JBMO ShortLists 2006

---

1. For an acute triangle ABC prove the inequality: $\sum_{cyclic} \frac{m_a^2}{-a^2+b^2+c^2} \geq \frac{9}{4}$ where m_a, m_b, m_c are lengths of corresponding medians.

2. Let x, y, z be positive real numbers such that $x + 2y + 3z = \frac{11}{12}$. Prove the inequality $6(3xy + 4xz + 2yz) + 6x + 3y + 4z + 72xyz \leq \frac{107}{18}$.

3. Let $n \geq 3$ be a natural number. A set of real numbers $\{x_1, x_2, \ldots, x_n\}$ is called *summable* if $\sum_{i=1}^{n} \frac{1}{x_i} = 1$. Prove that for every $n \geq 3$ there always exists a *summable* set which consists of n elements such that the biggest element is: a) bigger than 2^{2n-2} b) smaller than n^2

4. Determine the biggest possible value of m for which the equation $2005x + 2007y = m$ has unique solution in natural numbers.

5. Determine all pairs (m, n) of natural numbers for which $m^2 = nk + 2$ where $k = \overline{n1}$.

   EDIT. It has been discovered the correct statement is with $k = \overline{1n}$.

6. Prove that for every composite number $n > 4$, numbers kn divides $(n-1)!$ for every integer k such that $1 \leq k \leq \lfloor \sqrt{n-1} \rfloor$.

7. Determine all numbers $\overline{abcd}$ such that $\overline{abcd} = 11(a + b + c + d)^2$.

8. Prove that there do not exist natural numbers $n \geq 10$ such that every n's digit is not zero, and all numbers which are obtained by permutating its digits are perfect squares.

9. Let $ABCD$ be a trapezoid with $AB \parallel CD, AB > CD$ and $\angle A + \angle B = 90^\circ$. Prove that the distance between the midpoints of the bases is equal to the semidifference of the bases.

10. Let $ABCD$ be a trapezoid inscribed in a circle $\mathcal{C}$ with $AB \parallel CD$, $AB = 2CD$. Let $\{Q\} = AD \cap BC$ and let P be the intersection of tangents to $\mathcal{C}$ at B and D. Calculate the area of the quadrilateral $ABPQ$ in terms of the area of the triangle PDQ.

11. Circles $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at A and B. Let $M \in AB$. A line through M (different from AB) cuts circles $\mathcal{C}_1$ and $\mathcal{C}_2$ at Z, D, E, C respectively such that $D, E \in ZC$. Perpendiculars at B to the lines EB, ZB and AD respectively cut circle $\mathcal{C}_2$ in F, K and N. Prove that $KF = NC$.

12. Let ABC be an equilateral triangle of center O, and $M \in BC$. Let K, L be projections of M onto the sides AB and AC respectively. Prove that line OM passes through the midpoint of the segment KL.

13. Let A be a subset of the set $\{1, 2, \ldots, 2006\}$, consisting of 1004 elements. Prove that there exist 3 distinct numbers $a, b, c \in A$ such that $gcd(a, b)$: a) divides c b) doesn't divide c

14. Let $n \geq 5$ be a positive integer. Prove that the set $\{1, 2, \ldots, n\}$ can be partitioned into two non-zero subsets S_n and P_n such that the sum of elements in S_n is equal to the product of elements in P_n.
