Art of Problem Solving
2006 USA Team Selection Test

USA Team Selection Test 2006

Day 1

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1 A communications network consisting of some terminals is called a [i]3-connector[/i] if among any three terminals, some two of them can directly communicate with each other. A communications network contains a *windmill* with n blades if there exist n pairs of terminals $\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_n, y_n\}$ such that each x_i can directly communicate with the corresponding y_i and there is a *hub* terminal that can directly communicate with each of the $2n$ terminals $x_1, y_1, \ldots, x_n, y_n$. Determine the minimum value of $f(n)$, in terms of n, such that a 3 -connector with $f(n)$ terminals always contains a windmill with n blades.

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2 In acute triangle ABC , segments $AD; BE$, and CF are its altitudes, and H is its orthocenter. Circle ω, centered at O, passes through A and H and intersects sides AB and AC again at Q and P (other than A), respectively. The circumcircle of triangle OPQ is tangent to segment BC at R. Prove that $\frac{CR}{BR} = \frac{ED}{FD}$.

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3 Find the least real number k with the following property: if the real numbers $x; y$, and z are not all positive, then

$$k(x^2 - x + 1)(y^2 - y + 1)(z^2 - z + 1) \geq (xyz)^2 - xyz + 1.$$

Day 2

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4 Let n be a positive integer. Find, with proof, the least positive integer d_n which cannot be expressed in the form

$$\sum_{i=1}^{n} (-1)^{a_i} 2^{b_i},$$

where a_i and b_i are nonnegative integers for each i.

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5 Let n be a given integer with n greater than 7 , and let $\mathcal{P}$ be a convex polygon with n sides. Any set of $n-3$ diagonals of $\mathcal{P}$ that do not intersect in the interior of the polygon determine a triangulation of $\mathcal{P}$ into $n-2$ triangles. A triangle in the triangulation of $\mathcal{P}$ is an interior triangle if all of its sides are diagonals of

$\mathcal{P}$. Express, in terms of n, the number of triangulations of $\mathcal{P}$ with exactly two interior triangles, in closed form.

6 Let ABC be a triangle. Triangles PAB and QAC are constructed outside of triangle ABC such that $AP = AB$ and $AQ = AC$ and $\angle BAP = \angle CAQ$. Segments BQ and CP meet at R. Let O be the circumcenter of triangle BCR. Prove that $AO \perp PQ$.