Art of Problem Solving

1998 Iran MO (2nd round)

National Math Olympiad (Second Round) 1998

Day 1

1 If $a_1 < a_2 < \cdots < a_n$ be real numbers, prove that:

$$a_1a_2^4 + a_2a_3^4 + \cdots + a_{n-1}a_n^4 + a_na_1^4 \geq a_2a_1^4 + a_3a_2^4 + \cdots + a_na_{n-1}^4 + a_1a_n^4.$$

2 Let ABC be a triangle. I is the incenter of ΔABC and D is the meet point of AI and the circumcircle of ΔABC. Let E, F be on BD, CD, respectively such that IE, IF are perpendicular to BD, CD, respectively. If $IE + IF = \frac{AD}{2}$, find the value of $\angle BAC$.

3 Let n be a positive integer. We call $(a_1, a_2, \cdots, a_n)$ a *good* $n-$tuple if $\sum_{i=1}^{n} a_i = 2n$ and there doesn't exist a set of a_is such that the sum of them is equal to n. Find all *good* $n-$tuple.
(For instance, $(1, 1, 4)$ is a *good* $3-$tuple, but $(1, 2, 1, 2, 4)$ is not a *good* $5-$tuple.)

Day 2

1 Let the positive integer n have at least for positive divisors and $0 < d_1 < d_2 < d_3 < d_4$ be its least positive divisors. Find all positive integers n such that:

$$n = d_1^2 + d_2^2 + d_3^2 + d_4^2.$$

2 Let ABC be a triangle and $AB < AC < BC$. Let D, E be points on the side BC and the line AB, respectively (A is between B, E) such that $BD = BE = AC$. The circumcircle of ΔBED meets the side AC at P and BP meets the circumcircle of ΔABC at Q. Prove that:

$$AQ + CQ = BP.$$

3 If $A = (a_1, \cdots, a_n)$, $B = (b_1, \cdots, b_n)$ be 2 $n-$tuple that $a_i, b_i = 0$ *or* 1 for $i = 1, 2, \cdots, n$, we define $f(A, B)$ the number of $1 \leq i \leq n$ that $a_i \neq b_i$.
For instance, if $A = (0, 1, 1)$, $B = (1, 1, 0)$, then $f(A, B) = 2$.

Now, let $A = (a_1, \cdots, a_n)$, $B = (b_1, \cdots, b_n)$, $C = (c_1, \cdots, c_n)$ be 3 $n-$tuple, such that for $i = 1, 2, \cdots, n$, $a_i, b_i, c_i = 0$ *or* 1 and $f(A, B) = f(A, C) = f(B, C) = d$. $a)$ Prove that d is even. $b)$ Prove that there exists a $n-$tuple $D = (d_1, \cdots, d_n)$ that $d_i = 0$ *or* 1 for $i = 1, 2, \cdots, n$, such that $f(A, D) = f(B, D) = f(C, D) = \frac{d}{2}$.