Iran Team Selection Test 2015

– TST 1

Day 1

1 Find all polynomials $P,Q\in \mathbb{Q}\left[x\right]$ such that
$$P(x)^3+Q(x)^3=x^{12}+1.$$

2 I_b is the B-excenter of the triangle ABC and ω is the circumcircle of this triangle. M is the middle of arc BC of ω which doesn't contain A. MI_b meets ω at $T\not=M$. Prove that
$$TB\cdot TC=TI_b^2.$$

3 Let $b_1<b_2<b_3<\dots$ be the sequence of all natural numbers which are sum of squares of two natural numbers.
Prove that there exists infinite natural numbers like m which $b_{m+1}-b_m=2015$.

Day 2

4 n is a fixed natural number. Find the least k such that for every set A of k natural numbers, there exists a subset of A with an even number of elements which the sum of it's members is divisible by n.

5 Let A be a subset of the edges of an $n\times n$ table. Let $V(A)$ be the set of vertices from the table which are connected to at least on edge from A and $j(A)$ be the number of the connected components of graph G which it's vertices are the set $V(A)$ and it's edges are the set A. Prove that for every natural number l:
$$\frac{l}{2}\leq min_{|A|\geq l}(|V(A)|-j(A)) \leq \frac{l}{2}+\sqrt{\frac{l}{2}}+1$$

6 $ABCD$ is a circumscribed and inscribed quadrilateral. O is the circumcenter of the quadrilateral. E,F and S are the intersections of AB,CD , AD,BC and AC,BD respectively. E' and F' are points on AD and AB such that $A\hat{E}E'=E'\hat{E}D$ and $A\hat{F}F'=F'\hat{F}B$. X and Y are points on OE' and OF' such that $\frac{XA}{XD}=\frac{EA}{ED}$ and $\frac{YA}{YB}=\frac{FA}{FB}$. M is the midpoint of arc BD of (O) which contains A.
Prove that the circumcircles of triangles OXY and OAM are coaxal with the circle with diameter OS.

– TST 2

Day 1

1 a,b,c,d are positive numbers such that $\sum_{cyc}\frac{1}{ab}=1$. Prove that : $abcd+16\geq 8\sqrt{(a+c)(\frac{1}{a}+\frac{1}{c})}+8\sqrt{(b+d)(\frac{1}{b}+\frac{1}{d})}$

2 In triangle ABC(with incenter I) let the line parallel to BC from A intersect circumcircle of $\triangle ABC$ at A_1 let $AI\cap BC=D$ and E is tangency point of incircle with BC let $EA_1\cap \odot(\triangle ADE)=T$ prove that $AI=TI$.

Day 2

4 Let $\triangle ABC$ be an acute triangle. Point Z is on A altitude and points X and Y are on the B and C altitudes out of the triangle respectively, such that: $\angle AYB=\angle BZC=\angle CXA=90$
Prove that X,Y and Z are collinear, if and only if the length of the tangent drawn from A to the nine point circle of $\triangle ABC$ is equal with the sum of the lengths of the tangents drawn from B and C to the nine point circle of $\triangle ABC$.

5 We call a permutation $(a_1,a_2,\cdots,a_n)$ of the set $\{1,2,\cdots,n\}$ "good" if for any three natural numbers $i<j<k$, $n\nmid a_i+a_k-2a_j$ find all natural numbers $n\geq 3$ such that there exist a "good" permutation of a set $\{1,2,\cdots,n\}$.

6 If a,b,c are positive real numbers such that $a+b+c=abc$ prove that

$$\frac{abc}{3\sqrt{2}}\left(\sum_{cyc}\frac{\sqrt{a^3+b^3}}{ab+1}\right)\geq\sum_{cyc}\frac{a}{a^2+1}$$

– TST 3

Day 1

1 Point A is outside of a given circle ω. Let the tangents from A to ω meet ω at S, T points X, Y are midpoints of AT, AS let the tangent from X to ω meet ω at $R \neq T$. points P, Q are midpoints of XT, XR let $XY \cap PQ = K, SX \cap TK = L$ prove that quadrilateral $KRLQ$ is cyclic.

2 Assume that a_1, a_2, a_3 are three given positive integers consider the following sequence: $a_{n+1} = \text{lcm}[a_n, a_{n-1}] - \text{lcm}[a_{n-1}, a_{n-2}]$ for $n \geq 3$
Prove that there exist a positive integer k such that $k \leq a_3 + 4$ and $a_k \leq 0$.
($[a, b]$ means the least positive integer such that$a \mid [a, b], b \mid [a, b]$ also because $\text{lcm}[a, b]$ takes only nonzero integers this sequence is defined until we find a zero number in the sequence)

3 $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ are $2n$ positive real numbers such that $a_1, a_2, \cdots, a_n$ aren't all equal. And assume that we can divide $a_1, a_2, \cdots, a_n$ into two subsets with equal sums.similarly $b_1, b_2, \cdots, b_n$ have these two conditions. Prove that there exist a simple $2n$-gon with sides $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ and parallel to coordinate axises Such that the lengths of horizontal sides are among $a_1, a_2, \cdots, a_n$ and the lengths of vertical sides are among $b_1, b_2, \cdots, b_n$.(simple polygon is a polygon such that it doesn't intersect itself)

Day 2

5 Prove that for each natural number d, There is a monic and unique polynomial of degree d like P such that $P(1)0$ and for each sequence like a_1,a_2, ... of real numbers that the recurrence relation below is true for them, there is a natural number k such that $0 = a_k = a_{k+1} = \ldots :$ $P(n)a_1 + P(n-1)a_2 + \ldots + P(1)a_n = 0$ $n > 1$

6 AH is the altitude of triangle ABC and H' is the reflection of H trough the midpoint of BC. If the tangent lines to the circumcircle of ABC at B and C, intersect each other at X and the perpendicular line to XH' at H', intersects AB and AC at Y and Z respectively, prove that $\angle ZXC = \angle YXB$.