All-Russian Olympiad 2007

– Grade level 8

Day 1

1 Given reals numbers a, b, c. Prove that at least one of three equations $x^2+(a-b)x+(b-c)=0$, $x^2+(b-c)x+(c-a)=0$, $x^2+(c-a)x+(a-b)=0$ has a real root.
O. Podlipsky

2 The numbers $1,2,\dots,100$ are written in the cells of a 10×10 table, each number is written once. In one move, Nazar may interchange numbers in any two cells. Prove that he may get a table where the sum of the numbers in every two adjacent (by side) cells is composite after at most 35 such moves.
N. Agakhanov

3 Given a rhombus $ABCD$. A point M is chosen on its side BC. The lines, which pass through M and are perpendicular to BD and AC, meet line AD in points P and Q respectively. Suppose that the lines PB, QC, AM have a common point. Find all possible values of a ratio $\frac{BM}{MC}$.
S. Berlov, F. Petrov, A. Akopyan

4 *A. Akopyan, A. Akopyan, A. Akopyan, I. Bogdanov*

A conjurer Arutyun and his assistant Amayak are going to show following super-trick. A circle is drawn on the board in the room. Spectators mark 2007 points on this circle, after that Amayak
removes one of them. Then Arutyun comes to the room and shows a semicircle, to which the removed point belonged. Explain, how Arutyun and Amayak may show this super-trick.

Day 2

5 The distance between Maykop and Belorechensk is 24 km. Two of three friends need to reach Belorechensk from Maykop and another friend wants to reach Maykop from Belorechensk. They have only one bike, which is initially in Maykop. Each guy may go on foot (with velocity at most 6 kmph) or on a bike (with velocity at most 18 kmph). It is forbidden to leave a bike on a road.

Prove that all of them may achieve their goals after 2 hours 40 minutes. (Only one guy may seat on the bike simultaneously).

Folclore

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6 A line, which passes through the incentre I of the triangle ABC, meets its sides AB and BC at the points M and N respectively. The triangle BMN is acute. The points K, L are chosen on the side AC such that $\angle ILA = \angle IMB$ and $\angle KC = \angle INB$. Prove that $AM + KL + CN = AC$.
S. Berlov

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7 For an integer $n > 3$ denote by $n?$ the product of all primes less than n. Solve the equation $n? = 2n + 16$.

V. Senderov

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8 Given a matrix $\{a_{ij}\}_{i,j=0}^{9}$, $a_{ij} = 10i + j + 1$. Andrei is going to cover its entries by 50 rectangles 1×2 (each such rectangle contains two adjacent entries) so that the sum of 50 products in these rectangles is minimal possible. Help him.
A. Badzyan

– Grade level 9

Day 1

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1 Unitary quadratic trinomials $f(x)$ and $g(x)$ satisfy the following interesting condition: $f(g(x)) = 0$ and $g(f(x)) = 0$ do not have real roots. Prove that at least one of equations $f(f(x)) = 0$ and $g(g(x)) = 0$ does not have real roots too.

S. Berlov

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2 100 fractions are written on a board, their numerators are numbers from 1 to 100 (each once) and denominators are also numbers from 1 to 100 (also each once). It appears that the sum of these fractions equals to $a/2$ for some odd a. Prove that it is possible to interchange numerators of two fractions so that sum becomes a fraction with odd denominator.
N. Agakhanov, I. Bogdanov

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3 Two players by turns draw diagonals in a regular $(2n + 1)$-gon $(n > 1)$. It is forbidden to draw a diagonal, which was already drawn, or intersects an odd number of already drawn diagonals. The player, who has no legal move, loses. Who has a winning strategy?
K. Sukhov

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4 BB_1 is a bisector of an acute triangle ABC. A perpendicular from B_1 to BC meets a smaller arc BC of a circumcircle of ABC in a point K. A perpendicular from B to AK meets AC in a point L. BB_1 meets arc AC in T. Prove that K, L, T are collinear.
V. Astakhov

Day 2

5 Two numbers are written on each vertex of a convex 100-gon. Prove that it is possible to remove a number from each vertex so that the remaining numbers on any two adjacent vertices are different.
F. Petrov

6 Let ABC be an acute triangle. The points M and N are midpoints of AB and BC respectively, and BH is an altitude of ABC. The circumcircles of AHN and CHM meet in P where $P \neq H$. Prove that PH passes through the midpoint of MN.
V. Filimonov

7 Given a matrix $\{a_{ij}\}_{i,j=0}^{9}$, $a_{ij} = 10i + j + 1$. Andrei is going to cover its entries by 50 rectangles 1×2 (each such rectangle contains two adjacent entries) so that the sum of 50 products in these rectangles is minimal possible. Help him.
A. Badzyan

8 Dima has written number $1/80!, 1/81!, \dots, 1/99!$ on 20 infinite pieces of papers as decimal fractions (the following is written on the last piece: $\frac{1}{99!} = 0{,}00\dots0010715\dots$, 155 0-s before 1). Sasha wants to cut a fragment of N consecutive digits from one of pieces without the comma. For which maximal N he may do it so that Dima may not guess, from which piece Sasha has cut his fragment?

A. Golovanov

– Grade level 10

Day 1

1 Faces of a cube $9 \times 9 \times 9$ are partitioned onto unit squares. The surface of a cube is pasted over by 243 strips 2×1 without overlapping. Prove that the number of bent strips is odd.

A. Poliansky

2 Given polynomial $P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$. Put $m = \min\{a_0, a_0 + a_1, \ldots, a_0 + a_1 + \cdots + a_n\}$. Prove that $P(x) \geq mx^n$ for $x \geq 1$.
A. Khrabrov

3 BB_1 is a bisector of an acute triangle ABC. A perpendicular from B_1 to BC meets a smaller arc BC of a circumcircle of ABC in a point K. A perpendicular from B to AK meets AC in a point L. BB_1 meets arc AC in T. Prove that K, L, T are collinear.
V. Astakhov

4 Arutyun and Amayak show another effective trick. A spectator writes down on a board a sequence of N (decimal) digits. Amayak closes two adjacent digits by a black disc. Then Arutyun comes and says both closed digits (and their order). For which minimal N they may show such a trick?
K. Knop, O. Leontieva

Day 2

5 Given a set of $n > 2$ planar vectors. A vector from this set is called *long*, if its length is not less than the length of the sum of other vectors in this set. Prove that if each vector is long, then the sum of all vectors equals to zero.
N. Agakhanov

6 Two circles ω_1 and ω_2 intersect in points A and B. Let PQ and RS be segments of common tangents to these circles (points P and R lie on ω_1, points Q and S lie on ω_2). It appears that $RB \parallel PQ$. Ray RB intersects ω_2 in a point $W \neq B$. Find RB/BW.

S. Berlov

7 Given a convex polyhedron F. Its vertex A has degree 5, other vertices have degree 3. A colouring of edges of F is called nice, if for any vertex except A all three edges from it have different colours. It appears that the number of nice colourings is not divisible by 5. Prove that there is a nice colouring, in which some three consecutive edges from A are coloured the same way.
D. Karpov

8 Dima has written number $1/80!$, $1/81!$, $\ldots, 1/99!$ on 20 infinite pieces of papers as decimal fractions (the following is written on the last piece: $\frac{1}{99!} = 0{,}00\ldots0010715\ldots$, 155 0-s before 1). Sasha wants to cut a fragment of N

consecutive digits from one of pieces without the comma. For which maximal N he may do it so that Dima may not guess, from which piece Sasha has cut his fragment?

A. Golovanov

– Grade level 11

Day 1

1 Prove that for $k > 10$ Nazar may replace in the following product some one cos by sin so that the new function $f_1(x)$ would satisfy inequality $|f_1(x)| \le 3 \cdot 2^{-1-k}$ for all real x.

$$f(x) = \cos x \cos 2x \cos 3x \ldots \cos 2^k x$$

N. Agakhanov

2 The incircle of triangle ABC touches its sides BC, AC, AB at the points A_1, B_1, C_1 respectively. A segment AA_1 intersects the incircle at the point $Q \neq A_1$. A line ℓ through A is parallel to BC. Lines A_1C_1 and A_1B_1 intersect ℓ at the points P and R respectively. Prove that $\angle PQR = \angle B_1QC_1$.
A. Polyansky

3 Arutyun and Amayak show another effective trick. A spectator writes down on a board a sequence of N (decimal) digits. Amayak closes two adjacent digits by a black disc. Then Arutyun comes and says both closed digits (and their order). For which minimal N they may show such a trick?
K. Knop, O. Leontieva

4 An infinite sequence (x_n) is defined by its first term $x_1 > 1$, which is a rational number, and the relation $x_{n+1} = x_n + \frac{1}{[x_n]}$ for all positive integers n. Prove that this sequence contains an integer.
A. Golovanov

Day 2

5 Two numbers are written on each vertex of a convex 100-gon. Prove that it is possible to remove a number from each vertex so that the remaining numbers on any two adjacent vertices are different.
F. Petrov

6 Do there exist non-zero reals a, b, c such that, for any $n > 3$, there exists a polynomial $P_n(x) = x^n + \cdots + ax^2 + bx + c$, which has exactly n (not necessary distinct) integral roots?
N. Agakhanov, I. Bogdanov

7 Given a tetrahedron T. Valentin wants to find two its edges a, b with no common vertices so that T is covered by balls with diameters a, b. Can he always find such a pair?

A. Zaslavsky

8 Given an undirected graph with N vertices. For any set of k vertices, where $1 \le k \le N$, there are at most $2k-2$ edges, which join vertices of this set. Prove that the edges may be coloured in two colours so that each cycle contains edges of both colours. (Graph may contain multiple edges).
I. Bogdanov, G. Chelnokov