USA Team Selection Test 2002

Day 1 June 21st

1 Let ABC be a triangle, and A, B, C its angles. Prove that

$$\sin\frac{3A}{2}+\sin\frac{3B}{2}+\sin\frac{3C}{2}\le\cos\frac{A-B}{2}+\cos\frac{B-C}{2}+\cos\frac{C-A}{2}.$$

2 Let $p > 5$ be a prime number. For any integer x, define

$$f_p(x)=\sum_{k=1}^{p-1}\frac{1}{(px+k)^2}$$

Prove that for any pair of positive integers x, y, the numerator of $f_p(x)-f_p(y)$, when written as a fraction in lowest terms, is divisible by p^3.

3 Let n be an integer greater than 2, and $P_1, P_2, \cdots, P_n$ distinct points in the plane. Let $\mathcal{S}$ denote the union of all segments $P_1P_2, P_2P_3, \dots, P_{n-1}P_n$. Determine if it is always possible to find points A and B in $\mathcal{S}$ such that $P_1P_n \parallel AB$ (segment AB can lie on line P_1P_n) and $P_1P_n = kAB$, where (1) $k = 2.5$; (2) $k = 3$.

Day 2 June 22nd

4 Let n be a positive integer and let S be a set of $2^n + 1$ elements. Let f be a function from the set of two-element subsets of S to $\{0, \dots, 2^{n-1} - 1\}$. Assume that for any elements x, y, z of S, one of $f(\{x, y\}), f(\{y, z\}), f(\{z, x\})$ is equal to the sum of the other two. Show that there exist a, b, c in S such that $f(\{a, b\}), f(\{b, c\}), f(\{c, a\})$ are all equal to 0.

5 Consider the family of nonisosceles triangles ABC satisfying the property $AC^2 + BC^2 = 2AB^2$. Points M and D lie on side AB such that $AM = BM$ and $\angle ACD = \angle BCD$. Point E is in the plane such that D is the incenter of triangle CEM. Prove that exactly one of the ratios

$$\frac{CE}{EM},\quad \frac{EM}{MC},\quad \frac{MC}{CE}$$

is constant.

6 Find in explicit form all ordered pairs of positive integers (m, n) such that $mn - 1$ divides $m^2 + n^2$.