2-nd Czech–Polish–Slovak Match 2002

Zwardoń, Poland
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1. Let a, b be distinct real numbers and k, m be positive integers $k + m = n \geq 3$, $k \leq 2m$, $m \leq 2k$. Consider sequences $x_1, \dots, x_n$ with the following properties:

   (i) k terms x_i, including x_1, are equal to a;

   (ii) m terms x_i, including x_n, are equal to b;

   (iii) no three consecutive terms are equal.

   Find all possible values of $x_n x_1 x_2 + x_1 x_2 x_3 + \cdots + x_{n-1} x_n x_1$.

2. A triangle ABC has sides $BC = a$, $CA = b$, $AB = c$ with $a < b < c$ and area S. Determine the largest number u and the least number v such that, for every point P inside $\triangle ABC$, the inequality $u \leq PD + PE + PF \leq v$ holds, where D, E, F are the intersection points of AP, BP, CP with the opposite sides.

3. Let $S = \{1, 2, \dots, n\}$, $n \in \mathbb{N}$. Find the number of functions $f : S \to S$ with the property that $x + f(f(f(f(x)))) = n + 1$ for all $x \in S$?

4. An integer $n > 1$ and a prime p are such that n divides $p - 1$, and p divides $n^3 - 1$. Prove that $4p - 3$ is a perfect square.

5. In an acute-angled triangle ABC with circumcenter O, points P and Q are taken on sides AC and BC respectively such that $\frac{AP}{PQ} = \frac{BC}{AB}$ and $\frac{BQ}{PQ} = \frac{AC}{AB}$. Prove that the points O, P, Q, C lie on a circle.

6. Let $n \geq 2$ be a fixed even integer. We consider polynomials of the form

   $$P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + 1$$

   with real coefficients, having at least one real roots. Find the least possible value of $a_1^2 + a_2^2 + \cdots + a_{n-1}^2$.

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