Psets and Exams of BdMO Camps

Zawadx

21/2/2018

Contents

1 | Problem Sets

1.1 15 Natcamp Combinatorics Pset 1

Problem 1.1.1. Prove that

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}$$

Problem 1.1.2. What is the average size of a subset of $1, 2, \ldots, n$

Problem 1.1.3. Let n be an odd positive integer. Prove that among

$$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{\frac{n-1}{2}}$$

there are an even number of odd numbers.

Problem 1.1.4. Simplify:

$$\frac{\binom{n}{0}}{1} + \frac{\binom{n}{1}}{2} + \cdots + \frac{\binom{n}{n}}{n+1}$$

Problem 1.1.5. Each of ten segments are longer than 1 but shorter than 55. Prove that it is possible to construct a triangle taking three segments from the ten.

Problem 1.1.6. 342 points are selected inside a cube with edge length 7. Can you place a small cube with edge 1 inside the big cube such that the interior of the small cube does not contain one of the selected points?

Problem 1.1.7. Prove that from any 52 integers we can find two such that their sum or difference is divisible by 100.

Problem 1.1.8. Let n be an integer not divisible by 2 or 5. Prove that there is a multiple of n with every digit 2 or 5.

Problem 1.1.9. Suppose that $P_1P_2 \ldots P_{325}$ be the regular 325 sided polygon. Find the number of incongruent triangles whose vertices are also vertices of the polygon.

Problem 1.1.10. The numbers 1 to 81 are written on a 9×9 board. Prove that there exist two neighbours which differ by at least 6

1.2 15 Natcamp Combinatorics Pset 2

Problem 1.2.1. Let G be a bipartite graph and H be one of its subgraphs. Prove that H is also bipartite.

Problem 1.2.2. Prove that there exists only one simple path from one vertex of a tree to another.

Problem 1.2.3. Supose that, in a bipartite graph, all vertices except possibly one has same degree d, and the remaining vertex has an unknown degree x. Prove that $d|x$

Problem 1.2.4. Every participant in a tournament plays with every other participant exactly once. No game is a draw. After the tournament, players make a list of

1. Players beaten by him
2. Players beaten by players who were beaten by him

Prove that there is a participant whose list contains the name of all other players.

Problem 1.2.5. Consider a group of $n > 1$ members. Any two members of this group are friends or enemies. Enemy of an enemy is a friend, and friend of a friend is a friend. Find all n for which the number of friendships might equal the number of enmities.

Problem 1.2.6. 12% of a sphere is colored black. Prove that it is possible to inscribe a rectangular box with all white vertices in the sphere.

Problem 1.2.7. Prove that in any convex 2n-gon, there is a diagonal not parallel to any of the sides.

Problem 1.2.8. Let A and B be finite disjoint sets of points in the plane such that any three distinct $A \bigcup B$ are not collinear. Assume that at least one of the two sets contains at least 5 points. Show that there exists a triangle, all of whose vertices are contained in either A or B, that does not contain any point from the other set in its interior.

Problem 1.2.9. An Art Gallery is in the form of a simple (but not necessarily convex) n-gon, with its sides forming walls. A watchman can see any point in the plane that isn't blocked from his view by a wall. Find the minimum number of watchmed needed to survey the building, no matter how complicated its shape.

Problem 1.2.10. Let n people take part in a party. Any two are either acquainted or they are not. What is the maximum number of pairs of participants for which the two are not acquainted but have a common acquaintance among the partygoers.

1.3 15 Excamp Pset 1

Problem 1.3.1. Let ABC be a triangle with incentre I. A point P in the interior of the triangle satisfies $\angle PBA + \angle PCA = \angle PBC + \angle PCB$. Show that $AP \geq AI$ with equality if and only $P = I$.

Problem 1.3.2. In triangle ABC, $AB = AC$. A circle is tangent to circumcircle of ABC and also to sides AB, AC at P, Q. Prove that the midpoint of PQ is the incentre of ABC.

Problem 1.3.3. Convex quadrilateral $ABCD$ is inscribed in a circle with center O. Diagonals AC and BD meet at P. The circumcircles of triangles ABP and CDP meet at P, Q. Assume that O, P, Q are distinct. Prove that $\angle OQP = \pi/2$

Problem 1.3.4. Triangle ABC is inscribed in circle ω with center O. The tangent lines to ω at B and C meet at T. Point S lies on ray BC such that $AS \perp AT$. Points B_1, C_1 lie on ray ST such that $B_1T = BT = C_1T$. Prove that the triangles ABC and AB_1C_1 are similar.

Problem 1.3.5. Let $ABCD$ be a convex quadrilateral with sides BC and AD equal and not parallel. Let E and F be interior points of sides BC and AD such that $BE = DF$. The lines AC and BD meet at P, BD and EF meet at Q and AC and EF meet at R. Show that as E and F vary, the circumcircles of all the triangles PQR have a common point other than P.

Problem 1.3.6. Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of ABC, ADC by ω_1, ω_2. Suppose that there exists a circle ω tangent to ray BA beyond A, the ray BC beyond C and also tangent to lines AD, CD. Prove that the external common tangents of ω_1, ω_2 intersect on ω.

1.4 15 Excamp Pset 2

Problem 1.4.1. We have $n \geq 2$ lamps $L_1, ..., L_n$ in a row, each of them being either on or off. Every second we simultaneously modify the state of each lamp as follows: if the lamp L_i and its neighbours (only one neighbour for $i = 1$ or $i = n$, two neighbours for other i) are in the same state, then L_i is switched off; âĂŞ otherwise, L_i is switched on. Initially all the lamps are off except the leftmost one which is on.

(a) Prove that there are infinitely many integers n for which all the lamps will eventually be off. (b) Prove that there are infinitely many integers n for which the lamps will never be all off.

Problem 1.4.2. If a polynomial $P(x)$ with an integer coefficients is a square for all integers n prove that $P(x) = Q(x)^2$, where $Q(x)$ is some polynomial with integer coefficients.

Problem 1.4.3. Solve in integers:

$$x^2 + 2014y^2 = z^2 + 3t^2$$
$$2014x^2 + y^2 = 3z^2 + t^2$$

Problem 1.4.4. Triangle ABC has circumcircle Γ with centre O. Assume that $\angle ABC > 90^\circ$. Let D be the point of intersection of the line AB with the line perpendicular to AC at C. Let ℓ be the line through D which is perpendicular to AO. Let E be the point of intersection of ℓ with AC and F the intersection of Γ and ℓ between D, E. Prove that the circumcircles of triangles BFE and CFD are tangent at F.

Problem 1.4.5. Let $ABCD$ be a cyclic quadrilateral with $AB \cap CD = N, BC \cap AD = M$ and $\angle ABC > 90^\circ$. Let X be a point on segment CD and O_1, O_2 be the circumcenter of $\Delta MCX, \Delta MDX$. If $\angle BNM = \angle BNC$, prove that the intersection point of $\bigcirc MAB, \bigcirc MO_1O_2$ is the circumcenter of ΔMCD.

2 | 13 National Camp

2.1 NT

Problem 2.1.1. Let k be a positive odd integer. Is $\sum_{i=0}^{n} i^k$ divisible by $\sum_{i=0}^{n} i$?

Problem 2.1.2. Is it possible to have a perfect square whose only digits are 1 (which has more than one digit)?

Problem 2.1.3. Find all positive integers (a, b, n) that satisfy the equation $a! + b! = 2^n$

Problem 2.1.4. By connecting the vertices of a regular n-gon we obtain an n-mon. Prove that if n is even, the n-mon contains two parallel lines, and if n is odd it's impossible that there are exactly two parallel lines in the n-mon.

Problem 2.1.5. Let $a_{n+1} = a_n^3 - 2a_n^2 + 2$ for all $n \geq 1$ and $a_1 = 5$. Prove that if $p \equiv 3(mod4)$ is a prime divisor of $a_2011 + 1$, then $p = 3$.

2.2 Easy Mock

Problem 2.2.1. Triangle ABC has incentre I and touchpoints D, E, F on BC, CA, AB. The line ID intersects EF at K. Prove that A, K, M are collinear where M is the midpoint of BC.

Problem 2.2.2. Find all functions $f\mathbb{R} \to \mathbb{R}$ so that

$$(x + y)(f(x) - f(y)) = (x - y)(f(x) + f(y))$$

for all $x, y \in \mathbb{R}$

Problem 2.2.3. For each $i = 1, 2, \ldots, N$, let a_i, b_i, c_i be integers such that at least one of them is odd. Show that one can find integers x, y, z such that $xa_i + yb_i + zc_i$ is odd for at least $4N/7$ different values of i.

Problem 2.2.4. Let p be an odd prime. Prove that,

$$1^{p-2} + 2^{p-2} + \cdots + \left(\frac{p-1}{2}\right)^{p-2} \equiv \frac{2 - 2^p}{p}(modp)$$

Problem 2.2.5. There are 2012 distinct points in the plane each of which is to be coloured using one of n colours so that the number of points of each colour are distinct. A set is said to be multi-coloured if it has n points and its colours are distinct. Determine n that maximizes the number of multi-coloured sets.

Problem 2.2.6. A circle passing through B, C in ΔABC meets AB, AC at D, E. Let B', E' be the projections of B, E on CD. Let C', D' be the projections of C, D on BE. Prove that B', C', D', E' are concyclic.

2.3 Hard Mock

Problem 2.3.1. Let ABC be a triangle such that $AB \neq AC$ and $\angle B \neq \pi/2$. Let I be the incentre of ABC, and D, E, F the incircle touchpoints with BC, CA, AB. Let S be the intersection of AB and DI, and T the intersection of DE and the line passing through F perpendicular to DF. Let R be the intersection of ST and EF. Let P_{ABC} be the intersection of the circle with diameter IR and the incircle of ABC which is on the other side of A with respect to line IR. Let XYZ be an isosceles triangle with $XZ = YZ > XY$ and W be a point on side YZ with $WY < XY$. For $K = P_{YXW}$, $L = P_{ZXW}$, show that $XY \geq 2KL$.

Problem 2.3.2. For an acute triangle ABC, let H be the perpendicular foot to the side BC from A. Points D and E are on AB and AC. F, G are the perpendicular feet on BC from D, E. Suppose DG, EF, AH are concurrent. Find k for which $\angle APE = k\angle CPE$

Problem 2.3.3. Let n be a positive integer. Prove that there are infinitely many triples of mutually co-prime integers (x, y, z) satisfying $nx^2 + y^3 = z^4$.

Problem 2.3.4. Find all prime numbers p such that the number $p^2 - p - 1$ is a cube of some positive integer.

Problem 2.3.5. Let a_n be a sequence with the property that for every prime p and for every positive integer k the following relationship holds:

$$a_{kp+1} = pa_k - 3a_p + 13$$

Find a_{2013}.

2.4 Geo Hard

Problem 2.4.1. In an acute triangle ABC, let D be a point on the side BC different from the vertices. Let M_1, M_2, M_3, M_4, M_5 be the midpoints of the line segments AD, AB, AC, BD, CD. Let O_1, O_2, O_3, O_4 be the circumcentres of the triangles $ABD, ACD, M_1M_2M_4, M_1M_3M_5$. Let S, T be the midpoints of the line segments AO_1 and AO_2 respectively. Prove that SO_3O_4T is an isosceles trapezoid.

Problem 2.4.2. The incircle of a triangle ABC touches the sides BC, CA, AB at points D, E, F. The circle passing through the point A and touching BC at D intersects segments BF and CE at K, L. The line passing through E and parallel to DL and the line passing through F and parallel to DK intersect at P. Let R_1, R_2, R_3, R_4 denote the circumradii of the triangles AFD, AED, FPD, EPD. Find the value of k for which $R_1R_4 = kR_2R_3$.

Problem 2.4.3. Let A, B, C, A', B', C' be distinct points on the plane satisfying $\Delta ABC \cong \Delta A'B'C'$ and let the point G be the centroid of the triangle ABC. Suppose the circle with center A' passing through G and the circle of diameter AA' intersect at point A_1. B_1, C_1 are defined analogously. Show that $AA_1^2 + BB_1^2 + CC_1^2 \leq AB^2 + BC^2 + CA^2$.

3 | 14 National Camp

3.1 Dummy Exam

Problem 3.1.1. Suppose P, Q are two points on the same side of the line AB, R is a point on the segment PQ such that $PR = \gamma PQ$. Let (ABC) denote the area of ΔABC. Prove that $(ABR) = (1-\gamma)(ABP) + \gamma(ABQ)$.

Problem 3.1.2. Let M be an interior point of triangle ABC, AM meets BC at D, BM meets CA at E, CM meets AB at F. Prove that $(DEF) \leq \frac{1}{4}(ABC)$.

Problem 3.1.3. Prove for all $n \in \mathbb{N}$,

$$\sum_{i=0}^{n} \frac{1}{\sqrt{i}} \geq \sqrt{n}$$

3.2 Khichuri Exam

Problem 3.2.1. Given the set $M{=}1, 2, \ldots, 21$ is there a decomposition into subsets such that the greatest number in any subset is the sum of the other numbers in this subset?

Problem 3.2.2. Is it possible to dissect any triangle into exactly five non degenerate isosceles triangles?

Problem 3.2.3. Given a triangle ABC, let A_1, B_1, C_1 be the points of tangency of the incircle with the sides BC, CA, AB. Let D be the point such that C_1D is a diameter of the incircle and E the intersection of B_1C_1 and A_1D. Find the ratio of the segments CE/CB_1

Problem 3.2.4. We have n bowls around a circular table. Aastogadha walks clockwise around the table from bowl to bowl and fills them with marbles. First, he places a marble on some bowl. Then he places a marble in the next bowl. Then he skips one bowl before placing a marble. Next he skips two bowls. He skips $k-1$ bowls before placing the k-th marble. He stops when every bowl contains at least one marble. For which n does this occur?

4 | 15 National Camp

4.1 Combi

Problem 4.1.1. In a picnic, 1^2 students are from class One, 2^2 students are from class Two and so on till class Five. How many students must be picked before you can be certain of picking at least 10 from the same class.

Problem 4.1.2. n people go to a party. When they leave, they randomly take a left shoe and a right shoe. What is the probability that none of them got both shoes of the same pair?

Problem 4.1.3. You have n jewels, but exactly one of them is fake. You know that the fake jewel is lighter. With a scale balance, what is the minimum number of measurements sufficient to find the fake jewel.

Problem 4.1.4. There are 2015 ants on a 56 meter rope, each walking at $7m/s$. When two ants collide, they turn around and continue to move the way they came from at the same speed. When an ant reaches the end of the rope they fall off. Find the greatest amount of time after which every single ant must fall of the rope, with an arrangement where this happens.

Problem 4.1.5. We have 2015 points in the plane such that no three are collinear. Prove that there is a circle which contains 1007 points in its interior and 1007 in its exterior.

Problem 4.1.6. Is it possible to choose 2015 integers smaller than 10^5such that no three are in arithmetic progression.

Problem 4.1.7. Prove that for any integer $n > 2$, there exists a set of 2^{n-1} points in the plane such that no 3 are collinear and no $2n$ are vertices of a convex $2n$-gon.

4.2 Geo

Problem 4.2.1. A point P is chosen in the interior of ΔABC so that when lines are drawn through P parallel to the sides, the resulting smaller triangles have areas 4,9,49. Find the area of ΔABC.

Problem 4.2.2. A convex hexagon $ABCDEF$ is inscribed in a circle such that $AB = CD = EF$ and diagonals AD, BE, CF are concurrent. Let P be the intersection of AD and CE. Prove that

$$\frac{CP}{PE} = \left(\frac{AC}{CE}\right)^2$$

Problem 4.2.3. Let $ABCD$ be a convex quadrilateral such that the diagonals AC and BD intersect at right angles at E. Prove that the reflections of E across AB, BC, CD, DA are concyclic.

Problem 4.2.4. The diagonals AC and BD of cyclic quadrilateral $ABCD$ meet at P. The circumcircle of ΔPDC meets BC, AD at E, F. The circumcircle of ΔPAB cuts BC, AD at H, G. Prove that $EFGH$ is cyclic with centre P.

Problem 4.2.5. Let O be the circumcentre of ΔABC and let ℓ be the be the line going through the midpoint of BC which is perpendicular to the internal bisector of $\angle BAC$. Find the value of $\angle BAC$ if ℓ bisects AO.

Problem 4.2.6. Given a point P_0 in the plane of the triangle $A_1A_2A_3$. Define $A_s = A_{s-3}$ for all $s \geq 4$. Construct a set of points $P_1, P_2, P_3, \ldots$ such that P_{k+1} is the image of P_k under a rotation center A_{k+1} through an angle 120^o clockwise for $k = 0, 1, 2, \ldots$. Prove that if $P_{1986} = P_0$, then the triangle $A_1A_2A_3$ is equilateral.

4.3 Mock 1

Problem 4.3.1. Find all (m, n, p) that satisfy

$$m^2 - 3mn + p^2n^2 = 12p$$

where m, n are integers and p is a prime.

Problem 4.3.2. In a convex quadrilateral $ABCD$, the diagonals AC and BD intersect at E and $\angle AEB = \pi/2$. A point P is chosen on AD other than A such that $PE = EC$. The circumcircle of ΔBCD intersect AD at Q other than A. The circle passing through A and tangent to EP at P intersects AC at R. If B, Q, R are collienar, then show that $\angle BCD = \pi/2$

Problem 4.3.3. Suppose there are 2012 bags containing colored balls. Each bag contains a finite numer of coloured balls. Suppose now the bags are distributed into k boxes such that for each box at least one of the following two conditions holds:

1. all of the bags in a box contain a ball of the same color
2. each bag of a box contains a ball colored differently from all balls of all other bags of this box.

Find the smallest value of k for which this is always possible.

Problem 4.3.4. In equilateral ΔABC, let D be a point on side BC other than the vertices. Let I be the excenter of ΔABD opposite the side AB and let J be the excenter of ΔACD opposite the side AC. Let the circumcircles of triangles ΔAIB and ΔAJC intersect at point E and A. Prove that A is the incenter of ΔEIJ.

4.4 Mock 2

Problem 4.4.1. Determine all triples of positive integers (k, m, n) for which $2^k + 3^m + 1 = 6^n$.

Problem 4.4.2. Let Γ be the circumcircle of ΔABC. Let ℓ be the line tangent to Γ at A. Let D, E be interior points of sides AB, AC such that $\frac{BD}{DA} = \frac{AE}{EC}$. Let F, G be the intersections of line DE and Γ.Let H be the point of intersiont of ℓ and the line through D parallel to AC. Let I be the intersection of ℓ and the line through E parallel to AB. Prove that F, G, H, I lie on a circle tangent to BC.

Problem 4.4.3. Let n be a positive integer. For every pair of students enrolled in a certain school having n students, either the pair are mutual friends or not. Let N be the smallest possible sum, $a + b$, of a, b satisfying the following conditions:

1. It is possible to divide the students into a teams such that any pair of students in the same team are mutual friends.
2. It is possible to divide the students into b teams such that any pair of students in the same team are not mutual friends.

Assume that every student will belong to one and only one team when students are divided into teams for the above conditions. A team of one person satisfies both conditions. Determine in terms of n the maximum possible value of N.

5 | 16 SSC Camp

5.1 Geo

Problem 5.1.1. Let $ABCD$ be a parallelogram. Let E be a point on the internal bisector of $\angle BAD$. BE, CD meet at X, DE, CB meet at Y. Prove that $BY = DX$.

Problem 5.1.2. Let the tangents to the circumcircle of ΔABC at B, C meet at P. Let S, T be the feet of the perpendiculars of AB, AC from P. Prove that the orthocentre of ΔAST lies on BC.

Problem 5.1.3. Let E be a point on the altitude BD of ΔABC so that $\angle AEC = 90°$. O_1, O_2 are the circumcenters of $\Delta AEB, \Delta CEB$. F, L are the midpoints of AC, O_1O_2. Prove that F, E, L are collinear.

Problem 5.1.4. Let O, H be the circumcenter and orthocenter of acute angled ΔABC. Prove that if $AO \perp OH$, then the perpendicular bisector of AO touches the nine point circle.

5.2 NT

Problem 5.2.1. Find all primes p for which $17p^2 + 1$ is also a prime number

Problem 5.2.2. Prove that there exists infinitely many positive integers n for which $4n^2 + 1$ is divisible by both 5 and 13.

Problem 5.2.3. Find all pairs of positive integers (a, b) for which $\frac{a}{b} + \frac{21b}{25a}$ is a positive integer.

Problem 5.2.4. For all positive integers n prove that

$$(2^n - 1)^2 \quad | \quad 2^{(2^n-1)n} - 1$$

5.3 Mock

Problem 5.3.1. Find all finite sets of positive integers with at least two elements such that for any two numbers $a, b (a > b)$ belonging to the set, the number $\frac{b^2}{a-b}$ belongs to the set too.

Problem 5.3.2. Two circles ω and Γ intersect at A, B. An arbitrary line through B meets ω, Γ at C, D. The tangent to ω at C and the tangent to Γ at D intersects at M. MA, CD meet at P and K is on AC such that $PK \parallel MC$. Prove that KB touches Γ at B.

Problem 5.3.3. Consider an $n \times n$ square grid, divided into n^2 unit squares. Every unit square is to be coloured using either red or blue colour. find the number of such colourings for which every 2×2 square contains exactly 2 red squares.

6 | 17 National Camp

6.1 Combi

Problem 6.1.1. At a gathering of 30 people, there are 20 people who all know each other and 10 people who know no one. People who know each other hug, and people who do not know each other shake hands. How many handshakes occur?

Problem 6.1.2. The number 912837465 is an example of a 9 digit number that contains each of the digits 1 to 9 exactly once. It also has the property that the digits 1 to 5 occur in natural order, while the digits 1 to 6 do not. How many such numbers are there?

Problem 6.1.3. In the land of Hexagonia, 6 cities are connected by a rail network with a direct line between any two cities. On Sundays, some lines may be closed for repair. The government has passed a law stating that any city must be accessible from any other city. In how many ways can some lines be closed subject to this condition?

Problem 6.1.4. Prove that if u is a vertex of odd degree in a graph, then there exists a path from u to another vertex v of the graph where v also has odd degree.

Problem 6.1.5. Let S be a subset of $1, 2, \ldots, 2008$ with 756 elements. Prove that two elements have sum divisible by 8

6.2 NT

Problem 6.2.1. Show that $x^3 + x + a^2 = y^2$ has at least one pair of positive integer solution (x, y) for each positive integer a.

Problem 6.2.2. Find all $a, b, c \in \mathbb{Z}$, $c \geq 0$ such that $a^n + 2^n | b^n + c$ for all positive integers n where $2ab$ is non-square.

Problem 6.2.3. Does there exist a function f mapping the integers to the integers that cannot be expressed as a polynomial but satisfying the following condition: $a - b | f(a) - f(b)$ for integers $a \neq b$

6.3 Geo

Problem 6.3.1. In ABC, let P be the midpoint of BC and Q a point on CA such that $CQ = 2QA$. Let S be the intersection of BQ, AP. For what value of κ is $AS = \kappa SP$.

Problem 6.3.2. Let A be a point on the circle ω centered at B and Γ be a circle centered at A. For $i = 1, 2, 3$, a chord P_iQ_i of ω is tangent to Γ at S_i and another chord P_iR_i of ω is perpendicular to AB at M_i. Let Q_iT_i be the other tangent from Q_i to Γ at T_i and let N_i be the intersection of AQ_i with M_iTi. Prove N_1, N_2, N_3 are collinear.

Problem 6.3.3. Let ABC be an equilateral triangle and let D be a point on segment AB. Next, let E be the point on AC such that DE is parallel to BC. Furthermore, let F be the midpoint of CD and G the circumcentre of ADE. Determine the interior angles of ΔBFG.

Problem 6.3.4. Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The incircle ω of ΔBCD meets CD at E. Let F be a point on the internal angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumcircle of ΔACF meet line CD at C, G. Prove that the triangle AFG is isosceles.

7 | 14 TSTs

7.1 Khichuri

Problem 7.1.1. The incircle of a non-isosceles triangle ABC with the center I touches the sides BC, CA, AB at A_1, B_1, C_1 respectively. The line AI meets the circumcircle of ABC at A_2. The line B_1C_1 meets the line BC at A_3 and the line A_2A_3 meets the circumcircle of ABC at $A_4(\neq A_2)$. Define B_4, C_4 similarly. Prove that the lines AA_4, BB_4, CC_4 are concurrent.

Problem 7.1.2. The incircle ω of a quadrilateral $ABCD$ touches AB, BC, CD, DA at E, F, G, H, respectively. Choose an arbitrary point X on the segment AC inside ω. The segments XB, XD meet ω at I, J respectively. Prove that FJ, IG, AC are concurrent.

Problem 7.1.3. (N3) Prove that there exist infinitely many positive integers n such that the largest prime divisor of $n^4 + n^2 + 1$ is equal to the largest prime divisor of $(n+1)^4 + (n+1)^2 + 1$.

7.2 Geo 1

Problem 7.2.1. (G3) In a triangle ABC, let D and E be the feet of the angle bisectors of angles A and B, respectively. A rhombus is inscribed into the quadrilateral $AEDB$ (all vertices of the rhombus lie on different sides of $AEDB$). Let φ be the non-obtuse angle of the rhombus. Prove that $\varphi \leq \max\{\angle BAC, \angle ABC\}$.

Problem 7.2.2. (G4) Let ABC be a triangle with $\angle B > \angle C$. Let P and Q be two different points on line AC such that $\angle PBA = \angle QBA = \angle ACB$ and A is located between P and C. Suppose that there exists an interior point D of segment BQ for which $PD = PB$. Let the ray AD intersect the circle ABC at $R \neq A$. Prove that $QB = QR$.

7.3 Geo 2

Problem 7.3.1. (G2) Let ω be the circumcircle of a triangle ABC. Denote by M and N the midpoints of the sides AB and AC, respectively, and denote by T the midpoint of the arc BC of ω not containing A. The circumcircles of the triangles AMT and ANT intersect the perpendicular bisectors of AC and AB at points X and Y, respectively; assume that X and Y lie inside the triangle ABC. The lines MN and XY intersect at K. Prove that $KA = KT$.

Problem 7.3.2. (G5) Let $ABCDEF$ be a convex hexagon with $AB = DE$, $BC = EF$, $CD = FA$, and $\angle A - \angle D = \angle C - \angle F = \angle E - \angle B$. Prove that the diagonals AD, BE, and CF are concurrent.

8 | 15 TSTs

8.1 Geo

Problem 8.1.1. The circle ω_1 with diameter AB and the circle ω_2 with center A intersects at points C, D. Let E be a point on ω_2 outside of ω_1 and on the same side as C of line AB. Let the second intersection of BE with ω_2 be F. Suppose that a point $K \in \omega_1$ is on the same side as A with respect tot eh diameter of ω_1 through C and that $2CK \cdot AC = CE \cdot AB$. Let the second point of the intersection of the line KF with ω_1 be L. Show that the reflection of D over BE lies on the circumcircle of LFC.

Problem 8.1.2. Let Γ be the circumcircle of acute ΔABC with $AB < AC$. Let ℓ be the reflection of the line BC with respect to AB. ℓ intersects Γ at B, E. The tangent to Γ at A intersects ℓ at D. Let F be the reflection of D over A. The line CF intersects Γ at C, G. Prove that CE and GB are parallel.

Problem 8.1.3. (G4) Consider a fixed circle Γ with three fixed points A, B, and C on it. Also, let us fix a real number $\lambda \in (0, 1)$. For a variable point $P \notin \{A, B, C\}$ on Γ, let M be the point on the segment CP such that $CM = \lambda \cdot CP$. Let Q be the second point of intersection of the circumcircles of the triangles AMP and BMC. Prove that as P varies, the point Q lies on a fixed circle.

8.2 NT

Problem 8.2.1. (N1) Let $n \geq 2$ be an integer, and let A_n be the set

$$A_n = \{2^n - 2^k \mid k \in \mathbb{Z},\ 0 \leq k < n\}.$$

Determine the largest positive integer that cannot be written as the sum of one or more (not necessarily distinct) elements of A_n .

Problem 8.2.2. (N2) Determine all pairs (x, y) of positive integers such that

$$\sqrt[3]{7x^2 - 13xy + 7y^2} = |x - y| + 1.$$

Problem 8.2.3. (N4) Let $n > 1$ be a given integer. Prove that infinitely many terms of the sequence $(a_k)_{k \geq 1}$, defined by

$$a_k = \left\lfloor \frac{n^k}{k} \right\rfloor,$$

are odd. (For a real number x, $\lfloor x \rfloor$ denotes the largest integer not exceeding x.)

Problem 8.2.4. Prove that an infinite set of points have mutually integer distances only if they all lie on a straight line.

8.3 Combi

Problem 8.3.1. There are n cars, numbered from 1 to n and a row with n parking spots, numbered from 1 to n. Each car i has its favorite parking spot a_i (Different cars may have the same favorite spot). When it is time to park, it goes to its favorite parking spot. If it's free, it parks and if it's taken, it advances until the next free parking spot and parks there. If it can't find a spot this way, it leaves. First car 1 tries to park, then car 2, and so on till car n. find the number of lists of favorite spots $a_1, \ldots, a_n$ such that all cars can park.

Problem 8.3.2. Given a convex 2007-gon, find the smallest integer k such that among any k vertices of the polygon there are 4 vertices with the property that the convex quadrilateral they form share 3 sides with the polygon.

Problem 8.3.3. The entries of a $2 \times n$ matrix are positive real numbers. The sum of the numbers in each of the n columns sum to 1. Show that we can select one number in each column such that the sum of the selected numbers in each row is at most $(n + 1)/4$.

8.4 Geomock

Problem 8.4.1. (A3) For a sequence $x_1, x_2, \dots, x_n$ of real numbers, we define its *price* as

$$\max_{1 \le i \le n} |x_1 + \cdots + x_i|.$$

Given n real numbers, Dave and George want to arrange them into a sequence with a low price. Diligent Dave checks all possible ways and finds the minimum possible price D. Greedy George, on the other hand, chooses x_1 such that $|x_1|$ is as small as possible; among the remaining numbers, he chooses x_2 such that $|x_1 + x_2|$ is as small as possible, and so on. Thus, in the i-th step he chooses x_i among the remaining numbers so as to minimise the value of $|x_1 + x_2 + \cdots x_i|$. In each step, if several numbers provide the same value, George chooses one at random. Finally he gets a sequence with price G.

Find the least possible constant c such that for every positive integer n, for every collection of n real numbers, and for every possible sequence that George might obtain, the resulting values satisfy the inequality $G \le cD$.

Problem 8.4.2.

1. In acute ΔABC, let AD be the altitude from A on BC. Let P be a point on AD. Line PB meets AC at E and PC meets AB at F. Suppose that $AEDF$ is concyclic. Prove that $PA/PD = (tanB + tanC)cot(A/2)$

2. In acute ΔABC, H is the orthocentre and P is a point on line AH. The line perpendicular to AC at C cuts BP at M and the line perpendicular to AB at B cuts CP at N. Let K be the projection of A on MN. Prove that the value to $\angle BKC + \angle MAN$ does not depend on P.

Problem 8.4.3. Let ΔABC be an acute triangle inscribed in circle (O). Two points $P.Q$ lie on segments AB, AC. The circumcircle of APQ intersects (O) at A, M. N is the reflection of M over PQ. Prove that

1. $S_{AQP} + S_{BNP} + S_{CNQ} < S_{ABC}$ where S_X is the area of ΔX

2. If the point N lies on BC, then MN passes through a certain fixed point.

8.5 Mock 1

Problem 8.5.1. (G2) Let ABC be a triangle. The points K, L, and M lie on the segments BC, CA, and AB, respectively, such that the lines AK, BL, and CM intersect in a common point. Prove that it is possible to choose two of the triangles ALM, BMK, and CKL whose inradii sum up to at least the inradius of the triangle ABC.

Problem 8.5.2. (C2) We have 2^m sheets of paper, with the number 1 written on each of them. We perform the following operation. In every step we choose two distinct sheets; if the numbers on the two sheets are a and b, then we erase these numbers and write the number $a + b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least 4^m .

Problem 8.5.3. (N5) Find all triples (p, x, y) consisting of a prime number p and two positive integers x and y such that $x^{p-1} + y$ and $x + y^{p-1}$ are both powers of p.

8.6 Mock 2

Problem 8.6.1. (G3) Let Ω and O be the circumcircle and the circumcentre of an acute-angled triangle ABC with $AB > BC$. The angle bisector of $\angle ABC$ intersects Ω at $M \ne B$. Let Γ be the circle with diameter BM. The angle bisectors of $\angle AOB$ and $\angle BOC$ intersect Γ at points P and Q, respectively. The point R is chosen on the line PQ so that $BR = MR$. Prove that $BR \parallel AC$.

Problem 8.6.2. (A2) Define the function $f : (0, 1) \to (0, 1)$ by

$$f(x) = \begin{cases} x + \frac{1}{2} & \text{if } x < \frac{1}{2} \\ x^2 & \text{if } x \ge \frac{1}{2} \end{cases}$$

Let a and b be two real numbers such that $0 < a < b < 1$. We define the sequences a_n and b_n by $a_0 = a, b_0 = b$, and $a_n = f(a_{n-1})$, $b_n = f(b_{n-1})$ for $n > 0$. Show that there exists a positive integer n such that

$$(a_n - a_{n-1})(b_n - b_{n-1}) < 0.$$

Problem 8.6.3. (C1) Let n points be given inside a rectangle R such that no two of them lie on a line parallel to one of the sides of R. The rectangle R is to be dissected into smaller rectangles with sides parallel to the sides of R in such a way that none of these rectangles contains any of the given points in its interior. Prove that we have to dissect R into at least $n + 1$ smaller rectangles.

9 | 16 TSTs

9.1 AIME

Problem 9.1.1. Let a, b, c, d be positive numbers such that $\frac{1}{a^3} + \frac{512}{b^3} + \frac{125}{c^3} = \frac{d}{(a+b+c)^3}$

Problem 9.1.2. How many three-digit positive integers are there such that the digits form an arithmetic sequence?

Problem 9.1.3. Find $x = \sqrt{1 + \frac{1}{1^2} + \frac{1}{2^2}} + \sqrt{1 + \frac{1}{2^2} + \frac{1}{3^2}} + \cdots + \sqrt{1 + \frac{1}{2012^2} + \frac{1}{2013^2}}$.

Problem 9.1.4. Let x, y, z be nonnegative numbers such that $x^2 + y^2 + z^2 + x + 2y + 3z = \frac{13}{4}$. Find the minimum value of $x + y + z$

Problem 9.1.5. Peter, Paul, David join a table tennis tournament. On the first day, two of them were randomly chosen to play a game against each other. On each subsequent day, the loser of the game on the previous day would be benched and the other two would play a game. After a certain number of days, it was found that Peter had won 22 games, Paul had won 20 and David had won 32. How many games had Peter played?

Problem 9.1.6. The sequence $1, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 2, 1$ is formed as follows: write down infinitely many '1's and insert k '2's between the k-th and $(k+1)$-th '1's. If a_n denotes the n-th term of the sequence, find the value of $\sum_{i=1}^{2} 013 a_i a_{i+1}$.

Problem 9.1.7. There are n different positive integers, each not greater than 2013, such that the sum of any three of them is divisible by 39. Find the greatest value of n.

Problem 9.1.8. If x is a real number, find the smallest value of $\sqrt{x^2 + 4x + 5} + \sqrt{x^2 - 8x + 25}$

Problem 9.1.9. The equation $9x^3 - 3x^2 - 3x - 1 = 0$ has a real root of the form $\frac{\sqrt{3}a + \sqrt{3}b + 1}{c}$, where a, b, c are positive integers. Find $a + b + c$.

Problem 9.1.10. By permuting the digits of 20130518, how many different eight digit positive odd numbers can be formed?

Problem 9.1.11. Let a, b, c be the roots of the equation $8x^3 + 2012x + 2013 = 0$. Find the value of $(a + b)^3 + (b + c)^3 + (c + a)^3$.

Problem 9.1.12. $ABCD$ is a square on the coordinate plane, and $(31, 27), (42, 43), (60, 27), (46, 16)$ lie on its sides AB, BC, CD, DA. Find the area of the square.

Problem 9.1.13. In ΔABC, $AB = 8, AC = 13, BC = 15$. Let H, I, O be the orthocentre, incentre and circumcentre of ΔABC. Find $sin\angle HIO$.

Problem 9.1.14. Let $ABCD$ be a convex quadrilateral and E be a point on CD such that the circumcircle of ABE is tangent to CD. Suppose $AC \cap BE = F, BD \cap AE = G, AC \cap BD = H$. If $FG \parallel CD$, and the area of $\Delta ABH, \Delta BCE, \Delta ADE$ are 2,3,4, find the area of ΔABE.

Problem 9.1.15. Let I be the incentre of ΔABC. If $BC = AC + AI$ and $\angle ABC - \angle ACB = 13°$, find $\angle BAC$.

Problem 9.1.16. A, B, C, M, N are points on the circumference of a circle with MN as a diameter. A, B are on the same side of MN and C is on the opposite side. A is the midpoint of arc MN. CA and CB meet MN at P, Q. If $MN = 1, MB = \frac{12}{13}$, find the greatest length of PQ.

Problem 9.1.17. How many pairs (m, n) of nonnegative integers are there such that $m \neq n$ and $\frac{50688}{m+n}$ is an odd positive power of 2?

Problem 9.1.18. A positive integer is good if each digit is 1 or 2, without four consecutive 1s or three consecutive 2s. Let a_n denote the number of n-digit good numbers. Find $\frac{a_1 0 - a_8 - a_5}{a_7 + a_6}$

Problem 9.1.19. Let p and q be positive integers so that $\frac{p}{q} = 0.\overline{123456789}$ i.e. 123456789 repeats infinitely. Find $p + q$.

Problem 9.1.20. Let a, b be real numbers such that $17(a^2 + b^2) - 30ab - 16 = 0$. Find the maximum value of $\sqrt{16a^2 + 4b^2 - 16ab - 12a + 6b + 9}$

9.2 Geo 1

Problem 9.2.1. (2011 G1) Let ABC be an acute triangle. Let ω be a circle whose centre L lies on the side BC. Suppose that ω is tangent to AB at B' and AC at C'. Suppose also that the circumcentre O of triangle ABC lies on the shorter arc $B'C'$ of ω. Prove that the circumcircle of ABC and ω meet at two points.

Problem 9.2.2. (2011 G2) Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. Let O_1 and r_1 be the circumcentre and the circumradius of the triangle $A_2A_3A_4$. Define O_2, O_3, O_4 and r_2, r_3, r_4 in a similar way. Prove that

$$\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_3A_3^2 - r_3^2} + \frac{1}{O_4A_4^2 - r_4^2} = 0.$$

Problem 9.2.3. (2011 G3) Let $ABCD$ be a convex quadrilateral whose sides AD and BC are not parallel. Suppose that the circles with diameters AB and CD meet at points E and F inside the quadrilateral. Let ω_E be the circle through the feet of the perpendiculars from E to the lines AB, BC and CD. Let ω_F be the circle through the feet of the perpendiculars from F to the lines CD, DA and AB. Prove that the midpoint of the segment EF lies on the line through the two intersections of ω_E and ω_F.

9.3 Geo 2

Problem 9.3.1. (G1) Let ABC be an acute triangle with orthocenter H. Let G be the point such that the quadrilateral $ABGH$ is a parallelogram. Let I be the point on the line GH such that AC bisects HI. Suppose that the line AC intersects the circumcircle of the triangle GCI at C and J. Prove that $IJ = AH$.

Problem 9.3.2. (G3) Let ABC be a triangle with $\angle C = 90°$, and let H be the foot of the altitude from C. A point D is chosen inside the triangle CBH so that CH bisects AD. Let P be the intersection point of the lines BD and CH. Let ω be the semicircle with diameter BD that meets the segment CB at an interior point. A line through P is tangent to ω at Q. Prove that the lines CQ and AD meet on ω.

Problem 9.3.3. (G4) Let ABC be an acute triangle and let M be the midpoint of AC. A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC. Determine all possible values of $\frac{BT}{BM}$.

9.4 NT

Problem 9.4.1. (N1) Determine all positive integers M such that the sequence $a_0, a_1, a_2, \cdots$ defined by

$$a_0 = M + \frac{1}{2} \qquad \text{and} \qquad a_{k+1} = a_k \lfloor a_k \rfloor \quad \text{for } k = 0, 1, 2, \cdots$$

contains at least one integer term.

Problem 9.4.2. (N2) Let a and b be positive integers such that $a! + b!$ divides $a!b!$. Prove that $3a \geq 2b + 2$.

Problem 9.4.3. (N3) Let m and n be positive integers such that $m > n$. Define $x_k = \frac{m+k}{n+k}$ for $k = 1, 2, \ldots, n+1$. Prove that if all the numbers $x_1, x_2, \ldots, x_{n+1}$ are integers, then $x_1x_2 \ldots x_{n+1} - 1$ is divisible by an odd prime.

9.5 Alg

Problem 9.5.1. (A1) Suppose that a sequence $a_1, a_2, \ldots$ of positive real numbers satisfies

$$a_{k+1} \geq \frac{ka_k}{a_k^2 + (k-1)}$$

for every positive integer k. Prove that $a_1 + a_2 + \ldots + a_n \geq n$ for every $n \geq 2$.

Problem 9.5.2. (A2) Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ with the property that

$$f(x - f(y)) = f(f(x)) - f(y) - 1$$

holds for all $x, y \in \mathbb{Z}$.

Problem 9.5.3. (A3) Let n be a fixed positive integer. Find the maximum possible value of

$$\sum_{1 \le r < s \le 2n} (s - r - n) x_r x_s,$$

where $-1 \le x_i \le 1$ for all $i = 1, \cdots, 2n$.

9.6 Combi

Problem 9.6.1. (C1) In Lineland there are $n \ge 1$ towns, arranged along a road running from left to right. Each town has a left bulldozer (put to the left of the town and facing left) and a right bulldozer (put to the right of the town and facing right). The sizes of the $2n$ bulldozers are distinct. Every time when a left and right bulldozer confront each other, the larger bulldozer pushes the smaller one off the road. On the other hand, bulldozers are quite unprotected at their rears; so, if a bulldozer reaches the rear-end of another one, the first one pushes the second one off the road, regardless of their sizes.

Let A and B be two towns, with B to the right of A. We say that town A can sweep town B away if the right bulldozer of A can move over to B pushing off all bulldozers it meets. Similarly town B can sweep town A away if the left bulldozer of B can move over to A pushing off all bulldozers of all towns on its way.

Prove that there is exactly one town that cannot be swept away by any other one.

Problem 9.6.2. (C3) For a finite set A of positive integers, a partition of A into two disjoint nonempty subsets A_1 and A_2 is *good* if the least common multiple of the elements in A_1 is equal to the greatest common divisor of the elements in A_2. Determine the minimum value of n such that there exists a set of n positive integers with exactly 2015 good partitions.

Problem 9.6.3. (C4) Let n be a positive integer. Two players A and B play a game in which they take turns choosing positive integers $k \le n$. The rules of the game are:

(i) A player cannot choose a number that has been chosen by either player on any previous turn. (ii) A player cannot choose a number consecutive to any of those the player has already chosen on any previous turn. (iii) The game is a draw if all numbers have been chosen; otherwise the player who cannot choose a number anymore loses the game.

The player A takes the first turn. Determine the outcome of the game, assuming that both players use optimal strategies.

10 | 17 TSTs

10.1 Geo 1

Problem 10.1.1. **(G2)** Let ABC be a triangle with circumcircle Γ and incenter I and let M be the midpoint of $\overline{BC}$. The points D, E, F are selected on sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $\overline{ID} \perp \overline{BC}$, $\overline{IE} \perp \overline{AI}$, and $\overline{IF} \perp \overline{AI}$. Suppose that the circumcircle of $\triangle AEF$ intersects Γ at a point X other than A. Prove that lines XD and AM meet on Γ.

Problem 10.1.2. **(G4)** Let ABC be a triangle with $AB = AC \neq BC$ and let I be its incentre. The line BI meets AC at D, and the line through D perpendicular to AC meets AI at E. Prove that the reflection of I in AC lies on the circumcircle of triangle BDE.

10.2 Combi

Problem 10.2.1. **(C1)** The leader of an IMO team chooses positive integers n and k with $n > k$, and announces them to the deputy leader and a contestant. The leader then secretly tells the deputy leader an n-digit binary string, and the deputy leader writes down all n-digit binary strings which differ from the leaderâĂŹs in exactly k positions. (For example, if $n = 3$ and $k = 1$, and if the leader chooses 101, the deputy leader would write down $001, 111$ and 100.) The contestant is allowed to look at the strings written by the deputy leader and guess the leaderâĂŹs string. What is the minimum number of guesses (in terms of n and k) needed to guarantee the correct answer?

Problem 10.2.2. **(C3)** Let n be a positive integer relatively prime to 6. We paint the vertices of a regular n-gon with three colours so that there is an odd number of vertices of each colour. Show that there exists an isosceles triangle whose three vertices are of different colours.

10.3 NT

Problem 10.3.1. **(N1)** For any positive integer k, denote the sum of digits of k in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n \geq 2016$, the integer $P(n)$ is positive and

$$S(P(n)) = P(S(n)).$$

Problem 10.3.2. **(N2)** Let $\tau(n)$ be the number of positive divisors of n. Let $\tau_1(n)$ be the number of positive divisors of n which have remainders 1 when divided by 3. Find all positive integral values of the fraction $\frac{\tau(10n)}{\tau_1(10n)}$.

10.4 Alg

Problem 10.4.1. **(A2)** Find the smallest constant $C > 0$ for which the following statement holds: among any five positive real numbers a_1, a_2, a_3, a_4, a_5 (not necessarily distinct), one can always choose distinct subscripts i, j, k, l such that

$$\left|\frac{a_i}{a_j} - \frac{a_k}{a_l}\right| \leq C.$$

Problem 10.4.2. **(A4)** Find all functions $f : (0, \infty) \to (0, \infty)$ such that for any $x, y \in (0, \infty)$,

$$xf(x^2)f(f(y)) + f(yf(x)) = f(xy)\left(f(f(x^2)) + f(f(y^2))\right).$$

10.5 Geo 2

Problem 10.5.1. **(G5)** Let D be the foot of perpendicular from A to the Euler line (the line passing through the circumcentre and the orthocentre) of an acute scalene triangle ABC. A circle ω with centre

S passes through A and D, and it intersects sides AB and AC at X and Y respectively. Let P be the foot of altitude from A to BC, and let M be the midpoint of BC. Prove that the circumcentre of triangle XSY is equidistant from P and M.

Problem 10.5.2. (G6) Let $ABCD$ be a convex quadrilateral with $\angle ABC = \angle ADC < 90°$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at E and F respectively, and meet each other at point P. Let M be the midpoint of AC and let ω be the circumcircle of triangle BPD. Segments BM and DM intersect ω again at X and Y respectively. Denote by Q the intersection point of lines XE and YF. Prove that $PQ \perp AC$.

Problem 10.5.3. (G3) Let $B = (-1, 0)$ and $C = (1, 0)$ be fixed points on the coordinate plane. A nonempty, bounded subset S of the plane is said to be nice if

(i) there is a point T in S such that for every point Q in S, the segment TQ lies entirely in S; and

(ii) for any triangle $P_1P_2P_3$, there exists a unique point A in S and a permutation σ of the indices $\{1, 2, 3\}$ for which triangles ABC and $P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$ are similar.

Prove that there exist two distinct nice subsets S and S' of the set $\{(x, y) : x \geq 0, y \geq 0\}$ such that if $A \in S$ and $A' \in S'$ are the unique choices of points in (ii), then the product $BA \cdot BA'$ is a constant independent of the triangle $P_1P_2P_3$.

10.6 Easy Mock

Problem 10.6.1. Let ABC be an acute triangle. Let F be the foot on AB of the altitude through C. Suppose that $AF = 3BF$. Let M and N be the midpoints of the segments AB and AC. Let P be a point such that $NP = NC$ and $CP = CB$ and B and P lie on opposite sides of the line AC. Show that $\angle APM = \angle PBA$

Problem 10.6.2. Let n be a positive integer, and consider a square of dimensions $2^n \times 2^n$. We cover this square by at least two rectangles without overlap so that every rectangle has integer dimensions and a power of two as area. Show that two of the rectangle used must have the same width and same height, without rotating them.

Problem 10.6.3. Find all positive integers k for which the equation

$$lcm(m, n) - gcd(m, n) = k(m - n)$$

does not have any solutions (m, n) in positive integers with $m \neq n$

Problem 10.6.4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(xy - 1) + f(x)f(y) = 2xy - 1$$

for all $x, y \in \mathbb{R}$.