India

National Olympiad

2006

$\boxed{1}$ In a non equilateral triangle ABC the sides a, b, c form an arithmetic progression. Let I be the incentre and O the circumcentre of the triangle ABC. Prove that

(1) IO is perpendicular to BI;

(2) If BI meets AC in K, and D, E are the midpoints of BC, BA respectively then I is the circumcentre of triangle DKE.

$\boxed{2}$ Prove that for every positive integer n there exists a unique ordered pair (a, b) of positive integers such that

$$n = \frac{1}{2}(a+b-1)(a+b-2)+a.$$

$\boxed{3}$ Let $X = \mathbb{Z}^3$ denote the set of all triples (a, b, c) of integers. Define $f : X \to X$ by

$$f(a, b, c) = (a+b+c, ab+bc+ca, abc).$$

Find all triples (a, b, c) such that

$$f(f(a, b, c)) = (a, b, c).$$

$\boxed{4}$ Some 46 squares are randomly chosen from a 9×9 chess board and colored in red. Show that there exists a 2×2 block of 4 squares of which at least three are colored in red.

$\boxed{5}$ In a cyclic quadrilateral $ABCD$, $AB = a$, $BC = b$, $CD = c$, $\angle ABC = 120^\circ$ and $\angle ABD = 30^\circ$. Prove that

(1) $c \geq a + b$;

(2) $|\sqrt{c+a} - \sqrt{c+b}| = \sqrt{c-a-b}$.

$\boxed{6}$ (a) Prove that if n is a integer such that $n \geq 4011^2$ then there exists an integer l such that

$$n < l^2 < (1 + \frac{1}{2005})n.$$

(b) Find the smallest positive integer M for which whenever an integer n is such that $n \geq M$ then there exists an integer l such that

$$n < l^2 < (1 + \frac{1}{2005})n.$$