

APMO 2002

- 1** Let $a_1, a_2, a_3, \dots, a_n$ be a sequence of non-negative integers, where n is a positive integer. Let

$$A_n = \frac{a_1 + a_2 + \dots + a_n}{n}.$$

Prove that

$$a_1! a_2! \dots a_n! \geq (\lfloor A_n \rfloor!)^n$$

where $\lfloor A_n \rfloor$ is the greatest integer less than or equal to A_n , and $a! = 1 \times 2 \times \dots \times a$ for $a \geq 1$ (and $0! = 1$). When does equality hold?

- 2** Find all positive integers a and b such that

$$\frac{a^2 + b}{b^2 - a} \quad \text{and} \quad \frac{b^2 + a}{a^2 - b}$$

are both integers.

- 3** Let ABC be an equilateral triangle. Let P be a point on the side AC and Q be a point on the side AB so that both triangles ABP and ACQ are acute. Let R be the orthocentre of triangle ABP and S be the orthocentre of triangle ACQ . Let T be the point common to the segments BP and CQ . Find all possible values of $\angle CBP$ and $\angle BCQ$ such that the triangle TRS is equilateral.

- 4** Let x, y, z be positive numbers such that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1.$$

Show that

$$\sqrt{x + yz} + \sqrt{y + zx} + \sqrt{z + xy} \geq \sqrt{xyz} + \sqrt{x} + \sqrt{y} + \sqrt{z}$$

- 5** Let \mathbf{R} denote the set of all real numbers. Find all functions f from \mathbf{R} to \mathbf{R} satisfying:

(i) there are only finitely many s in \mathbf{R} such that $f(s) = 0$,



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and

(ii) $f(x^4 + y) = x^3 f(x) + f(f(y))$ for all x, y in \mathbf{R} .
