National Math Olympiad (3rd Round) 2004

1 We say $m \circ n$ for natural m,n $\Longleftrightarrow$
nth number of binary representation of m is 1 or mth number of binary representation of n is 1.
and we say $m \bullet n$ if and only if m, n doesn't have the relation $\circ$
We say $A \subset \mathbb{N}$ is golden $\Longleftrightarrow \forall U, V \subset A$ that are finite and arenot empty and $U \cap V = \emptyset$,There exist $z \in A$ that $\forall x \in U, y \in V$ we have $z \circ x, z \bullet y$
Suppose $\mathbb{P}$ is set of prime numbers.Prove if $\mathbb{P} = P_1 \cup ... \cup P_k$ and $P_i \cap P_j = \emptyset$ then one of $P_1, ..., P_k$ is golden.

2 A is a compact convex set in plane. Prove that there exists a point $O \in A$, such that for every line XX' passing through O, where X and X' are boundary points of A, then

$$\frac{1}{2} \leq \frac{OX}{OX'} \leq 2.$$

3 Suppose $V = \mathbb{Z}_2^n$ and for a vector $x = (x_1, ..x_n)$ in V and permutation σ.We have $x_\sigma = (x_{\sigma(1)}, ..., x_{\sigma(n)})$
Suppose $n = 4k+2, 4k+3$ and $f : V \to V$ is injective and if x and y differ in more than $n/2$ places then $f(x)$ and $f(y)$ differ in more than $n/2$ places.
Prove there exist permutaion σ and vector v that $f(x) = x_\sigma + v$

4 We have finite white and finite black points that for each 4 oints there is a line that white points and black points are at different sides of this line.Prove there is a line that all white points and black points are at different side of this line.

5 assume that k,n are two positive integer $k \leq n$count the number of permutation $\{\ 1, \ldots, n\}$ st for any $1 \leq i, j \leq k$and any positive integer m we have $f^m(i) \neq j$ (f^m meas iterarte function,)

6 assume that we have a n*n table we fill it with 1,...,n such that each number exists exactly n times prove that there exist a row or column such that at least $\sqrt{n}$ diffrent number are contained.

7 Suppose F is a polygon with lattice vertices and sides parralell to x-axis and y-axis.Suppose $S(F), P(F)$ are area and perimeter of F.
Find the smallest k that: $S(F) \leq k.P(F)^2$

8 $\mathbb{P}$ is a n-gon with sides $l_1, ..., l_n$ and vertices on a circle. Prove that no n-gon with this sides has area more than $\mathbb{P}$

9 Let ABC be a triangle, and O the center of its circumcircle.
Let a line through the point O intersect the lines AB and AC at the points M and N, respectively. Denote by S and R the midpoints of the segments BN and CM, respectively.
Prove that $\measuredangle ROS = \measuredangle BAC$.

10 $f : \mathbb{R}^2 \to \mathbb{R}^2$ is injective and surjective. Distance of X and Y is not less than distance of $f(X)$ and $f(Y)$. Prove for A in plane:

$$S(A) \geq S(f(A))$$

where $S(A)$ is area of A

11 assume that ABC is acute traingle and AA' is median we extend it until it meets circumcircle at A". let AP_a be a diameter of the circumcircle. the pependicular from A' to AP_a meets the tangent to circumcircle at A" in the point X_a; we define X_b, X_c similary . prove that X_a, X_b, X_c are one a line.

12 $\mathbb{N}_{10}$ is generalization of $\mathbb{N}$ that every hypernumber in $\mathbb{N}_{10}$ is something like: $\overline{...a_2a_1a_0}$ with $a_i \in 0, 1..9$
(Notice that $\overline{...000} \in \mathbb{N}_{10}$)
Also we easily have $+, *$ in $\mathbb{N}_{10}$.
first k number of $a * b$= first k nubmer of (first k number of a * first k number of b)
first k number of $a + b$= first k nubmer of (first k number of a + first k number of b)
Fore example $\overline{...999} + \overline{...0001} = \overline{...000}$
Prove that every monic polynomial in $\mathbb{N}_{10}[x]$ with degree d has at most d^2 roots.

13 Suppose f is a polynomial in $\mathbb{Z}[X]$ and m is integer .Consider the sequence a_i like this $a_1 = m$ and $a_{i+1} = f(a_i)$ find all polynomials f and alll integers m that for each i:

$$a_i | a_{i+1}$$

14 We define $f:\mathbb{N}\to\mathbb{N}$, $f(n)=\sum_{k=1}^{n}(k,n)$.

a) Show that if $\gcd(m,n)=1$ then we have $f(mn)=f(m)\cdot f(n)$;

b) Show that $\sum_{d|n} f(d)=nd(n)$.

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15 This problem is easy but nobody solved it.
point A moves in a line with speed v and B moves also with speed v' that at every time the direction of move of B goes from A.We know $v\geq v'$.If we know the point of beginning of path of A, then B must be where at first that B can catch A.

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16 Let ABC be a triangle . Let point X be in the triangle and AX intersects BC in Y . Draw the perpendiculars YP,YQ,YR,YS to lines CA,CX,BX,BA respectively. Find the necessary and sufficient condition for X such that $PQRS$ be cyclic .

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17 Let $p=4k+1$ be a prime. Prove that p has at least $\frac{\phi(p-1)}{2}$ primitive roots.

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18 Prove that for any n, there is a subset $\{a_1,\dots,a_n\}$ of $\mathbb{N}$ such that for each subset S of $\{1,\dots,n\}$, $\sum_{i\in S}a_i$ has the same set of prime divisors.

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19 Find all integer solutions of $p^3=p^2+q^2+r^2$ where p,q,r are primes.

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20 $p(x)$ is a polynomial in $\mathbb{Z}[x]$ such that for each $m,n\in\mathbb{N}$ there is an integer a such that $n\mid p(a^m)$. Prove that 0 or 1 is a root of $p(x)$.

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21 $a_1,a_2,\dots,a_n$ are integers, not all equal. Prove that there exist infinitely many prime numbers p such that for some k

$$p\mid a_1^k+\ldots+a_n^k.$$

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22 Suppose that $\mathcal{F}$ is a family of subsets of X. A,B are two subsets of X s.t. each element of $\mathcal{F}$ has non-empty intersection with A,B. We know that no subset of X with $n-1$ elements has this property. Prove that there is a representation A,B in the form $A=\{a_1,\dots,a_n\}$ and $B=\{b_1,\dots,b_n\}$, such that for each $1\leq i\leq n$, there is an element of $\mathcal{F}$ containing both a_i,b_i.

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23 $\mathcal{F}$ is a family of 3-subsets of set X. Every two distinct elements of X are exactly in k elements of $\mathcal{F}$. It is known that there is a partition of $\mathcal{F}$ to sets X_1, X_2 such that each element of $\mathcal{F}$ has non-empty intersection with both X_1, X_2. Prove that $|X| \leq 4$.

24 In triangle ABC, points M, N lie on line AC such that $MA = AB$ and $NB = NC$. Also K, L lie on line BC such that $KA = KB$ and $LA = LC$. It is know that $KL = \frac{1}{2}BC$ and $MN = AC$. Find angles of triangle ABC.

25 Finitely many convex subsets of $\mathbb{R}^3$ are given, such that every three have non-empty intersection. Prove that there exists a line in $\mathbb{R}^3$ that intersects all of these subsets.

26 Finitely many points are given on the surface of a sphere, such that every four of them lie on the surface of open hemisphere. Prove that all points lie on the surface of an open hemisphere.

27 $\Delta_1, \ldots, \Delta_n$ are n concurrent segments (their lines concur) in the real plane. Prove that if for every three of them there is a line intersecting these three segments, then there is a line that intersects all of the segments.

28 Find all prime numbers p such that $p = m^2 + n^2$ and $p \mid m^3 + n^3 - 4$.

29 Incircle of triangle ABC touches AB, AC at P, Q. BI, CI intersect with PQ at K, L. Prove that circumcircle of ILK is tangent to incircle of ABC if and only if $AB + AC = 3BC$.

30 Find all polynomials $p \in \mathbb{Z}[x]$ such that $(m, n) = 1 \Rightarrow (p(m), p(n)) = 1$