All-Russian Olympiad 2018

– Grade 9

1 Suppose $a_1, a_2, \dots$ is an infinite strictly increasing sequence of positive integers and $p_1, p_2, \dots$ is a sequence of distinct primes such that $p_n \mid a_n$ for all $n \ge 1$. It turned out that $a_n - a_k = p_n - p_k$ for all $n, k \ge 1$. Prove that the sequence $(a_n)_n$ consists only of prime numbers.

2 Circle ω is tangent to sides AB, AC of triangle ABC. A circle Ω touches the side AC and line AB (produced beyond B), and touches ω at a point L on side BC. Line AL meets ω, Ω again at K, M. It turned out that $KB \parallel CM$. Prove that $\triangle LCM$ is isosceles.

3 Suppose that $a_1, \cdots, a_{25}$ are non-negative integers, and k is the smallest of them. Prove that
the smallest of them. Prove that

$$\left[\sqrt{a_1}\right] + \left[\sqrt{a_2}\right] + \cdots + \left[\sqrt{a_{25}}\right] \ge \left[\sqrt{a_1 + a_2 + \cdots + a_{25} + 200k}\right].$$

(As usual, $[x]$ denotes the integer part of the number x , that is, the largest integer not exceeding x.)

4 On the $n \times n$ checker board, several cells were marked in such a way that lower lefft (L) and upper right(R) cells are not marked and that for any knight-tour from L to R, there is at least one marked cell. For which $n > 3$, is it possible that there always exists three consective cells going through diagonal for which at least two of them are marked?

5 On the circle, 99 points are marked, dividing this circle into 99 equal arcs. Petya and Vasya play the game, taking turns. Petya goes first; on his first move, he paints in red or blue any marked point. Then each player can paint on his own turn, in red or blue, any uncolored marked point adjacent to the already painted one. Vasya wins, if after painting all points there is an equilateral triangle, all three vertices of which are colored in the same color. Could Petya prevent him?

6 a and b are given positive integers. Prove that there are infinitely many positive integers n such that $n^b + 1$ doesn't divide $a^n + 1$.

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7 In a card game, each card is associated with a numerical value from 1 to 100, with each card beating less, with one exception: 1 beats 100. The player knows that 100 cards with different values lie in front of him. The dealer who knows the order of these cards can tell the player which card beats the other for any pair of cards he draws. Prove that the dealer can make one hundred such messages, so that after that the player can accurately determine the value of each card.

8 $ABCD$ is a convex quadrilateral. Angles A and C are equal. Points M and N are on the sides AB and BC such that $MN||AD$ and $MN = 2AD$. Let K be the midpoint of MN and H be the orthocenter of $\triangle ABC$. Prove that HK is perpendicular to CD.

– Grade 10

1 Determine the number of real roots of the equation

$$|x| + |x+1| + \cdots + |x+2018| = x^2 + 2018x - 2019$$

2 Let $\triangle ABC$ be an acute-angled triangle with $AB < AC$. Let M and N be the midpoints of AB and AC, respectively; let AD be an altitude in this triangle. A point K is chosen on the segment MN so that $BK = CK$. The ray KD meets the circumcircle Ω of ABC at Q. Prove that C, N, K, Q are concyclic.

3 A positive integer k is given. Initially, N cells are marked on an infinite checkered plane. We say that the cross of a cell A is the set of all cells lying in the same row or in the same column as A. By a turn, it is allowed to mark an unmarked cell A if the cross of A contains at least k marked cells. It appears that every cell can be marked in a sequence of such turns. Determine the smallest possible value of N.

4 Initially, a positive integer is written on the blackboard. Every second, one adds to the number on the board the product of all its nonzero digits, writes down the results on the board, and erases the previous number. Prove that there exists a positive integer which will be added inifinitely many times.

5 In a 10×10 table, positive numbers are written. It is known that, looking left-right, the numbers in each row form an arithmetic progression and, looking up-down, the numbers is each column form a geometric progression. Prove that all the ratios of the geometric progressions are equal.

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6	Same as Grade 9 P6
7	Same as Grade 9 P8
8	The board used for playing a game consists of the left and right parts. In each part there are several fields and therere several segments connecting two fields from different parts (all the fields are connected.) Initially, there is a violet counter on a field in the left part, and a purple counter on a field in the right part. Lyosha and Pasha alternatively play their turn, starting from Pasha, by moving their chip (Lyosha-violet, and Pasha-purple) over a segment to other field that has no chip. Its prohibited to repeat a position twice, i.e. cant move to position that already been occupied by some earlier turns in the game. A player losses if he cant make a move. Is there a board and an initial positions of counters that Pasha has a winning strategy?
–	Grade 11
1	The polynomial $P(x)$ is such that the polynomials $P(P(x))$ and $P(P(P(x)))$ are strictly monotone on the whole real axis. Prove that $P(x)$ is also strictly monotone on the whole thing axis.
2	Let $n \geq 2$ and $x_1, x_2, ..., x_n$ positive real numbers.Prove that $\frac{1+x_1^2}{1+x_1x_2} + \frac{1+x_2^2}{1+x_2x_3} + ... + \frac{1+x_n^2}{1+x_nx_1} \geq n$
3	Same as Grade 10 P3
4	On the sides AB and AC of the triangle ABC, the points P and Q are chosen, respectively, so that $PQ\|\|BC$. Segments of BQ and CP intersect at point O. Point A' is symmetric to point A relative to lineBC. The segment $A'O$ intersects circle w circumcircle of the triangle APQ, at the point S. Prove that circumcircle of BSC is tangent to the circle w.
5	On the table, there're 1000 cards arranged on a circle. On each card, a positive integer was written so that all 1000 numbers are distinct. First, Vasya selects one of the card, remove it from the circle, and do the following operation: If on the last card taken out was written positive integer k, count the k^{th} clockwise card not removed, from that position, then remove it and repeat the operation. This continues until only one card left on the table. Is it possible that, initially,

there's a card A such that, no matter what other card Vasya selects as first card, the one that left is always card A?

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6 Three diagonals of a regular n-gon prism intersect at an interior point O. Show that O is the center of the prism.
(The diagonal of the prism is a segment joining two vertices not lying on the same face of the prism.)

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7 Given a sequence of positive integers $a_1, a_2, a_3, ...$ defined by $a_n = \lfloor n^{\frac{2018}{2017}} \rfloor$. Show that there exists a positive integer N such that among any N consecutive terms in the sequence, there exists a term whose decimal representation contain digit 5.

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8 Initially, on the lower left and right corner of a 2018×2018 board, there're two horses, red and blue, respectively. A and B alternatively play their turn, A start first. Each turn consist of moving their horse (A-red, and B-blue) by, simultaneously, 20 cells respect to one coordinate, and 17 cells respect to the other; while preserving the rule that the horse can't occupied the cell that ever occupied by any horses in the game. The player who can't make the move loss, who has the winning strategy?