Art of Problem Solving
2007 Iran Team Selection Test

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Day 1

1 In triangle ABC, M is midpoint of AC, and D is a point on BC such that $DB = DM$. We know that $2BC^2 - AC^2 = AB.AC$. Prove that

$$BD.DC = \frac{AC^2.AB}{2(AB+AC)}$$

2 Let A be the largest subset of $\{1, \dots, n\}$ such that for each $x \in A$, x divides at most one other element in A. Prove that

$$\frac{2n}{3} \leq |A| \leq \left\lceil \frac{3n}{4} \right\rceil.$$

3 Find all solutions of the following functional equation:

$$f(x^2 + y + f(y)) = 2y + f(x)^2.$$

Day 2

1 In an isosceles right-angled triangle shaped billiards table , a ball starts moving from one of the vertices adjacent to hypotenuse. When it reaches to one side then it will reflect its path. Prove that if we reach to a vertex then it is not the vertex at initial position
By Sam Nariman

2 Find all monic polynomials $f(x)$ in $\mathbb{Z}[x]$ such that $f(\mathbb{Z})$ is closed under multiplication.
By Mohsen Jamali

3 Let ω be incircle of ABC. P and Q are on AB and AC, such that PQ is parallel to BC and is tangent to ω. AB, AC touch ω at F, E. Prove that if M is midpoint of PQ, and T is intersection point of EF and BC, then TM is tangent to ω.
By Ali Khezeli

Day 3

1 Does there exist a a sequence $a_0, a_1, a_2, \ldots$ in $\mathbb{N}$, such that for each $i \neq j, (a_i, a_j) = 1$, and for each n, the polynomial $\sum_{i=0}^{n} a_i x^i$ is irreducible in $\mathbb{Z}[x]$?
By Omid Hatami

2 Suppose n lines in plane are such that no two are parallel and no three are concurrent. For each two lines their angle is a real number in $[0, \frac{\pi}{2}]$. Find the largest value of the sum of the $\binom{n}{2}$ angles between line.
By Aliakbar Daemi

3 O is a point inside triangle ABC such that $OA = OB + OC$. Suppose B', C' be midpoints of arcs $\overset{\frown}{AOC}$ and AOB. Prove that circumcircles COC' and BOB' are tangent to each other.

Day 4

1 Find all polynomials of degree 3, such that for each $x, y \geq 0$:

$$p(x + y) \geq p(x) + p(y)$$

2 Triangle ABC is isosceles ($AB = AC$). From A, we draw a line ℓ parallel to BC. P, Q are on perpendicular bisectors of AB, AC such that $PQ \perp BC$. M, N are points on ℓ such that angles $\angle APM$ and $\angle AQN$ are $\frac{\pi}{2}$. Prove that

$$\frac{1}{AM} + \frac{1}{AN} \leq \frac{2}{AB}$$

3 Let P be a point in a square whose side are mirror. A ray of light comes from P and with slope α. We know that this ray of light never arrives to a vertex. We make an infinite sequence of $0, 1$. After each contact of light ray with a horizontal side, we put 0, and after each contact with a vertical side, we put 1. For each $n \geq 1$, let B_n be set of all blocks of length n, in this sequence.
a) Prove that B_n does not depend on location of P.
b) Prove that if $\frac{\alpha}{\pi}$ is irrational, then $|B_n| = n + 1$.