India
National Olympiad
1991

$\boxed{1}$ Find the number of positive integers n for which

(i) $n \leq 1991$;

(ii) 6 is a factor of $(n^2 + 3n + 2)$.

$\boxed{2}$ Given an acute-angled triangle ABC, let points A', B', C' be located as follows: A' is the point where altitude from A on BC meets the outwards-facing semicircle on BC as diameter. Points B', C' are located similarly. Prove that $A[BCA']^2 + A[CAB']^2 + A[ABC']^2 = A[ABC]^2$ where $A[ABC]$ is the area of triangle ABC.

$\boxed{3}$ Given a triangle ABC let

$$
\begin{aligned}
x &= \tan\left(\frac{B-C}{2}\right)\tan\left(\frac{A}{2}\right)\cancel{y} \\
\tan\left(\frac{C-A}{2}\right)\tan\left(\frac{B}{2}\right) z &= \tan\left(\frac{A-B}{2}\right)\tan\left(\frac{C}{2}\right).
\end{aligned}
$$

(0)

Prove that $x + y + z + xyz = 0$.

Let a, b, c be real numbers with $0 < a < 1$, $0 < b < 1$, $0 < c < 1$, and $a + b + c = 2$. Prove that $\frac{a}{1-a} \cdot \frac{b}{1-b} \cdot \frac{c}{1-c} \geq 8$.

Triangle ABC has an incenter I. Let points X, Y be located on the line segments AB, AC respectively, so that $BX \cdot AB = IB^2$ and $CY \cdot AC = IC^2$. Given that the points X, I, Y lie on a straight line, find the possible values of the measure of angle A.

(i) Determine the set of all positive integers n for which 3^{n+1} divides $2^{3^n} + 1$;
(ii) Prove that 3^{n+2} does not divide $2^{3^n} + 1$ for any positive integer n.

Solve the following system for real x, y, z

$$
\left\{
\begin{array}{ccc}
x + y - z & = & 4 \\
x^2 - y^2 + z^2 & = & -4 \\
xyz & = & 6.
\end{array}
\right.
$$

There are 10 objects of total weight 20, each of the weights being a positive integers. Given that none of the weights exceeds 10 , prove that the ten objects can be divided into two groups that balance each other when placed on 2 pans of a balance.

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Triangle ABC has an incenter I l its incircle touches the side BC at T. The line through T parallel to IA meets the incircle again at S and the tangent to the incircle at S meets AB, AC at points C', B' respectively. Prove that triangle $AB'C\Gamma issimilartotriangle$ABC.$Foranypositiveinteger$n,$let$s(n)$denotethenumbe
$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$. Determine the set of positive integers for which $s(n) = 5$