50 Functional Equations

1 Definitions

1. $\mathbb{N}$ is the set of positive integers.
2. $\mathbb{N}\cup\{0\}=\mathbb{N}_0$ is the set of non-negative integers.
3. $\mathbb{Z}$ is the set of integers.
4. $\mathbb{Q}$ is the set of rational numbers.
5. $\mathbb{R}_+$ is the set of positive real numbers.
6. $\mathbb{R}_0$ is the set of nonnegative real numbers.
7. $\mathbb{R}$ is the set of real numbers.
8. $\forall x$ is the short from of **for all** x.
9. $\exists x$ is the short from of **there exists** x.
10. $s.t.$ is the short form of **such that**.
11. WLOG is the short form of **without loss of generality**.
12. $[x]$ denotes the largest integer that is not greater than x.
13. For a function f and set S, $f(S)=\{f(x)\mid x\in S\}$
14. If a function f is defined on the set A to the set B, we write $f\colon A\longrightarrow B$ and read f **is a function from the set A to the set B.**
15. If $f\colon A\longrightarrow B$, then A and B are the **Domain** and **Range** of f, respectively. And $C=f(A)$ is called the **Co-domain** of f.
16. A function $f\colon A\longrightarrow B$ is called **surjectve** if $B=f(A)$, is called **injective** if for all $x,y\in A$, $f(x)=f(y)\Longleftrightarrow x=y$ and is called **bijective** if it is both injective and surjective.
17. $f\colon A\longrightarrow\mathbb{R}$ has a limit $y=\lim_{x\longrightarrow c}f(x)$ if $(\forall\varepsilon>0)(\exists\delta>0)(\forall x\in A)$, $|x-c|<\delta\Longrightarrow|f(x)-y|<\varepsilon$.
18. $f\colon[a,b]\longrightarrow\mathbb{R}$ is called **continuous** if $(\forall x\in[a,b])(\forall\varepsilon>0)(\exists\delta>0)$ s.t. $|x-y|<\delta\Longrightarrow|f(x)-f(y)|<\varepsilon$.
19. $f\colon A\longrightarrow\mathbb{R}$ is called **differentiable** at x if the limit $f'(x)=\lim_{h\longrightarrow 0}\frac{f(x+h)-f(x)}{h}$ exists. And if f is differentiable at x, $f'(x)$ is called it's derivative or first differential.

2 Basic(Trivial) Functions and Their Children

[In this section, assume that all functions are continuous]

1. $f:\mathbb{R}\longrightarrow\mathbb{R}$, $f(x+y)=f(x)+f(y)$ for all $x,y\in\mathbb{R}$
2. $f:\mathbb{R}\longrightarrow\mathbb{R}$, $f(x+y)=f(x)f(y)$ for all $x,y\in\mathbb{R}$
3. $f:\mathbb{R}_+\longrightarrow\mathbb{R}$, $f(xy)=f(x)+f(y)$ for all $x,y\in\mathbb{R}_+$
4. $f:\mathbb{R}\longrightarrow\mathbb{R}$, $f(xy)=f(x)f(y)$ for all $x,y\in\mathbb{R}$
5. $f:\mathbb{R}\longrightarrow\mathbb{R}$, $f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$ for all $x,y\in\mathbb{R}$
6. $f:\mathbb{R}\longrightarrow\mathbb{R}$, $f(x+y)+f(x-y)=2f(x)f(y)$ for all $x,y\in\mathbb{R}$
7. $f:\mathbb{R}_+\longrightarrow\mathbb{R}$, $f(x)f(y)=f(xy)+f\left(\frac{x}{y}\right)$ for all $x,y\in\mathbb{R}_+$

8. $f:\mathbb{R}\longrightarrow\mathbb{R},\ f\Big(\sqrt{x^2+y^2}\Big)=f(x)f(y)$ for all $x,y\in\mathbb{R}$
9. $f:\mathbb{R}\longrightarrow\mathbb{R},\ f(0)=0,\ f(1)=1,\ f(a)+f(b)=f(a)f(b)+f(a+b-ab)$ for all $a,b\in\mathbb{R}$
10. $f:\mathbb{R}_+\longrightarrow\mathbb{R}_+,\ f\big(\frac{x+y}{2}\big)=\frac{2f(x)f(y)}{f(x)+f(y)}$ for all $x,y\in\mathbb{R}_+$

3 Functions involving Natural Number

1. $f:\mathbb{N}\longrightarrow\mathbb{N},\ f(n+1)>f(f(n))$ for all $n\in\mathbb{N}$
2. Find a function $f:\mathbb{N}\longrightarrow\mathbb{N}$ such that, $f(f(n))=2n$ for all $n\in\mathbb{N}$
3. $f:\mathbb{N}\longrightarrow\mathbb{N},\ f(m)^2+f(n)\mid(m^2+n)^2$ for all $m,\ n\in\mathbb{N}$
4. $f:\mathbb{N}\longrightarrow\mathbb{N},\ f(f(a)+f(b))=a+b-1$ for all $a,b\in\mathbb{N}$
5. $f:\mathbb{N}\longrightarrow\mathbb{N},$ for all $a,b\in\mathbb{N}$ there exists a non-degenerate triangle with side length a, $f(b)$, $f(b+f(a)-1)$.

4 Elementary Non-trivial Functions

1. Let $g:\ \mathbb{R}\longrightarrow\mathbb{R}$, such that $g(x)=x-[x]$.
   Find all functions $f:\ \mathbb{R}\longrightarrow\mathbb{R}$ such that, $g(f(x+y))=g(f(x))+g(f(y))$ for all $x,y\in\mathbb{R}$
2. $f:\mathbb{R}\longrightarrow\mathbb{R},\ (f(x)+f(z))(f(y)+f(t))=f(xy+zt)+f(xt-yz)$ for all $x,y,z,t\in\mathbb{R}$
3. $f:\mathbb{R}_+\longrightarrow\mathbb{R}_+,\ f$ continuous, $f(x)+f(y)=f\big(\frac{x+y}{2}\big)+f\Big(\frac{2xy}{x+y}\Big)$ for all $x,y\in\mathbb{R}_+$
4. f: $\mathbb{R}\longrightarrow\mathbb{R},\ f(x+y)=\max\{f(x),y\}+\min\{f(y),x\}$ for all $x,y\in\mathbb{R}$
5. f: $\mathbb{R}\longrightarrow\mathbb{R},\ f(x-f(y))=f(f(y))+xf(y)+f(x)-1$ for all $x,y\in\mathbb{R}$
6. f: $\mathbb{R}\longrightarrow\mathbb{R},\ f(x)\geq e^x$ for all $x\in\mathbb{R}$ and $f(x+y)\geq f(x)f(y)$ for all $x,y\in\mathbb{R}$
7. Prove that there are no functions f,g such that, $f(g(x))=x^2$ and $g(f(x))=x^3$ for all $x\in\mathbb{R}$.
8. Does there exist any continuous function $f:\mathbb{R}\longrightarrow\mathbb{R}$ such that, $f(x)\in\mathbb{Q}\Longleftrightarrow x\notin\mathbb{Q}$?
9. $f:\mathbb{R}_+\longrightarrow\mathbb{R}_+,\ f\Big(yf\Big(\frac{x}{y}\Big)\Big)=\frac{x^4}{f(y)}$ for all $x,y\in\mathbb{R}_+$

5 Advanced Functions

1. $f:\mathbb{R}\longrightarrow\mathbb{R},\ |f(x)-f(y)|\leq(x-y)^2$ for all $x,y\in\mathbb{R}$
2. $f:\mathbb{R}\longrightarrow\mathbb{R},\ f(x+f(y))=y+f(x)$ for all $x,y\in\mathbb{R}$ and the set $\Big\{\frac{x}{f(x)}\ \Big|\ x\in\mathbb{R}\Big\}$ is finite.
3. $f:\mathbb{R}_+\longrightarrow\mathbb{R},\ f(x)f(y)=y^\alpha f\big(\frac{x}{2}\big)+x^\beta f\big(\frac{y}{2}\big)$ for some constant $\alpha,\beta\in R$ and for all $x,y\in\mathbb{R}_+$
4. $f:\mathbb{R}_+\longrightarrow\mathbb{R},\ xf(y)-yf(x)=f\Big(\frac{x}{y}\Big)$ for all $x,y\in\mathbb{R}_+$
5. $f:\mathbb{R}\longrightarrow\mathbb{R},\ f(f(x)-y^2)=f(x)^2-2f(x)y^2+f(f(y))$ for all $x,y\in\mathbb{R}$
6. $f:\mathbb{R}\longrightarrow\mathbb{R},\ f(f(x)+y)=xf(1+xy)$ for all $x,y\in\mathbb{R}$
7. $f:\mathbb{R}\longrightarrow\mathbb{R},\ f\Big(\frac{x+f(x)}{2}+y+f(2z)\Big)=2x-f(x)+f(f(y))+2f(z)$ for all $x,y,z\in\mathbb{R}$
8. $f:\mathbb{R}\longrightarrow\mathbb{R},\ f$ surjective and strictly increasing, $f(f(x))=f(x)+12x.$ for all $x\in\mathbb{R}$
9. $f:\mathbb{R}\longrightarrow\mathbb{R},\ f(x+y^2)\geq(y+1)f(x)^2$ for all $x,y\in\mathbb{R}$
10. $f:\mathbb{R}\longrightarrow\mathbb{R},\ f(y)f(xf(y))=f(xy)$ for all $x,y\in\mathbb{R}$

11. $f:\mathbb{R}\longrightarrow\mathbb{R},\ f(x)f(y)\le|x-y|$ for all $x\in\mathbb{Q}$ and $y\notin\mathbb{Q}$
12. $f:\mathbb{R}\longrightarrow\mathbb{R},\ f(x+y)\le yf(x)+f(f(x))$, Prove that $f(x)=0$ for all $x\le 0$
13. $f:\mathbb{R}\longrightarrow\mathbb{R},\ f((x+1)f(y))=y(f(x)+1)$ for all $x,y\in\mathbb{R}$
14. $f:\mathbb{R}_+\longrightarrow\mathbb{R}_+,\ f(x+y)\ge f(x)+yf(f(x))$ for all $x,y\in\mathbb{R}_+$
15. $f:\mathbb{R}_+\times\mathbb{R}_+\longrightarrow\mathbb{R}_+,\ xf(x,y)f\big(y,\frac{1}{x}\big)=yf(y,x)$ for all $x,y\in\mathbb{R}_+$
16. $f:\mathbb{R}_+\longrightarrow\mathbb{R}_+,\ f(x)^2\ge f(x+y)(f(x)+y)$ for all $x,y\in\mathbb{R}_+$
17. $f:\mathbb{R}\longrightarrow\mathbb{R},\ f(xy)(f(x)-f(y))=f(x)f(y)(x-y)$ for all $x,y\in\mathbb{R}$
18. Prove that there doesn't exist any function $f:\mathbb{R}\longrightarrow\mathbb{R}$, such that,
    - $f(1)=1$
    - $\exists M\in\mathbb{R}_+$ s.t. $|f(x)|\le M\ \forall x\in\mathbb{R}$
    - $f\big(x+\frac{1}{x^2}\big)=f(x)+f\big(\frac{1}{x}\big)^2$ for all $x\in\mathbb{R}$
19. $f:\mathbb{R}_0\longrightarrow\mathbb{R}$ and $f(x)\le\int_0^x f(t)\,\mathrm{d}t$ for all $x\ge 0$. Prove that, $f(x)=0$ for all $x\ge 0$
20. $f:\mathbb{R}_+\longrightarrow\mathbb{R}_+,\ f(x+y^n+f(x))=f(x)$, $\frac{f(x)+x^n}{f(y)+y^n}\in\mathbb{Q}$ for all $x,y\in\mathbb{R}_+$
21. $f:\mathbb{R}_+\longrightarrow\mathbb{R}_+,\ f(x+y^n+f(x))=f(x)$ for all $x,y\in\mathbb{R}_+$

6 Extreme Functions!

1. Find all functions $f:\mathbb{R}_+\longrightarrow\mathbb{R}_+$, such that,
   $f(x-f(y))=f(x+y^n)+f(y+f(y))$ for all $x,y\in\mathbb{R}_+$ and a fixed positive integer $n\ge 2$.
2. Find all continuous functions $f:\mathbb{R}_+\longrightarrow\mathbb{R}_+$, such that, $f(xf(y)+yf(x))=f(f(xy))$ for all $x,y\in\mathbb{R}_+$
3. f is a function such that $f'(x)=\frac{x^2-f(x)^2}{x^2(f(x)^2+1)}$ for all $x>1$. Prove that, $\lim_{x\longrightarrow\infty}f(x)=\infty$
4. Is there any strictly increasing function $f:\mathbb{R}_+\longrightarrow\mathbb{R}_+$, such that, $f'(x)=f(f(x))$?
5. $f:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ such that,
    - $f(x)=x$ if $x\le e$
    - $f(x)=xf(\ln x)$ if $x>e$

   Prove that $\sum_{n=1}^{\infty}\frac{1}{f(x)}$ diverges.

Dhananjoy Biswas
ebappa1971@gmail.com