SymmeTree Contest 1

Problem 1. Let $a_1, a_2, \cdots, a_n$ be positive integers with product P, where n is an odd positive integer. Prove that

$$gcd(a_1^n + P, a_2^n + P, \cdots, a_n^n + P) \leq 2gcd(a_1, ..., a_n)^n$$

Problem 2. ABC is a triangle with $\angle B > 45°, \angle C > 45°$.Draw the isosceles triangles CAM, BAN on the sides AC, AB and outside the triangle, respectively, such that $\angle CAM = \angle BAN = 90°$. Then draw isosceles triangle BPC on the side BC and inside the triangle such that $\angle BPC = 90°$. Prove that ΔMPN is an isosceles triangle and $\angle MPN = 90°$.

Problem 3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x^2 + y) = f(f(x) - y) + 4f(x)y$$

for all real numbers x and y.

Problem 4. Let $n \geq 3$ be a positive integer. In a game,n players sit in a circle in that order. Initially, a deck of $3n$ cards labeled $1, ..., 3n$ is shuffled and distributed among the players so that every player holds 3 cards in their hand. Then, every hour, each player simultaneously gives the smallest card in their hand to their left neighbor, and the largest card in their hand to their right neighbor. (Thus after each exchange, each player still has exactly 3 cards). Prove that each player's hand after the first $n - 1$ exchanges is their same as their hand after the first $2n - 1$ exchanges.