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2011

Set 1

1. Let ABC be a triangle in which $\angle BAC = 60^\circ$. Let P (similarly Q) be the point of intersection of the bisector of $\angle ABC$(similarly of $\angle ACB$) and the side AC(similarly AB). Let r_1 and r_2 be the in-radii of the triangles ABC and APQ, respectively. Determine the circum-radius of APQ in terms of r_1 and r_2.

2. Let $\tau(n)$ be the number of positive divisors of a natural number n, and $\sigma(n)$ be their sum. Find the largest real number α such that

$$\frac{\sigma(n)}{\tau(n)} \geq \alpha\sqrt{n}$$

for all $n \geq 1$.

3. Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function such that

$$2f(f(x)) = (x^2 - x)f(x) + 4 - 2x$$

for all real x. Find $f(2)$ and all possible values of $f(1)$. For each value of $f(1)$, construct a function achieving it and satisfying the given equation.

4. In a lottery, a person must select six distinct numbers from $1, 2, 3, \ldots, 36$ to put on a ticket. The lottery commitee will then draw six distinct numbers randomly from $1, 2, 3, \ldots, 36$. Any ticket with numbers not containing any of these 6 numbers is a winning ticket. Show that there is a scheme of buying 9 tickets guaranteeing at least one winning ticket, but 8 tickets are not enough to guarantee a winning ticket in general.

5. Let $< a_n >$ be a sequence of non-negative real numbers such that $a_{m+n} \leq a_m + a_n$ for all $m, n \in \mathbb{N}$. Prove that

$$\sum_{k=1}^{N} \frac{a_k}{k^2} \geq \frac{a_N}{4N} \ln N$$

for any $N \in \mathbb{N}$, where ln denotes the natural logarithm.

6. Let T be an isosceles right triangle. Let S be the circle such that the dierence in the areas of $T \cup S$ and $T \cap S$ is the minimal. Prove that the centre of S divides the altitude drawn on the hypotenuse of T in the golden ratio (i.e., $\frac{(1+\sqrt{5})}{2}$)

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Set 2

$\boxed{1}$ Let X be the set of all positive real numbers. Find all functions $f : X \longrightarrow X$ such that

$$f(x + y) \geq f(x) + yf(f(x))$$

for all x and y in X.

$\boxed{2}$ Let ABC be an acute triangle with $\angle BAC = 30^\circ$. The internal and external angle bisectors of $\angle ABC$ meet the line AC at B_1 and B_2 , respectively, and the internal and external angle bisectors of $\angle ACB$ meet the line AB at C_1 and C_2 , respectively. Suppose that the circles with diameters B_1B_2 and C_1C_2 meet inside the triangle ABC at point P . Prove that $\angle BPC = 90^\circ$.

$\boxed{3}$ Let C be a circle, $A_1, A_2, \ldots, A_n$ be distinct points inside C and $B_1, B_2, \ldots, B_n$ be distinct points on C such that no two of the segments $A_1B_1, A_2B_2, \ldots, A_nB_n$ intersect. A grasshopper can jump from A_r to A_s if the line segment A_rA_s does not intersect any line segment $A_tB_t (t \neq r, s)$. Prove that after a certain number of jumps, the grasshopper can jump from any A_u to any A_v .

$\boxed{4}$ For all $a, b, c > 0$ and $abc = 1$, prove that

$$\frac{1}{a(a+1) + ab(ab+1)} + \frac{1}{b(b+1) + bc(bc+1)} + \frac{1}{c(c+1) + ca(ca+1)} \geq \frac{3}{4}$$

$\boxed{5}$ Let $(a_n)_{n\geq 1}$ be a sequence of integers that satises

$$a_n = a_{n-1} - \min(a_{n-2}, a_{n-3})$$

for all $n \geq 4$. Prove that for every positive integer k, there is an n such that a_n is divisible by 3^k .

$\boxed{6}$ A positive integer is called *monotonic* if when written in base 10, the digits are weakly increasing. Thus 12226778 is monotonic. Note that a positive integer cannot have rst digit which is 0. Prove that for every positive integer n, there is an n-digit monotonic number which is a perfect square.

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Set 3

$\boxed{1}$ Let I be the incentre of a triangle ABC and Γ_a be the excircle opposite A touching BC at D. If ID meets Γ_a again at S, prove that DS bisects $\angle BSC$.

$\boxed{2}$ For a positive integer n, consider the set

$$S = \{0, 1, 1+2, 1+2+3, \ldots, 1+2+3+\ldots+(n-1)\}$$

Prove that the elements of S are mutually incongruent modulo n if and only if n is a power of 2.

$\boxed{3}$ Let $P(x)$ be a polynomial with integer coecients. Given that for some integer a and some positive integer n, where

$$\underbrace{P(P(\ldots P}_{\text{n times}}(a)\ldots)) = a,$$

is it true that $P(P(a)) = a$?

$\boxed{4}$ Consider 2011^2 points arranged in the form of a 2011×2011 grid. What is the maximum number of points that can be chosen among them so that no four of them form the vertices of either an isosceles trapezium or a rectangle whose parallel sides are parallel to the grid lines?

$\boxed{5}$ Let P be a point inside a triangle ABC such that

$$\angle PAB = \angle PBC = \angle PCA$$

Suppose AP, BP, CP meet the circumcircles of triangles PBC, PCA, PAB at X, Y, Z respectively ($\neq P$) . Prove that

$$[XBC] + [YCA] + [ZAB] \geq 3[ABC]$$

$\boxed{6}$ In a party among any four persons there are three people who are mutual acquaintances or mutual strangers. Prove that all the people can be separated into two groups A and B such that in A everybody knows everybody else and in B nobody knows anybody else.

Set 4

1 Prove that, for any positive integer n, there exists a polynomial $p(x)$ of degree at most n whose coecients are all integers such that, $p(k)$ is divisible by 2^n for every even integer k, and $p(k) - 1$ is divisible by 2^n for every odd integer k.

2 Let x be a positive real number and let k be a positive integer. Assume that $x^k + \frac{1}{x^k}$ and $x^{k+1} + \frac{1}{x^{k+1}}$ are both rational numbers. Prove that $x + \frac{1}{x}$ is also a rational number.

3 Construct a triangle, by straight edge and compass, if the three points where the extensions of the medians intersect the circumcircle of the triangle are given.

4 Let a, b, c be positive integers for which

$$ac = b^2 + b + 1$$

Prove that the equation

$$ax^2 - (2b+1)xy + cy^2 = 1$$

has an integer solution.

5 The seats in the Parliament of some country are arranged in a rectangle of 10 rows of 10 seats each. All the 100 MPs have dierent salaries. Each of them asks all his neighbours (sitting next to, in front of, or behind him, i.e. 4 members at most) how much they earn. They feel a lot of envy towards each other: an MP is content with his salary only if he has at most one neighbour who earns more than himself. What is the maximum possible number of MPs who are satised with their salaries?

6 On a circle there are n red and n blue arcs given in such a way that each red arc intersects each blue one. Prove that some point is contained by at least n of the given coloured arcs.

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Set 5

1. Does the sequence

$$11, 111, 1111, 11111, \ldots$$

contain any fth power of a positive integer? Justify your answer.

2. For which $n \geq 1$ is it possible to place the numbers $1, 2, \ldots, n$ in some order (a) on a line segment, or (b) on a circle so that for every s from 1 to $\frac{n(n+1)}{2}$, there is a connected subset of the segement or circle such that the sum of the numbers in that subset is s?

3. Let ABC be a scalene triangle. Let l_A be the tangent to the nine-point circle at the foot of the perpendicular from A to BC, and let l'_A be the tangent to the nine-point circle from the mid-point of BC. The lines l_A and l'_A intersect at A' . Dene B' and C' similarly. Show that the lines AA', BB' and CC' are concurrent.

4. Let $n > 1$ be a positive integer. Find all n-tuples $(a_1, a_2, \ldots, a_n)$ of positive integers which are pairwise distinct, pairwise coprime, and such that for each i in the range $1 \leq i \leq n$,

$$(a_1 + a_2 + \ldots + a_n)|(a_1^i + a_2^i + \ldots + a_n^i)$$

.

5. Let a, b and c be positive real numbers. Prove that

$$\frac{\sqrt{a^2+bc}}{b+c} + \frac{\sqrt{b^2+ca}}{c+a} + \frac{\sqrt{c^2+ab}}{a+b} \geq \sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}}$$

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Set 6

$\boxed{1}$ Let $ABCD$ be a quadrilateral with an inscribed circle, centre O. Let

$$AO = 5, BO = 6, CO = 7, DO = 8.$$

If M and N are the midpoints of the diagonals AC and BD, determine $\frac{OM}{ON}$.

$\boxed{2}$ Let $S(k)$ denote the digit-sum of a positive integer k(in base 10). Determine the smallest positive integer n such that

$$S(n^2) = S(n) - 7$$

$\boxed{3}$ Let $f : \mathbb{N} \longrightarrow \mathbb{N}$ be a function such that $(x + y)f(x) \leq x^2 + f(xy) + 110$, for all x, y in $\mathbb{N}$. Determine the minimum and maximum values of $f(23) + f(2011)$.

$\boxed{4}$ Suppose there are n boxes in a row and place n balls in them one in each. The balls are colored red, blue or green. In how many ways can we place the balls subject to the condition that any box B has at least one adjacent box having a ball of the same color as the ball in B? [Assume that balls in each color are available abundantly.]

$\boxed{5}$ Let H be the orthocentre and O be the circumcentre of an acute triangle ABC. Let AD and BE be the altitudes of the triangle with D on BC and E on CA. Let $K = OD \cap BE, L = OE \cap AD$. Let X be the second point of intersection of the circumcircles of triangles HKD and HLE, and let M be the midpoint of side AB. Prove that points K, L, M are collinear if and only if X is the circumcentre of triangle EOD.

$\boxed{6}$ Prove that there exist integers a, b, c all greater than 2011 such that

$$(a + \sqrt{b})^c = \ldots 2010 \cdot 2011 \ldots$$

[Decimal point separates an integer ending in 2010 and a decimal part beginning with 2011.]