Romanian Masters In Mathematics 2010

1 For a finite non empty set of primes P, let $m(P)$ denote the largest possible number of consecutive positive integers, each of which is divisible by at least one member of P.

(i) Show that $|P| \le m(P)$, with equality if and only if $\min(P) > |P|$.

(ii) Show that $m(P) < (|P|+1)(2^{|P|}-1)$.

(The number $|P|$ is the size of set P)

Dan Schwarz, Romania

2 For each positive integer n, find the largest real number C_n with the following property. Given any n real-valued functions $f_1(x), f_2(x), \cdots, f_n(x)$ defined on the closed interval $0 \le x \le 1$, one can find numbers $x_1, x_2, \cdots x_n$, such that $0 \le x_i \le 1$ satisfying

$$|f_1(x_1) + f_2(x_2) + \cdots f_n(x_n) - x_1x_2\cdots x_n| \ge C_n$$

Marko Radovanovi, Serbia

3 Let $A_1A_2A_3A_4$ be a quadrilateral with no pair of parallel sides. For each $i = 1, 2, 3, 4$, define ω_1 to be the circle touching the quadrilateral externally, and which is tangent to the lines $A_{i-1}A_i, A_iA_{i+1}$ and $A_{i+1}A_{i+2}$ (indices are considered modulo 4 so $A_0 = A_4, A_5 = A_1$ and $A_6 = A_2$). Let T_i be the point of tangency of ω_i with the side A_iA_{i+1}. Prove that the lines A_1A_2, A_3A_4 and T_2T_4 are concurrent if and only if the lines A_2A_3, A_4A_1 and T_1T_3 are concurrent.

Pavel Kozhevnikov, Russia

4 Determine whether there exists a polynomial $f(x_1, x_2)$ with two variables, with integer coefficients, and two points $A = (a_1, a_2)$ and $B = (b_1, b_2)$ in the plane, satisfying the following conditions:

(i) A is an integer point (i.e a_1 and a_2 are integers);

(ii) $|a_1 - b_1| + |a_2 - b_2| = 2010$;

(iii) $f(n_1, n_2) > f(a_1, a_2)$ for all integer points (n_1, n_2) in the plane other than A;

(iv) $f(x_1, x_2) > f(b_1, b_2)$ for all integer points (x_1, x_2) in the plane other than B.

Massimo Gobbino, Italy

5 Let n be a given positive integer. Say that a set K of points with integer coordinates in the plane is connected if for every pair of points $R, S \in K$, there exists a positive integer ℓ and a sequence $R = T_0, T_1, T_2, \ldots, T_\ell = S$ of points in K, where each T_i is distance 1 away from T_{i+1}. For such a set K, we define the set of vectors

$$\Delta(K) = \{\overrightarrow{RS} \mid R, S \in K\}$$

What is the maximum value of $|\Delta(K)|$ over all connected sets K of $2n+1$ points with integer coordinates in the plane?

Grigory Chelnokov, Russia

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6 Given a polynomial $f(x)$ with rational coefficients, of degree $d \geq 2$, we define the sequence of sets $f^0(\mathbb{Q}), f^1(\mathbb{Q}), \ldots$ as $f^0(\mathbb{Q}) = \mathbb{Q}$, $f^{n+1}(\mathbb{Q}) = f(f^n(\mathbb{Q}))$ for $n \geq 0$. (Given a set S, we write $f(S)$ for the set $\{f(x) \mid x \in S\}$).
Let $f^\omega(\mathbb{Q}) = \bigcap_{n=0}^{\infty} f^n(\mathbb{Q})$ be the set of numbers that are in all of the sets $f^n(\mathbb{Q})$, $n \geq 0$. Prove that $f^\omega(\mathbb{Q})$ is a finite set.

Dan Schwarz, Romania
