

IMO Shortlist 2010

- Algebra
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- 1** Find all function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ the following equality holds
- $$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$
- where $\lfloor a \rfloor$ is greatest integer not greater than a .
Proposed by Pierre Bornsztein, France
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- 2** Let the real numbers a, b, c, d satisfy the relations $a + b + c + d = 6$ and $a^2 + b^2 + c^2 + d^2 = 12$. Prove that
- $$36 \leq 4(a^3 + b^3 + c^3 + d^3) - (a^4 + b^4 + c^4 + d^4) \leq 48.$$
- Proposed by Nazar Serdyuk, Ukraine*
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- 3** Let x_1, \dots, x_{100} be nonnegative real numbers such that $x_i + x_{i+1} + x_{i+2} \leq 1$ for all $i = 1, \dots, 100$ (we put $x_{101} = x_1, x_{102} = x_2$). Find the maximal possible value of the sum $S = \sum_{i=1}^{100} x_i x_{i+2}$.
- Proposed by Sergei Berlov, Ilya Bogdanov, Russia*
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- 4** A sequence x_1, x_2, \dots is defined by $x_1 = 1$ and $x_{2k} = -x_k, x_{2k-1} = (-1)^{k+1} x_k$ for all $k \geq 1$. Prove that $\forall n \geq 1$ $x_1 + x_2 + \dots + x_n \geq 0$.
- Proposed by Gerhard Wglinger, Austria*
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- 5** Denote by \mathbb{Q}^+ the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}^+ \mapsto \mathbb{Q}^+$ which satisfy the following equation for all $x, y \in \mathbb{Q}^+$:
- $$f(f(x)^2 y) = x^3 f(xy).$$
- Proposed by Thomas Huber, Switzerland*
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- 6** Suppose that f and g are two functions defined on the set of positive integers and taking positive integer values. Suppose also that the equations $f(g(n)) = f(n) + 1$ and $g(f(n)) = g(n) + 1$ hold for all positive integers. Prove that $f(n) = g(n)$ for all positive integer n .
- Proposed by Alex Schreiber, Germany*
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- 7 Let a_1, a_2, a_3, \dots be a sequence of positive real numbers, and s be a positive integer, such that

$$a_n = \max\{a_k + a_{n-k} \mid 1 \leq k \leq n-1\} \text{ for all } n > s.$$

Prove there exist positive integers $\ell \leq s$ and N , such that

$$a_n = a_\ell + a_{n-\ell} \text{ for all } n \geq N.$$

Proposed by Morteza Saghafeian, Iran

- 8 Given six positive numbers a, b, c, d, e, f such that $a < b < c < d < e < f$. Let $a + c + e = S$ and $b + d + f = T$. Prove that

$$2ST > \sqrt{3(S+T)(S(bd+bf+df)+T(ac+ae+ce))}.$$

Proposed by Sung Yun Kim, South Korea

— Combinatorics

- 1 In a concert, 20 singers will perform. For each singer, there is a (possibly empty) set of other singers such that he wishes to perform later than all the singers from that set. Can it happen that there are exactly 2010 orders of the singers such that all their wishes are satisfied?

Proposed by Gerhard Wglinger, Austria

- 2 On some planet, there are 2^N countries ($N \geq 4$). Each country has a flag N units wide and one unit high composed of N fields of size 1×1 , each field being either yellow or blue. No two countries have the same flag. We say that a set of N flags is diverse if these flags can be arranged into an $N \times N$ square so that all N fields on its main diagonal will have the same color. Determine the smallest positive integer M such that among any M distinct flags, there exist N flags forming a diverse set.

Proposed by Toni Kokan, Croatia

- 3 2500 chess kings have to be placed on a 100×100 chessboard so that
- (i) no king can capture any other one (i.e. no two kings are placed in two squares sharing a common vertex);
 - (ii) each row and each column contains exactly 25 kings.

Find the number of such arrangements. (Two arrangements differing by rotation or symmetry are supposed to be different.)

Proposed by Sergei Berlov, Russia

- 4 Each of the six boxes $B_1, B_2, B_3, B_4, B_5, B_6$ initially contains one coin. The following operations are allowed
- Type 1) Choose a non-empty box B_j , $1 \leq j \leq 5$, remove one coin from B_j and add two coins to B_{j+1} ;
- Type 2) Choose a non-empty box B_k , $1 \leq k \leq 4$, remove one coin from B_k and swap the contents (maybe empty) of the boxes B_{k+1} and B_{k+2} .
- Determine if there exists a finite sequence of operations of the allowed types, such that the five boxes B_1, B_2, B_3, B_4, B_5 become empty, while box B_6 contains exactly $2010^{2010^{2010}}$ coins.

Proposed by Hans Zantema, Netherlands

- 5 $n \geq 4$ players participated in a tennis tournament. Any two players have played exactly one game, and there was no tie game. We call a company of four players *bad* if one player was defeated by the other three players, and each of these three players won a game and lost another game among themselves. Suppose that there is no bad company in this tournament. Let w_i and l_i be respectively the number of wins and losses of the i -th player. Prove that

$$\sum_{i=1}^n (w_i - l_i)^3 \geq 0.$$

Proposed by Sung Yun Kim, South Korea

- 6 Given a positive integer k and other two integers $b > w > 1$. There are two strings of pearls, a string of b black pearls and a string of w white pearls. The length of a string is the number of pearls on it. One cuts these strings in some steps by the following rules. In each step:
- (i) The strings are ordered by their lengths in a non-increasing order. If there are some strings of equal lengths, then the white ones precede the black ones. Then k first ones (if they consist of more than one pearl) are chosen; if there are less than k strings longer than 1, then one chooses all of them.
- (ii) Next, one cuts each chosen string into two parts differing in length by at most one. (For instance, if there are strings of 5, 4, 4, 2 black pearls, strings of 8, 4, 3 white pearls and $k = 4$, then the strings of 8 white, 5 black, 4 white and 4 black pearls are cut into the parts (4, 4), (3, 2), (2, 2) and (2, 2) respectively.)

The process stops immediately after the step when a first isolated white pearl appears.

Prove that at this stage, there will still exist a string of at least two black pearls.

Proposed by Bill Sands, Thao Do, Canada

- 7 Let P_1, \dots, P_s be arithmetic progressions of integers, the following conditions being satisfied:

- (i) each integer belongs to at least one of them;
- (ii) each progression contains a number which does not belong to other progressions.

Denote by n the least common multiple of the ratios of these progressions; let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ its prime factorization.

Prove that

$$s \geq 1 + \sum_{i=1}^k \alpha_i (p_i - 1).$$

Proposed by Dierk Schleicher, Germany

- Geometry

- 1 Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and the circumcircle is P . The lines BP and DF meet at point Q . Prove that $AP = AQ$.

Proposed by Christopher Bradley, United Kingdom

- 2 Let P be a point interior to triangle ABC (with $CA \neq CB$). The lines AP, BP and CP meet again its circumcircle Γ at K, L , respectively M . The tangent line at C to Γ meets the line AB at S . Show that from $SC = SP$ follows $MK = ML$.

Proposed by Marcin E. Kuczma, Poland

- 3 Let $A_1 A_2 \dots A_n$ be a convex polygon. Point P inside this polygon is chosen so that its projections P_1, \dots, P_n onto lines $A_1 A_2, \dots, A_n A_1$ respectively lie on the sides of the polygon. Prove that for arbitrary points X_1, \dots, X_n on sides $A_1 A_2, \dots, A_n A_1$ respectively,

$$\max \left\{ \frac{X_1 X_2}{P_1 P_1}, \dots, \frac{X_n X_1}{P_n P_1} \right\} \geq 1.$$

Proposed by Nairi Sedrakyan, Armenia

- 4 Given a triangle ABC , with I as its incenter and Γ as its circumcircle, AI intersects Γ again at D . Let E be a point on the arc BDC , and F a point on the segment BC , such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. If G is the midpoint of IF , prove that the meeting point of the lines EI and DG lies on Γ .

Proposed by Tai Wai Ming and Wang Chongli, Hong Kong

- 5 Let $ABCDE$ be a convex pentagon such that $BC \parallel AE$, $AB = BC + AE$, and $\angle ABC = \angle CDE$. Let M be the midpoint of CE , and let O be the circumcenter of triangle BCD . Given that $\angle DMO = 90^\circ$, prove that $2\angle BDA = \angle CDE$.

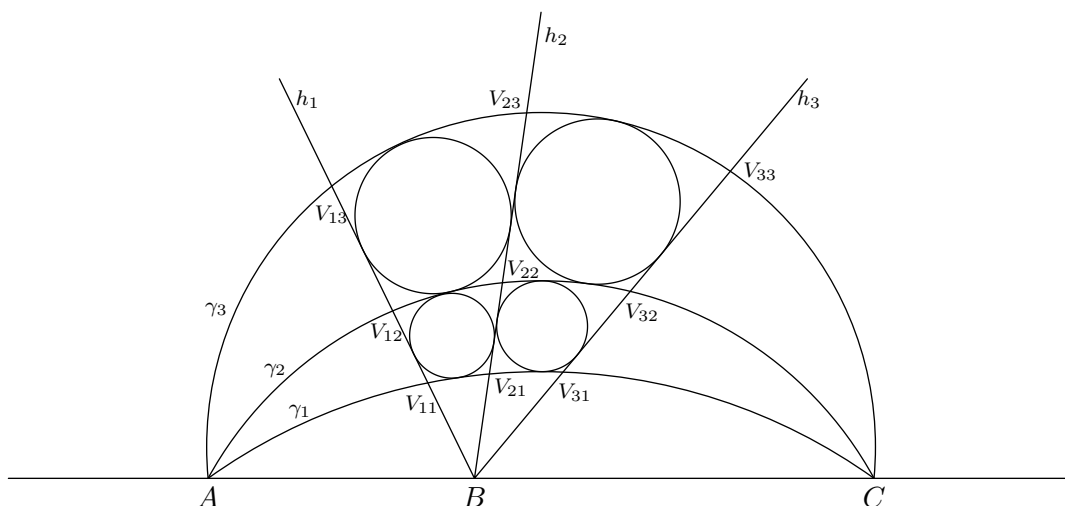
Proposed by Nazar Serdyuk, Ukraine

- 6 The vertices X, Y, Z of an equilateral triangle XYZ lie respectively on the sides BC, CA, AB of an acute-angled triangle ABC . Prove that the incenter of triangle ABC lies inside triangle XYZ .

Proposed by Nikolay Beluhov, Bulgaria

- 7 Three circular arcs γ_1, γ_2 , and γ_3 connect the points A and C . These arcs lie in the same half-plane defined by line AC in such a way that arc γ_2 lies between the arcs γ_1 and γ_3 . Point B lies on the segment AC . Let h_1, h_2 , and h_3 be three rays starting at B , lying in the same half-plane, h_2 being between h_1 and h_3 . For $i, j = 1, 2, 3$, denote by V_{ij} the point of intersection of h_i and γ_j (see the Figure below). Denote by $\widehat{V_{ij}V_{kj}V_{kl}V_{il}}$ the curved quadrilateral, whose sides are the segments $V_{ij}V_{il}$, $V_{kj}V_{kl}$ and arcs $V_{ij}V_{kj}$ and $V_{il}V_{kl}$. We say that this quadrilateral is *circumscribed* if there exists a circle touching these two segments and two arcs. Prove that if the curved quadrilaterals $\widehat{V_{11}V_{21}V_{22}V_{12}}, \widehat{V_{12}V_{22}V_{23}V_{13}}, \widehat{V_{21}V_{31}V_{32}V_{22}}$ are circumscribed, then the curved quadrilateral $\widehat{V_{22}V_{32}V_{33}V_{23}}$ is circumscribed, too.

Proposed by Gza Ks, Hungary



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Number Theory

1

Find the least positive integer n for which there exists a set $\{s_1, s_2, \dots, s_n\}$ consisting of n distinct positive integers such that

$$\left(1 - \frac{1}{s_1}\right) \left(1 - \frac{1}{s_2}\right) \cdots \left(1 - \frac{1}{s_n}\right) = \frac{51}{2010}.$$

Proposed by Daniel Brown, Canada

2

Find all pairs (m, n) of nonnegative integers for which

$$m^2 + 2 \cdot 3^n = m(2^{n+1} - 1).$$

Proposed by Angelo Di Pasquale, Australia

3

Find the smallest number n such that there exist polynomials f_1, f_2, \dots, f_n with rational coefficients satisfying

$$x^2 + 7 = f_1(x)^2 + f_2(x)^2 + \dots + f_n(x)^2.$$

Proposed by Mariusz Skaba, Poland

- 4 Let a, b be integers, and let $P(x) = ax^3 + bx$. For any positive integer n we say that the pair (a, b) is n -good if $n|P(m) - P(k)$ implies $n|m - k$ for all integers m, k . We say that (a, b) is *very good* if (a, b) is n -good for infinitely many positive integers n .

-(a) Find a pair (a, b) which is 51-good, but not very good.

-(b) Show that all 2010-good pairs are very good.

Proposed by Okan Tekman, Turkey

- 5 Find all functions $g : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$(g(m) + n)(g(n) + m)$$

is a perfect square for all $m, n \in \mathbb{N}$.

Proposed by Gabriel Carroll, USA

- 6 The rows and columns of a $2^n \times 2^n$ table are numbered from 0 to $2^n - 1$. The cells of the table have been coloured with the following property being satisfied: for each $0 \leq i, j \leq 2^n - 1$, the j -th cell in the i -th row and the $(i + j)$ -th cell in the j -th row have the same colour. (The indices of the cells in a row are considered modulo 2^n .) Prove that the maximal possible number of colours is 2^n .

Proposed by Hossein Dabirian, Sepehr Ghazi-nezami, Iran
