International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Fall 2001.**

1. Do there exist positive integers $a_1 < a_2 < \cdots < a_{100}$ such that for $2 \leq k \leq 100$, the greatest common divisor of a_{k-1} and a_k is greater than the greatest common divisor of a_k and a_{k+1}?

2. Let $n \geq 3$ be an integer. A circle is divided into $2n$ arcs by $2n$ points. Each arc has one of three possible lengths, and no two adjacent arcs have the same length. The $2n$ points are coloured alternately red and blue. Prove that the n-gon with red vertices and the n-gon with blue vertices have the same perimeter and the same area.

3. Let $n \geq 3$ be an integer. Each row in an $(n-2) \times n$ array consists of the numbers 1, 2, ..., n in some order, and the numbers in each column are all different. Prove that this array can be expanded into an $n \times n$ array such that each row and each column consists of the numbers 1, 2, ..., n.

4. Let $n \geq 2$ be an integer. A regular $(2n+1)$-gon is divided into $2n-1$ triangles by diagonals which do not meet except at the vertices. Prove that at least three of these triangles are isosceles.

5. Alex places a rook on any square of an empty 8×8 chessboard. Then he places additional rooks one rook at a time, each attacking an odd number of rooks which are already on the board. A rook attacks to the left, to the right, above and below, and only the first rook in each direction. What is the maximum number of rooks Alex can place on the chessboard?

6. Several numbers are written in a row. In each move, Robert chooses any two adjacent numbers in which the one on the left is greater than the one on the right, doubles each of them and then switches them around. Prove that Robert can make only a finite number of such moves.

7. It is given that 2^{333} is a 101-digit number whose first digit is 1. How many of the numbers 2^k, $1 \leq k \leq 332$, have first digit 4?

Note: The problems are worth 4, 5, 5, 5, 6, 8 and 8 points respectively.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2001.**

1. In the quadrilateral $ABCD$, AD is parallel to BC. K is a point on AB. Draw the line through A parallel to KC and the line through B parallel to KD. Prove that these two lines intersect at some point on CD.

2. Clara computed the product of the first n positive integers and Valerie computed the product of the first m even positive integers, where $m \geq 2$. They got the same answer. Prove that one of them had made a mistake.

3. Kolya is told that two of his four coins are fake. He knows that all real coins have the same weight, all fake coins have the same weight, and the weight of a real coin is greater than that of a fake coin. Can Kolya decide whether he indeed has exactly two fake coins by using a balance twice?

4. On an east-west shipping lane are ten ships sailing individually. The first five from the west are sailing eastwards while the other five ships are sailing westwards. They sail at the same constant speed at all times. Whenever two ships meets, each turns around and sails in the opposite direction. When all ships have returned to port, how many meetings of two ships have taken place?

5. On the plane is a set of at least four points. If any one point from this set is removed, the resulting set has an axis of symmetry. Is it necessarily true that the whole set also has an axis of symmetry?

Note: Each problem is worth 4 points.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Fall 2001.**

1. On the plane is a triangle with red vertices and a triangle with blue vertices. O is a point inside both triangles such that the distance from O to any red vertex is less than the distance from O to any blue vertex. Can the three red vertices and the three blue vertices all lie on the same circle?

2. Do there exist positive integers $a_1 < a_2 < \cdots < a_{100}$ such that for $2 \leq k \leq 100$, the least common multiple of a_{k-1} and a_k is greater than the least common multiple of a_k and a_{k+1}?

3. An 8×8 array consists of the numbers 1, 2, ..., 64. Consecutive numbers are adjacent along a row or a column. What is the minimum value of the sum of the numbers along a diagonal?

4. Let F_1 be an arbitrary convex quadrilateral. For $k \geq 2$, F_k is obtained by cutting F_{k-1} into two pieces along one of its diagonals, flipping one piece over and then glueing them back together along the same diagonal. What is the maximum number of non-congruent quadrilaterals in the sequence $\{F_k\}$?

5. Let a and d be positive integers. For any positive integer n, the number $a + nd$ contains a block of consecutive digits which constitute the number n. Prove that d is a power of 10.

6. In a row are 23 boxes such that for $1 \leq k \leq 23$, there is a box containing exactly k balls. In one move, we can double the number of balls in any box by taking balls from another box which has more. Is it always possible to end up with exactly k balls in the k-th box for $1 \leq k \leq 23$?

7. The vertices of a triangle have coordinates (x_1, y_1), (x_2, y_2) and (x_3, y_3). For any integers h and k, not both 0, the triangle whose vertices have coordinates (x_1+h, y_1+k), (x_2+h, y_1+k) and $(x_3 + h, y_3 + k)$ has no common interior points with the original triangle.
   (a) Is it possible for the area of this triangle to be greater than $\frac{1}{2}$?
   (b) What is the maximum area of this triangle?

Note: The problems are worth 4, 5, 6, 6, 7, 7 and 3+6 points respectively.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall, 2001.**

1. An altitude of a pentagon is the perpendicular drop from a vertex to the opposite side. A median of a pentagon is the line joining a vertex to the midpoint of the opposite side. If the five altitudes and the five medians all have the same length, prove that the pentagon is regular.

2. There exists a block of 1000 consecutive positive integers containing no prime numbers, namely, $1001! + 2$, $1001! + 3$, $\ldots$, $1001! + 1001$. Does there exist a block of 1000 consecutive positive integers containing exactly five prime numbers?

3. On an east-west shipping lane are ten ships sailing individually. The first five from the west are sailing eastwards while the other five ships are sailing westwards. They sail at the same constant speed at all times. Whenever two ships meets, each turns around and sails in the opposite direction. When all ships have returned to port, how many meetings of two ships have taken place?

4. On top of a thin square cake are triangular chocolate chips which are mutually disjoint. Is it possible to cut the cake into convex polygonal pieces each containing exactly one chip?

5. The only pieces on an 8×8 chessboard are three rooks. Each moves along a row or a column without running to or jumping over another rook. The white rook starts at the bottom left corner, the black rook starts in the square directly above the white rook and the red rook start in the square directly to the right of the white rook. The white rook is to finish at the top right corner, the black rook in the square directly to the left of the white rook and the red rook in the square directly below the white rook. At all times, each rook must be either in the same row or the same column as another rook. Is it possible to get the rooks to their destinations?

Note: Each problem is worth 4 points.

International Mathematics

22nd Tournament of Towns

Spring 2001, Advanced Level

Solutions

JUNIOR (GRADES 7, 8, 9 AND 10)

1. [3] In a certain country 10% of the employees get 90% of the total salary paid in this country. Supposing that the country is divided in several regions, is it possible that in every region the total salary of any 10% of the employees is no greater than 11% of the total salary paid in this region?

Solution. Yes, it is possible. Assume there are 100 employees and 2 regions A and B in the country. Assume also that there are 10 people in region A and 90 people in region B. Let the salary of each employee in region A be \$81,000 and the salary of each employee in region B be \$1,000.

The salary of 10 people (which is 10% of the employees) in region A is \$810,000 (which is 90% of the total salary). Also the salary of any 10% of employees in region A (i.e. of any person) is 10% of the salary paid in this region. Clearly, the same holds for region B.

2. [5] In three piles there are 51, 49, and 5 stones, respectively. You can combine any two piles into one pile or divide a pile consisting of an even number of stones into two equal piles. Is it possible to get 105 piles with one stone in each?

Solution. No, it is not. Note that if at one step the number of stones in each pile is divisible by an odd integer, then at the next step the number of stones in each pile is divisible by the same integer. Clearly, at the very first step we only can obtain either two piles of 100 and 5 stones, or two piles of 56 and 49 stones, or two piles of 54 and 51 stones. In each case the number of stones in two piles has an odd divisor (5, 7, and 3, respectively) greater than 1. Thus, we cannot obtain 105 piles of 1 stone each, since the common divisor in that case is 1.

International Mathematics

22nd Tournament of Towns

Spring 2001, Ordinary Level

Solutions

JUNIOR (GRADES 7, 8, 9 AND 10)

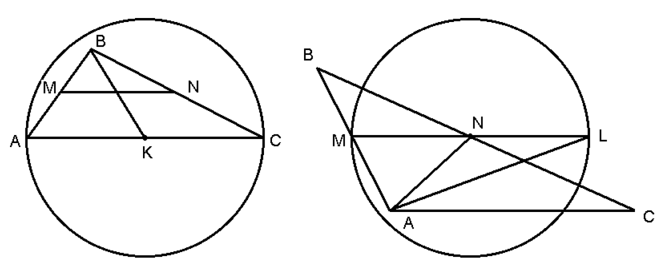
1. [3] The natural number n can be replaced by ab if $a + b = n$, where a and b are natural numbers. Can the number 2001 be obtained from 22 after a sequence of such replacements?

Solution. Yes, it can. In fact, there are infinitely many ways of obtaining 2001 from 22. First note that $n = (n - 1) + 1$, so from n we can obtain $n - 1$. Now it is enough to get any number larger than 2001 and then descend to 2001 one by one. For example we can do: $22 = 11 + 11 \to 121 = 60 + 61 \to 3660 \to 3659 \to \cdots \to 2001$.

2. [4] One of the midlines of triangle $\triangle ABC$ is longer than one of its medians. Prove that the triangle has an obtuse angle.

Solution. Let M and N be the midpoints of AB and BC, respectively. Assume first that midline MN is longer than median BK. Let us draw a circle centered at K with radius $AK = KC = MN$, so AC is its diameter. Since BK is shorter than the radius MN, point B lies inside the circle. Therefore, angle $\angle ABC$ is obtuse.

Now assume MN is longer than one of the other two medians, say $|MN| > |AN|$. Let us draw a circle centered at N with radius MN. Let ML be its diameter. Again since AN is shorter than the radius MN, point A lies inside the circle. Therefore angle $\angle MAL$ is obtuse and hence BAC is obtuse.



SOLUTIONS OF TOURNAMENT OF TOWNS

Spring 2001, Level A, Senior (grades 11-OAC)

Problem 1 [3] Find at least one polynomial $P(x)$ of degree 2001 such that $P(x) + P(1-x) = 1$ holds for all real numbers x.

Solution. It is easy to see that polynomial

$$P(x) = (1-x)^{2001} - x^{2001} + \frac{1}{2}$$

satisfies identity $P(x) + P(1-x) = 1$.

Problem 2 [5] At the end of the school year it became clear that for any arbitrarily chosen group of no less than 5 students, 80% of the marks "F" received by this group were given to no more than 20% of the students in the group. Prove that at least 3/4 of all "F" marks were given to the same student.

Solution. Let us arrange all the students in the school according to the number of "F"marks they received. So, $F_1 \geq F_2 \geq \cdots \geq F_n$ where F_j is the number of "F" received by j-th student, $1 \leq j \leq n$, $F_j \geq 0$ and $\sum_{j=1}^{n} F_j = F$ where F is a total number of "F" marks.

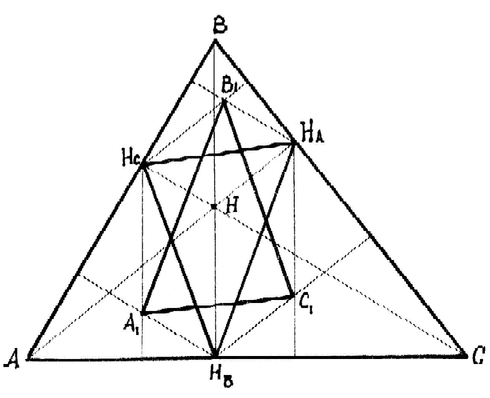
Now let us consider the first five students. According to the condition, one student (who has to be on top of the list) got at least 80% of "F" marks received by this group which leaves no more than 20% of "F" marks remaining for the other four students. So, $F_2 + F_3 + F_4 + F_5 \leq \frac{1}{4}F_4$ and we have an estimate $F_2 \leq \frac{1}{4}F_1$. Considering students from k-th to $k+4$-th ($k+4 \leq n$) we conclude that $F_{k+1} \leq \frac{1}{4}F_k$ which implies that $F_{k+1} \leq \frac{1}{4^k}F_1$ ($k \leq n-5$) and $F_{n-3} + F_{n-2} + F_{n-1} + F_n \leq \frac{1}{4}F_{n-4}$.

Now we have

$$F = F_1 + F_2 + \cdots + F_{n-4} + (F_{n-3} + \cdots + F_n) \leq$$
$$F_1 + \frac{1}{4}F_1 + \frac{1}{4^2}F_1 + \cdots + \frac{1}{4^{n-5}}F_1 + \frac{1}{4^{n-4}}F_1 < \sum_{k=0}^{\infty} \frac{1}{4^k}F_1 = \frac{F_1}{1-\frac{1}{4}} = \frac{4}{3}F_1;$$

Therefore $F_1 > \frac{3}{4}F$.

Problem 3 [5] Let AH_A, BH_B and CH_C be the altitudes of triangle $\triangle ABC$. Prove that the triangle whose vertices are the intersection points of the altitudes of $\triangle AH_BH_C$, $\triangle BH_AH_C$ and $\triangle CH_AH_B$ is congruent to $\triangle H_AH_BH_C$.



Solution. Let us notice that $H_CA_1H_BH$ and $HH_AC_1H_B$ are parallelograms (HH_A and H_BC_1 are perpendicular to BC; H_CA_1, HH_B and H_AC_1 are perpendicular to AC; HH_C and H_BA_1 are perpendicular to AB). Therefore $H_CA_1 = H_AC_1$ and since they are parallel we conclude that $H_CH_AC_1A_1$ is parallelogram, thus $H_CH_A = A_1C_1$. In a similar way we can prove that $H_BH_A = A_1B_1$ and $H_CH_B = B_1C_1$. Therefore $\triangle H_CH_AH_B \cong \triangle A_1B_1C_1$.

SOLUTIONS OF TOURNAMENT OF TOWNS

Spring 2001, Level 0, Senior (grades 11-OAC)

Problem 1 [3] A bus that moves along a 100 km route is equipped with a computer, which predicts how much more time is needed to arrive at its final destination. This prediction is made on the assumption that the average speed of the bus in the remaining part of the route is the same as that in the part already covered. Forty minutes after the departure of the bus, the computer predicts that the remaining travelling time will be 1 hour. And this predicted time remains the same for the next 5 hours. Could this possibly occur? If so, how many kilometers did the bus cover when these 5 hours passed? (Average speed is the number of kilometers covered divided by the time it took to cover them.)

Solution. Let $S(t)$ be a distance covered by the bus for a time t. If the described situation is possible then for any moment $t \geq \frac{2}{3}$ (in hours) we have

$$\frac{100 - S(t)}{1} = \frac{S(t)}{t}$$

or

$$S(t) = \frac{100t}{1+t}. \qquad (*)$$

It is easy to see that $S(t)$ is a continuous monotone increasing function on $(0, \infty)$; this means that the bus is moving toward its destination. Moreover the distance expressed by (*) means that at any moment t the estimated remaining time will be 1 hour. Substituting $t = 5\frac{2}{3}$ into $(*)$ we get that $S(t) = 85$ km.

Problem 2 [4] The decimal expression of the natural number a consists of n digits, while that of a^3 consists of m digits. Can $n + m$ be equal to 2001?

Solution. The fact that the decimal expression of a natural number a consists of n digits means that

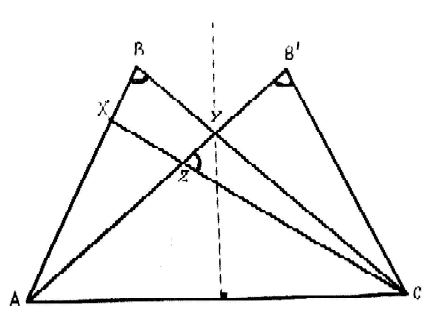
$$10^{n-1} \leq a < 10^n;$$

thus

$$10^{3n-3} \leq a^3 < 10^{3n}.$$

So, $m \in \{3n-2, 3n-1, 3n\}$ and $n + m \in \{4n-2, 4n-1, 4n\}$. Therefore $n + m \neq 1 \pmod 4$ and the answer is negative.

Problem 3 [4] Points X and Y are chosen on the sides AB and BC of the triangle $\triangle ABC$. The segments AY and CX intersect at the point Z. Given that $AY = YC$ and $AB = ZC$ prove that the points B, X, Z, and Y lie on the same circle.



Solution. Let us construct B' symmetrical to B with respect to the straight line passing through Y perpendicular to AC. We get that $\triangle ABY$ is congruent to $\triangle YB'C$; so $\angle ABC = \angle AB'C$. Since $ZC = AB = B'C$ we have $\angle AB'C = \angle B'CZ$. This implies that $\angle XZY = 180° - \angle XBY$ which means that the points B, X, Z, Y lie on the same circle.

Juniors

(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS: SOLUTIONS

A-Level Paper **Fall 2002.**

1 *Answer: \$ 3002*. First, let us prove that d (the difference in salaries) does not exceed 3002. Let us number employees in clock-wise direction starting from one with the minimal salary. Let n be the employee with the maximal salary. Then 1 and n are separated by $n-2$ employees in clock-wise and $(2002-n)$ counter-clockwise. So $d \leq 3(n-1)$ and $d \leq (2003-n)$. Then $d \leq 3(n-1+2003-n)/2 = 3003$. Note, that $d = 3003$ is only possible if the difference between any two neighbors is exactly 3, which contradicts to assumption that all employees have different salaries.

Let us construct an example with the difference 3002. Let $S(k)$ be a salary of k-th worker. Let $S(1) = 0$, $S(2) = 2$, $S(k) = S(k-1)+3$ for $k = 3, 4, \ldots, 1002$, $S(1003) = S(1002) - 2$, $S(k) = S(k-1) - 3$ for $k = 1004, \ldots, 2002$. Then $S(1002) - S(1) = 3002$.

2 *The answer is negative.* It is sufficient to give an example of two numbers which have the same gcd with all the other numbers from 2 to 20,000.

Examples: a) 2^{13} and 2^{14};

b) $19,993$ and $19,997$; both numbers are primes because they have no prime divisors less than 142.

3 Let AB be an arc from A to B in clock-wise direction. For any ordered pair of opposite arcs AB and CD we define $d(AB)$ equal to the difference between arc DA and arc BC. Obviously $d(AB)$ is divisible by 50 (because the difference between two opposite arcs is ± 25 and we have 24 pairs).

Now let us switch to next pair of opposite arcs in clock-wise direction. Note that the increment of $d(AB)$ is either 50, or -50, or 0. Also note that $d(CD) = -d(AB)$. Therefore at some moment we reach a pair of opposite arcs with difference 0.

Then corresponding sides of polygon are parallel.

4 Let us encircle $\triangle ABC$. Let K be an intersection point of continuation of BP and encircle. Then $\angle ABK = \angle ACK$ and $\angle CBK = \angle CAK$ (subtended by the same arc). Then $\triangle APC \cong \triangle AKC$ (A-S-A). Therefore $PK \perp AC$. Similarly, we prove that $AP \perp BC$ as well.

5 Since in N-gon the sum of all angles equals $(N-2)\cdot 180°$, then N-gon is split into $(N-2)$ triangles by $(N-3)$ diagonals, not intersecting inside of N-gon. Side of each white (black) triangle we call white (black); so diagonals are both black and white.

Then, there are at least $(N-3)$ white (black) sides; therefore there are at least $\lceil \frac{1}{3}(N-3) \rceil$ triangles of each color. Let $R(N)$ be the difference in question. Let us consider 3 cases:

Juniors

(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS: SOLUTIONS

O-Level Paper **Fall 2002.**

1 Consider a triangulation of 2002-gon satisfying the conditions. Triangles which contain at least one side of 2002-gon we call *exterior triangles*. So, our problem is reduced to the following question:

Is it possible to have exactly 1000 exterior triangles? (then we have exactly 1000 triangles which have diagonals for all three sides).

The answer is negative. Really, every exterior triangle contains at most 2 sides of 2002-gon and there should be at least 1001 of them. Contradiction.

2 Let j and m be numbers selected by J and M respectively. Note that $j|2002$; otherwise J would know that $m = 2002 - j$. Also $j \neq 2002$; otherwise $m = 1$ (since $m \neq 0$). So, $j \leq 1001$.

Further, the same is true for m. In addition, M knows that $j \leq 1001$. Therefore, $m = 1001$ (otherwise M would know $j = 2002 : m$).

So, $m = 1001$ is the only possible solution. One can check that it works.

3 Let N be the number of students in the class, M the number of the problems, P the number of passed students, H the number of hard problems. According to definition "a problem is hard" if it has not been solved by at least rN students; where $r = \frac{2}{3}, \frac{3}{4}, \frac{7}{10}$ in (a), (b),(c). Also, according to definition "a student passes" if he solves at least rM problems.

a) *It is possible.* Consider a class consisting of students S_1, S_2, S_3 and set of problems P_1, P_2, P_3. Let S_1 solve P_1 and P_3, S_2 solve P_2 and P_3 and S_3 solved neither P_1 nor P_2. Then S_1, S_2 pass and P_1, P_2 are hard problems.

b) *It is impossible.* Let us write down the results of the test ("+" or "−") into $N \times M$ table.

Let passed students be on the top and hard problems on the left of the table. Let us estimate K_+ and K_-, the numbers of "+" and "−" in the table. First,

$$K_+ \geq (\text{number of " + "got by students who passed}) \geq P \times rM \geq r^2MN$$

and

$$K_- \geq (\text{number of " − " got for hard problems}) \geq H \times rN \geq r^2MN.$$

Then $MN = K_+ + K_- \geq 2r^2MN$ which is impossible for $r = \frac{3}{4}$.

Seniors

(Grades 11 and up)

International Mathematics
TOURNAMENT OF THE TOWNS: SOLUTIONS

A-Level Paper **Fall 2002.**

1 *The answer is negative.* It is sufficient to give an example of two numbers which have the same gcd with all the other numbers from 2 to 20,000.

Examples: a) 2^{13} and 2^{14};

b) $19,993$ and $19,997$; both numbers are primes because they have no prime divisors less than 142.

2 Proof by a contradiction. Assume that pentagon has sides ranging from 0.8 to 1.2. To get a pentagon in cross-section of a cube, a plane has to cross five faces, two pairs of which are parallel. Therefore the pentagon has two pairs of parallel sides. Let us consider pentagon $BCDKL$ with $BC \parallel DK$ and $CD \parallel LB$. Then A be a point of intersection of BL and KD (extended). Note that $ABCD$ is a parallelogram. Due to triangle inequality $AL+AK > LK$, then $AB + AD > BL + LK + KD$. So, $BC + CD > BL + LK + KD$. Then even if BC and CD are two longest sides, $BC + CD \leq 2 \cdot 1.2 = 2.4$ while $BL + LK + KD \geq 3 \cdot 0.8 = 2.4$ which is contradiction.

3 Since in N-gon the sum of all angles equals $(N-2)\cdot 180°$, then N-gon is split into $(N-2)$ triangles by $(N-3)$ diagonals, not intersecting inside of N-gon. Side of each white (black) triangle we call white (black); so diagonals are both black and white.

Then, there are at least $(N-3)$ white (black) sides; therefore there are at least $\lceil \frac{1}{3}(N-3) \rceil$ triangles of each color. Let $R(N)$ be the difference in question. Let us consider 3 cases:

a) $N = 3k$. Then there are at least $k-1$ black triangles, at most $2k-1$ white triangles and thus $R(N) \leq k$.

b) $N = 3k+1$. Then there are at least k black triangles, at most $2k-1$ white triangles and thus $R(N) \leq k-1$.

c) $N = 3k+2$. Then there are at least k black triangles, at most $2k$ white triangles and thus $R(N) \leq k$.

Let us prove that all these estimates are sharp and equalities could be reached. For $N = 3, 4, 5$ $(k = 1)$ one can check it easily. For larger N one can construct example by induction by k.

Let us assume that for some k we have corresponding N-gon with the required difference (white triangles are in excess). Then we add a pentagon (2 white and 1 black triangles) to N-gon matching black side of pentagon with the white one of N-gon. Then N increases by 3, k increases by 1 and $R(N)$ increases by 1.

Seniors

(Grades 11 and up)

International Mathematics
TOURNAMENT OF THE TOWNS: SOLUTIONS

O-Level Paper **Fall 2002.**

1 Let j and m be numbers selected by J and M respectively. Note that $j|2002$; otherwise J would know that $m = 2002 - j$. Also $j \neq 2002$; otherwise $m = 1$ (since $m \neq 0$). So, $j \leq 1001$.

Further, the same is true for m. In addition, M knows that $j \leq 1001$. Therefore, $m = 1001$ (otherwise M would know $j = 2002 : m$).

So, $m = 1001$ is the only possible solution. One can check that it works.

2 Let N be the number of students in the class, M the number of the problems, P the number of passed students, H the number of hard problems. According to definition "a problem is hard" if it has not been solved by at least rN students; where $r = \frac{2}{3}, \frac{3}{4}, \frac{7}{10}$ in (a), (b),(c). Also, according to definition "a student passes" if he solves at least rM problems.

a) *It is possible.* Consider a class consisting of students S_1, S_2, S_3 and set of problems P_1, P_2, P_3. Let S_1 solve P_1 and P_3, S_2 solve P_2 and P_3 and S_3 solved neither P_1 nor P_2. Then S_1, S_2 pass and P_1, P_2 are hard problems.

b) *It is impossible.* Let us write down the results of the test ("+" or "−") into $N \times M$ table.

Let passed students be on the top and hard problems on the left of the table. Let us estimate K_+ and K_-, the numbers of "+" and "−" in the table. First,

$$K_+ \geq (\text{number of "+" got by students who passed}) \geq P \times rM \geq r^2MN$$

and

$$K_- \geq (\text{number of "−" got for hard problems}) \geq H \times rN \geq r^2MN.$$

Then $MN = K_+ + K_- \geq 2r^2MN$ which is impossible for $r = \frac{3}{4}$.

c) *It is impossible.* Arguments of (b) do not work here since $2r^2 \leq 1$. Now we denote by K_+ and K_- the numbers of "+" and "−" in the top-left $P \times H$ sub-table. Then

$$K_+ \geq (\text{minimal number of "+" for hard problems got by students who passed}) \geq P \times \frac{4}{7}H$$

(a student cannot pass if he solves less than $\frac{4}{7}H$ of hard problems even if he solves all the easy problems, the number of which does not exceed $\frac{3}{7}M$). On the other hand,

$$K_- \geq (\text{minimal number of "−" got by students who passed for hard problems}) \geq H \times \frac{4}{7}P.$$

So, $PH = K_+ + K_- \geq \frac{8}{7}PH$ which is impossible.

Solution to Junior A-Level Spring 2002

1. We have

$$\begin{aligned} a^3+b^3+3abc-c^3 &= a^3+b^3+(-c)^3-3ab(-c) \\ &= (a+b+9-c))(a^2+b^2+(-c)^2-b(-c)-(-c)a-ab) \\ &= \frac{1}{2}(a+b-c)((b+c)^2+(c+a)^2+(a-b)^2). \end{aligned}$$

   This is positive since $a+b-c>0$ in a non-degenerate triangle.

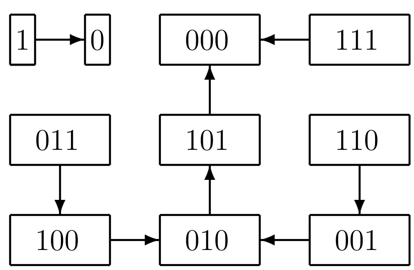
2. Initially, the four chips determine a rectangle, with chips of the same colour at opposite corners. After a move by the first player from such a position, there is no victory since the two white chips are in different rows and different columns. Moreover, the four chips will no longer determine a rectangle. However, the second player can restore this position in his move. Thus there is no victory for the first player.

3. Denote the area of the polygon P by $[P]$. Then

$$[BAD]=[ABEFD]-[BEFD]=[ABE]+[AEF]+[AFD]-3[BCD].$$

   In order to maximize $[BAD]$, BCD must have the smallest area among the four triangles whose area are four consecutive integers. The maximum value of $[BAD]$ is $[BCD]+1+[BCD]+2+[BCD]+3-3[BCD]=6$.

4. Denote by 0 a lamp which off and by 1 a lamp which is on. The following diagram shows that for $n=1$ or 3, there are no initial configurations which lead to perpetual light.



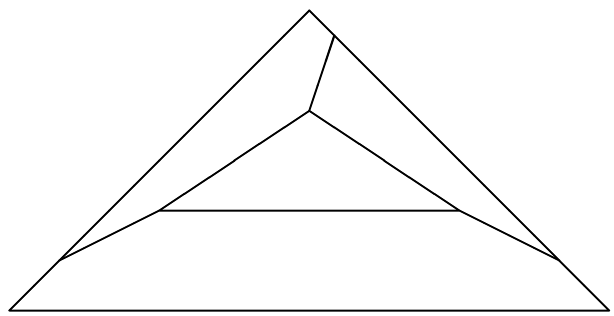
   For even n, the initial configuration 1001100110... will work since it will alternate with 0110011001.... For odd $n>3$, just add 010 to the previous configuration. It will alternate with 100 since the third light will not go on because of the fourth. Hence this part will alternate with 100, independent of the second part. In conclusion, perpetual light is possible for all n except 1 and 3.

Solution to Junior O-Level Spring 2002

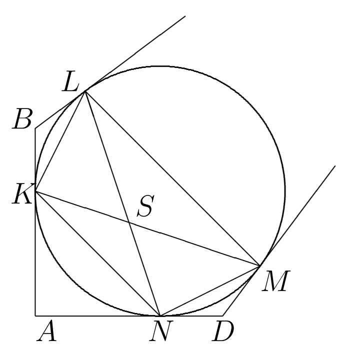
1. The only common divisors of $49 \times 51 = 3 \times 7^2 \times 17$ and $99 \times 101 = 3^2 \times 11 \times 101$ are 1 and 3. Since $a < b$, $ab > 1$. So $ab = 3$ and we must have $a = 1$ and $b = 3$.

2. If either x or y is odd, $x^2 + xy + y^2$ is also odd. Hence they are both even. If one is a multiple of 10 and the other is not, $x^2 + xy + y^2$ is not a multiple of 10. Suppose both x and y are not multiples of 10. Then x^2 and y^2 end in 4 or 6, while xy cannot end in 0. So we cannot have one ending in 4 and the other in 6. If x^2 and y^2 both end in 4 or both end in 6, then xy must also end in 4 or 6. It follows that the only possibility is for both x and y to be multiples of 10, so that $x^2 + xy + y^2$ will indeed be a multiple of 100.

3. One such dissection is shown in the diagram below.



4. Since BK and BL are tangents, $\angle BKL = \angle KML = \angle BLK$. Denote their common value by θ. Then $\angle BKL = 180° - 2\theta$. Similarly, $\angle DMN = \angle MLN = \angle DNM$. Denote their common value by ϕ. Then $\angle MDN = 180° - 2\phi$. Now $\angle KSL = \angle SLM + \angle SML = \theta + \phi$. Similarly, $\angle MSN = \theta + \phi$. Since $SKBL$ is cyclic, $\angle KBL + \angle KSL = 180°$, which implies that $\theta = \phi$. Then $\angle MDN + \angle MSN = 180°$, which implies that $SMDN$ is cyclic.

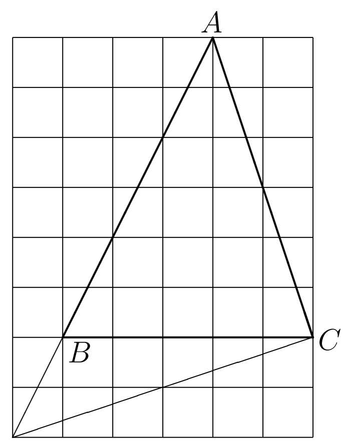


Solution to Senior A-Level Spring 2002

1. First, note that we have

$$\begin{aligned}\tan A+\tan B+\tan C &= \tan A+\tan B-\frac{\tan A+\tan B}{1-\tan A\tan B}\\ &= (\tan A+\tan B)\left(1-\frac{1}{1-\tan A\tan B}\right)\\ &= -\frac{\tan A+\tan B}{1-\tan A\tan B}\tan A\tan B\\ &= \tan A\tan B\tan C.\end{aligned}$$

Let $\tan A=a,\ \tan B=b$ and $\tan C=c$ where $a,\ b$ and c are integers such that $a+b+c=abc$. ABC cannot be a right triangle. Suppose $\angle A$ is obtuse. Then a is negative while b and c are positive. If $b=c=1$, then $abc=a<a+2=a+b+c$. Any increase in the values of b or c will increase that of $a+b+c$ while decrease that of abc. It follows that ABC is an acute triangle, so that $a,\ b$ and c are all positive. We may assume that $a\le b\le c$. Then $abc=a+b+c\le 3c$, so that $ab\le 3$. We cannot have $a=b=1$. Hence $a=1,\ b=2$ and $c=3$. Finally, the diagram below shows a triangle ABC with $\tan A=1,\ \tan B=2$ and $\tan C=3$.



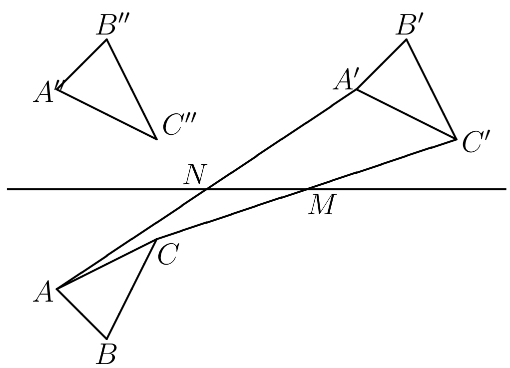
2. Consider the points $A(a,a^3)$ and $B(b,b^3+b+1)$ where $a>b>0$. We wish to choose a and b such that $a-b<\frac{1}{100}$ while $a^3=b^3+b+1$. Let $t=a-b>0$. ¿From $(b+t)^3=b^3+b+1$, we have $3tb^2-(1-3t^2)b-(1-t^3)=0$. If $t<\frac{1}{100}$, the constant term of this quadratic equation is negative, so that it has one positive root and one negative root. Thus a and b can be chosen so that $AB<\frac{1}{100}$.

3. Let the sequence be $\{a_n\}$ and let S_n denote the sum of all the terms up to but not including a_n. For $n\ge 2002$, a_n is a divisor of S_n. Hence there exists a positive integer d_n such that $a_n=\frac{S_n}{d_n}$. Then $S_{n+1}=S_n+a_n=\frac{(d_n+1)S_n}{d_n}$. If $d_{n+1}\ge d_n+1$, then $a_{n+1}\le\frac{S_n}{d_n}=a_n$, and this contradicts the hypothesis that $\{a_n\}$ is strictly increasing. Hence $\{d_n\}$ is non-decreasing for $n\ge 2002$. However, this sequence cannot maintain a value $k>1$ indefinitely as otherwise $\{S_n\}$ becomes a geometric progression with common ratio $\frac{k+1}{k}$ starting from some term. However, k and $k+1$ are relatively prime, and we can only divide the first term of the geometric progression by k finitely many times. It follows that $d_n=1$ eventually.

Solution to Senior O-Level Spring 2002

1. If either x or y is odd, x^2+xy+y^2 is also odd. Hence they are both even. If one is a multiple of 10 and the other is not, x^2+xy+y^2 is not a multiple of 10. Suppose both x and y are not multiples of 10. Then x^2 and y^2 end in 4 or 6, while xy cannot end in 0. So we cannot have one ending in 4 and the other in 6. If x^2 and y^2 both end in 4 or both end in 6, then xy must also end in 4 or 6. It follows that the only possibility is for both x and y to be multiples of 10, so that x^2+xy+y^2 will indeed be a multiple of 100.

2. Let M be the midpoint of AA' and N be the midpoint of CC'. Then A and A' are equidistant from MN, as are C and C'. Let $A''B''C''$ be the reflection of ABC across MN. Then A and A'' are equidistant from MN, as are C and C''. Hence $A'A''$ and $C'C''$ are both parallel to MN. Now $A''B''C''$ is congruent to ABC and opposite in orientation. Hence it is congruent to $A'B'C'$ and in the same orientation. It follows that $A'B'C'$ and $A''B''C''$ may be obtained from each other by a translation in the direction parallel to MN. Hence B' and B'' are equidistant from MN. It follows that so are B and B', so that the midpoint of BB' indeed lies on MN.



3. The only possible groupings are (126,345), (136,245), (146,235), (156,234) and (236,145). First weigh 146 against 235. If they balance, the task is accomplished. If 146 is heavier, then 156 will be heavier than 234. Then we weigh 136 against 245. If they balance, the task is accomplished. If 136 is heavier, then 236 will be heavier than 145. Hence 126 must balance 345. If in the first weighing 146 is lighter, then 136 will be lighter than 245, 126 will be lighter than 345 and 145 will be lighter than 236. Hence 156 must balance 234.

4. We first solve the problem for a 2×5 table. Each successful placement of the numbers is replaced with a continuous path from one number to the next. Suppose first that 1 and 10 are also adjacent, so that the path could have linked up to form a cycle. The cycle could be broken up in any of 10 places. Hence there are 10 paths of this kind. Suppose now that 1 and 10 are not adjacent, so that we have an open path. We classify them according to whether the vertical segments are in one, two or three groups, where vertical segments on adjacent columns are considered to be in the same group. Note that apart from a path obtained from the cycle, each end column must contain a vertical segment.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper[1] **Fall 2003.**

1. An increasing arithmetic progression consists of one hundred positive integers. Is it possible that every two of them are relatively prime?

2. Smallville is populated by unmarried men and women, some of them are mutual acquaintants. The City's two Official Matchmakers are aware of all the mutual acquaintances. One of them claimed: "I can arrange it so that every brown haired man will marry a woman with whom he is mutually acquainted." The other claimed, "I can arrange it so that every blond haired woman will marry a man with whom she is mutually acquainted." An amateur mathematician overheard their conversation and said, "Then both arrangements can be made at the same time!" Is he right?

3. Determine all positive integers k such that there exist positive integers m and n satisfying $m(m+k) = n(n+1)$.

4. In chess, a bishop attacks any square on the two diagonals that contain the square on which it stands, including that square itself. Several squares on a 15×15 chessboard are to be marked so that a bishop placed on any square of the board attacks at least two of the marked squares. Determine the minimal number of such marked squares.

5. Prove that $135° \leq \angle OAB + \angle OBC + \angle OCD + \angle ODA \leq 225°$ for any point O inside a square $ABCD$.

6. An ant crawls on the outer surface of a rectangular box. The distance between two points on a surface is defined as the length of the shortest path the ant needs to crawl to reach one point from the other. Is it true that if the ant is at a vertex, then the opposite vertex is the point on the surface which is at the greatest distance away?

7. In a game, Boris has 1000 cards numbered $2, 4, \ldots, 2000$ while Anna has 1001 cards numbered $1, 3, \ldots, 2001$. The game lasts 1000 rounds. In an odd-numbered round, Boris plays any card of his. Anna sees it and plays a card of hers. The player whose card has the larger number wins the round, and both cards are discarded. An even-numbered round is played in the same manner except that Anna plays first. At the end of the game, Anna discards her unused card. What is the maximal number of rounds each player can guarantee to win, regardless of how the opponent plays?

Note: The problems are worth 4, 5, 5, 6, 7, 7 and 8 points respectively.

[1] Courtesy of Andy Liu.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2003.**
Solutions

1. ANSWER: yes.

   EXAMPLE: Placing 5 into each square of two 4×3-faces, 8 into each square of two 5×3-faces and 9 into each square of two 4×5-faces , we satisfy all the given requirements.

2. Assume that a 7-gon is convex. In order for a polygon to be regular, one needs to prove equalities of all its sides and angles. Let us color the sides of the polygon with white, the first set of equal diagonals with blue, and the second set of equal diagonals with red. All triangles created by two blue and one red sides are equal (S-S-S). Therefore the next two sets of angles are equal:

   (a) Angles at the bases of isosceles triangles mentionned (let us denote them by α).

   (b) Angles at the vertices of isosceles triangles mentionned (let us denote them by β).

   Consider the triangles created by three different colors. All the angles opposed to white sides are equal (each equals $\alpha - \beta$). Then all sides of the polygon are equal (S-A-S).

   Now, all the triangles, created by two white sides and one blue side are equal (S-S-S). Thus all angles of the polygon are equal.

   Therefore, the polygon is regular.

3. Let us denote the greatest odd divisor of number K as $\text{god}(K)$. Obviously, $\text{god}(K) \leq K$. Assume, that $\text{god}(l) = \text{god}(m)$, $l, m \in [n+1; 2n]$, $l < m$. Then $m \geq 2l$, which is impossible.

   Therefore, for all (n) numbers from $[n+1, 2n]$ their corresponding greatest odd divisors are distinct. It means, that the set $\{\text{god}(m), n+1 \leq m \leq 2n\}$ coincides with $\{1, 3, \ldots, 2n-1\}$. Then, the sum in question is $1 + 3 + \cdots + 2n + 1 = n^2$.

4. Let us note that

   (a) from each point emanates exactly $(n-1)$ segments (each pair of points is connected).

   (b) from each point emanates either 2 blue/red segments or 2 blue and 2 red segments (each broken line is closed, without intersection and there are no isolated points).

   Therefore, we have 2 possible cases to consider:

   (i) n=3.

   (ii) n=5.

   Case (i) is trivial, corresponding to a triangle with all sides of the same color.

   Case (ii) is also possible:

   EXAMPLE. Consider points $A(0;4); B(-4;-4); C(4;-4); D(-1;0); E(1;0)$. Connect them by red segments in the order A, B, C, D, E, A and by blue segments in the order A, D, B, D, C, A.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper [1] **Fall 2003.**

1. Smallville is populated by unmarried men and women, some of them are mutual acquaintants. The City's two Official Matchmakers are aware of all the mutual acquaintances. One of them claimed: "I can arrange it so that every brown haired man will marry a woman with whom he is mutually acquainted." The other claimed, "I can arrange it so that every blond haired woman will marry a man with whom she is mutually acquainted." An amateur mathematician overheard their conversation and said, "Then both arrangements can be made at the same time!" Is he right?

2. Prove that one can represent every positive integer in the form $3^{u_1} \cdot 2^{v_1} + 3^{u_2} \cdot 2^{v_2} + \ldots + 3^{u_k} \cdot 2^{v_k}$ where $u_1 > u_2 > \cdots > u_k \geq 0$ and $0 \leq v_1 < v_2 < \cdots < v_k$ are integers.

3. An ant crawls on the outer surface of a rectangular box. The distance between two points on a surface is defined as the length of the shortest path the ant needs to crawl to reach one point from the other. Is it true that if the ant is at a vertex, then the opposite vertex is the point on the surface which is at the greatest distance away?

4. Triangle ABC has orthocentre H, incentre I and circumcentre O. K is the point where the incircle touches BC. If IO is parallel to BC, prove that AO is parallel to HK.

5. In a game, Boris has 1000 cards numbered $2, 4, \ldots, 2000$ while Anna has 1001 cards numbered $1, 3, \ldots, 2001$. The game lasts 1000 rounds. In an odd-numbered round, Boris plays any card of his. Anna sees it and plays a card of hers. The player whose card has the larger number wins the round, and both cards are discarded. An even-numbered round is played in the same manner except that Anna plays first. At the end of the game, Anna discards her unused card. What is the maximal number of rounds each player can guarantee to win, regardless of how the opponent plays?

6. Let O be the incentre of a tetrahedron $ABCD$ in which the sum of areas of the faces ABC and ABD is equal to the sum of areas of the faces CDA and CDB. Prove that midpoints of BC, AD, AC and BD lie on a plane passing through O.

7. Each cell of an $m \times n$ table is filled with a $+$ sign or a $-$ sign. Such a table is said to be *irreducible* if one cannot change all $-$ signs to $+$ signs by applying, as many times as desired, some permissible operation.

   (a) Suppose the permissible operation is to change the signs of all cells in a row or a column to the opposite signs. Prove that an irreducible table contains an irreducible 2×2 sub-table.

   (b) Suppose the permissible operation is to change the signs of all cells in a row, a column or a diagonal (which may be of any length, including those of length 1, consisting of a corner cell). Prove that an irreducible table contains an irreducible 4×4 sub-table.

Note: The problems are worth 4, 4, 6, 7, 7, 7 and 3+6 points respectively.

[1] Courtesy of Andy Liu

International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2003.**
Solutions

1. Let us denote the greatest odd divisor of number K as god(K). Obviously, god$(K) \leq K$. Assume, that god$(l) =$ god(m), $l, m \in [n+1; 2n]$, $l < m$. Then $m \geq 2l$, which is impossible.

   Therefore, for all (n) numbers from $[n+1, 2n]$ their corresponding greatest odd divisors are distinct. It means, that the set $\{\text{god}(m), n+1 \leq m \leq 2n\}$ coincides with $\{1, 3, \ldots, 2n-1\}$. Then, the sum in question is $1 + 3 + \cdots + 2n + 1 = n^2$.

2. Let us solve the problem for a $(2n+1) \times (2n+1)$-square. Note that all unit boundary squares $(8n)$ should be drawn (otherwise there would be "holes" in a frame). Now we have a $(2n-1) \times (2n-1)$ square with the frame. Let it be colored as a chess board, with black squares at the angles. Drawing only white squares would give us the whole picture in question. The number of white squares is equal to $\big((2n-1)^2 - 1\big)/2$. Then the total number of the unit squares used equals $8n + ((2n-1)^2 - 1)/2 = 360$, if $n = 25$.

   Let us show that it is indeed the minimal number. Let us tile a $(2n-1) \times (2n-1)$ square with dominos (2×1 rectangles). Then we use $\big((2n-1)^2 - 1\big)/2$ dominos with 1 unit square left. In each domino we have to draw at least one square, so the least number of squares drawn is $\big((2n-1)^2 - 1\big)/2$.

3. Let Customer give all his money to Salesman. The value of change could vary from 0 (if Cat is a "gift") to 1999 rubles and we show that any integer value could be created within a set of bills. It is enough to solve the problem in the case of the following (minimal) set of sixteen bills $\{1000, 500, 100, 100, 100, 100, 50, 10, 10, 10, 10, 5, 1, 1, 1, 1\}$. Actually, representing 1999 as 1000+900+90+9 one can see that in the "minimal" case we must have exactly one 1000 ruble bill, one 500 ruble bill, four 100 ruble bills, one 50 ruble bill, four 10 ruble bills, one 5 ruble bill and four 1 ruble bills. Now, notice that if a transaction could be made with the minimal set of bills, then it could be also made with any other set of bills. Actually, each smaller nomination divides every larger one. Therefore, if in an arbitrary set we have more than, say, four ruble bills, we wrap five rubles by a rubber band and consider it as a 5 ruble bill and so on. So, any set of bills could be reduced to the minimal set.

   It is easy to see that any value of change in the form $ABCD$, where $A = 0, 1$ and B, C, D are any digits from 0 to 9, could be paid with the minimal set of bills.

4. Lemma. The area of a quadrilateral, placed into a circle with radius R does not exceed $2R^2$.

   Proof. Let $ABCD$ be the quadrilateral in mention, and O the center of the circle. Then, Area$(\triangle ABO) = \frac{1}{2} AO \cdot BO \cdot \sin(\angle AOB) \leq \frac{1}{2} R^2$ and

$$\text{Area}(ABCD) = \text{Area}(\triangle ABO) + \text{Area}(\triangle BCO) + \text{Area}(\triangle CDO) + \text{Area}(\triangle DAO) \leq 2R^2.$$

   Let O be the center of the given square $KLMN$.

Juniors
(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS: SOLUTIONS

A-Level Paper **Spring 2003.**

1 ANSWER: Yes.

EXAMPLE. Consider quadratic equation $x^2+5x+6=0$. It could be transformed into one of the following four equations:

(a) $x^2+5x+6=0$ (roots $-2,-3$;

(b) $x^2+5x-6=0$ (roots $-6,1$);

(c) $x^2-5x+6=0$ (roots $2,3$);

(d) $x^2-5x-6=0$ (roots $6,-1$).

2 The longest side of the triangle is a chord of a circumscribed circle and thus it does not exceeds its diameter: $a \le 2R$. Projection of incircle onto the shortest altitude is contained strictly inside of the projection of the triangle onto this altitude. So $2r < h$. Since all the numbers are positive we can multiply these inequalities: $2r \cdot a < h \cdot 2R$ which implies $a/h < R/r$.

3

(a) ANSWER: Yes. Let us assign to i-th team a number $a_i = 0$, if prior to the game it already played even numbers of games and $a_i = 1$ otherwise. Note, that a_i changes after each game in which i-th team participated.

Assume, that all games were "even", meaning that prior to the game both teams had the same parity.

Consider the sum $A = a_1 + a_2 + \cdots + a_{15}$ of the parities of all teams. After each game played by two teams with the same parity A changes by $\pm 2 \equiv 2 \mod 4$.

Initially we had $a_1 = a_2 = \cdots = a_{15} = 0$, therefore $A = 0$. In the end we have $a_1 = a_2 = \cdots = a_{15} = 0$ (each team played an even number of games (14)) and again $A = 0$.

Since the total number of games $15 \cdot 14/2 = 105$ is odd, so in the end of the tournament $A \equiv 2 \mod 4$.

Contradiction.

(b) ANSWER: Yes. We will construct an example of a tournament with one "odd" game. Let us consider a graph, in which vertices represent teams and edges represent games. It is enough to draw edges in such a way that every time (but one) we connect the vertices of the same parity. Let us split all the vertices into three sets of five: $A_1, A_2, \ldots A_5$; $B_1, \ldots, B_5$; $C_1, \ldots, C_5$. We proceed in three steps:

Juniors

(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS: SOLUTIONS

O-Level Paper **Spring 2003.**

1 Let S be an entire amount of money (\$2003),

a_i be amount of money in i-pocket, $i = 1, 2, \ldots, M$. Then

$$a_i < N, \qquad S = \sum_{i=1}^{M} a_i < MN. \tag{1}$$

Let us assume that each purse contains no less than M dollars in it. Let b_i be amount of money in i-purse. Then

$$b_i \geq M, \qquad S = \sum_{i=1}^{N} b_i \geq MN. \tag{2}$$

Contradiction.

2 Consider three cases:

(a) $n > 4$. Let us show that the first player has a winning strategy. On each of his subsequent moves, the first player colours a side which is one space away from one of already coloured sides. Note, that doing this, he creates a "store", which he can use later; however, the second player can not, because of the nature of requirement. So, in the end of the game, after the first player's move, we are left with cases:

(i) One uncoloured side is left (plus "store"). The second player has no move.

(ii) Two uncoloured sides are left (plus "store"). After the second player's move, the first player wins.

(iii) Three uncoloured sides are left (plus "store"). After the second player's move, the first player uses his "store", and wins on his next move.

(b) From above, we can see that the only chance for the second player to win is in the case (iii), when "store" is not yet created. It corresponds to the case $n = 4$. Really, the first player can not produce his second move and loses.

(c) $n = 3$. The first player wins.

Seniors

(Grades 11 and up)

International Mathematics
TOURNAMENT OF THE TOWNS: SOLUTIONS

A-Level Paper **Spring 2003.**

1 **Solution 1.** The longest edge of the pyramid is a chord of the circumscribed sphere and thus it does not exceed diameter of the sphere: $a \le 2R$. Projection of insphere onto the shortest altitude of the pyramid is strictly contained in the projection of the pyramid onto this altitude. So, $2r < h$. Multiplying inequalities we get $2r \cdot a < h \cdot 2R$, which is equivalent to $\dfrac{a}{h} < \dfrac{R}{r}$.

Solution 2. Let us calculate the volume of the pyramid in two ways: $V = \frac{1}{3}H_jS_j$ and $V = \frac{1}{3}r(S_1 + S_2 + S_3 + S_4)$, where S_j is the area of j-th face, and H_j is a corresponding altitude. Thus $H_j = 3V/S_j$ and $h = 3V/S_{\max}$, where $S_{\max} = \max_j S_j$ is the area of the face with the largest area.

Therefore, $r = 3V/(S_1+S_2+S_3+S_4)$. Note that $(S_1+S_2+S_3+S_4) > 2S_{\max}$. Really, if we project the pyramid onto one of its faces (treated as a base) then a projections of the lateral faces will cover the base. Since area of projection is less than the area of the face itself (because none of lateral faces is parallel to the base) we get our inequality.

Then
$$\frac{R}{r} = \frac{R(S_1 + S_2 + S_3 + S_4)}{3V} > \frac{2RS_{\max}}{3V} = \frac{2R}{h} \ge \frac{a}{h}.$$

2 Answer: $\deg P = 1$.

Solution. We consider a more general problem when a_i are integers (not necessarily positive).

(i) $\deg P = 0$ then $P = c = \text{const}$ and all $a_i = P(a_{i+1})$ are equal which contradicts conditions.

(ii) $\deg P = 1$ is possible: for example, $a_i = i$, $P(x) = x - 1$.

Seniors

(Grades 11 and up)

International Mathematics
TOURNAMENT OF THE TOWNS: SOLUTIONS

O-Level Paper **Spring 2003.**

1 Let S be an entire amount of money (\$2003),

a_i be amount of money in i-pocket, $i = 1, 2, \ldots, M$. Then

$$a_i < N, \qquad S = \sum_{i=1}^{M} a_i < MN. \tag{1}$$

Let us assume that each purse contains no less than M dollars in it. Let b_i be amount of money in i-purse. Then

$$b_i \geq M, \qquad S = \sum_{i=1}^{N} b_i \geq MN. \tag{2}$$

Contradiction.

2 Yes, it could happen.

Example. Consider a 100-gon with sides:

$$1, 1, 2, 2^2, \ldots, 2^{98}, 2^{99} - 1.$$

Since $1 + 1 + 2 + \ldots + 2^{98} = 2^{99} > 2^{99} - 1$ it is possible to construct 100-gon with these sides. On the other hand, one cannot construct a polygon from any lesser number of sides. Really, consider two cases:

(a) Side $(2^{99} - 1)$ is among selected.
Then even if the shortest side is absent, $1 + 2 + \ldots + 2^{98} = 2^{99} - 1$.

(b) The longest selected side is 2^k, $1 \leq k \leq 2^{98}$.
Then $1 + 1 + \ldots + 2^{k-1} = 2^k$.

International Mathematics

TOURNAMENT OF THE TOWNS

Junior A-Level Paper[1] **Fall 2004.**

1. An angle is said to be rational if its measure in degrees is a rational number. A triangle is said to be rational if all its angles are rational. Prove that there exist at least three different points inside any acute rational triangle such that when each is connected to the three vertices of the original triangle, we obtain three rational triangles.

2. The incircle of triangle ABC touches the sides BC, CA and AB at D, E and F respectively. If $AD = BE = CF$, does it follow that ABC is equilateral?

3. What is the maximum number of knights that can be place on an 8×8 chessboard such that each attacks at most seven other knights?

4. On a blackboard are written four numbers. They are the values, in some order, of $x + y$, $x - y$, xy and $\frac{x}{y}$ where x and y are positive numbers. Prove that x and y are uniquely determined.

5. K is a point on the side BC of triangle ABC. The incircle of triangle BAK touches BC at M. The incircle of triangle CAK touches BC at N. Prove that $BM \cdot CN > KM \cdot KN$.

6. Two persons share a block of cheese as follows. They take turns cutting an existing block of cheese into two, until there are five blocks. Then they take turns choosing one block at a time. The person who makes the first cut also makes the first choice, and gets an extra block. Each wants to get as much cheese as possible. What is the optimal strategy for each, and how much is each guaranteed to get, regardless of the counter measures of the other?

7. We have many copies of each of two rectangles. If a rectangle similar to the first can be made by putting together copies of the second, prove that a rectangle similar to the second can be made by putting together copies of the first, with no overlapping in both instances.

Note: The problems are worth 4, 5, 6, 6, 7, 8 and 8 points respectively.

[1] Courtesy of Andy Liu.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper[1] **Fall 2004.**

1. Is it possible to arrange the numbers from 1 to 2004 inclusive in some order such that the sum of any ten adjacent numbers is divisible by 10?

2. A bag contains 111 balls, each of which is green, red, white or blue. If 100 balls are drawn at random, there will always be 4 balls of different colours among them. What is the smallest number of balls that must be drawn, at random, in order to guarantee that there will be 3 balls of different colours among them?

3. Various pairs of towns in Russia were linked by direct bus services with no intermediate stops. Alexei Frugal bought one ticket for each route, which allowed travel in either direction but not returning on the same route. He started from Moscow, used up all his tickets without buying any new ones, and finished at Kaliningrad. Boris Lavish bought n tickets for each route, and started from Moscow. However, after using some of his tickets, he got stuck in some town which he could not leave without buying a new ticket. Prove that he got stuck in either Moscow or Kaliningrad.

4. Given a line and a circle which do not intersect, use straight edge and compass to construct a square with two adjacent vertices on the line and the other two on the circle, assuming that such a square exists.

5. In how many ways can 2004 be expressed as the sum of one or more positive integers in non-decreasing order, such that the difference between the last term and the first term is at most 1?

Note: The problems are worth 3, 4, 4, 5 and 5 points respectively.

[1]Courtesy of Andy Liu.

International Mathematics

TOURNAMENT OF THE TOWNS

Senior A-Level Paper[1] **Fall 2004.**

1. The functions f and g are such that $g(f(x)) = x$ and $f(g(y)) = y$ for any real numbers x and y. If for all real numbers x, $f(x) = kx + h(x)$ for some constant k and some periodic function $h(x)$, prove that $g(x)$ can similarly be expressed as a sum of a linear function and a periodic function. A function h is said to be periodic if for any real number x, $h(x + p) = h(x)$ for some fixed real number p.

2. Two players alternately remove pebbles from a pile. In each move, the first player must remove either 1 or 10 pebbles, while the second player must remove either m or n pebbles. Whoever cannot make a move loses. If the first player can guarantee a win regardless of the initial number of pebbles in the pile, determine m and n.

3. On a blackboard are written four numbers. They are the values, in some order, of $x + y$, $x - y$, xy and $\frac{x}{y}$ where x and y are positive numbers. Prove that x and y are uniquely determined.

4. A circle with centre I is inside another circle with centre O. AB is a variable chord of the larger circle which is tangent to the smaller circle. Determine the locus of the circumcentre of triangle IAB.

5. We have many copies of each of two rectangles. If a rectangle similar to the first can be made by putting together copies of the second, prove that a rectangle similar to the second can be made by putting together copies of the first, with no overlapping in both instances.

6. Let $n \geq 5$ be a fixed odd prime number. A triangle is said to be admissible if the measure of each of its angles is of the form $\frac{m}{n}180°$ for some positive integer m. Initially, there is one admissible triangle on the table. In each move, one may pick up a triangle from the table and cut it into two admissible ones, neither of which is similar to any other triangle on the table. The two new triangles are put back on the table. After a while, no more moves can be made. Prove that at that point, every admissible triangle is similar to some triangle on the table.

7. From a point O are four rays OA, OC, OB and OD in that order, such that $\angle AOB = \angle COD$. A circle tangent to OA and OB intersects a circle tangent to OC and OD at E and F. Prove that $\angle AOE = \angle DOF$.

Note: The problems are worth 5, 5, 5, 6, 7, 8 and 8 points respectively.

[1]Courtesy of Andy Liu.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper[1] **Fall 2004.**

1. Three circles all passing through X intersect one another again pairwise at A, B and C respectively. The extension of the common chord AX of two of the circles intersects the third circle again at D. Similarly, the extensions of BX and CX yield the points E and F respectively. Prove that triangles BCD, CAE and ABF are similar to one another.

2. A bag contains 100 balls, each of which is red, white or blue. If 26 balls are drawn at random, there will always be 10 balls of the same colour among them. What is the smallest number of balls that must be drawn, at random, in order to guarantee that there will be 30 balls of the same colour among them?

3. $P(x)$ and $Q(x)$ are non-constant polynomials such that for all x, $P(P(x)) = Q(Q(x))$ and $P(P(P(x))) = Q(Q(Q(x)))$. Is it necessarily true that $P(x) = Q(x)$ for all x?

4. In how many ways can 2004 be expressed as the sum of one or more positive integers in non-decreasing order, such that the difference between the last term and the first term is at most 1?

5. For which positive integers n is it possible to arrange the numbers from 1 to n in some order, such that the average of any group of two or more adjacent numbers is not an integer?

Note: The problems are worth 3, 3, 4, 4 and 5 points respectively.

[1]Courtesy of Andy Liu.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Spring 2004.**

1. The sum of all terms of a finite arithmetical progression of integers is a power of two. Prove that the number of terms is also a power of two.

2. What is the maximal number of checkers that can be placed on an 8×8 checkerboard so that each checker stands on the middle one of three squares in a row diagonally, with exactly one of the other two squares occupied by another checker?

3. Each day, the price of the shares of the corporation "Soap Bubble, Limited" either increases or decreases by n percent, where n is an integer such that $0 < n < 100$. The price is calculated with unlimited precision. Does there exist an n for which the price can take the same value twice?

4. Two circles intersect in points A and B. Their common tangent nearer B touches the circles at points E and F, and intersects the extension of AB at the point M. The point K is chosen on the extention of AM so that $KM = MA$. The line KE intersects the circle containing E again at the point C. The line KF intersects the circle containing F again at the point D. Prove that the points A, C and D are collinear.

5. All sides of a polygonal billiard table are in one of two perpendicular directions. A tiny billiard ball rolls out of the vertex A of an inner 90° angle and moves inside the billiard table, bouncing off its sides according to the law "angle of reflection equals angle of incidence". If the ball passes a vertex, it will drop in and srays there. Prove that the ball will never return to A.

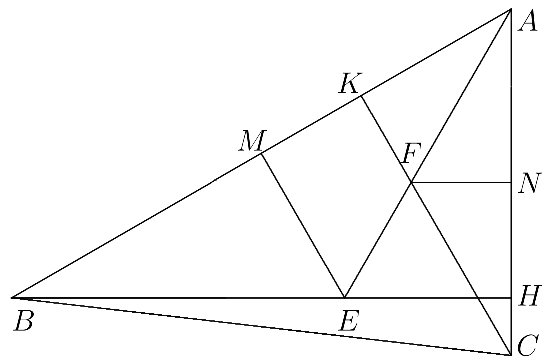
6. At the beginning of a two-player game, the number 2004! is written on the blackboard. The players move alternately. In each move, a positive integer smaller than the number on the blackboard and divisible by at most 20 different prime numbers is chosen. This is subtracted from the number on the blackboard, which is erased and replaced by the difference. The winner is the player who obtains 0. Does the player who goes first or the one who goes second have a guaranteed win, and how should that be achieved?

Note: The problems are worth 4, 5, 5, 6, 6 and 7 points respectively.

International Mathematics
TOURNAMENT OF THE TOWNS

Solution to Junior O-Level Spring 2004[1]

1. Let M and N be the respective midpoints of AB and AC. Let the extension of BE cut AC at H, and the extension of CF cut AB at K. Note that triangles AEH, AEM and BEM are congruent to one another. Hence $\angle BEM = \angle MEA = \angle AEH = 60°$. It follows that $\angle MAE = \angle EAH = 30°$. Since triangles AFN and CFN are congruent to each other, $\angle FCN = 30°$, so that $\angle CKA = 90°$. Thus CF is indeed perpendicular to AB.



2. Clearly, we can have $n = 1$ by taking any prime number. We can also have $n = 2$ since each odd prime is the sum of two consecutive numbers. Suppose $p = a + (a+1) + \cdots + (a+k)$ for some prime number p and positive integers a and $k \geq 2$. Then $2p = (k+1)(2a+k)$. Each of $k+1$ and $2a+k$ is greater than 2. This is a contradiciton since p is a prime number. Hence $n = 1$ or 2.

3. (a) We describe the process in the following chart.

Action Taken	Amount in Bucket A	Bucket B	Bucket C
Initial State	3	20	0
Pour from B into C until C=A	3	17	3
Pour away C	3	17	0
Pour from B into C until C=A	3	14	3
Pour away C	3	14	0
Pour from B into C until C=A	3	11	3
Pour away C	3	11	0
Pour from B into C until C=A	3	8	3
Pour away C	3	8	0
Pour from B into C until C=A	3	5	3
Pour from A into C until C=B	1	5	5
Pour from B into A until A=C	5	1	5
Pour from C into A	10	1	0

[1]Courtesy of Andy Liu.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Spring 2004.**

1. Each day, the price of the shares of the corporation "Soap Bubble, Limited" either increases or decreases by n percent, where n is an integer such that $0 < n < 100$. The price is calculated with unlimited precision. Does there exist an n for which the price can take the same value twice?

2. All angles of a polygonal billiard table have measures in integral numbers of degrees. A tiny billiard ball rolls out of the vertex A of an interior 1° angle and moves inside the billiard table, bouncing off its sides according to the law "angle of reflection equals angle of incidence". If the ball passes throigh a vertex, it will drop in and stays there. Prove that the ball will never return to A.

3. The perpendicular projection of a triangular pyramid on some plane has the largest possible area. Prove that this plane is parallel to either a face or two opposite edges of the pyramid.

4. At the beginning of a two-player game, the number 2004! is written on the blackboard. The players move alternately. In each move, a positive integer smaller than the number on the blackboard and divisible by at most 20 different prime numbers is chosen. This is subtracted from the number on the blackboard, which is erased and replaced by the difference. The winner is the player who obtains 0. Does the player who goes first or the one who goes second have a guaranteed win, and how should that be achieved?

5. The parabola $y = x^2$ intersects a circle at exactly two points A and B. If their tangents at A coincide, must their tangents at B also coincide?

6. The audience shuffles a deck of 36 cards, containing 9 cards in each of the suits spades, hearts, diamonds and clubs. A magician predicts the suit of the cards, one at a time, starting with the uppermost one in the face-down deck. The design on the back of each card is an arrow. An assistant examines the deck without changing the order of the cards, and points the arrow on the back each card either towards or away from the magician, according to some system agreed upon in advance with the magician. Is there such a system which enables the magician to guarantee the correct prediction of the suit of at least

   (a) 19 cards;

   (b) 20 cards?

Note: The problems are worth 4, 6, 6, 6, 7 and 3+5points respectively.

International Mathematics
TOURNAMENT OF THE TOWNS

Solution to Senior O-Level Spring 2004[1]

1. Let O be the centre of the circle, K be the point of tangency with BC and H be the point of intersection of AC and BD. Since $AB = BC$, AC is perpendicular to OB by symmetry. Similarly, BD is perpendicular to OC. Since AC intersects BD at H, H is the orthocentre of triangle OBC. Now the radius OK is perpendicular to the tangent BC. Hence the third altitude OK of triangle OBC passes through H.

2. Note that $b = a(10^n + 1)$ so that $\frac{b}{a^2} = \frac{10^n+1}{a}$. Suppose it is an integer d. Since a is an n-digit number, $1 < d < 11$. Since $10^n + 1$ is not divisible by 2, 3 or 5, the only possible value for d is 7. The example $a = 143$ and $b = 143143$ shows that we can indeed have $d = 7$.

3. Let the quadrilateral be $ABCD$ with $AC = 1001$ and $BD = n$. Note that $1002^2-1001^2 = 2003$ lies between 44^2 and 45^2. For $45 \leq n \leq 1001$, let M be the common midpoint of AC and BD. Initially, let B lie on AM, so that the degenerate quadrilateral $ABCD$ has perimeter 2002. Now rotate BD about M. When BD is perpendicular to AC, the perimeter of $ABCD$ will exceed 2004. Hence at some point during the rotation, the perimeter of $ABCD$ is exactly 2004. It follows that all values of n between 45 and 1001 inclusive are possible. For $2 \leq n \leq 44$, start with the rhombus $ABCD$ whose perimeter is less than 2004. Translate BD in the direction AC. When C is the midpoint of BD, both AB and AD are longer than 1001, so that the degenerate quadrilateral $ABCD$ has perimeter exceeding $2002 + n \geq 2004$. Hence at some point during the translation, the perimeter of $ABCD$ is exactly 2004. It follows that all values of n between 2 and 44 inclusive are possible. Finally, consider the case $n = 1$. Let M be the point of intersection of AC and BD. Then

$$\begin{aligned} 2004 &= AB + BC + CD + DA \\ &< MA + MB + MB + MC + MC + MD + MD + MA \\ &= 2(AC + BD) \\ &= 2004, \end{aligned}$$

   which is a contradiction. It follows that we cannot have $n = 1$.

4. Let the first three terms be $a_1 = a$, $a_2 = a + d$ and $a_3 = a + 2d$, where d is the common difference. Let $a_1^2 = a + kd$, $a_2^2 = a + md$ and $a_3^2 = a + nd$ for some positive integers k, m and n. Then $a^2 = a + kd$, $a^2 + 2ad + d^2 = a + md$ and $a^2 + 4ad + 4d^2 = a + nd$. It follows that $2ad + d^2 = nd - kd$ or $2a + d = m - k$, and $4ad + 4d^2 = nd - kd$ or $4a + 4d = n - k$. Eliminating d, we have $a = \frac{4m-n-3k}{4}$. Hence a is an integral multiple of $\frac{1}{4}$. Eliminating a, we have $d = \frac{n+k-2m}{2}$. Hence d is an integral multiple of $\frac{1}{2}$. Denote by $\{x\}$ the fractional part of x. We consider the following cases.
   (1) Let $\{a\} = 0$ and $\{d\} = \frac{1}{2}$. Every term of the progression is an integral multiple of $\frac{1}{2}$ but a_2^2 is not, a contradiction.

[1] Courtesy of Andy Liu.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Fall 2005.**[1]

1. A palindrome is a positive integer which reads the same from left to right and from right to left. For example, 1, 343 and 2002 are palindromes, while 2005 is not. Is it possible to find 2005 values of n such that both n and $n + 110$ are palindromes?

2. The extensions of the sides AB and DC of a convex quadrilateral $ABCD$ intersect at the point K. M and N are the midpoints of AB and CD, respectively. Prove that if $AD = BC$, then triangle MNK is obtuse.

3. Initially, there is a rook on each of the 64 squares of an 8×8 chessboard. Two rooks attack each other if they are in the same row or column, and there are no other rooks directly in between. In each move, one may take away any rook which attacks an odd number of other rooks still on the chessboard. What is the maximum number of rooks that can be removed?

4. Each side of a polygon is longer than 100 centimetres. Initially, two ants are on the same edge of the polygon, at a distance 10 centimetres from each other. They crawl along the perimeter of the polygon, maintaining the distance of 10 centimetres measured along a straight line.

   (a) Suppose the polygon is convex. Is it always possible for each point on the perimeter of the polygon to be visited by both ants?

   (b) Suppose the polygon is not necessarily convex. Is it always possible for each point on the perimeter of the polygon to be visited by at least one of the ants?

5. Determine the largest positive integer N for which there exist a unique triple (x, y, z) of positive integers such that $99x + 100y + 101z = N$.

6. There are 1000 pots each containing varying amounts of jam, not more than $\frac{1}{100}$-th of the total. Each day, exactly 100 pots are to be chosen, and from each chosen pot, the same amount of jam is eaten. Prove that it is possible to eat up all the jam in a finite number of days.

Note: The problems are worth 3, 5, 6, 2+4, 7 and 8 points respectively.

[1]Courtesy of Professor Andy Liu.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2005.**[1]

1. In triangle ABC, points D, E and F are the midpoints of BC, CA and AB respectively, while points L, M and N are the feet of the altitudes from A, B and C respectively. Prove that one can construct a triangle with the segments DN, EL and FM.

2. Each corner of a cube is labelled with a number. In each step, each number is replaced with the average of the labels of the three adjacent corners. All eight numbers are replaced simultaneously. After ten steps, all labels are the same as their respective initial values. Does it necessarily follow that all eight numbers are equal initially?

3. A segment of length 1 is cut into eleven shorter segments, each with length at most a. For what values of a will it be true that any three of the eleven segments can form a triangle, regardless of how the initial segment is cut?

4. A chess piece may start anywhere on a 15×15 chessboard. It can jump over 8 or 9 vacant squares either vertically or horizontally, but may not visit the same square twice. At most how many squares can it visit?

5. One of 6 coins is a fake. We do not know the weight of either a real coin or the fake coin, except that the real coins all weigh the same but different from the fake coin. Using a scale which shows the total weight of the coins being weighed, how can the fake coin be found in 3 weighings?

Note: The problems are worth 3, 3, 4, 4 and 5 points respectively.

[1]Courtesy of Professor Andy Liu.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Fall 2005.**

1. For which positive integers n can one find distinct positive integers $a_1, a_2, \ldots, a_n$ such that $\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_n}{a_1}$ is also an integer?

2. Each side of a polygon is longer than 100 centimetres. Initially, two ants are on the same edge of the polygon, at a distance 10 centimetres from each other. They crawl along the perimeter of the polygon, maintaining the distance of 10 centimetres measured along a straight line.

   (a) Suppose the polygon is convex. Is it always possible for each point on the perimeter of the polygon to be visited by both ants?

   (b) Suppose the polygon is not necessarily convex. Is it always possible for each point on the perimeter of the polygon to be visited by at least one of the ants?

3. Initially, there is a rook on each of the 64 squares of an 8×8 chessboard. Two rooks attack each other if they are in the same row or column, and there are no other rooks directly in between. In each move, one may take away any rook which attacks an odd number of other rooks still on the chessboard. What is the maximum number of rooks that can be removed?

4. On a circle are a finite number of red points. Each is labelled with a positive number less than or equal to 1. The circle is to be divided into three arcs so that each red point is in exactly one of them. The sum of the labels of all red points in each arc is computed. This is taken to be 0 if the arc contains no red points. Prove that it is always possible to find a division for which the sums on any two arcs will differ by at most 1.

5. In triangle ABC, $\angle A = 2\angle B = 4\angle C$. Their bisectors meet the opposite sides at D, E and F respectively. Prove that $DE = DF$.

6. A blackboard is initially empty. In each move, one may either add two 1s, or erase two copies of a number n and replace them with $n-1$ and $n+1$. What is the minimum number of moves needed to put 2005 on the blackboard?

Note: The problems are worth 3, 2+3, 5, 6, 7 and 8 points respectively.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2005.**[1]

1. Do there exist positive integers $a,\ b,\ n$ such that $n^2 < a^3 < b^3 < (n+1)^2$?
2. A segment of length $\sqrt{2}+\sqrt{3}+\sqrt{5}$ is given. Can a segment of length 1 be constructed using a straight-edge and a compass?
3. One of 6 coins is a fake. We do not know the weight of either a real coin or the fake coin, except that the eal coins all weigh the same, but different from the fake coin. Using a scale which shows the total weight of the coins being weighed, how can the fake coin be found in 3 weighings?
4. On all three sides of a right triangle ABC, external squares are constructed, their centres being $D,\ E$ and F. Prove that the ratio of the area of triangle DEF to the area of triangle ABC is
   (a) greater than 1;
   (b) at least 2.
5. A cube lies on the plane, with a letter A on its top face. In each move, it is rolled over one of its bottom edges onto the adjacent face. After a few moves, the cube returns to its initial position, again with the letter A on its top face. Is it possible for the letter A to have made a 90° turn?

Note: The problems are worth 3, 3, 4, 2+2 and 5 points respectively.

[1]Courtesy of Professor Andy Liu.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper[1] **Spring 2005.**

1. On the graph of a polynomial with integral coefficients are two points with integral coordinates. Prove that if the distance between these two points is integral, then the segment connecting them is parallel to the x-axis.

2. The altitudes AD and BE of triangle ABC meet at its orthocentre H. The midpoints of AB and CH are X and Y, respectively. Prove that XY is perpendicular to DE.

3. Baron Münchhausen's watch works properly, but has no markings on its face. The hour, minute and second hands have distinct lengths, and they move uniformly. The Baron claims that since none of the mutual positions of the hands is repeats twice in the period between 8:00 and 19:59, he can use his watch to tell the time during the day. Is his assertion true?

4. A 10×12 paper rectangle is folded along the grid lines several times, forming a thick 1×1 square. How many pieces of paper can one possibly get by cutting this square along the segment connecting
   (a) the midpoints of a pair of opposite sides;
   (b) the midpoints of a pair of adjacent sides?

5. In a rectangular box are a number of rectangular blocks, not necessarily identical to one another. Each block has one of its dimensions reduced. Is it always possible to pack these blocks in a smaller rectangular box, with the sides of the blocks parallel to the sides of the box?

6. John and James wish to divide 25 coins, of denominations 1, 2, 3, …, 25 kopeks. In each move, one of them chooses a coin, and the other player decides who must take this coin. John makes the initial choice of a coin, and in subsequent moves, the choice is made by the player having more kopeks at the time. In the event that there is a tie, the choice is made by the same player in the preceding move. After all the coins have been taken, the player with more kokeps wins. Which player has a winning strategy?

7. The squares of a chessboard are numbered in the following way. The upper left corner is numbered 1. The two squares on the next diagonal from top-right to bottom-left are numbered 2 and 3. The three squares on the next diagonal are numbered 4, 5 and 6, and so on. The two squares on the second-to-last diagonal are numbered 62 and 63, and the lower right corner is numbered 64. Peter puts eight pebbles on the squares of the chessboard in such a way that there is exactly one pebble in each column and each row. Then he moves each pebble to a square with a number greater than that of the original square. Can it happen that there is still exactly one pebble in each column and each row?

Note: The problems are worth 4, 5, 5, 2+4, 6, 6 and 8 points respectively.

[1]Courtesy of Andy Liu.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper[1] **Spring 2005.**

1. Anna and Boris move simultaneously towards each other, from points A and B respectively. Their speeds are constant, but not necessarily equal. Had Anna started 30 minutes earlier, they would have met 2 kilometers nearer to B. Had Boris started 30 minutes earlier instead, they would have met some distance nearer to A. Can this distance be uniquely determined?

2. Prove that one of the digits 1, 2 and 9 must appear in the base-ten expression of n or $3n$ for any positive integer n.

3. There are eight identical Black Queens in the first row of a chessboard and eight identical White Queens in the last row. The Queens move one at a time, horizontally, vertically or diagonally by any number of squares as long as no other Queens are in the way. Black and White Queens move alternately. What is the minimal number of moves required for interchanging the Black and White Queens?

4. M and N are the midpoints of sides BC and AD, respectively, of a square $ABCD$. K is an arbitrary point on the extension of the diagonal AC beyond A. The segment KM intersects the side AB at some point L. Prove that $\angle KNA = \angle LNA$.

5. In a certain big city, all the streets go in one of two perpendicular directions. During a drive in the city, a car does not pass through any place twice, and returns to the parking place along a street from which it started. If it has made 100 left turns, how many right turns must it have made?

Note: The problems are worth 3, 4, 5, 5 and 5 points respectively.

[1]Courtesy of Andy Liu.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper[1] **Spring 2005.**

1. On the graph of a polynomial with integral coefficients are two points with integral coordinates. Prove that if the distance between these two points is integral, then the segment connecting them is parallel to the x-axis.

2. A circle ω_1 with centre O_1 passes through the centre O_2 of a second circle ω_2. The tangent lines to ω_2 from a point C on ω_1 intersect ω_1 again at points A and B respectively. Prove that AB is perpendicular to O_1O_2.

3. John and James wish to divide 25 coins, of denominations 1, 2, 3, ..., 25 kopeks. In each move, one of them chooses a coin, and the other player decides who must take this coin. John makes the initial choice of a coin, and in subsequent moves, the choice is made by the player having more kopeks at the time. In the event that there is a tie, the choice is made by the same player in the preceding move. After all the coins have been taken, the player with more kokeps wins. Which player has a winning strategy?

4. For any function $f(x)$, define $f^1(x) = f(x)$ and $f^n(x) = f(f^{n-1}(x))$ for any integer $n \geq 2$. Does there exist a quadratic polynomial $f(x)$ such that the equation $f^n(x) = 0$ has exactly 2^n distinct real roots for every positive integer n?

5. Prove that if a regular icosahedron and a regular dodecahedron have a common circumsphere, then they have a common insphere.

6. A *lazy* rook can only move from a square to a vertical or a horizontal neighbour. It follows a path which visits each square of an 8×8 chessboard exactly once. Prove that the number of such paths starting at a corner square is greater than the number of such paths starting at a diagonal neighbour of a corner square.

7. Every two of 200 points in space are connected by a segment, no two intersecting each other. Each segment is painted in one colour, and the total number of colours is k. Peter wants to paint each of the 200 points in one of the colours used to paint the segments, so that no segment connects two points both in the same colour as the segment itself. Can Peter always do this if

   (a) $k = 7$;

   (b) $k = 10$?

Note: The problems are worth 4, 5, 5, 6, 7, 7 and 4+4 points respectively.

[1]Courtesy of Andy Liu.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper[1] **Spring 2005.**

1. The graphs of four functions of the form $y = x^2 + ax + b$, where a and b are real coefficients, are plotted on the coordinate plane. These graphs have exactly four points of intersection, and at each one of them, exactly two graphs intersect. Prove that the sum of the largest and the smallest x-coordinates of the points of intersection is equal to the sum of the other two.

2. The base-ten expressions of all the positive integers are written on an infinite ribbon without spacing: $1234567891011\ldots$. Then the ribbon is cut up into strips seven digits long. Prove that any seven digit integer will:

   (a) appear on at least one of the strips;

   (b) appear on an infinite number of strips.

3. M and N are the midpoints of sides BC and AD, respectively, of a square $ABCD$. K is an arbitrary point on the extension of the diagonal AC beyond A. The segment KM intersects the side AB at some point L. Prove that $\angle KNA = \angle LNA$.

4. In a certain big city, all the streets go in one of two perpendicular directions. During a drive in the city, a car does not pass through any place twice, and returns to the parking place along a street from which it started. If it has made 100 left turns, how many right turns must it have made?

5. The sum of several positive numbers is equal to 10, and the sum of their squares is greater than 20. Prove that the sum of the cubes of these numbers is greater than 40.

Note: The problems are worth 3, 3+1, 4, 4 and 5 points respectively.

[1]Courtesy of Andy Liu.

International Mathematics
TOURNAMENT OF THE TOWNS

Solutions[1] A-level, Juniors — Fall, 2006

1. Let $2a$ be the length of a side of a regular polygon, while r and R be the radii of its inscribed and circumscribed circles. Since the radius of the inscribed circle is perpendicular to the side of the polygon and touches it at its midpoint, then $a^2 + r^2 = R^2$. Therefore, the area of the ring between the circles is equal to $\pi(R^2 - r^2) = \pi a^2$. This implies the statement of the problem.

2. Counterexample. Consider a company: a host with three sons and three guests. The guests do not know each other, the host knows all the guests, while each son knows only two guests. No two sons know the same pair of the guests. It is clear, that the guests chords intersect the host chord in three distinct points; one point is between two others. So, the guest chord through this point separates two other guest chords. Therefore, the chord of the son who knows only two latter guests must intersect the guest chord in between. Contradiction.

3. a) Let S be a magic sum. Then

$$(a+b+c)+(a+d+g)+(c+f+i)+(g+h+i) = 4S = 2(b+e+h)+2(d+e+f). \quad (1)$$

Subtracting $(b+d+f+h)$ from both sides, we get $2(a+c+g+i)=b+d+f+h+4e$.

b) Let us notice that $a+i = c+g = b+h = d+f = S-e$. Combining with (1) we get $4(S-e) = 2(S-e)+4e$; therefore, $S = 3e$. Next, let us prove

$$2(a^2+c^2+g^2+i^2) = b^2+d^2+f^2+h^2+4e^2. \quad (2)$$

We have $a+c = S-b = h+e$, $c+i = S-f = d+e$, $g+i = S-h = b+e$, $a+g = S-d = f+e$. In addition, we have

$$ac+ci+ag+gi = (a+i)(c+g) = (S-e)^2 = 2e(S-e) = e(b+d+f+h).$$

Therefore,

$$\begin{aligned}
&2(a^2+c^2+g^2+i^2) = \\
&(a+c)^2+(c+i)^2+(a+g)^2+(g+i)^2-2(ac+ci+ag+gi) = \\
&(h+e)^2+(d+e)^2+(f+e)^2+(b+e)^2-2e(b+d+f+h) = \\
&b^2+d^2+f^2+h^2+4e^2.
\end{aligned}$$

To finish the proof let us notice that the statement of b) holds if we increase each entry of the table by the same value. Really,

$$\begin{aligned}
&2((a+t)^3+(c+t)^3+(g+t)^3+(i+t)^3) = \\
&\quad 2((a^3+c^3+g^3+i^3)+3t(a^2+c^2+g^2+i^2)+3t^2(a+c+g+i)+4t^3) = \\
&b^3+d^3+f^3+h^3+4e^3+3t(b^2+d^2+f^2+h^2+4e^2)+3t^2(b+d+f+h+4e)+8t^3 = \\
&\qquad (b+t)^3+(d+t)^3+(f+t)^3+(h+t)^3+4(e+t)^3.
\end{aligned}$$

Therefore, it is enough to consider the case $e = 0$. However, in this case the statement is obvious, since $a+i = c+g = b+h = d+f = 2e = 0$.

[1]by L. Mednikov, A. Shapovalov

International Mathematics

TOURNAMENT OF THE TOWNS.
Solutions

Junior O-Level Paper **Fall 2006**[1]

1. We claim that the sum of the numbers in Mary's notebook is equal to the product of the two numbers originally on the blackboard. We use induction on the number n of steps for Mary to reduce one of the numbers to 0. For $n = 1$, the two numbers on the blackboard must be equal to each other. In recording the square of the smaller number, Mary is in fact recording the product of the two numbers. Suppose the claim holds for some $n \geq 1$. Let the original numbers be x and y with $x < y$. Then Mary records x^2 in her notebook and replaces y by $y-x$. By the induction hypothesis, the sum of the remaining numbers in her notebook is equal to $x(y-x)$, so that the sum of all the numbers in her notebook is equal to $x^2+x(y-x) = xy$.

2. (a) Ask each of the three people: "Are you a Normal?" Since the Knight and the Knave will give opposite answers, the three answers consist of a matching pair and an odd one out. If the odd answer is "Yes", the replier is the Knight, and if the odd answer is "No", the replier is the Knave. From this person, we can learn the identity of all three people.

   (b) The first Normal will act as though he is a Knight while the second Normal will act as though he is a Knave. Then we cannot tell the difference between the first Normal and the Knight, nor between the second Normal and the Knave.

3. Suppose a number is expressible in the form $a^2 - b^2 = (a + b)(a - b)$. If a and b are of the same parity, then the product is divisible by 4. If they are of opposite parity, then the product is odd. Conversely, a number of the form $4n$ may be expressed as $(n + 1)^2 - (n - 1)^2$ while a number of the form $2n + 1$ may be expressed as $(n + 1)^2 - n^2$. Hence a number is not expressible in the form $a^2 - b^2$ if and only if it is of the form $4n + 2$. The only way in which a product takes the form $4n + 2$ is when exactly one of the factors is of that form, and the others are odd.

   (a) Suppose an even number of the 2007 numbers is of the form $4n+2$. Then there exists at least one number not of this form, and we choose this number. Suppose an odd number of the 2007 numbers is of the form $4n + 2$. Then we choose any of these. Among the remaining 2006 numbers, there will not be exactly one number of the form $4n+2$. Hence their product is expressible in the form $a^2 - b^2$.

   (b) If there is a number of the form $4n + 2$ other than 2006, then any of the other 2005 numbers may be chosen so that the product of the remaining 2006 numbers will not be of the form $4n + 2$. Hence the choice will not be unique. It follows that 2006 is the only number of the form $4n + 2$, and it must be the chosen number.

[1]Courtesy of Professor Andy Liu.

International Mathematics
TOURNAMENT OF THE TOWNS

Solutions[1] A-level, Seniors **Fall, 2006**

1. Counterexample. Consider a company: a host, his three sons and three guests. The guests do not know each other, the host knows all the guests, while each son knows only two guests. No two sons know the same pair of the guests. It is clear, that guests chords intersect the host chord in three distinct points; one point is between the others two. Further, this two guest chords lie on the different sides of the guest chord in between. Then the chord of the son who knows only these two guests must intersect the middle chord. Contradiction.

2. Consider triangle $A_1B_1C_1$. Let A_2 be intersection point of bisectors of exterior angles B_1 and C_1, while B_2 and C_2 be intersections of bisectors of exterior angles A_1 and C_1, and A_1 and B_1 respectively. Notice, that A_2 is equidistant from side B_1C_1, extension of side A_1B_1 and extension of side A_1C_1. Therefore, A_2 belongs to bisector A_1A; moreover, A_1A_2, B_1B_2, C_1C_2 are altitudes of triangle $A_2B_2C_2$. Let us prove that triangle $A_2B_2C_2$ and triangleABC coincide. Assume that A_2 is outside of triangle ABC. Note, that ray A_2B_2 intersects side AB of triangle ABB_1 at C_1 and does not intersect side AB_1 since sides AB and AB_1 are separated by A_2A_1. Therefore, B_2 is inside of triangle ABC. In the same way C_2 is inside of triangle ABC. However, segment B_2C_2 must intersect side BC at point A_1. Contradiction.

3. Let us assume that a is rational. Then a is periodic decimal fraction with period k. Then starting from some place the digits occupying the positions $k,\ 10k, \dots, 10^m k, \dots$ coincide. On the other hand, these are consecutive digits of representation $\sqrt{k}$. However, an irrational number cannot be represented by periodical fraction. Therefore, a is irrational.

4. Answer: no.

   The total sum of volumes of the pyramids with bases on the bottom base of the prism does not exceed one third of the prism volume. The same is true for the pyramids with bases on the top base of the prism. Therefore, the total sum of volumes of all the pyramids is less than the volume of the prism. Contradiction.

5. Let us consider $n = p(p-1) - 1$, where p is an odd prime number. Notice, that b_{n+1} is not divisible by p. Really, in the corresponding sum only denominators of the fractions $\dfrac{1}{p}, \dfrac{1}{2p}, \dots, \dfrac{1}{(p-1)p}$ are divisible by p.

   However, by regrouping the fractions in the following way:
$$\frac{1}{p} + \frac{1}{(p-1)p} = \frac{1}{(p-1)}, \quad \frac{1}{p} + \frac{1}{(p-2)p} = \frac{1}{2(p-2)}$$
   etc., we see that no factor of b_{n+1} is divisible by p.

   We have
$$\frac{a_n}{b_n} = \frac{a_{n+1}}{b_{n+1}} - \frac{1}{(p-1)p} = \frac{(a_{n+1}(p-1)p - b_{n+1})}{b_{n+1}(p-1)p}.$$

[1]by L. Mednikov, A. Shapovalov

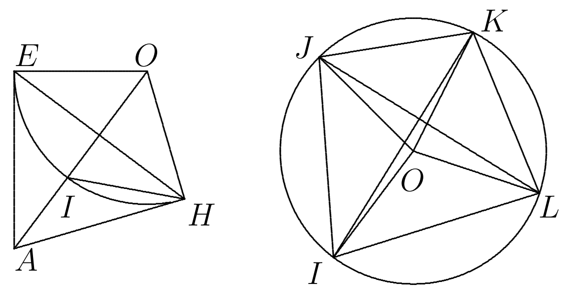
International Mathematics

TOURNAMENT OF THE TOWNS.
Solutions

Senior O-Level Paper **Fall 2006**[1]

1. We claim that the sum of the numbers in Mary's notebook is equal to the product of the three numbers originally on the blackboard. We use induction on the number n of steps for Mary to reduce one of the numbers to 0. For $n = 1$, one of the three numbers on the blackboard must be equal to 1 and is reduced to 0. In recording the product of the other two numbers, Mary is in fact recording the product of all three numbers. Suppose the claim holds for some $n \geq 1$. Let the original numbers be x, y and z. By symmetry, we may assume that Mary records xy in her notebook and replaces z by $z - 1$. By the induction hypothesis, the sum of the remaining numbers in her notebook is equal to $xy(z - 1)$, so that the sum of all the numbers in her notebook is equal to $xy + xy(z - 1) = xyz$.

2. Let O be the incentre of $ABCD$. Let AO intersect the incircle of $ABCD$ at I. Let $\angle AOH = \angle AOE = 2\alpha$. Since $\angle AHO = 90° = \angle AEO$, A, E, O and H are concyclic, so that $\angle AHE = \angle AOE = 2\alpha$. We have $\angle OAH = 180° - \angle AOH - \angle AHO = 90° - 2\alpha$ and since $OH = OI$, $\angle OIH = \frac{1}{2}(180° - \angle IOH) = 90° - \alpha$. It follows that $\angle AHI = \angle OIH - \angle OAH = \alpha = \frac{1}{2}\angle AHO$. Hence I is the incentre of triangle HAE. Similarly, the respective incentres J, K and L of triangles EBF, FCG and GDH all lie on the incircle of $ABCD$. Let $\angle BOE = \angle BOF = 2\beta$, $\angle COF = \angle COG = 2\gamma$ and $\angle DOG = \angle DOH = 2\delta$. Then $\angle IOJ + \angle KOL = \alpha + \beta + \gamma + \delta = 180°$. Now $\angle ILJ + \angle KIL = \frac{1}{2}(\angle IOJ + \angle KOL) = 90°$. Hence IK and JL are perpendicular to each other.



3. We can replace each number by the remainder obtained when it is divided by 4. Thus we have 1003^2 copies of each of 0, 1, 2 and 3. Divide the board into 1003^2 2×2 subboards. Each subboard may contain at most one 0 and at most one 2. Since we have exactly as many copies of each number as we have subboards, there is exactly one 0 and exactly one 2 in each subboard. The remaining two cells in each subboard must both contain copies of 1 or both contain copies of 3. However, this is impossible as we have an odd number of copies of each of 1 and 3.

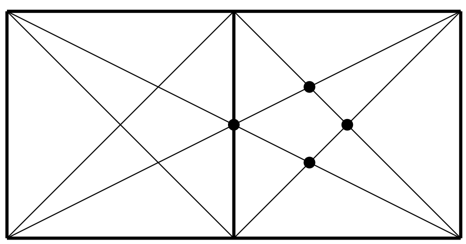
[1]Courtesy of Professor Andy Liu.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Spring 2006.**

1. A pool table has a shape of a 2×1 rectangle; there are six pockets: one in each corner, and one in the midpoint of each of the long sides of the table. What is the minimal number of balls one needs to put on the table so that every pocket lies on the same line with at least two balls? (Consider pockets and balls as points.) (B.R. Frenkin)

   Answer. 4 balls.

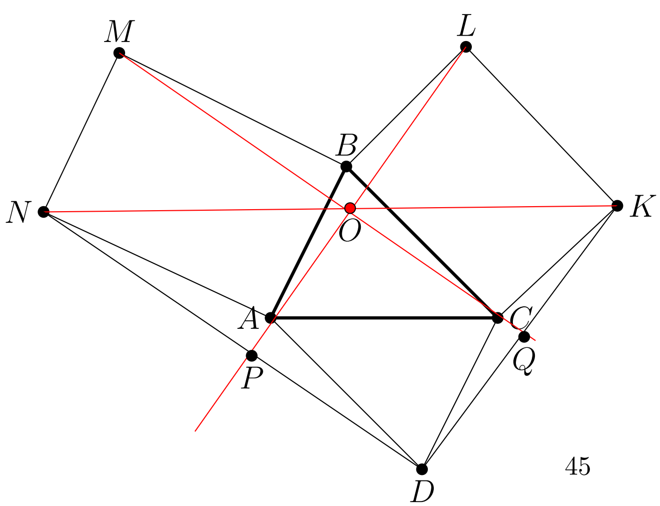


Solution. The example for 4 balls is given on the picture. Let us show that 3 balls are not enough. Straight line passing through two balls inside the rectangle intersects its boundary at exactly two points. We have 6 pockets, so we need at least 3 straight lines. Three balls creates three lines if and only if these balls form a triangle. However, all possible straight lines are drawn on the picture and none of them form a triangle with vertices inside the pool table.

2. Prove that one can find 100 pairs of integers with the following property: in the decimal representation of each integer, each digit is greater or equal to 6, and the product of the two integers in the pair is also an integer whose decimal representation has no digits less than 6. (S.I. Tokarev, A.V.Shapovalov)

   Solution. All pairs (7, 9...97) are in use to our problem since their products are equal to 67...79.

3. Assume an acute triangle ABC is given. Two equal rectangles, $ABMN$ and $LBCK$, are drawn on the sides AB and BC in the outside. Given that $AB = LB$, prove that the straight lines AL, NK, and MC are concurrent. (A.Gavriluk)

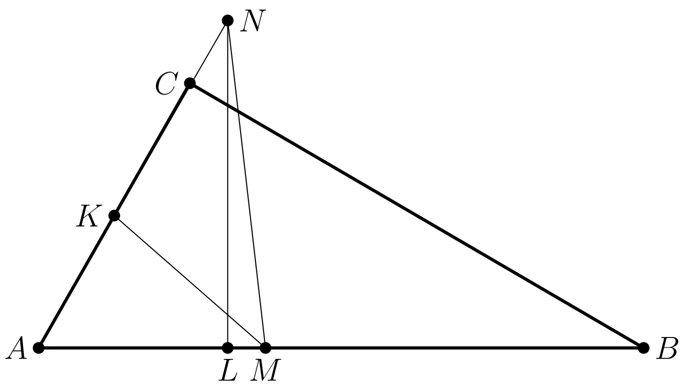


Solution 2. Draw a parallelogram $ABCD$. Then $ALKD$ and $CDNM$ are also parallelograms. Isosceles $\triangle CBM$ can be obtained from $\triangle ABL$ by the rotation by $90°$ and homothety, thus $CM \perp AL$, but then and $CM \perp KD$. Continuation of MC, height CQ in the isosceles triangle KCD is its median, consequently CM is the perpendicular from the midpoint of KD. Similarly AL is the perpendicular from the midpoint of ND. Parallelogram $OPDQ$ is a rectangle, hence triangle KDN is right-angled, and perpendiculars from the midpoints of its legs pass through the midpoint of the hypotenuse KN.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Spring 2006.**

1. In triangle ABC angle A is equal to $60°$. The perpendicular from the midpoint of side AB intersects AC at the point N. The perpendicular from the midpoint of side AC intersects AB at the point M. Prove that $CB = MN$. (R.G. Zhenodarov)



SOLUTION. By the property of the perpendicular from the midpoint $NA = NB$, thus triangle ANB is isosceles. Angle A is equal to $60°$, this means that triangle ANB is equilateral and $AN = AB$. Similarly, triangle AMC is equilateral, $AM = AC$. Triangles ACB and AMN are equal according to the equality of two sides and angle between them. Hence $BC = MN$.

2. Consider an $n \times n$ table. In each square of its first column someone has written the number 1, in each square of the second column, number 2, and so on. Then someone erased the numbers on the diagonal which connects top-left with bottom-right angle of the table. Prove that the sum of the numbers above the diagonal is twice the sum of the numbers under it. (S.A.Zaitsev)

SOLUTION 1. For each square on the diagonal compare the sums of the numbers situated to the left of it and situated above it. If the square is situated at the intersection of the k-th row and the k-th column the sum to the left is equal to $1 + 2 + \cdots + (k-1) = k(k-1)/2$, while the sum of the numbers above it is equal to k(k-1), that is two times more. Hence the sum of all numbers above the diagonal is two times more than the sum of the numbers situated to the left of it.

SOLUTION 2.

	2	3	4	...	n
1		3	4	...	n
1	2		4	...	n
1	2	3		...	n
...	...	...	...		...
1	2	3	4	...	

	1	2	3	...	$n-1$
1		1	2	...	$n-2$
1	2		1	...	$n-3$
1	2	3		...	$n-4$
...	...	...	...		...
1	2	3	4	...	

SOLUTION 3. In the original table (left picture) there are $(n-1)$ ones, $(n-2)$ twos, $(n-3)$ threes and so on. Let us subtract from each number above the diagonal the number symmetrical to it with respect to the diagonal. We get the picture to the right. It has equal numbers situated on the diagonals above the main one and parallel to it: $(n-1)$ ones, $(n-2)$ twos, $(n-3)$ threes and so on. We decreased the upper sum by the lower sum and got the

International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Spring 2006.**

1. Assume a convex polygon with 100 vertices is given. Prove that one can chose 50 points inside the polygon in such a way that every vertex lies on a line passing through two of the chosen points. (B.R.Frenkin)

   Solution. Enumerate vertices of the polygon in a clockwise order: 1, …, 100. Consider polygon consisting of 10 vertices: 1, 2, 21, 22, 41, 42, 61, 62, 81, 82. Its vertices lay on the 5 straight lines 1-22, 21-42, 41-62, 61-82, 81-2, which are given by 5 points of intersections (the first straight line with the second one, the second one with the third one … the fifth one with the second one, it is evident, that all these points are different). Repeat this for the decagons with numbers of vertices that can be obtained from the numbers of considered decagon by adding 2, 4, …, 18. This problem has lots of different solutions.

2. Do there exist positive integers n and k such that the decimal representation of 2^n contains the decimal representation of 5^k as its leftmost part, while the decimal representation of 5^n contains the decimal representation of 2^k as its leftmost part? (G.A.Galperin)

   Answer: No, they doesnt exist.

   Solution If for some positive integer n the number 2^n starts by 5^k and the number 5^n by 2^k then this means that $5^k \times 10^s < 2^n < (5^k + 1) \times 10^s$ and $2^k \times 10^l < 5^n < (2^k + 1) \times 10^l$, thus $10^{k+l+s} < 10^n < 10^{k+l+s+1}$, which is impossible. (Last inequality $10^n < 10^{k+l+s+1}$ is true, because $5^k + 1 < 2 \times 5^k$ and $2^k + 1 < 5 \times 2^k$).

3. Consider the polynomial $P(x) = x^4 + x^3 - 3x^2 + x + 2$. Prove that for every positive integer k, the polynomial $P(x)^k$ has at least one negative coefficient. (M.I.Malkin)

   Solution 1. Observe that for any polynomial $P(x)$ its value in the point $x = 1$ is equal to the sum of all coefficients. Consequently, the sum of the coefficients of the polynomial $P(x)^n$ is equal to $P(1)^n = (1+1-3+1+2)^n = 2^n$. But the free term of $P(x)^n$ is equal to $P(0)^n = 2^n$, while the coefficient at x^{4n} is equal to 1, and their sum is already $2^n + 1$. Hence one of the remaining coefficients of $P(x)^n$ is negative.

   Solution 2. The coefficient at x^3 for the polynomial $P(x)^n$ can be obtained by adding n items $2^{n-1}x^3$ and $n(n-1)$ items $-3x^2 \times x \times 2^{n-2}$, consequently this coefficient is equal

   $$n \cdot 2^{n-1} - 3n(n-1)2^{n-2} = 2^{n-2}(-3n^2 + 5n) = n \cdot 2^{n-2}(-3n + 5),$$

   which is negative number for an arbitrary integer $n \geq 2$.

   Solution 3. Observe that $P(0)^n = P(1)^n = 2^n$. But any polynomial F with positive coefficients is strongly monotonic when $x > 0$ (i.e. $x > y > 0 \implies F(x) > F(y) > 0$). This means that polynomial $P(x)^n$ has at least one negative coefficient.

4. Consider a triangle ABC, take the angle bisector AA', and assume given a point X on the interval AA'. Assume that the line BX intersects the line AC in a point denoted B', while the line CX intersects the line AB in a point denoted C'. Assume also that the intervals $A'B'$ and CC' meet in a point denoted P, and the intervals $A'C'$ and BB' meet in a point denoted Q. Prove that the angles PAC and QAB are equal. (M.A. Volchkevich)

International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Spring 2006.**

1. Consider a convex polyhedron with 100 edges. All its vertices were cut off near themselves using sharp knives planes (it was done in such a way that these planes have no intersections inside or on the boundary of the polyhedron). Find out for the resulting polyhedron:

   (a) number of vertices,

   (b) number of edges.

   (G.A.Galperin)

   Answer. a) 200; b) 300.

   Solution. Observe that there are two vertices of the new polyhedron on each edge of the given one, and there are 3 edges starting in each vertex of the new polyhedron. Consequently, there are $2 \cdot 100 = 200$ vertices and $\frac{100 \cdot 3}{2} = 300$ edges in the resulting polyhedron.

2. Is it possible to find two such functions $p(x)$ and $q(x)$ that $p(x)$ is an even function, while $p\big(q(x)\big)$ is an odd function (other than identically equal to 0) ? (A.D. Blinkov, V.M. Gurovic)

   Answer. Yes, it is possible.

   Solution. Consider functions $p(x) = \cos x$ and $q(x) = \frac{\pi}{2} - x$. It is evident that $p(x)$ is an even function, while $p(q(x)) = \sin x$ is odd an odd function. There are also lots of different solutions.

3. Consider an arbitrary number $a > 0$. We know that the inequality $10 < a^x < 100$ has exactly 5 positive integer solutions. How many solutions in positive integers may have the inequality $100 < a^x < 1000$?

   Find all possibilities. (A.K. Tolpygo)

   Answer. 4,5 or 6.

   Solution. The inequality $10 < a^x < 100$ can be rewritten as $10 < 10^{bx} < 100$ or $1 < bx < 2$. Similarly $100 < ax < 1000$ is equivalent to $2 < bx < 3$. If n is the minimal integer solution of $1 < bx < 2$, then $b(n-1) < 1 < bn$ and $b(n+4) < 2 < b(n+5)$. Summing up the first inequality with itself and with the second one we obtain $b(2n-2) < 2 < b(2n)$ and $b(2n+3) < 3 < b(2n+5)$. Hence the inequality $2 < bx < 3$ has from 4 and up to 6 integer solutions ($2n, \ldots, 2n+3$ are always solutions, while $2n-1$ and $2n+4$ may be and may not). Actually, all 3 cases are possible:

   - $b = \frac{5}{23}$; solutions of the first inequality are 5, 6, 7, 8, 9, solutions of the second one are 10, 11, 12, 13.
   - $b = \frac{5}{26}$; solutions of the first inequality are 6, 7, 8, 9, 10, solutions of the second one are 11, 12, 13, 14, 15.
   - $b = \frac{5}{27}$; solutions of the first inequality are 6, 7, 8, 9, 10, solutions of the second one are 11, 12, 13, 14, 15, 16.

International Mathematics

TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Fall 2007.**

1. Let $ABCD$ be a rhombus. Let K be a point on the line CD, other than C or D, such that $AD = BK$. Let P be the point of intersection of BD with the perpendicular bisector of BC. Prove that A, K and P are collinear.
2. (a) Each of Peter and Basil thinks of three positive integers. For each pair of his numbers, Peter writes down the greatest common divisor of the two numbers. For each pair of his numbers, Basil writes down the least common multiple of the two numbers. If both Peter and Basil write down the same three numbers, prove that these three numbers are equal to each other.
   (b) Can the analogous result be proved if each of Peter and Basil thinks of four positive integers instead?
3. Michael is at the centre of a circle of radius 100 metres. Each minute, he will announce the direction in which he will be moving. Catherine can leave it as is, or change it to the opposite direction. Then Michael moves exactly 1 metre in the direction determined by Catherine. Does Michael have a strategy which guarantees that he can get out of the circle, even though Catherine will try to stop him?
4. Two players take turns entering a symbol in an empty cell of a $1 \times n$ chessboard, where n is an integer greater than 1. Aaron always enters the symbol X and Betty always enters the symbol O. Two identical symbols may not occupy adjacent cells. A player without a move loses the game. If Aaron goes first, which player has a winning strategy?
5. Attached to each of a number of objects is a tag which states the correct mass of the object. The tags have fallen off and have been replaced on the objects at random. We wish to determine if by chance all tags are in fact correct. We may use exactly once a horizontal lever which is supported at its middle. The objects can be hung from the lever at any point on either side of the support. The lever either stays horizontal or tilts to one side. Is this task always possible?
6. The audience arranges n coins in a row. The sequence of heads and tails is chosen arbitrarily. The audience also chooses a number between 1 and n inclusive. Then the assistant turns one of the coins over, and the magician is brought in to examine the resulting sequence. By an agreement with the assistant beforehand, the magician tries to determine the number chosen by the audience.
   (a) Prove that if this is possible for some n, then it is also possible for $2n$.
   (b) Determine all n for which this is possible.
7. For each letter in the English alphabet, William assigns an English word which contains that letter. His first document consists only of the word assigned to the letter A. In each subsequent document, he replaces each letter of the preceding document by its assigned word. The fortieth document begins with "Till whatsoever star that guides my moving." Prove that this sentence reappears later in this document.

Note: The problems are worth 5, 3+3, 6, 7, 8, 4+5 and 9 points respectively.

International Mathematics

TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2007.**

1. Black and white checkers are placed on an 8×8 chessboard, with at most one checker on each cell. What is the maximum number of checkers that can be placed such that each row and each column contains twice as many white checkers as black ones?

2. Initially, the number 1 and a non-integral number x are written on a blackboard. In each step, we can choose two numbers on the blackboard, not necessarily different, and write their sum or their difference on the blackboard. We can also choose a non-zero number of the blackboard and write its reciprocal on the blackboard. Is it possible to write x^2 on the blackboard in a finite number of moves?

3. D is the midpoint of the side BC of triangle ABC. E and F are points on CA and AB respectively, such that BE is perpendicular to CA and CF is perpendicular to AB. If DEF is an equilateral triangle, does it follow that ABC is also equilateral?

4. Each cell of a 29×29 table contains one of the integers 1, 2, 3, $\ldots$, 29, and each of these integers appears 29 times. The sum of all the numbers above the main diagonal is equal to three times the sum of all the numbers below this diagonal. Determine the number in the central cell of the table.

5. The audience chooses two of five cards, numbered from 1 to 5 respectively. The assistant of a magician chooses two of the remaining three cards, and asks a member of the audience to take them to the magician, who is in another room. The two cards are presented to the magician in arbitrary order. By an arrangement with the assistant beforehand, the magician is able to deduce which two cards the audience has chosen only from the two cards he receives. Explain how this may be done.

Note: The problems are worth 3, 4, 4, 5 and 5 points respectively.

International Mathematics

TOURNAMENT OF THE TOWNS

Junior P-Level Paper **Fall 2007**

1 [**1**] (from The Good Soldier Švejk) Senior military doctor Bautze exposed *abccc* malingerers among *aabbb* draftees who claimed not to be fit for the military service. He managed to expose all but one draftees. (He would for sure expose this one too if the lucky guy was not taken by a stroke at the very moment when the doctor yelled at him "Turn around !. . . ") Now many malingerers were exposed by the vigilant doctor?

Each digit substitutes a letter. The same digits substitute the same letters, while distinct digits substitute distinct letters.

Solution Problem is equivalent to: $aabbb = abccc + 1$,

Then, $c = 9$ (otherwise, nothing is carried out to the second digit); therefore, $b = 0$ and $a = 1$.

2 [**2**] Let us call a triangle "almost right angle triangle" if one of its angles differs from $90°$ by no more than $15°$. Let us call a triangle "almost isosceles triangle" if two of its angles differs from each other by no more than $15°$. Is it true that that any acute triangle is either "almost right angle triangle" or "almost isosceles triangle"?

Answer: Yes, it is true.

Solution. Let $a \geq b \geq c$ be angles of a triangle. Let us assume that a triangle in not "almost isosceles". Then $a - b > 15°$ and $b - c > 15°$ (so $a - c > 30°$). Then $180° = a + b + c < a + a + 15° + a + 30°$ or $3a > 225°$; so $a > 75°$. That implies that the triangle is "almost right angle triangle".

3 [**2**] A triangle with sides a, b, c is folded along a line ℓ so that a vertex C is on side c. Find the segments on which point C divides c, given that the angles adjacent to ℓ are equal.

Solution. Let ABC be a given triangle. It is clear that the folding along line is equivalent to the mirror reflection with respect to this line. Let point C' (on side AB) be an image of vertex C under mirror reflection with respect to line ℓ; thus, CC' is perpendicular to ℓ. Let M and N be points of intersection of ℓ with sides AC and CB respectively. Since angles adjacent to ℓ are equal then $\angle CMN = \angle CNM$ and triangle CMN is isosceles. Therefore, line CC' is an altitude of isosceles triangle. Then, CC' is also a bisector of $\angle C$. By a property of bisector we have $AC'/C'B = AC/CB$ or $AC' - (c - AC') = b/a$ and we get $C'B = ac/(a + b)$.

4 [**3**] From the first 64 positive integers are chosen two subsets with 16 numbers in each. The first subset contains only odd numbers while the second one contains only even numbers. Total sums of both subsets are the same. Prove that among all the chosen numbers there are two whose sum equals 65.

Solution. Let us pair the first 64 positive integers: $(i, 65 - i)$. It is easy to see that we have one-to-one correspondence between all odd and all even numbers of $\{1, \ldots, 64\}$. Let us pick up any $F \subset \{1, 3, \ldots, 63\}$ consisting of 16 numbers. Let us also pick up any $S \subset \{2, 4, \ldots, 64\}$

International Mathematics

TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2007.**

1. Pictures are taken of 100 adults and 100 children, with one adult and one child in each, the adult being the taller of the two. Each picture is reduced to $\frac{1}{k}$ of its original size, where k is a positive integer which may vary from picture to picture. Prove that it is possible to have the reduced image of each adult taller than the reduced image of every child.

2. Initially, the number 1 and two positive numbers x and y are written on a blackboard. In each step, we can choose two numbers on the blackboard, not necessarily different, and write their sum or their difference on the blackboard. We can also choose a non-zero number of the blackboard and write its reciprocal on the blackboard. Is it possible to write on the blackboard, in a finite number of moves, the number
   (a) x^2;
   (b) xy?

3. Give a construction by straight-edge and compass of a point C on a line ℓ parallel to a segment AB, such that the product $AC \cdot BC$ is minimum.

4. The audience chooses two of twenty-nine cards, numbered from 1 to 29 respectively. The assistant of a magician chooses two of the remaining twenty-seven cards, and asks a member of the audience to take them to the magician, who is in another room. The two cards are presented to the magician in an arbitrary order. By an arrangement with the assistant beforehand, the magician is able to deduce which two cards the audience has chosen only from the two cards he receives. Explain how this may be done.

5. A square of side length 1 centimetre is cut into three convex polygons. Is it possible that the diameter of each of them does not exceed
   (a) 1 centimetre;
   (b) 1.01 centimetres;
   (c) 1.001 centimetres?

Note: The problems are worth 3, 2+2, 4, 4 and 1+2+2 points respectively.

International Mathematics

TOURNAMENT OF THE TOWNS

Senior P-Level Paper **Fall 2007.**

1 [**1**] A straight line is colored with two colors. Prove that there are three points A, B, C of the same color such that $AB = BC$.

Solution. Consider any two points of the same color; say white, W and W'; let $2d$ be the distance between them and W' be to the right of W. Consider two new points, on the distance $2d$ to the left of W and $2d$ to the of right of W'. Both of them must be black; otherwise, the problem is solved. Now, consider a midpoint between W and W'. It must be black as well; otherwise, the problem is solved. However, in this case this midpoint is equidistant from both black points. The statement is proven.

2 [**2**] A student did not notice multiplication sign between two three-digit numbers and wrote it as a six-digit number. Result was 7 times more that it should be. Find these numbers.

Solution. Problem is equivalent to find two 3-digit numbers u, v, so that $1000u + v = 7u \times v$. Therefore, $u = v/(7v - 1000)$. Since $100 \geq u \geq 999$ then $100 \geq v/(7v - 1000) \geq 999$. Solving the last inequality we get $v = 143$. Then we find corresponding $u = 143$.

3 [**3**] Two players in turns color the squares of a 4×4 grid, one square at the time. Player loses if after his move a square of 2×2 is colored completely. Which of the players has the winning strategy, First or Second?

Solution. Second Player has a strategy. On each move of First Player, Second Player corresponding move is two squares down (or two squares up) in the same column. It is easy to see that if First Player has a move, so does Second Player.

4 [**3**] There three piles of pebbles, containing 5, 49, and 51 pebbles respectively. It is allowed to combine any two piles into a new one or to split any pile consisting of even number of pebbles into two equal piles. Is it possible to have 105 piles with one pebble in each in the end?

Answer: it is not possible.

Solution. It is clear that the first operation can be one of the following:

a). Combining 5 and 49;

b). Combining 5 and 51;

c). Combining 49 and 51.

Let us consider case a). After the first opearation is applied we have two piles: 54 and and 51. Note, that both piles are multiple of 3. If a number is multiple of 3 then dividing it by 2 (coprime with 3) results in a number that is multiple of 3. Adding two numbers multiple of 3 results in a number that is multiple of 3. Therefore, no matter which operation we apply from now on we can get only piles that all are multiple of 3. But 1 is not a multiple of 3. Therefore, in case a) it is impossible to get piles with one pebble in each.

Cases b) and c) are dealt in similar way (piles in case b are multiple of 7 while in case c are multiple of 5).

International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper[1] **Spring 2007.**

1. Let n be a positive integer. In order to find the integer closest to $\sqrt{n}$, Mary finds a^2, the closest perfect square to n. She thinks that a is then the number she is looking for. Is she always correct?

2. K, L, M and N are points on sides AB, BC, CD and DA, respectively, of the unit square $ABCD$ such that KM is parallel to BC and LN is parallel to AB. The perimeter of triangle KLB is equal to 1. What is the area of triangle MND?

3. Anna's number is obtained by writing down 20 consecutive positive integers, one after another in arbitrary order. Bob's number is obtained in the same way, but with 21 consecutive positive integers. Can they obtain the same number?

4. Several diagonals (possibly intersecting each other) are drawn in a convex n-gon in such a way that no three diagonals intersect in one point. If the n-gon is cut into triangles, what is the maximum possible number of these triangles?

5. Find all (finite) increasing arithmetic progressions, consisting only of prime numbers, such that the number of terms is larger than the common difference.

6. In the quadrilateral $ABCD$, $AB = BC = CD$ and $\angle BMC = 90°$, where M is the midpoint of AD. Determine the acute angle between the lines AC and BD.

7. Nancy shuffles a deck of 52 cards and spreads the cards out in a circle face up, leaving one spot empty. Andy, who is in another room and does not see the cards, names a card. If this card is adjacent to the empty spot, Nancy moves the card to the empty spot, without telling Andy; otherwise nothing happens. Then Andy names another card and so on, as many times as he likes, until he says "stop."

   (a) Can Andy guarantee that after he says "stop," no card is in its initial spot?

   (b) Can Andy guarantee that after he says "stop," the Queen of Spades is not adjacent to the empty spot?

Note: The problems are worth 3, 4, 5, 6, 7, 8 and 5+5 points respectively.

[1] Courtesy of Professor Andy Liu

International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper[1] **Spring 2007.**

1. The sides of a convex pentagon are extended on both sides to form five triangles. If these triangles are congruent to one another, does it follow that the pentagon is regular?

2. Two 2007-digit numbers are given. It is possible to delete 7 digits from each of them to obtain the same 2000-digit number. Prove that it is also possible to insert 7 digits into the given numbers so as to obtain the same 2014-digit number.

3. What is the least number of rooks that can be placed on a standard 8×8 chessboard so that all the white squares are attacked? (A rook also attacks the square it is on, in addition to every other square in the same row or column.)

4. Three nonzero real numbers are given. If they are written in any order as coefficients of a quadratic trinomial, then each of these trinomials has a real root. Does it follow that each of these trinomials has a positive root?

5. A triangular pie has the same shape as its box, except that they are mirror images of each other. We wish to cut the pie in two pieces which can fit together in the box without turning either piece over. How can this be done if

   (a) one angle of the triangle is three times as big as another;

   (b) one angle of the triangle is obtuse and is twice as big as one of the acute angles?

Note: The problems are worth 4, 4, 4, 4 and 1+4 points respectively.

[1] Courtesy of Professor Andy Liu

International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper[1] **Spring 2007.**

1. A, B, C and D are points on the parabola $y = x^2$ such that AB and CD intersect on the y-axis. Determine the x-coordinate of D in terms of the x-coordinates of A, B and C, which are a, b and c respectively.

2. A convex figure F is such that any equilateral triangle with side 1 has a parallel translation that takes all its vertices to the boundary of F. Is F necessarily a circle?

3. Let $f(x)$ be a polynomial of nonzero degree. Can it happen that for any real number a, an even number of real numbers satisfy the equation $f(x) = a$?

4. Nancy shuffles a deck of 52 cards and spreads the cards out in a circle face up, leaving one spot empty. Andy, who is in another room and does not see the cards, names a card. If this card is adjacent to the empty spot, Nancy moves the card to the empty spot, without telling Andy; otherwise nothing happens. Then Andy names another card and so on, as many times as he likes, until he says "stop."

   (a) Can Andy guarantee that after he says "stop," no card is in its initial spot?

   (b) Can Andy guarantee that after he says "stop," the Queen of Spades is not adjacent to the empty spot?

5. From a regular octahedron with edge 1, cut off a pyramid about each vertex. The base of each pyramid is a square with edge $\frac{1}{3}$. Can copies of the polyhedron so obtained, whose faces are either regular hexagons or squares, be used to tile space?

6. Let a_0 be an irrational number such that $0 < a_0 < \frac{1}{2}$. Define $a_n = \min\{2a_{n-1}, 1 - 2a_{n-1}\}$ for $n \geq 1$.

   (a) Prove that $a_n < \frac{3}{16}$ for some n.

   (b) Can it happen that $a_n > \frac{7}{40}$ for all n?

7. T is a point on the plane of triangle ABC such that $\angle ATB = \angle BTC = \angle CTA = 120°$. Prove that the lines symmetric to AT, BT and CT with respect to BC, CA and AB, respectively, are concurrent.

Note: The problems are worth 3, 5, 5, 4+4, 8, 4+4 and 8 points respectively.

[1]Courtesy of Professor Andy Liu

International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper[1] **Spring 2007.**

1. A 9×9 chessboard with the standard checkered pattern has white squares at its four corners. What is the least number of rooks that can be placed on this board so that all the white squares are attacked? (A rook also attacks the square it is on, in addition to every other square in the same row or column.)

2. The polynomial $x^3 + px^2 + qx + r$ has three roots in the interval (0,2). Prove that $-2 < p + q + r < 0$.

3. B is a point on the line which is tangent to a circle at the point A. The line segment AB is rotated about the centre of the circle through some angle to the line segment $A'B'$. Prove that the line AA' passes through the midpoint of BB'.

4. A binary sequence is constructed as follows. If the sum of the digits of the positive integer k is even, the k-th term of the sequence is 0. Otherwise, it is 1. Prove that this sequence is not periodic.

5. A triangular pie has the same shape as its box, except that they are mirror images of each other. We wish to cut the pie in two pieces which can fit together in the box without turning either piece over. How can this be done if

   (a) one angle of the triangle is obtuse and is twice as big as one of the acute angles;

   (b) the angles of the triangle are $20°$, $30°$ and $130°$?

Note: The problems are worth 3, 4, 4, 4 and 3+3 points respectively.

[1]Courtesy of Professor Andy Liu

International Mathematics

TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Fall 2008.**

1. On a 100×100 chessboard, 100 Queens are placed so that no two attack each other. Prove that if the board is divided into four 50×50 subboards, then there is at least one Queen in each subboard.

2. Each of four stones weighs an integral number of grams. Available for use is a balance which shows the difference of the weights between the objects in the left pan and those in the right pan. Is it possible to determine the weight of each stone by using this balance four times, if it may make a mistake of 1 gram either way in at most one weighing?

3. Serge has drawn triangle ABC and one of its medians AD. When informed of the ratio $\frac{AD}{AC}$, Elias is able to prove that $\angle CAB$ is obtuse and $\angle BAD$ is acute. Determine the ratio $\frac{AD}{AC}$ and justify your result.

4. Baron Münchausen asserts that he has a map of Oz showing five towns and ten roads, each road connecting exactly two cities. A road may intersect at most one other road once. The four roads connected to each town are alternately red and yellow. Can this assertion be true?

5. Let $a_1,\ a_2,\ \ldots,\ a_n$ be positive numbers such that $a_1 + a_2 + \cdots + a_n \le \frac{1}{2}$. Prove that $(1 + a_1)(1 + a_2)\cdots(1 + a_n) < 2$.

6. ABC is a non-isosceles triangle. E and F are points outside triangle ABC such that $\angle ECA = \angle EAC = \angle FAB = \angle FBA = \theta$. The line through A perpendicular to EF intersects the perpendicular bisector of BC at D. Determine $\angle BDC$.

7. In the infinite sequence $\{a_n\}$, $a_0 = 0$. For $n \ge 1$, if the greatest odd divisor of n is congruent modulo 4 to 1, then $a_n = a_{n-1} + 1$, but if the greatest odd divisor of n is congruent modulo 4 to 3, then $a_n = a_{n-1} - 1$. The initial terms are 0, 1, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, 2, 3, 2 and 1.

   (a) Prove that the number 1 appears infinitely many times in this sequence.

   (b) Prove that every positive integer appears infinitely many times in this sequence.

Note: The problems are worth 4, 6, 6, 6, 8, 9 and 5+5 points respectively.

International Mathematics

TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2008.**

1. Arrange the boxes in a line so that the number of pencils in them increases from left to right. Then the first box from the left contains at least one pencil, the next one contains at least two pencils, ..., the tenth box from the left contains at least 10 pencils. Take any pencil from the first box. Since the second box contains pencils of at least two different colors, some of these pencils has color distinct from that of the chosen pencil. Take it. The third box contains pencils of at least three colors. Hence some of these pencils has color distinct from the colors of both chosen pencils. Take it. Proceeding in the same manner, we choose the required 10 pencils of different colors.

2. Subtract 50 from each given number exceeding 50. By the conditions of the problem, each of the resulting differences is distinct from any of 25 given numbers not exceeding 50. So these numbers and the differences form a set of 50 distinct positive integers not exceeding 50. Thus it contains all positive integers from 1 to 50. Their sum equals $51 \cdot 25$, hence the sum of the given numbers equals $51 \cdot 25 + 50 \cdot 25 = 101 \cdot 25 = 2525$.

3. Let B_1, B_2, B_3 be the midpoints of arcs A_1A_2, A_2A_3, A_3A_1, respectively. The area of hexagon $A_1B_1A_2B_2A_3B_3$ is the sum of the areas of quadrilaterals $OA_1B_1A_2$, $OA_2B_2A_3$, and $OA_3B_3A_1$. But each of these quadrilaterals has perpendicular diagonals, hence the area of each quadrilateral is the half-product of its diagonals. Therefore, the required sum is equal to $\frac{1}{2}OB_1 \cdot A_1A_2 + \frac{1}{2}OB_2 \cdot A_2A_3 + \frac{1}{2}OB_3 \cdot A_3A_1$. Since $OB_1 = OB_2 = OB_3 = 2$ by the conditions of the problem, this sum is numerically equal to $A_1A_2 + A_2A_3 + A_3A_1$, as required.

4. Answer. Yes, it can.

   Solution. First take any three distinct positive integers such that one of them is equal to the half-sum of the remaining two; for instance, 1, 2, and 3. Their product equals 6 and so is not 2008th power of a positive

International Mathematics

TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Fall 2008.**

1. A standard 8×8 chessboard is modified by varying the distances between parallel grid lines, so that the cells are rectangles which are not necessarily squares, and do not necessarily have constant area. The ratio between the area of any white cell and the area of any black cell is at most 2. Determine the maximum possible ratio of the total area of the white cells to the total area of the black cells.

2. Space is dissected into non-overlapping unit cubes. Is it necessarily true that for each of these cubes, there exists another one sharing a common face with it?

3. A two-player game has $n > 2$ piles each initially consisting of a single nut. The players take turns choosing two piles containing numbers of nuts relatively prime to each other, and merging the two piles into one. The player who cannot make a move loses the game. For each n, determine the player with a winning strategy, regardless of how the opponent may respond.

4. In the quadrilateral $ABCD$, AD is parallel to BC but $AB \neq CD$. The diagonal AC meets the circumcircle of triangle BCD again at A' and the circumcircle of triangle BAD again at C'. The diagonal BD meets the circumcircle of triangle ABC again at D' and the circumcircle of triangle ADC again at B'. Prove that the quadrilateral $A'B'C'D'$ also has a pair of parallel sides.

5. In the infinite sequence $\{a_n\}$, $a_0 = 0$. For $n \geq 1$, if the greatest odd divisor of n is congruent modulo 4 to 1, then $a_n = a_{n-1} + 1$, but if the greatest odd divisor of n is congruent modulo 4 to 3, then $a_n = a_{n-1} - 1$. The initial terms are 0, 1, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, 2, 3, 2 and 1. Prove that every positive integer appears infinitely many times in this sequence.

6. $P(x)$ is a polynomial with real coefficients such that there exist infinitely many pairs (m, n) of integers satisfying $P(m)+P(n) = 0$. Prove that the graph $y = P(x)$ has a centre of symmetry.

7. A contest consists of 30 true or false questions. Victor knows nothing about the subject matter. He may write the contest several times, with exactly the same questions, and is told the number of questions he has answered correctly each time. How can he be sure that he will answer all 30 questions correctly
   (a) on his 30th attempt;
   (b) on his 25th attempt?

Note: The problems are worth 4, 6, 6, 6, 8, 9 and 5+5 points respectively.

International Mathematics

TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2008.**

1. Arrange the boxes in a line so that the number of cookies in them decreases from left to right. On a sheet of squared paper, draw a "staircase" in which the height of the first column (square side in width) equals the number of cookies in the first box from the left, the height of the second column equals the number of cookies in the second box, and so on. Then the staircase divides into "footsteps": the first footstep (from the left) consists of the highest columns, the second footstep consists of the columns next to the highest, and so on. The last footstep (to the right) consists of the lowest columns. The number of different integers in Alex's records is equal to the number of footsteps of this staircase (the boxes with the maximal number of cookies correspond to the highest footstep, and so on). But this number is equal to the number of different integers in Serge's records. Indeed, choosing a cooky in each box may be described as cutting off the bottom row of cells in our staircase. Therefore, when we fill up the plates with the maximal number of cookies, several rows will be removed so that the lowest footstep will disappear, and thus the number of footsteps will decrease by 1. By filling up the plates with the next to maximal number of cookies, we remove the next footstep, and so on. Hence the number of footsteps equals the number of different integers in Serge's records as required.

2. Answer: $x_1 = 1, x_2 = \ldots = x_n = 0$.

   Solution. Let us square the equality $\sqrt{x_1} + \sqrt{x_2 + \ldots + x_n} = \sqrt{x_2} + \sqrt{x_3 + \ldots + x_n + x_1}$, subtract the sum $x_1 + \ldots + x_n$ from both sides, and square again. We obtain $x_1(x_2 + \ldots + x_n) = x_2(x_3 + \ldots + x_n + x_1)$, hence $(x_1 - x_2)(x_3 + \ldots + x_n) = 0$. Since $x_1 - x_2 = 1$, we have $x_3 + \ldots + x_n = 0$. Since our equations contain square roots of $x_3, \ldots, x_n$, these numbers are nonnegative, and since their sum is 0, each of them is 0.

   Suppose $x_2 \neq 0$, that is, $x_2 - x_3 \neq 0$. Considering the sums which contain $\sqrt{x_2}$ and $\sqrt{x_3}$ and arguing as above, we get $x_1 = 0$. Then

International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Spring 2008.**

1. An integer N is the product of two consecutive integers.
   (a) Prove that we can add two digits to the right of this number and obtain a perfect square.
   (b) Prove that this can be done in only one way if $N > 12$.
2. A line parallel to the side AC of triangle ABC cuts the side AB at K and the side BC at M. O is the point of intersection of AM and CK. If $AK = AO$ and $KM = MC$, prove that $AM = KB$.
3. Alice and Brian are playing a game on a $1 \times (N + 2)$ board. To start the game, Alice places a checker on any of the N interior squares. In each move, Brian chooses a positive integer n. Alice must move the checker to the n-th square on the left or the right of its current position. If the checker moves off the board, Alice wins. If it lands on either of the end squares, Brian wins. If it lands on another interior square, the game proceeds to the next move. For which values of N does Brian have a strategy which allows him to win the game in a finite number of moves?
4. Given are finitely many points in the plane, no three on a line. They are painted in four colours, with at least one point of each colour. Prove that there exist three triangles, distinct but not necessarily disjoint, such that the three vertices of each triangle have different colours, and none of them contains a coloured point in its interior.
5. Standing in a circle are 99 girls, each with a candy. In each move, each girl gives her candy to either neighbour. If a girl receives two candies in the same move, she eats one of them. What is the minimum number of moves after which only one candy remains?
6. Do there exist positive integers $a,\ b,\ c$ and d such that $\frac{a}{b} + \frac{c}{d} = 1$ and $\frac{a}{d} + \frac{c}{b} = 2008$?
7. A convex quadrilateral $ABCD$ has no parallel sides. The angles between the diagonal AC and the four sides are $55^\circ,\ 55^\circ, 19^\circ$ and 16° in some order. Determine all possible values of the acute angle between AC and BD.

Note: The problems are worth 2+2, 5, 6, 6, 7, 7 and 8 points respectively.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Spring 2008.**

1. In the convex hexagon $ABCDEF$, AB, BC and CD are respectively parallel to DE, EF and FA. If $AB = DE$, prove that $BC = EF$ and $CD = FA$.

2. There are ten congruent segments on a plane. Each point of intersection divides every segment passing through it in the ratio 3:4. Find the maximum number of points of intersection.

3. There are ten cards with the number a on each, ten with b and ten with c, where a, b and c are distinct real numbers. For every five cards, it is possible to add another five cards so that the sum of the numbers on these ten cards is 0. Prove that one of a, b and c is 0.

4. Find all positive integers n such that $(n+1)!$ is divisible by $1! + 2! + \cdots + n!$.

5. Each cell of a 10×10 board is painted red, blue or white, with exactly twenty of them red. No two adjacent cells are painted in the same colour. A domino consists of two adjacent cells, and it is said to be good if one cell is blue and the other is white.

   (a) Prove that it is always possible to cut out 30 good dominoes from such a board.

   (b) Give an example of such a board from which it is possible to cut out 40 good dominoes.

   (c) Give an example of such a board from which it is not possible to cut out more than 30 good dominoes.

Note: The problems are worth 4, 5, 5, 5 and 6 points respectively.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Spring 2008.**

1. A triangle has an angle of measure θ. It is dissected into several triangles. Is it possible that all angles of the resulting triangles are less than θ, if

   (a) $\theta = 70°$;

   (b) $\theta = 80°$?

2. Alice and Brian are playing a game on the real line. To start the game, Alice places a checker on a number x where $0 < x < 1$. In each move, Brian chooses a positive number d. Alice must move the checker to either $x + d$ or $x - d$. If it lands on 0 or 1, Brian wins. Otherwise the game proceeds to the next move. For which values of x does Brian have a strategy which allows him to win the game in a finite number of moves?

3. A polynomial $\quad x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n$

   has n distinct real roots $x_1,\ x_2,\ \ldots,\ x_n$, where $n > 1$. The polynomial

$$nx^{n-1} + (n-1)a_1x^{n-2} + (n-2)a_2x^{n-3} + \cdots + 2a_{n-2}x + a_{n-1}$$

   has roots $y_1,\ y_2,\ \ldots,\ y_{n-1}$. Prove that

$$\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n} > \frac{y_1^2 + y_2^2 + \cdots + y_{n-1}^2}{n-1}.$$

4. Each of Peter and Basil draws a convex quadrilateral with no parallel sides. The angles between a diagonal and the four sides of Peter's quadrilateral are $\alpha,\ \alpha,\ \beta$ and γ in some order. The angles between a diagonal and the four sides of Basil's quadrilateral are also $\alpha,\ \alpha,\ \beta$ and γ in some order. Prove that the acute angle between the diagonals of Peter's quadrilateral is equal to the acute angle between the diagonals of Basil's quadrilateral.

5. The positive integers are arranged in a row in some order, each occuring exactly once. Does there always exist an adjacent block of at least two numbers somewhere in this row such that the sum of the numbers in the block is a prime number?

6. Seated in a circle are 11 wizards. A different positive integer not exceeding 1000 is pasted onto the forehead of each. A wizard can see the numbers of the other 10, but not his own. Simultaneously, each wizard puts up either his left hand or his right hand. Then each declares the number on his forehead at the same time. Is there a strategy on which the wizards can agree beforehand, which allows each of them to make the correct declaration?

7. Each of three lines cuts chords of equal lengths in two given circles. The points of intersection of these lines form a triangle. Prove that its circumcircle passes through the midpoint of the segment joining the centres of the circles.

Note: The problems are worth 3+3, 6, 6, 7, 8, 8 and 8 points respectively.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Spring 2008.**

1. There are ten cards with the number a on each, ten with b and ten with c, where a, b and c are distinct real numbers. For every five cards, it is possible to add another five cards so that the sum of the numbers on these ten cards is 0. Prove that one of a, b and c is 0.

2. Can it happen that the least common multiple of 1, 2, $\ldots$, n is 2008 times the least common multiple of 1, 2, $\ldots$, m for some positive integers m and n?

3. In triangle ABC, $\angle A = 90°$. M is the midpoint of BC and H is the foot of the altitude from A to BC. The line passing through M and perpendicular to AC meets the circumcircle of triangle AMC again at P. If BP intersects AH at K, prove that $AK = KH$.

4. No matter how two copies of a convex polygon are placed inside a square, they always have a common point. Prove that no matter how three copies of the same polygon are placed inside this square, they also have a common point.

5. We may permute the rows and the columns of the table below. How may different tables can we generate?

1	2	3	4	5	6	7
7	1	2	3	4	5	6
6	7	1	2	3	4	5
5	6	7	1	2	3	4
4	5	6	7	1	2	3
3	4	5	6	7	1	2
2	3	4	5	6	7	1

Note: The problems are worth 4, 5, 5, 5 and 6 points respectively.

International Mathematics

TOURNAMENT OF THE TOWNS

Solutions to Junior A-Level Paper **Fall 2009**

1. Pour from each jar exactly one tenth of what it initially contains into each of the other nine jars. At the end of these ten operations, each jar will contain one tenth of what is inside each jar initially. Since the total amount of milk remains unchanged, each jar will contain one tenth of the total amount of milk.

2. Assign spatial coordinates to the unit cubes, each dimension ranging from 1 to 10. If all cubes are in the same colour orientation, there is nothing to prove. Hence we may assume that (i, j, k) and $(i + 1, j, k)$ do not. Since they share a left-right face, let the common colour be red. We may assign blue to the front-back faces of (i, j, k). Then its top-bottom faces are white, the front-back faces of $(i+1, j, k)$ are white and the top-bottom faces of $(i + 1, j, k)$ is blue.

   Now $(i, j + 1, k)$ share a white face with (i, j, k) while $(i + 1, j + 1, k)$ share a blue face with $(i+1, j, k)$. Since $(i, j+1, k)$ and $(i+1, j+1, k)$ share a left-right face, the only available colour is red. It follows that the $1\times2\times10$ block with $(i, 1, k)$ and $(i+1, 1, k)$ at one end and $(i, 10, k)$ and $(i + 1, 10, k)$ at the other end has 1×10 faces left and right which are all red.

   Similarly, if we carry out the expansion vertically, we obtain a $2\times10\times10$ black with 10×10 faces left and right which are all red. Finally, if we carry out the expansion sideways, we will have the left and right faces of the large cube all red.

3. Suppose $a = b$. Then $a+a^2 = a(a+1)$ is a power of 2, so that each of a and $a+1$ is a power of 2. This is only possible if $a = 1$. Suppose $a \neq b$. By symmetry, we may assume that $a > b$, so that $a^2+b > a+b^2$. Since their product is a power of 2, each is a power of 2.

   Let $a^2 + b = 2^r$ and $a + b^2 = 2^s$ with $r > s$. Then $2^s(2^{r-s} - 1) = 2^r - 2^s = a^2 + b - a - b^2 = (a - b)(a + b - 1)$.

   Now $a-b$ and $a+b-1$ have opposite parity. Hence one of them is equal to 2^s and the other to $2^{r-s} - 1$. If $a - b = 2^s = a + b^2$, then -$b = b^2$.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2009.**[1]

1. Is it possible to cut a square into nine squares and colour one of them white, three of them grey and five of them black, such that squares of the same colour have the same size and squares of different colours will have different sizes?

2. There are forty weights: 1, 2, ..., 40 grams. Ten weights with even masses were put on the left pan of a balance. Ten weights with odd masses were put on the right pan of the balance. The left and the right pans are balanced. Prove that one pan contains two weights whose masses differ by exactly 20 grams.

3. A cardboard circular disk of radius 5 centimetres is placed on the table. While it is possible, Peter puts cardboard squares with side 5 centimetres outside the disk so that:
(1) one vertex of each square lies on the boundary of the disk;
(2) the squares do not overlap;
(3) each square has a common vertex with the preceding one.
Find how many squares Peter can put on the table, and prove that the first and the last of them must also have a common vertex.

4. We only know that the password of a safe consists of 7 different digits. The safe will open if we enter 7 different digits, and one of them matches the corresponding digit of the password. Can we open this safe in less than 7 attempts?

5. A new website registered 2000 people. Each of them invited 1000 other registered people to be their friends. Two people are considered to be friends if and only if they have invited each other. What is the minimum number of pairs of friends on this website?

Note: The problems are worth 3, 4, 4, 5 and 5 points respectively.

[1] Courtesy of Andy Liu.

International Mathematics

TOURNAMENT OF THE TOWNS

Solutions to Seniors A-Level Paper **Fall 2009**

1. A pirate who owes money is put in group A, and the others are put in group B. Each pirate in group A puts the full amount of money he owes into a pot, and the pot is shared equally among all 100 pirates. For each pirate in group B, each of the 100 pirates puts 1/100-th of the amount owed to him in a pot, and this pirate takes the pot. We claim that all debts are then settled. Let a be the total amount of money the pirates in group A owe, and let b be the total amount of money owed by the pirates in group B. Clearly, $a = b$. Each pirate in group A pays off his debt, takes back a/100 and then pays out another $b/100$. Hence he has paid off his debt exactly. Each pirate in group B takes in $a/100$, pays out $b/100$ and then takes inwhat is owed him. Hence the debts to him have been settled too. (Wen-Hsien Sun)

2. Let the given rectangle R have length m and width n with $m > n$. Contract the length of R by a factor of n/m, resulting in an $n \times n$ square. For each of the N rectangle in R, the corresponding rectangle in S has the same width but shorter length. Thus we can cut the former into a primary piece congruent to the latter, plus a secondary piece. Using S as a model, the N primary pieces may be assembled into an $n \times n$ square while the N secondary pieces may be assembled into an $(n - m) \times n$ rectangle. (Rosu Cristina, Jonathan Zung)

3. Let the points of tangency to the sphere of AB, AC, DB and DC be K, L, M and N respectively. The line KL intersects the line BC at some point P not between B and C. By the converse of the undirected version of Menelaus Theorem, since $LA = AK$

$$1 = \frac{BP}{PC} \cdot \frac{CL}{LA} \cdot \frac{AK}{KB} = \frac{BP}{PC} \cdot \frac{CL}{KB}.$$

Since $CL = CN$, $KB = MB$ and $ND = DM$,

$$1 = \frac{BP}{PC} \cdot \frac{CN}{MB} = \frac{BP}{PC} \cdot \frac{CN}{ND} \cdot \frac{DM}{MB}.$$

International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2009.**[1]

1. A 7-digit passcode is called good if all digits are different. A safe has a good passcode, and it opens if seven digits are entered and one of the digits matches the corresponding digit of the passcode. Is there a method of opening the safe box with an unknown passcode using less than 7 attempts?

2. A, B, C, D, E and F are points in space such that AB is parallel to DE, BC is parallel to EF, CD is parallel to FA, but $AB \neq DE$. Prove that all six points lie in the same plane.

3. Are there positive integers a, b, c and d such that $a^3 + b^3 + c^3 + d^3 = 100^{100}$?

4. A point is chosen on each side of a regular 2009-gon. Let S be the area of the 2009-gon with vertices at these points. For each of the chosen points, reflect it across the midpoint of its side. Prove that the 2009-gon with vertices at the images of these reflections also has area S.

5. A country has two capitals and several towns. Some of them are connected by roads. Some of the roads are toll roads where a fee is charged for driving along them. It is known that any route from the south capital to the north capital contains at least ten toll roads. Prove that all toll roads can be distributed among ten companies so that anybody driving from the south capital to the north capital must pay each of these companies.

Note: The problems are worth 4, 4, 4, 4 and 5 points respectively.

[1] Courtesy of Andy Liu.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Spring**

1. Basil and Peter play the following game. Initially, there are two numbers on the blackboard, $\frac{1}{2009}$ and $\frac{1}{2008}$. At each move, Basil chooses an arbitrary positive number x, and Peter selects one of the two numbers on the blackboard and increases it by x. Basil wins if one of the numbers on the blackboard increases to 1. Does Basil have a winning strategy, regardless of what Peter does?

2. (a) Prove that there exists a polygon which can be dissected into two congruent parts by a line segment which cuts one side of the original polygon in half and another side in the ratio 1:2.

   (b) Can such a polygon be convex?

3. The central square of an 101×101 board is the bank. Every other square is marked S or T. A bank robber who enters a square marked S must go straight ahead in the same direction. A bank robber who enters a square marked T must make a right turn or a left turn. Is it possible to mark the squares in such a way that no bank robber can get to the bank?

4. In a sequence of distinct positive integers, each term except the first is either the arithmetic mean or the geometric mean of the term immediately before and the term immediately after. Is it necessarily true that from a certain point on, the means are either all arithmetic means or all geometric means?

5. A castle is surrounded by a circular wall with 9 towers. Some knights stand on guard on these towers. After every hour, each knight moves to a neighbouring tower. A knight always moves in the same direction, whether clockwise or counter-clockwise. At some hour, there are at least two knights on each tower. At another hour, there are exactly 5 towers each of which has exactly one knight on it. Prove that at some other hour, there is a tower with no knights on it.

6. In triangle ABC, $AB = AC$ and $\angle CAB = 120°$. D and E are points on BC, with D closer to B, such that $\angle DAE = 60°$. F and G are points on AB and AC respectively such that $\angle FDB = \angle ADE$ and $\angle GEC = \angle AED$. Prove that the area of triangle ADE is equal to the sum of the areas of triangles FBD and GCE.

7. Let $\binom{n}{k}$ be the number of ways of choosing a subset of k objects from a set of n objects. Prove that if k and ℓ are positive integers less than n, then $\binom{n}{k}$ and $\binom{n}{\ell}$ have a common divisor greater than 1.

Note: The problems are worth 3, 2+3, 5, 5, 6, 7 and 9 points respectively.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Spring 2009.**

1. In a convex 2009-gon, all diagonals are drawn. A line intersects the 2009-gon but does not pass through any of its vertices. Prove that the line intersects an even number of diagonals.

2. Let $a \wedge b$ denote the number a^b. The order of operations in the expresson $7 \wedge 7 \wedge 7 \wedge 7 \wedge 7 \wedge 7 \wedge 7$ must be determined by inserting five pairs of brackets. Is it possible to put brackets in two distinct ways so that the expressions have the same value?

3. Vlad is going to print a digit on each face of several unit cubes, in such a way that a 6 does not turn into a 9. If it is possible to form every 30-digit number with these blocks, what is the minimum number of the blocks?

4. When a positive integer is increased by 10%, the result is another positive integer whose digit-sum has decreased by 10%. Is this possible?

5. In the rhombus $ABCD$, $\angle A = 120°$. M is a point on BC and N is a point on CD such that $\angle MAN = 30°$. Prove that the circumcentre of triangle MAN lies on a diagonal of $ABCD$.

Note: The problems are worth 3, 4, 4, 4 and 5 points respectively.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Spring 2009.**

1. A rectangle is dissected into several smaller rectangles. Is it possible that for each pair of rectangles so obtained, the line segment joining their centres intersects some other rectangle?
2. In a sequence of distinct positive integers, each term except the first is either the arithmetic mean or the geometric mean of the term immediately before and the term immediately after. Is it necessarily true that from a certain point on, the means are either all arithmetic means or all geometric means?
3. There is a counter is each square of a 10×10 board. We may choose a diagonal containing an even number of countes and remove any counter from it. What is the maximum number of counters which can be removed from the board by these operations?
4. Three planes dissect a parallelepiped into eight hexahedra such that all of their faces are quadrilaterals. One of the hexahedra has a circumsphere. Prove that each of these hexahedra has a circumsphere.
5. Let $\binom{n}{k}$ be the number of ways of choosing a subset of k objects from a set of n objects. Prove that if k and ℓ are positive integers less than n, then $\binom{n}{k}$ and $\binom{n}{\ell}$ have a common divisor greater than 1.
6. A positive integer n is given. Two players take turns marking points on a circle. The first player uses the red colour while the second player uses the blue colour. When n points of each colour has been marked, the game is over,and the circle has been divided into $2n$ arcs. The winner is the player who has the longest arc both endpoints of which are of this player's colour. Which player can always win, regardless of any action of the opponent?
7. At step 1, the computer has the number 6 in a memory cell. In step n, it computes the greatest common divisor d of n and the number m currently in that cell, and replaces m with $m + d$. Prove that if $d > 1$, then d must be prime.

Note: The problems are worth 4, 4, 6, 6, 8, 9 and 9 points respectively.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Spring 2009.**

1. Let $a \wedge b$ denote the number a^b. The order of operations in the expresson $7 \wedge 7 \wedge 7 \wedge 7 \wedge 7 \wedge 7 \wedge 7$ must be determined by inserting five pairs of brackets. Is it possible to put brackets in two distinct ways so that the expressions have the same value?

2. Several points on the plane are such that no three lie on a straight line. Some pairs of points are connected by segments. If any line which does not pass through any of these points intersects an even number of these segments, prove that each of these points is connected to an even number of the other points.

3. Let a and b be arbitrary positive integers. The sequence $\{x_k\}$ is defined by $x_1 = a,\ x_2 = b$ and for $k \geq 3$, x_k is the greatest common divisor of $x_{k-1} + x_{k-2}$.

   (a) Prove that the sequence is eventually constant.

   (b) How can this constant value be determined from a and b?

4. In an arbitrary binary number, consider any digit 1 and any digit 0 which follows it, not necessarily immediately. They form an odd pair if the number of other digits in between is odd, and an even pair if this number is even. Prove that the number of even pairs is greater than or equal to the number of odd pairs.

5. X is an arbitrary point inside a tetrahedron. Through each of the vertices of the tetrahedron, draw a line parallel to the line joining X to the centroid of the opposite face. Prove that these four lines are concurrent.

Note: The problems are worth 3, 4, 2+2, 4 and 4 points respectively.

International Mathematics

TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Fall 2010.**[1]

1. A round coin may be used to construct a circle passing through one or two given points on the plane. Given a line on the plane, show how to use this coin to construct two points such that they define a line perpendicular to the given line. Note that the coin may not be used to construct a circle tangent to the given line.

2. Pete has an instrument which can locate the midpoint of a line segment, and also the point which divides the line segment into two segments whose lengths are in a ratio of $n : (n + 1)$, where n is any positive integer. Pete claims that with this instrument, he can locate the point which divides a line segment into two segments whose lengths are at any given rational ratio. Is Pete right?

3. At a circular track, 10 cyclists started from some point at the same time in the same direction with different constant speeds. If any two cyclists are at some point at the same time again, we say that they meet. No three or more of them have met at the same time. Prove that by the time every two cyclists have met at least once, each cyclist has had at least 25 meetings.

4. A rectangle is divided into 2×1 and 1×2 dominoes. In each domino, a diagonal is drawn, and no two diagonals have common endpoints. Prove that exactly two corners of the rectangle are endpoints of these diagonals.

5. For each side of a given pentagon, divide its length by the total length of all other sides. Prove that the sum of all the fractions obtained is less than 2.

6. In acute triangle ABC, an arbitrary point P is chosen on altitude AH. Points E and F are the midpoints of sides CA and AB respectively. The perpendiculars from E to CP and from F to BP meet at point K. Prove that $KB = KC$.

7. Merlin summons the n knights of Camelot for a conference. Each day, he assigns them to the n seats at the Round Table. From the second day on, any two neighbours may interchange their seats if they were not neighbours on the first day. The knights try to sit in some cyclic order which has already occurred before on an earlier day. If they succeed, then the conference comes to an end when the day is over. What is the maximum number of days for which Merlin can guarantee that the conference will last?

Note: The problems are worth 4, 5, 8, 8, 8, 8 and 12 points respectively.

[1] Courtesy of Andy Liu

International Mathematics

TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2010**[1]

1. In a multiplication table, the entry in the i-th row and the j-th column is the product ij. From an $m \times n$ subtable with both m and n odd, the interior $(m-2) \times (n-2)$ rectangle is removed, leaving behind a frame of width 1. The squares of the frame are painted alternately black and white. Prove that the sum of the numbers in the black squares is equal to the sum of the numbers in the white squares.

2. In a quadrilateral $ABCD$ with an incircle, $AB = CD$, $BC < AD$ and BC is parallel to AD. Prove that the bisector of $\angle C$ bisects the area of $ABCD$.

3. A $1\times1\times1$ cube is placed on an 8×8 chessboard so that its bottom face coincides with a square of the chessboard. The cube rolls over a bottom edge so that the adjacent face now lands on the chessboard. In this way, the cube rolls around the chessboard, landing on each square at least once. Is it possible that a particular face of the cube never lands on the chessboard?

4. In a school, more than 90% of the students know both English and German, and more than 90% of the students know both English and French. Prove that more than 90% of the students who know both German and French also know English.

5. A circle is divided by $2N$ points into $2N$ arcs of length 1. These points are joined in pairs to form N chords. Each chord divides the circle into two arcs, the length of each being an even integer. Prove that N is even.

Note: The problems are worth 4, 4, 4, 4 and 4 points respectively.

[1]Courtesy of Andy Liu

International Mathematics

TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Fall 2010.**[1]

1. There are 100 points on the plane. All 4950 pairwise distances between two points have been recorded.

   (a) A single record has been erased. Is it always possible to restore it using the remaining records?

   (b) Suppose no three points are on a line, and k records were erased. What is the maximum value of k such that restoration of all the erased records is always possible?

2. At a circular track, $2n$ cyclists started from some point at the same time in the same direction with different constant speeds. If any two cyclists are at some point at the same time again, we say that they meet. No three or more of them have met at the same time. Prove that by the time every two cyclists have met at least once, each cyclist has had at least n^2 meetings.

3. For each side of a given polygon, divide its length by the total length of all other sides. Prove that the sum of all the fractions obtained is less than 2.

4. Two dueling wizards are at an altitude of 100 above the sea. They cast spells in turn, and each spell is of the form "decrease the altitude by a for me and by b for my rival" where a and b are real numbers such that $0 < a < b$. Different spells have different values for a and b. The set of spells is the same for both wizards, the spells may be cast in any order, and the same spell may be cast many times. A wizard wins if after some spell, he is still above water but his rival is not. Does there exist a set of spells such that the second wizard has a guaranteed win, if the number of spells is

   (a) finite;

   (b) infinite?

5. The quadrilateral $ABCD$ is inscribed in a circle with center O. The diagonals AC and BD do not pass through O. If the circumcentre of triangle AOC lies on the line BD, prove that the circumcentre of triangle BOD lies on the line AC.

6. Each cell of a 1000×1000 table contains 0 or 1. Prove that one can either cut out 990 rows so that at least one 1 remains in each column, or cut out 990 columns so that at least one 0 remains in each row.

7. A square is divided into congruent rectangles with sides of integer lengths. A rectangle is important if it has at least one point in common with a given diagonal of the square. Prove that this diagonal bisects the total area of the important rectangles.

Note: The problems are worth 2+3, 6, 6, 2+5, 8, 12 and 14 points respectively.

[1]Courtesy of Andy Liu.

International Mathematics

TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2010.**[1]

1. The exchange rate in a Funny-Money machine is s McLoonies for a Loonie or $\frac{1}{s}$ Loonies for a McLoonie, where s is a positive real number. The number of coins returned is rounded off to the nearest integer. If it is exactly in between two integers, then it is rounded up to the greater integer.
   (a) Is it possible to achieve a one-time gain by changing some Loonies into McLoonies and changing all the McLoonies back to Loonies?
   (b) Assuming that the answer to (a) is "yes", is it possible to achieve multiple gains by repeating this procedure, changing all the coins in hand and back again each time?

2. The diagonals of a convex quadrilateral $ABCD$ are perpendicular to each other and intersect at the point O. The sum of the inradii of triangles AOB and COD is equal to the sum of the inradii of triangles BOC and DOA.
   (a) Prove that $ABCD$ has an incircle.
   (b) Prove that $ABCD$ is symmetric about one of its diagonals.

3. From a police station situated on a straight road infinite in both directions, a thief has stolen a police car. Its maximal speed equals 90% of the maximal speed of a police cruiser. When the theft is discovered some time later, a policeman starts to pursue the thief on a cruiser. However, he does not know in which direction along the road the thief has gone, nor does he know how long ago the car has been stolen. Is it possible for the policeman to catch the thief?

4. A square board is dissected into n^2 rectangular cells by $n-1$ horizontal and $n-1$ vertical lines. The cells are painted alternately black and white in a chessboard pattern. One diagonal consists of n black cells which are squares. Prove that the total area of all black cells is not less than the total area of all white cells.

5. In a tournament with 55 participants, one match is played at a time, with the loser dropping out. In each match, the numbers of wins so far of the two participants differ by not more than 1. What is the maximal number of matches for the winner of the tournament?

Note: The problems are worth 2+3, 2+3, 5, 5 and 5 points respectively.

[1] Courtesy of Andy Liu

International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Spring 2010.**[1]

1. Alex has a piece of cheese. He chooses a positive number $\alpha \neq 1$ and cut the piece into two, in the ratio $1 : \alpha$. He can then choose any piece and cut it in the same way. Is it possible for him to obtain, after a finite number of cuts, two piles of pieces each containing half the original amount of cheese?

2. M is the midpoint of the side CA of triangle ABC. P is some point on the side BC. AP and BM intersect at the point O. If $BO = BP$, determine $\frac{OM}{PC}$.

3. Along a circle are placed 999 numbers, each 1 or -1, and there is at least one of each. The product of each block of 10 adjacent numbers along the circle is computed. Let S denote the sum of these 999 products.
(a) What is the minimum value of S?
(b) What is the maximum value of S?

4. Is it possible that the sum of the digits of a positive integer n is 100 while the sum of the digits of the number n^3 is 100^3?

5. On a circular road are N horsemen, riding in the same direction, each at a different constant speed. There is only one point along the road at which a horseman is allowed to pass another horseman. Can they continue to ride for an arbitrarily long period if
(a) $N = 3$;
(b) $N = 10$?

6. A broken line consists of 31 segments joined end to end. It does not intersect itself, and has distinct end points. What is the smallest number of straight lines which can contain all segments of such a broken line?

7. A number of fleas are on a 10×10 chessboard, each in a different cell. Every minute, a flea jumps to the adjacent square either to the east, to the south, to the west or to the north. It continues to jump in the same direction as long as this is possible, but reverses direction if it has reached the edge of the chessboard. In one hour, no two fleas ever occupy the same cell. What is the maximum number of fleas on the chessboard?

Note: The problems are worth 3, 4, 3+3, 6, 3+5, 8 and 11 points respectively.

[1] Courtesy of Andy Liu.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Spring 2010.**[1]

1. Each of six baskets contains some pears, plums and apples. The number of plums in each basket is equal to the total number of apples in the other five baskets, and the number of apples in each basket is equal to the total number of pears in the other five baskets. Prove that the total number of fruit in the six baskets is a multiple of 31.

2. Karlsson and Lillebror are dividing a square cake. Karlsson chooses a point P of the cake which is not on the boundary. Lillebror makes a straight cut from P to the boundary of the cake, in any direction he chooses. Then Karlsson makes a straight cut from P to the boundary, at a right angle to the first cut. Lillebror will get the smaller of the two pieces. Can Karlsson prevent Lillebror from getting at least one quarter of the cake?

3. An angle is given in the plane, and a compass is the only available tool.
   (a) Use the compass the minimum number of times to determine if the angle is acute or obtuse.
   (b) Use the compass any number of times to determine if the angle is exactly $31°$.

4. At a party, each person knows at least three other people. Prove that an even number of them, at least four, can sit at a round table such that each knows both neighbours.

5. On the blackboard are the squares of the first 101 positive integers. In each move, we can replace two of them by the absolute value of their differences. After 100 moves, only one number remains. What is the minimum value of this number?

Note: The problems are worth 3, 3, 2+2, 5 and 5 points respectively.

[1]Courtesy of Andy Liu.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Spring 2010.**[1]

1. Is it possible to divide the lines in the plane into pairs of perpendicular lines so that every line belongs to exactly one pair?

2. Alex has a piece of cheese. He chooses a positive number α and cut the piece into two, in the ratio $1 : \alpha$. He can then choose any piece and cut it in the same way. Is it possible for him to obtain, after a finite number of cuts, two piles of pieces each containing half the original amount of cheese, if
(a) α is irrational;
(b) $\alpha \neq 1$ is rational?

3. Can we obtain the number 2010 from the number 1 by applying any combination of the functions sin, cos, tan, cot, arcsin, arccos, arctan and arccot?

4. At a convention, each of the 5000 participants watched at least one movie. Several participants can form a discussion group if either they had all watched the same movie, or each had watched a movie nobody else in the group had. A single participant may also form a group. Prove that the number of groups could always be exactly 100.

5. On a circular road are 33 horsemen, riding in the same direction, each at a different constant speed. There is only one point along the road at which a horseman is allowed to pass another horseman. Can they continue to ride for an arbitrarily long period?

6. A circle with centre I is tangent to all four sides of a convex quadrilateral $ABCD$. M and N are the midpoints of AB and CD respectively. If $\frac{IM}{AB} = \frac{IN}{CD}$, prove that $ABCD$ has a pair of parallel sides.

7. A multi-digit number is written on the blackboard. Susan puts in a number of plus signs between some pairs of adjacent digits. The addition is performed and the process is repeated with the sum. Prove that regardless of what number was initially on the blackboard, Susan can always obtain a single-digit number in at most ten steps.

Note: The problems are worth 3, 2+2, 6, 6, 7, 8 and 9 points respectively.

[1]Courtesy of Andy Liu.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Spring 2010.**[1]

1. Bananas, lemons and pineapples are being delivered by 2010 ships. The number of bananas in each ship is equal to the total number of lemons in the other 2009 ships, and the number of lemons in each ship is equal to the total number of pineapples in the other 2009 ships. Prove that the total number of fruit being delivered is a multiple of 31.

2. Each line in the coordinate plane has the same number of common points with the parabola $y = x^2$ and with the graph $y = f(x)$. Prove that $f(x) = x^2$.

3. Is it possible to cover the surface of a regular octahedron by several regular hexagons, without gaps or overlaps?

4. Baron Münchausen claims that a polynomial $P(x)$ with non-negative integers as coefficients is uniquely determined by the values of $P(2)$ and $P(P(2))$. Surely the Baron is wrong, isn't he?

5. A segment is given on the plane. In each move, it may be rotated about either of its endpoints in a $45°$ angle clockwise or counterclockwise. Is it possible that after a finite number of moves, the segment returns to its original position except that its endpoints are interchanged?

Note: The problems are worth 3, 4, 5, 5 and 6 points respectively.

[1]Courtesy of Andy Liu.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Fall 2011.**

1. An integer $n > 1$ is written on the board. Alex replaces it by $n + d$ or $n - d$, where d is any divisor of n greater than 1. This is repeated with the new value of n. Is it possible for Alex to write on the board the number 2011 at some point, regardless of the initial value of n?

2. P is a point on the side AB of triangle ABC such that $AP = 2PB$. If $CP = 2PQ$, where Q is the midpoint of AC, prove that ABC is a right triangle.

3. A set of at least two objects with pairwise different weights has the property that for any pair of objects from this set, we can choose a subset of the remaining objects so that their total weight is equal to the total weight of the given pair. What is the minimum number of objects in this set?

4. A game is played on a board with 2012 horizontal rows and $k > 2$ vertical columns. A marker is placed in an arbitrarily chosen cell of the left-most column. Two players move the marker in turns. During each move, the player moves the marker one cell to the right, or one cell up or down to a cell that has never been occupied by the marker before. The game is over when any of the players moves the marker to the right-most column. There are two versions of this game. In Version A, the player who gets the marker to the right-most column wins. In Version B, this player loses. However, it is only when the marker reaches the second column from the right that the players learn whether they are playing Version A or Version B. Does either player have a winning strategy?

5. Let $0 < a, b, c, d < 1$ be real numbers such that $abcd = (1-a)(1-b)(1-c)(1-d)$. Prove that $(a + b + c + d) - (a + c)(b + d) \geq 1$.

6. A car goes along a straight highway at the speed of 60 kilometres per hour. A 100 metre long fence is standing parallel to the highway. Every second, the passenger of the car measures the angle of vision of the fence. Prove that the sum of all angles measured by him is less than $1100°$.

7. Each vertex of a regular 45-gon is red, yellow or green, and there are 15 vertices of each colour. Prove that we can choose three vertices of each color so that the three triangles formed by the chosen vertices of the same color are congruent to one another.

Note: The problems are worth 3, 4, 5, 6, 6, 7 and 9 points respectively.

International Mathematics

TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2011.**

1. P and Q are points on the longest side AB of triangle ABC such that $AQ = AC$ and $BP = BC$. Prove that the circumcentre of triangle CPQ coincides with the incentre of triangle ABC.

2. Several guests at a round table are eating from a basket containing 2011 berries. Going in clockwise direction, each guest has eaten either twice as many berries as or six fewer berries than the next guest. Prove that not all the berries have been eaten.

3. From the 9×9 chessboard, all 16 unit squares whose row numbers and column numbers are both even have been removed. Disect the punctured board into rectangular pieces, with as few of them being unit squares as possible.

4. The vertices of a 33-gon are labelled with the integers from 1 to 33. Each edge is then labelled with the sum of the labels of its two vertices. Is it possible for the edge labels to consist of 33 consecutive numbers?

5. On a highway, a pedestrian and a cyclist were going in the same direction, while a cart and a car were coming from the opposite direction. All were travelling at different constant speeds. The cyclist caught up with the pedestrian at 10 o'clock. After a time interval, she met the cart, and after another time interval equal to the first, she met the car. After a third time interval, the car met the pedestrian, and after another time interval equal to the third, the car caught up with the cart. If the pedestrian met the car at 11 o'clock, when did he meet the cart?

Note: The problems are worth 3, 4, 4, 4 and 5 points respectively.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Fall 2011.**

1. Pete has marked at least 3 points in the plane such that all distances between them are different. A pair of marked points A and B will be called *unusual* if A is the furthest marked point from B, and B is the nearest marked point to A (apart from A itself). What is the largest possible number of unusual pairs that Pete can obtain?

2. Let $0 < a, b, c, d < 1$ be real numbers such that $abcd = (1-a)(1-b)(1-c)(1-d)$. Prove that $(a+b+c+d) - (a+c)(b+d) \geq 1$.

3. In triangle ABC, points D, E and F are bases of altitudes from vertices A, B and C respectively. Points P and Q are the projections of F to AC and BC respectively. Prove that the line PQ bisects the segments DF and EF.

4. Does there exist a convex n-gon such that all its sides are equal and all vertices lie on the parabola $y = x^2$, where

   (a) $n = 2011$;

   (b) $n = 2012$?

5. We will call a positive integer *good* if all its digits are nonzero. A good integer will be called *special* if it has at least k digits and their values are strictly increasing from left to right. Let a good integer be given. In each move, one may insert a special integer into the digital expression of the current number, on the left, on the right or in between any two of the digits. Alternatively, one may also delete a special number from the digital expression of the current number.What is the largest k such that any good integer can be turned into any other good integer by a finite number of such moves?

6. Prove that for $n > 1$, the integer $1^1 + 3^3 + 5^5 + \ldots + (2^n - 1)^{2^n - 1}$ is a multiple of 2^n but not a multiple of 2^{n+1}.

7. A blue circle is divided into 100 arcs by 100 red points such that the lengths of the arcs are the positive integers from 1 to 100 in an arbitrary order. Prove that there exist two perpendicular chords with red endpoints.

Note: The problems are worth 4, 4, 5, 3+4, 7, 7 and 9 points respectively.

International Mathematics

TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2011.**

1. Several guests at a round table are eating from a basket containing 2011 berries. Going in clockwise direction, each guest has eaten either twice as many berries as or six fewer berries than the next guest. Prove that not all the berries have been eaten.

2. Peter buys a lottery ticket on which he enters an n-digit number, none of the digits being 0. On the draw date, the lottery administrators will reveal an $n \times n$ table, each cell containing one of the digits from 1 to 9. A ticket wins a prize if it does *not* match any row or column of this table, read in either direction. Peter wants to bribe the administrators to reveal the digits on some cells chosen by Peter, so that Peter can guarantee to have a winning ticket. What is the minimum number of digits Peter has to know?

3. In a convex quadrilateral $ABCD$, $AB = 10$, $BC = 14$, $CD = 11$ and $DA = 5$. Determine the angle between its diagonals.

4. Positive integers $a < b < c$ are such that $b + a$ is a multiple of $b - a$ and $c + b$ is a multiple of $c - b$. If a is a 2011-digit number and b is a 2012-digit number, exactly how many digits does c have?

5. In the plane are 10 lines in general position, which means that no 2 are parallel and no 3 are concurrent. Where 2 lines intersect, we measure the smaller of the two angles formed between them. What is the maximum value of the sum of the measures of these 45 angles?

Note: The problems are worth 3, 4, 4, 4 and 5 points respectively.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Spring 2011.**

1. Does there exist a hexagon that can be divided into four congruent triangles by a straight cut?

2. Passing through the origin of the coordinate plane are 180 lines, including the coordinate axes, which form 1° angles with one another at the origin. Determine the sum of the x-coordinates of the points of intersection of these lines with the line $y = -x + 100$.

3. Baron Munchausen has a set of 50 coins. The mass of each is a distinct positive integer not exceeding 100, and the total mass is even. The Baron claims that it is not possible to divide the coins into two piles with equal total mass. Can the Baron be right?

4. Given an integer $n > 1$, prove that there exist distinct positive integers a, b, c and d such that $a + b = c + d$ and $\frac{a}{b} = \frac{nc}{d}$.

5. AD and BE are altitudes of an acute triangle ABC. From D, perpendiculars are dropped to AB at G and AC at K. From E, perpendiculars are dropped to AB at F and BC at H. Prove that FG is parallel to HK and $FK = GH$.

6. Two ants crawl along the sides of the 49 squares of a 7×7 board. Each ant passes through all 64 vertices exactly once and returns to its starting point. What is the smallest possible number of sides covered by both ants?

7. In every cell of a square table is a number. The sum of the largest two numbers in each row is a and the sum of the largest two numbers in each column is b. Prove that $a = b$.

Note: The problems are worth 4, 4, 5, 6, 7, 10 and 10 points respectively.

International Mathematics
TOURNAMENT OF THE TOWNS

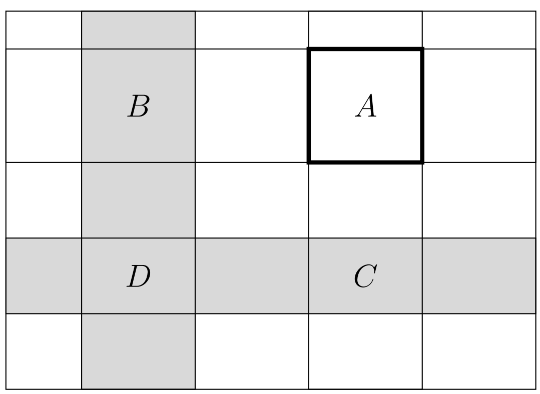
Solution to Junior O-Level Spring 2011

1. Suppose that we managed to place the numbers on a circle so that the difference between two adjacent numbers is odd. This means that odd and even numbers must alternate. From the condition follows that each number has both neighbours either both greater or both less than itself.

   Note that 1 is an odd number and it can only have the neighbours greater than itself. Since the numbers increase and decrease alternately, each odd number has the neighbours greater than itself and therefore each even number has the neighbours less than itself. However number 2 can have possibly only one number less than itself. Contradiction.

2. The number of cells with non integer perimeters is at most $121 - 111 = 10$. No matter how these cells are distributed in the rectangle, at least one row and one column consist of cells with integer perimeters. On the figure below these raw and column are shaded.

   We prove that any other cell has an integer perimeter. Let A be this shell and (a, b) be the dimensions its dimensions. Then it has a perimeter $p = 2a+2b = (2a+2c)+(2b+2d)-(2c+2d)$ where $(2a+2c)$, $(2b+2d)$, and $(2c+2d)$ are the perimeters of cells C, B, and D respectively. Since all these perimeters are integers, the perimeter $2a + 2b$ is also integer.



3. Yes, it is possible. (Actually one can grow any number of worms in one hour).

   We can take for granted that in time t the worm grows additional t in the length $(0 < t \leq 1)$.

   Assume that at moment 0 there was 1 fully grown worm. Let dissect it into two parts of lengths t and $(1 - t)$ respectively, $0 < t \leq \frac{1}{2}$.

   Then, at moment t the part that had the length t becomes $2t$, while another part becomes fully grown worm. We dissect it into parts of the lengths $(2t)$ and $(1 - 2t)$. respectively. Therefore, after the dissection we have two worms of length $2t$ and one of the length $(1 - 2t)$.

   In time $2t$ after the last dissection (or $t + 2t = 3t$) from the beginning, we have two worms of length $4t$ (each grown from the part $2t$) and one fully grown worm. Again, we dissect the

International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Spring 2011.**

1. Baron Munchausen has a set of 50 coins. The mass of each is a distinct positive integer not exceeding 100, and the total mass is even. The Baron claims that it is not possible to divide the coins into two piles with equal total mass. Can the Baron be right?

2. In the coordinate space, each of the eight vertices of a rectangular box has integer coordinates. If the volume of the solid is 2011, prove that the sides of the rectangular box are parallel to the coordinate axes.

3. (a) Does there exist an infinite triangular beam such that two of its cross-sections are similar but not congruent triangles?
   (b) Does there exist an infinite triangular beam such that two of its cross-sections are equilateral triangles of sides 1 and 2 respectively?

4. There are n red sticks and n blue sticks. The sticks of each colour have the same total length, and can be used to construct an n-gon. We wish to repaint one stick of each colour in the other colour so that the sticks of each colour can still be used to construct an n-gon. Is this always possible if
   (a) $n = 3$;
   (b) $n > 3$?

5. In the convex quadrilateral $ABCD$, BC is parallel to AD. Two circular arcs ω_1 and ω_3 pass through A and B and are on the same side of AB. Two circular arcs ω_2 and ω_4 pass through C and D and are on the same side of CD. The measures of ω_1, ω_2, ω_3 and ω_4 are α, β, β and α respectively. If ω_1 and ω_2 are tangent to each other externally, prove that so are ω_3 and ω_4.

6. In every cell of a square table is a number. The sum of the largest two numbers in each row is a and the sum of the largest two numbers in each column is b. Prove that $a = b$.

7. Among a group of programmers, every two either know each other or do not know each other. Eleven of them are geniuses. Two companies hire them one at a time, alternately, and may not hire someone already hired by the other company. There are no conditions on which programmer a company may hire in the first round. Thereafter, a company may only hire a programmer who knows another programmer already hired by that company. Is it possible for the company which hires second to hire ten of the geniuses, no matter what the hiring strategy of the other company may be?

Note: The problems are worth 4, 6, 3+4, 4+4, 8, 8 and 11 points respectively.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Spring 2011.**

1. The faces of a convex polyhedron are similar triangles. Prove that this polyhedron has two pairs of congruent faces.

2. Worms grow at the rate of 1 metre per hour. When they reach their maximum length of 1 metre, they stop growing. A full-grown worm may be dissected into two new worms of arbitrary lengths totalling 1 metre. Starting with 1 full-grown worm, can one obtain 10 full-grown worms in less than 1 hour?

3. Along a circle are 100 white points. An integer k is given, where $2 \leq k \leq 50$. In each move, we choose a block of k adjacent points such that the first and the last are white, and we paint both of them black. For which values of k is it possible for us to paint all 100 points black after 50 moves?

4. Four perpendiculars are drawn from four vertices of a convex pentagon to the opposite sides. If these four lines pass through the same point, prove that the perpendicular from the fifth vertex to the opposite side also passes through this point.

5. In a country, there are 100 towns. Some pairs of towns are joined by roads. The roads do not intersect one another except meeting at towns. It is possible to go from any town to any other town by road. Prove that it is possible to pave some of the roads so that the number of paved roads at each town is odd.

Note: The problems are worth 3, 4, 4, 5 and 5 points respectively.

International Mathematics
TOURNAMENT OF TOWNS

Junior A-Level Paper **Fall 2012.**

1. The decimal representation of an integer uses only two different digits. The number is at least 10 digits long, and any two neighbouring digits are distinct. What is the greatest power of two that can divide this number?

Solution. The answer is 6. Let N be the given number. Consider the case when the number of digits of N is $2m$. Then N can be represented as $N = xyxy \dots xy = 1010 \dots 1 \times xy$. Since the first factor is odd, the greatest power of two that can divide N coincides with the greatest power of two that can divide xy, which is 6 (then $xy = 64 = 2^6$).

If N contains $2m+1$ digits then $N = yxy \dots xy = y \cdot 10^{2m} + xy \dots xy$ where $m \geq 5$ and therefore $2m \geq 10$. Then $y \cdot 10^{2m}$ is divisible by 2^7; therefore if N is divisible by 2^7 so is $xy \dots xy$, but it is divisible by at most 2^6 as is shown above. Hence the answer is 6.

2. Chip and Dale play the following game. Chip starts by splitting 222 nuts between two piles, so Dale can see it. In response, Dale chooses some number N from 1 to 222. Then Chip moves nuts from the piles he prepared to a new (third) pile until there will be exactly N nuts in any one or two piles. When Chip accomplishes his task, Dale gets an exact amount of nuts that Chip moved. What is the maximal number of nuts that Dale can get for sure, no matter how Chip acts?

(Naturally, Dale wants to get as many nuts as possible, while Chip wants to lose as little as possible).

Solution The answer is 37. *Upper estimate.* If Chip puts 74 and 148 nuts in the piles, he can move not more 37 nuts for any N. Indeed, represent N as $74k + r$ where k equals 0, 1, 2, or 3, and $-37 \leq r < 37$. If $r = 0$ then $k > 0$, N equals 74, 148, or 222, and this is the number of nuts in one or two piles. If $r > 0$ then $74k$ equals 0, 74, or 148. Then Chip moves r nuts from an appropriate pile to a new pile and presents this pile to Dale, adding if necessary the pile that was not changed.

Lower estimate. For any initial splitting, there exists N such that at least 37 nuts must be moved. Indeed, let the numbers of nuts in the initial piles be p and q, $q \geq p$. If $p \geq 74$ then for $N = 37$ it is necessary to move 37 nuts. If $p < 74$ then $q > 148$. For $N = 111$ it is necessary either to add more than 37 to p nuts or to remove more than 37 from q nuts.

3. Some cells of a 11×11 table are filled with pluses. It is known that the total number of pluses in the given table and in any of its 2×2 sub-tables is even. Prove that the total number of pluses on the main diagonal of the given table is also even.

TOURNAMENT OF TOWNS

Junior O-Level Paper **Fall 2012.**

1. Five students have the first names Clark, Donald, Jack, Robin and Steve, and have the last names (in a different order) Clarkson, Donaldson, Jackson, Robinson and Stevenson. It is known that Clark is 1 year older than Clarkson, Donald is 2 years older than Donaldson, Jack is 3 years older than Jackson, Robin is 4 years older than Robinson.

Who is older, Steve or Stevenson and what is the difference in their ages?

Solution. The sum of ages of Clark, Donald, Jack, Robin and Steve is equal to the sum of ages of Clarkson, Donaldson, Jackson, Robinson and Stevenson. Hence Stevenson is older than Steve, and the difference is $1 + 2 + 3 + 4 = 10$ years.

2. Let $C(n)$ be the number of prime divisors of a positive integer n. (For example, $C(10) = 2$, $C(11) = 1$, $C(12) = 2$).

Consider set S of all pairs of positive integers (a, b) such that $a \neq b$ and

$$C(a + b) = C(a) + C(b).$$

Is set S finite or infinite?

Solution. The set of pairs is infinite. *Example 1*. $a = 2^k$, $b = 2^{k+1}$, $(a + b) = 3 \cdot 2^k$, $k = 1, 2, \ldots$. Then $C(a) = 1$, $C(b) = 1$, $C(a + b) = 2$.

Example 2 (based on different idea). Let $a = p$, $b = 5p$, $(a+b) = 6p = 2 \cdot 3 \cdot p$. Let $p \neq 2, 3, 5$ is a prime. Then, $C(a) = 1$, $C(b) = 2$, $C(a + b) = 3$.

3. A table 10×10 was filled according to the rules of the game “Bomb Squad”: several cells contain bombs (one bomb per cell) while each of the remaining cells contains a number, equal to the number of bombs in all cells adjacent to it by side or by vertex.

Then the table is rearranged in the “reverse” order: bombs are placed in all cells previously occupied with numbers and the remaining cells are filled with numbers according to the same rule. Can it happen that the total sum of the numbers in the table will increase in a result?

Solution. The answer is no. In a given table consider all unordered pairs of adjacent cells (by side or by vertex) one of which has a bomb and another is empty (two pairs are different it they differ in at least one cell). It is clear that the sum of all numbers in the table equals to the number of these pairs.

For the complementary table (bomb and no-bomb cells reversed) we have the same number of such pairs. Therefore the sum of the numbers in the complementary table will be the same as in the original table.

International Mathematics
TOURNAMENT OF TOWNS

Senior A-Level Paper **Fall 2012.**

1. Given an infinite sequence of numbers $a_1, a_2, a_3, \dots$ For each positive integer k there exists a positive integer $t = t(k)$ such that $a_k = a_{k+t} = a_{k+2t} = \dots$ Is this sequence necessarily periodic? That is, does a positive integer T exist such that $a_k = a_{k+T}$ for each positive integer k?

Solution. The answer is no. For example, let m_k be the highest degree of 2 that divides k, $a_k = 0$ if m_k is even and $a_k = 1$ if m_k is odd, and $t(k) = 2m_k$.

2. Chip and Dale play the following game. Chip starts by splitting 1001 nuts between three piles, so Dale can see it. In response, Dale chooses some number N from 1 to 1001. Then Chip moves nuts from the piles he prepared to a new (fourth) pile until there will be exactly N nuts in any one or more piles. When Chip accomplishes his task, Dale gets an exact amount of nuts that Chip moved. What is the maximal number of nuts that Dale can get for sure, no matter how Chip acts? (Naturally, Dale wants to get as many nuts as possible, while Chip wants to lose as little as possible).

Solution. Consider a line segment of length 1001 on which we mark points A, B and C corresponding to the piles with a, b and c nuts in them. Let us also mark the points $A + B$, $A + B$ and $B + C$, corresponding to two combined piles, the point O, corresponding to an empty pile, and the point $A + B + C$, corresponding to the pile of 1001 nuts. If Dale choses a number n then Chip's strategy is to look for the closest marked point to this number and to move nuts from the corresponding pile (or combined piles) to the pile 0. It is clear that if the points are marked uniformly (with the distance 143 between each pair of subsequent points) then the maximal difference between n and the closest number is 71, therefore Chip can loose at most 71 nuts.

On the other hand, since the maximal distance between the subsequent points is at least 143, Dale can always choose a number such that he can guarantee at least 71 nuts.

3. A car rides along a circular track in the clockwise direction. At noon Peter and Paul took their positions at two different points of the track. Some moment later they simultaneously ended their duties and compared their notes. The car passed each of them at least 30 times. Peter noticed that each circle was passed by the car 1 second faster than the preceding one while Paul's observation was opposite: each circle was passed 1 second slower than the preceding one.

Prove that their duty was at least an hour and a half long.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2012.**

1. A table 10×10 was filled according to the rules of the game "Bomb Squad": several cells contain bombs (one bomb per cell) while each of the remaining cells contains a number, equal to the number of bombs in all cells adjacent to it by side or by vertex.

Then the table is rearranged in the "reverse" order: bombs are placed in all cells previously occupied with numbers and the remaining cells are filled with numbers according to the same rule. Can it happen that the total sum of the numbers in the table will increase in a result?

Solution. The answer is no. In a given table consider all unordered pairs of adjacent cells (by side or by vertex) one of which has a bomb and another is empty (two pairs are different it they differ in at least one cell). It is clear that the sum of all numbers in the table equals to the number of these pairs.

For the complementary table (bomb and no-bomb cells reversed) we have the same number of such pairs. Therefore the sum of the numbers in the complementary table will be the same as in the original table.

2. Given a convex polyhedron and a sphere intersecting each its edge at two points so that each edge is trisected (divided into three equal parts). Is it necessarily true that all faces of the polyhedron are

(a) congruent polygons?

(b) regular polygons?

Solution. a) The answer is negative. Consider a regular prism with triangular base and square lateral faces. On each edge, mark the trisecting points. Clearly they are equidistant from the centre of the prism and thus belong to the corresponding sphere.

b) The answer is positive. Suppose $A_1 \dots A_n$ is a face of the polyhedron. All the points B_i, C_i trisecting its sides $A_{i-1}A_i$ lie on the same circle that is the intersection of the sphere with the plane of the face. Suppose $A_{i-1}A_i = 3a$, $A_iA_{i+1} = 3b$. By secant theorem, $A_iB_i \cdot A_iC_i = A_iB_{i+1} = A_iC_{i+1} \Leftrightarrow 2a^2 = 2b^2 \Leftrightarrow a = b$. Hence the face is equilateral. It remains to prove that it has equal angles. All segments B_iC_i are equal as well as isosceles triangles B_iOC_i where O is the centre of the circle. Hence the equality holds for all triangles B_iOA_i, all angles B_iA_iO and all angles $\angle A_{i-1}A_iA_{i+1} = 2\angle B_iA_iO$.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Spring 2012[1].**

1. It is possible to place an even number of pears in a row such that the masses of any two neighbouring pears differ by at most 1 gram. Prove that it is then possible to put the pears two in a bag and place the bags in a row such that the masses of any two neighbouring bags differ by at most 1 gram.

2. One hundred points are marked in the plane, with no three in a line. Is it always possible to connect the points in pairs such that all fifty segments intersect one another?

3. In a team of guards, each is assigned a different positive integer. For any two guards, the ratio of the two numbers assigned to them is at least 3:1. A guard assigned the number n is on duty for n days in a row, off duty for n days in a row, back on duty for n days in a row, and so on. The guards need not start their duties on the same day. Is it possible that on any day, at least one in such a team of guards is on duty?

4. Each entry in an $n \times n$ table is either $+$ or $-$. At each step, one can choose a row or a column and reverse all signs in it. From the initial position, it is possible to obtain the table in which all signs are $+$. Prove that this can be accomplished in at most n steps.

5. Let p be a prime number. A set of $p + 2$ positive integers, not necessarily distinct, is called *interesting* if the sum of any p of them is divisible by each of the other two. Determine all interesting sets.

6. A bank has one million clients, one of whom is Inspector Gadget. Each client has a unique PIN number consisting of six digits. Dr. Claw has a list of all the clients. He is able to break into the account of any client, choose any n digits of the PIN number and copy them. The n digits he copies from different clients need not be in the same n positions. He can break into the account of each client, but only once. What is the smallest value of n which allows Dr. Claw to determine the complete PIN number of Inspector Gadget?

7. Let AH be an altitude of an equilateral triangle ABC. Let I be the incentre of triangle ABH, and let L, K and J be the incentres of triangles ABI, BCI and CAI respectively. Determine $\angle KJL$.

Note: The problems are worth 4, 4, 6, 6, 8, 8 and 8 points respectively.

[1]Courtesy of Professor Andy Liu.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Spring 2012[1].**

1. A treasure is buried under a square of an 8×8 board, Under each other square is a message which indicates the minimum number of steps needed to reach the square with the treasure. Each step takes one from a square to another square sharing a common side. What is the minmum number of squares we must dig up in order to bring up the treasure for sure?

2. The number 4 has an odd number of odd positive divisors, namely 1, and an even number of even positive divisors, namely 2 and 4. Is there a number with an odd number of even positive divisors and an even number of odd positive divisors?

3. In the parallelogram $ABCD$, the diagonal AC touches the incircles of triangles ABC and ADC at W and Y respectively, and the diagonal BD touches the incircles of triangles BAD and BCD at X and Z respectively. Prove that either W, X, Y and Z coincide, or $WXYZ$ is a rectangle.

4. Brackets are to be inserted into the expression $10 \div 9 \div 8 \div 7 \div 6 \div 5 \div 4 \div 3 \div 2$ so that the resulting number is an integer.
   (a) Determine the maximum value of this integer.
   (b) Determine the minimum value of this integer.

5. RyNo, a little rhinoceros, has 17 scratch marks on its body. Some are horizontal and the rest are vertical. Some are on the left side and the rest are on the right side. If RyNo rubs one side of its body against a tree, two scratch marks, either both horizontal or both vertical, will disappear from that side. However, at the same time, two new scratch marks, one horizontal and one vertical, will appear on the other side. If there are less than two horizontal and less than two vertical scratch marks on the side being rubbed, then nothing happens. If RyNo continues to rub its body against trees, is it possible that at some point in time, the numbers of horizontal and vertical scratch marks have interchanged on each side of its body?

Note: The problems are worth 3, 4, 4, 2+3 and 5 points respectively.

[1]Courtesy of Professor Andy Liu.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Spring 2012[1].**

1. In a team of guards, each is assigned a different positive integer. For any two guards, the ratio of the two numbers assigned to them is at least 3:1. A guard assigned the number n is on duty for n days in a row, off duty for n days in a row, back on duty for n days in a row, and so on. The guards need not start their duties on the same day. Is it possible that on any day, at least one in such a team of guards is on duty?

2. One hundred points are marked inside a circle, with no three in a line. Prove that it is possible to connect the points in pairs such that all fifty lines intersect one another inside the circle.

3. Let n be a positive integer. Prove that there exist integers $a_1, a_2, \ldots, a_n$ such that for any integer x, the number $(\cdots(((x^2+a_1)^2+a_2)^2+\cdots)^2+a_{n-1})^2+a_n$ is divisible by $2n-1$.

4. Alex marked one point on each of the six interior faces of a hollow unit cube. Then he connected by strings any two marked points on adjacent faces. Prove that the total length of these strings is at least $6\sqrt{2}$.

5. Let ℓ be a tangent to the incircle of triangle ABC. Let ℓ_a, ℓ_b and ℓ_c be the respective images of ℓ under reflection across the exterior bisector of $\angle A$, $\angle B$ and $\angle C$. Prove that the triangle formed by these lines is congruent to ABC.

6. We attempt to cover the plane with an infinite sequence of rectangles, overlapping allowed.
   (a) Is the task always possible if the area of the nth rectangle is n^2 for each n?
   (b) Is the task always possible if each rectangle is a square, and for any number N, there exist squares with total area greater than N?

7. Konstantin has a pile of 100 pebbles. In each move, he chooses a pile and splits it into two smaller ones until he gets 100 piles each with a single pebble.
   (a) Prove that at some point, there are 30 piles containing a total of exactly 60 pebbles.
   (b) Prove that at some point, there are 20 piles containing a total of exactly 60 pebbles.
   (c) Prove that Konstantin may proceed in such a way that at no point, there are 19 piles containing a total of exactly 60 pebbles.

Note: The problems are worth 4, 5, 6, 6, 8, 3+6 and 6+3+3 points respectively.

[1]Courtesy of Professor Andy Liu

International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Spring 2012[1].**

1. Each vertex of a convex polyhedron lies on exactly three edges, at least two of which have the same length. Prove that the polyhedron has three edges of the same length.

2. The cells of a $1 \times 2n$ board are labelled $1, 2, \ldots, n, \ -n, \ \ldots, \ -2, \ -1$ from left to right. A marker is placed on an arbitrary cell. If the label of the cell is positive, the marker moves to the right a number of cells equal to the value of the label. If the label is negative, the marker moves to the left a number of cells equal to the absolute value of the label. Prove that if the marker can always visit all cells of the board, then $2n+1$ is prime.

3. Consider the points of intersection of the graphs $y = \cos x$ and $x = 100\cos(100y)$ for which both coordinates are positive. Let a be the sum of their x-coordinates and b be the sum of their y-coordinates. Determine the value of $\frac{a}{b}$.

4. A quadrilateral $ABCD$ with no parallel sides is inscribed in a circle. Two circles, one passing through A and B, and the other through C and D, are tangent to each other at X. Prove that the locus of X is a circle.

5. In an 8×8 chessboard, the rows are numbers from 1 to 8 and the columns are labelled from a to h. In a two-player game on this chessboard, the first player has a White Rook which starts on the square b2, and the second player has a Black Rook which starts on the square c4. The two players take turns moving their rooks. In each move, a rook lands on another square in the same row or the same column as its starting square. However, that square cannot be under attack by the other rook, and cannot have been landed on before by either rook. The player without a move loses the game. Which player has a winning strategy?

Note: The problems are worth 4, 4, 5, 5 and 5 points respectively.

[1]Courtesy of Professor Andy Liu.

International Mathematics

TOURNAMENT OF THE TOWNS

Junior A-Level Solutions **Fall 2013**

1. There are 100 red, 100 yellow and 100 green sticks. One can construct a triangle using any three sticks all of different colour. Prove that there is a colour such that one can construct a triangle using any three sticks of this colour.

SOLUTION. For each of three colours (a, b,and c) consider two smallest sticks and the largest stick and denote these (x_1, x_2, x) respectively. Assume that there is no such colour that one can always construct a triangle using any three sticks of this colour. This implies: $x_1 + x_2 \le x$. Without loss of generality assume that $a_1 \le b_1 \le c_1$. Then $a_1 + b_1 \le c_1 + c_2 \le c$.

Contradiction: one cannot construct a triangle using any three sticks all of different colours.

2. A math teacher chose 10 consequent positive integers and submitted them to Pete and Basil. Each boy should split these numbers in pairs and calculate the sum of products of numbers in pairs. Prove that the boys can pair the numbers differently so that the resulting sums are equal.

SOLUTION. Let consecutive numbers be in the form $n+1, n+2, n+3, \ldots, n+10$. One can check that $P_1 = P_2$, where

$$\begin{aligned} P_1 = (n+1)(n+8) + (n+2)(n+7) + (n+3)(n+6) \\ + (n+4)(n+5) + (n+9)(n+10) \end{aligned}$$

and

$$\begin{aligned} P_2 = (n+1)(n+10) + (n+2)(n+3) + (n+4)(n+5) \\ + (n+6)(n+7) + (n+8)(n+9). \end{aligned}$$

3. Assume that C is a right angle of triangle ABC and N is a midpoint of the semicircle, constructed on CB as on diameter externally. Prove that AN divides the bisector of angle C in halves.

SOLUTION. Extend segment BN to intersect line AC at some point K. In triangle BCK the altitude CN is also a bisector, thus $KN = NB$. Angles BCL and CBK are equal to $45°$, hence the bisector CL is parallel to BK. Therefore in triangle ABK the median AN bisects CL as well.

International Mathematics

TOURNAMENT OF THE TOWNS

Junior O-Level Paper Solutions **Fall 2013**

1 [**3**] In a wrestling tournament, there are 100 participants, all of different strengths. The stronger wrestler always wins over the weaker opponent. Each wrestler fights twice and those who win both of their fights are given awards. What is the least possible number of awardees?

ANSWER: 1.

SOLUTION. Arrange participants by strength a_1 (the weakest), $a_2, a_3, \dots a_{100}$ (the strongest).

Obviously, a_{100} is one of the winners. Let wrestlers in the first round be paired as follows: $a_{100} - a_{99}$, $a_{98} - a_{97}$,..., $a_2 - a_1$, then $a_1, a_3, a_5, \dots a_{99}$ are losers.

Let the second round be paired as follows: $a_{100} - a_1$, $a_{99} - a_{98}$,..., $a_3 - a_2$, then $a_2, a_4, a_6, \dots, a_{98}$ are losers. Therefore the only participant who won in both rounds is a_{100}.

2 [**4**] Does there exist a ten-digit number such that all its digits are different and after removing any six digits we get a composite four-digit number?

ANSWER: yes.

SOLUTION. Observe that a four-digit number 1379 is divided by 7 ($1379 = 7 \times 197$) We can consider a ten-digit number in the form $1379\dots$ where the tail is any combination of remaining digits $2, 4, 6, 8, 0, 5$. It is easy to see that this number satisfies the conditions: the remaining four digits form either 1379, either an even four-digit number, or a four-digit multiple of 5.

3 [**4**] Denote by (a, b) the greatest common divisor of a and b. Let n be a positive integer such that

$$(n, n+1) < (n, n+2) < \dots < (n, n+35). \tag{1}$$

Prove that $(n, n+35) < (n, n+36)$.

SOLUTION. First we need

Lemma. $(n, n+m) \le m$.

Proof. Indeed, if p divides both n and $(n+m)$ it also divides their difference which is m. □

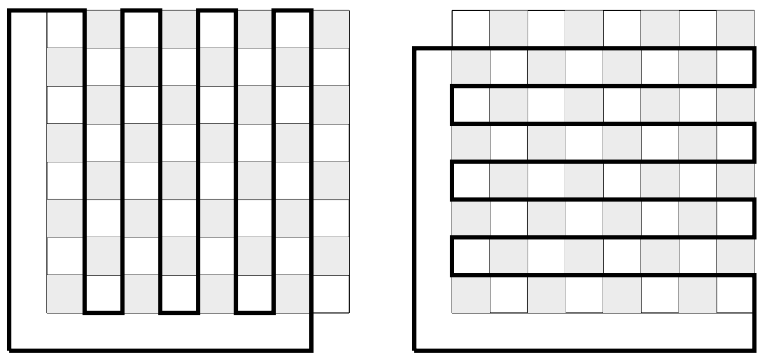
Since $(n, n+1) = 1$ and $(n, n+k)$ increases for $k = 1, \dots, 35$ then this lemma implies that for all $m = 1, \dots, 35$ we have $(n, n+m) = m$ and therefore n is divisible by m. In particular n is divisible by both 4 and 9 and therefore it is divisible by 36. Then $n+36$ is also divisible by 36 and $(n, n+36) = 36 > (n, n+35) = 35$.

International Mathematics

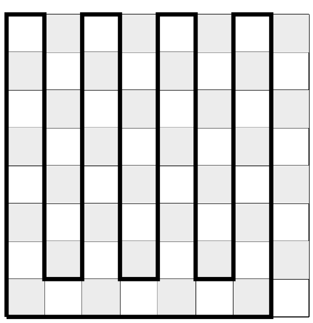
TOURNAMENT OF THE TOWNS

Senior A-Level Solutions **Fall 2013**

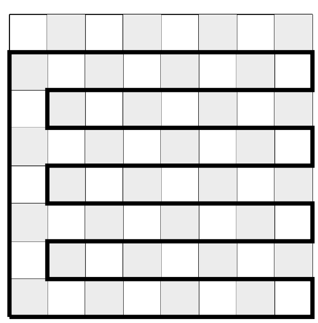
1. Pete drew a square in the plane, divided it into 64 equal square cells and painted it in a chess board fashion. He chose some cell and an interior point in it. Basil can draw any polygon (without self-intersections) in the plane and ask Pete whether the chosen point is inside or outside this polygon. What is the minimal number of questions sufficient to determine whether the chosen point is black or white?



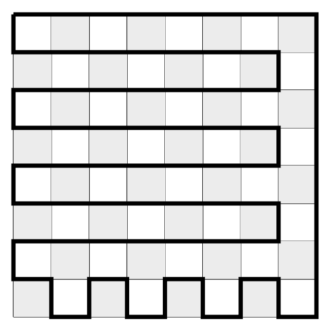
Solution 1. One question is not enough because a polygon containing all white points and no black point has to be self-intersecting. However two questions are enough: if a point belongs to just one polygon then it is white, and if a point belongs to both or none then it is black.



(a) Polygon 1



(b) Polygon 2b



(c) Polygin 2c

Solution 2 [Nikita Kapustin, gr. 11, Richmond Hill H.S.]. If the point is outside of the Polygon 1 then it is confined to verticals 2,4,6,8 and we determine the colour by drawing Polygon 2b.

International Mathematics

TOURNAMENT OF THE TOWNS

Senior O-Level Paper Solutions **Fall 2013**

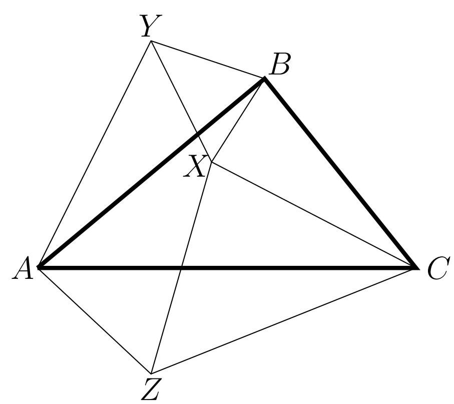
1 [**3**] Does there exist a ten-digit number such that all its digits are different and after removing any six digits we get a composite four-digit number?

ANSWER: yes.

SOLUTION. Observe that a four-digit number 1379 is divided by 7 ($1379 = 7 \times 197$) We can consider a ten-digit number in the form $1379\ldots$ where the tail is any combination of remaining digits $2, 4, 6, 8, 0, 5$. It is easy to see that this number satisfies the conditions: the remaining four digits form either 1379, either an even four-digit number, or a four-digit multiple of 5.

2 [**4**] On the sides of triangle ABC, three similar triangles are constructed with triangle YBA and triangle ZAC in the exterior and triangle XBC in the interior. (Above, the vertices of the triangles are ordered so that the similarities take vertices to corresponding vertices, for example, the similarity between triangle YBA and triangle ZAC takes Y to Z, B to A and A to C). Prove that $AYXZ$ is a parallelogram.

SOLUTION. Draw the following figure



For simplicity of notations let us denote: $\angle BAC = a$, $\angle ABC = b$, $\angle ACB = c$. Further, using similarity of triangles YBA, ZAC, and XBC let us denote

$$\angle YAB = \angle ZCA = \angle XCB = \alpha,$$
$$\angle YBA = \angle ZAC = \angle XBC = \beta$$

and

$$\angle AYB = \angle CZA = \angle CXB = \gamma.$$

Since triangle YBA is similar to triangle XBC we have $YB : AB = XB : BC$. It follows that $YB : BX = AB : BC$. Since $\angle YBA = \angle XBC$ we have $\angle YBX = b$.

International Mathematics
TOURNAMENT OF THE TOWNS SOLUTIONS

Junior A-Level Paper **Spring 2013.**

1 [**4**] Several positive integers are written on a blackboard. The sum of any two of them is some power of two (for example, $2, 4, 8, \ldots$). What is the maximal possible number of different integers on the blackboard?

ANSWER: Two.

SOLUTION 1. Let a be the greatest number written on a blackboard. There is an integer $n \geq 0$ such that $2^n \leq a < 2^{n+1}$. Then $2^n < a + b \leq 2a < 2^{n+2}$ where b is the other number on the board. Hence $a + b = 2^{n+1}$. Thus all the remaining integers are in the form $2^{n+1} - a$. Therefore the number of different integers on the board is no more than two.

Example of two integers: 1 and 3.

SOLUTION 2. We prove that the number of integers does not exceed 2. Assume that $a < b < c$ on the board. Then $a + b < a + c < b + c$ are different powers of 2 and therefore $b + c \geq 2(a + c)$. Then $b \geq 2a + c$ which is impossible. Example of two integers: 1 and 3.

2 [**4**] Twenty children, ten boys and ten girls, are standing in a line. Each boy counted the number of children standing to the right of him. Each girl counted the number of children standing to the left of her. Prove that the sums of numbers counted by the boys and the girls are the same.

SOLUTION 1. Assume that the children in a line stay to the right of the first person. Let a boy on the k-th position count the number $20 - k$ while a girl on the n-th position count the number $n - 1$. Therefore the total sums of numbers obtained by boys and girls are $200 - S_b$ and $S_g - 10$ respectively, where S_b is the sum of boys' positions and S_g is the sum of girls' positions. It remains to check that $200 - S_b = S_g - 10$. The latter follows from $S_b + S_g = 1 + 2 + \cdots + 20 = 210$.

SOLUTION 2. Let B and G be the sums counted by boys and girls respectively. Note that if a boy and a girl interchange their places in the line, both sums will increase or decrease on the same amount. Therefore the difference between B and G is always the same. However, in situation when ten girls are followed by ten boys it is obvious that both sums are the same.

3 [**5**] There is a 19×19 board. Is it possible to mark some 1×1 squares so that each of 10×10 squares contain different number of marked squares?

ANSWER. Yes, it is possible. SOLUTION. Observe that each of 100 of 10×10 squares in a 19×19 shares a common central cell (1×1 square). Assume that it is marked; otherwise, we can interchange marked and unmarked cells.

Let us mark every cell in each of nine bottom rows, the central cell and all cells in central row to the right of it. Consider a 10×10 square at the top left position. It has one marked cell. Let us move this square to the right, one column at time. In this way, each new 10×10 square will have one more marked cell than the previous one. Therefore we get squares with 1,2,. . . ,10 marked cells.

Now, move each of these ten squares down one row at time. It is easy to see that each new 10×10 square contains 10 more marked cells than the square one position above it. In this way, we get squares with 11, 21, 31,. . . ,91, 12, 22, 32,. . . ,92,. . . , 20, 30,. . . , 100 marked cells.

4 [**5**] On a circle, there are 1000 nonzero real numbers painted black and white in turn. Each black number is equal to the sum of two white numbers adjacent to it, and each white number is

International Mathematics
TOURNAMENT OF THE TOWNS SOLUTIONS

Junior O-Level Paper **Spring 2013.**

1 **[3]** There are six points on the plane such that one can split them into two triples each creating a triangle. Is it always possible to split these points into two triples creating two triangles with no common point (neither inside, nor on the boundary)?

ANSWER: No.

Example: Consider the vertices and the midpoints of a triangle.

2 **[4]** There is a positive integer A. Two operations are allowed: increasing this number by 9 and deleting a digit equal to 1 from any position. Is it always possible to obtain $A+1$ by applying these operations several times?

REMARK. If leading digit 1 is deleted, all leading zeros are deleted as well.

ANSWER: Yes.

SOLUTION. Given the number $A+1$ create a "new number" which starts with eight "1"s followed by the number $A+1$. Note that the new number and the number A have the same remainders when divided by 9. Therefore given the number A one can get the number $A+1$ by adding "9"s to A until one obtains the "new number". Then one removes eight leading "1"s.

3 **[4]** Each of 11 weights is weighing an integer number of grams. No two weights are equal. It is known that if all these weights or any group of them are placed on a balance then the side with a larger number of weights is always heavier. Prove that at least one weight is heavier than 35 grams.

SOLUTION. Let us arrange the weights in increasing order, $a_1 < a_2 < a_3 \cdots < a_{11}$. Note that the difference between any two consequent weights is at least 1. Therefore, $a_n \geq a_m + (n-m)$, if $m < n$. According to the given we have

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_2 + a_6 > a_7 + a_8 + a_9 + a_{10} + a_{11}$$

and since

$$a_7 + a_8 + a_9 + a_{10} + a_{11} \geq (a_2+5) + (a_3+5) + \ldots + (a_6+5) = a_2 + a_3 + a_4 + a_5 + a_6 + 25$$

we have $a_1 > 25$. Then $a_{11} \geq a_1 + 10 > 35$.

4 **[5]** Eight rooks are placed on a 8×8 chessboard, so that no two rooks attack one another. All squares of the board are divided between the rooks as follows. A square where a rook is placed belongs to it. If a square is attacked by two rooks then it belongs to the nearest rook; in case these two rooks are equidistant from this square each of them possesses a half of the square. Prove that every rook possesses the equal area.

SOLUTION. Observe that a rook attacks 15 squares in total, 7 squares in a column and 7 squares in a row where it stands plus a square where it stands.

International Mathematics
TOURNAMENT OF THE TOWNS SOLUTIONS

Senior A-Level Paper **Spring 2013.**

1 [**3**] Several positive integers are written on a blackboard. The sum of any two of them is a positive integer power of two (for example, $2, 4, 8, \ldots$). What is the maximal possible number of different integers on the blackboard?

Answer. Two.

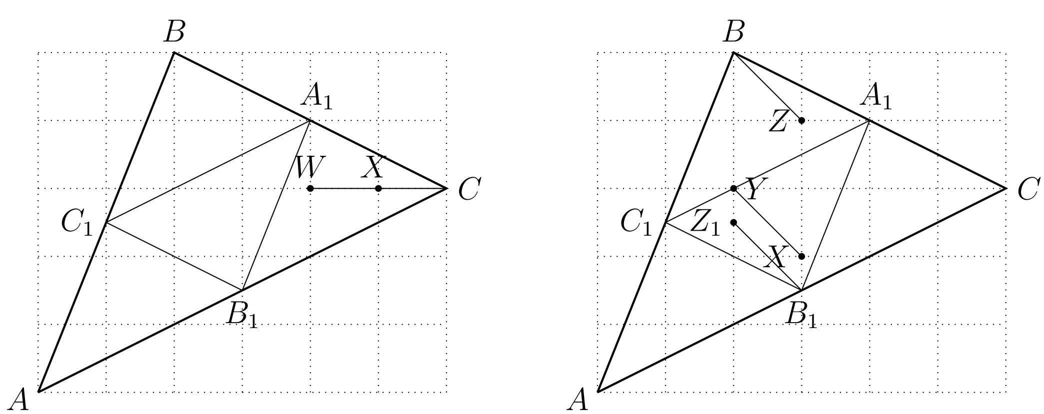
Solution. See Juniors 1.

2 [**4**] A boy and a girl were sitting on a long bench. Then twenty more children one after another came to sit on the bench, each taking a place between already sitting children. Let us call a girl brave if she sat down between two boys, and let us call a boy brave if he sat down between two girls. It happened, that in the end all girls and boys were sitting in the alternating order. Is it possible to uniquely determine the number of brave children?

Solution. Divide the bench into segments occupied by boys or girls only. These segments alternate. Notice that if a not brave child comes to the bench then the number of segments does not change. If a brave child comes to the bench then the number of segments increases by 2. Initially there were two segments. In the end there were 22 segments. Therefore, the number of brave children is $(22 - 2) : 2 = 10$.

3 [**6**] A point in the plane is called a node if both its coordinates are integers. Consider a triangle with vertices at nodes containing at least two nodes inside. Prove that there exists a pair of internal nodes such that a straight line connecting them either passes through a vertex or is parallel to side of the triangle.

Solution. Let A_1, B_1, and C_1 be midpoints of sides BC, CA, and AB of triangle ABC respectively.



(a) (b)

Consider two arbitrary nodes X and Y inside the triangle. Suppose one of them lies outside triangle $A_1B_1C_1$; assume that node X belongs to triangle A_1B_1C (see Figure (a)). Construct segment CW so that point X is the midpoint of CW. Note that point W is also a node, and that W belongs to the interior of triangle ABC. Hence there are two interior nodes, X and W such that line CW passes through a vertex (C).

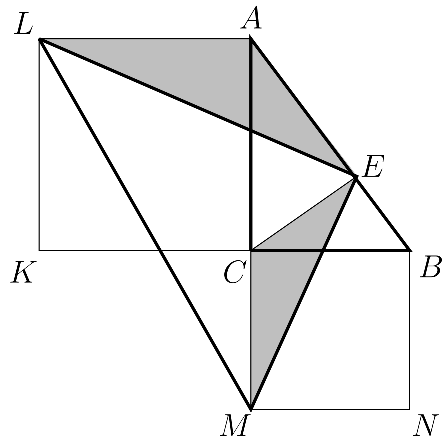
International Mathematics
TOURNAMENT OF THE TOWNS SOLUTIONS

Senior O-Level Paper **Spring 2013.**

1 [**3**] There is a positive integer A. Two operations are allowed: increasing this number by 9 and deleting a digit equal to 1 from any position. Is it always possible to obtain $A+1$ by applying these operations several times?

SOLUTION. See Junior 1.

2 [**4**] Let C be a right angle in triangle ABC. On legs AC and BC the squares $ACKL$, $BCMN$ are constructed outside of triangle. If CE is an altitude of the triangle prove that LEM is a right angle.



SOLUTION 1. Since ABC is right triangle and CE is perpendicular to AB, triangles CBE and ACE are similar. Then we have

(a) $\angle CAB = \angle ECB$ (and therefore, $\angle LAE = \angle MCE$) and also

(b) $CM/CE = AL/AE$ (it follows from $CB/CE = AC/AE$).

Therefore, triangles ALE and CME are similar. Then $\angle ALE = \angle EMC$ and therefore quadrilateral $LAEM$ is cyclic. This implies $\angle LEM = \angle LAM = 90°$.

SOLUTION 2. It is easy to see that AEC and ACB are similar, hence $\frac{CE}{EA} = \frac{CB}{CA} = \frac{CM}{AL}$. Thus the rotation by 90° followed by homothety with center E and factor $\frac{CE}{CA}$ transforms segment EA into segment EC and line AL into line CM. Then segment AL transforms into CM while segment EL into segment EM. Hence $\angle LEM = 90°$.

3 [**4**] Eight rooks are placed on a 8×8 chessboard, so no two rooks attack one another. All squares of the board are divided between the rooks as follows. A square where a rook is placed belongs to it. If a square is attacked by two rooks then it belongs to the nearest rook; in case these two rooks are equidistant from this square then each of them possesses a half of the square. Prove that every rook possesses the equal area.

SOLUTION. See Junior 4.

INTERNATIONAL MATHEMATICS TOURNAMENT OF TOWNS
Junior A-Level Paper, Spring 2014.

1. During Christmas party Santa handed out to the children 47 chocolates and 74 marmalades. Each girl got 1 more chocolate than each boy but each boy got 1 more marmalade than each girl. What was the number of the children?

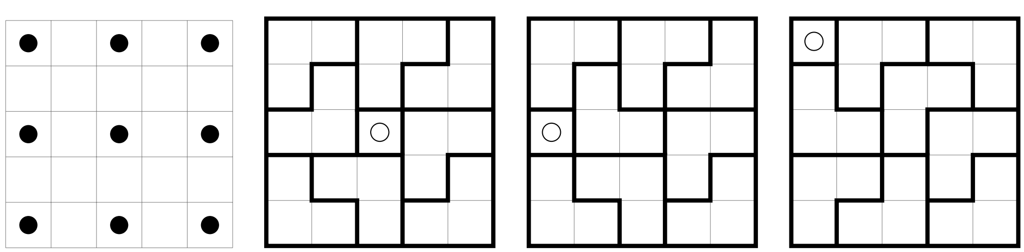
Solution. Each child got the same number of treats and the total number of treats is $74 + 47 = 121$. Therefore there could be either (a) 11 children, or (b) 121, or (c) just 1 child, and each child got 11, 1, or 121 treat respectively.

Remark. In case (a) let x denote the number of boys and c the number of chocolates each girl got. Then $(c-1)x + c(11-x) = 47$ or $11c = 47 + x$. The only integer solution with $0 \le x \le 11$ is $x = 8$, $c = 5$ (so, 8 boys, 3 girls). In case (b) each boy got just 1 marmalade, and each girl got just 1 chocolate (so, 74 boys and 47 girls). Case (c) is correct from the point of view of formal logic.

2. Peter marks several cells on a 5×5 board. Basil wins if he can cover all marked cells with three-cell corners. The corners must be inside the board and not overlap. What is the least number of cells Peter should mark to prevent Basil from winning? (Cells of the corners must coincide with the cells of the board).

Solution. If Peter marks 9 points as shown on (a) Basil cannot cover them. Indeed, no corner can cover more than one marked cell, so Basil needs 9 corners; but they contain 27 cells while the whole board contains only 25.

If Peter marks 8 cells Basil can cover all of them. Indeed, one of the cells shown on (a) is not marked. However the remaining 24 cells could be covered as shown on (b)–(d).



(a) (b) (c) (d)

INTERNATIONAL MATHEMATICS TOURNAMENT OF TOWNS

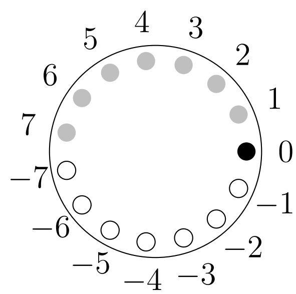
Junior O-Level, Spring 2014.

1. Each of given 100 numbers was increased by 1. Then each number was increased by 1 once more. Given that the first time the sum of the squares of the numbers was not changed find how this sum was changed the second time.

Solution. Given that the sum of the squares did not change when we added 1 to each number, we have $(a_1+1)^2+(a_2+1)^2+\cdots+(a_{100}+1)^2)-(a_1^2+a_2^2+\cdots+a_{100}^2)=0$ or $(2a_1+1)+(2a_2+1)+\cdots+(2a_{100}+1)=0$. Therefore, we have $a_1+a_2+\cdots+a_{100}=-50$. If we increase each number by 1 once more, the sum of squares will be change by $(a_1+2)^2+(a_2+2)^2+\cdots+(a_{100}+2)^2-(a_1^2+a_2^2+\cdots+a_{100}^2)=(4a_1+4)+(4a_2+4)+\cdots+(4a_{100}+4)=4\times(-50)+400=200$.

2. Mother baked 15 pasties. She placed them on a round plate in a circular way: 7 with cabbage, 7 with meat and one with cherries in that exact order and put the plate into a microwave. All pasties look the same but Olga knows the order. However she doesn't know how the plate has been rotated in the microwave. She wants to eat a pasty with cherries. Can Olga eat her favourite pasty for sure if she is not allowed to try more than three other pasties?

Solution. Denote the cherry pasty by 0, the cabbage pasties by 1,. . . , 7 and the meat pasties by $-1,\dots,-7$. If Olga does not get the cherry pasty on her first try, it must be either a cabbage pasty or a meat pasty. On her second try Olga takes the 4-th pasty from the first one in the direction of the cherry pasty. She gets either the cherry pasty 0, or the cabbage pasty 1,2,3, or the meat pasty $-1,-2,-3$.



On her last try Olga takes the second pasty from her second try in the direction of the cherry pasty and gets either the cherry pasty 0, or the cabbage pasty 1, or the meat pasty -1. Hence, after at most three tries Olga knows the position of the cherry pasty for sure.

3. The entries of a 7×5 table are filled with numbers so that in each 2×3 rectangle (vertical or horizontal) the sum of numbers is 0. For 100 dollars

INTERNATIONAL MATHEMATICS TOURNAMENT OF TOWNS
Senior A-Level Paper, Spring 2014.

1. Doono wrote several **1**s, placed signs "+" or "×" between every two of them, put several brackets and got 2014 in the result. His friend Dunno replaced all "+" by "×" and all "×" by "+" and also got 2014. Can this be true?

Solution. Yes, it could be true. For example, consider the following expression consisting of 4027 **1**s:

$$1+\underbrace{1\times 1+1\times 1+\ldots+1\times 1}_{\text{2013 terms}}$$

which obviously equals 2014. After Dunno changed signs it became

$$\underbrace{1\times 1+1\times 1+\ldots+1\times 1}_{\text{2013 terms}}+1$$

which also equals 2014.

2. Is it true that any convex polygon can be dissected by a straight line into two polygons with equal perimeters and

(a) equal greatest sides?

(b) equal smallest sides?

(a) Answer: Yes

Solution. Consider a convex polygon polygon and point M on its boundary. Consider its opposite point $N=N(M)$. It means that MN dissects polygon into two $MA_1\ldots A_mN$ and $NA_{m+1}\ldots A_nM$ with equal perimeters (it is possible that M and N are among vertices of the original polygon). Here $MA_1\ldots A_mN$ is in the counterclockwise direction. Define $f(M)$ as a greatest side of $MA_1\ldots A_mN$. Observe that $f(M)$ continuously depends on M. Then $g(M)=f(M)-f(N(M))$ also continuously depends on M. However as M changes from original point M_0 to its opposite point N_0, $g(M)$ changes from $g(M_0)$ to $g(N_0)=-g(M_0)$. Therefore $g(M)=0$ for some M.

Remark. $h(M)$ as the smallest side of $MA_1\ldots A_mN$ is not continuous and these arguments do not work for Part (b).

INTERNATIONAL MATHEMATICS TOURNAMENT OF TOWNS

Senior O-Level, Spring 2014.

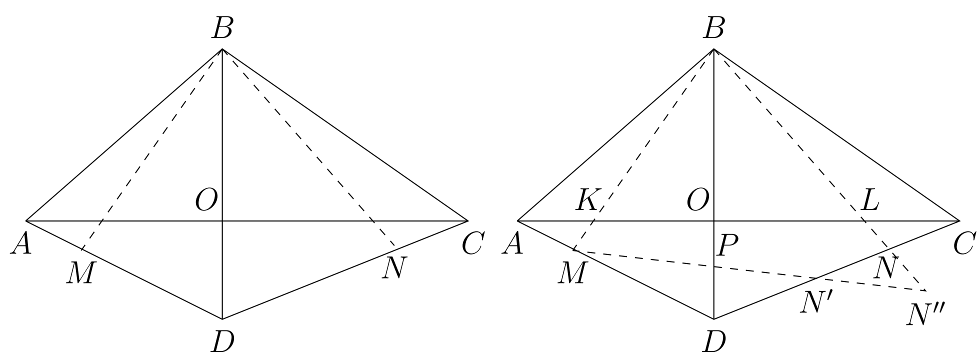
1. Inspector Gadget has 36 stones with masses 1 gram, 2 grams, ..., 36 grams. Doctor Claw has a superglue such that one drop of it glues two stones together (thus two drops glue 3 stones together and so on). Doctor Claw wants to glue some stones so that in obtained set Inspector Gadget cannot choose one or more stones with the total mass 37 grams. Find the least number of drops needed for Doctor Claw to fulfil his task.

Answer: 9

Solution. (a) Among the given stones there are 18 stones with odd masses which could be split into 9 pairs. To glue stones in pairs Doctor Claw needs 9 drops. In new group of stones there is no stone with odd weight. Therefore, Inspector Gadget cannot fulfil his task.

(b) Let us split all stones into 18 pairs so that in each pair a total weight of stones is 37. Then Doctor Claw needs to "spoil" at least one stone in each pair which is impossible with less than 9 drops.

2. In a convex quadrilateral $ABCD$ the diagonals are perpendicular. Points M and N are marked on sides AD and CD respectively. Prove that lines AC and MN are parallel given that angles ABN and CBM are right angles.



(a) (b)

37th International Mathematics

TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2015**

Problem 1. Is it true that every positive integer can be multiplied by one of the digits 1, 2, 3, 4 or 5 so that the resulting number starts with 1?

Solution. Let the number a start with digit x. We are going to check that for any a by appropriate choice of n from $\{1, 2, 3, 4, 5\}$ the product of $a \times n$ starts with 1.

(i) $x = 1$. We chose $n = 1$.

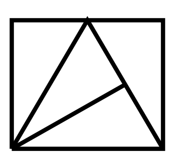
(ii) $x = 2, 3$. We choose $n = 5$. Indeed, the smallest value of the product $a \times 5$ is $20\dots0 \times 5$ while the largest value of the product does not exceed $39\dots9 \times 5 = (40\dots0 - 1) \times 5$. Both of these numbers start with digit 1, so do all the numbers in between.

(iii) $x = 4$. We can choose $n = 3$. The smallest value of $a \times n$ is no less than $40\dots0 \times 3$ while the largest number does not exceed $4\dots9 = (50\dots0 - 1) \times 3$. Both of these numbers as well as all numbers in between start with digit 1.

(iv) $x = 5, 6, 7, 8, 9$. We choose $n = 2$. The smallest value of $a \times n$ is no less than $50\dots0 \times 2$ while the largest number does not exceed $9\dots9 = (10\dots0 - 1) \times 2$. Both of these numbers as well all numbers in between start with digit 1. $\square$

Problem 2. A rectangle is split into equal non-isosceles right-angled triangles (without gaps or overlaps). Is it true that any such arrangement contains a rectangle made of two such triangles?

Solution. Counterexample. Rectangle with sides $2, \sqrt{3}$ can be split into four non-isosceles right angle triangles with angles 30° and 60° as shown on the picture. No two triangles are arranged into rectangle.



$\square$

Problem 3. Three players play the game "rock-paper-scissors". In every round, each player simultaneously shows one of these shapes. Rock beats scissors, scissors beat paper, while paper beats rock. If in a round exactly two distinct shapes are shown (and thus one of them is shown twice) then 1 point is added to the score of the player(s) who showed the winning shape, otherwise no point is added. After several rounds it occurred that each shape had been shown the same number of times. Prove that the total sum of points at this moment was a multiple of 3.

37th International Mathematics

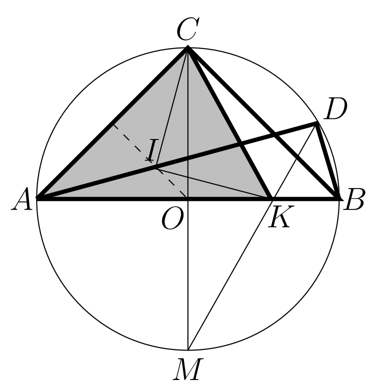
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2015**

Problem 1. Let p be a prime number. Determine the number of positive integers n such that pn is a multiple of $p+n$.

Solution. Let k be a positive integer such that $pn = k(n+p)$. Then $pn = kn+kp$ and $n-k = kn/p$. Note that $k < n$ and $k < p$ ($n-k$ must be positive while kn/p can not exceed n). It follows that $n = mp$, where m is a positive integer. Thus, $pm-k = km$ so that $p = k(m+1)/m$ and therefore $k = lm$, where l is a positive integer. Then $p = l(m+1)$ which implies that $l = 1$ (p is prime). Hence $m = p-1$ and $n = p(p-1)$ is the only one possible value. □

Problem 2. Suppose that ABC and ABD are right-angled triangles with common hypotenuse AB (D and C are on the same side of line AB). If $AC = BC$ and DK is a bisector of angle ADB, prove that the circumcenter of triangle ACK belongs to line AD.



Solution. Observe that triangles ABC and ADB share the same circumcircle with diameter AB. Denote by M a point symmetrical to C about the centre. Observe that K belongs to MD. (Indeed $AM = BM$ implies that $\angle ADM = \angle MDB$).

Let $\angle CAD = \alpha$. Then $\angle CMD = \angle CAD = \alpha$. Since triangle CMK is isosceles its altitude OK is also a bisector and therefore $\angle AKC = 90^\circ - \alpha$. Let I be a point on AD equidistant from A and C. Since $\angle ACI = \angle CAI = \alpha$, $\angle AIC = 180^\circ - 2\alpha$. Consider a circle with centre I and radius $AI = CI$. Since $\angle AKC = 1/2\angle AIC$, K belongs to this circle. Hence I is a centre of circumcircle of triangle ACK. □

Problem 3. Three players play the game "rock-paper-scissors". In every round, each player simultaneously shows one of these shapes. Rock beats scissors, scissors beat paper, while paper beats rock. If in a round exactly two distinct shapes are shown (and thus one of them is shown twice) then 1 point is added to the score of the

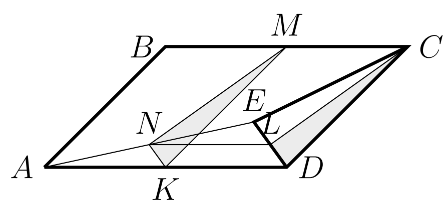
International Mathematics

TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Spring 2015**

Problem 1. A point is chosen inside a parallelogram $ABCD$ so that $CD = CE$. Prove that the segment DE is perpendicular to the segment connecting the midpoints of the segments AE and BC.

Solution. Denote by M, N, K and L the midpoints of BC, AE, AD and ED respectively. Since triangle ECD is isosceles, the median CL is also an altitude and therefore $\angle CLD = 90°$. Since NK is the midline of triangle AED, NK is parallel to ED and $NK = LD$. Then triangles MKN and CDL are congruent. ($NK = LD, MK = CD$ and $\angle MKN = \angle CDL$, as angles between parallel sides). Hence, $\angle MNK = 90°$ implying that ED is perpendicular to MN.

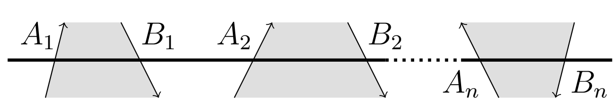


□

Problem 2. Area 51 has the shape of a non-convex polygon. It is protected by a chain fence along its perimeter and is surrounded by a minefield so that a spy can only move along the fence. The spy went around the Area once so that the Area was always on his right. A straight power line with 36 poles crosses this area so that some of the poles are inside the Area, and some are outside it. Each time the spy crossed the power line, he counted the poles to the left of him (he could see all the poles). Having passed along the whole fence, the spy had counted 2015 poles in total. Find the number of poles inside the fence.

Answer. 1.

Solution. Let $A_1, B_1, A_2, B_2, \dots, A_n, B_n$ be consecutive points where the power line enters and exits the Area; a Spy goes along the fence so that the Area is on his right. Let us orient the line so that when it enters the Area the Spy goes "up" and when it exits the Area, the Spy goes "down" (see the figure). Then passing through A_k and B_k (A_k and B_k are not necessary consecutive for the Spy) the Spy counts all poles to the left from A_k and all poles to the right of B_k, therefore he counts $36 - a_k$ poles (skipping the poles a_k between A_k and B_k). Then coming back to the point he started the Spy counts $36n - x = 2015$ poles where $x = a_1 + a_2 + \cdots + a_n$ $(0 \le x \le 36)$. Since$2016 = 2015 + 1$ is divisible by 36 this equation has an unique solution $x = 1$.



□

International Mathematics

TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Spring 2015**

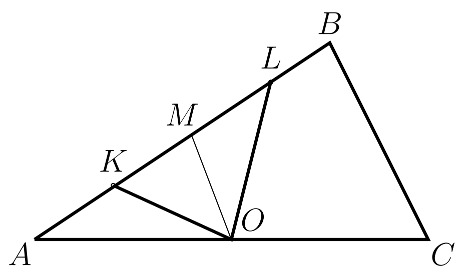
Problem 1. Is it possible to paint six faces of a cube into three colours so that each colour is present, but from any position one can see at most two colours?

Answer. Yes, it is possible.

Solution. Colour two opposite faces of the cube in red and blue, while the other faces in green. From any position one can not see red and blue faces at the same time. □

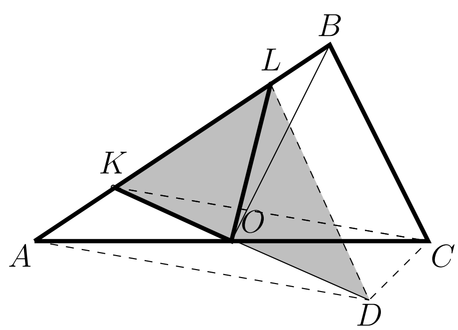
Problem 2. Points K and L are marked on side AB of triangle ABC so that $KL = BC$ and $AK = LB$. Given that O is the midpoint of side AC, prove that $\angle KOL = 90°$.

Solution 1. Let M be a midpoint of AB. Then $MO = 1/2BC = 1/2KL = KM = ML$. Therefore, points K, M, and O belong to a circle with radius KM and centre at M. Since KL is a diameter of this circle, $\angle KOL = 90°$.



□

Solution 2. Let us draw $LD \parallel BC$ and $CD \parallel AB$. Quadrilateral $AKCD$ is a parallelogram ($CD = LB = AK$ and $CD \parallel AK$). Then O, the midpoint of AC is the point of intersection of its diagonals and therefore $KO = OD$. Since the triangle KLD is isosceles ($LK = BC = LD$, its median LO is also an altitude. Hence $\angle KOL = 90°$.



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International Mathematics

TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Spring 2015**

Problem 1. (a) The integers x, x^2 and x^3 begin with the same digit. Does it imply that this digit is 1?

(b) The same question for the integers $x, x^2, x^3, \dots, x^{2015}$.

Answer. No.

Solution. (a) *Example:* $x = 99$, $x^2 = 99^2 = 9801$, $x^3 = 99^3 = 970299$.

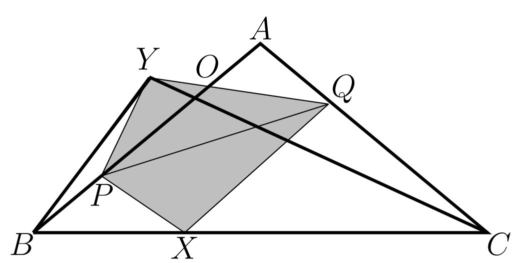
(b) *Solution (Ben Wei).* We use Bernoulli inequality $(1-\varepsilon)^k \geq (1-k\varepsilon)$ for $0 < \varepsilon < 1$ and $k \geq 1$ (It can be proved by induction). Consider $x = 99999$ in the form $x = 10^5(1-\varepsilon)$ with $\varepsilon = 10^{-5}$. Then $x^k = 10^{5k}(1-\varepsilon)^k \geq 10^{5k}(1-k\varepsilon) \geq 0.9 \cdot 10^{5k}$.

Therefore $10^{5k} > x^k \geq 0.9 \cdot 10^{5k}$ for all $k = 1, 2, \dots, 2015$, meaning that all given integers start with digit 9.

□

Problem 2. A point X is marked on the base BC of an isosceles triangle ABC, and points P and Q are marked on the sides AB and AC so that $APXQ$ is a parallelogram. Prove that the point Y symmetrical to X with respect to line PQ lies on the circumcircle of the triangle ABC.

Solution (Richard Chow). Consider triangles PYO and OAQ. Note that $\angle PYQ = \angle PXQ = \angle PAQ$ and $\angle YOP = \angle AOQ$. Then $\angle YPO = \angle AQO$ which implies that $\angle BPY = \angle YQC$. Since triangle ABC is isosceles and $PX \parallel AC$, triangle BPX is also isosceles and since $PX = PY$, triangle BPY is isosceles as well. In similar way we can rove that triangle YQC is also isosceles. Then triangles BPY and YQC are similar. If follows that $\angle BYC = \angle BAC$ and therefore quadrilateral $BYAC$ is cyclic.



□

Problem 3. (a) A $2 \times n$-table (with $n > 2$) is filled with numbers so that the sums in all the columns are different. Prove that it is possible to permute the numbers in the table so that the sums in the columns would still be different and the sums in the rows would also be different.

International Mathematics

TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Spring 2015**

Problem 1. Pete summed up 100 consecutive powers of 2, starting from some power, while Basil summed up several consecutive positive integers starting from 1. Can they get the same result?

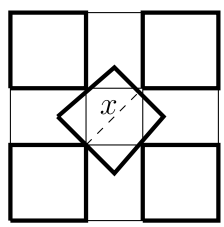
Answer. Yes, they can.

Solution. Indeed, let $2^{k+1} + \dots 2^{k+100} = 1 + 2 + \dots + n$. Simplifying we see that $2^{k+2}(2^{100} - 1) = n(n + 1)$ holds for $n = 2^{100} - 1$ and $k = 98$. $\square$

Problem 2. A moth made four small holes in a square carpet with a 275 cm side. Can one always cut out a square piece with a 1 m side without holes? (Consider holes as points).

Answer. One can always cut out a square piece with a 1 m side without holes.

Solution. On the picture one can see the positions of 5 non overlapping squares, four corner squares with side of 1 m and a central square with side $x = 0.75\sqrt{2} > 1$. Four moths can make holes in at most four of these pieces.



$\square$

Problem 3. Among $2n + 1$ positive integers there is exactly one 0, while each of the numbers $1, 2, ..., n$ is presented exactly twice. For which n can one line up these numbers so that for any $m = 1, \dots, n$ there are exactly m numbers between two m's?

Answer. For any n.

Solution. Observe that two sets of odd numbers, each set from from 1 to $2k + 1$ can be arranged according to the requirement with one empty space in the middle:

$$2k+1, 2k-1, \dots, 3, 1, \square, 1, 3, \dots, 2k-1, 2k+1$$

while two sets of even from from 1 to $2k$ can be arranged according to the requirement with two empty spaces in the middle:

$$2k, 2k-2, \dots, 2, 1, \square\square, 1, 2, \dots, 2k-2, 2k$$