National Math Olympiad (Second Round) 2005

Day 1

1 Let $n, p > 1$ be positive integers and p be prime. We know that $n|p-1$ and $p|n^3-1$. Prove that $4p-3$ is a perfect square.

2 In triangle ABC, $\angle A = 60^\circ$. The point D changes on the segment BC. Let O_1, O_2 be the circumcenters of the triangles $\Delta ABD, \Delta ACD$, respectively. Let M be the meet point of BO_1, CO_2 and let N be the circumcenter of ΔDO_1O_2. Prove that, by changing D on BC, the line MN passes through a constant point.

3 In one galaxy, there exist more than one million stars. Let M be the set of the distances between any 2 of them. Prove that, in every moment, M has at least 79 members. (Suppose each star as a point.)

Day 2

1 We have a $2 \times n$ rectangle. We call each 1×1 square a room and we show the room in the i^{th} row and j^{th} column as (i, j). There are some coins in some rooms of the rectangle. If there exist more than 1 coin in each room, we can delete 2 coins from it and add 1 coin to its right adjacent room OR we can delete 2 coins from it and add 1 coin to its up adjacent room. Prove that there exists a finite configuration of allowable operations such that we can put a coin in the room $(1, n)$.

2 BC is a diameter of a circle and the points X, Y are on the circle such that $XY \perp BC$. The points P, M are on XY, CY (or their stretches), respectively, such that $CY||PB$ and $CX||PM$. Let K be the meet point of the lines XC, BP. Prove that $PB \perp MK$.

3 Let $\mathbb{R}^+$ be the set of positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all positive real numbers x, y the equation holds:

$$(x+y)f(f(x)y) = x^2 f(f(x) + f(y))$$