– Time allowed: 4 hours
Each problem is worth 7 points

1 We say that a triangle ABC is great if the following holds: for any point D on the side BC, if P and Q are the feet of the perpendiculars from D to the lines AB and AC, respectively, then the reflection of D in the line PQ lies on the circumcircle of the triangle ABC. Prove that triangle ABC is great if and only if $\angle A = 90^\circ$ and $AB = AC$.

2 A positive integer is called *fancy* if it can be expressed in the form

$$2^{a_1} + 2^{a_2} + \cdots + 2^{a_{100}},$$

where $a_1, a_2, \cdots, a_{100}$ are non-negative integers that are not necessarily distinct. Find the smallest positive integer n such that no multiple of n is a *fancy* number.

3 Let AB and AC be two distinct rays not lying on the same line, and let ω be a circle with center O that is tangent to ray AC at E and ray AB at F. Let R be a point on segment EF. The line through O parallel to EF intersects line AB at P. Let N be the intersection of lines PR and AC, and let M be the intersection of line AB and the line through R parallel to AC. Prove that line MN is tangent to ω.

4 The country Dreamland consists of 2016 cities. The airline Starways wants to establish some one-way flights between pairs of cities in such a way that each city has exactly one flight out of it. Find the smallest positive integer k such that no matter how Starways establishes its flights, the cities can always be partitioned into k groups so that from any city it is not possible to reach another city in the same group by using at most 28 flights.

5 Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$(z+1)f(x+y) = f(xf(z)+y) + f(yf(z)+x),$$

for all positive real numbers x, y, z.

APMO 2015

1 Let ABC be a triangle, and let D be a point on side BC. A line through D intersects side AB at X and ray AC at Y. The circumcircle of triangle BXD intersects the circumcircle ω of triangle ABC again at point Z distinct from point B. The lines ZD and ZY intersect ω again at V and W respectively. Prove that $AB = VW$

Proposed by Warut Suksompong, Thailand

2 Let $S = \{2, 3, 4, \ldots\}$ denote the set of integers that are greater than or equal to 2. Does there exist a function $f : S \to S$ such that

$$f(a)f(b) = f(a^2b^2) \text{ for all } a, b \in S \text{ with } a \neq b?$$

3 A sequence of real numbers $a_0, a_1, \ldots$ is said to be good if the following three conditions hold.
(i) The value of a_0 is a positive integer.
(ii) For each non-negative integer i we have $a_{i+1} = 2a_i + 1$ or $a_{i+1} = \frac{a_i}{a_i+2}$
(iii) There exists a positive integer k such that $a_k = 2014$.

Find the smallest positive integer n such that there exists a good sequence $a_0, a_1, \ldots$ of real numbers with the property that $a_n = 2014$.

Proposed by Wang Wei Hua, Hong Kong

4 Let n be a positive integer. Consider $2n$ distinct lines on the plane, no two of which are parallel. Of the $2n$ lines, n are colored blue, the other n are colored red. Let $\mathcal{B}$ be the set of all points on the plane that lie on at least one blue line, and $\mathcal{R}$ the set of all points on the plane that lie on at least one red line. Prove that there exists a circle that intersects $\mathcal{B}$ in exactly $2n-1$ points, and also intersects $\mathcal{R}$ in exactly $2n-1$ points.

5 Determine all sequences $a_0, a_1, a_2, \ldots$ of positive integers with $a_0 \geq 2015$ such that for all integers $n \geq 1$:
(i) a_{n+2} is divisible by a_n ;
(ii) $|s_{n+1} - (n+1)a_n| = 1$, where $s_{n+1} = a_{n+1} - a_n + a_{n-1} - \cdots + (-1)^{n+1}a_0$.

APMO 2014

1 For a positive integer m denote by $S(m)$ and $P(m)$ the sum and product, respectively, of the digits of m. Show that for each positive integer n, there exist positive integers $a_1, a_2, \ldots, a_n$ satisfying the following conditions:

$$S(a_1) < S(a_2) < \cdots < S(a_n) \text{ and } S(a_i) = P(a_{i+1}) \quad (i = 1, 2, \ldots, n).$$

(We let $a_{n+1} = a_1$.)

2 Let $S = \{1, 2, \ldots, 2014\}$. For each non-empty subset $T \subseteq S$, one of its members is chosen as its representative. Find the number of ways to assign representatives to all non-empty subsets of S so that if a subset $D \subseteq S$ is a disjoint union of non-empty subsets $A, B, C \subseteq S$, then the representative of D is also the representative of one of A, B, C.

3 Find all positive integers n such that for any integer k there exists an integer a for which $a^3 + a - k$ is divisible by n.

4 Let n and b be positive integers. We say n is b-discerning if there exists a set consisting of n different positive integers less than b that has no two different subsets U and V such that the sum of all elements in U equals the sum of all elements in V.

(a) Prove that 8 is 100-discerning.
(b) Prove that 9 is not 100-discerning.

5 Circles ω and Ω meet at points A and B. Let M be the midpoint of the arc AB of circle ω (M lies inside Ω). A chord MP of circle ω intersects Ω at Q (Q lies inside ω). Let ℓ_P be the tangent line to ω at P, and let ℓ_Q be the tangent line to Ω at Q. Prove that the circumcircle of the triangle formed by the lines ℓ_P, ℓ_Q and AB is tangent to Ω.

APMO 2013

1 Let ABC be an acute triangle with altitudes AD, BE, and CF, and let O be the center of its circumcircle. Show that the segments OA, OF, OB, OD, OC, OE dissect the triangle ABC into three pairs of triangles that have equal areas.

2 Determine all positive integers n for which $\dfrac{n^2+1}{[\sqrt{n}]^2+2}$ is an integer. Here $[r]$ denotes the greatest integer less than or equal to r.

3 For $2k$ real numbers $a_1, a_2, ..., a_k$, $b_1, b_2, ..., b_k$ define a sequence of numbers X_n by

$$X_n = \sum_{i=1}^{k}[a_i n + b_i] \quad (n = 1, 2, ...).$$

If the sequence X_N forms an arithmetic progression, show that $\sum_{i=1}^{k} a_i$ must be an integer. Here $[r]$ denotes the greatest integer less than or equal to r.

4 Let a and b be positive integers, and let A and B be finite sets of integers satisfying
(i) A and B are disjoint;
(ii) if an integer i belongs to either to A or to B, then either $i+a$ belongs to A or $i-b$ belongs to B.
Prove that $a\,|A| = b\,|B|$. (Here $|X|$ denotes the number of elements in the set X.)

5 Let $ABCD$ be a quadrilateral inscribed in a circle ω, and let P be a point on the extension of AC such that PB and PD are tangent to ω. The tangent at C intersects PD at Q and the line AD at R. Let E be the second point of intersection between AQ and ω. Prove that B, E, R are collinear.

APMO 2012

1 Let P be a point in the interior of a triangle ABC, and let D, E, F be the point of intersection of the line AP and the side BC of the triangle, of the line BP and the side CA, and of the line CP and the side AB, respectively. Prove that the area of the triangle ABC must be 6 if the area of each of the triangles PFA, PDB and PEC is 1.

2 Into each box of a 2012×2012 square grid, a real number greater than or equal to 0 and less than or equal to 1 is inserted. Consider splitting the grid into 2 non-empty rectangles consisting of boxes of the grid by drawing a line parallel either to the horizontal or the vertical side of the grid. Suppose that for at least one of the resulting rectangles the sum of the numbers in the boxes within the rectangle is less than or equal to 1, no matter how the grid is split into 2 such rectangles. Determine the maximum possible value for the sum of all the 2012×2012 numbers inserted into the boxes.

3 Determine all the pairs (p, n) of a prime number p and a positive integer n for which $\frac{n^p+1}{p^n+1}$ is an integer.

4 Let ABC be an acute triangle. Denote by D the foot of the perpendicular line drawn from the point A to the side BC, by M the midpoint of BC, and by H the orthocenter of ABC. Let E be the point of intersection of the circumcircle Γ of the triangle ABC and the half line MH, and F be the point of intersection (other than E) of the line ED and the circle Γ. Prove that $\frac{BF}{CF} = \frac{AB}{AC}$ must hold.

(Here we denote XY the length of the line segment XY.)

5 Let n be an integer greater than or equal to 2. Prove that if the real numbers $a_1, a_2, \cdots, a_n$ satisfy $a_1^2 + a_2^2 + \cdots + a_n^2 = n$, then

$$\sum_{1 \le i < j \le n} \frac{1}{n - a_i a_j} \le \frac{n}{2}$$

must hold.

APMO 2011

1 Let a, b, c be positive integers. Prove that it is impossible to have all of the three numbers $a^2+b+c, b^2+c+a, c^2+a+b$ to be perfect squares.

2 Five points A_1, A_2, A_3, A_4, A_5 lie on a plane in such a way that no three among them lie on a same straight line. Determine the maximum possible value that the minimum value for the angles $\angle A_iA_jA_k$ can take where i, j, k are distinct integers between 1 and 5.

3 Let ABC be an acute triangle with $\angle BAC = 30°$. The internal and external angle bisectors of $\angle ABC$ meet the line AC at B_1 and B_2, respectively, and the internal and external angle bisectors of $\angle ACB$ meet the line AB at C_1 and C_2, respectively. Suppose that the circles with diameters B_1B_2 and C_1C_2 meet inside the triangle ABC at point P. Prove that $\angle BPC = 90°$.

4 Let n be a fixed positive odd integer. Take $m+2$ **distinct** points $P_0, P_1, \ldots, P_{m+1}$ (where m is a non-negative integer) on the coordinate plane in such a way that the following three conditions are satisfied:
1) $P_0 = (0, 1), P_{m+1} = (n+1, n)$, and for each integer $i, 1 \le i \le m$, both x- and y- coordinates of P_i are integers lying in between 1 and n (1 and n inclusive).
2) For each integer $i, 0 \le i \le m$, P_iP_{i+1} is parallel to the x-axis if i is even, and is parallel to the y-axis if i is odd.
3) For each pair i, j with $0 \le i < j \le m$, line segments P_iP_{i+1} and P_jP_{j+1} share at most 1 point.
Determine the maximum possible value that m can take.

5 Determine all functions $f : \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers, satisfying the following two conditions:
1) There exists a real number M such that for every real number $x, f(x) < M$ is satisfied.
2) For every pair of real numbers x and y,

$$f(xf(y)) + yf(x) = xf(y) + f(xy)$$

is satisfied.

APMO 2010

1 Let ABC be a triangle with $\angle BAC \neq 90^\circ$. Let O be the circumcenter of the triangle ABC and Γ be the circumcircle of the triangle BOC. Suppose that Γ intersects the line segment AB at P different from B, and the line segment AC at Q different from C. Let ON be the diameter of the circle Γ. Prove that the quadrilateral $APNQ$ is a parallelogram.

2 For a positive integer k, call an integer a *pure* $k-th$ *power* if it can be represented as m^k for some integer m. Show that for every positive integer n, there exists n distinct positive integers such that their sum is a pure $2009-$th power and their product is a pure $2010-$th power.

3 Let n be a positive integer. n people take part in a certain party. For any pair of the participants, either the two are acquainted with each other or they are not. What is the maximum possible number of the pairs for which the two are not acquainted but have a common acquaintance among the participants?

4 Let ABC be an acute angled triangle satisfying the conditions $AB > BC$ and $AC > BC$. Denote by O and H the circumcentre and orthocentre, respectively, of the triangle ABC. Suppose that the circumcircle of the triangle AHC intersects the line AB at M different from A, and the circumcircle of the triangle AHB intersects the line AC at N different from A. Prove that the circumcentre of the triangle MNH lies on the line OH.

5 Find all functions f from the set $\mathbb{R}$ of real numbers into $\mathbb{R}$ which satisfy for all $x, y, z \in \mathbb{R}$ the identity $f(f(x)+f(y)+f(z)) = f(f(x)-f(y)) + f(2xy + f(z)) + 2f(xz - yz)$

APMO 2009

1 Consider the following operation on positive real numbers written on a blackboard:
Choose a number r written on the blackboard, erase that number, and then write a pair of positive real numbers a and b satisfying the condition $2r^2 = ab$ on the board.

Assume that you start out with just one positive real number r on the blackboard, and apply this operation $k^2 - 1$ times to end up with k^2 positive real numbers, not necessarily distinct. Show that there exists a number on the board which does not exceed kr.

2 Let a_1, a_2, a_3, a_4, a_5 be real numbers satisfying the following equations:
$\frac{a_1}{k^2+1} + \frac{a_2}{k^2+2} + \frac{a_3}{k^2+3} + \frac{a_4}{k^2+4} + \frac{a_5}{k^2+5} = \frac{1}{k^2}$ for $k = 1, 2, 3, 4, 5$

Find the value of $\frac{a_1}{37} + \frac{a_2}{38} + \frac{a_3}{39} + \frac{a_4}{40} + \frac{a_5}{41}$ (Express the value in a single fraction.)

3 Let three circles $\Gamma_1, \Gamma_2, \Gamma_3$, which are non-overlapping and mutually external, be given in the plane. For each point P in the plane, outside the three circles, construct six points $A_1, B_1, A_2, B_2, A_3, B_3$ as follows: For each $i = 1, 2, 3$, A_i, B_i are distinct points on the circle Γ_i such that the lines PA_i and PB_i are both tangents to Γ_i. Call the point P exceptional if, from the construction, three lines A_1B_1, A_2B_2, A_3B_3 are concurrent. Show that every exceptional point of the plane, if exists, lies on the same circle.

4 Prove that for any positive integer k, there exists an arithmetic sequence $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, ..., \frac{a_k}{b_k}$ of rational numbers, where a_i, b_i are relatively prime positive integers for each $i = 1, 2, ..., k$ such that the positive integers $a_1, b_1, a_2, b_2, ..., a_k, b_k$ are all distinct.

5 Larry and Rob are two robots travelling in one car from Argovia to Zillis. Both robots have control over the steering and steer according to the following algorithm: Larry makes a 90 degrees left turn after every ℓ kilometer driving from start, Rob makes a 90 degrees right turn after every r kilometer driving from start, where ℓ and r are relatively prime positive integers.

In the event of both turns occurring simultaneously, the car will keep going without changing direction. Assume that the ground is flat and the car can move in any direction. Let the car start from Argovia facing towards Zillis. For which choices of the pair (ℓ, r) is the car guaranteed to reach Zillis, regardless of how far it is from Argovia?

APMO 2008

1 Let ABC be a triangle with $\angle A < 60^\circ$. Let X and Y be the points on the sides AB and AC, respectively, such that $CA + AX = CB + BX$ and $BA + AY = BC + CY$. Let P be the point in the plane such that the lines PX and PY are perpendicular to AB and AC, respectively. Prove that $\angle BPC < 120^\circ$.

2 Students in a class form groups each of which contains exactly three members such that any two distinct groups have at most one member in common. Prove that, when the class size is 46, there is a set of 10 students in which no group is properly contained.

3 Let Γ be the circumcircle of a triangle ABC. A circle passing through points A and C meets the sides BC and BA at D and E, respectively. The lines AD and CE meet Γ again at G and H, respectively. The tangent lines of Γ at A and C meet the line DE at L and M, respectively. Prove that the lines LH and MG meet at Γ.

4 Consider the function $f : \mathbb{N}_0 \to \mathbb{N}_0$, where $\mathbb{N}_0$ is the set of all non-negative integers, defined by the following conditions :

(i) $f(0) = 0$; (ii) $f(2n) = 2f(n)$ and (iii) $f(2n + 1) = n + 2f(n)$ for all $n \geq 0$.

(a) Determine the three sets $L = \{n|f(n) < f(n+1)\}$, $E = \{n|f(n) = f(n+1)\}$, and $G = \{n|f(n) > f(n + 1)\}$. (b) For each $k \geq 0$, find a formula for $a_k = \max\{f(n) : 0 \leq n \leq 2^k\}$ in terms of k.

5 Let a, b, c be integers satisfying $0 < a < c - 1$ and $1 < b < c$. For each k, $0 \leq k \leq a$, Let $r_k, 0 \leq r_k < c$
be the remainder of kb when divided by c. Prove that the two sets $\{r_0, r_1, r_2, \cdots , r_a\}$ and $\{0, 1, 2, \cdots , a\}$ are different.

APMO 2007

– March 12th

1 Let S be a set of 9 distinct integers all of whose prime factors are at most 3. Prove that S contains 3 distinct integers such that their product is a perfect cube.

2 Let ABC be an acute angled triangle with $\angle BAC = 60^\circ$ and $AB > AC$. Let I be the incenter, and H the orthocenter of the triangle ABC . Prove that $2\angle AHI = 3\angle ABC$.

3 Consider n disks $C_1; C_2; ...; C_n$ in a plane such that for each $1 \le i < n$, the center of C_i is on the circumference of C_{i+1}, and the center of C_n is on the circumference of C_1. Define the *score* of such an arrangement of n disks to be the number of pairs $(i; j)$ for which C_i properly contains C_j . Determine the maximum possible score.

4 Let $x; y$ and z be positive real numbers such that $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$. Prove that $\frac{x^2+yz}{\sqrt{2x^2(y+z)}} + \frac{y^2+zx}{\sqrt{2y^2(z+x)}} + \frac{z^2+xy}{\sqrt{2z^2(x+y)}} \ge 1$.

5 A regular (5×5)-array of lights is defective, so that toggling the switch for one light causes each adjacent light in the same row and in the same column as well as the light itself to change state, from on to off, or from off to on. Initially all the lights are switched off. After a certain number of toggles, exactly one light is switched on. Find all the possible positions of this light.

APMO 2006

1 Let n be a positive integer. Find the largest nonnegative real number $f(n)$ (depending on n) with the following property: whenever $a_1, a_2, ..., a_n$ are real numbers such that $a_1 + a_2 + \cdots + a_n$ is an integer, there exists some i such that $\left|a_i - \frac{1}{2}\right| \geq f(n)$.

2 Prove that every positive integer can be written as a finite sum of distinct integral powers of the golden ratio.

3 Let $p \geq 5$ be a prime and let r be the number of ways of placing p checkers on a $p \times p$ checkerboard so that not all checkers are in the same row (but they may all be in the same column). Show that r is divisible by p^5. Here, we assume that all the checkers are identical.

4 Let A, B be two distinct points on a given circle O and let P be the midpoint of the line segment AB. Let O_1 be the circle tangent to the line AB at P and tangent to the circle O. Let l be the tangent line, different from the line AB, to O_1 passing through A. Let C be the intersection point, different from A, of l and O. Let Q be the midpoint of the line segment BC and O_2 be the circle tangent to the line BC at Q and tangent to the line segment AC. Prove that the circle O_2 is tangent to the circle O.

5 In a circus, there are n clowns who dress and paint themselves up using a selection of 12 distinct colours. Each clown is required to use at least five different colours. One day, the ringmaster of the circus orders that no two clowns have exactly the same set of colours and no more than 20 clowns may use any one particular colour. Find the largest number n of clowns so as to make the ringmaster's order possible.

APMO 2005

1 Prove that for every irrational real number a, there are irrational real numbers b and b' so that $a+b$ and ab' are both rational while ab and $a+b'$ are both irrational.

2 Let a, b, c be positive real numbers such that $abc = 8$. Prove that

$$\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} + \frac{b^2}{\sqrt{(1+b^3)(1+c^3)}} + \frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \geq \frac{4}{3}$$

3 Prove that there exists a triangle which can be cut into 2005 congruent triangles.

4 In a small town, there are $n \times n$ houses indexed by (i, j) for $1 \leq i, j \leq n$ with $(1, 1)$ being the house at the top left corner, where i and j are the row and column indices, respectively. At time 0, a fire breaks out at the house indexed by $(1, c)$, where $c \leq \frac{n}{2}$. During each subsequent time interval $[t, t+1]$, the fire fighters defend a house which is not yet on fire while the fire spreads to all undefended *neighbors* of each house which was on fire at time t. Once a house is defended, it remains so all the time. The process ends when the fire can no longer spread. At most how many houses can be saved by the fire fighters?
A house indexed by (i, j) is a *neighbor* of a house indexed by (k, l) if $|i-k| + |j-l| = 1$.

5 In a triangle ABC, points M and N are on sides AB and AC, respectively, such that $MB = BC = CN$. Let R and r denote the circumradius and the inradius of the triangle ABC, respectively. Express the ratio MN/BC in terms of R and r.

APMO 2004

1 Determine all finite nonempty sets S of positive integers satisfying

$$\frac{i+j}{(i,j)} \qquad \text{is an element of S for all i,j in S,}$$

where (i,j) is the greatest common divisor of i and j.

2 Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Prove that the area of one of the triangles AOH, BOH and COH is equal to the sum of the areas of the other two.

3 Let a set S of 2004 points in the plane be given, no three of which are collinear. Let $\mathcal{L}$ denote the set of all lines (extended indefinitely in both directions) determined by pairs of points from the set. Show that it is possible to colour the points of S with at most two colours, such that for any points p, q of S, the number of lines in $\mathcal{L}$ which separate p from q is odd if and only if p and q have the same colour.

Note: A line ℓ separates two points p and q if p and q lie on opposite sides of ℓ with neither point on ℓ.

4 For a real number x, let $\lfloor x \rfloor$ stand for the largest integer that is less than or equal to x. Prove that

$$\left\lfloor \frac{(n-1)!}{n(n+1)} \right\rfloor$$

is even for every positive integer n.

5 Prove that the inequality

$$\left(a^2+2\right)\left(b^2+2\right)\left(c^2+2\right) \geq 9\left(ab+bc+ca\right)$$

holds for all positive reals a, b, c.

APMO 2003

1 Let a, b, c, d, e, f be real numbers such that the polynomial

$$p(x) = x^8 - 4x^7 + 7x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f$$

factorises into eight linear factors $x - x_i$, with $x_i > 0$ for $i = 1, 2, \dots, 8$. Determine all possible values of f.

2 Suppose $ABCD$ is a square piece of cardboard with side length a. On a plane are two parallel lines ℓ_1 and ℓ_2, which are also a units apart. The square $ABCD$ is placed on the plane so that sides AB and AD intersect ℓ_1 at E and F respectively. Also, sides CB and CD intersect ℓ_2 at G and H respectively. Let the perimeters of $\triangle AEF$ and $\triangle CGH$ be m_1 and m_2 respectively.

Prove that no matter how the square was placed, $m_1 + m_2$ remains constant.

3 Let $k \geq 14$ be an integer, and let p_k be the largest prime number which is strictly less than k. You may assume that $p_k \geq 3k/4$. Let n be a composite integer. Prove:
(a) if $n = 2p_k$, then n does not divide $(n-k)!$;
(b) if $n > 2p_k$, then n divides $(n-k)!$.

4 Let a, b, c be the sides of a triangle, with $a + b + c = 1$, and let $n \geq 2$ be an integer. Show that

$$\sqrt[n]{a^n + b^n} + \sqrt[n]{b^n + c^n} + \sqrt[n]{c^n + a^n} < 1 + \frac{\sqrt[n]{2}}{2}.$$

5 Given two positive integers m and n, find the smallest positive integer k such that among any k people, either there are $2m$ of them who form m pairs of mutually acquainted people or there are $2n$ of them forming n pairs of mutually unacquainted people.

APMO 2002

1 Let $a_1, a_2, a_3, \dots, a_n$ be a sequence of non-negative integers, where n is a positive integer. Let

$$A_n = \frac{a_1 + a_2 + \cdots + a_n}{n} .$$

Prove that

$$a_1! a_2! \dots a_n! \geq (\lfloor A_n \rfloor !)^n$$

where $\lfloor A_n \rfloor$ is the greatest integer less than or equal to A_n, and $a! = 1 \times 2 \times \cdots \times a$ for $a \geq 1$(and $0! = 1$). When does equality hold?

2 Find all positive integers a and b such that

$$\frac{a^2+b}{b^2-a} \quad \text{and} \quad \frac{b^2+a}{a^2-b}$$

are both integers.

3 Let ABC be an equilateral triangle. Let P be a point on the side AC and Q be a point on the side AB so that both triangles ABP and ACQ are acute. Let R be the orthocentre of triangle ABP and S be the orthocentre of triangle ACQ. Let T be the point common to the segments BP and CQ. Find all possible values of $\angle CBP$ and $\angle BCQ$ such that the triangle TRS is equilateral.

4 Let x, y, z be positive numbers such that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1.$$

Show that

$$\sqrt{x+yz} + \sqrt{y+zx} + \sqrt{z+xy} \geq \sqrt{xyz} + \sqrt{x} + \sqrt{y} + \sqrt{z}$$

5 Let $\mathbf{R}$ denote the set of all real numbers. Find all functions f from $\mathbf{R}$ to $\mathbf{R}$ satisfying:

(i) there are only finitely many s in $\mathbf{R}$ such that $f(s) = 0$, and

(ii) $f(x^4+y)=x^3f(x)+f(f(y))$ for all x,y in $\mathbf{R}$.

APMO 2001

1 For a positive integer n let $S(n)$ be the sum of digits in the decimal representation of n. Any positive integer obtained by removing several (at least one) digits from the right-hand end of the decimal representation of n is called a *stump* of n. Let $T(n)$ be the sum of all stumps of n. Prove that $n = S(n) + 9T(n)$.

2 Find the largest positive integer N so that the number of integers in the set $\{1, 2, \dots, N\}$ which are divisible by 3 is equal to the number of integers which are divisible by 5 or 7 (or both).

3 Two equal-sized regular n-gons intersect to form a $2n$-gon C. Prove that the sum of the sides of C which form part of one n-gon equals half the perimeter of C.

Alternative formulation:

Let two equal regular n-gons S and T be located in the plane such that their intersection $S \cap T$ is a $2n$-gon (with $n \geq 3$). The sides of the polygon S are coloured in red and the sides of T in blue.

Prove that the sum of the lengths of the blue sides of the polygon $S \cap T$ is equal to the sum of the lengths of its red sides.

4 A point in the plane with a cartesian coordinate system is called a *mixed point* if one of its coordinates is rational and the other one is irrational. Find all polynomials with real coefficients such that their graphs do not contain any mixed point.

5 Find the greatest integer n, such that there are $n + 4$ points A, B, C, D, $X_1, \dots,$ X_n in the plane with $AB \neq CD$ that satisfy the following condition: for each $i = 1, 2, \dots, n$ triangles ABX_i and CDX_i are equal.

APMO 2000

1 Compute the sum: $\sum_{i=0}^{101} \frac{x_i^3}{1-3x_i+3x_i^2}$ for $x_i = \frac{i}{101}$.

2 Find all permutations $a_1, a_2, \ldots, a_9$ of $1, 2, \ldots, 9$ such that

$$a_1 + a_2 + a_3 + a_4 = a_4 + a_5 + a_6 + a_7 = a_7 + a_8 + a_9 + a_1$$

and

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 = a_4^2 + a_5^2 + a_6^2 + a_7^2 = a_7^2 + a_8^2 + a_9^2 + a_1^2$$

3 Let ABC be a triangle. Let M and N be the points in which the median and the angle bisector, respectively, at A meet the side BC. Let Q and P be the points in which the perpendicular at N to NA meets MA and BA, respectively. And O the point in which the perpendicular at P to BA meets AN produced.

Prove that QO is perpendicular to BC.

4 Let n, k be given positive integers with $n > k$. Prove that:

$$\frac{1}{n+1} \cdot \frac{n^n}{k^k (n-k)^{n-k}} < \frac{n!}{k!(n-k)!} < \frac{n^n}{k^k (n-k)^{n-k}}$$

5 Given a permutation $(a_0, a_1, \ldots, a_n)$ of the sequence $0, 1, \ldots, n$. A transportation of a_i with a_j is called legal if $a_i = 0$ for $i > 0$, and $a_{i-1} + 1 = a_j$. The permutation $(a_0, a_1, \ldots, a_n)$ is called regular if after a number of legal transportations it becomes $(1, 2, \ldots, n, 0)$.
For which numbers n is the permutation $(1, n, n-1, \ldots, 3, 2, 0)$ regular?

APMO 1999

1 Find the smallest positive integer n with the following property: there does not exist an arithmetic progression of 1999 real numbers containing exactly n integers.

2 Let $a_1, a_2, \ldots$ be a sequence of real numbers satisfying $a_{i+j} \leq a_i + a_j$ for all $i, j = 1, 2, \ldots$. Prove that

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} \geq a_n$$

for each positive integer n.

3 Let Γ_1 and Γ_2 be two circles intersecting at P and Q. The common tangent, closer to P, of Γ_1 and Γ_2 touches Γ_1 at A and Γ_2 at B. The tangent of Γ_1 at P meets Γ_2 at C, which is different from P, and the extension of AP meets BC at R.
Prove that the circumcircle of triangle PQR is tangent to BP and BR.

4 Determine all pairs (a, b) of integers with the property that the numbers $a^2 + 4b$ and $b^2 + 4a$ are both perfect squares.

5 Let S be a set of $2n + 1$ points in the plane such that no three are collinear and no four concyclic. A circle will be called Good if it has 3 points of S on its circumference, $n - 1$ points in its interior and $n - 1$ points in its exterior.
Prove that the number of good circles has the same parity as n.

APMO 1998

1 Let F be the set of all n-tuples $(A_1, \ldots, A_n)$ such that each A_i is a subset of $\{1, 2, \ldots, 1998\}$. Let $|A|$ denote the number of elements of the set A. Find

$$\sum_{(A_1,\ldots,A_n)\in F} |A_1 \cup A_2 \cup \cdots \cup A_n|$$

2 Show that for any positive integers a and b, $(36a + b)(a + 36b)$ cannot be a power of 2.

3 Let a, b, c be positive real numbers. Prove that

$$\left(1 + \frac{a}{b}\right)\left(1 + \frac{b}{c}\right)\left(1 + \frac{c}{a}\right) \geq 2\left(1 + \frac{a+b+c}{\sqrt[3]{abc}}\right).$$

4 Let ABC be a triangle and D the foot of the altitude from A. Let E and F lie on a line passing through D such that AE is perpendicular to BE, AF is perpendicular to CF, and E and F are different from D. Let M and N be the midpoints of the segments BC and EF, respectively. Prove that AN is perpendicular to NM.

5 Find the largest integer n such that n is divisible by all positive integers less than $\sqrt[3]{n}$.

APMO 1997

1 Given:

$$S = 1 + \frac{1}{1+\frac{1}{3}} + \frac{1}{1+\frac{1}{3}+\frac{1}{6}} + \cdots + \frac{1}{1+\frac{1}{3}+\frac{1}{6}+\cdots+\frac{1}{1993006}}$$

where the denominators contain partial sums of the sequence of reciprocals of triangular numbers (i.e. $k = \frac{n(n+1)}{2}$ for $n = 1,\ 2,\ \ldots,1996$). Prove that $S > 1001$.

2 Find an integer n, where $100 \le n \le 1997$, such that

$$\frac{2^n+2}{n}$$

is also an integer.

3 Let ABC be a triangle inscribed in a circle and let

$$l_a = \frac{m_a}{M_a}\ ,\quad l_b = \frac{m_b}{M_b}\ ,\quad l_c = \frac{m_c}{M_c}\ ,$$

where m_a,m_b, m_c are the lengths of the angle bisectors (internal to the triangle) and M_a, M_b, M_c are the lengths of the angle bisectors extended until they meet the circle. Prove that

$$\frac{l_a}{\sin^2 A} + \frac{l_b}{\sin^2 B} + \frac{l_c}{\sin^2 C} \ge 3$$

and that equality holds iff ABC is an equilateral triangle.

4 Triangle $A_1A_2A_3$ has a right angle at A_3. A sequence of points is now defined by the following iterative process, where n is a positive integer. From A_n ($n \ge 3$), a perpendicular line is drawn to meet $A_{n-2}A_{n-1}$ at A_{n+1}.
(a) Prove that if this process is continued indefinitely, then one and only one point P is interior to every triangle $A_{n-2}A_{n-1}A_n$, $n \ge 3$.
(b) Let A_1 and A_3 be fixed points. By considering all possible locations of A_2 on the plane, find the locus of P.

5 Suppose that n people A_1, A_2, …, A_n, $(n \geq 3)$ are seated in a circle and that A_i has a_i objects such that

$$a_1 + a_2 + \cdots + a_n = nN$$

where N is a positive integer. In order that each person has the same number of objects, each person A_i is to give or to receive a certain number of objects to or from its two neighbours A_{i-1} and A_{i+1}. (Here A_{n+1} means A_1 and A_n means A_0.) How should this redistribution be performed so that the total number of objects transferred is minimum?

APMO 1996

1 Let $ABCD$ be a quadrilateral $AB = BC = CD = DA$. Let MN and PQ be two segments perpendicular to the diagonal BD and such that the distance between them is $d > \frac{BD}{2}$, with $M \in AD$, $N \in DC$, $P \in AB$, and $Q \in BC$. Show that the perimeter of hexagon $AMNCQP$ does not depend on the position of MN and PQ so long as the distance between them remains constant.

2 Let m and n be positive integers such that $n \leq m$. Prove that

$$2^n n! \leq \frac{(m+n)!}{(m-n)!} \leq (m^2+m)^n$$

3 If $ABCD$ is a cyclic quadrilateral, then prove that the incenters of the triangles ABC, BCD, CDA, DAB are the vertices of a rectangle.

4 The National Marriage Council wishes to invite n couples to form 17 discussion groups under the following conditions:

(1) All members of a group must be of the same sex; i.e. they are either all male or all female.

(2) The difference in the size of any two groups is 0 or 1.

(3) All groups have at least 1 member.

(4) Each person must belong to one and only one group.

Find all values of n, $n \leq 1996$, for which this is possible. Justify your answer.

5 Let a, b, c be the lengths of the sides of a triangle. Prove that

$$\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} \leq \sqrt{a} + \sqrt{b} + \sqrt{c}$$

and determine when equality occurs.

APMO 1995

1 Determine all sequences of real numbers $a_1, a_2, \ldots, a_{1995}$ which satisfy:

$$2\sqrt{a_n - (n-1)} \geq a_{n+1} - (n-1), \text{ for } n = 1, 2, \ldots 1994,$$

and

$$2\sqrt{a_{1995} - 1994} \geq a_1 + 1.$$

2 Let $a_1, a_2, \ldots, a_n$ be a sequence of integers with values between 2 and 1995 such that:
(i) Any two of the a_i's are relatively prime,
(ii) Each a_i is either a prime or a product of primes.
Determine the smallest possible values of n to make sure that the sequence will contain a prime number.

3 Let $PQRS$ be a cyclic quadrilateral such that the segments PQ and RS are not parallel. Consider the set of circles through P and Q, and the set of circles through R and S. Determine the set A of points of tangency of circles in these two sets.

4 Let C be a circle with radius R and centre O, and S a fixed point in the interior of C. Let AA' and BB' be perpendicular chords through S. Consider the rectangles $SAMB$, $SBN'A'$, $SA'M'B'$, and $SB'NA$. Find the set of all points M, N', M', and N when A moves around the whole circle.

5 Find the minimum positive integer k such that there exists a function f from the set $\mathbb{Z}$ of all integers to $\{1, 2, \ldots k\}$ with the property that $f(x) \neq f(y)$ whenever $|x - y| \in \{5, 7, 12\}$.