IMO Shortlist 1998

–	Geometry

1   A convex quadrilateral $ABCD$ has perpendicular diagonals. The perpendicular bisectors of the sides AB and CD meet at a unique point P inside $ABCD$. Prove that the quadrilateral $ABCD$ is cyclic if and only if triangles ABP and CDP have equal areas.

2   Let $ABCD$ be a cyclic quadrilateral. Let E and F be variable points on the sides AB and CD, respectively, such that $AE : EB = CF : FD$. Let P be the point on the segment EF such that $PE : PF = AB : CD$. Prove that the ratio between the areas of triangles APD and BPC does not depend on the choice of E and F.

3   Let I be the incenter of triangle ABC. Let K, L and M be the points of tangency of the incircle of ABC with AB, BC and CA, respectively. The line t passes through B and is parallel to KL. The lines MK and ML intersect t at the points R and S. Prove that $\angle RIS$ is acute.

4   Let M and N be two points inside triangle ABC such that

$$\angle MAB = \angle NAC \quad \text{and} \quad \angle MBA = \angle NBC.$$

Prove that

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1.$$

5   Let ABC be a triangle, H its orthocenter, O its circumcenter, and R its circumradius. Let D be the reflection of the point A across the line BC, let E be the reflection of the point B across the line CA, and let F be the reflection of the point C across the line AB. Prove that the points D, E and F are collinear if and only if $OH = 2R$.

6   Let $ABCDEF$ be a convex hexagon such that $\angle B + \angle D + \angle F = 360°$ and

$$\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1.$$

Prove that

$$\frac{BC}{CA} \cdot \frac{AE}{EF} \cdot \frac{FD}{DB} = 1.$$

7   Let ABC be a triangle such that $\angle ACB = 2\angle ABC$. Let D be the point on the side BC such that $CD = 2BD$. The segment AD is extended to E so that $AD = DE$. Prove that

$$\angle ECB + 180° = 2\angle EBC.$$

8   Let ABC be a triangle such that $\angle A = 90°$ and $\angle B < \angle C$. The tangent at A to the circumcircle ω of triangle ABC meets the line BC at D. Let E be the reflection of A in the line BC, let X be the foot of the perpendicular from A to BE, and let Y be the midpoint of the segment AX. Let the line BY intersect the circle ω again at Z.

Prove that the line BD is tangent to the circumcircle of triangle ADZ.

Edited by Orl.

–   Number Theory

1   Determine all pairs (x, y) of positive integers such that $x^2y + x + y$ is divisible by $xy^2 + y + 7$.

2   Determine all pairs (a, b) of real numbers such that $a\lfloor bn \rfloor = b\lfloor an \rfloor$ for all positive integers n. (Note that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

3   Determine the smallest integer $n \geq 4$ for which one can choose four different numbers a, b, c and d from any n distinct integers such that $a + b - c - d$ is divisible by 20.

4   A sequence of integers $a_1, a_2, a_3, \ldots$ is defined as follows: $a_1 = 1$ and for $n \geq 1$, a_{n+1} is the smallest integer greater than a_n such that $a_i + a_j \neq 3a_k$ for any i, j and k in $\{1, 2, 3, \ldots, n+1\}$, not necessarily distinct. Determine a_{1998}.

5   Determine all positive integers n for which there exists an integer m such that $2^n - 1$ is a divisor of $m^2 + 9$.

6   For any positive integer n, let $\tau(n)$ denote the number of its positive divisors (including 1 and itself). Determine all positive integers m for which there exists a positive integer n such that $\frac{\tau(n^2)}{\tau(n)} = m$.

7   Prove that for each positive integer n, there exists a positive integer with the following properties: It has exactly n digits. None of the digits is 0. It is divisible by the sum of its digits.

8   Let $a_0, a_1, a_2, \ldots$ be an increasing sequence of nonnegative integers such that every nonnegative integer can be expressed uniquely in the form $a_i + 2a_j + 4a_k$, where i, j and k are not necessarily distinct. Determine a_{1998}.

–   Algebra

1   Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n < 1$. Prove that

$$\frac{a_1 a_2 \cdots a_n \left[1 - (a_1 + a_2 + \cdots + a_n)\right]}{(a_1 + a_2 + \cdots + a_n)(1 - a_1)(1 - a_2) \cdots (1 - a_n)} \leq \frac{1}{n^{n+1}}.$$

2   Let $r_1, r_2, \ldots, r_n$ be real numbers greater than or equal to 1. Prove that

$$\frac{1}{r_1 + 1} + \frac{1}{r_2 + 1} + \cdots + \frac{1}{r_n + 1} \geq \frac{n}{\sqrt[n]{r_1 r_2 \cdots r_n} + 1}.$$

3   Let x, y and z be positive real numbers such that $xyz = 1$. Prove that

$$\frac{x^3}{(1 + y)(1 + z)} + \frac{y^3}{(1 + z)(1 + x)} + \frac{z^3}{(1 + x)(1 + y)} \geq \frac{3}{4}.$$

4   For any two nonnegative integers n and k satisfying $n \geq k$, we define the number $c(n, k)$ as follows:

- $c(n, 0) = c(n, n) = 1$ for all $n \geq 0$;

- $c(n + 1, k) = 2^k c(n, k) + c(n, k - 1)$ for $n \geq k \geq 1$.

Prove that $c(n, k) = c(n, n - k)$ for all $n \geq k \geq 0$.

5   Determine the least possible value of $f(1998)$, where $f : \mathbb{N} \to \mathbb{N}$ is a function such that for all $m, n \in \mathbb{N}$,

$$f\left(n^2 f(m)\right) = m\left(f(n)\right)^2.$$

–	Combinatorics

1   A rectangular array of numbers is given. In each row and each column, the sum of all numbers is an integer. Prove that each nonintegral number x in the array can be changed into either $\lceil x \rceil$ or $\lfloor x \rfloor$ so that the row-sums and column-sums remain unchanged. (Note that $\lceil x \rceil$ is the least integer greater than or equal to x, while $\lfloor x \rfloor$ is the greatest integer less than or equal to x.)

2   Let n be an integer greater than 2. A positive integer is said to be *attainable* if it is 1 or can be obtained from 1 by a sequence of operations with the following properties:

1.) The first operation is either addition or multiplication.

2.) Thereafter, additions and multiplications are used alternately.

3.) In each addition, one can choose independently whether to add 2 or n

4.) In each multiplication, one can choose independently whether to multiply by 2 or by n.

A positive integer which cannot be so obtained is said to be *unattainable*.

a.) Prove that if $n \geq 9$, there are infinitely many unattainable positive integers.

b.) Prove that if $n = 3$, all positive integers except 7 are attainable.

3   Cards numbered 1 to 9 are arranged at random in a row. In a move, one may choose any block of consecutive cards whose numbers are in ascending or descending order, and switch the block around. For example, 9 1 6 5 3 2 7 4 8 may be changed to 91 3 5 6 2 7 4 8. Prove that in at most 12 moves, one can arrange the 9 cards so that their numbers are in ascending or descending order.

4   Let $U = \{1, 2, \ldots, n\}$, where $n \geq 3$. A subset S of U is said to be *split* by an arrangement of the elements of U if an element not in S occurs in the

arrangement somewhere between two elements of S. For example, 13542 splits $\{1, 2, 3\}$ but not $\{3, 4, 5\}$. Prove that for any $n-2$ subsets of U, each containing at least 2 and at most $n-1$ elements, there is an arrangement of the elements of U which splits all of them.

5   In a contest, there are m candidates and n judges, where $n \geq 3$ is an odd integer. Each candidate is evaluated by each judge as either pass or fail. Suppose that each pair of judges agrees on at most k candidates. Prove that

$$\frac{k}{m} \geq \frac{n-1}{2n}.$$

6   Ten points are marked in the plane so that no three of them lie on a line. Each pair of points is connected with a segment. Each of these segments is painted with one of k colors, in such a way that for any k of the ten points, there are k segments each joining two of them and no two being painted with the same color. Determine all integers k, $1 \leq k \leq 10$, for which this is possible.

7   A solitaire game is played on an $m \times n$ rectangular board, using mn markers which are white on one side and black on the other. Initially, each square of the board contains a marker with its white side up, except for one corner square, which contains a marker with its black side up. In each move, one may take away one marker with its black side up, but must then turn over all markers which are in squares having an edge in common with the square of the removed marker. Determine all pairs (m, n) of positive integers such that all markers can be removed from the board.