Mathematical Contests
1995-1996

Olympiad Problems and Solutions from around the World

Published by the American Mathematics Competitions, 1997

Preface

This book is a supplement to *Mathematical Olympiads 1995: Olympiad Problems from Around the World,* published by the American Mathematics Competitions. It contains solutions to the problems from 28 national and regional contests featured in the earlier pamphlet, together with selected problems (without solutions) from national and regional contests given during the academic year 1995-1996.

This collection is intended as practice for the serious student who wishes to improve his or her performance on the USAMO. Some of the problems are comparable to the USAMO in that they came from national contests. Others are harder, as some country's first have a national olympiad, and later one or more exams to select a team for the IMO. Some problems come from regional international contests ("mini-IMO's").

Different nations have different mathematical cultures, so you will find some of these problems extremely hard and some rather easy. We have tried to present a wide variety of problems, especially from those countries that have often done well at the IMO.

Each contest has its own time limit. We have not furnished this information, because we have not always included complete contests. As a rule of thumb, most contests allow a time limit ranging between one-half to one full hour per problem.

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Contents

1 1995 Contests: Problems and Solutions

1.1 Austria

1. For how many a) even and b) odd numbers n, does n divide $3^{12}-1$, yet n does not divide 3^k-1 for $k=1,2,\ldots,11$.

Solution: We note

$$\begin{aligned} 3^{12}-1 &= (3^6-1)(3^6+1) \\ &= (3^2-1)(3^4+3^2+1)(3^2+1)(3^4-3^2+1) \\ &= (2^3)(7\cdot 13)(2\cdot 5)(73). \end{aligned}$$

Recall that the number of divisors of $p_1^{e_1}\cdots p_k^{e_k}$ is $(e_1+1)\cdots(e_k+1)$. Therefore $3^{12}-1$ has $2\cdot 2\cdot 2\cdot 2=16$ odd divisors and $4\cdots 16=64$ even divisors.

If $3^{12}\equiv 1 \pmod{m}$ for some integer m, then the smallest integer d such that $3^d\equiv 1 \pmod{m}$ divides 12. (Otherwise we could write $12=pq+r$ with $0<r<d$ and find $3^r\equiv 1 \pmod{m}$.) Hence to ensure $n \not| 3^k-1$ for $k=1,\ldots,11$, we need only check $k=1,2,3,4,6$. But

$$\begin{aligned} 3^1-1 &= 2 \\ 3^2-1 &= 2^3 \\ 3^3-1 &= 2\cdot 13 \\ 3^4-1 &= 2^4\cdot 5 \\ 3^6-1 &= 2^3\cdot 7\cdot 13. \end{aligned}$$

The odd divisors we throw out are $1,5,7,13,91$, while the even divisors are 2^i for $1\le i\le 4$, $2^i\cdot 5$ for $1\le i\le 4$, and each of $2^j\cdot 7$, $2^j\cdot 13$, and $2^j\cdot 7\cdot 13$ for $1\le i\le 3$. As we are discarding 17 even divisors and 5 odd ones, we remain with 47 even divisors and 11 odd ones.

2. Consider a triangle ABC. For each circle K which passes through A and B, define P_K, Q_K to be the intersections of K with BC and AC, respectively.

   (a) Show that for any circles K, K' passing through A and B, the lines P_KQ_K and $P_{K'}Q_{K'}$ are parallel.

(b) Determine the locus of the circumcircle of CP_KQ_K.

Solution:

(a) We show that for all K, the line P_KQ_K is parallel to the reflection of AB across the internal angle bisector at C. If P_K and Q_K lie on the segments BC and CA, respectively, we have

$$\angle Q_KP_KC = \pi - \angle BP_KQ_K = \angle Q_KAB = \angle CAB,$$

which implies the claim. The proof in case P_K or L_K lies outside its corresponding segment is similar (or can be subsumed by viewing the angles above as directed angles modulo π).

(b) From (a) we deduce that the circles CP_KQ_K are all homothetic; hence they form a family of circles having a common tangent at C.

3. The positive real numbers $x_1, \ldots, x_n$ ($n \geq 3$) satisfy $x_1 = 19, x_2 = 95$ and $x_k^3 \leq x_{k-1}^2 x_{k+1}$ for $2 \leq k \leq n-1$. Determine the largest of the numbers $x_1, \ldots, x_n$.

Solution: Note that if $x_{k-1} < x_k$, then

$$x_{k+1} \geq x_k(x_k/x_{k-1})^2 > x_k.$$

Hence by induction, $x_1 < x_2 < \cdots < x_n$ and x_n is the largest.

4. A cube is inscribed in a regular tetrahedron of unit length in such a way that each of the 8 vertices of the cube lie on faces of the tetrahedron. Determine the possible side lengths for such a cube.

Solution: Let A, B, C, D be the vertices of the tetrahedron and s the length of the side of the cube. We consider two cases, each leading to a unique side length s. In the first case, some three vertices lie on a single face of the tetrahedron, say ABC. This means an entire face of the cube must lie on ABC. The cross-section along the plane of the opposite face of the cube is an equilateral triangle, whose side length we call t, with a square inside having each vertex on some side of the triangle. Let PQR be the vertices of the triangle. One of the sides, say PQ, of the triangle must contain two vertices of the square. Then a homothety through P with ratio t/s carries the square to a square erected externally on PQ. The ratio t/s must also equal the ratio of the distances from P to PQ and the side of the external square opposite PQ, which is $(t + t\sqrt{3}/2)/t$. We conclude that $t = (1 + \sqrt{3}/2)s$.

On the other hand, the ratio of the height of the cube to the height of the tetrahedron is $1 - t$. The latter height is $\sqrt{2/3}$ (an easy computation), so $1 - t = \sqrt{3/2}s$. Adding these expressions for t yields $s = 1/(1 + \sqrt{3}/2 + \sqrt{3/2})$.

In the second case, no more than two vertices lie on each face. Hence each face must contain an edge of the tetrahedron. Of these four edges, two must be parallel, and so must be parallel to the intersection of the two corresponding faces of the tetrahedron, which is simply an edge of the tetrahedron. The only direction perpendicular to this edge which lies in either of the other two faces is along the opposite edge of the tetrahedron, so the edges of the cube lying on those faces must be parallel to that opposite edge.

By an easy calculation (or by inscribing the tetrahedron in a cube), the segment joining the midpoints of opposite edges has length $\sqrt{2}/2$, and by symmetry, two faces of the cube must be perpendicular to this segment. Then the distance from one of these faces to the closer edge parallel to it is $\sqrt{2}/4 - s/2$. However, the ratio of this distance to $\sqrt{2}/2$ must equal the ratio of the edges of the cube and the tetrahedron (by a homothety), which is s. We solve for s and find $s = (\sqrt{2} - 1)/2$ in this case.

1.2 Bulgaria

1. Let p and q be positive numbers such that the parabola $y = x^2 - 2px + q$ has no common point with the x-axis. Let O denote the origin $(0,0)$. Prove that there exist points A and B on the parabola such that the segment AB is parallel to the x-axis and $\angle AOB$ is a right angle if and only if $p^2 < q \leq 1/4$. Find the values of p and q for which the points A and B are defined uniquely.

   Solution: Since the parabola has no common point with the x-axis, the roots of the equation $x^2 - 2px + q = 0$ are not real and hence $q \leq p^2$. Let the points $A(x_1, y_0)$ and $B(x_2, y_0)$ have the required properties. Then x_1 and x_2 are the roots of the equation $x^2 - 2px + q - y_0 = 0$ and $y_0 > q - p^2$, because the vertex of the parabola has coordinates $(p, q - p^2)$. On the other hand $OA^2 = x_1^2 + y_0^2, OB^2 = x_2^2 + y_0^2, AB^2 = (x_1 - x_2)^2$ and it follows from the Pythagorean theorem that $y_0^2 + x_1x_2 = 0$. But $x_1x_2 = q - y_0$ and thus $y_0^2 - y_0 + q = 0$. Consequently the existence of the points A and B is equivalent to the assertion that the equation $f(y) = y^2 - y + q = 0$ has a solution $y_0 > q - p^2$. (A and B are defined in a unique way if this is the only solution.) A necessary condition is that the discriminant of the equation is not negative, i.e. $q \leq \frac{1}{4}$. The last condition is sufficient because $f(q - p^2) = q - p^2 + p^2 > 0$ and $\frac{1}{2} > \frac{1}{4} \geq q - p^2$. The corresponding solution y_0 is unique iff $q = \frac{1}{4}$.

2. Let $A_1A_2A_3A_4A_5A_6A_7, B_1B_2B_3B_4B_5B_6B_7, C_1C_2C_3C_4C_5C_6C_7$ be regular heptagons with areas S_A, S_B and S_C, respectively. Suppose that $A_1A_2 = B_1B_3 = C_1C_4$. Prove that

$$\frac{1}{2} < \frac{S_B + S_C}{S_A} < 2 - \sqrt{2}.$$

   Solution: Let $A_1A_2 = a, A_1A_3 = b, A_1A_4 = \mathrm{c}$. By Ptolemys theorem for the quadrangle $A_1A_3A_4A_5$ it follows that $ab + ac = bc$, i.e. $\frac{a}{b} + \frac{a}{c} = 1$. Since $\Delta A_1A_2A_3 \cong \Delta B_1B_2B_3$, then $\frac{B_1B_2}{B_1B_3} = \frac{a}{b}$ and hence $B_1B_2 = \frac{a^2}{b}$. Analogously $C_1C_2 = \frac{a^2}{c}$. Therefore $\frac{S_B+S_C}{S_A} = \frac{a^2}{b} + \frac{a^2}{c}$. Then $\frac{a^2}{b} + \frac{a^2}{c} > \frac{1}{2}(\frac{a}{b} + \frac{a}{c})^2 = \frac{1}{2}$ (equality is not possible because

$\frac{a}{b} \neq \frac{a}{c}$). On the other hand

$$\frac{a^2}{b^2} + \frac{a^2}{c^2} = (\frac{a}{b} + \frac{a}{c})^2 - \frac{2a^2}{bc} = 1 - \frac{2a^2}{bc}. \tag{1}$$

By the sine theorem we get $\frac{a^2}{bc} = \frac{\sin^2 \frac{\pi}{7}}{\sin^2 \frac{\pi}{7} \sin \frac{4\pi}{7}} = \frac{1}{4 \cos \frac{2\pi}{7}(1+\cos \frac{2\pi}{7})}$. Since $\cos \frac{2\pi}{7} < \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, then $\frac{a^2}{bc} > \frac{1}{4\frac{\sqrt{2}}{2}(1+\frac{\sqrt{2}}{2})} = \sqrt{2} - 1$. From here and from (1) we get the right hand side inequality of the problem.

3. Let n be an integer greater than 1. Find the number of permutations $(a_1, a_2, \ldots, a_n)$ of the numbers $1, 2, \ldots, n$ such that there exists only one index $i \in \{1, 2, \ldots, n-1\}$ with $a_i > a_{i+1}$.

Solution: Denote by p_n the number of the permutations we are looking for. Obviously $p_1 = 0$ and $p_2 = 1$. Let $n \geq 2$. The number of permutations with $a_n = n$ is equal to p_{n-1}. Consider all the permutations $(a_1, a_2, ..., a_n)$ with $a_i = n$, where $1 \leq i \leq n-1$ is fixed. There are $\binom{n-1}{i-1}$ of these. Consequently

$$p_n = p_{n-1} + \sum_{i=1}^{n-1} \binom{n-1}{i-1} = p_{n-1} + 2^{n-1} - 1 \ .$$

From here

$$p_n = (2^{n-1} - 1) + (2^{n-2} - 1) + ... + (2 - 1) = 2^n - n - 1 \ .$$

4. Let $n \geq 2$ and $0 \leq x_i \leq 1$ for all $i = 1, 2, \ldots, n$. Show that

$$(x_1 + x_2 + \cdots + x_n) - (x_1 x_2 + x_2 x_3 + \cdots + x_{n-1} x_n + x_n x_1) \leq \left\lfloor \frac{n}{2} \right\rfloor,$$

and determine when there is equality.

Solution: Denote by $S(x_1, x_2, \ldots, x_n)$ the left hand side of the inequality. This function is linear with respect to each of the variables x_i. In particular,

$$S(x_1, x_2, \ldots, x_n) \leq \max(S(0, x_2, \ldots, x_n), S(1, x_2, \ldots, x_n)) \ .$$

From here it follows by induction that it is enough to prove the inequality when the x_i's are equal to 0 or 1. On the other hand, for arbitrary x_i we have

$$\begin{aligned} 2S(x_1, x_2, ..., x_n) &= n - (1 - x_1)(1 - x_2) \\ &\quad - (1 - x_2)(1 - x_3) - \ldots - (1 - x_n)(1 - x_1) \\ &\quad - x_1x_2 - x_2x_3 - ... - x_nx_1, \end{aligned}$$

i.e. $S(x_1, x_2, ..., x_n) \le \frac{n}{2}$, when $x_i \in [0,1]$. In the case when the x_i's are equal to 0 or 1, the left hand side of the last inequality is an integer. Consequently $S(x_1, x_2, ..., x_n) \le \left[\frac{n}{2}\right]$. It follows that when n is even, the equality is satisfied iff $(x_1, x_2, ..., x_n) = (0, 1, 0, 1, ..., 0, 1)$ or $(x_1, x_2, ..., x_n) = (1, 0, 1, 0, ..., 1, 0)$ where $x \in [0, 1]$ is arbitrary.

5. On sides BC, CA, AB of triangle ABC choose points A_1, B_1, C_1 in such a way that the lines AA_1, BB_1, CC_1 have a common point M. Prove that if M is the centroid of triangle $A_1B_1C_1$, then M is the centroid of triangle ABC.

Solution: Let M be the medcenter of $\Delta A_1B_1C_1$. Take a point A_2 on $MA^{\rightarrow}$ in such a way that $B_1A_1C_1A_2$ is a parallelogram. Take points B_2 and C_2 analogously. Since $A_1C_1 \| A_1B_1 \| C_1B_2$, then the points A_2, C_1, B_2 are colinear and C_1 is the midpoint of A_2B_2. The same is true for the points A_2, B_1, C_2 and C_2, A_1, B_2. We shall prove that $A_2 = A, B_2 = B$ and $C_2 = C$, which will solve the problem.

Assume that $A_2 \neq A$ and let A be between A_2 and M. Then C_2 is between C and M, B is between B_2 and M, and consequently A_2 is between A and M, which is a contradiction.

6. Find all pairs of positive integers (x, y) for which $\dfrac{x^2 + y^2}{x - y}$ is an integer which divides 1995.

Solution: It is enough to find all pairs (x, y) for which $x > y$ and $x^2 + y^2 = k(x - y)$, where k divides $1995 = 3 \cdot 5 \cdot 7 \cdot 19$. We shall use the following well-known fact: if p is prime of the kind $4q + 3$ and if it divides $x^2 + y^2$ then p divides x and y. (For $p = 3, 7, 19$ the last statement can be proved directly.) If k is divisible by 3, then x and y

are divisible by 3 too. Simplifying by 9 we get an equality of the kind $x_1^2 + y_1^2 = k_1(x_1 - y_1)$, where k_1 divides $5 \cdot 7 \cdot 19$. Considering 7 and 19, analogously we get an equality of the kind $a^2 + b^2 = 5(a - b)$, where $a > b$. (It is not possible to get an equality of the kind $a^2+b^2 = a-b$.) From here $(2a-5)^2+(2b-5)^2 = 50$, i.e. $a = 3, b = 1$, or $a = 2, b = 1$. The above consideration implies that the pairs we are looking for are of the kind $(3c, c), (2c, c), (c, 3c), (c, 2c)$, where $c = 1, 3, 7, 19, 3 \cdot 7, 3 \cdot 19, 7 \cdot 19, 3 \cdot 7 \cdot 19$.

7. Find the number of integers $n > 1$ for which the number $a^{25} - a$ is divisible by n for each integer a.

Solution: Let n have the required property. Then p^2 (p a prime) does not divide n since p^2 does not divide $p^{25} - p$. Hence n is the multiple of different prime numbers. On the other hand $2^{25} - 2 = 2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 241$. But n is not divisible by 17 and 241 since $3^{25} \equiv -3 \pmod{17}$ and $3^{25} \equiv 32 \pmod{241}$. The Fermat Theorem implies that $a^{25} \equiv a \pmod{p}$ when $p = 2, 3, 5, 7, 13$. Thus n should be equal to the divisors of $2 \cdot 3 \cdot 5 \cdot 7 \cdot 13$ which are different from 1 and there are $2^5 - 1 = 31$ of them.

8. Let ABC be a triangle with semiperimeter s. Choose points E and F on AB such that $CE = CF = s$. Prove that the excircle K_1 of triangle ABC to side AB is tangent to the circumcircle of triangle EFC.

Solution: Let P and Q be the common points k_1 with the lines CA and CB, respectively. Since $CP = CQ = p$, then the points E, P, Q and F lie on the circle with center C and radius p. We denote by i the inversion defined by this circle. Since $i(P) = P, i(Q) = Q$, then $i(k_1) = k_1$. On the other hand $i(E) = E$ and $i(F) = F$. Hence $i(k)$ is the line AB. But k_1 touches AB and thus k touches k_1.

9. Two players A and B take stones one after the other from a heap with $n \geq 2$ stones. A begins the game and takes at least 1 but not more than $n - 1$ stones. Each player on his turn must take at least 1 but no more than the other player has taken before him. The player who takes the last stone is the winner. Which player has a winning strategy?

Solution: Consider the pair (m, l), where m is the number of stones in the heap and l is the maximal number of stones that could be taken by the player on turn. We must find for which n the position $(n, n-1)$ is winning (i.e. **A** wins), and for which n it is losing (**B** wins). We shall use the following assertion several times: If (m, l) is a losing position and $l_1 < l$, then (m, l_1) is also losing. Now we shall prove that $(n, n-1)$ is a losing position iff n is a power of 2.

Sufficiency: Let $n = 2^k, k \geq 1$. If $k = 1$ then **B** wins on his or her first move. Assume, that $(2^k, 2^k - 1)$ is a losing position, and consider $(2^{k+1}, 2^{k+1} - 1)$. If **A** takes at least 2^k stones on his/her first move, then **B** wins at once. Let **A** take l stones, where $1 \leq l < 2^k$. By inductive assumption **B** could play in such a way that he/she could win the game $(2^k, l)$ since $l \leq 2^k - 1$; the las move will be the move of **B**. After this move we get the position $(2^k, m)$ with $m \leq l$, which is losing for **A**, according to the inductive assumption.

Necessity: It is enough to prove that if n is a power of 2, then $(n, n-1)$ is a winning position. Let $n = 2^k + r$, where $1 \leq r \leq 2^k - 1$. On his/her first move **A** takes r stones and **B** is faced with the position $(2^k, r)$, which is losing for **B**.

10. On sides AB, BC and CA of equilateral triangle ABC, points C_1, A_1 and B_1 are chosen respectively in such a way that the radii of the incircles of the triangles C_1AB_1, B_1CA_1, A_1BC and $A_1B_1C_1$ are equal. Prove that A_1, B_1 and C_1 are midpoints of the corresponding sides.

Solution: We shall prove that $BA_1 = CB_1 = AC_1$. Assume the contrary and let $BA_1 > CB_1 > AC_1$. Let ρ be the rotation through $120°$ with a center that coincides with the center of the incircle of triangle ABC. This rotation transforms the incircles of the triangles C_1BA_1, B_1AC_1 and A_1CB_1 to the incircles of the triangles A_1CB_1, C_1BA_1 and B_1AC_1, respectively. Let $A_2 = \rho(A_1), B_2 = \rho(B_1)$ and $C_2 = \rho(C_1)$. It follows that $BB_2 < BC_1$ and $BC_2 < BA_1$. But B_2C_2 is tangent to the incircle of ΔC_1BA_1, which is a contradiction.

Let r be the radius of the incircles of the triangles C_1AB_1, B_1CA_1, A_1BC_1 and $A_1B_1C_1$. From the triangle B_1AC_1 we have $r = \frac{1-B_1C_1}{2}$.

$\frac{\sqrt{3}}{3}$, and from triangle $A_1B_1C_1$, which is equilateral, we have $r = B_1C_1 \cdot \frac{\sqrt{3}}{6}$. From here $B_1C_1 = \frac{1}{2}$ and consequently A_1, B_1, C_1 are midpoints of the corresponding sides.

11. Let $A = \{1, 2, \ldots, m+n\}$, where m and n are positive integers and let the function $f : A \to A$ be defined by

$$f(i) = i + 1 \quad \text{for} \quad i = 1, 2, \ldots, m-1, m+1, \ldots, m+n-1,$$

$$f(m) = 1 \quad \text{and} \quad f(m+n) = m+1.$$

(a) Prove that if m and n are odd, then there exists a function $g : A \to A$ such that $g(g(a)) = f(a)$ for all $a \in A$.

(b) Prove that if m is even, then $m = n$ if and only if there exists a function $g : A \to A$ such that $g(g(a)) = f(a)$ for all $a \in A$.

Solution:

(a) Let $m = 2p+1, n = 2q+1$ and $g(i) = p+i+1$ for $i = 1, 2, \ldots, p; g(i) = q+i+1$ for $i = m+1, m+2, \ldots, m+q; g(2p+1) = p+1; g(p+1) = 1; g(m+2q+1) = m+q+1; g(m+q+1) = m+1$.

(b) Let $m = n$ and $g(i) = m+i$ for $i = 1, 2, \ldots, m; g(m+i) = i+1$ for $i = 1, 2, \ldots, m-1; g(2m) = 1$.

For the converse let $M = \{1, 2, \ldots, m\}$. It follows by the definition of f that the elements of M remain in M after applying the powers of f with respect to superposition. Moreover, these powers give back the whole M. The same is true for the set $A \setminus M$. The function f is bijective in A, and if there exists g satisfying the conditions, then g is bijective as well. We shall prove that $g(M) \cap M = \emptyset$. It follows from the contrary that there exists $i \in M$ such that $g(i) \in M$. Consider the sequence $i, g(i), g^2(i), \ldots$ and the subsequence $i, f(i), f^2(i), \ldots$. It is easy to see that $g(M) = M$. We deduce that there exists a permutation $a_1, a_2, \ldots, a_m$ of elements of M, such that $g(a_i) = a_{i+1}$ for $i = 1, 2, \ldots, m-1; g(a_m) = a_1$ and $f(a_{2i-1}) = a_{2i+1}$ for $i = 1, 2, \ldots, s-1; f(a_{2s-1}) = a_1$, where $m = 2s$. The last statement contradicts to the properties of f that were mentioned already. Thus $g(M) \cap M = \emptyset$. Analogously $g(A \setminus M) = A \setminus M$, if $g(i) \in A \setminus M$ for $i \in A \setminus M$. At last, let us observe that when starting from an element of M and applying g we go to $A \setminus M$, but when applying g for a second time we go back to M. The same is true for the set $A \setminus M$. From here and from the bijectivity of g it follows that M and $A \setminus M$ have the same number of elements, i.e. $n = m$.

12. Let x and y be different real numbers such that $\dfrac{x^n - y^n}{x - y}$ is an integer for four consecutive positive integral values of n. Prove that $\dfrac{x^n - y^n}{x - y}$ is an integer for all positive integers n.

Solution: Let $t_n = \frac{x^n - y^n}{x-y}$. Then $t_{n+2} + b \cdot t_{n+1} + \mathrm{c} \cdot t_n = 0$ for $b = -(x+y), \mathrm{c} = xy$, where $t_0 = 0, t_1 = 1$. We shall show that $b, \mathrm{c}, \in \mathbf{Z}$. Let $t_n \in \mathbf{Z}$ for $n = m, m+1, m+2, m+3$. Since $\mathrm{c}^n = (xy)^n = t_{n+1}^2 - t_n \cdot t_{n+2} \in \mathbf{Z}$ when $n = m, m+1$, then $\mathrm{c}^m, \mathrm{c}^{m+1} \in \mathbf{Z}$. Therefore c is rational and from $\mathrm{c}^{m+1} \in \mathbf{Z}$ it follows that $\mathrm{c} \in \mathbf{Z}$. On the other hand

$$b = \frac{t_m t_{m+3} - t_{m+1} t_{m+2}}{\mathrm{c}^m},$$

i.e. b is rational. From the recurrence equation it follows by induction that t_n can be represented in the following way: $t_n = f_{n-1}(b)$, where $f_{n-1}(X)$ is a polynomial with integer coefficients, with $\deg f_{n-1} = n - 1$ and with coefficient of x^{n-1} equal to 1. Since b is a root of the equation $f_m(X) = t_{m+1}$, then $b \in \mathbf{Z}$. Now from the recurrence equation it follows that $t_n \in \mathbf{Z}$ for all n.

1.3 China

1. Suppose that $2n$ real numbers $a_1, \dots, a_n, b_1, \dots, b_n$ $(n \geq 3)$ satisfy the following conditions:

   (a) $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$;

   (b) $0 < a_1 = a_2, a_i + a_{i+1} = a_{i+2}, \quad i = 1, 2, \dots, n-2$;

   (c) $0 < b_1 \leq b_2, b_i + b_{i+1} \leq b_{i+2}, \quad i = 1, 2, \dots, n-2$.

   Prove that
   $$a_{n-1} + a_n \leq b_{n-1} + b_n.$$

   Solution: Let F_n be the n-th Fibonacci number (with $F_0 = 0, F_1 = 1$), so that $a_i = F_{i-1}a_1$. Put $d_2 = b_2 - b_1$ and $d_i = b_i - b_{i-1} - b_{i-2}$ for $i > 2$. Then it is easy to see that
   $$b_i = F_{i-1}b_1 + F_{i-2}d_2 + \cdots + F_1 d_i.$$
   Recall that $F_0 + \cdots + F_k = F_{k+2} - 1$ (proof by induction). Then
   $$\begin{aligned} \frac{b_{n-1} + b_n}{b_1 + \cdots + b_{n-2}} &= \frac{F_{n+1}b_1 + F_n d_2 + \cdots + F_2 d_n}{(F_n - 1)b_1 + (F_{n-1} - 1)d_2 + \cdots + (F_1 - 1)d_n} \\ &\geq \frac{F_{n+1}b_1}{(F_n - 1)b_1} = \frac{a_{n-1} + a_n}{a_1 + \cdots + a_{n-2}}. \end{aligned}$$
   The first inequality arises from the fact that if $a, b, c, d > 0$ and $a/b \leq c/d$, then $a/b \leq (a+c)/(b+d)$. Let $s = a_1 + \cdots + a_n$; then we have shown that
   $$\frac{a_{n-1} + a_n}{s - a_{n-1} - a_n} \leq \frac{b_{n-1} + b_n}{s - b_{n-1} - b_n}.$$
   Since $f(x) = x/(s - x)$ is an increasing function on $[0, s]$, this implies $a_{n-1} + a_n \leq b_{n-1} + b_n$.

2. Let $\mathbb{N}$ denote the set $\{1, 2, 3, \dots\}$. Suppose that $f : \mathbb{N} \to \mathbb{N}$ satisfies $f(1) = 1$, and, for any n in $\mathbb{N}$,

   (a) $3f(n)f(2n+1) = f(2n)(1 + 3f(n))$,

   (b) $f(2n) < 6f(n)$.

Find all solutions of the equation

$$f(k) + f(\ell) = 293, \quad k < \ell.$$

Solution: From (a), we see that $3f(n)$ divides $f(2n)(1+3f(n))$. On the other hand, $3f(n)$ is coprime to $1 + 3f(n)$, so it must actually divide $f(2n)$. Thus $f(2n)/(3f(n))$ is an integer, but by (b) it is strictly less than 2. We conclude $f(2n) = 3f(n)$ and $f(2n + 1) = 1 + 3f(n)$ for all n.

Based on these identities, it is easy to show by induction that $f(n)$ can be computed by writing out the base 2 expansion of n and reading the result as a base 3 expansion. That is, if $n = 2^{a_0} + \cdots + 2^{a_k}$ for $a_0 < \cdots < a_k$, then $f(n) = 3^{a_0} + \cdots + 3^{a_k}$.

To solve $f(k) + f(\ell) = 293$, we need to find numbers summing to 293 whose base 3 expansions only contain the digits 0 and 1. Since $293 = (101212)_3$, we get 4 decompositions:

$$\begin{aligned} 293 &= (101)_3 + (101111)_3 \\ &= (111)_3 + (101101)_3 \\ &= (1101)_3 + (100111)_3 \\ &= (1111)_3 + (100101)_3, \end{aligned}$$

depending on how we divide up the 3 places where one number must contain a 1 and the other a 0. We read off the solutions by pretending these numbers are written in base 2:

$$(k, \ell) = (5, 47), (7, 45), (13, 39), (15, 37).$$

3. Find the minimal value of

$$\sum_{i=1}^{10} \sum_{j=1}^{10} \sum_{k=1}^{10} |k(x + y - 10i)(3x - 6y - 36j)(19x + 95y - 95k)|,$$

where x and y are arbitrary real numbers.

Solution: The sum factors as

$$\sum_{i=1}^{10} |x + y - 10i| \sum_{j=1}^{10} |3x - 6y - 36j| \sum_{k=1}^{10} |k(19x + 95y - 95k)|.$$

Observe that a function of the form $f(x) = |x - a_1| + \cdots + |x - a_{2n}|$, with $a_1 < \cdots < a_{2n}$, takes its minimum value for all $x \in [a_n, a_{n+1}]$. (It's piecewise linear, and the slope at a non-corner x is $2m - 2n$, where m is the largest integer such that $a_m < x$.) So the first sum takes its minimum for $50 \leq x + y \leq 60$ and the second for $180 \leq 3x - 6y \leq 216$.

Where is the third sum minimized? Again, view each term as a function of $t = 19x + 95y$ and consider slopes. The slope is $-1 - 2 \cdots - 10$ up to $t = 95$, where the -1 becomes $+1$. Then at $t = 2 \cdot 95$, the -2 becomes $+2$, and so on. The turning point between positive and negative occurs at $t = 7 \cdot 95$. Before this the slope is $1+2+3+4+5+6-7-8-9-10 = -13$, after this the slope is $1+2+3+4+5+6+7-8-9-10 = +1$.

If we can simultaneously solve the inequalities $50 \leq x + y \leq 60$ and $180 \leq 3x - 6y \leq 216$ and the equation $19x + 95y = 7 \cdot 95$, the solution must be a minimum for the product, since it is a minimum for each term. The equation combined with the first inequality gives $53.75 \leq x \leq 66.25$, with the second inequality it gives $52.857 \leq x \leq 61.428$. Thankfully, these do have a common solution, so the minimum is the product of the minima of the three sums, which are $250, 900, 10640$, respectively. Their product is 2394000000.

4. Four balls in space have radii 2,2,3 and 3 respectively. Each ball is tangent to three others. There is another small ball which is tangent to all these four balls. What is the radius of this ball?

Solution: Let A, B be the centers of the balls of radius 2, C, D the centers of the balls of radius 3, O the center of the other ball, and M, N the midpoints of AB, CD, respectively. Then by symmetry, O lies on the perpendicular bisecting planes of AB and CD, which meet in the line MN. Since $AC = AD = 5$, we have $AN \perp CD$, so $AN^2 = AD^2 - DN^2 = 5^2 - 3^2 = 4^2$, and similarly $BN^2 = 4^2$. (That makes $\triangle ABN$ equilateral, but this doesn't seem to make things any easier.) In the same vein, $MN \perp AB$ (since the entire plane CDM is perpendicular to AB) and so $MN^2 = AN^2 - AM^2 = 4^2 - 2^2 = 12$.

Let r be the radius of the small ball; then $MO^2 = (r+2)^2 - 2^2 = r^2 + 4r$ and $NO^2 = (r+3)^2 - 3^2 = r^2 + 6r$. The fact that $MO + NO = MN$

implies

$$\sqrt{r^2+4r}+\sqrt{r^2+6r}=\sqrt{12}.$$

We now solve for r. First, we move the second radical to the right side and square both sides:

$$r^2+4r=r^2+6r+12-2\sqrt{12(r^2+6r)},$$

which simplifies to

$$r+6=\sqrt{12(r^2+6r)}.$$

Square both sides again and simplify:

$$11r^2+60r-36=0.$$

The unique positive root of this equation is $r=6/11$.

5. Suppose that $a_1, a_2, \ldots, a_{10}$ are arbitrary distinct natural numbers whose sum is 1995. What is the minimal value of

$$a_1a_2+a_2a_3+\cdots a_9a_{10}+a_{10}+a_1?$$

Solution: The minimum is 4069, achieved by

$$(a_1,\ldots,a_{10})=(1950,1,9,2,8,3,7,4,6,5).$$

We prove this by starting from an arbitrary configuration, and bringing it to this form by a series of steps that do not decrease the value of the function. First, we may assume a_1 is the largest of the a_i; if $a_i > a_1$, then we can replace $a_1, \ldots, a_i$ by $a_i, \ldots, a_1$ and the sum goes down by $(a_i - a_1)(a_{i+1}-1)$.

By a similar argument, we prove that the sum is not increased by reversing the segment between the second term and the smallest term, so we may assume a_2 is the smallest of the a_i.

Clearly $a_1 \leq 1950$ since $a_2+\cdots+a_10 \geq 1+\cdots+9=45$. If $a_1<1950$, then the numbers $0, a_2, \ldots, a_10$ are not consecutive, and so one of the a_i can be reduced by 1 while keeping all of them positive and distinct. Suppose a_k has this property. In the given expression, a_1 is multiplied by a_2, while a_k is multiplied by some integer M which is greater than a_2. (If $k = 10$, $M = a_9 + 1$, otherwise $M = a_{k-1} + a_{k+1}$. In either case, since a_2 is the smallest a_i, $M > a_2$.) Hence replacing a_1 by $a_1 + 1$ and a_k by $a_k - 1$ reduces the sum by the positive quantity $M - a_2$.

We are now reduced to the case $a_1 = 1950$, in which case $a_2, \ldots, a_{10}$ must equal $1, \ldots, 9$ in some order. Moreover, since we know a_2 is the smallest, $a_2 = 1$. We now resume the reversing procedure begun earlier, which guarantees that the numbers $1, \ldots, 9$ must appear in the order presented above. Hence this configuration achieves the minimum value.

6. Let n be an odd number greater than 1 and

$$X_0 = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) = (1, 0, \ldots, 0, 1).$$

Suppose that

$$x_i^{(k)} = \begin{cases} 0, & \text{if } x_i^{(k-1)} = x_{i+1}^{(k-1)}, \\ 1, & \text{if } x_i^{(k-1)} \neq x_{i+1}^{(k-1)}, \end{cases} \qquad i = 1, 2, \ldots, n,$$

where

$$x_{n+1}^{(k-1)} = x_1^{(k-1)}.$$

Denote

$$X_k = (x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}), \quad k = 1, 2, \ldots.$$

Prove that, if a positive integer m satisfies $X_m = X_n$, then m is a multiple of n.

Solution: More generally, we write $x_i^{(k)} = x_j^{(k)}$ whenever $i \equiv j \pmod{n}$. Let T be the transformation taking $(x_1, \ldots, x_n)$ to $(x_1 + x_2, \ldots, x_n + x_1)$, where the arithmetic is mod 2. Then for vectors v and w, $Tv = Tw$ if and only if v and w are either equal or complementary. Moreover, a vector is in the image of T if and only if it has an even number of 1s. (Clearly this is a necessary condition, but since T is two-to-one, every such vector must occur.) We have $TX_k = X_{k+1}$ by the definition of the X_k. It is easily shown by induction that

$$x_i^{(k+m)} \equiv \sum_{j=0}^{m} \binom{m}{j} x_{i+j}^{(k)} \pmod{2}.$$

Put $X_{-1} = (1, \ldots, 1, 0)$. Note that if $X_a = X_b$ for some $a, b \geq 0$, then X_{a-1} and X_{b-1} are both preimages of X_a under T, and since both are in the image of T, both have an even number of 1s. (Note that X_{-1} is chosen so as to also satisfy this property.) Hence they cannot be complementary, and so we must have $X_{a-1} = X_{b-1}$. Repeating this, we find that $X_{|a-b|-1} = X_{-1}$.

On the other hand, we can express X_{t-1} as $T^t X_{-1}$, or more conveniently as $T^t(0, \ldots, 0, 1)$. In terms of binomial coefficients,

$$\begin{aligned} x_i^{t-1} &\equiv \sum_{j \equiv -i \pmod n} \binom{t}{j} \\ &\equiv \sum_{j \equiv -i \pmod n} \binom{t}{t-j} \\ &\equiv \sum_{k \equiv t+i \pmod n} \binom{t}{j} \equiv x_{-t-i}^{t-1}. \end{aligned}$$

In particular, $x_0^i \equiv x_{-t}^{t-1}$. However, if $X_{t-1} = X_{-1}$, this implies that $x_{-t}^{-1} \equiv 0$, which is only the case when $n|t$. In particular, if $X_m \equiv X_n$, we must have $X_{|m-n|-1} = X_{-1}$, implying $n|m-n$ and so $n|m$, as desired.

1.4 Czech and Slovak Republics

1. Let $ABCD$ be a tetrahedron such that

$$\angle BAC + \angle CAD + \angle DAB = \angle ABC + \angle CBD + \angle DBA = 180^\circ.$$

Prove that $CD \geq AB$.

Solution: Imagine the tetrahedron as being made out of paper. Cut along the edges AD, BD, CD and lay the result flat in the plane. Let D_1, D_2, D_3 be the vertices of the resulting figure between A and B, B and C, and C and A, respectively. The assumption ensures that D_3, A, D_1 are collinear, as are D_1, B, D_2. Moreover, A is the midpoint of D_1D_3 (since both D_1A and D_3A are equal in length to DA), and similarly B is the midpoint of D_1D_2. Hence $D_3D_2 = 2AB$, but by the triangle inequality $D_2D_3 \leq CD_3 + CD_2 = 2CD$. Hence $AB \leq CD$ as desired.

2. Find the positive real numbers x, y given that the four means

$$\frac{x+y}{2}, \quad \sqrt{xy}, \quad \frac{2xy}{x+y}, \quad \sqrt{\frac{x^2+y^2}{2}}$$

are integers whose sum equals 66.

Solution: Label the four means

$$a = \frac{x+y}{2}, g = \sqrt{xy}, h = \frac{2xy}{x+y}, k = \sqrt{\frac{x^2+y^2}{2}}.$$

We cannot have a, g, h, k all equal, since their sum is not a multiple of 4. It easily follows that $h < g < a < k$. If $\mathrm{c} = (a, g)$, then we can write $a = \mathrm{c}a_1$ and $g = \mathrm{c}g_1$, where $(a_1, g_1) = 1$ and $g_1 < a_1$. Since $h = g^2/a = \mathrm{c}g_1^2/a_1$, we have $a_1|\mathrm{c}$, so we may write $\mathrm{c} = da_1$ for some natural number d. In terms of d, a_1, g_1, we now have

$$h = dg_1^2, g = da_1g_1, a = da_1^2, k = \sqrt{2a^2 - g^2} = da_1\sqrt{2a_1^2 - g_1^2}.$$

A rational square root of an integer must itself be an integer, so $\sqrt{2a_1^2 - g_1^2}$ is an integer strictly greater than a_1. Since $d \geq 1$, we have

$$66 = dg_1^2 + da_1g_1 + da_1^2 + da_1\sqrt{2a_1^2 - g_1^2} > 2da_1^2 \geq 2a_1^2.$$

Hence $a_1 < 6$. By explicit computation, the only integers $1 \leq g_1 < a_1 \leq 6$ such that $2a_1^2 - g_1^2$ is a perfect square are $a_1 = 5, g_1 = 1$. From the sum, we find $d = 1$, and so compute $(h, g, a, k) = (1, 5, 25, 35)$. Now x, y are the solutions of the quadratic $t^2 - 50t + 25 = 0$, giving $x, y = 25 \pm 10\sqrt{6}$.

3. In the plane there are given five distinct points and five distinct lines. Prove that two distinct points and two distinct lines can always be selected so that neither of the two selected points lies on any of the two selected lines.

Solution: Let $A_1, \ldots, A_5$ be the points and $p_1, \ldots, p_5$ the lines. First suppose each line contains at most two of the given points. If A_i lies on both p_j and p_k for some i, j, k, then p_j and p_k each contain at most one other point, so we may select the other two points together with p_j and p_k. If no such i, j, k exist, then each point lies on at most one line, so for any two points, at least three lines pass through neither point.

On the other hand, suppose that one of the lines, say p_1, contains three of the points, say A_1, A_2, A_3. Let M_1, M_2, M_3 be the two-element subsets of $\{A_1, A_2, A_3\}$. Since each of $p_2, \ldots, p_5$ meets p_1 in one point, there must be a two-element set M_i neither of whose elements is this intersection point. Since there are only three subsets, two of the lines must yield the same subset, and these two lines together with the two points of the subset give the deisred selection.

4. Decide whether there exist 10,000 ten-digit numbers divisible by seven, all of which can be obtained from one another by a reordering of their digits.

Solution: The number of ways to choose ten digits, with repetitions allowed, is $\binom{19}{9}$ by the usual "stars and bars" argument. This number is $92378 < 10^5$. Since the number of ten-digit multiples of seven exceeds 10^9, some collection of digits must occur as the digits of at least $10^9/10^5 = 10^4$ multiples of seven, as desired.

5. On a circle k with center S two points A, B are given such that $\angle ASB = 90°$. Circles k_1 and k_2 are internally tangent to k at A and B, respectively, and are also externally tangent to one another at Z.

Circle k_3 lies in the interior of $\angle ASB$ and is internally tangent to k at C and externally tangent to k_1 and k_2 at X and Y, repectively. Prove that $\angle XCY = 45°$.

Solution: Let S, S_1, S_2, S_3 be the centers and r, r_1, r_2, r_3 the radii of circles k, k_1, k_2, k_3, respectively. The lengths of the sides of ΔSS_1S_2 are

$$SS_1 = r - r_1, \quad SS_2 = r - r_2, \quad S_1S_2 = r_1 + r_2$$

and $\angle S_1SS_2 = \angle ASB = 90°$. If we complete this triangle to a rectangle SS_1DS_2, then clearly

$$DS_1 = r - r_2, \quad DS_2 = r - r_1, \quad DS = r_1 + r_2.$$

The triangle inequality $S_1S_2 < SS_1 + SS_2$ implies $r_1 + r_2 < 2r - r_1 - r_2$, or $r_1 + r_2 < r$. Hence we may draw a circle centered at D with radius $r - r_1 - r_2$. This circle lies in the interior of $\angle ASB$ and is internally tangent to k_2 and k_3, and hence coincides with k_3. Therefore

$$\angle XCY = \frac{1}{2}\angle S_1DS_2 = \frac{1}{2}90° = 45°,$$

as desired.

6. Find all real p for which the equation

$$x^3 - 2p(p+1)x^2 + (p^4 + 4p^3 - 1)x - 3p^3 = 0$$

has three distinct roots which are the lengths of the sides of a right triangle.

Solution: Let $0 < a < b < c$ be the three roots, and assume that $a^2 + b^2 = c^2$. Recall that the coefficients of the polynomial can be expressed in terms of the roots as follows:

$$\begin{aligned} 2p(p+1) &= a+b+c \\ p^4 + 4p^3 - 1 &= ab + bc + ca \\ 3p^3 &= abc. \end{aligned}$$

Therefore

$$\begin{aligned} 2c^2 &= a^2 + b^2 + c^2 \\ &= (a+b+c)^2 - 2(ab+bc+ca) \\ &= 4p^2(p+1)^2 - 2(p^4 + 4p^3 - 1) = 2(p^2+1)^2 \end{aligned}$$

and so $c = p^2 + 1$. Now we find

$$\begin{aligned} a+b &= 2p(p+1) - c = p^2 + 2p - 1 \\ ab &= p^4 + 4p^3 - 1 - c(a+b) = 2p^3 - 2p. \end{aligned}$$

Therefore a, b are the roots of the quadratic equation

$$u^2 - (p^2 + 2p - 1)u + (2p^3 - 2p) = 0.$$

The roots are easily computed to be $2p$ and $p^2 - 1$. We must also have

$$3p^3 = abc = 2p(p^2 - 1)(p^2 + 1).$$

Since all three roots must be positive, $p > 0$ and $p^2 > 1$, so $p > 1$. The only such root of the last equation is $p = \sqrt{2}$, which gives the roots $1, 2\sqrt{2}, 3$.

1.5 France

1. [Corrected] Let ABC be a triangle. For any line ℓ which is not parallel to any of its sides, consider the centroid G_ℓ of the triangle determined by the intersection of ℓ with AB, BC, and CA. (The centroid of a degenerate triangle is the vector average of the three points.) Find the locus of G_ℓ as ℓ varies.

Solution:

Since centroids are preserved under affine transformations, we reduce to the case where ABC is equilateral. Now fix a line t and consider the set of all lines ℓ parallel to t; we claim the set of G_ℓ for these ℓ is a line tangent to the incircle of ABC.

We show this using vectors. Introduce coordinates so that the sides of the triangle take the form $\vec{u} \cdot \vec{\ell}_i = 1$ for $i = 1, 2, 3$. Now fix a unit vector $\vec{x}$ and let $\vec{y}$ and $\vec{m}_i$ be the images of $\vec{x}$ and vl_i under a 90° counterclockwise rotation. If ℓ is the line $\vec{u} \cdot \vec{y} = k$, then one has $G_\ell = \vec{a} + k\vec{b}$, where

$$\begin{aligned} \vec{a} &= \frac{1}{3}\left(\sum_i \frac{1}{\vec{x} \cdot \vec{\ell}_i}\right)\vec{x} \\ \vec{b} &= \frac{1}{3}\sum_i \frac{1}{\vec{x} \cdot \vec{\ell}_i}\vec{m}_i. \end{aligned}$$

To see this, check that the point $\vec{x}/(\vec{x} \cdot \vec{\ell}_i) + k\vec{m}_i/(\vec{x} \cdot \vec{\ell}_i)$ lies on the lines $\vec{u} \cdot \vec{\ell}_i = 1$ and $\vec{u} \cdot \vec{y} = k$, using the fact that $\vec{x} \cdot \vec{\ell}_i = \vec{y} \cdot \vec{m}_i$.

Let θ be the angle between $\vec{x}$ and $\vec{\ell}_1$; then

$$\vec{a} = (1/3)\vec{x}[\sec\theta + \sec(\theta + 2\pi/3) + \sec(\theta + 4\pi/3)];$$

combining terms and expanding in terms of x, we find

$$\vec{a} = -\sec(3\theta)\vec{x}.$$

In the same fashion, we find

$$\begin{aligned} 3\vec{a} \cdot \vec{b} &= -\sec 3\theta[\tan\theta + \tan(\theta + 2\pi/3) + \tan(\theta + 4\pi/3)] \\ &= -3\sec(3\theta)\tan(3\theta) \end{aligned}$$

and

$$\begin{aligned} 9\vec{b}\cdot\vec{b} &= \sum_i \sec^2(\theta+2\pi i/3) - \sum_{i<j} \sec(\theta+2\pi i/3)\sec(\theta+2\pi j/3) \\ &= 9\sec^2(3\theta). \end{aligned}$$

Clearly as k varies, G_ℓ varies along a line, and the distance from this line to the origin is the length of the component of $\vec{a}$ perpendicular to $\vec{b}$, which by the Pythagorean theorem is

$$\vec{a}^2 - \frac{(\vec{a}\cdot\vec{b})^2}{\vec{b}^2} = \sec^2(3\theta) - \tan^2(3\theta) = 1.$$

The claim follows.

Since every point outside of a circle lies on some tangent to the circle, the locus for ABC equilateral is simply the set of points outside of the incircle of ABC, minus the lines AB, BC, CA. For a general triangle ABC, we deduce that the locus is the set of points outside of the unique ellipse centered at the centroid of ABC and tangent to all three sides, again minus AB, BC, CA.

2. Study the convergence of the sequence defined by $u_0 \geq 0$ and

$$u_{n+1} = \sqrt{u_n} + \frac{1}{n+1}$$

for all nonnegative integers n.

Solution: The sequence converges to 1 for all u_0. Since $u_1 = \sqrt{u_0}+1 \geq 1$, we have $u_i \geq 1$ for all $i \geq 1$. Note, however, that $x - \sqrt{x}$ is increasing for $x \geq 1$. For any $L > 1$, let $d = L - \sqrt{L}$ and choose N so that $1/(N+1) < d/2$. Then for $n \geq N$, as long as $u_n \geq L$, we have

$$u_n - u_{n+1} = u_n - \sqrt{u_n} - \frac{1}{n+1} > L - \sqrt{L} - d/2 = d/2.$$

On the other hand, if $u_n < L$, then $u_{n+1} < u_n + 1/(N+1) < L + d/2$. Thus eventually all values of the sequence are less than $L+d/2$. Since this holds for any $L > 1$, and $L + (L-\sqrt{L})/2 \to 1$ as $L \to 1$, the limit superior of the sequence can be no greater than 1. Since each term is at least 1, the sequence must indeed converge to 1.

3. For three coplanar congruent circles which pass through a common point, let S be the set of points which are interior to at least two of the circles. How should the three circles be placed so that the area of S is minimized?

Solution: The second intersection points of each pair of circles form a triangle XYZ with circumcenter P, the common intersection point of the three circles, and circumradius r, the commom radius of the circles. Since the common intersection of all three circles is just a point, the area interior to at least two circles is the disjoint union of the pairwise intersections of the circles.

Consider, for example, the two circles meeting at P and X. Their common area equals twice the area of the slice of one circle cut off by the chord PX. The area of this slice equals the area of the sector minus the area of the triangle formed by the chord and the center of the circle. Hence the area of the intersection is $r(\theta - \sin\theta)$, where $\theta = \sin^{-1} XY/2r$.

Clearly the minimum cannot occur when XYZ does not contain P, since we can rotate one circle by 180° around P and reduce the total shared area. Hence if $\alpha = \sin^{-1} XY/r$, $\beta = \sin^{-1} YZ/2r$, $\gamma = \sin^{-1} ZX/2r$, then $\alpha + \beta + \gamma = \pi$ and so the total area of intersection is $\pi r - (\sin\alpha + \sin\beta + \sin\gamma)/2$. Since sine is a concave function, this expression is minimized when $\alpha = \beta = \gamma = \pi/2$. In other words, the circles are centered at the vertices of an equilateral triangle of radius r.

4. Let $A_1, A_2, A_3, B_1, B_2, B_3$ be coplanar points such that $A_iB_j = i + j$ for any $i, j \in \{1, 2, 3\}$. What can one say about these six points?

Solution: One can say that all six of the points are collinear. Let C_i be a circle of radius i centered at A_i and let D_j be a circle of radius j centered at B_j; then C_i is externally tangent to D_j. Suppose no two of C_1, C_2, C_3 are tangent; then it is a classical theorem of Vieta that there is a unique circle externally tangent to all three, contradicting our assumptions. (A more modern proof can be given using inversion. Given one circle touching all three, invert through the tangency point of this circle with one of the given three and prove another suitable circle cannot be found in the inverted diagram.)

Hence two of the circles are tangent; one easily shows that they are internally tangent, and that all circles tangent to both touch at this point of tangency. That means five of the six points are collinear, and collinearity of the last point follows immediately.

Note that the conditions can be met by marking a point O on a line, then putting A_1, A_2, A_3 on the line on one side of O and B_1, B_2, B_3 on the other side, such that $OA_i = OB_i = i$.

5. Let f be a bijection from the set of nonnegative integers onto itself. Show that there exist nonnegative integers a, b, c such that $a < b < c$ and $f(a) + f(c) = 2f(b)$.

Solution: Let $a = 0$ and let b be the smallest number such that $f(b) > f(a)$. Then $2f(b) - f(a) = f(b) + (f(b) - f(a)) > f(b) > f(a)$, so $f(c) > f(a)$ and by the definition of b, $c \geq b$. Equality cannot occur since $f(c) > f(b)$ as well, so $c > b$, giving the desired inequalities.

1.6 Germany

1. Let x be a real number such that $x + 1/x$ is an integer. Prove that $x^n + 1/x^n$ is an integer, for all positive integers n.

Solution: Let $s_n = x^n + x^{-n}$; then

$$\begin{aligned} s_{n+1} + s_{n-1} &= x^{n+1} + x^{n-1} + x^{-n+1} + x^{-n-1} \\ &= (x + x^{-1})(x^n + x^{-n}) = s_1 s_n. \end{aligned}$$

Since $s_0 = 1$ and $s_1 = x + x^{-1}$ is assumed to be an integer, by induction s_n is an integer for all n.

2. Let ABC be an equilateral triangle and P a point in its interior. Let X, Y, Z be the projections of P onto BC, CA and AB, respectively. Prove that the sum of the areas of the triangles BXP, CYP and AZP does not depend on P.

Solution: Draw a line through P parallel to BC and let I, F be its intersections with AB, CA, respectively. Similarly, let E, H be the intersections of BC, AB, respectively, with the line through P parallel to CA, and let G, D be the intersections of CA, BC, respectively, with the line through P parallel to AB. Then the area of PXB is the sum of the areas of PBD and PDX, and the former is half of the area of the parallelogram $PIBD$, while the latter is half of the area of the equilateral triangle PDE. Hence the area of PXB is half of the area of the trapezoid $PIBE$. Similarly, the area of PYC is half of the area of $PECG$, and the area of PZA is half of the area of $PGAI$. Since the three trapezoids cover ABC exactly, the sum of the areas of PXB, PYC, PZA is half of the area of ABC, and so is independent of P.

3. Prove that for all integers k and n with $1 \le k \le 2n$,

$$\binom{2n+1}{k-1} + \binom{2n+1}{k+1} \ge 2 \cdot \frac{n+1}{n+2} \cdot \binom{2n+1}{k}.$$

Solution: By symmetry, it suffices to prove the result for $k \le n$. Recall that

$$\binom{m}{n+1} = \frac{m-n}{n+1}\binom{m}{n}.$$

Dividing the desired inequality by $\binom{2n+1}{k}$ and using the above relation, we reach the equivalent form

$$\frac{k}{2n+2-k}+\frac{2n+1-k}{k+1}\geq 2\frac{n+1}{n+2}.$$

Note that the function $f(x)=x/(2n+2-x)=(2n+2)/(2n+2-x)-1$ is convex on $[1,2n+1]$, and the left side of the inequality is just $f(k)+f(2n+1-k)$. By convexity and the assumption $k\leq n$, we have

$$f(k)+f(2n+1-k)\geq f(n)+f(n+1)=\frac{n}{n+2}+1=\frac{2(n+1)}{n+2},$$

as desired.

4. Let ABC be a triangle and let D and E be points on sides BC and CA, respectively such that DE passes through the incenter of ABC. Let S denote the area of triangle CDE and r the inradius of ABC. Prove that $S\geq 2r^2$.

Solution: Let I be the incenter of ABC, and draw the line through I perpendicular to AI. Let D',E' be the intersections of this line with BC,CA, respectively. We claim the area of $CD'E'$ is no greater than that of CDE. To see this, first note that either $CD\geq CD'$ or $CE\geq CE'$; without loss of generality, suppose $CD\geq CD'$. Then $DI\geq IE$ by the Law of Cosines. The area of triangle $DD'I$ equals $1/2DI\cdot D'I\sin\angle D'ID$, while the area of $EE'I$ equals $1/2EI\cdot E'I\sin\angle E'IE$. The two angles are equal, as are $D'I$ and $E'I$, so we deduce that the area of $D'DI$ is at least that of $EE'I$. Adding the area of the quadrilateral $CD'IE$ to both sides, we see that CDE has area no less than that of $CD'E'$.

Hence it suffices to show that the area of $CD'E'$ is at least $2r^2$. This area equals $1/2CI\cdot D'E'=CI\cdot D'I$. By similar triangles, $CI=r\csc A/2$ and $D'I=r\sec A/2$, so the area of $CD'E'$ equals

$$r^2\csc A/2\sec A/2=2r^2\csc A\geq 2r^2,$$

as desired.

5. Find all pairs of nonnegative integers (x,y) such that $x^3+8x^2-6x+8=y^3$.

Solution: Note that for all real x,

$$0 < 5x^2 - 9x + 7 = (x^3 + 8x^2 - 6x + 8) - (x + 1)^3.$$

Therefore if (x, y) is a solution, we must have $y \geq x + 2$. In the same vein, we note that for $x \geq 1$,

$$0 > -x^2 - 33x + 15 = (x^3 + 8x^2 - 6x + 8) - (x^3 + 9x^2 + 27x + 27).$$

Hence we either have $x = 0$, in which case $y = 2$ is a solution, or $x \geq 1$, in which case we must have $y = x + 2$. But this means

$$0 = (x^3 + 8x^2 - 6x + 8) - (x^3 + 6x^2 + 12x + 8) = 2x^2 - 18x.$$

Hence the only solutions are $(0, 2), (9, 11)$.

6. Let a and b be positive integers and let the sequence $(x_n)_{n \geq 0}$ be defined by $x_0 = 1$ and $x_{n+1} = ax_n + b$ for all nonnegative integers n. Prove that for any choice of a and b, the sequence $(x_n)_{n \geq 0}$ contains infinitely many composite numbers.

Solution: Assume on the contrary that x_n is composite for only finitely many n. Take N larger than all such n, so that x_m is prime for all $n > N$. Choose such a prime $x_m = p$ not dividing $a - 1$ (this excludes only finitely many candidates). Let t be such that $t(1 - a) \equiv b \,(\mathrm{mod}\ p)$; then

$$x_{n+1} - t \equiv ax_n + b - b + at = a(x_n - t) \,(\mathrm{mod}\ p).$$

In particular,

$$x_{m+p-1} = t + (x_{m+p-1} - t) \equiv t + a^{p-1}(x_m - t) \equiv 0 \,(\mathrm{mod}\ p).$$

However, x_{m+p-1} is a prime greater than p, yielding a contradiction. Hence infinitely many of the x_n are composite.

7. Two persons, P and Q play the following game. In the equation

$$x^3 + ax^2 + bx + \mathrm{c} = 0,$$

starting with P, the players alternately choose one of the coefficients a, b, c which has not been chosen before, and replace it with a real number. P wins if the resulting equation has 3 distinct real zeros. Determine whether P can win, no matter how Q plays.

Solution: Player P wins by setting $\mathrm{c} = 1$, and then responding to B's move at a or b by choosing the other number so that $a + b = -2$. Then if $f(x) = x^3 + ax^2 + bx + \mathrm{c}$, we have $f(0) = 1$ and $f(1) = -1$. Since $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$, by continuity there must be a root in each of the intervals $(-\infty, 0), (0, 1), (1, \infty)$, so there are three real roots and P wins.

1.7 Greece

1. Find all positive integers n for which $-5^4+5^5+5^n$ is a perfect square. Do the same for $2^4+2^7+2^n$.

   Solution: We are trying to find n such that $2500+5^n=m^2$, which we rewrite as $5^m=m^2-2500=(m+50)(m-50)$. This implies that both $m+50$ and $m-50$ are powers of 5, but the only powers of 5 that differ by 100 are 25 and 125. Hence $n=5$ is the only solution.

   Similarly, we need n such that $2^n+144=m^2$, which we rewrite as $2^n=m^2-144=(m+12)(m-12)$. Again, $m+12$ and $m-12$ differ by 24, but the only powers of 2 that differ by 24 are 8 and 32. Hence $n=8$ is the only solution.

2. Let ABC be an isosceles triangle ($AB=AC$) and let D be a point on BC, such that the incircle of triangle ABD is congruent to the excircle of triangle ADC tangent to DC. Show that the radius of these circles is equal to one quarter of the length of the altitude of triangle ABC drawn from B.

   Solution: By Stewart's theorem, $AD^2=AB^2-BD\cdot CD$. Now recall that the area of a triangle can be expressed either as rs, where r is the inradius and s the semiperimeter, or as $r_A(s-a)$, where r_A is the radius of the excircle opposite A. Let R be the common radius of the two circles; then the area of ABD is $R/2(AB+BD+AD)$ and the area of ADC is $R/2(AD+AC-CD)$. Since these two triangles have the same altitude from A, their areas are proportional to their bases. Therefore
   $$\frac{AB+BD+AD}{AC+AD-CD}=\frac{BD}{CD}.$$
   Multiplying this out and using the fact that $AB=AC$, we get
   $$\begin{aligned}(AB+AD)(BD-CD) &= 2BD\cdot CD\\ &= 2(AB^2-AD^2)\\ &= 2(AB+AD)(AB-AD).\end{aligned}$$
   Therefore $BD-CD=2(AB-AD)$.

   The area of ABC can be computed either as $(h_B\cdot AC)/2$, where h_B is the altitude from A, or as the sum of the areas of ABD and ADC.

Setting these equal and multiplying by 2, we get

$$\begin{aligned} h_B \cdot AC &= R(AB + BD + AD) + R(AD + AC - CD) \\ &= R(2AB + BD - CD + 2AD) \\ &= R(2AB + 2AB) = 4R \cdot AB. \end{aligned}$$

We conclude $R = h_B/4$, as desired.

3. If the equation $ax^2 + (c - b)x + (e - d) = 0$ has real roots greater than 1, show that the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ has at least one real root.

Solution: Assume without loss of generality $a > 0$, and suppose that $P(x) = ax^4 + bx^3 + cx^2 + dx + e$ has no real roots, or in other words that $P(x) > 0$ for all real x. Note that

$$P(x) = ax^4 + (c - b)x^2 + (e - d) + (x - 1)(bx^2 + d).$$

Let y be a root of $ay^2 + (c - b)y + (e - d)$ and let $x = \sqrt{y}$; since $y > 1$, we have x 1. Therefore $P(x) = (x - 1)(bx^2 + d)$ and we deduce that $bx^2 + d > 0$. However, $P(-x) = (-x - 1)(bx^2 + d)$ and this implies $bx^2 + d < 0$, a contradiction. Hence P has a real root.

4. Lines $\ell_1, \ell_2, \ldots, \ell_k$ are in general position in the plane (no two are parallel and no three are concurrent). For which values of k can we label the intersection points of these lines by the numbers $1, 2, \ldots, k-1$ so that in each of the lines $\ell_1, \ell_2, \ldots, \ell_k$ all the labels appear exactly once?

Solution: Such a labeling is possible if and only if k is even. If such a labeling exists, the label 1 must occur once on each line, and each point labeled 1 lies on 2 lines. Hence the number of 1's is $k/2$, and hence k must be even.

On the other hand, suppose k is even. Place the numbers $1, \ldots, k-1$ at the vertices of a regular $(k-1)$-gon and k at its center. Now label the intersection points as follows. The intersection point of ℓ_k with ℓ_m receives the label m for $m = 1, \ldots, k - 1$. If $i, j < k$, then the intersection of ℓ_i with ell_j receives the label m, where the segment from k to m is perpendicular to the diagonal of the $(k - 1)$-gon between i and j. One easily checks that no two points on the same line receive the same label.

1.8 Hungary

1. [Corrected] We cut a rectangle, whose vertices have integer coordinates and whose sides are parallel to the coordinate axes, into triangles with area $1/2$, whose vertices also have integer coordinates. Prove that among the triangles there are at least twice as many right triangles as the length of the shorter side of the rectangle.

   Solution: We start with some observations about triangles of area $1/2$ with integer vertices. A triangle of this form contains no lattice points (boundary included) aside from its vertices, by Pick's theorem. Moreover, given a segment PQ between two lattice points that passes through no others, of length strictly greater than 1, it is not hard to show that there exist exactly two lattice points R and S such that the triangles PQR and PQS have area $1/2$ and have longest side PQ. Obviously $PRQS$ must be a parallelogram.

   Now we reduce the original problem to the case where no triangle has a side longer than $\sqrt{5}$, or equivalently a dissection of the rectangle into right triangles and parallelograms formed from two right triangles. Suppose this is not the case; let PQ be a segment in the dissection of maximum length. In particular, it is the longest side of the two triangles it bounds, whose other vertices we call R and S. By the above, $PRQS$ is a parallelogram, so we can replace the triangles PQR and PQS by PRS and QRS, thus reducing the number of segments of the maximum length, or reducing the maximum itself if PQ was unique. Repeating eventually yields a dissection of the desired form.

2. Let $P(x_1, x_2, \ldots, x_n)$ be a polynomial with n variables. We know that by substituting $+1$ or -1 into all variables, the value of P will be positive if the number of -1's is even, and negative if the number of -1's is odd. Prove that the degree of P is at least n, i.e., it contains a term where the sum of the powers of the variables is at least n.

   Solution: In fact, we will prove that there must be a term that has a nonzero power of each variable (i.e. it is a multiple of $x_1 \cdots x_n$). We prove this by induction on n (easy base case $n = 1$). Viewing P as a polynomial in x_n whose coefficients are polynomials in the other variables, we can write it as $Q + R$, where Q has only terms of odd degree in x_n and R has only terms of even degree. Then

$$Q(x_1, \ldots, x_{n-1}, x_n) = \frac{1}{2}(P(x_1, \ldots, x_n) - P(x_1, \ldots, -x_n)).$$

Now consider the polynomial $T(x_1, \ldots, x_{n-1}) = Q(x_1, \ldots, x_{n-1}, 1)$. If we plug in an even number of -1's and the rest $+1$'s, both $P(x_1, \ldots, x_{n-1}, 1)$ and $-P(x_1, \ldots, x_{n-1}, -1)$ will be positive, so T will be as well. Similarly, if we plug in an odd number of -1's and the rest $+1$'s, T is negative. By the induction hypothesis, T contains a term divisible by $x_1 \ldots x_{n-1}$, so Q does also. But Q only contains terms of odd degree in x_n, so this term must actually be divisible by $x_1 \ldots x_n$, completing the induction.

3. Among points A, B, C, D no three are collinear. AB and CD intersect at E; BC and DA intersect at F. Prove that the circles with diameters AC, BD and EF either all pass through a common point, or no two of them have any common point.

Solution: In fact, we will prove the circles are coaxial (have a common radical axis), which proves the claim: if any two of the circles meet, every circle coaxial with these passes through the same two points. This is a result (and proof) from Coxeter and Greitzer's classic *Geometry Revisited.*

Let H be the orthocenter of triangle $\triangle ADE$. It is easy to show (chase angles) that the angle subtended at H by any side is supplementary to the corresponding angle of the triangle, which makes it equal to the angle inscribed in the arc subtended by that side. Hence the reflection of H across each side lies on the circumcircle. If A', D', E' are the feet of the altitudes from A, D, E, respectively, this means that $AH \cdot A'H$ equals half of the power of H with respect to the circumcircle, and similarly for the other two altitudes. In particular,

$$AH \cdot A'H = DH \cdot D'H = EH \cdot E'H.$$

What about the power of H with respect to the circle with diameter AC? Since A' lies on this circle, the power is again $AH \cdot A'H$. By a similar argument and the above equality, H has the same power with respect to the circles with diameters AC, BD, EF.

On the other hand, the same must be true of the orthocenters of the other three triangles formed by any three of the four lines AB, BC, CD, DA. Since these do not all coincide, the three circles must have a common radical axis.

4. Prove that if for x, y, z distinct real numbers

$$\frac{x}{y-z} + \frac{y}{z-x} + \frac{z}{x-y} = 0,$$

then

$$\frac{x}{(y-z)^2} + \frac{y}{(z-x)^2} + \frac{z}{(x-y)^2} = 0.$$

Solution: We have that

$$\begin{aligned} 0 &= \left(\frac{x}{y-z} + \frac{y}{z-x} + \frac{z}{x-y}\right)\left(\frac{1}{y-z} + \frac{1}{z-x} + \frac{1}{x-y}\right) \\ &= \frac{x}{(y-z)^2} + \frac{y}{(z-x)^2} + \frac{z}{(x-y)^2} \\ &\quad + \frac{y+z}{(z-x)(x-y)} + \frac{z+x}{(x-y)(y-z)} + \frac{x+y}{(y-z)(z-x)}. \end{aligned}$$

When we collect the last three fractions over a common denominator, the numerator becomes

$$(y^2 - z^2) + (z^2 - x^2) + (x^2 - y^2) = 0.$$

Hence as desired,

$$\frac{x}{(y-z)^2} + \frac{y}{(z-x)^2} + \frac{z}{(x-y)^2}.$$

5. Prove that if the vertices of the quadrilateral $PQRS$ are on different sides of a unit square then the perimeter of the quadrilateral $PQRS$ is at least $2\sqrt{2}$.

Solution: Reflect the square across the side containing Q, then reflect the result across the side containing the image of R, then reflect the result across the side containing the image of S. This "unfolds" the quadrilateral into a path between P and its image. Note, however, that the distance between these two is exactly $2\sqrt{2}$

(if we reflect a fourth time across the side containing the image of P, the result is simply a translation down 2 and over 2). Hence the perimeter of $PQRS$ is at least $2\sqrt{2}$ by the triangle inequality, with equality if the quadrilateral makes equal angles against each side.

6. Let k and n be positive integers such that

$$(n+2)^{n+2}, (n+4)^{n+4}, (n+6)^{n+6}, \ldots (n+2k)^{n+2k}$$

end in the same digit in decimal representation. At most how large is k?

Solution: We cannot have $k \geq 5$, since then one of the terms would be divisible by 5 and so would end in a different digit than those not divisible by 5. Hence $k \leq 4$. In fact, we shall see that $k = 3$ is best possible.

Since $x^5 \equiv x \,(\text{m}od\ 10)$ for all x, $x^x \,(\text{m}od\ 10)$ only depends on $x \,(\text{m}od\ 20)$. Hence it suffices to tabulate the last digit of x^x for $x = 0, \ldots, 19$ and look for the longest run. For the evens, we get

$$0, 4, 6, 6, 6, 0, 6, 6, 6, 4$$

while for the odds we get

$$1, 7, 5, 3, 9, 1, 3, 5, 7, 9.$$

Clearly a run of 3 is best possible.

7. [Corrected] Three married couples attend a dinner party. Each person arrives at a different time to the dinner place. Each new commer shakes hands at arrival with all the people present except his or her own spouse. After everybody sat down for dinner, one person asked from all the others the number of handshakes they had when they arrived. What is the rank in arrival of the person who asked the question, if he or she received five different answers?

Solution: There are $\binom{6}{3} - 3 = 12$ handshakes (since husband and wife do not shake hands). Since each person can only give answers between 0 and 4, inclusive, the answers received must be 0,1,2,3,4, and the person asking had 2 handshakes.

What about the order? The first person obviously had 0 handshakes, the second either 0 or 1 (the former if the first and second persons are a couple), the third either 1 or 2. Assuming the answers are as above, the first three answers must be 0,1,2. Working in the other direction, we similarly see that the last three answers must be 2,3,4. Therefore the asker must have entered either third or fourth (clearly both are indeed possible).

8. Given triangle ABC with shortest side BC and let P be a point of AB such that $\angle PCB = \angle BAC$, and Q be a point on AC such that $\angle QBC = \angle BAC$. Prove that the line through the centers of the circumcircles of triangles ABC and APQ is perpendicular to BC.

   Solution: Since the perpendicular from the circumcircle of $\triangle ABC$ to BC is simply the perpendicular bisector of BC, we need to show that the circumcircle O of $\triangle APQ$ also lies on this bisector, which would follow from $OB = OC$. Well, the fact that $\angle PCB = \angle BAC$ implies that triangles $\triangle ABC$ and $\triangle CBP$ are similar, so $BP/BC = BC/BA$ or $BC^2 = BP \cdot BA$. On the other hand, $BP \cdot BA$ is the power of the point B with respect to the circumcircle of $\triangle APQ$, so it equals $OB^2 - r^2$, where r is the circumradius of $\triangle APQ$. We find that $OB^2 = r^2 + BC^2$, and by the analogous argument $OC^2 = r^2 + BC^2$ as well, proving $OB = OC$ and the desired result.

9. The inscribed circle of a triangle cuts a median of the triangle into three pieces inn such a way that the two pieces outside the circle have equal length. Prove that then one side of the triangle must be twice as long as an other side.

   Solution: Let ABC be the triangle, AM the median in question, and X, Y the intersections of AM with the incircle, where $AX < AY$. Let P, Q be the points where the incircle touches BC, CA, respectively; assume without loss of generality that $BM < BP$. By assumption, $AX = YM$, which of course also implies $AY = XM$; by the power-of-a-point theorem,

$$AQ^2 = AX \cdot AY = MX \cdot MY = MP^2.$$

   Therefore $AQ = MP$ and $CA = AQ + QC = MP + PC = MC$ by equal tangents. Since M is the midpoint of BC, $BC = 2MC = CA$

as desired.

10. The product of a few primes is ten times as much as the sum of the primes. What are these (not necessarily distinct) primes?

Solution: Obviously 2 and 5 must be among the primes, and there must be at least one more. Let p be the largest of the remaining primes, and let σ and π be the sum and product, respectively, of the remaining primes. Then we must have $10(7+p+\sigma) = 10p\pi$, or more simply $p\pi = 7 + p + \sigma$.

The product of any collection of numbers, each at least 2, must be at least as large as their sum. For two numbers x and y this follows because

$$0 \leq (x-1)(y-1) - 1 = xy - x - y.$$

The general result follows by induction. In particular, we have $\pi \geq \sigma$. Therefore

$$7 + p + \sigma \geq p\sigma,$$

which in the same vein can be reexpressed as

$$(p-1)(\sigma - 1) \leq 8.$$

We can only have $\sigma = 0$ if there are no primes left, in which case $\pi = 1$ and we must have $p + 7 = p$, contradiction. Hence $\sigma \geq 2$ and so we must have $p - 1 \leq 8$. This leaves $p = 2, 3, 5$ as the only options.

If $p = 5$, we must have $\sigma - 1 \leq 2$ and so the remaining primes must be either a single 2 or a single 3. If $p = 3$, we get $\sigma - 1 \leq 4$ and the remaining primes can be one or two 2s, a 3, or a 2 and a 3. If $p = 2$, we get $\sigma - 1 \leq 8$ and the remaining primes can be at most four 2s.

Checking these possibilities, we find that only $p = 5, \sigma = 3$ gives a solution. Hence the primes in the collection are $2, 3, 5, 5$.

11. Let arbitrary natural numbers label the vertices of a square. The number at any vertex can be replaced by the remainder of the division when the product of the numbers labeling any two other vertices is divided by a given prime p. (The two numbers are freely chosen out of the three.) Prove that by applying the above rule for changing the numbers, using any prime p, in finitely many steps we can

reach the situation where the four numbers at the four vertices are the same.

Solution: If one of the numbers is divisible by p, we can set the other three numbers to 0 by using the multiple of p in our products, then set the fourth to 0. So let us assume instead that all of our numbers are coprime to p. Let a, b be two of the numbers; we begin by replacing the other two numbers by ab (actually its reduction modulo p, but that will be tacit hereafter). Now replace a by $(ab)^2$, b by $(ab)(ab)^2 = (ab)^3$, $(ab)^2$ by $(ab)(ab)^3 = (ab)^4$, and so on. At every stage we have two copies of ab, plus $(ab)^{n-1}$ and $(ab)^n$ for some n, and we can replace $(ab)^{n-1}$ with $(ab)(ab)^n = (ab)^{n+1}$. We do this until we reach an n such that $(ab)^n \equiv 1 \pmod{p}$. Then the other three numbers are $ab, ab, (ab)^{-1}$. Replace one of the copies of (ab) by $(ab)(ab)^{-1} \equiv 1$, and use the two 1s to replace the other two numbers with 1.

12. Show that

$$\left\lfloor \frac{n+2^0}{2^1} \right\rfloor + \left\lfloor \frac{n+2^1}{2^2} \right\rfloor + \left\lfloor \frac{n+2^2}{2^3} \right\rfloor + \cdots + \left\lfloor \frac{n+2^{n-1}}{2^n} \right\rfloor = n$$

for any positive integer n.

Solution: Using the identity

$$\left\lfloor x + \frac{1}{2} \right\rfloor = \lfloor 2x \rfloor - \lfloor x \rfloor$$

the left side becomes the telescoping sum

$$\sum_{i=0}^{n-1} \lfloor 2^{-i} n \rfloor - \lfloor 2^{-i-1} n \rfloor = \lfloor n \rfloor - \lfloor 2^{-n} n \rfloor = n.$$

(Note: this problem appeared as problem 6 on the 1968 IMO.)

13. Consider a finite set of squares, such that the side of every element of the set is less than 1. Prove that if the sum of the areas of these squares is at least 4, then we can arrange these squares to cover a unit square.

Solution: We cover the unit square using the following algorithm. First sort the squares in decreasing order of side length; let $s_1 \geq s_2 \geq \cdots$ be the side lengths. Now define the sequence i_n recursively as follows: $i_0 = 0$, and for $n > 0$, i_n is the smallest integer such that $s_{i_{n-1}+1} + \cdots + s_{i_n} \geq 1$. Now arrange squares $s_{i_{n-1}+1}, \cdots, s_{i_n}$ in a row so as to cover a $1 \times s_{i_{n+1}}$ rectangle. We claim these cover the unit square; to show this, we need only prove that $t_1 + t_2 + \cdots \geq 1$, where for convenience we put $t_n = s_{i_n}$.

Let k be the largest integer such that i_k is defined; then the remaining squares $s_{i_k+1}, \ldots$ have total side length less than 1, and hence their total area is less than 1. As for row n, each square in the row has side length at most t_{n-1}, where $t_0 = 1$ by convention. Hence the squares in the row, except for s_{i_n}, can be covered by a $1 \times t_{n-1}$ rectangle. (Remember that the total side lengths must have been less than 1 before s_{i_n} was added.) Since all side lengths are less than 1, the area of the last square in the row is $t_n^2 < t_n$. Hence the area of all of the squares is no greater than

$$1 + (t_0 + t_1) + (t_1 + t_2) + \cdots \leq 2 + 2 \sum t_i.$$

Since this area is at least 4, we deduce $\sum t_i \geq 1$, which as noted above completes the proof.

1.9 India

1. Let ABC be a triangle and let D, E be points on the sides AB, AC respectively such that DE is parallel to BC. Let P be any point interior to triangle ADE, and let the lines BP, CP meet the side DE at F, G respectively. Let the circumcircles of triangles PDG and PFE meet again at Q. Prove that the points A, P, Q lie on a straight line.

   Solution: (Note: angles will be directed, modulo π.) Let M be the second intersection of AB with the circumcircle of DPG, and let N be the second intersection of N with the circumcircle of EPF. Now $\angle DMP = \angle DGP$ by cyclicity, and $\angle DGP = \angle BCP$ by parallelism, so $\angle DMP = \angle BCP$ and the points B, C, P, M are concyclic. Analogously, B, C, P, N are concyclic. Therefore the points B, C, M, N are concyclic, so $\angle DMN = \angle BCN$. Again by parallels, $\angle BCN = \angle DEN$, so the points D, E, M, N are concyclic.

   We now apply the radical axis theorem to the circumcircles of DGP, EPF, and $DEMN$ to conclude that $DM \cap EN = A$ lies on the line PQ, as desired.

2. Let n be a possitive integer such that n is a divisor of the sum

   $$1 + \sum_{i=1}^{n-1} i^{n-1}$$

   Prove that n is not divisible by any square greater than 1.

   Solution: If $n = mp^2$ for some prime p, then

   $$\begin{aligned} 1 + \sum_{i=1}^{n-1} i^{n-1} &= 1 + \sum_{j=0}^{p-1} \sum_{k=0}^{mp-1} (kp + j)^{n-1} \\ &\equiv 1 + (mp) \left(\sum_{j=0}^{p-1} j^{p-1} \right) \equiv 1 \pmod{p} \end{aligned}$$

   and the sum is not even a multiple of p. Hence if the sum is a multiple of n, n must have no repeated prime divisors, or equivalently no square divisors greater than 1.

3. Let S be a set of cardinality n, and let $A_1, A_2, \ldots, A_n$ be distinct subsets of S. Prove that there exists an element $x \in S$, such that

$$A_1 - \{x\}, A_2 - \{x\}, \ldots, A_n - \{x\}$$

are all distinct.

Solution: Suppose, on the contrary, that upon removing any element of S, two of the A_i become identical. Draw a graph whose vertices are the A_i, and put in one edge corresponding to each $x \in S$ between some pair of A_i that become identical after removing x. Since this graph has n vertices and n edges, it must have a cycle. Without loss of generality, suppose the cycle contains $A_1, \ldots, A_k$ in that order, and let x_i be the element of S that identifies A_i with A_{i+1} (where for the moment we write $A_{k+1} = A_1$). Starting with A_1, and successively interchanging the status of $x_1, \ldots, x_k$, we obtain $A_2, \ldots, A_n$ and finally A_1. However, of these two copies of A_1, one contains x_1 and the other does not. Contradiction.

4. Let $x_1, x_2, \ldots, x_n$ be n positive numbers whose sum is 1. Prove that

$$\frac{x_1}{\sqrt{1-x_1}} + \frac{x_2}{\sqrt{1-x_1}} + \cdots + \frac{x_n}{\sqrt{1-x_1}} \geq \sqrt{\frac{n}{n-1}}.$$

Solution: Let $f(x) = x(1-x)^{-1/2}$; then $f'(x) = (1-x)^{-1/2} + x/2(1-x)^{-3/2}$, which is clearly increasing on $(0,1)$. Hence f is convex there, and by Jensen's inequality the left side is at least $nf(1/n) = (n/(n-1))^{1/2}$.

5. In triangle ABC, let J denote the point of concurrence of the lines that join A, B, C to the points of contact of the respective excircles with the sides BC, CA, AB. Suppose that J lies on the incircle of triangle ABC. Prove that the sum of some two of the sides of the triangle is equal to three times the third side.

Solution: Let a, b, c be the lengths of sides BC, CA, AB, respectively, and $s = (a+b+c)/2$ the semiperimeter. Using equal tangents, one computes that the length of the tangent from B to the excircle opposite A is $s - c$, and so forth. From this one determines (using Menelaos' theorem) that the point J is represented by the vector

$$\frac{(b+c-a)A + (c+a-b)B + (a+b-c)C}{a+b+c}.$$

Similarly, the incenter I is represented by the point

$$\frac{aA + bB + cC}{a+b+c}.$$

(As an aside, we note that $(2I + J)/3 = G$, the centroid of ABC.) If r is the inradius of ABC, then $r(a+b+c) = 2K$, where K is the area. The given condition is that $(I - J)^2 = r^2$, which after multiplying through by $(a+b+c)^2$ and rearranging becomes

$$[(c+a-2b)(B-A) + (a+b-2c)(C-A)]^2 = 4K^2.$$

Of course $(B-A)^2 = c^2$ and $(C-A)^2 = b^2$. From the Law of Cosines, $2(B-A)\cdot(C-A) = b^2 + c^2 - a^2$. The left side now becomes

$$(a-2b+c)^2c^2 + (a-2b+c)(a+b-2c)(b^2+c^2-a^2) + (a+b-2c)^2b^2.$$

From Heron's formula,

$$16K^2 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4.$$

Substituting these expressions into the above equation yields a rather large polynomial:

$$0 = -3(a^4+b^4+c^4) + 4(a^3b+a^3c+\cdots) + 14(a^2b^2+\cdots) - 20abc(a+b+c).$$

Fortunately, the right side factors:

$$0 = (a+b+c)(a+b-3c)(b+c-3a)(c+a-3b).$$

Since $a + b + c \neq 0$, we conclude that the sum of some two of the sides equals three times the third side.

6. Find all positive integers x, y such that $7^x - 3^y = 4$.

Solution: Clearly $y = 0$ does not yield a solution, while $x = y = 1$ is a solution. We show there are no solutions with $y \geq 2$. In this case $7^x \equiv 4 \pmod 9$, which implies $x \equiv 2 \pmod 4$. In particular, we can write $x = 2n$, so that

$$3^y = 7^{2n} - 4 = (7^n + 2)(7^n - 2).$$

Both factors on the right side must be powers of 3, but no two powers of 3 differ by 4. Hence there are no solutions other than $x = y = 1$.

7. Let twenty-one distinct points $P_1, P_2, \ldots, P_{21}$ be arbitrarily chosen on a circle. Prove that there exist at least one hundred arcs P_iP_j that subtend at most 120 degrees at the center of the circle.

Solution: View the points as the vertices of a graph, where two vertices are joined by an edge if their corresponding points subtend an arc of at least 120 degrees. Clearly this graph contains no three-cycles, so by Turán's theorem, it contains at most $\lfloor 21^2/4 \rfloor = 110$ edges, leaving $\binom{21}{2} - 110 = 100$ pairs of points that subtend an arc of at most 120 degrees. Equality is achieved by placing 20 points near one endpoint of a diameter and the other 21 points near the other endpoint of the diameter.

8. Let ABC be a triangle with inradius r Suppose D, E, F are points on the sides BC, CA, AB, respectively, such that the inradii of the triangles $AFE < BDF$ and CED are all equal to r'. Show that the inradius of triangle DEF is $r - r'$.

Solution: Let s and s' be the semiperimeter of ABC and DEF, respectively. Then the semiperimeters of AFE, BDF, CED add up to $s + s'$, since their perimeters cover ABC and DEF exactly once. The area of any triangles equals its inradius times its semiperimeter. Since the areas of AFE, BDF, CED and DEF add up to that of ABC, we have

$$rs = (s + s')(r - r') + ts',$$

where t is the inradius of DEF. A simple calculation yields $t = r - r'$.

9. Let X be finite set and $E(X)$ be the collection of all subsets of X with even cardinality. A real-valued function f is defined on $E(X)$ such that $f(D) > 1995$ for at least one D in $E(X)$ and $f(A \cup B) = f(A) + f(B) - 1995$ for all disjoint sets A and B in $E(X)$. Prove

that X may be partitioned into two disjoint subsets P and Q such that $f(S) > 1995$ for all non-empty S in $E(P)$ and $F(T) \leq 1995$ for all T in $E(Q)$.

Solution: Of all $P \subseteq X$ such that $f(S) > 1995$ for all nonempty $S \in E(P)$, choose P to have the maximum possible number of elements. I claim $f(T) \leq 1995$ for all $T \subseteq X - P$, so that P and $Q = X - P$ form the desired partition. Suppose that on the contrary, $f(T) > 1995$ some T; choose such a T with the minimum possible number of elements. In fact, T must only have two elements, or else we could write T as the disjoint of two nonempty subsets T' and T'', and we would have $f(T') \leq 1995$ and $f(T'') \leq 1995$, implying $f(T) \leq 1995$, contrary to assumption.

Let $T = \{u, v\}$. By our choice of P, neither $P \cup \{u\}$ nor $P \cup \{v\}$ has the property that $f(S) > 1995$ for all nonempty subsets S of even cardinality. Let $S \subseteq P \cup \{u\}$ be a subset with $f(S) \leq 1995$ and choose any $x \in S - \{u\}$. (Obviously S must contain u by the definition of P.) Then S can be written as the union of $\{x, u\}$ and $S - \{x, u\}$, and $f(S - \{x, u\}) > 1995$ if $S - \{u, x\}$ is nonempty. This implies that $f(\{x, u\}) \leq 1995$ (whether or not $S - \{u, x\}$ is nonempty). Similarly, if $y \in P$ is an element of some $S' \subseteq P \cup \{v\}$ such that $f(S') \leq 1995$, then $f(\{v, y\}) \leq 1995$.

However, we must now have $x = y$, or else $f(\{u, v, x, y\}$ would be at once no greater than 1995, by the splitting as $\{u, x\} \cup \{v, y\}$, and greater than 1995, by the splitting as $\{u, v\} \cup \{x, y\}$. Since x was an arbitrary element of S, we must have $S = \{u, x\}$ and similarly $S' = \{v, x\}$. In other words, these are the only nonempty subsets of $P \cup \{u\}$ and $P \cup \{v\}$, respectively, with values no greater than 1995.

We finally note that $P \cup \{u, v\} - \{x\}$ has the property that all of its nonempty subsets S have $f(S) > 1995$. We have checked this for subsets containing u but not v, and vice versa, and for sets containing neither u nor v this follows by the construction of P. As for sets containing both u and v, we can write these as the disjoint union of $\{u, v\}$ and a subset of P, which verifies the claim for these subsets also. However, this set has one more element than P, contradicting the assumption that P was the set with this property with the most elements. Therefore the existence of $T \subseteq X - P$ such that $f(T) > 1995$ is impossible, and the proof is complete.

10. Find all positive integer solutions x, y, z, p, with p a prime, of the equation $x^p + y^p = p^z$.

Solution: The only solutions are $(2^k)^2 + (2^k)^2 = 2^{2k+1}$. First we show there are no solutions if p is odd. If we had one, we could assume x and y were coprime, since their only common factor could be a power of p, and dividing it off would yield another solution. We factor the left side as $(x+y)(x^{p-1} - x^{p-2}y + \cdots - y^{p-1})$, and modulo $x+y$, the second term is congruent to px^{p-1}. Since we have assumed x, y are coprime, this means the only common factor of the first and second factor is p. Since their product is a power of p, one of the factors must equal 1, and since $x, y > 0$, we cannot have $x + y = 1$. Hence the other factor must be 1, i.e. $x^p + y^p = x + y$. But $x^p > x$ for $x > 1$, so we can only have $x = y = 1$, which is not a solution.

In case $p = 2$, write $x = ad, y = bd$, where $(a, b) = 1$. Then d must be a power of 2, and $a^2 + b^2$ is a power of 2. One solution is $a = b = 1$, yielding the solution given above. If this is not the case, $a^2 + b^2$ is a multiple of 4, but this is only possible if a and b are both even, contradicting the assumption that they are coprime.

11. From the sequence $1, 1/2, 1/3, \ldots, 1/n, \ldots$ show that one can choose, for each positive integer k, a subsequence of k distinct numbers such that each number in the subsequence, beginning with the third number, is equal to the difference between the two preceeding terms. Show also that no infinite subsequence with this property exists.

Solution: Suppose $a_1 < \ldots < a_k$ is a sequence of k numbers such that $1/a_1, \ldots, 1/a_k$ has the desired property; from these we will construct a similar sequence with $k + 1$ terms. Suppose that $1/a_1 + 1/a_2 = a/b$. Then the sequence $b, a_1 a, \ldots, a_k a$ has the desired property.

On the other hand, suppose $a_1, a_2, \ldots$ were an infinite sequence with the desired property. Let s_k be the greatest common divisor of $a_1, \ldots, a_k$. Since $1/a_i - 1/a_{i+1} = 1/a_{i+2}$, we have $a_{i+2}(a_{i+1} - a_i) = a_i a_{i+1}$. Let d be the greatest common divisor of a_i and a_{i+1}; then $(a_{i+1} - a_i)/d$ is an integer, so a_{i+2} divides $a_i a_{i+1}/d$, which is also the least common multiple of a_i and a_{i+1}. It follows that $s_1 = s_2 = \cdots$, but this common number has only finitely many divisors.

1.10 Iran

1. In triangle ABC, every side is divided into n equal parts and from each division point, two lines are drawn parallel to two other sides. Count the number of parallelograms that can be seen inside ABC.

   Solution: Each parallogram is determined by two pairs of lines, each pair parallel to a different side of the triangle. Thus the total number of parallelograms is 3 times the number determined by lines parallel to two given sides. We may assume, without loss of generality, that ABC is a right triangle. Let T_n be the number of rectangles formed by segments parallel to the sides when the sides of the triàngle are divided into n equal parts. Then $T_1 = 0$ and $T_2 = 1$. Suppose the sides of the triangle have been divided into n equal pieces and that the segments parallel to the sides of ABC have been drawn. Erasing the bottom row of rectangles gives the case in which the sides are divided into $n-1$ equal parts. The difference between T_n and T_{n-1} is the number of rectangles lost when the row of rectangles is erased. The number of rectangles lost in this process is

$$1(n-1) + 2(n-2) + \cdots + (n-1)1 = \binom{n+1}{3}.$$

   Thus we have

$$T_n = \binom{n+1}{3} + T_{n-1}$$

   which leads to

$$T_n = \binom{n+1}{3} + \binom{n}{3} + \cdots + \binom{3}{3} + T_1 = \binom{n+2}{4}.$$

   Hence the total number of parallelograms in the triangle is

$$3T_n = 3\binom{n+2}{4}.$$

2. [Corrected] Let A, B, and C be three points on a circle with center O. Line CO intersects AB in D and line BO intersects AC in E. If angles BAC, CDE, and ADE have the same angular measure α, find α.

Solution: We first calculate some angles. From the triangle CDA we find $\angle ACD = \pi - 3\alpha$. Now $OC = OA$, so $\angle COA = \pi - 2(\pi - 3\alpha) = 6\alpha - \pi$, and $\angle CBA = 3\alpha - \pi/2$. Since $\angle BOC = 2\angle BAC = 2\alpha$ and $BO = OC$, we have $\angle OCB = \angle CBO = \pi/2 - \alpha$, and $\angle EBA = (3\alpha - \pi/2) - (\pi/2 - \alpha) = 4\alpha - \pi$.

Now by the angle bisector theorem,

$$CD/DA = CE/EA = (CE/EB)(EB/EA).$$

Rewriting both sides of this equation using the Law of Sines, we get

$$\frac{\sin\alpha}{\sin(\pi - 3\alpha)} = \frac{\sin(\pi/2 - \alpha)}{\sin(3\pi/2 - 4\alpha)} \frac{\sin\alpha}{\sin(4\alpha - \pi)}.$$

Multiplying out and making elementary simplifications, we get

$$\sin 3\alpha \cos\alpha = \sin 4\alpha \cos 4\alpha,$$

or $\sin 2\alpha + \sin 4\alpha = \sin 8\alpha$. Rewrite this to get

$$\sin 2\alpha = \sin 8\alpha - \sin 4\alpha = 2\cos 6\alpha \sin 2\alpha$$

and set $\cos 6\alpha = 1/2$. The only root with $\pi - 3\alpha$ and $4\alpha - \pi$ (two of the angles in the diagram) lying in $(0, \pi)$ is $\alpha = 5\pi/18$.

3. Let Z and Q denote the sets of integers and rational numbers, respectively. Find all functions $f : Z - \{0\} \to Q$ such that

$$f\left(\frac{x+y}{3}\right) = \frac{f(x) + f(y)}{2}$$

for every $x, y \in Z - \{0\}$.

Solution: The only such functions are constant functions. We have

$$f(a) = f\left(\frac{a + 2a}{3}\right) = \frac{f(a) + f(2a)}{2},$$

so $f(a) = f(2a)$. In particular, $f(1) = f(2) = f(4)$. Also,

$$f(2) = f\left(\frac{3+3}{3}\right) = \frac{f(3) + f(3)}{2},$$

from which we get $f(2) = f(3)$.

Now suppose that $f(k) = f(1)$ has been established for $k = 1, 2, \ldots, n-1 \geq 4$. Take $i = 1$, 2 or 3 so that $n + i \equiv 0 \pmod 3$. Because $(n+i)/3 \leq n-1$ we have

$$f(1) = f\left(\frac{n+i}{3}\right) = \frac{f(n)+f(i)}{2} = \frac{f(n)+f(1)}{2}.$$

It follows that $f(n) = f(1)$. Hence, by induction, $f(n) = f(1)$ for all positive integers n.

If n is a negative integer, then

$$f(1) = f\left(\frac{n+(-n+3)}{3}\right) = \frac{f(n)+f(3-n)}{2} = \frac{f(n)+f(1)}{2},$$

and it follows that $f(n) = f(1)$.

4. Let $a_1 a_2 \ldots a_n$ be a string of symbols from the set $\{a, b, \ldots, j\}$. Prove that there is a one-to-one function

$$f : \{a, b, \ldots, j\} \to \{0, 1, \ldots, 9\}$$

such that the string $f(a_1)f(a_2)\ldots f(a_n)$ represents an n-digit integer in base 10 that is divisible by 3.

Solution: Let $S = \{a, b, \ldots, j\}$. Given $s \in S$, let n_s be the number of times that s occurs in the string $a_1 a_2 \ldots a_n$. For $i = 0$, 1, 2, define

$$S_i = \{s : n_s \equiv i \pmod 3\}.$$

If one of the sets S_i has six or more elements, assigen f values of 1, 2, 4, 5, 7, 8 to six of the elements of S_i. The remaining four elements of S can be assigned the values 0, 3, 6, 9 in any way we like.

If there is no set with six or more elements, then there must a set S_i with 4 or 5 elements, and some other set S_j with at least two elements. Assign f values of 1, 2, 4, 5 to four of the elements of S_i and value 7, 8 to two elements of S_j. The remaining four elements of S can again be assigned the values 0, 3, 6, 9 in any way we like.

With f values assigned as above we have

$$\sum_{s \in S_i} f(s) \equiv 0 \pmod 3$$

for $i = 0, 1, 2$. Then the sum of the digits of $f(a_1)f(a_2)\dots f(a_n)$ is

$$\sum_{k=1}^{n} f(a_k) \equiv 0 \cdot \sum_{s \in S_0} f(s) + 1 \cdot \sum_{s \in S_1} f(s) + 2 \cdot \sum_{s \in S_2} f(s) \equiv 0 \pmod 3.$$

Thus $f(a_1)f(a_2)\dots f(a_n)$ is divisible by 3.

5. Let M, N and P be points of intersection of the incircle of triangle ABC with sides AB, AC and BC respectively. Prove that the points of intersection of the altitudes of MNP, the circumcenter of ABC, and the incenter of ABC are collinear.

Solution: The orthocenter of $\triangle MNP$ and the incenter of $\triangle ABC$ lie on the Euler line of $\triangle ABC$, so it suffices to prove the circumcenter of $\triangle ABC$ also lies on this line. If we invert through the incircle of $\triangle ABC$, the circumcircle of $\triangle ABC$ inverts to another circle centered on the line through the incenter and the circumcenter of $\triangle ABC$. However, A, B, C invert to the midpoints of $\triangle MNP$, whose circumcircle is the nine-point circle of $\triangle MNP$, and this is indeed centered on the Euler line of $\triangle MNP$. Hence this line also contains the circumcenter of $\triangle ABC$, as desired.

An alternate solution uses complex numbers. Let the incircle of triangle MNP be the unit circle in the complex plane, with $M = 1$, $N = \alpha$, and $N = \beta$. The line we are seeking contains the orthocenter and circumcenter of triangle MNP so must be the Euler line of this triangle. Because the orthocenter is at the point $1 + \alpha + \beta$ and the circumcenter is at 0, the Euler line is given by $t(1 + \alpha + \beta)$, where $-\infty < t < \infty$.

Since side AB is tangent to the circle through M, N, and P at M, this side is on the line with equation $z + \overline{z} = 2$. By rotation, the equations for sides AC and BC are, respectively

$$\overline{\alpha}z + \alpha\overline{z} = 2 \qquad \text{and} \qquad \overline{\beta}z + \beta\overline{z} = 2.$$

The pairwise intersections of these three lines give the affixes for A, B, and C,

$$A = \frac{2\alpha}{1+\alpha}, \qquad B = \frac{2\beta}{1+\beta}, \qquad C = \frac{2\alpha\beta}{\alpha+\beta}.$$

The equation for the perpendicular bisector of AC is

$$\det \begin{pmatrix} z & \overline{z} & 1 \\ \frac{2\alpha}{1+\alpha} & \frac{2}{\alpha+\beta} & 1 \\ \frac{2\alpha\beta}{\alpha+\beta} & \frac{2}{1+\alpha} & 1 \end{pmatrix} = 0,$$

where we have used $\overline{\alpha} = 1/\alpha$ and $\overline{\beta} = 1/\beta$. When expanded and simplified this equation becomes

$$z - \alpha^2 \overline{z} = \frac{2\alpha(\beta - \alpha^2)}{(1+\alpha)(\alpha+\beta)}.$$

We are interested in finding out where this perpendicular bisector of AC meets the Euler line. Substituting $z = t(1+\alpha+\beta)$ and solving for t we find

$$t = \frac{2\alpha\beta}{(1+\alpha)(1+\beta)(\alpha+\beta)}.$$

This quantity is indeed real, and is symmetric in α and β. Hence the perpendicular bisector of BC will meet the line $t(1+\alpha+\beta)$ for the same t value. Thus the circumcenter of triangle ABC also lies on the Euler line of triangle MNP, showing the desired points are collinear.

6. Let $n > 3$ be an odd integer with prime factorization $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ (each p_i is prime.) If $m = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)$, prove that there is a prime p such that p divides $2^m - 1$, but does not divide m.

Solution: Because $m = \phi(n)$ is Euler's phi-function and n is odd, we know by Fermat's Theorem that n divides $2^m - 1$. We consider two cases.

First let $n = p^r > 3$ for some odd prime p. Then $m = p^k - p^{k-1}$ is even and $m \geq 4$. Since p divides

$$2^m - 1 = (2^{m/2} - 1)(2^{m/2} + 1),$$

it must also divide one of the factors on the right. Any prime divisor of the *other* factor (note this factor exceeds 1) will also divide $2^m - 1$ but will not divide $n = p^r$.

If n has at least two distinct prime factors, then $m \equiv 0 \pmod 4$ and $p-1$ divides $m/2$ for each prime factor of n. Hence, by Fermat's Theorem, p also divides $2^{m/2} - 1$. It follows that no prime factor of n divides $2^{m/2} + 1$. Hence any prime factor of $2^{m/2} + 1$ is a factor of $2^m - 1$ but not a factor of n.

1.11 Ireland

1. There are n^2 students in a class. Each week they participate in a contest. Their teacher arranges them into n teams of n players each. For as many weeks as possible, this arrangement is done in such a way that any pair of students who were members of the same team one week are not on the same team in subsequent weeks. Prove that after at most $n+2$ weeks, it is necessary for some pair of students to have been members of the same team on at least two different weeks.

Solution: Each student is on a team with $n-1$ others each week. Therefore, each student can avoid being on a team with another student twice only for $(n^2-1)/(n-1) = n+1$ weeks.

2. Determine with proof all integers a for which the equation

$$x^2 + axy + y^2 = 1$$

has infinitely many distinct integer solutions (x, y).

Solution: The equation has infinitely many solutions if and only if $a^2 \geq 4$. Rewrite the given equation in the form

$$(2x + ay)^2 - (a^2 - 4)y^2 = 4.$$

If $a^2 < 4$, the real solutions to this equation form an ellipse and so only finitely many integer solutions occur. If $a = \pm 2$, there are infinitely many solutions, since the left side factors as $(x \pm y)^2$. If $a^2 > 4$, then $a^2 - 4$ is not a perfect square and so the Pell equation $u^2 - (a^2 - 4)v^2 = 1$ has infinitely many solutions. But setting $x = u - av, y = 2v$ gives infinitely many solutions of the given equation.

3. Let A, X, D be points on a line with X between A and D. Let B be a point in the plane such that the arc $\widehat{ABX}$ subtends an angle greater than $120°$ and let C be a point on the line between B and X. Prove the inequality

$$2AD \geq \sqrt{3}(AB + BC + CD).$$

Solution: By the triangle inequality, $CD \leq CX + XD$. Also, one finds $AX = \sqrt{3}AB$. Therefore

$$\begin{aligned}
2AD &\geq 2AX + 2XD \\
&= \sqrt{3}(AB + BX) + 2XD \\
&\geq \sqrt{3}(AB + BC + CX) + \sqrt{3}XD \\
&\geq \sqrt{3}(AB + BC + CD).
\end{aligned}$$

Equality occurs only for $C = X = D$.

4. Consider the following one-person game played on the x-axis. For each integer k, let X_k be the point with coordinates $(k, 0)$. During the game disks are piled at some of the points X_k. To perform a move in the game, the player chooses a point X_j at which at least two disks are piled and then takes two disks from the pile at X_j and places one of them at x_{j-1} and one at X_{j+1}. To begin the game, $2n + 1$ disks are placed at X_0. The player then proceeds to perform moves in the game as long as possible. Prove that after $n(n + 1)(2n + 1)/6$ moves, no further moves are possible and that at this stage, one disk remains at each of the positions $X_{-n}, X_{-(n-1)}, \ldots, X_{-2}, X_{-1}, X_0, X_1, X_2, \ldots, X_n$.

Solution: The structure of the proof is as follows. We first exhibit an invariant, and a quantity that increases by 2 at each move. We then show the number of positions is finite, implying that the game ends. We show that only one final position is possible, then exhibit a game ending with the given distribution, and calculate the length of the game using the monovariant constructed at the outset.

Let n_k denote the number of disks at X_k. Then $\sum kn_k$ is unchanged by a move and hence always equals 0. On the other hand, $\sum k^2 n_k$ increases by 2 at each move, since

$$(x + 1)^2 + (x - 1)^2 = 2x^2 + 2.$$

Now note that no two piles can be separated by more than one empty point: at some time earlier there was a move either at one of the empty points or at the two points flanking the gap, and at the last such time this move would put a disk into the gap, contradicting the assumption.

From this it follows that the checkers span a range of no more than $4n$. This range must include 0 since $\sum kn_k = 0$. Hence the number of possible positions in the game is finite. Since $\sum k^2 n_k$ is always increasing, the game must eventually reach a terminating position.

Now we shall show that there is only one possible final position. Suppose, on the contrary, that from the initial setup it is possible to reach more than one final distribution. Consider the last move at which this is still possible, and suppose a move at X_m leads to final position A, while a move at X_n leads to final position B. After moving at X_m, there are still two disks on X_n, so a move there is possible, and since we moved first at X_m this must lead to final position B. But the position after moving at X_m and then at X_n is the same after moving at X_n and then at X_m, so it must also lead to position A, contradicting our assumption that we started at the last position from which multiple final positions could be reached. Hence only one final position exists.

We now exhibit a sequence of moves leaving one checker on each of $X_{-n}, \ldots, X_n$. In fact, we need only specify the numbers a_i, where a_i is the number of moves made at X_i. Then the number of disks left on X_i at the end of the game is $a_{i+1} + a_{i-1} - 2a_i - b_i$, where $b_0 = 2n+1$ and $b_i = 0$ for $i \neq 0$. One checks that putting $a_i = a_{-i} = (n-i+2)(n-i+1)/2$ for $i = 1, \ldots, n$ gives the desired distribution, and it is easy to construct a game with this distribution of moves.

In the final position with one checker on each of $X_{-n}, \ldots, X_n$, we have $\sum k^2 n_k = 2(1^2 + \cdots + n^2) = n(n+1)(2n+1)/3$, so the game lasts for $n(n+1)(2n+1)/6$ moves. The proof is complete.

5. Determine with proof all real-valued functions $f(x)$ satisfying the equation

$$xf(x) - yf(y) = (x-y)f(x+y)$$

for all real numbers x, y.

Solution: Any function of the form $f(x) = ax + b$ satisfies the equation. On the other hand, if f is a solution, so is

$$g(x) = f(x) - f(0) - (f(1) - f(0))x,$$

and $g(0) = g(1) = 0$. But then

$$xg(x) + xg(-x) = 2xg(0) = 0$$

and so $g(x) = -g(x)$ for all x. Moreover,

$$(x+y)g(x-y) = xg(x)+yg(-y) = xg(x)-yg(y) = (x-y)g(x+y).$$

Taking $y = x + 1$ yields

$$-g(1 + 2x) = (1 + 2x)g(-1) = 0$$

and we conclude that $g(x) = 0$. Hence the only solutions are of the form $f(x) = ax + b$.

6. Prove the inequalities

$$n^n \leq (n!)^2 \leq \left(\frac{(n+1)(n+2)}{6}\right)^n$$

for every positive integer n.

Solution: The first inequality follows by writing

$$(n!)^2 = \prod_{i=1}^{n} i(n+1-i)$$

and noting that the function $f(x) = x(n + 1 - x)$, being concave, achieves its minimum over the interval $[1, n]$ only at the endpoints. In the other direction, the AM-GM inequality gives

$$(n!)^{2/n} \leq \frac{1}{n}\sum_{k=1}^{n} k(n+1-k) = \frac{(n+1)(n+2)}{6},$$

which is equivalent to the second inequality.

7. Suppose that a, b, c are complex numbers and that all three roots z of the equation

$$x^3 + ax^2 + bx + c = 0$$

satisfy $|z| = 1$. Prove that all three roots w of the equation

$$x^3 + |a|x^2 + |b|x + |c| = 0$$

also satisfy $|w| = 1$.

Solution: Suppose the roots of $P(x) = x^3 + ax^2 + bx + c$ are p, q, r, with $|p| = |q| = |r| = 1$. Note that

$$b = (pq + qr + rp) = pqr(1/p + 1/q + 1/r) = pqr(\overline{p} + \overline{q} + \overline{r}) = -c\overline{a}.$$

Since $|c| = |pqr| = 1$, $|a| = |b|$ and so

$$x^3 + |a|x^2 + |b|x + |c| = (x+1)(x^2 + (|a| - 1)x + 1)$$

has -1 as a root. The other two roots have absolute value one unless they are real and distinct. But the latter only occurs if $||a| - 1| > 2$, while by the triangle inequality

$$||a| - 1| = ||a| - |p|| \leq |q + r| \leq |q| + |r| = 2,$$

so the other two roots also have absolute value 1.

8. Let S be the square consisting of all points (x, y) in the plane with $0 \leq x, y \leq 1$. For each real number t with $0 < t < 1$, let C_t denote the set of all points $(x, y) \in S$ such that (x, y) is on or above the line joining $(t, 0)$ to $(0, 1 - t)$. Prove that the points common to all C_t are those points in S which are on or above the curve $\sqrt{x} + \sqrt{y} = 1$.

   Solution: The line through $(t, 0)$ and $(0, 1 - t)$ has the equation $y = mx + b$, where $m = (t - 1)/t$ and $b = 1 - t$. For a given x, we have
   $$y = x - x/t + 1 - t$$
   and this is minimized when $t = \sqrt{x}$, in which case $y = x + 1 - 2\sqrt{x} = (1 - \sqrt{x})^2$. Therefore (x, y) lies above all of the C_t if and only if $\sqrt{y} \geq 1 - \sqrt{x}$, as claimed.

9. We are given three points P, Q, R in the plane. It is known that there is triangle ABC such that P is the midpoint of the side BC, Q is the point on the side CA with $CQ/QA = 2$ and R is the point on the side AB with $AR/RB = 2$. Determine with proof how the triangle ABC may be reconstructed from P, Q, R.

   Solution: Let S be the midpoint of PQ and construct T on RQ such that $RT : TQ = 2 : 1$. Let U be the intersection of RS and PT, and construct A on the extension of PU such that $PU : UA = 1 : 2$. Now B can be constructed on AR such that $BR : AR = 1 : 2$ and C can be constructed on AQ such that $CQ : AQ = 2 : 1$.

To verify that this works, start with ABC drawn, and add ines through Q and R parallel to BC. Since the line through R is midway between BC and the line through Q, the line through R meets PQ at the midpoint of PQ, which we called S. Similarly, AP meets the line through R at the point U such that $PU : UA$. Hence the constructed points do have the desired properties.

10. For each integer n such that $n = p_1 p_2 p_3 p_4$, where p_1, p_2, p_3, p_4 are distinct primes, let

$$d_1 = 1 < d_2 < d_3 < \cdots < d_{16} = n$$

be the sixteen positive integers which divide n. Prove that if $n < 1995$, then $d_9 - d_8 \neq 22$.

Solution: Note that $35 \cdot 57 = 1995 = 2 \cdot 3 \cdot 7 \cdot 19$. Suppose that $n < 1995$ and $d_9 - d_8 = 22$; then $d_8 d_9 = n$, so $d_8 < 35$. Moreover, d_8 cannot be even since that would make n divisible by 4, whereas n has distinct prime factors. Hence d_8, d_9 and n are odd.

The divisors $d_1, \ldots, d_8$ each are the product of distinct odd primes, since they divide n. Since $3 \cdot 5 \cdot 7 > 35$, none of $d_1, \ldots, d_8$ is large enough to have three odd prime factors, so each is either prime or the product of two primes. Since n only has four prime factors, four of the d_i must be the product of two odd primes. But the smallest such numbers are

$$15, 21, 33, 35, \ldots$$

and so we must have $d_8 \geq 35$, contrary to assumption.

1.12 Israel

1. The positive integers $d_1, d_2, \ldots, d_n$ divide 1995. Prove that there exist d_i and d_j among them, such that the numerator of the reduced fraction d_i/d_j is at least n.

   Solution: Note that $3 \cdot 5 \cdot 7 \cdot 19 = 1995$. If the chosen divisors include one divisible by 19 and another not divisible by 19, the quotient of the two has numerator divisible by 19, solving the problem since $n \leq 16$. If this is not the case, either all divisors are or are not divisible by 19, and in particular $n \leq 8$. Without loss of generality, assume the divisors are all not divisible by 19.

   Under this assumption, we are done if the divisors include one divisible by 7 and another not divisible by 7, unless $n = 8$. In the latter case all of the divisors not divisible by 19 occur, including 1 and $3 \cdot 5 \cdot 7$, so this case also follows. We now assume that none of the chosen divisors is divisible by 4, so that in particular $n \leq 4$.

   Again, we are done if the divisors include one divisible by 5 and another not divisible by 5. But this can only fail to occur if $n = 1$ or $n = 2$. The former case is trivial, while in the latter case we simply divide the larger divisor by the smaller one, and the resulting numerator has at least one prime divisor and so is at least 3. Hence the problem is solved in all cases.

2. Two players play a game on an infinte board that consists of 1×1 squares. Player I chooses a square and marks it with an O. Then, player II chooses another square and marks it with X. They play until one of the players marks a whole row or a whole column of 5 consecutive squares, and this player wins the game. If no player can achieve this, the result of the game is a tie. Show that player II can prevent player I from winning.

   Solution: We exhibit below a tiling of the plane with 1×2 rectangles, such that every row or column of 5 squares contains an entire rectangle. On each move, Player II plays in the other square of the rectangle Player I moved in. This guarantees that no 5 in a row can ever be completed by Player I.

A	A	B	C	D	D
E	F	F	C	G	H
E	I	J	J	G	K
L	I	M	N	N	K
O	O	M	P	Q	Q

3. Two thieves stole an open chain with $2k$ white beads and $2m$ black beads. They want to share the loot equally, by cutting the chain to pieces in such a way that each one gets k white beads and m black beads. What is the minimal number of cuts that is always sufficient?

   Solution: One cut will not suffice in general, since that cut must occur at the middle of the chain, and we can easily ensure that this does not yield a fair division. On the other hand, two cuts will suffice.

   Label the beads $1, \ldots, 2k + 2m$ in order along the chain, and by convention if $n > 2k + 2m$ then bead $n - 2k - 2m$ is also called bead n. For $0 \leq x \leq 2k + 2m - 1$, let $f(x)$ be the number of white beads among the beads numbered $x, x+1, \ldots, x+k+m$. Since every white bead is counted exactly $k+m$ out of $2k+2m$ times, the average value of $f(x)$ is k. Therefore f must take some value no less than k, as well as some value no greater than k. Somewhere between these two values, the value k must occur (since f changes by at most 1 at each step). We remove the corresponding segment with two cuts, give the segment to one thief and give the two ends of the chain to the other thief.

4. Let K, T be two given circles which intersect each other at two points. Find the locus of the centers of all circles that are orthogonal to both K and T.

   Solution: Recall that two circles are orthogonal if and only if the tangent to one circle at one of the points of intersection passes through the center of the other circle. In particular, if O is the center of a circle centered at K and T, the radius of the circle equals the length of a tangent from O to either circle. Therefore, O has the same power with respect to both circles, and hence lies on the radical axis of the circles. Of course, O must lie outside the circles as well.

Conversely, suppose O lies on the radical axis not interior to the circles. Then the tangents from O to the two circles have the same length, so the circle centered at O with radius equal to the length of the tangents is orthogonal to K and T. Hence the locus is the two rays of the radical axis formed by removing the portion lying inside both triangles.

5. Four points are given in space, in general position (i.e., they are not contained in a single plane). A plane π is called an *equalizing* plane if all four points have the same distance from π Find the number of equalizing planes.

Solution: The four points cannot all lie on one side of an equalizing plane, or else they would lie in a plane parallel to the equalizing plane. Hence either three lie on one side and one on the other, or two lie on each side.

We claim each of these divisions yields exactly one equalizing plane. First consider three on one side. This forces the plane to be parallel to the plane through those three points, and it must lie between that plane and the fourth point. As we move from the plane of the three points to the fourth point, there is a unique location at the same distance from the two. This yields exactly one equalizing plane.

The argument is analogous for the two and two case; the equalizing plane is then parallel to the two skew lines through points on the same side, and lies between the two lines. Hence we get 4 planes from the first case and 3 from the second, or 7 overall.

6. Let n be a positive integer and let A_n be the set of all points in ghe plane, whose x and y coordinates are positive integers between 0 and n. A point (i,j) is called *internal* if $0 < i,j < n$. A real function f, definined on A_n, is called a *good* function if it has the following property: for every internal point x, the value of $f(x)$ is the mean of its values on the four neighboring points (the four points whose distance form x equals 1). Let f and g be two given good functions such that $f(a) = g(a)$ for every point $a \in A_n$ which is not internal. Prove that $f \equiv g$.

Solution: Let M be the maximum value of $f - g$, and suppose it is achieved at the point a. If a is internal, the value of $f - g$ is the average at the four neighbors of a, and each of these has value at most M, so in fact each must also achieve the maximum. Repeating this process, we eventually conclude that the maximum is achieved at some noninternal point, which gives $M = 0$. Analogously, the minimum of $f - g$ is also achieved at a noninternal point and hence is also 0. We conclude $f = g$.

7. [Corrected] Find all real solutions of the system

$$\begin{aligned} x + \log(x + \sqrt{x^2+1}) &= y \\ y + \log(y + \sqrt{y^2+1}) &= z \\ z + \log(z + \sqrt{z^2+1}) &= x \end{aligned}$$

There is an obvious solution $x = y = z = 0$; we claim that there are no others. In fact, if $f(x) = x + \log(x + \sqrt{x^2+1})$, then we claim $f(x) > x$ for $x > 1$ and $f(x) < x$ for $x < 1$, which proves the result: if we had a solution with $x > 0$, we would deduce $x < y < z < x$, and a similar contradiction occurs for $x < 0$.

If $x > 0$, then clearly $x + \sqrt{x^2+1} > 1$, and we conclude $f(x) > x$ since the logarithm is an increasing function. If $x < 0$, we simply note that

$$x + \sqrt{x^2+1} = \frac{1}{\sqrt{x^2+1} - x} < 1$$

since $x^2 + 1 > 1$ and $-x > 0$. Hence $f(x) < x$ in this case.

8. Let PQ be the diameter of semicircle H. Circle O is internally tangent to H and tangent to PQ at C. Let A be a point on H and B a point on PQ such that AB is orthogonal to PQ and is also tangent to O. Prove that AC bisects $\angle PAB$.

Solution: Perform an inversion centered at C, and use primes to denote images under the inversion. Since $\angle PAC = \angle A'P'C$ and $\angle CAB = \angle CB'A'$, the original claim $\angle PAC = \angle CAB$ is equivalent to $\angle A'P'C = \angle CB'A'$.

The line PQ remains fixed, the circle O becomes a line H' parallel to PQ, while the semicircle and the line AB become a pair of circles tangent to O' and orthogonal to $P'Q'$. The orthogonality simply means both circles are centered on $P'Q'$, which together with the tangency to O' means they are congruent (their common radius is the distance between the lines). Hence the arcs $A'Q'$ and AC are equal, and so are their inscribed angles $\angle A'P'Q' = \angle A'P'C$ and $\angle AB'C'$.

9. Let α be a given real number. Find all functions $f : (0,\infty) \to (0,\infty)$ such that

$$\alpha x^2 f\left(\frac{1}{x}\right) + f(x) = \frac{x}{x+1}$$

holds for all real positive x.

Solution: Put $g(x) = f(x)/x$, so the given equation becomes

$$\alpha g(1/x) + g(x) = \frac{1}{x+1}.$$

Applying the equation twice, once with $x = y$ and once with $x = 1/y$, we find

$$\begin{aligned} \alpha g(1/y) + g(y) &= \frac{1}{y+1} \\ \alpha g(y) + g(1/y) &= \frac{1}{1+1/y} = \frac{y}{y+1}. \end{aligned}$$

If $\alpha = \pm 1$, these equations have no common solution. Otherwise, we can solve for $g(y)$:

$$g(y) = \frac{1-\alpha y}{(y+1)(1-\alpha^2)}.$$

For y close to 0, $1 - \alpha y > 0$, and since $g(y) > 0$, we must have $\alpha^2 < 1$. On the other hand, we must also then have $1 - \alpha y > 0$ for all y, which is only possible for $\alpha < 0$.

On the other hand, this expression does yield a valid solution in case $-1 < \alpha < 0$, which by our arguments must be unique.

1.13 Japan

1. Let n and r be positive integers such that $n \geq 2$ and $r \not\equiv 0 \pmod{n}$, and let g be the greatest comon divisor of n and r. Prove that

$$\sum_{i=1}^{n-1} \left\{ \frac{ri}{n} \right\} = \frac{1}{2}(n-g),$$

where $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x.

Solution: We have

$$\sum_{i=1}^{n-1} \left\{ \frac{ri}{n} \right\} = \sum_{i=1}^{n-1} \left\{ \frac{(r/g)i}{(n/g)} \right\}.$$

Because the GCD of n/g and r/g is 1, the numbers $(r/g)i$ run through a complete residue system modulo n/g when i runs through n/g consecutive integers, and for these values of i the numbers $\left\{ \frac{(r/g)i}{(n/g)} \right\}$ give the set

$$\left\{ \frac{0}{n/g}, \frac{1}{n/g}, \frac{2}{n/g}, \ldots, \frac{(n/g)-1}{n/g} \right\}.$$

Hence the desired sum is

$$g \sum_{i=0}^{(n/g)-1} \left\{ \frac{(r/g)i}{n/g} \right\} = g \sum_{k=0}^{(n/g)-1} \frac{k}{n/g} = g \frac{((n/g)-1)(n/g)}{2(n/g)} = \frac{1}{2}(n-g).$$

2. Let $f(x)$ be a rational function (i.e., a quotient of two polynomials) which is not a constant, and let a be a real number. Find all pairs of a and $f(x)$ which satisfy $f(x)^2 - a = f(x^2)$.

Solution: Let $f(x) = p(x)/q(x)$ where p and q are polynomials and q is monic. In the expression

$$\frac{p(x)^2 - aq(x)^2}{q(x)^2} = f(x)^2 - a = f(x^2) = \frac{p(x^2)}{q(x^2)},$$

the rational functions on the left and right have monic denominators of the same degree. It follows that $q(x)^2 = q(x^2)$ and hence, by coefficient matching, that $q(x) = x^n$ for some nonnegative integer n.

If $a = 0$, then $f(x)^2 = f(x^2)$ implies that $p(x)^2 = p(x^2)$ and again it follows that $p(x) = x^m$ for some nonnegative integer m. Then and $f(x) = x^m/x^n = x^k$ for some non-zero integer k.

If $a \neq 0$ and $q(x) = 1$, then it follows quickly that no solutions are possible. Hence we may assume that $q(x) = x^n$ with $n \neq 0$ and that $p(x)$ has a non-zero constant term. We then have

$$p(x)^2 - ax^{2n} = p(x^2).$$

Assume that $p(x) = a_m x^m + a_r x^r + \cdots + a_s x^s + a_0$ with a_m, $a_0 \neq 0$. By matching coefficients, we find that $a_m = a_0 = 1$, $a_r = a_s = 0$, $a = 2$, and $m = 2n$.

Hence the solutions $(a, f(x))$ are

$$(0, x^n) \qquad \text{and} \qquad \left(2,\ x^n + \frac{1}{x^n}\right),$$

n a non-zero integer.

3. Let $ABCDE$ be a convex pentagon. Let S, R be the points of intersection of BE with AC, AD respectively. Let T, P be the points of intersection of BD with CA, CE respectively. Let Q be the point of intersection of CE and AD. Assume that the areas of triangles ASR, BTS, CPT, DQP, ERQ are all equal to 1.

   (a) Determine the area of pentagon $PQRST$.

   (b) Determine the area of pentagon $ABCDE$.

Solution: In quadrilateral $AEQS$, let $\theta = \angle ARS = \angle ERQ$. Then

$$\frac{1}{2}(SR)(RA)\sin\theta = [SRA] = 1 = [ERQ] = \frac{1}{2}(ER)(RQ)\sin\theta.$$

Hence $SR/RQ = ER/RA$ and it follows that triangles ARE and QRS are similar and then that AE is parallel to SQ. Let $SR = x$ and $RQ = y$. Then $RE = \alpha x$ and $RA = \alpha y$ for some real number α. We then have

$$[SRQ] = \frac{1}{2}xy\sin(\angle SRQ) = \frac{1}{2}xy\sin(\angle QRE) = \frac{1}{\alpha}[QRE] = \frac{1}{\alpha},$$

and $[ARE] = \alpha$ follows from a similar calculation.

Now consider triangle ACE. Triangle ACE is similar to triangle SCQ and the ratio of similiarty is α. Hence

$$2 + \alpha + \frac{1}{\alpha} + [SQPT] + 1 = [ACE] = \alpha^2[SCQ] = \alpha^2([SQPT] + 1).$$

We can solve this last expression for $[SQPT]$ in terms of α. Then

$$[RSTPQ] = [SQPT] + [RST] = \frac{3 + \alpha + \frac{1}{\alpha} - \alpha^2}{\alpha^2 - 1} + \frac{1}{\alpha} = \frac{3 - \alpha}{\alpha - 1}.$$

The above argument can be repeated starting with quadrilateral $BTRA$, $CPSB$, $DQCT$, or $ERPD$. If β is the ratio for the similar triangles involved in the argument, we are lead to $[RSTPQ] = (3 - \beta)/(\beta - 1)$. Because the function f defined by $f(x) = (3 - x)/(x - 1)$ is a one-to-one function, it follows that $\beta = \alpha$. Hence

$$[ABS] = [BCT] = [CDP] = [DEQ] = [ERA] = \alpha$$

and $[PQRST] = (3 - \alpha)/(\alpha - 1)$.

We now know that $[ABE] = [ABC]$. Because these triangles share side AB they must have equal altitudes to this side. It follows that CE is parallel to AB, with similar parallel relationships holding for the other side-diagonal pairs. Thus $ABPE$ is a parallelogram. Because a diagonal of a parallelogram divides it into two triangles of equal area, we have $2\alpha + 1 = [ABE] = [BPE] = 2 + [PQRST]$. Solving this expression for $[PQRST]$ we find

$$2\alpha - 1 = [PQRST] = \frac{3 - \alpha}{\alpha - 1},$$

from which we obtain

$$\alpha = \frac{1 + \sqrt{5}}{2}.$$

Thus we have

(a) $[PQRST] = 2\alpha - 1 = \sqrt{5}$.

(b) $[ABCDE] = 5(1 + \alpha) + [PQRST] = \dfrac{15 + 7\sqrt{5}}{2}$.

4. [Corrected] Define a sequence $(a_i)_{i\geq 1}$ by $a_{2n} = a_n$ and $a_{2n+1} = (-1)^n$. A point P moves on the coordinate plane as follows:

   (a) Let P_0 be the origin. First P moves from P_0 to $(1,0)$. Denote this point by P_1.

   (b) After P has moved to P_i, it turns $90°$ to the left and moves forward 1 unit if $a_i = 1$, and turns $90°$ to the right and moves forward 1 unit if $a_i = -1$. Denote this point by P_{i+1}.

   Prove that P does not pass through the same segment twice.

Solution: We begin with an observation. From the path traced by P, we can construct a new path by joining $P_0, P_2, P_4, \ldots$. How does this path behave at P_{2n}? Suppose that P makes a left turn at P_{2n-1}. Then P must make a right turn at P_{2n+1}. If a left turn occurs at P_{2n}, the net move from P_{2n-2} to P_{2n} to P_{2n+2} is a left turn, and similarly if a right turn occurs at P_{2n}. The analysis also proceeds likewise if P makes a right turn at P_{2n-1}. In summary, the turn made in the new path at P_{2n} is in the same turn as at P_{2n} in the old path. However, since $a_{2n} = a_n$, this is also the same direction as the turn made at P_n in the old path. In other words, the new path can be obtained from the old by a homothety of ratio $\sqrt{2}$ followed by a $45°$ rotation, both centered at P_0.

Now suppose, by way of contradiction, that the segment from P_i to P_{i+1} is the same as the segment from P_j to P_{j+1}, and that i is the smallest number for which such j exists. The argument now splits into two cases. In the first case, $P_i = P_j$ and $P_{i+1} = P_{j+1}$, whence we must have $i \equiv j \pmod 4$, since between P_i and P_j we must make an even number of moves both horizontally and vertically. If i and j are even, then $a_{i+1} = a_{j+1}$, and so the abridged path traces the same segment between P_i and P_{i+2} as between P_j and P_{j+2}. By our observation, the original path must trace the same segment between $P_{i/2}$ and $P_{i/2+1}$ as between P_j and $P_{j/2+1}$, contradicting the minimality of i. Similarly, if i and j are odd, then $a_i = a_j$, and so the segment between $P_{(i-1)/2}$ and $P_{(i+1)/2}$ is retraced, again a contradiction.

In the second case, $P_i = P_{j+1}$ and $P_j = P_{i+1}$. Now i and $j+1$ must be congruent mod 4. If i is even, this means $a_{i+1} = -a_{j+2}$, and so the segment from P_i to P_{i+2} coincides with the segment from P_{j-1} to

P_{j+1}. Similarly, if i is odd, the segment from P_{i-1} to P_{i+1} coincides with the segment from P_j to P_{j+2}. As before, this contradicts the minimality of i by the observation.

We conclude that neither case can occur, and so P never traces the same segment twice.

5. Let k and n be integers such that $1 \leq k \leq n$, and assume that $a_1, a_2, \ldots, a_k$ satisfy

$$\begin{aligned} a_1 + a_2 + \cdots + a_k &= n \\ a_1^2 + a_2^2 + \cdots + a_k^2 &= n \\ &\vdots \\ a_1^k + a_2^k + \cdots + a_k^k &= n. \end{aligned}$$

Prove that

$$(x+a_1)(x+a_2)\cdots(x+a_k) = x^k + \binom{n}{1}x^{k-1} + \binom{n}{2}x_{k-2} + \cdots + \binom{n}{k}.$$

Solution: For positive integer ℓ let

$$S_\ell = a_1^\ell + a_2^\ell + \cdots + a_k^\ell,$$

and let

$$T_\ell = \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell \leq k} a_{i_1} a_{i_2} \cdots a_{i_\ell}$$

be the ℓ-th symmetric functions of $a_1, a_2, \ldots, a_k$. The well-known Newton formulae relate these power sums and symmetric sums by

$$S_\ell - T_1 S_{\ell-1} + - \cdots + (-1)^{\ell-1} S_1 T_{\ell-1} + (-1)^\ell \ell S_\ell = 0, \qquad \ell \leq k.$$

Now the T_ℓ's are the coefficients of the expanded polynomial, that is,

$$(x+a_1)(x+a_2)\cdots(x+a_k) = x^k + T_1 x^{k-1} + T_2 x^{k-2} + \cdots + T_k.$$

It is easy to show that under the conditions $S_1 = S_2 = \cdots = S_k = n$, we have $T_1 = \binom{n}{1}$ and $T_2 = \binom{n}{2}$. Using the Newton formulae, the identity

$$T_\ell = \binom{n}{\ell}, \qquad 1 \leq \ell \leq k,$$

follows easily by induction.

1.14 Korea

1. Consider finitely many points in the plane such that, if we choose any three points A, B, C among them, the area of triangle ABC is always less than 1. Show that all of these points lie within the interior or on the boundary of a triangle with area less than 4.

   Solution: Let triangle ABC have the largest area among all triangles whose vertices are taken from the given set of points. Then $[ABC] \leq 1$. Let triangle LMN be the triangle whose medial triangle is ABC. Then $[LMN] = 4[ABC] \leq 4$. We claim that the set of points must lie on the boundary or in the interior of LMN. Suppose a point P lies outside LMN. It is easy (draw a picture) to connect P with two of the vertices of ABC forming a triangle with larger area than ABC, contradicting the maximality of $[ABC]$.

2. For a given positive integer m, find all paris (n, x, y) of positive integers such that m, n are relatively prime and $(x^2 + y^2)^m = (xy)^n$, where n, x, y can be represented in terms of m.

   Solution: If (n, x, y) is a solution, then the AM-GM inequality yields
   $$(xy)^n = (x^2 + y^2)^m \geq (2xy)^m > (xy)^m,$$
   so $n > m$. Let p be a common prime divisor of x and y and let $p^a \| x, p^b \| y$. Then $p^{(a+b)n} \| (xy)^n = (x^2 + y^2)^m$. Suppose $b > a$. Since $p^{2a} \| x^2, p^{2b} \| y^2$, we see that $p^{2a} \| x^2 + y^2$ and $p^{2am} \| (x^2 + y^2)^m$. Thus $2am = (a + b)n > 2an$ and $m > n$, a contradiction. Likewise, $a > b$ produces a contradiction, so we must have $a = b$ and $x = y$. This quickly leads to $x = 2^t$ for some integer t and all solutions are of the form
   $$(n, x, y) = (2t + 1, 2^t, 2^t)$$
   for nonnegative integers t.

3. Let A, B, C be three points lying on a circle, and let P, Q, R be midpoints of arcs BC, CA, AB, respectively. AP, BQ, CR intersect BC, CA, AB at L, M, N, respectively. show that
   $$\frac{AL}{PL} + \frac{BM}{QM} + \frac{CN}{RN} \geq 9.$$

When does equality hold?

Solution: Note that lines AL, BM, CN are angle bisectors of angles A, B, C (of triangle ABC) respectively. Thus $BL = \frac{ca}{b+c}, CL = \frac{ba}{b+c}$. We also have $AL \cdot PL = BL \cdot CL = \frac{a^2bc}{(b+c)^2}$. Since $[ABL] + [ALC] = [ABC]$ we have (using the $\frac{1}{2}ab\sin C$ area formula)

$$AL = \frac{bc}{b+c}\frac{\sin A}{\sin \frac{A}{2}} = \frac{2bc}{b+c}\cos\frac{A}{2}.$$

Using the previous results and the law of cosines yields (after some work)

$$\frac{AL}{PL} = \left(\frac{b+c}{a}\right)^2 - 1,$$

with similar formulas for $\frac{BM}{QM}$ and $\frac{CN}{RN}$. We have

$$\frac{AL}{PL} + \frac{BM}{QM} + \frac{CN}{RN} = \left(\frac{b+c}{a}\right)^2 + \left(\frac{c+a}{b}\right)^2 + \left(\frac{a+b}{c}\right)^2 - 3.$$

Showing that this quantity is greater than or equal to 9 is pretty simple and can be done in many ways. One way is to use AM-GM on each term. Denoting the product abc by P, we have

$$\left(\frac{b+c}{a}\right)^2 = \frac{b^2+c^2}{a^2} + \frac{2bc}{a^2} \geq \frac{2bc}{a^2} + \frac{2bc}{a^2} = \frac{4P}{a^3}.$$

Hence the sum of the three terms is greater than or equal to

$$4P\left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}\right) \geq 3 \cdot 4P\sqrt[3]{\frac{1}{a^3b^3c^3}} = 12.$$

4. A partition of a positive integer n is a sequence $(u_1, u_2, \ldots, u_k)$ of positive integers such that $u_1+u_2+\cdots+u_k = n$ and $u_1 \geq u_2 \geq \cdots \geq 1$. Each u_i is called a summand. For example, (4,3,1) is a partition of 8 whose summands are distinct. Show that, for a positive integer m with $n > \frac{1}{2}m(m+1)$, the number of all partitions of n into m distinct summands is equal to the number of all partitions of $n - \frac{1}{2}m(m+1)$ into r summands ($r \leq m$).

Solution: Let $u_1 + u_2 + \cdots + u_m = n$ be a partition of n into m distinct summands. Distinctness and monotonicity imply that

$$u_m \geq 1, u_{m-1} \geq 2, \ldots, u_1 \geq m.$$

Define $v_k = u_k - (m - k + 1)$. The v_k are nonnegative, and sum to $n - \frac{1}{2}m(m+1)$. Moreover, each set of u_i correspond to a different set of v_i. This correspondence immediately yields the desired equality.

5. If we select at random three points on a given circle, find the probability that these three points lie on a semicircle.

Solution: The probability is $\frac{3}{4}$. The problem is equivalent to finding the probability that the triangle formed by the three points is obtuse. Imagine dropping the points down (at random) on the circle in the order A, B, C. Let us compute the probability that triangle ABC is obtuse. One possibility is that $\angle A$ is obtuse. Consider the diameter ℓ of the circle which is perpendicular to the diameter through A. The probability that $\angle A$ is obtuse is equal to the probability that both B and C lie on the same side of ℓ as A, which is $(\frac{1}{2})^2 = \frac{1}{4}$. The other possibility is that A is not obtuse, but one of B or C is instead. To compute this probability, consider the diameter through A. Point B can be placed anywhere, but C must lie on the same side of this diameter as B, which occurs with probability $\frac{1}{2}$. Hence the required probability is the sum, $\frac{3}{4}$.

6. Show that any positive integer $n > 1$ can be expressed by a finite sum of numbers satisfying the following conditions:

   (a) They do not have factors other than 2 or 3.

   (b) Any two of them are neither a factor nor a multiple of each other.

In other words, show that $n = \sum_{i=1}^{N} 2^{a_i} 3^{b_i}$, where a_i, b_i are nonnegative integers and $(a_i - a_j)(b_i - b_j) < 0$ whenever $i \neq j$.

Solution: Use (strong) induction on n. Assume that for every $m \leq n$, the required representation is possible. Note that if m

can be represented, then so can $2m$ (just multiply each term in the representation of m by 2). So if n is odd, then it is easy to see that $n+1$ has the required representation — use the fact that $(n+1)/2$ has a representation, and multiply it by 2.

If n is even, then if $n+1$ is a power of 3, we are done. Otherwise, there is a positive integer k such that $3^k < n+1 < 3^{k+1}$. Now $n+1-3^k$ is even and less than $2n$, so $n+1-3^k$ can be represented by the above argument. Thus $n+1 = 3^k +$ the representation for $n+1-3^k$. It is easy to check that this is a valid representation, and we are done.

7. Find all real-valued functions f defined on nonzero real numbers such that

$$\frac{1}{x}f(-x) + f\left(\frac{1}{x}\right) = x, \quad x \neq 0.$$

Solution: Substitute $x = -y$ into the given equation, and we have

$$-\frac{1}{y}f(y) + f\left(-\frac{1}{y}\right) = -y.$$

Substitute $x = 1/y$, and we get

$$yf(\frac{1}{y}) + f(y) = \frac{1}{y}.$$

Now solve for $f(y)$, and we get

$$f(y) = \frac{1}{2}\left(y^2 + \frac{1}{y}\right).$$

8. Two circles O_1, O_2 with radii r_1, r_2 $(r_1 < r_2)$, respectively, intersect at two points A and B. P is any point on circle O_1. Lines PA, PB and circle O_2 intersect at Q and R, respectively.

   (a) Express QR in terms of r_1, r_2 and $\angle APB$.

   (b) Show that $QR = 2r_2$ is a necessary and sufficient condition that circle O_1 is orthogonal to circle O_2.

Solution: There are two cases: either P lies in the exterior of circle O_2 or in the interior. We will do the first case; the other is quite similar.

(a) Let $\angle PAB = \alpha$ and $AB = a$. Then $a = 2r_1 \sin\theta$ and

$$\frac{PR}{PA} = \frac{PQ}{PB} = \frac{QR}{AB} = \frac{y}{a},$$

$BQ/\sin\alpha = 2r_2$, and

$$\frac{PB}{\sin\alpha} = \frac{a}{\sin\theta} = \frac{PA}{\sin(\theta+\alpha)} = 2r_1.$$

Applying the law of cosines to triangle PBQ yields

$$BQ^2 = PB^2\left(1 + \frac{y^2}{a^2} - 2\frac{y}{a}\cos(\angle BPQ)\right).$$

Some algebra gives us

$$y = 2\sin\theta\left(r_1\cos\theta + \sqrt{r_2^2 - r_1^2\sin^2\theta}\right).$$

(b) If the circles are orthogonal, then clearly $r_2 = r_1\tan\theta$. Conversely, if $y = 2r_2$, substituting into formula (a) quickly yields $r_2/r_1 = \tan\theta$.

9. For any positive integer m, show that there exist integers a, b satisfying

$$|a| \le m, \quad |B| \le m, \quad 0 < a + b\sqrt{2} \le \frac{1+\sqrt{2}}{m+2}.$$

Solution: Define $f(x,y) = x + y\sqrt{2}$ and let

$$S = \{f(a,b)|a,b, \text{ integers with } 0 \le a,b \le m\}.$$

Because $\sqrt{2}$ is irrational, S has $(m+1)^2$ distinct elements, the largest of which is $m(1+\sqrt{2})$. Divide the interval $[0, m(1+\sqrt{2})]$ into m^2+2m subintervals of length $\frac{1+\sqrt{2}}{m+2}$. By the pigeonhole principle, there exist two *distinct* $f(a_1,b_1) > f(a_2,b_2)$ in the same subinterval. Without loss of generality, $f(a_1,b_1) > f(a_2,b_2) > 0$. It is easy to check that $a = a_1 - a_2$ and $b = b_1 - b_2$ fit the bill.

10. Let A be the set of non-negative integers. Find all functions $f : A \to A$ satisfying the following two conditions:

(a) For any $m, n \in A$,

$$2f(m^2+n^2) = (f(m))^2 + (f(n))^2.$$

(b) For any $m, n \in A$ with $m \geq n$,

$$f(m^2) \geq f(n^2).$$

Solution: Substituting $m = 0$ and $n = 0$ in (a) yields

$$f(m)^2 - f(n)^2 = 2(f(m^2) - f(n^2)).$$

By (b), if follows that f is monotone increasing, i.e., if $m \geq n$ then $f(m) \geq f(n)$. Plugging in $m = n = 0$ into (a) yields $f(0) = 0$ or 1.

- Case I: $f(0) = 1$. Then $2f(m^2) = f(m)^2 + 1$ so $f(1) = 1$. Likewise, $f(2) = 1$. Also,

$$f(2^{2^n}) = \frac{1}{2}\left(f(2^{2^{n-1}})^2 + 1\right).$$

  This implies that $f(2^k) = 1$ for all non-negative integers k. By the monotonicity of f, we conclude that $f(n) = 1$ for all non-negative integers n.

- Case II. $f(0) = 0$. Then $2f(m^2) = f(m)^2$, or $f(m^2)/2 = (f(m)/2)^2$. Since $f(2) = f(1)^2$, so

$$\begin{aligned} \frac{f(2^{2^n})}{2} &= \left(\frac{f(2^{2^{n-1}})}{2}\right)^2 = \left(\frac{f(2^{2^{n-2}})}{2}\right)^{2^2} = \cdots \\ &= \left(\frac{f(2)}{2}\right)^{2^n} = \frac{f(1)^{2^{n+1}}}{2^{2^n}}. \end{aligned}$$

  Since $2f(1) = f(1)^2$, either $f(1) = 0$ or $f(1) = 2$. If $f(1) = 0$, the above chain of equalities implies that $f(2^{2^n}) = 0$ for $n \geq 0$. Thus $f(n) = 0$, using the previous argument from case I. If $f(1) = 2$, then $f(2^{2^n}) = 2 \cdot 2^{2^n}$. Since $f(m^2)/2 = (f(m)/2)^2$, $f(m)$ is always even. We have

$$f(m+1)^2 = 2f((m+1)^2) \geq 2f(m^2+1) = f(m)^2 + f(n)^2 > f(m)^2,$$

which implies that $f(m+1) > f(m)$. Consequently, $f(m+1) - f(m) - 2 \geq 0$. But

$$\sum_{m=0}^{2^{2^{n}-1}} (f(m+1) - f(m) - 2) = f(2^{2^n}) - f(0) - 2 \cdot 2^{2^n} = 0.$$

Since n is arbitrary, we conclude that $f(m+1) = f(m) + 2$ for all $m \geq 0$. Thus $f(n) = 2n$.

In conclusion, $f(n) \equiv 1$ or $f(n) \equiv 0$ or $f(n) \equiv 2n$ are the only possibilities.

11. Let ABC be an equilateral triangle with side length 1, D a point on BC, and let r_1, r_2 be inradii of triangles ABD, ADC, respectively. Express $r_1 r_2$ in terms of BD and find its maximum value.

Solution: Let $\ell = AD$, and $p = BD$. Law of Cosines yields $\ell^2 = p^2 - p + 1$. Since the inradius times the semiperimeter equals the area of a triangle, we have

$$r_1 \frac{1+p+\ell}{2} = \frac{\sqrt{3}}{4} p, \quad r_2 \frac{2-p+\ell}{2} = \frac{\sqrt{3}}{4}(1-p).$$

Multiplying these two expressions and substituting for ℓ^2 yields

$$\begin{aligned} r_1 r_2 &= \frac{1}{4}(1-\ell) = \frac{1}{4}\left(1 - \sqrt{p^2 - p + 1}\right) \\ &= \frac{1}{4}\left(1 - \sqrt{(p-1/2)^2 + 3/4}\right) \leq \frac{2-\sqrt{3}}{8}. \end{aligned}$$

This quantity then is the maximum, attained when $p = 1/2$.

12. Let O and R be the circumcenter and circumradius of triangle ABC, respectively, and let P be any point in the interior of triangle ABC. Let perpendiculars PA_1, PB_1, PC_1 be dropped to the three sides BC, CA, AB, respectively. Express $[A_1B_1C_1]/[ABC]$, in terms of R and OP, where $[ABC]$ denotes the area of triangle ABC.

Solution: Quadrilateral PA_1CB_1 is inscribed in a circle of diameter CP, so $A_1B_1 = PC \sin C$. Likewise, $B_1C_1 = AP \sin A$. Let

D be the intersection point between line CP and the circumcircle of triangle ABC. Since $\angle PB_1A_1 = \angle PCA_1 = \angle DAB$ and $\angle PB_1C_1 = \angle PAC_1$, we see that $\angle A_1B_1C_1 = \angle PB_1A_1 + \angle PB_1C_1 = \angle DAB + \angle PAC_1 = \angle PAD$. In triangle PAD we have $AP/\sin B = DP/\sin(\angle PAD) = DP/\sin(\angle A_1B_1C_1)$ because $\angle PDA = \angle B$. Thus $AP\sin(\angle A_1B_1C_1) = DP\sin B$. This yields

$$[A_1B_1C_1] = \frac{1}{2}A_1B_1 \cdot B_1C_1 \sin(\angle A_1B_1C_1) = \frac{1}{2}PC \cdot DP \sin A \sin B \sin C.$$

Since $[ABC] = 2R^2 \sin A \sin B \sin C$, we have

$$\frac{[A_1B_1C_1]}{[ABC]} = \frac{1}{4R^2}PC \cdot DP.$$

If P lies in the interior of circle O, then $PC \cdot DP = R^2 - d^2$, and if P lies in the exterior of circle O, then $PC \cdot DP = d^2 - R^2$. Finally, if P lies on the circle, then A_1, B_1, C_1 are collinear (Simson's line) and $[A_1B_1C_1] = 0$. In summary, we have

$$\frac{[A_1B_1C_1]}{[ABC]} = \frac{|R^2 - d^2|}{4R^2}.$$

13. Let p be a prime number such that

(a) p is the greatest common divisor of a and b;

(b) p^2 is a divisor of a.

Prove that the polynomial $x^{n+2} + ax^{n+1} + bx^n + a + b$ cannot be factored as a product of two polynomials with integral coefficients, whose degrees are greater than one.

Solution: This is a simple consequence of the *Eisenstein Irreducibility Criterion:*

> Let $f(x) = c_0x^n + c_1x^{n-1} + \cdots + c_n$ be a polynomial with integral coefficients. If there exists a prime p such that $p \not| c_0, p|c_i, (i = 1, 2, \ldots, n)$ and $p^2 \not| c_n$, then $f(x)$ is irreducible, i.e. cannot be expressed as the product of two polynomials with integer coeffecients of lower degrees.

To prove Eisenstein's Criterion, assume that

$$f(x) = \sum_{i=0}^{n+1} a_i x^i \sum_{j=0}^{n+1} b_j x^j = c_0 x^n + c_1 x^{n-1} + \cdots + c_n,$$

where it is understood that some of these coefficients may be zero. Then $a_0 b_0 = c_n$. Since $p^2 \not| c_n$, either $p|a_0$ or $p|b_0$, but not both. Assume without loss of generality that $p|a_0$ and $p \not| b_0$. Since $c_1 = a_0 b_b + a_1 b_0$, we have $p|a_1$. Continuing inductively, we see that $p|a_i, (i = 0, 1, 2, \ldots, n)$. Let $\deg(\sum_{i=0}^{n+1} a_i x^i) = s$. Then $\deg(\sum_{j=0}^{n+1} b_j x^j) = n - s$. Now we have $a_s b_{n-s} = c_0$, so $p|c_0$, a contradiction.

14. Let m, n be positive integers with $1 \leq n \leq m - 1$. A box is locked with several padlocks, all of which must be opened to open the box, and all of which have different keys. Each m people have keys to some of the locks. No n people can open the box but any $n + 1$ people can open the box. Find the smallest number ℓ of locks and in that case find the number of keys that each person has.

    Solution: Let the m people be denoted $A_1, A_2, \ldots, A_m$ and imagine writing on each lock the indices of the people who do not hold a key for that lock. Since any $n + 1$ people can open the box, no lock has more than $n + 1$ indices on it. For any n people, there is at least one lock with the corresponding n indices. Thus the smallest number ℓ of locks is $\binom{m}{n}$ and each person has $\binom{m-1}{n}$ keys.

1.15 Poland

1. Find the number of those subsets of $\{1, 2, \ldots, 2n\}$ in which the equation $x + y = 2n + 1$ has no solutions.

   Solution: We must ensure that x and $2n + 1 - x$ are not both chosen for $x = 1, \ldots, n$. We may either choose one, choose the other or choose neither. Hence we have three options for each of these n subsets, leaving 3^n subsets in total.

2. A convex pentagon is partitioned by its diagonals into eleven regions: one pentagon and ten triangles. What is the maximum number of those triangles that can have equal areas?

   Solution: Six of the areas can be equal, but seven cannot be. Let A, B, C, D, E be the vertices of the original pentagon, and F, G, H, I, J the vertices of the inner pentagon opposite A, B, C, D, E, respectively. We first construct a pentagon with six equal areas. Start with an equilateral triangle AIH, and mark points B, E on line IH such that $BI : IH : HE = 3 : 1 : 3$. Also mark points J, C on AI such that $AI = IJ = JC$ and mark G, D on AH such that $AH = HG = GD$. By Menelaos' theorem, C, G, E are collinear, so this does result in a pentagon, and the areas of triangles $BAI, BIJ, BJC, EAH, EHG, EGD$ are all equal.

   Now suppose seven areas are equal. We refer to $ABI, BJC, \ldots$ as the "outer" triangles, and $AHI, BIJ, \ldots$ as the "inner" triangles. We cannot have AIH, ABI, BIJ all with equal areas, for that implies $AI = IJ, BI = IH$, and $AHBJ$ must be a parallelogram, contradicting the fact that AH and BJ meet at D. Similarly, no two consecutive inner triangles can occur together with the outer triangle between them.

   How can we choose the seven equal areas subject to this restriction? We must choose at least two inner triangles; it is easily shown that choosing all five outer triangles forces the pentagon to be affinely equivalent to a regular pentagon, in which case no inner triangle has the same area as any outer triangle. Choosing four inner triangles eliminates three of the outer triangles, leaving a total of only six areas. Hence we must choose three inner triangles, and not all consecutive. We may as well choose AHI, CJF, DFG, and the four outer triangles must be AIB, BJC, DGE, EHA.

By an affine transformation, we may make AIB equilateral. Since $AI = CJ$, we must have FH parallel to IJ and FI parallel to GH, which means FIH is also equilateral. By checking angles, we find $BJ \perp AJ$, which means the area of BJC is equal to that of BJA, which is the sum of the areas of BJI and BIA. However, BJC and BIA were assumed to have the same area. This contradiction means seven areas cannot occur, so six is indeed the maximum.

3. Let $p \geq 3$ be a given prime number. Define a sequence (a_n) by $a_n = n$ for $n = 0, 1, 2, \ldots, p-1$, and $a_n = a_{n-1} + a_{n-p}$ for $n \geq p$. Determine the remainder when a_{p^3} is divided by p.

Solution: Let $u_1, \ldots, u_p$ be the roots of $x^p - x^{p-1} - 1$ over the field of p elements. The roots are stable under the map $x \mapsto x^p$, so

$$u_i^{p^2} = u_i^{p(p-1)} - 1.$$

On the other hand,

$$0 = u_i^{p(p-1)}(u_i^p - u_i^{p-1} - 1) = u_i^{p^2} - u_i^{p^2-1} - u_i^{p(p-1)}.$$

We conclude that

$$u_i^{p^2-1} = 1$$

and hence all of the u_i lie in the field of p^2 elements. We recall that any recurrent sequence of the form $a_n \equiv a_{n-1} + a_{n-p}$ can be expressed in the form $a_n \equiv c_1 u_1^n + \cdots + c_p u_p^n$ for some c_i. From this we note that $a_{p^2 n} \equiv a_n$, since $u_i^{p^2 n} \equiv u_i^n$. Hence $a_{p^3} \equiv a_p = p - 1$.

4. For a fixed integer $n \geq 1$ compute the minimum value of the sum

$$x_1 + \frac{x_2^2}{2} + \frac{x_3^3}{3} + \cdots + \frac{x_n^n}{n},$$

given that $x_1, x_2, \ldots, x_n$ are positive numbers satisfying the condition

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = n.$$

Solution: Let $M = 1 + \cdots + 1/n$. We shall prove M is the desired minimum; note that it is achieved by setting $x_1 = \cdots = x_n = 1$. By the weighted power mean inequality, $x_i^i + (i-1) \geq ix_i$. Therefore

$$x_1 + \frac{x_2^2}{2} + \cdots + \frac{x_n^n}{n} \geq x_1 + x_2 - \frac{1}{2} \cdots + x_n - \frac{n-1}{n} = x_1 + \cdots + x_n - n + M.$$

On the other hand, by the arithmetic-harmonic mean inequality,

$$\frac{x_1 + \cdots + x_n}{n} \geq \frac{n}{1/x_1 + \cdots + 1/x_n} = 1.$$

We conclude that the given expression is at least $n - n + M = M$. Since we have seen M is achieved, it is the desired minimum.

5. [Corrected] Let n and k be positive integers. From an urn containing n tokens numbered 1 through n, the tokens are drawn one by one without replacement, until a number divisible by k appears. For a given n, determine all those $k \leq n$ for which the expected number of draws equals k.

Solution: We begin by solving the question: if m of the n tokens are marked and we draw without replacement until a marked token appears, what is the expected number of draws? This number can be written as 1 plus the sum over k that at least $k+1$ draws are needed; this probability is

$$\frac{(n-m)\cdots(n-m-k)}{n\cdots(n-k)}.$$

Note, however, that this is equal to

$$\frac{1}{m+1}\left((n-k)\frac{(n-m)\cdots(n-m-k)}{n\cdots(n-k)} \right.$$
$$\left. -(n-m-k-1)\frac{(n-m)\cdots(n-m-k)}{n\cdots(n-k)}\right).$$

We thus have a telescoping sum, and the desired probability comes to $1 + (n-m)/(m+1) = (n+1)/(m+1)$.

In the original question, $m = \lfloor n/k \rfloor$, and the question becomes for which k do we have $n + 1 = k(\lfloor n/k \rfloor + 1)$. This happens precisely when $n \equiv -1 \pmod{k}$, and so the desired k are simply the divisors of $n+1$.

6. Let k, ℓ, m be three non-coplanar rays emanating from a common origin P and let A be a given point on k (other than P). Show that there exists exactly one pair of points B, C, with B lying on ℓ and C on m, such that

$$PA + AB = PC + CB \quad \text{and} \quad PB + BC = PA + AC.$$

Solution: We first rewrite the given conditions as

$$PA - BC = PB - CA = PC - AB$$

and note that they are equivalent to the existence of a sphere tangent to the rays k, ℓ, m, with A, B, C between P and the corresponding points of tangency, and also tangent to the segments AB, BC, CA. Indeed, if such a sphere exists, the equalities follow by the law of equal tangents. Conversely, if the equations are satisfied, draw the sphere tangent to k, ℓ, m with the length of the tangents from P equal to

$$PA + PB + AB = PB + PC + BC = PC + CA + AC.$$

This is also the length of the tangents from P to the excircle of PAB opposite P, and so AB is tangent to the sphere as well, and so forth.

Instead of fixing A, we will fix the radius of the sphere and show that there exist unique positions for A, B, C satisfying the conditions. This is equivalent to the given problem because we can rescale any such configuration to bring A to a particular point.

We now move A along k from P to the sphere, and move B along ℓ so that AB remains tangent to the sphere; clearly B moves from the sphere back to P. Similarly, if we keep C on m so that BC remains tangent to the sphere, C moves from P to the sphere, and if we keep D on k so that CD remains tangent to the sphere, D moves from the sphere to P. Since A and D are moving in opposite directions, there is a unique position at which they coincide, and this gives the unique configuration satisfying the given conditions.

1.16 Romania

1. Let ABC be a triangle, $D \in (BC)$ such that AD is perpendicular to BC and let E, F be the centers of the circles inscribed into triangles ABD and ACD, respectively. Line EF intersects AB and AC in K and L, respectively. Show that $AK = AL$ if and only if $AB = AC$ or $A = 90°$.

Solution: Let r_1 and r_2 be the radii of circles inscribed in triangles ABD and triangle ACD respectively. Then

$$r_1 = 4R_1 \sin(\angle BAD/2) \sin(\angle B/2) \sin(45°)$$

and, after a short computation, $r_1 = (\cos(B) + \sin(B) - 1)R\sin(C)$, $r_2 = (\cos(C)+\sin(C)-1)R\sin(B)$. Since AE and AF bisect $\angle KAT$ and $\angle TAL$ respectively, we have that $ET/EK = AT/AK$, and $FT/FL = AT/AL$. Suppose $AK = AL$. It follows that $ET/EK = FT/FL \Rightarrow ET/TK = FT/TL \Rightarrow ET/FT = KT/TL$. But, in $\triangle AKT$, we have:

$$\frac{KT}{\sin(90° - B)} = \frac{AT}{\sin(90° - (A/2))} \Rightarrow KT = \frac{AT\cos(B)}{\sin(90° - (A/2))}$$
$$\frac{LT}{sin(90° - C)} = \frac{AT}{\sin(90° - (A/2))} \Rightarrow KT = \frac{AT\cos(C)}{\sin(90° - (A/2))}$$

These two equations imply

$$\frac{KT}{TL} = \frac{\cos(B)}{\cos(C)}.$$

But $ET/FT = r_1/r_2$, so it follows that

$$\frac{\cos(B)}{\cos(C)} = \frac{(\cos(B) + \sin(B) - 1)\sin(C)}{(\cos(C) + \sin(C) - 1)\sin(B)}$$

or equivalently,

$$\frac{\sin(2B)}{\cos(2B)} = \frac{\sin(B) + \cos(B) - 1}{\sin(C) + \cos(C) - 1}.$$

Now, let $x = \sin(B) + \cos(B)$ and $y = \sin(C) + \cos(C)$. Then $x^2 = 1 + \sin(2B)$ and $y^2 = 1 + \sin(2C)$. Hence $\frac{x^2-1}{y^2-1} = \frac{x-1}{y-1} \Rightarrow x =$

y. It follows that $\sin(B) + \cos(B) = \sin(C) + \cos(C)$, so, after a trivial calculation, $B = C$ or $A = 90°$. Conversely, the calculation is straightforward.

Second Solution

If $\triangle ABC$ is isosceles, obviously KL is parallel to BC. Now suppose $A = 90°$. We will prove that $AK = AL = AD$. Let K' and L' be points on AB and AC respectively such that $AK' = AL' = AD$, and E', F' the intersection points of $K'L'$ with the bisectors of $\angle BAD$ and $\angle CAD$.

Clearly $\triangle AK'E' \equiv \triangle ADE'$, and $\triangle ADF' \equiv \triangle AL'F'$. Since $\angle AK'E' = \angle AL'F' = 45°$, it follows that $\angle ADE' = \angle ADF' = 45°$. Therefore $E' = E$ and $F' = F$, so $K' = K$ and $L' = L$.

Conversely, suppose $AK = AL$. Let D' be the point on AD such that $AK = AL = AD'$. If $D \neq D'$, then $\triangle AKE \equiv \triangle AD'E$, and $\triangle ALF \equiv \triangle AD'F$. So $\angle ED'A = \angle FD'A$, and $\angle ED'D = \angle FD'D$. Therefore $\triangle ED'D \equiv \triangle FD'D$. It follows that EF is parallel to BC, and $\triangle ABC$ is isosceles. If $D' = D$, we similarly obtain $\angle AKE = \angle ADE$. But $\angle ADE = 45°$, so $A = 90°$.

2. Find all integer and positive solutions (x, y, z, t) of the equation

$$(x + y)(y + z)(z + x) = txyz$$

such that $(x, y) = (y, z) = (z, x) = 1$.

Solution: It is obvious that $(x, x + y) = (x, x + z) = 1$, then x divides $y + z$, y divides $z + x$ and z divides $x + y$. Let a, b, and c be integers such that

$$\begin{aligned} x + y &= \mathrm{c}z \\ y + z &= ax \\ z + x &= by. \end{aligned}$$

If we consider these equalities in a system of linear equations having a non-zero solution, we get: $\Delta = ab\mathrm{c} - 2 - a - b - \mathrm{c} = 0$ which is the determinant of

$$\begin{pmatrix} 1 & 1 & -c \\ 1 & -b & 1 \\ -a & 1 & 1 \end{pmatrix}$$

The Diophantine equation $abc - 2 = a + b + c$ can be solved by considering the following cases:

(a) $a = b = c$. Then $a = 2$ and it follows that $x = y = z$, as $(x, z) = (y, z) = (z, x) = 1$. This means that $x = y = z = 1$ and $t = 8$. Therefore we have obtained the solution $(1, 1, 1, 8)$.

(b) $a = b$, $a \neq c$. The equation becomes:

$$a^2c - 2 = 2a + c \Leftrightarrow c(a^2 - 1) = 2(a + 1) \Leftrightarrow c(a - 1) = 2$$

If $c = 2$, it follows that $x = y = z$ (which is case a). So $c = 1$ and, immediately, $x = y = 1$ and $z = 2$. So the solution is $(1, 1, 2, 9)$.

(c) $a > b > c$. In this case, $abc - 2 = a + b + c < 3a$. Therefore $a(bc - 3) < 2$. It follows that $bc - 3 < 2 \Rightarrow bc < 5$. We have the following cases:

i. $b = 2$, $c = 1 \Rightarrow a = 3$ and we return to case b).
ii. $b = 3$, $c = 1 \Rightarrow a = 5$. We obtain the solution $(1, 2, 3, 10)$.
iii. $b = 4$, $c = 1 \Rightarrow 3a = 7$, impossible.

Finally the solutions are: $(1, 1, 1, 8), (1, 1, 2, 9), (1, 2, 3, 10)$ and those obtained by permutations of x, y, z.

3. Let n, p be natural numbers, $n \geq 6$ and $3 \leq p \leq n - p$. The vertices of a regular n-gon are colored with red and black: p red vertices and $n - p$ black vertices. Show that there exist two congruent polygons, having each at least $\lfloor p/2 \rfloor + 1$ vertices, one of them with red vertices and the other with black vertices.

Solution: Consider the rotations R_k, $k = 1, 2, \ldots, n - 1$ of the polygon by the angles $2k\pi/n$. Let a_i be the number of red vertices that coincide to black vertices after rotation R_i. It is sufficient to show that there exists an $i$so that $a_i \geq \lfloor p/2 \rfloor + 1$. Because after all

$n-1$ rotations each red vertex has coincided to black vertices $n-p$ times, the following equality holds:

$$a_1 + a_2 + \ldots + a_n = p(n-p)$$

From $n \geq 2p$ we deduce that $p(n-p)/(n-1) > (p/2)$. Therefore there exists an a_i so that $a_i > (p/2)$ and this implies $a_i \geq [p/2]+1$. In order to have a polygon, we need $[p/2]+1 \geq 3$. In the case $[p/2] < 3$, there is only the possibility $p=3$. In this case we need $p(n-p)/(n-1) > 2 \Leftrightarrow 3(n-3)/(n-1) > 2 \Leftrightarrow n \geq 8$. If $n=6$, $p=3$ we have only three possiblilties. In the case $n=7$, $p=3$ there are four situations.

4. Let m, n be natural numbers greater than or equal to 2. Find the number of polynomials of degree $2n-1$ having pairwise distinct coefficients in the set $\{1, 2, \ldots, m\}$ that are divisible by the polynomial

$$X^{n-1} + \ldots + X + 1.$$

Solution: Let $P(x) = a_{2n-1}x^{2n-1} + \ldots + a_0$. Now $x^{n-1} + x^{n-2} + \ldots + x + 1$ divides P if and only if $x^n - 1$ divides $(x-1)P(x)$. That is $x^n - 1$ divides the polynomial $a_{2n-1}x^{2n} + (a_{2n-2} - a_{2n-1})x^{2n-1} + \ldots + (a_0 - a_1)x - a_0$. Since $x^{n+i} \equiv x^i (mod x^n - 1)$, we deduce that $x^n - 1$ divides the polynomial

$$(a_{2n-2} - a_{2n-1} + a_{n-2} - a_{n-1})x^{n-1} + \ldots$$
$$+ (a_n - a_{n+1} + a_0 - a_1)x + (a_{2n-2} + a_{n-1} - a_n - a_0).$$

Therefore $a_{2n-1} + a_{n-1} = a_{2n-2} + a_{n-2} = \ldots = a_n + a_0$. It follows that we have to find the number of partitions of a positive integer k into n sums of two positive integers (all the $2n$ numbers being distinct elements of the set $\{1, 2, \ldots, m\}$). We deduce that $m \geq 2n$ and $k \geq (1 + 2 + \ldots + 2n)/n = 2n+1$. So the distinct partitions of k are of the type $\{i, k-i\}$, with $i = 1, 2, \ldots, [(k-1)/2]$ and can be chosen in $\binom{(k-1)/2}{n} 2^n n!$ ways. Finally the requested number is:

$$\sum\nolimits_{k=2n+1}^{m} \binom{(k-1)/2}{n} 2^n n! + \sum_{l=1}^{m-2n+1} \binom{(m-l-1)/2}{n} 2^n n!$$

5. On the surface of a polygon of area $n(n-1)^2/2$, n polygons of equal area $(n-1)^2$ are arranged. Show that there exist two such polygons whose common part has area at least 1.

Solution: Let S_k be the part of the surface of the given polygon which is covered by at least k covering polygons, $1 \leq k \leq n$. We will denote the area of S_k by s_k. It is obvious that $s_1 \geq s_2 \geq \ldots \geq s_n$ as a consequence of the inclusions $S_1 \supset S_2 \supset \ldots \supset S_n$. Also, we have the following equality:

$$ns = s_1 + s_2 + \ldots + s_n$$

Therefore $ns = s_1 + (s_2 + \ldots + s_n) \leq s_1 + (n-1)s_1$. Since $s_1 \leq S$, it follows that $s_2 \geq (ns - s_1)/(n-1) \geq (ns - S)/(n-1) = \frac{n(n-1)}{2}$. But S_2 is the union of $\binom{n}{2} = \frac{n(n-1)}{2}$ common parts of all intersections of two polygons. If we suppose that any two polygons have a common area σ less than 1, we find that $s_2 \leq \Sigma\sigma < \frac{n(n-1)}{2}$, which is a contradiction. Thus the assertion of the problem is proved.

6. Let $ABCD$ be a convex quadrilateral. On the segments AB, BC, CD, DA are chosen points M, N, P, Q, respectively, such that

$$AQ = DP = CN = BM.$$

Show that if $MNPQ$ is a square, then $ABCD$ is also a square.

Solution: Let $BM = x$, $MN = u$, $AB = a$ and $\angle PQD = \alpha$. Then $AM = a - x$ and $\angle MQA = 90° - \alpha$.

By the sine law in ΔPQD we get $\sin(\alpha) = x\sin(D)/u$. By the law of cosines in ΔAMQ we deduce

(1) $(a-x)^2 = u^2 + x^2 - 2u\sin(\alpha)$.

Hence $(a-x)^2 - u^2 = x^2 - 2x^2\sin(D)$. In ΔAMQ we have:

$u^2 = x^2 + (a-x)^2 - 2x(a-2)\cos(A) \Rightarrow$

(2) $\Rightarrow (a-x)^2 - u^2 = 2x(a-x)cos(A) - x^2$.

From (1) and (2) we obtain $\cos(A) = \frac{x(1-\sin(D))}{(a-x)} \geq 0$. Hence, $\angle A \leq 90°$, and similarly, $\angle B \leq 90°$, $\angle C \leq 90°$, $\angle D \leq 90°$. So it follows that $\angle A = \angle B = \angle C = \angle D = 90°$. In this case, from (1) it follows that $a = b = c = d$.

Second Solution

Let QR be perpendicular to AB and $S \in QR$ such that $AQ = QS$. Obviously $S = A \Leftrightarrow A = 90°$. Then $\Delta MSQ \equiv \Delta NBM$, because $MQ = MN$, $SQ = BM$ and $\angle MQS = 90° - \angle AMQ = \angle BMN$. Therefore $\angle SMQ = \angle BNM$. Hence $\angle AMQ \leq \angle BMN$, with equality if and only if $S = A$. Performing similar constructions on vertices B, C and D, we finally obtain $\angle AMQ \leq \angle BMN \leq \angle CPN \leq \angle DQP \leq \angle AMQ$. So all the inequalities must be equalities, therefore $\angle A = \angle B = \angle C = \angle D = 90°$. The conclusion is straightforward.

7. Let S be a convex set in the plane. The points of S are painted in p colors. Show that if S contains three noncollinear points, then for every $n, n \geq 3$, there exist infinitely many congruent polygons with n sides, having the vertices painted in the same color.

Solution: Let A, B, C be the three noncolinear points. Obviously ΔABC and its interior are in S. Consider a circle included in the interior of ΔABC, and a point O on this circle. We divide the circle into $m = np + 1$ equal arcs, $O = A_1, A_2, \ldots, A_m$. Clearly among these points there exist n having the same color. With the first n such points we form a polygon. Rotate the polygon from $O = A_1$ to A_2 with an angle of $0 < \alpha < 2\pi/m$. We thus obtain infinitely many polygons with n sides, all inscribed in the same circle and each having vertices of the same color. Since the number of colors is finite, there exists a color common to infinitely many polygons, althought these are not necessarily congruent. Nevertheless, since a n-sided convex polygon can be chosen in a finite number of ways from the m-sided initial polygon, one can find infinitely many congruent polygons having vertices with the same color.

8. A cube is dissected into a finite number of rectangular parallelepipeds such that the volume of the circumscribed sphere of the cube is equal to the sum of the volumes of all spheres circumscribed to the parallelepipeds. Prove that all the parallelepipeds are cubes.

Solution: We start with the following lemma: in a given sphere, the inscribed parallelepiped of maximal volume is the cube. Indeed if the edges of the parallelepiped P are of lengths a, b, c then the volume $V_p = abc$ is maximal. From the arithmetic-geometric mean inequality, we have

$$V_p^2 = a^2b^2c^2 \leq ((a^2+b^2+c^2)/3)^3$$

and the equality holds if and only if $a = b = c$.

In the sphere of radius R, $a^2 + b^2 + c^2 = 4R^2$. Therefore $V_p \leq ((4/3)R^2)^3$ and the equality holds if and only if P is a cube C. In this case, if the edge of the cube has length m, we get $4R^2 = 3m^2$ and the volume of the cube is $V_C = m^3 = ((2R)/\sqrt{3})^3 = \frac{2}{\pi\sqrt{3}} V_S$, where V_S is the volume of the sphere.

Now let $C = P_1 \cup P_2 \cup \ldots \cup P_k$ be a partition of the cube C into parallelepipeds $P_1, \ldots, P_k$. From the equality $V_C = \sum_{i=1}^{k} V_{P_i}$ we obtain:

$$\begin{aligned} V_S &= \frac{\pi\sqrt{3}}{2} V_C = \frac{\pi\sqrt{3}}{2} \sum_{i=1}^{k} V_{P_i} \\ &\leq \frac{\pi\sqrt{3}}{2} \left(\frac{2}{\pi\sqrt{3}} V_{S_1} + \ldots + \frac{2}{\pi\sqrt{3}} V_{S_k} \right) = \sum_{i=1}^{k} V_{S_i}, \end{aligned}$$

where S_i denotes the sphere circumscribed to P_i. From the hypothesis the above inequality is an equality, hence all P_i are cubes.

9. Let $f(x)$ be a monic irreducible polynomial with integer coefficients of odd degree greater than 3. Suppose that the absolute value of the roots of $f(x)$ are greater than 1 and that the coefficient $f(0)$ is square-free. Prove that the polynomial $g(x) = f(x^3)$ cannot be factored into two polynomials with integer coefficients.

Solution: Suppose $f(X^3)$ is a decomposable polynomial in $Z[X]$, and let us consider a decomposition $f(X) = g(X)h(X)$. As $\deg f(X)$ is an odd number, let x be a real root of the polynomial $g(X)$. We may suppose $g(X)$ is irreducible in $Z[X]$ (then also in $Q[X]$) and it follows that $g(X)$ is the minimal polynomial of the algebraic number x. Let ϵ be a cubic root of unity. Then

$$f(x^3) = g(x)h(x) = g(\epsilon x)h(\epsilon x) = g(\epsilon^2 x)h(\epsilon^2 x) = 0.$$

Consider the case $g(\epsilon x) = 0$. From the representation

$$(1) g(X) = a(X^3) + Xb(X^3) + X^2 c(X^3)$$

we obtain the equalities

$$\begin{aligned} g(x) &= a(x^3) + xb(x^3) + x^2 c(x^3) = 0 \\ g(\epsilon x) &= a(x^3) + \epsilon x b(x^3) + \epsilon^2 x^2 c(x^3) \\ g(\epsilon^2 x) &= a(x^3) + \epsilon^2 x b(x^3) + \epsilon x^2 c(x^3) = 0. \end{aligned}$$

Solving this system one obtains $a(x^3) = b(x^3) = c(x^3) = 0$. It follows that $X - x^3$ is a divisor of the polynomials $a(X), b(X), c(X)$ in the ring $\Re[X]$. But x^3 is an algebraic number. Let $q(X)$ be its minimal polynomial. From (1) it follows that $q(X)$ divides $g(X)$ and since g is irreducible it must be that $g = q$. But in this case, since $q(X)$ divides each of the polynomials $a(X^3), b(X^3), c(X^3)$, we obtain that $b(X^3) = c(X^3) = 0$. Hence $g(X) = a(X^3)$, so $h(X) \in Z[X]$ and this contradicts the irreducibility of $f(X)$.

The second possible case is $h(\epsilon x) = 0$. Consider the following formula for $h(x)$:

$$h(X) = a_1(X^3) + Xb_1(X^3) + X^2 c(X^3).$$

Since $h(\epsilon x) = 0$, we obtain $a_1(x^3) = x^2 c_1(x^3)$ and $b_1(x^3) = x c_1(x^3)$. It follows that $g(X)$ divides $a_1(X^3) - X^2 c(X^3)$, which means that $h(0) = a_1(0)$ is divisible by $g(0)$. From the decomposition of $f(X)$ we have $f(0) = g(0)h(0)$, and since $f(0)$ is a square-free number, we deduce that $g(0) = \pm 1$. Also by hypothesis, the roots of $f(X)$ have modulus ≥ 1. So $|x| = 1$ and finally $x = \pm 1$. Therefore $f(X) = g(X)$. This contradicts the degree of $f(X)$.

10. Let $A_1, A_2, \ldots, A_n$ be points on a circle. Find the number of possible colorings of these points with p colors, $p \geq 2$, such that any two neighboring points have distinct colors.

Solution: Let a_n be the desired number; clearly $n \geq 2$. We proceed by induction of n. It is clear that a_2 and a_3 represent the number of injective functions on a set of two or three elements respectively into a set of p elements. So $a_2 = p(p-1)$, and $a_3 = p(p-1)(p-2)$.

Let us consider a set of $n+1$ points. say $A_1, A_2, \ldots, A_n, A_{n+1}$. In order to obtain a desired coloring of this set we can proceed in two ways. If A_1 and A_n have different colors, then $A_1, \ldots, A_n$ can be colored in a_n ways and A_{n+1} in $p-2$ ways. Thus, we obtain $(p-2)a_n$ colorings. If $A-1$ and A_n have the same color, then by identifying $A_1 \equiv A_n$, one obtains that$A_1, \ldots, A_n$ can be colored in a_{n-2} ways while A_{n+1} in $p-1$ ways, obtaining in this case $(p-1)a_{n-2}$ colorings. Finally, one obtains the recursive relation:

$$a_{n+1} = (p-2)a_n + (p-1)a_{n-1},$$

for all $n \geq 3$. From the above relation, one has:

$$a_{n+1} + a_n = (p-1)(a_n + a_{n-1})$$

so inductively,

$$a_{n+1} + a_n = (p-1)^{n-2}(a_3 + a_2) = p(p-1)^n.$$

Now it is immediate that $a_n = (p-1)^n + (-1)^n(p-1)$.

11. Let $f : \mathbb{N} - \{0,1\} \to \mathbb{N}$ be the function defined by

$$f(n) = \text{lcm}[1, 2, \ldots, n].$$

 (a) Prove that for all $n, n \geq 2$, there exist n consecutive numbers for which f is constant.

 (b) Find the greatest number of elements of a set of consective integers on which f is strictly increasing, and determine all sets for which this maximum is realized.

Solution: Let us consider the following.

Lemma $f(n) < f(n+1)$ if and only if $n+1 = p^k$, where p is a prime number.

It is clear that $n+1 = p^k$ implies $f(n) < f(n+1)$. Conversely, suppose $f(n) < f(n+1)$ and $n+1 = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$, where $r \geq 2$ and $k_i \geq 1$. If $n_i = (n+1)/p_i^{k_i}$, one has $p_i^{k_i} \leq (n+1)/2 \leq n$, and the $p_i^{k_i}$ divides $f(n)$. It follows that $n+1$ divides $f(n)$ and then $f(n+1) = f(n)$.

(a) It is sufficient to show that for every $n \geq 2$ there exist n consecutive numbers, none of them being a power of a prime number. This is a well-known result. For instance, if $p_1, p_2, \dots, p_n$, $q_1, q_2, \dots, q_n$ are distinct prime numbers, consider the system of congruences:

$$\begin{aligned} x &\equiv -1 \pmod{p_1 q_1} \\ x &\equiv -2 \pmod{p_2 q_2} \\ &\vdots \\ x &\equiv -n \pmod{p_n q_n} \end{aligned}$$

By the chinese remainder theorem, a positive solution$x$ of the system gives numbers $x+1, x+2, \dots, x+n$ such that $p_i q_i$ divides $x+i$.

(b) The function f cannot be strictly increasing on a set of 5 consecutive numbers, otherwise the last 4 of the numbers must be powers of primes. But two of them must be even, so they are 2 and 2^2, which is impossible, since A contains numbers greater than 2 (A cannot be $\{1, 2, 3, 4, 5\}$). Thus the maximum number of such elements is four.
It is easy to prove that the only solutions are $\{2, 3, 4, 5\}$ and $\{6, 7, 8, 9\}$. Indeed, if f is strictly increasing on $\{a, b, c\}$ and a is even, then $\{a, b, c\} = \{2, 3, 4\}$. If a is odd, then $b = 2^s, a = p^r$, and $c = q^t$, hence $p^r + 1 = 2^s$, $2^s + 1 = q^t$. It follows that $\{a, b, c\} = \{3, 4, 5\}$ or $\{a, b, c\} = \{7, 8, 9\}$.

12. The lengths of the altitudes of a triangle are positive integers and the length of the radius of the incircle is a prime number. Find the lengths of the sides of the triangle.

Solution: Let ABC be a triangle, a, b, c the lengths of its sides, x, y, z the lengths of its altitudes and p the radius of the inscribed circle. Since $2S = ax + by + cz = (a + b + c)p$, we obtain the diophantine equation

$$\frac{1}{p} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}(1)$$

This is a particular case of the famous conjecture of Erdős-Strauss (see L. J. Mordell, Diophantine Equations, Acad. Press, 1969, ch. 30). The present situation is simplified by supplementary geometric conditions: obviously $x > 2p$ hence $x \geq 2p + 1$, and similarly $y \geq 2p + 1, z \geq 2p + 1$ (2). Also, the altitudes of the triangle satisfy: $1/x < 1/y + 1/z$, and the two similar conditions $1/y < 1/x + 1/z$, and $1/z < 1/x + 1/y$ (3).

Solving equation (1), we distinguish several cases:

Case1 p divides each of x, y, and z. If $x = x_1 p, y = y_1 p$ and $z = z_1 p$, we get

$$1 = \frac{1}{x_1} + \frac{1}{y_1} + \frac{1}{z_1}(4)$$

As $x = px_1 > 2p$, we get $x_1 > 2$, and similarly $y_1 > 2$ and $z_1 > 2$. Hence $x_1 \geq 3$, $y_1 \geq 3$ and $z_1 \geq 3$, and clearly (4) is verified only for $x_1 = y_1 = z_1 = 3$. In this case, we obtain $x = y = z = 3p$, and ABC is equilateral, with $a = b = c = 2\sqrt{3}p$.

Case2 p divides x but not y and z. If $x = px_1$, with $x_1 > 2$, we obtain:

$$\frac{1}{y} + \frac{1}{z} = \frac{1}{p} - \frac{1}{px_1} = \frac{x_1 - 1}{px_1} \Rightarrow px_1(y + z) = yz(x_1 - 1).$$

From $(p, yz) = 1$ we get $x_1 - 1 = ps$, where $s \geq 1$. Therefore

$$x = p(ps + 1)(4)$$

From (1) we obtain

$$\frac{1}{p} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq \frac{1}{x}\frac{1}{2p+1} + \frac{1}{2p+1} \Rightarrow x \leq p(2p+1)$$

so, from (4) we obtain $s \leq 2$.

Now, after a short computation, if $s = 2$ one obtains $x = p(2p+1)$, $y = p(2p+1)$, and so ABC is an isosceles triangle, and in the case $s = 1$, we find that $x = p(p+1), y = z = 2(p+1)$.

Case 3 p divides x and y but not z. Then $x = px_1$, $y = py_1$, where $x_1, y_1 \geq 3$. Introducing this into (1) we obtain:

$$px_1y_1 = z(x_1y_1 - x_1 - y_1),$$

where $x_1y_1 - x_1 - y_1 > 0$. It follows that z divides px_1y_1, hence z divides x_1y_1. Say

$$x_1y_1 = zq(1)$$

Then $pzq = z(x_1y_1 - x_1 - y_1)$, hence $pq = zq - x_1 - y_1$ and

$$q(z-p) = x_1 + y_1(6)$$

Suppose $y \geq x$. Then

$$\frac{1}{x} \leq \frac{1}{y} + \frac{1}{z} \Rightarrow \frac{1}{z} > \frac{y-x}{xy} = \frac{y_1 - x_1}{px_1y_1}$$

Using (5) we obtain:

$$\frac{1}{z} > \frac{y_1 - x_1}{pzq}$$

Therefore: $y_1 - x_1, pq$, hence $(x_1 + y_1)^2 - 4x_1y_1 < p^2q^2$. Using (5) and (6) we obtain:

$$\begin{aligned} q^2(z-p)^2 - 4zq < p^2q^2 \quad &\Rightarrow \quad q(z-p)^2 - 4z < p^2q \\ &\Rightarrow \quad q((z-p)^2 - p^2 < 4z \\ &\Rightarrow \quad q(z^2 - 2pz) < 4z \\ &\Rightarrow \quad q(z-2p) < 4. \end{aligned}$$

Because q and $z - 2p$ are positive integers, one has $q(z-2p) \leq 3$.

Now consider several subcases:

a) $q = 3, z - 2p = 1$, hence $x_1y_1 = 3(2p-1)$ and $x_1 + y_1 = 3(p+1)$ which leads to a contradiction.

b) $q = 2, z - 2p = 1$, also leads to a contradiction.

c) $q = 1, z - 2p = s$ where $s \in \{1, 2, 3\}$. In this case, $x_1y_1 = 2p + s$, and $x_1 + y_1 = p + s$, therefore x_1, y_1 are the roots of the equation

$$t^2 - (p+s)t + 2p + s = 0$$

The discriminant is $\Delta = p^2+2p(s-4)+s^2-4s = (p+(s-4))^2+4s-16$. Then $\Delta = u^2$ if and only if

$$(p+s-4-u)(p+s-4+u) = 4(4-s).(7)$$

If $s = 1$, (7) becomes $(p-3-u)(p-3+u) = 12$, hence $p-3-u = 2$ and $p-3+u = 6$. It follows that $p = 7$ and $x = 21, y = 35, z = 15$.

If $s = 2$, (7) becomes $(p-2-u)(p-2+u) = 8$. We obtain $p-2-u = 2$ and $p-2+u = 4$. It follows that $p = 5, x = 15, y = 20$, and $z = 12$.

If $s = 3$, we obtain case 1 again.

Computing the side lengths in all cases being a trivial matter, we only list the results:

Case 1: the equilateral triangle with $a = b = c = 2\sqrt{3}p$

Case 2:a) the isosceles triangle with sides

$$\frac{2p(2p+1)}{sqrt4p^2-1}; \frac{2p^2(2p+1)}{\sqrt{4p^2-1}}; \frac{2p^2(2p+1)}{\sqrt{4p^2-1}}$$

Case2:b) The isosceles triangle with sides

$$\frac{2p(p+1)}{\sqrt{p^2-1}}; \frac{p^2(p+1)}{\sqrt{p^2-1}}; \frac{p^2(p+1)}{\sqrt{p^2-1}}$$

Case 3:c) for $p = 7$, the obtuse triangle with sides

$$\frac{98}{3}\sqrt{3}; \frac{70}{3}\sqrt{3}; 14\sqrt{3}$$

For $p = 5$, the right triangle with sides

$$15, 20, 25$$

1.17 Russia

1. Prove that for all positive x and y,

$$\frac{x}{x^4+y^2}+\frac{y}{y^4+x^2}\leq\frac{1}{xy}.$$

Solution: We apply the arithmetic-geometric mean inequality to get $x^4+y^2\geq 2x^2y$, $x^2+y^4\geq 2xy^2$, and finally

$$\frac{x}{x^4+y^2}+\frac{y}{y^4+x^2}\leq\frac{x}{2x^2y}+\frac{y}{2xy^2}=\frac{1}{xy}.$$

Equality occurs where it occurs in the two AM-GM inequalities, which is to say, $x=y^2$ and $y=x^2$, which implies $x=y=1$.

2. Is it possible to place 1995 different natural numbers along a circle so that for any two of these numbers, the ratio of the greatest to the least is a prime?

Solution: No, this is impossible. Let $a_0,\ldots,a_{1995}=a_0$ be the integers. Then for $i=1,\ldots,1995$, a_{k-1}/a_k is either a prime or the reciprocal of a prime; suppose the former occurs m times and the latter $1995-m$ times. The product of all of these ratios is $a_0/a_{1995}=1$, but this means that the product of some m primes equals the product of some $1995-m$ primes. This can only occurs when the primes are the same (by unique factorization), and in particular there have to be the same number on both sides. But $m=1995-m$ is impossible since 1995 is odd, contradiction.

3. Two circles of radii R and r are tangent to line ℓ at points A and B respectively and intersect each other at C and D. Prove that the radius of the circumcircle of triangle ABC does not depend on the length of the segment AB.

Solution: By the Extended Law of Sines, the radius of the circumcircle of ΔABC is $AB/(2\sin\angle ACB)$. If we draw a line parallel to the common tangent through the center of one of the circles, we discover a right triangle with legs $|R-r|$ and AB and hypotenuse d, the distance between the centers. Therefore $AB^2=R^2-2rR+r^2-d^2$.

On the other hand, if O_1 and O_2 are the centers, then

$$\begin{aligned}\angle ACB &= (\pi/2 - \angle CBA) + (\pi/2 - \angle BAC) \\ &= \angle O_1AC + \angle O_2BC = \angle O_1CA + \angle O_1CB.\end{aligned}$$

Therefore $\angle O_1CO_2 = 2\pi - 2\angle ACB$ and in particular, $\cos\angle O_1CO_2 = \cos 2\angle ACB = 1 - 2\sin^2\angle ACB$. But by the Law of Cosines, $\cos\angle O_1CO_2 = (R^2 + r^2 - d^2)/(2rR)$. Therefore

$$\frac{AB^2}{\sin^2\angle ACB} = (R^2 - 2rR + r^2 - d^2)\frac{4rR}{2rR - r^2 - r^2 + d^2} = 4rR.$$

We deduce that the circumradius of $\triangle ABC$ is $\sqrt{rR}$ independent of the position of the circles.

4. All sides and diagonals of a regular 12-gon are painted in 12 colors (each segment is painted in one color). Is it possible that for any three colors there exist three vertices which are joined with each other by segments of these colors?

   Solution: No, this is impossible. There are $(12 \cdot 11)/2 = 66$ edges (sides and diagonals), since choosing an edge is the same as choosing an unordered pair of endpoints. If these are colored in 12 colors, by the Pigeonhole Principle, some color occurs on at most 5 edges. Call this color "blue". Each edge belongs to exactly 10 triangles, one for each choice of the third vertex, so the blue edges belong collectively to at most 50 triangles (some may contain more than one). However, if every three colors occurred as the colors of the sides of some triangle, this would in particular hold for the $(11 \cdot 10)/2 = 55$ triples of colors containing blue. Each of these triangles would have to be distinct, which is impossible since only 50 triangles have a blue edge. Hence the condition cannot be satisfied.

5. Find all primes p such the the number $p^2 + 11$ has exactly six different divisors (including 1 and the number itself).

   Solution: For $p \neq 3$, $p^2 \equiv 1 \,(\mathrm{m}od\ 3)$ and so $3 | p^2 + 11$. Similarly, for $p \neq 2$, $p^2 \equiv 1 \,(\mathrm{m}od\ 4)$ and so $4 | p^2 + 11$. Except in these two cases, then, $12 | p^2 + 11$; since 12 itself has 6 divisors (1,2,3,4,6,12) and $p^2 + 11 > 12$ for $p > 1$, $p^2 + 11$ must have more than 6 divisors.

The only cases to check are $p = 2, 3$. If $p = 2$, then $p^2 + 11 = 15$, which only has 4 divisors (1,3,5,15), while if $p = 3$, then $p^2+11 = 20$, which indeed has 6 divisors (1,2,4,5,10,20). Hence $p = 3$ is the only solution.

6. [Corrected] Circles S_1 and S_2 with centeres O_1 and O_2 intersect each other at points A and B. Let S_3 be the circle passing through O_1, O_2 and A. S_3 intersects S_1 at $D \neq A$, S_2 at E, and line AB at C. Prove that $CD = CB = CE$.

   The proliferation of circles in the diagram leads us naturally to an inversion centered at B, where primed letters denote images under the inversion. (The center of the inversion will be called O.) What does the resulting configuration of $A', C', D', E', O_1', O_2', O$ look like? Well, the first six still lie on a circle, namely S_3'. The fact that O_1 is the center of a circle passing through A, B, D means that $A'D'$ is the perpendicular bisector of OO_1', and similarly $A'E'$ is the perpendicular bisector of OO_2'.

   What we want to show is that C is the center of the circumcircle of BDE; in the inverted diagram, this says that $D'E'$ is the perpendicular bisector of OC'. For starters, we will need to know that the reflection of O across $D'E'$ is actually on S_3', which amounts to checking that $\angle E'OD' = \pi - \angle E'AD'$. However, the other two perpendicular bisectors give us the relations $\angle D'OA' = \pi - \angle D'E'A'$ and $\angle A'OE' = \pi - \angle A'D'E'$, and adding these gives the desired result, since the three angles around O sum to 2π. (This argument follows the directed angles mod π convention, so it works even if O is not inside $\Delta A'D'E'$. See for yourself!)

   To conclude, we need to know that $A'O$ is an altitude of $\Delta A'D'E'$. But the previous observation means that O is symmetrically defined in terms of A', D', E', so maybe O should lie on all three altitudes. In fact, if H is the orthocenter of $\Delta A'D'E'$ and A'', D'', E'' are the feet of the altitudes from A', D', E', respectively, then

$$\begin{aligned}
\angle D'HE' &= \pi - \angle E'HD'' \\
&= \pi - (\pi/2 - \angle D''E'H) \\
&= \pi/2 + \angle A'E'E'' \\
&= \pi/2 + (\pi/2 - \angle D'A'E') = \pi - \angle D'A'E'.
\end{aligned}$$

Thus $D'E'$ subtends equal angles at H and O, and a similar argument gives the same conclusion for $E'A'$ and $A'D'$. This implies $H = 0$, and so $A'O \perp D'E'$. Hence the reflection of O across $D'E'$ lies both on $A'O$ and on S_3', and so is indeed C'.

7. Consider all quadratic functions $f(x) = ax^2 + bx + \mathrm{c}$ such that $a < b$ and $f(x) \geq 0$ for all x. Determine the minimal value of the expression $(a + b + \mathrm{c})/(b - a)$.

Solution: The condition $f(x) \geq 0$ on a quadratic polynomial is equivalent to $a \geq 0$ and $b^2 - 4ac \leq 0$. (In other words, the polynomial is concave up and does not have distinct real roots.) Let $d = b - a$, so the condition $b > a$ becomes $d > 0$. Now $\mathrm{c} \geq (a + d)^2/4a$ and we must minimize $(2a + d + \mathrm{c})/d$. Certainly we might as well take $\mathrm{c} = (a + d)^2/4a$, so the problem becomes to minimize

$$\frac{(4a)(2a + d) + (a + d)^2}{4ad} = \frac{9a^2 + 6ad + d^2}{4ad} = \frac{9}{4}\frac{a}{d} + \frac{3}{2} + \frac{1}{4}\frac{d}{a}.$$

The sum of the first and third terms takes its minimum value $3/2$ when $a/d = 1/3$. Hence the minimum value of the given expression is 3, achieved with, for example, $a = 1, b = 4, \mathrm{c} = 4$.

8. Is it possible to write the numbers $1, 2, \ldots, 121$ in an 11×11 table so that any two consecutive numbers be written in cells with a common side and all perfect squares lie in a single column?

Solution: This cannot be done. The path traced out by the numbers would only cross the column at the squares, so the numbers between 1^2 and 2^2 would lie on one side, the numbers between 2^2 and 3^2 on the other side, and so forth. But then the number of columns on the side containing 2 would be

$$(2^2 - 1^2 - 1 + 4^2 - 3^2 - 1 + \cdots + 10^2 - 9^2 - 1)/11 = (2 + 6 + 10 + 14 + 18)/11,$$

which is not an integer.

9. If $f(x) = 1/\sqrt[3]{1 - x^3}$, find $\underbrace{f(\cdots f(f(19))\cdots)}_{95\text{ times}}$.

Solution: Let $x_0 = 19$ and $x_{n+1} = f(x_n)$, so that the quantity we seek is x_{95}. Now put $y_n = x_n^3$; then $y_{n+1} = g(y_n)$, where $g(x) =$

$1/(1-x)$. However, $g(g(x)) = (x-1)/x$ and $g(g(g(x))) = x$. Therefore $y_{95} = y_{92} = \cdots = y_2 = 1/(1-19^3)$ and so $x_{95} = (1-19^3)^{-1/3}$.

10. Let m and n be positive integers such that

$$\mathrm{lcm}[m,n] + \gcd[m,n] = m + n.$$

Prove that one of the two numbers is divisible by the other.

Solution: Let d be the greatest common divisor, and write $m = ad, n = bd$, where a, b are relatively prime. Then the equation becomes $d + abd = ad + bd$, or $d(ab - a - b + 1) = 0$. The left side factors as $d(a-1)(b-1)$, so we must have either $a = 0$, in which case $m = d$ and so $m|n$, or $b = 0$, in which case $n|m$.

11. Altitude BK of acute triangle ABC is a a diameter of a circle S that intersects the sides AB and BC at points E and F, respectively. Tangents to the circle are drawn through the points E and F. Prove that their common point lies on a median of the triangle passing through B.

Solution: Note that $\angle BEF = \angle BKF = \pi/2 - \angle FBK = \angle ACB$, and similarly $\angle BFE = \angle CAB$. Hence ΔBEF is oppositely similar to ΔBCA, so in particular the median of ΔBCA through A is also the symmedian of ΔBEF through A. (The symmedian of ΔPQR through P is defined as the reflection of the median through P across the angle bisector of $\angle RPQ$.)

The problem, then, is to show that for any triangle ΔBEF, the tangents to the circumcircle at E and F meet on the symmedian through B. Let M be the midpoint of EF, G the midpoint of the arc EF on the circumcircle, and H the intersection of the tangents at E and F. Then BM is the angle bisector of $\angle BEF$ since $\angle EBM$ and $\angle MBF$ each subtend half of arc EF. The fact that BH is a symmedian is then equivalent to BM being the angle bisector of $\angle GBH$.

We will show that $BG/BH = MG/MH$, which will imply that BM is the angle bisector of $\angle GBH$ by the Angle Bisector Theorem. One can certainly show this directly, but it helps to remember that for points P, Q in the plane and k a constant not equal to 1, the set of points R such that $PR/QR = k$ is a circle. (Such a circle is known as a circle of Apollonius. Believe it or not, the shortest proof of this fact is to write out the equation $PR^2 - kQR^2 = 0$ in coordinates!)

In particular, the set of all X such that $XG/XH = MG/MH$ is a circle. Certainly M lies on this circle; since $\angle HEM = \angle EBM = \pi/2 - \angle BME = \angle MEG$, we have $EG/EH = MG/MH$ by the Angle Bisector Theorem. Hence E lies on the circle also, and similarly for F. Since three points determine a circle, the circle of Apollonius is none other than the circumcircle of $\triangle BEF$, and so $BG/BH = MG/MH$ as desired.

12. Several identical square sheets of paper lie on a table so that their sides are parallel to the edges of the table (the sheets can overlap with one another). Prove that it is possible to stick several pins in the table so that each sheet be fixed to the table by just one pin.

    Solution: Imagine a lattice, with side length equal to the sides of the sheets, floating over the plane. If I slide the lattice a short distance vertically, in only finitely many positions will any lattice points lie on horizontal edges of the sheets, and similarly if I slide horizontally. So we can choose a position where no lattice points lie on edges of sheets. Put a pin down at every lattice point lying over at least one sheets; then clearly each sheet is fixed by exactly one pin.

13. [Corrected] $ABCD$ is a quadrilateral such that $AB = AD$ and B and D are right angles. Points F and E are chosen on BC and CD, respectively, so that $DF \perp AE$. Prove that $AF \perp BE$.

    Solution: With so many perpendiculars floating around, vectors are a natural choice here. Take the origin to be any point on AC. The fact that $DF \perp AE$ can be expressed in vector notation as $(D - F) \cdot (A - E) = 0$, or

    $$A \cdot D + E \cdot F = A \cdot F + D \cdot E.$$

    By symmetry, $A \cdot D = A \cdot B$. Since $AB \perp FB$, we have $(A - B) \cdot (B - F) = 0$, or $A \cdot B + B \cdot F = A \cdot F + B \cdot B$. Similarly, $AD \perp DE$ gives $A \cdot D + D \cdot E = A \cdot E + D \cdot D$. Subtracting these equations, and using the relations $B \cdot B = D \cdot D$ and $A \cdot B = A \cdot D$ (again consequences of symmetry), we find $D \cdot E + A \cdot F = B \cdot F + A \cdot E$. Putting everything together, we deduce

$$A \cdot B + E \cdot F = B \cdot F + A \cdot E,$$

or $(A - F) \cdot (B - E) = 0$, so $AF \perp BE$ as desired.

14. N^3 unit cubes are drilled through along their diagonals and strung on a thread whose ends are then connected to form a "necklace" such that the vertices of adjacent cubes touch each other. For what N is it possible to pack this "necklace" into an $N \times N \times N$ box?

    Solution: This can be done if and only if N is even. If N is odd, consider a projection of the box onto one edge. Suppose we put a tiny bead between every pair of cubes on the necklace; then the projections of adjacent beads are one unit apart, but the path traced by the sequence of beads returns to its starting point after an odd number of steps, namely N^3 steps. This is clearly impossible.

    If N is even, we build up the arrangement by splicing together smaller necklaces. The idea is that the two strings cross, we can make a new necklace that follows one of the originals to the juncture, traverses the second one, then finishes traversing the first one.

    Start by arranging 4 cubes in a $1 \times 2 \times 2$ square. If we put two of these together in any fashion, they can be spliced together. By the same token, we can get any $1 \times 2m \times 2n$ rectangle by splicing, so in particular we can get a $1 \times N \times N$ rectangle. Now take this arrangement and its mirror image; they can be put on top of each other and spliced to yield a $2 \times N \times N$ box. Any number of these can be spliced together, so in particular we can make an $N \times N \times N$ box.

    (Aside: for what integer dimensions is this possible if the box need not be cubical? We have only shown that one even dimension is necessary and that three are sufficient.)

15. Streets of a town are simple polygonal lines that do not intersect one another at internal points. Each street contains two crossings and is painted in one of three colors: red, white, or blue. Exactly three streets meet at each crossing, being all different colors. Assume that

we stand on a crossing and turn counterclockwise. If the streets that meet on this crossing appear in the order white, blue, and red, the crossing is said to be *positive*, and in the opposite case it is called *negative*. Prove that the difference between the numbers of positive and negative crossings is divisible by 4.

Solution: The set of blue and white edges enclose a certain set of polygons (each vertex has exactly two edges from this set), which we call "blue" polygons. Similarly, the red and white edges enclose "red" polygons.

How can a red polygon and a blue polygon intersect? Each intersection comprises an entire edge (they can't meet in a point because no point has four edges coming from it), and by the "what goes in must come out" rule (a/k/a the Jordan curve theorem), there must be an even number of such intersections between any given pair of polygons. On the other hand, an edge occurs as such an intersection if and only if it is white and connects two positive vertices or two negative vertices. (Draw the possible configurations and this becomes evident.) Hence the number of such edges is even.

Let P and N be the number of white edges between two positive vertices and two negative vertices, respectively. Then the difference between the number of positive and negative vertices is exactly $2P - 2N$. (Each vertex occurs on one white edge; P edges contribute $+2$, N edges contribute -2, and other edges contribute nothing.) But $P + N$ was just seen to be even, so $2(P - N)$ is a multiple of 4, as desired.

16. A section of a rectangular parallelepiped is a regular hexagon. Prove that this parallelepiped is a cube.

Solution: Let O be the center of the regular hexagon and let P and Q be adjacent vertices, which must lie on a pair of edges incident to a commom vertex T of the box (rectangular parallelepiped). There is a unique box with P, Q, T, O as vertices, PT and QT as edges and O antipodal to T. If a, b, c are the dimensions of this box, the fact that ΔOPQ is equilateral implies that $OP = PQ = QO$, which in turn implies $a^2 + b^2 = b^2 + c^2 = c^2 + a^2$. This forces $a = b = c$, so the small box is actually a cube. In particular, $PQ = \sqrt{2}TQ$.

Now let R be the other vertex of the hexagon adjacent to Q. If U is the other endpoint of the edge containing Q, then by the same argument, $QR = \sqrt{2}UQ$. Since $PQ = QR$, we must have $TQ = UQ$, so Q is the midpoint of TU. In the same way, we conclude that each of the six vertices of the hexagon is the midpoint of the edge it lies on. But now the same calculation as above shows that the dimensions of the large box must also be equal, i.e. it is a cube.

17. In a plane, consider a finite set of identical squares whose sides are parallel. For any $k+1$ squares, at least two of them contain points in common. Prove that this set can be divided into no more than $2k-1$ subsets so that in each subset all squares have a common point.

Solution: It might help to start with the one-dimensional version of this problem: consider a finite set of unit intervals on a line, such that no $k+1$ are mutually disjoint. Prove that they can be divided into at most k subsets, each having a common point.

The solution to the one-dimensional problem is by induction. Take the leftmost interval I; every interval that intersects I contains its right endpoint, so all of these (including I) have a common point. Of the remaining intervals, no k are disjoint (since all are disjoint from I), so they can be divided into $k-1$ subsets.

The two-dimensional problem is pretty similar (though finding the right generalization of the above argument is tricky). Choose a square S whose left edge is farthest to the left. If there are several such squares, choose the highest one. Any of the remaining squares meeting S cover its top right or bottom right corner, so these can be divided into two subsets, each having a common point. Of the remaining squares, no k are disjoint, so by induction they can be divided into $2k-3$ subsets, each with a common point.

18. Let x, y, z be such that $\sin x + \sin y + \sin z \geq 2$. Prove that $\cos x + \cos y + \cos z \leq \sqrt{5}$.

Solution: The key fact here is the inequality $(a+b+c)^2 \leq 3a^2 + 3b^2 + 3c^2$, which can be proved directly or deduced from Cauchy-

Schwarz (or the Power Mean inequality). The point is that

$$\begin{aligned}&(\sin x+\sin y+\sin z)^2+(\cos x+\cos y+\cos z)^2\\ \leq\ &3(\sin^2 x+\sin^2 y+\sin^2 z+\cos^2 x+\cos^2 y+\cos^2 z)=9.\end{aligned}$$

So if $\sin x+\sin y+\sin z\geq 2$, we must have $\cos x+\cos y+\cos z\leq\sqrt{5}$.

19. The sequence $a_0, a_1, a_2, \ldots$ satisfies

$$a_{m+n}+a_{m-n}=\frac{1}{2}(a_{2m}+a_{2n})$$

for all nonnegative integers m and n with $m\geq n$. If $a_1=1$, determine a_{1995}.

Solution: The relations $a_{2m}+a_{2m}=2(a_{2m}+a_0)=4(a_m+a_m)$ imply $a_{2m}=4a_m$, as well as $a_0=0$. Thus we compute $a_2=4, a_4=16$. Also, $a_1+a_3=(a_2+a_4)/2=10$ so $a_3=9$. At this point we guess that $a_i=i^2$ for all $i\geq 1$.

We prove our guess by induction on i. Suppose that $a_j=j^2$ for $j<i$. Then the given equation with $m=i-1, j=1$ gives

$$\begin{aligned}a_n &= \frac{1}{2}(a_{2n-2}+a_2)-a_{n-2}\\ &= 2a_{n-1}+2a_1-a_{n-2}\\ &= 2(n^2-2n+1)+2-(n^2-4n+4)=n^2.\end{aligned}$$

So in particular, $a_{1995}=1995^2$.

20. Circles S_1 and S_2 with centers O_1, O_2 respectively intersect each other at points A and B. Ray O_1B intersects S_2 at point F and ray O_2B intersects S_1 at point E. The line parallel to EF and passing through B intersects S_1 and S_2 at points M and N, respectively. Prove that $MN=AE+AF$.

Solution: We first focus on the triangle $\triangle AEF$. We have

$$\angle EAB=\frac{1}{2}\angle EO_1B=\frac{\pi}{2}-\angle O_1BE=\frac{\pi}{2}-\angle FBO_2=\angle BAF.$$

Therefore AB is the angle bisector of $\angle EAF$. Also

$$\begin{aligned}\angle EBA + \angle FBA &= \angle EBA + \pi - \angle O_1BA \\ &= \pi + \angle EBO_1 \\ &= \pi + \frac{\pi}{2} - \angle EAB.\end{aligned}$$

Similarly, the same quantity equals $3\pi/2 - \angle FAB$. This has two consequences. First, BA is the angle bisector of $\angle EAF$. Second,

$$\begin{aligned}\angle EBF &= 2\pi - \angle EBA - \angle FBA \\ &= 2\pi - \frac{1}{2}\left(\frac{3\pi}{2} - \angle EAB + \frac{3\pi}{2} - \angle FAB\right) \\ &= \frac{\pi + \angle EAF}{2}.\end{aligned}$$

Thus B is constrained to lie both on the angle bisector of $\angle EAF$ and on a certain arc, which have a unique intersection. However, the incenter of $\triangle EAF$ meets both of these constraints, so it must equal B.

This implies that $\angle AEB = \angle BEF = \angle EBM$ (the last by parallels) and so $EBAM$ is an isosceles trapezoid, whence $EA = MB$. Similarly $FA = NB$ and so $MN = MB + NB = AE + AF$.

21. A train left Moscow at x hours y minutes, reached Saratov at y hours z minutes. The time of travel was z hours x minutes. Find all possible values of x.

Solution: The problem statement requires $(60x + y) + (60z + x) = (60y + z)$. If $x = 0$, this is possible with $y = z$. If $x > 0$, then $y > z$ and so there must be a "carry in base 60" in the units digit. In other words, $y + x = 60 + z$ and $x + z + 1 = y$. Adding these equations yields $2x = 59$, which is impossible. Hence only $x = 0$ is possible.

22. Chord CD of a circle with center O is perpendicular to its diameter AB, while chord AE bisects the radius OC. Prove that chord DE bisects chord BC.

Solution: Let F be the point on the circle such that DF bisects BC. Note that $\triangle OAC$ and $\triangle BDC$ are isosceles triangles with the same vertex angle $\angle AOC = \angle BDC$. Hence they are similar, so the angle between BD and the median to DC equals the angle between OA and the median to OC. The former is just $\angle OAE = \angle BAE$ and the latter is $\angle BDF = \angle BAF$, so E and F must coincide. (A cute alternate solution can be obtained by applying Pascal's theorem to the hexagon $ABCFDE$; details left to the reader.)

23. Let $f(x), g(x), h(x)$ be quadratic trinomials. Is is possible for the equation $f(g(h(x))) = 0$ to have the roots $1, 2, 3, 4, 5, 6, 7$ and 8?

    Solution: No, this is not possible. Suppose this were the case. Put $f(x) = \alpha(x-a)(x-b)$. Then the polynomials $g(h(x)) - a$ and $g(h(x)) - b$ have 1, 2, 3, 4, 5, 6, 7, 8 as their roots. However, they only differ in their constant term, so in particular their sum of roots is the same. Hence the sum of roots of each is 18.

    Repeat the process, writing $g(x) - a$ as $\beta(x - \mathrm{c})(x - d)$. Then $h(x) - \mathrm{c}$ and $h(x) - d$ again have the same sum of roots, namely 9. We must then have $h(1) = h(8), h(2) = h(7), h(3) = h(6), h(4) = h(5)$. We must have $h(x) = px^2 - 9px + q$ for some p, q with $p \neq 0$.

    Now we are reduced to finding f, g such that $f(g(x)) = 0$ has roots $q - 8p, q - 14p, q - 18p, q - 20p$. Again, a necessary condition is that we can split the roots into two pairs with equal sum, but clearly we can't do that with these four numbers. We conclude such f, g, h do not exist.

24. [Corrected] Is it possible to fill in the cells of a 9×9 table with positive integers ranging form 1 to 81 in such a way that the sum of the elements of every 3×3 square is the same?

    Solution: This is possible. Start with a 9×9 table with first row $0, 1, 2, 3, 4, 5, 6, 7, 8$, second row $3, 4, 5, 6, 7, 8, 0, 1, 2$, third row $6, 7, 8, 0, 1, 2, 3, 4, 5$, and this pattern repeats two more times. Then the sum of each 3×3 square is 36 (check). Call this table A. Let table B be the same thing rotated by 90 degrees. Now each square can be labeled by a pair (a, b), where a is its label in table A and b its label in table B. It can be checked that no two squares have the same label, so giving the square the label $9a + b + 1$ assigns each square a different number between 1 and 81. By construction, the sum of each 3×3 square is $36 + 9 \cdot 36 + 9 = 369$.

25. Call positive integers *similar* if they are written using the same set of digits. For example, for the set $1, 1, 2$, the similar numbers are 112, 121 and 211. Prove that there exist 3 similar 1995-digit numbers containing no zeros, such that the sum of two of them equals the third.

Solution: Noting that 1995 is a multiple of 3, we might first trying to find 3 similar 3-digit numbers such that the sum of two of them equals the third. There are various digits arrangements to try, one of which is $abc + acb = cba$. The middle column must have a carry or else we would have $c = 0$ and no integer can begin with a 0. If there is a carry, we must have $c = 9$, which implies $a = 4$ by looking at the first column. From the third column, we find $b = 5$ and discover that indeed $459 + 495 = 954$. Now to solve the original problem, simply write $459 \cdots 459 + 495 \cdots 495 = 954 \cdots 954$, where each three-digit number is repeated $1995/3$ times.

26. Points A_2, B_2 and C_2 are the midpoints of the altitudes AA_1, BB_1 and CC_1 of acute triangle ABC. Find the sum of angles $\angle B_2A_1C_2$, $\angle C_2B_1A_2$ and $\angle A_2C_1B_2$.

Solution: Let H be the orthocenter of ΔABC and X, Y, Z the midpoints of BC, CA, AB, respectively. Then $\Delta AZA_2 \sim \Delta ABA_1$, so in particular, $ZA_2 \perp A_2H$. Therefore the circle with diameter ZH passes through A_2, and similarly it also passes through B_2. However, since $HC_1 \perp C_1Z$, the circle also passes through C_1. Therefore $\angle A_2C_1B_2 = \angle A_2ZB_2$; since A_2 lies on YZ and B_2 on XZ, this angle also equals $\angle YZX$. Hence the three angles we are adding are just the three angles of ΔXYZ, so their sum is π. (It wasn't needed, but you did notice that $\Delta XYZ \sim \Delta ABC$, right?)

27. Three piles of stones are given. Sisyphus carries them one-by-one from one pile to another pile. For each transfer of a stone, he receives from Zeus a number of coins equal to the difference of two numbers: the number of stones in the pile from which a stone is drawn is subtracted from the number of stones in the recipient pile (the stone in Sisyphus's hands is not counted). If this difference is negative, Sisyphus pays back the corresponding sum (the generous Zeus allows him to pay later if he is broke). At some point, all stones have been

returned to their piles. What is the maximum possible income for Sisyphus at this moment?

Solution: Recalling the myth of Sisyphus might suggest the answer: our poor friend will always end up with zero net income, no matter how he proceeds!

Each transaction can be thought of as a sum of contributions from each stone other than the moving one, where a stone in the originating pile contributes -1, a stone in the receiving pile contributes $+1$, and a stone in the third pile contributes 0. We organize Sisyphus' net income into contributions from each (unordered) pair of distinct stones. Then one can check that the net contribution of a pair of stones after any sequence of moves depends only on their initial and final relative position. Specifically, two stones starting in the same pile contribute 0 if they end up in the same pile and -1 otherwise, while two stones starting in different piles contribute 0 if they end up in different piles and 1 otherwise. Since all stones end up where they started, the net contribution of each pair is 0, and so is the total income.

Another way to put this is that $a(a-1)/2+b(b-1)/2+\mathrm{c}(\mathrm{c}-1)/2+s$ is invariant, where a, b, c are the sizes of the piles and s is Sisyphus' net income. (Sisyphus' income counts the change in the number of unordered pairs of stones that are in the same pile.)

28. The numbers 1 and -1 are written in the cells of a 2000×2000 table. If the sum of all the numbers is not negative, prove that there exist 1000 columns and 1000 rows such that the sum of the numbers written in the cells where they cross is not less than 1000.

Solution: The claim will follow by induction from the following statement: suppose the numbers 1 and -1 are written in the cells of an $x \times x$ table, where $x = n+t$ with $0 < t \leq n$, such that the sum of all entries is at least $n-t$. Then there must be an $(x-1) \times (x-1)$ subtable with sum at least $n-t+1$.

To show this, let s be the sum of the entries. Assume without loss of generality that the smallest row sum occurs in the first row; call this sum u. Also assume that when the first row is removed, the smallest column sum occurs in the first column; call it v.

We have $s < x$ unless every entry of the table is 1, in which case the claim is obvious. Therefore $u \le s/x < 1$ and u, being an integer, is at most 0. In case $(s - u) \ge x$, then the average sum of an $(x-1)\times(x-1)$ subtable of the original table with the first row deleted is $(s-u)(x-1)/x \ge x-1 \ge n-t+1$. Otherwise, $v \le (s-u)/x < 1$ and again v is at most 0. The sum of the first row and column is then an odd number (since it's the sum of $2x-1$ numbers each equal to ± 1) not greater than 0. Therefore $u + v \le -1$ and the sum with the first row and column deleted is $s - u - v \ge n - t + 1$, as desired.

29. Solve the equation $\cos(\cos(\cos(\cos x))) = \sin(\sin(\sin(\sin x)))$.

Solution: We will show that $\cos(\cos(x)) > \sin(\sin(x))$ for all x, from which it follows that

$$\begin{aligned}\cos(\cos(\cos(\cos(x)))) &> \sin(\sin(\cos(\cos(x)))) \\ &> \sin(\sin(\sin(\sin(x)))),\end{aligned}$$

and hence the given equation has no solutions. (The second inequality comes from the facts that $\cos(\cos(x))$ and $\sin(\sin(x))$ lie in $[0,1]$ and that the function $f(x) = \sin(\sin(x))$ is monotone increasing on this interval.)

Of course, we need only check the inequality for $x \in [0, 2\pi]$. However, for $x \in [\pi, 2\pi]$, $\cos(\cos(x)) > 0$ while $\sin(\sin(x)) < 0$, so we need only check $x \in [0, \pi]$. Moreover, neither expression changes upon replacing x by $\pi - x$, so it suffices to check $x \in [0, \pi/2]$.

Note that

$$\cos(\cos(x)) = \sin(\pi/2 - \cos(x))$$

and $\sin(x) + \cos(x) = \sqrt{2}\cos(x - \pi/4) \le \sqrt{2} < \pi/2$. Since sine is monotonic on $[0, \pi/2]$, we have

$$\sin(\pi/2 - \cos(x)) > \sin(\sin(x)),$$

implying the desired inequality.

30. Is there a sequence of natural numbers in which every natural number occurs just once and moreover, for any $k = 1, 2, 3, \ldots$ the sum of the first k terms is divisible by k?

Solution: We recursively construct such a sequence. Suppose $a_1, \ldots, a_m$ have been chosen, with $s = a_1 + \cdots + a_m$, and let n be the smallest number not yet appearing. By the Chinese Remainder Theorem, there exists t such that $t \equiv -s \pmod{m+1}$ and $t \equiv -s - n \pmod{m+2}$. We can increase t by a suitably large multiple of $(m+1)(m+2)$ to ensure it does not equal any of $a_1, \ldots, a_m$. Then $a_1, \ldots, a_m, t, n$ also has the desired property, and the construction assures that $1, \ldots, m$ all occur among the first $2m$ terms.

31. Given a convex equiangular polygon, prove that at least two of its sides have lengths which do not exceed the lengths of the adjacent sides.

Solution: Suppose a polygon exists failing this condition; since the shortest side certainly is not longer than its neighbors, every other side must be adjacent to a shorter side. In particular, the second longest side must be adjacent to the longest, the third must be adjacent to the first or the second, and so on. By induction, the k longest sides must form a consecutive block for all k. What this means is that there are vertices P and Q such that if we trace either path from P to Q, each side we cover is shorter than the previous one.

We now show this is impossible. Suppose the counterclockwise path from P to Q contains m edges and the clockwise path contains $n-m$ edges. Draw a large regular n-gon with one vertex at P, such that the two sides at P run along the sides of our original polygon. Let R be the vertex m away from P going counterclockwise. We can view the m edges from P to Q as vectors whose sum is $Q-P$; break these up into components parallel and perpendicular to RP. Let's call the direction of the perpendicular component of the first edge "down". The first and last of these edges make equal angles with RP, but the first is longer, so their total contribution in the perpendicular direction points down. The same is true of the second and next to last edges, and so on. Hence $Q-P$ has a perpendicular component pointing down; however, the analogous argument on the other $n-m$ edges shows that the same component points up, a contradiction.

32. The sequence $a_1, a_2, \ldots$ of natural numbers satisfies $(a_i, a_j) = (i, j)$

for all $i \neq j$. Prove that $a_i = i$ for all i.

Solution: For any integer m, we have $(a_m, a_{2m}) = (2m, m)$ and so $m | a_m$. This means that for any other integer n, m divides a_n if and only if it divides $(a_m, a_n) = (m, n)$. Hence a_n has exactly the same divisors as n and so must equal n for all n.

33. Given a semicircle with diameter AB and center O and a line which intersects the semicircle at C and D and line AB at M ($MB < MA, MD < MC$). Let K be the second point of intersection of the cicumcircles of triangles AOC and DOB. Prove that angle MKO is a right angle.

Solution: Invert the diagram through the circle with diameter AB, and use primes to denote images under the inversion. Of course, $A' = A, B' = B, C' = C, D' = D$, while K' is simply the intersection of the lines AC and BD, and M' is the intersection of the line AB with the circumcircle of $OC'D'$. The line MK inverts to the circumcircle of $OM'K'$; the claim that $MK \perp KO$ is equivalent to OK' being a diameter of the circumcircle, or in other words $OM' \perp M'K'$.

Let's stare at the triangle $A'B'M'$ for a moment. The points C' and D' are the feet of the altitudes from A' and B', respectively, and O is the midpoint of side $A'B'$. Hence the circumcircle of $OC'D'$ is none other than the nine-point circle of $\Delta A'B'M'$, whose second intersection with $A'B'$ is the foot of the altitude from M'. We conclude $M'K' \perp OM'$ as desired.

34. Given monic polynomials $P(x)$ and $Q(x)$, not both constant, prove that the sum of the squares of the coefficients of $P(x)Q(x)$ is not less that the sum of the squares of the constant terms of $P(x)$ and $Q(x)$.

Solution: Put

$$\begin{aligned}
P(x) &= p_0x^m + p_1x^{m-1} + \cdots + p_{m-1}x + p_m \\
Q(x) &= q_0x^n + q_1x^{n-1} + \cdots + q_{n-1}x + q_n \\
P(x)Q(x) &= r_0x^{m+n} + r_1x^{m+n-1} + \cdots + r_{m+n-1}x + r_{m+n} \\
P(x)Q(1/x) &= s_{-m}x^m + \cdots + s_nx^{-n},
\end{aligned}$$

where $p_0 = q_0 = 1$. I claim the following identity holds:

$$\sum_{i=0}^{m+n} r_i^2 = \sum_{j=-m}^{n} s_j^2.$$

Since $s_{-m} = q_n$ and $s_n = p_m$, this will imply the desired inequality.

Since r_i is the sum of $p_a q_b$ over those pairs a, b with $a + b = i$, r_i^2 is the sum of $p_a q_b p_c q_d$ over all a, b, c, d with $a + b = c + d = i$, and summing over i simply means we take all a, b, c, d with $a + b = c + d$.

On the other hand, s_j is the sum of $p_a q_d$ with $a - d = j$, so s_j^2 is the sum of $p_a q_d p_c q_b$ with $a - d = b - c = j$, and summing over j simply means we take all a, b, c, d with $a - d = b - c$.

The identity follows by simply noticing that these two paragraphs describe exactly the same sum! In symbols,

$$\sum r_i^2 = \sum_{a+b=c+d} p_a q_b p_c q_d = \sum_{a-d=b-c} p_a q_d p_c q_b = \sum s_j^2.$$

Note: the identity can be extended to complex polynomials as follows. Define the r_i as above, but now put

$$P(x)\overline{Q}(1/x) = s_{-m}x^m + \cdots + s_n x^{-n},$$

where $\overline{Q}$ is the polynomial whose coefficients are the complex conjugates of the coefficients of Q. Then it is easily verified that

$$\sum_{i=0}^{m+n} |r_i|^2 = \sum_{j=-m}^{n} |s_c|^2.$$

See Taiwan 1 for an application of this generalization.

35. Is it possible for the numbers $1, 2, 3, \ldots, 100$ to be the terms of 12 geometrical progressions?

Solution: No geometric progression of integers between 1 and 100 can have more than 7 terms. To see this, let m/n be the common ratio, where m, n are coprime. If the progression has k terms, the first term must be a multiple of n^{k-1} and the last must be a multiple of m^{k-1}. One of m, n must be at least 2, and if $k > 7$ then

$2^{k-1} > 100$, so it cannot divide a number between 1 and 100. Hence each progression has at most 7 terms. (Of course, $1, 2, 4, 8, 16, 32, 64$ achieves this bound.)

So 12 progressions, each with at most 7 terms, only can cover 84 or fewer of the numbers from 1 to 100.

Alternate solution: there are 25 primes from 1 to 100, and it's easy to prove that no three can occur in the same progression.

36. [Corrected] Prove that any function defined for all real numbers can be represented as a sum of two functions, each of whose graphs has an axis of symmetry.

Solution: We write our given function $f(x)$ as the sum of $g(x)$, a function symmetric across the axis $x = 0$, and $h(x)$, a function symmetric across the axis $x = a$ for some fixed (but arbitrary) $a > 0$. We will define the function successively on the intervals $[-a, a], [a, 3a], [-3a, -a], [3a, 5a]$, etc.

On $[-a, a]$, put $g(x) = 0$ and $h(x) = f(x)$. We are forced to put $h(x) = f(2a - x)$ for $x \in [a, 3a]$, so we put $g(x) = f(x) - f(2a - x)$. Now we are forced to put $g(x) = g(-x)$ for $x \in [-3a, -a]$, so we put $h(x) = f(x) - g(-x)$ there, and so forth.

37. Given two points on the plane, 1 unit apart, we construct more points by intersecting lines and circles in the following ways:

- Given any two already constructed points, we may draw a circle with radius equal to the distance between these points, with center at an already constructed point.
- Given any two already constructed points, we may draw a line though them.

Let $C(n)$ denote the least number of circles that must be drawn so that we can construct two points which are n units apart. Let $RC(n)$ denote the least number of curves (circles and/or lines) that must be drawn so that we can construct two points which are n units apart. Prove that the sequence $C(n)/RC(n)$ is unbounded.

Solution: We will show that $C(2^m) \geq m - 1$, but $RC(2^{2^r}) \leq dr + e$ for some constants d, e, which will prove the claim, since then

$C(n)/RC(n) \geq (2^r - 1)/(dr + e)$ for $n = 2^{2^r}$.

The claim $C(2^m) \geq m - 1$ follows immediately from the fact that after k circles have been drawn, all of the circles sit inside some circle of radius 2^k. Proof by induction: if the first k circles sit in a circle of radius 2^k, the distance between any two of the marked points is at most 2^{k+1}, so the circle centered at the center of the next circle of radius 2^{k+1} contains all of the old circles as well as the new one.

The claim $RC(2^{2^r}) \leq dr + e$ is demonstrated by the following algorithm. Let A, B be the given points. Draw the circle through A centered at B, and the circle through B centered at A, and let C be one of the intersections. Also draw the circle through A centered at C. Now draw the lines AB and AC, and let P_0 be the intersection of AC with the circle around C, so that $AP_0 = 2$.

We construct points P_i, Q_i recursively as follows. Let Q_i be the intersection of the ray AB with the circle through P_{i-1} centered at A, and construct P_i as the intersection of AP_{i-1} with the line through Q_i parallel to BP_{i-1}. The point is that $AQ_i = BP_{i-1}$ and $AP_i/AP_{i-1} = AQ_i/AB$, which implies that $AP_i = (AP_{i-1})^2$. If d is the number of steps in each iteration and e is the number of initial steps, then the distance $AP_r = 2^{2^r}$ is constructed in $dr + e$ steps, as claimed.

38. Prove that for any natural number $a_1 > 1$, there exists an increasing sequence of natural numbers $a_1, a_2, \ldots$ such that $a_1^2 + a_2^2 + \cdots + a_k^2$ is divisible by $a_1 + a_2 + \cdots + a_k$ for all $k \geq 1$.

Solution: We will prove in fact that any finite sequence $a_1, \ldots, a_k$ with the property can be extended by a suitable a_{k+1}. Let $s_k = a_1 + \cdots + a_k$ and $t_k = a_1^2 + \cdots + a_k^2$. Then we are seeking a_{k+1} such that $a_{k+1} + s_k | a_{k+1}^2 + t_k$. This is clearly equivalent to $a_{k+1} + s_k | s_k^2 + t_k$. Why not, then, choose $a_{k+1} = s_k^2 - s_k + t_k$? Certainly this is greater than a_k and ensures that the desired property is satisfied.

39. There are n seats at a merry-go-round. A boy takes n rides. Between each ride, he moves clockwise a certain number of places to a new horse. Each time he moves a different number of places. Find all n for which the boy ends up riding each horse.

Solution: This is only possible for n even. If n is odd, the boy's net travel between the first and last rides is $1 + \cdots + n - 1$, since each distance must occur exactly once. But this is $n(n-1)/2$, which is divisible by n.

On the other hand, if n is odd, the boy can accomplish his goal by moving forward $1, n-2, 3, n-4, \ldots, n-1$ places. (Draw a small example to see how this works.)

40. The altitudes of a tetrahedron intersect at one point. Prove that this point, the foot of one of the altitudes, and the 3 points which cut the other altitudes at the ratio 2:1 (measured from the vertices) lie on a sphere.

Solution: Let $ABCD$ be the tetrahedron, H the intersection of the altitudes and G the centroid of face BCD. The plane through G parallel to face ACD cuts the altitude from B in the same ratio in which G cuts the median from B to CD, namely $2:1$. Let B_2, C_2, D_2 be the points cutting the altitudes from B, C, D, respectively, in the ratio $2:1$; we have just shown that B_2, C_2, D_2 lie on the sphere with diameter HG. However, the foot of the altitude from A also lies on this sphere, so the five points in question are indeed cospherical. (Compare with Russia 26 and Vietnam 3.)

1.18 Singapore

1. Let $\mathbb{N}$ be the set of all positive integers and let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying $f(1) = 1, f(2n) = f(n)$ and $f(2n+1) = f(2n)+1$ for all positive integers n.

   (a) Calculate the maximum value M of $f(n)$ where $1 \le n \le 1994$.

   (b) Find all positive integers n, $1 \le n \le 1994$, such that $f(n) = M$.

   Solution: By induction, $f(n)$ is the number of ones in the binary representation of n. Since $2^{10} < 1994 < 2^{11}$, every $n \le 1994$ has at most 11 binary digits, not all of which can be 1. Hence $M \le 10$, which is achieved by all numbers of the form $2^{11} - 2^n - 1$ with $1 \le n \le 10$. Of these, the ones not exceeding 1994 are $1023, 1535, 1791, 1919, 1983$.

2. Let ABC be a triangle with $m\angle A > 90°$. On side BC, two distinct points P and Q are chosen such that $\angle BAP = \angle PAQ$ and $BP \cdot CQ = BC \cdot PQ$. Find $\angle PAC$.

   Solution: We will show $\angle PAC = \pi/2$. By the (internal) angle bisector theorem, $BP/PQ = BA/AQ$. Let D be the intersection of BC with the external angle bisector at A. Then BD/DQ is also equal to BA/AQ. The proof of the latter assertion is analogous to that of the angle bisector theorem:

$$\frac{BD}{DQ} = \frac{BD}{\sin \angle BAD} \frac{\sin \angle DAQ}{DQ} = \frac{BA}{\sin \angle BDA} \frac{\sin \angle QDA}{AQ} = \frac{BA}{AQ}.$$

   Therefore $BD/DQ = BA/AQ = BC/CQ$ and so $C = D$, proving the claim.

3. Let $f(x) = \frac{1}{x+1}$ where x is a positive real number, and for any positive integer n, define

$$g_n(x) = x + f(x) + f(f(x)) + \cdots + \underbrace{f(f(\cdots f(x)))}_{n \text{ times}}.$$

   Prove that

   (a) $g_n(x) > g_n(y)$ if $x > y > 0$.

(b) $g_n(1) = \frac{F_1}{F_2} + \frac{F_2}{F_3} + \cdots + \frac{F_{n+1}}{F_{n+2}}$, where $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$.

Solution:

(a) Note that f is decreasing, but $x + f(x)$ is increasing (its derivative is $1-(1+x)^{-2}$, which is positive). Similarly, $f(f(x))$ is increasing, as is $f(f(x)) + f(f(f(x)))$ (the latter is the composition of the increasing functions $x + f(x)$ and $f(f(x))$). Hence we can group $g_n(x)$ into pairs of terms whose sum is increasing, and if n is even, the term left over is increasing as well. Hence $g_n(x)$ is increasing in x.

(b) The result follows from the following calculation:

$$f(F_n/F_{n+1}) = \frac{1}{F_n/F_{n+1} + 1} = \frac{F_{n+1}}{F_n + F_{n+1}} = F_{n+1}/F_{n+2}.$$

4. Let ABC be an acute-angled triangle. Suppose that the altitude of triangle ABC at B intersects the circle with diameter AC at P and Q and the altitude at C intersects the circle with diameter AB at M and N. Prove that P, Q, M and N lie on a circle.

Solution: We will show that $AP = AQ = AM = AN$, proving that P, Q, M, N lie on a circle centered at A. By symmetry, $AP = AQ$ and $AM = AN$, so we need only show $AP = AM$.

Let B' and C' be the feet of the altitudes from B and C, respectively. By similar triangles, $AB' \cdot AC = AP^2$ and $AC' \cdot AB = AM^2$. However, since $\triangle ABB' \sim \triangle ACC'$, $AB'/AC' = AB/AC$, and so $AP^2 = AM^2$, giving the desired equality.

5. Show that a path consisting of only horizontal or vertical edges on a rectanglular grid which starts at the top-left corner, goes through each point on the grid exactly once, and ends at the bottom-right corner divides the grid into two portions of equal area as follows:

   (a) The part consisting of those regions bounded by the path and opening at the top or to the right;

   (b) The part consisting of those regions bounded by the path and opening at the bottom or to the left.

Solution: We start with a few observation. A shaded region of s squares is adjacent to $2s+1$ of the edges of the path (by induction). Moreover, the edges on the top and right sides of the grid not lying on the path correspond to the shaded regions. If there are k shaded regions containing $s_1, \ldots, s_k$ squares, the total number of edges in the path can be counted as the number touching each shaded region, plus those along the top and right sides. Since the path consists of $mn-1$ edges, we get

$$mn-1=\sum_{i=1}^{k}(2s_i+1)+[(m-1)+(n-1)-k]=m+n-2+2\sum_{i=1}^{k}s_i$$

and therefore the number of shaded squares

$$\sum_{i=1}^{k}s_i=\frac{mn-m-n+1}{2}=\frac{(m-1)(n-1)}{2}.$$

Note that this quantity must be even because no such path exists when m and n are both even.

1.19 Taiwan

1. Let $P(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + a_nx^n$ be a polynomial with complex coefficients. Suppose the roots of $P(x)$ are $\alpha_1, \alpha_2, \ldots, \alpha_n$ with

$$|\alpha_1| > 1, |\alpha_2| > 1, \ldots, |\alpha_j| > 1$$

and

$$|\alpha_{j+1}| \leq 1, |\alpha_{j+2}| \leq 1, \ldots, |\alpha_n| \leq 1.$$

Prove that

$$\prod_{i=1}^{j} |\alpha_i| \leq \sqrt{|a_0|^2 + |a_1|^2 + \cdots + |a_n|^2}.$$

Solution: This inequality follows from the identity stated at the end of the solution of Russia 34. Put $Q(x) = (x - \alpha_1)\cdots(x - \alpha_j)$ and $R(x) = (x - \alpha_{j+1})\cdots(x - \alpha_n)$. Then the identity states that $|a_0|^2 + \cdots + |a_n|^2$ is equal to the sum of the squares of the absolute values of the coefficients of $Q(x)\overline{R}(1/x)$. The coefficient of x^{j-n} in the latter is $\alpha_1 \cdots \alpha_j$, and hence

$$|\alpha_1 \cdots \alpha_j|^2 \leq |a_0|^2 + \cdots + |a_n|^2,$$

as desired.

2. Consider the operation which transforms the 8-term sequence x_1, x_2, ..., x_8 into the new 8-term sequence

$$|x_2 - x_1|, |x_3 - x_2|, \ldots, |x_8 - x_7|, |x_1 - x_8|.$$

Find all 8-term sequences of integers which have the property that after finitely many applications of this operation, one is left with a sequence, all of whose terms are equal.

Solution: We shall show that in fact eventually all terms become 0. We show this by induction on the maximum absolute value of the terms, which one notes does not increase after the transformation.

First we show that eventually all terms will be even. Since $|x_2 - x_1| \equiv x_2 + x_1 \pmod{2}$, we can show by induction that after k steps, the new value of x_i will be congruent to

$$\sum_{j=0}^{k} \binom{k}{j} x_{i+j},$$

where $x_a = x_b$ whenever $a \equiv b \pmod{8}$. Recall that $\binom{8}{j}$ is even for $1 \leq j \leq 7$, so after 8 steps all terms will be even.

At this point, divide each term by 2. The resulting sequence has smaller maximum absolute value, and so by assumption it eventually becomes all 0. Obviously the same holds whether or not we divide by 2, so we conclude the original sequence eventually becomes all 0, as desired.

3. Suppose that each of the n guests at a party is acquainted with exactly 8 other quests. Furthermore, suppose that each pair of guests who are acquainted with each other have 4 acquaintances in common at the party, and each pair of guests who are not acquainted have only 2 acquaintances in common. What are the possible values of n?

Solution: No such n exists. We will first reduce to the case $n = 21$ by counting the number of ordered triples (u, v, w), where u knows v and w but v does not know w.

For a given v, there are $n - 9$ guests w that v does not know, and by assumption there are two guests u who know both v and w. Hence the number of such triples is $2n(n-9)$. On the other hand, for a given u, the 8 people who u knows each have 4 common acquaintances with u. Hence of the $8 \cdot 7 = 56$ pairs among these people, 32 consist of pairs who know each other, leaving 24 unacquainted pairs. Hence the number of such triples is also $56n$. Solving $2n^2 - 18n = 24n$ for positive integers n yields only the solution $n = 21$.

Now suppose we have a configuration of 21 people meeting the given condition. Label the people $1, \ldots, 21$ and let A be the 21×21 matrix whose ij entry is 1 if guest i knows guest j, and 0 otherwise. (In particular, $A_{ii} = 0$.) Now A^2_{ij} equals the number of people who know guests i and j. If $i = j$, this counts the numbes of people guest i knows, so $A^2_{ii} = 8$. If i and j are acquainted, we have $A^2_{ij} = 4$, otherwise $A^2_{ij} = 2$.

Let I be the 21×21 identity matrix and J the matrix of all ones; then we have $A^2 = 6I + 4A + 2(J - A) = 6I + 2J + 2A$. We now apply some linear algebra to get a contradiction. Since A is symmetric, it can be diagonalized completely, and moreover since $AJ = JA$, we can diagonalize A and J simultaneously. However, J has one eigenvector of eigenvalue 21 (the all-ones vector) and the rest have eigenvalue 0. So if v is an eigenvector of A linearly independent from the all-ones vector (whose eigenvalue is 8), then $\lambda^2 - 2\lambda - 6 = 0$, where $Av = \lambda v$. These 20 eigenvalues must come in 10 conjugate pairs since A is defined over the reals, but then the trace of A is $8 + 10(-2) \neq 0$, a contradiction since A has zeroes on the main diagonal. Hence $n = 21$ is impossible as well.

4. Let n distinct integers $m_1, m_2, \ldots, m_n$ be given. Prove that there exists a polynomial $f(x)$ of degree n with integral coefficients which satisfies the following conditions:

   (a) $f(m_i) = -1, \quad i = 1, 2, \ldots, n.$

   (b) $f(x)$ cannot be factored into a product of two nonconstant polynomials with integral coefficients.

Solution: The polynomial $f(x) = (x - m_1) \cdots (x - m_n) - 1$ has the desired properties. To see this, assume, on the contrary, that $f(x) = g(x)h(x)$, where g, h are nonconstant polynomials with integer coefficients. Then $g(m_i)h(m_i) = -1$ for each i, so one of $g(m_i), h(m_i)$ equals 1 and the other equals -1. In particular, $g(m_i) + h(m_i) = 0$ for each i. On the other hand, since neither g nor h has degree n (else the other would be constant), $g + h$ is a polynomial of degree less than n, and we have exhibited n of its roots. Therefore $g(x) + h(x) = 0$ identically, and hence the leading coefficients of g and h sum to 0. However, their product is the leading coefficient of f, which is 1, a contradiction.

5. Let P be a point on the circumcircle of triangle $A_1A_2A_3$. Let H be the orthocenter of triangle $A_1A_2A_3$. Let B_1 (B_2, B_3, respectively) be the point of intersection of the perpendicular form P to A_2A_3 (A_3A_1, A_1A_2, respectively). Suppose that B_1, B_2 and B_3 are colinear. Prove the the line $B_1B_2B_3$ passes through the midpoint of line segment PH.

Solution: Note that $B_1B_2B_3$ are always collinear; the line through these points is the Simson line of P with respect to $A_1A_2A_3$ Let the circumcenter O be the origin of a vector system; then $H = A_1 + A_2 + A_3$ and the midpoint of PH is $A_1 + A_2 + A_3 + P$. By symmetry, if the result holds, then the Simson line of each of A_1, A_2, A_3, H with respect to the triangle formed by the other three must pass through $(A_1 + A_2 + A_3 + P)/2$. We prove the result in the latter form, relabeling the four points A, B, C, D for clarity.

Let C_{AD}, C_{BD} be the feet of perpendiculars from C to AD and BD, respectively, and let D_{AC}, D_{BC} be the feet of perpendiculars from D to AC and BC, respectively. Then the points $C_{AD}, C_{BD}, D_{AC}, D_{BC}$ all lie on the circle with diameter CD, and so we may apply Pascal's theorem to the cyclic hexagon $CC_{AD}C_{BD}DD_{AC}D_{BC}$ to deduce that

$$CC_{AD} \cap DD_{AC}, C_{AD}C_{BD} \cap D_{AC}D_{BC}, C_{BD}D \cap D_{BC}C$$

are collinear. The first point is the orthocenter of ACD. The second is the intersection of the Simson line of C with respect to ABD and the Simson line of D with respect to ABC. The third point is simply B. Hence the intersection of the Simson lines of C and D lie on the line through B and $A+C+D$; similarly, it lies on the line through A and $B+C+D$. These intersect precisely at $(A+B+C+D)/2$, and so both Simson lines pass through this point. The same argument applies to show that the other two Simson lines pass through this point.

6. Let a, b, c, d be integers such that $ad - b\mathrm{c} = k > 0, (a, b) = (\mathrm{c}, d) = 1$. Prove that there are exactly k ordered pairs of real numbers (x_1, x_2) satisfying $0 \leq x_1, x_2 < 1$ and both $ax_1 + bx_2$ and $\mathrm{c}x_1 + dx_2$ are integers.

Solution: The question asks how many lattice points lie in the half-open parallelogram $\{(ax_1 + bx_2, \mathrm{c}x_1 + dx_2) : 0 \leq x_1, x_2 < 1\}$. Pick's Theorem states that the area of this parallelogram equals the number of interior lattice points plus half the number of boundary lattice points. Since (x_1, x_2) yields a lattice point if and only if $(1 - x_1, 1 - x_2)$ does, we conclude that the number of lattice points in the half-open parallelogram is exactly the area, which is easily computed as $ad - b\mathrm{c} = k$.

1.20 Turkey

1. [Corrected] Find all real solutions of the system of equations

$$\begin{aligned} x_1^2 + 2ax_1 + b^2 &= x_2 \\ x_2^2 + 2ax_2 + b^2 &= x_3 \\ &\vdots \\ x_{n-1}^2 + 2ax_{n-1} + b^2 &= x_n \\ x_n^2 + 2ax_n + b^2 &= x_1, \end{aligned}$$

where $b \geq a > 0$ are given real numbers.

Solution: Let $y_i = x_i - a + 1/2$; then we find

$$y_i^2 + y_i - a^2 + a + b^2 - 1/4 = y_{i+1}.$$

Let $d = b^2 - a^2 + a - 1/4$. First suppose $d > 0$. Then $y_{i+1} - y_i = y_i^2 + d > 0$ and so $y_1 < y_2 < \cdots < y_n < y_1$, which is impossible. So there are no solutions in this case.

Now suppose $d \leq 0$; note that we must in any case have $d \geq -1/4$ since $a > 0$ and $b^2 > a^2$. We have two solutions $y_1 = \cdots = y_n = \pm\sqrt{-d}$; we shall prove there are no others. As before, we have no solutions with $y_1 > \sqrt{-d}$. If $-\sqrt{-d} < y_i < \sqrt{-d}$, we have $-\sqrt{-d} < y_{i+1} < y_i$, where the left side follows because $f(x) = x^2 + x + c$ is increasing for $x > -\sqrt{-d} \geq -1/2$. (Here is where $d \geq -1/4$ is crucial; the solutions get much more complicated otherwise, and have been studied extensively in the investigation of dynamical systems.) Hence in this case $y_1 > \cdots > y_n$ and we get no solutions. Moreover, starting with $y_1 < -\sqrt{-d}$, the sequence either increases or eventually exceeds $-\sqrt{-d}$, in which case it will not decrease below $-\sqrt{-d}$. Hence there are no solutions in this case.

In summary, we only have solutions when $d \leq 0$, in which case the two solutions (one if $d = 0$) are

$$x_1 = \cdots = x_n = \pm\sqrt{d} + a - 1/2.$$

2. Let n be a positive integer. Find the number of permutations $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that $\sigma(j) \geq j$ for exactly two js.

Solution: Every permutation can be written uniquely as a product of disjoint cycles, and $\sigma(j) \geq j$ for at least one j in every cycle (namely, the smallest element of the cycle). Hence σ can have only one or two cycles.

If σ has two cycles, each must feature its elements in descending order, starting from the largest. Thus the number of σ of this form is simply the number of ways to split $\{1, \ldots, n\}$ into two nonempty sets, which is $2^{n-1} - 1$.

Now suppose σ has one cycle. This consists of a sequence of elements decreasing to 1, followed by a sequence decreasing to some $k > 1$. The elements $2, \ldots, k-1$ must appear in the first sequence, so the only choices are whether to put each of $k+1, \ldots, n$ into the first or second sequence. This gives 2^{n-k} such permutation. Since k can be any of $2, \ldots, n$, we get $1 + \cdots + 2^{n-2} = 2^{n-1} - 1$ permutations with one cycle.

Hence we have $2(2^{n-1} - 1) = 2^n - 2$ such permutations in total.

3. Let D be a point on the (small) arc AC of the circumcircle of an equilateral triangle ABC (D is neither A nor C). Let E and F be the feet of perpendiculars dropped from D to the sides BC and AC. Find the locus of the intersection point of EF and OD, where O is the circumcenter of ABC.

Solution: The intersection point is the midpoint of OD, which thus traces an arc centered at O with radius half of the circumradius of $\triangle ABC$. For proof of this assertion, see the solution to Taiwan 5.

4. In a convex quadrilateral, $m\angle CAB = 40°$, $m\angle CAD = 30°$, $m\angle DBA = 75°$, $m\angle DBC = 25°$. Find $m\angle BDC$.

Solution: Let $\theta = \angle BDC$. We first calculate some angles: $\angle ABE = 75°$, $\angle ACB = 40°, \angle AEB = 65°$. In particular, $\triangle ACB$ is isosceles with $AB = BC$. By the Law of Sines,

$$
\begin{aligned}
\frac{\sin\theta}{\sin(115° - \theta)} &= \frac{EC}{ED} \\
&= \frac{EC}{BC}\frac{AB}{AE}\frac{AE}{ED}
\end{aligned}
$$

$$= \frac{\sin\angle EBC}{\sin\angle BEC}\frac{\sin\angle AEB}{\sin\angle ABE}\frac{\sin\angle ADE}{\sin\angle EAD}$$

$$= \frac{\sin 25^\circ}{\sin 115^\circ}\frac{\sin 65^\circ}{\sin 75^\circ}\frac{\sin 35^\circ}{\sin 30^\circ} = \frac{2\sin 25 \sin 35^\circ}{\sin 75^\circ}.$$

However,

$$\frac{\sin\theta}{\sin(115^\circ - \theta)} = \frac{1}{\sin 115^\circ \cot\theta - \cos 115^\circ}$$

is a decreasing function of θ, and so the solution is unique. Hence one needs only verify that $\theta = 30^\circ$ verifies the above equation (a straightforward computation) to deduce $\angle BDC = 30^\circ$.

5. Prove that the following are equivalent.

   (a) For any positive integer a, n divides $a^n - a$.

   (b) For any prime divisor p of n, p^2 does not divides n and $p-1$ divides $n-1$.

Solution: First assume (a). If $p^2|n$ for some prime p, we must have $p^2|(p+1)^{p^2} - (p+1)$. However,

$$(p+1)^{p^2} - (p+1) = p^2 - p + \sum_{k=2}^{p^2}\binom{p^2}{k}p^k.$$

All terms but the first are divisible by p^2, contradicting the assumption. Therefore $p^2 \not| n$. Moreover, if a is a primitive root modulo p, then $a^{n-1} \equiv 1 \pmod{p}$ implies $p-1|n-1$.

On the other hand, if n is square-free and $p-1|n-1$ for all primes $p|n$, then for any a, either $p|a$ or $a^{p-1} \equiv 1 \pmod{p}$; in either case $a^n \equiv a \pmod{p}$ for all p dividing n. Hence the conditions are equivalent.

6. Define a sequence (x_n) of real numbers by $x_1 = 1$ and

$$x_{n+1} = x_n + x_n^{1/3}, \quad n \geq 1.$$

Show that there exist real numbers a and b such that $\lim_{n\to\infty} \frac{x_n}{an^b} = 1$.

Solution: Let $c_n = x_n n^{-3/2}$; we prove that $c_n \to (2/3)^{3/2}$. Recall that $n[(1+1/n)^{3/2} - 1] \to 3/2$ as $n \to 0$; hence for any $\epsilon > 0$, there exists N such that for $n \geq N$,

$$3/2 - \epsilon < n[(1+1/n)^{3/2} - 1] < 3/2 + \epsilon.$$

Now suppose $n \geq N$ and $c_n < (3/2+\epsilon)^{-3/2}$. Then $c_n^{2/3} < (3/2+\epsilon)^{-1}$ and so $(3/2+\epsilon)c_n < c_n^{1/3}$.

$$\begin{aligned}
x_{n+1} &= x_n + x_n^{1/3} \\
&= c_n n^{3/2} + c_n^{1/3} n^{1/2} \\
&> c_n n^{3/2} + (3/2+\epsilon) c_n n^{1/2} \\
&> c_n n^{3/2} + c_n n((1+1/n)^{3/2} - 1) n^{1/2} \\
&= c_n n^{3/2} (1+1/n)^{3/2} \\
&= c_n (n+1)^{3/2}.
\end{aligned}$$

Therefore $c_{n+1} > c_n$. We conclude that the limit inferior of the sequence $\{c_n\}$ is at least $(3/2+\epsilon)^{-3/2}$ for every $\epsilon > 0$, and so is at least $(2/3)^{3/2}$. A similar argument shows that the limit superior of the sequence is at most $(3/2-\epsilon)^{-3/2}$ for every $\epsilon > 0$. We conclude that $c_n \to (2/3)^{3/2}$, as desired.

1.21 United Kingdom

1. (a) Find the first positive integer whose square ends in three 4's.
   (b) Find all positive integers whose squares end in three 4's.
   (c) Show that no perfect square ends with four 4's.

   Solution: It is easy to check that $38^2 = 1444$ is the first positive integer whose square ends in three 4's. Now let n be any such positive integer. Then $n^2 - 38^2 = (n-38)(n+38)$ is divisible by $1000 = 2^3 5^3$. Hence at least one of $n - 38, n + 38$ is divisible by 4, and thus both are, since their difference is $76 = 4 \cdot 19$. Since $5 \not| 76$, then 5 divides only one of the two factors. Consequently $n - 38$ or $n + 38$ is a multiple of $4 \cdot 5^3 = 500$, so we have $n = 500k \pm 38$. It is easy to check that the square of all numbers of this form (where k is a positive integer) end in three 4's.

   Finally, notice that $n^2 = 250000k^2 \pm 38000k + 1444$, and the thousands digit must be odd. Thus no perfect squares end in four 4's.

2. Let $ABCD$ and $FGHE$ be opposite faces of a cube, labeled so that AF, BG, CH and DE are edges of the cube.

   (a) Find the area of the quadrilateral $AMHN$, where M is the midpoint of BC and N is the midpoint of EF.

   (b) Let P be the midpoint of AB and Q the midpoint of HE. Let AM meet CP at X and HN meet FQ at Y. Find the length of XY.

   Solution:

(a) By the Pythagorean theorem, $AMHN$ is a rhombus with each side $\sqrt{5}$. The area is $\frac{1}{2}AH \cdot MN = \frac{1}{2}2\sqrt{3} \cdot 2\sqrt{3} = 2\sqrt{6}$.

(b) Let Y' be the projection of Y onto $ABCD$. Then X is the intersection of medians AM, CP of triangle ABC. Thus $BX = BD/3 = 2\sqrt{2}/3$. By symmetry, $DY' = BX$, so $XY' = 2\sqrt{2}/3$. Pythagorean theorem yields $XY = 2\sqrt{11}/3$.

3. (a) Find the maximum value of the expression $x^2y - y^2x$ when $0 \le x \le 1$ and $0 \le y \le 1$.

(b) Find the maximum value of the expression

$$x^2y + y^2z + z^2x - x^2z - y^2x - z^2y$$

when $0 \le x \le 1$, $0 \le y \le 1$ and $0 \le z \le 1$.

Solution:

(a) Clearly $x = 1$; now $y(1 - y)$ is maximzed when $y = 1/2$ (complete the square, use calculus, etc.)

(b) Since $x^2y+y^2z+z^2x-x^2z-y^2x-z^2y = (x-y)(x-z)(x-z)$, WLOG z must equal zero, which reduces the expression to (a) above. Hence the solution is any cyclic permutation of $x = 1, y = 1/2, z = 0$.

4. Let ABC be a triangle with right angle at C. The internal angle bisectors of $\angle BAC$ and $\angle ABC$ meet BC and CA at P and Q, respectively. Let M and N be the feet of the perpendiculars from P and Q to AB, respectively. Find the measure of $\angle MCN$.

Solution: Let CK be the altitude from C. In triangles ACP, AMP, $\angle CAP = \angle MAP$ and $\angle ACP = \angle AMP = 90°$, and side AP is common. Hence triangles ACP, AMP are congruent and $CP = PM$. Therefore $\angle MCP = \angle PMC = \angle MCK$ (since $CK \parallel PM$). Similarly, $\angle NCQ = \angle NCK$. Consequently, $\angle MCN = \frac{1}{2}\angle ACB = 45°$.

5. The seven dwarfs walk to work each morning in single file. As they go, they sing their famous song, *"High-low-high-low, it's off to work we go ..."*. Each day they line up so that no three successive dwarfs are either increasing or decreasing in height. Thus, the lineup must go *up-down-up-down- ...* or *down-up-down-up- ...* . If they all have different heights, for how many days can they go to work like this if they insist on using a different order each day? What if Snow White always came along too?

Solution: Let u_n be the number of ways of arranging n dwarfs in up-down-up-down- ...order by height. Now, replacing smallest by tallest, next-smallest by next tallest, ...etc., shows that u_n is also the number of ways of arranging them in down-up-down-up order.

Define $u_0 = 1$. For $n > 1$, consider any one of the $2u_n$ "alternating" arrangements. If the tallest dwarf is in position r, then the number of possibilities is $\binom{n-1}{r-1}u_r u_{n-r}$. Thus, $2u_n = \sum_{r=1}^{n}\binom{n-1}{r-1}u_r u_{n-r}$ for $n > 1$. Computing, $2u_7 = 544$ and $2u_8 = 2770$ (if Snow White is included).

6. Find all triples of positive integers (a, b, c) such that

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{\mathrm{c}}\right) = 2.$$

Solution: Without loss of generality, assume $a \geq b \geq \mathrm{c}$. Note that we must have $2 \leq (1 + 1/\mathrm{c})^3$. This implies that $\mathrm{c} \leq 3$.

- If $\mathrm{c} = 1$, then $(1+1/a)(1+1/b) = 1$ which is clearly impossible.
- $\mathrm{c} = 2$ leads to $(1 + 1/a)(1 + 1/b) \doteq 4/3$ which implies that $4/3 \leq (1 + 1/b)^2$ which forces $b < 7$. Since $(1 + 1/a) > 1$, we must also have $b > 3$. Plugging in values yields the solutions $(7, 6, 2), (9, 5, 2), (15, 4, 2)$.
- $\mathrm{c} = 3$ gives $(1 + 1/a)(1 + 1/b) = 3/2$. Similar analysis leads to $b < 5, b \geq \mathrm{c} = 3$. These values yield the solutions $(8, 3, 3)$ and $(5, 4, 3)$.

In conclusion, the solutions are all cyclic permutations of

$$(7, 6, 2), (9, 5, 2), (15, 4, 2), (8, 3, 3) \text{ and } (5, 4, 3).$$

7. Let ABC be a triangle, and D, E, F be the midpoints of BC, CA, AB, respectively. Prove that $\angle DAC = \angle ABE$ iy and only if $\angle AFC = \angle ADB$.

Solution: Let the medians AD, BE, CF intersect at G. Supoose that $\angle DAC = \angle ABE$. Now $DF \parallel CA$ and $\angle DAC = \angle ADF$. We thus have $\angle GBF = \angle GDF$ and it follows that F, G, D, B are concyclic. Then

$$\angle AFC = 180° - \angle BFG = 180° - (180° - \angle GDB) = \angle ADB.$$

This argument reverses, so the converse is true.

8. Let a, b, c be real numbers satisfying $a < b < c, a + b + c = 6$ and $ab + bc + ca = 9$. Prove that $0 < a < 1 < b < 3 < c < 4$.
Solution: The numbers a, b, c are the roots of the polynomial $f(x) = x^3 - 6x^2 + 9x - p$, where $p = abc$. $f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$. This implies that $f(1) > 0, f(3) < 0$ and $a < 1 < b < 3 < c$. Also, $f(1) = 4 - p > 0$ and $f(3) = -p < 0$, so $0 < p < 4$. This forces $a > 0$. Finally, $f(x)$ is strictly increasing when $x > 3$ and $f(4) = f(1) = 4 - p > 0$. This implies $c < 4$.

9. (a) Determine, with careful explanation, how many ways $2n$ people can be paired off to form n teams of 2.

   (b) Prove that $\{(mn)!\}^2$ is divisible by $(m!)^{n+1}(n!)^{m+1}$ for all positive integers m, n.

Solution:

(a) Consider the $2n$ people arranged in pairs in a row, which can be done in $(2n)!$ ways. We have to divide this number by 2^n to discount the orders of the people within each of the n pairs, and also by $n!$ to discount the order of the pairs themselves. Hence our answer is $\dfrac{(2n)!}{(2^n n!)}$.

(b) By similar reasoning mn people can be placed in n groups, each containing m people, in $\dfrac{(mn)!}{(m!)^n n!}$ ways. Thus

$$\frac{(mn)!}{(m!)^n n!}\frac{(nm)!}{(n!)^m m!}$$

is a product of two integers, which establishes that $((mn)!)^2$ is divisible by $(m!)^{n+1}(n!)^{m+1}$.

1.22 United States of America

1. Let p be an odd prime. The sequence $(a_n)_{n\geq 0}$ is defined as follows: $a_0 = 0$, $a_1 = 1, \dots, a_{p-2} = p-2$ and, for all $n \geq p-1$, a_n is the least integer greater than a_{n-1} that does not form an arithmetic sequence of length p with any of the preceding terms. Prove that, for all n, a_n is the number obtained by writing n in base $p-1$ and reading the result in base p.

Solution: Our proof uses the following result.

PROPOSITION *Let* $B = \{b_0, b_1, b_2, \dots\}$, *where* b_n *is the number obtained by writing* n *in base* $p-1$ *and reading the result in base* p. *Then (i) for every* $a \notin B$, *there exists* $d > 0$ *so that* $a - kd \in B$ *for* $k = 1, 2, \dots, p-1$, *and (ii)* B *contains no* p*-term arithmetic progression.*

Proof. Note that $b \in B$ if and only if the representation of b in base p does not use the digit $p-1$.

(i) Since $a \notin B$, when a is written in base p at least one digit is $p-1$. Let d be the positive integer whose representation in base p is obtained from that of a by replacing each $p-1$ by 1 and each digit other than $p-1$ by 0. Then none of the numbers $a-d$, $a-2d$, $\dots$, $a-(p-1)d$ has $p-1$ as a digit when written in base p, and the result follows.

(ii) Let a, $a+d$, $a+2d$, $\dots$, $a+(p-1)d$ be an arbitrary p-term arithmetic progression of nonnegative integers. Let δ be the rightmost nonzero digit when d is written in base p, and let α be the corresponding digit in the representation of a. Then $\alpha, \alpha+\delta, \dots, \alpha + (p-1)\delta$ is a complete set of residues modulo p. It follows that at least one of the numbers a, $a+d$, $\dots$, $a+(p-1)d$ has $p-1$ as a digit when written in base p. Hence at least one term of the given arithmetic progression does not belong to B.

Let $(a_n)_{n\geq 0}$ be the sequence defined in the problem. To prove that $a_n = b_n$ for all $n \geq 0$, we use mathematical induction. Clearly $a_0 = b_0 = 0$. Assume that $a_k = b_k$ for $0 \leq k \leq n-1$, where $n \geq 1$. Then a_n is the smallest integer greater than b_{n-1} such that $\{b_0, b_1, \dots, b_{n-1}, a_n\}$ contains no p-term arithmetic progression. By part (i) of the proposition, $a_n \in B$ so $a_n \geq b_n$. By part (ii) of the proposition, the choice of $a_n = b_n$ does not yield a p-term arithmetic progression with any of the preceding terms. It follows by induction that $a_n = b_n$ for all $n \geq 0$.

2. A calculator is broken so that the only keys that still work are the sin, cos, tan, $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ buttons. The display initially shows 0. Given any positive rational number q, show that pressing some finite sequence of buttons will yield q. Assume that the calculator does real number calculations with infinite precision. All functions are in terms of radians.

Solution: Since $\cos^{-1} \sin\theta = \pi/2 - \theta$ and $\tan(\pi/2 - \theta) = 1/\tan\theta$ for $0 < \theta < \pi/2$, we have for any $x > 0$,

$$\tan \cos^{-1} \sin \tan^{-1} x = \tan(\pi/2 - \tan^{-1} x) = 1/x. \qquad (1)$$

Also for $x \geq 0$,

$$\cos \tan^{-1} \sqrt{x} = 1/\sqrt{x+1},$$

so by (1),

$$\tan \cos^{-1} \sin \tan^{-1} \cos \tan^{-1} \sqrt{x} = \sqrt{x+1}. \qquad (2)$$

By (1) and (2), we can obtain $\sqrt{r}$ for any nonnegative rational number r that can be obtained from 0 using the operations

$$x \mapsto x+1 \qquad \text{and} \qquad x \mapsto 1/x.$$

We now prove that every nonnegative rational number r can be so obtained, by induction on the denominator of r. If the denominator is 1, we can obtain the nonnegative integer r by repeated application of $x \mapsto x+1$. Now assume we can get all r with denominator up to n. In particular, we can get any of

$$\frac{n+1}{1}, \frac{n+1}{2}, \ldots, \frac{n+1}{n},$$

so we can also get

$$\frac{1}{n+1}, \frac{2}{n+1}, \ldots, \frac{n}{n+1},$$

and any positive r of exact denominator $n+1$ can be obtained by repeatedly adding 1 to one of these.

Thus for any positive rational number r, we can obtain $\sqrt{r}$. In particular, we can obtain $\sqrt{q^2} = q$.

3. Given a nonisosceles, nonright triangle ABC, let O denote the center of its circumscribed circle, and let A_1, B_1, and C_1 be the midpoints of sides BC, CA, and AB, respectively. Point A_2 is located on the ray OA_1 so that ΔOAA_1 is similar to ΔOA_2A. Points B_2 and C_2 on rays OB_1 and OC_1, respectively, are defined similarly. Prove that lines AA_2, BB_2, and CC_2 are concurrent, i.e. these three lines intersect at a point.

Solution: Let Γ denote the circumcircle of ΔABC. Since ΔOAA_1 and ΔOA_2A are similar, we have

$$\frac{OA_1}{OA} = \frac{OA}{OA_2} \qquad \text{so} \qquad (OA_1)(OA_2) = (OA)^2 = (OB)^2 = (OC)^2.$$

(Thus A_2 is the image of A_1 with respect to inversion about Γ.) Note that OA_1 is perpendicular to chord BC, and that $OA_2/OB = OB/OA_1$, so ΔOA_2B is similar to ΔOBA_1. Hence the segment A_2B is tangent to Γ at B. Likewise, A_2C is tangent to Γ at C.

Similarly, B_2C and B_2A are tangents to Γ from B_2, and C_2A and C_2B are tangents to Γ from C_2. Thus Γ is the incircle of $\Delta A_2B_2C_2$ with A, B, C as the points of tangency. It is well known and easy to prove (using Ceva's theorem and equal tangents) that the cevians AA_2, BB_2, and CC_2 of $\Delta A_2B_2C_2$ are concurrent.

4. Suppose $q_0, q_1, q_2, \ldots$ is an infinite sequence of integers satisfying the following two conditions:

   (i) $m - n$ divides $q_m - q_n$ for $m > n \geq 0$,

   (ii) there is a polynomial P such that $|q_n| < P(n)$ for all n.

   Prove that there is a polynomial Q such that $q_n = Q(n)$ for all n.

Solution: Let d be the degree of P. There is a polynomial $Q(x)$ of degree at most d and having rational coefficients such that $Q(i) = q_i$ for $i = 0, 1, 2, \ldots, d$. An explicit expression for this polynomial is

$$Q(x) = q_0L_0(x) + q_1L_1(x) + \cdots + q_dL_d(x),$$

where

$$L_i(x) = \prod_{\substack{0 \leq j \leq d \\ j \neq i}} \left(\frac{x-j}{i-j}\right).$$

(This is the form of the interpolating polynomial due to Lagrange.) We shall prove that $q_n = Q(n)$ for all $n \geq 0$.

Let $k \geq 1$ be a common denominator for the coefficients of Q, and for $n = 0, 1, 2, \ldots$ set $r_n = k(Q(n) - q_n)$. Then $r_i = 0$ for $i = 0, 1, \ldots, d$. It is well known that for any polynomial $\mathcal{P}$ with integer coefficients, and any distinct integers m, n,

$$m - n \text{ divides } \mathcal{P}(m) - \mathcal{P}(n).$$

The polynomial $kQ(n)$ has integer coefficients and $m - n$ divides $q_m - q_n$, so $m - n$ divides $r_m - r_n$ for all $m > n \geq 0$. Also, there is a polynomial R of degree at most d such that $|r_n| < R(n)$ for all $n \geq 0$. This follows from the fact that P and Q have degree at most d and

$$|r_n| \leq k(|Q(n)| + |q_n|) < k(|Q(n)| + P(n)).$$

Thus $|r_n| < an^d + b$ for all $n \geq 0$ if a and b are appropriately large constants. For any $n > d$ and $0 \leq i \leq d$,

$$n - i \text{ divides } r_n - r_i = r_n,$$

so $\mathrm{LCM}(n, n-1, \ldots, n-d)$ divides r_n. We claim that

$$\mathrm{LCM}(n, n-1, \ldots, n-d) > R(n)$$

for all sufficiently large n, say for all $n \geq N$. Since $\mathrm{LCM}(n, n-1, \ldots, n-d)$ divides r_n and $R(n) > |r_n|$, the truth of this claim implies that $r_n = 0$ for all $n \geq N$. Now $r_n = 0$ for $n < N$ as well, since for arbitrary large $m \geq N$,

$$m - n \text{ divides } r_m - r_n = -r_n.$$

Hence $r_n = 0$ for all $n \geq 0$. By definition of r_n, we get $q_n = Q(n)$ for all $n \geq 0$.

Proof of the claim. We use the fact that for any finite sequence of positive integers $a_1, a_2, \ldots, a_m$,

$$\mathrm{LCM}(a_1, a_2, \ldots, a_m) \geq \frac{\prod_{i=1}^{m} a_i}{\prod_{1 \leq i < j \leq m} \mathrm{GCD}(a_i, a_j)}.$$

To see this, let p be an arbitrary prime and for $i = 1, 2, \ldots, m$ let e_i be the exponent of p in the prime factorization of a_i. Then the exponent of p in the prime factorization of $\mathrm{LCM}(a_1, a_2, \ldots, a_m)$ is $\max(e_1, e_2, \ldots, e_m)$, and this is clearly at least as large as $\sum_{i=1}^m e_i - \sum_{1 \le i < j \le m} \min(e_i, e_j)$, which is the exponent of p in the prime factorization of the right hand side. Since

$$\prod_{0 \le i < j \le d} \mathrm{GCD}(n-i, n-j) = \prod_{0 \le i < j \le d} \mathrm{GCD}(n-i, j-i) \le \prod_{0 \le i < j \le d} (j-i),$$

the degree of R is d, and the degree of $n(n-1)\cdots(n-d)$ is $d+1$, it follows that

$$\mathrm{LCM}(n, n-1, \ldots, n-d) \ge \frac{n(n-1)\cdots(n-d)}{\prod_{0 \le i < j \le d}(j-i)} > R(n)$$

for all sufficiently large n.

5. Suppose that in a certain society, each pair of persons can be classified as either *amicable* or *hostile.* We shall say that each member of an amicable pair is a *friend* of the other, and each member of a hostile pair is a *foe* of the other. Suppose that the society has n persons and q amicable pairs, and that for every set of three persons, at least one pair is hostile. Prove that there is at least one member of the society whose foes include $q(1 - 4q/n^2)$ or fewer amicable pairs.

Solution: Let S denote the set of persons in the society, let A denote the set of all amicable pairs, and let H denote the set of all hostile pairs. For each $x \in S$, let $f(x)$ denote the number of friends of x and let $F(x)$ denote the number of amicable pairs among the foes of x. Since an amicable pair $\{a, b\}$ is counted by both $f(a)$ and $f(b)$,

$$q = |A| = \sum_{\{a,b\} \in A} 1 = \frac{1}{2} \sum_{x \in S} f(x).$$

Also

$$\sum_{\{a,b\} \in A} (f(a) + f(b)) = \sum_{x \in S} f^2(x),$$

since for each $x \in S$ the term $f(x)$ occurs $f(x)$ times in the sum on the left.

The fact that each set of three persons contains at least one hostile pair implies that for each amicable pair $\{a, b\}$, the number of persons who are foes of both a and b is $(n-2)-(f(a)-1)-(f(b)-1)=n-f(a)-f(b)$. Consider $\sum_{x \in S} F(x)$. This sum counts the number of triples $\{x, y, z\}$ such that $\{x, y\} \in H$, $\{x, z\} \in H$, and $\{y, z\} \in A$. In view of the above remark, the same set of triples is counted by $\sum_{\{a,b\} \in A}(n-f(a)-f(b))$. Hence the average value of F satisfies

$$\begin{aligned}
\frac{1}{n} \sum_{x \in S} F(x) &= \frac{1}{n} \sum_{\{a,b\} \in A} (n-f(a)-f(b)) \\
&= q-\frac{1}{n} \sum_{x \in S} f^2(x) \\
&= q-\frac{4q^2}{n^2}-\frac{1}{n} \sum_{x \in S}(f(x)-2q/n)^2 \\
&\leq q-\frac{4q^2}{n^2}.
\end{aligned}$$

It follows that $F(x) \leq q(1-4q/n^2)$ for some $x \in S$.

1.23 Vietnam

1. [Corrected] Find all real solutions of the equation $x^3 - 3x^2 - 8x + 40 - 8\sqrt[4]{4x+4} = 0$.

   Solution: The left-hand side is only defined for $x \geq -1$, so we assume this throughout. A little experimentation shows that $x = 3$ is a solution, and we will show there are none others. In fact, the function is convex on $-1 \leq x < \infty$ with its minimum at $x = 3$.

   The easiest way to show these things is by taking derivates. The first derivative of the left side is $3x^2 - 6x - 8 - 8(4x+4)^{-3/4}$, which indeed vanishes at $x = 3$. The second derivative is $6x - 6 + 24(4x+4)^{-7/4}$. This is clearly positive for $x > 1$, so the first derivative is negative for $1 \leq x < 3$ and positive for $x > 3$. For $-1 \leq x \leq 1$, we can see by inspection that again the first derivative is negative; hence the function decreases to its minimum at $x = 3$ and then increases.

2. The sequence $(a_n)_{n \geq 0}$ is defined by $a_0 = 1, a_1 = 3$ and

$$a_{n+2} = \begin{cases} a_{n+1} + 9a_n & \text{if } n \text{ is even,} \\ 9a_{n+1} + 5a_n & \text{if } n \text{ is odd.} \end{cases}$$

   Prove that

   (a) $\sum_{k=1995}^{2000} a_k^2$ is divisible by 20,

   (b) a_{2n+1} is not a perfect square for every $n = 0, 1, 2, \ldots$.

   Solution:

(a) We will first prove the sum is divisible by 4, then by 5. Note that $a_{n+2} \equiv a_{n+1} + a_n \pmod 4$ whether n is odd or even. The sequence modulo 4 thus proceeds $1, 3, 0, 3, 3, 2, 1, 3, \cdots$ in a cycle of 6, so the sum of squares of any six consecutive terms is congruent to $1^2 + 3^2 + 0^2 + 3^2 + 3^2 + 2^2 \equiv 0 \pmod 4$.

   Now let us work modulo 5, in which case $a_{n+2} \equiv a_{n+1} + 4a_n$ if n is even and $a_{n+2} \equiv 4a_{n+1}$ if n is odd. Hence the sequence modulo 5 proceeds $1, 3, 2, 3, 1, 4, 3, 2, 4, 1, 2, 3, \cdots$ in a cycle of 8 beginning with a_2. This means

$$a_{1995}^2 + \cdots + a_{2000}^2 \equiv a_3^2 + \cdots + a_8^2 \equiv 3^2 + 1^2 + 4^2 + 3^2 + 2^2 + 4^2 \equiv 0 \pmod{5}.$$

(b) Notice that $a_{2n+1} \equiv 5a_{2n-1} \pmod{9}$. Since $a_1 = 3$, by induction $a_{2n+1} \equiv 3 \pmod{9}$ for all n. However, no perfect square is congruent to 3 modulo 9, since any square divisible by 3 is also divisible by 9. Hence a_{2n+1} is not a square.

3. Consider a nonequilateral triangle ABC with altitudes AD, BE, CF. For every $k \in \mathbb{R}$, $k \neq 0$, let A', B', C' be points defined by $AA' = kAD', BB' = kBE, CC' = kCF$.

   (a) Let $k = 2/3$. Prove that triangle $A'B'C'$ is similar to triangle ABC and calculate the coefficient of similitude in terms of the angles A, B, C.

   (b) Determine all $k \neq 0$ such that for every nonequilateral triangle ABC, triangle $A'B'C'$ is similar to ABC.

Solution:

(a) Note that the lines through the centroid G parallel to BC, CA, AB cut the altitudes at A', B', C', since the centroid cuts each median in the ratio $2 : 1$. Therefore A', B', C' each lie on the circle with diameter GH, so in particular

$$\angle A'B'C' = \pi - \angle C'HA' = \pi - \angle FHD = \angle DBF = \angle CBA.$$

Analogously, the other two angles are equal, so $\triangle A'B'C' \sim \triangle ABC$. The coefficient of similitude equals the ratio of the circumradii of the triangles. If R is the circumradius of $\triangle ABC$, this ratio is $GH/2R$. (Compare Russia 26 and 40.)

(b) Suppose $\triangle ABC$ is an isosceles right triangle with $AB = AC = s$. Then $E = F = A$ and D is the midpoint of BC. If $\triangle A'B'C' \sim \triangle ABC$, then we must have $(B'C')^2 = 2(A'B')^2$. The former is $2(1-k)^2 s^2$, while the latter is

$$2\left[\frac{s^2}{2}(1-k)^2 + \frac{s^2}{2}(1-2k)^2\right].$$

Equality occurs when $(1-k)^2 = (1-2k)^2$, or $3k^2 - 2k = 0$. This can only happen for $k = 0$ or $k = 2/3$, and the former is disallowed, so only $k = 2/3$ is possible.

4. Given tetrahedron $ABCD$ and let A', B', C', D' be the centers of the cirumcircles of triangles BCD, CDA, DAB, ABC, respectively. Suppose that A', B', C', D' are not coplanar. Prove that the following four planes have a common point:

   (a) Passing through A and perpendicular to $C'D'$.

   (b) Passing through B and perpendicular to $D'A'$.

   (c) Passing through C and perpendicular to $A'B'$.

   (d) Passing through D and perpendicular to $B'C'$.

Solution: We first note that the perpendicular bisecting plane of the segment AB contains both C' and D', so $C'D' \perp AB$. Hence (a) contains B as well, and so (a) and (b) meet in the line through B perpendicular to the plane $C'D'A'$. Similarly, (c) and (d) meet in the line through D perpendicular to $A'B'C'$.

Of these two lines, the former lies in the plane through B perpendicular to $A'C'$, and the second lies in the plane through D perpendicular to $A'C'$. However, A' and C' lie on the perpendicular bisecting plane of the segment BD, so $A'C' \perp BD$ and the aforementioned planes coincide. Since the two lines are not parallel (or else A', B', C', D' would be coplanar) and lie in a plane, they intersect, and their common point lies on all four of the given planes.

5. Determine all polynomials $P(x)$ satisfying the following condition: for every $a > 1995$, the number of real roots of the equation $P(x) = a$ (each root is counted with its multiplicity) is equal to the degree of the polynomial $P(x)$ and every real root of this equation is greater than 1995.

Solution: No such P can exist with degree greater than 2, since for large positive a, the line $y = a$ crosses the graph of P only once if the degree of P is odd, or twice if the degree is even. Moreover, if P is quadratic, for large a the smaller root of $P(x) = a$ will be much smaller than 1995. Hence we can only have $P(x) = cx + d$. Then for every $a > 1995$ we must have $(a - d)/c > 1995$ also. Necessary and sufficient conditions are $c > 0$ and $-d/c \geq 1995$.

6. Let $n \geq 2$ be an integer. Color all vertices of a regular $2n$-gon by n colors so that the following conditions are satisfied.

   (a) Each vertex is colored by exactly one color.

   (b) Each color is used to color exactly two nonadjacent vertices.

   Two such colorings are called *equivalent* if one coloring is obtained from the other by a rotation about the center of the regular polygon. Determine the number of nonequivalent colorings.

   Solution: We first solve a closely related problem. Suppose n married couples are to be seated around a table such that no two neighbors are from the same couple. How many ways is this possible? (Rotations are not to be considered distinct; we can achieve the same effect by requiring a certain person to occupy a particular chair.)

   We compute this using the principle of inclusion-exclusion (PIE). For any given subset of k of the couples, there are $2^k(2n - k - 1)!$ different arrangements keeping at least those couples together. (Each couple can be internally ordered in two ways. Viewing the couples as "blocks", we now have to arrange $2n - k$ blocks around the table.) Therefore the number of arrangements keeping no couples together is

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}2^i(2n - i - 1)!.$$

   To solve the original problem, we simply divide by 2^n since we no longer distinguish vertices of the same color.

1.24 Austrian-Polish Mathematics Competition

1. For a given integer $n \geq 3$ find all real solutions $(a_1, \ldots, a_n)$ of the system of equations

$$\begin{aligned} a_3 &= a_2 + a_1, \\ a_4 &= a_3 + a_2, \\ &\vdots \\ a_n &= a_{n-1} + a_{n-2}, \\ a_1 &= a_n + a_{n-1}, \\ a_2 &= a_1 + a_n. \end{aligned}$$

Solution: Let F_n be the n-th Fibonacci number (with $F_0 = 0, F_1 = 1$). Then by induction, we have $a_k = F_{k-1}a_2 + F_{k-2}a_1$ for $k = 2, \ldots, n$. Therefore a_1, a_2 form part of a solution if and only if $a_1 = F_n a_2 + F_{n-1}a_1$ and $a_2 = F_{n+1}a_2 + F_n a_1$. These two equations can be rewritten as $F_n a_2 = (1 - F_{n-1})a_1$ and $(1 - F_{n+1})a_2 = F_n a_1$. However, for $n \geq 3$, $F_n/(1 - F_{n-1}) \neq (1 - F_{n+1})/F_n$, so there are no solutions other than $(0, 0, \ldots, 0)$.

2. Let A_1, A_2, A_3, A_4 be four distinct points in the plane and let $X = \{A_1, A_2, A_3, A_4\}$. Show that there exists a subset Y of the set X with the following property: there is no disk K such that $K \cap X = Y$. *Note*: all points of the circle limiting a disk are considered to belong to the disk.

Solution: If one of the points, say A_4, lies in the triangle formed by the other three, then we may take $Y = \{A_1, A_2, A_3\}$. Otherwise, the four form a convex quadrilateral in some order. Without loss of generality, assume that they occur in the order $A_1A_2A_3A_4$.

The sum of the angles of this quadrilateral is 2π, so one of the pairs of opposite angles must sum to at least π. Suppose these are the angles at A_2 and A_4. Then the circumcircle of $A_1A_2A_3$ encloses A_4, and so any disk containing A_1, A_2, A_3 also contains A_4.

3. Let $P(x) = x^4 + x^3 + x^2 + x + 1$. Show that there exist polynomials $Q(y)$ and $R(y)$ of positive degrees, with integer coefficients, such that $Q(y) \cdot R(y) = P(5y^2)$ for all y.

Solution: It is reasonable to try a factorization with $R(y) = Q(-y)$. By direct calculation, one easily discovers the factorization

$$P(5y^2) = (25y^4 + 25y^3 + 15y^2 + 5y + 1)(25y^4 - 25y^3 + 15y^2 - 5y + 1).$$

4. Determine all polynomials $P(x)$ with real coefficients, such that

$$(P(x))^2 + (P(1/x))^2 = P(x^2)P(1/x^2) \qquad \text{for all} \quad x \neq 0.$$

Solution: The constant polynomials satisfying the equation are $P(x) = \mathrm{c}$ for those c such that $\mathrm{c}^2 = 2\mathrm{c}$, namely $\mathrm{c} = 0, 2$. We shall show there are no other solutions. Suppose, on the contrary, that there exists a polynomial of positive degree satisfying the equation; let $P(x) = a_0x^n + \cdots + a_n$ be such a polynomial with minimal degree. The leading coefficient of $P(x)^2 + P(1/x)^2$ is a_0^2, while the leading coefficient of $P(x^2)P(1/x^2)$ is a_0a_n. Therefore $a_0 = a_n$. In particular, the latter cannot be zero.

Now note that

$$P(x)^2 + P(1/x)^2 = P(x^2)P(1/x^2) = P(-x)^2 + P(-1/x)^2.$$

Rearranging this in the form

$$P(x)^2 - P(-x)^2 = P(-1/x)^2 - P(1/x)^2,$$

we observe that we have a polynomial in x equal to a polynomial in $1/x$. We conclude both must equal 0. (For example, as $x \to \infty$, the right side goes to $P(0)^2 - P(0)^2 = 0$, whereas the only polynomials that go to 0 as $x \to \infty$ are constants.) Therefore $P(x)^2 - P(-x)^2 = 0$, and the left side factors as $[P(x) + P(-x)][P(x) - P(-x)]$, so one of the factors must be the zero polynomial. However, if $P(x) + P(-x) = 0$, we have $P(0) = -P(0)$ and so $P(0) = 0$, whereas we noted above that the constant term of P must be nonzero. Therefore $P(x) - P(-x) = 0$, which is to say P is an even polynomial.

Every even polynomial P can be expressed as $Q(x^2)$ for some polynomial Q, and the given equation for P implies that

$$Q(x^2)^2 + Q(1/x^2)^2 = Q(x^4)Q(1/x^4).$$

Letting $x^2 = y$, we get the polynomial identity

$$Q(y)^2 + Q(1/y)^2 = Q(y^2)Q(1/y^2)$$

which means Q is also a polynomial satisfying the given equation. However, $\deg Q = (\deg P)/2$, whereas P was chosen to be a solution of the equation with minimal positive degree. This contradiction implies no such polynomial with positive degree exists, completing the solution.

5. An equilateral triangle ABC is given. Denote the midpoints of sides BC, CA, AB respectively by A_1, B_1, C_1. Three distinct parallel lines p, q, r are drawn through A_1, B_1, C_1, respectively. Line p cuts B_1C_1 at A_2; line q cuts C_1A_1 at B_2; line r cuts A_1B_1 at C_2. Prove that the lines AA_2, BB_2, CC_2 concur at a point D lying on the circumcircle of triangle ABC.

Solution: We use directed angles modulo π. Let D_A, D_B, D_C be the second intersections of AA_2, BB_2, CC_2 with the circumcircle of $\triangle ABC$. Then $\angle D_CBA = B_2BC_1$. Since $\triangle ABC$ is equilateral, B is the reflection of B_1 across A_1C_1, and hence $\angle B_2BC_1 = \angle C_1B_1B_2$. Similarly $\angle D_BCA = \angle C_2CB_1 = \angle B_1C_1C_2$. However, note that $\angle C_1B_1B_2 = \angle B_1C_1C_2$ since both of these are the angle between the line C_1B_1 and the pair of parallel lines B_1B_2 and C_1C_2. Hence $\angle D_CBA = \angle D_BCA$; since A, B, C, D_B are concyclic, we have $\angle D_BCA = \angle D_BBA$ and therefore D_B, D_C, B are collinear. Since all three of these lie on the circle, $D_B = D_C$. (We are using the assumption that D_B and D_C are distinct from B, which is clearly true except in certain degenerate cases. The result is obviously still true in those cases, by a limiting argument.)

By an analogous argument, we deduce $D_A = D_B$, and hence all three of AA_2, BB_2, CC_2 pass through a single point on the circumcircle of $\triangle ABC$, as claimed.

6. The Alpine Club consisting of n members organizes four mountain expeditions for its members. Let E_1, E_2, E_3, E_4 be the four teams participating in these expeditions. How many ways are there to form those teams, given the condition that $E_1 \cap E_2 \neq \emptyset$, $E_2 \cap E_3 \neq \emptyset$, $E_3 \cap E_4 \neq \emptyset$?

Solution: By the principle of inclusion and exclusion, we get the desired total by counting the number of formations without any constraints, then for each constraint subtracting the number of formations that fail to meet that constraint, then adding for each pair of constraints the number that fail to meet both constraints, then subtracting the number that fail all three constraints.

Since each person can choose whether or not to be on each team, there are 16^n formations in total. Suppose one constraint fails, say $E_1 \cap E_2 = \emptyset$. Then each person only has 12 choices, since four choices involve joining both E_1 and E_2. If $E_1 \cap E_2$ and $E_3 \cap E_4$ are both empty, each person has only 9 choices, while if either of the other pairs fail, each person has 10 choices. Finally, if all three constraints fail, each person has 8 choices. Hence the number of acceptable formations is

$$16^n - 3 \cdot 12^n + 9^n + 2 \cdot 10^n - 8^n.$$

7. For every integer c consider the equation $3y^4 + 4cy^3 + 2xy + 48 = 0$, with integer unknowns x and y. Determine all integers c for which the number of solutions (x, y) in pairs of integers satisfying the additional conditions (A) and (B) is a maximum:

(A) the number $|x|$ is the square of an integer;

(B) the number y is square-free (i.e., there is no prime p with p^2 dividing y).

Solution: Since $3y^4 = -4cy^3 - 2xy - 48$, y must be even; assumption (B) ensures that y is not divisible by 4. But now $2xy = -3y^4 - 4cy^3 - 48$ is a multiple of 16, so x must be a multiple of 4. If we put $x = 4a, y = 2b$, we have the new equation

$$3b^4 + 2cb^3 + ab + 3 = 0,$$

with $|a|$ a perfect square and b odd and squarefree. Now it is clear that $b|3$, so the only possibilities are $b = 1, -1, 3, -3$. Solving for a, we find the four solutions are

$$a = 6 - 2c, -6 - 2c, -82 - 18c, 82 - 18c.$$

Since c and $-$c give the same number of squares, we assume c ≥ 0. Looking mod 4, we see none of these can be ± 1 times a square if c is even. If c $= 2d+1$ is odd, we have

$$a/4 = 1-d, -2-d, -25-9d, 16-9d.$$

The first two differ by 3, so they can both have square absolute value if and only if $d=-3$ or $d=2$ (making the values 4 and 1). The last two differ by 41,so they can both have square absolute value if and only if $d=0$ (so the values are 16 and 25; setting the squares to 441 and 400 gives d not an integer).

Of these, only $d=0$ gives a third square. Hence the maximum is three solutions, achieved only when c $=1$.

8. Consider the cube with vertices $(\pm1,\pm1,\pm1)$, i.e., the set

$$\{(x,y,z)\colon \ |x|\leq 1, \ |y|\leq 1, \ |z|\leq 1\}.$$

Let $V_1,\ldots,V_{95}$ be points of that cube. Denote by $\mathbf{v}_i$ the vector from $(0,0,0)$ to V_i. Consider the 2^{95} vectors of the form $s_1\mathbf{v}_1+s_2\mathbf{v}_2+\cdots+s_{95}\mathbf{v}_{95}$, where $s_i=1$ or $s_i=-1$.

(a) Let $d=48$. Show that among all such vectors one can find a vector $\mathbf{w}=(a,b,\mathrm{c})$ with $a^2+b^2+\mathrm{c}^2\leq d$.

(b) Find a number $d<48$ with the same property. *Note*: the smaller the d, the better the grade.

Solution: We prove the claim for $d=12$. More precisely, we show by induction that for any $n\geq 4$, if $\mathbf{v}_1,\ldots,\mathbf{v}_n$ are vectors whose components have absolute value at most 1, there exist $s_1,\ldots,s_n\in\{1,-1\}$ such that $|s_1\mathbf{v}_1+\cdots+s_n\mathbf{v}_n|^2\leq 12$. (Eager readers may check whether this bound can also be improved.) To start with, if $n=4$, then by repeated applications of the parallelogram law, the average of $|s_1\mathbf{v}_1+\cdots+s_4\mathbf{v}_4|^2$ equals $|\mathbf{v}_1|^2+\cdots+|\mathbf{v}_4|^2\leq 4(1+1+1)=12$.

Now suppose $n>4$. Without loss of generality, we may assume that the x-component of each $\mathbf{v}_i$ is nonnegative (if $\mathbf{v}_i$ must be negated to ensure this, also negate s_i wherever it appears). We may divide the half-cube $x\geq 0$ into four cubes of side length 1 with the planes $y=0$ and $z=0$. By the pigeonhole principle, two of the $\mathbf{v}_i$ point from the origin into the same subcube; without loss of generality, assume these are $\mathbf{v}_{n-1}$ and $\mathbf{v}_n$. Now $\mathbf{v}_{n-1} - \mathbf{v}_n$ has components in $[-1, 1]$, and we can apply the induction hypothesis to $\mathbf{v}_1, \dots, (\mathbf{v}_{n-1} - \mathbf{v}_n)$ to prove the claim.

9. Prove that the following inequality holds for every integers $n, m \geq 1$ and every positive real numbers x, y:

$$\begin{aligned}(n-1)(m-1)(x^{n+m} + y^{n+m}) + (n+m-1)(x^n y^m + x^m y^n)\\ \geq nm(x^{n+m-1}y + xy^{n+m-1}).\end{aligned}$$

Solution: We rewrite the given inequality in the form

$$nm(x-y)(x^{n+m-1} - y^{n+m-1}) \geq (n+m-1)(x^n - y^n)(x^m - y^m)$$

and divide both sides by $(x-y)^2$ to get the equivalent form

$$\begin{aligned}&nm(x^{n+m-2} + x^{n+m-3}y + \cdots + y^{n+m-2})\\ \geq\ &(n+m-1)(x^{n-1} + \cdots + y^{n-1})(x^{m-1} + \cdots + y^{m-2}).\end{aligned}$$

We now prove a more general result. Suppose $P(x, y) = a_d x^d + \cdots + a_{-d} y^d$ is a homogeneous polynomial of degree d with the following properties.

(a) For $i = 1, \dots, d$, $a_i = a_{-i}$. (Equivalently, $P(x, y) = P(y, x)$.)

(b) $\sum_{i=-d}^{d} a_d = 0$. (Equivalently, $P(x, x) = 0$.)

(c) For $i = 0, \dots, d-1$, $a_d + \cdots + a_{d-i} \geq 0$.

Then $P(x, y) \geq 0$ for all $x, y > 0$. (The properties are easily verified for $P(x, y)$ equal to the difference of the two sides in our desired inequality. The third property follows from the fact that in this case, $a_d \geq a_{d-1} \geq \cdots \geq a_0$.)

We prove the general result by induction on d, as it is obvious for $d = 0$. Suppose P has the desired properties, and let

$$\begin{aligned}Q(x, y) &= (a_d + a_{d-1})x^{d-1} + a_{d-2}x^{d-2}y + \cdots\\ &\quad + a_{-d+2}xy^{d-2} + (a_{-d} + a_{-d+1})y^{d-1}.\end{aligned}$$

Then Q has smaller degree and satisfies the required properties, so by the induction hypothesis $Q(x, y) \geq 0$. Moreover,

$$\begin{aligned} P(x, y) - Q(x, y) &= a_d(x^d - x^{d-1}y - xy^{d-1} + y^d) \\ &= a_d(x - y)(x^d - y^d) \geq 0 \end{aligned}$$

since $a_d \geq 0$ and the sign of $x - y$ is the same as the sign of $x^d - y^d$. Adding these two inequalities give $P(x, y) \geq 0$, as desired.

1.25 Balkan Mathematical Olympiad

1. Find the value of the expression

$$(\cdots(((2\star 3)\star 4)\star 5)\star\cdots)\star 1995,$$

where $x\star y=\dfrac{x+y}{1+xy}$ for all positive x,y.

Solution: Let $f(x)=(1+x)/(1-x)$. Also note that

$$f(x\star y)=\frac{xy+x+y+1}{xy-x-y+1}=\frac{1+x}{1-x}\frac{1+y}{1-y}=f(x)f(y).$$

Thus the desired value is

$$f^{-1}\left(\prod_{n=2}^{1995}\frac{1+n}{1-n}\right)=f^{-1}\left(\frac{1996\cdot 1995}{2\cdot 1}\right).$$

If $y=f(x)$, then a little algebra gives $x=-1/f(y)$. Therefore $x=-1/f(1991010)=1991009/1991011$.

2. Consider two circles c_1 and c_2 with centers O_1,O_2 and radii r_1,r_2, respectively ($r_2>r_1$) which intersect at A and B such that $\angle O_1AO_2=90^\circ$. The line O_1O_2 intersects c_1 at C,D and c_2 at E,F, where E lies between C and D and D lies between E and F. Line BE meets c_1 at K and intersects line AC at M, and BD meets c_2 at L and AF at N. Show that

$$\frac{r_2}{r_1}=\frac{KE}{KM}\cdot\frac{LN}{LD}.$$

Solution: We perform an inversion about B. This carries the orthogonal circles $ADBCK$ and $AEBFL$ to perpendicular lines $C'K'A'D'$ and $E'A'L'F'$. The line $CEDF$ becomes a circle orthogonal to both of these lines; this implies that the circle $C'E'D'F'B$ is in fact centered at A. The point M' becomes the intersection of the line BE' with the circumcircle of $AB'C'$; from the circles we have the equal angles

$$\angle A'M'B=\angle A'C'B=\angle D'C'B=\angle D'E'B$$

and so $A'M'||D'E'$. Similarly $A'N'||D'E'$, and so M', N' can be characterized more simply as the intersections of BE', BF', respectively, with the line through A' parallel to $D'E'$.

We translate the desired result into the inverted diagram using the formula

$$PQ = P'Q'\frac{r^2}{BP' \cdot BQ'},$$

where r is the radius of inversion. Rewriting r_2/r_1 as the ratio of diameters CD/EF, we deduce that the desired result is equivalent to the equation

$$\frac{C'D' \cdot BE' \cdot BF'}{E'F' \cdot BC' \cdot BD'} = \frac{K'E' \cdot BK' \cdot BM'}{K'M' \cdot BK' \cdot BE'}\frac{L'N' \cdot BL' \cdot BD'}{L'D' \cdot BL' \cdot BN'}.$$

By parallels, $BM'/BN' = BE'/BD'$, and $C'D' = E'F'$ since both are diameters. Therefore, the desired equation can be rewritten

$$\frac{BE' \cdot BF'}{BC' \cdot BD'} = \frac{K'E' \cdot L'N'}{K'M' \cdot L'D'}.$$

Now $K'E'/K'A' = 1/\sin\angle BE'F'$ and $K'A'/K'M' = KD'/KE' = \sin\angle KE'D'/\sin 45°$. Therefore

$$K'E'/K'M' = \frac{1}{\sin 45°}\frac{\sin\angle BE'F'}{\sin\angle KE'D'} = \sqrt{2}\frac{BF'}{BD'}.$$

Similarly $L'N'/L'D' = (1/\sqrt{2})BE'/BC'$. Putting these two equations together yields the desired result.

3. Let a, b be positive integers such that $a > b$ and $a + b$ is even. Prove that the roots of the equation

$$x^2 - (a^2 - a + 1)(x - b^2 - 1) - (b^2 + 1)^2 = 0$$

are positive integers, none of which is a perfect square.

Solution: The roots of the quadratic can be found by the quadratic formula to be $x = b^2 + 1$ and $x = a^2 - a - b^2$. The former is obviously not a perfect square for $b > 0$. As for the latter, suppose $a^2 - a = b^2 + c^2$. Recall that a prime congruent to 3 mod 4 cannot occur in the factorization of $b^2 + c^2$ to an odd power. If a is even, then $a-1$ cannot be congruent to 3 mod 4, or else one of its prime factors is congruent to 3 mod 4 and occurs to an odd power. Since this factor cannot also divide a, it occurs in the factorization of b^2+c^2 to an odd power, a contradiction. Hence $a \equiv 2 \pmod 4$, and thus $b^2+c^2 \equiv 2 \pmod 4$. This implies b and c are both odd, since perfect squares are congruent to 0 or 1 mod 4. Similarly, if a is odd, then a cannot be congruent to 3 mod 4, so $a \equiv 1 \pmod 4$ and $b^2+c^2 \equiv 0 \pmod 4$, forcing b and c to be even. In neither case can $a+b$ be even, so if $a+b$ is even, neither root is a perfect square.

4. Let n be a positive integer and S the set of all points (x,y) where x and y are positive integers with $x \le n, y \le n$. Assume that T is the set of all squares whose vertices belong to S. Denote by a_k $(k \ge 0)$ the number of pairs of points in S which are the vertices of exactly k squares from T. Prove that $a_0 = a_2 + 2a_3$.

Solution: We may rewrite the desired equation as

$$a_0 + a_1 + a_2 + a_3 = a_1 + 2a_2 + 3a_3.$$

The left side counts the number of pairs of points, which is $\binom{n^2}{2} = n^2(n^2-1)/2$. The right side counts each pair appearing in each square. Each square contains $\binom{4}{2} = 6$ pairs of points. Moreover, each square can be inscribed in a square with sides parallel to the coordinate axes. If this square is $k \times k$, it can be placed within T in $(n-k)^2$ locations. Moreover, a $k \times k$ square contains k inscribed squares (since there are $4k$ points on the perimeter, each on one inscribed square). Hence the total number of squares in T is

$$\begin{aligned}\sum_{k=1}^{n-1} k(n-k)^2 &= \sum_{j=1}^{n-1}(n-j)j^2 \\ &= n\frac{(n-1)(n)(2n-1)}{6} - \frac{(n-1)^2}{n^2}4 \\ &= \frac{n^2(n-1)(n+1)}{12}.\end{aligned}$$

Since this quantity is one-sixth of the number of pairs of points in T, the desired equation holds.

1.26 Czech-Slovak Match

1. Let $a_1, a_2, \ldots$ be a sequence satisfying $a_1 = 2, a_2 = 5$ and

$$a_{n+2} = (2-n^2)a_{n+1} + (2+n^2)a_n$$

for all $n \geq 1$. Do there exist indices p, q and r such that $a_p a_q = a_r$?

Solution: No such p, q, r exist. We show that for all n, $a_n \equiv 2\,(\mathrm{mod}\ 3)$. This holds for $n = 1$ and $n = 2$ by assumption and follows for all n by induction:

$$\begin{aligned} a_{n+2} &= (2-n^2)a_{n+1} + (2+n^2)a_n \\ &\equiv 2(2-n^2) + 2(2+n^2) = 8 \equiv 2\,(\mathrm{mod}\ 3). \end{aligned}$$

Hence for any p, q, r, $a_p a_q \equiv 1\,(\mathrm{mod}\ 3)$ while $a_r \equiv 2\,(\mathrm{mod}\ 3)$, so $a_p a_q \neq a_r$.

2. Find all pairs of functions $f, g : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(g(x) + y) = g(f(y) + x)$$

holds for arbitrary integers x and y, and at the same time $g(x) = g(y)$ only for $x = y$.

Solution: The functions $f(x) = x + \mathrm{c}, g(x) = x + d$ satisfy the properties for any c, d. We shall show these are the only solutions. Let $\mathrm{c} = f(0)$ and $d = g(0)$. Substituting $x = 0$, then $y = 0$ in the given equation yields the two relations

$$\begin{aligned} g(f(y)) &= f(d+y) \\ f(g(x)) &= g(\mathrm{c}+x). \end{aligned}$$

Now we have two expressions for $g(f(g(x)))$. From the first of the above equations,

$$g(f(g(x))) = f(d + g(x)) = g(f(d) + x),$$

where the second equality comes from the original equation. From the second of the above equations,

$$g(f(g(x))) = g(g(\mathrm{c}+x)).$$

Therefore $g(f(d) + x) = g(g(\mathrm{c} + x))$. Since g is assumed to be injective, this implies $g(\mathrm{c}+x) = f(d)+x$. Replacing x by $x - \mathrm{c}$, this becomes $g(x) = f(d) + x - \mathrm{c}$. Now we have

$$f(f(d) + x - \mathrm{c}) = (\mathrm{c} + x) - \mathrm{c} + f(d)$$

and so $f(x) = x + \mathrm{c}$ for all x, and similarly $g(x) = x + d$.

3. In the plane with a Cartesian coordinate system, consider all triangles ABC all three of whose vertices are lattice points (points with integral coordinates) and whose interior contains exactly one lattice point P. Let E be the intersection of the side BC with the line AP. Determine the maximum possible value of the ratio AP/PE.

Solution: Without loss of generality, we may assume A lies at the origin of the coordinate system. Let $L = (B + C)/2, M = (C + A)/2, N = (A+B)/2$ be the midpoints of AB, BC, CA, respectively. Denote also

$$\begin{aligned} S &= \frac{1}{3}(2L + M) = \frac{1}{3}(B + C + M) \\ T &= \frac{1}{3}(B + C + N) \\ Q &= 2P - B \\ R &= 3P - B - C. \end{aligned}$$

Then Q and R are the lattice points obtained from P by the homothety with center B and ratio 2, and by the homothety with center L and ratio 3, respectively. Evidently $Q \neq P \neq R$. Therefore Q cannot lie in the interior of $\triangle ABC$, and hence P does not lie in the interior of $\triangle NBL$, since the homothety with center B and ratio 2 maps $\triangle NBL$ to $\triangle ABC$. Similarly (if we take instead $Q = 2P - C$) we find that P does not lie in the interior of $\triangle MLC$. Since the lattice point R cannot lie in the interior of $\triangle ABC$, the point P (which is the image of R under the homothety with center L and ratio $1/3$) cannot lie in the interior of $\triangle LST$. Finally, since the distance from A to the line ST is five times greater than the distance between lines ST and BC, we have the estimate

$$\frac{AP}{PE} \leq 5.$$

On the other hand, choosing $A = (0,0), B = (2,0), C = (0,3)$ gives a triangle containing a single lattice point $P = (1,1) = T$ in its interior and for which equality takes place in the above estimate. The maximum possible value of the ratio under consideration is thus equal to 5.

4. For each real number $r > 1$, find the maximum value of the sum $x + y$, where x and y are subject to the condition

$$(x + \sqrt{1 + x^2})(y + \sqrt{1 + y^2}) = r.$$

Solution: Put $t = x + \sqrt{x^2 + 1}$, so that $t > 0$ and $x = (t^2 - 1)/(2t)$. Then the constraint is equivalent to $y + \sqrt{y^2 + 1} = p/t$, which in turn is equivalent to

$$y = \frac{(p/t)^2 - 1}{2(p/t)} = \frac{p^2 - t^2}{2pt}.$$

Hence

$$x + y = \frac{t^2 - 1}{2t} + \frac{p^2 - t^2}{2pt} = \frac{p - 1}{2p}\left(t + \frac{p}{t}\right) \geq \frac{p - 1}{p}\sqrt{t \cdot \frac{p}{t}} = \frac{p - 1}{\sqrt{p}},$$

and equality occurs for $t = \sqrt{p}$, i.e.

$$x = y = \frac{p - 1}{2\sqrt{p}}.$$

5. Find all pairs of nonnegative integers x and y which solve the equation

$$p^x - y^p = 1$$

where p is a given odd prime.

Solution: If (x, y) is a solution, then

$$p^x = y^p + 1 = (y + 1)(y^{p-1} - \cdots + y^2 - y + 1)$$

and so $y + 1 = p^n$ for some n. If $n = 0$, then $x = y = 0$ and p may be arbitrary. Otherwise,

$$\begin{aligned} p^x &= (p^n - 1)^p + 1 \\ &= p^{np} - p \cdot p^{n(p-1)} + \binom{p}{2} p^{n(p-2)} + \cdots - \binom{p}{p-2} p^{2n} + p \cdot p^n. \end{aligned}$$

Since p is prime, all of the binomial coefficients are divisible by p. Hence all terms are divisible by p^{n+1}, and all but the last by p^{n+2}. Therefore the highest power of p dividing the right side is p^{n+1} and so $x = n + 1$. We also have

$$0 = p^{np} - p \cdot p^{n(p-1)} + \binom{p}{2} p^{n(p-2)} + \cdots - \binom{p}{p-2} p^{2n}.$$

For $p = 3$ this reads $0 = 3^{3n} - 3 \cdot 3^{2n}$, which only occurs for $n = 1$, yielding $x = y = 2$. For $p \geq 5$, the coefficient $\binom{p}{p-2}$ is not divisible by p^2, so every term but the last on the right side is divisible by p^{2n+2}, while the last term is not. Since the terms sum to 0, this is impossible.

Hence the only solutions are $x = y = 0$ for all p and $x = y = 2$ for $p = 3$.

1.27 Iberoamerican Olympiad

1. Determine all possible values of the sum of the digits of a perfect square.

   Solution: The sum of the digits of a number is congruent to the number modulo 9, and so for a perfect square this must be congruent to $0, 1, 4$ or 7. We show that all such numbers occur. (The cases $n = 1$ and $n = 4$ are trivial, so assume $n > 4$.

   If $n = 9m$, then n is the sum of the digits of $(10^m - 1)^2 = 10^m(10^m - 2) + 1$, which looks like $9\cdots980\cdots01$. If $n = 9m + 1$, consider $(10^m - 2)^2 = 10^m(10^m - 4) + 4$, which looks like $9\cdots960\cdots04$. If $n = 9m + 4$, consider $(10^m - 3)^2 = 10^m(10^m - 6) + 9$, which looks like $9\cdots94\cdots09$. Finally, if $n = 9m - 2$, consider $(10^m - 5)^2 = 10^m(10^m - 10) + 25$, which looks like $9\cdots900\cdots025$.

2. Let n be an integer greater than 1. Determine real numbers $x_1, x_2, \ldots, x_n \geq 1$ and $x_{n+1} > 0$ such that

   (a)
$$\frac{x_1^{1/2} + x_2^{1/3} + \cdots + x_n^{1/(n+1)}}{n} = x_{n+1}^{1/2},$$

   (b)
$$\frac{x_1 + x_2 + \cdots + x_n}{n} = x_{n+1}.$$

   Solution: We show that $x_1 = \cdots = x_n = 1$ is the only possibility. By the Power Mean inequality,

$$x_{n+1}^{1/2} = \left(\frac{x_1 + \cdots + x_n}{n}\right)^{1/2} \geq \frac{x_1^{1/2} + \cdots + x_n^{1/2}}{n},$$

   with equality for $x_1 = \cdots = x_n$. Note, however, that for $x_k \geq 1$, $x_k^{1/2} \geq x_k^{1/(k+1)}$ with equality if $k = 1$ or $x_k = 1$. Thus we get $x_{n+1}^{1/2} \geq x_{n+1}^{1/2}$ and so equality must occur everywhere, implying $x_1 = x_2$ and $x_2 = \cdots = x_n = 1$, and proving the claim.

3. Let r and s be two non-coplanar, orthogonal lines. Let AB be their common perpendicular, with A on r and B on s. Consider the sphere having AB as a diameter. The points M on r and N on s vary so that MN is tangent to the sphere at a point T. Determine the locus of T.

Solution: Let U and V be the intersections of r and s, respectively, with the tangent plane at a point T on the sphere. Then T lies on the locus if and only if T, U, V are collinear.

To describe the locus most easily, we introduce a coordinate system. Suppose r is the line $z = 1, x = 0$ and s is the line $z = -1, y = 0$, so that the sphere with diameter AB is the sphere $x^2 + y^2 + z^2 = 1$. If $T = (x, y, z)$, the equation of the tangent plane, in vector notation, is $\vec{v} \cdot T = 1$, and so we easily compute $U = (0, (1 - z)/y, 1)$ and $V = ((1+z)/x, 0, -1)$. Then T, U, V are collinear if and only if there exists a number t such that $tU + (1-t)V = T$. Using the coordinate expressions just found, this becomes

$$(x, y, z) = ((1 - t)(1 + z)/x, t(1 - z)/y, 2t - 1).$$

We must have $t = (z + 1)/2$ by comparing the third coordinates. The first two coordinates then both yield the equation $x^2 = y^2$. As long as $x, y \neq 0$ and $x = \pm y$, such t exists and T lies on the locus.

Hence the locus can be described as the union of the two great circles through A and B whose tangent lines at either point make angles of $45°$ with r and s.

4. [Corrected] Pieces are placed on an $n \times n$ board. Each piece "attacks" all squares that belong to its row, column, and the northwest-southeast diagonal which contains it. Determine the least number of pieces which are necessary to attack all the squares of the board.

Solution: The minimum number of pieces is $\lceil (2n - 1)/3 \rceil$. To see this many pieces are necessary, suppose k pieces suffice for an $n \times n$ board. There are at least $n - k$ rows and $n - k$ columns not containing a piece, and the squares lying at the intersections of these rows and columns lie on at leat $2(n - k) - 1$ diagonals. Hence each of these diagonals must contain a piece, yielding $k \geq 2(n - k) - 1$ or $k \geq (2n - 1)/3$.

To see this many pieces are sufficient, we first consider n of the form $3r+2$, in which case putting pieces on $(1,1),(2,3),\ldots,(r+1,2r+1)$ and $(r+2,2),(r+3,4),\ldots,(2r+1,2r)$ is easily seen to work. If $n = 3r+3$ or $3r+4$, we achieve the bound by adding a piece on $(3r+3,3r+3)$ and (in the latter case) on $(3r+4,3r+4)$.

5. The incircle of triangle ABC is tangent to BC, CA and AB at D, E and F, respectively. Suppose the incircle intersects again with AD at a point X such that $AX = XD$. XB and XC intersect again with the incircle at points Y and Z, respectively. Show that $EY = FZ$.

Solution: By the power-of-a-point theorem, $AF^2 = AX \cdot AD = 2AX^2$ and $BD^2 = BF^2 = BY \cdot BX$. Let $x = AX$ and $y = BF$, so that $AF = x\sqrt{2}, AB = y + x\sqrt{2}$. By the Law of Cosines,

$$\begin{aligned}
\cos\angle BDF &= \frac{AD^2 + BD^2 - AB^2}{2 \cdot AD \cdot BD} \\
&= \frac{(2x)^2 + y^2 - (y + x\sqrt{2})^2}{2(2x)y} \\
&= \frac{2x^2 - 2xy\sqrt{2}}{4xy} = \frac{x - y\sqrt{2}}{2y}.
\end{aligned}$$

With this, we again apply the Law of Cosines to compute

$$\begin{aligned}
BX^2 &= BD^2 + DX^2 - 2BD \cdot DX \cos\angle BDF \\
x^2 + y^2 - x(x - y\sqrt{2}) & \\
&= y(y + x\sqrt{2}) = BF \cdot BA.
\end{aligned}$$

Now

$$\frac{BY}{BX} = \frac{BX \cdot BY}{BX^2} = \frac{BF^2}{BF \cdot BA} = \frac{BF}{BA}$$

and hence $XA || YF$. Similarly $XA || EZ$, and so $FY || EZ$. The trapezoid $EZYF$, being cyclic, must be isosceles, and we conclude $EY = FZ$, as desired.

6. A function $f : \mathbb{N} \to \mathbb{N}$ is "circular" if for each $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ with $n \leq m$ such that $f^n(m) = m$, where f^n means that f is iterated n times. A circular function f has "level" $k, 0 < k < 1$, if for each $m \in \mathbb{N}$, $f^i(m) \neq m$ for $i = 1, 2, \ldots, \lfloor k/m \rfloor$. Determine the greatest level that a circular function may have.

Solution: The greatest possible level is $1/2$, which occurs for the function f such that $f(n) = n$ if $n + 1$ is not a power of 2 and $f(n) = (n + 1)/2$ if $n + 1$ is a power of 2. On the other hand, if n is the largest element of a cycle of length m, each element of the cycle is no less than m (since we assume that for each a there exists $b \leq a$ such that $f^a(b) = b$). This limits membership in the cycle to the numbers $m, \ldots, n$, so in particular $m \leq n - m + 1$ and the level of f cannot exceed $(n - 1/2)/n$. For n large, this tends to $1/2$, and so the level can be no greater than $1/2$.

1.28 UNESCO

1. Let $n \geq 2, a > 0$ be integers and p a prime such that $a^p \equiv 1 \pmod{p^n}$. Show that if $p > 2$, then $a \equiv 1 \pmod{p^{n-1}}$, and if $p = 2$, then $a \equiv \pm 1 \pmod{2^{n-1}}$.

Solution: We have $a^p \equiv 1 \,(\mathrm{mod}\, p)$ with $n \geq 2$, so $a^p \equiv 1 \,(\mathrm{mod}\, p)$. But, from Fermat's theorem, $a^p \equiv a \,(\mathrm{mod}\, p)$, hence $a \equiv 1 \,(\mathrm{mod}\, p)$. For $a = 1$, the result is obvious; otherwise, put $a = 1 + kp^d$, where $d \geq 1$ and $p \not| k$. Then for $p > 2$, $a^p = 1 + kp^{d+1} + Mp^{2d+1}$ for M an integer. Therefore $d + 1 \geq n$ and so $a \equiv 1 \,(\mathrm{mod}\, p^n - 1)$. In case $p = 2$, we have $2^n | a^2 - 1 = (a-1)(a+1)$. Since these differ by 2, both cannot be multiples of 4. Hence either $a+1$ or $a-1$ is divisible by 2^{n-1}, i.e. $a \equiv \pm 1 \,(\mathrm{mod}\, 2^{n-1})$, as desired.

2. Let p be an odd prime, $k \geq 2$ and $A_1, A_2, \ldots, A_k$ pairwise disjoint subsets of $\mathbb{N}$ such that $\mathbb{N} = A_1 \cup A_2 \cup \cdots \cup A_k$. Show that there exists $i \in \{1, 2, \ldots, k\}$ and infinitely many polynomials of degree $p-1$, with coefficients in A_i, pairwise distinct, that canot be factored into a product of polynomials with integer coefficients, having degree greater than or equal to one.

Solution: We may as well assume each A_i is nonempty. Choose $a_i \in A_i$ for $i = 1, \ldots, k$, and choose a prime p bigger than all of the a_i. The set of numbers divisible by p but not by p^2 is infinite, so for some i, A_i contains infinitely many such numbers $b_1, b_2, \ldots$. Now by Eisenstein's criterion, the polynomial

$$a_i x^{p-1} + b_n x^{p-2} + \cdots + b_{n+p-3} x + b_{n+p-2}$$

is irreducible for any n.

An alternate solution uses the fact that if $a_0, \ldots, a_{p-1}$ are primes with $a_{p-1} > a_0 + \cdots + a_{p-2}$, then the polynomial $a_0 x^{p-1} + a_1 x^{p-2} + \cdots + a_{p-1}$ is irreducible. (In a factorization, one of the factors would have constant term ± 1, and hence would have a root of magnitude at most 1, contradicting the assumed inequality.)

3. A rectangle $MNPQ$ "circumscribes" a triangle ABC if on each of the edges MN, NP, PQ, QM there is at least one vertex of ABC.

Find the locus of centers O of rectangles $MNPQ$ which circumscribe a given triangle ABC.

Solution: Let A', B', C' be the midpoints of sides BC, CA, AB, respectively. At least one vertex of the rectangle must be a vertex of $\triangle ABC$ whose angle is at most $\pi/2$. Suppose A is such a vertex. Then B' and C' each lie on one of the lines through O parallel to the sides of the rectangle, and therefore O lies on the circle with diameter $B'C'$.

In case $\triangle ABC$ is acute, we thus get as the locus three arcs, intersecting at the feet of the altitudes of $\triangle ABC$ and centered at the midpoints of $B'C', C'A', A'B'$. In case $\angle A$ is obtuse, we get only two arcs, from A to the foot of its altitude, centered at the midpoints of $A'B'$ and $C'A'$.

4. Let $p \geq 2$, $a_0, a_1, \ldots, a_n$ be nonnegative integers, and $f(x) = a_0 + a_1x + \cdots + a_nx^n$. Prove that if the numbers $\sqrt[p]{f(0)}$, $\sqrt[p]{f(1)}$, $\sqrt[p]{f(2)}, \ldots$ are all rational, then there exists a polynomial $g(x)$ with integer coefficients such that $f(x) = (g(x))^p$.

   Solution: Suppose f is not a perfect p-th power; then for some $k > 0$, we can write $f = h^p f_1^{a_1} \cdots f_k^{a_k}$, where $h, f_1, \ldots, f_k$ are integer polynomials, f_i is irreducible and $a_i < p$ for $i = 1, \ldots, k$. Since f is irreducible, it has no repeated roots, and so it is coprime to its derivative f' as well as to $f_2, \ldots, f_k$. Hence there exist polynomials r, s and an integer c such that $r(x)f_1(x) + s(x)f_1'(x)f_2(x) \cdots f_k(x) =$ c for all x.

   Let $q > |c|$ be a prime such that $q[f_1(x)$ for some x. It is easily checked that $f_1(x+q) - f_1(x) \equiv qf_1'(x) \,(\text{mod}\ q^2)$. We cannot have $q|f_1'(x)$ or $q|f_j(x)$ for any $j > 1$, or else we would also have $q|$c. Therefore, either $f_1(x)$ or $f_1(x+q)$ is divisible by q but not by q^2. Without loss of generality, suppose $q^2 \not| f_1(x)$. Then $f(x)/h(x)$ is a perfect p-th power, but it is a multiple of q and not of q^p, a contradiction. Hence f is a perfect p-th power, as claimed.

5. [Corrected] Let $n = p_1^{r_1} \cdots p_k^{r_k}$ be the prime factorization of the positive integer n and let $r \geq 2$ be an integer. Prove that the following are equivalent:

(a) The equation $x^r \equiv a \pmod{n}$ has a solution for every a.

(b) $r_1 = r_2 = \cdots = r_k = 1$ and $(p_i - 1, r) = 1$ for every $i \in \{1, 2, \ldots, k\}$.

Solution: If (b) holds, then $\phi(n) = (p_1 - 1) \cdots (p_k - 1)$ is coprime to r, so there exists s with $rs \equiv 1 \pmod{\phi(n)}$, and the unique solution of $x^r \equiv a \pmod{n}$ is $a = x^s$. Conversely, suppose $x^r \equiv a \pmod{n}$ has a solution for every a; then $x^r \equiv a \pmod{p_i^{r_i}}$ also has a solution for every a. However, if $r_1 >$ and a is a number divisible by p but not by p^2, then x^r cannot be congruent to a, since it is not divisible by p unless x is divisible by p, in which case it is already divisible by p^2. Hence $r_1 = 1$.

Let $d = (p_i - 1, r)$ and put $m = (p_i - 1)/d$. If $x^r \equiv a \pmod{p_i}$ and $a \not\equiv 0$, then

$$a^m \equiv x^{rm} = x^{p_i - 1} \equiv 1 \pmod{p_i}.$$

However, if a is a primitive root mod p_i, then this only occurs for $m \equiv 0 \pmod{p_i - 1}$, which implies $d = 1$. Hence $r_i = 1$ and $(r, p_i - 1) = 1$, as desired.

6. Given a seqment AB and a constant $k > 0$. Find the locus of points C in the plane such that in triangle ABC, the length of a side is k times the length of the corresponding altitude.

Solution: The locus comes in three pieces, one for each of the three sides AB, BC, CA that may be forced to equal k times the length of the corresponding altitude. Let A', B', C' be the feet of the altitudes from A, B, C, respectively. If $AB = kCC'$, then C is constrained to lie on a pair of lines parallel to AB at distance $1/k$ from it.

Now suppose $BC = kAA'$. Let D, D' be the points on the perpendicular to AB through B such that $kBD = kBD' = AB$. Then $BD/BC = AB/AA'$ and since either $\angle A'AB = \angle DBC$ or $\angle A'AB = \angle D'BC$, we either have $\Delta BCD \sim \Delta AA'B$ and so $BC \perp DC$, or analogously $BC \perp D'C$. Hence C is constrained to two circles, with diameters BD and BD'.

If $CA = kBB'$, we construct E, E' analogously and deduce that C lies on one of two circles, with diameters AE and AE'.

2 1996 National Contests: Problems

2.1 Bulgaria

1. Prove that for all natural numbers $n \geq 3$ there exist odd natural numbers x_n, y_n such that $7x_n^2 + y_n^2 = 2^n$.

2. The circles k_1 and k_2 with respective centers O_1 and O_2 are externally tangent at the point C, while the circle k with center O is externally tangent to k_1 and k_2. Let ℓ be the common tangent of k_1 and k_2 at the point C and let AB be the diameter of k perpendicular to ℓ. Assume that O and A lie on the same side of ℓ. Show that the lines AO_2, BO_1, ℓ have a common point.

3. Let a, b, c be real numbers and let M be the maximum of the function $y = |4x^3 + ax^2 + bx + \mathrm{c}|$ in the interval $[-1, 1]$. Show that $M \geq 1$ and find all cases where equality occurs.

4. The real numbers $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ form an arithmetic progression. There exists a permutation $a_{i_1}, a_{i_2}, \ldots, a_{i_n}$ of $a_1, a_2, \ldots, a_n$ which is a geometric progression. Find the numbers $a_1, a_2, \ldots, a_n$ if they are all different and the largest of them is equal to 1996.

5. A convex quadrilateral ABC is given for which $\angle ABC + \angle BCD < 180°$. The common point of the lines AB and CD is E. Prove that $\angle ABC = \angle ADC$ if and only if
$$AC^2 = CD \cdot CE - AB \cdot AE.$$

6. Find all prime numbers p, q for which pq divides $(5^p - 2^p)(5^q - 2^q)$.

7. Find the side length of the smallest equilateral triangle in which three discs of radii $2, 3, 4$ can be placed without overlap.

8. The quadratic polynomials f and g with real coefficients are such that if $g(x)$ is an integer for some $x > 0$, then so is $f(x)$. Prove that there exist integers m, n such that $f(x) = mg(x) + n$ for all x.

9. The sequence $\{a_n\}_{n=1}^{\infty}$ is defined by
$$a_1 = 1, a_{n+1} = \frac{a_n}{n} + \frac{n}{a_n}, \quad n \geq 1.$$
Prove that for $n \geq 4$, $\lfloor a_n^2 \rfloor = n$.

10. The quadrilateral $ABCD$ is inscribed in a circle. The lines AB and CD meet at E, while the diagonals AC and BD meet at F. The circumcircles of the triangles AFD and BFC meet again at H. Prove that $\angle EHF = 90°$.

11. A 7×7 chessboard is given with its four corners deleted.

    (a) What is the smallest number of squares which can be colored black so that an uncolored 5-square (Greek) cross cannot be found?

    (b) Prove that an integer can be written in each square such that the sum of the integers in each 5-square cross is negative while the sum of the numbers in all squares of the board is positive.

2.2 Canada

1. If α, β, γ are the roots of $x^3 - x - 1 = 0$, compute

$$\frac{1-\alpha}{1+\alpha} + \frac{1-\beta}{1+\beta} + \frac{1-\gamma}{1+\gamma}.$$

2. Find all real solutions to the following system of equations:

$$\begin{aligned} \frac{4x^2}{1+4x^2} &= y \\ \frac{4y^2}{1+4y^2} &= z \\ \frac{4z^2}{1+4z^2} &= x. \end{aligned}$$

3. Let $f(n)$ be the number of permutations $a_1, \ldots, a_n$ of the integers $1, \ldots, n$ such that

   (i) $a_1 = 1$;

   (ii) $|a_i - a_{i+1}| \leq 2$, $i = 1, \ldots, n-1$.

   Determine whether $f(1996)$ is divisible by 3.

4. Let $\triangle ABC$ be an isosceles triangle with $AB = AC$. Suppose that the angle bisector of $\angle B$ meets AC at D and that $BC = BD + AD$. Determine $\angle A$.

5. Let $r_1, r_2, \ldots, r_m$ be a given set of positive rational numbers whose sum is 1. Define the function f by $f(n) = n - \sum_{k=1}^{m} \lfloor r_k n \rfloor$ for each positive integer n. Determine the minimum and maximum values of $f(n)$.

2.3 China

1. Let H be the orthocenter of acute triangle ABC. The tangents from A to the circle with diameter BC touch the circle at P and Q. Prove that P, Q, H are collinear.

2. Find the smallest positive integer K such that every K-element subset of $\{1, 2, \ldots, 50\}$ contains two distinct elements a, b such that $a+b$ divides ab.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for all $x, y \in \mathbb{R}$,

$$f(x^3 + y^3) = (x + y)(f(x)^2 - f(x)f(y) + f(y)^2).$$

Prove that for all $x \in \mathbb{R}$, $f(1996x) = 1996f(x)$.

4. Eight singers participate in an art festival where m songs are performed. Each song is performed by 4 singers, and each pair of singers performs together in the same number of songs. Find the smallest m for which this is possible.

5. Suppose $n \in \mathbb{N}$, $x_0 = 0$, $x_i > 0$ for $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} x_i = 1$. Prove that

$$1 \leq \sum_{i=1}^{n} \frac{x_i}{\sqrt{1 + x_0 + \cdots + x_i - 1} \cdot \sqrt{x_i + \cdots + x_n}} < \frac{\pi}{2}.$$

6. In triangle ABC, $\angle C = 90°$, $\angle A = 30°$ and $BC = 1$. Find the minimum of the length of the longest side of a triangle inscribed in ABC (that is, one such that each side of ABC contains a different vertex of the triangle).

2.4 Czech and Slovak Republics

1. Prove that if a sequence $\{G(n)\}_{n=0}^{\infty}$ of integers satisfies

$$\begin{aligned} G(0) &= 0, \\ G(n) &= n - G(G(n)) \qquad (n = 1, 2, 3, \ldots), \end{aligned}$$

   then

   (a) $G(k) \geq G(k-1)$ for any positive integer k;

   (b) no integer k exists such that $G(k-1) = G(k) = G(k+1)$.

2. Let ABC be an acute triangle with altitudes AP, BQ, CR. Show that for any point P in the interior of the triangle PQR, there exists a tetrahedron $ABCD$ such that P is the point of the face ABC at the greatest distance (measured along the surface of the tetrahedron) from D.

3. Given six three-element subsets of a finite set X, show that it is possible to color the elements of X in two colors such that none of the given subsets is all in one color.

4. An acute angle XCY and points A and B on the rays CX and CY, respectively, are given such that $|CX| < |CA| = |CB| < |CY|$. Show how to construct a line meeting the ray CX and the segments AB, BC at the points K, L, M, respectively, such that

$$KA \cdot YB = XA \cdot MB = LA \cdot LB \neq 0.$$

5. For which integers k does there exist a function $f : \mathbb{N} \to \mathbb{Z}$ such that

   (a) $f(1995) = 1996$, and

   (b) $f(xy) = f(x) + f(y) + kf(\gcd(x, y))$ for all $x, y \in \mathbb{N}$?

6. A triangle ABC and points K, L, M on the sides AB, BC, CA, respectively, are given such that

$$\frac{AK}{AB} = \frac{BL}{BC} = \frac{CM}{CA} = \frac{1}{3}.$$

   Show that if the circumcircles of the triangles AKM, BLK, CML are congruent, then so are the incircles of these triangles.

2.5 France

1. Let ABC be a triangle and construct squares $ABED, BCGF, ACHI$ externally on the sides of ABC. Show that the points D, E, F, G, H, I are concyclic if and only if ABC is equilateral or isosceles right.

2. Let a, b be positive integers with a odd. Define the sequence $\{u_n\}$ as follows: $u_0 = b$, and for $n \in \mathbb{N}$,

$$u_n = \begin{cases} \frac{1}{2}u_n & \text{if } u_n \text{ is even} \\ u_n + a & \text{otherwise.} \end{cases}$$

   (a) Show that $u_n \le a$ for some $n \in \mathbb{N}$.

   (b) Show that the sequence $\{u_n\}$ is periodic from some point onwards.

3. (a) Find the minimum value of x^x for x a positive real number.

   (b) If x and y are positive real numbers, show that $x^y + y^x > 1$.

4. Let n be a positive integer. We say a positive integer k satisfies the condition C_n if there exist $2k$ distinct positive integers $a_1, b_1, \ldots, a_n, b_n$ such that the sums $a_1 + b_1, \ldots, a_k + b_k$ are all distinct and less than n.

   (a) Show that if k satisfies the condition C_n, then $k \le (2n-3)/5$.

   (b) Show that 5 satisfies the condition C_{14}.

   (c) Suppose $(2n-3)/5$ is an integer. Show that $(2n-3)/5$ satisfies the condition C_n.

2.6 Germany

1. Starting at $(1,1)$, a stone is moved in the coordinate plane according to the following rules:

   (i) From any point (a,b), the stone can move to $(2a,b)$ or $(a,2b)$.

   (ii) From any point (a,b), the stone can move to $(a-b,b)$ if $a > b$, or to $(a,b-a)$ if $a < b$.

   For which positive integers x, y can the stone be moved to (x,y)?

2. Suppose S is a union of finitely many disjoint subintervals of $[0,1]$ such that no two points in S have distance $1/10$. Show that the total length of the intervals comprising S is at most $1/2$.

3. Each diagonal of a convex pentagon is parallel to one side of the pentagon. Prove that the ratio of the length of a diagonal to that of its corresponding side is the same for all five diagonals, and compute this ratio.

4. Prove that every integer $k > 1$ has a multiple less than k^4 whose decimal expansion has at most four distinct digits.

2.7 Greece

1. In a triangle ABC the points D, E, Z, H, Θ are the midpoints of the segments BC, AD, BD, ED, EZ, respectively. If I is the point of intersection of BE and AC, and K is the point of intersection of $H\Theta$ and AC, prove that

   (a) $AK = 3CK$;

   (b) $HK = 3H\Theta$;

   (c) $BE = 3EI$;

   (d) the area of ABC is 32 times that of $E\Theta H$.

2. Let ABC be an acute triangle, AD, BE, CZ its altitudes and H its orthocenter. Let $AI, A\Theta$ be the internal and external bisectors of angle A. Let M, N be the midpoints of BC, AH, respectively. Prove that

   (a) MN is perpendicular to EZ;

   (b) if MN cuts the segments $AI, A\Theta$ at the points K, L, then $KL = AH$.

3. Given 81 natural numbers whose prime divisors belong to the set $\{2, 3, 5$, prove there exist 4 numbers whose product is the fourth power of an integer.

4. Determine the number of functions $f : \{1, 2, \ldots, n\} \to \{1995, 1996\}$ which satsify the condition that $f(1) + f(2) + \cdots + f(1996)$ is odd.

2.8 Iran

1. Prove the following inequality for positive real numbers x, y, z:

$$(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq \frac{9}{4}.$$

2. Prove that for every pair m, k of natural numbers, m has a unique representation in the form

$$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_t}{t},$$

where

$$a_k > a_{k-1} > \cdots > a_t \geq t \geq 1.$$

3. In triangle ABC, we have $\angle A = 60°$. Let O, H, I, I' be the circumcenter, orthocenter, incenter, and excenter opposite A, respectively, of ABC. Let B' and C' be points on the segments AC and AB such that $AB = AB'$ and $AC = AC'$. Prove that:

   (a) The eight points $B, C, H, O, I, I', B', C'$ are concyclic.

   (b) If OH intersects AB and AC at E and F, respectively, the perimeter of triangle AEF equals $AB + AC$.

   (c) $OH = |AB - AC|$.

4. Let ABC be a scalene triangle. The medians from A, B, C meet the circumcircle again at L, M, N, respectively. If $LM = LN$, prove that $2BC^2 = AB^2 + AC^2$.

5. The top and bottom edges of a chessboard are identified together, as are the left and right edges, yielding a torus. Find the maximum number of knights which can be placed so that no two attack each other.

6. Find all real numbers $a_1 \leq a_2 \leq \ldots \leq a_n$ satisfying

$$\sum_{i=1}^{n} a_i = 96, \quad \sum_{i=1}^{n} a_i^2 = 144, \quad \sum_{i=1}^{n} a_i^3 = 216.$$

7. Points D and E lie on sides AB and AC of triangle ABC such that $DE||BC$. Let P be an arbitrary point inside ABC. The lines PB and PC intersect DE at F and G, respectively. If O_1 is the circumcenter of PDG and O_2 is the circumcenter of PFE, show that $AP \perp O_1O_2$.

8. Let $P(x)$ be a polynomial with rational coefficients such that $P^{-1}(\mathbb{Q}) \subseteq \mathbb{Q}$. Show that P is linear.

9. For $S = \{x_1, x_2, \ldots, x_n\}$ a set of n real numbers, all at least 1, we count the number of reals of the form

$$\sum_{i=1}^{n} \epsilon_i x_i, \quad \epsilon_i \in \{0, 1\}$$

lying in an open interval I of length 1. Find the maximum value of this count over all I and S.

2.9 Ireland

1. For each positive integer n, find the greatest common divisor of $n!+1$ and $(n+1)!$.

2. For each positive integer n, let $S(n)$ be the sum of the digits in the decimal expansion of n. Prove that for all n,

$$S(2n) \leq 2S(n) \leq 10S(2n)$$

and show that there exists n such that $S(n) = 1996S(3n)$.

3. Let $f : [0,1] \to \mathbb{R}$ be a function such that

   (i) $f(1) = 1$,

   (ii) $f(x) \geq 0$ for all $x \in [0,1]$,

   (iii) if x, y and $x+y$ all lie in $[0,1]$, then $f(x+y) \geq f(x) + f(y)$.

   Prove that $f(x) \leq 2x$ for all $x \in K$.

4. Let F be the midpoint of side BC of triangle ABC. Construct isosceles right triangles ABD and ACE externally on sides AB and AC with the right angles at D and E, respectively. Show that DEF is an isosceles right triangle.

5. Show, with proof, how to dissect a square into at most five pieces in such a way that the pieces can be reassembled to form three squares no two of which have the same area.

6. Let F_n denote the Fibonacci sequence, so that $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. Prove that

   (i) The statement "$F_{n+k} - F_n$ is divisible by 10 for all positive integers n" is true if $k = 60$ and false for any positive integer $k < 60$;

   (ii) The statement "$F_{n+t} - F_n$ is divisible by 100 for all positive integers n" is true if $t = 300$ and false for any positive integer $t < 300$.

7. Prove that for all positive integers n,

$$2^{1/2} \cdot 4^{1/4} \cdots (2^n)^{1/2^n} < 4.$$

8. Let p be a prime number and a, n positive integers. Prove that if

$$2^p + 3^p = a^n,$$

then $n = 1$.

9. Let ABC be an acute triangle and let D, E, F be the feet of the altitudes from A, B, C, respectively. Let P, Q, R be the feet of the perpendiculars from A, B, C to EF, FD, DE, respectively. Prove that the lines AP, BQ, CR are concurrent.

10. On a 5×9 rectangular chessboard, the following game is played. Initially, a number of discs are randomly placed on some of the squares, no square containing more than one disc. A turn consists of moving all of the discs subject to the following rules:

    (i) each disc may be moved one square up, down, left, or right;

    (ii) if a disc moves up or down on one turn, it must move left or right on the next turn, and vice versa;

    (iii) at the end of each turn, no square can contain two or more discs.

    The game stops if it becomes impossible to complete another turn. Prove that if initially 33 discs are placed on the board, the game must eventually stop. Prove also that it is possible to place 32 discs on the board so that the game can continue forever.

2.10 Italy

1. Among triangles with one side of a given length ℓ and with given area S, determine all of those for which the product of the lengths of the three altitudes is maximum.

2. Prove that the equation $a^2 + b^2 = c^2 + 3$ has infinitely many integer solutions $\{a, b, c\}$.

3. Let A and B be opposite vertices of a cube of edge length 1. Find the radius of the sphere with center interior to the cube, tangent to the three faces meeting at A and tangent to the three edges meeting at B.

4. Given an alphabet with three letters a, b, c, find the number of words of n letters which contain an even number of a's.

5. Let C be a circle and A a point exterior to C. For each point P on C, construct the square $APQR$, where the vertices A, P, Q, R occur in counterclockwise order. Find the locus of Q as P runs over C.

6. Whas is the minimum number of squares that one needs to draw on a white sheet in order to obtain a complete grid with n squares on a side?

2.11 Japan

1. Consider a triangulation of the plane, i.e. a covering of the plane with triangles such that no two triangles have overlapping interiors, and no vertex lies in the interior of an edge of another triangle. Let A, B, C be three vertices of the triangulation and let θ be the smallest angle of the triangle $\triangle ABC$. Suppose no vertices of the triangulation lie inside the circumcircle of $\triangle ABC$. Prove there is a triangle σ in the triangulation such that $\sigma \cap \triangle ABC \neq \emptyset$ and every angle of σ is greater than θ.

2. Let m and n be positive integers with $\gcd(m, n) = 1$. Compute $gcd(5^m + 7^m, 5^n + 7^n)$.

3. Let $x > 1$ be a real number which is not an integer. For $n = 1, 2, 3, \ldots$, let $a_n = \lfloor x^{n+1} \rfloor - x\lfloor x^n \rfloor$. Prove that the sequence $\{a_n\}$ is not periodic.

4. Let θ be the maximum of the six angles between the edges of a regular tetrahedron and a given plane. Find the minimum value of θ over all positions of the plane.

5. Let q be a real number with $(1 + \sqrt{5})/2 < q < 2$. For a number n with binary representation

$$n = 2^k + a_{k-1} \cdot 2^{k-1} + \cdots + a_1 \cdot 2 + a_0$$

with $a_i \in \{0, 1\}$, we define p_n as follows:

$$p_n = q^k + a_{k-1}q^{k-1} + \cdots + a_1 q + a_0.$$

Prove that there exist infinitely many positive integers k for which there does not exist a positive integer l such that $p_{2k} < p_l < p_{2k+1}$.

2.12 Poland

1. Find all pairs (n, r), with n a positive integer and r a real number, for which the polynomial $(x+1)^n - r$ is divisible by $2x^2 + 2x + 1$.

2. Let ABC be a triangle and P a point inside it such that $\angle PBC = \angle PCA < \angle PAB$. The line PB cuts the circumcircle of ABC at B and E, and the line CE cuts the circumcircle of APE at E and F. Show that the ratio of the area of the quadrilateral $APEF$ to the area of the triangle ABP does not depend on the choice of P.

3. Let $n \geq 2$ be a fixed natural number and let $a_1, a_2, \ldots, a_n$ be positive numbers whose sum is 1. Prove that for any positive numbers $x_1, x_2, \ldots, x_n$ whose sum is 1,
$$2\sum_{i<j} x_i x_j \leq \frac{n-2}{n-1} + \sum_{i=1}^{n} \frac{a_i x_i^2}{1-a_i},$$
and determine when equality holds.

4. Let $ABCD$ be a tetrahedron with $\angle BAC = \angle ACD$ and $\angle ABD = \angle BDC$. Show that edges AB and CD have the same length.

5. For a natural number k, let $p(k)$ denote the smallest prime number which does not divide k. If $p(k) > 2$, define $q(k)$ to be the product of all primes less than $p(k)$, otherwise let $q(k) = 1$. Consider the sequence
$$x_0 = 1, \qquad x_{n+1} = \frac{x_n p(x_n)}{q(x_n)} \qquad n = 0, 1, 2, \ldots.$$
Determine all natural numbers n such that $x_n = 111111$.

6. From the set of all permutations f of $\{1, 2, \ldots, n\}$ that satisfy the condition
$$f(i) \geq i - 1 \qquad i = 1, 2, \ldots, n,$$
one is chosen uniformly at random. Let p_n be the probability that the chosen permutation f satisfies
$$f(i) \leq i + 1 \qquad i = 1, 2, \ldots, n.$$
Find all natural numbers n such that $p_n > 1/3$.

2.13 Romania

1. Let $n > 2$ be an integer and $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that for any regular n-gon $A_1A_2\ldots A_n$,

$$f(A_1) + f(A_2) + \cdots + f(A_n) = 0.$$

Prove that f is the zero function.

2. Find the greatest positive integer n for which there exist n nonnegative integers $x_1, x_2, \ldots, x_n$, not all zero, such that for any sequence $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ of elements of $\{-1, 0, 1\}$, not all zero, n^3 does not divide $\epsilon_1 x_1 + \epsilon_2 x_2 + \ldots + \epsilon_n x_n$.

3. Let x, y be real numbers. Show that if the set

$$\{\cos(n\pi x) + \cos(n\pi y) | n \in \mathbb{N}\}$$

is finite, then $x, y \in \mathbb{Q}$.

4. Let $ABCD$ be a cyclic quadrilateral and let M be the set of incenters and excenters of the triangles BCD, CDA, DAB, ABC (for a total of 16 points). Show that there exist two sets of parallel lines K and L, each consisting of four lines, such that any line of $K \cup L$ contains exactly four points of M.

5. Given $a \in \mathbb{R}$ and $f_1, f_2, \ldots, f_n : \mathbb{R} \to \mathbb{R}$ additive functions such that $f_1(x)f_2(x)\cdots f_n(x) = ax^n$ for all $x \in \mathbb{R}$. Prove that there exists $b \in \mathbb{R}$ and $i \in \{1, 2, \ldots, n\}$ such that $f_i(x) = bx$ for all $x \in \mathbb{R}$.

6. The sequence $\{a_n\}_{n\geq 2}$ is defined as follows: if $p_1, p_2, \ldots, p_k$ are the distinct prime divisors of n, then $a_n = p_1^{-1} + p_2^{-1} + \ldots + p_k^{-1}$. Show that for any positive integer $N \geq 2$,

$$\sum_{n=2}^{N} a_2 a_3 \cdots a_n < 1.$$

7. Let $n \geq 3$ be an integer and $x_1, x_2, \ldots, x_{n-1}$ positive integers such that

$$\begin{aligned} x_1 + x_2 + \ldots + x_{n-1} &= n \\ x_1 + 2x_2 + \ldots + (n-1)x_{n-1} &= 2n - 2. \end{aligned}$$

Find the minimum of the sum

$$F(x_1, \ldots, x_{n-1}) = \sum_{k=1}^{n-1} k x_k (2n - k).$$

8. Let n, r be positive integers and A a set of lattice points in the plane, such that any open disc of radius r contains a point of A. Show that for any coloring of the points of A using n colors, there exist four points of the same color which are the vertices of a rectangle.

9. Find all prime numbers p, q for which the congruence

$$\alpha^{3pq} \equiv \alpha \pmod{3pq}$$

holds for all integers α.

10. Let $n \geq 3$ be an integer and $p \geq 2n - 3$ a prime. Let M be a set of n points in the plane, no three collinear, and let $f : M \to \{0, 1, \ldots, p-1\}$ be a function such that:

 (i) only one point of M maps to 0, and
 (ii) if A, B, C are distinct points in M and k is the circumcircle of the triangle ABC, then

$$\sum_{P \in M \cap k} f(P) \equiv 0 \pmod{p}.$$

Show that all of the points of M lie on a circle.

11. Let $x_1, x_2, \ldots, x_n, x_{n+1}$ be positive reals such that $x_1 + x_2 + \cdots + x_n = x_{n+1}$. Prove that

$$\sum_{i=1}^{n} \sqrt{x_i(x_{n+i} - x_i)} \leq \sqrt{\sum_{i=1}^{n} x_{n+1}(x_{n+1} - x_i)}.$$

12. Let x, y, z be real numbers. Prove that the following conditions are equivalent.

 (i) $x, y, z > 0$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq 1$.
 (ii) For every quadrilateral with sides a, b, c, d, $a^2x + b^2y + c^2z > d^2$.

13. Let n be a positive integer and D a set of n concentric circles in the plane. Prove that if the function $f : D \to D$ satisfies $d(f(A), f(B)) \geq d(A, B)$ for all $A, B \in D$, then $d(f(A), f(B)) = d(A, B)$ for every $A, B \in D$.

14. Let $n \geq 3$ be an integer and $X \subseteq \{1, 2, \ldots, n^3\}$ a set of $3n^2$ elements. Prove that one can find nine distinct numbers $a_1, \ldots, a_9$ in X such that the system

$$\begin{aligned} a_1x + a_2y + a_3z &= 0 \\ a_4x + a_5y + a_6z &= 0 \\ a_7x + a_8y + a_9z &= 0 \end{aligned}$$

has a solution (x_0, y_0, z_0) in nonzero integers.

2.14 Russia

1. Which are there more of among the natural numbers from 1 to 1000000, inclusive: numbers that can be represented as the sum of a perfect square and a (positive) perfect cube, or numbers that cannot be?

2. The centers O_1, O_2, O_3 of three nonintersecting circles of equal radius are positioned at the vertices of a triangle. From each of the points O_1, O_2, O_3 one draws tangents to the other two given circles. It is known that the intersection of these tangents form a convex hexagon. The sides of the hexagon are alternately colored red and blue. Prove that the sum of the lengths of the red sides equals the sum of the lengths of the blue sides.

3. Let x, y, p, n, k be natural numbers such that
$$x^n + y^n = p^k.$$
Prove that if $n > 1$ is odd, and p is an odd prime, then n is a power of p.

4. In the Duma there are 1600 delegates, who have formed 16000 committees of 80 persons each. Prove that one can find two committees having no fewer than four common members.

5. Show that in the arithmetic progression with first term 1 and ratio 729, there are infinitely many powers of 10.

6. In the isosceles triangle ABC ($AC = BC$) point O is the circumcenter, I the incenter, and D lies on BC so that lines OD and BI are perpendicular. Prove that ID and AC are parallel.

7. Two piles of coins lie on a table. It is known that the sum of the weights of the coins in the two piles are equal, and for any natural number k, not exceeding the number of coins in either pile, the sum of the weights of the k heaviest coins in the first pile is not more than that of the second pile. Show that for any natural number x, if each coin (in either pile) of weight not less than x is replaced by a coin of weight x, the first pile will not be lighter than the second.

8. Can a 5×7 checkerboard be covered by L's (figures formed from a 2×2 square by removing one of its four 1×1 corners), not crossing its

borders, in several layers so that each square of the board is covered by the same number of L's?

9. Points E and F are given on side BC of convex quadrilateral $ABCD$ (with E closer than F to B). It is known that $\angle BAE = \angle CDF$ and $\angle EAF = \angle FDE$. Prove that $\angle FAC = \angle EDB$.

10. On a coordinate plane are placed four counters, each of whose centers has integer coordinates. One can displace any counter by the vector joining the centers of two of the other counters. Prove that any two preselected counters can be made to coincide by a finite sequence of moves.

11. Find all natural numbers n, such that there exist relatively prime integers x and y and an integer $k > 1$ satisfying the equation $3^n = x^k + y^k$.

12. Show that if the integers $a_1, \ldots, a_m$ are nonzero and for each $k = 0, 1, \ldots, m$ $(n < m - 1)$,

$$a_1 + a_2 2^k + a_3 3^k + \ldots + a_m m^k = 0,$$

then the sequence $a_1, \ldots, a_m$ contains at least $n + 1$ pairs of consecutive terms having opposite signs.

13. At the vertices of a cube are written eight pairwise distinct natural numbers, and on each of its edges is written the greatest common divisor of the numbers at the endpoints of the edge. Can the sum of the numbers written at the vertices be the same as the sum of the numbers written at the edges?

14. Three sergeants and several solders serve in a platoon. The sergeants take turns on duty. The commander has given the following orders:

    (a) Each day, at least one task must be issued to a soldier.

    (b) No soldier may have more than two task or receive more than one tasks in a single day.

    (c) The lists of soldiers receiving tasks for two different days must not be the same.

    (d) The first sergeant violating any of these orders will be jailed.

Can at least one of the sergeants, without conspiring with the others, give tasks according to these rules and avoid being jailed?

15. A convex polygon is given, no two of whose sides are parallel. For each side we consider the angle the side subtends at the vertex farthest from the side. Show that the sum of these angles equals 180°.

16. Goodnik writes 10 numbers on the board, then Nogoodnik writes 10 more numbers, all 20 of the numbers being positive and distinct. Can Goodnik choose his 10 numbers so that no matter what Nogoodnik writes, he can form 10 quadratic trinomials of the form $x^2 + px + q$, whose coefficients p and q run through all of the numbers written, such that the real roots of these trinomials comprise exactly 11 values?

17. Can the number obtained by writing the numbers from 1 to n in order ($n > 1$) be the same when read left-to-right and right-to-left?

18. Several hikers travel at fixed speeds along a straight road. It is known that over some period of time, the sum of their pairwise distances is monotonically decreasing. Show that there is a hiker, the sum of whose distances to the other hikers is monotonically decreasing over the same period.

19. Show that for $n \geq 5$, a cross-section of a pyramid whose base is a regular n-gon cannot be a regular $(n + 1)$-gon.

20. Do there exist three natural numbers greater than 1, such that the square of each, minus one, is divisible by each of the others?

21. In isosceles triangle ABC ($AB = BC$) one draws the angle bisector CD. The perpendicular to CD through the center of the circumcircle of ABC intersects BC at E. The parallel to CD through E meets AB at F. Show that $BE = FD$.

22. Does there exist a finite set M of nonzero real numbers, such that for any natural number n a polynomial of degree no less than n with coefficients in M, all of whose roots are real and belong to M?

23. The numbers from 1 to 100 are written in an unknown order. One may ask about any 50 numbers and find out their relative order. What is the fewest questions needed to find the order of all 100 numbers?

2.15 Spain

1. The natural numbers a and b are such that
$$\frac{a+1}{b}+\frac{b+1}{a}$$
is an integer. Show that the greatest common divisor of a and b is not greater than $\sqrt{a+b}$.

2. Let G be the centroid of the triangle ABC. Prove that if $AB+GC = AC+GB$, then ABC is isosceles.

3. Let a, b, c be real numbers. Consider the functions
$$f(x)=ax^2+bx+c, \qquad g(x)=cx^2+bx+a.$$
Given that
$$|f(-1)| \leq 1, \quad |f(0)| \leq 1, \quad |f(1)| \leq 1,$$
show that for $-1 \leq x \leq 1$,
$$|f(x)| \leq \frac{5}{4} \quad \text{and} \quad |g(x)| \leq 2.$$

4. Find all real solutions of the equation
$$\sqrt{x^2-p}+2\sqrt{x^2-1}=x$$
for each real value of p.

5. At Port Aventura there are 16 secret agents. Each agent is watching one or more other agents, but no two agents are both watching each other. Moreover, any 10 agents can be ordered so that the first is watching the second, the second is watching the third, etc., and the last is watching the first. Show that any 11 agents can also be so ordered.

6. A regular pentagon is constructed externally on each side of a regular pentagon of side 1. This figure is then folded and the two edges meeting at each vertex of the original pentagon but not belonging to the original pentagon are glued together. Determine the volume of water that can be poured into the resulting contained without spillage.

2.16 Turkey

1. Let

$$\prod_{n=1}^{1996}\left(1+nx^{3^n}\right)=1+a_1x^{k_1}+a_2x^{k_2}+\ldots+a_mx^{k_m},$$

where $a_1, a_2, \ldots, a_m$ are nonzero and $k_1 < k_2 < \ldots < k_m$. Find a_{1996}.

2. In a parallelogram $ABCD$ with $\angle A < 90°$, the circle with diameter AC meets the lines CB and CD again at E and F, respectively, and the tangent to this circle at A meets BD at P. Show that P, F, E are collinear.

3. Given real numbers $0 = x_1 < x_2 < \ldots < x_{2n} < x_{2n+1} = 1$ with $x_{i+1} - x_i \leq h$ for $1 \leq i \leq 2n$, show that

$$\frac{1-h}{2} < \sum_{i=1}^{n} x_{2i}(x_{2i+1} - x_{2i-1}) < \frac{1+h}{2}.$$

4. In a convex quadrilateral $ABCD$, triangles ABC and ADC have the same area. Let E be the the intersection of AC and BD, and let the parallels through E to the lines AD, DC, CB, BA meet AB, BC, CD, DA at K, L, M, N, respectively. Compute the ratio of the areas of the quadrilaterals $KLMN$ and $ABCD$.

5. Find the maximum number of pairwise disjoint sets of the form $S_{a,b} = \{n^2 + an + b : n \in \mathbb{Z}\}$ with $a, b \in \mathbb{Z}$.

6. For which ordered pairs of positive real numbers (a, b) is the limit of every sequence $\{x_n\}$ satisfying the condition

$$\lim_{n\to\infty}(ax_{n+1} - bx_n) = 0$$

zero?

2.17 United Kingdom

1. Consider the pair of four-digit positive integers

$$(M, N) = (3600, 2500).$$

Notice that M and N are both perfect squares, with equal digits in two places, and differing digits in the remaining two places. Moreover, when the digits differ, the digit in M is exactly one greater than the corresponding digit in N. Find all pairs of four-digit positive integers (M, N) with these properties.

2. A function f defined on the positive integers satisfies $f(1) = 1996$ and

$$f(1) + f(2) + \cdots + f(1996) = n^2 f(n) \quad (n > 1).$$

Calculate $f(1996)$.

3. Let ABC be an acute triangle and O its circumcenter. Let S denote the circle through A, B, O. The lines CA and CB meet S again at P and Q, respectively. Prove that the lines CO and PQ are perpendicular.

4. Define

$$q(n) = \left\lfloor \frac{n}{\lfloor \sqrt{n} \rfloor} \right\rfloor \quad (n = 1, 2, \ldots).$$

Determine all positive integers n for which $q(n) > q(n+1)$.

5. Let a, b, c be positive real numbers.

   (a) Prove that $4(a^3 + b^3) \geq (a+b)^3$.

   (b) Prove that $9(a^3 + b^3 + c^3) \geq (a+b+c)^3$.

6. Find all solutions in nonnegative integers x, y, z of the equation

$$2^x + 3^y = z^2.$$

7. The sides a, b, c and u, v, w of two triangles ABC and UVW are related by the equations

$$\begin{aligned} u(v + w - u) &= a^2, \\ v(w + u - v) &= b^2, \\ w(u + v - w) &= c^2. \end{aligned}$$

Prove that ABC is acute, and express the angles U, V, W in terms of A, B, C.

8. Two circles S_1 and S_2 touch each other externally at K; they also touch a circle S internally at A_1 and A_2, respectively. Let P be one point of intersection of S with the common tangent to S_1 and S_2 at K. The line PA_1 meets S_1 again at B_1, and PA_2 meets S_2 again at B_2. Prove that B_1B_2 is a common tangent to S_1 and S_2.

9. Find all solutions in positive real numbers a, b, c, d to the following system of equations:

$$\begin{aligned} a+b+c+d &= 12 \\ abcd &= 27+ab+ac+ad+bc+bd+cd. \end{aligned}$$

2.18 United States of America

1. Prove that the average of the numbers $n \sin n^\circ$ $(n = 2, 4, 6, \ldots, 180)$ is $\cot 1^\circ$.

2. For any nonempty set S of real numbers, let $\sigma(S)$ denote the sum of the elements of S. Given a set A of n positive integers, consider the collection of all distinct sums $\sigma(S)$ as S ranges over the nonempty subsets of A. Prove that this collection of sums can be partitioned into n classes so that in each class, the ratio of the largest sum to the smallest sum does not exceed 2.

3. Let ABC be a triangle. Prove that there is a line ℓ (in the plane of triangle ABC) such that the intersection of the interior of triangle ABC and the interior of its reflection $A'B'C'$ in ℓ has area more than $2/3$ the area of triangle ABC.

4. An n-term sequence $(x_1, x_2, \ldots, x_n)$ in which each term is either 0 or 1 is called a ***binary sequence of length n***. Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to 0, 1, 0 in that order. Let b_n be the number of binary sequences of length n that contain no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that $b_{n+1} = 2a_n$ for all positive integers n.

5. Triangle ABC has the following property: there is an interior point P such that $\angle PAB = 10^\circ$, $\angle PBA = 20^\circ$, $\angle PCA = 30^\circ$, and $\angle PAC = 40^\circ$. Prove that triangle ABC is isosceles.

6. Determine (with proof) whether there is a subset X of the integers with the following property: for any integer n there is exactly one solution of $a + 2b = n$ with $a, b \in X$.

2.19 Vietnam

1. Solve the system of equations:

$$\begin{aligned}\sqrt{3x}\left(1+\frac{1}{x+y}\right) &= 2\\ \sqrt{7y}\left(1-\frac{1}{x+y}\right) &= 4\sqrt{2}.\end{aligned}$$

2. Let $ABCD$ be a tetrahedron with $AB = AB = AD$ and circumcenter O. Let G be the centroid of triangle ACD, let E be the midpoint of BG, and let F be the midpoint of AE. Prove that OF is perpendicular to BG if and only if OD is perpendicular to AC.

3. Determine, as a function of n, the number of permutations of the set $\{1, 2, \ldots, n\}$ such that no three of $1, 2, 3, 4$ appear consecutively.

4. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying (for all $n \in \mathbb{N}$)

$$f(n) + f(n+1) = f(n+2)f(n+3) - 1996.$$

5. Consider triangles ABC where $BC = 1$ and $\angle BAC$ has a fixed measure $\alpha > \pi/3$. Determine which such triangle minimizes the distance between the incenter and centroid of ABC, and compute this distance in terms of α.

6. Let a, b, c, d be four nonnegative real numbers satisfying the condition

$$2(ab + ac + ad + bc + bd + cd) + abc + abd + acd + bcd = 16.$$

Prove that

$$a + b + c + d \geq \frac{2}{3}(ab + ac + ad + bc + bd + cd)$$

and determine when equality occurs.

3 1996 Regional Contests: Problems

3.1 Asian Pacific Mathematics Olympiad

1. Let $ABCD$ be a quadrilateral with $AB = BC = CD = DA$. Let MN and PQ be two segments perpendicular to the diagonal BD and such that the distance between them is $d > BD/2$, with $M \in AD$, $N \in DC$, $P \in AB$, and $Q \in BC$. Show that the perimeter of the hexagon $AMNCQP$ does not depend on the position of MN and PQ so long as the distance between them remains constant.

2. Let m and n be positive integers such that $n \leq M$. Prove that

$$2^n n! \leq \frac{(m+n)!}{(m-n)!} \leq (m^2+m)^n.$$

3. Let P_1, P_2, P_3, P_4 be four points on a circle, and let I_1 be the incenter of the triangle $P_2P_3P_4$, I_2 be the incenter of the triangle $P_1P_3P_4$, I_3 be the incentre of the triangle $P_1P_2P_4$, and I_4 be the incenter of the triangle $P_1P_2P_3$. Prove that I_1, I_2, I_3 and I_4 are the vertices of a rectangle.

4. The National Marriage Council wishes to invite n couples to form 17 discussion groups under the following conditions:

   (a) All members of the group must be of the same sex, i.e. they are either all male or all female.

   (b) The difference in the size of any two groups is either 0 or 1.

   (c) All groups have at least one member.

   (d) Each person must belong to one and only one group.

   Find all values of n, $n \leq 1996$, for which this is possible. Justify your answer.

5. Let a, b and c be the lengths of the sides of a triangle. Prove that

$$\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} \leq \sqrt{a} + \sqrt{b} + \sqrt{c}$$

   and determine when equality occurs.

3.2 Austrian-Polish Mathematics Competition

1. Let $k \geq 1$ be an integer. Show that there are exactly 3^{k-1} positive integers n with the following properties:

   (a) The decimal representation of n consists of exactly k digits.
   (b) All digits of k are odd.
   (c) The number n is divisible by 5.
   (d) The number $m = n/5$ has k odd (decimal) digits.

2. A convex hexagon $ABCDEF$ satisfies the following conditions:

   (a) Opposite sides are parallel (i.e. $AB \parallel DE$, $BC \parallel EF$, $CD \parallel FA$).
   (b) The distances between opposite sides are equal (i.e. $d(AB, DE) = d(BC, EF) = d(CD, FA)$, where $d(g, h)$ denotes the distance between lines g and h).
   (c) The angles $\angle FAB$ and $\angle CDE$ are right.

   Show that diagonals BE and CF intersect at an angle of $45°$.

3. The polynomials $P_n(x)$ are defined by $P_0(x) = 0$, $P_1(x) = x$ and

   $$P_n(x) = xP_{n-1}(x) + (1-x)P_{n-2}(x) \qquad n \geq 2.$$

   For every natural number $n \geq 1$, find all real numbers x satisfying the equation $P_n(x) = 0$.

4. The real numbers x, y, z, t satisfy the equalities $x + y + z + t = 0$ and $x^2 + y^2 + z^2 + t^2 = 1$. Prove that

   $$-1 \leq xy + yz + zt + tx \leq 0.$$

5. A convex polyhedron P and a sphere S are situated in space such that S intercepts on each edge AB of P a segment XY with $AX = XY = YB = \frac{1}{3}AB$. Prove that there exists a sphere T tangent to all edges of P.

6. Natural numbers k, n are given such that $1 < k < n$. Solve the system of n equations

   $$x_i^3(x_i^2 + \cdots + x_{i+k-1}^2) = x_{i-1}^2 \qquad 1 \leq i \leq n$$

   in n real unknowns $x_1, \ldots, x_n$. (Note: $x_0 = x_n, x_1 = x_{n+1}$, etc.)

7. Show that there do not exist nonnegative integers k and m such that $k! + 48 = 48(k+1)^m$.

8. Show that there is no polynomial $P(x)$ of degree 998 with real coefficients satisfying the equation $P(x)^2 - 1 = P(x^2+1)$ for all real numbers x.

9. We are given a collection of rectangular brikcs, no one of which is a cube. The edge lengths are integers. For every triple of positive integers (a, b, c), not all equal, there is a sufficient supply of $a \times b \times \mathrm{c}$ bricks. Suppose that the bricks completely tile a $10 \times 10 \times 10$ box.

   (a) Assume that at least 100 bricks have been used. Prove that there exist at least two parallel bricks, that is, if AB is an edge of one of the bricks, $A'B'$ is an edge of the other and $AB \parallel A'B'$, then $AB = A'B'$.

   (b) Prove the same statement with 100 replaced by a smaller number. The smaller the number, the better the solution.

3.3 Balkan Mathematical Olympiad

1. Let O and G be the circumcenter and centroid, respectively, of triangle ABC. If R is the circumradius and r the inradius of ABC, show that
$$OG \leq \sqrt{R(R-2r)}.$$

2. Let $p > 5$ be a prime number and $X = \{p - n^2 | n \in \mathbb{N}, n^2 < p\}$. Prove that X contains two distinct elements x, y such that $x \neq 1$ and x divides y.

3. Let $ABCDE$ be a convex pentagon, and let M, N, P, Q, R be the midpoints of sides AB, BC, CD, DE, EA, respectively. If the segments AP, BQ, CR, DM have a common point, show that this point also lies on EN.

4. Show that there exists a subset A of the set $\{1, 2, \ldots, 1996\}$ having the following properties:

   (a) $1, 2^{1996} - 1 \in A$;

   (b) every element of A, except 1, is the sum of two (not necessarily distinct) elements of A;

   (c) A contains at most 2012 elements.

3.4 Czech-Slovak Match

1. Let $\mathbb{Z}^*$ denote the set of nonzero integers. Show that an integer $p > 3$ is prime if and only if for any $a, b \in \mathbb{Z}^*$, exactly one of the numbers

$$N_1 = a + b - 6ab + \frac{p-1}{6}, \quad N_2 = a + b + 6ab + \frac{p+1}{6}$$

belongs to $\mathbb{Z}^*$.

2. Let M be a nonempty set and $*$ a binary operation on M. That is, to each pair $(a, b) \in M \times M$ one assigns an element $a * b$. Suppose further that for any $a, b \in M$,

$$(a * b) * b = a \qquad \text{and} \qquad a * (a * b) = b.$$

   (a) Show that $a * b = b * a$ for all $a, b \in M$.

   (n) For which finite sets M does such a binary operation exist?

3. A pyramid π is given whose base is a square of side $2a$ and whose lateral edges have length $a\sqrt{17}$. Let M be a point in the interior of the pyramid, and for each face of π, consider the pyramid similar to π whose vertex is M and whose base lies in the plane of the face. Show that the sum of the surface areas of these five pyramids is greater than or equal to one-fifth the surface area of π, and determine for which M equality holds.

4. Determine whether there exists a function $f : \mathbb{Z} \to \mathbb{Z}$ such that for each $k = 0, 1, \ldots, 1996$ and for each $m \in \mathbb{Z}$ the equation $f(x) + bx = m$ has at least one solution $x \in \mathbb{Z}$.

5. Two sets of intervals A, B on a line are given. The set A contains $2m - 1$ intervals, every two of which have a common interior point. Moreover, each interval in A contains at least two disjoint intervals of B. Show that there exists an interval in B which belongs to at least m intervals from A.

6. The points E and D lie in the interior of sides AC and BC, respectively, of a triangle ABC. Let F be the intersection of the lines AD and BE. Show that the area of the triangles ABC and ABF satisfies

$$\frac{S_{ABC}}{S_{ABF}} = \frac{|AC|}{|AE|} + \frac{|BC|}{|BD|} - 1.$$

3.5 Iberoamerican Olympiad

1. Let n be a natural number. A cube of side length n can be divided into 1996 cubes whose side lengths are also natural numbers. Determine the smallest possible value of n.

2. Let M be the midpoint of the median AD of triangle ABC. The line BM intersects side AC at the point N. Show that AB is tangent to the circumcircle of NB if and only if the following equality holds:
$$\frac{BM}{BN} = \frac{BC^2}{BN^2}.$$

3. We have a square table of $k^2 - k + 1$ rows and $k^2 - k + 1$ columns, where $k = p + 1$ and p is a prime number. For each prime p, give a method of distributing the numbers 0 and 1, one number in each square of the table, such that in each row and column there are exactly k zeroes, and moreover no rectangle with sides parallel to the sides of the table has zeroes at all four corners.

4. Given a natural number $n \geq 2$, consider all of the fractions of the form $\frac{1}{ab}$, where a and b are relatively prime natural numbers such that $a < b \leq n$ and $a + b > n$. Show that the sum of these fractions is $1/2$.

5. Three counters A, B, C are placed at the corners of an equilateral triangle of side n. The triangle is divided into triangles of side length 1. Initially all lines of the figure are painted blue. The counters move along the lines, painting their paths red, according to the following rules:

   (i) First A moves, then B, then C, then A, and so on in succesion. On each turn, each counter moves the full length of a side of one of the short triangles.

   (ii) No counter may retrace a segment already painted red, though it can stop on a red vertex, even if another counter is already there.

   Show that for all integers $n > 0$ it is possible to paint all of the segments red in this fashion.

6. In the plane are given n distinct points $A_1, \ldots, A_n$, and to each point A_i is assigned a nonzero real number λ_i such that $(A_iA_j)^2 = \lambda_i + \lambda_j$ for all $i \neq j$. Show that

   (a) $n \geq 4$;

   (b) If $n = 4$, then $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4} = 0$.

3.6 St. Petersburg City Mathematical Olympiad

1. Several one-digit numbers are written on a blackboard. One can replace any one of the numbers by the last digit of the sum of all of the numbers. Prove that the initial collection of numbers can be recovered by a sequence of such operations.

2. Fifty numbers are chosen from the set $\{1, \ldots, 99\}$, no two of which sum to 99 or 100. Prove that the chosen numbers must be $50, 51, \ldots, 99$.

3. Let M be the intersection of the diagonals of the trapezoid $ABCD$. A point P such that $\angle APM = \angle DPM$ is chosen on the base BC. Prove that the distance from C to the line AP is equal to the distance from B to the line DP.

4. In a group of several people, some are acquainted with each other and some are not. Every evening, one person invites all of his acquaintances to a party and introduces them to each other. Suppose that after each person has arranged at least one party, some two people are still unacquainted. Prove that they will not be introduced at the next party.

5. Let M be the intersection of the diagonals of a cyclic quadrilateral, N the intersections of the lines joining the midpoints of opposite sides, and O the circumcenter. Prove that $OM \geq ON$.

6. Prove that for every polynomial $P(x)$ of degree 10 with integer coefficients, there is an infinite (in both directions) arithmetic progression which does not contain $P(k)$ for any integer k.

7. There are n parking spaces along a one-way road down which n drivers are traveling. Each driver goes to his favorite parking space and parks there if it is free; otherwise, he parks at the nearest free place down the road. If there is no free space after his favorite, he drives away. How many lists $a_1, \ldots, a_n$ of favorite parking spaces are there which permit all of the drivers to park?

8. Find all positive integers n such that $3^{n-1} + 5^{n-1}$ divides $3^n + 5^n$.

9. Let M be the midpoint of side BC of triangle ABC, and let r_1 and r_2 be the radii of the incircles of triangles ABM and ACM. Prove that $r_1 < 2r_2$.

10. Several positive integers are written on a blackboard. One can erase any two distinct integers and write their greatest common divisor and least common multiple instead. Prove that eventually the numbers will stop changing.

11. No three diagonals of a convex 1996-gon meet in a point. Prove that the number of triangles lying in the interior of the 1996-gon and having sides on its diagonals is divisible by 11.

12. Prove that for every polynomial x^2+px+q with integer coefficients, there exists a polynomial $2x^2+rx+s$ with integer coefficiets such that the sets of values of the two polynomials on the integers are disjoint.

13. In a convex pentagon $ABCDE$, $AB = BC$, $\angle ABE + \angle DBC = \angle EBD$, and $\angle AEB + \angle BDE = \pi$. Prove that the orthocenter of triangle BDE lies on AC.

14. In a federation consisting of two republics, each pair of cities is linked by a one-way road, and each city can be reached from each other city by these roads. The Hamilton travel agency provides n different tours of the cities of the first republic (visiting each city once and returning to the starting city without leaving the republic) and m tours of the second republic. Prove that Hamilton can offer mn such tours around the whole federation.

15. Sergey found 11 different solutions to the equation $f(19x-96/x)=0$. Prove that if he had tried harder, he could have found at least one more solution.

16. The numbers $1,2,\dots,2n$ are divided into two groups of n numbers. Prove that the pairwise sums of numbers in each group (the sum of each number with itself included) have the same remainders upon division by $2n$.

17. The points A' and C' are chosen on the diagonal BD of a parallelogram $ABCD$ so that $AA' \parallel CC'$. The point K lies on the segment $A'C$, and the line AK meets CC' at L. A line parallel to BC is drawn through K, and a line parallel to BD is drawn through C; these meet at M. Prove that D, M, L are collinear.

18. Find all quadruples of polynomials $P_1(x), P_2(x), P_3(x), P_4(x)$ with real coefficients such that for each quadruple of integers x, y, z, t such that $xy - zt = 1$, one has

$$P_1(x)P_2(y) - P_3(z)P_4(t) = 1.$$

19. Two players play the following game on a 100×100 board. The first player marks a free square, then the second player puts a 1×2 domino down covering two free squares, one of which is marked. The first player wins if the entire board is covered, otherwise the second player wins. Which player has a winning strategy?

20. Let BD be the bisector of angle B in triangle ABC. The circumcircle of triangle BDC meets segment AB at E, while the circumcircle of triangle ABD meets segment BC at F. Prove that $AE = CF$.

21. A 10×10 table consists of positive integers such that for every five rows and five columns, the sum of the numbers at their intersections is even. Prove that all of the integers in the table are even.

22. Prove that there are no positive integers a and b such that for each pair p, q of distinct primes greater than 1000, the number $ap + bq$ is also prime.

23. In triangle ABC, the angle A is $60°$. A point O is taken inside the triangle such that $\angle AOB = \angle BOC = 120°$. The points D and E are the midpoints of sides AB and AC. Prove that the quadrilateral $ADOE$ is cyclic.

24. There are 2000 towns in a country, each pair of which is linked by a road. The Ministry of Reconstruction proposed all of the possible assignments of one-way traffic to each road. The Ministry of Transportation rejected each assignment that did not allow travel from any town to any other town. Prove that more of half of the assignments remained.

25. The positive integers m, n, m, n are written on a blackboard. A generalized Euclidean algorithm is applied to this quadruple as follows: if the numbers x, y, u, v appear on the board and $x > y$, then $x - y, y, u + v, v$ are written instead; otherwise $x, y - x, u, v + u$ are written instead. The algorithm stops when the numbers in the first

pair become equal (they will equal the greatest common divisor of m and n). Prove that the arithmetic mean of the numbers in the second pair at that moment equals the least common multiple of m and n.

26. A set of geometric figures consists of red equilateral triangles and blue quadrilaterals with all angles greater than 80° and less than 120°. A convex polygon with all of its angles greater than 60° is assembled from the figures in the set. Prove that the number of (entirely) red sides of the polygon is a multiple of 3.

27. The positive integers $1, 2, \ldots, n^2$ are placed in some fashion in the squares of an $n \times n$ table. As each number is placed in a square, the sum of the numbers already placed in the row and column containing that square is written on a blackboard. Give an arrangement of the numbers that minimizes the sum of the numbers written on the blackboard.