All-Russian Olympiad 1997

– Grade level 9

Day 1

1 Let $P(x)$ be a quadratic polynomial with nonnegative coeficients. Show that for any real numbers x and y, we have the inequality $P(xy)^2 \leqslant P(x^2)P(y^2)$.
E. Malinnikova

2 Given a convex polygon M invariant under a 90° rotation, show that there exist two circles, the ratio of whose radii is $\sqrt{2}$, one containing M and the other contained in M.
A. Khrabrov

3 The lateral sides of a box with base $a \times b$ and height c (where a; b;c are natural numbers) are completely covered without overlap by rectangles whose edges are parallel to the edges of the box, each containing an even number of unit squares. (Rectangles may cross the lateral edges of the box.) Prove that if c is odd, then
the number of possible coverings is even.
D. Karpov, C. Gukshin, D. Fon-der-Flaas

4 The Judgment of the Council of Sages proceeds as follows: the king arranges the sages in a line and places either a white hat or a black hat on each sage's head. Each sage can see the color of the hats of the sages in front of him, but not of his own hat or of the hats of the sages behind him. Then one by one (in an order of their choosing), each sage guesses a color. Afterward, the king executes those sages who did not correctly guess the color of their own hat. The day before, the Council meets and decides to minimize the number of executions. What is the smallest number of sages guaranteed to survive in this case?

See also `http://www.artofproblemsolving.com/Forum/viewtopic.php?f=42&t=530553`

Day 2

1 Do there exist real numbers b and c such that each of the equations $x^2+bx+c=0$ and $2x^2+(b+1)x+c+1=0$ have two integer roots?
N. Agakhanov

2 A class consists of 33 students. Each student is asked how many other students in the class have his first name, and how many have his last name. It turns out that each number from 0 to 10 occurs among the answers. Show that there are two students in the class with the same first and last name.
A. Shapovalov

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3 The incircle of triangle ABC touches sides AB;BC;CA at M;N;K, respectively.
The line through A parallel to NK meets MN at D.
The line through A parallel to MN meets NK at E.
Show that the line DE bisects sides AB and AC of triangle ABC.
M. Sonkin

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4 The numbers from 1 to 100 are arranged in a 10×10 table so that any two adjacent numbers have sum no larger than S. Find the least value of S for which this is possible.
D. Hramtsov

– Grade level 10

Day 1

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1 Find all integer solutions of the equation $(x^2 - y^2)^2 = 1 + 16y$.
M. Sonkin

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2 An $n \times n$ square grid ($n \geqslant 3$) is rolled into a cylinder. Some of the cells are then colored black. Show that there exist two parallel lines (horizontal, vertical or diagonal) of cells containing the same number of black cells.
E. Poroshenko

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3 Two circles intersect at A and B. A line through A meets the first circle again at C and the second circle again at D. Let M and N be the midpoints of the arcs BC and BD not containing A, and let K be the midpoint of the segment CD. Show that $\angle MKN = \pi/2$.
(You may assume that C and D lie on opposite sides of A.)
D. Tereshin

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4 A polygon can be divided into 100 rectangles, but not into 99. Prove that it cannot be divided into 100 triangles.
A. Shapovalov

Day 2

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1 Do there exist two quadratic trinomials ax^2+bx+c and $(a+1)x^2+(b+1)x+(c+1)$ with integer coeficients, both of which have two integer roots?
N. Agakhanov

2 A circle centered at O and inscribed in triangle ABC meets sides AC;AB;BC at K;M;N, respectively. The median BB_1 of the triangle meets MN at D. Show that O;D;K are collinear.
M. Sonkin

3 Find all triples m; n; l of natural numbers such that $m+n = gcd(m;n)^2$; $m+l = gcd(m;l)^2$; $n+l = gcd(n;l)^2$:
S. Tokarev

4 On an infinite (in both directions) strip of squares, indexed by the integers, are placed several stones (more than one may be placed on a single square). We perform a sequence of moves of one of the following types:
(a) Remove one stone from each of the squares $n-1$ and n and place one stone on square $n+1$.
(b) Remove two stones from square n and place one stone on each of the squares $n+1$, $n-2$.
Prove that any sequence of such moves will lead to a position in which no further moves can be made, and moreover that this position is independent of the sequence of moves.
D. Fon-der-Flaas

– Grade level 11

Day 1

1 Find all integer solutions of the equation $(x^2-y^2)^2 = 1+16y$.
M. Sonkin

2 The Judgment of the Council of Sages proceeds as follows: the king arranges the sages in a line and places either a white hat, black hat or a red hat on each sage's head. Each sage can see the color of the hats of the sages in front of him, but not of his own hat or of the hats of the sages behind him. Then one by one (in an order of their choosing), each sage guesses a color. Afterward, the king executes those sages who did not correctly guess the color of their own hat. The day before, the Council meets and decides to minimize the number of executions. What is the smallest number of sages guaranteed to survive in this case?
K. Knop

P.S. Of course, the sages hear the previous guesses.

See also http://www.artofproblemsolving.com/Forum/viewtopic.php?f=42&t=530552

3 Two circles intersect at A and B. A line through A meets the first circle again at C and the second circle again at D. Let M and N be the midpoints of the arcs BC and BD not containing A, and let K be the midpoint of the segment CD. Show that $\angle MKN = \pi/2$.
(You may assume that C and D lie on opposite sides of A.)
D. Tereshin

4 An $n \times n \times n$ cube is divided into unit cubes. We are given a closed non-self-intersecting polygon (in space), each of whose sides joins the centers of two unit cubes sharing a common face. The faces of unit cubes which intersect the polygon are said to be distinguished. Prove that the edges of the unit cubes may be colored in two colors so that each distinguished face has an odd number of edges of each color, while each nondistinguished face has an even number of edges of each color.
M. Smurov

Day 2

1 Of the quadratic trinomials $x^2 + px + q$ where p; q are integers and $1 \leqslant p, q \leqslant 1997$, which are there more of: those having integer roots or those not having real roots?
M. Evdokimov

2 We are given a polygon, a line l and a point P on l in general position: all lines containing a side of the polygon meet l at distinct points diering from P.
We mark each vertex of the polygon the sides meeting which, extended away from the vertex, meet the
line l on opposite sides of P. Show that P lies inside the polygon if and only if on each side of l there are an odd number of marked vertices.
O. Musin

3 A sphere inscribed in a tetrahedron touches one face at the intersection of its angle bisectors, a second face at the intersection of its altitudes, and a third face at the intersection of its medians. Show that the tetrahedron is regular.
N. Agakhanov

4 In an $m \times n$ rectangular grid, where m and n are odd integers, 1×2 dominoes are initially placed so as to exactly cover all but one of the 1×1 squares at one

corner of the grid.
It is permitted to slide a domino towards the empty square, thus exposing another square.
Show that by a sequence of such moves, we can move the empty square to any corner of the rectangle.
A. Shapovalov