JBMO ShortLists 2002

$\boxed{1}$ A student is playing computer. Computer shows randomly 2002 positive numbers. Game's rules let do the following operations - to take 2 numbers from these, to double first one, to add the second one and to save the sum. - to take another 2 numbers from the remainder numbers, to double the first one, to add the second one, to multiply this sum with previous and to save the result. - to repeat this procedure, until all the 2002 numbers won't be used. Student wins the game if final product is maximum possible. Find the winning strategy and prove it.

$\boxed{2}$ Positive real numbers are arranged in the form: 1 3 6 10 15... 2 5 9 14... 4 8 13... 5 12... 11... Find the number of the line and column where the number 2002 stays.

$\boxed{3}$ Let a, b, c be positive real numbers such that $abc = \frac{9}{4}$. Prove the inequality: $a^3 + b^3 + c^3 > a\sqrt{b+c} + b\sqrt{c+a} + c\sqrt{a+b}$

Jury's variant: Prove the same, but with $abc = 2$

$\boxed{5}$ Let a, b, c be positive real numbers. Prove the inequality: $\frac{a^3}{b^2} + \frac{b^3}{c^2} + \frac{c^3}{a^2} \geq \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}$

$\boxed{6}$ Let $a_1, a_2, ..., a_6$ be real numbers such that: $a_1 \neq 0, a_1a_6 + a_3 + a_4 = 2a_2a_5$ and $a_1a_3 \geq a_2^2$ Prove that $a_4a_6 \leq a_5^2$. When does equality holds?

$\boxed{7}$ Consider integers $a_i, i = \overline{1, 2002}$ such that $a_1^{-3} + a_2^{-3} + \ldots + a_{2002}^{-3} = \frac{1}{2}$ Prove that at least 3 of the numbers are equal.

$\boxed{8}$ Let ABC be a triangle with centroid G and A_1, B_1, C_1 midpoints of the sides BC, CA, AB. A paralel through A_1 to BB_1 intersects B_1C_1 at F. Prove that triangles ABC and FA_1A are similar if and only if quadrilateral AB_1GC_1 is cyclic.

$\boxed{9}$ In triangle ABC, H, I, O are orthocenter, incenter and circumcenter, respectively. CI cuts circumcircle at L. If $AB = IL$ and $AH = OH$, find angles of triangle ABC.

$\boxed{10}$ Let ABC be a triangle with area S and points D, E, F on the sides BC, CA, AB. Perpendiculars at points D, E, F to the BC, CA, AB cut circumcircle of the triangle ABC at points $(D_1, D_2), (E_1, E_2), (F_1, F_2)$. Prove that: $|D_1B \cdot D_1C - D_2B \cdot D_2C| + |E_1A \cdot E_1C - E_2A \cdot E_2C| + |F_1B \cdot F_1A - F_2B \cdot F_2A| > 4S$

$\boxed{11}$ Let ABC be an isosceles triangle with $AB = AC$ and $\angle A = 20°$. On the side AC consider point D such that $AD = BC$. Find $\angle BDC$.

$\boxed{12}$ Let $ABCD$ be a convex quadrilateral with $AB = AD$ and $BC = CD$. On the sides AB, BC, CD, DA we consider points K, L, L_1, K_1 such that quadrilateral KLL_1K_1 is rectangle. Then consider rectangles $MNPQ$ inscribed in the triangle BLK, where $M \in KB, N \in BL, P, Q \in LK$ and $M_1N_1P_1Q_1$ inscribed in triangle DK_1L_1 where P_1 and Q_1 are situated on the L_1K_1, M on the DK_1 and N_1 on the DL_1. Let S, S_1, S_2, S_3 be the areas of the

$ABCD, KLL_1K_1, MNPQ, M_1N_1P_1Q_1$ respectively. Find the maximum possible value of the expression: $\frac{S_1+S_2+S_3}{S}$

$\boxed{13}$ Let $A_1, A_2, ..., A_{2002}$ be the arbitrary points in the plane. Prove that for every circle of the radius 1 and for every rectangle inscribed in this circle, exist 3 vertexes M, N, P of the rectangle such that: $MA_1 + MA_2 + ... + MA_{2002} + NA_1 + NA_2 + ... + NA_{2002} + PA_1 + PA_2 + ... + PA_{2002} \geq 6006$

