IMO 2003

Tokio, Japan

Day 1 - *13 July 2003*

$\boxed{1}$ Let A be a 101-element subset of the set $S = \{1, 2, \ldots, 1000000\}$. Prove that there exist numbers t_1, $t_2, \ldots, t_{100}$ in S such that the sets

$$A_j = \{x + t_j \mid x \in A\}, \qquad j = 1, 2, \ldots, 100$$

are pairwise disjoint.

$\boxed{2}$ Determine all pairs of positive integers (a, b) such that

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer.

$\boxed{3}$ Each pair of opposite sides of a convex hexagon has the following property: the distance between their midpoints is equal to $\dfrac{\sqrt{3}}{2}$ times the sum of their lengths. Prove that all the angles of the hexagon are equal.

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Day 2 - *14 July 2003*

$\boxed{4}$ Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA, AB, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC.

$\boxed{5}$ Let n be a positive integer and let $x_1 \leq x_2 \leq \cdots \leq x_n$ be real numbers. Prove that

$$\left(\sum_{i,j=1}^{n} |x_i - x_j|\right)^2 \leq \frac{2(n^2-1)}{3} \sum_{i,j=1}^{n} (x_i - x_j)^2.$$

Show that the equality holds if and only if $x_1, \ldots, x_n$ is an arithmetic sequence.

$\boxed{6}$ Let p be a prime number. Prove that there exists a prime number q such that for every integer n, the number $n^p - p$ is not divisible by q.