equal angles

geometry geometric transformation reflection

Source: Iranian third round 2015 geometry problem 3

andria
721 posts

Sep 10, 2015, 5:52 pm PM #1

Let ABC be a triangle. consider an arbitrary point P on the plain of $\triangle ABC$. Let R,Q be the reflections of P wrt AB,AC respectively. Let $RQ\cap BC=T$. Prove that $\angle APB=\angle APC$ if and only if $\angle APT=90^{\circ}$.

This post has been edited 1 time. Last edited by andria, Sep 11, 2015, 3:12 pm

Luis Gonz...
3881 pos...

Sep 11, 2015, 3:44 am • 1 PM #2

Let the perpendicular to PA at P cut AC,AB,BC at Y,Z,T^*, resp and let $X\equiv RZ\cap QY$. Since AC,AB are perpendicular bisectors of PQ,PR, then A is the center of $\odot(PQR)\Longrightarrow YZ,QY,RZ$ are tangents at P,Q,R $\Longrightarrow XP$ is the polar of T^* WRT $\odot(PQR)\Longrightarrow X(Z,Y,P,T^*)=-1\Longrightarrow A(B,C,P,T^*)=-1$ or $P(B,C,A,T^*)=-1$. As a result, $\angle APT=90^{\circ}\Longleftrightarrow T\equiv T^*\Longleftrightarrow PA$ bisect $\angle BPC\Longleftrightarrow \angle APB=\angle APC$.

Luis Gonz...
3881 pos...

Sep 11, 2015, 9:43 pm PM #3

Sorry there is a flaw in the previous resolution. T^* is the pole of XP WRT $\odot(PQR)$ only when $T\equiv T^*$, though this can be easily fixed keeping the same notations.

If $\angle APT=90^{\circ}\Longrightarrow T$ is the pole of XP WRT $\odot(PQR)\Longrightarrow X(Z,Y,P,T)=-1\Longrightarrow P(B,C,A,T)=-1\Longrightarrow$ AP bisects $\angle BPC$. Conversely if AP bisects $\angle BPC\Longrightarrow P(B,C,A,T^*)=-1$ or $A(B,C,P,T^*)=-1\Longrightarrow$ perpendiculars from P to AB,AC,AP,AT^* form a harmonic pencil as well. If $D\in\odot(PQR)$ is the reflection of P on AT^*, then $P(Q,R,D,T^*)=-1\Longrightarrow PQDR$ is harmonic $\Longrightarrow Q,R,T^*$ are collinear $\Longrightarrow T\equiv T^*\Longrightarrow$ $\angle APT=90^{\circ}$.

TelvCohl
1990 pos...

Sep 12, 2015, 12:34 am PM #4

After performing the Inversion with center A we get the following equivalent problem :

Given a $\triangle ABC$ and an arbitrary point P. Let Q,R be the reflection of P in CA,AB, respectively. Let T be the second intersection of $\odot(ABC)$ and $\odot(AP)$. Prove that A,Q,R,T are concyclic if and only if $\angle PBA=\angle PCA$.

Proof :

Let Y,Z be the projection of P on CA,AB, respectively. Since A lies on the perpendicular bisector CA,AB of PQ,PR, so A is the circumcenter of $\triangle PQR\Longrightarrow\angle AQR=\angle ARQ=90^{\circ}-\angle BAC$, hence T lies on $\odot(AQR)$ iff $\angle QTP=\angle RTP=\angle BAC$. On the other hand, since T is the Miquel point of the complete quadrilateral $\{BC,CA,AB,YZ\}$, so $\triangle TYC\sim\triangle TZB\Longrightarrow\angle QTP=\angle RTP=\angle BAC$ iff $\triangle TYC\cup(P,Q)\sim\triangle TZB\cup(R,P)$ (notice $PQ\perp CY,RP\perp BZ$ and Y,Z is the midpoint of PQ,RP, respectively.) iff $\angle PBA=\angle PBZ=\angle RBZ=\angle QCY=\angle PCY=\angle PCA$.

andria
721 posts

Sep 12, 2015, 1:21 am PM #5

Different solution by inversion:
Consider an inversion Ψ with center P. After performing Ψ we get the following problem:
Problem:
Let PBC be a triangle. Consider an arbitrary point A. Let R,Q be the circumcenters of $\triangle PAB,\triangle PAC$ respectively. Let T be a point on circumcircle of $\triangle ABC$ such that $PT\perp PA$ then $RPTQ$ is cyclic if and only if $\angle APB=\angle APC$.
Proof:
Let O be the circumcenter of $\triangle PBC$ then RO,OQ are perpendicular bisectors of PB,PC respectively. since RQ is perpendicular bisector of PA so $PT\parallel RQ$. also $OP=OT$. Now if:

1) $RPTQ$ is cyclic then it is isosceles trapezoid so $PR=QT$ and $\angle OPR=\angle OTQ$ hence $\triangle OTQ=\triangle OPR\Longrightarrow OR=OQ\Longrightarrow\angle APC=\angle APB$.

2) $\angle APC=\angle APB$ then $\angle OQR=\angle ORQ\Longrightarrow OR=OQ$ Also since T is midpoint of arc BPC of $\odot(PBC)$ we get $\angle TOQ=\angle POR=\angle C$ hence $\triangle OTQ=\triangle OPR\Longrightarrow TQ=PR\Longrightarrow RPTQ$ is isosceles trapezoid so it is cyclic.
DONE

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