China Girls Math Olympiad 2005

Day 1

1 As shown in the following figure, point P lies on the circumcicle of triangle ABC. Lines AB and CP meet at E, and lines AC and BP meet at F. The perpendicular bisector of line segment AB meets line segment AC at K, and the perpendicular bisector of line segment AC meets line segment AB at J. Prove that

$$\left(\frac{CE}{BF}\right)^2 = \frac{AJ \cdot JE}{AK \cdot KF}.$$

2 Find all ordered triples (x, y, z) of real numbers such that

$$5\left(x+\frac{1}{x}\right) = 12\left(y+\frac{1}{y}\right) = 13\left(z+\frac{1}{z}\right),$$

and

$$xy + yz + zy = 1.$$

3 Determine if there exists a convex polyhedron such that

(1) it has 12 edges, 6 faces and 8 vertices;

(2) it has 4 faces with each pair of them sharing a common edge of the polyhedron.

4 Determine all positive real numbers a such that there exists a positive integer n and sets $A_1, A_2, \ldots, A_n$ satisfying the following conditions:

(1) every set A_i has infinitely many elements;
(2) every pair of distinct sets A_i and A_j do not share any common element
(3) the union of sets $A_1, A_2, \ldots, A_n$ is the set of all integers;
(4) for every set A_i, the positive difference of any pair of elements in A_i is at least a^i.

Day 2

5 Let x and y be positive real numbers with $x^3 + y^3 = x - y$. Prove that

$$x^2 + 4y^2 < 1.$$

6 An integer n is called good if there are $n \geq 3$ lattice points $P_1, P_2, \ldots, P_n$ in the coordinate plane satisfying the following conditions: If line segment P_iP_j has a rational length, then there is P_k such that both line segments P_iP_k and P_jP_k have irrational lengths; and if line segment P_iP_j has an irrational length, then there is P_k such that both line segments P_iP_k and P_jP_k have rational lengths.

(1) Determine the minimum good number.

(2) Determine if 2005 is a good number. (A point in the coordinate plane is a lattice point if both of its coordinate are integers.)

7 Let m and n be positive integers with $m > n \geq 2$. Set $S = \{1, 2, \ldots, m\}$, and $T = \{a_l, a_2, \ldots, a_n\}$ is a subset of S such that every number in S is not divisible by any two distinct numbers in T. Prove that

$$\sum_{i=1}^{n} \frac{1}{a_i} < \frac{m+n}{m}.$$

8 Given an $a \times b$ rectangle with $a > b > 0$, determine the minimum side of a square that covers the rectangle. (A square covers the rectangle if each point in the rectangle lies inside the square.)