Problems On Divisibility and Congruence

Masum Billal

February 21, 2015

These are the problems that were posed as homework problems in number theory for math campers in 2014. The problems are divided into divisibility and congruence catagory, but that does not mean that you can't use congruence to solve divisibility problems and vice-versa. And I suggest, you solve the problems here and the ones in **Problem Solving Strategies** by *Arthur Engel* simultaneously.

Divisibility Problems

1. Find all prime p such that $17p + 1$ is a prime.

2. Find all positive integers d such that it divides $n + 4$ and $n + 2014$ for some positive integer n.

3. Find all n so that $7n + 1|8n + 55$.

4. Find the maximum positive integer n such that $\dfrac{n^3 + 10}{n + 10}$ is an integer.

5. The difference of two odd numbers is divisible by 2 but not by 4. Prove that their sum is divisible by 4.

6. Decide if a number ending in some zeros followed by some 2s is a square? In explanation, the number is like 22...2200...00.

7. The sum of two positive integers is a prime. Find their greatest common divisor.

8. If n is odd, then prove that, $n^2 - 1$ is divisible by 8.

9. For all odd n, prove that, $n^2|1^3 + 2^3 + \ldots + n^3$.

Hint. You know any formula for right side of the divisibility?

10. Find the maximum value of n so that, n consecutive integers are pairwise co-prime.

11. Every prime greater than 3 can be written as $6k + 1$ or $6k - 1$.

12. Use the previous problem as a lemma to prove that $24|p^2 - 1$.

13. Two primes p, q are called *Sophie Germain Prime* if $p = 2q+1$. Prove that, p and q must be of the form $6k-1$. Hence, $6|p-q$.

14. Find all positive integers d so that d divides n^2+1 and $(n+1)^2+1$ for some positive integer n.

15. if $2n+1$ and $3n+1$ are both perfect squares, then $8|n$.

16. Prove that, $(a, bc) = (a, (a, b) \cdot c)$.

17 (IMO)**.** Prove that, the fraction $\dfrac{14n+3}{21n+4}$ is irreducible.

18. Show that, $641|2^{32}+1$. No congruence, please!

Hint. Use the fact that, $641 = 5^4 + 2^4$.

19. For all positive integers n and odd positive integer k,

$$1+2+\ldots+n|1^k+2^k+\ldots+n^k$$

20. Prove that, for all $n > 1$, n^4+4^n is composite.

21. Find all primes p so that, $p-4$ is a perfect $4-th$ power i.e. it is k^4 for some positive integer k.s

22. Find all positive integer n such that, $5n+1|n^6+n^4$.

23. Find all n such that n^5+n^4+1 is a prime.

24. $1 - \dfrac{1}{2} + \dfrac{1}{3} - ... + \dfrac{1}{1319} = \dfrac{a}{b}$ where $(a, b) = 1$. Prove that $1979|a$.

25. Find all n such that $2^n+n|8^n+n$.

26. If $\dfrac{a+1}{b} + \dfrac{b+1}{a}$ is a positive integer for natural number a, b, prove that, $a+b \geq (a, b)^2$.

27. Find all positive integers (a, b) so that, $\dfrac{a}{b} + \dfrac{b}{a}$ is a positive integer.

28. Prove that, for odd n, $n|1^n+2^n+\ldots+n^n$.

29. Is $4^{449}+625$ a prime?

30. Prove that, $341|2^{341}-2$.

31. Find all integers x, y with $12x+54y = 65464$.

32. Find all positive integer m so that $1008^m-1|2014^m-1$.

33 (IMO)**.** Find all positive integers (a, b) such that $ab^2+b+7|a^2b+a+b$.

34. Count (x^2-x+1, x^2+x+1).

35. Does there exist three distinct primes p, q, r so that $p|q+r, q|r+p, r|p+q$.

36. For a positive integer n, $\varphi(n)$ is the number of positive integers less than or equal to n and co-prime to n. For example, $\varphi(6) = 2$ because from $1, 2, 3, 4, 5, 6$ only 1 and 5 are co-prime to 6. Similarly, you can see, $\varphi(12) = 4$ since only $1, 5, 7, 11$ are less than or equal to 12 and co-prime to 12. Find all n so that $\varphi(n)$ is odd.

Hint. Look at k and $n - k$.

37. Find the general solution to $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$.

38. Find all positive integer m so that $2^{m+1} - m^2$ is a prime.

39. Prove that, if $a^m - 1 | a^n - 1$, then $m|n$.

Note. The converse is true also.

40. Find if the same is true for odd m, n i.e. $a^m + 1|a^n + 1$ implies $m|n$ for odd m, n.

41. Find all positive integers a, m, n so that, $a^m - 1|a^n + 1$.

42. Find all positive integers a, b, c such that, $2^a - 1|2^b + 2^c + 1$.

43. Is $2014^{2014^{2014}+2014} + 1$ a prime?

44. Prove that, if $a^n + 1$ is a prime, n is a power of 2.

45. Prove that, $(a^m - 1, a^n - 1) = a^{(m,n)} - 1$.

46. Find all sequence $\{a_i\}$ such that, $(a_{i+2}, a_{i+1}) = [a_i, a_{i-1}]$.

47 (Russia)**.** $a \leq b$ are positive integers. Prove that, if $a + b = (a, b) + [a, b]$, then $a|b$.

48 (Romania)**.** Find all co-prime positive integers a, b such that, $a + b|a^n + b^n$ where n is even.

49. a, b, c, d are integers. Prove that, $(a - b)(a - c)(a - d)(b - c)(b - d)(c - d)$ is divisible by 12.

50 (Generalization of previous problem)**.** For $2n$ arbitrary integers $a_1, a_2, ..., a_{2n}$, take all pairs (a_i, a_j) of them. Take all $a_i - a_j$ and multiply them. Prove that, the product is divisible by $2^{n^2-n}(2n - 1)(2n - 3)(2n - 5)$.

51. Find infinite pair of positive integers (a, b) such that $a + b^2|a^3 + b^3$.

52. Find infinite quadruple (a, b, c, d) so that $ac - bd = 1$ and they are pair-wise co-prime.

53. Prove that, if $2^n - 1$ is a prime, then n must be a prime.

54. When can $2^n - 1$ and $2^n + 1$ be prime at the same time?

55. $m = 4a + 3$ is divisible by 11. Find the remainder of a^4 upon division by 11.

56. Prove that, there exists infinite x, y such that $x + y|x^2 + y^2 + x + 1$.

57. For odd a, prove that, $2^{k+1}|a^{2^k}-1$.

Hint. Use that $a+1$ or $a-1$ is divisible by 2 and that $a^2-1=(a+1)(a-1)$.

58. Find all natural solutions to $n^a+n^b=n^c$.

Hint. Can we assume that $a \geq b$? If so, what gives us the advantage?

59. Find infinite triplets (a,b,c) so that, $a+b+c|3abc$.

60. Find all positive integer n so that, $3|n \cdot 2^n - 1$.

61. Show that, the sum of divisors of a perfect square i.e. k^2 is odd.

62. Show the same for $2k^2$.

63. Now, show that, the sum of divisor of n is even otherwise i.e. n has an odd sum of divisor if n is of the form k^2 and $2k^2$.

64. Find infinite n so that $n|3^n+1$.

Hint. Once again, use smallest prime factor trick!

65. Find infinite n so that, $n|2^n+1$.

Hint. This time there is no hint. And that is your hint. ;)

66 (IMO)**.** Find all positive integer n so that $n^2|2^n+1$.

Hint. First find the smallest prime factor of n and then use **Lifting The Exponent Lemma**. You will have to do the same again!.

Congruence Problems

67. Is $1^1+2^2+\ldots+2014^{2014}$ even or odd?

68. What is the remainder of 2^{2014} when divided by 7?

69. What is the last digit of 3^{81}? Generalize.

70. Prove that, for all integer a, $561|a^{561}-a$.

71. Without Fermat's Little Theorem, prove that, $13|a^{13}-a$ and $10|a^5-a$ for integer a.

72. Show that, a number n is divisible by 2^k if and only if the number formed by the last k digits of n is divisible by 2^k.

73. Prove that, for prime p and positive integers (a,b), $p|ab^p-ba^p$.

Hint. Can you use **Fermat's Little Theorem**?

74. Find a prime factor of $2010^{2010}-1$.

Hint. What about 2011? Is it a prime? Remind you of anything?

75. Find all n such that 7 divides 2^n-1.

76. Find all integer solutions to $15x^2 - y^2 = 1234$.

77. Find all integer solution to $x^2 + y^2 = 72435653473322$.

Note. Don't be afraid after you see such a huge number on the right side! ;)

78. Prove that, if there are integers x, y, z with $x^3 + y^3 = z^3$, then $7|xyz$.

79. Show that, 7 is a prime factor of $2222^{5555} + 5555^{2222}$.

80. Find the last 3 digits of 7^{9999}.

81. Let $a_n = 6^n + 8^n$. Find the remainder of a_{49} upon division by 49.

82. We could restate the problem as : Prove that there exists at least one positive integer n such that $n|a_n$. Though the solution would remain same, this problem will be definitely be a harder one than the previous.

83. Remember how to divide in congruence? You can't say that if $ac \equiv bc \pmod{n}$, then $a \equiv b \pmod{n}$. But you can say, $a \equiv b \pmod{\frac{n}{(n,c)}}$. Prove it.

84. Prove that, for prime $p > 3$, $p|(p-2)! - 1$.

Hint. Do you remember **Wilson's Theorem**?

85. Find all n such that 44...44(n times is a perfect square.

86. Find all positive integer solution to $2^n - 1 = 3^m$.

87. Do the same for $2^n + 1 = 3^m$.

Hint. Use that, $2^n \equiv 0 \pmod{4}$ for $n \geq 2$ and $2^n \equiv 0 \pmod{8}$ for $n \geq 3$. Also, use $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + b^{n-1}$.

88. Prove that, if p and $8p^2 + 1$ both are primes, then $8p^2 + 1$ is a prime too.

89. Find all primes p so that $2^p + p^2$ is a prime.

90. Find all primes p such that $p + 2$ and $p + 4$ are prime also.

Hint. Consider the numbers $\pmod{3}$.

91. Find all primes p so that $p + 2i$ is a prime for all $0 \leq i < p$.

Hint. Same trick. But this time, you have to be careful!

92. Prove that the sum of squares 3 consecutive integers is not a square.

93. Prove that the sum of squares 5 consecutive integers is not a square.

Hint. For both problems, use something like $x^2 \equiv 0, 1 \pmod{3}$. In other words, it is not possible to find $x^2 \equiv 2 \pmod{3}$.

94. Show that, if $a^x \equiv b^x \pmod{n}$ and $a^y \equiv b^y \pmod{n}$, then $a^{(x,y)} \equiv b^{(x,y)} \pmod{n}$.

Hint. Can you use *Bezout's Identity*? Or Euclidean Algorithm?

95. Find all positive integer n such that $n|2^n - 1$.

Hint. Remember the smallest prime factor trick?

96. Find all positive integer solutions to $3^x - 2^y = 7$.

97. Find all primes p, q such that $pq|(5^q - 2^q)(5^p - 2^p)$.

Hint. Take one prime at once, say p. And also, assume $p \geq q$. Then $p|(5^p - 2^p)(5^q - 2^q)$. And make two cases. $p|5^p - 2^p$ and $p|5^q - 2^q$.

98. Prove that, $77|36^{36} + 41^{41}$.

99. Find all integer x so that, $x^{24} + 7x \equiv 2 \pmod{13}$.

Hint. The number 24 rings a bell?

100. Prove that every prime factor of $a^2 + b^2$ is of the form $4k + 1$ where a is co-prime to b.

Hint. Use Fermat's Little Theorem to show that $4|p - 1$.

101. Use the previous problem to show that if p is of the form $4k + 3$ and $p|a^2 + b^2$, then $p|a, p|b$.

102 (Iran)**.** Find all positive integer x, y so that $4xy - x - y$ is a square.

Hint. Use the previous two problems.

103 (IMO)**.** Let A be the sum of the digits of the number 4444^{4444} and B the sum of the digits of the number A. Find the sum of the digits of the number B.