Mock Olympiad

Canada Winter Camp 2015

1. Let $n \geq 2$. Given n positive real numbers $x_1, \ldots, x_n$ with $x_1 + x_2 + \cdots + x_n = 1$ prove that
$$\left(\frac{1}{x_1^2} - 1\right)\left(\frac{1}{x_2^2} - 1\right)\cdots\left(\frac{1}{x_n^2} - 1\right) \geq (n^2 - 1)^n$$

2. Recall that for any positive integer m, $\phi(m)$ denotes the number of positive integers less than m which are relatively prime to m. Let n be an odd positive integer such that both $\phi(n)$ and $\phi(n+1)$ are powers of two. Prove $n+1$ is power of two or $n = 5$.

3. Let ω be a semicircle with diameter AB and center O. A line intersects ω at C and D and intersects the line AB at M with $|MB| < |MA|$ and $|MD| < |MC|$. The circumcircles of triangles ΔAOC and ΔDOB meet again at K. Prove that $\angle MKO = 90$.

4. A $2^n \times n$ matrix of 1's and -1's is such that its 2^n rows are pairwise distinct. An arbitrary subset of the entries of the matrix are changed to 0. Prove that there is a nonempty subset of the rows of the altered matrix that sum to the zero vector.