Tournament Of Towns 2008

Day 1

1. 100 Queens are placed on a 100×100 chessboard so that no two attack each other. Prove that each of four 50×50 corners of the board contains at least one Queen.

2. Each of 4 stones weights the integer number of grams. A balance with arrow indicates the difference of weights on the left and the right sides of it. Is it possible to determine the weights of all stones in 4 weighings, if the balance can make a mistake in 1 gram in at most one weighing?

3. In his triangle ABC Serge made some measurements and informed Ilias about the lengths of median AD and side AC. Based on these data Ilias proved the assertion: angle CAB is obtuse, while angle DAB is acute. Determine a ratio AD/AC and prove Ilias' assertion (for any triangle with such a ratio).

4. Baron Munchausen claims that he got a map of a country that consists of

   five cities. Each two cities are connected by a direct road. Each road intersects no more than one another road (and no more than once). On the map, the roads are colored in yellow or red, and while circling any city (along its border) one can notice that the colors of crossed roads alternate. Can Baron's claim be true?

5. Let $a_1, a_2, \cdots, a_n$ be a sequence of positive numbers, so that $a_1 + a_2 + \cdots + a_n \le \frac{1}{2}$. Prove that
$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) < 2.$$

   [hide="Remark"]Remark. I think this problem was posted before, but I can't find the link now.

6. Let ABC be a non-isosceles triangle. Two isosceles triangles $AB'C$ with base AC and $CA'B$ with base BC are constructed outside of triangle ABC. Both triangles have the same base angle φ. Let C_1 be a point of intersection of the perpendicular from C to $A'B'$ and the perpendicular bisector of the segment AB. Determine the value of $\angle AC_1B$.

7. In an in

   finite sequence $a_1, a_2, a_3, \cdots$, the number a_1 equals 1, and each $a_n, n > 1$, is obtained from a_{n-1} as follows:

   - if the greatest odd divisor of n has residue 1 modulo 4, then $a_n = a_{n-1} + 1$,

   - and if this residue equals 3, then $a_n = a_{n-1} - 1$.

   Prove that in this sequence

   (a) the number 1 occurs infi

nitely many times;

(b) each positive integer occurs infi

nitely many times.

(The initial terms of this sequence are $1, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, \cdots$)

Day 2

1. Each of ten boxes contains a different number of pencils. No two pencils in the same box are of the same colour. Prove that one can choose one pencil from each box so that no two are of the same colour.

2. Twenty-fi

ve of the numbers $1, 2, \cdots, 50$ are chosen. Twenty-

five of the numbers$51, 52, \cdots, 100$ are also chosen. No two chosen numbers differ by 0 or 50. Find the sum of all 50 chosen numbers.

3. Acute triangle $A_1A_2A_3$ is inscribed in a circle of radius 2. Prove that one can choose points B_1, B_2, B_3 on the arcs A_1A_2, A_2A_3, A_3A_1 respectively, such that the numerical value of the area of the hexagon $A_1B_1A_2B_2A_3B_3$ is equal to the numerical value of the perimeter of the triangle $A_1A_2A_3$.

4. Given three distinct positive integers such that one of them is the average of the two others. Can the product of these three integers be the perfect 2008th power of a positive integer?

5. On a straight track are several runners, each running at a different constant speed. They start at one end of the track at the same time. When a runner reaches any end of the track, he immediately turns around and runs back with the same speed (then he reaches the other end and turns back again, and so on). Some time after the start, all runners meet at the same point. Prove that this will happen again.

Day 3

$\boxed{1}$ A square board is divided by lines parallel to the board sides (7 lines in each direction, not necessarily equidistant) into 64 rectangles. Rectangles are colored into white and black in alternating order. Assume that for any pair of white and black rectangles the ratio between area of white rectangle and area of black rectangle does not exceed 2. Determine the maximal ratio between area of white and black part of the board. White (black) part of the board is the total sum of area of all white (black) rectangles.

$\boxed{2}$ Space is dissected into congruent cubes. Is it necessarily true that for each cube there exists another cube so that both cubes have a whole face in common?

$\boxed{3}$ There are N piles each consisting of a single nut. Two players in turns play the following game. At each move, a player combines two piles that contain coprime numbers of nuts into a new pile. A player who can not make a move, loses. For every $N > 2$ defi

ne which of the players, the

first or the second has a winning strategy.

$\boxed{4}$ Let $ABCD$ be a non-isosceles trapezoid. De

fine a point $A1$ as intersection of circumcircle of triangle BCD and line AC. (Choose A_1 distinct from C). Points B_1, C_1, D_1 are de

fined in similar way. Prove that $A_1B_1C_1D_1$ is a trapezoid as well.

$\boxed{5}$ In an in

finite sequence $a_1, a_2, a_3, \cdots$, the number a_1 equals 1, and each $a_n, n > 1$, is obtained from a_{n-1} as follows:

- if the greatest odd divisor of n has residue 1 modulo 4, then $a_n = a_{n-1} + 1$,

- and if this residue equals 3, then $a_n = a_{n-1} - 1$.

Prove that in this sequence

(a) the number 1 occurs infi

nitely many times;

(b) each positive integer occurs infi

nitely many times.

(The initial terms of this sequence are $1, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, \cdots$)

6 Let $P(x)$ be a polynomial with real coefficients so that equation $P(m) + P(n) = 0$ has infi nitely many pairs of integer solutions (m, n). Prove that graph of $y = P(x)$ has a center of symmetry.

7 A test consists of 30 true or false questions. After the test (answering all 30 questions), Victor gets his score: the number of correct answers. Victor is allowed to take the test (the same questions) several times. Can Victor work out a strategy that insure him to get a perfect score after

(a) 30th attempt?

(b) 25th attempt?

(Initially, Victor does not know any answer)

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Day 4

1. Alex distributes some cookies into several boxes and records the number of cookies in each box. If the same number appears more than once, it is recorded only once. Serge takes one cookie from each box and puts them on the

   first plate. Then he takes one cookie from each box that is still non-empty and puts the cookies on the second plate. He continues until all the boxes are empty. Then Serge records the number of cookies on each plate. Again, if the same number appears more than once, it is recorded only once. Prove that Alex's record contains the same number of numbers as Serge's record.

2. Solve the system of equations ($n > 2$)

$$\sqrt{x_1} + \sqrt{x_2 + x_3 + \cdots + x_n} = \sqrt{x_2} + \sqrt{x_3 + x_4 + \cdots + x_n + x_1} = \cdots = \sqrt{x_n} + \sqrt{x_1 + x_2 + \cdots + x_{n-1}} \ ,$$

$$x_1 - x_2 = 1.$$

3. A 30-gon $A_1A_2 \cdots A_{30}$ is inscribed in a circle of radius 2. Prove that one can choose a point B_k on the arc A_kA_{k+1} for $1 \le k \le 29$ and a point B_{30} on the arc $A_{30}A_1$, such that the numerical value of the area of the 60-gon $A_1B_1A_2B_2 \ldots A_{30}B_{30}$ is equal to the numerical value of the perimeter of the original 30-gon.

4. Five distinct positive integers form an arithmetic progression. Can their product be equal to a^{2008} for some positive integer a ?

5. On the in

   finite chessboard several rectangular pieces are placed whose sides run along the grid lines. Each two have no squares in common, and each consists of an odd number of squares. Prove that these pieces can be painted in four colours such that two pieces painted in the same colour do not share any boundary points.