Balkan MO 2005

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1 Let ABC be an acute-angled triangle whose inscribed circle touches AB and AC at D and E respectively. Let X and Y be the points of intersection of the bisectors of the angles $\angle ACB$ and $\angle ABC$ with the line DE and let Z be the midpoint of BC. Prove that the triangle XYZ is equilateral if and only if $\angle A = 60^\circ$.

2 Find all primes p such that $p^2 - p + 1$ is a perfect cube.

3 Let a, b, c be positive real numbers. Prove the inequality

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c + \frac{4(a-b)^2}{a+b+c}.$$

When does equality occur?

4 Let $n \geq 2$ be an integer. Let S be a subset of $\{1, 2, \ldots, n\}$ such that S neither contains two elements one of which divides the other, nor contains two elements which are coprime. What is the maximal possible number of elements of such a set S?