USA Team Selection Test 2015

Dec	December 11th, 2014
1	Let ABC be a non-isosceles triangle with incenter I whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Denote by M the midpoint of $\overline{BC}$. Let Q be a point on the incircle such that $\angle AQD = 90^\circ$. Let P be the point inside the triangle on line AI for which $MD = MP$. Prove that either $\angle PQE = 90^\circ$ or $\angle PQF = 90^\circ$. *Proposed by Evan Chen*
2	Prove that for every $n \in \mathbb{N}$, there exists a set S of n positive integers such that for any two distinct $a, b \in S$, $a - b$ divides a and b but none of the other elements of S. *Proposed by Iurie Boreico*
3	A physicist encounters 2015 atoms called usamons. Each usamon either has one electron or zero electrons, and the physicist can't tell the difference. The physicist's only tool is a diode. The physicist may connect the diode from any usamon A to any other usamon B. (This connection is directed.) When she does so, if usamon A has an electron and usamon B does not, then the electron jumps from A to B. In any other case, nothing happens. In addition, the physicist cannot tell whether an electron jumps during any given step. The physicist's goal is to isolate two usamons that she is sure are currently in the same state. Is there any series of diode usage that makes this possible? *Proposed by Linus Hamilton*
Jan	January 15, 2015
1	Let $f : \mathbb{Q} \to \mathbb{Q}$ be a function such that for any $x, y \in \mathbb{Q}$, the number $f(x+y) - f(x) - f(y)$ is an integer. Decide whether it follows that there exists a constant c such that $f(x) - cx$ is an integer for every rational number x. *Proposed by Victor Wang*
2	A tournament is a directed graph for which every (unordered) pair of vertices has a single directed edge from one vertex to the other. Let us define a proper directed-edge-coloring to be an assignment of a color to every (directed) edge, so that for every pair of directed edges $\overrightarrow{uv}$ and $\overrightarrow{vw}$, those two edges are in different colors. Note that it is permissible for $\overrightarrow{uv}$ and $\overrightarrow{uw}$ to be the same color. The

directed-edge-chromatic-number of a tournament is defined to be the minimum total number of colors that can be used in order to create a proper directed-edge-coloring. For each n, determine the minimum directed-edge-chromatic-number over all tournaments on n vertices.

Proposed by Po-Shen Loh

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3 Let ABC be a non-equilateral triangle and let M_a, M_b, M_c be the midpoints of the sides BC, CA, AB, respectively. Let S be a point lying on the Euler line. Denote by X, Y, Z the second intersections of M_aS, M_bS, M_cS with the nine-point circle. Prove that AX, BY, CZ are concurrent.
