China Girls Math Olympiad 2015

Day 1

1 Let $\triangle ABC$ be an acute-angled triangle with $AB > AC$, O be its circumcenter and D the midpoint of side BC. The circle with diameter AD meets sides AB, AC again at points E, F respectively. The line passing through D parallel to AO meets EF at M. Show that $EM = MF$.

2 Let $a \in (0,1)$,$f(x) = ax^3 + (1-4a)x^2 + (5a-1)x - 5a + 3$, $g(x) = (1-a)x^3 - x^2 + (2-a)x - 3a - 1$.
Prove that:For any real number x ,at least one of $|f(x)|$ and $|g(x)|$ not less than $a+1$.

3 In a 12×12 grid, colour each unit square with either black or white, such that there is at least one black unit square in any 3×4 and 4×3 rectangle bounded by the grid lines. Determine, with proof, the minimum number of black unit squares.

4 Let $g(n)$ be the greatest common divisor of n and 2015. Find the number of triples (a, b, c) which satisfies the following two conditions: 1) $a, b, c \in \{1, 2, ..., 2015\}$; 2) $g(a), g(b), g(c), g(a+b), g(b+c), g(c+a), g(a+b+c)$ are pairwise distinct.

Day 2

5 Determine the number of distinct right-angled triangles such that its three sides are of integral lengths, and its area is 999 times of its perimeter.
(Congruent triangles are considered identical.)

6 Let Γ_1 and Γ_2 be two non-overlapping circles. A, C are on Γ_1 and B, D are on Γ_2 such that AB is an external common tangent to the two circles, and CD is an internal common tangent to the two circles. AC and BD meet at E. F is a point on Γ_1, the tangent line to Γ_1 at F meets the perpendicular bisector of EF at M. MG is a line tangent to Γ_2 at G. Prove that $MF = MG$.

7 Let $x_1, x_2, \cdots, x_n \in (0,1)$, $n \geq 2$. Prove that

$$\frac{\sqrt{1-x_1}}{x_1} + \frac{\sqrt{1-x_2}}{x_2} + \cdots + \frac{\sqrt{1-x_n}}{x_n} < \frac{\sqrt{n-1}}{x_1 x_2 \cdots x_n}.$$

8 Let $n \geq 2$ be a given integer. Initially, we write n sets on the blackboard and do a sequence of moves as follows: choose two sets A and B on the blackboard such that none of them is a subset of the other, and replace A and B by $A \cap B$ and $A \cup B$. This is called a *move*.
Find the maximum number of moves in a sequence for all possible initial sets.