Mathematical Olympiads 2001–2002

Problems and Solutions From Around the World

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2001 National Contests: Problems and Solutions

1.1 Belarus

Problem 1 The problem committee of a mathematical olympiad prepares some variants of the contest. Each variant contains 4 problems, chosen from a shortlist of n problems, and any two variants have at most one problem in common.

(a) If $n = 14$, determine the largest possible number of variants the problem committee can prepare.

(b) Find the smallest value of n such that it is possible to prepare ten variants of the contest.

Solution:

(a) The problem committee can prepare 14 variants, and no more.

We prove that given a shortlist of n problems, the committee prepares at most $\lfloor \frac{n-1}{3} \rfloor \cdot \frac{n}{4}$ variants. Consider any one of the n problems, and suppose that k variants contain that problem. The other $3k$ problems in these variants are distinct from each other and from the chosen problem, implying that $3k \leq n - 1$ and $k \leq \lfloor \frac{n-1}{3} \rfloor$. Now, summing the number of variants containing each problem over the n possible problems, we obtain a maximum count of $\lfloor \frac{n-1}{3} \rfloor \cdot n$ problems in all the variants combined. Because each variant has 4 problems, there are at most $\lfloor \frac{n-1}{3} \rfloor \cdot \frac{n}{4}$ variants.

In particular, when $n = 14$, the problem committee can prepare at most $\lfloor \frac{14}{3} \rfloor \cdot \frac{14}{4} = 14$ variants. We now show that this is indeed possible. Label the problems $1, \ldots, 14$, with labels taken modulo 14. Then consider the following fourteen variants for $t = 0, 1, \ldots, 13$:

$$\{1 + t, 2 + t, 5 + t, 7 + t\}.$$

Take any pair A, B of distinct problems. It suffices to show that the pair A, B appears in at most one variant; i.e., that there is at most one way to write $(A, B) \equiv (a + t, b + t) \pmod{14}$ with $a, b \in \{1, 2, 5, 7\}$ and $0 \leq t \leq 13$.

Consider the 12 pairs (a, b) with $a, b \in \{1, 2, 5, 7\}$ and $a \neq b$. The differences $b - a$ take on 12 distinct values $\pm 1, \pm 2, \ldots, \pm 6$ over these 12 pairs. Thus, there is at most one pair of values $a, b \in \{1, 2, 5, 7\}$ with $b - a \equiv B - A \pmod{14}$. With this pair, there is at most one value $t \in \{0, 1, \ldots, 13\}$ with $A \equiv a + t \pmod{14}$.

This completes the proof.

(b) Using the result in part (a), if $n \le 12$, then there are at most $\lfloor \frac{n-1}{3} \rfloor \cdot \frac{n}{4} \le \lfloor \frac{11}{3} \rfloor \cdot \frac{12}{4} = 9$ variants. Hence, $n \ge 13$.

Indeed, $n = 13$ problems suffice: take the 14 variants described in part (a) and remove the four variants that contain problem 14. We are left with 10 variants, as required.

Problem 2 Let x_1, x_2, and x_3 be real numbers in $[-1, 1]$, and let y_1, y_2, and y_3 be real numbers in $[0, 1)$. Find the maximum possible value of the expression

$$\frac{1-x_1}{1-x_2y_3} \cdot \frac{1-x_2}{1-x_3y_1} \cdot \frac{1-x_3}{1-x_1y_2}.$$

Solution: The maximum possible value of the expression is 8. We first rewrite the expression as follows:

$$\frac{1-x_1}{1-x_1y_2} \cdot \frac{1-x_2}{1-x_2y_3} \cdot \frac{1-x_3}{1-x_3y_1}. \qquad (*)$$

Under the given restraints, the three numerators are nonnegative and the three denominators are positive. Thus, the three fractions in the above product are nonnegative.

By the given inequalities,

$$x_1(2y_2 - 1) \le |x_1||2y_2 - 1| \le 1,$$

or

$$1 - x_1 \le 2(1 - x_1y_2).$$

Dividing by $1 - x_1y_2$ (which is positive under the given restraints), we find that $\frac{1-x_1}{1-x_1y_2} \le 2$. Applying similar reasoning shows that all three fractions in $(*)$ are at most 2.

Therefore, the three fractions in $(*)$ are between 0 and 2, implying that their product is at most 8. When $x_1 = x_2 = x_3 = -1$ and $y_1 = y_2 = y_3 = 0$, this bound is attained.

Problem 3 Let $ABCD$ be a convex quadrilateral circumscribed about a circle. Lines AB and DC intersect at E, and B and C lie on $\overline{AE}$ and $\overline{DE}$, respectively; lines DA and CB intersect at F, and A and B lie on $\overline{DF}$ and $\overline{CF}$, respectively. Let I_1, I_2, and I_3 be the incenters of triangles AFB, BEC, and ABC, respectively. Line I_1I_3 intersects lines EA and ED at K and L, respectively, and line I_2I_3

intersects lines FC and FD at M and N, respectively. Prove that $EK = EL$ if and only if $FM = FN$.

Solution: Let I be the incenter of quadrilateral $ABCD$.

Observe that $EK = EL$ if and only if line KL is perpendicular to the internal angle bisector of angle AED. Line KL is the same as line I_1I_3, and the internal angle bisector of angle AED is line II_2. Thus, $EK = EL$ if and only if $\overline{I_1I_3} \perp \overline{II_2}$. Likewise, $FM = FN$ if and only if $\overline{I_2I_3} \perp \overline{II_1}$. Hence, it suffices to show that $\overline{I_1I_3} \perp \overline{II_2}$ if and only if $\overline{I_2I_3} \perp \overline{II_1}$.

Observe that lines II_3 and I_1I_2 are the angle bisectors of the pair of vertical angles formed at B. Hence, $\overline{II_3} \perp \overline{I_1I_2}$. Thus, if $\overline{I_1I_3} \perp \overline{II_2}$, then I_3 is the orthocenter of triangle II_1I_2, implying that $\overline{I_2I_3} \perp \overline{II_1}$. Likewise, if $\overline{I_2I_3} \perp \overline{II_1}$, then $\overline{I_1I_3} \perp \overline{II_2}$.

Problem 4 On the Cartesian coordinate plane, the graph of the parabola $y = x^2$ is drawn. Three distinct points A, B, and C are marked on the graph with A lying between B and C. Point N is marked on $\overline{BC}$ so that $\overline{AN}$ is parallel to the y-axis. Let K_1 and K_2 be the areas of triangles ABN and ACN, respectively. Express AN in terms of K_1 and K_2.

Solution: We will show that $AN = \sqrt[3]{4K_1K_2}$. Let $A = (a, a^2)$, $B = (b, b^2)$, and $C = (c, c^2)$. Without loss of generality, assume that $b < c$.

It is easy to verify that the point

$$(a, (a-b)(b+c) + b^2) = (a, ab + ca - bc) = (a, (a-c)(b+c) + c^2)$$

is on $\overline{BC}$, implying that this point is N. Thus,

$$AN = ab + ca - bc - a^2 = (a-b)(c-a).$$

Also, we have $K_1 = \frac{1}{2}AN(a-b)$ and $K_2 = \frac{1}{2}AN(c-a)$. Thus, $a - b = 2K_1/AN$ and $c - a = 2K_2/AN$. Combining this with our above result gives

$$AN = (a-b)(c-a) = \frac{2K_1}{AN} \cdot \frac{2K_2}{AN},$$

or $AN = \sqrt[3]{4K_1K_2}$.

Problem 5 Prove that for every positive integer n and every positive real a,

$$a^n + \frac{1}{a^n} - 2 \geq n^2 \left(a + \frac{1}{a} - 2 \right).$$

Solution: By the AM-GM inequality, we know that $x^{n-k} + \frac{1}{x^{n-k}} \geq 2$ when $0 < k < n$ and $x > 0$. If n is even, we sum the inequalities for $k = 1, 3, \ldots, n-1$ to obtain

$$x^{n-1} + x^{n-3} + \cdots + \frac{1}{x^{n-1}} \geq \frac{n}{2} \cdot 2 = n.$$

When n is odd, we add one to the sum of the inequalities for $k = 1, 3, \ldots, n-2$ to obtain

$$x^{n-1} + x^{n-3} + \cdots + x^2 + 1 + \frac{1}{x^2} + \cdots + \frac{1}{x^{n-1}} \geq \frac{n-1}{2} \cdot 2 + 1 = n.$$

In either case, we have

$$\frac{x^n - 1/x^n}{x - 1/x} = x^{n-1} + x^{n-3} + \cdots + \frac{1}{x^{n-1}} \geq n.$$

In particular, setting $x = a^{1/2}$ yields

$$\frac{a^{n/2} - 1/a^{n/2}}{a^{1/2} - 1/a^{1/2}} \geq n.$$

Squaring both sides and rearranging, we obtain

$$\left(a^{n/2} - \frac{1}{a^{n/2}} \right)^2 \geq n^2 \left(a^{1/2} - \frac{1}{a^{1/2}} \right)^2,$$

which, when expanded, is exactly the desired inequality.

Problem 6 Three distinct points A, B, and N are marked on the line ℓ, with B lying between A and N. For an arbitrary angle $\alpha \in (0, \frac{\pi}{2})$, points C and D are marked in the plane on the same side of ℓ such that N, C, and D are collinear; $\angle NAD = \angle NBC = \alpha$; and A, B, C, and D are concyclic. Find the locus of the intersection points of the diagonals of $ABCD$ as α varies between 0 and $\frac{\pi}{2}$.

Solution: Let R be the point between A and B satisfying $AR/RB = AN/NB$. The locus is the circle ω with diameter $\overline{NR}$, with N and R removed.

We first show that given points C, D satisfying the conditions, the intersection P of the diagonals of quadrilateral $ABCD$ lies on ω.

Because quadrilateral $ABCD$ is a trapezoid (with $\overline{AD} \parallel \overline{BC}$) and cyclic, it is an isosceles trapezoid symmetric about the perpendicular bisector of $\overline{AD}$ and $\overline{BC}$. By symmetry, N and P lie on this line, and line NP is the internal angle bisector of angle BPC (and the external angle bisector of angle BPA).

Draw the line through P parallel to $\overline{BC}$ and perpendicular to $\overline{PN}$, and let it intersect $\overline{AB}$ at R'. Because line PR' is perpendicular to line PN (the external angle bisector of angle BPA), line PR' must be the *internal* angle bisector of angle BPA. By the Internal and External Angle Bisector Theorems, we have

$$\frac{AR'}{R'B} = \frac{AP}{PB} = \frac{AN}{NB},$$

implying that $R = R'$. Because $\angle NPR = \angle NPR' = \pi/2$, P lies on the circle with diameter $\overline{NR}$, as claimed.

(Alternatively, it is easy to show that line PR' is the image of N under the polar transformation through circle $ABCD$. Hence, A, R', B, N are a harmonic range and $AR'/R'B = AN/NB$.)

It remains to show that any point P on $\omega \setminus \{N, R\}$ is in the locus. This is simple: given such a P, reflect $\overline{AB}$ across line NP to form segment $\overline{CD}$; C, D satisfy the required conditions. Let P' be the intersection of diagonals $\overline{AC}$ and $\overline{BD}$. Then P, P' lie on the same line through N, and $\angle NPR = \angle NP'R = \pi/2$, implying that $P = P'$.

Problem 7 In the increasing sequence of positive integers a_1, a_2, $\ldots$, the number a_k is said to be *funny* if it can be represented as the sum of some other terms (not necessarily distinct) of the sequence.

(a) Prove that all but finitely terms of the sequence are funny.

(b) Does the result in (a) always hold if the terms of the sequence can be any positive rational numbers?

Solution:

(a) Without loss of generality, 1 is the greatest integer that divides a_k for all k. (Otherwise, if $d > 1$ divides every term of the sequence, then dividing each term by d does not change the problem.)

Let $A > 1$ be some term in the sequence, and let $p_1, \ldots, p_n$ be the primes that divide A. For each $k = 1, 2, \ldots, n$, we can find a term A_k in the sequence such that $p_k \nmid A_k$. We claim that all but finitely terms in the sequence can be written as the sum of various terms (with repetition allowed) in $\{A, A_1, A_2, \ldots, A_n\}$.

Let B equal the following sum of n values:

$$B = \frac{\prod_{k=1}^{n} p_k}{p_1} A_1 + \frac{\prod_{k=1}^{n} p_k}{p_2} A_2 + \cdots + \frac{\prod_{i=1}^{n} p_k}{p_n} A_n.$$

Observe that $p_1 \not| B$, because p_1 divides all n summands except the first. Likewise, $p_k \not| B$ for $k = 2, 3, \ldots, n$. Therefore, A and B are relatively prime.

Let C be any integer greater than $2AB$. There exist integers $0 \le x < A$, $0 \le y < B$ such that $xB \equiv C \pmod{A}$ and $yA \equiv C \pmod{B}$. Then $xB + yA \equiv C \pmod{\operatorname{lcm}(A, B)}$, implying that $C = xB + yA + zAB$ for some integer z. Because $C > 2AB > xB + yA$, z is positive. Thus, $C = (x + zA)B + yA$ is a linear combination of A and B with non-negative integer coefficients.

All but finitely may of the a_k satisfy $a_k > 2AB$. From above, any such a_k can be written as a linear combination of A and B with non-negative integer coefficients. Because each of A and B is the sum of various terms (with repetition allowed) in $\{A, A_1, A_2, \ldots, A_n\}$, it follows that any a_k greater than $2AB$ can also be expressed as a sum of this form.

(b) The result does not hold if we allow the a_k to be rational numbers. One counterexample is the sequence $a_k = \frac{k}{k+1}$. The sequence is clearly increasing, with each term in $[1/2, 1)$. Thus, the sum of two or more terms of the sequence is always at least 1, so no terms of the sequence are funny.

Problem 8 Let n be a positive integer. Each square of a $(2n-1) \times (2n-1)$ square board contains an arrow, either pointing up, down, to the left, or to the right. A beetle sits in one of the cells. Each year it creeps from one square in the direction of the arrow in that square, either reaching another square or leaving the board. Each time the beetle moves, the arrow in the square it leaves turns $\pi/2$ clockwise. Prove that the beetle leaves the board in at most $2^{3n-1}(n-1)! - 4$ years after it first moves.

Solution: In other words, we wish to prove that the beetle leaves a $(2n-1) \times (2n-1)$ board after at most $2^{3n-1}(n-1)! - 3$ moves. We prove this by induction on n. The base case $n = 1$ is trivial — the beetle is guaranteed to leave the board on its first move.

Now assume that we have proved the claim for $n = k$; we show that the statement is true for the case $n = k + 1$. Call the squares on the edge of the $(2k+1) \times (2k+1)$ board *boundary squares* and call the rest of the squares *interior squares.*

Observe that the beetle cannot visit the same boundary square twice and encounter an arrow pointing in the same direction both times. Otherwise, between these two visits, the arrow must turn through at least one full rotation. Then at some point the arrow pointed off the board and the beetle should have left, a contradiction.

Thus, the beetle moves at most:

(i) once from any non-corner boundary square to an interior square,

(ii) three times from any non-corner boundary square to another square on the board, and

(iii) twice from any corner square to another square on the board.

From (i), the beetle can move from a boundary square to an interior square at most $4(2k-1) = 8k-4$ times, the number of boundary squares that are adjacent to interior squares. Adding 1 for the case in which the beetle started out in the interior, we see that the beetle stays consecutively on interior squares for at most $8k-3 < 8k$ periods. By the induction hypothesis, each period lasts at most $2^{3k-1}(k-1)!-3$ moves. Thus, the beetle makes fewer than $2^{3(k+1)-1}k! - 24k$ or

$$[2^{3(k+1)-1}k! - 3] - (24k-3)$$

moves from interior squares.

From (ii) and (iii), the beetle makes at most $2 \cdot 4 + 3 \cdot 4(2k-1) = 24k-4$ moves from boundary squares. Thus, in total the beetle makes fewer than

$$2^{3(k+1)-1}k! - 4$$

times. This completes the inductive step and the proof.

Problem 9 The convex quadrilateral $ABCD$ is inscribed in the circle S_1. Let O be the intersection of $\overline{AC}$ and $\overline{BD}$. Circle S_2 passes through D and O, intersecting $\overline{AD}$ and $\overline{CD}$ at M and N, respectively. Lines OM and AB intersect at R, lines ON and BC intersect at T, and R and T lie on the same side of line BD as A. Prove that O, R, T, and B are concyclic.

Solution: Because quadrilateral $ABCD$ is cyclic in circle S_1, we have $\angle TBR = \angle CDA$. Furthermore, quadrilateral $MOND$ is cyclic in S_2, implying that $\angle CDA = \angle TOR$. Hence, $\angle TBR = \angle TOR$. Because T and R lie on the same side of line OB, it follows that O, R, T, and B are concyclic.

Problem 10 There are n aborigines on an island. Any two of them are either friends or enemies. One day, the chieftain orders that all citizens (including himself) make and wear a necklace with zero or more stones so that (i) given a pair of friends, there exists a color such that each has a stone of that color; (ii) given a pair of enemies, there does *not* exist a color such that each has a stone of that color.

(a) Prove that the aborigines can carry out the chieftain's order.

(b) What is the minimum number of colors of stones required for the aborigines to carry out the chieftain's order?

Solution:

(a) Assign to each pair of friends a distinct color, and have each member of the pair add a stone of that color to his necklace. This arrangement clearly satisfies both required conditions.

(b) The minimum number of colors required in the worst case is $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$. We introduce graph-theoretic notation: define a graph on n vertices, each vertex corresponding to a different aborigine, so that an edge exists between a pair of vertices if and only if the corresponding aborigines are friends.

   First we show that in one scenario, at least $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ colors are required. Suppose the aborigines form a bipartite graph with parts of size $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$ and an edge between each pair of vertices from opposite parts. In this case, no color can be shared by more than two aborigines, because there does not exist a triple of mutual friends (i.e., the graph does not contain a triangle). It follows that each of the $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ pairs of friends requires a distinct color.

   We are left with proving that no case requires more than the above number of colors. We show by induction on n that no graph with n vertices requires more than $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ colors. The base cases $n = 1, 2, 3$ are easily checked. Now suppose that $k \geq 4$ and that the claim is true for $n = k-3$. We prove that it holds for $n = k$.

If the graph has no triangles, let m be the maximum degree of the vertices. Let V be a vertex adjacent to $V_1, V_2, \dots V_m$. The latter m vertices cannot be adjacent to each other, so each has degree at most $k-m$ and their degrees sum to at most $m(k-m)$. The other $k-m$ vertices each has degree at most m, so their degrees sum to at most $m(k-m)$. The number of edges equals half the sum of all the degrees, or at most $m(k-m) \le \lfloor \frac{k}{2} \rfloor \lceil \frac{k}{2} \rceil$. Thus, we may assign each pair of friends a distinct color of beads to use at most $\lfloor \frac{k}{2} \rfloor \lceil \frac{k}{2} \rceil$ beads.

If the graph does have a triangle, then we need only the following colors of beads:

- one color shared by the three aborigines in the triangle;
- $k-3$ colors, one for each aborigine not in the triangle, to use whenever that aborigine is friends with an aborigine in the triangle; and
- $\lfloor \frac{k-3}{2} \rfloor \lceil \frac{k-3}{2} \rceil$ colors for the friendships among aborigines outside the triangle (this is possible by the induction hypothesis).

Therefore, we need at most $\lfloor \frac{k-3}{2} \rfloor \lceil \frac{k-3}{2} \rceil + k - 2 < \lfloor \frac{k}{2} \rfloor \lceil \frac{k}{2} \rceil$ colors in total, completing the induction.

1.2 Bulgaria

Problem 1 Diagonals $\overline{AC}$ and $\overline{BD}$ of a cyclic quadrilateral $ABCD$ intersect at point E. Prove that if $\angle BAD = \pi/3$ and $AE = 3CE$, then the sum of some two sides of the quadrilateral equals the sum of the other two.

Solution: Because quadrilateral $ABCD$ is cyclic, angles BAD and BCD are supplementary. Applying the Law of Cosines to triangles ABD and CBD gives

$$AB^2 + AD^2 - BD^2 = 2AB \cdot AD \cos \angle BAD = AB \cdot AD$$

and

$$CB^2 + CD^2 - BD^2 = 2CB \cdot CD \cos \angle BCD = -CB \cdot CD.$$

We also have

$$\frac{AB \cdot AD}{CB \cdot CD} = \frac{1/2 \cdot AD \cdot AD \sin \angle BAD}{1/2 \cdot CB \cdot CD \sin \angle DCB} = \frac{[ABD]}{[BCD]} = \frac{AE}{EC} = 3.$$

Therefore,

$$\begin{aligned}(AB - DA)^2 &= AB^2 + AD^2 - 2AB \cdot AD \\ &= BD^2 - AB \cdot AD = BD^2 - 3CB \cdot CD \\ &= CB^2 + CD^2 - 2CB \cdot CD = (BC - CD)^2,\end{aligned}$$

implying the desired result.

Problem 2 Find the least positive integer n such that it is possible for a set of n people to have the following properties: (i) among any four of the n people, some two are not friends with each other; (ii) given any $k \geq 1$ of the n people among whom there is no pair of friends, there exists three people among the remaining $n - k$ people such that every two of the three are friends. (If a person A is a friend of a person B, then B is a friend of A as well.)

Solution: The answer is $n = 7$.

A situation in which 7 people are standing in a circle and two people are friends if they are neighbors or if there is only one person between them demonstrates that $n = 7$ is possible.

Call a group of three people of which every two are friends a *triangle*.

It is not difficult to see that n cannot be 4 or less.

Suppose for the purpose of contradiction that there is some scenario, satisfying (i) and (ii), in which $n = 5$. Consider, then, the person A with the most friends. If A has four friends, then by (ii) some triangle must exist among the four people other than A, and (i) is thus violated. If A has three friends, then by (ii) some triangle must exist among the three people other than A and the person who is not one of A's friends, and (i) is again violated. It is not difficult to see that A cannot have only two friends or fewer. Therefore, n cannot be 5.

Now suppose for the purpose of contradiction that there is some scenario, satisfying (i) and (ii), in which $n = 6$. Consider, then, the person A with the most friends. The possibility of A having five or four friends is ruled out in a way similar to the reasoning presented in the case $n = 5$. Suppose that A has three friends, B, C, and D, and that A is not a friend of either E or F. Then (i) implies that some two of the friends of A are not friends with each other. Assume without loss of generality that B, C are such. Then, by (ii), there exists a triangle among A, D, E, F, and this triangle can only be D, E, F. Then D is friends with A, E, F, and, because A was assumed to be the one with the most friends, D cannot be friends with either B or C. Then (ii) implies that A, E, F is a triangle, which is a contradiction. Thus, A cannot have three friends. It is not difficult to see that A cannot have only two or fewer friends, either. Therefore, n cannot be 6.

Problem 3 Let ABC be a right triangle with hypotenuse $\overline{AB}$. A point D distinct from A and C is chosen on ray $\overrightarrow{AC}$ such that the line through the incenter of triangle ABC parallel to the internal bisector of angle ADB is tangent to the incircle of triangle BCD. Prove that $AD = BD$.

Solution: Let the incircle ω of triangle BCD have center J and radius ρ. Let lines g, γ be the internal and external angle bisectors of $\angle BCD$, respectively. Let lines τ_B, τ_C be the tangents to ω parallel to the internal bisector of angle BDC closer to B and to C, respectively. Let lines t_B, t_D be the tangents to ω parallel to the external bisector of angle BDC closer to B and to D, respectively.

Let I be the incenter of triangle ABC. Then we distinguish between four cases:

1. *D lies on segment AC and I is the intersection between g and t_B.* Actually, this is not possible because the intersection between g and t_B is on the opposite side of line BC as point D, but A is supposed to be on the same side of line BC as point D.

2. *D lies on segment AC and I is X, the intersection between g and t_C.* This is not possible. Note that the angle ϕ between t_C and g is equal to the angle between lines JB and CB. Therefore, $JX = \rho \csc \phi = JB$, and $\angle CXB = \angle XBJ < 90^\circ$, so X is too far away from C to be the incenter of any triangle with B and C as two of its vertices.

3. *D does not lie on segment AC and I is Ξ, the intersection between γ and τ_B.* This is not possible. Let K be the intersection of lines DJ and γ. Note that $CK > CJ$ because in triangle CJK, angle C is right and angle J is equal to $\angle JCD + \angle CDJ$, which is greater than $\pi/4$. On the other hand, $K\Xi = JX$ because the angle between γ and τ_B is equal to the angle between g and t_D. Therefore, $C\Xi > CX$, and as X is too far away from C to be the incenter of any triangle with B and C as two of its vertices, Ξ is also too far.

4. *D does not lie on segment AC and I is the intersection between γ and τ_D.* This is possible. Let A' be the point on ray DC such that $DA' = DB$. Let the incircle of triangle $A'BC$ have center I' and touch $A'B$ at P.

   Let $b = CD$, $c = BD$, $d = BC$, and $e = A'B$. Note that $A'C = c - b$. Say the directed distance from A' to line DJ is $+e/2$. Then the directed distance from B to that line is $-e/2$. Now $2A'P = e+c-b-d$, and $2PB = e-c+b+d$, so the directed distance from I' to line DJ, which is equal to the directed distance from P to line DJ, is

   $$\frac{PB \cdot +e/2 + A'P \cdot -e/2}{e} = \frac{-c+b+d}{2} = +\rho,$$

   so I' lies on τ_C as well as γ, so $I' = I$, and $A' = A$, and $DA = DB$, as desired.

Problem 4 Find all triples of positive integers (a, b, c) such that $a^3 + b^3 + c^3$ is divisible by a^2b, b^2c, and c^2a.

Solution: Answer: triples of the form (k, k, k) or $(k, 2k, 3k)$ or permutations thereof.

Let g be the positive greatest common divisor of a and b. Then g^3 divides a^2b, so g^3 divides $a^3 + b^3 + c^3$, and g divides c. Thus, the gcd of any two of a, b, c is the gcd of all three.

Let $(l, m, n) = (a/g, b/g, c/g)$. Then (l, m, n) is a triple satisfying the conditions of the problem, and l, m, n are pairwise relatively prime. Because l^2, m^2, and n^2 all divide $l^3 + m^3 + n^3$, we have

$$l^2m^2n^2 \mid (l^3 + m^3 + n^3).$$

We will prove that (l, m, n) is either $(1, 1, 1)$ or a permutation of $(1, 2, 3)$.

Assume without loss of generality that $l \geq m \geq n$. Because a positive integer is at least as great as any of its divisors, we have

$$3l^3 \geq l^3 + m^3 + n^3 \geq l^2m^2n^2,$$

and, therefore, $l \geq m^2n^2/3$. Because $l^2 \mid (m^3 + n^3)$, we also have

$$2m^3 \geq m^3 + n^3 \geq l^2 \geq m^4n^4/9.$$

If $n \geq 2$, then $m \leq 2 \cdot 9/2^4 < 2 \leq n$, which contradicts the assumption that $m \geq n$. Therefore, n must be 1. It is not difficult to see that $(1, 1, 1)$ is the unique solution with $m = 1$.

If $m \geq 2$, then $l > m$ because l and m are relatively prime, so

$$2l^3 > l^3 + m^3 + 1 \geq l^2m^2,$$

and $l > m^2/2$, so

$$m^3 + 1 \geq l^2 > m^4/4,$$

and $m \leq 4$. It is not difficult to check that the only solution here is $(3, 2, 1)$.

Problem 5 Consider the sequence $\{a_n\}$ such that $a_0 = 4$, $a_1 = 22$, and $a_n - 6a_{n-1} + a_{n-2} = 0$ for $n \geq 2$. Prove that there exist sequences $\{x_n\}$ and $\{y_n\}$ of positive integers such that

$$a_n = \frac{y_n^2 + 7}{x_n - y_n}$$

for any $n \geq 0$.

Solution: Consider the sequence $\{c_n\}$ of positive integers such that $c_0 = 2$, $c_1 = 1$, and $c_n = 2c_{n-1} + c_{n-2}$ for $n \geq 2$.

We prove by induction that $a_n = c_{2n+2}$ for $n \geq 0$. We check the base cases of $a_0 = 4 = c_2$ and $a_1 = 9 = c_4$. Then, for any $k \geq 2$, assuming the claim holds for $n = k-2$ and $n = k-1$,

$$\begin{aligned} c_{2k+2} &= 2c_{2k+1} + c_{2k} \\ &= 2(2c_{2k} + c_{2k-1}) + a_{k-1} \\ &= 4c_{2k} + (c_{2k} - c_{2k-2}) + a_{k-1} \\ &= 6a_{k-1} - a_{k-2} \\ &= a_k, \end{aligned}$$

so the claim holds for $n = k$ as well, and induction is complete.

For $n \geq 1$,

$$\begin{pmatrix} a_{n+1} & a_n \\ a_{n+2} & a_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a_n & a_{n-1} \\ a_{n+1} & a_n \end{pmatrix},$$

and

$$\begin{vmatrix} a_{n+1} & a_n \\ a_{n+2} & a_{n+1} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} \begin{vmatrix} a_n & a_{n-1} \\ a_{n+1} & a_n \end{vmatrix} = - \begin{vmatrix} a_n & a_{n-1} \\ a_{n+1} & a_n \end{vmatrix}$$

Thus, for $n \geq 0$,

$$c_{n+1}^2 - c_n c_{n+2} = (-1)^n (c_1^2 - c_0 c_2) = (-1)^n (1^2 - 2 \cdot 4) = (-1)^n \cdot -7.$$

In particular, for all $n \geq 0$,

$$c_{2n+1}^2 - c_{2n} a_n = c_{2n+1}^2 - c_{2n} c_{2n+2} = (-1)^{2n} \cdot -7 = -7,$$

and

$$a_n = \frac{c_{2n+1}^2 + 7}{c_{2n}}.$$

We may therefore take $y_n = c_{2n+1}$ and $x_n = c_{2n} + y_n$.

Problem 6 Let I be the incenter and k be the incircle of nonisosceles triangle ABC. Let k intersect $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at A_1, B_1, and C_1, respectively. Let $\overline{AA_1}$ intersect k again at A_2, and define B_2 and C_2 similarly. Finally, choose A_3 and B_3 on $\overline{B_1C_1}$ and $\overline{A_1C_1}$, respectively, such that $\overline{A_1A_3}$ and $\overline{B_1B_3}$ are angle bisectors in triangle $A_1B_1C_1$. Prove that (a) $\overline{A_2A_3}$ bisects angle $B_1A_2C_1$; (b) if the circumcircles of triangles $A_1A_2A_3$ and $B_1B_2B_3$ intersect at P and Q, then I lies on line $\overleftrightarrow{PQ}$.

Solution: Because triangles AB_1A_2 and AA_1B_1 are similar, we have

$$\frac{A_2B_1}{A_1B_1} = \frac{AA_1}{AB_1} = \frac{AA_1}{AC_1} = \frac{A_2C_1}{A_1C_1},$$

and $A_2B_1/A_2C_1 = A_1B_1/A_1C_1$. By the angle bisector theorem, $A_1B_1/A_1C_1 = A_3B_1/A_3C_1$, so $A_2B_1/A_2C_1 = A_3B_1/A_3C_1$, and, by the converse of the angle bisector theorem, $\overline{A_2A_3}$ bisects angle $B_1A_2C_1$, proving (a).

Let ω be the circumcircle of triangle $A_1A_2A_3$, and let O be its center and R its radius. Denote by $P(X)$ the power of any point X with respect to ω. Let A_4 be the second intersection of ray A_1A_3 with k, and let A_5 be the second intersection of ray A_2A_3 with k. Note that A_4, A_5 are the midpoints of the two arcs B_1C_1 in k. [Here we used the result from (a).] Therefore, $\overline{A_4A_5}$ is a diameter of k, and, by the median formula,

$$IO^2 = \frac{A_4O^2 + A_5O^2}{2} - r^2,$$

where r is the radius of k. It follows that

$$\begin{aligned} P(I) = IO^2 - R^2 &= \frac{A_4O^2 + A_5O^2}{2} - r^2 - R^2 \\ &= \frac{P(A_4) + P(A_5)}{2} - r^2. \end{aligned}$$

Because A_4 is the midpoint of the arc B_1C_1 in k not containing A_1, we have $\angle A_4A_1B_1 = \angle A_4C_1B_1 = \angle C_1B_1A_4$, and, therefore, triangles $A_4B_1A_3$ and $A_4A_1B_1$ are similar. Thus, $P(A_4) = A_4A_3 \cdot A_4A_1 = (A_4B_1)^2$. Similarly, $P(A_5) = (B_1A_5)^2$. Therefore, the power of I with respect to the circumcircle of triangle $A_1A_2A_3$ is

$$P(I) = \frac{(A_4B_1)^2 + (B_1A_5)^2}{2} - r^2 = \frac{(2r)^2}{2} - r^2 = 3r^2.$$

The same holds for the power of I with respect to the circumcircle of triangle $B_1B_2B_3$. Therefore, I lies on the radical axis of those two circumcircles, that is, the line PQ, proving (b).

Problem 7 Given a permutation $(a_1, a_2, \ldots, a_n)$ of the numbers $1, 2, \ldots, n$, one may interchange any two consecutive "blocks" — that is, one may transform

$$(a_1, \ldots, a_i, \underbrace{a_{i+1}, \ldots, a_{i+p}}_{A}, \underbrace{a_{i+p+1}, \ldots, a_{i+q}}_{B}, a_{i+q+1}, \ldots, a_n)$$

into

$$(a_1, \ldots, a_i, \underbrace{a_{i+p+1}, \ldots, a_{i+q}}_{B}, \underbrace{a_{i+1}, \ldots, a_{i+p}}_{A}, a_{i+q+1}, \ldots, a_n)$$

by interchanging the "blocks" A and B. Find the least number of such changes which are needed to transform $(n, n-1, \ldots, 1)$ into $(1, 2, \ldots, n)$.

Solution: The answer is 0 for $n = 1$, 1 for $n = 2$, and $\lceil (n+1)/2 \rceil$ for $n \geq 3$. The cases of $n = 1$ and $n = 2$ are not difficult to show, so assume from now on that $n \geq 3$.

We first show that $\lceil (n+1)/2 \rceil$ is possible.

If n is even, then write $n = 2m$, and for the first m moves, swap block $a_i, \ldots, a_{i+m-2}$ with $a_{i+m-1}, \ldots, a_{i+m}$ for $i = 1, 2, \ldots, m$. After this, the sequence is

$$m, m-1, m-2, \ldots, 1; n, n-1, n-2, \ldots, m+1.$$

Next swap block $a_1, \ldots, a_m$ with $a_{m+1}, \ldots, a_n$. The total number of moves is $m+1$, as desired.

If n is odd, then write $n = 2m+1$, and for the first m moves, swap block $a_i, \ldots, a_{i+m-1}$ with a_{i+m}, a_{i+m+1} for $i = 1, 2, \ldots, m$. After this, the sequence is

$$m+1, m, m-1, \ldots, 2; n, n-1, n-2, \ldots, m+2; 1.$$

Next swap block $a_1, \ldots, a_m$ with $a_{m+1}, \ldots, a_{2m}$. The total number of moves is $m+1$, as desired.

Now we show that $\lceil (n+1)/2 \rceil$ is the minimum possible number of moves. Consider the number X of neighboring terms of the sequence that are in increasing order. For $n \geq 3$, at least 2 swaps are necessary. The first and last swaps increase X by exactly one. For any other swap, say from

$$\ldots, a, \underbrace{b, \ldots, e}, \underbrace{f, \ldots, c}, d, \ldots,$$

to

$$\ldots, a, \underbrace{f, \ldots, c}, \underbrace{b, \ldots, e}, d, \ldots,$$

if X were to increase by 3, then it would have to be the case that

$$a > b,\ b > c,\ c > d,\ d > e,\ e > f,\ f > a,$$

which is not possible. Therefore, X increases by at most 2 with any given move. Because X starts at 0 and must finish at $n-1$, it is not difficult to see that the number of moves must be at least $\lceil (n+1)/2 \rceil$.

Problem 8 Let $n \geq 2$ be a fixed integer. At any lattice point (i, j) we write the unique integer $k \in \{0, 1, \ldots, n-1\}$ such that $i + j \equiv k \pmod{n}$. Find all pairs a, b of positive integers such that the rectangle with vertices $(0,0)$, $(a,0)$, (a,b), and $(0,b)$ has the following properties: (i) the numbers $0, 1, \ldots, n-1$ appear in its interior an equal number of times; (ii) the numbers $0, 1, \ldots, n-1$ appear on its boundary an equal number of times.

Solution: The necessary and sufficient condition is

- a and b are not both even, if $n = 2$, or
- one of a and b is one more than a multiple of n and the other is one less than a multiple of n, if $n > 2$.

Erase the label of each point (i, j) and relabel it with the number ζ^{i+j} instead, where $\zeta = e^{2\pi i/n}$.

The condition (ii) implies that

$$\sum_{i=1}^{a-1} \sum_{j=1}^{b-1} \zeta^{i+j} = \frac{(a-1)(b-1)}{n} \cdot \sum_{k=0}^{n-1} \zeta^k = 0.$$

Because the left hand side is none other than the product of $\sum_{i=1}^{a-1} \zeta^i$ and $\sum_{j=1}^{b-1} \zeta^j$, one of these two factors must equal 0. Assume that it is the former; the other case is similar. Then a is one more than a multiple of n.

The conditions (i) and (ii) together imply that the n n-th roots of unity appear an equal number of times in the entire rectangle (i.e., boundary and interior), so

$$\sum_{i=0}^{a} \sum_{j=0}^{b} \zeta^{i+j} = \frac{(a+1)(b+1)}{n} \cdot \sum_{k=0}^{n-1} \zeta^k = 0.$$

It follows that either $\sum_{i=0}^{a} \zeta^i$ or $\sum_{j=0}^{b} \zeta^j$ must equal zero, and except when $n = 2$ the former cannot, so for $n > 2$ the latter must; i.e., b must be one less than a multiple of n.

It is not difficult to see that the conditions established are not only necessary, but also sufficient.

Problem 9 Find all real numbers t for which there exist real numbers x, y, z such that

$$\begin{aligned} 3x^2+3xz+z^2&=1,\\ 3y^2+3yz+z^2&=4,\\ x^2-xy+y^2&=t. \end{aligned}$$

Solution: Answer: t can be any number in the interval $[1/3, 3]$. If x, y, z must all be nonnegative, then t is restricted to the interval $[(3-\sqrt{5})/2, 1]$.

Note that, whenever (x, y, z) is a solution, so is $(-x, -y, -z)$, so we can assume without loss of generality that $z \geq 0$.

In the plane, let X be the point with polar coordinates $(\sqrt{3}x, 150^\circ)$. [If x is negative, then X will be $(-\sqrt{3}x, -30^\circ)$.] Let Y be the point $(\sqrt{3}y, -150^\circ)$. [If y is negative, then Y will be $(-\sqrt{3}y, 30^\circ)$.] Let Z be the point $(z, 0^\circ)$. Let O be the origin.

By the law of cosines,

$$\begin{aligned} XZ &= \sqrt{OX^2 - 2OX\cdot OZ\cos\angle ZOX + OZ^2}\\ &= \sqrt{3x^2+3xz+z^2} = 1. \end{aligned}$$

(Note that this holds regardless of the sign of x.) Similarly, $YZ = 2$, and $XY = \sqrt{3t}$, so $t = XY^2/3$.

If we restrict our attention to nonnegative x, y, $\angle XZY$ can range from its minimum when $z = 1$ to its maximum when $z = 0$. When $z = 1$, we have $x = 0$, and, in the triangle XZY,

$$XY^2 + \sqrt{3}XY - 1 = 4,$$

so $XY = (-\sqrt{3}+\sqrt{15})/2$, and $t = XY^2/3 = (3-\sqrt{5})/2$. When $z = 0$, we have $OX = 1$, and $OY = 2$, so because $\angle XOY = 60^\circ$, $XY = \sqrt{3}$, and $t = 1$. Therefore, if $x, y \geq 0$, then $t \in [(3-\sqrt{5})/2, 1]$, and because z is allowed any value in $[0, 1]$ and t is a continuous function of z, any value of t in that interval is possible.

If, on the other hand, we allow x, y to be negative, then XY is at a minimum when Z, X, Y lie on a line in that order. This actually happens when

$$(x, y, z) = \left(\frac{-1}{\sqrt{21}}, \frac{2}{\sqrt{21}}, \frac{2\sqrt{3}}{\sqrt{7}} \right),$$

and $t = XY^2/3 = 1^2/3 = 1/3$. From this situation we can slide X, Y, and Z continuously so that x and y increase while z decreases until $(x, y, z) = (0, (-1+\sqrt{5})/2, 1)$, where we have as before $t = (3-\sqrt{5})/2$. Therefore, t can be any value in the interval $[1/3, (3-\sqrt{5})/2]$. It can also take any value in $[(3-\sqrt{5})/2, 1]$, as shown above.

When X, Z, Y lie on a line in that order, XY reaches a maximum; this happens when $(x, y, z) = (-1, -2, 2)$, and $t = 3$ here. From this situation we can slide X, Y, and Z continuously so that x and y increase while z decreases until $(x, y, z) = (-1/\sqrt{3}, -2/\sqrt{3}, 0)$, where $t = 1$ as before. Therefore t can take any value in $[1, 3]$.

Putting it all together, we see that if x, y, z are allowed to be negative, then t can take any value in $[1/3, 3]$.

Problem 10 Let p be a prime number congruent to 3 modulo 4, and consider the equation

$$(p+2)x^2 - (p+1)y^2 + px + (p+2)y = 1.$$

Prove that this equation has infinitely many solutions in positive integers, and show that if $(x, y) = (x_0, y_0)$ is a solution of the equation in positive integers, then $p \mid x_0$.

Solution: We show first that $p|x$. Substituting $y = z + 1$ and rewriting, we obtain

$$x^2 = (z - x)((p+1)(z + x) + p).$$

Let $q = \gcd(z - x, (p+1)(z + x) + p)$. Then $q|x$, therefore $q|z$, and therefore $q|p$. On the other hand, $q \neq 1$, because otherwise both factors on the right hand side must be perfect squares, yet $(p+1)(z+x)+p \equiv 3 \pmod 4$. Thus $q = p$ and $p|x$ as desired.

Now, write $x = px_1$ and $z = pz_1$ to obtain

$$x_1^2 = (z_1 - x_1)((p+1)(z_1 + x_1) + 1).$$

By what we showed above, the two terms on the right are coprime and must be perfect squares. Therefore, for some a, b we have

$$z_1 - x_1 = a^2, \quad (p+1)(z_1 + x_1) + 1 = b^2, \quad x_1 = ab$$

The above equality implies

$$b^2 = (p+1)((a^2 + 2ab) + 1, \ i.e.$$

$$(p+2)b^2 - (p+1)(a+b)^2 = 1.$$

Vice versa, given a and b satisfying the last equation, there exists a unique pair (x_1, y_1) satisfying the equation above, and hence a unique pair (x, y) satisfying the original equation.

Thus, we reduced the original equation to a “Pell-type” equation. To get some solutions, look at the odd powers of $\sqrt{p+2} + \sqrt{p+1}$. It follows easily that

$$(\sqrt{p+2} + \sqrt{p+1})^{2k+1} = m_k\sqrt{p+2} + n_k\sqrt{p+1}$$

for some positive integers m_k, n_k. Then

$$(\sqrt{p+2} - \sqrt{p+1})^{2k+1} = m_k\sqrt{p+2} - n_k\sqrt{p+1},$$

and, multiplying the left and right sides gives

$$(p+2)m_k^2 - (p+1)n_k^2 = 1.$$

Clearly, $n_k > m_k$, so setting $b_k = m_k$, $a_k = n_k - m_k$ gives a solution for (a, b). Finally, it is easy to see that the sequences $\{m_k\}$, $\{n_k\}$ are strictly increasing, so we obtain infinitely many solutions this way.

1.3 Canada

Problem 1 Let ABC be a triangle with $AC > AB$. Let P be the intersection point of the perpendicular bisector of $\overline{BC}$ and the internal angle bisector of angle CAB. Let X and Y be the feet of the perpendiculars from P to lines AB and AC, respectively. Let Z be the intersection point of lines XY and BC. Determine the value of $\frac{BZ}{ZC}$.

Solution: We denote the foot of the perpendicular from P to line BC by Z', and the intersection of line $Z'P$ with the circumcircle of ABC by P'. Then $\widehat{P'B} = \widehat{P'C}$, implying that P' is on the angle bisector of angle CAB and is therefore equal to P. We see that X, Y, and Z', being the feet of the perpendiculars from P to lines AB, AC, and BC, respectively, make up the Simson line of triangle ABC and must therefore be collinear. Since Z' is on both XY and BC, we must have $Z' = Z$. Thus $\frac{BZ}{ZC} = 1$.

Problem 2 Let n be a positive integer. Nancy is given a matrix in which each entry is a positive integer. She is permitted to make either of the following two moves:

(i) select a row and multiply each entry in this row by n;

(ii) select a column and subtract n from each entry in this column.

Find all possible values of n for which given any matrix, it is possible for Nancy to perform a finite sequence of moves to obtain a matrix in which each entry is 0.

Solution: First we give an example of a matrix that will not satisfy the conditions for $n \geq 3$. We examine the $(n-1) \times 1$ matrix with first entry n and all other entries equal to $(n-1)$, and consider how the sum of all entries changes with respect to each of the operations. The first operation, multiplying a row by n, increases the total sum by $(n-1)$ times the sum of the entries in the selected row. The second operation decreases the total sum by $n(n-1)$. In either case, the sum of all entries is invariant modulo $(n-1)$. Since the sum of the entries in the given matrix is congruent to 1 modulo $(n-1)$, we see that it is impossible to obtain the matrix in which all entries are 0.

For the case $n = 1$, we examine the 2×1 matrix with first entry 2 and second entry 1. The first operation has no effect, and it is clear that the zero matrix cannot be obtained by using only the second operation.

In the case $n = 2$, we describe an algorithm for attaining the zero matrix from any given matrix. Working from left to right with respect to the columns in the matrix, we see that because the first operation has no effect on entries that are already equal to 0, it is enough to devise a strategy for converting all entries of a given column to 0's. With this in mind, we first multiply each entry of the column by 2, by applying the second operation to each row of the matrix. Since each of the original entries is a positive integer, each of the entries becomes an even integer greater than or equal to 2. We now work from the top of the column to the bottom, our aim being to convert each entry to 2. If an entry is greater than 2, say $(2k+2)$ for $k \geq 1$, we see that it can be reduced to 2 by repeating the second operation k times. We want to do this, however, without making any other entries in the column negative or changing an entry that is already 2 into another number. Thus, whenever this is in danger of happening—that is, whenever we have an entry equal to 2—we multiply the entry's row by 2 before performing the subtraction on the column. In this way, each entry that is already equal to 2 will again be equal to 2 after applying both operations ($2 \cdot 2 - 2 = 2$). After performing the algorithm on all entries of the column, we are left with a column of 2's, which can easily be reduced to a column of 0's by applying the first operation. Thus we see that $n = 2$ is the only solution.

Problem 3 Let P_0, P_1, and P_2 be three points on a circle with radius 1, where $P_1P_2 = t < 2$. Define the sequence of points $P_3, P_4, \ldots$ recursively by letting P_i be the circumcenter of triangle $P_{i-1}P_{i-2}P_{i-3}$ for each integer $i \geq 3$.

(a) Prove that the points $P_1, P_5, P_9, P_{13}, \ldots$ are collinear.

(b) Let $x = P_1P_{1001}$ and $y = P_{1001}P_{2001}$. Prove that $\sqrt[500]{x/y}$ depends only on t, not on the position of P_0, and determine all values of t for which $\sqrt[500]{x/y}$ is an integer.

Solution: (a) From the definition of circumcenter, we know that line P_3P_4 is perpendicular to P_1P_2, line P_4P_5 is perpendicular to P_2P_3, and triangle $P_3P_4P_5$ is isosceles. Therefore triangle $P_3P_4P_5$ is a spiral similarity of $P_1P_2P_3$ through an angle of rotation of $\pi/2$. Since the position of each point is determined solely by the positions of the three previous points, we can deduce that triangle $P_5P_6P_7$ will also be a spiral similarity of $P_3P_4P_5$ through an angle of $\pi/2$, making it a spiral similarity through an angle π of triangle $P_1P_2P_3$, and therefore a homothety. Because the composition of two homotheties is still a homothety, we see that any triangle $P_{4n+1}P_{4n+2}P_{4n+3}$, n a positive integer, must then be a homothety of $P_1P_2P_3$. Since the center of a homothetic relation is mapped to itself from one figure to the next, we also see that the center of homothety must be the same for all triangles. This in turn implies that all points $P_1, P_5, P_9, P_{13}, \ldots$ are collinear, on a line that passes through the common center of homothety.

(b) The point P_3 is easily seen to be the center of the given circle, and each point thereafter depends only on the positions of the three previous points. Thus we see that none of the points P_1, P_{1001}, P_{2001} depend on the location of P_0, making the values of x and y independent of P_0 as well.

Since the family of homothetic triangles $P_{4n+1}P_{4n+2}P_{4n+3}$ is related through a constant ratio of dilation, we may write

$$\frac{P_{4n-3}P_{4n+1}}{P_{4n+1}P_{4n+5}} = \frac{1}{k}$$

for some nonzero constant k. Since applying the homothety 250 times takes $\overline{P_1P_{1001}}$ to $\overline{P_{1001}P_{2001}}$, we see that $x/y = k^{250}$. The value we seek is $|k|^{1/2}$.

Recall that we found the homothety taking P_i to P_{i+4} (for all i) by composing the pair of identical spiral similarities taking $P_1P_2P_3$ to $P_3P_4P_5$ and $P_3P_4P_5$ to $P_5P_6P_7$. Thus, $\sqrt{|k|}$ is equal to the ratio of similarity in each of the spiral similarities. This in turn is equal to $\frac{P_3P_4}{P_1P_2}$. Now, $P_1P_2 = t$ by definition, and P_3P_4 is the circumradius of $P_1P_2P_3$, which is

$$\frac{1 \cdot 1 \cdot t}{4 \cdot [P_1P_2P_3]}.$$

Thus

$$\sqrt[500]{\frac{x}{y}} = \sqrt{|k|} = 4 \cdot [P_1P_2P_3] = 4 \cdot \frac{1}{2} \cdot 1 \cdot 1 \cdot \sin\angle P_1P_3P_2.$$

For $0 < \angle P_1P_3P_2 < \pi$, $0 < \sin\angle P_1P_3P_2 \leq 1$. In order for $\sqrt[500]{x/y}$ to be an integer, we must have $\sin\angle P_1P_3P_2 = 1/2$ or 1, generating the solutions $\angle P_1P_3P_2 = \pi/6, 5\pi/6$, or $\pi/2$. Since

$$t = 2 \cdot \sin \frac{\angle P_1P_3P_2}{2},$$

we find the corresponding values of t to be $(\sqrt{6}-\sqrt{2})/2, (\sqrt{6}+\sqrt{2})/2$, and $\sqrt{2}$.

1.4 China

Problem 1 Let a be a fixed real number with $\sqrt{2} < a < 2$, and let $ABCD$ be a convex cyclic quadrilateral whose circumcenter O lies in its interior. The quadrilateral's circumcircle ω has radius 1, and the longest and shortest sides of the quadrilateral have lengths a and $\sqrt{4-a^2}$, respectively. Lines $\ell_A, \ell_B, \ell_C, \ell_D$ are tangent to ω at A, B, C, D, respectively. Let lines ℓ_A and ℓ_B, ℓ_B and ℓ_C, ℓ_C and ℓ_D, ℓ_D and ℓ_A intersect at A', B', C', D', respectively. Determine the minimum value of

$$\frac{[A'B'C'D']}{[ABCD]}.$$

Solution: The minimum is $\frac{4}{a\sqrt{4-a^2}}$.

Observe that the areas of triangles AOB and $AA'B$ are determined solely by the measure of central angle $\angle AOB$, and hence by the length of side $\overline{AB}$. Likewise, $[BOC]$ and $[BB'C]$ are determined by the length of $\overline{BC}$, and so on. Because O lies within quadrilateral $ABCD$, we have $[ABCD] = [AOB] + [BOC] + [COD] + [DOA]$ and $[A'B'C'D'] = [AOB] + [AA'B] + \cdots + [DOA] + [DD'A]$. Thus, these areas depend only on the lengths of the sides of $ABCD$ and not on their order. Hence, we may assume without loss of generality that $AB = a$, $BC = \sqrt{4-a^2}$. Because the length of the diameter of ω is 2, it then follows that AC is a diameter. Therefore, ℓ_A and ℓ_C are each perpendicular to AC and hence parallel to each other.

We must now choose D to complete cyclic quadrilateral $ABCD$ and minimize the ratio of $[A'B'C'D']$ and $[ABCD]$, obeying $\sqrt{4-a^2} \leq CD, DA \leq a$. We claim that the minimal choice of D is the midpoint of arc $\widehat{AC}$ on the opposite side of AC as B. This choice of D clearly satisfies the latter restriction, because $CD = DA = \sqrt{2}$ in this case. To show that it is indeed minimal, we decompose the areas $[A'B'C'D']$ and $[ABCD]$ into $[A'ACB'] + [D'ACC']$ and $[ABC] + [ADC]$, respectively. The first summand in each expression is fixed, and only the second depends on D. Furthermore, $[D'ACC'] = \frac{1}{2}AC(AD' + CC') = \frac{1}{2} \cdot 2(DD' + DC') = D'C'$, and because $\overline{AC}$ is the projection of $\overline{D'C'}$ onto line AC, $D'C' \geq AC$ with equality when $D'C' \parallel AC$. This occurs when $DO \perp AC$, so we see that our choice of D minimizes $[D'ACC']$ and hence $[A'B'C'D']$. On the other hand, our choice of D clearly maximizes the length of the altitude from D

to AC, and hence the area $[ADC]$. Thus, it maximizes $[ABCD]$. It follows that this D minimizes $[A'B'C'D']/[ABCD]$, as claimed.

It remains to calculate the minimal ratio. Because $AB \perp BC$, we have $[ABC] = \frac{1}{2}a\sqrt{4-a^2}$. Because $OA' \perp AB$, $\angle A'OA = \angle BCA$, and hence triangles ABC and $A'AO$ are similar. It follows that $AA' = \frac{a}{\sqrt{4-a^2}}$ and likewise $CB' = \frac{\sqrt{4-a^2}}{a}$. Thus, we compute $[A'ACB'] = \frac{1}{2}AC(AA'+CB') = \frac{a}{\sqrt{4-a^2}} + \frac{\sqrt{4-a^2}}{a}$. We also easily have $[ADC] = 1$, $[D'ACC'] = 2$. Substituting these into $[A'B'C'D'] = [A'ACB'] + [D'ACC']$ and $[ABCD] = [ABC] + [ADC]$, taking the ratio, and doing some algebra yield the answer of $\frac{4}{a\sqrt{4-a^2}}$.

Problem 2 Determine the smallest positive integer m such that for any m-element subsets W of $X = \{1, 2, \ldots, 2001\}$, there are two elements u and v (not necessarily distinct) in W with $u + v = 2^n$ for some positive integer n.

Solution: The smallest m is 999. Our approach is to partition X into subsets of size 1 and 2, with the singleton sets containing powers of 2 and the doubleton sets of the form $\{u, v\}$, with $u + v = 2^n$ for some n. We begin by forming sets $\{u, v\}$ with $u+v = 2048$. We work down from $u = 2001$ to 1025, forming the doubleton sets

$$\{2001, 47\}, \{2000, 48\}, \ldots, \{1025, 1023\}$$

and then the singleton set $\{1024\}$. Having used up all the numbers greater than 46, we now repeat the procedure with $u + v = 64$, the smallest power of 2 greater than 46. This time we obtain

$$\{46, 18\}, \{45, 19\}, \ldots, \{33, 31\}, \{32\}.$$

Continuing in the same manner starting from 17, we form $\{17, 15\}$ and $\{16\}$. Finally, beginning with 14, we produce

$$\{14, 2\}, \{13, 3\}, \ldots, \{9, 7\}, \{8\},$$

with $\{1\}$ as our final set.

Now, consider any subset $W \subset X$ with $|W| = 999$. If any of the singletons we formed (1, 8, 16, 32, and 1024) are in W, then we immediately have two (equal) elements of W that sum to a power of 2. On the other hand, there are $\frac{2001-5}{2} = 998$ doubleton sets, and if both elements of any doubleton are in W, we again have two elements that sum to a power of 2. Hence by the pigeonhole principle, $m = 999$ works.

It remains to exhibit a set of S of size 998 that has no two elements summing to a power of 2. We do this by putting the larger element of each doubleton in S. That is, letting

$$S_3 = \{9, \ldots, 14\}, S_4 = \{17\}, S_5 = \{33, \ldots, 46\},$$

$$S_{10} = \{1025, \ldots, 2001\},$$

we have

$$S = S_3 \cup S_4 \cup S_5 \cup S_{10}.$$

Now it is easy to see that if $u \in S_j, v \in S_k$, $j < k$, we have $2^k < u + v < 2^{k+1}$, while if $j = k$, we have $2^{k+1} < u + v < 2^{k+2}$. Thus no two elements of S sum to a power of 2, as wanted.

Problem 3 Two triangle are said to be *of the same type* if they are both acute triangles, both right triangles, or both obtuse triangles. Let n be a positive integer and let $\mathcal{P}$ be a n-sided regular polygon. Exactly one magpie sits at each vertex of $\mathcal{P}$. A hunter passes by, and the magpies fly away. When they return, exactly one magpie lands on each vertex of $\mathcal{P}$, not necessarily in its original position. Find all n for which there must exist three magpies with the following property: the triangle formed by the vertices the magpies originally sit at, and the triangle formed by the vertices they return to after the hunter passes by, are of the same type.

Solution: The property holds for all n but 5. We first consider even n. In this case, for each vertex of $\mathcal{P}$, there is another vertex diametrically opposite it. Moreover, any pair of diametrically opposite vertices forms only right triangles with other vertices of $\mathcal{P}$. So, let A be any vertex of $\mathcal{P}$ and let B be the vertex opposite it. For any vertex V of $\mathcal{P}$, let V' denote the vertex that the magpie originally on V lands on after the hunter passes. Then if A' and B' are still diametrically opposite after the permutation of vertices, take C to be any vertex of $\mathcal{P}$. Then both triangles ABC and $A'B'C'$ are right, so we have found two triangles of the same type. Otherwise, if B' is not opposite A', let C be the vertex such that C' is diametrically opposite A'. Then we again have that ABC and $A'B'C'$ are right. This completes the proof of the even case.

For odd n, we first handle the small cases. The property holds trivially for $n = 3$. For $n = 5$, the property does not hold, as can be seen from the following example. Label the vertices of $\mathcal{P}$ $A_1, A_2, \ldots, A_5$ in clockwise order. Then the permutation that sends A_i to A_{i+2} (mod 5) sends all acute triangles to obtuse triangles, and vice versa.

We are left with showing that all odd $n \geq 7$ satisfy the property. Begin with any vertex A. Let B, C, and D be the next three vertices in clockwise order. Then all of the triangles ABC, ABD, ACD are obtuse. Now consider the positions of B', C', D' relative to A'. Let ℓ be the line through A and the center of $\mathcal{P}$; then two of B', C', D' must be on the same side of ℓ. Without loss of generality, let these be B' and C'. Then $A'B'C'$ is obtuse, as is ABC, completing the proof.

Problem 4 We are given three integers a, b, c such that a, b, c, $a + b - c$, $a + c - b$, $b + c - a$, and $a + b + c$ are seven distinct primes. Let d be the difference between the largest and smallest of these seven primes. Suppose that $800 \in \{a+b, b+c, c+a\}$. Determine the maximum possible value of d.

Solution: Answer: 1594.

First, observe that a, b, c must all be odd primes; this follows from the assumption that the seven quantities listed are distinct primes and the fact that there is only one even prime, 2. Therefore, the smallest of the seven primes is at least 3. Next, assume without loss of generality that $a + b = 800$. Because $a + b - c > 0$, we must have $c < 800$. We also know that c is prime; therefore, since $799 = 17 \cdot 47$, we have $c \leq 797$. It follows that the largest prime, $a + b + c$, is no more than 1597. Combining these two bounds, we can bound d by $d \leq 1597 - 3 = 1594$. It remains to observe that we can choose $a = 13, b = 787, c = 797$ to achieve this bound. The other four primes are then $3, 23, 1571$, and 1597.

Problem 5 Let $P_1P_2 \ldots P_{24}$ be a regular 24-sided polygon inscribed in a circle ω with circumference 24. Determine the number of ways to choose sets of eight distinct vertices $\{P_{i_1}, P_{i_2}, \ldots, P_{i_8}\}$ such that none of the arcs $P_{i_j}P_{i_k}$ has length 3 or 8.

Solution: There are 258 ways to choose the vertices. We begin by observing that the condition that none of the arcs has length 8 means that at most one vertex from each of the equilateral triangles $\{P_i, P_{i+8}, P_{i+16}\}$ can be chosen. Since there are only 8 such triangles, *exactly* one vertex from each triangle is chosen. Next, label each vertex P_i with the least residue of i mod 3. Then each equilateral triangle has one vertex labeled with each of 0, 1, 2. Label the triangles themselves $T_1, T_2, \ldots, T_8$, so that $P_1 \in T_1$, $P_4 \in T_2$, $P_7 \in T_3$, and so on, skipping 3 vertices each time. Thus, for each choice of 8 vertices satisfying the given conditions, we can create an ordered 8-tuple such that the j^{th} coordinate is the label (0, 1, or 2) of the vertex selected from T_j. Furthermore, the condition that no arc has length 3 reduces to the statement that no two consecutive coordinates of the 8-tuple (where the 8^{th} and 1^{st} are also considered to be "consecutive") can be the same. Our task now is to find the number of legal 8-tuples of the above form.

We do so by creating a generating function. For any legal 8-tuple $(x_1, \ldots, x_8)$, consider the differences $d_i = x_{i+1} - x_i$, $i = 1, \ldots, 8$ between consecutive coordinates. Since no two consecutive coordinates are the same, for all d_i, we have $d_i \equiv \pm 1 \pmod 3$. Clearly, we must also have $d_1 + \cdots + d_8 = 0$. Going in the other direction, given the 8 differences $d_i = \pm 1$, which are understood to be taken modulo 3, and the first coordinate x_1, we can reconstruct the rest of the x_i's to form a legal 8-tuple, provided $3 \mid d_1 + \cdots + d_8$.

Representing our choices $d_i = 1$ and $d_i = -1$ by x^1 and x^{-1}, we now construct the generating function

$$g(x) = (x + x^{-1})^8.$$

Then the coefficient of x^n in $g(x)$ is the number of choices of d_i's that sum to n. It follows that the total number of legal choices—those that sum to a multiple of 3—is the sum of the coefficients of the x^{3k} terms.

To find this sum, we evaluate $g(x)$ at the cube roots of unity. Letting $\omega = e^{2\pi i/3}$, the sum

$$g(1) + g(\omega) + g(\omega^2)$$

causes all terms x^n with $3 \nmid n$ to disappear (since $1 + \omega + \omega^2 = 0$), while all terms x^{3k} evaluate to 1 each time, so that the coefficients of x^{3k} terms are multiplied by 3. As we have not yet chosen x_1 and there are 3 choices for it, we need to multiply by 3 anyway; hence our answer is in fact $g(1) + g(\omega) + g(\omega^2) = 2^8 + (-1)^8 + (-1)^8 = 258$.

Problem 6 Let $a = 2001$. Consider the set A of all pairs of positive integers (m, n) such that

(i) $m < 2n$;

(ii) $2am - m^2 + n^2$ is divisible by $2n$;

(iii) $n^2 - m^2 + 2mn \leq 2a(n - m)$.

For $(m, n) \in A$, let

$$f(m, n) = \frac{2am - m^2 - mn}{n}.$$

Determine the maximum and minimum values of f, respectively.

Solution: The maximum is 3750 and the minimum is 2.

We begin by proving that $m < n$. Rearranging condition (iii), we have $2mn \leq (n-m)(2a-n-m)$. On the other hand, (i) multiplied by m gives us $m^2 \leq 2mn$. Thus, we have $m^2 \leq (n-m)(2a-n-m)$. Now if $m \geq n$, we can write $(n-m)(2a-n-m) = (m-n)(m+n-2a) \leq (m-n)(m+n) = m^2 - n^2 < m^2$, a contradiction. Hence, $m < n$, and it follows from $0 < m^2 \leq (n-m)(2a-n-m)$ that $m + n < 2a$.

Next, we prove that 2 is the minimum value of f. By condition (ii), $2am - m^2 + n^2$ is divisible by $2n$. It follows that $n^2 - m^2$ is even, so m and n must be of the same parity. Thus, $2 \mid (m + n)$, so that $2n \mid (n^2 + mn)$. Subtracting, we have $2n \mid (2am - m^2 - mn)$, the numerator of the expression for f. Thus, $f(m, n)$ is an even integer for all $(m, n) \in A$.

Factoring out m from the numerator, we have

$$f(m, n) = \frac{(2a - m - n)m}{n}.$$

We saw earlier that $2a - m - n > 0$. We also are given that $m, n > 0$, so $f(m, n) > 0$, from which we obtain $f(m, n) \geq 2$ because f is even. It is now easy to check that the lower bound of 2 is achieved when $(m, n) = (2, 2000) \in A$.

To prove the upper bound, first set $m = n - k$, with k a positive even integer from the previous. Substituting into the expression for

f, we have

$$\begin{aligned} f(m,n) &= \frac{(2a-n-(n-k))(n-k)}{n} \\ &= \frac{-2n^2+(2a+3k)n-k(2a+k)}{n} \\ &= 2a+3k-2\left(n+\frac{\frac{k}{2}(2a+k)}{n}\right). \end{aligned}$$

Next, we translate conditions (ii) and (iii) in terms of n and k. ¿From (ii), we know that $2am-m^2+n^2 = 2a(n-k)-(n-k)^2+n^2 = 2n(a+k)-k(2a+k)$ is divisible by $2n$, so that $2n \mid k(2a+k)$, or equivalently,

$$n \mid \frac{k}{2}(2a+k).$$

Condition (iii) becomes $n^2-(n-k)^2+2(n-k)n \leq 2ak$, so that $2n^2-k^2 \leq 2ak$, or

$$n^2 \leq \frac{k}{2}(2a+k).$$

Now, for k fixed, maximizing f is equivalent to minimizing $n+\frac{k}{2}(2a+k)/n$. The product of these two terms is fixed at $\frac{k}{2}(2a+k)$. Thus, their sum will be minimized when n is as close as possible to $\sqrt{\frac{k}{2}(2a+k)}$, subject to restrictions (ii) and (iii). It follows that for fixed k, the choice of n that maximizes f is the largest factor of $\frac{k}{2}(2a+k)$ less than $\sqrt{\frac{k}{2}(2a+k)}$.

When $k=2$, $\frac{k}{2}(2a+k)=4004$, and the best choice of n is 52. Then $m=n-k=50$, which yields $f(50,52)=3750$. Otherwise, because k is even, we must have $k \geq 4$. Also, because $m+n<2a$, we must have $k<2a=4002$. For these k we can bound f from above by substituting $n=\sqrt{\frac{k}{2}(2a+k)}$, producing

$$f(m,n) \leq 2a+3k-4\sqrt{\frac{k}{2}(2a+k)}.$$

We wish to show that $2a+3k-4\sqrt{\frac{k}{2}(2a+k)} \leq 3750$ for $4 \leq k < 4002$. Rearranging terms and substituting $a=2001$, we must show that

$$3k+252 \leq 4\sqrt{\frac{k}{2}(4002+k)}.$$

Squaring both sides, we need

$$9k^2 + 6 \cdot 252k + 252^2 \le 8k^2 + 8 \cdot 4002k,$$

or, rearranging, $k^2 - 30504 + 252^2 \le 0$. Now it is easy to check by the quadratic formula that this is indeed true for $4 \le k < 4002$, proving that 3750 is the required maximum.

Problem 7 For each integer $k > 1$, find the smallest integer m greater than 1 with the following property: there exists a polynomial $f(x)$ with integer coefficients such that $f(x) - 1$ has at least 1 integer root and $f(x) - m$ has exactly k distinct integer roots.

Solution: The smallest m is $\lfloor k/2 \rfloor! \lceil k/2 \rceil! + 1$. Let $x_1, \ldots, x_k$ be the integer roots of $f(x) - m$. Then

$$f(x) - m = (x - x_1) \cdots (x - x_k) q(x)$$

for some integer-coefficient polynomial $q(x)$. Thus,

$$m = (f(x) - 1) + 1 - (x - x_1) \cdots (x - x_k) q(x).$$

Now let a be an integer root of $f(x) - 1$. Plugging it into the above equation, we have

$$m = 1 - (a - x_1) \cdots (a - x_k) q(a).$$

Because $m > 1$, we must have

$$-(a - x_1) \cdots (a - x_k) q(a) > 0.$$

Then, since all terms involved are integers, it is not hard to see that $|a - x_1| \cdots |a - x_k|$ is minimized when $a, x_1, \ldots, x_k$ are $k+1$ consecutive numbers with a as close as possible to the middle. This gives us

$$|a - x_1| \cdots |a - x_k| \ge \lfloor k/2 \rfloor! \lceil k/2 \rceil!.$$

Combining this with $|q(a)| \ge 1$, we have

$$-(a - x_1) \cdots (a - x_k) q(a) \ge \lfloor k/2 \rfloor! \lceil k/2 \rceil!,$$

from which the desired result follows.

Problem 8 Given positive integers k, m, n such that $k \le m \le n$, express

$$\sum_{i=0}^{n} (-1)^i \frac{1}{n+k+i} \cdot \frac{(m+n+i)!}{i!(n-i)!(m+i)!}$$

in closed form.

Solution: The answer is 0.

Lemma. *If $f(x)$ is a polynomial of degree less than n, define $\Delta^n f = \sum_{i=0}^{n}(-1)^i\binom{n}{i}f(i)$ (the n^{th} finite difference of f). Then $\Delta^n f = 0$.*

Proof. Induct on n. If $n = 1$, then f has degree 0, so it is a constant. Hence, $\Delta^1 f = -f(1) + f(0) = 0$.

For the induction step, assume $\Delta^{n-1}g = 0$ for all polynomials g of degree less than $n-1$. For a polynomial f of degree n, we have

$$\begin{aligned}\Delta^n f &= \sum_{i=0}^{n}(-1)^i\binom{n}{i}f(i)\\ &= \sum_{i=0}^{n}(-1)^i\left(\binom{n-1}{i-1}+\binom{n-1}{i}\right)f(i)\\ &= \sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}(-f(i+1)+f(i)).\end{aligned}$$

Let $g(i) = -f(i+1) + f(i)$. Then $g(i)$ is a polynomial of degree less than $n-1$. Therefore, by the induction hypothesis, $\Delta^n f = \Delta^{n-1}g = 0$. $\square$

Returning to the problem at hand, we observe that we can write the given expression as follows:

$$\sum_{i=0}^{n}(-1)^i\frac{1}{n+k+i}\cdot\frac{(m+n+i)!}{i!(n-i)!(m+i)!} = \frac{1}{n!}\sum_{i=0}^{n}(-1)^i\binom{n}{i}(m{+}1{+}i)(m{+}2{+}i)\cdots(n{+} \qquad (*)$$

Note that $n+k+i \geq m+1+i$ because $n \geq m$ and $k \geq 1$, while $n+k+i \leq m+n+i$ because $k \leq m$, so that the products on the right make sense.

Now consider the polynomial

$$f(x) = (m{+}1{+}x)(m{+}2{+}x)\cdots(n{+}k{-}1{+}x)(n{+}k{+}1{+}x)\cdots(m{+}n{+}x).$$

Observe that f is a polynomial of degree $n-1$. Furthermore, $\Delta^n f$ is exactly the sum that appears on the right side of $(*)$. Thus, the sum is indeed equal to 0, as claimed.

Problem 9 Let a be a positive integer with $a \geq 2$, and let N_a be the number of positive integers k such that

$$k_{n-1}^2 + k_{n-2}^2 + \cdots + k_0^2 = k,$$

where $k_{n-1}k_{n-2}\ldots k_0$ is the base a representation of k. Prove that:

(a) N_a is odd;

(b) for any positive integer M, there is some a for which $N_a \geq M$.

Solution: We first prove that any k satisfying the given conditions has no more than two digits in base a. The condition translates to

$$\sum_{i=0}^{n-1} k_i^2 = \sum_{i=0}^{n-1} a^i k_i,$$

from which we obtain

$$\sum_{i=0}^{n-1} k_i(k_i - a^i) = 0.$$

Then only the $i = 0$ term, $k_0(k_0 - 1)$, can be nonnegative. Also, since $k_0 < a$, it is bounded above by $(a-1)(a-2) \leq a(a-2) < a^2 - 2$. Now if $n \geq 3$, we consider the $i = 2$ term, $k_2(k_2 - a^2)$. If $k_2 = 1$, this term is $1 - a^2$. Otherwise, $k_2 \geq 2$, so $k_2^2 < a^2$ while $k_2 a^2 \geq 2a^2$, so $k_2(k_2 - a^2) < -a^2$. Thus we see in either case that the $i = 2$ term is at most $1 - a^2$, from which it follows that the whole sum is negative, a contradiction.

Clearly, the only positive one-digit number that works is 1. Thus we turn our attention to the only remaining possibility, the two-digit numbers. To prove part (a), we must show that the number of these solutions is even. The equation the digits must satisfy for $n = 2$ is

$$k_1^2 + k_0^2 = ak_1 + k_0.$$

Rearranging, we have

$$k_0(k_0 - 1) = k_1(a - k_1).$$

Since $1 \leq k_1 \leq a - 1$, we see that the number k_1k_0 (in base a) works if and only if the number $(a - k_1)k_0$ works. Furthermore, these two numbers are distinct, because if $k_1 = a - k_1$, then the left side of our equation above is the product of two consecutive numbers while the right side is a positive perfect square, an impossibility. Thus, we can split the set of two-digit solutions into pairs as indicated above, and the number of such solutions must indeed be even.

For part (b), we wish to find an a such that our equation

$$k_0(k_0 - 1) = k_1(a - k_1).$$

has a large number of solutions. We consider possible solutions (k_0, k_1) of the form $(hz + 1, h)$, where $h > 0$. Substituting into the equation above, we need

$$(hz + 1)hz = h(a - h).$$

Cancelling h and rearranging, this is equivalent to

$$a = h(z^2 + 1) + z.$$

Thus, if we fix z, any a larger than $z^2 + 1$ satisfying

$$a \equiv z \pmod{z^2 + 1}$$

will produce an h forming a solution $(hz + 1, h)$. Furthermore, the ratio of the two digits forming the solution is $\frac{hz+1}{h} = z + \frac{1}{h} \in (z, z+1)$, so two distinct values of z are guaranteed to produce distinct solutions. Thus, if we can find M relatively prime numbers of the form $z^2 + 1$, we will be done by applying the Chinese Remainder Theorem to find a. We can do this by constructing the sequence

$$z_1 = 2, \quad z_{n+1} = (z_1 \cdots z_n)^2 + 1.$$

Clearly, each term z_{n+1} is relatively prime to all of the terms before it.

Problem 10 Let n be a positive integer, and define

$$M = \{(x, y) \mid x, y \in \mathbb{N}, 1 \le x, y \le n\}.$$

Determine the number of functions f defined on M such that

(i) $f(x, y)$ is a nonnegative integer for any $(x, y) \in M$;

(ii) for $1 \le x \le n$, $\sum_{y=1}^{n} f(x, y) = n - 1$;

(iii) if $f(x_1, y_1)f(x_2, y_2) > 0$, then $(x_1 - x_2)(y_1 - y_2) \ge 0$.

Solution: There are $\binom{n^2-1}{n-1}$ possible functions f. We treat a function f on M as an $n \times n$ matrix M_f. Condition (i) requires that all of the entries of M_f be nonnegative integers, while condition (ii) means that the sum of the entries in each row of M_f must be

$n-1$. Condition (iii) asserts that all of the positive entries of M_f can be traversed along a path from the northwest entry to the southeast entry by only moving south or east at each step. With this in mind, we consider the following scenario.

A park is divided into an $n \times n$ grid of unit squares. The park gardener must plant $n-1$ trees in each row of the grid. The gardener works his way from the northwest corner of the park to the southeast corner. He plants one row of trees at a time, and once he finishes a row, he automatically moves south one square to the next row. Thus, at any stage, he has two options: to plant a tree in the square he is in, or to move one square east. He stops once he reaches the southeast corner.

Now we consider the number of each type of "operation" the gardener performs. He plants $n-1$ trees in each row for a total of $n(n-1)$ trees planted. As for eastward moves, he travels from the western edge of the park to the eastern edge, never moving back west, so he makes a total of $n-1$ eastward moves. Since he can perform these operations in whatever order he chooses, the number of ways in which he can complete his task is $\binom{n^2-1}{n-1}$.

Taking the number of trees planted in each grid square to be the corresponding entry of the matrix M_f, it is not hard to see that there is a one-to-one correspondence between the legal matrices M_f and the gardener's tree-planting options. Therefore, the number of matrices M_f, and hence the number of functions f, is also equal to $\binom{n^2-1}{n-1}$.

1.5 Czech and Slovak Republics

Problem 1 Find all triples a, b, c of real numbers for which a real number x satisfies

$$\sqrt{2x^2 + ax + b} > x - c$$

if and only if $x \leq 0$ or $x > 1$.

Solution: The appropriate triples (a, b, c) are those with $a = (1-c)^2 - 2$, $b = 0$ and $0 < c \leq 1$.

Suppose that a, b, c satisfy the given conditions — that is,

$$\sqrt{2x^2 + ax + b} > x - c \tag{1}$$

if and only if $x \leq 0$ or $x > 1$.

First, we analyze (1) near $x = 0$. Because (1) holds for $x = 0$ (that is, $\sqrt{b} > -c$) we must have $b \geq 0$.

Suppose for sake of contradiction that $b > 0$. Then $2x^2 + ax + b > 0$ for all small positive x, so that $\sqrt{2x^2 + ax + b}$ is well-defined. Because a, b, c satisfy the given conditions, we then have $\sqrt{2x^2 + ax + b} - (x - c) \leq 0$ for all small positive x; by continuity, $\sqrt{2x^2 + ax + b} - (x - c) \leq 0$ for $x = 0$ as well, a contradiction. Hence, our assumption that $b > 0$ was false, and instead $b = 0$.

Because $\sqrt{b} > -c$, it also follows that $0 > -c$, or $c > 0$.

Next, we analyze (1) near $x = 1$. Because all $x > 1$ satisfy (1), we have $2x^2 + ax + b \geq 0$ for all $x > 1$; it follows that $2x^2 + ax + b \geq 0$ for $x = 1$ as well. We already know that $b = 0$, so this implies that $2 + a \geq 0$. Thus, $a \geq -2$.

If $c > 1$, then at $x = 1$ we have $\sqrt{2x^2 + ax + b} = \sqrt{2 + a} \geq 0 > 1 - c = x - c$, a contradiction. Hence, $c \leq 1$.

For $x \geq 1$, note that when $a \geq -2$, $b = 0$, and $x \geq 1$, (1) is equivalent to

$$2x^2 + ax + b > (x - c)^2. \tag{2}$$

This is because $2x^2 + ax + b$ and $x - c$ are both non-negative, so that we can square both sides of (1) to obtain the equivalent inequality (2). Because (2) must hold for $x > 1$ but not $x = 1$, the two sides of (2) must be equal when $x = 1$. That is, $2 + a = (1 - c)^2$.

Combining all of the above results, we find that a, b, c satisfy:

$$a = (1-c)^2 - 2, \quad b = 0, \quad 0 < c \le 1.$$

Now we prove that any triple a, b, c satisfying these constraints also satisfies the given conditions.

Suppose that $x \le 0$. Because $0 < c \le 1$, we have $a < -1 < 0$. Hence, $2x^2 + ax + b = x(2x+a)$ is the product of two non-positive numbers, implying that $\sqrt{2x^2+ax+b}$ is well-defined and non-negative. Also, $x \le 0$ and $c > 0$, implying that $x - c < 0$. Therefore, $\sqrt{2x^2+ax+b} \ge 0 > x - c$.

Next suppose that $x > 1$. As we argued above, when $a \ge -2$, $b = 0$, and $x > 1$, inequalities (1) and (2) are equivalent. Because $a = (1-c)^2 - 2$, inequality (2) fails for $x = 1$ and holds for $x > 1$, as desired.

Third, suppose that $0 \le x \le 1$. Along this interval, $(2x^2 + ax + b) - (x-c)^2$ (a convex function) attains a maximum at either $x = 0$ or $x = 1$. Thus, for $x \in [-0, 1]$, we have $(2x^2 + ax + b) - (x-c)^2 \le \max\{-c^2, 0\} = 0$. Therefore, $2x^2 + ax + b \le (x-c)^2$ for $0 \le x \le 1$.

- If $0 < x < -\frac{a}{2}$, then $\sqrt{2x^2+ax+b}$ is undefined because $2x^2 + ax + b = x(2x+a)$ is negative (it is the product of a positive number and a negative number). Hence, the given inequality fails.
- If $-\frac{a}{2} \le x < 1$, then we claim that $x - c \ge 0$. Indeed, $x - c \ge -\frac{a}{2} - c = \frac{1}{2}(2 - (1-c)^2) - c = \frac{1}{2}(1 - c^2)$, which is non-negative because $0 < c \le 1$. From $(2x^2 + ax + b) - (x-c)^2 \le 0$ and $x - c \ge 0$, we deduce that $\sqrt{2x^2+ax+b} \le \sqrt{(x-c)^2} = x - c$. Hence, the given inequality fails.

Problem 2 In a certain language there are n letters. A sequence of letters is called a *word* if and only if between any pair of identical letters, there is no other pair of equal letters. Prove that there exists a word of maximum possible length, and find the number of words which have that length.

Solution: Every word contains at most $3n$ letters, and there are $n! \cdot 2^{n-1}$ words with $3n$ letters.

Suppose a letter appears 4 or more times in a sequence; then between the outer two identical letters there are at least two equal letters, and the sequence is not a word. So each letter appears at most 3 times, and every word contains at most $3n$ letters.

We claim that the words with $3n$ letters are precisely those of the form

$$1, 1, \begin{bmatrix} 1,2 \\ \text{or} \\ 2,1 \end{bmatrix}, 2, \begin{bmatrix} 2,3 \\ \text{or} \\ 3,2 \end{bmatrix}, 3, \ldots, n-1, \begin{bmatrix} n-1,n \\ \text{or} \\ n,n-1 \end{bmatrix}, n, n.$$

(Here, all the letters labelled k are identical.) Once we prove this, we know that there are $n! \cdot 2^{n-1}$ words with $3n$ letters: there are $n!$ ways to label the n letters $1, 2, \ldots, n$, and then 2^{n-1} ways to order the $n-1$ pairs $\begin{bmatrix} k,k+1 \\ \text{or} \\ k+1,k \end{bmatrix}$.

First, note that any sequence of the above form is a word, because between any two identical letters k, there is at most one $k-1$, at most one k, at most one $k+1$, and no other letters.

Conversely, suppose that we have a word with $3n$ letters. Label the letters $1^-, 2^-, \ldots, n^-$, $1, 2, \ldots, n$, $1^+, 2^+, \ldots, n^+$, where k^-, k, k^+ are identical and appear in the word in that order, and where $1, 2, \ldots, n$ appear in the word in that order:

$$\ldots, 1, \ldots, 2, \ldots, 3, \ldots, \ldots, \ldots, n-1, \ldots, n, \ldots.$$

We claim that k^+ appears between k and $k+1$ for each $k = 1, 2, \ldots, n-1$. Suppose not, for the sake of contradiction. Then k^+ appears after $k+1$:

$$\ldots, k, \ldots, k+1, \ldots, k^+, \ldots.$$

Because $(k+1)^-$ and $k+1$ cannot both lie between the identical letters k, k^+, we know that $(k+1)^-$ appears before k:

$$\ldots, (k+1)^-, \ldots, k, \ldots, k+1, \ldots, k^+, \ldots.$$

But now k^- can neither come before $(k+1)^-$ (otherwise, between k^- and k^+ would come identical letters $(k+1)^-$ and $k+1$), and nor can k^- come after $(k+1)^-$ (otherwise, between $(k+1)^-$ and $k+1$ would come identical letters k^- and k). Hence, our assumption was false, and k^+ appears between k and $k+1$.

Likewise, $(k+1)^-$ appears between k and $k+1$ for each $k = 1, 2, \ldots, n-1$.

It easily follows that every $3n$-letter word is of the form described earlier, as desired.

Problem 3 Let $n \geq 1$ be an integer, and let $a_1, a_2, \ldots, a_n$ be positive integers. Let $f : \mathbb{Z} \to \mathbb{R}$ be a function such that $f(x) = 1$ for each integer $x < 0$ and

$$f(x) = 1 - f(x - a_1)f(x - a_2) \cdots f(x - a_n)$$

for each integer $x \geq 0$. Show that there exist positive integers s and t such that $f(x + t) = f(x)$ for any integer $x > s$.

Solution: First we claim that $f(x)$ equals 0 or 1 for all integers x. This is clearly true for $x < 0$. If it is true for all $x < k$, then $f(k - a_1)f(k - a_2) \cdots f(k - a_n)$ equals 0 or 1, so that

$$f(k) = 1 - f(k - a_1)f(k - a_2) \cdots f(k - a_n) \in \{1 - 0, 1 - 1\} = \{0, 1\}.$$

Hence, by induction, $f(x) \in \{0, 1\}$ for all integers x.

Let N be the largest of the a_k, and write $F(x) = (f(x + 1), f(x + 2), \ldots, f(x + N))$. Note that each $F(x)$ is one of the finitely many vectors in $\{0, 1\}^N$. Hence, some two of $F(1), F(2), \ldots$ are the same — say, $F(s) = F(s + t)$ for positive integers s, t. It follows easily by induction on x that $F(x) = F(x + t)$ for all integers $x \geq s$. Hence, $f(x) = f(x + t)$ for all integers $x > s$.

1.6 Hungary

Problem 1 Let x, y, and z be positive real numbers smaller than 4. Prove that among the numbers

$$\frac{1}{x}+\frac{1}{4-y}, \quad \frac{1}{y}+\frac{1}{4-z}, \quad \frac{1}{z}+\frac{1}{4-x},$$

there is at least one which is greater than or equal to 1.

Solution: Note that

$$\frac{1}{x}+\frac{1}{4-x}=\frac{4}{x(4-x)}=\frac{4}{4-(x-2)^2}\geq 1.$$

Similar inequalities hold for y and z. Thus, the sum of the three given quantities $\frac{1}{x}+\frac{1}{4-y}$, $\frac{1}{y}+\frac{1}{4-z}$, $\frac{1}{z}+\frac{1}{4-x}$ is greater than or equal to 3, implying that at least one of them is greater than or equal to 1.

Problem 2 Find all integers x, y, and z such that $5x^2-14y^2=11z^2$.

Solution: The only solution is $(0,0,0)$.

Assume, for sake of contradiction, that there is a triple of integers $(x,y,z)\neq(0,0,0)$ satisfying the given equation, and let $(x,y,z)=(x_0,y_0,z_0)$ be a nonzero solution that minimizes $|x+y+z|>0$.

Because $5x_0^2-14y_0^2=11z_0^2$, we have

$$-2x_0^2\equiv 4z_0^2 \pmod 7,$$

or $x_0^2\equiv -2z_0^2\equiv 5z_0^2 \pmod 7$. Therefore, we have $z_0\equiv 0 \pmod 7$, because otherwise we have

$$5\equiv\left(x_0z_0^{-1}\right)^2 \pmod 7,$$

which is impossible because 5 is not a quadratic residue modulo 7. (The squares modulo 7 are 0, 1, 2, and 4.)

It follows that x_0 and z_0 are divisible by 7, so that $14y^2=5x^2-11z^2$ is divisible by 49. Therefore, $7 \mid y_0$. Then $\left(\frac{x_0}{7},\frac{y_0}{7},\frac{z_0}{7}\right)$ is also a solution, but $\left|\frac{x_0}{7}+\frac{y_0}{7}+\frac{z_0}{7}\right|<|x_0+y_0+z_0|$, contradicting the minimality of (x_0,y_0,z_0).

Therefore, our original assumption was false, and the only integer solution is $(0,0,0)$.

Problem 3 Find all triangles ABC for which it is true that the median from A and the altitude from A are reflections of each other across the internal angle bisector from A.

Solution: It is easy to check that all triangles in which $AB = AC$ or $\angle CAB = \pi/2$ have the required property.

We now prove the converse: suppose we are given a triangle ABC with the required property. Let $\overline{AM}, \overline{AD}, \overline{AH}$ be the median, angle bisector, and altitude from A respectively. Without loss of generality, assume that $AB \geq AC$. Then it is well-known that B, M, D, H, C lie on $\overline{BC}$ in that order.

Let $\alpha = \angle BAC$, $\beta = \angle ABC$, $\gamma = \angle CAB$, and let $\theta = \angle HAC = \frac{\pi}{2} - \gamma$. Because lines AM and AH are reflections of each other across line AD, $\angle BAM = \theta$.

Because M is the midpoint of BC, $[ABM] = [ACM]$, or

$$\frac{1}{2}AB \cdot AM \sin\theta = \frac{1}{2}AC \cdot AM \sin(\alpha - \theta)$$

Also, from the Law of Sines, we have

$$\frac{AB}{AC} = \frac{\sin\gamma}{\sin\beta}$$

Combining these two equations gives

$$\begin{aligned}
\sin\gamma\sin\theta &= \sin\beta\sin(\alpha - \theta) \\
\sin\gamma\cos(\pi/2 - \theta) &= \sin\beta\cos(\pi/2 - \alpha + \theta) \\
\sin\gamma\cos\gamma &= \sin\beta\cos\beta \\
\sin 2\gamma &= \sin 2\beta,
\end{aligned}$$

so that $2\beta = 2\gamma$ or $2\beta + 2\gamma = \pi$, i.e., so that $\beta = \gamma$ or $\alpha = \pi/2$, as claimed.

Problem 4 Let m and n be integers such that $1 \leq m \leq n$. Prove that m is a divisor of

$$n\sum_{k=0}^{m-1}(-1)^k\binom{n}{k}.$$

Solution: We can rewrite the given expression as follows:

$$\begin{aligned}
n\sum_{k=0}^{m-1}(-1)^k\binom{n}{k} &= n\sum_{k=0}^{m-1}(-1)^k\left(\binom{n-1}{k}+\binom{n-1}{k-1}\right)\\
&= n\sum_{k=0}^{m-1}(-1)^k\binom{n-1}{k}+n\sum_{k=1}^{m-1}(-1)^k\binom{n-1}{k-1}\\
&= n\sum_{k=0}^{m-1}(-1)^k\binom{n-1}{k}-n\sum_{k=0}^{m-2}(-1)^k\binom{n-1}{k}\\
&= n(-1)^{m-1}\binom{n-1}{m-1}\\
&= m(-1)^{m-1}\binom{n}{m}.
\end{aligned}$$

The final expression is clearly divisible by m.

Problem 5 Find all real numbers c with the following property: Given any triangle, one can find two points A and B on its perimeter so that they divide the perimeter in two parts of equal length and so that AB is at most c times the perimeter.

Solution: Answer: All $c \geq \frac{1}{4}$.

Suppose that A and B lie on the perimeter of triangle XYZ and divide the perimeter in two parts of equal length. Let s and p be the semiperimeter and perimeter of the triangle. We claim that if A and B lie on $\overline{XY}$, $\overline{XZ}$, then

$$AB^2 \geq \frac{1}{2}s^2(1-\cos\angle X), \tag{$*$}$$

with equality when $XA = XB = \frac{s}{2}$. Let $a = XA$, $XB = b$. Then we have

$$s^2 = (a+b)^2 = a^2 + 2ab + b^2$$

Also, from the Law of Cosines, we have

$$AB^2 = a^2 + b^2 - 2ab\cos\angle X$$

Therefore, we have

$$\begin{aligned} s^2 - AB^2 &= 2ab + 2ab\cos\angle X \\ &= 2ab(\cos\angle X + 1) \\ &\le 2\left(\frac{a+b}{2}\right)^2 (\cos\angle X + 1) \\ &= \frac{1}{2}s^2(\cos\angle X + 1), \end{aligned}$$

with equality when $XA = XB = \frac{s}{2}$. Rearranging this final inequality gives the desired inequality.

Suppose now that triangle XYZ is equilateral, and that A and B lie on the perimeter and divide it into two equal pieces. A and B cannot lie on the same side of the triangle, since each side has length less than $s/2$. Then from $(*)$, we have

$$AB^2 \ge \frac{1}{2}s^2\left(1 - \cos\frac{\pi}{3}\right) = \frac{1}{4}s^2 = \frac{1}{16}p^2,$$

so that $AB \ge \frac{1}{4}p$. Therefore, any c with the required property must be at least $\frac{1}{4}$.

Conversely, given any $c \ge \frac{1}{4}$ and any triangle XYZ, let $x = YZ$, $y = ZX$, $z = XY$. Without loss of generality assume that $x \le y, z$, so that $\angle X \le \pi/3$. Then (using the inequalities $x+z > y$ and $z \ge x$), $z - \frac{s}{2} = \frac{(x+z-y)+2(z-x)}{4} > 0$, so we may locate a point A on $\overline{XY}$ such that $XA = \frac{s}{2}$. Similarly, we may locate a point B on $\overline{XZ}$ such that $XB = \frac{s}{2}$. Then A and B divide the perimeter into two equal pieces, and

$$AB^2 = \frac{1}{2}s^2(1 - \cos\angle X) \le \frac{1}{2}s^2(1 - \cos\angle\pi/3) = \frac{1}{4}s^2 = \frac{1}{16}p^2,$$

so that $AB \le \frac{1}{4}p \le cp$, as desired.

Problem 6 The circles k_1 and k_2 and the point P lie in a plane. There exists a line ℓ and points $A_1, A_2, B_1, B_2, C_1, C_2$ with the following properties: ℓ passes through P and intersects k_i at A_i and B_i for $i = 1, 2$; C_i lies on k_i for $i = 1, 2$; C_1 and C_2 lie on the same side of ℓ; and $A_1C_1 = B_1C_1 = A_2C_2 = B_2C_2$. Describe how to construct such a line and such points given only k_1, k_2, and P.

Solution: Let O_i be the center of k_i and let D_i be the midpoint of $\overline{A_iB_i}$. Also let r_i be the radius of k_i and let $d_i = D_iC_i$, the distance from C_i to ℓ.

Because $\overline{O_iD_iC_i} \perp \overline{A_iB_i}$, we can apply the Pythagorean Theorem twice to find that

$$\begin{aligned} A_iC_i^2 &= A_iD_i^2 + D_iC_i^2 = (A_iO_i^2 - D_iO_i^2) + D_iC_i^2 \\ &= r_i^2 - (r_i - d_i)^2 + d_i^2 = 2r_id_i. \end{aligned}$$

Because $A_1C_1 = A_2C_2$, this implies that $r_1d_1 = r_2d_2$, or $d_1 = d_2\frac{r_2}{r_1}$.

Dilate k_2 about P with ratio $\frac{r_2}{r_1}$ to produce k_2'. The distance from the image C_2' of C_2 to ℓ is $d_2\frac{r_2}{r_1} = d_1$. Also, because C_1 and C_2 are on the same side of ℓ, so are C_1 and C_2'. Hence, there is a line m parallel to ℓ that passes through C_1 and C_2'.

Temporarily assume, without loss of generality, that ℓ is vertical and that the C_i lie to its right. Then the tangent line to k_i at C_i is vertical and lies to the right of k_i. With $i = 1$, this line must be m because m passes through C_1 and is parallel to ℓ. With $i = 2$, this implies that the tangent to k_2' at C_2' is vertical and lies to the right of k_2'; again, this line must be m because m passes through C_2' and is parallel to ℓ. Hence, there exists a common external tangent to k_1 and k_2' that is parallel to ℓ.

This gives us the following method of constructing ℓ if we are only given k_1, k_2, and P. First, dilate k_2 about P with ratio $\frac{r_2}{r_1}$ to produce k_2'. Draw the two external tangents to k_1 and k_2' (we showed above that one must exist). Then, draw the two lines through P parallel to these external tangents. It is easy to check each line to see if it has the required properties, and at least one (if not both) must because one of the lines is ℓ.

Problem 7 Let k and m be positive integers, and let $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_m$ be distinct integers greater than 1. Each a_i is the product of an even number of primes, not necessarily distinct, while each b_i is the product of an odd number of primes, again not necessarily distinct. How many ways can we choose several of the $k+m$ given numbers such that each b_i has an even number of divisors among the chosen numbers?

Solution: The answer is 2^k. We show that for any given $T \subseteq \{a_1, \ldots, a_k\}$, there exists exactly one $S \subseteq \{b_1, \ldots, b_m\}$ such that each b_i has an even number of divisors among the numbers in $T \cup S$.

Fix T. Without loss of generality, assume $b_i < b_j$ for $i < j$. Therefore, $b_j \not| \, b_i$ for $i < j$. Thus, for any S such that $T \cup S$ satisfies the given condition, all divisors of b_i that are in $T \cup S$ are in $T \cup \{b_j \mid j \leq i\}$. From the given assumption that the a_i and b_i are distinct, we have $b_i \notin \{b_j \mid j < i\}$ and $b_i \notin T$. Also, $b_i \mid b_i$. Therefore, if T and $S \cap \{b_j \mid j < i\}$ have been chosen, exactly one of the choices $b_i \in S$, $b_i \notin S$ will result in b_i having an even number of divisors in $S \cup T$. Therefore, as i runs from 1 to m, each choice between $b_i \in S$, $b_i \notin S$ is forced — and if we construct S this way, then each b_i will in fact have an even number of divisors in $S \cup T$.

Thus, for T fixed, there exists exactly one set S that works. Because for each a_i we can have $a_i \in T$ or $a_i \notin T$, there are 2^k possible sets T. Therefore 2^k possible sets $T \cup S$ that satisfy the given condition.

1.7 India

Problem 1 Every vertex of the unit squares on an $m \times n$ chessboard is colored either blue, green, or red, such that all the vertices on the boundary of the board are colored red. We say that a unit square of the board is *properly colored* if exactly one pair of adjacent vertices of the square are the same color. Show that the number of properly colored squares is even.

Solution: We introduce a coloring of the segments forming the edges of the grid squares. Color an edge black if its endpoints are of the same color, and white otherwise. Then a square is properly colored if and only if exactly one of its four sides is colored black. Let s_i be the number of squares with i sides colored black, $i = 0, 1, \ldots, 4$. Then the sum

$$0 \cdot s_0 + 1 \cdot s_1 + 2 \cdot s_2 + 3 \cdot s_3 + 4 \cdot s_4$$

counts each black edge on the boundary of the board once and each black edge in the interior twice. Let k be the number of black edges in the interior. All $2(m+n)$ edges on the boundary of the board are black, so we have

$$\sum_{i=0}^{4} i \cdot s_i = 2(m+n) + 2k.$$

Hence,

$$s_1 + 3s_3 \equiv 0 \pmod 2.$$

It is impossible for exactly three edges of a square to be colored black, because then all vertices of the square must be the same color, implying that all four edges of the square should be black. Thus $s_3 = 0$, and it follows that s_1, the number of properly colored squares, is even.

Problem 2 Let $ABCD$ be a rectangle, and let Γ be an arc of a circle passing through A and C. Let Γ_1 be a circle which is tangent to lines CD and DA as well as tangent to Γ. Similarly, let Γ_2 be a circle lying completely inside rectangle $ABCD$ which is tangent to lines AB and BC as well as tangent to Γ. Suppose that Γ_1 and Γ_2 both lie completely in the closed region bounded by rectangle $ABCD$.

Let r_1 and r_2 be the radii of Γ_1 and Γ_2, respectively, and let r be the inradius of triangle ABC.

(a) Prove that $r_1 + r_2 = 2r$.

(b) Show that one of the common internal tangents to Γ_1 and Γ_2 is parallel to $\overline{AC}$ and has length $|AB - BC|$.

Solution:

(a) Let E denote the center of $ABCD$. We introduce a coordinate system centered at E, so that A, B, C, D have coordinates $(-x, y), (x, y), (x, -y), (-x, -y)$ respectively. We suppose without loss of generality that the center of Γ is on the same side of line AC as D. Letting that center be O, we know that O is on the perpendicular bisector of $\overline{AC}$, so its coordinates are $(-ty, -tx)$ for some $t \geq 0$. Let O_1 and O_2 be the respective centers of Γ_1 and Γ_2. Then $O_1 = (-x + r_1, -y + r_1)$ and $O_2 = (x - r_2, y - r_2)$.

We now proceed to compute r_1 and r_2 in terms of t, x, and y. Letting the radius of Γ be R, we have

$$OO_1 = R - r_1.$$

Calculating OO_1 and $R = OA$ with the distance formula, we have

$$\sqrt{(r_1 + ty - x)^2 + (r_1 + tx - y)^2} = \sqrt{(ty - x)^2 + (tx + y)^2} - r_1$$

Squaring the left side gives

$$2r_1^2 + 2r_1(t - 1)(x + y) + (t^2 + 1)(x^2 + y^2) - 4txy.$$

Squaring the right side gives

$$r_1^2 - 2r_1\sqrt{(-x + ty)^2 + (y + tx)^2} + (t^2 + 1)(x^2 + y^2).$$

This two expressions are equal, so after simplifying we obtain the quadratic equation in r_1

$$r_1^2 + 2r_1\left[(t - 1)(x + y) + \sqrt{(t^2 + 1)(x^2 + y^2)}\right] - 4txy = 0.$$

Letting D denote the discriminant, we have

$$\begin{aligned}\frac{D}{4} &= \left((t - 1)(x + y) + \sqrt{(t^2 + 1)(x^2 + y^2)}\right)^2 + 4txy \\ &= (t - 1)^2(x + y)^2 + (t^2 + 1)(x^2 + y^2) + 4txy \\ &\quad + 2(t - 1)(x + y)\sqrt{(t^2 + 1)(x^2 + y^2)}.\end{aligned}$$

Hence, $\frac{D}{4} - 2(t-1)(x+y)\sqrt{(t^2+1)(x^2+y^2)}$ equals to

$$(t^2+1)(x+y)^2 - 2t(x+y)^2 + (t^2+1)(x^2+y^2) + 4txy,$$

which can be rewritten as

$$(t^2+1)(x+y)^2 - 2t(x^2+y^2) - 4txy + (t^2+1)(x^2+y^2) + 4txy.$$

Finally after simplifying we come to

$$(t^2+1)(x+y)^2 + (t-1)^2(x^2+y^2),$$

so that

$$\frac{D}{4} = \left(\sqrt{t^2+1}(x+y) + (t-1)\sqrt{x^2+y^2}\right)^2.$$

Observing that the product of the roots of our quadratic is $-4txy < 0$, we must take r_1 to be the positive root. Thus, applying the quadratic formula, we obtain

$$\begin{aligned} r_1 = &- \left[(t-1)(x+y) + \sqrt{(t^2+1)(x^2+y^2)}\right] \\ &+ \left[\sqrt{t^2+1}(x+y) + (t-1)\sqrt{x^2+y^2}\right]. \end{aligned}$$

The calculation of r_2 is similar; the quadratic we obtain is

$$r_2^2 - 2r_2\left[(t+1)(x+y) + \sqrt{(t^2+1)(x^2+y^2)}\right] + 4xyt = 0,$$

and the discriminant again simplifies to a square, ultimately yielding

$$\begin{aligned} r_2 = &\left[(t+1)(x+y) + \sqrt{(t^2+1)(x^2+y^2)}\right] \\ &- \left[\sqrt{t^2+1}(x+y) + (t+1)\sqrt{x^2+y^2}\right]. \end{aligned}$$

Note that this time we took the negative square root of the discriminant, because the first term on the right, $(t+1)(x+y)$, is already larger than $x+y$, while the radius r_2 is bounded by $\frac{1}{2}x$.

Finally, adding our expressions for r_1 and r_2, we obtain

$$r_1 + r_2 = 2(x+y) - 2\sqrt{x^2+y^2}.$$

On the other hand, equating the area formulas $A = rs = \frac{1}{2}bh$ for triangle ABC, we obtain

$$r(x+y+\sqrt{x^2+y^2}) = \frac{(2x)(2y)}{2}.$$

Multiplying through by $x + y - \sqrt{x^2 + y^2}$ and simplifying gives

$$r = x + y - \sqrt{x^2 + y^2},$$

giving $r_1 + r_2 = 2r$, as wanted.

(b) Let C_1 and C_2 be the incircles of triangles ADC and ABC, and let T_1 and T_2 be their points of tangency with $\overline{AC}$. Let U_1 be the image of T_1 under the homothety H_1 about D that takes C_1 to Γ_1, and define U_2 and H_2 analogously. We claim that the vectors $\overrightarrow{T_1T_2}$ and $\overrightarrow{U_1U_2}$ are equal.

Because $\overrightarrow{DT_1} + \overrightarrow{T_1T_2} + \overrightarrow{T_2B} = \overrightarrow{DB} = \overrightarrow{DU_1} + \overrightarrow{U_1U_2} + \overrightarrow{U_2B}$, it suffices to show that $\overrightarrow{DT_1} + \overrightarrow{T_2B} = \overrightarrow{DU_1} + \overrightarrow{U_2B}$. Observe that all of these vectors are parallel and oriented in the same direction: in fact, $\overrightarrow{DT_1} = \overrightarrow{T_2B}$ by symmetry, and $\overrightarrow{DU_1}$ and $\overrightarrow{U_2B}$ are the images of $\overrightarrow{DT_1}$ and $\overrightarrow{T_2B}$ under homothety. Hence, it suffices to show that the sums of the lengths of the vectors on each side are the same. Now, because H_1, which takes T_1 to U_1, also takes C_1 to Γ_1, its ratio is r_1/r. Hence,

$$DU_1 = \frac{r_1}{r} \cdot DT_1.$$

Likewise,

$$U_2B = \frac{r_2}{r} \cdot T_2B.$$

But because $DT_1 = T_2B$, we have

$$DU_1 + U_2B = \frac{r_1 + r_2}{r} \cdot DT_1 = 2DT_1 = DT_1 + T_2B,$$

substituting $r_1 + r_2 = 2r$ from part (a). This proves the claim.

To complete the proof, observe first that $\overrightarrow{U_1U_2} = \overrightarrow{T_1T_2}$ implies that lines U_1U_2 and AC are parallel. It follows that line U_1U_2 is the image of line AC under homothety H_1. Because C_1 was tangent to AC, Γ_1 must therefore be tangent to U_1U_2. Likewise, Γ_2 is tangent to U_1U_2 as well. Furthermore, again by considering the homotheties, U_1 and U_2 must be the points of tangency of Γ_1 and Γ_2 with line U_1U_2. Hence $\overline{U_1U_2}$ is a common internal tangent to Γ_1 and Γ_2, and is parallel to line AC. All that remains to be shown now is that $U_1U_2 = |AB - AC|$.

Reusing our result $\overrightarrow{U_1U_2} = \overrightarrow{T_1T_2}$, we have $U_1U_2 = T_1T_2$. By a standard computation involving equal tangents, we compute

$AT_2 = CT_1 = \frac{1}{2}(AB + AC - BC)$. Hence,

$$T_1T_2 = |AT_2 + CT_1 - AC| = |AB - BC|,$$

as desired.

Problem 3 Let a_1, a_2, ... be a strictly increasing sequence of positive integers such that $\gcd(a_m, a_n) = a_{\gcd(m,n)}$ for all positive integers m and n. There exists a least positive integer k for which there exist positive integers $r < k$ and $s > k$ such that $a_k^2 = a_r a_s$. Prove that r divides k and that k divides s.

Solution: We begin by proving a lemma:

Lemma. *If positive integers a, b, c satisfy $b^2 = ac$, then*

$$\gcd(a, b)^2 = \gcd(a, c) \cdot a.$$

Proof. Consider any prime p. Let e be the highest exponent such that p^e divides b, and let e_1 and e_2 be the corresponding highest exponents for a and c, respectively. Because $b^2 = ac$, we have $2e = e_1 + e_2$. If $e_1 \geq e$, then the highest powers of p that divide $\gcd(a, b), \gcd(a, c)$, and a are e, e_2, and e_1, respectively. Otherwise, these highest powers are all e_1. Therefore, in both cases, the exponent of p on the left side of the desired equation is the same as the exponent of p on the right side. The desired result follows. $\square$

Applying the lemma to the given equation $a_k^2 = a_r a_s$, we have

$$\gcd(a_r, a_k)^2 = \gcd(a_r, a_s)a_r.$$

It now follows from the given equation that

$$a_{\gcd(r,k)}^2 = a_{\gcd(r,s)}a_r.$$

Assume, for sake of contradiction, that $\gcd(r, k) < r$, so that $a_{\gcd(r,k)} < a_r$. Then from the above equation, it follows that $a_{\gcd(r,k)} > a_{\gcd(r,s)}$, so that $\gcd(r, k) > \gcd(r, s)$. But then we have $(k_0, r_0, s_0) = (\gcd(r, k), \gcd(r, s), r)$ satisfies $a_{k_0}^2 = a_{r_0}a_{s_0}$ with $r_0 < k_0 < s_0$ and $k_0 < r < k$, contradicting the minimality of k.

Thus, we must have $\gcd(r, k) = r$, implying that $r \mid k$. Then

$$\gcd(a_r, a_k) = a_{\gcd(r,k)} = a_r,$$

so $a_r \mid a_k$. Thus $a_s = a_k \frac{a_k}{a_r}$ is an integer multiple of a_k, and

$$a_{\gcd(k,s)} = \gcd(a_k, a_s) = a_k.$$

Because $a_1, a_2, \ldots$ is increasing, it follows that $\gcd(k, s) = k$. Therefore, $k \mid s$, completing the proof.

Problem 4 Let $a \geq 3$ be a real number and $p(x)$ be a polynomial of degree n with real coefficients. Prove that

$$\max_{0 \leq j \leq n+1} \{|a^j - p(j)|\} \geq 1.$$

Solution: Let $y_j = p(j)$ for $j = 0, 1, \ldots, n+1$. Then $p(x)$ is the unique polynomial of degree at most $n+1$ that passes through all of the points (j, y_j). By Lagrange Interpolation, this polynomial is

$$p(x) = \sum_{j=0}^{n+1} y_j \cdot \frac{(x-0)\cdots(x-(j-1))(x-(j+1))\cdots(x-(n+1))}{(j-0)\cdots(j-(j-1))(j-(j+1))\cdots(j-(n+1))}.$$

Because p has degree n, the coefficient of x^{n+1} is

$$\sum_{j=0}^{n+1} \frac{y_j}{j!(n+1-j)!(-1)^{n+1-j}} = 0.$$

Multiplying by $(n+1)!$, we have

$$\sum_{j=0}^{n+1} \binom{n+1}{j} (-1)^{n+1-j} y_j = 0. \qquad (*)$$

Now assume, for sake of contradiction, that

$$a^j - 1 < y_j < a^j + 1$$

for all j. Then

$$(-1)^{n+1-j} a^j - 1 < (-1)^{n+1-j} y_j < (-1)^{n+1-j} a^j + 1$$

for all j. Therefore,

$$\begin{aligned}
\sum_{j=0}^{n+1} \binom{n+1}{j} (-1)^{n+1-j} y_j &> \sum_{j=0}^{n+1} \binom{n+1}{j} ((-1)^{n+1-j} a^j - 1) \\
&= (a-1)^{n+1} - 2^{n+1} \\
&\geq (3-1)^{n+1} - 2^{n+1} \\
&= 0,
\end{aligned}$$

which contradicts our earlier calculation $(*)$. Therefore, our initial assumption was false. The desired result follows.

1.8 Iran

Problem 1 Let α be a real number between 1 and 2, exclusive. Prove that α has a unique representation as an infinite product

$$\alpha = \prod_{k=1}^{\infty} \left(1 + \frac{1}{n_k}\right),$$

where each n_k is a natural number and $n_k^2 \leq n_{k+1}$ for all $k \geq 1$.

Solution:

Lemma. *If the integers $n_1, n_2, \ldots$ satisfy $n_k > 1$ and $n_k^2 \leq n_{k+1}$ for all $k \geq 1$, then*

$$\prod_{k=j}^{\infty} \left(1 + \frac{1}{n_k}\right) \in \left(1 + \frac{1}{n_j}, 1 + \frac{1}{n_j - 1}\right]$$

for each j.

Proof. The lower bound is clear. For the upper bound, observe that

$$\prod_{k=j}^{\infty} \left(1 + \frac{1}{n_k}\right) \leq \prod_{k=0}^{\infty} \left(1 + \frac{1}{n_j^{2^k}}\right) = \prod_{k=0}^{\infty} \left(1 + \left(\frac{1}{n_j}\right)^{2^k}\right).$$

Observe that for $0 < a < 1$, we have

$$(1+a)(1+a^2)(1+a^4)\cdots(1+a^{2^k})\cdots = 1 + a + a^2 + a^3 + \cdots = \frac{1}{1-a}.$$

(The first equality holds because the monomial a^j can be written uniquely as a product of finitely many, distinct a^{2^k}, according to the binary representation of j.) Applying this with $a = \frac{1}{n_j}$ above, we find that

$$\prod_{k=j}^{\infty} \left(1 + \frac{1}{n_k}\right) \leq \frac{1}{1 - \frac{1}{n_j}} = 1 + \frac{1}{n_j - 1}.$$

$\square$

Fix $\alpha \in (1, 2)$. Suppose that we can write

$$\alpha = \prod_{k=1}^{\infty} \left(1 + \frac{1}{n_k}\right),$$

where $n_k^2 \leq n_{k+1}$ for all $k \geq 1$. Because $\alpha < 2$, we have $n_k > 1$ for all k. Also observe that for any $x \in (1, 2)$, there exists a unique value $n > 1$ such that $x \in (1 + \frac{1}{n}, 1 + \frac{1}{n-1}]$.

Define $\alpha_1, \alpha_2, \ldots$ recursively by setting $\alpha_1 = \alpha$ and

$$\alpha_{k+1} = \frac{\alpha_k}{1 + \frac{1}{n_k}} \tag{$*$}$$

for $k \geq 1$. We know that $\alpha_1 \in (1, 2)$, and for $k \geq 1$ we have $\alpha_{k+1} \in (1, 2)$ because $1 + \frac{1}{n_k} < \alpha_k \leq 2 < 2(1 + \frac{1}{n_k})$.

By the lemma, for each $k \geq 1$ we have

$$\alpha_k \in \left(1 + \frac{1}{n_k}, 1 + \frac{1}{n_k - 1}\right]. \tag{$\dagger$}$$

There is exactly one value $n_k > 1$ such that the above condition holds. Therefore, α can be written in the desired form in at most one way.

Even if we do not know that α can be written in the desired form, we can set α_1 and recursively define $\alpha_1, n_1, \alpha_2, n_2, \ldots$ using ($*$) and ($\dagger$) above. Then for $k \geq 1$, we have

$$1 + \frac{1}{n_{k+1}} < \alpha_{k+1} = \frac{\alpha_k}{1 + \frac{1}{n_k}} \leq \frac{1 + \frac{1}{n_k - 1}}{1 + \frac{1}{n_k}} = 1 + \frac{1}{n_k^2 - 1}.$$

Thus, $n_{k+1} > n_k^2 - 1$, or $n_{k+1} \geq n_k^2$, as required.

By the definition of the n_k and by the lemma,

$$1 < \frac{\alpha}{\prod_{k=1}^{N}\left(1 + \frac{1}{n_k}\right)} = \prod_{k=N+1}^{\infty}\left(1 + \frac{1}{n_k}\right) \leq 1 + \frac{1}{n_{N+1} - 1}.$$

As $N \to \infty$, $n_{N+1} \to \infty$. Hence, the partial product $\prod_{k=1}^{N}\left(1 + \frac{1}{n_k}\right)$ converges to α.

Therefore, any $\alpha \in (1, 2)$ can be written in the desired form, and in exactly one way.

Problem 2 We flip a fair coin repeatedly until encountering three consecutive flips of the form (i) two tails followed by heads, or (ii) heads, followed by tails, followed by heads. Which sequence, (i) or (ii), is more likely to occur first?

Solution: For either (i) or (ii) to occur, the coin must show tails at some point. The first time tails comes up, exactly one of the following cases occurs: (a) the flip immediately before was a heads; (b) the tails is the first flip, and it is followed by heads; (c) the tails was the first flip, and it is followed by tails.

In case (a), if the next flip is a heads, then (ii) occurs before (i). However, if the next flip is a tails, then (i) will occur before (ii). So, (i) and (ii) are equally likely to occur first in this case.

In case (b), we are forced into position (a) eventually before either (i) or (ii) occurs, so again (i) and (ii) are equally likely to occur first in this case.

In case (c), (i) will occur before (ii).

Therefore, (i) is more likely to occur first.

Problem 3 Suppose that x, y, and z are natural numbers such that $xy = z^2 + 1$. Prove that there exist integers a, b, c, and d such that $x = a^2 + b^2$, $y = c^2 + d^2$, and $z = ac + bd$.

Solution: We prove the claim by strong induction on z. For $z = 1$, we have $(x, y) = (1, 2)$ or $(2, 1)$; in the former (resp. latter) case, we can set $(a, b, c, d) = (1, 0, 1, 1)$ (resp. $(0, 1, 1, 1)$).

Suppose that the claim is true whenever $z < z_0$, and that we wish to prove it for $(x, y, z) = (x_0, y_0, z_0)$ where $x_0 y_0 = z_0^2 + 1$. Without loss of generality, assume that $x_0 \le y_0$. Consider the triple $(x_1, y_1, z_1) = (x_0, x_0 + y_0 - 2z_0, z_0 - x_0)$, so that $(x_0, y_0, z_0) = (x_1, x_1 + y_1 + 2z_1, x_1 + z_1)$.

First, using the fact that $x_0 y_0 = z_0^2 + 1$, it is easy to check that $(x, y, z) = (x_1, y_1, z_1)$ satisfies $xy = z^2 + 1$.

Second, we claim that $x_1, y_1, z_1 > 0$. This is obvious for x_1. Next, note that $y_1 = x_0 + y_0 - 2z_0 \ge 2\sqrt{x_0 y_0} - 2z_0 > 2z_0 - 2z_0 = 0$. Finally, because $x_0 \le y_0$ and $x_0 y_0 = z_0^2 + 1$, we have $x_0 \le \sqrt{z_0^2 + 1}$, or $x_0 \le z_0$. However, $x_0 \ne z_0$, because this would imply that $z_0 y_0 = z_0^2 + 1$, but $z_0 \not| (z_0^2 + 1)$ when $z_0 > 1$. Thus, $z_0 - x_0 > 0$, or $z_1 > 0$.

Therefore, (x_1, y_1, z_1) is a triple of positive integers (x, y, z) satisfying $xy = z^2 + 1$ and with $z < z_0$. By the induction hypothesis, we can write $x_1 = a^2 + b^2$, $y_1 = c^2 + d^2$, and $z_1 = ac + bd$. Then

$$\begin{aligned}
(ac + bd)^2 &= z_1^2 = x_1 y_1 - 1 \\
&= (a^2 + b^2)(c^2 + d^2) - 1 \\
&= (a^2c^2 + b^2d^2 + 2abcd) + (a^2d^2 + b^2c^2 - 2abcd) - 1 \\
&= (ac + bd)^2 + (ad - bc)^2 - 1,
\end{aligned}$$

so that $|ad - bc| = 1$.

Now, note that $x_0 = x_1 = a^2 + b^2$ and $y_0 = x_1 + y_1 + 2z_1 = a^2 + b^2 + c^2 + d^2 + 2(ac + bd) = (a + c)^2 + (b + d)^2$; in other words, $x_0 = a'^2 + b'^2$ and $y_0 = c'^2 + d'^2$ for $(a', b', c', d') = (a, b, a + c, b + d)$. Then $|a'd' - b'c'| = |ad - bc| = 1$, implying (by logic analogous to the reasoning in the previous paragraph) that $z_0 = a'c' + b'd'$, as desired. This completes the inductive step, and the proof.

Problem 4 Let ACE be a triangle, B be a point on $\overline{AC}$, and D be a point on $\overline{AE}$. Let F be the intersection of $\overline{CD}$ and $\overline{BE}$. If $AB + BF = AD + DF$, prove that $AC + CF = AE + EF$.

Solution: Let ω be the excircle of triangle BCF opposite B, tangent to line CD at V, line BFE at W, and line ABC at X.

We show that ω is also the excircle of triangle DEF opposite D (i.e., that ω is tangent to line ADE). By equal tangents, we have

$$\begin{aligned} AX &= AB + BX = AB + BW = AB + BF + FW \\ &= AD + DF + FV = AD + DV. \end{aligned}$$

Consider the following two distinct circles: the circle centered at A with radius AD, and the circle concentric to ω, consisting of the points P such that the tangent from P to ω has length DV. Because the length of each tangent from A to ω is $AX = AD + DV$, these two circles intersect on each tangent from A to ω (at the point a distance AD from A and a distance DV from the tangent point). These circles also intersect at D. But two distinct circles intersect in at most two points, implying that D must lie on a tangent from A to ω. That is, line AD is tangent to ω at some point Y, as claimed.

Therefore,

$$AC + CF = AC + (CV + VF) = (AC + CX) + VF = AX + VF$$

and

$$AE + EF = AE + (EW + WF) = (AE + EY) + WF = AY + WF.$$

Because $AX = AY$ and $VF = WF$, we have $AC + CF = AE + EF$, as desired.

Problem 5 Suppose that $a_1, a_2, \ldots$ is a sequence of natural numbers such that for all natural numbers m and n, $\gcd(a_m, a_n) =$

$a_{\gcd(m,n)}$. Prove that there exists a sequence $b_1, b_2, \ldots$ of natural numbers such that $a_n = \prod_{d|n} b_d$ for all integers $n \geq 1$.

Solution: For each n, let $\operatorname{rad}(n)$ denote the largest square-free divisor of n (i.e., the product of all distinct prime factors of n). We let b_n equal to the ratio of the following two numbers:

- E_n, the product of all $a_{n/d}$ such that d is square-free, divides n, and has an even number of prime factors.
- O_n, the product of all $a_{n/d}$ such that d is square-free, divides n, and has an odd number of prime factors.

Lemma. $\prod_{d|a_n} b_d = a_n$.

Proof. Fix n, and observe that $\prod_{d|n} b_n$ equals

$$\frac{\prod_{d|n} E_d}{\prod_{d|n} O_d}. \tag{$*$}$$

In the numerator of $(*)$, each E_d is the product of a_m such that $m \mid d$. Also, $d \mid n$, implying that the numerator is the product of various a_m such that $m \mid n$. For fixed m that divides n, how many times does a_m appear in the numerator $\prod_{d|n} E_d$ of $(*)$?

If a_m appears in E_d and $d \mid n$, then let $t = d/m$. By the definition of E_d, we know that (i) t is square-free and (ii) t has an even number of prime factors. Because $d \mid n$ and $t = d/m$, we further know that (iii) t divides n/m.

Conversely, suppose that t is any positive integer satisfying (i), (ii), and (iii), and write $d = tm$. By (iii), d is a divisor of n. Also, t is square-free by (i), is a divisor of d, and has an even number of prime factors by (ii). Thus, a_m appears in E_d.

Suppose that n/m has ℓ distinct prime factors. Then it has $\binom{\ell}{0} + \binom{\ell}{2} + \cdots$ factors t satisfying (i), (ii), and (iii), implying that a_m appears in the numerator of $(*)$ exactly

$$\binom{\ell}{0} + \binom{\ell}{2} + \cdots$$

times. Similarly, a_m appears in the denominator of $(*)$ exactly

$$\binom{\ell}{1} + \binom{\ell}{3} + \cdots$$

times. If $m < n$, then $\ell \geq 1$ and these expressions are equal, so that the a_m's in the numerator and denominator of $(*)$ cancel each other out. If $m = n$, then $\ell = 0$, so that a_n appears in the numerator once and in the denominator zero times. Therefore,

$$\prod_{d|n} b_d = \frac{\prod_{d|n} E_d}{\prod_{d|n} O_d} = a_n,$$

as desired. $\square$

Lemma. *For any integer α that divides some term in $a_1, a_2, \ldots,$ there exists an integer d such that*

$$\alpha \mid a_n \iff d \mid n.$$

Proof. Of all the integers n such that $\alpha \mid a_n$, let d be the smallest.

If $\alpha \mid a_n$, then $\alpha \mid \gcd(a_d, a_n) = a_{\gcd(d,n)}$. By the minimal definition of d, $\gcd(d, n) \geq d$. But $\gcd(d, n) \mid n$ as well, implying that $\gcd(d, n) = d$. Hence, $d \mid n$.

If $d \mid n$, then $\gcd(a_d, a_n) = a_{\gcd(d,n)} = a_d$. Thus, $a_d \mid a_n$. Because $\alpha \mid a_d$, it follows that $\alpha \mid a_n$ as well. $\square$

Lemma. *For each positive integer n, $b_n = E_n/O_n$ is an integer.*

Proof. Fix n. Call an integer d a *top divisor* (resp. a *bottom divisor*) if $d \mid n$, n/d is square-free, and n/d has an even (resp. odd) number of prime factors. By definition, E_d is the product of a_d over all top divisors d, and O_d is the product of a_d over all bottom divisors d.

Fix any prime p. We show that p divides E_n at least as many times as it divides O_n. To do this, it suffices to show the following for any positive integer k:

> (†) The number of top divisors d with $a_{n/d}$ divisible by p^k is greater than or equal to the number of bottom divisors d with $a_{n/d}$ divisible by p^k.

Let k be any positive integer. If p^k divides none of $a_1, a_2, \ldots$, then (†) holds trivially. Otherwise, by the previous lemma, there exists an integer d_0 such that

$$p^k \mid a_m \iff d_0 \mid m.$$

Hence, to show (†) it suffices to show:

> (‡) The number of top divisors d such that $d_0 \mid (n/d)$, is greater than or equal to the number of bottom divisors d such that $d_0 \mid (n/d)$.

If $d_0 \not| \, n$, then (‡) holds because d_0 does not divide n/d for *any* divisor d of n, including top or bottom divisors.

Otherwise, $d_0 \mid n$. For which top and bottom divisors d does d_0 divide n/d? Precisely those for which d divides n/d_0. If n/d_0 has $\ell \geq 1$ distinct prime factors, then there are as many top divisors with this property as there are bottom divisors, namely

$$\binom{\ell}{0} + \binom{\ell}{2} + \cdots = 2^{\ell-1} = \binom{\ell}{1} + \binom{\ell}{3} + \cdots .$$

If instead $d_0 = n$ and $\ell = 0$, then the top divisor 1 is the only value d with $d \mid (n/d_0)$. In either case, there are at least as many top divisors d with $d \mid (n/d_0)$ as there are bottom divisors with the same property. Therefore, (‡) holds. This completes the proof. □

Therefore, $a_n = \prod_{d|n} b_d$, and $b_n = E_n/O_n$ is an integer for each n.

Problem 6 Let a *generalized diagonal* in an $n \times n$ matrix be a set of entries which contains exactly one element from each row and one element from each column. Let A be an $n \times n$ matrix filled with 0s and 1s which contains exactly one generalized diagonal whose entries are all 1. Prove that it is possible to permute the rows and columns of A to obtain an *upper-triangular matrix*, a matrix $(b_{ij})_{1 \leq i,j \leq n}$ such that $b_{ij} = 0$ whenever $1 \leq j < i \leq n$.

Solution: Because there is a generalized diagonal, every column has at least one 1.

First we claim that some column of A contains only one 1. Suppose, for sake of contradiction, that every column has at least two 1's. We then permute all of the rows and columns to let the generalized diagonal occupy (k, k) for all $k \leq n$, and so that 1's occupy (d_k, k) for some $d_1 \neq 1, \ldots, d_k \neq k$. We claim that some combination of the (d_k, k) and (k, k) give a second generalized diagonal. Draw a directed graph on the n vertices (d_k, k), drawing directed edges from (d_k, k) to (d_{d_k}, d_k). Every vertex of this graph has outdegree 1, so there is a cycle. (If we follow the vertices along the edges, eventually there must be a repeated vertex because we never reach a dead end). Letting C be the set of squares in this cycle, note that $C \cup \{(k, k) \mid (d_k, k) \notin C\}$ forms a generalized diagonal, as desired. This contradicts the given condition that there is only one generalized diagonal. Therefore, our original assumption was false, and some column contains only one 1.

We now prove the claim by induction on n. Clearly, when $n = 1$, the problem statement's permutation can be satisfied. Now, we look at $n = k + 1$ for some positive integer k. We find a column with a single 1, and permute rows and columns so that the 1 lies in the upper-left hand corner and the remainder of the first column contains 0's. Then the bottom-right $k \times k$ submatrix contains exactly one generalized diagonal, and applying the induction hypothesis we can permute rows and columns (without changing the first column) to make that $k \times k$ submatrix upper-triangular. Doing so makes the entire $n \times n$ matrix upper-triangular, as desired.

Problem 7 Let O and H be the circumcenter and orthocenter, respectively, of scalene triangle ABC. The *nine-point circle* of triangle ABC is the circle passing through the midpoints of the sides, the feet of the altitudes, and the midpoints of $\overline{AH}$, $\overline{BH}$, and $\overline{CH}$. Let N be the center of this circle, and assume that N does not lie on any of the lines AB, BC, CA. Let N' be the point such that

$$\angle N'BA = \angle NBC \qquad \text{and} \qquad \angle N'AB = \angle NAC.$$

Let the perpendicular bisector of $\overline{OA}$ intersect line BC at A', and define B' and C' similarly. Prove that A', B', and C' lie on a line ℓ which is perpendicular to line ON'.

Solution: We use directed angles modulo π. Let D and M be the midpoints of $\overline{BC}$ and $\overline{AO}$, respectively. Let P be the foot of the altitude from A to $\overline{BC}$. It is well-known that $OH = 2ON$, $AH = 2OD$, and $\angle BAP = \angle OAC$.

It is also well-known that if X does not lie on any of the lines AB, BC, CA, then there is a unique point X' (the *isogonal conjugate* of X) that satisfies any given two of the equalities $\angle X'AB = \angle XAC$, $\angle X'BC = \angle XBA$, $\angle X'CA = \angle XAB$. It is easy to see that N' is the isogonal conjugate of N.

We have $OA = 2OM$ and (by a well-known fact) $OH = 2ON$, implying that triangles OAH and OMN are similar with ratio 2. Thus, $\overline{MN} \parallel \overline{AH}$ and $2MN = AH$.

In addition, $\overline{AH} \parallel \overline{OD}$ and $AH = 2OD$ (the latter is a well-known fact). Hence, $\overline{MN} \parallel \overline{OD}$ and $MN = OD$, and quadrilateral $MNDO$ is a parallelogram.

Because $OH = 2ON$, N lies halfway between lines AHP and OD. Hence,

$$\angle NPA = \angle ODN = \angle OMN = \angle MAP.$$

(The first equality holds by symmetry because N lies halfway between lines AP and OD; the second equality holds because quadrilateral $MNDO$ is a parallelogram; and the third equality holds because line OMA cuts parallel lines MN and AP.) Therefore, quadrilateral $AMNP$ is an isosceles trapezoid with $\overline{PA} \parallel \overline{MN}$ and $AM = NP$.

Because A, M, N, P form an isosceles triangle, they lie on a single circle. Also, $\angle AMA' = \angle APA' = \pi/2$, implying that A' lies on the same circle. Because $MA = PN$, it follows that $\angle MA'A = \angle PA'N$. Hence,

$$\pi/2 - \angle AOA' = \angle MA'A = \angle PA'N = \angle PAN. \qquad (*)$$

(The first equality comes from the properties of isosceles triangle $A'AO$, and the third equality holds because A, A', M, N, P are concyclic.)

Now, the equality $\angle BAN' = \angle NAC$ and the well-known equality $\angle BAP = \angle OAC$ together imply $\angle PAN = \angle NA'O$. Substituting this into $(*)$ yields

$$\pi/2 = \angle AOA' + \angle NA'O,$$

implying that $\overline{A'O} \perp \overline{AN'}$.

Lemma. *If $\overline{WX} \perp \overline{YZ}$, then $WY^2 - WZ^2 = XY^2 - XZ^2$.*

Proof. If D is the intersection of lines WX and YZ, then repeatedly applying the Pythagorean Theorem shows that both sides of the desired equality equal $DY^2 - DZ^2$. $\square$

Applying the lemma to $\overline{A'O}$ and $\overline{AN}$ yields

$$OA^2 - ON'^2 = A'A^2 - A'N'^2 = A'O^2 - A'N'^2.$$

Writing $r = OA = OB = OC$, we have

$$A'O^2 - A'N'^2 = r^2 - ON'^2.$$

Previously we observed that N' is the isogonal conjugate of N, so both N and N' are defined symmetrically with respect to A, B, C. Hence, the above proof and two proofs analogous to the one above yield

$$A'O^2 - A'N'^2 = B'O^2 - B'N'^2 = C'O^2 - C'N'^2 = r^2 - ON'^2.$$

Therefore, A', B', C' lie in the set of points X with $XO^2 - XN'^2$ equal to the constant $R^2 - O'N^2$. This set of points is a line perpendicular to $\overline{ON'}$, as desired.

Problem 8 Let $n = 2^m + 1$ for some positive integer m. Let $f_1, f_2, \ldots, f_n : [0,1] \to [0,1]$ be increasing functions. Suppose that for $i = 1, 2, \ldots, n$, $f_i(0) = 0$ and

$$|f_i(x) - f_i(y)| \leq |x - y|$$

for all $x, y \in [0,1]$. Prove that there exist distinct integers i and j between 1 and n, inclusive, such that

$$|f_i(x) - f_j(x)| \leq \frac{1}{m}$$

for all $x \in [0,1]$.

Solution:

Lemma. *There exist distinct* i_1, i_2 *and integers* $\sigma_0, \sigma_1, \ldots, \sigma_{m+1}$ *such that*

$$f_{i_1}\left(\frac{j}{m+1}\right), f_{i_2}\left(\frac{j}{m+1}\right) \in \left[\frac{\sigma_j}{m+1}, \frac{\sigma_j + 1}{m+1}\right]$$

for $j = 0, 1, \ldots, m+1$.

Proof. Consider any $i \in \{1, 2, \ldots, 2^m + 1\}$. Define the sequence

$$\sigma_0, \sigma_1, \ldots, \sigma_m, \sigma_{m+1}$$

as follows: σ_j is the smallest integer $t \geq 0$ such that

$$f_i\left(\frac{j}{m+1}\right) \in \left[\frac{t}{m+1}, \frac{t+1}{m+1}\right].$$

With the given conditions, it is easy to show that $\sigma_0 = \sigma_1 = 0$ independent of the choice of i.

By the restrictions that f_i is increasing and that $|f_i(x) - f_i(y)| \leq |x - y|$, we have $\sigma_{j+1} - \sigma_j \in \{0, 1\}$ for $j = 1, 2, \ldots, m$. Thus,

$$(\sigma_2 - \sigma_1, \sigma_3 - \sigma_2, \ldots, \sigma_{m+1} - \sigma_m) \qquad (*)$$

is one of the 2^m m-tuples in $\{0, 1\}^m$.

By the Pigeonhole Principle, some two values $i = i_1$ and $i = i_2$ give rise to the same m-tuples in $(*)$. These values of i also give rise to the same values $\sigma_0 = 0$ and $\sigma_1 = 0$. Hence, $i = i_1$ and $i = i_2$ correspond to the same

$$(\sigma_0, \sigma_1, \ldots, \sigma_{m+1}),$$

as desired. $\square$

Choose i_1, i_2, and the σ_j as given by the lemma. Without loss of generality, assume that $i_1 = 1$ and $i_2 = 2$. We claim that

$$|f_1(x) - f_2(x)| \le \frac{1}{m+1} \qquad (*)$$

for all $x \in [0, 1]$.

Connect the points $(\frac{j}{m+1}, \frac{\sigma_j}{m+1})$ with one piecewise linear path $\mathcal{P}^+$ and the points $(\frac{j}{m+1}, \frac{\sigma_j+1}{m+1})$ with another piecewise linear path $\mathcal{P}^-$. We show that the graphs of f_1, f_2 lie within the region between the two paths so they never differ by more than 1, the vertical distance between the paths.

We first prove that any point $(x_0, f_1(x_0))$ lies on or above the lower path $\mathcal{P}^-$. This is trivial when x_0 is a multiple of $\frac{1}{m+1}$, so assume otherwise. Write $\frac{j}{m+1} < x_0 < \frac{j+1}{m+1}$, and let A_1, A_2, A_3 be the points on the graph of f_1 at $x = \frac{j}{m+1}, x_0, \frac{j}{m+1}$, respectively. Also write $P = (\frac{j}{m+1}, \frac{\sigma_j}{m+1})$ and $Q = (\frac{j+1}{m+1}, \frac{\sigma_{j+1}}{m+1})$, so that $\overline{PQ}$ is one piece of the lower path $\mathcal{P}^-$.

Suppose that $\sigma_{j+1} = \sigma_j$. By the given conditions, line A_1A_2 has non-negative slope, so A_2 lies on or above the horizontal line through A_1. This line, in turn, lies on or above the horizontal line PQ through P, because A_1 lies directly above P (or $A_1 = P$). Hence A_2 lies on or above $\overline{PQ}$ in this case.

Otherwise, $\sigma_{j+1} = \sigma_j + 1$. By the given conditions, line A_2A_3 has slope at most 1, so A_2 lies on or above the line through A_3 with slope 1. This line, in turn, lies on or above the line PQ through Q with slope 1, because A_3 lies directly above Q (or $A_3 = Q$). Hence, A_2 lies on or above $\overline{PQ}$ in this case as well.

Therefore, f_1 lies on or above the lower path $\mathcal{P}^-$. Analogous proofs show that f_1 lies on or below the upper path $\mathcal{P}^+$, and that f_2 lies between the two paths. This completes the proof.

Problem 9 In triangle ABC, let I be the incenter and let I_a be the excenter opposite A. Suppose that $\overline{II_a}$ meets $\overline{BC}$ and the circumcircle of triangle ABC at A' and M, respectively. Let N be the midpoint of arc MBA of the circumcircle of triangle ABC. Let lines NI and NI_a intersect the circumcircle of triangle ABC again at S and T, respectively. Prove that S, T, and A' are collinear.

Solution: We use directed angles modulo π (where arc measures are directed modulo 2π). Note that $\angle ICI_a = \pi/2 = \angle IBI_a$, so that I, I_a, B, and C are cyclic on some circle ω_1. Since angle $\angle TI_aI$ cuts off ω_1 at the arcs NA and TM, it satisfies

$$\angle TI_aI = \frac{1}{2}(\widehat{NA} - \widehat{MT}) = \frac{1}{2}(\widehat{MN} - \widehat{MT}) = \frac{1}{2}\widehat{TN} = \angle TSN = \angle TSI.$$

Thus, S, T, I, I_a lie on a single circle ω_2.

Because I and I_a lie on both circles ω_1, ω_2, line II_a is the radical axis of these circles. Thus, A' lies on the radical axis of ω_1 and ω_2. It also lies on line BC, the radical axis of ω_1 and the circumcircle of ABC. Therefore, by the Radical Axis Theorem, A' also lies on the radical axis of ω_2 and the circumcircle of ABC. This radical axis is line ST, implying that S, T, and A' are collinear.

Problem 10 The set of *n-variable formulas* is a subset of the functions of n variables $x_1, \ldots, x_n$, and it is defined recursively as follows: the formulas $x_1, \ldots, x_n$ are n-variable formulas, as is any formula of the form

$$(x_1, \ldots, x_n) \mapsto \max\{f_1(x_1, \ldots, x_n), \ldots, f_k(x_1, \ldots, x_n)\}$$

or

$$(x_1, \ldots, x_n) \mapsto \min\{f_1(x_1, \ldots, x_n), \ldots, f_k(x_1, \ldots, x_n)\},$$

where each f_i is an n-variable formula. For example,

$$\max(x_2, x_3, \min(x_1, \max(x_4, x_5)))$$

is a 5-variable formula. Suppose that P and Q are two n-variable formulas such that

$$P(x_1, \ldots, x_n) = Q(x_1, \ldots, x_n) \qquad (*)$$

for all $x_1, \ldots, x_n \in \{0, 1\}$. Prove that $(*)$ also holds for all $x_1, \ldots, x_n \in \mathbb{R}$.

Solution: Consider the function $p(a, x)$, which takes the value 0 if $x < a$ and 1 otherwise. We can see that $p(a, \max\{x_1, x_2, \ldots x_n\}) =$

$\max\{p(a,x_1),p(a,x_2),\dots p(a,x_n)\}$. The similar result holds if "max" is replaced by "min".

Lemma. *If f is an n-variable formula, then*

$$f(p(a,x_1),p(a,x_2),\dots p(a,x_n)) = p(a,f(x_1,x_2,\dots x_n)).$$

Proof. We induct on the "depth" of f, defined as the maximum number of levels of nesting in the definition of f.

The claim is trivial if the depth of f is 0, so now suppose that f has depth $d>0$. If f is of the form

$$f(\mathbf{x}) = \max\{f_1(\mathbf{x}),\dots f_k(\mathbf{x})\},$$

where we write $\mathbf{x}=(x_1,\dots,x_n)$ for convenience, then

$$\begin{aligned} &p(a,f(\mathbf{x})) = p(a,\max\{f_1(\mathbf{x}),\dots,f_k(\mathbf{x})\})\\ &\quad= \max\{p(a,f_1(\mathbf{x})),\dots,p(a,f_k(\mathbf{x}))\}\\ &\quad= \max\{f_1(p(a,x_1),\dots,p(a,x_n)),\dots,f_k(p(a,x_1),\dots,p(a,x_n))\}\\ &\quad= f(p(a,x_1),\dots,p(a,x_n)), \end{aligned}$$

where we have applied the induction hypothesis to the formulas f_i of depth less than d. If f is of the form

$$\min\{f_1(\mathbf{x}),\dots f_k(\mathbf{x})\},$$

a similar argument can be used. This completes the inductive step. $\square$

Suppose we are given P and Q that satisfy $(*)$. Suppose that $\mathbf{x}=(x_1,\dots,x_n)$ and let a be a real number. By the lemma,

$$p(a,P(\mathbf{x})) = P(p(a,x_1),\dots p(a,x_n)),$$

$$p(a,Q(\mathbf{x})) = Q(p(a,x_1),\dots p(a,x_n)).$$

Because the $p(a,x_i)$ all equal 0 or 1, by $(*)$ we have

$$P(p(a,x_1),\dots p(a,x_n)) = Q(p(a,x_1),\dots p(a,x_n)).$$

Thus, $p(a,P(\mathbf{x})) = p(a,Q(\mathbf{x}))$. Varying a shows that $P(\mathbf{x})=Q(\mathbf{x})$. This holds for all $\mathbf{x}$, as desired.

1.9 Japan

Problem 1 Each square of an $m \times n$ chessboard is painted black or white. Each black square is adjacent to an odd number of black squares. Prove that the number of black squares is even. (Two squares are adjacent if they are different and share a common edge.)

Solution: Construct a graph whose vertices are the black squares, where two vertices are adjacent if the corresponding squares border each other. We are given that every vertex has odd degree. The sum of all degrees in any graph is twice the number of edges of the graph. So this number must be even. Since in our graph all vertices have odd degrees, its number of vertices must be even. Thus, the board has an even number of black squares.

Problem 2 Find all positive integers n such that

$$n = \prod_{k=0}^{m} (a_k + 1),$$

where $\overline{a_m a_{m-1} \ldots a_0}$ is the decimal representation of n — that is, where $a_0, a_1, \ldots, a_m$ is the unique sequence of integers in $\{0, 1, \ldots, 9\}$ such that $n = \sum_{k=0}^{m} a_k 10^k$ and $a_m \neq 0$.

Solution: We claim that the only such n is 18. If $n = \overline{a_m \cdots a_1 a_0}$, then let

$$P(n) = \prod_{j=0}^{m} (a_j + 1).$$

Note that if $s \geq 1$ and t is a single-digit number, then $P(10s + t) = (s + 1)P(t)$. Using this we will prove two following statements.

Lemma. *If $P(s) \leq s$, then $P(10s + t) < 10s + t$.*

Proof. Indeed, if $P(s) \leq s$, then

$$10s + t \geq 10s \geq 10P(s) \geq (t + 1)P(s) = P(10s + t).$$

Equality must fail either in the first inequality (if $t \neq 0$) or in the third inequality (if $t \neq 9$). $\square$

Lemma. *$P(n) \leq n + 1$ for all n.*

Proof. We prove this by induction on the number of digits of n. First, we know that for all one-digit n, $P(n) = n + 1$. Now suppose that $P(n) \leq n+1$ for all m-digit numbers n. Any $(m+1)$-digit number n is of the form $10s + t$, where s is an m-digit number. Then

$$\begin{aligned} t(P(s) - 1) &\leq 9((s+1) - 1) \\ tP(s) - 10s - t &\leq -s \\ P(s)(t+1) - 10s - t &\leq P(s) - s \\ P(10s+t) - (10s+t) &\leq P(s) - s \leq 1, \end{aligned}$$

completing the inductive step. Thus, $P(n) \leq n + 1$ for all n. □

If $P(n) = n$, then n has more than one digit and we may write $n = 10s + t$. From the first statement, we have $P(s) \geq s + 1$. From the second one, we have $P(s) \leq s + 1$. Thus, $P(s) = s + 1$. Hence,

$$\begin{aligned} (t+1)P(s) = P(10s+t) &= 10s + t \\ (t+1)(s+1) &= 10s + t \\ 1 &= (9-t)s. \end{aligned}$$

This is only possible if $t = 8$ and $s = 1$, so the only possible n such that $P(n) = n$ is 18. Indeed, $P(18) = (1+1)(8+1) = 18$.

Problem 3 Three real numbers $a, b, c \geq 0$ satisfy the inequalities $a^2 \leq b^2 + c^2$, $b^2 \leq c^2 + a^2$, and $c^2 \leq a^2 + b^2$. Prove that

$$(a+b+c)(a^2+b^2+c^2)(a^3+b^3+c^3) \geq 4(a^6+b^6+c^6),$$

and determine when equality holds.

Solution: We claim that equality holds if and only if $a = b = c = 0$, or two of a, b, c are equal and the third is zero.

Without loss of generality, suppose that $c \geq b \geq a$. Then we know that

$$4c^6 + 4a^2c^4 + 4b^2c^4 \geq 4(a^6 + b^6 + c^6).$$

Thus,

$$\begin{aligned} 4(a^6+b^6+c^6) &\leq 4c^6 + 4a^2c^4 + 4b^2c^4 \\ &= 2c^2((c^2)^2 + c^4 + 2a^2c^2 + 2b^2c^2). \end{aligned}$$

Applying the given inequality $c^2 \leq a^2+b^2$ to the right hand side gives

$$\begin{aligned}4(a^6+b^6+c^6) &\leq 2c^2((a^2+b^2)^2+c^4+2a^2c^2+2b^2c^2)\\ &= 2c^2(a^4+b^4+c^4+2a^2b^2+2a^2c^2+2b^2c^2)\\ &= 2c^2(a^2+b^2+c^2)^2.\end{aligned}$$

By Cauchy-Schwartz,

$$2c^2(a^2+b^2+c^2)^2 \leq 2c^2(a+b+c)(a^3+b^3+c^3). \qquad (*)$$

Because $c^2 \leq a^2+b^2$,

$$2c^2(a+b+c)(a^3+b^3+c^3) \leq (a+b+c)(a^2+b^2+c^2)(a^3+b^3+c^3). \quad (\dagger)$$

Combining all these inequalities yields the desired result. Equality holds only if equality holds in $(*)$ and $(\dagger)$. For equality to hold in $(*)$, (a^3, b^3, c^3) must be a multiple of (a, b, c) — that is, either $a = b = c$, $a = 0 < b = c$, or $a = b = 0 < c$. For equality to hold in $(\dagger)$, we must have either $(a+b+c)(a^3+b^3+c^3) = 0$ or $c^2 = a^2+b^2$, i.e., either $a = b = c = 0$ or $c^2 = a^2+b^2$. Combining these conditions yields $a = b = c = 0$ or $a = 0 < b = c$, and indeed equality holds in both these cases.

Removing the constraint $c \leq b \leq a$ and permuting the equality cases, gives the conditions for equality presented at the beginning of the solution.

Problem 4 Let p be a prime number and m be a positive integer. Show that there exists a positive integer n such that there exist m consecutive zeroes in the decimal representation of p^n.

Solution: It is well-known that if $\gcd(s,t) = 1$, then $s^k \equiv 1 \pmod{t}$ for some k: indeed, of all the positive powers of s, some two $s^{k_1} < s^{k_2}$ must be congruent modulo t, and then $s^{k_2-k_1} \equiv 1 \pmod{t}$.

First suppose that $p \neq 2, 5$. Then $\gcd(p, 10^{m+1}) = 1$, so there exists such k that $p^k \equiv 1 \pmod{10^{m+1}}$. Then $p^k = a \cdot 10^{m+1} + 1$, so there are m consecutive zeroes in the decimal representation of p^k.

Now suppose that $p = 2$. We claim that for any a, some power of 2 has the following final a digits: $a - \lceil \log 2^a \rceil$ zeroes, followed by the $\lceil \log 2^a \rceil$ digits of 2^a. Because $\gcd(2, 5^a) = 1$, there exists k such that $2^k \equiv 1 \pmod{5^a}$. Let $b = k + a$. Then $2^b \equiv 2^a \pmod{5^a}$, and $2^b \equiv 0 \equiv 2^a \pmod{2^a}$. Hence, by the Chinese Remainder Theorem,

$2^b \equiv 2^a \pmod{10^a}$. Because $2^a < 10^a$, it follows that 2^b has the required property.

Now, simply choose a such that $a - \lceil \log 2^a \rceil \geq m$ (for instance, we could choose $a = \lceil \frac{m+1}{1-\log 2} \rceil$). Then 2^b contains at least m consecutive zeroes, as desired.

Finally, the case $p = 5$ is done analogously to the case $p = 2$.

Problem 5 Two triangles ABC and PQR satisfy the following properties: A and P are the midpoints of $\overline{QR}$ and $\overline{BC}$, respectively, and lines QR and BC are the internal angle bisectors of angles BAC and QPR, respectively. Prove that $AB + AC = PQ + PR$.

Solution: Let X be the intersection of lines BC and QR, let the circumcircle of triangle PQR intersect BC at D, and let the circumcircle of triangle ABC intersect QR at S. Then D is the midpoint of arc $\widehat{QR}$ and S is the midpoint of arc $\widehat{BC}$, so $DA \perp QR$ and $SP \perp BC$. Therefore, quadrilateral $PADS$ is cyclic. Now, $QX \cdot XR = PX \cdot XD = AX \cdot XS = BX \cdot XC$, so $BQCR$ is cyclic. Let O be the circumcenter of $BQCR$. Then O lies on AD and PS, and since $OP \perp PD$, $OPQDR$ must be inscribable in a circle with diameter OD. Likewise, $OACSB$ is cyclic.

Reflect C about OD to get C' and Q about OS to get Q'. Then R, P, Q' are collinear, B, A, C' are collinear, and $RCC'QQ'B$ is cyclic. But $\angle OBC' = \angle OSA = \angle ODP = \angle ORQ'$, so $BC' = RQ'$. This means that

$$BA + AC = BA + AC' = BC' = RQ' = RP + PQ' = RP + PQ,$$

as desired.

1.10 Korea

Problem 1 Given an odd prime p, find all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying the following two conditions:

(i) $f(m) = f(n)$ for all $m, n \in \mathbb{Z}$ such that $m \equiv n \pmod{p}$;

(ii) $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{Z}$.

Solution: The only such functions are:

- $f(x) = 1$;
- $f(x) = 0$;
- $f(x) = \begin{cases} 0 & \text{if } p \mid x, \\ 1 & \text{otherwise;} \end{cases}$
- $f(x) = \begin{cases} 0 & \text{if } p \mid x, \\ 1 & \text{if } x \text{ is a quadratic residue modulo } p, \\ -1 & \text{otherwise.} \end{cases}$

It is easy to verify that these satisfy conditions (i) and (ii).

Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ satisfies (i) and (ii). Setting $m = n = 0$ in (ii) yields $f(0) \cdot f(0) = f(0)$, so that $f(0) = 0$ or $f(0) = 1$.

If $f(0) = 1$, then for all x we may set $(m, n) = (x, 0)$ in condition (ii) to find $f(0) = f(x)f(0)$, or $1 = f(x)$. This yields one of our solutions.

Otherwise, $f(0) = 0$. Let g be a primitive root modulo p. By (i), $f(x) = 0$ for all $x \equiv 0 \pmod{p}$. For all other values x, $f(x)$ must equal $f(g)^t$, where $x \equiv g^t \pmod{p}$; i.e., f is determined by the value $f(g)$. Note that $f(g) = f(g^p)$, and (applying (ii) repeatedly) $f(g^p) = (f(g))^p$; that is, $f(g) = (f(g))^p$. Therefore, $f(g) = 0$, $f(g) = 1$, or $f(g) = -1$. These three cases yield the other three presented solutions.

Problem 2 Let P be a point inside convex quadrilateral $O_1O_2O_3O_4$, where we write $O_0 = O_4$ and $O_5 = O_1$. For each $i = 1, 2, 3, 4$, consider the lines ℓ that pass through P and meet the rays O_iO_{i-1} and O_iO_{i+1} at distinct points $A_i(\ell)$ and $B_i(\ell)$. Let $f_i(\ell) = PA_i(\ell) \cdot PB_i(\ell)$. Among all such lines ℓ, let m_i be a line for which f_i is the minimum. Show that if $m_1 = m_3$ and $m_2 = m_4$, then the quadrilateral $O_1O_2O_3O_4$ is a parallelogram.

Solution: Fix i. We claim that m_i is the line ℓ through P perpendicular to the angle bisector of angle $O_{i-1}O_iO_{i+1}$. This line intersects rays O_iO_{i-1} and O_iO_{i+1} at two points A and B equidistant from O_i, so that there is a circle ω_i tangent to rays O_iO_{i-1} and O_iO_{i+1} at A and B. Given any other line ℓ' intersecting ray O_iO_{i-1} and O_iO_{i+1} at A' and B', respectively, let ℓ' intersect ω at A'' and B'', with A', A'', P, B'', B on a line in that order. Then $OA \cdot OB = OA'' \cdot OB''$ by Power of a Point, and $OA'' \cdot OB'' < OA' \cdot OB'$. Hence, the given ℓ minimizes $f_i(\ell)$, as claimed.

Suppose now that $m_1 = m_3$. From above, this line is perpendicular to the angle bisector n_1 of angle $O_4O_1O_2$ and perpendicular to the angle bisector n_3 of angle $O_2O_3O_1$. Let n_1 intersect ray O_4O_3 at C, and let n_3 intersect ray O_2O_1 at D. Then (using angle bisectors and parallel lines)

$$\angle O_2O_3D = \angle DO_3O4 = \angle O_1CO_4,$$

$$\angle O_3DO_2 = \angle CO_1O_2 = \angle O_4O_1C.$$

Hence, $\angle DO_2O_3 = \angle CO_4O_1$, or $\angle O_1O_2O_3 = \angle O_3O_4O_1$.

Similarly, if $m_2 = m_4$, then $\angle O_2O_3O_4 = \angle O_4O_1O_2$. Therefore, if $m_1 = m_3$ *and* $m_2 = m_4$, then the opposite angles of quadrilateral $O_1O_2O_3O_4$ are congruent, and the quadrilateral is a parallelogram.

Problem 3 Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be real numbers satisfying $\sum_{i=1}^n x_i^2 = \sum_{i=1}^n y_i^2 = 1$. Show that

$$(x_1y_2 - x_2y_1)^2 \le 2\left|1 - \sum_{i=1}^n x_iy_i\right|,$$

and determine when equality holds.

Solution: By Cauchy-Schwarz,

$$1 - \sum_{i=1}^n x_iy_i \ge 1 - \sqrt{\sum_{i=1}^n x_i^2 \sum_{i=1}^n y_i^2} = 0,$$

so that

$$\left|1 - \sum_{i=1}^n x_iy_i\right| = 1 - \sum_{i=1}^n x_iy_i.$$

Hence, applying Cauchy-Schwarz again (but to fewer of the numbers),

$$
\begin{aligned}
2\left|1-\sum_{i=1}^{n} x_i y_i\right| &= 2-2(x_1y_1+x_2y_2+\sum_{i=3}^{n} x_i y_i) \\
&\geq 2-2\left(x_1y_1+x_2y_2+\sqrt{\sum_{i=3}^{n} x_i^2 \sum_{i=3}^{n} y_i^2}\right).
\end{aligned}
$$

Consider the vectors $\mathbf{x} = (x_1, x_2, \sqrt{\sum_{i=3}^{n} x_i^2})$ and $\mathbf{y} = (y_1, y_2, \sqrt{\sum_{i=3}^{n} y_i^2})$, with dot product $\mathbf{x} \cdot \mathbf{y}$ and cross product $\mathbf{x} \times \mathbf{y}$. We are given that $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$, and the above inequality gives

$$
2\left|1-\sum_{i=1}^{n} x_i y_i\right| \geq 2-2\mathbf{x}\cdot\mathbf{y}. \tag{$*$}
$$

Notice that the third coordinate of $\mathbf{x} \times \mathbf{y}$ is $x_1y_2 - x_2y_1$, so that

$$
(x_1y_2 - x_2y_1)^2 \leq \|\mathbf{x} \times \mathbf{y}\|^2. \tag{$\dagger$}
$$

Combining $(*)$ and $(\dagger)$, we see that to prove the desired claim it suffices to prove that

$$
2(1-\mathbf{x}\cdot\mathbf{y}) \geq \|\mathbf{x} \times \mathbf{y}\|^2. \tag{$\ddagger$}
$$

Letting θ be the angle between $\mathbf{x}$ and $\mathbf{y}$, this inequality becomes

$$
\begin{aligned}
2(1-\cos\theta) &\geq \sin^2\theta \\
2-2\cos\theta &\geq 1-\cos^2\theta \\
1-2\cos\theta+\cos^2\theta &\geq 0 \\
(1-\cos\theta)^2 &\geq 0.
\end{aligned}
$$

The last inequality is clearly true, and hence $(\ddagger)$ indeed holds.

Equality holds in the desired inequality only if $\cos\theta = 1$, i.e., $\mathbf{x} = \mathbf{y}$. In this case, $x_1 = y_1$, $x_2 = y_2$. Also, in order for equality to hold in the desired inequality, equality must hold in the application of Cauchy-Schwarz; that is, $(x_3, x_4, \ldots, x_n)$ and $(y_3, y_4, \ldots, y_n)$ must be nonnegative multiples of each other. Because $\sum_{i=3}^{n} x_i^2 = \sum_{i=3}^{n} y_i^2$, we must actually have $(x_3, x_4, \ldots, x_n) = (y_3, y_4, \ldots, y_n)$.

Hence, equality holds only if $x_i = y_i$ for all i, and it is easy to check that in this case both sides of the given inequality equal 0.

Problem 4 Given positive integers n and N, let $\mathcal{P}_n$ be the set of all polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$ with integer coefficients satisfying the following two conditions:

(i) $|a_j| \le N$ for $j = 0, 1, \ldots, n$;

(ii) at most two of $a_0, a_1, \ldots, a_n$ equal N.

Find the number of elements in the set $\{f(2N) \mid f(x) \in \mathcal{P}_n\}$.

Solution: The answer is $(2N)^{n-1} + (2N)^n + (2N)^{n+1}$.

Let $\overline{b_k \ldots b_1 b_0}$ denote the base-$2N$ representation of $\sum_{j=0}^{k} b_j (2N)^j$.

Let $\mathcal{Q}_n$ be the set of degree-n polynomials in x satisfying the following conditions:

(i′) each coefficient is an integer in $[0, 2N]$;

(ii′) at most two coefficients equal $2N$.

Notice that $f(x) \in \mathcal{P}_n$ if and only if $f(x) + N\sum_{j=0}^{n} x^j \in \mathcal{Q}_n$. Therefore, the size of

$$\{f(2N) \mid f(x) \in \mathcal{P}_n\}$$

equals the size of

$$\{f(2N) \mid f(x) \in \mathcal{Q}_n\}.$$

It suffices, then, to prove the following lemma.

Lemma.

$$\{f(2N) \mid f(x) \in \mathcal{Q}_n\} = \{0, 1, \ldots, \overline{111\underbrace{00\ldots0}_{n-1}} - 1\}.$$

Proof. Suppose that $f(x) \in \mathcal{Q}_n$. Then

$$\begin{aligned} 0 &\le f(2N) \\ &\le (2N)\cdot(2N)^n + (2N)\cdot(2N)^{n-1} \\ &\qquad + \overline{\underbrace{(2N-1)(2N-1)\ldots(2N\_1)}_{n-1}} \\ &= \overline{111\underbrace{00\ldots0}_{n-1}} - 1. \end{aligned}$$

Conversely, we now prove that any $B = \overline{b_{n+1}b_n \ldots b_0}$ satisfying

$$0 \le B \le \overline{111\underbrace{00\ldots0}_{n-1}} - 1$$

can be written in the form $f(2N)$ for some $f(x) = \sum_{j=0}^{n} a_j x^j$ in $\mathcal{Q}_n$. There are three cases:

- $b_{n+1} = b_n = 1$. By the upper bound on B, we have $b_{n-1} = 0$. Then

$$\begin{aligned} B &= \overline{110b_{n-2}b_{n-3}\dots b_0} \\ &= (2N)\cdot(2N)^n + (2N)\cdot(2N)^{n-1} + \overline{b_{n-2}b_{n-3}\dots b_0}. \end{aligned}$$

  Hence, we may set $a_j = b_j$ for $j \le n-2$ and set $a_{n-1} = a_n = 2N$.

- $b_{n+1} = 1$, $b_n = 0$. Then

$$\begin{aligned} B &= \overline{10b_{n-1}b_{n-2}\dots b_0} \\ &= (2N)\cdot(2N)^n + \overline{b_{n-1}b_{n-2}\dots b_0}. \end{aligned}$$

  Hence, we may set $a_j = b_j$ for $j \le n-1$ and set $a_n = 2N$.

- $b_{n+1} = 0$. Then we may set $a_j = b_j$ for $j \le n$.

$\square$

Problem 5 In isosceles triangle ABC, $AB = BC$ and $\angle ABC < \pi/3$. Point D lies on $\overline{BC}$ so that the incenter of triangle ABD coincides with the circumcenter O of triangle ABC. Let ω be the circumcircle of triangle AOC. Let P be the point of intersection of the two tangent lines to ω at A and C. Let Q be the point of intersection of lines AD and CO, and let X be the point of intersection of line PQ and the tangent line to ω at O. Show that $XO = XD$.

Solution:

Lemma. *$AD = BD$.*

Proof. Let the incircle of triangle ABD be tangent to side $\overline{AB}$ at M. Because the center of this circle is O, we have $\overline{OM} \perp \overline{AB}$. We also know that $OA = OB$, implying that M is the midpoint of $\overline{AB}$. Then

$$\frac{1}{2}(AB + AD - BD) = AM = BM = \frac{1}{2}(BA + BD - AD),$$

implying that $AD = BD$. $\square$

By the lemma, $\angle ABD = \angle DAB$. Then $\angle ADC = \angle ABD + \angle BAD = 2\angle ABD = 2\angle ABC = \angle AOC$. Hence, quadrilateral $ACDO$ is cyclic (and its circumcircle is ω).

Let X' be the intersection of the tangents at O and D to ω; we claim that $X = X'$. Let Y be the intersection of lines AO and CD. By Pascal's Theorem applied to the degenerate hexagon $ADDCOO$, points Q, X', and Y are collinear. By Pascal's Theorem applied to the degenerate hexagon $AADCCO$, P, Q and Y are collinear. Therefore, both X' and P lie on line QY. Therefore, X' is the intersection of line PQ and the tangent line to ω at O; that is, $X' = X$.

Thus $\overline{XO}$ and $\overline{XD}$ (which equal $\overline{X'O}$ and $\overline{X'D}$) are tangent to ω by the definition of X', implying that $XO = XD$.

Problem 6 Let $n \geq 5$ be a positive integer, and let a_1, b_1, a_2, b_2, $\dots$, a_n, b_n be integers satisfying the following two conditions:

(i) the pairs (a_i, b_i) are all distinct for $i = 1, 2, \dots, n$;

(ii) $|a_i b_{i+1} - a_{i+1} b_i| = 1$ for $i = 1, 2, \dots, n$, where $(a_{n+1}, b_{n+1}) = (a_1, b_1)$.

Show that there exist i, j with $1 \leq i, j \leq n$ such that $1 < |i-j| < n-1$ and $|a_i b_j - a_j b_i| = 1$.

Solution: All indices are taken modulo n.

Let $V_i = (a_i, b_i)$, and think of the V_i as points in a plane with origin $O = (0,0)$. Observe that no V_i equals $(0,0)$. Also notice that $|a_i b_j - a_j b_i| = 1$ if and only if the (unsigned) area of triangle OV_iV_j equals $\frac{1}{2}$.

Suppose that some two points V_j and V_k are reflections of each other across the origin. Then $[OV_iV_j] = [OV_iV_k]$ for all i. Thus, $[OV_iV_j] = \frac{1}{2}$ for $i = j-1, j+1, k-1, k+1$. Because $n \geq 5$, one of these i satisfies $1 < |i-j| < n-1$ and $[OV_iV_j] = \frac{1}{2}$, as needed.

Otherwise, assume that (iii) no two V_i are reflections of each other across the origin. Then we can replace any V_i by its reflection across the origin without affecting the problem: if we do so, then conditions (i) and (iii) still hold; and the $|a_i b_j - a_j b_i|$ do not change, so condition (ii) and the desired result are unaffected.

Without loss of generality, assume V_2 has maximum distance from the origin among the V_i. Because $[OV_1V_2], [OV_2V_3] > 0$, V_1 and V_3 do not lie on line OV_2. Without loss of generality, assume that V_1 and V_3 are on different sides of line OV_2 (otherwise, we could reflect one of these points across O). Let P be the intersection of lines OV_2 and V_1V_3; without loss of generality, assume that P lies on ray OV_2 (otherwise, we could reflect V_2 across O). Because P lies on the interior of $\overline{V_1V_3}$, we have $OP < \max\{OV_1, OV_3\} \le OV_2$. Hence, $\overline{OV_2}$ and $\overline{V_1V_3}$ intersect at a point different from V_2.

Furthermore, we claim that $\overline{OV_2}$ and $\overline{V_1V_3}$ intersect at a point different from O. Otherwise, $[OV_1V_2] = [OV_2V_3]$ implies $OV_1 = OV_3$. This is impossible because V_1, V_3 are not equal, and nor are they reflections of each other across O.

Therefore, $\overline{OV_2}$ and $\overline{V_1V_3}$ intersect at a point in their interiors, implying that quadrilateral $OV_1V_2V_3$ is convex and non-degenerate. Triangles OV_1V_2 and OV_2V_3 each have area $\frac{1}{2}$, so the total area of quadrilateral $OV_1V_2V_3$ is 1. Therefore, the area $\frac{1}{2}|a_1b_3 - a_3b_1|$ of triangle OV_1V_3 is strictly between 0 and 1. Because this area is half of an integer, it must be $\frac{1}{2}$. Therefore, $|a_1b_3 - a_3b_1| = 1$, as desired.

1.11 Poland

Problem 1 Let $n \geq 2$ be an integer. Show that

$$\sum_{k=1}^{n} kx_k \leq \binom{n}{2} + \sum_{k=1}^{n} x_k^k$$

for all nonnegative reals $x_1, x_2, \ldots, x_n$.

Solution: Note that

$$\binom{n}{2} + \sum_{k=1}^{n} x_k^k = \sum_{k=1}^{n} \left(x_k^k + (k-1)\right).$$

By the AM-GM inequality applied to the k terms $x_k^k, 1, 1, \ldots, 1$, we have

$$x_k^k + (k-1) \geq k \cdot \sqrt[k]{x_k^k \cdot 1 \cdot 1 \cdot \cdots \cdot 1} = kx_k$$

Summing each side from $k = 1$ to n, we get the desired result.

Problem 2 Let P be a point inside a regular tetrahedron whose edges have length 1. Show that the sum of the distances from P to the vertices of the tetrahedron is at most 3.

Solution:

Lemma. *Given a point inside a unit equilateral triangle, the sum of the distances from the point to the vertices is at most 2.*

Proof. Let the point be P_0. If P_0 is on the boundary of the triangle, then the result is clear. Otherwise, consider the ellipse $\{P \mid AP + BP = AP_0 + BP_0\}$ with foci A and B. Let P_1 be the intersection of the ellipse with side $\overline{BC}$. Then $CP_0 \leq CP_1$, so that

$$\begin{aligned} AP_0 + BP_0 + CP_0 &= AP_1 + BP_1 + CP_0 \\ &\leq AP_1 + BP_1 + CP_1 = AP_1 + 1 \leq 2, \end{aligned}$$

as desired. $\square$

We now extend this proof to three dimensions for a unit regular tetrahedron $ABCD$. Because the space of points P in the interior of, or on the boundary of, tetrahedron $ABCD$ is compact, there is a maximum value of $AP + BP + CP + DP$ at some point $P = P_0$.

If P_0 is in the interior of the tetrahedron, then consider the ellipsoids $\mathcal{E}_1 = \{P \mid AP + BP = AP_0 + BP_0\}$ and $\mathcal{E}_2 = \{P \mid CP + DP = CP_0 + DP_0\}$. If we take a slightly larger ellipsoid $\mathcal{E}_2' = \{P \mid CP + DP = CP_0 + DP_0 + \epsilon\}$, then $\mathcal{E}_1$ and $\mathcal{E}_2'$ would intersect at a point P_1 in the interior of the tetrahedron with $AP_1 + BP_1 + CP_1 + DP_1 = AP_0 + BP_0 + CP_0 + DP_0 + \epsilon$, a contradiction.

Thus, P_0 is on the boundary of the tetrahedron — without loss of generality, on face ABC. By the lemma, $AP_0 + BP_0 + CP_0 \leq 2$. Also, $DP_0 \leq 1$. Hence, $AP_0 + BP_0 + CP_0 + DP_0 \leq 3$, as desired.

Problem 3 The sequence $x_1, x_2, x_3, \ldots$ is defined recursively by $x_1 = a$, $x_2 = b$, and $x_{n+2} = x_{n+1} + x_n$ for $n = 1, 2, \ldots$, where a and b are real numbers. Call a number c a *repeated value* if $x_k = x_\ell = c$ for some two distinct positive integers k and ℓ. Prove that one can choose the initial terms a and b so that there are more than 2000 repeated values in the sequence $x_1, x_2, \ldots$, but that it is impossible to choose a and b so that there are infinitely many repeated values.

Solution: Define the Fibonacci sequence $\{F_n\}$ by $F_1 = F_2 = 1$ and the recursive relation $F_{n+1} = F_n + F_{n-1}$ for $n \in \mathbb{Z}$. Note that we define this for negative indices by running the recurrence relation backwards: $F_0 = 0$, and it is easy to prove by induction that $F_{-n} = F_n$ for odd n and $F_{-n} = -F_n$ for even n.

If we set $a = F_{-4001}$ and $b = F_{-4000}$, then $x_n = F_{-4002+n}$. This yields a sequence with the 2001 repeated values

$$\begin{gathered} x_1 = F_{-4001} = F_{4001} = x_{8003}, \\ x_3 = F_{-3999} = F_{3999} = x_{8001}, \\ \ldots, \\ x_{4001} = F_{-1} = F_1 = x_{4003}. \end{gathered}$$

Now we show there cannot be infinitely many repeated values. Note that $x_1 = a + 0 \cdot b = aF_{-1} + bF_0$, and $x_2 = 0 \cdot a + b = aF_0 + bF_1$. Because $x_n = x_{n-1} + x_{n-2}$, it follows by induction that

$$x_n = aF_{n-2} + bF_{n-1}$$

Writing $r = \frac{1}{2}(1 + \sqrt{5})$ and $s = \frac{1}{2}(1 - \sqrt{5})$, Binet's Formula states that $F_n = \frac{r^n - s^n}{\sqrt{5}}$. Thus,

$$\begin{aligned} x_{n+2} = aF_n + bF_{n+1} &= a\frac{r^n - s^n}{\sqrt{5}} + b\frac{r^{n+1} - s^{n+1}}{\sqrt{5}} \\ &= \frac{r^n(a+br)}{\sqrt{5}} - \frac{s^n(a+bs)}{\sqrt{5}}. \end{aligned}$$

First suppose that $a + br \neq 0$. Because $|r| > 1$ and $|s| < 1$, we find that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n}$ equals

$$\lim_{x\to\infty} \frac{\frac{r^{n+1}(a+br)}{\sqrt{5}} - \frac{s^{n+1}(a+bs)}{\sqrt{5}}}{\frac{r^n(a+br)}{\sqrt{5}} - \frac{s^n(a+bs)}{\sqrt{5}}} = \lim_{n\to\infty} \frac{r^{n+1}(a+br)}{r^n(a+br)} = r > 1$$

This implies that the sequence is strictly increasing in absolute value after some finite number of terms, and so no two terms after that point can be equal. Thus there cannot be any repeated values after that point, and there are finitely many repeated values.

If instead $a + br = 0$ and $a + bs \neq 0$, then

$$\lim_{n\to\infty} \frac{\frac{r^{n+1}(a+br)}{\sqrt{5}} - \frac{s^{n+1}(a+bs)}{\sqrt{5}}}{\frac{r^n(a+br)}{\sqrt{5}} - \frac{s^n(a+bs)}{\sqrt{5}}} = \lim_{n\to\infty} \frac{s^{n+1}(a+bs)}{s^n(a+bs)} = s \in (0,1).$$

In this case, the sequence must have strictly decreasing absolute value after a finite number of terms, and again there are finitely many repeated values.

Finally, if $a + br = a + bs = 0$, then $b = 0$ and the terms of the sequence alternate between a and 0. Again, there are finitely many repeated values.

Problem 4 The integers a and b have the property that for every nonnegative integer n, the number $2^n a + b$ is a perfect square. Show that $a = 0$.

Solution: If $a \neq 0$ and $b = 0$, then at least one of $2^1 a + b$ and $2^2 a + b$ is not a perfect square, a contradiction.

If $a \neq 0$ and $b \neq 0$, then each $(x_n, y_n) = (2\sqrt{2^n a + b}, \sqrt{2^{n+2} a + b})$ satisfies

$$(x_n + y_n)(x_n - y_n) = 3b.$$

Hence, $(x_n + y_n) \mid 3b$ for each n. But this is impossible because $3b \neq 0$ but $|x_n + y_n| > |3b|$ for large enough n.

Therefore, $a = 0$.

Problem 5 Let $ABCD$ be a parallelogram, and let K and L be points lying on $\overline{BC}$ and $\overline{CD}$, respectively, such that $BK \cdot AD = DL \cdot AB$. Let $\overline{DK}$ and $\overline{BL}$ intersect at P. Show that $\angle DAP = \angle BAC$.

Solution: Draw R, S, T, U on sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$, respectively, so that $\overline{RT} \parallel \overline{AD}$, $\overline{SU} \parallel \overline{AB}$, and $\overline{RT} \cap \overline{SU} = P$. Let $UP = DT = a$ and $RP = BS = b$. Also let $AB = CD = x$, $BC = DA = y$, and $DL = my$. We are given $BK \cdot AD = DL \cdot AB$, so $BK = mx$.

Segments $\overline{RB}$, $\overline{BP}$, $\overline{PR}$ are parallel to $\overline{TL}$, $\overline{LP}$, $\overline{PT}$, implying that $\triangle RBP \sim \triangle TLP$. Note that $RB = x - a$, $RP = b$, $TL = a - my$, and $TP = y - b$. Hence,

$$\frac{x-a}{a-my} = \frac{RB}{TL} = \frac{RP}{TP} = \frac{b}{y-b},$$

or

$$(x-a)(y-b) = (a-my)b.$$

Analogous calculations with similar triangles UDP and SKP yield

$$(x-a)(y-b) = (b-mx)a.$$

Hence, $(a - my)b = (b - mx)a$, or $\frac{a}{b} = \frac{y}{x}$. Thus, parallelogram $AUPR$ is similar to parallelogram $ABCD$. Therefore, corresponding angles UAP and BAC are congruent, so that $\angle DAP = \angle BAC$, as desired.

Problem 6 Let $n_1 < n_2 < \cdots < n_{2000} < 10^{100}$ be positive integers. Prove that one can find two nonempty disjoint subsets A and B of $\{n_1, n_2, \ldots, n_{2000}\}$ such that $|A| = |B|$, $\sum_{x \in A} x = \sum_{x \in B} x$, and $\sum_{x \in A} x^2 = \sum_{x \in B} x^2$.

Solution: Given any subset $S \subseteq \{n_1, n_2, \cdots n_{2000}\}$ of size 1000, we have

$$0 < \textstyle\sum_{x \in S} x < 1000 \cdot 10^{100},$$
$$0 < \textstyle\sum_{x \in S} x^2 < 1000 \cdot 10^{200}.$$

Thus, as S varies, there are fewer than $(1000 \cdot 10^{100})(1000 \cdot 10^{200}) = 10^{306}$ values of $(\sum_{x \in S} x, \sum_{x \in S} x^2)$.

Because $\sum_{k=0}^{2000} \binom{2000}{k} = 2^{2000}$ and $\binom{2000}{1000}$ is the biggest term in the sum, $\binom{2000}{1000} > \frac{2^{2000}}{2001}$. There are

$$\binom{2000}{1000} > \frac{2^{2000}}{2001} > \frac{10^{600}}{2001} > 10^{306}$$

distinct subsets of size 1000. By the Pigeonhole Principle, there exist distinct subsets C and D of size 1000, such that $\sum_{x \in C} x^2 = \sum_{x \in D} x^2$ and $\sum_{x \in C} x = \sum_{x \in D} x$. Removing the common elements from C and D yields sets A and B with the required properties.

1.12 Romania

Problem 1 Determine the ordered systems (x, y, z) of positive rational numbers for which $x+\frac{1}{y}$, $y+\frac{1}{z}$, and $z+\frac{1}{x}$ are integers.

Solution: We claim that the desired ordered triples are

$$(1,1,1), \quad \left(2,1,\frac{1}{2}\right), \quad \left(\frac{3}{2},2,\frac{1}{3}\right), \quad \left(3,\frac{1}{2},\frac{2}{3}\right),$$

and their cyclic permutations. A simple calculation confirms that all these triples have the required property.

Let $x=\frac{a}{b}$, $y=\frac{c}{d}$, and $z=\frac{e}{f}$ for the pairs (a, b), (c, d), and (e, f), each consisting of two relatively prime positive integers. We are given that $\frac{ac+bd}{bc}$, $\frac{ce+df}{de}$, and $\frac{ea+fb}{fa}$ are all positive integers. In other words,

$$bc \mid (ac+bd), \quad de \mid (ce+df), \quad fa \mid (ea+fb). \qquad (*)$$

Because $bc \mid (ac+bd)$, we deduce that $b \mid ac$ and $c \mid bd$. But b is relatively prime to a, so we must have $b \mid c$ and similarly $c \mid b$ as c is relatively prime to d. This can only happen if $b=c$. Similarly, $d=e$ and $a=f$. Writing the relations in $(*)$ in terms of a, c, e and simplifying yields $c \mid (e+a)$, $e \mid (a+c)$, and $a \mid (c+e)$. Thus, a, c, and e all divide their sum $S=a+c+e$.

We will assume for the time being that $a \geq c \geq e$ and take into account the possible permutations later. Then $a \geq \frac{S}{3}$ and $a<S$. Because $a \mid S$, we have $a=\frac{S}{3}$ or $\frac{S}{2}$.

The first case is $a=\frac{S}{3}$. Then from $S=a+c+e \leq 3a=S$ we obtain $a=c=e$. Thus, $x=y=z=1$, and this solution has the required properties.

Otherwise, we have $a=\frac{S}{2}$. Then $\frac{S}{2}=S-a>c$ and $2c \geq e+c=\frac{S}{2}$. Because $c \mid S$, either (i) $c=\frac{S}{4}$ or (ii) $c=\frac{S}{3}$. In the first case, $(a, c, e)=\left(\frac{S}{2}, \frac{S}{4}, \frac{S}{4}\right)$. Removing the constraint $a \geq c \geq e$, we find that (x, y, z) equals $\left(2,1, \frac{1}{2}\right)$, $\left(1, \frac{1}{2}, 2\right)$, or $\left(\frac{1}{2}, 2,1\right)$.

In case (ii), $(a, c, e)=\left(\frac{S}{2}, \frac{S}{3}, \frac{S}{6}\right)$. This triple and its permutations give the solutions $(x, y, z)=\left(\frac{3}{2}, 2, \frac{1}{3}\right)$, $\left(2, \frac{1}{3}, \frac{3}{2}\right)$, $\left(\frac{1}{3}, \frac{3}{2}, 2\right)$, $\left(3, \frac{1}{2}, \frac{2}{3}\right)$, $\left(\frac{1}{2}, \frac{2}{3}, 3\right)$, $\left(\frac{2}{3}, 3, \frac{1}{2}\right)$. This completes the proof.

Problem 2 Let m and k be positive integers such that $k<m$, and let M be a set with m elements. Let p be an integer such that there exist subsets $A_1, A_2, \ldots, A_p$ of M for which $A_i \cap A_j$ has at most k elements for each pair of distinct numbers $i, j \in\{1,2, \ldots, p\}$, $i \neq j$.

Prove that the maximum possible value of p is

$$p_{\max} = \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \cdots + \binom{m}{k+1}.$$

Solution: We begin by showing that $p = p_{\max}$ is achievable. We take the collection of sets $\{A_1, A_2, \ldots, A_p\}$ to be all of the subsets of M with at most $k+1$ elements. Clearly, there are $p_{\max}$ such sets, because each term $\binom{m}{r}$ in the definition of $p_{\max}$ corresponds to the number of subsets of M of size r. Also, for distinct A_i, A_j we have $|A_i \cap A_j| < \max\{|A_i|, |A_j|\} \le k+1$.

Now assume, for sake of contradiction, that there exist p sets satisfying the given conditions with $p > p_{max}$. Among the p sets, at most $\binom{m}{0} + \binom{m}{1} + \cdots + \binom{m}{k}$ have k or fewer elements; thus, there are at least $\binom{m}{k+1}+1$ sets in the collection with $k+1$ or more elements. We may associate with each such set A_i an arbitrary subset $A_i' \subseteq A_i$ with exactly $k+1$ elements. But there are only $\binom{m}{k+1}$ subsets of M with $k+1$ elements. Therefore, two subsets A_i' and A_j' must be the same, and $A_i \cap A_j$ has at least $k+1$ elements, a contradiction. Thus, p_{max} is the desired maximum.

Problem 3 Let $n \ge 2$ be an even integer, and let a and b be real numbers such that $b^n = 3a + 1$. Show that the polynomial $p(x) = (x^2 + x + 1)^n - x^n - a$ is divisible by $q(x) = x^3 + x^2 + x + b$ if and only if $b = 1$.

Solution: We first prove the "if" direction. If $b = 1$, then $q(x) = x^3 + x^2 + x + 1 = (x+1)(x+i)(x-i)$. Also, $a = 0$, so $p(x) = (x^2+x+1)^n - x^n$, and it is easy to verify that -1 and $\pm i$ are indeed roots of $p(x)$. Thus, $q(x) \mid p(x)$.

For the "only if" direction, assume that $q(x) \mid p(x)$. Then $q(x)$ divides $x^n p(x) = (x^3+x^2+x)^n - x^{2n} - ax^n$. In addition, $q(x)$ divides

$$(x^3+x^2+x)^n - (-b)^n = (x^3+x^2+x+b) \sum_{i=0}^{n-1} \left((x^3+x^2+x)^i (-b)^{n-1-i} \right).$$

Subtracting $x^n p(x)$ from this polynomial and noticing that n is even and $(-b)^n = b^n$, we have

$$q(x) \mid (x^{2n} + ax^n - b^n).$$

Call the polynomial on the right side $f(x)$. We now consider the roots of $f(x)$. Substituting $y = x^n$, we have $f(x) = y^2 + ay - b^n$; let this quadratic in y be $g(y)$. Because n is even, b^n is nonnegative, and it follows that the discriminant of $g(y)$ is nonnegative. Thus, $g(y)$ has two roots u and v in $\mathbb{R}$, with

$$u + v = -a, \quad uv = -b^n.$$

It follows that the roots of $f(x)$ are the roots of the equations $x^n = u$, $x^n = v$, which lie on two (possibly coinciding) circles on the complex plane centered around the origin. Let the radii of these two circles be

$$r_1 = |u|^{1/n}, \quad r_2 = |v|^{1/n}.$$

Taking absolute values of both sides of the equation $uv = -b^n$, we have $|u||v| = |b|^n$, so that

$$r_1 r_2 = |b|.$$

Because $q(x) \mid f(x)$, the three complex roots of $q(x)$ must be among the roots of $f(x)$ and thus also lie on the two circles. Also, the product of the roots is $-b$ (the negative of the constant coefficient of $q(x)$). We now consider two cases.

Case 1: Two of the roots α, β of $q(x)$ lie on one circle and one root γ lies on the other circle. Without loss of generality, assume that α and β are on the circle with radius r_1. Then taking absolute values in the relation $\alpha\beta\gamma = -b$, we have $|\alpha||\beta||\gamma| = |b|$, so that $r_1^2 r_2 = |b|$. Combining this with our earlier equation $r_1 r_2 = |b|$, we obtain $r_1 = 1, r_2 = |b|$.

Returning to the roots of the quadratic $g(y)$ and using the fact that n is even, we have $|u| = 1, |v| = |b|^n = b^n$. Because $uv = -b^n$ and u, v are real, we either have $u = 1, v = -b^n$ or $u = -1, v = b^n$. Thus, $u + v = \pm(1 - b^n) = \pm(1 - (3a + 1)) = \pm 3a$. On the other hand, we saw earlier that $u + v = -a$; thus, either case yields $a = 0$, from which it follows that $b = \pm 1$.

If $b = -1$, then $q(x) = x^3 + x^2 + x - 1$ while $f(x) = x^{2n} - 1$. We know that $q(x)$ has at least one real root because it has odd degree; however, the only real roots of $f(x)$ are ± 1, and neither is a root of $q(x)$, contradicting the fact that $q(x) \mid f(x)$. Thus, $b = 1$ in this case.

Case 2: All three roots of $q(x)$ lie on one circle. Because their product is $-b$, each root has absolute value $|b|^{1/3}$. On the other hand, at least one root is real because $q(x)$ has odd degree. Thus, either $b^{1/3}$ or $-b^{1/3}$ must be a root of $q(x)$. Also, it cannot be the case that $b = 0$, for then all three roots of $q(x)$ would have to be 0, which is not the case. Let $c = b^{1/3} \neq 0$. Then either c or $-c$ is a root of $q(x) = x^3 + x^2 + x + c^3$. In the first case, we have $2c^3 + c^2 + c = 0$. Dividing by c, we obtain $2c^2 + c + 1 = 0$. This quadratic in c has no real roots, a contradiction. In the second case, we have $c^2 - c = 0$, and we may again divide by c to obtain $c = 1$. Thus, $b = c^3 = 1$.

Therefore, in both cases we have $b = 1$, as desired.

Problem 4 Show that if a, b, and c are complex numbers such that

$$(a+b)(a+c) = b,$$

$$(b+c)(b+a) = c,$$

$$(c+a)(c+b) = a,$$

then a, b, and c are real numbers.

Solution: We make the substitution $x = b + c$, $y = c + a$, $z = a + b$, so that $a = \frac{-x+y+z}{2}$, $b = \frac{x-y+z}{2}$, $c = \frac{x+y-z}{2}$. Upon clearing denominators, our equations become

$$2yz = x - y + z,$$

$$2zx = x + y - z,$$

$$2xy = -x + y + z.$$

Because $2yz = x - y + z$, we have $4yz + 2y - 2z - 1 = 2x - 1$, or $(2y-1)(2z+1) = 2x - 1$. Similarly, $(2z-1)(2x+1) = 2y - 1$ and $(2x-1)(2y+1) = 2z - 1$. We make another substitution: $p = 2x + 1$, $q = 2y + 1$, and $r = 2z + 1$. This gives us

$$\begin{aligned} r(q-2) &= p - 2, \\ p(r-2) &= q - 2, \\ q(p-2) &= r - 2. \end{aligned} \tag{$*$}$$

To prove that a, b, c are real, it suffices to prove that p, q, r are real.

First we dispose of the cases in which one of $p-2$, $q-2$, and $r-2$ is 0. If $p-2=0$, then $r-2=q(p-2)=0$ and $q-2=p(r-2)=0$, so that $p=q=r=2$. Similarly, $p=q=r=2$ if either $q-2=0$ or $r-2=0$. Thus, if one of $p-2$, $q-2$, and $r-2$ is 0, then p, q, and r are all real, as desired.

If $p,q,r \neq 2$, then $(*)$ gives $p,q,r \neq 0$. Also, multiplying the equations in $(*)$ together and dividing both sides by $(p-2)(q-2)(r-2)$ gives $pqr=1$. Thus, $p=\frac{1}{qr}$. Applying this to the second equation gives $\frac{r-2}{r}\cdot\frac{1}{q}=q-2$, or $1-\frac{2}{r}=q(q-2)$. Solving for r gives

$$r=\frac{2}{1-q(q-2)}.$$

Plugging $r=\frac{2}{1-q(q-2)}$ into the last equation in $(*)$ yields

$$q(p-2)=\frac{2q(q-2)}{1-q(q-2)}.$$

Cancelling the q's and substituting in

$$p=\frac{1}{qr}=\frac{1-q(q-2)}{2q}$$

gives

$$\frac{1-q(q-2)}{2q}-2=\frac{2(q-2)}{1-q(q-2)}.$$

Clearing denominators yields

$$(1-q(q-2))^2-4q(1-q(q-2))=4q(q-2),$$

or

$$q^4-10q^2+8q+1=0.$$

Inspection shows that $q=1$ is a root of this equation, and the above equation factors as

$$(q-1)(q^3+q^2-9q-1)=0.$$

We claim that q is real. If $q=1$, this is clear. Otherwise, q is a root of $P(x)=x^3+x^2-9x-1$, which has three complex roots. A simple calculation shows that $P(-1)=8$, $P(0)=-1$, and $P(3)=8$. Thus, by the Intermediate Value Theorem, $P(x)$ must have one real root between -1 and 0 and another between 0 and 3. But $P(x)$ is a polynomial of odd degree and so must have an odd number of real roots. Thus, all three of its roots must be real, and in particular q is real.

If q is real, then $r = \frac{2}{1-q(q-2)}$ is real and $p = \frac{1}{qr}$ is real as well. Therefore, p, q, r are all real, as desired.

Problem 5

(a) Let $f,\ g\colon\ \mathbb{Z} \to \mathbb{Z}$ be injective maps. Show that the function $h : \mathbb{Z} \to \mathbb{Z}$, defined by $h(x) = f(x)g(x)$ for all $x \in \mathbb{Z}$, cannot be surjective.

(b) Let $f\colon\ \mathbb{Z} \to \mathbb{Z}$ be a surjective map. Show that there exist surjective functions $g, h\colon\ \mathbb{Z} \to \mathbb{Z}$ such that $f(x) = g(x)h(x)$ for all $x \in \mathbb{Z}$.

Solution: First we prove part (a) by way of contradiction. Suppose h is surjective. Then there exist distinct integers $a_1, a_2, \ldots$ such that $h(a_n) = p_n$, where p_n is the n^{th} prime number. Therefore, $f(a_n)g(a_n) = h(a_n) = p_n$, so that one of $f(a_n)$ and $g(a_n)$ equals ± 1. Because this is true for infinitely many n, and because the a_n are distinct, one of the functions f and g takes on one of the values 1 and -1 infinitely many times. However, this is not possible because f and g are surjective, a contradiction.

To solve (b), we define functions g, h as follows:

- When $f(n) = m^2$ for some $m \geq 0$, let $g(n) = h(n) = m$.
- When $f(n) = 2m^2$ for some $m > 0$, let $g(n) = -m$, $h(n) = -2m$.
- When $f(n) = -m^2$ for some $m > 0$, let $g(n) = m$, $h(n) = -m$.
- Otherwise, let $g(n) = 1$, $h(n) = f(n)$.

This function is well-defined because the square roots of 2, -1, and -2 are irrational, so that no two of the first three conditions occur simultaneously. For all n these definitions clearly satisfy the relation $f(n) = g(n)h(n)$. Also, we have $g(f^{-1}(k^2)) = h(f^{-1}(k^2)) = k$ for each integer $k > 0$, and $g(f^{-1}(2k^2)) = h(f^{-1}(-k^2)) = k$ for each integer $k < 0$. Thus, g and h are both surjective.

Problem 6 Three schools each have 200 students. Every student has at least one friend in each school. (If student a is a friend of student b, then b is a friend of a; also, for the purposes of this problem, no student is a friend of himself.) There exists a set E of 300 students (chosen from among the 600 students at the three schools) with the following property: for any school S and any two students $x, y \in E$ who are not in the school S, x and y do not have the same number of friends in S. Show that one can find three students, one in each school, such that any two are friends with each other.

Solution: We name the sets of students in each of the three schools S_1, S_2, and S_3. There are 600 students among the three schools, with 300 students in E. Thus, one of the schools must have at most $\frac{300}{3} = 100$ students in E. Without loss of generality, assume that this school is S_1. Then consider the 200 or more students in $E \setminus S_1$. We are given that every student has at least one friend in S_1, so in particular every student in $E \setminus S_1$ has between 1 and 200 friends in S_1. Also, by the restriction on E, no two students in $E \setminus S_1$ have the same number of friends in S_1. Thus, because there are at least 200 students in $E \setminus S_1$, one student $a \in S \setminus E$ must have exactly 200 friends in S_1.

Without loss of generality, assume $a \in S_2$. Every student has a friend in S_3, so a is friends with some student b in S_3; every student has a friend in S_1, so b is friends with some student c in S_1. By our choice of a, a knows all students in S_1, and so a knows c. Thus, of the three students a, b, and c, any two are friends with each other.

Problem 7 The vertices of square $ABCD$ lie outside a circle centered at M. Let $\overline{AA'}, \overline{BB'}, \overline{CC'}, \overline{DD'}$ be tangents to the circle. We assume that these segments can be arranged to be the four consecutive sides of a quadrilateral p in which some circle is inscribed. Prove that p has an axis of symmetry.

Solution: Let r be the radius of the circle centered at M, and let a, b, c, d denote the lengths AA', BB', CC', DD', respectively. Then we have $a^2 = AM^2 - r^2$, and we have similar expressions for b^2, c^2, d^2.

Let P, R be the feet of the altitudes from M to lines AB and CD, respectively. Because quadrilateral $ABCD$ is a square, we have $AP = DR$ and $CR = BP$. Hence, $a^2 + c^2 = AM^2 - r^2 + CM^2 - r^2$, and then we have

$$\begin{aligned} a^2 + c^2 &= AP^2 + PM^2 - r^2 + CR^2 + RM^2 - r^2 \\ &= DR^2 + RM^2 - r^2 + BP^2 + PM^2 - r^2 \\ &= DM^2 - r^2 + BM^2 - r^2 = b^2 + d^2 \end{aligned}$$

Furthermore, because a, b, c, d are the lengths of the consecutive sides of the circumscribed quadrilateral p, we have

$$a + c = b + d.$$

Squaring both sides and subtracting from $2(a^2 + c^2) = 2(b^2 + d^2)$, we obtain

$$(a - c)^2 = (b - d)^2.$$

Hence,

$$a - c = \pm(b - d).$$

Combined with $a+c = b+d$, this equation yields either $(a, c) = (b, d)$ or $(a, c) = (d, b)$. Either way, the quadrilateral p is a kite and has an axis of symmetry.

Problem 8 Find the least number n with the following property: given any n rays in three-dimensional space sharing a common endpoint, the angle between some two of these rays is acute.

Solution: The least n is 7. First observe that seven rays are necessary, because the six unit vectors in the directions of the positive and negative x-, y-, and z-axes form no acute angles.

We now prove that given any 7 vectors emanating from the origin, some two of them form an acute angle. Without loss of generality, assume that all the vectors have unit length. For each vector, consider the closed unit hemisphere centered around it. Each hemisphere has surface area equal to half the total surface area of the unit sphere, so the sum of all the surface areas is $7/2$ the total area. It follows that there exists a point closed region of positive surface area in which 4 or more of the hemispheres intersect. Let P be a point in the *interior* of this region, and choose coordinate axes so that $P = (0, 0, 1)$; then 4 of the 7 vectors, say (x_i, y_i, z_i) for $1 \leq i \leq 4$, have endpoints on the open hemisphere $\{(x, y, z) \mid x^2+y^2+z^2 = 1, z > 0\}$. Some two of the four projections (x_i, y_i) — without loss of generality, (x_1, y_1) and (x_2, y_2) — form an angle less than or equal to $\pi/2$. Then $0 \leq (x_1, y_1) \cdot (x_2, y_2) = x_1x_2 + y_1y_2$. Thus, because z_1 and z_2 are positive, we have $0 < x_1x_2 + y_1y_2 + z_1z_2 = (x_1, y_1, z_1) \cdot (x_2, y_2, z_2)$, so that vectors (x_1, y_1, z_1) and (x_2, y_2, z_2) meet an acute angle. This completes the proof.

Problem 9 Let $f(x) = a_0 + a_1x + \cdots + a_mx^m$, with $m \geq 2$ and $a_m \neq 0$, be a polynomial with integer coefficients. Let n be a positive integer, and suppose that:

(i) $a_2, a_3, \ldots, a_m$ are divisible by all the prime factors of n;

(ii) a_1 and n are relatively prime.

Prove that for any positive integer k, there exists a positive integer c such that $f(c)$ is divisible by n^k.

Solution: Consider any integers c_1, c_2 such that $c_1 \not\equiv c_2 \pmod{n^k}$. Observe that if $n^k \mid st$ for some integers s, t where t is relatively prime to n, then $n^k \mid s$. In particular, $n^k \not| \, (c_1 - c_2)t$ if t is relatively prime to n.

Note that

$$\begin{aligned} f(c_1) - f(c_2) &= (c_1 - c_2)a_1 + \sum_{i=2}^{m} a_i(c_1^i - c_2^i) \\ &= (c_1 - c_2)\underbrace{\left(a_1 + \sum_{i=2}^{m}\left(a_i \sum_{j=0}^{m-1}(c_1^j c_2^{m-1-j})\right)\right)}_{t}. \end{aligned}$$

For any prime p dividing n, p divides $a_2, \ldots, a_m$ but not a_1. Hence, p does not divide the second factor t in the expression above. This implies that t is relatively prime to n, so n^k does not divide the product $(c_1 - c_2)t = f(c_1) - f(c_2)$.

Therefore, $f(0), f(1), \ldots, f(n^k - 1)$ are distinct modulo n^k, and one of them — say, $f(c)$ — must be congruent to 0 modulo n^k; that is, $n^k \mid f(c)$, as desired.

Problem 10 Find all pairs (m, n) of positive integers, with $m, n \geq 2$, such that $a^n - 1$ is divisible by m for each $a \in \{1, 2, \ldots, n\}$.

Solution: The solution set is the set of all $(p, p-1)$, for odd primes p. The fact that all of these pairs are indeed solutions follows immediately from Fermat's Little Theorem. Now we show that no other solutions exist.

Suppose that (m, n) is a solution. Let p be a prime dividing m. We first observe that $p > n$. Otherwise, we could take $a = p$, and then $p^n - 1$ would not be divisible by p, let alone m. Then because $n \geq 2$, we have $p \geq 3$ and hence p is odd.

Now we prove that $p < n + 2$. Suppose on the contrary that $p \geq n+2$. If n is odd, then $n+1$ is even and less than p. Otherwise, if n is even, then $n+2$ is even and hence less than p as well, because p is odd. In either case, there exists an even d such that $n < d < p$ with $\frac{d}{2} \leq n$. Setting $a = 2$, $\frac{d}{2}$ in the given condition, we find that

$$d^n \equiv 2^n \cdot \left(\frac{d}{2}\right)^n \equiv 1 \cdot 1 \equiv 1 \pmod{m},$$

so that $d^n - 1 \equiv 0 \pmod{m}$ as well. Because $n < d < p < m$, we see that $1, 2, \ldots, n, d$ are $n+1$ distinct roots of the polynomial congruence $x^n - 1 \equiv 0 \pmod{p}$. By Lagrange's theorem, however, this congruence can have at most n roots, a contradiction.

Thus, we have sandwiched p between n and $n+2$, and the only possibility is that $p = n+1$. Therefore, all solutions are of the form $(p^k, p-1)$ with p an odd prime. It remains to prove that $k = 1$. Using $a = n = p-1$, it suffices to prove that

$$p^k \not| \left((p-1)^{p-1} - 1\right).$$

Expanding the term $(p-1)^{p-1}$ modulo p^2, and recalling that p is odd, we have

$$\begin{aligned}
(p-1)^{p-1} &= \sum_{i=0}^{p-1} \binom{p-1}{i} (-1)^{p-1-i} p^i \\
&\equiv \binom{p-1}{0}(-1)^{p-1} + \binom{p-1}{1}(-1)^{p-2} p \\
&\equiv 1 - p(p-1) \\
&\equiv 2 \not\equiv 0 \pmod{p^2}.
\end{aligned}$$

It follows immediately that k cannot be greater than 1, completing the proof.

Problem 11 Prove that there is no function $f : (0, \infty) \to (0, \infty)$ such that

$$f(x+y) \geq f(x) + yf(f(x))$$

for all $x, y \in (0, \infty)$.

Solution: Assume, for sake of contradiction, that there does exist such a function f. Fix x in the given inequality. Then as y varies, $f(x+y)$ is bounded below by a linear function in y which has a positive coefficient of y. Thus, $f(x) \to \infty$ as $x \to \infty$. It follows that $f(f(x)) \to \infty$ as $x \to \infty$. Hence there exists an a such that $c = f(f(a)) > 1$. Let $b = f(a)$; then setting $x = a$ in the given inequality, we have

$$f(a + y) \geq b + cy$$

for all y. Because $c > 1$, there exists a y_0 such that $b + cy_0 \geq a + y_0 + 1$. Letting $x_0 = a + y_0$, we have $f(x_0) \geq x_0 + 1$.

Now set $x = x_0$ and $y = f(x_0) - x_0$ in the given inequality. We obtain:

$$f(f(x_0)) \geq f(x_0) + (f(x_0) - x_0)f(f(x_0)).$$

But because $f(x_0) \geq x_0 + 1$, the right hand side is strictly greater than $f(f(x_0))$, a contradiction. Therefore, our original assumption was false, and no function f satisfies the given condition.

Problem 12 Let P be a convex polyhedron with vertices V_1, V_2, $\ldots$, V_p. Two vertices V_i and V_j are called *neighbors* if they are distinct and belong to the same face of the polyhedron. The p sequences $(v_i(n))_{n \geq 0}$, for $i = 1, 2, \ldots, p$, are defined recursively as follows: the $v_i(0)$ are chosen arbitrarily; and for $n \geq 0$, $v_i(n+1)$ is the arithmetic mean of the numbers $v_j(n)$ for all j such that V_i and V_j are neighbors. Suppose that $v_i(n)$ is an integer for all $1 \leq i \leq p$ and $n \in \mathbb{N}$. Prove that there exist $N \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that $v_i(n) = k$ for all $n \geq N$ and $i = 1, 2, \ldots, p$.

Solution: For each $n \geq 0$, let $m(n) = \min\{v_1(n), v_2(n), \ldots, v_p(n)\}$ and let $M(n) = \max\{v_1(n), v_2(n), \ldots, v_p(n)\}$. Clearly $m(n) \leq M(n)$ for all n.

Observe that $v_i(n+1)$ is the average of $v_j(n)$ over all neighbors V_j of V_i. Because $v_j(n) \geq m(n)$, we have

$$v_i(n+1) \geq m(n), \qquad (*)$$

with equality if and only if $v_j(n) = m(n)$ for all neighbors V_j of V_i.

From $(*)$ it follows that $m(n+1) \geq m(n)$ for all n. Likewise, $M(n+1) \leq M(n)$ for all n. These inequalities, together with $m(n) \leq M(n)$ and the fact that all the $m(n), M(n)$ are integers, imply that there exist integers $m \leq M$ with the following property: $m(n) = m$ and $M(n) = M$ for all sufficiently large n, i.e., for all $n \geq N$ for some N.

Suppose, for sake of contradiction, that $m < M$. For any $n \geq N$, some vertices V_i of the polyhedron have $v_i(n) = m$, and other vertices V_j have $v_j(n) > m$. At least one vertex V_i from the former set is adjacent to a vertex V_j from the latter set. Then equality cannot hold in $(*)$, so $v_i(n+1) > m$. In other words, the number of vertices V_i such that $v_i(n) = m$, decreases by at least 1 whenever n increases by 1. Thus, for large enough n there are no vertices V_i for which $v_i(n) = m$, a contradiction. Hence, our original assumption was false, and $m = M$.

Therefore, $v_i(n) = m = M$ for all $1 \leq i \leq p$ and $n \geq N$, as desired.

1.13 Russia

Problem 1 Peter and Alex play a game starting with an ordered pair of integers (a, b). On each turn, the current player increases or decreases either a or b: Peter by 1, and Alex by 1 or 3. Alex wins if at some point in the game the roots of $x^2 + ax + b$ are integers. Is it true that given any initial values a and b, Alex can guarantee that he wins?

Solution: Yes, Alex can win.

First observe that if $|a| > 3$ or $|b| > 3$ immediately before one of Alex's turns, then Alex can decrease $|a| + |b|$ by 3 during his turn: if $|a| > 3$, then he can decrease $|a|$ by 3; if $|b| > 3$, then he can decrease $|b|$ by 3. On the next turn, Peter can increase $|a|+|b|$ by at most 1, for a net decrease of at least 2. Because $|a| + |b|$ cannot decrease forever, using this strategy eventually Alex forces $-3 \leq a, b \leq 3$ immediately before one of his turns.

At this point in the game, one of the following four cases holds:

(1) $b \in \{-3, -1, 0, 1, 3\}$. Then either b already equals 0, or Alex can set $b = 0$. Once $b = 0$, Alex wins.

(2) $b = -2$ and $a \in \{-3, -1, 1, 3\}$. Alex sets $a = 0$ to make the polynomial $x^2 - 2$. Peter has four possible moves, each resulting in one of the following polynomials:

$$\begin{gathered} x^2 - x - 2, \\ x^2 + x - 2, \\ x^2 - 1, \\ x^2 - 3. \end{gathered}$$

The first three polynomials result in Peter's immediate defeat; if Peter makes the last polynomial, then Alex sets $b = 0$ to win.

(3) $b = -2$ and $a = \pm 2$. Then Alex sets $b = -3$ and wins.

(4) $b = 2$. Alex sets $b = -1$. If Peter lets b remain equal to -1 on his next turn, then Alex wins by setting $b = 0$ on his next turn. Otherwise, after Peter's turn, $b \in \{-2, 0\}$ and a remains in $[-3, 3]$, reducing this case to either (1) or (2). Hence, Alex can win in this case as well.

Therefore, Alex can guarantee that he wins in all cases, as claimed.

Problem 2 Let M and N be points on sides $\overline{AB}$ and $\overline{BC}$, respectively, of parallelogram $ABCD$ such that $AM = NC$. Let Q be the intersection of $\overline{AN}$ and $\overline{CM}$. Prove that $\overline{DQ}$ is an angle bisector of angle CDA.

First Solution: Let line DQ intersect line BC at T. From similar triangles ADQ and NTQ, we have

$$\frac{DA}{TN} = \frac{AQ}{NQ}.$$

By Menelaus' Theorem, $\frac{MB}{AM} \cdot \frac{AQ}{NQ} \cdot \frac{NC}{CB} = 1$. Combined with $AM = NC$, this gives

$$\frac{AQ}{NQ} = \frac{CB}{MB}.$$

Finally,

$$\frac{CB}{MB} = \frac{DA}{MB}$$

because sides $\overline{CB}$ and $\overline{DA}$ of quadrilateral $ABCD$ are congruent.

Combining the above three equations, we find that $TN = MB$. Hence,

$$TC = TN + NC = AM + MB = AB = DC,$$

leading to the conclusion $\angle CDT = \angle DTC = \angle TDA$. In other words, $\overline{DQ}$ bisects angle CDA, as desired.

Second Solution: Using parallel lines AB and CD, we have

$$QC \sin \angle QCD = QC \sin(\pi - \angle QMA) = QC \sin \angle QMA.$$

Similarly,

$$QA \sin \angle QAD = QA \sin \angle(\pi - QNC) = QA \sin \angle QNC.$$

Also, using the Law of Sines and the given condition $AM = CN$, we have

$$\frac{QC}{\sin \angle QNC} = \frac{CN}{\sin \angle CQN} = \frac{AM}{\sin \angle AQM} = \frac{QA}{\sin \angle QMA}.$$

Combining these equations gives

$$QC \sin \angle QCD = QA \sin \angle QAD.$$

In other words, Q is equidistant from lines DC and DA. Because Q lies between rays DA and DC, it follows that it lies on the internal angle bisector of angle ADC.

Problem 3 A target consists of an equilateral triangle broken into 100 equilateral triangles of unit side length by three sets of parallel lines. A sniper shoots at the target repeatedly as follows: he aims at one of the small triangles and then hits either that triangle or one of the small triangles which shares a side with it. He may choose to stop shooting at any time. What is the greatest number of triangles that he can be sure to hit exactly five times?

Solution: The answer is 25.

More generally, we prove that given any $n, k \in \mathbb{Z}^+$ and a target consisting of $(2n)^2$ small triangles, n^2 is the greatest number of triangles that the shooter can be sure to hit exactly k times.

We can divide the target into n^2 *target areas*, triangles with side length 2 units (consists 4 unit triangles). For each of these target areas, we color the unit triangle in the middle black. Notice every triangle on the target is either a black triangle, or next to one. Therefore, it is possible that the sniper always end up shooting a black triangle regardless of where he aims. Therefore, he cannot be sure that he will hit more than n^2 triangles exactly k times.

Now we want to show the sniper can always shoot n^2 targets exactly k times. To do so, for each black triangle he aims at it until one of the triangles in the corresponding target area is hit exactly k times. Because the target areas do not overlap, the sniper is guaranteed to hit one triangle in each of the n^2 target areas exactly k times.

Problem 4 Two points are selected inside a convex pentagon. Prove that it is possible to select four of the pentagon's vertices so that the quadrilateral they form contains both points.

Solution: Call the two points X and Y, and let the pentagon's vertices be A, B, C, D, E in that order. It is clear that any point in pentagon $ABCDE$ must lie in at least one of the triangles ABD, BCE, CDA, DEB, and EAC. Without loss of generality, assume that point X lies in triangle ABD. If point Y does too, then X and Y lie in quadrilateral $ABDE$. If Y does not, then Y lies in either triangle ABC or triangle ADE, in which case X and Y lie in quadrilateral $ABCD$ or quadrilateral $ABDE$, respectively.

Problem 5 Does there exist a positive integer such that the product of its proper divisors ends with exactly 2001 zeroes?

Solution: Yes. Given an integer n with $\tau(n)$ positive divisors, the product of all positive divisors of n is equal to

$$\sqrt{\left(\prod_{d|n} d\right)\left(\prod_{d|n}(n/d)\right)} = \sqrt{\prod_{d|n} d \cdot (n/d)} = \sqrt{n^{\tau(n)}}.$$

Thus, the product of all *proper* positive divisors of n equals

$$n^{\frac{1}{2}\tau(n)-1}.$$

If $n = \prod_{i=1}^{k} p_i{}^{q_i}$ with the p_i's distinct primes and the q_i's positive integers, then $\tau(n) = \prod_{i=1}^{k}(q_i + 1)$. Hence, if we set $n = 2^1 \cdot 5^1 \cdot 7^6 \cdot 11^{10} \cdot 13^{12}$, then

$$\frac{1}{2}\tau(n) - 1 = \frac{1}{2}(2 \cdot 2 \cdot 7 \cdot 11 \cdot 13) - 1 = 2001.$$

Thus, the product of the proper divisors of n is equal to $2^{2001} \cdot 5^{2001} \cdot 7^{6\cdot 2001} \cdot 11^{10\cdot 2001} \cdot 13^{12\cdot 2001}$, an integer ending in exactly 2001 zeroes.

Problem 6 A circle is tangent to rays OA and OB at A and B, respectively. Let K be a point on minor arc AB of this circle. Let L be a point on line OB such that $\overline{OA} \parallel \overline{KL}$. Let M be the intersection (distinct from K) of line AK and the circumcircle ω of triangle KLB. Prove that line OM is tangent to ω.

Solution: All angles are directed modulo π. Because K, L, M, B are concyclic and $\overline{KL} \parallel \overline{OA}$, we have $\angle AMB = \angle KMB = \angle KLB = \angle AOB$. Thus, A, B, M, O are concyclic and hence

$$\angle AMO = \angle ABO, \quad \angle MAO = \angle MBO.$$

Because K, L, M, B are concyclic,

$$\angle KML = \angle KBL.$$

Also, because $\overline{OA}$ is tangent to the circumcircle of triangle AKB,

$$\angle ABK = \angle KAO = \angle MAO.$$

From these we have

$$\angle LMO = \angle AMO - \angle KML = \angle ABO - \angle KBL$$
$$= \angle ABK = \angle MAO = \angle MBO.$$

Therefore, $\overline{OM}$ is tangent to ω.

Problem 7 Let $a_1, a_2, \ldots, a_{10^6}$ be nonzero integers between 1 and 9, inclusive. Prove that at most 100 of the numbers $\overline{a_1 a_2 \ldots a_k}$ $(1 \le k \le 10^6)$ are perfect squares. (Here, $\overline{a_1 a_2 \ldots a_k}$ denotes the decimal number with the k digits $a_1, a_2, \ldots, a_k$.)

Solution: For each positive integer x, let $d(x)$ be the number of decimal digits in x.

Lemma. *Suppose that $y > x$ are perfect squares such that $y = 10^{2b}x + c$ for some positive integers b, c with $c < 10^{2b}$. Then*

$$d(y) - 1 \ge 2(d(x) - 1).$$

Proof. Because $y > 10^{2b}x$, we have $\sqrt{y} > 10^b\sqrt{x}$. Because $\sqrt{y}$ and $10^b\sqrt{x}$ are both integers, $\sqrt{y} \ge 10^b\sqrt{x} + 1$, so that $10^{2b}x + c = y \ge 10^{2b}x + 2 \cdot 10^b\sqrt{x} + 1$. Thus, $c \ge 2 \cdot 10^b\sqrt{x} + 1$.

Also, $10^{2b} > c$ by assumption, implying that

$$10^{2b} > c \ge 2 \cdot 10^b\sqrt{x} + 1.$$

Hence, $10^b > 2\sqrt{x}$. It follows that

$$y > 10^{2b}x > 4x^2.$$

Therefore,

$$d(y) \ge 2d(x) - 1,$$

as desired. $\square$

We claim that there are at most 36 perfect squares $\overline{a_1 a_2 \ldots a_k}$ with an even (resp. odd) number of digits. Let $s_1 < s_2 < \cdots < s_n$ be these perfect squares. Clearly $d(s_n) \le 10^6$. We now prove that if $n > 1$, then $d(s_n) \ge 1 + 2^{n-1}$.

Because $s_1, s_2, \cdots, s_n$ all have an even (resp. odd) number of digits, for each $i = 1, 2, \ldots, n-1$, we can write $s_{i+1} = 10^{2b}s_i + c$ for some integers $b > 0$ and $0 \le c < 10^{2b}$. Because no a_i equals 0, we further know that $0 < c$. Hence, by our lemma,

$$d(s_{i+1}) - 1 \geq 2(d(s_i) - 1)$$

for each $i = 1, 2, \ldots, n-1$. Because $d(s_2) - 1 \geq 2$, we thus have $d(s_n) - 1 \geq 2^{n-1}$, as desired.

Thus, if $n > 1$,

$$1 + 2^{n-1} \leq d(s_n) \leq 10^6,$$

and

$$n \leq \left\lfloor \frac{\log(10^6 - 1)}{\log 2} \right\rfloor + 1 = 20.$$

Hence, there are at most 20 perfect squares $\overline{a_1 a_2 \ldots a_k}$ with an even (resp. odd) number of digits.

Therefore, there are at most $40 < 100$ perfect squares $\overline{a_1 a_2 \ldots a_k}$.

Problem 8 The lengths of the sides of an n-gon equal $a_1, a_2, \ldots, a_n$. If f is a quadratic such that

$$f(a_k) = f\left(\left(\sum_{i=1}^{n} a_i\right) - a_k\right)$$

for $k = 1$, prove that this equality holds for $k = 2, 3, \ldots, n$ as well.

Solution: Write $s = \sum_{i=1}^{n} a_i$. Defining f to be the general quadratic $ax^2 + bx + c$, some algebra shows that the condition

$$f(a_k) = f\left(\left(\sum_{i=1}^{n} a_i\right) - a_k\right), \tag{1}$$

or $aa_k^2 + ba_k + c = a(s - a_k)^2 + b(s - a_k) + c$, is equivalent to the condition

$$2a_k(b + as) = s(b + as).$$

Because $s > 2a_k$ by the triangle inequality, this last condition is equivalent to the condition that $b + as = 0$.

If (1) holds for $k = 1$, then $b + as = 0$; it follows that (1) holds for $k = 2, 3, \ldots, n$, as desired.

Problem 9 Given any point K in the interior of diagonal $\overline{AC}$ of parallelogram $ABCD$, construct the line ℓ_K as follows. Let s_1 be the circle tangent to lines AB and AD such that of s_1's two intersection points with $\overline{AC}$, K is the point farther from A. Similarly, let s_2 be the circle tangent to lines CB and CD such that of s_2's two intersection points with $\overline{CA}$, K is the point farther from C. Then let ℓ_K be the line connecting the centers of s_1 and s_2. Prove that as K varies along $\overline{AC}$, all the lines ℓ_K are parallel to each other.

Solution: Let r_i be the radius and O_i be the center of s_i for $i = 1, 2$. Note that

$$\begin{aligned} r_1 + r_2 &= AK \sin \frac{\angle A}{2} + KC \sin \frac{\angle C}{2} \\ &= AK \sin \frac{\angle A}{2} + KC \sin \frac{\angle A}{2} = AC \sin \frac{\angle A}{2}, \end{aligned}$$

a constant.

Now we set up a coordinate system with vectors $(\vec{x}, \vec{y})$, such that $\vec{x}$, $\vec{y}$ are perpendicular to lines AD, AB, respectively. Let $O_i = (x_i, y_i)$ in this coordinate system; we claim that $(x_1 - x_2, y_1 - y_2)$ is a constant. Let the distance between lines AD and BC be t. Because the distance between O_1 and line AD is r_1 and the distance between O_2 and line BC is r_2, we have $x_1 - x_2 = t - (r_1 - r_2)$, a constant. Similarly, $y_1 - y_2$ is also constant. Therefore, the lines ℓ_K are all parallel to each other.

Problem 10 Describe all possible ways to color each positive integer in one of three colors such that any positive integers a, b, c (not necessarily distinct) which satisfy $2000(a + b) = c$ are colored either in one color or in three different colors.

Solution: Either all integers are colored the same color; or, 1, 2, and 3 are colored differently and any number n is colored the same color as the $k \in \{1, 2, 3\}$ for which $n \equiv k \pmod{3}$.

The monochrome coloring clearly satisfies the given conditions; we now check that the other coloring does as well. Suppose that $2000(a + b) = c$. Then $2(a + b) \equiv c \pmod{3}$, or $a + b + c \equiv 0 \pmod{3}$. Thus, either a, b, c are all congruent modulo 3, in which case they are colored the same color; or, they are pairwise distinct modulo 3, in which case they are colored three different colors.

It remains to prove that these are the only possible colorings. Suppose we are given any three consecutive positive integers $x - 1$, x, $x + 1$. Setting $(a, b, c) = (x, x, 4000x)$ in the given condition shows that $4000x$ is the same color as x. Setting $(a, b, c) = (x - 1, x + 1, 4000x)$ shows that $x - 1$, $x + 1$, and $4000x$ are all the same color or are all different colors; i.e., $x - 1$, $x + 1$, and x are all the same color or are all different colors. Hence, two consecutive positive integers uniquely determine the color of the next greatest positive integer. The coloring of the positive integers is therefore determined uniquely by the colors of 1 and 2; there are nine possible colorings of the numbers 1 and 2, and these give rise to the colorings described above.

Problem 11 Three sets of ten parallel lines are drawn. Find the greatest possible number of triangles whose sides lie along the lines but whose interiors do not intersect any of the lines.

Solution: Define a *proper* triangle to be a triangle that satisfies the given conditions. We claim that the greatest possible number of proper triangles is 150. We will find a formula for the more general case, where we have three sets of n parallel lines for some even integer n.

Through an affine transformation, we reposition the first two sets of lines so that they lie horizontally and vertically on a coordinate plane, with equations $x = y_i, y = x_i, i = 1, 2, \ldots, n$, where the x_i and y_i lie in increasing order. We call the (x_i, y_i) *grid points*. We also transform the third set of lines so that they are given by $x + y = z_i$ for $i = 1, 2, \ldots, n$. Observe that any triangle bounded by lines of the form $x = x_i$, $y = y_j$, $x + y = z_k$ has its right-angled vertex pointing to either the lower-left (if $x_i + y_j < z_k$) or the upper-right (if $x_i + y_j > z_k$).

Lemma. *There are at most $\frac{3n^2}{4}$ proper triangles with right angles in their upper-right corner.*

Proof. Let S be any set of points with the following property:

> $(*)$: If (α_1, β_1), (α_2, β_2) lie in S, then either (i) $\alpha_1 \le \alpha_2$ and $\beta_1 \le \beta_2$; or (ii) $\alpha_2 \le \alpha_1$ and $\beta_2 \le \beta_1$. (In other words, any line connecting two points in S is vertical or has nonnegative slope.)

We claim that no two proper triangles with upper-right vertices in S are bounded by the same line $x + y = z_k$. Suppose, for sake of contradiction, that $x + y = z_k$ formed a proper triangle $\mathcal{T}_1$ with upper-right vertex $(\alpha_1, \beta_1) \in S$ and another proper triangle $\mathcal{T}_2$ with upper-right vertex $(\alpha_2, \beta_2) \in S$. Without loss of generality, assume that $\alpha_1 \le \alpha_2$ and $\beta_1 \le \beta_2$. The interior of $\mathcal{T}_1$ consists of the points (x, y) such that

- $x+y>z_k$;
- $x<\alpha_2$;
- $y<\beta_2$.

Let ϵ be a tiny positive value. Because $\mathcal{T}_2$ is bounded from the lower-left by the line $x+y=z_k$, both $(\alpha_1-\epsilon,\beta_1)$ and $(\alpha_1,\beta_1-\epsilon)$ satisfy the first criterion. At least one of these points also satisfies the latter two criteria, because $\alpha_1\le\alpha_2$ and $\beta_1\le\beta_2$. Hence, at least one of the lines $y=\beta_1$ or $x=\alpha_1$ intersects the interior of $\mathcal{T}_1$, contradicting the assumption that $\mathcal{T}_1$ is proper. Therefore, our original assumption was false, and each line $x+y=z_k$ bounds at most one proper triangle whose upper-right vertex is in S.

For $k=1,2,\ldots,\frac{n}{2}$, let S_k consist of (x_{n+1-k},y_k), along with every grid point due north or due west of (x_{n+1-k},y_k). Each S_k has property $(*)$, so for fixed k, at most n proper triangles have an upper-right vertex in S_k. Also, for each of the remaining $\frac{n^2}{4}$ grid points P not in any S_k — namely, (x_i,y_j) where either $i\le\frac{n}{2}$ or $j\ge\frac{n+2}{2}$ — at most one proper triangle has P as its upper-right vertex. Therefore, at most

$$n\cdot\frac{n}{2}+\frac{n^2}{4}=\frac{3n^2}{4}$$

proper triangles have a right angle as an upper-right vertex. □

Similarly, at most $\frac{3n^2}{4}$ proper triangles have a right angle as a lower-left vertex. Therefore, there are at most $\frac{3n^2}{4}+\frac{3n^2}{4}=\frac{3n^2}{2}$ proper triangles, as claimed.

It remains to show that this bound can be achieved. Let two sets of parallel lines be $x=k$ and $y=k$ for $k=1,2,\ldots,n$, so that their intersections form an $n\times n$ grid. Let the final set of parallel lines be $x+y=k$ (with $\epsilon<1$) for $k=\frac{n+2}{2}+\frac{1}{2},\frac{n+4}{2}+\frac{1}{2},\ldots,\frac{3n}{2}+\frac{1}{2}$, i.e., lines between the $n+1$ longest diagonals of the $n\times n$ grid. Each of the $\frac{3n^2}{4}$ grid points on the lower n of these $n+1$ diagonals, is the lower-left vertex of a proper triangle; each of the $\frac{3n^2}{4}$ grid points on the upper n of these $n+1$ diagonals, is the upper-right vertex of a proper triangle. In total, there are thus $\frac{3n^2}{2}$ proper triangles, as desired.

For the case $n=10$, our formula shows that the maximum number of proper triangles is 150.

Note: For odd n, the argument above gives an upper bound of $\lceil \frac{3n^2}{2} \rceil$; however, the actual maximum is $\lfloor \frac{3n^2}{2} \rfloor$. The above argument can be sharpened to produce this bound.

Problem 12 Let a, b, and c be integers such that $b \neq c$. If ax^2+bx+c and $(c-b)x^2+(c-a)x+(a+b)$ have a common root, prove that $a+b+2c$ is divisible by 3.

Solution: Let $f(x) = ax^2+bx+c$ and $g(x) = (c-b)x^2+(c-a)x+(a+b)$. Let ζ be their common root. Then r is also a root of the difference

$$f(x) - g(x) = (a+b-c)(x^2+x-1).$$

Observe that $a+b+2c \equiv a+b-c \pmod 3$. Thus, if $a+b-c=0$, then we are done. Otherwise, ζ is a root of x^2+x-1 and hence is irrational.

By the Euclidean algorithm, we can write $f(x) = q(x)(x^2+x-1)+r(x)$ for polynomials $q(x)$ and $r(x)$ with integer coefficients, where $\deg r < 2$. Setting $x = \zeta$ yields $r(\zeta) = 0$; because ζ is irrational, $r(x)$ must equal 0.

Hence, x^2+x-1 divides $f(x)$. Therefore, $a = b = -c$, so that $a+b+2c = 0$ and $3 \mid (a+b+2c)$, as desired.

Problem 13 Let ABC be a triangle with $AC \neq AB$, and select point B_1 on ray AC such that $AB = AB_1$. Let ω be the circle passing through C, B_1, and the foot of the internal bisector of angle CAB. Let ω intersect the circumcircle of triangle ABC again at Q. Prove that $\overline{AC}$ is parallel to the tangent to ω at Q.

Solution: All angles are directed modulo π. Let the angle bisector from A intersect $\overline{BC}$ at E and the circumcircle of triangle ABC at D. Because $AB_1 = AB$, $AE = AE$, and $\angle B_1AE = \angle EAB$, by SAS we have that triangles B_1AE and BAE are congruent with opposite orientations. Thus,

$$\angle EDC = \angle ADC = \angle ABC = \angle ABE = \angle EB_1A = \angle EB_1C,$$

implying that E, B_1, C, D are concyclic. Thus, $Q = D$, implying that Q, E, A are collinear.

Now, let line TQ be the line tangent to ω at Q, so that $\angle TQE = \angle QCE$. Marshalling much of our information so far — Q lies on line

EA, and E lies on line BC, quadrilateral $ABQC$ is cyclic, and line AQ bisects angle CAB — we thus have

$$\angle TQA = \angle TQE = \angle QCE = \angle QCB = \angle QAB = \angle CAQ.$$

Therefore, $\overline{TQ} \parallel \overline{AC}$, as desired.

Problem 14 We call a set of squares in a checkerboard plane *rook-connected* if it is possible to travel between any two squares in the set by moving finitely many times like a rook, without visiting any square outside of the set. (One moves "like a rook" by moving between two distinct — not necessarily adjacent — squares which lie in the same row or column.) Prove that any rook-connected set of 100 squares can be partitioned into fifty pairs of squares, such that the two squares in each pair lie in the same row or column.

Solution: Let all the squares in S be white. Then this coloring has the following property:

$(*)$ Every row contains an even number of black squares.

Given a coloring satisfying $(*)$, suppose that some column C_1 contains an odd number of white squares. By $(*)$, the total number of black squares is even. Because S contains an even number of squares, the total number of white squares is even as well, implying that another column C_2 also contains an odd number of white squares. Choose two squares $s_1, s_2 \in S$ in these columns. By the given condition, one can travel from s_1 to s_2 via some sequence $\mathtt{s}$ of rook moves in S. Without loss of generality, assume that $\mathtt{s}$ does not pass through the same square twice. Then for each horizontal rook move in $\mathtt{s}$, change the color of the origin and destination squares of that move.

Upon this recoloring, note that $(*)$ still holds, because squares are recolored in twos, where each recolored pair contains two squares in the same row. Also, in every column C except C_1, C_2, the parity of the number of white squares stays constant; the number of horizontal rook moves entering C must equal the number of horizontal rook moves leaving C, so that an even number of squares in C change color. On the other hand, a similar argument shows that after the recoloring, C_1 and C_2 contain an *even* number of white squares instead of an odd number. Thus, the recoloring preserves $(*)$ while increasing the number of columns that contain an even number of white squares.

Therefore, repeating this algorithm eventually yields a coloring satisfying $(*)$, such that every column contains an even number of white squares.

Given this coloring, the black squares can be partitioned into pairs, such that the two squares in each pair lie in the same row; the white squares can be partitioned into pairs, such that the two squares in each pair lie in the same column. This completes the proof.

Problem 15 At each of one thousand distinct points on a circle are written two positive integers. The sum of the numbers at each point P is divisible by the product of the numbers on the point which is the clockwise neighbor of P. What is the maximum possible value of the greatest of the 2000 numbers?

Solution: The maximum value is 2001. One possible configuration with maximum value 2001 is as follows: label the points with the pairs $(1001, 2)$, $(1003, 1)$, $(1004, 1)$, $\ldots$, $(2001, 1)$ in clockwise order, so that the sum of the numbers at each point *equals* the product of the numbers at the its clockwise neighbor.

To show that every values is at most 2001, we assume the opposite, for sake of contradiction. Let $m > 2001$ be the maximum value of the 2000 numbers, and let P_1 be one point at which m occurs. Let the points be $P_1, P_2, \ldots, P_{1000}$ in counterclockwise order, where indices are taken modulo 1000.

First we see that the other number at P_1 must be 1. If this were not the case, the sum of the numbers at P_2 would be at least $2m$; thus (using the maximal definition of m), the pair at P_2 would be (m, m). This would in turn make the sum of the numbers at P_3 at least m^2, which is impossible if both numbers at P_3 are at most m.

Knowing that the numbers at P_1 are m and 1, we can now consider the following cases:

Case 1: The number 1 is included in each of the 1000 pairs. In this case, we can prove by induction that P_n is labelled with $(m+1-n, 1)$ for $n = 1, 2, \ldots, 1000$. Appyling the given condition to points P_1, P_{1000}, we find that $(m - 999) \mid (m + 1)$. This means that $(m - 999) \mid 1000$, from which we find that $m \leq 1999$, a contradiction.

Case 2: The number 1 is not included in all of the 1000 pairs. Let k be the smallest positive integer such that 1 does not appear at P_k, and let $(q, 1)$ be the pair of integers at P_{k-1}. Then $q = m + 1 - (k-1) \geq m - 998$. The sum of the numbers at P_k is at least q, and the product is at least

$$2 \cdot (q-2) \geq 2 \cdot (m-1000) \geq m + m - 2000 \geq m + 2.$$

The sum of the numbers at P_{k+1} is divisible by the product of the numbers at P_k, and hence is also at least $m+2$. Then neither number at P_{k+1} equals 1, because otherwise the other number would be at least $m+1$, contradicting the maximal definition of m. It follows that the product of the numbers at P_{k+1} is at least $2((m+2)-2) = 2m$.

The sum of the numbers at P_{k+2} is divisible by the product of the numbers at P_{k+1}, and hence is also at least $2m$. Because both numbers at P_{k+2} are at most m, both numbers must in fact *equal* m. But then the sum of the two numbers at P_{k+3} is at least m^2, which is impossible because both numbers at P_{k+3} are at most m. Hence, Case 2 is impossible.

Hence, neither case is possible, and our original assumption was false; the maximum value of the 2000 numbers is at most 2001.

Problem 16 Find all primes p and q such that $p + q = (p-q)^3$.

Solution: The only such primes are $p = 5$, $q = 3$.

Because $(p-q)^3 = p + q \neq 0$, p and q are distinct and hence relatively prime.

Taking the given equation modulo $p+q$ gives $0 \equiv 8p^3$. Because p and q are relatively prime, so are p and $p+q$. Thus, $(p+q) \mid 8$.

Taking the given equation modulo $p-q$ gives $2p \equiv 0$. Because p and q are relatively prime, so are p and $p-q$. Thus, $(p-q) \mid 2$.

It easily follows that (p, q) equals $(3, 5)$ or $(5, 3)$; only the latter satisfies the given equation.

Problem 17 The monic polynomial $f(x)$ with real coefficients has exactly two distinct real roots. Suppose that $f(f(x))$ has exactly three distinct real roots. Is it possible that $f(f(f(x)))$ has exactly seven distinct real roots?

Solution: Yes, it is possible. We first seek a monic quartic that contains five points (x_1, y_1), $\ldots$, (x_5, y_5) (with $x_1 < \cdots < x_5$) with the following properties:

(a) The quartic curves down from (x_1, y_1) to a global minimum at (x_2, y_2), up to a local maximum at (x_3, y_3) with $y_3 < y_1$, down to a local non-global minimum at (x_4, y_4), and back up to (x_5, y_5).

(b) $y_1 = y_5$ and $x_5 - x_1 = y_1 - y_2$. (In other words, the graph of the quartic for $x_1 \le x \le x_5$ can be inscribed in a square, with (x_1, y_1) and (x_5, y_5) in the upper corners, (x_2, y_2) on the bottom side, and (x_3, y_3) and (x_4, y_4) in the interior.)

(c) $y_4 - y_2 < x_2 - x_1 < y_3 - y_2$. (In other words, if we change coordinates so that $(x_5, y_5) = 0$, then the line $y = x_2$ lies between the lines $y = y_4$ and $y = y_3$, so it intersects the quartic in four distinct points.)

If we can find one such quartic, then by changing coordinates we can find another such quartic $f(x)$ for which $(x_5, y_5) = 0$. Then the roots of $f(x) = 0$ are x_1 and $x_5 = 0$; the sole root of $f(x) = x_1$ is x_2; and $f(x) = x_2$ has exactly four roots. It follows easily that $f(f(x))$ has exactly four real roots and that $f(f(f(x)))$ has exactly seven real roots.

We now find a monic quartic g with the four properties described above. Let $\epsilon_1 \in (0, 1)$ be an extremely small positive values; how small, we later describe with two conditions $(*)$ and $(\dagger)$.

Step One: Find a Polynomial That Almost Works. To find g, we will alter the polynomial

$$g_1(x) = (x-1)^2(x+1)^2.$$

We claim that there exists fixed $-2 < a_1 < b_1 < -\frac{3}{2}$ such that:

(1) There is exactly one value $a_2 \neq -a_1$ for which $g_1(a_1) = g_1(a_2)$. For this a_2, we have $a_2 - a_1 < g(a_1)$.

(2) There is exactly one value $b_2 \neq -b_1$ for which $g_1(b_1) = g_1(b_2)$. For this b_2, we have $b_2 - b_1 > g(b_1)$.

For any $x \in (-2, -\frac{3}{2})$, $g_1(x)$ is greater than the value of g_1 at the local maximum $(0, 1)$, so there is exactly one value $x' \neq x$ such that $g_1(x) = g_1(x')$: namely, $x' = -x$. Because $2 - (-2) < 9 = g(-2)$, (1) holds for some a_1 close to -2; because $\frac{3}{2} - (-\frac{3}{2}) > 1 = g(-\frac{3}{2})$, (2) holds for some b_1 close to $-\frac{3}{2}$.

(To motivate the remainder of this solution, note that $g_1(x)$ nearly has all the required properties (a)-(c). If we choose $(x_2, y_2) = (-1, 0)$, $(x_3, y_3) = (0, 1)$, and $(x_4, y_4) = (1, 0)$, then (1) and (2) imply that we can choose (x_1, y_1) and (x_5, y_5) properly so that $g_1(x)$ has property (b). Unfortunately, (x_4, y_4) is a global minimum, which is not allowed in (a). We now change g_1 slightly so that this is not the case, and then argue that the modified polynomial has the required properties.)

Step Two: Find a New Polynomial That Satisfies (a) and (b). Note that $g_1'(x) = 4(x-1)x(x+1)$. For a fixed ϵ_1, define $g(x)$ such that

$$g'(x) = 4(x-1)x(x+1+\epsilon_1), \quad \min g(x) = 0$$

Let $x_2 = -1-\epsilon_1$, $x_3 = 0$, $x_4 = 1$. Observe that $g(x)$ dips down until $x = x_2$, rises up until $x = x_3$, then dips back down until $x = x_4$, and finally rises up again. Also, it is easy to confirm that $g_2'(x) + g_2'(-x) = 8x^2\epsilon_1 > 0$ for all x, so that $g(x) > g(-x)$ for $x > 0$. Hence, $g(1) > g(-1)$, implying that $g(x)$ does not have a global minimum at $x = 1$ but rather only at $(x_2, 0)$.

Before we found fixed $a_1 \approx -2, b_1 \approx -\frac{3}{2}$ such that (1) and (2) hold for $g_1(x)$; with these same a_1, b_1, for all small ϵ_1 we have:

($*$) Conditions (1) and (2) continue to hold for $g(x)$.

Hence, there exists $x_1 \in (a_1, b_1)$ such that:

There is exactly one value $x_5 \neq x_1$ such that $g(x_1) = g(x_5)$.
For this x_5, we have $x_5 - x_1 = g(x_1)$.

In other words, the graph of $g(x)$ for $x_1 \leq x \leq x_5$ can be inscribed in a square, with $(x_1, g(x_1))$ and $(x_5, g(x_5))$ in the upper corners and $(x_2, 0)$ at the bottom.

Step Three: Show That the New Polynomial Satisfies (c). So far, we have confirmed that $g(x)$ has properties (a) and (b); we now show that it has property (c).

Because $g_1(0) = 1$ and $g_1(1) = 0$, for all small ϵ_1 we have $g(0) \approx 1$ and $g(1) \approx 0$. That is (recalling that $-2 > a_1 > b_1 > -\frac{3}{2}$), for all small ϵ_1 we have

($\dagger$) $g(0) > (-1-a_1)-\epsilon_1 = x_2 - a_1$, and $g(1) < (-1-b_1)-\epsilon_1 = x_2 - b_1$.

Hence, $g(1) < x_2 - b_1 < x_2 - x_1$ and $x_2 - x_1 < x_2 - a_1 < g(0)$, implying that g has property (c). This completes the proof.

Problem 18 Let $\overline{AD}$ be the internal angle bisector of A in triangle BAC, with D on $\overline{BC}$. Let M and N be points on the circumcircles of triangles ADB and ADC, respectively, so that $\overline{MN}$ is tangent to these two circles. Prove that line MN is tangent to the circle passing through the midpoints of $\overline{BD}$, $\overline{CD}$, and $\overline{MN}$.

Solution: All angles are directed modulo π.

Let lines BM and CN meet at X. Let E, F, G be the midpoints of $\overline{BD}$, $\overline{CD}$, $\overline{MN}$, respectively.

Observe that $\angle XMD = \pi - \angle DMB = \pi - \angle DAB = \pi - \angle CAD = \pi - \angle CND = \angle DNX$, and $\angle MDN = \pi - \angle DNM - \angle NMD = \pi - \angle DCN - \angle MBD = \pi - \angle BCX - \angle XBC = \angle CXB = \angle NXM$. Hence, quadrilateral $MDNX$ is a parallelogram, so that $\overline{MB} \parallel \overline{ND}$ and $\overline{NC} \parallel \overline{MD}$.

Because E and G are the midpoints of $\overline{MN}$ and $\overline{BD}$, and $\overline{MB} \parallel \overline{ND}$, we have $\overline{EG} \parallel \overline{MB} \parallel \overline{ND}$. Similarly, $\overline{FG} \parallel \overline{NC} \parallel \overline{MD}$. Thus,

$$\angle MGE = \angle MND = \angle NCD = \angle GFE.$$

Therefore, $\overline{MG}$ is tangent to the circumcircle of triangle EFG, as desired.

Problem 19 In tetrahedron $A_1A_2A_3A_4$, let ℓ_k be the line connecting A_k with the incenter of the opposite face. If ℓ_1 and ℓ_2 intersect, prove that ℓ_3 and ℓ_4 intersect.

Solution: We denote by I_k the incenter of the face opposite the vertex A_k. Since ℓ_1 and ℓ_2 intersect, we know that A_1, A_2, I_1, and I_2 lie on a single plane $\mathcal{P}$. Because line A_3A_4 is obviously not on $\mathcal{P}$, it intersects $\mathcal{P}$ in at most one point.

Line A_1I_2 is the internal angle bisector of angle $\overline{A_3IA_4}$, so it intersects line A_3A_4; because line A_1I_2 lies in $\mathcal{P}$, so does the intersection point Q_1. Similarly, line A_2I_1 intersects line A_3A_4 at a point $Q_2 \in \mathcal{P}$. Because line A_3A_4 and $\mathcal{P}$ intersect in at most one point, we have $Q_1 = Q_2$; that is, lines A_1I_2, A_2I_1, and A_3A_4 concur.

Applying the angle bisector theorem to triangles $A_1A_3A_4$ and

$A_2A_3A_4$, we see that

$$\frac{A_1A_3}{A_1A_4} = \frac{Q_1A_3}{Q_1A_4} = \frac{Q_2A_3}{Q_2A_4} = \frac{A_2A_3}{A_2A_4},$$

or

$$\frac{A_1A_3}{A_2A_3} = \frac{A_1A_4}{A_2A_4}.$$

Lines A_3I_4 and A_4I_3 intersect $\overline{A_1A_2}$ at some points Q_3 and Q_4, respectively. Then the above equation and the angle bisector theorem gives $\frac{A_1Q_3}{A_2Q_3} = \frac{A_1Q_4}{A_2Q_4}$, so that $Q_3 = Q_4$; that is, lines A_3I_4, A_4I_3, and A_1A_2 concur at some point Q. Then I_3 and I_4 lie on sides $\overline{QA_4}$ and $\overline{QA_3}$, respectively, of triangle QA_3A_4, implying that lines A_3I_3 and A_4I_4 — cevians of this triangle — intersect.

Problem 20 An infinite set S of points on the plane has the property that no 1×1 square of the plane contains infinitely many points from S. Prove that there exist two points A and B from S such that $\min\{XA, XB\} \geq 0.999AB$ for any other point X in S.

Solution: Let P_1 be any point in S. Given P_k, let P_{k+1} be a point in S with minimal distance from P_k. Such a point must exist, because otherwise some circle around P_k contains infinitely many points in S — but this circle can be covered with 1×1 squares each containing finitely many points in S, a contradiction.

We claim that $P_{k+1}P_{k+2} \geq 0.999P_kP_{k+1}$ for some k. Suppose not; then the P_k are all distinct, and

$$P_1P_{k+1} \leq \sum_{k=1}^{\infty} P_kP_{k+1} < \sum_{k=0}^{\infty} 0.999^k P_1P_2 = \frac{1}{1-0.999}P_1P_2 = 1000P_1P_2.$$

Hence, the circle of radius $1000P_1P_2$ centered at P_1 contains infinitely many points in S, a contradiction.

Hence, $P_{k+1}P_{k+2} \geq 0.999P_kP_{k+1}$ for some k, and we can set $(A, B) = (P_k, P_{k+1})$.

Problem 21 Prove that from any set of 117 pairwise distinct three-digit numbers, it is possible to select 4 pairwise disjoint subsets such that the sums of the numbers in each subset are equal.

Solution: We examine subsets of exactly two numbers. Clearly, if two distinct subsets have the same sum, they must be disjoint. The number of two-element subsets is $\binom{117}{2} = 6786$. Furthermore, the lowest attainable sum is $100 + 101 = 201$, while the highest sum is $998 + 999 = 1997$, for a maximum of 1797 different sums. By the Pigeonhole Principle and the fact that $1797 \cdot 3 + 1 = 5392 < 6786$, we see that there are 4 two-element subsets with the required property.

Problem 22 The numbers from 1 to 999999 are divided into two groups. For each such number n, if the square closest to n is odd, then n is placed in the first group; otherwise, n is placed in the second group. The sum of the numbers in each group is computed. Which group yields the larger sum?

Solution: Both groups yield equal sums. First we will prove the following result.

Lemma. *If the numbers in $[k^2, (k+1)^2 - 1]$, k a positive integer, are divided according to the rules, then the sums of the numbers in the two groups are the same.*

Proof. The numbers will be divided depending on whether they are closer to k^2 or $(k+1)^2$. We call the former group A and the latter group B. The largest number in A is then $\left\lfloor \frac{k^2+(k+1)^2}{2} \right\rfloor = k^2 + k$, while the smallest number in B is $k^2 + k + 1$.

Thus, A has $k+1$ elements with average value $\frac{1}{2}(k^2 + (k^2 + k)) = \frac{1}{2}k(2k+1)$, so the sum of the elements in A is

$$\frac{1}{2}k(k+1)(2k+1).$$

B has k elements with average value $\frac{1}{2}((k^2 + k + 1) + (k^2 + 2k)) = \frac{1}{2}(2k+1)(k+1)$, so the sum of the elements in B is

$$\frac{1}{2}k(2k+1)(k+1).$$

We see that the two sums are equal, proving the lemma. □

By applying the lemma for $k = 1, 2, 3, \ldots, 999$, we see that the two given total sums are equal, as claimed.

Problem 23 Two polynomials $P(x) = x^4 + ax^3 + bx^2 + cx + d$ and $Q(x) = x^2 + px + q$ take negative values on some common real interval I of length greater than 2, and outside of I they take on nonnegative values. Prove that $P(x_0) < Q(x_0)$ for some real number x_0.

Solution: We first shift the polynomials so that the smaller root is at the origin and the other root is at r, for $r > 2$. Thus $Q(x) = x(x-r)$ and $P(x) = Q(x)R(x)$ for some monic quadratic $R(x)$. Note that

$$R(x) = P(x)/Q(x) > 0 \qquad (*)$$

for all $x \neq 0, r$.

We claim that $R(0) \neq 1$ or $R(r) \neq 1$. If instead $R(0) = R(r) = 1$, then $R(x) = x(x-r)+1 = x^2-rx+1$. But then $R(r/2) = -\frac{r^2}{4}+1 < 0$, contradicting $(*)$.

If $R(0) \neq 1$, then for small $\epsilon > 0$ either

(i) $R(-\epsilon) < 1$, or

(ii) $R(\epsilon) > 1$.

Also, for $\epsilon < r$ note that $Q(-\epsilon) > 0$ and $Q(\epsilon) < 0$ for small $\epsilon > 0$. Hence, either

(i) $P(-\epsilon) = Q(-\epsilon)R(-\epsilon) < Q(-\epsilon)$, or

(ii) $P(-\epsilon) = Q(-\epsilon)R(-\epsilon) < Q(-\epsilon)$.

Similarly, if $R(r) \neq 1$, then for small $\epsilon > 0$ either $P(r-\epsilon) < Q(r-\epsilon)$ or $P(r+\epsilon) < Q(r+\epsilon)$.

Problem 24 The point K is selected inside parallelogram $ABCD$ such that the midpoint of $\overline{AD}$ is equidistant from K and C and such that the midpoint of $\overline{CD}$ is equidistant from K and A. Let N be the midpoint of $\overline{BK}$. Prove that $\angle NAK = \angle NCK$.

Solution: We denote the midpoint of $\overline{CK}$ by P and the midpoint of $\overline{AD}$ by Q. Because $QC = QK$, $\overline{PQ} \perp \overline{CK}$.

Because $\overline{NP}$ is the midline of triangle KBC, $NP = \frac{1}{2} \cdot BC = AQ$ and $\overline{NP} \parallel \overline{BC} \parallel \overline{AQ}$. Thus quadrilateral $ANPQ$ is a parallelogram. Because $\overline{PQ} \perp \overline{CK}$, $\overline{AN} \perp \overline{CK}$ as well. Similarly, $\overline{CN} \perp \overline{AK}$. If we denote by R the intersection of line AN with line CK and by S the intersection of line CN with line AK, we see that $\triangle ANS \sim \triangle CNR$. It follows that $\angle NAK = \angle NCK$.

Problem 25 We are given a 2000-sided polygon in which no three diagonals are concurrent. Each diagonal is colored in one of 999 colors. Prove that there exists a triangle whose sides lie entirely on diagonals of one color. (The triangle's vertices need not be vertices of the 2000-sided polygon.)

Solution: We disregard one of the 2000 vertices and focus on the 1999-gon formed among the other vertices. Consider the 1999 diagonals of the new polygon that split the 1999-gon into two halves with 999 and 1000 sides each. By the Pigeonhole principle, we know that at least three of these diagonals must be the same color. Because every two of these diagonals intersect, the three diagonals must form a triangle with sides of the same color.

Problem 26 Jury lays 2001 coins, each worth 1, 2, or 3 kopecks, in a row. Between any two k-kopeck coins lie at least k coins for $k = 1, 2, 3$. For which n is it possible that Jury lays down exactly n 3-kopeck coins?

Solution: It is possible for Jury to lay down n 3-kopeck coins for precisely $n = 500$ and 501. We consider the row of coins as a sequence of 2001 numbers each of which is 1, 2, or 3.

Suppose that some 3 in the sequence has at least four neighbors a_1, a_2, a_3, a_4 to its right (resp. left). If $a_1 = 2$, then $a_2 = 1$ and a_3 cannot equal 3, 2, or 1, a contradiction. Hence, $a_1 \neq 2$, and certainly $a_1 \neq 3$. Thus, $a_1 = 1$, and applying the given condition we find that $(a_1, a_2, a_3, a_4) = (1, 2, 1, 3)$.

Therefore, the 3-kopeck coins occupy every fourth slot in the row, starting with either the first, second, third, or fourth slot. In these cases, there are $\lfloor 2004/4 \rfloor$, $\lfloor 2003/4 \rfloor$, $\lfloor 2002/4 \rfloor$, or $\lfloor 2001/4 \rfloor$ 3-kopeck coins, respectively; that is, there are either 500 or 501 3-kopeck coins. These values of n are attainable, by laying out 500 sets of coins with values 3, 1, 2, 1 from left to right, and then either (i) placing a final 1-kopeck coin to the left, or (ii) placing a final 3-kopeck coin to the right.

Problem 27 A company of $2n + 1$ people has the property that for each group of n people, there is a person among the other $n + 1$ who knows everybody in that group. Prove that some person in the company knows everybody else. (If a person A knows a person B, then B knows A as well.)

Solution: We begin with the following lemma.

Lemma. *Given a graph where every vertex has positive degree, one may choose a set S of at most half of the vertices, such that every* *vertex not in S is adjacent to at least one vertex in S.*

Proof. We prove the claim by strong induction on m, the number of vertices. For $m = 2, 3$, the claim is obvious. Assume that the claim holds for $m < k$. Then given a graph with k vertices satisfying the given conditions, choose any vertex v and remove all vertices adjacent to it, leaving a graph with at most $k-2$ vertices. The remaining graph satisfies the condition in the induction hypothesis, so we may apply the induction hypothesis to choose a set S' with $|S'| \leq \frac{k-2}{2}$. Thus, $S = S \cup \{v\}$ contains at most $k/2$ vertices, and any vertex not in S is either adjacent to v or adjacent to some vertex in S'. This completes the inductive step, and the proof. $\square$

Assume, for the sake of contradiction, that no person knows everybody else. Construct a graph whose vertices are the $2n + 1$ people, where two people are connected by an edge if and only if they do not know each other. Applying the lemma, it follows that one may choose n of the $2n+1$ vertices such that each of the remaining $n+1$ vertices are adjacent to one of the n chosen vertices. In other words, among the group of the corresponding n people, nobody among the other $n + 1$ people knows all the people in that group — contradicting the conditions given in the problem statement. Therefore, our original assumption was false, and some person knows everybody else.

Problem 28 Side $\overline{AC}$ is the longest of the three sides in triangle ABC. Let N be a point on $\overline{AC}$. Let the perpendicular bisector of $\overline{AN}$ intersect line AB at K, and let the perpendicular bisector of $\overline{CN}$ intersect line BC at M. Prove that the circumcenter of triangle ABC lies on the circumcircle of triangle KBM.

Solution: We denote the circumcenter of triangle ABC by O, the projections of O onto lines AB, BC by C_1, A_1, and the projection of B onto line AC by P. If $N = P$, then $\angle OKB = \angle OMM = \pi/2$, so that O, K, B, M are concyclic. Assume now that $N \neq P$.

Because $\angle C_1OB = \angle C$ and $\angle A_1OB = \angle A$, we have

$$\frac{OC_1}{OA_1} = \frac{OB \cdot \cos \angle C}{OB \cdot \cos \angle A} = \frac{\cos \angle C}{\cos \angle A}.$$

Next, observe that — using signed distances — the signed length of the projection of $\overline{C_1K}$ onto line AC equals $\frac{1}{2}NA$ (the distance from the midpoint of $\overline{NA}$ to A), minus $\frac{1}{2}PA$ (the distance from the projection of K to A), or $\frac{1}{2}NA - \frac{1}{2}PA = \frac{1}{2}NP$. Similarly, the length of the projection of $\overline{A_1M}$ equals $\frac{1}{2}NP$. Thus,

$$C_1K = \frac{NP}{2\cos\angle A}$$

and

$$A_1M = \frac{NP}{2\cos\angle C},$$

or

$$\frac{C_1K}{A_1M} = \frac{\cos\angle C}{\cos\angle A} = \frac{OC_1}{OA_1}.$$

Because we also have $\angle OC_1K = \pi/2 = \angle OA_1M$, this means that $\triangle OC_1K \sim \triangle OA_1M$. Hence, $\angle OKC_1 = \angle OMA_1$.

Because $\overline{AC}$ is the longest side of triangle ABC, O lies in between rays BA and BC. This, combined with the fact that the projections of $\overline{KC_1}$ and $\overline{MA_1}$ onto line AC have the same signed length (so that if $\overline{AC}$ is horizontal, then K is to the left of C_1 if and only if M is to the left of A_1), implies that $\angle OKC_1$ and $\angle OMA_1$ are equal as *directed* angles. Then using directed angles modulo π, we have

$$\angle OKB = \angle OKC_1 = \angle OMA_1 = \angle OMB,$$

so that O, K, M, B are indeed concyclic.

Problem 29 Find all odd positive integers n greater than 1 such that for any relatively prime divisors a and b of n, the number $a+b-1$ is also a divisor of n.

Solution: We will call a number *good* if it satisfies the conditions given. It is not difficult to see that all prime powers are good. Suppose n is a good number that has at least two distinct prime factors. Let $n = p^r s$, where p is the smallest prime dividing n and s is not divisible by p. Because n is good, $p + s - 1$ must divide n. For any prime q dividing s, $s < p + s - 1 < s + q$, so q does not divide $p + s - 1$. Therefore, the only prime factor of $p+s-1$ is p. Then $s = p^c - p + 1$ for some $c > 1$. Because p^c must also divide n, $p^c + s - 1 = 2p^c - p$ divides n. Because $2p^{c-1} - 1$ has no factors of p, it must divide s. But $\frac{p-1}{2}2p^{c-1} - 1 = p^c - p^{c-1} - \frac{p-1}{2} < p^c - p + 1 < \frac{p+1}{2}2p^{c-1} - 1 = p^c + p^{c-1} - \frac{p+1}{2}$. This is a contradiction, so the only good numbers are prime powers.

Problem 30 Each of the subsets $A_1, A_2, \ldots, A_{100}$ of a line is the union of 100 pairwise disjoint closed intervals. Prove that the intersection of these 100 sets is the union of no more than 9901 disjoint closed intervals. (A closed interval is a single point or a segment.)

Solution: We prove by induction that for $m \geq 1$, $A_1 \cap \cdots \cap A_m$ is the union of no more than $99m+1$ disjoint closed intervals. The base case $m=1$ is true by the given condition. Now assuming that our assertion is true for $m=k$, we prove it is true for $m=k+1$. We begin by using the induction hypothesis to write

$$A_1 \cap \cdots \cap A_k = \bigcup_{i=1}^{n} S_i,$$

where $n \leq 99k+1$ and the S_i are disjoint closed intervals.

Consider what happens to each S_i when we take the intersection of A_{k+1} with the set of intervals we have so far. Some S_i's may disappear, some may be reduced to smaller intervals, and some may be split into more than one interval. Of these possibilities, the only one that increases the number of intervals in our intersection is the last one. Suppose that $A_{k+1} \cap S_i$ consists of $t+1$ intervals, thus increasing the number of intervals by t. Then the t "gaps" between these intervals must correspond to t "gaps" between intervals of A_{k+1}. Now, the total number of gaps between intervals in A_{k+1} is at most 99, because the total number of intervals in A_{k+1} is 100. Thus, summing over all i, the total number of intervals in $A_{k+1} \cap (\bigcup_{i=1}^{n} S_i)$ is at most 99 more than the number of intervals in $\bigcup_{i=1}^{n} S_i$, completing the induction.

Problem 31 Two circles are internally tangent at a point N, and a point K different from N is chosen on the smaller circle. A line tangent to the smaller circle at K intersects the larger circle at A and B. Let M be the midpoint of the arc AB of the larger circle not containing N. Prove that the circumradius of triangle BMK is constant as K varies along the smaller circle (and regardless of which arc MN point B lies on.

Solution: Let the small circles and large circles be ω_1 and ω_2, with radii r_1 and r_2, respectively. Notice that the homothety about N of ratio R/r takes ω_1 to ω_2. Letting O be the radius of ω_2, line OM is perpendicular to line AB (because $\overline{OM}$ is the perpendicular bisector of $\overline{AB}$) and also to the tangent to ω_2 at M. Hence, line AB is parallel to the tangent at M. Thus, the homothety of ratio R/r about N not only takes ω_1 to ω_2, it takes K to M as well.

So, N, K, and M are collinear. This gives us $\angle BMN = \angle BMK$. Because M is the midpoint of the arc AB, we also have $\angle BNM = \angle MNA = \angle MBA = \angle MBK$. Therefore, $\triangle BMN \sim \triangle KMB$. Then

$$\frac{BM}{KM} = \frac{MN}{MB},$$

or

$$\frac{MB}{MN} = \sqrt{\frac{MK}{MN}}.$$

Our homothety sends $\overline{NK}$ to $\overline{NM}$, so $\frac{KN}{MN} = \frac{r}{R}$ and $\frac{MK}{MN} = 1 - \frac{r}{R}$. Thus, the ratio of triangle KMB to triangle BMN is $\frac{MB}{MN} = \sqrt{\frac{MK}{MN}} = \sqrt{1 - \frac{r}{R}}$. Because the circumradius of triangle BMN is R, the circumradius of triangle KMB is $R\sqrt{1 - \frac{r}{R}}$, a constant.

Problem 32 In a country, two-way roads connect some cities in pairs such that given two cities A and B, there exists a unique path from A to B which does not pass through the same city twice. It is known that exactly 100 cities in the country have exactly one outgoing road. Prove that it is possible to construct 50 new two-way roads so that if any single road were closed, it would still be possible to travel from any city to any other.

Solution: We prove the claim in the more general situation with 100 replaced by $2n$ and 50 replaced by n, where n is a positive integer.

Consider the network of roads as a graph $\mathcal{G}$ with vertices representing cities and edges representing roads. Then the condition that a unique non-self-intersecting path exists between each pair of cities A and B means that the graph is connected and contains no cycles; that is, $\mathcal{G}$ is a tree. Call a vertex of $\mathcal{G}$ a *leaf* if it has degree 1 (so that $\mathcal{G}$ has exactly $2n$ leaves) and a *branch point* if it has degree at least 3. If $n > 1$, from any given leaf L, we can follow edges of $\mathcal{G}$ until we reach a branch point B — otherwise, we would ultimately reach another leaf and be unable to reach the other $2n - 2$ leaves of $\mathcal{G}$, contradicting the connectedness of $\mathcal{G}$. Call the set of vertices and edges on the path from L to B, *excluding* B, the *twig* corresponding to L. Finally, call a graph that remains connected if any single edge is removed *2-connected.*

In this terminology, we are to show that given a tree $\mathcal{G}$ with $2n$ leaves, we can add n edges to make it 2-connected.

We induct on n. In the case $n = 1$, adding an edge between the two leaves of $\mathcal{G}$ suffices: this causes $\mathcal{G}$ to become a single cycle, which is clearly 2-connected. Otherwise, assume the claim is true for graphs with $2n - 2$ leaves, $n \geq 2$; we prove that it is true for graphs with $2n$ leaves.

Let L_1 be any leaf of $\mathcal{G}$. Because $n \geq 2$, there exists a twig T_1 corresponding to L_1. Let B_1 be the branch point at the other end of T_1. Now, let $\mathcal{G}' = G \setminus T_1$, the subgraph of $\mathcal{G}$ with T_1 removed. Note that $\mathcal{G}'$ now has $2n - 1$ leaves, and that B_1 is not necessarily still a branch point, but certainly does not become a leaf. We claim that we can find another leaf L_2 such that its twig T_2 does not contain B_1.

In the case that B_1 is the only branch point of $\mathcal{G}'$, we can choose L_2 to be any of the remaining leaves: each leaf will have its twig end at B_1, and since a twig by definition does not include its branch point, any leaf suffices.

Otherwise, $\mathcal{G}'$ has some other branch point B. In this case B has 3 or more neighbors, one of which is along the unique path from B to B_1. We choose any neighbor other than that one, and follow edges of $\mathcal{G}'$ until we reach a leaf. We claim that we can take this leaf to be L_2. Indeed, its twig is a subset of path from L_2 to B, and hence cannot include B_1.

Thus, we now have a second leaf L_2 of $\mathcal{G}'$ such that its twig T_2 does not contain B_1. Therefore, T_2 is also the twig of L_2 in our original graph $\mathcal{G}$; we will use this fact later. Cut off T_2 from $\mathcal{G}'$, leaving the subgraph $\mathcal{G}'' = G' \setminus T_2$. Now $\mathcal{G}''$ is a tree with $2n-2$ leaves, so we can apply the induction hypothesis to it to obtain a 2-connected graph $\mathcal{G}_2''$ at the cost of $n - 1$ added edges.

Going back to $\mathcal{G}$, we add the $n - 1$ edges that the induction hypothesis prescribed for $\mathcal{G}''$ plus one additional edge between L_1 and L_2. Call this new graph $\mathcal{G}_2$; we claim that $\mathcal{G}_2$ is 2-connected. Consider the path between the branch points B_1 and B_2 (possibly the same) associated with L_1 and L_2 in the original graph $\mathcal{G}$. This path cannot intersect either T_1 or T_2, because neither T_1 nor T_2 contains a branch point. Hence, this path, along with T_1, T_2, and the added edge between L_1 and L_2, forms a cycle $\mathcal{C}$. Furthermore, $\mathcal{G}_2$ is the union of $\mathcal{C}$ and $\mathcal{G}_2''$. Now, because $\mathcal{C}$ and $\mathcal{G}_2''$ share at least one vertex and each is 2-connected, it follows that $\mathcal{G}_2$ is also 2-connected.

Problem 33 The polynomial $P(x) = x^3 + ax^2 + bx + c$ has three distinct real roots. The polynomial $P(Q(x))$, where $Q(x) = x^2 + x + 2001$, has no real roots. Prove that $P(2001) > \frac{1}{64}$.

Solution: We denote the three roots of $P(x)$ by r_1, r_2, and r_3. Thus we may write $P(x) = (x - r_1)(x - r_2)(x - r_3)$, or $P(2001) = (2001 - r_1)(2001 - r_2)(2001 - r_3)$. We consider the case where one or more of the r_k's, say r_1, is greater than or equal to $(2001 - \frac{1}{4})$. Because the minimum of $Q(x)$ is at the point $(-\frac{1}{2}, 2001 - \frac{1}{4})$, we will have $Q(x) = r_1$ for some x_0. Then $P(Q(x_0)) = 0$, a contradiction. Thus we see that $r_k < 2001 - \frac{1}{4}$ for $k = 1, 2, 3$, and $P(2001) > \left(\frac{1}{4}\right)^3 = \frac{1}{64}$, as wanted.

Problem 34 Each number $1, 2, \ldots, n^2$ is written once in an $n \times n$ grid such that each square contains one number. Given any two squares in the grid, a vector is drawn from the center of the square containing the larger number to the center of the other square. If the sums of the numbers in each row or column of the grid are equal, prove that the sum of the drawn vectors is zero.

Solution: Take coordinates so that the bottom-left square is $(1, 1)$ and the top right square is (n, n). Let $S = \{(a, b) \mid 1 \leq a, b \leq n\}$ be our set of squares, and for $\mathbf{x} \in S$ let $w(\mathbf{x})$ denote the number written in square $\mathbf{x}$. The sum P in question satisfies

$$P = \sum_{\substack{\mathbf{x},\mathbf{y} \in S \\ w(\mathbf{x}) < w(\mathbf{y})}} (\mathbf{y} - \mathbf{x}).$$

Given $\mathbf{y} \in S$, we have $w(\mathbf{y}) - 1$ choices of $\mathbf{x}$ for which $w(\mathbf{x}) < w(\mathbf{y})$, and $n^2 - w(\mathbf{y})$ choices of $\mathbf{x}$ for which $w(\mathbf{x}) > w(\mathbf{y})$. Thus, $\mathbf{y}$ appears in the sum with a coefficient of $(w(\mathbf{y}) - 1) - (n^2 - w(\mathbf{y})) = 2w(\mathbf{y}) - (n^2 + 1)$. Therefore,

$$P = \sum_{\mathbf{y} \in S} \left(2w(\mathbf{y}) - (n^2 + 1)\right) \mathbf{y}.$$

We claim that the x-coordinate of P equals zero. Let T_k consist of the squares in the k-th column. Then the x-coordinate of

$$\sum_{\mathbf{y} \in T_k} \left(2w(\mathbf{y}) - (n^2+1)\right) \mathbf{y} \tag{1}$$

equals

$$k \sum_{\mathbf{y} \in T_k} \left(2w(\mathbf{y}) - (n^2+1)\right) = k \left(2 \sum_{\mathbf{y} \in T_k} w(\mathbf{y}) - n(n^2+1) \right). \tag{2}$$

Now, the sum σ of all the numbers in the grid equals $\frac{n^2(n^2+1)}{2}$. The sum of the numbers in each column is the same, implying that the numbers in each column have sum $\frac{\sigma}{n} = \frac{n(n^2+1)}{2}$. Specifically, $\sum_{\mathbf{y} \in T_k} w(\mathbf{y}) = \frac{n(n^2+1)}{2}$. Thus, the expression in the right hand side of (2) is zero, i.e., the x-coordinate of (1) is zero.

Summing (1) for $k = 1, 2, \ldots, n$, we find that P has x-coordinate 0. Similarly, P has y-coordinate 0. This completes the proof.

Problem 35 Distinct points A_1, B_1, C_1 are selected inside triangle ABC on the altitudes from A, B, and C, respectively. If $[ABC_1] + [BCA_1] + [CAB_1] = [ABC]$, prove that the circumcircle of triangle $A_1B_1C_1$ passes through the orthocenter H of triangle ABC.

Solution: Angles are directed modulo π, lengths are directed (with $HA, HB, HC > 0$), and triangle areas a re directed.

We begin by proving a strengthened version of the converse. Assume that A_1, B_1, C_1 are distinct points on lines AH, BH, CH (not necessarily lying inside triangle ABC). Assume that (i) points A_1, B_1, C_1, H are distinct and concyclic, or that (ii) triangle $A_1B_1C_1$ is tangent to one of the lines AA_1, BB_1, CC_1 at H. In case (i), because quadrilateral $A_1B_1C_1H$ are concyclic, two of $\overline{HA_1}, \overline{HB_1}, \overline{HC_1}$ lie on the boundary of the quadrilateral, and the two corresponding lengths from HA_1, HB_1, HC_1 will have the same sign. Then by Ptolemy's Theorem,

$$|B_1C_1| \cdot HA_1 + |A_1C_1| \cdot HB_1 + |A_1B_1| \cdot HC_1 = 0. \tag{$*$}$$

This equation also holds in case (ii): if, for instance, $C_1 = H$, then HA_1, HB_1 must have opposite same sign, and the left hand side of $(*)$ is $|B_1H| \cdot HA_1 + |A_1H| \cdot HB_1 = 0$.

Let E and F be the feet of the altitudes from B and C, respectively. Using directed angles modulo π, note that $\angle CAB = \pi - \angle EHF =$

$\angle FHB = \angle C_1HB_1 = \angle C_1A_1B_1$. Similarly, we find that $\angle CBA = \angle C_1B_1A_1$, implying that $\triangle ABC \sim \triangle A_1B_1C_1$. Together with $(*)$, this gives

$$|BC| \cdot HA_1 + |AC| \cdot HB_1 + |AB| \cdot HC_1 = 0. \tag{$\dagger$}$$

Furthermore, we have

$$\begin{aligned} |BC| \cdot HA_1 &= 2 \cdot ([CA_1B] - [CHB]), \\ |AC| \cdot HB_1 &= 2 \cdot ([AB_1C] - [AHC]), \\ |AB| \cdot HC_1 &= 2 \cdot ([BC_1A] - [BHA]). \end{aligned}$$

Substituting into $(\dagger)$ above yields

$$[ABC_1] + [BCA_1] + [CAB_1] = [CHB] + [AHC] + [BHA] = [ABC],$$

as desired.

Now suppose we do not know that A_1, B_1, C_1, H are concyclic. We draw the circumcircle of triangle A_1B_1H and denote its second intersection with line CH by C'. (If the circle is tangent to line CH, then $C' = H$.) By the above result, we know that $[ABC'] + [BCA_1] + [CAB_1] = [ABC]$. Only one point C_1 on line CH that satisfies the area equation, so we see that $C' = C_1$. Thus A_1, B_1, C_1, and H must be concyclic.

Problem 36 We are given a set of 100 stones with total weight $2S$. Call an integer k *average* if it is possible to select k of the 100 stones whose total weight equals S. What is the maximum possible number of integers which are average?

Solution: Observe that $k = 0$ and 100 cannot be average; and if the values $k = 1$ or $k = 99$ are average then no other values can be average. Thus, at most 97 integers are average.

Indeed, 97 can be attained: We claim that one set that yields 97 average integers contains four stones of weight 1 and two stones of weight 2^i for $i = 1, 2, \ldots, 48$. In this case, $2S = 2(2^{49} - 1) + 2 = 2^{50}$.

For $k = 2, 3, \ldots, 98$, we can find a set T_k of k stones with total weight $S = 2^{49}$ as follows. For $k = 2$, we let T_2 contain the two stones of weight 2^{48}. For $3 < k \leq 50$, we let T_k contain one stone of each weight $2^{48}, 2^{47}, \cdots, 2^{51-k}$, and two stones of weight 2^{50-k}, for a total weight of

$$2^{50-k} + \sum_{i=50-k}^{48} 2^i = 2^{50-k} + \sum_{i=50-k}^{48} (2^{i+1} - 2^i)$$
$$= 2^{50-k} + (2^{49} - 2^{50-k}) = 2^{49}.$$

(In other words, given $k-1$ stones of weights $2^{48}, 2^{47}, \ldots, 2^{51-k}, 2^{51-k}$, we replace one stone of weight 2^{51-k} with two stones of weight 2^{50-k}.) Finally, for $51 \le k \le 98$, we simply take all the stones not in T_{100-k}.

Problem 37 Two finite sets S_1 and S_2 of convex polygons in the plane are given with the following properties: (i) given any polygon from S_1 and any polygon from S_2, the two polygons have a common point; (ii) each of the two sets contains a pair of disjoint polygons. Prove that there exists a line which intersects all the polygons in both sets.

Solution: In this solution, we treat the polygons as two-dimensional regions rather than the one-dimensional boundaries of such regions. (The problem statement is the same regardless of which definition of "polygon" we use.)

We begin by proving the following lemma:

Lemma 1. *Given two closed, convex polygons that are disjoint, some line ℓ separates the polygons into two open half-planes. (That is, each open half-plane bounded by ℓ contains one of the polygons.)*

Proof. Choose point A in one polygon $\mathcal{P}_1$ and point B in the second polygon $\mathcal{P}_2$ so that the distance between them is minimized, and let ℓ be the perpendicular bisector of $\overline{AB}$. Without loss of generality, ℓ is vertical with A on its left. Suppose, for sake of contradiction, that some point C in the first polygon is to the right of A. Segment $\overline{AC}$ passes through the interior of the circle centered at B with radius AB, so some point $D \in \overline{AC}$ lies closer to B than A does. However, by convexity, D lies inside $\mathcal{P}_1$, contradicting the minimal choice of A and B.

Thus, $\mathcal{P}_1$ lies entirely to the left of ℓ. Likewise, $\mathcal{P}_2$ lies entirely to the right of ℓ. $\square$

Call the polygons in S_1 *red*, and call the polygons in S_2 *blue.* Define a *red skewer* (resp. *blue skewer*) to be any line that intersects each polygon in S_1 (resp. S_2).

Lemma 2. *If a line ℓ separates two red (resp. blue) polygons into two closed half-planes, then it is a blue (resp. red) skewer.*

Proof. We prove the claim when ℓ separates two red polygons. Let the two red polygons be $\mathcal{P}_1$ and $\mathcal{P}_2$, and take any blue polygon $\mathcal{Q}$. By assumption, $\mathcal{Q}$ intersects polygons $\mathcal{P}_1, \mathcal{P}_2$ at some points A_1, A_2, respectively. By convexity, segment $\overline{A_1A_2}$ lies within $\mathcal{Q}$. Because ℓ separates the $\mathcal{P}_i$, it must intersect $\overline{A_1A_2}$; i.e., ℓ must intersect $\mathcal{Q}$. Because this holds for every blue polygon, ℓ must be a blue skewer. $\square$

We are given that there are two disjoint red polygons. By Lemma 1, some line ℓ separates them into two open half-planes. Set up coordinate axes where ℓ is given by the line $x = 0$. Consider the set $S \subseteq \mathbb{R}^2$ of order pairs (m, b) such that $y = mx + b$ is a red skewer. We claim that it is nonempty, closed, and bounded.

We first prove that S is nonempty. We are given that there are two disjoint blue polygons. By Lemma 1, some line separates them into two open half-planes; by Lemma 2, this line is a red skewer. Hence, there exists a red skewer, and S is nonempty.

Next, we prove that S is bounded. Of all the points (x, y) in red polygons, we have $b_1 \le y \le b_2$ for some b_1, b_2; it is easy to see that $b_1 \le b \le b_2$ if $(m, b) \in S$. Hence, b is bounded. As for m, we have $m = \frac{y_1 - y_2}{x_1 - x_2}$ for some (x_1, y_1) in a red polygon strictly to the left of ℓ and some (x_2, y_2) in a red polygon strictly to the right of ℓ. Over all such pairs of points, $|x_1 - x_2|$ attains some positive minimum value and $|y_1 - y_2|$ attains some maximum value. Hence, m is bounded as well.

Finally, we prove that S is closed. For any red polygon $\mathcal{P}$, consider the set $S_{\mathcal{P}} \subseteq \mathbb{R}^2$ of ordered pairs (m, b) such that $y = mx + b$ passes through $\mathcal{P}$. For each fixed b, the set of m such that $(m, b) \in S_{\mathcal{P}}$ is a closed interval $[m_1, m_2]$ with $m_1 < m_2$; also, m_1 and m_2 are continuous functions of b. Thus, $S_{\mathcal{P}}$ is a closed (infinite) region bounded by the continuous curves $m = m_1(b)$ and $m = m_2(b)$. S is simply the intersection of the sets $S_{\mathcal{P}}$ over all (finitely many) red polygons $\mathcal{P}$. Because each $S_{\mathcal{P}}$ is closed, S is closed as well.

Therefore, S is nonempty, closed, and bounded. It follows that there is a red skewer $y = mx + b = m_0x + b_0$ with maximal m. We claim that $y = m_0x + b_0$ is a blue skewer as well.

Suppose, for sake of contradiction, that every polygon passes into the open half-plane above (resp. below) $y = m_0x + b_0$. Fix some point to the far left (resp. far right) on this line, and then rotate the line counter-clockwise by a tiny amount. Then the new line also passes through every red polygon, giving a red skewer with higher slope — contradicting the maximal definition of m_0. Therefore, some polygon lies in the closed half-plane below $y = m_0x + b_0$, and some polygon lies in the closed half-plane above $y = m_0x + b_0$. By Lemma 2, $y = m_0x + b_0$ is a blue skewer. Therefore, $y = m_0x + b_0$ is both a red skewer and a blue skewer, as desired.

Problem 38 In a contest consisting of n problems, the jury defines the difficulty of each problem by assigning it a positive integral number of points. (The same number of points may be assigned to different problems.) Any participant who answers the problem correctly receives that number of points for that problem; any other participants receive 0 points. After the participants submitted their answers, the jury realizes that given any ordering of the participants (where ties are not permitted), it could have defined the problems' difficulty levels to make that ordering coincide with the participants' ranking according to their total scores. Determine, in terms of n, the maximum number of participants for which such a scenario could occur.

Solution: The maximum is n. Label the problems $1, 2, \dots, n$.

Suppose that there are exactly n participants labelled $1, 2, \dots, n$, where participant i solves problem i and no other problems. It is clear that the jury can choose the problem weights to produce any ordering of the participants.

Now assume, for sake of contradiction, that there exists a scenario involving $m > n$ participants which meets the given conditions. Assign to participant i the n-tuple $\mathbf{x}_i$, where the j^{th} coordinate of $\mathbf{x}_i$ is 1 if participant i answered question j correctly and 0 otherwise. Also, to each possible choice of problem weights, associate an n-tuple containing in its j^{th} entry the weight of problem j. Then for a given problem-weight vector p, participant i scores $\mathbf{p} \cdot \mathbf{x}_i$ points.

Because there are $m > n$ participants and the dimension of the vector space containing the $\mathbf{x}_i$'s is n, the $\mathbf{x}_i$ are linearly dependent. That is, there exist constants a_i, not all 0, such that $\sum_{i=1}^{m} a_i\mathbf{x}_i = \mathbf{0}$. Because all coefficients of each $\mathbf{x}_i$ are either 0 or 1, there must be at least one $a_i > 0$ and at least one $a_i < 0$. Without loss of generality, assume that $a_1, \dots, a_k \geq 0$ and $a_{k+1}, \dots, a_m \leq 0$

where $1 < k < m$; also assume that $\sum_{i=1}^{k} |a_i| \geq \sum_{i=k+1}^{m} |a_i|$. (If either of these conditions were not true, we could simply relabel the participants and the indices of the $\mathbf{x}_i$.) Writing $b_i = |a_i|$, we have

$$\sum_{i=1}^{k} b_i \mathbf{x}_i = \sum_{i=k+1}^{m} b_i \mathbf{x}_i,$$

$$\sum_{i=1}^{k} b_i \geq \sum_{i=k+1}^{m} b_i. \qquad (*)$$

Also, $b_i \geq 0$ for $i = 1, 2, \ldots, n$, and $b_i > 0$ for some $1 \leq i \leq k$ and for some $k + 1 \leq i \leq m$.

By assumption, there exists a choice of problem weights such that the participants are ranked $1, 2, \ldots, m$. That is, there exists $\mathbf{p}$ such that

$$\mathbf{p} \cdot \mathbf{x}_1 > \mathbf{p} \cdot \mathbf{x}_2 > \cdots > \mathbf{p} \cdot \mathbf{x}_m.$$

Taking the dot product of both sides of our equation with $\mathbf{p}$, we obtain

$$\sum_{i=1}^{k} b_i (\mathbf{p} \cdot \mathbf{x}_i) = \sum_{i=k+1}^{m} b_i (\mathbf{p} \cdot \mathbf{x}_i).$$

Each expression $\mathbf{p} \cdot \mathbf{x}_i$ on the left side is equal to at least $\mathbf{p} \cdot \mathbf{x}_k$, while each such expression on the right side is strictly less. Thus, because $b_i \geq 0$ for all i and there is at least one nonzero b_i on each side, we have

$$\sum_{i=1}^{k} b_i (\mathbf{p} \cdot \mathbf{x}_k) \leq \sum_{i=1}^{k} b_i (\mathbf{p} \cdot \mathbf{x}_i) = \sum_{i=k+1}^{m} b_i (\mathbf{p} \cdot \mathbf{x}_i) < \sum_{i=k+1}^{m} b_i (\mathbf{p} \cdot \mathbf{x}_k).$$

Also, because $k < m$, participant k beat at least one other participant, implying that $\mathbf{p} \cdot \mathbf{x}_k > 0$. But then we can cancel this term from both sides of the above inequality, leaving

$$\sum_{i=1}^{k} b_i < \sum_{i=k+1}^{m} b_i,$$

which contradicts $(*)$. Therefore, our original assumption was incorrect, and n is indeed the largest possible number of participants.

Problem 39 The monic quadratics f and g take negative values on disjoint nonempty intervals of the real numbers, and the four endpoints of these intervals are also distinct. Prove that there exist positive numbers α and β such that

$$\alpha f(x) + \beta g(x) > 0$$

for all real numbers x.

First Solution: Let $(a-u, a+u)$ and $(b-v, b+v)$ be the intervals on which f and g are negative, and assume without loss of generality that $a+u < b-v$. We claim that setting $(\alpha, \beta) = (u, v)$ suffices.

Suppose that $x \in (b-v, b+v)$. Then $g'(x) = 2(x-b) > -2\beta v$, and because $x > a+u$ we have $f'(x) = 2(x-a) > 2\alpha u$. Hence, the derivative of $\alpha f(x) + \beta g(x)$ is greater than $2\alpha u - 2\beta v = 0$. Because $\alpha f(x) + \beta g(x)$ is positive at $x = b-v$, it must be positive along the entire interval $[b-v, b+v]$.

Likewise, $\alpha f(x) + \beta g(x)$ is positive along the interval $[a-u, a+u]$. And because $\alpha f(x) + \beta g(x)$ is clearly positive outside $[a-u, a+u] \cup [b-v, b+v]$, it is positive everywhere.

Second Solution: Let (r_1, r_2) and (s_1, s_2) be the intervals on which f and g are negative, and assume without loss of generality that $r_2 < s_1$. Then we have $f(x) = (x-r_1)(x-r_2)$ and $g(x) = (x-s_1)(x-s_2)$, so

$$\begin{aligned}\alpha f(x) + \beta g(x) &= \alpha(x-r_1)(x-r_2) + \beta(x-s_1)(x-s_2) \\ &= (\alpha+\beta)x^2 - (\alpha(r_1+r_2) + \beta(s_1+s_2))x \\ &\qquad + \alpha r_1 r_2 + \beta s_1 s_2.\end{aligned}$$

The leading coefficient of this quadratic is positive for any $\alpha, \beta > 0$, so the quadratic is always positive if and only if the discriminant D is negative. Let $u = r_2 - r_1$, $v = s_2 - s_1$, and $w = s_1 - r_2$. Then we have

$$\begin{aligned}
D &= (\alpha(r_1+r_2)+\beta(s_1+s_2))^2 - 4(\alpha+\beta)(\alpha r_1 r_2 + \beta s_1 s_2) \\
&= \alpha^2(r_1-r_2)^2 + \beta^2(s_1-s_2)^2 \\
&\quad + 2\alpha\beta((r_1+r_2)(s_1+s_2) - 2r_1r_2 - 2s_1s_2) \\
&= \alpha^2u^2 + \beta^2v^2 + 2\alpha\beta((r_1-r_2)(s_1-s_2) - 2(r_1-s_1)(r_2-s_2)) \\
&= \alpha^2u^2 + \beta^2v^2 + 2\alpha\beta(uv - 2(u+w)(v+w)) \\
&< \alpha^2u^2 + \beta^2v^2 + 2\alpha\beta(uv - 2uv) \\
&= \alpha^2u^2 + \beta^2v^2 - 2\alpha\beta uv,
\end{aligned}$$

where the inequality follows from the fact that all variables involved are positive. Choosing $\alpha = v$ and $\beta = u$ causes the last quantity to be 0, and hence D to be negative.

Problem 40 Let a and b be distinct positive integers such that $ab(a+b)$ is divisible by a^2+ab+b^2. Prove that $|a-b| > \sqrt[3]{ab}$.

Solution: We have that a^2+ab+b^2 divides

$$(a^2+ab+b^2)a - ab(a+b) = a^3,$$

and similarly that a^2+ab+b^2 divides

$$(a^2+ab+b^2)b - ab(a+b) = b^3.$$

Write $a = x \cdot g$ and $b = y \cdot g$ with $\gcd(x,y) = 1$. Then the above results imply that

$$(x^2+xy+y^2) \mid gx^3 \qquad \text{and} \qquad (x^2+xy+y^2) \mid gy^3.$$

Because x and y are relatively prime,

$$(x^2+xy+y^2) \mid g,$$

implying that $g \geq x^2+xy+y^2$.

Hence,

$$\begin{aligned}
|a-b|^3 &= g \cdot g^2 \cdot |x-y|^3 \\
&\geq (x^2+xy+y^2) \cdot g^2 \cdot 1 \\
&> xy \cdot g^2 = ab.
\end{aligned}$$

It follows that $|a-b| > \sqrt[3]{ab}$.

Problem 41 In a country of 2001 cities, some cities are connected in pairs by two-way roads. We call two cities which are connected by a road *adjacent*. Each city is adjacent to at least one other city, and no city is adjacent to every other city. A set D of cities is called *dominating* if any city not included in D is adjacent to some city in D. It is known that any dominating set contains at least k cities. Prove that the country can be divided into $2001 - k$ republics such that no two cities in any single republic are adjacent.

Solution: First observe that any city C is adjacent to at most $2001 - k$ other cities. Otherwise, there are $t < k - 1$ cities that C is *not* adjacent to; taking C together with these t cities gives a dominating set of fewer than k cities, which is impossible.

We consider two cases:

Case 1: There exists a city C that is adjacent to exactly $2001 - k$ others. Let S be the set of $2001 - k$ cities that C is adjacent to, and let T consist of the remaining k cities not in S.

Observe that no two cities A, B in T are adjacent, because otherwise $T - \{A\}$ is a dominating set with fewer than k cities.

Also, we claim that some two cities X, Y in S are not adjacent. Otherwise, take any city A in T; it is adjacent to *some* city B. From the previous paragraph, $B \in S$. Then B is adjacent to more than $2001 - k$ cities: it is adjacent to the $2001 - k$ cities in S, and it is adjacent to A. This contradicts our initial observation.

Hence, we can form $2001 - k$ republics with the required property as follows: T is one republic; X and Y form another republic; and the remaining $1999 - k$ cities in S each lie in their own republic.

Case 2: Each city is adjacent to fewer than $2001 - k$ others. In this case, we start with $2001 - k$ empty republics and then add cities one by one. By the time we add a city C, because C is adjacent to fewer than $2001 - k$ others, some republic does not yet contain any of C's neighbors; we place C in that republic. When all 2001 cities are placed, no two cities in a republic are adjacent, as desired.

Problem 42 Let $SABC$ be a tetrahedron. The circumcircle of ABC is a great circle of a sphere ω, and ω intersects $\overline{SA}$, $\overline{SB}$, and $\overline{SC}$ again at A_1, B_1, and C_1, respectively. The planes tangent to

ω at A_1, B_1, and C_1 intersect at a point O. Prove that O is the circumcenter of tetrahedron $SA_1B_1C_1$.

Solution: Consider the inversion about point S that fixes ω. This inversion interchanges A with A_1, B with B_1, and C with C_1. Thus, it takes the circumsphere of $SA_1B_1C_1$ to the plane ABC, which passes through the center of ω by assumption. It follows that the circumcenter of $SA_1B_1C_1$ is taken to the reflection S' of S across plane ABC.

It suffices, then, to show that S' is the image of O with respect to the inversion. O lies on the three given planes tangent to ω; these three planes can meet only at one point, however. Thus, it suffices to show that S' lies on the images of the three given planes under the inversion.

The plane tangent to ω at A_1 inverts to the sphere ω_A which passes through S and which is tangent to ω at A. Because ω is symmetric about plane ABC, it follows that ω_A is also symmetric with respect to plane ABC. Hence, it passes through S'. Likewise, we see that ω_B and ω_C, the corresponding images of the planes tangent to ω at A_1 and B_1, pass through S' as well. This completes the proof.

1.14 Taiwan

Problem 1 Let O be the excenter of triangle ABC opposite A. Let M be the midpoint of $\overline{AC}$, and let P be the intersection point of $\overline{MO}$ and $\overline{BC}$. Prove that $AB = BP$ if $\angle BAC = 2\angle ACB$.

First Solution: We let a, b, c be the lengths of segments $\overline{BC}, \overline{CA}, \overline{AB}$ respectively. Let Γ be the excircle of triangle ABC opposite A and T, U, and V the points of tangency of Γ to lines AC, AB and BC respectively. Let X be the intersection of line AO and BC, N the foot of the perpendicular from P to AC, and Q the intersection of MO with the perpendicular to AC from C. Finally, let $\theta = \angle ACB$ so $\angle BAC = 2\theta$.

First we prove some relations among the lengths a, b and c and the angle θ that will later be useful in calculating the length BP. If R is the circumradius of ABC, by the extended law of sines $\sin\theta = \frac{c}{2R}$ and $\sin 2\theta = 2\sin\theta\cos\theta = \frac{a}{2R}$. Dividing these two equations yields $\cos\theta = \frac{a}{2c}$.

Now, construct point D such that $ABDC$ is an isosceles trapezoid with AB and DC as equal legs. Then

$$\angle BAD = \angle BAC - \angle DAC = \angle BAC - \angle ACB = 2\theta - \theta = \theta.$$

Now, $ABDC$ is a cyclic quadrilateral (as it is an isosceles trapezoid) and chords AB and BD both subtend angles of θ. Therefore they are equal in length and so $BD = AB = DC = c$. Also $AD = BC = a$. Now, by Ptolemy's Theorem, $AB \cdot DC + AC \cdot AB = AD \cdot BC$, or $c^2 + cb = a^2$. This can be rewritten as $b + c = \frac{a^2}{c}$ or $b = \frac{a^2 - c^2}{c}$.

By equal tangents, we have $AT = AU$, but

$$AT = AC + CT = AC + CV \quad \text{and} \quad AU = AB + BU = AB + BV$$

by two more applications of equal tangents. So

$$2AT = AB + AC + BV + CV = AB + AC + BC = a + b + c.$$

It implies $AT = \frac{a+b+c}{2}$. But as $b + c = \frac{a^2}{c}$, $AT = \frac{a(a+c)}{2c}$.

Because $\angle XAC = \angle XCA = \theta$, triangle AXC is isosceles with $AX = XC$. This means that the median XM is perpendicular to side AC. Thus, triangle AMX has a right angle at M, and so

$$XM = AM \tan\angle XAM = \frac{b}{2}\tan\theta = \frac{(a^2 - c^2)\tan\theta}{2c}.$$

As well, since T is the point of tangency of Γ and AC, $OT \perp AC$. Thus we also have $OT = AT \tan\theta = \frac{a(a+c)(\tan\theta)}{2c}$. Now, triangles MCQ and MTO are similar since they share the angle OAM and their bases, being both perpendicular to AT, are parallel. Hence $QC = OT\frac{MC}{MT}$. Now, $MC = \frac{b}{2} = \frac{a^2-c^2}{2c}$ as M bisects AC, and $MT = AT - AM = \frac{a+b+c}{2} - \frac{b}{2} = \frac{a+c}{2}$. So,

$$\begin{aligned} QC &= \frac{OT \cdot MC}{MT} \\ &= \frac{a(a+c)(a^2-c^2)\tan\theta}{2c^2(a+c)} \\ &= \frac{a(a^2-c^2)\tan\theta}{2c^2} \end{aligned}$$

Now we wish to calculate PN. Because triangles PMN and QMC are similar, $\frac{PN}{QC} = \frac{MN}{MC}$. But also triangles PCN and XCM are similar, so that $\frac{PN}{XM} = \frac{CN}{MC}$. Summing these two equations gives $\frac{PN}{QC} + \frac{PN}{XM} = \frac{MN+CN}{MC} = 1$. Dividing by PN gives $\frac{1}{QC} + \frac{1}{XM} = \frac{1}{PN}$. We plug in our previously calculated values for QC and XM, and get

$$\begin{aligned} \frac{1}{PN} &= \frac{1}{QC} + \frac{1}{XM} \\ &= \frac{2c^2}{a(a^2-c^2)\tan\theta} + \frac{2c}{(a^2-c^2)\tan\theta} \\ &= \frac{2c(c+a)}{a(a^2-c^2)\tan\theta} \\ &= \frac{2c}{a(a-c)\tan\theta} \end{aligned}$$

and so $PN = \frac{a(a-c)\tan\theta}{2c}$. Now, since angle CNP is right, we have $CP = \frac{PN}{\sin\angle PCN} = \frac{PN}{\sin\theta}$. Substituting in our value for PN into this equation yields $CP = \frac{a(a-c)\tan\theta}{2c\sin\theta} = \frac{a(a-c)}{2c\cos\theta)}$. But as previously calculated, $\cos\theta = \frac{a}{2c}$, so

$$CP = \frac{a(a-c)}{2c\frac{a}{2c}} = a - c.$$

This implies that $BP = BC - CP = a - (a-c) = c = AB$, which is what we wanted.

Second Solution: As before, let $\angle ACB = \theta$, so that $\angle BAC = 2\theta$. Let T be the intersection of lines AO and BC. Then we have

$\angle BAT = \angle TAC = \theta$, so $\angle BTA = 2\theta$. It follows that triangles ABC and BTA are similar and hence

$$\frac{AT}{AC} = \frac{BT}{AB}.$$

Applying the exterior angle bisector theorem to triangle ABT and bisector BO, we obtain

$$\frac{BT}{AB} = \frac{TO}{AO}.$$

Combining this with the previous equation and rearranging gives

$$\frac{AT}{TO} = \frac{AC}{AO}.$$

Let $d(X, PQ)$ denote the distance from point X to line PQ. Then we have

$$\frac{MP}{PO} = \frac{d(M,BC)}{d(O,BC)} = \frac{1}{2} \cdot \frac{d(A,BC)}{d(O,BC)} = \frac{1}{2} \cdot \frac{AT}{TO},$$

where in the second step we used the fact that M is the midpoint of $\overline{AC}$. We also have

$$\frac{AM}{AO} = \frac{1}{2} \cdot \frac{AC}{CO}.$$

Thus, we see that

$$\frac{MP}{PO} = \frac{AM}{AO},$$

from which it follows that AP is the angle bisector of $\angle OAM$. Hence,

$$\angle BAP = \angle BAT + \angle TAP = \theta + \frac{1}{2}\theta = \frac{3}{2}\theta.$$

Since

$$\angle ABC = 180 - \angle BAC - \angle ACB = 180 - 3\theta,$$

we have $\angle BPA = \frac{3}{2}\theta$ as well. This proves that $AB = BP$ as wanted.

Problem 2 Let $n \geq 3$ be an integer, and let A be a set of n distinct integers. Let the minimal and maximal elements of A be m and M, respectively. Suppose that there exists a polynomial p with integer coefficients such that (i) $m \leq p(a) < M$ for all $a \in A$, and (ii) $p(m) < p(a)$ for all $a \in A - \{m, M\}$. Show that $n \leq 5$, and prove that there exist integers b and c such that each element of A is a solution to the equation $p(x) + x^2 + bx + c = 0$.

Solution: We begin by proving a lemma.

Lemma. *If p is a polynomial with integer coefficients, then for any integers a, b, we have $(a-b) \mid (p(a)-p(b))$.*

Proof. Let $p(x) = \sum_{i=0}^{d} c_i x^i$, where d is the degree of p. Then

$$\begin{aligned} p(a)-p(b) &= \sum_{i=0}^{d} c_i(a^i - b^i) \\ &= \sum_{i=0}^{d} c_i(a-b)(a^{i-1} + a^{i-2}b + \cdots + b^{i-1}) \\ &= (a-b)\sum_{i=0}^{d} c_i((a^{i-1} + a^{i-2}b + \cdots + b^{i-1}), \end{aligned}$$

so we have expressed $p(a)-p(b)$ as an integer multiple of $a-b$. $\square$

Applying the lemma to our polynomial p at points m and M, we obtain $(M-m) \mid (p(M)-p(m))$. On the other hand, by condition (i), $m \le p(m) < M$ and $m \le p(M) < M$, so $|p(M)-p(m)| < M-m$. Thus, we must have $p(M)-p(m) = 0$, or $p(m) = p(M)$.

Now the polynomial $f(x) = p(x) - p(m)$ has m and M as roots, so we can write $f(x) = (x-m)(M-x)q(x)$, where q is another polynomial with integer coefficients. Let a be any element of set $A - \{m, M\}$. Then by (ii), we have $p(a) > p(m)$, so $f(a) > 0$. Because $m < f(a) < M$, both $x-m$ and $M-x$ are positive; hence, $q(a) > 0$ as well. On the other hand, q has integer coefficients, so we have $q(a) \ge 1$ and hence $f(a) \ge (a-m)(M-a)$. Also, $f(a) = p(a) - p(m) < M - p(m) \le M - m$, by applying (ii) in the first inequality and (i) in the second. Therefore, we have $(a-m)(M-a) \le M-m-1$. The left side is a concave quadratic in a which equals $M-m-1$ when $a = m+1$ or $a = M-1$. It follows that these are the only possible values for a. Thus, $n \le 4$: A contains m, M, and possibly $m+1$ and $M-1$. Furthermore, $f(x)$ matches the quadratic $(x-m)(M-x)$ at m and M, and it also does at $m+1$ and $M-1$ if either is in A. Because $f(x) = p(x)-p(m)$, we have $p(x)-(x-m)(M-x)-p(m) = 0$ at all elements of A, satisfying the second claim.

Problem 3 Let $n \ge 3$ be an integer and let $A_1, A_2, \ldots, A_n$ be n distinct subsets of $S = \{1, 2, \ldots, n\}$. Show that there exists an element $x \in S$ such that the n subsets $A_1 \setminus \{x\}$, $A_2 \setminus \{x\}$, $\ldots$, $A_n \setminus \{x\}$ are also distinct.

Solution: We construct a graph whose vertices are $A_1, A_2, \ldots, A_n$ as follows. For each x, if there exist distinct sets A_i, A_j such that $A_i \setminus \{x\} = A_j \setminus \{x\}$, then choose *one* such pair of sets A_i, A_j and draw an edge between them. (Even if there are multiple pairs of sets A_i, A_j with the required property, we only draw one edge.)

For sake of contradiction, suppose that the resulting graph contains a cycle — without loss of generality, suppose that $A_1, A_2, \ldots, A_k$ form a cycle and

$$A_1 \setminus \{x_1\} = A_2 \setminus \{x_1\}, A_2 \setminus \{x_2\} = A_3 \setminus \{x_2\}, \ldots, A_k \setminus \{x_k\} = A_1 \setminus \{x_1\}.$$

By construction, $x_1, x_2, \ldots, x_k$ are distinct. Without loss of generality, assume that $x_1 \notin A_1$ and $x_1 \in A_2$. Because $A_2 \setminus \{x_2\} = A_3 \setminus \{x_2\}$ and $x_1 \in A_2$, we have $x_1 \in A_3$ as well. Similarly, x_1 lies in A_4, A_5, $\ldots$, A_k, and finally $x_1 \in A_1$, a contradiction.

Thus, the resulting graph is a tree with n vertices. Any such graph has at most $n-1$ edges. Therefore, for some $x \in S$, there do not exist distinct sets A_i, A_j such that $A_i \setminus \{x\} = A_j \setminus \{x\}$, as desired.

Problem 4 Let Γ be the circumcircle of a fixed triangle ABC. Suppose that M and N are the midpoints of arcs $\widehat{BC}$ and $\widehat{CA}$, respectively, and let X be any point on arc $\widehat{AB}$. (Here, arc $\widehat{AB}$ refers to the arc not containing C; analogous statements hold for arcs $\widehat{BC}$ and $\widehat{CA}$.) Let O_1 and O_2 be the incenters of triangles XAC and XBC, respectively. Let Γ and the circumcircle of triangle XO_1O_2 intersect at Q. Prove that $\triangle QNO_1 \sim \triangle QMO_2$, and determine the locus of Q.

Solution: We claim that the locus of Q consists of a single point. To locate this point, we let Γ_1 be the circle centered at M with radius $MB = MC$, let Γ_2 be the circle centered at N with radius $NA = NC$, and let T be the intersection of Γ_1 and Γ_2. We will prove the Q is the center of the unique spiral similarity taking M to N and B to T.

First, we prove that triangles QNO_1 and QMO_2 are similar. Because arcs $\widehat{AN}$ and $\widehat{NC}$ of Γ are equal, the angles AXN and NXC that subtend them are also equal. Thus XN bisects AXC, and since O_1 is the incenter of XAC, O_1 lies on XN. Thus X, O_1, and N are collinear, and similarly X, O_2, and M are collinear.

Now, since X, N, M, and Q are concyclic,

$$\angle O_1NQ = \angle XNQ = \angle XMQ = \angle O_2MQ.$$

As well, since X, O_1, O_2, and Q are concyclic,

$$\angle QO_1N = \pi - \angle QO_1X = \pi - \angle QO_2X = \angle QO_2M.$$

So because $\angle O_1NQ = \angle O_2MQ$ and $\angle QO_1N = \angle QO_2M$, triangles QNO_1 and QMO_2 are similar, proving the first half of the problem. As well, it implies that Q is the center of a spiral similarity taking M to N and O_2 to O_1. It suffices to show that this similarity takes B to T to complete the solution.

First, we show that B, N, and T are collinear. To do this, let T' be the intersection of BN with Γ_1. We must show that $T' = T$. Let $\alpha = \angle CBM$. Then as $BM = CM$, $\angle BCM = \angle CBM = \alpha$ and

$$\angle BMC = \pi - \angle CBM - \angle BCM = \pi - 2\alpha.$$

Now, $\angle BT'C = \frac{2\pi - \angle BMC}{2} = \frac{\pi}{2} + \alpha$. So,

$$\angle NT'C = \pi - \angle BT'C = \frac{\pi}{2} - \alpha.$$

Also, quadrilateral $BMCN$ is cyclic and so

$$\angle T'NC = \angle BNC = \pi - \angle CMB = 2\alpha.$$

Because the sum of the angles of triangle $NT'C$ is π,

$$\angle NCT' = \pi - \angle T'NC - \angle NT'C = \pi - 2\alpha - (\frac{\pi}{2} - \alpha) = \frac{\pi}{2} - \alpha = \angle NT'C.$$

Hence, triangle $NT'C$ is isosceles with $NT' = NC$. Hence, T' lies on Γ_2, and so $T' = T$.

We now show that O_1 lies on Γ_2 and O_2 lies on Γ_1. Because AO_1 bisects $\angle XAC$, $\angle O_1AC = \frac{\angle XAC}{2}$ and also $\angle CAN = \angle CXN = \frac{\angle CXA}{2}$ as XN bisects $\angle CXA$. Putting these two together gives

$$\angle O_1AN = \frac{\angle CXA + \angle XAC}{2} = \frac{\pi - \angle ACX}{2}.$$

Now, $\angle ANO_1 = \angle ANX = \angle ACX$. Because the sum of the angles of triangle ANO_1 is π, we have

$$\begin{aligned}\angle NO_1A &= \pi - \angle O_1AN - \angle ANO_1 \\ &= \pi - \frac{\pi - \angle ACX}{2} - \angle ACX \\ &= \frac{\pi - \angle ACX}{2} = \angle O_1AN.\end{aligned}$$

This means that triangle ANO_1 is isosceles with $NO_1 = NA$, so O_1 lies on Γ_2 and similarly O_2 lies on Γ_1.

Now, $\angle O_1NT = \angle XNB = \angle XMB = \angle O_2MB$ by equal inscribed angles. Because O_1N and NT are both radii of Γ_2, triangle O_1NT is isosceles and similarly triangle O_2MB is also isosceles. This implies that triangles O_1NT and O_2MB are similar. Thus the spiral similarity around Q taking M to N and O_2 to O_1 must also take B to T. Thus Q is the center of the unique spiral similarity taking M to N and B to T. Because this does not depend on X, the locus of Q is that single point, as desired.

Problem 5 Let x, y be distinct real numbers, and let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = \sum_{k=0}^{n-1} y^k x^{n-1-k}$ for all $n \in \mathbb{N}$. Suppose that $f(m)$, $f(m+1)$, $f(m+2)$, and $f(m+3)$ are integers for some positive integer m. Prove that $f(n)$ is an integer for all $n \in \mathbb{N}$.

Solution:

First of all, we note that $f(n) = \sum_{k=0}^{n-1} y^k x^{n-1-k} = \frac{x^n - y^n}{x-y}$. Denote $x + y$ and xy by a and b respectively. Consider two functions from $\mathbb{N}$ to $\mathbb{R}$: $g(n) = x^n$ and $h(n) = y^n$. Since x and y are the roots of $q(t) = t^2 - at + b$, $f(n)$ and $g(n)$ satisfy $g(n+1) = ag(n) - bg(n-1)$ and $h(n+1) = ah(n) - bh(n-1)$. Function $f(n)$ is a linear combination of $g(n)$ and $h(n)$, so it satisfies the same condition:

$$f(n+1) = af(n) - bf(n-1). \tag{*}$$

Moreover, we have $f(1) = 1$ and $f(2) = a$. Hence, it suffices to prove that a and b are integers.

Consider $d = f(m)f(m+2) - f(m+1)^2$, which is an integer. On the other hand

$$\begin{aligned} d &= \frac{(x^m - y^m)(x^{m+2} - y^{m+2}) - (x^{m+1} - y^{m+1})^2}{(x-y)^2} \\ &= \frac{-x^m y^{m+2} - y^m x^{m+2}) + 2x^{m+1} y^{m+1}}{(x-y)^2} = -(xy)^m. \end{aligned}$$

It means that b^m is an integer. Similarly through the calculation of $f(m+1)f(m+3) - f(m+2)^2$, we obtain that b^{m+1} is integer too. If $b^m = 0$, then $b = 0$ too. Otherwise $b = b^{m+1}/b^m$ is rational. Since b is rational and b^m is an integer, it follows that b is an integer too. We have $f(m+2) = af(m+1) - bf(m)$ and $a = \frac{f(m+2)+bf(m)}{f(m+1)}$ is rational ($f(m+1) \neq 0$ because $x \neq y$).

We claim that if we define $f(n)$ as

$$f(1) = 1, f(2) = a, f(n+1) = af(n) - bf(n-1)$$

with b fixed integer, $f(k)$ is a polynomial of a of degree $k-1$ with 1 as the coefficient of a^{k-1}. We proceed by induction on k. For $k = 1, 2$ the statement is true. Now let $f(k)$ and $f(k-1)$ satisfy the desired condition. Then $f(k+1) = af(k) - bf(k-1)$ which is a polynomial with integer coefficients of degree k and coefficient of x^k the same as the coefficient of x^{k-1} in $f(k)$ that is 1. This completes the induction step. Note that from $(*)$ this definition agrees with the definition in the problem statement.

Since $l = f(m+1)$ is an integer, a is a root of $f(m+1) - l$, which is a polynomial with integer coefficients and 1 as the coefficient of x^m. Because a is rational root, it must be an integer by the Gauss' lemma.

So, a and b are integers as desired.

Problem 6 We are given n stones $A_1, A_2 \ldots, A_n$ labeled with distinct real numbers. We may *compare* two stones by asking what the order of their corresponding numbers are. We are given that the numbers on $A_1, A_2, \ldots, A_{n-1}$ are increasing in that order; the n orderings of the numbers on $A_1, A_2, \ldots, A_n$ which satisfy this condition are assumed to be equally likely. Based on this information, an algorithm is created that minimizes the expected number of comparisons needed to determine the order of the numbers on $A_1, A_2, \ldots, A_n$. What is this expected number?

Solution: Let $a = \lceil \log_2 n \rceil$. Then the answer is $a + 1 - \frac{2^a}{n}$.

Let $f(n)$ be the minumum expected number of comparisons for the case n. We will derive a recursive formula for $f(n)$ and then prove that its explicit formula is the answer above.

Denote the number on stone A_n by a_n. By the given condition, $a_1 < a_2 < \cdots < a_{n-1}$ and a_n is equally likely to fall in each of the n intervals $(-\infty, a_1), (a_1, a_2), \ldots, (a_{n-1}, \infty)$. Thus, if we start by comparing stone A_n to stone A_k, the probability that $a_n < a_k$ is $\frac{k}{n}$ and the probability that $a_n > a_k$ is $\frac{n-k}{n}$. In the first case, we are left with k equally likely intervals for a_n to fall in, a situation analogous to having started with k stones. Likewise, the second case is analogous to having started with $n-k$ stones. Thus, by the definition of f, the minimum expected number of comparisons for the remainder of the first case is $f(k)$, and for the second, $f(n-k)$. Remembering to count in our first comparison, the total expected number of comparisons with this strategy is $1+\frac{k}{n}f(k)+\frac{n-k}{n}f(n-k)$. Finally, minimizing this quantity over all choices for k gives us our recursive formula:

$$f(n)=1+\min_{1\le k\le n-1}\left\{\frac{k}{n}f(k)+\frac{n-k}{n}f(n-k)\right\}.$$

Before we prove the explicit formula, we convert our recursive formula into a nicer form by setting $g(n)=nf(n)$. Then we obtain

$$g(n)=n+\min_{1\le k\le n-1}\{g(k)+g(n-k)\}.$$

We wish to show that $f(n)=a+1-\frac{2^a}{n}$, where $a=\lceil\log_2 n\rceil$. In terms of g, we want $g(n)=n(a+1)-2^a$. To prove this, we use strong induction on n. The base case, $n=1$, is trivial: there is only one stone, so we do not need any comparisons. This matches $1(0+1)-2^0=0$.

Now we assume that the formula for g holds for $1,2,\ldots,n-1$ stones. We wish to show that it also holds for n, $n>1$. To do this, we will show that $k=\lfloor\frac{n}{2}\rfloor$ minimizes $g(k)+g(n-k)$. First, we consider the consecutive differences $g(x)-g(x-1)$ and prove that they are nondecreasing for $x=2,3,\ldots,n-1$. Indeed, if $x-1$ is not a power of 2, then $\lceil\log_2 x\rceil=\lceil\log_2(x-1)\rceil=a$, so

$$\begin{aligned}g(x)-g(x-1)&=(x(a+1)-2^a)-((x-1)(a+1)-2^a)\\&=a+1=\lceil\log_2 x\rceil+1.\end{aligned}$$

Otherwise, $x-1=2^a$, from which we have $g(x-1)=2^a(a+1)-2^a$ and $g(x)=(2^a+1)(a+2)-2^{a+1}$. Subtracting now gives

$$g(x)-g(x-1)=a+2=\lceil\log_2 x\rceil+1$$

again. Hence, because $\log_2 x$ is increasing and $\lceil x\rceil$ is nondecreasing, we see that $g(x)-g(x-1)$ is nondecreasing as x ranges from 2 to $n-1$.

It follows now that $g(x)+g(y)\ge g(x+1)+g(y-1)$ holds for all $1\le x<y\le n-1$. Indeed, this is equivalent to inequality $g(x+1)-g(x)\le g(y)-g(y-1)$, which is true because $x+1\le y$. Thus, applying this repeatedly, we have

$$g(1)+g(n-1)\ge g(2)+g(n-2)\ge\cdots\ge g\left(\left\lfloor\frac{n}{2}\right\rfloor\right)+g\left(\left\lceil\frac{n}{2}\right\rceil\right),$$

which proves that $k=\lfloor\frac{n}{2}\rfloor$ minimizes $g(k)+g(n-k)$.

To complete the proof, we need only check that $n+g(\lfloor \frac{n}{2} \rfloor)+g(\lceil \frac{n}{2} \rceil)$ coincides with our formula for $g(n)$. Letting $a = \lceil \log_2 n \rceil$, we have $a-1 = \lceil \log_2 \lfloor \frac{n}{2} \rfloor \rceil = \lceil \log_2 \lceil \frac{n}{2} \rceil \rceil$. Thus,

$$\begin{aligned} n+g\left(\left\lfloor \frac{n}{2} \right\rfloor\right)+g\left(\left\lceil \frac{n}{2} \right\rceil\right) &= n+\left\lfloor \frac{n}{2} \right\rfloor ((a-1)+1)-2^{a-1} \\ &\quad +\left\lceil \frac{n}{2} \right\rceil ((a-1)+1)-2^{a-1} \\ &= n(a+1)-2^a, \end{aligned}$$

completing the induction and the proof.

1.15 United States of America

Problem 1 Each of eight boxes contains six balls. Each ball has been colored with one of n colors, such that no two balls in the same box are the same color, and no two colors occur together in more than one box. Determine, with justification, the smallest integer n for which this is possible.

First Solution: The smallest such n is 23.

We first show that $n = 22$ cannot be achieved.

Assume that some color, say red, occurs four times. Then the first box containing red contains 6 colors, the second contains red and 5 colors not mentioned so far, and likewise for the third and fourth boxes. A fifth box can contain at most one color used in each of these four, so must contain 2 colors not mentioned so far, and a sixth box must contain 1 color not mentioned so far, for a total of 6+5+5+5+2+1=24, a contradiction.

Next, assume that no color occurs four times; this forces at least four colors to occur three times. In particular, there are two colors that occur at least three times and which both occur in a single box, say red and blue. Now the box containing red and blue contains 6 colors, the other boxes containing red each contain 5 colors not mentioned so far, and the other boxes containing blue each contain 3 colors not mentioned so far (each may contain one color used in each of the boxes containing red but not blue). A sixth box must contain one color not mentioned so far, for a total of 6+5+5+3+3+1=23, again a contradiction.

We now give a construction for $n = 23$. We still cannot have a color occur four times, so at least two colors must occur three times. Call these red and green. Put one red in each of three boxes, and fill these with 15 other colors. Put one green in each of three boxes, and fill each of these boxes with one color from each of the three boxes containing red and two new colors. We now have used $1 + 15 + 1 + 6 = 23$ colors, and each box contains two colors that have only been used once so far. Split those colors between the last two boxes. The resulting arrangement is:

1	3	4	5	6	7
1	8	9	10	11	12
1	13	14	15	16	17
2	3	8	13	18	19
2	4	9	14	20	21
2	5	10	15	22	23
6	11	16	18	20	22
7	12	17	19	21	23

Note that the last 23 can be replaced by a 22.

Now we present a few more methods of proving $n \geq 23$.

Second Solution: As in the first solution, if $n = 22$ is possible, it must be possible with no color appearing four or more times. By the Inclusion-Exclusion Principle, the number of colors (call it C) equals the number of balls (48), minus the number of pairs of balls of the same color (call it P), plus the number of triples of balls of the same color (call it T); that is,

$$C = 48 - P + T.$$

For every pair of boxes, at most one color occurs in both boxes, so $P \leq \binom{8}{2} = 28$. Also, if $n \leq 22$, there must be at least $48 - 2(22) = 4$ colors that occur three times. Then $C \geq 48 - 28 + 4 = 24$, a contradiction.

Third Solution: Assume $n = 22$ is possible. By the **Pigeonhole Principle**, some color occurs three times; call it color 1. Then there are three boxes containing 1 and fifteen other colors, say colors 2 through 16. The other five boxes each contain at most three colors in common with the first three boxes, so they contain at least three colors from 17 through 22.

Since $5 \times 3 > 2 \times 6$, one color from 17 to 22 occurs at least three times in the last five boxes; say it's color 17. Then two balls in each of those three boxes have colors among those labeled 18 through 22. But then one of these colors must appear together with 17, a contradiction.

Fourth Solution: Label the colors $1, 2, \ldots, n$, and let $a_1, a_2, \ldots,$

a_n be the number of balls of color $1, 2, \ldots, n$, respectively. Then

$$\sum_{i=1}^{n} a_i = 48.$$

Since $\binom{a_i}{2}$ is the number of boxes sharing color i and there are $\binom{8}{2} = 28$ pairs of boxes, each of which can only share at most one color,

$$\begin{aligned} 28 = \binom{8}{2} \geq \sum_{i=1}^{n} \binom{a_i}{2} &= \sum_{i=1}^{n} \frac{a_i(a_i-1)}{2} \\ &= \frac{1}{2}\sum_{i=1}^{n} a_i^2 - \frac{1}{2}\sum_{i=1}^{n} a_i = \frac{1}{2}\sum_{i=1}^{n} a_i^2 - 24, \end{aligned}$$

or $\sum_{i=1}^{n} a_i^2 \leq 104$. By the **RMS-AM Inequality**,

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} \geq \frac{1}{n}\sum_{i=1}^{n} a_i.$$

It follows that

$$104n \geq 48^2 \quad \text{or} \quad n \geq \frac{288}{13} > 22.$$

Fifth Solution: Let $m_{i,j}$ be the number of balls which are the same color as the j^{th} ball in box i (including that ball). For a fixed box i, $1 \leq i \leq 8$, consider the sums

$$S_i = \sum_{j=1}^{6} m_{i,j} \quad \text{and} \quad s_i = \sum_{j=1}^{6} \frac{1}{m_{i,j}}.$$

For each fixed i, since no pair of colors is repeated, each of the remaining seven boxes can contribute at most one ball to S_i. Thus $S_i \leq 13$. It follows by the **convexity** of $f(x) = 1/x$ (and consequently, by the **Jensen's Inequality**) that s_i is minimized when one of the $m_{i,j}$ is equal to 3 and the other five equal 2. Hence $s_i \geq 17/6$. Note that

$$n = \sum_{i=1}^{8}\sum_{j=1}^{6} \frac{1}{m_{i,j}} \geq 8 \cdot \frac{17}{6} = \frac{68}{3} = 22\frac{2}{3}.$$

Hence there must be at least 23 colors.

Problem 2 Let ABC be a triangle and let ω be its incircle. Denote by D_1 and E_1 the points where ω is tangent to sides $\overline{BC}$ and $\overline{AC}$, respectively. Denote by D_2 and E_2 the points on sides $\overline{BC}$ and $\overline{AC}$, respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by P the point of intersection of $\overline{AD_2}$ and $\overline{BE_2}$. Circle ω intersects $\overline{AD_2}$ at two points, the closer of which to the vertex A is denoted by Q. Prove that $AQ = D_2P$.

First Solution: The key observation is the following Lemma.

Lemma *Segment D_1Q is a diameter of circle ω.*

Proof: Let I be the center of circle ω, that is, I is the incenter of triangle ABC. Extend segment D_1I through I to intersect circle ω again at Q', and extend segment AQ' through Q' to intersect segment BC at D'. We show that $D_2 = D'$, which in turn implies that $Q = Q'$, that is, D_1Q is a diameter of ω.

Let ℓ be the line tangent to circle ω at Q', and let ℓ intersect segments AB and AC at B_1 and C_1, respectively. Then ω is an **excircle** of triangle AB_1C_1. Let $\mathbf{H}_1$ denote the dilation with center A and ratio AD'/AQ'. Since $\ell \perp D_1Q'$ and $BC \perp D_1Q$, $\ell \parallel BC$. Hence, $AB/AB_1 = AC/AC_1 = AD'/AQ'$. Thus, $\mathbf{H}_1(Q') = D'$, $\mathbf{H}_1(B_1) = B$, and $\mathbf{H}_1(C_1) = C$. It also follows that the excircle Ω of triangle ABC opposite vertex A is tangent to side BC at D'.

It is well known that

$$CD_1 = \frac{1}{2}(BC + CA - AB). \tag{1}$$

We compute BD'. Let X and Y denote the points of tangency of circle Ω with rays AB and AC, respectively. Then by equal tangents, $AX = AY$, $BD' = BX$, and $D'C = YC$. Hence,

$$\begin{aligned} AX = AY &= \frac{1}{2}(AX + AY) \\ &= \frac{1}{2}(AB + BX + YC + CA) \\ &= \frac{1}{2}(AB + BC + CA). \end{aligned}$$

It follows that

$$BD' = BX = AX - AB = \frac{1}{2}(BC + CA - AB). \tag{2}$$

Combining (1) and (2) yields $BD' = CD_1$. Thus,

$$BD_2 = BD_1 - D_2D_1 = D_2C - D_2D_1 = D_1C = BD',$$

that is, $D' = D_2$, as desired. ■

Now we prove our main result. Let M_1 and M_2 be the midpoints of segments BC and CA, respectively. Then M_1 is also the midpoint of segment D_1D_2, from which it follows that IM_1 is the midline of triangle D_1QD_2. Hence,

$$QD_2 = 2IM_1 \tag{3}$$

and $AD_2 \parallel M_1I$. Similarly, we can prove that $BE_2 \parallel M_2I$.

Let G be the centroid of triangle ABC. Thus, segments AM_1 and BM_2 intersect at G. Define transformation $\mathbf{H}_2$ as the **dilation** with center G and ratio $-1/2$. Then $\mathbf{H}_2(A) = M_1$ and $\mathbf{H}_2(B) = M_2$. Under the dilation, parallel lines go to parallel lines and the intersection of two lines goes to the intersection of their images. Since $AD_2 \parallel M_1I$ and $BE_2 \parallel M_2I$, $\mathbf{H}$ maps lines AD_2 and BE_2 to lines M_1I and M_2I, respectively. It also follows that $\mathbf{H}_2(P) = I$ and that

$$\frac{IM_1}{AP} = \frac{GM_1}{AG} = \frac{1}{2}$$

or

$$AP = 2IM_1. \tag{4}$$

Combining (3) and (4) yields

$$AQ = AP - QP = 2IM_1 - QP = QD_2 - QP = PD_2,$$

as desired.

Second Solution: From the Lemma, we have

$$\frac{AQ}{AD_2} = \frac{r}{r_a},$$

where r and r_a are the radii of circles ω and Ω, respectively. Note that

$$r(AB + BC + CA) = 2[ABC]$$

and that

$$\begin{aligned} r_a(AB + AC - BC) &= 2[I_aAB] + 2[I_aAC] - 2[I_aBC] \\ &= 2[I_aBAC] - 2[I_aBC] = 2[ABC], \end{aligned}$$

where I_a is the center of Ω and $[\mathcal{R}]$ is the area of region $\mathcal{R}$. Thus,

$$\frac{AQ}{AD_2} = \frac{AB + AC - BC}{AB + BC + CA}. \tag{5}$$

Applying the **Menelaus's Theorem** to triangle AD_2C and line BE_2 gives

$$\frac{AP}{PD_2} \cdot \frac{D_2B}{BC} \cdot \frac{CE_2}{E_2A} = 1,$$

or

$$\begin{aligned}\frac{AP}{PD_2} &= \frac{BC \cdot E_2A}{D_2B \cdot CE_2} = \frac{BC \cdot CE_1}{CD_1 \cdot AE_1} \\ &= \frac{BC}{AE_1} = \frac{2BC}{AB + AC - BC}.\end{aligned}$$

Hence,

$$\frac{AD_2}{PD_2} = 1 + \frac{AP}{PD_2} = \frac{AB + AC + BC}{AB + AC - BC},$$

or

$$\frac{PD_2}{AD_2} = \frac{AB + AC - BC}{AB + AC + BC}. \tag{6}$$

The desired result now follows from (5) and (6).

Problem 3 Let a, b, and c be nonnegative real numbers such that

$$a^2 + b^2 + c^2 + abc = 4.$$

Prove that

$$0 \le ab + bc + ca - abc \le 2.$$

First Solution: From the condition, at least one of a, b, and c does not exceed 1, say $a \le 1$. Then

$$ab + bc + ca - abc = a(b + c) + bc(1 - a) \ge 0.$$

To obtain equality, we have $a(b + c) = bc(1 - a) = 0$. If $a = 1$, then $b + c = 0$ or $b = c = 0$, which contradicts the given condition $a^2 + b^2 + c^2 + abc = 4$. Hence $1 - a \neq 0$ and only one of b and c is 0. Without loss of generality, say $b = 0$. Therefore $b + c > 0$ and $a = 0$. Plugging $a = b = 0$ back into the given condition gives $c = 2$. By permutation, the lower bound holds if and only if (a, b, c) is one of the triples $(2, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$.

Now we prove the upper bound. Let us note that at least two of the three numbers a, b, and c are both greater than or equal to 1 or less than or equal to 1. Without loss of generality, we assume that the numbers with this property are b and c. Then we have

$$(1 - b)(1 - c) \ge 0. \tag{1}$$

The given equality $a^2+b^2+c^2+abc=4$ and the inequality $b^2+c^2 \geq 2bc$ imply

$$a^2+2bc+abc \leq 4, \quad \text{or} \quad bc(2+a) \leq 4-a^2.$$

Dividing both sides of the last inequality by $2+a$ yields

$$bc \leq 2-a. \tag{2}$$

Combining (1) and (2) gives

$$\begin{aligned} ab+bc+ac-abc &\leq ab+2-a+ac(1-b) \\ &= 2-a(1+bc-b-c) \\ &= 2-a(1-b)(1-c) \leq 2, \end{aligned}$$

as desired.

The last equality holds if and only if $b=c$ and $a(1-b)(1-c)=0$. Hence, equality for the upper bound holds if and only if (a,b,c) is one of the triples $(1,1,1)$, $(0,\sqrt{2},\sqrt{2})$, $(\sqrt{2},0,\sqrt{2})$, and $(\sqrt{2},\sqrt{2},0)$.

Second Solution: We prove only the upper bound here. Either two of a,b,c are less than or equal to 1, or two are greater than or equal to 1. Assume b and c have this property. Then

$$b+c-bc = 1-(1-b)(1-c) \leq 1. \tag{3}$$

Viewing the given equality as a quadratic equation in a and solving for a yields

$$a = \frac{-bc \pm \sqrt{(b^2-4)(c^2-4)}}{2}.$$

Note that

$$\begin{aligned} (b^2-4)(c^2-4) &= b^2c^2-4(b^2+c^2)+16 \\ &\leq b^2c^2-8bc+16 = (4-bc)^2. \end{aligned}$$

For the given equality to hold, we must have $b,c \leq 2$ so that $4-bc \geq 0$. Hence,

$$a \leq \frac{-bc+|4-bc|}{2} = \frac{-bc+4-bc}{2} = 2-bc,$$

or

$$2-bc \geq a. \tag{4}$$

Combining (3) and (4) gives

$$2 - bc \geq a(b + c - bc) = ab + ac - abc,$$

or

$$ab + ac + bc - abc \leq 2,$$

as desired.

Third Solution: We prove only the upper bound here. Define functions f, g as

$$f(x, y, z) = x^2 + y^2 + z^2 + xyz = (x + y)^2 + z^2 - (2 - z)xy,$$
$$g(x, y, z) = xy + yz + zx - xyz = z(x + y) + (1 - z)xy$$

for all nonnegative numbers x, y, z. Observe that if $z \leq 1$, then both f and g are unbounded, increasing functions of x and y.

Assume that $f(a, b, c) = 4$ and, without loss of generality, that $a \geq b \geq c \geq 0$. Then $c \leq 1$.

Let $a' = (a + b)/2$. Because $a + b = a' + a'$ and $ab \leq \left(\frac{a-b}{2}\right)^2 + ab = a'^2$, we have

$$f(a', a', c) \leq f(a, b, c) = 4 \quad \text{and} \quad g(a', a', c) \geq g(a, b, c).$$

Now increase a' to $e \geq 0$ such that $f(e, e, c) = 4$. Note that $g(e, e, c) \geq g(a', a', c)$. It suffices to prove that $g(e, e, c) \leq 2$.

Since $f(e, e, c) = 2e^2 + c^2 + e^2 c = 4$, $e^2 = (4 - c^2)/(2 + c) = 2 - c$. We obtain that

$$\begin{aligned} g(e, e, c) &= 2ec + (1 - c)e^2 \leq e^2 + c^2 + (1 - c)e^2 \\ &= (2 - c)e^2 + c^2 = (2 - c)^2 + c^2 \\ &= 2(2 - 2c + c^2) = 2[1 + (1 - c)^2] \leq 2, \end{aligned}$$

as desired.

Problem 4 Let P be a point in the plane of triangle ABC such that there exists an obtuse triangle whose sides are congruent to $\overline{PA}$, $\overline{PB}$, and $\overline{PC}$. Assume that in this triangle the obtuse angle opposes the side congruent to $\overline{PA}$. Prove that angle BAC is acute.

Solution: By the **Cauchy-Schwarz Inequality**,

$$\sqrt{PB^2 + PC^2}\sqrt{AB^2 + AC^2} \geq PB \cdot AC + PC \cdot AB.$$

Applying the **(Generalized Ptolemy's Inequality** to quadrilateral $ABPC$ yields

$$PB \cdot AC + PC \cdot AB \geq PA \cdot BC.$$

Because PA is the longest side of an obtuse triangle with side lengths PA, PB, PC, we have $PA > \sqrt{PB^2 + PC^2}$ and hence

$$PA \cdot BC \geq \sqrt{PB^2 + PC^2} \cdot BC.$$

Combining these three inequalities yields $\sqrt{AB^2 + AC^2} > BC$, implying that angle BAC is acute.

Problem 5 Let S be a set of integers (not necessarily positive) such that

(a) there exist $a, b \in S$ with $\gcd(a, b) = \gcd(a-2, b-2) = 1$;

(b) if x and y are elements of S (possibly equal), then $x^2 - y$ also belongs to S.

Prove that S is the set of all integers.

First Solution: In the solution below we use the expression S *is stable under* $x \mapsto f(x)$ to mean that if t belongs to S, then $f(t)$ also belongs to S. If $c, d \in S$, then by condition (b), S is stable under $x \mapsto c^2 - x$ and $x \mapsto d^2 - x$. Hence, it is stable under $x \mapsto c^2 - (d^2 - x) = x + (c^2 - d^2)$. Similarly, S is stable under $x \mapsto x + (d^2 - c^2)$. Hence, S is stable under $x \mapsto x + n$ and $x \mapsto x - n$, whenever n is an integer linear combination of finitely many numbers in $T = \{\, c^2 - d^2 \mid c, d \in S \,\}$.

By condition (a), $S \neq \emptyset$ and hence $T \neq \emptyset$ as well. For the sake of contradiction, assume that some p divides every element in T. Then $c^2 - d^2 \equiv 0 \pmod{p}$ for all $c, d \in S$. In other words, for each $c, d \in S$, either $d \equiv c \pmod{p}$ or $d \equiv -c \pmod{p}$. Given $c \in S$, $c^2 - c \in S$, by condition (b), so $c^2 - c \equiv c \pmod{p}$ or $c^2 - c \equiv -c \pmod{p}$. Hence,

$$c \equiv 0 \,(\text{mod } p) \text{ or } c \equiv 2 \,(\text{mod } p) \qquad (*)$$

for each $c \in S$. By condition (a), there exist some a and b in S such that $\gcd(a, b) = 1$, that is, at least one of a or b cannot be divisible by p. Denote such an element of S by α; thus, $\alpha \not\equiv 0 \pmod{p}$. Similarly, by condition (a), $\gcd(a-2, b-2) = 1$, so p cannot divide both $a-2$ and $b-2$. Thus, there is an element of S, call it β, such that $\beta \not\equiv 2 \pmod{p}$. By $(*)$, $\alpha \equiv 2 \pmod{p}$ and $\beta \equiv 0 \pmod{p}$. By condition (b), $\beta^2 - \alpha \in S$. Taking $c = \beta^2 - \alpha$ in $(*)$ yields either $-2 \equiv 0 \pmod{p}$ or $-2 \equiv 2 \pmod{p}$, so $p = 2$. Now $(*)$ says that all elements of S are even, contradicting condition (a). Hence, our assumption is false and no prime divides every element in T.

It follows that $T \neq \{0\}$. Let x be an arbitrary nonzero element of T. For each prime divisor of x, there exists an element in T which is not divisible by that prime. The set A consisting of x and each of these elements is finite. By construction, $m = \gcd\{y \mid y \in A\} = 1$, and m can be written as an integer linear combination of finitely many elements in A and hence in T. Therefore, S is stable under $x \mapsto x+1$ and $x \mapsto x-1$. Because S is nonempty, it follows that S is the set of all integers.

Second Solution: Define T, a, and b as in the first solution. We present another proof that no prime divides every element in T. Suppose, for sake of contradiction, that such a prime p does exist. By condition (b), $a^2 - a, b^2 - b \in S$. Therefore, p divides $a^2 - b^2$, $x_1 = (a^2 - a)^2 - a^2$, and $x_2 = (b^2 - b)^2 - b^2$. Because $\gcd(a, b) = 1$, both $\gcd(a^2 - b^2, a^3)$ and $\gcd(a^2 - b^2, b^3)$ equal 1, so p does not divide a^3 or b^3. But p does divide $x_1 = a^3(a-2)$ and $x_2 = b^3(b-2)$, so it must divide $a - 2$ and $b - 2$. Because $\gcd(a - 2, b - 2) = 1$ by condition (a), this implies $p \mid 1$, a contradiction. Therefore our original assumption was false, and no such p exists.

Problem 6 Each point in the plane is assigned a real number such that, for any triangle, the number at the center of its inscribed circle is equal to the arithmetic mean of the three numbers at its vertices. Prove that all points in the plane are assigned the same number.

Solution: Let each lowercase letter denote the number assigned to the point labeled with the corresponding uppercase letter. Let A, B be arbitrary distinct points, and consider a regular hexagon $ABCDEF$ in the plane. Let lines CD and FE intersect at G. Let ℓ be the line through G perpendicular to line ED. Then A, F, E and B, C, D are symmetric to each other, respectively, with respect to line ℓ. Hence triangles CEG and DFG share the same incenter, i.e., $c + e = d + f$; triangles ACE and BDF share the same incenter, i.e., $a + c + e = b + d + f$. Therefore, $a = b$, and we are done.

1.16 Vietnam

Problem 1 The sequence of integers $a_0, a_1, \ldots$ is defined recursively by the initial condition $a_0 = 1$ and the recursive relation $a_n = a_{n-1} + a_{\lfloor n/3 \rfloor}$ for all integers $n \geq 1$. (Here, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.) Prove that for every prime number $p \leq 13$, there exists an infinite number of natural numbers k such that a_k is divisible by p.

Solution: We proceed by way of contradiction: Suppose that for some prime $p \leq 13$, there were only a finite number of values of n with $p \mid a_n$. There is at least one positive n with $p \mid a_n$, as straightforward computation confirms:

$$\begin{array}{lll} 2 \mid a_6 = 12, & 3 \mid a_2 = 3, & 5 \mid a_3 = 5, \\ 7 \mid a_4 = 7, & 11 \mid a_{11} = 23, & 13 \mid a_{20} = 117. \end{array}$$

Thus, the set of a_n such that $p \mid a_n$ is nonempty and contains an element greater than 2. Because we assumed that this set is finite, it must have a greatest element a_m. Then for all $n > m$, p does not divide a_n.

By the recurrence for a_n, we have $a_i = a_{i-1} + a_m$ for $3m \leq i \leq 3m+2$. Because $p \mid a_m$, we have $a_i \equiv a_{i-1} \pmod{p}$ for such i, i.e.,

$$a_{3m-1} \equiv a_{3m} \equiv a_{3m+1} \equiv a_{3m+2} \pmod{p}.$$

By the maximal definition of m, $k = a_{3m}$ is not congruent to 0 modulo p.

Now, if $9m - 3 \leq i \leq 9m + 8$, then $3m - 1 \leq \lfloor i/3 \rfloor \leq 3m + 2$, and so by the definition of a_n,

$$a_i - a_{i-1} = a_{\lfloor i/3 \rfloor} \equiv k \pmod{p}.$$

Thus,

$$a_{9m-4+j} \equiv a_{9m+4} + jk \pmod{p} \tag{$*$}$$

for $0 \leq j \leq 13$. However, because p does not divide k, there exists a j_0 with $0 \leq j \leq p$ such that $j_0 k \equiv -a_{9m+4} \pmod{p}$. Then $j_0 \leq p \leq 13$, and substituting $j = j_0$ into ($*$) gives

$$a_{9m-4+j_0} \equiv a_{9m+4} + (-a_{9m+4}) \equiv 0 \pmod{p}.$$

Thus, there exists an $n > m$ with $a_n \equiv 0 \pmod{p}$, a contradiction. Therefore, our original assumption was false, and there must be infinitely many n with $p \mid a_n$.

Problem 2 In the plane, two circles intersect at A and B, and a common tangent intersects the circles at P and Q. Let the tangents at P and Q to the circumcircle of triangle APQ intersect at S, and let H be the reflection of B across line PQ. Prove that the points A, S, and H are collinear.

First Solution: All angles are directed modulo 2π except where otherwise indicated. We will perform calculations in terms of the following four angles:

$$\angle APQ = x, \quad \angle PQA = y, \quad \angle PAB = m, \quad \angle BAQ = n.$$

Some of these calculations depend on whether (i) line PQ is closer to A, or (ii) line PQ is closer to B. Many angles in the diagram can be parameterized using just these four angles, and there are many triples of concurrent lines in the diagram, suggesting that there is a relatively straightforward solution involving applications of the trigonometric form of Ceva's Theorem.

First we prove that $\frac{\sin m}{\sin n} = \frac{\sin x}{\sin y}$. (Incidentally, with a bit of effort this equation gives $PA/PB = QA/QB$.) In triangle AQB, cevians $\overline{AP}$, $\overline{QP}$, $\overline{BP}$ concur at P. By the trigonometric form of Ceva's Theorem,

$$1 = \frac{\sin \angle BQP}{\sin \angle PQA} \frac{\sin \angle QAP}{\sin \angle PAB} \frac{\sin \angle ABP}{\sin \angle PBQ}.$$

Suppose that (i) holds. Because line PQ is tangent to the circumcircles of triangles ABP and ABQ, we have $\angle ABP = x$, $\angle QBA = y$, $\angle QPB = \pi - m$, and $\angle BQP = \pi - n$. Also, $\angle QAP = \pi - x - y$ and $\angle QBP = x + y$. Hence, the above equation then becomes

$$1 = \frac{-\sin n}{\sin y} \frac{\sin(\pi - x - y)}{\sin m} \frac{\sin x}{-\sin(x+y)},$$

or $\frac{\sin m}{\sin n} = \frac{\sin x}{\sin y}$. We get the same final result in case (ii), although the angles are different: $\angle ABP = \pi - x, \angle QBA = \pi - y, \angle BPQ = m, \angle PQB = n, \angle QAP = \pi - x - y, \angle QBP = x + y$.

We now quickly calculate $\frac{\sin \angle QAH}{\sin \angle PQB}$. Using directed angles modulo π temporarily, note that by the definition of H, $\angle QPH = \angle BPQ$, or $\angle QPH = \pi - m$. Similarly, $\angle HQP = \angle PQB = \pi - n$. Also, note that

$$\angle PHQ = \angle QBP = \angle QBA + \angle ABP = \angle PQA + \angle APQ = \pi - \angle QAP.$$

Hence, quadrilateral $APHQ$ is cyclic. Using directed angles modulo 2π again, because quadrilateral $APHQ$ is convex in that order, we have $\angle QAH = \angle QPH = \angle BPQ$ and $\angle HAP = \angle HQP = \angle PQB$. Earlier we showed that $(\angle QPB, \angle BQP) = (\pi - m, \pi - n)$ in case (i) and that $(\angle BPQ, \angle PQB) = (m, n)$ in case (ii). Hence, in either case, $\frac{\sin \angle QAH}{\sin \angle HAP} = \frac{\sin m}{\sin n}$.

Now, look at triangle PQA and cevians PS, QS, and AH. We claim they concur. By the trigonometric form of Ceva's theorem, this is true if

$$1 = \frac{\sin \angle APS}{\sin \angle SPQ} \frac{\sin \angle PQS}{\sin \angle SQA} \frac{\sin \angle QAH}{\sin \angle HAP}.$$

Substituting in various angle measures and $\frac{\sin \angle QAH}{\sin \angle HAP} = \frac{\sin m}{\sin n}$, this latter equation holds if

$$1 = \frac{\sin(-y)}{\sin(x+y)} \frac{\sin(x+y)}{\sin(-x)} \frac{\sin m}{\sin n}$$

Indeed, the right hand side of this last equation simplifies to

$$\frac{\sin y}{\sin x} \frac{\sin m}{\sin n},$$

which (as we previously showed) equals 1.

Therefore, lines PS, QS, and AH concur. The first two intersect at S, implying that S lies on line AH, as desired.

Second Solution: All angles are directed modulo π. Using the fact that line PQ is tangent to the circumcircles of triangles ABP and ABQ, we have

$$\angle QHP = \angle PBQ = \angle PBA + \angle ABQ$$
$$= \angle QPA + \angle PQA = \pi - \angle PAQ.$$

Hence, A, P, Q, H are concyclic.

Because $\overline{SP}$ and $\overline{SQ}$ are tangent to the circumcircle of triangle APQ, we have $SP = SQ$ and $\angle SPQ = \angle PQS$.

Now, we perform an inversion with center P and arbitrary radius. Let A', B', Q', S', and H' be the images of A, B, Q, S, and H, respectively, under the given inversion. We wish to show that quadrilateral $A'S'H'P$ is cyclic.

Because A, P, Q, H are concyclic, (i) A', Q', H' are collinear. Next, because line PS is tangent to the circumcircle of triangle APQ, (ii) line PS' is parallel to line $Q'A'$. And because $\angle SPQ = \angle PQS$, we have $\angle S'PQ' = \angle Q'S'P$, or (iii) $Q'P = Q'S$.

We now prove that (iv) $Q'A' = Q'H'$. Because line PQ is tangent to the circumcircle of triangle ABP, line PQ' is parallel to line $A'B'$, and $\angle B'A'Q' = \angle PQ'A'$. Also, because line PQ' is tangent to the circumcircle of triangle $Q'A'B$', we have $\angle PQ'A' = \angle Q'B'A'$. Thus, $\angle B'A'Q' = \angle Q'B'A'$, or $A'Q' = B'Q'$. (Incidentally, $A'Q'/B'Q' = \frac{QA}{QB} \cdot \frac{PB}{AB}$, so this implies that $\frac{PA}{PB} = \frac{QA}{QB}$ — making the result $A'Q' = B'Q'$ analogous to the result $\frac{\sin x}{\sin y} = \frac{\sin m}{\sin n}$ in the first solution.) Because B and H are reflections of each other across line PQ, B' and H' are reflections of each other across line PQ'. Therefore, $A'Q' = B'Q' = H'Q'$, as desired.

By (i) and (ii), $\overline{PS'} \parallel \overline{H'A'}$, and quadrilateral $PH'A'S'$ is a trapezoid. By (iii) and (iv), Q' lies on the perpendicular bisectors of the parallel sides $\overline{PS'}$, $\overline{H'A'}$, so in fact these sides have the same perpendicular bisector ℓ. Then trapezoid $PH'A'S'$ is symmetric about ℓ, and $PH' = SA'$, $PA' = S'H'$. Thus, quadrilateral $PH'A'S'$ is an isosceles trapezoid and therefore cyclic.

Inverting again, it follows that H, A, S are collinear, as desired.

Problem 3 A club has 42 members. Among each group of 31 members, there is at least one pair of participants — one male, one female — who know each other. (Person A knows person B if and only if person B knows person A.) Prove that there exist 12 distinct males $a_1, \ldots, a_{12}$ and 12 distinct females $b_1, \ldots, b_{12}$ such that a_i knows b_i for all i.

Solution: Let A be the set of all males in the club and B be the set of females in the club, with $|A| + |B| = 42$. By the given condition, there is no group of 31 members of the same sex, so $|A|, |B| \leq 30$.

We claim that for each nonempty $S \subseteq A$, at least $|S| + 12 - |A|$ females know at least one male in S. Otherwise, the set T of females who do *not* know any male in S, has more than $|B| - (|S| + 12 - |A|) = 30 - |S|$ females. But then $S \cup T$ has at least 31 members, among whom no male knows a female, a contradiction.

Define G to be a bipartite graph whose vertices are the members of the club, together with a set C of $|A| - 12$ additional females who know every single male in A; a vertex $a \in A$ is adjacent to $b \in B \cup C$

if a and b know each other. For every nonempty $S \subseteq A$, at least $(|S| + 12 - |A|) + (|A| - 12) = |S|$ vertices in $B \cup C$ are adjacent to at least one vertex in S. Thus, by the Marriage Lemma, we can choose disjoint edges (a_1, b_1), (a_2, b_2), $\ldots$, $(a_{|A|}, b_{|A|})$ in $A \times (B \cup C)$. At most $|A| - 12$ of these edges connects a male in A to a female in C, so at least twelve of them connect a male in A to a female in B. Without loss of generality, a_i knows b_i for $1 \le i \le 12$, as desired.

Problem 4 The positive real numbers a, b, and c satisfy the condition $21ab + 2bc + 8ca \le 12$. Find the least possible value of the expression $\frac{1}{a} + \frac{2}{b} + \frac{3}{c}$.

Solution: We claim that

$$(x + 2y + 3z)^2(2x + 8y + 21z) \ge 675xyz \qquad (*)$$

for all positive x, y, z, with equality when $10x = 24y = 45z$. In particular, this holds for $(x, y, z) = (bc, ca, ab)$, in which case

$$\frac{1}{a} + \frac{2}{b} + \frac{3}{c} = \frac{x + 2y + 3z}{\sqrt{xyz}} \ge \frac{\sqrt{675}}{\sqrt{2x + 8y + 21z}} \ge \frac{\sqrt{675}}{\sqrt{12}} = \frac{15}{2}.$$

Equality can be achieved: when $a = \frac{1}{3}$, $b = \frac{4}{5}$, $c = \frac{3}{2}$, we have $21ab + 2bc + 8ca = 12$ and $\frac{1}{a} + \frac{2}{b} + \frac{3}{c} = \frac{15}{2}$.

To prove $(*)$, let (α, β, γ) and (A, B, C) be triples of positive numbers with sum 1, to be determined more precisely later. By the weighted arithmetic mean-geometric mean inequality, we have

$$\begin{aligned} x + 2y + 3z &= \alpha\frac{x}{\alpha} + \beta\frac{2y}{\beta} + \gamma\frac{3z}{\gamma} \\ &\ge \left(\frac{x}{\alpha}\right)^{\alpha}\left(\frac{2y}{\beta}\right)^{\beta}\left(\frac{3z}{\gamma}\right)^{\gamma} = \frac{1^{\alpha}2^{\beta}3^{\gamma}}{\alpha^{\alpha}\beta^{\beta}\gamma^{\gamma}}x^{\alpha}y^{\beta}z^{\gamma}, \end{aligned}$$

with equality when $\frac{x}{\alpha} = \frac{2y}{\beta} = \frac{3z}{\gamma}$. Similarly,

$$2x + 8y + 21z \ge \frac{2^A 8^B 21^C}{A^A B^B C^C}x^A y^B z^C,$$

with equality when $\frac{2x}{A} = \frac{8y}{B} = \frac{21z}{C}$.

Therefore,

$$\begin{aligned} &(x + 2y + 3z)^2(2x + 8y + 21z) \\ &\quad \ge \left(\left(\tfrac{2^{\beta}3^{\gamma}}{\alpha^{\alpha}\beta^{\beta}\gamma^{\gamma}}\right)^2 \tfrac{2^A 8^B 21^C}{A^A B^B C^C}\right) x^{2\alpha+A}y^{2\beta+B}z^{2\gamma+C}. \end{aligned} \qquad (\dagger)$$

We now find (α, β, γ) and (A, B, C) such that:

- the equality conditions $\frac{x}{\alpha} = \frac{2y}{\beta} = \frac{3z}{\gamma}$ and $\frac{2x}{A} = \frac{8y}{B} = \frac{21z}{C}$ are the same;
- the exponents $2\alpha + A$, $2\beta + B$, $2\gamma + C$ in (†) all equal 1.

In fact, it is easy to verify that $(\alpha, \beta, \gamma) = (2/5, 1/3, 4/15)$ and $(A, B, C) = (1/5, 1/3, 7/15)$ satisfy these conditions.

By our choice of (α, β, γ) and (A, B, C), equality in (†) holds when $10x = 24y = 45z$. In particular, equality holds when $(x, y, z) = (36, 15, 8)$. For this (x, y, z), we have

$$(x + 2y + 3z)^2(2x + 8y + 21z) = 2916000 = 675 \cdot 4320 = 675xyz.$$

Thus, the coefficient of xyz in (†) equals 675, proving $(*)$. This completes the proof.

Note: The values for (α, β, γ) and (A, B, C) do not appear from nowhere. Although unnecessary for presenting the solution, a method of determining these values is certainly necessary for finding the solution in the first place. Using the two conditions on (α, β, γ) and (A, B, C), it is easy to show that

$$\beta = \frac{\alpha}{2 - 2\alpha}, \quad \gamma = \frac{2\alpha}{7 - 10\alpha}.$$

The equation $\alpha + \beta + \gamma = 1$ thus becomes

$$\alpha + \frac{\alpha}{2 - 2\alpha} + \frac{2\alpha}{7 - 10\alpha} = 1.$$

Clearing denominators and simplifying yields

$$20\alpha^3 - 68\alpha^2 + 59\alpha - 14 = 0.$$

If we are optimistic, then we hope that this equation has a rational solution $\alpha = \frac{p}{q} \in (0, 1)$. It is well-known that any such root satisfies $p \mid 14$ and $q \mid 20$. Trial and error then yields that $\alpha = \frac{2}{5}$ is a solution, and the values of β, γ, A, B, C follow easily. (There are two other roots to the above cubic polynomial, namely $\alpha = \frac{3 \pm \sqrt{2}}{2}$, but our proof requires only one value α that yields $0 < \alpha, \beta, \gamma, A, B, C < 1$; in fact, the logic of our proof guarantees that only one such α exists.)

Problem 5 Let $n > 1$ be an integer, and let T be the set of points (x, y, z) in three-dimensional space such that x, y, and z are integers between 1 and n, inclusive. We color the points in T so that if $x_0 \leq x_1$,

$y_0 \le y_1$, and $z_0 \le z_1$, then (x_0, y_0, z_0) and (x_1, y_1, z_1) are either equal or not both colored. At most how many points in T can be colored?

Solution: The answer is $\lceil \frac{3n^2}{4} \rceil$, or equivalently $3k^2$ if $n = 2k$ and $3k^2 + 3k + 1$ if $n = 2k + 1$.

Given $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$, we write $a \prec b$ if $a \neq b$ and $a_i \le b_i$ for $1 \le i \le n$. We wish to find the size of the largest possible subset S of T such that $a \not\prec b$ for all $a, b \in S$.

We will first show that at most $\lfloor \frac{3n^2}{4} \rfloor$ points of T can be colored. For $1 \le m \le n$, we define the "m^{th} shell" T_m to be the points (x, y, z) with the following property: (x, y) coincides with, lies due south of, or lies due east of $(n + 1 - m, m)$ in the xy-plane.

Lemma. *If $S \subseteq T$ such that $a \not\prec b$ for all $a, b \in S$, then $|S \cap T_m| \le \min\{2m - 1, n\}$ for all m.*

Proof. Suppose that $|S \cap T_m| > 2m - 1$. For $(x, y, z) \in T_m$, there are only $2m - 1$ possible values of (x, y): namely, $(n + 1 - m, 1), (m, 2), \dots, (n + 1 - m, m)$ and $(n + 2 - m, m), (n + 3 - m, m), \dots, (n, m)$. Hence, two distinct points $a, b \in S$ have the same x- and y- coordinates, implying that either $a \prec b$ or $b \prec a$, a contradiction.

Instead suppose that $|S \cap T_m| > n$. For $(x, y, z) \in T_m$, there are only n possible values of z. Hence, two distinct points $a = (x_1, y_1, z_1)$ and $b = (x_2, y_2, z_2)$ in S have the same z-coordinate $z_1 = z_2$. Because a and b lie in the same shell, it is easy to show that either (i) $x_1 \le x_2$ and $y_1 \le y_2$; or (ii) $x_2 \le x_1$ and $y_2 \le y_1$. In either case, $a \prec b$ or $b \prec a$, a contradiction.

Hence, $|S \cap T_m|$ is less than or equal to both $2m - 1$ and n, as claimed. $\square$

Note that the T_m partition T: given $(x, y, z) \in T$, it either lies in T_{n+1-x} (if $x \le y$) or T_y (if $x \ge y$). Hence, $|S| = \sum_{m=1}^{n} |S \cap T_m|$.

Applying the lemma, we thus have

$$\begin{aligned}|S| &= \sum_{m=1}^{\lfloor n/2\rfloor} |S\cap T_m| + \sum_{m=\lfloor n/2\rfloor+1}^{n} |S\cap T_m| \\ &\le \sum_{m=1}^{\lfloor n/2\rfloor} (2m-1) + \sum_{m=\lfloor n/2\rfloor+1}^{n} n \\ &= \lfloor n/2\rfloor^2 + n(n-\lfloor n/2\rfloor).\end{aligned}$$

If $n = 2k$, then this upper bound equals $k^2 + 2k(k) = 3k^2 = \lceil \frac{3n^2}{4}\rceil$; if $n = 2k+1$, then this upper bound equals $k^2 + (2k+1)(k+1) = 3k^2+3k+1 = \lceil \frac{3n^2}{4}\rceil$. Hence, $|S| \le \lceil \frac{3n^2}{4}\rceil$, as claimed.

Now we construct a subset S of T with $|S| = \lceil \frac{3n^2}{4}\rceil$ such that $a \not\prec b$ for all $a, b \in S$. Namely, we let $S = \{(x,y,z) \in T \mid x+y+z = \lfloor \frac{3n+3}{2}\rfloor\}$. If $(x_1,y_1,z_1) \prec (x_2,y_2,z_2)$ lie in S, then $x_1 \le x_2$, $y_1 \le y_2$, $z_1 \le z_2$. But equality must hold in these three inequalities because $x_1+y_1+z_1 = x_2+y_2+z_2$. Thus, $(x_1,y_1,z_1) = (x_2,y_2,z_2)$, a contradiction. Thus, S has the required property.

Next, we prove that $|S| = \lceil \frac{3n^2}{4}\rceil$. For any positive integer t, we claim that the number of triples of positive integers (x,y,z) for which $x+y+z = t$ is $\frac{(t-1)(t-2)}{2}$. This is clearly true if $t = 1$ or 2; otherwise, for each value of $z = 1, 2, \ldots, t-2$, there are $t-1-z$ pairs (x,y) such that $x+y = t-z$, for a total of $(t-2)+(t-3)+\cdots+1 = \frac{(t-1)(t-2)}{2}$.

Thus, there are

$$\frac{(\lfloor \frac{3n+3}{2}\rfloor - 1)(\lfloor \frac{3n+3}{2}\rfloor - 2)}{2} \qquad (*)$$

triples of positive integers (x,y,z) with $x+y+z = \lfloor \frac{3n+3}{2}\rfloor$.

Of such triples, those with $x > n$ are in one-to-one correspondence with triples (x',y,z) of positive integers with $x'+y+z = \lfloor \frac{3n+3}{2}\rfloor - n = \lfloor \frac{n+3}{2}\rfloor$: simply set $x' = x-n$. Hence, among the triples counted in $(*)$, exactly

$$\frac{(\lfloor \frac{n+3}{2}\rfloor - 1)(\lfloor \frac{n+3}{2}\rfloor - 2)}{2}$$

have $x > n$. The same count holds given either $y > n$ or $z > n$. Furthermore, no triple counted in $(*)$ has two of x, y, z greater than

n, because $\lfloor \frac{3n+3}{2} \rfloor < 2n$. Thus,

$$3\frac{(\lfloor \frac{n+3}{2} \rfloor - 1)(\lfloor \frac{n+3}{2} \rfloor - 2)}{2}$$

triples counted in $(*)$ are not in T.

Therefore, the number of triples $(x, y, z) \in T$ with $x+y+z = \lfloor \frac{3n+3}{2} \rfloor$ equals

$$\frac{(\lfloor \frac{3n+3}{2} \rfloor - 1)(\lfloor \frac{3n+3}{2} \rfloor - 2)}{2} - 3\frac{(\lfloor \frac{n+3}{2} \rfloor - 1)(\lfloor \frac{n+3}{2} \rfloor - 2)}{2}.$$

For $n = 2k$ this simplifies to $\frac{3k(3k-1)}{2} - \frac{3k(k-1)}{2} = 3k^2$, and for $n = 2k+1$ it simplifies to $\frac{(3k+2)(3k+1)}{2} - \frac{3(k+1)k}{2} = 3k^2 + 3k + 1$. Thus, $|S| = \lceil \frac{3n^2}{4} \rceil$, as claimed.

Problem 6 Let $a_1, a_2, \ldots$ be a sequence of positive integers satisfying the condition $0 < a_{n+1} - a_n \leq 2001$ for all integers $n \geq 1$. Prove that there exist an infinite number of ordered pairs (p, q) of distinct positive integers such that a_p is a divisor of a_q.

Solution: Consider all pairs (p, q) of distinct positive integers such that a_p is a divisor of a_q. Assume, by way of contradiction, that there exists a positive N such that $q < N$ for all such pairs.

We prove by induction on k that for each $k \geq 1$, there exist

- a finite set $S_k \subset \{a_N, a_{N+1}, \ldots\}$, and
- a set T_k of 2001 consecutive positive integers greater than or equal to a_N,

such that at least k elements of T_k are divisible by some element of S_k.

For $k = 1$, the sets $S_1 = \{a_N\}$ and $T_1 = \{a_N, a_{N+1}, \ldots, a_{N+2000}\}$ suffice.

Given S_k and T_k (with $k \geq 1$), define

$$T_{k+1} = \{t + \prod_{s \in S_k} s \mid t \in T_k\}.$$

T_{k+1}, like T_k, consists of 2001 consecutive positive integers greater than or equal to a_n — in fact, greater than or equal to $\max S_k$. Also, at least k elements of T_{k+1} are divisible by some element of S_k: namely, $t + \prod_{s \in S_k} s$ for each of the elements $t \in T_k$ which are divisible by some element of S_k.

By the given condition $0 < a_{n+1} - a_n \leq 2001$, and because the elements of T_{k+1} are greater than or equal to a_N, we have that $a_q \in T_{k+1}$ for some $q \geq N$. Because the elements of T_{k+1} are greater than $\max S_k$, we have $a_q \notin S_k$. Thus, by the definition of N, no element of S_k divides a_q.

Hence, at least $k+1$ elements of T_{k+1} are divisible by some element of $S_k \cup \{a_q\}$: at least k elements of T_{k+1} are divisible by some element of S_k, and in addition a_q is divisible by itself. Therefore, setting $S_{k+1} = S_k \cup \{a_q\}$ completes the inductive step.

Setting $k = 2002$, we have the absurd result that T_{2002} is a set of 2001 elements, at least 2002 of which are divisible by some element of S_{2002}. Therefore, our original assumption was false, and for each N there exists $q > N$ and $p \neq q$ such that $a_p \mid a_q$. It follows that there are infinitely many ordered pairs (p, q) with $p \neq q$ and $a_p \mid a_q$.

2
2001 Regional Contests: Problems and Solutions

2.1 Asian Pacific Mathematical Olympiad

Problem 1 For each positive integer n, let $S(n)$ be the sum of digits in the decimal representation of n. Any positive integer obtained by removing several (at least one) digits from the right-hand end of the decimal representation of n is called a *stump* of n. Let $T(n)$ be the sum of all stumps of n. Prove that $n = S(n) + 9T(n)$.

Solution: Let d_i be the digit associated with 10^i in the base-10 representation of n, so that $n = \overline{d_m d_{m-1} \ldots d_0}$ for some integer $m \geq 0$ (where $d_m \neq 0$). The stumps of n are $\sum_{j=k}^{m} d_j 10^{j-k}$ for $k = 1, 2, \ldots, m$, and their sum is

$$\begin{aligned} T(n) &= \sum_{k=1}^{m} \sum_{j=k}^{m} d_j 10^{j-k} = \sum_{j=1}^{m} d_j \sum_{k=1}^{j} 10^{j-k} \\ &= \sum_{j=1}^{m} d_j \sum_{k=0}^{j-1} 10^k = \sum_{j=1}^{m} d_j \frac{10^j - 1}{10 - 1}. \end{aligned}$$

Hence,

$$\begin{aligned} 9T(n) &= \sum_{j=1}^{m} d_j (10^j - 1) = \sum_{j=1}^{m} 10^j d_j - \sum_{j=1}^{m} d_j \\ &= \sum_{j=0}^{m} 10^j d_j - \sum_{j=0}^{m} d_j = n - S(n), \end{aligned}$$

as desired.

Problem 2 Find the largest positive integer N so that the number of integers in the set $\{1, 2, \ldots, N\}$ which are divisible by 3 is equal to the number of integers which are divisible by 5 or 7 (or both).

Solution: Answer: the largest such N is 65.

The number of positive integers less than or equal to n that are divisible by m is $\lfloor \frac{n}{m} \rfloor$. Thus, the number of integers less than or equal to n that are divisible by 7 but not 5 (i.e., divisible by 7 but not 35) is $\lfloor \frac{n}{7} \rfloor - \lfloor \frac{n}{35} \rfloor$; so, the number of integers less than or equal to n that are divisible by 5 or 7 (or both) is $\lfloor \frac{n}{7} \rfloor + \lfloor \frac{n}{5} \rfloor - \lfloor \frac{n}{35} \rfloor$.

Write
$$f(n) = \left\lfloor \frac{n}{3} \right\rfloor - \left\lfloor \frac{n}{5} \right\rfloor - \left\lfloor \frac{n}{7} \right\rfloor + \left\lfloor \frac{n}{35} \right\rfloor.$$
We seek the largest positive integer N such that $f(N) = 0$.

We claim that $f(n) > 0$ for $0 \le n < 70$. Observe that
$$\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{7} \right\rfloor \le \left\lfloor \frac{n}{5} + \frac{n}{7} \right\rfloor = \left\lfloor \frac{12n}{35} \right\rfloor = \left\lfloor \frac{n}{3} + \frac{n}{105} \right\rfloor.$$
For $0 \le n < 35$, the above inequality gives
$$\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{7} \right\rfloor \le \left\lfloor \frac{n}{3} + \frac{34}{105} \right\rfloor = \left\lfloor \frac{n}{3} \right\rfloor.$$
Hence, $f(n) \ge 0 + \lfloor \frac{n}{35} \rfloor = 0$. For $35 \le n < 70$, the above inequality gives $\lfloor \frac{n}{5} \rfloor + \lfloor \frac{n}{7} \rfloor \le \lfloor \frac{n}{3} \rfloor + 1$. Hence, $f(n) \ge -1 + \lfloor \frac{n}{35} \rfloor = 0$.

We also claim that $f(n) > 0$ for $n \ge 70$. Observe that
$$\begin{aligned} f(n+70) &= \left(23 + \left\lfloor \frac{n+1}{3} \right\rfloor\right) - \left(10 + \left\lfloor \frac{n}{7} \right\rfloor\right) \\ &\quad - \left(14 + \left\lfloor \frac{n}{5} \right\rfloor\right) + \left(2 + \left\lfloor \frac{n}{35} \right\rfloor\right) \\ &= 1 + \left\lfloor \frac{n+1}{3} \right\rfloor - \left\lfloor \frac{n}{5} \right\rfloor - \left\lfloor \frac{n}{7} \right\rfloor + \left\lfloor \frac{n}{35} \right\rfloor \\ &\ge f(n) + 1. \end{aligned}$$
Because $f(n) \ge 0$ for $0 \le n < 70$, it follows that $f(n) > 0$ for $n \ge 70$.

Finally, it is easy to calculate $f(69) = 2$, $f(68) = 1$, $f(67) = 1$, $f(66) = 1$, and $f(65) = 0$. Therefore, the largest N such that $f(N) = 0$ is $N = 65$.

Problem 3 Let two congruent regular n-sided ($n \ge 3$) polygonal regions S and T be located in the plane such that their intersection is a $2n$-sided polygonal region P. The sides of S are colored red and the sides of T are colored blue. Prove that the sum of the lengths of the blue sides of P is equal to the sum of the lengths of its red sides.

Solution: In this solution, all indices are taken modulo n.

Suppose we know that some (connected) polygonal regions S and T intersect in a nonempty (connected) polygonal region P. Any side of P must lie along a side of S or T.

Using the information that S and T each have n sides, we then know that every side of P must lie along one of the $2n$ segments that are sides of S or T. Furthermore, because P is convex (it is the intersection of two convex regions), no two sides of P lie along the same of these $2n$ segments. We are given that P has $2n$ sides, so it follows that every side of S or T contains a segment that is also a side of P. We let the vertices of P, S and T be, in counterclockwise order, $P_1, P_2, \ldots, P_{2n}$; $S_1, S_2, \ldots, S_{2n}$ and $T_1, T_2, \ldots, T_{2n}$ respectively.

Lemma. *No vertex of S or T is a vertex of P.*

Proof. We prove that no vertex of S is a vertex of P; an analogous proof shows that no vertex of T is a vertex of P. Suppose, for sake of contradiction, that one vertex — say, P_1 — of P is also a vertex of S. If a vertex of P is also a vertex of S, then each neighboring vertex of P either: (i) is also a vertex of S, or (ii) lies on the boundary of T. Suppose that $P_1, \ldots, P_k$ satisfy (i); because there are only n vertices of S, we have $k \leq n$. Then P_k satisfies (i), and P_{k+1} satisfies (ii). Relabelling, we can assume that P_1 satisfies (i) while P_2 satisfies (ii).

From P_2, we travel clockwise along the boundary of T until reaching a vertex of T — without loss of generality, T_1. □

No two adjacent sides of P can both lie on the boundary of S, because the vertex between these sides would then also be a vertex of S — contradicting the above Lemma. Likewise, no two adjacent sides of P can both lie on the boundary of T.

Hence, the sides of P alternate between portions of sides of S and portions of sides of T. We must alternate between the S_i and the T_i as we trace around P. Without loss of generality, the order is $S_1, T_1, S_2, T_2, \ldots, S_n, T_n$. Each side of P forms a triangle with one of the S_i or T_i, so let the side forming a triangle with S_i be s_i, and similarly for t_i. We wish to show the sum of the s_i equals the sum of the t_i. Now, in of these triangles the angle at S_i or T_i has measure $180 - 360/n$ degrees. Adjacent triangles have vertical angles which are thus equal, so all these triangles are similar. Because the triangles alternate in orientation, all those that have S_i as one of their points are oriented the same. Then the sides of such any such triangle are s_i, as_i and bs_i. Similarly the sides of triangles containing T_i are t_i, bt_i, and at_i. Then $S_iS_{i+1} = bs_i + t_i + as_{i+1}$ and $T_iT_{i+1} = at_i + s_{i+1} + at_{i+1}$. Then the perimeter of S is (taking indices of $n+1$ as 1)

$$\sum_{i=1}^{n} S_iS_{i+1} = \sum_{i=1}^{n} (bs_i + t_i + as_{i+1}) = \sum_{i=1}^{n} (bs_i + as_i + t_i)$$

While the perimeter of T is

$$\sum_{i=i}^{n} T_iT_{i+1} = \sum_{i=1}^{n} (at_i + s_{i+1} + bt_{i+1}) = \sum_{i=1}^{n} (at_i + bt_i + s_i)$$

Since these two values are equal, we have

$$0 = \sum_{i=1}^{n} ((a+b)t_i + s_i - (a+b)s_i - t_i) = \sum_{i=1}^{n} (a+b-1)(t_i - s_i)$$

We divide both sides by (a+b-1), which is not 0 by the triangle inequality, to obtain

$$\sum_{i=1}^{n} (t_i - s_i) = 0, \text{ or } \sum_{i=1}^{n} s_i = \sum_{i=1}^{n} t_i$$

as desired.

Problem 4 A point in the Cartesian coordinate plane is called a *mixed point* if one of its coordinates is rational and the other one is irrational. Find all polynomials with real coefficients such that their graphs do not contain any mixed point.

Solution: Answer: All (non-constant) linear polynomials with rational coefficients. From here, we call a polynomial *pure* if it has no mixed points.

Lemma. *If the polynomial $P(x)$ assumes rational values for infinitely many rational values x, then every coefficient of $P(x)$ is rational.*

Proof. Let n be the degree of $P(x)$ and $x_0, x_1, \ldots, x_n$, $y_0, y_1, \ldots, y_n$ be such rational numbers, that $P(x_i) = y_i$ for all i and all x_i are different. For $i = 0, 1, \ldots, n$ consider the following polynomials:

$$Q_i(x) = \frac{\prod_{j \neq i} (x - x_j)}{\prod_{j \neq i} (x_i - x_j)}$$

We have $Q_i(x_j) = 1$ if $j = i$, and $Q_i(x_j) = 0$ otherwise. Let $P'(x) = \sum_{i=0}^{n} y_i Q_i(x)$. Then, $P'(x_i) = y_i$ for $i = 0, 1, \ldots, n$ and $\deg P' = n$. We have $P(x) - P'(x) = 0$ for $n + 1$ different values of x. Since $P(x) - P'(x)$ has degree at most n, it must be 0. So $P(x) = P'(x)$. As constructed, all $Q_i(x)$ have rational coefficients. Hence, all coefficients of $P(x)$ are rational. $\square$

For sake of contradiction, assume that there exists at least one pure polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

with degree $n \geq 2$. By the Lemma, the a_k are rational. If the least common denominator of the a_k is the integer m, then $P(mx)$ and $-P(mx)$ are pure polynomials with integer coefficients; and at least one of $P(mx)$ and $-P(mx)$ has positive leading coefficient. Thus, we may assume without loss of generality that $a_n > 0$ and that the a_k are integers.

Let p be a prime so that $p \nmid a_n$. Let r be an integer large enough that there exists a positive x_0 such that $P(x_0) = \frac{pr+1}{p}$. (Because the leading coefficient of P is positive, such an r exists.) Because $P(x_0)$ is rational, so is x_0, and we may write $x_0 = \frac{s}{t}$ for relatively prime positive integers s, t. Then

$$\frac{pr+1}{p} = P(x_0) = \frac{a_n s^n + a_{n-1}s^{n-1}t + \cdots + a_0 t^n}{t^n},$$

or

$$(pr+1)t^n = p(a_n s^n + a_{n-1}s^{n-1}t + \cdots + a_0 t^n).$$

Because p divides the right hand side, it must divide the left hand side, implying that $p \mid t$. Furthermore, because $n \geq 2$, the left hand side is divisible by p^2. Thus, the right hand side is divisible by p^2, implying that

$$p \mid (a_n s^n + a_{n-1}s^{n-1}t + \cdots + a_0 t^n).$$

Because $p \mid t$, p divides $a_{n-1}s^{n-1}t + \cdots + a_0 t^n$. Thus the above relation implies

$$p \mid a_n s^n.$$

Because $p \mid t$ and $\gcd(s,t) = 1$, we have $p \nmid s$; and by the definition of p, we have $p \nmid a_n$. Hence, $p \nmid a_n s^n$, a contradiction. Thus, our original assumption was false, and there is no pure polynomial of degree greater than 1.

Therefore, every pure polynomial is of degree 0 or 1. Any constant polynomial has mixed points, and it is easy to check that any non-constant linear polynomial with rational coefficients has no mixed points.

Problem 5 Find the greatest integer n, such that there are $n+4$ points $A, B, C, D, X_1, \ldots, X_n$ in the plane with the following properties: the lengths AB and CD are distinct; and for each $i = 1, 2, \ldots, n$, triangles ABX_i and CDX_i are congruent (although not necessarily in that order).

Solution: Answer: $n = 4$. One example is

$$A = (-1, \sqrt{3}), B = (1, \sqrt{3}), C = (-2, 0), D = (2, 0),$$
$$X_1 = (-3, -\sqrt{3}), X_2 = (-1, -\sqrt{3}), X_3 = (1, -\sqrt{3}), X_4 = (3, -\sqrt{3}).$$

When we consider the possible orderings of the corresponding vertices of congruent triangles, there are six possible permutations of C, D, X to match with ABX. Of these, two require $AB = CD$: CDX and DCX. Thus, there are four we need to consider. We will show that there is only one point Z such that ABZ is congruent to CZD. The result will follow for the other three valid arrangements of C, D, Z, as they result from swapping A with B and/or C with D in the arrangement.

Suppose two points Z and Z' satisfy conditions $\triangle ABZ \cong \triangle CZD$, $\triangle ABZ' \cong \triangle CZ'D$. We have $CZ = AB, CD = AZ, BZ = ZD$, and likewise for Z'. Since $BZ = DZ$, Z is on the perpendicular bisector of BD, as is Z'. Draw a circle with center A and radius CD. That circle can intersect with the perpendicular bisector of $\overline{BD}$ in at most two points. Thus, these points are Z and Z'. The circle centered at C with radius AB also intersects the perpendicular bisector of $\overline{BD}$ at Z and Z'. Thus, A and C are both on the perpendicular bisector of $\overline{ZZ'}$, and lines AC and BD are parallel.

Reflect triangle CZD across line ZZ' to obtain triangle $C'ZD'$. Now, $B = D'$ because ZZ' is the perpendicular bisector of $\overline{BD}$. Thus, we have $\triangle CZD \cong \triangle C'ZB$. We also know that $\triangle CZD \cong \triangle ABZ$. Thus, $\triangle C'ZB \cong \triangle ABZ$. Draw altitudes from A and C' to line BZ; they must have the same length because $\overline{BZ}$ is the common base of two congruent triangles. Therefore, $BZ \parallel AC'$. However, AC was parallel to BD, and C' is on line AC and distinct from A (because $AB \neq CD$). Hence, AC', and consequently BZ, are also parallel to BD. So, Z must be the midpoint of BD. But the same holds for Z', so $Z' = Z$, a contradiction. Therefore, there is only one choice for Z per configuration, and we can have at most 4 points X_i.

2.2 Austrian-Polish Mathematics Competition

Problem 1 Let k be a fixed positive integer. Consider the sequence defined recursively by $a_0 = 1$ and

$$a_{n+1} = a_n + \lfloor \sqrt[k]{a_n} \rfloor$$

for $n = 0, 1, \ldots$. (Here, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.) For each k, find the set A_k consisting of all integers in the sequence $\sqrt[k]{a_0}, \sqrt[k]{a_1}, \ldots$.

Solution: We claim that A_k is the set of all powers of 2, regardless of k. $1 \in A_k$ because the first term of every sequence is one. Suppose that $n \in A_k$. Then there must exist i such that $a_i = n^k$. For all such j that $n^k \le a_j < (n+1)^k$, we have $a_{j+1} = a_j + n$. Therefore, for these j, $a_{j+1} \equiv a_j \pmod{n}$. Because congruence is transitive, it follows that $a_{j+1} \equiv 0 \pmod{n}$. Eventually, it must be the case that $a_{j+1} \ge (n+1)^k$ but $a_j < (n+1)^k$. Let $a_{j+1} = (n+1)^k + m_1$. We know that $0 \le m_1 < n$ because $a_j < (n+1)^k$. We have

$$0 \equiv a_{j+1} \equiv (n+1)^k + m_1 \equiv 1^k + m_1 \equiv m_1 + 1 \pmod{n}$$

Therefore, $m_1 = n - 1$. Similarly, we add $n+1$ until we obtain a term of the form $(n+2)^k + m_2$ with $0 \le m_2 < n+1$. Hence,

$$n - 1 \equiv (n+2)^k + m_2 \equiv 1^k + m_2 \equiv m_2 + 1 \pmod{n+1}$$

Then $m_2 = n - 2$. In general,

$$m_i \equiv (n+i+1)^k + m_{i+1} \equiv 1^k + m_{i+1} \pmod{n+i}.$$

Then $m_i = n - i$ for $0 < i \le n$. Therefore, m_n is the first m_i that equals zero, so $(n+n)^k = (2n)^k$ is the next k^{th} power in the set. By induction, A_k is the set of all powers of two.

Problem 2 Consider the set A of all positive integers n with the following properties: the decimal expansion contains no 0, and the sum of the (decimal) digits of n divides n.

(a) Prove that there exist infinitely many elements in A with the following property: the digits that appear in the decimal expansion of A appear the same number of times.

(b) Show that for each positive integer k, there exists an element in A with exactly k digits.

Solution:

(a) Let $n_1 = 3$ and $n_2 = 111$. Suppose that the decimal expansion of n_{i+1} contains n_i ones, and n_i divides n_{i+1}. We define n_{i+2} the following way: $n_{i+2} = n_{i+1} \sum_{j=0}^{n_{i+1}/n_i - 1} 10^{n_i j}$ Then n_{i+2} is the integer whose decimal expansion consists of n_{i+1} ones, and it is divisible by n_{i+1}. Therefore, the $n_i's$ are an infinite family of positive integers that satisfy the desired condition.

(b) We will need the following lemmas.

Lemma. *For every $d > 0$ there exists a d-digit number that contains only ones and twos in its decimal expansion and is a multiple of 2^d.*

Proof. We will prove the following statement: for every $l > 0$ there exists a positive integer divisible by 2^d whose l rightmost digits are only ones and twos. Proceed by induction on l. There exists such a, that $2^d a \equiv 2 \pmod{10}$, so for $l = 1$ the statement is true. Suppose it holds for l and let b be a multiple of 2^d satisfying the condition. Let c be the $(l+1)^{\text{st}}$ digit of b from the right side. There exists x such that $2^d x + c \equiv 1 \text{ or } 2 \pmod{10}$. Hence, $b + 2^d 10^l x$ has only ones and twos among its $l+1$ rightmost digits, completing the step of induction. Now, putting $l = d$, we obtain some multiple of 2^d, say N, such that all its $l+1$ rightmost digits are only ones and twos. Considering $N \bmod 10^d$ proves the original claim. $\square$

Lemma. *For each $k > 2$ there exists $d \leq k$ such that the following inequality holds: $k + d \leq 2^d \leq 9k - 8d$.*

Proof. For $3 \leq k \leq 5$, $d = 3$ satisfies the inequalities. For $5 \leq k \leq 10$, $d = 4$ satisfies the inequalities. We will show that $d = \lfloor \log_2 4k \rfloor$ satisfies for all $k > 10$. If $k > 3$, then $\log_2 4k \leq 2^k$, so $d < k$. Additionally, $k + d \leq 2k \leq 2^d$. If $k > 10$, then $16k^2 \leq 2^k$, so $4k \leq 2^{k/2} \leq 2^{5k/8}$, $d \leq \log_2 4k \leq \frac{5}{8}n$, and $9k - 8d \geq 4k \geq 2^d$. $\square$

Now, return to the original problem. For $k = 1$, $n = 1$ has the desired property. For $k = 2$, $n = 12$ has the desired property. Now, for each $k > 2$ we have some number d satisfying the condition of the second Lemma. Consider a k-digit integer n such that the last d digits of n have the property described in the first Lemma. We can choose each of the other digits of n to be any number between zero and nine. We know that the sum of the last d digits of n is between d and $2d$, and we can choose the sum of the other $k-d$ digits to be any number between $k-d$ and $9(k-d)$. Since $k-d+2d \leq 2^d \leq 9(k-d)+d$, we can choose the other digits such that the sum of the digits of n is 2^d. It completes the proof because n is a multiple of 2^d.

Problem 3 We are given a right prism with a regular octagon for its base, whose edges all have length 1. The points $M_1, M_2, \ldots, M_{10}$ are the centers of the faces of the prism. Let P be a point inside the prism, and let P_i denote the second intersection of line M_iP with the surface of the prism. Suppose that the interior of each face contains exactly one of $P_1, P_2, \ldots, P_{10}$. Prove that $\sum_{i=1}^{10} \frac{M_iP}{M_iP_i} = 5$.

Solution: Suppose that M_n is the center of a base and P_n is on a lateral face. Project the prism onto a plane perpendicular to both the base and the face containing P_n. The bases and two faces become edges of the large rectangle. The other six faces become smaller rectangles. In the projection, P must lie in the triangle formed by the projection of M_n and the endpoints of the projection of the face containing P_n. There are six remaining $M_i's$ outside the triangle and only five remaining faces that intersect the triangle. Therefore, one of the M_iP_i segments must lie entirely outside the triangle, which is impossible, because P is inside the triangle. Then the assumption is false, so P_n must be on one of the bases.

If M_i is on a lateral face, then P_i must also be on a lateral face. Suppose that the face containing P_i is not opposite the face containing M_i. Project the prism onto one of the bases. Then, M_iP_i divides this base onto two parts. One of the parts have more remaining M_j's than the number of remaining faces intersecting the second one. Therefore one of the M_jP_j do not intersect M_iP_i, which is impossible. Then the assumption is false, so every P_i must be on the face opposite M_i.

Let M_i and M_j be on opposite faces. M_i, M_j, P_i, and P_j are in the same plane because lines M_iP_i and M_jP_j intersect. Line segments M_iP_j and M_jP_i do not intersect because they belong to parallel planes. Therefore, these lines must be parallel. Angles M_iP_jP and P_iM_jP are alternate interior angles, as are angles M_jP_iP

and P_jM_iP. Triangles M_iP_jP and P_iM_jP are similar by AA. Then $\frac{M_iP}{M_iP_i} + \frac{M_jP}{M_jP_j} = \frac{M_iP}{M_iP_i} + \frac{P_iP}{M_iP_i} = 1$. There are five such pairs of points, so $\sum_{i=1}^{10} \frac{M_iP}{M_iP_i} = 5$.

Problem 4 Let $n > 10$ be a positive integer and let A be a set containing $2n$ elements. The family $\{A_i \mid i = 1, 2, \ldots, m\}$ of subsets of the set A is called *suitable* if:

- for each $i = 1, 2, \ldots, m$, the set A_i contains n elements;
- for all $1 \leq i < j < k \leq m$, the set $A_i \cap A_j \cap A_k$ contains at most one element.

For each n, determine the largest m for which there exists a suitable family of m sets.

Solution: We claim that $m = 4$. Choose any two distinct n-element subsets A_1 and A_2 that are not complements of each other. Let A_3 be the complement of A_1, and let A_4 be the complement of A_2. All three-element intersections are empty because each intersection either contains either A_1 and A_3 or A_2 and A_4.

Now, suppose $\{A_i\}$ has five members. Let I_i denote the sum of the orders of all intersections of n distinct members of $\{A_i\}$, and let U_i denote the sum of the orders of all unions of n distinct members of $\{A_i\}$ By the inclusion-exclusion principle,

$$U_4 = 4I_1 - 3I_2 + 2I_3 - I_4$$

$$U_5 = I_1 - I_2 + I_3 - I_4 + I_5$$

Each union of four sets is no larger than the union of all five sets. Therefore, $5U_5 \geq U_4$, or

$$5I_1 - 5I_2 + 5I_3 - 5I_4 + 5I_5 \geq 4I_1 - 3I_2 + 2I_3 - I_4$$

$$I_1 - 2I_2 + 3I_3 - 4I_4 + 5I_5 \geq 0$$

We know that I_1 is $5n$ because each of the five A_i's has n elements.

$$5n - 2I_2 + 3I_3 - 4I_4 + 5I_5 \geq 0$$

Additionally, U_5 is at most $2n$, so

$$2n \geq 5n - I_2 + I_3 - I_4 + I_5$$

If we multiply by two and add to the previous inequality, we obtain

$$9n - 2I_2 + 3I_3 - 4I_4 + 5I_5 \geq 10n - 2I_2 + 2I_3 - 2I_4 + 2I_5$$

which implies

$$I_3 - 2I_4 + 3I_5 \geq n$$

But $I_3 \leq 10 < n$ because all of the intersections of three subsets have at most one element. The intersection of all five subsets is not larger than the intersection of any four of them, so $2I_4 \geq 10I_5 \geq 3I_5$. This is a contradiction, so we cannot find a suitable family of five subsets.

2.3 Balkan Mathematical Olympiad

Problem 1 Let n be a positive integer. Show that if a and b are integers greater than 1 such that $2^n - 1 = ab$, then $ab - (a - b) - 1$ can be written as $k \cdot 2^{2m}$ for some odd integer k and some positive integer m.

Solution: Note that $ab - (a - b) - 1 = (a + 1)(b - 1)$. We shall show that the highest powers of two dividing $(a + 1)$ and $(b - 1)$ are the same. Let 2^s and 2^t be the highest powers of 2 dividing $(a + 1)$ and $(b - 1)$, respectively. Because $a + 1, b + 1 \leq ab + 1 = 2^n$, we have $s, t \leq n$.

Note that 2^s divides $2^n = ab + 1$ and $a + 1$, so that

$$ab \equiv a \equiv -1 \pmod{2^s}.$$

Hence, $b \equiv 1 \pmod{2^s}$, or $2^s \mid (b - 1)$, so that $s \leq t$.

Similarly, $ab \equiv -b \equiv -1 \pmod{2^t}$, so $a \equiv -1 \pmod{2^t}$, and $2^t \mid (a + 1)$. Thus, $t \leq s$.

Therefore, $s = t$, the highest power of two dividing $(a + 1)(b - 1)$ is $2s$, and $ab - (a - b) - 1 = k \cdot 2^{2s}$ for some odd k.

Problem 2 Prove that if a convex pentagon satisfies the following conditions, then it is a regular pentagon:

(i) all the interior angles of the pentagon are congruent;

(ii) the lengths of the sides of the pentagon are rational numbers.

Solution: Let the pentagon have side lengths $AB = a_1, BC = a_2$, $CD = a_3, DE = a_4$ and $EA = a_5$. Let $\zeta = e^{\frac{2\pi i}{5}}$. Placing the pentagon in the complex plane, with $\overrightarrow{AB}$ aligned along the positive real axis, we have $\overrightarrow{AB} = a_1$, $\overrightarrow{BC} = a_2 \cdot \zeta$, $\overrightarrow{CD} = a_3 \cdot \zeta^2$, $\overrightarrow{DE} = a_4 \cdot \zeta^3$ and $\overrightarrow{EA} = a_5 \cdot \zeta^4$. These five vectors have sum zero; that is,

$$a_1 + a_2 \cdot \zeta + a_3 \cdot \zeta^2 + a_4 \cdot \zeta^3 + a_5 \cdot \zeta^4 = 0.$$

In other words, ζ satisfies the equation

$$a_1 + a_2 x + a_3 x^2 + a_4 x^3 + a_5 x^4 = 0.$$

Let $f(x) = x^4 + x^3 + x^2 + x + 1$. Note that $f(x) = \frac{x^5-1}{x-1}$ for $x \neq 1$, so that $f(\zeta) = \frac{0}{\zeta-1} = 0$. Note that

$$f(x+1) = \frac{(x+1)^5 - 1}{x-1} = x^4 + 5x^3 + 10x^2 + 10x + 5.$$

This polynomial is irreducible over the rationals, by Eisenstein's irreducibility criterion: the leading coefficient is 1, the other coefficients are divisible by the same prime (5), and the constant coefficient is not divisible by the square of that prime (25). Hence, $f(x)$ is irreducible over the rationals.

Consider all polynomials with rational coefficients that have ζ as a root; there is at least one such polynomial different from 0, namely $f(x)$. Then it is well-known that there is one such polynomial $p_0(x)$ that is monic and divides all the rest. (To prove this, let $p_0(x)$ be the nonzero monic polynomial of smallest degree. By the Euclidean algorithm, any other polynomial $p(x)$ with root ζ can be written in the form $p(x) = p_0(x) \cdot q(x) + r(x)$ with $\deg r < \deg p$. Plugging in $x = \zeta$ yields $0 = 0 \cdot q(\zeta) + r(\zeta)$, so that $r(x) = 0$ by the minimal definition of p_0. Thus, $p(x) = p_0(x)q(x)$, as desired.) Because $p_0(x)$ must divide the monic, irreducible polynomial $f(x)$, we have $p_0(x) = f(x)$. Thus, $f(x)$ divides $a_1 + a_2x + a_3x^2 + a_4x^3 + a_5x^4$, implying that $a = b = c = d = e$. Thus, pentagon $ABCDE$ must be regular.

Note: In general, every number α that is the root of a nonzero polynomial with rational coefficients has a unique minimal polynomial — the lowest-degree monic polynomial with rational coefficients that has α as a root. Moreover, this polynomial is unique and divides any polynomial that has α as a root. For prime p, the p^{th} roots of unity have minimal polynomial $1 + x + \cdots + x^{p-1} = \frac{x^p-1}{x-1}$.

Problem 3 A $3 \times 3 \times 3$ cube is divided into 27 congruent $1 \times 1 \times 1$ cells. One of these cells is empty, and the others are filled with unit cubes labelled $1, 2, \ldots, 26$ in some order. An *admissible move* consists of moving a unit cube which shares a face with the empty cell into the empty cell. Does there always exist — for any initial empty cell and any labelling of the 26 cubes — a finite sequence of admissible moves after which each unit cube labelled with k is in the cell originally containing the unit cube labelled with $27-k$, for each $k = 1, 2, \ldots, 26$?

Solution: No; in fact, we claim that regardless of the initial configuration, such a sequence never exists. It is well-known that all permutations of $\{1, 2, \ldots, n\}$ can be assigned a parity — even or odd — so that any permutation obtained by an even (resp. odd) number of transpositions is even (resp. odd). Treat the empty cell as if it contained a cube labelled 27, so that any admissible move is a transposition, namely of cube 27 with a neighboring cube. The desired permutation of the cubes can be achieved with an odd number of transpositions, namely the the 13 transpositions switching cubes 1 and 26, cubes 2 and 25, and so on. Thus, the desired permutation is odd, and any finite sequence of admissible moves ending in the desired configuration must have an odd number of admissible moves.

Now color the cells black and white "checkerboard" style, so that no two white cells or two black cells are adjacent. In any admissible move, the empty cell changes color. In the desired configuration, the empty cell lies in its original position, so any finite sequence of admissible moves ending in this configuration must contain an *even* number of admissible moves. This contradicts the result in the previous paragraph, so no such finite sequence exists.

2.4 Baltic Mathematics Competition

Problem 1 Let 2001 given points on a circle be colored either red or green. In one *step* all points are recolored simultaneously in the following way: If before the recoloring, both neighbors of a point P have the same color as P, then the color of P remains unchanged; otherwise, the color of P is changed. Starting with an initial coloring F_1, we obtain the colorings $F_2, F_3, \ldots$ after several steps. Prove that there is a number $n_0 \leq 1000$ such that $F_{n_0} = F_{n_0+2}$. Is this assertion also true if 1000 is replaced by 999?

Solution: The assertion is true for $n_0 \leq 1000$ but not for $n_0 \leq 999$.

We first prove that $F_{n_0} = F_{n_0+2}$ for some $n_0 \leq 1000$. Observe that if at any time neighbors A, B are colored differently from each other, then from that time on they change color during each step.

Now we introduce some notation. Label the points $P_1, P_2, \ldots, P_{2001}$ in clockwise order, with indices taken modulo 2001. Call a point *unstable* if it is adjacent to a point of a different color; otherwise, call it *stable*. If all points are initially stable, then all points are the same color and the result is trivial. From henceforth, we assume that not all points are initially stable.

Denote by s_n the number of stable points in the coloring F_n. We claim that $s_{n+1} \leq s_n - 2$ or $s_{n+1} = 0$ for each n. From the definition of a step, all unstable points change color and all stable points stay the same. We initially observe that any point that becomes unstable, remains unstable. As for stable points, consider any block of stable points $P_i, \ldots, P_j$ surrounded by unstable points P_{i-1}, P_{j+1}. After the step, $P_i, \ldots, P_j$ remain the same color but P_{i-1} and P_{j+1} change to the opposite color. The result is that the (formerly stable) points P_i and P_j become unstable, while $P_{i+1}, \ldots, P_{j-1}$ remain stable. Hence, in any stable block of length 2 or more, two points become unstable during the next step; in any stable block of length 1, the stable point becomes unstable during the next step. It follows that for any n, if $2 \leq s_n$, then $s_{n+1} \leq s_n - 2$; otherwise, $s_{n+1} = 0$.

Because we are assuming that not all the points are initially the same color, some three adjacent points are initially colored red, green, red or initially colored green, red, green; they are unstable, so initially $s_1 \leq 1998$. Because $s_{n+1} \leq s_n - 2$ or s_{n+1} for each n, we conclude that $s_{1000} = 0$. Then all points are unstable by the stage F_{1000}, and it easily follows that $F_{1000} = F_{1002}$. This completes the first part of the problem.

We now show that there does not necessarily exist $n_0 \leq 999$ with $F_{n_0} = F_{n_0+2}$. Suppose that the initial coloring F_1 is as follows: one point P_1 is red, and all other points are green. Then we begin with three consecutive unstable points (P_{2001}, P_1, and P_2) and a block of 1998 stable points. During each step, the number of stable points decreases by 2 (the two endpoints of the stable block become unstable), so that $s_{999} = 2$ while $s_{1000} = s_{1001} = 0$. Thus, $s_n \neq s_{n+2}$ for all $n \leq 999$, from which it follows that $F_n \neq F_{n+2}$ for $n \leq 999$.

Problem 2 In a triangle ABC, the bisector of angle BAC meets $\overline{BC}$ at D. Suppose that $BD \cdot CD = AD^2$ and $\angle ADB = \pi/4$. Determine the angles of triangle ABC.

Solution: The angle measures are $\angle A = \pi/3, \angle B = 7\pi/12, \angle C = \pi/12$.

Let O be the circumcenter of ABC and let E be the midpoint of the arc BC not containing A, i.e., the second intersection of line AD with the circumcircle of triangle ABC. By Power of a Point, $AD \cdot ED = BD \cdot CD$, from which it follows that $ED = AD$.

Observe that

$$\begin{aligned}\angle ACE &= \angle ACB + \angle BCE = \angle ACB + \angle BAE \\ &= \angle ACB + \angle EAC = \angle ADB = \pi/4,\end{aligned}$$

so that $\angle AOE = 2\angle ACE = \pi/2$ and $\overline{OE} \perp \overline{OA}$.

Orienting our diagram properly, we have A at the north pole of the circumcircle of triangle ABC and E at the east pole. Because E is the midpoint of arc BC, $\overline{BC}$ is vertical; and it passes through D, the midpoint of $\overline{AE}$. It follows that B and C lie $\pi/3$ counterclockwise and clockwise from E, respectively. The angle measures $\angle A = \pi/3$, $\angle B = 7\pi/12$, $\angle C = \pi/12$ follow easily.

Problem 3 Let $a_0, a_1, \ldots$ be a sequence of positive real numbers satisfying

$$i \cdot a_i^2 \geq (i+1) \cdot a_{i-1}a_{i+1}$$

for $i = 1, 2, \ldots$. Furthermore, let x and y be positive reals, and let $b_i = xa_i + ya_{i-1}$ for $i = 1, 2, \ldots$. Prove that

$$i \cdot b_i^2 > (i+1) \cdot b_{i-1}b_{i+1}$$

for all integers $i \geq 2$.

Solution: Fix $k \geq 2$. We wish to show that

$$k \cdot (xa_k + ya_{k-1})^2 > (k+1) \cdot (xa_{k-1} + ya_{k-2})(xa_{k+1} + ya_k),$$

or equivalently that

$$\begin{aligned}&(ka_k^2 - (k+1)a_{k-1}a_{k+1})x^2\\&\quad +((k-1)a_k a_{k-1} - (k+1)a_{k-2}a_{k_1})xy\\&\quad +(ka_{k-1}^2 - (k+1)a_{k-2}a_k)y^2\end{aligned}$$

is positive. Because $x, y > 0$, it suffices to prove that the coefficients of x^2, xy, y^2 above are nonnegative and not all zero.

From the given inequality with $i = k$, the coefficient of x^2 is nonnegative.

Next, we take the given inequality with $i = k-1$:

$$(k-1) \cdot a_{k-1}^2 \geq k \cdot a_{k-2}a_k.$$

Multiplying this by the inequality $\frac{k}{k-1} > \frac{k+1}{k}$ shows that the coefficient of y^2 is strictly positive.

Finally, we take the given inequality for $i = k-1$ and $i = k$:

$$\begin{aligned}(k-1) \cdot a_{k-1}^2 &\geq k \cdot a_{k-2}a_k,\\ k \cdot a_k^2 &\geq (k+1) \cdot a_{k-1}a_{k+1}.\end{aligned}$$

Multiplying these two inequalities and cancelling like terms, we obtain a third inequality

$$(k-1) \cdot a_{k-1}a_k \geq (k+1) \cdot a_{k-2}a_{k+1},$$

which implies that the coefficient of xy is nonnegative. This completes the proof.

Problem 4 Let a be an odd integer. Prove that $a^{2^n} + 2^{2^n}$ and $a^{2^m} + 2^{2^m}$ are relatively prime for all positive integers n and m with $n \neq m$.

Solution: Without loss of generality, assume that $m > n$. For any prime p dividing $a^{2^n} + 2^{2^n}$, we have

$$a^{2^n} \equiv -2^{2^n} \pmod{p}.$$

We square both sides of the equation $m - n$ times to obtain

$$a^{2^m} \equiv 2^{2^m} \pmod{p}.$$

Because a is odd, we have $p \neq 2$. Thus, $2^{2^m} + 2^{2^m} = 2^{2^m+1} \not\equiv 0 \pmod{p}$ so that

$$a^{2^m} \equiv 2^{2^m} \not\equiv -2^{2^m} \pmod{p}.$$

Therefore, $p \nmid (a^{2^m} + 2^{2^m})$, proving the desired result.

2.5 St. Petersburg City Mathematical Olympiad (Russia)

Problem 1 In the parliament of the country Alternativia, for any two deputies there exists a third who is acquainted with exactly one of the two. Each deputy belongs to one of two parties. Each day the president (not a member of the parliament) selects a group of deputies and orders them to change parties, at which time each deputy acquainted with at least one member of the group also changes parties. Prove that the president can arrange that at some point, every deputy belongs to a single party.

Solution: Let D be the set of deputies in the parliament of Alternativia. For a set $S \subset D$ of deputies, we define $A(S)$ to be the set of deputies $d \in D$ such that either $D \in S$ or D is acquainted with at least one member of S. We state the following lemma:

Lemma. *For any set $F \subset D$ there exists a set $S_0 \subset D$ such that $A(S_0) \cap F$ is equal to $F - \{d_0\}$ for some deputy d_0.*

Proof. Consider the set of subsets $S \subset D$ such that $A(S) \cap F \neq F$. Choose S_0 to be such a subset with the maximal number of elements. Then if S_0 is a proper subset of some other set $S_1 \subset D$, $A(S_1) \cap F = F$, or equivalently $F \subset A(S_1)$ for $A(S_1) \cap F \neq F$ would contradict the maximality of S_0.

We now claim that the set S_0 thus defined is our desired subset of D. For suppose that $A(S_0)$ did not equal $D - \{d_0\}$ for any $d_0 \in D$. Then there would have to exist distinct deputies d_1, d_2 such that $d_1, d_2 \in F$ but $d_1, d_2 \notin A(S_0)$. By the condition given in the problem there must exist a deputy $d_3 \in D$ such that exactly one of d_1, d_2 is acquainted with d_3, or since d_3 is necessarily distinct from d_1 and d_2, exactly one is a member of $A(\{d_3\})$. Without loss of generality assume that $d_1 \in A(\{d_3\})$, $d_2 \notin A(\{d_3\})$. Consider the set $S_1 = S_0 \cup \{d_3\}$. Now, by definition d_1 is not acquainted with any member of S_0. However d_1 is acquainted with d_3: we conclude that $d_3 \notin S_0$ and so $S_0 \neq S_1$ and S_0 is a proper subset of S_1.

We also know that d_2 is not a member of $A(S_0)$, nor is it a member of $A(\{d_3\})$. This means that it cannot be a member of $A(S_0 \cup \{d_3\}) = A(S_1)$ either but $d_2 \in F$, implying that F is not a subset of $A(S_1)$. Combined with the fact that S_1 is a proper superset of S_0, this contradicts the maximality of S_0. We are forced to conclude that our assumption was false and that $A(S_0) = D - \{d_0\}$ for some $d_0 \in D$. This proves the lemma. $\square$

We now proceed to prove by induction on $|F|$ that for any parliament D and for any subset F of deputies in the parliament, all the deputies in F can be made to belong to a single party.

Base Case: $|F| = 1$. In this case the set F contains only one deputy. Hence all deputies in F must of necessity be in the same party to begin with, and there is nothing to be shown.

Inductive Step: Assume that for all subsets F of D with $|F| = n$, for any initial configuration of deputies all deputies can be moved to a single party. We must show that for all parliaments F with $|F| = n + 1$, for any initial configuration of deputies all deputies can be moved to a single party.

We apply the lemma. Let S_0 be a subset of D that satisfies $A(S_0) \cap F = F - \{d_0\}$ condition for some $d_0 \in D$. Remove deputy d_0 from the set F to produce a new set $F' = F - \{d_0\}$ with $|F'| = n$. By the induction hypothesis, there is some sequence of permitted operations that will produce a configuration where all deputies in F' are in the same party (call it party 1). If d_0 belongs to party 1, we are done.

If not, make all the deputies in $A(S_0) \supset F'$ change parties. Now all members of F' belong to party 2, as will d_0 so we have reached the desired configuration. This means that if the president can always make all the deputies in F belong to one party whenever $|F| = n$, he can still make the deputies belong to one party whenever $|F| = n+1$. Thus by induction for any set $F \subset D$ the president can arrange for all the members of F to belong to the same party.

This holds even if $F = D$: that is, the president can arrange for all the members of the parliament to belong to a single party.

Problem 2 Do there exist distinct numbers $x, y, z \in [0, \pi/2]$ such that the six numbers $\sin x, \sin y, \sin z, \cos x, \cos y, \cos z$ can be divided into three pairs with equal sum?

Solution: Assume for sake of contradiction that this can be done. Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6$ be $\sin x, \sin y, \sin z, \cos x, \cos y, \cos z$, written in the nonincreasing order. As supposed, these numbers can be partitioned onto three pairs with equal sums. Let a_i be paired with a_1. If $a_i > a_6$, then a_6 with some a_j form a pair with the sum $a_6 + a_j < a_i + a_j \leq a_i + a_1$. Hence, because these pairs must have equal sums, there is a contradiction and $a_j > a_6$ cannot be true. So, $a_6 = a_j$ and without loss of generality we can conclude that a_6 is paired with a_1. Let a_k be paired with a_2. If $a_k > a_5$, then we similarly have $a_2 + a_k > a_2 + a_5 \geq a_l + a_5$, where a_l is paired with a_5. It proves that without loss of generality the pairs are $\{a_1, a_6\}, \{a_2, a_5\}, \{a_3, a_4\}$.

On the other hand, these six numbers can be partitioned onto the following pairs: $\{\sin x, \cos x\}, \{\sin y, \cos y\}, \{\sin z, \cos z\}$. The sum of squares of the numbers in each pair is 1. Let a_x, a_y, a_v, a_w be the numbers paired with a_1, a_2, a_6, a_5 respectively. Applying similar approach, we conclude that $a_x > a_6$ implies

$$1 = a_1^2 + a_x^2 > a_1^2 + a_6^2 \geq a_v^2 + a_w^2 = 1.$$

This cannot be true. Hence, without loss of generality a_1 is paired with a_6. Therefore, similarly $a_y > a_5$ implies

$$1 = a_2^2 + a_y^2 > a_2^2 + a_5^2 \geq a_w^2 + a_5^2 = 1.$$

This is false too. As above, without loss of generality we conclude that these pairs are $\{a_1, a_6\}, \{a_2, a_5\}, \{a_3, a_4\}$ too.

That two statements gives us

$$\sin x + \cos x = \sin y + \cos y = \sin z + \cos z.$$

Squaring, subtracting 1 and multiplying by 2 give

$$2 \sin x \cos x = 2 \sin y \cos y = 2 \sin z \cos z.$$

It is equivalent to $\sin 2x = \sin 2y = \sin 2z$. Numbers $2x$ and $2y$ are different and lie in $[0, \pi]$, so $\sin 2x = \sin 2y$ implies $2x + 2y = \pi$. Similar equation for x and z shows that $z = y$ and gives a contradiction with our first assumption. Hence such numbers do not exist.

Problem 3 A country has 2000 cities and a complete lack of roads. Show that it is possible to join pairs of cities by (two-way) roads so that for $n = 1, \ldots, 1000$, there are exactly two cities where exactly n roads meet.

Solution: This problem can be reformulated in terms of graph theory. If we let G be the graph whose vertices are the cities of the country and whose edges are the roads, our task becomes to prove the existence of a graph G with 1000 vertices such that for $n = 1, \ldots, 1000$ exactly two vertices of G have degree n.

We will prove a stronger statement by induction: namely, that for any nonnegative integer k there exists a graph G with $4k$ vertices such that for $n = 1, \ldots, 2k$ exactly two vertices of G have degree n.

Base Case: $k = 0$. Let G be the graph whose vertex set is empty. This vacuously satisfies the above condition.

Inductive Step. We assume that there exists a graph H with $4k$ vertices such that for $n = 1, \ldots, 2k$ exactly two vertices of H have degree n. We must construct a graph G with $4(k+1)$ vertices such that for $n = 1, \ldots, 2(k+1)$ exactly two vertices of G have degree n.

For $n = 1, \ldots, 2k$ label the two vertices of H with degree n a_n and b_n. We define G to be a graph with a vertex set equal to the vertex set of H along with four additional vertices a_0, b_0, a_{2k+1} and b_{2k+1}. The edges of G are the edges of H along with edges connecting a_{2k+1} to a_n and b_{2k+1} to b_n for $n = 0, \ldots, 2k$ and an edge connecting a_{2k+1} with b_{2k+1}.

We now prove that for $n = 0, \ldots, 2k+1$, a_n has degree $n+1$ in G. For $n = 0$ this is clear, because the only edge incident to a_0 is the edge connecting a_0 to a_{2k+1}. For $n = 1, \ldots, 2k$ a_n had degree n in H, and the only edge in G incident to a_n not in H is the edge connecting a_n to a_{2k+1}, so that a_n has total degree $n+1$ in G. Finally, a_{2n+1} is connected to the vertices $a_0, \ldots, a_{2n}, b_{2n+1}$, giving it degree $2n+2$. Similarly for $n = 0, \ldots, 2k+1$, b_n has degree $n+1$ in G.

So for $n = 1, \ldots, 2(k+1)$ the vertices a_{n-1}, b_{n-1} and no others have degree n in G, and so G is the desired graph. By induction, we can construct such a graph G for any value of k, particularly $k = 500$, which gives a graph with 2000 vertices satisfying the conditions of the problem, as wanted.

Problem 4 The points A_1, B_1, C_1 are the midpoints of sides $\overline{BC}, \overline{CA}, \overline{AB}$ of acute triangle ABC. On lines B_1C_1 and A_1B_1 are chosen points E and F such that line BE bisects angle AEB_1 and line BF bisects angle CFB_1. Prove that $\angle BAE = \angle BCF$.

Solution:

Lemma. *Given $\overline{AB}$ with midpoint M and any ray not on line AB extending from M to O, a point at infinity, there is exactly one point E on ray $\overrightarrow{MO}$ such that line BE bisects $\angle AEO$*

Proof. Let E be an arbitrary point on ray $\overrightarrow{MO}$, and let P be a point past point E on ray $\overrightarrow{BE}$. Then as E moves from point M to O, $\angle AEP$ strictly increases from 0 to π, while $\angle PEO$ strictly decreases from $\angle AMO$ to 0. Therefore at exactly one point E, line BE will bisect $\angle AEO$. □

Now, consider the transformation that reflects across the bisector of $\angle ABC$ and scales upward by a factor of $\frac{BC}{AB}$. This transformation will take point A to point C and point C_1 to point A_1. Furthermore, because this transformation preserves all angles, and because $\angle AC_1B_1 = \angle CA_1B_1$, ray $\overrightarrow{C_1B_1}$ maps to ray $\overrightarrow{A_1B_1}$. Letting point E map to point E' on ray $\overrightarrow{A_1B_1}$, we see that line BE' also bisects $\angle CE'B_1$. Therefore, by the lemma, points E' and F are in fact the same, and because the transformation preserves all angles, $\angle BAE = \angle BCF$.

Problem 5 For all positive integers $m > n$, prove that

$$\operatorname{lcm}(m,n) + \operatorname{lcm}(m+1,n+1) > \frac{2mn}{\sqrt{m-n}}.$$

Solution: Let $m = n + k$. Then

$$\begin{aligned}
\operatorname{lcm}(m,n) + \operatorname{lcm}(m+1,n+1) &= \frac{mn}{\gcd(m,n)} + \frac{(m+1)(n+1)}{\gcd(m+1,n+1)} \\
&> \frac{mn}{\gcd(n+k,n)} + \frac{mn}{\gcd(m+1,n+1)} \\
&= \frac{mn}{\gcd(k,n)} + \frac{mn}{\gcd(n+k+1,n+1} \\
&= \frac{mn}{\gcd(k,n)} + \frac{mn}{\gcd(k,n+1)}
\end{aligned}$$

Now, $\gcd(k,n) \mid k$, and $\gcd(k,n+1) \mid k$. We conclude that $\gcd(k,n)$ has no common prime factor with $\gcd(k,n+1)$ because if they did, $n+1$ would have a common prime factor with n, which is impossible. Since both divide k, $\gcd(k,n+1) \le \frac{k}{\gcd(k,n)}$. We have

$$\begin{aligned}
\operatorname{lcm}(m,n) + \operatorname{lcm}(m+1,n+1) &> \frac{mn}{\gcd(k,n)} + \frac{mn}{\gcd(k,n+1)} \\
&\ge mn\left(\frac{1}{\gcd(k,n)} + \frac{\gcd(k,n)}{k}\right).
\end{aligned}$$

Then by the AM-GM inequality,

$$\operatorname{lcm}(m,n)+\operatorname{lcm}(m+1,n+1) > mn\left(\frac{2}{\sqrt{k}}\right)=\frac{2mn}{\sqrt{m-n}}$$

Problem 6 Acute triangle ABC has incenter I and orthocenter H. The point M is the midpoint of minor arc AC of the circumcircle of ABC. Given that $MI = MH$, find $\angle ABC$.

Solution: The answer is $\frac{\pi}{3}$. To show this, first note that because $\angle MBA = \angle MBC$, $MA = MC$. Furthermore

$$\begin{aligned}\angle MIA &= \angle IAB + \angle IBA = \angle IAC + \angle IBC \\ &= \angle IAC + \angle MAC = \angle MAI\end{aligned}$$

Therefore $MC = MA = MI = MH$ and $ACIH$ is cyclic, that implies $\angle AHC = \angle AIC$. Now,

$$\begin{aligned}\angle AIC = \pi - \angle ICA - \angle IAC &= \pi - \frac{\angle BAC + \angle BCA}{2} \\ = \pi - \frac{\pi - \angle ABC}{2} &= \frac{\pi + \angle ABC}{2}\end{aligned}$$

Also, extending $\overline{AH}$ to meet $\overline{BC}$ at H_a and extending $\overline{CH}$ to meet $\overline{AB}$ at H_c, we see that because $\angle HH_aB = \angle HH_cB = \frac{\pi}{2}$, HH_aBH_c is cyclic. Therefore $\angle AHC = \angle H_aHH_c = \pi - \angle ABC$. It follows that $\pi - \angle ABC = \frac{\pi + \angle ABC}{2}$, $3\angle ABC = \pi$, and $\angle ABC = \frac{\pi}{3}$

Problem 7 Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(x+y+f(y)) = f(x) + 2y$$

for all integers x, y.

Solution: The only answers are $f(x) \equiv x$ and $f(x) = -2x$. Using $x = -f(y)$, we obtain

$$f(y) = f(-f(y)+y+f(y)) = f(-f(y)) + 2y \tag{1}$$

From the given with arbitrary x, y, z, we have

$$f((x+y+f(y))+z+f(z)) = f(x+y+f(y))+2z = f(x)+2y+2z \tag{2}$$

and for $z = -f(y)$, this gives

$$f(x+y+f(y)-f(y)+f(-f(y))) = f(x)+2y-2f(y)$$

On the other hand

$$\begin{aligned} f(x+y+f(y)-f(y)+f(-f(y))) &= f(x+y+f(-f(y))) \\ &= f(x+y+f(y)-2y) \\ &= f((x-2y)+y+f(y)) \\ &= f(x-2y)+2y. \end{aligned}$$

Combining these two equations, we obtain $f(x)+2y-2f(y) = f(x-2y)+2y$ and

$$f(x)-2f(y) = f(x-2y) \tag{3}$$

Hence we now have (substituting $y=1, x=n+2$) $f(n) = f(n+2) - 2f(1)$. Putting $x=y=0$ into (3) gives $f(0) = f(0)-2f(0) = -f(0)$, so

$$f(0) = 0 \tag{4}$$

Using $f(1) = 1 \cdot f(1)$ and equations (3) and (4), we induct on n to prove $f(n) = nf(1)$ for all n in $\mathbb{Z}$. Substituting that result into the given relation gives us:

$$xf(1)+yf(1)+yf(1)^2 = f(x+y+yf(1)) = xf(1)+2y$$

$$y(f(1)+f(1)^2-2) = 0$$

which, for $y \neq 0$ gives $f(1) = 1, -2$. Thus, the only possible solutions are $f(n) = n$ and $f(n) = -2n$.

Problem 8 From a 20×20 grid are removed 20 rectangles of sizes $1 \times 20, 1 \times 19, \ldots, 1 \times 1$, where the sides of the rectangle lie along gridlines. Prove that at least 85 1×2 rectangles can be removed from the remainder.

Solution: Let's call removed rectangles *strips.* Without loss of generality assume that 1×20 stripe is vertical. Construct the following partition of the grid: left and right 1×20 edges of the grid are divided onto ten 1×2 rectangles each and the rest is 18×20 rectangle, which is partitioned onto 90 squares with sides equal to 2. Look at each pair of 1×2 rectangles from opposite edges, that are symmetric with respect to the vertical axis of the grid. Call each such pair and each 2×2 square *general square.* If at least one cell of a general square is removed when removing $1 \times k$ strip, say that the strip and the general square intersect. Call a pair of such strip and general square *intersection*

It is not hard to observe that if a general square intersects at most one strip, then a 1×2 rectangle can be removed from its remaining part. Hence, if no more than k 1×2 rectangles can be removed from the remaining part of a general square, then it is intersected by at least k strips. Because there are 100 general squares, at least 200 minus the total number of intersections.

Now, look at the partition which is derived from constructed one by rotating on $90°$. It is also true that if no more than k 1×2 rectangles can be removed from the remaining part of a general square, then it intersects at least k strips, except for the pair of 1×2 rectangles, forming general square, which intersects at 1×20 strip. For convenience, say that the strip intersects this pair twice.

The next step is to find the total number of intersections for two that partitions. For this purpose find the number of general squares that $1 \times k$ strip intersects. Without loss of generality the strip is vertical. If $k = 2m - 1$, then the strip intersects m general squares of the first partition and m general squares of the second one. If $k = 2m < 20$, then the strip intersects m general squares of the first partition and $m + 1$ of the second one. So it intersects $2m + 1$ general squares. It is true for $k = 20$ too, because it intersects one of the general squares twice. Thus, the total number of intersections is

$$2 + 3 + 4 + 5 + \ldots + 21 = 230.$$

So, by pigeonhole principle, for one of the partitions the total number of intersections is no more than 115. Consider that partition. The number of 1×2 rectangles, which we can remove is at least twice number of general squares (that equals to 200) minus the total number of intersections for that partition (that is at most 115). Hence, 85 rectangles can be removed.

Problem 9 In a 10×10 table are written natural numbers not exceeding 10. Any two numbers that appear in adjacent or diagonally adjacent spaces of the table are relatively prime. Prove that some number appears in the table at least 17 times.

Solution: In any 2×2 square, only one of the numbers can be divisible by 2 and only one can be divisible by 3, so if we tile the table with these 2×2 squares, at most 50 of the numbers in the table are divisible by 2 or 3. The remaining 50 numbers must be divided among the integers not divisible by 2 or 3, and thus only ones available are 1, 5, and 7. By the pigeonhole principle, one of the these numbers appears at least 17 times.

Problem 10 The bisectors of angles A and B of convex quadrilateral $ABCD$ meet at P, and the bisectors of angles C and D meet at Q. Suppose that $P \neq Q$ and that line PQ passes through the midpoint of side $\overline{AB}$. Prove that $\angle ABC = \angle BAD$ or $\angle ABC + \angle BAD = \pi$.

Solution: If lines BC and AD are parallel, it immediately follows that $\angle ABC + \angle BAD = \pi$. Assume instead that lines BC and AD intersect at X, and let line l be the angle bisector of $\angle AXB$. Because P lies on the angle bisectors of $\angle ABC$ and $\angle BAD$, P is equidistant from lines BC, AB, and AD, so P lies on line l. Similarly, Q is equidistant from lines BC, CD, and AD, so Q also lies on line l. Therefore lines PQ and l are the same, and the midpoint of $\overline{AB}$, call it M, lies on line l. Since $\overline{XM}$ is the bisector of $\angle AXB$, $\frac{AX}{BX} = \frac{AM}{BM} = 1$, $AX = BX$, and so $\angle XAB = \angle XBA$. If C lies between X and A, then D lies between X and B, and

$$\angle ABC = \angle ABX = \angle BAX = \angle BAD.$$

Otherwise, A lies between X and C, B lies between X and D, and

$$\angle ABC = \pi - \angle ABX = \pi - \angle BAX = \angle BAD.$$

In all of these cases, either $\angle ABC = \angle BAD$ or $\angle ABC + \angle BAD = \pi$.

Problem 11 Do there exist quadratic polynomials f and g with leading coefficients 1, such that for any integer n, $f(n)g(n)$ is an integer but none of $f(n)$, $g(n)$, and $f(n) + g(n)$ are integers?

Solution: There do exist such polynomials. One example is:

$$f(x) = x^2 + (\sqrt{3} - \sqrt{2})x - \sqrt{6}x, \; g(x) = x^2 + (\sqrt{2} - \sqrt{3})x - \sqrt{6}x.$$

Then $f(n) + g(n) = 2x^2 - 2\sqrt{6}$, and clearly $f(n)$ and $g(n)$ are never integers. But $f(n)g(n) = x^4 - 5x^2 + 6$, always an integer.

We will provide a method for motivating such solutions. Since the polynomials are monic, $f(x) = (x-a)(x-b)$ and $g(x) = (x-c)(x-d)$, and $f(x)g(x) = (x-a)(x-b)(x-c)(x-d)$. We thus want to choose a, b, c, d as irrational numbers such that $(x-a)(x-c)$ and $(x-b)(x-d)$

are polynomials with integer coefficients. So we choose b and d as a conjugate pair $(r + s\sqrt{t}, r - s\sqrt{t})$ for t an non-square integer, and a and c as a conjugate pair, and with the two conjugate pair choices distinct. Then $f(n)$ and $g(n)$ are never integers, and $f(n)g(n)$ is an integer for all n. In order for $f(n) + g(n)$ to not be an integer, the square root parts in a and b must have opposite signs.

Problem 12 Ten points, labelled 1 to 10, are chosen in the plane. Permutations of $\{1, \ldots, 10\}$ are obtained as follows: for each rectangular coordinate system in which the ten points have distinct first coordinates, the labels of the points are listed in increasing order of the first coordinates of the points. Over all sequences of 10 labelled points, what is the maximum number of permutations of $\{1, \ldots, 10\}$ obtained in this fashion?

Solution: The maximum number of permutations of $\{1, \ldots, 10\}$ is 90. Since the order of the points is the same in two coordinate systems that are translations of each other, the order of the points is completely determined by the angle of the y-axis. Therefore choosing an an arbitrary y-axis and rotating it through 360 degrees will produce all possible orders of the points. During this rotation, between every change in the order of points, at least two points must have the same x-coordinate, and the line between these points will be parallel to the y-axis. This will happen twice for each of the $\binom{10}{2} = 45$ pairs of points, so the order will change at most 90 times during the rotation. Since the rotation starts and ends on the same permutation, there can be at most 90 different orders of the points.

To show that 90 permutations can always be attained, consider n distinct points, no three of which are collinear. It will suffice to prove that the 90 permutations obtained from these points as in the previous paragraph are all distinct. Consider any two possible y-axes Y_1 and Y_2 which yield the permutations P_1 and P_2, such that the order of the n points changes at least once in the rotation between Y_1 and Y_2, and also in the rotation from Y_2 to Y_1. Assume without loss of generality that the angle rotated through to get from Y_1 to Y_2 is less than or equal to 180 degrees. Since there is a change in the order of the points between Y_1 and Y_2, there must be some y-axis Y_3 in between Y_1 and Y_2 which is parallel to the line between some two points X and Y. Because the only other y-axis parallel to the line between X and Y is 180 degrees opposite Y_3, and because the angle rotated through to get from Y_1 to Y_2 is less than or equal to 180 degrees, points X and Y can switch orders only one in the rotation between Y_1 and Y_2. Therefore P_1 and P_2 are different permutations, and there are a total of 90 different permutations.

Problem 13 A natural number is written on a chalkboard. Two players take turns, each turn consisting of replacing the number n with either $n-1$ or $\lfloor (n+1)/2 \rfloor$. The first player to write the number 1 wins. If the starting number is 1000000, which player wins with correct play?

Solution: The first player wins. We define a winning position as a numbers x such that with optimal play on both sides, the player whose turn it is can force a win. Similarly, a losing position is where the opposing player can force a win. We will first show that every number is either losing or winning. We note that 2 is a winning position because the player can write 1. Suppose that $1, \ldots, n-1$ are either winning or losing positions. Both $\lfloor (n+1)/2 \rfloor$ and $n-1$ are less than n. If either n or $\lfloor (n+1)/2 \rfloor$ is a losing position, then n is a winning position because the player can write that losing position. If both are winning positions, n is a losing position because the player can only write winning positions for the other player. Hence all positions are either winning or losing by induction on n.

We will show by induction on n that all even numbers are winning positions. The base case is 2, which we have already determined is winning. Suppose the current number is $2n$, and $2, 4, \ldots, 2n-2$ are all winning positions. The first player can write either $2n-1$ or n. If n is a losing position, then the first player writes n and thus the first player wins. If n is a winning position, then the first player goes to $2n-1$. The second player now may write $2n-2$ or n. Since n is a winning position, the second player loses if he writes n. But if the second player writes $2n-2$, that is a smaller even number, and player 1 wins. Thus player 1 wins for even n by induction. Since 1000000 is even, player 1 wins.

Problem 14 The altitudes of triangle ABC meet at H. Point K is chosen such that the circumcircles of BHK and CHK are tangent to line BC. Point D is the foot of the altitude from B. Prove that A is equidistant from lines KB and KD.

Solution: All angles are directed modulo π. Let E be the foot of the altitude from C. Since $\angle AEH = \angle ADH = \frac{\pi}{2}$, $ADHE$ is cyclic. Since BC is tangent to the circumcircle of triangle HKB, $\angle HKB = \angle HBC$, and since BC is tangent to the circumcircle of triangle HKC, $\angle HKC = \angle HCB$. Therefore

$$\begin{aligned}\angle CKB = \angle CKH + \angle HKB &= \angle BCH + \angle HBC \\ &= \angle BHC = \angle DHE = \angle DAE = \angle CAB\end{aligned}$$

so $ABCK$ is cyclic, and $\angle AKB = \angle ACB$. In fact

$$\begin{aligned}\angle ACB + \angle HKA = \angle AKB + \angle HKA &= \angle HKB \\ &= \angle HBC = \angle DBC = \angle DCB + \angle BDC \\ &= \angle ACB + \frac{\pi}{2}\end{aligned}$$

so $\angle HKA = \frac{\pi}{2}$ and $AKDHE$ is cyclic. Letting F be the foot of the altitude from A, we see that

$$\begin{aligned}\angle DKA = \angle DHA &= \angle DAH + \angle HDA \\ &= \angle CAF + \angle AFC = \angle ACF \\ &= \angle ACB = \angle AKB\end{aligned}$$

Therefore line KA bisects lines KB and KD, so A is equidistant from lines KB and KD.

Problem 15 Let m, n, k be positive integers with $n > 1$. Show that $\sigma(n)^k \neq n^m$, where $\sigma(n)$ is the sum of the positive integers dividing n.

Solution: Let $n = p_1^{e_1} p_2^{e_2} ... p_k^{e_k}$. Because $\sigma(n) > n$, if $\sigma(n)^k = n^m$, then $\sigma(n) = p_1^{f_1} p_2^{f_2} ... p_k^{f_k}$ where $f_i > e_i$. This implies $f_i >= e_i + 1$, for all i and

$$\begin{aligned}&\sigma(n) \geq p_1^{1+e_1} p_2^{1+e_2} ... p_k^{1+e_k} > \frac{p_1^{1+e_1} - 1}{p_1 - 1} \frac{p_2^{1+e_2} - 1}{p_2 - 1} ... \frac{p_k^{1+e_k} - 1}{p_k - 1} \\ &= (1 + p_1 + ... p_1^{e_1})(1 + p_2 + ... p_2^{e_2}) ... (1 + p_k + ... p_k^{e_k}) = \sigma(n).\end{aligned}$$

This is a contradiction.

Problem 16 At a chess club, players may play against each other or against the computer. Yesterday there were n players at the club.

Each player played at most n games, and every pair of players that did not play each other played at most n games in total. Prove that at most $n(n+1)/2$ games were played.

Solution: Let us call a pair of player untried if they did not play each other. We proceed by induction on n. For $n=1$, the one player present indeed played at most one game. Now assume that this is true for $k-1$ players ($k \geq 2$). For any gathering of k players, consider the following cases:

Case 1: All players played at most $\frac{k}{2}$ games. In this case, at most $\frac{k^2}{2} < \frac{k(k+1)}{2}$ total games have been played.

Case 2: There is a player X who has played more than $\frac{k}{2}$ games. In this case, let us consider the subset S of players and games excluding player X and all of the games he has played. Of the $k-1$ players in S, those who played a game with player X played at most $k-1$ games in S, and those who did not play player X played less than $k-\frac{k}{2}<k$ games. Therefore, in S, no player played more than $k-1$ games. Since every untried pair played at most k games total, every untried pair of players in S, at least one of whom played player X, played at most $k-1$ games in S. Furthermore, for every untried pair of players Y and Z, neither of whom played player X, each of Y and Z played less than $\frac{k}{2}$ games, for a total of less than k games. Therefore, in S, no untried pair of players played more than $k-1$ games in total, and by the induction hypothesis, at most $\frac{k(k-1)}{2}$ games have been played in S. Since player X played at most k games, this yields at most $\frac{k(k+1)}{2}$ total games played.

Problem 17 Show that there exist infinitely many positive integers n such that the largest prime divisor of n^4+1 is greater than $2n$.

Solution: First we prove the following result.

Lemma. *There are infinitely many numbers being prime divisors of m^4+1 for some m.*

Proof. Suppose that there are only finite number of such primes. Let $p_1, p_2, \ldots, p_k$ be all of them. Let p be any prime divisor of $(p_1 p_2 \cdots p_k)^4+1$. This number cannot equal to any p_i. It makes a contradiction with our assumption, and proves the lemma. $\square$

Let $\mathcal{P}$ be the set of all numbers being prime divisors of $m^4 + 1$ for some m. Pick any p from $\mathcal{P}$ and m from $\mathbb{Z}$, such that p divides m^4+1. Let r be the residue of m modulo p. We have $r < p$, $p \mid r^4+1$ and $p \mid (p-r)^4+1$. Let n be the minimum of r and $p-r$. It follows that $n < p/2$ and $p > 2n$. If n can be obtained using the construction above, then it satisfies the desired condition. If it is constructed using prime p, then $p \mid n^4+1$. Thus, any such number n can be constructed with only finite number of primes p. Since $\mathcal{P}$ is infinite, and for each integer m such number n can be constructed, there are infinite number of integers n, satisfying the desired condition.

Problem 18 In the interior of acute triangle ABC is chosen a point M such that $\angle AMC+\angle ABC = \pi$. Line AM meets side $\overline{BC}$ at D, and line CM meets side $\overline{AB}$ at E. Show that the circumcircle of triangle BDE passes through some fixed point different from B, independent of the choice of M.

Solution: All angles are directed modulo π. First of all, note that for any such M, $\angle EBD = \pi - \angle AMC = \angle EMD$, so $EMDB$ is cyclic. Now, the locus of all possible points M form an arc inside the triangle between points A and C. As M moves from A to C, $\frac{MD}{AD}$ strictly decreases from 1 to 0, while $\frac{ME}{CE}$ strictly increases from 0 to 1. Therefore for some M', with line AM' meeting $\overline{BC}$ at D' and line CM' meeting $\overline{AB}$ at E', $\frac{M'D'}{AD'} = \frac{M'E'}{CE'}$. Now consider any possible M. Since $EMDB$ is cyclic, we have

$$\angle CEE' = \angle MEB = \angle MDB = \angle ADD'.$$

Similarly, since $E'M'D'B$ is cyclic, we have

$$\angle CE'E = \angle M'E'B = \angle M'D'B = \angle AD'D,$$

and so triangles CEE' and ADD' are similar. Furthermore, since $\angle M'E'E = \angle M'D'D$ and $\frac{M'D'}{M'E'} = \frac{AD'}{CE'} = \frac{D'D}{E'E}$, triangles $M'D'D$ and $M'E'E$ are similar. Therefore, we have

$$\angle BDM' = \angle D'DM' = \angle E'EM' = \angle BEM'.$$

It implies that $BEM'D$ is cyclic for any given M.

Problem 19 Given are real numbers $x_1, \ldots, x_{10}$ in the interval $[0, \pi/2]$ such that $\sin^2 x_1 + \sin^2 x_2 + \cdots + \sin^2 x_{10} = 1$. Prove that

$$3(\sin x_1 + \cdots + \sin x_{10}) \le \cos x_1 + \cdots + \cos x_{10}.$$

Solution: Since $\sin^2 x_1 + \sin^2 x_2 + \cdots + \sin^2 x_{10} = 1$,

$$\cos x_i = \sqrt{1 - \sin^2 x_i} = \sqrt{\sum_{j \neq i} \sin^2 x_j}.$$

By the Power Mean Inequality,

$$\cos x_i = \sqrt{\sum_{j \neq i} \sin^2 x_j} \geq \frac{\sum_{j \neq i} \sin x_j}{3}.$$

which summed over all the $\cos x_i$ gives

$$\sum_{i=1}^{10} \cos x_i \geq \sum_{i=1}^{10} \sum_{j \neq i} \frac{\sin x_j}{3} = \sum_{i=1}^{10} 9 \frac{\sin x_i}{3} = 3 \sum_{i=1}^{10} \sin x_i,$$

as desired.

Problem 20 The convex 2000-gon $\mathcal{M}$ satisfies the following property: the maximum distance between two vertices is equal to 1. It is known that among all convex 2000-gons with this same property, $\mathcal{M}$ has maximal area. Prove that some two diagonals of $\mathcal{M}$ are perpendicular.

Solution: Assume for sake of contradiction that no two diagonals of $\mathcal{M}$ are perpendicular. Let us label the vertices $X_1, \ldots, X_{2000}$ consecutively in a clockwise direction, and consider an arbitrary point, say X_1. Letting Y be the intersection of lines X_2X_3 and $X_{1999}X_{2000}$, we see that moving X_1 to any other point within triangle $X_{2000}YX_2$ will not change the convexity of $\mathcal{M}$.

Now assume that X_1 is exactly one unit away from at most one vertex, and let the furthest vertex from X_1 be X_i. By assumption, lines X_1X_i and $X_{2000}X_2$ are not perpendicular. Therefore one can rotate X_1X_i about X_i by a small enough angle so that: (a) The angle between lines X_1X_i and $X_{2000}X_2$ becomes closer to $\frac{\pi}{2}$, (b) Every vertex besides X_i is still less than one unit away from X_1, and (c) X_1 still lies inside triangle $X_{2000}X_2$. By (b) and (c), $\mathcal{M}$ will still be convex with no vertices more than one unit apart, but by (a), the area of $\mathcal{M}$ will still have increased. Therefore every vertex of $\mathcal{M}$ must be exactly one unit from at least two other vertices.

Lemma. *Given a convex quadrilateral $ABCD$ with $AB = CD = 1$, if $AC \leq 1$, then $BD > 1$.*

Proof. Since $AC \leq AB$, $\angle ABC \leq \angle ACB$. Therefore, we have $\angle DBC < \angle ABC \leq \angle ACB < \angle DCB$ and $BD > CD = 1$. $\square$

Now let X_1 be exactly 1 unit from X_i and X_j $(i < j)$, and assume that $j > i+1$. Let X_k be one of the vertices different from X_1 that is exactly one unit away from X_{i+1}. If $1 < k < i+1$, then $X_1X_jX_{i+1}X_k$ is a convex quadrilateral with $X_1X_j = X_{i+1}X_k = 1$ and $X_1X_{i+1} \leq 1$, so by the lemma, $X_jX_k > 1$ Similarly, if $i+1 < k \leq 2000$, then $X_1X_iX_{i+1}X_k$ is a convex quadrilateral with $X_1X_i = X_{i+1}X_k = 1$ and $X_1X_{i+1} \leq 1$, so by the lemma, $X_iX_k > 1$. Therefore the assumption that $j > i+1$ is false, and every vertex must be exactly one unit from exactly two consecutive vertices.

Now let X_i be exactly 1 unit from X_j and X_{j+1}. X_{j+1} must be exactly one unit away from X_i and an adjacent vertex, either X_{i-1} or X_{i+1}. If $X_{i-1}X_{j+1} = 1$, then $X_iX_jX_{j+1}X_{i+1}$ is a convex quadrilateral with $X_iX_j = X_{j+1}X_{i-1} = 1$ and $X_iX_{j+1} \leq 1$, so by the lemma, $X_jX_{i-1} > 1$. So, $X_{j+1}X_{i+1} = 1$. Assuming $X_1X_k = X_1X_{k+1} = 1$, applying this to $i = 1, j = k$ yields $X_2X_{k+1} = 1$. Applying this to $i = k+1$, $j = 1$ yields $X_2X_{k+2} = 1$ Repeating this process $k-1$ more times will yield $X_{k+1}X_{2k} = X_{k+1}X_{2k+1} = 1$. But this implies that $2k \equiv 1 \pmod{2000}$, which has no solutions for integral k. Therefore our initial assumption that no two diagonals of $\mathcal{M}$ are perpendicular has proven to be false.

Problem 21 Let a, b be integers greater than 1. The sequence $x_1, x_2, \ldots$ is defined by the initial conditions $x_0 = 0, x_1 = 1$ and the recursion

$$x_{2n} = ax_{2n-1} - x_{2n-2}, \qquad x_{2n+1} = bx_{2n} - x_{2n-1}$$

for $n \geq 1$. Prove that for any natural numbers m and n, the product $x_{n+m}x_{n+m-1}\cdots x_{n+1}$ is divisible by x_mx_{m-1}.

Solution: We will show that $x_m \mid x_{km}$, and then show that $\gcd(x_m, x_{m-1}) = 1$.

First, consider our sequence modulo x_m for some m. Each x_{k+1} is uniquely determined by x_k, x_{k-1} and the parity of k. Express each x_i as a function $f_i(a, b)$. We have $x_i \equiv f_i(a,b)x_1 \pmod{x_m}$. Suppose $x_r \equiv 0 \pmod{x_m}$ for some r. Since each term is a linear combination of two preceding ones,

$$x_{i+r} \equiv f_i(a,b)x_{r+1} \pmod{x_m} \quad \text{if } m \text{ is even,} \tag{*}$$

$$x_{i+r} \equiv f_i(b,a)x_{r+1} \pmod{x_m} \quad \text{if } m \text{ is odd.} \tag{†}$$

Now we need to prove the following statement.

Lemma. *The function $f_i(a,b)$ is symmetric for any odd i.*

Proof. We will prove also that $f_i(a,b)$ is symmetric function multiplies by a. Now, we are to prove that $f_{2k-1}(a,b)$ is symmetric and $f_{2k-2}(a,b) = ag_{2k-2}(a,b)$, where g_{2k-2} is symmetric too, for any positive integer k. Proceed by induction on k. For $k=1$ we have $f_1(a,b)=1$ and $g_0(a,b)=0$. Suppose that $f_{2k-1}(a,b)$ is symmetric and $f_{2k-2}(a,b) = ag_{2k-2}(a,b)$ where $g_{2k-2}(a,b)$ is symmetric too. Then we can write

$$\begin{aligned} f_{2k}(a,b) = x_{2k} = ax_{2k-1} - x_{2k-2} &= a(x_{2k-1} - g(a,b)) \\ &= a(f_{2k-1}(a,b) - g_{2k-2}(a,b)) \end{aligned}$$

and

$$\begin{aligned} f_{2k+1}(a,b) = x_{2k+1} &= abx_{2k-1} - bx_{2k-2} - x_{2k-1} \\ &= abx_{2k-1} - abq - x_{2k-1} \\ &= (ab-1)f_{2k-1}(a,b) - abg_{2k-2}(a,b). \end{aligned}$$

It shows that f_{2k+1} and g_{2k} are symmetric too and completes the step of induction. □

Now we are to prove that $x_m \mid x_{km}$. Proceed by induction on k. For $k=1$ this statement is true. Let $x_m \mid x_{km}$. Then from $(*)$ and (†) putting $r=km$ and $i=m$, we obtain the following. If km is even, then

$$x_{m(k+1)} \equiv f_m(a,b)x_{km+1} \equiv x_m xkm + 1 \equiv 0 \pmod{x_m}.$$

For km odd m is odd too and $f_m(a,b) = f_m(b,a)$. Hence, we have

$$x_{m(k+1)} \equiv f_m(b,a)x_{km+1} \equiv f_m(a,b)x_{km+1} \equiv x_m x_{km+1} \equiv 0 \pmod{x_m}.$$

So, for each nonnegative integers k, m $x_m \mid x_{km}$.

Since the product $x_{n+1}x_{n+2}\cdots x_{n+m}$ has m terms, one of their indices is divisible by m and another's index is divisible by $m-1$. Thus both x_m and x_{m-1} divide the product. If we can show that x_m is relatively prime to x_{m-1}, we would be done. We will prove this by induction. For the base case, x_0 is relatively prime to x_1. Now, $x_{2n} = ax_{2n-1} - x_{2n-2}$. Any prime factor common to x_{2n} and x_{2n-1} must also divide x_{2n-2}, but because x_{2n-2} is relatively prime to x_{2n-1}, there is no such prime factor. A similar argument holds for x_{2n+1} because $x_{2n+1} = bx_{2n} - x_{2n-1}$. Thus $x_m x_{m-1} \mid (x_{n+1}x_{n+2}\cdots x_{n+m})$.

3
2002 National Contests: Problems

3.1 Belarus

Problem 1 We are given a partition of $\{1, 2, \ldots, 20\}$ into nonempty sets. Of the sets in the partition, k have the following property: for each of the k sets, the product of the elements in that set is a perfect square. Determine the maximum possible value of k.

Problem 2 The rational numbers $\alpha_1, \ldots, \alpha_n$ satisfy

$$\sum_{i=1}^{n} \{k\alpha_i\} < \frac{n}{2}$$

for any positive integer k. (Here, $\{x\}$ denotes the fractional part of x, the unique number in $[0, 1)$ such that $x - \{x\}$ is an integer.)

(a) Prove that at least one of $\alpha_1, \ldots, \alpha_n$ is an integer.

(b) Do there exist $\alpha_1, \ldots, \alpha_n$ that satisfy $\sum_{i=1}^{n} \{k\alpha_i\} \geq \frac{n}{2}$, such that no α_i is an integer?

Problem 3 There are 20 cities in Wonderland. The company Wonderland Airways establishes 18 air routes between them. Each of the routes is a closed loop that passes through exactly five different cities. Each city belongs to at least three different routes. Also, for any two cities, there is at most one route in which the two cities are neighboring stops. Prove that using the airplanes of Wonderland Airways, one can fly from any city of Wonderland to any other city.

Problem 4 Determine whether there exists a three-dimensional solid with the following property: for any natural $n \geq 3$, there is a plane such that the orthogonal projection of the solid onto the plane is a convex n-gon.

Problem 5 Prove that there exist infinitely many positive integers that cannot be written in the form

$$x_1^3 + x_2^5 + x_3^7 + x_4^9 + x_5^11$$

for some positive integers x_1, x_2, x_3, x_4, x_5.

Problem 6 The altitude $\overline{CH}$ of the right triangle ABC ($\angle C = \pi/2$) intersects the angle bisectors $\overline{AM}$ and $\overline{BN}$ at points P and Q, respectively. Prove that the line passing through the midpoints of segments $\overline{QN}$ and $\overline{PM}$ is parallel to line AB.

Problem 7 On a table lies a point X and several face clocks, not necessarily identical. Each face clock consists of a fixed center, and two hands (a minute hand and an hour hand) of equal length. (The hands rotate around the center at a fixed rate; each hour, a minute hand completes a full revolution while an hour hand completes $1/12$ of a revolution.) It is known that at some moment, the following two quantities are distinct:

- the sum of the distances between X and the end of each minute hand; and
- the sum of the distances between X and the end of each hour hand.

Prove that at some moment, the former sum is greater than the latter sum.

Problem 8 A set S of three-digit numbers formed from the digits $1, 2, 3, 4, 5, 6$ (possibly repeating one of these six digits) is called *nice* if it satisfies the following condition: for any two distinct digits from $1, 2, 3, 4, 5, 6$, there exists a number in S which contains both of the chosen digits. For each nice set S, we calculate the sum of all the elements in S; determine, over all nice sets, the minimum value of this sum.

3.2 Bulgaria

Problem 1 Let $a_1, a_2, \ldots$ be a sequence of real numbers such that

$$a_{n+1} = \sqrt{a_n^2 + a_n - 1}$$

for $n \geq 1$. Prove that $a_1 \notin (-2, 1)$.

Problem 2 Consider the feet of the orthogonal projections of A, B, C of triangle ABC onto the external angle bisectors of angles BCA, BCA, and ABC, respectively. Let d be the length of the diameter of the circle passing through these three points. Also, let r and s be the inradius and semiperimeter, respectively, of triangle ABC. Prove that $r^2 + s^2 = d^2$.

Problem 3 Given are n^2 points in the plane, no three of them collinear, where $n = 4k + 1$ for some positive integer k. Find the minimum number of segments that must be drawn connecting pairs of points, in order to ensure that among any n of the n^2 points, some 4 of the n chosen points are connected to each other pairwise.

Problem 4 Let I be the incenter of non-equilateral triangle ABC, and let T_1, T_2, T_3 be the tangency points of the incircle with sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, respectively. Prove that the orthocenter of triangle $T_1T_2T_3$ lies on line OI, where O is the circumcenter of triangle ABC.

Problem 5 Let b, c be positive integers, and define the sequence $a_1, a_2, \ldots$ by $a_1 = b$, $a_2 = c$, and

$$a_{n+2} = |3a_{n+1} - 2a_n|$$

for $n \geq 1$. Find all such (b, c) for which the sequence $a_1, a_2, \ldots$ has only a finite number of composite terms.

Problem 6 In a triangle ABC, let $a = BC$ and $b = CA$, and let ℓ_a and ℓ_b be the lengths of the internal angle bisectors from A and B, respectively. Find the smallest number k such that

$$\frac{\ell_a + \ell_b}{a + b} \leq k$$

for all such triangles ABC.

3.3 Canada

Problem 1 Let a, b, c be positive real numbers. Prove that

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c,$$

and determine when equality holds.

Problem 2 Let Γ be a circle with radius r. Let A and B be distinct points on Γ such that $AB < \sqrt{3}r$. Let the circle with center B and radius AB meet Γ again at C. Let P be the point inside Γ such that triangle ABP is equilateral. Finally, let line CP meet Γ again at Q. Prove that $PQ = r$.

Problem 3 Determine all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that

$$xf(y) + yf(x) = (x + y)f(x^2 + y^2)$$

for all positive integers x, y.

3.4 Czech and Slovak Republics

Problem 1 Find all integers x, y such that

$$\langle 4x \rangle_5 + 7y = 14,$$

$$\langle 2y \rangle_5 - \langle 3x \rangle_7 = 74,$$

where $\langle n \rangle_k$ denotes the multiple of k closest to the number n.

Problem 2 Let $ABCD$ be a square. Let KLM be an equilateral triangle such that K, L, M lie on sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, respectively. Find the locus of the midpoint of segment $\overline{KL}$ for all such triangles KLM.

Problem 3 Show that a given positive integer m is a perfect square if and only if for each positive integer n, at least one of the differences

$$(m+1)^2 - m, (m+2)^2 - m, \ldots, (m+n)^2 - m$$

is divisible by n.

Problem 4 Find all pairs of real numbers a, b such that the equation

$$\frac{ax^2 - 24x + b}{x^2 - 1} = x$$

has exactly two real solutions, and such that the sum of these two real solutions is 12.

Problem 5 In the plane is given a triangle KLM. Point A lies on line KL, on the opposite side of K as L. Construct a rectangle $ABCD$ whose vertices B, C, and D lie on lines KM, KL, and LM, respectively.

Problem 6 Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$f(xf(y)) = f(xy) + x$$

for all positive reals x, y.

3.5 Germany

Problem 1 Determine all ordered pairs (a, b) of real numbers that satisfy

$$2a^2 - 2ab + b^2 = a$$
$$4a^2 - 5ab + 2b^2 = b.$$

Problem 2

(a) Prove that there exist eight points on the surface of a sphere with radius R, such that all the pairwise distances between these points are greater than $1.15R$.

(b) Do there exist nine points with this property?

Problem 3 Let p be a prime. Prove that

$$\sum_{k=1}^{p-1} \left\lfloor \frac{k^3}{p} \right\rfloor = \frac{(p-2)(p-1)(p+1)}{4}.$$

Problem 4 Let a_1 be a positive real number, and define $a_2, a_3, \ldots$ recursively by setting $a_{n+1} = 1 + a_1 a_2 \cdots a_n$ for $n \geq 1$. In addition, define $b_n = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$ for all $n \geq 1$. Prove that $b_n < x$ holds for all n if and only if $x \geq \frac{2}{a_1}$.

Problem 5 Prove that a triangle is a right triangle if and only if its angles α, β, γ satisfy

$$\frac{\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma}{\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma} = 2.$$

Problem 6 Ralf Reisegern explains to his friend Markus, a mathematician, that he has visited eight EURO-counties this year. In order to motivate his five children to use the new Cent- and Euro-coins, he brought home five coins (not necessarily with distinct values) from each country. Because his children can use the new coins in Germany, Ralf made sure that among the 40 coins, each of the eight values (1, 2, 5, 10, 20, and 50 Cents; 1 and 2 Euros) appeared on exactly five coins. Now Ralf wonders whether he will be able to present each child eight coins, one from each country, such that the total value of the coins that each child receives is 3,88 Euro. (1 Euro equals 100 Cents, and 3,88 Euro equals 3 Euro and 88 Cents.) “That must be possible!” says Markus, without looking more carefully at the coins. Prove or disprove Markus’ statement.

3.6 Iran

Problem 1 Find all functions f from the nonzero reals to the reals such that

$$xf\left(x+\frac{1}{y}\right)+yf(y)+\frac{y}{x}=yf\left(y+\frac{1}{x}\right)+xf(x)+\frac{x}{y}$$

for all nonzero reals x, y.

Problem 2 Let segment $\overline{AB}$ be a diameter of a circle ω. Let ℓ_a, ℓ_b be the lines tangent to ω at A and B, respectively. Let C be a point on ω such that line BC meets ℓ_a at a point K. The angle bisector of angle CAK meets line CK at H. Let M be the midpoint of arc CAB, and let S be the second intersection of line HM and ω. Let T be the intersection of ℓ_b and the line tangent to ω at M. Show that S, T, K are collinear.

Problem 3 Let $k \geq 0$ and $n \geq 1$ be integers, and let $a_1, a_2, \ldots, a_n$ be distinct integers such that there are at least $2k$ different integers modulo $n+k$ among them. Prove that there is a subset of $\{a_1, a_2, \ldots, a_n\}$ whose sum of elements is divisible by $n+k$.

Problem 4 The sequence $x_1, x_2, \ldots$ is defined by $x_1 = 1$ and

$$x_{n+1} = \left\lfloor x_n! \sum_{k=1}^{\infty} \frac{1}{k!} \right\rfloor.$$

Prove that $\gcd(x_m, x_n) = x_{\gcd(m,n)}$ for all positive integers m, n.

Problem 5 Distinct points B, M, N, C lie on a line in that order such that $BM = CN$. A is a point not on the same line, and P, Q are points on segments $\overline{AN}, \overline{AM}$, respectively, such that $\angle PMC = \angle MAB$ and $\angle QNB = \angle NAC$. Prove that $\angle QBC = \angle PCB$.

Problem 6 A *strip* of width w is the closed region between two parallel lines a distance w apart. Suppose that the unit disk $\{(x, y) \in \mathbb{R}^2, x^2+y^2 \leq 1\}$ is covered by strips. Show that the sum of the widths of these strips is at least 2.

Problem 7 Given a permutation $(a_1, a_2, \ldots, a_n)$ of $1, 2, \ldots, n$, we call the permutation *quadratic* if there is at least one perfect square among the numbers $a_1, a_1+a_2, \ldots, a_1+a_2+\cdots+a_n$. Find all positive integers n such that every permutation of $1, 2, \ldots, n$ is quadratic.

Problem 8 A $10 \times 10 \times 10$ cube is divided into $1000\ 1 \times 1 \times 1$ blocks. 500 of the blocks are black and the others are white. Show that there exists at least 100 unit squares which are a shared face of a black block and a white block.

Problem 9 Let ABC be a triangle. The incircle of triangle ABC touches side $\overline{BC}$ at A'. Let segment $\overline{AA'}$ meet the incircle again at P. Segments $\overline{BP}, \overline{CP}$ meet the incircle at M, N, respectively. Show that lines AA', BN, CM are concurrent.

Problem 10 Let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $\sum_{i=1}^n x_i^2 = n$. Write $S = \sum_{i=1}^n x_i$. Show that for any real λ with $0 \le \lambda \le 1$, at least

$$\left\lceil \frac{S^2(1-\lambda)^2}{n} \right\rceil$$

of the x_i are greater than $\frac{\lambda S}{n}$.

Problem 11 Around a circular table sit n people labelled $1, 2, \ldots, n$. Some pairs of them are friends, where if A is a friend of B, then B is a friend of A. Each minute, one pair of neighbor friends exchanges seats. What is the necessary and sufficient condition about the friendship relations among the people, such that it is possible to form any permutation of the initial seating arrangement?

Problem 12 Circle ω_1 is internally tangent to the circumcircle of triangle ABC at point M. Assume that ω_1 is tangent to sides $\overline{AB}$ and $\overline{AC}$ as well. Let H be the point where the incircle of triangle ABC touches side $\overline{BC}$, and let A' be a point on the circumcircle for which we have $\overline{AA'} \parallel \overline{BC}$. Show that points M, H, A' are collinear.

3.7 Japan

Problem 1 On a circle ω_0 are given three distinct points A, M, B with $AM = MB$. Let P be a variable point on the arc AB not containing M. Denote by ω_1 the circle inscribed in ω_0 that is tangent to ω_0 at P and also tangent to chord $\overline{AB}$. Let Q be the point where ω_0 intersects chord $\overline{AB}$. Prove that $MP \cdot MQ$ is constant, independent of the choice of P.

Problem 2 There are $n \geq 3$ coins are placed along a circle, with one showing heads and the others showing tails. An *operation* consists of simultaneously turning over each coin that satisfies the following condition: among the coin and its two neighbors, there is an odd number of heads among the three.

(a) Prove that if n is odd, then the coins will never become all tails.

(b) For what values of n will the coins eventually show all tails? For those n, how many operations are required to make all the coins show tails?

Problem 3 Let $n \geq 3$ be an integer. Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be positive real numbers with

$$a_1 + a_2 + \cdots + a_n = 1 \quad \text{and} \quad b_1^2 + b_2^2 + \cdots + b_n^2 = 1.$$

Prove that

$$a_1(b_1 + a_2) + a_2(b_2 + a_3) + \cdots + a_n(b_n + a_1) < 1.$$

Problem 4 A set S of 2002 distinct points in the xy-plane is chosen. We call a rectangle *proper* if its sides are parallel to the coordinate axes and if the endpoints of at least one diagonal lie in S. Find the largest N such that, no matter how the points of S are chosen, at least one proper rectangle contains $N + 2$ points on or within its boundary.

3.8 Korea

Problem 1 Let p be a prime of the form $12k+1$ for some positive integer k, and write $\mathbb{Z}_p\{0,1,2,\ldots,p-1\}$. Let $\mathbb{E}_p$ consist of all (a,b) such that $a,b\in\mathbb{Z}_p$ and $p\nmid(4a^3+27b^2)$. For $(a,b),(a',b')\in\mathcal{E}_p$, we say that (a,b) and (a',b') are *equivalent* if there is a nonzero element $c\in\mathbb{Z}_p$ such that

$$p\mid(a'-ac^4)\quad\text{and}\quad p\mid(b'-bc^6).$$

Find the maximal number of elements in $\mathbb{E}_p$ such that no two of the chosen elements are equivalent.

Problem 2 Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$f(x-f(y))=f(x)+xf(y)+f(f(y))$$

for all $x,y\in\mathbb{R}$.

Problem 3 Find the minimum value of n such that in any mathematics contest satisfying the following conditions, there exists a contestant who solved all the problems:

(i) The contest contains $n\geq 4$ problems, each of which is solved by exactly four contestants.

(ii) For each pair of problems, there is exactly one contest who solved both problems.

(iii) There are at least $4n$ contestants.

Problem 4 Let $n\geq 3$ be an integer. Let $a_1,a_2,\ldots,a_n,b_1,b_2,\ldots,b_n$ be positive real numbers, where the b_i are pairwise distinct.

(a) Find the number of distinct real zeroes of the polynomial

$$f(x)=(x-b_1)(x-b_2)\cdots(x-b_n)\sum_{j=1}^{n}\frac{a_j}{x-b_j}.$$

(b) Writing $S=a_1+a_2+\cdots+a_n$ and $T=b_1b_2\cdots b_n$, prove that

$$\frac{1}{n-1}\sum_{j=1}^{n}\left(1-\frac{a_j}{S}\right)b_j>\left(\frac{T}{S}\sum_{j=1}^{n}\frac{a_j}{b_j}\right)^{1/(n-1)}.$$

Problem 5 Let ABC be an acute triangle and let ω be its circumcircle. Let the perpendicular from A to line BC meet ω at D. Let P

be a point on ω, and let Q be the foot of the perpendicular from P to line AB. Suppose that Q lies outside ω and that $2\angle QPB = \angle PBC$. Prove that D, P, Q are collinear.

Problem 6 Let $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$ be the sequence of primes in increasing order.

(a) Let $n \geq 10$ be a fixed integer. Let r be the smallest integer satisfying

$$2 \leq r \leq n-2 \quad \text{and} \quad n-r+1 < p_r.$$

For $s = 1, 2, \ldots, p_r$, define $N_s = sp_1p_2\cdots p_{r-1} - 1$. Prove that there exists j, with $1 \leq j \leq p_r$, such that none of $p_1, p_2, \ldots, p_n$ divides N_j.

(b) Using the result from (a), find all positive integers m for which

$$p_{m+1}^2 < p_1p_2\cdots p_m.$$

3.9 Poland

Problem 1 Determine all triples of positive integers a, b, c such that a^2+1, b^2+1 are prime and $(a^2+1)(b^2+1) = c^2+1$.

Problem 2 On sides $\overline{AC}, \overline{BC}$ of acute triangle ABC are constructed rectangles $ACPQ$ and $BKLC$. The rectangles lie outside triangle ABC and have equal areas. Prove that a single line passes through C, the midpoint of segment $\overline{PL}$, and the circumcenter of triangle ABC.

Problem 3 On a board are written three nonnegative integers. Each minute, one erases two of the numbers k, m, replacing them with their sum $k+m$ and their positive difference $|k-m|$. Determine whether it is always possible to eventually obtain a triple of numbers such that at least two of them are zeroes.

Problem 4 Let $n \geq 3$ be an integer. Let $x_1, x_2, \ldots, x_n$ be positive integers, where indices are taken modulo n. Prove that one of the following inequalities holds:

$$\sum_{i=1}^{n} \frac{x_i}{x_{i+1}+x_{i+2}} \geq \frac{n}{2} \quad \text{or} \quad \sum_{i=1}^{n} \frac{x_i}{x_{i-1}+x_{i-2}} \geq \frac{n}{2}.$$

Problem 5 In three-dimensional space are given a triangle ABC and a sphere ω, such that ω does not intersect plane (ABC). Lines AK, BL, CM are tangent to ω at K, L, M, respectively. There exists a point P on ω such that

$$\frac{AK}{AP} = \frac{BL}{BP} = \frac{CM}{CP}.$$

The circumcircle of triangle of ABC is the great circle of some sphere ω'. Prove that ω and ω' are tangent.

Problem 6 Let k be a fixed positive integer. We define the sequence $a_1, a_2, \ldots$ by $a_1 = k+1$ and the recursion $a_{n+1} = a_n^2 - ka_n + k$ for $n \geq 1$. Prove that a_m and a_n are relatively prime for distinct positive integers m and n.

3.10 Romania

Problem 1 Find all pairs of sets A, B, which satisfy the conditions:

(i) $A \cup B = \mathbb{Z}$;

(ii) if $x \in A$, then $x - 1 \in B$;

(iii) if $x \in B$ and $y \in B$, then $x + y \in A$.

Problem 2 Let $a_0, a_1, a_2, \ldots$ be the sequence defined as follows: $a_0 = a_1 = 1$ and $a_{n+1} = 14a_n - a_{n-1}$ for any $n \geq 1$. Show that the number $2a_n - 1$ is a perfect square for all positive integers n.

Problem 3 Let ABC be an acute triangle. Segment $\overline{MN}$ is the midline of the triangle that is parallel to side $\overline{BC}$, and P is the orthogonal projection of point N onto side $\overline{BC}$. Let A_1 be the midpoint of segment $\overline{MP}$. Points B_1 and C_1 are constructed in a similar way. Show that if lines AA_1, BB_1, and CC_1 are concurrent, then triangle ABC has two congruent sides.

Problem 4 For any positive integer n, let $f(n)$ be the number of possible choices of signs $+$ or $-$ in the algebraic expression $\pm 1 \pm 2 \pm \cdots \pm n$, such that the obtained sum is zero. Show that $f(n)$ satisfies the following conditions:

(i) $f(n) = 0$, if $n \equiv 1 \pmod 4$ or $n \equiv 2 \pmod 4$;

(ii) $2^{\frac{n}{2}-1} \leq f(n) < 2^n - 2^{\lfloor \frac{n}{2} \rfloor + 1}$, if $n \equiv 0 \pmod 4$ or $n \equiv 3 \pmod 4$.

Problem 5 Let $ABCD$ be a unit square. For any interior points M, N such that line MN does not contain a vertex of the square, we denote by $s(M, N)$ the minimum area of all the triangles whose vertices lie in the set of points $\{A, B, C, D, M, N\}$. Find the least number k such that $s(M, N) \leq k$ for all such points M, N.

Problem 6 Let $p(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0$ and $q(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0$, where each coefficient a_i and b_i equals either 1 or 2002. Assuming that $p(x)$ divides $q(x)$, show that $m + 1$ is a divisor of $n + 1$.

Problem 7 Let a, b be positive real numbers. For any positive integer n, denote by x_n the sum of the digits of $\lfloor an + b \rfloor$ (written in its decimal representation). Show that $x_1, x_2, \cdots$ contains a constant (infinite) subsequence.

Problem 8 At an international conference there are four official languages. Any two participants can talk to each other in one of these languages. Show that some language is spoken by at least 60% of the participants.

Problem 9 Let $ABCDE$ by a convex pentagon inscribed in a circle of center O, such that $\angle B = 120^\circ$, $\angle C = 120^\circ$, $\angle D = 130^\circ$, and $\angle E = 100^\circ$. Show that diagonals $\overline{BD}$ and $\overline{CE}$ meet at a point on diameter $\overline{AO}$.

Problem 10 Let $n \geq 4$ be an integer and let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1^2 + a_2^2 + \cdots + a_n^2 = 1$. Show that

$$\frac{a_1}{a_2^2+1} + \frac{a_2}{a_3^2+1} + \cdots + \frac{a_n}{a_1^2+1} \geq \frac{4}{5}\left(a_1\sqrt{a_1} + a_2\sqrt{a_2} + \cdots + a_n\sqrt{a_n}\right)^2.$$

Problem 11 Let n be a positive integer. Let S be the set of all positive integers a such that $1 < a < n$ and $n \mid (a^{a-1} - 1)$. Show that if $S = \{n-1\}$, then n is twice a prime number.

Problem 12 Show that there does not exist a function $f : \mathbb{Z} \to \{1, 2, 3\}$ satisfying $f(x) \neq f(y)$ for all $x, y \in \mathbb{Z}$ such that $|x - y| \in \{2, 3, 5\}$.

Problem 13 Let $a_1, a_2, \ldots$ be a sequence of positive integers defined as follows:

- $a_1 > 0$, $a_2 > 0$;
- a_{n+1} is the smallest prime divisor of $a_{n-1} + a_n$, for all $n \geq 2$.

The digits of the decimal representations of $a_1, a_2, \ldots$ are written in that order after a decimal point to form a real (decimal) number x. Prove that x is rational.

Problem 14 Let r be a positive number and let $A_1A_2A_3A_4$ be a unit square. Given any four discs $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$ centered at A_1, A_2, A_3, A_4 with radii whose sum is r, we are given that there exists an equilateral triangle whose vertices lie in three of the four discs. (That is, there is an equilateral triangle BCD and three distinct discs $\mathcal{D}_i, \mathcal{D}_j, \mathcal{D}_k$ such that $B \in \mathcal{D}_i$, $C \in \mathcal{D}_j$, $D \in \mathcal{D}_k$.) Find the smallest positive number r with this property.

Problem 15 Elections occur and every member of parliament is assigned a positive number, his or her *absolute rating.* On the first day of parliament, the members are partitioned into groups. In each group, every member of that group receives a *relative rating*: the ratio of his or her absolute rating to the sum of the absolute ratings of all members of that group. From time to time, a member of parliament decides to move to a different group, and immediately after the switch each member's relative rating changes accordingly. No two members can move at the same time, and members only make moves that will increase their relative ratings. Show that only a finite number of moves can be made.

Problem 16 Let m, n be positive integers of distinct parities such that $m < n < 5m$. Show that there exists a partition of $\{1, 2, \ldots, 4mn\}$ into two-element subsets, such that the sum of the numbers in each pair is a perfect square.

Problem 17 Let ABC be a triangle such that $AC \neq BC$ and $AB < AC$. Let Γ be its circumcircle. Let D be the intersection of line BC with the tangent line to Γ at A. Let Γ_1 be the circle tangent to Γ, segment $\overline{AD}$, and segment $\overline{BD}$. We denote by M the point where Γ_1 touches segment $\overline{BD}$. Show that $AC = MC$ if and only if line AM is the angle bisector of angle DAB.

Problem 18 There are $n \geq 2$ players who are playing a card game with np cards. The cards are colored in n colors, and there are p cards of each color, labelled $1, 2, \ldots, p$. They play a game according to the following rules:

- Each player receives p cards.
- During each round, one player throws a card (say, with the color c) on the table. Every other player also throws a card on the table; if it is possible to throw down a card of color c, then the player must do so. The winner is the player who puts down the card of color c labelled with the highest number.
- A person is randomly chosen to start the first round. Thereafter, the winner of each round starts the next round.
- All the cards thrown on the table during one round are removed from the game at the end of the round, and the game ends after p rounds.

At the end of the game, it turns out that all cards labeled 1 won some round. Prove that $p \geq 2n$.

3.11 Russia

Problem 1 Each cells in a 9×9 grid is painted either blue or red. Two cells are called *diagonal neighbors* if their intersection is exactly a point. Show that some cell has exactly two red neighbors, or exactly two blue neighbors, or both.

Problem 2 A monic quadratic polynomial f with integer coefficients attains prime values at three consecutive integer points. Show that it attains a prime value at some other integer point as well.

Problem 3 Let O be the circumcenter of an acute triangle ABC with $AB = AC$. Point M lies on segment $\overline{BO}$, and point M' is the reflection of M across the midpoint of side $\overline{AB}$. Point K is the intersection of lines $M'O$ and AB. Point L lies on side $\overline{BC}$ such that $\angle CLO = \angle BLM$. Show that O, K, B, L are concyclic.

Problem 4 There are $\lfloor \frac{4}{3}n \rfloor$ rectangles on the plane whose sides are parallel to the coordinate axes. It is known that any rectangle intersects at least n other rectangles. Show that one of the rectangles intersects all the other rectangles.

Problem 5 Around a circle are written the numbers $a_1, a_2, \ldots, a_{60}$, a permutation of the numbers $1, 2, \ldots, 60$. (All indices are taken modulo 60.) Is it possible that $2 \mid (a_n + a_{n+2})$, $3 \mid (a_n + a_{n+3})$, and $7 \mid (a_n + a_{n+7})$ for all n?

Problem 6 Let $ABCD$ be a trapezoid with $\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \nparallel \overline{DA}$. Let A' be the point on the boundary of the trapezoid such that line AA' splits the trapezoid into two halves with the same area. The points B', C', D' are defined similarly. Let P be the intersection of the diagonals of quadrilateral $ABCD$, and let P' be the intersection of the diagonals of quadrilateral $A'B'C'D'$. Prove that P and P' are reflections of each other across the midpoint of the midline of trapezoid $ABCD$. (The midline of the trapezoid is the line connecting the midpoints of sides $\overline{BC}$ and $\overline{DA}$.)

Problem 7 18 stones are arranged on a line. It is known that there are 3 consecutive stones that weigh 99 grams each, whereas all the other stones weigh 100 grams each. You are allowed to perform the following operation twice: choose a subset of the 18 stones, then weigh that collection of stones. Describe a method for determining which three stones weigh 99 grams each.

Problem 8 What is the largest possible length of an arithmetic progression of positive integers $a_1, a_2, \dots, a_n$ with difference 2, such that $a_k^2 + 1$ is prime for $k = 1, 2, \dots, n$?

Problem 9 A convex polygon on the plane contains at least m^2+1 lattice points strictly in its interior. Show that one some $m+1$ lattice points strictly inside the polygon lie on the same line.

Problem 10 The perpendicular bisector of side $\overline{AC}$ of a triangle ABC meets side $\overline{BC}$ at a point M. The ray bisecting angle AMB intersects the circumcircle of triangle ABC at K. Show that the line passing through the incenters of triangles AKM and BKM is perpendicular to the angle bisector of angle AKB.

Problem 11

(a) The sequence $a_0, a_1, a_2, \dots$ satisfies $a_0 = 0$ and $0 \le a_{k+1} - a_k \le 1$ for $k \ge 1$. Prove that

$$\sum_{k=0}^{n} a_k^3 \le \left(\sum_{k=0}^{n} a_k \right)^2 .$$

(b) If the sequence $a_0, a_1, a_2, \dots$ instead satisfies $a_0 = 0$ and $a_{k+1} \ge a_k + 1$ for $k \ge 1$, prove the reverse of the inequality in (a).

Let $n \ge 3$ be an integer. On the x-axis have been chosen pairwise distinct points $X_1, X_2, \dots, X_n$. Let $f_1, f_2, \dots, f_m$ be the monic quadratic polynomials that have two distinct X_i as roots. Prove that $y = f_1(x) + \cdots + f_m(x)$ crosses the x-axis at exactly two points.

Problem 12 What is the largest number of colors in which one can paint all the squares of a 10×10 checkerboard so that each of its columns, and each of its rows, is painted in at most 5 different colors?

Problem 13 Real numbers x and y have the property that $x^p + y^q$ is rational for any distinct odd primes p, q. Prove that x and y are rational.

Problem 14 The altitude from S of pyramid $SABCD$ passes through the intersection of the diagonals of base $ABCD$. Let $\overline{AA_1}$, $\overline{BB_1}$, $\overline{CC_1}$, $\overline{DD_1}$ be the perpendiculars to lines SC, SD, SA, and SB, respectively (where A_1 lies on line SC, etc.). It is known that S, A_1, B_1, C_1, D_1 are distinct and lie on the same sphere. Show that lines AA_1, BB_1, CC_1, DD_1 are concurrent.

Problem 15 The plane is divided into 1×1 cells. Each cell is colored in one of n^2 colors so that any $n \times n$ grid of cells contains one cell of each color. Show that there exists an (infinite) column colored in exactly n colors.

Problem 16 Let $p(x)$ be a polynomial of odd degree. Show that the equation $p(p(x)) = 0$ has at least as many real roots as the equation $p(x) = 0$.

Problem 17 There are $n > 1$ points on the plane. Two players choose in turn a pair of points and draw a vector from one to the other. It is forbidden to choose points already connected by a vector. If at a certain moment the sum of all drawn vectors is zero, then the second player wins. If at a certain moment it is impossible to draw a new vector and the sum of the existing vectors is not zero, then the first player wins. As a function of the choice of n points, which player has a winning strategy?

Problem 18 Let $ABCD$ be a convex quadrilateral, and let ℓ_A, ℓ_B, ℓ_C, ℓ_D be the bisectors of its external angles. Lines ℓ_A and ℓ_B meet at a point K, ℓ_B and ℓ_C meet at a point L, ℓ_C and ℓ_D meet at a point M, and ℓ_D and ℓ_A meet at a point N. Show that if the circumcircles of triangles ABK and CDM are externally tangent to each other, then the same is true for the circumcircles of triangles BCL and DAN.

Problem 19 Let n be a fixed integer between 2 and 2002, inclusive. On the segment $[0, 2002]$ are marked $n + 1$ points with integer coordinates, including the two endpoints of the segment. These points divide $[0, 2002]$ into n segments, and we are given that the lengths of these segments are pairwise relatively prime. One is allowed to choose any segment whose endpoints are already marked, divide it into n equal parts, and mark the endpoints of all these parts — provided that these new marked points all have integer coordinates. (One is allowed to mark the same point twice.)

(a) By repeating this operation, is it always possible — for fixed n, but regardless of the choice of initial markings — to mark all the points on the segment with integer coordinates?

(b) Suppose that $n = 3$, and that when we divide any segment into 3 parts we must erase one of its endpoints. By repeating the modified operation, is it always possible — regardless of the choice of initial markings — to mark any given single point of $[0, N]$?

Problem 20 Distinct points O, B, C lie on a line in that order, and point A lies off the line. Let O_1 be the incenter of triangle OAB, and let O_2 be the excenter of triangle OAC opposite A. If $O_1A = O_2A$, show that triangle ABC is isosceles.

Problem 21 Six red, six blue, and six green points are marked on the plane. No three of these points are collinear. Show that the sum of the areas of those triangles whose vertices are marked points of the same color, does not exceed one quarter of the sum of the areas of *all* the triangles whose vertices are marked points.

Problem 22 A mathematical *hydra* consists of heads and necks, where any neck joins exactly two heads, and where each pair of heads is joined by exactly 0 or 1 necks. With a stroke of a sword, Hercules can destroy all the necks coming out of some head A of the hydra. Immediately after that, new necks appear joining A with all the heads that were not joined with A immediately before the stroke. To defeat a hydra, Hercules needs to chop it into two parts not joined by necks (that is, given any two heads, one from each part, they are not joined by a neck). Find the minimal N for which he can defeat any hydra with 100 necks by making at most N strokes.

Problem 23 There are 8 rooks on a chessboard, no two of which lie in the same column or row. We define the distance between two rooks to be the distance between the centers of the squares that they lie on. Prove that among all the distances between rooks, there are at least two distances that are equal.

Problem 24 There are $k > 1$ blue boxes, one red box, and a stack of $2n$ cards numbered from 1 to $2n$. Originally, the cards in the stack are in some arbitrary order, and the stack is in the red box. One is allowed to take the top card from any box; say that the card's label is m. Then the card is put either (i) in an empty box, or (ii) in a box whose top card is labelled $m + 1$. What is the maximal n for which it is possible to move all of the cards into one blue box?

Problem 25 Let O be the circumcenter of triangle ABC. On sides $\overline{AB}$ and $\overline{BC}$ there have been chosen points M and N, respectively, such that $2\angle MON = \angle AOC$. Show that the perimeter of triangle MBN is at least AC.

Problem 26 Let $n \geq 1$ be an integer. $2^{2n-1}+1$ odd numbers are chosen from the interval $(2^{2n}, 2^{3n})$. Show that among these numbers, one can find two numbers a, b for which $a \not| \, b^2$ and $b \not| \, a^2$.

Problem 27 Let p, q, r be polynomials with real coefficients, such that at least one of the polynomials has degree 2 and at least one of the polynomials has degree 3. Assume that

$$p^2 + q^2 = r^2.$$

Show that at least one of the polynomials both has degree 3 and has 3 (not necessarily distinct) real roots.

Problem 28 Quadrilateral $ABCD$ is inscribed in circle ω at A intersects the extension of side $\overline{BC}$ past B at a point K. The tangent line to ω at B meets the extension of side $\overline{AD}$ past A at a point M. If $AM = AD$ and $BK = BC$, show that quadrilateral $ABCD$ is a trapezoid.

Problem 29 Show that for any positive integer $n > 10000$, there exists a positive integer m that is a sum of two squares and such that $0 < m - n < 3\sqrt[4]{n}$.

Problem 30 Once upon a time, there were 2002 cities in a kingdom. The only way to travel between cities was to travel between two cities that are connected by a (two-way) road. In fact, the road system was such that even if it had been forbidden to pass through any one of the cities, it would still have been possible to get from any remaining city to any other remaining city. On year, the king decided to modify the road system from this initial set-up. Each year, the king chose a loop of roads that did not intersect itself, and then ordered:

(i) to build a new city,

(ii) to construct roads from this new city to any city on the chosen loop, and

(iii) to destroy all the roads of the loop, as they were no longer useful.

As a result, at a certain moment there no longer remained any loops of roads. Show that at this moment, there must have been at least 2002 cities accessible by exactly one road.

Problem 31 Let a, b, c be positive numbers with sum 3. Prove that

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \geq ab + bc + ca.$$

Problem 32 The excircle of triangle ABC opposite A touches side $\overline{BC}$ at A'. Line ℓ_A passes through A' and is parallel to the angle bisector of angle CAB. The lines ℓ_B and ℓ_C are defined similarly. Prove that ℓ_A, ℓ_B, ℓ_C are concurrent.

Problem 33 A finite number of red and blue lines are drawn on the plane. No two of the lines are parallel to each other, and through any point where two lines of the same color meet, there also passes a line of the other color. Show that all the lines have a common point.

Problem 34 Some points are marked on the plane in such a way that for any three marked points, there exists a Cartesian coordinate system in which these three points are lattice points. (A Cartesian coordinate system is a coordinate system with perpendicular coordinate axes with the same scale.) Show that there exists a Cartesian coordinate system in which *all* the marked points have integer coordinates.

Problem 35 Show that

$$2|\sin^n x = \cos^n x| \leq 3|\sin^m x - \cos^m x|$$

for all $x \in (0, \pi/2)$ and for all positive integers $n > m$.

Problem 36 In a certain city, there are several squares. All streets are one-way and start or terminate only in squares; any two squares are connected by at most one road. It is known that there are exactly two streets that go out of any given square. Show that one can divide the city into 1014 districts so that (i) no street connects two cities in the same district, and (ii) for any two districts, all the streets that connect them have the same direction (either all the streets go from the first district to the second, or vice versa).

Problem 37 Find the smallest positive integer which can be written both as (i) a sum of 2002 positive integers (not necessarily distinct), each of which has the same sum of digits; and (ii) as a sum of 2003 positive integers (not necessarily distinct), each of which has the same sum of digits.

Problem 38 Let $ABCD$ be a quadrilateral inscribed in a circle, and let O be the intersection point of diagonals $\overline{AC}$ and $\overline{BD}$. The circumcircles of triangles ABO and COD meet again at K. Point L has the property that triangles BLC and AKD are similar (with the similarity respecting this order of vertices). Show that if quadrilateral $BLCK$ is convex, then it is circumscribed about some circle.

Problem 39 Show that there are infinitely many positive integers n for which the numerator of the irreducible fraction equal to $1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is not a positive integer power of a prime number.

3.12 Taiwan

Problem 1 For each n, determine all n-tuples of nonnegative integers $x_1, x_2, \ldots, x_n$ such that

$$\sum_{i=1}^{n} x_i^2 = 1 + \frac{4}{4n+1}\left(\sum_{i=1}^{n} x\right)^2.$$

Problem 2 We call a lattice point X in the plane *visible* from the origin O if the segment $\overline{OX}$ does not contain any other lattice points besides O and X. Show that for any positive integer n, there exists an square of n^2 lattice points (with sides parallel to the coordinate axes) such that none of the lattice points inside the square is visible from the origin.

Problem 3 Let $x, y, z, a, b, c, d, e, f$ be real numbers satisfying

$$\max\{a,0\} + \max\{b,0\} < x + ay + bz < 1 + \min\{a,0\} + \min\{b,0\},$$
$$\max\{c,0\} + \max\{d,0\} < cx + y + dz < 1 + \min\{c,0\} + \min\{d,0\},$$
$$\max\{e,0\} + \max\{f,0\} < ex + fy + z < 1 + \min\{e,0\} + \min\{f,0\}.$$

Show that $0 < x, y, z < 1$.

Problem 4 Suppose that $0 < x_1, x_2, x_3, x_4 \le \frac{1}{2}$. Prove that

$$\frac{x_1x_2x_3x_4}{(1-x_1)(1-x_2)(1-x_3)(1-x_4)}$$

is less than or equal to

$$\frac{x_1^4 + x_2^4 + x_3^4 + x_4^4}{(1-x_1)^4 + (1-x_2)^4 + (1-x_3)^4 + (1-x_4)^4}.$$

Problem 5 The 2002 real numbers $a_1, a_2, \ldots, a_{2002}$ satisfy

$$\frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_{2002}}{2003} = \frac{4}{3},$$
$$\frac{a_1}{3} + \frac{a_2}{4} + \cdots + \frac{a_{2002}}{2004} = \frac{4}{5},$$
$$\vdots$$
$$\frac{a_1}{2003} + \frac{a_2}{2004} + \cdots + \frac{a_{2002}}{4004} = \frac{4}{4005}.$$

Evaluate

$$\frac{a_1}{3} + \frac{a_2}{5} + \frac{a_3}{7} + \cdots + \frac{a_{2002}}{4005}.$$

Problem 6 Given three fixed points A, B, C in a plane, let D be a variable point different from A, B, C such that A, B, C, D are concylic. Let ℓ_A be the Simson line of A with respect to triangle BCD, and define ℓ_B, ℓ_C, ℓ_D analogously. (It is well known that if W is a point on the circumcircle of triangle XYZ, then the feet of the perpendiculars from W to lines XY, YZ, ZX lie on a single line. This line is called the *Simson line* of W with respect to triangle BCD.) As D varies, find the locus of all possible intersections of some two of $\ell_A, \ell_B, \ell_C, \ell_D$.

3.13 United States of America

Problem 1 Let S be a set with 2002 elements, and let N be an integer with $0 \leq N \leq 2^{2002}$. Prove that it is possible to color every subset of S either blue or red so that the following conditions hold:

(a) the union of any two red subsets is red;

(b) the union of any two blue subsets is blue;

(c) there are exactly N red subsets.

Problem 2 Let ABC be a triangle such that

$$\left(\cot \frac{A}{2}\right)^2 + \left(2\cot \frac{B}{2}\right)^2 + \left(3\cot \frac{C}{2}\right)^2 = \left(\frac{6s}{7r}\right)^2,$$

where s and r denote its semiperimeter and its inradius, respectively. Prove that triangle ABC is similar to a triangle T whose side lengths are all positive integers with no common divisor and determine these integers.

Problem 3 Prove that any monic polynomial (a polynomial with leading coefficient 1) of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

Problem 4 Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all pairs of real numbers x and y.

Problem 5 Let a, b be integers greater than 2. Prove that there exists a positive integer k and a finite sequence $n_1, n_2, \ldots, n_k$ of positive integers such that $n_1 = a, n_k = b$, and $n_i n_{i+1}$ is divisible by $n_i + n_{i+1}$ for each i $(1 \leq i < k)$.

Problem 6 I have an $n \times n$ sheet of stamps, from which I've been asked to tear out blocks of three adjacent stamps in a single row or column. (I can only tear along the perforations separating adjacent stamps, and each block must come out of a sheet in one piece.) Let $b(n)$ be the smallest number of blocks I can tear out and make it impossible to tear out any more blocks. Prove that there are constants c and d such that

$$\frac{1}{7}n^2 - cn \leq b(n) \leq \frac{1}{5}n^2 + dn$$

for all $n > 0$.

Problem 7 Let ABC be a triangle. Prove that

$$\sin\frac{3A}{2} + \sin\frac{3B}{2} + \sin\frac{3C}{2} \le \cos\frac{A-B}{2} + \cos\frac{B-C}{2} + \cos\frac{C-A}{2}.$$

Problem 8 Let n be an integer greater than 2, and $P_1, P_2, \cdots, P_n$ distinct points in the plane. Let $\mathcal{S}$ denote the union of the segments $P_1P_2, P_2P_3, \ldots, P_{n-1}P_n$. Determine whether it is always possible to find points A and B in $\mathcal{S}$ such that $P_1P_n \parallel AB$ (segment AB can lie on line P_1P_n) and $P_1P_n = kAB$, where (1) $k = 2.5$; (2) $k = 3$.

Problem 9 Let n be a positive integer and let S be a set of $2^n + 1$ elements. Let f be a function from the set of two-element subsets of S to $\{0, \ldots, 2^{n-1} - 1\}$. Assume that for any elements x, y, z of S, one of $f(\{x, y\}), f(\{y, z\}), f(\{z, x\})$ is equal to the sum of the other two. Show that there exist a, b, c in S such that $f(\{a, b\}), f(\{b, c\}), f(\{c, a\})$ are all equal to 0.

Problem 10 Consider the family of non-isosceles triangles ABC satisfying the property $AC^2 + BC^2 = 2AB^2$. Points M and D lie on side AB such that $AM = BM$ and $\angle ACD = \angle BCD$. Point E is in the plane such that D is the incenter of triangle CEM. Prove that exactly one of the ratios

$$\frac{CE}{EM}, \quad \frac{EM}{MC}, \quad \frac{MC}{CE}$$

is constant (i.e., is the same for all triangles in the family).

Problem 11 Find in explicit form all ordered pairs of positive integers (m, n) such that $mn - 1$ divides $m^2 + n^2$.

3.14 Vietnam

Problem 1 Let ABC be a triangle such that angle BCA is acute. Let the perpendicular bisector of side $\overline{BC}$ intersect the rays that trisect angle BAC at K and L, so that $\angle BAK = \angle KAL = \angle LAC = \frac{1}{3}\angle BAC$. Also let M be the midpoint of side $\overline{BC}$, and let N be the foot of the perpendicular from A to line BC. Find all such triangles ABC for which $AB = KL = 2MN$.

Problem 2 A positive integer is written on a board. Two players alternate performing the following operation until 0 appears on the board: the current player erases the existing number N from the board and replaces it with either $N-1$ or $\lfloor N/3 \rfloor$. Whoever writes the number 0 on the board first wins. Determine who has the winning strategy when the initial number equals (a) 120, (b) $(3^{2002}-1)/2$, and (c) $(3^{2002}+1)/2$.

Problem 3 The positive integer m has a prime divisor greater than $\sqrt{2m}+1$. Find the smallest positive integer M such that there exists a finite set T of distinct positive integers satisfying: (i) m and M are the least and greatest elements, respectively, in T, and (ii) the product of all the numbers in T is a perfect square.

Problem 4 On an $n \times 2n$ rectangular grid of squares ($n \geq 2$) are marked n^2 of the $2n^2$ squares. Prove that for each $k = 2, 3, \ldots, \lfloor n/2 \rfloor + 1$, there exists k rows of the board and

$$\left\lceil \frac{k!(n-2k+2)}{(n-k+1)(n-k+2)\cdots(n-1)} \right\rceil$$

columns, such that the intersection of each chosen row and each chosen column is a marked square.

Problem 5 Find all polynomials $p(x)$ with integer coefficients such that

$$q(x) = (x^2+6x+10)(p(x))^2 - 1$$

is the square of a polynomial with integer coefficients.

Problem 6 Prove that there exists an integer $m \geq 2002$ and m distinct positive integers $a_1, a_2, \ldots, a_m$ such that

$$\prod_{i=1}^{m} a_i^2 - 4\sum_{i=1}^{m} a_i^2$$

is a perfect square.

4
2002 Regional Contests: Problems

4.1 Asian Pacific Mathematics Olympia d

Problem 1 Let $a_1, a_2, \ldots, a_n$ be a sequence of non-negative integers, where n is a positive integer. Let

$$A_n = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Prove that

$$a_1! a_2! \cdots a_n! \geq \left(\lfloor A_n \rfloor !\right)^n,$$

and determine when equality holds. (Here, $\lfloor A_n \rfloor$ denotes the greatest integer less than or equal to A_n, $a! = 1 \times 2 \times \cdots \times a$ for $a \geq 1$, and $0! = 1$.)

Problem 2 Find all positive integers a and b such that

$$\frac{a^2+b}{b^2-a} \quad \text{and} \quad \frac{b^2+a}{a^2-b}$$

are both integers.

Problem 3 Let ABC be an equilateral triangle. Let P be a point on side $\overline{AC}$ and let Q be a point on side $\overline{AB}$ so that both triangles ABP and ACQ are acute. Let R be the orthocenter of triangle ABP and let S be the orthocenter of triangle ACQ. Let T be the intersection of segments $\overline{BP}$ and $\overline{CQ}$. Find all possible values of $\angle CBP$ and $\angle BCQ$ such that triangle TRS is equilateral.

Problem 4 Let x, y, z be positive numbers such that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1.$$

Show that

$$\sqrt{x+yz} + \sqrt{y+zx} + \sqrt{z+xy} \geq \sqrt{xyz} + \sqrt{x} + \sqrt{y} + \sqrt{z}.$$

Problem 5 Find all functions $f : \mathbb{R} \to \mathbb{R}$ with the following properties:

(i) there are only finitely many s in $\mathbb{R}$ such that $f(s) = 0$, and

(ii) $f(x^4 + y) = x^3 f(x) + f(f(y))$ for all $x, y \in \mathbb{R}$.

4.2 Austrian-Polish Mathematics Olympiad

Problem 1 Let $A = \{2, 7, 11, 13\}$. A polynomial f with integer coefficients has the property that for each integer n, there exists $p \in A$ such that $p \mid f(n)$. Prove that there exists $p \in A$ such that $p \mid f(n)$ for all integers n.

Problem 2 The diagonals of a convex quadrilateral $ABCD$ intersect at the point E. Let triangle ABE have circumcenter U and orthocenter H. Similarly, let triangle CDE have circumcenter V and orthocenter K. Prove that E lies on line UK if and only if it lies on line VH.

Problem 3 Find all functions $f : \mathbb{Z}^+ \to \mathbb{R}$ such that $f(x + 22) = f(x)$ and $f(x^2 y) = (f(x))^2 f(y)$ for all positive integers x and y.

Problem 4 Determine the number of real solutions of the system

$$x_1 = \cos x_n, \quad x_2 = \cos x_1, \quad \ldots, \quad x_n = \cos x_{n-1}.$$

Problem 5 For every real number x, let $F(x)$ be the family of real sequences $a_1, a_2, \ldots$ satisfying the recursion

$$a_{n+1} = x - \frac{1}{a_n}$$

for $n \geq 1$. The family $F(x)$ has *minimal period* p if (i) each sequence in $F(x)$ is periodic with period p, and (ii) for each $0 < q < p$, some sequence in $F(x)$ is not periodic with period q. Prove or disprove the following claim: for each positive integer P, there exists a real number x such that the family $F(x)$ has minimal period $p > P$.

4.3 Balkan Mathematical Olympiad

Problem 1 Let $A_1, A_2, \ldots, A_n$ $(n \geq 4)$ be points in the plane such that no three of them are collinear. Some pairs of distinct points among $A_1, A_2, \ldots, A_n$ are connected by line segments, such that every point is directly connected to at least three others. Prove that from among these points can be chosen an even number of distinct points $X_1, X_2, \ldots, X_{2k}$ $(k \geq 1)$ such that X_i is directly connected to X_{i+1} for $i = 1, 2, \ldots, 2k$. (Here, we write $X_{2k+1} = X_1$.)

Problem 2 The sequence $a_1, a_2, \ldots$ is defined by the initial conditions $a_1 = 20$, $a_2 = 30$ and the recursion $a_{n+2} = 3a_{n+1} - a_n$ for $n \geq 1$. Find all positive integers n for which $1 + 5a_na_{n+1}$ is a perfect square.

Problem 3 Two circles with different radii intersect at two points A and B. The common tangents of these circles are segments $\overline{MN}$ and $\overline{ST}$, where M, S lie on one circle while N, T lie on the other. Prove that the orthocenters of triangles AMN, AST, BMN, and BST are the vertices of a rectangle.

Problem 4 Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that

$$2n + 2001 \leq f(f(n)) + f(n) \leq 2n + 2003$$

for all positive integers n.

4.4 Baltic Team Contest

Problem 1 A spider is sitting on a cube. A fly lands on the cube, hoping to maximize the length of the shortest path to the spider along the surface of the cube. Can the fly guarantee doing so by choosing the point directly opposite the spider (i.e., the point that is the reflection of the spider's position across the cube's center)?

Problem 2 Find all nonnegative integers m such that $(2^{2m+1})^2 + 1$ is divisible by at most two different primes.

Problem 3 Show that the sequence

$$\binom{2002}{2002}, \binom{2003}{2002}, \binom{2004}{2002}, \ldots,$$

considered modulo 2002, is periodic.

Problem 4 Find all integers $n > 1$ such that any prime divisor of $n^6 - 1$ is a divisor of $(n^3 - 1)(n^2 - 1)$.

Problem 5 Let n be a positive integer. Prove that the equation

$$x + y + \frac{1}{x} + \frac{1}{y} = 3n$$

does not have solutions in positive rational numbers.

Problem 6 Does there exist an infinite, non-constant arithmetic progressions, each term of which is of the form a^b where a and b are positive integers with $b \geq 2$?

4.5 Czech-Polish-Slovak Mathematical C ompetition

Problem 1 Let a and b be distinct real numbers, and let k and m be positive integers with $k+m=n\geq 3$, $k\leq 2m$, and $m\leq 2k$. We consider n-tuples $(x_1,x_2,\ldots,x_n)$ with the following properties:

(i) k of the x_i are equal to a, and in particular $x_1=a$;

(ii) m of the x_i are equal to b, and in particular $x_n=b$;

(iii) no three consecutive terms of $x_1,x_2,\ldots,x_n$ are equal.

Determine all possible values of the sum

$$x_nx_1x_2+x_1x_2x_3+\cdots+x_{n-1}x_nx_1.$$

Problem 2 Given is a triangle ABC with side lengths $BC=a\leq CA=b\leq AB=c$ and area S. Let P be a variable point inside triangle ABC, and let D,E,F be the intersections of rays AP,BP,CP with the opposite sides of the triangle. Determine (as a function of a, b, c, and S) the greatest number u and the least number v such that $u\leq PD+PE+PF\leq v$ for all such P.

Problem 3 Let n be a given positive integer, and let $S=\{1,2,\ldots,n\}$. How many functions $f:S\to S$ are there such that $x+f(f(f(f(x))))=n+1$ for all $x\in S$?

Problem 4 Let n,p be integers such that $n>1$ and p is a prime. If $n\mid(p-1)$ and $p\mid(n^3-1)$, show that $4p-3$ is a perfect square.

Problem 5 In acute triangle ABC with circumcenter O, points P and Q lie on sides $\overline{AC}$ and $\overline{BC}$, respectively. Suppose that

$$\frac{AP}{PQ}=\frac{BC}{AB}\quad\text{and}\quad\frac{BQ}{PQ}=\frac{AC}{AB}.$$

Show that O, P, Q, and C are concyclic.

Problem 6 Let $n\geq 2$ be a fixed even integer. We consider polynomials of the form

$$x^n+a_{n-1}x^{n-1}+\cdots+a_1x+1$$

with real coefficients, having at least one real root. Determine the least possible value of the sum $a_1^2+\cdots+a_{n-1}^2$.

4.6 Mediterranean Mathematical Comp etition

Problem 1 Find all positive integers x, y such that $y \mid (x^2+1)$ and $x^2 \mid (y^3+1)$.

Problem 2 Let x, y, a be real numbers such that

$$x + y = x^3 + y^3 = x^5 + y^5 = a.$$

Determine all positive values of a.

Problem 3 Let ABC be an acute triangle. Let M and N be points on the interiors of sides $\overline{AC}$ and $\overline{BC}$, respectively, and let K be the midpoint of segment $\overline{MN}$. The circumcircles of triangles CAN and BCM meet at C and at a second point D. Prove that line CD passes through the circumcircle of triangle ABC if and only if the perpendicular bisector of segment $\overline{AB}$ passes through K.

4.7 St. Petersburg City Mathematical Olympiad (Russia)

Problem 1 Positive numbers a, b, c, d, x, y, and z satisfy $a + x = b + y = c + z = 1$. Prove that

$$(abc + xyz)\left(\frac{1}{ay} + \frac{1}{bz} + \frac{1}{cx}\right) \geq 3.$$

Problem 2 Let $ABCD$ be a convex quadrilateral such that $\angle ABC = 90°$, $AC = CD$, and $\angle BCA = \angle ACD$. Let E be the midpoint of segment AD, and L be the intersection point of segments BF and AC. Prove that $BC = CL$.

Problem 3 One can perform the following operations on a positive integer:

(i) raise it to any positive integer power;

(ii) cut out the last two digits of the integer, multiply the obtained two-digit number by 3, and add it to the number formed by the remaining digits of the initial integer. (For example, from 3456789 one can get $34567 + 3 \cdot 89$.)

Is it possible to obtain 82 from 81 by using operations (i) and (ii)?

Problem 4 Points M and N are marked on diagonals AC and BD of cyclic quadrilateral $ABCD$. Given that $\frac{BM}{DN} = \frac{AM}{CM}$ and $\angle BAD = \angle BMC$, prove that $\angle ANB = \angle ADC$.

Problem 5 A country consists of no fewer than 100,000 cities, where 2001 paths are outgoing from each city. Each path connects two cities, and every pair of cities is connected by no more than one path. The government decides to close some of the paths (at least one but not all) so that the number of paths outgoing from each city is the same. Is this always possible?

Problem 6 Let ABC be a triangle and let I be the center of its incircle ω. The circle Γ passes through I and is tangent to AB and AC at points X and Y, respectively. Prove that segment $\overline{XY}$ is tangent to ω.

Problem 7 Several 1×3 rectangles and 100 L-shaped figures formed by three unit squares ("corners") are situated on a grid plane. It is known that these figures can be shifted parallel to themselves so that the resulting figure is a rectangle. A student Olya can translate 96 corners to form 48 2×3 rectangles. Prove that the remaining four corners can be translated to form two additional 2×3 rectangles.

Problem 8 The sequence $\{a_n\}$ is given by the following relation:

$$a_{n+1} = \begin{cases} (a_n - 1)/2 & \text{if } a_n \geq 1, \\ 2a_n/(1 - a_n) & \text{if } a_n < 1. \end{cases}$$

Given that a_0 is a positive integer, $a_n \neq 2$ for each $n = 1, 2, \dots, 2001$, and $a_{2002} = 2$, find a_0.

Problem 9 There are two 2-pan balances in a zoo for weighing animals. An elephant is on a pan of the first balance and a camel is on a pan of the second balance. The weights of both animals are whole numbers, and their total does not exceed 2000. A set of weights, totaling 2000, have been delivered to the zoo, where each weight is a whole number. It turns out that no matter what the elephant's and the camel's weights are, one can distribute some of the weights over the balances' four pans so that both balances are in equilibrium. Find the minimum number of weights that could have been delivered to the zoo.

Problem 10 The integer $N = \overline{a0a0\ldots a0b0c0c0\ldots c0}$, where the digits a and c are written 1001 times each, is divisible by 37. Prove that $b = a + c$.

Problem 11 Let $ABCD$ be a trapezoid such that the length of lateral side $\overline{AB}$ equals the sum of the lengths of bases $\overline{AD}$ and $\overline{BC}$. Prove that the bisectors of angles A and B meet at a point on side $\overline{CD}$.

Problem 12 Can the sum of the pairwise distances between the vertices of a 25-vertex tree be equal to 1225?

Problem 13 The integers from 5 to 10 are written on a blackboard. Each minute, Kolya erases three or four of the smallest integers and writes down seven or eight consecutive integers following the largest integer on the board. Prove that the sum of all the integers on the blackboard is never a power of 3.

Problem 14 Find the maximal value of $\alpha > 0$ for which any set of eleven real numbers,

$$0 = a_1 \leq a_2 \leq \cdots \leq a_{11} = 1,$$

can be split into two disjoint subsets with the following property: the arithmetic mean of the numbers in the first subset differs from the arithmetic mean of the numbers in the second subset by at most α.

Problem 15 Let O be the circumcenter of acute scalene triangle ABC, C_1 be the point symmetric to C with respect to O, D be the midpoint of side AB, and K be the circumcenter of triangle ODC_1. Prove that point O divides into two equal halves the segment of line OK that lies inside angle ACB.

Problem 16 Polygon $\mathcal{P}$ has the following two properties: (i) no three vertices of $\mathcal{P}$ are collinear; and (ii) there are at least two ways that $\mathcal{P}$ can be dissected into triangles by drawing non-intersecting diagonals of $\mathcal{P}$. Prove that some four vertices of $\mathcal{P}$ form a convex quadrilateral lying entirely inside $\mathcal{P}$.

Problem 17 Let p be a prime number. Given that the equation

$$p^k + p^l + p^m = n^2$$

has an integer solution, prove that $p + 1$ is divisible by 8.

Problem 18 An alchemist has 50 different substances. He can convert any 49 substances taken in equal quantities into the remaining substance without changing the total mass. Prove that, after a finite number of manipulations, the alchemist can obtain the same amount of each of the 50 substances.

Problem 19 Let $ABCD$ be a cyclic quadrilateral. Points X and Y are marked on sides $\overline{AB}$ and $\overline{BC}$ such that quadrilateral $XBYD$ is a parallelogram. Points M and N are the midpoints of diagonals $\overline{AC}$ and $\overline{BD}$, and lines AC and XY meet at point L. Prove that points M, N, L, and D are concyclic.

Problem 20 Two players play the following game. There are 64 vertices on the plane at the beginning. On each turn, the first player picks any two vertices that do not yet have an edge between them and connects them with an edge, and the second player introduces a direction on this edge. The second player wins if the graph obtained after 1959 turns is connected; otherwise the first player wins. Which player has a winning strategy?

Problem 21 The shape of a lakeside is a convex centrally-symmetric 100-gon $A_1A_2\ldots A_{100}$ with center of symmetry O. There is a polygonal island $B_1B_2\ldots B_{100}$ in the lake whose vertices B_i are the midpoints of the segments $\overline{OA_i}$, $i = 1, 2, \ldots, 100$. There is a jail on the island surrounded with a high fence along its perimeter. Two security guards are situated at the opposite points on the lakeside. Prove that every point on the lakeside can be observed by at least one of the guards.

Problem 22 Each of the FBI's safes has a secret code that is a positive integer between 1 and 1700, inclusive. Two spies learn the codes of two different safes and decide to exchange their information. Coordinating beforehand, they meet at the shore of a river near a pile of 26 rocks. The first spy throws several rocks into the water, then the second, then the first, and so on until all the rocks are used. The spies leave after that, without having said a word to each other. How could the information have been transmitted?

Problem 23 A flea jumps along integer points on the real line, starting from the origin. The length of each its jumps is 1. During each jump, the flea sings one of $(p-1)/2$ songs, where p is an odd prime. Consider all of the flea's musical paths from the origin back to the origin consisting of no more than $p-1$ jumps. Prove that the number of such paths is divisible by p.

Problem 24 Let $ABCD$ be a circumscribed quadrilateral with O the center of its inscribed circle. A line ℓ passes through O and meets sides $\overline{AB}$ and $\overline{CD}$ at point X and Y, respectively. Given that $\angle AXY = \angle DYX$, prove that $AX/BX = CY/DY$.

Problem 25 Let $a_n = F_n^n$, where F_n is the n^{th} Fibonacci number ($F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$). Is the sequence $b_n = \sqrt{a_1 + \sqrt{a_2 + \cdots + \sqrt{a_n}}}$ bounded above?

Problem 26 Let a and b be positive integers such that $2a+1$ and $2b+1$ are relatively prime. Find all possible values of $\gcd(2^{2a+1} + 2^{2a+1} + 1, 2^{2b+1} + 2^{2b+1} + 1)$.

Problem 27 Let O be the center of the incircle ω of triangle ABC. Let the tangency points of ω with $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ be A_1, B_1, and C_1, respectively. The perpendicular to line AA_1 at A_1 meets line

B_1C_1 at X. Prove that line BC passes through the midpoint of segment $\overline{AX}$.

Problem 28 A positive integer is written on a blackboard. Dima and Sasha play the following game. Dima calls some positive integer x, and Sasha adds $\pm x$ to the number on the blackboard. They repeat this procedure many times. Dima's goal is to get a nonnegative power of a particular positive integer k on the board. Find all possible values of k for which Dima will be able to do this regardless of the initial number written on the board.

Problem 29 Find all continuous functions $f : (0, \infty) \to (0, \infty)$ such that for all positive real numbers x and y,

$$f(x)f(y) = f(xy) + f\left(\frac{x}{y}\right).$$