International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Fall 2001.**

1. Do there exist positive integers $a_1 < a_2 < \cdots < a_{100}$ such that for $2 \leq k \leq 100$, the greatest common divisor of a_{k-1} and a_k is greater than the greatest common divisor of a_k and a_{k+1}?

2. Let $n \geq 3$ be an integer. A circle is divided into $2n$ arcs by $2n$ points. Each arc has one of three possible lengths, and no two adjacent arcs have the same length. The $2n$ points are coloured alternately red and blue. Prove that the n-gon with red vertices and the n-gon with blue vertices have the same perimeter and the same area.

3. Let $n \geq 3$ be an integer. Each row in an $(n-2) \times n$ array consists of the numbers 1, 2, ..., n in some order, and the numbers in each column are all different. Prove that this array can be expanded into an $n \times n$ array such that each row and each column consists of the numbers 1, 2, ..., n.

4. Let $n \geq 2$ be an integer. A regular $(2n+1)$-gon is divided into $2n-1$ triangles by diagonals which do not meet except at the vertices. Prove that at least three of these triangles are isosceles.

5. Alex places a rook on any square of an empty 8×8 chessboard. Then he places additional rooks one rook at a time, each attacking an odd number of rooks which are already on the board. A rook attacks to the left, to the right, above and below, and only the first rook in each direction. What is the maximum number of rooks Alex can place on the chessboard?

6. Several numbers are written in a row. In each move, Robert chooses any two adjacent numbers in which the one on the left is greater than the one on the right, doubles each of them and then switches them around. Prove that Robert can make only a finite number of such moves.

7. It is given that 2^{333} is a 101-digit number whose first digit is 1. How many of the numbers 2^k, $1 \leq k \leq 332$, have first digit 4?

Note: The problems are worth 4, 5, 5, 5, 6, 8 and 8 points respectively.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2001.**

1. In the quadrilateral $ABCD$, AD is parallel to BC. K is a point on AB. Draw the line through A parallel to KC and the line through B parallel to KD. Prove that these two lines intersect at some point on CD.

2. Clara computed the product of the first n positive integers and Valerie computed the product of the first m even positive integers, where $m \geq 2$. They got the same answer. Prove that one of them had made a mistake.

3. Kolya is told that two of his four coins are fake. He knows that all real coins have the same weight, all fake coins have the same weight, and the weight of a real coin is greater than that of a fake coin. Can Kolya decide whether he indeed has exactly two fake coins by using a balance twice?

4. On an east-west shipping lane are ten ships sailing individually. The first five from the west are sailing eastwards while the other five ships are sailing westwards. They sail at the same constant speed at all times. Whenever two ships meets, each turns around and sails in the opposite direction. When all ships have returned to port, how many meetings of two ships have taken place?

5. On the plane is a set of at least four points. If any one point from this set is removed, the resulting set has an axis of symmetry. Is it necessarily true that the whole set also has an axis of symmetry?

Note: Each problem is worth 4 points.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Fall 2001.**

1. On the plane is a triangle with red vertices and a triangle with blue vertices. O is a point inside both triangles such that the distance from O to any red vertex is less than the distance from O to any blue vertex. Can the three red vertices and the three blue vertices all lie on the same circle?

2. Do there exist positive integers $a_1 < a_2 < \cdots < a_{100}$ such that for $2 \leq k \leq 100$, the least common multiple of a_{k-1} and a_k is greater than the least common multiple of a_k and a_{k+1}?

3. An 8×8 array consists of the numbers 1, 2, ..., 64. Consecutive numbers are adjacent along a row or a column. What is the minimum value of the sum of the numbers along a diagonal?

4. Let F_1 be an arbitrary convex quadrilateral. For $k \geq 2$, F_k is obtained by cutting F_{k-1} into two pieces along one of its diagonals, flipping one piece over and then glueing them back together along the same diagonal. What is the maximum number of non-congruent quadrilaterals in the sequence $\{F_k\}$?

5. Let a and d be positive integers. For any positive integer n, the number $a + nd$ contains a block of consecutive digits which constitute the number n. Prove that d is a power of 10.

6. In a row are 23 boxes such that for $1 \leq k \leq 23$, there is a box containing exactly k balls. In one move, we can double the number of balls in any box by taking balls from another box which has more. Is it always possible to end up with exactly k balls in the k-th box for $1 \leq k \leq 23$?

7. The vertices of a triangle have coordinates (x_1, y_1), (x_2, y_2) and (x_3, y_3). For any integers h and k, not both 0, the triangle whose vertices have coordinates $(x_1 + h, y_1 + k)$, $(x_2 + h, y_1 + k)$ and $(x_3 + h, y_3 + k)$ has no common interior points with the original triangle.

   (a) Is it possible for the area of this triangle to be greater than $\frac{1}{2}$?

   (b) What is the maximum area of this triangle?

Note: The problems are worth 4, 5, 6, 6, 7, 7 and 3+6 points respectively.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall, 2001.**

1. An altitude of a pentagon is the perpendicular drop from a vertex to the opposite side. A median of a pentagon is the line joining a vertex to the midpoint of the opposite side. If the five altitudes and the five medians all have the same length, prove that the pentagon is regular.

2. There exists a block of 1000 consecutive positive integers containing no prime numbers, namely, $1001! + 2$, $1001! + 3$, $\ldots$, $1001! + 1001$. Does there exist a block of 1000 consecutive positive integers containing exactly five prime numbers?

3. On an east-west shipping lane are ten ships sailing individually. The first five from the west are sailing eastwards while the other five ships are sailing westwards. They sail at the same constant speed at all times. Whenever two ships meets, each turns around and sails in the opposite direction. When all ships have returned to port, how many meetings of two ships have taken place?

4. On top of a thin square cake are triangular chocolate chips which are mutually disjoint. Is it possible to cut the cake into convex polygonal pieces each containing exactly one chip?

5. The only pieces on an 8×8 chessboard are three rooks. Each moves along a row or a column without running to or jumping over another rook. The white rook starts at the bottom left corner, the black rook starts in the square directly above the white rook and the red rook start in the square directly to the right of the white rook. The white rook is to finish at the top right corner, the black rook in the square directly to the left of the white rook and the red rook in the square directly below the white rook. At all times, each rook must be either in the same row or the same column as another rook. Is it possible to get the rooks to their destinations?

Note: Each problem is worth 4 points.

March 4, 2001

TOURNAMENT OF TOWNS

Spring 2001, Level A, Junior (grades 8-10)

Your total score is based on the three problems for which you earn the most points; the scores for the individual parts of a single problem are summed. Points for each problem are shown in brackets [ ].

1. [3] In a certain country 10% of the employees get 90% of the total salary paid in this country. Supposing that the country is divided in several regions, is it possible that in every region the total salary of any 10% of the employees is no greater than 11% of the total salary paid in this region?

2. [5] In three piles there are 51, 49, and 5 stones, respectively. You can combine any two piles into one pile or divide a pile consisting of an even number of stones into two equal piles. Is it possible to get 105 piles with one stone in each?

3. [5] Point A lies inside an angle with vertex M. A ray issuing from point A is reflected in one side of the angle at point B, then in the other side at point C and then returns back to point A (the ordinary rule of reflection holds). Prove that the center of the circle circumscribed about triangle $\triangle BCM$ lies on line AM.

4. [5] Several non-intersecting diagonals divide a convex polygon into triangles. At each vertex of the polygon the number of triangles adjacent to it is written. Is it possible to reconstruct all the diagonals using these numbers if the diagonals are erased?

5.

(a) [3] One black and one white pawn are placed on a chessboard. You may move the pawns in turn to the neighbouring empty squares of the chessboard using vertical and horizontal moves. Can you arrange the moves so that every possible position of the two pawns will appear on the chessboard exactly once?

(b) [4] Same question, but you don't have to move the pawns in turn.

6. [7] Let AH_A, BH_B and CH_C be the altitudes of triangle $\triangle ABC$. Prove that the triangle whose vertices are the intersection points of the altitudes of triangles $\triangle AH_BH_C$, $\triangle BH_AH_C$ and $\triangle CH_AH_B$ is equal to triangle $\triangle H_AH_BH_C$.

7. Alex thinks of a two-digit integer (any integer between 10 and 99). Greg is trying to guess it. If the number Greg names is correct, or if one of its digits is equal to the corresponding digit of Alex's number and the other digit differs by one from the corresponding digit of Alex's number, then Alex says "hot"; otherwise, he says "cold". (For example, if Alex's number was 65, then by naming any of 64, 65, 66, 55 or 75 Greg will be answered "hot", otherwise he will be answered "cold".)

(a) [2] Prove that there is no strategy which guarantees that Greg will guess Alex's number in no more than 18 attempts.

(b) [3] Find a strategy for Greg to find out Alex's number (regardless of what the chosen number was) using no more than 24 attempts.

(c) [3] Is there a 22 attempt winning strategy for Greg?

International Mathematics

22nd Tournament of Towns

Ordinary Level

February 25, 2001

JUNIOR (GRADES 7, 8, 9 AND 10)

Your total score is based on the three problems for which you earn the most points; the scores for the individual parts of a single problem are added. Points for each problem are shown in brackets ( ).

1. (3) The natural number n can be replaced by ab if $a + b = n$, where a and b are natural numbers. Can the number 2001 be obtained from 22 after a sequence of such replacements?

2. (4) One of the midlines of a triangle is longer than one of its medians. Prove that the triangle has an obtuse angle.

3. (4) Twenty kilograms of cheese are on sale in a grocery store. Several customers are lined up to buy this cheese. After a while, having sold the demanded portion of cheese to the next customer, the salesgirl calculates the average weight of the portions of cheese already sold and declares the number of customers for whom there is exactly enough cheese if each customer will buy a portion of cheese of weight exactly equal to the average weight of the previous purchases. Could it happen that the salesgirl can declare, after each of the first 10 customers has made their purchase, that there just enough cheese for the next 10 customers? If so, how much cheese will be left in the store after the first 10 customers have made their purchases? (The average weight of a series of purchases is the total weight of the cheese sold divided by the number of purchases.)

4. a. (2) There are 5 identical paper triangles on the table. Each can be moved in any direction parallel to itself (i.e., without rotating it). Is it true that then any one of them can be covered by the 4 others?

   b. (3) There are 5 identical equilateral paper triangles on the table. Each can be moved in any direction parallel to itself. Prove that any one of them can be covered by the 4 others in this way.

5. (5) On a square board divided into 15×15 little squares there are 15 rooks that do not attack each other. Then each rook makes one move like that of a knight. Prove that after this is done a pair of rooks will necessarily attack each other.

PROBLEMS OF TOURNAMENT OF TOWNS

Spring 2001, Level A, Senior (grades 11-OAC)

Problem 1 [3] Find at least one polynomial $P(x)$ of degree 2001 such that $P(x) + P(1-x) = 1$ holds for all real numbers x.

Problem 2 [5] At the end of the school year it became clear that for any arbitrarily chosen group of no less than 5 students, 80% of the marks "F" received by this group were given to no more than 20% of the students in the group. Prove that at least 3/4 of all "F" marks were given to the same student.

Problem 3 [5] Let AH_A, BH_B and CH_C be the altitudes of triangle $\triangle ABC$. Prove that the triangle whose vertices are the intersection points of the altitudes of $\triangle AH_BH_C$, $\triangle BH_AH_C$ and $\triangle CH_AH_B$ is congruent to $\triangle H_AH_BH_C$.

Problem 4 [5] There are two matrices A and B of size $m \times n$ each filled only by "0"s and "1"s. It is given that along any row or column its elements do not decrease (from left to right and from top to bottom). It is also given that the numbers of "1"s in both matrices are equal and for any $k = 1, \dots, m$ the sum of the elements in the top k rows of the matrix A is no less than that of the matrix B. Prove for any $l = 1, \dots, n$ the sum of the elements in left l columns of the matrix A is no greater than that of the matrix B.

Problem 5 In a chess tournament, every participant played with each other exactly once, receiving 1 point for a win, 1/2 for a draw and 0 for a loss.

(a) [4] Is it possible that for every player P, the sum of points of the players who were beaten by P is greater than the sum of points of the players who beat P?

(b) [4] Is it possible that for every player P, the first sum is less than the second one?

Problem 6 [8] Prove that there exist 2001 convex polyhedra such that any three of them do not have any common points but any two of them touch each other (i.e., have at least one common boundary point but no common inner points).

Problem 7 Several boxes are arranged in a circle. Each box may be empty or may contain one or several chips. A move consists of taking all the chips from some box and distributing them one by one into subsequent boxes clockwise starting from the next box in the clockwise direction.

(a) [4] Suppose that on each move (except for the first one) one must take the chips from the box where the last chip was placed on the previous move. Prove that after several moves the initial distribution of the chips among the boxes will reappear.

(b) [4] Now, suppose that in each move one can take the chips from any box. Is it true that for every initial distribution of the chips you can get any possible distribution?

PROBLEMS OF TOURNAMENT OF TOWNS

Spring 2001, Level 0, Senior (grades 11-OAC)

Problem 1 [3] A bus that moves along a 100 km route is equipped with a computer, which predicts how much more time is needed to arrive at its final destination. This prediction is made on the assumption that the average speed of the bus in the remaining part of the route is the same as that in the part already covered. Forty minutes after the departure of the bus, the computer predicts that the remaining travelling time will be 1 hour. And this predicted time remains the same for the next 5 hours. Could this possibly occur? If so, how many kilometers did the bus cover when these 5 hours passed? (Average speed is the number of kilometers covered divided by the time it took to cover them.)

Problem 2 [4] The decimal expression of the natural number a consists of n digits, while that of a^3 consists of m digits. Can $n + m$ be equal to 2001?

Problem 3 [4] Points X and Y are chosen on the sides AB and BC of the triangle $\triangle ABC$. The segments AY and CX intersect at the point Z. Given that $AY = YC$ and $AB = ZC$ prove that the points B, X, Z, and Y lie on the same circle.

Problem 4 [5] Two persons play a game on a board divided into 3×100 squares. They move in turn: the first places tiles of size 1×2 lengthwise (along the long axis of the board), the second, in the perpendicular direction. The loser is the one who cannot make a move. Which of the players can always win (no matter how his opponent plays), and what is the winning strategy?

Problem 5 [5] Nine points are drawn on the surface of a regular tetrahedron with an edge of 1 cm. Prove that among these points there are two located at a distance (in space) no greater than 0.5 cm.

Juniors

(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS

A-Level Paper **Fall 2002.**

1 [**4**] There are 2002 employees in a bank. All the employees came to celebrate the bank's jubilee and were seated around one round table. It is known that the difference in salaries of any two employees sitting next to each other is 2 or 3 dollars. Find the maximal difference in salaries of two employees, if it is known that all the salaries are different.

2 [**5**] All the species of plants existing in Russia are catalogued (numbered by integers from 2 to 20 000; one after another, without omissions or repetitions). For any pair of species, the greatest common divisor of their catalogue numbers was calculated and recorded, but the catalogue numbers themselves were lost computer error). Is it possible to restore the catalogue number for each specie from that data?

3 [**6**] The vertices of a 50-gon divide a circumference into 50 arcs, whose lengths are 1, 2, 3,. . . , 50, in some order. It is known that lengths of any pair of "opposite" arcs (corresponding to opposite sides of the polygon) differ by 25. Prove that the polygon has two parallel sides.

4 [**6**] Point P is chosen in triangle ABC so that $\angle ABP$ is congruent to $\angle ACP$, while $\angle CBP$ is congruent to $\angle CAP$. Prove that P is the intersection point of the altitudes of the triangle.

5 [**7**] A convex N-gon is divided by diagonals into triangles so that no two diagonals intersect inside of the polygon. The triangles are painted in black and white so that any two triangles with common side are painted in different colors. For each N, find the maximal difference between the numbers of black and white triangles.

6 [**9**] There is a large pile of cards. On each card one of the numbers $1, 2, \ldots, n$ is written. It is known that the sum of all numbers of all the cards is equal to $k \cdot n!$ for some integer k. Prove that it is possible to arrange cards into k stacks so that the sum of numbers written on the cards in each stack is equal to $n!$.

7 a) [**5**] A power grid has the shape of a 3×3 lattice with 16 nodes (vertices of the lattice) joined by wires (along the sides of the squares). It may have happened that some of the wires are burned out. In one test technician can choose any pair of nodes and check if electrical current circulates between them (that is, check if there is a chain of intact wires joining the chosen nodes). Technician knows that current will circulate from any node to any other node. What is the least number of tests which is required to demonstrate this?

7 b) [**5**] The same question for a grid in the shape of a 5×5 lattice (36 nodes).

Keep the problem set.
Visit: `http://www.math.toronto.edu/oz/turgor/`

Juniors

(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS

O-Level Paper **Fall 2002.**

1 [**4**] In a convex 2002-gon several diagonals are drawn so that they do not intersect inside of the polygon. As a result, the polygon splits into 2000 triangles.

Is it possible that exactly 1000 triangles have diagonals for all of their three sides?

2 [**5**] Each of two children (John and Mary) selected a natural number and communicated it to Bill. Bill wrote down the sum of these numbers on one card and their product on another, hid one card and showed the other to John and Mary.

John looked at the number (which was 2002) and declared that he was not able to determine the number chosen by Mary. Knowing this, Mary said that she was also not able to determine the number chosen by John.

What was the number chosen by Mary?

3

a) [**1**] A test was conducted in a class. It is known that at least $\dfrac{2}{3}$ of the problems were hard: each such problem was not solved by at least $\dfrac{2}{3}$ of the students. It is also known that at least $\dfrac{2}{3}$ of students passed the test: each such student solved at least $\dfrac{2}{3}$ of the suggested problems. *Is this situation possible?*

b) [**2**] The same question with $\dfrac{2}{3}$ replaced by $\dfrac{3}{4}$.

c) [**2**] The same question with $\dfrac{2}{3}$ replaced by $\dfrac{7}{10}$.

4) [**5**] 2002 cards with the numbers 1, 2, 3,...,2002 written on them are put on a table face up. Two players in turns pick up a card from the table until all cards are gone. The player who gets the last digit of the sum of all numbers on his cards larger than his opponent, wins.

Who has a winning strategy and how one should play to win?

5) [**5**] An angle and a point A inside of it are given. Is it possible to draw through A three straight lines so that on either side of the angle one of three points of intersection of these lines be the midpoint between two other points of intersection with that side?

Keep the problem set.
Visit: `http://www.math.toronto.edu/oz/turgor/`

Seniors

(Grades 11 and up)

International Mathematics
TOURNAMENT OF THE TOWNS

A-Level Paper **Fall 2002.**

1 [4] All the species of plants existing in Russia are catalogued (numbered by integers from 2 to 20 000; one after another, without omissions or repetitions). For any pair of species, the greatest common divisor of their catalogue numbers was calculated and recorded, but the numbers themselves were lost (as the result of a computer error). Is it possible to restore the catalogue number for each specie from that data?

2 [**6**] A cube is cut by a plane so that the cross-section is a pentagon. Prove that the length of one of the sides of the pentagon differs from 1 meter by at least 20 centimeters.

3 [6] A convex N-gon is divided by diagonals into triangles so that no two diagonals intersect inside of the polygon. The triangles are painted in black and white so that any two triangles with common side are painted in different colors. For each N, find the maximal difference between the numbers of black and white triangles.

4 [8] There is a large pile of cards. On each card one of the numbers $\{1, 2, \ldots, n\}$ is written. It is known that the sum of all numbers of all the cards is equal to $k \cdot n!$ for some integer k. Prove that it is possible to arrange cards into k stacks so that the sum of numbers written on the cards in each stack is equal to $n!$.

5 Two circles intersect at points A and B. Through point B a straight line is drawn, intersecting the first and second circle at points K and M (different from B) respectively. Line ℓ_1 is tangent to the first circle at point Q and parallel to line AM. Line QA intersects the second circle at point R (different from A). Further, line ℓ_2 is tangent to the second circle at point R. Prove that

a) [4] ℓ_2 is parallel to AK;

b) [4] Lines ℓ_1, ℓ_2 and KM have a common point.

6 [8] A sequence with first two terms equal 1 and 2 respectively is defined by the following rule: each subsequent term is equal to the smallest positive integer which has not yet occurred in the sequence and is not coprime with the previous term. Prove that all positive integers occur in this sequence.

7 a) [4] A power grid has the shape of a 3×3 lattice with 16 nodes (vertices of the lattice) joined by wires (along the sides of the squares). It may have happened that some of the wires are burned out. In one test technician can choose any pair of nodes and check if electrical current circulates between them (that is, check if there is a chain of intact wires joining the chosen nodes). Technician knows that current will circulate from any node to any other node. What is the least number of tests which is required to demonstrate this?

7 b) [5] The same question for a grid in the shape of a 7×7 lattice (36 nodes).

Keep the problem set.
Visit: `http://www.math.toronto.edu/oz/turgor/`

Seniors

(Grades 11 and up)

International Mathematics
TOURNAMENT OF THE TOWNS

O-Level Paper **Fall 2002.**

1 [**4**] Each of two children (John and Mary) selected a natural number and communicated it to Bill. Bill wrote down the sum of these numbers on one card and their product on another, hid one card and showed the other to John and Mary.

John looked at the number (which was 2002) and declared that he was not able to determine the number chosen by Mary. Knowing this, Mary said that she was also not able to determine the number chosen by John.

What was the number chosen by Mary?

2

a [**1**] A test was conducted in a class. It is known that at least $\frac{2}{3}$ of the problems were hard: each such problem was not solved by at least $\frac{2}{3}$ of the students. It is also known that at least 2/3 of students passed the test: each such student solved at least $\frac{2}{3}$ of the suggested problems.

Is this situation possible?

b [**1**] The same question with $\frac{2}{3}$ replaced by $\frac{3}{4}$.

c [**2**] The same question with $\frac{2}{3}$ replaced by $\frac{7}{10}$.

3 [**5**] Several straight lines such that no two of them are parallel, cut the plane into several regions. A point A is marked inside of one region. Prove that a point, separated from A by each of these lines, exists if and only if A belongs to unbounded region.

4 [**5**] Let x, y, z be any three numbers from the open interval $(0, \pi/2)$. Prove the inequality

$$\frac{x \cdot \cos x + y \cdot \cos y + z \cdot \cos z}{x + y + z} \leq \frac{\cos x + \cos y + \cos z}{3}.$$

5 [**5**] Each term of an infinite sequence of natural numbers is obtained from the previous term by adding to it one of its nonzero digits. Prove that this sequence contains an even number.

Keep the problem set.
Visit: `http://www.math.toronto.edu/oz/turgor/`

International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Spring 2002.**

1. Let $a,\ b$ and c be the sides of a triangle. Prove that $a^3 + b^3 + 3abc > c^3$.

2. A game is played on a 23×23 board. The first player controls two white chips which start in the bottom-left and the top-right corners. The second player controls two black ones which start in the bottom-right and the top-left corners. The players move alternately. In each move, a player can move one of the chips under control to a vacant square which shares a common side with its current location. The first player wins if the two white chips are located on two squares sharing a common side. Can the second player prevents the first player from winning?

3. Let E and F be the respective midpoints of sides BC and CD of a convex quadrilateral $ABCD$. Segments AE, AF and EF cut $ABCD$ into four triangles whose areas are four consecutive positive integers. Determine the maximal area of triangle BAD.

4. There are n lamps in a row, some of which are on. Every minute, all the lamps already on will go off. Those which were off and were adjacent to exactly one lamp that was on will go on. For which n can one find an initial configuration of which lamps are on, such that at least one lamp will be on at any time?

5. An acute triangle was dissected by a straight cut into two pieces which are not necessarily triangles. Then one of the pieces was dissected by a straight cut into two pieces, and so on. After a few dissections, it turned out that all the pieces are triangles. Can all of them be obtuse?

6. In an increasing infinite sequence of positive integers, every term starting from the 2002-th term divides the sum of all preceding terms. Prove that every term starting from some term is equal to the sum of all preceding terms.

7. Some domino pieces are placed in a chain according to the standard rules. In each move, we may remove a sub-chain with equal numbers at its ends, turn the whole sub-chain around, and put it back in the same place. Prove that for every two legal chains formed from the same pieces and having the same numbers at their ends, we can transform one to the other in a finnite sequence of moves.

Note: The problems are worth 4, 4, 6, 7, 7, 7 and 8 points respectively.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper (Grades up to 10) **Spring 2002.**

1 [**4**] There are many $a \times b$-rectangular cardboard pieces (a and b are positive integers, and $a < b$). It is given that by putting such rectangles together (without overlapping) one can make 49×51-rectangle, and 99×101-rectangle. Can one uniquely determine values of a and b from these conditions?

2 [**5**] Can any triangle be cut into four convex figures: a triangle, a quadrilateral, a pentagon, and a hexagon?

3 [**5**] The last digit of the number $x^2 + xy + y^2$ is zero (where x and y are positive integers). Prove that two last digits of this number are zeroes.

4 [**5**] Quadrilateral $ABCD$ is circumscribed about some circle and K, L, M, N are points of tangency of sides AB, BC, CD and DA respectively, S is an intersection point of the segments KM and LN. It is known that the quadrilateral $SKBL$ is cyclic. Prove that the quadrilateral $SNDM$ is also cyclic.

5

a) [**3**] There are 128 coins of two different weights, 64 of each. How can one always find two different coins by performing no more than 7 weightings on a regular balance?

b) [**3**] There are eight coins of two different weights, four of each. How can one always find two different coins by performing two weightings on a regular balance?

International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Spring 2002.**

1. In triangle ABC, $\tan A$, $\tan B$ and $\tan C$ are integers. Find their values.

2. Does there exist a point A on the graph of $y = x^3$ and a point B on the graph $y = x^3 + |x| + 1$ such that the distance between A and B does not exceed $\frac{1}{100}$?

3. In an increasing infinite sequence of positive integers, every term starting from the 2002-th term divides the sum of all preceding terms. Prove that every term starting from some term is equal to the sum of all preceding terms.

4. The spectators are seated in a row with no empty places. Each is in a seat which does not match the spectator's ticket. An usher can order two spectators in adjacent seats to trade places unless one of them is already seated correctly. Is it true that from any initial arrangement, the usher can place all the spectators in their correct seats?

5. Let AA_1, BB_1 and CC_1 be the altitudes of an acute triangle ABC. Let O_A, O_B and O_C be the respective incentres of triangles AB_1C_1, BA_1C_1 and CA_1B_1. Let T_A, T_B and T_C be the points of tangency of the incircle of ABC with sides BC, CA and AB respectively. Prove that $T_AO_CT_BO_AT_CO_B$ is an equilateral hexagon.

6. The 52 cards in a standard deck are placed in a 13×4 array. If every two adjacent cards, vertically or horizontally, have either the same suit or the same value, prove that all 13 cards of the same suit are in the same row.

7. Do there exist irrational numbers a and b such that $a > 1$, $b > 1$ and $\lfloor a^m \rfloor$ differs $\lfloor b^n \rfloor$ for any two positive integers m and n?

Note: The problems are worth 4, 4, 5, 5, 6, 7 and 8 points respectively.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper (Grades 11-OAC) **Spring 2001.**

1 [**4**] The last digit of the number x^2+xy+y^2 is zero (where x and y are positive integers). Prove that two last digits of this number are zeroes.

2 [**5**] Triangle ABC and its mirror reflection $A'B'C'$ are arbitrarily placed on the plane. Prove that the midpoints of the segments AA', BB' and CC' lie on the same straight line.

3 [**5**] There are 6 pieces of cheese all of different weight. For any two of them one can determine, just by looking at them, which of them is the heaviest.

Given that it is possible to divide them into two groups of equal weights (three pieces in each group) demonstrate how to find these groups by performing two weightings on the regular balance.

4 [**5**] In how many ways can one place the numbers from 1 to 100 in a 2×50-rectangle (divided into 100 little squares) so that any two consecutive numbers are always placed in squares with a common side?

5 [**6**] Does there exist a regular triangular prism that can be covered (without overlapping) by different equilateral triangles? (One is allowed to bend the triangles around the edges of the prism.)

Juniors

(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS

A-Level Paper **Fall 2003.**

1 [**4**] An increasing arithmetic progression consists of one hundred positive integers. Is it possible that every two of them are relatively prime?

2 [**5**] Smallville is populated by unmarried men and women, some of them are acquainted. Two city's matchmakers are aware of all acquaintances. Once, one of matchmakers claimed: "I could arrange that every brunette man would marry a woman he was acquainted with". The other matchmaker claimed "I could arrange that every blonde woman would marry a man she was acquainted with". An amateur mathematician overheard their conversation and said "Then both arrangements could be done at the same time! " Is he right?

3 [**5**] Find all positive integers k such that there exist two positive integers m and n satisfying $m(m+k) = n(n+1)$.

4 [**6**] Several squares on a 15×15 chessboard are marked so that a bishop placed on any square of the board attacks at least two of marked squares. Find the minimal number of marked squares.

5 [**7**] A point O lies inside of the square $ABCD$. Prove that the difference between the sum of angles OAB, OBC, OCD , ODA and $180°$ does not exceed $45°$.

6 [**7**] An ant crawls on the outer surface of the box in a shape of rectangular parallelepiped. From ant's point of view, the distance between two points on a surface is defined by the length of the shortest path ant need to crawl to reach one point from the other. Is it true that if ant is at vertex then from ant's point of view the opposite vertex be the most distant point on the surface?

7 [**8**] Two players in turns play a game. Each player has 1000 cards with numbers written on them; namely, First Player has cards with numbers $2, 4, \ldots, 2000$ while Second Player has cards with numbers $1, 3, \ldots, 2001$. In each his turn, a player chooses one of his cards and puts it on a table; the opponent sees it and puts his card next to the first one. Player, who put the card with a larger number, scores 1 point. Then both cards are discarded. First Player starts. After 1000 turns the game is over; First Player has used all his cards and Second Player used all but one. What are the maximal scores, that players could guarantee for themselves, no matter how the opponent would play?

Juniors

(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS

O-Level Paper **Fall 2003.**

1 [**3**] There is $3 \times 4 \times 5$ - box with its faces divided into 1×1 - squares. Is it possible to place numbers in these squares so that the sum of numbers in every stripe of squares (one square wide) circling the box, equals 120?

2 [**4**] In 7-gon $A_1A_2A_3A_4A_5A_6A_7$ diagonals A_1A_3, A_2A_4, A_3A_5, A_4A_6, A_5A_7, A_6A_1 and A_7A_2 are congruent to each other and diagonals A_1A_4, A_2A_5, A_3A_6, A_4A_7, A_5A_1, A_6A_2 and A_7A_3 are also congruent to each other. Is the polygon necessarily regular?

3 [**4**] For any integer $n+1, \dots, 2n$ (n is a natural number) consider its greatest odd divisor. Prove that the sum of all these divisors equals n^2.

4 [**4**] There are N points on the plane; no three of them belong to the same straight line. Every pair of points is connected by a segment. Some of these segments are colored in red and the rest of them in blue. The red segments form a closed broken line without self-intersections (each red segment having only common endpoints with its two neighbors and no other common points with the other segments), and so do the blue segments. Find all possible values of N for which such a disposition of N points and such a choice of red and blue segments are possible.

5 [**5**] 25 checkers are placed on 25 leftmost squares of $1 \times N$ board. Checker can either move to the empty adjacent square to its right or jump over adjacent right checker to the next square if it is empty. Moves to the left are not allowed. Find minimal N such that all the checkers could be placed in the row of 25 successive squares but in the reverse order.

Seniors

(Grades 11,12)

International Mathematics
TOURNAMENT OF THE TOWNS

A-Level Paper **Fall 2003.**

1 [**4**] Smallville is populated by unmarried men and women, some of them are acquainted. Two city's matchmakers are aware of all acquaintances. Once, one of matchmakers claimed: "I could arrange that every brunette man would marry a woman he was acquainted with". The other matchmaker claimed "I could arrange that every blonde woman would marry a man she was acquainted with". An amateur mathematician overheard their conversation and said "Then both arrangements could be done at the same time!" Is he right?

2 [**4**] Prove that every positive integer can be represented in the form

$$3^{u_1} \cdot 2^{v_1} + 3^{u_2} \cdot 2^{v_2} + \ldots + 3^{u_k} \cdot 2^{v_k}$$

with integers $u_1, u_2, \ldots, u_k, v_1, \ldots, v_k$ such that $u_1 > u_2 > \ldots > u_k \geq 0$ and $0 \leq v_1 < v_2 < \ldots < v_k$.

3 [**6**] An ant crawls on the outer surface of the box in a shape of rectangular parallelepiped. From ant's point of view, the distance between two points on a surface is defined by the length of the shortest path ant need to crawl to reach one point from the other. Is it true that if ant is at vertex then from ant's point of view the opposite vertex be the most distant point on the surface?

4 [**7**] In a triangle ABC let H be the point of intersection of altitudes , I the center of incircle, O the center of encircle, K the point where incircle touches BC. Given, that IO is parallel to BC, prove that AO is parallel to HK.

5 [**7**] Two players in turns play a game. Each player has 1000 cards with numbers written on them; namely, First Player has cards with numbers $2, 4, \ldots, 2000$ while Second Player has cards with numbers $1, 3, \ldots, 2001$. In each his turn, a player chooses one of his cards and puts it on a table; the opponent sees it and puts his card next to the first one. Player, who put the card with a larger number, scores 1 point. Then both cards are discarded. First Player starts. After 1000 turns the game is over; First Player has used all his cards and Second Player used all but one. What are the maximal scores, that players could guarantee for themselves, no matter how the opponent would play?

6 [**7**] Let O be the center of insphere of a tetrahedron $ABCD$. The sum of areas of faces ABC and ABD equals the sum of areas of faces CDA and CDB. Prove that O and midpoints of BC, AD, AC and BD belong to the same plane.

7 A $m \times n$ table is filled with signs "+" and "−". A table is called irreducible if one cannot reduce it to the table filled with "+", applying the following operations (as many times as one wishes).

a) [**3**] It is allowed to flip all the signs in a row or in a column. Prove that an irreducible table contains an irreducible 2×2 sub table.

b) [**6**] It is allowed to flip all the signs in a row or in a column or on a diagonal (corner cells are diagonals of length 1). Prove that an irreducible table contains an irreducible 4×4 sub table.

Seniors

(Grades 11,12)

International Mathematics
TOURNAMENT OF THE TOWNS

O-Level Paper **Fall 2003.**

1 [3] For any integer $n+1, \dots, 2n$ (n is a natural number) consider its greatest odd divisor. Prove that the sum of all these divisors equals n^2.

2 [4] What least possible number of unit squares (1×1) must be drawn in order to get a picture of 25×25-square divided into 625 of unit squares?

3 [5] A salesman and a customer altogether have 1999 rubles in coins and bills of 1, 5, 10, 50, 100, 500 , 1000 rubles. The customer has enough money to buy a Cat in the Bag which costs the integer number of rubles. Prove that the customer can buy the Cat and get the correct change.

4 Each side of 1×1 square is a hypothenuse of an exterior right triangle. Let A, B, C, D be the vertices of the right angles and O_1, O_2, O_3, O_4 be the centers of the incircles of these triangles. Prove that

a) [3] The area of quadrilateral $ABCD$ does not exceed 2;

b) [3] The area of quadrilateral $O_1O_2O_3O_4$ does not exceed 1.

6 [5] A paper tetrahedron is cut along some of so that it can be developed onto the plane. Could it happen that this development cannot be placed on the plane in one layer?

Juniors

(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS

A-Level Paper **Spring 2003.**

1 [**4**] Johnny writes down quadratic equation

$$ax^2 + bx + c = 0$$

with positive integer coefficients a, b, c. Then Pete changes one, two, or none "+" signs to "−". Johnny wins, if both roots of the (changed) equation are integers. Otherwise (if there are no real roots or at least one of them is not an integer), Pete wins.

Can Johnny choose the coefficients in such a way that he will always win?

2 [**4**] $\triangle ABC$ is given. Prove that $R/r > a/h$, where R is the radius of the circumscribed circle, r is the radius of the inscribed circle, a is the length of the longest side, h is the length of the shortest altitude.

3 In a tournament, each of 15 teams played with each other exactly once. Let us call the game *"odd"* if the total number of games previously played by both competing teams was odd.

(a) [**4**] Prove that there was at least one "odd" game.

(b) [**3**] Could it happen that there was exactly one "odd" game?

4 [**7**] A chocolate bar in the shape of an equilateral triangle with side of the length n, consists of triangular chips with sides of the length 1, parallel to sides of the bar. Two players take turns eating up the chocolate.

Each player breaks off a triangular piece (along one of the lines), eats it up and passes leftovers to the other player (as long as bar contains more than one chip, the player is not allowed to eat it completely).

A player who has no move or leaves exactly one chip to the opponent, loses.

For each n, find who has a winning strategy.

5 [**7**] What is the largest number of squares on 9×9 square board that can be cut along their both diagonals so that the board does not fall apart into several pieces?

6 [**7**] A trapezoid with bases AD and BC is circumscribed about a circle, E is the intersection point of the diagonals. Prove that $\angle AED$ is not acute.

Juniors

(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS

O-Level Paper **Spring 2003.**

1 [**4**] 2003 dollars are placed into N purses, and the purses are placed into M pockets. It is known that N is greater than the number of dollars in any pocket. Is it true that there is a purse with less than M dollars in it?

2 [**4**] Two players in turns colour the sides of an n-gon. The first player colours any side that has 0 or 2 common vertices with already coloured sides. The second player colours any side that has exactly 1 common vertex with already coloured sides. The player who cannot move, loses. For which n the second player has a winning strategy?

3 [**5**] Points K and L are chosen on the sides AB and BC of the isosceles $\triangle ABC$ ($AB = BC$) so that $AK + LC = KL$. A line parallel to BC is drawn through midpoint M of the segment KL, intersecting side AC at point N. Find the value of $\angle KNL$.

4 [**5**] Each term of a sequence of natural numbers is obtained from the previous term by adding to it its largest digit. What is the maximal number of successive odd terms in such a sequence?

5 [**5**] Is it possible to tile 2003×2003 board by 1×2 dominoes placed horizontally and 1×3 rectangles placed vertically?

Seniors

(Grades 11 and up)

International Mathematics
TOURNAMENT OF THE TOWNS

A-Level Paper **Spring 2003.**

1 [**4**] A triangular pyramid $ABCD$ is given. Prove that $R/r > a/h$, where R is the radius of the circumscribed sphere, r is the radius of the inscribed sphere, a is the length of the longest edge, h is the length of the shortest altitude (from a vertex to the opposite face).

2 [**5**] $P(x)$ is a polynomial with real coefficients such that $P(a_1) = 0$, $P(a_{i+1}) = a_i$ $(i = 1, 2, \ldots)$ where $\{a_i\}_{i=1,2,\ldots}$ is an infinite sequence of distinct natural numbers. Determine the possible values of degree of $P(x)$.

3 [**5**] Can one cover a cube by three paper triangles (without overlapping)?

4 [**6**] A right $\triangle ABC$ with hypothenuse AB is inscribed in a circle. Let K be the midpoint of the arc BC not containing A, N the midpoint of side AC, and M a point of intersection of ray KN with the circle. Let E be a point of intersection of tangents to the circle at points A and C.

Prove that $\angle EMK = 90°$.

5 [**6**] Prior to the game John selects an integer greater than 100.

Then Mary calls out an integer d greater than 1. If John's integer is divisible by d, then Mary wins. Otherwise, John subtracts d from his number and the game continues (with the new number). Mary is not allowed to call out any number twice. When John's number becomes negative, Mary loses. Does Mary have a winning strategy?

6 [**7**] The signs "+" or "-" are placed in all cells of a 4×4 square table. It is allowed to change a sign of any cell altogether with signs of all its adjacent cells (i.e. cells having a common side with it). Find the number of different tables that could be obtained by iterating this procedure.

7 [**8**] A square is triangulated in such way that no three vertices are colinear. For every vertex (including vertices of the square) the number of sides issuing from it is counted. Can it happen that all these numbers are even?

Keep the problem set.
Visit: `http://www.math.toronto.edu/oz/turgor/`

Seniors

(Grades 11 and up)

International Mathematics
TOURNAMENT OF THE TOWNS

O-Level Paper **Spring 2003.**

1 [**3**] 2003 dollars are placed into N purses, and the purses are placed into M pockets. It is known that N is greater than the number of dollars in any pocket. Is it (always) true that there is a purse with less than M dollars in it?

2 [**3**] 100-gon made of 100 sticks. Could it happen that it is not possible to construct a polygon from any lesser number of these sticks?

3 [**4**] Point M is chosen in $\triangle ABC$ so that the radii of the circumcircles of $\triangle AMC$, $\triangle BMC$, and $\triangle BMA$ are no smaller than the radius of the circumcircle of $\triangle ABC$. Prove that all four radii are equal.

4 [**5**] In the sequence 00, 01, 02, 03,. . . , 99 the terms are rearranged so that each term is obtained from the previous one by increasing or decreasing one of its digits by 1 (for example, 29 can be followed by 19, 39, or 28, but not by 30 or 20). What is the maximal number of terms that could remain on their places?

5 [**5**] Prove that one can cut $a \times b$ rectangle, $\frac{b}{2} < a < b$, into three pieces and rearrange them into a square (without overlaps and holes).

Keep the problem set.
Visit: `http://www.math.toronto.edu/oz/turgor/`

Juniors
(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS

A-Level Paper **Fall 2004.**[1]

1 [**4**] A triangle is called *rational* if all its angles (measured in grades) are rational. An interior point of the triangle is called *rational* if all three triangles obtained by connecting this point with three vertices are rational.

Prove that every acute rational triangle has at least three distinct rational points.

2 [**5**] An incircle of triangle ABC touches the sides BC, CA and AB at points A', B' and C' respectively.

Is it necessarily true that triangle ABC is equilateral if $AA' = BB' = CC'$?

3 [**6**] What is the maximal number of knights one can place on 8×8 chessboard so that each knight attacks no more than 7 other knights?

4 [**6**] The results of operations

$$x + y, \qquad x - y, \qquad xy, \qquad x/y$$

are written on four cards that are placed on a table in random order. Prove that one can restore both x and y given that x and y are positive numbers.

5 [**7**] Point K belongs to side BC of triangle ABC. Incircles of triangles ABK and ACK touch BC at points M and N respectively. Prove that $BM \cdot CN > KM \cdot KN$.

6 [**8**] Joe and Pete, in turns, divide a piece of cheese. At first, Joe cuts the cheese into two pieces, then Pete chooses one of them and cuts it into two pieces. Then Joe chooses and cuts. The procedure continues until they get 5 pieces.

Now starting from Joe they, in turns, pick up pieces until nothing is left. What is the maximal amount of cheese that each of them can guarantee for himself (no matter how his opponent plays)?

7 [**8**] There are two rectangles A and B. It is known that one can tile a rectangle similar to B using copies of A. Prove that one can tile a rectangle similar to A using copies of B.

[1] Your total score is based on the three problems for which you earn the most points. Points for each problem are shown in brackets [ ].

Juniors

(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS

O-Level Paper **Fall 2004.**[1]

1 **[3]** Is it possible to arrange integers from 1 to 2004 in some order so that the sum of any 10 consecutive numbers is divisible by 10?

2 [**4**] A box contains red, green, blue, and white balls; 111 balls in total. It is known that among any 100 of them there are always balls of all 4 colors in mention.

Find the minimal number N such that among any N balls there are always balls of at least 3 different colors.

3 [**4**] A country consists of several cities; some of them are connected by Direct Express buses (each route connects two cities without intermediate stops).

Mr. Poor bought one ticket for every bus route while Mr. Rich bought n tickets for every bus route (a ticket allows a single one-way travel in either direction). Both Mr. Poor and Mr. Rich started from town A. Mr. Poor finished his travel in town B using up all his tickets without buying extra ones. Mr. Rich, after using some of his tickets, got stuck in town X: he cannot leave it without buying a new ticket. Prove that X is either A or B.

4 [**5**] A circle and a straight line with no common points are given. With compass and straightedge construct a square with two adjacent vertices on the circle and two other vertices on the line (it is known that such a square exists).

5 [**5**] Find the number of ways to decompose 2004 into a sum of positive integers (one or more) that all are "approximately equal".

Decompostions obtained from one another by permutations are not considered as different.

Two numbers are called *approximately equal* if their difference is at most 1.

[1]Your total score is based on the three problems for which you earn the most points. Points for each problem are shown in brackets [ ].

Seniors

(Grades 11 & 12)

International Mathematics
TOURNAMENT OF THE TOWNS

A-Level Paper **Fall 2004.**[1]

1 **[5]** Functions $f(x)$ and $g(y)$, defined for all real x and y satisfy conditions

$$f\big(g(y)\big) = y, \qquad g\big(f(x)\big) = x \qquad \text{for all } x \text{ and } y.$$

It is known that $f(x) = kx + h(x)$, where k is a coefficient and $h(x)$ is a periodic function. Prove that $g(y)$ is also a sum of a linear and a periodic function.

Function h is called *periodic* if there exists $d \neq 0$ such that $h(x + d) = h(x)$ for all x.

2 **[5]** In turns Joe and Pete pick up pebbles from the pile. Joe starts. On his turn he takes either 1 or 10 pebbles. On his turn Pete takes either m or n pebbles.

The player who cannot move, loses. It is known that Joe has a winning strategy for any initial number of pebbles in the pile (he can win no matter how Pete plays). Find possible values of m and n.

3 **[5]** The results of operations

$$x + y, \qquad x - y, \qquad xy, \qquad x/y$$

are written on four cards that are placed on a table in random order. Prove that one can restore both x and y given that x and y are positive numbers.

4 **[6]** A circle with the center I is entirely inside of a circle with center O. Consider all possible chords AB of the larger circle which are tangent to the smaller one. Find the locus of the centers of the circles circumscribed about the triangle AIB.

5 **[7]** here are two rectangles A and B. It is known that one can tile a rectangle similar to B using copies of A. Prove that one can tile a rectangle similar to A using copies of B.

6 **[8]** Let n be an integer divisible by neither 2 nor 3. Let us call a triangle *admissible* if all its angles are in the form $\frac{m}{n} \cdot 180^\circ$ where m is an integer. Triangles which are not similar we call *essentially different*.

In the beginning there is one admissible triangle. The following procedure is applied: we chose a triangle from the the set obtained on the previous stage and cut it into two admissible triangles so that all the triangles in the new set are essentially different.

This procedure repeats itself until it is possible. Prove that in the end we get all possible admissible triangles.

7 **[8]** Let $\angle AOB$ be obtained from $\angle COD$ by rotation (ray AO transforms into ray CO). Let E and F be the points of intersection of the circles inscribed into these angles.

Prove that $\angle AOE = \angle DOF$.

[1]Your total score is based on the three problems for which you earn the most points. Points for each problem are shown in brackets [ ].

Seniors

(Grades 11 & 12)

International Mathematics
TOURNAMENT OF THE TOWNS

O-Level Paper **Fall 2004.**[1]

1 [3] Three circles pass through point X and A, B, C are their intersection points (other than X). Let A' be the second point of intersection of straight line AX and the circle circumscribed around triangle BCX. Define simiarly points B', C'. Prove that triangles ABC', $AB'C$, and $A'BC$ are similar.

2 [3] A box contains red, blue, and white balls; 100 balls in total. It is known that among any 26 of them there are always 10 balls of the same color.

Find the minimal number N such that among any N balls there are always 30 balls of the same color.

3 [4] $P(x)$ and $Q(x)$ are polynomials of positive degree such that

$$P(P(x)) = Q(Q(x)) \qquad \text{and} \qquad P(P(P(x))) = Q(Q(Q(x))) \qquad\qquad \text{for all } x.$$

Does this necessarily mean that $P(x) = Q(x)$?

4 [4] Find the number of ways to decompose 2004 into a sum of positive integers (one or more) that all are "approximately equal".

Decompostions obtained from one another by permutations are not considered as different.

Two numbers are called *approximately equal* if their difference is at most 1.

5 [5] Find all values N such that it is possible to arrange all integers from 1 to N in a way that for any group of two or more consecutive numbers the arithmetic mean of this group is not an integer.

[1] Your total score is based on the three problems for which you earn the most points. Points for each problem are shown in brackets [ ].

International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Spring 2004.**

1. The sum of all terms of a finite arithmetical progression of integers is a power of two. Prove that the number of terms is also a power of two.

2. What is the maximal number of checkers that can be placed on an 8×8 checkerboard so that each checker stands on the middle one of three squares in a row diagonally, with exactly one of the other two squares occupied by another checker?

3. Each day, the price of the shares of the corporation "Soap Bubble, Limited" either increases or decreases by n percent, where n is an integer such that $0 < n < 100$. The price is calculated with unlimited precision. Does there exist an n for which the price can take the same value twice?

4. Two circles intersect in points A and B. Their common tangent nearer B touches the circles at points E and F, and intersects the extension of AB at the point M. The point K is chosen on the extention of AM so that $KM = MA$. The line KE intersects the circle containing E again at the point C. The line KF intersects the circle containing F again at the point D. Prove that the points A, C and D are collinear.

5. All sides of a polygonal billiard table are in one of two perpendicular directions. A tiny billiard ball rolls out of the vertex A of an inner 90° angle and moves inside the billiard table, bouncing off its sides according to the law "angle of reflection equals angle of incidence". If the ball passes a vertex, it will drop in and srays there. Prove that the ball will never return to A.

6. At the beginning of a two-player game, the number 2004! is written on the blackboard. The players move alternately. In each move, a positive integer smaller than the number on the blackboard and divisible by at most 20 different prime numbers is chosen. This is subtracted from the number on the blackboard, which is erased and replaced by the difference. The winner is the player who obtains 0. Does the player who goes first or the one who goes second have a guaranteed win, and how should that be achieved?

Note: The problems are worth 4, 5, 5, 6, 6 and 7 points respectively.

Juniors
(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS

O-Level Paper **Spring 2004.**

1 [**3**] In triangle ABC the bisector of angle A, the perpendicular to side AB from its midpoint, and the altitude from vertex B, intersect in the same point. Prove that the bisector of angle A, the perpendicular to side AC from its midpoint, and the altitude from vertex C also intersect in the same point.

2 [**3**] Find all possible values of $n \geq 1$ for which there exist n consecutive positive integers whose sum is a prime number.

3 Bucket A contains 3 litres of syrup. Bucket B contains n litres of water. Bucket C is empty. We can perform any combination of the following operations:

- Pour away the entire amount in bucket X,
- Pour the entire amount in bucket X into bucket Y,
- Pour from bucket X into bucket Y until buckets Y and Z contain the same amount.

(a) [**3**] How can we obtain 10 litres of 30% syrup if $n = 20$?

(b) [**2**] Determine all possible values of n for which the task in (a) is possible.

4 [**5**] A positive integer $a > 1$ is given (in decimal notation). We copy it twice and obtain a number $b = \overline{aa}$ which happened to be a multiple of a^2. Find all possible values of b/a^2.

5 [**6**] Two 10-digit integers are called neighbours if they differ in exactly one digit (for example, integers 1234567890 and 1234507890 are neighbours). Find the maximal number of elements in the set of 10-digit integers with no two integers being neighbours.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Spring 2004.**

1. Each day, the price of the shares of the corporation "Soap Bubble, Limited" either increases or decreases by n percent, where n is an integer such that $0 < n < 100$. The price is calculated with unlimited precision. Does there exist an n for which the price can take the same value twice?

2. All angles of a polygonal billiard table have measures in integral numbers of degrees. A tiny billiard ball rolls out of the vertex A of an interior $1°$ angle and moves inside the billiard table, bouncing off its sides according to the law "angle of reflection equals angle of incidence". If the ball passes throigh a vertex, it will drop in and stays there. Prove that the ball will never return to A.

3. The perpendicular projection of a triangular pyramid on some plane has the largest possible area. Prove that this plane is parallel to either a face or two opposite edges of the pyramid.

4. At the beginning of a two-player game, the number 2004! is written on the blackboard. The players move alternately. In each move, a positive integer smaller than the number on the blackboard and divisible by at most 20 different prime numbers is chosen. This is subtracted from the number on the blackboard, which is erased and replaced by the difference. The winner is the player who obtains 0. Does the player who goes first or the one who goes second have a guaranteed win, and how should that be achieved?

5. The parabola $y = x^2$ intersects a circle at exactly two points A and B. If their tangents at A coincide, must their tangents at B also coincide?

6. The audience shuffles a deck of 36 cards, containing 9 cards in each of the suits spades, hearts, diamonds and clubs. A magician predicts the suit of the cards, one at a time, starting with the uppermost one in the face-down deck. The design on the back of each card is an arrow. An assistant examines the deck without changing the order of the cards, and points the arrow on the back each card either towards or away from the magician, according to some system agreed upon in advance with the magician. Is there such a system which enables the magician to guarantee the correct prediction of the suit of at least

   (a) 19 cards;

   (b) 20 cards?

Note: The problems are worth 4, 6, 6, 6, 7 and 3+5points respectively.

Seniors

(Grades 11 and up)

International Mathematics
TOURNAMENT OF THE TOWNS

O-Level Paper **Spring 2004.**

1 [**4**] Segments AB, BC and CD of the broken line $ABCD$ are equal and are tangent to a circle with centre at the point O. Prove that the point of contact of this circle with BC, the point O and the intersection point of AC and BD are collinear.

2 [**4**] A positive integer $a > 1$ is given (in decimal notation). We copy it twice and obtain a number $b = \overline{aa}$ which happened to be a multiple of a^2. Find all possible values of $\dfrac{b}{a^2}$.

3 [**4**] Perimeter of a convex quadrilateral is 2004 and one of its diagonals is 1001. Can another diagonal be 1? 2? 1001?

4 [**5**] Arithmetical progression $a_1, a_2, a_3, a_4, \ldots$ contains a_1^2, a_2^2 and a_3^2 at some positions. Prove that all terms of this progression are integers.

5 [**5**] Two 10-digit integers are called neighbours if they differ in exactly one digit (for example, integers 1234567890 and 1234507890 are neighbours). Find the maximal number of elements in the set of 10-digit integers with no two integers being neighbours.

Juniors

(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS

A-Level Paper **Fall 2005.**[1]

1 [**3**] A palindrome is a positive integer which reads in the same way in both directions (for example, 1, 343 and 2002 are palindromes, while 2005 is not). Is it possible to find 2005 pairs in the form of $(n, n+110)$ where both numbers are palindromes?

2 [**5**] The extensions of sides AB and CD of a convex quadrilateral $ABCD$ intersect at K. It is known that $AD = BC$. Let M and N be the midpoints of sides AB and CD. Prove that the triangle MNK is obtuse.

3 [**6**] Originally, every square of 8×8 chessboard contains a rook. One by one, rooks which attack an odd number of others are removed. Find the maximal number of rooks that can be removed. (A rook attacks another rook if they are on the same row or column and there are no other rooks between them.)

4 [**6**] Two ants crawl along the perimeter of a polygonal table, so that the distance between them is always 10 cm. Each side of the table is more than 1 meter long. At the initial moment both ants are on the same side of the table.

(a) [**2**] Suppose that the table is a convex polygon. Is it always true that both ants can visit each point on the perimeter?

(b) [**4**] Is it always true (this time without assumption of convexity) that each point on the perimeter can be visited by at least one ant?

5 [**7**] Find the largest positive integer N such that the equation $99x + 100y + 101z = N$ has an unique solution in the positive integers x, y, z.

6 [**8**] Karlsson-on-the-Roof has 1000 jars of jam. The jars are not necessarily identical; each contains no more than $\frac{1}{100}$-th of the total amount of the jam. Every morning, Karlsson chooses any 100 jars and eats the same amount of the jam from each of them. Prove that Karlsson can eat all the jam.

[1] Your total score is based on the three problems for which you earn the most points. Points for each problem are shown in brackets [ ].

Juniors
(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS

O-Level Paper **Fall 2005.**[1]

1 [**3**] In triangle ABC, points M_1, M_2 and M_3 are midpoints of sides AB, BC and AC, respectively, while points H_1, H_2 and H_3 are bases of altitudes drawn from C, A and B, respectively. Prove that one can construct a triangle from segments H_1M_2, H_2M_3 and H_3M_1.

2 [**3**] A number is written in each corner of the cube. On each step, each number is replaced with the average of three numbers in the three adjacent corners (all the numbers are replaced simultaneously). After ten such steps, every number returns to its initial value. Must all numbers have been originally equal?

3 [4] A segment of unit length is cut into eleven smaller segments, each with length of no more than a. For what values of a, can one guarantee that any three segments form a triangle?

4 [4] A chess piece moves as follows: it can jump 8 or 9 squares either vertically or horizontally. It is not allowed to visit the same square twice. At most, how many squares can this piece visit on a 15×15 board (it can start from any square)?

5 [5] Among 6 coins one is counterfeit (its weight differs from that real one and neither weights is known). Using scales that show the total weight of coins placed on the cup, find the counterfeit coin in 3 weighings.

[1] Your total score is based on the three problems for which you earn the most points. Points for each problem are shown in brackets [ ].

Seniors

(Grades 11, 12)

International Mathematics
TOURNAMENT OF THE TOWNS

A-Level Paper **Fall 2005.**[1]

1 **[3]** For which $n \geq 2$ can one find a sequence of distinct positive integers $a_1, a_2, \ldots, a_n$ so that the sum

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + ... + \frac{a_n}{a_1}$$

is an integer?

2 **[5]** Two ants crawl along the perimeter of a polygonal table, so that the distance between them is always 10 cm. Each side of the table is more than 1 meter long. At the initial moment both ants are on the same side of the table.

(a) **[2]** Suppose that the table is a convex polygon. Is it always true that both ants can visit each point on the perimeter?

(b) **[3]** Is it always true (this time without assumption of convexity) that each point on the perimeter can be visited by at least one ant?

3 **[5]** Originally, every square of 8×8 chessboard contains a rook. One by one, rooks which attack an odd number of others are removed. Find the maximal number of rooks that can be removed. (A rook attacks another rook if they are on the same row or column and there are no other rooks between them.)

4 **[6]** Several positive numbers each not exceeding 1 are written on the circle. Prove that one can divide the circle into three arcs so that the sums of numbers on any two arcs differ by no more than 1. (If there are no numbers on an arc, the sum is equal to zero.)

5 **[7]** In triangle ABC bisectors AA_1, BB_1 and CC_1 are drawn. Given $\angle A : \angle B : \angle C = 4 : 2 : 1$, prove that $A_1B_1 = A_1C_1$.

6 **[8]** Two operations are allowed:

(i) to write two copies of number 1;

(ii) to replace any two identical numbers n by $(n + 1)$ and $(n - 1)$.

Find the minimal number of operations that required to produce the number 2005 (at the beginning there are no numbers).

[1]Your total score is based on the three problems for which you earn the most points. Points for each problem are shown in brackets [ ].

Seniors

(Grades 11, 12)

International Mathematics
TOURNAMENT OF THE TOWNS

O-Level Paper **Fall 2005.**[1]

1 [**3**] Can two perfect cubes fit between two consecutive perfect squares? In other words, do there exist positive integers a, b, n such that $n^2 < a^3 < b^3 < (n+1)^2$?

2 [**3**] A segment of length $\sqrt{2}+\sqrt{3}+\sqrt{5}$ is drawn. Is it possible to draw a segment of unit length using a compass and a straightedge?

3 [**4**] Among 6 coins one is counterfeit (its weight differs from that real one and neither weights is known). Using scales that show the total weight of coins placed on the cup, find the counterfeit coin in 3 weighings.

4 [**4**] On all three sides of a right triangle ABC external squares are constructed; their centers denoted by D, E, F. Show that the ratio of the area of triangle DEF to the area of triangle ABC is:

a) [**2**] greater than 1;

b) [**2**] at least 2.

5 [**5**] A cube lies on the plane. After being rolled a few times (over its edges), it is brought back to its initial location with the same face up. Could the top face have been rotated by 90 degrees?

[1]Your total score is based on the three problems for which you earn the most points. Points for each problem are shown in brackets [ ].

International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper[1] **Spring 2005.**

1. On the graph of a polynomial with integral coefficients are two points with integral coordinates. Prove that if the distance between these two points is integral, then the segment connecting them is parallel to the x-axis.

2. The altitudes AD and BE of triangle ABC meet at its orthocentre H. The midpoints of AB and CH are X and Y, respectively. Prove that XY is perpendicular to DE.

3. Baron Münchhausen's watch works properly, but has no markings on its face. The hour, minute and second hands have distinct lengths, and they move uniformly. The Baron claims that since none of the mutual positions of the hands is repeats twice in the period between 8:00 and 19:59, he can use his watch to tell the time during the day. Is his assertion true?

4. A 10×12 paper rectangle is folded along the grid lines several times, forming a thick 1×1 square. How many pieces of paper can one possibly get by cutting this square along the segment connecting
   (a) the midpoints of a pair of opposite sides;
   (b) the midpoints of a pair of adjacent sides?

5. In a rectangular box are a number of rectangular blocks, not necessarily identical to one another. Each block has one of its dimensions reduced. Is it always possible to pack these blocks in a smaller rectangular box, with the sides of the blocks parallel to the sides of the box?

6. John and James wish to divide 25 coins, of denominations 1, 2, 3, ..., 25 kopeks. In each move, one of them chooses a coin, and the other player decides who must take this coin. John makes the initial choice of a coin, and in subsequent moves, the choice is made by the player having more kopeks at the time. In the event that there is a tie, the choice is made by the same player in the preceding move. After all the coins have been taken, the player with more kokeps wins. Which player has a winning strategy?

7. The squares of a chessboard are numbered in the following way. The upper left corner is numbered 1. The two squares on the next diagonal from top-right to bottom-left are numbered 2 and 3. The three squares on the next diagonal are numbered 4, 5 and 6, and so on. The two squares on the second-to-last diagonal are numbered 62 and 63, and the lower right corner is numbered 64. Peter puts eight pebbles on the squares of the chessboard in such a way that there is exactly one pebble in each column and each row. Then he moves each pebble to a square with a number greater than that of the original square. Can it happen that there is still exactly one pebble in each column and each row?

Note: The problems are worth 4, 5, 5, 2+4, 6, 7 and 8 points respectively.

[1] Courtesy of Andy Liu.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper[1] **Spring 2005.**

1. Anna and Boris move simultaneously towards each other, from points A and B respectively. Their speeds are constant, but not necessarily equal. Had Anna started 30 minutes earlier, they would have met 2 kilometers nearer to B. Had Boris started 30 minutes earlier instead, they would have met some distance nearer to A. Can this distance be uniquely determined?

2. Prove that one of the digits 1, 2 and 9 must appear in the base-ten expression of n or $3n$ for any positive integer n.

3. There are eight identical Black Queens in the first row of a chessboard and eight identical White Queens in the last row. The Queens move one at a time, horizontally, vertically or diagonally by any number of squares as long as no other Queens are in the way. Black and White Queens move alternately. What is the minimal number of moves required for interchanging the Black and White Queens?

4. M and N are the midpoints of sides BC and AD, respectively, of a square $ABCD$. K is an arbitrary point on the extension of the diagonal AC beyond A. The segment KM intersects the side AB at some point L. Prove that $\angle KNA = \angle LNA$.

5. In a certain big city, all the streets go in one of two perpendicular directions. During a drive in the city, a car does not pass through any place twice, and returns to the parking place along a street from which it started. If it has made 100 left turns, how many right turns must it have made?

Note: The problems are worth 3, 4, 5, 5 and 5 points respectively.

[1]Courtesy of Andy Liu.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper[1] **Spring 2005.**

1. On the graph of a polynomial with integral coefficients are two points with integral coordinates. Prove that if the distance between these two points is integral, then the segment connecting them is parallel to the x-axis.

2. A circle ω_1 with centre O_1 passes through the centre O_2 of a second circle ω_2. The tangent lines to ω_2 from a point C on ω_1 intersect ω_1 again at points A and B respectively. Prove that AB is perpendicular to O_1O_2.

3. John and James wish to divide 25 coins, of denominations 1, 2, 3, …, 25 kopeks. In each move, one of them chooses a coin, and the other player decides who must take this coin. John makes the initial choice of a coin, and in subsequent moves, the choice is made by the player having more kopeks at the time. In the event that there is a tie, the choice is made by the same player in the preceding move. After all the coins have been taken, the player with more kokeps wins. Which player has a winning strategy?

4. For any function $f(x)$, define $f^1(x) = f(x)$ and $f^n(x) = f(f^{n-1}(x))$ for any integer $n \geq 2$. Does there exist a quadratic polynomial $f(x)$ such that the equation $f^n(x) = 0$ has exactly 2^n distinct real roots for every positive integer n?

5. Prove that if a regular icosahedron and a regular dodecahedron have a common circumsphere, then they have a common insphere.

6. A *lazy* rook can only move from a square to a vertical or a horizontal neighbour. It follows a path which visits each square of an 8×8 chessboard exactly once. Prove that the number of such paths starting at a corner square is greater than the number of such paths starting at a diagonal neighbour of a corner square.

7. Every two of 200 points in space are connected by a segment, no two intersecting each other. Each segment is painted in one colour, and the total number of colours is k. Peter wants to paint each of the 200 points in one of the colours used to paint the segments, so that no segment connects two points both in the same colour as the segment itself. Can Peter always do this if

   (a) $k = 7$;

   (b) $k = 10$?

Note: The problems are worth 4, 5, 5, 6, 7, 7 and 4+4 points respectively.

[1]Courtesy of Andy Liu.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper[1] **Spring 2005.**

1. The graphs of four functions of the form $y = x^2 + ax + b$, where a and b are real coefficients, are plotted on the coordinate plane. These graphs have exactly four points of intersection, and at each one of them, exactly two graphs intersect. Prove that the sum of the largest and the smallest x-coordinates of the points of intersection is equal to the sum of the other two.

2. The base-ten expressions of all the positive integers are written on an infinite ribbon without spacing: 1234567891011.... Then the ribbon is cut up into strips seven digits long. Prove that any seven digit integer will:

   (a) appear on at least one of the strips;

   (b) appear on an infinite number of strips.

3. M and N are the midpoints of sides BC and AD, respectively, of a square $ABCD$. K is an arbitrary point on the extension of the diagonal AC beyond A. The segment KM intersects the side AB at some point L. Prove that $\angle KNA = \angle LNA$.

4. In a certain big city, all the streets go in one of two perpendicular directions. During a drive in the city, a car does not pass through any place twice, and returns to the parking place along a street from which it started. If it has made 100 left turns, how many right turns must it have made?

5. The sum of several positive numbers is equal to 10, and the sum of their squares is greater than 20. Prove that the sum of the cubes of these numbers is greater than 40.

Note: The problems are worth 3, 3+1, 4, 4 and 5 points respectively.

[1]Courtesy of Andy Liu.

Juniors

(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS

A-Level Paper **Fall 2006.**[1]

1 [3] Two regular polygons, a 7-gon and a 17-gon are given. For each of them two circles are drawn, an inscribed circle and a circumscribed circle. It happened that rings containing the polygons have equal areas. Prove that sides of the polygons are equal.

2 [5] When Ann meets new people, she tries to find out who is acquainted with who. In order to memorize it she draws a circle in which each person is depicted by a chord; moreover, chords corresponding to acquainted persons intersect (possibly at the ends), while the chords corresponding to non-acquainted persons do not. Ann believes that such set of chords exists for any company. Is her judgement correct?

3 A 3×3 square is filled with numbers: a, b, c, d, e, f, g, h, i in the following way: Given that the square is magic (sums of the numbers in each row, column and each of two diagonals are the same), show that

a	b	c
d	e	f
g	h	i

a) **[3]** $2(a + c + g + i) = b + d + f + h + 4e$.

b) **[3]** $2(a^3 + c^3 + g^3 + i^3) = b^3 + d^3 + f^3 + h^3 + 4e^3$.

4 [6] A circle of radius R is inscribed into an acute triangle. Three tangents to the circle split the triangle into three right angle triangles and a hexagon that has perimeter Q. Find the sum of diameters of circles inscribed into the three right triangles.

5 Consider a square painting of size 1×1. A rectangular sheet of paper of area 2 is called its "envelope" if one can wrap the painting with it without cutting the paper. (For instance, a 2×1 rectangle and a square with side $\sqrt{2}$ are envelopes.)

a) **[4]** Show that there exist other envelopes.

b) **[3]** Show that there exist infinitely many envelopes.

6 [8] Let $1 + 1/2 + 1/3 + \cdots + 1/n = a_n/b_n$, where a_n and b_n are relatively prime. Show that there exist infinitely many positive integers n, such that $b_{n+1} < b_n$.

7 [9] A Magician has a deck of 52 cards. Spectators want to know the order of cards in the deck (without specifying face-up or face-down). They are allowed to ask the questions "How many cards are there between such-and-such card and such-and-such card?" One of the spectators knows the card order. Find the minimal number of questions he needs to ask to be sure that the other spectators can learn the card order.

[1] Your total score is based on the three problems for which you earn the most points. Points for each problem are shown in brackets [ ].

International Mathematics

TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2006**[1]

1. Two positive integers are written on the blackboard. Mary records in her notebook the square of the smaller number and replaces the larger number on the blackboard by the difference of the two numbers. With the new pair of numbers, she repeats the process, and continues until one of the numbers on the blackboard becomes zero. What will be the sum of the numbers in Mary's notebook at that point?

2. A Knight always tells the truth. A Knave always lies. A Normal may either lie or tell the truth. You are allowed to ask questions that can be answered with "yes" or "no", such as "Is this person a Normal?"

   (a) There are three people in front of you. One is a Knight, another one is a Knave, and the third one is a Normal. They all know the identities of one another. How can you too learn the identity of each?

   (b) There are four people in front of you. One is a Knight, another one is a Knave, and the other two are Normals. They all know the identities of one another. Prove that the Normals may agree in advance to answer your questions in such a way that you will not be able to learn the identity of any of the four people.

3. (a) Prove that from 2007 given positive integers, one of them can be chosen so the product of the remaining numbers is expressible in the form $a^2 - b^2$ for some positive integers a and b.

   (b) One of 2007 given positive integers is 2006. Prove that if there is a unique number among them such that the product of the remaining numbers is expressible in the form $a^2 - b^2$ for some positive integers a and b, then this unique number is 2006.

4. Given triangle ABC, BC is extended beyond B to the point D such that $BD = BA$. The bisectors of the exterior angles at vertices B and C intersect at the point M. Prove that quadrilateral $ADMC$ is cyclic.

5. A square is dissected into n congruent non-convex polygons whose sides are parallel to the sides of the square, and no two of these polygons are parallel translates of each other. What is the maximum value of n?

Note: The problems are worth 4, 1+3, 2+2, 4 and 4 points respectively.

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[1]Courtesy of Professor Andy Liu.

Seniors

(Grades 11 and 12)

International Mathematics
TOURNAMENT OF THE TOWNS

A-Level Paper **Fall 2006.**[2]

1 [**4**] When Ann meets new people, she tries to find out who is acquainted with who. In order to memorize it she draws a circle in which each person is depicted by a chord; moreover, chords corresponding to acquainted persons intersect (possibly at the ends), while the chords corresponding to non-acquainted persons do not. Ann believes that such set of chords exists for any company. Is her judgement correct?

2 [**6**] Suppose ABC is an acute triangle. Points A_1, B_1 and C_1 are chosen on sides BC, AC and AB respectively so that the rays A_1A, B_1B and C_1C are bisectors of triangle $A_1B_1C_1$. Prove that AA_1, BB_1 and CC_1 are altitudes of triangle ABC.

3 [**6**] The n-th digit of number $a = 0.12457\ldots$ equals the first digit of the integer part of the number $n\sqrt{2}$. Prove that a is irrational number.

4 [**6**] Is it possible to split a prism into disjoint set of pyramids so that each pyramid has its base on one base of the prism, while its vertex on another base of the prism ?

5 [**7**] Let $1 + 1/2 + 1/3 + \cdots + 1/n = a_n/b_n$, where a_n and b_n are relatively prime. Show that there exist infinitely many positive integers n, such that $b_{n+1} < b_n$.

6 Let us say that a deck of 52 cards is arranged in a "regular" way if the ace of spades is on the very top of the deck and any two adjacent cards are either of the same value or of the same suit (top and bottom cards regarded adjacent as well). Prove that the number of ways to arrange a deck in regular way is

a) [**3**] divisible by 12!

b) [**5**] divisible by 13!

7 Positive numbers x_1, ..., x_k satisfy the following inequalities:

$$x_1^2 + \cdots + x_k^2 < \frac{x_1 + \cdots + x_k}{2} \quad \text{and} \quad x_1 + \cdots + x_k < \frac{x_1^3 + \cdots + x_k^3}{2}.$$

a) [**3**] Show that $k > 50$;

b) [**3**] Give an example of such numbers for some value of k;

c) [**3**] Find minimum k, for which such an example exists.

[2] Your total score is based on the three problems for which you earn the most points. Points for each problem are shown in brackets [ ].

International Mathematics

TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2006**[1]

1. Three positive integers x and y are written on the blackboard. Mary records in her notebook the product of any two of them and reduces the third number on the blackboard by 1. With the new trio of numbers, she repeats the process, and continues until one of the numbers on the blackboard becomes zero. What will be the sum of the numbers in Mary's notebook at that point?

2. The incircle of the quadrilateral $ABCD$ touches AB, BC, CD and DA at E, F, G and H respectively. Prove that the line joining the incentres of triangles HAE and FCG is perpendicular to the line joining the incentres of triangles EBF and GDH.

3. Each of the numbers 1, 2, 3, $\ldots$, 2006^2 is placed at random into a cell of a 2006×2006 board. Prove that there exist two cells which share a common side or a common vertex such that the sum of the numbers in them is divisible by 4.

4. Every term of an infinite geometric progression is also a term of a given infinite arithmetic progression. Prove that the common ratio of the geometric progression is an integer.

5. Can a regular octahedron be inscribed in a cube in such a way that all vertices of the octahedron are on cube's edges?

Note: The problems are worth 4, 4, 4, 4 and 5 points respectively.

[1]Courtesy of Professor Andy Liu.

Juniors
(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS

A-Level Paper **Spring 2006.**[1]

1 **[4]** There is a billiard table in shape of rectangle 2×1, with pockets at its corners and at midpoints of its two largest sizes. Find the minimal number of balls one has to place on the table interior so that any pocket is on a straight line with some two balls. (Assume that pockets and balls are points).

2 **[4]** Prove that one can find 100 distinct pairs of integers such that every digit of each number is no less than 6 and the product of the numbers in each pair is also a number with all its digits being no less than 6.

3 **[5]** On sides AB and BC of an acute triangle ABC two congruent rectangles $ABMN$ and $LBCK$ are constructed (outside of the triangle), so that $AB = LB$. Prove that straight lines AL, CM and NK intersect at the same point.

4 **[5]** Is there exist some positive integer n, such that the first decimal of 2^n (from left to the right) is 5 while the first decimal of 5^n is 2?

5 **[6]** Numbers 0, 1 and 2 are placed in a table 2005×2006 so that total sums of the numbers in each row and in each column are factors of 3. Find the maximal possible number of 1-s that can be placed in the table.

6 **[7]** Let us call a pentagon curved, if all its sides are arcs of some circles. Are there exist a curved pentagon P and a point A on its boundary so that any straight line passing through A divides perimeter of P into two parts of the same length?

7 Anna and Boris have the same copy of 5×5 table filled with 25 distinct numbers. After choosing the maximal number in the table, Anna erases the row and the column that contain this number. Then she continue the same operations with a smaller table till it is possible. Boris basically does the same; however, each time choosing the minimal number in a table. Can it happen that the total sum of the numbers chosen by Boris

a) **[6]** is greater than the total sum of the numbers chosen by Anna?

b) **[2]** is greater than the total sum of any 5 numbers of initial table given that no two of the numbers are in the same row or in the same column?

[1] Your total score is based on the three problems for which you earn the most points. Points for each problem are shown in brackets [ ].

Juniors
(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS

O-Level Paper **Spring 2006.**[1]

1 **[3]** Let $\angle A$ in a triangle ABC be $60°$. Let point N be the intersection of AC and perpendicular bisector to the side AB while point M be the intersection of AB and perpendicular bisector to the side AC. Prove that $CB = MN$.

2 **[3]** A $n \times n$ table is filled with the numbers as follows: the first column is filled with 1's, the second column with 2's, and so on. Then, the numbers on the main diagonal (from top-left to bottom-right) are erased. Prove that the total sums of the numbers on both sides of the main diagonal differ in exactly two times.

3 **[4]** Let a be some positive number. Find the number of integer solutions x of inequality $2 < xa < 3$ given that inequality $1 < xa < 2$ has exactly 3 integer solutions. Consider all possible cases.

4 Anna, Ben and Chris sit at the round table passing and eating nuts. At first only Anna has the nuts that she divides equally between Ben and Chris, eating a leftover (if there is any). Then Ben does the same with his pile. Then Chris does the same with his pile. The process repeats itself: each of the children divides his/her pile of nuts equally between his/her neighbours eating the leftovers if there are any. Initially, the number of nuts is large enough (more than 3). Prove that

a) **[3]** at least one nut is eaten;

b) **[3]** all nuts cannot be eaten.

5 Pete has n^3 white cubes of the size $1 \times 1 \times 1$. He wants to construct a $n \times n \times n$ cube with all its faces being completely white. Find the minimal number of the faces of small cubes that Basil must paint (in black colour) in order to prevent Pete from fulfilling his task. Consider the cases:

a) **[2]** $n = 2$;

b) **[4]** $n = 3$.

[1]Your total score is based on the three problems for which you earn the most points. Points for each problem are shown in brackets [ ].

Seniors

(Grades 11 and 12)

International Mathematics
TOURNAMENT OF THE TOWNS

A-Level Paper **Spring 2006.**[2]

1 [**4**] Prove that one can always mark 50 points inside of any convex 100-gon, so that each its vertix is on a straight line connecting some two marked points.

2 [**5**] Are there exist some positive integers n and k, such that the first decimals of 2^n (from left to the right) represent the number 5^k while the first decimals of 5^n represent the number 2^k?

3 [**5**] Consider a polynomial $P(x) = x^4 + x^3 - 3x^2 + x + 2$. Prove that at least one of the coefficients of $\big(P(x)\big)^k$, (k is any positive integer) is negative.

4 [**6**] In triangle ABC let X be some fixed point on bisector AA' while point B' be intersection of BX and AC and point C' be intersection of CX and AB. Let point P be intersection of segments $A'B'$ and CC' while point Q be intersection of segments $A'C'$ and BB'. Prove that $\angle PAC = \angle QAB$.

5 [**6**] Prove that one can find infinite number of distinct pairs of integers such that every digit of each number is no less than 7 and the product of two numbers in each pair is also a number with all its digits being no less than 7.

7 On a circumference at some points sit 12 grasshoppers. The points divide the circumference into 12 arcs. By a signal each grasshopper jumps from its point to the midpoint of its arc (in clockwise direction). In such way new arcs are created. The process repeats for a number of times. Can it happen that at least one of the grasshoppers returns to its initial point after

a) [**4**] 12 jumps?

a) [**3**] 13 jumps?

8 [**8**] An ant craws along a closed route along the edges of a dodecahedron, never going backwards. Each edge of the route is passed exactly twice. Prove that one of the edges is passed both times in the same direction. (Dodecahedron has 12 faces in the shape of pentagon, 30 edges and 20 vertices; each vertex emitting 3 edges).

[2]Your total score is based on the three problems for which you earn the most points. Points for each problem are shown in brackets [ ].

Seniors

(Grades 11 and 12)

International Mathematics
TOURNAMENT OF THE TOWNS

O-Level Paper **Spring 2006.**[2]

1 All vertices of a convex polyhedron with 100 edges are cut off by some planes. The planes do not intersect either inside or on the surface of the polyhedron. For this new polyhedron find

a) [**1**] the number of vertices;

b) [**2**] the number of edges.

2 [**3**] Do there exist functions $p(x)$ and $q(x)$, such that $p(x)$ is an even function while $p\big(q(x)\big)$ is an odd function (different from 0)?

3 [**4**] Let a be some positive number. Find the number of integer solutions x of inequality $100 < xa < 1000$ given that inequality $10 < xa < 100$ has exactly 5 integer solutions. Consider all possible cases.

4 [**5**] Quadrilateral $ABCD$ is a cyclic, $AB = AD$. Points M and N are chosen on sides BC and CD respectfully so that $\angle MAN$ =1/2 ($\angle BAD$). Prove that $MN = BM + ND$.

5 Pete has n^3 white cubes of the size $1 \times 1 \times 1$. He wants to construct a $n \times n \times n$ cube with all its faces being completely white. Find the minimal number of the faces of small cubes that Basil must paint (in black colour) in order to prevent Pete from fulfilling his task. Consider the cases:

a) [**3**] $n = 3$;

b) [**3**] $n = 1000$.

[2]Your total score is based on the three problems for which you earn the most points. Points for each problem are shown in brackets [ ].

International Mathematics

TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Fall 2007.**

1. Let $ABCD$ be a rhombus. Let K be a point on the line CD, other than C or D, such that $AD = BK$. Let P be the point of intersection of BD with the perpendicular bisector of BC. Prove that A, K and P are collinear.
2. (a) Each of Peter and Basil thinks of three positive integers. For each pair of his numbers, Peter writes down the greatest common divisor of the two numbers. For each pair of his numbers, Basil writes down the least common multiple of the two numbers. If both Peter and Basil write down the same three numbers, prove that these three numbers are equal to each other.
   (b) Can the analogous result be proved if each of Peter and Basil thinks of four positive integers instead?
3. Michael is at the centre of a circle of radius 100 metres. Each minute, he will announce the direction in which he will be moving. Catherine can leave it as is, or change it to the opposite direction. Then Michael moves exactly 1 metre in the direction determined by Catherine. Does Michael have a strategy which guarantees that he can get out of the circle, even though Catherine will try to stop him?
4. Two players take turns entering a symbol in an empty cell of a $1 \times n$ chessboard, where n is an integer greater than 1. Aaron always enters the symbol X and Betty always enters the symbol O. Two identical symbols may not occupy adjacent cells. A player without a move loses the game. If Aaron goes first, which player has a winning strategy?
5. Attached to each of a number of objects is a tag which states the correct mass of the object. The tags have fallen off and have been replaced on the objects at random. We wish to determine if by chance all tags are in fact correct. We may use exactly once a horizontal lever which is supported at its middle. The objects can be hung from the lever at any point on either side of the support. The lever either stays horizontal or tilts to one side. Is this task always possible?
6. The audience arranges n coins in a row. The sequence of heads and tails is chosen arbitrarily. The audience also chooses a number between 1 and n inclusive. Then the assistant turns one of the coins over, and the magician is brought in to examine the resulting sequence. By an agreement with the assistant beforehand, the magician tries to determine the number chosen by the audience.
   (a) Prove that if this is possible for some n, then it is also possible for $2n$.
   (b) Determine all n for which this is possible.
7. For each letter in the English alphabet, William assigns an English word which contains that letter. His first document consists only of the word assigned to the letter A. In each subsequent document, he replaces each letter of the preceding document by its assigned word. The fortieth document begins with "Till whatsoever star that guides my moving." Prove that this sentence reappears later in this document.

Note: The problems are worth 5, 3+3, 6, 7, 8, 4+5 and 9 points respectively.

International Mathematics

TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2007**[1]

1 [3] Black and white checkers are placed on an 8×8 chessboard, with at most one checker on each cell. What is the maximum number of checkers that can be placed such that each row and each column contains twice as many white checkers as black ones?

2 [4] Initially, the number 1 and a non-integral number x are written on a blackboard. In each step, we can choose two numbers on the blackboard, not necessarily different, and write their sum or their difference on the blackboard. We can also choose a non-zero number of the blackboard and write its reciprocal on the blackboard. Is it possible to write x^2 on the blackboard in a finite number of moves?

3 [4] D is the midpoint of the side BC of triangle ABC. E and F are points on CA and AB respectively, such that BE is perpendicular to CA and CF is perpendicular to AB. If DEF is an equilateral triangle, does it follow that ABC is also equilateral?

4 [5] Each cell of a 29×29 table contains one of the integers $1, 2, 3, \ldots, 29$, and each of these integers appears 29 times. The sum of all the numbers above the main diagonal is equal to three times the sum of all the numbers below this diagonal. Determine the number in the central cell of the table.

5 [5] The audience chooses two of five cards, numbered from 1 to 5 respectively. The assistant of a magician chooses two of the remaining three cards, and asks a member of the audience to take them to the magician, who is in another room. The two cards are presented to the magician in arbitrary order. By an arrangement with the assistant beforehand, the magician is able to deduce which two cards the audience has chosen only from the two cards he receives. Explain how this may be done.

[1] Courtesy of Professor Andy Liu.

International Mathematics

TOURNAMENT OF THE TOWNS

Junior P-Level Paper **Fall 2007**

1 [**1**] (from The Good Soldier Švejk) Senior military doctor Bautze exposed *abccc* malingerers among *aabbb* draftees who claimed not to be fit for the military service. He managed to expose all but one draftees. (He would for sure expose this one too if the lucky guy was not taken by a stroke at the very moment when the doctor yelled at him "Turn around !. . . ") Now many malingerers were exposed by the vigilant doctor?

Each digit substitutes a letter. The same digits substitute the same letters, while distinct digits substitute distinct letters.

2 [**2**] Let us call a triangle "almost right angle triangle" if one of its angles differs from $90°$ by no more than $15°$. Let us call a triangle "almost isosceles triangle" if two of its angles differs from each other by no more than $15°$. Is it true that that any acute triangle is either "almost right angle triangle" or "almost isosceles triangle"?

3 [**2**] A triangle with sides a, b, c is folded along a line ℓ so that a vertex C is on side c. Find the segments on which point C divides c, given that the angles adjacent to ℓ are equal.

4 [**3**] From the first 64 positive integers are chosen two subsets with 16 numbers in each. The first subset contains only odd numbers while the second one contains only even numbers. Total sums of both subsets are the same. Prove that among all the chosen numbers there are two whose sum equals 65.

5 [**4**] Two players in turns color the squares of a 4×4 grid, one square at the time. Player loses if after his move a square of 2×2 is colored completely. Which of the players has the winning strategy, First or Second?

International Mathematics

TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Fall 2007.**

1. (a) Each of Peter and Basil thinks of three positive integers. For each pair of his numbers, Peter writes down the greatest common divisor of the two numbers. For each pair of his numbers, Basil writes down the least common multiple of the two numbers. If both Peter and Basil write down the same three numbers, prove that these three numbers are equal to each other.
   (b) Can the analogous result be proved if each of Peter and Basil thinks of four positive integers instead?
2. Let K, L, M and N be the midpoints of the sides AB, BC, CD and DA of a cyclic quadrilateral $ABCD$. Let P be the point of intersection of AC and BD. Prove that the circumradii of triangles PKL, PLM, PMN and PNK are equal to one another.
3. Determine all finite increasing arithmetic progressions in which each term is the reciprocal of a positive integer and the sum of all the terms is 1.
4. Attached to each of a number of objects is a tag which states the correct mass of the object. The tags have fallen off and have been replaced on the objects at random. We wish to determine if by chance all tags are in fact correct. We may use exactly once a horizontal lever which is supported at its middle. The objects can be hung from the lever at any point on either side of the support. The lever either stays horizontal or tilts to one side. Is this task always possible?
5. The audience arranges n coins in a row. The sequence of heads and tails is chosen arbitrarily. The audience also chooses a number between 1 and n inclusive. Then the assistant turns one of the coins over, and the magician is brought in to examine the resulting sequence. By an agreement with the assistant beforehand, the magician tries to determine the number chosen by the audience.
   (a) Prove that if this is possible for some n_1 and n_2, then it is also possible for $n_1 n_2$.
   (b) Determine all n for which this is possible.
6. Let P and Q be two convex polygons. Let h be the length of the projection of Q onto a line perpendicular to a side of P which is of length p. Define $f(P,Q)$ to be the sum of the products hp over all sides of P. Prove that $f(P,Q) = f(Q,P)$.
7. There are 100 boxes, each containing either a red cube or a blue cube. Alex has a sum of money initially, and places bets on the colour of the cube in each box in turn. The bet can be anywhere from 0 up to everything he has at the time. After the bet has been placed, the box is opened. If Alex loses, his bet will be taken away. If he wins, he will get his bet back, plus a sum equal to the bet. Then he moves onto the next box, until he has bet on the last one, or until he runs out of money. What is the maximum factor by which he can guarantee to increase his amount of money, if he knows that the exact number of blue cubes is
   (a) 1;
   (b) some integer k, $1 < k \leq 100$.

Note: The problems are worth 2+2, 6, 6, 6, 4+4, 8 and 3+5 points respectively.

International Mathematics

TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2007**[2]

1 [3] Pictures are taken of 100 adults and 100 children, with one adult and one child in each, the adult being the taller of the two. Each picture is reduced to $\frac{1}{k}$ of its original size, where k is a positive integer which may vary from picture to picture. Prove that it is possible to have the reduced image of each adult taller than the reduced image of every child.

2 Initially, the number 1 and two positive numbers x and y are written on a blackboard. In each step, we can choose two numbers on the blackboard, not necessarily different, and write their sum or their difference on the blackboard. We can also choose a non-zero number of the blackboard and write its reciprocal on the blackboard. Is it possible to write on the blackboard, in a finite number of moves, the number

a) [2] x^2;

b) [2] xy?

3 [4] Give a construction by straight-edge and compass of a point C on a line ℓ parallel to a segment AB, such that the product $AC \cdot BC$ is minimum.

4 [4] The audience chooses two of twenty-nine cards, numbered from 1 to 29 respectively. The assistant of a magician chooses two of the remaining twenty-seven cards, and asks a member of the audience to take them to the magician, who is in another room. The two cards are presented to the magician in an arbitrary order. By an arrangement with the assistant beforehand, the magician is able to deduce which two cards the audience has chosen only from the two cards he receives. Explain how this may be done.

5 A square of side length 1 centimetre is cut into three convex polygons. Is it possible that the diameter of each of them does not exceed

a) [1] 1 centimetre;

b) [2] 1.01 centimetres;

c) [2] 1.001 centimetres?

[2]Courtesy of Professor Andy Liu.

International Mathematics

TOURNAMENT OF THE TOWNS

Senior P-Level Paper **Fall 2007.**

1 [**1**] A straight line is colored with two colors. Prove that there are three points A, B, C of the same color such that $AB = BC$.

2 [**2**] A student did not notice multiplication sign between two three-digit numbers and wrote it as a six-digit number. Result was 7 times more that it should be. Find these numbers.

3 [**3**] Two players in turns color the squares of a 4×4 grid, one square at the time. Player loses if after his move a square of 2×2 is colored completely. Which of the players has the winning strategy, First or Second?

4 [**3**] There three piles of pebbles, containing 5, 49, and 51 pebbles respectively. It is allowed to combine any two piles into a new one or to split any pile consisting of even number of pebbles into two equal piles. Is it possible to have 105 piles with one pebble in each in the end?

5 [**4**] Jim and Jane divide a triangular cake between themselves. Jim choses any point in the cake and Jane makes a straight cut through this point and choses the piece. Find the size of the piece that each of them can guarantee for himself/herself (both of them want to get as much as possible).

International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper[1] **Spring 2007.**

1. Let n be a positive integer. In order to find the integer closest to $\sqrt{n}$, Mary finds a^2, the closest perfect square to n. She thinks that a is then the number she is looking for. Is she always correct?

2. K, L, M and N are points on sides AB, BC, CD and DA, respectively, of the unit square $ABCD$ such that KM is parallel to BC and LN is parallel to AB. The perimeter of triangle KLB is equal to 1. What is the area of triangle MND?

3. Anna's number is obtained by writing down 20 consecutive positive integers, one after another in arbitrary order. Bob's number is obtained in the same way, but with 21 consecutive positive integers. Can they obtain the same number?

4. Several diagonals (possibly intersecting each other) are drawn in a convex n-gon in such a way that no three diagonals intersect in one point. If the n-gon is cut into triangles, what is the maximum possible number of these triangles?

5. Find all (finite) increasing arithmetic progressions, consisting only of prime numbers, such that the number of terms is larger than the common difference.

6. In the quadrilateral $ABCD$, $AB = BC = CD$ and $\angle BMC = 90°$, where M is the midpoint of AD. Determine the acute angle between the lines AC and BD.

7. Nancy shuffles a deck of 52 cards and spreads the cards out in a circle face up, leaving one spot empty. Andy, who is in another room and does not see the cards, names a card. If this card is adjacent to the empty spot, Nancy moves the card to the empty spot, without telling Andy; otherwise nothing happens. Then Andy names another card and so on, as many times as he likes, until he says "stop."

   (a) Can Andy guarantee that after he says "stop," no card is in its initial spot?

   (b) Can Andy guarantee that after he says "stop," the Queen of Spades is not adjacent to the empty spot?

Note: The problems are worth 3, 4, 5, 6, 7, 8 and 5+5 points respectively.

[1]Courtesy of Professor Andy Liu

International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper[1] **Spring 2007.**

1. The sides of a convex pentagon are extended on both sides to form five triangles. If these triangles are congruent to one another, does it follow that the pentagon is regular?

2. Two 2007-digit numbers are given. It is possible to delete 7 digits from each of them to obtain the same 2000-digit number. Prove that it is also possible to insert 7 digits into the given numbers so as to obtain the same 2014-digit number.

3. What is the least number of rooks that can be placed on a standard 8×8 chessboard so that all the white squares are attacked? (A rook also attacks the square it is on, in addition to every other square in the same row or column.)

4. Three nonzero real numbers are given. If they are written in any order as coefficients of a quadratic trinomial, then each of these trinomials has a real root. Does it follow that each of these trinomials has a positive root?

5. A triangular pie has the same shape as its box, except that they are mirror images of each other. We wish to cut the pie in two pieces which can fit together in the box without turning either piece over. How can this be done if

   (a) one angle of the triangle is three times as big as another;

   (b) one angle of the triangle is obtuse and is twice as big as one of the acute angles?

Note: The problems are worth 4, 4, 4, 4 and 1+4 points respectively.

[1]Courtesy of Professor Andy Liu

International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper[1] **Spring 2007.**

1. A, B, C and D are points on the parabola $y = x^2$ such that AB and CD intersect on the y-axis. Determine the x-coordinate of D in terms of the x-coordinates of A, B and C, which are a, b and c respectively.

2. A convex figure F is such that any equilateral triangle with side 1 has a parallel translation that takes all its vertices to the boundary of F. Is F necessarily a circle?

3. Let $f(x)$ be a polynomial of nonzero degree. Can it happen that for any real number a, an even number of real numbers satisfy the equation $f(x) = a$?

4. Nancy shuffles a deck of 52 cards and spreads the cards out in a circle face up, leaving one spot empty. Andy, who is in another room and does not see the cards, names a card. If this card is adjacent to the empty spot, Nancy moves the card to the empty spot, without telling Andy; otherwise nothing happens. Then Andy names another card and so on, as many times as he likes, until he says "stop."
   (a) Can Andy guarantee that after he says "stop," no card is in its initial spot?
   (b) Can Andy guarantee that after he says "stop," the Queen of Spades is not adjacent to the empty spot?

5. From a regular octahedron with edge 1, cut off a pyramid about each vertex. The base of each pyramid is a square with edge $\frac{1}{3}$. Can copies of the polyhedron so obtained, whose faces are either regular hexagons or squares, be used to tile space?

6. Let a_0 be an irrational number such that $0 < a_0 < \frac{1}{2}$. Define $a_n = \min\{2a_{n-1}, 1 - 2a_{n-1}\}$ for $n \geq 1$.
   (a) Prove that $a_n < \frac{3}{16}$ for some n.
   (b) Can it happen that $a_n > \frac{7}{40}$ for all n?

7. T is a point on the plane of triangle ABC such that $\angle ATB = \angle BTC = \angle CTA = 120°$. Prove that the lines symmetric to AT, BT and CT with respect to BC, CA and AB, respectively, are concurrent.

Note: The problems are worth 3, 5, 5, 4+4, 8, 4+4 and 8 points respectively.

[1]Courtesy of Professor Andy Liu

International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper[1] **Spring 2007.**

1. A 9×9 chessboard with the standard checkered pattern has white squares at its four corners. What is the least number of rooks that can be placed on this board so that all the white squares are attacked? (A rook also attacks the square it is on, in addition to every other square in the same row or column.)

2. The polynomial $x^3 + px^2 + qx + r$ has three roots in the interval (0,2). Prove that $-2 < p + q + r < 0$.

3. B is a point on the line which is tangent to a circle at the point A. The line segment AB is rotated about the centre of the circle through some angle to the line segment $A'B'$. Prove that the line AA' passes through the midpoint of BB'.

4. A binary sequence is constructed as follows. If the sum of the digits of the positive integer k is even, the k-th term of the sequence is 0. Otherwise, it is 1. Prove that this sequence is not periodic.

5. A triangular pie has the same shape as its box, except that they are mirror images of each other. We wish to cut the pie in two pieces which can fit together in the box without turning either piece over. How can this be done if

   (a) one angle of the triangle is obtuse and is twice as big as one of the acute angles;

   (b) the angles of the triangle are $20°$, $30°$ and $130°$?

Note: The problems are worth 3, 4, 4, 4 and 3+3 points respectively.

[1]Courtesy of Professor Andy Liu

International Mathematics

TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Fall 2008**

1 [**4**] 100 Queens are placed on a 100×100 chessboard so that no two attack each other. Prove that each of four 50×50 corners of the board contains at least one Queen.

2 [**6**] Each of 4 stones weights the integer number of grams. A balance with arrow indicates the difference of weights on the left and the right sides of it. Is it possible to determine the weights of all stones in 4 weighings, if the balance can make a mistake in 1 gram in at most one weighing?

3 [**6**] In his triangle ABC Serge made some measurements and informed Ilias about the lengths of median AD and side AC. Based on these data Ilias proved the assertion: angle CAB is obtuse, while angle DAB is acute. Determine a ratio AD/AC and prove Ilias' assertion (for any triangle with such a ratio).

4 [**6**] Baron Münchausen claims that he got a map of a country that consists of five cities. Each two cities are connected by a direct road. Each road intersects no more than one another road (and no more than once). On the map, the roads are colored in yellow or red, and while circling any city (along its border) one can notice that the colors of crossed roads alternate. Can Baron's claim be true?

5 [**8**] Let $a_1, \dots, a_n$ be a sequence of positive numbers, so that $a_1 + a_2 + \dots + a_n \leq 1/2$. Prove that
$(1 + a_1)(1 + a_2) \dots (1 + a_n) < 2$.

6 [**9**] Let ABC be a non-isosceles triangle. Two isosceles triangles $AB'C$ with base AC and $CA'B$ with base BC are constructed outside of triangle ABC. Both triangles have the same base angle φ. Let C_1 be a point of intersection of the perpendicular from C to $A'B'$ and the perpendicular bisector of the segment AB. Determine the value of $\angle AC_1B$.

7 In an infinite sequence $a_1, a_2, a_3, \dots$, the number a_1 equals 1, and each $a_n,\ n > 1$, is obtained from a_{n-1} as follows:

- if the greatest odd divisor of n has residue 1 modulo 4, then $a_n = a_{n-1} + 1$,
- and if this residue equals 3, then $a_n = a_{n-1} - 1$.

Prove that in this sequence

(a) [5] the number 1 occurs infinitely many times;

(b) [5] each positive integer occurs infinitely many times.
(The initial terms of this sequence are 1, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, ...)

International Mathematics

TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2008**

1 [**3**] Each of ten boxes contains a different number of pencils. No two pencils in the same box are of the same colour. Prove that one can choose one pencil from each box so that no two are of the same colour.

2 [**3**] Twenty-five of the numbers 1, 2, ..., 50 are chosen. Twenty-five of the numbers 51, 52, ..., 100 are also chosen. No two chosen numbers differ by 0 or 50. Find the sum of all 50 chosen numbers.

3 [**4**] Acute triangle $A_1A_2A_3$ is inscribed in a circle of radius 2. Prove that one can choose points B_1, B_2, B_3 on the arcs A_1A_2, A_2A_3, A_3A_1 respectively, such that the numerical value of the area of the hexagon $A_1B_1A_2B_2A_3B_3$ is equal to the numerical value of the perimeter of the triangle $A_1A_2A_3$.

4 [**4**] Given three distinct positive integers such that one of them is the average of the two others. Can the product of these three integers be the perfect 2008th power of a positive integer?

5 [**4**] On a straight track are several runners, each running at a different constant speed. They start at one end of the track at the same time. When a runner reaches any end of the track, he immediately turns around and runs back with the same speed (then he reaches the other end and turns back again, and so on). Some time after the start, all runners meet at the same point. Prove that this will happen again.

International Mathematics

TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Fall 2008**

1 [4] A square board is divided by lines parallel to the board sides (7 lines in each direction, not necessarily equidistant) into 64 rectangles. Rectangles are colored into white and black in alternating order. Assume that for any pair of white and black rectangles the ratio between area of white rectangle and area of black rectangle does not exceed 2. Determine the maximal ratio between area of white and black part of the board. White (black) part of the board is the total sum of area of all white (black) rectangles.

2 [6] Space is dissected into congruent cubes. Is it necessarily true that for each cube there exists another cube so that both cubes have a whole face in common?

3 [6] There are N piles each consisting of a single nut. Two players in turns play the following game. At each move, a player combines two piles that contain coprime numbers of nuts into a new pile. A player who can not make a move, loses. For every $N > 2$ define which of the players, the first or the second has a winning strategy.

4 [6] Let $ABCD$ be a non-isosceles trapezoid. Define a point A_1 as intersection of circumcircle of triangle BCD and line AC. (Choose A_1 distinct from C). Points B_1, C_1, D_1 are defined in similar way. Prove that $A_1B_1C_1D_1$ is a trapezoid as well.

5 [8] In an infinite sequence $a_1, a_2, a_3, \ldots,$ the number a_1 equals 1, and each $a_n,\ n > 1$, is obtained from a_{n-1} as follows:

- if the greatest odd divisor of n has residue 1 modulo 4, then $a_n = a_{n-1} + 1$,
- and if this residue equals 3, then $a_n = a_{n-1} - 1$.

Prove that in this sequence each positive integer occurs infinitely many times.
(The initial terms of this sequence are 1, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, ...)

6 [9] Let $P(x)$ be a polynomial with real coefficients so that equation $P(m) + P(n) = 0$ has infinitely many pairs of integer solutions (m, n). Prove that graph of $y = P(x)$ has a center of symmetry.

7 A test consists of 30 true or false questions. After the test (answering all 30 questions), Victor gets his score: the number of correct answers. Victor is allowed to take the test (the same questions) several times. Can Victor work out a strategy that insure him to get a perfect score after

(a) [5] 30th attempt?

(b) [5] 25th attempt?

(Initially, Victor does not know any answer)

International Mathematics

TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2008**

1 [3] Alex distributes some cookies into several boxes and records the number of cookies in each box. If the same number appears more than once, it is recorded only once. Serge takes one cookie from each box and puts them on the first plate. Then he takes one cookie from each box that is still non-empty and puts the cookies on the second plate. He continues until all the boxes are empty. Then Serge records the number of cookies on each plate. Again, if the same number appears more than once, it is recorded only once. Prove that Alex's record contains the same number of numbers as Serge's record.

2 [3] Solve the system of equations ($n > 2$)

$$\sqrt{x_1} + \sqrt{x_2 + \cdots + x_n} = \sqrt{x_2} + \sqrt{x_3 + \cdots + x_n + x_1} = \cdots = \sqrt{x_n} + \sqrt{x_1 + \cdots + x_{n-1}};$$
$$x_1 - x_2 = 1.$$

3 [4] A 30-gon $A_1A_2 \dots A_{30}$ is inscribed in a circle of radius 2. Prove that one can choose a point B_k on the arc A_kA_{k+1} for $1 \le k \le 29$ and a point B_{30} on the arc $A_{30}A_1$, such that the numerical value of the area of the 60-gon $A_1B_1A_2B_2 \dots A_{30}B_{30}$ is equal to the numerical value of the perimeter of the original 30-gon.

4 [4] Five distinct positive integers form an arithmetic progression. Can their product be equal to a^{2008} for some positive integer a?

5 [4] On the infinite chessboard several rectangular pieces are placed whose sides run along the grid lines. Each two have no squares in common, and each consists of an odd number of squares. Prove that these pieces can be painted in four colours such that two pieces painted in the same colour do not share any boundary points.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Spring 2008.**

1. An integer N is the product of two consecutive integers.

   (a) Prove that we can add two digits to the right of this number and obtain a perfect square.

   (b) Prove that this can be done in only one way if $N > 12$.

2. A line parallel to the side AC of triangle ABC cuts the side AB at K and the side BC at M. O is the point of intersection of AM and CK. If $AK = AO$ and $KM = MC$, prove that $AM = KB$.

3. Alice and Brian are playing a game on a $1 \times (N + 2)$ board. To start the game, Alice places a checker on any of the N interior squares. In each move, Brian chooses a positive integer n. Alice must move the checker to the n-th square on the left or the right of its current position. If the checker moves off the board, Alice wins. If it lands on either of the end squares, Brian wins. If it lands on another interior square, the game proceeds to the next move. For which values of N does Brian have a strategy which allows him to win the game in a finite number of moves?

4. Given are finitely many points in the plane, no three on a line. They are painted in four colours, with at least one point of each colour. Prove that there exist three triangles, distinct but not necessarily disjoint, such that the three vertices of each triangle have different colours, and none of them contains a coloured point in its interior.

5. Standing in a circle are 99 girls, each with a candy. In each move, each girl gives her candy to either neighbour. If a girl receives two candies in the same move, she eats one of them. What is the minimum number of moves after which only one candy remains?

6. Do there exist positive integers $a,\ b,\ c$ and d such that $\frac{a}{b} + \frac{c}{d} = 1$ and $\frac{a}{d} + \frac{c}{b} = 2008$?

7. A convex quadrilateral $ABCD$ has no parallel sides. The angles between the diagonal AC and the four sides are $55°,\ 55°, 19°$ and $16°$ in some order. Determine all possible values of the acute angle between AC and BD.

Note: The problems are worth 2+2, 5, 6, 6, 7, 7 and 8 points respectively.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Spring 2008.**

1. In the convex hexagon $ABCDEF$, AB, BC and CD are respectively parallel to DE, EF and FA. If $AB = DE$, prove that $BC = EF$ and $CD = FA$.

2. There are ten congruent segments on a plane. Each point of intersection divides every segment passing through it in the ratio 3:4. Find the maximum number of points of intersection.

3. There are ten cards with the number a on each, ten with b and ten with c, where a, b and c are distinct real numbers. For every five cards, it is possible to add another five cards so that the sum of the numbers on these ten cards is 0. Prove that one of a, b and c is 0.

4. Find all positive integers n such that $(n + 1)!$ is divisible by $1! + 2! + \cdots + n!$.

5. Each cell of a 10×10 board is painted red, blue or white, with exactly twenty of them red. No two adjacent cells are painted in the same colour. A domino consists of two adjacent cells, and it is said to be good if one cell is blue and the other is white.
   (a) Prove that it is always possible to cut out 30 good dominoes from such a board.
   (b) Give an example of such a board from which it is possible to cut out 40 good dominoes.
   (c) Give an example of such a board from which it is not possible to cut out more than 30 good dominoes.

Note: The problems are worth 4, 5, 5, 5 and 6 points respectively.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Spring 2008.**

1. A triangle has an angle of measure θ. It is dissected into several triangles. Is it possible that all angles of the resulting triangles are less than θ, if

   (a) $\theta = 70°$;

   (b) $\theta = 80°$?

2. Alice and Brian are playing a game on the real line. To start the game, Alice places a checker on a number x where $0 < x < 1$. In each move, Brian chooses a positive number d. Alice must move the checker to either $x + d$ or $x - d$. If it lands on 0 or 1, Brian wins. Otherwise the game proceeds to the next move. For which values of x does Brian have a strategy which allows him to win the game in a finite number of moves?

3. A polynomial $\quad x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-2} x^2 + a_{n-1} x + a_n$

   has n distinct real roots $x_1,\ x_2,\ \ldots,\ x_n$, where $n > 1$. The polynomial

   $$nx^{n-1} + (n-1)a_1 x^{n-2} + (n-2)a_2 x^{n-3} + \cdots + 2a_{n-2}x + a_{n-1}$$

   has roots $y_1,\ y_2,\ \ldots,\ y_{n-1}$. Prove that

   $$\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n} > \frac{y_1^2 + y_2^2 + \cdots + y_{n-1}^2}{n-1}.$$

4. Each of Peter and Basil draws a convex quadrilateral with no parallel sides. The angles between a diagonal and the four sides of Peter's quadrilateral are $\alpha,\ \alpha,\ \beta$ and γ in some order. The angles between a diagonal and the four sides of Basil's quadrilateral are also $\alpha,\ \alpha,\ \beta$ and γ in some order. Prove that the acute angle between the diagonals of Peter's quadrilateral is equal to the acute angle between the diagonals of Basil's quadrilateral.

5. The positive integers are arranged in a row in some order, each occuring exactly once. Does there always exist an adjacent block of at least two numbers somewhere in this row such that the sum of the numbers in the block is a prime number?

6. Seated in a circle are 11 wizards. A different positive integer not exceeding 1000 is pasted onto the forehead of each. A wizard can see the numbers of the other 10, but not his own. Simultaneously, each wizard puts up either his left hand or his right hand. Then each declares the number on his forehead at the same time. Is there a strategy on which the wizards can agree beforehand, which allows each of them to make the correct declaration?

7. Each of three lines cuts chords of equal lengths in two given circles. The points of intersection of these lines form a triangle. Prove that its circumcircle passes through the midpoint of the segment joining the centres of the circles.

Note: The problems are worth 3+3, 6, 6, 7, 8, 8 and 8 points respectively.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Spring 2008.**

1. There are ten cards with the number a on each, ten with b and ten with c, where a, b and c are distinct real numbers. For every five cards, it is possible to add another five cards so that the sum of the numbers on these ten cards is 0. Prove that one of a, b and c is 0.

2. Can it happen that the least common multiple of 1, 2, ..., n is 2008 times the least common multiple of 1, 2, ..., m for some positive integers m and n?

3. In triangle ABC, $\angle A = 90°$. M is the midpoint of BC and H is the foot of the altitude from A to BC. The line passing through M and perpendicular to AC meets the circumcircle of triangle AMC again at P. If BP intersects AH at K, prove that $AK = KH$.

4. No matter how two copies of a convex polygon are placed inside a square, they always have a common point. Prove that no matter how three copies of the same polygon are placed inside this square, they also have a common point.

5. We may permute the rows and the columns of the table below. How may different tables can we generate?

1	2	3	4	5	6	7
7	1	2	3	4	5	6
6	7	1	2	3	4	5
5	6	7	1	2	3	4
4	5	6	7	1	2	3
3	4	5	6	7	1	2
2	3	4	5	6	7	1

Note: The problems are worth 4, 5, 5, 5 and 6 points respectively.

International Mathematics

TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Fall 2009** [1]

1 [**4**] Each of 10 identical jars contains some milk, up to 10 percent of its capacity. At any time, we can tell the precise amount of milk in each jar. In a move, we may pour out an exact amount of milk from one jar into each of the other 9 jars, the same amount in each case. Prove that we can have the same amount of milk in each jar after at most 10 moves.

2 [**6**] Mike has 1000 unit cubes. Each has 2 opposite red faces, 2 opposite blue faces and 2 opposite white faces. Mike assembles them into a $10 \times 10 \times 10$ cube. Whenever two unit cubes meet face to face, these two faces have the same colour. Prove that an entire face of the $10 \times 10 \times 10$ cube has the same colour.

3 [**6**] Find all positive integers a and b such that $(a + b^2)(b + a^2) = 2^m$ for some integer m.

4 [**6**] Let $ABCD$ be a rhombus. P is a point on side BC and Q is a point on side CD such that $BP = CQ$. Prove that centroid of triangle APQ lies on the segment BD.

5 We have N objects with weights 1, 2, …, N grams. We wish to choose two or more of these objects so that the total weight of the chosen objects is equal to average weight of the remaining objects. Prove that

(a) [**2**] if $N + 1$ is a perfect square, then the task is possible;

(b) [**6**] if the task is possible, then $N + 1$ is a perfect square.

6 [**9**] On an infinite chessboard are placed 2009 $n \times n$ cardboard pieces such that each of them covers exactly n^2 cells of the chessboard. Prove that the number of cells of the chessboard which are covered by odd numbers of cardboard pieces is at least n^2.

7 [**12**] Anna and Ben decided to visit Archipelago with 2009 islands. Some pairs of islands are connected by boats which run both ways. Anna and Ben are playing during the trip:

Anna chooses the first island on which they arrive by plane. Then Ben chooses the next island which they could visit. Thereafter, the two take turns choosing an island which they have not yet visited. When they arrive at an island which is connected only to islands they had already visited, whoever's turn to choose next would be the loser. Prove that Anna could always win, regardless of the way Ben played and regardless of the way the islands were connected.

[1] Courtesy of Andy Liu

International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2009.**[1]

1. Is it possible to cut a square into nine squares and colour one of them white, three of them grey and five of them black, such that squares of the same colour have the same size and squares of different colours will have different sizes?

2. There are forty weights: 1, 2, ..., 40 grams. Ten weights with even masses were put on the left pan of a balance. Ten weights with odd masses were put on the right pan of the balance. The left and the right pans are balanced. Prove that one pan contains two weights whose masses differ by exactly 20 grams.

3. A cardboard circular disk of radius 5 centimetres is placed on the table. While it is possible, Peter puts cardboard squares with side 5 centimetres outside the disk so that:
(1) one vertex of each square lies on the boundary of the disk;
(2) the squares do not overlap;
(3) each square has a common vertex with the preceding one.
Find how many squares Peter can put on the table, and prove that the first and the last of them must also have a common vertex.

4. We only know that the password of a safe consists of 7 different digits. The safe will open if we enter 7 different digits, and one of them matches the corresponding digit of the password. Can we open this safe in less than 7 attempts?

5. A new website registered 2000 people. Each of them invited 1000 other registered people to be their friends. Two people are considered to be friends if and only if they have invited each other. What is the minimum number of pairs of friends on this website?

Note: The problems are worth 3, 4, 4, 5 and 5 points respectively.

[1]Courtesy of Andy Liu

International Mathematics

TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Fall 2009.**[1]

1 [**4**] One hundred pirates played cards. When the game was over, each pirate calculated the amount he won or lost. The pirates have a gold sand as a currency; each has enough to pay his debt.

Gold could only change hands in the following way. Either one pirate pays an equal amount to every other pirate, or one pirate receives the same amount from every other pirate.

Prove that after several such steps, it is possible for each winner to receive exactly what he has won and for each loser to pay exactly what he has lost.

2 [**6**] A non-square rectangle is cut into N rectangles of various shapes and sizes. Prove that one can always cut each of these rectangles into two rectangles so that one can construct a square and rectangle, each figure consisting of N pieces.

3 [**7**] Every edge of a tetrahedron is tangent to a given sphere. Prove that the three line segments joining the points of tangency of the three pairs of opposite edges of the tetrahedron are concurrent.

4 [**8**] Denote by $[n]!$ the product $1 \cdot 11 \cdot \ldots \cdot \underbrace{11\ldots1}_{n \text{ ones}}$ (n factors in total). Prove that $[n+m]!$ is divisible by $[n]! \times [m]!$.

5 [**8**] Let XYZ be a triangle. The convex hexagon $ABCDEF$ is such that AB, CD and EF are parallel and equal to XY, YZ and ZX, respectively. Prove that area of triangle with vertices at the midpoints of BC, DE and FA is no less than area of triangle XYZ.

6 [**10**] Anna and Ben decided to visit Archipelago with 2009 islands. Some pairs of islands are connected by boats which run both ways. Anna and Ben are playing during the trip:

Anna chooses the first island on which they arrive by plane. Then Ben chooses the next island which they could visit. Thereafter, the two take turns choosing an island which they have not yet visited. When they arrive at an island which is connected only to islands they had already visited, whoever's turn to choose next would be the loser. Prove that Anna could always win, regardless of the way Ben played and regardless of the way the islands were connected.

7 [**11**] At the entrance to a cave is a rotating round table. On top of the table are n identical barrels, evenly spaced along its circumference. Inside each barrel is a herring either with its head up or its head down. In a move, Ali Baba chooses from 1 to n of the barrels and turns them upside down. Then the table spins around. When it stops, it is impossible to tell which barrels have been turned over. The cave will open if the heads of the herrings in all n barrels are up or are all down. Determine all values of n for which Ali Baba can open the cave in a finite number of moves.

[1]Courtesy of Andy Liu

International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2009.**[1]

1. A 7-digit passcode is called good if all digits are different. A safe has a good passcode, and it opens if seven digits are entered and one of the digits matches the corresponding digit of the passcode. Is there a method of opening the safe box with an unknown passcode using less than 7 attempts?

2. A, B, C, D, E and F are points in space such that AB is parallel to DE, BC is parallel to EF, CD is parallel to FA, but $AB \neq DE$. Prove that all six points lie in the same plane.

3. Are there positive integers a, b, c and d such that $a^3 + b^3 + c^3 + d^3 = 100^{100}$?

4. A point is chosen on each side of a regular 2009-gon. Let S be the area of the 2009-gon with vertices at these points. For each of the chosen points, reflect it across the midpoint of its side. Prove that the 2009-gon with vertices at the images of these reflections also has area S.

5. A country has two capitals and several towns. Some of them are connected by roads. Some of the roads are toll roads where a fee is charged for driving along them. It is known that any route from the south capital to the north capital contains at least ten toll roads. Prove that all toll roads can be distributed among ten companies so that anybody driving from the south capital to the north capital must pay each of these companies.

Note: The problems are worth 4, 4, 4, 4 and 5 points respectively.

[1]Courtesy of Andy Liu.

International Mathematics

TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Spring 2009.**[1]

1 **[3]** There are two numbers on a board, 1/2009 and 1/2008. Alex and Ben play the following game. At each move, Alex names a number x (of his choice), while Ben responds by increasing one of the numbers on the board (of his choice) by x. Alex wins if at some moment one of the numbers on the board becomes 1. Can Alex win (no matter how Ben plays)?

2 **(a)** **[2]** Find a polygon which can be cut by a straight line into two congruent parts so that one side of the polygon is divided in half while another side at a ratio of $1 : 2$.

(b) **[3]** Does there exist a convex polygon with this property?

3 **[5]** In each square of a 101×101 board, except the central one, is placed either a sign "turn" or a sign "straight". The chess piece "car" can enter any square on the boundary of the board from outside (perpendicularly to the boundary). If the car enters a square with the sign "straight" then it moves to the next square in the same direction, otherwise (in case it enters a square with the sign "turn") it turns either to the right or to the left (its choice). Can one place the signs in such a way that the car never enter the central square?

4 **[5]** Consider an infinite sequence consisting of distinct positive integers such that each term (except the first one) is either an arithmetic mean or a geometric mean of two neighboring terms. Does it necessarily imply that starting at some point the sequence becomes either arithmetic progression or a geometric progression?

5 **[6]** A castle is surrounded by a circular wall with 9 towers which are guarded by knights during the night. Every hour the castle clock strikes and the guards shift to the neighboring towers; each guard always moves in the same direction (either clockwise or counterclockwise). Given that (i) during the night each knight guards every tower (ii) at some hour each tower was guarded by at least two knights (iii) at some hour exactly 5 towers were guarded by single knights, prove that at some hour one of the towers was unguarded.

6 **[7]** Angle C of an isosceles triangle ABC equals 120°. Each of two rays emitting from vertex C (inwards the triangle) meets AB at some point (P_i) reflects according to the rule "the angle of incidence equals the angle of reflection" and meets lateral side of triangle ABC at point Q_i ($i = 1, 2$). Given that angle between the rays equals 60°, prove that area of triangle P_1CP_2 equals the sum of areas of triangles AQ_1P_1 and BQ_2P_2 ($AP_1 < AP_2$).

7 **[9]** Let $\binom{n}{k}$ be the number of ways that k objects can be chosen (regardless of order) from a set of n objects. Prove that if positive integers k and l are greater than 1 and less than n, then integers $\binom{n}{k}$ and $\binom{n}{l}$ have a common divisor greater than 1.

[1]Courtesy of Andy Liu

International Mathematics

TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Spring 2009.**[1]

1 [**3**] In a convex 2009-gon, all diagonals are drawn. A line intersects the 2009-gon but does not pass through any of its vertices. Prove that the line intersects an even number of diagonals.

2 [**4**] Let a^b denote the number a^b. The order of operations in the expression 7^7^7^7^7^7^7 must be determined by parentheses (5 pairs of parentheses are needed). Is it possible to put parentheses in two distinct ways so that the value of the expression be the same?

3 [**4**] Alex is going to make a set of cubical blocks of the same size and to write a digit on each of their faces so that it would be possible to form every 30-digit integer with these blocks. What is the minimal number of blocks in a set with this property? (The digits 6 and 9 do not turn one into another.)

4 [**4**] We increased some positive integer by 10% and obtained a positive integer. Is it possible that in doing so we decreased the sum of digits exactly by 10%?

5 [**5**] In rhombus $ABCD$, angle A equals $120°$. Points M and N are chosen on sides BC and CD so that angle NAM equals $30°$. Prove that the circumcenter of triangle NAM lies on a diagonal of of the rhombus.

[1]Courtesy of Andy Liu

International Mathematics

TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Spring 2009.**[1]

1 [**4**] A rectangle is dissected into several smaller rectangles. Is it possible that for each pair of these rectangles, the line segment connecting their centers intersects some third rectangle?

2 [**4**] Consider an infinite sequence consisting of distinct positive integers such that each term (except the first one) is either an arithmetic mean or a geometric mean of two neighboring terms. Does it necessarily imply that starting at some point the sequence becomes either arithmetic progression or a geometric progression?

3 [**6**] Each square of a 10×10 board contains a chip. One may choose a diagonal containing an even number of chips and remove any chip from it. Find the maximal number of chips that can be removed from the board by these operations.

4 [**6**] Three planes dissect a parallelepiped into eight hexahedrons such that all of their faces are quadrilaterals (each plane intersects two corresponding pairs of opposite faces of the parallelepiped and does not intersect the remaining two faces). One of the hexahedrons has a circumscribed sphere. Prove that each of these hexahedrons has a circumscribed sphere.

5 [**8**] Let $\binom{n}{k}$ be the number of ways that k objects can be chosen (regardless of order) from a set of n objects. Prove that if positive integers k and l are greater than 1 and less than n, then integers $\binom{n}{k}$ and $\binom{n}{l}$ have a common divisor greater than 1.

6 [**9**] An integer $n > 1$ is given. Two players in turns mark points on a circle. First Player uses red color while Second Player uses blue color. The game is over when each player marks n points. Then each player finds the arc of maximal length with ends of his color, which does not contain any other marked points. A player wins if his arc is longer (if the lengths are equal, or both players have no such arcs, the game ends in a draw). Which player has a winning strategy?

7 [**9**] Initially a number 6 is written on a blackboard. At n-th step an integer k on the blackboard is replaced by $k+\gcd(k, n)$. Prove that at each step the number on the blackboard increases either by 1 or by a prime number.

[1] Courtesy of Andy Liu

International Mathematics

TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Spring 2009.**[1]

1 **[3]** Let a^b denote the number a^b. The order of operations in the expression 7^7^7^7^7^7^7 must be determined by parentheses (5 pairs of parentheses are needed). Is it possible to put parentheses in two distinct ways so that the value of the expression be the same?

2 **[4]** Several points on the plane are given; no three of them lie on the same line. Some of these points are connected by line segments. Assume that any line that does not pass through any of these points intersects an even number of these segments. Prove that from each point exits an even number of the segments.

3 For each positive integer n, denote by $O(n)$ its greatest odd divisor. Given any positive integers $x_1 = a$ and $x_2 = b$, construct an infinite sequence of positive integers as follows: $x_n = O(x_{n-1} + x_{n-2})$, where $n = 3, 4, \dots$.

(a) **[2]** Prove that starting from some place, all terms of the sequence are equal to the same integer.

(b) **[2]** Express this integer in terms of a and b.

4 **[4]** Several zeros and ones are written down in a row. Consider all pairs of digits (not necessarily adjacent) such that the left digit is 1 while the right digit is 0. Let M be the number of the pairs in which 1 and 0 are separated by an even number of digits (possibly zero), and let N be the number of the pairs in which 1 and 0 are separated by an odd number of digits. Prove that $M \geq N$.

5 **[4]** Suppose that X is an arbitrary point inside a tetrahedron. Through each vertex of the tetrahedron, draw a straight line that is parallel to the line segment connecting X with the intersection point of the medians of the opposite face. Prove that these four lines meet at the same point.

[1]Courtesy of Andy Liu

International Mathematics

TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Fall 2010.**[1]

1. A round coin may be used to construct a circle passing through one or two given points on the plane. Given a line on the plane, show how to use this coin to construct two points such that they define a line perpendicular to the given line. Note that the coin may not be used to construct a circle tangent to the given line.

2. Pete has an instrument which can locate the midpoint of a line segment, and also the point which divides the line segment into two segments whose lengths are in a ratio of $n:(n+1)$, where n is any positive integer. Pete claims that with this instrument, he can locate the point which divides a line segment into two segments whose lengths are at any given rational ratio. Is Pete right?

3. At a circular track, 10 cyclists started from some point at the same time in the same direction with different constant speeds. If any two cyclists are at some point at the same time again, we say that they meet. No three or more of them have met at the same time. Prove that by the time every two cyclists have met at least once, each cyclist has had at least 25 meetings.

4. A rectangle is divided into 2×1 and 1×2 dominoes. In each domino, a diagonal is drawn, and no two diagonals have common endpoints. Prove that exactly two corners of the rectangle are endpoints of these diagonals.

5. For each side of a given pentagon, divide its length by the total length of all other sides. Prove that the sum of all the fractions obtained is less than 2.

6. In acute triangle ABC, an arbitrary point P is chosen on altitude AH. Points E and F are the midpoints of sides CA and AB respectively. The perpendiculars from E to CP and from F to BP meet at point K. Prove that $KB = KC$.

7. Merlin summons the n knights of Camelot for a conference. Each day, he assigns them to the n seats at the Round Table. From the second day on, any two neighbours may interchange their seats if they were not neighbours on the first day. The knights try to sit in some cyclic order which has already occurred before on an earlier day. If they succeed, then the conference comes to an end when the day is over. What is the maximum number of days for which Merlin can guarantee that the conference will last?

Note: The problems are worth 4, 5, 8, 8, 8, 8 and 12 points respectively.

[1]Courtesy of Andy Liu

International Mathematics

TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2010**[1]

1. In a multiplication table, the entry in the i-th row and the j-th column is the product ij. From an $m \times n$ subtable with both m and n odd, the interior $(m-2) \times (n-2)$ rectangle is removed, leaving behind a frame of width 1. The squares of the frame are painted alternately black and white. Prove that the sum of the numbers in the black squares is equal to the sum of the numbers in the white squares.

2. In a quadrilateral $ABCD$ with an incircle, $AB = CD$, $BC < AD$ and BC is parallel to AD. Prove that the bisector of $\angle C$ bisects the area of $ABCD$.

3. A $1 \times 1 \times 1$ cube is placed on an 8×8 chessboard so that its bottom face coincides with a square of the chessboard. The cube rolls over a bottom edge so that the adjacent face now lands on the chessboard. In this way, the cube rolls around the chessboard, landing on each square at least once. Is it possible that a particular face of the cube never lands on the chessboard?

4. In a school, more than 90% of the students know both English and German, and more than 90% of the students know both English and French. Prove that more than 90% of the students who know both German and French also know English.

5. A circle is divided by $2N$ points into $2N$ arcs of length 1. These points are joined in pairs to form N chords. Each chord divides the circle into two arcs, the length of each being an even integer. Prove that N is even.

Note: The problems are worth 4, 4, 4, 4 and 4 points respectively.

[1]Courtesy of Andy Liu

International Mathematics

TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Fall 2010.**[1]

1. There are 100 points on the plane. All 4950 pairwise distances between two points have been recorded.
   (a) A single record has been erased. Is it always possible to restore it using the remaining records?
   (b) Suppose no three points are on a line, and k records were erased. What is the maximum value of k such that restoration of all the erased records is always possible?

2. At a circular track, $2n$ cyclists started from some point at the same time in the same direction with different constant speeds. If any two cyclists are at some point at the same time again, we say that they meet. No three or more of them have met at the same time. Prove that by the time every two cyclists have met at least once, each cyclist has had at least n^2 meetings.

3. For each side of a given polygon, divide its length by the total length of all other sides. Prove that the sum of all the fractions obtained is less than 2.

4. Two dueling wizards are at an altitude of 100 above the sea. They cast spells in turn, and each spell is of the form "decrease the altitude by a for me and by b for my rival" where a and b are real numbers such that $0 < a < b$. Different spells have different values for a and b. The set of spells is the same for both wizards, the spells may be cast in any order, and the same spell may be cast many times. A wizard wins if after some spell, he is still above water but his rival is not. Does there exist a set of spells such that the second wizard has a guaranteed win, if the number of spells is
   (a) finite;
   (b) infinite?

5. The quadrilateral $ABCD$ is inscribed in a circle with center O. The diagonals AC and BD do not pass through O. If the circumcentre of triangle AOC lies on the line BD, prove that the circumcentre of triangle BOD lies on the line AC.

6. Each cell of a 1000×1000 table contains 0 or 1. Prove that one can either cut out 990 rows so that at least one 1 remains in each column, or cut out 990 columns so that at least one 0 remains in each row.

7. A square is divided into congruent rectangles with sides of integer lengths. A rectangle is important if it has at least one point in common with a given diagonal of the square. Prove that this diagonal bisects the total area of the important rectangles.

Note: The problems are worth 2+3, 6, 6, 2+5, 8, 12 and 14 points respectively.

[1]Courtesy of Andy Liu

International Mathematics

TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2010.**[1]

1. The exchange rate in a Funny-Money machine is s McLoonies for a Loonie or $\frac{1}{s}$ Loonies for a McLoonie, where s is a positive real number. The number of coins returned is rounded off to the nearest integer. If it is exactly in between two integers, then it is rounded up to the greater integer.
   (a) Is it possible to achieve a one-time gain by changing some Loonies into McLoonies and changing all the McLoonies back to Loonies?
   (b) Assuming that the answer to (a) is "yes", is it possible to achieve multiple gains by repeating this procedure, changing all the coins in hand and back again each time?
2. The diagonals of a convex quadrilateral $ABCD$ are perpendicular to each other and intersect at the point O. The sum of the inradii of triangles AOB and COD is equal to the sum of the inradii of triangles BOC and DOA.
   (a) Prove that $ABCD$ has an incircle.
   (b) Prove that $ABCD$ is symmetric about one of its diagonals.
3. From a police station situated on a straight road infinite in both directions, a thief has stolen a police car. Its maximal speed equals 90% of the maximal speed of a police cruiser. When the theft is discovered some time later, a policeman starts to pursue the thief on a cruiser. However, he does not know in which direction along the road the thief has gone, nor does he know how long ago the car has been stolen. Is it possible for the policeman to catch the thief?
4. A square board is dissected into n^2 rectangular cells by $n-1$ horizontal and $n-1$ vertical lines. The cells are painted alternately black and white in a chessboard pattern. One diagonal consists of n black cells which are squares. Prove that the total area of all black cells is not less than the total area of all white cells.
5. In a tournament with 55 participants, one match is played at a time, with the loser dropping out. In each match, the numbers of wins so far of the two participants differ by not more than 1. What is the maximal number of matches for the winner of the tournament?

Note: The problems are worth 2+3, 2+3, 5, 5 and 5 points respectively.

[1]Courtesy of Andy Liu

International Mathematics

TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Spring 2010**

1 [**3**] Alex has a piece of cheese. He chooses a positive number $a \neq 1$ and cuts the piece into several pieces one by one. Every time he choses a piece and cuts it in the same ratio $1 : a$. His goal is to divide the cheese into two piles of equal masses. Can he do it?

2 [**4**] Let M be the midpoint of side AC of the triangle ABC. Let P be a point on the side BC. If O is the point of intersection of AP and BM and $BO = BP$, determine the ratio OM/PC .

3 Each of 999 numbers placed in a circular way is either 1 or -1. (Both values appear). Consider the total sum of the products of every 10 consecutive numbers.

(a) [**3**] Find the minimal possible value of this sum.

(b) [**3**] Find the maximal possible value of this sum.

4 [**6**] Can it happen that the sum of digits of some positive integer n equals 100 while the sum of digits of number n^3 equals 100^3?

5 N horsemen are riding in the same direction along a circular road. Their speeds are constant and pairwise distinct. There is a single point on the road where the horsemen can surpass one another. Can they ride in this fashion for arbitrarily long time ? Consider the cases:

(a) [**3**] $N = 3$;

(b) [**5**] $N = 10$.

6 [**8**] A broken line consists of 31 segments. It has no self intersections, and its start and end points are distinct. All segments are extended to become straight lines. Find the least possible number of straight lines.

7 [**11**] Several fleas sit on the squares of a 10×10 chessboard (at most one flea per square). Every minute, all fleas simultaneously jump to adjacent squares. Each flea begins jumping in one of four directions (up, down, left, right), and keeps jumping in this direction while it is possible; otherwise, it reverses direction on the opposite. It happened that during one hour no two fleas ever occupied the same square. Find the maximal possible number of fleas on the board.

International Mathematics

TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Spring 2010**

1 [**3**] Each of six fruit baskets contains pears, plums and apples. The number of plums in each basket equals the total number of apples in all other baskets combined while the number of apples in each basket equals the total number of pears in all other baskets combined. Prove that the total number of fruits is a multiple of 31.

2 [**3**] Karlson and Smidge divide a cake in a shape of a square in the following way. First, Karlson places a candle on the cake (chooses some interior point). Then Smidge makes a straight cut from the candle to the boundary in the direction of his choice. Then Karlson makes a straight cut from the candle to the boundary in the direction perpendicular to Smidge's cut. As a result, the cake is split into two pieces; Smidge gets the smaller one. Smidge wants to get a piece which is no less than a quarter of the cake. Can Karlson prevent Smidge from getting the piece of that size?

3 An angle is given in a plane. Using only a compass, one must find out

- **(a)** [**2**] if this angle is acute. Find the minimal number of circles one must draw to be sure.
- **(b)** [**2**] if this angle equals 31°. (One may draw as many circles as one needs.)

4 [**5**] At the math contest each participant met at least 3 pals who he/she already knew. Prove that the Jury can choose an even number of participants (more than two) and arrange them around a table so that each participant be set between these who he/she knows.

5 [**5**] 101 numbers are written on a blackboard: $1^2, 2^2, \dots, 101^2$. Alex choses any two numbers and replaces them by their positive difference. He repeats this operation until one number is left on the blackboard. Determine the smallest possible value of this number.

International Mathematics

TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Spring 2009**

1 **[3]** Is it possible to split all straight lines in a plane into the pairs of perpendicular lines, so that every line belongs to a single pair?

2 Alex has a piece of cheese. He chooses a positive number a and cuts the piece into several pieces one by one. Every time he choses a piece and cuts it in the same ratio $1 : a$. His goal is to divide the cheese into two piles of equal masses. Can he do it if

(a) **[2]** a is irrational?

(b) **[2]** a is rational, $a \neq 1$?

3 **[6]** Consider a composition of functions sin, cos, tan, cot, arcsin, arccos, arctan, arccot, applied to the number 1. Each function may be applied arbitrarily many times and in any order. (ex: sin cos arcsin cos sin ...1). Can one obtain the number 2010 in this way?

4 **[6]** 5000 movie fans gathered at a convention. Each participant had watched at least one movie. The participants should be split into discussion groups of two kinds. In each group of the first kind, the members would discuss a movie they all watched. In each group of the second kind, each member would tell about the movie that no one else in this group had watched. Prove that the chairman can always split the participants into exactly 100 groups. (A group consisting of one person is allowed; in this case this person submits a report).

5 **[7]** 33 horsemen are riding in the same direction along a circular road. Their speeds are constant and pairwise distinct. There is a single point on the road where the horsemen can surpass one another. Can they ride in this fashion for arbitrarily long time ?

6 **[8]** Quadrilateral $ABCD$ is circumscribed around the circle with centre I. Let points M and N be the midpoints of sides AB and CD respectively and let $IM/AB = IN/CD$. Prove that $ABCD$ is either a trapezoid or a parallelogram.

7 **[9]** Peter writes some positive integer on a blackboard. Susan can place pluses between some of its digits; then the children calculate the resulting sum (for example, starting from 123456789 one may obtain $12345 + 6 + 789 = 13140$). Susan is allowed to apply this procedure to the resulting number up to ten times. Prove that she can always end up with one-digit number.

International Mathematics

TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Spring 2010**

1 [**3**] 2010 ships deliver bananas, lemons and pineapples from South America to Russia. The total number of bananas on each ship equals the number of lemons on all other ships combined, while the total number of lemons on each ship equals the total number of pineapples on all other ships combined. Prove that the total number of fruits is a multiple of 31.

2 [**4**] Let $f(x)$ be a function such that every straight line has the same number of intersection points with the graph $y = f(x)$ and with the graph $y = x^2$. Prove that $f(x) = x^2$.

3 [**5**] Is it possible to cover the surface of a regular octahedron by several regular hexagons without gaps and overlaps? (A regular octahedron has 6 vertices, each face is an equilateral triangle, each vertex belongs to 4 faces.)

4 [**5**] Assume that $P(x)$ is a polynomial with integer nonnegative coefficients, different from constant. Baron Münchausen claims that he can restore $P(x)$ provided he knows the values of $P(2)$ and $P(P(2))$ only. Is the baron's claim valid?

5 [**6**] A needle (a segment) lies on a plane. One can rotate it 45° around any of its endpoints. Is it possible that after several rotations the needle returns to initial position with the endpoints interchanged?

INTERNATIONAL MATHEMATICS TOURNAMENT OF TOWNS

Junior A-Level Paper, Fall 2011.

Grades 8 – 10

(The result is computed from the three problems with the highest scores.)

points	problems
3	1. An integer $N > 1$ is written on the board. Alex writes a sequence of positive integers, obtaining new integers in the following manner: he takes any divisor greater than 1 of the last number and either adds it to, or subtracts it from the number itself. Is it always (for all $N > 1$) possible for Alex to write the number 2011 at some point?
4	2. On side AB of triangle ABC a point P is taken such that $AP = 2PB$. It is known that $CP = 2PQ$ where Q is the midpoint of AC. Prove that ABC is a right triangle.
5	3. A balance and a set of pairwise different weights are given. It is known that for any pair of weights from this set put on the left pan of the balance, one can counterbalance them by one or several of the remaining weights put on the right pan. Find the least possible number of weights in the set.
6	4. A checkered table consists of 2012 rows and $k > 2$ columns. A marker is placed in a cell of the left-most column. Two players move the marker in turns. During each move, the player moves the marker by 1 cell to the right, up or down to a cell that had never been occupied by the marker before. The game is over when any of the players moves the marker to the right-most column. However, whether this player is to win or to lose, the players are advised only when the marker reaches the second column from the right. Can any player secure his win?
6	5. Given that $0 < a, b, c, d < 1$ and $abcd = (1-a)(1-b)(1-c)(1-d)$, prove that $$(a+b+c+d) - (a+c)(b+d) \geq 1.$$
7	6. A car goes along a straight highway at the speed of 60 km per hour. A 100 meter long fence is standing parallel to the highway. Every second, the passenger of the car measures the angle of vision of the fence. Prove that the sum of all angles measured by him is less than 1100 degrees.
9	7. The vertices of a regular 45-gon are painted into three colors so that the number of vertices of each color is the same. Prove that three vertices of each color can be selected so that three triangles formed by the chosen vertices of the same color are all equal.

International Mathematics

TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2011.**

1. P and Q are points on the longest side AB of triangle ABC such that $AQ = AC$ and $BP = BC$. Prove that the circumcentre of triangle CPQ coincides with the incentre of triangle ABC.

2. Several guests at a round table are eating from a basket containing 2011 berries. Going in clockwise direction, each guest has eaten either twice as many berries as or six fewer berries than the next guest. Prove that not all the berries have been eaten.

3. From the 9×9 chessboard, all 16 unit squares whose row numbers and column numbers are both even have been removed. Disect the punctured board into rectangular pieces, with as few of them being unit squares as possible.

4. The vertices of a 33-gon are labelled with the integers from 1 to 33. Each edge is then labelled with the sum of the labels of its two vertices. Is it possible for the edge labels to consist of 33 consecutive numbers?

5. On a highway, a pedestrian and a cyclist were going in the same direction, while a cart and a car were coming from the opposite direction. All were travelling at different constant speeds. The cyclist caught up with the pedestrian at 10 o'clock. After a time interval, she met the cart, and after another time interval equal to the first, she met the car. After a third time interval, the car met the pedestrian, and after another time interval equal to the third, the car caught up with the cart. If the pedestrian met the car at 11 o'clock, when did he meet the cart?

Note: The problems are worth 3, 4, 4, 4 and 5 points respectively.

INTERNATIONAL MATHEMATICS TOURNAMENT OF TOWNS

Senior A-Level Paper, Fall 2011.

Grades 11 – 12

(The result is computed from the three problems with the highest scores, the scores for the individual parts of a single problem are summed.)

points	problems
4	1. Pete has marked several (three or more) points in the plane such that all distances between them are different. A pair of marked points $A; B$ will be called unusual if A is the furthest marked point from B, and B is the nearest marked point to A (apart from A itself). What is the largest possible number of unusual pairs that Pete can obtain?
4	2. Given that $0 < a, b, c, d < 1$ and $abcd = (1-a)(1-b)(1-c)(1-d)$, prove that $$(a+b+c+d) - (a+c)(b+d) \geq 1.$$
5	3. In triangle ABC, points A_1, B_1, C_1 are bases of altitudes from vertices A, B, C, and points C_A, C_B are the projections of C_1 to AC and BC respectively. Prove that line C_AC_B bisects the segments C_1A_1 and C_1B_1.
	4. Does there exist a convex N-gon such that all its sides are equal and all vertices belong to the parabola $y = x^2$ for
3	a) $N = 2011$;
4	b) $N = 2012$?
7	5. We will call a positive integer *good* if all its digits are nonzero. A good integer will be called *special* if it has at least k digits and their values strictly increase from left to right. Let a good integer be given. At each move, one may either add some special integer to its digital expression from the left or from the right, or insert a special integer between any two its digits, or remove a special number from its digital expression.What is the largest k such that any good integer can be turned into any other good integer by such moves?
7	6. Prove that the integer $1^1 + 3^3 + 5^5 + \ldots + (2^n - 1)^{2^n - 1}$ is a multiple of 2^n but not a multiple of 2^{n+1}.
9	7. 100 red points divide a blue circle into 100 arcs such that their lengths are all positive integers from 1 to 100 in an arbitrary order. Prove that there exist two perpendicular chords with red endpoints.

International Mathematics

TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2011.**

1. Several guests at a round table are eating from a basket containing 2011 berries. Going in clockwise direction, each guest has eaten either twice as many berries as or six fewer berries than the next guest. Prove that not all the berries have been eaten.

2. Peter buys a lottery ticket on which he enters an n-digit number, none of the digits being 0. On the draw date, the lottery administrators will reveal an $n \times n$ table, each cell containing one of the digits from 1 to 9. A ticket wins a prize if it does *not* match any row or column of this table, read in either direction. Peter wants to bribe the administrators to reveal the digits on some cells chosen by Peter, so that Peter can guarantee to have a winning ticket. What is the minimum number of digits Peter has to know?

3. In a convex quadrilateral $ABCD$, $AB = 10$, $BC = 14$, $CD = 11$ and $DA = 5$. Determine the angle between its diagonals.

4. Positive integers $a < b < c$ are such that $b + a$ is a multiple of $b - a$ and $c + b$ is a multiple of $c - b$. If a is a 2011-digit number and b is a 2012-digit number, exactly how many digits does c have?

5. In the plane are 10 lines in general position, which means that no 2 are parallel and no 3 are concurrent. Where 2 lines intersect, we measure the smaller of the two angles formed between them. What is the maximum value of the sum of the measures of these 45 angles?

Note: The problems are worth 3, 4, 4, 4 and 5 points respectively.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Spring 2011.**

1. Does there exist a hexagon that can be divided into four congruent triangles by a straight cut?

2. Passing through the origin of the coordinate plane are 180 lines, including the coordinate axes, which form 1° angles with one another at the origin. Determine the sum of the x-coordinates of the points of intersection of these lines with the line $y = -x + 100$.

3. Baron Munchausen has a set of 50 coins. The mass of each is a distinct positive integer not exceeding 100, and the total mass is even. The Baron claims that it is not possible to divide the coins into two piles with equal total mass. Can the Baron be right?

4. Given an integer $n > 1$, prove that there exist distinct positive integers a, b, c and d such that $a + b = c + d$ and $\frac{a}{b} = \frac{nc}{d}$.

5. AD and BE are altitudes of an acute triangle ABC. From D, perpendiculars are dropped to AB at G and AC at K. From E, perpendiculars are dropped to AB at F and BC at H. Prove that FG is parallel to HK and $FK = GH$.

6. Two ants crawl along the sides of the 49 squares of a 7×7 board. Each ant passes through all 64 vertices exactly once and returns to its starting point. What is the smallest possible number of sides covered by both ants?

7. In every cell of a square table is a number. The sum of the largest two numbers in each row is a and the sum of the largest two numbers in each column is b. Prove that $a = b$.

Note: The problems are worth 4, 4, 5, 6, 7, 10 and 10 points respectively.

International Mathematics

TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Spring 2011**

1 [**3**] The numbers from 1 to 2010 inclusive are placed along a circle so that if we move along the circle in clockwise order, they increase and decrease alternately. Prove that the difference between some two adjacent integers is even.

2 [**4**] A rectangle is divided by 10 horizontal and 10 vertical lines into 121 rectangular cells. If 111 of them have integer perimeters, prove that they all have integer perimeters.

3 [**5**] Worms grow at the rate of 1 metre per hour. When they reach their maximal length of 1 metre, they stop growing. A full-grown worm may be dissected into two not necessarily equal parts. Each new worm grows at the rate of 1 metre per hour. Starting with 1 full-grown worm, can one obtain 10 full-grown worms in less than 1 hour?

4 [**5**] Each diagonal of a convex quadrilateral divides it into two isosceles triangles. The two diagonals of the same quadrilateral divide it into four isosceles triangles. Must this quadrilateral be a square?

5 A dragon gave a captured knight 100 coins. Half of them are magical, but only dragon knows which are. Each day, the knight should divide the coins into two piles (not necessarily equal in size). The day when either magic coins or usual coins are spread equally between the piles, the dragon set the knight free. Can the knight guarantee himself a freedom in at most

(a) [**2**] 50 days?

(b) [**3**] 25 days?

International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Spring 2011.**

1. Baron Munchausen has a set of 50 coins. The mass of each is a distinct positive integer not exceeding 100, and the total mass is even. The Baron claims that it is not possible to divide the coins into two piles with equal total mass. Can the Baron be right?

2. In the coordinate space, each of the eight vertices of a rectangular box has integer coordinates. If the volume of the solid is 2011, prove that the sides of the rectangular box are parallel to the coordinate axes.

3. (a) Does there exist an infinite triangular beam such that two of its cross-sections are similar but not congruent triangles?

   (b) Does there exist an infinite triangular beam such that two of its cross-sections are equilateral triangles of sides 1 and 2 respectively?

4. There are n red sticks and n blue sticks. The sticks of each colour have the same total length, and can be used to construct an n-gon. We wish to repaint one stick of each colour in the other colour so that the sticks of each colour can still be used to construct an n-gon. Is this always possible if

   (a) $n = 3$;

   (b) $n > 3$?

5. In the convex quadrilateral $ABCD$, BC is parallel to AD. Two circular arcs ω_1 and ω_3 pass through A and B and are on the same side of AB. Two circular arcs ω_2 and ω_4 pass through C and D and are on the same side of CD. The measures of ω_1, ω_2, ω_3 and ω_4 are α, β, β and α respectively. If ω_1 and ω_2 are tangent to each other externally, prove that so are ω_3 and ω_4.

6. In every cell of a square table is a number. The sum of the largest two numbers in each row is a and the sum of the largest two numbers in each column is b. Prove that $a = b$.

7. Among a group of programmers, every two either know each other or do not know each other. Eleven of them are geniuses. Two companies hire them one at a time, alternately, and may not hire someone already hired by the other company. There are no conditions on which programmer a company may hire in the first round. Thereafter, a company may only hire a programmer who knows another programmer already hired by that company. Is it possible for the company which hires second to hire ten of the geniuses, no matter what the hiring strategy of the other company may be?

Note: The problems are worth 4, 6, 3+4, 4+4, 8, 8 and 11 points respectively.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Spring 2011.**

1. The faces of a convex polyhedron are similar triangles. Prove that this polyhedron has two pairs of congruent faces.

2. Worms grow at the rate of 1 metre per hour. When they reach their maximum length of 1 metre, they stop growing. A full-grown worm may be dissected into two new worms of arbitrary lengths totalling 1 metre. Starting with 1 full-grown worm, can one obtain 10 full-grown worms in less than 1 hour?

3. Along a circle are 100 white points. An integer k is given, where $2 \leq k \leq 50$. In each move, we choose a block of k adjacent points such that the first and the last are white, and we paint both of them black. For which values of k is it possible for us to paint all 100 points black after 50 moves?

4. Four perpendiculars are drawn from four vertices of a convex pentagon to the opposite sides. If these four lines pass through the same point, prove that the perpendicular from the fifth vertex to the opposite side also passes through this point.

5. In a country, there are 100 towns. Some pairs of towns are joined by roads. The roads do not intersect one another except meeting at towns. It is possible to go from any town to any other town by road. Prove that it is possible to pave some of the roads so that the number of paved roads at each town is odd.

Note: The problems are worth 3, 4, 4, 5 and 5 points respectively.

International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Spring 2012**[1]**.**

1. It is possible to place an even number of pears in a row such that the masses of any two neighbouring pears differ by at most 1 gram. Prove that it is then possible to put the pears two in a bag and place the bags in a row such that the masses of any two neighbouring bags differ by at most 1 gram.

2. One hundred points are marked in the plane, with no three in a line. Is it always possible to connect the points in pairs such that all fifty segments intersect one another?

3. In a team of guards, each is assigned a different positive integer. For any two guards, the ratio of the two numbers assigned to them is at least 3:1. A guard assigned the number n is on duty for n days in a row, off duty for n days in a row, back on duty for n days in a row, and so on. The guards need not start their duties on the same day. Is it possible that on any day, at least one in such a team of guards is on duty?

4. Each entry in an $n \times n$ table is either $+$ or $-$. At each step, one can choose a row or a column and reverse all signs in it. From the initial position, it is possible to obtain the table in which all signs are $+$. Prove that this can be accomplished in at most n steps.

5. Let p be a prime number. A set of $p+2$ positive integers, not necessarily distinct, is called *interesting* if the sum of any p of them is divisible by each of the other two. Determine all interesting sets.

6. A bank has one million clients, one of whom is Inspector Gadget. Each client has a unique PIN number consisting of six digits. Dr. Claw has a list of all the clients. He is able to break into the account of any client, choose any n digits of the PIN number and copy them. The n digits he copies from different clients need not be in the same n positions. He can break into the account of each client, but only once. What is the smallest value of n which allows Dr. Claw to determine the complete PIN number of Inspector Gadget?

7. Let AH be an altitude of an equilateral triangle ABC. Let I be the incentre of triangle ABH, and let L, K and J be the incentres of triangles ABI, BCI and CAI respectively. Determine $\angle KJL$.

Note: The problems are worth 4, 4, 6, 6, 8, 8 and 8 points respectively.

[1]Courtesy of Professor Andy Liu

International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Spring 2012**[1]**.**

1. A treasure is buried under a square of an 8×8 board, Under each other square is a message which indicates the minimum number of steps needed to reach the square with the treasure. Each step takes one from a square to another square sharing a common side. What is the minmum number of squares we must dig up in order to bring up the treasure for sure?

2. The number 4 has an odd number of odd positive divisors, namely 1, and an even number of even positive divisors, namely 2 and 4. Is there a number with an odd number of even positive divisors and an even number of odd positive divisors?

3. In the parallelogram $ABCD$, the diagonal AC touches the incircles of triangles ABC and ADC at W and Y respectively, and the diagonal BD touches the incircles of triangles BAD and BCD at X and Z respectively. Prove that either W, X, Y and Z coincide, or $WXYZ$ is a rectangle.

4. Brackets are to be inserted into the expression $10 \div 9 \div 8 \div 7 \div 6 \div 5 \div 4 \div 3 \div 2$ so that the resulting number is an integer.

   (a) Determine the maximum value of this integer.

   (b) Determine the minimum value of this integer.

5. RyNo, a little rhinoceros, has 17 scratch marks on its body. Some are horizontal and the rest are vertical. Some are on the left side and the rest are on the right side. If RyNo rubs one side of its body against a tree, two scratch marks, either both horizontal or both vertical, will disappear from that side. However, at the same time, two new scratch marks, one horizontal and one vertical, will appear on the other side. If there are less than two horizontal and less than two vertical scratch marks on the side being rubbed, then nothing happens. If RyNo continues to rub its body against trees, is it possible that at some point in time, the numbers of horizontal and vertical scratch marks have interchanged on each side of its body?

Note: The problems are worth 3, 4, 4, 2+3 and 5 points respectively.

[1]Courtesy of Professor Andy Liu.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Spring 2012[1].**

1. In a team of guards, each is assigned a different positive integer. For any two guards, the ratio of the two numbers assigned to them is at least 3:1. A guard assigned the number n is on duty for n days in a row, off duty for n days in a row, back on duty for n days in a row, and so on. The guards need not start their duties on the same day. Is it possible that on any day, at least one in such a team of guards is on duty?

2. One hundred points are marked inside a circle, with no three in a line. Prove that it is possible to connect the points in pairs such that all fifty lines intersect one another inside the circle.

3. Let n be a positive integer. Prove that there exist integers $a_1, a_2, \ldots, a_n$ such that for any integer x, the number $(\cdots(((x^2 + a_1)^2 + a_2)^2 + \cdots)^2 + a_{n-1})^2 + a_n$ is divisible by $2n - 1$.

4. Alex marked one point on each of the six interior faces of a hollow unit cube. Then he connected by strings any two marked points on adjacent faces. Prove that the total length of these strings is at least $6\sqrt{2}$.

5. Let ℓ be a tangent to the incircle of triangle ABC. Let ℓ_a, ℓ_b and ℓ_c be the respective images of ℓ under reflection across the exterior bisector of $\angle A$, $\angle B$ and $\angle C$. Prove that the triangle formed by these lines is congruent to ABC.

6. We attempt to cover the plane with an infinite sequence of rectangles, overlapping allowed.

   (a) Is the task always possible if the area of the nth rectangle is n^2 for each n?

   (b) Is the task always possible if each rectangle is a square, and for any number N, there exist squares with total area greater than N?

7. Konstantin has a pile of 100 pebbles. In each move, he chooses a pile and splits it into two smaller ones until he gets 100 piles each with a single pebble.

   (a) Prove that at some point, there are 30 piles containing a total of exactly 60 pebbles.

   (b) Prove that at some point, there are 20 piles containing a total of exactly 60 pebbles.

   (c) Prove that Konstantin may proceed in such a way that at no point, there are 19 piles containing a total of exactly 60 pebbles.

Note: The problems are worth 4, 5, 6, 6, 8, 3+6 and 6+3+3 points respectively.

[1]Courtesy of Professor Andy Liu.

International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Spring 2012[1].**

1. Each vertex of a convex polyhedron lies on exactly three edges, at least two of which have the same length. Prove that the polyhedron has three edges of the same length.

2. The cells of a $1 \times 2n$ board are labelled $1, 2, \ldots, n, \ -n, \ \ldots, \ -2, \ -1$ from left to right. A marker is placed on an arbitrary cell. If the label of the cell is positive, the marker moves to the right a number of cells equal to the value of the label. If the label is negative, the marker moves to the left a number of cells equal to the absolute value of the label. Prove that if the marker can always visit all cells of the board, then $2n + 1$ is prime.

3. Consider the points of intersection of the graphs $y = \cos x$ and $x = 100\cos(100y)$ for which both coordinates are positive. Let a be the sum of their x-coordinates and b be the sum of their y-coordinates. Determine the value of $\frac{a}{b}$.

4. A quadrilateral $ABCD$ with no parallel sides is inscribed in a circle. Two circles, one passing through A and B, and the other through C and D, are tangent to each other at X. Prove that the locus of X is a circle.

5. In an 8×8 chessboard, the rows are numbers from 1 to 8 and the columns are labelled from a to h. In a two-player game on this chessboard, the first player has a White Rook which starts on the square b2, and the second player has a Black Rook which starts on the square c4. The two players take turns moving their rooks. In each move, a rook lands on another square in the same row or the same column as its starting square. However, that square cannot be under attack by the other rook, and cannot have been landed on before by either rook. The player without a move loses the game. Which player has a winning strategy?

Note: The problems are worth 4, 4, 5, 5 and 5 points respectively.

[1]Courtesy of Professor Andy Liu.

International Mathematics

TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Fall 2013**

1 [**5**] There are 100 red, 100 yellow and 100 green sticks. One can construct a triangle using any three sticks all of different colours (one red, one yellow and one green) . Prove that there is a colour such that one can construct a triangle using any three sticks of this colour.

2 [**5**] A math teacher chose 10 consequtive numbers and submitted them to Pete and Basil. Each boy should split these numbers in pairs and calculate the sum of products of numbers in pairs. Prove that the boys can pair the numbers differently so that the resulting sums are equal.

3 [**6**] Assume that C is a right angle of triangle ABC and N is a midpoint of the semicircle, constructed on CB as on diameter externally. Prove that AN divides the bisector of angle C in half.

4 [**7**] There is a 8×8 table, drawn in a plane and painted in a chess board fashion. Peter mentally chooses a square and an interior point in it. Basil can draws any polygon (without self-intersections) in the plane and ask Peter whether the chosen point is inside or outside this polygon. What is the minimal number of questions sufficient to determine whether the chosen point is black or white?

5 [**9**] A 101-gon is inscribed in a circle. From each vertex of this polygon a perpendicular is dropped to the opposite side or its extension. Prove that at least one perpendicular drops to the side.

6 [**10**] The number

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{2n-1} - \frac{1}{2n}$$

is represented as an irreducible fraction. If $3n+1$ is a prime number, prove that the numerator of this fraction is a multiple of $3n+1$.

7 [**12**] On a table, there are 11 piles of ten stones each. Pete and Basil play the following game. In turns they take 1, 2 or 3 stones at a time: Pete takes stones from any single pile while Basil takes stones from different piles but no more than one from each. Pete moves first. The player who cannot move, loses.

Which of the players, Pete or Basil, has a winning strategy?

International Mathematics

TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2013**

1 [**3**] In a wrestling tournament, there are 100 participants, all of different strengths. The stronger wrestler always wins over the weaker opponent. Each wrestler fights twice and those who win both of their fights are given awards. What is the least possible number of awardees?

2 [**4**] Does there exist a ten-digit number such that all its digits are different and after removing any six digits we get a composite four-digit number?

3 [**4**] Denote by (a, b) the greatest common divisor of a and b. Let n be a positive integer such that

$$(n, n+1) < (n, n+2) < \cdots < (n, n+35).$$

Prove that $(n, n+35) < (n, n+36)$.

4 [**5**] Let ABC be an isosceles triangle. Suppose that points K and L are chosen on lateral sides AB and AC respectively so that $AK = CL$ and $\angle ALK + \angle LKB = 60°$. Prove that $KL = BC$.

5 [**6**] Eight rooks are placed on a chessboard so that no two rooks attack each other. Prove that one can always move all rooks, each by a move of a knight so that in the final position no two rooks attack each other as well. (In intermediate positions several rooks can share the same square).

International Mathematics

TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Fall 2013**

1 **[5]** There is a 8×8 table, drawn in a plane and painted in a chess board fashion. Peter mentally chooses a square and an interior point in it. Basil can draws any polygon (without self-intersections) in the plane and ask Peter whether the chosen point is inside or outside this polygon. What is the minimal number of questions sufficient to determine whether the chosen point is black or white?

2 **[6]** Find all positive integers n for which the following statement holds:

For any two polynomials $P(x)$ and $Q(x)$ of degree n there exist monomials ax^k and bx^ℓ, $0 \le k, \ell \le n$, such that the graphs of $P(x) + ax^k$ and $Q(x) + bx^\ell$ have no common points.

3 **[6]** Let ABC be an equilateral triangle with centre O. A line through C meets the circumcircle of triangle AOB at points D and E. Prove that points A, O and the midpoints of segments BD, BE are concyclic.

4 **[7]** Is it true that every integer is a sum of finite number of cubes of distinct integers?

5 Do there exist two integer-valued functions f and g such that for every integer x we have

(a) **[3]** $f(f(x)) = x, \quad g(g(x)) = x, \quad f(g(x)) > x, \quad g(f(x)) > x$?

(b) **[5]** $f(f(x)) < x, \quad g(g(x)) < x, \quad f(g(x)) > x, \quad g(f(x)) > x$?

6 **[9]** On a table, there are 11 piles of ten stones each. Pete and Basil play the following game. In turns they take 1, 2 or 3 stones at a time: Pete takes stones from any single pile while Basil takes stones from different piles but no more than one from each. Pete moves first. The player who cannot move, loses.

Which of the players, Pete or Basil, has a winning strategy?

7 **[14]** A closed broken self-intersecting line is drawn in the plane. Each of the links of this line is intersected exactly once and no three links intersect at the same point. Further, there are no self-intersections at the vertices and no two links have a common segment. Can it happen that every point of self-intersection divides both links in halves?

International Mathematics

TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2013**

1 [3] Does there exist a ten-digit number such that all its digits are different and after removing any six digits we get a composite four-digit number?

2 [4] On the sides of triangle ABC, three similar triangles are constructed with triangle YBA and triangle ZAC in the exterior and triangle XBC in the interior. (Above, the vertices of the triangles are ordered so that the similarities take vertices to corresponding vertices, for example, the similarity between triangle YBA and triangle ZAC takes Y to Z, B to A and A to C). Prove that $AYXZ$ is a parallelogram.

3 [4] Denote by $[a, b]$ the least common multiple of a and b. Let n be a positive integer such that

$$[n, n + 1] > [n, n + 2] > \cdots > [n, n + 35].$$

Prove that $[n, n + 35] > [n, n + 36]$.

4 [5] Eight rooks are placed on a chessboard so that no two rooks attack each other. Prove that one can always move all rooks, each by a move of a knight so that in the final position no two rooks attack each other as well. (In intermediate positions several rooks can share the same square).

5 [6] A spacecraft landed on an asteroid. It is known that the asteroid is either a ball or a cube. The rover started its route at the landing site and finished it at the point symmetric to the landing site with respect to the center of the asteroid. On its way, the rover transmitted its spatial coordinates to the spacecraft on the landing site so that the trajectory of the rover movement was known. Can it happen that this information is not sufficient to determine whether the asteroid is a ball or a cube?

International Mathematics

TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Fall 2014**

1 **[4]** Half of entries of a square table are filled with pluses, and the remaining half are filled with minuses. Prove that either two rows, or two columns contain the same number of pluses.

2 **[5]** Prove that any circumscribed polygon has three sides that can form a triangle.

3 **[6]** Is it possible to divide all positive divisors of 100! (including 100! and 1) into two groups so that each group contains the same number of integers and the product of numbers in the first group is equal to the product of numbers in the second group?

4 **[7]** On a circular road there are 25 police posts equally distant. Every policeman (one at each post) has a badge with a unique number, from 1 to 25. The policemen are ordered to switch their posts so that the numbers on the badges would be in the consecutive order, from 1 to 25 clockwise. If the total sum of distances walked by the policemen along the road is minimal possible, prove that one of them remains at his initial position.

5 **[8]** In a right-angled triangle, two equal circles are constructed so that they touch one another and each one touches hypotenuse and one leg. Consider a segment connecting the points of tangency of the circles and the hypotenuse. Prove that the midpoint of this segment belongs to bisector of right angle of the triangle.

6 **[8]** Let us call a positive integer *plain* if it consists of the same digits (examples: 4, 111, 999999). Prove that any n-digit integer can be represented as a sum of at most $n+1$ plain integers.

7 A spiderweb is a square with 100×100 nodes. 100 flies caught into the web stacked at 100 different nodes. A spider which was originally at the corner of the web crawls from a node to an adjacent node counting moves and eating flies on its way. Can the spider eat all flies in no more than

- **(a)** **[5]** 2100 moves;
- **(b)** **[5]** 2000 moves?

International Mathematics

TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2014**

1 **[3]** There are 99 sticks of lengths 1, 2, 3, ..., 99. Is it possible to arrange them in a shape of a rectangle?

2 Do there exist ten distinct positive integers such that their arithmetic mean is equal to

(a) **[2]** their greatest common divisor multiplied by 6?

(b) **[2]** their greatest common divisor multiplied by 5?

3 **[5]** Points K and L are marked on sides AB and BC of square $ABCD$ respectively so that $KB = LC$. Let P be a point of intersection of segments AL and CK. Prove that segments DP and KL are perpendicular. 1.07) L;

4 **[5]** During his last school year, Andrew recorded his marks in maths. He called his upcoming mark (2, 3, 4, or 5) *unexpected* if until this moment it appeared less often than any other possible mark. (For instance, if he had marks 3,4,2,5,5,5,2,3,4,3 on his list then unexpected marks would be the first 5 and the second 4). It happened that at the end of the year Andrew had on his record list forty marks and each possible mark was repeated exactly 10 times (the order of marks is unknown). Is it possible to determine the number of unexpected marks?

5 There are N right-angled triangles. In every given triangle Adam chose a leg and calculated the sum of the lengths of the selected legs. Then he found the total sum of the lengths of the remaining legs. Finally, he found the total sum of the hypotenuses. Given that these three numbers create a right-angled triangle, prove that in every given triangle the ratio of the greater leg to the smaller leg is the same. Consider the cases:

(a) **[2]** $N = 2$;

(b) **[3]** $N \geq 2$ is any positive integer.

International Mathematics

TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Fall 2014**

1 [**4**] Prove that any circumscribed polygon has three sides that can form a triangle.

2 [**6**] On a circular road there are 25 police posts equally distant. Every policeman (one at each post) has a badge with a unique number, from 1 to 25. The policemen are ordered to switch their posts so that the numbers on the badges would be in the consecutive order, from 1 to 25 clockwise. If the total sum of distances walked by the policemen along the road is minimal possible, prove that one of them remains at his initial position.

3 [**6**] Gregory wrote 100 numbers on a blackboard and calculated their product. Then he increased each number by 1 and observed that the product didn't change. He increased the numbers in the same way again, and again the product didn't change. He performed this procedure k times, each time having the same product. Find the greatest possible value of k.

4 [**7**] The circle inscribed in triangle ABC touches the sides BC, CA, AB at points A', B', C' respectively. Three lines, AA', BB' and CC' meet at point G. Define the points C_A and C_B as points of intersection of the circle circumscribed about triangle $GA'B'$ with lines AC and BC, different from B' and A'. In similar way define the points A_B, A_C, B_C, B_A. Prove that the points C_A, C_B, A_B, A_C, B_C, and B_A belong to the same circle.

5 [**7**] Pete counted all possible words consisting of m letters, such that each letter can be only one of T, O, W or N and each word contains as many T as O. Basil counted all possible words consisting of $2m$ letters such that each letter is either T or O and each word contains as many T as O. Which of the boys obtained the greater number of words?

6 [**8**] There is a wire triangle with angles $x°$, $y°$, $z°$. Mischievous Nick bent every side of the triangle at some point by 1 degree. In the result he got a non convex hexagon with angles $(x-1)°$, $181°$, $(y-1)°$, $181°$, $(z-1)°$, $181°$. Prove that the points that became the new vertices split the sides of the initial triangle in the same ratio.

7 [**10**] In one kingdom gold and platinum sands are used as currency. Exchange rate is defined by two positive integers g and p; namely, x grams of gold sand are equivalent to y grams of platinum sand if $x : y = p : g$ (x and y are not necessarily integers). At the day when the numbers were $g = p = 1001$, the Treasury announced that every following day one of the numbers, either g or p would be decreased by 1 so that after 2000 days both numbers would become equal to 1. However, the exact order in which the numbers would be decreasing was not announced. At that moment a banker had 1 kg of gold sand and 1 kg of platinum sand. The banker's goal is to perform exchanges so that by the end he would have at least 2 kg of gold sand and 2 kg of platinum sand. Can the banker fulfil his goal for certain?

International Mathematics

TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2014**

1 Do there exist ten distinct positive integers such that their arithmetic mean is equal to

(a) [**1**] their greatest common divisor multiplied by 6?

(b) [**2**] their greatest common divisor multiplied by 5?

2 [**4**] The vertices of a triangle are marked by A, B, C clockwise. The triangle is rotated clockwise in a sequence: about A by $\angle A$, about B by $\angle B$, about C by $\angle C$, and so on (each time the rotation is performed about the current position of the vertex in the sequence). Prove that after six rotations the triangle returns to its initial position.

3 [**5**] From a set of 15 distinct integers Pete selects 7 numbers in all possible ways and for every selection he writes down the sum of the selected numbers. Basil, in his turn, selects 8 numbers in all possible ways and each time writes down the sum of his selected numbers. Can it happen that Pete and Basil will obtain the same set of numbers? (Each integer must be repeated in Pete's set as many times as it is repeated in Basil's set.)

4 [**5**] There are N right-angled triangles. In every given triangle Adam chose a leg and calculated the sum of the lengths of the selected legs. Then he found the total sum of the lengths of the remaining legs. Finally, he found the total sum of the hypotenuses. Given that these three numbers create a right-angled triangle, prove that all initial triangles are similar.

5 [**5**] Originally there was a pile of silver coins on a table. One can either add a gold coin and record the number of silver coins on the first list or remove a silver coin and record the number of gold coins on the second list. It happened that after several such operations only gold coins remained on the table. Prove that at that moment the sums of the numbers on the two lists were equal. We see that both sums are the same.

International Mathematics

35th TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Spring 2014**

1 [3] During Christmas party Santa handed out to the children 47 chocolates and 74 marmalades. Each girl got 1 more chocolate than each boy but each boy got 1 more marmalade than each girl. What was the number of the children?

2 [5] Peter marks several cells on a 5×5 board. Basil wins if he can cover all marked cells with three-cell corners. The corners must be inside the board and not overlap. What is the least number of cells Peter should mark to prevent Basil from winning? (Cells of the corners must coincide with the cells of the board).

3 [6] A square table is covered with a square cloth (may be of a different size) without folds and wrinkles. All corners of the table are left uncovered and all four hanging parts are triangular. Given that two adjacent hanging parts are equal prove that two other parts are also equal.

4 [7] The King called two wizards. He ordered First Wizard to write down 100 positive integers (not necessarily distinct) on cards without revealing them to Second Wizard. Second Wizard must correctly determine all these integers, otherwise both wizards will lose their heads. First Wizard is allowed to provide Second Wizard with a list of distinct integers, each of which is either one of the integers on the cards or a sum of some of these integers. He is not allowed to tell which integers are on the cards and which integers are their sums. If Second Wizard correctly determines all 100 integers the King tears as many hairs from each wizard's beard as the number of integers in the list given to Second Wizard. What is the minimal number of hairs each wizard should sacrifice to stay alive?

5 [7] There are several white and black points. Every white point is connected with every black point by a segment. Each segment is equipped with a positive integer. For any closed circuit the product of the integers on the segments passed in the direction from white to black point is equal to the product of the integers on the segments passed in the opposite direction. Can one always place the integer at each point so that the integer on each segment is the product of the integers at its ends?

6 [9] A $3\times3\times3$ cube is made of $1\times1\times1$ cubes glued together. What is the maximal number of small cubes one can remove so the remaining solid has the following features:

1) Projection of this solid on each face of the original cube is a 3×3 square;

2) The resulting solid remains face-connected (from each small cube one can reach any other small cube along a chain of consecutive cubes with common faces).

7 [9] Points A_1, A_2, …, A_{10} are marked on a circle clockwise. It is known that these points can be divided into pairs of points symmetric with respect to the centre of the circle. Initially at each marked point there was a grasshopper. Every minute one of the grasshoppers jumps over its neighbour along the circle so that the resulting distance between them doesn't change. It is not allowed to jump over any other grasshopper and to land at a point already occupied. It occurred that at some moment nine grasshoppers were found at points A_1, A_2, …, A_9 and the tenth grasshopper was on arc $A_9A_{10}A_1$. Is it necessarily true that this grasshopper was exactly at point A_{10}?

International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Spring 2014**

1 [3] Each of given 100 numbers was increased by 1. Then each number was increased by 1 once more. Given that the first time the sum of the squares of the numbers was not changed find how this sum was changed the second time.

2 [4] Mother baked 15 pasties. She placed them on a round plate in a circular way: 7 with cabbage, 7 with meat and one with cherries in that exact order and put the plate into a microwave. All pasties look the same but Olga knows the order. However she doesn't know how the plate has been rotated in the microwave. She wants to eat a pasty with cherries. Can Olga eat her favourite pasty for sure if she is not allowed to try more than three other pasties?

3 [4] The entries of a 7×5 table are filled with numbers so that in each 2×3 rectangle (vertical or horizontal) the sum of numbers is 0. For 100 dollars Peter may choose any single entry and learn the number in it. What is the least amount of dollars he should spend in order to learn the total sum of numbers in the table for sure?

4 [5] Point L is marked on side BC of triangle ABC so that AL is twice as long as the median CM. Given that angle ALC is equal to 45° prove that AL is perpendicular to CM.

5 [6] Ali Baba and the 40 thieves want to cross Bosporus strait. They made a line so that any two people standing next to each other are friends. Ali Baba is the first; he is also a friend with the thief next to his neighbour. There is a single boat that can carry 2 or 3 people and these people must be friends. Can Ali Baba and the 40 thieves always cross the strait if a single person cannot sail?

35th International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Spring 2014**

1 [**3**] Doono wrote several **1**s, placed signs "+" or "×" between every two of them, put several brackets and got 2014 in the result. His friend Dunno replaced all "+" by "×" and all "×" by "+" and also got 2014. Can this be true?

2 Is it true that any convex polygon can be dissected by a straight line into two polygons with equal perimeters and

(**a**) [**4**] equal greatest sides?

(**b**) [**4**] equal smallest sides?

3 [**6**] The King called two wizards. He ordered First Wizard to write down 100 positive real numbers (not necessarily distinct) on cards without revealing them to Second Wizard. Second Wizard must correctly determine all these numbers, otherwise both wizards will lose their heads. First Wizard is allowed to provide Second Wizard with a list of distinct numbers, each of which is either one of the numbers on the cards or a sum of some of these numbers. He is not allowed to tell which numbers are on the cards and which numbers are their sums. If Second Wizard correctly determines all 100 numbers the King tears as many hairs from each wizard's beard as the number of numbers in the list given to Second Wizard. What is the minimal number of hairs each wizard should sacrifice to stay alive?

4 [**7**] In the plane are marked all points with integer coordinates (x, y), $0 \le y \le 10$. Consider a polynomial of degree 20 with integer coefficients. Find the maximal possible number of marked points which can lie on its graph.

5 [**8**] There is a scalene triangle. Peter and Basil play the following game. On each his turn Peter chooses a point in the plane. Basil responds by painting it into red or blue. Peter wins if some triangle similar to the original one has all vertices of the same colour. Find the minimal number of moves Peter needs to win no matter how Basil would play (independently of the shape of the given triangle)?

6 [**9**] In some country every town has a unique number. In a flight directory for any two towns there is an indication whether or not they are connected by a direct non-stop flight. It is known that for any two assigned numbers M and N one can change the numeration of towns so that the town with number M gets the number N but the directory remains correct. Is it always true that for any two assigned numbers M and N one can change the numeration of towns so that the towns with numbers M and N interchange their numbers but the directory is still correct?

7 [**10**] Consider a polynomial $P(x)$ such that

$$P(0) = 1; \qquad (P(x))^2 = 1 + x + x^{100}Q(x), \text{ where } Q(x) \text{ is also a polynomial.}$$

Prove that in the polynomial $(P(x) + 1)^{100}$ the coefficient at x^{99} is zero.

International Mathematics

TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Spring 2014**

1 [**4**] Inspector Gadget has 36 stones with masses 1 gram, 2 grams, ..., 36 grams. Doctor Claw has a superglue such that one drop of it glues two stones together (thus two drops glue 3 stones together and so on). Doctor Claw wants to glue some stones so that in obtained set Inspector Gadget cannot choose one or more stones with the total mass 37 grams. Find the least number of drops needed for Doctor Claw to fulfil his task.

2 [**4**] In a convex quadrilateral $ABCD$ the diagonals are perpendicular. Points M and N are marked on sides AD and CD respectively. Prove that lines AC and MN are parallel given that angles ABN and CBM are right angles.

3 [**5**] Ali Baba and the 40 thieves want to cross Bosporus strait. They made a line so that any two people standing next to each other are friends. Ali Baba is the first; he is also a friend with the thief next to his neighbour. There is a single boat that can carry 2 or 3 people and these people must be friends. Can Ali Baba and the 40 thieves always cross the strait if a single person cannot sail?

4 [**5**] Positive integers a, b, c, d are pairwise coprime and satisfy the equation

$$ab + cd = ac - 10bd.$$

Prove that one can always choose three numbers among them such that one number equals the sum of two others.

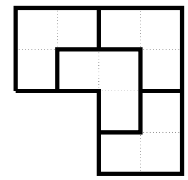
5 [**5**] Park's paths go along sides and diagonals of the convex quadrilateral $ABCD$. Alex starts at A and hikes along $AB-BC-CD$. Ben hikes along AC; he leaves A simultaneously with Alex and arrives to C simultaneously with Alex. Chris hikes along BD; he leaves B at the same time as Alex passes B and arrives to D simultaneously with Alex. Can it happen that Ben and Chris arrive at point O of intersection of AC and BD at the same time? The speeds of the hikers are constant.

37th International Mathematics

TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Fall 2015**

1 A polygon on a grid is called *amazing* if it is not a rectangle and several its copies form a polygon similar to it. For instance, a corner consisting of three cells is an amazing polygon (see the figure at the right).



(a) [**2**] Find an amazing polygon consisting of 4 cells.

(b) [**3**] For which $n > 4$ there exists an amazing polygon consisting of n cells?

2 From a set of integers $\{1, \ldots, 100\}$, k integers were deleted. Is it always possible to choose k distinct integers from the remaining set such that their sum is 100 if

(a) [**2**] $k = 9$?

(b) [**4**] $k = 8$?

3 Let P be the perimeter of an arbitrary triangle. Prove that sum of the lengths of any two medians is

(a) [**3**] no greater than $\frac{3P}{4}$;

(b) [**4**] no less than $\frac{3P}{8}$.

4 [**8**] A 9×9 grid square is made of matches, so that the side of any cell is one match. In turns Pete and Basil remove matches, one match at a time. The player who destroys the last cell wins. Pete starts first. Which of the players has a winning strategy, no matter how his opponent plays? (A cell is destroyed if it has less than 4 matches on its perimeter).

5 [**8**] In triangle ABC, medians AA_0, BB_0 and CC_0 intersect at point M. Prove that the circumcenters of triangles MA_0B_0, MCB_0, MA_0C_0, MBC_0 and point M are concyclic.

6 Several distinct real numbers are written on a blackboard. Peter wants to create an algebraic expression such that among its values there would be these and only these numbers. He may use any real numbers, brackets, signs $+$, $-$, $\times$ and a special sign $\pm$. Usage of $\pm$ is equivalent to usage of $+$ and $-$ in all possible combinations. For instance, the expression 5 ± 1 results in $\{4, 6\}$, while $(2 \pm 0.5) \pm 0.5$ results in $\{1, 2, 3\}$. Can Peter construct an expression if the numbers on the blackboard are:

(a) [**3**] 1, 2, 4?

(b) [**7**] any 100 distinct numbers (not necessarily integer)?

7 [**9**] Santa Claus had n sorts of candies, k candies of each sort. He distributed them at random between k gift bags, n candies per a bag and gave a bag to everyone of k children at Christmas party. The children learned what they had in their bags and decided to trade. Two children trade one candy for one candy in case if each of them gets the candy of the sort which was absent in his/her bag. Prove that they can organize a sequence of trades so that finally every child would have candies of each sort.

37th International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2015**

1 [**4**] Is it true that every positive integer can be multiplied by one of integers 1, 2, 3, 4 or 5 so that the resulting number starts with 1?

2 [**4**] A rectangle is split into equal non-isosceles right-angled triangles (without gaps or overlaps). Is it true that any such arrangement contains a rectangle made of two such triangles?

3 [**5**] Three players play the game "rock-paper-scissors". In every round, each player simultaneously shows one of these shapes. Rock beats scissors, scissors beat paper, while paper beats rock. If in a round exactly two distinct shapes are shown (and thus one of them is shown twice) then 1 point is added to the score of the player(s) who showed the winning shape, otherwise no point is added. After several rounds it occurred that each shape had been shown the same number of times. Prove that the total sum of points at this moment was a multiple of 3.

4 [**5**] In a right-angled triangle ABC ($\angle C = 90°$) points K, L and M are chosen on sides AC, BC and AB respectively so that $AK = BL = a$, $KM = LM = b$ and $\angle KML = 90°$. Prove that $a = b$.

5 In a country there are 100 cities. Every two cities are connected by direct flight (in both directions). Each flight costs a positive (not necessarily integer) number of doubloons. The flights in both directions between two given cities are of the same cost. The average cost of a flight is 1 doubloon. A traveller plans to visit any m cities for m flights, starting and ending at his native city (which is one of these m cities). Can the traveller always fulfil his plans given that he can spend at most m doubloons if

a [**3**] $m = 99$;

a [**3**] $m = 100$?

37th International Mathematics

TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Fall 2015**

1 [**3**] A geometrical progression consists of 37 positive integers. The first and the last terms are relatively prime numbers. Prove that the 19th term of the progression is the 18th power of some positive integer.

2 [**6**] A 10×10 square on a grid is split by 80 unit grid segments into 20 polygons of equal area (no one of these segments belongs to the boundary of the square). Prove that all polygons are congruent.

3 [**6**] Each coefficient of a polynomial is an integer with absolute value not exceeding 2015. Prove that every positive root of this polynomial exceeds $1/2016$.

4 [**7**] Let $ABCD$ be a cyclic quadrilateral, K and N be the midpoints of the diagonals and P and Q be points of intersection of the extensions of the opposite sides. Prove that $\angle PKQ + \angle PNQ = 180°$.

5 Several distinct real numbers are written on a blackboard. Peter wants to create an algebraic expression such that among its values there would be these and only these numbers. He may use any real numbers, brackets, signs $+$, $-$, $\times$ and a special sign $\pm$. Usage of $\pm$ is equivalent to usage of $+$ and $-$ in all possible combinations. For instance, the expression 5 ± 1 results in $\{4, 6\}$, while $(2 \pm 0.5) \pm 0.5$ results in $\{1, 2, 3\}$. Can Peter construct an expression if the numbers on the blackboard are:

(a) [**2**] 1, 2, 4?

(b) [**6**] any 100 distinct real numbers?

6 Basil has a melon in a shape of a ball, 20 cm in diameter. Using a long knife, Basil makes three mutually perpendicular cuts. Each cut carves a circular segment in a plane of the cut, h cm deep (h is a height of the segment). Does it necessarily follow that the melon breaks into two or more pieces if

(a) [**6**] $h = 17$?

(b) [**6**] $h = 18$?

7 [**12**] N children no two of the same height stand in a line. The following two-step procedure is applied: first, the line is split into the least possible number of groups so that in each group all children are arranged from the left to the right in ascending order of their heights (a group may consist of a single child). Second, the order of children in each group is reversed, so now in each group the children stand in descending order of their heights. Prove that in result of applying this procedure $(N - 1)$ times the children in the line would stand from the left to the right in descending order of their heights.

37th International Mathematics

TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2015**

1 [**3**] Let p be a prime number. Determine the number of positive integers n such that pn is a multiple of $p+n$.

2 [**4**] Suppose that ABC and ABD are right-angled triangles with common hypotenuse AB (D and C are on the same side of line AB). If $AC = BC$ and DK is a bisector of angle ADB, prove that the circumcenter of triangle ACK belongs to line AD.

3 [**4**] Three players play the game "rock-paper-scissors". In every round, each player simultaneously shows one of these shapes. Rock beats scissors, scissors beat paper, while paper beats rock. If in a round exactly two distinct shapes are shown (and thus one of them is shown twice) then 1 point is added to the score of the player(s) who showed the winning shape, otherwise no point is added. After several rounds it occurred that each shape had been shown the same number of times. Prove that the total sum of points at this moment was a multiple of 3.

4 In a country there are 100 cities. Every two cities are connected by direct flight (in both directions). Each flight costs a positive (not necessarily integer) number of doubloons. The flights in both directions between two given cities are of the same cost. The average cost of a flight is 1 doubloon. A traveller plans to visit any m cities for m flights, starting and ending at his native city (which is one of these m cities). Can the traveller always fulfil his plans given that he can spend at most m doubloons if

a [**2**] $m = 99$;

a [**2**] $m = 100$?

5 [**5**] An infinite increasing arithmetical progression is given. A new sequence is constructed in the following way: its first term is the sum of several first terms of the original sequence, its second term is the sum of several next terms of the original sequence and so on. Is it possible that the new sequence is a geometrical progression?

International Mathematics

TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Spring 2015**

1 **[4]** A point is chosen inside a parallelogram $ABCD$ so that $CD = CE$. Prove that the segment DE is perpendicular to the segment connecting the midpoints of the segments AE and BC.

2 **[6]** Area 51 has the shape of a non-convex polygon. It is protected by a chain fence along its perimeter and is surrounded by a minefield so that a spy can only move along the fence. The spy went around the Area once so that the Area was always on his right. A straight power line with 36 poles crosses this area so that some of the poles are inside the Area, and some are outside it. Each time the spy crossed the power line, he counted the poles to the left of him (he could see all the poles). Having passed along the whole fence, the spy had counted 2015 poles in total. Find the number of poles inside the fence.

3 **(a)** **[3]** The integers x, x^2 and x^3 begin with the same digit. Does it imply that this digit is 1?

(b) **[4]** The same question for the integers $x, x^2, x^3, \dots, x^{2015}$.

4 For each side of some polygon, the line containing it contains at least one more vertex of this polygon. Is it possible that the number of vertices of this polygon is

(a) **[4]** ≤ 9?

(b) **[5]** ≤ 8?

5 **(a)** **[4]** A $2 \times n$-table (with $n > 2$) is filled with numbers so that the sums in all the columns are different. Prove that it is possible to permute the numbers in the table so that the sums in the columns would still be different and the sums in the rows would also be different.

(b) **[5]** A 10×10-table is filled with numbers such that the sums in all the columns are different. Is it always possible to permute the numbers in the table so that the sums in the columns would still be different and the sums in the rows would also be different?

6 **[9]** A convex N-gon with equal sides is located inside a circle. Each side is extended in both directions up to the intersection with the circle so that it contains two new segments outside the polygon. Prove that one can paint some of these new $2N$ segments in red and the rest in blue so that the sum of lengths of all the red segments would be the same as for the blue ones.

7 **[10]** An Emperor invited 2015 wizards to a festival. Each of the wizards knows who of them is good and who is evil, however the Emperor doesn't know this. A good wizard always tells the truth, while an evil wizard can tell the truth or lie at any moment. The Emperor asks each wizard (in an order of his choice) a single question, maybe different for different wizards, and listens to the answer which is either "yes" or "no". Having listened to all the answers, the Emperor expels a single wizard through a magic door which shows if this wizard is good or evil. Then the Emperor repeats the procedure with the remaining wizards, and so on. The Emperor may stop at any moment, and after this the Emperor may expel or not expel a wizard. Prove that the Emperor can expel all the evil wizards having expelled at most one good wizard.

International Mathematics

TOURNAMENT OF THE TOWNS

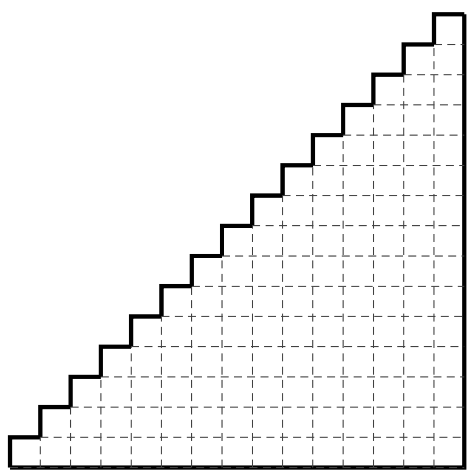
Junior O-Level Paper **Spring 2015**

1 **[3]** Is it possible to paint six face of a cube into three colours so that each colour is present, but from any position one can see at most two colours?

2 **[4]** Points K and L are marked on side AB of triangle ABC so that $KL = BC$ and $AK = LB$. Given that O is the midpoint of side AC, prove that $\angle KOL = 90°$.

3 **[4]** Pete summed up 10 consecutive powers of two, while Basil summed up several first consecutive positive integers. Can they get the same result?

4 **[4]** A figure, given on the grid, consists of a 15-step staircase and horizontal and vertical bases (see the figure). What is the least number of squares one can split this figure into? (Splitting is allowed only along the grid).



5 **[5]** Among $2n+1$ positive integers there is exactly one 0, while each of the numbers $1, 2, ..., n$ is presented exactly twice. For which n can one line up these numbers so that for any $m = 1, \ldots, n$ there are exactly m numbers between two m's?

International Mathematics

TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Spring 2015**

1 **(a)** **[2]** The integers x, x^2 and x^3 begin with the same digit. Does it imply that this digit is 1?

(b) **[3]** The same question for the integers $x, x^2, x^3, \dots, x^{2015}$.

2 **[5]** A point X is marked on the base BC of an isosceles triangle ABC, and points P and Q are marked on the sides AB and AC so that $APXQ$ is a parallelogram. Prove that the point Y symmetrical to X with respect to line PQ lies on the circumcircle of the triangle ABC.

3 **(a)** **[2]** A $2 \times n$-table (with $n > 2$) is filled with numbers so that the sums in all the columns are different. Prove that it is possible to permute the numbers in the table so that the sums in the columns would still be different and the sums in the rows would also be different.

(b) **[6]** A 100×100-table is filled with numbers such that the sums in all the columns are different. Is it always possible to permute the numbers in the table so that the sums in the columns would still be different and the sums in the rows would also be different?

4 **[8]** A convex N-gon with equal sides is located inside a circle. Each side is extended in both directions up to the intersection with the circle so that it contains two new segments outside the polygon. Prove that one can paint some of these new $2N$ segments in red and the rest in blue so that the sum of lengths of all the red segments would be the same as for the blue ones.

5 **[10]** Do there exist two polynomials with integer coefficients such that each polynomial has a coefficient with an absolute value exceeding 2015 but all coefficients of their product have absolute values not exceeding 1?

6 **[10]** An Emperor invited 2015 wizards to a festival. Each of the wizards knows who of them is good and who is evil, however the Emperor doesn't know this. A good wizard always tells the truth, while an evil wizard can tell the truth or lie at any moment. The Emperor gives each wizard a card with a single question, maybe different for different wizards, and after that listens to the answers of all wizards which are either "yes" or "no". Having listened to all the answers, the Emperor expels a single wizard through a magic door which shows if this wizard is good or evil. Then the Emperor makes new cards with questions and repeats the procedure with the remaining wizards, and so on. The Emperor may stop at any moment, and after this the Emperor may expel or not expel a wizard. Prove that the Emperor can expel all the evil wizards having expelled at most one good wizard.

7 **[10]** It is well-known that if a quadrilateral has the circumcircle and the incircle with the same centre then it is a square. Is the similar statement true in 3 dimensions: namely, if a cuboid is inscribed into a sphere and circumscribed around a sphere and the centres of the spheres coincide, does it imply that the cuboid is a cube? (A cuboid is a polyhedron with 6 quadrilateral faces such that each vertex belongs to 3 edges.)

International Mathematics

TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Spring 2015**

1 [**3**] Pete summed up 100 consecutive powers of two, while Basil summed up several first consecutive positive integers. Can they get the same result?

2 [**4**] A moth made four small holes in a square carpet with a 275 cm side. Can one always cut out a square piece with a 1 m side without holes? (Consider holes as points).

3 [**5**] Among $2n+1$ positive integers there is exactly one 0, while each of the numbers $1, 2, ..., n$ is presented exactly twice. For which n can one line up these numbers so that for any $m = 1, \ldots, n$ there are exactly m numbers between two m's?

5 [**5**] Points K and L are marked on the median AM of triangle ABC, so that $AK = KL = LM$. Point P is chosen so that triangles KPL and ABC are similar (the corresponding vertices are listed in the same order). Given that points P and C are on the same side of line AM, prove that point P lies on line AC.

5 [**5**] 2015 positive integers are arranged in a circular order. The difference between any two adjacent numbers coincides with their greatest common divisor. Determine the maximal value of N which divides the product of the numbers, regardless of their choice.