# Art of Problem Solving
## 2005 USAMO

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| **Day 1** | April 19th |
| **1** | Determine all composite positive integers $n$ for which it is possible to arrange all divisors of $n$ that are greater than 1 in a circle so that no two adjacent divisors are relatively prime. |
| **2** | Prove that the system $$x^6 + x^3 + x^3y + y = 147^{157}$$ $$x^3 + x^3y + y^2 + y + z^9 = 157^{147}$$ has no solutions in integers $x$, $y$, and $z$. |
| **3** | Let $ABC$ be an acute-angled triangle, and let $P$ and $Q$ be two points on its side $BC$. Construct a point $C_1$ in such a way that the convex quadrilateral $APBC_1$ is cyclic, $QC_1 \parallel CA$, and $C_1$ and $Q$ lie on opposite sides of line $AB$. Construct a point $B_1$ in such a way that the convex quadrilateral $APCB_1$ is cyclic, $QB_1 \parallel BA$, and $B_1$ and $Q$ lie on opposite sides of line $AC$. Prove that the points $B_1$, $C_1$, $P$, and $Q$ lie on a circle. |
| **Day 2** | April 20th |
| **4** | Legs $L_1, L_2, L_3, L_4$ of a square table each have length $n$, where $n$ is a positive integer. For how many ordered 4-tuples $(k_1, k_2, k_3, k_4)$ of nonnegative integers can we cut a piece of length $k_i$ from the end of leg $L_i$ $(i = 1, 2, 3, 4)$ and still have a stable table?<br><br>(The table is *stable* if it can be placed so that all four of the leg ends touch the floor. Note that a cut leg of length 0 is permitted.) |
| **5** | Let $n$ be an integer greater than 1. Suppose $2n$ points are given in the plane, no three of which are collinear. Suppose $n$ of the given $2n$ points are colored blue and the other $n$ colored red. A line in the plane is called a *balancing line* if it passes through one blue and one red point and, for each side of the line, the number of blue points on that side is equal to the number of red points on the same side.<br>Prove that there exist at least two balancing lines. |

**6** For $m$ a positive integer, let $s(m)$ be the sum of the digits of $m$. For $n \geq 2$, let $f(n)$ be the minimal $k$ for which there exists a set $S$ of $n$ positive integers such that $s\left(\sum_{x\in X} x\right) = k$ for any nonempty subset $X \subset S$. Prove that there are constants $0 < C_1 < C_2$ with

$$C_1 \log_{10} n \leq f(n) \leq C_2 \log_{10} n.$$