JBMO ShortLists 2000

$\boxed{1}$ Prove that there are at least 666 positive composite numbers with 2006 digits, having a digit equal to 7 and all the rest equal to 1.

$\boxed{2}$ Find all the positive perfect cubes that are not divisible by 10 such that the number obtained by erasing the last three digits is also a perfect cube.

$\boxed{3}$ Find the greatest positive integer x such that 23^{6+x} divides 2000!

$\boxed{4}$ Find all the integers written as $\overline{abcd}$ in decimal representation and $\overline{dcba}$ in base 7.

$\boxed{5}$ Find all pairs of integers (m, n) such that the numbers $A = n^2 + 2mn + 3m^2 + 2$, $B = 2n^2 + 3mn + m^2 + 2$, $C = 3n^2 + mn + 2m^2 + 1$ have a common divisor greater than 1.

$\boxed{6}$ Find all four-digit numbers such that when decomposed into prime factors, each number has the sum of its prime factors equal to the sum of the exponents.

$\boxed{7}$ Find all the pairs of positive integers (m, n) such that the numbers $A = n^2 + 2mn + 3m^2 + 3n$, $B = 2n^2 + 3mn + m^2$, $C = 3n^2 + mn + 2m^2$ are consecutive in some order.

$\boxed{8}$ Find all positive integers a, b for which $a^4 + 4b^4$ is a prime number.

$\boxed{9}$ Find all the triples (x, y, z) of positive integers such that $xy + yz + zx - xyz = 2$.

$\boxed{10}$ Prove that there are no integers x, y, z such that

$$x^4 + y^4 + z^4 - 2x^2y^2 - 2y^2z^2 - 2z^2x^2 = 2000$$

$\boxed{11}$ Prove that for any integer n one can find integers a and b such that

$$n = \left[a\sqrt{2}\right] + \left[b\sqrt{3}\right]$$

$\boxed{12}$ Consider a sequence of positive integers x_n such that:

$$\text{(A) } x_{2n+1} = 4x_n + 2n + 2$$

$$\text{(B) } x_{3n+2} = 3x_{n+1} + 6x_n$$

for all $n \geq 0$. Prove that

$$\text{(C) } x_{3n-1} = x_{n+2} - 2x_{n+1} + 10x_n$$

for all $n \geq 0$.

13 Prove that

$$\sqrt{(1^k+2^k)(1^k+2^k+3^k)\ldots(1^k+2^k+\ldots+n^k)}$$
$$\geq 1^k+2^k+\ldots+n^k-\frac{2^{k-1}+2\cdot 3^{k-1}+\ldots+(n-1)\cdot n^{k-1}}{n}$$

for all integers $n, k \geq 2$.

14 Let m and n be positive integers with $m \leq 2000$ and $k = 3 - \frac{m}{n}$. Find the smallest positive value of k.

15 Let x, y, a, b be positive real numbers such that $x \neq y$, $x \neq 2y$, $y \neq 2x$, $a \neq 3b$ and $\frac{2x-y}{2y-x} = \frac{a+3b}{a-3b}$. Prove that $\frac{x^2+y^2}{x^2-y^2} \geq 1$.

16 Find all the triples (x, y, z) of real numbers such that

$$2x\sqrt{y-1}+2y\sqrt{z-1}+2z\sqrt{x-1} \geq xy+yz+zx$$

17 A triangle ABC is given. Find all the pairs of points X, Y so that X is on the sides of the triangle, Y is inside the triangle, and four non-intersecting segments from the set $\{XY, AX, AY, BX, BY, CX, CY\}$ divide the triangle ABC into four triangles with equal areas.

18 A triangle ABC is given. Find all the segments XY that lie inside the triangle such that XY and five of the segments XA, XB, XC, YA, YB, YC divide the triangle ABC into 5 regions with equal areas. Furthermore, prove that all the segments XY have a common point.

19 Let ABC be a triangle. Find all the triangles XYZ with vertices inside triangle ABC such that XY, YZ, ZX and six non-intersecting segments from the following $AX, AY, AZ, BX, BY, BZ, CX, CY, C$ divide the triangle ABC into seven regions with equal areas.

20 Let ABC be a triangle and let a, b, c be the lengths of the sides BC, CA, AB respectively. Consider a triangle DEF with the side lengths $EF = \sqrt{au}$, $FD = \sqrt{bu}$, $DE = \sqrt{cu}$. Prove that $\angle A > \angle B > \angle C$ implies $\angle A > \angle D > \angle E > \angle F > \angle C$.

21 All the angles of the hexagon $ABCDEF$ are equal. Prove that

$$AB - DE = EF - BC = CD - FA$$

22 Consider a quadrilateral with $\angle DAB = 60^\circ$, $\angle ABC = 90^\circ$ and $\angle BCD = 120^\circ$. The diagonals AC and BD intersect at M. If $MB = 1$ and $MD = 2$, find the area of the quadrilateral $ABCD$.

23 The point P is inside of an equilateral triangle with side length 10 so that the distance from P to two of the sides are 1 and 3. Find the distance from P to the third side.