# Art of Problem Solving
## 2016 IMO Shortlist

Problems from the 2016 IMO Shortlist

– Algebra (A)

**A1** Let $a$, $b$, $c$ be positive real numbers such that $\min(ab, bc, ca) \geq 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leq \left(\frac{a+b+c}{3}\right)^2 + 1.$$

**A2** Find the smallest constant $C > 0$ for which the following statement holds: among any five positive real numbers $a_1, a_2, a_3, a_4, a_5$ (not necessarily distinct), one can always choose distinct subscripts $i, j, k, l$ such that

$$\left|\frac{a_i}{a_j} - \frac{a_k}{a_l}\right| \leq C.$$

**A3** Find all positive integers $n$ such that the following statement holds: Suppose real numbers $a_1$, $a_2$, $\dots$, $a_n$, $b_1$, $b_2$, $\dots$, $b_n$ satisfy $|a_k| + |b_k| = 1$ for all $k = 1, \dots, n$. Then there exists $\varepsilon_1$, $\varepsilon_2$, $\dots$, $\varepsilon_n$, each of which is either $-1$ or 1, such that

$$\left|\sum_{i=1}^{n} \varepsilon_i a_i\right| + \left|\sum_{i=1}^{n} \varepsilon_i b_i\right| \leq 1.$$

**A4** Find all functions $f : (0, \infty) \to (0, \infty)$ such that for any $x, y \in (0, \infty)$,

$$xf(x^2)f(f(y)) + f(yf(x)) = f(xy)\left(f(f(x^2)) + f(f(y^2))\right).$$

**A5** Consider fractions $\frac{a}{b}$ where $a$ and $b$ are positive integers.
(a) Prove that for every positive integer $n$, there exists such a fraction $\frac{a}{b}$ such that $\sqrt{n} \leq \frac{a}{b} \leq \sqrt{n+1}$ and $b \leq \sqrt{n} + 1$.
(b) Show that there are infinitely many positive integers $n$ such that no such fraction $\frac{a}{b}$ satisfies $\sqrt{n} \leq \frac{a}{b} \leq \sqrt{n+1}$ and $b \leq \sqrt{n}$.

**A6** The equation

$$(x-1)(x-2)\cdots(x-2016)=(x-1)(x-2)\cdots(x-2016)$$

is written on the board, with 2016 linear factors on each side. What is the least possible value of $k$ for which it is possible to erase exactly $k$ of these 4032 linear factors so that at least one factor remains on each side and the resulting equation has no real solutions?

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**A7** Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(0)\neq 0$ and for all $x,y\in\mathbb{R}$,

$$f(x+y)^2=2f(x)f(y)+\max\left\{f(x^2+y^2),f(x^2)+f(y^2)\right\}.$$

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**A8** Find the largest real constant $a$ such that for all $n\geq 1$ and for all real numbers $x_0,x_1,...,x_n$ satisfying $0=x_0<x_1<x_2<\cdots<x_n$ we have

$$\frac{1}{x_1-x_0}+\frac{1}{x_2-x_1}+\cdots+\frac{1}{x_n-x_{n-1}}\geq a\left(\frac{2}{x_1}+\frac{3}{x_2}+\cdots+\frac{n+1}{x_n}\right)$$

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– Combinatorics (C)

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**C1** The leader of an IMO team chooses positive integers $n$ and $k$ with $n>k$, and announces them to the deputy leader and a contestant. The leader then secretly tells the deputy leader an $n$-digit binary string, and the deputy leader writes down all $n$-digit binary strings which differ from the leaders in exactly $k$ positions. (For example, if $n=3$ and $k=1$, and if the leader chooses $101$, the deputy leader would write down $001, 111$ and $100$.) The contestant is allowed to look at the strings written by the deputy leader and guess the leaders string. What is the minimum number of guesses (in terms of $n$ and $k$) needed to guarantee the correct answer?

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**C2** Find all positive integers $n$ for which all positive divisors of $n$ can be put into the cells of a rectangular table under the following constraints:

-each cell contains a distinct divisor;
-the sums of all rows are equal; and
-the sums of all columns are equal.

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**C3** Let $n$ be a positive integer relatively prime to 6. We paint the vertices of a regular $n$-gon with three colours so that there is an odd number of vertices of each colour. Show that there exists an isosceles triangle whose three vertices are of different colours.

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**C4** Find all integers $n$ for which each cell of $n \times n$ table can be filled with one of the letters $I, M$ and $O$ in such a way that:

- in each row and each column, one third of the entries are $I$, one third are $M$ and one third are $O$; and
-in any diagonal, if the number of entries on the diagonal is a multiple of three, then one third of the entries are $I$, one third are $M$ and one third are $O$.

**Note.** The rows and columns of an $n \times n$ table are each labelled 1 to $n$ in a natural order. Thus each cell corresponds to a pair of positive integer $(i, j)$ with $1 \le i, j \le n$. For $n > 1$, the table has $4n - 2$ diagonals of two types. A diagonal of first type consists all cells $(i, j)$ for which $i + j$ is a constant, and the diagonal of this second type consists all cells $(i, j)$ for which $i - j$ is constant.

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**C5** Let $n \ge 3$ be a positive integer. Find the maximum number of diagonals in a regular $n$-gon one can select, so that any two of them do not intersect in the interior or they are perpendicular to each other.

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**C6** There are $n \ge 3$ islands in a city. Initially, the ferry company offers some routes between some pairs of islands so that it is impossible to divide the islands into two groups such that no two islands in different groups are connected by a ferry route.

After each year, the ferry company will close a ferry route between some two islands $X$ and $Y$. At the same time, in order to maintain its service, the company will open new routes according to the following rule: for any island which is connected to a ferry route to exactly one of $X$ and $Y$, a new route between this island and the other of $X$ and $Y$ is added.

Suppose at any moment, if we partition all islands into two nonempty groups in any way, then it is known that the ferry company will close a certain route connecting two islands from the two groups after some years. Prove that after some years there will be an island which is connected to all other islands by ferry routes.

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**C7** There are $n \ge 2$ line segments in the plane such that every two segments cross and no three segments meet at a point. Geoff has to choose an endpoint of each segment and place a frog on it facing the other endpoint. Then he will clap his hands $n - 1$ times. Every time he claps,each frog will immediately jump forward to the next intersection point on its segment. Frogs never change the direction of their jumps. Geoff wishes to place the frogs in such a way that no two of them will ever occupy the same intersection point at the same time.

(a) Prove that Geoff can always fulfill his wish if $n$ is odd.

(b) Prove that Geoff can never fulfill his wish if $n$ is even.

**C8** Let $n$ be a positive integer. Determine the smallest positive integer $k$ with the following property: it is possible to mark $k$ cells on a $2n \times 2n$ board so that there exists a unique partition of the board into $1 \times 2$ and $2 \times 1$ dominoes, none of which contain two marked cells.

– Geometry (G)

**G1** Triangle $BCF$ has a right angle at $B$. Let $A$ be the point on line $CF$ such that $FA = FB$ and $F$ lies between $A$ and $C$. Point $D$ is chosen so that $DA = DC$ and $AC$ is the bisector of $\angle DAB$. Point $E$ is chosen so that $EA = ED$ and $AD$ is the bisector of $\angle EAC$. Let $M$ be the midpoint of $CF$. Let $X$ be the point such that $AMXE$ is a parallelogram. Prove that $BD, FX$ and $ME$ are concurrent.

**G2** Let $ABC$ be a triangle with circumcircle $\Gamma$ and incenter $I$ and let $M$ be the midpoint of $\overline{BC}$. The points $D$, $E$, $F$ are selected on sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $\overline{ID} \perp \overline{BC}$, $\overline{IE} \perp \overline{AI}$, and $\overline{IF} \perp \overline{AI}$. Suppose that the circumcircle of $\triangle AEF$ intersects $\Gamma$ at a point $X$ other than $A$. Prove that lines $XD$ and $AM$ meet on $\Gamma$.

*Proposed by Evan Chen, Taiwan*

**G3** Let $B = (-1, 0)$ and $C = (1, 0)$ be fixed points on the coordinate plane. A nonempty, bounded subset $S$ of the plane is said to be *nice* if

(i) there is a point $T$ in $S$ such that for every point $Q$ in $S$, the segment $TQ$ lies entirely in $S$; and

(ii) for any triangle $P_1P_2P_3$, there exists a unique point $A$ in $S$ and a permutation $\sigma$ of the indices $\{1, 2, 3\}$ for which triangles $ABC$ and $P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$ are similar.

Prove that there exist two distinct nice subsets $S$ and $S'$ of the set $\{(x, y) : x \geq 0, y \geq 0\}$ such that if $A \in S$ and $A' \in S'$ are the unique choices of points in (ii), then the product $BA \cdot BA'$ is a constant independent of the triangle $P_1P_2P_3$.

**G4** Let $ABC$ be a triangle with $AB = AC \neq BC$ and let $I$ be its incentre. The line $BI$ meets $AC$ at $D$, and the line through $D$ perpendicular to $AC$ meets $AI$ at $E$. Prove that the reflection of $I$ in $AC$ lies on the circumcircle of triangle $BDE$.

**G5** Let $D$ be the foot of perpendicular from $A$ to the Euler line (the line passing through the circumcentre and the orthocentre) of an acute scalene triangle $ABC$. A circle $\omega$ with centre $S$ passes through $A$ and $D$, and it intersects sides $AB$ and $AC$ at $X$ and $Y$ respectively. Let $P$ be the foot of altitude from $A$ to $BC$, and let $M$ be the midpoint of $BC$. Prove that the circumcentre of triangle $XSY$ is equidistant from $P$ and $M$.

**G6** Let $ABCD$ be a convex quadrilateral with $\angle ABC = \angle ADC < 90^\circ$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet $AC$ at $E$ and $F$ respectively, and meet each other at point $P$. Let $M$ be the midpoint of $AC$ and let $\omega$ be the circumcircle of triangle $BPD$. Segments $BM$ and $DM$ intersect $\omega$ again at $X$ and $Y$ respectively. Denote by $Q$ the intersection point of lines $XE$ and $YF$. Prove that $PQ \perp AC$.

**G7** Let $I$ be the incentre of a non-equilateral triangle $ABC$, $I_A$ be the $A$-excentre, $I'_A$ be the reflection of $I_A$ in $BC$, and $l_A$ be the reflection of line $AI'_A$ in $AI$. Define points $I_B$, $I'_B$ and line $l_B$ analogously. Let $P$ be the intersection point of $l_A$ and $l_B$.

- Prove that $P$ lies on line $OI$ where $O$ is the circumcentre of triangle $ABC$.
- Let one of the tangents from $P$ to the incircle of triangle $ABC$ meet the circumcircle at points $X$ and $Y$. Show that $\angle XIY = 120^\circ$.

**G8** Let $A_1, B_1$ and $C_1$ be points on sides $BC$, $CA$ and $AB$ of an acute triangle $ABC$ respectively, such that $AA_1$, $BB_1$ and $CC_1$ are the internal angle bisectors of triangle $ABC$. Let $I$ be the incentre of triangle $ABC$, and $H$ be the orthocentre of triangle $A_1B_1C_1$. Show that

$$AH + BH + CH \geq AI + BI + CI.$$

– Number Theory (N)

**N1** For any positive integer $k$, denote the sum of digits of $k$ in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n \geq 2016$, the integer $P(n)$ is positive and

$$S(P(n)) = P(S(n)).$$

*Proposed by Warut Suksompong, Thailand*

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**N2** Let $\tau(n)$ be the number of positive divisors of $n$. Let $\tau_1(n)$ be the number of positive divisors of $n$ which have remainders 1 when divided by 3. Find all positive integral values of the fraction $\frac{\tau(10n)}{\tau_1(10n)}$.

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**N3** A set of postive integers is called *fragrant* if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n) = n^2 + n + 1$. What is the least possible positive integer value of $b$ such that there exists a non-negative integer $a$ for which the set

$$\{P(a+1), P(a+2), \ldots, P(a+b)\}$$

is fragrant?

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**N4** Let $n, m, k$ and $l$ be positive integers with $n \neq 1$ such that $n^k + mn^l + 1$ divides $n^{k+l} - 1$. Prove that

-$m = 1$ and $l = 2k$; or
-$l|k$ and $m = \frac{n^{k-l}-1}{n^l-1}$.

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**N5** Let $a$ be a positive integer which is not a perfect square, and consider the equation

$$k = \frac{x^2 - a}{x^2 - y^2}.$$

Let $A$ be the set of positive integers $k$ for which the equation admits a solution in $\mathbb{Z}^2$ with $x > \sqrt{a}$, and let $B$ be the set of positive integers for which the equation admits a solution in $\mathbb{Z}^2$ with $0 \leq x < \sqrt{a}$. Show that $A = B$.

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**N6** Denote by $\mathbb{N}$ the set of all positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers $m$ and $n$, the integer $f(m) + f(n) - mn$ is nonzero and divides $mf(m) + nf(n)$.

*Proposed by Dorlir Ahmeti, Albania*

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**N7** Let $P = A_1A_2 \cdots A_k$ be a convex polygon in the plane. The vertices $A_1, A_2, \ldots, A_k$ have integral coordinates and lie on a circle. Let $S$ be the area of $P$. An odd positive integer $n$ is given such that the squares of the side lengths of $P$ are integers divisible by $n$. Prove that $2S$ is an integer divisible by $n$.

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**N8** Find all polynomials $P(x)$ of odd degree $d$ and with integer coefficients satisfying the following property: for each positive integer $n$, there exists $n$ positive integers $x_1, x_2, \dots, x_n$ such that $\frac{1}{2} < \frac{P(x_i)}{P(x_j)} < 2$ and $\frac{P(x_i)}{P(x_j)}$ is the $d$-th power of a rational number for every pair of indices $i$ and $j$ with $1 \le i, j \le n$.

IMO Shortlist 2015

– Algebra

**A1** Suppose that a sequence $a_1, a_2, \ldots$ of positive real numbers satisfies

$$a_{k+1} \geq \frac{ka_k}{a_k^2 + (k-1)}$$

for every positive integer $k$. Prove that $a_1 + a_2 + \ldots + a_n \geq n$ for every $n \geq 2$.

**A2** Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ with the property that

$$f(x - f(y)) = f(f(x)) - f(y) - 1$$

holds for all $x, y \in \mathbb{Z}$.

**A3** Let $n$ be a fixed positive integer. Find the maximum possible value of

$$\sum_{1 \le r < s \le 2n} (s - r - n)x_r x_s,$$

where $-1 \le x_i \le 1$ for all $i = 1, \cdots, 2n$.

**A4** Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the equation

$$f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x)$$

for all real numbers $x$ and $y$.

Proposed by Dorlir Ahmeti, Albania

**A5** Let $2\mathbb{Z} + 1$ denote the set of odd integers. Find all functions $f : \mathbb{Z} \mapsto 2\mathbb{Z} + 1$ satisfying

$$f(x + f(x) + y) + f(x - f(x) - y) = f(x + y) + f(x - y)$$

for every $x, y \in \mathbb{Z}$.

**A6** Let $n$ be a fixed integer with $n \geq 2$. We say that two polynomials $P$ and $Q$ with real coefficients are *block-similar* if for each $i \in \{1, 2, \ldots, n\}$ the sequences

$$P(2015i), P(2015i-1), \ldots, P(2015i-2014) \quad \text{and}$$
$$Q(2015i), Q(2015i-1), \ldots, Q(2015i-2014)$$

are permutations of each other.

(a) Prove that there exist distinct block-similar polynomials of degree $n+1$.
(b) Prove that there do not exist distinct block-similar polynomials of degree $n$.

*Proposed by David Arthur, Canada*

– Combinatorics

**C1** In Lineland there are $n \geq 1$ towns, arranged along a road running from left to right. Each town has a *left bulldozer* (put to the left of the town and facing left) and a *right bulldozer* (put to the right of the town and facing right). The sizes of the $2n$ bulldozers are distinct. Every time when a left and right bulldozer confront each other, the larger bulldozer pushes the smaller one off the road. On the other hand, bulldozers are quite unprotected at their rears; so, if a bulldozer reaches the rear-end of another one, the first one pushes the second one off the road, regardless of their sizes.

Let $A$ and $B$ be two towns, with $B$ to the right of $A$. We say that town $A$ can *sweep* town $B$ *away* if the right bulldozer of $A$ can move over to $B$ pushing off all bulldozers it meets. Similarly town $B$ can sweep town $A$ away if the left bulldozer of $B$ can move over to $A$ pushing off all bulldozers of all towns on its way.

Prove that there is exactly one town that cannot be swept away by any other one.

**C2** We say that a finite set $\mathcal{S}$ of points in the plane is *balanced* if, for any two different points $A$ and $B$ in $\mathcal{S}$, there is a point $C$ in $\mathcal{S}$ such that $AC = BC$. We say that $\mathcal{S}$ is *centre-free* if for any three different points $A$, $B$ and $C$ in $\mathcal{S}$, there is no points $P$ in $\mathcal{S}$ such that $PA = PB = PC$.

(a) Show that for all integers $n \geq 3$, there exists a balanced set consisting of $n$ points.

(b) Determine all integers $n \geq 3$ for which there exists a balanced centre-free set consisting of $n$ points.

Proposed by Netherlands

**C3** For a finite set $A$ of positive integers, a partition of $A$ into two disjoint nonempty subsets $A_1$ and $A_2$ is *good* if the least common multiple of the elements in $A_1$ is equal to the greatest common divisor of the elements in $A_2$. Determine the minimum value of $n$ such that there exists a set of $n$ positive integers with exactly 2015 good partitions.

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**C4** Let $n$ be a positive integer. Two players $A$ and $B$ play a game in which they take turns choosing positive integers $k \leq n$. The rules of the game are:

(i) A player cannot choose a number that has been chosen by either player on any previous turn.
(ii) A player cannot choose a number consecutive to any of those the player has already chosen on any previous turn.
(iii) The game is a draw if all numbers have been chosen; otherwise the player who cannot choose a number anymore loses the game.

The player $A$ takes the first turn. Determine the outcome of the game, assuming that both players use optimal strategies.

*Proposed by Finland*

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**C5** The sequence $a_1, a_2, \ldots$ of integers satisfies the conditions:

(i) $1 \leq a_j \leq 2015$ for all $j \geq 1$,
(ii) $k + a_k \neq \ell + a_\ell$ for all $1 \leq k < \ell$.

Prove that there exist two positive integers $b$ and $N$ for which

$$\left| \sum_{j=m+1}^{n} (a_j - b) \right| \leq 1007^2$$

for all integers $m$ and $n$ such that $n > m \geq N$.

Proposed by Ivan Guo and Ross Atkins, Australia

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**C6** Let $S$ be a nonempty set of positive integers. We say that a positive integer $n$ is *clean* if it has a unique representation as a sum of an odd number of distinct elements from $S$. Prove that there exist infinitely many positive integers that are not clean.

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**C7** In a company of people some pairs are enemies. A group of people is called *unsociable* if the number of members in the group is odd and at least 3, and it is possible to arrange all its members around a round table so that every two neighbors are enemies. Given that there are at most 2015 unsociable groups, prove that it is possible to partition the company into 11 parts so that no two enemies are in the same part.

*Proposed by Russia*

– Geometry

**G1** Let $ABC$ be an acute triangle with orthocenter $H$. Let $G$ be the point such that the quadrilateral $ABGH$ is a parallelogram. Let $I$ be the point on the line $GH$ such that $AC$ bisects $HI$. Suppose that the line $AC$ intersects the circumcircle of the triangle $GCI$ at $C$ and $J$. Prove that $IJ = AH$.

**G2** Triangle $ABC$ has circumcircle $\Omega$ and circumcenter $O$. A circle $\Gamma$ with center $A$ intersects the segment $BC$ at points $D$ and $E$, such that $B$, $D$, $E$, and $C$ are all different and lie on line $BC$ in this order. Let $F$ and $G$ be the points of intersection of $\Gamma$ and $\Omega$, such that $A$, $F$, $B$, $C$, and $G$ lie on $\Omega$ in this order. Let $K$ be the second point of intersection of the circumcircle of triangle $BDF$ and the segment $AB$. Let $L$ be the second point of intersection of the circumcircle of triangle $CGE$ and the segment $CA$.

Suppose that the lines $FK$ and $GL$ are different and intersect at the point $X$. Prove that $X$ lies on the line $AO$.

Proposed by Greece

**G3** Let $ABC$ be a triangle with $\angle C = 90^\circ$, and let $H$ be the foot of the altitude from $C$. A point $D$ is chosen inside the triangle $CBH$ so that $CH$ bisects $AD$. Let $P$ be the intersection point of the lines $BD$ and $CH$. Let $\omega$ be the semicircle with diameter $BD$ that meets the segment $CB$ at an interior point. A line through $P$ is tangent to $\omega$ at $Q$. Prove that the lines $CQ$ and $AD$ meet on $\omega$.

**G4** Let $ABC$ be an acute triangle and let $M$ be the midpoint of $AC$. A circle $\omega$ passing through $B$ and $M$ meets the sides $AB$ and $BC$ at points $P$ and $Q$ respectively. Let $T$ be the point such that $BPTQ$ is a parallelogram. Suppose that $T$ lies on the circumcircle of $ABC$. Determine all possible values of $\frac{BT}{BM}$.

**G5** Let $ABC$ be a triangle with $CA \neq CB$. Let $D$, $F$, and $G$ be the midpoints of the sides $AB$, $AC$, and $BC$ respectively. A circle $\Gamma$ passing through $C$ and tangent to $AB$ at $D$ meets the segments $AF$ and $BG$ at $H$ and $I$, respectively. The points $H'$ and $I'$ are symmetric to $H$ and $I$ about $F$ and $G$, respectively. The line $H'I'$ meets $CD$ and $FG$ at $Q$ and $M$, respectively. The line $CM$ meets $\Gamma$ again at $P$. Prove that $CQ = QP$.

*Proposed by El Salvador*

**G6** Let $ABC$ be an acute triangle with $AB > AC$. Let $\Gamma$ be its cirumcircle, $H$ its orthocenter, and $F$ the foot of the altitude from $A$. Let $M$ be the midpoint of $BC$. Let $Q$ be the point on $\Gamma$ such that $\angle HQA = 90^\circ$ and let $K$ be the point on $\Gamma$ such that $\angle HKQ = 90^\circ$. Assume that the points $A$, $B$, $C$, $K$ and $Q$ are all different and lie on $\Gamma$ in this order.

Prove that the circumcircles of triangles $KQH$ and $FKM$ are tangent to each other.

Proposed by Ukraine

**G7** Let $ABCD$ be a convex quadrilateral, and let $P$, $Q$, $R$, and $S$ be points on the sides $AB$, $BC$, $CD$, and $DA$, respectively. Let the line segment $PR$ and $QS$ meet at $O$. Suppose that each of the quadrilaterals $APOS$, $BQOP$, $CROQ$, and $DSOR$ has an incircle. Prove that the lines $AC$, $PQ$, and $RS$ are either concurrent or parallel to each other.

**G8** A *triangulation* of a convex polygon $\Pi$ is a partitioning of $\Pi$ into triangles by diagonals having no common points other than the vertices of the polygon. We say that a triangulation is a *Thaiangulation* if all triangles in it have the same area.

Prove that any two different Thaiangulations of a convex polygon $\Pi$ differ by exactly two triangles. (In other words, prove that it is possible to replace one pair of triangles in the first Thaiangulation with a different pair of triangles so as to obtain the second Thaiangulation.)

*Proposed by Bulgaria*

– Number Theory

**N1** Determine all positive integers $M$ such that the sequence $a_0, a_1, a_2, \cdots$ defined by

$$a_0 = M + \frac{1}{2} \qquad \text{and} \qquad a_{k+1} = a_k \lfloor a_k \rfloor \quad \text{for } k = 0, 1, 2, \cdots$$

contains at least one integer term.

**N2** Let $a$ and $b$ be positive integers such that $a! + b!$ divides $a!b!$. Prove that $3a \geq 2b + 2$.

**N3** Let $m$ and $n$ be positive integers such that $m > n$. Define $x_k = \frac{m+k}{n+k}$ for $k = 1, 2, \ldots, n+1$. Prove that if all the numbers $x_1, x_2, \ldots, x_{n+1}$ are integers, then $x_1 x_2 \ldots x_{n+1} - 1$ is divisible by an odd prime.

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**N4** Suppose that $a_0, a_1, \cdots$ and $b_0, b_1, \cdots$ are two sequences of positive integers such that $a_0, b_0 \geq 2$ and

$$a_{n+1} = \gcd(a_n, b_n) + 1, \qquad b_{n+1} = \operatorname{lcm}(a_n, b_n) - 1.$$

Show that the sequence $a_n$ is eventually periodic; in other words, there exist integers $N \geq 0$ and $t > 0$ such that $a_{n+t} = a_n$ for all $n \geq N$.

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**N5** Find all postive integers $(a, b, c)$ such that

$$ab - c, \quad bc - a, \quad ca - b$$

are all powers of 2.

*Proposed by Serbia*

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**N6** Let $\mathbb{Z}_{>0}$ denote the set of positive integers. Consider a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$. For any $m, n \in \mathbb{Z}_{>0}$ we write $f^n(m) = \underbrace{f(f(\ldots f}_{n}(m)\ldots))$. Suppose that $f$ has the following two properties:

(i) if $m, n \in \mathbb{Z}_{>0}$, then $\frac{f^n(m) - m}{n} \in \mathbb{Z}_{>0}$;
(ii) The set $\mathbb{Z}_{>0} \setminus \{f(n) \mid n \in \mathbb{Z}_{>0}\}$ is finite.

Prove that the sequence $f(1) - 1, f(2) - 2, f(3) - 3, \ldots$ is periodic.

*Proposed by Ang Jie Jun, Singapore*

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**N7** Let $\mathbb{Z}_{>0}$ denote the set of positive integers. For any positive integer $k$, a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ is called [i]$k$-good[/i] if $\gcd(f(m) + n, f(n) + m) \leq k$ for all $m \neq n$. Find all $k$ such that there exists a $k$-good function.

*Proposed by James Rickards, Canada*

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**N8** For every positive integer $n$ with prime factorization $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, define

$$\mho(n) = \sum_{i:\, p_i > 10^{100}} \alpha_i.$$

That is, $\mho(n)$ is the number of prime factors of $n$ greater than $10^{100}$, counted with multiplicity.

Find all strictly increasing functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$\mho(f(a) - f(b)) \leq \mho(a - b) \quad \text{for all integers } a \text{ and } b \text{ with } a > b.$$

*Proposed by Rodrigo Sanches Angelo, Brazil*

IMO Shortlist 2014

– Algebra

**A1** Let $a_0 < a_1 < a_2 \ldots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + a_2 + \cdots + a_n}{n} \leq a_{n+1}.$$

*Proposed by Gerhard Wginger, Austria.*

**A2** Define the function $f : (0, 1) \to (0, 1)$ by

$$f(x) = \begin{cases} x + \frac{1}{2} & \text{if } \ x < \frac{1}{2} \\ x^2 & \text{if } \ x \geq \frac{1}{2} \end{cases}$$

Let $a$ and $b$ be two real numbers such that $0 < a < b < 1$. We define the sequences $a_n$ and $b_n$ by $a_0 = a, b_0 = b$, and $a_n = f(a_{n-1})$, $b_n = f(b_{n-1})$ for $n > 0$. Show that there exists a positive integer $n$ such that

$$(a_n - a_{n-1})(b_n - b_{n-1}) < 0.$$

*Proposed by Denmark*

**A3** For a sequence $x_1, x_2, \ldots, x_n$ of real numbers, we define its *price* as

$$\max_{1 \leq i \leq n} |x_1 + \cdots + x_i|.$$

Given $n$ real numbers, Dave and George want to arrange them into a sequence with a low price. Diligent Dave checks all possible ways and finds the minimum possible price $D$. Greedy George, on the other hand, chooses $x_1$ such that $|x_1|$ is as small as possible; among the remaining numbers, he chooses $x_2$ such that $|x_1 + x_2|$ is as small as possible, and so on. Thus, in the $i$-th step he chooses $x_i$ among the remaining numbers so as to minimise the value of $|x_1 + x_2 + \cdots x_i|$. In each step, if several numbers provide the same value, George chooses one at random. Finally he gets a sequence with price $G$.

Find the least possible constant $c$ such that for every positive integer $n$, for every collection of $n$ real numbers, and for every possible sequence that George might obtain, the resulting values satisfy the inequality $G \leq cD$.

*Proposed by Georgia*

**A4** Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f\big(f(m)+n\big)+f(m)=f(n)+f(3m)+2014$$

for all integers $m$ and $n$.

*Proposed by Netherlands*

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**A5** Consider all polynomials $P(x)$ with real coefficients that have the following property: for any two real numbers $x$ and $y$ one has

$$|y^2 - P(x)| \le 2|x| \quad \text{if and only if} \quad |x^2 - P(y)| \le 2|y|.$$

Determine all possible values of $P(0)$.

*Proposed by Belgium*

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**A6** Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$n^2 + 4f(n) = f(f(n))^2$$

for all $n \in \mathbb{Z}$.

*Proposed by Sahl Khan, UK*

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– Combinatorics

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**C1** Let $n$ points be given inside a rectangle $R$ such that no two of them lie on a line parallel to one of the sides of $R$. The rectangle $R$ is to be dissected into smaller rectangles with sides parallel to the sides of $R$ in such a way that none of these rectangles contains any of the given points in its interior. Prove that we have to dissect $R$ into at least $n+1$ smaller rectangles.

*Proposed by Serbia*

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**C2** We have $2^m$ sheets of paper, with the number 1 written on each of them. We perform the following operation. In every step we choose two distinct sheets; if the numbers on the two sheets are $a$ and $b$, then we erase these numbers and write the number $a+b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least $4^m$ .

*Proposed by Abbas Mehrabian, Iran*

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**C3** Let $n \ge 2$ be an integer. Consider an $n \times n$ chessboard consisting of $n^2$ unit squares. A configuration of $n$ rooks on this board is *peaceful* if every row and every column contains exactly one rook. Find the greatest positive integer $k$

such that, for each peaceful configuration of $n$ rooks, there is a $k \times k$ square which does not contain a rook on any of its $k^2$ unit squares.

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**C4** Construct a tetromino by attaching two $2 \times 1$ dominoes along their longer sides such that the midpoint of the longer side of one domino is a corner of the other domino. This construction yields two kinds of tetrominoes with opposite orientations. Let us call them $S$- and $Z$-tetrominoes, respectively.
Assume that a lattice polygon $P$ can be tiled with $S$-tetrominoes. Prove that no matter how we tile $P$ using only $S$- and $Z$-tetrominoes, we always use an even number of $Z$-tetrominoes.

*Proposed by Tamas Fleiner and Peter Pal Pach, Hungary*

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**C5** A set of lines in the plane is in *general position* if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite area; we call these its *finite regions*. Prove that for all sufficiently large $n$, in any set of $n$ lines in general position it is possible to colour at least $\sqrt{n}$ lines blue in such a way that none of its finite regions has a completely blue boundary.

*Note*: Results with $\sqrt{n}$ replaced by $c\sqrt{n}$ will be awarded points depending on the value of the constant $c$.

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**C6** We are given an infinite deck of cards, each with a real number on it. For every real number $x$, there is exactly one card in the deck that has $x$ written on it. Now two players draw disjoint sets $A$ and $B$ of 100 cards each from this deck. We would like to define a rule that declares one of them a winner. This rule should satisfy the following conditions:
1. The winner only depends on the relative order of the 200 cards: if the cards are laid down in increasing order face down and we are told which card belongs to which player, but not what numbers are written on them, we can still decide the winner.
2. If we write the elements of both sets in increasing order as $A = \{a_1, a_2, \ldots, a_{100}\}$ and $B = \{b_1, b_2, \ldots, b_{100}\}$, and $a_i > b_i$ for all $i$, then $A$ beats $B$.
3. If three players draw three disjoint sets $A, B, C$ from the deck, $A$ beats $B$ and $B$ beats $C$ then $A$ also beats $C$.
How many ways are there to define such a rule? Here, we consider two rules as different if there exist two sets $A$ and $B$ such that $A$ beats $B$ according to one rule, but $B$ beats $A$ according to the other.

*Proposed by Ilya Bogdanov, Russia*

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**C7** Let $M$ be a set of $n \geq 4$ points in the plane, no three of which are collinear. Initially these points are connected with $n$ segments so that each point in $M$

is the endpoint of exactly two segments. Then, at each step, one may choose two segments $AB$ and $CD$ sharing a common interior point and replace them by the segments $AC$ and $BD$ if none of them is present at this moment. Prove that it is impossible to perform $n^3/4$ or more such moves.

*Proposed by Vladislav Volkov, Russia*

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**C8** A card deck consists of 1024 cards. On each card, a set of distinct decimal digits is
written in such a way that no two of these sets coincide (thus, one of the cards is empty). Two players alternately take cards from the deck, one card per turn. After the deck is empty, each player checks if he can throw out one of his cards so that each of the ten digits occurs on an even number of his remaining cards. If one player can do this but the other one cannot, the one who can is the winner; otherwise a draw is declared.
Determine all possible first moves of the first player after which he has a winning strategy.

*Proposed by Ilya Bogdanov & Vladimir Bragin, Russia*

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**C9** There are $n$ circles drawn on a piece of paper in such a way that any two circles intersect in two points, and no three circles pass through the same point. Turbo the snail slides along the circles in the following fashion. Initially he moves on one of the circles in clockwise direction. Turbo always keeps sliding along the current circle until he reaches an intersection with another circle. Then he continues his journey on this new circle and also changes the direction of moving, i.e. from clockwise to anticlockwise or *vice versa*.
Suppose that Turbos path entirely covers all circles. Prove that $n$ must be odd.

*Proposed by Tejaswi Navilarekallu, India*

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– Geometry

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**G1** Let $P$ and $Q$ be on segment $BC$ of an acute triangle $ABC$ such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let $M$ and $N$ be the points on $AP$ and $AQ$, respectively, such that $P$ is the midpoint of $AM$ and $Q$ is the midpoint of $AN$. Prove that the intersection of $BM$ and $CN$ is on the circumference of triangle $ABC$.

*Proposed by Giorgi Arabidze, Georgia.*

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**G2** Let $ABC$ be a triangle. The points $K, L$, and $M$ lie on the segments $BC, CA$, and $AB$, respectively, such that the lines $AK, BL$, and $CM$ intersect in a common point. Prove that it is possible to choose two of the triangles $ALM, BMK$, and $CKL$ whose inradii sum up to at least the inradius of the triangle $ABC$.

*Proposed by Estonia*

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**G3** Let $\Omega$ and $O$ be the circumcircle and the circumcentre of an acute-angled triangle $ABC$ with $AB > BC$. The angle bisector of $\angle ABC$ intersects $\Omega$ at $M \neq B$. Let $\Gamma$ be the circle with diameter $BM$. The angle bisectors of $\angle AOB$ and $\angle BOC$ intersect $\Gamma$ at points $P$ and $Q$, respectively. The point $R$ is chosen on the line $PQ$ so that $BR = MR$. Prove that $BR \parallel AC$.
(Here we always assume that an angle bisector is a ray.)

*Proposed by Sergey Berlov, Russia*

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**G4** Consider a fixed circle $\Gamma$ with three fixed points $A, B,$ and $C$ on it. Also, let us fix a real number $\lambda \in (0, 1)$. For a variable point $P \notin \{A, B, C\}$ on $\Gamma$, let $M$ be the point on the segment $CP$ such that $CM = \lambda \cdot CP$ . Let $Q$ be the second point of intersection of the circumcircles of the triangles $AMP$ and $BMC$. Prove that as $P$ varies, the point $Q$ lies on a fixed circle.

*Proposed by Jack Edward Smith, UK*

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**G5** Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90^\circ$. Point $H$ is the foot of the perpendicular from $A$ to $BD$. Points $S$ and $T$ lie on sides $AB$ and $AD$, respectively, such that $H$ lies inside triangle $SCT$ and

$$\angle CHS - \angle CSB = 90^\circ, \quad \angle THC - \angle DTC = 90^\circ.$$

Prove that line $BD$ is tangent to the circumcircle of triangle $TSH$.

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**G6** Let $ABC$ be a fixed acute-angled triangle. Consider some points $E$ and $F$ lying on the sides $AC$ and $AB$, respectively, and let $M$ be the midpoint of $EF$ . Let the perpendicular bisector of $EF$ intersect the line $BC$ at $K$, and let the perpendicular bisector of $MK$ intersect the lines $AC$ and $AB$ at $S$ and $T$ , respectively. We call the pair $(E, F)$ *interesting*, if the quadrilateral $KSAT$ is cyclic.

Suppose that the pairs $(E_1, F_1)$ and $(E_2, F_2)$ are interesting. Prove that $\frac{E_1E_2}{AB} = \frac{F_1F_2}{AC}$

*Proposed by Ali Zamani, Iran*

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**G7** Let $ABC$ be a triangle with circumcircle $\Omega$ and incentre $I$. Let the line passing through $I$ and perpendicular to $CI$ intersect the segment $BC$ and the arc $BC$

(not containing $A$) of $\Omega$ at points $U$ and $V$ , respectively. Let the line passing through $U$ and parallel to $AI$ intersect $AV$ at $X$, and let the line passing through $V$ and parallel to $AI$ intersect $AB$ at $Y$ . Let $W$ and $Z$ be the midpoints of $AX$ and $BC$, respectively. Prove that if the points $I, X$, and $Y$ are collinear, then the points $I, W$, and $Z$ are also collinear.

*Proposed by David B. Rush, USA*

– Number Theory

**N1** Let $n \geq 2$ be an integer, and let $A_n$ be the set

$$A_n = \{2^n - 2^k \mid k \in \mathbb{Z},\, 0 \leq k < n\}.$$

Determine the largest positive integer that cannot be written as the sum of one or more (not necessarily distinct) elements of $A_n$ .

*Proposed by Serbia*

**N2** Determine all pairs $(x, y)$ of positive integers such that

$$\sqrt[3]{7x^2 - 13xy + 7y^2} = |x - y| + 1.$$

*Proposed by Titu Andreescu, USA*

**N3** For each positive integer $n$, the Bank of Cape Town issues coins of denomination $\frac{1}{n}$. Given a finite collection of such coins (of not necessarily different denominations) with total value at most most $99 + \frac{1}{2}$, prove that it is possible to split this collection into 100 or fewer groups, such that each group has total value at most 1.

**N4** Let $n > 1$ be a given integer. Prove that infinitely many terms of the sequence $(a_k)_{k\geq 1}$, defined by

$$a_k = \left\lfloor \frac{n^k}{k} \right\rfloor,$$

are odd. (For a real number $x$, $\lfloor x \rfloor$ denotes the largest integer not exceeding $x$.)

*Proposed by Hong Kong*

**N5** Find all triples $(p, x, y)$ consisting of a prime number $p$ and two positive integers $x$ and $y$ such that $x^{p-1} + y$ and $x + y^{p-1}$ are both powers of $p$.

*Proposed by Belgium*

**N6** Let $a_1 < a_2 < \cdots < a_n$ be pairwise coprime positive integers with $a_1$ being prime and $a_1 \ge n+2$. On the segment $I = [0, a_1 a_2 \cdots a_n]$ of the real line, mark all integers that are divisible by at least one of the numbers $a_1, \ldots, a_n$ . These points split $I$ into a number of smaller segments. Prove that the sum of the squares of the lengths of these segments is divisible by $a_1$.

*Proposed by Serbia*

**N7** Let $c \ge 1$ be an integer. Define a sequence of positive integers by $a_1 = c$ and

$$a_{n+1} = a_n^3 - 4c \cdot a_n^2 + 5c^2 \cdot a_n + c$$

for all $n \ge 1$. Prove that for each integer $n \ge 2$ there exists a prime number $p$ dividing $a_n$ but none of the numbers $a_1, \ldots, a_{n-1}$ .

*Proposed by Austria*

**N8** For every real number $x$, let $||x||$ denote the distance between $x$ and the nearest integer.

Prove that for every pair $(a, b)$ of positive integers there exist an odd prime $p$ and a positive integer $k$ satisfying

$$\left|\left|\frac{a}{p^k}\right|\right| + \left|\left|\frac{b}{p^k}\right|\right| + \left|\left|\frac{a+b}{p^k}\right|\right| = 1.$$

*Proposed by Geza Kos, Hungary*

IMO Shortlist 2013

– Algebra

**A1** Let $n$ be a positive integer and let $a_1, \dots, a_{n-1}$ be arbitrary real numbers. Define the sequences $u_0, \dots, u_n$ and $v_0, \dots, v_n$ inductively by $u_0 = u_1 = v_0 = v_1 = 1$, and $u_{k+1} = u_k + a_k u_{k-1}$, $v_{k+1} = v_k + a_{n-k} v_{k-1}$ for $k = 1, \dots, n-1$.

Prove that $u_n = v_n$.

**A2** Prove that in any set of 2000 distinct real numbers there exist two pairs $a > b$ and $c > d$ with $a \neq c$ or $b \neq d$, such that

$$\left| \frac{a-b}{c-d} - 1 \right| < \frac{1}{100000}.$$

**A3** Let $\mathbb{Q}_{>0}$ be the set of all positive rational numbers. Let $f : \mathbb{Q}_{>0} \to \mathbb{R}$ be a function satisfying the following three conditions:

(i) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x)f(y) \geq f(xy)$;
(ii) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x+y) \geq f(x) + f(y)$;
(iii) there exists a rational number $a > 1$ such that $f(a) = a$.

Prove that $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$.

*Proposed by Bulgaria*

**A4** Let $n$ be a positive integer, and consider a sequence $a_1, a_2, \cdots, a_n$ of positive integers. Extend it periodically to an infinite sequence $a_1, a_2, \cdots$ by defining $a_{n+i} = a_i$ for all $i \geq 1$. If

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq a_1 + n$$

and

$$a_{a_i} \leq n + i - 1 \quad \text{for} \quad i = 1, 2, \cdots, n,$$

prove that

$$a_1 + \cdots + a_n \leq n^2.$$

**A5** Let $\mathbb{Z}_{\ge 0}$ be the set of all nonnegative integers. Find all the functions $f : \mathbb{Z}_{\ge 0} \rightarrow \mathbb{Z}_{\ge 0}$ satisfying the relation

$$f(f(f(n))) = f(n+1) + 1$$

for all $n \in \mathbb{Z}_{\ge 0}$.

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**A6** Let $m \neq 0$ be an integer. Find all polynomials $P(x)$ with real coefficients such that

$$(x^3 - mx^2 + 1)P(x+1) + (x^3 + mx^2 + 1)P(x-1) = 2(x^3 - mx + 1)P(x)$$

for all real number $x$.

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– Combinatorics

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**C1** Let $n$ be an positive integer. Find the smallest integer $k$ with the following property; Given any real numbers $a_1, \cdots, a_d$ such that $a_1 + a_2 + \cdots + a_d = n$ and $0 \le a_i \le 1$ for $i = 1, 2, \cdots, d$, it is possible to partition these numbers into $k$ groups (some of which may be empty) such that the sum of the numbers in each group is at most 1.

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**C2** A configuration of 4027 points in the plane is called Colombian if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is good for a Colombian configuration if the following two conditions are satisfied:

i) No line passes through any point of the configuration.

ii) No region contains points of both colors.

Find the least value of $k$ such that for any Colombian configuration of 4027 points, there is a good arrangement of $k$ lines.

Proposed by *Ivan Guo* from *Australia.*

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**C3** A crazy physicist discovered a new kind of particle wich he called an imon, after some of them mysteriously appeared in his lab. Some pairs of imons in the lab can be entangled, and each imon can participate in many entanglement relations. The physicist has found a way to perform the following two kinds of operations with these particles, one operation at a time.
(i) If some imon is entangled with an odd number of other imons in the lab, then the physicist can destroy it.
(ii) At any moment, he may double the whole family of imons in the lab by

creating a copy $I'$ of each imon $I$. During this procedure, the two copies $I'$ and $J'$ become entangled if and only if the original imons $I$ and $J$ are entangled, and each copy $I'$ becomes entangled with its original imon $I$; no other entanglements occur or disappear at this moment.

Prove that the physicist may apply a sequence of much operations resulting in a family of imons, no two of which are entangled.

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**C4** Let $n$ be a positive integer, and let $A$ be a subset of $\{1, \cdots, n\}$. An $A$-partition of $n$ into $k$ parts is a representation of n as a sum $n = a_1 + \cdots + a_k$, where the parts $a_1, \cdots, a_k$ belong to $A$ and are not necessarily distinct. The number of different parts in such a partition is the number of (distinct) elements in the set $\{a_1, a_2, \cdots, a_k\}$.
We say that an $A$-partition of $n$ into $k$ parts is optimal if there is no $A$-partition of $n$ into $r$ parts with $r < k$. Prove that any optimal $A$-partition of $n$ contains at most $\sqrt[3]{6n}$ different parts.

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**C5** Let $r$ be a positive integer, and let $a_0, a_1, \cdots$ be an infinite sequence of real numbers. Assume that for all nonnegative integers $m$ and $s$ there exists a positive integer $n \in [m+1, m+r]$ such that

$$a_m + a_{m+1} + \cdots + a_{m+s} = a_n + a_{n+1} + \cdots + a_{n+s}$$

Prove that the sequence is periodic, i.e. there exists some $p \geq 1$ such that $a_{n+p} = a_n$ for all $n \geq 0$.

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**C6** In some country several pairs of cities are connected by direct two-way flights. It is possible to go from any city to any other by a sequence of flights. The distance between two cities is defined to be the least possible numbers of flights required to go from one of them to the other. It is known that for any city there are at most 100 cities at distance exactly three from it. Prove that there is no city such that more than 2550 other cities have distance exactly four from it.

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**C7** Let $n \geq 3$ be an integer, and consider a circle with $n+1$ equally spaced points marked on it. Consider all labellings of these points with the numbers $0, 1, ..., n$ such that each label is used exactly once; two such labellings are considered to be the same if one can be obtained from the other by a rotation of the circle. A labelling is called *beautiful* if, for any four labels $a < b < c < d$ with $a+d = b+c$, the chord joining the points labelled $a$ and $d$ does not intersect the chord joining the points labelled $b$ and $c$.

Let $M$ be the number of beautiful labelings, and let N be the number of ordered pairs $(x, y)$ of positive integers such that $x + y \leq n$ and $\gcd(x, y) = 1$. Prove that

$$M = N + 1.$$

**C8** Players $A$ and $B$ play a "paintful" game on the real line. Player $A$ has a pot of paint with four units of black ink. A quantity $p$ of this ink suffices to blacken a (closed) real interval of length $p$. In every round, player $A$ picks some positive integer $m$ and provides $1/2^m$ units of ink from the pot. Player $B$ then picks an integer $k$ and blackens the interval from $k/2^m$ to $(k+1)/2^m$ (some parts of this interval may have been blackened before). The goal of player $A$ is to reach a situation where the pot is empty and the interval $[0,1]$ is not completely blackened.
Decide whether there exists a strategy for player $A$ to win in a finite number of moves.

– Geometry

**G1** Let $ABC$ be an acute triangle with orthocenter $H$, and let $W$ be a point on the side $BC$, lying strictly between $B$ and $C$. The points $M$ and $N$ are the feet of the altitudes from $B$ and $C$, respectively. Denote by $\omega_1$ is the circumcircle of $BWN$, and let $X$ be the point on $\omega_1$ such that $WX$ is a diameter of $\omega_1$. Analogously, denote by $\omega_2$ the circumcircle of triangle $CWM$, and let $Y$ be the point such that $WY$ is a diameter of $\omega_2$. Prove that $X, Y$ and $H$ are collinear.

*Proposed by Warut Suksompong and Potcharapol Suteparuk, Thailand*

**G2** Let $\omega$ be the circumcircle of a triangle $ABC$. Denote by $M$ and $N$ the midpoints of the sides $AB$ and $AC$, respectively, and denote by $T$ the midpoint of the arc $BC$ of $\omega$ not containing $A$. The circumcircles of the triangles $AMT$ and $ANT$ intersect the perpendicular bisectors of $AC$ and $AB$ at points $X$ and $Y$, respectively; assume that $X$ and $Y$ lie inside the triangle $ABC$. The lines $MN$ and $XY$ intersect at $K$. Prove that $KA = KT$.

**G3** In a triangle $ABC$, let $D$ and $E$ be the feet of the angle bisectors of angles $A$ and $B$, respectively. A rhombus is inscribed into the quadrilateral $AEDB$ (all vertices of the rhombus lie on different sides of $AEDB$). Let $\varphi$ be the non-obtuse angle of the rhombus. Prove that $\varphi \le \max\{\angle BAC, \angle ABC\}$.

**G4** Let $ABC$ be a triangle with $\angle B > \angle C$. Let $P$ and $Q$ be two different points on line $AC$ such that $\angle PBA = \angle QBA = \angle ACB$ and $A$ is located between $P$ and $C$. Suppose that there exists an interior point $D$ of segment $BQ$ for which

$PD = PB$. Let the ray $AD$ intersect the circle $ABC$ at $R \neq A$. Prove that $QB = QR$.

**G5** Let $ABCDEF$ be a convex hexagon with $AB = DE$, $BC = EF$, $CD = FA$, and $\angle A - \angle D = \angle C - \angle F = \angle E - \angle B$. Prove that the diagonals $AD$, $BE$, and $CF$ are concurrent.

**G6** Let the excircle of triangle $ABC$ opposite the vertex $A$ be tangent to the side $BC$ at the point $A_1$. Define the points $B_1$ on $CA$ and $C_1$ on $AB$ analogously, using the excircles opposite $B$ and $C$, respectively. Suppose that the circumcentre of triangle $A_1B_1C_1$ lies on the circumcircle of triangle $ABC$. Prove that triangle $ABC$ is right-angled.

*Proposed by Alexander A. Polyansky, Russia*

– Number Theory

**N1** Let $\mathbb{Z}_{>0}$ be the set of positive integers. Find all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that

$$m^2 + f(n) \mid mf(m) + n$$

for all positive integers $m$ and $n$.

**N2** Assume that $k$ and $n$ are two positive integers. Prove that there exist positive integers $m_1, \ldots, m_k$ such that

$$1 + \frac{2^k - 1}{n} = \left(1 + \frac{1}{m_1}\right) \cdots \left(1 + \frac{1}{m_k}\right).$$

*Proposed by Japan*

**N3** Prove that there exist infinitely many positive integers $n$ such that the largest prime divisor of $n^4 + n^2 + 1$ is equal to the largest prime divisor of $(n+1)^4 + (n+1)^2 + 1$.

**N4** Determine whether there exists an infinite sequence of nonzero digits $a_1, a_2, a_3, \cdots$ and a positive integer $N$ such that for every integer $k > N$, the number $\overline{a_k a_{k-1} \cdots a_1}$ is a perfect square.

**N5** Fix an integer $k > 2$. Two players, called Ana and Banana, play the following game of numbers. Initially, some integer $n \geq k$ gets written on the blackboard. Then they take moves in turn, with Ana beginning. A player making a move erases the number $m$ just written on the blackboard and replaces it by some number $m'$ with $k \leq m' < m$ that is coprime to $m$. The first player who cannot move anymore loses.

An integer $n \geq k$ is called good if Banana has a winning strategy when the initial number is $n$, and bad otherwise.

Consider two integers $n, n' \geq k$ with the property that each prime number $p \leq k$ divides $n$ if and only if it divides $n'$. Prove that either both $n$ and $n'$ are good or both are bad.

**N6** Determine all functions $f : \mathbb{Q} \to \mathbb{Z}$ satisfying

$$f\left(\frac{f(x)+a}{b}\right) = f\left(\frac{x+a}{b}\right)$$

for all $x \in \mathbb{Q}$, $a \in \mathbb{Z}$, and $b \in \mathbb{Z}_{>0}$. (Here, $\mathbb{Z}_{>0}$ denotes the set of positive integers.)

**N7** Let $\nu$ be an irrational positive number, and let $m$ be a positive integer. A pair of $(a, b)$ of positive integers is called *good* if

$$a\lceil b\nu \rceil - b\lfloor a\nu \rfloor = m.$$

A good pair $(a, b)$ is called *excellent* if neither of the pair $(a-b, b)$ and $(a, b-a)$ is good.

Prove that the number of excellent pairs is equal to the sum of the positive divisors of $m$.

IMO Shortlist 2012

| | |
|---|---|
| – | Algebra |
| **A1** | Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers $a, b, c$ that satisfy $a + b + c = 0$, the following equality holds: $$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$ (Here $\mathbb{Z}$ denotes the set of integers.) *Proposed by Liam Baker, South Africa* |
| **A2** | Let $\mathbb{Z}$ and $\mathbb{Q}$ be the sets of integers and rationals respectively.<br>a) Does there exist a partition of $\mathbb{Z}$ into three non-empty subsets $A, B, C$ such that the sets $A+B, B+C, C+A$ are disjoint?<br>b) Does there exist a partition of $\mathbb{Q}$ into three non-empty subsets $A, B, C$ such that the sets $A+B, B+C, C+A$ are disjoint?<br>Here $X + Y$ denotes the set $\{x + y : x \in X, y \in Y\}$, for $X, Y \subseteq \mathbb{Z}$ and for $X, Y \subseteq \mathbb{Q}$. |
| **A3** | Let $n \geq 3$ be an integer, and let $a_2, a_3, \ldots, a_n$ be positive real numbers such that $a_2a_3 \cdots a_n = 1$. Prove that $$(1 + a_2)^2(1 + a_3)^3 \cdots (1 + a_n)^n > n^n.$$ *Proposed by Angelo Di Pasquale, Australia* |
| **A4** | Let $f$ and $g$ be two nonzero polynomials with integer coefficients and $\deg f > \deg g$. Suppose that for infinitely many primes $p$ the polynomial $pf + g$ has a rational root. Prove that $f$ has a rational root. |
| **A5** | Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the conditions $$f(1 + xy) - f(x + y) = f(x)f(y) \quad \text{for all } x, y \in \mathbb{R},$$ and $f(-1) \neq 0$. |
| **A6** | Let $f : \mathbb{N} \to \mathbb{N}$ be a function, and let $f^m$ be $f$ applied $m$ times. Suppose that for every $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $f^{2k}(n) = n + k$, and let $k_n$ be the smallest such $k$. Prove that the sequence $k_1, k_2, \ldots$ is unbounded. |

**A7** We say that a function $f : \mathbb{R}^k \to \mathbb{R}$ is a metapolynomial if, for some positive integers $m$ and $n$, it can be represented in the form

$$f(x_1, \cdots, x_k) = \max_{i=1,\cdots,m} \min_{j=1,\cdots,n} P_{i,j}(x_1, \cdots, x_k),$$

where $P_{i,j}$ are multivariate polynomials. Prove that the product of two metapolynomials is also a metapolynomial.

– Combinatorics

**C1** Several positive integers are written in a row. Iteratively, Alice chooses two adjacent numbers $x$ and $y$ such that $x > y$ and $x$ is to the left of $y$, and replaces the pair $(x, y)$ by either $(y+1, x)$ or $(x-1, x)$. Prove that she can perform only finitely many such iterations.

*Proposed by Warut Suksompong, Thailand*

**C2** Let $n \geq 1$ be an integer. What is the maximum number of disjoint pairs of elements of the set $\{1, 2, \ldots, n\}$ such that the sums of the different pairs are different integers not exceeding $n$?

**C3** In a $999 \times 999$ square table some cells are white and the remaining ones are red. Let $T$ be the number of triples $(C_1, C_2, C_3)$ of cells, the first two in the same row and the last two in the same column, with $C_1, C_3$ white and $C_2$ red. Find the maximum value $T$ can attain.

*Proposed by Merlijn Staps, The Netherlands*

**C4** Players $A$ and $B$ play a game with $N \geq 2012$ coins and 2012 boxes arranged around a circle. Initially $A$ distributes the coins among the boxes so that there is at least 1 coin in each box. Then the two of them make moves in the order $B, A, B, A, \ldots$ by the following rules:
**(a)** On every move of his $B$ passes 1 coin from every box to an adjacent box.
**(b)** On every move of hers $A$ chooses several coins that were *not* involved in $B$'s previous move and are in different boxes. She passes every coin to and adjacent box.
Player $A$'s goal is to ensure at least 1 coin in each box after every move of hers, regardless of how $B$ plays and how many moves are made. Find the least $N$ that enables her to succeed.

**C5** The columns and the row of a $3n \times 3n$ square board are numbered $1, 2, \ldots, 3n$. Every square $(x, y)$ with $1 \leq x, y \leq 3n$ is colored asparagus, byzantium or citrine according as the modulo 3 remainder of $x + y$ is $0, 1$ or $2$ respectively.

One token colored asparagus, byzantium or citrine is placed on each square, so that there are $3n^2$ tokens of each color.
Suppose that one can permute the tokens so that each token is moved to a distance of at most $d$ from its original position, each asparagus token replaces a byzantium token, each byzantium token replaces a citrine token, and each citrine token replaces an asparagus token. Prove that it is possible to permute the tokens so that each token is moved to a distance of at most $d+2$ from its original position, and each square contains a token with the same color as the square.

**C6** The *liar's guessing game* is a game played between two players $A$ and $B$. The rules of the game depend on two positive integers $k$ and $n$ which are known to both players.

At the start of the game $A$ chooses integers $x$ and $N$ with $1 \le x \le N$. Player $A$ keeps $x$ secret, and truthfully tells $N$ to player $B$. Player $B$ now tries to obtain information about $x$ by asking player $A$ questions as follows: each question consists of $B$ specifying an arbitrary set $S$ of positive integers (possibly one specified in some previous question), and asking $A$ whether $x$ belongs to $S$. Player $B$ may ask as many questions as he wishes. After each question, player $A$ must immediately answer it with *yes* or *no*, but is allowed to lie as many times as she wants; the only restriction is that, among any $k+1$ consecutive answers, at least one answer must be truthful.

After $B$ has asked as many questions as he wants, he must specify a set $X$ of at most $n$ positive integers. If $x$ belongs to $X$, then $B$ wins; otherwise, he loses. Prove that:

1. If $n \ge 2^k$, then $B$ can guarantee a win.
2. For all sufficiently large $k$, there exists an integer $n \ge (1.99)^k$ such that $B$ cannot guarantee a win.

*Proposed by David Arthur, Canada*

**C7** There are given $2^{500}$ points on a circle labeled $1, 2, \dots, 2^{500}$ in some order. Prove that one can choose 100 pairwise disjoint chords joining some of theses points so that the 100 sums of the pairs of numbers at the endpoints of the chosen chord are equal.

– Geometry

**G1** Given triangle $ABC$ the point $J$ is the centre of the excircle opposite the vertex $A$. This excircle is tangent to the side $BC$ at $M$, and to the lines $AB$ and $AC$ at $K$ and $L$, respectively. The lines $LM$ and $BJ$ meet at $F$, and the lines $KM$

and $CJ$ meet at $G$. Let $S$ be the point of intersection of the lines $AF$ and $BC$, and let $T$ be the point of intersection of the lines $AG$ and $BC$. Prove that $M$ is the midpoint of $ST$.

(The *excircle* of $ABC$ opposite the vertex $A$ is the circle that is tangent to the line segment $BC$, to the ray $AB$ beyond $B$, and to the ray $AC$ beyond $C$.)

*Proposed by Evangelos Psychas, Greece*

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**G2** Let $ABCD$ be a cyclic quadrilateral whose diagonals $AC$ and $BD$ meet at $E$. The extensions of the sides $AD$ and $BC$ beyond $A$ and $B$ meet at $F$. Let $G$ be the point such that $ECGD$ is a parallelogram, and let $H$ be the image of $E$ under reflection in $AD$. Prove that $D, H, F, G$ are concyclic.

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**G3** In an acute triangle $ABC$ the points $D, E$ and $F$ are the feet of the altitudes through $A, B$ and $C$ respectively. The incenters of the triangles $AEF$ and $BDF$ are $I_1$ and $I_2$ respectively; the circumcenters of the triangles $ACI_1$ and $BCI_2$ are $O_1$ and $O_2$ respectively. Prove that $I_1I_2$ and $O_1O_2$ are parallel.

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**G4** Let $ABC$ be a triangle with $AB \neq AC$ and circumcenter $O$. The bisector of $\angle BAC$ intersects $BC$ at $D$. Let $E$ be the reflection of $D$ with respect to the midpoint of $BC$. The lines through $D$ and $E$ perpendicular to $BC$ intersect the lines $AO$ and $AD$ at $X$ and $Y$ respectively. Prove that the quadrilateral $BXCY$ is cyclic.

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**G5** Let $ABC$ be a triangle with $\angle BCA = 90^\circ$, and let $D$ be the foot of the altitude from $C$. Let $X$ be a point in the interior of the segment $CD$. Let $K$ be the point on the segment $AX$ such that $BK = BC$. Similarly, let $L$ be the point on the segment $BX$ such that $AL = AC$. Let $M$ be the point of intersection of $AL$ and $BK$.

Show that $MK = ML$.

*Proposed by Josef Tkadlec, Czech Republic*

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**G6** Let $ABC$ be a triangle with circumcenter $O$ and incenter $I$. The points $D, E$ and $F$ on the sides $BC, CA$ and $AB$ respectively are such that $BD + BF = CA$ and $CD + CE = AB$. The circumcircles of the triangles $BFD$ and $CDE$ intersect at $P \neq D$. Prove that $OP = OI$.

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**G7** Let $ABCD$ be a convex quadrilateral with non-parallel sides $BC$ and $AD$. Assume that there is a point $E$ on the side $BC$ such that the quadrilaterals $ABED$ and $AECD$ are circumscribed. Prove that there is a point $F$ on the side $AD$ such that the quadrilaterals $ABCF$ and $BCDF$ are circumscribed if and only if $AB$ is parallel to $CD$.

**G8** Let $ABC$ be a triangle with circumcircle $\omega$ and $\ell$ a line without common points with $\omega$. Denote by $P$ the foot of the perpendicular from the center of $\omega$ to $\ell$. The side-lines $BC, CA, AB$ intersect $\ell$ at the points $X, Y, Z$ different from $P$. Prove that the circumcircles of the triangles $AXP$, $BYP$ and $CZP$ have a common point different from $P$ or are mutually tangent at $P$.

*Proposed by Cosmin Pohoata, Romania*

– Number Theory

**N1** Call admissible a set $A$ of integers that has the following property:
If $x, y \in A$ (possibly $x = y$) then $x^2 + kxy + y^2 \in A$ for every integer $k$.
Determine all pairs $m, n$ of nonzero integers such that the only admissible set containing both $m$ and $n$ is the set of all integers.

*Proposed by Warut Suksompong, Thailand*

**N2** Find all triples $(x, y, z)$ of positive integers such that $x \leq y \leq z$ and

$$x^3(y^3 + z^3) = 2012(xyz + 2).$$

**N3** Determine all integers $m \geq 2$ such that every $n$ with $\frac{m}{3} \leq n \leq \frac{m}{2}$ divides the binomial coefficient $\binom{n}{m-2n}$.

**N4** An integer $a$ is called friendly if the equation $(m^2 + n)(n^2 + m) = a(m - n)^3$ has a solution over the positive integers.
**a)** Prove that there are at least 500 friendly integers in the set $\{1, 2, \ldots, 2012\}$.
**b)** Decide whether $a = 2$ is friendly.

**N5** For a nonnegative integer $n$ define $\operatorname{rad}(n) = 1$ if $n = 0$ or $n = 1$, and $\operatorname{rad}(n) = p_1 p_2 \cdots p_k$ where $p_1 < p_2 < \cdots < p_k$ are all prime factors of $n$. Find all polynomials $f(x)$ with nonnegative integer coefficients such that $\operatorname{rad}(f(n))$ divides $\operatorname{rad}(f(n^{\operatorname{rad}(n)}))$ for every nonnegative integer $n$.

**N6** Let $x$ and $y$ be positive integers. If $x^{2^n} - 1$ is divisible by $2^n y + 1$ for every positive integer $n$, prove that $x = 1$.

**N7** Find all positive integers $n$ for which there exist non-negative integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} = 1.$$

*Proposed by Dusan Djukic, Serbia*

**N8** Prove that for every prime $p > 100$ and every integer $r$, there exist two integers $a$ and $b$ such that $p$ divides $a^2 + b^5 - r$.

IMO Shortlist 2011

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|---|---|
| – | Algebra |
| **1** | Given any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers, we denote the sum $a_1 + a_2 + a_3 + a_4$ by $s_A$. Let $n_A$ denote the number of pairs $(i, j)$ with $1 \le i < j \le 4$ for which $a_i + a_j$ divides $s_A$. Find all sets $A$ of four distinct positive integers which achieve the largest possible value of $n_A$. *Proposed by Fernando Campos, Mexico* |
| **2** | Determine all sequences $(x_1, x_2, \ldots, x_{2011})$ of positive integers, such that for every positive integer $n$ there exists an integer $a$ with $$\sum_{j=1}^{2011} j x_j^n = a^{n+1} + 1$$ *Proposed by Warut Suksompong, Thailand* |
| **3** | Determine all pairs $(f, g)$ of functions from the set of real numbers to itself that satisfy $$g(f(x+y)) = f(x) + (2x+y)g(y)$$ for all real numbers $x$ and $y$. *Proposed by Japan* |
| **4** | Determine all pairs $(f, g)$ of functions from the set of positive integers to itself that satisfy $$f^{g(n)+1}(n) + g^{f(n)}(n) = f(n+1) - g(n+1) + 1$$ for every positive integer $n$. Here, $f^k(n)$ means $\underbrace{f(f(\ldots f)}_{k}(n)\ldots))$. *Proposed by Bojan Bai, Serbia* |
| **5** | Prove that for every positive integer $n$, the set $\{2, 3, 4, \ldots, 3n+1\}$ can be partitioned into $n$ triples in such a way that the numbers from each triple are the lengths of the sides of some obtuse triangle. *Proposed by Canada* |

**6** Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function defined on the set of real numbers that satisfies
$$f(x+y) \le yf(x) + f(f(x))$$
for all real numbers $x$ and $y$. Prove that $f(x) = 0$ for all $x \le 0$.

*Proposed by Igor Voronovich, Belarus*

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**7** Let $a, b$ and $c$ be positive real numbers satisfying $\min(a+b, b+c, c+a) > \sqrt{2}$ and $a^2 + b^2 + c^2 = 3$. Prove that
$$\frac{a}{(b+c-a)^2} + \frac{b}{(c+a-b)^2} + \frac{c}{(a+b-c)^2} \ge \frac{3}{(abc)^2}.$$

*Proposed by Titu Andreescu, Saudi Arabia*

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– Combinatorics

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**1** Let $n > 0$ be an integer. We are given a balance and $n$ weights of weight $2^0, 2^1, \cdots, 2^{n-1}$. We are to place each of the $n$ weights on the balance, one after another, in such a way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, until all of the weights have been placed.
Determine the number of ways in which this can be done.

*Proposed by Morteza Saghafian, Iran*

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**2** Suppose that 1000 students are standing in a circle. Prove that there exists an integer $k$ with $100 \le k \le 300$ such that in this circle there exists a contiguous group of $2k$ students, for which the first half contains the same number of girls as the second half.

*Proposed by Gerhard Wginger, Austria*

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**3** Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A *windmill* is a process that starts with a line $\ell$ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the *pivot* $P$ until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, $Q$, takes over as the new pivot, and the line now rotates clockwise about $Q$, until it next meets a point of $\mathcal{S}$. This process continues indefinitely.
Show that we can choose a point $P$ in $\mathcal{S}$ and a line $\ell$ going through $P$ such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.

*Proposed by Geoffrey Smith, United Kingdom*

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**4** Determine the greatest positive integer $k$ that satisfies the following property: The set of positive integers can be partitioned into $k$ subsets $A_1, A_2, \ldots, A_k$ such that for all integers $n \geq 15$ and all $i \in \{1, 2, \ldots, k\}$ there exist two distinct elements of $A_i$ whose sum is $n$.

*Proposed by Igor Voronovich, Belarus*

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**5** Let $m$ be a positive integer, and consider a $m \times m$ checkerboard consisting of unit squares. At the centre of some of these unit squares there is an ant. At time 0, each ant starts moving with speed 1 parallel to some edge of the checkerboard. When two ants moving in the opposite directions meet, they both turn $90^\circ$ clockwise and continue moving with speed 1. When more than 2 ants meet, or when two ants moving in perpendicular directions meet, the ants continue moving in the same direction as before they met. When an ant reaches one of the edges of the checkerboard, it falls off and will not re-appear.

Considering all possible starting positions, determine the latest possible moment at which the last ant falls off the checkerboard, or prove that such a moment does not necessarily exist.

*Proposed by Toomas Krips, Estonia*

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**6** Let $n$ be a positive integer, and let $W = \ldots x_{-1}x_0x_1x_2\ldots$ be an infinite periodic word, consisting of just letters $a$ and/or $b$. Suppose that the minimal period $N$ of $W$ is greater than $2^n$.

A finite nonempty word $U$ is said to *appear* in $W$ if there exist indices $k \leq \ell$ such that $U = x_kx_{k+1}\ldots x_\ell$. A finite word $U$ is called *ubiquitous* if the four words $Ua$, $Ub$, $aU$, and $bU$ all appear in $W$. Prove that there are at least $n$ ubiquitous finite nonempty words.

*Proposed by Grigory Chelnokov, Russia*

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**7** On a square table of 2011 by 2011 cells we place a finite number of napkins that each cover a square of 52 by 52 cells. In each cell we write the number of napkins covering it, and we record the maximal number $k$ of cells that all contain the same nonzero number. Considering all possible napkin configurations, what is the largest value of $k$?

*Proposed by Ilya Bogdanov and Rustem Zhenodarov, Russia*

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– Geometry

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**1** Let $ABC$ be an acute triangle. Let $\omega$ be a circle whose centre $L$ lies on the side $BC$. Suppose that $\omega$ is tangent to $AB$ at $B'$ and $AC$ at $C'$. Suppose also that the circumcentre $O$ of triangle $ABC$ lies on the shorter arc $B'C'$ of $\omega$. Prove that the circumcircle of $ABC$ and $\omega$ meet at two points.

*Proposed by Hrmel Nestra, Estonia*

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**2** Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. Let $O_1$ and $r_1$ be the circumcentre and the circumradius of the triangle $A_2A_3A_4$. Define $O_2, O_3, O_4$ and $r_2, r_3, r_4$ in a similar way. Prove that

$$\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_3A_3^2 - r_3^2} + \frac{1}{O_4A_4^2 - r_4^2} = 0.$$

*Proposed by Alexey Gladkich, Israel*

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**3** Let $ABCD$ be a convex quadrilateral whose sides $AD$ and $BC$ are not parallel. Suppose that the circles with diameters $AB$ and $CD$ meet at points $E$ and $F$ inside the quadrilateral. Let $\omega_E$ be the circle through the feet of the perpendiculars from $E$ to the lines $AB, BC$ and $CD$. Let $\omega_F$ be the circle through the feet of the perpendiculars from $F$ to the lines $CD, DA$ and $AB$. Prove that the midpoint of the segment $EF$ lies on the line through the two intersections of $\omega_E$ and $\omega_F$.

*Proposed by Carlos Yuzo Shine, Brazil*

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**4** Let $ABC$ be an acute triangle with circumcircle $\Omega$. Let $B_0$ be the midpoint of $AC$ and let $C_0$ be the midpoint of $AB$. Let $D$ be the foot of the altitude from $A$ and let $G$ be the centroid of the triangle $ABC$. Let $\omega$ be a circle through $B_0$ and $C_0$ that is tangent to the circle $\Omega$ at a point $X \neq A$. Prove that the points $D, G$ and $X$ are collinear.

*Proposed by Ismail Isaev and Mikhail Isaev, Russia*

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**5** Let $ABC$ be a triangle with incentre $I$ and circumcircle $\omega$. Let $D$ and $E$ be the second intersection points of $\omega$ with $AI$ and $BI$, respectively. The chord $DE$ meets $AC$ at a point $F$, and $BC$ at a point $G$. Let $P$ be the intersection point of the line through $F$ parallel to $AD$ and the line through $G$ parallel to $BE$. Suppose that the tangents to $\omega$ at $A$ and $B$ meet at a point $K$. Prove that the three lines $AE, BD$ and $KP$ are either parallel or concurrent.

*Proposed by Irena Majcen and Kris Stopar, Slovenia*

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**6** Let $ABC$ be a triangle with $AB = AC$ and let $D$ be the midpoint of $AC$. The angle bisector of $\angle BAC$ intersects the circle through $D, B$ and $C$ at the point $E$ inside the triangle $ABC$. The line $BD$ intersects the circle through $A, E$ and $B$ in two points $B$ and $F$. The lines $AF$ and $BE$ meet at a point $I$, and the lines $CI$ and $BD$ meet at a point $K$. Show that $I$ is the incentre of triangle $KAB$.

*Proposed by Jan Vonk, Belgium and Hojoo Lee, South Korea*

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**7** Let $ABCDEF$ be a convex hexagon all of whose sides are tangent to a circle $\omega$ with centre $O$. Suppose that the circumcircle of triangle $ACE$ is concentric with $\omega$. Let $J$ be the foot of the perpendicular from $B$ to $CD$. Suppose that the perpendicular from $B$ to $DF$ intersects the line $EO$ at a point $K$. Let $L$ be the foot of the perpendicular from $K$ to $DE$. Prove that $DJ = DL$.

*Proposed by Japan*

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**8** Let $ABC$ be an acute triangle with circumcircle $\Gamma$. Let $\ell$ be a tangent line to $\Gamma$, and let $\ell_a, \ell_b$ and $\ell_c$ be the lines obtained by reflecting $\ell$ in the lines $BC$, $CA$ and $AB$, respectively. Show that the circumcircle of the triangle determined by the lines $\ell_a, \ell_b$ and $\ell_c$ is tangent to the circle $\Gamma$.

*Proposed by Japan*

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– Number Theory

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**1** For any integer $d > 0$, let $f(d)$ be the smallest possible integer that has exactly $d$ positive divisors (so for example we have $f(1) = 1$, $f(5) = 16$, and $f(6) = 12$). Prove that for every integer $k \geq 0$ the number $f\left(2^k\right)$ divides $f\left(2^{k+1}\right)$.

*Proposed by Suhaimi Ramly, Malaysia*

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**2** Consider a polynomial $P(x) = \prod_{j=1}^{9}(x+d_j)$, where $d_1, d_2, \ldots d_9$ are nine distinct integers. Prove that there exists an integer $N$, such that for all integers $x \geq N$ the number $P(x)$ is divisible by a prime number greater than 20.

*Proposed by Luxembourg*

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**3** Let $n \geq 1$ be an odd integer. Determine all functions $f$ from the set of integers to itself, such that for all integers $x$ and $y$ the difference $f(x) - f(y)$ divides $x^n - y^n$.

*Proposed by Mihai Baluna, Romania*

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**4** For each positive integer $k$, let $t(k)$ be the largest odd divisor of $k$. Determine all positive integers $a$ for which there exists a positive integer $n$, such that all the differences

$$t(n+a)-t(n); t(n+a+1)-t(n+1), \ldots, t(n+2a-1)-t(n+a-1)$$

are divisible by 4.

*Proposed by Gerhard Wginger, Austria*

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**5** Let $f$ be a function from the set of integers to the set of positive integers. Suppose that, for any two integers $m$ and $n$, the difference $f(m)-f(n)$ is divisible by $f(m-n)$. Prove that, for all integers $m$ and $n$ with $f(m) \leq f(n)$, the number $f(n)$ is divisible by $f(m)$.

*Proposed by Mahyar Sefidgaran, Iran*

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**6** Let $P(x)$ and $Q(x)$ be two polynomials with integer coefficients, such that no nonconstant polynomial with rational coefficients divides both $P(x)$ and $Q(x)$. Suppose that for every positive integer $n$ the integers $P(n)$ and $Q(n)$ are positive, and $2^{Q(n)}-1$ divides $3^{P(n)}-1$. Prove that $Q(x)$ is a constant polynomial.

*Proposed by Oleksiy Klurman, Ukraine*

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**7** Let $p$ be an odd prime number. For every integer $a$, define the number $S_a = \sum_{j=1}^{p-1} \frac{a^j}{j}$. Let $m, n \in \mathbb{Z}$, such that $S_3+S_4-3S_2 = \frac{m}{n}$. Prove that $p$ divides $m$.

*Proposed by Romeo Metrovi, Montenegro*

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**8** Let $k \in \mathbb{Z}^+$ and set $n = 2^k+1$. Prove that $n$ is a prime number if and only if the following holds: there is a permutation $a_1, \ldots, a_{n-1}$ of the numbers $1, 2, \ldots, n-1$ and a sequence of integers $g_1, \ldots, g_{n-1}$, such that $n$ divides $g_i^{a_i} - a_{i+1}$ for every $i \in \{1, 2, \ldots, n-1\}$, where we set $a_n = a_1$.

*Proposed by Vasily Astakhov, Russia*

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IMO Shortlist 2010

– Algebra

**1** Find all function $f:\mathbb{R}\rightarrow\mathbb{R}$ such that for all $x,y\in\mathbb{R}$ the following equality holds

$$f(\lfloor x\rfloor y)=f(x)\lfloor f(y)\rfloor$$

where $\lfloor a\rfloor$ is greatest integer not greater than $a$.

*Proposed by Pierre Bornsztein, France*

**2** Let the real numbers $a,b,c,d$ satisfy the relations $a+b+c+d=6$ and $a^2+b^2+c^2+d^2=12$. Prove that

$$36 \leq 4 \left(a^3+b^3+c^3+d^3\right) - \left(a^4+b^4+c^4+d^4\right) \leq 48.$$

*Proposed by Nazar Serdyuk, Ukraine*

**3** Let $x_1, \ldots , x_{100}$ be nonnegative real numbers such that $x_i + x_{i+1} + x_{i+2} \leq 1$ for all $i = 1, \ldots , 100$ (we put $x_{101} = x_1, x_{102} = x_2$). Find the maximal possible value of the sum $S = \sum^{100}_{i=1} x_i x_{i+2}$.

*Proposed by Sergei Berlov, Ilya Bogdanov, Russia*

**4** A sequence $x_1, x_2, \ldots$ is defined by $x_1 = 1$ and $x_{2k} = -x_k, x_{2k-1} = (-1)^{k+1}x_k$ for all $k \geq 1$. Prove that $\forall n \geq 1$ $x_1 + x_2 + \ldots + x_n \geq 0$.

*Proposed by Gerhard Wginger, Austria*

**5** Denote by $\mathbb{Q}^+$ the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}^+ \mapsto \mathbb{Q}^+$ which satisfy the following equation for all $x, y \in \mathbb{Q}^+ :$

$$f\left( f(x)^2y \right) = x^3 f(xy).$$

*Proposed by Thomas Huber, Switzerland*

**6** Suppose that $f$ and $g$ are two functions defined on the set of positive integers and taking positive integer values. Suppose also that the equations $f(g(n)) = f(n) + 1$ and $g(f(n)) = g(n) + 1$ hold for all positive integers. Prove that $f(n) = g(n)$ for all positive integer $n$.

*Proposed by Alex Schreiber, Germany*

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**7** Let $a_1, a_2, a_3, \dots$ be a sequence of positive real numbers, and $s$ be a positive integer, such that

$$a_n = \max\{a_k + a_{n-k} \mid 1 \le k \le n-1\} \text{ for all } n > s.$$

Prove there exist positive integers $\ell \le s$ and $N$, such that

$$a_n = a_\ell + a_{n-\ell} \text{ for all } n \ge N.$$

*Proposed by Morteza Saghafiyan, Iran*

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**8** Given six positive numbers $a, b, c, d, e, f$ such that $a < b < c < d < e < f$. Let $a + c + e = S$ and $b + d + f = T$. Prove that

$$2ST > \sqrt{3(S+T)\left(S(bd+bf+df) + T(ac+ae+ce)\right)}.$$

*Proposed by Sung Yun Kim, South Korea*

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– Combinatorics

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**1** In a concert, 20 singers will perform. For each singer, there is a (possibly empty) set of other singers such that he wishes to perform later than all the singers from that set. Can it happen that there are exactly 2010 orders of the singers such that all their wishes are satisfied?

*Proposed by Gerhard Wginger, Austria*

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**2** On some planet, there are $2^N$ countries ($N \ge 4$). Each country has a flag $N$ units wide and one unit high composed of $N$ fields of size $1 \times 1$, each field being either yellow or blue. No two countries have the same flag. We say that a set of $N$ flags is diverse if these flags can be arranged into an $N \times N$ square so that all $N$ fields on its main diagonal will have the same color. Determine the smallest positive integer $M$ such that among any $M$ distinct flags, there exist $N$ flags forming a diverse set.

*Proposed by Toni Kokan, Croatia*

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**3** 2500 chess kings have to be placed on a $100 \times 100$ chessboard so that

**(i)** no king can capture any other one (i.e. no two kings are placed in two squares sharing a common vertex);
**(ii)** each row and each column contains exactly 25 kings.

Find the number of such arrangements. (Two arrangements differing by rotation or symmetry are supposed to be different.)

*Proposed by Sergei Berlov, Russia*

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**4** Each of the six boxes $B_1$, $B_2$, $B_3$, $B_4$, $B_5$, $B_6$ initially contains one coin. The following operations are allowed

Type 1) Choose a non-empty box $B_j$, $1 \leq j \leq 5$, remove one coin from $B_j$ and add two coins to $B_{j+1}$;

Type 2) Choose a non-empty box $B_k$, $1 \leq k \leq 4$, remove one coin from $B_k$ and swap the contents (maybe empty) of the boxes $B_{k+1}$ and $B_{k+2}$.

Determine if there exists a finite sequence of operations of the allowed types, such that the five boxes $B_1$, $B_2$, $B_3$, $B_4$, $B_5$ become empty, while box $B_6$ contains exactly $2010^{2010^{2010}}$ coins.

*Proposed by Hans Zantema, Netherlands*

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**5** $n \geq 4$ players participated in a tennis tournament. Any two players have played exactly one game, and there was no tie game. We call a company of four players *bad* if one player was defeated by the other three players, and each of these three players won a game and lost another game among themselves. Suppose that there is no bad company in this tournament. Let $w_i$ and $l_i$ be respectively the number of wins and losses of the $i$-th player. Prove that

$$\sum_{i=1}^{n} (w_i - l_i)^3 \geq 0.$$

*Proposed by Sung Yun Kim, South Korea*

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**6** Given a positive integer $k$ and other two integers $b > w > 1$. There are two strings of pearls, a string of $b$ black pearls and a string of $w$ white pearls. The length of a string is the number of pearls on it. One cuts these strings in some steps by the following rules. In each step:

**(i)** The strings are ordered by their lengths in a non-increasing order. If there are some strings of equal lengths, then the white ones precede the black ones. Then $k$ first ones (if they consist of more than one pearl) are chosen; if there are less than $k$ strings longer than 1, then one chooses all of them.
**(ii)** Next, one cuts each chosen string into two parts differing in length by at most one. (For instance, if there are strings of $5, 4, 4, 2$ black pearls, strings of $8, 4, 3$ white pearls and $k = 4$, then the strings of 8 white, 5 black, 4 white and 4 black pearls are cut into the parts $(4, 4), (3, 2), (2, 2)$ and $(2, 2)$ respectively.)

The process stops immediately after the step when a first isolated white pearl appears.

Prove that at this stage, there will still exist a string of at least two black pearls.

*Proposed by Bill Sands, Thao Do, Canada*

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**7** Let $P_1, \dots, P_s$ be arithmetic progressions of integers, the following conditions being satisfied:

**(i)** each integer belongs to at least one of them;
**(ii)** each progression contains a number which does not belong to other progressions.

Denote by $n$ the least common multiple of the ratios of these progressions; let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ its prime factorization.

Prove that

$$s \geq 1 + \sum_{i=1}^{k} \alpha_i (p_i - 1).$$

*Proposed by Dierk Schleicher, Germany*

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– Geometry

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**1** Let $ABC$ be an acute triangle with $D, E, F$ the feet of the altitudes lying on $BC, CA, AB$ respectively. One of the intersection points of the line $EF$ and the circumcircle is $P$. The lines $BP$ and $DF$ meet at point $Q$. Prove that $AP = AQ$.

*Proposed by Christopher Bradley, United Kingdom*

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**2** Let $P$ be a point interior to triangle $ABC$ (with $CA \neq CB$). The lines $AP$, $BP$ and $CP$ meet again its circumcircle $\Gamma$ at $K$, $L$, respectively $M$. The tangent line at $C$ to $\Gamma$ meets the line $AB$ at $S$. Show that from $SC = SP$ follows $MK = ML$.

*Proposed by Marcin E. Kuczma, Poland*

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**3** Let $A_1A_2 \dots A_n$ be a convex polygon. Point $P$ inside this polygon is chosen so that its projections $P_1, \dots, P_n$ onto lines $A_1A_2, \dots, A_nA_1$ respectively lie on the sides of the polygon. Prove that for arbitrary points $X_1, \dots, X_n$ on sides $A_1A_2, \dots, A_nA_1$ respectively,

$$\max \left\{ \frac{X_1X_2}{P_1P_2}, \dots, \frac{X_nX_1}{P_nP_1} \right\} \geq 1.$$

*Proposed by Nairi Sedrakyan, Armenia*

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**4** Given a triangle $ABC$, with $I$ as its incenter and $\Gamma$ as its circumcircle, $AI$ intersects $\Gamma$ again at $D$. Let $E$ be a point on the arc $BDC$, and $F$ a point on the segment $BC$, such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. If $G$ is the midpoint of $IF$, prove that the meeting point of the lines $EI$ and $DG$ lies on $\Gamma$.

*Proposed by Tai Wai Ming and Wang Chongli, Hong Kong*

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**5** Let $ABCDE$ be a convex pentagon such that $BC \parallel AE$, $AB = BC + AE$, and $\angle ABC = \angle CDE$. Let $M$ be the midpoint of $CE$, and let $O$ be the circumcenter of triangle $BCD$. Given that $\angle DMO = 90^\circ$, prove that $2\angle BDA = \angle CDE$.

*Proposed by Nazar Serdyuk, Ukraine*

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**6** The vertices $X, Y, Z$ of an equilateral triangle $XYZ$ lie respectively on the sides $BC, CA, AB$ of an acute-angled triangle $ABC$. Prove that the incenter of triangle $ABC$ lies inside triangle $XYZ$.

*Proposed by Nikolay Beluhov, Bulgaria*

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**7** Three circular arcs $\gamma_1, \gamma_2$, and $\gamma_3$ connect the points $A$ and $C$. These arcs lie in the same half-plane defined by line $AC$ in such a way that arc $\gamma_2$ lies between the arcs $\gamma_1$ and $\gamma_3$. Point $B$ lies on the segment $AC$. Let $h_1, h_2$, and $h_3$ be three rays starting at $B$, lying in the same half-plane, $h_2$ being between $h_1$ and $h_3$. For $i, j = 1, 2, 3$, denote by $V_{ij}$ the point of intersection of $h_i$ and $\gamma_j$ (see the Figure below). Denote by $\widehat{V_{ij}V_{kj}}\widehat{V_{kl}V_{il}}$ the curved quadrilateral, whose sides are the segments $V_{ij}V_{il}$, $V_{kj}V_{kl}$ and arcs $V_{ij}V_{kj}$ and $V_{il}V_{kl}$. We say that this quadrilateral is *circumscribed* if there exists a circle touching these two segments and two arcs. Prove that if the curved quadrilaterals $\widehat{V_{11}V_{21}}\widehat{V_{22}V_{12}}$, $\widehat{V_{12}V_{22}}\widehat{V_{23}V_{13}}$, $\widehat{V_{21}V_{31}}\widehat{V_{32}V_{22}}$ are circumscribed, then the curved quadrilateral $\widehat{V_{22}V_{32}}\widehat{V_{33}V_{23}}$ is circumscribed, too.

*Proposed by Gza Ks, Hungary*

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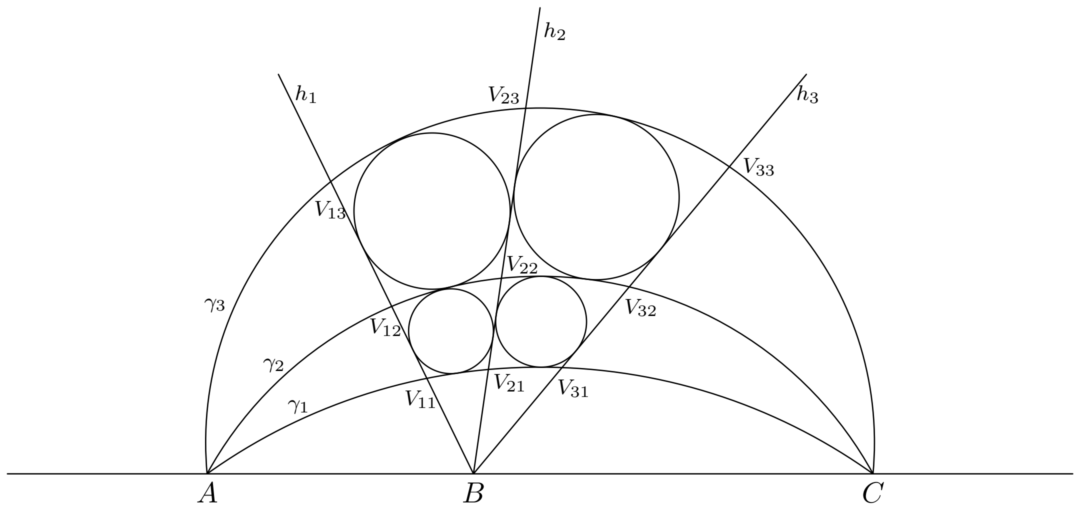


| – | Number Theory |
|---|---|
| **1** | Find the least positive integer $n$ for which there exists a set $\{s_1, s_2, \ldots, s_n\}$ consisting of $n$ distinct positive integers such that $$\left(1-\frac{1}{s_1}\right)\left(1-\frac{1}{s_2}\right)\cdots\left(1-\frac{1}{s_n}\right)=\frac{51}{2010}.$$ *Proposed by Daniel Brown, Canada* |
| **2** | Find all pairs $(m, n)$ of nonnegative integers for which $$m^2+2\cdot 3^n=m\left(2^{n+1}-1\right).$$ *Proposed by Angelo Di Pasquale, Australia* |
| **3** | Find the smallest number $n$ such that there exist polynomials $f_1, f_2, \ldots, f_n$ with rational coefficients satisfying $$x^2+7=f_1(x)^2+f_2(x)^2+\ldots+f_n(x)^2.$$ *Proposed by Mariusz Skaba, Poland* |

**4** Let $a, b$ be integers, and let $P(x) = ax^3 + bx$. For any positive integer $n$ we say that the pair $(a, b)$ is $n$-good if $n|P(m) - P(k)$ implies $n|m - k$ for all integers $m, k$. We say that $(a, b)$ is *very good* if $(a, b)$ is $n$-good for infinitely many positive integers $n$.
-**(a)** Find a pair $(a, b)$ which is 51-good, but not very good.
-**(b)** Show that all 2010-good pairs are very good.

*Proposed by Okan Tekman, Turkey*

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**5** Find all functions $g : \mathbb{N} \to \mathbb{N}$ such that

$$(g(m) + n)\,(g(n) + m)$$

is a perfect square for all $m, n \in \mathbb{N}$.

*Proposed by Gabriel Carroll, USA*

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**6** The rows and columns of a $2^n \times 2^n$ table are numbered from 0 to $2^n - 1$. The cells of the table have been coloured with the following property being satisfied: for each $0 \leq i, j \leq 2^n - 1$, the $j$-th cell in the $i$-th row and the $(i + j)$-th cell in the $j$-th row have the same colour. (The indices of the cells in a row are considered modulo $2^n$.) Prove that the maximal possible number of colours is $2^n$.

*Proposed by Hossein Dabirian, Sepehr Ghazi-nezami, Iran*

IMO Shortlist 2009

– Algebra

**1** Find the largest possible integer $k$, such that the following statement is true: Let 2009 arbitrary non-degenerated triangles be given. In every triangle the three sides are coloured, such that one is blue, one is red and one is white. Now, for every colour separately, let us sort the lengths of the sides. We obtain

$$\begin{array}{rl} b_1 \leq b_2 \leq \ldots \leq b_{2009} & \text{the lengths of the blue sides} \\ r_1 \leq r_2 \leq \ldots \leq r_{2009} & \text{the lengths of the red sides} \\ \text{and} \quad w_1 \leq w_2 \leq \ldots \leq w_{2009} & \text{the lengths of the white sides} \end{array}$$

Then there exist $k$ indices $j$ such that we can form a non-degenerated triangle with side lengths $b_j$, $r_j$, $w_j$.

*Proposed by Michal Rolinek, Czech Republic*

**2** Let $a$, $b$, $c$ be positive real numbers such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=a+b+c$. Prove that:

$$\frac{1}{(2a+b+c)^2}+\frac{1}{(a+2b+c)^2}+\frac{1}{(a+b+2c)^2}\leq\frac{3}{16}.$$

*Proposed by Juhan Aru, Estonia*

**3** Determine all functions $f$ from the set of positive integers to the set of positive integers such that, for all positive integers $a$ and $b$, there exists a non-degenerate triangle with sides of lengths

$$a, f(b) \text{ and } f(b+f(a)-1).$$

(A triangle is non-degenerate if its vertices are not collinear.)

*Proposed by Bruno Le Floch, France*

**4** Let $a$, $b$, $c$ be positive real numbers such that $ab+bc+ca\leq 3abc$. Prove that

$$\sqrt{\frac{a^2+b^2}{a+b}}+\sqrt{\frac{b^2+c^2}{b+c}}+\sqrt{\frac{c^2+a^2}{c+a}}+3\leq\sqrt{2}\left(\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}\right)$$

*Proposed by Dzianis Pirshtuk, Belarus*

**5** Let $f$ be any function that maps the set of real numbers into the set of real numbers. Prove that there exist real numbers $x$ and $y$ such that

$$f(x - f(y)) > yf(x) + x$$

*Proposed by Igor Voronovich, Belarus*

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**6** Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive integers such that the sub-sequences

$$s_{s_1},\, s_{s_2},\, s_{s_3},\, \ldots \qquad \text{and} \qquad s_{s_1+1},\, s_{s_2+1},\, s_{s_3+1},\, \ldots$$

are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \ldots$ is itself an arithmetic progression.

*Proposed by Gabriel Carroll, USA*

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**7** Find all functions $f$ from the set of real numbers into the set of real numbers which satisfy for all $x$, $y$ the identity

$$f(xf(x+y)) = f(yf(x)) + x^2$$

*Proposed by Japan*

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– Combinatorics

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**1** Consider 2009 cards, each having one gold side and one black side, lying on parallel on a long table. Initially all cards show their gold sides. Two player, standing by the same long side of the table, play a game with alternating moves. Each move consists of choosing a block of 50 consecutive cards, the leftmost of which is showing gold, and turning them all over, so those which showed gold now show black and vice versa. The last player who can make a legal move wins.
(a) Does the game necessarily end?
(b) Does there exist a winning strategy for the starting player?

*Proposed by Michael Albert, Richard Guy, New Zealand*

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**2** For any integer $n \geq 2$, let $N(n)$ be the maxima number of triples $(a_i, b_i, c_i)$, $i = 1, \ldots, N(n)$, consisting of nonnegative integers $a_i$, $b_i$ and $c_i$ such that the following two conditions are satisfied:
- $a_i + b_i + c_i = n$ for all $i = 1, \ldots, N(n)$,

- If $i \neq j$ then $a_i \neq a_j$, $b_i \neq b_j$ and $c_i \neq c_j$
Determine $N(n)$ for all $n \geq 2$.

*Proposed by Dan Schwarz, Romania*

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**3** Let $n$ be a positive integer. Given a sequence $\varepsilon_1, \ldots, \varepsilon_{n-1}$ with $\varepsilon_i = 0$ or $\varepsilon_i = 1$ for each $i = 1, \ldots, n-1$, the sequences $a_0, \ldots, a_n$ and $b_0, \ldots, b_n$ are constructed by the following rules:

$$a_0 = b_0 = 1, \quad a_1 = b_1 = 7,$$

$$a_{i+1} = \begin{cases} 2a_{i-1} + 3a_i, & \text{if } \varepsilon_i = 0, \\ 3a_{i-1} + a_i, & \text{if } \varepsilon_i = 1, \end{cases} \qquad \text{for each } i = 1, \ldots, n-1,$$

$$b_{i+1} = \begin{cases} 2b_{i-1} + 3b_i, & \text{if } \varepsilon_{n-i} = 0, \\ 3b_{i-1} + b_i, & \text{if } \varepsilon_{n-i} = 1, \end{cases} \qquad \text{for each } i = 1, \ldots, n-1.$$

Prove that $a_n = b_n$.

*Proposed by Ilya Bogdanov, Russia*

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**4** For an integer $m \geq 1$, we consider partitions of a $2^m \times 2^m$ chessboard into rectangles consisting of cells of chessboard, in which each of the $2^m$ cells along one diagonal forms a separate rectangle of side length 1. Determine the smallest possible sum of rectangle perimeters in such a partition.

*Proposed by Gerhard Woeginger, Netherlands*

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**5** Five identical empty buckets of 2-liter capacity stand at the vertices of a regular pentagon. Cinderella and her wicked Stepmother go through a sequence of rounds: At the beginning of every round, the Stepmother takes one liter of water from the nearby river and distributes it arbitrarily over the five buckets. Then Cinderella chooses a pair of neighbouring buckets, empties them to the river and puts them back. Then the next round begins. The Stepmother goal's is to make one of these buckets overflow. Cinderella's goal is to prevent this. Can the wicked Stepmother enforce a bucket overflow?

*Proposed by Gerhard Woeginger, Netherlands*

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**6** On a $999 \times 999$ board a *limp rook* can move in the following way: From any square it can move to any of its adjacent squares, i.e. a square having a common side with it, and every move must be a turn, i.e. the directions of any two consecutive moves must be perpendicular. A *non-intersecting route* of the limp rook consists of a sequence of pairwise different squares that the limp rook can

visit in that order by an admissible sequence of moves. Such a non-intersecting route is called *cyclic*, if the limp rook can, after reaching the last square of the route, move directly to the first square of the route and start over.
How many squares does the longest possible cyclic, non-intersecting route of a limp rook visit?

*Proposed by Nikolay Beluhov, Bulgaria*

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**7** Let $a_1, a_2, \ldots, a_n$ be distinct positive integers and let $M$ be a set of $n-1$ positive integers not containing $s = a_1 + a_2 + \ldots + a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making $n$ jumps to the right with lengths $a_1, a_2, \ldots, a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in $M$.

*Proposed by Dmitry Khramtsov, Russia*

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**8** For any integer $n \geq 2$, we compute the integer $h(n)$ by applying the following procedure to its decimal representation. Let $r$ be the rightmost digit of $n$.
-If $r = 0$, then the decimal representation of $h(n)$ results from the decimal representation of $n$ by removing this rightmost digit 0.
-If $1 \leq r \leq 9$ we split the decimal representation of $n$ into a maximal right part $R$ that solely consists of digits not less than $r$ and into a left part $L$ that either is empty or ends with a digit strictly smaller than $r$. Then the decimal representation of $h(n)$ consists of the decimal representation of $L$, followed by two copies of the decimal representation of $R-1$. For instance, for the number $17,151,345,543$, we will have $L = 17,151$, $R = 345,543$ and $h(n) = 17,151,345,542,345,542$.
Prove that, starting with an arbitrary integer $n \geq 2$, iterated application of $h$ produces the integer 1 after finitely many steps.

*Proposed by Gerhard Woeginger, Austria*

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– Geometry

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**1** Let $ABC$ be a triangle with $AB = AC$ . The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides $BC$ and $CA$ at $D$ and $E$ , respectively. Let $K$ be the incentre of triangle $ADC$. Suppose that $\angle BEK = 45^\circ$ . Find all possible values of $\angle CAB$ .

*Jan Vonk, Belgium, Peter Vandendriessche, Belgium and Hojoo Lee, Korea*

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**2** Let $ABC$ be a triangle with circumcentre $O$. The points $P$ and $Q$ are interior points of the sides $CA$ and $AB$ respectively. Let $K, L$ and $M$ be the midpoints of the segments $BP, CQ$ and $PQ$. respectively, and let $\Gamma$ be the circle passing through $K, L$ and $M$. Suppose that the line $PQ$ is tangent to the circle $\Gamma$. Prove that $OP = OQ$.

*Proposed by Sergei Berlov, Russia*

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**3** Let $ABC$ be a triangle. The incircle of $ABC$ touches the sides $AB$ and $AC$ at the points $Z$ and $Y$, respectively. Let $G$ be the point where the lines $BY$ and $CZ$ meet, and let $R$ and $S$ be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelogram.
Prove that $GR = GS$.

*Proposed by Hossein Karke Abadi, Iran*

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**4** Given a cyclic quadrilateral $ABCD$, let the diagonals $AC$ and $BD$ meet at $E$ and the lines $AD$ and $BC$ meet at $F$. The midpoints of $AB$ and $CD$ are $G$ and $H$, respectively. Show that $EF$ is tangent at $E$ to the circle through the points $E$, $G$ and $H$.

*Proposed by David Monk, United Kingdom*

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**5** Let $P$ be a polygon that is convex and symmetric to some point $O$. Prove that for some parallelogram $R$ satisfying $P \subset R$ we have

$$\frac{|R|}{|P|} \leq \sqrt{2}$$

where $|R|$ and $|P|$ denote the area of the sets $R$ and $P$, respectively.

*Proposed by Witold Szczechla, Poland*

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**6** Let the sides $AD$ and $BC$ of the quadrilateral $ABCD$ (such that $AB$ is not parallel to $CD$) intersect at point $P$. Points $O_1$ and $O_2$ are circumcenters and points $H_1$ and $H_2$ are orthocenters of triangles $ABP$ and $CDP$, respectively. Denote the midpoints of segments $O_1H_1$ and $O_2H_2$ by $E_1$ and $E_2$, respectively. Prove that the perpendicular from $E_1$ on $CD$, the perpendicular from $E_2$ on $AB$ and the lines $H_1H_2$ are concurrent.

*Proposed by Eugene Bilopitov, Ukraine*

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**7** Let $ABC$ be a triangle with incenter $I$ and let $X$, $Y$ and $Z$ be the incenters of the triangles $BIC$, $CIA$ and $AIB$, respectively. Let the triangle $XYZ$ be equilateral. Prove that $ABC$ is equilateral too.

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*Proposed by Mirsaleh Bahavarnia, Iran*

**8** Let $ABCD$ be a circumscribed quadrilateral. Let $g$ be a line through $A$ which meets the segment $BC$ in $M$ and the line $CD$ in $N$. Denote by $I_1$, $I_2$ and $I_3$ the incenters of $\triangle ABM$, $\triangle MNC$ and $\triangle NDA$, respectively. Prove that the orthocenter of $\triangle I_1I_2I_3$ lies on $g$.

*Proposed by Nikolay Beluhov, Bulgaria*

– Number Theory

**1** Let $n$ be a positive integer and let $a_1, a_2, a_3, \ldots, a_k$ $(k \geq 2)$ be distinct integers in the set $1, 2, \ldots, n$ such that $n$ divides $a_i(a_{i+1} - 1)$ for $i = 1, 2, \ldots, k - 1$. Prove that $n$ does not divide $a_k(a_1 - 1)$.

*Proposed by Ross Atkins, Australia*

**2** A positive integer $N$ is called *balanced*, if $N = 1$ or if $N$ can be written as a product of an even number of not necessarily distinct primes. Given positive integers $a$ and $b$, consider the polynomial $P$ defined by $P(x) = (x + a)(x + b)$.
(a) Prove that there exist distinct positive integers $a$ and $b$ such that all the number $P(1)$, $P(2)$,. . ., $P(50)$ are balanced.
(b) Prove that if $P(n)$ is balanced for all positive integers $n$, then $a = b$.

*Proposed by Jorge Tipe, Peru*

**3** Let $f$ be a non-constant function from the set of positive integers into the set of positive integer, such that $a - b$ divides $f(a) - f(b)$ for all distinct positive integers $a$, $b$. Prove that there exist infinitely many primes $p$ such that $p$ divides $f(c)$ for some positive integer $c$.

*Proposed by Juhan Aru, Estonia*

**4** Find all positive integers $n$ such that there exists a sequence of positive integers $a_1$, $a_2$,. . ., $a_n$ satisfying:

$$a_{k+1} = \frac{a_k^2 + 1}{a_{k-1} + 1} - 1$$

for every $k$ with $2 \leq k \leq n - 1$.

*Proposed by North Korea*

**5** Let $P(x)$ be a non-constant polynomial with integer coefficients. Prove that there is no function $T$ from the set of integers into the set of integers such that the number of integers $x$ with $T^n(x) = x$ is equal to $P(n)$ for every $n \geq 1$, where $T^n$ denotes the $n$-fold application of $T$.

*Proposed by Jozsef Pelikan, Hungary*

**6** Let $k$ be a positive integer. Show that if there exists a sequence $a_0, a_1, \ldots$ of integers satisfying the condition

$$a_n = \frac{a_{n-1} + n^k}{n} \text{ for all } n \geq 1,$$

then $k - 2$ is divisible by 3.

*Proposed by Okan Tekman, Turkey*

**7** Let $a$ and $b$ be distinct integers greater than 1. Prove that there exists a positive integer $n$ such that $(a^n - 1)(b^n - 1)$ is not a perfect square.

*Proposed by Mongolia*

IMO Shortlist 2008

– Algebra

**1** Find all functions $f : (0,\infty) \mapsto (0,\infty)$ (so $f$ is a function from the positive real numbers) such that

$$\frac{\left(f(w)\right)^2+\left(f(x)\right)^2}{f(y^2)+f(z^2)}=\frac{w^2+x^2}{y^2+z^2}$$

for all positive real numbers $w, x, y, z$, satisfying $wx = yz$.

*Author: Hojoo Lee, South Korea*

**2** **(a)** Prove that

$$\frac{x^2}{(x-1)^2}+\frac{y^2}{(y-1)^2}+\frac{z^2}{(z-1)^2}\geq 1$$

for all real numbers $x$, $y$, $z$, each different from 1, and satisfying $xyz = 1$.

**(b)** Prove that equality holds above for infinitely many triples of rational numbers $x$, $y$, $z$, each different from 1, and satisfying $xyz = 1$.

*Author: Walther Janous, Austria*

**3** Let $S \subseteq \mathbb{R}$ be a set of real numbers. We say that a pair $(f, g)$ of functions from $S$ into $S$ is a *Spanish Couple* on $S$, if they satisfy the following conditions:
(i) Both functions are strictly increasing, i.e. $f(x) < f(y)$ and $g(x) < g(y)$ for all $x$, $y \in S$ with $x < y$;

(ii) The inequality $f\left(g\left(g\left(x\right)\right)\right) < g\left(f\left(x\right)\right)$ holds for all $x \in S$.

Decide whether there exists a Spanish Couple - on the set $S = \mathbb{N}$ of positive integers; - on the set $S = \{a - \frac{1}{b} : a, b \in \mathbb{N}\}$

*Proposed by Hans Zantema, Netherlands*

**4** For an integer $m$, denote by $t(m)$ the unique number in $\{1, 2, 3\}$ such that $m+t(m)$ is a multiple of 3. A function $f : \mathbb{Z} \to \mathbb{Z}$ satisfies $f(-1) = 0$, $f(0) = 1$, $f(1) = -1$ and $f\left(2^n + m\right) = f\left(2^n - t(m)\right) - f(m)$ for all integers $m$, $n \geq 0$ with $2^n > m$. Prove that $f(3p) \geq 0$ holds for all integers $p \geq 0$.

*Proposed by Gerhard Woeginger, Austria*

**5** Let $a$, $b$, $c$, $d$ be positive real numbers such that $abcd = 1$ and $a + b + c + d > \frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}$. Prove that

$$a + b + c + d < \frac{b}{a} + \frac{c}{b} + \frac{d}{c} + \frac{a}{d}$$

*Proposed by Pavel Novotn, Slovakia*

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**6** Let $f : \mathbb{R} \to \mathbb{N}$ be a function which satisfies $f\left(x + \frac{1}{f(y)}\right) = f\left(y + \frac{1}{f(x)}\right)$ for all $x$, $y \in \mathbb{R}$. Prove that there is a positive integer which is not a value of $f$.

*Proposed by ymantas Darbnas (Zymantas Darbenas), Lithuania*

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**7** Prove that for any four positive real numbers $a$, $b$, $c$, $d$ the inequality

$$\frac{(a-b)(a-c)}{a+b+c} + \frac{(b-c)(b-d)}{b+c+d} + \frac{(c-d)(c-a)}{c+d+a} + \frac{(d-a)(d-b)}{d+a+b} \geq 0$$

holds. Determine all cases of equality.

*Author: Darij Grinberg (Problem Proposal), Christian Reiher (Solution), Germany*

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– Combinatorics

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**1** In the plane we consider rectangles whose sides are parallel to the coordinate axes and have positive length. Such a rectangle will be called a *box*. Two boxes *intersect* if they have a common point in their interior or on their boundary. Find the largest $n$ for which there exist $n$ boxes $B_1$, $\ldots$, $B_n$ such that $B_i$ and $B_j$ intersect if and only if $i \not\equiv j \pm 1 \pmod{n}$.

*Proposed by Gerhard Woeginger, Netherlands*

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**2** Let $n \in \mathbb{N}$ and $A_n$ set of all permutations $(a_1, \ldots, a_n)$ of the set $\{1, 2, \ldots, n\}$ for which

$$k|2(a_1 + \cdots + a_k), \text{ for all } 1 \leq k \leq n.$$

Find the number of elements of the set $A_n$.

*Proposed by Vidan Govedarica, Serbia*

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3 In the coordinate plane consider the set $S$ of all points with integer coordinates. For a positive integer $k$, two distinct points $a$, $B \in S$ will be called *$k$-friends* if there is a point $C \in S$ such that the area of the triangle $ABC$ is equal to $k$. A set $T \subset S$ will be called *$k$-clique* if every two points in $T$ are $k$-friends. Find the least positive integer $k$ for which there exits a $k$-clique with more than 200 elements.

*Proposed by Jorge Tipe, Peru*

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4 Let $n$ and $k$ be positive integers with $k \geq n$ and $k - n$ an even number. Let $2n$ lamps labelled 1, 2, ..., $2n$ be given, each of which can be either *on* or *off*. Initially all the lamps are off. We consider sequences of steps: at each step one of the lamps is switched (from on to off or from off to on).

Let $N$ be the number of such sequences consisting of $k$ steps and resulting in the state where lamps 1 through $n$ are all on, and lamps $n+1$ through $2n$ are all off.

Let $M$ be number of such sequences consisting of $k$ steps, resulting in the state where lamps 1 through $n$ are all on, and lamps $n + 1$ through $2n$ are all off, but where none of the lamps $n + 1$ through $2n$ is ever switched on.

Determine $\frac{N}{M}$.

*Author: Bruno Le Floch and Ilia Smilga, France*

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5 Let $S = \{x_1, x_2, \ldots, x_{k+l}\}$ be a $(k + l)$-element set of real numbers contained in the interval $[0, 1]$; $k$ and $l$ are positive integers. A $k$-element subset $A \subset S$ is called *nice* if

$$\left| \frac{1}{k} \sum_{x_i \in A} x_i - \frac{1}{l} \sum_{x_j \in S \setminus A} x_j \right| \leq \frac{k+l}{2kl}$$

Prove that the number of nice subsets is at least $\dfrac{2}{k+l}\dbinom{k+l}{k}$.

*Proposed by Andrey Badzyan, Russia*

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6 For $n \geq 2$, let $S_1$, $S_2$, $\ldots$, $S_{2^n}$ be $2^n$ subsets of $A = \{1, 2, 3, \ldots, 2^{n+1}\}$ that satisfy the following property: There do not exist indices $a$ and $b$ with $a < b$ and elements $x$, $y$, $z \in A$ with $x < y < z$ and $y$, $z \in S_a$, and $x$, $z \in S_b$. Prove that at least one of the sets $S_1$, $S_2$, $\ldots$, $S_{2^n}$ contains no more than $4n$ elements.

*Proposed by Gerhard Woeginger, Netherlands*

– Geometry

**1** Let $H$ be the orthocenter of an acute-angled triangle $ABC$. The circle $\Gamma_A$ centered at the midpoint of $BC$ and passing through $H$ intersects the sideline $BC$ at points $A_1$ and $A_2$. Similarly, define the points $B_1$, $B_2$, $C_1$ and $C_2$.

Prove that the six points $A_1$, $A_2$, $B_1$, $B_2$, $C_1$ and $C_2$ are concyclic.

*Author: Andrey Gavrilyuk, Russia*

**2** Given trapezoid $ABCD$ with parallel sides $AB$ and $CD$, assume that there exist points $E$ on line $BC$ outside segment $BC$, and $F$ inside segment $AD$ such that $\angle DAE = \angle CBF$. Denote by $I$ the point of intersection of $CD$ and $EF$, and by $J$ the point of intersection of $AB$ and $EF$. Let $K$ be the midpoint of segment $EF$, assume it does not lie on line $AB$. Prove that $I$ belongs to the circumcircle of $ABK$ if and only if $K$ belongs to the circumcircle of $CDJ$.

*Proposed by Charles Leytem, Luxembourg*

**3** Let $ABCD$ be a convex quadrilateral and let $P$ and $Q$ be points in $ABCD$ such that $PQDA$ and $QPBC$ are cyclic quadrilaterals. Suppose that there exists a point $E$ on the line segment $PQ$ such that $\angle PAE = \angle QDE$ and $\angle PBE = \angle QCE$. Show that the quadrilateral $ABCD$ is cyclic.

*Proposed by John Cuya, Peru*

**4** In an acute triangle $ABC$ segments $BE$ and $CF$ are altitudes. Two circles passing through the point $A$ anf $F$ and tangent to the line $BC$ at the points $P$ and $Q$ so that $B$ lies between $C$ and $Q$. Prove that lines $PE$ and $QF$ intersect on the circumcircle of triangle $AEF$.

*Proposed by Davood Vakili, Iran*

**5** Let $k$ and $n$ be integers with $0 \le k \le n-2$. Consider a set $L$ of $n$ lines in the plane such that no two of them are parallel and no three have a common point. Denote by $I$ the set of intersections of lines in $L$. Let $O$ be a point in the plane not lying on any line of $L$. A point $X \in I$ is colored red if the open line segment $OX$ intersects at most $k$ lines in $L$. Prove that $I$ contains at least $\frac{1}{2}(k+1)(k+2)$ red points.

*Proposed by Gerhard Woeginger, Netherlands*

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**6** There is given a convex quadrilateral $ABCD$. Prove that there exists a point $P$ inside the quadrilateral such that

$\angle PAB + \angle PDC = \angle PBC + \angle PAD = \angle PCD + \angle PBA = \angle PDA + \angle PCB = 90^\circ$

if and only if the diagonals $AC$ and $BD$ are perpendicular.

*Proposed by Dusan Djukic, Serbia*

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**7** Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of triangles $ABC$ and $ADC$ by $\omega_1$ and $\omega_2$ respectively. Suppose that there exists a circle $\omega$ tangent to ray $BA$ beyond $A$ and to the ray $BC$ beyond $C$, which is also tangent to the lines $AD$ and $CD$. Prove that the common external tangents to $\omega_1$ and $\omega_2$ intersect on $\omega$.

*Author: Vladimir Shmarov, Russia*

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– Number Theory

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**1** Let $n$ be a positive integer and let $p$ be a prime number. Prove that if $a$, $b$, $c$ are integers (not necessarily positive) satisfying the equations

$$a^n + pb = b^n + pc = c^n + pa$$

then $a = b = c$.

*Proposed by Angelo Di Pasquale, Australia*

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**2** Let $a_1$, $a_2$, $\ldots$, $a_n$ be distinct positive integers, $n \ge 3$. Prove that there exist distinct indices $i$ and $j$ such that $a_i + a_j$ does not divide any of the numbers $3a_1$, $3a_2$, $\ldots$, $3a_n$.

*Proposed by Mohsen Jamaali, Iran*

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**3** Let $a_0$, $a_1$, $a_2$, $\ldots$ be a sequence of positive integers such that the greatest common divisor of any two consecutive terms is greater than the preceding term; in symbols, $\gcd(a_i, a_{i+1}) > a_{i-1}$. Prove that $a_n \ge 2^n$ for all $n \ge 0$.

*Proposed by Morteza Saghafian, Iran*

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**4** Let $n$ be a positive integer. Show that the numbers

$$\binom{2^n-1}{0}, \binom{2^n-1}{1}, \binom{2^n-1}{2}, \ldots, \binom{2^n-1}{2^{n-1}-1}$$

are congruent modulo $2^n$ to 1, 3, 5, $\ldots$, $2^n-1$ in some order.

*Proposed by Duskan Dukic, Serbia*

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**5** For every $n \in \mathbb{N}$ let $d(n)$ denote the number of (positive) divisors of $n$. Find all functions $f : \mathbb{N} \to \mathbb{N}$ with the following properties: - $d\left(f(x)\right) = x$ for all $x \in \mathbb{N}$. - $f(xy)$ divides $(x-1)y^{xy-1}f(x)$ for all $x$, $y \in \mathbb{N}$.

*Proposed by Bruno Le Floch, France*

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**6** Prove that there are infinitely many positive integers $n$ such that $n^2+1$ has a prime divisor greater than $2n+\sqrt{2n}$.

*Author: Kestutis Cesnavicius, Lithuania*

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IMO Shortlist 2007

– Algebra

**1** Real numbers $a_1$, $a_2$, …, $a_n$ are given. For each $i$, $(1 \leq i \leq n)$, define

$$d_i = \max\{a_j \mid 1 \leq j \leq i\} - \min\{a_j \mid i \leq j \leq n\}$$

and let $d = \max\{d_i \mid 1 \leq i \leq n\}$.

(a) Prove that, for any real numbers $x_1 \leq x_2 \leq \cdots \leq x_n$,

$$\max\{|x_i - a_i| \mid 1 \leq i \leq n\} \geq \frac{d}{2}. \qquad (*)$$

(b) Show that there are real numbers $x_1 \leq x_2 \leq \cdots \leq x_n$ such that the equality holds in (*).

*Author: Michael Albert, New Zealand*

**2** Consider those functions $f : \mathbb{N} \mapsto \mathbb{N}$ which satisfy the condition

$$f(m+n) \geq f(m) + f(f(n)) - 1$$

for all $m, n \in \mathbb{N}$. Find all possible values of $f(2007)$.

*Author: Nikolai Nikolov, Bulgaria*

**3** Let $n$ be a positive integer, and let $x$ and $y$ be a positive real number such that $x^n + y^n = 1$. Prove that

$$\left(\sum_{k=1}^{n} \frac{1+x^{2k}}{1+x^{4k}}\right) \cdot \left(\sum_{k=1}^{n} \frac{1+y^{2k}}{1+y^{4k}}\right) < \frac{1}{(1-x)\cdot(1-y)}.$$

*Author: Juhan Aru, Estonia*

**4** Find all functions $f : \mathbb{R}^{+} \to \mathbb{R}^{+}$ satisfying $f\left(x + f\left(y\right)\right) = f\left(x + y\right) + f\left(y\right)$ for all pairs of positive reals $x$ and $y$. Here, $\mathbb{R}^{+}$ denotes the set of all positive reals.

*Proposed by Paisan Nakmahachalasint, Thailand*


Contributors: Valentin Vornicu, orl, the.sceth, freemind, Omid Hatami, delegat, April, discredit, pohoatza, Dida Drogbier, ashwath.rabindranath

**5** Let $c > 2$, and let $a(1), a(2), \ldots$ be a sequence of nonnegative real numbers such that

$$a(m+n) \leq 2 \cdot a(m) + 2 \cdot a(n) \text{ for all } m, n \geq 1,$$

and $a\left(2^k\right) \leq \frac{1}{(k+1)^c}$ for all $k \geq 0$. Prove that the sequence $a(n)$ is bounded.

*Author: Vjekoslav Kova, Croatia*

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**6** Let $a_1, a_2, \ldots, a_{100}$ be nonnegative real numbers such that $a_1^2 + a_2^2 + \ldots + a_{100}^2 = 1$. Prove that

$$a_1^2 \cdot a_2 + a_2^2 \cdot a_3 + \ldots + a_{100}^2 \cdot a_1 < \frac{12}{25}.$$

*Author: Marcin Kuzma, Poland*

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**7** Let $n$ be a positive integer. Consider

$$S = \{(x, y, z) \mid x, y, z \in \{0, 1, \ldots, n\}, x + y + z > 0\}$$

as a set of $(n+1)^3 - 1$ points in the three-dimensional space. Determine the smallest possible number of planes, the union of which contains $S$ but does not include $(0, 0, 0)$.

*Author: Gerhard Wginger, Netherlands*

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**–** Combinatorics

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**1** Let $n > 1$ be an integer. Find all sequences $a_1, a_2, \ldots a_{n^2+n}$ satisfying the following conditions:

$$\text{(a) } a_i \in \{0, 1\} \text{ for all } 1 \leq i \leq n^2 + n;$$

$$\text{(b) } a_{i+1} + a_{i+2} + \ldots + a_{i+n} < a_{i+n+1} + a_{i+n+2} + \ldots + a_{i+2n} \text{ for all } 0 \leq i \leq n^2 - n.$$

*Author: Dusan Dukic, Serbia*

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**2** A rectangle $D$ is partitioned in several ($\geq 2$) rectangles with sides parallel to those of $D$. Given that any line parallel to one of the sides of $D$, and having common points with the interior of $D$, also has common interior points with the interior of at least one rectangle of the partition; prove that there is at least one rectangle of the partition having no common points with $D$'s boundary.

*Author: Kei Irie, Japan*

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**3** Find all positive integers $n$ for which the numbers in the set $S = \{1, 2, \ldots, n\}$ can be colored red and blue, with the following condition being satisfied: The set $S \times S \times S$ contains exactly 2007 ordered triples $(x, y, z)$ such that:

**(i)** the numbers $x$, $y$, $z$ are of the same color,
and
**(ii)** the number $x + y + z$ is divisible by $n$.

*Author: Gerhard Wginger, Netherlands*

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**4** Let $A_0 = (a_1, \ldots, a_n)$ be a finite sequence of real numbers. For each $k \geq 0$, from the sequence $A_k = (x_1, \ldots, x_k)$ we construct a new sequence $A_{k+1}$ in the following way.
1. We choose a partition $\{1, \ldots, n\} = I \cup J$, where $I$ and $J$ are two disjoint sets, such that the expression

$$\left| \sum_{i \in I} x_i - \sum_{j \in J} x_j \right|$$

attains the smallest value. (We allow $I$ or $J$ to be empty; in this case the corresponding sum is 0.) If there are several such partitions, one is chosen arbitrarily.
2. We set $A_{k+1} = (y_1, \ldots, y_n)$ where $y_i = x_i + 1$ if $i \in I$, and $y_i = x_i - 1$ if $i \in J$.
Prove that for some $k$, the sequence $A_k$ contains an element $x$ such that $|x| \geq \frac{n}{2}$.

*Author: Omid Hatami, Iran*

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**5** In the Cartesian coordinate plane define the strips $S_n = \{(x, y) | n \leq x < n + 1\}$, $n \in \mathbb{Z}$ and color each strip black or white. Prove that any rectangle which is not a square can be placed in the plane so that its vertices have the same color.

**IMO Shortlist 2007 Problem C5 as it appears in the official booklet:** In the Cartesian coordinate plane define the strips $S_n = \{(x, y) | n \leq x < n + 1\}$ for every integer $n$. Assume each strip $S_n$ is colored either red or blue, and let $a$ and $b$ be two distinct positive integers. Prove that there exists a rectangle with side length $a$ and $b$ such that its vertices have the same color.
(*Edited by Orlando Dhring*)

*Author: Radu Gologan and Dan Schwarz, Romania*

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**6** In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a *clique* if each two of them are friends. (In particular, any group of fewer than two competitions is a clique.) The number of members of a clique is called its *size*.

Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged into two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

*Author: Vasily Astakhov, Russia*

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**7** Let $\alpha < \frac{3-\sqrt{5}}{2}$ be a positive real number. Prove that there exist positive integers $n$ and $p > \alpha \cdot 2^n$ for which one can select $2 \cdot p$ pairwise distinct subsets $S_1, \ldots, S_p, T_1, \ldots, T_p$ of the set $\{1, 2, \ldots, n\}$ such that $S_i \cap T_j \neq \emptyset$ for all $1 \leq i, j \leq p$

*Author: Gerhard Wginger, Austria*

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**8** Given is a convex polygon $P$ with $n$ vertices. Triangle whose vertices lie on vertices of $P$ is called *good* if all its sides are equal in length. Prove that there are at most $\frac{2n}{3}$ *good* triangles.

*Author: Vyacheslav Yasinskiy, Ukraine*

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– Geometry

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**1** In triangle $ABC$ the bisector of angle $BCA$ intersects the circumcircle again at $R$, the perpendicular bisector of $BC$ at $P$, and the perpendicular bisector of $AC$ at $Q$. The midpoint of $BC$ is $K$ and the midpoint of $AC$ is $L$. Prove that the triangles $RPK$ and $RQL$ have the same area.

*Author: Marek Pechal, Czech Republic*

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**2** Denote by $M$ midpoint of side $BC$ in an isosceles triangle $\triangle ABC$ with $AC = AB$. Take a point $X$ on a smaller arc $\overset{\frown}{MA}$ of circumcircle of triangle $\triangle ABM$. Denote by $T$ point inside of angle $BMA$ such that $\angle TMX = 90$ and $TX = BX$.

Prove that $\angle MTB - \angle CTM$ does not depend on choice of $X$.

*Author: Farzan Barekat, Canada*

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**3** The diagonals of a trapezoid $ABCD$ intersect at point $P$. Point $Q$ lies between the parallel lines $BC$ and $AD$ such that $\angle AQD = \angle CQB$, and line $CD$ separates points $P$ and $Q$. Prove that $\angle BQP = \angle DAQ$.

*Author: Vyacheslav Yasinskiy, Ukraine*

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**4** Consider five points $A$, $B$, $C$, $D$ and $E$ such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let $\ell$ be a line passing through $A$. Suppose that $\ell$ intersects the interior of the segment $DC$ at $F$ and intersects line $BC$ at $G$. Suppose also that $EF = EG = EC$. Prove that $\ell$ is the bisector of angle $DAB$.

*Author: Charles Leytem, Luxembourg*

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**5** Let $ABC$ be a fixed triangle, and let $A_1$, $B_1$, $C_1$ be the midpoints of sides $BC$, $CA$, $AB$, respectively. Let $P$ be a variable point on the circumcircle. Let lines $PA_1$, $PB_1$, $PC_1$ meet the circumcircle again at $A'$, $B'$, $C'$, respectively. Assume that the points $A$, $B$, $C$, $A'$, $B'$, $C'$ are distinct, and lines $AA'$, $BB'$, $CC'$ form a triangle. Prove that the area of this triangle does not depend on $P$.

*Author: Christopher Bradley, United Kingdom*

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**6** Determine the smallest positive real number $k$ with the following property. Let $ABCD$ be a convex quadrilateral, and let points $A_1$, $B_1$, $C_1$, and $D_1$ lie on sides $AB$, $BC$, $CD$, and $DA$, respectively. Consider the areas of triangles $AA_1D_1$, $BB_1A_1$, $CC_1B_1$ and $DD_1C_1$; let $S$ be the sum of the two smallest ones, and let $S_1$ be the area of quadrilateral $A_1B_1C_1D_1$. Then we always have $kS_1 \geq S$.

*Author: Zuming Feng and Oleg Golberg, USA*

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**7** Given an acute triangle $ABC$ with $\angle B > \angle C$. Point $I$ is the incenter, and $R$ the circumradius. Point $D$ is the foot of the altitude from vertex $A$. Point $K$ lies on line $AD$ such that $AK = 2R$, and $D$ separates $A$ and $K$. Lines $DI$ and $KI$ meet sides $AC$ and $BC$ at $E, F$ respectively. Let $IE = IF$.

Prove that $\angle B \leq 3\angle C$.

*Author: Davoud Vakili, Iran*

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**8** Point $P$ lies on side $AB$ of a convex quadrilateral $ABCD$. Let $\omega$ be the incircle of triangle $CPD$, and let $I$ be its incenter. Suppose that $\omega$ is tangent to the incircles of triangles $APD$ and $BPC$ at points $K$ and $L$, respectively. Let lines $AC$ and $BD$ meet at $E$, and let lines $AK$ and $BL$ meet at $F$. Prove that points $E$, $I$, and $F$ are collinear.

*Author: Waldemar Pompe, Poland*

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– Number Theory

**1** Find all pairs of natural numbers $(a, b)$ such that $7^a - 3^b$ divides $a^4 + b^2$.

*Author: Stephan Wagner, Austria*

**2** Let $b, n > 1$ be integers. Suppose that for each $k > 1$ there exists an integer $a_k$ such that $b - a_k^n$ is divisible by $k$. Prove that $b = A^n$ for some integer $A$.

*Author: Dan Brown, Canada*

**3** Let $X$ be a set of 10,000 integers, none of them is divisible by 47. Prove that there exists a 2007-element subset $Y$ of $X$ such that $a - b + c - d + e$ is not divisible by 47 for any $a, b, c, d, e \in Y$.

*Author: Gerhard Wginger, Netherlands*

**4** For every integer $k \geq 2$, prove that $2^{3k}$ divides the number

$$\binom{2^{k+1}}{2^k} - \binom{2^k}{2^{k-1}}$$

but $2^{3k+1}$ does not.

*Author: Waldemar Pompe, Poland*

**5** Find all surjective functions $f : \mathbb{N} \to \mathbb{N}$ such that for every $m, n \in \mathbb{N}$ and every prime $p$, the number $f(m+n)$ is divisible by $p$ if and only if $f(m) + f(n)$ is divisible by $p$.

*Author: Mohsen Jamaali and Nima Ahmadi Pour Anari, Iran*

**6** Let $k$ be a positive integer. Prove that the number $(4 \cdot k^2 - 1)^2$ has a positive divisor of the form $8kn - 1$ if and only if $k$ is even.

Actual IMO 2007 Problem, posed as question 5 in the contest, which was used as a lemma in the official solutions for problem N6 as shown above. (`http://www.mathlinks.ro/viewtopic.php?p=89465#894656`)

*Author: Kevin Buzzard and Edward Crane, United Kingdom*

**7** For a prime $p$ and a given integer $n$ let $\nu_p(n)$ denote the exponent of $p$ in the prime factorisation of $n!$. Given $d \in \mathbb{N}$ and $\{p_1, p_2, \ldots, p_k\}$ a set of $k$ primes, show that there are infinitely many positive integers $n$ such that $d \mid \nu_{p_i}(n)$ for all $1 \leq i \leq k$.

*Author: Tejaswi Navilarekkallu, India*

IMO Shortlist 2006

– Algebra

**1** A sequence of real numbers $a_0,\ a_1,\ a_2,\dots$ is defined by the formula

$$a_{i+1} = \lfloor a_i \rfloor \cdot \langle a_i \rangle \qquad \text{for} \quad i \geq 0;$$

here $a_0$ is an arbitrary real number, $\lfloor a_i \rfloor$ denotes the greatest integer not exceeding $a_i$, and $\langle a_i \rangle = a_i - \lfloor a_i \rfloor$. Prove that $a_i = a_{i+2}$ for $i$ sufficiently large.

*Proposed by Harmel Nestra, Estionia*

**2** The sequence of real numbers $a_0, a_1, a_2, \dots$ is defined recursively by

$$a_0 = -1, \qquad \sum_{k=0}^{n} \frac{a_{n-k}}{k+1} = 0 \quad \text{for} \quad n \geq 1.$$

Show that $a_n > 0$ for all $n \geq 1$.

*Proposed by Mariusz Skalba, Poland*

**3** The sequence $c_0, c_1, ..., c_n, ...$ is defined by $c_0 = 1, c_1 = 0$, and $c_{n+2} = c_{n+1} + c_n$ for $n \geq 0$. Consider the set $S$ of ordered pairs $(x, y)$ for which there is a finite set $J$ of positive integers such that $x = \sum_{j \in J} c_j$, $y = \sum_{j \in J} c_{j-1}$. Prove that there exist real numbers $\alpha$, $\beta$, and $M$ with the following property: An ordered pair of nonnegative integers $(x, y)$ satisfies the inequality

$$m < \alpha x + \beta y < M$$

if and only if $(x, y) \in S$.

*Remark:* A sum over the elements of the empty set is assumed to be 0.

**4** Prove the inequality:

$$\sum_{i<j} \frac{a_i a_j}{a_i + a_j} \leq \frac{n}{2(a_1 + a_2 + \cdots + a_n)} \cdot \sum_{i<j} a_i a_j$$

for positive reals $a_1, a_2, \dots, a_n$.

*Proposed by Dusan Dukic, Serbia*

**5** If $a, b, c$ are the sides of a triangle, prove that

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}}+\frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}}+\frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \leq 3$$

*Proposed by Hojoo Lee, Korea*

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**6** Determine the least real number $M$ such that the inequality

$$|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)| \leq M(a^2+b^2+c^2)^2$$

holds for all real numbers $a$, $b$ and $c$.

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– Combinatorics

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**1** We have $n \geq 2$ lamps $L_1, ..., L_n$ in a row, each of them being either on or off. Every second we simultaneously modify the state of each lamp as follows: if the lamp $L_i$ and its neighbours (only one neighbour for $i = 1$ or $i = n$, two neighbours for other $i$) are in the same state, then $L_i$ is switched off; otherwise, $L_i$ is switched on.
Initially all the lamps are off except the leftmost one which is on.
$(a)$ Prove that there are infinitely many integers $n$ for which all the lamps will eventually be off. $(b)$ Prove that there are infinitely many integers $n$ for which the lamps will never be all off.

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**2** Let $P$ be a regular 2006-gon. A diagonal is called *good* if its endpoints divide the boundary of $P$ into two parts, each composed of an odd number of sides of $P$. The sides of $P$ are also called *good*.
Suppose $P$ has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of $P$. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.

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**3** Let $S$ be a finite set of points in the plane such that no three of them are on a line. For each convex polygon $P$ whose vertices are in $S$, let $a(P)$ be the number of vertices of $P$, and let $b(P)$ be the number of points of $S$ which are outside $P$. A line segment, a point, and the empty set are considered as convex polygons of 2, 1, and 0 vertices respectively. Prove that for every real number $x$

$$\sum_P x^{a(P)}(1-x)^{b(P)} = 1,$$

where the sum is taken over all convex polygons with vertices in $S$.

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*Alternative formulation*:

Let $M$ be a finite point set in the plane and no three points are collinear. A subset $A$ of $M$ will be called round if its elements is the set of vertices of a convex $A-$gon $V(A)$. For each round subset let $r(A)$ be the number of points from $M$ which are exterior from the convex $A-$gon $V(A)$. Subsets with $0,1$ and 2 elements are always round, its corresponding polygons are the empty set, a point or a segment, respectively (for which all other points that are not vertices of the polygon are exterior). For each round subset $A$ of $M$ construct the polynomial

$$P_A(x) = x^{|A|}(1-x)^{r(A)}.$$

Show that the sum of polynomials for all round subsets is exactly the polynomial $P(x) = 1$.

*Proposed by Federico Ardila, Colombia*

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**4** A cake has the form of an $n$ x $n$ square composed of $n^2$ unit squares. Strawberries lie on some of the unit squares so that each row or column contains exactly one strawberry; call this arrangement $\mathcal{A}$.

Let $\mathcal{B}$ be another such arrangement. Suppose that every grid rectangle with one vertex at the top left corner of the cake contains no fewer strawberries of arrangement $\mathcal{B}$ than of arrangement $\mathcal{A}$. Prove that arrangement $\mathcal{B}$ can be obtained from $\mathcal{A}$ by performing a number of switches, defined as follows:

A switch consists in selecting a grid rectangle with only two strawberries, situated at its top right corner and bottom left corner, and moving these two strawberries to the other two corners of that rectangle.

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**5** An $(n,k)-$ tournament is a contest with $n$ players held in $k$ rounds such that:

$(i)$ Each player plays in each round, and every two players meet at most once.
$(ii)$ If player $A$ meets player $B$ in round $i$, player $C$ meets player $D$ in round $i$, and player $A$ meets player $C$ in round $j$, then player $B$ meets player $D$ in round $j$.

Determine all pairs $(n,k)$ for which there exists an $(n,k)-$ tournament.

*Proposed by Carlos di Fiore, Argentina*

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**6** A holey triangle is an upward equilateral triangle of side length $n$ with $n$ upward unit triangular holes cut out. A diamond is a $60^\circ - 120^\circ$ unit rhombus.
Prove that a holey triangle $T$ can be tiled with diamonds if and only if the

following condition holds: Every upward equilateral triangle of side length $k$ in $T$ contains at most $k$ holes, for $1 \leq k \leq n$.

*Proposed by Federico Ardila, Colombia*

**7** Consider a convex polyhedron without parallel edges and without an edge parallel to any face other than the two faces adjacent to it. Call a pair of points of the polyhedron *antipodal* if there exist two parallel planes passing through these points and such that the polyhedron is contained between these planes. Let $A$ be the number of antipodal pairs of vertices, and let $B$ be the number of antipodal pairs of midpoint edges. Determine the difference $A - B$ in terms of the numbers of vertices, edges, and faces.

*Proposed by Kei Irei, Japan*

– Geometry

**1** Let $ABC$ be triangle with incenter $I$. A point $P$ in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

**2** Let $ABC$ be a trapezoid with parallel sides $AB > CD$. Points $K$ and $L$ lie on the line segments $AB$ and $CD$, respectively, so that $AK/KB = DL/LC$. Suppose that there are points $P$ and $Q$ on the line segment $KL$ satisfying

$$\angle APB = \angle BCD \qquad \text{and} \qquad \angle CQD = \angle ABC.$$

Prove that the points $P$, $Q$, $B$ and $C$ are concyclic.

*Proposed by Vyacheslev Yasinskiy, Ukraine*

**3** Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \qquad \text{and} \qquad \angle ABC = \angle ACD = \angle ADE.$$

The diagonals $BD$ and $CE$ meet at $P$. Prove that the line $AP$ bisects the side $CD$.

*Proposed by Zuming Feng, USA*

**4** A point $D$ is chosen on the side $AC$ of a triangle $ABC$ with $\angle C < \angle A < 90^\circ$ in such a way that $BD = BA$. The incircle of $ABC$ is tangent to $AB$ and $AC$ at points $K$ and $L$, respectively. Let $J$ be the incenter of triangle $BCD$. Prove that the line $KL$ intersects the line segment $AJ$ at its midpoint.

**5** In triangle $ABC$, let $J$ be the center of the excircle tangent to side $BC$ at $A_1$ and to the extensions of the sides $AC$ and $AB$ at $B_1$ and $C_1$ respectively. Suppose that the lines $A_1B_1$ and $AB$ are perpendicular and intersect at $D$. Let $E$ be the foot of the perpendicular from $C_1$ to line $DJ$. Determine the angles $\angle BEA_1$ and $\angle AEB_1$.

*Proposed by Dimitris Kontogiannis, Greece*

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**6** Circles $w_1$ and $w_2$ with centres $O_1$ and $O_2$ are externally tangent at point $D$ and internally tangent to a circle $w$ at points $E$ and $F$ respectively. Line $t$ is the common tangent of $w_1$ and $w_2$ at $D$. Let $AB$ be the diameter of $w$ perpendicular to $t$, so that $A, E, O_1$ are on the same side of $t$. Prove that lines $AO_1$, $BO_2$, $EF$ and $t$ are concurrent.

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**7** In a triangle $ABC$, let $M_a$, $M_b$, $M_c$ be the midpoints of the sides $BC$, $CA$, $AB$, respectively, and $T_a$, $T_b$, $T_c$ be the midpoints of the arcs $BC$, $CA$, $AB$ of the circumcircle of $ABC$, not containing the vertices $A$, $B$, $C$, respectively. For $i \in \{a, b, c\}$, let $w_i$ be the circle with $M_iT_i$ as diameter. Let $p_i$ be the common external common tangent to the circles $w_j$ and $w_k$ (for all $\{i, j, k\} = \{a, b, c\}$) such that $w_i$ lies on the opposite side of $p_i$ than $w_j$ and $w_k$ do.
Prove that the lines $p_a$, $p_b$, $p_c$ form a triangle similar to $ABC$ and find the ratio of similitude.

*Proposed by Tomas Jurik, Slovakia*

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**8** Let $ABCD$ be a convex quadrilateral. A circle passing through the points $A$ and $D$ and a circle passing through the points $B$ and $C$ are externally tangent at a point $P$ inside the quadrilateral. Suppose that

$$\angle PAB + \angle PDC \leq 90^\circ \qquad \text{and} \qquad \angle PBA + \angle PCD \leq 90^\circ.$$

Prove that $AB + CD \geq BC + AD$.

*Proposed by Waldemar Pompe, Poland*

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**9** Points $A_1$, $B_1$, $C_1$ are chosen on the sides $BC$, $CA$, $AB$ of a triangle $ABC$ respectively. The circumcircles of triangles $AB_1C_1$, $BC_1A_1$, $CA_1B_1$ intersect the circumcircle of triangle $ABC$ again at points $A_2$, $B_2$, $C_2$ respectively ($A_2 \neq A, B_2 \neq B, C_2 \neq C$). Points $A_3$, $B_3$, $C_3$ are symmetric to $A_1$, $B_1$, $C_1$ with respect to the midpoints of the sides $BC$, $CA$, $AB$ respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

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**10** Assign to each side $b$ of a convex polygon $P$ the maximum area of a triangle that has $b$ as a side and is contained in $P$. Show that the sum of the areas assigned to the sides of $P$ is at least twice the area of $P$.

– Number Theory

**1** Determine all pairs $(x, y)$ of integers such that

$$1 + 2^x + 2^{2x+1} = y^2.$$

**2** For $x \in (0, 1)$ let $y \in (0, 1)$ be the number whose $n$-th digit after the decimal point is the $2^n$-th digit after the decimal point of $x$. Show that if $x$ is rational then so is $y$.

*Proposed by J.P. Grossman, Canada*

**3** We define a sequence $(a_1, a_2, a_3, \ldots)$ by

$$a_n = \frac{1}{n}\left(\left\lfloor\frac{n}{1}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + \cdots + \left\lfloor\frac{n}{n}\right\rfloor\right),$$

where $\lfloor x \rfloor$ denotes the integer part of $x$.

**a)** Prove that $a_{n+1} > a_n$ infinitely often.
**b)** Prove that $a_{n+1} < a_n$ infinitely often.

*Proposed by Johan Meyer, South Africa*

**4** Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let $k$ be a positive integer. Consider the polynomial $Q(x) = P(P(\ldots P(P(x))\ldots))$, where $P$ occurs $k$ times. Prove that there are at most $n$ integers $t$ such that $Q(t) = t$.

**5** Find all integer solutions of the equation

$$\frac{x^7 - 1}{x - 1} = y^5 - 1.$$

**6** Let $a > b > 1$ be relatively prime positive integers. Define the weight of an integer $c$, denoted by $w(c)$ to be the minimal possible value of $|x| + |y|$ taken over all pairs of integers $x$ and $y$ such that

$$ax + by = c.$$

An integer $c$ is called a *local champion* if $w(c) \geq w(c \pm a)$ and $w(c) \geq w(c \pm b)$.

Find all local champions and determine their number.

*Proposed by Zoran Sunic, USA*

**7** For all positive integers $n$, show that there exists a positive integer $m$ such that $n$ divides $2^m + m$.

*Proposed by Juhan Aru, Estonia*

IMO Shortlist 2005

– Algebra

**1** Find all pairs of integers $a, b$ for which there exists a polynomial $P(x) \in \mathbb{Z}[X]$ such that product $(x^2 + ax + b) \cdot P(x)$ is a polynomial of a form

$$x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$$

where each of $c_0, c_1, \ldots, c_{n-1}$ is equal to 1 or $-1$.

**2** We denote by $\mathbb{R}^+$ the set of all positive real numbers.

Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ which have the property:

$$f(x)f(y) = 2f(x + yf(x))$$

for all positive real numbers $x$ and $y$.

*Proposed by Nikolai Nikolov, Bulgaria*

**3** Four real numbers $p$, $q$, $r$, $s$ satisfy $p+q+r+s = 9$ and $p^2+q^2+r^2+s^2 = 21$. Prove that there exists a permutation $(a, b, c, d)$ of $(p, q, r, s)$ such that $ab-cd \geq 2$.

**4** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) + f(x)f(y) = f(xy) + 2xy + 1$ for all real numbers $x$ and $y$.

*Proposed by B.J. Venkatachala, India*

**5** Let $x, y, z$ be three positive reals such that $xyz \geq 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0.$$

*Hojoo Lee, Korea*

– Combinatorics

**1** A house has an even number of lamps distributed among its rooms in such a way that there are at least three lamps in every room. Each lamp shares a switch with exactly one other lamp, not necessarily from the same room. Each change in the switch shared by two lamps changes their states simultaneously. Prove that for every initial state of the lamps there exists a sequence of changes in some of the switches at the end of which each room contains lamps which are on as well as lamps which are off.

*Proposed by Australia*

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**2** This ISL 2005 problem has not been used in any TST I know. A pity, since it is a nice problem, but in its shortlist formulation, it is absolutely incomprehensible. Here is a mathematical restatement of the problem:

Let $k$ be a nonnegative integer.

A forest consists of rooted (i. e. oriented) trees. Each vertex of the forest is either a leaf or has two successors. A vertex $v$ is called an *extended successor* of a vertex $u$ if there is a chain of vertices $u_0 = u$, $u_1$, $u_2$, ..., $u_{t-1}$, $u_t = v$ with $t > 0$ such that the vertex $u_{i+1}$ is a successor of the vertex $u_i$ for every integer $i$ with $0 \leq i \leq t-1$. A vertex is called *dynastic* if it has two successors and each of these successors has at least $k$ extended successors.

Prove that if the forest has $n$ vertices, then there are at most $\frac{n}{k+2}$ dynastic vertices.

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**3** Consider a $m \times n$ rectangular board consisting of $mn$ unit squares. Two of its unit squares are called *adjacent* if they have a common edge, and a *path* is a sequence of unit squares in which any two consecutive squares are adjacent. Two parths are called *non-intersecting* if they don't share any common squares.

Each unit square of the rectangular board can be colored black or white. We speak of a *coloring* of the board if all its $mn$ unit squares are colored.

Let $N$ be the number of colorings of the board such that there exists at least one black path from the left edge of the board to its right edge. Let $M$ be the number of colorings of the board for which there exist at least two non-intersecting black paths from the left edge of the board to its right edge.

Prove that $N^2 \geq M \cdot 2^{mn}$.

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**4** Let $n \geq 3$ be a fixed integer. Each side and each diagonal of a regular $n$-gon is labelled with a number from the set $\{1; 2; ...; r\}$ in a way such that the following two conditions are fulfilled:

**1.** Each number from the set $\{1; 2; ...; r\}$ occurs at least once as a label.

**2.** In each triangle formed by three vertices of the $n$-gon, two of the sides are labelled with the same number, and this number is greater than the label of the third side.

**(a)** Find the maximal $r$ for which such a labelling is possible.

**(b)** *Harder version (IMO Shortlist 2005):* For this maximal value of $r$, how many such labellings are there?

*Easier version (5th German TST 2006):* Show that, for this maximal value of $r$, there are exactly $\frac{n!(n-1)!}{2^{n-1}}$ possible labellings.

*Proposed by Federico Ardila, Colombia*

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**5** There are $n$ markers, each with one side white and the other side black. In the beginning, these $n$ markers are aligned in a row so that their white sides are all up. In each step, if possible, we choose a marker whose white side is up (but not one of the outermost markers), remove it, and reverse the closest marker to the left of it and also reverse the closest marker to the right of it. Prove that, by a finite sequence of such steps, one can achieve a state with only two markers remaining if and only if $n-1$ is not divisible by 3.

*Proposed by Dusan Dukic, Serbia*

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**6** In a mathematical competition, in which 6 problems were posed to the participants, every two of these problems were solved by more than $\frac{2}{5}$ of the contestants. Moreover, no contestant solved all the 6 problems. Show that there are at least 2 contestants who solved exactly 5 problems each.

*Radu Gologan and Dan Schwartz*

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**7** Suppose that $a_1$, $a_2$, $\ldots$, $a_n$ are integers such that $n \mid a_1 + a_2 + \ldots + a_n$. Prove that there exist two permutations $(b_1, b_2, \ldots, b_n)$ and $(c_1, c_2, \ldots, c_n)$ of $(1, 2, \ldots, n)$ such that for each integer $i$ with $1 \le i \le n$, we have

$$n \mid a_i - b_i - c_i$$

*Proposed by Ricky Liu & Zuming Feng, USA*

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**8** Suppose we have a $n$-gon. Some $n-3$ diagonals are coloured black and some other $n-3$ diagonals are coloured red (a side is not a diagonal), so that no two diagonals of the same colour can intersect strictly inside the polygon, although they can share a vertex. Find the maximum number of intersection points between diagonals coloured differently strictly inside the polygon, in terms of $n$.

*Proposed by Alexander Ivanov, Bulgaria*

– Geometry

**1** Given a triangle $ABC$ satisfying $AC + BC = 3 \cdot AB$. The incircle of triangle $ABC$ has center $I$ and touches the sides $BC$ and $CA$ at the points $D$ and $E$, respectively. Let $K$ and $L$ be the reflections of the points $D$ and $E$ with respect to $I$. Prove that the points $A$, $B$, $K$, $L$ lie on one circle.

*Proposed by Dimitris Kontogiannis, Greece*

**2** Six points are chosen on the sides of an equilateral triangle $ABC$: $A_1$, $A_2$ on $BC$, $B_1$, $B_2$ on $CA$ and $C_1$, $C_2$ on $AB$, such that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths.

Prove that the lines $A_1B_2$, $B_1C_2$ and $C_1A_2$ are concurrent.

*Bogdan Enescu, Romania*

**3** Let $ABCD$ be a parallelogram. A variable line $g$ through the vertex $A$ intersects the rays $BC$ and $DC$ at the points $X$ and $Y$, respectively. Let $K$ and $L$ be the $A$-excenters of the triangles $ABX$ and $ADY$. Show that the angle $\measuredangle KCL$ is independent of the line $g$.

*Proposed by Vyacheslev Yasinskiy, Ukraine*

**4** Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and $BC$ not parallel with $DA$. Let two variable points $E$ and $F$ lie of the sides $BC$ and $DA$, respectively and satisfy $BE = DF$. The lines $AC$ and $BD$ meet at $P$, the lines $BD$ and $EF$ meet at $Q$, the lines $EF$ and $AC$ meet at $R$.

Prove that the circumcircles of the triangles $PQR$, as $E$ and $F$ vary, have a common point other than $P$.

**5** Let $\triangle ABC$ be an acute-angled triangle with $AB \neq AC$. Let $H$ be the orthocenter of triangle $ABC$, and let $M$ be the midpoint of the side $BC$. Let $D$ be a point on the side $AB$ and $E$ a point on the side $AC$ such that $AE = AD$ and the points $D$, $H$, $E$ are on the same line. Prove that the line $HM$ is perpendicular to the common chord of the circumscribed circles of triangle $\triangle ABC$ and triangle $\triangle ADE$.


Contributors: Megus, Nima Ahmadi Pour, who, andre.l, Valentin Vornicu, darij grinberg, Pentakratie, rohitsingh0812, epitomy01, Michal Marcinkowski, madatmath, BogG, Arne, Amir.S, Yimin Ge, k2c901_1, ZetaX

**6** Let $ABC$ be a triangle, and $M$ the midpoint of its side $BC$. Let $\gamma$ be the incircle of triangle $ABC$. The median $AM$ of triangle $ABC$ intersects the incircle $\gamma$ at two points $K$ and $L$. Let the lines passing through $K$ and $L$, parallel to $BC$, intersect the incircle $\gamma$ again in two points $X$ and $Y$. Let the lines $AX$ and $AY$ intersect $BC$ again at the points $P$ and $Q$. Prove that $BP = CQ$.

**7** In an acute triangle $ABC$, let $D$, $E$, $F$ be the feet of the perpendiculars from the points $A$, $B$, $C$ to the lines $BC$, $CA$, $AB$, respectively, and let $P$, $Q$, $R$ be the feet of the perpendiculars from the points $A$, $B$, $C$ to the lines $EF$, $FD$, $DE$, respectively.

Prove that $p\left(ABC\right)p\left(PQR\right) \geq \left(p\left(DEF\right)\right)^2$, where $p\left(T\right)$ denotes the perimeter of triangle $T$ .

*Proposed by Hojoo Lee, Korea*

**–** Number Theory

**1** Determine all positive integers relatively prime to all the terms of the infinite sequence
$$a_n = 2^n + 3^n + 6^n - 1,\ n \geq 1.$$

**2** Let $a_1, a_2, \ldots$ be a sequence of integers with infinitely many positive and negative terms. Suppose that for every positive integer $n$ the numbers $a_1, a_2, \ldots, a_n$ leave $n$ different remainders upon division by $n$.

Prove that every integer occurs exactly once in the sequence $a_1, a_2, \ldots$.

**3** Let $a$, $b$, $c$, $d$, $e$, $f$ be positive integers and let $S = a+b+c+d+e+f$. Suppose that the number $S$ divides $abc+def$ and $ab+bc+ca-de-ef-df$. Prove that $S$ is composite.

**4** Find all positive integers $n$ such that there exists a unique integer $a$ such that $0 \leq a < n!$ with the following property:
$$n! \mid a^n + 1$$

*Proposed by Carlos Caicedo, Colombia*

**5** Denote by $d(n)$ the number of divisors of the positive integer $n$. A positive integer $n$ is called highly divisible if $d(n) > d(m)$ for all positive integers $m < n$. Two highly divisible integers $m$ and $n$ with $m < n$ are called consecutive if there exists no highly divisible integer $s$ satisfying $m < s < n$.

(a) Show that there are only finitely many pairs of consecutive highly divisible
integers of the form $(a, b)$ with $a \mid b$.

(b) Show that for every prime number $p$ there exist infinitely many positive highly divisible integers $r$ such that $pr$ is also highly divisible.

**6** Let $a$, $b$ be positive integers such that $b^n + n$ is a multiple of $a^n + n$ for all positive integers $n$. Prove that $a = b$.

*Proposed by Mohsen Jamali, Iran*

**7** Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$, where $a_0, \ldots, a_n$ are integers, $a_n > 0$, $n \geq 2$. Prove that there exists a positive integer $m$ such that $P(m!)$ is a composite number.


Contributors: Megus, Nima Ahmadi Pour, who, andre.l, Valentin Vornicu, darij grinberg, Pentakratie, rohitsingh0812, epitomy01, Michal Marcinkowski, madatmath, BogG, Arne, Amir.S, Yimin Ge, k2c901_1, ZetaX

IMO Shortlist 2004

– Geometry

**1** 1. Let $ABC$ be an acute-angled triangle with $AB \neq AC$. The circle with diameter $BC$ intersects the sides $AB$ and $AC$ at $M$ and $N$ respectively. Denote by $O$ the midpoint of the side $BC$. The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect at $R$. Prove that the circumcircles of the triangles $BMR$ and $CNR$ have a common point lying on the side $BC$.

**2** Let $\Gamma$ be a circle and let $d$ be a line such that $\Gamma$ and $d$ have no common points. Further, let $AB$ be a diameter of the circle $\Gamma$; assume that this diameter $AB$ is perpendicular to the line $d$, and the point $B$ is nearer to the line $d$ than the point $A$. Let $C$ be an arbitrary point on the circle $\Gamma$, different from the points $A$ and $B$. Let $D$ be the point of intersection of the lines $AC$ and $d$. One of the two tangents from the point $D$ to the circle $\Gamma$ touches this circle $\Gamma$ at a point $E$; hereby, we assume that the points $B$ and $E$ lie in the same halfplane with respect to the line $AC$. Denote by $F$ the point of intersection of the lines $BE$ and $d$. Let the line $AF$ intersect the circle $\Gamma$ at a point $G$, different from $A$.

Prove that the reflection of the point $G$ in the line $AB$ lies on the line $CF$.

**3** Let $O$ be the circumcenter of an acute-angled triangle $ABC$ with $\angle B < \angle C$. The line $AO$ meets the side $BC$ at $D$. The circumcenters of the triangles $ABD$ and $ACD$ are $E$ and $F$, respectively. Extend the sides $BA$ and $CA$ beyond $A$, and choose on the respective extensions points $G$ and $H$ such that $AG = AC$ and $AH = AB$. Prove that the quadrilateral $EFGH$ is a rectangle if and only if $\angle ACB - \angle ABC = 60^\circ$.

*Proposed by Hojoo Lee, Korea*

**4** In a convex quadrilateral $ABCD$, the diagonal $BD$ bisects neither the angle $ABC$ nor the angle $CDA$. The point $P$ lies inside $ABCD$ and satisfies

$$\angle PBC = \angle DBA \quad \text{and} \quad \angle PDC = \angle BDA.$$

Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

**5** Let $A_1A_2A_3 \ldots A_n$ be a regular $n$-gon. Let $B_1$ and $B_n$ be the midpoints of its sides $A_1A_2$ and $A_{n-1}A_n$. Also, for every $i \in \{2, 3, 4, \ldots, n-1\}$. Let $S$ be the point of intersection of the lines $A_1A_{i+1}$ and $A_nA_i$, and let $B_i$ be the point of intersection of the angle bisector bisector of the angle $\measuredangle A_iSA_{i+1}$ with the segment $A_iA_{i+1}$.

Prove that $\sum_{i=1}^{n-1} \measuredangle A_1B_iA_n = 180^\circ$.

*Proposed by Dusan Dukic, Serbia and Montenegro*

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**6** Let $P$ be a convex polygon. Prove that there exists a convex hexagon that is contained in $P$ and whose area is at least $\frac{3}{4}$ of the area of the polygon $P$.

*Alternative version.* Let $P$ be a convex polygon with $n \geq 6$ vertices. Prove that there exists a convex hexagon with

**a)** vertices on the sides of the polygon (or)
**b)** vertices among the vertices of the polygon

such that the area of the hexagon is at least $\frac{3}{4}$ of the area of the polygon.

*Proposed by Ben Green and Edward Crane, United Kingdom*

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**7** For a given triangle $ABC$, let $X$ be a variable point on the line $BC$ such that $C$ lies between $B$ and $X$ and the incircles of the triangles $ABX$ and $ACX$ intersect at two distinct points $P$ and $Q$. Prove that the line $PQ$ passes through a point independent of $X$.

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**8** Given a cyclic quadrilateral $ABCD$, let $M$ be the midpoint of the side $CD$, and let $N$ be a point on the circumcircle of triangle $ABM$. Assume that the point $N$ is different from the point $M$ and satisfies $\frac{AN}{BN} = \frac{AM}{BM}$. Prove that the points $E$, $F$, $N$ are collinear, where $E = AC \cap BD$ and $F = BC \cap DA$.

*Proposed by Dusan Dukic, Serbia and Montenegro*

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– Number Theory

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**1** Let $\tau(n)$ denote the number of positive divisors of the positive integer $n$. Prove that there exist infinitely many positive integers $a$ such that the equation $\tau(an) = n$ does not have a positive integer solution $n$.

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**2** The function $f$ from the set $\mathbb{N}$ of positive integers into itself is defined by the equality

$$f(n) = \sum_{k=1}^{n} \gcd(k, n), \qquad n \in \mathbb{N}.$$

a) Prove that $f(mn) = f(m)f(n)$ for every two relatively prime $m, n \in \mathbb{N}$.

b) Prove that for each $a \in \mathbb{N}$ the equation $f(x) = ax$ has a solution.


Contributors: Valentin Vornicu, darij grinberg, Sasha, Pascual2005, Chang Woo-JIn, silouan, orl, pohoatza, grobber, vinoth_90_2004, Tales, ZetaX, pigfly, pbornsztein, Zorro, Peter Scholze, matematikator, jmerry, K09, pleurestique, Severius

c) Find all $a \in \mathbb{N}$ such that the equation $f(x) = ax$ has a unique solution.

**3** Find all functions $f : \mathbb{N}^* \to \mathbb{N}^*$ satisfying

$$\left(f^2\left(m\right)+f\left(n\right)\right) \mid \left(m^2+n\right)^2$$

for any two positive integers $m$ and $n$.

*Remark.* The abbreviation $\mathbb{N}^*$ stands for the set of all positive integers: $\mathbb{N}^* = \{1, 2, 3, ...\}$.
By $f^2\left(m\right)$, we mean $\left(f\left(m\right)\right)^2$ (and not $f\left(f\left(m\right)\right)$).

*Proposed by Mohsen Jamali, Iran*

**4** Let $k$ be a fixed integer greater than 1, and let $m = 4k^2 - 5$. Show that there exist positive integers $a$ and $b$ such that the sequence $(x_n)$ defined by

$$x_0 = a, \quad x_1 = b, \quad x_{n+2} = x_{n+1} + x_n \quad \text{for} \quad n = 0, 1, 2, \dots,$$

has all of its terms relatively prime to $m$.

*Proposed by Jaroslaw Wroblewski, Poland*

**5** We call a positive integer *alternating* if every two consecutive digits in its decimal representation are of different parity.

Find all positive integers $n$ such that $n$ has a multiple which is alternating.

**6** Given an integer $n > 1$, denote by $P_n$ the product of all positive integers $x$ less than $n$ and such that $n$ divides $x^2 - 1$. For each $n > 1$, find the remainder of $P_n$ on division by $n$.

*Proposed by John Murray, Ireland*

**7** Let $p$ be an odd prime and $n$ a positive integer. In the coordinate plane, eight distinct points with integer coordinates lie on a circle with diameter of length $p^n$. Prove that there exists a triangle with vertices at three of the given points such that the squares of its side lengths are integers divisible by $p^{n+1}$.

*Proposed by Alexander Ivanov, Bulgaria*

– Algebra

**1** Let $n \geq 3$ be an integer. Let $t_1$, $t_2$, ..., $t_n$ be positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + \cdots + t_n)\left(\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n}\right).$$

Show that $t_i$, $t_j$, $t_k$ are side lengths of a triangle for all $i$, $j$, $k$ with $1 \leq i < j < k \leq n$.

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**2** Let $a_0$, $a_1$, $a_2$, ... be an infinite sequence of real numbers satisfying the equation $a_n = |a_{n+1} - a_{n+2}|$ for all $n \geq 0$, where $a_0$ and $a_1$ are two different positive reals.

Can this sequence $a_0$, $a_1$, $a_2$, ... be bounded?

*Proposed by Mihai Blun, Romania*

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**3** Does there exist a function $s\colon \mathbb{Q} \rightarrow \{-1, 1\}$ such that if $x$ and $y$ are distinct rational numbers satisfying $xy = 1$ or $x + y \in \{0, 1\}$, then $s(x)s(y) = -1$? Justify your answer.

*Proposed by Dan Brown, Canada*

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**4** Find all polynomials $f$ with real coefficients such that for all reals $a, b, c$ such that $ab + bc + ca = 0$ we have the following relations

$$f(a-b) + f(b-c) + f(c-a) = 2f(a+b+c).$$

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**5** If $a$, $b$ ,$c$ are three positive real numbers such that $ab + bc + ca = 1$, prove that

$$\sqrt[3]{\frac{1}{a} + 6b} + \sqrt[3]{\frac{1}{b} + 6c} + \sqrt[3]{\frac{1}{c} + 6a} \leq \frac{1}{abc}.$$

---

**6** Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying the equation

$$f(x^2 + y^2 + 2f(xy)) = (f(x+y))^2.$$

for all $x, y \in \mathbb{R}$.

---

**7** Let $a_1, a_2, \ldots, a_n$ be positive real numbers, $n > 1$. Denote by $g_n$ their geometric mean, and by $A_1, A_2, \ldots, A_n$ the sequence of arithmetic means defined by

$$A_k = \frac{a_1 + a_2 + \cdots + a_k}{k}, \qquad k = 1, 2, \ldots, n.$$


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Let $G_n$ be the geometric mean of $A_1, A_2, \ldots, A_n$. Prove the inequality

$$n \sqrt[n]{\frac{G_n}{A_n}} + \frac{g_n}{G_n} \le n + 1$$

and establish the cases of equality.

*Proposed by Finbarr Holland, Ireland*

– Combinatorics

**1** There are 10001 students at an university. Some students join together to form several clubs (a student may belong to different clubs). Some clubs join together to form several societies (a club may belong to different societies). There are a total of $k$ societies. Suppose that the following conditions hold:

*i.)* Each pair of students are in exactly one club.

*ii.)* For each student and each society, the student is in exactly one club of the society.

*iii.)* Each club has an odd number of students. In addition, a club with $2m+1$ students ($m$ is a positive integer) is
in exactly $m$ societies.

Find all possible values of $k$.

*Proposed by Guihua Gong, Puerto Rico*

**2** Let $n$ and $k$ be positive integers. There are given $n$ circles in the plane. Every two of them intersect at two distinct points, and all points of intersection they determine are pairwise distinct (i. e. no three circles have a common point). No three circles have a point in common. Each intersection point must be colored with one of $n$ distinct colors so that each color is used at least once and exactly $k$ distinct colors occur on each circle. Find all values of $n \ge 2$ and $k$ for which such a coloring is possible.

*Proposed by Horst Sewerin, Germany*

**3** The following operation is allowed on a finite graph: Choose an arbitrary cycle of length 4 (if there is any), choose an arbitrary edge in that cycle, and delete it from the graph. For a fixed integer $n \ge 4$, find the least number of edges of a graph that can be obtained by repeated applications of this operation from the complete graph on $n$ vertices (where each pair of vertices are joined by an edge).

*Proposed by Norman Do, Australia*

**4** Consider a matrix of size $n \times n$ whose entries are real numbers of absolute value not exceeding 1. The sum of all entries of the matrix is 0. Let $n$ be an even positive integer. Determine the least number $C$ such that every such matrix necessarily has a row or a column with the sum of its entries not exceeding $C$ in absolute value.

*Proposed by Marcin Kuczma, Poland*

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**5** $A$ and $B$ play a game, given an integer $N$, $A$ writes down 1 first, then every player sees the last number written and if it is $n$ then in his turn he writes $n+1$ or $2n$, but his number cannot be bigger than $N$. The player who writes $N$ wins. For which values of $N$ does $B$ win?

*Proposed by A. Slinko & S. Marshall, New Zealand*

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**6** For an $n \times n$ matrix $A$, let $X_i$ be the set of entries in row $i$, and $Y_j$ the set of entries in column $j$, $1 \leq i, j \leq n$. We say that $A$ is *golden* if $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ are distinct sets. Find the least integer $n$ such that there exists a $2004 \times 2004$ golden matrix with entries in the set $\{1, 2, \ldots, n\}$.

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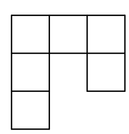
**7** Define a "hook" to be a figure made up of six unit squares as shown below in the picture, or any of the figures obtained by applying rotations and reflections to this figure.



Determine all $m \times n$ rectangles that can be covered without gaps and without overlaps with hooks such that

- the rectangle is covered without gaps and without overlaps
- no part of a hook covers area outside the rectangle.

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**8** For a finite graph $G$, let $f(G)$ be the number of triangles and $g(G)$ the number of tetrahedra formed by edges of $G$. Find the least constant $c$ such that

$$g(G)^3 \leq c \cdot f(G)^4$$

for every graph $G$.

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*Proposed by Marcin Kuczma, Poland*

IMO Shortlist 2003

– Geometry

**1** Let $ABCD$ be a cyclic quadrilateral. Let $P$, $Q$, $R$ be the feet of the perpendiculars from $D$ to the lines $BC$, $CA$, $AB$, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with $AC$.

**2** Given three fixed pairwisely distinct points $A$, $B$, $C$ lying on one straight line in this order. Let $G$ be a circle passing through $A$ and $C$ whose center does not lie on the line $AC$. The tangents to $G$ at $A$ and $C$ intersect each other at a point $P$. The segment $PB$ meets the circle $G$ at $Q$.

Show that the point of intersection of the angle bisector of the angle $AQC$ with the line $AC$ does not depend on the choice of the circle $G$.

**3** Let $ABC$ be a triangle, and $P$ a point in the interior of this triangle. Let $D$, $E$, $F$ be the feet of the perpendiculars from the point $P$ to the lines $BC$, $CA$, $AB$, respectively. Assume that

$AP^2 + PD^2 = BP^2 + PE^2 = CP^2 + PF^2$.

Furthermore, let $I_a$, $I_b$, $I_c$ be the excenters of triangle $ABC$. Show that the point $P$ is the circumcenter of triangle $I_aI_bI_c$.

*Proposed by C.R. Pranesachar, India*

**4** Let $\Gamma_1$, $\Gamma_2$, $\Gamma_3$, $\Gamma_4$ be distinct circles such that $\Gamma_1$, $\Gamma_3$ are externally tangent at $P$, and $\Gamma_2$, $\Gamma_4$ are externally tangent at the same point $P$. Suppose that $\Gamma_1$ and $\Gamma_2$; $\Gamma_2$ and $\Gamma_3$; $\Gamma_3$ and $\Gamma_4$; $\Gamma_4$ and $\Gamma_1$ meet at $A$, $B$, $C$, $D$, respectively, and that all these points are different from $P$. Prove that

$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}.$$

**5** Let $ABC$ be an isosceles triangle with $AC = BC$, whose incentre is $I$. Let $P$ be a point on the circumcircle of the triangle $AIB$ lying inside the triangle $ABC$. The lines through $P$ parallel to $CA$ and $CB$ meet $AB$ at $D$ and $E$, respectively. The line through $P$ parallel to $AB$ meets $CA$ and $CB$ at $F$ and $G$, respectively. Prove that the lines $DF$ and $EG$ intersect on the circumcircle of the triangle $ABC$.

*Proposed by Hojoo Lee, Korea*

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**6** Each pair of opposite sides of a convex hexagon has the following property: the distance between their midpoints is equal to $\dfrac{\sqrt{3}}{2}$ times the sum of their lengths. Prove that all the angles of the hexagon are equal.

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**7** Let $ABC$ be a triangle with semiperimeter $s$ and inradius $r$. The semicircles with diameters $BC$, $CA$, $AB$ are drawn on the outside of the triangle $ABC$. The circle tangent to all of these three semicircles has radius $t$. Prove that

$$\frac{s}{2} < t \le \frac{s}{2} + \left(1 - \frac{\sqrt{3}}{2}\right) r.$$

*Alternative formulation.* In a triangle $ABC$, construct circles with diameters $BC$, $CA$, and $AB$, respectively. Construct a circle $w$ externally tangent to these three circles. Let the radius of this circle $w$ be $t$.
Prove: $\frac{s}{2} < t \le \frac{s}{2} + \frac{1}{2}\left(2 - \sqrt{3}\right) r$, where $r$ is the inradius and $s$ is the semiperimeter of triangle $ABC$.

*Proposed by Dirk Laurie, South Africa*

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– Number Theory

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**1** Let $m$ be a fixed integer greater than 1. The sequence $x_0$, $x_1$, $x_2$, $\ldots$ is defined as follows:

$x_i = 2^i$ if $0 \le i \le m-1$ and $x_i = \sum_{j=1}^{m} x_{i-j}$, if $i \ge m$.

Find the greatest $k$ for which the sequence contains $k$ consecutive terms divisible by $m$ .

*Proposed by Marcin Kuczma, Poland*

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**2** Each positive integer $a$ is subjected to the following procedure, yielding the number $d = d(a)$:

**(a)** The last digit of $a$ is moved to the first position. The resulting number is called $b$.
**(b)** The number $b$ is squared. The resulting number is called $c$.
**(c)** The first digit of $c$ is moved to the last position. The resulting number is called $d$.

(All numbers are considered in the decimal system.) For instance, $a = 2003$ gives $b = 3200$, $c = 10240000$ and $d = 02400001 = 2400001 = d(2003)$.

Find all integers a such that $d(a) = a^2$.

*Proposed by Zoran Sunic, USA*

**3** Determine all pairs of positive integers $(a, b)$ such that

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer.

**4** Let $b$ be an integer greater than 5. For each positive integer $n$, consider the number

$$x_n = \underbrace{11\cdots1}_{n-1}\underbrace{22\cdots2}_{n}5,$$

written in base $b$.

Prove that the following condition holds if and only if $b = 10$: [i]there exists a positive integer $M$ such that for any integer $n$ greater than $M$, the number $x_n$ is a perfect square.[/i]

*Proposed by Laurentiu Panaitopol, Romania*

**5** An integer $n$ is said to be *good* if $|n|$ is not the square of an integer. Determine all integers $m$ with the following property: $m$ can be represented, in infinitely many ways, as a sum of three distinct good integers whose product is the square of an odd integer.

*Proposed by Hojoo Lee, Korea*

**6** Let $p$ be a prime number. Prove that there exists a prime number $q$ such that for every integer $n$, the number $n^p - p$ is not divisible by $q$.

**7** The sequence $a_0, a_1, a_2, \ldots$ is defined as follows: $a_0 = 2, \quad a_{k+1} = 2a_k^2 - 1$ for $k \geq 0$. Prove that if an odd prime $p$ divides $a_n$, then $2^{n+3}$ divides $p^2 - 1$.

Hi guys ,

Here is a nice problem:

Let be given a sequence $a_n$ such that $a_0 = 2$ and $a_{n+1} = 2a_n^2 - 1$ . Show that if $p$ is an odd prime such that $p|a_n$ then we have $p^2 \equiv 1 \pmod{2^{n+3}}$

Here are some futher question proposed by me :Prove or disprove that :
1) $gcd(n, a_n) = 1$

2) for every odd prime number $p$ we have $a_m \equiv \pm 1 \pmod{p}$ where $m = \frac{p^2-1}{2^k}$ where $k = 1$ or $2$

Thanks kiu si u

*Edited by Orl.*

**8** Let $p$ be a prime number and let $A$ be a set of positive integers that satisfies the following conditions: **(1)** the set of prime divisors of the elements in $A$ consists of $p-1$ elements; **(2)** for any nonempty subset of $A$, the product of its elements is not a perfect $p$-th power. What is the largest possible number of elements in $A$ ?

– Algebra

**1** Let $a_{ij}$ (with the indices $i$ and $j$ from the set $\{1,\ 2,\ 3\}$) be real numbers such that

$a_{ij} > 0$ for $i = j$;
$a_{ij} < 0$ for $i \neq j$.

Prove the existence of positive real numbers $c_1$, $c_2$, $c_3$ such that the numbers

$a_{11}c_1 + a_{12}c_2 + a_{13}c_3$,
$a_{21}c_1 + a_{22}c_2 + a_{23}c_3$,
$a_{31}c_1 + a_{32}c_2 + a_{33}c_3$

are either all negative, or all zero, or all positive.

*Proposed by Kiran Kedlaya, USA*

**2** Find all nondecreasing functions $f : \mathbb{R} \to \mathbb{R}$ such that
(i) $f(0) = 0, f(1) = 1$;
(ii) $f(a) + f(b) = f(a)f(b) + f(a + b - ab)$ for all real numbers $a, b$ such that $a < 1 < b$.

*Proposed by A. Di Pisquale & D. Matthews, Australia*

**3** Consider two monotonically decreasing sequences $(a_k)$ and $(b_k)$, where $k \geq 1$, and $a_k$ and $b_k$ are positive real numbers for every k. Now, define the sequences

$c_k = \min{(a_k, b_k)}$;
$A_k = a_1 + a_2 + ... + a_k$;
$B_k = b_1 + b_2 + ... + b_k$;
$C_k = c_1 + c_2 + ... + c_k$

for all natural numbers k.

**(a)** Do there exist two monotonically decreasing sequences $(a_k)$ and $(b_k)$ of positive real numbers such that the sequences $(A_k)$ and $(B_k)$ are not bounded, while the sequence $(C_k)$ is bounded?

**(b)** Does the answer to problem **(a)** change if we stipulate that the sequence $(b_k)$ must be $b_k = \frac{1}{k}$ for all k ?

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**4** Let $n$ be a positive integer and let $x_1 \le x_2 \le \cdots \le x_n$ be real numbers. Prove that

$$\left(\sum_{i,j=1}^{n} |x_i - x_j|\right)^2 \le \frac{2(n^2-1)}{3} \sum_{i,j=1}^{n} (x_i - x_j)^2.$$

Show that the equality holds if and only if $x_1, \dots, x_n$ is an arithmetic sequence.

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**5** Let $\mathbb{R}^+$ be the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ that satisfy the following conditions:

- $f(xyz) + f(x) + f(y) + f(z) = f(\sqrt{xy})f(\sqrt{yz})f(\sqrt{zx})$ for all $x, y, z \in \mathbb{R}^+$;

- $f(x) < f(y)$ for all $1 \le x < y$.

*Proposed by Hojoo Lee, Korea*

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**6** Let $n$ be a positive integer and let $(x_1, \dots, x_n)$, $(y_1, \dots, y_n)$ be two sequences of positive real numbers. Suppose $(z_2, \dots, z_{2n})$ is a sequence of positive real numbers such that $z_{i+j}^2 \ge x_i y_j$ for all $1 \le i, j \le n$.

Let $M = \max\{z_2, \dots, z_{2n}\}$. Prove that

$$\left(\frac{M + z_2 + \cdots + z_{2n}}{2n}\right)^2 \ge \left(\frac{x_1 + \cdots + x_n}{n}\right)\left(\frac{y_1 + \cdots + y_n}{n}\right).$$

*Edited by Orl.*

*Proposed by Reid Barton, USA*

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Contributors: iandrei, sebadollahi, darij grinberg, grobber, vinoth_90_2004, orl, Valiowk, Fedor Petrov, Anonymous, iura, Myth, pluricomplex, flip2004, rope0811, Dapet, jmerry, heartwork, hxtung

– Combinatorics

**1** Let $A$ be a 101-element subset of the set $S = \{1, 2, \ldots, 1000000\}$. Prove that there exist numbers $t_1$, $t_2, \ldots, t_{100}$ in $S$ such that the sets

$$A_j = \{x + t_j \mid x \in A\}, \qquad j = 1, 2, \ldots, 100$$

are pairwise disjoint.

**2** Let $D_1$, $D_2$, ..., $D_n$ be closed discs in the plane. (A closed disc is the region limited by a circle, taken jointly with this circle.) Suppose that every point in the plane is contained in at most 2003 discs $D_i$. Prove that there exists a disc $D_k$ which intersects at most $7 \cdot 2003 - 1 = 14020$ other discs $D_i$.

**3** Let $n \geq 5$ be an integer. Find the maximal integer $k$ such that there exists a polygon with $n$ vertices (convex or not, but not self-intersecting!) having $k$ internal $90^\circ$ angles.

*Proposed by Juozas Juvencijus Macys, Lithuania*

**4** Given $n$ real numbers $x_1$, $x_2$, ..., $x_n$, and $n$ further real numbers $y_1$, $y_2$, ..., $y_n$. The entries $a_{ij}$ (with $1 \leq i \leq n$ and $1 \leq j \leq n$) of an $n \times n$ matrix $A$ are defined as follows:

$$a_{ij} = \begin{cases} 1 & \text{if} \quad x_i + y_j \geq 0; \\ 0 & \text{if} \quad x_i + y_j < 0. \end{cases}$$

Further, let $B$ be an $n \times n$ matrix whose elements are numbers from the set $\{0;\ 1\}$ satisfying the following condition: The sum of all elements of each row of $B$ equals the sum of all elements of the corresponding row of $A$; the sum of all elements of each column of $B$ equals the sum of all elements of the corresponding column of $A$. Show that in this case, $A = B$.

**5** Regard a plane with a Cartesian coordinate system; for each point with integer coordinates, draw a circular disk centered at this point and having the radius $\frac{1}{1000}$.

a) Prove the existence of an equilateral triangle whose vertices lie in the interior of different disks;

b) Show that every equilateral triangle whose vertices lie in the interior of different disks has a sidelength ¿ 96.

*Radu Gologan, Romania*

The "¿ 96" in **(b)** can be strengthened to "¿ 124". By the way, part **(a)** of this problem is the place where I used the well-known "Dedekind" theorem (`http://mathlinks.ro/viewtopic.php?t=5537`).

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**6** Let $f(k)$ be the number of all non-negative integers $n$ satisfying the following conditions:

**(1)** The integer $n$ has exactly $k$ digits in the decimal representation (where the first digit is not necessarily non-zero!), i. e. we have $0 \leq n < 10^k$.

**(2)** These $k$ digits of n can be permuted in such a way that the resulting number is divisible by 11.

Show that for any positive integer number $m$, we have $f\left(2m\right) = 10f\left(2m-1\right)$.

*Proposed by Dirk Laurie, South Africa*

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Contributors: iandrei, sebadollahi, darij grinberg, grobber, vinoth_90_2004, orl, Valiowk, Fedor Petrov, Anonymous, iura, Myth, pluricomplex, flip2004, rope0811, Dapet, jmerry, heartwork, hxtung

IMO Shortlist 2002

– Geometry

**1** Let $B$ be a point on a circle $S_1$, and let $A$ be a point distinct from $B$ on the tangent at $B$ to $S_1$. Let $C$ be a point not on $S_1$ such that the line segment $AC$ meets $S_1$ at two distinct points. Let $S_2$ be the circle touching $AC$ at $C$ and touching $S_1$ at a point $D$ on the opposite side of $AC$ from $B$. Prove that the circumcentre of triangle $BCD$ lies on the circumcircle of triangle $ABC$.

**2** Let $ABC$ be a triangle for which there exists an interior point $F$ such that $\angle AFB = \angle BFC = \angle CFA$. Let the lines $BF$ and $CF$ meet the sides $AC$ and $AB$ at $D$ and $E$ respectively. Prove that

$$AB + AC \geq 4DE.$$

**3** The circle $S$ has centre $O$, and $BC$ is a diameter of $S$. Let $A$ be a point of $S$ such that $\angle AOB < 120^\circ$. Let $D$ be the midpoint of the arc $AB$ which does not contain $C$. The line through $O$ parallel to $DA$ meets the line $AC$ at $I$. The perpendicular bisector of $OA$ meets $S$ at $E$ and at $F$. Prove that $I$ is the incentre of the triangle $CEF$.

**4** Circles $S_1$ and $S_2$ intersect at points $P$ and $Q$. Distinct points $A_1$ and $B_1$ (not at $P$ or $Q$) are selected on $S_1$. The lines $A_1P$ and $B_1P$ meet $S_2$ again at $A_2$ and $B_2$ respectively, and the lines $A_1B_1$ and $A_2B_2$ meet at $C$. Prove that, as $A_1$ and $B_1$ vary, the circumcentres of triangles $A_1A_2C$ all lie on one fixed circle.

**5** For any set $S$ of five points in the plane, no three of which are collinear, let $M(S)$ and $m(S)$ denote the greatest and smallest areas, respectively, of triangles determined by three points from $S$. What is the minimum possible value of $M(S)/m(S)$ ?

**6** Let $n \geq 3$ be a positive integer. Let $C_1, C_2, C_3, \ldots, C_n$ be unit circles in the plane, with centres $O_1, O_2, O_3, \ldots, O_n$ respectively. If no line meets more than two of the circles, prove that

$$\sum_{1 \leq i < j \leq n} \frac{1}{O_iO_j} \leq \frac{(n-1)\pi}{4}.$$

**7** The incircle $\Omega$ of the acute-angled triangle $ABC$ is tangent to its side $BC$ at a point $K$. Let $AD$ be an altitude of triangle $ABC$, and let $M$ be the midpoint of the segment $AD$. If $N$ is the common point of the circle $\Omega$ and the line $KM$ (distinct from $K$), then prove that the incircle $\Omega$ and the circumcircle of triangle $BCN$ are tangent to each other at the point $N$.

**8** Let two circles $S_1$ and $S_2$ meet at the points $A$ and $B$. A line through $A$ meets $S_1$ again at $C$ and $S_2$ again at $D$. Let $M$, $N$, $K$ be three points on the line segments $CD$, $BC$, $BD$ respectively, with $MN$ parallel to $BD$ and $MK$ parallel to $BC$. Let $E$ and $F$ be points on those arcs $BC$ of $S_1$ and $BD$ of $S_2$ respectively that do not contain $A$. Given that $EN$ is perpendicular to $BC$ and $FK$ is perpendicular to $BD$ prove that $\angle EMF = 90^\circ$.

**–** Number Theory

**1** What is the smallest positive integer $t$ such that there exist integers $x_1, x_2, \ldots, x_t$ with

$$x_1^3 + x_2^3 + \ldots + x_t^3 = 2002^{2002}\,?$$

**2** Let $n \geq 2$ be a positive integer, with divisors $1 = d_1 < d_2 < \ldots < d_k = n$. Prove that $d_1d_2 + d_2d_3 + \ldots + d_{k-1}d_k$ is always less than $n^2$, and determine when it is a divisor of $n^2$.

**3** Let $p_1, p_2, \ldots, p_n$ be distinct primes greater than 3. Show that $2^{p_1p_2\cdots p_n} + 1$ has at least $4^n$ divisors.

**4** Is there a positive integer $m$ such that the equation

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} = \frac{m}{a+b+c}$$

has infinitely many solutions in positive integers $a, b, c$?

**5** Let $m, n \geq 2$ be positive integers, and let $a_1, a_2, \ldots, a_n$ be integers, none of which is a multiple of $m^{n-1}$. Show that there exist integers $e_1, e_2, \ldots, e_n$, not all zero, with $|\,e_i\,| < m$ for all $i$, such that $e_1a_1 + e_2a_2 + \ldots + e_na_n$ is a multiple of $m^n$.

**6** Find all pairs of positive integers $m, n \geq 3$ for which there exist infinitely many positive integers $a$ such that

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is itself an integer.

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– Algebra

**1** Find all functions $f$ from the reals to the reals such that

$$f\left(f(x)+y\right)=2x+f\left(f(y)-x\right)$$

for all real $x,y$.

**2** Let $a_1, a_2, \dots$ be an infinite sequence of real numbers, for which there exists a real number $c$ with $0 \leq a_i \leq c$ for all $i$, such that

$$\left|a_i-a_j\right| \geq \frac{1}{i+j} \quad \text{for all } i,\ j \text{ with } i \neq j.$$

Prove that $c \geq 1$.

**3** Let $P$ be a cubic polynomial given by $P(x) = ax^3+bx^2+cx+d$, where $a,b,c,d$ are integers and $a \neq 0$. Suppose that $xP(x) = yP(y)$ for infinitely many pairs $x,y$ of integers with $x \neq y$. Prove that the equation $P(x) = 0$ has an integer root.

**4** Find all functions $f$ from the reals to the reals such that

$$\left(f(x)+f(z)\right)\left(f(y)+f(t)\right)=f(xy-zt)+f(xt+yz)$$

for all real $x,y,z,t$.

**5** Let $n$ be a positive integer that is not a perfect cube. Define real numbers $a,b,c$ by

$$a=\sqrt[3]{n}\,,\qquad b=\frac{1}{a-[a]}\,,\qquad c=\frac{1}{b-[b]}\,,$$

where $[x]$ denotes the integer part of $x$. Prove that there are infinitely many such integers $n$ with the property that there exist integers $r,s,t$, not all zero, such that $ra+sb+tc=0$.

**6** Let $A$ be a non-empty set of positive integers. Suppose that there are positive integers $b_1, \dots b_n$ and $c_1, \dots, c_n$ such that

- for each $i$ the set $b_iA + c_i = \{b_ia + c_i \colon a \in A\}$ is a subset of $A$, and

- the sets $b_iA + c_i$ and $b_jA + c_j$ are disjoint whenever $i \neq j$

Prove that
$$\frac{1}{b_1} + \dots + \frac{1}{b_n} \leq 1.$$

**–** Combinatorics

**1** Let $n$ be a positive integer. Each point $(x, y)$ in the plane, where $x$ and $y$ are non-negative integers with $x + y < n$, is coloured red or blue, subject to the following condition: if a point $(x, y)$ is red, then so are all points $(x', y')$ with $x' \leq x$ and $y' \leq y$. Let $A$ be the number of ways to choose $n$ blue points with distinct $x$-coordinates, and let $B$ be the number of ways to choose $n$ blue points with distinct $y$-coordinates. Prove that $A = B$.

**2** For $n$ an odd positive integer, the unit squares of an $n \times n$ chessboard are coloured alternately black and white, with the four corners coloured black. A it tromino is an $L$-shape formed by three connected unit squares. For which values of $n$ is it possible to cover all the black squares with non-overlapping trominos? When it is possible, what is the minimum number of trominos needed?

**3** Let $n$ be a positive integer. A sequence of $n$ positive integers (not necessarily distinct) is called **full** if it satisfies the following condition: for each positive integer $k \geq 2$, if the number $k$ appears in the sequence then so does the number $k-1$, and moreover the first occurrence of $k-1$ comes before the last occurrence of $k$. For each $n$, how many full sequences are there ?

**4** Let $T$ be the set of ordered triples $(x, y, z)$, where $x, y, z$ are integers with $0 \leq x, y, z \leq 9$. Players $A$ and $B$ play the following guessing game. Player $A$ chooses a triple $(x, y, z)$ in $T$, and Player $B$ has to discover $A$*'s* triple in as few moves as possible. A *move* consists of the following: $B$ gives $A$ a triple $(a, b, c)$ in $T$, and $A$ replies by giving $B$ the number $|x + y - a - b| + |y + z - b - c| + |z + x - c - a|$. Find the minimum number of moves that $B$ needs to be sure of determining $A$*'s* triple.

**5** Let $r \geq 2$ be a fixed positive integer, and let $F$ be an infinite family of sets, each of size $r$, no two of which are disjoint. Prove that there exists a set of size $r-1$ that meets each set in $F$.

**6** Let $n$ be an even positive integer. Show that there is a permutation $(x_1, x_2, \ldots, x_n)$ of $(1,\ 2,\ \ldots, n)$ such that for every $i \in \{1,\ 2,\ ...,\ n\}$, the number $x_{i+1}$ is one of the numbers $2x_i$, $2x_i - 1$, $2x_i - n$, $2x_i - n - 1$. Hereby, we use the cyclic subscript convention, so that $x_{n+1}$ means $x_1$.

**7** Among a group of 120 people, some pairs are friends. A *weak quartet* is a set of four people containing exactly one pair of friends. What is the maximum possible number of weak quartets ?

IMO Shortlist 2001

| | |
|---|---|
| – | Geometry |
| **1** | Let $A_1$ be the center of the square inscribed in acute triangle $ABC$ with two vertices of the square on side $BC$. Thus one of the two remaining vertices of the square is on side $AB$ and the other is on $AC$. Points $B_1,\ C_1$ are defined in a similar way for inscribed squares with two vertices on sides $AC$ and $AB$, respectively. Prove that lines $AA_1,\ BB_1,\ CC_1$ are concurrent. |
| **2** | Consider an acute-angled triangle $ABC$. Let $P$ be the foot of the altitude of triangle $ABC$ issuing from the vertex $A$, and let $O$ be the circumcenter of triangle $ABC$. Assume that $\angle C \geq \angle B + 30^\circ$. Prove that $\angle A + \angle COP < 90^\circ$. |
| **3** | Let $ABC$ be a triangle with centroid $G$. Determine, with proof, the position of the point $P$ in the plane of $ABC$ such that $AP{\cdot}AG + BP{\cdot}BG + CP{\cdot}CG$ is a minimum, and express this minimum value in terms of the side lengths of $ABC$. |
| **4** | Let $M$ be a point in the interior of triangle $ABC$. Let $A'$ lie on $BC$ with $MA'$ perpendicular to $BC$. Define $B'$ on $CA$ and $C'$ on $AB$ similarly. Define $$p(M) = \frac{MA' \cdot MB' \cdot MC'}{MA \cdot MB \cdot MC}.$$ Determine, with proof, the location of $M$ such that $p(M)$ is maximal. Let $\mu(ABC)$ denote this maximum value. For which triangles $ABC$ is the value of $\mu(ABC)$ maximal? |
| **5** | Let $ABC$ be an acute triangle. Let $DAC, EAB$, and $FBC$ be isosceles triangles exterior to $ABC$, with $DA = DC, EA = EB$, and $FB = FC$, such that $$\angle ADC = 2\angle BAC, \quad \angle BEA = 2\angle ABC, \quad \angle CFB = 2\angle ACB.$$ Let $D'$ be the intersection of lines $DB$ and $EF$, let $E'$ be the intersection of $EC$ and $DF$, and let $F'$ be the intersection of $FA$ and $DE$. Find, with proof, the value of the sum $$\frac{DB}{DD'} + \frac{EC}{EE'} + \frac{FA}{FF'}.$$ |

**6** Let $ABC$ be a triangle and $P$ an exterior point in the plane of the triangle. Suppose the lines $AP$, $BP$, $CP$ meet the sides $BC$, $CA$, $AB$ (or extensions thereof) in $D$, $E$, $F$, respectively. Suppose further that the areas of triangles $PBD$, $PCE$, $PAF$ are all equal. Prove that each of these areas is equal to the area of triangle $ABC$ itself.

**7** Let $O$ be an interior point of acute triangle $ABC$. Let $A_1$ lie on $BC$ with $OA_1$ perpendicular to $BC$. Define $B_1$ on $CA$ and $C_1$ on $AB$ similarly. Prove that $O$ is the circumcenter of $ABC$ if and only if the perimeter of $A_1B_1C_1$ is not less than any one of the perimeters of $AB_1C_1, BC_1A_1$, and $CA_1B_1$.

**8** Let $ABC$ be a triangle with $\angle BAC = 60^\circ$. Let $AP$ bisect $\angle BAC$ and let $BQ$ bisect $\angle ABC$, with $P$ on $BC$ and $Q$ on $AC$. If $AB + BP = AQ + QB$, what are the angles of the triangle?

**–** Number Theory

**1** Prove that there is no positive integer $n$ such that, for $k = 1, 2, \dots, 9$, the leftmost digit (in decimal notation) of $(n + k)!$ equals $k$.

**2** Consider the system

$$x + y = z + u,$$
$$2xy = zu.$$

Find the greatest value of the real constant $m$ such that $m \leq x/y$ for any positive integer solution $(x, y, z, u)$ of the system, with $x \geq y$.

**3** Let $a_1 = 11^{11}$, $a_2 = 12^{12}$, $a_3 = 13^{13}$, and $a_n = |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}|, n \geq 4$. Determine $a_{14^{14}}$.

**4** Let $p \geq 5$ be a prime number. Prove that there exists an integer $a$ with $1 \leq a \leq p - 2$ such that neither $a^{p-1} - 1$ nor $(a + 1)^{p-1} - 1$ is divisible by $p^2$.

**5** Let $a > b > c > d$ be positive integers and suppose that

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.

**6** Is it possible to find 100 positive integers not exceeding $25,000$, such that all pairwise sums of them are different?

– Algebra

**1** Let $T$ denote the set of all ordered triples $(p, q, r)$ of nonnegative integers. Find all functions $f : T \to \mathbb{R}$ satisfying

$$f(p,q,r) = \begin{cases} 0 & \text{if } pqr = 0, \\ 1 + \frac{1}{6}(f(p+1,q-1,r) + f(p-1,q+1,r) & \\ +f(p-1,q,r+1) + f(p+1,q,r-1) & \\ +f(p,q+1,r-1) + f(p,q-1,r+1)) & \text{otherwise} \end{cases}$$

for all nonnegative integers $p$, $q$, $r$.

**2** Let $a_0, a_1, a_2, \ldots$ be an arbitrary infinite sequence of positive numbers. Show that the inequality $1 + a_n > a_{n-1}\sqrt[n]{2}$ holds for infinitely many positive integers $n$.

**3** Let $x_1, x_2, \ldots, x_n$ be arbitrary real numbers. Prove the inequality

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \cdots + \frac{x_n}{1+x_1^2+\cdots+x_n^2} < \sqrt{n}.$$

**4** Find all functions $f : \mathbb{R} \to \mathbb{R}$, satisfying

$$f(xy)(f(x) - f(y)) = (x - y)f(x)f(y)$$

for all $x, y$.

**5** Find all positive integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{99}{100} = \frac{a_0}{a_1} + \frac{a_1}{a_2} + \cdots + \frac{a_{n-1}}{a_n},$$

where $a_0 = 1$ and $(a_{k+1} - 1)a_{k-1} \geq a_k^2(a_k - 1)$ for $k = 1, 2, \ldots, n-1$.

**6** Prove that for all positive real numbers $a, b, c$,

$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1.$$

– Combinatorics

**1** Let $A = (a_1, a_2, \ldots, a_{2001})$ be a sequence of positive integers. Let $m$ be the number of 3-element subsequences $(a_i, a_j, a_k)$ with $1 \le i < j < k \le 2001$, such that $a_j = a_i + 1$ and $a_k = a_j + 1$. Considering all such sequences $A$, find the greatest value of $m$.

**2** Let $n$ be an odd integer greater than 1 and let $c_1, c_2, \ldots, c_n$ be integers. For each permutation $a = (a_1, a_2, \ldots, a_n)$ of $\{1, 2, \ldots, n\}$, define $S(a) = \sum_{i=1}^n c_i a_i$. Prove that there exist permutations $a \ne b$ of $\{1, 2, \ldots, n\}$ such that $n!$ is a divisor of $S(a) - S(b)$.

**3** Define a $k$-*clique* to be a set of $k$ people such that every pair of them are acquainted with each other. At a certain party, every pair of 3-cliques has at least one person in common, and there are no 5-cliques. Prove that there are two or fewer people at the party whose departure leaves no 3-clique remaining.

**4** A set of three nonnegative integers $\{x, y, z\}$ with $x < y < z$ is called *historic* if $\{z - y, y - x\} = \{1776, 2001\}$. Show that the set of all nonnegative integers can be written as the union of pairwise disjoint historic sets.

**5** Find all finite sequences $(x_0, x_1, \ldots, x_n)$ such that for every $j$, $0 \le j \le n$, $x_j$ equals the number of times $j$ appears in the sequence.

**6** For a positive integer $n$ define a sequence of zeros and ones to be *balanced* if it contains $n$ zeros and $n$ ones. Two balanced sequences $a$ and $b$ are *neighbors* if you can move one of the $2n$ symbols of $a$ to another position to form $b$. For instance, when $n = 4$, the balanced sequences 01101001 and 00110101 are neighbors because the third (or fourth) zero in the first sequence can be moved to the first or second position to form the second sequence. Prove that there is a set $S$ of at most $\frac{1}{n+1}\binom{2n}{n}$ balanced sequences such that every balanced sequence is equal to or is a neighbor of at least one sequence in $S$.

**7** A pile of $n$ pebbles is placed in a vertical column. This configuration is modified according to the following rules. A pebble can be moved if it is at the top of a column which contains at least two more pebbles than the column immediately to its right. (If there are no pebbles to the right, think of this as a column with 0 pebbles.) At each stage, choose a pebble from among those that can be moved (if there are any) and place it at the top of the column to its right. If no pebbles can be moved, the configuration is called a *final configuration*. For each $n$, show that, no matter what choices are made at each stage, the final configuration obtained is unique. Describe that configuration in terms of $n$.

IMO ShortList 2001, combinatorics problem 7, alternative (http://www.mathlinks.ro/Forum/viewtopic.php?p=119189)

**8** Twenty-one girls and twenty-one boys took part in a mathematical competition. It turned out that each contestant solved at most six problems, and for each pair of a girl and a boy, there was at least one problem that was solved by both the girl and the boy. Show that there is a problem that was solved by at least three girls and at least three boys.

IMO Shortlist 2000

– Algebra

**1** Let $a, b, c$ be positive real numbers so that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

**2** Let $a, b, c$ be positive integers satisfying the conditions $b > 2a$ and $c > 2b$. Show that there exists a real number $\lambda$ with the property that all the three numbers $\lambda a, \lambda b, \lambda c$ have their fractional parts lying in the interval $\left(\frac{1}{3}, \frac{2}{3}\right]$.

**3** Find all pairs of functions $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + g(y)) = xf(y) - yf(x) + g(x) \quad \text{for all } x, y \in \mathbb{R}.$$

**4** The function $F$ is defined on the set of nonnegative integers and takes nonnegative integer values satisfying the following conditions: for every $n \geq 0$,

(i) $F(4n) = F(2n) + F(n)$,
(ii) $F(4n + 2) = F(4n) + 1$,
(iii) $F(2n + 1) = F(2n) + 1$.

Prove that for each positive integer $m$, the number of integers $n$ with $0 \leq n < 2^m$ and $F(4n) = F(3n)$ is $F(2^{m+1})$.

**5** Let $n \geq 2$ be a positive integer and $\lambda$ a positive real number. Initially there are $n$ fleas on a horizontal line, not all at the same point. We define a move as choosing two fleas at some points $A$ and $B$, with $A$ to the left of $B$, and letting the flea from $A$ jump over the flea from $B$ to the point $C$ so that $\frac{BC}{AB} = \lambda$.

Determine all values of $\lambda$ such that, for any point $M$ on the line and for any initial position of the $n$ fleas, there exists a sequence of moves that will take them all to the position right of $M$.

**6** A nonempty set $A$ of real numbers is called a $B_3$-set if the conditions $a_1, a_2, a_3, a_4, a_5, a_6 \in A$ and $a_1 + a_2 + a_3 = a_4 + a_5 + a_6$ imply that the sequences $(a_1, a_2, a_3)$ and $(a_4, a_5, a_6)$ are identical up to a permutation. Let $A = \{a_0 = 0 < a_1 < a_2 < \cdots\}$, $B = \{b_0 = 0 < b_1 < b_2 < \cdots\}$ be infinite sequences of real numbers with $D(A) = D(B)$, where, for a set $X$ of real numbers, $D(X)$ denotes the difference set $\{|x - y| \mid x, y \in X\}$. Prove that if $A$ is a $B_3$-set, then $A = B$.

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**7** For a polynomial $P$ of degree 2000 with distinct real coefficients let $M(P)$ be the set of all polynomials that can be produced from $P$ by permutation of its coefficients. A polynomial $P$ will be called [b]$n$-independent[/b] if $P(n) = 0$ and we can get from any $Q \in M(P)$ a polynomial $Q_1$ such that $Q_1(n) = 0$ by interchanging at most one pair of coefficients of $Q$. Find all integers $n$ for which $n$-independent polynomials exist.

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– Geometry

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**1** In the plane we are given two circles intersecting at $X$ and $Y$. Prove that there exist four points with the following property:

(P) For every circle touching the two given circles at $A$ and $B$, and meeting the line $XY$ at $C$ and $D$, each of the lines $AC$, $AD$, $BC$, $BD$ passes through one of these points.

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**2** Two circles $G_1$ and $G_2$ intersect at two points $M$ and $N$. Let $AB$ be the line tangent to these circles at $A$ and $B$, respectively, so that $M$ lies closer to $AB$ than $N$. Let $CD$ be the line parallel to $AB$ and passing through the point $M$, with $C$ on $G_1$ and $D$ on $G_2$. Lines $AC$ and $BD$ meet at $E$; lines $AN$ and $CD$ meet at $P$; lines $BN$ and $CD$ meet at $Q$. Show that $EP = EQ$.

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**3** Let $O$ be the circumcenter and $H$ the orthocenter of an acute triangle $ABC$. Show that there exist points $D$, $E$, and $F$ on sides $BC$, $CA$, and $AB$ respectively such that

$$OD + DH = OE + EH = OF + FH$$

and the lines $AD$, $BE$, and $CF$ are concurrent.

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**4** Let $A_1A_2 \dots A_n$ be a convex polygon, $n \geq 4$. Prove that $A_1A_2 \dots A_n$ is cyclic if and only if to each vertex $A_j$ one can assign a pair $(b_j, c_j)$ of real numbers, $j = 1, 2, \dots, n$, so that $A_iA_j = b_jc_i - b_ic_j$ for all $i, j$ with $1 \leq i < j \leq n$.

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**5** The tangents at $B$ and $A$ to the circumcircle of an acute angled triangle $ABC$ meet the tangent at $C$ at $T$ and $U$ respectively. $AT$ meets $BC$ at $P$, and $Q$ is the midpoint of $AP$; $BU$ meets $CA$ at $R$, and $S$ is the midpoint of $BR$. Prove that $\angle ABQ = \angle BAS$. Determine, in terms of ratios of side lengths, the triangles for which this angle is a maximum.

**6** Let $ABCD$ be a convex quadrilateral. The perpendicular bisectors of its sides $AB$ and $CD$ meet at $Y$. Denote by $X$ a point inside the quadrilateral $ABCD$ such that $\measuredangle ADX = \measuredangle BCX < 90^\circ$ and $\measuredangle DAX = \measuredangle CBX < 90^\circ$. Show that $\measuredangle AYB = 2 \cdot \measuredangle ADX$.

**7** Ten gangsters are standing on a flat surface, and the distances between them are all distinct. At twelve oclock, when the church bells start chiming, each of them fatally shoots the one among the other nine gangsters who is the nearest. At least how many gangsters will be killed?

**8** Let $AH_1, BH_2, CH_3$ be the altitudes of an acute angled triangle $ABC$. Its incircle touches the sides $BC, AC$ and $AB$ at $T_1, T_2$ and $T_3$ respectively. Consider the symmetric images of the lines $H_1H_2, H_2H_3$ and $H_3H_1$ with respect to the lines $T_1T_2, T_2T_3$ and $T_3T_1$. Prove that these images form a triangle whose vertices lie on the incircle of $ABC$.

– Number Theory

**1** Determine all positive integers $n \geq 2$ that satisfy the following condition: for all $a$ and $b$ relatively prime to $n$ we have

$$a \equiv b \pmod{n} \qquad \text{if and only if} \qquad ab \equiv 1 \pmod{n}.$$

**2** For a positive integer $n$, let $d(n)$ be the number of all positive divisors of $n$. Find all positive integers $n$ such that $d(n)^3 = 4n$.

**3** Does there exist a positive integer $n$ such that $n$ has exactly 2000 prime divisors and $n$ divides $2^n + 1$?

**4** Find all triplets of positive integers $(a, m, n)$ such that $a^m + 1 \mid (a+1)^n$.

**5** Prove that there exist infinitely many positive integers $n$ such that $p = nr$, where $p$ and $r$ are respectively the semiperimeter and the inradius of a triangle with integer side lengths.


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**6** Show that the set of positive integers that cannot be represented as a sum of distinct perfect squares is finite.

– Combinatorics

**1** A magician has one hundred cards numbered 1 to 100. He puts them into three boxes, a red one, a white one and a blue one, so that each box contains at least one card. A member of the audience draws two cards from two different boxes and announces the sum of numbers on those cards. Given this information, the magician locates the box from which no card has been drawn.

How many ways are there to put the cards in the three boxes so that the trick works?

**2** A staircase-brick with 3 steps of width 2 is made of 12 unit cubes. Determine all integers $n$ for which it is possible to build a cube of side $n$ using such bricks.

**3** Let $n \geq 4$ be a fixed positive integer. Given a set $S = \{P_1, P_2, \ldots, P_n\}$ of $n$ points in the plane such that no three are collinear and no four concyclic, let $a_t$, $1 \leq t \leq n$, be the number of circles $P_iP_jP_k$ that contain $P_t$ in their interior, and let

$$m(S) = a_1 + a_2 + \cdots + a_n.$$

Prove that there exists a positive integer $f(n)$, depending only on $n$, such that the points of $S$ are the vertices of a convex polygon if and only if $m(S) = f(n)$.

**4** Let $n$ and $k$ be positive integers such that $\frac{1}{2}n < k \leq \frac{2}{3}n$. Find the least number $m$ for which it is possible to place $m$ pawns on $m$ squares of an $n \times n$ chessboard so that no column or row contains a block of $k$ adjacent unoccupied squares.

**5** In the plane we have $n$ rectangles with parallel sides. The sides of distinct rectangles lie on distinct lines. The boundaries of the rectangles cut the plane into connected regions. A region is *nice* if it has at least one of the vertices of the $n$ rectangles on the boundary. Prove that the sum of the numbers of the vertices of all nice regions is less than $40n$. (There can be nonconvex regions as well as regions with more than one boundary curve.)

**6** Let $p$ and $q$ be relatively prime positive integers. A subset $S$ of $\{0, 1, 2, \ldots\}$ is called **ideal** if $0 \in S$ and for each element $n \in S$, the integers $n + p$ and $n + q$ belong to $S$. Determine the number of ideal subsets of $\{0, 1, 2, \ldots\}$.


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IMO Shortlist 1999

– Geometry

**1** Let ABC be a triangle and $M$ be an interior point. Prove that

$$\min\{MA, MB, MC\} + MA + MB + MC < AB + AC + BC.$$

**2** A circle is called a **separator** for a set of five points in a plane if it passes through three of these points, it contains a fourth point inside and the fifth point is outside the circle. Prove that every set of five points such that no three are collinear and no four are concyclic has exactly four separators.

**3** A set $S$ of points from the space will be called **completely symmetric** if it has at least three elements and fulfills the condition that for every two distinct points $A$ and $B$ from $S$, the perpendicular bisector plane of the segment $AB$ is a plane of symmetry for $S$. Prove that if a completely symmetric set is finite, then it consists of the vertices of either a regular polygon, or a regular tetrahedron or a regular octahedron.

**4** For a triangle $T = ABC$ we take the point $X$ on the side $(AB)$ such that $AX/AB = 4/5$, the point $Y$ on the segment $(CX)$ such that $CY = 2YX$ and, if possible, the point $Z$ on the ray $(CA$ such that $\widehat{CXZ} = 180 - \widehat{ABC}$. We denote by $\Sigma$ the set of all triangles $T$ for which $\widehat{XYZ} = 45$. Prove that all triangles from $\Sigma$ are similar and find the measure of their smallest angle.

**5** Let $ABC$ be a triangle, $\Omega$ its incircle and $\Omega_a, \Omega_b, \Omega_c$ three circles orthogonal to $\Omega$ passing through $(B, C), (A, C)$ and $(A, B)$ respectively. The circles $\Omega_a$ and $\Omega_b$ meet again in $C'$; in the same way we obtain the points $B'$ and $A'$. Prove that the radius of the circumcircle of $A'B'C'$ is half the radius of $\Omega$.

**6** Two circles $\Omega_1$ and $\Omega_2$ touch internally the circle $\Omega$ in M and N and the center of $\Omega_2$ is on $\Omega_1$. The common chord of the circles $\Omega_1$ and $\Omega_2$ intersects $\Omega$ in $A$ and $B$. $MA$ and $MB$ intersects $\Omega_1$ in $C$ and $D$. Prove that $\Omega_2$ is tangent to $CD$.

**7** The point $M$ is inside the convex quadrilateral $ABCD$, such that

$$MA = MC, \quad \widehat{AMB} = \widehat{MAD} + \widehat{MCD} \quad \text{and} \quad \widehat{CMD} = \widehat{MCB} + \widehat{MAB}.$$

Prove that $AB \cdot CM = BC \cdot MD$ and $BM \cdot AD = MA \cdot CD$.

**8** Given a triangle $ABC$. The points $A$, $B$, $C$ divide the circumcircle $\Omega$ of the triangle $ABC$ into three arcs $BC$, $CA$, $AB$. Let $X$ be a variable point on the arc $AB$, and let $O_1$ and $O_2$ be the incenters of the triangles $CAX$ and $CBX$. Prove that the circumcircle of the triangle $XO_1O_2$ intersects the circle $\Omega$ in a fixed point.

– Number Theory

**1** Find all the pairs of positive integers $(x,p)$ such that p is a prime, $x \leq 2p$ and $x^{p-1}$ is a divisor of $(p-1)^x + 1$.

**2** Prove that every positive rational number can be represented in the form $\dfrac{a^3+b^3}{c^3+d^3}$ where a,b,c,d are positive integers.

**3** Prove that there exists two strictly increasing sequences $(a_n)$ and $(b_n)$ such that $a_n(a_n+1)$ divides $b_n^2+1$ for every natural n.

**4** Denote by S the set of all primes such the decimal representation of $\frac{1}{p}$ has the fundamental period divisible by 3. For every $p \in S$ such that $\frac{1}{p}$ has the fundamental period $3r$ one may write

$$\frac{1}{p} = 0, a_1a_2 \ldots a_{3r}a_1a_2 \ldots a_{3r} \ldots,$$

where $r = r(p)$; for every $p \in S$ and every integer $k \geq 1$ define $f(k,p)$ by

$$f(k,p) = a_k + a_{k+r(p)} + a_{k+2.r(p)}$$

a) Prove that $S$ is infinite.
b) Find the highest value of $f(k,p)$ for $k \geq 1$ and $p \in S$

**5** Let $n, k$ be positive integers such that n is not divisible by 3 and $k \geq n$. Prove that there exists a positive integer $m$ which is divisible by $n$ and the sum of its digits in decimal representation is $k$.

**6** Prove that for every real number $M$ there exists an infinite arithmetic progression such that:

- each term is a positive integer and the common difference is not divisible by 10
- the sum of the digits of each term (in decimal representation) exceeds $M$.

– Algebra

**1** Let $n \geq 2$ be a fixed integer. Find the least constant $C$ such the inequality

$$\sum_{i<j} x_i x_j \left(x_i^2 + x_j^2\right) \leq C \left(\sum_i x_i\right)^4$$

holds for any $x_1, \ldots, x_n \geq 0$ (the sum on the left consists of $\binom{n}{2}$ summands). For this constant $C$, characterize the instances of equality.

**2** The numbers from 1 to $n^2$ are randomly arranged in the cells of a $n \times n$ square ($n \geq 2$). For any pair of numbers situated on the same row or on the same column the ratio of the greater number to the smaller number is calculated. Let us call the **characteristic** of the arrangement the smallest of these $n^2 (n-1)$ fractions. What is the highest possible value of the characteristic ?

**3** A game is played by $n$ girls ($n \geq 2$), everybody having a ball. Each of the $\binom{n}{2}$ pairs of players, is an arbitrary order, exchange the balls they have at the moment. The game is called nice **nice** if at the end nobody has her own ball and it is called **tiresome** if at the end everybody has her initial ball. Determine the values of $n$ for which there exists a nice game and those for which there exists a tiresome game.

**4** Prove that the set of positive integers cannot be partitioned into three nonempty subsets such that, for any two integers $x, y$ taken from two different subsets, the number $x^2 - xy + y^2$ belongs to the third subset.

**5** Find all the functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all $x, y \in \mathbb{R}$.

**6** For $n \geq 3$ and $a_1 \leq a_2 \leq \ldots \leq a_n$ given real numbers we have the following instructions:

- place out the numbers in some order in a ring;
- delete one of the numbers from the ring;
- if just two numbers are remaining in the ring: let $S$ be the sum of these two numbers. Otherwise, if there are more the two numbers in the ring, replace

Afterwards start again with the step (2). Show that the largest sum $S$ which can result in this way is given by the formula

$$S_{max} = \sum_{k=2}^{n} \binom{n-2}{[\frac{k}{2}]-1} a_k.$$

**–** Combinatorics

**1** Let $n \geq 1$ be an integer. A **path** from $(0,0)$ to $(n,n)$ in the $xy$ plane is a chain of consecutive unit moves either to the right (move denoted by $E$) or upwards (move denoted by $N$), all the moves being made inside the half-plane $x \geq y$. A **step** in a path is the occurence of two consecutive moves of the form $EN$. Show that the number of paths from $(0,0)$ to $(n,n)$ that contain exactly $s$ steps $(n \geq s \geq 1)$ is

$$\frac{1}{s}\binom{n-1}{s-1}\binom{n}{s-1}.$$

**2** If a $5 \times n$ rectangle can be tiled using $n$ pieces like those shown in the diagram, prove that $n$ is even. Show that there are more than $2 \cdot 3^{k-1}$ ways to file a fixed $5 \times 2k$ rectangle $(k \geq 3)$ with $2k$ pieces. (symmetric constructions are supposed to be different.)

**3** A biologist watches a chameleon. The chameleon catches flies and rests after each catch. The biologist notices that:
-the first fly is caught after a resting period of one minute;
-the resting period before catching the $2m^{\text{th}}$ fly is the same as the resting period before catching the $m^{\text{th}}$ fly and one minute shorter than the resting period before catching the $(2m+1)^{\text{th}}$ fly;
-when the chameleon stops resting, he catches a fly instantly.

-How many flies were caught by the chameleon before his first resting period of 9 minutes in a row?
-After how many minutes will the chameleon catch his $98^{\text{th}}$ fly?
-How many flies were caught by the chameleon after 1999 minutes have passed?

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**4** Let $A$ be a set of $N$ residues $\pmod{N^2}$. Prove that there exists a set $B$ of of $N$ residues $\pmod{N^2}$ such that $A+B=\{a+b|a\in A, b\in B\}$ contains at least half of all the residues $\pmod{N^2}$.

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**5** Let $n$ be an even positive integer. We say that two different cells of a $n \times n$ board are **neighboring** if they have a common side. Find the minimal number of cells on the $n \times n$ board that must be marked so that any cell (marked or not marked) has a marked neighboring cell.

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**6** Suppose that every integer has been given one of the colours red, blue, green or yellow. Let $x$ and $y$ be odd integers so that $|x| \neq |y|$. Show that there are two integers of the same colour whose difference has one of the following values: $x, y, x+y$ or $x-y$.

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**7** Let $p>3$ be a prime number. For each nonempty subset $T$ of $\{0,1,2,3,\ldots,p-1\}$, let $E(T)$ be the set of all $(p-1)$-tuples $(x_1,\ldots,x_{p-1})$, where each $x_i\in T$ and $x_1+2x_2+\ldots+(p-1)x_{p-1}$ is divisible by $p$ and let $|E(T)|$ denote the number of elements in $E(T)$. Prove that

$$|E(\{0,1,3\})| \geq |E(\{0,1,2\})|$$

with equality if and only if $p=5$.

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IMO Shortlist 1998

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|---|---|
| – | Geometry |
| **1** | A convex quadrilateral $ABCD$ has perpendicular diagonals. The perpendicular bisectors of the sides $AB$ and $CD$ meet at a unique point $P$ inside $ABCD$. Prove that the quadrilateral $ABCD$ is cyclic if and only if triangles $ABP$ and $CDP$ have equal areas. |
| **2** | Let $ABCD$ be a cyclic quadrilateral. Let $E$ and $F$ be variable points on the sides $AB$ and $CD$, respectively, such that $AE:EB=CF:FD$. Let $P$ be the point on the segment $EF$ such that $PE:PF=AB:CD$. Prove that the ratio between the areas of triangles $APD$ and $BPC$ does not depend on the choice of $E$ and $F$. |
| **3** | Let $I$ be the incenter of triangle $ABC$. Let $K,L$ and $M$ be the points of tangency of the incircle of $ABC$ with $AB,BC$ and $CA$, respectively. The line $t$ passes through $B$ and is parallel to $KL$. The lines $MK$ and $ML$ intersect $t$ at the points $R$ and $S$. Prove that $\angle RIS$ is acute. |
| **4** | Let $M$ and $N$ be two points inside triangle $ABC$ such that $$\angle MAB=\angle NAC \quad \text{and} \quad \angle MBA=\angle NBC.$$ Prove that $$\frac{AM\cdot AN}{AB\cdot AC}+\frac{BM\cdot BN}{BA\cdot BC}+\frac{CM\cdot CN}{CA\cdot CB}=1.$$ |
| **5** | Let $ABC$ be a triangle, $H$ its orthocenter, $O$ its circumcenter, and $R$ its circumradius. Let $D$ be the reflection of the point $A$ across the line $BC$, let $E$ be the reflection of the point $B$ across the line $CA$, and let $F$ be the reflection of the point $C$ across the line $AB$. Prove that the points $D$, $E$ and $F$ are collinear if and only if $OH=2R$. |
| **6** | Let $ABCDEF$ be a convex hexagon such that $\angle B+\angle D+\angle F=360^\circ$ and $$\frac{AB}{BC}\cdot\frac{CD}{DE}\cdot\frac{EF}{FA}=1.$$ Prove that $$\frac{BC}{CA}\cdot\frac{AE}{EF}\cdot\frac{FD}{DB}=1.$$ |

**7** Let $ABC$ be a triangle such that $\angle ACB = 2\angle ABC$. Let $D$ be the point on the side $BC$ such that $CD = 2BD$. The segment $AD$ is extended to $E$ so that $AD = DE$. Prove that

$$\angle ECB + 180^\circ = 2\angle EBC.$$

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**8** Let $ABC$ be a triangle such that $\angle A = 90^\circ$ and $\angle B < \angle C$. The tangent at $A$ to the circumcircle $\omega$ of triangle $ABC$ meets the line $BC$ at $D$. Let $E$ be the reflection of $A$ in the line $BC$, let $X$ be the foot of the perpendicular from $A$ to $BE$, and let $Y$ be the midpoint of the segment $AX$. Let the line $BY$ intersect the circle $\omega$ again at $Z$.

Prove that the line $BD$ is tangent to the circumcircle of triangle $ADZ$.

*Edited by Orl.*

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– Number Theory

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**1** Determine all pairs $(x, y)$ of positive integers such that $x^2y + x + y$ is divisible by $xy^2 + y + 7$.

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**2** Determine all pairs $(a, b)$ of real numbers such that $a\lfloor bn\rfloor = b\lfloor an\rfloor$ for all positive integers $n$. (Note that $\lfloor x\rfloor$ denotes the greatest integer less than or equal to $x$.)

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**3** Determine the smallest integer $n \geq 4$ for which one can choose four different numbers $a, b, c$ and $d$ from any $n$ distinct integers such that $a + b - c - d$ is divisible by 20.

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**4** A sequence of integers $a_1, a_2, a_3, \ldots$ is defined as follows: $a_1 = 1$ and for $n \geq 1$, $a_{n+1}$ is the smallest integer greater than $a_n$ such that $a_i + a_j \neq 3a_k$ for any $i, j$ and $k$ in $\{1, 2, 3, \ldots, n+1\}$, not necessarily distinct. Determine $a_{1998}$.

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**5** Determine all positive integers $n$ for which there exists an integer $m$ such that $2^n - 1$ is a divisor of $m^2 + 9$.

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**6** For any positive integer $n$, let $\tau(n)$ denote the number of its positive divisors (including 1 and itself). Determine all positive integers $m$ for which there exists a positive integer $n$ such that $\frac{\tau(n^2)}{\tau(n)} = m$.

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**7** Prove that for each positive integer $n$, there exists a positive integer with the following properties: It has exactly $n$ digits. None of the digits is 0. It is divisible by the sum of its digits.

**8** Let $a_0, a_1, a_2, \ldots$ be an increasing sequence of nonnegative integers such that every nonnegative integer can be expressed uniquely in the form $a_i + 2a_j + 4a_k$, where $i, j$ and $k$ are not necessarily distinct. Determine $a_{1998}$.

**–** Algebra

**1** Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n < 1$. Prove that

$$\frac{a_1 a_2 \cdots a_n \left[1 - (a_1 + a_2 + \cdots + a_n)\right]}{(a_1 + a_2 + \cdots + a_n)(1 - a_1)(1 - a_2) \cdots (1 - a_n)} \leq \frac{1}{n^{n+1}}.$$

**2** Let $r_1, r_2, \ldots, r_n$ be real numbers greater than or equal to 1. Prove that

$$\frac{1}{r_1 + 1} + \frac{1}{r_2 + 1} + \cdots + \frac{1}{r_n + 1} \geq \frac{n}{\sqrt[n]{r_1 r_2 \cdots r_n} + 1}.$$

**3** Let $x, y$ and $z$ be positive real numbers such that $xyz = 1$. Prove that

$$\frac{x^3}{(1 + y)(1 + z)} + \frac{y^3}{(1 + z)(1 + x)} + \frac{z^3}{(1 + x)(1 + y)} \geq \frac{3}{4}.$$

**4** For any two nonnegative integers $n$ and $k$ satisfying $n \geq k$, we define the number $c(n, k)$ as follows:

- $c\left(n, 0\right) = c\left(n, n\right) = 1$ for all $n \geq 0$;

- $c\left(n + 1, k\right) = 2^k c\left(n, k\right) + c\left(n, k - 1\right)$ for $n \geq k \geq 1$.

Prove that $c\left(n, k\right) = c\left(n, n - k\right)$ for all $n \geq k \geq 0$.

**5** Determine the least possible value of $f(1998)$, where $f : \mathbb{N} \to \mathbb{N}$ is a function such that for all $m, n \in \mathbb{N}$,

$$f\left(n^2 f(m)\right) = m\left(f(n)\right)^2.$$

– Combinatorics

**1** A rectangular array of numbers is given. In each row and each column, the sum of all numbers is an integer. Prove that each nonintegral number $x$ in the array can be changed into either $\lceil x \rceil$ or $\lfloor x \rfloor$ so that the row-sums and column-sums remain unchanged. (Note that $\lceil x \rceil$ is the least integer greater than or equal to $x$, while $\lfloor x \rfloor$ is the greatest integer less than or equal to $x$.)

**2** Let $n$ be an integer greater than 2. A positive integer is said to be *attainable* if it is 1 or can be obtained from 1 by a sequence of operations with the following properties:

1.) The first operation is either addition or multiplication.

2.) Thereafter, additions and multiplications are used alternately.

3.) In each addition, one can choose independently whether to add 2 or $n$

4.) In each multiplication, one can choose independently whether to multiply by 2 or by $n$.

A positive integer which cannot be so obtained is said to be *unattainable*.

**a.)** Prove that if $n \geq 9$, there are infinitely many unattainable positive integers.

**b.)** Prove that if $n = 3$, all positive integers except 7 are attainable.

**3** Cards numbered 1 to 9 are arranged at random in a row. In a move, one may choose any block of consecutive cards whose numbers are in ascending or descending order, and switch the block around. For example, 9 1 $\underline{6\ 5\ 3}$ 2 7 4 8 may be changed to 91 $\underline{3\ 5\ 6}$ 2 7 4 8. Prove that in at most 12 moves, one can arrange the 9 cards so that their numbers are in ascending or descending order.

**4** Let $U = \{1, 2, \ldots, n\}$, where $n \geq 3$. A subset $S$ of $U$ is said to be *split* by an arrangement of the elements of $U$ if an element not in $S$ occurs in the arrangement somewhere between two elements of $S$. For example, 13542 splits $\{1, 2, 3\}$ but not $\{3, 4, 5\}$. Prove that for any $n-2$ subsets of $U$, each containing at least 2 and at most $n-1$ elements, there is an arrangement of the elements of $U$ which splits all of them.

**5** In a contest, there are $m$ candidates and $n$ judges, where $n \geq 3$ is an odd integer. Each candidate is evaluated by each judge as either pass or fail. Suppose that each pair of judges agrees on at most $k$ candidates. Prove that

$$\frac{k}{m} \geq \frac{n-1}{2n}.$$

**6** Ten points are marked in the plane so that no three of them lie on a line. Each pair of points is connected with a segment. Each of these segments is painted with one of $k$ colors, in such a way that for any $k$ of the ten points, there are $k$ segments each joining two of them and no two being painted with the same color. Determine all integers $k$, $1 \leq k \leq 10$, for which this is possible.

**7** A solitaire game is played on an $m \times n$ rectangular board, using $mn$ markers which are white on one side and black on the other. Initially, each square of the board contains a marker with its white side up, except for one corner square, which contains a marker with its black side up. In each move, one may take away one marker with its black side up, but must then turn over all markers which are in squares having an edge in common with the square of the removed marker. Determine all pairs $(m, n)$ of positive integers such that all markers can be removed from the board.

IMO Shortlist 1997

**1** In the plane the points with integer coordinates are the vertices of unit squares. The squares are coloured alternately black and white (as on a chessboard). For any pair of positive integers $m$ and $n$, consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths $m$ and $n$, lie along edges of the squares. Let $S_1$ be the total area of the black part of the triangle and $S_2$ be the total area of the white part. Let $f(m,n) = |S_1 - S_2|$.

a) Calculate $f(m,n)$ for all positive integers $m$ and $n$ which are either both even or both odd.

b) Prove that $f(m,n) \leq \frac{1}{2}\max\{m,n\}$ for all $m$ and $n$.

c) Show that there is no constant $C \in \mathbb{R}$ such that $f(m,n) < C$ for all $m$ and $n$.

**2** Let $R_1, R_2, \ldots$ be the family of finite sequences of positive integers defined by the following rules: $R_1 = (1)$, and if $R_{n-1} = (x_1, \ldots, x_s)$, then

$$R_n = (1, 2, \ldots, x_1, 1, 2, \ldots, x_2, \ldots, 1, 2, \ldots, x_s, n).$$

For example, $R_2 = (1,2)$, $R_3 = (1,1,2,3)$, $R_4 = (1,1,1,2,1,2,3,4)$. Prove that if $n > 1$, then the $k$th term from the left in $R_n$ is equal to 1 if and only if the $k$th term from the right in $R_n$ is different from 1.

**3** For each finite set $U$ of nonzero vectors in the plane we define $l(U)$ to be the length of the vector that is the sum of all vectors in $U$. Given a finite set $V$ of nonzero vectors in the plane, a subset $B$ of $V$ is said to be maximal if $l(B)$ is greater than or equal to $l(A)$ for each nonempty subset $A$ of $V$.

(a) Construct sets of 4 and 5 vectors that have 8 and 10 maximal subsets respectively.

(b) Show that, for any set $V$ consisting of $n \geq 1$ vectors the number of maximal subsets is less than or equal to $2n$.

**4** An $n \times n$ matrix whose entries come from the set $S = \{1, 2, \ldots, 2n-1\}$ is called a *silver matrix* if, for each $i = 1, 2, \ldots, n$, the $i$-th row and the $i$-th column together contain all elements of $S$. Show that:

(a) there is no silver matrix for $n = 1997$;

(b) silver matrices exist for infinitely many values of $n$.

**5** Let $ABCD$ be a regular tetrahedron and $M, N$ distinct points in the planes $ABC$ and $ADC$ respectively. Show that the segments $MN, BN, MD$ are the sides of a triangle.

**6** (a) Let $n$ be a positive integer. Prove that there exist distinct positive integers $x, y, z$ such that

$$x^{n-1} + y^n = z^{n+1}.$$

(b) Let $a, b, c$ be positive integers such that $a$ and $b$ are relatively prime and $c$ is relatively prime either to $a$ or to $b$. Prove that there exist infinitely many triples $(x, y, z)$ of distinct positive integers $x, y, z$ such that

$$x^a + y^b = z^c.$$

**7** The lengths of the sides of a convex hexagon $ABCDEF$ satisfy $AB = BC$, $CD = DE$, $EF = FA$. Prove that:

$$\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \geq \frac{3}{2}.$$

**8** It is known that $\angle BAC$ is the smallest angle in the triangle $ABC$. The points $B$ and $C$ divide the circumcircle of the triangle into two arcs. Let $U$ be an interior point of the arc between $B$ and $C$ which does not contain $A$. The perpendicular bisectors of $AB$ and $AC$ meet the line $AU$ at $V$ and $W$, respectively. The lines $BV$ and $CW$ meet at $T$.

Show that $AU = TB + TC$.

*Alternative formulation:*

Four different points $A, B, C, D$ are chosen on a circle $\Gamma$ such that the triangle

$BCD$ is not right-angled. Prove that:

(a) The perpendicular bisectors of $AB$ and $AC$ meet the line $AD$ at certain points $W$ and $V$, respectively, and that the lines $CV$ and $BW$ meet at a certain point $T$.

(b) The length of one of the line segments $AD, BT$, and $CT$ is the sum of the lengths of the other two.

**9** Let $A_1A_2A_3$ be a non-isosceles triangle with incenter $I$. Let $C_i$, $i = 1, 2, 3$, be the smaller circle through $I$ tangent to $A_iA_{i+1}$ and $A_iA_{i+2}$ (the addition of indices being mod 3). Let $B_i, i = 1, 2, 3$, be the second point of intersection of $C_{i+1}$ and $C_{i+2}$. Prove that the circumcentres of the triangles $A_1B_1I, A_2B_2I, A_3B_3I$ are collinear.

**10** Find all positive integers $k$ for which the following statement is true: If $F(x)$ is a polynomial with integer coefficients satisfying the condition $0 \leq F(c) \leq k$ for each $c \in \{0, 1, \ldots, k+1\}$, then $F(0) = F(1) = \ldots = F(k+1)$.

**11** Let $P(x)$ be a polynomial with real coefficients such that $P(x) > 0$ for all $x \geq 0$. Prove that there exists a positive integer n such that $(1+x)^n \cdot P(x)$ is a polynomial with nonnegative coefficients.

**12** Let $p$ be a prime number and $f$ an integer polynomial of degree $d$ such that $f(0) = 0, f(1) = 1$ and $f(n)$ is congruent to 0 or 1 modulo $p$ for every integer $n$. Prove that $d \geq p - 1$.

**13** In town $A$, there are $n$ girls and $n$ boys, and each girl knows each boy. In town $B$, there are $n$ girls $g_1, g_2, \ldots, g_n$ and $2n-1$ boys $b_1, b_2, \ldots, b_{2n-1}$. The girl $g_i$, $i = 1, 2, \ldots, n$, knows the boys $b_1, b_2, \ldots, b_{2i-1}$, and no others. For all $r = 1, 2, \ldots, n$, denote by $A(r), B(r)$ the number of different ways in which $r$ girls from town $A$, respectively town $B$, can dance with $r$ boys from their own town, forming $r$ pairs, each girl with a boy she knows. Prove that $A(r) = B(r)$ for each $r = 1, 2, \ldots, n$.

**14** Let $b, m, n$ be positive integers such that $b > 1$ and $m \neq n$. Prove that if $b^m - 1$ and $b^n - 1$ have the same prime divisors, then $b + 1$ is a power of 2.

**15** An infinite arithmetic progression whose terms are positive integers contains the square of an integer and the cube of an integer. Show that it contains the sixth power of an integer.

**16** In an acute-angled triangle $ABC$, let $AD, BE$ be altitudes and $AP, BQ$ internal bisectors. Denote by $I$ and $O$ the incenter and the circumcentre of the triangle, respectively. Prove that the points $D, E$, and $I$ are collinear if and only if the points $P, Q$, and $O$ are collinear.

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**17** Find all pairs $(a, b)$ of positive integers that satisfy the equation: $a^{b^2} = b^a$.

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**18** The altitudes through the vertices $A, B, C$ of an acute-angled triangle $ABC$ meet the opposite sides at $D, E, F$, respectively. The line through $D$ parallel to $EF$ meets the lines $AC$ and $AB$ at $Q$ and $R$, respectively. The line $EF$ meets $BC$ at $P$. Prove that the circumcircle of the triangle $PQR$ passes through the midpoint of $BC$.

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**19** Let $a_1 \geq \cdots \geq a_n \geq a_{n+1} = 0$ be real numbers. Show that

$$\sqrt{\sum_{k=1}^{n} a_k} \leq \sum_{k=1}^{n} \sqrt{k}(\sqrt{a_k} - \sqrt{a_{k+1}}).$$

*Proposed by Romania*

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**20** Let $ABC$ be a triangle. $D$ is a point on the side $(BC)$. The line $AD$ meets the circumcircle again at $X$. $P$ is the foot of the perpendicular from $X$ to $AB$, and $Q$ is the foot of the perpendicular from $X$ to $AC$. Show that the line $PQ$ is a tangent to the circle on diameter $XD$ if and only if $AB = AC$.

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**21** Let $x_1$, $x_2$, $\ldots$, $x_n$ be real numbers satisfying the conditions:

$$\begin{cases} |x_1 + x_2 + \cdots + x_n| & = & 1 \\ |x_i| & \leq & \dfrac{n+1}{2} \qquad \text{for } i = 1, 2, \ldots, n. \end{cases}$$

Show that there exists a permutation $y_1$, $y_2$, $\ldots$, $y_n$ of $x_1$, $x_2$, $\ldots$, $x_n$ such that

$$|y_1 + 2y_2 + \cdots + ny_n| \leq \frac{n+1}{2}.$$

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**22** Does there exist functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = x^2$ and $g(f(x)) = x^k$ for all real numbers $x$

a) if $k = 3$?

b) if $k = 4$?

**23** Let $ABCD$ be a convex quadrilateral. The diagonals $AC$ and $BD$ intersect at $K$. Show that $ABCD$ is cyclic if and only if $AK \sin A + CK \sin C = BK \sin B + DK \sin D$.

**24** For each positive integer $n$, let $f(n)$ denote the number of ways of representing $n$ as a sum of powers of 2 with nonnegative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For instance, $f(4) = 4$, because the number 4 can be represented in the following four ways: 4; 2+2; 2+1+1; 1+1+1+1.

Prove that, for any integer $n \geq 3$ we have $2^{\frac{n^2}{4}} < f(2^n) < 2^{\frac{n^2}{2}}$.

**25** Let $X, Y, Z$ be the midpoints of the small arcs $BC, CA, AB$ respectively (arcs of the circumcircle of $ABC$). $M$ is an arbitrary point on $BC$, and the parallels through $M$ to the internal bisectors of $\angle B, \angle C$ cut the external bisectors of $\angle C, \angle B$ in $N, P$ respectively. Show that $XM, YN, ZP$ concur.

**26** For every integer $n \geq 2$ determine the minimum value that the sum $\sum_{i=0}^{n} a_i$ can take for nonnegative numbers $a_0, a_1, \ldots, a_n$ satisfying the condition $a_0 = 1$, $a_i \leq a_{i+1} + a_{i+2}$ for $i = 0, \ldots, n-2$.

IMO Shortlist 1996

– Algebra

**1** Suppose that $a, b, c > 0$ such that $abc = 1$. Prove that

$$\frac{ab}{ab + a^5 + b^5} + \frac{bc}{bc + b^5 + c^5} + \frac{ca}{ca + c^5 + a^5} \leq 1.$$

**2** Let $a_1 \geq a_2 \geq \ldots \geq a_n$ be real numbers such that for all integers $k > 0$,

$$a_1^k + a_2^k + \ldots + a_n^k \geq 0.$$

Let $p = \max\{|a_1|, \ldots, |a_n|\}$. Prove that $p = a_1$ and that

$$(x - a_1) \cdot (x - a_2) \cdots (x - a_n) \leq x^n - a_1^n$$

for all $x > a_1$.

**3** Let $a > 2$ be given, and starting $a_0 = 1, a_1 = a$ define recursively:

$$a_{n+1} = \left(\frac{a_n^2}{a_{n-1}^2} - 2\right) \cdot a_n.$$

Show that for all integers $k > 0$, we have: $\sum_{i=0}^{k} \frac{1}{a_i} < \frac{1}{2} \cdot (2 + a - \sqrt{a^2 - 4})$.

**4** Let $a_1, a_2 ... a_n$ be non-negative reals, not all zero. Show that that
(a) The polynomial $p(x) = x^n - a_1 x^{n-1} + ... - a_{n-1}x - a_n$ has preceisely 1 positive real root $R$.
(b) let $A = \sum_{i=1}^{n} a_i$ and $B = \sum_{i=1}^{n} i a_i$. Show that $A^A \leq R^B$.

**5** Let $P(x)$ be the real polynomial function, $P(x) = ax^3 + bx^2 + cx + d$. Prove that if $|P(x)| \leq 1$ for all $x$ such that $|x| \leq 1$, then

$$|a| + |b| + |c| + |d| \leq 7.$$

**6** Let $n$ be an even positive integer. Prove that there exists a positive inter $k$ such that

$$k = f(x) \cdot (x+1)^n + g(x) \cdot (x^n + 1)$$

for some polynomials $f(x), g(x)$ having integer coefficients. If $k_0$ denotes the least such $k$, determine $k_0$ as a function of $n$, i.e. show that $k_0 = 2^q$ where $q$ is the odd integer determined by $n = q \cdot 2^r, r \in \mathbb{N}$.

Note: This is variant A6' of the three variants given for this problem.

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**7** Let $f$ be a function from the set of real numbers $\mathbb{R}$ into itself such for all $x \in \mathbb{R}$, we have $|f(x)| \leq 1$ and

$$f\left(x + \frac{13}{42}\right) + f(x) = f\left(x + \frac{1}{6}\right) + f\left(x + \frac{1}{7}\right).$$

Prove that $f$ is a periodic function (that is, there exists a non-zero real number $c$ such $f(x+c) = f(x)$ for all $x \in \mathbb{R}$).

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**8** Let $\mathbb{N}_0$ denote the set of nonnegative integers. Find all functions $f$ from $\mathbb{N}_0$ to itself such that

$$f(m + f(n)) = f(f(m)) + f(n) \qquad \text{for all } m, n \in \mathbb{N}_0.$$

---

**9** Let the sequence $a(n), n = 1, 2, 3, \ldots$ be generated as follows with $a(1) = 0$, and for $n > 1:$

$$a(n) = a\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + (-1)^{\frac{n(n+1)}{2}}.$$

1.) Determine the maximum and minimum value of $a(n)$ over $n \leq 1996$ and find all $n \leq 1996$ for which these extreme values are attained.

2.) How many terms $a(n), n \leq 1996$, are equal to 0?

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– Geometry

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**1** Let $ABC$ be a triangle, and $H$ its orthocenter. Let $P$ be a point on the circumcircle of triangle $ABC$ (distinct from the vertices $A$, $B$, $C$), and let $E$ be the foot of the altitude of triangle $ABC$ from the vertex $B$. Let the parallel to the line $BP$ through the point $A$ meet the parallel to the line $AP$ through the point $B$ at a point $Q$. Let the parallel to the line $CP$ through the point $A$ meet the parallel to the line $AP$ through the point $C$ at a point $R$. The lines $HR$ and $AQ$ intersect at some point $X$. Prove that the lines $EX$ and $AP$ are parallel.

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**2** Let $P$ be a point inside a triangle $ABC$ such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let $D$, $E$ be the incenters of triangles $APB$, $APC$, respectively. Show that the lines $AP$, $BD$, $CE$ meet at a point.

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**3** Let $O$ be the circumcenter and $H$ the orthocenter of an acute-angled triangle $ABC$ such that $BC > CA$. Let $F$ be the foot of the altitude $CH$ of triangle $ABC$. The perpendicular to the line $OF$ at the point $F$ intersects the line $AC$ at $P$. Prove that $\measuredangle FHP = \measuredangle BAC$.

---

**4** Let $ABC$ be an equilateral triangle and let $P$ be a point in its interior. Let the lines $AP$, $BP$, $CP$ meet the sides $BC$, $CA$, $AB$ at the points $A_1$, $B_1$, $C_1$, respectively. Prove that
$A_1B_1 \cdot B_1C_1 \cdot C_1A_1 \geq A_1B \cdot B_1C \cdot C_1A$.

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**5** Let $ABCDEF$ be a convex hexagon such that $AB$ is parallel to $DE$, $BC$ is parallel to $EF$, and $CD$ is parallel to $FA$. Let $R_A, R_C, R_E$ denote the circumradii of triangles $FAB, BCD, DEF$, respectively, and let $P$ denote the perimeter of the hexagon. Prove that

$$R_A + R_C + R_E \geq \frac{P}{2}.$$

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**6** Let the sides of two rectangles be $\{a, b\}$ and $\{c, d\}$, respectively, with $a < c \leq d < b$ and $ab < cd$. Prove that the first rectangle can be placed within the second one if and only if

$$\left(b^2 - a^2\right)^2 \leq (bc - ad)^2 + (bd - ac)^2.$$

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Contributors: Arne, orl, RealZeta, Behsan, grobber, paladin8, Yimin Ge, cancer, lomos_lupin

**7** Let $ABC$ be an acute triangle with circumcenter $O$ and circumradius $R$. $AO$ meets the circumcircle of $BOC$ at $A'$, $BO$ meets the circumcircle of $COA$ at $B'$ and $CO$ meets the circumcircle of $AOB$ at $C'$. Prove that

$$OA' \cdot OB' \cdot OC' \geq 8R^3.$$

Sorry if this has been posted before since this is a very classical problem, but I failed to find it with the search-function.

**8** Let $ABCD$ be a convex quadrilateral, and let $R_A, R_B, R_C, R_D$ denote the circumradii of the triangles $DAB, ABC, BCD, CDA$, respectively. Prove that $R_A + R_C > R_B + R_D$ if and only if $\angle A + \angle C > \angle B + \angle D$.

**9** In the plane, consider a point $X$ and a polygon $\mathcal{F}$ (which is not necessarily convex). Let $p$ denote the perimeter of $\mathcal{F}$, let $d$ be the sum of the distances from the point $X$ to the vertices of $\mathcal{F}$, and let $h$ be the sum of the distances from the point $X$ to the sidelines of $\mathcal{F}$. Prove that $d^2 - h^2 \geq \frac{p^2}{4}$.

– Number Theory

**1** Four integers are marked on a circle. On each step we simultaneously replace each number by the difference between this number and next number on the circle, moving in a clockwise direction; that is, the numbers $a, b, c, d$ are replaced by $a-b, b-c, c-d, d-a$. Is it possible after 1996 such to have numbers $a, b, c, d$ such the numbers $|bc-ad|, |ac-bd|, |ab-cd|$ are primes?

**2** The positive integers $a$ and $b$ are such that the numbers $15a+16b$ and $16a-15b$ are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?

**3** A finite sequence of integers $a_0, a_1, \ldots, a_n$ is called quadratic if for each $i$ in the set $\{1, 2\ldots, n\}$ we have the equality $|a_i - a_{i-1}| = i^2$.

a.) Prove that any two integers $b$ and $c$, there exists a natural number $n$ and a quadratic sequence with $a_0 = b$ and $a_n = c$.

b.) Find the smallest natural number $n$ for which there exists a quadratic sequence with $a_0 = 0$ and $a_n = 1996$.

**4** Find all positive integers $a$ and $b$ for which

$$\left\lfloor \frac{a^2}{b} \right\rfloor + \left\lfloor \frac{b^2}{a} \right\rfloor = \left\lfloor \frac{a^2+b^2}{ab} \right\rfloor + ab.$$

| | |
|---|---|
| **5** | Show that there exists a bijective function $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that for all $m, n \in \mathbb{N}_0$: $$f(3mn + m + n) = 4f(m)f(n) + f(m) + f(n).$$ |
| – | Combinatorics |
| **1** | We are given a positive integer $r$ and a rectangular board $ABCD$ with dimensions $AB = 20, BC = 12$. The rectangle is divided into a grid of $20 \times 12$ unit squares. The following moves are permitted on the board: one can move from one square to another only if the distance between the centers of the two squares is $\sqrt{r}$. The task is to find a sequence of moves leading from the square with $A$ as a vertex to the square with $B$ as a vertex.<br><br>(a) Show that the task cannot be done if $r$ is divisible by 2 or 3.<br><br>(b) Prove that the task is possible when $r = 73$.<br><br>(c) Can the task be done when $r = 97$? |
| **2** | A square $(n-1) \times (n-1)$ is divided into $(n-1)^2$ unit squares in the usual manner. Each of the $n^2$ vertices of these squares is to be coloured red or blue. Find the number of different colourings such that each unit square has exactly two red vertices. (Two colouring schemse are regarded as different if at least one vertex is coloured differently in the two schemes.) |
| **3** | Let $k, m, n$ be integers such that $1 < n \leq m - 1 \leq k$. Determine the maximum size of a subset $S$ of the set $\{1, 2, 3, \ldots, k-1, k\}$ such that no $n$ distinct elements of $S$ add up to $m$. |
| **4** | Determine whether or nor there exist two disjoint infinite sets $A$ and $B$ of points in the plane satisfying the following conditions:<br><br>a.) No three points in $A \cup B$ are collinear, and the distance between any two points in $A \cup B$ is at least 1.<br><br>b.) There is a point of $A$ in any triangle whose vertices are in $B$, and there is a point of $B$ in any triangle whose vertices are in $A$. |

**5** Let $p, q, n$ be three positive integers with $p+q<n$. Let $(x_0, x_1, \cdots, x_n)$ be an $(n+1)$-tuple of integers satisfying the following conditions :

(a) $x_0 = x_n = 0$, and

(b) For each $i$ with $1 \leq i \leq n$, either $x_i - x_{i-1} = p$ or $x_i - x_{i-1} = -q$.

Show that there exist indices $i < j$ with $(i, j) \neq (0, n)$, such that $x_i = x_j$.

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**6** A finite number of coins are placed on an infinite row of squares. A sequence of moves is performed as follows: at each stage a square containing more than one coin is chosen. Two coins are taken from this square; one of them is placed on the square immediately to the left while the other is placed on the square immediately to the right of the chosen square. The sequence terminates if at some point there is at most one coin on each square. Given some initial configuration, show that any legal sequence of moves will terminate after the same number of steps and with the same final configuration.

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**7** let $V$ be a finitive set and $g$ and $f$ be two injective surjective functions from $V \text{to} V$.let $T$ and $S$ be two sets such that they are defined as following" $S = \{w \in V : f(f(w)) = g(g(w))\}$ $T = \{w \in V : f(g(w)) = g(f(w))\}$
we know that $S \cup T = V$, prove:
for each $w \in V : f(w) \in S$ if and only if $g(w) \in S$

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IMO Shortlist 1995

– Algebra

**1** Let $a$, $b$, $c$ be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

**2** Let $a$ and $b$ be non-negative integers such that $ab \geq c^2$, where $c$ is an integer. Prove that there is a number $n$ and integers $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ such that

$$\sum_{i=1}^{n} x_i^2 = a, \sum_{i=1}^{n} y_i^2 = b, \text{ and } \sum_{i=1}^{n} x_i y_i = c.$$

**3** Let $n$ be an integer, $n \geq 3$. Let $a_1, a_2, \ldots, a_n$ be real numbers such that $2 \leq a_i \leq 3$ for $i = 1, 2, \ldots, n$. If $s = a_1 + a_2 + \ldots + a_n$, prove that

$$\frac{a_1^2 + a_2^2 - a_3^2}{a_1 + a_2 - a_3} + \frac{a_2^2 + a_3^2 - a_4^2}{a_2 + a_3 - a_4} + \ldots + \frac{a_n^2 + a_1^2 - a_2^2}{a_n + a_1 - a_2} \leq 2s - 2n.$$

**4** Find all of the positive real numbers like $x, y, z$, such that :

1.) $x + y + z = a + b + c$

2.) $4xyz = a^2x + b^2y + c^2z + abc$

Proposed to Gazeta Matematica in the 80s by VASILE CRTOAJE and then by Titu Andreescu to IMO 1995.

**5** Let $\mathbb{R}$ be the set of real numbers. Does there exist a function $f : \mathbb{R} \mapsto \mathbb{R}$ which simultaneously satisfies the following three conditions?

**(a)** There is a positive number $M$ such that $\forall x : -M \leq f(x) \leq M$.
**(b)** The value of $f(1)$ is 1.
**(c)** If $x \neq 0$, then

$$f\left(x + \frac{1}{x^2}\right) = f(x) + \left[f\left(\frac{1}{x}\right)\right]^2$$


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**6** Let $n$ be an integer,$n \geq 3$. Let $x_1, x_2, \ldots, x_n$ be real numbers such that $x_i < x_{i+1}$ for $1 \leq i \leq n-1$. Prove that

$$\frac{n(n-1)}{2} \sum_{i<j} x_i x_j > \left(\sum_{i=1}^{n-1} (n-i) \cdot x_i\right) \cdot \left(\sum_{j=2}^{n} (j-1) \cdot x_j\right)$$

**–** Geometry

**1** Let $A, B, C, D$ be four distinct points on a line, in that order. The circles with diameters $AC$ and $BD$ intersect at $X$ and $Y$. The line $XY$ meets $BC$ at $Z$. Let $P$ be a point on the line $XY$ other than $Z$. The line $CP$ intersects the circle with diameter $AC$ at $C$ and $M$, and the line $BP$ intersects the circle with diameter $BD$ at $B$ and $N$. Prove that the lines $AM, DN, XY$ are concurrent.

**2** Let $A, B$ and $C$ be non-collinear points. Prove that there is a unique point $X$ in the plane of $ABC$ such that

$$XA^2 + XB^2 + AB^2 = XB^2 + XC^2 + BC^2 = XC^2 + XA^2 + CA^2.$$

**3** The incircle of triangle $\triangle ABC$ touches the sides $BC$, $CA$, $AB$ at $D, E, F$ respectively. $X$ is a point inside triangle of $\triangle ABC$ such that the incircle of triangle $\triangle XBC$ touches $BC$ at $D$, and touches $CX$ and $XB$ at $Y$ and $Z$ respectively.
Show that $E, F, Z, Y$ are concyclic.

**4** An acute triangle $ABC$ is given. Points $A_1$ and $A_2$ are taken on the side $BC$ (with $A_2$ between $A_1$ and $C$), $B_1$ and $B_2$ on the side $AC$ (with $B_2$ between $B_1$ and $A$), and $C_1$ and $C_2$ on the side $AB$ (with $C_2$ between $C_1$ and $B$) so that

$$\angle AA_1A_2 = \angle AA_2A_1 = \angle BB_1B_2 = \angle BB_2B_1 = \angle CC_1C_2 = \angle CC_2C_1.$$

The lines $AA_1, BB_1$, and $CC_1$ bound a triangle, and the lines $AA_2, BB_2$, and $CC_2$ bound a second triangle. Prove that all six vertices of these two triangles lie on a single circle.

**5** Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$ and $DE = EF = FA$, such that $\angle BCD = \angle EFA = \frac{\pi}{3}$. Suppose $G$ and $H$ are points in the interior of the hexagon such that $\angle AGB = \angle DHE = \frac{2\pi}{3}$. Prove that $AG + GB + GH + DH + HE \geq CF$.

**6** Let $A_1A_2A_3A_4$ be a tetrahedron, $G$ its centroid, and $A_1', A_2', A_3'$, and $A_4'$ the points where the circumsphere of $A_1A_2A_3A_4$ intersects $GA_1, GA_2, GA_3$, and $GA_4$, respectively. Prove that

$$GA_1 \cdot GA_2 \cdot GA_3 \cdot GA.4 \leq GA_1' \cdot GA_2' \cdot GA_3' \cdot GA_4'$$

and

$$\frac{1}{GA_1'} + \frac{1}{GA_2'} + \frac{1}{GA_3'} + \frac{1}{GA_4'} \leq \frac{1}{GA_1} + \frac{1}{GA_2} + \frac{1}{GA_3} + \frac{1}{GA_4}.$$

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**7** Let ABCD be a convex quadrilateral and O a point inside it. Let the parallels to the lines BC, AB, DA, CD through the point O meet the sides AB, BC, CD, DA of the quadrilateral ABCD at the points E, F, G, H, respectively. Then, prove that $\sqrt{|AHOE|} + \sqrt{|CFOG|} \leq \sqrt{|ABCD|}$, where $|P_1P_2...P_n|$ is an abbreviation for the non-directed area of an arbitrary polygon $P_1P_2...P_n$.

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**8** Suppose that $ABCD$ is a cyclic quadrilateral. Let $E = AC \cap BD$ and $F = AB \cap CD$. Denote by $H_1$ and $H_2$ the orthocenters of triangles $EAD$ and $EBC$, respectively. Prove that the points $F$, $H_1$, $H_2$ are collinear.

Original formulation:

Let $ABC$ be a triangle. A circle passing through $B$ and $C$ intersects the sides $AB$ and $AC$ again at $C'$ and $B'$, respectively. Prove that $BB'$, $CC'$ and $HH'$ are concurrent, where $H$ and $H'$ are the orthocentres of triangles $ABC$ and $AB'C'$ respectively.

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**–** NT, Combs

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**1** Let $k$ be a positive integer. Show that there are infinitely many perfect squares of the form $n \cdot 2^k - 7$ where $n$ is a positive integer.

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**2** Let $\mathbb{Z}$ denote the set of all integers. Prove that for any integers $A$ and $B$, one can find an integer $C$ for which $M_1 = \{x^2 + Ax + B : x \in \mathbb{Z}\}$ and $M_2 = 2x^2 + 2x + C : x \in \mathbb{Z}$ do not intersect.

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**3** Determine all integers $n > 3$ for which there exist $n$ points $A_1, \cdots, A_n$ in the plane, no three collinear, and real numbers $r_1, \cdots, r_n$ such that for $1 \leq i < j < k \leq n$, the area of $\triangle A_iA_jA_k$ is $r_i + r_j + r_k$.

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**4** Find all $x, y$ and $z$ in positive integer: $z + y^2 + x^3 = xyz$ and $x = \gcd(y, z)$.

**5** At a meeting of $12k$ people, each person exchanges greetings with exactly $3k+6$ others. For any two people, the number who exchange greetings with both is the same. How many people are at the meeting?

**6** Let $p$ be an odd prime number. How many $p$-element subsets $A$ of $\{1, 2, \dots, 2p\}$ are there, the sum of whose elements is divisible by $p$?

**7** Does there exist an integer $n > 1$ which satisfies the following condition? The set of positive integers can be partitioned into $n$ nonempty subsets, such that an arbitrary sum of $n-1$ integers, one taken from each of any $n-1$ of the subsets, lies in the remaining subset.

**8** Let $p$ be an odd prime. Determine positive integers $x$ and $y$ for which $x \leq y$ and $\sqrt{2p} - \sqrt{x} - \sqrt{y}$ is non-negative and as small as possible.

– Sequences

**1** Does there exist a sequence $F(1), F(2), F(3), \dots$ of non-negative integers that simultaneously satisfies the following three conditions?

**(a)** Each of the integers $0, 1, 2, \dots$ occurs in the sequence.
**(b)** Each positive integer occurs in the sequence infinitely often.
**(c)** For any $n \geq 2$,

$$F(F(n^{163})) = F(F(n)) + F(F(361)).$$

**2** Find the maximum value of $x_0$ for which there exists a sequence $x_0, x_1 \cdots, x_{1995}$ of positive reals with $x_0 = x_{1995}$, such that

$$x_{i-1} + \frac{2}{x_{i-1}} = 2x_i + \frac{1}{x_i},$$

for all $i = 1, \cdots, 1995$.

**3** For an integer $x \geq 1$, let $p(x)$ be the least prime that does not divide $x$, and define $q(x)$ to be the product of all primes less than $p(x)$. In particular, $p(1) = 2$.

For $x$ having $p(x) = 2$, define $q(x) = 1$. Consider the sequence $x_0, x_1, x_2, \ldots$ defined by $x_0 = 1$ and

$$x_{n+1} = \frac{x_n p(x_n)}{q(x_n)}$$

for $n \geq 0$. Find all $n$ such that $x_n = 1995$.

**4** Suppose that $x_1, x_2, x_3, \ldots$ are positive real numbers for which

$$x_n^n = \sum_{j=0}^{n-1} x_n^j$$

for $n = 1, 2, 3, \ldots$ Prove that $\forall n$,

$$2 - \frac{1}{2^{n-1}} \leq x_n < 2 - \frac{1}{2^n}.$$

**5** For positive integers $n$, the numbers $f(n)$ are defined inductively as follows: $f(1) = 1$, and for every positive integer $n$, $f(n+1)$ is the greatest integer $m$ such that there is an arithmetic progression of positive integers $a_1 < a_2 < \ldots < a_m = n$ for which

$$f(a_1) = f(a_2) = \ldots = f(a_m).$$

Prove that there are positive integers $a$ and $b$ such that $f(an+b) = n+2$ for every positive integer $n$.

**6** Let $\mathbb{N}$ denote the set of all positive integers. Prove that there exists a unique function $f : \mathbb{N} \mapsto \mathbb{N}$ satisfying

$$f(m + f(n)) = n + f(m + 95)$$

for all $m$ and $n$ in $\mathbb{N}$. What is the value of $\sum_{k=1}^{19} f(k)$?

IMO Shortlist 1994

– Algebra

**1** Let $a_0 = 1994$ and $a_{n+1} = \frac{a_n^2}{a_n+1}$ for each nonnegative integer $n$. Prove that $1994 - n$ is the greatest integer less than or equal to $a_n$, $0 \leq n \leq 998$

**2** Let $m$ and $n$ be two positive integers. Let $a_1$, $a_2$, $\ldots$, $a_m$ be $m$ different numbers from the set $\{1, 2, \ldots, n\}$ such that for any two indices $i$ and $j$ with $1 \leq i \leq j \leq m$ and $a_i + a_j \leq n$, there exists an index $k$ such that $a_i + a_j = a_k$. Show that

$$\frac{a_1 + a_2 + ... + a_m}{m} \geq \frac{n+1}{2}.$$

**3** Let $S$ be the set of all real numbers strictly greater than 1. Find all functions $f : S \to S$ satisfying the two conditions:

(a) $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all $x, y$ in $S$;

(b) $\frac{f(x)}{x}$ is strictly increasing on each of the two intervals $-1 < x < 0$ and $0 < x$.

**4** Let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{R}^+$ the subset of all positive ones. Let $\alpha$ and $\beta$ be given elements in $\mathbb{R}$, not necessarily distinct. Find all functions $f : \mathbb{R}^+ \mapsto \mathbb{R}$ such that

$$f(x)f(y) = y^\alpha f\left(\frac{x}{2}\right) + x^\beta f\left(\frac{y}{2}\right) \forall x, y \in \mathbb{R}^+.$$

**5** Let $f(x) = \frac{x^2+1}{2x}$ for $x \neq 0$. Define $f^{(0)}(x) = x$ and $f^{(n)}(x) = f(f^{(n-1)}(x))$ for all positive integers $n$ and $x \neq 0$. Prove that for all non-negative integers $n$ and $x \neq \{-1, 0, 1\}$

$$\frac{f^{(n)}(x)}{f^{(n+1)}(x)} = 1 + \frac{1}{f\left(\left(\frac{x+1}{x-1}\right)^{2n}\right)}.$$

– Geometry


Contributors: Jan, darij grinberg, galois, orl, Fermat -Euler, Valentin Vornicu, chaogold, grobber, Pascual2005

**1** $C$ and $D$ are points on a semicircle. The tangent at $C$ meets the extended diameter of the semicircle at $B$, and the tangent at $D$ meets it at $A$, so that $A$ and $B$ are on opposite sides of the center. The lines $AC$ and $BD$ meet at $E$. $F$ is the foot of the perpendicular from $E$ to $AB$. Show that $EF$ bisects angle $CFD$

**2** $ABCD$ is a quadrilateral with $BC$ parallel to $AD$. $M$ is the midpoint of $CD$, $P$ is the midpoint of $MA$ and $Q$ is the midpoint of $MB$. The lines $DP$ and $CQ$ meet at $N$. Prove that $N$ is inside the quadrilateral $ABCD$.

**3** A circle $C$ has two parallel tangents $L'$ and$L"$. A circle $C'$ touches $L'$ at $A$ and $C$ at $X$. A circle $C"$ touches $L"$ at $B$, $C$ at $Y$ and $C'$ at $Z$. The lines $AY$ and $BX$ meet at $Q$. Show that $Q$ is the circumcenter of $XYZ$

**4** Let $ABC$ be an isosceles triangle with $AB = AC$. $M$ is the midpoint of $BC$ and $O$ is the point on the line $AM$ such that $OB$ is perpendicular to $AB$. $Q$ is an arbitrary point on $BC$ different from $B$ and $C$. $E$ lies on the line $AB$ and $F$ lies on the line $AC$ such that $E, Q, F$ are distinct and collinear. Prove that $OQ$ is perpendicular to $EF$ if and only if $QE = QF$.

**5** A circle $C$ with center $O$. and a line $L$ which does not touch circle $C$. $OQ$ is perpendicular to $L$, $Q$ is on $L$. $P$ is on $L$, draw two tangents $L_1, L_2$ to circle $C$. $QA, QB$ are perpendicular to $L_1, L_2$ respectively. ($A$ on $L_1$, $B$ on $L_2$). Prove that, line $AB$ intersect $QO$ at a fixed point.

*Original formulation:*

A line $l$ does not meet a circle $\omega$ with center $O$. $E$ is the point on $l$ such that $OE$ is perpendicular to $l$. $M$ is any point on $l$ other than $E$. The tangents from $M$ to $\omega$ touch it at $A$ and $B$. $C$ is the point on $MA$ such that $EC$ is perpendicular to $MA$. $D$ is the point on $MB$ such that $ED$ is perpendicular to $MB$. The line $CD$ cuts $OE$ at $F$. Prove that the location of $F$ is independent of that of $M$.

– Number Theory

**1** $M$ is a subset of $\{1, 2, 3, \ldots, 15\}$ such that the product of any three distinct elements of $M$ is not a square. Determine the maximum number of elements in $M$.

**2** Find all ordered pairs $(m, n)$ where $m$ and $n$ are positive integers such that $\frac{n^3+1}{mn-1}$ is an integer.

**3** Show that there exists a set $A$ of positive integers with the following property: for any infinite set $S$ of primes, there exist *two* positive integers $m$ in $A$ and $n$ not in $A$, each of which is a product of $k$ distinct elements of $S$ for some $k \geq 2$.

**4** Define the sequences $a_n, b_n, c_n$ as follows. $a_0 = k, b_0 = 4, c_0 = 1$.
If $a_n$ is even then $a_{n+1} = \frac{a_n}{2}$, $b_{n+1} = 2b_n$, $c_{n+1} = c_n$.
If $a_n$ is odd, then $a_{n+1} = a_n - \frac{b_n}{2} - c_n$, $b_{n+1} = b_n$, $c_{n+1} = b_n + c_n$.
Find the number of positive integers $k < 1995$ such that some $a_n = 0$.

**5** For any positive integer $k$, let $f_k$ be the number of elements in the set $\{k + 1, k + 2, \dots, 2k\}$ whose base 2 representation contains exactly three 1s.

(a) Prove that for any positive integer $m$, there exists at least one positive integer $k$ such that $f(k) = m$.

(b) Determine all positive integers $m$ for which there exists *exactly one* $k$ with $f(k) = m$.

**6** Define the sequence $a_1, a_2, a_3, ...$ as follows. $a_1$ and $a_2$ are coprime positive integers and $a_{n+2} = a_{n+1}a_n + 1$. Show that for every $m > 1$ there is an $n > m$ such that $a_m^m$ divides $a_n^n$. Is it true that $a_1$ must divide $a_n^n$ for some $n > 1$?

**7** A wobbly number is a positive integer whose digits are alternately zero and non-zero with the last digit non-zero (for example, 201). Find all positive integers which do not divide any wobbly number.

**–** Combinatorics

**1** Two players play alternately on a $5 \times 5$ board. The first player always enters a 1 into an empty square and the second player always enters a 0 into an empty square. When the board is full, the sum of the numbers in each of the nine $3 \times 3$ squares is calculated and the first player's score is the largest such sum. What is the largest score the first player can make, regardless of the responses of the second player?

**2** In a certain city, age is reckoned in terms of real numbers rather than integers. Every two citizens $x$ and $x'$ either know each other or do not know each other. Moreover, if they do not, then there exists a chain of citizens $x = x_0, x_1, \dots, x_n = x'$ for some integer $n \geq 2$ such that $x_{i-1}$ and $x_i$ know each other. In a census, all male citizens declare their ages, and there is at least one male citizen. Each female citizen provides only the information that her age is the average of the ages of all the citizens she knows. Prove that this is enough to determine uniquely the ages of all the female citizens.

**3** Peter has three accounts in a bank, each with an integral number of dollars. He is only allowed to transfer money from one account to another so that the amount of money in the latter is doubled. Prove that Peter can always transfer all his money into two accounts. Can Peter always transfer all his money into one account?

**4** There are $n+1$ cells in a row labeled from 0 to $n$ and $n+1$ cards labeled from 0 to $n$. The cards are arbitrarily placed in the cells, one per cell. The objective is to get card $i$ into cell $i$ for each $i$. The allowed move is to find the smallest $h$ such that cell $h$ has a card with a label $k > h$, pick up that card, slide the cards in cells $h+1$, $h+2$, ... , $k$ one cell to the left and to place card $k$ in cell $k$. Show that at most $2^n - 1$ moves are required to get every card into the correct cell and that there is a unique starting position which requires $2^n - 1$ moves. [For example, if $n = 2$ and the initial position is 210, then we get 102, then 012, a total of 2 moves.]

**5** 1994 girls are seated at a round table. Initially one girl holds $n$ tokens. Each turn a girl who is holding more than one token passes one token to each of her neighbours.

a.) Show that if $n < 1994$, the game must terminate.
b.) Show that if $n = 1994$ it cannot terminate.

**6** Two players play alternatively on an infinite square grid. The first player puts an $X$ in an empty cell and the second player puts an $O$ in an empty cell. The first player wins if he gets 11 adjacent $X$'s in a line - horizontally, vertically or diagonally. Show that the second player can always prevent the first player from winning.

**7** Let $n > 2$. Show that there is a set of $2^{n-1}$ points in the plane, no three collinear such that no $2n$ form a convex $2n$-gon.

IMO Shortlist 1993

– Algebra

**1** Define a sequence $< f(n) >_{n=1}^{\infty}$ of positive integers by

$$f(1) = 1$$

and

$$f(n) = \begin{cases} f(n-1) - n & \text{if } f(n-1) > n; \\ f(n-1) + n & \text{if } f(n-1) \leq n, \end{cases}$$

for $n \geq 2$. Let $S = \{n \in \mathbb{N} | f(n) = 1993\}$.

**(i)** Prove that $S$ is an infinite set.
**(ii)** Find the least positive integer in $S$.
**(iii)** If all the elements of $S$ are written in ascending order as

$$n_1 < n_2 < n_3 < \dots,$$

show that

$$\lim_{i \to \infty} \frac{n_{i+1}}{n_i} = 3.$$

**2** Show that there exists a finite set $A \subset \mathbb{R}^2$ such that for every $X \in A$ there are points $Y_1, Y_2, \dots, Y_{1993}$ in $A$ such that the distance between $X$ and $Y_i$ is equal to 1, for every $i$.

**3** Prove that

$$\frac{a}{b+2c+3d} + \frac{b}{c+2d+3a} + \frac{c}{d+2a+3b} + \frac{d}{a+2b+3c} \geq \frac{2}{3}$$

for all positive real numbers $a, b, c, d$.

**4** Solve the following system of equations, in which $a$ is a given number satisfying $|a| > 1$:

$$x_1^2 = ax_2 + 1$$

$$x_2^2 = ax_3 + 1$$

$$\dots$$

$$x_{999}^2 = ax_{1000} + 1$$

$$x_{1000}^2 = ax_1 + 1$$

**5** $a > 0$ and $b$, $c$ are integers such that $ac\ \ b^2$ is a square-free positive integer P. P could be 3*5, but not 3ˆ2*5. Let $f(n)$ be the number of pairs of integers $d, e$ such that $ad^2 + 2bde + ce^2 = n$. Show that$f(n)$ is finite and that $f(n) = f(P^k n)$ for every positive integer $k$.

**Original Statement:**

Let $a, b, c$ be given integers $a > 0$, $ac - b^2 = P = P_1 \cdots P_n$ where $P_1 \cdots P_n$ are (distinct) prime numbers. Let $M(n)$ denote the number of pairs of integers $(x, y)$ for which

$$ax^2 + 2bxy + cy^2 = n.$$

Prove that $M(n)$ is finite and $M(n) = M(P_k \cdot n)$ for every integer $k \geq 0$. Note that the "$n$" in $P_N$ and the "$n$" in $M(n)$ do not have to be the same.

**6** Let $\mathbb{N} = \{1, 2, 3, \ldots\}$. Determine if there exists a strictly increasing function $f : \mathbb{N} \mapsto \mathbb{N}$ with the following properties:

(i) $f(1) = 2$;

(ii) $f(f(n)) = f(n) + n, (n \in \mathbb{N})$.

**7** Let $n > 1$ be an integer and let $f(x) = x^n + 5 \cdot x^{n-1} + 3$. Prove that there do not exist polynomials $g(x), h(x)$, each having integer coefficients and degree at least one, such that $f(x) = g(x) \cdot h(x)$.

**8** Let $c_1, \ldots, c_n \in \mathbb{R}$ with $n \geq 2$ such that

$$0 \leq \sum_{i=1}^{n} c_i \leq n.$$

Show that we can find integers $k_1, \ldots, k_n$ such that

$$\sum_{i=1}^{n} k_i = 0$$

and

$$1 - n \leq c_i + n \cdot k_i \leq n$$

for every $i = 1, \ldots, n$.

Let $x_1, \ldots, x_n$, with $n \geq 2$ be real numbers such that

$$|x_1 + \ldots + x_n| \leq n.$$

Show that there exist integers $k_1, \ldots, k_n$ such that

$$|k_1 + \ldots + k_n| = 0.$$

and

$$|x_i + 2 \cdot n \cdot k_i| \leq 2 \cdot n - 1$$

for every $i = 1, \ldots, n$. In order to prove this, denote $c_i = \frac{1+x_i}{2}$ for $i = 1, \ldots, n$, etc.

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**9** Let $a, b, c, d$ be four non-negative numbers satisfying

$$a + b + c + d = 1.$$

Prove the inequality

$$a \cdot b \cdot c + b \cdot c \cdot d + c \cdot d \cdot a + d \cdot a \cdot b \leq \frac{1}{27} + \frac{176}{27} \cdot a \cdot b \cdot c \cdot d.$$

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– Combinatorics

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**1** a) Show that the set $\mathbb{Q}^+$ of all positive rationals can be partitioned into three disjoint subsets. $A, B, C$ satisfying the following conditions:

$$BA = B; \& B^2 = C; \& BC = A;$$

where $HK$ stands for the set $\{hk : h \in H, k \in K\}$ for any two subsets $H, K$ of $\mathbb{Q}^+$ and $H^2$ stands for $HH$.

b) Show that all positive rational cubes are in $A$ for such a partition of $\mathbb{Q}^+$.

c) Find such a partition $\mathbb{Q}^+ = A \cup B \cup C$ with the property that for no positive integer $n \leq 34$, both $n$ and $n + 1$ are in $A$, that is,

$$\min\{n \in \mathbb{N} : n \in A, n + 1 \in A\} > 34.$$

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**2** Let $n, k \in \mathbb{Z}^+$ with $k \leq n$ and let $S$ be a set containing $n$ distinct real numbers. Let $T$ be a set of all real numbers of the form $x_1+x_2+\ldots+x_k$ where $x_1, x_2, \ldots, x_k$ are distinct elements of $S$. Prove that $T$ contains at least $k(n-k)+1$ distinct elements.

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**3** Let $n > 1$ be an integer. In a circular arrangement of $n$ lamps $L_0, \ldots, L_{n-1}$, each of of which can either ON or OFF, we start with the situation where all lamps are ON, and then carry out a sequence of steps, $Step_0, Step_1, \ldots$. If $L_{j-1}$ ($j$ is taken mod $n$) is ON then $Step_j$ changes the state of $L_j$ (it goes from ON to OFF or from OFF to ON) but does not change the state of any of the other lamps. If $L_{j-1}$ is OFF then $Step_j$ does not change anything at all. Show that:

(i) There is a positive integer $M(n)$ such that after $M(n)$ steps all lamps are ON again,

(ii) If $n$ has the form $2^k$ then all the lamps are ON after $n^2-1$ steps,

(iii) If $n$ has the form $2^k+1$ then all lamps are ON after $n^2-n+1$ steps.

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**4** Let $n \geq 2, n \in \mathbb{N}$ and $A_0 = (a_{01}, a_{02}, \ldots, a_{0n})$ be any $n-$tuple of natural numbers, such that $0 \leq a_{0i} \leq i-1$, for $i = 1, \ldots, n$. $n-$tuples $A_1 = (a_{11}, a_{12}, \ldots, a_{1n}), A_2 = (a_{21}, a_{22}, \ldots, a_{2n}), \ldots$ are defined by: $a_{i+1,j} = Card\{a_{i,l} | 1 \leq l \leq j-1, a_{i,l} \geq a_{i,j}\}$, for $i \in \mathbb{N}$ and $j = 1, \ldots, n$. Prove that there exists $k \in \mathbb{N}$, such that $A_{k+2} = A_k$.

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**5** Let $S_n$ be the number of sequences $(a_1, a_2, \ldots, a_n)$, where $a_i \in \{0,1\}$, in which no six consecutive blocks are equal. Prove that $S_n \to \infty$ when $n \to \infty$.

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– Geometry

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**1** Let $ABC$ be a triangle, and $I$ its incenter. Consider a circle which lies inside the circumcircle of triangle $ABC$ and touches it, and which also touches the sides $CA$ and $BC$ of triangle $ABC$ at the points $D$ and $E$, respectively. Show that the point $I$ is the midpoint of the segment $DE$.

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**2** A circle $S$ bisects a circle $S'$ if it cuts $S'$ at opposite ends of a diameter. $S_A$, $S_B$,$S_C$ are circles with distinct centers $A, B, C$ (respectively).
Show that $A, B, C$ are collinear iff there is no unique circle $S$ which bisects each of $S_A$, $S_B$,$S_C$ . Show that if there is more than one circle $S$ which bisects each of $S_A$, $S_B$,$S_C$ , then all such circles pass through two fixed points. Find these

points.

**Original Statement:**

A circle $S$ is said to cut a circle $\Sigma$ **diametrically** if and only if their common chord is a diameter of $\Sigma$.
Let $S_A, S_B, S_C$ be three circles with distinct centres $A, B, C$ respectively. Prove that $A, B, C$ are collinear if and only if there is no unique circle $S$ which cuts each of $S_A, S_B, S_C$ diametrically. Prove further that if there exists more than one circle $S$ which cuts each $S_A, S_B, S_C$ diametrically, then all such circles $S$ pass through two fixed points. Locate these points in relation to the circles $S_A, S_B, S_C$.

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**3** Let triangle $ABC$ be such that its circumradius is $R = 1$. Let $r$ be the inradius of $ABC$ and let $p$ be the inradius of the orthic triangle $A'B'C'$ of triangle $ABC$. Prove that

$$p \leq 1 - \frac{1}{3 \cdot (1 + r)^2}.$$

Let $ABC$ be a triangle with circumradius $R$ and inradius $r$. If $p$ is the inradius of the orthic triangle of triangle $ABC$, show that $\frac{p}{R} \leq 1 - \frac{\left(1+\frac{r}{R}\right)^2}{3}$.
*Note.* The orthic triangle of triangle $ABC$ is defined as the triangle whose vertices are the feet of the altitudes of triangle $ABC$.

**SOLUTION 1 by mecrazywong:**

$p = 2R\cos A\cos B\cos C, 1+\frac{r}{R} = 1+4\sin A/2\sin B/2\sin C/2 = \cos A+\cos B+\cos C$.
Thus, the ineqaulity is equivalent to $6\cos A\cos B\cos C + (\cos A + \cos B + \cos C)^2 \leq 3$. But this is easy since $\cos A+\cos B+\cos C \leq 3/2, \cos A\cos B\cos C \leq 1/8$.

**SOLUTION 2 by Virgil Nicula:**

I note the inradius $r'$ of a orthic triangle.

Must prove the inequality $\frac{r'}{R} \leq 1 - \frac{1}{3}\left(1 + \frac{r}{R}\right)^2$.

From the wellknown relations $r' = 2R\cos A\cos B\cos C$

and $\cos A\cos B\cos C \leq \frac{1}{8}$ results $\frac{r'}{R} \leq \frac{1}{4}$.

But $\frac{1}{4} \leq 1 - \frac{1}{3}\left(1 + \frac{r}{R}\right)^2 \Longleftrightarrow \frac{1}{3}\left(1 + \frac{r}{R}\right)^2 \leq \frac{3}{4} \Longleftrightarrow$

$\left(1+\frac{r}{R}\right)^2 \le \left(\frac{3}{2}\right)^2 \iff 1+\frac{r}{R} \le \frac{3}{2} \iff \frac{r}{R} \le \frac{1}{2} \iff 2r \le R$ (true).

Therefore, $\frac{r'}{R} \le \frac{1}{4} \le 1-\frac{1}{3}\left(1+\frac{r}{R}\right)^2 \implies \frac{r'}{R} \le 1-\frac{1}{3}\left(1+\frac{r}{R}\right)^2$.

**SOLUTION 3 by darij grinberg:**

I know this is not quite an ML reference, but the problem was discussed in Hyacinthos messages #6951, #6978, #6981, #6982, #6985, #6986 (particularly the last message).

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**4** Given a triangle $ABC$, let $D$ and $E$ be points on the side $BC$ such that $\angle BAD = \angle CAE$. If $M$ and $N$ are, respectively, the points of tangency of the incircles of the triangles $ABD$ and $ACE$ with the line $BC$, then show that

$$\frac{1}{MB}+\frac{1}{MD}=\frac{1}{NC}+\frac{1}{NE}.$$

---

**5** On an infinite chessboard, a solitaire game is played as follows: at the start, we have $n^2$ pieces occupying a square of side $n$. The only allowed move is to jump over an occupied square to an unoccupied one, and the piece which has been jumped over is removed. For which $n$ can the game end with only one piece remaining on the board?

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**6** For three points $A, B, C$ in the plane, we define $m(ABC)$ to be the smallest length of the three heights of the triangle $ABC$, where in the case $A$, $B$, $C$ are collinear, we set $m(ABC)=0$. Let $A$, $B$, $C$ be given points in the plane. Prove that for any point $X$ in the plane,

$$m(ABC) \le m(ABX)+m(AXC)+m(XBC).$$

---

**7** Let $A$, $B$, $C$, $D$ be four points in the plane, with $C$ and $D$ on the same side of the line $AB$, such that $AC \cdot BD = AD \cdot BC$ and $\angle ADB = 90^\circ + \angle ACB$. Find the ratio

$$\frac{AB \cdot CD}{AC \cdot BD},$$

and prove that the circumcircles of the triangles $ACD$ and $BCD$ are orthogonal. (Intersecting circles are said to be orthogonal if at either common point their tangents are perpendicuar. Thus, proving that the circumcircles of the triangles $ACD$ and $BCD$ are orthogonal is equivalent to proving that the tangents to the circumcircles of the triangles $ACD$ and $BCD$ at the point $C$ are perpendicular.)

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**8** The vertices $D, E, F$ of an equilateral triangle lie on the sides $BC, CA, AB$ respectively of a triangle $ABC$. If $a, b, c$ are the respective lengths of these sides, and $S$ the area of $ABC$, prove that

$$DE \geq \frac{2 \cdot \sqrt{2} \cdot S}{\sqrt{a^2 + b^2 + c^2 + 4 \cdot \sqrt{3} \cdot S}}.$$

– Number Theory

**2** A natural number $n$ is said to have the property $P$, if, for all $a, n^2$ divides $a^n - 1$ whenever $n$ divides $a^n - 1$.

a.) Show that every prime number $n$ has property $P$.

b.) Show that there are infinitely many composite numbers $n$ that possess property $P$.

**3** Let $a, b, n$ be positive integers, $b > 1$ and $b^n - 1 \mid a$. Show that the representation of the number $a$ in the base $b$ contains at least $n$ digits different from zero.

**4** Show that for any finite set $S$ of distinct positive integers, we can find a set $T$ $S$ such that every member of $T$ divides the sum of all the members of $T$.

**Original Statement:**

A finite set of (distinct) positive integers is called a **DS-set** if each of the integers divides the sum of them all. Prove that every finite set of positive integers is a subset of some **DS-set**.

**5** Let $S$ be the set of all pairs $(m, n)$ of relatively prime positive integers $m, n$ with $n$ even and $m < n$. For $s = (m, n) \in S$ write $n = 2^k \cdot n_o$ where $k, n_0$ are positive integers with $n_0$ odd and define

$$f(s) = (n_0, m + n - n_0).$$

Prove that $f$ is a function from $S$ to $S$ and that for each $s = (m, n) \in S$, there exists a positive integer $t \leq \frac{m+n+1}{4}$ such that

$$f^t(s) = s,$$

where

$$f^t(s) = \underbrace{(f \circ f \circ \cdots \circ f)}_{t \text{ times}}(s).$$

If $m+n$ is a prime number which does not divide $2^k-1$ for $k = 1, 2, \ldots, m+n-2$, prove that the smallest value $t$ which satisfies the above conditions is $\left[\frac{m+n+1}{4}\right]$ where $[x]$ denotes the greatest integer $\leq x$.

IMO Shortlist 1992

**1** Prove that for any positive integer $m$ there exist an infinite number of pairs of integers $(x, y)$ such that

*(i)* $x$ and $y$ are relatively prime;
*(ii)* $y$ divides $x^2 + m$;
*(iii)* $x$ divides $y^2 + m$.
*(iv)* $x + y \leq m + 1 -$ (optional condition)

**2** Let $\mathbb{R}^+$ be the set of all non-negative real numbers. Given two positive real numbers $a$ and $b$, suppose that a mapping $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ satisfies the functional equation:

$$f(f(x)) + af(x) = b(a + b)x.$$

Prove that there exists a unique solution of this equation.

**3** The diagonals of a quadrilateral $ABCD$ are perpendicular: $AC \perp BD$. Four squares, $ABEF, BCGH, CDIJ, DAKL$, are erected externally on its sides. The intersection points of the pairs of straight lines $CL, DF, AH, BJ$ are denoted by $P_1, Q_1, R_1, S_1$, respectively (left figure), and the intersection points of the pairs of straight lines $AI, BK, CEDG$ are denoted by $P_2, Q_2, R_2, S_2$, respectively (right figure). Prove that $P_1Q_1R_1S_1 \cong P_2Q_2R_2S_2$ where $P_1, Q_1, R_1, S_1$ and $P_2, Q_2, R_2, S_2$ are the two quadrilaterals.

*Alternative formulation:* Outside a convex quadrilateral $ABCD$ with perpendicular diagonals, four squares $AEFB, BGHC, CIJD, DKLA$, are constructed (vertices are given in counterclockwise order). Prove that the quadrilaterals $Q_1$ and $Q_2$ formed by the lines $AG, BI, CK, DE$ and $AJ, BL, CF, DH$, respectively, are congruent.

**4** Consider 9 points in space, no four of which are coplanar. Each pair of points is joined by an edge (that is, a line segment) and each edge is either colored blue or red or left uncolored. Find the smallest value of $n$ such that whenever exactly $n$ edges are colored, the set of colored edges necessarily contains a triangle all of whose edges have the same color.

**5** A convex quadrilateral has equal diagonals. An equilateral triangle is constructed on the outside of each side of the quadrilateral. The centers of the triangles on opposite sides are joined. Show that the two joining lines are perpendicular.

*Alternative formulation.* Given a convex quadrilateral $ABCD$ with congruent diagonals $AC = BD$. Four regular triangles are errected externally on its sides. Prove that the segments joining the centroids of the triangles on the opposite sides are perpendicular to each other.

*Original formulation:* Let $ABCD$ be a convex quadrilateral such that $AC = BD$. Equilateral triangles are constructed on the sides of the quadrilateral. Let $O_1, O_2, O_3, O_4$ be the centers of the triangles constructed on $AB, BC, CD, DA$ respectively. Show that $O_1O_3$ is perpendicular to $O_2O_4$.

---

**6** Let $\mathbb{R}$ denote the set of all real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(x^2 + f(y)\right) = y + (f(x))^2 \quad \text{for all } x, y \in \mathbb{R}.$$

---

**7** Two circles $\Omega_1$ and $\Omega_2$ are externally tangent to each other at a point $I$, and both of these circles are tangent to a third circle $\Omega$ which encloses the two circles $\Omega_1$ and $\Omega_2$.
The common tangent to the two circles $\Omega_1$ and $\Omega_2$ at the point $I$ meets the circle $\Omega$ at a point $A$. One common tangent to the circles $\Omega_1$ and $\Omega_2$ which doesn't pass through $I$ meets the circle $\Omega$ at the points $B$ and $C$ such that the points $A$ and $I$ lie on the same side of the line $BC$.
Prove that the point $I$ is the incenter of triangle $ABC$.

*Alternative formulation.* Two circles touch externally at a point $I$. The two circles lie inside a large circle and both touch it. The chord $BC$ of the large circle touches both smaller circles (not at $I$). The common tangent to the two smaller circles at the point $I$ meets the large circle at a point $A$, where the points $A$ and $I$ are on the same side of the chord $BC$. Show that the point $I$ is the incenter of triangle $ABC$.

---

**8** Show that in the plane there exists a convex polygon of 1992 sides satisfying the following conditions:

*(i)* its side lengths are $1, 2, 3, \ldots, 1992$ in some order;
*(ii)* the polygon is circumscribable about a circle.

---

*Alternative formulation:* Does there exist a 1992-gon with side lengths $1, 2, 3, \ldots, 1992$ circumscribed about a circle? Answer the same question for a 1990-gon.

---

**9** Let $f(x)$ be a polynomial with rational coefficients and $\alpha$ be a real number such that

$$\alpha^3 - \alpha = [f(\alpha)]^3 - f(\alpha) = 33^{1992}.$$

Prove that for each $n \geq 1$,

$$[f^n(\alpha)]^3 - f^n(\alpha) = 33^{1992},$$

where $f^n(x) = f(f(\cdots f(x)))$, and $n$ is a positive integer.

---

**10** Let $S$ be a finite set of points in three-dimensional space. Let $S_x$, $S_y$, $S_z$ be the sets consisting of the orthogonal projections of the points of $S$ onto the $yz$-plane, $zx$-plane, $xy$-plane, respectively. Prove that

$$|S|^2 \leq |S_x| \cdot |S_y| \cdot |S_z|,$$

where $|A|$ denotes the number of elements in the finite set $A$.

Note: The orthogonal projection of a point onto a plane is the foot of the perpendicular from that point to the plane.

---

**11** In a triangle $ABC$, let $D$ and $E$ be the intersections of the bisectors of $\angle ABC$ and $\angle ACB$ with the sides $AC, AB$, respectively. Determine the angles $\angle A, \angle B, \angle C$ if $\angle BDE = 24^\circ$, $\angle CED = 18^\circ$.

---

**12** Let $f, g$ and $a$ be polynomials with real coefficients, $f$ and $g$ in one variable and $a$ in two variables. Suppose

$$f(x) - f(y) = a(x, y)(g(x) - g(y)) \forall x, y \in \mathbb{R}$$

Prove that there exists a polynomial $h$ with $f(x) = h(g(x))$ $\forall x \in \mathbb{R}$.

---

**13** Find all integers $a, b, c$ with $1 < a < b < c$ such that

$$(a-1)(b-1)(c-1)$$

is a divisor of $abc - 1$.

---

**14** For any positive integer $x$ define $g(x)$ as greatest odd divisor of $x$, and

$$f(x) = \begin{cases} \frac{x}{2} + \frac{x}{g(x)} & \text{if } x \text{ is even,} \\ 2^{\frac{x+1}{2}} & \text{if } x \text{ is odd.} \end{cases}$$

Construct the sequence $x_1 = 1, x_{n+1} = f(x_n)$. Show that the number 1992 appears in this sequence, determine the least $n$ such that $x_n = 1992$, and determine whether $n$ is unique.

---

**15** Does there exist a set $M$ with the following properties?

*(i)* The set $M$ consists of 1992 natural numbers.
*(ii)* Every element in $M$ and the sum of any number of elements have the form $m^k$ $(m, k \in \mathbb{N}, k \geq 2)$.

---

**16** Prove that $\frac{5^{125}-1}{5^{25}-1}$ is a composite number.

---

**17** Let $\alpha(n)$ be the number of digits equal to one in the binary representation of a positive integer $n$. Prove that:

(a) the inequality $\alpha(n)(n^2) \leq \frac{1}{2}\alpha(n)(\alpha(n)+1)$ holds;
(b) the above inequality is an equality for infinitely many positive integers, and
(c) there exists a sequence $(n_i)_1^\infty$ such that $\frac{\alpha(n_i^2)}{\alpha(n_i}$
goes to zero as $i$ goes to $\infty$.

*Alternative problem:* Prove that there exists a sequence a sequence $(n_i)_1^\infty$ such that $\frac{\alpha(n_i^2)}{\alpha(n_i)}$

(d) $\infty$;
(e) an arbitrary real number $\gamma \in (0, 1)$;
(f) an arbitrary real number $\gamma \geq 0$;

as $i$ goes to $\infty$.

---

**18** Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to $x$. Pick any $x_1$ in $[0, 1)$ and define the sequence $x_1, x_2, x_3, \ldots$ by $x_{n+1} = 0$ if $x_n = 0$ and $x_{n+1} = \frac{1}{x_n} - \left\lfloor \frac{1}{x_n} \right\rfloor$ otherwise. Prove that

$$x_1 + x_2 + \ldots + x_n < \frac{F_1}{F_2} + \frac{F_2}{F_3} + \ldots + \frac{F_n}{F_{n+1}},$$

where $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$.

---

**19** Let $f(x) = x^8 + 4x^6 + 2x^4 + 28x^2 + 1$. Let $p > 3$ be a prime and suppose there exists an integer $z$ such that $p$ divides $f(z)$. Prove that there exist integers $z_1, z_2, \ldots, z_8$ such that if

$$g(x) = (x - z_1)(x - z_2) \cdot \ldots \cdot (x - z_8),$$

then all coefficients of $f(x) - g(x)$ are divisible by $p$.

---

**20** In the plane let $C$ be a circle, $L$ a line tangent to the circle $C$, and $M$ a point on $L$. Find the locus of all points $P$ with the following property: there exists two points $Q, R$ on $L$ such that $M$ is the midpoint of $QR$ and $C$ is the inscribed circle of triangle $PQR$.

---

**21** For each positive integer $n$, $S(n)$ is defined to be the greatest integer such that, for every positive integer $k \leq S(n)$, $n^2$ can be written as the sum of $k$ positive squares.

**a.)** Prove that $S(n) \leq n^2 - 14$ for each $n \geq 4$.
**b.)** Find an integer $n$ such that $S(n) = n^2 - 14$.
**c.)** Prove that there are infintely many integers $n$ such that $S(n) = n^2 - 14$.

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IMO Shortlist 1991

**1** Given a point $P$ inside a triangle $\triangle ABC$. Let $D$, $E$, $F$ be the orthogonal projections of the point $P$ on the sides $BC$, $CA$, $AB$, respectively. Let the orthogonal projections of the point $A$ on the lines $BP$ and $CP$ be $M$ and $N$, respectively. Prove that the lines $ME$, $NF$, $BC$ are concurrent.

*Original formulation:*

Let $ABC$ be any triangle and $P$ any point in its interior. Let $P_1, P_2$ be the feet of the perpendiculars from $P$ to the two sides $AC$ and $BC$. Draw $AP$ and $BP$, and from $C$ drop perpendiculars to $AP$ and $BP$. Let $Q_1$ and $Q_2$ be the feet of these perpendiculars. Prove that the lines $Q_1P_2, Q_2P_1$, and $AB$ are concurrent.

**2** $ABC$ is an acute-angled triangle. $M$ is the midpoint of $BC$ and $P$ is the point on $AM$ such that $MB = MP$. $H$ is the foot of the perpendicular from $P$ to $BC$. The lines through $H$ perpendicular to $PB$, $PC$ meet $AB, AC$ respectively at $Q, R$. Show that $BC$ is tangent to the circle through $Q, H, R$ at $H$.

*Original Formulation:*

For an acute triangle $ABC, M$ is the midpoint of the segment $BC, P$ is a point on the segment $AM$ such that $PM = BM, H$ is the foot of the perpendicular line from $P$ to $BC, Q$ is the point of intersection of segment $AB$ and the line passing through $H$ that is perpendicular to $PB$, and finally, $R$ is the point of intersection of the segment $AC$ and the line passing through $H$ that is perpendicular to $PC$. Show that the circumcircle of $QHR$ is tangent to the side $BC$ at point $H$.

**3** Let $S$ be any point on the circumscribed circle of $PQR$. Then the feet of the perpendiculars from S to the three sides of the triangle lie on the same straight line. Denote this line by $l(S, PQR)$. Suppose that the hexagon $ABCDEF$ is inscribed in a circle. Show that the four lines $l(A, BDF)$, $l(B, ACE)$, $l(D, ABF)$, and $l(E, ABC)$ intersect at one point if and only if $CDEF$ is a rectangle.

**4** Let $ABC$ be a triangle and $P$ an interior point of $ABC$. Show that at least one of the angles $\angle PAB$, $\angle PBC$, $\angle PCA$ is less than or equal to $30^\circ$.

**5** In the triangle $ABC$, with $\angle A = 60^\circ$, a parallel $IF$ to $AC$ is drawn through the incenter $I$ of the triangle, where $F$ lies on the side $AB$. The point $P$ on the side $BC$ is such that $3BP = BC$. Show that $\angle BFP = \frac{\angle B}{2}$.

---

**7** $ABCD$ is a terahedron: $AD+BD = AC+BC$, $BD+CD = BA+CA$, $CD+AD = CB+AB$, $M, N, P$ are the mid points of $BC, CA, AB$. $OA = OB = OC = OD$. Prove that $\angle MOP = \angle NOP = \angle NOM$.

---

**8** $S$ be a set of $n$ points in the plane. No three points of $S$ are collinear. Prove that there exists a set $P$ containing $2n-5$ points satisfying the following condition: In the interior of every triangle whose three vertices are elements of $S$ lies a point that is an element of $P$.

---

**9** In the plane we are given a set $E$ of 1991 points, and certain pairs of these points are joined with a path. We suppose that for every point of $E$, there exist at least 1593 other points of $E$ to which it is joined by a path. Show that there exist six points of $E$ every pair of which are joined by a path.

*Alternative version:* Is it possible to find a set $E$ of 1991 points in the plane and paths joining certain pairs of the points in $E$ such that every point of $E$ is joined with a path to at least 1592 other points of $E$, and in every subset of six points of $E$ there exist at least two points that are not joined?

---

**10** Suppose $G$ is a connected graph with $k$ edges. Prove that it is possible to label the edges $1, 2, \ldots, k$ in such a way that at each vertex which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is equal to 1.

**Note: Graph-Definition**. A **graph** consists of a set of points, called vertices, together with a set of edges joining certain pairs of distinct vertices. Each pair of vertices $u, v$ belongs to at most one edge. The graph $G$ is connected if for each pair of distinct vertices $x, y$ there is some sequence of vertices $x = v_0, v_1, v_2, \cdots, v_m = y$ such that each pair $v_i, v_{i+1}$ $(0 \leq i < m)$ is joined by an edge of $G$.

---

**11** Prove that $\sum_{k=0}^{995} \frac{(-1)^k}{1991-k} \binom{1991-k}{k} = \frac{1}{1991}$

---

**12** Let $S = \{1, 2, 3, \cdots, 280\}$. Find the smallest integer $n$ such that each $n$-element subset of $S$ contains five numbers which are pairwise relatively prime.

---

**13** Given any integer $n \geq 2$, assume that the integers $a_1, a_2, \ldots, a_n$ are not divisible by $n$ and, moreover, that $n$ does not divide $\sum_{i=1}^n a_i$. Prove that there exist at least $n$ different sequences $(e_1, e_2, \ldots, e_n)$ consisting of zeros or ones such $\sum_{i=1}^n e_i \cdot a_i$ is divisible by $n$.

---

**14** Let $a, b, c$ be integers and $p$ an odd prime number. Prove that if $f(x) = ax^2 + bx + c$ is a perfect square for $2p-1$ consecutive integer values of $x$, then $p$ divides $b^2 - 4ac$.

---

**15** Let $a_n$ be the last nonzero digit in the decimal representation of the number $n!$. Does the sequence $a_1, a_2, \ldots, a_n, \ldots$ become periodic after a finite number of terms?

---

**16** Let $n > 6$ be an integer and $a_1, a_2, \cdots, a_k$ be all the natural numbers less than $n$ and relatively prime to $n$. If

$$a_2 - a_1 = a_3 - a_2 = \cdots = a_k - a_{k-1} > 0,$$

prove that $n$ must be either a prime number or a power of $2$.

---

**17** Find all positive integer solutions $x, y, z$ of the equation $3^x + 4^y = 5^z$.

---

**18** Find the highest degree $k$ of 1991 for which $1991^k$ divides the number

$$1990^{1991^{1992}} + 1992^{1991^{1990}}.$$

---

**19** Let $\alpha$ be a rational number with $0 < \alpha < 1$ and $\cos(3\pi\alpha) + 2\cos(2\pi\alpha) = 0$. Prove that $\alpha = \frac{2}{3}$.

---

**20** Let $\alpha$ be the positive root of the equation $x^2 = 1991x + 1$. For natural numbers $m$ and $n$ define

$$m * n = mn + \lfloor \alpha m \rfloor \lfloor \alpha n \rfloor.$$

Prove that for all natural numbers $p$, $q$, and $r$,

$$(p * q) * r = p * (q * r).$$

---

**21** Let $f(x)$ be a monic polynomial of degree 1991 with integer coefficients. Define $g(x) = f^2(x) - 9$. Show that the number of distinct integer solutions of $g(x) = 0$ cannot exceed 1995.

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**22** Real constants $a, b, c$ are such that there is exactly one square all of whose vertices lie on the cubic curve $y = x^3 + ax^2 + bx + c$. Prove that the square has sides of length $\sqrt[4]{72}$.

**23** Let $f$ and $g$ be two integer-valued functions defined on the set of all integers such that

*(a)* $f(m + f(f(n))) = -f(f(m+1) - n$ for all integers $m$ and $n$;
*(b)* $g$ is a polynomial function with integer coefficients and g(n) $= g(f(n))$ $\forall n \in \mathbb{Z}$.

**24** An odd integer $n \geq 3$ is said to be nice if and only if there is at least one permutation $a_1, \cdots, a_n$ of $1, \cdots, n$ such that the $n$ sums $a_1 - a_2 + a_3 - \cdots - a_{n-1} + a_n$, $a_2 - a_3 + a_3 - \cdots - a_n + a_1$, $a_3 - a_4 + a_5 - \cdots - a_1 + a_2$, $\cdots$, $a_n - a_1 + a_2 - \cdots - a_{n-2} + a_{n-1}$ are all positive. Determine the set of all 'nice' integers.

**25** Suppose that $n \geq 2$ and $x_1, x_2, \dots, x_n$ are real numbers between 0 and 1 (inclusive). Prove that for some index $i$ between 1 and $n - 1$ the inequality

$$x_i(1 - x_{i+1}) \geq \frac{1}{4} x_1(1 - x_n)$$

**26** Let $n \geq 2, n \in \mathbb{N}$ and let $p, a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in \mathbb{R}$ satisfying $\frac{1}{2} \leq p \leq 1$, $0 \leq a_i$, $0 \leq b_i \leq p$, $i = 1, \dots, n$, and

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i.$$

Prove the inequality:

$$\sum_{i=1}^{n} b_i \prod_{j=1, j \neq i}^{n} a_j \leq \frac{p}{(n-1)^{n-1}}.$$

**27** Determine the maximum value of the sum

$$\sum_{i<j} x_i x_j (x_i + x_j)$$

over all $n-$tuples $(x_1, \dots, x_n)$, satisfying $x_i \geq 0$ and $\sum_{i=1}^{n} x_i = 1$.

**28** An infinite sequence $x_0, x_1, x_2, \ldots$ of real numbers is said to be **bounded** if there is a constant $C$ such that $|x_i| \leq C$ for every $i \geq 0$. Given any real number $a > 1$, construct a bounded infinite sequence $x_0, x_1, x_2, \ldots$ such that

$$|x_i - x_j||i - j|^a \geq 1$$

for every pair of distinct nonnegative integers $i, j$.

**29** We call a set $S$ on the real line $\mathbb{R}$ *superinvariant* if for any stretching $A$ of the set by the transformation taking $x$ to $A(x) = x_0 + a(x - x_0), a > 0$ there exists a translation $B$, $B(x) = x + b$, such that the images of $S$ under $A$ and $B$ agree; i.e., for any $x \in S$ there is a $y \in S$ such that $A(x) = B(y)$ and for any $t \in S$ there is a $u \in S$ such that $B(t) = A(u)$. Determine all *superinvariant* sets.

**30** Two students $A$ and $B$ are playing the following game: Each of them writes down on a sheet of paper a positive integer and gives the sheet to the referee. The referee writes down on a blackboard two integers, one of which is the sum of the integers written by the players. After that, the referee asks student $A$ : Can you tell the integer written by the other student? If A answers no, the referee puts the same question to student $B$. If $B$ answers no, the referee puts the question back to $A$, and so on. Assume that both students are intelligent and truthful. Prove that after a finite number of questions, one of the students will answer yes.

IMO Shortlist 1990

**1** The integer 9 can be written as a sum of two consecutive integers: $9 = 4 + 5$. Moreover, it can be written as a sum of (more than one) consecutive positive integers in exactly two ways: $9 = 4 + 5 = 2 + 3 + 4$. Is there an integer that can be written as a sum of 1990 consecutive integers and that can be written as a sum of (more than one) consecutive positive integers in exactly 1990 ways?

**2** Given $n$ countries with three representatives each, $m$ committees $A(1), A(2), \ldots, A(m)$ are called a cycle if

*(i)* each committee has $n$ members, one from each country;
*(ii)* no two committees have the same membership;
*(iii)* for $i = 1, 2, \ldots, m$, committee $A(i)$ and committee $A(i+1)$ have no member in common, where $A(m+1)$ denotes $A(1)$;
*(iv)* if $1 < |i-j| < m-1$, then committees $A(i)$ and $A(j)$ have at least one member in common.

Is it possible to have a cycle of 1990 committees with 11 countries?

**3** Let $n \geq 3$ and consider a set $E$ of $2n-1$ distinct points on a circle. Suppose that exactly $k$ of these points are to be colored black. Such a coloring is **good** if there is at least one pair of black points such that the interior of one of the arcs between them contains exactly $n$ points from $E$. Find the smallest value of $k$ so that every such coloring of $k$ points of $E$ is good.

**4** Assume that the set of all positive integers is decomposed into $r$ (disjoint) subsets $A_1 \cup A_2 \cup \ldots \cup A_r = \mathbb{N}$. Prove that one of them, say $A_i$, has the following property: There exists a positive $m$ such that for any $k$ one can find numbers $a_1, a_2, \ldots, a_k$ in $A_i$ with $0 < a_{j+1} - a_j \leq m$, $(1 \leq j \leq k-1)$.

**5** Given a triangle $ABC$. Let $G$, $I$, $H$ be the centroid, the incenter and the orthocenter of triangle $ABC$, respectively. Prove that $\angle GIH > 90°$.

**6** Given an initial integer $n_0 > 1$, two players, $\mathcal{A}$ and $\mathcal{B}$, choose integers $n_1$, $n_2$, $n_3$, $\ldots$ alternately according to the following rules :

**I.)** Knowing $n_{2k}$, $\mathcal{A}$ chooses any integer $n_{2k+1}$ such that

$$n_{2k} \leq n_{2k+1} \leq n_{2k}^2.$$

**II.)** Knowing $n_{2k+1}$, $\mathcal{B}$ chooses any integer $n_{2k+2}$ such that

$$\frac{n_{2k+1}}{n_{2k+2}}$$

is a prime raised to a positive integer power.

Player $\mathcal{A}$ wins the game by choosing the number 1990; player $\mathcal{B}$ wins by choosing the number 1. For which $n_0$ does :

**a.)** $\mathcal{A}$ have a winning strategy?
**b.)** $\mathcal{B}$ have a winning strategy?
**c.)** Neither player have a winning strategy?

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**7** Let $f(0) = f(1) = 0$ and

$$f(n+2) = 4^{n+2} \cdot f(n+1) - 16^{n+1} \cdot f(n) + n \cdot 2^{n^2}, \quad n = 0, 1, 2, \ldots$$

Show that the numbers $f(1989), f(1990), f(1991)$ are divisible by 13.

---

**8** For a given positive integer $k$ denote the square of the sum of its digits by $f_1(k)$ and let $f_{n+1}(k) = f_1(f_n(k))$. Determine the value of $f_{1991}(2^{1990})$.

---

**9** The incenter of the triangle $ABC$ is $K$. The midpoint of $AB$ is $C_1$ and that of $AC$ is $B_1$. The lines $C_1K$ and $AC$ meet at $B_2$, the lines $B_1K$ and $AB$ at $C_2$. If the areas of the triangles $AB_2C_2$ and $ABC$ are equal, what is the measure of angle $\angle CAB$?

---

**10** A plane cuts a right circular cone of volume $V$ into two parts. The plane is tangent to the circumference of the base of the cone and passes through the midpoint of the altitude. Find the volume of the smaller part.

*Original formulation:*

A plane cuts a right circular cone into two parts. The plane is tangent to the circumference of the base of the cone and passes through the midpoint of the altitude. Find the ratio of the volume of the smaller part to the volume of the whole cone.

---

**11** Chords $AB$ and $CD$ of a circle intersect at a point $E$ inside the circle. Let $M$ be an interior point of the segment $EB$. The tangent line at $E$ to the circle through $D$, $E$, and $M$ intersects the lines $BC$ and $AC$ at $F$ and $G$, respectively. If

$$\frac{AM}{AB} = t,$$

find $\frac{EG}{EF}$ in terms of $t$.

---

**12** Let $ABC$ be a triangle, and let the angle bisectors of its angles $CAB$ and $ABC$ meet the sides $BC$ and $CA$ at the points $D$ and $F$, respectively. The lines $AD$ and $BF$ meet the line through the point $C$ parallel to $AB$ at the points $E$ and $G$ respectively, and we have $FG = DE$. Prove that $CA = CB$.

*Original formulation:*

Let $ABC$ be a triangle and $L$ the line through $C$ parallel to the side $AB$. Let the internal bisector of the angle at $A$ meet the side $BC$ at $D$ and the line $L$ at $E$ and let the internal bisector of the angle at $B$ meet the side $AC$ at $F$ and the line $L$ at $G$. If $GF = DE$, prove that $AC = BC$.

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**13** An eccentric mathematician has a ladder with $n$ rungs that he always ascends and descends in the following way: When he ascends, each step he takes covers $a$ rungs of the ladder, and when he descends, each step he takes covers $b$ rungs of the ladder, where $a$ and $b$ are fixed positive integers. By a sequence of ascending and descending steps he can climb from ground level to the top rung of the ladder and come back down to ground level again. Find, with proof, the minimum value of $n$, expressed in terms of $a$ and $b$.

---

**14** In the coordinate plane a rectangle with vertices $(0, 0)$, $(m, 0)$, $(0, n)$, $(m, n)$ is given where both $m$ and $n$ are odd integers. The rectangle is partitioned into triangles in such a way that

*(i)* each triangle in the partition has at least one side (to be called a good side) that lies on a line of the form $x = j$ or $y = k$, where $j$ and $k$ are integers, and the altitude on this side has length 1;

*(ii)* each bad side (i.e., a side of any triangle in the partition that is not a good one) is a common side of two triangles in the partition.

Prove that there exist at least two triangles in the partition each of which has two good sides.

**15** Determine for which positive integers $k$ the set

$$X = \{1990, 1990+1, 1990+2, \ldots, 1990+k\}$$

can be partitioned into two disjoint subsets $A$ and $B$ such that the sum of the elements of $A$ is equal to the sum of the elements of $B$.

**16** Prove that there exists a convex 1990-gon with the following two properties :

**a.)** All angles are equal.
**b.)** The lengths of the 1990 sides are the numbers $1^2$, $2^2$, $3^2$, $\cdots$, $1990^2$ in some order.

**17** Unit cubes are made into beads by drilling a hole through them along a diagonal. The beads are put on a string in such a way that they can move freely in space under the restriction that the vertices of two neighboring cubes are touching. Let $A$ be the beginning vertex and $B$ be the end vertex. Let there be $p \times q \times r$ cubes on the string $(p, q, r \geq 1)$.

*(a)* Determine for which values of $p, q$, and $r$ it is possible to build a block with dimensions $p, q$, and $r$. Give reasons for your answers.
*(b)* The same question as (a) with the extra condition that $A = B$.

**18** Let $a, b \in \mathbb{N}$ with $1 \leq a \leq b$, and $M = \left[\frac{a+b}{2}\right]$. Define a function $f : \mathbb{Z} \mapsto \mathbb{Z}$ by

$$f(n) = \begin{cases} n+a, & \text{if } n \leq M, \\ n-b, & \text{if } n \geq M. \end{cases}$$

Let $f^1(n) = f(n)$, $f_{i+1}(n) = f(f^i(n))$, $i = 1, 2, \ldots$ Find the smallest natural number $k$ such that $f^k(0) = 0$.

**19** Let $P$ be a point inside a regular tetrahedron $T$ of unit volume. The four planes passing through $P$ and parallel to the faces of $T$ partition $T$ into 14 pieces. Let $f(P)$ be the joint volume of those pieces that are neither a tetrahedron nor a parallelepiped (i.e., pieces adjacent to an edge but not to a vertex). Find the exact bounds for $f(P)$ as $P$ varies over $T$.

**20** Prove that every integer $k$ greater than 1 has a multiple that is less than $k^4$ and can be written in the decimal system with at most four different digits.

**21** Let $n$ be a composite natural number and $p$ a proper divisor of $n$. Find the binary representation of the smallest natural number $N$ such that

$$\frac{(1+2^p+2^{n-p})N-1}{2^n}$$

is an integer.

**22** Ten localities are served by two international airlines such that there exists a direct service (without stops) between any two of these localities and all airline schedules offer round-trip service between the cities they serve. Prove that at least one of the airlines can offer two disjoint round trips each containing an odd number of landings.

**23** Determine all integers $n > 1$ such that

$$\frac{2^n+1}{n^2}$$

is an integer.

**24** Let $w, x, y, z$ are non-negative reals such that $wx + xy + yz + zw = 1$.
Show that $\frac{w^3}{x+y+z} + \frac{x^3}{w+y+z} + \frac{y^3}{w+x+z} + \frac{z^3}{w+x+y} \geq \frac{1}{3}$.

**25** Let $\mathbb{Q}^+$ be the set of positive rational numbers. Construct a function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that

$$f(xf(y)) = \frac{f(x)}{y}$$

for all $x$, $y$ in $\mathbb{Q}^+$.

**26** Let $p(x)$ be a cubic polynomial with rational coefficients. $q_1$, $q_2$, $q_3$, ... is a sequence of rationals such that $q_n = p(q_{n+1})$ for all positive $n$. Show that for some $k$, we have $q_{n+k} = q_n$ for all positive $n$.

**27** Find all natural numbers $n$ for which every natural number whose decimal representation has $n-1$ digits 1 and one digit 7 is prime.

**28** Prove that on the coordinate plane it is impossible to draw a closed broken line such that

*(i)* the coordinates of each vertex are rational;

*(ii)* the length each of its edges is 1;
*(iii)* the line has an odd number of vertices.

IMO Shortlist 1989

**1** $ABC$ is a triangle, the bisector of angle $A$ meets the circumcircle of triangle $ABC$ in $A_1$, points $B_1$ and $C_1$ are defined similarly. Let $AA_1$ meet the lines that bisect the two external angles at $B$ and $C$ in $A_0$. Define $B_0$ and $C_0$ similarly. Prove that the area of triangle $A_0B_0C_0 = 2\cdot$ area of hexagon $AC_1BA_1CB_1 \geq 4\cdot$ area of triangle $ABC$.

**2** Ali Barber, the carpet merchant, has a rectangular piece of carpet whose dimensions are unknown. Unfortunately, his tape measure is broken and he has no other measuring instruments. However, he finds that if he lays it flat on the floor of either of his storerooms, then each corner of the carpet touches a different wall of that room. If the two rooms have dimensions of 38 feet by 55 feet and 50 feet by 55 feet, what are the carpet dimensions?

**3** Ali Barber, the carpet merchant, has a rectangular piece of carpet whose dimensions are unknown. Unfortunately, his tape measure is broken and he has no other measuring instruments. However, he finds that if he lays it flat on the floor of either of his storerooms, then each corner of the carpet touches a different wall of that room. He knows that the sides of the carpet are integral numbers of feet and that his two storerooms have the same (unknown) length, but widths of 38 feet and 50 feet respectively. What are the carpet dimensions?

**4** Prove that $\forall n > 1, n \in \mathbb{N}$ the equation

$$\sum_{k=1}^{n} \frac{x^k}{k!} + 1 = 0$$

has no rational roots.

**5** Find the roots $r_i \in \mathbb{R}$ of the polynomial

$$p(x) = x^n + n \cdot x^{n-1} + a_2 \cdot x^{n-2} + \ldots + a_n$$

satisfying

$$\sum_{k=1}^{16} r_k^{16} = n.$$

**6** For a triangle $ABC$, let $k$ be its circumcircle with radius $r$. The bisectors of the inner angles $A, B$, and $C$ of the triangle intersect respectively the circle $k$ again at points $A', B'$, and $C'$. Prove the inequality

$$16Q^3 \geq 27r^4P,$$

where $Q$ and $P$ are the areas of the triangles $A'B'C'$ and $ABC$ respectively.

**7** Show that any two points lying inside a regular $n-$gon $E$ can be joined by two circular arcs lying inside $E$ and meeting at an angle of at least $\left(1 - \frac{2}{n}\right) \cdot \pi$.

**8** Let $R$ be a rectangle that is the union of a finite number of rectangles $R_i$, $1 \leq i \leq n$, satisfying the following conditions:

**(i)** The sides of every rectangle $R_i$ are parallel to the sides of $R$.
**(ii)** The interiors of any two different rectangles $R_i$ are disjoint.
**(iii)** Each rectangle $R_i$ has at least one side of integral length.

Prove that $R$ has at least one side of integral length.

*Variant:* Same problem but with rectangular parallelepipeds having at least one integral side.

**9** $\forall n > 0, n \in \mathbb{Z}$, there exists uniquely determined integers $a_n, b_n, c_n \in \mathbb{Z}$ such

$$\left(1 + 4 \cdot \sqrt[3]{2} - 4 \cdot \sqrt[3]{4}\right)^n = a_n + b_n \cdot \sqrt[3]{2} + c_n \cdot \sqrt[3]{4}.$$

Prove that $c_n = 0$ implies $n = 0$.

**10** Let $g : \mathbb{C} \to \mathbb{C}$, $\omega \in \mathbb{C}$, $a \in \mathbb{C}$, $\omega^3 = 1$, and $\omega \neq 1$. Show that there is one and only one function $f : \mathbb{C} \to \mathbb{C}$ such that

$$f(z) + f(\omega z + a) = g(z), z \in \mathbb{C}$$

**11** Define sequence $(a_n)$ by $\sum_{d|n} a_d = 2^n$. Show that $n|a_n$.

**12** There are n cars waiting at distinct points of a circular race track. At the starting signal each car starts. Each car may choose arbitrarily which of the two possible directions to go. Each car has the same constant speed. Whenever two cars meet they both change direction (but not speed). Show that at some time each car is back at its starting point.

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**13** Let $ABCD$ be a convex quadrilateral such that the sides $AB, AD, BC$ satisfy $AB = AD + BC$. There exists a point $P$ inside the quadrilateral at a distance $h$ from the line $CD$ such that $AP = h + AD$ and $BP = h + BC$. Show that:

$$\frac{1}{\sqrt{h}} \geq \frac{1}{\sqrt{AD}} + \frac{1}{\sqrt{BC}}$$

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**14** A bicentric quadrilateral is one that is both inscribable in and circumscribable about a circle, i.e. both the incircle and circumcircle exists. Show that for such a quadrilateral, the centers of the two associated circles are collinear with the point of intersection of the diagonals.

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**15** Let $a, b, c, d, m, n \in \mathbb{Z}^+$ such that

$$a^2 + b^2 + c^2 + d^2 = 1989,$$

$$a + b + c + d = m^2,$$

and the largest of $a, b, c, d$ is $n^2$. Determine, with proof, the values of $m$ and $n$.

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**16** The set $\{a_0, a_1, \ldots, a_n\}$ of real numbers satisfies the following conditions:

**(i)** $a_0 = a_n = 0$,
**(ii)** for $1 \leq k \leq n-1$,

$$a_k = c + \sum_{i=k}^{n-1} a_{i-k} \cdot (a_i + a_{i+1})$$

Prove that $c \leq \frac{1}{4n}$.

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**17** Given seven points in the plane, some of them are connected by segments such that:

**(i)** among any three of the given points, two are connected by a segment;

**(ii)** the number of segments is minimal.

How many segments does a figure satisfying **(i)** and **(ii)** have? Give an example of such a figure.

**18** Given a convex polygon $A_1A_2\dots A_n$ with area $S$ and a point $M$ in the same plane, determine the area of polygon $M_1M_2\dots M_n$, where $M_i$ is the image of $M$ under rotation $R^{\alpha}_{A_i}$ around $A_i$ by $\alpha_i, i = 1, 2, \dots, n$.

**19** A natural number is written in each square of an $m \times n$ chess board. The allowed move is to add an integer $k$ to each of two adjacent numbers in such a way that non-negative numbers are obtained. (Two squares are adjacent if they have a common side.) Find a necessary and sufficient condition for it to be possible for all the numbers to be zero after finitely many operations.

**20** Let $n$ and $k$ be positive integers and let $S$ be a set of $n$ points in the plane such that

**i.)** no three points of $S$ are collinear, and

**ii.)** for every point $P$ of $S$ there are at least $k$ points of $S$ equidistant from $P$.

Prove that:

$$k < \frac{1}{2} + \sqrt{2 \cdot n}$$

**21** Prove that the intersection of a plane and a regular tetrahedron can be an obtuse-angled triangle and that the obtuse angle in any such triangle is always smaller than $120^\circ$.

**22** Prove that in the set $\{1, 2, \dots, 1989\}$ can be expressed as the disjoint union of subsets $A_i, \{i = 1, 2, \dots, 117\}$ such that

**i.)** each $A_i$ contains 17 elements

**ii.)** the sum of all the elements in each $A_i$ is the same.

**23** A permutation $\{x_1, x_2, \dots, x_{2n}\}$ of the set $\{1, 2, \dots, 2n\}$ where $n$ is a positive integer, is said to have property $T$ if $|x_i - x_{i+1}| = n$ for at least one $i$ in $\{1, 2, \dots, 2n - 1\}$. Show that, for each $n$, there are more permutations with property $T$ than without.

**24** For points $A_1, \ldots, A_5$ on the sphere of radius 1, what is the maximum value that $min_{1 \leq i,j \leq 5} A_iA_j$ can take? Determine all configurations for which this maximum is attained. (Or: determine the diameter of any set $\{A_1, \ldots, A_5\}$ for which this maximum is attained.)

**25** Let $a, b \in \mathbb{Z}$ which are not perfect squares. Prove that if

$$x^2 - ay^2 - bz^2 + abw^2 = 0$$

has a nontrivial solution in integers, then so does

$$x^2 - ay^2 - bz^2 = 0.$$

**26** Let $n \in \mathbb{Z}^+$ and let $a, b \in \mathbb{R}$. Determine the range of $x_0$ for which

$$\sum_{i=0}^{n} x_i = a \text{ and } \sum_{i=0}^{n} x_i^2 = b,$$

where $x_0, x_1, \ldots, x_n$ are real variables.

**27** Let $m$ be a positive odd integer, $m > 2$. Find the smallest positive integer $n$ such that $2^{1989}$ divides $m^n - 1$.

**28** Consider in a plane $P$ the points $O, A_1, A_2, A_3, A_4$ such that

$$\sigma(OA_iA_j) \geq 1 \quad \forall i, j = 1, 2, 3, 4, i \neq j.$$

where $\sigma(OA_iA_j)$ is the area of triangle $OA_iA_j$. Prove that there exists at least one pair $i_0, j_0 \in \{1, 2, 3, 4\}$ such that

$$\sigma(OA_iA_j) \geq \sqrt{2}.$$

**29** 155 birds $P_1, \ldots, P_{155}$ are sitting down on the boundary of a circle $C$. Two birds $P_i, P_j$ are mutually visible if the angle at centre $m(\cdot)$ of their positions $m(P_iP_j) \leq 10^\circ$. Find the smallest number of mutually visible pairs of birds, i.e. minimal set of pairs $\{x, y\}$ of mutually visible pairs of birds with $x, y \in \{P_1, \ldots, P_{155}\}$. One assumes that a position (point) on $C$ can be occupied simultaneously by several birds, e.g. all possible birds.

**30** Prove that for each positive integer $n$ there exist $n$ consecutive positive integers none of which is an integral power of a prime number.

**31** Let $a_1 \geq a_2 \geq a_3 \in \mathbb{Z}^+$ be given and let $\mathrm{N}(a_1, a_2, a_3)$ be the number of solutions $(x_1, x_2, x_3)$ of the equation

$$\sum_{k=1}^{3} \frac{a_k}{x_k} = 1.$$

where $x_1, x_2,$ and $x_3$ are positive integers. Prove that

$$N(a_1, a_2, a_3) \leq 6a_1a_2(3 + ln(2a_1)).$$

**32** The vertex $A$ of the acute triangle $ABC$ is equidistant from the circumcenter $O$ and the orthocenter $H$. Determine all possible values for the measure of angle $A$.