India
Regional Mathematical Olympiad
2012

Region 1

1. Let $ABCD$ be a unit square. Draw a quadrant of the a circle with A as centre and B, D as end points of the arc. Similarly, draw a quadrant of a circle with B as centre and A, C as end points of the arc. Inscribe a circle Γ touching arcs AC and BD both externally and also touching the side CD. Find the radius of Γ.

2. Let a, b, c be positive integers such that $a|b^5, b|c^5$ and $c|a^5$. Prove that $abc|(a+b+c)^{31}$.

3. Let a and b be positive real numbers such that $a+b=1$. Prove that $a^a b^b + a^b b^a \leq 1$.

4. Let $X = \{1, 2, 3, ..., 10\}$. Find the number of pairs of $\{A, B\}$ such that $A \subseteq X, B \subseteq X, A \neq B$ and $A \cap B = \{5, 7, 8\}$.

5. Let ABC be a triangle. Let D, E be points on the segment BC such that $BD = DE = EC$. Let F be the mid-point of AC. Let BF intersect AD in P and AE in Q respectively. Determine the ratio of the area of the triangle APQ to that of the quadrilateral $PDEQ$.

6. Find all positive integers such that $3^{2n} + 3n^2 + 7$ is a perfect square.

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Region 2

$\boxed{1}$ Let $ABCD$ be a convex quadrilateral such that $\angle ADC = \angle BCD > 90^\circ$. Let E be the point of intersection of AC and the line through B parallel to AD; let F be the point of intersection of BD and the line through A parallel to BC. Prove that $EF \parallel CD$.

$\boxed{2}$ Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial of degree $n \geq 3$. Knowing that $a_{n-1} = -\binom{n}{1}$ and $a_{n-2} = \binom{n}{2}$, and that all the roots of P are real, find the remaining coefficients. Note that $\binom{n}{r} = \frac{n!}{(n-r)!r!}$.

$\boxed{3}$ Find all natural numbers x, y, z such that

$$(2^x - 1)(2^y - 1) = 2^{2^z} + 1.$$

$\boxed{4}$ Let a, b, c be positive real numbers such that $abc(a + b + c) = 3$. Prove that we have

$$(a + b)(b + c)(c + a) \geq 8.$$

Also determine the case of equality.

$\boxed{5}$ Let AL and BK be the angle bisectors in a non-isosceles triangle ABC, where L lies on BC and K lies on AC. The perpendicular bisector of BK intersects the line AL at M. Point N lies on the line BK such that LN is parallel to MK. Prove that $LN = NA$.

$\boxed{6}$ A computer program generated 175 positive integers at random, none of which had a prime divisor grater than 10. Prove that there are three numbers among them whose product is the cube of an integer.