is a square, we have $I_AA_0' = B_2C$. It is known that $B_2C = AB_1$, thus $I_AA_0' = AB_1$. Then $A_0'B_1 \parallel I_AA$, but $I_AA \parallel B_1C_1$, therefore, A_0', B_1, C_1 are collinear and A_0' coincides with A_0, thus BA_0 as a tangent to the excircle is equal to the semiperimeter of ABC. Similarly we obtain that AB_0 is equal to the semiperimeter, therefore $AB_0 = BA_0$.

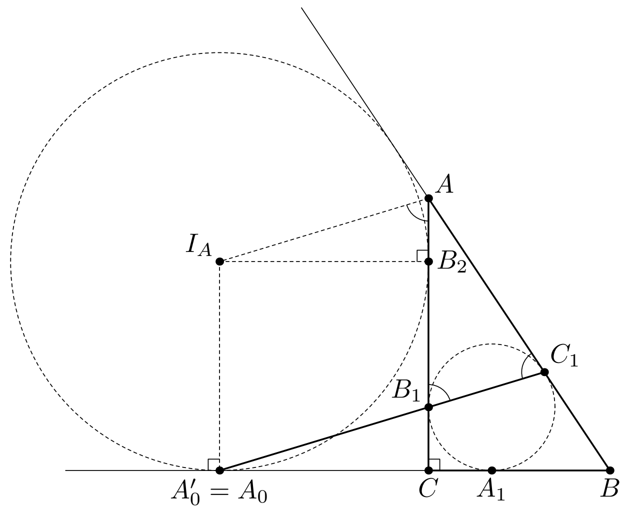


Fig. 8.1

Second solution. Since segments CA_1 and CB_1 are equal to the radius r of the incircle, and lines C_1A_1, C_1B_1 are perpendicular to the bisectors of angles B and A respectively, we obtain from right-angled triangles CA_0B_1 and CB_0A_1 that $A_0C = \frac{r}{\tan\frac{A}{2}}$, $B_0C = \frac{r}{\tan\frac{B}{2}}$. On the other hand $AC = r + \frac{r}{\tan\frac{A}{2}}$, $BC = r + \frac{r}{\tan\frac{B}{2}}$. Therefore $AB_0 = AC + CB_0 = BC + CA_0 = BA_0$.

8.2. (*B. Frenkin*) Let AH_a and BH_b be altitudes, AL_a and BL_b be angle bisectors of a triangle ABC. It is known that $H_aH_b \parallel L_aL_b$. Is it necessarily true that $AC = BC$?

Answer: yes.

First solution. Since triangles H_aH_bC and ABC are similar, triangles L_aL_bC and ABC are also similar, i.e $L_aC/AC = L_bC/BC$. Thus triangles AL_aC and BL_bC are similar. Thus, $\angle L_aBL_b = \angle L_bAL_a$, but these angles are equal to the halves of angles A and B. Therefore $AC = BC$.

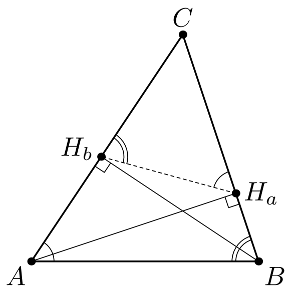


Fig. 8.2a

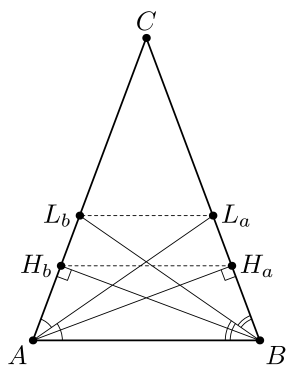


Fig. 8.2b

Second solution. Since H_aH_b and AB are antiparallel wrt AC and BC, L_aL_b and AB are also antiparallel wrt AC and BC, thus quadrilateral AL_bL_aB is cyclic. Then $\angle L_aBL_b = \angle L_bAL_a$ and $AC = BC$.

8.3. (*A. Blinkov*) Points M and N are the midpoints of sides AC and BC of a triangle ABC. It is known that $\angle MAN = 15°$ and $\angle BAN = 45°$. Find the value of angle ABM.

Answer: $75°$.

First solution. Extend segment MN and consider such points K and L that $KM = MN = NL$ (fig. 8.3Ю). Since M is the midpoint of segments AC and KN, we obtain that $AKCN$ is a parallelogram. then $\angle CKM = 45°$, $\angle KCM = 15°$. Consider such point P on segment CM that $\angle CKP = 15°$. Segment

KP divides triangle KCM into two isosceles triangles. Also $\angle PMN = 60°$, hence triangle MPN is regular. Triangles PLN and PKM are equal, triangle CPL is isosceles and right-angled, thus $\angle CLN = \angle CLP + \angle MLP = 75° = \angle ABM$, because $CLBM$ is a parallelogram.

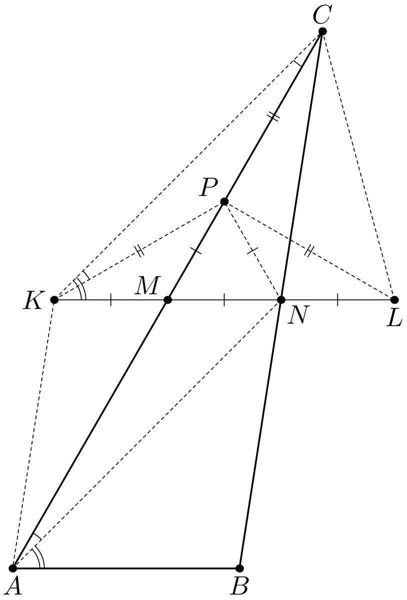


Fig. 8.3a

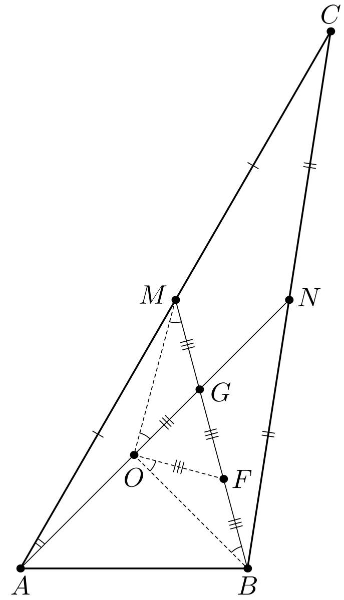


Fig. 8.3b

Second solution. Let G be the centroid of ABC, F be the midpoint GB, and GFO be the regular triangle such that points O and A lie in the same semiplane wrt MB. Since $\angle MOB = 120°$, O is the circumcenter of triangle MAB, also we have $\angle MOG = 30° = 2\angle MAG$, therefore AG meet OG on the circumcircle of AMB, i.e. A, O, G are collinear. Then $75° = \angle MOA/2 = \angle ABN$.

8.4. (*T. Kazitsyna*) Tanya has cut out a triangle from checkered paper as shown in the picture. The lines of the grid have faded. Can Tanya restore them without any instruments only folding the triangle (she remembers the triangle sidelengths)?

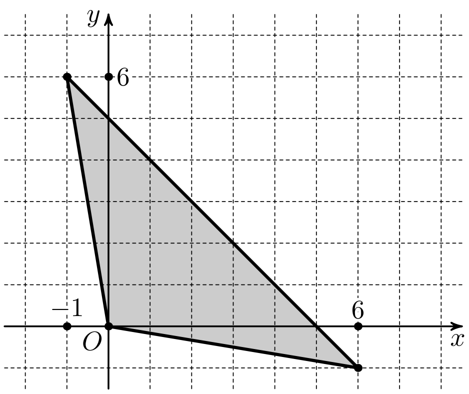


Fig. 8.4a

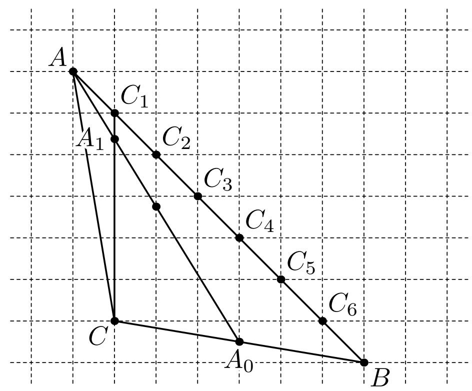


Fig. 8.4b

Solution. Let ABC be the given triangle ($AC = BC$). It is evident that we can find the midpoint of an arbitrary segment. Construct the median AA_0, and find on it such point A_1 that $AA_1 = AA_0/4$. By Thales theorem line CA_1 is the grid line intersecting AB at point C_1 such that $AC_1 = AB/7$ (fig.). Now constructing segments $C_1C_2 = C_2C_3 = \cdots = C_5C_6 = AC_1$, we find all nodes lying on AB. Folding the triangle by the line passing through C_2 in such way that C_3 be on CC_1, we restore the grid line passing through C_2, etc. The perpendicular lines can be restored similarly.

X Geometrical Olympiad in honour of I.F.Sharygin

Final round. Ratmino, 2014, August 1

Solutions

Second day. 8 grade

8.5. (*A. Shapovalov*) A triangle with angles of 30, 70 and 80 degrees is given. Cut it by a straight line into two triangles in such a way that an angle bisector in one of these triangles and a median in the other one drawn from two endpoints of the cutting segment are parallel to each other. (It suffices to find one such cutting.)

Solution. Let in triangle ABC $\angle A = 30°$, $\angle B = 70°$, $\angle C = 80°$. Take an altitude AH. Then $\angle CAH = \angle MHA = 10°$, where M is the midpoint of AC. Also $\angle HAL = 10°$, where L is the foot of the bisector of triangle HAB from vertex A. Therefore the median of triangle AHC from H and the bisector of triangle BAH from A are parallel, and AH is the desired cutting segment.

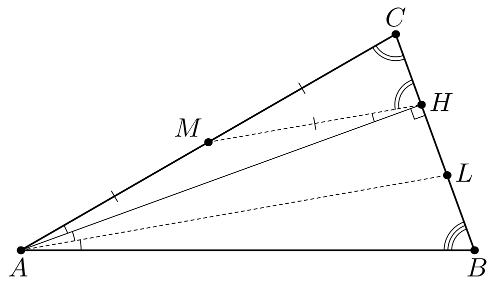


Fig. 8.5

8.6. (*V. Yasinsky*) Two circles k_1 and k_2 with centers O_1 and O_2 are tangent to each other externally at point O. Points X and Y on k_1 and k_2 respectively are such that rays O_1X and O_2Y are parallel and codirectional. Prove that two tangents from X to k_2 and two tangents from Y to k_1 touch the same circle passing through O.

Solution. Let S be the common point of XO_2 and YO_1. Let r_1 and r_2 be the radii of the corresponding circles. Then $\frac{XS}{SO_2} = \frac{O_1S}{SY} = \frac{r_1}{r_2} = \frac{O_1O}{OO_2}$. Thus $SO = \frac{r_1}{r_1+r_2}O_2Y = \frac{r_1r_2}{r_1+r_2}$.

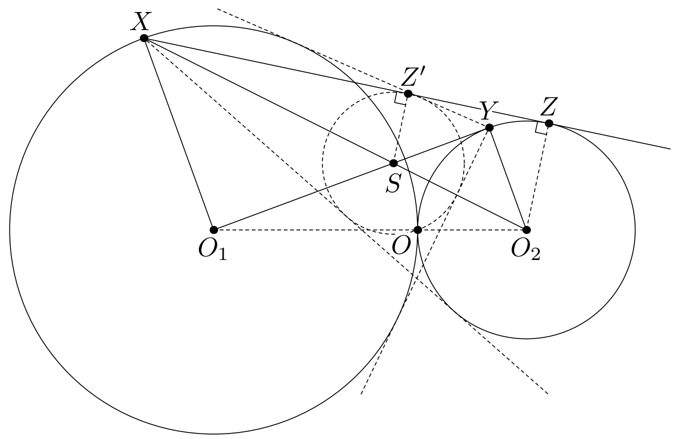


Fig. 8.6

Let XZ be a tangent from X to, and Z' be the projection of S to XZ. Then $SZ' = \frac{r_1}{r_1+r_2}O_2Z = \frac{r_1r_2}{r_1+r_2} = SO$. Similarly the distance from S to three remaining tangents is equal to SO, i.e. S is the center of the desired circle.

8.7. (*Folklor*) Two points on a circle are joined by a broken line shorter than the diameter of the circle. Prove that there exists a diameter which does not intersect this broken line.

Solution. Let A and B be the endpoints of the broken line. Consider the diameter XY parallel to AB. Let C be the reflection of B in XY, then AC is a diameter of the circle. Consider an arbitrary point Z on XY. Since $AZ + BZ = AZ + CZ \geq AC$, Z can not lie on the broken line, therefore XY is the desired diameter.

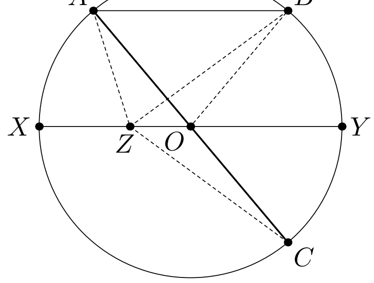


Fig. 8.7

8.8. (*Tran Quang Hung*) Let M be the midpoint of the chord AB of a circle centered at O. Point K is symmetric to M with respect to O, and point P is chosen arbitrarily on the circle. Let Q be the intersection of the line perpendicular to AB through A and the line perpendicular to PK through P. Let H be the projection of P onto AB. Prove that QB bisects PH.

First solution Let QA intersect the circle (O) at C which is distinct from A. Since BC is the diameter of the circle (O), we obtain that BC and MK bisect each other at the center of the circle, which implies that the quadrilateral $CKBM$ is a parallelogram. Furthermore, M is the midpoint of AB, then $CKMA$ is a rectangle since one of its angles is right. We shall prove that MQ is perpendicular to PC. We have

$$MC^2 - MP^2 - QC^2 + QP^2 = (CK^2 + MK^2) - (2PO^2 + 2OK^2 - PK^2) - (QK^2 - CK^2) + \\ + (QK^2 - PK^2) = 2CK^2 + 4OK^2 - 2PO^2 - 2OK^2 = 2CK^2 + 2OK^2 - 2OC^2 = 0.$$

Hence, MQ is perpendicular to PC. Let BP meet QA at R. Notice that CB is a diameter of (O), then BR is perpendicular to PC. Thus, it follows that MQ is parallel to BR. Q is the midpoint of AR, which follows from the fact that M is the midpoint of AB. Hence, QB bisects PH.

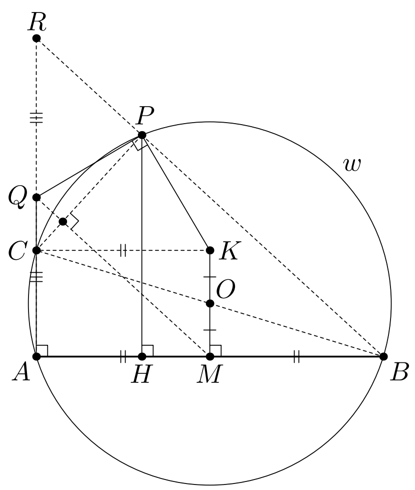


Fig. 8.8a

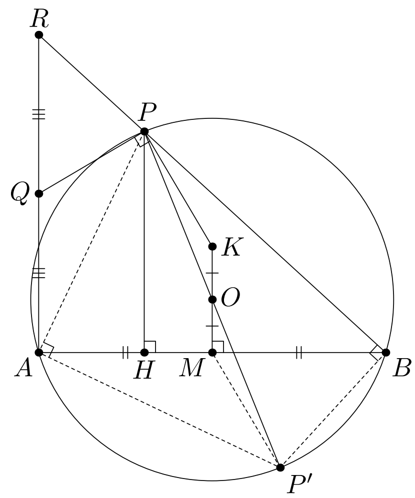


Fig. 8.8b

Second solution. Note that $\angle PBA \neq 90°$; in the other case $PK \parallel AB$, and point Q doesn't exist. Then BP meets AQ at point R. Since triangles BPH and BRA are homothetic, we have to prove that Q is the midpoint of AR.

Let point P' be opposite to P. Then $PA \perp P'A$, $PR \perp P'B$, $AR \perp AB$, i.e. the correspondent sides of triangles $P'AB$ and PAR are perpendicular. Thus these triangles are similar and their medians from P and P' are also perpendicular. Using the symmetry wrt O we obtain that $P'M \parallel PK \perp PQ$. Therefore PQ is the median in $\triangle PAR$.

X Geometrical Olympiad in honour of I.F.Sharygin
Final round. Ratmino, 2014, July 31

Solutions
First day. 9 grade

9.1. (*V. Yasinsky*) Let $ABCD$ be a cyclic quadrilateral. Prove that $AC > BD$ if and only if

$$(AD - BC)(AB - CD) > 0.$$

First solution. Without loss of generality we can suppose that arcs ABC and BCD are not greater than a cemicircle. Then $\smile AD = 2\pi - \smile ABC - \smile BCD + \smile BC > \smile BC$. Since arc $ABCD$ is also greater than arc BC, we obtain that $AD > BC$.

Now if $AC > BD$, then $\smile ABC > \smile BCD$, $\smile AB > \smile CD$ and $AB > CD$. If $AC < BD$ all inequalities are opposite.

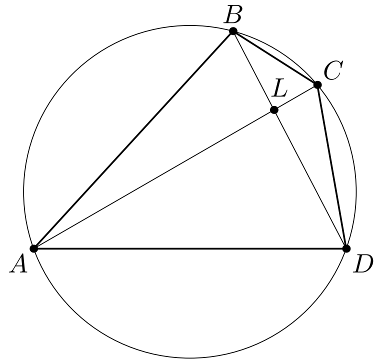


Fig. 9.1

Second solution. Let M, N be the midpoints of AC and BD, L be their common point, and O be the circumcenter. Let AL be the longest of segments AL, BL, CL, DL. Since $AL \cdot CL = BL \cdot DL$, CL is the shortest of these segments. Then $LM > LN$, $OM < ON$ and $AC > BD$. Also since triangles ALB and DLC are similar we obtain that $\frac{AB}{CD} = \frac{AL}{DL}$, i.e $AB > CD$. By the same way using the similarity of triangles ALD and BLC we obtain $AD > BC$.

Third solution. Note that $AC = 2R\sin B$ and $BD = 2R\sin A$, thus inequality $AC > BD$ is equivalent to $\sin B > \sin A$.

Now $(AD - BC)(AB - CD) > 0 \Leftrightarrow AD \cdot AB + BC \cdot CD > AD \cdot CD + BC \cdot AB$, which is equivalent to (multiply to $\frac{1}{2}\sin A \sin B = \frac{1}{2}\sin A \sin D = \frac{1}{2}\sin C \sin B$).

$$\left(\frac{AD \cdot AB \sin A}{2} + \frac{BC \cdot CD \sin C}{2}\right)\sin B > \left(\frac{AD \cdot CD \sin D}{2} + \frac{BC \cdot AB \sin B}{2}\right)\sin A \Leftrightarrow$$
$$\Leftrightarrow (S(DAB) + S(BCD))\sin B > (S(CDA) + S(ABC))\sin A \Leftrightarrow$$
$$\Leftrightarrow S(ABCD)\sin B > S(ABCD)\sin A \Leftrightarrow \sin B > \sin A.$$

9.2. (*F. Nilov*) In a quadrilateral $ABCD$ angles A and C are right. Two circles with diameters AB and CD meet at points X and Y. Prove that line XY passes through the midpoint of AC.

Solution. Let M, N, K be the midpoints of AB, CD and AC respectively. Then the degree of point K wrt the circle with diameter AB is equal to $KM^2 - MA^2 = \frac{CB^2 - AB^2}{4}$, and its degree wrt the circle with diameter CD is equal to $\frac{AD^2 - CD^2}{4}$. Since $AB^2 + AD^2 = BD^2 = BC^2 + CD^2$, we obtain that these degrees are equal.

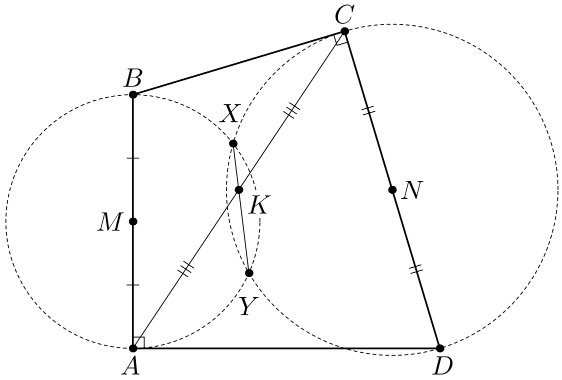


Fig. 9.2

9.3. (*E. Diomidov*) An acute angle A and a point E inside it are given. Construct points B, C on the sides of the angle such that E is the center of the Euler circle of triangle ABC.

First solution. Let l_1 and l_2 be the arms of $\angle A$ so that rotating l_1 about A to an angle $\alpha < 90°$ maps it onto l_2. Rotate l_2 about E to an angle 2α and let its image meet l_1 at M_b and B be the reflection of A in M_a. The vertex C is constructed analogously.

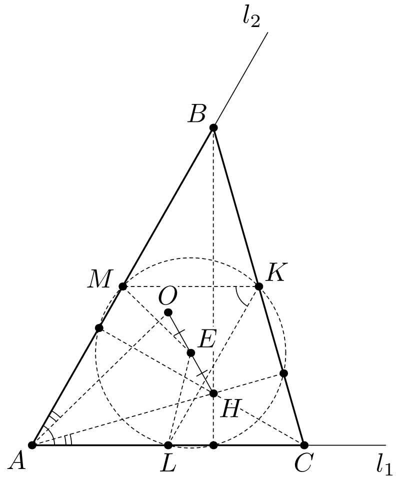


Fig. 9.3

Second solution. Let O and H be the circumcenter and the orthocenter of the sought triangle. Then E is the midpoint of OH, $\angle BAO = \angle HAC$ and $AH = 2AO\cos\angle A$. Therefore the composition of the reflection about the bisector of angle A, the homothety with center A and the coefficient equal to $2\cos\angle A$ and the reflection around E is a similarity with center O. Thus finding the center of this similarity we can construct B and C as the second common points of the arms of the given angle and the circle with center O, passing through A.

Note. If $\angle A = 60°$ the considered similariry is the reflection about the line passing through E and perpendicular to the bisector of angle A. Thus we can take as O an arbitrary point of this line. In the other cases the solution is unique.

9.4. (*Mahdi Etesami Fard*) Let H be the orthocenter of a triangle ABC. Given that H lies on the incircle of ABC , prove that three circles with centers A, B, C and radii AH, BH, CH have a common tangent.

First solution. Let H_a, H_b, H_c be the feet of the altitudes. Since $AH \cdot HH_a = BH \cdot HH_b = CH \cdot HH_c$, there exists an inversion about a circle with center H, transforming A, B, C to H_a, H_b, H_c respectively (if the triangle is acute-angled take a composition of the inversion and the reflection around H). This inversion transforms the sidelines of the triangle to the circles with diameters AH, BH, CH, and it transforms the incircle to the line touching these three circle. The homothety with center H and the coefficient 2 transforms this line to the sought one.

Second solution. Let I be the center of the incircle, A_1, B_1, C_1 be its touching points with BC, AC, AB respectively, and A_2, B_2, C_2 be such points on three circles that $\triangle A_1IH \sim \triangle HAA_2$, $\triangle B_1IH \sim \triangle HBB_2$

and $\triangle C_1IH \sim \triangle HCC_2$. The tangents to the circles in these points and the tangent to the incircle in H are parallel; prove that these three tangents coincide, i.e. the projections of vectors $\overrightarrow{HA_2}$, $\overrightarrow{HB_2}$ and $\overrightarrow{HC_2}$ to IH are equal. It is evident that they are codirectional. Since the angles formed by HA_2 with IH and IA_1 are equal, the first projection are equal to the projection of HA_2 to AH, i.e. $\frac{AH}{r} \cdot HH_a$. Find similarly the remaining projections and note that $AH \cdot HH_a = BH \cdot HH_b = CH \cdot HH_c$.

Solutions

Second day. 9 grade

9.5. (*D. Shvetsov*) In triangle ABC $\angle B = 60^\circ$, O is the circumcenter, and L is the foot of an angle bisector of angle B. The circumcirle of triangle BOL meets the circumcircle of ABC at point $D \neq B$. Prove that $BD \perp AC$.

Solution. Let H be the orthocenter of ABC, and D' be the reflection of H in AC. Then D' lies on the circumcircle, and since $\angle B = 60^\circ$, we have $BO = BH$. Thus, since BL is the bisector of angle OBH, then $LO = LH = LD'$. Therefore $BOLD'$ is a cyclic quadrilateral, i.e. D' coincides with D.

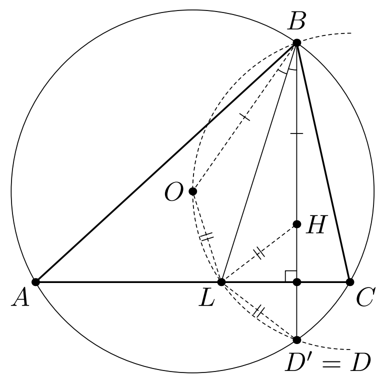


Fig. 9.5

9.6. (*A. Polyansky*) Let I be the incenter of triangle ABC, and M, N be the midpoints of arcs ABC and BAC of its circumcircle. Prove that points M, I, N are collinear if and only if $AC + BC = 3AB$.

First solution. Let A_1, B_1, C_1 be the midpoints of arcs BC, CA, AB of the circumcircle, not containing the other vertices of ABC. It is evident that MN and A_1B_1 are equal and parallel. Therefore they cut equal segments CC_2 and IC_1, where C_2 is the midpoint of CI, on the line CC_1, perpendicular to MN. Since C_1 is the circumcenter of triangle AIB we obtain that $C_2A_0 = C_2C = IC_1 = C_1A = C_1B$ (A_0 and B_0 are the touching points of the incircle with BC and CA respectively). Thus triangles C_2CA_0 and C_1AB are equal ($AB = CA_0$). From this $AC + CB = AB_0 + B_0C + CA_0 + A_0B = 2AB + AB_0 + A_0B = 3AB$. Similarly we obtain the opposite assertion.

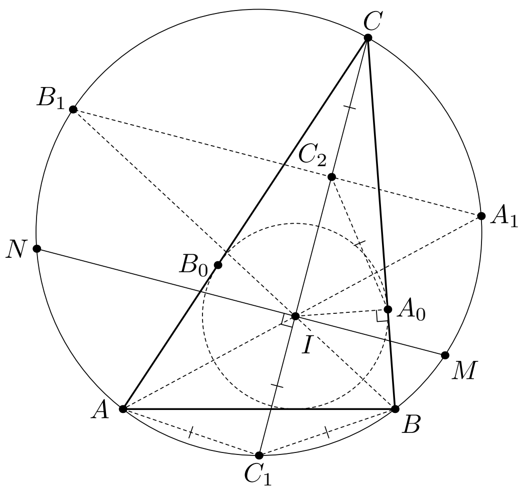


Fig. 9.6a

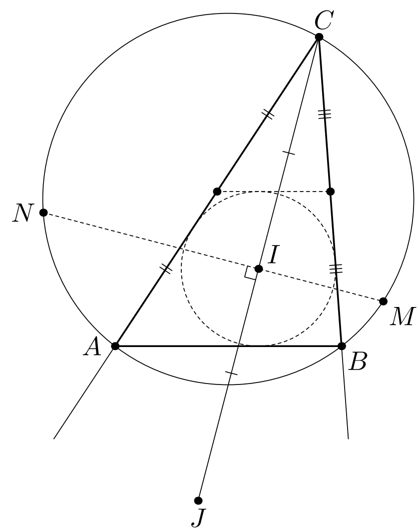


Fig. 9.6b

Second solution. Let J be the center of the excircle touching side AB. Then M and N are the centers of circles ACJ and BCJ, and therefore MN is the perpendicular bisector to segment CJ, i.e. I is the midpoint of CJ. Using the homothety with center C and the coefficient $1/2$ we obtain that the incircle touches the medial line parallel to AB. The trapezoid formed by this medial line and the sidelines of ABC is circumscribed if the sought equality is correct.

9.7. (*N. Beluhov*) Nine circles are drawn around an arbitrary triangle as in the figure. All circles tangent to the same side of the triangle have equal radii. Three lines are drawn, each one connecting one of the triangle's vertices to the center of one of the circles touching the opposite side, as in the figure. Show that the three lines are concurrent.

Solution. Introduce the following notation. Let r_a, r_b, r_c be the radii of the circles centered at O_a, O_b, O_c, respectively. Let $d_a(X)$ be the distance from X to BC, and define d_b and d_c analogously.

The figure composed of the lines CA and CB and the first three circles in the chain tangent to CA, counting from C, is similar to the figure composed of the lines CB and CA and the chain tangent to CB. Therefore, $d_a(O_b) : r_b = d_b(O_a) : r_a$. Analogous reasoning applies to the vertices A and B.

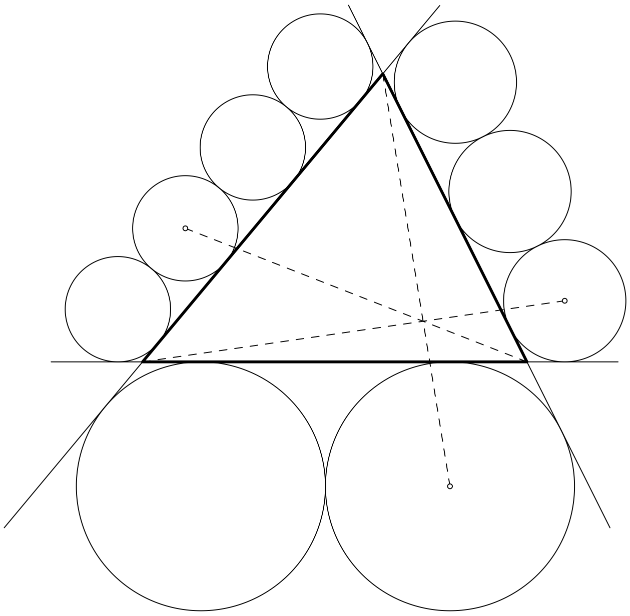


Fig. 9.7a

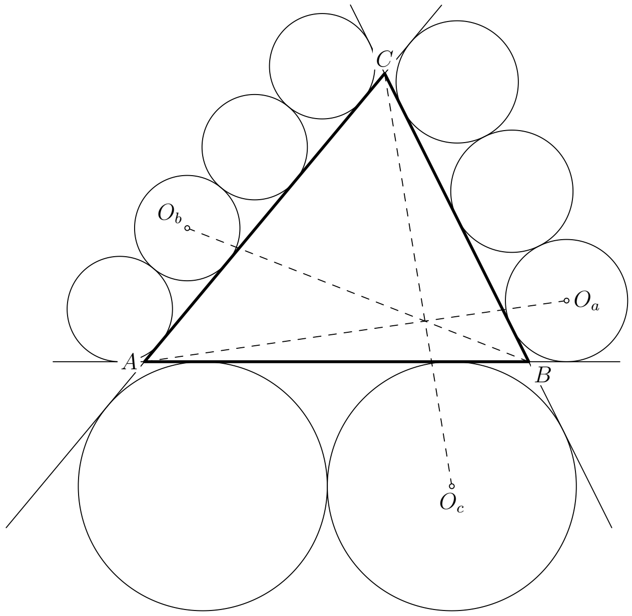


Fig. 9.7b

We have, therefore,

$$\frac{d_c(O_a)}{d_b(O_a)} \cdot \frac{d_a(O_b)}{d_c(O_b)} \cdot \frac{d_b(O_c)}{d_a(O_c)} = \frac{r_a}{r_c} \cdot \frac{r_c}{r_b} \cdot \frac{r_b}{r_a} = 1,$$

and the claim follows.

9.8. (*N. Beluhov, S. Gerdgikov*) A convex polygon P lies on a flat wooden table. You are allowed to drive some nails into the table. The nails must not go through P, but they may touch its boundary. We say that a set of nails blocks P if the nails make it impossible to move P without lifting it off the table. What is the minimum number of nails that suffices to block any convex polygon P?

Solution. If P is a parallelogram, then you need at least four nails to block it. Indeed, if there is a side s of P such that no nail touches the interior of s, then you can slide P in the direction determined by the two sides adjacent to s.

Now let P be an arbitrary convex polygon. We will show that four nails suffice to block P.

A set of nails blocks P if and only if, for every sufficiently small movement f (i.e., for every translation to a sufficiently small distance and every rotation to a sufficiently small angle), the interior of the image $f(P)$ of P covers some nail.

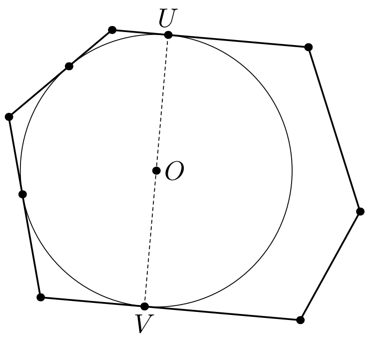


Fig. 9.8a

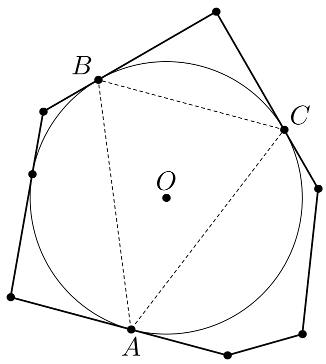


Fig. 9.8b

Let the circle c of center O be one of the largest circles contained within P. Let A_1, A_2, ... A_k be the points at which c touches P's boundary, and let H be their convex hull.

Suppose that there are two vertices U and V of H such that UV is a diameter of c. Place two nails at U and V. It is easy to see that, since the sides of P that contain U and V are parallel, the only movements still permitted to P are the translations in a direction perpendicular to UV. (Indeed, all other directions of translation would cause P to cover either U or V when the translation distance is small enough; all clockwise rotations whose center lies to the left of the ray $\overrightarrow{UV}$ would cause P to cover V when the rotation angle is small enough; all clockwise rotations whose center lies to the right of $\overrightarrow{UV}$ would cause P to cover U when the rotation angle is small enough; and so on.) A third nail prevents P from sliding to the left of $\overrightarrow{UV}$, and a fourth one prevents it from sliding to the right.

We are left to consider the case when no side or diagonal of H contains O.

Suppose that $O \notin H$. Let PQ be that side of H which separates H and O and let the tangents to c at P and Q meet in T. Then a homothety of center T and ratio larger than and sufficiently close to one maps c onto a larger circle contained within P: a contradiction.

Therefore, $O \in H$. Consider an arbitrary triangulation π of H and let ABC be that triangle in π which contains O. (A, B, and C being three of the contact points of H with the boundary of P.)

Since no side or diagonal of H contains O, O lies in the interior of $\triangle ABC$. It is easy to see, then — as above — that three nails placed at A, B, and C block P.

X Geometrical Olympiad in honour of I.F.Sharygin

Final round. Ratmino, 2014, July 31

Solutions

First day. 10 grade

10.1. (*I. Bogdanov, B. Frenkin*) The vertices and the circumcenter of an isosceles triangle lie on four different sides of a square. Find the angles of this triangle.

Answer. 15°, 15° and 150°.

Solution. Let the circumcenter O of triangle XYZ lie on side AB, and its vertices X, Y, Z lie on sides BC, CD, DA of square $ABCD$. Since segment OY intersect segment XZ, angle XYZ is obtuse, thus XZ is the base of the triangle. Then $OY \perp XZ$; since segments OY and XZ are perpendicular and their projections to perpendicular lines BC and AB respectively are equal, we obtain that these segments are also equal, i.e. the side of the triangle is equal to its circumradius. Since angle XYZ is obtuse, we obtain that $\angle XYZ = 150^\circ$, then two remaining angles are equal to 15°.

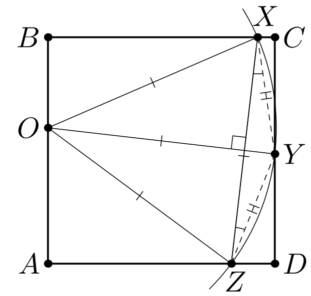


Fig. 10.1

10.2. (*A. Zertsalov, D. Skrobot*) A circle, its chord AB and the midpoint W of the minor arc AB are given. Take an arbitrary point C on the major arc AB. The tangent to the circle at C meets the tangents at A and B at points X and Y respectively. Lines WX and WY meet AB at points N and M respectively. Prove that the length of segment NM does not depend on point C.

First solution. Let T be the common point of AB and CW. Then AT and AC are antiparallel wrt angle AWC. Since WX is the symedian of triangle CAW, it is the median of triangle ATW, Thus N is the midpoint of AT. Similarly M is the midpoint of BT, i.e. $MN = AB/2$.

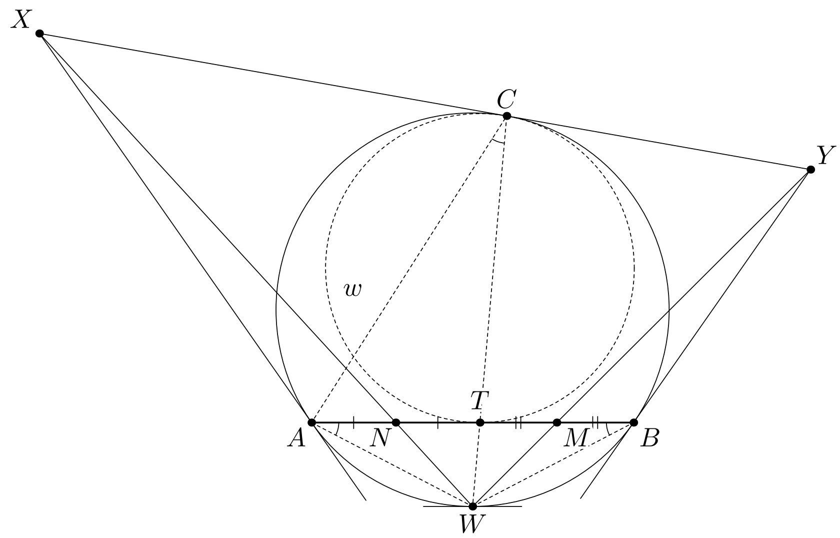


Рис. 10.2

Second solution. Consider circle w, touching XY at C and touching AB (at point T). It is easy to see that WX is the radical axis of A and w, i.e. it passes through the midpoint N of segment AT, Similarly WY passes through the midpoint M of segment ZB. Thus $MN = AB/2$.

10.3. (*A. Blinkov*) Do there exist convex polyhedra with an arbitrary number of diagonals (a *diagonal* is a segment joining two vertices of a polyhedron and not lying on the surface of this polyhedron)?

Answer. Yes.

Solution. Let $SA_1 \dots A_{n+2}$ be a $(n+2)$-gon pyramid and $TSA_{n+1}An+2$ be a pyramid with base $SA_{n+1}An+2$ and sufficiently small altitude. Then the diagonals of polyhedron $TSA_1 \dots A_{n+2}$ are segments $TA_1, \dots, TA_n$.

10.4. (*A. Garkavyj, A. Sokolov*) Let ABC be a fixed triangle in the plane. Let D be an arbitrary point in the plane. The circle with center D, passing through A, meets AB and AC again at points A_b and A_c respectively. Points B_a, B_c, C_a and C_b are defined similarly. A point D is called *good* if the points A_b, A_c, B_a, B_c, C_a, and C_b are concyclic. For a given triangle ABC, how many good points can there be?

Answer. 4.

Solution. It is evident that the circumcenter O satisfies the condition. Now let D does not coincide with O. Let A', B', C' be the projections of D to BC, CA, AB respectively. Then the midpoints of segments AB and A_bB_a are symmetric wrt C', therefore the perpendicular bisector to A_bB_a passes through point O', symmetric to O wrt D. The perpendicular bisectors to A_cC_a and B_cC_b also pass through O', thus O' is the center of the circle passing through six points.

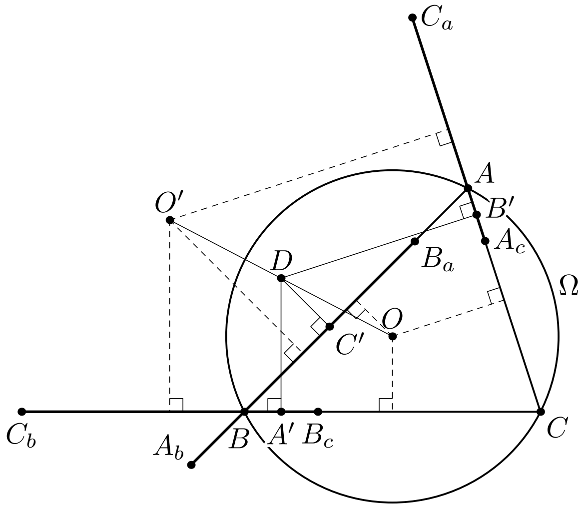


Fig. 10.4a

Since points D and O' are on equal distances from A_b and A_c, line DO' is the perpendicular bisector to A_bA_c. But $A_bA_c \parallel B'C'$, therefore $DO' \perp B'C'$. Similarly $DO' \perp A'B'$, i.e. points A', B', C' are collinear. Thus, D lies on the circumcircle of ABC and its Simson line $A'B'C'$ is perpendicular to radius OD. When D moves on the circle its Simson line rotates in the opposite direction with twice as smaller velocity, therefore there exists exactly three points with such property (these points form a regular triangle).

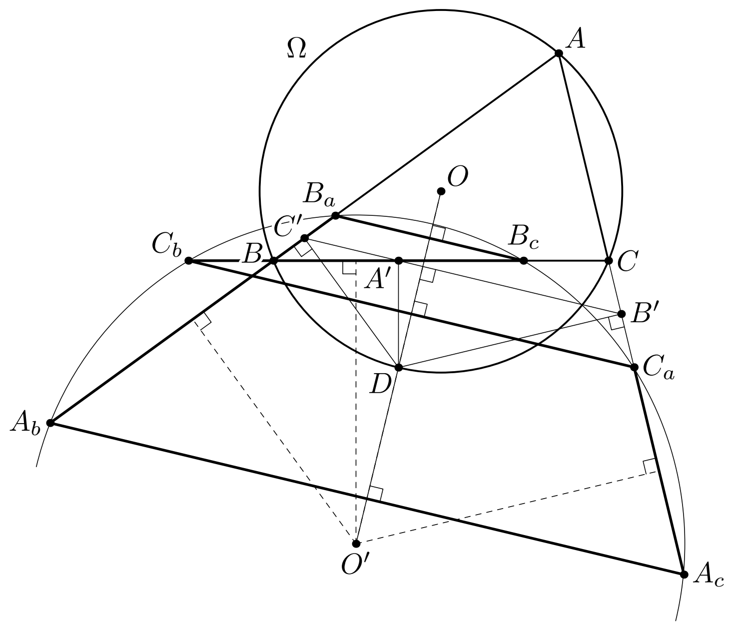


Fig. 10.4b

But several of these points can coincide with the vertices of the given triangle. Since the Simson line of the vertex A coincide with the corresponding altitude, that happens when the radius OA is parallel to BC, i.e. $|\angle B - \angle C| = 90°$. This is true for two vertices iff the angles of the given triangle are equal to $30°$, $30°$ and $120°$. From this the answer follows.

X Geometrical Olympiad in honour of I.F.Sharygin

Final round. Ratmino, 2014, August 1

Solutions

Second day. 10 grade

10.5. (*A. Zaslavsky*) The altitude from one vertex of a triangle, the bisector from the another one and the median from the remaining vertex were drawn, the common points of these three lines were marked, and after this everything was erased except three marked points. Restore the triangle. (For every two erased segments, it is known which of the three points was their intersection point.)

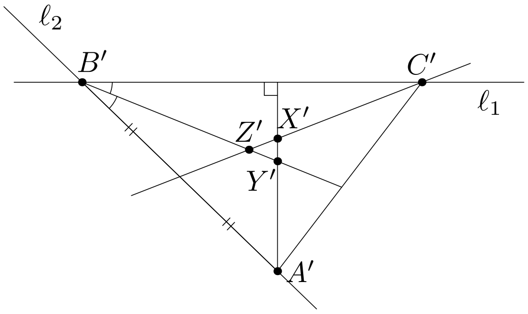


Fig. 10.5

Solution. Let X, Y, Z be the marked points. Then we have to find points A, B, C on lines XY, YZ, ZX respectively such that XY, YZ, ZX be the altitude, the bisector and the median of triangle ABC. From an arbitrary point B' draw a ray l_1 perpendicular to XY, and such ray l_2, that the bisector of the angle formed by these rays be parallel to YZ. Take an arbitrary point A' on l_2 and draw through the midpoint of $A'B'$ the line parallel to ZX meeting l_1 at point C'. Triangle $A'B'C'$ is homothetic to the desired one. Constructing the points corresponding to X, Y, Z, find the center and the coefficient of the homothety.

10.6. (*E. H. Garsia*) The incircle of a non-isosceles triangle ABC touches AB at point C'. The circle with diameter BC' meets the incircle and the bisector of angle B again at points A_1 and A_2 respectively. The circle with diameter AC' meets the incircle and the bisector of angle A again at points B_1 and B_2 respectively. Prove that lines AB, A_1B_1, A_2B_2 concur.

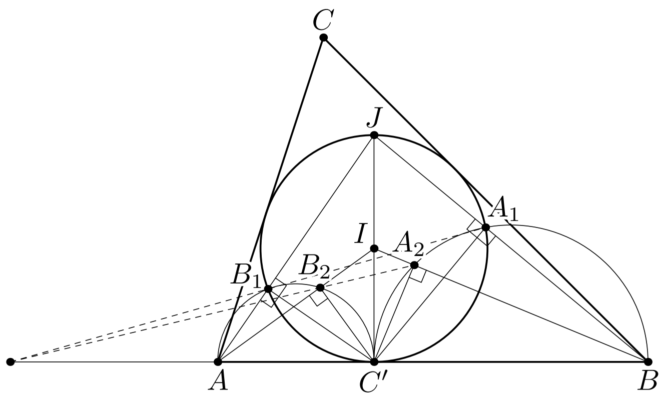


Fig. 10.6

Solution. Let I be the center of the incircle, and J be its point opposite to C'. Then A_1 and B_1 are the common points of AJ, BJ with the incircle (because $\angle AB_1C' = \angle C'B_1J = \angle BA_1C' = \angle C'A_1J = 90°$). From right-angled triangles $AC'I$, $BC'I$, $AC'J$ and $BC'J$ with altitudes $C'B_2$, $C'A_2$, $C'B_1$ and $C'A_1$ we obtain

$$\frac{AB_2}{B_2I}\cdot\frac{IA_2}{A_2B}=\frac{AC'^2}{C'I^2}\cdot\frac{IC'^2}{C'B^2}=\frac{AC'^2}{C'J^2}\cdot\frac{JC'^2}{C'B^2}=\frac{AB_1}{B_1J}\cdot\frac{JA_1}{A_1B},$$

i.e. by Menelaos theorem A_1B_1 and A_2B_2 meet AB at the same point.

10.7. (*S. Shosman, O. Ogievetsky*) Prove that the smallest dihedral angle between faces of an arbitrary tetrahedron is not greater than the dihedral angle between faces of a regular tetrahedron.

Solution. Let the greatest area of the faces of the tetrahedron is equal to 1. Let S_1, S_2, S_3 be the areas of the remaining faces, and α_1, α_2, α_3 be the angles between these faces and the greatest face. Then $S_1 \cos\alpha_1 + S_2 \cos\alpha_2 + S_3 \cos\alpha_3 = 1$ and, therefore, one of angles α_1, α_2, α_3 is not greater than $\arccos\frac{1}{3}$.

10.8. (*N. Beluhov*) Given is a cyclic quadrilateral $ABCD$. The point L_a lies in the interior of $\triangle BCD$ and is such that its distances to the sides of this triangle are proportional to the lengths of corresponding sides. The points L_b, L_c, and L_d are defined analogously. Given that the quadrilateral $L_aL_bL_cL_d$ is cyclic, prove that the quadrilateral $ABCD$ has two parallel sides.

Solution. If $ABCD$ is an isosceles trapezoid, then so is $L_aL_bL_cL_d$.

Suppose, then, that $L_aL_bL_cL_d$ is cyclic and that $ABCD$ has no parallel sides. Let $P = AB \cap CD$, $Q = AD \cap BC$, and $R = AC \cap BD$. Furthermore, let the tangents at A and B to the circumcircle of $ABCD$ meet in S, those at B and C meet in T, those at C and D – in U, and those at D and A – in V.

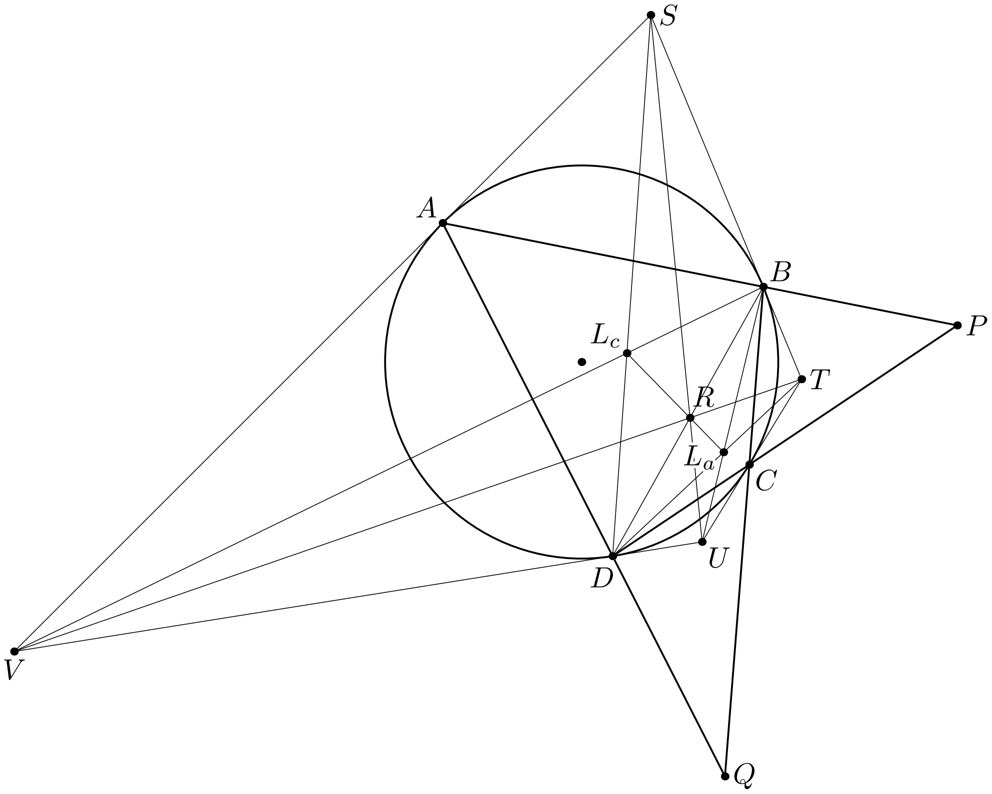


Fig. 10.8

It is well-known that $R = SU \cap TV$ and that $L_a = BU \cap DT$ and $L_c = BV \cap DS$. By Pappus's theorem for the hexagon $BUSDTV$, we see that R lies on L_aL_c. Similarly, R lies on L_bL_d and, therefore, $R = L_aL_c \cap L_bL_d$. Analogously, $P = L_aL_b \cap L_cL_d$ and $Q = L_aL_d \cap L_bL_c$.

Since the vertices of $\triangle PQR$ are the intersections of the diagonals and opposite sides of $ABCD$, the circumcircle k of $ABCD$ has the property that the polar of any vertex of $\triangle PQR$ with respect to k is the side opposite to that vertex. Analogously, the circumcircle s of $L_aL_bL_cL_d$ has the same property. Given $\triangle PQR$, however, there is exactly one such circle. It follows that $k \equiv s$, and this is a contradiction because $L_aL_bL_cL_d$ lies in the interior of $ABCD$.

XI Geometrical Olympiad in honour of I.F.Sharygin
Final round. Grade 8. First day. Solutions

Ratmino, 2015, July 30.

1. (V. Yasinsky) In trapezoid $ABCD$ angles A and B are right, $AB = AD$, $CD = BC + AD$, $BC < AD$. Prove that $\angle ADC = 2\angle ABE$, where E is the midpoint of segment D.

   First solution. Choose a point K on the side CD so that $CK = CB$; let M be the common point of AB and the perpendicular from K to CD. The right triangles BCM and KCM are congruent by hypothenuse and leg, so $BM = MK$, and CM bisects the angle C. By the problem condition we get $KD = AD$, and in a similar way we obtain that $AM = MK$ and that DM bisects the angle D. Thus $AM = AB/2 = AE$, i.e., the triangles ABE and ADM are congruent. Therefore, $\angle ADC = 2\angle ADM = 2\angle ABE$ (Fig. 8.1).

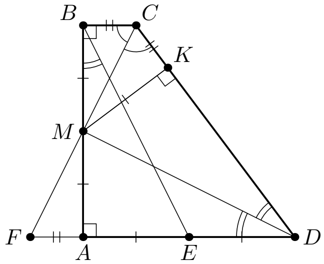
   

   Fig. 8.1

   Second solution. Choose a point F on the extension of DA beyond point A so that $AF = BC$. Then we have $DF = DC$. Let M be the common point of AB and CF, i.e., M is the midpoint of AB. The right triangles ABE and ADM are congruent, and DM is a median in the triangle CDF, so it bisects the angle CDA; therefore, $\angle CDA/2 = \angle ABE$.

2. (A. Blinkov) A circle passing through A, B and the orthocenter of triangle ABC meets sides AC, BC at their inner points. Prove that $60° < \angle C < 90°$.

   First solution. Let A' and B' be the second meeting points of the circle with BC and AC, respectively. Then $\angle C = (\smile AB - \smile A'B')/2$. Since the angle between the altitudes is equal to $180° - \angle C = \smile AB/2$, we obtain that $180° - \angle C > \angle C$ and thus $\angle C < 90°$.

   On the other hand, the angle C is greater than the angle between the tangents to the circle at A and B; the latter angle is equal to $180° - 2\angle C$. Therefore, $\angle C > 60°$.

   Second solution. If the angle C is not acute, then H either lies outside the triangle or coincides with C. In both cases the intersection points do not belong to the interiors of the sides.

Since $\angle AA'B = \angle BB'A = \angle AHB = 180^\circ - \angle C$, we have $\angle AA'C = \angle BB'C = \angle C$; but these angles are greater than the angles A and B as they are external angles of triangles $AA'B$ and $BB'A$. Therefore, C is the largest angle of the triangle ABC, i.e. $\angle C > 60^\circ$.

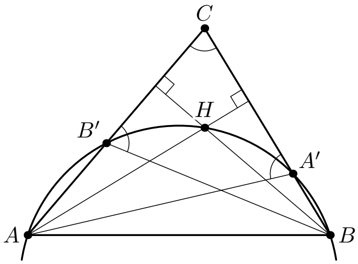


Fig 8.2

3. (M. Yevdokimov) In triangle ABC we have $AB = BC$, $\angle B = 20^\circ$. Point M on AC is such that $AM : MC = 1 : 2$, point H is the projection of C to BM. Find angle AHB.

   Answer. 100°.

   First solution. Construct a point D such that the quadrilateral $ABCD$ is a rhombus. Let O be the center of the rhombus. Then the line BM divides the median AO of the triangle ABD in ratio $2 : 1$. Thus this line is also a median, i.e., it passes through the midpoint K of the segment AD. Since the points O and H lie on the circle with diameter BC, we have $\angle KHO = \angle BCO = \angle KAO$. Therefore, the quadrilateral $AHOK$ is cyclic, so $\angle AHK = \angle AOK = 80^\circ$, and hence $\angle AHB = 100^\circ$ (Fig. 8.3).

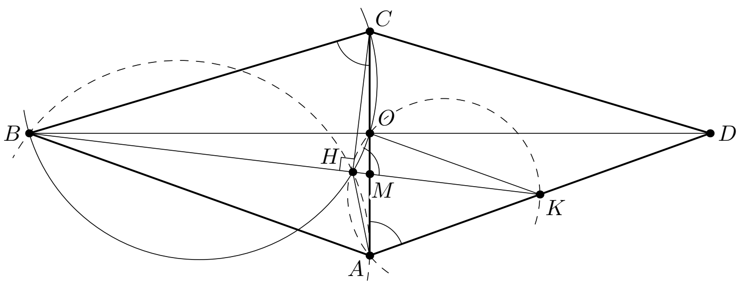


Fig. 8.3

   Second solution. As in the previous solution, we notice that the quadrilateral $BCOH$ is cyclic. So, $MH \cdot MB = MO \cdot MC = MA^2$. Thus the circle AHB is tangent to the line AC, which yields $\angle AHB = 180^\circ - \angle BAC = 100^\circ$.

4. (N. Belukhov) Prove that an arbitrary convex quadrilateral can be divided into five polygons having symmetry axes.

   Solution. Let $ABCD$ be a given quadrilateral. If $AB + CD = AD + BC$, then the quadrilateral is circumscribed. The radii of its incircle joining the center with the points of tangency with the sides divide the quadrilateral into four symmetric quadrilaterals. It remains to divide one of them into two isosceles triangles.

Now assume that $AB + CD > AD + BC$. Construct a circle with center O_1 tangent to sides AB, AD, and CD at points P_1, Q_1, and R_1 respectively, and also construct a circle with center O_2 tangent to the sides AB, BC, and CD at points P_2, Q_2, R_2 respectively (Fig. 8.4). The radii passing to these six points divide $ABCD$ into the quadrilaterals $AP_1O_1Q_1$, $BP_2O_2Q_2$, $CQ_2O_2R_2$, $DQ_1O_1R_1$ and the hexagon $P_1P_2O_2R_2R_1O_1$. Their symmetry axes are respectively the bisectors of the angles A, B, C, D, and the bisector of the angle between the lines AB and CD.

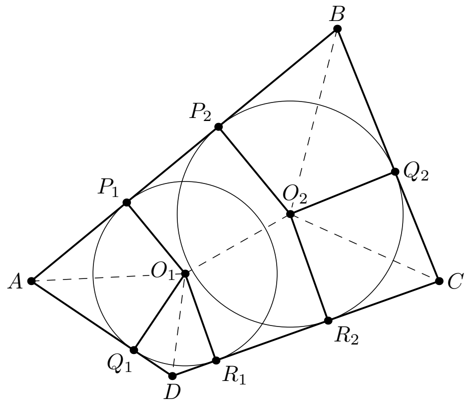


Fig. 8.4

XI Geometrical Olympiad in honour of I.F.Sharygin
Final round. Grade 8. Second day

Ratmino, 2015, July 31.

5. (E. Bakayev, A. Zaslavsky) Two equal hard triangles are given. One of their angles is equal to α (these angles are marked). Dispose these triangles on the plane in such a way that the angle formed by some three vertices would be equal to $\alpha/2$. (*No instruments are allowed, even a pencil.*)

   Solution. The required configuration is shown in Fig 8.5. The triangle BCC' is isosceles, with $BC = C'C$ and $\angle C'CB = 180^\circ - \alpha$. Thus $\angle C'BC = \alpha/2$.

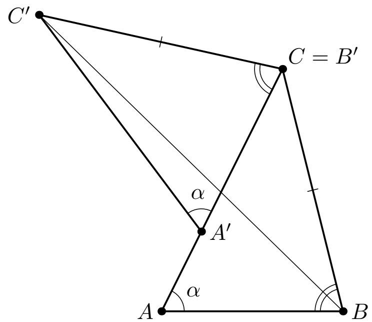


Fig. 8.5

6. (D. Prokopenko) Lines b and c passing through vertices B and C of triangle ABC are perpendicular to sideline BC. The perpendicular bisectors to AC and AB meet b and c at points P and Q respectively. Prove that line PQ is perpendicular to median AM of triangle ABC.

   First solution. Let M be the midpoint of BC. It suffices to prove that $AP^2 - AQ^2 = MP^2 - MQ^2$.

   Since the points P and Q lie on the perpendicular bisectors to AC and BC, respectively, we have $AP^2 - AQ^2 = CP^2 - BQ^2 = (BC^2 + BP^2) - (BC^2 + CQ^2) = (MB^2 + BP^2) - (MC^2 + CQ^2) = MP^2 - MQ^2$.

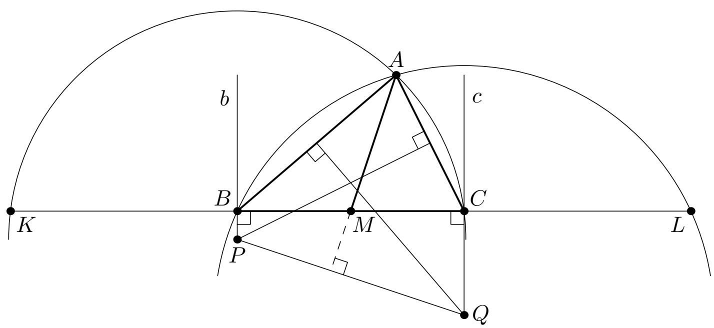


Fig. 8.6

Second solution. Construct a circle centered at P and passing through A. It meets BC at C and also at a point K symmetric to C in B. Similarly, the circle centered at Q and passing through A meets BC at B and at a point L symmetric to B in C. The powers of M with respect to these circles are equal, so the radical axis of these circles is AM, and it is perpendicular to the line of centers PQ (Fig. 8.6).

7. (M. Kungozhin) Point M on side AB of quadrilateral $ABCD$ is such that quadrilaterals $AMCD$ and $BMDC$ are circumscribed around circles centered at O_1 and O_2 respectively. Line O_1O_2 cuts an isosceles triangle with vertex M from angle CMD. Prove that $ABCD$ is a cyclic quadrilateral.

   Solution. If $AB \parallel CD$ then the incircles of $AMCD$ and $BMDC$ have equal radii; now the problem conditions imply that the whole picture is symmetric about the perpendicular from M to O_1O_2, and hence $ABCD$ is an iosceles trapezoid (or a rectangle).

   Now suppose that the lines AB and CD meet at a point K; we may assume that A lies between K and B. The points O_1 and O_2 lie on the bisector of the angle BKC. By the problem condition, this angle bisector forms equal angles with the lines CM and DM; this yields $\angle DMK = \angle KCM$ (Fig. 8.7). Since O_1 and O_2 are the incenter of $\triangle KMC$ and an excenter of $\triangle KDM$, respectively, we have $\angle DO_2K = \angle DMK/2 = \angle KCM/2 = \angle DCO_1$, so the quadrilateral CDO_1O_2 is cyclic. Next, the same points are an excenter of $\triangle AKD$ and the incenter of $\triangle KBC$, respectively, so $\angle KAD = 2\angle KO_1D = 2\angle DCO_2 = \angle KCB$; this implies the desired cyclicity of the quadrilateral $ABCD$.

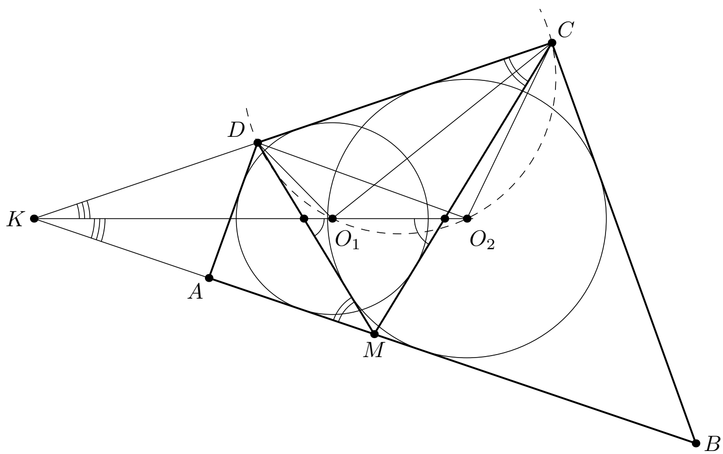


Fig. 8.7

8. (A. Antropov, A. Yakubov) Points C_1, B_1 on sides AB, AC respectively of triangle ABC are such that $BB_1 \perp CC_1$. Point X lying inside the triangle

is such that $\angle XBC = \angle B_1BA$, $\angle XCB = \angle C_1CA$. Prove that $\angle B_1XC_1 = 90^\circ - \angle A$.

First solution. Let X_1 be the projection of X onto BC, and let O be the meeting point of the lines BB_1 and CC_1. Then the triangle C_1BO is similar to the triangle XBX_1 by two angles, thus $\frac{BC_1}{BX} = \frac{BO}{BX_1}$. This implies that the triangles BC_1X and BOX_1 are also similar by two proportional sides and the angle between them. Therefore, $\angle BXC_1 = \angle BX_1O$. Similarly we obtain $\angle B_1XC = \angle OX_1C$ (Fig. 8.8). Hence $\angle BXC_1 + \angle CXB_1 = \angle BX_1O + \angle OX_1C = 180^\circ$. Therefore, $\angle C_1XB_1 = 180^\circ - \angle BXC = \angle XBC + \angle XCB = \angle ABB_1 + \angle ACC_1 = \angle BOC - \angle BAC = 90^\circ - \angle A$, as required.

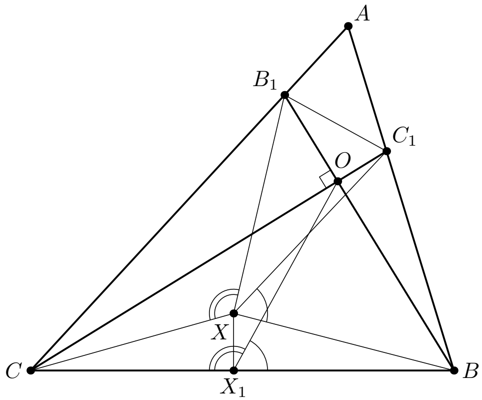


Fig. 8.8

Second solution. We start with two lemmas.

Lemma 1. If the diagonals of a quadrilateral are perpendicular, then the projections of their meeting point to the sidelines are concyclic.

Proof. Let the diagonals of a quadrilateral $ABCD$ meet at O, and let K, L, M, and N be the projections of O onto AB, BC, CD, and DA, respectively. Since the quadrilaterals $OKBL$, $OLCM$, $OMDN$, and $ONAK$ are cyclic, we have $\angle LKN + \angle LMN = \angle OBC + \angle OCB + \angle OAD + \angle ODA = 180^\circ$.

Lemma 2. If the projections of a point P onto the sidelines of $ABCD$ are concyclic, then the reflections of the lines AP, BP, CP, and DP about the bisectors of the corresponding angles are concurrent.

Proof. Since the projections of P to the sidelines are concyclic, the points K, L, M, and N which are symmetric to P about the lines AB, BC, CD, and DA, respectively, are also concyclic. Since $AK = AP = AN$, the perpendicular bisector to the segment KN coincides with the bisectrix of the angle KAN; this bisectrix is symmetric to AP about the bisectrix of the angle BAD. Therefore, all of these four lines pass through the circumcenter of $KLMN$.

To solve the problem, apply these two lemmas to the quadrilateral BCC_1B_1. Since the lines BX and CX are symmetric to BB_1 and CC_1 about the bisectors of the angles B and C respectively, the lines B_1X and C_1X are also symmetric to B_1B and C_1C about the bisectors of the angles CB_1C_1 and BC_1B_1 respectively. This yields that $\angle B_1XC_1 = 180° - \angle XB_1C_1 - \angle XC_1B_1 = 180° - \angle CB_1O - \angle BC_1O = 180° - (90° + \angle A) = 90° - \angle A$.

XI Geometrical Olympiad in honour of I.F.Sharygin
Final round. Grade 9. First day

Ratmino, 2015, July 30.

1. (D. Mukhin) Circles α and β pass through point C. The tangent to α at this point meets β at point B, and the tangent to β at C meets α at point A so that A and B are distinct from C and angle ACB is obtuse. Line AB meets α and β for the second time at points N and M respectively. Prove that $2MN < AB$.

   Solution. Since AC and BC are tangent to β and α, respectively, we have $\angle ACM = \angle CBA$ and $\angle BCN = \angle CAB$. Since the angle ACB is obtuse, the points A, M, N, B are arranged on AB in this order. Using the tangency again, we get $AM = AC^2/AB$ and $BN = BC^2/AB$. Using the AM–QM inequality and the triangle inequality in this order, we obtain $2(AM+BN) \geqslant \frac{(AC+BC)^2}{AB} > AB$; this is equivalent to the desired inequality.

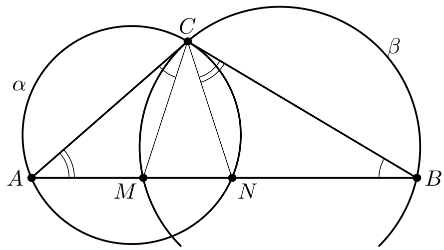


Fig. 9.1

2. (A. Zaslavsky) A convex quadrilateral is given. Using a compass and a ruler construct a point such that its projections to the sidelines of this quadrilateral are the vertices of a parallelogram.

   Solution. All the angles in the solution are directed.

   Let K, L, M, and N be the projections of the point P onto AB, BC, CD, and DA, respectively. The condition $KL \parallel MN$ is equivalent to $\angle BKL + \angle MND = \angle BAD$. Since $PKBL$ and $PMDB$ are cyclic quadrilaterals, we have $\angle BKL = \angle BPL$ and $\angle MND = \angle MPD$. Consequently, the condition $KL \parallel MN$ is equivalent to

$$\angle BPD = (\angle BPL + \angle MPD) + \angle LPM = \angle BAD + (180^\circ - \angle DCB).$$

   Thus, we can construct a circle passing through B and D and containing P (Fig. 9.2).

   Similarly, the condition $KN \parallel LM$ is equivalent to $\angle CPA = 180^\circ + \angle CBA - \angle ADC$, so we can construct a circle through A and C containing P. One of the meeting points of the two constructed circles is the Miquel point of the lines AB, BC, CD, and DA (its projections are collinear by the Simson theorem). The other point is the desired one.

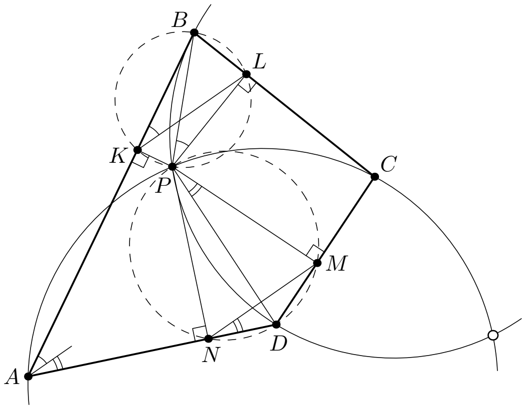


Fig. 9.2

3. (M. Kharitonov, A. Polyansky) Let 100 discs lie on the plane in such a way that each two of them have a common point. Prove that there exists a point lying inside at least 15 of these discs.

Solution. Let K be the smallest of the given discs; we may suppose that its radius is equal to 1. Let O be the center of K, and let $A_1A_2A_3A_4A_5A_6$ be a regular hexagon with center O and side length $\sqrt{3}$. We will prove that each of given discs contains one of points $O, A_1, \ldots, A_6$; by pigeonhole principle, this implies the problem statement.

Let O' be the center of some disc K'. If O' lies in K, then K' contains O, because the radius of K' is at least 1. So we assume henceforth that $OO' > 1$.

The angle between the ray OO' and one of the rays OA_i (say, with OA_1) is at most $30°$. Thus we have

$$O'A_1^2 = O'O^2 + OA_1^2 - 2O'O \cdot OA_1 \cos \angle O'OA_1 \leqslant O'O^2 + 3 - 3O'O.$$

If $1 < O'O \leq 2$, then $O'A_1 \leq 1$, so K' contains A_1. Otherwise, we have $O'O > 2$, and this implies that $O'A_1 < O'O - 1$. On the other hand, the radius of K' is not less than $OO' - 1$ because this disc intersects K, so in this case K' also contains A_1.

4. (R. Krutovsky, A. Yakubov) A fixed triangle ABC is given. Point P moves on its circumcircle so that segments BC and AP intersect. Line AP divides triangle BPC into two triangles with incenters I_1 and I_2. Line I_1I_2 meets BC at point Z. Prove that all lines ZP pass through a fixed point.

First solution. It is known that the centers of two arbitrary circles, together with their internal and external homothety centers, form a harmonic quadruple. For two given circles centered at I_1 and I_2, the external homothety center is Z, and the internal one lies on the line AP, since BZ and AP are an external and an internal common tangents, respectively. Thus, the projection mapping centered at P acting from the line I_1I_2 to the circumcircle Ω of $\triangle ABC$ maps the internal homothety center to A, and the points I_1 and I_2 to the midpoints of the arcs AB and AC, respectively. These three points are fixed, so the image of Z is also fixed, and all lines PQ pass through this fixed point on Ω.

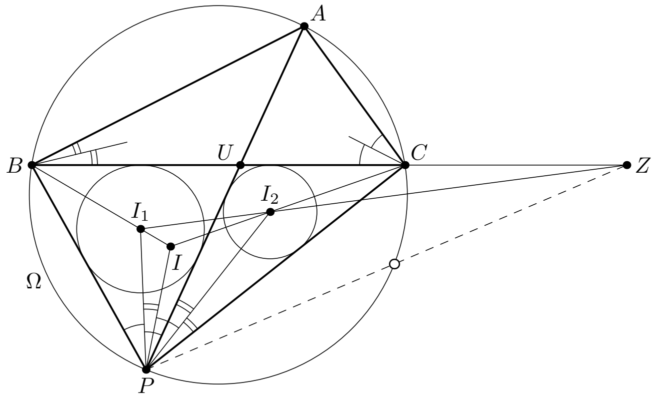


Fig. 9.4

Second solution. Let U be the common point of AP and BC. We prove that the cross-ratio $(BC; ZU)$ is independent of P; again, after projecting the line BC from P to the circumcircle Ω of $\triangle ABC$ this will yield that the line PZ meets Ω at a fixed point.

Let I, I_1, and I_2 be the incenters of the triangles PBC, PBU, and PCU respectively. Applying Menelaus' theorem to the triangle BIC we obtain

$$\frac{BZ}{CZ} = \frac{BI_1}{I_1I} \cdot \frac{II_2}{I_2C}.$$

Since PI, PI_1, and PI_2 are the bisectors of the angles BPC, BPU, and CPU respectively, we have $\angle BPI_1 = \angle IPI_2 = \angle C/2$ and $\angle I_1PI = \angle I_2PC = \angle B/2$. Hence, applying the sine law to the triangles BPI_1, I_1PI, IPI_2, and I_2PC we obtain

$$\frac{BI_1}{I_1I} = \frac{BP}{IP} \cdot \frac{\sin(\angle C/2)}{\sin(\angle B/2)} \quad \text{and} \quad \frac{II_2}{I_2C} = \frac{IP}{CP} \cdot \frac{\sin(\angle C/2)}{\sin(\angle B/2)}.$$

Similarly, applying the sine law to the triangles BPC, ABU, and ACU, we get

$$\frac{BP}{CP} = \frac{\sin\angle BCP}{\sin\angle CBP} = \frac{\sin\angle BAU}{\sin\angle CAU} = \frac{BU}{AB} \cdot \frac{AC}{CU}.$$

Multiplying the four obtained equalities, we conclude that

$$(BC; ZU) = \frac{\sin^2(\angle C/2)}{\sin^2(\angle B/2)} \cdot \frac{AC}{AB} = \frac{\mathrm{tg}(\angle C/2)}{\mathrm{tg}(\angle B/2)}.$$

Remark. The value of $(BC; ZU)$ obtained above may be implemented to show that the line PZ meets the circumcircle at the point collinear with the incenter of $\triangle ABC$ and the midpoint of the arc CAB. One may also see that there exists a circle tangent to the circumcircle at that point and also tangent to the segments AB and AC.

XI Geometrical Olympiad in honour of I.F.Sharygin
Final round. Grade 9. Second day

Ratmino, 2015, July 31.

5. (D. Svhetsov) Let BM be a median of nonisosceles right-angled triangle ABC ($\angle B = 90^\circ$), and H_a, H_c be the orthocenters of triangles ABM, CBM respectively. Prove that lines AH_c and CH_a meet on the medial line of triangle ABC.

   Solution. Let A' and C' be the midpoints of the legs AB and BC, respectively. Since the triangles AMB and BMC are isosceles, their altitudes from M pass through A' and C', respectively. Then $AA' \perp BC \perp H_cC'$, $AH_a \perp BM \perp H_cC$, and $A'H_a \perp AB \perp C'C$. This shows that the corresponding sides of the triangles $AA'H_a$ and $H_cC'C$ are parallel, i.e., these triangles are homothetic. Therefore, the lines AH_c, H_aC, and $A'C'$ concur at the homothety center, and it lies on the midline $A'C'$ (Fig. 9.5).

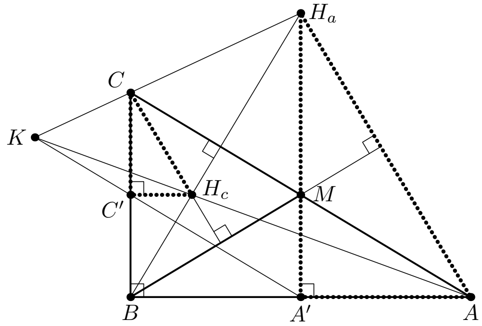


Fig. 9.5

6. (A. Zaslavsky) The diagonals of convex quadrilateral $ABCD$ are perpendicular. Points A', B', C', D' are the circumcenters of triangles ABD, BCA, CDB, DAC respectively. Prove that lines AA', BB', CC', DD' concur.

   Solution. We start with two lemmas.

   Lemma 1. If the diagonals of a quadrilateral are perpendicular, then the projections of their meeting point to the sidelines are concyclic.

   Proof. Let the diagonals of a quadrilateral $ABCD$ meet at O, and let K, L, M, and N be the projections of O onto AB, BC, CD, and DA, respectively. Since the quadrilaterals $OKBL$, $OLCM$, $OMDN$, and $ONAK$ are cyclic, we have $\angle LKN + \angle LMN = \angle OBC + \angle OCB + \angle OAD + \angle ODA = 180^\circ$.

   Lemma 2. If the projections of a point P onto the sidelines of $ABCD$ are concyclic, then the reflections of the lines AP, BP, CP, and DP about the bisectors of the corresponding angles are concurrent.

Proof. Since the projections of P to the sidelines are concyclic, the points K, L, M, and N which are symmetric to P about the lines AB, BC, CD, and DA, respectively, are also concyclic. Since $AK = AP = AN$, the perpendicular bisector to the segment KN coincides with the bisectrix of the angle KAN; this bisectrix is symmetric to AP about the bisectrix of the angle BAD. Therefore, all of these four lines pass through the circumcenter of $KLMN$.

Now we pass to the solution. Note that the line AC contains the altitude of the triangle DAB, so the line AA' is its reflection about the bisector of the angle A; similar arguments work for the other vertices of $ABCD$. Thus the four lines under consideration are concurrent.

Remark. One may show that, if three of the lines AA', BB', CC', and DD' are concurrent, then either the quadrilateral $ABCD$ is cyclic or its diagonals are perpendicular. In both cases the fourth line also passes through the concurrency point.

7. (D. Krekov) Let ABC be an acute-angled, nonisosceles triangle. Altitudes AA' and BB' meet at point H, and the medians of triangle AHB meet at point M. Line CM bisects segment $A'B'$. Find angle C.

Answer. $45°$.

Solution. Let C_0 be the midpoint of AB, and let H' be the point symmetric to H in C_0; it is well known that H' is the point opposite to C on the circumcircle of $\triangle ABC$. The medians CC_0 and CM of similar triangles ABC and $A'B'C$ are symmetric about the bisector of the angle C. An altitude CH and the diameter CH' of the circumcircle are also symmetric about this bisector. Therefore, $\angle H'CC_0 = \angle MCH$, i.e., CM is a symmedian in the triangle CHH' (Fig. 9.7). Thus we have $(CH'/CH)^2 = H'M/MH = 2$; now, from $CH = CH' \cos \angle C$ we obtain $\angle C = 45°$.

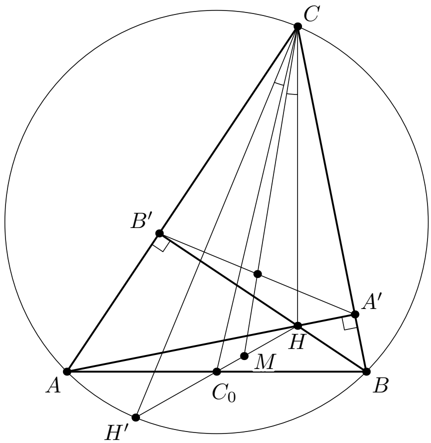


Fig. 9.7

8. (I.Frolov) A perpendicular bisector to side BC of triangle ABC meets lines AB and AC at points A_B and A_C respectively. Let O_a be the circumcenter of triangle AA_BA_C. Points O_b and O_c are defined similarly. Prove that the circumcircle of triangle $O_aO_bO_c$ touches the circumcircle of the original triangle.

   Solution. Let A', B', and C' be the pairwise meeting points of the tangents to the circumcircle of $\triangle ABC$ at its vertices (so, A' is the meeting point of the tangents at B and C, and so on).

   Consider an arbitrary triangle formed by the lines CA, CB, and any line ℓ perpendicular to AB. All such triangles are pairwise homothetic at C. Moreover, if the line ℓ moves with a constant speed, then the circumcenter of the triangle also moves with a constant speed along some line passing through C.

   Consider two specific positions of ℓ, when it passes through B and A. For the first position, the circumcenter of the obtained triangle CC'_AB is A', since $CA' = BA'$ and $\angle CA'B = 180° - 2\angle A = 2\angle CC'_AB$ (Fig. 9.8). Similarly, the circumcenter for the second position in B'. Hence, the circumcenter O_c of the triangle CC_AC_B is the midpoint of $A'B'$. (Different cases of mutual positions of the points can be treated analogously.)

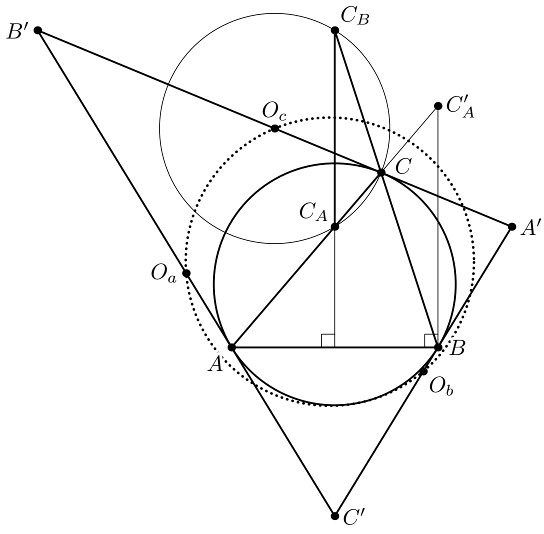


Fig. 9.8

   In a similar way we obtain that O_a and O_b are the midpoints of $B'C'$ and $C'A'$ respectively. Therefore, the circumcircle of $\triangle O_aO_bO_c$ is the Euler circle of the triangle $A'B'C'$, and the circumcircle of $\triangle ABC$ is either its incircle (if $\triangle ABC$ is acute-angled) or its excircle (otherwise). In any case, the Feuerbach theorem shows that these two circles are tangent to each other.

XI Geometrical Olympiad in honour of I.F.Sharygin
Final round. Grade 10. First day

Ratmino, 2015, July 30.

1. (A. Karlyuchenko) Let K be an arbitrary point on side BC of triangle ABC, and KN be a bisector of triangle AKC. Lines BN and AK meet at point F, and lines CF and AB meet at point D. Prove that KD is a bisector of triangle AKB.

   Solution. By the bisector property, we have $\frac{CN}{NA} = \frac{CK}{KA}$. Now by Ceva's theorem we obtain
   $$\frac{BD}{DA} = \frac{BK}{KC} \cdot \frac{CN}{NA} = \frac{BK}{KA},$$
   which means exactly that KD is a bisector of the angle AKB.

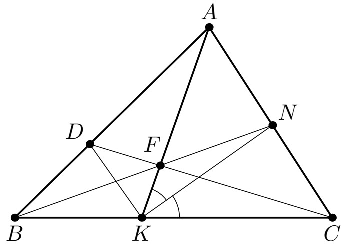


Fig. 10.1

2. (A. Shapovalov) Prove that an arbitrary triangle with area 1 can be covered by an isosceles triangle with area less than $\sqrt{2}$.

   Solution. Let ABC be a given triangle with $AB \geq AC \geq BC$. Let CH be its altitude. Let A' be the reflection of A in H, and let B' be the reflection of B about the bisector of angle A. Then each of the isosceles triangles ACA' and ABB' covers $\triangle ABC$, and we have $S_{ACA'}/S_{ABC} = AA'/AB = 2AH/AB$ and $S_{ABB'}/S_{ABC} = AB'/AC = AB/AC$. The product of these two ratios is $2AH/AC < 2$; therefore, one of them is less than $\sqrt{2}$.

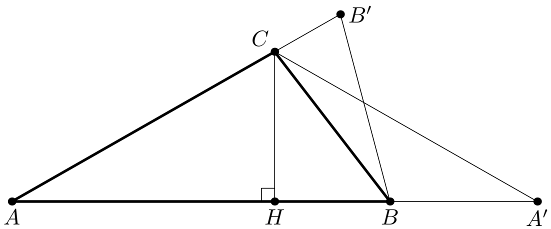


Fig. 10.2

3. (A. Akopyan) Let A_1, B_1 and C_1 be the midpoints of sides BC, CA and AB of triangle ABC. Points B_2 and C_2 are the midpoints of segments BA_1 and CA_1 respectively. Point B_3 is symmetric to C_1 wrt B, and C_3 is symmetric

to B_1 wrt C. Prove that one of common points of circles BB_2B_3 and CC_2C_3 lies on the circumcircle of triangle ABC.

First solution. Choose a point X on the circumcircle Ω of $\triangle ABC$ so that $\angle XA_1C = \angle CA_1A$. Then X is symmetric to the second common point of AA_1 and Ω about the perpendicular bisector of BC; hence, $A_1X \cdot A_1A = A_1B \cdot A_1C = A_1C^2$. This implies that the triangles XA_1C and CA_1A are similar.

Let T be the midpoint of AA_1. Then XC_2 and CT are corresponding medians in similar triangles, thus $\angle CXC_2 = \angle ACT$ (Fig. 10.3.1). On the other hand, the quadrilateral CTC_2C_3 is a parallelogram, i.e., $\angle CC_3C_2 = \angle ACT = \angle CXC_2$. Hence, X lies on the circumcircle of $\triangle CC_2C_3$.

Similarly, X lies on the circumcircle of $\triangle BB_2B_3$. Therefore, X is a desired point.

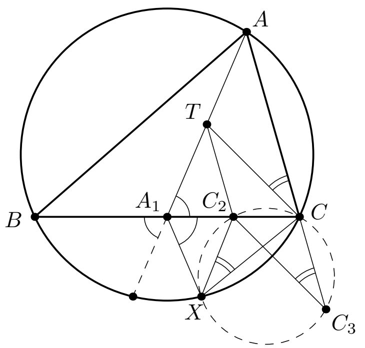


Fig. 10.3.1

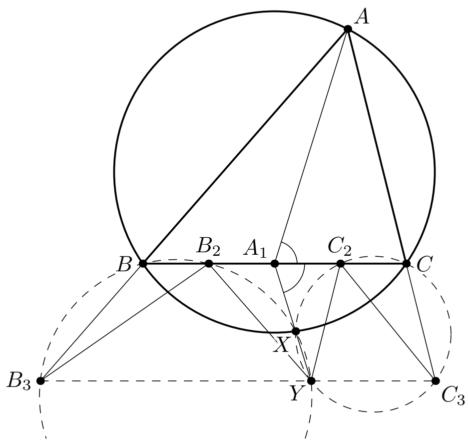


Fig. 10.3.2

Second solution. Let us construct a triangle YB_2C_2 similar to $\triangle ABC$ and separated from that by the line BC (Fig, 10.3.2). Since $B_2C_2 = BC/2$, the points B_3, Y, and C_3 are equidistant from the line BC (this common distance is half the distance of A from BC). Moreover, we have $YB_2 = AB/2 = BB_3$ and $YC_2 = AC/2 = CC_3$. Altogether this means that BB_2YB_3 and CC_2YC_3 are isosceles trapezoids, and hence Y is one of the common points of the circles BB_2B_3 and CC_2C_3.

Since the powers of A_1 with respect to these circles are equal, their second common point lies on A_1Y. Let the ray A_1Y meet the circumcircle of $\triangle ABC$ at X. Since AA_1 and YA_1 are corresponding medians in similar triangles ABC and YB_2C_2, we obtain $\angle XA_1C = \angle CA_1A$; thus, as in the previous solution, we have $A_1X \cdot A_1A = A_1B \cdot A_1C$. Therefore, $A_1X \cdot A_1Y = A_1X \cdot A_1A/2 = A_1B^2/2 = A_1B \cdot A_1B_2$, and so X is the second common point of the two circles under consideration.

4. (I. Yakovlev) Let AA_1, BB_1, CC_1 be the altitudes of an acute-angled, non-isosceles triangle ABC, and A_2, B_2, C_2 be the touching points of sides BC, CA, AB with the correspondent excircles. It is known that line B_1C_1 touches the incircle of ABC. Prove that A_1 lies on the circumcircle of $A_2B_2C_2$.

First solution. Let H, I, and O be the orthocenter, the incenter, and the circumcenter of $\triangle ABC$, respectively. Let Ω, ω, and r denote its circumcircle, incircle, and inradius, respectively. Let A', B', and C' be the tangency points of ω with the sides BC, AC, and AB, and let I_A, I_B, and I_C be the excenters corresponding to these sides, respectively. Finally, let M_A be the midpoint of BC.

The problem condition says that the quadrilateral BC_1B_1C is circumscribed around ω. This quadrilateral is also inscribed into a circle with diameter BC. It is known that in such a quadrilateral, the circumcenter, the incenter, and the meeting point of the diagonals are collinear (e.g., this follows from the fact that the polar line of the meeting point of the diagonals with respect to both the incircle and the circumcircle is the line passing through the meeting points of the extensions of opposite sides). Thus the points H, I, and M_A are collinear.

Let A_3 be the point opposite to A' on ω. It is known that the points A, A_3, and A_2 are collinear. Furthermore, the point M_A is the midpoint of A_2A'. Therefore, IM_A is a midline in the triangle $A_2A'A_3$. this yields that $HI \parallel AA_3$, i.e., the quadrilateral AA_3IH is a parallelogram, and so $r = A_3I = AH = 2OM_A$ (this quadrilateral is non-degenerate, otherwise $\triangle ABC$ would be isosceles). Therefore, M_AO is a midline in the triangle $IA'A_2$, since it passes through the midpoint of $A'A_2$ and is parallel to IA'; moreover, since $M_AO = r/2 = IA'/2$, we conclude that O is the midpoint of IA_2.

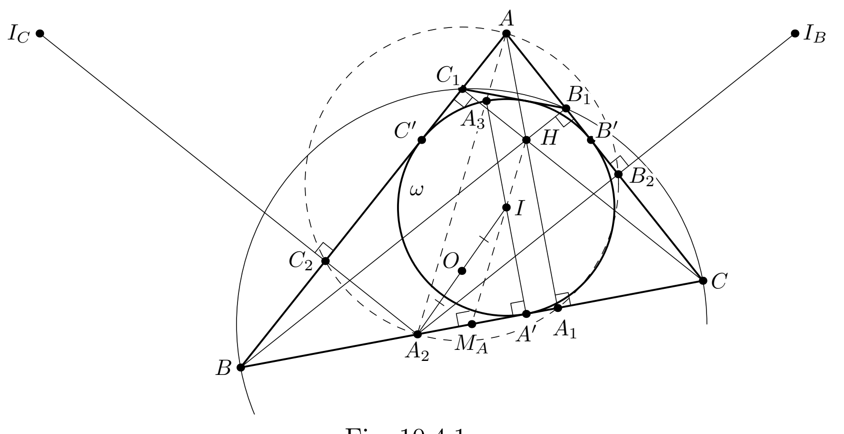


Fig. 10.4.1

Thus, the point A_2 is symmetric to I in O. Finally, the lines I_CC_2 and IC', as well as the lines I_BB_2 and IB', are symmetric in O; therefore, both these lines pass through A_2. Hence $\angle A_2B_2B = \angle A_2C_2C = 90^\circ = \angle AA_1B$. This means that all five points A, A_2, B_2, C_2, and A_1 lie on a circle with diameter AA_2.

Second solution. We present a different proof of the fact that $OM_A = r/2$; after that, one may finish the solution as shown above. The notation from the previous solution is still in force. Additionally, we denote by T the tangency point of the excircle ω_A with AB.

By the problem condition, the circle ω_A is an excircle of $\triangle AB_1C_1$. Therefore, the similarity transform mapping $\triangle AB_1C_1$ to $\triangle ABC$ sends ω to ω_A. Thus, $AB'/AT = \cos\angle A$. This means that $TB' \perp AC$, and hence TB' passes through I. Since $\angle TIC' = \angle A = \angle COM_A$, the right triangles $TC'I$ and COM_A are similar (Fig. 10.4.2). Moreover, we have $TC' = TB + BC' = BA_2 + CA_2 = BC = 2CM_A$, so the similarity ratio of these two triangles is 2; thus $OM_A = IC'/2 = r/2$.

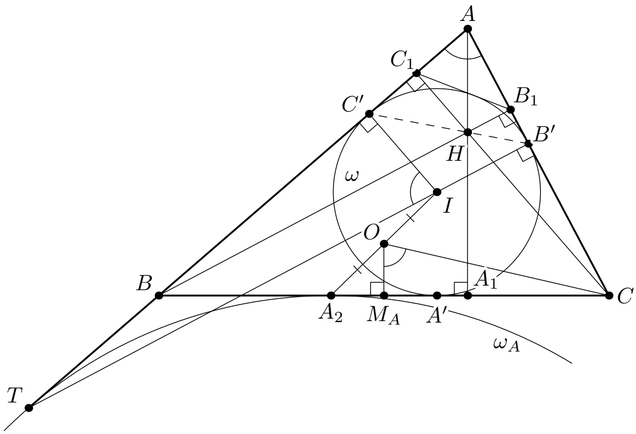


Fig. 10.4.2

Remark 1. In any triangle ABC, the circle $A_2B_2C_2$ is a pedal circle of a point K symmetric to I in O. This circle is also a pedal circle of a point K' isogonally conjugate to K. Thus, A_1 lies on the circle $A_2B_2C_2$ if and only if K' lies on AA_1, i.e., if either K lies on AO or $K' = A$. In the first case the triangle has to be isosceles, and in the second case we have $A_2 = K$.

Remark 2. One can also show that under the problem conditions, the orthocenter lies on the line $B'C'$ (Fig. 10.4.2), and the excircle tangent to the side BC is orthogonal to the circumcircle.

XI Geometrical Olympiad in honour of I.F.Sharygin
Final round. Grade 10. Second day
Ratmino, 2015, July 31.

5. (D. Shvetsov) Let BM be a median of right-angled nonisosceles triangle ABC ($\angle B = 90°$), and H_a, H_c be the orthocenters of triangles ABM, CBM respectively. Lines AH_c and CH_a meet at point K. Prove that $\angle MBK = 90°$.

   Solution. Since the lines AH_a and CH_c are perpendicular to BM, the quadrilateral AH_cCH_a is a trapezoid, and K is the common point of its lateral sidelines. Moreover, since the triangles ABM and CBM are isosceles, we have $H_aA = H_aB$ and $H_cC = H_cB$. Therefore, $KC/KH_a = CH_c/AH_a = BH_c/BH_a$, i.e. $KB \parallel CH_c \perp BM$ (Fig. 10.5).

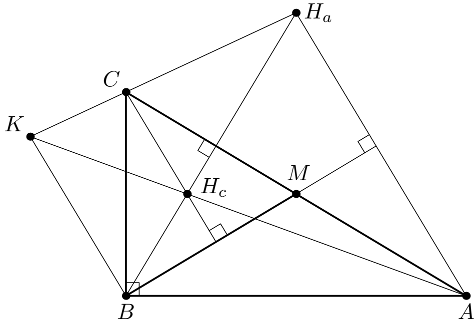


Fig. 10.5

6. (A. Sokolov) Let H and O be the orthocenter and the circumcenter of triangle ABC. The circumcircle of triangle AOH meets the perpendicular bisector to BC at point A_1. Points B_1 and C_1 are defined similarly. Prove that lines AA_1, BB_1 and CC_1 concur.

   Solution. We will make use of the following fact.

   Lemma. Assume that a line ℓ passes through the orthocenter of a triangle; then the reflections of ℓ about the sidelines are concurrent.

   Proof. Let H be the orthocenter of a given triangle ABC. Then the points H_a, H_b, and H_c symmetric to H about BC, CA, and AB, respectively, lie on the circumcircle of the triangle. Furthermore, the angle subtended by the arc H_aH_b is equal to $2\angle C$, i.e., it is equal to the reflections of ℓ about BC and CA. Thus these reflections meet on the circumcircle. Clearly, the third line also passes through the same point. This finishes the proof of the lemma.

   Back to the problem. Consider the triangle $A'B'C'$ formed by the reflections of O about the sides of the triangle ABC. Its vertices are the circumcenters of the triangles HBC, HCA, and HAB; thus its sides are the perpendicular bisectors of HA, HB, and HC. Therefore, the sides of $\triangle A'B'C'$ are parallel to those of $\triangle ABC$, and O is its orthocenter (Fig. 10.6).

Now, the sides AH and A_1O of the cyclic quadrilateral $AHOA_1$ are parallel, so the lines AA_1 and OH are symmetric about the perpendicular bisector of AH, i.e., about $B'C'$. The similar statement holds for the lines BB_1 and CC_1. Hence these lines are concurrent due to the lemma applied to the triangle $A'B'C'$.

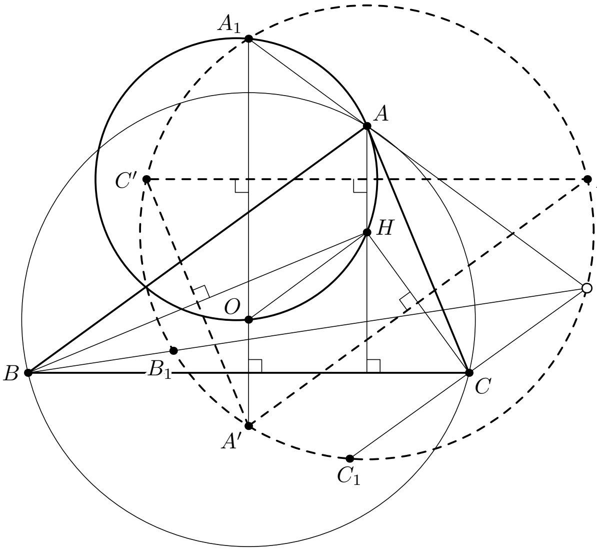


Fig. 10.6

7. (I.I.Bogdanov) Let $SABCD$ be an inscribed pyramid, and AA_1, BB_1, CC_1, DD_1 be the perpendiculars from A, B, C, D to lines SC, SD, SA, SB respectively. Points S, A_1, B_1, C_1, D_1 are distinct and lie on a sphere. Prove that points A_1, B_1, C_1 and D_1 are complanar.

   Solution. Since AA_1 and CC_1 are altitudes of the triangle SAC, the points A, C, A_1, and C_1 are concyclic, i.e., $SC \cdot SA_1 = SA \cdot SC_1$. Therefore, there exists an inversion centered at S interchanging A_1 with C and C_1 with A. Since $SB \cdot SD_1 = SD \cdot SB_1$, this inversion maps B_1 and D_1 to some points B_2 and D_2 on the rays SD and SB, respectively, such that $B_2D_2 \parallel BD$.

   On the other hand, the points A, C, B_2, and D_2 must be coplanar (as they are the images of the points lying on a sphere passing through S). However, if the line B_2D_2 does not lie in the plane $ABCD$, then the lines B_2D_2 and AC are skew. Thus, we are left with the only option that $B_2 = D$ and $D_2 = B$. Therefore, the points A_1, B_1, C_1, and D_1 lie in a plane which is the image of the sphere $SABCD$.

8. (M. Artemyev) Does there exist a rectangle which can be divided into a

regular hexagon with sidelength 1 and several equal right-angled triangles with legs 1 and $\sqrt{3}$?

Answer. No.

Solution. Suppose that such partition of some rectangle exists. Note that the area of each triangle in the partition is $S = \sqrt{3}/2$, and the area of the hexagon is equal to $3S$. Each side of the rectangle is partitioned into segments with lengths 1, 2, and $\sqrt{3}$, i.e., the lengths of these sides have the form $a+b\sqrt{3}$ and $c + d\sqrt{3}$ with nonnegative integers a, b, c, and d. Thus, the area of the rectangle equals

$$(a + b\sqrt{3})(c + d\sqrt{3}) = (ac + 3bd) + (ad + bc)\sqrt{3}.$$

On the other hand, this area is a multiple of S, therefore $ac + 3bd = 0$, i.e., $ac = 0$ and $bd = 0$.

This yields that one of these sides (say, vertical) has an integer length, while the other one (say, horizontal) has a length which is a multiple of $\sqrt{3}$. Thus the area of the rectangle is a multiple of $2S$. Since the area of the hexagon equals $3S$, the number of the triangles in the partition is odd. Now we prove that this is impossible.

Each (non-extendable) segment in the partition is covered by the segments of integer lengths and sides of length $\sqrt{3}$ on both sides. Thus the number of the segments of length $\sqrt{3}$ adjoining the segment is even. Next, none of the vertical sides of the rectangle adjoins such segments, while the horizontal sides of the rectangle are partitioned into such segments, and hence they also adjoin an even number of segments of length $\sqrt{3}$ in total. Thus the total number of sides of length $\sqrt{3}$ is even; but any triangle in the partition contains exactly one such side. This is a contradiction.

XII Geometrical Olympiad in honour of I.F.Sharygin
Final round. Solutions. First day. 8 grade

Ratmino, 2016, July 31

1. (Yu.Blinkov) An altitude AH of triangle ABC bisects a median BM. Prove that the medians of triangle ABM are sidelengths of a right-angled triangle.

   Solution. Let AH and BM meet at point K, let L be the midpoint of AM, and let N and P be the projections of L and M respectively to BC (fig.8.1). Since K is the midpoint of BM, it follows that KH is a midline of triangle BMP, i.e. $PH = HB$. On the other hand, by the Thales theorem $CP = PH$ and $PN = NH$, hence N is the midpoint of BC. Therefore NK is a medial line of triangle BMC, i.e. $NK \parallel AC$ and $ALNK$ is a parallelogram. Hence $LN = AK$. Also the median from M in triangle AMB is a midline of ABC, hence it is congruent to BN. Therefore the sides of right-angled triangle BNL are congruent to the medians of ABM.

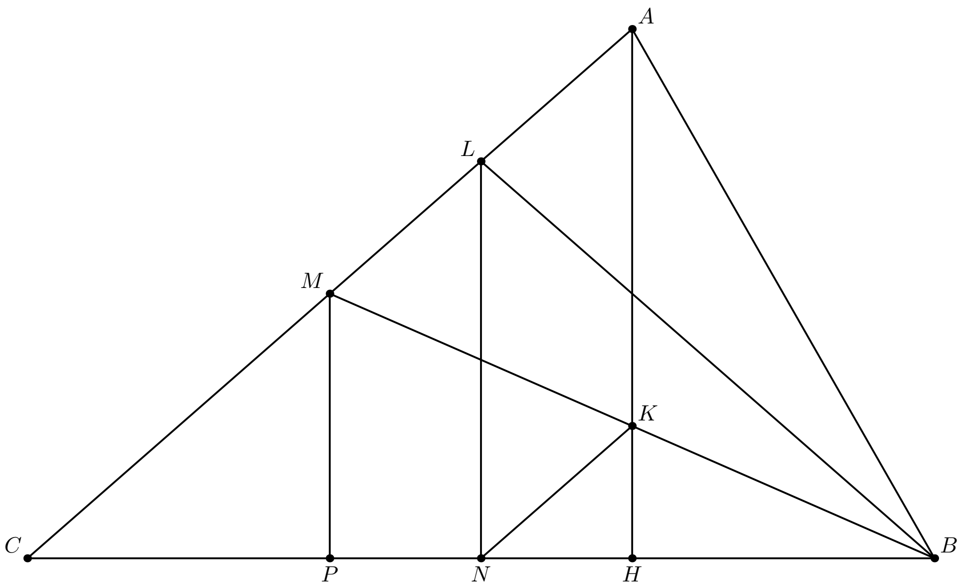


Fig. 8.1

2. (E.Bakaev) A circumcircle of triangle ABC meets the sides AD and CD of a parallelogram $ABCD$ at points K and L respectively. Let M be the midpoint of arc KL not containing B. Prove that $DM \perp AC$.

   First solution. By the assumption we obtain that $ALCB$ is an isosceles trapezoid, i.e $AL = AD$ (fig.8.2). Now AM is the bisector of isosceles triangle ALD, thus AM is also its altitude. Hence $AM \perp CD$. Similarly $CM \perp AD$. Therefore M is the orthocenter of triangle ACD and $DM \perp AC$.

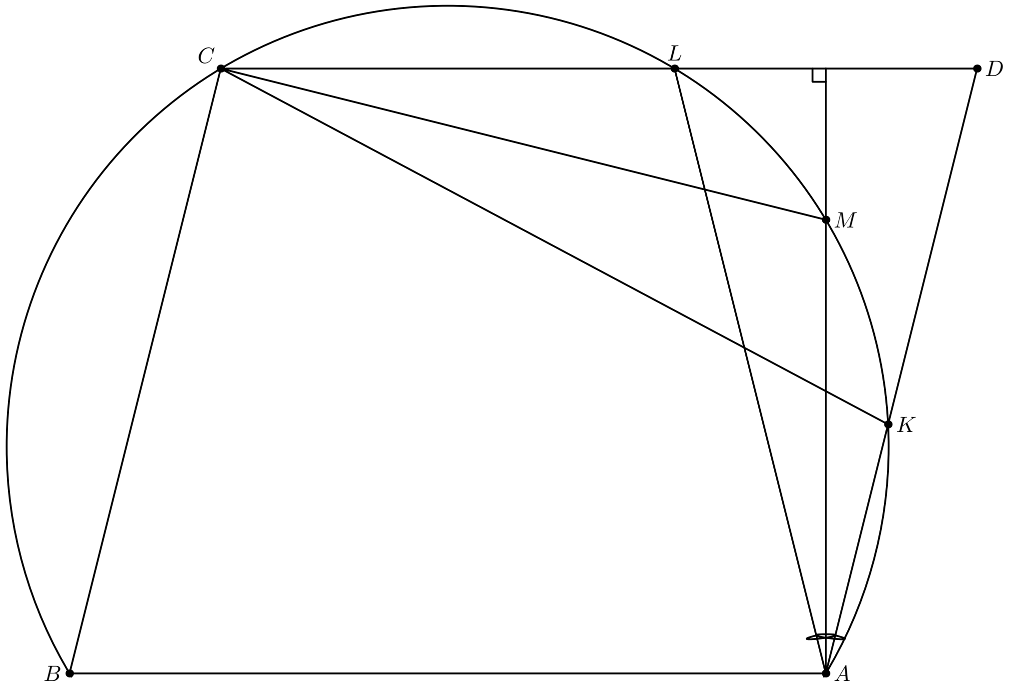


Fig. 8.2

Second solution. Consider the circumcircle of triangle ABC. The equality of angles BAK and BCL yields the equality of arcs BAK and BCL. The arcs LM and KM are also equal, and since the sum of these four arcs is the whole circle, we obtain that BM is a diameter. Then the triangles BAM and BCM are right-angled, i.e $BA^2 + AM^2 = BM^2 = BC^2 + CM^2$. Rewrite this equality as $BA^2 - BC^2 = CM^2 - AM^2$ and modify left part using the equality of opposite sides of parallelogram: $CD^2 - AD^2 = CM^2 - AM^2$. By the Carnot principle we obtain that $DM \perp AC$.

3. (D.Prokopenko) A trapezoid $ABCD$ and a line l perpendicular to its bases AD and BC are given. A point X moves along l. The perpendiculars from A to BX and from D to CX meet at point Y. Find the locus of Y.

   Answer. The line l' that is perpendicular to the bases of the trapezoid and divides AD in the same ratio as l divides CB.

   First solution. Let XU, YV be the altitudes of triangles BXC, AYD (fig.8.3.1). Then $\angle YAV = \angle BXU$ and $\angle YAD = \angle CXU$ because the sides of these angles are perpendicular. Therefore the triangle AVY is similar to XUB, and the triangle DVY is similar to XUC. From this we obtain that the ratio $AV : VD = CU : UB$ does not depend on X, i.e. Y lies on l'. It is clear that all points of this line are in the required locus.

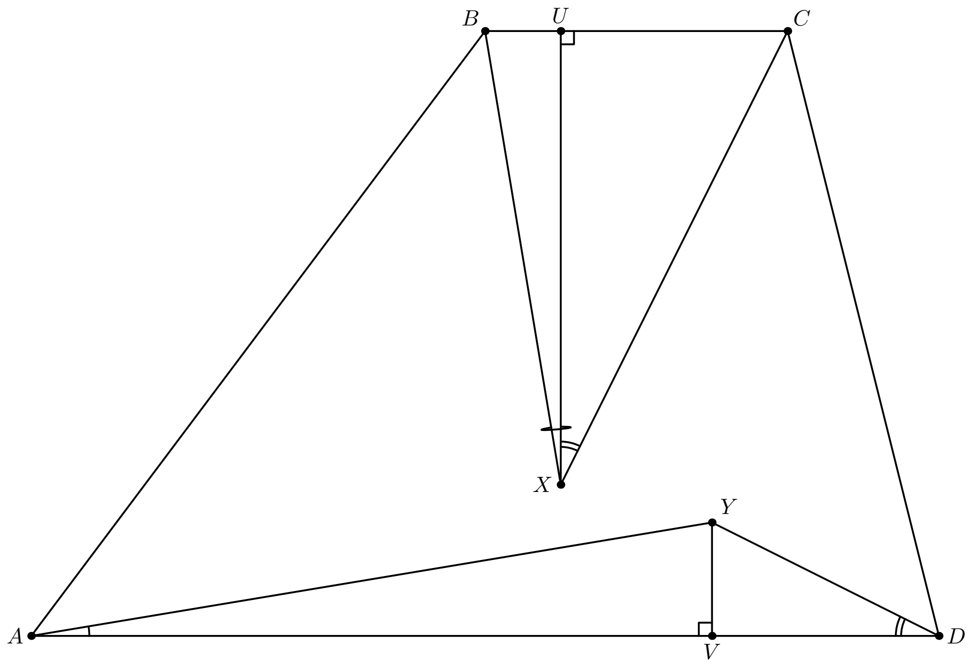


Fig. 8.3

Second solution. The locus of points with the constant difference of squares of the distances from the endpoints of a segment is a line perpendicular to this segment. Hence it is sufficient to prove that the difference $YB^2 - YC^2$ is constant.

Since the lines BX and AY are perpendicular, we have $YB^2 - AB^2 = YX^2 - AX^2$. Similarly $DC^2 - YC^2 = DX^2 - YX^2$. Summing these equalities we obtain that $YB^2 - YC^2 = (DX^2 - AX^2) + (AB^2 - DC^2)$. The first difference is constant by the definition of X. Therefore all points Y lie on the line perpendicular to BC.

Third solution. Let the lines AB and CD meet at point P. Consider the homothety with center P mapping the segment BC to AD. Let X' be the image of X. The homothety maps BX and CX to parallel lines AX' and DX'. Therefore the angles $X'AY$ and $X'DY$ are right and the quadrilateral $X'AYD$ is cyclic. We obtain also that X' moves along a fixed line l' parallel to l.

Let Q, R be the projections of X' and Y to AD (fig. 8.3.2). Since the midpoint of diameter $X'Y$ is projected to the midpoint of chord AD, we obtain by the Thales theorem that $AQ = DR$. The point Q is fixed, hence Y moves along the line passing through R and parallel to the bases.

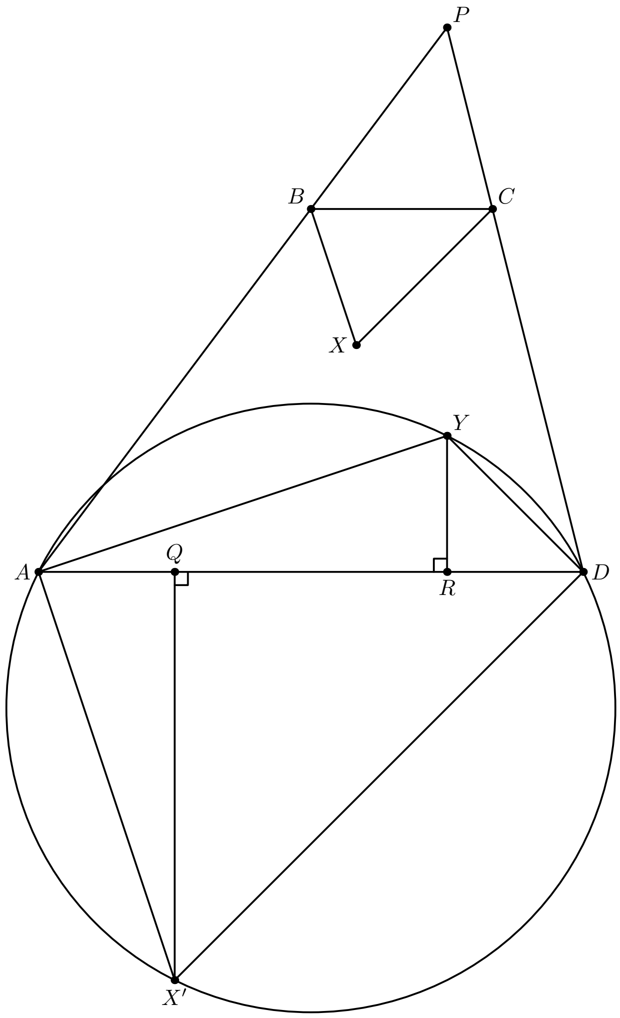


Fig. 8.3.2

4. (N.Beluhov) Is it possible to dissect a regular decagon along some of its diagonals so that the resulting parts can form two regular polygons?

   Answer. Yes, see fig.8.4

   Remark. This construction works for all regular $2n$-gons with $n \geq 3$.

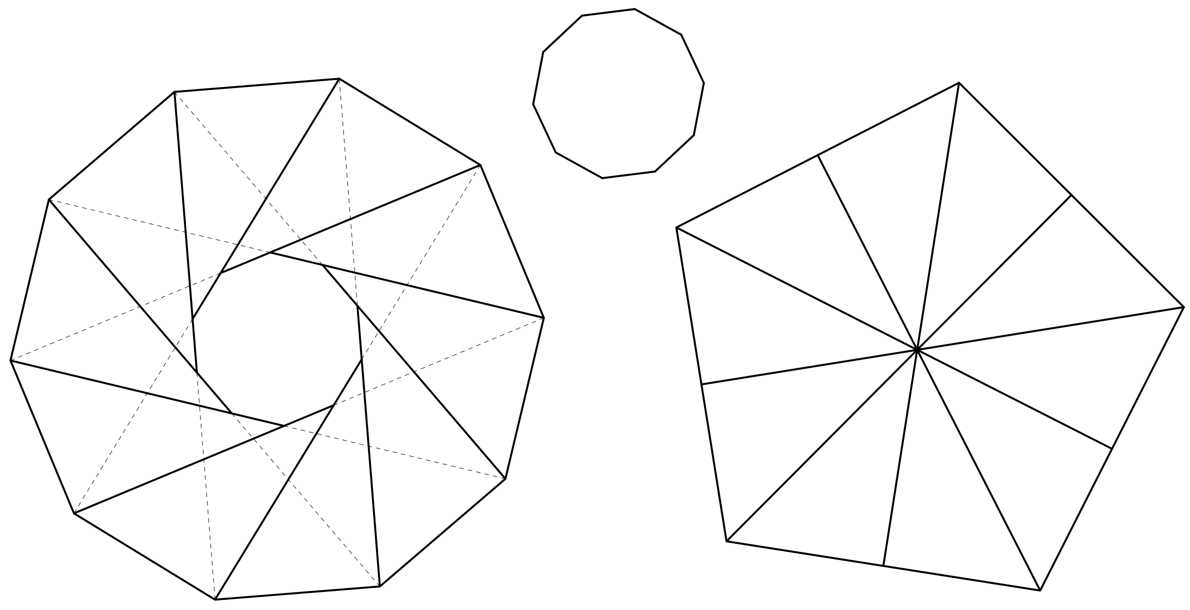


Fig.8.4

XII Geometrical Olympiad in honour of I.F.Sharygin
Final round. Solutions. Second day. 8 grade

Ratmino, 2016, August 1

5. (A.Khachaturyan) Three points are marked on the transparent sheet of paper. Prove that the sheet can be folded along some line in such a way that these points form an equilateral triangle.

   Solution. Let A, B, C be the given points, AB be the smallest side of triangle ABC, D be the vertex of an equilateral triangle ABD, l be the perpendicular bisector to segment CD. Since $AD = AB \leq AC$ and $BD = AB \leq BC$, the points A, B lie on the same side from l as D. Thus if we fold the sheet along l then A and B do not move, and C maps to D.

6. (E.Bakaev) A triangle ABC with $\angle A = 60°$ is given. Points M and N on AB and AC respectively are such that the circumcenter of ABC bisects segment MN. Find the ratio $AN : MB$.

   Answer. 2.

   First solution. Let P, Q be the projections of N and the circumcenter O respectively to AB (fig.8.6). From the condition we have $MQ = QP$. On the other hand Q is the midpoint of AB, thus $BM = AP$. But in the right-angled triangle APN we have $\angle A = 60°$. Therefore $BM = AP = AN/2$.

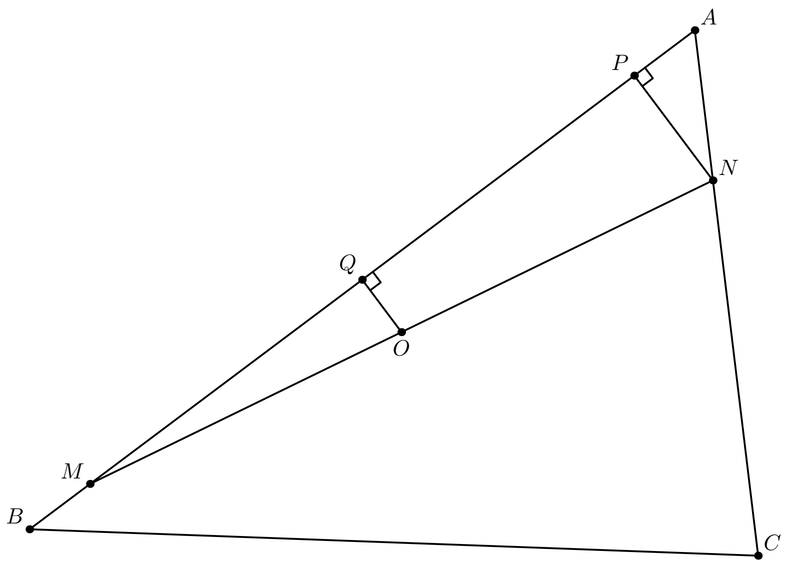


Fig. 8.6

   Second solution. Let P be the point on the circumcircle of ABC opposite to A. Since O bisects the segments AP and MN, we have that $AMPN$ is a parallelogram. The angles BAC and BMP are equal because $AC \parallel MP$. The angle ABP is right because AP is a diameter. Thus BMP is a right-angled triangle with $\angle M = 60°$, therefore $MP : MB = 2$. The segments MP and AN are the opposite sides of parallelogram, hence $AN : MB = 2$.

7. (A.Zaslavsky) Diagonals of a quadrilateral $ABCD$ are equal and meet at point O. The perpendicular bisectors to segments AB and CD meet at point P, and the perpendicular bisectors to BC and AD meet at point Q. Find angle POQ.

   Answer. $90°$.

   Solution. Since $PA = PB$ and $PC = PD$, the triangles PAC and PBD are congruent (fig.8.7). Therefore the distances from P to the lines AC and BD are equal, i.e. P lies on the bisector of some angle formed by these lines. Similarly Q also lies on the bisector of some of these angles. Let us prove that these points lie on different bisectors. The bisector of angle AOB meets the perpendicular bisector to AB at the midpoint of arc AB of the circle AOB. Also this bisector meets the perpendicular bisector to CD at the midpoint of arc CD of circle COD. These two points lie on the different sides from O, hence P lies on the bisector of angle AOD. Similarly Q lies on the bisector of angle AOB. It is evident that these bisectors are perpendicular.

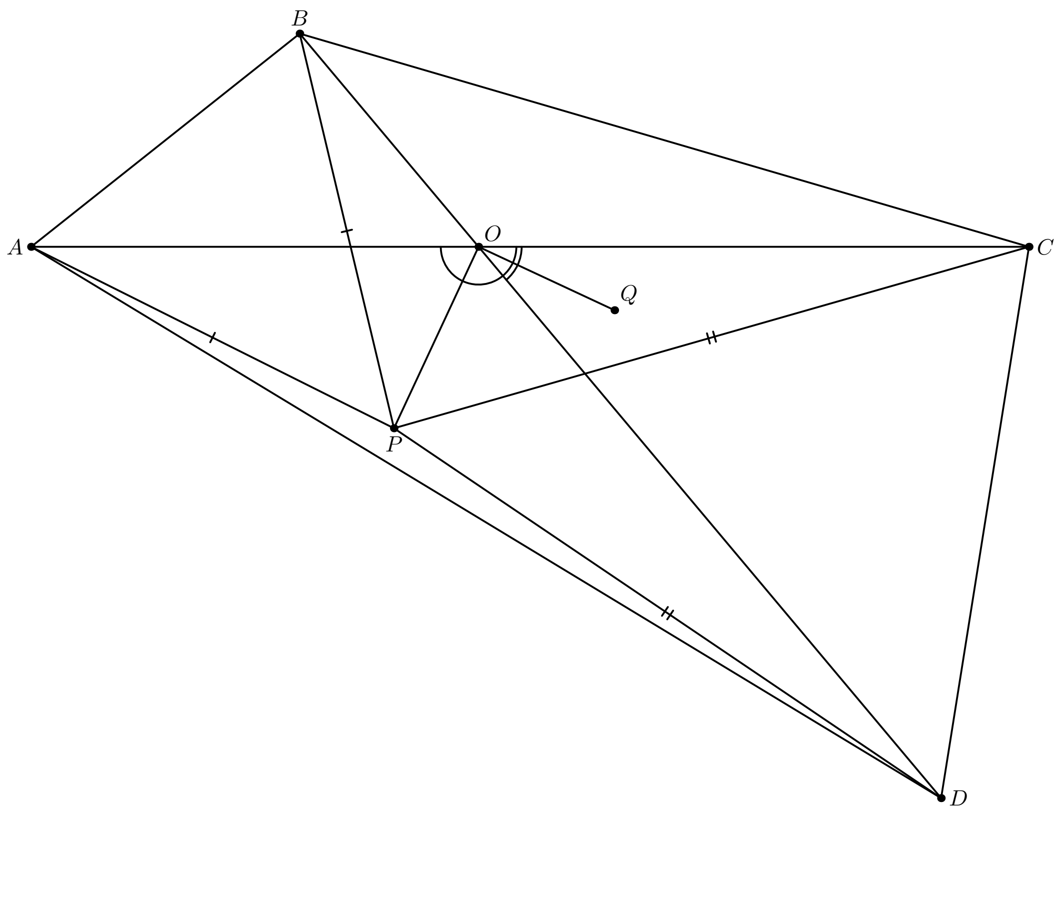


Fig. 8.7

8. (V.Protasov) A criminal is at point X, and three policemen at points A, B and C block him up, i.e. the point X lies inside the triangle ABC. Each evening one of the policemen is replaced in the following way: a new policeman takes the position equidistant from three former policemen, after this one of the former policemen goes away so that three remaining policemen block up the criminal too. May the policemen after some time occupy again the points A, B and C (it is known that at any moment X does not lie on a side of the triangle)?

Answer. No.

First solution. It is evident that all triangles formed by the policemen after the first evening are isosceles. Thus we can suppose that in the original triangle $AC = BC$. Let O, R be the circumcenter and the circumradius of triangle ABC. Then since $OC \perp AB$ and X lies inside ABC, we obtain that the projection of X to the altitude CD lies between C and D. Hence $XC^2 - XO^2 < CD^2 - DO^2 = AC^2 - AO^2$ or $XC^2 - AC^2 < XO^2 - R^2$. Similarly $O'X^2 - R'^2 < OX^2 - R^2$, where O', R' are the circumcenter and the circumradius of the new triangle formed by the policemen. Therefore the degree of X wrt the circumcircle of policemen's triangle decreases each evening and the policemen cannot occupy the initial points.

Second solution. Let A be the vertex of the triangle nearest to X, and O be the circumcenter. It is clear that X cannot lie inside the triangle OBC, i.e. A is a vertex of the new triangle containing X. Therefore the distance from X to the nearest vertex does not increase. This is also correct for the further steps. If the sequence of triangles is periodic then this distance is constant and A is the vertex of all triangles containing X. These triangles are isosceles and A is the vertex at the base, i.e. the angle at this vertex is acute. Hence one of rays BO, CO passes through the triangle. Let the extension of segment AX meet BC at point Y . Since one of rays BO, CO intersects the segment AY, we obtain that the distance XY decreases at each step, therefore the policemen cannot occupy the initial points again.

XII Geometrical Olympiad in honour of I.F.Sharygin
Final round. Solutions. First day. 9 grade

Ratmino, 2016, July 31

1. (D.Shvetsov) The diagonals of a parallelogram $ABCD$ meet at point O. The tangent to the circumcircle of triangle BOC at O meets the ray CB at point F. The circumcircle of triangle FOD meets BC for the second time at point G. Prove that $AG = AB$.

   Solution. From the tangency we have $\angle FOB = \angle BCO = \angle GCA$, and since $FGOD$ is cyclic, $\angle FOB = \angle DGC$.

   We obtain that $\angle GCA = \angle DGC$, hence $AGCD$ is an isosceles trapezoid and $AG = DC = AB$.

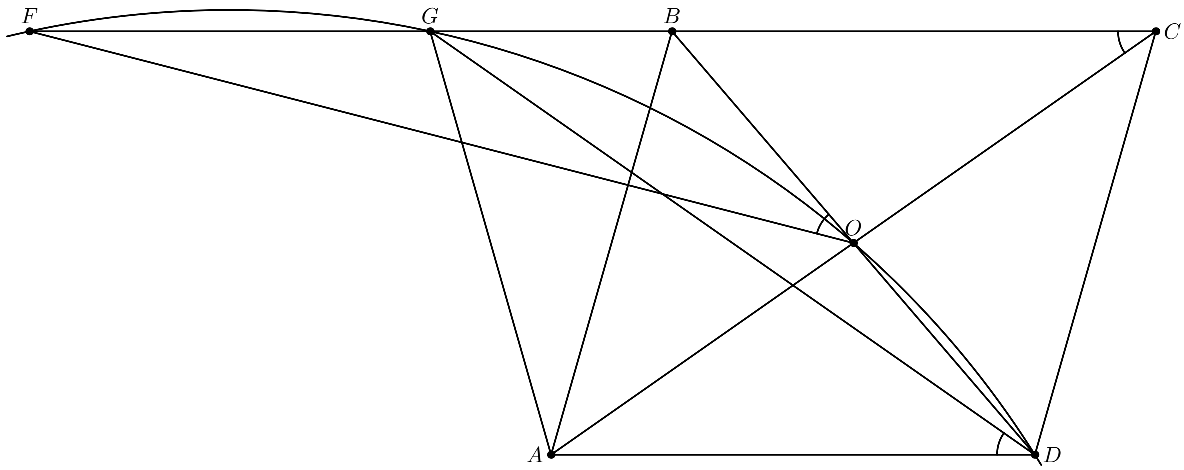


Fig. 9.1

2. (D.Khilko) Let H be the orthocenter of an acute-angled triangle ABC. Point X_A lying on the tangent at H to the circumcircle of triangle BHC is such that $AH = AX_A$ and $H \neq X_A$. Points X_B and X_C are defined similarly. Prove that the triangle $X_AX_BX_C$ and the orthotriangle of ABC are similar.

   Solution. Let O be the circumcenter of ABC (fig. 9.2). Let us prove that $AO \perp HX_A$. In fact, the translation by vector AH maps the circle ABC to the circle BHC. Hence the tangent at H is parallel to the tangent at A and perpendicular to the radius OA. Since HAX_A is an isosceles triangle, its altitude coincides with the median. Thus AO is the perpendicular bisector to HX_A. Similarly BO, CO are the perpendicular bisectors to HX_B, HX_C respectively. Therefore H, X_A, X_B, X_C lie on a circle centered at O. Now we have $\angle X_AX_CX_B = \angle X_AHX_B = \angle CHX_A + \angle X_BHC = 2(90° -$

$\angle C) = \angle H_1H_3H_2$. Similarly $\angle X_AX_BX_C = \angle H_1H_2H_3$ and $\angle X_BX_AX_C = \angle H_2H_1H_3$. Since the correspondent angles of triangles $X_AX_BX_C$ and $H_1H_2H_3$ are equal, these triangles are similar.

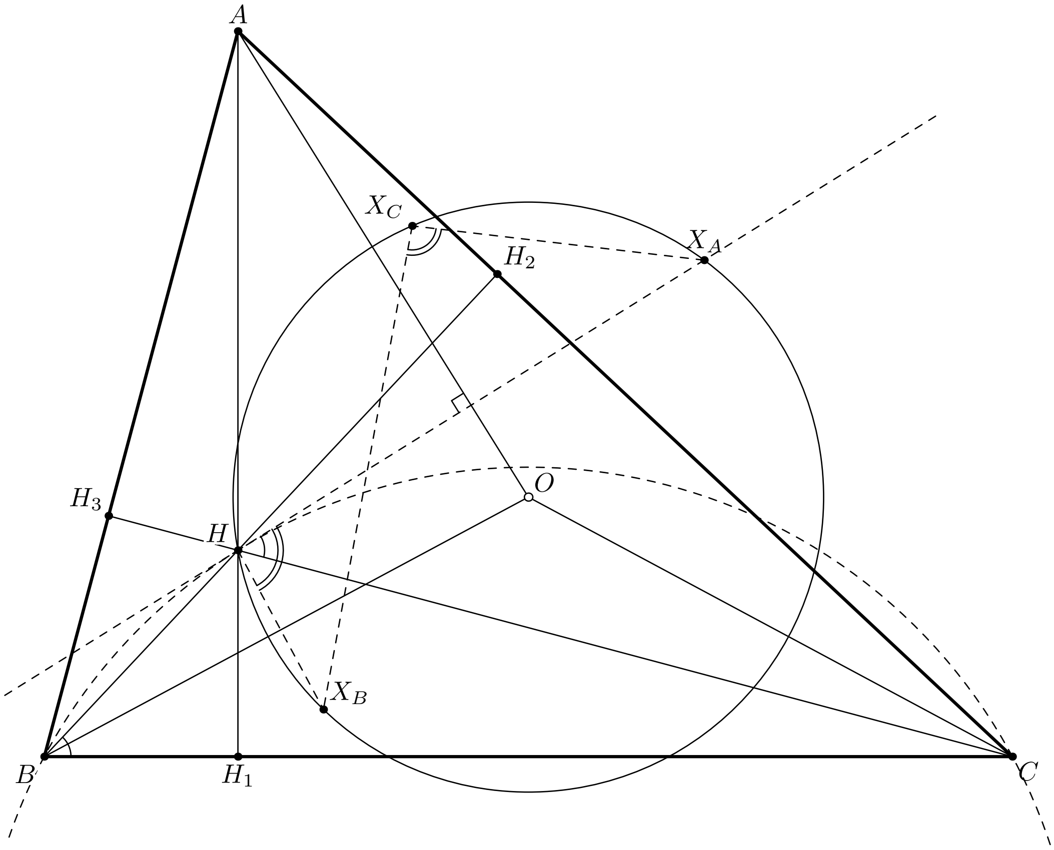


Fig. 9.2.

Remark. This solution can be modified. The midpoints of segments HX_A, HX_B, HX_C lie on the circle with diameter OH and form a triangle similar to the orthotriangle (this can be proved as above). This reasoning allows to prove a general assertion: if P and Q are isogonally conjugated, and A_1, B_1, C_1 are the projections of P to AQ, BQ and CQ, then the triangle $A_1B_1C_1$ is similar to the pedal triangle of P.

3. (V.Kalashnikov) Let O and I be the circumcenter and the incenter of triangle ABC. The perpendicular from I to OI meets AB and the external bisector of angle C at points X and Y respectively. In what ratio does I divide the segment XY?

Answer. $1 : 2$.

First solution. Let I_a, I_b, I_c be the excenters of ABC. Then ABC and its circumcircle are the orthotriangle and the nine-points circle of triangle

$I_aI_bI_c$. Hence the circumcenter of $I_aI_bI_c$ is symmetric to I wrt O, and its circumradius is equal to the double circumradius of ABC. The triangle $A'B'C'$ homothetic to ABC with center I and coefficient 2 has the same circumcircle. The line l passing through I and perpendicular to OI carves the chord of this circle with midpoint I, the chords I_aA' and I_bB' also pass through it (fig.9.3). By the butterfly theorem I_aI_b and $A'B'$ meet l at two points symmetric about I, therefore $IX : IY = 1 : 2$.

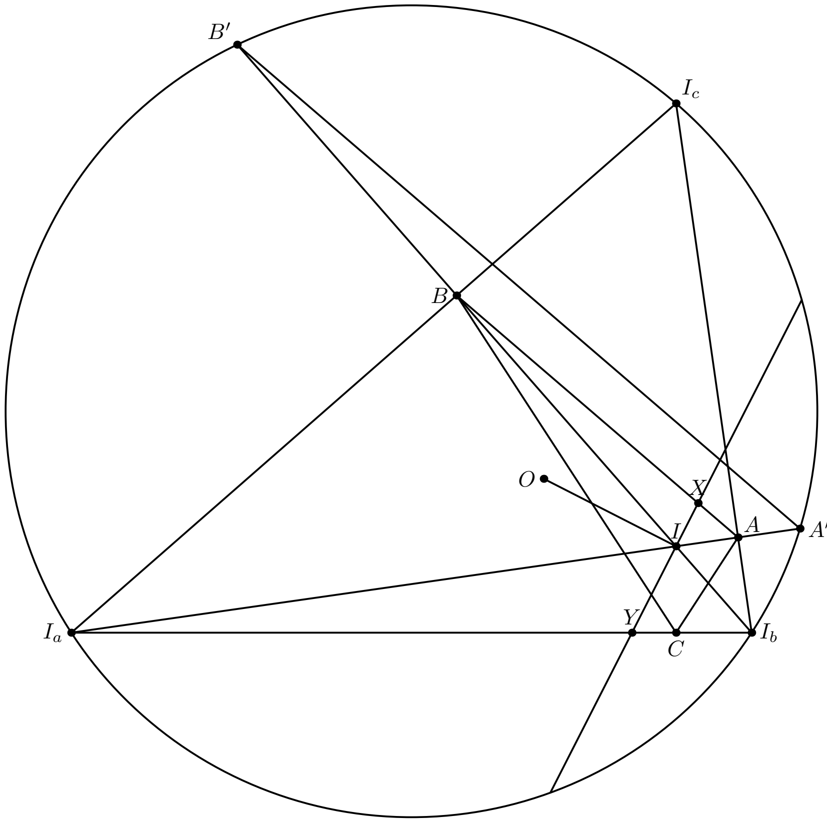


Fig. 9.3

Second solution. Consider the points such that the sum of oriented distances from them to the sidelines of ABC is equal to $3r$, where r is the radius of the incircle. Since the distance is a linear function, the locus of such points is a line passing through I. Since the sum of the projections of vector OI to the lines AB, BC, CA is zero, this line is perpendicular to OI. Since Y lies on the external bisector of angle C, the sum of distances from Y to AC and BC is zero. Thus the distance from Y to AB is equal to $3r$, i.e. $YX = 3IX$.

4. (N.Beluhov) One hundred and one beetles are crawling in the plane. Some of the beetles are friends. Every one hundred beetles can position themselves so that two of them are friends if and only if they are at the unit distance

from each other. Is it always true that all one hundred and one beetles can do the same?

Answer. No.

First solution. Let two beetles be friends if and only if they are connected by a solid line in the fig. 9.4.

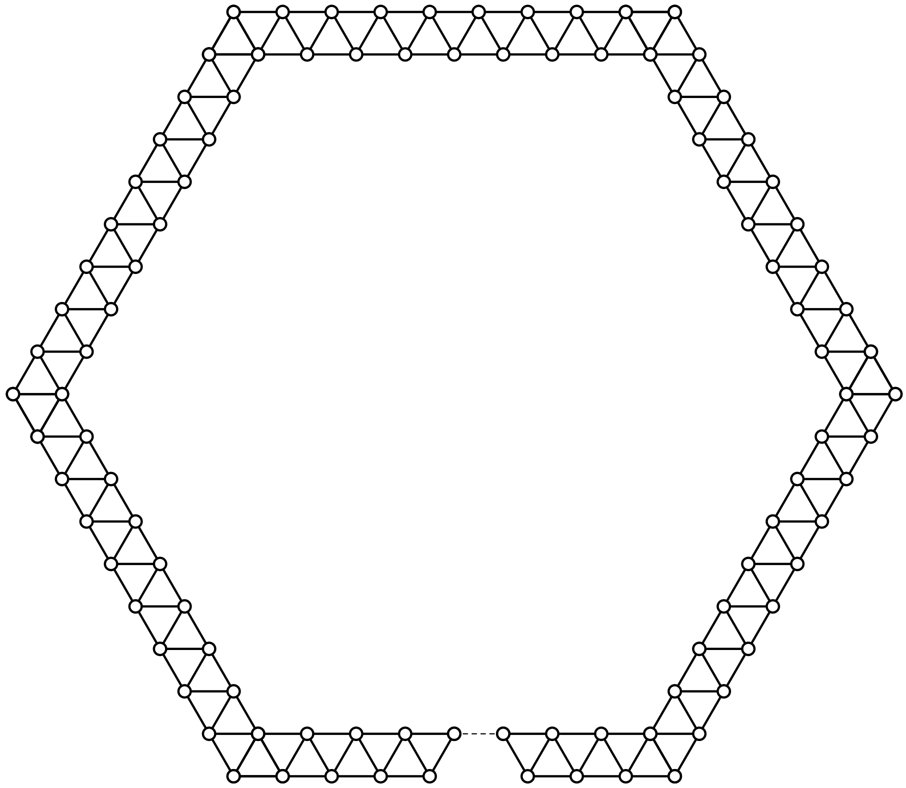


Fig. 9.4.

Suppose that all one hundred and one beetles have positioned themselves so that the only if part is satisfied (if two beetles are friends then they are the unit distance apart). If two beetles occupy the same position then the if part (if two beetles are the unit distance apart then they are friends) fails. Otherwise, friendships determine a unique structure which forces the two beetles connected by a dashed line to be the unit distance apart without being friends and the if part fails again.

If we temporarily forget about any one beetle, the structure becomes flexible enough so that both the if and the only if part can be satisfied.

Second solution. Consider the following graph: the trapezoid $ABCD$ with the bases $BC = 33$ and $AD = 34$ and the altitude $\sqrt{3}/2$ composed from 67 regular triangles with side 1, and the path with length 33 joining A and

D. It is clear that this graph can not be drawn on the plane satisfying the condition of the problem, but we can do it if an arbitrary vertex is removed.

XII Geometrical Olympiad in honour of I.F.Sharygin
Final round. Solutions. Second day. 9 grade

Ratmino, 2016, August 1

5. (F.Nilov) The center of a circle ω_2 lies on a circle ω_1. Tangents XP and XQ to ω_2 from an arbitrary point X of ω_1 (P and Q are the touching points) meet ω_1 for the second time at points R and S. Prove that the line PQ bisects the segment RS.

   First solution. Let O be the center of ω_2. Since XO is the bisector of angle PXQ, we have $OR = OS$. Thus the right-angled triangles OPR and OQS are congruent by a cathetus and the hypothenuse, i.e. $PR = QS$ (fig.9.5). Since $\angle XPQ = \angle XQP$, we obtain that R and S lie at equal distances from the line PQ, which is equivalent to the required assertion.

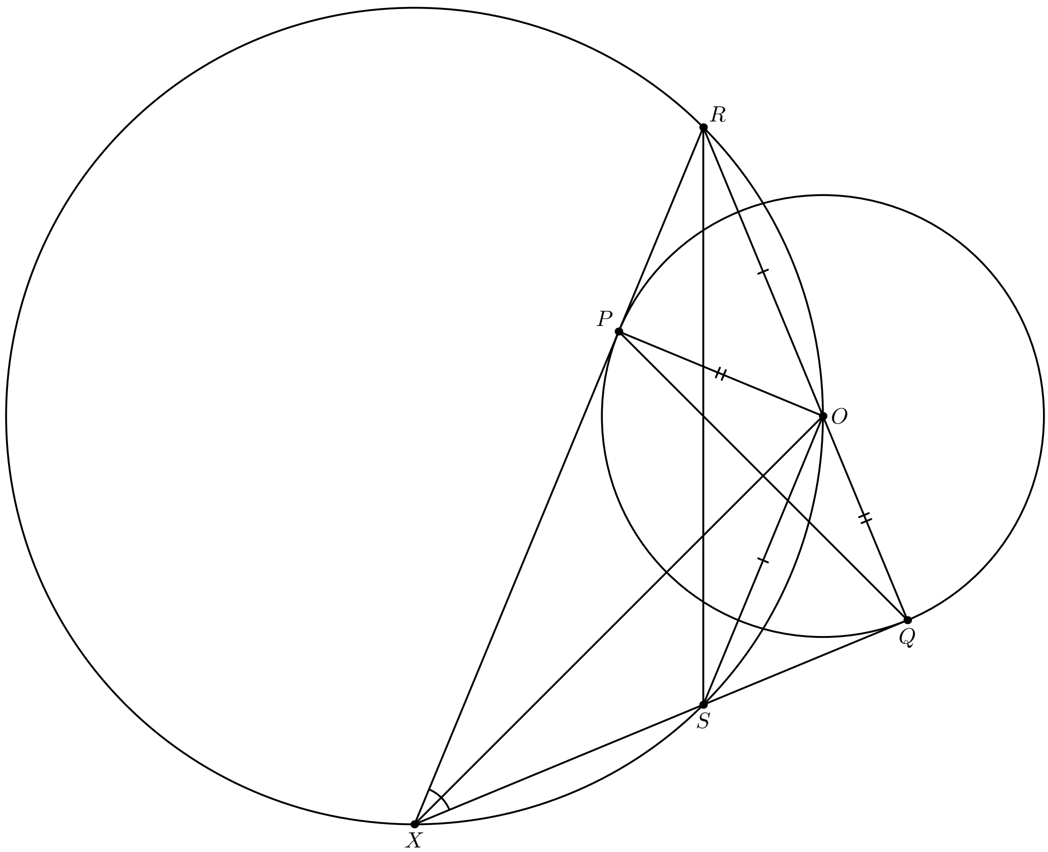


Fig. 9.5

   Second solution. Let O be the center of ω_2. Since XO bisects the angle PXQ, we obtain that O is the midpoint of arc RS. Hence the midpoint K of segment RS is the projection of O to RS. Therefore P, Q and K lie on the Simson line of point O.

6. (M.Timokhin) The sidelines AB and CD of a trapezoid $ABCD$ meet at point P, and the diagonals of this trapezoid meet at point Q. Point M on the smallest base BC is such that $AM = MD$. Prove that $\angle PMB = \angle QMB$.

   First solution. Let the lines PM, QM meet AD at points X, Y respectively, and let U be the midpoint of AD. Since $AX : XD = BM : MC = YD : AY$, we obtain that $AX = YD$ and $XU = UY$ (fig. 9.6). Hence the perpendicular bisector UM of segment AD is also the bisector of isosceles triangle XMY, and BC is the bisector of angle PMQ.

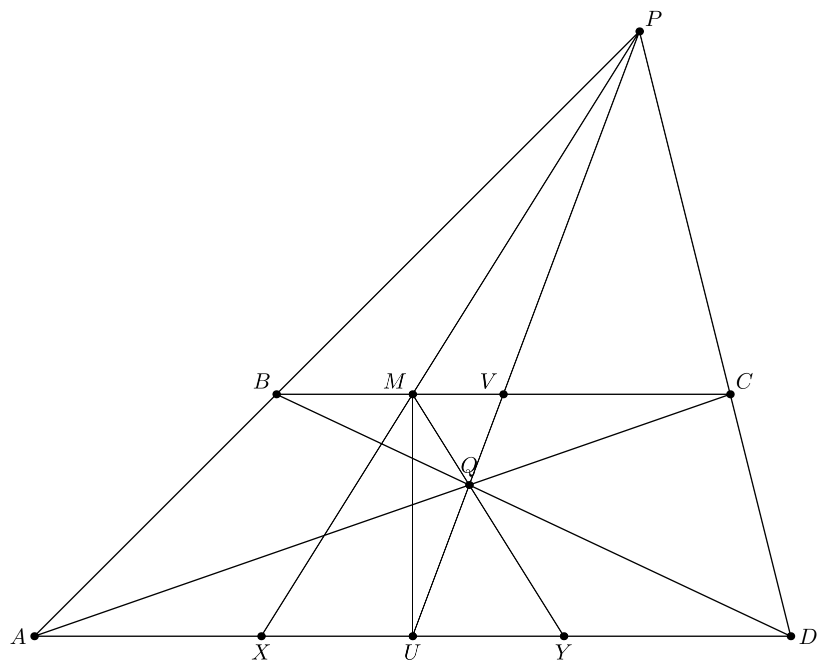


Fig. 9.6

   Second solution. The line PQ passes through U and the midpoint V of segment BC (fig. 9.6) so that the quadruple P, Q, U, V is harmonic. Since the lines MU and MV are perpendicular they are the external and the internal bisectors of angle PMQ.

7. (A.Zaslavsky) From the altitudes of an acute-angled triangle, a triangle can be composed. Prove that a triangle can be composed from the bisectors of this triangle.

   Solution. Let in a triangle ABC be $\angle A \geq \angle B \geq \angle C$. Then the altitudes h_a, h_b, h_c satisfy the inequality $h_a \leq h_b \leq h_c$, and the similar inequality holds for the bisectors l_a, l_b, l_c. Consider two cases.

   1) $\angle B \geq 60^\circ$. Then $\angle A - \angle B \leq \angle B - \angle C$. Hence $h_c/l_c = \cos(\angle A - \angle B)/2 \geq h_a/l_a = \cos(\angle B - \angle C)/2$. Also $h_c/l_c > h_b/l_b$. Now from $h_c < h_a + h_b$ we obtain that $l_c < l_a + l_b$.

2) $\angle B \leq 60°$. Then since $\angle A < 90°$, we have $\angle C > 30°$. Thus $l_a \geq h_a = AC \sin \angle C > AC/2$ and $l_b > BC/2$. But l_c is not greater than the corresponding median, which is less than the half-sum of AC and BC. Therefore $l_c < l_a + l_b$.

Remark. Note that in the first case we did not use that the triangle is acute-angled, and in the second case we did not use that a triangle can be composed from the altitudes. But both conditions are necessary. An example of an obtuse-angled triangle, for which a triangle can be composed from the altitudes but not from the bisectors is constructed in the solution of problem 9.5 of VII Sharygin Olympiad.

8. (I.Frolov) The diagonals of a cyclic quadrilateral $ABCD$ meet at point M. A circle ω touches segments MA and MD at points P, Q respectively and touches the circumcircle of $ABCD$ at point X. Prove that X lies on the radical axis of circles ACQ and BDP.

   First solution. The inversion with the center at X maps the lines AC and BD to the circles ω_1 and ω_2 intersecting at points X and M'. Furthemore this inversion maps ω to a line touching these circles at points P', Q' respectively. Finally it maps the circle $ABCD$ to a line parallel to $P'Q'$, meeting ω_1 at points A', C', and meeting ω_2 at points B', D' (fig. 9.8). Since M lies on the radical axis of circles ACQ and BDP, we have to prove that the radical axis of $A'C'Q'$ and $B'D'P'$ coincides with the line XM'.

   Let K be the common point of XM' and $A'D'$. Since $A'K \cdot KC' = XK \cdot KM' = B'K \cdot KD'$, we obtain that K lies on the radical axis of circles $A'C'Q'$ and $B'D'P'$. Also the circle $A'C'Q'$ meets $P'Q'$ for the second time at the point symmetric to Q' about P', and the circle $B'D'P'$ meets it at the point symmetric to P' about Q'. Thus the degrees of the midpoint of $P'Q'$ lying on $M'X$, about these circles are also equal, and this completes the proof.

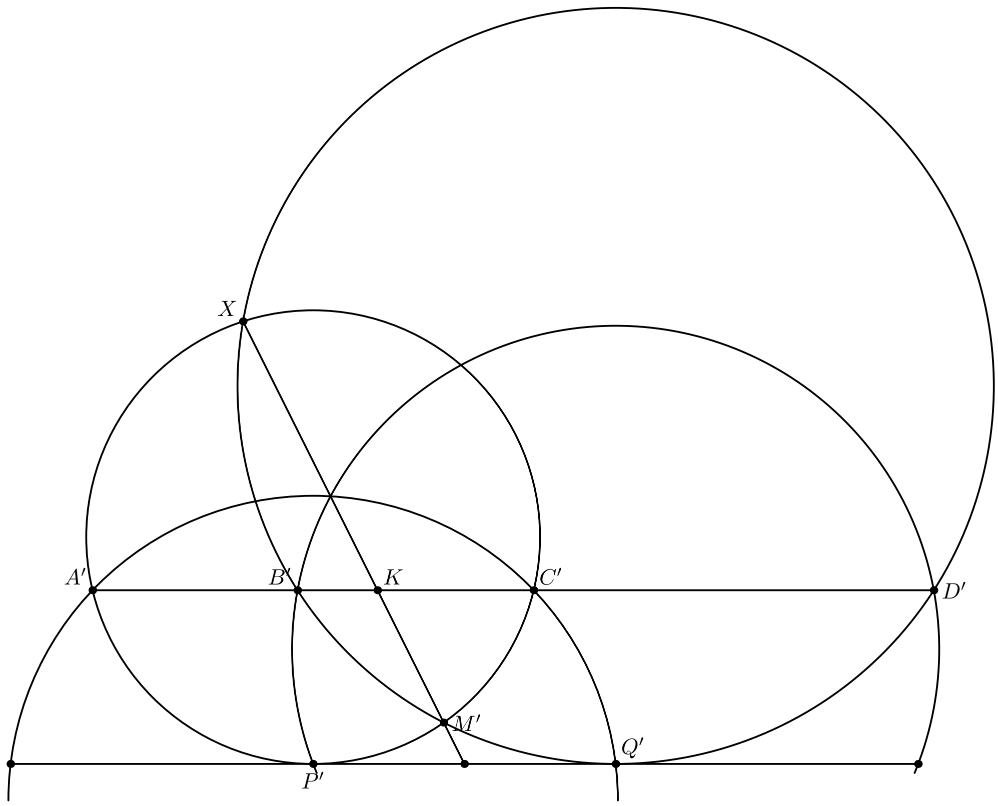


Fig. 9.8

Second solution. Let l be the tangent at X to the circle $ABCD$; and let l meet AC and BD at the points S and T respectively. Then SM is the radical axis of circles $ABCD$ and ACQ, ST is the radical axis of circles $ABCD$ and ω, i.e. S is the radical center of circles $ABCD$, ACQ and ω, hence SQ is the radical axis of circles ACQ and ω (because Q lies on the circles ACQ and ω). Similarly TP is the radical axis of circles BDP and ω. Therefore the common point G of SQ and TP is the radical center of circles ACQ, BDP and ω. On the other hand M is the radical center of circles ACQ, BDP and $ABCD$, i.e. MG is the radical center of circles ACQ and BDP, also MG passes through X, because G is the Gergonne point of triangle MST.

XII Geometrical Olympiad in honour of I.F.Sharygin
Final round. Solutions. First day. 10 grade

Ratmino, 2016, July 31

1. V.Yasinsky A line parallel to the side BC of a triangle ABC meets the sides AB and AC at points P and Q, respectively. A point M is chosen inside the triangle APQ. The segments MB and MC meet the segment PQ at E and F, respectively. Let N be the second intersection point of the circumcircles of the triangles PMF and QME. Prove that the points A, M, and N are collinear.

First solution. Let P' and Q' be the second intersection points of the circle (PMF) with AB and of the circle (QME) with AC. We have $\angle MP'A = \angle MFP = \angle MCB$, so the point P' lies on the circle (BMC). Similarly, the point Q' also lies on the same circle. Therefore, we have $AP'/AQ' = AC/AB = AQ/AP$, which means that the powers of the point A with respect to the two given circles are equal. This yields that A lies on the line MN.

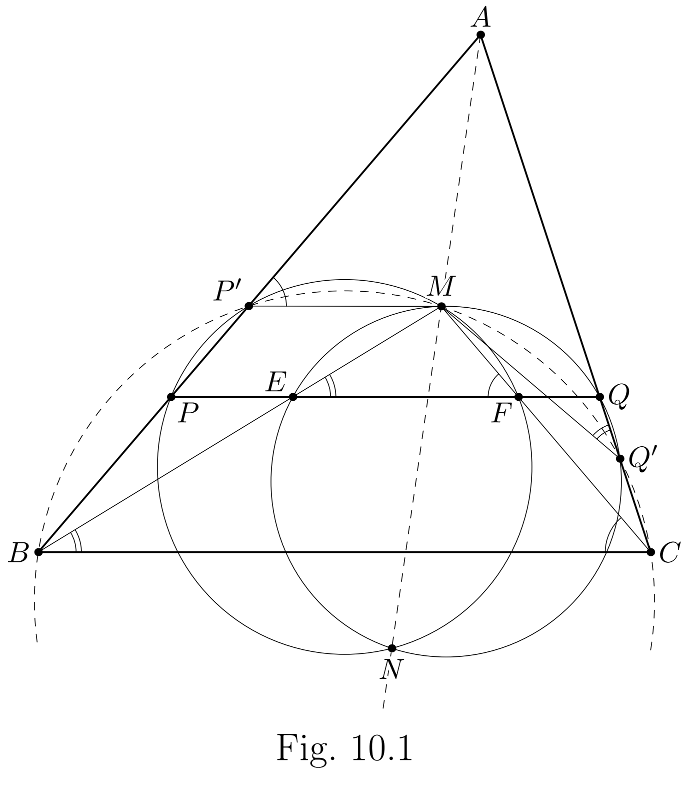


Fig. 10.1

Second solution. Let AM meet PQ and BC at points K and L respectively. Then $EK : FK = BL : CL = PK : QK$. Therefore, $PK \cdot FK = QK \cdot EK$ and both circles meet AM at the same point.

2. (P.Kozhevnikov) Let I and I_a be the incenter and the excenter of a triangle ABC; let A' be the point of its circumcircle opposite to A, and A_1 be the base of the altitude from A. Prove that $\angle IA'I_a = \angle IA_1I_a$.

   Solution. Since $\angle A_1AB = \angle CAA'$ and $\angle ACA' = 90°$, the triangles ACA' and AA_1B are similar. Hence $AA_a \cdot AA' = AB \cdot AC$. On the other hand $\angle AI_aC = \angle ABI$, thus the triangles AIB and ACI_a are similar and $AI \cdot AI_a = AB \cdot AC$.

   Let A_2 be the reflection of A_1 about the bisector of angle A. Then A_2 lies on AA' and as is proved above $AA_2 \cdot AA' = AI \cdot AI_a$. Therefore $IA_2A'I_a$ is a cyclic quadrilateral and $\angle IA'I_a = \angle IA_2I_a = \angle IA_1I_a$ (fig. 10.2).

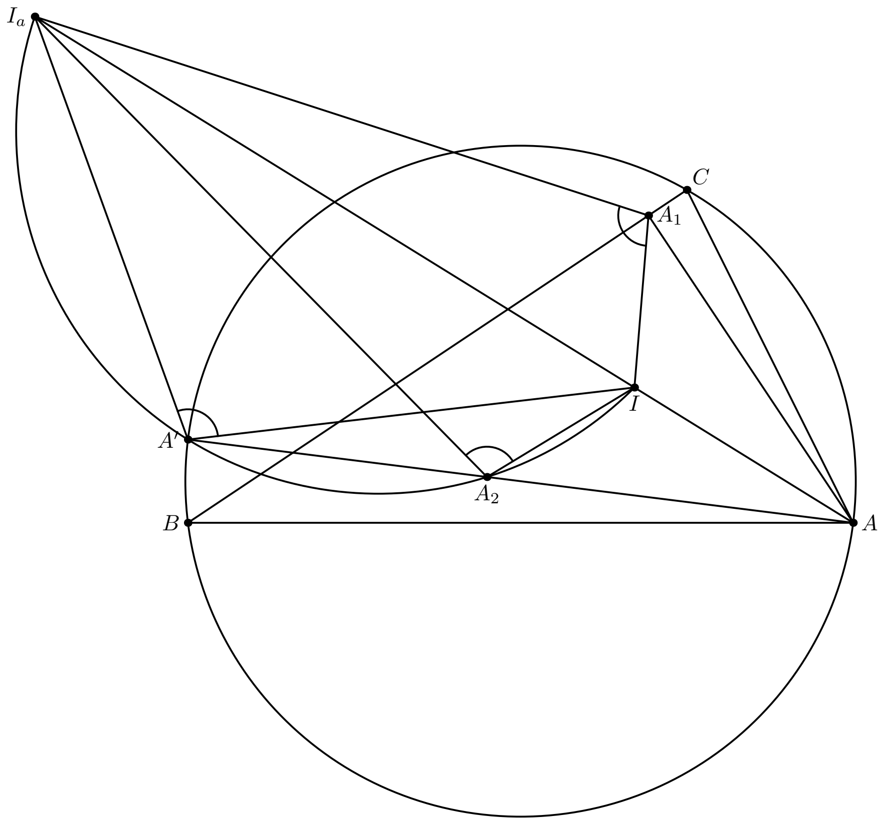


Fig. 10.2

3. (V.Kalashnikov) Let two triangles ABC and $A'B'C'$ have the common incircle and circumcircle, and let a point P lie inside both triangles. Prove that the sum of distances from P to the sidelines of ABC is equal to the sum of distances from P to the sidelines of $A'B'C'$.

   Solution. As is proved in the solution of problem 9.3, the locus of the points with constant sum of oriented distances to the sidelines of triangle ABC is a line perpendicular to OI, where O, I are the circumcenter and the incenter respectively. Also the sum of distances from I to the sidelines of both triangles is equal to $3r$, and the sum of the corresponding distances from O is equal to $R+r$ (the Carnot formula), where R and r are the radii of the circumcircle and the incircle. Therefore these sums are equal for all points of the plane.

Remark. The assertion remains true if we replace the triangles by two bicentric n-gons with an arbitrary n.

4. (N.Beluhov) Devil and Man are playing a game. Initially, Man pays some sum s and lists 97 triples (not necessarily distinct) $A_iA_jA_k$, $1 \le i < j < k \le 100$. After this Devil draws some convex 100-gon $A_1A_2 \ldots A_{100}$ of area 100 and pays the total area of 97 triangles $A_iA_jA_k$ to Man. For which maximal s this game is profitable for Man?

   Answer. For $s = 0$.

   First solution. *Lemma.* Let T be a set of at most $n-3$ triangles with the vertices chosen among those of the convex n-gon $P = A_1A_2 \ldots A_n$. Then the vertices of P can be coloured in three colours so that every colour occurs at least once, the vertices of every colour are successive, and T contains no triangle whose vertices have three different colours.

   Proof of the lemma. We proceed by induction on n.

   When $n = 3$, T is empty and the claim holds.

   Suppose $n > 3$. If A_1A_n is not a side of any triangle in T, then we colour A_1 and A_n in two different colours and all other vertices in the remaining colour.

   Now suppose that A_1A_n is a side of at least one triangle in T and the set U is obtained from T by removing all these triangles and replacing A_n by A_1 in all the remaining ones. By the induction hypothesis for the polygon $Q = A_1A_2 \ldots A_{n-1}$ and the set U, there is an appropriate colouring of the vertices of Q. By further colouring A_n in the colour of A_1, we get an appropriate colouring of P. $\square$

   Now imagine that Devil has chosen a convex 100-gon P of area 100 such that P is inscribed in a circle k, all vertices of P of colour i lie within the arc c_i of this circle with central angle ε°, and the midpoints of c_1, c_2, and c_3 form an equilateral triangle. When ε tends to zero, the areas of all triangles listed by the Man also tend to zero, and so does their sum.

Second solution. For each triple (i, j, k) let the vertex A_i be labelled by the number of sides covered by the angle $A_jA_iA_k$ (it is the same for all 100-gons), and do the same operation with A_j and A_k. The sum of these numbers is 100 for each triple, therefore the total sum is equal to $97 \cdot 100$, thus the sum in some vertex (for example A_1) is not greater than 97; from this we obtain that there exists a side A_kA_{k+1} not containing A_1 and such that the angles with vertex A_1 do not cover this side. Now the Devil can draw a 100-gon, in which the vertices $A_2, \ldots, A_{k-1}$ are close to A_k, and the vertices $A_{k+2}, \ldots, A_{100}$ are close to A_{k+1}, and make the areas of all 97 triangles arbitrary small.

XII Geometrical Olympiad in honour of I.F.Sharygin
Final round. Solutions. Second day. 10 grade

Ratmino, 2016, August 1

5. (A.Blinkov) Does there exist a convex polyhedron having equal numbers of edges and diagonals? (*A diagonal of a polyhedron is a segment between two vertices not lying in one face.*)

   Answer. Yes. For example each vertex of the upper base of a hexagonal prism is the endpoint of three diagonals joining it with the vertices of the lower base. Hence the common number of diagonals is 18 as well as the number of edges.

6. (I.I.Bogdanov) A triangle ABC is given. The point K is the base of an external bisector of angle A. The point M is the midpoint of arc AC of the circumcircle. The point N on the bisector of angle C is such that $AN \parallel BM$. Prove that M, N and K are collinear.

   First solution. Let I be the incenter. Then K, M, N lie on the sidelines of triangle BIC (fig. 10.6). We have $KB/KC = AB/AC$, $NC/IN = AC/AB' = (BC + AB)/AB$ (where B' is the base of bisector from B), $MI/MB = MC/MB = AB'/AB = AC/(AB + BC)$ (the second equality follows from the similarity of triangles BMC and BAB'). By the Menelaus theorem we obtain the required assertion.

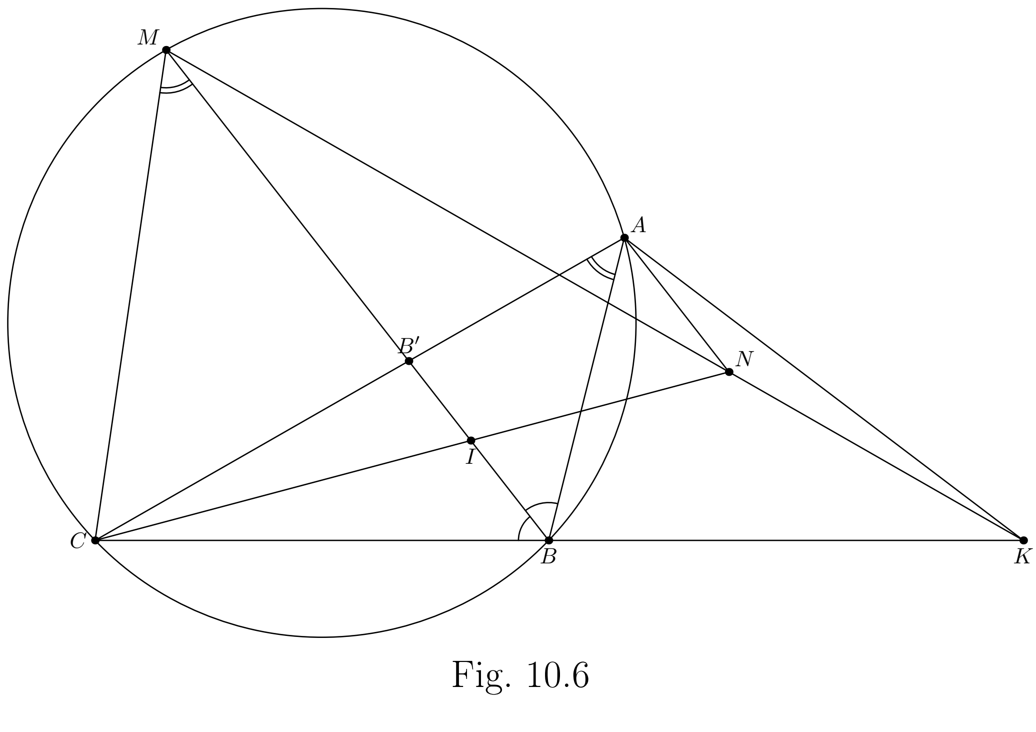


Fig. 10.6

Second solution. Note that $\angle MAC = \angle MBC = \angle ABM = \angle BAN$, i.e. the lines AI and AK are the internal and the external bisector of triangle AMN. Let AI meet MN and BC at points P and Q respectively, and let AK meet MN at K'. Then the quadruples (B, C, K, Q) and (M, N, K', P) are harmonic, therefore projecting MN to BC from I, we obtain that K' coincides with K.

7. (A.Zaslavsky) Restore a triangle by one of its vertices, the circumcenter and the Lemoine point. (*The Lemoine point is the common point of the lines symmetric to the medians about the correspondent bisectors.*)

   First solution. Since the vertex A and the circumcenter O are given, we can construct the circumcircle. Let XY be the chord of this circle with the midpoint at the Lemoine point L, let UV be the diameter parallel to this chord, and let the diagonals of the trapezoid with bases XY and UV meet at point K. Consider a transformation that maps each point P of the circumcircle to the second common point P' of the circle and the line KP. This transformation preserves the cross-ratios, thus it can be extended to a projective transformation of the plane. Since this transformation maps L to O, it maps the triangle in question to a triangle with coinciding Lemoine point and circumcenter. This triangle is regular. From this we obtain the following construction.

   Draw line AK and find its second common point A' with the circumcircle. Inscribe a regular triangle $A'B'C'$ into the circle and find the second common points B, C of lines BK, CK with the circle. Then ABC is the required triangle.

   Second solution. We use the following assertion.

   Lemma. Let a triangle ABC and a point P be given. An inversion with center A maps B, C, P to B', C', P' respectively. Let the circle $B'C'P'$ meet AP for the second time at Q. Then the similarity transforming the triangle $AC'B'$ to ABC maps Q to the point isogonally conjugated to P.

   The assertion of this lemma clearly follows from the equalities $\angle ABP = \angle B'P'A = \angle B'C'Q$.

   Let us return to the problem. Let an inversion with center A map L and the circumcircle to L' and line l respectively. Let AL meet l at point T, and let point M divide the segment AT in ratio $2 : 1$. Then M is the centroid of triangle $AB'C'$, where B', C' are the images of B and C. By the lemma M

lies on the circle $B'C'L'$, therefore $KB'^2 = KC'^2 = KM \cdot KL'$. Thus we can construct B', C', and B, C.

8. (S.Novikov) Let ABC be a nonisosceles triangle, let AA_1 be its bisector, and let A_2 be the touching point of BC with the incircle. The points B_1, B_2, C_1, C_2 are defined similarly. Let O and I be the circumcenter and the incenter of the triangle. Prove that the radical center of the circumcircles of triangles AA_1A_2, BB_1B_2, CC_1C_2 lies on OI.

   First solution. Let A' be the midpoint of an arc BC not containing A. Since the inversion with center A' and radius $A'B$ transposes the line BC and the circumcircle, it maps A_1 and A_2 to A and the common point A'' of $A'A_2$ and the circumcircle. Therefore the points A, A_1, A_2 and A'' are concyclic. Furthermore since $OA' \parallel IA_2$, the lines OI and $A'A_2$ meet at the point K which is the homothety center of the circumcircle and the incircle (fig. 10.8). Hence the degree of K wrt the circle AA_1A_2 is equal to

$$(\vec{KA_2}, \vec{KA''}) = \frac{r}{R}(\vec{KA'}, \vec{KA''}) = -\frac{r^3R}{(R-r)^2},$$

   because $(\vec{KA'}, \vec{KA''})$ is the degree of K wrt the circumcircle, equal to $-R^2r^2/(R-r)^2$.

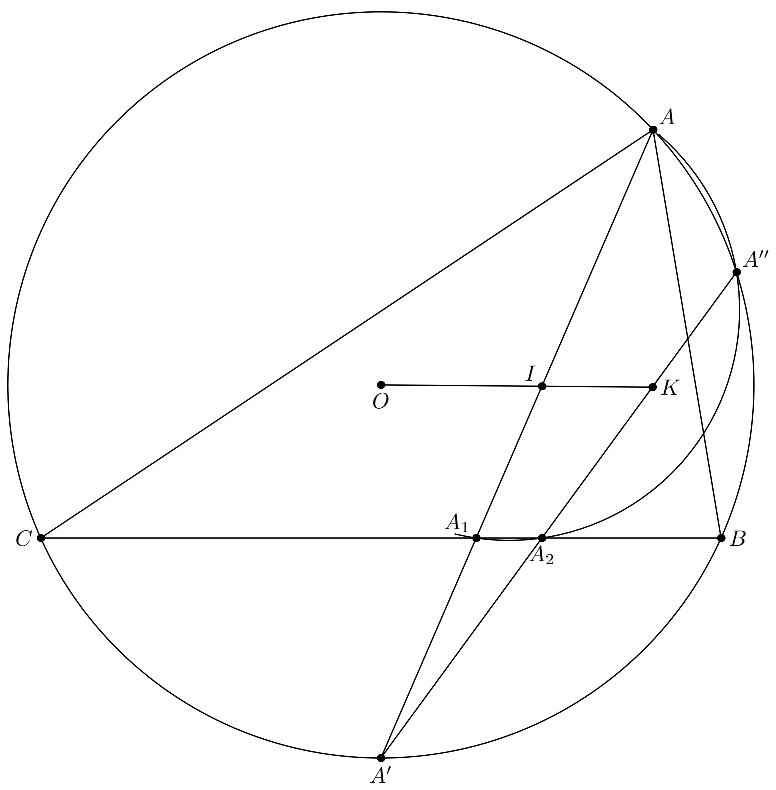


Fig. 10.8

Clearly the degrees of K wrt the circles BB_1B_2 and CC_1C_2 are the same, i.e. K is the radical center.

Second solution. Let A', B', C' be the midpoints of the arcs BC, CA, AB. Then the triangles $A'B'C'$ and $A_2B_2C_2$ are homothetic with a positive coefficient and center K, i.e. $KA_2/A'A_2 = KB_2/B'B_2 = KC_2/C'C_2 = k$. For the points of line $A'A_2$ consider the difference of the degrees wrt AA_1A_2 and the incircle. This is a linear function. In A_2 this function is equal to zero, and in A' it is equal to r^2 because $A'A_1 \cdot A'A = A'B^2 = A'I^2$. Thus in K this difference is equal to $-kr^2$. Two similar differences in K are also equal to $-kr^2$, and we obtain the required assertion.

XIII Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2017, July 31

Problems

First day. 8 grade

8.1. Let $ABCD$ be a cyclic quadrilateral with $AB = BC$ and $AD = CD$. A point M lies on the minor arc CD of its circumcircle. The lines BM and CD meet at point P, the lines AM and BD meet at point Q. Prove that $PQ \parallel AC$.

8.2. Let H and O be the orthocenter and the circumcenter of an acute-angled triangle ABC, respectively. The perpendicular bisector to segment BH meets AB and BC at points A_1 and C_1, respectively. Prove that OB bisects the angle A_1OC_1.

8.3. Let AD, BE and CF be the medians of triangle ABC. The points X and Y are the reflections of F about AD and BE, respectively. Prove that the circumcircles of triangles BEX and ADY are concentric.

8.4. Alex dissects a paper triangle into two triangles. Each minute after this he dissects one of obtained triangles into two triangles. After some time (at least one hour) it appeared that all obtained triangles were congruent. Find all initial triangles for which this is possible.

XIII Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2017, July 31

Problems

First day. 8 grade

8.1. Let $ABCD$ be a cyclic quadrilateral with $AB = BC$ and $AD = CD$. A point M lies on the minor arc CD of its circumcircle. The lines BM and CD meet at point P, the lines AM and BD meet at point Q. Prove that $PQ \parallel AC$.

8.2. Let H and O be the orthocenter and the circumcenter of an acute-angled triangle ABC, respectively. The perpendicular bisector to segment BH meets AB and BC at points A_1 and C_1, respectively. Prove that OB bisects the angle A_1OC_1.

8.3. Let AD, BE and CF be the medians of triangle ABC. The points X and Y are the reflections of F about AD and BE, respectively. Prove that the circumcircles of triangles BEX and ADY are concentric.

8.4. Alex dissects a paper triangle into two triangles. Each minute after this he dissects one of obtained triangles into two triangles. After some time (at least one hour) it appeared that all obtained triangles were congruent. Find all initial triangles for which this is possible.

XIII Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2017, August 1

Problems

Second day. 8 grade

8.5. A square $ABCD$ is given. Two circles are inscribed into angles A and B, and the sum of their diameters is equal to the sidelength of the square. Prove that one of their common tangents passes through the midpoint of AB.

8.6. A median of an acute-angled triangle dissects it into two triangles. Prove that each of them can be covered by a semidisc congruent to a half of the circumdisc of the initial triangle.

8.7. Let $A_1A_2\dots A_{13}$ and $B_1B_2\dots B_{13}$ be two regular 13-gons in the plane such that the points B_1 and A_{13} coincide and lie on the segment A_1B_{13}, and both polygons lie in the same semiplane with respect to this segment. Prove that the lines A_1A_9, $B_{13}B_8$ and A_8B_9 are concurrent.

8.8. Let $ABCD$ be a square, and let P be a point on the minor arc CD of its circumcircle. The lines PA, PB meet the diagonals BD, AC at points K, L respectively. The points M, N are the projections of K, L respectively to CD, and Q is the common point of lines KN and ML. Prove that PQ bisects the segment AB.

XIII Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2017, August 1

Problems

Second day. 8 grade

8.5. A square $ABCD$ is given. Two circles are inscribed into angles A and B, and the sum of their diameters is equal to the sidelength of the square. Prove that one of their common tangents passes through the midpoint of AB.

8.6. A median of an acute-angled triangle dissects it into two triangles. Prove that each of them can be covered by a semidisc congruent to a half of the circumdisc of the initial triangle.

8.7. Let $A_1A_2\dots A_{13}$ and $B_1B_2\dots B_{13}$ be two regular 13-gons in the plane such that the points B_1 and A_{13} coincide and lie on the segment A_1B_{13}, and both polygons lie in the same semiplane with respect to this segment. Prove that the lines A_1A_9, $B_{13}B_8$ and A_8B_9 are concurrent.

8.8. Let $ABCD$ be a square, and let P be a point on the minor arc CD of its circumcircle. The lines PA, PB meet the diagonals BD, AC at points K, L respectively. The points M, N are the projections of K, L respectively to CD, and Q is the common point of lines KN and ML. Prove that PQ bisects the segment AB.

XIII Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2017, July 31

Problems

First day. 9 grade

9.1. Let ABC be a regular triangle. The line passing through the midpoint of AB and parallel to AC meets the minor arc AB of its circumcircle at point K. Prove that the ratio $AK : BK$ is equal to the ratio of the side and the diagonal of a regular pentagon.

9.2. Let I be the incenter of triangle ABC, M be the midpoint of AC, and W be the midpoint of arc AB of its circumcircle not containing C. It is known that $\angle AIM = 90^\circ$. Find the ratio $CI : IW$.

9.3. The angles B and C of an acute-angled triangle ABC are greater than 60°. Points P and Q are chosen on the sides AB and AC, respectively, so that the points A, P, Q are concyclic with the orthocenter H of the triangle ABC. Point K is the midpoint of PQ. Prove that $\angle BKC > 90^\circ$.

9.4. Points M and K are chosen on lateral sides AB and AC, respectively, of an isosceles triangle ABC, and point D is chosen on its base BC so that $AMDK$ is a parallelogram. Let the lines MK and BC meet at point L, and let X, Y be the intersection points of AB, AC respectively with the perpendicular line from D to BC. Prove that the circle with center L and radius LD and the circumcircle of triangle AXY are tangent.

XIII Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2017, July 31

Problems

First day. 9 grade

9.1. Let ABC be a regular triangle. The line passing through the midpoint of AB and parallel to AC meets the minor arc AB of its circumcircle at point K. Prove that the ratio $AK : BK$ is equal to the ratio of the side and the diagonal of a regular pentagon.

9.2. Let I be the incenter of triangle ABC, M be the midpoint of AC, and W be the midpoint of arc AB of its circumcircle not containing C. It is known that $\angle AIM = 90^\circ$. Find the ratio $CI : IW$.

9.3. The angles B and C of an acute-angled triangle ABC are greater than 60°. Points P and Q are chosen on the sides AB and AC, respectively, so that the points A, P, Q are concyclic with the orthocenter H of the triangle ABC. Point K is the midpoint of PQ. Prove that $\angle BKC > 90^\circ$.

9.4. Points M and K are chosen on lateral sides AB and AC, respectively, of an isosceles triangle ABC, and point D is chosen on its base BC so that $AMDK$ is a parallelogram. Let the lines MK and BC meet at point L, and let X, Y be the intersection points of AB, AC respectively with the perpendicular line from D to BC. Prove that the circle with center L and radius LD and the circumcircle of triangle AXY are tangent.

XIII Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2017, August 1

Problems

Second day. 9 grade

9.5. Let BH_b, CH_c be altitudes of an acute-angled triangle ABC. The line H_bH_c meets the circumcircle of ABC at points X and Y. Points P and Q are the reflections of X and Y about AB and AC, respectively. Prove that $PQ \parallel BC$.

9.6. Let ABC be a right-angled triangle ($\angle C = 90°$) and D be the midpoint of an altitude from C. The reflections of the line AB about AD and BD, respectively, meet at point F. Find the ratio $S_{ABF} : S_{ABC}$.

9.7. Let a and b be parallel lines with 50 distinct points marked on a and 50 distinct points marked on b. Find the greatest possible number of acute-angled triangles all whose vertices are marked.

9.8. Let AK and BL be the altitudes of an acute-angled triangle ABC, and let ω be the excircle of ABC touching the side AB. The common internal tangents to circles CKL and ω meet AB at points P and Q. Prove that $AP = BQ$.

XIII Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2017, August 1

Problems

Second day. 9 grade

9.5. Let BH_b, CH_c be altitudes of an acute-angled triangle ABC. The line H_bH_c meets the circumcircle of ABC at points X and Y. Points P and Q are the reflections of X and Y about AB and AC, respectively. Prove that $PQ \parallel BC$.

9.6. Let ABC be a right-angled triangle ($\angle C = 90°$) and D be the midpoint of an altitude from C. The reflections of the line AB about AD and BD, respectively, meet at point F. Find the ratio $S_{ABF} : S_{ABC}$.

9.7. Let a and b be parallel lines with 50 distinct points marked on a and 50 distinct points marked on b. Find the greatest possible number of acute-angled triangles all whose vertices are marked.

9.8. Let AK and BL be the altitudes of an acute-angled triangle ABC, and let ω be the excircle of ABC touching the side AB. The common internal tangents to circles CKL and ω meet AB at points P and Q. Prove that $AP = BQ$.

XIII Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2017, August 1

Problems

First day. 10 grade

10.1. Let A and B be the common points of two circles, and CD be their common tangent (C and D are the tangency points). Let O_a, O_b be the circumcenters of triangles CAD, CBD respectively. Prove that the midpoint of segment O_aO_b lies on the line AB.

10.2. Prove that the distance from any vertex of an acute-angled triangle to the corresponding excenter is less than the sum of two greatest sidelengths.

10.3. Let $ABCD$ be a convex quadrilateral, and let $\omega_A, \omega_B, \omega_C, \omega_D$ be the circumcircles of triangles BCD, ACD, ABD, ABC, respectively. Denote by X_A the product of the power of A with respect to ω_A and the area of triangle BCD. Define X_B, X_C, X_D similarly. Prove that $X_A + X_B + X_C + X_D = 0$.

10.4. A scalene triangle ABC and its incircle ω are given. Using only a ruler and drawing at most eight lines, rays or segments, construct points A', B', C' on ω such that the rays $B'C'$, $C'A'$, $A'B'$ pass through A, B, C, respectively.

XIII Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2017, August 1

Problems

First day. 10 grade

10.1. Let A and B be the common points of two circles, and CD be their common tangent (C and D are the tangency points). Let O_a, O_b be the circumcenters of triangles CAD, CBD respectively. Prove that the midpoint of segment O_aO_b lies on the line AB.

10.2. Prove that the distance from any vertex of an acute-angled triangle to the corresponding excenter is less than the sum of two greatest sidelengths.

10.3. Let $ABCD$ be a convex quadrilateral, and let $\omega_A, \omega_B, \omega_C, \omega_D$ be the circumcircles of triangles BCD, ACD, ABD, ABC, respectively. Denote by X_A the product of the power of A with respect to ω_A and the area of triangle BCD. Define X_B, X_C, X_D similarly. Prove that $X_A + X_B + X_C + X_D = 0$.

10.4. A scalene triangle ABC and its incircle ω are given. Using only a ruler and drawing at most eight lines, rays or segments, construct points A', B', C' on ω such that the rays $B'C'$, $C'A'$, $A'B'$ pass through A, B, C, respectively.

XIII Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2017, August 1

Problems

Second day. 10 grade

10.5. Let BB', CC' be the altitudes of an acute-angled triangle ABC. Two circles passing through A and C' are tangent to BC at points P and Q. Prove that A, B', P, Q are concyclic.

10.6. Let the insphere of a pyramid $SABC$ touch the faces SAB, SBC, SCA at points D, E, F respectively. Find all possible values of the sum of angles SDA, SEB and SFC.

10.7. A quadrilateral $ABCD$ is circumscribed around circle ω centered at I and inscribed into circle Γ. The lines AB and CD meet at point P, the lines BC and AD meet at point Q. Prove that the circles PIQ and Γ are orthogonal.

10.8. Suppose S is a set of points in the plane, $|S|$ is even; no three points of S are collinear. Prove that S can be partitioned into two sets S_1 and S_2 so that their convex hulls have equal number of vertices.

XIII Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2017, August 1

Problems

Second day. 10 grade

10.5. Let BB', CC' be the altitudes of an acute-angled triangle ABC. Two circles passing through A and C' are tangent to BC at points P and Q. Prove that A, B', P, Q are concyclic.

10.6. Let the insphere of a pyramid $SABC$ touch the faces SAB, SBC, SCA at points D, E, F respectively. Find all possible values of the sum of angles SDA, SEB and SFC.

10.7. A quadrilateral $ABCD$ is circumscribed around circle ω centered at I and inscribed into circle Γ. The lines AB and CD meet at point P, the lines BC and AD meet at point Q. Prove that the circles PIQ and Γ are orthogonal.

10.8. Suppose S is a set of points in the plane, $|S|$ is even; no three points of S are collinear. Prove that S can be partitioned into two sets S_1 and S_2 so that their convex hulls have equal number of vertices.

XIII Geometrical Olympiad in honour of I.F.Sharygin
Solutions. Final round. First day. 8 grade

1. (D.Mukhin, D.Shiryaev) Let $ABCD$ be a cyclic quadrilateral with $AB = BC$ and $AD = CD$. A point M lies on the minor arc CD of its circumcircle. The lines BM and CD meet at point P, the lines AM and BD meet at point Q. Prove that $PQ \parallel AC$.

   Solution. The angle MPD cuts the arcs MD and BC, and the angle MQD cuts the arcs MD and AB. Therefore these angles are equal and $MPQD$ is a cyclic quadrilateral (fig.8.1). Now since $\angle DMP = \angle DMB = 90°$, we have $PQ \perp BD$, thus $PQ \parallel AC$.

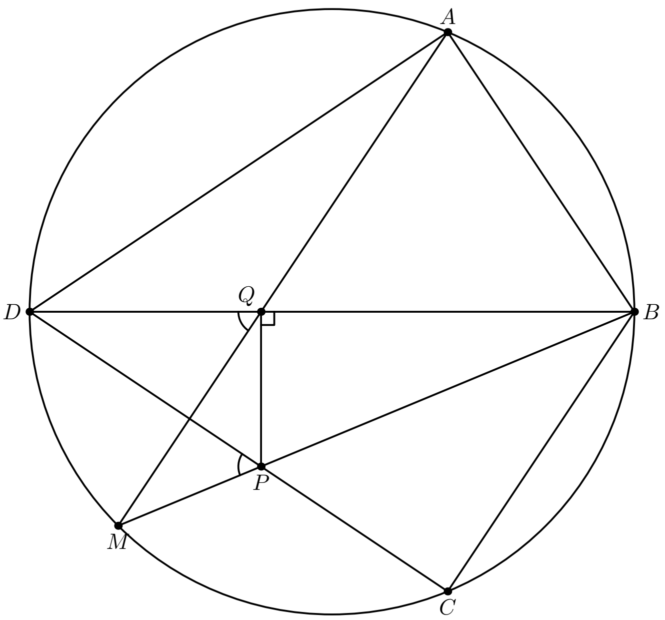


Fig. 8.1

2. (A.Sokolov) Let H and O be the orthocenter and the circumcenter of an acute-angled triangle ABC, respectively. The perpendicular bisector to segment BH meets AB and BC at points A_1 and C_1, respectively. Prove that OB bisects the angle A_1OC_1.

   First solution. Since $\angle HBC = \angle ABO = 90° - \angle C$, isosceles triangles HBC_1 and ABO are similar. Hence triangles OBC_1 and ABH are also similar, i.e., $\angle C_1OB = \angle HAB = 90° - \angle B$ (fig.8.2). Similarly $\angle A_1OB = \angle HCB = 90° - \angle B$.

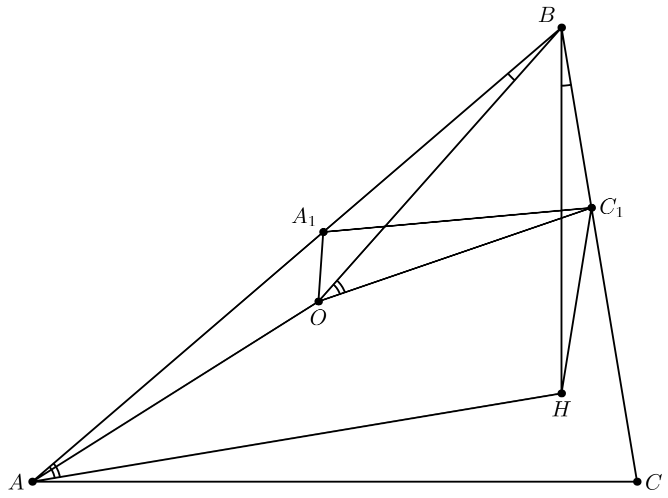


Fig. 8.2

Second solution. Use the following assertion.

Let A', B', C' be the reflections of point P about the sidelines of triangle ABC. Then the circumcenter of $A'B'C'$ is isogonally conjugated to P with respect to ABC.

Consider the triangle A_1BC_1. The reflections of H about BA_1 and BC_1 lie on the circumcircle of ABC, and the reflection of H about A_1C_1 coincide with B, thus, O and H are isogonally conjugated with respect to A_1BC_1. Then $\angle AA_1O = \angle HA_1C = \angle C_1A_1B$, i.e. C_1B is the external bisector of angle OA_1C_1. Similarly A_1B is the external bisector of angle C_1OA_1. Therefore B is the excenter of triangle A_1OC_1 and OB is the bisector of angle A_1OC_1.

3. (M.Kyranbai, Kazakhstan) Let AD, BE and CF be the medians of triangle ABC. The points X and Y are the reflections of F about AD and BE, respectively. Prove that the circumcircles of triangles BEX and ADY are concentric.

   Solution. Since $AFDE$ is a parallelogram, the midpoints of segments FE and AD coincide, therefore $EX \parallel AD$. Since FEX is a right-angled triangle, the perpendicular bisector to EX passes through the midpoint of EF, thus it coincides with the perpendicular bisector to AD (fig.8.3). Similarly we obtain that the perpendicular bisectors to DY and BE coincide.

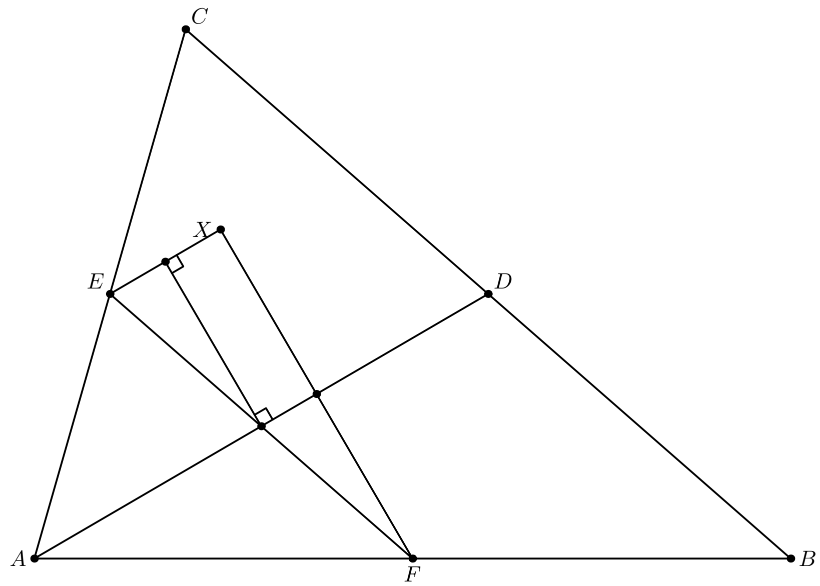


Fig. 8.3

4. (A.Shapovalov) Alex dissects a paper triangle into two triangles. Each minute after this he dissects one of obtained triangles into two triangles. After several time (at least one hour) it appeared that all obtained triangles are congruent. Find all initial triangles for which this is possible.

   Answer. Isosceles or right-angled.

   Solution. Sufficiency. Cutting an isosceles triangle by its median we obtain two congruent right-angled triangles, and cutting a right-angled triangle by the median from the right angle we obtain two isosceles triangles. Cutting each of them into two congruent triangles we obtain four congruent right-angled triangles. Similarly we can cut off each of these triangles into four congruent triangles etc.

   Necessity. After the last cutting we obtain two congruent triangles having two adjacent angles. Each of these angles is greater than a non-adjacent angle of the other triangle, thus it is equal to the adjacent angle, i.e, it is right. So the initial triangle is divided into right-angled triangles. Let their angles be α, $\beta = 90° - \alpha$ and $90°$, where $\alpha \leq \beta$. If $\alpha = 45°$ or $\alpha = 30°$, then all angles of the initial triangle are divisible by α, and direct listing of the alternatives shows that they are equal to $(45°, 45°, 90°)$, $(30°, 60°, 90°)$, $(30°, 30°, 120°)$ or $(60°, 60°, 60°)$, i.e., the triangle is right-angled or isosceles.

   For the remaining values of α the list α, β, 2α, $90°$, 2β does not contain equal angles, and all pairs of adjacent angles from this list are $(90°, 90°)$ or $(2\alpha, 2\beta)$. Let the area of all resulting triangles be 1, then the area s of the initial triangles and all areas of intermediate triangles are natural. Let us prove by induction on s that the angles of these triangles are $(\alpha, \beta, 90°)$, $(\alpha, \alpha, 2\beta)$ or $(\beta, \beta, 2\alpha)$. The base $s = 1$ is proved. A triangle T with $s > 1$ was divided into two triangles with smaller area. By induction, each of these triangles has one of three sets of angles from the list and they have two adjacent angles. If these angles are right then both triangles have a common cathetus. There are three possible cases: both opposite angles are equal to α; both angles are equal to β, or one angle is α, and the second one is β. In all these cases T belongs to one of three types. And if two adjacent angles are equal to 2α and 2β then T is right-angled with angles $(\alpha, \beta, 90°)$.

XIII Geometrical Olympiad in honour of I.F.Sharygin
Solutions. Final round. Second day. 8 grade

5. (E.Bakaev) A square $ABCD$ is given. Two circles are inscribed into angles A and B, and the sum of their diameters is equal to the sidelength of the square. Prove that one of their common tangents passes through the midpoint of AB.

   Solution. Let O_a, O_b be the centers of the circles, T_a, T_b be their touching points with AB, and M be the midpoint of AB (fig.8.5). By the assumption of the problem, we obtain that $T_aM = MO_b$, $T_bM = MO_a$. Therefore $\angle O_aMT_a + \angle O_bMT_b = 90^\circ$, i.e., $O_aM \perp O_bM$. Thus the line l symmetric to AB wrt O_aM is also symmetric to AB wrt O_bM. Since the distances from the centers of both circles to l are equal to their radii, we obtain that l is the common tangent.

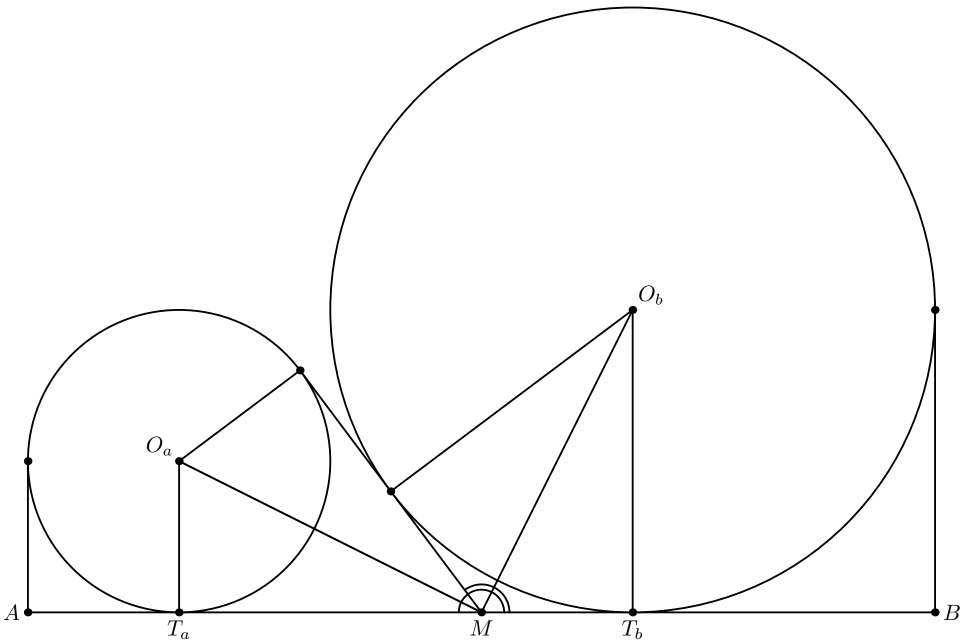


Fig. 8.5

6. (A.Shapovalov) A median of an acute-angled triangle dissects it into two triangles. Prove that each of them can be covered by a semidisc congruent to a half of the circumdisc of the initial triangle.

   Solution. Let CD be the median of triangle ABC, angle ADC be non-acute, and angle BDC be non-obtuse. Then all vertices of triangle BDC lie on the same semiplane wrt the perpendicular bisector to AB, which is the diameter of the circumcircle of ABC, therefore this triangle lies inside the corresponding semidisc. Furthermore the triangle ACD can be covered by the semidisc wuth diameter AC, hence it can be covered by a greater semidisc.

7. (E.Bakaev) Let $A_1A_2\dots A_{13}$ and $B_1B_2\dots B_{13}$ be two regular 13-gons in the plane such that the points B_1 and A_{13} coincide and lie on the segment A_1B_{13}, and both polygons lie in the same semiplane with respect to this segment. Prove that the lines A_1A_9, $B_{13}B_8$ and A_8B_9 are concurrent.

   Solution. Consider the regular polygon $C_1C_2\dots C_{13}$, where $C_1 = A_1$, $C_{13} = B_{13}$. Clearly the lines A_1A_9 and B_1B_8 coincide with C_1C_9 and $C_{13}C_8$ respectively. Furthermore $C_1C_{13} = C_8C_9$, thus C_1C_8 and $C_{13}C_9$ are the bases of an isosceles trapezoid. The points A_8 and B_9 lie on these bases and $A_1A_8 : A_8C_8 = A_1A_{13} : B_1B_{13} = C_9B_9 : B_9B_{13}$. Therefore the line A_8B_9 passes through the common point of the diagonals of the trapezoid (fig.8.7).

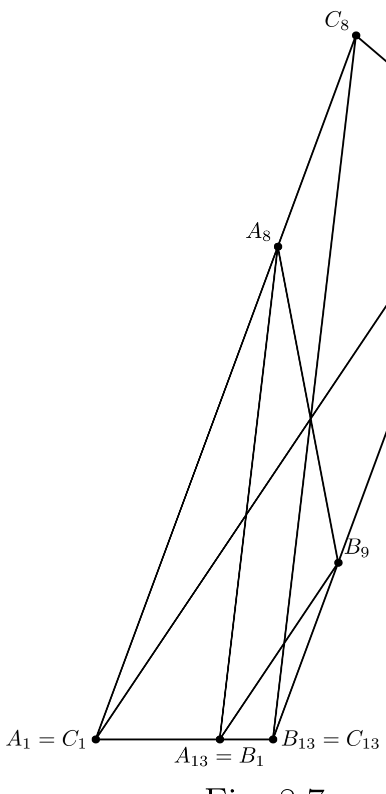


Fig. 8.7

8. (Tran Quang Hung, Vietnam) Let $ABCD$ be a square, and let P be a point on the minor arc CD of its circumcircle. The lines PA, PB meet the diagonals BD, AC at points K, L respectively. The points M, N are the

projections of K, L respectively to CD, and Q is the common point of lines KN and ML. Prove that PQ bisects the segment AB.

Solution. Firstly prove the following assertion.

Lemma. Let $AP = AC$ and $BQ = BC$ be the perpendiculars to the hypothenuse AB of a right-angled triangle ABC lying on the outside of the triangle. The lines AQ and BP meet at point R, and the lines CP and CQ meet AB at points M and N respectively. Then CR bisects the segment MN.

Proof. Since $\angle CAP = 90^\circ + \angle CAB = 180^\circ - \angle CBA$, we have $\angle ACP = \frac{\angle B}{2}$. Hence $BM = BC = BQ$ and similarly $AN = AC = AP$. Let the line passing through R and parallel to AB meet CP, CQ at points X, Y respectively, and let Z be the projection of R to AB (fig.8.8). Then $RX : BM = PR : PB = AR : AQ = RZ : QB$, Therefore $RX = RZ$. Similarly $RY = RZ$ (fig.8.8.1). Thus CR bisects XY, and hence it bisects MN.

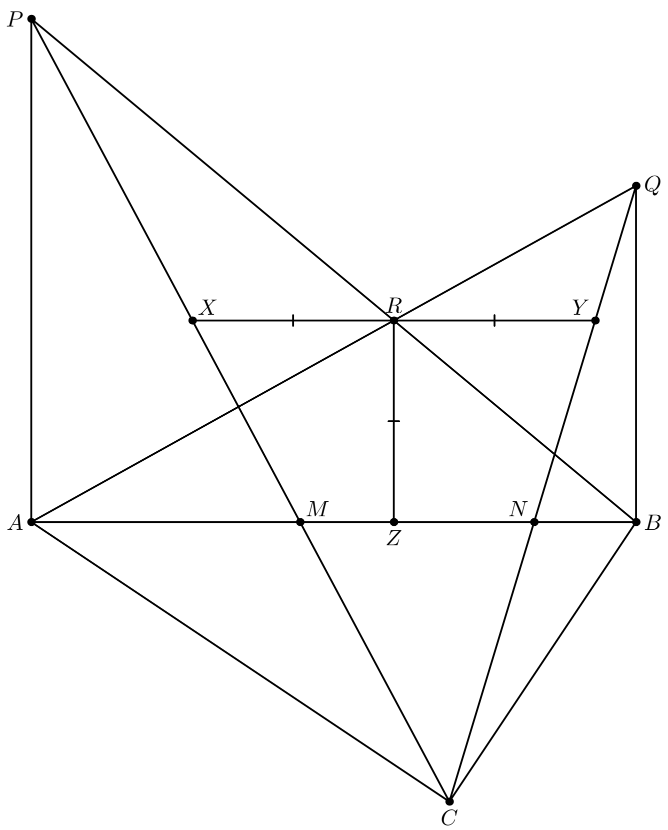


Fig. 8.8.1

Note. It it easy to see that CZ is the bisectrix of ABC and CR passes through the touching point of its incircle with the hypothenuse.

Return to the problem. Since KMD is an isosceles right-angled triangle, and $\angle KPD = 45°$, we obtain that M is the circumcenter of triangle KPD. Similarly N is the circumcenter of PCL. Furthermore $\angle MPN = 45° + (90° - \frac{\angle BDP}{2}) + (90° - \frac{\angle ACP}{2}) = 90°$. Applying the lemma to the points P, M, N, K, L we obtain the required assertion (fig.8.8.2).

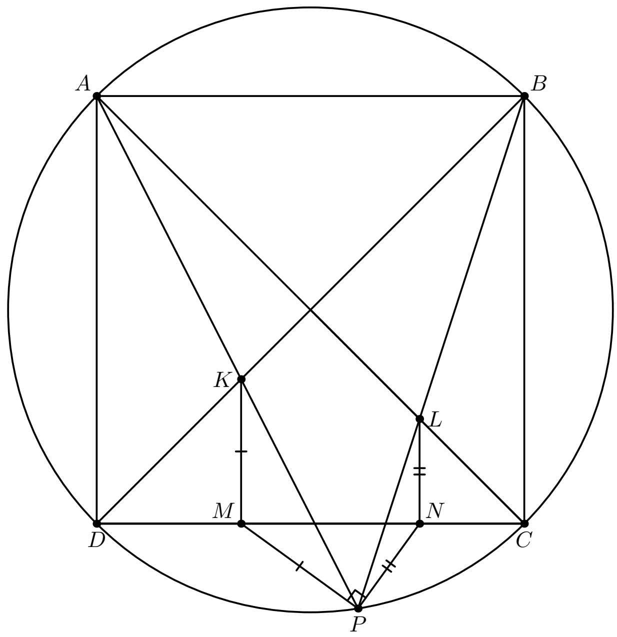


Fig. 8.8.2

XIII Geometrical Olympiad in honour of I.F.Sharygin
Solutions. Final round. First day. 9 grade

1. (A.Zaslavsky) Let ABC be a regular triangle. The line passing through the midpoint of AB and parallel to AC meets the minor arc AB of its circumcircle at point K. Prove that the ratio $AK : BK$ is equal to the ratio of the side and the diagonal of a regular pentagon.

   Solution. Let L be the second common point of the line and the circle (fig.9.1). Since ABC is a regular triangle we have $AL = BL + CL = BK + AK$. On the other hand KL bisects AB, thus the areas of triangles AKL and BKL are equal, i.e. $AK \cdot AL = BK \cdot BL = BK^2$. Therefore $t = AK/BK$ is the root of the equation $t^2 + t - 1 = 0$, equal to the ratio of the side and the diagonal of a regular pentagon.

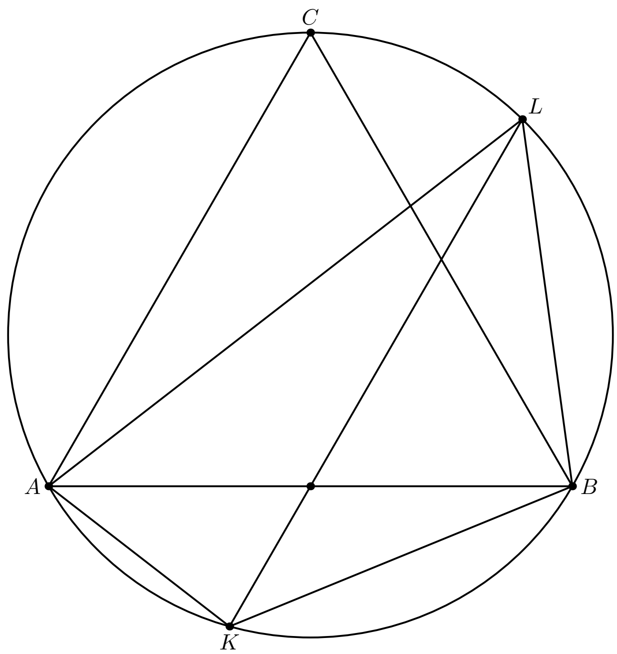


Fig. 9.1

2. (S.Berlov, A.Polyanskii) Let I be the incenter of triangle ABC, M be the midpoint of AC, and W be the midpoint of arc AB of its circumcircle not containing C. It is known that $\angle AIM = 90^\circ$. Find the ratio $CI : IW$.

   Answer. $2 : 1$.

   Solution. Let I_c be the center of the excircle touching the side AB. Since $AI_c \perp AI$ we obtain that $IM \parallel AI_c$, i.e. IM is a medial line of triangle ACI_c. Also W is the midpoint of II_c, therefore $CI = II_c = 2IW$ (fig.9.2).

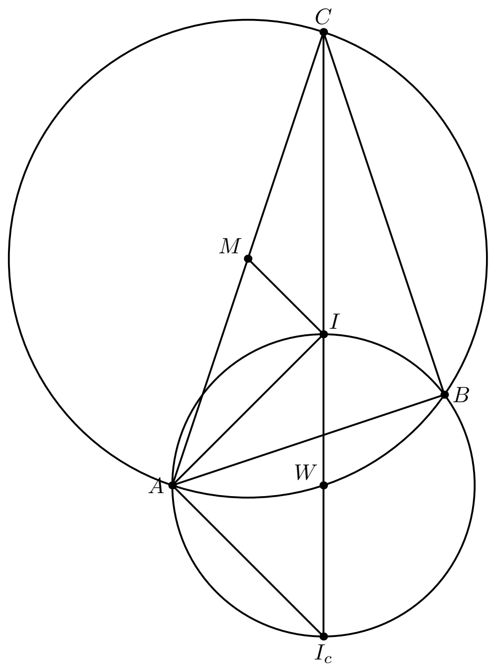


Fig. 9.2

3. (A.Mudgal, India) The angles B and C of an acute-angled triangle ABC are greater than $60°$. Points P and Q are chosen on the sides AB and AC, respectively, so that the points A, P, Q are concyclic with the orthocenter H of the triangle ABC. Point K is the midpoint of PQ. Prove that $\angle BKC > 90°$.

   Solution. Let BB', CC' be the altitudes of triangle ABC. Since $\angle PHQ = 180° - \angle A = \angle B'HC'$ we obtain that the triangles $HB'Q$ and $HC'P$ are similar. Thus, when P moves uniformly along AB the point Q also moves uniformly along AC and K moves along some segment. Since $\angle AHC' > \angle CHB'$ and $\angle AHB' > \angle BHC'$, the endpoints of this segment correspond to the coincidence of P with B or Q with C, let us consider the last case (fig.9.3).

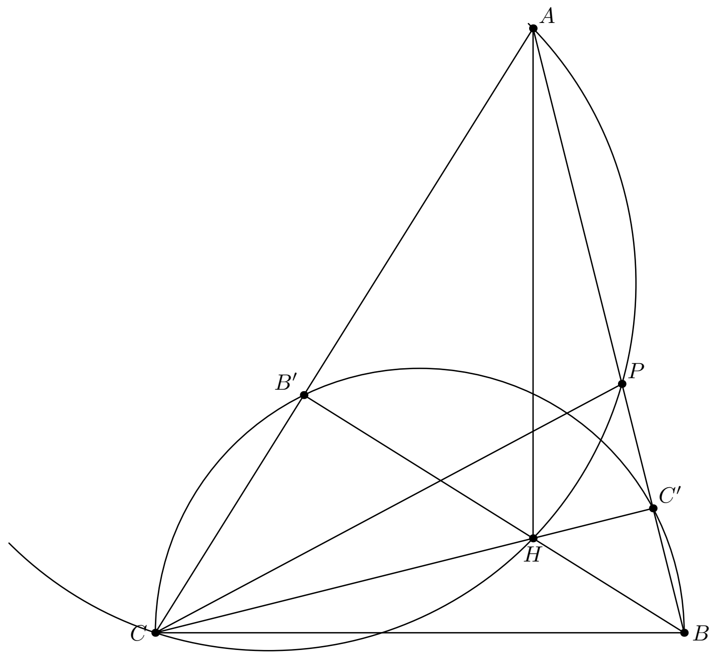


Fig. 9.3

If $Q = C$ then $\angle HCP = \angle HAP = \angle HCB$, i.e. BCP is non-isosceles triangle and the distance from K to the midpoint of BC is equal to BC'. Since $\angle B > 60^\circ$, we have $BC' < BC/2$, thus K lies inside the circle with diameter BC. Similarly the second endpoint of the segment passed by K lies inside this circle, therefore all points of the segment also lie inside it.

4. (M.Etesamifard, Iran) Points M and K are chosen on lateral sides AB and AC, respectively, of an isosceles triangle ABC, and point D is chosen on its base BC so that $AMDK$ is a parallelogram. Let the lines MK and BC meet at point L, and let X, Y be the intersection points of AB, AC respectively with the perpendicular line from D to BC. Prove that the circle with center L and radius LD and the circumcircle of triangle AXY are tangent.

   Solution. It is clear that the circumcircles of triangles ABC and AXY are perpendicular. Let E be their second common point. Since E is the center of spiral similarity mapping X and Y to B and C respectively, the triangles EXB and EYC are similar, i.e. $EB : EC = XB : YC = BD : CD$. On the other hand $LB : LD = LM : LK = LD : LC$. Hence B and C are inverse wrt the circle centered at L with radius LD i.e. this circle is also perpendicular to the circumcircle of ABC. Also the ratio of distances from B and C is the same for all points of this circle (an Apollonius circle), thus it passes through E (fig.9.4). Since both circles are perpendicular to the circumcircle of ABC and meet it at the same point we conclude that they are tangent.

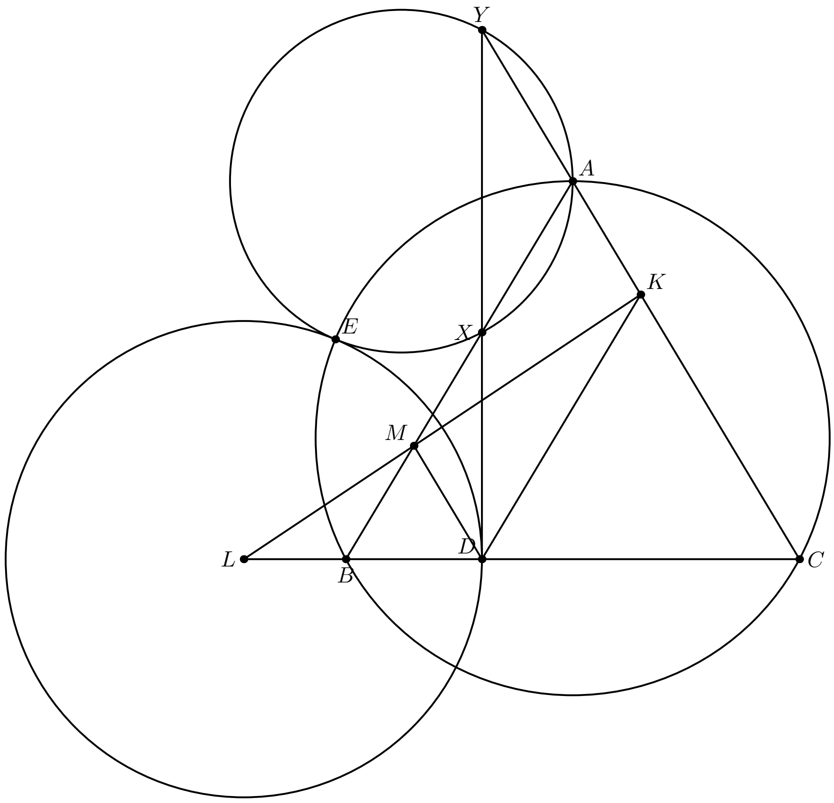


Fig. 9.4

XIII Geometrical Olympiad in honour of I.F.Sharygin
Solutions. Final round. Second day. 9 grade

5. (P.Kozhevnikov) Let BH_b, CH_c be altitudes of a triangle ABC. The line H_bH_c meets the circumcircle of ABC at points X and Y. Points P and Q are the reflections of X and Y about AB and AC, respectively. Prove that $PQ \parallel BC$.

   Solution. Let O be the circumcenter of triangle ABC. Since the line AO is the reflection of the altitude from A about the bisector from the same vertex, and $\angle AH_bH_c = \angle ABC$, we obtain that $AO \perp H_bH_c$, i.e. AO is the perpendicular bisector to the segment XY. Therefore, $AP = AX = AY = AQ$ and $XPQY$ is a cyclic quadrilateral (fig.9.5). Hence the lines XY and PQ are antiparallel with respect to the lines XP and YQ which are parallel to the altitudes of the triangle. But BC and H_bH_c are also antiparallel with respect to the altitudes, thus $PQ \parallel BC$.

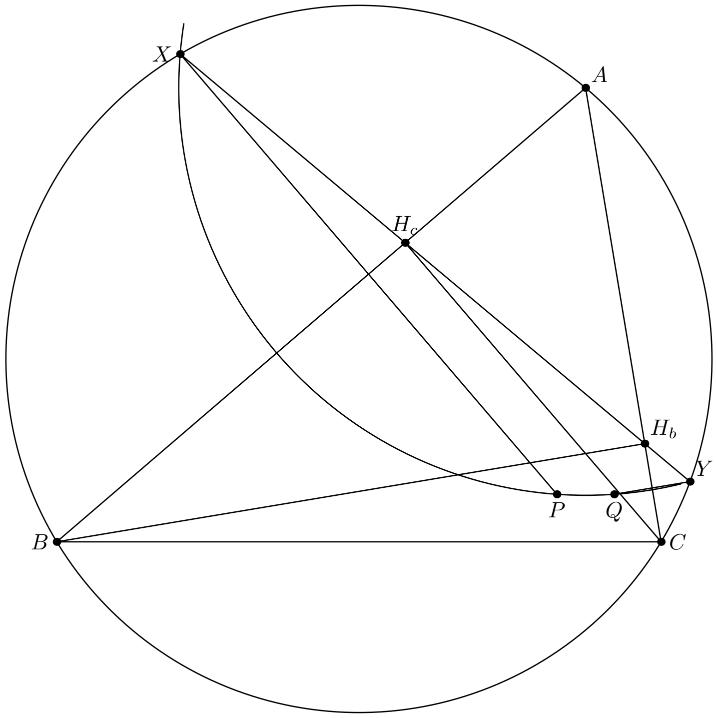


Fig. 9.5

6. (M.Etesamifard, Iran) Let ABC be a right-angled triangle ($\angle C = 90^\circ$) and D be the midpoint of an altitude from C. The reflections of the line AB about AD and BD, respectively, meet at point F. Find the ratio $S_{ABF} : S_{ABC}$.

   Answer. 4/3.

Solution. Let CH be the altitude and K, L be the common points of the line passing through C and parallel to AB with AF and BF respectively (fig.9.6). Since the trapezoid $AKLB$ is circumscribed around the circle with diameter CH, we obtain that KD and LD are the bisectors of angles AKL and BLK respectively. Hence $\angle CKD = 90^\circ - \angle HAD$, i.e. the triangles KCD and DHA are similar, and $KC = CD^2/AH = CH^2/(4AH) = BH/4$. Similarly $CL = AH/4$. Therefore the ratio of the altitudes of similar triangles FKL and FAB is equal to $1/4$, and the ratio of the altitudes of triangles AFB and ABC is $4/3$.

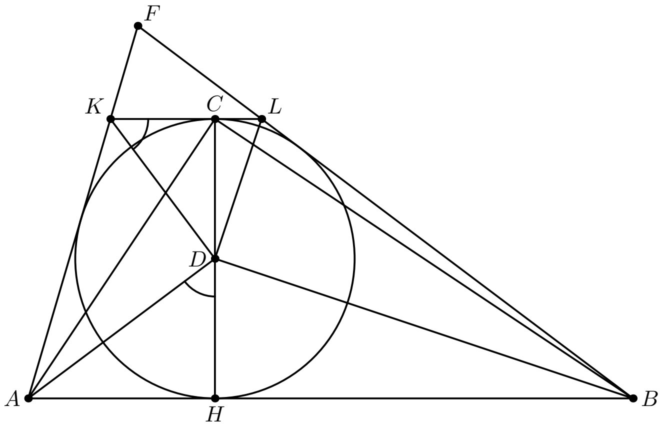


Fig. 9.6

7. (P.Kozhevnikov) Let a and b be parallel lines with 50 distinct points marked on a and 50 distinct points marked on b. Find the greatest possible number of acute-angled triangles all whose vertices are marked.

Answer. 41650.

First solution. Let $n = 50$.

Introduce coordinates so that lines a and b are given by equations $y = 0$ and $y = 1$ respectively. Denote by $A_1, A_2, \ldots, A_n$ marked points on a so that their x-coordinates $a_1, a_2, \ldots, a_n$ are ordered: $a_1 < \ldots < a_n$. Similarly define $B_1, \ldots, B_n$ with x-coordinates $b_1 < \ldots < b_n$. Let A^- and A^+ be points on a such that their x-coordinates satisfy conditions $a^- < a_1$ and $a^+ > a_n$. Similarly define B^- and B^+.

Estimate. The total number of triangles with all vertices marked is $T = 2\binom{n}{2}n = n^2(n-1)$. The number of non-acute triangles is not less than the number N of non-acute angles among angles $A_iA_jB_k$ and $B_iB_jA_k$. Let us estimate N.

Fix $t \in \{1, 2, \ldots, n\}$ and $s \in \{1, 2, \ldots, n\}$, then WLOG $t \le s$, and consider the segment A_tB_s (similarly consider B_tA_s). The segment A_tB_s forms two pairs of equal angles with lines a and b. Note that either $\angle A^-A_tB_s$ and $\angle A_tB_sB^+$ both are non-acute, or $\angle A^+A_tB_s$ and $\angle A_tB_sB^-$ both are non-acute. In the first case all angles $\angle A_iA_tB_s$ and $\angle A_tB_sB_j$ with $i < t$ and $j > s$ are non-acute; the number of such angles is $(t-1)+(n-s) = n-1-(s-t)$. In the second case all angles $\angle A_iA_tB_s$ and $\angle A_tB_sB_j$ with $i > t$ and $j < s$ are non-acute; the number of such angles is $(n-t)+(s-1) = n-1+(s-t)$. Anyway the total number of non-acute angles of the form $\angle A_iA_tB_s$ or $\angle A_tB_sB_j$ is not less than $n-1-(s-t)$. Thus we have the estimate $N \ge n(n-1) + 2\sum\limits_{1 \le t < s \le n}(n-1-(s-t)) = \frac{(n-1)(2n^2-n)}{3}$ (here the summand $n(n-1)$ corresponds to n segments A_sB_t with $t = s$). A direct calculation shows that the number of acute triangles with all vertices marked is not greater than $T - N \le \frac{(n-1)n(n+1)}{3} = 41650$.

Example. Let us mark points so that $0 < a_1 < b_1 < a_2 < b_2 < \ldots < a_n < b_n < 1/10$. Note that in this example all angles $\angle A_iB_kA_j$ and $\angle B_iA_kB_j$ are acute. Moreover, for each segment A_tB_s (or A_sB_t) with $t \le s$, among the angles of the form $\angle A_iA_tB_s$ and $\angle A_tB_sB_j$ exactly $n-(s-t)+1$ are non-acute. This means that this example is optimal, since our estimate in this case is sharp.

Second solution. Clearly, the maximum is achieved when the points of both sides are located sufficiently closely, in such a way that all angles having the vertex on one line and the sides passing through two points of the second one are acute. Let the points on one line be colored blue and the projections to this line of the points on the second one be colored red. Then we obtain the following reformulation of the problem.

Suppose 50 blue and 50 red points are marked on a line. Find the maximal number of triples having the medial point of one color and two extreme points of the other color.

Let $A_1, \ldots, A_{50}$, $B_1, \ldots, B_{50}$ be the red and the blue points respectively, ordered from left to right. Consider two adjacent points A_i and B_j. If A_i lies on the left side from B_j, these two points form a good triple with $B_1, \ldots, B_{j-1}$ and $A_{i+1}, \ldots, A_{50}$, i.e. we have $n-1+(j-i)$ good triples. When $i > j$ we can transpose A_i and B_j and increase the number of such good triples, and this operation retains all remaining good triples. Hence in the optimal disposition any point A_i lies to the right from B_{i-1}, but to the left from B_{i+1} (the order of A_i and B_i can be arbitrary). In particular, the alternating disposition is optimal. The number of acute-angled triangles in this disposition is calculated in the previous solution.

8. (I.Frolov) Let AK and BL be the altitudes of an acute-angled triangle ABC, and let ω be the excircle of ABC touching the side AB. The common internal tangents to circles CKL and ω meet AB at points P and Q. Prove that $AP = BQ$.

   First solution. Let R be the internal center of similitude of CKL and ω, the incircle of ABC touch AB at C_1, the incircle of triangle PQR touch AB at C_1', ω touch AB at C_2, and C_2C_3 be a diameter in ω. Then C, R, C_3 are collinear. Furthermore C, C_1, C_3 are collinear because C is the homothety center of the incircle and the excircle. Analogously, R, C_1', C_3 are collinear. So C_1' coincides with C_1. Thus the midpoints of AB, C_1C_2, $C_1'C_2$, and PQ coincide, as needed.

   Second solution. Let us prove that the assertion of the problem is correct when we replace CKL by an arbitrary circle centered on the altitude and passing through C. As in the first solution we obtain that the common point R of internal common tangents lies on CC_1. So we have the following reformulation of the problem.

   The tangents from an arbitrary point R of line CC_1 to the excircle meet AB at points P and Q. Prove that these points are symmetric wrt the midpoint of AB.

   It can be proved that the correspondence between P and Q preserves the cross-ratios. Hence it is sufficient to find two pairs P, Q symmetric wrt AB. For $R = C$ the points P, Q coincide with A, B, and for $R = C_1$ they coincide with C_1, C_2. In both cases they are symmetric.

   Note. In the formulation of the problem, we can replace the excircle by the incircle and the internal common tangents by the external ones. Moreover, arguing as in the second solution we obtain that the common points of tangents with AB are the same in both cases.

XIII Geometrical Olympiad in honour of I.F.Sharygin
Solutions. Final round. First day. 10 grade

1. (D.Shvetsov) Let A and B be the common points of two circles, and CD be their common tangent (C and D are the tangency points). Let O_a, O_b be the circumcenters of triangles CAD, CBD respectively. Prove that the midpoint of segment O_aO_b lies on the line AB.

   Solution. Let C', D' be the touching points of the circles with the second common tangent. The angles ACD and ADC are equal to the halves of arcs AC and AD of the corresponidng circles, and the angles BCD and BDC are equal to the halves of arcs BC and BD which are equal to $C'A$ and $D'A$. Therefore, the sum of four angles is equal to the half-sum of arcs $C'AC$ and $D'AD$. Since the last arc is homothetic to $C'C$, this half-sum is equal to π. Thus the circumcenters of triangles CAD and CBD are symmetric with respect to CD, i.e., the midpoint of O_aO_b coincides with the midpoint of CD which lies on the radical axis AB of the circles (fig.10.1).

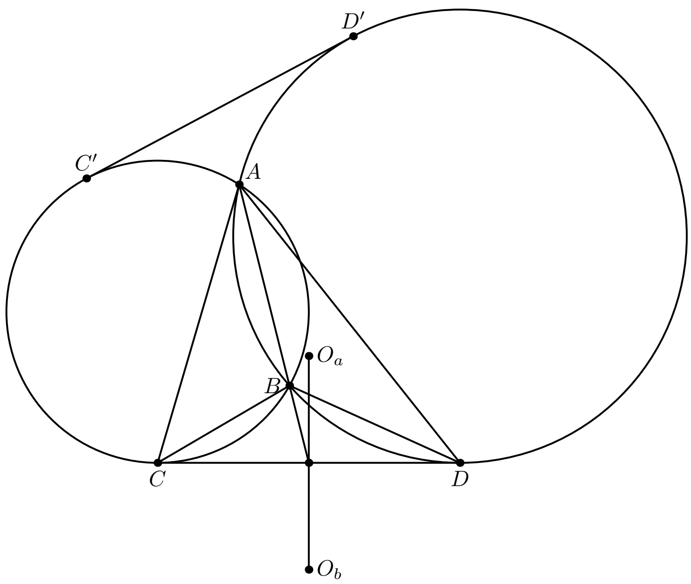


Fig. 10.1

2. (A.Peshnin) Prove that the distance from any vertex of an acute-angled triangle to the corresponding excenter is less than the sum of two greatest sidelengths.

   Solution. Let in a triangle ABC $\angle A = 2\alpha$, $\angle B = 2\beta$, $\angle C = 2\gamma$ and $\alpha \geq \beta \geq \gamma$. Let I_a, I_b, I_c be the excenters and p be the semiperimeter of the triangle .Then the inequalities $2\alpha < 90^\circ < 2\beta + 2\gamma$ and $\beta \geq \gamma$ yield that $2\beta > \alpha$. Also since $AI_a \cos\alpha = BI_b \cos\beta = CI_c \cos\gamma = p$, we have $AI_a \geq BI_b \geq CI_c$, and it is sufficient to prove that $AI_a < AC + BC$. We can obtain this in several ways.

First way. Note that a point K symmetric to B with respect to the external bisector CI_a lies on AC, and $CK = CB$. Since BI_a is the external bisector of angle B, we have $\angle I_aKA = \alpha+\gamma$, and since $\angle I_aAK = \alpha$, we obtain that $\angle AI_aK = 2\beta+\gamma$ (fig.10.2.1). Since $2\beta > \alpha$ we have $\angle AKI_a < \angle AI_aK$, i.e. $AI_a < AK = AC + BC$.

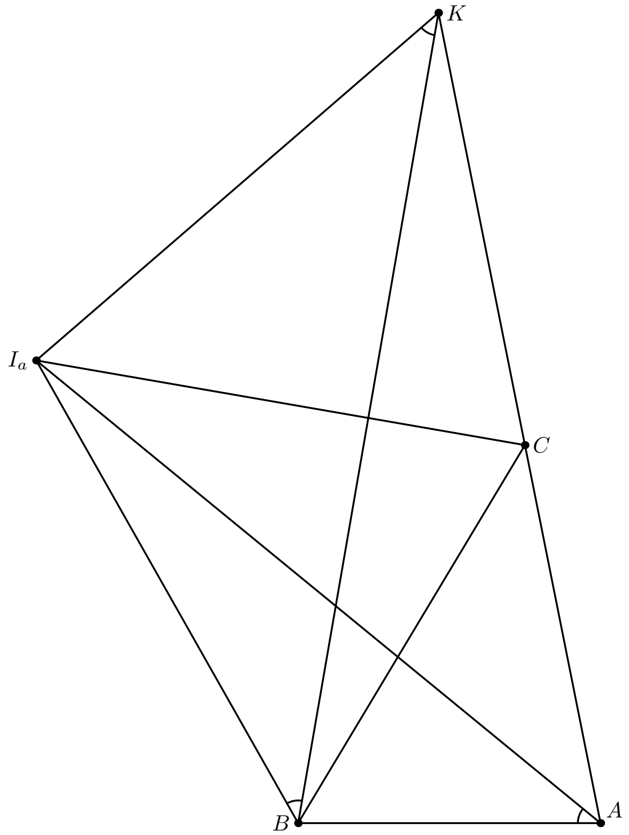


Fig. 10.2.1

Second way. Let the excircle touches AB at point T, and let U be the reflection of B about T. Since $AT = p$, we have $AU = 2p-AB = AC+BC$. Also, in the triangle AUI_a we have $\angle UAI_a = \alpha$, $\angle AUI_a = \angle I_aBT = \pi/2 - \beta$, thus $\angle AI_aU = \pi/2-\alpha+\beta$. Since $2\beta > \alpha$ we obtain that $\angle AI_aU > \angle AUI_a$ and $AU > AI_a$ (fig.10.2.2).

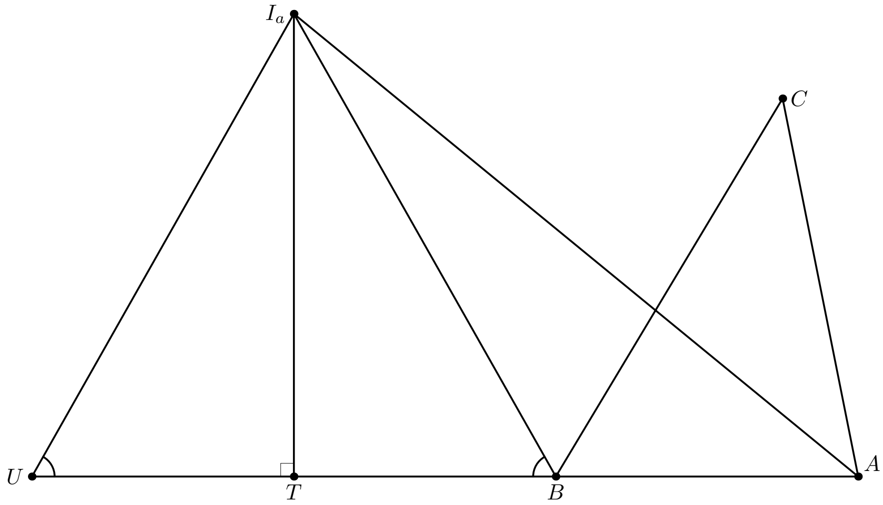


Fig. 10.2.2

Third way. We will use the following facts.

1. **The trident theorem.** The circumcenter of triangle BCI_a coincide with the midpoint W of the arc BC of circle ABC.

2. Let a circle be given, its chord AB be fixed, and point X move on an arc AB. Then the sum $AX + BX$ increases while X comes closer to the midpoint of the arc.

By the trident theorem we have $AI_a = AW+WB$. Since $\angle WBA-\angle WAB = 2\beta > 2(\alpha-\beta) = \angle CAB-\angle CBA$ we obtain that C is closer to the midpoint of arc ACB than W. Therefore, $AW + BW < AC + BC$.

Note. In fact, we proved that the segment joining any vertex to the corresponding excenter is less than the sum of the opposite and the greatest of two adjacent sidelengths.

3. (A.Sokolov) Let $ABCD$ be a convex quadrilateral, and let $\omega_A, \omega_B, \omega_C, \omega_D$ be the circumcircles of triangles BCD, ACD, ABD, ABC, respectively. Denote by X_A the product of the power of A with respect to ω_A and the area of triangle BCD. Define X_B, X_C, X_D similarly. Prove that $X_A + X_B + X_C + X_D = 0$.

Solution. If the quadrilateral is cyclic then the assertion is evident. Now note that D lies outside ω_D iff $\angle A + \angle C > \angle B + \angle D$, i.e. iff C lies inside ω_C. So the signs of X_C and X_D are opposite.

Let CD meet AB at point P and meet ω_C, ω_D for the second time at C', D' respectively. Then the ratio of areas of triangles ABC and ABD is equal to the ratio of their altitudes, which is equal to $\frac{PC}{PD}$. Since $PC{\cdot}PD' = PA{\cdot}PB =$

$PC' \cdot PD$, this ratio is equal to $\frac{PC}{PD} = \frac{PC'}{PD'} = \frac{PC-PC'}{PD-PD'} = \frac{CC'}{DD'}$. On the other hand, the ratio of the absolute values of the powers of C and D with respect to the corresponding circles is $\frac{CD\cdot CC'}{CD\cdot DD'} = \frac{CC'}{DD'}$ (fig.10.3). Therefore, $|X_C| = |X_D|$ and $X_C + X_D = 0$. Similarly $X_A + X_B = 0$.

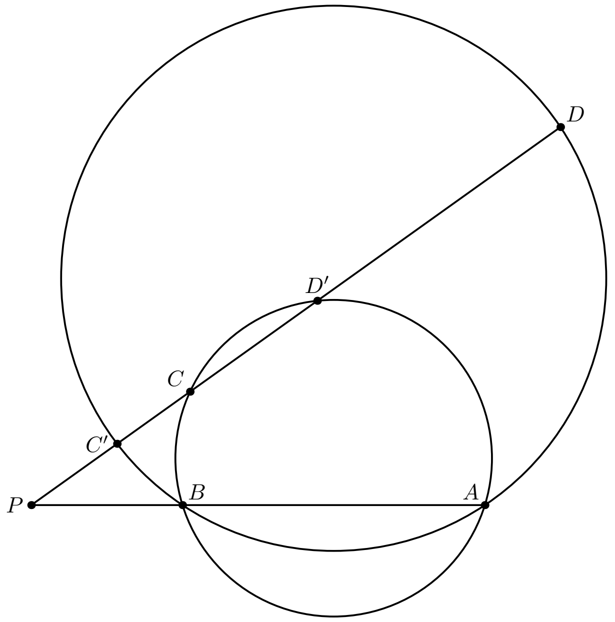


Fig. 10.3

If $AB \parallel CD$ then $S_{ABC} = S_{ABD}$, $CC' = DD'$ and we also obtain that $X_C + X_D = 0$.

Note. The equality $|X_A| = |X_B| = |X_C| = |X_D|$ is also valid when the four points do not form a convex quadrilateral.

4. (A.Zaslavsky) A scalene triangle ABC and its incircle ω are given. Using only a ruler and drawing at most eight lines, rays or segments, construct points A', B', C' on ω such that the rays $B'C'$, $C'A'$, $A'B'$ pass through A, B, C, respectively.

Solution. Let A_0, B_0, C_0 be the touching points of the incircle with BC, CA, AB respectively. Then the required points A', B', C' are such that $A'A_0C'C_0$, $B'B_0A'A_0$ and $C'C_0B'B_0$ are harmonic quadrilaterals. Consider a projective transform preserving ω and mapping the common point of AA_0, BB_0, CC_0 to its center. This transform maps ABC to a regular triangle. Then the triangles $A_0B_0C_0$ and $A'B'C'$ are also regular and $A'A_0C'C_0$ is an isosceles trapezoid. Let K be a midpoint of A_0C_0. The harmonicity condition $\angle C_0A'C' = \angle KA'A_0$ now reads $\angle KA'A_0 = \angle A_0C_0A'$, i.e., $\triangle KA'A_0 \sim \triangle A'C_0A_0$, whence $\angle A'KA_0 = 2\pi/3$ and $A'K \parallel BC \parallel B_0C_0$ (fig. 10.4).

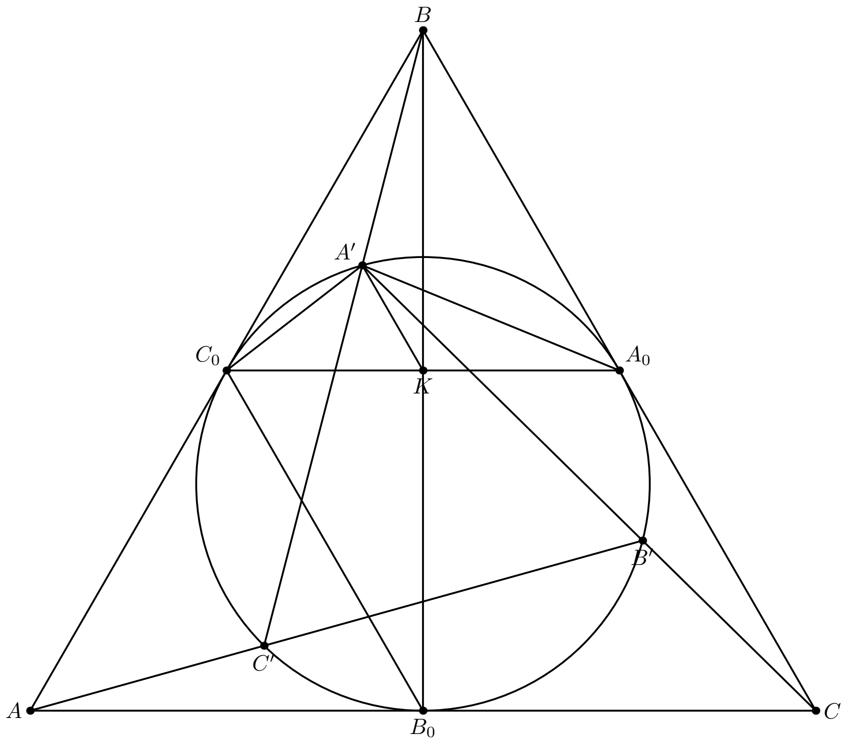


Fig. 10.4

Now, after the inverse transform we obtain the following construction.

1–2. Draw A_0C_0, BB_0 and find their common point K.

3–4. Draw BC, B_0C_0 and find their common point L.

5. Draw KL and find its common point A' with the arc A_0C_0.

6. Draw CA' and find its second common point B' with ω.

7. Draw AB' and find its second common point C' with ω.

XIII Geometrical Olympiad in honour of I.F.Sharygin
Solutions. Final round. Second day. 10 grade

5. (A.Garkavy) Let BB', CC' be the altitudes of an acute-angled triangle ABC. Two circles passing through A and C' are tangent to BC at points P and Q. Prove that A, B', P, Q are concyclic.

   First solution. Since $BP^2 = BQ^2 = BA \cdot BC'$ and the quadrilaterals $AC'A'C$, $AB'A'B$ are cyclic (AA' is the altitude) we have $CP \cdot CQ = CB^2 - BP^2 = CB^2 - BA \cdot BC' = BC^2 - BC \cdot BA' = BC \cdot CA' = CA \cdot CB'$. Clearly this is equivalent to the required assertion.

   Second solution. Let C_0 be the reflection of C' about B. Then $BC_0 \cdot BA = BC' \cdot BA = BP^2 = BP \cdot BQ$, so the points A, P, C_0, Q lie on some circle ω. Let H_0 be the point on ω opposite to A. Then $H_0C_0 \perp BC$. Hence, the reflection of $H - 0$ about B (whic is the midpoint of PQ), lies on the altitude CC'; on the other hand, this reflection also lies on the altitude AA' of the triangle APQ. Thus, the point H_0 is symmetric to the orthocenter H of ABC about B. Therefore, $BH_0 \cdot BB' = BH \cdot BB' = BC' \cdot BA = BC_0 \cdot BA$, which shows that B' also lies on ω (fig.10.5).

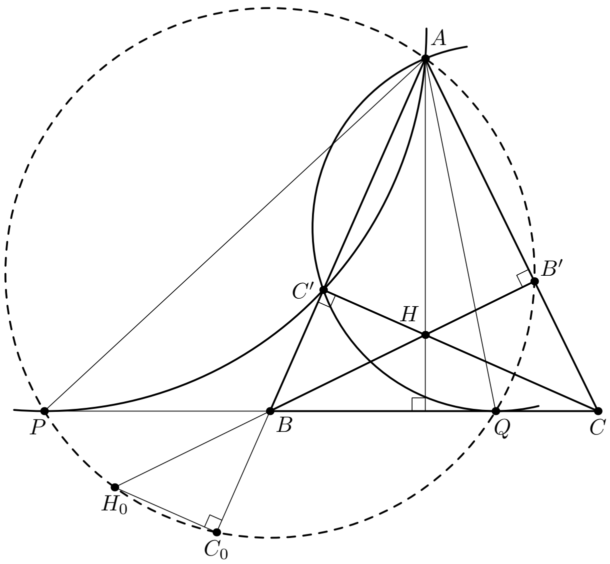


Fig. 10.5

Note. In fact, we implemented a well-known fact that C' is the projection of the orthocenter of the triangle APQ onto its median AB. This yields, in particular, that the triangles ABC and APQ have a common orthocenter H.

6. (I.I.Bogdanov) Let the insphere of a pyramid $SABC$ touch the faces SAB, SBC, SCA at points D, E, F respectively. Find all possible values of the sum of angles SDA, SEB and SFC.

   Answer. 2π.

   Solution. Since the triangles SCE and SCF are congruent we have $\angle SFC = \angle SEC$. Similarly $\angle SEB = \angle SDB$ and $\angle SDA = \angle SFA$. Hence $\angle SDA + \angle SEB + \angle SFC = \angle SFA + \angle SDB + \angle SEC = \frac{6\pi-(\angle ADB+\angle BEC+\angle CFA)}{2}$. But the angles ADB, BEC, CFA are equal to the angles AGB, BGC, CGA, where G is the tangency point of the insphere with the face ABC. Therefore their sum is equal to 2π.

   Note. One can show that, in fact, each of the triples of angles $(\angle SDA, \angle SDB, \angle ADB)$, $(\angle SEB, \angle SEC, \angle BEC)$, $(\angle SFC, \angle SFA, \angle AFC)$, and $(\angle AGB, \angle BGC, \angle CGA)$ contains the same three angles, perhaps permuted.

7. (I.Frolov) A quadrilateral $ABCD$ is circumscribed around circle ω centered at I and inscribed into circle Γ. The lines AB and CD meet at point P, the lines BC and AD meet at point Q. Prove that the circles PIQ and Γ are orthogonal.

   Solution. Since $ABCD$ is cyclic, the bisectors of the angles formed by its opposite sidelines are perpendicular. Thus $\angle PIQ = 90^\circ$, and PQ is a diameter of circle PIQ. Let R be the common point of the diagonals. Then the circle PIQ meets PR in a point S such that $PR \perp QS$. Since PR is the polar of Q with respect to Γ, we obtain that Q and S are inverse with respect to this circle, thus any circle passing through these two points is orthogonal to Γ (fig.10.7).

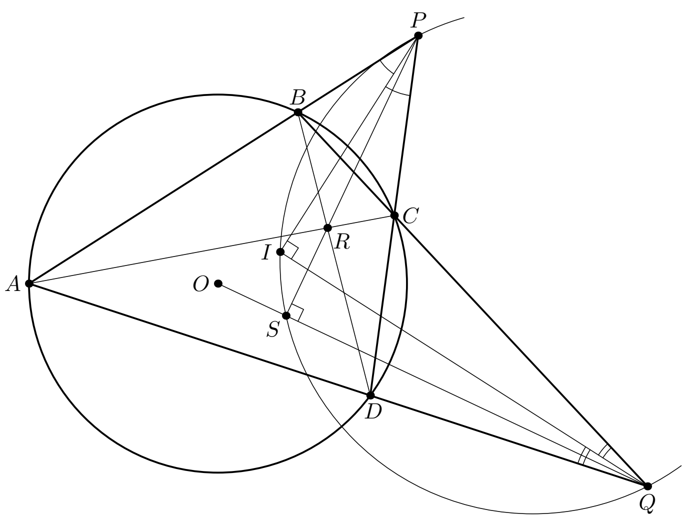


Fig. 10.7

Note. The assertion of the problem is valid for an arbitrary cyclic quadrilateral if we define I as the common point of the bisectors of angles APC and AQC.

8. (M.Saghafian,Iran; I.I.Bogdanov) Let S be a set of points in the plane, $|S|$ is even; no three points of S are collinear. Prove that S can be partitioned into two sets S_1 and S_2 so that their convex hulls have equal number of vertices.

Solution

Denote by $k(X)$ the number of vertices in the convex hull $\operatorname{conv} X$ of X.

Let $A = A_1A_2 \dots A_n = \operatorname{conv} S$, and let T be the set of all points in S lying (strictly) inside A. Set $X_i = \{A_1, \dots, A_i\} \cup T$, $Y_i = \{A_{i+1}, \dots, A_n\}$.

Let i be the minimal index such that $k(X_i) \geq k(Y_i)$. Clearly, $i < n$. If $i = 0$, then we may find a subset $T' \subseteq T$ such that $k(T') = n$ (removing the points from T one by one). Then $T' \sqcup (S \setminus T')$ is a required partition.

Assume now that $1 \leq i \leq n-1$. By the minimality of i, we get

$$k(X_i) - 1 \leq k(X_{i-1}) \leq k(Y_{i-1}) - 1 \leq k(Y_i).$$

So, either $k(X_i) = k(Y_i)$ (and they form a required partition), or

$$k(X_i) - 1 = k(X_{i-1}) = k(Y_{i-1}) - 1 = k(Y_i).$$

Let us consider the latter case.

Set $X = X_i$, $Y = Y_i$. Since $k(X) + k(Y)$ is odd, there exists at least one *extra* point $M \in X$ not on the boundary of $\operatorname{conv} X$ and $\operatorname{conv} Y$. If M is outside $\operatorname{conv} Y$, simply move it from X to Y to obtain the required partition. Otherwise all such extra points lie in $\operatorname{conv} X \cap \operatorname{conv} Y$. In particular, this intersection is nonempty.

Now let $X' = X \setminus \operatorname{conv} Y$. Then all points of X' lie on the boundary of $\operatorname{conv} X$ (all inner points of $\operatorname{conv} X$ lie also inside $\operatorname{conv} Y$), hence $k(X') < k(X)$ and so $k(X') \leq k(Y)$. If $k(X') = k(Y)$ then X' and $S \setminus X'$ form the required partition. Otherwise add to X' points from $X \cap \operatorname{conv} Y$ one by one until we get the set X'' with $k(X'') = k(Y)$. Then X'' and $S \setminus X''$ form the required partition.

XIV Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2018, July 31

Problems

First day. 8 grade

8.1. The incircle of right-angled triangle ABC ($\angle C = 90^\circ$) touches BC at point K. Prove that the chord of the incircle cut by the line AK is twice as large as the distance from C to that line.

8.2. A rectangle $ABCD$ and its circumcircle are given. Let E be an arbitrary point lying on the minor arc BC. The tangent to the circle at B meets CE at point G. The segments AE and BD meet at point K. Prove that GK and AD are perpendicular.

8.3. Let ABC be a triangle with $\angle A = 60^\circ$, and AA', BB', CC' be its internal angle bisectors. Prove that $\angle B'A'C' \leqslant 60^\circ$.

8.4. Find all sets of six points in the plane, no three collinear, such that if we partition such set arbitrarily into two sets of three points, then two obtained triangles are congruent.

XIV Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2018, July 31

Problems

First day. 8 grade

8.1. The incircle of right-angled triangle ABC ($\angle C = 90^\circ$) touches BC at point K. Prove that the chord of the incircle cut by the line AK is twice as large as the distance from C to that line.

8.2. A rectangle $ABCD$ and its circumcircle are given. Let E be an arbitrary point lying on the minor arc BC. The tangent to the circle at B meets CE at point G. The segments AE and BD meet at point K. Prove that GK and AD are perpendicular.

8.3. Let ABC be a triangle with $\angle A = 60^\circ$, and AA', BB', CC' be its internal angle bisectors. Prove that $\angle B'A'C' \leqslant 60^\circ$.

8.4. Find all sets of six points in the plane, no three collinear, such that if we partition such set arbitrarily into two sets of three points, then two obtained triangles are congruent.

XIV Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2018, August 1

Problems

Second day. 8 grade

8.5. The side AB of a square $ABCD$ is a base of an isosceles triangle ABE ($AE = BE$) lying outside the square. Let M be the midpoint of AE, O be the common point of AC and BD, and K be the common point of ED and OM. Prove that $EK = KO$.

8.6. The corresponding angles of quadrilaterals $ABCD$ and $A_1B_1C_1D_1$ are equal. Also $AB = A_1B_1$, $AC = A_1C_1$, $BD = B_1D_1$. Are the quadrilaterals $ABCD$ and $A_1B_1C_1D_1$ necessarily congruent?

8.7. Let ω_1, ω_2 be two circles centered at O_1, O_2 and lying each outside the other. Points C_1, C_2 lie on these circles in the same semiplane with respect to O_1O_2. The ray O_1C_1 meets ω_2 at points A_2, B_2, and the ray O_2C_2 meets ω_1 at points A_1, B_1. Prove that $\angle A_1O_1B_1 = \angle A_2B_2C_2$ if and only if $C_1C_2 \parallel O_1O_2$.

8.8. Let I be the incenter of fixed triangle ABC, and D be an arbitrary point on side BC. The perpendicular bisector of AD meets BI and CI at points F and E, respectively. Find the locus of orthocenters of triangles EIF.

XIV Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2018, August 1

Problems

Second day. 8 grade

8.5. The side AB of a square $ABCD$ is a base of an isosceles triangle ABE ($AE = BE$) lying outside the square. Let M be the midpoint of AE, O be the common point of AC and BD, and K be the common point of ED and OM. Prove that $EK = KO$.

8.6. The corresponding angles of quadrilaterals $ABCD$ and $A_1B_1C_1D_1$ are equal. Also $AB = A_1B_1$, $AC = A_1C_1$, $BD = B_1D_1$. Are the quadrilaterals $ABCD$ and $A_1B_1C_1D_1$ necessarily congruent?

8.7. Let ω_1, ω_2 be two circles centered at O_1, O_2 and lying each outside the other. Points C_1, C_2 lie on these circles in the same semiplane with respect to O_1O_2. The ray O_1C_1 meets ω_2 at points A_2, B_2, and the ray O_2C_2 meets ω_1 at points A_1, B_1. Prove that $\angle A_1O_1B_1 = \angle A_2B_2C_2$ if and only if $C_1C_2 \parallel O_1O_2$.

8.8. Let I be the incenter of fixed triangle ABC, and D be an arbitrary point on side BC. The perpendicular bisector of AD meets BI and CI at points F and E, respectively. Find the locus of orthocenters of triangles EIF.

XIV Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2018, July 31

Problems

First day. 9 grade

9.1. Let M be the midpoint of AB in a right-angled triangle ABC with $\angle C = 90°$. A circle passing through C and M intersects BC and AC at P and Q, respectively. Let c_1, c_2 be circles with centers P, Q and radii BP, AQ, respectively. Prove that c_1, c_2 and the circumcircle of ABC are concurrent.

9.2. A triangle ABC is given. A circle γ centered at A meets segments AB and AC. The common chord of γ and the circumcircle of ABC meets AB and AC at points X and Y respectively. The segments CX and BY meet γ at points S and T respectively. The circumcircles of triangles ACT and BAS meet at points A and P. Prove that CX, BY and AP concur.

9.3. The vertices of triangle DEF lie on different sides of triangle ABC. The lengths of the segments of the tangents from the incenter of DEF to the excircles of ABC are equal. Prove that $4S_{DEF} \geqslant S_{ABC}$. (By S_{XYZ} we denote the area of triangle XYZ.)

9.4. Let BC be a fixed chord of a given circle ω. Let A be a variable point on the major arc BC of ω. Let H be the orthocenter of triangle ABC. Points D and E lying on lines AB and AC respectively are such that H is the midpoint of segment DE . Let O_A be the circumcenter of triangle ADE. Prove that, as A varies, all points O_A lie on a fixed circle.

XIV Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2018, July 31

Problems

First day. 9 grade

9.1. Let M be the midpoint of AB in a right-angled triangle ABC with $\angle C = 90°$. A circle passing through C and M intersects BC and AC at P and Q, respectively. Let c_1, c_2 be circles with centers P, Q and radii BP, AQ, respectively. Prove that c_1, c_2 and the circumcircle of ABC are concurrent.

9.2. A triangle ABC is given. A circle γ centered at A meets segments AB and AC. The common chord of γ and the circumcircle of ABC meets AB and AC at points X and Y respectively. The segments CX and BY meet γ at points S and T respectively. The circumcircles of triangles ACT and BAS meet at points A and P. Prove that CX, BY and AP concur.

9.3. The vertices of triangle DEF lie on different sides of triangle ABC. The lengths of the segments of the tangents from the incenter of DEF to the excircles of ABC are equal. Prove that $4S_{DEF} \geqslant S_{ABC}$. (By S_{XYZ} we denote the area of triangle XYZ.)

9.4. Let BC be a fixed chord of a given circle ω. Let A be a variable point on the major arc BC of ω. Let H be the orthocenter of triangle ABC. Points D and E lying on lines AB and AC respectively are such that H is the midpoint of segment DE . Let O_A be the circumcenter of triangle ADE. Prove that, as A varies, all points O_A lie on a fixed circle.

XIV Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2018, August 1

Problems

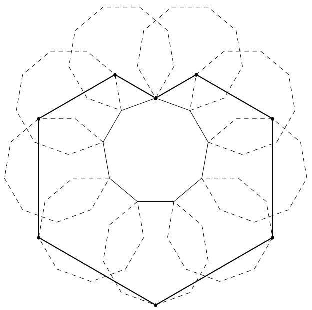
Second day. 9 grade

9.5. Let $ABCD$ be a cyclic quadrilateral, BL and CN be the internal angle bisectors in triangles ABD and ACD respectively. The circumcircles of triangles ABL and CDN meet at points P and Q. Prove that the line PQ passes through the midpoint of the arc AD not containing B.

9.6. Let $ABCD$ be a circumscribed quadrilateral. Prove that the common point of its diagonals, the incenter of triangle ABC and the center of excircle of triangle CDA touching the side AC are collinear.

9.7. Let B_1, C_1 be the midpoints of sides AC, AB of a triangle ABC, respectively. The tangents to the circumcircle at B and C meet the rays CC_1, BB_1 at points K and L respectively. Prove that $\angle BAK = \angle CAL$.

9.8. Consider a fixed regular n-gon of unit side. When a second regular n-gon of unit side rolls around the first one, one of its vertices successively pinpoints the vertices of a closed broken line κ as in the figure.



Let A be the area of a regular n-gon of unit side, and let B be the area of a regular n-gon of unit circumradius. Prove that the area enclosed by κ equals $6A - 2B$.

XIV Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2018, August 1

Problems

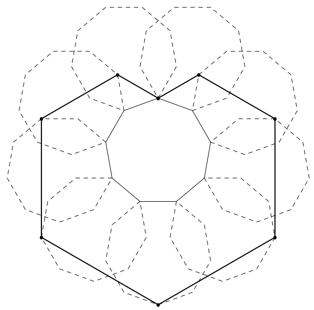
Second day. 9 grade

9.5. Let $ABCD$ be a cyclic quadrilateral, BL and CN be the internal angle bisectors in triangles ABD and ACD respectively. The circumcircles of triangles ABL and CDN meet at points P and Q. Prove that the line PQ passes through the midpoint of the arc AD not containing B.

9.6. Let $ABCD$ be a circumscribed quadrilateral. Prove that the common point of its diagonals, the incenter of triangle ABC and the center of excircle of triangle CDA touching the side AC are collinear.

9.7. Let B_1, C_1 be the midpoints of sides AC, AB of a triangle ABC, respectively. The tangents to the circumcircle at B and C meet the rays CC_1, BB_1 at points K and L respectively. Prove that $\angle BAK = \angle CAL$.

9.8. Consider a fixed regular n-gon of unit side. When a second regular n-gon of unit side rolls around the first one, one of its vertices successively pinpoints the vertices of a closed broken line κ as in the figure.



Let A be the area of a regular n-gon of unit side, and let B be the area of a regular n-gon of unit circumradius. Prove that the area enclosed by κ equals $6A - 2B$.

XIV Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2018, July 31

Problems

First day. 10 grade

10.1. The altitudes AH, CH of an acute-angled triangle ABC meet the internal bisector of angle B at points L_1, P_1, and the external bisector of this angle at points L_2, P_2. Prove that the orthocenters of triangles HL_1P_1, HL_2P_2 and the vertex B are collinear.

10.2. A fixed circle ω is inscribed into an angle with vertex C. An arbitrary circle passes through C, touches ω externally and meets the sides of the angle at points A and B. Prove that the perimeters of all triangles ABC are equal.

10.3. A cyclic n-gon is given. The midpoints of all its sides are concyclic. The sides of the n-gon cut n arcs of this circle lying outside the n-gon. Prove that these arcs can be coloured red and blue in such a way that the sum of the lengths of red arcs is equal to the sum of the lengths of blue arcs.

10.4. We say that a finite set S of red and green points in the plane is *separable* if there exists a triangle δ such that all points of one colour lie strictly inside δ and all points of the other colour lie strictly outside of δ. Let A be a finite set of red and green points in the plane, in general position. Is it always true that if every 1000 points in A form a separable set then A is also separable?

XIV Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2018, July 31

Problems

First day. 10 grade

10.1. The altitudes AH, CH of an acute-angled triangle ABC meet the internal bisector of angle B at points L_1, P_1, and the external bisector of this angle at points L_2, P_2. Prove that the orthocenters of triangles HL_1P_1, HL_2P_2 and the vertex B are collinear.

10.2. A fixed circle ω is inscribed into an angle with vertex C. An arbitrary circle passes through C, touches ω externally and meets the sides of the angle at points A and B. Prove that the perimeters of all triangles ABC are equal.

10.3. A cyclic n-gon is given. The midpoints of all its sides are concyclic. The sides of the n-gon cut n arcs of this circle lying outside the n-gon. Prove that these arcs can be coloured red and blue in such a way that the sum of the lengths of red arcs is equal to the sum of the lengths of blue arcs.

10.4. We say that a finite set S of red and green points in the plane is *separable* if there exists a triangle δ such that all points of one colour lie strictly inside δ and all points of the other colour lie strictly outside of δ. Let A be a finite set of red and green points in the plane, in general position. Is it always true that if every 1000 points in A form a separable set then A is also separable?

XIV Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2018, August 1

Problems

Second day. 10 grade

10.5. Let w be the incircle of a triangle ABC. The line passing through the incenter I and parallel to BC meets w at points A_B an A_C (A_B lies in the same semiplane with respect to AI as B). The lines BA_B and CA_C meet at point A_1. The points B_1 and C_1 are defined similarly. Prove that AA_1, BB_1 and CC_1 concur.

10.6. Let ω be the circumcircle of a triangle ABC, and KL be the diameter of ω passing through the midpoint M of AB (K and C lie on different sides of AB). A circle passing through L и M meets segment CK at points P and Q (Q lies on the segment KP). Let LQ meet the circumcircle of triangle KMQ again at point R. Prove that the quadrilateral $APBR$ is cyclic.

10.7. A convex quadrilateral $ABCD$ is circumscribed about a circle of radius r. What is the maximum possible value of $\frac{1}{AC^2} + \frac{1}{BD^2}$?

10.8. Two triangles ABC and $A'B'C'$ are given. The lines AB and $A'B'$ meet at point C_1, and the lines parallel to them and passing through C and C', respectively, meet at point C_2. The points A_1, A_2, B_1, B_2 are defined similarly. Prove that A_1A_2, B_1B_2 and C_1C_2 are either concurrent or parallel.

XIV Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2018, August 1

Problems

Second day. 10 grade

10.5. Let w be the incircle of a triangle ABC. The line passing through the incenter I and parallel to BC meets w at points A_B an A_C (A_B lies in the same semiplane with respect to AI as B). The lines BA_B and CA_C meet at point A_1. The points B_1 and C_1 are defined similarly. Prove that AA_1, BB_1 and CC_1 concur.

10.6. Let ω be the circumcircle of a triangle ABC, and KL be the diameter of ω passing through the midpoint M of AB (K and C lie on different sides of AB). A circle passing through L и M meets segment CK at points P and Q (Q lies on the segment KP). Let LQ meet the circumcircle of triangle KMQ again at point R. Prove that the quadrilateral $APBR$ is cyclic.

10.7. A convex quadrilateral $ABCD$ is circumscribed about a circle of radius r. What is the maximum possible value of $\frac{1}{AC^2} + \frac{1}{BD^2}$?

10.8. Two triangles ABC and $A'B'C'$ are given. The lines AB and $A'B'$ meet at point C_1, and the lines parallel to them and passing through C and C', respectively, meet at point C_2. The points A_1, A_2, B_1, B_2 are defined similarly. Prove that A_1A_2, B_1B_2 and C_1C_2 are either concurrent or parallel.

XIV Geometrical Olympiad in honour of I.F.Sharygin
Final round. Solutions. First day. 8 grade

1. (M.Volchkevich) The incircle of right-angled triangle ABC ($\angle C = 90°$) touches BC at point K. Prove that the chord of the incicle cutting by the line AK is twice as large than the distance from C to this line.

   Solution. Let I be the incenter of ABC, and P, Q be the projections of I, C respectively to AK (fig.8.1). Since $\angle IKC = 90°$, $\angle ICK = 45°$, we obtain that IKC is an isosceles triangle, i.e. $IK = KC$. Also $\angle IKP = \angle KCQ$ because the corresponding sides of these angles are perpendicular. Therefore triangles IKP and KCQ are congruent, i.e. $KP = CQ$. Since P is the midpoint of the chord which is cut by AK, we obtain the required assertion.

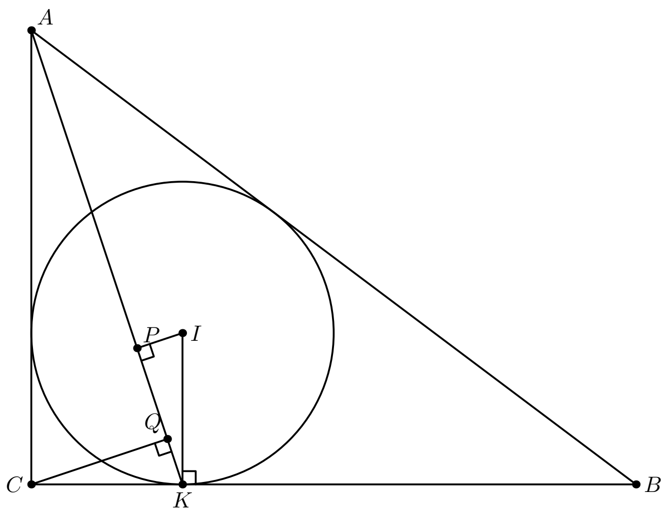


Fig. 8.1

2. (N.Moskvitin) A rectangle $ABCD$ and its circumcircle are given. Let E be an arbitrary point lying on the minor arc BC. The tangent to the circle at B meets CE at point G. The segments AE and BD meet at point K. Prove that GK and AD are perpendicular.

   Solution. Since $\angle DBG = \angle AEC = 90°$, we obtain that $BGEK$ is cyclic (fig.8.2). Hence $\angle BGK = \angle BEA = \angle DBC$ and $GK \perp BC$, which is equivalent to the required assertion.

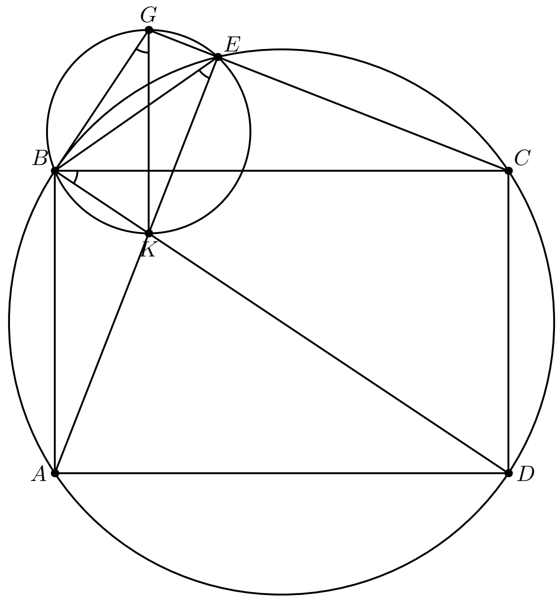


Fig. 8.2

3. (G.Feldman) Let ABC be a triangle with $\angle A = 60^\circ$, and AA', BB', CC' be its bisectors. Prove that $\angle B'A'C' \le 60^\circ$.

   Solution. If ABC is regular then the assertion is evident, thus we can suppose that $AC > AB$. Let I be the incenter. Then $\angle BIC = 120^\circ$, therefore $AB'IC'$ is cyclic, and since AI is the bisector, we obtain that $B'I = C'I$. Let $\angle ACB = 2\gamma$, then $\gamma < 30^\circ$ and $IA' = \frac{r}{\sin \angle AA'B} = \frac{r}{\sin(2\gamma+30^\circ)} > \frac{r}{\sin(\gamma+60^\circ)} = \frac{r}{\sin \angle CC'B} = IC'$. Hence A' lies outside the circle with center I and radius IC' (fig.8.3), i.e. $\angle B'A'C' < 60^\circ$.

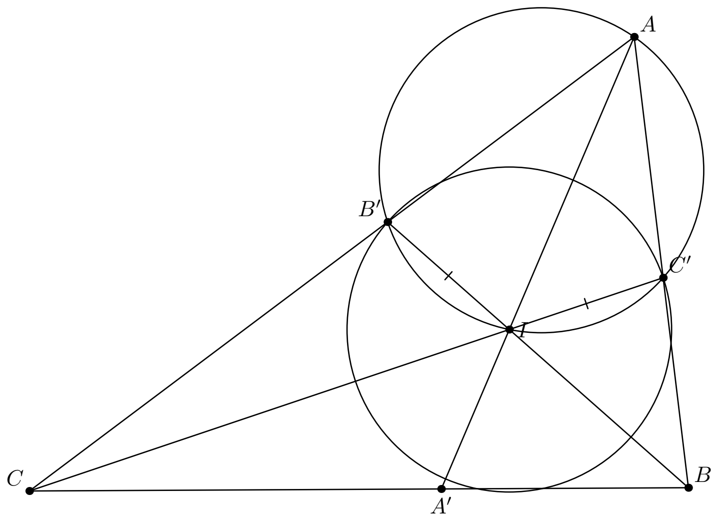


Fig. 8.3

4. (M.Saghafian) Find all sets of six points in the plane, no three collinear, such that if we divide them arbitrarily into two sets of three points, then two obtained triangles are equal.

Answer. Two regular triangles with common circumcircle.

First solution. Let D be the multiset of 15 distances between points $A_1, \dots, A_6$ (if there are n congruent segments then the multiset contains n corresponding numbers), and let D_i be the multiset of 5 distances from A_i to the remaining points. Consider the multiset of 30 sidelengths of triangles having A_i as one of vertices. This multiset contains four times each number from D_i and one time each number from $D \setminus D_i$. By the assumption, the sidelengths of triangles not having A_i as a vertex form the same multiset of 30 numbers, i.e. this multiset contains three times each number from $D \setminus D_i$. Therefore $D = 3D_i$ and all D_i coincide.

Introduce an arbitrary Бartesian coordinate system, and let M be the point such that each coordinate of M is the average of the corresponding coordinates of A_i. Let X be an arbitrary point, and $x, m, a_1, \dots, a_6$ be first coordinates of $X, M, A_1, \dots, A_6$ respectively. Then we have $(x-a_1)^2+\dots+(x-a_6)^2 = ((x-m)+(m-a_1))^2+\dots+((x-m)+(m-a_6))^2 = 6(x-m)^2+(m-a_1)^2+\dots+(m-a_6)^2$. Using the similar equality for the second coordinates and Pythagorean theorem we obtain

$$XA_1^2+\dots+XA_6^2 = 6XM^2+MA_1^2+\dots+MA_6^2$$

(this equality is a partial case of the Leibnitz theorem). Substituting $A_1, \dots, A_6$ for X we obtain that $MA_1 = \dots = MA_6$, i.e. all given points are concyclic. Suppose that they form a cyclic hexagon $A_1 \dots A_6$. Let A_1A_2 be its minor side. Since all multisets D_i are equal, we obtain that $A_1A_2 = A_3A_4 = A_5A_6$. Similarly $A_2A_3 = A_4A_5 = A_6A_1$. It is easy to see that these conditions are sufficient.

Second solution. Let A_1, A_2,…, A_6 be the given points. Here triangle, segment and length mean a triangle with vertices A_i, a segment with endpoints A_i and a length of such segment respectively. Let us prove several lemmas.

1) For each length x one of the following assertions is true:

(A) there exists a regular triangle with sidelength x;

(B) there exist three segments with sidelength x having six different endpoints.

In fact, let $A_1A_2 = A_3A_4 = x$. Since $\triangle A_1A_2A_4 = \triangle A_3A_5A_6$, there is a side with length x in the $\triangle A_3A_5A_6$. If this is A_5A_6, we obtain (B), else we have two adjacent segments with length x. Let (after renumeration) $A_1A_2 = A_2A_3 = x$ and $A_4A_5 = A_5A_6 = x$. Since $\triangle A_2A_3A_5 = \triangle A_1A_4A_6$, there is a side with length x in the $\triangle A_1A_4A_6$. If this is A_4A_6, we have (A), else we have a broken line with five links of length x. Its extreme and medial links satisfy (B).

2) Let x be the maximal length. Then (A) is not correct, and thus (B) is true.

In fact if $\triangle A_1A_2A_3$ is regular and x is its sidelength, then the vertices of a congruent $\triangle A_4A_5A_6$ lie inside the corresponding Reuleaux triangle which is impossible.

3) Two segments with maximal length x intersect.

In fact let $A_1A_2 = x$, draw two lines to A_1 and A_2 perpendicular to A_1A_2. The remaining points lie inside the strip between this lines. Constructing the similar strip for $A_3A_4 = x$, we obtain that A_1A_2 and A_3A_4 are the altitudes of a rhombus, joining inner points of its sides. It is clear that such altitudes intersect.

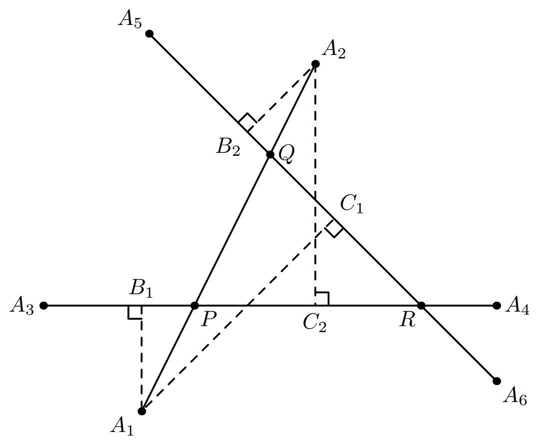


Fig. 8.4

4) Let segments A_1A_2, A_3A_4 and A_5A_6 intersect at three points (fig.8.4). Then the perpendiculars A_1B_1 and A_2B_2 are equal as altitudes of congruent triangles $A_1A_3A_4$ and $A_2A_5A_6$. Similarly the perpendiculars A_1C_1 and A_2C_2

are equal. Therefore $\dfrac{A_1P}{PA_2} = \dfrac{A_1B_1}{A_2C_2} = \dfrac{A_2B_2}{A_1C_1} = \dfrac{A_2Q}{QA_1}$ and $A_1P = QA_2$. This yields that $\triangle A_1PB_1 = \triangle A_2QB_2$, i.e. $\angle P = \angle Q$. Similarly we obtain that $\triangle PQR$ is regular, and thus $A_1P = QA_2 = A_3P = RA_4 = A_6R = QA_5$. It is easy to see that the obtained configuration satisfies the condition.

If three maximal segments concur then we similarly obtain that the angles between them are equal $60°$ and their common point bisects them.

XIV Geometrical Olympiad in honour of I.F.Sharygin
Final round. Solutions. Second day. 8 grade

5. (S.Sevastianov) The side AB of a square $ABCD$ is a base of an isosceles triangle ABE ($AE = BE$) lying outside the square. Let M be the midpoint of AE, O be the common point of AC and BD, and K be the common point of ED and OM. Prove that $EK = KO$.

   Solution. Since OM is a medial line of triangle ACE, $OM \parallel EC$, therefore $\angle KOE = \angle OEC$ (fig.8.5). But it is clear that EO bisects angle CED. Thus $\angle EOK = \angle OEK$ and OKE is an isosceles triangle.

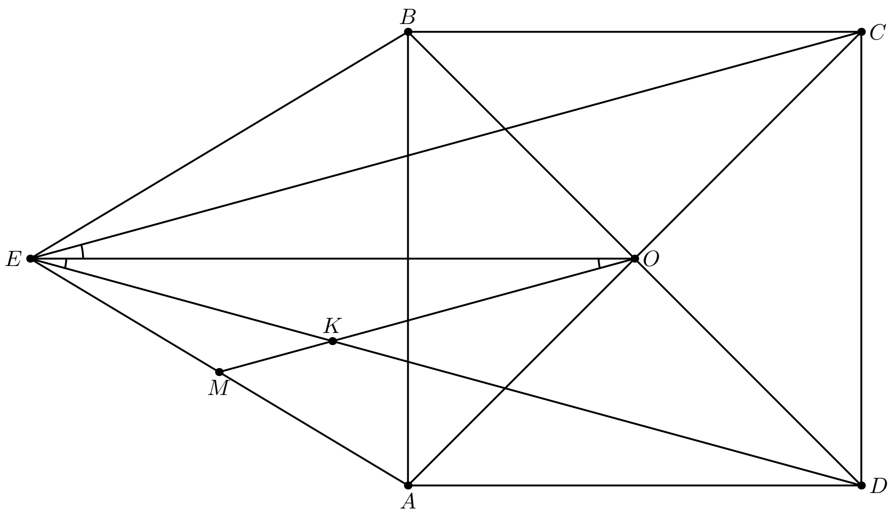


Fig. 8.5

6. (D.Shnol) The corresponding angles of quadrilaterals $ABCD$ and $A_1B_1C_1D_1$ are equal. Also $AB = A_1B_1$, $AC = A_1C_1$, $BD = B_1D_1$. Are the quadrilaterals $ABCD$ and $A_1B_1C_1D_1$ congruent?

   Answer. No.

   Solution. Let $A = A_1$, $B = B_1$, AXB be an isosceles triangle, AA', BB' be its altitudes. Let C, C_1 lie on BX and D, D_1 lie on AX in such a way that $CA' = C_1A' = DB' = D_1B'$. Then $AC = AC_1 = BD = BD_1$ and two isosceles trapezoids $ABCD$, $A_1B_1C_1D_1$ satisfy all conditions but are not congruent (fig.8.6).

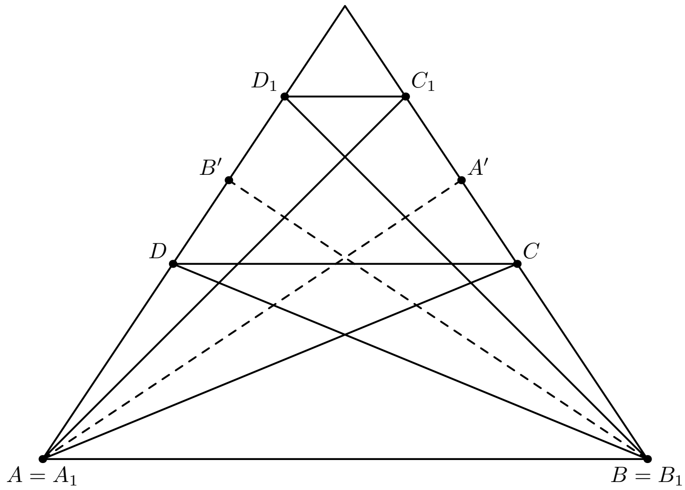


Fig. 8.6

7. (F.Nilov) Let ω_1, ω_2 be two circles centered at O_1, O_2 and lying each outside the other. Points C_1, C_2 lie on these circles in the same semiplane with respect to O_1O_2. The ray O_1C_1 meets ω_2 at points A_2, B_2, and the ray O_2C_2 meets ω_1 at points A_1, B_1. Prove that $\angle A_1O_1B_1 = \angle A_2O_2B_2$ if and only if $C_1C_2 \parallel O_1O_2$.

   First solution. Let R_1, R_2 be the radii of the circles, M_1, M_2 be the midpoints of A_1B_1, A_2B_2 respectively, and H_1, H_2 be the projections of C_1, C_2 to O_1O_2. The equality $\angle A_1O_1B_1 = \angle A_2O_2B_2$ is equivalent to $O_1M_1/R_1 = O_2M_2/R_2$. Since the triangle $O_1O_2M_2$ is similar to $O_1C_1H_1$, we have $O_2M_2/R_2 = (C_1H_1{\cdot}O_1O_2)/(R_1R_2)$ (fig.8.7). Similarly $O_1M_1/R_1 = (C_2H_2{\cdot}O_1O_2)/(R_1R_2)$. Therefore the equality $O_1M_1/R_1 = O_2M_2/R_2$ is equivalent to $C_1H_1 = C_2H_2$, which is equivalent to $C_1C_2 \parallel O_1O_2$.

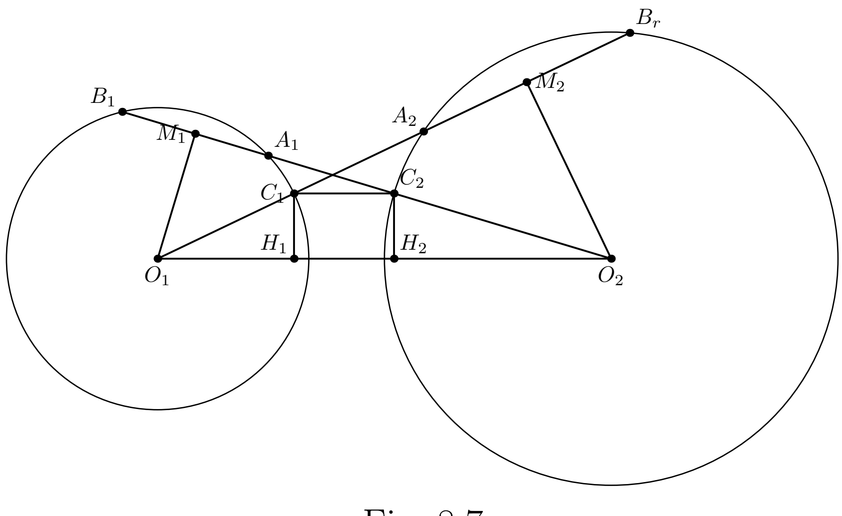


Fig. 8.7

Second solution. The equality $\angle A_1O_1B_1 = \angle A_2O_2B_2$ is equivalent to $\angle O_1A_1O_2 = \angle O_1A_2O_2$, i.e. $O_1A_1A_2O_2$ is cyclic. Let us prove that this is equivalent to $C_1C_2 \parallel O_1O_2$.

If $O_1A_1A_2O_2$ is cyclic then $\angle A_1O_1C_1 = \angle A_2O_2C_2$, $\angle O_1C_1A_1 = \angle O_2C_2A_2$ and $C_1A_1A_2C_2$ is cyclic. Therefore O_1O_2 and C_1C_2 are antiparallel to A_1A_2 with respect to O_1A_2 and O_2A_1. Hence these lines are parallel.

If $C_1C_2 \parallel O_1O_2$ then consider a common point X of ray O_1C_1 and circle $A_1O_1O_2$. Since $A_1C_1C_2X$ is cyclic we have $\angle A_1O_1X = \angle XO_2A_1$ and $\angle O_1C_1A_1 = \angle O_2C_2X$. Thus $\angle O_2XC_2 = \angle O_1A_1C_1 = \angle O_1C_1A_1 = \angle O_2C_2X$, i.e. $O_X = O_2C_2$ and X coincides with A_2.

8. (I.Kukharchuk) Let I be the incenter of triangle, and D be an arbitrary point of side BC. The perpendicular bisector to AD meets BI and CI at points F and E respectively. Find the locus of orthocenters of triangles EIF.

   Answer. The segment of line BC between its common points with two lines passing through I and parallel to AB, AC, probably without one or two points.

   Solution. Let G, H be the orthocenters of triangles DEF, IEF respectively. Since the triangles DEF and AEF are symmetric with respect to EF, we obtain that G is the reflection of the orthocenter of AEF and thus G lies on the circumcircle of this triangle.

   The common point E of the perpendicular bisector to AD and the bisectrix of angle C lies on the circumcircle of ACD. Hence $\angle AEF = \angle AED/2 = 90° - \angle C/2 = \angle A/2 + \angle B/2 = \angle AIF$ (because AIF is an external angle of triangle AIB), i.e. I lies on the circle AEF. Then, since $AEDC$ and $AEIG$ are cyclic, we obtain that $IG \parallel CD$.

   Since $\angle EHF = 180° - \angle EIF = \angle EAF = \angle EDF$, the points E, F, D, H are concyclic, therefore $IH = DG$. Also it is clear that $DG \parallel IH$. Thus $IGDH$ is a parallelogram and H lies on BC (fig.8.8). If for example D coincides with C then DG coincides with AC and $IH \parallel AC$. If BC is the smallest side of the triangle then all points of the obtained segment lie on the required locus. If for example $BC \geq AB$ then the reflection of A about the bisector of angle B lies on the segment BC. When D coincides with this point the perpendicular bisector to AD coincides with BI and the point F is not defined. Hence the corresponding point H has to be eliminated from the locus.

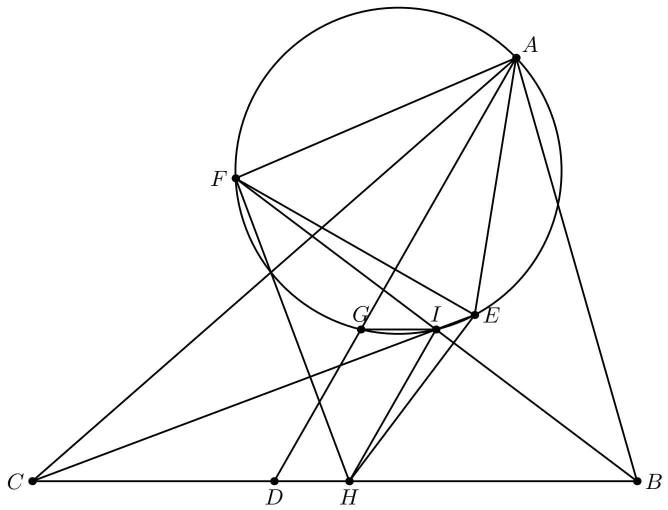


Fig. 8.8

Note. We can prove that H lies on BC in another way. The projections of A to the bisectors of angles B and C lie on the medial line of the triangle. The midpoint of AD which is the projection of A to EF also lies on this medial line. Hence the medial line is the Simson line of A with respect to triangle IEF, and the homothetic line BC passes through the orthocenter of this triangle.

XIV Geometrical Olympiad in honour of I.F.Sharygin
Final round. Solutions. First day. 9 grade

1. (M.Etesamifard) Let M be the midpoint of AB in a right-angled triangle ABC with $\angle C = 90^\circ$. A circle passing through C and M intersects the segments BC and AC at P and Q, respectively. Let c_1, c_2 be circles with centers P, Q and radii BP, CQ, respectively. Prove that c_1, c_2 and the circumcircle of ABC are concurrent.

   Solution. Let N be the second common point of circle MPQ with AB. Then $\angle QNA = \angle QPM = \angle ACM = \angle CAM$ (fig.9.1). Therefore $QA = QN$ and N lies on c_2. Similarly N lies on c_1. Now if D is the second common point of c_1 and c_2 then $\angle ADB = \angle ADN + \angle NDB = (\angle AQN + \angle NPB)/2 = 90^\circ$, i.e. D lies on the circumcircle of ABC.

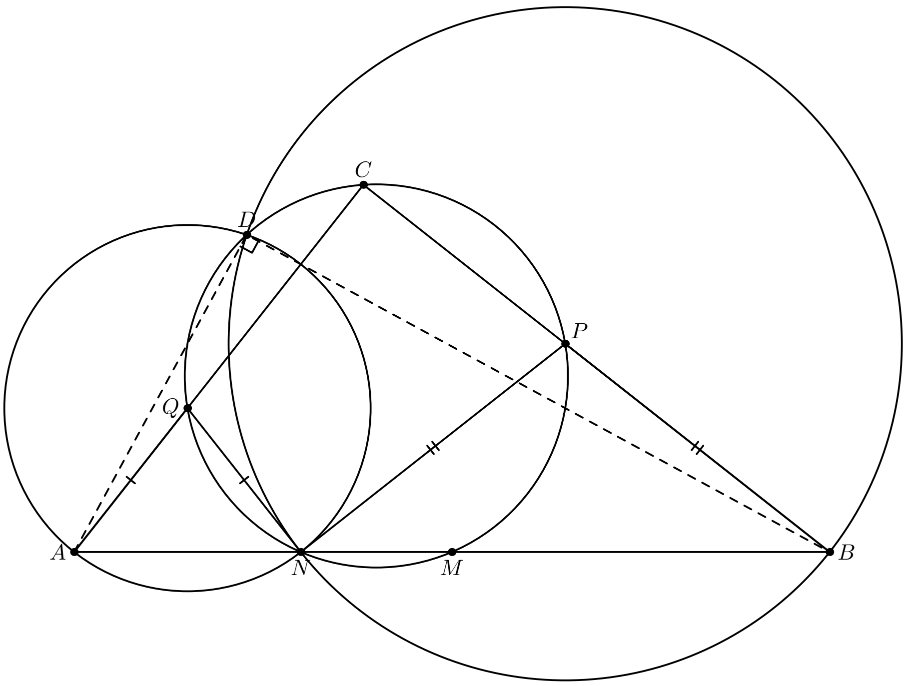


Fig. 9.1

2. (G.Naumenko) A triangle ABC is given. A circle γ centered at A meets segments AB and AC. The common chord of γ and the circumcircle of ABC meets AB and AC at points X and Y respectively. The segments CX and BY meet γ at points S and T respectively. The circumcircles of triangles ACT and BAS meet at points A and P. Prove that CX, BY and AP concur.

   Solution. Let U be the second common point of BY and γ. Since TU, AC and the common chord of circles ABC and γ meet at Y, we have $AY \cdot CY =$

$TY \cdot UY$, i.e. A, U, C, T are concyclic (fig.9.2). Similarly A, B, S and the second common point of CX with γ are concyclic. Therefore CX, BY and AP concur as the radical axes of circles γ, ACT and BAS.

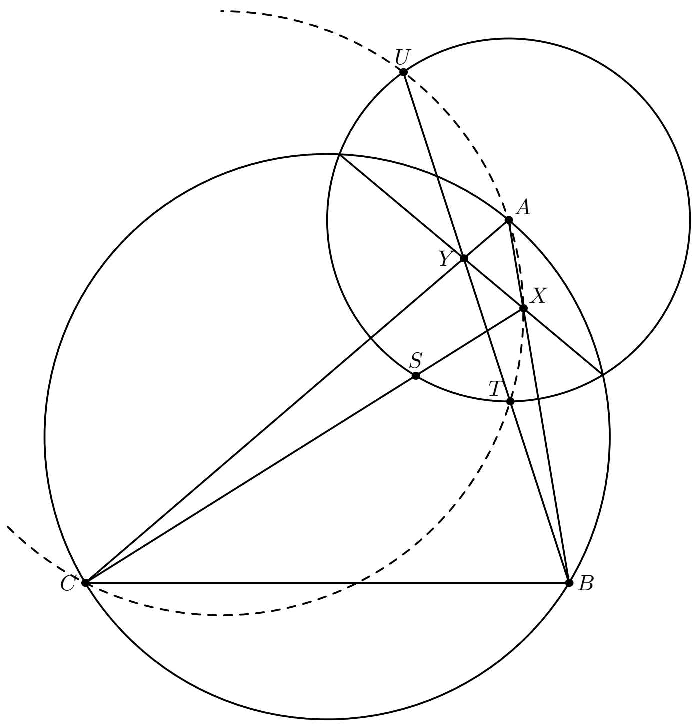


Fig. 9.2

3. (N.Beluhov) The vertices of triangle DEF lie on the different sides of triangle ABC. The tangents from the incenter of DEF to the excircles of ABC are equal. Prove that $4S_{DEF} \geq S_{ABC}$.

   Solution. Let A_0, B_0, C_0 be the midpoints of BC, CA, AB, and U, V be the tangency points of AB with the excircles touching the sides AC and BC respectively. Since $AV = BU = p$ (semiperimeter of ABC), the tangents from C_0 to these two excircles are equal. Furthemore the centers of these circles lie on the external bisector of angle C perpendicular to the bisector of angle $A_0C_0B_0$, hence this bisector is the radical axis of two excircles. Similarly the bisectors of angles $C_0A_0B_0$ and $B_0A_0C_0$ are the radical axes of two remaining pairs of excircles, thus the incenters of triangles DEF and $A_0B_0C_0$ coincide. Suppose that D lies on the segment CA_0. Now if the inradius r' of DEF is greater than the inradius r of $A_0B_0C_0$ then F lies on the segment BC_0, and thus E lies on AB_0. Furthermore if $r' < r$ then E lies on AB_0, and thus F lies on BC_0. Hence the distance from F to ED is not less than the distance from C_0 to this line, i.e. $S_{DEF} \geq S_{C_0DE}$. Similarly $S_{C_0DE} \geq S_{B_0C_0D} = S_{A_0B_0C_0} = S_{ABC}/4$.

4. (A.Mudgal, India) Let BC be a fixed chord of a given circle ω. Let A be a variable point on the major arc BC of ω. Let H be the orthocenter of triangle ABC. Points D and E lying on lines AB and AC respectively are such that H is the midpoint of segment XY . Let O_A be the circumcenter of triangle AXY. Prove that all points O_A lie on a fixed circle.

   Solution Denote by α the constant angle $90^\circ - \angle BAC$. Let P, Q be the midpoints of AD, AE, and R, S be points on BC such that $PR \perp AB$, $SQ \perp AC$ (fig. 9.4). Let us prove that R, S do not depend from A.

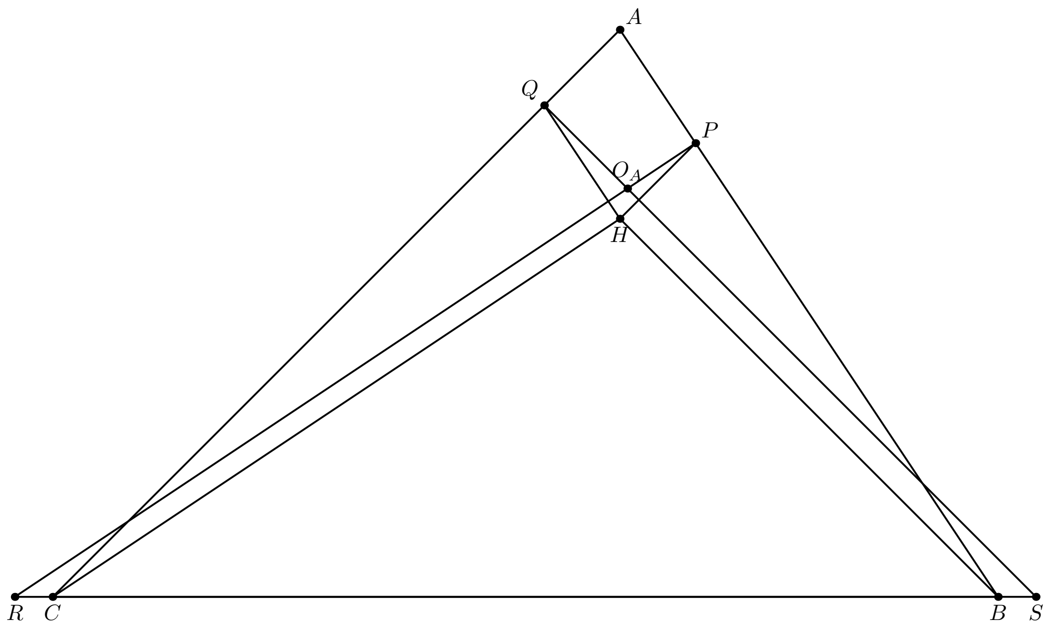


Fig. 9.4

   Note that $HQ \parallel AB$, i.e. $\angle CHQ = 90^\circ$ and $\angle CQH = \angle CAB$. Furthermore H moves aloong the circle symmetric to ω with respect to BC. Since Q is the image of H in the spiral similarity with center C, rotation angle α and coefficient $1/\cos\alpha$, we obtain that Q also moves along some circle which we denote by ω_C.

   Let O be the center of ω. Since $\angle OCB = \alpha$, the center of ω_C lies on BC. Since $\angle CQS = 90^\circ$, we obtain that S is opposite to C on ω_C. Therefore S does not depend from A. The proof for R is similar.

   Since O_A is the common point of PR and QS, and $\angle RO_AS = 90^\circ + \alpha$, we obtain that O_A moves along the arc of the circle passing through R and S.

XIV Geometrical Olympiad in honour of I.F.Sharygin
Final round. Solutions. Second day. 9 grade

5. (D.Prokopenko) Let $ABCD$ be a cyclic quadrilateral, BL and CN be the bisectors of triangles ABD and ACD respectively. The circumcircles of triangles ABL and CDN meet at point P and Q. Prove that the line PQ passes through the midpoint of the arc AD not containing B.

   Solution. Let M be the midpoint of arc AD. Then BL and CN pass through M. Also since $\smile AM = \smile DM$, we have $\angle ALB = (\smile AB + \smile DM)/2 = \smile BAM/2 = \angle BCM$, and thus $BCNL$ is cyclic (fig.9.5). Therefore $ML \cdot MB = MN \cdot MC$, and M lies on the radical axis PQ of circles ABL and CDN.

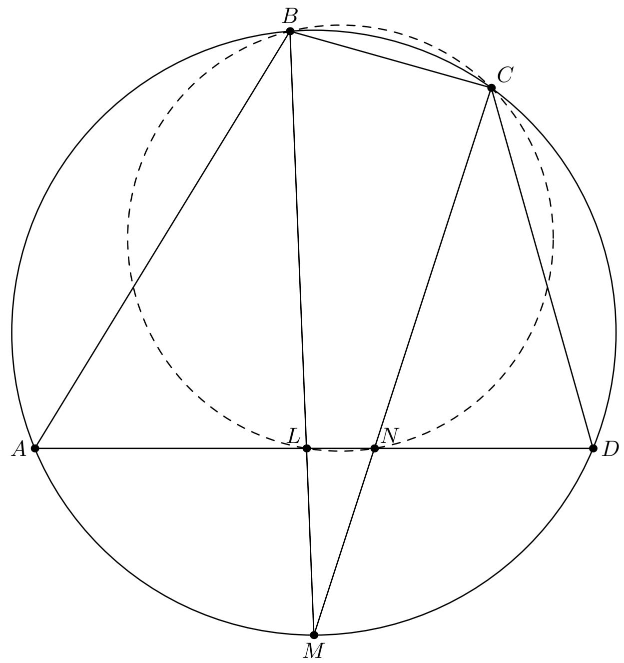


Fig. 9.5

6. (F.Ivlev) Let $ABCD$ be a circumscribed quadrilateral. Prove that the common point of its diagonals, the incenter of triangle ABC and the center of excircle of triangle CDA touching the side AC are collinear.

   First solution. Applying the three homothety centers to the incircle of $ABCD$, the incircle ω of triangle ABC and the excircle Ω of triangle ACD, we obtain that the common external tangents to ω and Ω meet on BD, and since AC is one of these two tangents, they meet at the common point of diagonals of $ABCD$. Thus this point lies on the centerline of ω and Ω.

Second solution. Let L be the common point of the diagonals of $ABCD$, I be its incenter, I_B be the incenter of triangle ABC, and I_D be the excenter of triangle ADC. Clearly I_B lies on the segment BI, and the ratio $BI_B : BI$ is equal to the ratio $r_B : r$ of the inradii of ABC and $ABCD$ respectively. Since $S_{ABCD} = (AB+BC+CD+DA)r/2$, $S_{ABC} = (AB+BC+CA)r_B/2$ and $S_{ABC} : S_{ABCD} = BL : BD$, we obtain that

$$\frac{I_BI}{I_BB} = \frac{DL(AB+BC+CA) - BL(AD+CD-AC)}{BL(AB+BC+CD+DA)}.$$

Similarly for the point I_D lying on the ray DI we have

$$\frac{I_DI}{I_DD} = \frac{DL(AB+BC+AB) - BL(AD+CD-AC)}{DL(AB+BC+CD+DA)}.$$

Applying Menelaos theorem to the triangle IBD, we obtain the required assertion (fig.9.6).

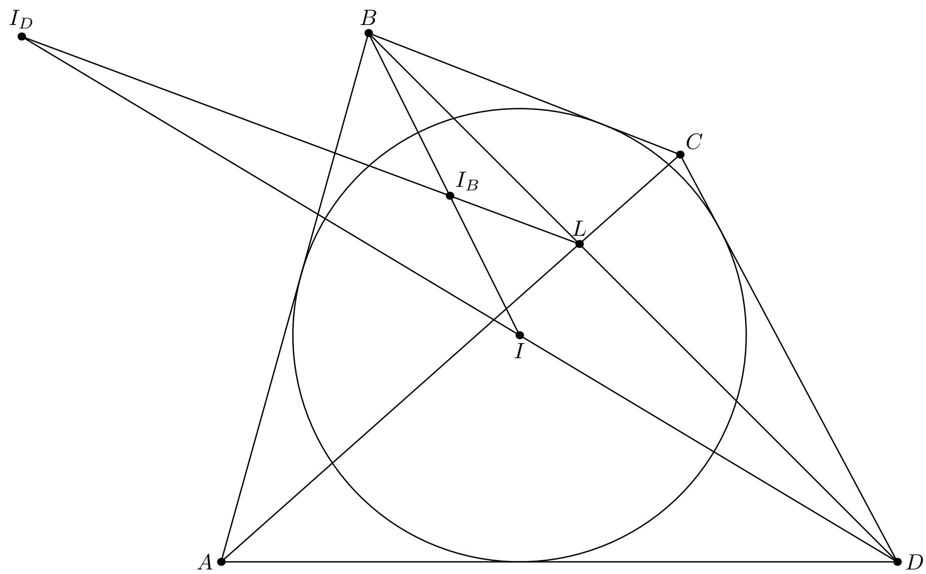


Fig. 9.6

7. (A.Kulikova) Let B_1, C_1 be the midpoints of sides AC, AB of a triangle ABC. The rays CC_1, BB_1 meet the tangents to the circumcircle at B and C at K and L respectively. Prove that $\angle BAK = \angle CAL$.

Solution. Use **the isogonals theorem**.

Let ℓ be a line passing through a point O. Let points A, A', B, B' be given and $X = AB \cap A'B'$, $X' = AB' \cap A'B$. Let OA and OA' be symmetric with respect to ℓ, OB and OB' be also symmetric with respect to ℓ. Then OX and OX' are symmetric with respect to ℓ,

Return to the problem. Let M be the centroid of ABC,and P be the common point of two tangents. Since AP is a symedian, the lines AP and AM are the isogonals with respect to angle BAC (fig.9.7). By the isogonals theorem, AK and AL are also isogonals.

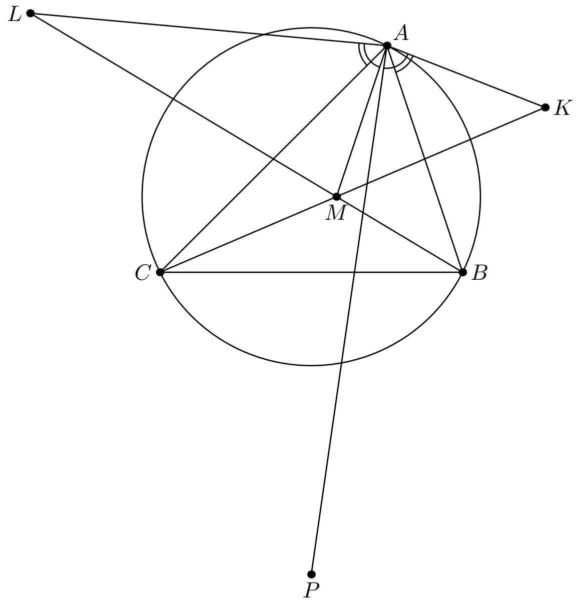
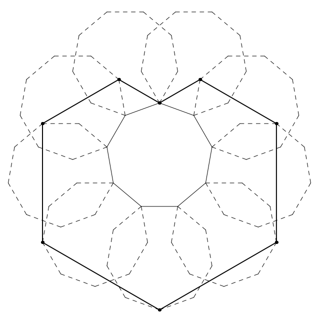


Fig. 9.7

8. (N.Beluhov) Consider a fixed regular n-gon of unit side. As a second regular n-gon of unit side rolls around the first one, one of its vertices successively pinpoints the vertices of a closed broken line κ as in the figure.



Let A be the area of a regular n-gon of unit side and let B be the area of a regular n-gon of unit circumradius. Prove that the area enclosed by κ equals $6A - 2B$.

Solution. Dissect the area enclosed by κ into triangles as in Fig.9.8.1.

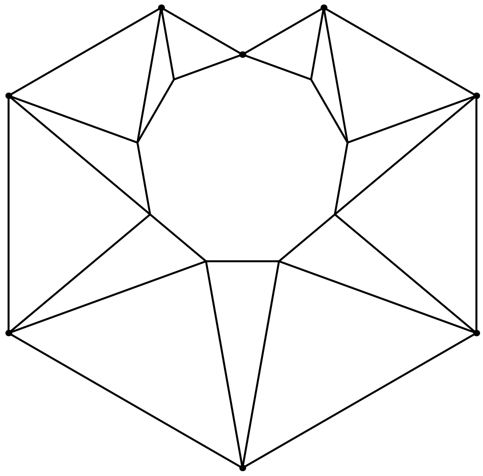


Fig.9.8.1

The triangles whose bases are sides $2, \ldots, n-1$ of a fixed regular n-gon come together to form a regular n-gon of unit side as in Figure 9.8.2.

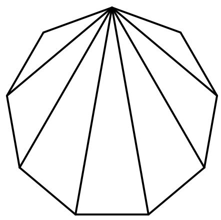


Fig.9.8.2

Dissect two regular n-gons of unit circumradius as in Figure 9.8.2, rearrange the resulting pieces into $n-1$ similar isosceles triangles with base angle $\frac{180°}{n}$ as in Figure 9.8.3, and adjoin the triangles thus obtained to the remaining triangles of Figure 9.8.1 as in Figure 9.8.4.

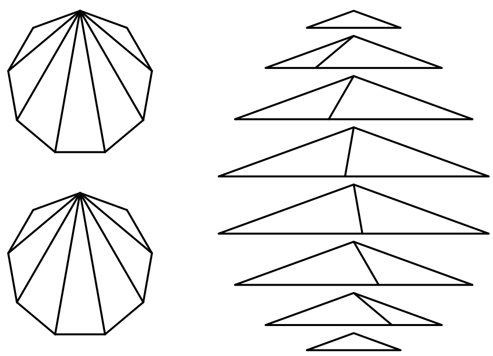


Fig.9.8.3

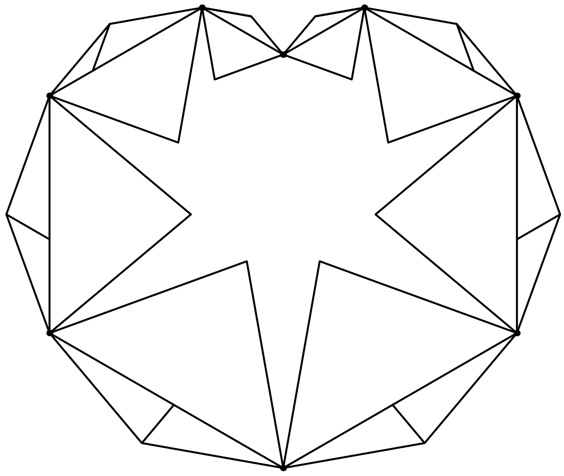


Fig.9.8.4

Dissect each one of the $n-1$ quadrilaterals in Figure 4 into two similar isosceles triangles with base angle $\frac{180^\circ}{n}$ as in Figure 9.8.5.

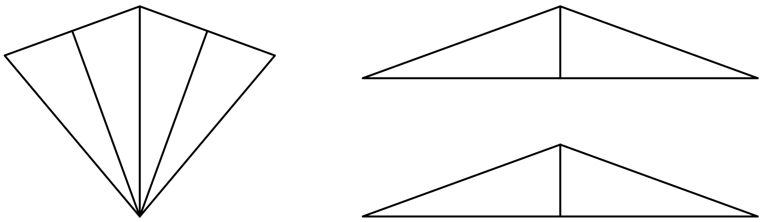


Fig.9.8.5

Lastly, dissect all $2n-2$ triangles thus obtained into four regular n-gons of unit side by the reverse of the process in Figure 9.8.3.

Eventually we adjoined two regular n-gons of unit circumradius to the area enclosed by κ and then dissected the resulting shape into six regular n-gons with unit side. This completes the solution.

XIV Geometrical Olympiad in honour of I.F.Sharygin
Final round. Solutions. First day. 10 grade

1. (D.Shvetsov) The altitudes AH, CH of an acute-angled triangle ABC meet the internal bisector of angle B an points L_1, P_1, and the external bisector of this angle at points L_2, P_2. Prove that the orthocenters of triangles HL_1P_1, HL_2P_2 and the vertex B are collinear.

   First solution. Note that HL_1P_1 and HL_2P_2 are isosceles triangles with angles at H equal to B and $\pi - B$ respectively. Let H_1, H_2 be the orthocenters of these triangles, and M_1, M_2 be the midpoints of L_1P_1, L_2P_2 respectively. Then the triangles HL_2P_2, $H_1L_1P_1$ are similar and H_2, H are their orthocenters, therefore $HH_1 : M_2B = HH_1 : HM_1 = H_2H : H_2M_2$, which is equal to the required assertion (fig.10.1).

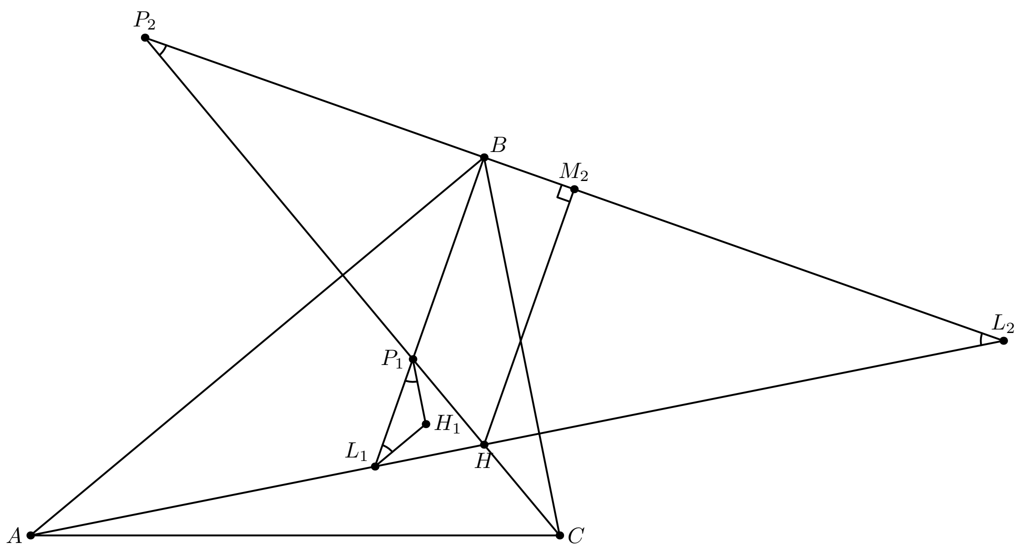


Fig. 10.1

   Second solution. Use next fact.

   The orthocenters of four triangles formed by four lines in general position are collinear (**the Aubert line**).

   In the given case the altitudes from A, C, the internal and the external bisectors of angle B form four four triangles, two of them are right-angled with right angle B. Thus B is also the orthocenter of these triangle, therefore B and the orthocenters of two remaining triangles are collinear.

2. (D.Krekov) A circle ω is inscribed into an angle with vertex C. An arbitrary circle passes through C, touches ω externally and meets the sides of the angle at points A and B. Prove that the perimeters of all triangles ABC are equal.

First solution. Let the length of the tangent from C to the given circle is 1. The inversion about the unit circle centered at C preserves the sides of the angle and the given circle, and maps A, B to points A', B' such that the triangle $A'B'C$ is circumscribed around the given circle. Now we have $AC = 1/A'C$, $BC = 1/B'C$, $AB = A'B'/(A'C \cdot B'C)$. Hence the perimeter of ABC is equal to

$$\frac{A'B' + A'C + B'C}{A'C \cdot B'C} = \frac{2p_{A'B'C} \sin \angle C}{2S_{A'B'C}} = \frac{\sin \angle C}{r_{A'B'C}}.$$

But the inradius of $A'B'C$ does not depend from A, B.

Second solution. Since ω is the semiexcircle of triangle ABC, the excenter ot these triangle coincide with the midpoint of the segment between the touching points of ω with the sidelines of the given angle, т.e this excenter is the same for all triangles. Therefore the touching points of the excircle with the sidelines do not depend on the triangle, hence its perimeter is also constant.

3. (F.Nilov) A cyclic n-gon is given. The midpoints of all its sides are concyclic. The sides of n-gon cut n arcs of this circle ling outside the n-gon. Prove that these arcs can be colored red and blue in such a way that the sum of red arcs is equal to the sum of blue arcs.

   Solution. Let M_1, M_2 be the midpoints of sides A_1A_2, A_2A_3 of polygon $A_1 \dots A_n$, O be the circumcenter of this polygon, and H_1, H_2 be the second common points of the sides with the circle passing through the midpoints. Then the sum of directed arcs $\smile M_1H_1 + \smile M_2H_2 = \smile M_1H_2 + \smile M_2H_1 = 2(\angle A_2M_2M_1 + \angle A_2M_1M_2) = 2(\angle OM_2M_1 + \angle OM_1M_2) = 2(\angle OA_2M_1 + \angle OA_2M_2)$ (the last equality holds because $OM_1A_2M_2$ is cyclic). Summing up such equalities we obtain that the directed sum of arcs M_iH_i is zero, therefore we can color the arcs in correspondence with their directions.

   Note. We can modify this argumentation as follows. The projections M_i of the circumcenter O to the sides are concyclic. Therefore the second common points H_i of the sides and these circles are the projections of some point H, and the rays A_iO and A_iH are symmetric with respect to the bisector of angle $A_{i-1}A_iA_{i+1}$. Now it is easy to see that the directed angle between M_1M_2 and H_1H_2 is equal to the directed angle HA_2O, and the sum of such angles is zero.

4. (N.Beluhov) We say that a finite set S of red and green points in the plane is *orderly* if there exists a triangle δ such that all points of one colour lie

strictly inside δ and all points of the other colour lie strictly outside of δ. Let A be a finite set of red and green points in the plane, in general position. Is it always true that if any 1000 points in A form an orderly set then A is also orderly?

Solution. At first let us consider a slightly different problem, in which "orderly" is replaced with "red-orderly": there exists a triangle δ such that all red points lie strictly inside δ and all green points lie strictly outside of δ.

Let A be a finite set of red and green points in the plane, in general position, and let P be the convex hull of some red points in A. Let also Q be a subset of the green points in A. How can we check if a triangle δ exists that separates P from Q?

Without loss of generality, the sides of δ are supporting lines for P.

Let c be some fixed circle. To each supporting line l of P assign the unique point $T(l)$ on c such that the tangent $t(T)$ to c at $T(l)$ is parallel to l and P lies on the same side from l as c does of $t(T)$.

Let X be any point in Q. Let $l_1(X)$ and $l_2(X)$ be the two supporting lines of P through X (if X lies inside P, then P cannot be separated from Q), and let $a(X)$ be the arc of c with endpoints $T(l_1(X))$ and $T(l_2(X))$.

If a separating triangle δ exists, then the three points on c assigned to its sides nail down all arcs $a(X)$ where X ranges over Q. The converse statement is slightly more subtle: if there exist three points on c that nail down all arcs $a(X)$, and are the vertices of an acute-angled triangle, then they give us a triangle δ that separates P from Q.

So we are looking for a system of arcs on c such that it is possible to nail down all its large subsystems by means of three points but this is impossible for the complete system. (It is not clear yet whether finding such a system or showing that none exist for some sense of "large" would solve the problem, but it should surely shed some light.)

We construct such a system as follows: we consider a large number of equal, equally spaced arcs set up in such a way that any point nails down nearly but not quite a third of them.

More precisely, let n be a positive integer and let $T_1, T_2, \ldots, T_{3n+1}$ be the vertices of a regular $(3n+1)$-gon inscribed in c, with $T_{i+3n+1} \equiv T_i$ for all i. For all i let a_i be the open arc T_iT_{i+n}. Then any point on c nails down at most n arcs, so it is impossible to nail down all arcs by means of three points.

On the other hand, remove any one arc, say T_1T_{n+1}, and three midpoints of $T_{n+1}T_{n+2}$, $T_{2n+1}T_{2n+2}$, and $T_{3n+1}T_1$ do the job.

This provides a counterexample for the original problem. Consider a regular 3001-gon $Y_1Y_2\dots Y_{3001}$ inscribed in a circle k of center O, where $Y_{i+3001} \equiv Y_i$ for all i. For all i let X_i be the intersection of the tangents to k at Y_i and Y_{i+1000}. Slide each X_i very slightly towards O so that all points are in general position. Colour all X_i green and all Y_i red, and let A be the set of all X_i and Y_i.

Since the convex hull of the X_i contains the Y_i, the set A can be orderly only if it is red-orderly. However, by the previous discussion, it is not.

Remove any X_i or any Y_i. Again, by the previous discussion, A becomes red-orderly and, therefore, orderly.

Notes. We say that a finite set S of red and green points in the plane is *line-orderly* if there exists a line l such that all points of one colour lie strictly on one side of l and all points of the other colour lie strictly on the other side of l. Let A be a finite set of red and green points in the plane, in general position. Then A is line-orderly if and only if every four-point subset of A is line-orderly.

We say that a finite set S of red and green points in the plane is *circle-orderly* if there exists a circle c such that all points of one colour lie strictly inside c and all points of the other colour lie strictly outside of c. Let A be a finite set of red and green points in the plane, in general position. Then A is circle-orderly if and only if every five-point subset of A is circle-orderly.

XIV Geometrical Olympiad in honour of I.F.Sharygin
Final round. Solutions. Second day. 10 grade

5. (A.Polyanskii) Let w be the incircle of a triangle ABC. The line passing through the incenter I and parallel to BC meets w at points A_B an A_C (A_B lies in the same semiplane with respect to AI that B). The lines BA_B and CA_C meet at point A_1. The points B_1 and C_1 are defines similarly. Prove that AA_1, BB_1 and CC_1 concur.

   First solution. Since segments A_BA_C and BC are homothetic with respect to A_1, the line A_1I passes through the midpoint M of BC and $A_1I : A_1M = 2r : BC$. Hence the distance from A_1 to AC is equal to $r(BC - h_b)/(BC - 2r)$, where h_b is the length of the altitude from B. Similarly the distance from A_1 to AB is equal to $r(BC - h_c)/(BC - 2r)$. Therefore $\sin\angle A_1AC : \sin\angle A_1AB = (1 - \sin\angle C) : (1 - \sin\angle B)$. Using the similar equalities for B_1, C_1 and Ceva theorem we obtain the required assertion.

   Second solution. Since $\angle A_BIB = \angle IBC = \angle IBA = \angle C_BIB$, the points A_B and C_B are symmetric with respect to the bisector of angle B (fig.10.5).

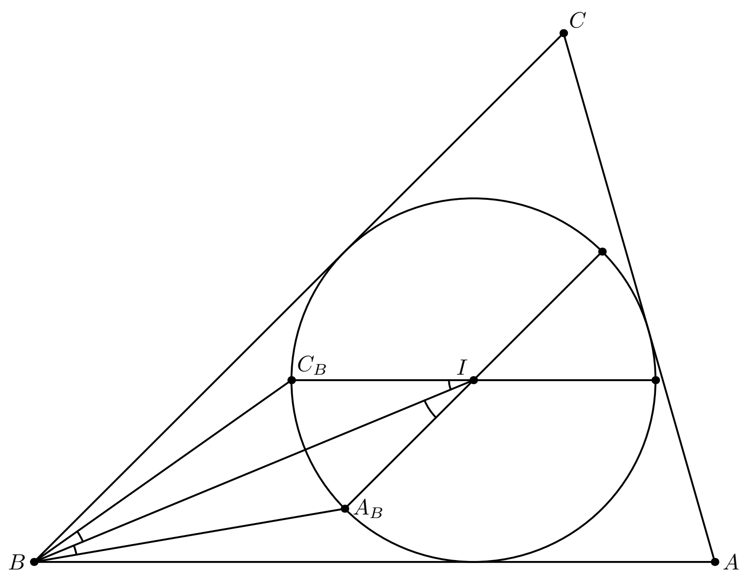


Fig. 10.5

By Ceva theorem

$$\frac{\sin\angle CAA_1}{\sin\angle BAA_1}\frac{\sin\angle ABA_B}{\sin\angle CBA_B}\frac{\sin\angle BCA_C}{\sin\angle ACA_C} = 1.$$

Multiplying this and two similar equalities we obtain the required assertion.

6. (M.Kungozhin) Let ω be the circumcircle of a triangle ABC, and KL be the diameter of ω passing through the midpoint M of AB (K and C lies on the different sides from AB). A circle passing through L и M meets segment CK at points P and Q (Q lies on the segment KP). Let LQ meet the circumcircle of triangle KMQ at point R. Prove that the quadrilateral $APBR$ is cyclic.

   Solution. Note that $\angle PML = \angle PQL = \angle KQR = \angle KMR$. Also $\angle PLM = \angle KQM = \angle KRM$, therefore the triangles PLM and KRM are similar, i.e. $PM \cdot RM = LM \cdot KM = AM^2$ (fig.10.6).

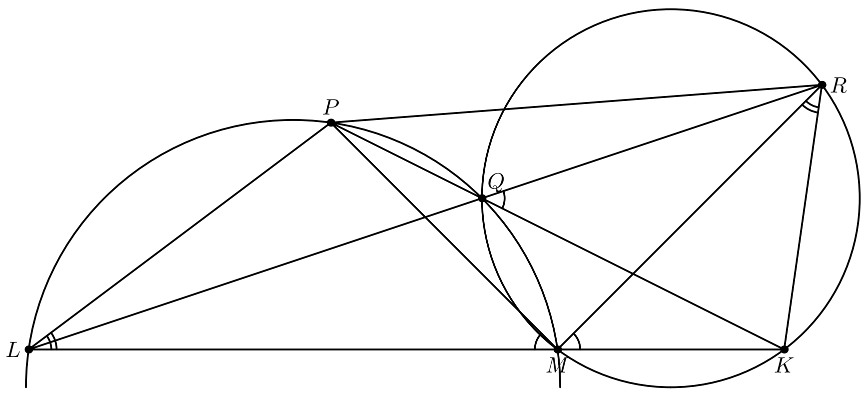


Fig. 10.6

   Let P' be the reflection of P about KL. The points A, B, P, P' are concyclic as the vertices of an isosceles trapezoid. Since P', M, R are collinear and $P'M \cdot RM = AM \cdot BM$, we obtain that R also lies on this circle.

7. (N.Beluhov) A convex quadrilateral $ABCD$ is circumscribed about a circle of radius r. What is the maximum possible value of $\frac{1}{AC^2} + \frac{1}{BD^2}$?

   First solution. Let $AC \cap BD = O$ and suppose without loss of generality that $\angle AOB \geq 90^\circ$. Construct E so that $BECD$ is a parallelogram (fig.10.7).

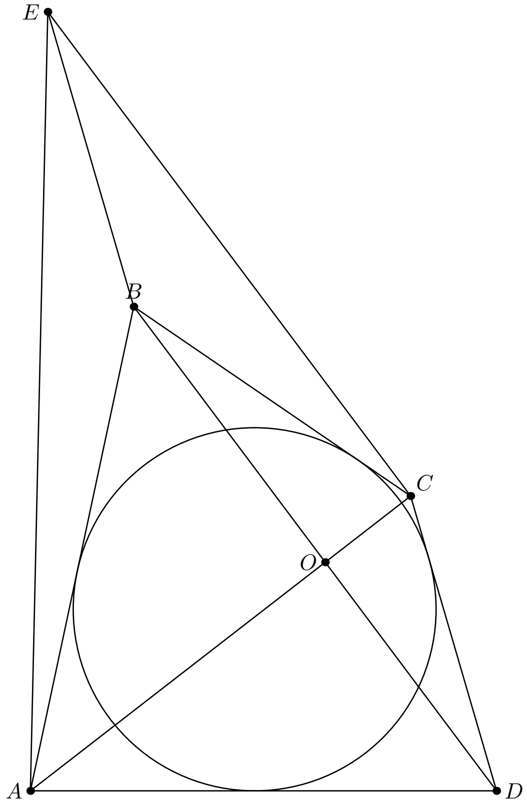


Fig. 10.7

We have

$$AC \cdot BD \geq 2S_{ABCD} = r \cdot P_{ABCD} = 2r \cdot (AB + CD).$$

Furthermore

$$AB + CD = AB + BE \geq AE$$

and (since $\angle ECA \geq 90°$)

$$AE^2 \geq AC^2 + CE^2 = AC^2 + BD^2.$$

Hence

$$AC^2 \cdot BD^2 \geq 4r^2 \cdot (AC^2 + BD^2) \Rightarrow \frac{1}{AC^2} + \frac{1}{BD^2} \leq \frac{1}{4r^2}.$$

Equality is attained just for $AC \cdot BD = 2S_{ABCD} \Leftrightarrow AC \perp BD$ and $AB + BE = AE \Leftrightarrow AB \| CD$, that is, when $ABCD$ is a rhombus.

Second solution. We begin by altering $ABCD$ continuously so that its incircle remains the same but its diagonals become shorter.

Let the circle ω with center I be the incircle of $ABCD$. Fix ω, the line l determined by the points A and C, and the line m through B parallel to l. Let B vary along m. What happens to the length of AC?

Suppose the tangent n to ω parallel to both l and m and separating them, meets AB and BC at P and Q, and let a circle ω' of center I' and radius r' be the incircle of $\triangle PBQ$.

When B varies, the ratio of similitude of $\triangle PBQ$ and $\triangle ABC$ remains constant. This means that the ratio $PQ : AC$, the ratio $r' : r$, and r' all remain constant too.

Furthermore, PQ equals the common external tangent of ω' and ω. Since r' and r are constant, this common external tangent is shortest when II' is shortest, i.e., when $BI \perp l$. Since $PQ : AC$ is constant, AC is also shortest in this case.

Slide B along m until it reaches a position B_1 with $IB_1 \perp l$, then slide it along IB_1 towards I until it reaches a position B_2 such that the length of A_2C_2 equals the original length of AC. Do the same with D. Then $A_2B_2C_2D_2$ is circumscribed about ø, symmetric about B_2D_2, and satisfies $A_2C_2 = AC$ and $B_2D_2 \leq BD$.

Repeat this procedure with A_2 and C_2: the result is a rhombus $A_3B_3C_3D_3$ circumscribed about ω which satisfies $A_3C_3 \leq AC$ and $B_3D_3 \leq BD$. For a rhombus, though, we have

$$\frac{1}{A_3C_3^2} + \frac{1}{B_3D_3^2} = \frac{1}{4r^2}$$

and if $A_3B_3C_3D_3 \neq ABCD$ then at least one of inequalities $A_3C_3 \leq AC$ and $B_3D_3 \leq BD$ is strict.

8. (A.Zaslavsky) Two triangles ABC and $A'B'C'$ are given. The lines AB and $A'B'$ meet at point C_1, and the lines parallel to them and passing through C and C' respectively meet at point C_2. The points A_1, A_2, B_1, B_2 are defined similarly. Prove that A_1A_2, B_1B_2 and C_1C_2 concur.

   First solution Apply a polar transform with center O. Now we have (we use new denotations) two triangles $A_1B_1C_1$ and $A_2B_2C_2$ such that the cevians A_1A_1', B_1B_1', and C_1C_1' in the first triangle and the cevians A_2A_2', B_2B_2', and C_2C_2' in the second triangle are all concurrent in O. Let P_a be the intersection of A_1A_2 and $A_1'A_2'$, define P_b and P_c similarly, now we wish to prove that P_a, P_b, and P_c are collinear.

   To this end, apply a projective transform that maps P_a and P_b to infinity. Then $OA_1' : A_1'A_1 = OA_2' : A_2'A_2$ and $OB_1' : B_1'B_1 = OB_2' : B_2'B_2$. However, $OA_1'/A_1'A_1 + OB_1'/B_1'B_1 + OC_1'/C_1'C_1 = S_{OB_1C_1}/S_{A_1B_1C_1} + S_{OC_1A_1}/S_{A_1B_1C_1} +$

$S_{OA_1B_1}/S_{A_1B_1C_1} = 1$ (signed areas) and similarly for the second triangle, so $OC_1' : C_1'C_1 = OC_2' : C_2'C_2$ and P_c is at infinity, too.

Second solution. Note that for any point X lying on C_1C_2 we have (the areas are directed)

$$S_{XAB}S_{A'B'C'} = S_{XA'B'}S_{ABC}.$$

To prove this equality it is sufficient to note that it is correct for C_1, C_2. Also it is easy to see that this equality is not true for all points of the plane, therefore this is the equation of line C_1C_2. Similarly we can find the equations of lines A_1A_2 and B_1B_2. It is evident that the point satisfying two of these three equations satisfy the third one.

XV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
Final round. First day. 8 form
Ratmino, July 30, 2019

1. A trapezoid with bases AB and CD is inscribed into a circle centered at O. Let AP and AQ be the tangents from A to the circumcircle of triangle CDO. Prove that the circumcircle of triangle APQ passes through the midpoint of AB.

2. A point M inside triangle ABC is such that $AM = AB/2$ and $CM = BC/2$. Points C_0 and A_0 lying on AB and CB respectively are such that $BC_0 : AC_0 = BA_0 : CA_0 = 3$. Prove that the distances from M to C_0 and to A_0 are equal.

3. Construct a regular triangle using a plywood square. (*You can draw lines through pairs of points lying on the distance not greater than the side of the square, construct the perpendicular from a point to a line if the distance between them does not exceed the side of the square, and measure segments on the constructed lines equal to the side or to the diagonal of the square.*)

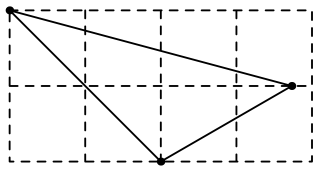
4. Let O and H be the circumcenter and the orthocenter of an acute-angled triangle ABC with $AB < AC$. Let K be the midpoint of AH. The line through K perpendicular to OK meets AB and the tangent to the circumcircle at A at points X and Y respectively. Prove that $\angle XOY = \angle AOB$.

XV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Second day. 8 form

Ratmino, July 31, 2019

5. A triangle having one angle equal to 45° is drawn on the chequered paper (see.fig.). Find the values of its remaining angles.



6. A point H lies on the side AB of regular pentagon $ABCDE$. A circle with center H and radius HE meets the segments DE and CD at points G and F respectively. It is known that $DG = AH$. Prove that $CF = AH$.

7. Let points M and N lie on the sides AB and BC of triangle ABC in such a way that $MN \parallel AC$. Points M' and N' are the reflections of M and N about BC and AB respectively. Let $M'A$ meet BC at X, and $N'C$ meet AB at Y. Prove that A, C, X, Y are concyclic.

8. What is the least positive integer k such that, in every convex 1001-gon, the sum of any k diagonals is greater than or equal to the sum of the remaining diagonals?

XV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
Final round. First day. 9 form
Ratmino, July 30, 2019

1. A triangle OAB with $\angle A = 90^\circ$ lies inside a right angle with vertex O. The altitude of OAB from A is extended beyond A until it intersects the side of angle O at M. The distances from M and B to the second side of angle O are equal to 2 and 1 respectively. Find the length of OA.

2. Let P lie on the circumcircle of triangle ABC. Let A_1 be the reflection of the orthocenter of triangle PBC about the perpendicular bisector to BC. Points B_1 and C_1 are defined similarly. Prove that A_1, B_1, and C_1 are collinear.

3. Let $ABCD$ be a cyclic quadrilateral such that $AD = BD = AC$. A point P moves along the circumcircle ω of $ABCD$. The lines AP and DP meet the lines CD and AB at points E and F respectively. The lines BE and CF meet at point Q. Find the locus of Q.

4. A ship tries to land in the fog. The crew does not know the direction to the land. They see a lighthouse on a little island, and they understand that the distance to the lighthouse does not exceed 10 km (the precise distance is not known). The distance from the lighthouse to the land equals 10 km. The lighthouse is surrounded by reefs, hence the ship cannot approach it. Can the ship land having sailed the distance not greater than 75 km? (The waterside is a straight line, the trajectory has to be given before the beginning of the motion, after that the autopilot navigates the ship according to it.)

XV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Second day. 9 form

Ratmino, July 31, 2019

5. Let R be the circumradius of a cyclic quadrilateral $ABCD$. Let h_1 and h_2 be the altitudes from A to BC and CD respectively. Similarly h_3 and h_4 are the altitudes from C to AB and AD. Prove that
$$\frac{h_1 + h_2 - 2R}{h_1 h_2} = \frac{h_3 + h_4 - 2R}{h_3 h_4}.$$

6. A non-convex polygon has the property that every three consecutive its vertices form a right-angled triangle. Is it true that this polygon has always an angle equal to $90°$ or to $270°$?

7. Let the incircle ω of triangle ABC touch AC and AB at points E and F respectively. Points X, Y of ω are such that $\angle BXC = \angle BYC = 90°$. Prove that EF and XY meet on the medial line of ABC.

8. A hexagon $A_1A_2A_3A_4A_5A_6$ has no four concyclic vertices, and its diagonals A_1A_4, A_2A_5 and A_3A_6 concur. Let l_i be the radical axis of circles $A_iA_{i+1}A_{i-2}$ and $A_iA_{i-1}A_{i+2}$ (the points A_i and A_{i+6} coincide). Prove that l_i, $i = 1, \dots, 6$, concur.

XV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. First day. 10 form

Ratmino, July 30, 2019

1. Given a triangle ABC with $\angle A = 45°$. Let A' be the antipode of A in the circumcircle of ABC. Points E and F on segments AB and AC respectively are such that $A'B = BE$, $A'C = CF$. Let K be the second intersection of circumcircles of triangles AEF and ABC. Prove that EF bisects $A'K$.

2. Let A_1, B_1, C_1 be the midpoints of sides BC, AC and AB of triangle ABC, AK be its altitude from A, and L be the tangency point of the incircle γ with BC. Let the circumcircles of triangles LKB_1 and A_1LC_1 meet B_1C_1 for the second time at points X and Y respectively and γ meet this line at points Z and T. Prove that $XZ = YT$.

3. Let P and Q be isogonal conjugates inside triangle ABC. Let ω be the circumcircle of ABC. Let A_1 be a point on arc BC of ω satisfying $\angle BA_1P = \angle CA_1Q$. Points B_1 and C_1 are defined similarly. Prove that AA_1, BB_1, and CC_1 are concurrent.

4. Prove that the sum of two nagelians is greater than the semiperimeter of the triangle. (A nagelian is the segment between a vertex of a triangle and the tangency point of the opposite side with the corresponding excircle.)

XV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Second day. 10 form

Ratmino, July 31, 2019

5. Let AA_1, BB_1, CC_1 be the altitudes of triangle ABC; and A_0, C_0 be the common points of the circumcircle of triangle A_1BC_1 with the lines A_1B_1 and C_1B_1 respectively. Prove that AA_0 and CC_0 meet on the median of ABC or are parallel to it.

6. Let AK and AT be the bisector and the median of an acute-angled triangle ABC with $AC > AB$. The line AT meets the circumcircle of ABC at point D. Point F is the reflection of K about T. If the angles of ABC are known, find the value of angle FDA.

7. Let P be an arbitrary point on side BC of triangle ABC. Let K be the incenter of triangle PAB. Let the incircle of triangle PAC touch BC at F. Point G on CK is such that $FG \parallel PK$. Find the locus of G.

8. Several points and planes are given in the space. It is known that for any two of given points there exist exactly two planes containing them, and each given plane contains at least four of given points. Is it true that all given points are collinear?

XV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
Final round. Solutions. First day. 8 form
Ratmino, July 30, 2019

1. (F.Ivlev) A trapezoid with bases AB and CD is inscribed into a circle centered at O. Let AP and AQ be the tangents from A to the circumcircle of triangle CDO. Prove that the circumcircle of triangle APQ passes through the midpoint of AB.

   Solution. Let O' be the circumcenter of triangle OCD. Then AO' is a diameter of circle APQ. Since O' lies on the perpendicular bisector of segment AB, the midpoint of segment AB also lies on this circle (fig/ 8.1).

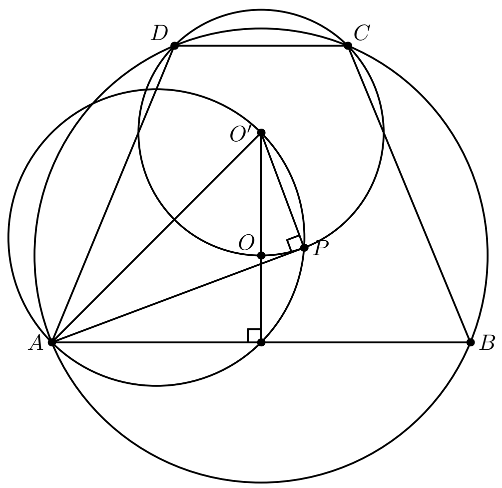


Fig. 8.1.

2. (P.Ryabov) A point M inside triangle ABC is such that $AM = AB/2$ and $CM = BC/2$. Points C_0 and A_0 lying on AB and CB respectively are such that $BC_0 : AC_0 = BA_0 : CA_0 = 3$. Prove that the distances from M to C_0 and to A_0 are equal.

   Solution. Let K, L, U, and V be the midpoints of segments AB, BC, AM and MC respectively. Then since AMK, CML are isosceles triangles, and KU, LV are medial lines of triangles ABM, CBM respectively, we have $MA_0 = LV = BM/2 = KU = MC_0$ (fig. 8.2).

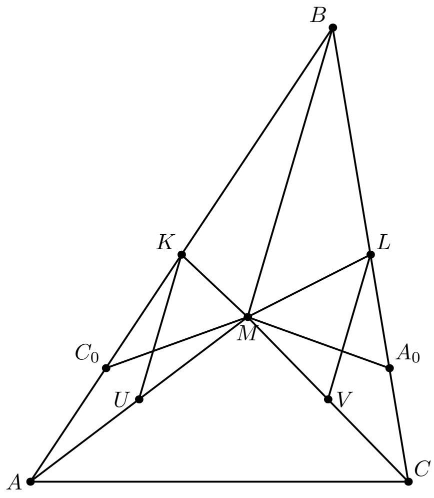


Fig. 8.2.

3. (M.Plotnikov) Construct a regular triangle using a plywood square. (*You can draw lines through pairs of points lying on the distance not greater than the side of the square, construct the perpendicular from a point to a line if the distance between them does not exceed the side of the square, and measure segments on the constructed lines equal to the side or to the diagonal of the square.*)

   Solution. Let the side of the square equal one. First we show how to construct the midpoint of any segment PQ such that the length of PQ does not exceed one. Construct any line ℓ through P such that ℓ is distinct from PQ and the angle between ℓ and PQ is distinct from 90°. Let R be the foot of the perpendicular from Q onto ℓ. Let the line through P perpendicular to PR and the line through Q perpendicular to QR meet at S. Then line RS bisects segment PQ.

   Now to solve the problem draw two perpendicular lines through A. Plot two segments $AB = AC = 1/2$ onto them. (First plot $AB' = AC' = 1$, then halve them.) Draw line BC and the line through C perpendicular to it. Construct D so that $\angle BCD = 90^\circ$ and $CD = 1/2$. Draw line BD and the line through D perpendicular to it. Construct E and F so that $\angle BDE = \angle BDF = 90^\circ$ and $DE = DF = 1/2$. Draw lines BE and BF. Then triangle BEF is the desired equilateral triangle with base $EF = 1$, and with altitude and median $BD = \sqrt{3}/2$.

   Remark. We can replace the segment with length $1/2$ by an arbitrary segment with sufficiently small length, for example $3 - 2\sqrt{2}$.

4. (M.Didin, I.Frolov) Let O and H be the circumcenter and the orthocenter of an acute-angled triangle ABC with $AB < AC$. Let K be the midpoint of

AH. The line through K perpendicular to OK meets AB and the tangent to the circumcircle at A at points X and Y respectively. Prove that $\angle XOY = \angle AOB$.

Solution. Since $\angle OKY = \angle OAY = 90°$, points K and A lie on the circle with diameter OY, i.e. $\angle OYX = \angle OAK = \angle B - \angle C$. Now let M be the midpoint of BC. Then $KHMO$ is a parallelogram, i.e. the corresponding sidelines of triangles AKX and CMH are perpendicular. Therefore these triangles are similar, and $KX/OK = KX/HM = AK/CM = OM/CM$. Thus the right-angled triangles OKX and CMO are similar, and $\angle OXK = \angle COM = \angle A$ (fig 8.4). Hence $\angle XOY = 2\angle C = \angle AOB$.

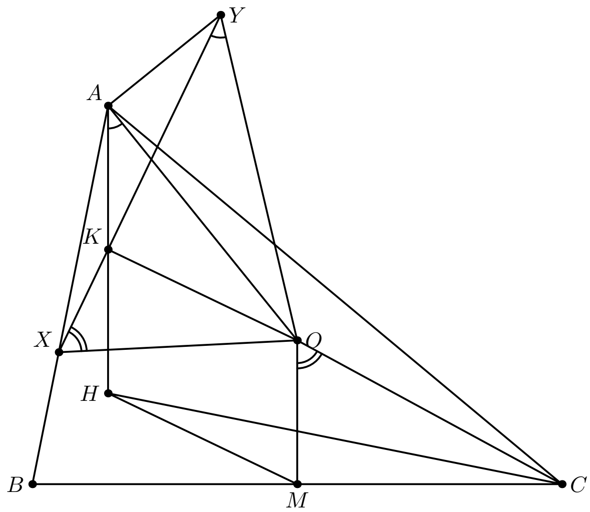


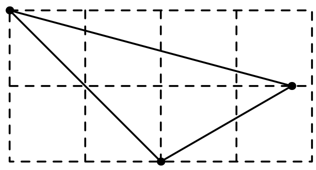
Fig. 8.4.

XV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Solutions. Second day. 8 form

Ratmino, July 31, 2019

5. (M.Volchkevich) A triangle having one angle equal to 45° is drawn on the chequered paper (see.fig.). Find the values of its remaining angles.



Answer. 30° and 105°.

First solutions. Denote the points as in the fig. 8.5.

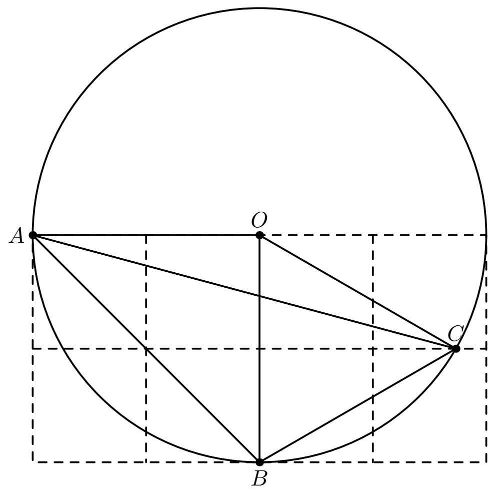


Fig. 8.5.

Since $\angle A < \angle OAB = 45^\circ = \angle OBA < \angle B$, we obtain that $\angle C = 45^\circ$. Since $OA = OB$, and $\angle AOB = 90^\circ = 2\angle ACB$, we obtain that O is the circumcenter of ABC, and $OC = OB$. But C lies on the perpendicular bisector to segment BO, thus $OC = BC$, triangle OBC is equilateral, and $\angle BOC = 60^\circ$. Hence $\angle A = 30^\circ$, $\angle B = 105^\circ$.

Second solution. Let M be the midpoint of AB. Then $\angle CMB = 45^\circ$ and we obtain that triangles ABC and CBM are similar. Hence $AB/BC = \sqrt{2}$ and by the sine law $\angle A = 30^\circ$.

6. (K.Knop) A point H lies on the side AB of regular pentagon $ABCDE$. A circle with center H and radius HE meets the segments DE and CD at points G and F respectively. It is known that $DG = AH$. Prove that $CF = AH$.

   Solution. Let F' lie on segment CD so that $CF' = AH$. Then quadrilaterals $AHGE$ and $CF'HB$ are congruent by three equal sides and two equal angles, thus $HF' = HG$. To see that F' coincides with F, which would solve the problem, it suffices to verify that the second common point of line CD with the circle lies outside segment CD. To this end, prove that $\angle DCH$ is right.

   Note that there exists a unique pair of points H and G lying on AB and ED respectively and such that $AH = DG$ and $HE = HG$. In fact, when H moves to A, and G moves to D, then the angle GEH increases, and the angle EGH decreases, therefore the equality $HE = HG$ is obtained in the unique position. Now let K be the common point of diagonals AD and CE, let the line passing through K and parallel to AE meet AB at H', and let the line passing through K and parallel to CD meet ED at G' (fig.8.6). Then $\angle DG'K = \angle DKG' = 72°$, i.e. $DG' = DK = EK = AH'$. Also $KH' = EA = CD = KC$ and $\angle G'KC = \angle G'KH' = 144°$. Therefore triangles CKG' and $H'KG'$ are congruent, i.e. $G'H' = G'C = H'E$ and H', G' coincide with H, G. Also $HC \perp GK \parallel CD$, q.e.d.

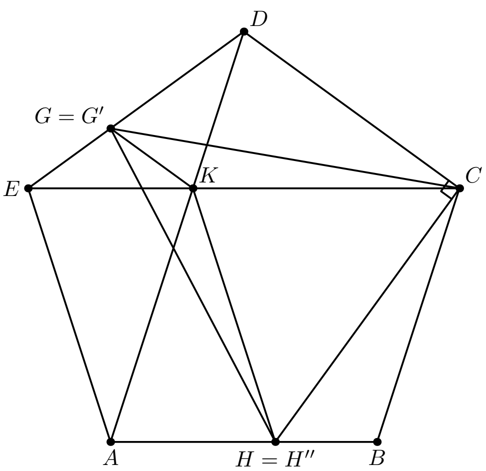


Fig. 8.6.

7. (P.Ryabov, T.Ryabova) Let points M and N lie on the sides AB and BC of triangle ABC in such a way that $MN \parallel AC$. Points M' and N' are the reflections of M and N about BC and AB respectively. Let $M'A$ meet BC at X, and $N'C$ meet AB at Y. Prove that A, C, X, Y are concyclic.

Solution. Let A' be the reflection of A about BC, and let C' be the reflection of C about AB. Let AA_1 and CC_1 be altitudes of triangle ABC. By Menelaus theorem for triangle $A'BA_1$ and line AXM', we obtain that $BX : XA_1 = 2 \cdot (BM' : M'A') = 2 \cdot (BM : MA)$. Similarly, $BY : YC_1 = 2 \cdot (BN : NC)$. Since $MN \parallel AC$, we have $BM : MA = BN : NC$, so $BX : XA_1 = BY : YC_1$, thus $XY \parallel A_1C_1$, and we are done (fig. 8.7).

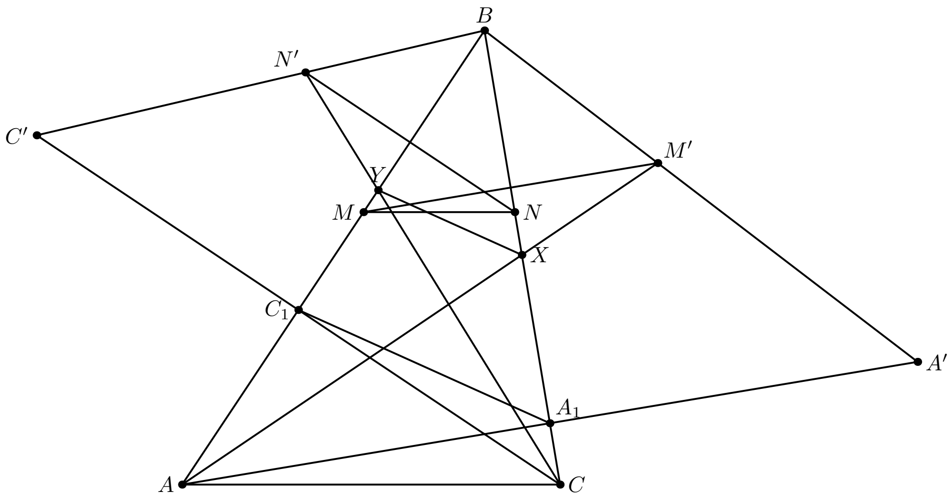


Fig. 8.7.

8. (N.Beluhov) What is the least positive integer k such that, in every convex 1001-gon, the sum of any k diagonals is greater than or equal to the sum of the remaining diagonals?

Answer. $k = 499000$.

Solution. Let $AB = 1$. Consider a convex 1001-gon such that one of its vertices is at A and the remaining 1000 vertices are within ε of B, where ε is small. Let $k + \ell$ equal the total number $\frac{1001 \cdot 998}{2} = 499499$ of diagonals. When $k \geq 498501$, the sum of the k shortest diagonals is approximately $k - 498501 = 998 - \ell$ and the sum of the remaining diagonals is approximately ℓ. Therefore, $\ell \leq 499$ and so $k \geq 499000$.

We proceed to show that $k = 499000$ works. To this end, colour all $\ell = 499$ remaining diagonals green. To each green diagonal AB apart from, possibly, two last ones, we will assign two red diagonals AC and CB so that no green diagonal is ever coloured red and no diagonal is coloured red twice.

Suppose that we have already done this for $0 \leq i \leq 498$ green diagonals (thus forming i red-green triangles) and let AB be up next. Let D be the set of all diagonals emanating from A or B and distinct from AB; we have that

$|D| = 2 \cdot 997 = 1994$. Every red-green triangle formed thus far has at most two sides in D and there are $499 - (i + 1)$ green diagonals distinct from AB for which the triangles are not constructed. Therefore, the subset E of all as-of-yet uncoloured diagonals in D contains at least $1994 - 2i - (499 - (i+1)) = 1496 - i$ elements.

When $i \leq 498$, we have that $|E| \geq 998$. The total number of endpoints distinct from A and B of diagonals in D, however, is 999. Therefore, there exist two diagonals in E having a common endpoint C and we can assign AC and CB to AB or no two diagonals in E have a common endpoint other than A and B, but if so then there are two diagonals in E that intersect. Otherwise, at least one of the two vertices adjacent to A (say a) is cut off from B by the diagonals emanating from A and at least one of the two vertices adjacent to B (say b) is cut off from A by the diagonals emanating from B (and $a \neq b$). This leaves us with at most 997 suitable endpoints and at least 998 diagonals in E, a contradiction.

By the triangle inequality, this completes the solution.

XV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Solutions. First day. 9 form

Ratmino, July 30, 2019

1. (V.Protasov) A triangle OAB with $\angle A = 90^\circ$ lies inside a right angle with vertex O. The altitude of OAB from A is extended beyond A until it intersects the side of angle O at M. The distances from M and B to the second side of angle O are equal to 2 and 1 respectively. Find the length of OA.

   Answer. $\sqrt{2}$.

   First solution. Let S be the projection of B onto line OM. Then quadrilateral $ABOS$ is *cyclic*, so $\angle OAS = \angle OBS = 90^\circ - \angle BOM = \angle OMA$, thus triangles AOS and MOA are similar (fig. 9.1). Therefore $OA^2 = OS \cdot OM = 1 \cdot 2$, and $OA = \sqrt{2}$.

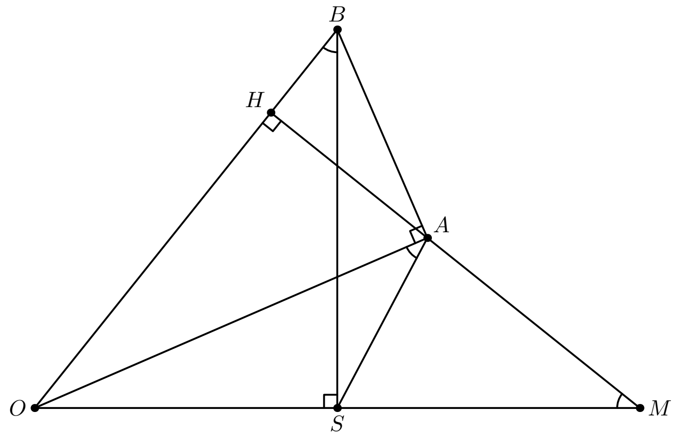


Fig. 9.1.

   Second solution. Let AH be the altitude of the triangle. Then $BHSM$ is a cyclic quadrilateral, therefore $OH \cdot OB = OS \cdot OM = 2$. But $OH \cdot OB = OA^2$ by the property of a right-angled triangle.

2. (D.Prokopenko) Let P lie on the circumcircle of triangle ABC. Let A_1 be the reflection of the orthocenter of triangle PBC about the perpendicular bisector to BC. Points B_1 and C_1 are defined similarly. Prove that A_1, B_1, and C_1 are collinear.

   First solution. Let H be the orthocenter of triangle ABC. Let P move along the circumcircle of ABC with a constant velocity. Then points A_1,

B_1 and C_1 move along circles BHC, CHA and AHB respectively with the same velocity. So it suffices to find a particular case when A_1, B_1, C_1 and H are collinear; then they will always be collinear. For example, the special case when AP is a diameter is very easy to verify.

Second solution. Let P' be the point opposite to P on the circumcircle of ABC. Then A_1 is the reflection of P' about BC, therefore A_1 lies on the Steiner line of P'. Similarly we obtain that B_1 and C_1 lie on the same line.

3. (I.Kukharchuk) Let $ABCD$ be a cyclic quadrilateral such that $AD = BD = AC$. A point P moves along the circumcircle ω of $ABCD$. The lines AP and DP meet the lines CD and AB at points E and F respectively. The lines BE and CF meet at point Q. Find the locus of Q.

Answer. A circle k passing through B, C and touching AB, CD.

Solution. Let S be the intersection point of segments AC and BD. Then S is the interior center of similarity for k and ω (because the tangent to ω at D is parallel to AB). Let ray SP meet k at Q'. We are going to prove that lines AP and BQ' meet on line CD. Then it would follow just in the same way that lines CQ' and DP meet on line AB, and, therefore, that Q' coincides with Q (fig. 9.3).

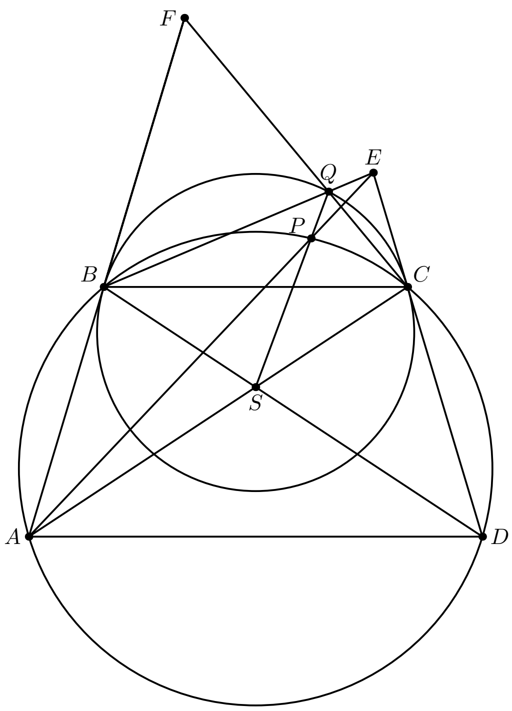


Fig. 9.3.

Let line SP meet ω for the second time at R, let lines AB and CD meet at T, and let line BD meet k for the second time at U. Let lines BQ' and AP meet line CD at E' and E'', respectively. We need to show that E' and E'' coincide. We will use cross-ratios.

We have that $(C, D, T, E') = (BC, BD, BT, BE') = (C, U, B, Q') = (A, B, D, R) = (C, D, B, P) = (AC, AD, AB, AP) = (C, D, T, E'')$ (the second equality is obtained by the projection from B onto k, and the third one follows from the homothety with center S between k and ω). This completes the solution.

Remark. We can also prove that B, Q' and E are collinear by the following way. Let R' be the common point of k and the line SP distinct from Q'. Since S is the homothety center of k and ω, we have $AP \parallel CR'$ and $BQ' \parallel DR$. Hence $\angle Q'CE = \angle Q'BC = \angle Q'R'C = \angle Q'PE$, i.e. $PQ'EC$ is a cyclic quadrilateral. Also $\angle BQ'R = \angle Q'RD = \angle PCE$. Therefore $\angle PQ'E + \angle BQ'P = 180°$, q.e.d.

4. (V.Protasov) A ship tries to land in the fog. The crew does not know the direction to the land. They see a lighthouse on a little island, and they understand that the distance to the lighthouse does not exceed 10 km (the precise distance is not known). The distance from the lighthouse to the land equals 10 km. The lighthouse is surrounded by reefs, hence the ship cannot approach it. Can the ship land having sailed the distance not greater than 75 km? (The waterside is a straight line, the trajectory has to be given before the beginning of the motion, after that the autopilot navigates the ship according to it.)

   Answer. Yes, it can.

   Solution. Let the ship be at point K, the lighthouse be at point M, and K' be the point of ray KM such that $KK' = 10$ km. To guarantee the attainment of the land, the convex hull of the trajectory has to contain the disc centered at M with radius KK', but since the position of M on segment KK' is not known, this convex hull has to contain the union of all such disks centered at KK'. It is clear that this condition is also sufficient.

   Let ω, ω' be circles centered at K, K' respectively with radii equal to KK', let CC' and DD' be the common tangents to these circles, X be the point of line CC' such that $\angle XKC = 30°$, XA be the tangent to ω, B be the midpoint of arc $C'D'$ lying outside ω, and Y be the projection of B onto CC' (fig. 9.4). Then the trajectory $KXADD'BY$ satisfies the condition, and its length equals $10(\sqrt{3} + 2\pi/3 + 1 + \pi/2 + 1) < 74$ km.

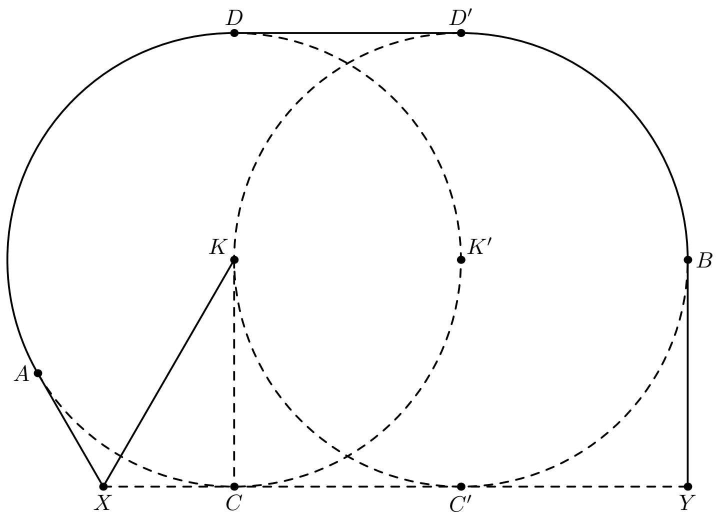


Fig. 9.4.

XV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Second day. 9 form

Ratmino, July 31, 2019

5. (A.Akopyan). Let R be the inradius of a circumscribed quadrilateral $ABCD$. Let h_1 and h_2 be the altitudes from A to BC and CD respectively. Similarly h_3 and h_4 are the altitudes from C to AB and AD. Prove that
$$\frac{h_1+h_2-2R}{h_1h_2}=\frac{h_3+h_4-2R}{h_3h_4}.$$

Solution. Let a be the length of the tangent from A to the incircle, and define b, c and d similarly. Then, by calculating the area of $ABCD$ in three different ways, we obtain $h_1(b+c)+h_2(c+d)=h_3(a+b)+h_4(a+d)=2R(a+b+c+d)$. Multiply both sides of the desired identity by $a+b+c+d$. Then the left-hand side numerator simplifies to $h_1(a+d)+h_2(a+b)$, and similarly for the right-hand side. So we are left to prove that $(a+b)/h_1+(a+d)/h_2=(b+c)/h_3+(c+d)/h_4$. This is clear since we have that $h_1(b+c)=h_3(a+b)$ by calculating the area of triangle ABC in two different ways, and similarly for h_2 and h_4.

6. (M.Saghafian) A non-convex polygon has the property that every three consecutive its vertices form a right-angled triangle. Is it true that this polygon has always an angle equal to 90° or to 270°?

First solution. (N.Beluhov). Let $A=(0,1)$, $B=(1,0)$, $C=(1,1)$, $D=(2,0)$, $E=(2,1)$, $F=(3,0)$, and let G be the intersection point of line BE and the line through F perpendicular to AF (fig. 9.6). Then heptagon $ABCDEFG$ does work.

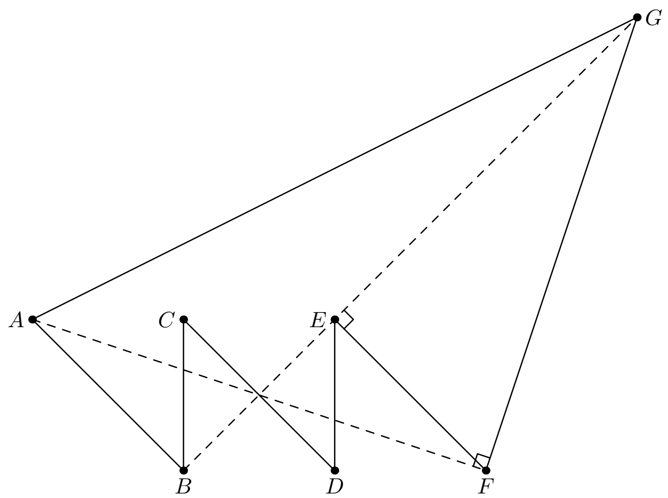


Fig. 9.6.

Second solution. Take a rectangle with sides equal to 2 and $\sqrt{3}$, and on each its side construct outside it a trapezoid with the ratio of sides equal to $1:1:1:2$, in such a way that the smallest base of the trapezoid coincides with the side of the rectangle. Every three consecutive vertices of obtained 12-gon form a triangle with angles equal to 30°, 60° and 90°, but every angle of 12-gon is equal to 60° or to 330°.

Third solution. (Found by the participants of the olympiad.) Fix two points A_4, A_5 and some point A_3 lying on the circle with diameter A_4A_5 and such that $A_3A_4 < A_3A_5$. Let A_2 be an arbitrary point inside triangle $A_3A_4A_5$ such that $\angle A_3A_2A_4 = 90^\circ$, and A_1 be such that $A_3A_1 \parallel A_4A_2$ and $\angle A_4A_1A_5 = 90^\circ$. If A_2 lies near the segment A_4A_5 we have $\angle A_1A_2A_5 < 90^\circ$. And if the angle between A_2A_4 and the tangent to circle $A_3A_4A_5$ at A_3 is small we have $\angle A_1A_2A_5 > 90^\circ$. Hence there exists such position of A_2 that $\angle A_1A_2A_5 = 90^\circ$. The corresponding pentagon $A_1A_2A_3A_4A_5$ is the required one.

7. (F.Yudin) Let the incircle ω of triangle ABC touch AC and AB at points E and F respectively. Points X, Y of ω are such that $\angle BXC = \angle BYC = 90^\circ$. Prove that EF and XY meet on the medial line of ABC. t **First solution.** Let A_0, B_0, C_0 be the midpoint of BC, CA, AB respectively. Let EF meet B_0C_0, A_0B_0, and A_0C_0 at points Z, M, and N respectively. Then M and N are the projections of C and B to the bisectors of angles B and C respectively, hence M and N lie on the circle $BXYC$. Also since $A_0C_0 \parallel AC$ and $A_0B_0 \parallel AB$ we obtain that $ZE/ZN = ZB_0/ZC_0 = ZM/ZF$, i.e. the powers of Z with respect to ω and the circle $BXYC$ are equal, thus Z lie on XY.

   Second solution. Let I be the incenter of triangle ABC, let H be the orthocenter of triangle BIC, let k be the circle with diameter IH, and let Γ be the circle with diameter BC.

   Observe that line XY is the radical axis of circles ω and Γ.

   Let K and L be the projections of B and C onto lines CI and BI respectively. It is well-known that K and L lie on line EF. Therefore, line EF is the radical axis of circles k and Γ.

   We are left to show that the midline ℓ of triangle ABC opposite to A is the radical axis of circles k and ω.

Let M and N be the projections of A onto lines BI and CI, respectively. It is well-known that M and N lie on line ℓ. We are going to show that the powers of M with respect to circles k and ω are equal. Then we would have similarly that the powers of N with respect to circles k and ω are equal as well.

Observe that the polar of A with respect to ω is line EF, which passes through L. So the polar of L with respect to ω passes through A. On the other hand, the polar of L with respect to ω is perpendicular to IL. So the polar of L with respect to ω is line AM. Consequently, the polar of M with respect to ω is line CL. Let MP and MQ be the tangents from M to ω. Then P and Q lie on line CL (fig.9.7). Therefore, $ML \cdot MI = MP^2$ and so the powers of M with respect to circles k and ω are equal, as needed.

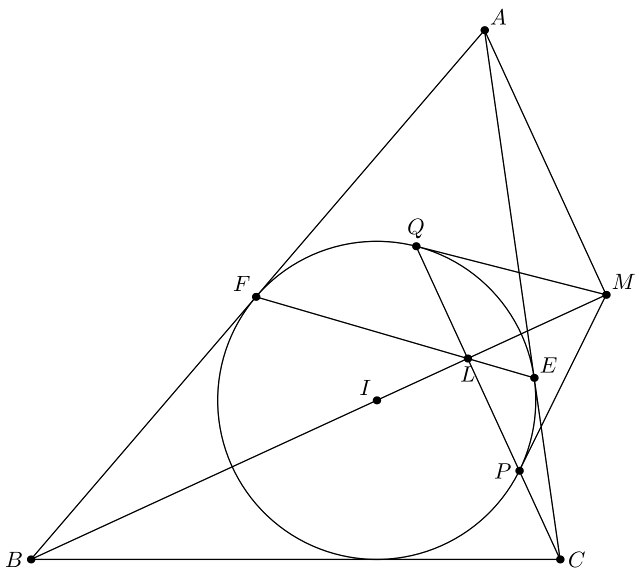


Fig. 9.7.

8. (I.Frolov) A hexagon $A_1A_2A_3A_4A_5A_6$ has no four concyclic vertices, and its diagonals A_1A_4, A_2A_5, and A_3A_6 concur. Let l_i be the radical axis of circles $A_iA_{i+1}A_{i-2}$ and $A_iA_{i-1}A_{i+2}$ (the points A_i and A_{i+6} coincide). Prove that l_i, $i = 1, \dots, 6$, concur.

Solution. Let $A_1, \dots, A_5$ be fixed and A_6 move along the line passing through A_3 and the common point of the diagonals of quadrilateral $A_1A_2A_4A_5$. Then the center O of circle $A_1A_2A_5$ is fixed, and the center O' of circle $A_1A_3A_6$ moves along the perpendicular bisector to segment A_1A_3 in such a way that the correspondence between A_6 and O' is projective (because $\angle O'A_1A_6 = \pi/2 - \angle A_6A_3A_1 = const$). Since the radical axis l_1 is perpendicular to OO', we obtain that the correspondence between A_6 and l_1 is also projective, thus the correspondence between lines l_1 and l_2 rotating around A_1 and A_2 is projective too. Therefore the common point of these lines moves along some conic. Since both lines coincide with A_1A_2, when A_6 meets the circle $A_1A_2A_3$, this conic degenerates to A_1A_2 and another line passing through A_3. Also when A_6 meets the circle $A_2A_3A_5$ then the common point lies on l_3, therefore it lies on l_3 for all positions of A_6. So l_1, l_2 and l_3 concur. Similarly we obtain that three remaining radical axes pass through the same point.

XV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
Final round. Solutions. First day. 10 form

Ratmino, July 30, 2019

1. (A.Dadgarnia) Given a triangle ABC with $\angle A = 45^\circ$. Let A' be the antipode of A in the circumcircle of ABC. Points E and F on segments AB and AC respectively are such that $A'B = BE$, $A'C = CF$. Let K be the second intersection of circumcircles of triangles AEF and ABC. Prove that EF bisects $A'K$.

   Solution. Let K' be the reflection of A' about the line EF. Since $\angle BA'E = \angle CA'F = 45^\circ$, we have that $\angle EK'F = \angle EA'F = 45^\circ$, and thus $AK'EF$ is a cyclic quadrilateral. Then $\angle K'EB = \angle K'FC$. Furthermore, $K'E : EB = A'E : EB = \sqrt{2} = A'F : FC = K'F : FC$, thus triangles $K'EB$ and $K'FC$ are similar. Then $\angle BK'C = 45^\circ$, so $AK'BC$ is a cyclic quadrilateral and K' coincides with K (fig. 10.1).

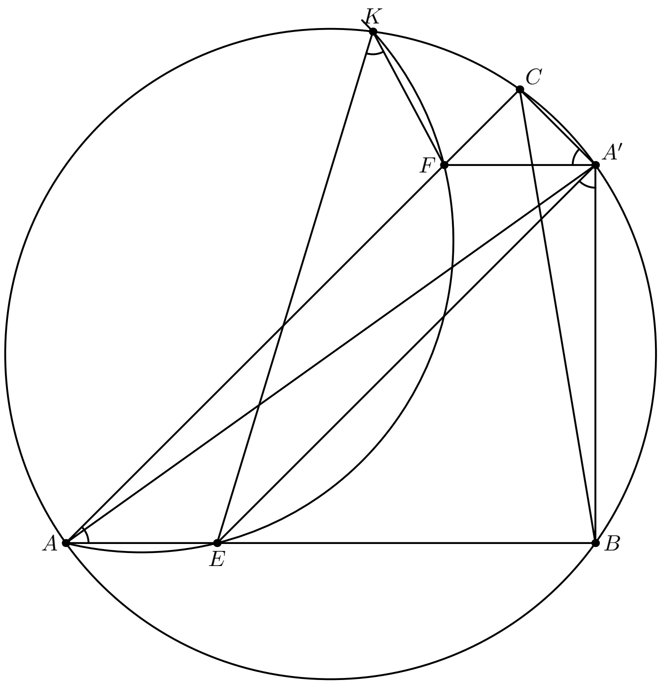


Fig. 10.1.

   Second solution. Let O be the circumcenter of ABC. Note that EBA' and FCA' are isosceles right-angled triangles, thus $AEA'F$ is a parallelogram, and O is the midpoint of EF. Also O lies on the perpendicular bisector to AK, but O does not coincide with the circumcenter of $AKEF$. Therefore

$EF \parallel AK$, i.e. EF is the medial line of triangle $AA'K$, which yields the required assertion.

2. (F.Ivlev) Let A_1, B_1, C_1 be the midpoints of sides BC, AC and AB of triangle ABC, AK be its altitude from A, and L be the tangency point of the incircle γ with BC. Let the circumcircles of triangles LKB_1 and A_1LC_1 meet B_1C_1 for the second time at points X and Y respectively and γ meet this line at points Z and T. Prove that $XZ = YT$.

   Solution. Since $BC \parallel B_1C_1$, both of KB_1XL and A_1LYC_1 are isosceles trapezoids. Then $\angle BLX = \angle CKB_1 = \angle BA_1C_1 = \angle CLY$, thus X and Y are symmetric with respect to line IL, where I is the incenter of triangle ABC (fig. 10.2). It is clear that Z and T are also symmetric with respect to IL, therefore $XZ = YT$, as needed.

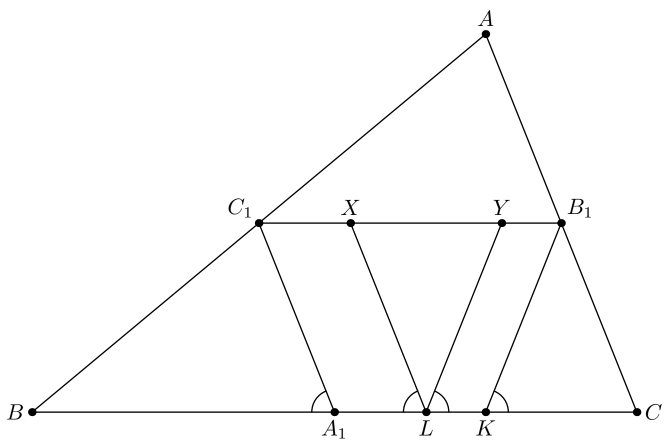


Fig. 10.2.

3. (A.Bhattacharya) Let P and Q be isogonal conjugates inside triangle ABC. Let ω be the circumcircle of ABC. Let A_1 be a point on arc BC of ω satisfying $\angle BA_1P = \angle CA_1Q$. Points B_1 and C_1 are defined similarly. Prove that AA_1, BB_1, and CC_1 are concurrent.

   First solution. Let A_1' be the the Miquel point for lines BP, BQ, CP, and CQ. Then $\angle BA_1'C = (\pi - \angle BPC) + (\pi - \angle BQC) = \pi - \angle A$ (fig.10.3), hence A_1' lies on ω. Also A_1' is the center of the similarity that maps B onto P and Q onto C. (Then it is also the center of the similarity that maps B onto Q and P onto C). Thus $\angle BA_1'P = \angle CA_1'Q$ and A_1' coincides with A_1 (a unique point A_1 satisfies to $\angle BA_1P = \angle CA_1Q$ because when the point moves along the arc BC one of these angles increases and the second one decreases). Then since triangle A_1BP is similar to triangle A_1QC, and triangle A_1BQ is similar to triangle A_1PC, we have $BA_1 : A_1C = (BA_1 : A_1P) \cdot (PA_1 : A_1C) = (BQ : PC) \cdot (BP : QC) = (BP \cdot BQ) : (CP \cdot CQ)$. Express ratios $(CB_1 : B_1A)$ and $(AC_1 : C_1B)$ similarly, and then multiply all three to obtain one. It follows that the main diagonals of inscribed hexagon $AC_1BA_1CB_1$ are concurrent.

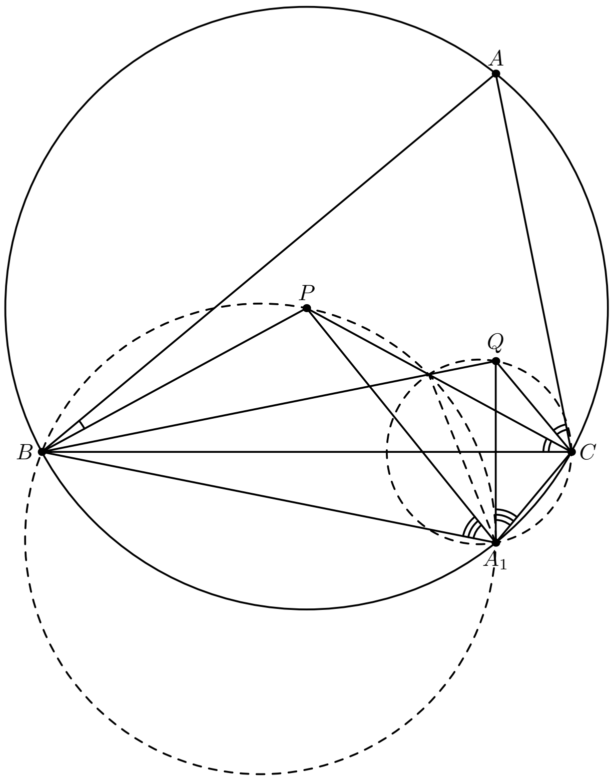


Fig. 10.3.

Second solution. Since the bisectors of angles BA_1C and PA_1Q coincide we have that the reflections of BA_1, CA_1, PA_1 and QA_1 with respect to the bisectors of angles PBQ, PCQ, BPC and BQC respectively are concurrent or parallel. Since the bisectors of angles PBQ and PCQ coincide with the bisectors of angles B and C of triangle ABC and A_1 lies on the circumcircle jf this triangle we obtain that these four lines are parallel, i.e. A_1 is isogonally conjugated with respect to quadrilateral $BPCQ$ to the infinite point of its Gauss line (i.e. coincide with the Miquel point of lines BP, BQ, CP и CQ). But the lines passing through A, B, C and parallel to the Gauss lines of quadrilaterals $BPCQ$, $APCQ$, $APBQ$ respectively concur at the point anticiomplimentar to the midpoint of PQ with respect to ABC. Therefore AA_1, BB_1, CC_1 concur at the isogonally conjugated point.

4. (L.Emelyanov) Prove that the sum of two nagelians is greater than the semiperimeter of the triangle. (A nagelian is the segment between a vertex of a triangle and the tangency point of the opposite side with the corresponding excircle.)

Solution. Let the incircle of ABC touch BC, CA, AB at points A', B', C' respectively, and let the correspondent excircles touch these sides at A'', B'', C''. Suppose that $\angle A \leq \angle B \leq \angle C$. Then $AA'' \geq BB'' \geq CC''$ and we have to prove that the sum of BB'' and CC'' is greater than the semiperimeter p. Let CH be the altitude of the triangle, and A_1 be the point of ray BA such that $BA_1 = p$ (fig.10.4.1). Then $AB'' = AA_1 = p - c$ and $p < BB'' + B''A_1$. Proving that $B''A_1 < CH$ we obtain that $p < BB'' + CH < BB'' + CC''$.

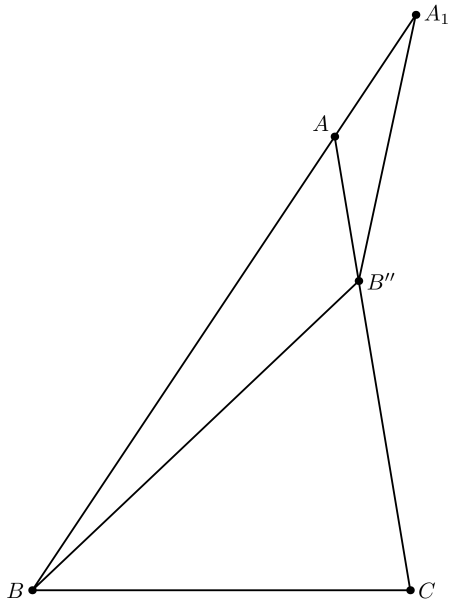


Fig. 10.4.1

Since $A_1B'' = 2(p-c)\cos\frac{\angle A}{2}$, $CH = AC\sin\angle A = 2AC\sin\frac{\angle A}{2}\cos\frac{\angle A}{2}$, we have to prove that $AC\sin\frac{\angle A}{2} > p - c$.

Let P be the projection of C to the bisector of angle A. Then P lies on segment $A'C'$ because $\angle C \geq \angle B$ (fig.10.4.2). Also $PC = AC\sin\frac{\angle A}{2}$, $A'C = p-c$ and $\angle PA'C = (\pi + \angle B)/2$, therefore CP is the greatest side of triangle $A'CP$, as needed.

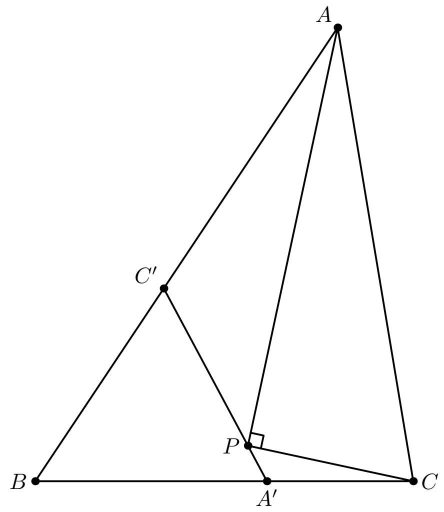


Fig. 10.4.2

XV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Solutions. Second day. 10 form

Ratmino, July 31, 2019

5. (D.Shvetsov) Let AA_1, BB_1, CC_1 be the altitudes of triangle ABC; and A_0, C_0 be the common points of the circumcircle of triangle A_1BC_1 with the lines A_1B_1 and C_1B_1 respectively. Prove that AA_0 and CC_0 meet on the median of ABC or are parallel to it.

Solution. Let lines AA_0 and BC meet at X and let lines CC_0 and AB meet at Y. It suffices to show that $BX : XC = BY : YA$. Observe that points A_0 and C_1 are symmetric with respect to line BB_1, as are points A_1 and C_0 (fig. 10.5). Let lines BA_0 and AC meet at Z. Then, by Menelaus theorem for triangle BCZ and line AA_0X, we obtain that $BX : XC = (BA_0 : A_0Z) \cdot (ZA : AC) = (2/AC) \cdot (BC_1 : C_1A) \cdot AB_1$. Similarly for $BY : YA$, and we are left to verify that $(BC_1 : C_1A) \cdot AB_1 = (BA_1 : A_1C) \cdot CB_1$. But this is just Ceva's theorem for the orthocenter.

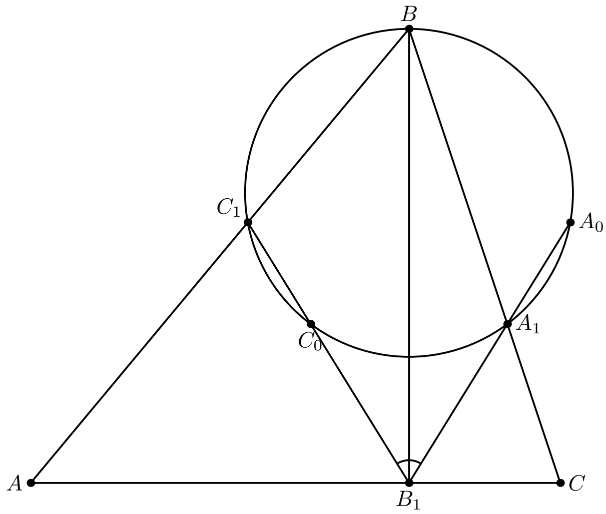


Fig. 10.5

Second solution. Since A_0, C_0 are the reflections about BH of C_1, A_1 respectively and triangles HAC_1, HCA_1 are similar we obtain that triangles HAA_0 and HCC_0 are also similar. Thus the common point of AA_0 and CC_0 coincide with the common point of circles HAC and HA_0C_0 distinct from H, i.e. with the projection of H to the median.

6. (A.Mostovoy) Let AK and AT be the bisector and the median of an acute-angled triangle ABC with $AC > AB$. The line AT meets the circumcircle of ABC at point D. Point F is the reflection of K about T. If the angles of ABC are known, find the value of angle FDA.

   First solution. Let M be the midpoint of arc BC. Then $\angle MFT = \angle MKT = \angle MKC = \alpha/2 + \gamma = \angle ACM = \angle ADM = \angle TDM$, thus quadrilateral $MDFT$ is cyclic (fig/ 10/6). Then $\angle ADF = \angle TDF = \angle TMF = \angle TMK = (\beta - \gamma)/2$.

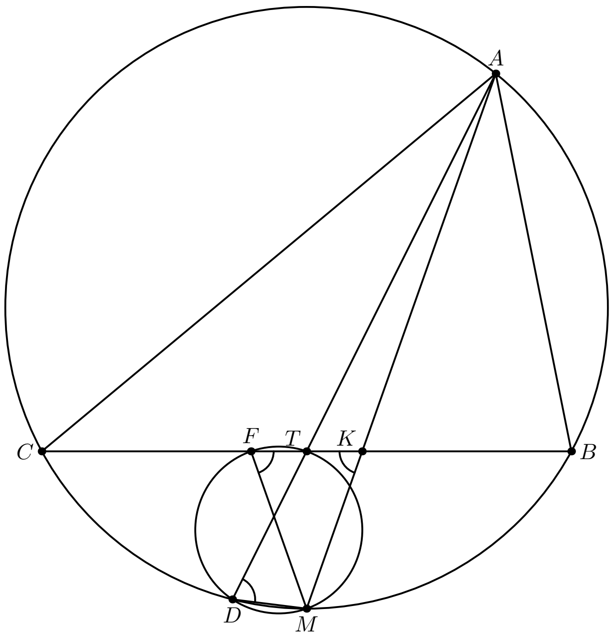


Fig. 10.6

   Second solution. Let the symmedian from A meet the circumcircle at Q. Since D and Q are symmetric about the perpendicular bisector to BC, we can replace required angle FDA by equal angle KQT. Since AK and TK are bisectors of triangle AQT, we obtain that $\angle KQT = \angle KQA/2$, but the last angle equals to $\angle B = \angle C$, because the ray QK meets the circumcircle at a point forming an isosceles trapezoid with the vertices of the given triangle.

7. (Tran Quang Hung) Let P be an arbitrary point on side BC of triangle ABC. Let K be the incenter of triangle PAB. Let the incircle of triangle PAC touch BC at F. Point G on CK is such that $FG \parallel PK$. Find the locus of G.

   Solution. Lemma. In triangle ABC, let I_B and I_C be the excenters opposite to B and C. Let the excircle opposite to B touch line BC at T and let ℓ be the line through T parallel to BI_C. Let P be any point on line BC and let line PI_C meet line ℓ at Q. Then $CQ \perp PI_B$.

Proof of the lemma. Let R be the intersection point of line PI_B and the line through T perpendicular to ℓ (and parallel to BI_B). Then $TQ : BI_C = TP : PB = TR : BI_B$, thus $TQ : TR = BI_C : BI_B = TC : TI_B$, i.e. triangles CTI_B and QTR are similar. It follows that triangles CTQ and I_BTR are similar as well. The angle of rotation of the similarity centered at T that maps one triangle onto the other equals $\angle CTI_B = \angle QTR = 90°$, so CQ is perpendicular to PI_B, as needed.

Return to the problem. Let the incircle touch sides AC and BC at X and Y respectively, and let Z be the midpoint of segment XY. We claim that the desired locus is the segment YZ.

To see this, observe first that the second common interior tangent to the incircles of triangles ABP and ACP passes through Y; this is well-known. Then apply the lemma to the triangle formed by the two common interior tangents to the incircles of triangles ABP and ACP and their common exterior tangent BC, and to point C on side PY of this triangle. We obtain that G lies on line XY (fig. 10.7). When P approaches B, G approaches Y; and when P approaches C, G approaches Z.

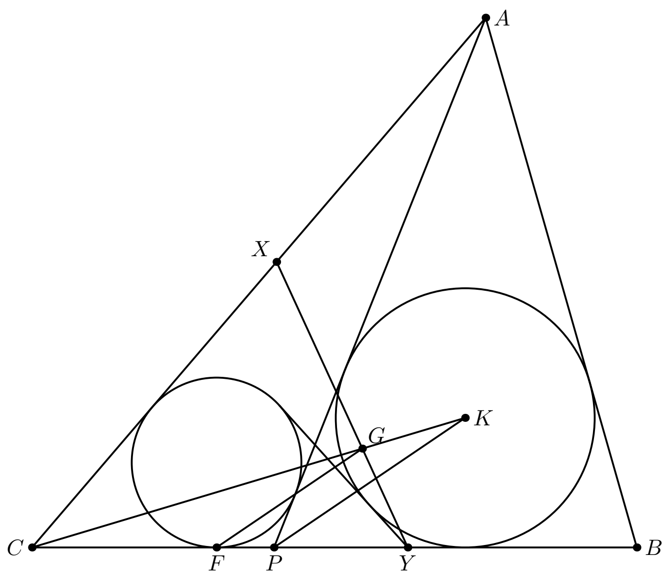


Fig. 10.7

8. (F.Nilov, Ye.Morozov) Several points and planes are given in the space. It is known that for any two of given points there exist exactly two planes

containing them, and each given plane contains at least four of given points. Is it true that all given points are collinear?

Answer. No.

Solution. Take 12 points — the midpoints of edges of cube $ABCDA'B'C'D'$, and 16 planes such that four of them pass through the center of cube and are perpendicular to its diagonals (each of these planes intersects the cube by a regular hexagon), and each of the remaining planes passes through the midpoints of four edges adjacent to the same edge of the cube (for example the midpoints of edges AB, BC, $A'B'$, and $B'C'$). It is clear that each plane contains at least four of the given points. Also for any two of the given points there exist exactly two planes containing them: the midpoints of two perpendicular edges lie on one rectangular and one hexagonal section, the midpoints of two parallel edges lying on the same face lie on two rectangular sections, and the midpoints of two opposite sections of the cube lie on two hexagonal sections.