APMO 1999

1 Find the smallest positive integer n with the following property: there does not exist an arithmetic progression of 1999 real numbers containing exactly n integers.

2 Let $a_1, a_2, \ldots$ be a sequence of real numbers satisfying $a_{i+j} \leq a_i + a_j$ for all $i, j = 1, 2, \ldots$. Prove that

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} \geq a_n$$

for each positive integer n.

3 Let Γ_1 and Γ_2 be two circles intersecting at P and Q. The common tangent, closer to P, of Γ_1 and Γ_2 touches Γ_1 at A and Γ_2 at B. The tangent of Γ_1 at P meets Γ_2 at C, which is different from P, and the extension of AP meets BC at R.
Prove that the circumcircle of triangle PQR is tangent to BP and BR.

4 Determine all pairs (a, b) of integers with the property that the numbers $a^2 + 4b$ and $b^2 + 4a$ are both perfect squares.

5 Let S be a set of $2n + 1$ points in the plane such that no three are collinear and no four concyclic. A circle will be called Good if it has 3 points of S on its circumference, $n - 1$ points in its interior and $n - 1$ points in its exterior.
Prove that the number of good circles has the same parity as n.