IMO 1960

Sinaia, Romania

Day 1

1 Determine all three-digit numbers N having the property that N is divisible by 11, and $\frac{N}{11}$ is equal to the sum of the squares of the digits of N.

2 For what values of the variable x does the following inequality hold:

$$\frac{4x^2}{(1-\sqrt{2x+1})^2} < 2x+9 \ ?$$

3 In a given right triangle ABC, the hypotenuse BC, of length a, is divided into n equal parts (n and odd integer). Let α be the acute angel subtending, from A, that segment which contains the mdipoint of the hypotenuse. Let h be the length of the altitude to the hypotenuse fo the triangle. Prove that:

$$\tan\alpha = \frac{4nh}{(n^2-1)a}.$$

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Day 2

4 Construct triangle ABC, given h_a, h_b (the altitudes from A and B), and m_a, the median from vertex A.

5 Consider the cube $ABCDA'B'C'D'$ (with face $ABCD$ directly above face $A'B'C'D'$).

a) Find the locus of the midpoints of the segments XY, where X is any point of AC and Y is any piont of $B'D'$;

b) Find the locus of points Z which lie on the segment XY of part a) with $ZY = 2XZ$.

6 Consider a cone of revolution with an inscribed sphere tangent to the base of the cone. A cylinder is circumscribed about this sphere so that one of its bases lies in the base of the cone. let V_1 be the volume of the cone and V_2 be the volume of the cylinder.

a) Prove that $V_1 \neq V_2$;

b) Find the smallest number k for which $V_1 = kV_2$; for this case, construct the angle subtended by a diamter of the base of the cone at the vertex of the cone.

7 An isosceles trapezoid with bases a and c and altitude h is given.

a) On the axis of symmetry of this trapezoid, find all points P such that both legs of the trapezoid subtend right angles at P;

b) Calculate the distance of p from either base;

c) Determine under what conditions such points P actually exist. Discuss various cases that might arise.