APMO 1997

1 Given:

$$S = 1 + \frac{1}{1+\frac{1}{3}} + \frac{1}{1+\frac{1}{3}+\frac{1}{6}} + \cdots + \frac{1}{1+\frac{1}{3}+\frac{1}{6}+\cdots+\frac{1}{1993006}}$$

where the denominators contain partial sums of the sequence of reciprocals of triangular numbers (i.e. $k = \frac{n(n+1)}{2}$ for $n = 1,\ 2,\ \ldots,1996$). Prove that $S > 1001$.

2 Find an integer n, where $100 \le n \le 1997$, such that

$$\frac{2^n+2}{n}$$

is also an integer.

3 Let ABC be a triangle inscribed in a circle and let

$$l_a = \frac{m_a}{M_a}\ ,\quad l_b = \frac{m_b}{M_b}\ ,\quad l_c = \frac{m_c}{M_c}\ ,$$

where m_a,m_b, m_c are the lengths of the angle bisectors (internal to the triangle) and M_a, M_b, M_c are the lengths of the angle bisectors extended until they meet the circle. Prove that

$$\frac{l_a}{\sin^2 A} + \frac{l_b}{\sin^2 B} + \frac{l_c}{\sin^2 C} \ge 3$$

and that equality holds iff ABC is an equilateral triangle.

4 Triangle $A_1A_2A_3$ has a right angle at A_3. A sequence of points is now defined by the following iterative process, where n is a positive integer. From A_n ($n \ge 3$), a perpendicular line is drawn to meet $A_{n-2}A_{n-1}$ at A_{n+1}.
(a) Prove that if this process is continued indefinitely, then one and only one point P is interior to every triangle $A_{n-2}A_{n-1}A_n$, $n \ge 3$.
(b) Let A_1 and A_3 be fixed points. By considering all possible locations of A_2 on the plane, find the locus of P.

5 Suppose that n people A_1, A_2, ..., A_n, ($n \geq 3$) are seated in a circle and that A_i has a_i objects such that

$$a_1 + a_2 + \cdots + a_n = nN$$

where N is a positive integer. In order that each person has the same number of objects, each person A_i is to give or to receive a certain number of objects to or from its two neighbours A_{i-1} and A_{i+1}. (Here A_{n+1} means A_1 and A_n means A_0.) How should this redistribution be performed so that the total number of objects transferred is minimum?