Romania Team Selection Test 2011

Day 1

1 Determine all real-valued functions f on the set of real numbers satisfying

$$2f(x) = f(x+y) + f(x+2y)$$

for all real numbers x and all non-negative real numbers y.

2 Prove that the set $S = \{\lfloor n\pi \rfloor \mid n = 0, 1, 2, 3, \ldots\}$ contains arithmetic progressions of any finite length, but no infinite arithmetic progressions.

Vasile Pop

3 Let ABC be a triangle such that $AB < AC$. The perpendicular bisector of the side BC meets the side AC at the point D, and the (interior) bisectrix of the angle ADB meets the circumcircle ABC at the point E. Prove that the (interior) bisectrix of the angle AEB and the line through the incentres of the triangles ADE and BDE are perpendicular.

4 Given an integer $n \geq 2$, compute $\sum_\sigma \operatorname{sgn}(\sigma) n^{\ell(\sigma)}$, where all n-element permutations are considered, and where $\ell(\sigma)$ is the number of disjoint cycles in the standard decomposition of σ.

Day 2

1 Suppose a square of sidelengh l is inside an unit square and does not contain its centre. Show that $l \leq 1/2$.

Marius Cavachi

2 In triangle ABC, the incircle touches sides BC, CA and AB in D, E and F respectively. Let X be the feet of the altitude of the vertex D on side EF of triangle DEF. Prove that AX, BY and CZ are concurrent on the Euler line of the triangle DEF.

3 Given a positive integer number n, determine the maximum number of edges a simple graph on n vertices may have such that it contain no cycles of even length.

4 Show that:
a) There are infinitely many positive integers n such that there exists a square equal to the sum of the squares of n consecutive positive integers (for instance, 2 is one such number as $5^2 = 3^2 + 4^2$).
b) If n is a positive integer which is not a perfect square, and if x is an integer number such that $x^2 + (x+1)^2 + ... + (x+n-1)^2$ is a perfect square, then there are infinitely many positive integers y such that $y^2 + (y+1)^2 + ... + (y+n-1)^2$ is a perfect square.

Day 3

1 Let $ABCD$ be a cyclic quadrilateral which is not a trapezoid and whose diagonals meet at E. The midpoints of AB and CD are F and G respectively, and ℓ is the line through G parallel to AB. The feet of the perpendiculars from E onto the lines ℓ and CD are H and K, respectively. Prove that the lines EF and HK are perpendicular.

2 Given real numbers x, y, z such that $x + y + z = 0$, show that

$$\frac{x(x+2)}{2x^2+1} + \frac{y(y+2)}{2y^2+1} + \frac{z(z+2)}{2z^2+1} \geq 0$$

When does equality hold?

3 Let S be a finite set of positive integers which has the following property:if x is a member of S,then so are all positive divisors of x. A non-empty subset T of S is *good* if whenever $x, y \in T$ and $x < y$, the ratio y/x is a power of a prime number. A non-empty subset T of S is *bad* if whenever $x, y \in T$ and $x < y$, the ratio y/x is not a power of a prime number. A set of an element is considered both *good* and *bad*. Let k be the largest possible size of a *good* subset of S. Prove that k is also the smallest number of pairwise-disjoint *bad* subsets whose union is S.

4 Let $ABCDEF$ be a convex hexagon of area 1, whose opposite sides are parallel. The lines AB, CD and EF meet in pairs to determine the vertices of a triangle. Similarly, the lines BC, DE and FA meet in pairs to determine the vertices of another triangle. Show that the area of at least one of these two triangles is at least $3/2$.

Day 4

1 Let $ABCD$ be a cyclic quadrilateral. The lines BC and AD meet at a point P. Let Q be the point on the line BP, different from B, such that $PQ = BP$. Consider the parallelograms $CAQR$ and $DBCS$. Prove that the points C, Q, R, S lie on a circle.

2 Let $ABCD$ be a convex quadrangle such that $AB = AC = BD$ (vertices are labelled in circular order). The lines AC and BD meet at point O, the circles ABC and ADO meet again at point P, and the lines AP and BC meet at the point Q. Show that the angles COQ and DOQ are equal.

3 Given a triangle ABC, let D be the midpoint of the side AC and let M be the point that divides the segment BD in the ratio $1/2$; that is, $MB/MD = 1/2$. The rays AM and CM meet the sides BC and AB at points E and F, respectively. Assume the two rays perpendicular: $AM \perp CM$. Show that the quadrangle $AFED$ is cyclic if and only if the median from A in triangle ABC meets the line EF at a point situated on the circle ABC.

Day 5

1 Show that there are infinitely many positive integer numbers n such that n^2+1 has two positive divisors whose difference is n.

2 Let n be an integer number greater than 2, let $x_1, x_2, \ldots, x_n$ be n positive real numbers such that

$$\sum_{i=1}^{n} \frac{1}{x_i+1} = 1$$

and let k be a real number greater than 1. Show that:

$$\sum_{i=1}^{n} \frac{1}{x_i^k+1} \geq \frac{n}{(n-1)^k+1}$$

and determine the cases of equality.

3 Given a set L of lines in general position in the plane (no two lines in L are parallel, and no three lines are concurrent) and another line ℓ, show that the total number of edges of all faces in the corresponding arrangement, intersected by ℓ, is at most $6|L|$.

Chazelle et al., Edelsbrunner et al.

Day 6

1 Given a positive integer number k, define the function f on the set of all positive integer numbers to itself by

$$f(n) = \begin{cases} 1, & \text{if } n \leq k+1 \\ f(f(n-1)) + f(n-f(n-1)), & \text{if } n > k+1 \end{cases}$$

Show that the preimage of every positive integer number under f is a finite non-empty set of consecutive positive integers.

2 Given a prime number p congruent to 1 modulo 5 such that $2p+1$ is also prime, show that there exists a matrix of 0s and 1s containing exactly $4p$ (respectively, $4p+2$) 1s no sub-matrix of which contains exactly $2p$ (respectively, $2p+1$) 1s.

3 The incircle of a triangle ABC touches the sides BC, CA, AB at points D, E, F, respectively. Let X be a point on the incircle, different from the points D, E, F. The lines XD and EF, XE and FD, XF and DE meet at points J, K, L, respectively. Let further M, N, P be points on the sides BC, CA, AB, respectively, such that the lines AM, BN, CP are concurrent. Prove that the lines JM, KN and LP are concurrent.

Dinu Serbanescu