Romania National Olympiad 2012

– IX

1 The altitude $[BH]$ dropped onto the hypotenuse of a triangle ABC intersects the bisectors $[AD]$ and $[CE]$ at Q and P respectively. Prove that the line passing through the midpoints of the segments $[QD]$ and $[PE]$ is parallel to the line AC .

2 Find all functions $f : \mathbb{R} \to \mathbb{R}$ with the following property: for any open bounded interval I, the set $f(I)$ is an open interval having the same length with I .

3 Prove that if $n \geq 2$ is a natural number and $x_1, x_2, \ldots, x_n$ are positive real numbers, then:

$$4\left(\frac{x_1^3 - x_2^3}{x_1 + x_2} + \frac{x_2^3 - x_3^3}{x_2 + x_3} + \ldots + \frac{x_{n-1}^3 - x_n^3}{x_{n-1} + x_n} + \frac{x_n^3 - x_1^3}{x_n + x_1}\right) \leq \leq (x_1 - x_2)^2 + (x_2 - x_3)^2 + \ldots + (x_{n-1} - x$$

4 On a table there are $k \geq 2$ piles having $n_1, n_2, \ldots, n_k$ pencils respectively. A *move* consists in choosing two piles having a and b pencils respectively, $a \geq b$ and transferring b pencils from the first pile to the second one. Find the necessary and sufficient condition for $n_1, n_2, \ldots, n_k$, such that there exists a succession of moves through which all pencils are transferred to the same pile.

– X

1 Let $M = \{x \in \mathbb{C} \,|\, |z| = 1,\ \mathrm{Re}\, z \in \mathbb{Q}\}$. Prove that there exist infinitely many equilateral triangles in the complex plane having all affixes of their vertices in the set M .

2 Let a , b and c be three complex numbers such that $a + b + c = 0$ and $|a| = |b| = |c| = 1$. Prove that:

$$3 \leq |z - a| + |z - b| + |z - c| \leq 4,$$

for any $z \in \mathbb{C}$, $|z| \leq 1$.

3 Let $a, b \in \mathbb{R}$ with $0 < a < b$. Prove that:

a) $2\sqrt{ab} \leq \frac{x+y+z}{3} + \frac{ab}{\sqrt[3]{xyz}} \leq a + b$, for $x, y, z \in [a, b]$.

b) $\left\{ \frac{x+y+z}{3} + \frac{ab}{\sqrt[3]{xyz}} \mid x, y, z \in [a, b] \right\} = [2\sqrt{ab}, a + b]$.

4 Let n and m be two natural numbers, $m \geq n \geq 2$. Find the number of injective functions

$$f\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$$

such that there exists a unique number $i \in \{1, 2, \ldots, n-1\}$ for which $f(i) > f(i+1)$.

– XI

1 Let $f, g\colon [0, 1] \to [0, 1]$ be two functions such that g is monotonic, surjective and $|f(x) - f(y)| \leq |g(x) - g(y)|$, for any $x, y \in [0, 1]$.

a) Prove that f is continuous and that there exists some $x_0 \in [0, 1]$ with $f(x_0) = g(x_0)$.
b) Prove that the set $\{x \in [0, 1] \mid f(x) = g(x)\}$ is a closed interval.

2 Let n and k be two natural numbers such that $n \geq 2$ and $1 \leq k \leq n-1$. Prove that if the matrix $A \in \mathcal{M}_n(\mathbb{C})$ has exactly k minors of order $n-1$ equal to 0 , then $\det(A) \neq 0$.

3 Let $A, B \in \mathcal{M}_4(\mathbb{R})$ such that $AB = BA$ and $\det(A^2 + AB + B^2) = 0$. Prove that:

$$\det(A + B) + 3\det(A - B) = 6\det(A) + 6\det(B) \ .$$

4 Find all differentiable functions $f\colon [0, \infty) \to [0, \infty)$ for which $f(0) = 0$ and $f'(x^2) = f(x)$ for any $x \in [0, \infty)$.

– XII

1 Let $f\colon [0,\infty) \to \mathbb{R}$ be a continuous function such that $\int_0^n f(x)f(n-x)\ \mathrm{d}x = \int_0^n f^2(x)\ \mathrm{d}x$, for any natural number $n \geq 1$. Prove that f is a periodic function.

2 Let $(R,+,\cdot)$ be a ring and let f be a surjective endomorphism of R such that $[x,f(x)]=0$ for any $x\in R$, where $[a,b]=ab-ba$, $a,b\in R$. Prove that:

a) $[x,f(y)]=[f(x),y]$ and $x[x,y]=f(x)[x,y]$, for any $x,y\in R$;
b) If R is a division ring and f is different from the identity function, then R is commutative.

3 Let $\mathcal{C}$ be the set of integrable functions $f\colon [0,1]\to\mathbb{R}$ such that $0\leq f(x)\leq x$ for any $x\in[0,1]$. Define the function $V\colon \mathcal{C}\to\mathbb{R}$ by

$$V(f)=\int_0^1 f^2(x)\ \mathrm{d}x-\left(\int_0^1 f(x)\ \mathrm{d}x\right)^2\ ,\ f\in\mathcal{C}\ .$$

Determine the following two sets:

a) $\{V(f_a)\,|\,0\leq a\leq 1\}$, where $f_a(x)=0$, if $0\leq x\leq a$ and $f_a(x)=x$, if $a<x\leq 1$;
b) $\{V(f)\,|\,f\in\mathcal{C}\}$.

4 Let m and n be two nonzero natural numbers. Determine the minimum number of distinct complex roots of the polynomial $\prod_{k=1}^m (f+k)$, when f covers the set of n^{th} - degree polynomials with complex coefficients.