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EGMO 2016

Day 1 April 12

1 Let n be an odd positive integer, and let $x_1, x_2, \cdots, x_n$ be non-negative real numbers. Show that

$$\min_{i=1,\ldots,n}(x_i^2 + x_{i+1}^2) \leq \max_{j=1,\ldots,n}(2x_j x_{j+1})$$

where $x_{n+1} = x_1$.

2 Let $ABCD$ be a cyclic quadrilateral, and let diagonals AC and BD intersect at X.Let C_1, D_1 and M be the midpoints of segments CX, DX and CD, respecctively. Lines AD_1 and BC_1 intersect at Y, and line MY intersects diagonals AC and BD at different points E and F, respectively. Prove that line XY is tangent to the circle through E, F and X.

3 Let m be a positive integer. Consider a $4m \times 4m$ array of square unit cells. Two different cells are *related* to each other if they are in either the same row or in the same column.No cell is related to itself.Some cells are coloured blue, such that every cell is related to at lest two blue cells.Determine the minimum number of blue cells.

Day 2 April 13

4 Two circles ω_1 and ω_2, of equal radius intersect at different points X_1 and X_2. Consider a circle ω externally tangent to ω_1 at T_1 and internally tangent to ω_2 at point T_2. Prove that lines X_1T_1 and X_2T_2 intersect at a point lying on ω

5 Let k and n be integers such that $k \geq 2$ and $k \leq n \leq 2k - 1$. Place rectangular tiles, each of size $1 \times k$, or $k \times 1$ on a $n \times n$ chessboard so that each tile covers exactly k cells and no two tiles overlap. Do this until no further tile can be placed in this way. For each such k and n, determine the minimum number of tiles that such an arrangement may contain.

6 Let S be the set of all positive integers n such that n^4 has a divisor in the range $n^2 + 1, n^2 + 2, ..., n^2 + 2n$. Prove that there are infinitely many elements of S of each of the forms $7m, 7m + 1, 7m + 2, 7m + 5, 7m + 6$ and no elements of S of the form $7m + 3$ and $7m + 4$, where m is an integer.