Winter Camp 2008 Pre-camp Problem Set

1. The is a checker at point $(1,1)$ of the lattice grid. At each step, suppose that the checker is at (x,y), then it may move to $(2x,y)$ or $(x,2y)$, and additionally, if $x>y$, then it may also move to $(x-y,y)$ and if $x<y$ then it may move to $(x,y-x)$. Which points of the lattice can the checker reach?

2. If $a \equiv b \pmod{n}$, show that $a^n \equiv b^n \pmod{n^2}$. Is the converse true?

3. (USAMO 2002) Let S be a set with 2002 elements, and let N be an integer with $0 \le N \le 2^{2002}$. Prove that it is possible to color every subset of S either black or white so that the following conditions hold:

   (a) the union of any two white subsets is white;

   (b) the union of any two black subsets is black;

   (c) there are exactly N white subsets.

4. (Canada 1986) The endpoints of a chord ST with constant length are moving along a semi-circle with diameter AB. Let M be the midpoint of ST and P the foot of the perpendicular from S to AB. Prove that angle SPM is is independent of the location of ST.

5. (Bulgaria 1984) Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be nonnegative real numbers such that $x_i + y_i = 1$ for each $i = 1, 2, \ldots, n$. Prove that

$$(1 - x_1x_2\cdots x_n)^m + (1-y_1^m)(1-y_2^m)\cdots(1-y_n^m) \ge 1,$$

   where m is an arbitrary positive integer. (Hint: use probability.)

6. Let n be a positive integer. Is it possible to arrange the numbers $1, 2, \cdots, n$ in a row so that the arithmetic mean of any two of these numbers is not equal to some number between them? (That is, is there a permutation $a_1, a_2, \ldots, a_n$ of $\{1, 2, \ldots, n\}$ such that $2a_k \ne a_i + a_j$ for all $i < k < j$?)

7. Let $P(x)$ be a monic polynomial of degree n and such that all its coefficients are positive. Also, it is known that $P(0) = 1$ and all the zeros of $P(x)$ are real. Show that $P(2) \ge 3^n$.

8. The entry in the ith row and the jth column of an $n \times n$ matrix equals $a_i + b_j$, where $a_1, a_2, \ldots, a_n, b_1, b_2, \cdots, b_n$ are distinct real numbers. The products of the numbers in each row of the matrix are equal. Prove that the products of the numbers in each column are also equal.

9. Let G be a simple graph with n vertices such that there does not exist three mutually connected vertices (i.e., no triangles). Show that G has at most $n^2/4$ edges.

10. In a banquet, $2n$ ambassadors are invited, and every ambassador has at most $n-1$ enemies. Prove that the ambassadors can be seated around a round table, so that nobody sits next to an enemy.

11. Let $x_1, x_2, \ldots, x_n > 0$. Prove that

$$\frac{x_1^3}{x_1^2 + x_1x_2 + x_2^2} + \frac{x_2^3}{x_2^2 + x_2x_3 + x_3^2} + \cdots + \frac{x_n^3}{x_n^2 + x_nx_1 + x_1^2} \geq \frac{1}{3}(x_1 + x_2 + \cdots + x_n).$$

12. If X is an arbitrary point interior to the triangle ABC then show that the sum $AX+BX+CX$ is greater than the semiperimeter of the triangle and less than its perimeter.

13. Let a, b, c be rational numbers such that $a + b\sqrt[3]{2} + c\sqrt[3]{4} = 0$. Prove that $a = b = c = 0$.

14. Solve $\sqrt{5-x} = 5 - x^2$. (Don't cheat with a computer!)

15. Prove that the equation $4xy - x - y = z^2$ has no positive integer solutions.

16. (Balkan 1997 and 2000!) Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy $f(xf(x) + f(y)) = f(x)^2 + y$ for all real numbers x, y.

17. A 6×6 rectangle is tiled by 2×1 dominoes. Prove that it has always at least one fault-line, i.e., a line cutting the rectangle without cutting any domino.

18. (USAMO 2005) Let ABC be an acute-angled triangle, and let P and Q be two points on side BC. Construct point C_1 in such a way that convex quadrilateral $APBC_1$ is cyclic, $QC_1 \parallel CA$, and C_1 and Q lie on opposite sides of line AB. Construct point B_1 in such a way that convex quadrilateral $APCB_1$ is cyclic, $QB_1 \parallel BA$, and B_1 and Q lie on opposite sides of line AC. Prove that points B_1, C_1, P, and Q lie on a circle.

19. (Canada 2006) In a rectangular array of nonnegative real numbers with m rows and n columns, each row and each column contains at least one positive element. Moreover, if a row and a column intersect in a positive element, then the sums of their elements are the same. Prove that $m = n$.

20. (IMO 1985) A circle with center O passes through the vertices A and C of the triangle ABC and intersects the line segments AB and BC again at distinct points K and N, respectively. The circumcircles of the triangles ABC and KBN intersect at exactly two distinct points B and M. Prove that angle OMB is a right angle.