Romania National Olympiad 2004

– Grade level 7

– April 5th

1 On the sides AB, AD of the rhombus $ABCD$ are the points E, F such that $AE = DF$. The lines BC, DE intersect at P and CD, BF intersect at Q. Prove that:

(a) $\frac{PE}{PD} + \frac{QF}{QB} = 1$;

(b) P, A, Q are collinear.

Virginia Tica, Vasile Tica

2 The sidelengths of a triangle are a, b, c.

(a) Prove that there is a triangle which has the sidelengths $\sqrt{a}, \sqrt{b}, \sqrt{c}$.

(b) Prove that $\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \leq a + b + c < 2\sqrt{ab} + 2\sqrt{bc} + 2\sqrt{ca}$.

3 Let $ABCD$ be an orthodiagonal trapezoid such that $\measuredangle A = 90^\circ$ and AB is the larger base. The diagonals intersect at O, $(OE$ is the bisector of $\measuredangle AOD$, $E \in (AD)$ and $EF \| AB$, $F \in (BC)$. Let P, Q the intersections of the segment EF with AC, BD. Prove that:

(a) $EP = QF$;

(b) $EF = AD$.

Claudiu-Stefan Popa

4 Let $\mathcal{U} = \{(x, y) \,|\, x, y \in \mathbb{Z},\ 0 \leq x, y < 4\}$.

(a) Prove that we can choose 6 points from $\mathcal{U}$ such that there are no isosceles triangles with the vertices among the chosen points.

(b) Prove that no matter how we choose 7 points from $\mathcal{U}$, there are always three which form an isosceles triangle.

Contributors: perfect_radio, Farenhajt, Jan, Gabriel(fr), scorpius119, paladin8, enescu, Myth, mecrazywong, grobber, xirti, Kuba, Kent Merryfield, ZetaX

Radu Gologan, Dinu Serbanescu

– Grade level 8

– April 5th

1 Find all non-negative integers n such that there are $a, b \in \mathbb{Z}$ satisfying $n^2 = a+b$ and $n^3 = a^2 + b^2$.

Lucian Dragomir

2 Prove that the equation $x^2 + y^2 + z^2 + t^2 = 2^{2004}$, where $0 \leq x \leq y \leq z \leq t$, has exactly 2 solutions in $\mathbb{Z}$.

Mihai Baluna

3 Let $ABCDA'B'C'D'$ be a truncated regular pyramid in which BC' and DA' are perpendicular.

(a) Prove that $\measuredangle (AB', DA') = 60^\circ$;

(b) If the projection of B' on (ABC) is the center of the incircle of ABC, then prove that $d(CB', AD') = \frac{1}{2}BC'$.

Mircea Fianu

4 In the interior of a cube of side 6 there are 1001 unit cubes with the faces parallel to the faces of the given cube. Prove that there are 2 unit cubes with the property that the center of one of them lies in the interior or on one of the faces of the other cube.

Dinu Serbanescu

– Grade level 9

– April 5th

1 Find the strictly increasing functions $f : \{1, 2, \ldots, 10\} \to \{1, 2, \ldots, 100\}$ such that $x + y$ divides $xf(x) + yf(y)$ for all $x, y \in \{1, 2, \ldots, 10\}$.

Cristinel Mortici

2 Let $P(n)$ be the number of functions $f : \mathbb{R} \to \mathbb{R}$, $f(x) = ax^2 + bx + c$, with $a, b, c \in \{1, 2, \ldots, n\}$ and that have the property that $f(x) = 0$ has only integer solutions. Prove that $n < P(n) < n^2$, for all $n \geq 4$.

Laurentiu Panaitopol

3 Let H be the orthocenter of the acute triangle ABC. Let BB' and CC' be altitudes of the triangle ($B' \in AC$, $C' \in AB$). A variable line ℓ passing through H intersects the segments $[BC']$ and $[CB']$ in M and N. The perpendicular lines of ℓ from M and N intersect BB' and CC' in P and Q. Determine the locus of the midpoint of the segment $[PQ]$.

Gheorghe Szolosy

4 Let $p, q \in \mathbb{N}^*$, $p, q \geq 2$. We say that a set X has the property $(\mathcal{S})$ if no matter how we choose p subsets $B_i \subset X$, $i = \overline{1, n}$, not necessarily distinct, each with q elements, there is a subset $Y \subset X$ with p elements s.t. the intersection of Y with each of the B_i's has an element at most, $i = \overline{1, p}$. Prove that:

(a) if $p = 4, q = 3$ then any set composed of 9 elements doesn't have $(\mathcal{S})$;

(b) any set X composed of $pq - q$ elements doesn't have the property $(\mathcal{S})$;

(c) any set X composed of $pq - q + 1$ elements has the property $(\mathcal{S})$.

Dan Schwarz

– Grade level 10

– April 5th

1 Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $|f(x) - f(y)| \leq |x - y|$, for all $x, y \in \mathbb{R}$.

Prove that if for any real x, the sequence $x, f(x), f(f(x)), \ldots$ is an arithmetic progression, then there is $a \in \mathbb{R}$ such that $f(x) = x + a$, for all $x \in \mathbb{R}$.

2 Let $ABCD$ be a tetrahedron in which the opposite sides are equal and form equal angles.

Prove that it is regular.

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3 Let $n > 2, n \in \mathbb{N}$ and $a > 0, a \in \mathbb{R}$ such that $2^a + \log_2 a = n^2$. Prove that:

$$2 \cdot \log_2 n > a > 2 \cdot \log_2 n - \frac{1}{n}.$$

Radu Gologan

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4 Let $(P_n)_{n \geq 1}$ be an infinite family of planes and $(X_n)_{n \geq 1}$ be a family of non-void, finite sets of points such that $X_n \subset P_n$ and the projection of the set X_{n+1} on the plane P_n is included in the set X_n, for all n.

Prove that there is a sequence of points $(p_n)_{n \geq 1}$ such that $p_n \in P_n$ and p_n is the projection of p_{n+1} on the plane P_n, for all n.

Does the conclusion of the problem remain true if the sets X_n are infinite?

Claudiu Raicu

– Grade level 11

– April 5th

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1 Let $n \geq 3$ be an integer and F be the focus of the parabola $y^2 = 2px$. A regular polygon $A_1 A_2 \dots A_n$ has the center in F and none of its vertices lie on Ox. $(FA_1 , (FA_2 , \dots , (FA_n$ intersect the parabola at $B_1, B_2, \dots, B_n$.

Prove that

$$FB_1 + FB_2 + \ldots + FB_n > np.$$

Calin Popescu

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2 Let $n \in \mathbb{N}$, $n \geq 2$.

(a) Give an example of two matrices $A, B \in \mathcal{M}_n(\mathbb{C})$ such that

$$\operatorname{rank}(AB) - \operatorname{rank}(BA) = \left\lfloor \frac{n}{2} \right\rfloor.$$

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(b) Prove that for all matrices $X, Y \in \mathcal{M}_n(\mathbb{C})$ we have

$$\operatorname{rank}(XY) - \operatorname{rank}(YX) \leq \left\lfloor \frac{n}{2} \right\rfloor .$$

Ion Savu

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3 Let $f : (a, b) \to \mathbb{R}$ be a function with the property that for all $x \in (a, b)$ there is a non-degenerated interval $[a_x, b_x]$ with $a < a_x \leq x \leq b_x < b$ such that f is constant on $[a_x, b_x]$.

(a) Prove that $\operatorname{Im} f$ is finite or numerable.

(b) Find all continuous functions which have the property mentioned in the hypothesis.

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4 (a) Build a function $f : \mathbb{R} \to \mathbb{R}_+$ with the property $(\mathcal{P})$, i.e. all $x \in \mathbb{Q}$ are local, strict minimum points.

(b) Build a function $f : \mathbb{Q} \to \mathbb{R}_+$ such that every point is a local, strict minimum point and such that f is unbounded on $I \cap \mathbb{Q}$, where I is a non-degenerate interval.

(c) Let $f : \mathbb{R} \to \mathbb{R}_+$ be a function unbounded on every $I \cap \mathbb{Q}$, where I is a non-degenerate interval. Prove that f doesn't have the property $(\mathcal{P})$.

– Grade level 12

– April 5th

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1 Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}^*$ we have

$$n^2 \int_x^{x+\frac{1}{n}} f(t)\, dt = n f(x) + \frac{1}{2}.$$

Mihai Piticari

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2 Let $f \in \mathbb{Z}[X]$. For an $n \in \mathbb{N}$, $n \geq 2$, we define $f_n : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ through $f_n\left(\widehat{x}\right) = \widehat{f\left(x\right)}$, for all $x \in \mathbb{Z}$.

(a) Prove that f_n is well defined.

(b) Find all polynomials $f \in \mathbb{Z}[X]$ such that for all $n \in \mathbb{N}$, $n \geq 2$, the function f_n is surjective.

Bogdan Enescu

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3 Let $f : [0,1] \to \mathbb{R}$ be an integrable function such that

$$\int_0^1 f(x)\,dx = \int_0^1 xf(x)\,dx = 1.$$

Prove that

$$\int_0^1 f^2(x)\,dx \geq 4.$$

Ion Rasa

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4 Let $\mathcal{K}$ be a field of characteristic p, $p \equiv 1\,(\mathrm{mod} 4)$.

(a) Prove that -1 is the square of an element from $\mathcal{K}$.

(b) Prove that any element $\neq 0$ from $\mathcal{K}$ can be written as the sum of three squares, each $\neq 0$, of elements from $\mathcal{K}$.

(c) Can 0 be written in the same way?

Marian Andronache

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