IMO 1971

Day 1

1 Let

$$E_n = (a_1-a_2)(a_1-a_3)\ldots(a_1-a_n)+(a_2-a_1)(a_2-a_3)\ldots(a_2-a_n)+\ldots+(a_n-a_1)(a_n-a_2)\ldots(a_n-a_{n-1}).$$

Let S_n be the proposition that $E_n \geq 0$ for all real a_i. Prove that S_n is true for $n = 3$ and 5, but for no other $n > 2$.

2 Let P_1 be a convex polyhedron with vertices $A_1, A_2, \ldots, A_9$. Let P_i be the polyhedron obtained from P_1 by a translation that moves A_1 to A_i. Prove that at least two of the polyhedra $P_1, P_2, \ldots, P_9$ have an interior point in common.

3 Prove that we can find an infinite set of positive integers of the from $2^n - 3$ (where n is a positive integer) every pair of which are relatively prime.

Day 2

1. All faces of the tetrahedron $ABCD$ are acute-angled. Take a point X in the interior of the segment AB, and similarly Y in BC, Z in CD and T in AD.

   a.) If $\angle DAB + \angle BCD \neq \angle CDA + \angle ABC$, then prove none of the closed paths $XYZTX$ has minimal length;

   b.) If $\angle DAB + \angle BCD = \angle CDA + \angle ABC$, then there are infinitely many shortest paths $XYZTX$, each with length $2AC \sin k$, where $2k = \angle BAC + \angle CAD + \angle DAB$.

2. Prove that for every positive integer m we can find a finite set S of points in the plane, such that given any point A of S, there are exactly m points in S at unit distance from A.

3. Let $A = (a_{ij})$, where $i, j = 1, 2, \ldots, n$, be a square matrix with all a_{ij} non-negative integers. For each i, j such that $a_{ij} = 0$, the sum of the elements in the ith row and the jth column is at least n. Prove that the sum of all the elements in the matrix is at least $\frac{n^2}{2}$.