Winter Camp 2008 Mock Olympiad

Monday, January 7, 2008

Solutions

1. Find all triples of positive integers x, y, z satisfying

$$1 + 2^x 3^y = z^2.$$

Solution: Since $z = 1, 2$ or 3 does not yield a solution, we assume $z \geq 4$. Factoring we obtain $(z+1)(z-1) = 2^x 3^y$. Since the numbers $z-1$ and $z+1$ differ by 2, at most one of them is divisible by 3. Also, since the product is even, the factors are both even, and exactly one of them is divisible by 4. So either $z+1 = 2 \cdot 3^y$ and $z-1 = 2^{x-1}$ or $z+1 = 2^{x-1}$ and $z-1 = 2 \cdot 3^y$.

By subtracting the two equations, the first case yields $2 = 2 \cdot 3^y - 2^{x-1}$; hence $3^y - 2^{x-2} = 1$. This has a solution $x = 3, y = 1$, for which $z = 5$. For solutions with $x \geq 4$, looking mod 4 we conclude that y must be even, say $y = 2y_1$. Factoring yields $(3^{y_1} + 1)(3^{y_1} - 1) = 2^{x-2}$. So both factors on the left are powers of two, and since they differ by two, the first one is 4 and the second is 2. This gives the solution $x = 5, y = 2$, and consequently $z = 17$.

The second case leads to the equation $2^{x-2} - 3^y = 1$. By looking at mod 3 we see that x is even, so let $x - 2 = 2x_1$. Then we have $(2^{x_1} - 1)(2^{x_1} + 1) = 3^y$. So $2^{x_1} - 1$ and $2^{x_1} + 1$ are two powers of 3 that differ by 2, which can only be 1 and 3. From this we get the solution $x = 4, y = 1, z = 7$.

In summary, the solutions are $(x, y, z) = (3, 1, 5), (5, 2, 17), (4, 1, 7)$.

Source: Romanian TST 1984

2. For positive real numbers a, b, c such that $abc \leq 1$, prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq a + b + c.$$

First solution: Using the weighted AM-GM, we have

$$\frac{2a}{b} + \frac{b}{c} \geq 3\sqrt[3]{\frac{a^2}{bc}} \geq 3a.$$

Similarly, $\frac{2b}{c} + \frac{c}{a} \geq 3b$ and $\frac{2c}{a} + \frac{a}{b} \geq 3c$. Adding these three inequalities gives the result.

Second solution: Replacing a, b, c by ta, tb, tc with $t = 1/\sqrt[3]{abc}$ preserves the LHS while increases the RHS and makes $at \cdot bt \cdot ct = abct^3 = 1$. Hence we may assume wolog that $abc = 1$. Then there exists positive real numbers x, y, z such that

$$a = \frac{y}{x}, \quad b = \frac{z}{y}, \quad , c = \frac{x}{z}.$$

The Rearrangement Inequality gives

$$x^3 + y^3 + z^3 \geq x^2 y + y^2 z + z^2 x.$$

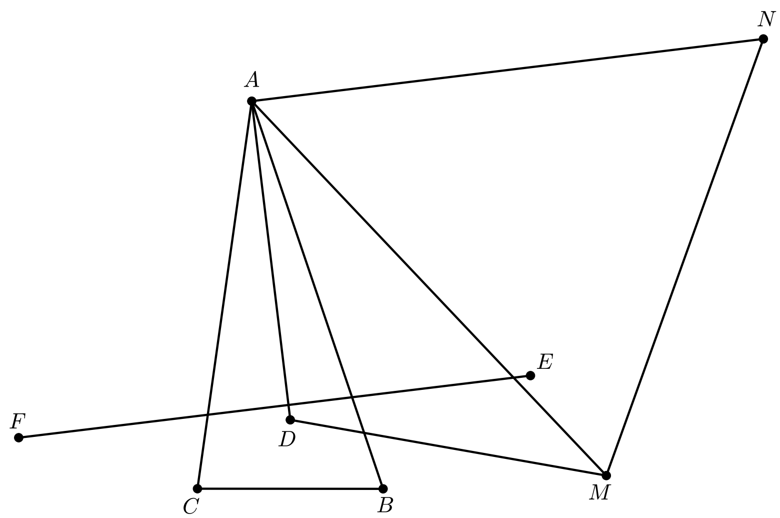
Thus

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=\frac{x^3+y^3+z^3}{xyz}\geq\frac{x^2y+y^2z+z^2x}{xyz}=a+b+c,$$

as desired.

3. In acute triangle ABC, $\angle A < 45^\circ$. Point D lies in the interior of triangle ABC such that $BD = CD$ and $\angle BDC = 4\angle A$. Point E is the reflection of C across line AB, and point F is the reflection of B across line AC. Prove that $AD \perp EF$.

   First solution: All angles are directed. Let $\mathbf{R}_A$ be the rotation by angle $2\angle CAB$ centered at A and let $\mathbf{R}_D$ be the rotation by angle $\angle BDC$ centered at D. Then $\mathbf{T} = \mathbf{R}_A \circ \mathbf{R}_D \circ \mathbf{R}_A$ is a translation because the sum of angles of $\mathbf{R}_A, \mathbf{R}_D, \mathbf{R}_A$ is zero.



   Note that $\mathbf{R}_A(F) = B$, $\mathbf{R}_D(B) = C$, $\mathbf{R}_A(C) = E$, so composing them gives $\mathbf{T}(F) = E$. Therefore, $\mathbf{T}$ is the translation by $\overrightarrow{FE}$. Let $M = \mathbf{R}_D(A)$, $N = \mathbf{R}_A(M)$. Then because triangle ADM is isosceles and $\angle MDA = 4\angle CAB$, we have $\angle DAM = 90^\circ - 2\angle CAB$. Therefore,

$$\angle DAN = \angle DAM + \angle MAN = 90^\circ - 2\angle CAB + 2\angle CAB = 90^\circ.$$

   Since $\mathbf{T}(A) = N$, we have $\overrightarrow{AN} = \overrightarrow{FE}$, and therefore $AD \perp EF$.

   Second solution: We use complex numbers. Use lower case letters to denote the complex number corresponding to a point. Place D at the center, and let B and C be on the unit circle so that B and C are symmetric about the real axis. Let α denote the unit complex number with argument $\angle CAB$. Then, $b = \alpha^2$, $c = \alpha^{-2}$. Since $AE = AC$ and $\angle CAE = 2\angle CAB$, we see that $e - a = \alpha^2(c - a)$, which gives us

$$e = a + \alpha^2(c-a) = a(1-\alpha^2) + c\alpha^2 = a(1-\alpha^2) + 1.$$

   Similarly, $f = a(1-\alpha^{-2})+1$. So $f - e = a(\alpha^2 - \alpha^{-2})$. Note that $\alpha^2 - \alpha^{-2}$ is purely imaginary as α lies on the unit circle. It follows that EF is perpendicular to AD.

   Source: MOP 2007

4. Let m, n be two positive integers with $m \geq n$. Prove that

$$\sum_{k=0}^{n}(-1)^k \binom{m-k}{n}\binom{n}{k} = 1.$$

First solution: (Finite differences) Let f be the polynomial

$$f(x) = \binom{m-x}{n} = \frac{(m-x)(m-1-x)\cdots(m-n+1-x)}{n!}.$$

Then

$$LHS = \sum_{k=0}^{n}(-1)^k \binom{n}{k} f(k) = (-1)^n \Delta^n f(0),$$

where Δ denotes the finite difference operator, which transforms a function $g(x)$ into $g(x+1) - g(x)$. For polynomials, if P is a polynomial of degree d, then ΔP is a polynomial of degree $d-1$. Since $\deg(f) = n$, we see that $\Delta^n f(x)$ is a constant. So

$$LHS = (-1)^n \Delta^n f(0) = (-1)^n \Delta^n f(m-n) = \sum_{k=0}^{n}(-1)^k \binom{n}{k} f(m-n+k).$$

Since $f(m-n+1) = f(m-n+2) = \cdots = f(m) = 0$, the only nonzero term is $f(m-n) = 1$, and thus the original sum equals to 1.

Second solution: (Combinatorial) For $i = 1, 2, \ldots, n$, let S_i denote the set of all n-element subsets of $\{1, 2, \ldots, m\}$ that does not contain the element i. That is

$$S_i = \{A \subset \{1, 2, \ldots, m\} : |A| = n, i \notin A\}.$$

Since $S_1 \cup S_2 \cup \cdots \cup S_n$ contains all n-element subset of $\{1, 2, \ldots, m\}$ except for the subset $\{1, 2, \ldots, n\}$, we have

$$|S_1 \cup S_2 \cup \cdots \cup S_n| = \binom{m}{n} - 1.$$

Also, the intersection $S_{a_1} \cap S_{a_2} \cap \cdots \cap S_{a_i}$, where $1 \leq a_1 < a_2 < \cdots < a_i \leq n$, contains all n-element subsets of $\{1, 2, \ldots, m\} \backslash \{a_1, a_2, \ldots a_i\}$. So

$$S_{a_1} \cap S_{a_2} \cap \cdots \cap S_{a_i} = \binom{m-i}{n}.$$

By the Inclusion-Exclusion Principle,

$$|S_1 \cup S_2 \cup \cdots \cup S_n| = \sum_{1 \leq i \leq n} |S_i| - \sum_{1 \leq i < j \leq n} |S_i \cap S_j| + \sum_{1 \leq i < j < k \leq n} |S_i \cap S_j \cap S_k| - \cdots - (-1)^n |S_1 \cap S_2 \cap \cdots \cap S_n|,$$

which is equivalent to

$$\binom{m}{n} - 1 = \binom{m-1}{n}\binom{n}{1} - \binom{m-2}{n}\binom{n}{2} + \binom{m-3}{n}\binom{n}{3} - \cdots - (-1)^n \binom{m-n}{n}\binom{n}{n},$$

and rearranging gives

$$\sum_{k=0}^{n}(-1)^k \binom{m-k}{n}\binom{n}{k} = 1,$$

as desired.

Third solution: (Generating function) We see that given sum equals to the coefficient of x^m in the product $F(x)G(x)$, where

$$F(x) = \sum_{k=0}^{\infty}(-1)^k \binom{n}{k} x^k, \quad \text{and} \quad G(x) = \sum_{k=0}^{\infty} \binom{k}{n} x^k.$$

Using binomial theorem, we have

$$F(x) = \sum_{k=0}^{n}(-1)^k \binom{n}{k} x^k = (1-x)^n,$$

and

$$G(x) = \sum_{k=n}^{\infty} \binom{k}{n} x^k = x^n \sum_{k=0}^{\infty} \binom{n+k}{n} x^k = x^n \sum_{k=0}^{\infty}(-1)^k \binom{-n-1}{k} x^k = x^n(1-x)^{-n-1}.$$

And thus

$$F(x)G(x) = x^n(1-x)^{-1} = x^n + x^{n+1} + x^{n+2} + \cdots.$$

So the coefficient of x^m is 1, and thus the sum equals to 1.

Source: MathLinks Contest