

Romania Team Selection Test 2009

Day 1

- 1 For non-empty subsets $A, B \subset \mathbb{Z}$ define

$$A + B = \{a + b : a \in A, b \in B\}, \quad A - B = \{a - b : a \in A, b \in B\}.$$

In the sequel we work with non-empty finite subsets of \mathbb{Z} .

Prove that we can cover B by at most $\frac{|A+B|}{|A|}$ translates of $A - A$, i.e. there exists $X \subset \mathbb{Z}$ with $|X| \leq \frac{|A+B|}{|A|}$ such that

$$B \subseteq \bigcup_{x \in X} (x + (A - A)) = X + A - A.$$

- 2 Consider a matrix whose entries are integers. Adding a same integer to all entries on a same row, or on a same column, is called an operation. It is given that, for infinitely many positive integers n , one can obtain, through a finite number of operations, a matrix having all entries divisible by n . Prove that, through a finite number of operations, one can obtain the null matrix.

- 3 Some $n > 2$ lamps are cyclically connected: lamp 1 with lamp 2, ..., lamp k with lamp $k + 1$, ..., lamp $n - 1$ with lamp n , lamp n with lamp 1. At the beginning all lamps are off. When one pushes the switch of a lamp, that lamp and the two ones connected to it change status (from off to on, or vice versa). Determine the number of configurations of lamps reachable from the initial one, through some set of switches being pushed.

Day 2

- 1 We call Golomb ruler a ruler of length l , bearing $k + 1 \geq 2$ marks $0 < a_1 < \dots < a_{k+1} < l$, such that the lengths that can be measured using marks on the ruler are consecutive integers starting with 1, and each such length be measurable between just two of the gradations of the ruler. Find all Golomb rulers.

- 2 A square of side $N = n^2 + 1$, $n \in \mathbb{N}^*$, is partitioned in unit squares (of side 1), along N rows and N columns. The N^2 unit squares are colored using N colors, N squares with each color. Prove that for any coloring there exists a row or a column containing unit squares of at least $n + 1$ colors.

3 Prove that pentagon $ABCDE$ is cyclic if and only if

$$d(E, AB) \cdot d(E, CD) = d(E, AC) \cdot d(E, BD) = d(E, AD) \cdot d(E, BC)$$

where $d(X, YZ)$ denotes the distance from point X to the line YZ .

Day 3

1 Let $ABCD$ be a circumscribed quadrilateral such that $AD > \max\{AB, BC, CD\}$, M be the common point of AB and CD and N be the common point of AC and BD . Show that

$$90^\circ < m(\angle AND) < 90^\circ + \frac{1}{2}m(\angle AMD).$$

Fixed, thank you Luis.

2 Prove that the circumcircle of a triangle contains exactly 3 points whose Simson lines are tangent to the triangle's Euler circle and these points are the vertices of an equilateral triangle.

3 Let ABC be a non-isosceles triangle, in which X, Y , and Z are the tangency points of the incircle of center I with sides BC, CA and AB respectively. Denoting by O the circumcircle of $\triangle ABC$, line OI meets BC at a point D . The perpendicular dropped from X to YZ intersects AD at E . Prove that YZ is the perpendicular bisector of $[EX]$.

Day 4

1 Given an integer $n \geq 2$, determine the maximum value the sum $x_1 + \dots + x_n$ may achieve, as the x_i run through the positive integers, subject to $x_1 \leq x_2 \leq \dots \leq x_n$ and $x_1 + \dots + x_n = x_1 x_2 \dots x_n$.

2 Let $m < n$ be two positive integers, let I and J be two index sets such that $|I| = |J| = n$ and $|I \cap J| = m$, and let $u_k, k \in I \cup J$ be a collection of vectors in the Euclidean plane such that

$$\left| \sum_{i \in I} u_i \right| = 1 = \left| \sum_{j \in J} u_j \right|.$$

Prove that

$$\sum_{k \in I \cup J} |u_k|^2 \geq \frac{2}{m+n}$$

and find the cases of equality.

Day 5

1 Given two (identical) polygonal domains in the Euclidean plane, it is not possible in general to superpose the two using only translations and rotations. Prove that this can however be achieved by splitting one of the domains into a finite number of polygonal subdomains which then fit together, via translations and rotations in the plane, to recover the other domain.

2 Let a and n be two integers greater than 1. Prove that if n divides $(a-1)^k$ for some integer $k \geq 2$, then n also divides $a^{n-1} + a^{n-2} + \cdots + a + 1$.

3 Given two integers $n \geq 1$ and $q \geq 2$, let $A = \{(a_1, \dots, a_n) : a_i \in \{0, \dots, q-1\}, i = 1, \dots, n\}$. If $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ are two elements of A , let $\delta(a, b) = \#\{i : a_i \neq b_i\}$. Let further t be a non-negative integer and B a non-empty subset of A such that $\delta(a, b) \geq 2t + 1$, whenever a and b are distinct elements of B . Prove that the two statements below are equivalent:

a) For any $a \in A$, there is a unique $b \in B$, such that $\delta(a, b) \leq t$;

b) $|B| \cdot \sum_{k=0}^t \binom{n}{k} (q-1)^k = q^n$

Day 6

2 Let n and k be positive integers. Find all monic polynomials $f \in \mathbb{Z}[X]$, of degree n , such that $f(a)$ divides $f(2a^k)$ for $a \in \mathbb{Z}$ with $f(a) \neq 0$.

3 Given an integer $n \geq 2$ and a closed unit disc, evaluate the maximum of the product of the lengths of all $\frac{n(n-1)}{2}$ segments determined by n points in that disc.

Day 7

1 The quadrilateral $ABCD$ inscribed in a circle with diameter BD . Let A', B' are symmetric to A, B with respect to the line BD and AC respectively. If $A'C \cap BD = P$ and $AC \cap B'D = Q$ then prove that $PQ \perp AC$

2 Prove that the edges of a finite simple planar graph (with no loops, multiple edges) may be oriented in such a way that at most three fourths of the total number of edges of any cycle share the same orientation. Moreover, show that this is the best global bound possible.



Art of Problem Solving

2009 Romania Team Selection Test

Comment: The actual problem in the TST asked to prove that the edges can be 2-colored so that the same conclusion holds. Under this circumstances, the problem is wrong and a counterexample was found in the contest by Marius Tiba.

- 3** Show that there are infinitely many pairs of prime numbers (p, q) such that $p \mid 2^{q-1} - 1$ and $q \mid 2^{p-1} - 1$.
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