1 Combinatorics

1. Let X be a subset of $\mathbb{Z}$. Denote $X + a = \{x + a | x \in X\}$. Show that if there exist nonnegative integers $a_1, a_2, ..., a_n$ such that $X + a_1, X + a_2, ..., X + a_n$ form a partition of $\mathbb{Z}$, then there is a non-zero integer N such that $X = X + N$.

2. An evil wizard places n people in a room and forces them to play the following game. He places on each person's head either a red hat or a blue hat independently with probability $1/2$. Each person can see the colors of the hats of all other $n - 1$ people, but not the color of his own hat. Simultaneously, each person must say a real number. They win if i) the sum of the numbers they say is strictly positive and there are an even number of red hats, or ii) the sum of the numbers they say is strictly negative and there are an odd number of red hats. What is their maximum probability of success? (They can decide on a strategy beforehand).

3. Consider a 6×6 grid. Define a *diagonal* to be the six squares whose co-ordinates $(i, j), 1 \leq i, j \leq 6$ satisfy $i - j = k \mod 6$ for some fixed k. Hence there are 6 diagonals. Prove that it is impossible to fill the grid with the numbers $1, 2, \ldots, 36$ such that each number appears exactly once and the sum of the numbers in each row, column, and diagonal are the same.

2 Algebra

1. Jacob and David play a game. We start with the polynomial $x^3 + x + 2015$. They alternated moves, with David going first. On his turn, Jacob can raise the coefficient of x by 1 or lower it by 1. David can raise the constant term by 1 or lower it by 1. Jacob wins if the polynomial acquires an integer root at any point. Prove that Jacob has a winning strategy.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ we have
$$\big(f(x^2) - yf(y)\big) \cdot \big(x^2 + f(y^2)\big) = f(x^4) - f(y^4).$$

3. Let n be a positive integer. Consider integers $0 < a_k \leq k$ for $1 \leq k \leq n$. Suppose that the sum $a_1 + a_2 + \cdots + a_n$ is even. Prove that one can choose signs $+$ or $-$ in the following expression such that it becomes valid:
$$a_1 \pm a_2 \pm a_3 \pm \cdots \pm a_n = 0.$$

3 Number Theory

1. Let N, k be positive integers with $10^{k-1} \leq N < 10^k$. Prove there exists a positive integer x such that if one writes out x^2 in base 10 then the first k digits represent N in base 10. For example, if $N = 14$, one can take $x = 12$.

2. Prove that for every integer a, there exist infinitely many primes p, such that there exist integers m, n with both $n^2 + 3$ and $m^3 - a$ divisible by p.

3. For a positive integer n, define the set A_n to consist of al positive integers x with $x \leq n$ such that $n \mid x^n + 1$. Does there exist an n such that A_n contains precisely 130 elements?

4 Geometry

1. Find the locus of points P in the plane of a square $ABCD$ such that
$$\max(PA, PC) = \frac{1}{\sqrt{2}}(PB + PD).$$

2. The quadrilateral $ABCD$ is inscribed in a circle. The lines AB and CD meet at E, and the diagonals AC and BD meet at F. The circumcircles of the triangles AFD and BFC meet again at H. Prove that $\angle EHF = 90°$.

3. Can the incenters of the four faces of a tetrahedron lie on a plane? *"What? 3D geometry? Not fair!" On the contrary, dear winter campers, it is the fairest of them all.*