India
Regional Mathematical Olympiad
2001

$\boxed{1}$ Let BE and CF be the altitudes of an acute triangle ABC with E on AC and F on AB. Let O be the point of intersection of BE and CF. Take any line KL through O with K on AB and L on AC. Suppose M and N are located on BE and CF respectively. such that KM is perpendicular to BE and LN is perpendicular to CF. Prove that FM is parallel to EN.

$\boxed{2}$ Find all primes p and q such that $p^2 + 7pq + q^2$ is a perfect square.

$\boxed{3}$ Find the number of positive integers x such that

$$\left[\frac{x}{99}\right] = \left[\frac{x}{101}\right].$$

$\boxed{4}$ Consider an $n \times n$ array of numbers a_{ij} (standard notation). Suppose each row consists of the n numbers $1, 2, \dots n$ in some order and $a_{ij} = a_{ji}$ for $i, j = 1, 2, \dots n$. If n is odd, prove that the numbers $a_{11}, a_{22}, \dots a_{nn}$ are $1, 2, 3, \dots n$ in some order.

$\boxed{5}$ In a triangle ABC, D is a point on BC such that AD is the internal bisector of $\angle A$. Suppose $\angle B = 2\angle C$ and $CD = AB$. prove that $\angle A = 72^\circ$.

$\boxed{6}$ If x, y, z are sides of a triangle, prove that

$$|x^2(y-z) + y^2(z-x) + z^2(x-y)| < xyz.$$

$\boxed{7}$ Prove that the product of the first 1000 positive even integers differs from the product of the first 1000 positive odd integers by a multiple of 2001.