Maximum number of perfect squares

Source: AllRussian-2014, Grade 9, day1, P2

mathuz
1229 posts

May 3, 2014, 9:21 pm PM #1

Sergei chooses two different natural numbers a and b. He writes four numbers in a notebook: a, $a+2$, b and $b+2$. He then writes all six pairwise products of the numbers of notebook on the blackboard. Let S be the number of perfect squares on the blackboard. Find the maximum value of S.

S. Berlov

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joybangla
836 posts

May 3, 2014, 10:35 pm PM #2

This is a number theory problem actually. Anyways we show the maximum is 2. Clearly, suppose (a_0,b_0) are two solutions of the Pell's equation $4x^2-y^2=3$. Now
$a=8(a_0^2-1), b=2(a_0^2-1)$ gives
$ab=(4(a_0^2-1))^2, (a+2)(b+2)=(8a_0^2-6)(2a_0^2)=(2a_0b_0)^2$. So 2 is achieved.
6 or 5 is impossible:
Let $a(a+2), b(b+2)$ none of them are squares since they are one less than a square. Thus we cannot have them being a natural square.
4 is impossible:
$ab=m^2, a(b+2)=n^2, (a+2)b=p^2, (a+2)(b+2)=q^2 \implies mq=pn$ also see that
$p^2+n^2+4=m^2+q^2 \implies (m-n)^2=(p-q)^2+4(\text{since } mq=pn) \implies p=q$
But then $a=b$ contradicting distinctness.
3 is impossible:
For this to happen one of these pairs must be simultaneously squares:
$\{(ab,a(b+2)),(ab,(a+2)b),(a+2)(b+2),a(b+2)),((a+2)(b+2),(a+2)b)\}$
but multiply any two of them and since their product is a perfect square then a^2+2a or b^2+2b must be a perfect square as well. Which is impossible.
Hence $\max(S)\le 2$.
$\blacksquare$

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Dec 26, 2014, 4:03 am • 1 PM #3

The only solutions to $4x^2-y^2=3$ are $(x,y)=(1,1)$, so that your values of a,b are 0. However, replacing your equation with $\boxed{16x^2-y^2=15}$, which has roots $(a_0,b_0)=(2,7)$ we can set $a=32(a_0^2-1), b=2(a_0^2-1)$, assuring us that $S=2$ works, then continue as before.

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Mar 5, 2015, 5:35 pm PM #4

It is clear that $a(a+2)=(a+1)^2-1, b(b+2)=(b+1)^2-1$ are not perfect squares .

Note that if ab and $a(b+2)$ were both perfect squares, then $(ab)*a(b+2)$ must be a perfect square , which in turn means that $b(b+2)$ must be a perfect square, which is not possible. Similarly at most one number of each of the pairs $(ab,b(a+2)),((a+2)(b+2),a(b+2)),((a+2)(b+2),b(a+2))$ can be a perfect square.

Joining the bits of information, we conclude that at most two perfect squares can be found. Note that for $(a,b)=(25,1)$, we have $ab=5^2, (a+2)(b+2)=9^2$, so 2 is achievable.

Therefore the maximum value of S is 2.

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quangmi...
608 posts

Sep 3, 2016, 10:44 am PM #5

My solution