Winter Camp 2008 Buffet Contest

Saturday, January 5, 2008

List of problems

A1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,

$$f(xf(y) + x) = xy + f(x).$$

A2. Let x, y, z be positive real numbers. Prove that

$$\frac{x}{x+\sqrt{(x+y)(x+z)}} + \frac{y}{y+\sqrt{(y+z)(y+x)}} + \frac{z}{z+\sqrt{(z+x)(z+y)}} \leq 1.$$

A3. Let $p(x)$ be a polynomial with integer coefficients. Does there always exist a positive integer k such that $p(x) - k$ is irreducible?

(An integer polynomial is *irreducible* if it cannot be written as a product of two nonconstant integer polynomials.)

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C1. Let X be a finite set of positive integers and A a subset of X. Prove that there exists a subset B of X such that A equals the set of elements of X which divide an odd number of elements of B.

C2. Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form $(\pm 1, \pm 1, \ldots, \pm 1)$ in n-dimensional space with $n \geq 3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle.

C3. Let S be a set of n points on a plane, no three collinear. A subset of these points is called *polite* if they are the vertices of a convex polygon with no points of S in the interior. Let c_k denote the number of polite sets with k points. Show that the sum

$$\sum_{i=3}^{n} (-1)^i c_i$$

depends only on n and not on the configuration of the points.

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G1. Let ABC be an acute triangle. The points M and N are taken on the sides AB and AC, respectively. The circles with diameters BN and CM intersect at points P and Q respectively. Prove that P, Q and the orthocenter H are collinear.

G2. Let ABC be a triangle with $AC \neq AB$, and select point B_1 on ray AC such that $AB = AB_1$. Let ω be the circle passing through C, B_1, and the foot of the internal bisector of angle CAB. Let ω intersect the circumcircle of triangle ABC again at Q. Prove that AC is parallel to the tangent to ω at Q.

G3. Let OAB and OCD be two directly similar triangles (i.e., CD can be obtained from AB by some rotation and dilatation both centered at O). Suppose their incircles meet at E and F. Prove that $\angle AOE = \angle DOF$.

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N1. Let $n > 1$ be an odd integer. Prove that n does not divide $3^n + 1$.

N2. Let S be a finite set of integers, each greater than 1. Suppose that for each integer n there is some $s \in S$ such that $\gcd(s, n) = 1$ or $\gcd(s, n) = s$. Show that there exist $s, t \in S$ such that $\gcd(s, t)$ is prime.

N3. Let a positive integer k be given. Prove that there are infinitely many pairs of integers (a, b) with $|a| > 1$ and $|b| > 1$ such that $ab + a + b$ divides $a^2 + b^2 + k$.