USA Team Selection Test 2005

Day 1

1 Let n be an integer greater than 1. For a positive integer m, let $S_m = \{1, 2, \ldots, mn\}$. Suppose that there exists a $2n$-element set T such that
(a) each element of T is an m-element subset of S_m;
(b) each pair of elements of T shares at most one common element;
and
(c) each element of S_m is contained in exactly two elements of T.

Determine the maximum possible value of m in terms of n.

2 Let $A_1A_2A_3$ be an acute triangle, and let O and H be its circumcenter and orthocenter, respectively. For $1 \le i \le 3$, points P_i and Q_i lie on lines OA_i and $A_{i+1}A_{i+2}$ (where $A_{i+3} = A_i$), respectively, such that OP_iHQ_i is a parallelogram. Prove that

$$\frac{OQ_1}{OP_1} + \frac{OQ_2}{OP_2} + \frac{OQ_3}{OP_3} \ge 3.$$

3 We choose random a unitary polynomial of degree n and coefficients in the set $1, 2, ..., n!$. Prove that the probability for this polynomial to be special is between 0.71 and 0.75, where a polynomial g is called special if for every $k > 1$ in the sequence $f(1), f(2), f(3), ...$ there are infinitely many numbers relatively prime with k.

Day 2

4 Consider the polynomials

$$f(x) = \sum_{k=1}^{n} a_k x^k \quad \text{and} \quad g(x) = \sum_{k=1}^{n} \frac{a_k}{2^k - 1} x^k,$$

where $a_1, a_2, \ldots, a_n$ are real numbers and n is a positive integer. Show that if 1 and 2^{n+1} are zeros of g then f has a positive zero less than 2^n.

5 Find all finite sets S of points in the plane with the following property: for any three distinct points A, B, and C in S, there is a fourth point D in S such that A, B, C, and D are the vertices of a parallelogram (in some order).

6 Let ABC be an acute scalene triangle with O as its circumcenter. Point P lies inside triangle ABC with $\angle PAB = \angle PBC$ and $\angle PAC = \angle PCB$. Point Q lies on line BC with $QA = QP$. Prove that $\angle AQP = 2\angle OQB$.