

# Tournament Of Towns 2007

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## Fall - Junior A-Level

- [1] Let  $ABCD$  be a rhombus. Let  $K$  be a point on the line  $CD$ , other than  $C$  or  $D$ , such that  $AD = BK$ . Let  $P$  be the point of intersection of  $BD$  with the perpendicular bisector of  $BC$ . Prove that  $A, K$  and  $P$  are collinear.
- [2] (a) Each of Peter and Basil thinks of three positive integers. For each pair of his numbers, Peter writes down the greatest common divisor of the two numbers. For each pair of his numbers, Basil writes down the least common multiple of the two numbers. If both Peter and Basil write down the same three numbers, prove that these three numbers are equal to each other.  
(b) Can the analogous result be proved if each of Peter and Basil thinks of four positive integers instead?
- [3] Michael is at the centre of a circle of radius 100 metres. Each minute, he will announce the direction in which he will be moving. Catherine can leave it as is, or change it to the opposite direction. Then Michael moves exactly 1 metre in the direction determined by Catherine. Does Michael have a strategy which guarantees that he can get out of the circle, even though Catherine will try to stop him?
- [4] Two players take turns entering a symbol in an empty cell of a  $1 \times n$  chessboard, where  $n$  is an integer greater than 1. Aaron always enters the symbol  $X$  and Betty always enters the symbol  $O$ . Two identical symbols may not occupy adjacent cells. A player without a move loses the game. If Aaron goes first, which player has a winning strategy?
- [5] Attached to each of a number of objects is a tag which states the correct mass of the object. The tags have fallen off and have been replaced on the objects at random. We wish to determine if by chance all tags are in fact correct. We may use exactly once a horizontal lever which is supported at its middle. The objects can be hung from the lever at any point on either side of the support. The lever either stays horizontal or tilts to one side. Is this task always possible?
- [6] The audience arranges  $n$  coins in a row. The sequence of heads and tails is chosen arbitrarily. The audience also chooses a number between 1 and  $n$  inclusive. Then the assistant turns one of the coins over, and the magician is brought in to examine the resulting sequence. By an agreement with the assistant beforehand, the magician tries to determine the number chosen by the audience. (a) Prove that if this is possible for some  $n$ , then it is also possible for  $2n$ .  
(b) Determine all  $n$  for which this is possible.

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- 7 For each letter in the English alphabet, William assigns an English word which contains that letter. His first document consists only of the word assigned to the letter  $A$ . In each subsequent document, he replaces each letter of the preceding document by its assigned word. The fortieth document begins with Till whatsoever star that guides my moving. Prove that this sentence reappears later in this document.

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## Fall - Junior O-Level

- 1 Black and white checkers are placed on an  $8 \times 8$  chessboard, with at most one checker on each cell. What is the maximum number of checkers that can be placed such that each row and each column contains twice as many white checkers as black ones?
- 2 Initially, the number 1 and a non-integral number  $x$  are written on a blackboard. In each step, we can choose two numbers on the blackboard, not necessarily different, and write their sum or their difference on the blackboard. We can also choose a non-zero number of the blackboard and write its reciprocal on the blackboard. Is it possible to write  $x^2$  on the blackboard in a finite number of moves?
- 3  $D$  is the midpoint of the side  $BC$  of triangle  $ABC$ .  $E$  and  $F$  are points on  $CA$  and  $AB$  respectively, such that  $BE$  is perpendicular to  $CA$  and  $CF$  is perpendicular to  $AB$ . If  $DEF$  is an equilateral triangle, does it follow that  $ABC$  is also equilateral?
- 4 Each cell of a  $29 \times 29$  table contains one of the integers  $1, 2, 3, \dots, 29$ , and each of these integers appears 29 times. The sum of all the numbers above the main diagonal is equal to three times the sum of all the numbers below this diagonal. Determine the number in the central cell of the table.
- 5 The audience chooses two of five cards, numbered from 1 to 5 respectively. The assistant of a magician chooses two of the remaining three cards, and asks a member of the audience to take them to the magician, who is in another room. The two cards are presented to the magician in arbitrary order. By an arrangement with the assistant beforehand, the magician is able to deduce which two cards the audience has chosen only from the two cards he receives. Explain how this may be done.

## Fall - Senior A-Level

- 1 (a) Each of Peter and Basil thinks of three positive integers. For each pair of his numbers, Peter writes down the greatest common divisor of the two numbers. For each pair of his numbers, Basil writes down the least common multiple of the two numbers. If both Peter and Basil write down the same three numbers, prove that these three numbers are equal to each other.

(b) Can the analogous result be proved if each of Peter and Basil thinks of four positive integers instead?
- 2 Let  $K, L, M$  and  $N$  be the midpoints of the sides  $AB, BC, CD$  and  $DA$  of a cyclic quadrilateral  $ABCD$ . Let  $P$  be the point of intersection of  $AC$  and  $BD$ . Prove that the circumradii of triangles  $PKL, PLM, PMN$  and  $PNK$  are equal to one another.
- 3 Determine all finite increasing arithmetic progressions in which each term is the reciprocal of a positive integer and the sum of all the terms is 1.
- 4 Attached to each of a number of objects is a tag which states the correct mass of the object. The tags have fallen off and have been replaced on the objects at random. We wish to determine if by chance all tags are in fact correct. We may use exactly once a horizontal lever which is supported at its middle. The objects can be hung from the lever at any point on either side of the support. The lever either stays horizontal or tilts to one side. Is this task always possible?
- 5 The audience arranges  $n$  coins in a row. The sequence of heads and tails is chosen arbitrarily. The audience also chooses a number between 1 and  $n$  inclusive. Then the assistant turns one of the coins over, and the magician is brought in to examine the resulting sequence. By an agreement with the assistant beforehand, the magician tries to determine the number chosen by the audience. (a) Prove that if this is possible for some  $n$ , then it is also possible for  $2n$ . (b) Determine all  $n$  for which this is possible.
- 6 Let  $P$  and  $Q$  be two convex polygons. Let  $h$  be the length of the projection of  $Q$  onto a line perpendicular to a side of  $P$  which is of length  $p$ . Define  $f(P, Q)$  to be the sum of the products  $hp$  over all sides of  $P$ . Prove that  $f(P, Q) = f(Q, P)$ .
- 7 There are 100 boxes, each containing either a red cube or a blue cube. Alex has a sum of money initially, and places bets on the colour of the cube in each box in turn. The bet can be anywhere from 0 up to everything he has at the time. After the bet has been placed, the box is opened. If Alex loses, his bet will be taken away. If he wins, he will get his bet back, plus a sum equal to the bet. Then he moves onto the next box, until he has bet on the last

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one, or until he runs out of money. What is the maximum factor by which he can guarantee to increase his amount of money, if he knows that the exact number of blue cubes is **(a)** 1; **(b)** some integer  $k$ ,  $1 < k \leq 100$ .

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## Fall - Senior O-Level

- 1 Pictures are taken of 100 adults and 100 children, with one adult and one child in each, the adult being the taller of the two. Each picture is reduced to  $\frac{1}{k}$  of its original size, where  $k$  is a positive integer which may vary from picture to picture. Prove that it is possible to have the reduced image of each adult taller than the reduced image of every child.
- 2 Initially, the number 1 and two positive numbers  $x$  and  $y$  are written on a blackboard. In each step, we can choose two numbers on the blackboard, not necessarily different, and write their sum or their difference on the blackboard. We can also choose a non-zero number of the blackboard and write its reciprocal on the blackboard. Is it possible to write on the blackboard, in a finite number of moves, the number **a)**  $x^2$ ; **b)**  $xy$ ?
- 3 Give a construction by straight-edge and compass of a point  $C$  on a line  $\ell$  parallel to a segment  $AB$ , such that the product  $AC \cdot BC$  is minimum.
- 4 The audience chooses two of twenty-nine cards, numbered from 1 to 29 respectively. The assistant of a magician chooses two of the remaining twenty-seven cards, and asks a member of the audience to take them to the magician, who is in another room. The two cards are presented to the magician in an arbitrary order. By an arrangement with the assistant beforehand, the magician is able to deduce which two cards the audience has chosen only from the two cards he receives. Explain how this may be done.
- 5 A square of side length 1 centimeter is cut into three convex polygons. Is it possible that the diameter of each of them does not exceed **a)** 1 centimeter; **b)** 1.01 centimeters; **c)** 1.001 centimeters?

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## Spring - Junior A-Level

- [1] Let  $n$  be a positive integer. In order to find the integer closest to  $\sqrt{n}$ , Mary finds  $a^2$ , the closest perfect square to  $n$ . She thinks that  $a$  is then the number she is looking for. Is she always correct?
- [2]  $K, L, M$  and  $N$  are points on sides  $AB, BC, CD$  and  $DA$ , respectively, of the unit square  $ABCD$  such that  $KM$  is parallel to  $BC$  and  $LN$  is parallel to  $AB$ . The perimeter of triangle  $KLB$  is equal to 1. What is the area of triangle  $MND$ ?
- [3] Anna's number is obtained by writing down 20 consecutive positive integers, one after another in arbitrary order. Bob's number is obtained in the same way, but with 21 consecutive positive integers. Can they obtain the same number?
- [4] Several diagonals (possibly intersecting each other) are drawn in a convex  $n$ -gon in such a way that no three diagonals intersect in one point. If the  $n$ -gon is cut into triangles, what is the maximum possible number of these triangles?
- [5] Find all (finite) increasing arithmetic progressions, consisting only of prime numbers, such that the number of terms is larger than the common difference.
- [6] In the quadrilateral  $ABCD$ ,  $AB = BC = CD$  and  $\angle BMC = 90^\circ$ , where  $M$  is the midpoint of  $AD$ . Determine the acute angle between the lines  $AC$  and  $BD$ .
- [7] Nancy shuffles a deck of 52 cards and spreads the cards out in a circle face up, leaving one spot empty. Andy, who is in another room and does not see the cards, names a card. If this card is adjacent to the empty spot, Nancy moves the card to the empty spot, without telling Andy; otherwise nothing happens. Then Andy names another card and so on, as many times as he likes, until he says "stop." (a) Can Andy guarantee that after he says "stop," no card is in its initial spot? (b) Can Andy guarantee that after he says "stop," the Queen of Spades is not adjacent to the empty spot?

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## Spring - Junior O-Level

- 1 The sides of a convex pentagon are extended on both sides to form five triangles. If these triangles are congruent to one another, does it follow that the pentagon is regular?
- 2 Two 2007-digit numbers are given. It is possible to delete 7 digits from each of them to obtain the same 2000-digit number. Prove that it is also possible to insert 7 digits into the given numbers so as to obtain the same 2014-digit number.
- 3 What is the least number of rooks that can be placed on a standard  $8 \times 8$  chessboard so that all the white squares are attacked? (A rook also attacks the square it is on, in addition to every other square in the same row or column.)
- 4 Three nonzero real numbers are given. If they are written in any order as coefficients of a quadratic trinomial, then each of these trinomials has a real root. Does it follow that each of these trinomials has a positive root?
- 5 A triangular pie has the same shape as its box, except that they are mirror images of each other. We wish to cut the pie in two pieces which can  
t together in the box without turning either piece over. How can this be done if **(a)** one angle of the triangle is three times as big as another; **(b)** one angle of the triangle is obtuse and is twice as big as one of the acute angles?



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## Spring - Senior A-Level

- [1]  $A, B, C$  and  $D$  are points on the parabola  $y = x^2$  such that  $AB$  and  $CD$  intersect on the  $y$ -axis. Determine the  $x$ -coordinate of  $D$  in terms of the  $x$ -coordinates of  $A, B$  and  $C$ , which are  $a, b$  and  $c$  respectively.
- [2] A convex figure  $F$  is such that any equilateral triangle with side 1 has a parallel translation that takes all its vertices to the boundary of  $F$ . Is  $F$  necessarily a circle?
- [3] Let  $f(x)$  be a polynomial of nonzero degree. Can it happen that for any real number  $a$ , an even number of real numbers satisfy the equation  $f(x) = a$ ?
- [4] Nancy shuffles a deck of 52 cards and spreads the cards out in a circle face up, leaving one spot empty. Andy, who is in another room and does not see the cards, names a card. If this card is adjacent to the empty spot, Nancy moves the card to the empty spot, without telling Andy; otherwise nothing happens. Then Andy names another card and so on, as many times as he likes, until he says "stop." **(a)** Can Andy guarantee that after he says "stop," no card is in its initial spot? **(b)** Can Andy guarantee that after he says "stop," the Queen of Spades is not adjacent to the empty spot?
- [5] From a regular octahedron with edge 1, cut off a pyramid about each vertex. The base of each pyramid is a square with edge  $\frac{1}{3}$ . Can copies of the polyhedron so obtained, whose faces are either regular hexagons or squares, be used to tile space?
- [6] Let  $a_0$  be an irrational number such that  $0 < a_0 < \frac{1}{2}$ . Define  $a_n = \min\{2a_{n-1}, 1 - 2a_{n-1}\}$  for  $n \geq 1$ . **(a)** Prove that  $a_n < \frac{3}{16}$  for some  $n$ . **(b)** Can it happen that  $a_n > \frac{7}{40}$  for all  $n$ ?
- [7]  $T$  is a point on the plane of triangle  $ABC$  such that  $\angle ATB = \angle BTC = \angle CTA = 120^\circ$ . Prove that the lines symmetric to  $AT, BT$  and  $CT$  with respect to  $BC, CA$  and  $AB$ , respectively, are concurrent.

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## Spring - Senior O-Level

- [1] A  $9 \times 9$  chessboard with the standard checkered pattern has white squares at its four corners. What is the least number of rooks that can be placed on this board so that all the white squares are attacked? (A rook also attacks the square it is on, in addition to every other square in the same row or column.)
- [2] The polynomial  $x^3 + px^2 + qx + r$  has three roots in the interval  $(0, 2)$ . Prove that  $-2 < p + q + r < 0$ .
- [3]  $B$  is a point on the line which is tangent to a circle at the point  $A$ . The line segment  $AB$  is rotated about the centre of the circle through some angle to the line segment  $A'B'$ . Prove that the line  $AA'$  passes through the midpoint of  $BB'$ .
- [4] A binary sequence is constructed as follows. If the sum of the digits of the positive integer  $k$  is even, the  $k$ -th term of the sequence is 0. Otherwise, it is 1. Prove that this sequence is not periodic.
- [5] A triangular pie has the same shape as its box, except that they are mirror images of each other. We wish to cut the pie in two pieces which can  
t together in the box without turning either piece over. How can this be done if **(a)** one angle of the triangle is three times as big as another; **(b)** one angle of the triangle is obtuse and is twice as big as one of the acute angles?