Cono Sur Olympiad 2014

Day 1

1 Numbers 1 through 2014 are written on a board. A valid operation is to erase two numbers a and b on the board and replace them with the greatest common divisor and the least common multiple of a and b.

Prove that, no matter how many operations are made, the sum of all the numbers that remain on the board is always larger than $2014 \times \sqrt[2014]{2014!}$

2 A pair of positive integers (a, b) is called *charrua* if there is a positive integer c such that $a+b+c$ and $a \times b \times c$ are both square numbers; if there is no such number c, then the pair is called *non-charrua*.

a) Prove that there are infinite *non-charrua* pairs.
b) Prove that there are infinite positive integers n such that $(2, n)$ is *charrua*.

3 Let $ABCD$ be a rectangle and P a point outside of it such that $\angle BPC = 90^\circ$ and the area of the pentagon $ABPCD$ is equal to AB^2.

Show that $ABPCD$ can be divided in 3 pieces with straight cuts in such a way that a square can be built using those 3 pieces, without leaving any holes or placing pieces on top of each other.

Note: the pieces can be rotated and flipped over.

Day 2

4 Show that the number $n^2 - 2^{2014} \times 2014n + 4^{2013}(2014^2 - 1)$ is not prime, where n is a positive integer.

5 Let $ABCD$ be an inscribed quadrilateral in a circumference with center O such that it lies inside $ABCD$ and $\angle BAC = \angle ODA$. Let E be the intersection of AC with BD. Lines r and s are drawn through E such that r is perpendicular to BC, and s is perpendicular to AD. Let P be the intersection of r with AD, and M the intersection of s with BC. Let N be the midpoint of EO.

Prove that M, N, and P lie on a line.

6 Let F be a family of subsets of $S = \{1, 2, ..., n\}$ $(n \geq 2)$. A valid play is to choose two disjoint sets A and B from F and add $A \cup B$ to F (without removing A and B).

Initially, F has all the subsets that contain only one element of S. The goal is to have all subsets of $n-1$ elements of S in F using valid plays.

Determine the lowest number of plays required in order to achieve the goal.