China Girls Math Olympiad 2010

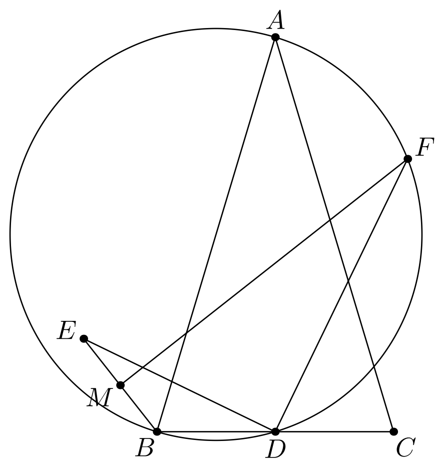
Day 1

1 Let n be an integer greater than two, and let $A_1, A_2, \cdots, A_{2n}$ be pairwise distinct subsets of $\{1, 2, , n\}$. Determine the maximum value of

$$\sum_{i=1}^{2n} \frac{|A_i \cap A_{i+1}|}{|A_i| \cdot |A_{i+1}|}$$

Where $A_{2n+1} = A_1$ and $|X|$ denote the number of elements in X.

2 In triangle ABC, $AB = AC$. Point D is the midpoint of side BC. Point E lies outside the triangle ABC such that $CE \perp AB$ and $BE = BD$. Let M be the midpoint of segment BE. Point F lies on the minor arc $\widehat{AD}$ of the circumcircle of triangle ABD such that $MF \perp BE$. Prove that $ED \perp FD$.



3 Prove that for every given positive integer n, there exists a prime p and an integer m such that (a) $p \equiv 5 \pmod 6$ (b) $p \nmid n$ (c) $n \equiv m^3 \pmod p$

4 Let $x_1, x_2, \cdots, x_n$ be real numbers with $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Prove that

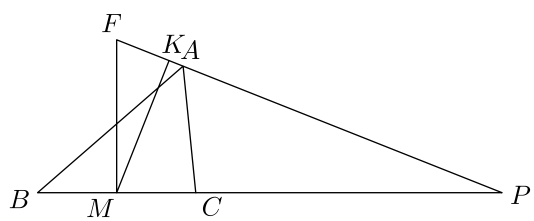
$$\sum_{k=1}^{n}\left(1 - \frac{k}{\sum\limits_{i=1}^{n} i x_i^2}\right)^2 \cdot \frac{x_k^2}{k} \le \left(\frac{n-1}{n+1}\right)^2 \sum_{k=1}^{n} \frac{x_k^2}{k}$$

Determine when does the equality hold?

Day 2

5 Let $f(x)$ and $g(x)$ be strictly increasing linear functions from $\mathbb{R}$ to $\mathbb{R}$ such that $f(x)$ is an integer if and only if $g(x)$ is an integer. Prove that for any real number x, $f(x) - g(x)$ is an integer.

6 In acute triangle ABC, $AB > AC$. Let M be the midpoint of side BC. The exterior angle bisector of $\widehat{BAC}$ meet ray BC at P. Point K and F lie on line PA such that $MF \perp BC$ and $MK \perp PA$. Prove that $BC^2 = 4PF \cdot AK$.



7 For given integer $n \ge 3$, set $S = \{p_1, p_2, \cdots, p_m\}$ consists of permutations p_i of $(1, 2, \cdots, n)$. Suppose that among every three distinct numbers in $\{1, 2, \cdots, n\}$, one of these number does not lie in between the other two numbers in every permutations p_i $(1 \le i \le m)$. (For example, in the permutation $(1, 3, 2, 4)$, 3 lies in between 1 and 4, and 4 does not lie in between 1 and 2.) Determine the maximum value of m.

8 Determine the least odd number $a > 5$ satisfying the following conditions: There are positive integers m_1, m_2, n_1, n_2 such that $a = m_1^2 + n_1^2$, $a^2 = m_2^2 + n_2^2$, and $m_1 - n_1 = m_2 - n_2$.

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