Romanian Masters In Mathematics 2009

1 For $a_i \in \mathbb{Z}^+$, $i = 1, \ldots, k$, and $n = \sum_{i=1}^k a_i$, let $d = \gcd(a_1, \ldots, a_k)$ denote the greatest common divisor of $a_1, \ldots, a_k$.
Prove that $\frac{d}{n} \cdot \frac{n!}{\prod\limits_{i=1}^k (a_i!)}$ is an integer.

Dan Schwarz, Romania

2 A set S of points in space satisfies the property that all pairwise distances between points in S are distinct. Given that all points in S have integer coordinates (x, y, z) where $1 \le x, y, z \le n$, show that the number of points in S is less than $\min\left((n+2)\sqrt{\frac{n}{3}}, n\sqrt{6}\right)$.

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3 Given four points A_1, A_2, A_3, A_4 in the plane, no three collinear, such that

$$A_1A_2 \cdot A_3A_4 = A_1A_3 \cdot A_2A_4 = A_1A_4 \cdot A_2A_3,$$

denote by O_i the circumcenter of $\triangle A_jA_kA_l$ with $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Assuming $\forall i A_i \neq O_i$, prove that the four lines A_iO_i are concurrent or parallel.

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4 For a finite set X of positive integers, let $\Sigma(X) = \sum_{x \in X} \arctan \frac{1}{x}$. Given a finite set S of positive integers for which $\Sigma(S) < \frac{\pi}{2}$, show that there exists at least one finite set T of positive integers for which $S \subset T$ and $\Sigma(S) = \frac{\pi}{2}$.

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