Balkan MO 2016

– May 7th

1 Find all injective functions $f:\mathbb{R}\to\mathbb{R}$ such that for every real number x and every positive integer n,

$$\left|\sum_{i=1}^{n} i\left(f(x+i+1)-f(f(x+i))\right)\right| < 2016$$

(Macedonia)

2 Let $ABCD$ be a cyclic quadrilateral with $AB<CD$. The diagonals intersect at the point F and lines AD and BC intersect at the point E. Let K and L be the orthogonal projections of F onto lines AD and BC respectively, and let M, S and T be the midpoints of EF, CF and DF respectively. Prove that the second intersection point of the circumcircles of triangles MKT and MLS lies on the segment CD.

(Greece - Silouanos Brazitikos)

3 Find all monic polynomials f with integer coefficients satisfying the following condition: there exists a positive integer N such that p divides $2(f(p)!)+1$ for every prime $p>N$ for which $f(p)$ is a positive integer.

Note: A monic polynomial has a leading coefficient equal to 1.

(Greece - Panagiotis Lolas and Silouanos Brazitikos)

4 The plane is divided into squares by two sets of parallel lines, forming an infinite grid. Each unit square is coloured with one of 1201 colours so that no rectangle with perimeter 100 contains two squares of the same colour. Show that no rectangle of size 1×1201 or 1201×1 contains two squares of the same colour.

Note: Any rectangle is assumed here to have sides contained in the lines of the grid.

(Bulgaria - Nikolay Beluhov)