IMO 1959

Brasov and Bucharest, Romania

Day 1

$\boxed{1}$ Prove that the fraction $\dfrac{21n+4}{14n+3}$ is irreducible for every natural number n.

$\boxed{2}$ For what real values of x is

$$\sqrt{x+\sqrt{2x-1}}+\sqrt{x-\sqrt{2x-1}}=A$$

given

a) $A=\sqrt{2}$;

b) $A=1$;

c) $A=2$,

where only non-negative real numbers are admitted for square roots?

$\boxed{3}$ Let a,b,c be real numbers. Consider the quadratic equation in $\cos x$

$$a\cos x^2+b\cos x+c=0.$$

Using the numbers a,b,c form a quadratic equation in $\cos 2x$ whose roots are the same as those of the original equation. Compare the equation in $\cos x$ and $\cos 2x$ for $a=4$, $b=2$, $c=-1$.

Day 2

4. Construct a right triangle with given hypotenuse c such that the median drawn to the hypotenuse is the geometric mean of the two legs of the triangle.

5. An arbitrary point M is selected in the interior of the segment AB. The square $AMCD$ and $MBEF$ are constructed on the same side of AB, with segments AM and MB as their respective bases. The circles circumscribed about these squares, with centers P and Q, intersect at M and also at another point N. Let N' denote the point of intersection of the straight lines AF and BC.

   a) Prove that N and N' coincide;

   b) Prove that the straight lines MN pass through a fixed point S independent of the choice of M;

   c) Find the locus of the midpoints of the segments PQ as M varies between A and B.

6. Two planes, P and Q, intersect along the line p. The point A is given in the plane P, and the point C in the plane Q; neither of these points lies on the straight line p. Construct an isosceles trapezoid $ABCD$ (with $AB \parallel CD$) in which a circle can be inscribed, and with vertices B and D lying in planes P and Q respectively.