Romania National Olympiad 2009

–	Grade level 7
–	Grade level 8
–	Grade level 9
–	Grade level 10
–	Grade level 11
1	Let $(t_n)_n$ a convergent sequence of real numbers, $t_n \in (0,1)$, $(\forall)n \in \mathbb{N}$ and $\lim_{n\to\infty} t_n \in (0,1)$. Define the sequences $(x_n)_n$ and $(y_n)_n$ by $$x_{n+1} = t_n x_n + (1-t_n)y_n,\ y_{n+1} = (1-t_n)x_n + t_n y_n,\ (\forall)n \in \mathbb{N}$$ and x_0, y_0 are given real numbers. a) Prove that the sequences $(x_n)_n$ and $(y_n)_n$ are convergent and have the same limit. b) Prove that if $\lim_{n\to\infty} t_n \in \{0,1\}$, then the question is false.
2	Let $f : \mathbb{R} \to \mathbb{R}$ a continuous function such that for any $x \in \mathbb{R}$, the limit $\lim_{h\to 0} \left\lvert \frac{f(x+h)-f(x)}{h} \right\rvert$ exists and it is finite. Prove that in any real point, f is differentiable or it has finite one-side derivates, of the same modul, but different signs.
3	Let $A, B \in \mathcal{M}_n(\mathbb{C})$ such that $AB = BA$ and $\det B \neq 0$. a) If $\lvert\det(A+zB)\rvert = 1$ for any $z \in \mathbb{C}$ such that $\lvert z\rvert = 1$, then $A^n = O_n$. b) Is the question from a) still true if $AB \neq BA$?
4	Let $f, g, h : \mathbb{R} \to \mathbb{R}$ such that f is differentiable, g and h are monotonic, and $f' = f + g + h$. Prove that the set of the points of discontinuity of g coincides with the respective set of h.
–	Grade level 12