How to use AM-GM like a boss

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Introduction

AM-GM inequality is one of the most frequently used inequality in olympiads. Not only in inequalities, but it is also used in equations. In this note we will actually exploit weighted AM-GM with rational weights, but in a more intuitive way where we don't have to deal with scary notations.

Here is the normal formulation of AM-GM.

For positive reals $a_1, a_2, \ldots, a_n$

$$a_1^n + a_2^n + \ldots + a_n^n \geq n a_1 a_2 \ldots a_n$$

Some problems

Problem 1 *For positive reals a, b, c prove that*

$$a^2 + b^2 + c^2 \geq ab + bc + ca$$

Solution 1 *Using AM-GM for two variables, we get,*

$$a^2 + b^2 \geq 2ab$$

$$b^2 + c^2 \geq 2bc$$

$$c^2 + a^2 \geq 2ca$$

Summing up and dividing by 2, we get,

$$a^2 + b^2 + c^2 \geq ab + bc + ca$$

Which was what we wanted.

Problem 2 *For positive reals a, b, c prove that*

$$a^4 + b^4 + c^4 \geq a^2bc + b^2ca + c^2ab$$

Solution 2 *Using AM-GM for four variables, we get,*

$$a^4 + a^4 + b^4 + c^4 \geq 4a^2bc$$

$$b^4 + b^4 + c^4 + a^4 \geq 4b^2ca$$

$$c^4 + c^4 + a^4 + b^4 \geq 4c^2ab$$

Summing up and dividing by 4*, we get the desired inequality.*

Problem 3 *For positive reals a, b, c so that $abc = 1$, prove that*

$$a^2 + b^2 + c^2 \geq a + b + c$$

Solution 3 *Using AM-GM on 6 variables, we get*

$$a^2 + a^2 + a^2 + a^2 + b^2 + c^2 \geq 6\sqrt[6]{a^8b^2c^2}$$

But $a^8b^2c^2 = a^6 \times a^2b^2c^2 = a^6$ Therefore,

$$a^2 + a^2 + a^2 + a^2 + b^2 + c^2 \geq 6a$$

Similarly,

$$b^2 + b^2 + b^2 + b^2 + c^2 + a^2 \geq 6b$$

$$c^2 + c^2 + c^2 + c^2 + a^2 + b^2 \geq 6c$$

Adding up and dividing by 6, we get the desired inequality.

Note that in this problem(and also in problem 2), the positive condition is redundant(Why?).

Problem 4 *For positive reals a, b, c, d prove that*

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{d} + \frac{d^2}{a} \geq a + b + c + d$$

Solution 4 *We use AM-GM with* 15 *variables.*

$$\frac{a^2}{b} + \frac{a^2}{b} + \frac{a^2}{b} + \frac{a^2}{b} + \frac{a^2}{b} + \frac{a^2}{b} + \frac{a^2}{b} + \frac{a^2}{b} + \frac{b^2}{c} + \frac{b^2}{c} + \frac{b^2}{c} + \frac{b^2}{c} + \frac{c^2}{d} + \frac{c^2}{d} + \frac{d^2}{a} \geq 15a$$

Similarly,

$$\frac{b^2}{c} + \frac{b^2}{c} + \frac{b^2}{c} + \frac{b^2}{c} + \frac{b^2}{c} + \frac{b^2}{c} + \frac{b^2}{c} + \frac{b^2}{c} + \frac{c^2}{d} + \frac{c^2}{d} + \frac{c^2}{d} + \frac{c^2}{d} + \frac{d^2}{a} + \frac{d^2}{a} + \frac{a^2}{b} \geq 15b$$

$$\frac{c^2}{d} + \frac{c^2}{d} + \frac{c^2}{d} + \frac{c^2}{d} + \frac{c^2}{d} + \frac{c^2}{d} + \frac{c^2}{d} + \frac{c^2}{d} + \frac{d^2}{a} + \frac{d^2}{a} + \frac{d^2}{a} + \frac{d^2}{a} + \frac{a^2}{b} + \frac{a^2}{b} + \frac{b^2}{c} \geq 15c$$

$$\frac{d^2}{a} + \frac{d^2}{a} + \frac{d^2}{a} + \frac{d^2}{a} + \frac{d^2}{a} + \frac{d^2}{a} + \frac{d^2}{a} + \frac{d^2}{a} + \frac{a^2}{b} + \frac{a^2}{b} + \frac{a^2}{b} + \frac{a^2}{b} + \frac{b^2}{c} + \frac{b^2}{c} + \frac{c^2}{d} \geq 15d$$

Summing up and dividing by 15*, we get the desired inequality.*

Cyclic sum

At this point, you are probably fed up with me for showing you overhead AM-GM solutions with no explanation whatsoever. Not to worry, I am now going to provide you with the motivation for these approaches. But before that, we need to define cyclic sums.

The cyclic sum of $P(a, b, c)$ will be defined as

$$P(a, b, c) + P(b, c, a) + P(c, a, b)$$

And will be denoted by

$$\sum_{cyc} P(a, b, c)$$

Now, time for some examples.

$$P(a, b, c) = a^2 \Longrightarrow \sum_{cyc} a^2 = a^2 + b^2 + c^2$$

$$P(a, b, c) = a^2bc \Longrightarrow \sum_{cyc} a^2bc = a^2bc + b^2ca + c^2ab$$

$$P(a, b, c) = abc \Longrightarrow \sum_{cyc} abc = abc + bca + cab = 3abc$$

$$P(a, b, c, d) = a^3b^2c \Longrightarrow \sum_{cyc} a^3b^2c = a^3b^2c + b^3c^2d + c^3d^2a + d^3a^2b$$

Do some more playing with this notation until you get the hang of it.

Explanation

Did you see a pattern behind the solutions? Note that we create an inequality regarding one term of the right side and cyclically sum them up. So, lets assume we take the first term n_1 times, the second term n_2 times and so on. If we don't use the ith term, we take $n_i = 0$. Now recall something from your textbook:

$$(a^m)^n = a^{mn}$$

That is the principle we use while finding the desired inequality. Now, we certainly remember the definition of power, don't we? So, lets start with the first problem.

For positive reals a, b, c prove that

$$a^2 + b^2 + c^2 \geq ab + bc + ca$$

Lets assume there l a^2's, m b^2's, n c^2's. We want to make them greater than ab. So the product would be $a^{2l}b^{2m}c^{2n}$. But this equals ab. So, $2l = 1$,$2m = 1$.

So,$l = \dfrac{1}{2}$,$m = \dfrac{1}{2}$.It is also clear that $n = 0$. However $l : m : n$ is what truly matters. That's why we scale them by 2 and get $(l, m, n) = (1, 1, 0)$. So, we have to take 1 a^2 and 1 b^2. And that is precisely what we did.

Now, for the second problem. Setting up equations like the previous one lets us assume,

$$4l = 2, 4m = 1, 4n = 1$$

Therefore $l = \dfrac{1}{2}, m = \dfrac{1}{4}, n = \dfrac{1}{4}$. Scaling up by 4, we get $l = 2, m = 1, n = 1$. So, the desired inequality would be $a^4 + a^4 + b^4 + c^4 \geq 4a^2bc$

Now why does this work out? That is for you to discover. Now, we move on to P3, which is a bit trickier because of the $abc = 1$ condition. However, for those acquinted with the homogenization technique, this should not be difficult. Here, like before, lets take l, m, n and set up equations. Here we get: $l = m = n = 1$. But that doesn't work!!!

What we have to do here is to use the $abc = 1$ condition. Notice that due to symmetry reasons, $m = n$. Also note that, since $abc = 1$, $a^m b^m c^m$=1. So $a^l b^m c^m = a^{l-m} a^m b^m c^m = a^{l-m}$. From these we get $2l - 2m = 1$. Here we evoke the $l + m + n = 1$ condition(Why?). So $l + 2m = 1$. So $l = 2/3$, and $m = n = 1/6$. Scaling up by 6, We get $l = 4, m = n = 1$. From there we get the desired inequality.

For P4, let us set up equations with p, q, r, s. Since $\dfrac{a^2}{b} = a^2b^{-1}$, we get,

$$2p - s = 1, 2q - p = 0, 2r - q = 0.2s - r = 0$$

Solving these and scaling yield $p = 8, q = 4, r = 2, s = 1$.

Now, we yet haven't explained why we took $l + m + n = 1$. Actually, before scaling, this was true in every problem. The reason is we directly took the products equal to equate powers. However, we neglected the factor in front of the product. So, we actually took it as 1. However, that is actually the number of terms. So, we are taking the number of terms equal to 1(!). That is also the reason we get fraction numbers. Note that, I am not giving any rigorous proof of this because you don't need to show these in your actual rigorous solutions. These are simply roughs. However, you are welcome to try and prove.

Now we present a little more difficult problem.

Problem 5 *Let a, b, c be positive reals with $a + b + c = 1$. Prove that,*

$$3(a^5 + b^5 + c^5) \geq a^2b^2 + b^2c^2 + c^2a^2$$

Solution 5 *We note that the degree of RHS is 4 whereas the degree of LHS is 5. However, if we multiply RHS with $a + b + c = 1$, then the degrees would be* *equal. So,*

$$3a^5 + 3b^5 + 3c^5 \geq (a+b+c)(a^2b^2 + b^2c^2 + c^2a^2)$$

is an equivalent statement. Now, expanding the RHS, we get,

$$\sum_{cyc} a^3b^2 + \sum_{cyc} a^2b^3 + \sum_{cyc} a^2b^2c$$

Clearly,

$$a^5 + a^5 + a^5 + b^5 + b^5 \geq 5a^3b^2$$
$$a^5 + a^5 + b^5 + b^5 + b^5 \geq 5a^2b^3$$
$$a^5 + a^5 + b^5 + b^5 + c^5 \geq 5a^2b^2c$$

Summing each of them cyclically and adding them up and dividing by 5, we get the desired inequaltiy.

So, yeah. If the RHS seems too unsymmetric, then divide it into symmetric parts and then apply AM-GM like a boss.

Practice problems

Exercise 1 *Create as many new inequality problems as you can.*

Exercise 2 *For positive reals a, b, c with $abc = 1$, Prove that,*

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq a + b + c$$

Exercise 3 *For positive reals a, b, c with $a + b + c = 1$, prove that*

$$a^3 + b^3 + c^3 \geq \frac{a^2 + b^2 + c^2}{3}$$

Exercise 4 *For positive reals a, b, c, prove that*

$$3(a^3 + b^3 + c^3) \geq (a+b+c)(a^2 + b^2 + c^2)$$

Exercise 5 *For positive reals a, b, c, prove that*

$$a^3 + b^3 + c^3 + ab^2 + bc^2 + ca^2 \geq 2(a^2b + b^2c + c^2a)$$