

APMO 2012

- 1 Let  $P$  be a point in the interior of a triangle  $ABC$ , and let  $D, E, F$  be the point of intersection of the line  $AP$  and the side  $BC$  of the triangle, of the line  $BP$  and the side  $CA$ , and of the line  $CP$  and the side  $AB$ , respectively. Prove that the area of the triangle  $ABC$  must be 6 if the area of each of the triangles  $PFA, PDB$  and  $PEC$  is 1.
- 2 Into each box of a  $2012 \times 2012$  square grid, a real number greater than or equal to 0 and less than or equal to 1 is inserted. Consider splitting the grid into 2 non-empty rectangles consisting of boxes of the grid by drawing a line parallel either to the horizontal or the vertical side of the grid. Suppose that for at least one of the resulting rectangles the sum of the numbers in the boxes within the rectangle is less than or equal to 1, no matter how the grid is split into 2 such rectangles. Determine the maximum possible value for the sum of all the  $2012 \times 2012$  numbers inserted into the boxes.
- 3 Determine all the pairs  $(p, n)$  of a prime number  $p$  and a positive integer  $n$  for which  $\frac{n^p+1}{p^n+1}$  is an integer.
- 4 Let  $ABC$  be an acute triangle. Denote by  $D$  the foot of the perpendicular line drawn from the point  $A$  to the side  $BC$ , by  $M$  the midpoint of  $BC$ , and by  $H$  the orthocenter of  $ABC$ . Let  $E$  be the point of intersection of the circumcircle  $\Gamma$  of the triangle  $ABC$  and the half line  $MH$ , and  $F$  be the point of intersection (other than  $E$ ) of the line  $ED$  and the circle  $\Gamma$ . Prove that  $\frac{BF}{CF} = \frac{AB}{AC}$  must hold.  
(Here we denote  $XY$  the length of the line segment  $XY$ .)
- 5 Let  $n$  be an integer greater than or equal to 2. Prove that if the real numbers  $a_1, a_2, \dots, a_n$  satisfy  $a_1^2 + a_2^2 + \dots + a_n^2 = n$ , then
 
$$\sum_{1 \leq i < j \leq n} \frac{1}{n - a_i a_j} \leq \frac{n}{2}$$
 must hold.