National Math Olympiad (Second Round) 2001

Day 1

1 Let n be a positive integer and p be a prime number such that $np+1$ is a perfect square. Prove that $n+1$ can be written as the sum of p perfect squares.

2 Let ABC be an acute triangle. We draw 3 triangles $B'AC, C'AB, A'BC$ on the sides of ΔABC at the out sides such that:

$$\angle B'AC = \angle C'BA = \angle A'BC = 30^\circ \quad , \quad \angle B'CA = \angle C'AB = \angle A'CB = 60^\circ$$

If M is the midpoint of side BC, prove that $B'M$ is perpendicular to $A'C'$.

3 Find all positive integers n such that we can put n equal squares on the plane that their sides are horizontal and vertical and the shape after putting the squares has at least 3 axises.

Day 2

1 Find all polynomials P with real coefficients such that $\forall x \in \mathbb{R}$ we have:

$$P(2P(x)) = 2P(P(x)) + 2(P(x))^2.$$

2 In triangle ABC, $AB > AC$. The bisectors of $\angle B, \angle C$ intersect the sides AC, AB at P, Q, respectively. Let I be the incenter of ΔABC. Suppose that $IP = IQ$. How much isthe value of $\angle A$?

3 Suppose a table with one row and infinite columns. We call each 1×1 square a *room*. Let the table be finite from left. We number the rooms from left to ∞. We have put in some rooms some coins (A room can have more than one coin.). We can do 2 below operations: a) If in 2 adjacent rooms, there are some coins, we can move one coin from the left room 2 rooms to right and delete one room from the right room. b) If a room whose number is 3 or more has more than 1 coin, we can move one of its coins 1 room to right and move one other coin 2 rooms to left.

i) Prove that for any initial configuration of the coins, after a finite number of movements, we cannot do anything more. ii) Suppose that there is exactly one coin in each room from 1 to n. Prove that by doing the allowed operations, we cannot put any coins in the room $n+2$ or the righter rooms.