ELMO Shortlist 2012

– Algebra

1 Let $x_1, x_2, x_3, y_1, y_2, y_3$ be nonzero real numbers satisfying $x_1+x_2+x_3=0, y_1+y_2+y_3=0$. Prove that

$$\frac{x_1x_2+y_1y_2}{\sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}}+\frac{x_2x_3+y_2y_3}{\sqrt{(x_2^2+y_2^2)(x_3^2+y_3^2)}}+\frac{x_3x_1+y_3y_1}{\sqrt{(x_3^2+y_3^2)(x_1^2+y_1^2)}}\ge -\frac{3}{2}.$$

Ray Li, Max Schindler.

2 Let a, b, c be three positive real numbers such that $a\le b\le c$ and $a+b+c=1$. Prove that

$$\frac{a+c}{\sqrt{a^2+c^2}}+\frac{b+c}{\sqrt{b^2+c^2}}+\frac{a+b}{\sqrt{a^2+b^2}}\le\frac{3\sqrt{6}(b+c)^2}{\sqrt{(a^2+b^2)(b^2+c^2)(c^2+a^2)}}.$$

Owen Goff.

3 Prove that any polynomial of the form $1+a_nx^n+a_{n+1}x^{n+1}+\cdots+a_kx^k$ $(k\ge n)$ has at least $n-2$ non-real roots (counting multiplicity), where the a_i $(n\le i\le k)$ are real and $a_k\ne 0$.

David Yang.

4 Let a_0, b_0 be positive integers, and define $a_{i+1}=a_i+\lfloor\sqrt{b_i}\rfloor$ and $b_{i+1}=b_i+\lfloor\sqrt{a_i}\rfloor$ for all $i\ge 0$. Show that there exists a positive integer n such that $a_n=b_n$.

David Yang.

5 Prove that if m, n are relatively prime positive integers, x^m-y^n is irreducible in the complex numbers. (A polynomial $P(x,y)$ is irreducible if there do not exist nonconstant polynomials $f(x,y)$ and $g(x,y)$ such that $P(x,y)=f(x,y)g(x,y)$ for all x, y.)

David Yang.

6 Let $a, b, c\ge 0$. Show that $(a^2+2bc)^{2012}+(b^2+2ca)^{2012}+(c^2+2ab)^{2012}\le(a^2+b^2+c^2)^{2012}+2(ab+bc+ca)^{2012}$.

Calvin Deng.

7 Let f, g be polynomials with complex coefficients such that $\gcd(\deg f, \deg g) = 1$. Suppose that there exist polynomials $P(x, y)$ and $Q(x, y)$ with complex coefficients such that $f(x) + g(y) = P(x, y)Q(x, y)$. Show that one of P and Q must be constant.

Victor Wang.

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8 Find all functions $f : \mathbb{Q} \to \mathbb{R}$ such that $f(x)f(y)f(x+y) = f(xy)(f(x) + f(y))$ for all $x, y \in \mathbb{Q}$.

Sammy Luo and Alex Zhu.

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9 Let a, b, c be distinct positive real numbers, and let k be a positive integer greater than 3. Show that

$$\left|\frac{a^{k+1}(b-c)+b^{k+1}(c-a)+c^{k+1}(a-b)}{a^k(b-c)+b^k(c-a)+c^k(a-b)}\right| \ge \frac{k+1}{3(k-1)}(a+b+c)$$

and

$$\left|\frac{a^{k+2}(b-c)+b^{k+2}(c-a)+c^{k+2}(a-b)}{a^k(b-c)+b^k(c-a)+c^k(a-b)}\right| \ge \frac{(k+1)(k+2)}{3k(k-1)}(a^2+b^2+c^2).$$

Calvin Deng.

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10 Let $A_1A_2A_3A_4A_5A_6A_7A_8$ be a cyclic octagon. Let B_i by the intersection of A_iA_{i+1} and $A_{i+3}A_{i+4}$. (Take $A_9 = A_1$, $A_{10} = A_2$, etc.) Prove that $B_1, B_2, \dots, B_8$ lie on a conic.

David Yang.

– Combinatorics

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1 Let $n \ge 2$ be a positive integer. Given a sequence (s_i) of n distinct real numbers, define the "class" of the sequence to be the sequence $(a_1, a_2, \dots, a_{n-1})$, where a_i is 1 if $s_{i+1} > s_i$ and -1 otherwise.

Find the smallest integer m such that there exists a sequence (w_i) of length m such that for every possible class of a sequence of length n, there is a subsequence of (w_i) that has that class.

David Yang.

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2 Determine whether it's possible to cover a K_{2012} with

a) 1000 K_{1006}'s;

b) 1000 $K_{1006,1006}$'s.

David Yang.

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3 Find all ordered pairs of positive integers (m, n) for which there exists a set $C = \{c_1, \ldots, c_k\}$ $(k \geq 1)$ of colors and an assignment of colors to each of the mn unit squares of a $m \times n$ grid such that for every color $c_i \in C$ and unit square S of color c_i, exactly two direct (non-diagonal) neighbors of S have color c_i.

David Yang.

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4 A tournament on $2k$ vertices contains no 7-cycles. Show that its vertices can be partitioned into two sets, each with size k, such that the edges between vertices of the same set do not determine any 3-cycles.

Calvin Deng.

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5 Form the infinite graph A by taking the set of primes p congruent to $1 \pmod 4$, and connecting p and q if they are quadratic residues modulo each other. Do the same for a graph B with the primes $1 \pmod 8$. Show A and B are isomorphic to each other.

Linus Hamilton.

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6 Consider a directed graph G with n vertices, where 1-cycles and 2-cycles are permitted. For any set S of vertices, let $N^+(S)$ denote the out-neighborhood of S (i.e. set of successors of S), and define $(N^+)^k(S) = N^+((N^+)^{k-1}(S))$ for $k \geq 2$.

For fixed n, let $f(n)$ denote the maximum possible number of distinct sets of vertices in $\{(N^+)^k(X)\}_{k=1}^{\infty}$, where X is some subset of $V(G)$. Show that there exists $n > 2012$ such that $f(n) < 1.0001^n$.

Linus Hamilton.

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7 Consider a graph G with n vertices and at least $n^2/10$ edges. Suppose that each edge is colored in one of c colors such that no two incident edges have the same color. Assume further that no cycles of size 10 have the same set of colors. Prove that there is a constant k such that c is at least $kn^{\frac{8}{5}}$ for any n.

David Yang.

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8 Consider the equilateral triangular lattice in the complex plane defined by the Eisenstein integers; let the ordered pair (x, y) denote the complex number $x+y\omega$ for $\omega = e^{2\pi i/3}$. We define an ω-chessboard polygon to be a (non self-intersecting) polygon whose sides are situated along lines of the form $x = a$ or $y = b$, where a and b are integers. These lines divide the interior into unit triangles, which are shaded alternately black and white so that adjacent triangles have different colors. To tile an ω-chessboard polygon by lozenges is to exactly cover the polygon by non-overlapping rhombuses consisting of two bordering triangles. Finally, a *tasteful tiling* is one such that for every unit hexagon tiled by three lozenges, each lozenge has a black triangle on its left (defined by clockwise orientation) and a white triangle on its right (so the lozenges are BW, BW, BW in clockwise order).

a) Prove that if an ω-chessboard polygon can be tiled by lozenges, then it can be done so tastefully.

b) Prove that such a tasteful tiling is unique.

Victor Wang.

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9 For a set A of integers, define $f(A) = \{x^2 + xy + y^2 : x, y \in A\}$. Is there a constant c such that for all positive integers n, there exists a set A of size n such that $|f(A)| \leq cn$?

David Yang.

– Geometry

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1 In acute triangle ABC, let D, E, F denote the feet of the altitudes from A, B, C, respectively, and let ω be the circumcircle of $\triangle AEF$. Let ω_1 and ω_2 be the circles through D tangent to ω at E and F, respectively. Show that ω_1 and ω_2 meet at a point P on BC other than D.

Ray Li.

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2 In triangle ABC, P is a point on altitude AD. Q, R are the feet of the perpendiculars from P to AB, AC, and QP, RP meet BC at S and T respectively. the circumcircles of BQS and CRT meet QR at X, Y.

a) Prove SX, TY, AD are concurrent at a point Z.

b) Prove Z is on QR iff $Z = H$, where H is the orthocenter of ABC.

Ray Li.

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3 ABC is a triangle with incenter I. The foot of the perpendicular from I to BC is D, and the foot of the perpendicular from I to AD is P. Prove that $\angle BPD = \angle DPC$.

Alex Zhu.

4 Circles Ω and ω are internally tangent at point C. Chord AB of Ω is tangent to ω at E, where E is the midpoint of AB. Another circle, ω_1 is tangent to Ω, ω, and AB at D, Z, and F respectively. Rays CD and AB meet at P. If M is the midpoint of major arc AB, show that $\tan \angle ZEP = \frac{PE}{CM}$.

Ray Li.

5 Let ABC be an acute triangle with $AB < AC$, and let D and E be points on side BC such that $BD = CE$ and D lies between B and E. Suppose there exists a point P inside ABC such that $PD \parallel AE$ and $\angle PAB = \angle EAC$. Prove that $\angle PBA = \angle PCA$.

Calvin Deng.

6 In $\triangle ABC$, H is the orthocenter, and AD, BE are arbitrary cevians. Let ω_1, ω_2 denote the circles with diameters AD and BE, respectively. HD, HE meet ω_1, ω_2 again at F, G. DE meets ω_1, ω_2 again at P_1, P_2 respectively. FG meets ω_1, ω_2 again Q_1, Q_2 respectively. P_1H, Q_1H meet ω_1 at R_1, S_1 respectively. P_2H, Q_2H meet ω_2 at R_2, S_2 respectively. Let $P_1Q_1 \cap P_2Q_2 = X$, and $R_1S_1 \cap R_2S_2 = Y$. Prove that X, Y, H are collinear.

Ray Li.

7 Let $\triangle ABC$ be an acute triangle with circumcenter O such that $AB < AC$, let Q be the intersection of the external bisector of $\angle A$ with BC, and let P be a point in the interior of $\triangle ABC$ such that $\triangle BPA$ is similar to $\triangle APC$. Show that $\angle QPA + \angle OQB = 90^\circ$.

Alex Zhu.

– Number Theory

1 Find all positive integers n such that $4^n + 6^n + 9^n$ is a square.

David Yang, Alex Zhu.

2 For positive rational x, if x is written in the form p/q with p, q positive relatively prime integers, define $f(x) = p + q$. For example, $f(1) = 2$.

a) Prove that if $f(x) = f(mx/n)$ for rational x and positive integers m, n, then $f(x)$ divides $|m - n|$.

b) Let n be a positive integer. If all x which satisfy $f(x) = f(2^n x)$ also satisfy $f(x) = 2^n - 1$, find all possible values of n.

Anderson Wang.

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3 Let $s(k)$ be the number of ways to express k as the sum of distinct 2012^{th} powers, where order does not matter. Show that for every real number c there exists an integer n such that $s(n) > cn$.

Alex Zhu.

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4 Do there exist positive integers $b, n > 1$ such that when n is expressed in base b, there are more than n distinct permutations of its digits? For example, when $b = 4$ and $n = 18$, $18 = 102_4$, but 102 only has 6 digit arrangements. (Leading zeros are allowed in the permutations.)

Lewis Chen.

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5 Let $n > 2$ be a positive integer and let p be a prime. Suppose that the nonzero integers are colored in n colors. Let $a_1, a_2, \ldots, a_n$ be integers such that for all $1 \le i \le n$, $p^i \nmid a_i$ and $p^{i-1} \mid a_i$. In terms of n, p, and $\{a_i\}_{i=1}^n$, determine if there must exist integers $x_1, x_2, \ldots, x_n$ of the same color such that $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$.

Ravi Jagadeesan.

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6 Prove that if a and b are positive integers and $ab > 1$, then

$$\left\lfloor \frac{(a-b)^2 - 1}{ab} \right\rfloor = \left\lfloor \frac{(a-b)^2 - 1}{ab - 1} \right\rfloor.$$

Here $\lfloor x \rfloor$ denotes the greatest integer not exceeding x.

Calvin Deng.

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7 A diabolical combination lock has n dials (each with c possible states), where $n, c > 1$. The dials are initially set to states $d_1, d_2, \ldots, d_n$, where $0 \le d_i \le c - 1$ for each $1 \le i \le n$. Unfortunately, the actual states of the dials (the d_i's) are concealed, and the initial settings of the dials are also unknown. On a given turn, one may advance each dial by an integer amount c_i ($0 \le c_i \le c - 1$), so that every dial is now in a state $d_i' \equiv d_i + c_i \pmod{c}$ with $0 \le d_i' \le c - 1$. After each turn, the lock opens if and only if all of the dials are set to the zero state;

otherwise, the lock selects a random integer k and cyclically shifts the d_i's by k (so that for every i, d_i is replaced by d_{i-k}, where indices are taken modulo n).

Show that the lock can always be opened, regardless of the choices of the initial configuration and the choices of k (which may vary from turn to turn), if and only if n and c are powers of the same prime.

Bobby Shen.

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8 Fix two positive integers $a, k \geq 2$, and let $f \in \mathbb{Z}[x]$ be a nonconstant polynomial. Suppose that for all sufficiently large positive integers n, there exists a rational number x satisfying $f(x) = f(a^n)^k$. Prove that there exists a polynomial $g \in \mathbb{Q}[x]$ such that $f(g(x)) = f(x)^k$ for all real x.

Victor Wang.

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9 Are there positive integers m, n such that there exist at least 2012 positive integers x such that both $m - x^2$ and $n - x^2$ are perfect squares?

David Yang.
