
Baltic Way 2016

- 1 Find all pairs of primes (p, q) such that

$$p^3 - q^5 = (p + q)^2.$$

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- 2 Prove or disprove the following hypotheses.

- a) For all $k \geq 2$, each sequence of k consecutive positive integers contains a number that is not divisible by any prime number less than k .
b) For all $k \geq 2$, each sequence of k consecutive positive integers contains a number that is relatively prime to all other members of the sequence.
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- 3 For which integers $n = 1, \dots, 6$ does the equation

$$a^n + b^n = c^n + n$$

have a solution in integers?

- 4 Let n be a positive integer and let a, b, c, d be integers such that $n|a + b + c + d$ and $n|a^2 + b^2 + c^2 + d^2$.
Show that

$$n|a^4 + b^4 + c^4 + d^4 + 4abcd.$$

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- 5 Let $p > 3$ be a prime such that $p \equiv 3 \pmod{4}$. Given a positive integer a_0 define the sequence a_0, a_1, \dots of integers by $a_n = a_{n-1}^{2^n}$ for all $n = 1, 2, \dots$. Prove that it is possible to choose a_0 such that the subsequence $a_N, a_{N+1}, a_{N+2}, \dots$ is not constant modulo p for any positive integer N .
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- 6 The set $\{1, 2, \dots, 10\}$ is partitioned to three subsets A, B and C . For each subset the sum of its elements, the product of its elements and the sum of the digits of all its elements are calculated.
Is it possible that A alone has the largest sum of elements, B alone has the largest product of elements, and C alone has the largest sum of digits?
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- 7 Find all positive integers n for which

$$3x^n + n(x + 2) - 3 \geq nx^2$$

holds for all real numbers x .

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- 8 Find all real numbers a for which there exists a non-constant function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying the following two equations for all $x \in \mathbb{R}$:
- i) $f(ax) = a^2 f(x)$ and
 - ii) $f(f(x)) = af(x)$.
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- 9 Find all quadruples (a, b, c, d) of real numbers that simultaneously satisfy the following equations:
- $$\begin{cases} a^3 + c^3 = 2 \\ a^2b + c^2d = 0 \\ b^3 + d^3 = 1 \\ ab^2 + cd^2 = -6. \end{cases}$$
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- 10 Let $a_{0,1}, a_{0,2}, \dots, a_{0,2016}$ be positive real numbers. For $n \geq 0$ and $1 \leq k < 2016$ set
- $$a_{n+1,k} = a_{n,k} + \frac{1}{2a_{n,k+1}} \quad \text{and} \quad a_{n+1,2016} = a_{n,2016} + \frac{1}{2a_{n,1}}.$$
- Show that $\max_{1 \leq k \leq 2016} a_{2016,k} > 44$.
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- 11 Set A consists of 2016 positive integers. All prime divisors of these numbers are smaller than 30. Prove that there are four distinct numbers a, b, c and d in A such that $abcd$ is a perfect square.
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- 12 Does there exist a hexagon (not necessarily convex) with side lengths 1, 2, 3, 4, 5, 6 (not necessarily in this order) that can be tiled with a) 31 b) 32 equilateral triangles with side length 1?
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- 13 Let n numbers all equal to 1 be written on a blackboard. A move consists of replacing two numbers on the board with two copies of their sum. It happens that after h moves all n numbers on the blackboard are equal to m . Prove that $h \leq \frac{1}{2}n \log_2 m$.
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- 14 A cube consists of 4^3 unit cubes each containing an integer. At each move, you choose a unit cube and increase by 1 all the integers in the neighbouring cubes having a face in common with the chosen cube. Is it possible to reach a position where all the 4^3 integers are divisible by 3, no matter what the starting position is?
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- 15 The Baltic Sea has 2016 harbours. There are two-way ferry connections between some of them. It is impossible to make a sequence of direct voyages $C_1 - C_2 -$
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... – C_{1062} where all the harbours C_1, \dots, C_{1062} are distinct. Prove that there exist two disjoint sets A and B of 477 harbours each, such that there is no harbour in A with a direct ferry connection to a harbour in B .

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- 16** In triangle ABC , the points D and E are the intersections of the angular bisectors from C and B with the sides AB and AC , respectively. Points F and G on the extensions of AB and AC beyond B and C , respectively, satisfy $BF = CG = BC$. Prove that $FG \parallel DE$.
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- 17** Let $ABCD$ be a convex quadrilateral with $AB = AD$. Let T be a point on the diagonal AC such that $\angle ABT + \angle ADT = \angle BCD$. Prove that $AT + AC \geq AB + AD$.
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- 18** Let $ABCD$ be a parallelogram such that $\angle BAD = 60^\circ$. Let K and L be the midpoints of BC and CD , respectively. Assuming that $ABKL$ is a cyclic quadrilateral, find $\angle ABD$.
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- 19** Consider triangles in the plane where each vertex has integer coordinates. Such a triangle can be *legally transformed* by moving one vertex parallel to the opposite side to a different point with integer coordinates. Show that if two triangles have the same area, then there exists a series of legal transformations that transforms one to the other.
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- 20** Let $ABCD$ be a cyclic quadrilateral with AB and CD not parallel. Let M be the midpoint of CD . Let P be a point inside $ABCD$ such that $PA = PB = CM$. Prove that AB, CD and the perpendicular bisector of MP are concurrent.
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