USA Team Selection Test 2004

**Day 1**

**1** Suppose $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers such that

$$(a_1^2 + a_2^2 + \cdots + a_n^2 - 1)(b_1^2 + b_2^2 + \cdots + b_n^2 - 1) > (a_1b_1 + a_2b_2 + \cdots + a_nb_n - 1)^2.$$

Prove that $a_1^2 + a_2^2 + \cdots + a_n^2 > 1$ and $b_1^2 + b_2^2 + \cdots + b_n^2 > 1$.

**2** Assume $n$ is a positive integer. Considers sequences $a_0, a_1, \ldots, a_n$ for which $a_i \in \{1, 2, \ldots, n\}$ for all $i$ and $a_n = a_0$.

(a) Suppose $n$ is odd. Find the number of such sequences if $a_i - a_{i-1} \not\equiv i \pmod{n}$ for all $i = 1, 2, \ldots, n$.

(b) Suppose $n$ is an odd prime. Find the number of such sequences if $a_i - a_{i-1} \not\equiv i, 2i \pmod{n}$ for all $i = 1, 2, \ldots, n$.

**3** Draw a $2004 \times 2004$ array of points. What is the largest integer $n$ for which it is possible to draw a convex $n$-gon whose vertices are chosen from the points in the array?

**Day 2**

**4** Let $ABC$ be a triangle. Choose a point $D$ in its interior. Let $\omega_1$ be a circle passing through $B$ and $D$ and $\omega_2$ be a circle passing through $C$ and $D$ so that the other point of intersection of the two circles lies on $AD$. Let $\omega_1$ and $\omega_2$ intersect side $BC$ at $E$ and $F$, respectively. Denote by $X$ the intersection of $DF$, $AB$ and $Y$ the intersection of $DE, AC$. Show that $XY \parallel BC$.

**5** Let $A = (0, 0, 0)$ in 3D space. Define the *weight* of a point as the sum of the absolute values of the coordinates. Call a point a *primitive lattice point* if all of its coordinates are integers whose gcd is 1. Let square $ABCD$ be an *unbalanced primitive integer square* if it has integer side length and also, $B$ and $D$ are primitive lattice points with different weights. Prove that there are infinitely many unbalanced primitive integer squares such that the planes containing the squares are not parallel to each other.

**6** Define the function $f : \mathbb{N} \cup \{0\} \to \mathbb{Q}$ as follows: $f(0) = 0$ and

$$f(3n+k) = -\frac{3f(n)}{2} + k,$$

for $k = 0, 1, 2$. Show that $f$ is one-to-one and determine the range of $f$.