Canada National Olympiad 2010

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1 For all natural n, an n-staircase is a figure consisting of unit squares, with one square in the first row, two squares in the second row, and so on, up to n squares in the n^{th} row, such that all the left-most squares in each row are aligned vertically.
Let $f(n)$ denote the minimum number of square tiles requires to tile the n-staircase, where the side lengths of the square tiles can be any natural number. e.g. $f(2) = 3$ and $f(4) = 7$.
(a) Find all n such that $f(n) = n$.
(b) Find all n such that $f(n) = n + 1$.

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2 Let A, B, P be three points on a circle. Prove that if a, b are the distances from P to the tangents at A, B respectively, and c is the distance from P to the chord AB, then $c^2 = ab$.

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3 Three speed skaters have a friendly "race" on a skating oval. They all start from the same point and skate in the same direction, but with different speeds that they maintain throughout the race. The slowest skater does 1 lap per minute, the fastest one does 3.14 laps per minute, and the middle one does L laps a minute for some $1 < L < 3.14$. The race ends at the moment when all three skaters again come together to the same point on the oval (which may differ from the starting point.) Determine the number of different choices for L such that exactly 117 passings occur before the end of the race.
Note: A passing is defined as when one skater passes another one. The beginning and the end of the race when all three skaters are together are not counted as passings.

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4 Each vertex of a finite graph can be coloured either black or white. Initially all vertices are black. We are allowed to pick a vertex P and change the colour of P and all of its neighbours. Is it possible to change the colour of every vertex from black to white by a
sequence of operations of this type?
Note: A finite graph consists of a finite set of vertices and a finite set of edges between vertices. If there is an edge between vertex A and vertex B, then A and B are neighbours of each other.

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5 Let $P(x)$ and $Q(x)$ be polynomials with integer coefficients. Let $a_n = n! + n$. Show that if $\frac{P(a_n)}{Q(a_n)}$ is an integer for every n, then $\frac{P(n)}{Q(n)}$ is an integer for every integer n such that $Q(n) \neq 0$.
