USA Team Selection Test 2009

Day 1

1 Let m and n be positive integers. Mr. Fat has a set S containing every rectangular tile with integer side lengths and area of a power of 2. Mr. Fat also has a rectangle R with dimensions $2^m \times 2^n$ and a 1×1 square removed from one of the corners. Mr. Fat wants to choose $m+n$ rectangles from S, with respective areas $2^0, 2^1, \ldots, 2^{m+n-1}$, and then tile R with the chosen rectangles. Prove that this can be done in at most $(m+n)!$ ways.

Palmer Mebane.

2 Let ABC be an acute triangle. Point D lies on side BC. Let O_B, O_C be the circumcenters of triangles ABD and ACD, respectively. Suppose that the points B, C, O_B, O_C lies on a circle centered at X. Let H be the orthocenter of triangle ABC. Prove that $\angle DAX = \angle DAH$.

Zuming Feng.

3 For each positive integer n, let $c(n)$ be the largest real number such that

$$c(n) \le \left| \frac{f(a) - f(b)}{a - b} \right|$$

for all triples (f, a, b) such that

–f is a polynomial of degree n taking integers to integers, and
–a, b are integers with $f(a) \neq f(b)$.

Find $c(n)$.

Shaunak Kishore.

Day 2

4 Let ABP, BCQ, CAR be three non-overlapping triangles erected outside of acute triangle ABC. Let M be the midpoint of segment AP. Given that $\angle PAB = \angle CQB = 45^\circ$, $\angle ABP = \angle QBC = 75^\circ$, $\angle RAC = 105^\circ$, and $RQ^2 = 6CM^2$, compute AC^2/AR^2.

Zuming Feng.

5 Find all pairs of positive integers (m, n) such that $mn - 1$ divides $(n^2 - n + 1)^2$.

Aaron Pixton.

6 Let $N > M > 1$ be fixed integers. There are N people playing in a chess tournament; each pair of players plays each other once, with no draws. It turns out that for each sequence of $M+1$ distinct players $P_0, P_1, \dots P_M$ such that P_{i-1} beat P_i for each $i = 1, \dots, M$, player P_0 also beat P_M. Prove that the players can be numbered $1, 2, \dots, N$ in such a way that, whenever $a \geq b + M - 1$, player a beat player b.

Gabriel Carroll.

Day 3

7 Find all triples (x, y, z) of real numbers that satisfy the system of equations

$$\begin{cases} x^3 = 3x - 12y + 50, \\ y^3 = 12y + 3z - 2, \\ z^3 = 27z + 27x. \end{cases}$$

Razvan Gelca.

8 Fix a prime number $p > 5$. Let a, b, c be integers no two of which have their difference divisible by p. Let i, j, k be nonnegative integers such that $i + j + k$ is divisible by $p - 1$. Suppose that for all integers x, the quantity

$$(x-a)(x-b)(x-c)[(x-a)^i(x-b)^j(x-c)^k - 1]$$

is divisible by p. Prove that each of i, j, k must be divisible by $p - 1$.

Kiran Kedlaya and Peter Shor.

9 Prove that for positive real numbers x, y, z,

$$x^3(y^2+z^2)^2+y^3(z^2+x^2)^2+z^3(x^2+y^2)^2 \geq xyz\left[xy(x+y)^2+yz(y+z)^2+zx(z+x)^2\right].$$

Zarathustra (Zeb) Brady.

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