Art of Problem Solving

2000 USA Team Selection Test

USA Team Selection Test 2000

Day 1 June 10th

1 Let a, b, c be nonnegative real numbers. Prove that

$$\frac{a+b+c}{3} - \sqrt[3]{abc} \leq \max\{(\sqrt{a}-\sqrt{b})^2, (\sqrt{b}-\sqrt{c})^2, (\sqrt{c}-\sqrt{a})^2\}.$$

2 Let $ABCD$ be a cyclic quadrilateral and let E and F be the feet of perpendiculars from the intersection of diagonals AC and BD to AB and CD, respectively. Prove that EF is perpendicular to the line through the midpoints of AD and BC.

3 Let p be a prime number. For integers r, s such that $rs(r^2 - s^2)$ is not divisible by p, let $f(r, s)$ denote the number of integers $n \in \{1, 2, \ldots, p-1\}$ such that $\{rn/p\}$ and $\{sn/p\}$ are either both less than $1/2$ or both greater than $1/2$. Prove that there exists $N > 0$ such that for $p \geq N$ and all r, s,

$$\left\lceil \frac{p-1}{3} \right\rceil \leq f(r, s) \leq \left\lfloor \frac{2(p-1)}{3} \right\rfloor.$$

Day 2 June 11th

4 Let n be a positive integer. Prove that

$$\binom{n}{0}^{-1} + \binom{n}{1}^{-1} + \cdots + \binom{n}{n}^{-1} = \frac{n+1}{2^{n+1}} \left(\frac{2}{1} + \frac{2^2}{2} + \cdots + \frac{2^{n+1}}{n+1} \right).$$

5 Let n be a positive integer. A *corner* is a finite set S of ordered n-tuples of positive integers such that if $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are positive integers with $a_k \geq b_k$ for $k = 1, 2, \ldots, n$ and $(a_1, a_2, \ldots, a_n) \in S$, then $(b_1, b_2, \ldots, b_n) \in S$. Prove that among any infinite collection of corners, there exist two corners, one of which is a subset of the other one.

6 Let ABC be a triangle inscribed in a circle of radius R, and let P be a point in the interior of triangle ABC. Prove that

$$\frac{PA}{BC^2} + \frac{PB}{CA^2} + \frac{PC}{AB^2} \geq \frac{1}{R}.$$

Alternative formulation: If ABC is a triangle with sidelengths $BC = a$, $CA = b$, $AB = c$ and circumradius R, and P is a point inside the triangle ABC, then prove that
$\frac{PA}{a^2} + \frac{PB}{b^2} + \frac{PC}{c^2} \geq \frac{1}{R}$.

–

USA Team Selection Test 2001

Day 1 June 9th

1 Let $\{a_n\}_{n\geq 0}$ be a sequence of real numbers such that $a_{n+1} \geq a_n^2 + \frac{1}{5}$ for all $n \geq 0$. Prove that $\sqrt{a_{n+5}} \geq a_{n-5}^2$ for all $n \geq 5$.

2 Express

$$\sum_{k=0}^{n}(-1)^k(n-k)!(n+k)!$$

in closed form.

3 For a set S, let $|S|$ denote the number of elements in S. Let A be a set of positive integers with $|A| = 2001$. Prove that there exists a set B such that

(i) $B \subseteq A$;
(ii) $|B| \geq 668$;
(iii) for any $u, v \in B$ (not necessarily distinct), $u + v \notin B$.

Day 2 June 10th

4 There are 51 senators in a senate. The senate needs to be divided into n committees so that each senator is on one committee. Each senator hates exactly three other senators. (If senator A hates senator B, then senator B does *not* necessarily hate senator A.) Find the smallest n such that it is always possible to arrange the committees so that no senator hates another senator on his or her committee.

5 In triangle ABC, $\angle B = 2\angle C$. Let P and Q be points on the perpendicular bisector of segment BC such that rays AP and AQ trisect $\angle A$. Prove that $PQ < AB$ if and only if $\angle B$ is obtuse.

6 Let a, b, c be positive real numbers such that

$$a + b + c \geq abc.$$

Prove that at least two of the inequalities

$$\frac{2}{a} + \frac{3}{b} + \frac{6}{c} \geq 6, \quad \frac{2}{b} + \frac{3}{c} + \frac{6}{a} \geq 6, \quad \frac{2}{c} + \frac{3}{a} + \frac{6}{b} \geq 6$$

are true.

Day 3

7 Let $ABCD$ be a convex quadrilateral such that $\angle ABC = \angle ADC = 135^\circ$ and

$$AC^2 \cdot BD^2 = 2 \cdot AB \cdot BC \cdot CD \cdot DA.$$

Prove that the diagonals of the quadrilateral $ABCD$ are perpendicular.

8 Find all pairs of nonnegative integers (m, n) such that

$$(m + n - 5)^2 = 9mn.$$

9 Let A be a finite set of positive integers. Prove that there exists a finite set B of positive integers such that $A \subseteq B$ and

$$\prod_{x \in B} x = \sum_{x \in B} x^2.$$

–

USA Team Selection Test 2002

Day 1 June 21st

1 Let ABC be a triangle, and A, B, C its angles. Prove that

$$\sin\frac{3A}{2}+\sin\frac{3B}{2}+\sin\frac{3C}{2}\leq\cos\frac{A-B}{2}+\cos\frac{B-C}{2}+\cos\frac{C-A}{2}.$$

2 Let $p>5$ be a prime number. For any integer x, define

$$f_p(x)=\sum_{k=1}^{p-1}\frac{1}{(px+k)^2}$$

Prove that for any pair of positive integers x, y, the numerator of $f_p(x)-f_p(y)$, when written as a fraction in lowest terms, is divisible by p^3.

3 Let n be an integer greater than 2, and $P_1,P_2,\cdots,P_n$ distinct points in the plane. Let $\mathcal{S}$ denote the union of all segments $P_1P_2,P_2P_3,\dots,P_{n-1}P_n$. Determine if it is always possible to find points A and B in $\mathcal{S}$ such that $P_1P_n\parallel AB$ (segment AB can lie on line P_1P_n) and $P_1P_n=kAB$, where (1) $k=2.5$; (2) $k=3$.

Day 2 June 22nd

4 Let n be a positive integer and let S be a set of 2^n+1 elements. Let f be a function from the set of two-element subsets of S to $\{0,\dots,2^{n-1}-1\}$. Assume that for any elements x,y,z of S, one of $f(\{x,y\}),f(\{y,z\}),f(\{z,x\})$ is equal to the sum of the other two. Show that there exist a,b,c in S such that $f(\{a,b\}),f(\{b,c\}),f(\{c,a\})$ are all equal to 0.

5 Consider the family of nonisosceles triangles ABC satisfying the property $AC^2+BC^2=2AB^2$. Points M and D lie on side AB such that $AM=BM$ and $\angle ACD=\angle BCD$. Point E is in the plane such that D is the incenter of triangle CEM. Prove that exactly one of the ratios

$$\frac{CE}{EM},\quad\frac{EM}{MC},\quad\frac{MC}{CE}$$

is constant.

6 Find in explicit form all ordered pairs of positive integers (m, n) such that $mn - 1$ divides $m^2 + n^2$.

USA Team Selection Test 2004

Day 1

1 Suppose $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ are real numbers such that

$$(a_1^2 + a_2^2 + \cdots + a_n^2 - 1)(b_1^2 + b_2^2 + \cdots + b_n^2 - 1) > (a_1b_1 + a_2b_2 + \cdots + a_nb_n - 1)^2.$$

Prove that $a_1^2 + a_2^2 + \cdots + a_n^2 > 1$ and $b_1^2 + b_2^2 + \cdots + b_n^2 > 1$.

2 Assume n is a positive integer. Considers sequences $a_0, a_1, \dots, a_n$ for which $a_i \in \{1, 2, \dots, n\}$ for all i and $a_n = a_0$.

(a) Suppose n is odd. Find the number of such sequences if $a_i - a_{i-1} \not\equiv i \pmod{n}$ for all $i = 1, 2, \dots, n$.

(b) Suppose n is an odd prime. Find the number of such sequences if $a_i - a_{i-1} \not\equiv i, 2i \pmod{n}$ for all $i = 1, 2, \dots, n$.

3 Draw a 2004×2004 array of points. What is the largest integer n for which it is possible to draw a convex n-gon whose vertices are chosen from the points in the array?

Day 2

4 Let ABC be a triangle. Choose a point D in its interior. Let ω_1 be a circle passing through B and D and ω_2 be a circle passing through C and D so that the other point of intersection of the two circles lies on AD. Let ω_1 and ω_2 intersect side BC at E and F, respectively. Denote by X the intersection of DF, AB and Y the intersection of DE, AC. Show that $XY \parallel BC$.

5 Let $A = (0, 0, 0)$ in 3D space. Define the *weight* of a point as the sum of the absolute values of the coordinates. Call a point a *primitive lattice point* if all of its coordinates are integers whose gcd is 1. Let square $ABCD$ be an *unbalanced primitive integer square* if it has integer side length and also, B and D are primitive lattice points with different weights. Prove that there are infinitely many unbalanced primitive integer squares such that the planes containing the squares are not parallel to each other.

6 Define the function $f : \mathbb{N} \cup \{0\} \to \mathbb{Q}$ as follows: $f(0) = 0$ and

$$f(3n+k) = -\frac{3f(n)}{2} + k,$$

for $k = 0, 1, 2$. Show that f is one-to-one and determine the range of f.

USA Team Selection Test 2005

Day 1

1 Let n be an integer greater than 1. For a positive integer m, let $S_m = \{1, 2, \ldots, mn\}$. Suppose that there exists a $2n$-element set T such that
(a) each element of T is an m-element subset of S_m;
(b) each pair of elements of T shares at most one common element;
and
(c) each element of S_m is contained in exactly two elements of T.

Determine the maximum possible value of m in terms of n.

2 Let $A_1A_2A_3$ be an acute triangle, and let O and H be its circumcenter and orthocenter, respectively. For $1 \le i \le 3$, points P_i and Q_i lie on lines OA_i and $A_{i+1}A_{i+2}$ (where $A_{i+3} = A_i$), respectively, such that OP_iHQ_i is a parallelogram. Prove that

$$\frac{OQ_1}{OP_1} + \frac{OQ_2}{OP_2} + \frac{OQ_3}{OP_3} \ge 3.$$

3 We choose random a unitary polynomial of degree n and coefficients in the set $1, 2, ..., n!$. Prove that the probability for this polynomial to be special is between 0.71 and 0.75, where a polynomial g is called special if for every $k > 1$ in the sequence $f(1), f(2), f(3), ...$ there are infinitely many numbers relatively prime with k.

Day 2

4 Consider the polynomials

$$f(x) = \sum_{k=1}^{n} a_k x^k \quad \text{and} \quad g(x) = \sum_{k=1}^{n} \frac{a_k}{2^k - 1} x^k,$$

where $a_1, a_2, \ldots, a_n$ are real numbers and n is a positive integer. Show that if 1 and 2^{n+1} are zeros of g then f has a positive zero less than 2^n.

5 Find all finite sets S of points in the plane with the following property: for any three distinct points A, B, and C in S, there is a fourth point D in S such that A, B, C, and D are the vertices of a parallelogram (in some order).

6 Let ABC be an acute scalene triangle with O as its circumcenter. Point P lies inside triangle ABC with $\angle PAB = \angle PBC$ and $\angle PAC = \angle PCB$. Point Q lies on line BC with $QA = QP$. Prove that $\angle AQP = 2\angle OQB$.

USA Team Selection Test 2006

Day 1

1 A communications network consisting of some terminals is called a [i]3-connector[/i] if among any three terminals, some two of them can directly communicate with each other. A communications network contains a *windmill* with n blades if there exist n pairs of terminals $\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_n, y_n\}$ such that each x_i can directly communicate with the corresponding y_i and there is a *hub* terminal that can directly communicate with each of the $2n$ terminals $x_1, y_1, \ldots, x_n, y_n$. Determine the minimum value of $f(n)$, in terms of n, such that a 3 -connector with $f(n)$ terminals always contains a windmill with n blades.

2 In acute triangle ABC , segments $AD; BE$, and CF are its altitudes, and H is its orthocenter. Circle ω, centered at O, passes through A and H and intersects sides AB and AC again at Q and P (other than A), respectively. The circumcircle of triangle OPQ is tangent to segment BC at R. Prove that $\frac{CR}{BR} = \frac{ED}{FD}$.

3 Find the least real number k with the following property: if the real numbers $x; y$, and z are not all positive, then

$$k(x^2 - x + 1)(y^2 - y + 1)(z^2 - z + 1) \geq (xyz)^2 - xyz + 1.$$

Day 2

4 Let n be a positive integer. Find, with proof, the least positive integer d_n which cannot be expressed in the form

$$\sum_{i=1}^{n} (-1)^{a_i} 2^{b_i},$$

where a_i and b_i are nonnegative integers for each i.

5 Let n be a given integer with n greater than 7 , and let $\mathcal{P}$ be a convex polygon with n sides. Any set of $n-3$ diagonals of $\mathcal{P}$ that do not intersect in the interior of the polygon determine a triangulation of $\mathcal{P}$ into $n-2$ triangles. A triangle in the triangulation of $\mathcal{P}$ is an interior triangle if all of its sides are diagonals of

$\mathcal{P}$. Express, in terms of n, the number of triangulations of $\mathcal{P}$ with exactly two interior triangles, in closed form.

6 Let ABC be a triangle. Triangles PAB and QAC are constructed outside of triangle ABC such that $AP = AB$ and $AQ = AC$ and $\angle BAP = \angle CAQ$. Segments BQ and CP meet at R. Let O be the circumcenter of triangle BCR. Prove that $AO \perp PQ$.

USA Team Selection Test 2007

Day 1

1	Circles ω_1 and ω_2 meet at P and Q. Segments AC and BD are chords of ω_1 and ω_2 respectively, such that segment AB and ray CD meet at P. Ray BD and segment AC meet at X. Point Y lies on ω_1 such that $PY \parallel BD$. Point Z lies on ω_2 such that $PZ \parallel AC$. Prove that points Q, X, Y, Z are collinear.
2	Let n be a positive integer and let $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$ be two nondecreasing sequences of real numbers such that $$a_1 + \cdots + a_i \le b_1 + \cdots + b_i \text{ for every } i = 1, \ldots, n$$ and $$a_1 + \cdots + a_n = b_1 + \cdots + b_n.$$ Suppose that for every real number m, the number of pairs (i, j) with $a_i - a_j = m$ equals the numbers of pairs (k, ℓ) with $b_k - b_\ell = m$. Prove that $a_i = b_i$ for $i = 1, \ldots, n$.
3	Let θ be an angle in the interval $(0, \pi/2)$. Given that $\cos\theta$ is irrational, and that $\cos k\theta$ and $\cos[(k+1)\theta]$ are both rational for some positive integer k, show that $\theta = \pi/6$.

Day 2

4	Determine whether or not there exist positive integers a and b such that a does not divide $b^n - n$ for all positive integers n.
5	Triangle ABC is inscribed in circle ω. The tangent lines to ω at B and C meet at T. Point S lies on ray BC such that $AS \perp AT$. Points B_1 and C_1 lie on ray ST (with C_1 in between B_1 and S) such that $B_1T = BT = C_1T$. Prove that triangles ABC and AB_1C_1 are similar to each other.
6	For a polynomial $P(x)$ with integer coefficients, $r(2i-1)$ (for $i = 1, 2, 3, \ldots, 512$) is the remainder obtained when $P(2i-1)$ is divided by 1024. The sequence $$(r(1), r(3), \ldots, r(1023))$$

is called the *remainder sequence* of $P(x)$. A remainder sequence is called *complete* if it is a permutation of $(1, 3, 5, \ldots, 1023)$. Prove that there are no more than 2^{35} different complete remainder sequences.

–

USA Team Selection Test 2008

Day 1

1 There is a set of n coins with distinct integer weights $w_1, w_2, \ldots, w_n$. It is known that if any coin with weight w_k, where $1 \le k \le n$, is removed from the set, the remaining coins can be split into two groups of the same weight. (The number of coins in the two groups can be different.) Find all n for which such a set of coins exists.

2 Let P, Q, and R be the points on sides BC, CA, and AB of an acute triangle ABC such that triangle PQR is equilateral and has minimal area among all such equilateral triangles. Prove that the perpendiculars from A to line QR, from B to line RP, and from C to line PQ are concurrent.

3 For a pair $A = (x_1, y_1)$ and $B = (x_2, y_2)$ of points on the coordinate plane, let $d(A, B) = |x_1 - x_2| + |y_1 - y_2|$. We call a pair (A, B) of (unordered) points *harmonic* if $1 < d(A, B) \le 2$. Determine the maximum number of harmonic pairs among 100 points in the plane.

Day 2

4 Prove that for no integer n is $n^7 + 7$ a perfect square.

5 Two sequences of integers, $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$, satisfy the equation

$$(a_n - a_{n-1})(a_n - a_{n-2}) + (b_n - b_{n-1})(b_n - b_{n-2}) = 0$$

for each integer n greater than 2. Prove that there is a positive integer k such that $a_k = a_{k+2008}$.

6 Determine the smallest positive real number k with the following property. Let $ABCD$ be a convex quadrilateral, and let points A_1, B_1, C_1, and D_1 lie on sides AB, BC, CD, and DA, respectively. Consider the areas of triangles AA_1D_1, BB_1A_1, CC_1B_1 and DD_1C_1; let S be the sum of the two smallest ones, and let S_1 be the area of quadrilateral $A_1B_1C_1D_1$. Then we always have $kS_1 \ge S$.

Author: Zuming Feng and Oleg Golberg, USA

Day 3

7	Let ABC be a triangle with G as its centroid. Let P be a variable point on segment BC. Points Q and R lie on sides AC and AB respectively, such that $PQ \parallel AB$ and $PR \parallel AC$. Prove that, as P varies along segment BC, the circumcircle of triangle AQR passes through a fixed point X such that $\angle BAG = \angle CAX$.
8	Mr. Fat and Ms. Taf play a game. Mr. Fat chooses a sequence of positive integers $k_1, k_2, \ldots, k_n$. Ms. Taf must guess this sequence of integers. She is allowed to give Mr. Fat a red card and a blue card, each with an integer written on it. Mr. Fat replaces the number on the red card with k_1 times the number on the red card plus the number on the blue card, and replaces the number on the blue card with the number originally on the red card. He repeats this process with number k_2. (That is, he replaces the number on the red card with k_2 times the number now on the red card plus the number now on the blue card, and replaces the number on the blue card with the number that was just placed on the red card.) He then repeats this process with each of the numbers $k_3, \ldots k_n$, in this order. After has has gone through the sequence of integers, Mr. Fat then gives the cards back to Ms. Taf. How many times must Ms. Taf submit the red and blue cards in order to be able to determine the sequence of integers $k_1, k_2, \ldots k_n$?
9	Let n be a positive integer. Given an integer coefficient polynomial $f(x)$, define its [i]signature modulo n[/i] to be the (ordered) sequence $f(1), \ldots, f(n)$ modulo n. Of the n^n such n-term sequences of integers modulo n, how many are the signature of some polynomial $f(x)$ if a) n is a positive integer not divisible by the square of a prime. b) n is a positive integer not divisible by the cube of a prime.
–	MAA MATHEMATICAL ASSOCIATION OF AMERICA

USA Team Selection Test 2009

Day 1

1 Let m and n be positive integers. Mr. Fat has a set S containing every rectangular tile with integer side lengths and area of a power of 2. Mr. Fat also has a rectangle R with dimensions $2^m \times 2^n$ and a 1×1 square removed from one of the corners. Mr. Fat wants to choose $m+n$ rectangles from S, with respective areas $2^0, 2^1, \ldots, 2^{m+n-1}$, and then tile R with the chosen rectangles. Prove that this can be done in at most $(m+n)!$ ways.

Palmer Mebane.

2 Let ABC be an acute triangle. Point D lies on side BC. Let O_B, O_C be the circumcenters of triangles ABD and ACD, respectively. Suppose that the points B, C, O_B, O_C lies on a circle centered at X. Let H be the orthocenter of triangle ABC. Prove that $\angle DAX = \angle DAH$.

Zuming Feng.

3 For each positive integer n, let $c(n)$ be the largest real number such that

$$c(n) \le \left| \frac{f(a) - f(b)}{a - b} \right|$$

for all triples (f, a, b) such that

–f is a polynomial of degree n taking integers to integers, and
–a, b are integers with $f(a) \neq f(b)$.

Find $c(n)$.

Shaunak Kishore.

Day 2

4 Let ABP, BCQ, CAR be three non-overlapping triangles erected outside of acute triangle ABC. Let M be the midpoint of segment AP. Given that $\angle PAB = \angle CQB = 45^\circ$, $\angle ABP = \angle QBC = 75^\circ$, $\angle RAC = 105^\circ$, and $RQ^2 = 6CM^2$, compute AC^2/AR^2.

Zuming Feng.

5 Find all pairs of positive integers (m, n) such that $mn-1$ divides $(n^2-n+1)^2$.

Aaron Pixton.

6 Let $N > M > 1$ be fixed integers. There are N people playing in a chess tournament; each pair of players plays each other once, with no draws. It turns out that for each sequence of $M+1$ distinct players $P_0, P_1, \ldots P_M$ such that P_{i-1} beat P_i for each $i = 1, \ldots, M$, player P_0 also beat P_M. Prove that the players can be numbered $1, 2, \ldots, N$ in such a way that, whenever $a \geq b+M-1$, player a beat player b.

Gabriel Carroll.

Day 3

7 Find all triples (x, y, z) of real numbers that satisfy the system of equations

$$\begin{cases} x^3 = 3x - 12y + 50, \\ y^3 = 12y + 3z - 2, \\ z^3 = 27z + 27x. \end{cases}$$

Razvan Gelca.

8 Fix a prime number $p > 5$. Let a, b, c be integers no two of which have their difference divisible by p. Let i, j, k be nonnegative integers such that $i+j+k$ is divisible by $p-1$. Suppose that for all integers x, the quantity

$$(x-a)(x-b)(x-c)[(x-a)^i(x-b)^j(x-c)^k - 1]$$

is divisible by p. Prove that each of i, j, k must be divisible by $p-1$.

Kiran Kedlaya and Peter Shor.

9 Prove that for positive real numbers x, y, z,

$$x^3(y^2+z^2)^2+y^3(z^2+x^2)^2+z^3(x^2+y^2)^2 \geq xyz\left[xy(x+y)^2 + yz(y+z)^2 + zx(z+x)^2\right].$$

Zarathustra (Zeb) Brady.

USA Team Selection Test 2010

Day 1

1 Let P be a polynomial with integer coefficients such that $P(0) = 0$ and

$$\gcd(P(0), P(1), P(2), \ldots) = 1.$$

Show there are infinitely many n such that

$$\gcd(P(n) - P(0), P(n+1) - P(1), P(n+2) - P(2), \ldots) = n.$$

2 Let a, b, c be positive reals such that $abc = 1$. Show that

$$\frac{1}{a^5(b+2c)^2} + \frac{1}{b^5(c+2a)^2} + \frac{1}{c^5(a+2b)^2} \geq \frac{1}{3}.$$

3 Let h_a, h_b, h_c be the lengths of the altitudes of a triangle ABC from A, B, C respectively. Let P be any point inside the triangle. Show that

$$\frac{PA}{h_b + h_c} + \frac{PB}{h_a + h_c} + \frac{PC}{h_a + h_b} \geq 1.$$

Day 2

4 Let ABC be a triangle. Point M and N lie on sides AC and BC respectively such that $MN||AB$. Points P and Q lie on sides AB and CB respectively such that $PQ||AC$. The incircle of triangle CMN touches segment AC at E. The incircle of triangle BPQ touches segment AB at F. Line EN and AB meet at R, and lines FQ and AC meet at S. Given that $AE = AF$, prove that the incenter of triangle AEF lies on the incircle of triangle ARS.

5 Define the sequence $a_1, a_2, a_3, \ldots$ by $a_1 = 1$ and, for $n > 1$,

$$a_n = a_{\lfloor n/2 \rfloor} + a_{\lfloor n/3 \rfloor} + \ldots + a_{\lfloor n/n \rfloor} + 1.$$

Prove that there are infinitely many n such that $a_n \equiv n \pmod{2^{2010}}$.

6 Let T be a finite set of positive integers greater than 1. A subset S of T is called *good* if for every $t \in T$ there exists some $s \in S$ with $\gcd(s, t) > 1$. Prove that the number of good subsets of T is odd.

Day 3

7 In triangle ABC, let P and Q be two interior points such that $\angle ABP = \angle QBC$ and $\angle ACP = \angle QCB$. Point D lies on segment BC. Prove that $\angle APB + \angle DPC = 180^\circ$ if and only if $\angle AQC + \angle DQB = 180^\circ$.

8 Let m, n be positive integers with $m \ge n$, and let S be the set of all n-term sequences of positive integers $(a_1, a_2, \dots a_n)$ such that $a_1 + a_2 + \cdots + a_n = m$. Show that

$$\sum_S 1^{a_1} 2^{a_2} \cdots n^{a_n} = \binom{n}{n} n^m - \binom{n}{n-1}(n-1)^m + \cdots + (-1)^{n-2}\binom{n}{2}2^m + (-1)^{n-1}\binom{n}{1}.$$

9 Determine whether or not there exists a positive integer k such that $p = 6k+1$ is a prime and

$$\binom{3k}{k} \equiv 1 \pmod{p}.$$

–

USA Team Selection Test 2011

Day 1 June 9th

1 In an acute scalene triangle ABC, points D, E, F lie on sides BC, CA, AB, respectively, such that $AD \perp BC, BE \perp CA, CF \perp AB$. Altitudes AD, BE, CF meet at orthocenter H. Points P and Q lie on segment EF such that $AP \perp EF$ and $HQ \perp EF$. Lines DP and QH intersect at point R. Compute HQ/HR.

Proposed by Zuming Feng

2 In the nation of Onewaynia, certain pairs of cities are connected by roads. Every road connects exactly two cities (roads are allowed to cross each other, e.g., via bridges). Some roads have a traffic capacity of 1 unit and other roads have a traffic capacity of 2 units. However, on every road, traffic is only allowed to travel in one direction. It is known that for every city, the sum of the capacities of the roads connected to it is always odd. The transportation minister needs to assign a direction to every road. Prove that he can do it in such a way that for every city, the difference between the sum of the capacities of roads entering the city and the sum of the capacities of roads leaving the city is always exactly one.

Proposed by Zuming Feng and Yufei Zhao

3 Let p be a prime. We say that a sequence of integers $\{z_n\}_{n=0}^\infty$ is a [i]p-pod[/i] if for each $e \geq 0$, there is an $N \geq 0$ such that whenever $m \geq N$, p^e divides the sum

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} z_k.$$

Prove that if both sequences $\{x_n\}_{n=0}^\infty$ and $\{y_n\}_{n=0}^\infty$ are p-pods, then the sequence $\{x_n y_n\}_{n=0}^\infty$ is a p-pod.

Day 2 June 10th

4 Find a real number t such that for any set of 120 points $P_1, \ldots P_{120}$ on the boundary of a unit square, there exists a point Q on this boundary with $|P_1Q| + |P_2Q| + \cdots + |P_{120}Q| = t$.

5 Let c_n be a sequence which is defined recursively as follows: $c_0 = 1$, $c_{2n+1} = c_n$ for $n \geq 0$, and $c_{2n} = c_n + c_{n-2^e}$ for $n > 0$ where e is the maximal nonnegative integer such that 2^e divides n. Prove that

$$\sum_{i=0}^{2^n-1} c_i = \frac{1}{n+2}\binom{2n+2}{n+1}.$$

6 A polynomial $P(x)$ is called *nice* if $P(0) = 1$ and the nonzero coefficients of $P(x)$ alternate between 1 and -1 when written in order. Suppose that $P(x)$ is nice, and let m and n be two relatively prime positive integers. Show that

$$Q(x) = P(x^n) \cdot \frac{(x^{mn}-1)(x-1)}{(x^m-1)(x^n-1)}$$

is nice as well.

Day 3 June 11th

7 Let ABC be an acute scalene triangle inscribed in circle Ω. Circle ω, centered at O, passes through B and C and intersects sides AB and AC at E and D, respectively. Point P lies on major arc BAC of Ω. Prove that lines BD, CE, OP are concurrent if and only if triangles PBD and PCE have the same incenter.

8 Let $n \geq 1$ be an integer, and let S be a set of integer pairs (a, b) with $1 \leq a < b \leq 2^n$. Assume $|S| > n \cdot 2^{n+1}$. Prove that there exists four integers $a < b < c < d$ such that S contains all three pairs (a, c), (b, d) and (a, d).

9 Determine whether or not there exist two different sets A, B, each consisting of at most 2011^2 positive integers, such that every x with $0 < x < 1$ satisfies the following inequality:

$$\left|\sum_{a\in A} x^a - \sum_{b\in B} x^b\right| < (1-x)^{2011}.$$

USA Team Selection Test 2012

– December TST

1 In acute triangle ABC, $\angle A < \angle B$ and $\angle A < \angle C$. Let P be a variable point on side BC. Points D and E lie on sides AB and AC, respectively, such that $BP = PD$ and $CP = PE$. Prove that as P moves along side BC, the circumcircle of triangle ADE passes through a fixed point other than A.

2 Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that for every pair of real numbers x and y,

$$f(x + y^2) = f(x) + |yf(y)|.$$

3 Determine, with proof, whether or not there exist integers $a, b, c > 2010$ satisfying the equation

$$a^3 + 2b^3 + 4c^3 = 6abc + 1.$$

4 There are 2010 students and 100 classrooms in the Olympiad High School. At the beginning, each of the students is in one of the classrooms. Each minute, as long as not everyone is in the same classroom, somebody walks from one classroom into a different classroom with at least as many students in it (prior to his move). This process will terminate in M minutes. Determine the maximum value of M.

– January TST

1 Consider (3-variable) polynomials

$$P_n(x, y, z) = (x - y)^{2n}(y - z)^{2n} + (y - z)^{2n}(z - x)^{2n} + (z - x)^{2n}(x - y)^{2n}$$

and

$$Q_n(x, y, z) = [(x - y)^{2n} + (y - z)^{2n} + (z - x)^{2n}]^{2n}.$$

Determine all positive integers n such that the quotient $Q_n(x, y, z)/P_n(x, y, z)$ is a (3-variable) polynomial with rational coefficients.

2 In cyclic quadrilateral $ABCD$, diagonals AC and BD intersect at P. Let E and F be the respective feet of the perpendiculars from P to lines AB and CD. Segments BF and CE meet at Q. Prove that lines PQ and EF are perpendicular to each other.

3 Determine all positive integers n, $n \geq 2$, such that the following statement is true:
If $(a_1, a_2, ..., a_n)$ is a sequence of positive integers with $a_1+a_2+\cdots+a_n = 2n-1$, then there is block of (at least two) consecutive terms in the sequence with their (arithmetic) mean being an integer.

4 Find all positive integers $a, n \geq 1$ such that for all primes p dividing $a^n - 1$, there exists a positive integer $m < n$ such that $p \mid a^m - 1$.

–

USA Team Selection Test 2013

– December TST

1 A social club has $2k+1$ members, each of whom is fluent in the same k languages. Any pair of members always talk to each other in only one language. Suppose that there were no three members such that they use only one language among them. Let A be the number of three-member subsets such that the three distinct pairs among them use different languages. Find the maximum possible value of A.

2 Find all triples (x, y, z) of positive integers such that $x \le y \le z$ and

$$x^3(y^3 + z^3) = 2012(xyz + 2).$$

3 Let ABC be a scalene triangle with $\angle BCA = 90^\circ$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. The circumcircle of triangle DKL intersects segment AB at a second point T (other than D). Prove that $\angle ACT = \angle BCT$.

4 Let $f : \mathbb{N} \to \mathbb{N}$ be a function, and let f^m be f applied m times. Suppose that for every $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $f^{2k}(n) = n + k$, and let k_n be the smallest such k. Prove that the sequence $k_1, k_2, \ldots$ is unbounded.

– January TST

1 Two incongruent triangles ABC and XYZ are called a pair of *pals* if they satisfy the following conditions:
(a) the two triangles have the same area;
(b) let M and W be the respective midpoints of sides BC and YZ. The two sets of lengths $\{AB, AM, AC\}$ and $\{XY, XW, XZ\}$ are identical 3-element sets of pairwise relatively prime integers.

Determine if there are infinitely many pairs of triangles that are pals of each other.

2 Let ABC be an acute triangle. Circle ω_1, with diameter AC, intersects side BC at F (other than C). Circle ω_2, with diameter BC, intersects side AC at

E (other than C). Ray AF intersects ω_2 at K and M with $AK < AM$. Ray BE intersects ω_1 at L and N with $BL < BN$. Prove that lines AB, ML, NK are concurrent.

3 In a table with n rows and $2n$ columns where n is a fixed positive integer, we write either zero or one into each cell so that each row has n zeros and n ones. For $1 \le k \le n$ and $1 \le i \le n$, we define $a_{k,i}$ so that the i^{th} zero in the k^{th} row is the $a_{k,i}^{\text{th}}$ column. Let $\mathcal{F}$ be the set of such tables with $a_{1,i} \ge a_{2,i} \ge \cdots \ge a_{n,i}$ for every i with $1 \le i \le n$. We associate another $n \times 2n$ table $f(C)$ from $C \in \mathcal{F}$ as follows: for the k^{th} row of $f(C)$, we write n ones in the columns $a_{n,k} - k + 1, a_{n-1,k} - k + 2, \ldots, a_{1,k} - k + n$ (and we write zeros in the other cells in the row).

(a) Show that $f(C) \in \mathcal{F}$.
(b) Show that $f(f(f(f(f(f(C)))))) = C$ for any $C \in \mathcal{F}$.

4 Determine if there exists a (three-variable) polynomial $P(x, y, z)$ with integer coefficients satisfying the following property: a positive integer n is *not* a perfect square if and only if there is a triple (x, y, z) of positive integers such that $P(x, y, z) = n$.

–

USA Team Selection Test 2014

– December TST

1 Let ABC be an acute triangle, and let X be a variable interior point on the minor arc BC of its circumcircle. Let P and Q be the feet of the perpendiculars from X to lines CA and CB, respectively. Let R be the intersection of line PQ and the perpendicular from B to AC. Let ℓ be the line through P parallel to XR. Prove that as X varies along minor arc BC, the line ℓ always passes through a fixed point. (Specifically: prove that there is a point F, determined by triangle ABC, such that no matter where X is on arc BC, line ℓ passes through F.)

Robert Simson et al.

2 Let $a_1, a_2, a_3, \ldots$ be a sequence of integers, with the property that every consecutive group of a_i's averages to a perfect square. More precisely, for every positive integers n and k, the quantity

$$\frac{a_n + a_{n+1} + \cdots + a_{n+k-1}}{k}$$

is always the square of an integer. Prove that the sequence must be constant (all a_i are equal to the same perfect square).

Evan O'Dorney and Victor Wang

3 Let n be an even positive integer, and let G be an n-vertex graph with exactly $\frac{n^2}{4}$ edges, where there are no loops or multiple edges (each unordered pair of distinct vertices is joined by either 0 or 1 edge). An unordered pair of distinct vertices $\{x, y\}$ is said to be *amicable* if they have a common neighbor (there is a vertex z such that xz and yz are both edges). Prove that G has at least $2\binom{n/2}{2}$ pairs of vertices which are amicable.

Zoltn Fredi (suggested by Po-Shen Loh)

– January TST

1 Let n be a positive even integer, and let $c_1, c_2, \ldots, c_{n-1}$ be real numbers satisfying

$$\sum_{i=1}^{n-1} |c_i - 1| < 1.$$

Prove that

$$2x^n - c_{n-1}x^{n-1} + c_{n-2}x^{n-2} - \cdots - c_1x^1 + 2$$

has no real roots.

---

2 Let $ABCD$ be a cyclic quadrilateral, and let E, F, G, and H be the midpoints of AB, BC, CD, and DA respectively. Let W, X, Y and Z be the orthocenters of triangles AHE, BEF, CFG and DGH, respectively. Prove that the quadrilaterals $ABCD$ and $WXYZ$ have the same area.

---

3 For a prime p, a subset S of residues modulo p is called a *sum-free multiplicative subgroup* of $\mathbb{F}_p$ if • there is a nonzero residue α modulo p such that $S = \left\{1, \alpha^1, \alpha^2, \dots\right\}$ (all considered mod p), and • there are no $a, b, c \in S$ (not necessarily distinct) such that $a + b \equiv c \pmod{p}$.
Prove that for every integer N, there is a prime p and a sum-free multiplicative subgroup S of $\mathbb{F}_p$ such that $|S| \geq N$.

Proposed by Noga Alon and Jean Bourgain

– MAA
MATHEMATICAL ASSOCIATION OF AMERICA

USA Team Selection Test 2015

Dec December 11th, 2014

1 Let ABC be a non-isosceles triangle with incenter I whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Denote by M the midpoint of $\overline{BC}$. Let Q be a point on the incircle such that $\angle AQD = 90^\circ$. Let P be the point inside the triangle on line AI for which $MD = MP$. Prove that either $\angle PQE = 90^\circ$ or $\angle PQF = 90^\circ$.

Proposed by Evan Chen

2 Prove that for every $n \in \mathbb{N}$, there exists a set S of n positive integers such that for any two distinct $a, b \in S$, $a - b$ divides a and b but none of the other elements of S.

Proposed by Iurie Boreico

3 A physicist encounters 2015 atoms called usamons. Each usamon either has one electron or zero electrons, and the physicist can't tell the difference. The physicist's only tool is a diode. The physicist may connect the diode from any usamon A to any other usamon B. (This connection is directed.) When she does so, if usamon A has an electron and usamon B does not, then the electron jumps from A to B. In any other case, nothing happens. In addition, the physicist cannot tell whether an electron jumps during any given step. The physicist's goal is to isolate two usamons that she is sure are currently in the same state. Is there any series of diode usage that makes this possible?

Proposed by Linus Hamilton

Jan January 15, 2015

1 Let $f : \mathbb{Q} \to \mathbb{Q}$ be a function such that for any $x, y \in \mathbb{Q}$, the number $f(x+y) - f(x) - f(y)$ is an integer. Decide whether it follows that there exists a constant c such that $f(x) - cx$ is an integer for every rational number x.

Proposed by Victor Wang

2 A tournament is a directed graph for which every (unordered) pair of vertices has a single directed edge from one vertex to the other. Let us define a proper directed-edge-coloring to be an assignment of a color to every (directed) edge, so that for every pair of directed edges $\overrightarrow{uv}$ and $\overrightarrow{vw}$, those two edges are in different colors. Note that it is permissible for $\overrightarrow{uv}$ and $\overrightarrow{uw}$ to be the same color. The directed-edge-chromatic-number of a tournament is defined to be the minimum total number of colors that can be used in order to create a proper directed-edge-coloring. For each n, determine the minimum directed-edge-chromatic-number over all tournaments on n vertices.

Proposed by Po-Shen Loh

3 Let ABC be a non-equilateral triangle and let M_a, M_b, M_c be the midpoints of the sides BC, CA, AB, respectively. Let S be a point lying on the Euler line. Denote by X, Y, Z the second intersections of M_aS, M_bS, M_cS with the nine-point circle. Prove that AX, BY, CZ are concurrent.

USA Team Selection Test 2016

Dec December 10th, 2015

1 Let $S = \{1, \ldots, n\}$. Given a bijection $f : S \to S$ an *orbit* of f is a set of the form $\{x, f(x), f(f(x)), \ldots\}$ for some $x \in S$. We denote by $c(f)$ the number of distinct orbits of f. For example, if $n = 3$ and $f(1) = 2$, $f(2) = 1$, $f(3) = 3$, the two orbits are $\{1, 2\}$ and $\{3\}$, hence $c(f) = 2$.

Given k bijections $f_1, \ldots, f_k$ from S to itself, prove that

$$c(f_1) + \cdots + c(f_k) \le n(k-1) + c(f)$$

where $f : S \to S$ is the composed function $f_1 \circ \cdots \circ f_k$.

Proposed by Maria Monks Gillespie

2 Let ABC be a scalene triangle with circumcircle Ω, and suppose the incircle of ABC touches BC at D. The angle bisector of $\angle A$ meets BC and Ω at E and F. The circumcircle of $\triangle DEF$ intersects the A-excircle at S_1, S_2, and Ω at $T \neq F$. Prove that line AT passes through either S_1 or S_2.

Proposed by Evan Chen

3 Let p be a prime number. Let $\mathbb{F}_p$ denote the integers modulo p, and let $\mathbb{F}_p[x]$ be the set of polynomials with coefficients in $\mathbb{F}_p$. Define $\Psi : \mathbb{F}_p[x] \to \mathbb{F}_p[x]$ by

$$\Psi\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} a_i x^{p^i}.$$

Prove that for nonzero polynomials $F, G \in \mathbb{F}_p[x]$,

$$\Psi(\gcd(F, G)) = \gcd(\Psi(F), \Psi(G)).$$

Here, a polynomial Q divides P if there exists $R \in \mathbb{F}_p[x]$ such that $P(x) - Q(x)R(x)$ is the polynomial with all coefficients 0 (with all addition and multiplication in the coefficients taken modulo p), and the gcd of two polynomials is the highest degree polynomial with leading coefficient 1 which divides both of them. A non-zero polynomial is a polynomial with not all coefficients 0. As an example of multiplication, $(x + 1)(x + 2)(x + 3) = x^3 + x^2 + x + 1$ in $\mathbb{F}_5[x]$.

Proposed by Mark Sellke

USA Team Selection Test 2017

TST#1 December 8th, 2016

1 In a sports league, each team uses a set of at most t signature colors. A set S of teams is *color-identifiable* if one can assign each team in S one of their signature colors, such that no team in S is assigned any signature color of a different team in S.

For all positive integers n and t, determine the maximum integer $g(n,t)$ such that: In
any sports league with exactly n distinct colors present over all teams, one can always
find a color-identifiable set of size at least $g(n,t)$.

2 Let ABC be an acute scalene triangle with circumcenter O, and let T be on line BC such that $\angle TAO = 90^\circ$. The circle with diameter $\overline{AT}$ intersects the circumcircle of $\triangle BOC$ at two points A_1 and A_2, where $OA_1 < OA_2$. Points B_1, B_2, C_1, C_2 are defined analogously.
- Prove that $\overline{AA_1}$, $\overline{BB_1}$, $\overline{CC_1}$ are concurrent.
- Prove that $\overline{AA_2}$, $\overline{BB_2}$, $\overline{CC_2}$ are concurrent on the Euler line of triangle ABC.
Evan Chen

3 Let $P, Q \in \mathbb{R}[x]$ be relatively prime nonconstant polynomials. Show that there can be at most three real numbers λ such that $P + \lambda Q$ is the square of a polynomial.

Alison Miller

TST#2 January 19th, 2017

1 You are cheating at a trivia contest. For each question, you can peek at each of the $n > 1$ other contestants' guesses before writing down your own. For each question, after all guesses are submitted, the emcee announces the correct answer. A correct guess is worth 0 points. An incorrect guess is worth -2 points for other contestants, but only -1 point for you, since you hacked the scoring system. After announcing the correct answer, the emcee proceeds to read the next question. Show that if you are leading by 2^{n-1} points at any time, then you can surely win first place.

Linus Hamilton

2 Let ABC be a triangle with altitude $\overline{AE}$. The A-excircle touches $\overline{BC}$ at D, and intersects the circumcircle at two points F and G. Prove that one can select points V and N on lines DG and DF such that quadrilateral $EVAN$ is a rhombus.

Danielle Wang

3 Prove that there are infinitely many triples (a, b, p) of positive integers with p prime, $a < p$, and $b < p$, such that $(a+b)^p - a^p - b^p$ is a multiple of p^3.