IMO 2002

Glasgow, United Kingdom

Day 1 - *24 July 2002*

$\boxed{1}$ Let n be a positive integer. Each point (x, y) in the plane, where x and y are non-negative integers with $x + y < n$, is coloured red or blue, subject to the following condition: if a point (x, y) is red, then so are all points (x', y') with $x' \leq x$ and $y' \leq y$. Let A be the number of ways to choose n blue points with distinct x-coordinates, and let B be the number of ways to choose n blue points with distinct y-coordinates. Prove that $A = B$.

$\boxed{2}$ The circle S has centre O, and BC is a diameter of S. Let A be a point of S such that $\angle AOB < 120°$. Let D be the midpoint of the arc AB which does not contain C. The line through O parallel to DA meets the line AC at I. The perpendicular bisector of OA meets S at E and at F. Prove that I is the incentre of the triangle CEF.

$\boxed{3}$ Find all pairs of positive integers $m, n \geq 3$ for which there exist infinitely many positive integers a such that

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is itself an integer.

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Day 2 - *25 July 2002*

$\boxed{4}$ Let $n \geq 2$ be a positive integer, with divisors $1 = d_1 < d_2 < \ldots < d_k = n$. Prove that $d_1 d_2 + d_2 d_3 + \ldots + d_{k-1} d_k$ is always less than n^2, and determine when it is a divisor of n^2.

$\boxed{5}$ Find all functions f from the reals to the reals such that

$$\left(f(x) + f(z)\right)\left(f(y) + f(t)\right) = f(xy - zt) + f(xt + yz)$$

for all real x, y, z, t.

$\boxed{6}$ Let $n \geq 3$ be a positive integer. Let $C_1, C_2, C_3, \ldots, C_n$ be unit circles in the plane, with centres $O_1, O_2, O_3, \ldots, O_n$ respectively. If no line meets more than two of the circles, prove that

$$\sum_{1 \leq i < j \leq n} \frac{1}{O_i O_j} \leq \frac{(n-1)\pi}{4}.$$