SOLUTIONS TO MOCK OLYMPIAD

(1) Multiplying through by $\prod_{i=1}^n x_i^2$ we obtain the equivalent inequality

$$\prod_{i=1}^n (1-x_i^2) \geq (\prod_{i=1}^n x_i^2)(n^2-1)^n.$$

The left hand side can be factored as

$$\prod_{i=1}^n (1-x_i^2) = \prod_{x=1}^n (1-x_i) \cdot \prod_{x=1}^n (1+x_i).$$

Now, by AM-GM,

$$1 - x_i = \sum_{j \neq i} x_j \geq (n-1) \sqrt[n-1]{\prod_{j \neq i} x_j}$$

and

$$1 + x_i = x_i + \sum_{j=1}^n x_j \geq (n+1) \sqrt[n+1]{x_i \prod_{j=1}^n x_j}.$$

Multiplying all these inequalities, we obtain the desired result.

(2) Factor n into prime powers $n = \prod_{i=1}^k p_i^{m_i}$ for p_i odd primes. Then it is well known that $\phi(n) = \prod_{i=1}^m \left(p_i^{m_i-1} \cdot (p_i - 1)\right)$.

now, if $m_i > 1$ for some i, then $p_i \mid \phi(n)$, which is a contradiction since $\phi(n)$ is a power of 2. For a prime $p_i \neq 2$, we learn that $p_i - 1$ is a power of 2, and hence we can write $p_i = 2^{c_i} + 1$.

We claim that c_i has no odd prime factor. Suppose not, so that $c_i = qh$ for q an odd prime. Then $p_i = (2^h)^q + 1^q$ and is therefore divisible by $2^h + 1$. But p_i is prime, so we have a contradiction.

Thus, we can write $c_i = 2^{d_i}$ for some non-negative integers d_i. Wlog, we can assume $d_1 < d_2 < \cdots < d_k$.

Now, expanding the product for n, we get a sum of terms of the form $2^{m_0 + \sum_{i \in S} 2^{d_i}}$ where S is a (possibly empty) subset of $\{1, \ldots, k\}$. Since every positive integer can be written as a sum of distinct powers of 2 in a unique way, we can just read off the binary expansion for n. In particular, we learn that the binary expansion of n has 1's showing up in consecutive blocks of size 2^r, where r is the maximal non-negative integer for which $d_1 = 0, d_2 = 1, \ldots, d_r = r-1$. Moreover, the number of such blocks is a power of 2. Note that the same conclusion follows for $n+1$ as well, by an identical analysis.

Now, the binary expansion for $n+1$ takes the right-most block of 1's in the binary expansion of n and turns them into 0, and changes the 0 immediately to the left into a 1, so that

$$\ldots 011 \ldots 1 \to \ldots 100 \ldots 0.$$

This either keeps the number of blocks the same, or decrease it by 1.

If it decreases the number of blocks by 1, then since the numbers of blocks in both n and $n+1$ are powers of 2, we must have that $n+1$ has exactly one such block and n has exactly 2 such blocks. However, if the sizes of the blocks for n are 2^r, then the gaps between these blocks is also of size at least 2^r. Thus we must have $2^r = 1$, and so n must be 101 in binary, and so $n = 5$.

Now assume that n and $n+1$ have the same number of blocks. If this number is 1, then we have that $n+1$ is a power of 2, as desired. If the number of blocks is larger than 1, then as the rightmost block of $n+1$ is now of size 1, all the blocks are of size 1. Thus $d_1 > 0$. Now write $n+1 = 2 \cdot \prod_{j=1}^{r}(2^{2^{e_j}+1})$ where e_j are distinct, increasing positive integers (positive since the rightmost 1 in the binary expansion of $n+1$ is isolated.) Then the second right-most 1 in $n+1$ is in position $2^{e_1}+1$, but it also is in position 2^{d_1}. But since $e_1, d_1 > 0$, this is a contradiction. This completes the proof.

(3) We claim that $MCKB$ is a cyclic quadrilateral. To see this, note

$$\begin{aligned}
\angle KCM &= \angle KCA + \angle ACM \\
&= \pi - \angle KOA + \pi - \angle ACD \text{ } (ACKO \text{ is cyclic }) \\
&= \angle KOB + \angle DBO \text{ } (ACDB \text{ is cyclic }) \\
&= \pi - \angle KDB + \angle BDO \text{ } (OKDB \text{ is cyclic and } |OB| = |OD|) \\
&= \pi - \angle KDO \\
&= \pi - \angle KBM \text{ } (OKDB \text{ is cyclic })
\end{aligned}$$

Chasing angle again, we have

$$\begin{aligned}
\angle MKO &= \angle MKB - \angle BKO \\
&= \angle MCB - \angle ODB \text{ } (OKDB \text{ and } MCKB \text{ are cyclic }) \\
&= \angle MCA + \angle ACB - \angle OBD \text{ } (|OB| = |OD|) \\
&= \angle MAC + \pi/2 - \pi + \angle DCA \text{ } (ABDC \text{ is cyclic }) \\
&= \pi/2
\end{aligned}$$

which proves $\angle MKO$ is a right angle.

(4) First, note that every string v of size n consisting of 1 and -1 exists exactly once in the original matrix. Let $f(v)$ be the changed string, where some number of entries become 0. Now, consider the set C consisting of the strings of size n consisting of 0 and 1. For each vertex s in C, let v_s denote the string where its i'th entry is $1-2s_i$, where s_i is the i'th entry of s. Note that $s + v_s$ is still a string in C, and so is $s + f(v_s)$. Set $g(s) = s + f(v_s)$. Now, starting at an arbitrary vertex s_0 of G, define $s_i = g(s_{i-1})$ for $i \geq 1$. Then as C is finite, we must get $s_a = s_{a+m}$ for some minimal positive integer m. But then

$$s_a = s_{a+m} = s_a + f(v_{s_a}) + f(v_{s_{a+1}}) + \cdots + f(v_{s_{a+m-1}})$$

and so
$$0 = f(v_{s_a}) + f(v_{s_{a+1}}) + \cdots + f(v_{s_{a+m-1}}).$$
since m was chosen to be minimal, we must have that this is a sum of distinct rows of the altered matrix, and this completes the proof.