India
Regional Mathematical Olympiad
2002

---

1. In an acute triangle ABC points D, E, F are located on the sides BC, CA, AB such that

$$\frac{CD}{CE} = \frac{CA}{CB}, \frac{AE}{AF} = \frac{AB}{AC}, \frac{BF}{FD} = \frac{BC}{BA}$$

Prove that AD, BE, CF are altitudes of triangle ABC.

2. Solve for real x :
$$(x^2 + x - 2)^3 + (2x^2 - x - 1)^3 = 27(x^2 - 1)^3.$$

3. Let a, b, c be positive integers such that a divides b^2, b divides c^2 and c divides a^2. Prove that abc divides $(a + b + c)^7$.

4. Suppose the integers $1, 2, \ldots 10$ are split into two disjoint collections $a_1, a_2, \ldots a_5$ and $b_1, \ldots b_5$ such that $a_1 < a_2 < a_3 < a_4 < a_5, b_1 < b_2 < b_3 < b_4 < b_5$ (i) Show that the larger number in any pair $\{a_j, b_j\}$, $1 \le j \le 5$ is at least 6. (ii) Show that $\sum_{i=1}^{5} |a_i - b_i| = 25$ for every such partition.

5. The circumference of a circle is divided into eight arcs by a convex quadrilateral $ABCD$ with four arcs lying inside the quadrilateral and the remaining four lying outside it. The lengths of the arcs lying inside the quadrilateral are denoted by p, q, r, s in counter-clockwise direction. Suppose $p + r = q + s$. Prove that $ABCD$ is cyclic.

6. Prove that for any natural number $n > 1$,

$$\frac{1}{2} < \frac{1}{n^2 + 1} + \frac{2}{n^2 + 2} + \ldots + \frac{n}{n^2 + n} < \frac{1}{2} + \frac{1}{2n}.$$

7. Find all integers a, b, c, d such that

(i) $1 \le a \le b \le c \le d$;

(ii) $ab + cd = a + b + c + d + 3$.