China Girls Math Olympiad 2008

Day 1

1 *(a)* Determine if the set $\{1, 2, \ldots, 96\}$ can be partitioned into 32 sets of equal size and equal sum.
(b) Determine if the set $\{1, 2, \ldots, 99\}$ can be partitioned into 33 sets of equal size and equal sum.

2 Let $\varphi(x) = ax^3 + bx^2 + cx + d$ be a polynomial with real coefficients. Given that $\varphi(x)$ has three positive real roots and that $\varphi(0) < 0$, prove that

$$2b^3 + 9a^2d - 7abc \le 0.$$

3 Determine the least real number a greater than 1 such that for any point P in the interior of the square $ABCD$, the area ratio between two of the triangles PAB, PBC, PCD, PDA lies in the interval $\left[\frac{1}{a}, a\right]$.

4 Equilateral triangles ABQ, BCR, CDS, DAP are erected outside of the convex quadrilateral $ABCD$. Let X, Y, Z, W be the midpoints of the segments PQ, QR, RS, SP, respectively. Determine the maximum value of

$$\frac{XZ + YW}{AC + BD}.$$

Day 2

5 In convex quadrilateral $ABCD$, $AB = BC$ and $AD = DC$. Point E lies on segment AB and point F lies on segment AD such that B, E, F, D lie on a circle. Point P is such that triangles DPE and ADC are similar and the corresponding vertices are in the same orientation (clockwise or counterclockwise). Point Q is such that triangles BQF and ABC are similar and the corresponding vertices are in the same orientation. Prove that points A, P, Q are collinear.

6 Let $(x_1, x_2, \cdots)$ be a sequence of positive numbers such that $(8x_2 - 7x_1)x_1^7 = 8$ and

$$x_{k+1}x_{k-1} - x_k^2 = \frac{x_{k-1}^8 - x_k^8}{x_k^7 x_{k-1}^7} \text{ for } k = 2, 3, \ldots$$

Determine real number a such that if $x_1 > a$, then the sequence is monotonically decreasing, and if $0 < x_1 < a$, then the sequence is not monotonic.

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7 On a given 2008×2008 chessboard, each unit square is colored in a different color. Every unit square is filled with one of the letters C, G, M, O. The resulting board is called *harmonic* if every 2×2 subsquare contains all four different letters. How many harmonic boards are there?

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8 For positive integers n, $f_n = \lfloor 2^n \sqrt{2008} \rfloor + \lfloor 2^n \sqrt{2009} \rfloor$. Prove there are infinitely many odd numbers and infinitely many even numbers in the sequence $f_1, f_2, \ldots$.