USAMO 1996

Day 1	May 2nd	

1 Prove that the average of the numbers $n \sin n^\circ$ $(n = 2, 4, 6, \ldots, 180)$ is $\cot 1^\circ$.

2 For any nonempty set S of real numbers, let $\sigma(S)$ denote the sum of the elements of S. Given a set A of n positive integers, consider the collection of all distinct sums $\sigma(S)$ as S ranges over the nonempty subsets of A. Prove that this collection of sums can be partitioned into n classes so that in each class, the ratio of the largest sum to the smallest sum does not exceed 2.

3 Let ABC be a triangle. Prove that there is a line ℓ (in the plane of triangle ABC) such that the intersection of the interior of triangle ABC and the interior of its reflection $A'B'C'$ in ℓ has area more than $\frac{2}{3}$ the area of triangle ABC.

Day 2 May 2nd

4 An n-term sequence $(x_1, x_2, \ldots, x_n)$ in which each term is either 0 or 1 is called a *binary sequence of length n*. Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to 0, 1, 0 in that order. Let b_n be the number of binary sequences of length n that contain no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that $b_{n+1} = 2a_n$ for all positive integers n.

5 Let ABC be a triangle, and M an interior point such that $\angle MAB = 10^\circ$, $\angle MBA = 20^\circ$, $\angle MAC = 40^\circ$ and $\angle MCA = 30^\circ$. Prove that the triangle is isosceles.

6 Determine (with proof) whether there is a subset X of the integers with the following property: for any integer n there is exactly one solution of $a + 2b = n$ with $a, b \in X$.

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