All-Russian Olympiad 1996

– Grade level 9

Day 1

1 Which are there more of among the natural numbers from 1 to 1000000, inclusive: numbers that can be represented as the sum of a perfect square and a (positive) perfect cube, or numbers that cannot be?

A. Golovanov

2 The centers O_1; O_2; O_3 of three nonintersecting circles of equal radius are positioned at the vertices of a triangle. From each of the points O_1; O_2; O_3 one draws tangents to the other two given circles. It is
known that the intersection of these tangents form a convex hexagon. The sides of the hexagon are alternately colored red and blue. Prove that the sum of the lengths of the red sides equals the sum of the lengths of the blue sides.

D. Tereshin

3 Let x, y, p, n, and k be positive integers such that $x^n + y^n = p^k$. Prove that if $n > 1$ is odd, and p is an odd prime, then n is a power of p.

A. Kovaldji, V. Senderov

4 In the Duma there are 1600 delegates, who have formed 16000 committees of 80 persons each. Prove that one can find two committees having no fewer than four common members.

A. Skopenkov

Day 2

5 Show that in the arithmetic progression with first term 1 and ratio 729, there are infinitely many powers of 10.

L. Kuptsov

6 In the isosceles triangle ABC ($AC = BC$) point O is the circumcenter, I the incenter, and D lies on BC so that lines OD and BI are perpendicular. Prove that ID and AC are parallel.

M. Sonkin

7 Two piles of coins lie on a table. It is known that the sum of the weights of the coins in the two piles are equal, and for any natural number k, not exceeding the number of coins in either pile, the sum of the weights of the k heaviest coins in the first pile is not more than that of the second pile. Show that for any natural number x, if each coin (in either pile) of weight not less than x is replaced by a coin of weight x, the first pile will not be lighter than the second.

D. Fon-der-Flaas

8 Can a 5×7 checkerboard be covered by L's (figures formed from a 2×2 square by removing one of its four 1×1 corners), not crossing its borders, in several layers so that each square of the board is covered by the same number of L's?

M. Evdokimov

– Grade level 10

Day 1

1 Points E and F are given on side BC of convex quadrilateral $ABCD$ (with E closer than F to B). It is known that $\angle BAE = \angle CDF$ and $\angle EAF = \angle FDE$. Prove that $\angle FAC = \angle EDB$.

M. Smurov

2 On a coordinate plane are placed four counters, each of whose centers has integer coordinates. One can displace any counter by the vector joining the centers of two of the other counters. Prove that any two preselected counters can be made to coincide by a finite sequence of moves.

. Sadykov

3 Find all natural numbers n, such that there exist relatively prime integers x and y and an integer $k > 1$ satisfying the equation $3^n = x^k + y^k$.

A. Kovaldji, V. Senderov

4 Show that if the integers $a_1; \dots\ a_m$ are nonzero and for each $k = 0; 1; \dots; n$ ($n < m - 1$), $a_1 + a_2 2^k + a_3 3^k + \cdots + a_m m^k = 0$; then the sequence $a_1, \dots, a_m$ contains at least $n + 1$ pairs of consecutive terms having opposite signs.

O. Musin

Day 2

5 At the vertices of a cube are written eight pairwise distinct natural numbers, and on each of its edges is written the greatest common divisor of the numbers at the endpoints of the edge. Can the sum of the numbers written at the vertices be the same as the sum of the numbers written at the edges?

A. Shapovalov

6 Three sergeants and several solders serve in a platoon. The sergeants take turns on duty. The commander has given the following orders:
(a) Each day, at least one task must be issued to a soldier.
(b) No soldier may have more than two task or receive more than one tasks in a single day.
(c) The lists of soldiers receiving tasks for two different days must not be the same.
(d) The first sergeant violating any of these orders will be jailed.
Can at least one of the sergeants, without conspiring with the others, give tasks according to these rules and avoid being jailed?

M. Kulikov

7 A convex polygon is given, no two of whose sides are parallel. For each side we consider the angle the side subtends at the vertex farthest from the side. Show that the sum of these angles equals 180°.

M. Smurov

8 Goodnik writes 10 numbers on the board, then Nogoodnik writes 10 more numbers, all 20 of the numbers being positive and distinct. Can Goodnik choose his 10 numbers so that no matter what Nogoodnik writes, he can form 10 quadratic trinomials of the form $x^2 + px + q$, whose coeficients p and q run through all of the numbers written, such that the real roots of these trinomials comprise exactly 11 values?

I. Rubanov

– Grade level 11

Day 1

1 Can the number obtained by writing the numbers from 1 to n in order $(n > 1)$ be the same when read left-to-right and right-to-left?

N. Agakhanov

2 Several hikers travel at fixed speeds along a straight road. It is known that over some period of time, the sum of their pairwise distances is monotonically decreasing. Show that there is a hiker, the sum of whose distances to the other hikers is monotonically decreasing over the same period.

A. Shapovalov

---

3 Show that for $n \geq 5$, a cross-section of a pyramid whose base is a regular n-gon cannot be a regular $(n+1)$-gon.

N. Agakhanov, N. Tereshin

---

4 Show that if the integers $a_1; \dots a_m$ are nonzero and for each $k = 0; 1; \dots; n$ $(n < m-1)$, $a_1 + a_2 2^k + a_3 3^k + \dots + a_m m^k = 0$; then the sequence $a_1, \dots, a_m$ contains at least $n+1$ pairs of consecutive terms having opposite signs.

O. Musin

Day 2

---

5 Do there exist three natural numbers greater than 1, such that the square of each, minus one, is divisible by each of the others?

A. Golovanov

---

6 In isosceles triangle ABC $(AB = BC)$ one draws the angle bisector CD. The perpendicular to CD through the center of the circumcircle of ABC intersects BC at E. The parallel to CD through E meets AB at F. Show that $BE = FD$.

M. Sonkin

---

7 Does there exist a finite set M of nonzero real numbers, such that for any natural number n a polynomial of degree no less than n with coeficients in M, all of whose roots are real and belong to M?

E. Malinnikova

---

8 The numbers from 1 to 100 are written in an unknown order. One may ask about any 50 numbers and find out their relative order. What is the fewest questions needed to find the order of all 100 numbers?

S. Tokarev
