Balkan MO 2002

– April 27th

**1** Consider $n$ points $A_1, A_2, A_3, \ldots, A_n$ $(n \geq 4)$ in the plane, such that any three are not collinear. Some pairs of distinct points among $A_1, A_2, A_3, \ldots, A_n$ are connected by segments, such that every point is connected with at least three different points. Prove that there exists $k > 1$ and the distinct points $X_1, X_2, \ldots, X_{2k}$ in the set $\{A_1, A_2, A_3, \ldots, A_n\}$, such that for every $i \in \overline{1, 2k-1}$ the point $X_i$ is connected with $X_{i+1}$, and $X_{2k}$ is connected with $X_1$.

**2** Let the sequence $\{a_n\}_{n\geq 1}$ be defined by $a_1 = 20$, $a_2 = 30$ and $a_{n+2} = 3a_{n+1} - a_n$ for all $n \geq 1$. Find all positive integers $n$ such that $1 + 5a_n a_{n+1}$ is a perfect square.

**3** Two circles with different radii intersect in two points $A$ and $B$. Let the common tangents of the two circles be $MN$ and $ST$ such that $M, S$ lie on the first circle, and $N, T$ on the second. Prove that the orthocenters of the triangles $AMN$, $AST$, $BMN$ and $BST$ are the four vertices of a rectangle.

**4** Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that for every positive integer $n$ we have:

$$2n + 2001 \leq f(f(n)) + f(n) \leq 2n + 2002.$$