Balkan MO 2012

– April 28th

1 Let A, B and C be points lying on a circle Γ with centre O. Assume that $\angle ABC > 90$. Let D be the point of intersection of the line AB with the line perpendicular to AC at C. Let l be the line through D which is perpendicular to AO. Let E be the point of intersection of l with the line AC, and let F be the point of intersection of Γ with l that lies between D and E.
Prove that the circumcircles of triangles BFE and CFD are tangent at F.

2 Prove that

$$\sum_{cyc}(x+y)\sqrt{(z+x)(z+y)} \geq 4(xy+yz+zx),$$

for all positive real numbers x, y and z.

3 Let n be a positive integer. Let $P_n = \{2^n, 2^{n-1}\cdot 3, 2^{n-2}\cdot 3^2, \dots, 3^n\}$. For each subset X of P_n, we write S_X for the sum of all elements of X, with the convention that $S_\emptyset = 0$ where $\emptyset$ is the empty set. Suppose that y is a real number with $0 \leq y \leq 3^{n+1} - 2^{n+1}$.
Prove that there is a subset Y of P_n such that $0 \leq y - S_Y < 2^n$

4 Let $\mathbb{Z}^+$ be the set of positive integers. Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that the following conditions both hold:
(i) $f(n!) = f(n)!$ for every positive integer n,
(ii) $m - n$ divides $f(m) - f(n)$ whenever m and n are different positive integers.