USAMO 2008

Day 1 April 29th

1 Prove that for each positive integer n, there are pairwise relatively prime integers $k_0, k_1, \dots, k_n$, all strictly greater than 1, such that $k_0 k_1 \dots k_n - 1$ is the product of two consecutive integers.

2 Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle.

3 Let n be a positive integer. Denote by S_n the set of points (x, y) with integer coordinates such that

$$|x| + \left| y + \frac{1}{2} \right| < n.$$

A path is a sequence of distinct points $(x_1, y_1), (x_2, y_2), \dots, (x_\ell, y_\ell)$ in S_n such that, for $i = 2, \dots, \ell$, the distance between (x_i, y_i) and (x_{i-1}, y_{i-1}) is 1 (in other words, the points (x_i, y_i) and (x_{i-1}, y_{i-1}) are neighbors in the lattice of points with integer coordinates). Prove that the points in S_n cannot be partitioned into fewer than n paths (a partition of S_n into m paths is a set $\mathcal{P}$ of m nonempty paths such that each point in S_n appears in exactly one of the m paths in $\mathcal{P}$).

Day 2 April 30th

4 Let $\mathcal{P}$ be a convex polygon with n sides, $n \geq 3$. Any set of $n-3$ diagonals of $\mathcal{P}$ that do not intersect in the interior of the polygon determine a *triangulation* of $\mathcal{P}$ into $n-2$ triangles. If $\mathcal{P}$ is regular and there is a triangulation of $\mathcal{P}$ consisting of only isosceles triangles, find all the possible values of n.

5 Three nonnegative real numbers r_1, r_2, r_3 are written on a blackboard. These numbers have the property that there exist integers a_1, a_2, a_3, not all zero, satisfying $a_1 r_1 + a_2 r_2 + a_3 r_3 = 0$. We are permitted to perform the following operation: find two numbers x, y on the blackboard with $x \leq y$, then erase y and write $y - x$ in its place. Prove that after a finite number of such operations, we can end up with at least one 0 on the blackboard.

6 At a certain mathematical conference, every pair of mathematicians are either friends or strangers. At mealtime, every participant eats in one of two large dining rooms. Each mathematician insists upon eating in a room which contains an even number of his or her friends. Prove that the number of ways that the mathematicians may be split between the two rooms is a power of two (i.e., is of the form 2^k for some positive integer k).