APMO 2014

1 For a positive integer m denote by $S(m)$ and $P(m)$ the sum and product, respectively, of the digits of m. Show that for each positive integer n, there exist positive integers $a_1, a_2, \ldots, a_n$ satisfying the following conditions:

$$S(a_1) < S(a_2) < \cdots < S(a_n) \text{ and } S(a_i) = P(a_{i+1}) \quad (i = 1, 2, \ldots, n).$$

(We let $a_{n+1} = a_1$.)

2 Let $S = \{1, 2, \ldots, 2014\}$. For each non-empty subset $T \subseteq S$, one of its members is chosen as its representative. Find the number of ways to assign representatives to all non-empty subsets of S so that if a subset $D \subseteq S$ is a disjoint union of non-empty subsets $A, B, C \subseteq S$, then the representative of D is also the representative of one of A, B, C.

3 Find all positive integers n such that for any integer k there exists an integer a for which $a^3 + a - k$ is divisible by n.

4 Let n and b be positive integers. We say n is b-discerning if there exists a set consisting of n different positive integers less than b that has no two different subsets U and V such that the sum of all elements in U equals the sum of all elements in V.

(a) Prove that 8 is 100-discerning.
(b) Prove that 9 is not 100-discerning.

5 Circles ω and Ω meet at points A and B. Let M be the midpoint of the arc AB of circle ω (M lies inside Ω). A chord MP of circle ω intersects Ω at Q (Q lies inside ω). Let ℓ_P be the tangent line to ω at P, and let ℓ_Q be the tangent line to Ω at Q. Prove that the circumcircle of the triangle formed by the lines ℓ_P, ℓ_Q and AB is tangent to Ω.