India
International Mathematical Olympiad Training Camp
2011

Day 1

$\boxed{1}$ Let ABC be a triangle each of whose angles is greater than 30°. Suppose a circle centered with P cuts segments BC in $T, Q; CA$ in K, L and AB in M, N such that they are on a circle in counterclockwise direction in that order.Suppose further PQK, PLM, PNT are equilateral. Prove that:

$a)$ The radius of the circle is $\frac{2abc}{a^2+b^2+c^2+4\sqrt{3}S}$ where S is area.

$b) a \cdot AP = b \cdot BP = c \cdot PC$.

$\boxed{2}$ Let the real numbers a, b, c, d satisfy the relations $a+b+c+d = 6$ and $a^2+b^2+c^2+d^2 = 12$. Prove that

$$36 \leq 4\left(a^3+b^3+c^3+d^3\right) - \left(a^4+b^4+c^4+d^4\right) \leq 48.$$

Proposed by Nazar Serdyuk, Ukraine

$\boxed{3}$ A set of n distinct integer weights $w_1, w_2, \ldots, w_n$ is said to be *balanced* if after removing any one of weights, the remaining $(n-1)$ weights can be split into two subcollections (not necessarily with equal size)with equal sum.

$a)$ Prove that if there exist *balanced* sets of sizes k, j then also a *balanced* set of size $k+j-1$.
$b)$ Prove that for all *odd* $n \geq 7$ there exist a *balanced* set of size n.

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Day 2

$\boxed{1}$ Find all positive integer n satisfying the conditions

$a)n^2 = (a+1)^3 - a^3$

$b)2n + 119$ is a perfect square.

$\boxed{2}$ Suppose $a_1, \ldots, a_n$ are non-integral real numbers for $n \geq 2$ such that ${a_1}^k + \ldots + {a_n}^k$ is an integer for all integers $1 \leq k \leq n$. Prove that none of $a_1, \ldots, a_n$ is rational.

$\boxed{3}$ Let T be a non-empty finite subset of positive integers ≥ 1. A subset S of T is called **good** if for every integer $t \in T$ there exists an s in S such that $gcd(t, s) > 1$. Let

$$A = (X, Y) \mid X \subseteq T, Y \subseteq T, gcd(x, y) = 1 \text{for all} x \in X, y \in Y$$

Prove that : $a)$ If X_0 is not **good** then the number of pairs (X_0, Y) in A is **even**. $b)$ the number of good subsets of T is **odd**.

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Day 3

1. Let $ABCDE$ be a convex pentagon such that $BC \parallel AE$, $AB = BC + AE$, and $\angle ABC = \angle CDE$. Let M be the midpoint of CE, and let O be the circumcenter of triangle BCD. Given that $\angle DMO = 90^\circ$, prove that $2\angle BDA = \angle CDE$.

   Proposed by Nazar Serdyuk, Ukraine

2. Prove that for no integer n is $n^7 + 7$ a perfect square.

3. Consider a $n \times n$ square grid which is divided into n^2 unit squares(think of a chess-board). The set of all unit squares intersecting the main diagonal of the square or lying under it is called an n-staircase. Find the number of ways in which an n-stair case can be partitioned into several rectangles, with sides along the grid lines, having mutually distinct areas.

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Day 4

1. Let ABC be an acute-angled triangle. Let AD, BE, CF be internal bisectors with D, E, F on BC, CA, AB respectively. Prove that
$$\frac{EF}{BC} + \frac{FD}{CA} + \frac{DE}{AB} \geq 1 + \frac{r}{R}$$

2. Find all pairs (m, n) of nonnegative integers for which
$$m^2 + 2 \cdot 3^n = m\left(2^{n+1} - 1\right).$$
Proposed by Angelo Di Pasquale, Australia

3. Let $\{a_0, a_1, \ldots\}$ and $\{b_0, b_1, \ldots\}$ be two infinite sequences of integers such that
$$(a_n - a_{n-1})(a_n - a_{n-2}) + (b_n - b_{n-1})(b_n - b_{n-2}) = 0$$
for all integers $n \geq 2$. Prove that there exists a positive integer k such that
$$a_{k+2011} = a_{k+2011^{2011}}.$$