All-Russian Olympiad 2010

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| – | Grade level 9 |

**Day 1**

**1** There are 24 different pencils, 4 different colors, and 6 pencils of each color. They were given to 6 children in such a way that each got 4 pencils. What is the least number of children that you can randomly choose so that you can guarantee that you have pencils of all colors.

P.S. for 10 grade gives same problem with 40 pencils, 10 of each color and 10 children.

**2** There are 100 random, distinct real numbers corresponding to 100 points on a circle. Prove that you can always choose 4 consecutive points in such a way that the sum of the two numbers corresponding to the points on the outside is always greater than the sum of the two numbers corresponding to the two points on the inside.

**3** Lines tangent to circle $O$ in points $A$ and $B$, intersect in point $P$. Point $Z$ is the center of $O$. On the minor arc $AB$, point $C$ is chosen not on the midpoint of the arc. Lines $AC$ and $PB$ intersect at point $D$. Lines $BC$ and $AP$ intersect at point $E$. Prove that the circumcentres of triangles $ACE$, $BCD$, and $PCZ$ are collinear.

**4** There are 100 apples on the table with total weight of 10 kg. Each apple weighs no less than 25 grams. The apples need to be cut for 100 children so that each of the children gets 100 grams. Prove that you can do it in such a way that each piece weighs no less than 25 grams.

**Day 2**

**1** Let $a \neq ba, b \in \mathbb{R}$ such that $(x^2 + 20ax + 10b)(x^2 + 20bx + 10a) = 0$ has no roots for $x$. Prove that $20(b - a)$ is not an integer.

**2** Each of 1000 elves has a hat, red on the inside and blue on the outside or vise versa. An elf with a hat that is red outside can only lie, and an elf with a hat that is blue outside can only tell the truth. One day every elf tells every other elf, Your hat is red on the outside. During that day, some of the elves turn their hats inside out at any time during the day. (An elf can do that more than once per day.) Find the smallest possible number of times any hat is turned inside out.

**3** Let us call a natural number *unlucky* if it cannot be expressed as $\frac{x^2-1}{y^2-1}$ with natural numbers $x, y > 1$. Is the number of *unlucky* numbers finite or infinite?

**4** In a acute triangle $ABC$, the median, $AM$, is longer than side $AB$. Prove that you can cut triangle $ABC$ into 3 parts out of which you can construct a rhombus.

– Grade level 10

**Day 1**

**1** There are 24 different pencils, 4 different colors, and 6 pencils of each color. They were given to 6 children in such a way that each got 4 pencils. What is the least number of children that you can randomly choose so that you can guarantee that you have pencils of all colors.

P.S. for 10 grade gives same problem with 40 pencils, 10 of each color and 10 children.

**2** There are 100 random, distinct real numbers corresponding to 100 points on a circle. Prove that you can always choose 4 consecutive points in such a way that the sum of the two numbers corresponding to the points on the outside is always greater than the sum of the two numbers corresponding to the two points on the inside.

**3** Let $O$ be the circumcentre of the acute non-isosceles triangle $ABC$. Let $P$ and $Q$ be points on the altitude $AD$ such that $OP$ and $OQ$ are perpendicular to $AB$ and $AC$ respectively. Let $M$ be the midpoint of $BC$ and $S$ be the circumcentre of triangle $OPQ$. Prove that $\angle BAS = \angle CAM$.

**4** In each unit square of square $100 * 100$ write any natural number. Called rectangle with sides parallel sides of square *good* if sum of number inside rectangle divided by 17. We can painted all unit squares in *good* rectangle. One unit square cannot painted twice or more.
Find maximum $d$ for which we can guaranteed paint at least $d$ points.

**Day 2**

**1** Let $a \neq ba, b \in \mathbb{R}$ such that $(x^2+20ax+10b)(x^2+20bx+10a)=0$ has no roots for $x$. Prove that $20(b-a)$ is not an integer.

**2** Into triangle $ABC$ gives point $K$ lies on bisector of $\angle BAC$. Line $CK$ intersect circumcircle $\omega$ of triangle $ABC$ at $M \neq C$. Circle $\Omega$ passes through $A$, touch $CM$ at $K$ and intersect segment $AB$ at $P \neq A$ and $\omega$ at $Q \neq A$.
Prove, that $P$, $Q$, $M$ lies at one line.

**3** Given $n \geq 3$ pairwise different prime numbers $p_1, p_2, ...., p_n$. Given, that for any $k \in \{1, 2, ...., n\}$ residue by division of $\prod_{i \neq k} p_i$ by $p_k$ equals one number $r$. Prove, that $r \leq n-2$.

**4** In the county some pairs of towns connected by two-way non-stop flight. From any town we can flight to any other (may be not on one flight). Gives, that if we consider any cyclic (i.e. beginning and finish towns match) route, consisting odd number of flights, and close all flights of this route, then we can found two towns, such that we can't fly from one to other.
Proved, that we can divided all country on 4 regions, such that any flight connected towns from other regions.

– Grade level 11

**Day 1**

**1** Do there exist non-zero reals numbers $a_1, a_2, ....., a_{10}$ for which

$$(a_1+\frac{1}{a_1})(a_2+\frac{1}{a_2})\cdots(a_{10}+\frac{1}{a_{10}})=(a_1-\frac{1}{a_1})(a_2-\frac{1}{a_2})\cdots(a_{10}-\frac{1}{a_{10}})\ ?$$

**2** On an $n \times n$ chart, where $n \geq 4$, stand "+" signs in the cells of the main diagonal and "$-$" signs in all the other cells. You can change all the signs in one row or in one column, from $-$ to $+$ or from $+$ to $-$. Prove that you will always have $n$ or more $+$ signs after finitely many operations.

**3** Quadrilateral $ABCD$ is inscribed into circle $\omega$, $AC$ intersect $BD$ in point $K$. Points $M_1$, $M_2$, $M_3$, $M_4$-midpoints of arcs $AB$, $BC$, $CD$, and $DA$ respectively. Points $I_1$, $I_2$, $I_3$, $I_4$-incenters of triangles $ABK$, $BCK$, $CDK$, and $DAK$ respectively. Prove that lines $M_1I_1$, $M_2I_2$, $M_3I_3$, and $M_4I_4$ all intersect in one point.

**4** Given is a natural number $n \geq 3$. What is the smallest possible value of $k$ if the following statements are true?
For every $n$ points $A_i = (x_i, y_i)$ on a plane, where no three points are collinear, and for any real numbers $c_i$ ($1 \leq i \leq n$) there exists such polynomial $P(x, y)$, the degree of which is no more than $k$, where $P(x_i, y_i) = c_i$ for every $i = 1, \dots, n$.

(The degree of a nonzero monomial $a_{i,j}x^iy^j$ is $i + j$, while the degree of polynomial $P(x, y)$ is the greatest degree of the degrees of its monomials.)

**Day 2**

**1** If $n \in \mathbb{N} n > 1$ prove that for every $n$ you can find $n$ consecutive natural numbers the product of which is divisible by all primes not exceeding $2n + 1$, but is not divisible by any other primes.

**2** Could the four centers of the circles inscribed into the faces of a tetrahedron be coplanar?

(vertexes of tetrahedron not coplanar)

**3** Polynomial $P(x)$ with degree $n \geq 3$ has $n$ real roots $x_1 < x_2 < x_3 < ... < x_n$, such that $x_2 - x_1 < x_3 - x_2 < .... < x_n - x_{n-1}$. Prove that the maximum of the function $y = |P(x)|$ where $x$ is on the interval $[x_1, x_n]$, is in the interval $[x_n - 1, x_n]$.

**4** In a board school, there are 9 subjects, 512 students, and 256 rooms (two people in each room.) For every student there is a set (a subset of the 9 subjects) of subjects the student is interested in. Each student has a different set of subjects, (s)he is interested in, from all other students. (Exactly one student has no subjects (s)he is interested in.)
Prove that the whole school can line up in a circle in such a way that every pair of the roommates has the two people standing next to each other, and those pairs of students standing next to each other that are not roommates, have the following properties. One of the two students is interested in all the subjects that the other student is interested in, and also exactly one more subject.