Balkan MO 2015

– May 5th

1 If a, b and c are positive real numbers, prove that

$$a^3b^6 + b^3c^6 + c^3a^6 + 3a^3b^3c^3 \geq abc\left(a^3b^3 + b^3c^3 + c^3a^3\right) + a^2b^2c^2\left(a^3 + b^3 + c^3\right).$$

(Montenegro).

2 Let $\triangle ABC$ be a scalene triangle with incentre I and circumcircle ω. Lines AI, BI, CI intersect ω for the second time at points D, E, F, respectively. The parallel lines from I to the sides BC, AC, AB intersect EF, DF, DE at points K, L, M, respectively. Prove that the points K, L, M are collinear.
(Cyprus)

3 A committee of 3366 film critics are voting for the Oscars. Every critic voted just an actor and just one actress. After the voting, it was found that for every positive integer $n \in \{1, 2, \ldots, 100\}$, there is some actor or some actress who was voted exactly n times. Prove that there are two critics who voted the same actor and the same actress.
(Cyprus)

4 Prove that among 20 consecutive positive integers there is an integer d such that for every positive integer n the following inequality holds

$$n\sqrt{d}\left\{n\sqrt{d}\right\} > \frac{5}{2}$$

where by $\{x\}$ denotes the fractional part of the real number x. The fractional part of the real number x is defined as the difference between the largest integer that is less than or equal to x to the actual number x.

(Serbia)