

India
National Olympiad
2013

- [1] Let Γ_1 and Γ_2 be two circles touching each other externally at R . Let O_1 and O_2 be the centres of Γ_1 and Γ_2 , respectively. Let ℓ_1 be a line which is tangent to Γ_2 at P and passing through O_1 , and let ℓ_2 be the line tangent to Γ_1 at Q and passing through O_2 . Let $K = \ell_1 \cap \ell_2$. If $KP = KQ$ then prove that the triangle PQR is equilateral.

- [2] Find all $m, n \in \mathbb{N}$ and primes $p \geq 5$ satisfying

$$m(4m^2 + m + 12) = 3(p^n - 1).$$

- [3] Let $a, b, c, d \in \mathbb{N}$ such that $a \geq b \geq c \geq d$. Show that the equation $x^4 - ax^3 - bx^2 - cx - d = 0$ has no integer solution.
- [4] Let N be an integer greater than 1 and let T_n be the number of non empty subsets S of $\{1, 2, \dots, n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.
- [5] In an acute triangle ABC , let O, G, H be its circumcentre, centroid and orthocenter. Let $D \in BC, E \in CA$ and $OD \perp BC, HE \perp CA$. Let F be the midpoint of AB . If the triangles ODC, HEA, GFB have the same area, find all the possible values of $\angle C$.
- [6] Let a, b, c, x, y, z be six positive real numbers satisfying $x + y + z = a + b + c$ and $xyz = abc$. Further, suppose that $a \leq x < y < z \leq c$ and $a < b < c$. Prove that $a = x, b = y$ and $c = z$.