USA Team Selection Test 2010

Day 1

1 Let P be a polynomial with integer coefficients such that $P(0) = 0$ and

$$\gcd(P(0), P(1), P(2), \ldots) = 1.$$

Show there are infinitely many n such that

$$\gcd(P(n) - P(0), P(n+1) - P(1), P(n+2) - P(2), \ldots) = n.$$

2 Let a, b, c be positive reals such that $abc = 1$. Show that

$$\frac{1}{a^5(b+2c)^2} + \frac{1}{b^5(c+2a)^2} + \frac{1}{c^5(a+2b)^2} \geq \frac{1}{3}.$$

3 Let h_a, h_b, h_c be the lengths of the altitudes of a triangle ABC from A, B, C respectively. Let P be any point inside the triangle. Show that

$$\frac{PA}{h_b + h_c} + \frac{PB}{h_a + h_c} + \frac{PC}{h_a + h_b} \geq 1.$$

Day 2

4 Let ABC be a triangle. Point M and N lie on sides AC and BC respectively such that $MN || AB$. Points P and Q lie on sides AB and CB respectively such that $PQ || AC$. The incircle of triangle CMN touches segment AC at E. The incircle of triangle BPQ touches segment AB at F. Line EN and AB meet at R, and lines FQ and AC meet at S. Given that $AE = AF$, prove that the incenter of triangle AEF lies on the incircle of triangle ARS.

5 Define the sequence $a_1, a_2, a_3, \ldots$ by $a_1 = 1$ and, for $n > 1$,

$$a_n = a_{\lfloor n/2 \rfloor} + a_{\lfloor n/3 \rfloor} + \ldots + a_{\lfloor n/n \rfloor} + 1.$$

Prove that there are infinitely many n such that $a_n \equiv n \pmod{2^{2010}}$.

6 Let T be a finite set of positive integers greater than 1. A subset S of T is called *good* if for every $t \in T$ there exists some $s \in S$ with $\gcd(s,t) > 1$. Prove that the number of good subsets of T is odd.

Day 3

7 In triangle ABC, let P and Q be two interior points such that $\angle ABP = \angle QBC$ and $\angle ACP = \angle QCB$. Point D lies on segment BC. Prove that $\angle APB + \angle DPC = 180^\circ$ if and only if $\angle AQC + \angle DQB = 180^\circ$.

8 Let m, n be positive integers with $m \geq n$, and let S be the set of all n-term sequences of positive integers $(a_1, a_2, \ldots a_n)$ such that $a_1 + a_2 + \cdots + a_n = m$. Show that

$$\sum_S 1^{a_1} 2^{a_2} \cdots n^{a_n} = \binom{n}{n} n^m - \binom{n}{n-1} (n-1)^m + \cdots + (-1)^{n-2} \binom{n}{2} 2^m + (-1)^{n-1} \binom{n}{1}.$$

9 Determine whether or not there exists a positive integer k such that $p = 6k+1$ is a prime and

$$\binom{3k}{k} \equiv 1 \pmod{p}.$$

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