All-Russian Olympiad 2000

– Grade level 9

Day 1

1 Let a, b, c be distinct numbers such that the equations $x^2 + ax + 1 = 0$ and $x^2 + bx + c = 0$ have a common real root, and the equations $x^2 + x + a = 0$ and $x^2 + cx + b$ also have a common real root. Compute the sum $a + b + c$.

2 Tanya chose a natural number $X \le 100$, and Sasha is trying to guess this number. He can select two natural numbers M and N less than 100 and ask about $\gcd(X + M, N)$. Show that Sasha can determine Tanya's number with at most seven questions.

3 Let O be the center of the circumcircle ω of an acute-angle triangle ABC. A circle ω_1 with center K passes through A, O, C and intersects AB at M and BC at N. Point L is symmetric to K with respect to line NM. Prove that $BL \perp AC$.

4 Some pairs of cities in a certain country are connected by roads, at least three roads going out of each city. Prove that there exists a round path consisting of roads whose number is not divisible by 3.

Day 2

5 The sequence $a_1 = 1, a_2, a_3, \cdots$ is defined as follows: if $a_n - 2$ is a natural number not already occurring on the board, then $a_{n+1} = a_n - 2$; otherwise, $a_{n+1} = a_n + 3$. Prove that every nonzero perfect square occurs in the sequence as the previous term increased by 3.

6 On some cells of a $2n \times 2n$ board are placed white and black markers (at most one marker on every cell). We first remove all black markers which are in the same column with a white marker, then remove all white markers which are in the same row with a black one. Prove that either the number of remaining white markers or that of remaining black markers does not exceed n^2.

7 Let E be a point on the median CD of a triangle ABC. The circle $\mathcal{S}_1$ passing through E and touching AB at A meets the side AC again at M. The circle

S_2 passing through E and touching AB at B meets the side BC at N. Prove that the circumcircle of $\triangle CMN$ is tangent to both $\mathcal{S}_1$ and $\mathcal{S}_2$.

8 One hundred natural numbers whose greatest common divisor is 1 are arranged around a circle. An allowed operation is to add to a number the greatest common divisor of its two neighhbors. Prove that we can make all the numbers pairwise copirme in a finite number of moves.

– Grade level 10

Day 1

1 Evaluate the sum

$$\left\lfloor \frac{2^0}{3} \right\rfloor + \left\lfloor \frac{2^1}{3} \right\rfloor + \left\lfloor \frac{2^2}{3} \right\rfloor + \cdots + \left\lfloor \frac{2^{1000}}{3} \right\rfloor.$$

2 Let $-1 < x_1 < x_2, \cdots < x_n < 1$ and $x_1^{13} + x_2^{13} + \cdots + x_n^{13} = x_1 + x_2 + \cdots + x_n$. Prove that if $y_1 < y_2 < \cdots < y_n$, then

$$x_1^{13}y_1 + \cdots + x_n^{13}y_n < x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

3 In an acute scalene triangle ABC the bisector of the acute angle between the altitudes AA_1 and CC_1 meets the sides AB and BC at P and Q respectively. The bisector of the angle B intersects the segment joining the orthocenter of ABC and the midpoint of AC at point R. Prove that P, B, Q, R lie on a circle.

4 We are given five equal-looking weights of pairwise distinct masses. For any three weights A, B, C, we can check by a measuring if $m(A) < m(B) < m(C)$, where $M(X)$ denotes the mass of a weight X (the answer is *yes* or *no*.) Can we always arrange the masses of the weights in the increasing order with at most nine measurings?

Day 2

5 Let M be a finite sum of numbers, such that among any three of its elements there are two whose sum belongs to M. Find the greatest possible number of elements of M.

6 A perfect number, greater than 6, is divisible by 3. Prove that it is also divisible by 9.

7 Two circles are internally tangent at N. The chords BA and BC of the larger circle are tangent to the smaller circle at K and M respectively. Q and P are midpoint of arcs AB and BC respectively. Circumcircles of triangles BQK and BPM are intersect at L. Show that $BPLQ$ is a parallelogram.

8 Some paper squares of k distinct colors are placed on a rectangular table, with sides parallel to the sides of the table. Suppose that for any k squares of distinct colors, some two of them can be nailed on the table with only one nail. Prove that there is a color such that all squares of that color can be nailed with $2k-2$ nails.

– Grade level 11

Day 1

1 Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(x+y)+f(y+z)+f(z+x) \geq 3f(x+2y+3z)$$

for all $x, y, z \in \mathbb{R}$.

2 Prove that one can partition the set of natural numbers into 100 nonempty subsets such that among any three natural numbers a, b, c satisfying $a+99b=c$, there are two that belong to the same subset.

3 A convex pentagon $ABCDE$ is given in the coordinate plane with all vertices in lattice points. Prove that there must be at least one lattice point in the pentagon determined by the diagonals AC, BD, CE, DA, EB or on its boundary.

4 Let $a_1, a_2, \cdots, a_n$ be a sequence of nonnegative integers. For $k = 1, 2, \cdots, n$ denote

$$m_k = \max_{1 \leq l \leq k} \frac{a_{k-l+1}+a_{k-l+2}+\cdots+a_k}{l}.$$

Prove that for every $\alpha > 0$ the number of values of k for which $m_k > \alpha$ is less than $\frac{a_1+a_2+\cdots+a_n}{\alpha}$.

Day 2

5 Prove the inequality

$$\sin^n(2x) + (\sin^n x - \cos^n x)^2 \le 1.$$

6 A perfect number, gerater than 28 is divisible by 7. Prove that it is also divisible by 49.

7 A quadrilateral $ABCD$ is circumscribed about a circle ω. The lines AB and CD meet at O. A circle ω_1 is tangent to side BC at K and to the extensions of sides AB and CD, and a circle ω_2 is tangent to side AD at L and to the extensions of sides AB and CD. Suppose that points O, K, L lie on a line. Prove that the midpoints of BC and AD and the center of ω also lie on a line.

8 All points in a 100×100 array are colored in one of four colors red, green, blue or yellow in such a way that there are 25 points of each color in each row and in any column. Prove that there are two rows and two columns such that their four intersection points are all in different colors.