Romania National Olympiad 2010

–	Grade level 7
1	Let S be a subset with 673 elements of the set $\{1, 2, \ldots, 2010\}$. Prove that one can find two distinct elements of S, say a and b, such that 6 divides $a + b$.
2	Let $ABCD$ be a rectangle of centre O, such that $\angle DAC = 60^\circ$. The angle bisector of $\angle DAC$ meets DC at S. Lines OS and AD meet at L, and lines BL and AC meet at M. Prove that lines SM and CL are parallel.
3	Each of the small squares of a 50×50 table is coloured in red or blue. Initially all squares are red. A *step* means changing the colour of all squares on a row or on a column. a) Prove that there exists no sequence of steps, such that at the end there are exactly 2011 blue squares. b) Describe a sequence of steps, such that at the end exactly 2010 squares are blue. *Adriana & Lucian Dragomir*
4	In the isosceles triangle ABC, with $AB = AC$, the angle bisector of $\angle B$ meets the side AC at B'. Suppose that $BB' + B'A = BC$. Find the angles of the triangle ABC. *Dan Nedeianu*
–	Grade level 8
1	Let a, b, c be integers larger than 1. Prove that $$a(a-1) + b(b-1) + c(c-1) \le (a+b+c-4)(a+b+c-5) + 4.$$
2	How many four digit numbers $\overline{abcd}$ simultaneously satisfy the equalities $a + b = c + d$ and $a^2 + b^2 = c^2 + d^2$?
3	Let $VABCD$ be a regular pyramid, having the square base $ABCD$. Suppose that on the line AC lies a point M such that $VM = MB$ and $(VMB) \perp (VAB)$. Prove that $4AM = 3AC$. *Mircea Fianu*

4 Let a, b, c, d be positive integers, and let $p = a + b + c + d$. Prove that if p is a prime, then p is not a divisor of $ab - cd$.

Marian Andronache

– Grade level 9

1 In a triangle ABC denote by D, E, F the points where the angle bisectors of $\angle CAB, \angle ABC, \angle BCA$ respectively meet it's circumcircle.
a) Prove that the orthocenter of triangle DEF coincides with the incentre of triangle ABC.
b) Prove that if $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = 0$, then the triangle ABC is equilateral.

Marin Ionescu

2 Prove that there is a similarity between a triangle ABC and the triangle having as sides the medians of the triangle ABC if and only if the squares of the lengths of the sides of triangle ABC form an arithmetic sequence.

Marian Teler & Marin Ionescu

3 For any integer $n \geq 2$ denote by A_n the set of solutions of the equation

$$x = \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x}{3} \right\rfloor + \cdots + \left\lfloor \frac{x}{n} \right\rfloor.$$

a) Determine the set $A_2 \cup A_3$.
b) Prove that the set $A = \bigcup_{n \geq 2} A_n$ is finite and find $\max A$.

Dan Nedeianu & Mihai Baluna

4 Consider the set $\mathcal{F}$ of functions $f : \mathbb{N} \to \mathbb{N}$ (where $\mathbb{N}$ is the set of non-negative integers) having the property that

$$f(a^2 - b^2) = f(a)^2 - f(b)^2, \text{ for all } a, b \in \mathbb{N},\ a \geq b.$$

a) Determine the set $\{f(1) \mid f \in \mathcal{F}\}$.
b) Prove that $\mathcal{F}$ has exactly two elements.

Nelu Chichirim

– Grade level 10

1 Let $(a_n)_{n\geq 0}$ be a sequence of positive real numbers such that

$$\sum_{k=0}^{n} C_n^k a_k a_{n-k} = a_n^2, \text{ for any } n \geq 0.$$

Prove that $(a_n)_{n\geq 0}$ is a geometric sequence.

Lucian Dragomir

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2 Consider v, w two distinct non-zero complex numbers. Prove that

$$|zw + \bar{w}| \leq |zv + \bar{v}|,$$

for any $z \in \mathbb{C}, |z| = 1$, if and only if there exists $k \in [-1, 1]$ such that $w = kv$.

Dan Marinescu

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3 In the plane are given 100 points, such that no three of them are on the same line. The points are arranged in 10 groups, any group containing at least 3 points. Any two points in the same group are joined by a segment.
a) Determine which of the possible arrangements in 10 such groups is the one giving the minimal numbers of triangles.
b) Prove that there exists an arrangement in such groups where each segment can be coloured with one of three given colours and no triangle has all edges of the same colour.

Vasile Pop

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4 On the exterior of a non-equilateral triangle ABC consider the similar triangles ABM, BCN and CAP, such that the triangle MNP is equilateral. Find the angles of the triangles ABM, BCN and CAP.

Nicolae Bourbacut

– Grade level 11

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1 Let $a, b \in \mathbb{R}$ such that $b > a^2$. Find all the matrices $A \in \mathcal{M}_2(\mathbb{R})$ such that $\det(A^2 - 2aA + bI_2) = 0$.

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2 Let $A, B, C \in \mathcal{M}_n(\mathbb{R})$ such that $ABC = O_n$ and rank $B = 1$. Prove that $AB = O_n$ or $BC = O_n$.

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3 Let $f:\mathbb{R}\to[0,\infty)$. Prove that $f(x+y)\geq(y+1)f(x),\ (\forall)x\in\mathbb{R}$ if and only if the function $g:\mathbb{R}\to[0,\infty),\ g(x)=e^{-x}f(x),\ (\forall)x\in\mathbb{R}$ is increasing.

4 Let $a\in\mathbb{R}_+$ and define the sequence of real numbers $(x_n)_n$ by $x_1=a$ and $x_{n+1}=\left|x_n-\frac{1}{n}\right|,\ n\geq 1$. Prove that the sequence is convergent and find it's limit.

– Grade level 12

1 Let $f:\mathbb{R}\to\mathbb{R}$ be a monotonic function and $F:\mathbb{R}\to\mathbb{R}$ given by

$$F(x)=\int_0^x f(t)\ \mathrm{d}t.$$

Prove that if F has a finite derivative, then f is continuous.

Dorin Andrica & Mihai Piticari

2 We say that a ring A has property (P) if any non-zero element can be written uniquely as the sum of an invertible element and a non-invertible element.
a) If in A, $1+1=0$, prove that A has property (P) if and only if A is a field.
b) Give an example of a ring that is not a field, containing at least two elements, and having property (P).

Dan Schwarz

3 Let G be a finite group of order n. Define the set

$$H=\{x : x\in G \text{ and } x^2=e\},$$

where e is the neutral element of G. Let $p=|H|$ be the cardinality of H. Prove that
a) $|H\cap xH|\geq 2p-n$, for any $x\in G$, where $xH=\{xh : h\in H\}$.
b) If $p>\frac{3n}{4}$, then G is commutative.
c) If $\frac{n}{2}<p\leq\frac{3n}{4}$, then G is non-commutative.

Marian Andronache

4 Let $f:[-1,1]\to\mathbb{R}$ be a continuous function having finite derivative at 0, and

$$I(h)=\int_{-h}^{h} f(x)\ \mathrm{d}x,\ \ h\in[0,1].$$

Prove that

a) there exists $M > 0$ such that $|I(h) - 2f(0)h| \leq Mh^2$, for any $h \in [0,1]$.
b) the sequence $(a_n)_{n\geq 1}$, defined by $a_n = \sum_{k=1}^{n} \sqrt{k}|I(1/k)|$, is convergent if and only if $f(0) = 0$.

Calin Popescu