Balkan MO 2004

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1 The sequence $\{a_n\}_{n\geq 0}$ of real numbers satisfies the relation:

$$a_{m+n}+a_{m-n}-m+n-1=\frac{1}{2}(a_{2m}+a_{2n})$$

for all non-negative integers m and n, $m\geq n$. If $a_1=3$ find a_{2004}.

2 Solve in prime numbers the equation $x^y-y^x=xy^2-19$.

3 Let O be an interior point of an acute triangle ABC. The circles with centers the midpoints of its sides and passing through O mutually intersect the second time at the points K, L and M different from O. Prove that O is the incenter of the triangle KLM if and only if O is the circumcenter of the triangle ABC.

4 The plane is partitioned into regions by a finite number of lines no three of which are concurrent. Two regions are called "neighbors" if the intersection of their boundaries is a segment, or half-line or a line (a point is not a segment). An integer is to be assigned to each region in such a way that:

i) the product of the integers assigned to any two neighbors is less than their sum;
ii) for each of the given lines, and each of the half-planes determined by it, the sum of the integers, assigned to all of the regions lying on this half-plane equal to zero.

Prove that this is possible if and only if not all of the lines are parallel.

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