LUIS GONZÁLEZ POSTS COLLECTION

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High School Olympiads

Perpendicular lines

Source: Own

buratinogigle

#1 May 25, 2016, 8:50 am • 2

Let ABC be a triangle and P is any point. Let K,L be the orthocenters of triangles PAB,PAC. PA cuts (PBC) again at Q. H is the projection of Q on BC. Prove that $AH\perp KL$.

TelvCohl

#2 May 25, 2016, 10:02 am • 5

Let $\triangle P_AP_BP_C$ be the antipedal triangle of P WRT $\triangle ABC$ and H_A,H_P,H_Q be the orthocenter of $\triangle P_ABC, \triangle P_AP_BP_C$, $\triangle QBC$, respectively. Clearly, $\triangle QBC$ and $\triangle P_AP_BP_C$ are inversely similar, so notice $QH_Q \stackrel{\parallel}{=} P_AH_A$ we get

$$\frac{QH}{QA}=\frac{\operatorname{dist}(Q,BC)}{\operatorname{dist}(P_A,P_BP_C)}=\frac{P_AH_A}{P_AH_P}.$$

Combining $QA \parallel P_AH_P$ and $QH \parallel P_AH_A$ we get $\triangle AHQ$ and $\triangle H_PH_AP_A$ are homothetic $\Longrightarrow AH \parallel H_AH_P$.

Let B^*,C^* be the isotomic conjugate of B,C WRT $(P_C,P_A),(P_A,P_B)$, respectively. Since $P_AB^* \stackrel{\parallel}{=} P_CB \stackrel{\parallel}{=} AK, P_AC^* \stackrel{\parallel}{=} P_BC \stackrel{\parallel}{=} AL$, so $\triangle AKL, \triangle P_AB^*C^*$ are homothetic (also congruent) $\Longrightarrow KL \parallel B^*C^*$, hence KL is parallel to the Newton line of the complete quadrilateral $\mathcal{Q}$ formed by $\triangle P_AP_BP_C, BC$. Notice H_AH_P is the Steiner line of $\mathcal{Q}$ we get $AH\perp KL$.

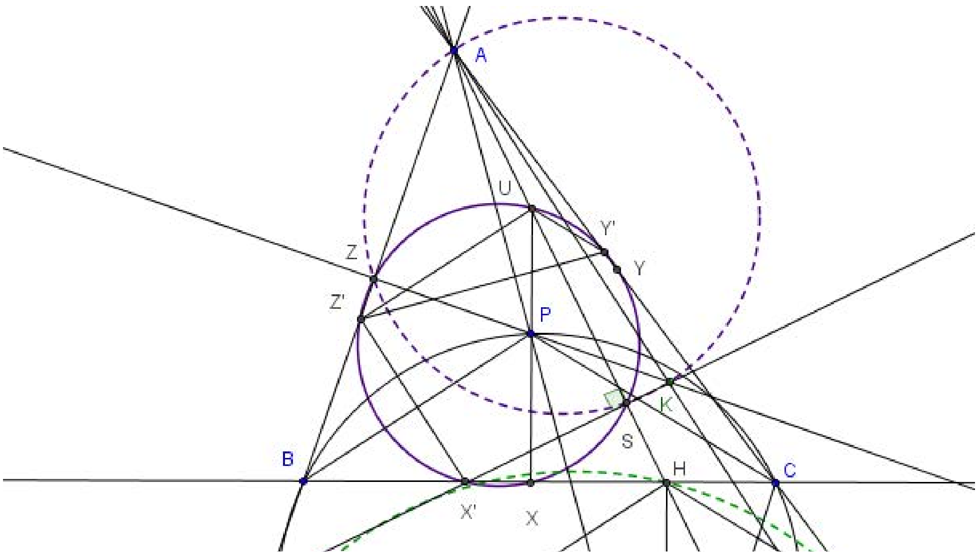
Luis González

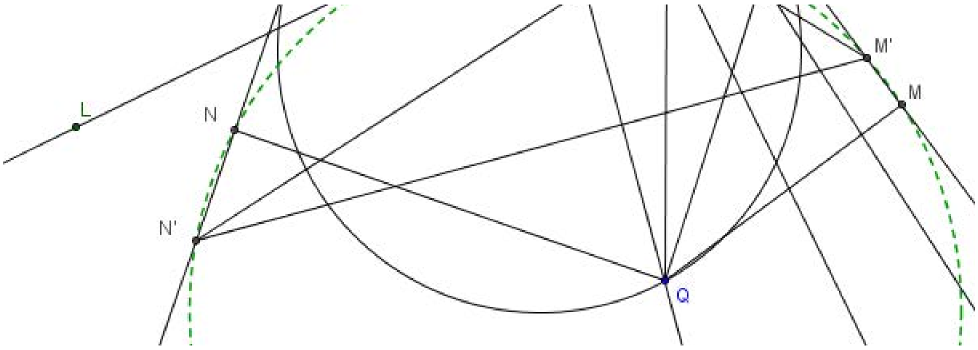
#3 Thursday at 9:14 AM • 6

Let $\triangle XYZ$ and $\triangle HMN$ be the pedal triangles of P and Q WRT $\triangle ABC$. $\odot(XYZ)$ cuts BC,CA,AB again at X',Y',Z', and $\odot(HMN)$ cuts CA,AB again at M',N'. PX cuts $\odot(XYZ)$ again at U. Clearly $Y'Z' \parallel M'N'$ and $\angle UZ'Y'=90^\circ-\angle X'Z'Y'=\angle APB-90^\circ=90^\circ-\angle BCQ=\angle HQC=\angle HN'M' \Longrightarrow UZ' \parallel HN'$ and similarly $UY' \parallel HM' \Longrightarrow \triangle UY'Z'$ and $\triangle HM'N'$ are homothetic with center $A \Longrightarrow A,U,H$ are collinear.

Let AH cut $\odot(XYZ)$ again at S and $K^*\equiv PZ\cap SX'$. Since $AZSK^*$ is cyclic on account of the right angles at S,Z, then by Reim's theorem $AK^* \parallel X'Z' \Longrightarrow AK^*\perp PB \Longrightarrow K\equiv K^*$. Analogously, $L\in SX' \Longrightarrow ASH\perp KL$.

Attachments:



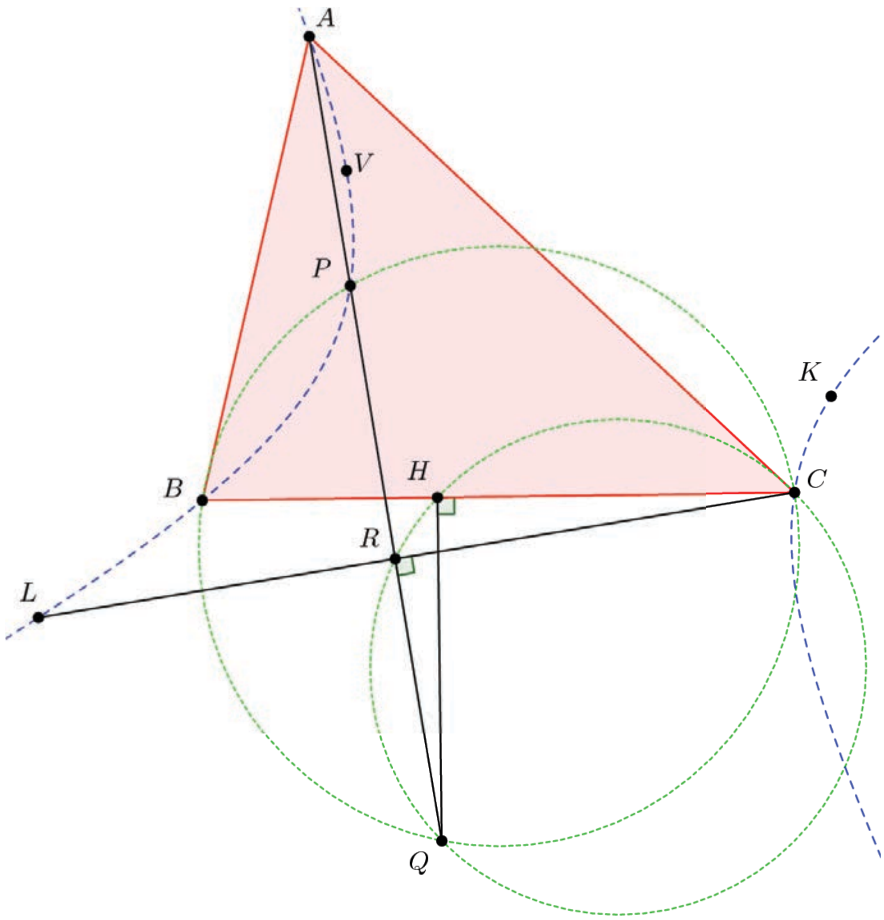


Dukejukem

#4 Thursday at 3:28 PM • 4

Let R be the projection of C onto AP and note that $R \in CL$. Let V be the orthocenter of $\triangle AKL$ and denote $H^* \equiv AV \cap BC$.

From Reim's Theorem for $\odot(BCPQ), \odot(CHQR)$ cut by BH, PR, we get $BP \parallel HR$.



On the other hand, if $\mathcal{H}$ is the rectangular circumhyperbola of $\triangle ABC$ passing through P, then then $K, L \in \mathcal{H}$ on account of rectangularity, which in turn gives $V \in \mathcal{H}$. Then since $(VL \parallel BP) \perp AK$, Pascal's Theorem for $AVLCBP$ gives $BP \parallel H^*R$. Hence, $H^* \equiv H$, so $AH \perp KL$.

This post has been edited 3 times. Last edited by Dukejukem, Thursday at 3:51 PM
Reason: Fix diagram

jayme

#5 Yesterday at 4:43 PM

Dear Mathlinkers,
just to say that another synthetic proof is possible... next on my site...

Sincerely
Jean-Louis

Quick Reply

Concurrent lines with pedal circle

Reply

Source: Own

buratinogigle

#1 May 24, 2016, 1:20 am • 1

Let ABC be a triangle and DEF is pedal triangle of a point P. Circle diameter AD cuts (DEF) again at X. Similarly, we have Y, Z. Prove that AX, BY, CZ are concurrent.

Luis González

#2 May 24, 2016, 5:03 am • 2

If D', E, F' are the antipodes of D, E, F on $\odot(DEF)$, then obviously $D' \in AX, F' \in BY$ and $E' \in CZ$. Now the configuration is a particular case of Concurrent 14 (Problem b).

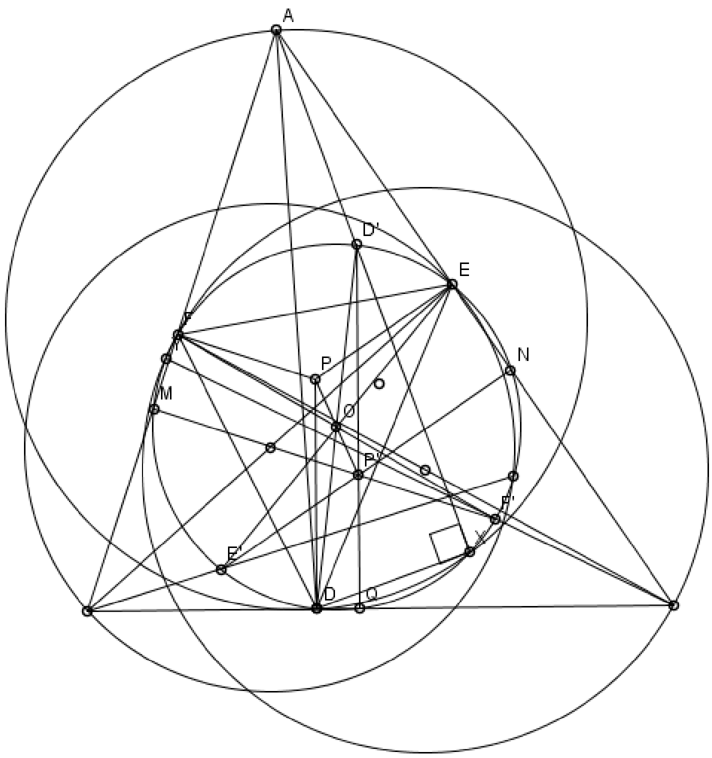
uraharakisuke_hsgs

#3 May 24, 2016, 9:56 pm

My soution

Lemma For any $\triangle ABC$ with arbitrary point P in the triangle
Let D, E, F is projection of P on BC, CA, AB respectively
Let ω is circumcircle of $\triangle DEF$ and PD, PE, PF intersect ω again at X, Y, Z respectively
Then AX, BY, CZ concurrent
Prove : this lemma is proved here at ***post 2***
Back to main problem : Let (O) be the circumcenter of $\triangle DEF$
D' reflects with D about O then $\angle DXD' = 90^\circ$. It follows that A, D' and X are collinear
Similary with E' and F'
Q, N, M are the projections of D', E', F' on BC, CA, AB respectively so Q, N, M lies on (O)
P' reflects with P about O, Obviously $P' \in D'Q$
$\implies D'Q, E'N, F'M$ are concurrent at P'.
By applying the lemma we have AX, BY, CZ are concurrent

Attachments:



Quick Reply

High School Olympiads

Concurrent 14

Reply

buratinogigle

#1 Sep 16, 2011, 4:06 pm • 3

Let ABC be a triangle and a point P. $A_1B_1C_1$ is pedal triangle of P. P^* is isogonal conjugate of P. $A_2B_2C_2$ is pedal triangle of P^*. Q is a point on line $PP^*.A_2Q, B_2Q, C_2Q$ cut cirumcircle $(A_1B_1C_1)$ again at A_3, B_3, C_3, respectively.

a) Prove that A_1A_3, B_1B_3, C_1C_3 are concurrent on line PP^*.

b) Prove that AA_3, BB_3, CC_3 are concurrent.

When $P, Q \equiv H$ we have problem in the post concurrent lines formed by incenter and excenters.

Luis González

#2 Sep 17, 2011, 12:36 am • 2

a) K is the center of the pedal circle of P, P^* (midpoint of PP*) and PA_1, PB_1 cut (K) again at U, V. Since $\angle UA_1A_2$ and $\angle VB_1B_2$ are right, it follows that $K \equiv UA_2 \cap VB_2$. By Pascal theorem for the cyclic hexagon $A_1UA_2B_1VB_2$, the intersections $P \equiv A_1U \cap B_1V, K \equiv UA_2 \cap VB_2$ and $R \equiv A_2B_1 \cap B_2A_1$ are collinear $(\star)$. By Pascal theorem for the cyclic hexagon $A_1A_3A_2B_1B_3B_2$, the intersections $S \equiv A_1A_3 \cap B_1B_3, Q \equiv A_3A_2 \cap B_3B_2$ and $R \equiv A_2B_1 \cap B_2A_1$ are collinear. Together with $(\star)$, we conclude that A_1A_3, B_1B_3 and PP^* concur at S. By similar reasoning, we get that C_1C_3 goes through S.

b) Using the result of the problem Collinearity and points on the circumcenter (post #2) for $\triangle A_3B_3C_3$ and the circumcevian triangles $\triangle A_1B_1C_1, \triangle A_2B_2C_2$ of S, Q WRT $\triangle A_3B_3C_3$, we deduce that the intersections $A_4 \equiv BC \cap B_3C_3$, $B_4 \equiv CA \cap C_3A_3$ and $C_4 \equiv AB \cap A_3B_3$ are collinear, i.e. $\triangle ABC$ and $\triangle A_3B_3C_3$ are perspective through $\overline{A_4B_4C_4}$ $\Longrightarrow AA_3, BB_3, CC_3$ concur.

buratinogigle

#3 Sep 17, 2011, 4:18 pm • 1

Thank you, very nice proof with Pascal theorem. With your proof, we can prove the following problem

Let ABC be a triangle, circumcricle (O). P, Q are two arbitrary point. PA, PB, PC cut (O) again at A_1, B_1, C_1, resp. QA, QB, QC cut (O) again at A_2, B_2, C_3, resp. R is a point on PQ. A_2R, B_2R, C_2R cuts (O) again at A_3, B_3, C_3. Prove that A_1A_3, B_1B_3, C_1C_3 concur at a point on PQ.

drmzjoseph

#4 Jun 15, 2015, 8:23 am

a) Let γ be the circle passes through $A_1, B_1, C_1, A_2, B_2, C_2$, if A_1P cut again to γ at A^* then A_2A^* cut to $PP*$ in the center L of γ. Analogously is define B^* and C^*. The composition of involutions tha fixed γ with poles P, L, Q, so changes A_1 with A_3, further interchanges the antipode points of cut PP^* with γ. hence is a involution too, so $A_1A_3, B_1B_3, C_1C_3, PP^*$ are concurrent.

b) **General problem:**

Let $(A_1, A_2), (B_1, B_2), (C_1, C_2)$ three pairs of points at a conic γ, and on the lines of sides BC, CA, AB of $\triangle ABC$, if A_3, B_3, C_3 lies on γ, suppose that $\triangle A_1B_1C_1$ is perspective to $\triangle A_3B_3C_3$ and $\triangle A_2B_2C_2$ is perspective to $\triangle A_3B_3C_3$. Prove that $\triangle A_3B_3C_3$ is perspective to $\triangle ABC$

Proof

Is sufficient works with γ a circle, because exist a homology between all conics.

Is sufficient prove that $\prod \frac{\sin \angle BAA_3}{\sin \angle CAA_3} = 1 \Longleftrightarrow \prod \frac{\sin \angle C_1B_1A_3}{\sin \angle B_1C_1A_3} \cdot \prod \frac{\sin \angle AC_1A_3}{\sin \angle AB_1A_3} = 1$ this is ceva in the triangles ABC and AB_1C_1.

$\prod \frac{\sin \angle C_1B_1A_3}{\sin \angle B_1C_1A_3} = \prod \frac{\sin \angle C_1A_1A_3}{\sin \angle B_1A_1A_3} = 1$
$\prod \frac{\sin \angle AC_1A_3}{\sin \angle AB_1A_3} = \prod \frac{\sin \angle C_2A_2A_3}{\sin \angle B_2A_2A_3} = 1$ is sufficient.

This post has been edited 1 time. Last edited by drmzjoseph, Jun 15, 2015, 9:37 am

Luis González

#5 Jun 15, 2015, 11:32 am

> *drmzjoseph wrote:*
> b) **General problem:**
>
> Let $(A_1, A_2), (B_1, B_2), (C_1, C_2)$ three pairs of points at a conic γ, and on the lines of sides BC, CA, AB of $\triangle ABC$, if A_3, B_3, C_3 lies on γ, suppose that $\triangle A_1B_1C_1$ is perspective to $\triangle A_3B_3C_3$ and $\triangle A_2B_2C_2$ is perspective to $\triangle A_3B_3C_3$. Prove that $\triangle A_3B_3C_3$ is perspective to $\triangle ABC$

The proof I gave to b) in my previous post still works for this general problem. Again using the projective version of Collinearity and points on the circumcenter (post #2) for $\triangle A_3B_3C_3$, we get that $A_1A_2 \cap B_3C_3, B_1B_2 \cap C_3A_3, C_1C_2 \cap A_3B_3$ are collinear, i.e. $\triangle ABC$ and $\triangle A_3B_3C_3$ are perspective.

Quick Reply

High School Olympiads

Circle is tangent to fixed circle

Source: Own, Vietnam IMO team training 2016

buratinogigle

#1 May 20, 2016, 6:00 pm • 3

Let ABC be the triangle with $AB=AC$ and circumcenter O. M is midpoint of BC and H is projection of M on AB. P lies on perpendicular bisector of AH. Q lies on PO such that $AQ\perp AC$. E,F are projections of P,Q on CA,AB, reps. Prove that circle (AEF) is always tangent to fixed circle when P moves.

Luis González

#2 May 21, 2016, 12:06 am • 3

Let N,L be the midpoints of AC,AB and denote ω the circle with center O tangent to AC,AB at N,L. It's known that if a variable circle passing through A tangent to ω cuts AC,AB again at E',F', then $E'\mapsto F'$ is a proyectivity between AB and AC. Since $P\mapsto Q$ is perspectivity through O, then obviously $E\mapsto F$ is a proyectivity between AB and AC. Thus it suffices to show that $\odot(AEF)$ is tangent to ω for 3 particular positions of P.

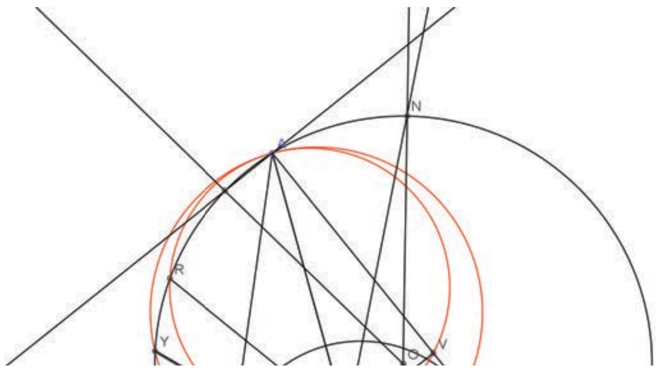
Let the perpendicular bisector τ of AH cut AC at J. When $P\equiv\tau\cap ON$ and P is at infinity, clearly $\odot(AEF)$ denegerates into AC and AB, respectively, trivially tangent to ω. When $P\equiv E\equiv J$, then $F\equiv H$. In this case, notice that $K\equiv AO\cap\tau$ is the incenter of the J-isosceles $\triangle AHJ$, which is clearly midpoint of $AM\Longrightarrow K\in NL\Longrightarrow\omega$ is the A-mixtilinear incircle of $\triangle AHJ$, i.e. $\odot(AEF)$ is tangent to ω. Consequently, $\odot(AEF)$ is tangent to ω for any P.

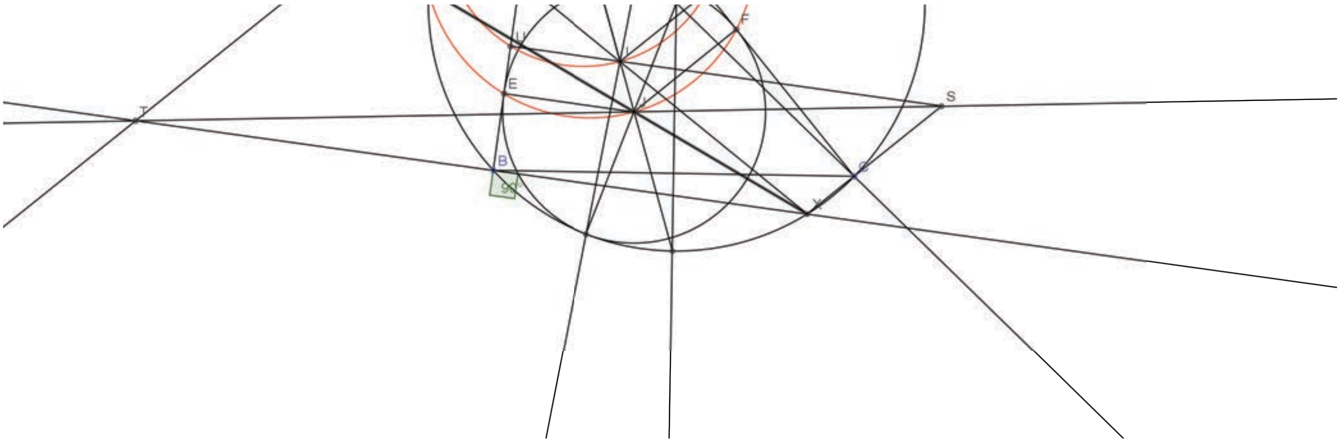
hoangA1K44PBC

#3 May 21, 2016, 9:01 am • 1

Dear Mr. Hung here is my solution.
Because I think it circle is A- Mixtilinear incircle of AEF, so I rewrite the problem as following.
Let triangle ABC with circumcircle (O). J is circumcenter of A- Mixtilinear incircle of ABC. I is it incenter. The line pass through I and perpendicular to AB cut the line pass through C perpendicular to AC at S. SJ cut the line pass through A perpendicular to AC at T. we need to prove that TB perpen to BA.(1)
Solution
We need two lemmas : (I think it has appear at problem weekly)
Lemma 1(well known): With point we sign in (1) then call X is symmetry of A through O. E, F is the tangents points of (J) at AB, AC. (AEF) cut (O) again at Y then Y, I, X are conlinear.
Lemma 2 : With point we sign in (1) then call X is symmetry of A through O. U,V is the tangents point of (I) at AB, AC then (AUV) cut (O) at R then R, I, X are conlinear.
Back to this problem :
call G is symmetry of C through O then we need to prove AG, JS, XB concurrent. So we need to prove X(BJAS) = A(GJXS) .
From IU // AG // CX so A(GJXS) = I(VJXS) but IV perpen to AC, IS perpen to AB. N is midpoint of arc BAC. then I(VJXS) = A(CNRB). projected onto the circle (AUV). Call t is tangent at A of (O). then X(BJAS) = A(BYtC). projected onto the circle (AEF) we have X(BJAS) = A(GJXS). So we have Q.E.D

Attachments:





This post has been edited 1 time. Last edited by hoangA1K44PBC, May 21, 2016, 9:07 am
Reason: sorry

buratinogigle

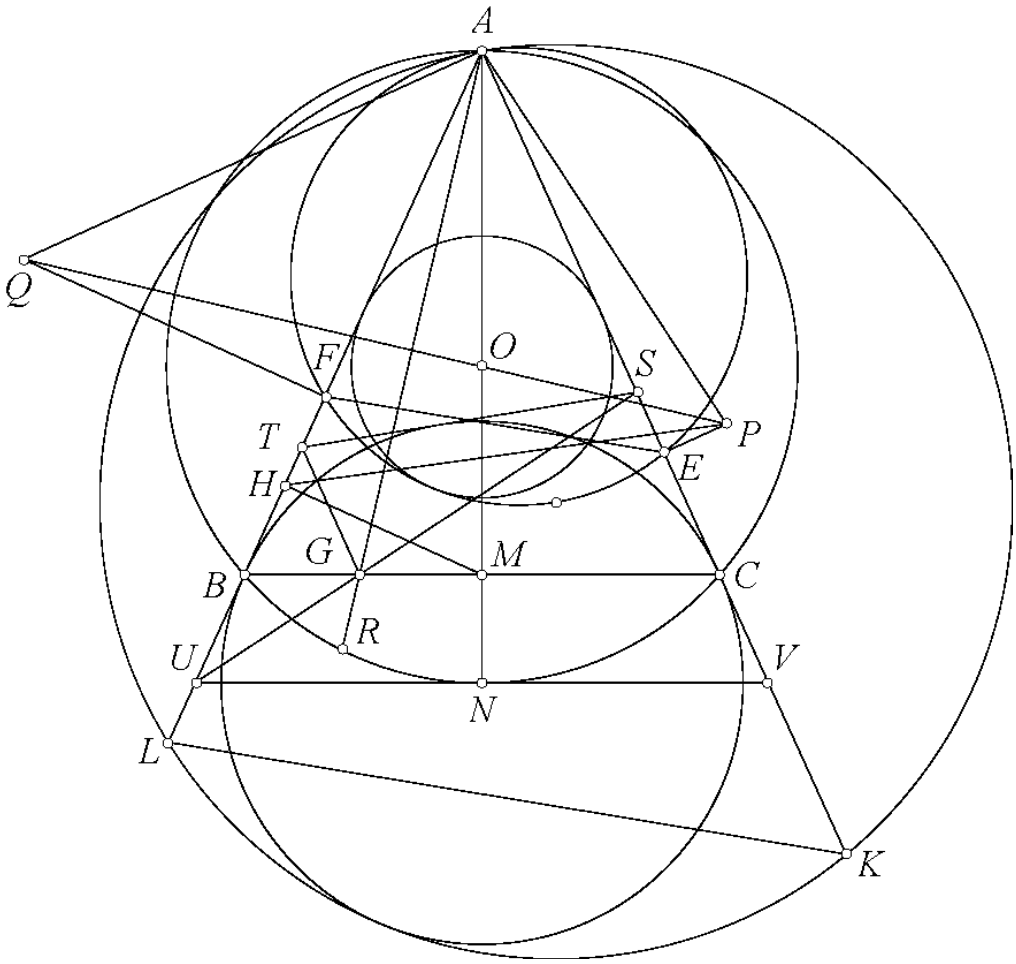
#4 May 21, 2016, 10:32 am • 3

Thank you dear Luis and Hoang for your interest, here is my solution.

Using dilation center A ratio 2 then E, F transform to K, L. Let R be the reflection of A through PQ the R lies on (O). Thus K is the intersection of circle (P) circumcircle of triangle ARH and AC and L is the intersection of circle (Q) which passes through A, R and is tangent to AC, with AB. Now using inversion center A with power AB^2 then R transform to G the intersection of AR and BC. M transform to N the intersection of AM and (O). H transform to U the intersection of tangent at N of (O) with AB. Thus, K transform to S the intersection of UG and AC. L transform to T on AB such that $GT \parallel AC$. We will prove that ST is tangent to fixed circle. Now NU cuts AC at V. We have, $\frac{UT}{UA} = \frac{UG}{US} = \frac{VC}{VS}$, we deduce, $UT.VS = UA.VC = \frac{UV^2}{4}$. We easily seen ST is tangent to circle (N) which is tangent to AB, AC. We are done.

See APMO 2016 P3.

Attachments:



Quick Reply

High School Olympiads

parabola

Reply

shmm

#1 May 19, 2016, 3:50 pm

A parabola is drawn on the plane.
Build its axis of symmetry
with the help of a
compass (drawing tool) and a ruler.

Luis González

#2 May 19, 2016, 7:54 pm

In general, given a conic on the plane (parabola, ellipse or hyperbola), its main axes can be drawn with ruler and compass. See the general construction in the topic how to find the foci of an ellipse (post #5).

Quick Reply

High School Olympiads

how to find the foci of an ellipse

Reply

fmasroor
#1 Oct 4, 2013, 8:02 am

In the plane I have drawn an ellipse. How do I construct its foci?

Dr Sonnhard Graubner
#2 Oct 5, 2013, 12:38 am

hello, see here:
http://answers.yahoo.com/question/index?qid=20070301184409AAXPGhg
Sonnhard.

fmasroor
#3 Oct 5, 2013, 3:56 am

I don't have the equation of the ellipse. I only have an ellipse, oriented in any way. I want to find, without any algebra (ie. purely geometrical), the foci.

mavropnevma
#4 Oct 5, 2013, 4:53 am

Draw the long axis, of length L, and the short axis, of length ℓ. Let O be their meeting point. The foci will be points on the long axis, symmetrical with respect to O, at distance $\frac{1}{2}\sqrt{L^2 - \ell^2}$ from O.

Luis González
#5 Oct 5, 2013, 5:08 am • 2

mavropnevma, actually drawing the main axes is what the problem is.

Center O can be found using conjugate diameters. Draw two any parallel chords in the conic, the line connecting their midpoints is the conjugate diameter of their direction, passing through $O.$ Repeat this for two more parallel chords to get another diameter and then $O.$ Conjugate diameters u,v in any conic clearly form an involutive pencil, hence main axes are nothing but the perpendicular rays in the referred involution.

Label $\mathcal{C}$ the given conic with center $O.$ Draw abitrary circle (K) with center K that goes through $O.$ If u,v cuts (K) again at $U,V,$ then $U \mapsto V$ is an involution on $(K).$ Construct the fixed pole I of this involution (two pairs of homologous points are sufficient). KI cuts (K) at $X,Y \Longrightarrow OX,OY$ are the axes of $\mathcal{C}.$

Quick Reply

High School Olympiads

Isogonal conjugate of inversion

Reply

Source: Goormaghtigh's Generalization of Musselman's Theorem, VMO team trainning 2016

buratinogigle

#1 May 17, 2016, 9:27 pm • 1

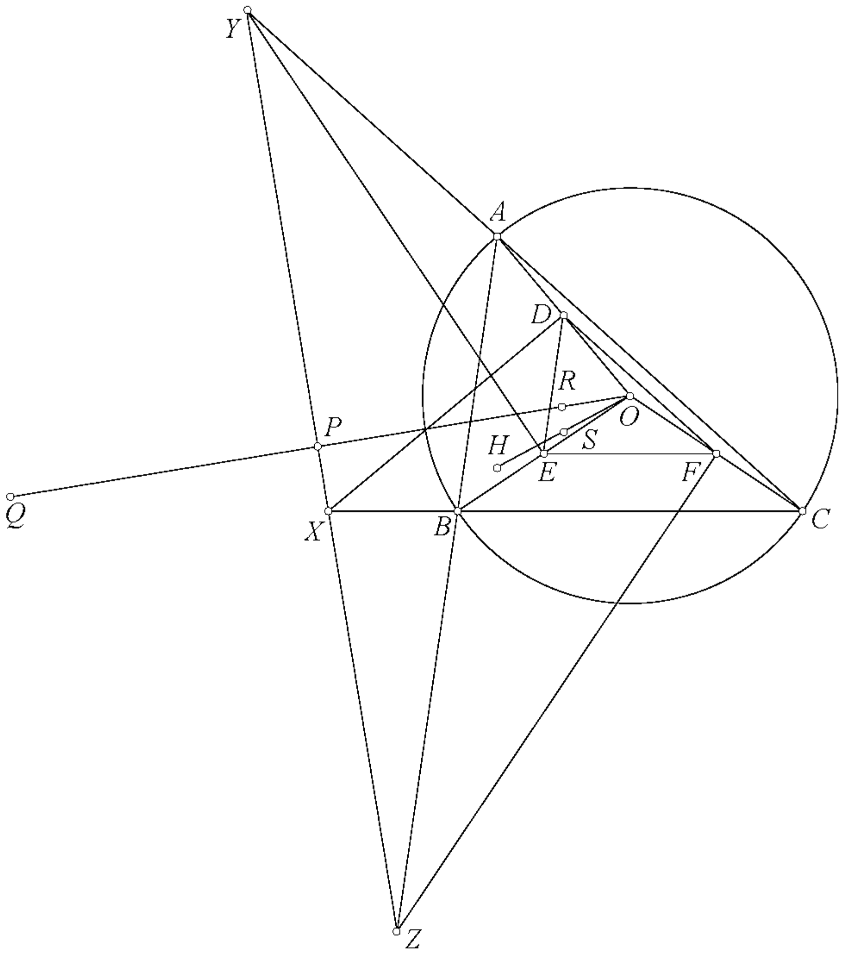
Let ABC be a triangle inscribed in circle (O). D, E, F devide segment OA, OB, OC, reps in the same ratio. X, Y, Z lie on BC, CA, AB, reps such that $DX \perp OA, EY \perp OB, FZ \perp OC$.

a) Prove that X, Y, Z are collinear on line d.

b) Let P be the projection of O on d. Q lie on OP such that $AQ \parallel DP$. Prove that isogonal conjugate with triangle ABC of inversion of Q through (O) lies on Euler of triangle ABC.

Reference

Attachments:



TelvCohl

#2 May 18, 2016, 4:19 am • 2

(a) Let $\triangle A_1B_1C_1$ be the tangential triangle of $\triangle DEF$. From Kariya's theorem $\Longrightarrow AA_1, BB_1, CC_1$ are concurrent, so from Desargues' theorem we get X, Y, Z are collinear at the perspectrix of $\triangle ABC$ and $\triangle A_1B_1C_1$.

(b) Let $\triangle M_aM_bM_c$ be the medial triangle of $\triangle ABC$ and let V^* be the image of an arbitrary point V under the inversion WRT

$\odot(DEF)$. Since $\triangle DEF, \triangle M_a^* M_b^* M_c^*$ are bilogic with common orthology center O, so DM_a^*, EM_b^*, FM_c^* are concurrent on the circum-rectangular hyperbola of $\triangle DEF$ passing through O (isogonal conjugate of Euler line).

Since OX is the diameter of $\odot(DOM_a)$, so we get X^* is the projection of O on DM_a^*. Analogously, we can prove Y^*, Z^* is the projection of O on EM_b^*, FM_c^*, respectively, so we get P^* is the perspector of $\triangle DEF, \triangle M_a^* M_b^* M_c^*$ and the isogonal conjugate of P^* WRT $\triangle DEF$ lies on the Euler line of $\triangle DEF$ ... $(\spadesuit)$.

Finally, it is obvious that $\triangle ABC \cup Q$ and $\triangle DEF \cup P$ are homothetic, so from $(\spadesuit)$ we conclude that the isogonal conjugate (WRT $\triangle ABC$) of the image of Q under the inversion WRT $\odot(ABC)$ lies on the Euler line of $\triangle ABC$.

Luis González

#3 May 18, 2016, 7:58 am • 3

a) Animate D, E, F on OA, OB, OC. Series D, E, F are similar and so are X, Y, Z, because the directions XD, YE, ZF are fixed, therefore $\tau \equiv YZ$ envelopes a parabola $\mathcal{P}$ tangent to AB, AC. When D, E, F coincide with A, B, C, then τ becomes the Lemoine axis of $\triangle ABC$ and when $D \equiv E \equiv F \equiv O$, then τ is the orthotransversal of O WRT $\triangle ABC \Longrightarrow \mathcal{P}$ is the Kiepert parabola of $\triangle ABC$. Similarly XY touches $\mathcal{P} \Longrightarrow X, Y, Z$ are collinear on a tangent τ of $\mathcal{P}$.

b) Let $\triangle A'B'C'$ be the tangential triangle of $\triangle ABC$ and the perpendicular τ' from Q to OP cuts $B'C', C'A', A'B'$ at X', Y', Z', Since $OP : OQ = OD : OA = OE : OB = OF : OC$, then by homothety it follows that X', Y', Z' lie on OX, OY, OZ, resp, thus the series X', Y', Z' are also projective, even similar, as X', Y', Z' go to infinity when τ is the line at infinity $\Longrightarrow \tau'$ envelopes a parabola $\mathcal{P}'$ inscribed in $\triangle A'B'C'$. Hence the inverse Q^* of Q on (O) (pole of τ' WRT (O)) describes the dual $\mathcal{J}$ of $\mathcal{P}'$ WRT (O). As $\mathcal{P}'$ is a parabola, then it follows that $\mathcal{J}$ passes through A, B, C and O. When D, E, F coincide with A, B, C, then clearly $\tau \equiv \tau' \Longrightarrow Q^*$ becomes the symmedian point K of $\triangle ABC \Longrightarrow K \in \mathcal{J} \Longrightarrow \mathcal{J}$ is the Jerabek hyperbola of $\triangle ABC$; isogonal conjugate of its Euler line $\Longrightarrow$ isogonal conjugate of the inverse of Q is then on the Euler line of $\triangle ABC$.

Quick Reply

High School Olympiads

A-symedian

Reply

Source: Own from some results.

doxuanlong15052000

#1 May 15, 2016, 8:05 pm

Let ABC be a triangle with (N) be the Euler-circle. (N) cuts (BOC) at X and Y. AX cuts BC at L. Let T be a point on BC such that $BL = CT$. The conjugate equiangular of AT cuts (O) at R. AY cuts (O) at H. S is the intersection of the tangent of (O) at R and H. Then AS is A-symedian.

Luis González

#2 May 15, 2016, 9:12 pm • 1

It's well-known that AX, AY are isogonals WRT $\angle BAC$ (note that $\odot(BOC)$ is the image of (N) under inversion with center A, power $\frac{1}{2}AB \cdot AC$ followed by reflection on the bisector of $\angle BAC$). Thus, the problem is a particular case of the following configuration:

Let U, V be the isotomic points on the side BC of $\triangle ABC$. The isogonals of AU, AV WRT $\angle BAC$ cut $\odot(ABC)$ again at M, N. The tangents of $\odot(ABC)$ at M, N meet at S. Then AS is the A-symmedian of $\triangle ABC$.

Let the tangent of $\odot(ABC)$ at A cut BC at L and let the parallel from A to BC cut $\odot(ABC)$ again at P. Inversion with center A, power $AB \cdot AC$, followed by reflection on the bisector of $\angle BAC$ swaps $\odot(ABC)$ and BC. Consequently, $U \mapsto M, V \mapsto N$ and $L \mapsto P$. By symmetry $APVU$ is an isosceles trapezoid (cyclic), hence it follows that L, M, N are collinear $\Longrightarrow AS$ is the polar of L WRT $\odot(ABC)$, i.e. the A-symmedian of $\triangle ABC$.

Quick Reply

High School Olympiads

Conics in triangle

Source: maybe well-known

jlammy

#1 May 14, 2016, 9:04 pm

Consider a triangle ABC and three conics, each with focus at one vertex and directrix at the opposite side to that vertex. Prove that these conics intersect the sides of $\triangle ABC$ at six points which lie on a conic.

Luis González

#2 May 14, 2016, 9:43 pm • 1

Denote $\epsilon_A, \epsilon_B, \epsilon_C$ the eccentricities of the conics $\mathcal{C}_A, \mathcal{C}_B, \mathcal{C}_B$ with foci A, B, C and directrices $BC, CA, AB,$ respectively. $\mathcal{C}_B$ and $\mathcal{C}_C$ cut $\overline{BC}$ at A_1 and A_2 and we define $\{B_1, B_2\} \in \overline{CA}$ and $\{C_1, C_2\} \in \overline{AB}$ similarly. Then

$$\frac{A_1B}{\text{dist}(A_1, AC)} = \epsilon_B \Longrightarrow \frac{A_1B}{A_1C} = \epsilon_B \cdot \sin C. \text{ Similarly } \frac{A_2C}{A_2B} = \epsilon_C \cdot \sin B.$$

$$\Longrightarrow \frac{A_1B}{A_1C} \cdot \frac{A_2B}{A_2C} = \frac{\epsilon_B}{\epsilon_C} \cdot \frac{\sin C}{\sin B}.$$

Multiplying the cyclic expressions together, we deduce by Carnot's theorem that $A_1, A_2, B_1, B_2, C_1, C_2$ lie on a same conic.

Concurrent related to three Symmedian points

Reply

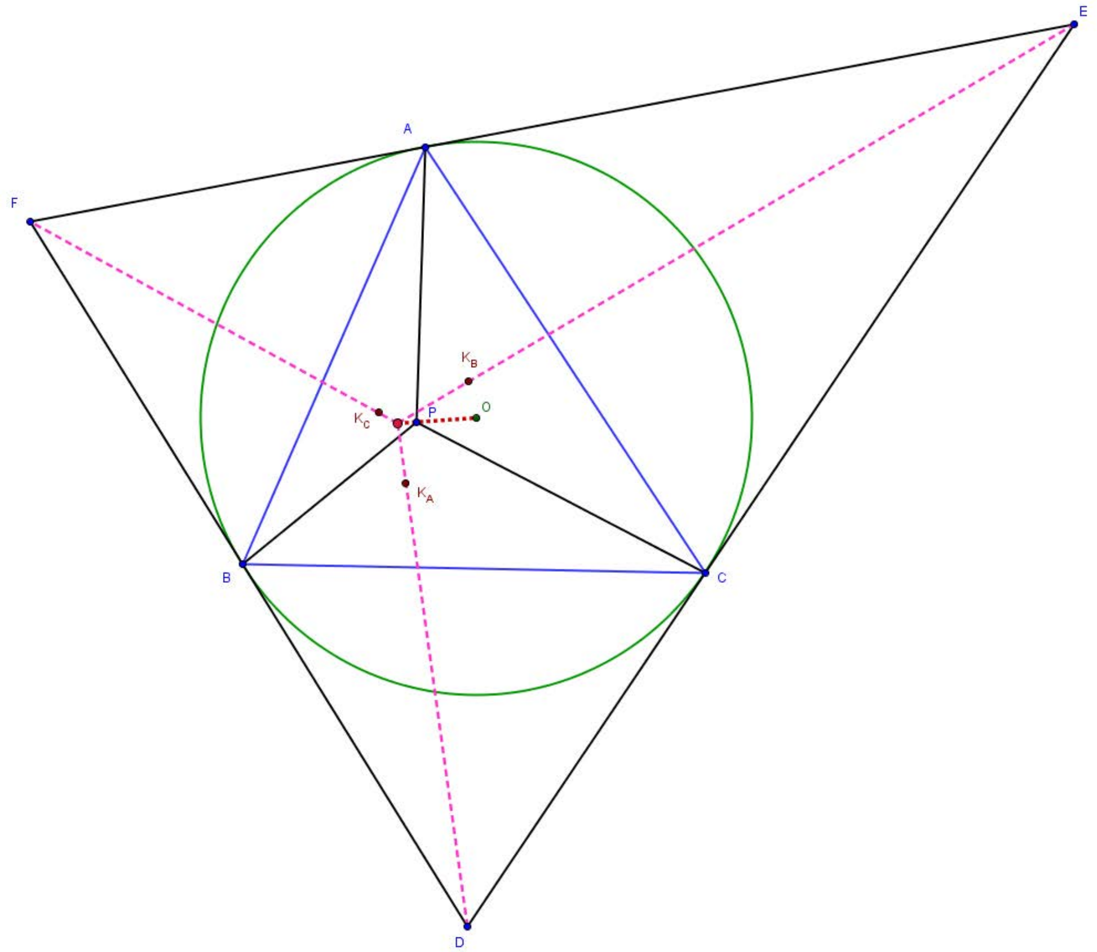
Source: Own (Inspired from mineiraojose's problem)

TelvCohl

#1 May 14, 2016, 2:33 am • 3

Given a $\triangle ABC$ inscribed in $\odot(O)$ and a point P. Let $\triangle DEF$ be the tangential triangle of $\triangle ABC$ and let K_A, K_B, K_C be the symmedian point of $\triangle BPC, \triangle CPA, \triangle APB$, respectively. Prove that OP, DK_A, EK_B, FK_C are concurrent.

Attachments:



Luis González

#2 May 14, 2016, 4:41 am • 5

Let S_A, S_B, S_C be the feet of the P-symmedians of $\triangle PBC, \triangle PCA, \triangle PAB$, respectively.Tangents of $\odot(PBC)$, $\odot(PCA), \odot(PAB)$ at P cut BC, CA, AB at U_A, U_B, U_C, respectively. Since $U_AP^2 = U_AB \cdot U_AC \Longrightarrow U_A$ has equal WRT (O) and the degenerate circle P. Likewise, U_B and U_C have equal powers WRT (O) and $P \Longrightarrow U_A, U_B, U_C$ are collinear on the radical axis of $(O), P \Longrightarrow \overline{U_AU_BU_C} \perp PO$ $(\star)$.

On the other hand, since $(S_A, B, C, U_A) = -1$ and DB, DC are tangents of (O), it follows that DS_A is the polar of U_A WRT (O) and analogously ES_B and FS_C are the polars of U_B and U_C WRT $(O) \Longrightarrow DS_A, ES_B, FS_C$ concur at the pole Q of $\overline{U_AU_BU_C}$ WRT $(O) \Longrightarrow OQ \perp \overline{U_AU_BU_C}$. Together with $(\star)$, we deduce that O, P, Q are collinear.

If the perpendicular bisector DO of BC cuts the P-symmedian PS_A of $\triangle PBC$ at D', then $D'B, D'C$ are tangents of $\odot(PBC) \Longrightarrow (K_A, P, S_A, D') = -1$. Hence, if DK_A cuts OP at R, we have $(R, P, Q, O) = D(K_A, P, S_A, D') = -1 \Longrightarrow R$ is fixed on $\overline{OPQ}$. Similarly EK_B and FK_C hit OP at the same point R.

Quick Reply

High School Olympiads

Concurrent on Simson line

Reply

langkhach11112

#1 May 13, 2016, 6:39 pm

Given ΔABC and its circumcircle (O). Point P lies on (O). d is an arbitrary line go through P. Points A_1, B_1, C_1 lies on d such that $AA_1, BB_1, CC_1 \perp d$. Points A_2, B_2, C_2 lies on BC, CA, AB, respectively, such that $A_1A_2 \perp BC, B_1B_2 \perp CA, C_1C_2 \perp AB$. Prove that A_1A_2, B_1B_2, C_1C_2 are concurrent on the Simson line of P wrt ΔABC.

Luis González

#2 May 13, 2016, 7:41 pm

This is well-known; A_1A_2, B_1B_2, C_1C_2 concur at the orthopole of d WRT $\triangle ABC$. Now see the previous lemma at Six orthopoles lie on a circle.

jam10307

#3 May 13, 2016, 8:42 pm

This is also pretty nice with complex numbers, when we set the circumcircle (denoted as ω) as the unit circle. Let P, P' be the points that line d intersects ω. Then we just spam

$$\ell_1 \cap \ell_2 = \frac{ab(c+d) - cd(a+b)}{ab - cd}, c \perp ab = \frac{1}{2}(a + b + c - ab\bar{c}).$$

Quick Reply

High School Olympiads

Six orthopoles lie on a circle

Reply

buratinogigle

#1 Sep 3, 2012, 12:00 am

Let ABC and $A'B'C'$ be two triangles inscribed circle (O). Prove that orthopoles of $B'C', C'A', A'B'$ with respect to triangle ABC and orthopoles of BC, CA, AB with respect to triangle $A'B'C'$ lie on a circle.

Luis González

#2 Sep 3, 2012, 2:58 am • 3

Lemma. Line τ cuts the circumcircle of $\triangle ABC$ at P, Q. Then Simson lines of P, Q with respect to $\triangle ABC$ meet at the orthopole of τ with respect to $\triangle ABC$.

Proof. Let A' be the orthogonal projection of A on τ. S, T, X are the orthogonal projections of Q on CB, BA, AC and M, N, Y the orthogonal projections of P on BC, CA, AB. Thus, A, Q, X, A' and A, P, Y, A' are concyclic. Using directed angles $\pmod{\pi}$, we have

$$\angle AA'X = \frac{\pi}{2} + \angle QA'X = \frac{\pi}{2} + \angle QAC \ , \ \angle AA'N = \angle APN = \frac{\pi}{2} - \angle PAC$$

$$\angle NAX' = \angle AA'X - \angle AA'N = \frac{\pi}{2} + \angle QAC - \left(\frac{\pi}{2} - \angle PAC\right) = \angle PAQ$$

Let $K \equiv SX \cap MY$. Since the angle between the Simson lines of P, Q is half the measure of the arc PQ of the circumcircle $(*)$, it follows that $\angle NAX' = \angle NKX \Longrightarrow N, K, A', X$ are concyclic. Consequently

$$\angle A'KX = \angle A'NX = \angle APQ = \angle ACQ = \angle XSQ \Longrightarrow KA' \parallel QS$$

That is, $KA' \perp BC$. Similarly, if B' denotes the orthogonal projection of B on τ, we have $KB' \perp CA \Longrightarrow K$ is the orthopole of τ with respect to $\triangle ABC$.

Based on the previous lemma, the pairwise Simson lines of A', B', C' WRT $\triangle ABC$ meet at the orthopoles A_0, B_0, C_0 of $B'C', C'A', A'B'$ WRT $\triangle ABC$. In addition, $\triangle A_0B_0C_0 \sim \triangle A'B'C'$, because of the assertion $(*)$. Let H, H' be the orthocenters of $\triangle ABC, \triangle A'B'C'$. Then B_0C_0, C_0A_0, A_0B_0 pass through the midpoints X_0, Y_0, Z_0 of HA', HB', HC', lying on the 9-point circle $(N, \frac{R}{2})$ of $\triangle ABC \Longrightarrow \triangle X_0Y_0Z_0 \sim \triangle A'B'C' \sim \triangle A_0B_0C_0$. If U is midpoint of $\overline{HH'}$, then UY_0 is H-midline of $\triangle HH'B' \Longrightarrow UY_0 \parallel B'H' \perp A'C'$. Similarly, $UZ_0 \perp A'B' \Longrightarrow$ $\angle Y_0UZ_0 = \angle B'A'C' = \angle B_0A_0C_0 \pmod{\pi} \Longrightarrow U \in \odot(A_0Y_0Z_0)$. Likewise, $U \in \odot(B_0Z_0X_0) \Longrightarrow U$ is the Miquel point of $\triangle X_0Y_0Z_0$ WRT $\triangle A_0B_0C_0 \Longrightarrow U$ is orthocenter of $\triangle X_0Y_0Z_0$ and circumcenter of $\triangle A_0B_0C_0$ (see the topic Own problem and elsewhere) $\Longrightarrow \odot(AY_0Z_0) \cong (N, \frac{R}{2})$.

Now, easy angle chase reveals that the angle between a sideline of ABC (A'B'C') and the Simson line of its opposite vertex WRT A'B'C' (ABC) is constant. Let λ be the measure of this angle. From $UY_0 \parallel B'H'$, we have then $UA_0 = R \cdot \sin\angle(B'H', A_0C_0) = R \cdot \sin(\frac{\pi}{2} - \lambda) = R \cdot \cos\lambda$. As a result, orthopoles of $B'C', C'A', A'B'$ WRT $\triangle ABC$ lie on a circle (U, ϱ) centered at the midpoint U of $\overline{HH'}$ with radius $\varrho = R \cdot \cos\lambda$. By similar reasoning, the orthopoles of BC, CA, AB WRT $\triangle A'B'C'$ lie on the same circle (U, ϱ).

Quick Reply

High School Olympiads

Concurrent lines

Reply

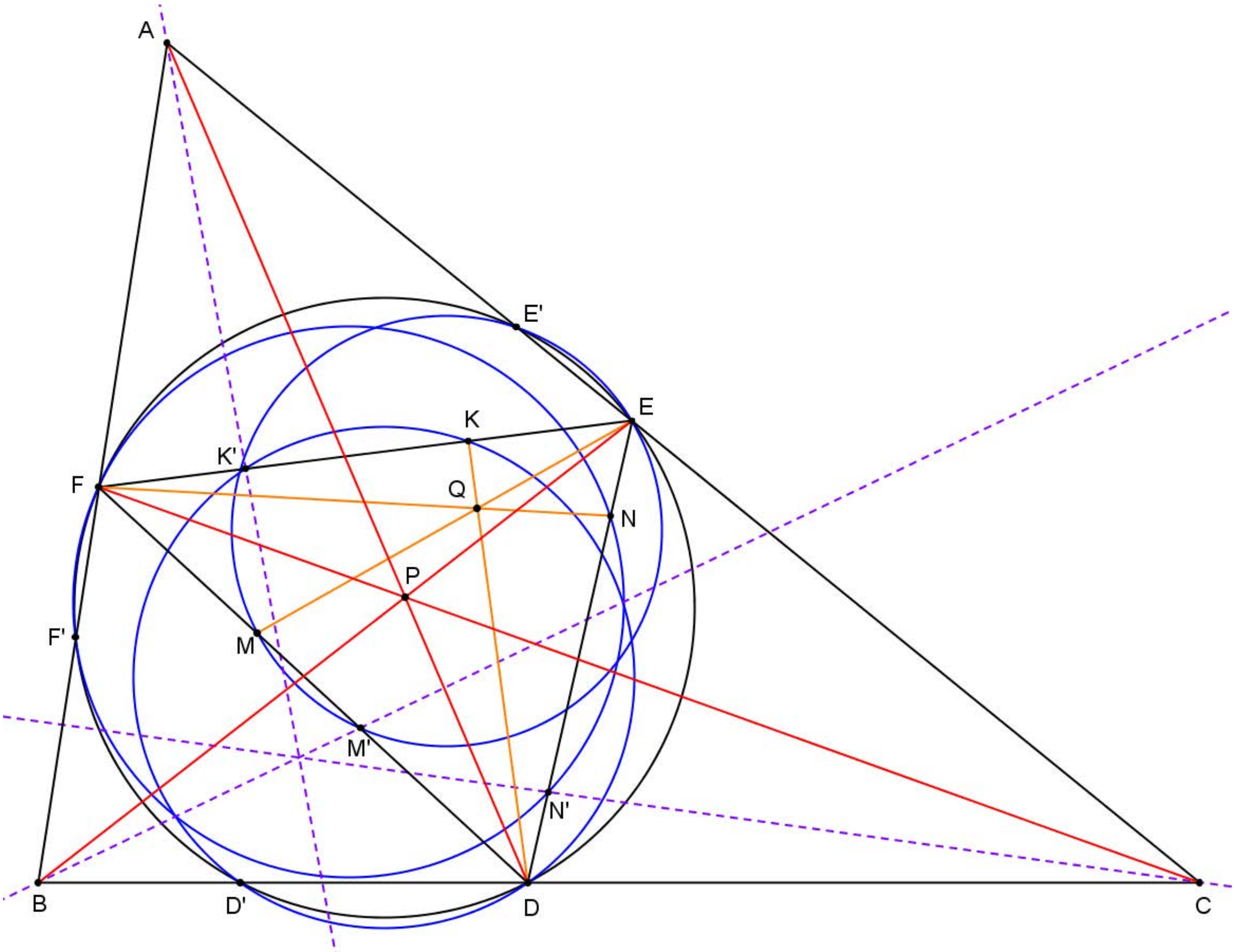
mjuk

#1 May 13, 2016, 12:40 am • 2

Let $\triangle ABC$ be a triangle, let P, Q be points. Let $\triangle DEF$ be cevian triangle of P wrt. $\triangle ABC$. Let $\odot DEF$ intersect BC, CA, AB at D', E', F' respectively. Let $\triangle KMN$ be cevian triangle of Q wrt. $\triangle DEF$. Let $K' = EF \cap \odot KDD'$, $M' = DF \cap \odot MEE'$ and $N' = DE \cap \odot NFF'$.

Prove that AK', BM', CN' are concurrent.

Attachments:



This post has been edited 1 time. Last edited by mjuk, May 13, 2016, 12:42 am
Reason: picture edit

Luis González

#2 May 13, 2016, 4:33 am • 2

Let EF, FD, DE cut BC, CA, AB at U, V, W and let EF, FD, DE cut PA, PB, PC at A', B', C', respectively. Then $UE \cdot UF = UD \cdot UD' = UK \cdot UK' \Longrightarrow U$ is the center of the involution that swaps E, F and $K, K' \Longrightarrow$

$$(K, F, E, U) = (K', E, F, \infty) \Longrightarrow \frac{K'E}{K'F} = \frac{KF}{KE} \cdot \frac{UE}{UF} = \frac{KF}{KE} \cdot \frac{A'E}{A'F}.$$

Hence, multiplying the cyclic expressions together, we deduce by Ceva's theorem that DK', EM', FN' concur. Thus by Cevian Nest Theorem, it follows that AK', BM', CN' concur.

Quick Reply

High School Math

A strange geometry problem

Reply

finn123

#1 May 11, 2016, 9:24 pm

For each point O on diameter AB of a circle, perform the following construction: Let the perpendicular to AB at O meet the circle at point P. Inscrible circles in the figures bounded by the circle and the lines AB, OP. Let the R and S be the points at which the two incircles to the curvilinear triangles AOP and BOP are tangent to the diameter AB. Show that angle RPS is independent of the position of O.

Luis González

#2 May 12, 2016, 9:58 pm

According to http://www.artofproblemsolving.com/community/c6h416223, PR and PS bisect $\angle OPA$ and $\angle OPB$, respectively. Since $\angle APB = 90^\circ$, it follows that $\angle RPS = 45^\circ$.

Quick Reply

High School Olympiads

AC bisects angle PAB

Reply

Source: Israel 1995

sergei93
#1 Jul 4, 2011, 5:11 am • 2

Let PQ be the diameter of semicircle H. The circle O is internally tangent to H and is tangent to PQ at C. Let A be a point on H, and let B be a point on PQ such that AB is perpendicular to PQ and is also tangent to O. Prove that AC bisects $\angle PAB$.

I'm looking particularly for non-inversive solutions. The natural solution uses inversion.

Luis González
#2 Jul 4, 2011, 5:48 am • 2

We assume that C is between P and B. Let A' and (O') be the reflections of A and (O) about PQ. Obviously, (O') is also tangent to $\odot(PAA')$. Therefore, $(O), (O)'$ are the Thebault circles of the angle bisector PB of the P-isosceles $\triangle APA'$. Hence, $C \equiv PB \cap OO'$ is the incenter of $\triangle PAA'$, i.e. AC bisects $\angle PAB$.

See also http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=146656

sunken rock
#3 Jul 15, 2011, 12:20 am

Take A' the reflection of A in PQ, D and E the common points of circle O with the semicircle and AB, respectively. Inversion of pole Q and power $\rho = QB \cdot PQ$ sends the circle $\odot APA'Q$ into the line AA', hence the point D to E, i.e. it preserves the circle O.
That means $\rho = QA^2 = QE \cdot QD = QC^2$ and C is the incenter of $\triangle APA'$.

Note: in the link proposed by Luis we can see something similar to this one; it does not, however, make the connection to inversion (it does show that the circle $\odot CDE$ remains unchanged through that inversion)!

Best regards,
sunken rock

Farenhajt
#4 Jul 15, 2011, 4:12 am

This is metric solution.

Let $PB = p, QB = q, CB = r$ and M the midpoint of PQ. Then $AB = \sqrt{pq}, AP = \sqrt{p^2 + pq}$. By Pythagoras

$$MO^2 - MC^2 = OC^2 \implies \left(\frac{p+q}{2} + r\right)^2 - \left(\frac{q-p}{2} + r\right)^2 = r^2$$

$$r^2 + 2rq - pq = 0 \implies r = \sqrt{pq + q^2} - q \implies PC = p + q - \sqrt{pq + q^2}$$

Hence $\dfrac{PC}{CB} = \dfrac{\sqrt{p+q}(\sqrt{p+q} - \sqrt{q})}{\sqrt{q}(\sqrt{p+q} - \sqrt{q})} = \dfrac{\sqrt{p^2 + pq}}{\sqrt{pq}} = \dfrac{PA}{AB}$ and the conclusion follows.

Quick Reply

High School Olympiads

Tripolar of Infinity point passes through Fe

Reply

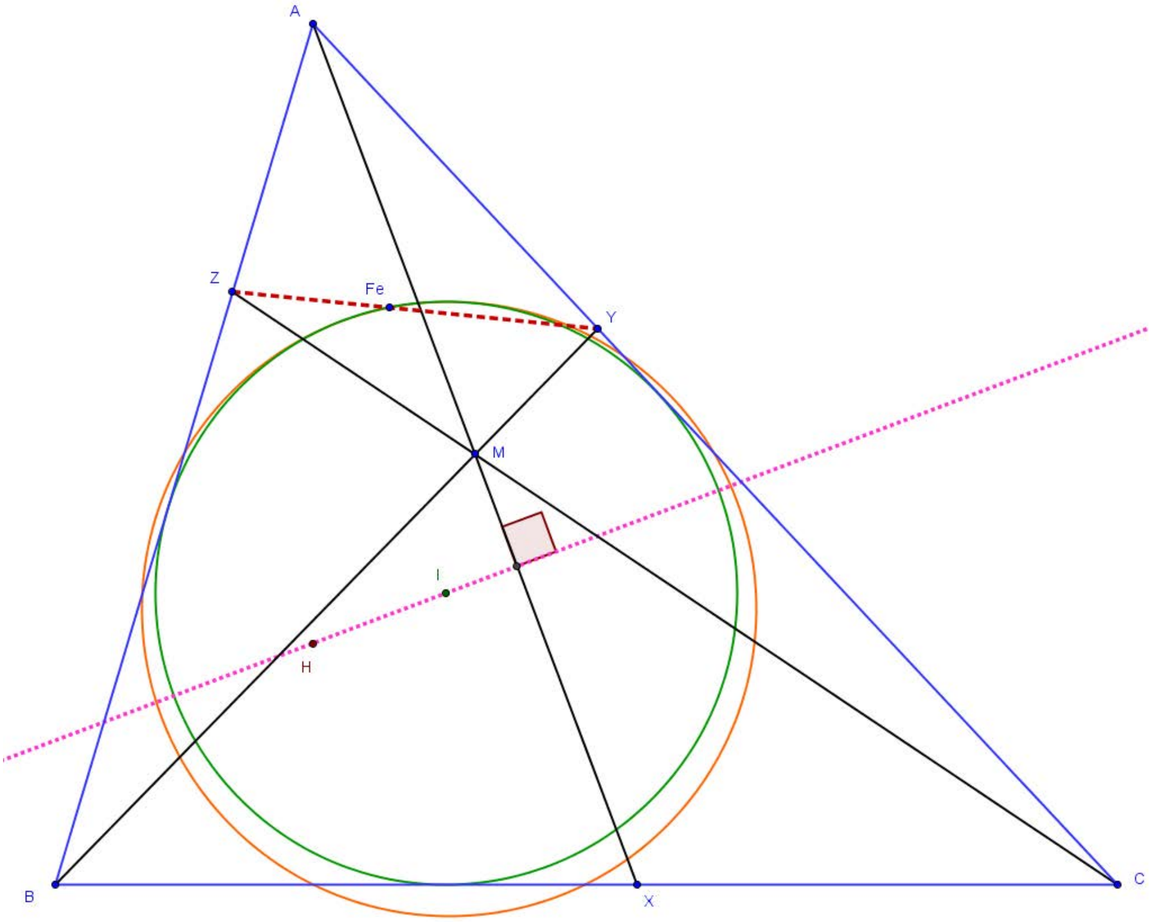
Source: Own

TelvCohl

#1 May 9, 2016, 6:56 pm • 4

Given a $\triangle ABC$ with incenter I, orthocenter H and the Feuerbach point F_e. Let X be the point on BC such that $AX \perp IH$. Let M be the midpoint of AX and let $Y \equiv BM \cap CA, Z \equiv CM \cap AB$. Prove that F_e, Y, Z are collinear.

Attachments:



Luis González

#2 May 10, 2016, 8:17 am • 6

Let D, E, F be the midpoints of BC, CA, AB and let the excircles againts A, B, C touch BC, CA, AB at U, V, W, respectively. EF, FE, DE cut VW, WU, UV at A', B', C'. $\triangle UVW$ is pedal triangle of Bevan point Be of $\triangle ABC$, lying on the line connecting I and the circumcenter O of $\triangle ABC$. Thus, the isogonal conjugate Be^* of Be WRT $\triangle ABC$ is on its Feuerbach hyperbola with center $Fe \Longrightarrow Fe$ is the Poncelet point of A, B, C, Be^*. Consequently, by 1st Fontené theorem, we deduce that Fe, U, A' are collinear and similarly Fe, V, B' and Fe, W, C' are collinear i.e. $Fe \equiv UA' \cap VB' \cap WC'$. But if $\mathcal{M}$ is the inconic tangent to BC, CA, AB at U, V, W (Mandart inellipse of $\triangle ABC$), then according to concurrent at one point in (I) (see the generalization at post #4), it follows that $Fe \equiv UA' \cap VB' \cap WC' \in \mathcal{M}$.

On the other hand, it's known that, in general AA', BB', CC' concur at the intersection of the trilinear polars of the Nagel point $Na \equiv AU \cap BV \cap CW$ and the centroid $G \equiv AD \cap BE \cap CF, \Longrightarrow AA'$ is parallel to the trilinear polar τ of Na. But from the problem hard geometry problem, we have $IH \perp \tau \Longrightarrow AA' \perp IH \Longrightarrow AA' \equiv AX \Longrightarrow M \equiv A'$. Now by the converse of Newton theorem, it follows that YZ is tangent to $\mathcal{M}$ at a point collinear with U and A', which is none other than Fe $\Longrightarrow Fe \in YZ$.

Quick Reply

High School Olympiads

concurrent at one point in (I)

Reply

DonaldLove

#1 Sep 11, 2013, 3:42 pm

consider triangle ABC, the inscribed circle (I) is tangent to BC, CA, AB at A_0, B_0, C_0. AI, BI, CI intersect BC, CA, AB at A_1, B_1, C_1. B_0C_0, C_0A_0, A_0B_0 intersect B_1C_1, C_1A_1, A_1B_1 at A_2, B_2, C_2. prove that A_0A_2, B_0B_2, C_0C_2 intersect at one point on (I)

jayme

#2 Sep 11, 2013, 9:27 pm • 1

Dear Mathlinkers,
only to begin, A2B2C2 is the Pelletier triangle of ABC.
Now look that A0B0C0 and A1B1C1 are two cevian triangle of ABC...
Sincerely
Jean-Louis

DonaldLove

#3 Sep 11, 2013, 9:49 pm

dear jayme,
can you explain more simple please?

Luis González

#4 Sep 12, 2013, 1:13 am • 5

Generalization: P_0, P_1 are two points on the plane of $\triangle ABC$. $\triangle A_0B_0C_0$ and $\triangle A_1B_1C_1$ are the cevian triangles of P_0, P_1 WRT $\triangle ABC$. $A_2 \equiv B_0C_0 \cap B_1C_1$ and B_2, C_2 are defined similarly. Then A_0A_2, B_0B_2, C_0C_2 concur on the inconic $\mathcal{K}$ with perspector P_0.

Let $\mathcal{C}$ be the conic through A, B, C, P_0, P_1. Then B_0C_0 and B_1C_1 are the polars of A_0 and A_1 WRT $\mathcal{C} \Longrightarrow A_0A_1 \equiv BC$ is the polar of $A_2 \equiv B_0C_0 \cap B_1C_1$ WRT $\mathcal{C} \Longrightarrow A_2B, A_2C$ are tangents of $\mathcal{C}$. By similar reasoning B_2C, B_2A and C_2A, C_2B are tangents of $\mathcal{C} \Longrightarrow A \in B_2C_2, B \in C_2A_2, C \in A_2B_2$. Further $\triangle A_2B_2C_2$ is then polar triangle of $\mathcal{C}$ WRT $\triangle ABC \Longrightarrow \triangle A_2B_2C_2$ and $\triangle ABC$ are perspective $\Longrightarrow$ the line through $Y \equiv BA_2C_2 \cap CA$ and $Z \equiv CA_2B_2 \cap AB$ is their perspetrix τ. Hence if $P \equiv \tau \cap A_0A_2$, then by Brianchon's theorem there is a conic tangent to BC, CY, YZ, ZB at A_0, B_0, P, C_0, respectively $\Longrightarrow$ it coincides with $\mathcal{K} \Longrightarrow P \in \mathcal{K}$. Analogously, B_0B_2 and C_0C_2 go through P.

P.S. When P_0 and P_1 coincide with the Gergonne point and the incenter, as the proposed problem, then $\mathcal{K}$ becomes (I) and P is then the tangency point of (I) with the perspectrix of $\triangle ABC$ and its Pelletier triangle $\triangle A_2B_2C_2$, the 6th Stevanovic point of $\triangle ABC$.

Quick Reply

High School Olympiads

hard geometry problem

Reply

Source: own

littlepascal
#1 Oct 24, 2014, 7:41 am

Given a triangle ABC, let N is the Nagel Point, I is the incenter, and H is the orthocenter. Let AN meets BC on D, BN meets AC on E, CN meets AB on F.

Prove that perspective axis of triangle ABC and DEF is orthogonal to IH.

Sorry for my bad English @@

Luis González
#2 Oct 24, 2014, 8:40 am • 1

$\triangle ABC$ and $\triangle DEF$ are perspective through the Nagel point N and orthologic, being the Bevan Be of $\triangle ABC$ one of the orthology center (concurrency point of the normals to BC,CA,AB at D,E,F). By Sondat's theorem NBe is perpendicular to their perspectrix. Thus, it suffices to show that $IH \parallel NBe.$

Let $\triangle A_0B_0C_0$ be the antimedial triangle of $\triangle ABC.$ Its orthocenter is the reflection L of H on O (De Longchamps point of ABC) and its incenter is $N.$ Thus, lines IH and NL are parallel, as they are homologues under the homothety that takes $\triangle ABC$ into $\triangle A_0B_0C_0.$ Be is reflection of I on $O,$ falling then on the parallel NL from L to $IH,$ i.e. $NBe \parallel IH.$

littlepascal
#3 Oct 24, 2014, 8:48 am

What is Sondat's theorem??

TelvCohl
#4 Oct 24, 2014, 10:49 am • 1

Another approach:

Let D', E', F' be the tangent point of the $\odot(I)$ with BC, CA, AB, respectively .
Let $X = E'F' \cap BC, Y = F'D' \cap CA, Z = D'E' \cap AB$ and l be the line pass through X, Y, Z .
Let D'', E'', F'' be the midpoint of $E'F', F'D', D'E'$.

Easy to see D'' lie on $\odot(AX)$, E'' lie on $\odot(BY)$, F'' lie on $\odot(CZ)$.

Since $IA \cdot ID'' = IB \cdot IE'' = IC \cdot IF''$,
so IH is the radical axis of $\odot(AX), \odot(BY), \odot(CZ)$,
hence the Newton line of complete quadrilateral form by $\triangle ABC$ and l is perpendicular to IH .
By a well known property of complete quadrilateral (see **Remark**) we get
the perspective axis of $\triangle ABC$ and $\triangle DEF$ is perpendicular to IH .

Q.E.D

Remark:

Let d' be the isotomic line of d .

Then the Newton line of complete quadrilateral form by $\triangle ABC$ and d is parallel to d' .

Proof **:**

Let D, E, F be the intersection of d and BC, CA, AB.
Let D', E', F' be the intersection of d' and BC, CA, AB.
Let X, Y, Z be the midpoint of BC, CA, AB and $X' = AD \cap YZ, Y' = BE \cap ZX, Z' = \cap XY$.

Easy to see $\triangle XYZ \cap X' \cap Y' \cap Z' \sim \triangle ABC \cap D' \cap E' \cap F'$,
so the Newton line $X'Y'Z'$ of $\triangle ABC \cap d$ is parallel to $d' = D'E'F'$.

This post has been edited 1 time. Last edited by TelvCohl, Mar 31, 2016, 3:39 am

Quick Reply

High School Olympiads

Nice geometry: EMMO 2016, Jr. 5

Reply

Source: Based largely on a problem of Tran Quang Hung (buratinogigle) and its solution by Telv.

AdithyaBhaskar
#1 May 7, 2016, 9:40 pm • 2

Let $\triangle ABC$ be a triangle with circumcenter O and circumcircle Γ. The point X lies on Γ such that AX is the A- symmedian of triangle $\triangle ABC$. The line through X perpendicular to AX intersects AB, AC in F, E, respectively. Denote by γ the nine-point circle of triangle $\triangle AEF$, and let Γ and γ intersect again in $P \neq X$. Further, let the tangent to Γ at A meet the line BC in Y, and let Z be the antipode of A with respect to circle Γ. Prove that the points Y, P, Z are collinear.
Notes: 1. The A-symmedian of triangle $\triangle ABC$ is the reflection of the A-median in the A-angle bisector.
2. The antipode of a point with respect to a circle is the point on the circle diametrically opposite to it.

Proposed by Adithya Bhaskar

This post has been edited 2 times. Last edited by AdithyaBhaskar, May 7, 2016, 10:55 pm

aopser123
#2 May 7, 2016, 10:52 pm • 1

> *AdithyaBhaskar wrote:*
> The point X lies on Γ such that AP is the A- symmedian of triangle $\triangle ABC$.

?

AdithyaBhaskar
#3 May 7, 2016, 10:54 pm • 1

Sorry, edited.

Luis González
#4 May 8, 2016, 2:25 am • 5

Obviously $Z \in EF$ and since AX is the polar of Y WRT Γ, then AX is perpendicular to OY at S. Therefore, OY is the A-midline of $\triangle AEF$, cutting AE, AF at their midpoints $U, V \Longrightarrow \odot(XUV)$ is 9-point circle of $\triangle AEF$.

Let P^* be the 2nd intersection of YZ with Γ. Then $\angle XP^*Z = \angle XAZ = \angle AYO = \angle AP^*S = \angle XYO \Longrightarrow \odot(P^*XY)$ is tangent to OY and $\angle SP^*X = \angle AP^*Z = 90^\circ$. Thus if $M \equiv P^*X \cap OY$, we have $MS^2 = MP^* \cdot MX = MY^2 \Longrightarrow M$ is midpoint of YS. Together with $(S, U, V, Y) = -1$, we have $MS^2 = MY^2 = MU \cdot MV \Longrightarrow MP^* \cdot MX = MU \cdot MV \Longrightarrow P^* \in \odot(XUV) \Longrightarrow P \equiv P^* \Longrightarrow P, Y, Z$ are collinear.

anantmudgal09
#5 May 24, 2016, 1:42 am

Here is a quick sketch: a standard $\sqrt{bc}$ inversion reduces this to an intermixing of Russian MO 2015 10th grade P7 and IMO Shortlist 2011 G4.

pi37
#6 May 24, 2016, 6:18 am • 1

Let M, N be the midpoints of AE, AF. Note that since $(A, X; B, C)$ is harmonic, YX is tangent to Γ, so MN, the perpendicular bisector of AX, is the line OY. Now let MN intersect AX at D, and note that $(D, Y; M, N) = (AD, AY; AC, AB) = -1$. Consider an inversion about Y preserving Γ. It fixes A, X and swaps B, C, and we wish to show it swaps P, Z. It suffices to show that the image of γ passes through Z. Let M', N', D' be the inverses of M, N, D. Since $(Y, D; M, N)$ is harmonic, D' is the midpoint of $M'N'$. But since D' is the midpoint of AX, $D' = O$. Thus the center of γ' lies on the perpendicular bisector of $M'N'$, which, since $XZ \parallel MN$, is the perpendicular bisector of XZ. This implies γ' passes through Z, as desired.

Quick Reply

High School Olympiads

On the circumcircle

Reply

Source: EMMO 2016, Sr

Ankoganit

#1 May 7, 2016, 9:45 pm • 1

In $\triangle ABC$, points A_1, B_1, C_1 are feet of alititudes opposite to vertices A, B, C respectively. Let points A_b, A_c be on the lines AB, AC such that $\angle BA_bB_1 = \angle ABC$ and $\angle CA_cC_1 = \angle ACB$. Let $A_bA_c \cap B_1C_1 = X_A$. Analogously define X_B, X_C are defined. Prove that lines AX_A, BX_B, CX_C concur on circumcircle of $\triangle ABC$.

Anant Mudgal, IMOTC-2016, Senior Batch

This post has been edited 1 time. Last edited by Ankoganit, May 8, 2016, 8:33 pm

Luis González

#2 May 8, 2016, 12:17 am • 3

Since $\angle BA_bB_1 = \angle ABC = \angle AB_1C_1$, then $\odot(AA_bB_1)$ is tangent to B_1C_1 and similarly $\odot(AA_cC_1)$ is tangent to B_1C_1. Thus using the result of the problem AF perpendicular to Euler line on $\triangle AB_1C_1$, we deduce that AX_A is perpendicular to the Euler line of $\triangle AB_1C_1$. But since B_1C_1 is antiparallel to BC WRT AB, AC, then it follows that AX_A is the isogonal (WRT $\angle BAC$) of the perpendicular from A to the Euler line OH of $\triangle ABC \Longrightarrow AX_A$ cuts $\odot(ABC)$ again at the anti-Steiner point of OH WRT $\triangle ABC$, i.e. it's Euler reflection point. Similarly BX_B and CX_C pass through this point.

anantmudgal09

#3 May 8, 2016, 6:07 pm

Proposed by Me.

P.S.- Did anyone do it in Jr batch?

Ankoganit

#4 May 8, 2016, 8:34 pm

> *anantmudgal09 wrote:*
> P.S.- Did anyone do it in Jr batch?

I guess you expect a 'no', and your guess is right AFAIK.

anantmudgal09

#5 May 8, 2016, 8:56 pm

> *Ankoganit wrote:*
> > *anantmudgal09 wrote:*
> > P.S.- Did anyone do it in Jr batch?
>
> I guess you expect a 'no', and your guess is right AFAIK.

Hmm, I did expect a No indeed, but expected some reasonable progress since my solution is highly motivated.

WizardMath

#6 May 17, 2016, 7:49 pm

No, Arya solved it using Cevian Nest and some angle chasing(as far as he claims).

Quick Reply

High School Olympiads

AF perpendicular to Euler line

Source: CHKMO 2014

boblimb

#1 Feb 8, 2015, 9:43 pm • 1

Let $\triangle ABC$ be a scalene triangle, and let D and E be points on sides AB and AC respectively such that the circumcircles of triangles $\triangle ACD$ and $\triangle ABE$ are tangent to BC. Let F be the intersection point of BC and DE. Prove that AF is perpendicular to the Euler line of $\triangle ABC$.

This post has been edited 1 time. Last edited by boblimb, Nov 10, 2015, 6:47 pm
Reason: Fix latex

Luis González

#2 Feb 8, 2015, 10:26 pm • 4

Let Q, R be the projections of B, C on $AC, AB. H \equiv BQ \cap CR$ is the orthocenter of $\triangle ABC$. Inversion with center A and power $AB \cdot AR = AC \cdot AQ$ takes BC into $\omega_A \equiv \odot(AQHR)$ and $\odot(ABE), \odot(ACD)$ into the the tangents of ω_A at R, Q. They pass through the midpoint M of BC (well-known) and cut AC, AB at the inverses E', D' of E, D. Thus DE is taken into $\odot(AD'E')$ cutting ω_A again at the inverse F' of F. Now from the problem Concyclic Quadrilateral, F' is on Euler line OH of $\triangle ABC$, i.e. $OHF' \perp AF'F$, as desired.

TelvCohl

#3 Jan 24, 2016, 7:57 pm • 3

Let O, H be the circumcenter, orthocenter of $\triangle ABC$, respectively. Let $\odot(H, HA)$ cuts CA, AB again at Y, Z, respectively. Clearly, YZ is antiparallel to BC WRT $\angle BAC$, so B, C, Y, Z lie on a circle Ω. From $\angle DCB = \angle BAC = \angle CZB$ we get CD is tangent to Ω at C. Similarly, we can prove BE is tangent to Ω at B, so from Pascal theorem (for $BBCCYZ$) $\Longrightarrow F$ lies on YZ, hence F is the radical center of $\Omega, \odot(O), \odot(H, HA) \Longrightarrow AF$ is the radical axis of $\odot(O)$ and $\odot(H, HA) \Longrightarrow$ AF is perpendicular to the Euler line OH of $\triangle ABC$.

Aiscrim

#5 Feb 23, 2016, 6:19 pm • 1

Let D, E, F be the projections of A, B, C on BC, CA, AB and let $\{X_B\} = DF \cap AC,\ \{X_c\} = DE \cap AB$. As $X_BA \cdot X_BC = X_BF \cdot X_BD$, we get that X_B is on the radical axis of the nine point circle and the circumcircle, whence X_BX_C is perpendicular on the Euler line. It is therefore enough to prove that $AF \parallel X_BX_C$, which is a relatively simple bary bash.

Dukejukem

#6 Feb 24, 2016, 8:18 am

Let G, H, N be the centroid, orthocenter, nine-point center of $\triangle ABC$, respectively. Let $\triangle M_aM_bM_c$ be the medial triangle and $\triangle H_aH_bH_c$ be the orthic triangle. Let Y, Z be the reflections of A in H_b, H_c respectively. Denote $P \equiv M_bM_c \cap H_bH_c$.

Lemma: AP is perpendicular to the Euler line e.
Proof: Take $Q \equiv M_bH_c \cap M_cH_b$. Then $Q \in e \equiv GH$ by Pappus' Theorem for $M_bBH_bM_cCH_c$. Hence, $e \equiv QN$, which is perpendicular to AP by Brokard's Theorem for $M_bM_cH_bH_c$. $\blacksquare$

Dilating the above result with center A and ratio 2, it follows that the line connecting A and $F^* \equiv YZ \cap BC$ is perpendicular to e. We will show that $F^* \equiv F$.

Note that $\triangle ABY \sim \triangle ZCA$. Moreover, $\measuredangle BCA = \measuredangle CDB = \measuredangle CDZ$, which entails $ABYC \sim ZCAD$. Therefore, $CA/CY = DZ/DA$. Similarly, $BA/BZ = EY/EA$. Thus, Menelaus' Theorem for $\triangle AYZ$ cut by $\overline{DEF}$ and $\overline{BCF^*}$

yields $F'Y/F'Z = F''Y/F''Z$, i.e. $F'' \equiv F'$, as desired. $\square$

This post has been edited 1 time. Last edited by Dukejukem Feb 24, 2016, 8:21 am

Complex2Liu

#7 Feb 25, 2016, 9:29 pm

Here is my solution with complex bashing. Let x, y, z denote the length of $\overline{AB}, \overline{BC}, \overline{CA}$, and a, b, c be the coordinate of A, B, C respectively.

Applying Menelaus's Theorem from DEF and $\triangle ABC$ yields that

$$\frac{BD}{DA} \cdot \frac{AE}{EC} \cdot \frac{CF}{FB} = 1. \qquad (1)$$

Since $\overline{BC}$ is tangent to $\odot(ADC)$, so $BC^2 = BD \cdot BA \implies BD = \frac{y^2}{x}, DA = \frac{x^2-y^2}{x} \implies \frac{BD}{DA} = \frac{y^2}{x^2-y^2}$, similarly $\frac{AE}{EC} = \frac{y^2-z^2}{y^2}$. Combining (1) we get $\frac{CF}{FB} = \frac{x^2-y^2}{y^2-z^2}$.

Clearly $x^2 = (a-b)(\overline{a}-\overline{b}) = 2 - \frac{a^2+b^2}{ab}$, similarly $y^2 = 2 - \frac{b^2+c^2}{bc}, z^2 = 2 - \frac{c^2+a^2}{ac}$. Thus we can calculate the coordinate of F

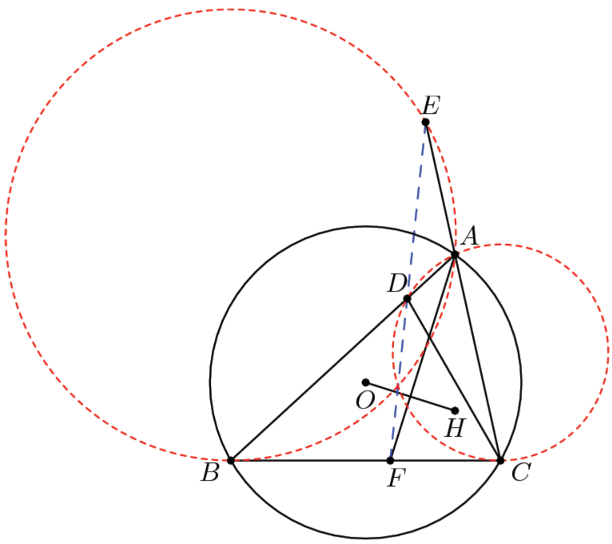
$$f = \frac{x^2-y^2}{x^2-z^2}b + \frac{y^2-z^2}{x^2-z^2}c = \frac{(a-c)(b^2-ac)}{(b-c)(a^2-bc)}b + \left(1 - \frac{(a-c)(b^2-ac)}{(b-c)(a^2-bc)}\right)c. \qquad (2)$$

Define $f' = kb + (1-k)c$ is a point on $\overline{BC}$ such that $AF' \perp OH$, where k is a constant. It suffices to prove that $f \equiv f'$. We have

$$\begin{aligned} AF' \perp OH \iff 0 &= \frac{a-f'}{a+b+c} + \overline{\left(\frac{a-f'}{a+b+c}\right)} \\ &= \frac{a-kb-c+kc}{a+b+c} + \frac{\frac{1}{a} - \frac{k}{b} - \frac{1}{c} + \frac{k}{c}}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \\ &= \frac{(a-kb-c+kc)\cdot(ab+bc+ca)}{(ab+bc+ca)(a+b+c)} + \frac{(bc-kac-ab+kab)\cdot(a+b+c)}{(ab+bc+ca)(a+b+c)} \\ &= \frac{k(a^2b-a^2c+bc^2-b^2c)-(ab^2+ac^2-b^2c-a^2c)}{(a+b+c)(ab+bc+ca)} \\ &= \frac{k(a^2-bc)(b-c)-(b^2-ac)(a-c)}{(a+b+c)(ab+bc+ca)} \end{aligned}$$

Hence $k = \frac{(b^2-ac)(a-c)}{(a^2-bc)(b-c)}$. Compare with (2) we conclude that $f \equiv f'$, as desired. We're done.

A diagram below:



EulerMacaroni

#8 May 8, 2016, 1:34 am

Notice that the problem is equivalent to showing that line AF is the radical axis of the circles with diameters $\overline{AH}$ and $\overline{AO}$. Consider the transformation which is a composition of inversion about A with power $r^2 = AB \cdot AC$, reflection about the A-angle bisector, and dilation from A with ratio $\frac{1}{2}$. We can now recast the problem as the following:

In triangle ABC, suppose the tangents to $\odot(ABC)$ at B and C intersect AC and AB at D and E, respectively. Furthermore, the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect AC and AB at P and Q, respectively. If $\odot(ADE)$ intersects $\odot(ABC)$ again at F, show that lines AF, PQ, and BC are concurrent.

Since $\angle BPC = 180^\circ - 2\angle BAC = \angle BQC$, it follows that $\odot(BQPC)$ is cyclic, and $X \equiv BD \cap CE$ lies on this circle. By angle chasing, we get that $PX \parallel BC$ and $QX \parallel AC$. Then

$$(D, C; P, P_{AC\infty}) \stackrel{X}{=} (B, E; P_{AB\infty}, Q) \implies \frac{PC}{PD} = \frac{QB}{QE}$$

which implies that F is the spiral center sending DE to PQ, hence $\odot(AFPQ)$ is cyclic. The conclusion now follows from the radical axis theorem on $\odot(ABC)$, $\odot(BPQC)$, and $\odot(AFPQ)$.

Quick Reply

High School Olympiads

Concurrency from exterior equilateral triangles

Reply

Source: Baltic Way 2003

WakeUp

#1 Nov 7, 2010, 8:22 pm

Equilateral triangles AMB, BNC, CKA are constructed on the exterior of a triangle ABC. The perpendiculars from the midpoints of MN, NK, KM to the respective lines CA, AB, BC are constructed. Prove that these three perpendiculars pass through a single point.

Luis González

#2 Nov 7, 2010, 11:48 pm

Let L be the reflection of N across line BC. Since $\angle LBC = \angle MBA = 60^\circ$, we deduce that $\angle MBL = \angle ABC$ and because of $MB = MA$ and $BL = BC$, it follows that $\triangle ABC \cong \triangle MBL$ by SAS criterion $\Longrightarrow ML = AC = AK$. Similar reasoning yields $KL = AM \Longrightarrow AMLK$ is a parallelogram, thus AL and MK bisect each other at U. If H_a, M_a denote the foot of the A-altitude and midpoint of BC, then perpendicular ℓ_a from U to BC is the midline of the trapezoid $ALM_aH_a \Longrightarrow \ell_a$ passes through the 9-point center of $\triangle ABC$. Likewise, perpendiculars from midpoints of MN, NK to CA, AB pass through the 9-point center of $\triangle ABC$.

P.S. The result is still true for similar isosceles triangles AMB,BNC,CKA and the proof is similar.

Quick Reply

High School Olympiads

Similar Isosceles Triangles Median Concurrence

Reply

Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Jan 27, 2013, 6:41 am

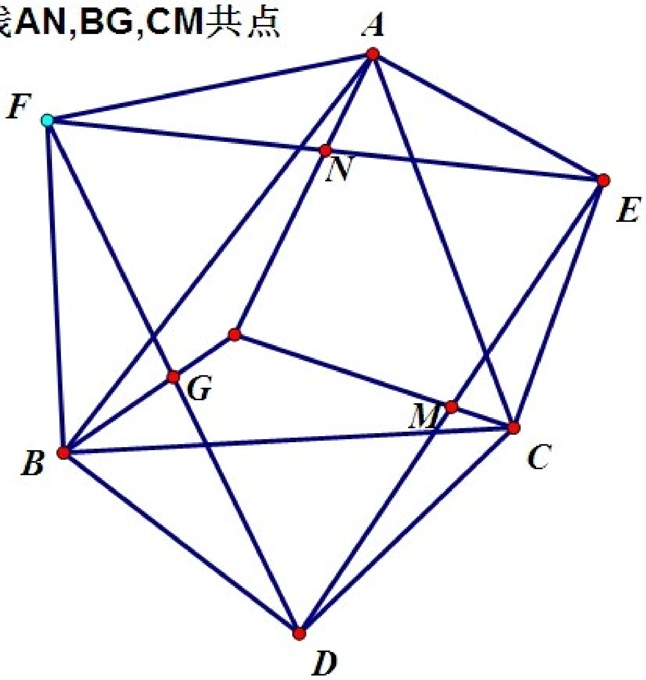
See Attachment.
This problem is proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:

已知（文武光华数学工作室 南京 潘成华）△AFB~△AEC ~BDC,AF=BF,点,M,N,G分别是ED,EF,DF中点

(2013 1 17 19:46)

求证 直线AN,BG,CM共点



△AFB~△AEC~△BDC, AF=BF,M,N,G are the midpoints of ED,EF,DF.
Prove: AN,BG,CM are concurrent.

Luis González

#2 Jan 27, 2013, 8:58 am

Let D', E', F' be the reflections of D, E, F about BC, CA, AB. $\triangle FAB$ and $\triangle D'CB$ are then spirally similar with center $B \Longrightarrow \triangle FBD'$ and $\triangle ABC$ are also spirally similar with center $B \Longrightarrow \frac{FD'}{AC} = \frac{FB}{AB} = \frac{AE}{AC} \Longrightarrow FD' = AE$. Similarly, $ED' = AF \Longrightarrow AED'F$ is a parallelogram with diagonal intersection $N \equiv AD' \cap EF$. Likewise, BG and CM pass through E' and F', thus $AN \equiv AD', BG \equiv BE'$ and $CM \equiv CF'$ concur at the inner Kiepert perspector of $\triangle ABC$ relative to $\angle DBC = \angle ECA = \angle FAB$.

Quick Reply

High School Olympiads

Kiepert center lies on the Euler line of Kiepert triangle

Reply

Source: Own

TelvCohl

#1 May 4, 2016, 8:53 pm • 2

Given a $\triangle ABC$ and an acute angle θ. Construct isosceles triangles $\triangle BDC, \triangle CEA$ and $\triangle AFB$ with base angle θ on the sides of $\triangle ABC$. Let H_D, H_E, H_F be the orthocenter of $\triangle BDC, \triangle CEA, \triangle AFB$, respectively. Prove that AH_D, BH_E, CH_F and the Euler line of $\triangle DEF$ are concurrent.

Scorpion.k48

#2 May 4, 2016, 10:22 pm • 1

Solution:
Let X, Y, Z be the circumcenter of $\triangle AFB, \triangle AEC, \triangle BH_DC$. We have $(XB, XA) \equiv (YA, YC) \equiv (ZB, ZC) \equiv 2\theta(mod\pi)$ and $\triangle AXB, \triangle AYC, \triangle BZC$ are isosceles, so $\triangle XBA \overset{+}{\sim} \triangle ZBC \overset{+}{\sim} \triangle YAC$. Therefore, $\triangle XBZ \overset{+}{\sim} \triangle ABC \overset{+}{\sim} \triangle YZC$. Because $ZB = ZC$, we have $\triangle XBZ \overset{+}{=} \triangle YZC$.
Therefore, $ZY = XB = XA$ and $XZ = YC = YA$. We have $AXZY$ is parallelogram.
We have $(XF, XZ) \equiv (XF, XA) + (XA, XZ) \equiv (YA, YE) + (YZ, YA) \equiv (YZ, YE)(mod\pi)$ and $XF = YZ, XZ = YE$, so $\triangle XFZ \overset{+}{=} \triangle YZE$ and we have $ZF = ZE$. Let O, H be circumcenter, orthocenter of $\triangle DEF$, we have $ZO \perp EF, DH \perp EF$
We have a **Lemma:** $AH_D \perp EF$. So $ZO//DH//AH_D$. Let $S = AH_D \cap OH$ and X', Y' be circumcenter of $\triangle AH_FB, \triangle AH_EC$. Because $\triangle BDC \overset{+}{\sim} \triangle CEA \overset{+}{\sim} \triangle AFB$, so $\frac{\overline{ZH_D}}{\overline{ZD}} = \frac{\overline{Y'H_E}}{\overline{Y'E}} = \frac{\overline{X'H_F}}{\overline{X'F}} = k$. Using Thales $\frac{\overline{OS}}{\overline{OH}} = \frac{\overline{ZH_D}}{\overline{ZD}} = k$. Similarly, we have BH_E, CH_F goes through S and we have done.

This post has been edited 1 time. Last edited by Scorpion.k48, May 4, 2016, 10:32 pm

Luis González

#3 May 5, 2016, 10:55 am • 2

Let X, Y, Z be the midpoints of BC, CA, AB. If AH_D cuts $\odot(BCH_D)$ again at J, we get $\angle BJH_D = \angle CJH_D = \angle BDX$. If the parallels from Y, Z to JC, JB meet at L, then $\triangle LYZ$ and $\triangle JCB$ are homothetic with center $A \Longrightarrow L \in AH_D$ and $\angle ZLJ = \angle YLJ = \angle BJH_D = \angle BDX = \angle AEY = \angle AFZ \Longrightarrow AEYL$ and $AFZL$ are cyclic $\Longrightarrow L$ coincides with the projection of A on $EF \Longrightarrow AH_D \perp EF$.

On the other hand, it's known that $\triangle ABC$ and $\triangle DEF$ have the same centroid G (see prove that two triangles share their centroid), thus if T is the orthocenter of $\triangle DEF$, then TG is Euler line of $\triangle DEF$ cutting AH_D at P. If $U \equiv AX \cap DT$, then from $(DT \parallel H_DA) \perp EF$, we obtain

$$k = \frac{XD}{DH_D} = \frac{UX}{AU} = \frac{\frac{1}{2}AG + UG}{AU} = \frac{1}{2} \cdot \frac{PG}{PT} + \frac{TG}{PT} = \frac{1}{2} + \frac{3}{2} \cdot \frac{TG}{PT} \Longrightarrow \frac{TG}{PT} = \frac{2}{3}\left(k - \frac{1}{2}\right).$$

Since k is clearly a constant, then it follows that BH_E and CH_F hit TG at the same point P.

Luis González

#4 May 6, 2016, 11:21 pm

Another property of the Kiepert's perspector $K(90^\circ - \theta) \equiv AH_D \cap BH_E \cap CH_F$. This point is collinear with the Kiepert's perspector $K(-\theta)$ and the 9-point center N of $\triangle ABC$.

Proof: Let A', B', C' be the midpoints of EF, FD, DE, respectively. From Concurrency from exterior equilateral triangles (see post #2), we get that the perpendiculars from A', B', C' to BC, CA, AB concur at the 9-point center N of $\triangle ABC$ and from Similar Isosceles Triangles Median Concurrence, AA', BB', CC' concur at $K(-\theta)$. Thus $\triangle ABC$ and $\triangle A'B'C'$ are orthologic with orthology centers $N, K(90^\circ - \theta)$ and perspective with perspector $K(-\theta)$. By Sondat's theorem, $N, K(-\theta)$ and $K(90^\circ - \theta)$ are collinear.

Quick Reply

High School Olympiads

Nine-point center lies on Euler line

Reply

Scorpion.k48

#1 May 6, 2016, 8:17 pm

Let $\triangle ABC$ with nine-point center N. Let $\triangle N_aN_bN_c$ is an antipedal triangle of N WRT $\triangle ABC$. Prove that nine-point center of $\triangle N_aN_bN_c$ lies on Euler line of $\triangle ABC$

Luis González

#2 May 6, 2016, 8:36 pm • 1

This was discussed before at Nine-point Center Lies On OI. See lemma 2 at post #7 and the previous lemma in my solution at post #12.

Quick Reply

High School Olympiads

Nine-point Center Lies On OI

Reply

Arab

#1 Aug 20, 2013, 6:20 pm

Let O, I be the circumcenter and incenter of $\triangle ABC$ and $D = AI \cap BC, E = BI \cap CA, F = CI \cap AB.$ Then the nine-point center of $\triangle DEF$ lies on $OI.$

Or

Let $\triangle A'B'C'$ be the pedal triangle of $\triangle ABC$ where $A' \in BC, B' \in CA, C' \in AB.$ If $X = AA' \cap B'C'$, $Y = BB' \cap C'A', Z = CC' \cap A'B'$, then the nine-point center of $\triangle XYZ$ lies on the Euler Line of $\triangle ABC.$

Is it well-known?

jayme

#2 Aug 20, 2013, 9:09 pm

Dear Mathlinkers,
I think through my tool that there is a typo... you mind perhaps that DEF is the i-pedal triangle of ABC
Sincerely
Jean-Louis

Math-lover123

#3 Aug 20, 2013, 11:30 pm

If $\triangle DEF$ is intouch triangle of ABC then the result is well known and can be proved using homotethy or inversion.

XmL

#4 Aug 21, 2013, 5:06 am • 1

> Arab wrote:
> Let O, I be the circumcenter and incenter of $\triangle ABC$ and $D = AI \cap BC, E = BI \cap CA, F = CI \cap AB.$ Then the nine-point center of $\triangle DEF$ lies on $OI.$
>
> Or
>
> Let $\triangle A'B'C'$ be the pedal triangle of $\triangle ABC$ where $A' \in BC, B' \in CA, C' \in AB.$ If $X = AA' \cap B'C'$, $Y = BB' \cap C'A', Z = CC' \cap A'B'$, then the nine-point center of $\triangle XYZ$ lies on the Euler Line of $\triangle ABC.$
>
> Is it well-known?

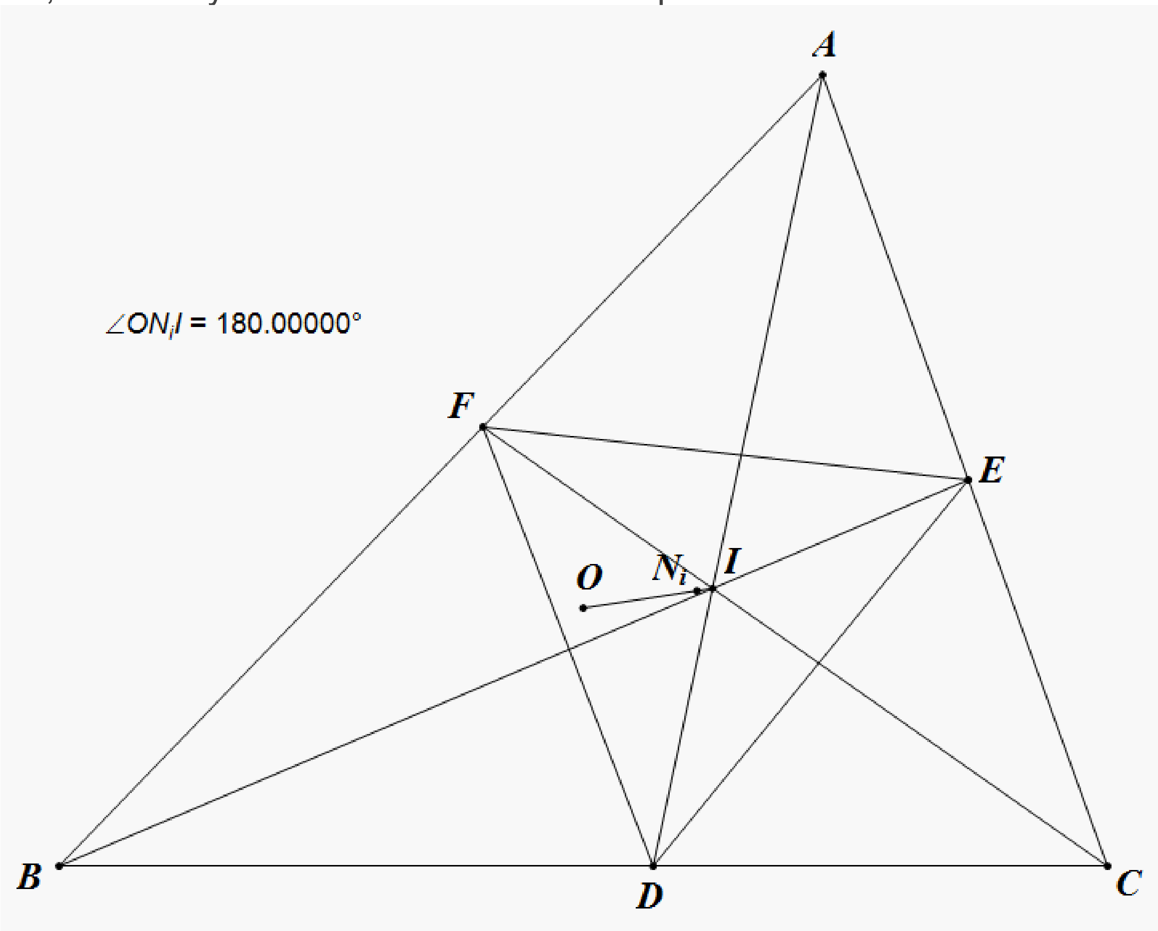
The first one is one of the equivalences I talked about! I will try these again.

Arab

#5 Aug 21, 2013, 6:33 am

@jayme and all mathlinkers
There are no typos.Please have a check again.

@XmL
Oh,that's really nice.In fact these two are equal.



duanby
#6 Nov 10, 2013, 1:33 pm • 1

Here's my solution:
use the first notion, let L, M, N be the midpoint of EF, DF, DE
to prove the nine point center of $\triangle ABC$ collinear with O, I
Use barycentric coordinate
it's sufficient to prove the radical axis of $(O), (DEF)$ is paraell to that of $(O), (I)$
the coordinate $D(0, b, c)\, E(a, 0, c)\, F(a, b, 0)$
$L(a(2a+b+c), b(a+c), c(a+b))$
$M(a(b+c), b(a+2b+c), c(a+b))$
$N(a(b+c), b(a+c), c(a+b+2c))$
Assume $(LMN) : -\sum a^2yz + (x+y+z)(\frac{\alpha}{a}x + \frac{\beta}{b}y + \frac{\gamma}{c}z) = 0$
The radical axis of $(O), (I) : \sum (s-a)^2x = 0$
It's sufficient to prove $\sum_{cyc} \alpha(b-c) = 0$
For α, β, γ we have
$2(a+b)(a+c)[\alpha(2a+b+c) + \beta(a+c) + \gamma(a+b)]$
$= abc[a(a+c)(a+b) + b(a+b)(2a+b+c) + c(a+c)(2a+b+c)]$ and two other similar equation by cycling the letter
Or in short,

$$\alpha(2a+b+c) + \beta(a+c) + \gamma(a+b) = P$$

$$\alpha(b+c) + \beta(a+2b+c) + \gamma(a+b) = Q$$

$$\alpha(b+c) + \beta(a+c) + \gamma(a+b+2c) = R$$

$\implies 2(a+b)(b+c)(c+a)\alpha = P(b+c)(a+b+c) - Qc(a+c) + Rb(a+b)$

Therefore $\sum_{cyc} \alpha(b-c)=0$

$\iff \sum_{cyc} (b-c)(b+c)(a+b+c)P - Qc(a+c)(b-c) - Rb(a+b)(b-c)=0$

$\iff \sum_{cyc} P(b+c)(b^2-c^2)=0$

$\iff \sum_{cyc} (b+c)^2(b^2-c^2)[a(a+c)(a+b)+b(a+b)(2a+b+c)+c(a+c)(2a+b+c)]=0$

$\iff \sum_{cyc}(a^3b^4-b^3c^4)+(b^4c^3-a^3c^4)+(3a^2b^5-3b^2c^5)+(3c^2b^5-3a^2c^5)+(3b^6c-3ac^6)+(3ab^6-3bc^6)+(b^7-c^7)+(9a^2b^4c-9ab^2c^4)+$
$(9ab^4c^2-9a^2bc^4)+(2a^3b^3c-2a^3bc^3)+(6a^2b^3c^2-6a^2b^2c^3)+(9ab^5c-9abc^5)=0$

which is true.

IDMasterz

#7 Jul 17, 2014, 2:20 am • 10

XML and I have found a synthetic solution to this problem, and will be outlined in an article we hope to be completed in due course.

Lemma 1:

Let ABC be a triangle with H its orthocentre, O its circumcentre, K its kosnita point, and N is its ninepoint centre. Let $BN \cap AC = N_B, CN \cap AB = N_C$, and $N_BN_C \cap BC = A_1$. Let K_A be the reflection of K over BC. Then, $HK_A \perp AA_1$.

Proof:

Since $BH \perp AC$, therefore it suffices to prove $\angle(HK_A, HC) = \angle(AA_1, AB)$. Let H' denote the reflection of H over BC, which lies on (O), define $H'K \cap (O) = A_2, AK \cap (O) = A_3$, we will prove that A_2CA_3B is a harmonic quadrilateral.

Let O_B, O_C denote the reflections of O over AC, AB respectively. It's well known that N is the midpoint of BO_B, CO_C, hence BCO_BO_C is a parallelogram. Consider the isogonal conjugate of K wrt CBH', denoted by K'. Since $\angle CBH' = 90 - \angle C = \angle ABO_C, \angle K'BH' = \angle KBC = \angle ABN$, therefore $\angle CBK' = \angle ABN \Rightarrow BK', BO_C$ are isogonals wrt $\angle NBC$. Symmetrical result can be obtain on $\angle NCB$, and since $BO_C \parallel CO_B \parallel ND$ where D is the midpoint of BC, therefore the isogonal conjugate of K' lies on ND and moreover $K' \in (NBC)$. Hence ND, NK' are isogonals wrt $\angle BNC \Rightarrow BNCK'$ is a harmonic quadrilateral.

Let $AN \cap (O) = A_4, K'H' \cap (NBC) = K*$. Note that $\angle(AA_4, H'A_4) = \angle(AC, CH') = \angle(NC, CK') = \angle(NK*, K*K')$, therefore $N, A_4, K*, H'$ are concyclic. By the radical axes theorem among $(O), (NBC), (NA_4H'K*)$, $NK*, H'A_4$ concur on BC. This means $H'(C, B; K', A_4) = K*(C, B; K', N) = -1$ since $BHCK'$ is harmonic. Now apply a reflection over the angle bisector of $BH'C$. We have $H'(C, B; A_2, A_3) = -1 \Rightarrow A_2CA_3B$ is a harmonic quadrilateral.

Now A_2CA_3B is a harmonic quadrilateral$\Rightarrow$ $(AC, AB; AK, AA_2) = -1$. When we reflect these pencils over the angle bisector of $\angle BAC$, $AB \to AC, AC \to AB, AK \to AN$. Let $AA_2 \to j$. Since the reflection preserves the cross ratio of these pencils, therefore $(AC, AB; AN, j) = -1$ Since by properties of complete quadrilaterals $(AC, AB; AN, AA_1) = -1$, therefore $j \equiv AA_1 \Rightarrow AA_2, AA_1$ are isogonals wrt $\angle BAC$ $\Rightarrow \angle(HK_A, HC) = \angle(CH', H'A_2) = \angle(AC, AA_2) = \angle(AA_1, AB)$ and we are done.

Corollary:

In the same configutation, define B_1 similarly to A_1, and K_B similarly to K_A. If the perpendiculars form O to AA_1, BB_1 intersect NK_A, NK_B at X, Y, then $XY \parallel K_AK_B$.

Lemma 2:

Let ABC be a triangle and N its ninepoint centre, and its antipedal triangle be $A^*B^*C^*$. The ninepoint centre, N', of $A^*B^*C^*$ lies on the euler line of ABC.

Proof:

Let $AN \cap \odot NBC = T, BN \cap \odot NAC = U, CN \cap NAB = V$. We have $\angle ATA^* = \frac{\pi}{2} = \angle(AN, B^*C^*) \implies TA^* \parallel B^*C^*$. Let $M_AM_BM_C$ be the medial triangle $A^*B^*C^*$, and notice that N' is the centre of $M_AM_BM_C$. Consider the circles with diametres NM_A, NM_B, NM_C, and denote them by $\Gamma_A, \Gamma_B, \Gamma_C$. Notice the circles contain A, B, C respectively by definition. Let $\Gamma_A \cap \odot NBC = P$. As $A^*T \parallel AM_A$, by the converse of Reim's theorem, $P \in A^*M_A$.

It follows that $(A*M, A^*T, A^*B^*, A^*C^*) = -1 \implies PBTC$ is a harmonic quadrilateral. Thus,

$(NP, NT, NB, NC) = -1 \implies NP$ intersects BC at the trillinear polar of N with respect to ABC. By radical axis theorem on $\Gamma_A, \odot NBC, ABC$, it follows that if Γ_A intersects $\odot ABC$ at A_2, then $AA_2 \cap NP \in BC = A_1$ where A_1 is defined as in lemma 1.

Suppose we let O_1, O_2, O_3 be the centres of $\Gamma_A, \Gamma_B, \Gamma_C$. The homothety centered at N with ratio 2 takes $O_1O_2O_3 \mapsto M_AM_BM_C$, so if we let the centre of $O_1O_2O_3$ be O_4, we have $O_4 \mapsto O$. Clearly, it suffice to prove that O_4 lies on the euler line of ABC. Due to AA_1 being the radical axis of ABC and Γ_A, it follows if O is the centre of ABC that $OO_1 \perp AA_1$ and similarly, $OO_2 \perp BB_1, OO_3 \perp CC_1$.

If K_A is defined as in lemma 1 and similarly K_B, K_C, then $K_AK_BK_C$ is homothetic to $O_1O_2O_3$, and it is well known that N is the circumcentre of $K_AK_BK_C$ meaning $O_4O_1O_2$ is therefore homothetic to NK_AK_B.

Let the centre of homothety be denoted by H_O. Obviously, $H_O = O_1K_A \cap O_2K_B$, H_O lies on the euler line of ABC. Considering triangles OO_1O_2 and NK_BK_C, it suffice to show $NO \cap K_AO_1 \cap K_BO_2$ exists, or that the triangles are perspective from a point. By Desargues' theorem, it suffice to show that these triangles are perspective from a point, or indeed that $OO_1 \cap NK_A = X', OO_2 \cap NK_B = Y', K_AK_B \cap O_1O_2 = P_\infty$ are collinear, or that $X'Y' \parallel K_AK_B$. But as $OO_1 \perp AA_1, OO_2 \perp BB_1$, $X' = X, Y = Y'$, so we are done the corollary.

Lemma 3:

Let $\triangle A'B'C'$ be the incentral triangle of the orthic triangle of ABC. Prove that the center of homothety of $\triangle A'B'C', \triangle A^*B^*C^*$ (A^* is defined as in previous lemmas) lies on the Euler line of $\triangle ABC$.

Proof:

It suffices to prove $C'C^*, B'B^*, HN$ concur. Let $NB^* \cap BH = E, NC^* \cap CH = F$. By Desargue's theorem on $\triangle HB'C', \triangle NB^*C^*$, the concurrence is equivalent to $EF, B'C', B^*C^*$ concur or simply $EF \parallel B^*C^*$.

Let the lines through B^*, C^* perpendicular to AC, AB respectively meet at M, hence N, M are isogonal conjugates with respect to $A^*B^*C^* \Rightarrow$ the circumcenter of ABC, O lies on NM(well known property of isogonal conjugates) $\Rightarrow M$ lies on the Euler line of ABC.

Since H, N, M are collinear, therefore $\dfrac{NE}{NB^*} = \dfrac{HN}{MN} = \dfrac{NF}{NC^*} \Rightarrow EF \parallel B^*C^*$ and we are done.

Main Proof:

Combining both lemmas 2.5 and 2.4 means the ninepoint centre of $A'B'C'$ lies on the euler line of ABC, as a homothety sends ninepoint centre to ninepoint centre. The orthocentre H of ABC is the incentre of orthic triangle, and the ninepoint centre of ABC is the circumcentre of the orthic triangle. Thus, taking the orthic triangle of ABC as a reference, we have the ninepoint centre of the incentral triangle lies on the line adjoining the circumcentre and incentre, which is the isogonal conjugate of the feurerbach hyperbola.

The problem is solved.

This post has been edited 4 times. Last edited by IDMasterz, Aug 17, 2014, 5:57 pm

XmL

#8 Jul 17, 2014, 2:37 am • 1

IDMasterz is too pro.

This post has been edited 2 times. Last edited by XmL, Jul 17, 2014, 6:23 am

IDMasterz

#9 Jul 17, 2014, 2:40 am • 1

I love you man, just remember that you're more pro though.

Arab

#10 Jul 18, 2014, 6:56 pm • 2

I love you two!Thank you for the exquisite work!

XmL

#11 Aug 16, 2014, 9:01 am • 2

I've found a second synthetic proof for this while investigating the nine-point center, it's now posted on my blog

Luis González

#12 Sep 25, 2015, 5:37 am • 2

Lemma: N is the 9-point center of $\triangle ABC$ and $\triangle N_aN_bN_c$ is the antipedal triangle of N WRT $\triangle ABC.$ Then the 9-point center of $\triangle N_aN_bN_c$ is on the Euler line of $\triangle ABC.$

Proof: Let O, K be the circumcenter and Kosnita point of $\triangle ABC$ (isogonal conjugate of N WRT $\triangle ABC$). $\triangle K_aK_bK_c$ is the pedal triangle of K WRT $\triangle ABC$ with circumcenter U (midpoint of NK) and orthocenter $T.$ Midpoint S of UT is then 9-point center of $\triangle K_aK_bK_c.$ From the thread HK passes through the circumcenter, we get that O, K, T are collinear such that $OK = 2 \cdot KT,$ hence if X is midpoint of $OK,$ we have $KS \parallel UX \parallel ON.$

On the other hand, if J is the isogonal conjugate of N WRT $\triangle N_aN_bN_c$ (reflection of N on O), we have $(K_bK_c \parallel N_bN_c) \perp AN, (KK_a \parallel JN_a) \perp BC,$ etc $\Longrightarrow \triangle K_aK_bK_c \cup K \sim \triangle N_aN_bN_c \cup J$ are homothetic. Thus if V is the 9-point center of $\triangle N_aN_bN_c \Longrightarrow JV \parallel KS \parallel ON \Longrightarrow V \in ON,$ i.e. V is on Euler line of $\triangle ABC.$

Back to the problem, let $\triangle I_aI_bI_c$ be the excentral triangle of $\triangle ABC$ with circumcenter $Be \in OI;$ the Bevan point of $\triangle ABC.$ $\triangle O_aO_bO_c$ is the antipedal triangle of O WRT $\triangle I_aI_bI_c$ and J is the isogonal conjugate of O WRT $\triangle O_aO_bO_c$ (reflection of O on Be). Since $OI_a \perp EF, OI_b \perp FD, OI_c \perp DE$ (well-known), then it follows that $\triangle O_aO_bO_c$ are $\triangle DEF$ are homothetic having parallel sides. Moreover $(AI \parallel JO_a) \perp I_bI_c,$ etc $\Longrightarrow \triangle DEF \cup I \sim \triangle O_aO_bO_c \cup J \Longrightarrow$ OIJ is double under the homothety that takes $\triangle DEF$ into $\triangle O_aO_bO_c,$ i.e. it contains their homothety center. Using the previous lemma for $\triangle I_aI_bI_c$ with 9-point center $O,$ we get that the 9-point center of $\triangle O_aO_bO_c$ is on its Euler line $OI \Longrightarrow$ 9-point center of $\triangle DEF$ is on OI as well.

Quick Reply

High School Olympiads

prove that two triangles share their centroid

Reply

minhtue0605
#1 Jun 29, 2012, 6:15 pm

Let be given triangle ABC. X, Y, Z are points outside triangle ABC such that $\widehat{XBC} = \widehat{YCA} = \widehat{ZAB}$ and $\widehat{XCB} = \widehat{YAC} = \widehat{ZBA}$. If G is the centroid of triangle ABC, prove that G is also the centroid of triangle XYZ.

Luis González
#2 Jun 30, 2012, 1:05 am

M is the midpoint of BC and P is the reflection of X on $M.$ $BPCX$ is obviously a parallelogram $\Longrightarrow$ $\angle XCB = \angle PBC = \angle ZBA \Longrightarrow \angle ABC = \angle ZBL.$ But $\triangle ABZ \sim \triangle BCX$ gives $\frac{BZ}{BP} = \frac{BZ}{CX} = \frac{BA}{BC}$, thus $\triangle ABC \sim \triangle ZBL$ by SAS $\Longrightarrow \frac{PZ}{CA} = \frac{BZ}{BA} = \frac{AY}{CA} \Longrightarrow AY = PZ.$ Similarly, $AZ = PY.$ Thus $AYPZ$ is a parallelogram $\Longrightarrow PA$ cuts YZ at its midpoint $N.$ Therefore, $L \equiv AM \cap XN$ is the centroid of $\triangle PAX \Longrightarrow \overline{LM} : \overline{LA} : -1 : 2$ and $\overline{LN} : \overline{LX} : -1 : 2 \Longrightarrow L \equiv G$ is the centroid of $\triangle ABC$ and $\triangle XYZ.$

Quick Reply

High School Olympiads

Perpendicular lines

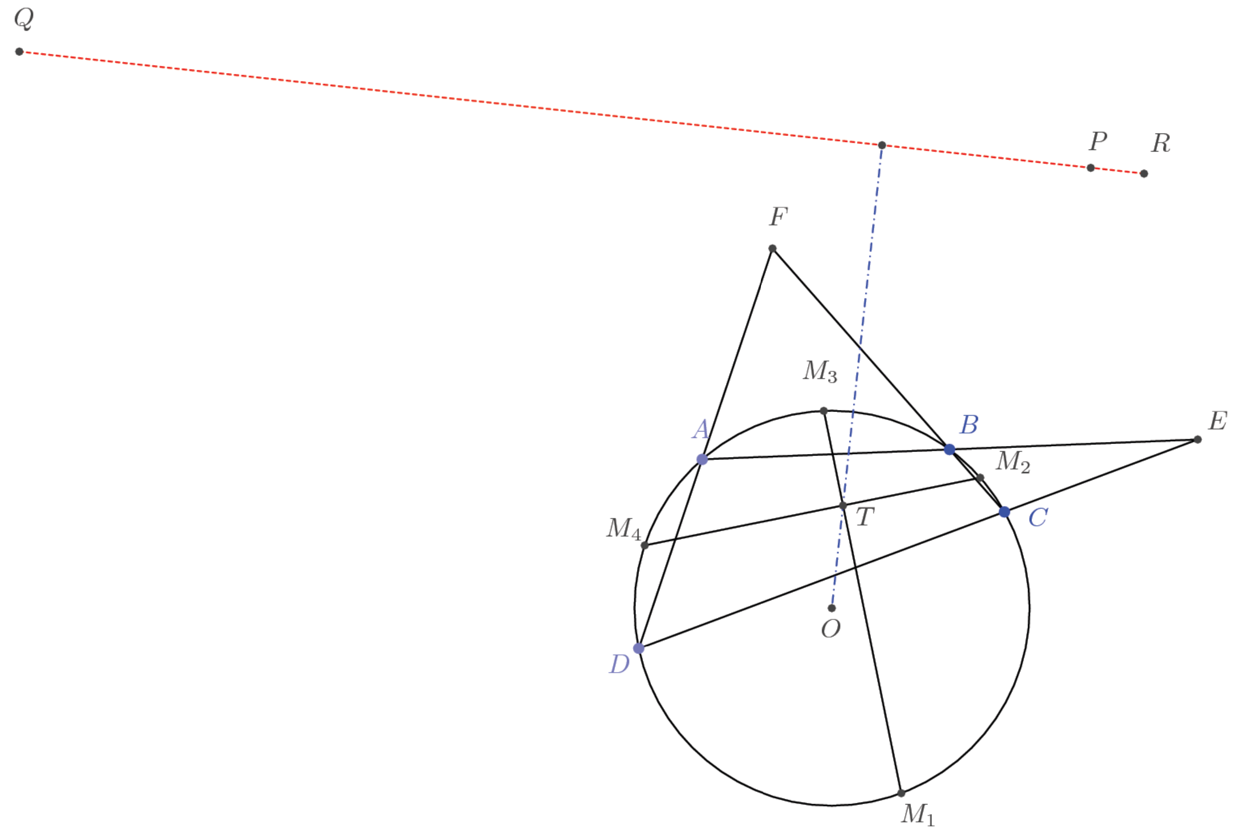
Reply

Source: Own

andria

#1 May 2, 2016, 1:59 pm

Let $ABCD$ be a cyclic quadrilateral with circumcenter O. Let $AB \cap CD = E$ and $AD \cap BC = F$. External bisectors of $\angle DAB$ and $\angle DCB$ intersect at P, external bisectors of $\angle ABC$ and $\angle ADC$ intersect at Q and external bisectors of $\angle AED$ and $\angle AFB$ intersect at R. Let M_1, M_2, M_3, M_4 be midpoints of inner arcs CD, BC, AB, AD respectively. Let $M_1M_3 \cap M_2M_4 = T$.
Prove that P, Q, R lie on a line perpendicular to OT.



Luis González

#2 May 2, 2016, 8:02 pm • 1

Let I and J the incenters of $\triangle FAB$ and $\triangle EBC$ and let K and L be the F-excenter and E-excenter of $\triangle FCD$ and $\triangle EDA$. Let $S \equiv IK \cap JL$. For any quadrilateral $ABCD$ (not necessarily cyclic) P, Q, R are collinear on the polar of S WRT $\odot(IJKL)$ (see Prove that P, Q , R are collinear, post #6). Hence if U is the center of $\odot(IJKL)$, then $US \perp \overline{PQR}$.

Particularly, when $ABCD$ is cyclic, easy angle chase reveals that M_1M_3, M_2M_4 are parallel to the internal bisectors IK, JL of $\angle AFB, \angle BEC$ and M_1M_2 is parallel to the external bisector JK of $\angle BCD$, etc. Thus $M_1M_2M_3M_4 \cup O$ and $KJIL \cup U$ are homothetic $\Longrightarrow OT \parallel US \Longrightarrow OT \perp \overline{PQR}$.

andria

#3 May 2, 2016, 8:19 pm

My solution:
Since the sum of distances of P, Q, R from the sides of $ABCD$ is zero from Distance sum (post 8) we get that they lie on a line perpendicular to $\frac{\overrightarrow{OM_4}}{|OM_4|} + \frac{\overrightarrow{OM_3}}{|OM_3|} + \frac{\overrightarrow{OM_2}}{|OM_2|} + \frac{\overrightarrow{OM_1}}{|OM_1|} \parallel OT$.

Q.E.D

This post has been edited 1 time. Last edited by andria, May 2, 2016, 8:22 pm

Quick Reply

High School Olympiads

Prove that P, Q , R are collinear

Reply

Amir Hossein
#1 May 18, 2010, 1:08 am

Let $ABCD$ be a quadrilateral, and E, F be intersection points of AB, CD and AD, BC respectively.
External bisectors of $D\hat{A}B$ and $D\hat{C}B$ intersect at P, external bisectors of $A\hat{B}C$ and $A\hat{D}C$ intersect at Q and external bisectors of $A\hat{E}D$ and $A\hat{F}B$ intersect at R. Prove that P, Q, R are collinear.

yetti
#2 May 18, 2010, 10:59 am • 1

Let external bisectors a, b, c, d of the angles $\widehat{DAB}, \widehat{ABC}, \widehat{BCD}, \widehat{CDA}$ pairwise intersect at $X \equiv a \cap b, Y \equiv b \cap c, Z \equiv c \cap d, T \equiv d \cap a.$ Likewise, let internal bisectors a', b', c', d' of these angles pairwise intersect at $X' \equiv a' \cap b', Y' \equiv b' \cap c', Z' \equiv c' \cap d', T' \equiv d' \cap a'.$ Let e, f be external and e', f' internal bisectors of the angles $\widehat{DEA}, \widehat{AFB}.$ Points $E, Y, Y', T', T \in e'$ are collinear as incenters/E-excenters of $\triangle ECB, \triangle EDA$ and similarly, points F, Z, Z', X', X are collinear as incenters/F-excenters of $\triangle FDC, \triangle FAB.$ Thus the quadrilaterals $XYZT$ and $X'Y'Z'T'$ have common diagonals $XZ \equiv X'Z'$ and $YT \equiv Y'T'$ intersecting at $S.$ Intersections $I \equiv a' \cap b \equiv T'X' \cap XY,$ $J \equiv a \cap b' \equiv X'Y' \cap TX, K \equiv c' \cap d \equiv Y'Z' \cap ZT, L \equiv c \cap d' \equiv YZ \cap Z'T'$ are all on f as A-, B-, C-, D-excenters of $\triangle FAB, \triangle FDC. P \equiv a \cap c \equiv TX \cap YZ$ and $Q \equiv b \cap d \equiv XY \cap ZT$ are intersections of the opposite sides of $XYZT.$ Similarly, let $P' \equiv a' \cap c' \equiv T'X' \cap Y'Z'$ and $Q' \equiv b' \cap d' \equiv X'Y' \cap Z'T'$ be intersections of the opposite sides of $X'Y'Z'T'.$

$\triangle QTX \equiv \triangle(abd), \triangle P'Y'X' \equiv \triangle(b'a'c')$ are perspective, because intersections $J \equiv a \cap b', I \equiv b \cap a', K \equiv d \cap c'$ of their corresponding sides are collinear $\Longrightarrow$ by Desargues theorem, QP', TY', XX' are concurrent at $S.$ Similarly, $\triangle PXY \equiv \triangle(bca), \triangle Q'X'T' \equiv \triangle(a'd'b')$ are perspective, because intersections $I \equiv b \cap a', L \equiv c \cap d', J \equiv a \cap b'$ of their corresponding sides are collinear $\Longrightarrow PQ', XX', YT'$ are also concurrent at $S. XX', PQ', QP'$ are concurrent at $S \Longrightarrow \triangle XPQ, \triangle X'Q'P'$ are perspective $\Longrightarrow$ intersections $R \equiv PQ \cap Q'P', J \equiv a \cap b' \equiv XP \cap X'Q',$ $I \equiv b \cap a' \equiv XQ \cap X'P'$ of their corresponding sides are collinear. As a result, $R \equiv PQ \cap P'Q'$ is on $f \equiv IJ.$ In exactly the same way, $R \in e \Longrightarrow R \equiv e \cap f \cap PQ \cap P'Q'. \square$

Amir Hossein
#3 May 18, 2010, 4:20 pm

> yetti wrote:
>
> Let external bisectors a, b, c, d of the angles $\widehat{DAB}, \widehat{ABC}, \widehat{BCD}, \widehat{CDA}$ pairwise intersect at $X \equiv a \cap b, Y \equiv b \cap c, Z \equiv c \cap d, T \equiv d \cap a.$ Likewise, let internal bisectors a', b', c', d' of these angles pairwise intersect at $X' \equiv a' \cap b', Y' \equiv b' \cap c', Z' \equiv c' \cap d', T' \equiv d' \cap a'.$ Let e, f be external and e', f' internal bisectors of the angles $\widehat{DEA}, \widehat{AFB}.$ Points $E, Y, Y', T', T \in e'$ are collinear as incenters/E-excenters of $\triangle ECB, \triangle EDA$ and similarly, points F, Z, Z', X', X are collinear as incenters/F-excenters of $\triangle FDC, \triangle FAB.$ Thus the quadrilaterals $XYZT$ and $X'Y'Z'T'$ have common diagonals $XZ \equiv X'Z'$ and $YT \equiv Y'T'$ intersecting at $S.$ Intersections $I \equiv a' \cap b \equiv T'X' \cap XY, J \equiv a \cap b' \equiv X'Y' \cap TX, K \equiv c' \cap d \equiv Y'Z' \cap ZT,$ $L \equiv c \cap d' \equiv YZ \cap Z'T'$ are all on f as A-, B-, C-, D-excenters of $\triangle FAB, \triangle FDC. P \equiv a \cap c \equiv TX \cap YZ$ and $Q \equiv b \cap d \equiv XY \cap ZT$ are intersections of the opposite sides of $XYZT.$ Similarly, let $P' \equiv a' \cap c' \equiv T'X' \cap Y'Z'$ and $Q' \equiv b' \cap d' \equiv X'Y' \cap Z'T'$ be intersections of the opposite sides of $X'Y'Z'T'.$
>
> $\triangle QTX \equiv \triangle(abd), \triangle P'Y'X' \equiv \triangle(b'a'c')$ are perspective, because intersections $J \equiv a \cap b', I \equiv b \cap a',$ $K \equiv d \cap c'$ of their corresponding sides are collinear $\Longrightarrow$ by Desargues theorem, QP', TY', XX' are concurrent at $S.$ Similarly, $\triangle PXY \equiv \triangle(bca), \triangle Q'X'T' \equiv \triangle(a'd'b')$ are perspective, because intersections $I \equiv b \cap a',$ $L \equiv c \cap d', J \equiv a \cap b'$ of their corresponding sides are collinear $\Longrightarrow PQ', XX', YT'$ are also concurrent at $S.$ XX', PQ', QP' are concurrent at $S \Longrightarrow \triangle XPQ, \triangle X'Q'P'$ are perspective $\Longrightarrow$ intersections $R \equiv PQ \cap Q'P',$ $J \equiv a \cap b' \equiv XP \cap X'Q', I \equiv b \cap a' \equiv XQ \cap X'P'$ of their corresponding sides are collinear. As a result,

$R \equiv PQ \cap P'Q'$ is on $f \equiv IJ.$ In exactly the same way, $R \in e \Longrightarrow R \equiv e \cap f \cap PQ \cap P'Q'.\square$

Nice soln , Thank you!

vittasko

#4 May 19, 2010, 2:13 am

A magic **Desarques**, by **Vladimir**.

Kostas Vittas.

Attachments:

t=349520.pdf (10kb)

yetti

#5 May 19, 2010, 2:42 am

Thanks, Kostas. I actually did almost the same problem before at http://www.artofproblemsolving.com/Forum/viewtopic.php?p=1352360#p1352360, using central projection. And of course, I remembered your proof of the underlying (more general) lemma at http://www.artofproblemsolving.com/Forum/viewtopic.php?p=1504248#p1504248 by Desargues theorem. In the special case of 2 common diagonals of the 2 quadrilaterals, some intersections are lost, and I did not see the correct perspective triangles right away. And if I don't try to use Desargues theorem, I would never learn how.

Luis González

#6 Jun 8, 2010, 12:34 pm • 1

Let the external bisectors of $\angle DAB, \angle ABC, \angle BCD, \angle CDA$ pairwise meet at $I, J, K, L.$ This is, according to my drawing that I, J are incenters of $\triangle AEB, \triangle BFC$ and K, L are E- and F- excenters of $\triangle EDC$ and $\triangle FAD.$ Let KC, KD cut ER at $M, N.$ Then $S \equiv KE \cap MD \cap NC$ is the incenter of $\triangle EDC.$ Angle Bisectors AI, DS of $\angle(AE, AF)$ and $\angle(DE, DF)$ meet at the D-excenter U of $\triangle ADF \Longrightarrow U \in FR$ and angle bisectors BJ, SC of $\angle(BC, BF)$ and $\angle(CE, CD)$ meet at the B-excenter V of $\triangle BCF \Longrightarrow V \in FR.$ In other words, the lines VN, UM, IK concur at $S \Longrightarrow \triangle VUI$ and $\triangle NMK$ are perspective through perspector $S \Longrightarrow$ By Desargues theorem, the intersections $P \equiv MK \cap IU, Q \equiv VI \cap KN$ and $R \equiv UV \cap MN$ are collinear. $\square$

• For instance, it is easy to see that the circle ω passing through E, F, R and $T \equiv IK \cap JL$ is orthogonal to the circumcircle (O, R) of the cyclic quadrilateral $IJKL.$ Line PQ is the polar of (O) with respect to $T,$ thus $OT \perp PQ$ at $H.$ If OT cuts (O) at $M_0, N_0,$ then $(M_0, N_0, T, H) = -1 \Longrightarrow {OM_0}^2 = {ON_0}^2 = R^2 = OT \cdot OH \Longrightarrow (O, R)$ is orthogonal to the pencil of circles passing through $T, H.$ Particularly $(O) \perp \omega.$

Quick Reply

High School Olympiads

Benelux Mathematical Olympiad 2016, Problem 4

socrates

#1 May 2, 2016, 1:31 am

A circle ω passes through the two vertices B and C of a triangle ABC. Furthermore, ω intersects segment AC in $D \neq C$ and segment AB in $E \neq B$. On the ray from B through D lies a point K such that $|BK| = |AC|$, and on the ray from C through E lies a point L such that $|CL| = |AB|$. Show that the circumcentre O of triangle AKL lies on ω.

Luis González

#2 May 2, 2016, 2:25 am • 2

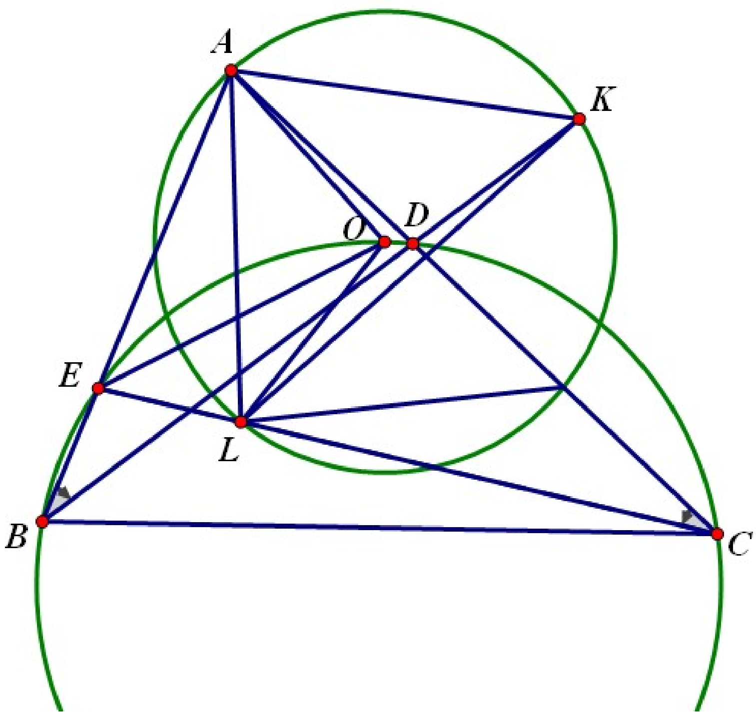
Since $CL = BA$, then $\omega \equiv \odot(EBC)$ and $\odot(AEL)$ meet again at the center M of the rotation that swaps $\overline{CL}$ and $\overline{BA}$. Thus, it follows that $MB = MC$ (M is the midpoint of the arc $BEDC$) and $MA = ML$. Similarly $MK = MA \Longrightarrow MA = MK = ML \Longrightarrow M$ is the circumcenter of $\triangle AKL$.

Vietnamisalwaysinmyheart

#3 May 2, 2016, 9:59 am

My solution:
Let O be circumcentre of (AKL).
We have: $\Delta BAK = \Delta CLA$, so: $AK = AL$ and: $\widehat{BAK} = \widehat{CLA}$, $\widehat{AKB} = \widehat{ALC}$
Since: $\widehat{AEL} = 180^\circ - \widehat{BAC} - \widehat{ACE} = \widehat{LAK}$ (easy by angle-chasing)
$\widehat{AOL} = 2\widehat{AKL} = 180^\circ - \widehat{LAK}$
Hence, A, O, L, E are concyclics, likewise, A, O, D, K are concyclics.
We have that:
$\widehat{EOD} = 360^\circ - \widehat{AOE} - \widehat{AOD} = 360^\circ - \widehat{ALE} - \widehat{AOD} = \widehat{ALC} + \widehat{AKB} = \widehat{ALC} + \widehat{LAC} = 180^\circ - \widehat{DCE}$.
This implies O lies on ω, as desired.

Attachments:



Konigsberg

#4 May 19, 2016, 7:23 pm

> *Luis González wrote:*
> Since $CL = BA$, then $\omega \equiv \odot(EBC)$ and $\odot(AEL)$ meet again at the center M of the rotation that swaps $\overline{CL}$ and $\overline{BA}.
>
> How do we know this?

Quick Reply

High School Olympiads

The sum not depends to x

Reply

Source: All russian olympiad 2016,Day2,grade 9,P7

MRF2017

#1 May 1, 2016, 4:47 pm

In triangle ABC,$AB < AC$ and ω is incirle.The A-excircle is tangent to BC at A'.Point X lies on AA' such that segment $A'X$ doesn't intersect with ω.The tangents from X to ω intersect with BC at Y, Z.Prove that the sum $XY + XZ$ not depends to point X.(Mitrofanov)

Luis González

#2 May 1, 2016, 10:11 pm

Let $\omega \equiv (I)$ touch BC at D and let D' be the antipode of D on (I). It's well-known that $D' \in AA'$. WLOG assume that Z is between D and Y and let the Z-excircle (J) of $\triangle XYZ$ touch BC at A''. Since X is the insimilicenter of $(I) \sim (J)$ and $ID' \parallel JA''$, then X, D', A'' are collinear $\Longrightarrow A'' \equiv A'$. Therefore $XY + XZ = DA' = AC - AB = \text{const}.$

FabrizioFelen

#3 May 1, 2016, 10:28 pm

Define Ω the A-excircle of $\triangle ABC$, Γ the Y-excircle of $\triangle XYZ$ and let $D \equiv BC \cap \omega$ and D, Y, Z lie on BC in this order. $\Longrightarrow$ The insimilicenter of ω and Γ is X, the insimilicenter of Ω and Γ lie in BC, the exsimilicenter of Ω and ω is $A \Longrightarrow$ by Monge-D'alemberg theorem we get the insimilicenter of Ω and Γ is $AX \cap BC = A'$ hence the Γ is tangent to BC in $A' \Longrightarrow ZA' = YD \Longrightarrow$ since $ZA' = YD$ we get $XZ + XY = A'D$ which it is fixed.

This post has been edited 1 time. Last edited by FabrizioFelen, May 1, 2016, 10:35 pm

shinichiman

#4 May 2, 2016, 8:11 am • 2

I think the problem is based on the following lemma.
Lemma. Triangle ABC with ω is incircle. The A-excircle is tangent to BC at A'. X is in line AA'. Tangents from X to ω intersect BC at Y, Z then $BY = CZ$.

I have a proof for this but it requires to consider the position of point X on line AA', which gives a lot of case work. I hope someone can have a better proof for this.

Quick Reply

High School Olympiads

Concurrence Point on Euler Line

Reply

ABCDE

#1 May 1, 2016, 1:54 am • 2

Let ABC be a triangle with orthocenter H, nine-point-center N, and altitudes AD, BE, and CF. Suppose that O_A, O_B, and O_C are the circumcenters of BNC, CNA, and ANB respectively. Let AD and EF intersect at X, BE and FD intersect at Y, and CF and DE intersect at Z. Prove that NH, O_AX, O_BY, and O_CZ concur.

Luis González

#2 May 1, 2016, 3:09 am • 1

Let O be the circumcenter of $\triangle ABC$. It's well-known that YZ, ZX, XY are perpendicular to AN, BN, CN, respectively. Thus since $(O_BO_C \parallel YZ) \perp AN$, etc, and $(OO_A \parallel HX) \perp BC$, etc, then it follows that $\triangle XYZ \cup H$ and $\triangle O_AO_BO_C \cup O$ are homothethic $\Longrightarrow OH, O_AX, O_BY, O_CZ$ concur at their homothety center.

Quick Reply

User Profile

Luis González

Joined: February 14, 2009

Status: Nobody untrained in geometry may enter my house. ~Plato.

Location: Venezuela

Occupation: Petroleum engineer

Activity

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High School Olympiads

Collnear

Reply

Source: Own from Y&M

doxuanlong15052000

#1 Apr 19, 2016, 4:18 pm

Let ABC be a triangle with circumcircle (O), nine points circle w, othocenter H. P is a point on OH. AP, BP, CP cut (O) at A_1, B_1, C_1.HA cuts (O) at X. HA_1 cuts w at A_2 such that A_2 and A_1 on the other side of BC. The line pass through H and parallel to XA_1 cuts (AA_2H) at A_3. Similarly, we have B_3,C_3. Prove that A_3, B_3, C_3 are collinear.

Luis González

#2 Apr 30, 2016, 10:03 pm • 1

Let D, E, F be the feet of the altitudes on BC, CA, AB and let DA_1, EB_1, FC_1 cut HA_3, HB_3, HC_3 at A_4, B_4, C_4, respectively. Inversion with center H and power $HA \cdot HD = HB \cdot HE = HC \cdot HF$ takes (O) into $w \Longrightarrow X, A_1$ are the inverses of $A, A_2 \Longrightarrow DA_1$ is the inverse of $\odot(AA_2H) \Longrightarrow A_4$ is the is inverse of A_3 and likewise B_4 and C_4 are the inverses of B_3 and C_3.

Since D is the midpoint of HX and $XA_1 \parallel HA_4$, it follows that XA_1PA_4 is a parallelogram $\Longrightarrow A_4$ is the reflection of A_1 on D and similarly B_4, C_4 are the reflections of B_1, C_1 on E, F. According to the problem Similar Hagge circle, H, A_4, B_4, C_4 lie on a circle with center on $OH \Longrightarrow A_3, B_3, C_3$ are collinear on the inverse of $\odot(HA_4B_4C_4)$. Thus in addition $\overline{A_3B_3C_3} \perp OH.$

Quick Reply

High School Olympiads

Similar Hagge circle

Reply

Scorpion.k48

#1 Sep 21, 2015, 7:56 pm

Let $\triangle ABC$ with circumcircle $\odot(O)$ and orthocenter H. P is a point lies on Euler line of $\triangle ABC$. $\triangle A_0B_0C_0$ is orthic triangle of $\triangle ABC$. AP, BP, CP cut $\odot(O)$ at A_1, B_1, C_1, res. Let A_2, B_2, C_2 be the reflection of A_1, B_1, C_1 to A_0, B_0, C_0. Prove that H, A_2, B_2, C_2 are cyclic in circle $\odot(K)$ and K lies on Euler line of $\triangle ABC$.

This post has been edited 1 time. Last edited by Scorpion.k48, Sep 21, 2015, 7:56 pm

TelvCohl

#3 Sep 21, 2015, 9:20 pm • 1

Let $\triangle A_3B_3C_3$ be the circumcevian triangle of H WRT $\triangle ABC$. Let $A_4, B_4, C_4, A_5, B_5, C_5$ be the antipode of $A_1, B_1, C_1, A_3, B_3, C_3$ in $\odot(O)$, respectively. From Pascal theorem (for $B_1C_3CC_1B_3B$) we get $B_1C_3 \cap C_1B_3$ lies on the Euler line of $\triangle ABC$, so from Pascal theorem (for $B_1C_3C_4C_1B_3B_4$) $\Longrightarrow Q \equiv B_3B_4 \cap C_3C_4 \in OH$. Similarly, we can prove Q lies on $A_3A_4 \Longrightarrow A_3A_4, B_3B_4, C_3C_4$ and OH are concurrent at Q. Analogously, we can prove A_1A_5, B_1B_5, C_1C_5 and OH are concurrent at R.

Let S be the midpoint of HQ. Since A_0 is the midpoint of HA_3, so we get $HA_2 \overset{\parallel}{=} A_1A_3$ and $A_0S \parallel A_3Q \parallel A_1R$ ($\perp A_1A_3$) $\Longrightarrow$ the perpendicular from A_2 to A_2H passes through the reflection T of R in S, hence A_2 lie on a circle with diameter HT. Analogously, we can prove B_2 and C_2 lie on $\odot(HT) \Longrightarrow H, A_2, B_2, C_2$ lie on a circle $\odot(K)$ with diameter HT and K lies on the Euler line of $\triangle ABC$ due to K is the midpoint of HT.

Luis González

#4 Sep 22, 2015, 4:21 am • 2

Let $\triangle A'B'C'$ be the antimedial triangle of $\triangle ABC$ whose orthocenter is H'. U and V are projections of A' and B' on $B'C'$ and $C'A'$. Since $A_0B_0 \parallel UV$ and $A_0B_0 = \frac{1}{2}UV$, then $Z \equiv UB_0 \cap VA_0$ is common reflection of U and V on B_0 and A_0. $F \equiv UV \cap AB$ is on radical axis of $(S) \equiv \odot(H'UC'V)$ and $(T) \equiv \odot(HBC'A) \Longrightarrow C'F$ is their radical axis perpendicular to their center line $\Longrightarrow C'F \perp ST \parallel HOH'$. Since $C'F$ is the polar of $R \equiv AV \cap BU$ WRT $(O) \Longrightarrow RO \perp C'F \Longrightarrow R \in OH.$

Animate P on OH. A_2, B_2 move on reflections $(O_A), (O_B)$ of (O) on A_0, B_0 that meet at H, Z and $P \barwedge A_1 \barwedge B_1 \barwedge A_2 \barwedge B_2$. Thus if H_A, H_B are the antipodes of H on $(O_A), (O_B)$, the pencils H_AA_2 and H_BB_2 are perspective because $H_AA_2 \equiv H_AZ \equiv H_BB_2 \equiv H_BZ$ when $P \equiv R$. Since $A_2 \equiv B_2 \equiv H$ when $P \equiv H$, then H_AA_2, H_BB_2, OH concur at a point M and likewise H_CC_2 goes through $M \Longrightarrow A_2, B_2, C_2$ lie on circle with diameter HM along the Euler line.

jayme

#5 Sep 22, 2015, 10:18 am

Dear Mathlinkers,

just a reference in order to put this particular problem in an historical perspective

http://www.mathlinks.ro/Forum/viewtopic.php?t=319520

Sincerely
Jean-Louis

Scorpion.k48

#6 Sep 22, 2015, 12:37 pm

> Luis González wrote:
>
> Animate P on OH. A_2, B_2 move on reflections $(O_A), (O_B)$ of (O) on A_0, B_0 that meet at H, Z and $P \barwedge A_1 \barwedge B_1 \barwedge A_2 \barwedge B_2$. Thus if H_A, H_B are the antipodes of H on $(O_A), (O_B)$, the pencils H_AA_2 and H_BB_2 are perspective because $H_AA_2 \equiv H_AZ \equiv H_BB_2 \equiv H_BZ$ when $P \equiv R$. Since $A_2 \equiv B_2 \equiv H$ when $P \equiv H$, then H_AA_2, H_BB_2, OH concur at a point M and likewise H_CC_2 goes through $M \Longrightarrow A_2, B_2, C_2$ lie on circle with diameter HM along the Euler line.

I don't understand this dear Mr. Luis

livetolove212

#7 Sep 28, 2015, 9:01 am

My old problem
http://artofproblemsolving.com/community/q3h345850p1857304

Quick Reply

High School Olympiads

convex equilateral pentagon

Reply

Pirkuliyev Rovsen

#1 Apr 5, 2016, 12:42 am

Prove that from any three diagonals of convex equilateral pentagon can make a triangle.

Luis González

#2 Apr 28, 2016, 9:01 pm

Label $ABCDE$ the vertices of the given pentagon and assume WLOG that EC is its longest diagonal. If $P \equiv AC \cap BE$, then P is strictly inside $ABCE$ as the pentagon is convex $\Longrightarrow PE < BE$ and $PC < AC$. Hence by triangle inequality on $\triangle PCE$, we get $EC < PE + PC < BE + AC \Longrightarrow AC, BE, CE$ make a triangle.

Quick Reply

High School Olympiads

Locus of a point

Reply

ThE-dArK-lOrD

#1 Apr 28, 2016, 6:51 pm

Given $\triangle ABC$ with altitude BE, CF
Let M is midpoint of BC, and P is arbitrary point on BC
Reflect P over M to get point Q
Let $AP, AQ \cap EF = U, V$ respectively
Let $(AUV) \cap (APQ) = T \neq A$, prove that T lie on a fixed circle when P varies on BC

Luis González

#2 Apr 28, 2016, 8:38 pm

More general: Let E, F be arbitrary points on AC, AB. $\{P, Q\} \in BC$, such that $P \mapsto Q$ is an involution on BC. AP, AQ cut EF at U, V. Then the 2nd intersection of $\odot(AUV)$ and $\odot(APQ)$ moves on a fixed circle.

Inverting with center A, we get the equivalent problem: E, F are arbitrary points on AC, AB. $\{P, Q\} \in \odot(ABC)$, such that $P \mapsto Q$ is an involution on $\odot(ABC)$. AP, AQ cut $\odot(AEF)$ again at U, V. Then $T \equiv UV \cap PQ$ moves on a fixed circle.

Since $AP \mapsto AQ$ is an involution, then $U \mapsto V$ is an involution on $\odot(AEF)$ as well $\Longrightarrow PQ$ and UV go through fixed points R and S, respectively (poles of the involutions). If $K \equiv UF \cap PB$, then $\odot(KFB), \odot(ABC), \odot(AEF)$ concur at the Miquel point L of the complete $PBFU, BCEF$. Therefore $\angle VTQ = \angle AQP + \angle UVQ = \angle KBF + \angle KFB = \angle FLB = \text{const}$ (mod 180°). Since R, S are fixed, then it follows that T moves on a fixed circle passing through R and S.

Quick Reply

High School Olympiads

Symedian and collinear.

Source: Own.

doxuanlong15052000

#1 Apr 3, 2016, 11:16 pm • 1

Let ABC be a triangle with AD be A-symedian ($D \in (O)$), t be the tangent at A WRT (O) and the diameter AU. R is a point on AC and L is the intersection of BR and AD such that $RA^2 = RL.RB$. T is a point on AC such that $(TR, AC) = -1$. The line passes through A and perpendicular to AB cuts UC at S. Draw $SM \perp t$ and AS cuts BC at J. Prove that T, M, J are collinear.

Luis González

#2 Apr 26, 2016, 10:04 am • 3

Since $RA^2 = RL \cdot RB$, then $\odot(BAL)$ is tangent to $AC \Longrightarrow L$ is midpoint of AD (well-known property of the symmedian line), hence $\odot(CAL)$ is tangent to AB. Thus $S' \equiv OL \cap CU$ is the antipode of A on $\odot(CAL) \Longrightarrow AS' \perp AB \Longrightarrow S \equiv S'$. If $E \equiv BC \cap AM$ and X is the 2nd intersection of BL and $\odot(CAL)$, then $L(X, C, S, A) = L(B, C, E, A) = -1$ and since $LS \perp LA$, it follows that LS, LA bisect $\angle CLX$, i.e. S is midpoint of the arc CX of $\odot(CAL) \Longrightarrow MS$ is bisector of $\angle CMX$.

Let $J' \equiv MX \cap BC$. Since $\angle SCJ' \equiv \angle BCU = \angle BAO = 90^\circ - \angle ACB = \angle SAM = \angle SCM$, then CS is bisector of $\angle MCJ' \Longrightarrow S$ is incenter of $\triangle MCJ'$, thus by symmetry $J'S$ is the perpendicular bisector of CX. But as A is the midpoint of the arc CMX, then AS is the perpendicular bisector of $CX \Longrightarrow J' \in AS \Longrightarrow J' \equiv J$. Now let $T' \equiv MXJ \cap AC$. Since $S(L, M, C, A) = A(L, E, C, B) = -1 \Longrightarrow LMCA$ is harmonic $\Longrightarrow X(L, M, C, A) = X(R, T', C, A) = -1 \Longrightarrow T \equiv T' \Longrightarrow M, J, T$ are collinear.

uraharakisuke_hsgs

#4 Apr 26, 2016, 8:58 pm

Lemma : $\triangle ABC$ with symmedian line AD . L is the midpoint of AD so AC is the tangent of (LAB)

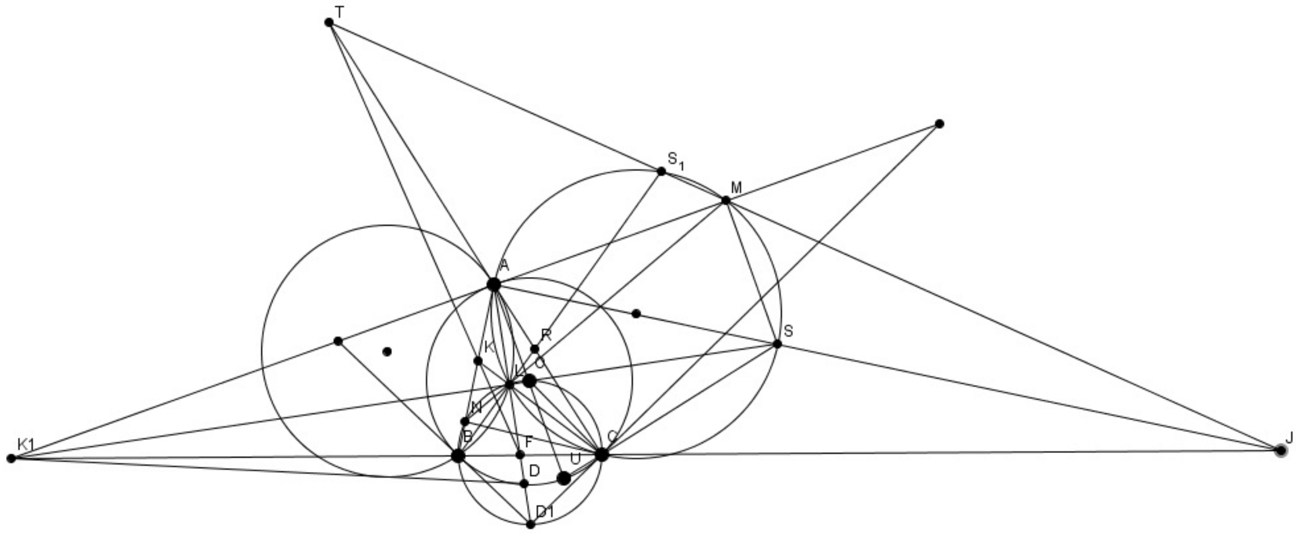
+**Prove** :
L is the midpoint of AD => $OL \perp AD$
D_1 is the intersection of 2 tangent from B and C of (O)
(OBC) cuts AB at G => $\angle BGD = \angle BCD = \angle BAC$ => GD // AC
=>$\angle DAC = \angle ADG = \angle ABL$ => AC is the tangent of (LAB)

Comeback with the problem :

We have :$RA^2 = RL.RB$ => AC is the tangent of (ALB)
Let D_1 be the intersection of 2 tangents from B and C of (O)
=>$OLBD_1C$ is cyclic => $OL \perp AD$ => L is the midpoint of AD => AB is the tangent of (LAC)
$AS \perp AB$ => the center X of (LAC) lies on AS
$CS \perp CA$ => S lies on (LAC) (because S is reflect with A about X)
$SM \perp t$ => M lies on X
N is the intersection of (BOC) and AB
=> $\angle NLC = 180 - \angle NBC = 180 - \angle MAC = 180 - \angle MLC$
=> N, L, M are collinear
We also have : $\angle NCL = \angle NBL = \angle LAC$ => NC is the tangent of (LAC)
=> tangents at A, C of $(ALCM)$ and ML are concurrent => $(ACML)$ = -1
Let BL cuts (X) at S_1 => $S_1(ACML) = -1$
So, if S_1M cuts AC at T_1 => $(T_1RAC) = -1$
=> $T_1 \equiv T$ => S_1, M, T are collinear
We need to prove that S_1, M, J are collinear

By **Pascal**
$(S_1\ S\ A$
$A\ M\ L)$
=> B, J_1 and K_1 are concurrent (K_1 is the intersection of OL with t, J_1 is the intersection of S_1M and AS) (1)
We have : K_1A is the tangent of (O) and $K_1O \perp AD$
=>K_1D is also a tangent of of (O) => K_1, B, C are collinear (2)
From (1) and (2) => $J_1 \equiv J$
=> T, M, J are collinear

Attachments:



This post has been edited 2 times. Last edited by uraharakisuke_hsgs, Apr 26, 2016, 10:19 pm

Quick Reply

High School Olympiads

Line is parallel to Brocard axis

Source: Own

buratinogigle

#1 Apr 17, 2016, 10:09 pm • 2

Let ABC be a triangle inscribed in circle (O) and orthocenter H. Let X,Y,Z lie on (O) such that $AX \parallel BC, BY \parallel CA, CZ \parallel AB$. Let XH, YH, ZH cut BC, CA, AB at D, E, F, reps. Prove that AD, BE, CF are concurrent at point P and line HP is parallel to Brocard axis of triangle ABC$

buratinogigle

#2 Apr 18, 2016, 5:46 pm • 2

This problem is true for any point on Euler line.

Let ABC be a triangle inscribed in circle (O) with orthocenter H. P is any point on Euler of triangle ABC. Let X,Y,Z lie on (O) such that $AX \parallel BC, BY \parallel CA, CZ \parallel AB$. Let XP, YP, ZP cut BC, CA, AB at D, E, F, reps. Prove that AD, BE, CF are concurrent at point Q and line HQ is parallel to Brocard axis of triangle ABC.

TelvCohl

#3 Apr 18, 2016, 7:48 pm • 3

> *buratinogigle wrote:*
> Let ABC be a triangle inscribed in circle (O) and orthocenter H. Let X,Y,Z lie on (O) such that $AX \parallel BC, BY \parallel CA, CZ \parallel AB$. Let XH, YH, ZH cut BC, CA, AB at D, E, F, reps. Prove that AD, BE, CF are concurrent at point P and line HP is parallel to Brocard axis of triangle ABC

Lemma : Given a $\triangle ABC$ and two points P and Q. Let $\triangle P_aP_bP_c, \triangle Q_aQ_bQ_c$ be the cevian triangle of P, Q WRT $\triangle ABC$, respectively. Let T be a point on PQ and Q_aT, Q_bT, Q_cT cuts P_bP_c, P_cP_a, P_aP_b at D, E, F, respectively. Then $\triangle DEF$ is the cevian triangle of S WRT $\triangle P_aP_bP_c$ and TS is tangent to the circumconic $\mathcal{C}$ of $\triangle ABC$ passing through P, T.

Proof : Let $S \equiv P_bE \cap P_cF$ and $V \in PQ$ be the intersection of P_bQ_c, P_cQ_b. From the dual of Desargues' involution theorem (for $P_bQ_bP_cQ_c$ and T) we get there exists an involution which swaps $(TA, TP), (TP_b, TP_c)$ and (TQ_b, TQ_c), but from the dual of Desargues' involution theorem (for BP_bCP_c and T) we get the aforementioned involution also swaps TB, TC. Note P_a is the pole of the involution on $\mathcal{C}$ that swaps $(B, C), (A, P)$, so if τ_t is the tangent of $\mathcal{C}$ passing through T, then we get there is an involution that swaps $(TB, TC), (TA, TP), (TP_b, TP_c), (TQ_b, TQ_c)$ and (τ_t, TP_a).

From the dual of Desargues' involution theorem (for EP_bFP_c and T) we get there exists an involution that swaps (TP_b, TP_c), (TQ_b, TQ_c) and (TS, TP_a), so S lies on τ_t. Similarly, we can prove the line connecting T and the intersection of P_cF, P_aD is tangent to $\mathcal{C}$ at T, so we conclude that P_aD, P_bE, P_cF are concurrent at $S \in \tau_t$.

Remark : The locus of S is a line when T varies on PQ.

Proof : Clearly, pencil $P_bE \mapsto P_cF$ is a homography, so notice $E \equiv P_c, F \equiv P_b$ when T coincide with V we conclude that the locus of S (intersection of P_bE and P_cF) is a line.

Back to the main problem :

Let $\triangle A^*B^*C^*$ be the antimedial triangle of $\triangle ABC$ and let G be the centroid of $\triangle ABC$. From **Lemma** $\Longrightarrow AD, BE, CF$ are concurrent at P and HP is tangent to the circumconic of $\triangle A^*B^*C^*$ passing through G and H, so notice the polar of any point WRT the conic through A^*, B^*, C^*, G passes through its isotomic conjugate WRT $\triangle ABC$ we get HP passes through the isotomic conjugate of H WRT $\triangle ABC$ which is the anticomplement of the symmedian point of $\triangle ABC$ WRT $\triangle ABC$ $\Longrightarrow HP$ is parallel to the Brocard axis of $\triangle ABC$.

> *buratinogigle wrote:*
> This problem is true for any point on Euler line.
>
> Let ABC be a triangle inscribed in circle (O) with orthocenter H. P is any point on Euler of triangle ABC. Let X, Y, Z lie on (O) such that $AX \parallel BC, BY \parallel CA, CZ \parallel AB$. Let XP, YP, ZP cut BC, CA, AB at D, E, F, reps. Prove that AD, BE, CF are concurrent at point Q and line HQ is parallel to Brocard axis of triangle ABC.

From the original problem and the remark of the **Lemma** we can get this generalization.

Luis González

#4 Apr 24, 2016, 9:02 am • 3

> *buratinogigle wrote:*
> This problem is true for any point on Euler line.
>
> Let ABC be a triangle inscribed in circle (O) with orthocenter H. P is any point on Euler of triangle ABC. Let X, Y, Z lie on (O) such that $AX \parallel BC, BY \parallel CA, CZ \parallel AB$. Let XP, YP, ZP cut BC, CA, AB at D, E, F, reps. Prove that AD, BE, CF are concurrent at point Q and line HQ is parallel to Brocard axis of triangle ABC.

Let $\triangle A'B'C'$ be the antimedial triangle of $\triangle ABC$. It's well-known that $U \equiv BZ \cap CY$ is on the Euler line e of $\triangle A'B'C', \triangle ABC$. When P runs on e, clearly $E \mapsto F$ is a proyectivity between AC and AB, even a perspectivity, because when $P \equiv U$, then $E \equiv C, F \equiv B \Longrightarrow Q \equiv BE \cap CF$ moves on a line τ.

When P is the orthocenter of $\triangle A'B'C'$, then BE, CF become Schwatt lines of $\triangle A'B'C'$ meeting at its symmedian point and when P is the centroid of $\triangle A'B'C'$, then clearly BE, CF become perpendicular bisectors of $A'C', A'B'$ meeting at H $\Longrightarrow \tau$ is the Brocard axis of $\triangle A'B'C'$. Similarly AD goes through $\tau \cap BE \cap CF$. Hence AD, BE, CF concur at a point Q on the Brocard axis of $\triangle A'B'C'$, thus parallel to the Brocard axis of $\triangle ABC$.

Quick Reply

High School Olympiads

Geometry

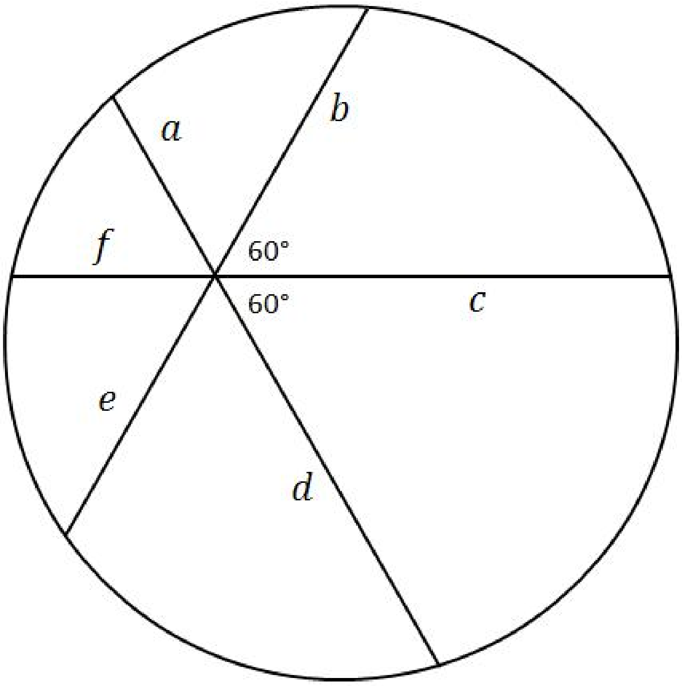
Reply

soroush.MG

#1 Apr 23, 2016, 5:42 pm

Prove that: $a + c + e = b + d + f$

Attachments:



Luis González

#2 Apr 23, 2016, 8:17 pm

See http://www.artofproblemsolving.com/community/c6h441896.

Quick Reply

High School Olympiads

chords

Reply

elegant

#1 Oct 29, 2011, 6:55 pm

Three chords AA1, BB1, CC1 of a circle (with the order on the circle A, B1, C, A1, B, C1) pass through a common point K and form six angles of 60°. Prove that KA + KB + KC = KA1 + KB1 + KC1.

yetti

#2 Oct 30, 2011, 5:13 am

Let (O) be the circle with the chords AA_1, BB_1, CC_1 concurrent at K.
Common internal bisector AKA_1 of $\angle CKB, \angle C_1KB_1$ meets circles $\odot(KBC), \odot(KB_1C_1)$ again at Z, Z_1.
$ZB = ZC$ and $\angle BZC = 180^\circ - \angle CKB = 60^\circ \Longrightarrow \triangle BZC$ is equilateral and similarly, $\triangle B_1Z_1C_1$ is equilateral.
By Ptolemy for cyclic $KBZC$, $KB \cdot ZC + KC \cdot ZB = ZK \cdot BC \Longrightarrow ZK = KB + KC$ and similarly,
$Z_1K = KB_1 + KC_1$.
Perpendicular bisectors of BC, B_1C_1 through Z, Z_1 meet at O. Since $KBCZ \sim KB_1Z_1C_1$ are similar,
$\angle OZK = \angle OZ_1K \Longrightarrow$
$\triangle OZZ_1$ is O-isosceles, just like $\triangle OAA_1 \Longrightarrow$
$\boxed{KA + KB + KC = ZK + KA = ZA = Z_1A_1 = Z_1K + KA_1 = KA_1 + KB_1 + KC_1}$

newsun

#3 Oct 30, 2011, 7:30 pm

I see this problem from another way..
Assume that this circle centered at O and which has radius R.
We have
$R^2 = OA^2 = (\vec{OK} + \vec{KA})^2 = OK^2 + KA^2 + 2\vec{OK}\vec{KA}$.
therefore, $R^2 - OK^2 = KA^2 + 2\vec{OK}\vec{KA}$
hence $KA.KA_1 = KA^2 + 2\vec{OK}\vec{KA}$
so, $KA_1 = KA + 2\vec{OK}.\dfrac{\vec{KA}}{KA}(1)$
Similarly, $KB_1 = KB + 2\vec{OK}.\dfrac{\vec{KB}}{KB}(2)$
$KC_1 = KC + 2\vec{OK}.\dfrac{\vec{KC}}{KC}(3)$
On the other hand, $\dfrac{\vec{KA}}{KA}, \dfrac{\vec{KB}}{KB}, \dfrac{\vec{KC}}{KC}$ are unit vectors and each pair of them forms angle of 60° then
$\dfrac{\vec{KA}}{KA} + \dfrac{\vec{KB}}{KB} + \dfrac{\vec{KC}}{KC} = \vec{0}$.
Combining $(1)(2)(3)and(4)$ we have the conclusion.

Luis González

#4 Oct 31, 2011, 7:54 am

Antipedal triangles $\triangle PQR$ and $\triangle P_1Q_1R_1$ of K with respect to $\triangle ABC$ and $\triangle A_1B_1C_1$ are clearly equilateral with parallel sides. But since QR, RP, PQ cut the given circle (O) again at the antipodes of A_1, B_1, C_1, we deduce that $\triangle PQR$ and $\triangle P_1Q_1R_1$ are symmetric about O. Thus, by Viviani's theorem we get

$$KA + KB + KC = \frac{\sqrt{3}}{2}PQ = \frac{\sqrt{3}}{2}P_1Q_1 = KA_1 + KB_1 + KC_1.$$

Quick Reply

High School Olympiads

Point on Euler line

Reply

ThE-dArK-lOrD
#1 Apr 22, 2016, 8:53 pm

Given $\triangle ABC$ with orthocenter H
Let D, E, F are midpoints of BC, CA, AB respectively
Let T be arbitrary point on Euler line of $\triangle ABC$
Let $AT \cap HD = X$, similarly define Y, Z
Prove that $\frac{HX}{XD} = \frac{HY}{YE} = \frac{HZ}{ZF}$

Luis González
#2 Apr 22, 2016, 10:03 pm

We use oriented segments throughout the proof. Let O and G be the circumcenter and centroid of $\triangle ABC$ and let $U \equiv AP \cap OD$. From $AH \parallel OU$, we get

$$\frac{\overline{XD}}{\overline{XH}} = \frac{\overline{DU}}{\overline{HA}} = \frac{\overline{OU} - \overline{OD}}{\overline{HA}} = \frac{\overline{OU}}{\overline{HA}} - \frac{\overline{GO}}{\overline{GH}} = \frac{\overline{PO}}{\overline{PH}} + \frac{1}{2}.$$

Similarly we get $\frac{\overline{YE}}{\overline{YH}} = \frac{\overline{ZF}}{\overline{ZH}} = \frac{\overline{PO}}{\overline{PH}} + \frac{1}{2}$ and the conclusion follows.

Quick Reply

High School Olympiads

Kosnita points lies on OI line

Source: Own

buratinogigle
#1 Apr 21, 2016, 11:54 am

Prove that OI line of a triangle passes through three Kosnita points of three triangles divided by its incenter.

mineiraojose
#2 Apr 21, 2016, 12:29 pm • 1

Let M be the midpoint of arc BC in $\odot(ABC)$. Since M is the circumcenter of $\odot(IBC)$ so O is the circumcenter of $\odot(MBC)$ so IO pass through to the Kosnita point of $\triangle IBC$. Analogously the others.

buratinogigle
#3 Apr 21, 2016, 12:43 pm • 1

Thank you for nice explanation. The following is general problem

Let ABC be a triangle inscribed in circle (O) and DEF is circumcevian of any point P. X,Y,Z are reflections of D,E,F through BC,CA,AB, reps. Let U,V,W,Q are isogonal conjugate of X,Y,Z,P with respect to triangles PBC,PCA,PAB,DEF. Prove that P,Q,U,V,W are collinear.

Luis González
#4 Apr 21, 2016, 8:06 pm • 3

> *buratinogigle wrote:*
> Let ABC be a triangle inscribed in circle (O) and DEF is circumcevian of any point P. X,Y,Z are reflections of D,E,F through BC,CA,AB, reps. Let U,V,W,Q are isogonal conjugate of X,Y,Z,P with respect to triangles PBC,PCA,PAB,DEF. Prove that P,Q,U,V,W are collinear.

Obviously $\triangle PBC \sim \triangle PFE$ and $\angle FEQ = \angle DEP = \angle DCB = \angle BCX$ and likewise $\angle EFQ = \angle CBX \Longrightarrow \triangle QEF \sim \triangle XCB \Longrightarrow PQEF \sim PXCB \Longrightarrow \angle CPX = \angle QPE \Longrightarrow PQ, PX$ are isogonals WRT $\angle BPC \Longrightarrow U \in PQ$ and similarly V and W lie on PQ.

High School Olympiads

Source: Own

buratinogigle

#1 Apr 21, 2016, 12:59 am

Let ABC be a triangle inscribed in circle (O), incenter I and orthocenter H. A_1, A_2 lie on BC such that $IA_1 \perp IB, IA_2 \perp IC$. AA_1, AA_2 cut (O) again at A_3, A_4. A_3A_4 cuts BC at A_0. Similarly, we have B_0, C_0. Prove that A_0, B_0, C_0 are collinear on a line which is perpendicular to line IH.

Luis González

#2 Apr 21, 2016, 7:43 am • 2

Moreover, we'll prove that A_0, B_0, C_0 are collinear on the trilinear polar of X_{85} of $\triangle ABC$ (isotomic conjugate of its Mittenpunkt). But first we introduce a lemma:

Lemma: P is a point inside $\triangle ABC$ and Q is the isotomic conjugate of the isotomcomplement of P WRT $\triangle ABC$. Then the trilinear polars of P and Q WRT $\triangle ABC$ are parallel.

Consider an affine homology carrying $\triangle ABC \cup P$ into a $\triangle ABC$ with orthocenter P. The trilinear polar of P is the orthic axis of $\triangle ABC$ and Q becomes isotomic conjugate of the symmedian point, whose trilinear polar, the De Longchamps line of $\triangle ABC$, is perpendicular to the Euler line, as it is the radical axis of $\odot(ABC)$ and the circumcircle of the antimedial triangle (well-known). Therefore the trilinear polars of P and Q WRT $\triangle ABC$ are paralell.

Corollary: When P is the Nagel point Na of $\triangle ABC$, then its isotomcomplement is the Mittenpunkt Mt (well-known). Thus from the previous lemma, it follows that the trilinear polar τ of the isotomic conjugate of Mt is parallel to the trilinear polar of Na. But according to the problem hard geometry problem, the trilinear polar of Na is perpendicular to IH, therefore $IH \perp \tau$.

Back to the main problem, it's well-known that A_1, A_2 are the tangency points of the B- and C- mixtilinear incircles with BC, thus we have the ratios $\frac{A_1C}{A_1B} = \frac{b}{s-b}$ and $\frac{A_2B}{A_2C} = \frac{c}{s-c} \Longrightarrow$

$$\frac{A_4B}{A_4C} = \frac{b}{c} \cdot \frac{A_2B}{A_2C} = \frac{b}{c} \cdot \frac{c}{s-c} = \frac{b}{s-c} \text{ and likewise } \frac{A_5C}{A_5B} = \frac{c}{s-b}$$

$$\Longrightarrow \frac{A_0B}{A_0C} = \frac{A_4B}{A_4C} \cdot \frac{A_5B}{A_5C} = \frac{b(s-b)}{c(s-c)}.$$

It's known that the A-cevian of Mt divides $\overline{BC}$ in the ratio $\frac{c(s-c)}{b(s-b)}$, thus from the latter expression we deduce that A_0 is on the trilinear polar of the isotomic conjugate of Mt and analogously B_0 and C_0 fall on this line. Now from the previous corollary, we get $IH \perp \overline{A_0B_0C_0}$.

TelvCohl

#3 Apr 21, 2016, 1:28 pm • 2

Let $\triangle DEF$ be the intouch triangle of $\triangle ABC$ and let AI, AD cuts $\odot(ABC)$ again at M, X, respectively. Clearly, IA and ID are isogonal conjugate WRT $\angle BIC$, so there exists an involution that swaps $(B, C), (A_1, A_2)$ and $(D, AI \cap BC) \Longrightarrow$ there is an involution that swaps $(AB, AC), (AA_3, AA_4)$ and (AM, AX), hence BC, A_3A_4, MX are concurrent at A_0.

Let I_a, I_b, I_c be the A-excenter, B-excenter, C-excenter of $\triangle ABC$, respectively. Let the isogonal conjugate of AD WRT $\angle A$ cuts $\odot(ABC)$ again at X^* and $A_0^* \equiv BC \cap MX^*$. Note X^* is the tangency point of $\odot(ABC)$ and the A-mixtilinear excircle of $\triangle ABC$, so $AI_a \perp I_aA_0^*$. Similarly, if B_0^*, C_0^* is the isotomic conjugate of B_0, C_0 WRT $(C, A), (A, B)$, respectively, then we get $BI_b \perp I_bB_0^*$ and $CI_c \perp I_cC_0^*$.

Obviously, IH is the radical axis of $\odot(AA_0^*I_a), \odot(BB_0^*I_b), \odot(CC_0^*I_c)$, so notice A_0, B_0, C_0, is the anticomplement (WRT $\triangle ABC$) of the center of these three circles we conclude that A_0, B_0, C_0 are collinear and $\overline{A_0B_0C_0} \perp IH$.

Quick Reply

High School Olympiads

Line is tangent to incircle

Reply

Source: Own

buratinogigle

#1 Apr 3, 2016, 11:58 pm

Let ABC be a triangle and incircle (I). DEF is cevian triangle of any point P. Let X,Y,Z are poles of EF,FD,DE with respect to (I). Prove that AX,BY,CZ intersect BC,CA,AB follow three collinear points on a tangent of (I).

Luis González

#2 Apr 4, 2016, 12:39 am • 1

This follows from your previous problem Poles with cevian triangle. If (I) touches BC,CA,AB at D',E',F', we already know that $U \equiv E'F' \cap EF \cap BY \cap CZ.$ Thus if AX,BY,CZ cut BC,CA,AB at X',Y',Z', then by the converse of Newton theorem, it follows that $Y'Z'$ touches (I) and similarly $X'Y'$ touches $(I) \Longrightarrow X',Y',Z'$ are collinear on a tangent of (I).

Quick Reply

High School Olympiads

Poles with cevian triangle

Source: Own

buratinogigle

#1 Oct 8, 2015, 11:02 pm

Let ABC be a triangle with incirlce (I) and DEF is cevian triangle of a point P. Let X,Y,Z be the poles of lines EF,FD,DE with respect to (I).

a) Prove that BY,CZ and EF are concurrent at point U.

b) Similarly, we have point U,V. Prove that AU,BV,CW are concurrent.

Luis González

#2 Oct 9, 2015, 12:31 am • 1

a) Let (I) touch BC,CA,AB at D',E',F'. Then $U \equiv EF \cap E'F', V \equiv FD \cap F'D', W \equiv DE \cap D'E'$ are the poles of AX,BY,CZ WRT (I). If $\mathcal{C}$ is the circum-conic through P and $AD' \cap BE' \cap CF'$, then A,X,V,W are collinear on a tangent of $\mathcal{C}$ (for a proof see collinear 4), thus their polars EF,BY,CZ WRT (I) concur at U.

b) $\triangle ABC$ and the polar triangle $\triangle UVW$ of $\mathcal{C}$ WRT $\triangle ABC$ are perspective (well-known), their perspector being $AU \cap BV \cap CW$.

High School Olympiads

A nice concurrency

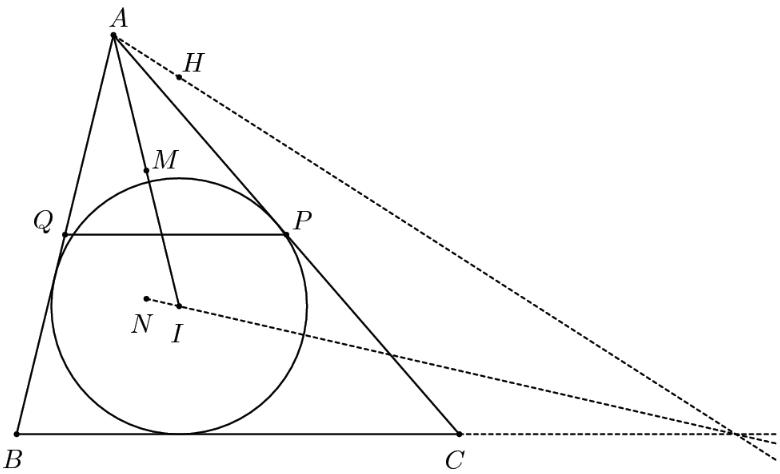
Reply

Source: own

mjuk

#1 Apr 2, 2016, 5:27 am • 1

Let $\triangle ABC$ be a triangle with incenter I. Let H be orthocenter of $\triangle BIC$, let P, Q, M be midpoints of AC, AB, AI. Let N be reflection of M over PQ. Prove that AH, NI, BC are concurrent.



This post has been edited 1 time. Last edited by mjuk, Apr 3, 2016, 6:14 pm
Reason: added latex figure

Luis González

#2 Apr 2, 2016, 7:42 am • 1

Let R be the midpoint of BC and let D, E, F be the tangency points of the incircle (I, r) with BC, CA, AB. $Fe \equiv \odot(PQR) \cap \odot(DEF)$ is the Feuerbach point of $\triangle ABC$.

By 1st Fontené theorem, DFe, EF, PQ concur at a point U and if L is the antipode of D on (I), then M, L, Fe are collinear (see Intersect on circle post #4 for a proof of the general case). Since M is clearly the incenter of $\triangle APQ$, it follows that $MN = r = IL \Longrightarrow MNIL$ is a parallelogram $\Longrightarrow (IN \parallel MLFe) \perp DFe \Longrightarrow X \equiv BC \cap IN$ is the pole of DFe WRT (I). As H is the pole of PQ WRT (I) (well-known), then we deduce that U is the pole of AH WRT $(I) \Longrightarrow$ $X \equiv AH \cap NI \cap BC$. Moreover XFe is tangent of (I) at Fe.

FabrizioFelen

#3 Apr 2, 2016, 10:07 am

My solution:
Let R the point of tangency of incircle and BC, $S = AH \cap BC, T = MN \cap BC$ and D is the foot of perpencicular of A in BC
Define: $AB = a + c$, $BC = a + b$, $CA = b + c$, $IR = r$, $AD = h$, $DR = 2y$ and $RS = z \Longrightarrow RB = a$, $RC = b$ and $MN = r$

Since H is the ortocenter of $\triangle BIC \Longrightarrow RI.RH = RB.RC \Longrightarrow HR = \frac{ab}{r}$

Since M is the midpoint of $AI \Longrightarrow 2MT = AD + IR \Longrightarrow MT = \frac{h+r}{2} \Longrightarrow NT = \frac{h-r}{2}$

Since $AD \parallel RH \Longrightarrow \frac{AD}{DS} = \frac{RH}{RS} \Longrightarrow \frac{h}{z+2y} = \frac{ab}{zr} \Longrightarrow z = \frac{2yab}{hr-ab} \Longrightarrow \frac{y+z}{z} = \frac{ab+hr}{2ab} ...(\star)$

But it's well know that $hr = \frac{2abc}{\ldots}$ and $\frac{h}{\ldots} = \frac{2(a+b+c)}{\ldots}$ replacing in $(\star)$ we get $\frac{h-r}{\ldots} = \frac{y+z}{\ldots} = \frac{a+b+2c}{\ldots} \Longrightarrow$

$\frac{h-r}{2r}=\frac{y+z}{z}\Longrightarrow\frac{NT}{IR}=\frac{TS}{RS}$...$(\star\star)$. Since $NT\parallel IR$ and $(\star\star)$ we get N,I,S are collinear hence AH,NI,BC are concurrent.

mjuk

#4 Apr 2, 2016, 1:50 pm

My solution:

Let $\triangle DEF$ be intouch triangle, let K be projection of A on BC, let $L=MN\cap BC$, $R=MN\cap PQ$, $S=NI\cap AK$, $X=DH\cap PQ$. $KA\parallel DH$, so suffices to show $\frac{KS}{KA}=\frac{DI}{DH}$. Let $\boxed{KA=h}$, $\boxed{DI=r}$.

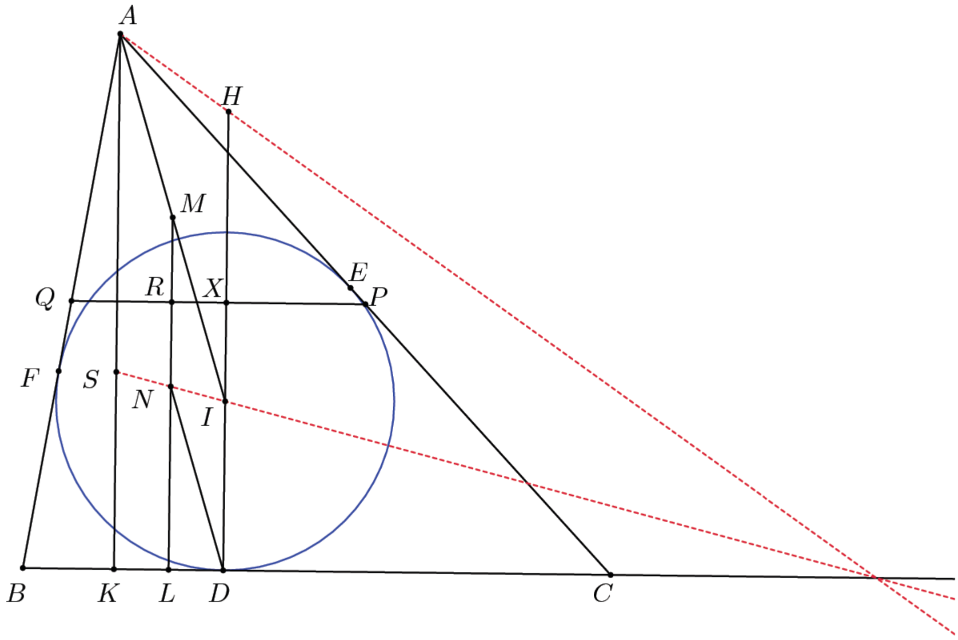
Since M is midpoint of AI, M is center of incircle of $\triangle APQ$ with radius $\frac{r}{2}$.

$\rightarrow KS=2LN-DI=2(LR-RN)-r=2\left(\frac{h}{2}-\frac{r}{2}\right)-r\rightarrow\boxed{KS=h-2r}$.

Well-known lemma: PQ is polar of H wrt. $(I)\rightarrow X,H$ are inverses wrt. (I)

$\rightarrow ID^2=IX\cdot IH\rightarrow r^2=\left(\frac{h}{2}-r\right)(DH-r)\rightarrow\boxed{DH=\frac{hr}{h-2r}}$.

And finally $\frac{KS}{KA}=\frac{h-2r}{h}=\frac{DI}{DH}$.



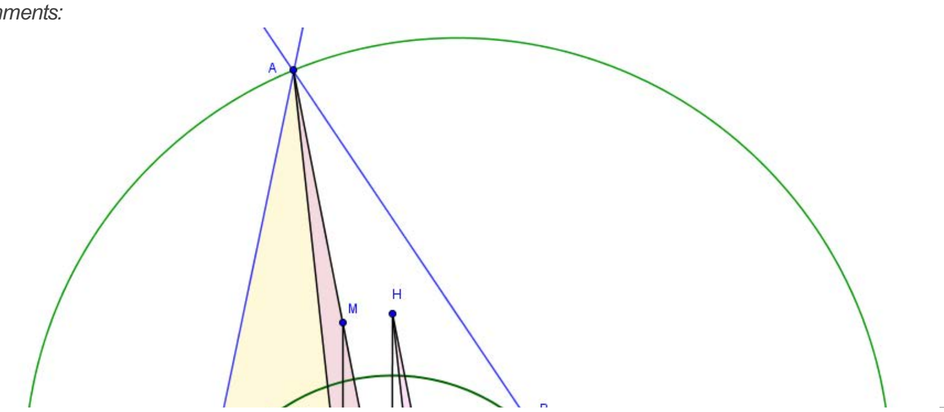
This post has been edited 4 times. Last edited by mjuk, Apr 3, 2016, 1:45 am
Reason: added figure

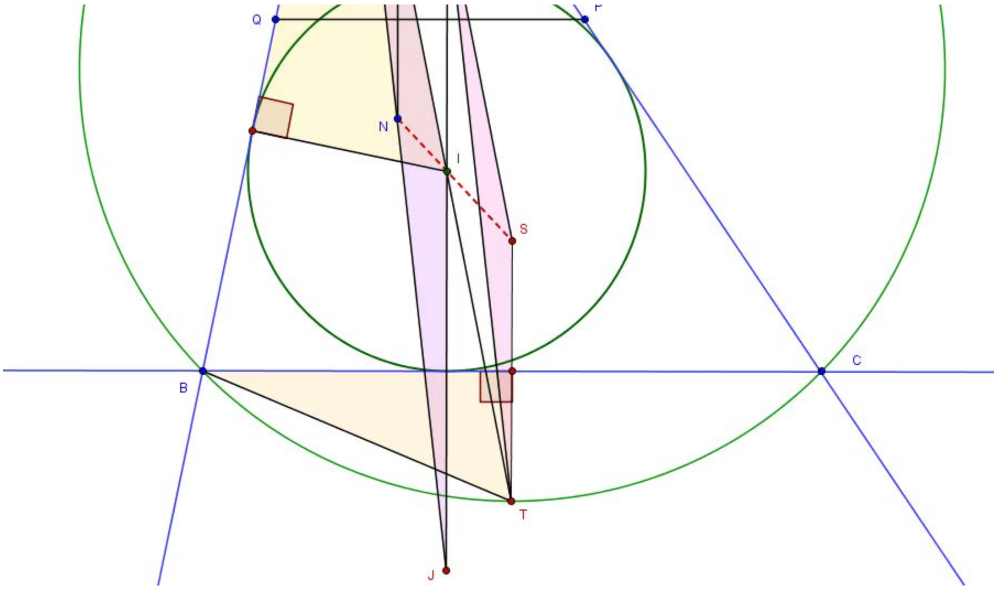
TelvCohl

#5 Apr 2, 2016, 2:38 pm • 2

Let T be the midpoint of arc BC in $\odot(ABC)$ and let S be the reflection of T in BC. Since $\frac{MI}{MN}=\frac{1}{2}\cdot\frac{AI}{r}=\frac{TI}{2\text{dist}(T,BC)}=\frac{TI}{TS}$, so I,N,S are collinear. Let J be the reflection of I in BC. Since N is the midpoint of AJ and $HI\overset{\parallel}{=}ST$, so $HS\parallel AI$ and $\frac{AI}{IJ}=\frac{MI}{MN}=\frac{TI}{TS}=\frac{HS}{ST}\Longrightarrow\triangle AIJ$ and $\triangle HST$ are homothetic, hence AH,NI,BC are concurrent.

Attachments:





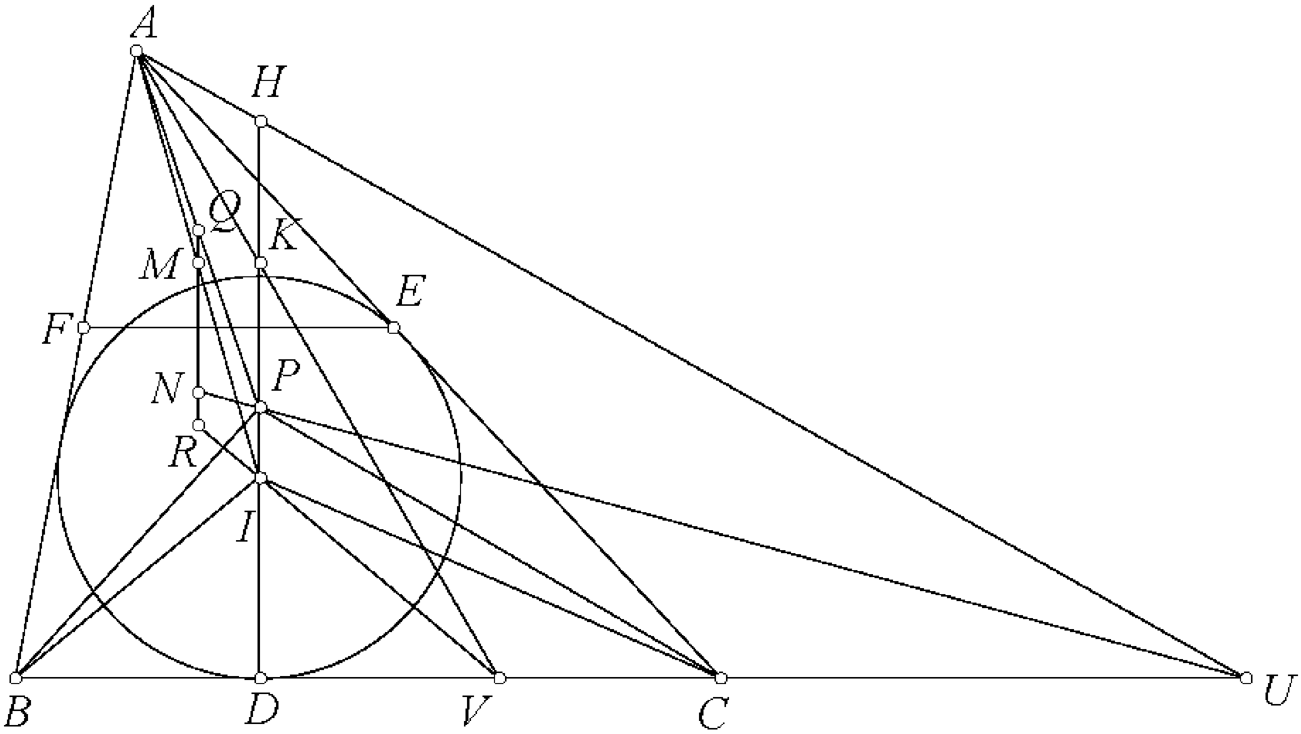
buratinogigle

#6 Apr 2, 2016, 11:46 pm

I see a problem as following

Let ABC be a triangle with incircle (I). E, F lie on CA, AB such that $EF \parallel BC$. H is orthocenter of triangle IBC. AH cuts BC at U. M is midpoint of AI. N is reflection of M through EF. NU cuts IH at P. K is orthocenter of triangle PBC. AK cuts BC at V. Q is midpoint of AP. R is reflection of Q through EF. Prove that V, I, R are collinear.

Attachments:



Luis González

#7 Apr 3, 2016, 4:24 am • 1

> *buratinogigle wrote:*
>
> I see a problem as following
>
> Let ABC be a triangle with incircle (I). E, F lie on CA, AB such that $EF \parallel BC$. H is orthocenter of triangle IBC. AH cuts BC at U. M is midpoint of AI. N is reflection of M through EF. NU cuts IH at P. K is orthocenter of triangle PBC. AK cuts BC at V. Q is midpoint of AP. R is reflection of Q through EF. Prove that V, I, R are collinear.

Animate E, F on AC, AB. Obviously Q, N, R move on the A-midline ℓ of $\triangle AIH$ and the series N, P, Q, K, V are clearly all projective. By symmetry $\overrightarrow{MQ} = \overrightarrow{RN}$, thus as the series N, Q are similar, then it follows that the series N, R are similar as well. Consequently $R \mapsto V$ is a projectivity between ℓ and BC. So all we need to prove is that this proyectivity is a perspectivity through I, in other words, that R, I, V are collinear for 3 particular cases.

When E, F go to infinity, Q, N, R, P go to the point at infinity of $\perp BC$, thus V coincides then with the projection of I on BC $\Longrightarrow R, I, V$ are collinear. When $P \equiv H$, then obviously R, I, V are collinear on the angle bisector of $\angle BAC$. Finally when E, F coincide with the midpoints of AC, AB, then from the original problem we get that R, I, V are collinear and we are done.

navi_09220114
#8 Apr 3, 2016, 8:27 am

I am sorry to ask but why it is sufficient to prove V, I, R is colinear in 3 cases after showing they are projective? And how the points N, P, Q, K, V are projective? Thanks in advance!

wwwrqnojcm
#9 Apr 3, 2016, 12:21 pm

> mjuk wrote:
> **My solution:**
>
> Well-known lemma: PQ is polar of H wrt. $(I) \rightarrow X, H$ are inverses wrt. (I)
> $\rightarrow ID^2 = IX \cdot IH \rightarrow r^2 = \left(\frac{h}{2} - r\right)(DH - r) \rightarrow \boxed{DH = \frac{hr}{h-2r}}$.
> And finally $\frac{KS}{KA} = \frac{h-2r}{h} = \frac{DI}{DH}$.

Could someone please specifically explain this lemma? Thanks.

mjuk
#10 Apr 3, 2016, 5:47 pm

> wwwrqnojcm wrote:
> Could someone please specifically explain this lemma? Thanks.

You can see post #2 here.

EinsteinXXI
#11 Apr 3, 2016, 11:07 pm

Here is my solution :
Let AH meets BC at G.
Suppose that (I) touches BC, CA, AB at D, E, F respectively.
K is the intersection point of EF and PQ.
We know that A is the polar point of EF WRT (I), H is the polar point of PQ WRT (I) so K is the polar point of AH WRT (I).
But D is the polar point of BC WRT (I) so G is the polar point of DK WRT (I).
Hence IG is perpendicular DK.
Let T, S is the intersection points of the line passing A perpendicular BC with BC, PQ respectively, R is the intersection point of ID and PQ.
We have $MN = AS - IR = ST - IR = DR - IR = ID$ so $MNDI$ is a parallelogram. So $IM = ND$.
We have $IK^2 - ID^2 = IK^2 - r^2 = -KE.KF = MK^2 - MI^2 = NK^2 - ND^2$ so $IN \perp DK$.
Hence I, N, G is collinear so AH, NI, BC are concurrent.

This post has been edited 3 times. Last edited by EinsteinXXI, Apr 5, 2016, 9:54 am

urahara...

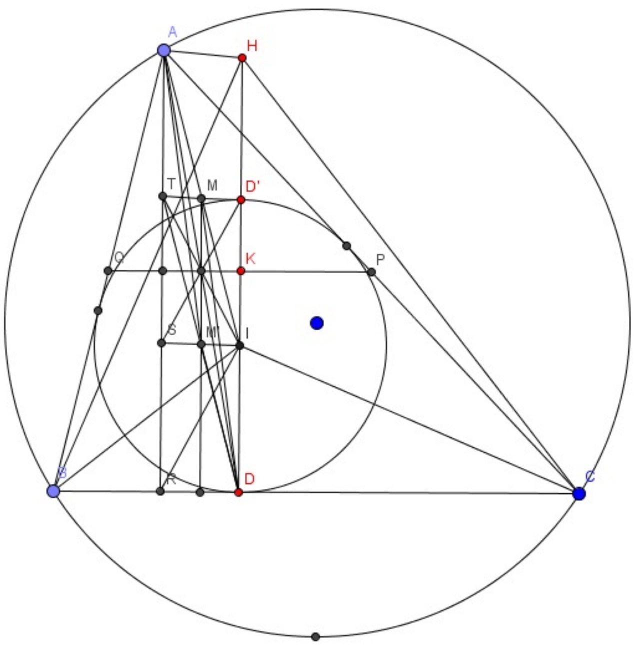
uraharakisuke_hsgs

#12 Apr 5, 2016, 1:14 am

Lemma : ΔABC with incenter I. H is the orthocenter of ΔIBC. HD cuts (I) at K . We have $HK.HI = r^2$

AR □ BC , D' reflect with D about I . OM cuts AR at S, $D'M$ cuts AR at T
=> $MM'DI$ is a parallelogram
=> D, M, T collinear
$ID^2 = IK.IH$ => $r^2 = (DH - r)(\frac{h}{2} - r)$
=> $DH = \frac{hr}{(h-2r)}$
=> $\frac{DH}{DI} = \frac{RA}{RS}$
=> AH, SI, DR concurrent

Attachments:



This post has been edited 5 times. Last edited by uraharakisuke_hsgs, Apr 24, 2016, 6:04 pm

High School Olympiads

Intersect on circle

Reply

buratinogigle
#1 Jan 17, 2010, 10:24 pm

Let $A'B'C'$ be a pedal triangle of arbitrary point P with respect to triangle ABC. P^* is isogonal conjugate of P with respect to triangle ABC. R_p is circumradii of triangle $A'B'C'$. The rays $[B'P$, $[C'P$ intersect circle $(P^*, 2R_p)$ at B'', C'', respectively. Prove that BB'', CC'' intersect on circle $(P^*, 2R_p)$.

Note that, this is generalization of the problem on the post Nice problem about concurrent:D

livetolove212
#2 Jan 18, 2010, 9:03 pm • 1

Nice problem dear Mr.**Hung** :)

Attachments:
bai9.PDF (47kb)

buratinogigle
#3 Jan 19, 2010, 3:13 pm

Great work dear Linh :D , all my problems always wait for your nice solution :)!

Luis González
#4 Jul 10, 2011, 8:57 pm • 5

Proposition: P is a point on the plane of $\triangle ABC$ and $\triangle P_AP_BP_C$ is the pedal triangle of P WRT $\triangle ABC$. A_1, B_1, C_1 are the midpoints of PA, PB, PC. Lines PP_A, PP_B, PP_C cut the pedal circle $\odot(P_AP_BP_C)$ again at P_1, P_2, P_3. Then P_1A_1, P_2B_1 and P_3C_1 concur on $\odot(P_AP_BP_C)$.

Proof: Let U be the Poncelet point of $ABCP$, which lies on the pedal circle of P WRT $\triangle ABC$ (for a proof, see post #2 here). $\odot(P_AP_BP_C)$ cuts BC again at D. R is the orthogonal projection of A onto $PC \Longrightarrow \odot(URA_1C_1)$ is 9-point circle of $\triangle APC$.

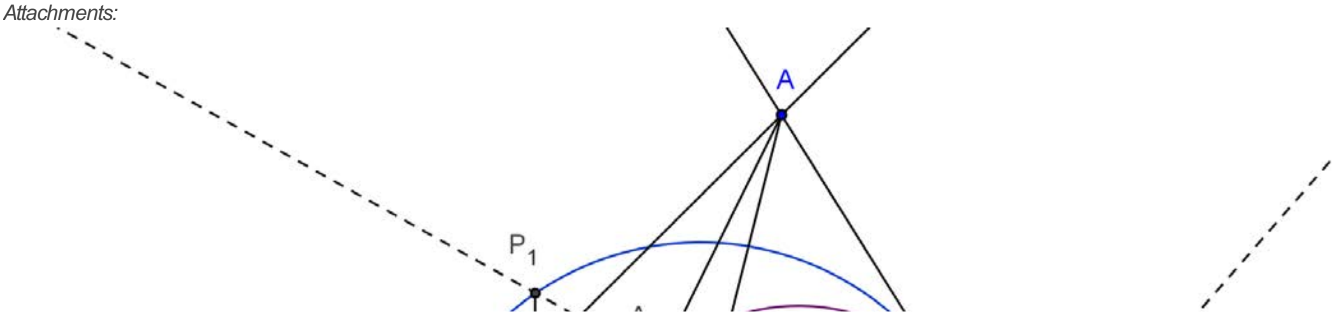
$\angle DUC_1 = \angle DUP_B - \angle C_1UP_B = 180^\circ - \angle CPP_B - \angle PRP_B$

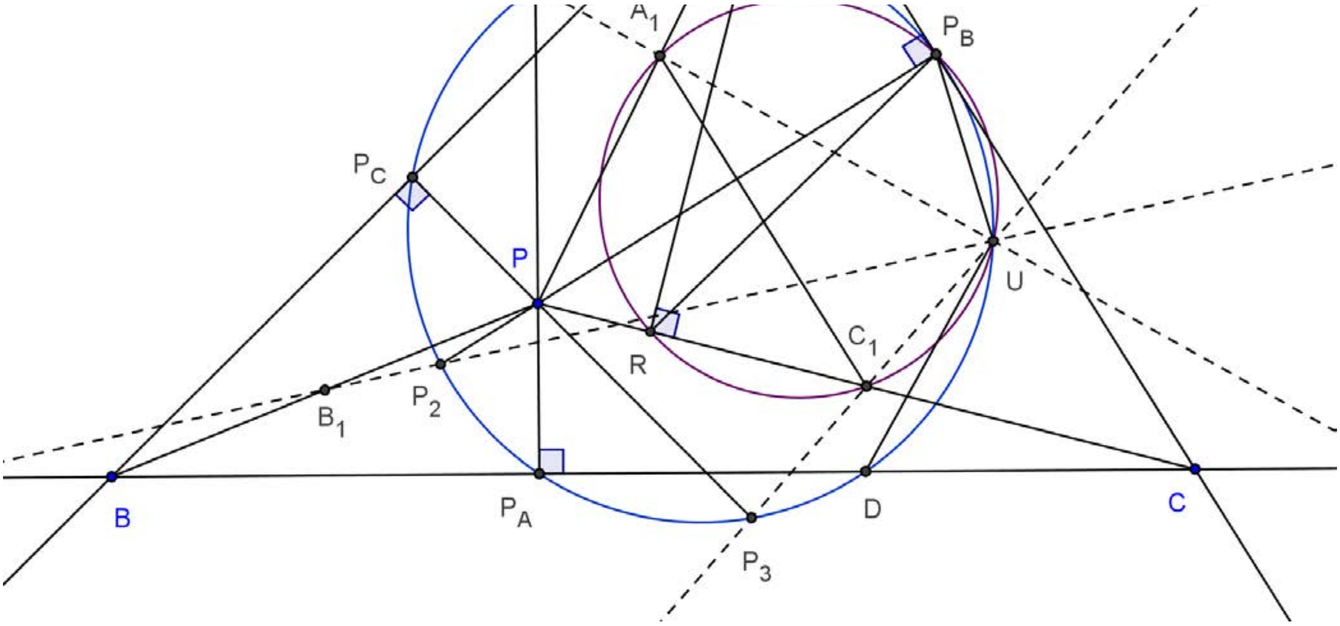
$\angle DUC_1 = \angle PAC - \angle CPP_B = \angle PAC - \angle RAC = 90^\circ - \angle APC$

Thus, $\angle DUA_1 = \angle DUC_1 + \angle C_1UA_1 = 90^\circ - \angle APC + \angle APC = 90^\circ$. Since $\angle DUP_1 = 90^\circ$, then $U \in P_1A_1$. Likewise, P_2B_1 and P_3C_1 pass through U.

P.S. Note that a dilatation with center P and factor 2 produces the configuration of the proposed problem. P-Pedal circle $\odot(P_AP_BP_C)$ with radius ϱ goes to the circle $\odot(P^*, 2\varrho)$ centered at the isogonal conjugate P^* of P.

Attachments:





buratinogigle

#5 Jul 11, 2011, 10:52 am

There is a generalization for above

Problem. Let ABC be a triangle and A', B', C' lies on BC, CA, AB, resp. P is Miquel point of $A'B'C'$ (the common point of circumcircles $(AB'C'), (BC'A'), (CA'B')$). PA', PB', PC' cut the circumcircle $(A'B'C')$ again at P_a, P_b, P_c. Let O_a, O_b, O_c be circumcenter of triangles $AB'C', BC'A', CA'B'$, resp. Prove that O_aP_a, O_bP_b, O_cP_c concur on $(A'B'C')$.

skytin

#6 Jul 11, 2011, 2:24 pm

> buratinogigle wrote:
>
> There is a generalization for above
>
> **Problem.** Let ABC be a triangle and A', B', C' lies on BC, CA, AB, resp. P is Miquel point of $A'B'C'$ (the common point of circumcircles $(AB'C'), (BC'A'), (CA'B')$). PA', PB', PC' cut the circumcircle $(A'B'C')$ again at P_a, P_b, P_c. Let O_a, O_b, O_c be circumcenter of triangles $AB'C', BC'A', CA'B'$, resp. Prove that O_aP_a, O_bP_b, O_cP_c concur on $(A'B'C')$.

Use homotety with center at P and coefficient 1/2 , then Luisgeometra's problem for O_aO_bO_c and P

Quick Reply

High School Olympiads

geometry

Locked

phantranhuongth

#1 Apr 1, 2016, 10:48 pm

ABC circumscribed circle (I), inscribed (O). (I) cut BC at D. M midpoint longbow BC.
A line through I perpendicular IO cut AO, MD at X, Y
Prove IX=IY

Luis González

#2 Apr 1, 2016, 11:36 pm

Discussed before at http://www.artofproblemsolving.com/community/c6h347299

High School Olympiads

An application of the butterfly property.

Reply

Virgil Nicula
#1 Apr 29, 2010, 8:24 pm

Let ABC be a triangle with incircle $C(I)$ and circumcircle $C(O)$. The incircle touches the side $[BC]$ in a point D and the lines AI , AO cut again the circumcircle in the points M , S respectively.
Let $X \in DM$, $Y \in AO$ be two points so that $I \in XY$. Prove that $IX = IY \iff OI \perp XY$.

Luis González
#2 May 1, 2010, 5:19 am

Let P be the midpoint of the arc ABC and $T \equiv MD \cap (O)$, different from M. Inversion in the circle (M) with radius $MB = MC = MI$ swaps (O) and the sideline BC, I is double and $D \mapsto T$, thus $\odot(IDT)$ is double $\Longrightarrow MI$ is tangent to $\odot(IDT)$. Hence, it follows that $\angle ITD = \angle DIM = \angle AMP = \angle ATP$, but $\angle MTP$ is right since M, P are antipodal. Then $\angle ATI = MTP = 90^\circ$, which implies that chords TS and AM meet at I. Now, by Butterfly theorem we conclude that $IX = IY \iff OI \perp XY$.

Quick Reply

High School Olympiads

Geometry

Reply

Source: It's a well-known result(i forgot the proof)

den_thewhitelion

#1 Mar 31, 2016, 8:50 pm

Let $\triangle ABC$ and M=midpoint of (BC),MD⊥AB,ME⊥AC;D∈(AB) and E∈(AC)
Then the ⊙(BAE)and ⊙(CAD) intersect on a point on the A-altitude of the triangle

babu2001

#2 Mar 31, 2016, 9:44 pm • 1

Let the orthocenter of $\triangle ABC$ be H. Let $BH \cap AC \equiv X$ & $CH \cap AB \equiv Y$. Then $BX||ME$ & $BC = 2MC$ hence $CX = 2CE$. Let the orthocenter of $\triangle BAE$ be H_1. Then $EH_1 \perp AB$ so $EH_1||CH$ & $CX = 2CE$ so $HX = 2H_1X$ as H, H_1, X are collinear. Let $BHH_1 \cap \odot(BAE) \equiv P$. By a well known result P is the reflection of H_1 in AE. Thus $HP = HX + XP = \frac{3}{2}HX$. Thus we have $HB \cdot HP = \frac{3}{2}HB \cdot HX$. Similarly let $CH \cap \odot(CAD) \equiv Q$, then $HC \cdot HQ = \frac{3}{2}HC \cdot HY$. From the cyclic quadrilateral $BYXC$ we have $BH \cdot HX = CH \cdot HY$. Hence we have $HB \cdot HP = HC \cdot HQ$, so H lies on the radical axis of $\odot(BAE)$ & $\odot(CAD) \Rightarrow AH$ is the radical axis of $\odot(BAE)$ & $\odot(CAD)$. Thus $\odot(BAE) \cap \odot(CAD) \in AH$, as required.

This post has been edited 2 times. Last edited by babu2001, Apr 1, 2016, 2:07 am
Reason: Improved Latex

den_thewhitelion

#3 Mar 31, 2016, 10:16 pm

Thanks @babu2001

Luis González

#4 Mar 31, 2016, 11:07 pm

See the topic http://www.artofproblemsolving.com/community/c6h497171.

Quick Reply

High School Olympiads

The line passing through centers is parallel to one side

Reply

Narcissus

#1 Sep 5, 2012, 10:14 am

Given a triangle ABC and AM is one of its medians. Let E and F be feets of M onto sides AB and AC respectively. Let (O_1) be the circumcenter of triangle ACE and (O_2) be the circumcenter of triangle ABF. Prove that O_1O_2 is parallel to BC

Luis González

#2 Sep 5, 2012, 12:18 pm

Let H be the orthocenter of $\triangle ABC. H_A, H_B$ are the feet of the A- and B- altitude. $MF \parallel BH_B$ is C-midline of $\triangle BCH_B$, i.e. F is midpoint of $\overline{CH_B}$. Let P be the midpoint of $\overline{HH_A}$. Then $\triangle BHH_A$ and $\triangle BCH_B$ are similar with corresponding B-medians BP and BF. Thus if Q is the reflection of P about BC, we have $\angle BQA = \angle BPH_A = \angle BFA \Longrightarrow Q \in (O_2)$. Similarly, $Q \in (O_1) \Longrightarrow$ A-altitude AQ is radical axis of $(O_1), (O_2) \Longrightarrow O_1O_2 \perp AQ$, or $O_1O_2 \parallel BC$.

Quick Reply

High School Olympiads

Concurrent and collinear

Reply

Source: Own

buratinogigle

#1 Mar 31, 2016, 10:05 pm • 1

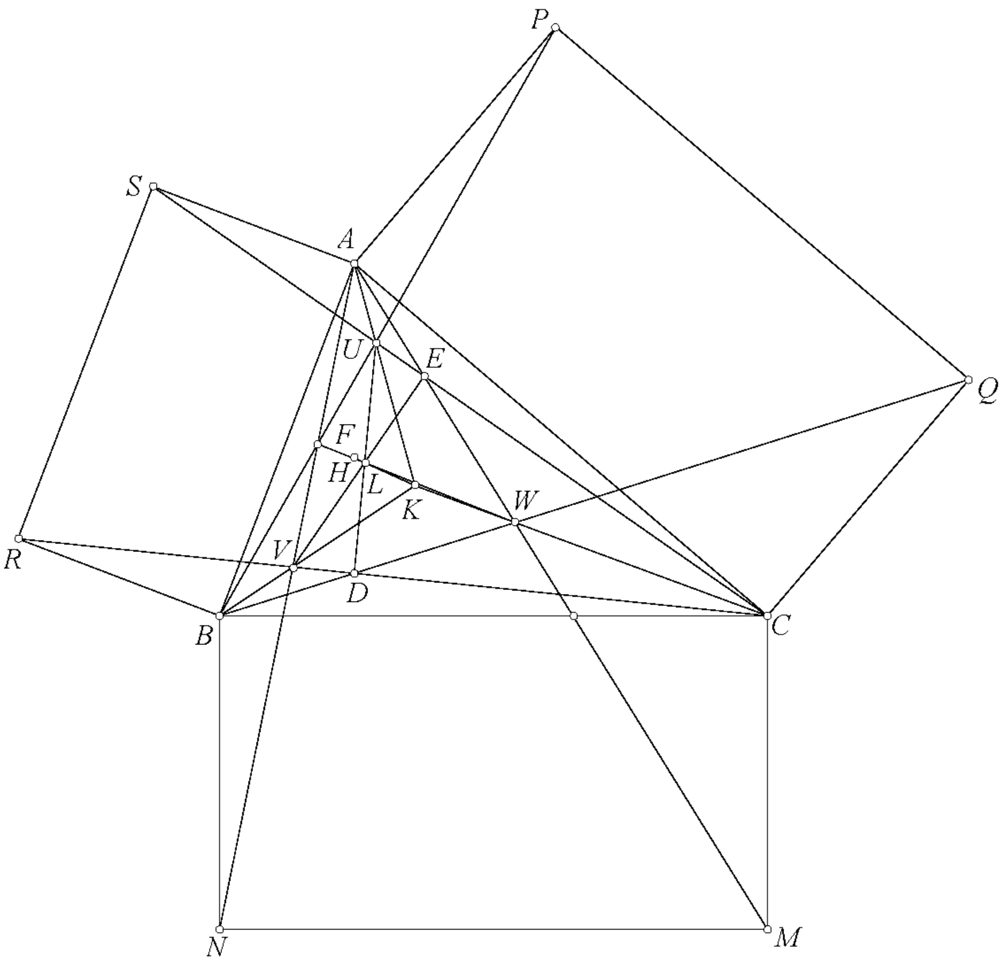
Let ABC be a triangle. Construct outside triangle three similar rectangles $BCMN, CAPQ, ABRS$. Let BQ cuts CR at D. Similar, we have E, F. Let BP cuts CS at U. Similarly, we have V, W.

a) Prove that AU, BV, CW are concurrent at K.

b) Prove that DU, EV, FW are concurrent at L.

c) Prove that KL passes through orthocenter H of triangle ABC.

Attachments:



Luis González

#2 Mar 31, 2016, 10:40 pm • 2

Denote O_1, O_2, O_3 the centers of the rectangles $BCMN, CAPQ, ABRS$, respectively and denote θ the base angle of the isosceles $\triangle O_1BC, \triangle O_2CA, \triangle O_3AB$. Since $\angle O_1BC = \angle O_1CB = \angle ABS = \angle ACP$ and $\angle BAS = \angle CAP = 90^\circ$, then by Jacobi's theorem AO_1, BP, CS concur, i.e. $AU \equiv AO_1$. Similary $BV \equiv BO_2$ and $CW \equiv CO_3 \Longrightarrow AU, BV, CW$ concur at the Kiepert perspector $K \equiv K(\theta)$.

Since $\angle BAR = \angle CAQ$ and $\angle ABR = \angle ACQ = 90^\circ$, the by Jacobi's theorem, it follows that BQ, CR and the A-altitude of $\triangle ABC$ concur, i.e. AD is the A-altitude of $\triangle ABC$ and similarly BE and CF are the altitudes issuing from B and C. Now see the general configuration at the topic The Equilateral triangle (post #3 and #4). We conclude that DU, EV, FW concur at L and K, H, L are collinear

Quick Reply

High School Olympiads

The Equilateral triangle

Reply

Source: vankhea

vankhea

#1 Jan 23, 2015, 12:23 pm

Let I be incenter of equilateral triangle ΔABC and let K be point lie on circle (I). Let X, Y, Z be midpoints of KA, KB, KC and let $D = BZ \cap CY, E = CX \cap AZ, F = AY \cap BX$. Let T be concurrent point of AD, BE, CF. Prove that $IT = \frac{1}{4}r$, with r be inradii.

Luis González

#2 Jan 23, 2015, 7:38 pm

Let (I) touch BC at its midpoint M. Thus KM, BZ and CY are medians of $\triangle KBC$ concurring at its centroid D. If AD cuts IK at T', then by Menelaus' theorem for $\triangle KIM$ cut by $\overline{AT'D}$, we get $\frac{T'K}{T'I} = \frac{AM}{AI} \cdot \frac{DK}{DM} = -\frac{3}{2} \cdot 2 = -3 \Longrightarrow IT' = \frac{1}{4}IK$. By similar reasoning, BE and CF cut IK at the same point $T' \Longrightarrow T \equiv T' \Longrightarrow IT = \frac{1}{4}r$.

In general we have the following: If P is an arbitrary point on the plane of $\triangle ABC$, the lines joining A, B, C with the centroids of $\triangle PBC, \triangle PCA, \triangle PAB$, resp, concur at the complement of the complement of P WRT $\triangle ABC$.

vankhea

#3 Jan 24, 2015, 10:01 am

Similarly the above problem
Let P be an arbitrary point on the plan of triangle ΔABC. Let X, Y, Z be points on PA, PB, PC respectively.
Let $D = BZ \cap CY, E = CX \cap AZ, F = AY \cap BX$.
Prove that AD, BE, CF concurrent.

Luis González

#4 Jan 28, 2015, 10:49 am

In responce to the latter message: The homography $\{B, Y, Z, C\} \mapsto \{B, F, E, C\}$ is nothing but a homology that fixes BC and the line pencil through $A \equiv FY \cap EF$, thus since $P \mapsto X$ and $D \mapsto Q \equiv BE \cap CF \Longrightarrow A \in QD$, in other words AD, BE, CF concur at Q, as desired.

Bonus: Since $YZ \mapsto EF \Longrightarrow YZ, EF, BC$ concur at U and likewise ZX, FD, CA concur at V and XY, DE, AB concur at $W \Longrightarrow \triangle ABC, \triangle DEF$ and $\triangle XYZ$ are perspective with common perspectrix $\overline{UVW}$, thus by the 3 homologies theorem, DX, EY, FZ concur at R and P, Q, R are collinear.

Quick Reply

High School Olympiads

Bisect the segment

Reply

Source: OWN

LeVietAn
#1 Mar 30, 2016, 8:06 am

Dear Mathlinkers,
Let $\triangle ABC$ with incenter I. The points P and Q lie on the side BC such that AP, AQ are isogonal conjugate WRT $\angle BAC$. The tangent line at B of $\odot(ABP)$ and the tangent line at C of $\odot(ACQ)$ intersect at D. Let $E = AP \cap IB, F = AQ \cap IC$. Prove that DI bisects the segment EF.

Luis González
#2 Mar 30, 2016, 9:10 am

Let M and N be the midpoints of BC and the arc BAC of $\odot(ABC)$. As P moves on BC, clearly D moves on the perpendicular bisector ℓ_A of $\overline{BC}$ and since $\angle DBC = \angle BAE = \angle CAF$, we deduce that the series P, Q, D, E, F, U are all projective, where U is the midpoint of EF. $\Longrightarrow E \mapsto F$ is an proyectivity between IB and $IC \Longrightarrow U$ describes a hyperbola $\mathcal{H}$ with asymptotes parallel to IB, IC and passing through I, because when $P \equiv Q \equiv AI \cap BC$, we have $E \equiv F \equiv U \equiv I$.

When $\{P \equiv B, Q \equiv C\}$, then $U \equiv M$ and when $P \equiv Q$ is identical with the foot of the external bisector of $\angle BAC$, then E and F become excenters of $\triangle ABC$ againts $B, C \Longrightarrow U \equiv N$. Therefore $U \mapsto D$ is a stereographic projection of $\mathcal{H}$ onto ℓ_A $\Longrightarrow I, U, D$ are collinear for any pair P, Q.

doxuanlong15052000
#3 Mar 30, 2016, 9:45 pm • 2

My solution:
We have $\angle DBC = \angle BAP = \angle CAQ = \angle DCB \Longrightarrow DB = DC \Longrightarrow$
$\frac{sinDIB}{sinDIC} = \frac{sinDBI}{sinDCI} = \frac{sinIEA}{sinIFA} = \frac{sinIEA}{sinIAE}.\frac{sinIAF}{sinIFA} = \frac{IE}{IF} \Longrightarrow DI$ bisects the segment EF.

This post has been edited 1 time. Last edited by doxuanlong15052000, Mar 30, 2016, 9:47 pm

PROF65
#4 Mar 30, 2016, 10:12 pm

@doxuanlong15
Good job just rectify a typo:
$\frac{IF}{IE}$ instead of $\frac{IE}{IF}$

Quick Reply

High School Olympiads

Line joining circumcenters passes through A

Reply

Source: Own

livetolove212

#1 Mar 29, 2016, 10:25 am

Given triangle ABC. Let P and Q be arbitrary points on BC such that $AP = AQ$. (APQ) intersects AC, AB at E, F, respectively. Let O_1, O_2 be the circumcenters of triangles BPE, CPF. Prove that O_1O_2 passes through A.

Luis González

#2 Mar 29, 2016, 10:43 am

Inversion WRT $\odot(A, AP)$ swaps $\odot(APQ)$ and PQ, therefore it swaps B, F and $C, E \Longrightarrow \odot(BPE)$ goes to $\odot(CPF)$ $\Longrightarrow \odot(BPE)$ and $\odot(CPF)$ are coaxal with their inversion circle $\odot(A, AP)$ (well-known) $\Longrightarrow$ their centers A, O_1, O_2 ar collinear.

Quick Reply

trumpeter's Problem of the Day Forum

Problem of the Day: 1/15/16

Reply

trumpeter

#1 Jan 16, 2016, 12:16 pm

Problem of the Day: 1/15/16

In $\triangle ABC$ with circumcircle Γ, suppose that the tangents to Γ from B and C intersect AC and AB at P and Q, respectively. Similarly, let the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect AC and AB at R and S, respectively. Show that Γ, the circumcircle of $\triangle APQ$ and the circumcircle of $\triangle ARS$ concur at a point other than A.

Source: Original
Author: Gunmay Handa

anantmudgal09

#2 Jan 17, 2016, 2:17 am

This was a very nice problem, I wonder the motivation behind it. Here is a hint to the solution. I would post my solution which I by far rate to be in the final monthly compilation.

The main idea I had was to use

Click to reveal hidden text

The rest is some nice trigonometric computation which I would post soon.

Although, this gives technically no information about the point of concurrence of these circles.

EulerMacaroni

#3 Jan 17, 2016, 2:38 am

> *anantmudgal09 wrote:*
> The main idea I had was to use
> Click to reveal hidden text

This is indeed my solution, but I am curious as to whether there is a better one?

anantmudgal09

#4 Jan 17, 2016, 2:58 am

Here is my solution in it's full glory

Solution

Luis González

#5 Mar 29, 2016, 9:10 am • 1

Let O be the circumcenter of $\triangle ABC$ and let $D \equiv BP \cap CQ$. From R-isosceles $\triangle RAB$, we get $\angle BRC = 2\angle BAC = \angle BOC \Longrightarrow R \in \odot(OBDC)$ and likewise $S \in \odot(OBDC)$. Thus $\angle BSD = \angle BCD = \angle BAC \Longrightarrow DS \parallel AC$ and similarly $DR \parallel AB$. Hence if B_∞ and C_∞ denote the points at infinity of AC and AB, we get $D(S, B, Q, C_\infty) = D(B_\infty, P, C, R) \Longrightarrow \frac{SB}{SQ} = \frac{RC}{RP} \Longrightarrow \overline{SBQ} \sim \overline{RCP} \Longrightarrow \odot(ABC)$, $\odot(APQ)$ and $\odot(ARS)$ meet again at the center of the spiral similarity that swaps $\overline{SBQ}$ and $\overline{RCP}$

Quick Reply

trumpeter's Problem of the Day Forum

Problem of the Day: 10/12/15

Reply

trumpeter
#1 Oct 12, 2015, 10:26 am

Problem of the Day: 10/12/15

$ABCD$ is a quadrilateral (not necessarily convex) with M, N midpoints of AB, CD respectively. Let $P = AC \cap BD$ and $Q = (ABP) \cap (CDP) \neq P$. Prove that (AMQ), (CNQ), and AC are concurrent. Note: (XYZ) is the circumcircle of $\triangle XYZ$.

Source: Original
Author: Daniel Liu

Luis González
#3 Mar 28, 2016, 11:06 am

This holds for all points M, N on AB, CD, such that $\overline{MA} : \overline{MB} = \overline{NC} : \overline{ND}$.

Let $X \equiv MN \cap AC$. Since Q is the center of the spiral similarity that swaps $\overline{AB}$ and $\overline{CD}$, it follows that $\triangle QAB \cup M \sim \triangle QCD \cup N \Longrightarrow \triangle QAM \sim \triangle QCN \Longrightarrow Q$ is center of the spiral similarity that swaps $\overline{AM}$ and $\overline{CN}$ $\Longrightarrow X \in \odot(AMQ) \cap \odot(CNQ)$ and the conclusion follows.

Quick Reply

trumpeter's Problem of the Day Forum

Problem of the Day: 1/30/16

Reply

trumpeter

#1 Jan 31, 2016, 11:48 am

Problem of the Day: 1/30/16

Let ω be a circle and let A, B, C, D be any four points on it such that all points in this problem exist. Let the tangents to ω at A, B meet at X and those at C, D meet at Y. Let AB and CD meet at P and let AC, BD meet at R and AD, BC meet at Q. Then, if QR meets ω again at X_1, X_2 and QR meets the circles (ABY) and (CDX) again at S, T respectively then prove that (OX_1X_2) and (OST) are tangent to one another at point O where O is the centre of ω. Here, (MNP) means the circumcircle of $\triangle MNP$.

Source: Original
Author: Anant Mudgal

Luis González

#2 Mar 28, 2016, 10:24 am

Clearly X,Y,Q,R are collinear on the polar of P WRT $\omega \equiv (O)$. Thus the inversion WRT (O) takes QR into the circle with Ω with diameter $\overline{OP}$ and takes X,Y into the midpoints M,N of $\overline{AB},\overline{CD}$, respectively. Hence $\odot(ABY)$ and $\odot(CDX)$ go to $\odot(ABN)$ and $\odot(CDM) \Longrightarrow S,T$ go to the second intersections U,V of $\Omega \equiv \odot(OMPN)$ with $\odot(ABN)$ and $\odot(CDM)$, respectively. If $\odot(OX_1X_2)$ and $\odot(OST)$ are tangent, then we need to show that their inverse lines X_1X_2 and UV are parallel.

Let X_1X_2 cut AB,CD at L,K, respectively. Since $(L,A,B,P)=(K,D,C,P)=-1$, it follows that $PM \cdot PL = PA \cdot PB = PD \cdot PC = PN \cdot PK \Longrightarrow MNKL$ is cyclic. X_1X_2, AB, NU are pairwise radical axes of $(O), \Omega, \odot(ABN)$ concurring at their radical center L, i.e. N,L,U are collinear and similarly M,K,V are collinear $\Longrightarrow$ $\angle UVM = \angle LNM = \angle LKM \Longrightarrow UV \parallel X_1X_2$, as desired.

Quick Reply

High School Olympiads

Circumcenter

Reply

ThE-dArK-lOrD

#1 Mar 26, 2016, 11:23 pm

For given triangle ABC with circumcircle ω and incenter I, let incicle touch AB, AC at E, F respectively
Let $AI \cap BC = M$
Extend line $EF \cap \omega = P, Q$ such that P lie on different side of C respect to AB
Let $(PEM) \cap \omega = K \neq P$ and $(QFM) \cap \omega = L \neq Q$
Prove that circumcenter of $\triangle KML$ lie on AM

Luis González

#2 Mar 27, 2016, 8:04 am • 1

After inverting WRT $\odot(A, AE)$ we get the following equivalent configuration:

A-mixtilinear excircle of $\triangle ABC$ touches AB, AC at E, F and $\odot(AEF)$ cuts BC at P, Q, such that B is between C, P. M is the midpoint of the arc BC of $\odot(ABC)$ and $\odot(PME), \odot(QFM)$ cut BC again at K, L. Then the circumcenter of $\triangle KML$ lies on AM.

Proof: WLOG assume that $AC > AB$ and let X, Y be projections of M and A on BC. Since $\triangle AEF$ is A-isosceles, its circumcenter is on $AM \Longrightarrow AM$ is the A-circumdiameter of $\triangle APQ \Longrightarrow$
$\angle XMA = \angle YAM = \angle APQ - \angle AQP = \angle AFQ - \angle AEP$. But by symmetry $\angle AEM = \angle AFM \Longrightarrow$
$\angle XMA = \angle MFQ - \angle MEP = \angle MLK - \angle MKL \Longrightarrow MA$ is the M-circumdiameter of $\triangle MKL$, i.e. the circumcenter of $\triangle KML$ is on AM, as desired.

Luis González

#3 Mar 27, 2016, 8:27 am

Notice in the previous proof we have only used M as any point on the angle bisector of $\angle BAC$ and $AE = AF$. Thus in the original problem, E, F can be any two points on $\overline{AB}, \overline{AC}$ verifying $AE = AF$ and M can be any point on AI.

Quick Reply

High School Olympiads

Concurrent lines

Reply

mjuk

#1 Mar 26, 2016, 6:36 pm

Let $\triangle ABC$ be a triangle with altitudes AD, BE, CF and orthocenter H. Let X be H-excenter of $\triangle HEF$. Define Y, Z similarily. Prove that AX, BY, CZ are concurrent.

Luis González

#2 Mar 26, 2016, 11:22 pm

This can be proved exactly in the same way as Baltic way 2009 (P13).

Quick Reply

High School Olympiads

Concurrence

Source: Baltic Way 2009

Sassha

#1 Dec 13, 2009, 2:19 am

In a triangle ABC, draw altitudes AD, BE, CF and let H be the orthocenter. Let O_1, O_2, O_3 be the incenters of $\triangle EHF, \triangle FHD, \triangle DHE$ respectively. Prove that AO_1, BO_2, CO_3 are concurrent.

Luis González

#2 Dec 13, 2009, 4:54 am

Construct the triangle $\triangle A'BC$ outside of $\triangle ABC$, such that $\triangle A'BC \sim \triangle O_1EF$ and define analogously the points B', C'. Since EF is antiparallel to BC with respect to AB, AC, the rays AO_1 and AA' are isogonals with respect to AB, AC. Similarly, BB' and CC' are the isogonals of BO_2 and CO_3. By angle chase, we have then

$$\angle CBA' = \angle ABC' = \frac{\pi - 2\angle B}{4}$$

$$\angle BCA' = \angle ACB' = \frac{\pi - 2\angle C}{4}$$

$$\angle CAB' = \angle BAC' = \frac{\pi - 2\angle A}{4}$$

By Jacobi's theorem, the rays AA', BB', CC' concur at the Jacobi's perspector J relative to the triple $\left(\frac{\pi-2\angle A}{4}, \frac{\pi-2\angle B}{4}, \frac{\pi-2\angle C}{4}\right)$. Therefore, the rays AO_1, BO_2, CO_3 concur at the isogonal conjugate J^* of J.

Sassha

#3 Dec 29, 2009, 9:46 pm

It could also be proven in the following way making use of another theorem:

Let ABC be a triangle, and let D, E, F be points on sides BC, CA, AB, respectively, such that the cevians AD, BE, CF are concurrent. If M, N, P are points on EF, FD, DE, respectively, then the lines AM, BN, CP are concurrent if and only if the lines DM, EN, FP concur. We can use this to solve the problem.

We know from triangle geometry that the orthocenter of a triangle is the incenter of its orthic triangle, therefore $\angle EFO_1 = \angle DFO_2, \angle FDO_2 = \angle EDO_3, \angle DEO_3 = \angle FEO_1$.

Then, according to the same Jacobi theorem, the lines DO_1, EO_2, FO_3 are concurrent, and because of the fact that the heights of any triangle are concurrent, the lines AO_1, BO_2, CO_3 meet in one point by the theorem above.

Does anybody know if that theorem has a name? And/or has a proof that does involve barycentric coordinates or complex numbers?

By the way, it was the hardest problem of BW 2009. Only one team solved it.

Luis González

#4 Jan 1, 2010, 1:15 am

> Sassha wrote:

It could also be proven in the following way making use of another theorem:

"Let $\triangle ABC$ be a triangle, and let D, E, F be points on sides BC, CA, AB, respectively, such that the cevians AD, BE, CF are concurrent. If M, N, P are points on EF, FD, DE, respectively, then the lines AM, BN, CP are concurrent if and only if the lines DM, EN, FP concur."

Does anybody know if that theorem has a name? And/or has a proof that does involve barycentric coordinates or complex numbers?

Dear Sassha the theorem you mentioned is known as Cevian Nest Theorem, you can see a synthetic proof in http://pagesperso-orange.fr/jl.ayme/ (Volume 3). But I'm afraid that this is not a direct application of the Cevian Nest Theorem as you claimed, since O_1, O_2, O_3 do not lie on the sidelines of the orthic triangle $\triangle DEF$.

Sung-yoon Kim

#5 Jan 13, 2010, 10:44 pm

I solved this using barycentric coordinates.

Given a triangle ABC, let $A_1 = (a_1, a_2, a_3), B_1 = (b_1, b_2, b_3), C_1 = (c_1, c_2, c_3)$.
$(a_1 + a_2 + a_3 = b_1 + b_2 + b_3 = c_1 + c_2 + c_3 = 1)$ Then if the barycentric coordinate of P with regard to the triangle $A_1B_1C_1$ is (x, y, z), the barycentric coordinate of P with regard to the triangle ABC is $(xa_1 + yb_1 + zc_1, xa_2 + yb_2 + zc_2, xa_3 + yb_3 + zc_3)$. Since $H = \frac{1}{\tan A + \tan B + \tan C}(\tan A, \tan B, \tan C)$, $E = \frac{1}{\tan A + \tan C}(\tan A, 0, \tan C)$, $F = \frac{1}{\tan A + \tan B}(\tan A, \tan B, 0)$, we can calculate the coordinate of O_1, and hence the coordinate of $P_1 = AO_1 \cap BC$. Apply Ceva's theorem.

jayme

#7 Oct 23, 2011, 8:48 pm

For a synthetic proof of the Jacobi's theorem, you can see
http://perso.orange.fr/jl.ayme vol. 5 Le theoreme de Jacobi
Sincerely
Jean-Louis

MathTwo

#8 Oct 23, 2011, 9:00 pm

There is also a succint solution using the trigonometric version of Ceva's theorem. First of all, we see that <DEF=180-2B, and because the altitude from B bisects this angle, we have that <FEH=90-B. Thus, <O_1EF=45-B/2, and <O_1FE=45-C/2.

Now, we see that [FAO_1]/[O_1AE] = AF*sin<FAO_1/AE*sin<EAO_1. However, we see that [FAO_1]/[O_1AE] is also equal to AF*FO_1*sin<AFO_1/AE*EO_1*sin<AEO_1. Thus, sin<FAO_1/sin<EAO_1= FO_1*sin<AFO_1/EO_1*sin<AEO_1. Now, because triangles AFE and ACB are similar <AFE=<ACB. Thus AFO_1=45-C/2+C=45+C/2.Similarly, <AEO_1=45+B/2. Using the law of sines in triangle FO_1E, we get that O_1E/O_1F=sin<45-C/2 / sin<45-B/2. Putting everything together, we have sin<FAO_1/sin<EAO_1 = sin(45-C/2)*sin(45+B/2)/sin(45-B/2)*sin(45+C/2). Taking the cyclic product of the LHS will then yield the desired result.

Quick Reply

High School Olympiads

Ratio segments

Reply

Source: Own

buratinogigle
#1 Mar 24, 2016, 10:22 pm • 1

Let ABC be a triangle with circumcenter O and P is any point. PB,PC cut CA,AB at E,F. The lines passes through P and is perpendicular to OP cuts BC,EF at M,N. XYZ symmetric of pedal triangle of P through P. Prove that if AX,BY,CZ are concurrent then $PM=2PN$.

Luis González
#2 Mar 24, 2016, 10:30 pm • 1

AX,BY,CZ are concurrent $\Longleftrightarrow$ P is on the Thomson cubic of $\triangle ABC$ $\Longleftrightarrow$ polar of P WRT $\odot(ABC)$ and trilinear polar of P WRT $\triangle ABC$ are parallel. Now use the result of the problem Metric relation XP=2YP.

Quick Reply

High School Olympiads

Metric relation XP=2YP

Source: Own

TelvCohl

#1 Dec 14, 2014, 8:50 pm • 3

Let O be the circumcenter of $\triangle ABC$.
Let ℓ_p be the trilinear polar of P WRT $\triangle ABC$.
Let $\triangle DEF$ be the cevian triangle of P WRT $\triangle ABC$.
Let a line through P perpendicular to OP intersect BC, EF at X, Y, respectively .

Prove that $OP \perp \ell_p \iff XP = 2YP$

This post has been edited 1 time. Last edited by TelvCohl, Aug 2, 2015, 12:06 am

shinichiman

#2 Dec 23, 2014, 4:31 pm • 3

It should be $XP = 2YP \iff \ell_P \parallel XY$. For convenience, I edited the problem:
Let ℓ_p be the trilinear polar of P with respect to $\triangle ABC$. Let $\triangle DEF$ be the cevian triangle of P with respect to $\triangle ABC$. EF, FD, DE cut BC, CA, AB at A_1, B_1, C_1 respectively. A line d through P cuts BC, CA, AB, EF, FD, DE at $A_2, B_2, C_2, A', B', C'$ respectively. Prove that $\ell_P \parallel d \iff PA_2 = 2PA'$.

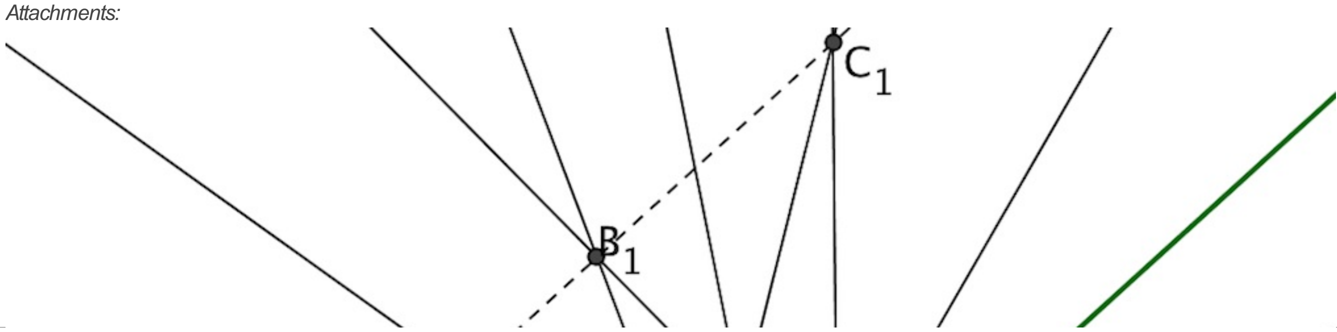
($\Longrightarrow$) If $\ell_P \parallel d$. It is easy to prove that $\frac{\overline{PB_2}}{\overline{PC_2}} = \frac{\overline{PB'}}{\overline{PC'}}$. Similarly, we can get $\frac{\overline{PB_2}}{\overline{PB'}} = \frac{\overline{PC_2}}{\overline{PC'}} = \frac{\overline{PA_2}}{\overline{PA'}} = -k$. We can see that $k \geq 0$.
We have $(BA, FC_1) = -1$ so $D(BA, FE) = D(A_2P, B'C') = -1$ or $(A_2P, B'C') = -1$. By Descartes' identity we have $\frac{2}{\overline{PA_2}} = \frac{1}{\overline{PB'}} + \frac{1}{\overline{PC'}}$ or $\frac{1}{\overline{PB'}} + \frac{1}{\overline{PC'}} + \frac{2}{k\overline{PA'}} = 0$. Similarly, we also obtain $\frac{1}{\overline{PB'}} + \frac{1}{\overline{PA'}} + \frac{2}{k\overline{PC'}} = 0$ and $\frac{1}{\overline{PA'}} + \frac{1}{\overline{PC'}} + \frac{2}{k\overline{PB'}} = 0$. Therefore

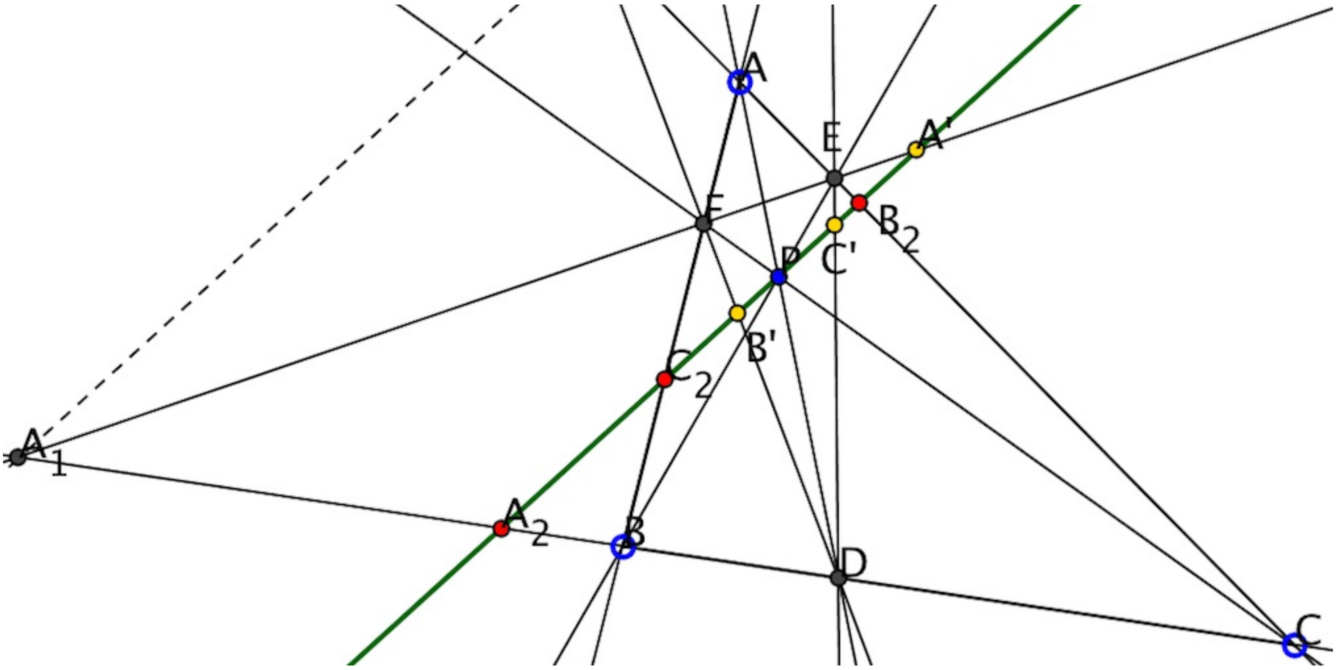
$$\left(\frac{2}{k} + 2\right)\left(\frac{1}{\overline{PB'}} + \frac{1}{\overline{PC'}} + \frac{1}{\overline{PA'}}\right) = 0.$$

Hence, $\frac{1}{\overline{PB'}} + \frac{1}{\overline{PC'}} + \frac{1}{\overline{PA'}} = 0$. We also get that $\frac{2}{\overline{PA_2}} = \frac{1}{\overline{PB'}} + \frac{1}{\overline{PC'}}$ so $\frac{2}{\overline{PA_2}} = \frac{-1}{\overline{PA'}}$ or $\overline{PA_2} = -2\overline{PA'}$. Thus, $PA_2 = 2PA'$.

($\Longleftarrow$). From $PA_2 = 2PA'$ we get $\frac{1}{\overline{PB'}} + \frac{1}{\overline{PC'}} + \frac{1}{\overline{PA'}} = 0$. And from here, we can easily obtain that $PB_2 = PB', PC_2 = PC'$ or $\frac{\overline{PB_2}}{\overline{PB'}} = \frac{\overline{PC_2}}{\overline{PC'}} = \frac{\overline{PA_2}}{\overline{PA'}} = -2$.We get $\frac{\overline{PB_2}}{\overline{PC_2}} = \frac{\overline{PB'}}{\overline{PC'}}$. From here we can find out that $\ell_P \parallel d$ by drawing a line $d' \parallel d$ through B_1 and prove that line also passes through C_1.

Attachments:





Luis González

#3 Dec 24, 2014, 10:58 am • 3

We work with the same initial notations of shinichiman's. The image of BC under homothety with center P and coefficient $-\frac{1}{2}$ will cut EF at only one point, thus it suffices to prove that $\ell_P \parallel d \Longrightarrow \overline{PA_2} : \overline{PA'} = -2.$ The converse is guarateed by the uniqueness of A'.

Let B_0, C_0 be the midpoints of $AC, AB. B_0C_0$ cuts EF at U and AU cuts BC at V. Since $(A, B, F, C_1) = -1 \Longrightarrow \overline{FB} : \overline{FC_0} = \overline{FC_1} : \overline{FA}$, but from $B_0UC_0 \parallel BC$, we get $\overline{FB} : \overline{FC_0} = \overline{FA_1} : \overline{FU} \Longrightarrow \overline{FC_1} : \overline{FA} = \overline{FA_1} : \overline{FU}$ $\Longrightarrow AU \parallel \ell_P \parallel d$. Now if AA_1 cuts ℓ at Q, we get $\overline{QA_2} : \overline{QA'} = \overline{AV} : \overline{AU} = 2$, but since $A_1(E, C, P, A) = (A', A_2, P, Q) = -1$, then $\overline{PA_2} : \overline{PA'} = -\overline{QA_2} : \overline{QA'} = -2.$

junior2001

#4 Dec 24, 2014, 11:39 am

wath is trilinear polar ?

TelvCohl

#5 Feb 20, 2015, 7:33 pm

> *shinichiman wrote:*
> It should be $XP = 2YP \iff \ell_P \parallel XY$.

Thank you **shinichiman** for point out this and thanks for **shinichiman** and **Luis González** for your nice solution
After thinking this problem again today, I found this problem is quite easy . I'll prove $\ell_P \parallel d \Longrightarrow \overline{PA_2} : \overline{PA'} = -2$:

Let $X = AA_1 \cap d$.

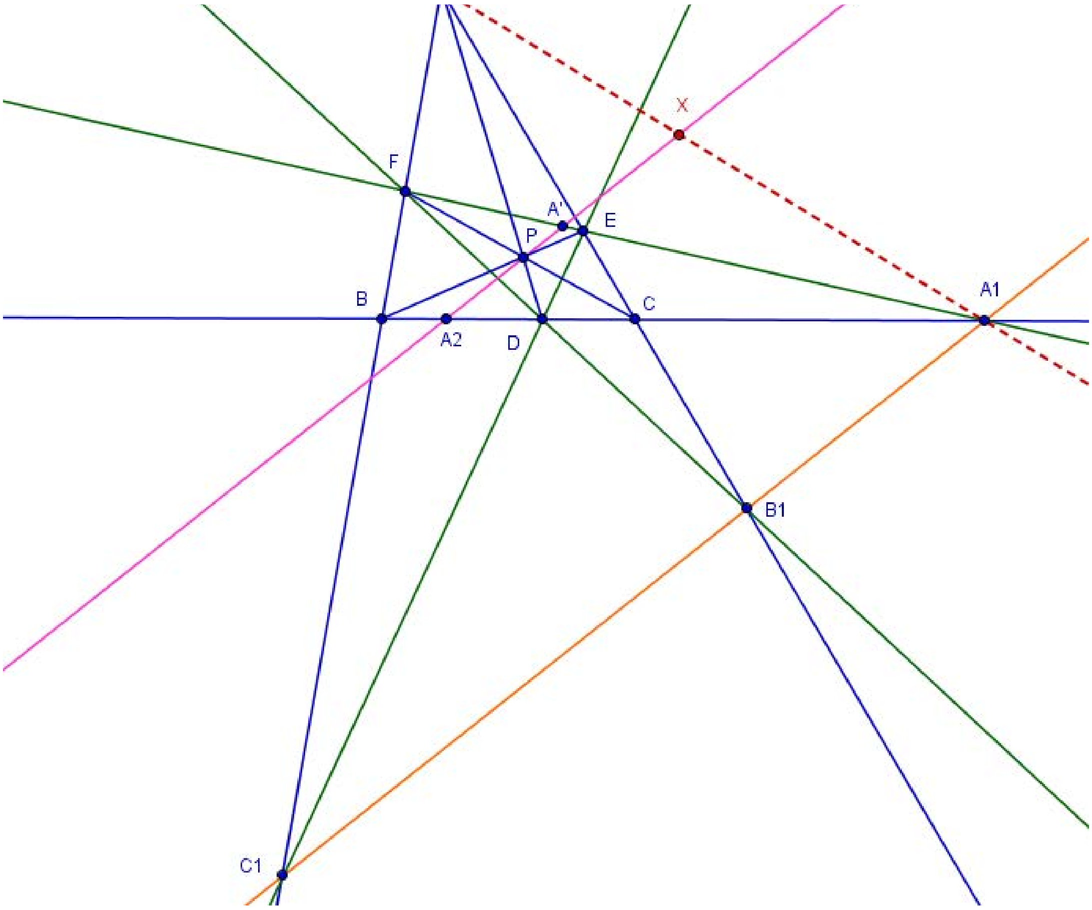
From $(A_1A', \ell_P; A_1X, A_1A_2) = -1$ and $\ell_P \parallel d \Longrightarrow A'$ is the midpoint of A_2X ,
so combine with $(X, P; A', A_2) = A_1(X, P; A', A_2) = -1$ we get $PA_2 = 2A'P$.

Done

> *junior2001 wrote:*
> wath is trilinear polar ?

See http://mathworld.wolfram.com/TrilinearPolar.html

Attachments:



toto1234567890

#6 Feb 21, 2015, 7:41 pm

Yes. It's quite easy but it is very interesting.

TelvCohl

#7 Feb 21, 2015, 7:59 pm • 1

Thank you **toto1234567890**

I found this problem when I tried to solve the problem A perspectrix perpendicular to OI (own)

This post has been edited 1 time. Last edited by TelvCohl, Aug 2, 2015, 12:07 am

toto1234567890

#8 Feb 22, 2015, 4:41 pm

At first I tried to use butterfly theorem and Pascal's Theorem and some harmonic Divisions but the first two wasn't needed.

It's quite funny that the intersection points of the lines and the line parallel to the trilinear polar and going through P are almost Homothetilike(?) .

Quick Reply

High School Olympiads

3 circles with common tangency point

Reply

Source: me

yetti
#1 Apr 10, 2012, 3:19 am

Let $ABCD$ be cyclic quadrilateral with circumcircle (O) and diagonal intersection $E \equiv AC \cap BD$.
Let the quadrilateral opposite sides meet at $F \equiv BC \cap AD$. WLOG, assume that points C, D, F are on the same side of the line AB.
Let $(J), (K)$ be incircles of curvilinear $\triangle ABE, \triangle ABF$, internally tangent to the arc $\overset{\frown}{AB}$ of circle (O).
Show that circles $(J), (K), (O)$ have common tangency point T.
Similarly...

Luis González
#2 Apr 10, 2012, 8:43 am • 1

Let U, V, W be the incenters of $\triangle ABC, \triangle ABD, \triangle ABE. UV$ cuts BC, AD at $P, Q. (J)$ touches EA, EB at M, N and (O) internally at T. According to the topic incenter of triangle, W, T, B, U, N and W, T, A, V, M are concyclic. Hence $\angle TVP = \angle TAC = \angle TBP \pmod \pi) \Longrightarrow T, B, P, V$ are concyclic. Likewise, T, A, Q, U are concyclic $\Longrightarrow$ $\angle TPB = \angle TVW = \angle TAU = \angle TQP \Longrightarrow BC$ is tangent to $\odot(TPQ)$ at P. Similarly, AD is tangent to $\odot(TPQ)$ at Q. Furthermore, $\angle BTP = \angle WVM = \angle WAC = \frac{1}{2}\angle BTC \Longrightarrow TP$ bisects $\angle BTC$, thus $\odot(TPQ)$ is tangent to (O) at $T \Longrightarrow (K) \equiv \odot(TPQ)$.

yetti
#3 Apr 10, 2012, 12:21 pm

I_C, I_D are incenters of $\triangle ABC, \triangle ABD$. AI_C, BI_C cut (O) again at X_C, Y_C and AI_D, BI_D cut (O) again at X_D, Y_D.
CI_C, DI_D meet at $Z \in (O)$.
I_CI_D cuts AB at S, AC, BD at P_J, Q_J and AD, BC at P_K, Q_K, respectively. SZ cuts (O) again at T.

Intersections $S \equiv AB \cap TZ, I_D \equiv DZ \cap BY_D$ and point $P_K \in AD$ are collinear $\Longrightarrow P_K \in ZY_D$ by Pascal theorem for $AZTBY_DD$.
Intersections $S \equiv AB \cap TZ, I_C \equiv CZ \cap AX_C$ and point $Q_K \in BC$ are collinear $\Longrightarrow Q_K \in ZX_C$ by Pascal theorem for $AZTBX_CC$.
Similarly, $P_J \in TY_C$ and $Q_J \in TX_D$.

ABI_CI_D is cyclic with circumcircle (Z) and $I_C \in AX_C, I_D \in BY_D \Longrightarrow$
$\angle AI_CI_D = \angle ABI_D \equiv \angle ABY_D = \angle AX_CY_D \Longrightarrow X_CY_D \parallel I_CI_D$.
Similarly, $X_DY_C \parallel I_CI_D$.

$\triangle TP_JQ_J \sim \triangle TY_CX_D$ are centrally similar with similarity center $T \in (O)$ and so are their circumcircles
$(J) \equiv \odot(TP_JQ_J), (O) \equiv \odot(TY_CX_D) \Longrightarrow$ they are internally tangent at T.
Tangents of (O) at Y_C, X_D are parallel to $AC, BD \Longrightarrow (J)$ is tangent to AEC, BED at P_J, Q_J.

$\triangle TP_KQ_K \sim \triangle TY_DX_C$ are centrally similar with similarity center $T \in (O)$ and so are their circumcircles
$(K) \equiv \odot(TP_KQ_K), (O) \equiv \odot(TY_DX_C) \Longrightarrow$ they are internally tangent at T.
Tangents of (O) at Y_D, X_C are parallel to $AD, BC \Longrightarrow (K)$ is tangent to ADF, BCF at P_K, Q_K.

Combined, $(J), (K), (O)$ are internally tangent at T.

The proposed problem was intended to provide an alternative solution of the problem at
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=463508.

armpist

#4 Apr 10, 2012, 6:49 pm

Dear Yetti, Luis and MLs,

Proposed problem is Poncelet theorem for a pencil
of circles. Amazing what a tender Frenchman can discover
when in a POW camp in Russian winter!

On top of what posted here, it also mentions another tangent circle
(for it would be frivolous and lighthearted to think that
the last pair of quadrilateral sides does not have a circle
with the same point touching qualities).

It also states that all 6 tangent points are collinear.

Now the question is : What are the incenters that this line
spears through (a la Sawayama) ?

Friendly,

M.T.

yetti

#5 Apr 11, 2012, 4:24 am

Assuming cyclic $ABCD$, with points $C, D, F \equiv BC \cap AD$ on the same side of the line AB.
Incenters I_C, I_D of $\triangle ABC, \triangle ABD$, C-excenter L_C of $\triangle BCD$ and D-excenter L_D of $\triangle ADC$ are collinear.

Suppose $(J), (K), (O)$ are 3 coaxal circles, forming intersecting, tangent, or non-intersecting pencil. $A \in (O)$ is arbitrary point.
Tangent to (J) from A cuts (O) again at C. Tangent to (K) from C cuts (O) again at B. Tangent to (J) from B cuts (O) again at D.
Then tangent to (K) from D always cuts (O) again at $A \Longrightarrow$ infinite number of solutions always exist for any 3 coaxal circles $(J), (K), (O)$, one for any $A \in (O)$.

I have never heard about Poncelet porism applied to a pencil of 3 circles. Perhaps you could give a referrence for your claim.
For now, I would say that Poncelet had something different in mind.

drmzjoseph

#6 Mar 18, 2015, 6:51 pm

General Problem
Let $ABCD$ be cyclic quadrilateral with circumcircle (O) and diagonal intersection $E \equiv AC \cap BD$.
Let (J) a circle tangent $\overrightarrow{EB}$ and $\overrightarrow{EA}$ in M and N respectively.$X \equiv BC \cap MN; Y \equiv AD \cap MN$
Let (K) the circle tangente to BC and AD in X and Y respectively.
Show that $(J), (K)$ and (O) are coaxial circles.

TelvCohl

#8 Mar 18, 2015, 7:25 pm • 1

> *drmzjoseph wrote:*
> **General Problem**
> Let $ABCD$ be cyclic quadrilateral with circumcircle (O) and diagonal intersection $E \equiv AC \cap BD$.
> Let (J) a circle tangent $\overrightarrow{EB}$ and $\overrightarrow{EA}$ in M and N respectively.$X \equiv BC \cap MN; Y \equiv AD \cap MN$
> Let (K) the circle tangente to BC and AD in X and Y respectively.
> Show that $(J), (K)$ and (O) are coaxial circles.

My solution:

Let $\mathcal{P}(P, \odot)$ be the power of a point P WRT circle $\odot$.

From Menelaus theorem we get $\frac{AY}{AN} = \frac{DY}{DM}$ and $\frac{BX}{BM} = \frac{CX}{CN}$ . ... $(\star)$

From $\angle YNA = \angle XMB, \angle NAY = \angle MBX \Longrightarrow \triangle AYN \sim \triangle BXM$,

so we get $\frac{\mathcal{P}(A, \odot(K))}{\mathcal{P}(A, \odot(J))} = \frac{AY^2}{AN^2} = \frac{BX^2}{BM^2} = \frac{\mathcal{P}(B, \odot(K))}{\mathcal{P}(B, \odot(J))}$,

hence combine with $(\star) \Longrightarrow \frac{\mathcal{P}(D, \odot(K))}{\mathcal{P}(D, \odot(J))} = \frac{\mathcal{P}(A, \odot(K))}{\mathcal{P}(A, \odot(J))} = \frac{\mathcal{P}(B, \odot(K))}{\mathcal{P}(B, \odot(J))} = \frac{\mathcal{P}(C, \odot(K))}{\mathcal{P}(C, \odot(J))}$.

i.e. $\odot(J), \odot(K), \odot(ABCD) \equiv \odot(O)$ are coaxial

Done

buratinogigle

#9 May 9, 2015, 6:54 pm • 1

Here is an extension of Poncelet theorem http://www.artofproblemsolving.com/community/c6h1076831p4710069

Acutally, I found this extension two years a ago, Tran Minh Ngoc wrote an aticle about it on Mathematics and Young of Vietnam, this is link of his article

https://tranminhngocctlhp.wordpress.com/2015/03/14/bo-de-poncelet-mo-rong-va-ung-dung/

Quick Reply

High School Olympiads

Mixtlinear Geometry Problem

Reply

comboisreallyhard

#1 Mar 23, 2016, 5:49 am • 2

In $\triangle ABC$, let M be the midpoint of the feet of the A-angle bisectors. Suppose T is the touchpoint of the A-mixtilinear incircle on $\odot(ABC)$; put M_a as the midpoint of major arc BAC and R as the midpoint of the midpoints of major arcs ABC and ACB. Show that lines MT and M_aR intersect on $\odot(ABC)$.

Luis González

#2 Mar 23, 2016, 6:26 am • 1

It's well-known that MA is tangent to $(O) \equiv \odot(ABC)$ and T, I, M_a are collinear. If I_a, I_b, I_c are the excenters of $\triangle ABC$ againts A, B, C, then clearly $I_aR \equiv I_aM_a$ is the I_a-median of $\triangle I_aI_bI_c \Longrightarrow I_aM_a$ cuts (O) again at the tangency point S of the A-mixtilinear excircle of $\triangle ABC$ with (O) (extraversion of the mixtilinear incircle case).

Let X and Y be the tangency points of the incircle (I) and A-excircle (I_a) with BC. The inversion with center A, power $AB \cdot AC$ followed by reflection on AI swaps (O) and BC, thus it swaps (I) and the A-mixtilinear excircle and (I_a) and the A-mixtilinear incircle. This inversion then swaps (X, S), (Y, T) and (M, U), being U the 2nd intersection of (O) with the parallel from A to BC. By symmetry $AUYX$ is an isosceles trapezoid (cyclic), thus S, T, M are collinear and the conclusion follows.

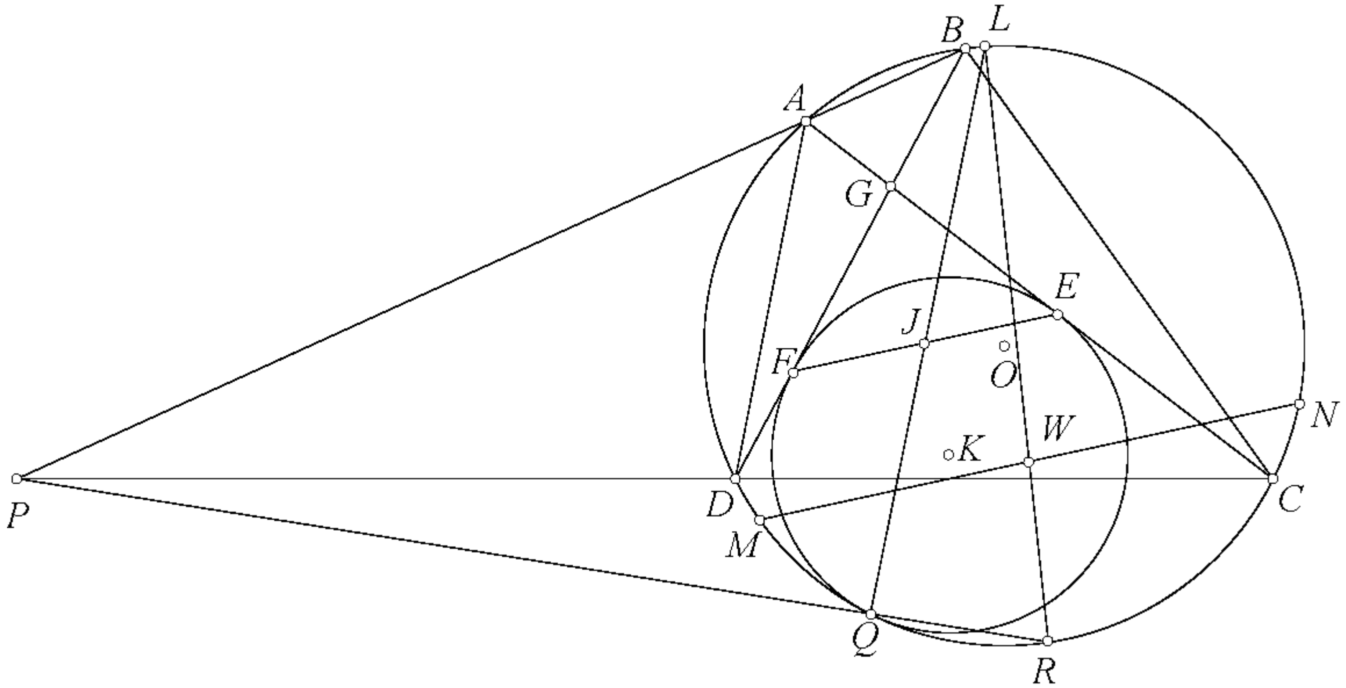
buratinogigle

#3 Mar 23, 2016, 10:51 pm • 1

Nice problem, this is general problem.

Let $ABCD$ be cyclic quadrilateral inscribed in circle (O) with AC cuts BD at G, AB cuts CD at P. Circle (K) is tangent to segment GC, GD at E, F and (O) at Q. M, N are midpoints of major arcs BC, AD. J is midpoint of EF. QJ cuts (O) again at L. PQ cuts (O) again at R. Prove that LR bisects segment MN.

Attachments:



Luis González

#4 Mar 24, 2016, 2:26 am • 3

> *buratinogigle wrote:*
>
> Nice problem, this is general problem.
>
> Let $ABCD$ be cyclic quadrilateral inscribed in circle (O) with AC cuts BD at G, AB cuts CD at P. Circle (K) is tangent to segment GC, GD at E, F and (O) at Q. M, N are midpoints of major arcs BC, AD. J is midpoint of EF. QJ cuts (O) again at L. PQ cuts (O) again at R. Prove that LR bisects segment MN.

Let $H \equiv AD \cap BC$. ω_1 is the circle tangent to $\overline{HA}, \overline{HB}$ and externally tangent to (O) at S. ω_2 is the circle tangent to the extensions of $\overline{AD}, \overline{BC}$ and externally tangent to (O) at R^*. Using the result of the problem 3 circles with common tangency point, we get that there are circles λ_1, λ_2 tangent to AC, BD and externally tangent to (O) at S, R^*, resp. Similarly there are circles ϵ_1, ϵ_2 tangent to AB, CD and externally tangent to (O) at Q, R^*. Since P is the exsimilicenter of $\epsilon_1 \sim \epsilon_2$ and Q, R^* are the insimilicenters of $(O) \sim \epsilon_1$ and $(O) \sim \epsilon_2$, then P, Q, R^* are collinear $\Longrightarrow R \equiv R^*$. Since G, S are the insimilicenters of $(K) \sim \lambda_1, (O) \sim \lambda_1$ and Q is the exsimilicenter of $(K) \sim (O)$, then Q, G, S are collinear. Since H is the exsimilicenter of $\omega_1 \sim \omega_2$ and S, R are the insimilicenters of $(O) \sim \omega_1$ and $(O) \sim \omega_2$, then H, S, R are collinear.

Let U and V be the tangency points of ω_1 and ω_2 with AD. From the external tangency of (O) and ω_1, we deduce that SU is external bisector of $\angle ASD$, i.e. $N \in SU$ and likewise $N \in RV$. As the exsimilicenter H of $\omega_1 \sim \omega_2$ is also center of their direct inversion, it follows that $HU \cdot HV = HS \cdot HR \Longrightarrow UVRS$ is cyclic $\Longrightarrow NU \cdot NS = NR \cdot NV \Longrightarrow N$ has equal power WRT ω_1, ω_2. Similarly M has equal power WRT $\omega_1, \omega_2 \Longrightarrow MN$ is the radical axis of $\omega_1, \omega_2 \Longrightarrow MN$ and the internal common tangents of $(O), \omega_1$ and $(O), \omega_2$ concur at the radical center of $(O), \omega_1, \omega_2 \Longrightarrow SMRN$ is harmonic $(\star)$.

Let QE, QF cut (O) again at X, Y. Since Q is the exsimilicenter of $(K) \sim (O)$, then $XY \parallel EF \parallel MN$. QG, being the Q-symmedian of $\triangle QEF$, is the isogonal of its Q-median QJ WRT $\angle EQF \equiv \angle XQF \Longrightarrow LS \parallel XY \parallel MN$. Together with $(\star)$, we get $L(R, M, N, S) = -1 \Longrightarrow LR$ passes through the midpoint of MN, as desired.

Quick Reply

High School Olympiads

Line passes through centroid

Source: Own

buratinogigle

#1 Mar 22, 2016, 8:27 pm • 1

Let ABC be a triangle inscribed in circle (O) and incenter I. Circle (ω_a) touches (O) at A and touches BC at D. Similarly, we have circles $(\omega_b), (\omega_c)$ and E, F. Let K be radical center of $(\omega_a), (\omega_b), (\omega_c)$. Prove that line IK passes through centroid of triangle DEF.

Luis González

#2 Mar 23, 2016, 5:23 am • 2

You rediscovered the 2nd Hatzipolakis-Yiu point; X_{595} on ETC, which was exactly defined as the radical center of those 3 circles. Circle ω_a passing through $A \equiv (1:0:0)$ and tangent to BC at $(0:b:c)$ has barycentric equation

$$\omega_a \equiv a^2yz + b^2zx + c^2xy - \frac{a^2(x+y+z)}{(b+c)^2}(c^2y + b^2z) = 0.$$

Similarly, the equations of ω_b and ω_c are given by

$$\omega_b \equiv a^2yz + b^2zx + c^2xy - \frac{b^2(x+y+z)}{(c+a)^2}(a^2z + c^2x) = 0$$

$$\omega_c \equiv a^2yz + b^2zx + c^2xy - \frac{c^2(x+y+z)}{(a+b)^2}(b^2x + a^2y) = 0.$$

Their radical center is then $K \equiv X_{595} \equiv (a^2(a^2 + ab + ac - bc))$. Thus easy algebraic manipulation shows that K is indeed on the line through $I \equiv (a:b:c)$ and the centroid $(a(b+c)(2a+b+c) : b(c+a)(2b+a+c) : c(a+b)(2c+a+b))$ of $\triangle DEF$. Furthermore this lines also passes throuh the Schiffler point X_{21} of $\triangle ABC$.

High School Olympiads

Radical center and collinear !

Locked

Source: own

TacH

#1 Mar 22, 2016, 11:07 pm

Let ABC be a scalene triangle I, O be Incenter and circumcenter
Let the Incenter tangent to BC, CA, AB at D, E, F respectively
Line EF cut the circumcircle of $\triangle ABC$ at D_1, D_2
Line ED cut the circumcircle of $\triangle ABC$ at F_1, F_2
Line DF cut the circumcircle of $\triangle ABC$ at E_1, E_2
Let S be the radical center of circumcircle of $\triangle DD_1D_2, \triangle EE_1E_2$ and$\triangle FF_1F_2$
Prove that S, I, O are collinear

Luis González

#2 Mar 23, 2016, 12:25 am

It's an old problem. See Radical center of DEF and Radical center and line OI (second problem).

High School Olympiads

Radical center of DEF

Reply

mathVNpro

#1 Jan 26, 2009, 8:04 am

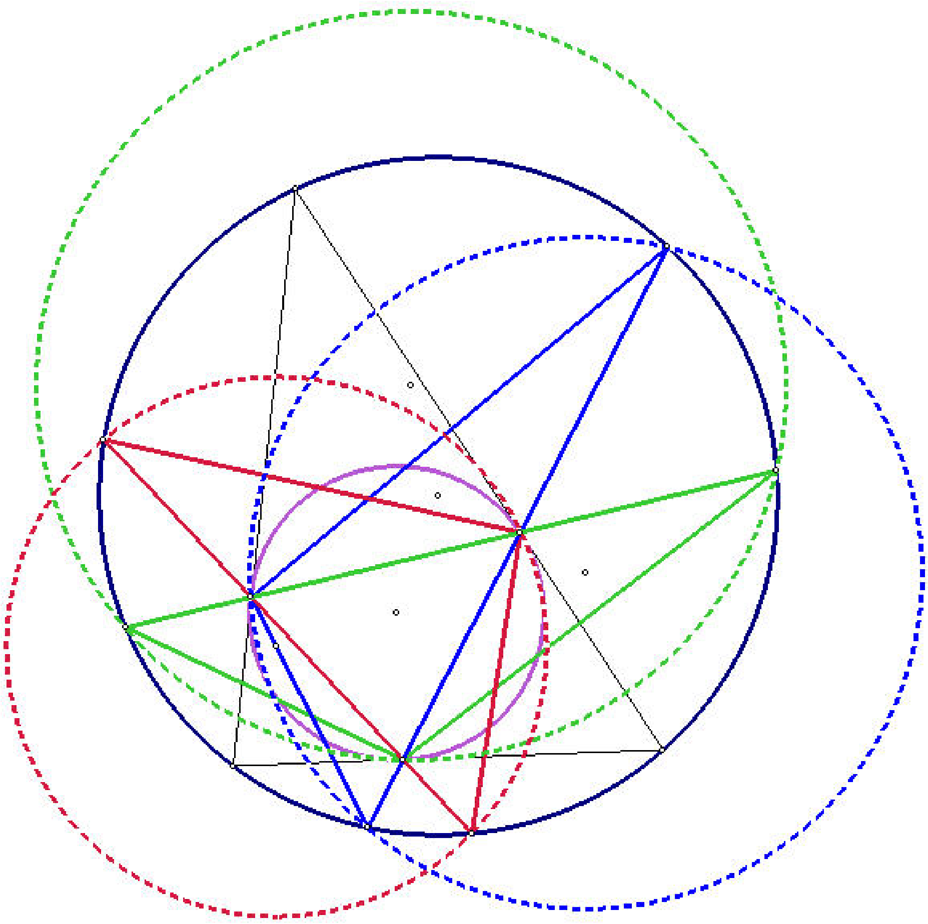
Let triangle ABC inscribed in circumcircle (O), (I) is its incenter. Let D, E, F perspectively be the tagency point of (I) with BC, CA, AB. EF intersects (O) at X_ 1, X_2. Similar for Y_1, Y_2; Z_1, Z_2. Prove that the radical center of (DX_1X_2), (EY_1Y_2), (FZ_1Z_2) lies on the Euler line of triangle DEF.

mathVNpro

#2 Jan 26, 2009, 8:22 am

Here is the figure

Attachments:



mathVNpro

#3 Jan 26, 2009, 11:07 pm

Mashimaru has had the solution, , he will posted it later

The QuattoMaster 6000

#4 Jan 28, 2009, 12:21 pm • 1

Solution

yetti

#5 Jan 28, 2009, 12:49 pm

Let the angle bisectors AI, BI, CI cut the circumcircle (O) again at $X, Y, Z.$ Let XX' be a diameter of $(O).$ I is orthocenter of $\triangle XYZ \Longrightarrow IYX'Z$ is a parallelogram, its diagonals IX', YZ cutting each other in half at $K.$ Let (K) be a circle with center K and radius $KD.$ IX' cuts (O) again at $T,$ which is tangency point of the mixtilinear A-incircle (J_a) of the $\triangle ABC$ with its circumcircle $(O).$

Parallel to XT through D cuts (I) again at $P.$ The angle $\angle X'TX$ is right, $IK \perp XT.$ Therefore, $DP \perp IK$ is radical axis of $(I), (K).$ Perpendicular to AI through I cuts CA, AB at E_a, F_a; these are tangency points of (J_a) with $CA, AB.$ The parallels $EF \parallel E_aF_a$ are similar with center A and coefficient $k = \dfrac{AI}{AJ_a}.$ XT cuts (J_a) again at $D_a.$ T is the external similarity center of $(O), (J_a) \Longrightarrow$ tangent of (J_a) at D_a is parallel to tangent of (O) at $X,$ which is parallel to tangent of (I) at $D.$ Since A is the external similarity center of $(I), (J_a),$ it follows that A, D, D_a are collinear. But then the parallels $DP \parallel XT$ are similar with the same center A and coefficient $k = \dfrac{AI}{AJ_a}.$

$(U), (V)$ are circles with diameters IA, IX and common tangent E_aF_a at $I.$ (U) goes through E, F and (V) goes through T on account of the right angles at these points. (U) cuts (O) again at $S.$ Pairwise radical axes AS, E_aF_a, XT of $(O), (U), (V)$ meet at their radical center $Q.$ Pairwise radical axes AS, EF of $(O), (U)$ and $(U), (I)$ meet at the radical center R of $(O), (U), (I).$ Radical axis of $(O), (I)$ goes through $R.$ From similarity of the parallels $EF \parallel E_aF_a$ and $DP \parallel XT$ with the same center A and coefficient $k,$ it follows that the radical axis DP of $(I), (K)$ also goes through $R.$ But then R is also radical center of $(O), (I), (K)$ and the radical axis of $(O), (K)$ also goes through $R.$ But this radical axis is perpendicular to the center line $OK,$ i.e., parallel to $EF,$ hence it is identical with $EF.$ As a result, (K) is the circle $\odot(DX_1X_2).$ In exactly the same way, circles $(L), (M)$ centered at the midpoints L, M of ZX, XY and with radii LE, MF are the circles $\odot(EY_1Y_2), \odot(FZ_1Z_2).$

D is radical center of the circles $(O), (L), (M).$ Radical axis of $(L), (M)$ goes through D and is perpendicular to their center line $LM \parallel YX \parallel EF,$ it is the D-altitude of the $\triangle DEF.$ Similarly, radical axes of $(M), (K)$ and $(K), (L)$ are the E- and F-altitudes, they meet at the orthocenter of the $\triangle DEF$ on its Euler line.

mathVNpro

#6 Jan 29, 2009, 10:17 pm

This beautiful geometry problem can be stated in another way, this statement is a little more complicated than the one I had posted: "Prove that the radical center of (DX_1X_2), (EY_1Y_2), (FZ_1Z_2) lies on OI." Actually, to prove this, it is not really hard . We just have to prove that O lies on the Euler line of triangle DEF by using the inversion with Pole and Polar. You can try

April

#7 Jan 30, 2009, 5:56 am • 1

This problem was proposed by **Cosmin Pohoata** and **Darij Grinberg** on *Mathematical Reflections*. And here is my solution:

Let S be the intersection of EF with the sideline BC, and M is the mid-point of BC. We have the division $(BDCS)$ is a harmonic, so that $SD \cdot SM = SB \cdot SC = SX_1 \cdot SX_2$. Thus, M lies on the circumcircle of triangle DX_1X_2.

Let us call P, Q the mid-points of arcs BC and AB of the circle $\rho(O)$ (not containing A, C), respectively. Let N the mid-point of the arc BC containing A of $\rho(O)$, O_1 the mid-point of the segment IN. We have OO_1 is parallel to IP, it means OO_1 is perpendicular to X_1X_2, and so it's also the perpendicular bisector of X_1X_2. On the other hand, we have the quadrilateral $IDMN$ is a trapezoid, so O_1 lies on the perpendicular bisector of DM. Now, we have the quadrilateral DMX_2X_1 is cyclic and O_1 is the intersection of the perpendicular bisectors of the segments DM and X_1X_2, so that O_1 is the center of the circumcircle of triangle DX_1X_2.

We can easily to see that OO_1 is parallel to IP and $OO_1 = \dfrac{1}{2} \cdot IP$. Similarly, OO_3 is parallel to IQ and $OO_3 = \dfrac{1}{2} \cdot IQ$, where O_3 is the center of the circumcircle of triangle FZ_1Z_2. Hence, we obtain O_1O_3 is parallel to PQ. Now, let's take a look at the circumcircles of triangles DX_1X_2 and FZ_1Z_2, we have the radical exits of the circles is the line d_2, which passes through E and perpendicular to O_1O_3, i.e. $d_2 \perp PQ$. It follows d_2 is the E-altitude of triangle DEF. Similarly, we have the radical axis d_3 of the circumcircles of triangles DX_1X_2 and EY_1Y_2 is the F-altitude of triangle DEF. Hence, we deduce the radical center of the circles DX_1X_2, EY_1Y_2, and FZ_1Z_2 is the orthocentre of triangle DEF, which has known that lies on OI. The solution is completed.

Hope that I didn't make any mistake!

mathVNpro

#8 Jan 30, 2009, 7:20 am

When I proved this problem, I didn't use any midpoints of AB, BC, CA. I noticed that if we let C1, B1 respectively be the midpoint of arc AB, AC then the midpoint of B1C1 are the center of the circumcircle of triangle DX1X2 . To prove this I use the similarity that maps (I) to (O), and directed angle

Then I also noticed that the radical center of (DX_1X_2), (EY_1Y_2), (FZ_1Z_2) are the orthocenter of triangle DEF, hence it lies on the Euler line of triangle DEF.

Proof is done then!!!

To April, Yetti, and The QuattoMaster 6000: Thanks for your all nice solution, you have widened my eyes

To April: May I ask your in Vietnamese???? I am just curious

Quick Reply

High School Olympiads

Radical center and line OI

Reply

Source: Own

livetolove212

#1 Aug 22, 2009, 4:32 pm

Given triangle ABC with incircle (I), circumcircle (O). (I) touches BC, AC, AB at A', B', C'. Prove that the radical center L of $(A, AA'), (B, BB'), (C, CC')$ lies on the line OI. Moreover, if $B'C'$ cuts (O) at A_1, A_2, similar for B_1, B_2, C_1, C_2 then L is the radical center of $(A'A_1A_2), (B'B_1B_2)$ and $(C'C_1C_2)$

Luis González

#2 Sep 3, 2009, 1:00 pm • 1

Let L be the orthocenter of the intouch triangle $\triangle A'B'C'$ lying on its Euler line IO. H is the foot of the perpendicular from A' to $B'C'$ and S is the intersection of $A'H$ with the external bisector of $\angle BAC$. Let $\mathcal{P}_a$ denote the power of L WRT (A, AA'). Then we have $\mathcal{P}_a = (AA')^2 - AL^2$, but since $AS \perp SA' \Longrightarrow \mathcal{P}_a = (SA')^2 - LS^2$

$$\mathcal{P}_a = (LS + LA')^2 - LS^2 = (LA')^2 + 2LS \cdot LA'$$

$$\mathcal{P}_a = (LA')^2 + 2LA'(SH + LH) = (LA')^2 + 2LA' \cdot SH + 2LA' \cdot LH$$

Denote $J \equiv AI \cap B'C'$. We have then

$$\mathcal{P}_a = (2IJ)^2 + 4IJ \cdot SH + 2LA' \cdot LH = 4IJ(IJ + SH) + 2LA' \cdot LH$$

$$\mathcal{P}_a = 4IJ \cdot IA + 2LA' \cdot LH = 4r^2 + 2LA' \cdot LH$$

$LA' \cdot LH$ equals the power of the inversion with center L taking (I) into the nine-point circle of $\triangle A'B'C'$. Hence, $\mathcal{P}_a$ is symmetric in terms of $a, b, c \Longrightarrow L$ has equal power WRT the circles $(A, AA'), (B, BB'), (C, CC')$. $\blacksquare$

Let $D \equiv B'C' \cap BC$ and M_a be the midpoint of BC. From the harmonic cross ratio (B, C, A', D), it follows that $DB \cdot DC = DM_a \cdot DA'$, but D has equal power WRT (O) and $\odot(A'A_1A_2) \Longrightarrow DB \cdot DC = DA_1 \cdot DA_2 \Longrightarrow DA_1 \cdot DA_2 = DM_a \cdot DA \Longrightarrow M_a \in \odot(A'A_1A_2)$. Analogously M_b, M_c lie on $\odot(B'B_1B_2), \odot(C'C_1C_2)$, respectively. Center O_a of $\odot(A'A_1A_2)$ is the intersection of the perpendicular bisector of M_aA' with the perpendicular to A_1A_2 through O. In other words, if A'' is the intersection of IA' with the perpendicular to A_1A_2 from O, then O_a is the midpoint of OA''. Similarly, O_b, O_c are midpoints of OB'', OC''. Note that $\triangle OA''B''$ and $\triangle LA'B'$ are homothetic with homothety center I, since $OA'' \parallel LA', OB'' \parallel LB'$ and $OL, A'A'', B'B''$ concur at I. Thereby, $A''B'' \parallel A'B' \parallel O_aO_b$. Hence, radical axis $\mathcal{L}_c$ of $(O_a), (O_b)$ goes through C', due to $C'A_1 \cdot C'A_2 = C'B_1 \cdot C'B_2$ and it is orthogonal to $O_aO_b \parallel A'B' \Longrightarrow \mathcal{L}_c$ is identical with the C'-altitude of $\triangle A'B'C'$. Analogously, $\mathcal{L}_a, \mathcal{L}_b$ are identical with the A'- and B'- altitudes of $\triangle A'B'C' \Longrightarrow L$ is the radical center of $(O_a), (O_b), (O_c)$. $\blacksquare$

livetolove212

#3 Sep 3, 2009, 2:13 pm

Dear **Luis**,
Thanks you very much for your nice solution! I have learned so much from you
The orthocenter of triangle $A'B'C'$ is the radical center of three circles $(A'A_1A_2), (B'B_1B_2), (C'C_1C_2)$ was mentioned in a nice problem which was proposed by **Cosmin Pohoata** and **Darij Grinberg**.
My solution is base on this lemma: "Given triangle ABC and circumcenter O, incenter I. M is inside triangle ABC. $MA' \perp BC, MB' \perp AC, MC' \perp AB$ then M lies on OI iff $AC' + BA' + CB' = p$"

Quick Reply

High School Olympiads

Major - ization

Reply

Source: Euler - SIN

armpist

#1 Jun 27, 2009, 6:56 am

It seems that the Euler triangle identity, although has been very very good,
still does not go far enough. The following is not full generalization,
but maybe can claim a majorization status.

The product of distances from circumcenter to two isogonal conjugate points
equals the product of distance from their midpoint to nine-point center and
circumdiameter.

With the diameter being part of the equality, it could be viewed as some expansion
of the SIN theorem as well.

M.T.

Attachments:
Euler-SIN.doc (25kb)

Luis González

#2 Mar 22, 2016, 1:10 pm • 1

Problem: Let $\odot(O,R)$ and N be the circumcircle and 9-point center of $\triangle ABC$. P,Q are isogonal conjugates WRT $\triangle ABC$ and T is the midpoint of $\overline{PQ}$ (center of the pedal circle of P,Q), then $NT=\frac{OP\cdot OQ}{2R}$.

Proof: Let $\triangle DEF$ be the antipedal triangle of P WRT $\triangle ABC$ with circumricle $\odot(U,\rho)$ and let $\triangle Q_AQ_BQ_C$ be the pedal triangle of Q WRT $\triangle ABC$. P^* is the isogonal conjugate of P WRT $\triangle DEF$ and Q^* is the isogonal conjugate of Q WRT $\triangle Q_AQ_BQ_C$. X,Y,Z are the circumcenters of $\triangle PBC, \triangle PCA, \triangle PAB$ and X',Y',Z' are the reflections of X,Y,Z on BC,CA,AB.

Clearly the 9-point centers A_P,B_P,C_P of $\triangle PBC, \triangle PCA, \triangle PAB$ are the midpoints of $PX',PY',PZ' \Longrightarrow$ $\triangle A_PB_PC_P$ is the image of $\triangle X'Y'Z'$ under homothety $\mathcal{H}(P,\tfrac{1}{2}) \Longrightarrow$ radius of $\odot(A_PB_PC_P)$ is half the radius ϱ of $\odot(A'B'C')$, but $\varrho=\tfrac{1}{2}PU$ (see area of triangle ABC; remark at post #2). Moreover, we have $\triangle A_PB_PC_P \cup N \cup T \sim \triangle Q_AQ_BQ_C \cup Q \cup Q^* \sim \triangle DEF \cup P^* \cup P$ (see Similar triangle related to 9-point center). Consequently

$$\frac{NT}{PP^*}=\frac{\frac{1}{2}\varrho}{\rho} \Longrightarrow NT=\frac{2\cdot OP\cdot \frac{1}{4}PU}{\rho}=\frac{OP\cdot PU}{2\rho} \quad (1).$$

On the other hand, since $\triangle DEF$ and $\triangle Q_AQ_BQ_C$ are homothetic, then by Gergonne-Ann theorem, we get $[ABC]^2=[DEF]\cdot[Q_AQ_BQ_C]$. Using Euler's pedal triangle theorem this expression is equivalent to

$$\frac{[ABC]}{[DEF]}=\frac{[Q_AQ_BQ_C]}{[ABC]} \Longrightarrow \frac{\rho^2-PU^2}{4\rho^2}=\frac{R^2-OQ^2}{4R^2} \Longrightarrow \frac{PU}{\rho}=\frac{OQ}{R} \quad (2).$$

Combining (1) and (2) yields $NT=\dfrac{OP\cdot OQ}{2R}$.

TelvCohl

#3 Mar 22, 2016, 3:01 pm • 1

> *Luis González wrote:*
> **Problem:** Let $\odot(O, R)$ and N be the circumcircle and 9-point center of $\triangle ABC$. P, Q are isogonal conjugates WRT $\triangle ABC$ and T is the midpoint of $\overline{PQ}$ (center of the pedal circle of P, Q), then $NT = \frac{OP \cdot OQ}{2R}$.

Let $\triangle A_1B_1C_1$ be the circumcevian triangle of P WRT $\triangle ABC$ and A_2, B_2, C_2 be the reflection of A_1, B_1, C_1 in BC, CA, AB, respectively. From Some properties of Hagge circle we get A_2, B_2, C_2 and the orthocenter H of $\triangle ABC$ are concyclic. Furthermore, if Q^* is the reflection of Q in N, then $\triangle A_1B_1C_1 \cup O \cup P \sim \triangle A_2B_2C_2 \cup Q^* \cup P$, so

$$NT = \frac{Q^*P}{2} = \frac{OP}{2} \cdot \frac{\text{radius of} \odot (A_2B_2C_2)}{\text{radius of} \odot (A_1B_1C_1)} = \frac{OP}{2} \cdot \frac{HQ^*}{R} = \frac{OP \cdot OQ}{2R}.$$

Quick Reply

High School Olympiads

area of triangle ABC

Reply

Source: vankhea

vankhea

#1 Dec 26, 2013, 7:46 am

Let P be any point in the plane of triangle ΔABC. Let X, Y, Z be circumcenter of triangles $\Delta APB, \Delta BPC, \Delta CPA$ respectively. X', Y', Z' be points of reflection X, Y, Z with mirror AB, BC, CA respectively.

Prove that $\left|[XYZ] - [X'Y'Z']\right| = [ABC]$

Luis González

#2 Mar 22, 2016, 1:09 pm

Relabel X, Y, Z the circumcenters of $\triangle PBC, \triangle PCA, \triangle PAB$, thus X', Y', Z' are the reflections of X, Y, Z on BC, CA, AB. Let $\triangle DEF$ be the antipedal triangle of P WRT $\triangle ABC$ (D, E, F againts A, B, C). Denote $\widehat{A}, \widehat{B}, \widehat{C}$ and $\widehat{D}, \widehat{E}, \widehat{F}$ the angles of $\triangle ABC$ and $\triangle DEF$. Easy angle chase reveals that $\widehat{Y'AZ'} = |\widehat{A} - \widehat{D}|$ and $\widehat{EPF} = \widehat{A} + \widehat{D}$, thus by cosine theorem, we obtain:

$$Y'Z'^2 = AY'^2 + AZ'^2 - 2 \cdot AY' \cdot AZ' \cdot \cos(\widehat{A} - \widehat{D}) = AY^2 + AZ^2 - 2 \cdot AY \cdot AZ \cdot \cos(\widehat{A} - \widehat{D}) =$$

$$= \tfrac{1}{4}(PE^2 + PF^2) - \tfrac{1}{2}PE \cdot PF \cos(\widehat{A} - \widehat{D}) = \tfrac{1}{4}(EF^2 + 2 \cdot PE \cdot PF \cdot \cos \widehat{EPF}) - \tfrac{1}{2}PE \cdot PF \cos(\widehat{A} - \widehat{D}) =$$

$$= \tfrac{1}{4}EF^2 - \tfrac{1}{2}PE \cdot PF[\cos(\widehat{A} - \widehat{D}) - \cos(\widehat{A} + \widehat{D})] = \tfrac{1}{4}EF^2 - PE \cdot PF \cdot \sin \widehat{A} \cdot \sin \widehat{D} =$$

$$= \tfrac{1}{4}EF^2 - AB \cdot AC \cdot \frac{\sin \widehat{A} \cdot \sin \widehat{D}}{\sin \widehat{E} \cdot \sin \widehat{F}} = \tfrac{1}{4}EF^2 - \frac{[ABC]}{[DEF]} \cdot EF^2 \Longrightarrow \frac{Y'Z'^2}{EF^2} = \frac{1}{4} - \frac{[ABC]}{[DEF]} \ (\star).$$

Analogously we get the same expression for the ratios $\tfrac{Z'X'^2}{FD^2}$ and $\tfrac{X'Y'^2}{DE^2}$, which means that $\triangle X'Y'Z' \sim \triangle DEF$ and since $\triangle XYZ$ is the image of $\triangle DEF$ under homothety $\mathcal{H}(P, \tfrac{1}{2})$, then $[XYZ] = \tfrac{1}{4}[DEF]$. Therefore the expression $(\star)$ becomes

$$\frac{[X'Y'Z']}{[DEF]} = \frac{[XYZ]}{[DEF]} - \frac{[ABC]}{[DEF]} \Longrightarrow [ABC] = [XYZ] - [X'Y'Z'].$$

Remark: Let ρ, ϱ denote the radii of the circumcircles of $\triangle DEF, \triangle X'Y'Z'$, resp and let U denote the circumcenter of $\triangle DEF$. Using Euler's pedal triangle theorem, the latter area relation is equivalent to

$$\frac{\varrho^2}{\rho^2} = \frac{1}{4} - \frac{\rho^2 - PU^2}{4\rho^2} \Longrightarrow PU = 2\varrho.$$

TelvCohl

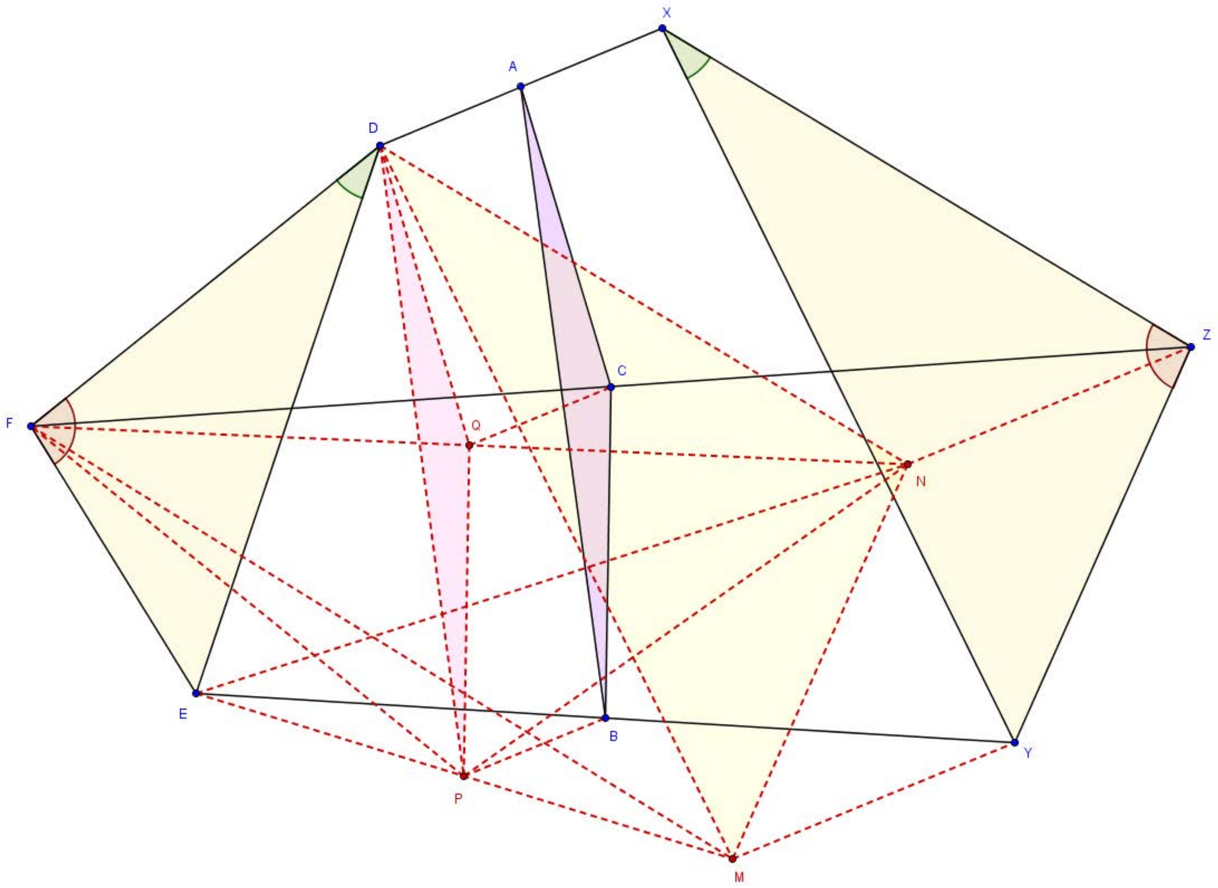
#3 Mar 22, 2016, 8:37 pm

Generalization : Given two triangles $\triangle DEF, \triangle XYZ$ such that $\triangle DEF \overset{-}{\sim} \triangle XYZ$. Let A, B, C be the midpoint of DX, EY, FZ, respectively. Then $|[DEF] - [XYZ]| = 4[ABC]$.

Proof : Let M, N be the image of Y, Z under translation $\mathbf{T}(\overrightarrow{XD})$, resp. and P, Q be the midpoint of EM, FN, respectively. Obviously, $\triangle DPQ$ is the image of $\triangle ABC$ under the translation $\mathbf{T}(\tfrac{1}{2}\overrightarrow{XD})$. From $DE \cdot DN = DF \cdot DM$ and $\angle EDN = \angle FDM$ we get $[DEN] = [DFM]$, so we conclude that

$$[DMN]-[DEF] = [DMN]+[DEN]-[DEF]-[DFM] = 2[DPN]-2[DPF] = 4[DPQ].$$

Attachments:



Quick Reply

High School Olympiads

Similar triangle related to 9-point center

Reply

Source: Own

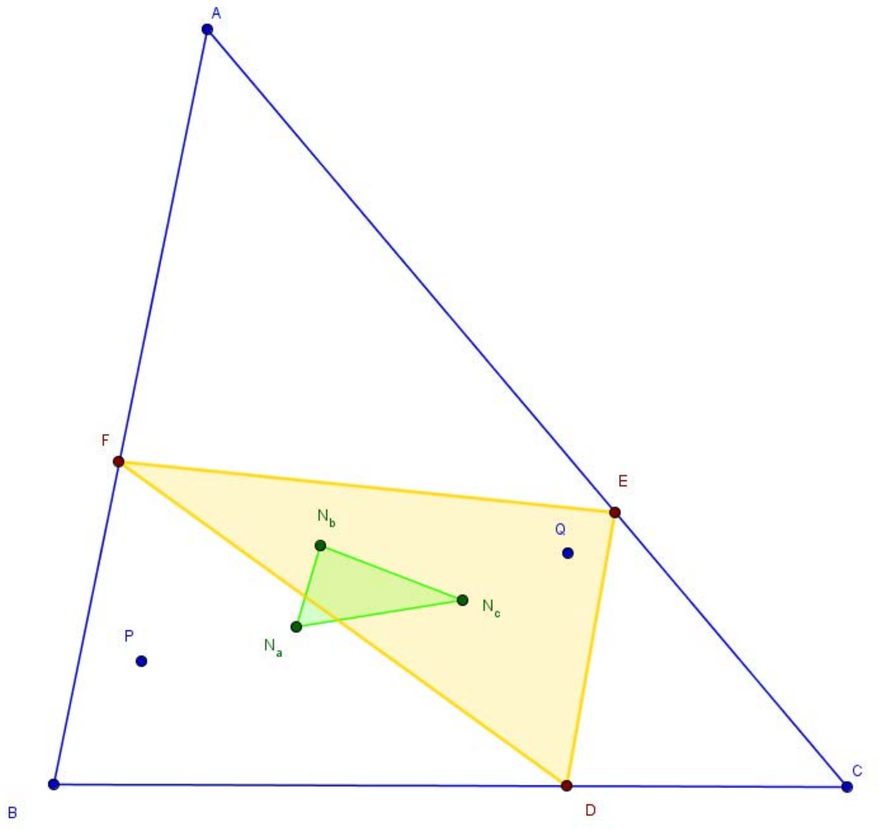
TelvCohl

#1 Jan 29, 2015, 8:04 pm

Let P, Q be the isogonal conjugate of $\triangle ABC$.
Let D, E, F be the projection of Q on $BC, CA, AB,$ respectively .
Let N_a, N_b, N_c be the 9-point center of $\triangle BPC, \triangle CPA, \triangle APB,$ respectively .

Prove that $\triangle DEF \sim \triangle N_aN_bN_c$

Attachments:



Luis González

#2 Jan 30, 2015, 1:48 am • 1

Let M, N, L and X, Y, Z denote the midpoints of BC, CA, AB and $PA, PB, PC.$ 9-point circles $(N_a), (N_b), (N_c)$ of $\triangle PBC, \triangle PCA, \triangle PAB$ concur at Poncelet point T of $ABCP,$ thus $N_bN_c \perp TX, N_cN_a \perp TY$ and $N_aN_b \perp TZ$ $\Longrightarrow \angle(N_aN_c, N_aN_b) = \angle(TY, TZ) = \angle(MY, MZ) = \angle(PC, PB) = \angle(DE, DF)$ and similarly we'll have $\angle(N_bN_a, N_bN_c) = \angle(EF, ED) \Longrightarrow \triangle N_aN_bN_c \sim \triangle DEF.$

Quick Reply

High School Olympiads

Perpendicular following tangent circles

Reply

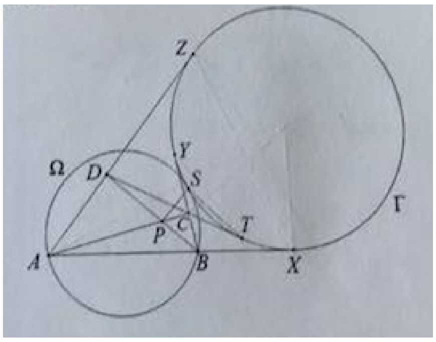
Source: China Team Selection Test 2016 Test 2 Day 2 Q6

leeky

#1 Mar 21, 2016, 11:53 am • 2

The diagonals of a cyclic quadrilateral $ABCD$ intersect at P, and there exist a circle Γ tangent to the extensions of AB, BC, AD, DC at X, Y, Z, T respectively. Circle Ω passes through points A, B, and is externally tangent to circle Γ at S. Prove that $SP \perp ST$.

Attachments:



ABCDE

#2 Mar 21, 2016, 12:42 pm • 3

Let Q be the intersection of XY and ZT and P' be the intersection of YZ and XT. By Pascal on $XYYZTT$, Q, C, and P' are collinear. By Pascal on $XXYZZT$, A, Q, and P' are collinear. Hence, P' is on AC. Similarly, we can show that P' is on BD, so $P' = P$.

Now, invert about P fixing Γ, and denote by K' the image of point K under this inversion. Since line ABX is tangent to Γ, the circumcircle of $PA'B'T$ is tangent to Γ. Because $ABCD$ is cyclic, $A'B'$ is parallel to line CD, which is the tangent to Γ at T. This implies that $TA' = TB'$. If T^* is the antipode of T with respect to Γ, then clearly the circumcircle of $T^*A'B'$ is also tangent to Γ at T^*, so $T^* = S'$. But this means that $\angle PST = \angle PXS' = TXT^* = 90^\circ$, as desired.

TelvCohl

#3 Mar 21, 2016, 12:54 pm • 2

Let J be the pole of SX WRT Γ and let $V \equiv AB \cap YZ$. It's well-known that P lies on XT, YZ and $(A, B; V, X) = -1$, so from $JX^2 = JA \cdot JB$ we get J is the midpoint of VX, hence J is the circumcenter of $\triangle SVX$. Since TX, YZ is parallel to the bisector of $\angle(AB, CD), \angle(BC, DA)$, respectively, so $\angle XPV = 90^\circ \Longrightarrow P, S, V, X$ lie on a circle with diameter VX, hence we conclude that $\measuredangle TSP = \measuredangle XSP - \measuredangle XST = \measuredangle XVP - \measuredangle VXP = 90^\circ$.

Luis González

#4 Mar 21, 2016, 1:10 pm • 2

Let O and K be the centers of $\odot(ABCD)$ and Γ. $E \equiv AD \cap BC$ and $F \equiv AB \cap CD$. Since $E(F, Y, Z, P) = -1$, then the pole of EF WRT Γ is on EP and similarly it must be on $FP \Longrightarrow P$ is the pole of EF WRT $\Gamma \Longrightarrow X, T, P$ are collinear and $EF \perp PK$ at Q, i.e. $K \in OP$. Since the polar of Q WRT (O) passes through P, then it follows that P, Q are the limiting points of $\Gamma, (O)$. Thus if the common tangent of Γ, Ω cuts AB at M, we get $MS^2 = MX^2 = MA \cdot MB \Longrightarrow$ $\odot(M, MS)$ is orthogonal to (O) and $\Gamma \Longrightarrow P \in \odot(M, MS) \Longrightarrow$ $\angle PST = \angle MSP + \angle MST = 90^\circ - \angle SXT + \angle SXT = 90^\circ$.

leeky

#5 Mar 21, 2016, 4:15 pm

Let $AD \cap BC = E, AB \cap CD = F, XY \cap TZ = G, XZ \cap TY = H, YZ \cap AB = Q$. Note that polar of H wrt Γ is AC, polar of G is BD, hence polar of GH must be $P = XT \cap YZ$. YZ, XT are parallel to the internal angle bisectors of $\angle AEB, \angle AFB$, which are also parallel to the internal bisectors of $\angle APB, \angle APD$ since $ABCD$ is cyclic $\implies PZ \perp PX$, and PQ is the angle bisector of $\angle APB \implies (A, B; Q, X) = -1$.

Let the tangent at S to the two circles meet AB at M. $MX^2 = MS^2 = MA \times MB$, so M is the midpoint of QX, combining with $MS = MX \implies QS \perp SX$. Hence Q, P, S, X lie on a circle with diameter $QX \implies \angle PSQ = \angle PXQ = \angle TSX \implies \angle PST = \angle QSX = 90°$.

Dukejukem

#6 Mar 22, 2016, 5:19 am

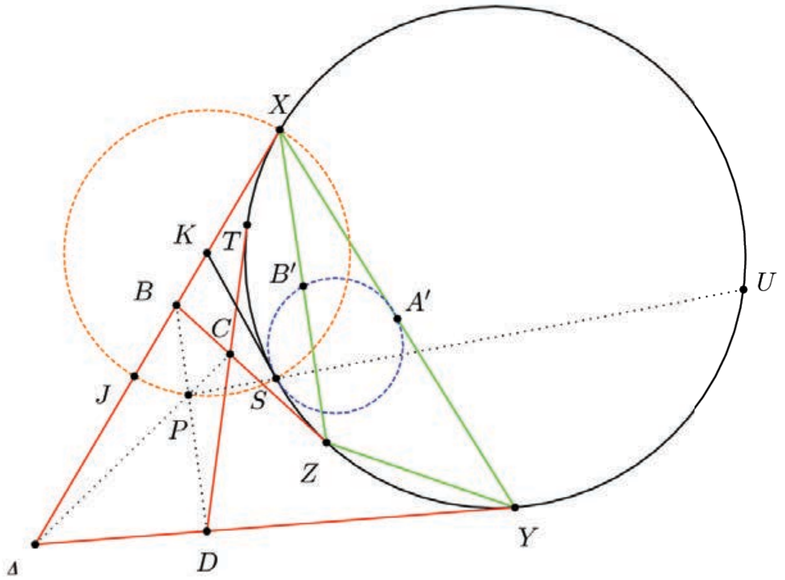
Let O be the center of Γ and let U be the antipode of T WRT Γ. Set A', B' as the midpoints of $\overline{XY}, \overline{XZ}$ respectively. It's enough to show that U, P, S are collinear.

Since A, C, P are collinear, their polars WRT Γ are concurrent. Hence, the polar of P passes through $XZ \cap YT$. Similarly, the polar of P passes through $XY \cap ZT$. Thus, by Brokard's Theorem, $P \equiv XT \cap YZ$.

On the other hand, since $ABCD, AXOY, CZOT$ are cyclic, we have

$$\angle XOY = 180° - \angle XAY = 180° - \angle BAD = \angle BCD = \angle ZCT = 180° - \angle ZOT = \angle ZOU.$$

It follows that $XYZU$ is an isosceles trapezoid. In particular, $XU \parallel YZ$ and consequently $XT \perp YZ$.



Meanwhile, inversion in Γ sends $\odot(ABS) \mapsto \odot(A'B'S)$. Therefore, $\odot(A'B'S)$ is tangent to Γ as well. Hence, $K \equiv XX \cap SS \cap A'B'$ is the radical center of $\Gamma, \odot(A'B'S), \odot(A'B'X)$. Clearly $KS = KX$ by equal tangents; also $KX = KP$ because $A'B'$ is the perpendicular bisector of $\overline{XP}$. Therefore, K is the circumcenter of $\triangle SXP$. Thus, if J is the reflection of X in K, we have $(JP \parallel XU) \perp XP$. By the converse of Reim's Theorem for XXJ and UPS, it follows that U, P, S are collinear, as desired.

Remark: This problem is essentially 2011 ISL G4, with $\triangle XYZ$ as the reference triangle.

This post has been edited 3 times. Last edited by Dukejukem, Mar 22, 2016, 8:51 pm

suli

#7 Mar 22, 2016, 11:16 am • 1

Projective KO:

1. Lemma: P is YZ intersect TX.
Proof.
1a. With respect to Γ, the pole of P is the line through the polars of BD and AC.
1b. But the polar of BD is the intersection of the poles of B and D, or XY and TZ.
1c. Similarly AC polar is intersection of XZ and YT.
1d. Thus pole of P is line through aforementioned intersections. But by Brokard's theorem the polar of this line is also the intersection of YZ and TX.

2. YZ intersect AB at M such that $A, B; M, X$ harmonic.
Proof.
2a. BY and BX are tangents, so ZA, BY, ZB, ZX form harmonic bundle.
2b. Thus ZA, ZM, ZB, ZX are harmonic, so $A, B; M, X$ harmonic.

3. $\angle ZPX = 90^\circ$ by easy angle chasing because $ABCD$ is cyclic, so $\angle A + \angle D = 180^\circ$.
4. Because the two circles are tangent at S, easy angle chasing show that SX is external angle bisector of ABS.
5. Thus by famous lemma $\angle MSX = 90^\circ$.

6. $MPSX$ is cyclic.
7. PST and MSX are spirally similar, so $\angle PST = \angle MSX = 90^\circ$.

v_Enhance
#8 Apr 2, 2016, 6:03 am • 2

Solution with Danielle Wang: Ignore $ABCD$ cyclic for now, and focus entirely on Γ.

Let Q be the inverse of P with respect to Γ. Since $P = AC \cap BD$, it follows P lies on the polars of $\overline{TY} \cap \overline{XZ}$ and $\overline{YZ} \cap \overline{TX}$. By Brokard's Theorem, this implies $P = \overline{YZ} \cap \overline{XT}$. Therefore Q is the Miquel point of cyclic quadrilateral $YTXZ$. Next let γ be the circumcircle of $\triangle PQX$, and M its center. Thus γ is orthogonal to Γ. So if W is the second intersection of Γ and γ, then $\overline{OW}$ and $\overline{OX}$ are tangents to γ. Angle chasing,

$$\angle WPT = \frac{1}{2}\angle WMX = 90^\circ - \angle WOX = \angle WTX - 90^\circ \implies \angle PWT = 90^\circ.$$

Thus we have shown that a circle centered at $M \in AB$ passes through P, W, Q, X with $\angle PWT = 90^\circ$.

Now suppose $ABCD$ is cyclic, centered at N with circumcircle ω. If $E = AB \cap CD$, $F = BC \cap DA$, then $Q \in EF$, $PQ \perp EF$, so Q is the Miquel point of cyclic quadrilateral $ABCD$. Consequently, N, P, Q, O collinear, and P and Q are inverses with respect to both ω and Γ. Now the circle with diameter PQ is orthogonal to both ω and Γ, thus the midpoint H of PQ is on the radical axis of ω and Γ. Thus from $HM \perp PQ$, M lies on this radical axis as well. Then $MA \cdot MB = MW^2$, so $W = S$ and we're done.