IMO Shortlist 1990

1 The integer 9 can be written as a sum of two consecutive integers: $9 = 4 + 5$. Moreover, it can be written as a sum of (more than one) consecutive positive integers in exactly two ways: $9 = 4 + 5 = 2 + 3 + 4$. Is there an integer that can be written as a sum of 1990 consecutive integers and that can be written as a sum of (more than one) consecutive positive integers in exactly 1990 ways?

2 Given n countries with three representatives each, m committees $A(1), A(2), \dots, A(m)$ are called a cycle if

(i) each committee has n members, one from each country;
(ii) no two committees have the same membership;
(iii) for $i = 1, 2, \dots, m$, committee $A(i)$ and committee $A(i + 1)$ have no member in common, where $A(m + 1)$ denotes $A(1)$;
(iv) if $1 < |i - j| < m - 1$, then committees $A(i)$ and $A(j)$ have at least one member in common.

Is it possible to have a cycle of 1990 committees with 11 countries?

3 Let $n \geq 3$ and consider a set E of $2n - 1$ distinct points on a circle. Suppose that exactly k of these points are to be colored black. Such a coloring is **good** if there is at least one pair of black points such that the interior of one of the arcs between them contains exactly n points from E. Find the smallest value of k so that every such coloring of k points of E is good.

4 Assume that the set of all positive integers is decomposed into r (disjoint) subsets $A_1 \cup A_2 \cup \ldots \cup A_r = \mathbb{N}$. Prove that one of them, say A_i, has the following property: There exists a positive m such that for any k one can find numbers $a_1, a_2, \dots, a_k$ in A_i with $0 < a_{j+1} - a_j \leq m$, $(1 \leq j \leq k - 1)$.

5 Given a triangle ABC. Let G, I, H be the centroid, the incenter and the orthocenter of triangle ABC, respectively. Prove that $\angle GIH > 90°$.

6 Given an initial integer $n_0 > 1$, two players, $\mathcal{A}$ and $\mathcal{B}$, choose integers n_1, n_2, n_3, $\ldots$ alternately according to the following rules :

I.) Knowing n_{2k}, $\mathcal{A}$ chooses any integer n_{2k+1} such that

$$n_{2k} \leq n_{2k+1} \leq n_{2k}^2.$$

II.) Knowing n_{2k+1}, $\mathcal{B}$ chooses any integer n_{2k+2} such that

$$\frac{n_{2k+1}}{n_{2k+2}}$$

is a prime raised to a positive integer power.

Player $\mathcal{A}$ wins the game by choosing the number 1990; player $\mathcal{B}$ wins by choosing the number 1. For which n_0 does :

a.) $\mathcal{A}$ have a winning strategy?
b.) $\mathcal{B}$ have a winning strategy?
c.) Neither player have a winning strategy?

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7 Let $f(0) = f(1) = 0$ and

$$f(n+2) = 4^{n+2} \cdot f(n+1) - 16^{n+1} \cdot f(n) + n \cdot 2^{n^2}, \quad n = 0, 1, 2, \ldots$$

Show that the numbers $f(1989), f(1990), f(1991)$ are divisible by 13.

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8 For a given positive integer k denote the square of the sum of its digits by $f_1(k)$ and let $f_{n+1}(k) = f_1(f_n(k))$. Determine the value of $f_{1991}(2^{1990})$.

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9 The incenter of the triangle ABC is K. The midpoint of AB is C_1 and that of AC is B_1. The lines C_1K and AC meet at B_2, the lines B_1K and AB at C_2. If the areas of the triangles AB_2C_2 and ABC are equal, what is the measure of angle $\angle CAB$?

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10 A plane cuts a right circular cone of volume V into two parts. The plane is tangent to the circumference of the base of the cone and passes through the midpoint of the altitude. Find the volume of the smaller part.

Original formulation:

A plane cuts a right circular cone into two parts. The plane is tangent to the circumference of the base of the cone and passes through the midpoint of the altitude. Find the ratio of the volume of the smaller part to the volume of the whole cone.

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11 Chords AB and CD of a circle intersect at a point E inside the circle. Let M be an interior point of the segment EB. The tangent line at E to the circle through D, E, and M intersects the lines BC and AC at F and G, respectively. If

$$\frac{AM}{AB} = t,$$

find $\frac{EG}{EF}$ in terms of t.

12 Let ABC be a triangle, and let the angle bisectors of its angles CAB and ABC meet the sides BC and CA at the points D and F, respectively. The lines AD and BF meet the line through the point C parallel to AB at the points E and G respectively, and we have $FG = DE$. Prove that $CA = CB$.

Original formulation:

Let ABC be a triangle and L the line through C parallel to the side AB. Let the internal bisector of the angle at A meet the side BC at D and the line L at E and let the internal bisector of the angle at B meet the side AC at F and the line L at G. If $GF = DE$, prove that $AC = BC$.

13 An eccentric mathematician has a ladder with n rungs that he always ascends and descends in the following way: When he ascends, each step he takes covers a rungs of the ladder, and when he descends, each step he takes covers b rungs of the ladder, where a and b are fixed positive integers. By a sequence of ascending and descending steps he can climb from ground level to the top rung of the ladder and come back down to ground level again. Find, with proof, the minimum value of n, expressed in terms of a and b.

14 In the coordinate plane a rectangle with vertices $(0,0)$, $(m,0)$, $(0,n)$, (m,n) is given where both m and n are odd integers. The rectangle is partitioned into triangles in such a way that

(i) each triangle in the partition has at least one side (to be called a good side) that lies on a line of the form $x = j$ or $y = k$, where j and k are integers, and the altitude on this side has length 1;

(ii) each bad side (i.e., a side of any triangle in the partition that is not a good one) is a common side of two triangles in the partition.

Prove that there exist at least two triangles in the partition each of which has two good sides.

15 Determine for which positive integers k the set

$$X = \{1990, 1990+1, 1990+2, \ldots, 1990+k\}$$

can be partitioned into two disjoint subsets A and B such that the sum of the elements of A is equal to the sum of the elements of B.

16 Prove that there exists a convex 1990-gon with the following two properties :

a.) All angles are equal.
b.) The lengths of the 1990 sides are the numbers 1^2, 2^2, 3^2, $\cdots$, 1990^2 in some order.

17 Unit cubes are made into beads by drilling a hole through them along a diagonal. The beads are put on a string in such a way that they can move freely in space under the restriction that the vertices of two neighboring cubes are touching. Let A be the beginning vertex and B be the end vertex. Let there be $p \times q \times r$ cubes on the string ($p, q, r \geq 1$).

(a) Determine for which values of p, q, and r it is possible to build a block with dimensions p, q, and r. Give reasons for your answers.
(b) The same question as (a) with the extra condition that $A = B$.

18 Let $a, b \in \mathbb{N}$ with $1 \leq a \leq b$, and $M = \left[\frac{a+b}{2}\right]$. Define a function $f : \mathbb{Z} \mapsto \mathbb{Z}$ by

$$f(n) = \begin{cases} n+a, & \text{if } n \leq M, \\ n-b, & \text{if } n \geq M. \end{cases}$$

Let $f^1(n) = f(n)$, $f_{i+1}(n) = f(f^i(n))$, $i = 1, 2, \ldots$ Find the smallest natural number k such that $f^k(0) = 0$.

19 Let P be a point inside a regular tetrahedron T of unit volume. The four planes passing through P and parallel to the faces of T partition T into 14 pieces. Let $f(P)$ be the joint volume of those pieces that are neither a tetrahedron nor a parallelepiped (i.e., pieces adjacent to an edge but not to a vertex). Find the exact bounds for $f(P)$ as P varies over T.

20 Prove that every integer k greater than 1 has a multiple that is less than k^4 and can be written in the decimal system with at most four different digits.

21 Let n be a composite natural number and p a proper divisor of n. Find the binary representation of the smallest natural number N such that

$$\frac{(1+2^p+2^{n-p})N-1}{2^n}$$

is an integer.

22 Ten localities are served by two international airlines such that there exists a direct service (without stops) between any two of these localities and all airline schedules offer round-trip service between the cities they serve. Prove that at least one of the airlines can offer two disjoint round trips each containing an odd number of landings.

23 Determine all integers $n > 1$ such that

$$\frac{2^n+1}{n^2}$$

is an integer.

24 Let w, x, y, z are non-negative reals such that $wx + xy + yz + zw = 1$.
Show that $\frac{w^3}{x+y+z} + \frac{x^3}{w+y+z} + \frac{y^3}{w+x+z} + \frac{z^3}{w+x+y} \geq \frac{1}{3}$.

25 Let $\mathbb{Q}^+$ be the set of positive rational numbers. Construct a function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that

$$f(xf(y)) = \frac{f(x)}{y}$$

for all x, y in $\mathbb{Q}^+$.

26 Let $p(x)$ be a cubic polynomial with rational coefficients. q_1, q_2, q_3, ... is a sequence of rationals such that $q_n = p(q_{n+1})$ for all positive n. Show that for some k, we have $q_{n+k} = q_n$ for all positive n.

27 Find all natural numbers n for which every natural number whose decimal representation has $n-1$ digits 1 and one digit 7 is prime.

28 Prove that on the coordinate plane it is impossible to draw a closed broken line such that

(i) the coordinates of each vertex are rational;

(ii) the length each of its edges is 1;
(iii) the line has an odd number of vertices.