High School Olympiads

A problem about the circumscribed quadriateral

Locked

TheBeatlesVN

#1 Aug 9, 2015, 12:12 pm

Two diagonal of a circumscribed quadrilateral ABCD interects at O. E, F, G, H are centres of inscribed circle of AOB, BOC, COD, DOA, respectively. Prove that EFGH is a iscribed quadrilateral.

Luis González

#2 Aug 9, 2015, 12:19 pm • 1

Posted many times before, e.g. http://www.artofproblemsolving.com/community/c6h21758

High School Olympiads

$I_1I_2I_3I_4$ inscribed <==> $ABCD$ circumscribed

Source: Unknown

pigfly
#1 Dec 24, 2004, 6:02 pm

Let $ABCD$ be a quadrilateral. Denote by I the point of intersection of the lines AC and BD. Let I_1, I_2, I_3, I_4 be the centers of the incircles of the triangles IAB, IBC, ICD, IDA, respectively. Prove that the quadrilateral $I_1I_2I_3I_4$ has a circumscribed circle if and only if the quadrilateral $ABCD$ has an inscribed circle.

This post has been edited 2 times. Last edited by pigfly, Dec 24, 2004, 8:47 pm

darij grinberg
#2 Dec 24, 2004, 6:29 pm • 1

This is a hard problem. You can find a solution in

Toshio Seimiya, Peter Y. Woo, *Solution of problem 2338*, Crux Mathematicorum 4 / 25 (1999) pages 243-245.

See the attached PDF file.

EDIT: See also http://www.mathlinks.ro/Forum/viewtopic.php?t=21768 .

Darij

Attachments:
crux2338.pdf (105kb)

yetti
#3 Jan 3, 2005, 3:06 pm

Let I be the incenter of $ABCD$ and J_1, J_3 the incenters of the triangles $\triangle ABD, \triangle BCD$. The diagonal AC is the (external) homothety axis of the incircles $(I), (J_1), (J_3)$. Let K be the external homothety center of $(J_1), (J_3)$ on AC. Then the line BK is the (external) homothety axis of $(I_1), (J_1), (J_3)$, of $(I_2), (J_1), (J_3)$, of $(I_1), (I_2), (J_1)$ and of $(I_1), I_2), (J_3)$. The external homothety center of $(I_1), (I_2)$ is therefore on BK and on also their their common external tangent AC, i.e., at K. Similarly, the external homothety center of $(I_3), (I_4)$ is also at K. The lines I_1I_2 and I_3I_4 meet at K on AC.

Since $ABCD$ is tangential, the circles $(J_1), (J_3)$ touch at a point M on BD and $KM \perp BD$. Let T_2, T_3 be the tangency points of $(I_2), (I_3)$ with AC. Denote s the semiperimeter of a triangle, X the diagonal intersection of $AC \cap BD$ and assume that the points B, M, X, D follow on the line BD in this order.

$$XT_2 = s(\triangle BCX) - BC$$
$$XT_3 = s(\triangle DCX) - CD$$
$$T_2T_3 = s(\triangle BCX) - s(\triangle DCX) - (BC - DC) = \frac{BX - DX}{2} - \frac{BC - DC}{2}$$

$$BX = BM + XM = s(\triangle BCD) - DC + XM$$
$$DX = DM - XM = s(\triangle BCD) - BC - XM$$
$$BX - DX = 2 \cdot XM + BC - DC$$

$$T_2T_3 = XM$$

Suppose we move a point C' on the line AC and draw the external tangents to the circles $(I_2), (I_3)$ (other than AC) intersecting BD at points B', D'. The incenter J_3' of the triangle $\triangle B'C'D'$ then moves on the normal KM to BD, because the distance of the tangency point M form X, equal to T_2T_3, does not depend of the position of the point C' on the line AC. If we move the point C' to coincide with the point K, then $C'M \equiv KM$ becomes the bisector of the $\measuredangle B'C'D'$, perpendicular to the side $B'D'$. It follows that the $\triangle B'C'D'$ becomes isosceles and consequently, the lines KM, KX are isogonals of the lines KI_2, KI_3. Let XI_1 cut $KM \perp BD$ at U. From here, $\angle KUI_1 = 90^\circ - \frac{AXB}{2} = \frac{\angle AXD}{2} = \angle KXI_4 \Longrightarrow$ $\triangle KI_1N \sim \triangle KXI_4$ by ASA $\Longrightarrow$ angles $\angle XI_1K = \angle XI_4K$ are equal $\Longrightarrow$ the right angle triangles $\triangle XI_1I_2 \sim \triangle XI_3I_4$ are similar and

$\frac{XI_1}{XI_2} = \frac{XI_4}{XI_3}$ and $XI_1 \cdot XI_3 = XI_2 \cdot XI_4$.

In addition, a convex tangential quadrilateral has one concyclic quadruple of triangle incenters and three concyclic quadruples of triangle excenters. Four concyclic quadruples of triangle incenters/excenters exist even in a concave tangential quadrilateral with an incircle or in a tangential quadrilateral (convex or concave) with an excircle. The triangle incenter and excenter pairs just have to be mixed up.

This post has been edited 1 time. Last edited by yetti, Feb 7, 2005, 11:22 pm

juancarlos

#4 Jun 5, 2005, 5:58 am

Let E the cut point MD and I_3I_4, now E is the insimilicenter of I_3 and I_4, further K, E are harmonic conjugates of I_3 and I_4. Also K, M are harmonic conjugates of J_1 and J_3. Now I_3I_4 is transversal wrt triangle J_1DJ_3 then J_3I_4 cut J_1I_3 at F on the altitude DM, hence angle I_4MD=angle I_3MD by Blanchet´s theorem. Follow I_1XI_4 and I_2XI_3 are similar, now: $I_1X.I_3X = I_2X.I_4X$, so $I_1I_2I_3I_4$ are cyclic. QED.

Quick Reply

High School Olympiads

Parallelogram to find angle

Reply

Source: All Russian MO 2015, grade 10, problem 2

silouan

#1 Aug 7, 2015, 5:38 pm

Given is a parallelogram $ABCD$, with $AB < AC < BC$. Points E and F are selected on the circumcircle ω of ABC so that the tangenst to ω at these points pass through point D and the segments AD and CE intersect.
It turned out that $\angle ABF = \angle DCE$. Find the angle $\angle ABC$.
A. Yakubov, S. Berlov

vanstraelen

#2 Aug 7, 2015, 10:20 pm

> *silouan wrote:*
> Given is a parallelogram $ABCD$, with $AB < AC < BC$. Points E and F are selected on the circumcircle ω of ABC so that the tangenst to ω at these points pass through point D and the segments AD and CE intersect.
> It turned out that $\angle ABF = \angle DCE$. Find the angle $\angle ABC$.

Can this text be read as

Points E and F are selected on the circumcircle ω of ABC so that the tangents to ω at these points pass through point D and through the intersection of the segments AD and CE ?

silouan

#3 Aug 8, 2015, 4:26 am

No, the text is as it was given. The meaning of the condition that AD and CE intersect is to avoid different configurations.

Luis González

#4 Aug 9, 2015, 7:36 am

Denote by O the center of ω. Parallel from B to AC cuts DC, DA at $Y, Z.$ Since ω becomes 9-point circle of $\triangle DYZ,$ then it cuts DC again at the projection P of Z on $DC.$ Since $\angle ABF = \angle DCE \Longrightarrow AF = EP \Longrightarrow AEFP$ is isosceles trapexoid with bases $EF \parallel AP \Longrightarrow DO \perp (EF \parallel AP) \Longrightarrow DO$ is perpendicular bisector of $\overline{AP} \Longrightarrow$ $DP = DA = AZ = AP \Longrightarrow \triangle DAP$ is equilateral $\Longrightarrow \angle ABC = \angle ADP = 60^\circ.$

Quick Reply

High School Math

Find angle in terms of another angle

Reply

SMOJ

#1 Aug 8, 2015, 10:13 am

Given an acute-angled triangle ABC with $AB < AC$ with incentre I and orthocentre H. Point J on side AB and K on side BC satisfy $\angle HIK = 90^\circ$ and $AC = AJ + CK$. Express $\angle HJK$ in terms of $\angle BCA$.

Luis González

#2 Aug 9, 2015, 5:04 am

Take point P on $\overline{AC}$ such that $CK = CP \Longrightarrow AP = AJ \Longrightarrow P$ is reflection of K on CI and J is reflection of P on AI $\Longrightarrow IK = IP = IJ.$ Together with $\angle IBK = \angle IBJ,$ it follows that I is the midpoint of the arc JK of $\odot(BJK).$ Thus by Miquel theorem $\odot(AJI)$ and $\odot(CKI)$ meet again at a point Q on $AC.$

If D is the projection of A on $BC,$ then from cyclic quadrilateral $IHKD,$ we get $\angle IHD = \angle IKC = \angle IQA \Longrightarrow AQIH$ is cyclic, i.e. $H \in \odot(AJIQ) \Longrightarrow \angle HJQ = \angle HAC = 90^\circ - \angle BCA.$ But
$\angle QJK = \angle IJQ + \angle IJK = \angle IAC + \angle IBC = 90^\circ - \tfrac{1}{2}\angle BCA \Longrightarrow$
$\angle HJK = \angle QJK - \angle HJQ = 90^\circ - \tfrac{1}{2}\angle BCA - (90^\circ - \angle BAC) = \tfrac{1}{2}\angle BCA.$

Quick Reply

High School Olympiads

Show two lengths are equal

Locked

SMOJ

#1 Aug 8, 2015, 8:05 pm

In acute-angled triangle ABC, CD is the altitude. A line through the midpoint M of side AB meets the rays CA, CB at K and L respectively such that $CK = CL$. Point S is the circumcenter of triangle CKL. Prove that $SD = SM$.

Luis González

#3 Aug 8, 2015, 11:37 pm

Posted before at http://www.artofproblemsolving.com/community/c6h25424.

High School Olympiads

Interesting [triangle ABC, prove that OD = OM]

Source: 54th Polish MO 2003

Herodatviet

#1 Feb 2, 2005, 8:39 am • 1

Let ABC be a triangle, and let M be the midpoint of its side AB. Let a line through the point M meet the lines CA and BC at the points K and L such that $CK = CL$. Denote by D the foot of the perpendicular from the point C to the line AB, and denote by O the circumcenter of triangle CKL.

Prove that $OD = OM$.

darij grinberg

#2 Feb 2, 2005, 5:08 pm

Problem. Let ABC be a triangle, and let M be the midpoint of its side AB. Let a line through the point M meet the lines CA and BC at the points K and L such that CK = CL. Denote by D the foot of the perpendicular from the point C to the line AB, and denote by O the circumcenter of triangle CKL.

Prove that OD = OM.

Solution. Since CK = CL, the triangle KCL is isosceles. Hence, the angle bisector of its angle KCL is the symmetry axis of this isosceles triangle, and thus it is perpendicular to its base KL. In other words, the line KL is perpendicular to the angle bisector of the angle KCL. But the angle bisector of the angle KCL is simultaneously the angle bisector of the angle BCA. Thus, the line KL is perpendicular to the angle bisector of the angle BCA. Since this line KL passes through the point M, we thus can summarize: The line KL is the perpendicular to the angle bisector of the angle BCA through the point M.

Let the angle bisector of the angle BCA meet the circumcircle of triangle ABC at a point V (apart from the point C). Then, < VCA = < VCB. On the other hand, since the point V lies on the circumcircle of triangle ABC, we have < VCA = < VBA and < VCB = < VAB. Thus, < VCA = < VCB becomes < VBA = < VAB. This means that the triangle AVB is isosceles, so that AV = BV, and thus, the point V lies on the perpendicular bisector of the segment AB. Consequently, the orthogonal projection of the point V on the line AB is the midpoint M of the segment AB.

Let K' and L' be the orthogonal projections of the point V on the lines CA and BC. By the Simson line theorem, since the point V lies on the circumcircle of triangle ABC, the orthogonal projections L', K' and M of this point V on the sidelines BC, CA and AB of triangle ABC lie on one line (namely, the Simson line of the point V with respect to the triangle ABC). In other words, the line K'L' passes through the point M.

Now, remember that the point V lies on the angle bisector of the angle BCA, and that the points K' and L' are the orthogonal projections of this point V on the lines CA and BC. Thus, it follows from symmetry that the points K' and L' are symmetric to each other with respect to the angle bisector of the angle BCA. Hence, the line K'L' is perpendicular to the angle bisector of the angle BCA. Since we know that this line K'L' passes through the point M, it follows that the line K'L' is the perpendicular to the angle bisector of the angle BCA through the point M. But we have previously shown that the line KL is the perpendicular to the angle bisector of the angle BCA through the point M. Thus, the lines K'L' and KL are one and the same line. Consequently, the points $K = KL \cap CA$ and $K' = K'L' \cap CA$ coincide; similarly, the points L and L' coincide.

Since we have defined the points K' and L' as the orthogonal projections of the point V on the lines CA and BC, we thus obtain that the points K and L are the orthogonal projections of the point V on the lines CA and BC. Thus, < CKV = 90° and < CLV = 90°. This implies that the points K and L lie on the circle with diameter CV. In other words, the circumcircle of triangle CKL is the circle with diameter CV. Consequently, the circumcenter O of triangle CKL is just the center of the circle with diameter CV, thus the midpoint of the segment CV.

Let T be the midpoint of the segment DM.

Since the point V lies on the perpendicular bisector of the segment AB, we have $VM \perp AB$ (since the point M is the midpoint of the segment AB). Also, clearly $CD \perp AB$, since the point D is the foot of the perpendicular from the point C to the line AB.

Hence, the lines VM and CD are both perpendicular to the line AB, so that the quadrilateral VMDC is a trapezoid with its bases VM and CD both perpendicular to the line AB. Thus, the line joining the midpoints O and T of its legs CV and DM must also be perpendicular to the line AB. In other words, we have $OT \perp AB$, so that the point O lies on the perpendicular to the line AB at the point T. But since the point T is the midpoint of the segment DM, the perpendicular to the line AB at the point T is the perpendicular bisector of the segment DM. Thus, the point O lies on the perpendicular bisector of the segment DM, and we get OD = OM, completing our solution.

Sorry for the length of the above solution – it's the first that came into my mind and I didn't have time to look for a simpler one or to optimise the above writeup. Nice problem, by the way.

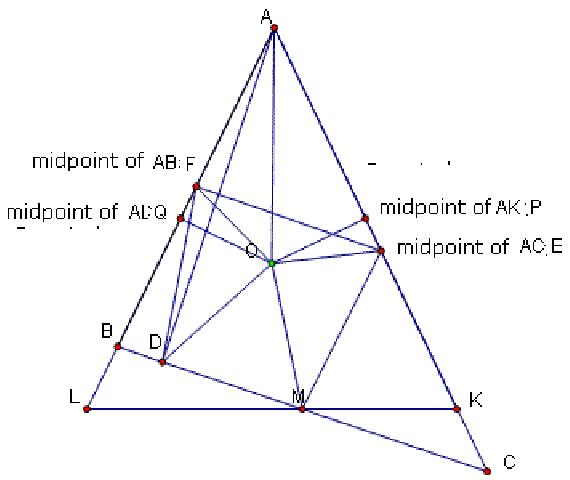
darij

lym

#3 Apr 3, 2009, 9:53 am

Just see the Isosceles trapezoid $EFDM$.

Attachments:



Luis González

#4 Apr 3, 2009, 10:17 am

Since $MA = MB$ and $CK = CL$, by Menelaus' theorem for $\triangle ABC$ cut by $\overline{KML}$, we get $AK = BL$. Thus, $E \equiv \odot(CKL) \cap \odot(ABC) \not\equiv C$ is the center of the rotation carrying AK into BL, and due to $EA = EB$, then E is the midpoint of the arc AB of $\odot(ABC)$.

Clearly E lies on the internal bisector of $\angle ACB$ and since CE is the circumdiameter of $\triangle CKL$ perpendicular to KL, it follows that O is the midpoint of CE. Moreover, if A', B' denote the midpoints of CB, CA, then O is the intersection of CE with the perpendicular bisector of segment $A'B'$, because the positive homothety with center C and factor $\frac{1}{2}$ transforms A, B into A', B', E into O and the perpendicular bisector EM of AB into the perpendicular bisector of $A'B'$. Now, it remains to note that $A'B'DM$ is an isosceles trapezoid, in which O lies on its symmetry axis. Indeed, $\triangle MA'B'$ and $\triangle DB'A'$ are symmetric about the perpendicular bisector of $A'B'$, hence $OD = OM$.

whisper00

#5 Apr 3, 2009, 12:08 pm

Let P be the intersection the line passing through K perpendicular to CK and the line passing through L perpendicular to CL. Since $\angle CKP + \angle CLP = 90^o + 90^o = 180^o$, we have that $CKPL$ is cyclic. Furthermore, the line KL is Simson Line for P. The line KL also contains M, which lies on AB. This implies PM is perpendicular to AB (by the property of the Simson Line). Since CD is also perpendicular to AB, we have that $CD||MP$.

But the quadrilateral $CKPL$ is cyclic, whose corresponding circle has center O (since O is the circumcenter of ΔCKL). Since $\angle CKP = 90^o$, CP is a diameter of this circle, which implies C, O, P are collinear and $|OC| = |OP|$. Since $CD||MP$, we conclude that O is equidistant to the lines CD and MP. This combined with the fact that MD is perpendicular to CD and MP implies $|OD| = |OM|$, as desired.

sunken rock

#6 Apr 3, 2009, 5:30 pm

Take A' on (CB with CA' = CA, call V the middle of the arc AB not containing C on (ABC); obviously, CV is the angle bisector of <ACB and, due to symmetry, A'V = AV (1), but AV = BV, so A'V = BV (2). It's obvious also that AA' is perpendicular to CV, hence AA'||ML and, M being the middle of AB, L is the middle of A'B; the triangle A'VB being isosceles - see relation (2) - it follows that VL is its altitude. Analogously prove VK and AC perpendicular, hence CKVL cyclic, and the circle of diameter CV is its circumcircle. DM being the orthogonal projection of the segment CV on AB, its extremities, D and M are equally apart of the middle of CV.

Best regards,
sunken rock

Virgil Nicula

#7 Apr 4, 2009, 7:31 am

> Herodatviet wrote:
> Let ABC be a triangle, and let M be the midpoint of $[AB]$. A line through M meet CA, CB at K, L respectively such that
>
> $CK = CL$. Denote $D \in AB$ for which $CD \perp AB$ and the circumcircle $C(R,\rho)$ of $\triangle KCL$. Prove that $RD = RM$.

Proof I (synthetic). Suppose w.l.o.g. that $b < a$. In this case $A \in (CK)$ and $L \in (CB)$. Let $w = C(O,R)$ be the circumcircle of $\triangle ABC$

with the diameter $[SN]$,where $M \in SN$ and the line AB separates the points S, C. Construct $\left\|\begin{array}{c} K' \in CA\ ;\ SK' \perp CA \\ L' \in CB\ ;\ SL' \perp AB \end{array}\right\|$. Observe that

$\left\|\begin{array}{c} CK' = CL' \\ SK' = SL' \\ SA = SB \end{array}\right\| \Longrightarrow$ the right triangles $K'AS$ and $L'BS$ are equivalently. Thus, $K'A = L'B$ and $\frac{K'A}{K'C}\cdot\frac{L'C}{L'B}\cdot\frac{MB}{MA} = 1$. Therefore,

$M \in K'L'$ and $CK' = CL' \quad\Longrightarrow\quad K' \equiv K$, $L' \equiv L$ and the quadrilateral $CKSL$ is a kite inscribed in a circle with the diameter $[CS]$.

In conclusion, $RC = RS = \rho$ and $RD = RM$ because $CD \parallel MS \perp AB$.

Proof II (metric). Suppose w.l.o.g. that $b < a$. In this case $A \in (CK)$ and $L \in (CB)$. Denote the midpoint X of $[KL]$, i.e. $CX \perp KL$ and the second intersection S of the line CX with the circumcircle $C(O,R)$ of $\triangle ABC$, i.e. $SM \perp AB$. Apply the **Menelaus' theorem** to the transversal $\overline{KML}$ and $\triangle ABC$: $\frac{KA}{KC}\cdot\frac{LC}{LB}\cdot\frac{MC}{MB} = 1 \Longrightarrow KA = LB$. Since $CK = CL$ obtain $KA + b = a - LB \Longrightarrow KA = LB = \frac{a-b}{2}$ and $CK = CL = \frac{a+b}{2}$. Thus, $CK = 2\rho\cos\frac{C}{2}$ $\Longrightarrow \rho = \frac{a+b}{4\cos\frac{C}{2}} = R\cos\frac{A-B}{2} = \frac{CS}{2}$. Therefore, $CR = RS = \rho$, i.e. the quadrilateral $CKSL$ is a kite with $KC \perp KS$ and its circumcenter R is the midpoint of the diagonal $[CS]$ in the trapezoid $CDSM$, where $CD \parallel MS \perp AB$. In conclusion, $FD = FM$.

Bandera

#8 Aug 9, 2015, 3:01 pm

WLOG we can assume that $AC > BC$. Let P be the point diametrically opposite to C on the circumcenter of $\triangle CKL$. Then $\angle PKC = \angle PLC = 90^\circ$. Also let E, N, F be the feet of the perpendiculars from A, P, B respectively to the line KL. $\triangle AEM = \triangle BFM$ as $AM = MB$, so $AE = BF$. $\angle EKA = \angle LKC = \angle KLC = \angle FLB$, so $\triangle AEK = \triangle BFL$ and $EK = FL$.
Let a circle built on AP as on a diameter intersects KL at M_1 besides K. If Q is the projection of the circle's center on EN, then $KQ = QM_1$. Moreover, $EQ = QN$ because $AENP$ is a trapezoid. That's why $EK = M_1N$. Similarly, if a circle built on PB as on a diameter intersects KL at M_2 besides L, then $M_2N = FL$. We have: $M_1N = EK = FL = M_2N$,

so $M_1 = M_2$. But $\angle AM_1P = \angle PM_2B = 90^{\circ}$. Therefore, AM_1B is a straight line. As M_1 lies both on EL and on AB, it is therefore the point of intersection of these lines and $M_1 = M$.
Conclusion: $\angle AMP = 90^{\circ}$, O is the midpoint of PC, so its projection on MD is the midpoint of latter, that's why $OD = OM$.

Quick Reply

High School Olympiads

Isotomic point of the height foot

Reply

Source: All Russian MO 2015, grade 10, problem 7

silouan
#1 Aug 7, 2015, 6:43 pm

In an acute-angled and not isosceles triangle ABC, we draw the median AM and the height AH.
Points Q and P are marked on the lines AB and AC, respectively, so that the $QM \perp AC$ and $PM \perp AB$.
The circumcircle of PMQ intersects the line BC for second time at point X. Prove that $BH = CX$.
M. Didin

TelvCohl
#2 Aug 7, 2015, 8:09 pm

My solution :

Let the tangent of $\odot(ABC)$ passing through B, C meets each other at T .
Let S be the isogonal conjugate of T WRT $\triangle ABC$ ($SB \parallel CA, SC \parallel AB$) .

From $PM \perp AB, TM \perp BC \Longrightarrow \angle TMP = \angle CBA = \angle TCP$,
so C, P, T, M lie on a circle with diameter $CT \Longrightarrow P$ is the projection of T on CA .
Similarly, Q is the projection of T on $AB \Longrightarrow \odot(PMQ)$ is the pedal circle of T WRT $\triangle ABC$,
so X is the projection of S on BC due to T, S share the same pedal circle (WRT $\triangle ABC$) $\Longrightarrow BH = CX$.

Q.E.D

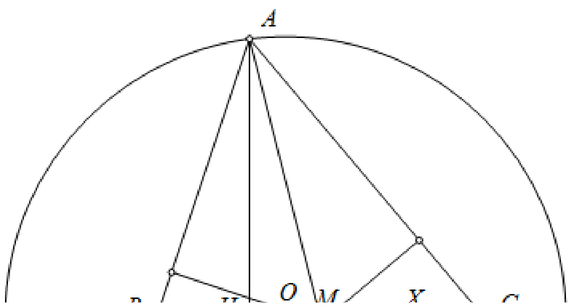
Luis González
#3 Aug 8, 2015, 3:58 am

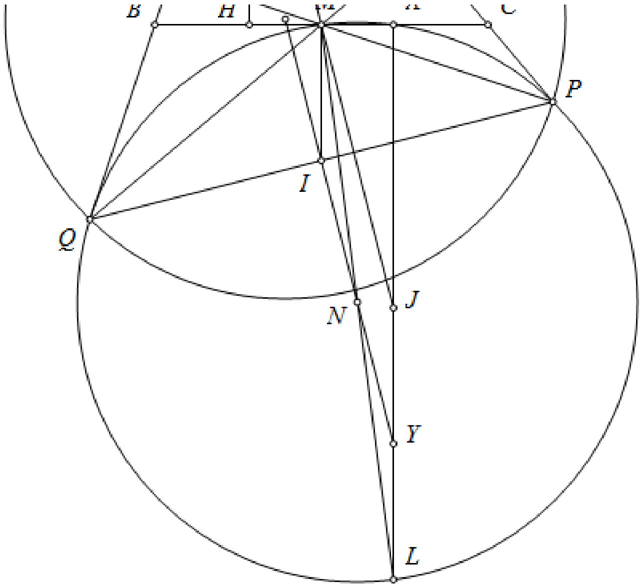
Let $U \equiv QM \cap AC, V \equiv PM \cap AB$. T is the midpoint of PQ and AM cuts $\odot(APQ)$ again at N (reflection of orthocenter M of $\triangle APQ$ on PQ). By symmetry, reflections Y and Z of X on PQ and T lie on $\odot(APQ)$ and $PQ \parallel YZ$ $\Longrightarrow \angle(XZ, XM) = \angle(ZN, ZX) = \angle ANZ = \angle AYZ \Longrightarrow AHY \perp BC$. But by Butterfly theorem for the cyclic $PQVU$, it follows that $TM \perp BC \Longrightarrow TM$ is X-midline of $\triangle XHY \Longrightarrow MX = MH$ or $BH = CX$.

livetolove212
#4 Nov 17, 2015, 11:03 am

Let I be the midpoint of PQ. Since M is the orthocenter of triangle APQ and $MB = MC$ then applying butterfly theorem for the circle with diameter PQ, $IM \perp BC$. Let J be the reflection of A wrt M, O be the circumcenter of triangle APQ, N is the center of (MPQ), L be the antipode of M wrt (MPQ), Y be the midpoint of LJ.
We have $NY \parallel= \frac{1}{2}MJ \parallel= \frac{1}{2}AM \parallel= IN$ hence N is the midpoint of IY. This means $IY \parallel= MJ$ or $JY \parallel IM$.
Therefore $LJ \perp BC$ at X. But A and J are symmetric wrt M hence $MH = MX$ or $BH = CX$.

Attachments:





This post has been edited 2 times. Last edited by livetolove212, Nov 17, 2015, 11:06 am

Quick Reply

High School Olympiads

Ptolemy's triangle

Source: Own, maybe known

Bandera

#1 Aug 7, 2015, 9:52 pm

Let $ABCD$ be a convex quadrangle. Assume that $\angle ADB \geq \angle ACB$. Then a triangle (or a degenerate triangle) having side lengths $AD \cdot BC, BD \cdot AC, CD \cdot AB$ has angles $\angle BDC - \angle BAC, \angle BAD + \angle DCB, \angle ADB - \angle ACB$ opposite to these sides, respectively.

Similarly, if D is a point inside a triangle ABC or on one of its sides, then a triangle having side lengths $AD \cdot BC$, $BD \cdot AC$, $CD \cdot AB$ has angles $\angle BDC - \angle BAC$, $\angle CDA - \angle CBA$, $\angle ADB - \angle ACB$ opposite to these sides, respectively.

Prove it.

Luis González

#2 Aug 8, 2015, 12:49 am • 1

We'll prove the second version of the problem. The first can be proved similarly with appropriate choice of signs.

Let X, Y, Z be the projections of D on $BC, CA, AB.$ From cyclic $DXCY$ and $DXBZ,$ we have $\angle DXY = \angle DCA$ and $\angle DXZ = \angle DBA \Longrightarrow \angle YXZ = \angle DCA + \angle DBA = \angle BDC - \angle BAC.$ Similarly we have $\angle ZYX = \angle CDA - \angle CBA$ and $\angle YZX = \angle ADB - \angle ACB.$ Now $YZ = AD \cdot \sin \angle BAC = \frac{AD \cdot BC}{2R}$ and similarly $ZX = \frac{BD \cdot AC}{2R}$ and $XY = \frac{CD \cdot AB}{2R} \Longrightarrow$ the sides of $\triangle XYZ$ are proportional to $AD \cdot BC, BD \cdot AC, CD \cdot AB$ and the conclusion follows.

Quick Reply

High School Olympiads

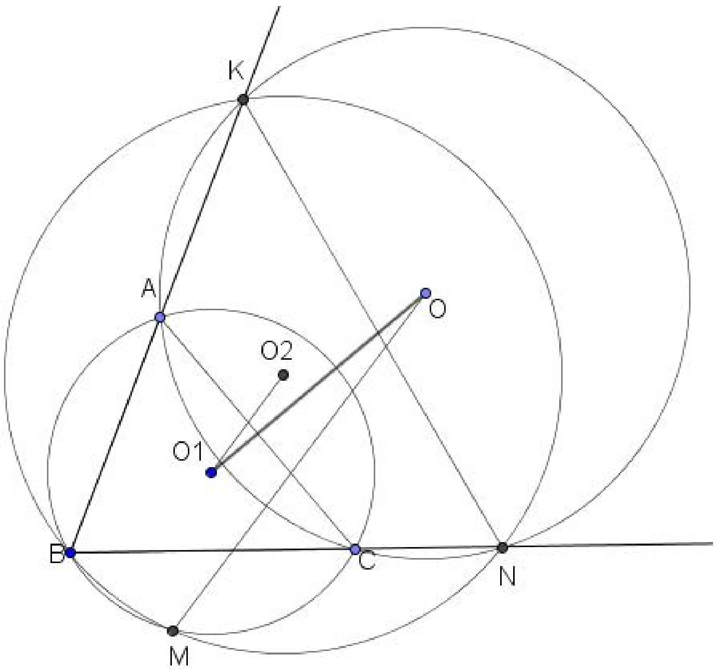
Prove parallel (P4)

Reply

PhuongMath
#1 Aug 7, 2015, 9:58 pm

Let $\odot(O_1)$, the triangle ABC is inscribed that circle.$\odot(O)$ is go through two points A,C cut AB, BC at K, N, resp. $\odot(O_2)$ is the circumcircle of triangle BKN. $\odot(O_1)\cap\odot(O_2)=B;N$
Prove: $O_1O_2||OM$

Attachments:



Luis González
#2 Aug 7, 2015, 10:44 pm

Let $P\equiv AN\cap CK$ and $Q\equiv AC\cap NK.$ Since BP is the polar of Q WRT $(O) \Longrightarrow PB\perp OQ$ at $T.$ BM,AC,NK are pairwise radical axes of $(O),(O_1),(O_2)$ concurring at their radical center $Q \Longrightarrow QM\cdot QB=QC\cdot QA=QO\cdot QT$ $\Longrightarrow BMTO$ is cyclic $\Longrightarrow OM\perp QB \Longrightarrow (OM \parallel O_1O_2)\perp QB.$

hayoola
#3 Aug 18, 2015, 11:56 am

o1o2 is perpendiqular to BM so we must prove that OM is perpendicular to BM so we must prove that we know that BM ,AC,KN are concurrent let Z be the interestion pont between AC and BM ZO is perpendicular to the line throte B and the interestion point between AN and CK name D so we have 4 cyclice points

Quick Reply

High School Olympiads

A simple property of McCay cubic

Reply

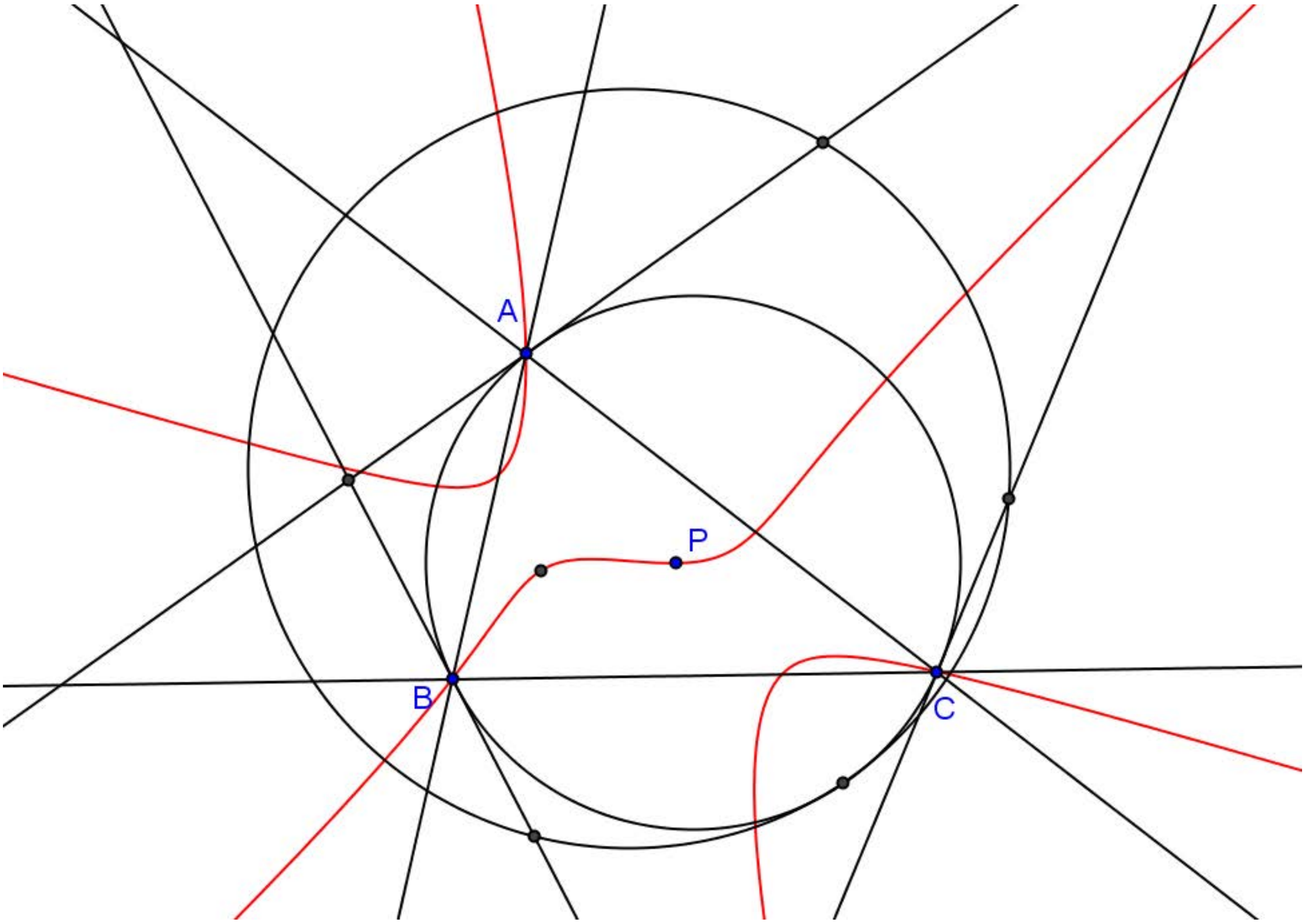
Source: Own

A-B-C

#1 Aug 7, 2015, 9:25 pm

Given $\triangle ABC$. P is a point such that P, circumcenter O and isogonal conjugate of P are collinear.
Prove that nine-point circle of antipedal triangle of P WRT $\triangle ABC$ is tangent to $\odot(ABC)$.

Attachments:



Luis González

#2 Aug 7, 2015, 10:11 pm

It simply follows from the 3rd Fontené theorem. See the topic two Yango's problems (post #2) and elsewhere.

TelvCohl

#3 Aug 7, 2015, 10:39 pm • 1

1st solution :

Let $\triangle DEF$ be the antipedal triangle of P WRT $\triangle ABC$. From B, P, C, D are concyclic $\Longrightarrow \measuredangle PCB = \measuredangle PDF$. Similarly, $\measuredangle PAC = \measuredangle PED$, $\measuredangle PBA = \measuredangle PFE$, so $\measuredangle PDF + \measuredangle PED + \measuredangle PFE = \measuredangle PCB + \measuredangle PAC + \measuredangle PBA = 90^\circ \Longrightarrow P$ lie on the McCay cubic of $\triangle DEF \Longrightarrow \odot(ABC)$ (pedal circle of P WRT $\triangle DEF$) is tangent to the 9-point circle of $\triangle DEF$.

2nd solution **:**

Let $\triangle DEF$ be the antipedal triangle of P WRT $\triangle ABC$. Let Q, R be the isogonal conjugate of P WRT $\triangle ABC, \triangle DEF$, respectively. From circumcevian triangles and isogonal conjugates (lemma at post #4 and post #5) we get $PQ \parallel TR$ where T is the circumcenter of $\triangle DEF$, so from $O \in PQ$ and $O \in PR$ ($\because R$ is the reflection of P in O) $\Longrightarrow T \in PR$, hence P lie on the McCay cubic of $\triangle DEF \Longrightarrow \odot(ABC)$ (pedal circle of P WRT $\triangle DEF$) is tangent to the 9-point circle of $\triangle DEF$.

Quick Reply

High School Olympiads

Three collinear points

Reply

Source: Own

andria

#1 Aug 7, 2015, 12:03 pm

Let O be the circumcenter of $\triangle ABC$. An arbitrary line ℓ passes through O. Let $\ell \cap AC = E, \ell \cap AB = F$. LET B', C' be the antipodes of B, C WRT $\triangle ABC$ respectively.
A) Prove that $B'E, C'F$ intersect each other on the circumcircle of $\triangle ABC$.
B) let P be the orthopole of ℓ WRT $\triangle ABC$ and $B'E \cap C'F = S$. let H be the orthocenter of $\triangle AEF$. Prove that S, H, P are collinear.

Luis González

#2 Aug 7, 2015, 12:16 pm

Discussed at http://www.artofproblemsolving.com/community/c6h495587. See lemma at post #3 and the subsequent replies.

Quick Reply

High School Olympiads

I count on it as a hard problem

Reply

mlm95

#1 Aug 25, 2012, 10:39 am

Let ABC be a triangle with circumcircle ω. Let O and I be it's circumcenter and incenter, respectively. Suppose that OI intersects AC at Y. BO cuts ω again at M. MY intersects ω again at N. Prove that NH passes through Feuerbach point F. (here H is the orthocenter of ABC)

Luis González

#2 Aug 29, 2012, 11:37 pm

I hate using brute force on the Feuerbach point, but here it goes. We resort to calculations with barycentric coordinates. Ray FH cuts the circumcircle of $\triangle ABC$ at the orthoassociate X_{108} of the Feuerbach point $F \equiv X_{11}$. Thus, we prove that X_{108}, the antipode M of B and the intersection $Y \equiv OI \cap AC$ are collinear.

Reflection M of B about $O(a^2S_A : b^2S_B : c^2S_C)$ is $M(a^2S_A : -S_AS_C : c^2S_C)$

$$OI \equiv bc(bS_B - cS_C)x + ca(cS_C - aS_A)y + ab(aS_A - bS_B)z = 0 \Longrightarrow$$

$$Y \equiv OI \cap CA \equiv (a(bS_B - aS_A) : 0 : c(bS_B - cS_C))$$

Now, we verify that M, Y and X_{108} are indeed collinear.

$$\begin{bmatrix} a^2S_A & -S_AS_C & c^2S_C \\ a(bS_B - aS_A) & 0 & c(bS_B - cS_C) \\ \frac{a}{(b-c)(b+c-a)S_A} & \frac{b}{(c-a)(c+a-b)S_B} & \frac{c}{(a-b)(a+b-c)S_C} \end{bmatrix} = 0$$

Luis González

#3 Sep 5, 2012, 2:35 am • 3

The proposed problem is just a particular case of the following configuration:

Lemma. Arbitrary line τ through circumcenter O of $\triangle ABC$ cuts AC at Y. BO cuts circumcircle (O) again at M. MY cuts (O) again at N. Then N, the orthocenter H of $\triangle ABC$ and the orthopole of τ WRT $\triangle ABC$ are collinear.

Let A_0, B_0, C_0 be the midpoints of BC, CA, AB. H_B is the foot of the B-altitude. Since $\angle BNM$ and $\angle BH_BY$ are right, then N, H_B lie on the circle (O_B) with diameter $\overline{BY}$. Let $T \in \odot(A_0B_0C_0)$ be the inverse of N under the inversion with center H that takes (O) into $\odot(A_0B_0C_0)$. Then $\overline{HN} \cdot \overline{HT} = \overline{HB} \cdot \overline{HH_B} \Longrightarrow T \in (O_B)$. Let E be the 2nd intersection of (O_B) with τ. Since E is on $\odot(BA_0C_0)$, its reflection T^* about A_0C_0 is on reflection $\odot(A_0B_0C_0)$ of $\odot(BA_0C_0)$ about A_0C_0. But $O_B \in A_0C_0 \Longrightarrow T^* \in (O_B) \Longrightarrow T \equiv T^*$. Consequently, T is anti-Steiner point of τ WRT $\triangle A_0B_0C_0$, i.e. the orthopole of τ WRT $\triangle ABC$.

When τ passes through the incenter I of $\triangle ABC$, its orthopole is the Feuerbach point F of $\triangle ABC$ (well-known).

mlm95

#4 Sep 5, 2012, 5:45 pm

Thanks. Same as what we did.

TelvCohl

#5 Oct 19, 2014, 2:01 pm • 1

> *Luis González wrote:*
> The proposed problem is just a particular case of the following configuration:
>
> **Lemma.** Arbitrary line τ through circumcenter O of $\triangle ABC$ cuts AC at Y. BO cuts circumcircle (O) again at M. MY cuts (O) again at N. Then N, the orthocenter H of $\triangle ABC$ and the orthopole of τ WRT $\triangle ABC$ are collinear.

My proof:

Let D,E,F be the projection of A,B,C on BC,CA,AB .
Invert with center H(factor $-\overline{AH}\cdot\overline{HD}$) and denote N' as the image of N

Since $\angle BNY=\angle BEY=90$ and $N'H\cdot HN=BH\cdot HE$,
so B,N,N',E,Y are concyclic. i.e. N' lie on the pedal circle ω of Y WRT $\triangle ABC$
Since N' is the second intersection ($\neq E$)of the nine point circle and ω ,
so N' is the Orthopole of OY which is lie on NH .

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, May 4, 2015, 9:33 pm

jayme
#6 Sep 22, 2015, 6:46 pm

Dear Mathlinkers,
a sythetic proof without inversion can be seen on

http://jl.ayme.pagesperso-orange.fr/Docs/Symetriques%20de%20OI%20par%20rapport%20au%20triangle%20de%20contact.pdf
p. 37-38

Sincerely
Jean-Louis

ATimo
#8 Sep 23, 2015, 3:55 am • 1

My solution:
I lies on OI and F lies on the nine point circle and the incircle of triangle $\triangle ABC$, so using fontene theorem for I we get that OI is the steiner line of F respect to the median triangle. Let P and Q be the foots of the perpendicular lines from Y to AB and BC respectively. Y lies on OI, so using fontene theorem for Y, we get that $YQPF$ is cyclic. So $\angle BFY=90$. Let K be the intersection point of FH and the circumcircle of $\triangle ABC$. We will prove that KY passes trough M. Let R be the foot of the perpendicular line from B to AC. We know that R and F lie on the nine point circle of triangle $\triangle ABC$. H is the exsimiliar center of ω and the nine point circle of triangle $\triangle ABC$. So we have $HF.HK=HR.HB$. So $BFRK$ is cyclic. We know that $\angle BFY=\angle BRY=90$, so $BFRY$ is cyclic. So $BFRYK$ is cyclic. So we have $\angle BKY=90$. So NY passes trough M.

Quick Reply

High School Olympiads

Prove parallel (P3)

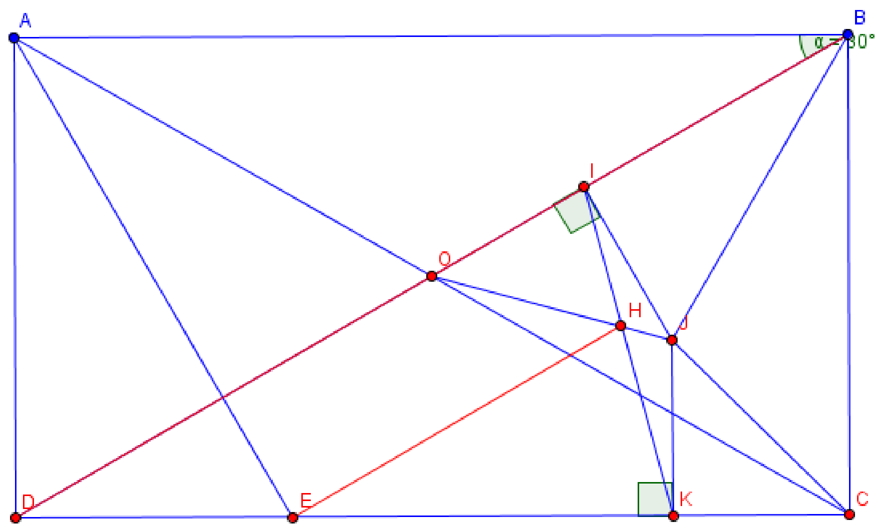
Reply

PhuongMath

#1 Aug 6, 2015, 9:56 pm

Let rectangle ABCD has $\angle ABD = 30^o$ and the point O is the center. Median line from the point A of triangle OAD contact CD at the point E,The point J is incircle center of triangle DBC,(J) contact with BD,CD at I,K, resp.$IK \cap OJ = H$
Prove: $HE||DB$

Attachments:



Luis González

#2 Aug 7, 2015, 1:50 am • 1

We rephrase the problem as follows: $\triangle ABC$ is right angled at A with $\angle ACB = 30^\circ$. It's incircle (I) touches BC, CA, AB at D, E, F and O is the midpoint of BC. IO cuts DE at M and the perpendicular bisector of OC cuts AC at N. Then $MN \parallel BC$.

$ID = AE = \frac{1}{2}(AC + AB - BC) = \frac{1}{2}(AC + AB - 2AB) = \frac{1}{2}(AC - AB) = OD \Longrightarrow \triangle IDO$ is right isosceles at $D \Longrightarrow \angle IOD = 45^\circ \Longrightarrow \angle MON = 180^\circ - 45^\circ - 30^\circ = 105^\circ \Longrightarrow ONEM$ is cyclic. But if Y is the midpoint of AC, we have $EY = \frac{1}{2}(BC - BA) = \frac{1}{2}BA = OY \Longrightarrow \triangle OYE$ is right isosceles at $Y \Longrightarrow \angle OEN = 45^\circ \Longrightarrow \angle OMN = \angle OEN = 45^\circ = \angle IOD \Longrightarrow MN \parallel BC.$

Quick Reply

High School Olympiads

Prove parallel (P2)

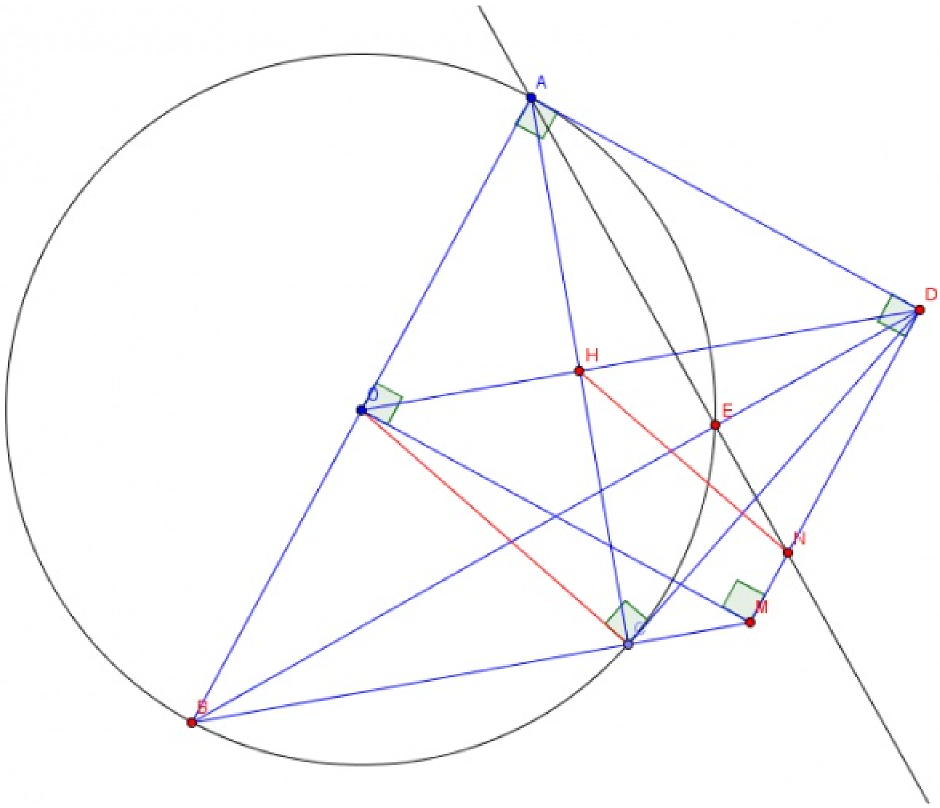
Reply

PhuongMath

#1 Aug 6, 2015, 9:43 pm

Let $\odot(O;\dfrac{AB}{2})$. The point C is on that circle such that AC>BC. Two tagent at A,C of that circle interest at the point D. $BD\cap\odot(O)=E$. From the pont O draw a line parallel with AD interest BC at M.
Prove: $HN||OC$

Attachments:



Luis González

#2 Aug 6, 2015, 11:36 pm • 1

Let $X\equiv AD\cap BM$. Since $OM\parallel AX$ and $OD\parallel BX\Longrightarrow DM\parallel AB$ is the X-midline of $\triangle ABX$. Together with $A(B,E,C,D)=-1$ (due to the harmonic quadrilateral $ABCE$), it follows that AC hits DM at F, such that N is midpoint of $DF\Longrightarrow\triangle NDH$ is N-isosceles $\Longrightarrow\angle NHD=\angle NDH=\angle DAH=\angle DCH\Longrightarrow NH\perp DC\Longrightarrow NH\parallel OC$.

Quick Reply

High School Olympiads

Prove parallel

Reply

PhuongMath
#1 Aug 6, 2015, 9:34 pm

Let a convex quadrilateral (not have any pairs of parallel edges). $AC \cap BC = I$. The point P is on AC. Draw $PM||CB(M \in AB; PN||AD(N \in CD)$. $CM \cap BN = L$. The points E,F are midpoint of AD, BC, resp.
Prove: $EF||IL$

sunken rock
#2 Aug 6, 2015, 10:03 pm • 1

See post #3 at http://www.artofproblemsolving.com/community/q1h496141p2786570, it may help.

Best regards,
sunken rock

TelvCohl
#4 Aug 6, 2015, 10:17 pm • 2

> *PhuongMath wrote:*
> Let a convex quadrilateral (not have any pairs of parallel edges). $AC \cap BD = I$. The point P is on AC. Draw $PM \parallel CB\, (M \in AB)$; $PN \parallel AD\, (N \in CD)$. $CM \cap BN = L$. The points E, F are midpoint of AD, BC, resp.
> Prove: $EF \parallel IL$

My solution **:**

Let $X \equiv AB \cap CD$ and Y be the point such that $YC \parallel AB, YB \parallel CD$.

Since EF is the Newton line of the complete quadrilateral $\{AB, CD, AC, BD\}$,
so EF passes through the midpoint of $XI \Longrightarrow EF$ is X-midline of $\triangle XYI \Longrightarrow EF \parallel IY\ (\star)$.

From $CN : DN = CP : AP = BM : AM \Longrightarrow B(C, D; N, Y) = C(B, A; M, Y)$,
so $I \equiv BD \cap CA, L \equiv BN \cap CM, Y$ are collinear $\Longrightarrow$ combine with $(\star)$ we conclude that $EF \parallel IL$.

Q.E.D

Luis González
#5 Aug 6, 2015, 10:56 pm • 2

Let $J \equiv AB \cap CD$ and let $G \in EF$ be the midpoint of IJ. As P varies, the series $\{B, M, A\}$ and $\{C, N, D\}$ are similar $\Longrightarrow$ pencils CM and BN are perspective $\Longrightarrow L$ moves on a line through I. When P is at infinity, $BLCJ$ becomes parallelogram $\Longrightarrow L$ is reflection of J on $F \Longrightarrow IL \parallel GEF \Longrightarrow IL \parallel EF$ for any P.

Quick Reply

High School Olympiads

Cevian triangle, incircle and tangents: collinear points

Reply

vnmathboy

#1 Dec 3, 2005, 12:35 pm

Let (I) be inscribed circle of triangle $A_1A_2A_3$. P_i lies on side $A_{i+1}A_{i+2}$ sastifying A_1P_1, A_2P_2, A_3P_3 concurent. t_i be tangent of (I) through P_i ($\neq A_{i+1}A_{i+2}$) meets $P_{i+1}P_{i+2}$ at Q_i. Proved Q_1, Q_2, Q_3 coliner

darij grinberg

#2 Dec 4, 2005, 10:43 pm

First I change the notations:

Problem. Let i be the incircle of a triangle ABC, and let D, E, F be three points on the sidelines BC, CA, AB of triangle ABC such that the lines AD, BE, CF are concurrent. Let d, e, f be the tangents to the circle i through the points D, E, F (different from the lines BC, CA, AB). Prove that the points $d \cap EF$, $e \cap FD$, $f \cap DE$ are collinear.

Solution. In the following solution, we will use the polar transformation with respect to circles, the Desargues theorem, and the cyclocevian conjugate theorem; the latter states:

Cyclocevian conjugate theorem. Let ABC be a triangle, let X and X' be two points on its side BC, let Y and Y' be two points on its side CA, and let Z and Z' be two points on its side AB such that the points X, X', Y, Y', Z, Z' lie on one circle. Assume that the lines AX, BY, CZ concur. Then, the lines AX', BY', CZ' concur.

This theorem was problem 1 of the MathLinks Contest, 5th edition, 6th round. Here is a quick *proof of this theorem*:

We are going to use directed segments. Since the points Y, Y', Z and Z' lie on one circle, the intersecting chords theorem yields $YA \cdot Y'A = AZ \cdot AZ'$. Similarly, $ZB \cdot Z'B = BX \cdot BX'$ and $XC \cdot X'C = CY \cdot CY'$. Hence,

$$\left(\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB}\right) \cdot \left(\frac{BX'}{X'C} \cdot \frac{CY'}{Y'A} \cdot \frac{AZ'}{Z'B}\right) = \frac{BX \cdot BX'}{XC \cdot X'C} \cdot \frac{CY \cdot CY'}{YA \cdot Y'A} \cdot \frac{AZ \cdot AZ'}{ZB \cdot Z'B}$$
$$= \frac{BX \cdot BX'}{CY \cdot CY'} \cdot \frac{CY \cdot CY'}{AZ \cdot AZ'} \cdot \frac{AZ \cdot AZ'}{BX \cdot BX'} = 1.$$

Now, since the lines AX, BY, CZ concur, the Ceva theorem yields $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1$. Hence, $\frac{BX'}{X'C} \cdot \frac{CY'}{Y'A} \cdot \frac{AZ'}{Z'B} = 1$, so that, by the Ceva theorem, the lines AX', BY', CZ' also concur, and the cyclocevian conjugate theorem is proven.

Now to the solution of the problem. Let the incircle i of triangle ABC touch the sides BC, CA, AB at the points X, Y, Z and the tangents d, e, f at the points X', Y', Z'. Let $U = YY' \cap ZZ'$, $V = ZZ' \cap XX'$, $W = XX' \cap YY'$.

The polar of a point P outside of a circle k with respect to this circle k is the line joining the points where the two tangents from the point P to the circle k touch k. Hence, the polar of the point A with respect to the circle i is the line YZ, and the polar of the point D with respect to the circle i is the line XX'. Hence, the pole of the line AD with respect to the circle i is the point $YZ \cap XX'$. Since the line XX' coincides with the line VW, we can thus state that the pole of the line AD with respect to the circle i is the point $YZ \cap VW$. Similarly, the poles of the lines BE and CF with respect to the circle i are the points $ZX \cap WU$ and $XY \cap UV$. Since the lines AD, BE, CF concur, their poles with respect to the circle i are collinear; in other words, the points $YZ \cap VW$, $ZX \cap WU$, $XY \cap UV$ are collinear. Hence, by the Desargues theorem, the triangles UVW and XYZ are perspective, i. e. the lines UX, VY, WZ concur. But now we have the triangle UVW and the points X and X' on its side VW, the points Y and Y' on its side WU, and the points Z and Z' on its side UV, and we know that these points X, X', Y, Y', Z, Z' lie on one circle (namely, on the circle i) and that the lines UX, VY, WZ concur. By the cyclocevian conjugate theorem, this yields that the lines UX', VY', WZ' also concur.

We proved that the polar of the point D with respect to the circle i is the line XX'. Similarly, the polars of the points E and F with respect to the circle i are the lines YY' and ZZ'. Thus, the pole of the line EF with respect to the circle i is the point $YY' \cap ZZ' = U$. On the other hand, the pole of the line d with respect to the circle i is the point X' (since the line d is tangent to the circle i at the point X'). Hence, the polar of the point $d \cap EF$ with respect to the circle i is the line X'U = UX'. Thus, the point $d \cap EF$ is the pole of the line UX' with respect to the circle i. Similarly, the points $e \cap FD$ and $f \cap DE$ are the poles of the lines VY' and WZ' with respect to the circle i. Now, since the lines UX', VY', WZ' concur, their poles $d \cap EF$, $e \cap FD$, $f \cap DE$ must be collinear, and the problem is solved.

Darij

Quick Reply

High School Olympiads

[Geometry]Concurrency problem

Reply

RHM
#1 Jul 13, 2013, 9:58 pm

In ΔABC, AD, BE, CF are concurrent lines. P, Q, R are points on EF, FD, DE such that DP, EQ, FR are concurrent. Prove that (using *plane geometry*) AP, BQ, CR are concurrent.

BBAI
#2 Jul 14, 2013, 5:06 pm • 1

It is the statement of Cevian nest theorem..

IDMasterz
#3 Jul 14, 2013, 5:27 pm • 2

It would be pretty calculative intensive doing this under plane geometry. But, regardless, here is a proof I found:

Let the perspector of $\triangle DEF, \triangle ABC$ be G. Let $AP \cap BC = X$ and define Y, Z similarly. Under a projective transformation make $\triangle ABC$ equilateral and G its centre. Then $\triangle DEF$ is its medial triangle and hence homothetic through a ratio $-\frac{1}{2}$ about G (the centroid). Now, the desired result can be concluded by Ceva's theorem with $\frac{EP}{PF} = \frac{BX}{XC}$, or this way (I like better):

Under a homothety taking $\triangle DEF$ to $\triangle ABC$, we take DP to AP' where P' is a point on sideline BC and similarly define Q', R'. So, if we let $S \mapsto S'$ under the homothety, then AP', BQ', CR' concur at S'. But then note that AX and AP' are isotomic, so AP, BQ, CR concur at the isotomic conjugate of S'.

sunken rock
#4 Jul 14, 2013, 6:08 pm • 3

1) From Ceva for AD, BE, CF in ΔABC: $\frac{AF}{BF} \cdot \frac{BD}{CD} \cdot \frac{CE}{AE} = 1 \ (* \ )$.
2) Similarly, $\frac{PE}{PF} \cdot \frac{QF}{QD} \cdot \frac{DR}{ER} = 1 \ (** \ )$.
3) Let $P' = BC \cap AP$ and let $\{E', F'\} \in AP$ so that $EE' \parallel FF' \parallel BC$.
Then $\frac{FF'}{BP'} = \frac{AF}{AB} \ (\ 1 \ )$ and $\frac{EE'}{CP'} = \frac{AE}{AC} \ (\ 2 \ )$. Divide side by side the last 2 equalities, arranging conveniently:
$\frac{FF'}{EE'} \cdot \frac{CP'}{BP'} = \frac{AF}{AE} \cdot \frac{AC}{AB} \implies \frac{FP}{PE} \cdot \frac{CP'}{BP'} = \frac{AF}{AE} \cdot \frac{AC}{AB} \ (\ 3 \ )$ and other 2 similar relations for the points Q', R' obtained as P' which, multiplied side by side, taking into account previous relations $(*), (**)$ will give $\frac{BP'}{CP'} \cdot \frac{CQ'}{AQ'} \cdot \frac{AR'}{BR'} = 1$, i.e. AP, BQ, CR concurrent.

Best regards,
sunken rock

RHM
#5 Jul 14, 2013, 8:36 pm

sunken rock's proof is simple and nice.

fmasroor

#6 Sep 6, 2013, 1:01 am

Even quicker solution:
\frac{FP}{PE}=\frac{AF}{AE}\frac{\sin{PAF}}{\sin{PAE}}$
Multiply these symmetrically, LHS =1 bc. DP, EQ, FR are concurrent
Product of the length ratios on the RHS is 1 bc. AD, BE, CF concurrent, meaning that the ratio of the sines on the RHS equals 1.
This implies AP, BQ, CR concurrent as desired.

sunken rock

#7 Sep 6, 2013, 10:22 pm

For a purely synthetic solution, see http://jl.ayme.pagesperso-orange.fr/vol3.html - 'The Cevian Nest Theorem'.

Best regards,
sunken rock

Quick Reply

High School Olympiads

The Begonia Point

Reply

Sutuxam

#1 Oct 17, 2011, 11:10 am

Let ABC be a triangle and let P be a point. Let A'B'C' be the Cevian triangle of P, which means that A', B', and C' are the intersections of AP, BP, and CP with BC, CA, and AB, respectively. Let X, Y, and Z be the reflections of P in B'C', C'A', and A'B', respectively. Prove that AX, BY, and CZ are concurrent.

Luis González

#2 Oct 17, 2011, 10:30 pm • 3

Let A_1, B_1, C_1 be the orthogonal projections of A', B', C' on $B'C', C'A', A'B'$, respectively. Pencils $A_1(P, X, C'A')$ and $A_1(P, A, C'A')$ are harmonic $\Longrightarrow X \in AA_1$. Similarly, $Y \in BB_1$ and $Z \in CC_1$. Thus, by **Cevian Nest Theorem**, AX, BY, CZ concur. If H' denotes the orthocenter of $\triangle A'B'C'$, then the Begonia point of P is just the cevian quotient H'/P with respect to $\triangle A_1B_1C_1$.

Proof of the Cevian Nest Theorem

Quick Reply

High School Olympiads

Concurrent Lines

Reply

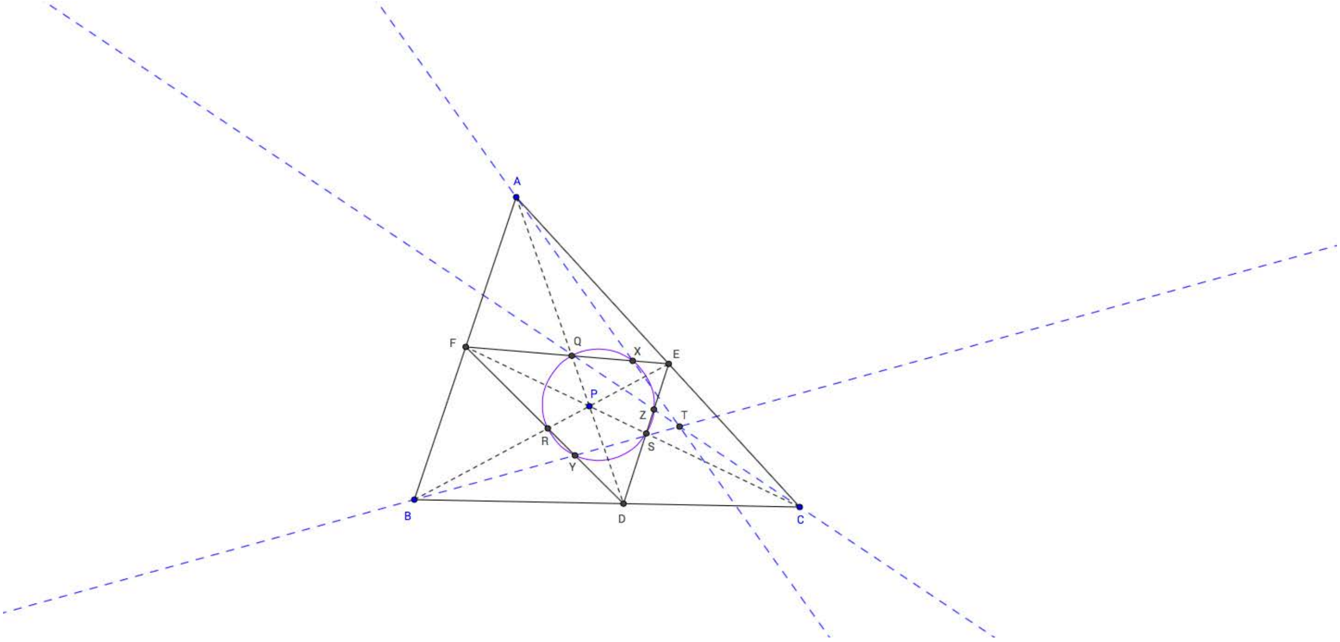
Source: Own?

tkhalid

#1 Aug 6, 2015, 9:10 am

Let P be a random point in the interior of $\triangle ABC$. Let $\triangle DEF$ be the P-cevian triangle of $\triangle ABC$ as shown. Let AP, BP, CP intersect EF, FD, DE at Q, R, S respectively. Let the circumcircle of $\triangle QRS$ intersect sides EF, FD, DE again at X, Y, and Z respectively. Prove lines AX, BY, and CZ are concurrent.

Attachments:



Luis González

#2 Aug 6, 2015, 9:20 am

DX, EY, FZ concur at the cyclocevian conjugate Q of P WRT $\triangle DEF$. Now by Cevian Nest Theorem, AX, BY, CZ concur at the cevian quotient Q/P WRT $\triangle DEF$.

tkhalid

#3 Aug 6, 2015, 9:34 am

Hi Luis, just to make sure your Q is the same as my T right?

Luis González

#4 Aug 6, 2015, 9:52 am

Sorry, I just realized you already used Q to denote the intersection of AP and EF. The lines DX, EY, FZ concur at a point called cyclocevian conjugate of P WRT $\triangle DEF$. This can be proved easily by Ceva's theorem and power of point (see for instance Cevian triangle, incircle and tangents: collinear points). Now the concurrency of AX, BY, CZ follows by the celebrated Cevian Nest Theorem (see [Geometry] Concurrency problem, The Begonia Point and elsewhere).

tkhalid

#5 Aug 6, 2015, 9:57 am

Thanks, I had the same solution

Quick Reply

High School Olympiads

Concurrent lines with mixtilinear incircles 2

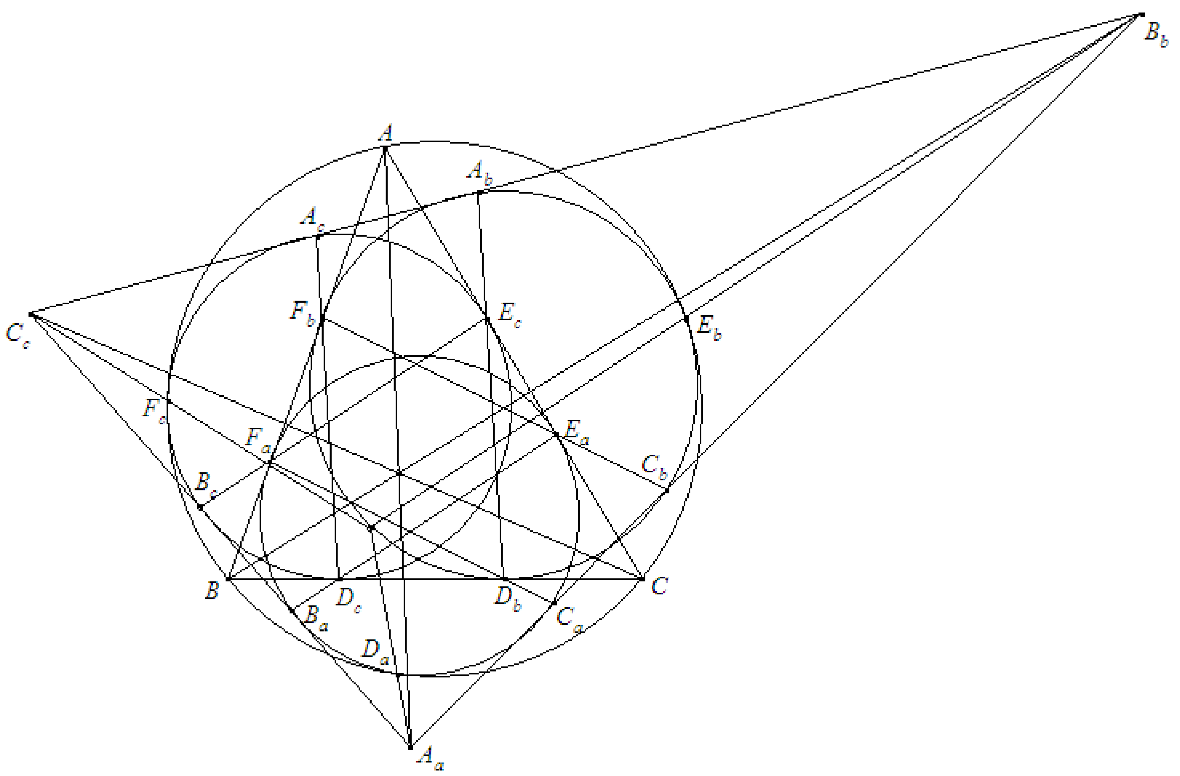
Source: Own

buratinogigle

#1 Aug 6, 2015, 8:50 am

Let ABC be a triangle inscribed in circle (O). A-mixtilinear incircle (K_a) touches $(O), CA, AB$ at D_a, E_a, F_a, reps. B-mixtilinear incircle (K_b) touches $(O), AB, BC$ at E_b, F_b, D_b, reps. C-mixtilinear incircle (K_c) touches $(O), BC, CA$ at D_c, F_c, E_c, reps. Let D_bE_c, D_cF_b cut $(K_b), (K_c)$ again at A_b, A_c, reps. Simiarly, we have B_c, B_a, C_a, C_b. Let C_aC_b cuts B_c, B_a at A_a. Similarly, we have B_b, C_c. Prove that triangle $A_aB_bC_c$ are both perpsective with triangle ABC and $D_aE_bF_c$.

Attachments:



Luis González

#2 Aug 6, 2015, 9:14 am

Let I be the incenter of $\triangle ABC$. It's well known that I is midpoint of D_bF_b and $D_cE_c \Longrightarrow E_cF_bD_cD_b$ is parallelogram $\Longrightarrow D_bE_c \parallel D_cF_b \Longrightarrow A_b$ and A_c are homologous points in the direct homothety that takes (K_b) to $(K_c) \Longrightarrow A_bA_C$ goes through the exsimilicenter A_0 of $(K_b) \sim (K_c)$ and by Monge's theorem for $(O), (K_b), (K_c)$, it follows that E_b, F_c, A_0 are collinear. Thus $A_0 \equiv E_bF_c \cap A_bA_c \cap BC$. Similarly, we have $B_0 \equiv F_cD_a \cap B_cB_a \cap CA$ and $C_0 \equiv D_aE_b \cap C_aC_b \cap AB$. A_0, B_0, C_0 are collinear on the positive homothety axis of $(K_a), (K_b), (K_c) \Longrightarrow \triangle ABC$ is perspective to $\triangle D_aE_bF_c$ and $\triangle A_aB_bC_c$ through $A_0B_0C_0$.

High School Olympiads

Geometry Problem (23)

Reply

vladimir92

#1 Oct 24, 2010, 8:21 pm

Problem. let O be the circumcenter of $\triangle ABC$. the perpendicular bissector of AO cut BC at A_1. Analogously, we define B_1 and C_1. Prove that A_1 , B_1 and C_1 lie all in a line that is perpendiclar with ON, where N is the isogonal conjugates of the nine-point center of $\triangle ABC$.

Luis González

#2 Oct 24, 2010, 9:45 pm

For convenience, denote the 9-point center and its isogonal conjugate as N, N_0. Let $\triangle H_aH_bH_c$ be the orthic triangle and let O_a, O_b, O_c be the circumcenters of $\triangle OBC, \triangle OCA, \triangle OAB$.

Since the isosceles triangles $\triangle NH_bH_c$ and $\triangle O_aBC$ are directly similar and BC, H_bH_c are antiparallel WRT AC, AB, then it follows that quadrangles AH_bNH_c and ABO_aC are similar, i.e. $\angle ABN = \angle CAO_a \Longrightarrow AO_a$ and AN are isogonal WRT A. Analogously, rays AO_b, BN and AO_c, CN are isogonal WRT B and $C \Longrightarrow AO_a, BO_b, CO_c$ concur at N_0. Therefore, $\triangle ABC$ and $\triangle O_aO_bO_c$ are perspective through N_0 and orthologic through the commom orthology center O. Thus, intersections $A_1 \equiv O_bO_c \cap BC, B_1 \equiv O_cO_a \cap CA$ and $C_1 \equiv O_aO_b \cap AB$ are collinear on their perspectrix. By Sondat's theorem, perspectrix $A_1B_1C_1$ is perpendicular to the line connecting the perspector N_0 with the orthology center O.

Quick Reply

High School Olympiads

3 lines are concurent

Locked

Scorpion.k48

#1 Aug 5, 2015, 11:47 am

Let $\triangle ABC$ with incircle $\odot(I)$. $\odot(I)$ touches BC, CA, AB at D, E, F. Let AH, BK, CL are altitudes of $\triangle ABC$ and X, Y, Z is midpoint of AH, BK, CL. Prove that DX, EY, FZ are concurent.

Luis González

#2 Aug 5, 2015, 12:11 pm

Posted before at http://www.artofproblemsolving.com/community/c6h42412.

High School Olympiads

incircle with center I of triangle ABC touches the side BC

Reply

Source: Vietnam TST 2003 for the 44th IMO, problem 2

orl

#1 Jun 26, 2005, 6:16 pm

Given a triangle ABC. Let O be the circumcenter of this triangle ABC. Let H, K, L be the feet of the altitudes of triangle ABC from the vertices A, B, C, respectively. Denote by A_0, B_0, C_0 the midpoints of these altitudes AH, BK, CL, respectively. The incircle of triangle ABC has center I and touches the sides BC, CA, AB at the points D, E, F, respectively. Prove that the four lines A_0D, B_0E, C_0F and OI are concurrent. (When the point O concides with I, we consider the line OI as an arbitrary line passing through O.)

mecrazywong

#2 Jun 26, 2005, 7:28 pm

$A_0D\cap(I)$ is the tangency point of the circle passing through B,C tangent to the incircle.
Then we consider the Apollonian circles with the ratio s-a/s-b,etc. It's easy to prove they intersect at exactly 2 points.
The concurrency of A_0D,B_0E,C_0F can be proved by radical axis.
The remaining is to show that I,O have the same power wrt the 3 Apollonian circles. For I it's easy; for O it's simple computation.

I know I have over-compressed the solution...

darij grinberg

#3 Jun 27, 2005, 9:27 pm • 2

You can find an alternative solution in Hyacinthos message #9551:

> [url=http://groups.yahoo.com/group/Hyacinthos/message/9551?expand=1]Hyacinthos message #9551[/url] (CORRECTED) wrote:
>
> **From:** Darij Grinberg
> **Subject:** Re: midpoints of AH, BK, CL
>
> Dear Orlando,
>
> In Hyacinthos message #9539, you wrote:
>
> > orl wrote:
> >
> > Given a triangle ABC inscribed in a circle with center O. Let H, K, L be the feet of the altitudes of triangle ABC from the vertices A, B, C, respectively. Denote by A_0, B_0, C_0 the midpoints of these altitudes AH, BK, CL, respectively. The incircle of triangle ABC has center I and touches the sides BC, CA, AB at the points D, E, F, respectively. Prove that the four lines A_0D, B_0E, C_0F and OI are concurrent. (When the point O concides with I, we consider the line OI as an arbitrary line passing through O.)
>
> The first three lines A_0D, B_0E, C_0F pass through the excenters I_a, I_b, I_c of triangle ABC (proof: see below). Hence, these three lines coincide with the lines DI_a, EI_b, FI_c, respectively. But triangles DEF and $I_aI_bI_c$ are homothetic (since their corresponding sidelines are parallel, what can be easily seen); hence, the lines DI_a, EI_b, FI_c concur at the homothetic center T of the two triangles DEF and $I_aI_bI_c$.
>
> Now, O is the nine-point center of triangle $I_aI_bI_c$, and I is the orthocenter of triangle $I_aI_bI_c$. Hence, the circumcenter of triangle $I_aI_bI_c$ is the reflection of the orthocenter in the nine-point center, i. e. the reflection of I in O.
>
> Obviously, the circumcenter of triangle DEF is I.
>
> Since the circumcenters of two homothetic triangles lie on one line with the homothetic center, it follows that the reflection

of I in O and the point I lie on one line with the homothetic center T of triangles DEF and $I_aI_bI_c$. In other words, the point T lies on the line OI.

Hence, altogether, the point T lies on the lines DI_a, EI_b, FI_c (i. e., on the lines A_0D, B_0E, C_0F), and on the line OI. The proof is complete.

Two notes:

(1) In the above, I have left out the proof that the lines A_0D, B_0E, C_0F pass through the points I_a, I_b, I_c.
Here is this proof:

The "semiprojection" of a point on a line will mean the midpoint between the point and its orthogonal projection on the line. Of course, if three points are collinear, their semiprojections on any line are collinear, too (in fact, point $\mapsto$ its semiprojection on a given line is an affine mapping).

If D' is the point of tangency of the A-excircle of triangle ABC with the side BC, and D" is the point diametrically opposite to D' on the A-excircle, then the tangent to the A-excircle at D" is parallel to BC; hence, if this tangent meets AB and AC at B" and C", respectively, the triangles ABC and AB"C" are homothetic. The homothetic center of these two triangles is A, of course.

Now, the incircle of triangle ABC touches BC at D; the incircle of triangle AB"C" is the A-excircle of triangle ABC and touches B"C" at D". Hence, the points D and D" are collinear with the homothetic center A of the two triangles ABC and AB"C".

Now, since the points D, D" and A are collinear, the points D, I_a, A_0 - being their semiprojections on the line BC - are collinear, too, i. e. the line A_0D passes through I_a. Similarly, the lines B_0E and C_0F pass through I_b and I_c, respectively, qed..

(2) The point T is the triangle center X(57) in Kimberling's ETC; in fact, T is the isogonal conjugate of the Mitten point X(9).

Sincerely,
Darij Grinberg

This post has been edited 4 times. Last edited by darij grinberg, Jul 29, 2007, 9:13 pm

Remike
#4 Sep 11, 2005, 5:51 pm

> *darij grinberg wrote:*
> But triangles A0B0C0 and IaIbIc are homothetic (since their corresponding sidelines are parallel, what can be easily seen);

What????

This is obviously an incorrect statement (a counterexample can be constructed very easyly). May be you meant that triangles DEF and $I_AI_BI_C$ are homothetic...

darij grinberg
#5 Sep 11, 2005, 6:18 pm

> *Remike wrote:*
> What????
>
> This is obviously an incorrect statement (a counterexample can be constructed very easyly). May be you meant that triangles DEF and $I_AI_BI_C$ are homothetic...

Okay, I'm sorry, I confused some notations (mainly DEF and A0B0C0). Now it should be correct.

Darij

Virgil Nicula
#6 Sep 16, 2005, 10:02 pm

Remark. The exincircles with the centers $I_a,\ I_b,\ I_c$ of the $\triangle ABC$ touch the sides $BC,\ CA,\ AB$ in the points $M,\ N,\ P$ respectively. Then $I\in MA_0\cap NB_0\cap PC_0$.

plane geometry
#7 Mar 19, 2009, 2:51 pm

> Virgil Nicula wrote:
> **Remark.** The exincircles with the centers $I_a,\ I_b,\ I_c$ of the $\triangle ABC$ touch the sides $BC,\ CA,\ AB$ in the points $M,\ N,\ P$ respectively. Then $I\in MA_0\cap NB_0\cap PC_0$.

dear Virgil Nicula,do you have a proof for this nice problem?

Luis González
#8 Mar 19, 2009, 8:40 pm

> Virgil Nicula wrote:
> **Remark.** The exincircles with the centers $I_a,\ I_b,\ I_c$ of the $\triangle ABC$ touch the sides $BC,\ CA,\ AB$ in the points $M,\ N,\ P$ respectively. Then $I\in MA_0\cap NB_0\cap PC_0$.

Basically, note that $\triangle AA_0I\sim\triangle I_aMI\Longrightarrow \frac{AA_0}{r_a}=\frac{IA}{II_a}$

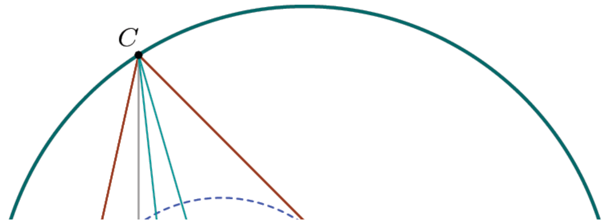
mathVNpro
#10 Apr 21, 2009, 10:57 am

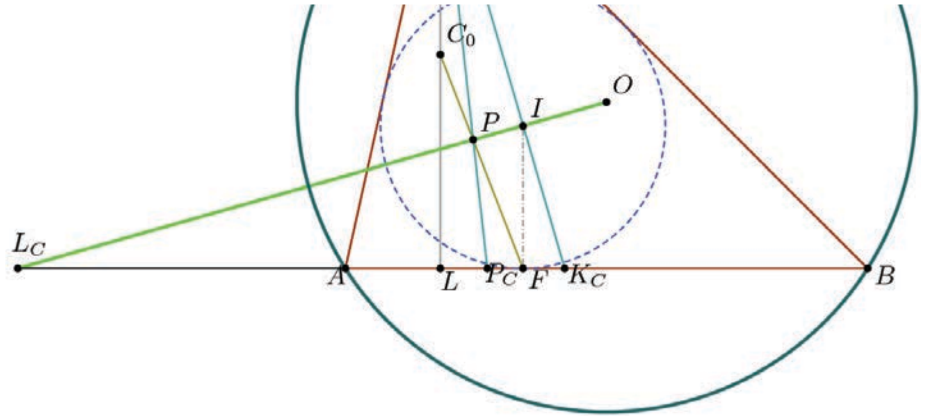
Let me restatement this problem so that it can fits my solution :
"let ABC be the acute triangle with incircle (I). Denote D, E, F the tagent points of (I) with BC, CA, AB, respectively. Let A_1, B_1, C_1 be the projections of A, B, C onto BC, CA, AB, respectively. Suppose A_2, B_2, C_2 are the midpoints of AA_1, BB_1, CC_1, respectively.
Prove that: DA_2, EB_2, FC_2, OI are concurent, where O is the circumcenter of triangle ABC."

Proof:
Let X be the intersection of EF with BC. It is well- known that $(X,D,B,C)=-1$ (1). Denote X' the intersection of AD with EF. From (1), we also get that $(X,X',E,F)=-1$. Therefore X is the pole of AD wrt (I). Hence, IX is perpendicular to AD. It is easy to notice that triangle IXD is similar to triangle DAA_1. Now, let me call A_3 is the midpoint of XD, from the notice above, we get, IA_3 is also perpendicular to DA_2. Therefore, A_3 is the pole of DA_2 wrt (I). Define the same for B_3, C_3.
So now, in order to prove DA_2, EB_2, FC_2 are concurent, we need to prove that A_3, B_3, C_3 are collinear.
Indeed, because, $(X,D,B,C)=-1$, A_3 is the midpoint of XD, we get, $A_3D^2=\overline{A_3B}.\overline{A_3C}$, which also means that the power of A_3 wrt (I) and (ABC) are equal. With the same argument for B_3, C_3. Hence, $(A_3B_3C_3)$ is the radical axis wrt (I) and (ABC), further, A_3, B_3, C_3 are collinear. As the result, DA_2, EB_2, FC_2 are concurent.
Let P be the intersection of these lines, we get that, P is the pole of $(A_3B_3C_3)$ wrt (I). Therefore, IP is perpendicular to $(A_3B_3C_3)$. OI is also perpendicular to $(A_3B_3C_3)$ (Because $(A_3B_3C_3)$ is the radical axis wrt (I) and (ABC)), where O is the circumcenter of triangle ABC. Hence, O,I,P are collinear. Which leads to the result that DA_2, EB_2, FC_2, OI are concurent.
Our proof is completed.

v_Enhance
#11 Jul 31, 2012, 3:50 am • 2

Here is a solution combining projective geometry and homogeneous/barycentric coordinates. The calculations fit on two pages when I performed them by hand.





Let $P = C_0F \cap IO$, $L_C = IO \cap AB$ and $K_C = AI \cap AB$. Define $P_C = AP \cap AB$. Let C_∞ be a point at infinity on line AL. Our goal is to compute the barycentric coordinates of P_C with respect to $\triangle ABC$.

We know that $K_C = (a : b : 0)$ and $F = (s-b : s-a : 0)$. The coordinates of L_C can be computed and simplified using the identities $aS_A - cS_C = s(c-a)(s-b)$, $I = (a : b : c)$ and $O = (a^2S_A : b^2S_B : c^2S_C)$ as

$$\begin{aligned} L_C &= \left(\det\begin{bmatrix} a^2S_A & c^2S_C \\ a & c \end{bmatrix} : \det\begin{bmatrix} b^2S_B & c^2S_C \\ b & c \end{bmatrix} : 0\right) \\ &= (a(aS_A - cS_C) : b(bS_B - cS_C) : 0) \\ &= (a(a-c)(s-b) : b(b-c)(s-a) : 0) \end{aligned}$$

Now, note that $(C, L; C_\infty, C_0) = -1$ is a harmonic bundle. Taking a perspective at F with onto line IO, we see that $(CF \cap IO, L_C; I, P) = -1$ is a harmonic bundle. Taking perspective at C onto line AB, we see that $(F, L_C; K_C, P_C) = -1$. There's a lemma that states that if we have $K_C = F + L_C$ when adding componentwise, then $P_C = -F + L_C$. So, we want to find a real r such that

$$\frac{r(s-b) + a(a-c)(s-b)}{r(s-a) + b(b-c)(s-a)} = \frac{a}{b}$$

because this will give us the coordinates of P_C for free. Solving for r (this is the hardest part),

$$\begin{aligned} r(b(s-b) - a(s-a)) &= ab((b-c)(s-a) - (a-c)(s-b)) \\ \implies r(a-b)(s-c) &= \frac{1}{2}ab((b-c)(b+c-a) - (a-c)(a+c-b)) \\ &= \frac{1}{2}ab((b^2-c^2) - (a^2-c^2) + b(a-c) - a(b-c)) \\ &= ab(b-a)(b+c-a) \\ \implies r &= -ab \end{aligned}$$

So what of P_C? We get

$$\frac{-r(s-b) + a(a-c)(s-b)}{-r(s-a) + b(b-c)(s-a)} = \frac{ab(s-b) + a(a-c)(s-b)}{ab(s-a) + b(b-c)(s-a)} = \frac{a(s-b)}{b(s-a)}.$$

Hence, $P_C = (a(s-b) : b(s-a) : 0)$.

If we define P_A and P_B analogously, we see that they concur at the point

$$P = (a(s-b)(s-c) : b(s-c)(s-a) : c(s-a)(s-b)).$$

It remains to show that this point actually lies on IO, which is surprisingly easy, as we have

$$\det\begin{bmatrix} a^2S_A & b^2S_B & c^2S_C \\ a & b & c \\ a(s-b)(s-c) & b(s-c)(s-a) & c(s-a)(s-b) \end{bmatrix}$$

which simplifies as

$$\sum_{\text{cyc}} a(s-b)(s-c)\,(bc(bS_B - cS_C)) = abc(s-a)(s-b)(s-c)\sum_{\text{cyc}}(b-c) = 0.$$

This implies the points are collinear, so we're done.

Virgil Nicula
#12 Jul 31, 2012, 8:58 am

> Quote:
> **Remark.** The exincircles with the centers $I_a,\ I_b,\ I_c$ of the $\triangle ABC$ touch the sides $BC,\ CA,\ AB$ in the points $M,\ N,\ P$ respectively. Then $I\in MA_0\cap NB_0\cap PC_0$.

$I\in MA_0\iff\frac{ID}{A_0H}=\frac{MD}{MH}\iff\frac{r}{\frac{h_a}{2}}=\frac{|b-c|}{\frac{s|b-c|}{a}}\iff\frac{2r}{h_a}=\frac{a}{s}\iff 2sr=ah_a(=2S)$ **O.K.**

thecmd999
#13 Jan 24, 2014, 6:17 am

Solution

jayme
#14 Sep 18, 2015, 5:27 pm

Dear Mathlinkers,
also a link

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=497757

Sincerely
Jean-Louis

Stranger8
#15 May 21, 2016, 7:07 pm

I use Three line Coordinates maybe this way is similar to Evan chen's solution,but still....
I use $A\ B\ C$ to represent the $cosA\ cosB\ cosC$
because A_c is the midpoint of the altitudes AH,then we can get $A_c\ (1,C,B)$,and we can get $D(0,1+C,1+B)$
then line A_0D is $(C-B,-1-B,1+C)$ similarily we can get line B_0E is $(1+A,A-C,-1-C)$ and line C_0F is $(-1-C,1+B,B-A)$
notice that incenter I is $(1,1,1)$ and circumcentre O is (A,B,C) then line OI is $(C-B,A-C,B-A)$ the rest part is to calculate the Det which is easy only takes 5minutes

navi_09220114
#16 May 21, 2016, 8:12 pm

It is well known (and proveable using cross ratio) that A_0D passes through the A-excenter, call it I_a. Now we want to prove DI_a, EI_b, F_c concur at OI, where I_a, I_b, I_c are the excenters. But just note that $I_bI_c\parallel EF$ and similarly to other two sides, so $\triangle I_aI_bI_c$ and $\triangle DEF$ are homothetic. So the three lines concur at the center of homothety, say P. P also lie on the line through circumcenter of the two triangles, which is line IX_{40}, which coincide with OI, because O is midpoint of IX_{40}. So P,O,I colinear and we are done.

Quick Reply

High School Olympiads

Concurrent lines with excircles

Reply

Source: Own

buratinogigle
#1 Jul 29, 2015, 2:32 am

Let ABC be a triangle with excircles $(I_a),(I_b),(I_c)$ touch BC,CA,AB at D,E,F, reps. Let X,Y,Z divide segments AI_a,BI_b,CI_c in the same ratio. Prove that DX,EY,FZ are concurrent.

mcdonalds106_7
#2 Jul 29, 2015, 3:56 am

Um, I'm pretty sure this is not true.

Luis González
#5 Aug 5, 2015, 11:58 am • 2

The series X,Y,Z are similar as the points X,Y,Z vary, thus the pencils EY,FZ are homographic $\Longrightarrow P \equiv EY \cap FZ$ moves on a fixed conic $\mathcal{H}$ through $E,F.$ When $Y \equiv B, Z \equiv C,$ then P is the Nagel point Na of $\triangle ABC.$ When $Y \equiv I_b, Z \equiv I_c,$ then P is the Bevan point Be of $\triangle ABC$ and when Y,Z are at infinity, then P becomes the orthology center U of $\triangle DEF$ WRT $\triangle I_aI_bI_c.$ Thus $N_a,Be,U,E,F \in \mathcal{H}.$

Since $\triangle DEF$ and $\triangle I_aI_bI_c$ are perspective through Be and orthologic, the conic through D,E,F,Be and the orthology center U of $\triangle DEF$ WRT $\triangle I_aI_bI_c$ is a rectangular hyperbola. Likewise since $\triangle DEF$ and $\triangle ABC$ are perspective through Na and orthologic, the conic through D,E,F,Na and the orthology center Be of $\triangle DEF$ WRT $\triangle ABC$ is a rectangular hyperbola $\Longrightarrow$ the conic through D,E,F,Be,Na,U is then a rectangular hyperbola, which is none other than $\mathcal{H}.$ Similarly the intersection $DX \cap EY$ is on the same conic $\mathcal{H} \Longrightarrow DX,EY,FZ$ concur at a point on $\mathcal{H}.$

Quick Reply

High School Olympiads

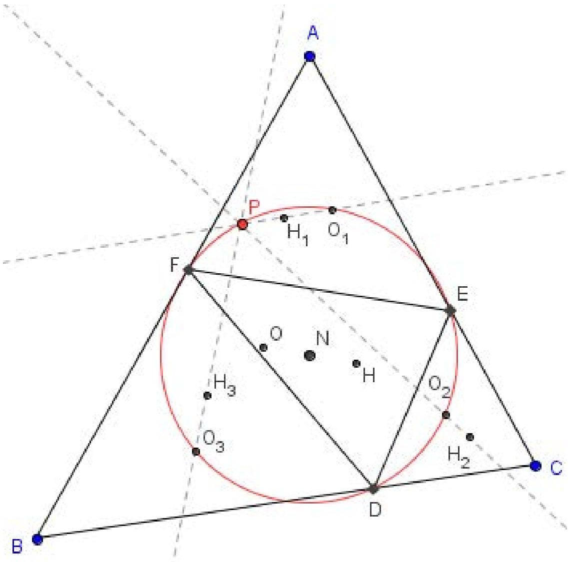
intersection of 3 Euler Line st is point on 9-Point Circle

Reply

khayyam-guilan
#1 Aug 23, 2013, 8:30 pm

Let D, E, F the foots of altitudes of A, B, C in triangle ABC.
Prove that Euler Lines of triangles AEF, CDE and BDF are congruent at a point on 9-Point Circle of triangle ABC.

Attachments:



IDMasterz
#2 Aug 23, 2013, 9:24 pm

Note that $\triangle EDC \sim \triangle AEF$, so a spiral similarity takes $\triangle EDC \to \triangle EFA$. It also then takes $\triangle EO_2H_2 \to \triangle EO_1H_1$.
If P is the intersection of O_1H_1 and O_2H_2 then EO_1PO_2 is cyclic. So, done.

jayme
#3 Aug 23, 2013, 9:47 pm

Dear Mathlinkers,
1. Note abc the Euler triangle of ABC
2. the isogonal line of (O1H1) wrt the a-inner bisector of abc is parallel to the Euler's line of ABC and circularly
3. According to the Beltrami's theorem, these isogonal line concur at the Thébault point X125 of ABC
Sincerely
Jean-Louis

Burii
#4 Aug 23, 2013, 11:23 pm

P is the center of Jerabek hyperbola.

XmL
#5 Aug 24, 2013, 12:28 am

Here's a property I found concerning this point:
P is collinear with the Centroid and Tarry point of ABC!

khayyam-guilan

#6 Aug 26, 2013, 8:25 pm

Please give a basic solution.

TelvCohl

#7 Oct 18, 2014, 12:59 pm • 1

In general :

Let D, E, F be the foot of altitude of A, B, C in $\triangle ABC$, respectively .
Let O, O_a, O_b, O_c be the circumcenter of $\triangle ABC, \triangle AEF, \triangle BFD, \triangle CDE$, respectively .
Let P, P_a, P_b, P_c be the points s.t. $\triangle ABC \cap P$ ~ $\triangle AEF \cap P_a$ ~ $\triangle BFD \cap P_b$ ~ $\triangle CDE \cap P_c$.

Then O_aP_a, O_bP_b, O_cP_c are concurrent at the orthopole of OP WRT $\triangle ABC$
(the center of the circum-rectangular hyperbola of $\triangle ABC$ passing through the isogonal conjugate of P WRT $\triangle ABC$)

In the original problem, three Euler lines are concurrent at the center of the Jerabek hyperbola of $\triangle ABC$ (X_{125} in ETC) .

This post has been edited 1 time. Last edited by TelvCohl, Aug 5, 2015, 5:41 am

Luis González

#8 Aug 5, 2015, 3:08 am

Here is a proof to the generalization above:

Let S, T be the midpoints of AC, AB and let OP cut AC, AB at Y, Z. If the circle with diameter $\overline{YZ}$ cuts AC, AB again at MN, then from Small problem about orthopole, we know that MN goes through the orthocenter O_a of $\triangle AST$ and the orthopole J of OP WRT $\triangle ABC \Longrightarrow MN \equiv JO_a$ is antiparallel to OP WRT AB, AC. But since $\triangle ABC \cup P \sim \triangle AEF \cup P_a$, it follows that O_aP_a is antiparallel to OP WRT $AB, AC \Longrightarrow J \in O_aP_a$. Likewise J lies on O_bP_b and O_cP_c.

Quick Reply

High School Olympiads

Small problem about orthopole

Source: Own

Luis González
#1 Aug 2, 2015, 6:33 am • 1

Arbitrary line ℓ through the circumcenter O of $\triangle ABC$ cuts AC, AB at $Y, Z,$ respectively. The circle with diameter $\overline{YZ}$ cuts AC, AB again at $M, N,$ respectively. Show that MN passes through the orthopole of ℓ WRT $\triangle ABC.$

A-B-C
#2 Aug 2, 2015, 9:10 am • 1

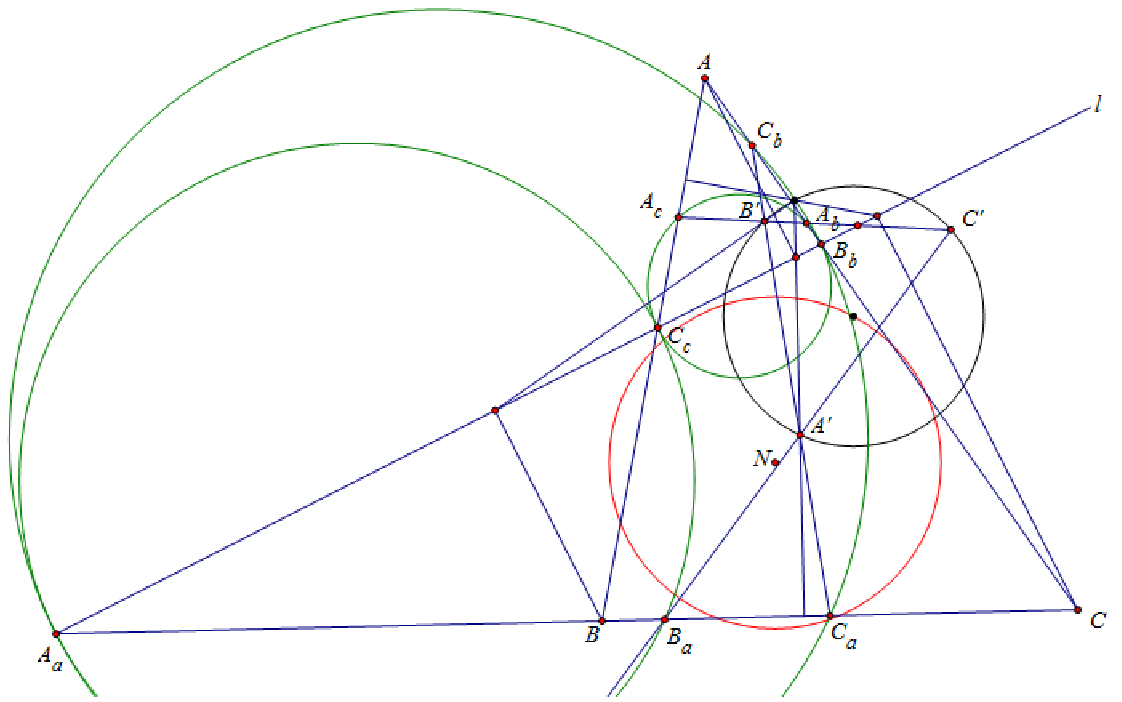
Let B', C' be midpoint of AC, AB, A_1 be orthogonal projection of A on ℓ
According to proof of the problem here http://artofproblemsolving.com/community/u245853h1121540p5161374
A_1 is Miquel point of complete quadrilateral formed by arbitary 4 in 5 lines: $(AY, AZ, ZM, YN, B'C')$
So reflection of A_1 in $B'C'$ lies on Steiner line of complete quadrilateral (AY, AZ, ZN, YM) - this coincides with MN
Reflection of A_1 in $B'C'$ is also orthopole of d WRT $\triangle ABC$ so we are done.

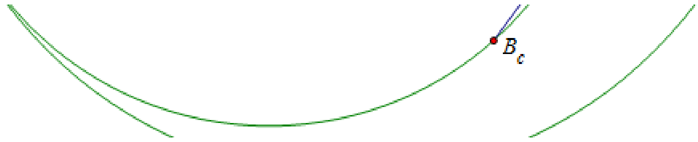
This post has been edited 1 time. Last edited by A-B-C, Aug 2, 2015, 9:10 am
Reason: typo

LeVietAn
#3 Aug 2, 2015, 11:19 am • 2

Dear Mathlinkers,
Note that orthopole of ℓ WRT $\triangle ABC$ lies on nine-point circle of $\triangle ABC$.
And I have found generalization as following
Let ABC be a triangle has nine-point circle (N). A line ℓ which intersects BC, CA, AB respectively at A_a, B_b, C_c. The circle with diameter B_bC_c intersects CA, AB again at A_b, A_c. Define similarly B_c, B_a and C_a, C_b. Suppose that the lines A_bA_c, B_cB_a and C_aC_b intersect in the points A', B' and C'.
a) Prove that the circumcenter of $\triangle A'B'C'$ lies on (N).
b) Prove that the orthopole of ℓ WRT $\triangle ABC$ lies on circumcircle of $\triangle A'B'C'$.

Attachments:





This post has been edited 1 time. Last edited by LeVietAn, Aug 2, 2015, 11:40 am

TelvCohl

#5 Aug 2, 2015, 3:22 pm

> *Luis González wrote:*
> Arbitrary line ℓ through the circumcenter O of $\triangle ABC$ cuts AC, AB at $Y, Z,$ respectively. The circle with diameter $\overline{YZ}$ cuts AC, AB again at $M, N,$ respectively. Show that MN passes through the orthopole of ℓ WRT $\triangle ABC.$

Since $\odot(BY), \odot(CZ)$ is the pedal circle of Y, Z WRT $\triangle ABC$, respectively, so one of the intersection T of these two circles is the orthopole of ℓ WRT $\triangle ABC$ and the other intersection S lie on the the circumcircle of $\triangle ABC$ (well-known), hence we get $\measuredangle NTM = \measuredangle NTS + \measuredangle STM = \measuredangle NBS + \measuredangle SCM = \measuredangle ABS + \measuredangle SCA = 180^\circ \Longrightarrow T \in MN.$

P.S. This problem is the particular case of A generalization of the Simson line theorem

tranquanghuy7198

#6 Aug 2, 2015, 3:55 pm

> *Luis González wrote:*
> Arbitrary line ℓ through the circumcenter O of $\triangle ABC$ cuts AC, AB at $Y, Z,$ respectively. The circle with diameter $\overline{YZ}$ cuts AC, AB again at $M, N,$ respectively. Show that MN passes through the orthopole of ℓ WRT $\triangle ABC.$

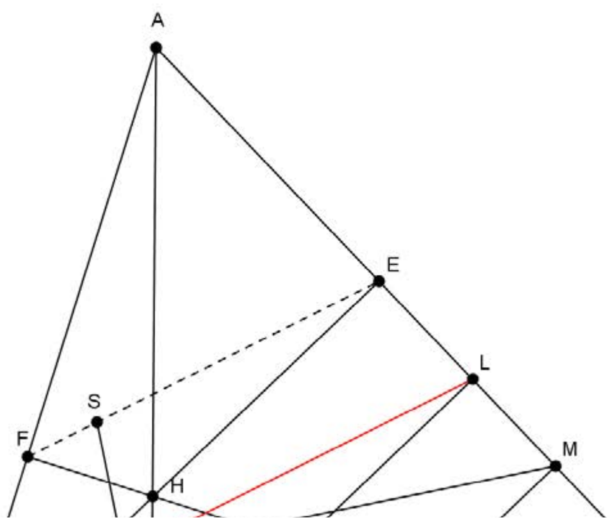
My solution:

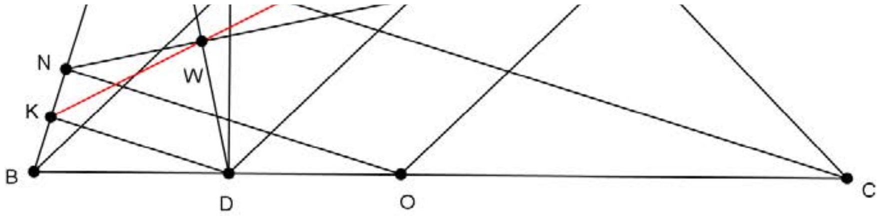
Lemma.
Given $\triangle ABC$ together with its altitudes AD, BE, CF and its orthocenter H. O varies on BC, M, N are the projections of O on CA, AB, resp. Prove that $S = R_{MN}(D) \in EF$

Proof.
W, K, L are the projections of D on $MN, AB, AC \Rightarrow W$ is the midpoint of $DS(\because S = R_{MN}(D))$
It's well-known that: $H_D^2(K), H_D^2(L) \in EF$
$\Rightarrow H_D^2 : KL \mapsto EF$
On the other hand: $D \in (AMN)(\because A, M, N, D \in (AO))$
$\Rightarrow \overline{W, K, L}$ (Simson line)
Now: $H_D^2 : W \mapsto S, KL \mapsto EF \Rightarrow S \in EF$

Back to our main problem.
Apply the lemma in $\triangle AYZ$ and we are done!!!

Attachments:





TelvCohl
#7 Aug 2, 2015, 4:29 pm • 1

> *LeVietAn wrote:*
> And I have found generalization as following
> Let ABC be a triangle has nine-point circle (N). A line ℓ which intersects BC, CA, AB respectively at A_a, B_b, C_c. The circle with diameter B_bC_c intersects CA, AB again at A_b, A_c. Define similarly B_c, B_a and C_a, C_b. Suppose that the lines A_bA_c, B_cB_a and C_aC_b intersect in the points A', B' and C'.
> a) Prove that the circumcenter of $\triangle A'B'C'$ lies on (N).
> b) Prove that the orthopole of ℓ WRT $\triangle ABC$ lies on circumcircle of $\triangle A'B'C'$.

My solution **:**

Let D, E, F be the midpoint of BC, CA, AB, respectively .
Let A_1, B_1, C_1 be the projection of A, B, C on ℓ, respectively .
Let X, Y, Z be the projection of A, B, C on BC, CA, AB, respectively .
Let ℓ' be the line passing through the circumcenter of $\triangle ABC$ and parallel to ℓ .
Let T be the orthopole of ℓ WRT $\triangle ABC$ and S be the orthopole of ℓ' WRT $\triangle ABC$.

Since B_cB_a, C_aC_b is anti-parallel to ℓ WRT $\angle CBA, \angle ACB$, respectively ,
so $\measuredangle C_bA'B_c = \measuredangle(C_aC_b, B_cB_a) = \measuredangle CAB = \measuredangle C_bAB_c \Longrightarrow A' \in \odot(AB_cC_b) \equiv \odot(AA_a)$,
hence combine $\measuredangle AC_bA' = \measuredangle(CA, C_aC_b) = \measuredangle(\ell, BC) = \measuredangle A_1A_aX \Longrightarrow A'$ is the reflection of A_1 in EF .
Similarly, B' is the reflection of B_1 in FD and C' is the reflection of C_1 in $DE \Longrightarrow T \equiv A'A_1 \cap B'B_1 \cap C'C_1$.

Since $\measuredangle B'TC' = \measuredangle B_1TC_1 = \measuredangle CAB = \measuredangle(C_aC_b, B_cB_a) = \measuredangle B'A'C'$,
so $T \in \odot(A'B'C')$. i.e. the orthopole of ℓ WRT $\triangle ABC$ lie on the circumcircle of $\triangle A'B'C'$

Since the reflection of S in EF, FD, DE is the projection of A, B, C on ℓ', respectively ,
so $SA' = SB' = SC' = \text{dist}(\ell, \ell') \Longrightarrow S$ (lie on $\odot(N)$) is the circumcenter of $\triangle A'B'C'$.

Q.E.D

A-B-C
#8 Aug 2, 2015, 5:18 pm

An observation: *Tangent line at A, B, C of $\odot(AB_bC_c), \odot(BC_cA_a), \odot(CA_aB_b)$ are concurrent at a point M on $\odot(ABC)$. H is orthocenter of $\triangle ABC$ then S is midpoint of HM.*(My notation is the same as Telv)
Proof
From Telv's proof, the reflections of S in EF, FD, DE lie on ℓ'
Hence if ℓ varies such that it always parallel to a fixed direction, then S is fixed. So giving a proof in case ℓ passes through O is enough.
D_1 is midpoint of HA. d, OD is parallel to Simson line of T, D WRT $\triangle DEF$. then

$$(D_1T, D_1D) = (OD, d) = (D_1T, OA)$$

$$(MA, OA) = (AM, AC) + (AC, AO) = (AB, d) + (AC, OE) + (OE, AO) = (AB, d) + 90^o + (BC, BA) = (BC, d) + (OD, BC) = (OD, d)$$

$\Rightarrow AM \parallel TD_1$
The homothety $\mathcal{H}_H^2 : \odot(DEF) \to \odot(ABC)$
$\Rightarrow H, T, M$ are collinear and $\dfrac{HT}{HM} = \dfrac{1}{2}$.

This post has been edited 2 times. Last edited by A-B-C, Aug 2, 2015, 5:19 pm
Reason: typo

Luis González

#9 Aug 3, 2015, 12:45 am

Thank you all for your solutions and remarks. This is what I did:

Let P be an abritrary point on ℓ and let U, V be the projections of P on AC, AB. R is the orthocenter of $\triangle AUV$. As P varies, the series U, V are clearly similar and since UR and VR have fixed directions, then R runs on a line. Making $P \equiv Y$ and $P \equiv Z$, it follows that $R \in MN$. When $P \equiv O$ and when P coincides with the projection A' of A on ℓ, then we deduce that MN is the Steiner line of the quadrangle bounded by AC, AB, the A-midline and the line joining the projections of A' on $AC, AB \Longrightarrow MN$ goes through the reflection of A' on the A-midline; the orthopole of ℓ WRT $\triangle ABC$.

Luis González

#10 Aug 3, 2015, 3:33 am

> *LeVietAn wrote:*
>
> Let ABC be a triangle has nine-point circle (N). A line ℓ which intersects BC, CA, AB respectively at A_a, B_b, C_c. The circle with diameter B_bC_c intersects CA, AB again at A_b, A_c. Define similarly B_c, B_a and C_a, C_b. Suppose that the lines A_bA_c, B_cB_a and C_aC_b intersect in the points A', B' and C'.
> a) Prove that the circumcenter of $\triangle A'B'C'$ lies on (N).
> b) Prove that the orthopole of ℓ WRT $\triangle ABC$ lies on circumcircle of $\triangle A'B'C'$.

Let D, E, F denote the feet of the altitudes on BC, CA, AB. Animate the line ℓ so that it has fixed direction. As A_bA_c is antiparallel to ℓ WRT AB, AC, then A_bA_c has fixed direction and similarly B_cB_a, C_aC_b have fixed directions. Moreover trivial angle chase reveals that $\triangle ABC$ and $\triangle A'B'C'$ are inversely similar. When $O \in \ell$, then $\triangle A'B'C'$ degenerates into the orthopole R of the parallel through O to $\ell \Longrightarrow$ all $\triangle A'B'C'$ are homothethic with center R. When $\ell \in A$, then $A' \equiv B_a \equiv C_a \equiv D$ and similarly when $B \in \ell$ and $C \in \ell$ we have $B' \equiv E$ and $C' \in F$, respectively $\Longrightarrow A', B', C'$ lie on RD, RE, RF, respectively. Hence $\angle(RC', RB') = \angle(DF, DE) = 2\angle(AC, AB) = 2\angle(A'C', A'B')$ and likewise $\angle(RC', RA') = 2\angle(B'C', B'A') \Longrightarrow R$ is circumcenter of $\triangle A'B'C'$ lying on 9-point circle $\odot(DEF)$.

Let T denote the orthopole of ℓ WRT $\triangle ABC$ and let U denote the projection of A on ℓ. Since the series U, B_b, C_c, A' are similar, then $A'U$ has fixed direction. When $A \in \ell$, then $U \equiv A$ and $A' \equiv D \Longrightarrow A'U \perp BC \Longrightarrow TA' \perp BC$ and likewise $TB' \perp CA \Longrightarrow \angle(TB', TA') = \angle(CA, CB) = \angle(C'B', C'A') \Longrightarrow T \in \odot(A'B'C')$.

Quick Reply

High School Olympiads

Clocks with hand endpoint forming an equilateral triangle

Source: 1961 All-Russian Olympiad

randomusername

#1 Aug 4, 2015, 3:35 pm

Points A and B move on circles centered at O_A and O_B such that O_AA and O_BB rotate at the same speed. Prove that vertex C of the equilateral triangle ABC moves along a certain circle at the same angular velocity. (The vertices of ABC are oriented clockwise.)

Luis González

#2 Aug 4, 2015, 10:43 pm • 1

Posted before at http://www.artofproblemsolving.com/community/c6h619257.

High School Olympiads

All Russian olympiad 1961

Reply

maths_lover5
#1 Dec 30, 2014, 7:23 pm

Points A and B move uniformly and with equal angle speed along the circumferences with Oa and Ob centres (both clockwise). Prove that a vertex C of the equilateral triangle ABC also moves along a certain circumference uniformly.

mavropnevma
#2 Dec 30, 2014, 8:57 pm • 2

Complex numbers completely kill it.

Luis González
#3 Dec 30, 2014, 11:17 pm

In fact, we can look for the locus of a point C verifying that all $\triangle ABC$ are directly similar. The reasoning is the same.

Let $A_0 \in (O_a)$ and $B_0 \in (O_b)$ be starting positions of A, B. Let $P \equiv O_aA \cap O_bB$ and $Q \equiv O_aA_0 \cap O_bB_0$. Since $\angle A_0O_aA = \angle B_0O_bB$, then P moves on fixed circle $\odot(QO_aO_b)$. Second intersection M of $\odot(PAB)$ and $\odot(PO_aO_b)$ is center of the spiral similarity that takes $AO_a \mapsto BO_b$ and $AB \mapsto O_aO_b$. Hence $\frac{MO_a}{MO_b} = \frac{AO_a}{BO_b} = \text{const} \Longrightarrow M$ is fixed. Furthermore $\triangle MAB \sim \triangle MO_aO_b$, thus $\triangle MAB \cup C$ are all directly similar $\Longrightarrow \angle(MA, MC)$ is constant and $\frac{MA}{MC}$ is constant $\Longrightarrow C$ runs on the circle image of (O_a) under spiral similarity with center M, rotational angle $\angle(MA, MC)$ and coefficient $\frac{MA}{MC}$.

Quick Reply

High School Olympiads

2014 China Second Round Olympiad Second Part Problem 2

Reply

Source: 2014 China Second Round Olympiad

abccsss

#1 Aug 4, 2015, 9:57 pm

Let ABC be an acute triangle such that $\angle BAC \neq 60^{\circ}$. Let D,E be points such that BD,CE are tangent to the circumcircle of ABC and $BD=CE=BC$ (A is on one side of line BC and D,E are on the other side). Let F,G be intersections of line DE and lines AB,AC. Let M be intersection of CF and BD, and N be intersection of CE and BG. Prove that $AM=AN$.

Luis González

#2 Aug 4, 2015, 10:41 pm

It's a particular case of http://www.artofproblemsolving.com/community/c6h610465, when the points M and N in that configuration coincide with B and C.

Quick Reply

High School Olympiads

AK=AL

Source: HSGS TST 2014-2015, Viet Nam

shinichiman

#1 Oct 18, 2014, 4:22 pm • 1

Let (O) be a circumcircle of triangle ABC. M, N are point on arc BC not containing A such that $MN \parallel BC$ (ray AM is between ray AB and ray AC). Let P, Q are points on ray BM, CN respectively such that $BP = BN = CM = CQ$. AM, AN intersect PQ at S, T, respectively. BT, CS intersect CQ, BP at L, K respectively. Prove that $AK = AL$.

TelvCohl

#2 Dec 4, 2014, 4:54 pm • 4

My solution:

WLOG $BN > BM, CM > CN$.
Let $M' = AM \cap BC, N' = AN \cap BC$.
Let X be the intersection of BC and the bisector of $\angle BAC$.

Since $\angle SMP = \angle BNA, \angle MPS = \angle MBC = \angle BAN$,
so we get $\triangle MPS \sim \triangle NAB \sim \triangle CAM'$.
Similarly, we can prove $\triangle NQT \sim \triangle MAC \sim \triangle BAN'$.

From Ptolemy theorem we get $AC \cdot BN + AB \cdot CN = BC \cdot AN$ ... (1)

Since $PM = BP - BM = BN - CN$,
so combine with (1) we get $AB \cdot PM = (AB + AC) \cdot BN - BC \cdot AN$.

$$\Longleftrightarrow AB + AC = BC \cdot \frac{AN}{BN} + PM \cdot \frac{AB}{BN} .$$

$$\Longleftrightarrow (AB + AC) \cdot BN = BC \cdot AN + AN \cdot PS \text{ . ... } (2)$$

Since $CX = \dfrac{AC}{AB + AC} \cdot BC, CM' = \dfrac{CA}{NA} \cdot BN$,

so from (2) we get $\dfrac{CX}{CM'} = \dfrac{BC}{BC + PS} = \dfrac{CK}{CS}$. ie. $KX \parallel SM'$

Similarly, we can prove $LX \parallel SM'$,

so we get $\dfrac{KX}{LX} = \dfrac{SM' \cdot \frac{CX}{CM'}}{TN' \cdot \frac{BX}{BN'}} = \dfrac{AM' \cdot \frac{AC}{CM'}}{AN' \cdot \frac{AB}{BN'}} = \dfrac{AC \cdot \frac{AB}{BN}}{AB \cdot \frac{AC}{CM}} = 1$. ie. $XK = XL$

Since $\angle AXK = 180^\circ - \angle MAX = 180^\circ - \angle XAN = \angle AXL$,
so we get $\triangle AXK \cong \triangle AXL$ and $AK = AL$.

Q.E.D

Attachments:

Luis González

#3 Dec 23, 2014, 3:25 am • 2

Let D be the midpoint of the arc BC of (O). AD, AM cut BC at E, F, resp. $\widehat{BPD} = \widehat{BND} = \widehat{CBD} \Longrightarrow \odot(PDB)$ is tangent to BC. If $U \equiv AC \cap PS$, then $SPU \parallel BC$ and PMB is antiparallel to AU WRT SA, SU. Thus if the isogonal of SC WRT $\angle ASU$ cuts AU at J, from Steiner theorem we get

$\dfrac{KM}{KP} = \dfrac{JU}{JA} = \dfrac{SU^2}{SA^2} \cdot \dfrac{CA}{CU} = \dfrac{FC^2}{FA^2} \cdot \dfrac{FA}{FS} = \dfrac{FC^2}{FA \cdot FS}$. But $\dfrac{FS}{BP} = \dfrac{FS}{CM} = \dfrac{FM}{BM}$

$$\Longrightarrow \frac{KM}{KP} = \frac{FC^2 \cdot BM}{FA \cdot FM \cdot CM} = \frac{FC^2 \cdot BM}{FB \cdot FC \cdot CM} = \frac{FC}{FB} \cdot \frac{BM}{CM} = \frac{AC}{AB} = \frac{EC}{EB}$$

$$\Longrightarrow \frac{KM \cdot KB}{KB \cdot KP} = \frac{EB \cdot EC}{EB^2}.$$

Thus, ratio of the powers of K, E WRT $(O), \odot(PDB)$ are equal $\Longrightarrow K, E$ lie on a circle coaxal with (O) and $\odot(PDB)$, i.e. $K \in \odot(BED)$ and similarly $L \in \odot(CED)$. Further, $\odot(BED) \cup K$ and $\odot(CED) \cup L$ are symmetric about AD because $\widehat{DBE} = \widehat{DCE}$ and $\widehat{EDK} = \widehat{EDL} = \pi - \widehat{CBM}$, i.e. K, L are symmetric about $AD \Longrightarrow AK = AL$.

buratinogigle

#4 Dec 27, 2014, 1:17 pm • 2

I proposed this problem for HSGS TST 2014, here is my solution

Let AM, AN cut BC at E, F. We have $\angle SMP = \angle BMA = \angle BNA$ and $\angle MSP = \angle AMN = \angle ABN$. From this $\triangle MSP \sim \triangle NBA$. Bisector of angle A cuts BC at D. We have $\dfrac{KS}{KC} = \dfrac{SP}{BC} = \dfrac{SP}{MP} \cdot \dfrac{MP}{BC} = \dfrac{AB}{AN} \cdot \dfrac{BP - BM}{BC} = \dfrac{AB}{AN} \cdot \dfrac{BN - CN}{BC}$. We will prove that $\dfrac{KS}{KC} = \dfrac{DE}{DC}$ by the way to show $\dfrac{AB}{AN} \cdot \dfrac{BN - CN}{BC} = \dfrac{DE}{DC}$, it is equivalent to

$$\frac{AB.BN - AB.CN}{AN.BC} = \frac{DE}{DC}$$

$\dfrac{AB.BN - AN.BC + AC.BN}{AN.BC} = \dfrac{DE}{DC}$ (Apply Ptolemy's theorem)

$$\frac{(AB + AC)BN}{AN.BC} = \frac{DE}{DC} + 1$$

$$\frac{(AB + AC)BN}{AN.BC} = \frac{CE}{CD}$$

$\dfrac{BN}{AN} CE = \dfrac{CD.BC}{AB + AC}$ (Because of $\triangle ANB \sim \triangle ACE$)

$$\frac{AC}{BC} = \frac{CD}{AB + AC}.$$

The last equality is true from AD is bisector of ABC. Thus, $\dfrac{KS}{KC} = \dfrac{DE}{DC}$ deduced $KD \parallel AE$. Similarly, $LD \parallel AF$. From this we have

$$\frac{DK}{DL} = \frac{DK}{SE} \cdot \frac{SE}{TF} \cdot \frac{TF}{DL} = \frac{CD}{CE} \cdot \frac{AE}{AF} \cdot \frac{FB}{BD} = \frac{CD}{BD} \cdot \frac{FB}{CE} \cdot \frac{AE}{AF} \quad (1).$$

Note that, $\triangle ACE \sim \triangle ANB$ và $\triangle ABF \sim \triangle AMC$ therefore $\dfrac{CE}{BF} = \dfrac{CE}{BN} \cdot \dfrac{CM}{BF} = \dfrac{AE}{AB} \cdot \dfrac{AC}{AF} = \dfrac{DC}{DB} \cdot \dfrac{AE}{AF} \quad (2).$

From $(1), (2)$ we have $\dfrac{DK}{DL} = 1$ hay $DK = DL$. Easily seen $\angle ADK = 180^\circ - \angle DAE = 180^\circ - \angle DAF = \angle ADL$. From this $\triangle ADK = \triangle ADL$ $(c.g.c)$ deduced $AK = AL$.

Attachments:

Figure2580.pdf (11kb)

Quick Reply

[illegible]

High School Olympiads

Pentagram

Reply

Source: Own

Bandera

#1 Aug 4, 2015, 9:45 am

Five points A, B, C, D, E are taken on a plane so that $BC \parallel AD$ and $ED \parallel AC$. If $F = AC \cap BD$ and $G = AD \cap EC$, prove that $EB \parallel GF$.

Luis González

#2 Aug 4, 2015, 10:30 am

More general: $ABCDE$ is a pentagon and $X \equiv AC \cap ED, Y \equiv BC \cap AD, F \equiv AC \cap BD, G \equiv AD \cap EC$ and $Z \equiv EB \cap GF$. Then X, Y, Z are collinear. When XY is at infinity, we get the proposed problem.

By dual of Pappus theorem for the pencils DE, DA, DB and CA, CB, CE, it follows that XY, BE, GF are concurrent at their projective center, i.e. X, Y, Z are collinear.

Quick Reply

High School Olympiads

Triangle and a line

Reply

Source: Own

Bandera

#1 Aug 3, 2015, 4:44 am

Given a triangle ABC and a point D on the side BC. A circle through A,B,D cuts AC again at E. A circle through A,C,D cuts AB again at F. Let K and L be the reflections of D over the perpendicular bisectors of CE and BF respectively. Prove that the line KL:
(a) is concurrent with the lines CF and BE;
(b) passes through A after having been reflected across the midpoint of BC.

Luis González

#2 Aug 3, 2015, 5:32 am • 1

If $P \equiv BE \cap CF$, then D is Miquel point of $AEPF \Longrightarrow DPEC$ and $DPFB$ are cyclic. Moreover $\angle AEB = \angle ADB = \angle CFB \Longrightarrow AEPF$ is cyclic. As $\triangle DEC \sim \triangle DBF$ are directly similar, the isosceles trapezoids $DECK \sim DBFL$ are directly similar as well $\Longrightarrow \angle DPK = \angle DPL \Longrightarrow K, L, P$ are collinear, i.e. KL, CF, BE concur.

Let X be the reflection of P on the midpoint of BC. Since $\angle(PB,PC) = \angle(AC,AB)$ (due to cyclic $AEPF$), then $\odot(PBC) \cong \odot(ABC) \Longrightarrow X \in \odot(ABC) \Longrightarrow \angle CAX = \angle CBX = \angle PCB \Longrightarrow$ $\angle(AX,BC) = \angle PCB + \angle ACB$. But $\angle(PL,BC) = \angle PDB - \angle DPL = \angle PEC - \angle DCK = \angle PCB + \angle ACB \Longrightarrow AX \parallel PL \Longrightarrow AX$ is reflection of $\overline{PKL}$ on the midpoint of BC.

TelvCohl

#3 Aug 3, 2015, 8:48 am • 1

Let $X \equiv BE \cap CF$ and T be the reflection of A in the midpoint of BC. Since D is the Miquel point of $AEXF$, so D lie on $\odot(BFX)$ and $\odot(CEX) \Longrightarrow B, D, F, L, X$ are concyclic and C, D, E, K, X are concyclic, hence $\measuredangle KXE = \measuredangle KDE$ $= \measuredangle AED = \measuredangle ABD = \measuredangle FBD = \measuredangle LDB = \measuredangle LXB \Longrightarrow L, X, K$ are collinear. i.e. BE, CF, KL are concurrent

Since $\measuredangle BXC = \measuredangle BXD + \measuredangle DXC = \measuredangle ACB + \measuredangle CBA = \measuredangle BTC$, so B, C, T, X are concyclic, hence $\measuredangle CXK = \measuredangle CEK = \measuredangle BCA = \measuredangle CBT = \measuredangle CXT \Longrightarrow T \in XK \Longrightarrow K, L, T, X$ are collinear $\Longrightarrow$ the reflection of the line KL in the midpoint of BC passes through A.

Quick Reply

High School Olympiads

Golden ratio in arbitrary triangle

Source: Own, Inspire from IMO SL 2011 G1

buratinogigle

#1 Jun 30, 2015, 9:42 pm • 1

Let ABC be a triangle inscribed in circle (O), with bisector AL, incenter I. d is diameter of (O) and h is length of altitude from A to BC. (L) is the circle center L and is tangent to AB, AC. Assume that (L) passes through O and $OI\perp AL$. Prove that ratio $\frac{d}{h}$ is golden section.

This post has been edited 1 time. Last edited by Kunihiko_Chikaya, Aug 2, 2015, 6:27 am
Reason: The correction of the problem

Luis González

#2 Aug 2, 2015, 5:02 am • 3

We use standard triangle notation: $BC=a, CA=b, AB=c, s=\tfrac{1}{2}(a+b+c)$ and R,r denote the radii of the circumcircle and incircle. It's known (posted before) that $OI \perp AL \Longleftrightarrow b+c=2a$. If (I,r) touches AB at Z and M is the projection of L on AB, we have

$$\frac{LM}{IZ}=\frac{LM}{r}=\frac{AL}{AI}=\frac{a+b+c}{b+c} \Longrightarrow LM=\frac{2\cdot r\cdot s}{b+c}=\frac{2[ABC]}{b+c}=\frac{a\cdot h}{2a}=\frac{h}{2}.$$

On the other hand, by PoP we have $R^2-LO^2=R^2-LM^2=LB\cdot LC \Longrightarrow$

$$R^2-\frac{h^2}{4}=\frac{ac}{b+c}\cdot\frac{ab}{b+c}=\frac{a^2bc}{(b+c)^2}=\frac{a^2bc}{4a^2}=\frac{bc}{4}=\frac{R\cdot h}{2} \Longrightarrow$$

$$4R^2-h^2-2R\cdot h=0 \Longrightarrow \left(\frac{d}{h}\right)^2-\frac{d}{h}-1=0 \Longrightarrow \frac{d}{h}=\frac{1+\sqrt{5}}{2}.$$

This post has been edited 1 time. Last edited by Kunihiko_Chikaya, Aug 2, 2015, 6:27 am

Quick Reply

High School Math

Help me !!

Locked

vuduyvy211

#1 Aug 1, 2015, 10:39 pm

Given a triangle ABC inscribed circle (O). P is any point. Line through P perpendicular to BC cut CA, AB at A1, A2. Call (Ka) is a circle of a triangle AA1A2. Similarly there (Kb), (Kc). Call (K) is in the circle tangent (Ka), (Kb), (Kc). Call (L) is exposed outer circle with (Ka), (Kb), (Kc). Prove that the circle (O), (K), (L) alignment centered

Luis González

#2 Aug 1, 2015, 11:39 pm

In fact, the circles (O),(K),(L) are coaxal (they have common radical axis), thus their centers are collinears. See http://www.artofproblemsolving.com/community/c6h539871

High School Olympiads

Coaxal circles

Reply

Source: Own

buratinogigle

#1 Jun 21, 2013, 5:17 pm

Let ABC be a triangle with circumcircle (O) and P is a point. Perpendicular line from P to BC cuts CA, AB at A_1, A_2. (K_a) is circumcircle of triangle AA_1A_2. Similarly we have circles $(K_b), (K_c)$. Let (K) be the circle touches $(K_a), (K_b), (K_c)$ internally and (L) be the circle touches $(K_a), (K_b), (K_c)$ externally. Prove that the circles $(K), (L)$ and (O) are coaxal.

Attachments:

Figure1070.pdf (9kb)

Luis González

#2 Jun 22, 2013, 10:56 pm • 1

Angle chase gives $\angle K_aAA_1 = 90^\circ - \angle AA_2A_1 = \angle ABC \Longrightarrow AK_a$ is tangent of (O), i.e. (O) is orthogonal to (K_a). Similarly, (O) is orthogonal to $(K_b), (K_c)$, hence O is radical center of $(K_a), (K_b), (K_c)$. Inversion WRT (O) takes then $(K_a), (K_b), (K_c)$ into themselves and swaps (L) and (K) due to conformity.

$(L), (K)$ and their circle of direct inversion (O) are always coaxal. If (O) cuts (L), say at X_1, X_2, then the pencil is obviously elliptic, since (K) passes through the double points X_1, X_2. If (O) does not cut (L), the pencil is then hyperbolic. Let U be the intersection of OL and the radical axis of $(O), (L)$. Let (U) be the circle orthogonal to $(O), (L)$. Inversion WRT (O) takes $(O), (U)$ into themselves and carries (L) into (K), centered on OL and orthogonal to (U) by conformity $\Longrightarrow U$ has equal power WRT $(O), (L), (K) \Longrightarrow (O), (L), (K)$ are coaxal.

Quick Reply

High School Olympiads

MN bisects arc DE

Reply

Source: KöMaL

MRF2017

#1 Jul 21, 2015, 7:40 pm • 1

A circle passing through vertices B and C of triangle ABC intersects side AB at D, and side AC at E. The intersection of lines CD and BE is O. Let M denote the centre of the inscribed circle of triangle ADE, and let N denote the centre of the inscribed circle of triangle ODE. Prove that line MN bisects the smaller arc DE.

This post has been edited 1 time. Last edited by math_explorer, Jul 22, 2015, 8:38 am
Reason: more descriptive title

Luis González

#2 Aug 1, 2015, 12:56 pm

Circle ω_1 is tangent to $\overline{OB}, \overline{OC}$ and internally tangent to $\odot(BCED)$ at P. According to 3 circles with common tangency point, there is a circle ω_2 tangent to $\overline{CE}, \overline{BD}$ and internally tangent to $\odot(BCED)$ at P. Now using the result of the problem incenter of triangle for ω_1, N and ω_2, M, we get that PM and PN bisect the arc DE of $\odot(BCED)$, i.e. MN bisects the arc DE of $\odot(BCED)$.

Quick Reply

High School Olympiads

tangents circles and isogonal conjugate points

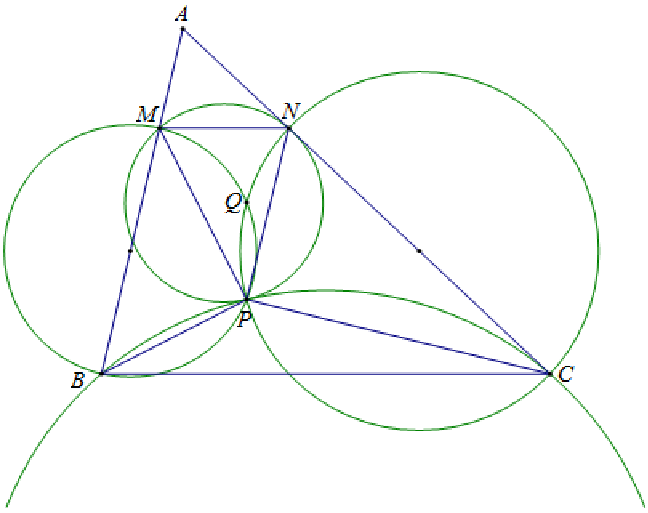
Reply

Source: OWN

LeVietAn
#1 Aug 1, 2015, 11:42 am

Dear Mathlinkers,
Let MN be a line parallel to the side BC of a triangle ABC, with M on the side AB and N on the side AC. Suppose that the circles with diameters BM and CN meet at two distinct points P and Q.
a) Prove that the circumcircles of triangles MNP and BCP are tangent to each other.
b) Prove that the points P, Q are isogonal conjugate points WRT $\triangle ABC$.

Attachments:



This post has been edited 2 times. Last edited by LeVietAn, Aug 1, 2015, 4:42 pm

tkhalid
#2 Aug 1, 2015, 11:46 am • 1

> *LeVietAn wrote:*
> Dear Mathlinkers,
> Let MN be a line parallel to the side BC of a triangle ABC, with M on the side AB and N on the side AC. Suppose that the circles with diameters BM and CN meet at two distinct points P and Q.
> a) Prove that the circumcircles of triangles MNQ and BCM are tangent to each other.
> b) Prove that the points P, Q are isogonal conjugate points WRT $\triangle ABC$.

I think you meant the above that I rewrote in red, correct?

TelvCohl

#3 Aug 1, 2015, 12:34 pm • 1

My solution :

Let $U \equiv PQ \cap MN, V \equiv PQ \cap BC$.
Let P_b, Q_b be the projection of P, Q on CA, respectively .
Let P_c, Q_c be the projection of P, Q on AB, respectively .

Since Q, U, N, Q_b are concyclic and Q, V, C, Q_b are concyclic ,
so $\angle VUQ_b + \angle UVQ_b = \angle QNC + \angle QCN = 90^\circ \Longrightarrow Q_b \in \odot(UV)$.
Similarly, we can prove $Q_c, P_b, P_c \in \odot(UV) \Longrightarrow U, V, P_b, P_c, Q_b, Q_c$ are concyclic ,
so P, Q are isogonal conjugate WRT $BCNM \Longrightarrow P, Q$ are isogonal conjugate WRT $\triangle ABC$.

Since $\angle MNQ + \angle BCQ = \angle UQ_bQ + \angle VQ_bQ = \angle UQ_bV = 90^\circ$,
so $\angle MNQ + \angle BCQ = \angle MQB \Longrightarrow \odot(MNQ)$ and $\odot(BCQ)$ are tangent at Q .

Q.E.D

Luis González

#4 Aug 1, 2015, 12:35 pm • 1

Let PQ cut MN, BC at $U, V.$ Obviously UV is perpendicular to $BC \parallel MN.$ Let X be the projection of P on $NC.$ From cyclic $PUNX$ and $PVCX,$ we get $\angle PUX = \angle PNC$ and $\angle PVX = \angle PCN \Longrightarrow \angle UXV = \angle NPC = 90^\circ,$ i.e. X is on circle with diameter $\overline{UV}$ and by similar reasoning the projection of P on MB and the projections of Q on NC, MB lie on the circle with diameter $\overline{UV}$, i.e. projections of P, Q on the sides of $BCNM$ are concyclic $\Longrightarrow P, Q$ are isogonal conjugates WRT $BCNM \Longrightarrow P, Q$ are isogonal conjugates WRT $\triangle ABC.$

On the other hand, since $\angle PMN + \angle PBC = \angle BPM = 90^\circ = \angle NPC,$ it follows that $\odot(MNP)$ and $\odot(BCP)$ are tangent at $P.$

Quick Reply

High School Olympiads

Concyclic Points (easy)

Reply

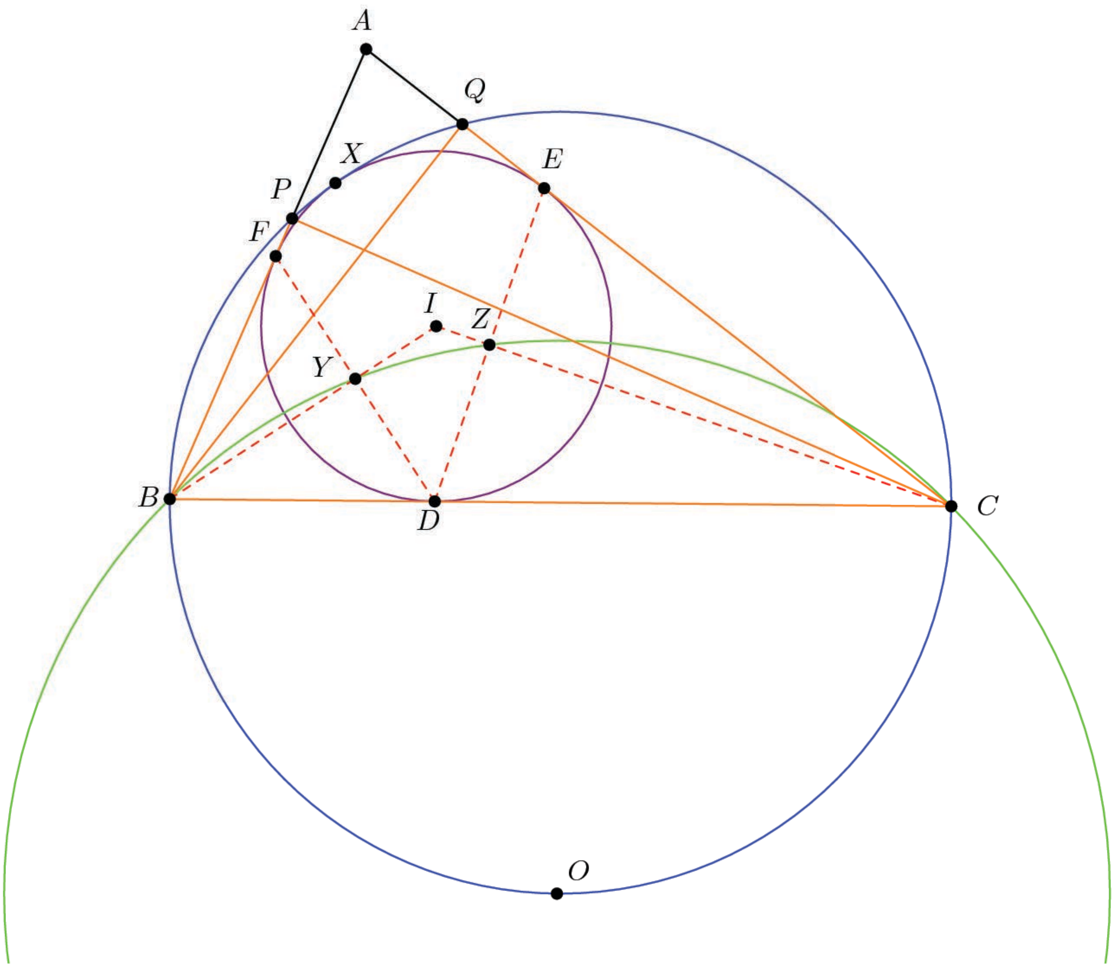
Source: own

tkhalid

#1 Aug 1, 2015, 11:42 am • 1

Let ABC be a triangle with incircle ω. Let I be the center of ω and let D, E, and F be the points of contact of ω with BC, CA, and AB respectively. Let Ω be the circle passing through B and C tangent to ω at a point X. Furthermore let Ω intersect AB and AC at P and Q respectively. Let DF intersect BI at Y and CI intersect DE at Z. Finally let PY intersect QZ at O. Prove that B, Y, Z, C are concyclic in a circle with center O.

Attachments:



Luis González

#2 Aug 1, 2015, 12:06 pm • 1

Since (I) is the B-mixtilinear incircle of $\triangle PBC$, then it follows that Y is the incenter of $\triangle PBC$ and similarly Z is the incenter of $\triangle QBC \Longrightarrow PY, QZ$ are internal bisectors of $\angle BPC, \angle BQC$ meeting at the midpoint O of the arc BC of Ω. Moreover, by incenter property we get $OB = OC = OY = OZ \Longrightarrow O$ is the center of $\odot(BYZC)$.

Phie11

#3 Aug 1, 2015, 1:41 pm

I forgot how to prove Y is incenter, can you help me? I can only prove XD bisects BXC.

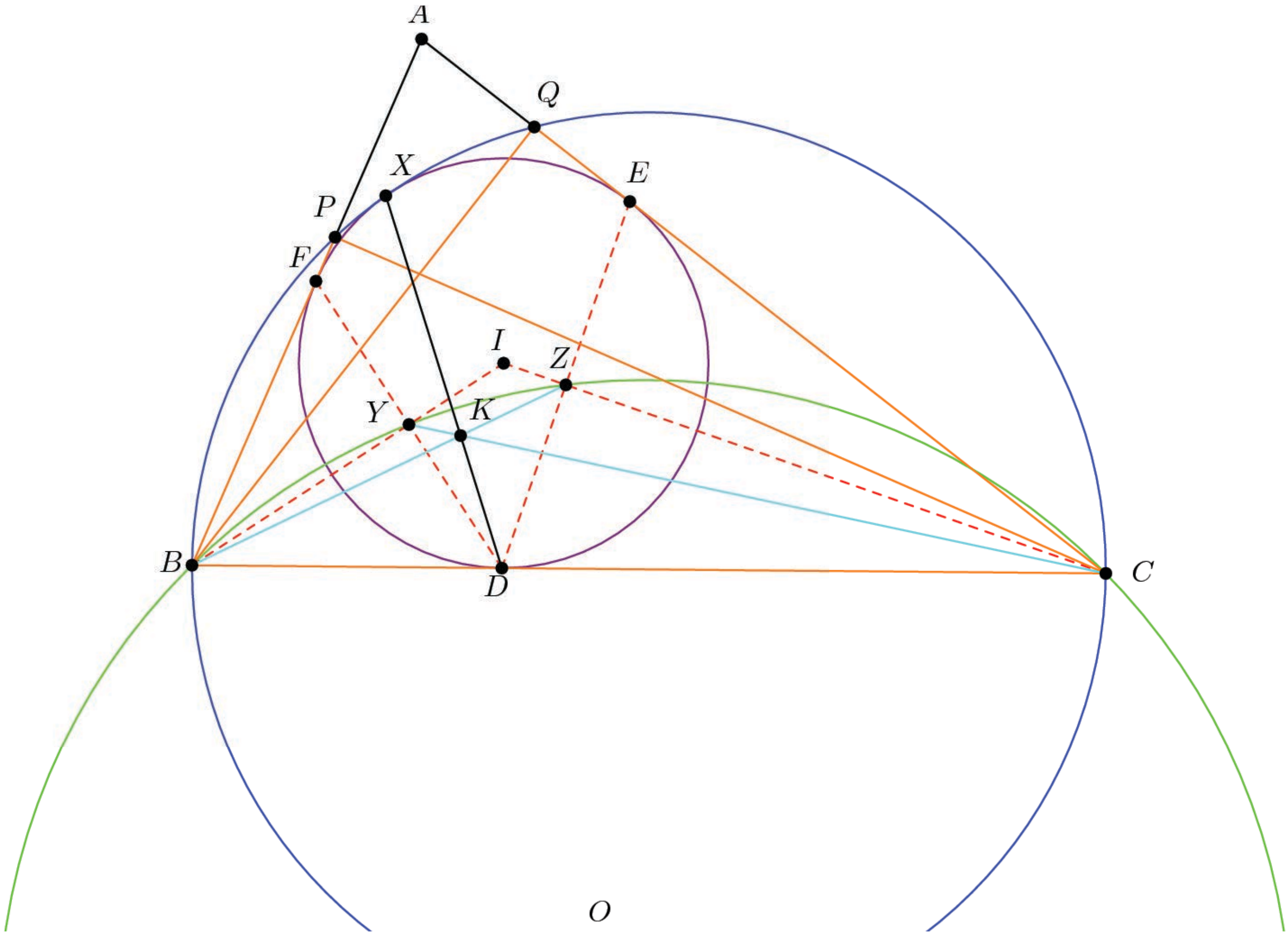
tkhalid

#4 Aug 2, 2015, 11:54 am

My solution was the same as Luis' solution. Here is an extension:

Let $CY \cap BZ = K$, then prove $X, K, D,$ and O are collinear.

Attachments:



LeVietAn

#5 Aug 2, 2015, 12:12 pm

Applying http://www.artofproblemsolving.com/community/u233841h1083361p4772869 and remarks(#9), we have $\odot(YKZX)$ touches (I) at X and $\overline{DKXO}$. Done

This post has been edited 1 time. Last edited by LeVietAn, Aug 2, 2015, 12:12 pm

jayme

#6 Aug 2, 2015, 4:29 pm

Dear Mathlinkers,
an outline of my proof...

1. the tangent to the circle (BIC) is parallel to EF
2. With anharmonic pencil, EF isparallel to YZ
3. by a converse of the Reim theorem, we are done...

Sincerely
Jean-Louis

tkhalid

#7 Aug 4, 2015, 5:41 pm

Nice solutions, there is also one with homothety that I found.

We can prove that there is a circle Γ passing through $X, Y, K,$ and Z internally tangent to ω at X. Then we consider the line parallel to BC passing through K. Because $\angle ZYK = \angle ZBC$ we see that the line must be tangent to Γ. So there exists a homothety centered at X which takes K to D, implying they are collinear.

Although there isn't really any need to do so, as once we've shown $XKYZ$ is cyclic simple angle chasing does the trick.

This post has been edited 1 time. Last edited by tkhalid, Aug 4, 2015, 7:13 pm

Quick Reply

High School Olympiads

Three parallel lines

Source: Own

andria

#1 Jul 31, 2015, 10:23 pm

Let P be any point in plain of $\triangle ABC$. Let P_a, P_b, P_c be the reflections of P WRT BC, CA, AB respectively. Let $A_0B_0C_0$ be the circumcevian triangle of P WRT $\triangle ABC$. Let $A_0P_a \cap \odot(\triangle ABC) = A_1$ we define B_1, C_1 similarly.
1) prove that $AA_1 \parallel BB_1 \parallel CC_1$.
2) assume that P is incenter of $\triangle ABC$ then prove that $AA_1 \perp OP$.

TelvCohl

#3 Jul 31, 2015, 11:33 pm • 4

My solution **:**

Lemma :

Let H be the orthocenter of $\triangle ABC$ and S be the anti-Steiner point of HP WRT $\triangle ABC$.
Let $\mathcal{H}$ be the circum-rectangular hyperbola of $\triangle ABC$ passing through P and $T \equiv \mathcal{H} \cap \odot(ABC)$.

Then S, P, T are collinear

Proof :

Since HA, HB, HC, HP is the Steiner line of A, B, C, S WRT $\triangle ABC$, resp ,
so we get $H(A, P; B, C) = (A, S; B, C) \Longrightarrow H(A, P; B, C) = T(A, S; B, C)$,
hence combine $T(A, P; B, C) = H(A, P; B, C)$ (cross ratio on $\mathcal{H}$) $\Longrightarrow T(A, P; B, C) = T(A, S; B, C)$.
i.e. S, P, T are collinear

Back to the main problem :

Let Q be the isogonal conjugate of P WRT $\triangle ABC$.
Let $\mathcal{H}$ be the circum-rectangular hyperbola of $\triangle ABC$ passing through P .
Let H be the orthocenter of $\triangle ABC$ and S be the anti-Steiner point of HP WRT $\triangle ABC$.
Let $D \equiv AH \cap \odot(ABC)$ and $T \equiv \mathcal{H} \cap \odot(ABC)$ (pole of the Simson line of $\triangle ABC$ with direction $\perp OQ$) .

From the lemma $\Longrightarrow S, P, T$ are collinear ,
so combine $S \in \odot(PP_aA_0)$ (well-known) $\Longrightarrow \measuredangle TAD = \measuredangle PSD = \measuredangle PA_0A_1$,
hence ATA_1D is an isosceles trapezoid $\Longrightarrow TA_1 \parallel AD \Longrightarrow TA_1 \perp BC \Longrightarrow AA_1 \perp OQ$.
Similarly, we can prove $BB_1 \perp OQ$ and $CC_1 \perp OQ \Longrightarrow AA_1 \parallel BB_1 \parallel CC_1$ (all perpendicular to OQ) .

If P coincide with the incenter of $\triangle ABC$, then $P \equiv Q \Longrightarrow AA_1 \perp OP, BB_1 \perp OP, CC_1 \perp OP$.

Q.E.D

Luis González

#4 Aug 1, 2015, 10:26 am • 1

Let (O) be the circumcircle of $\triangle ABC$ and let Q be the isogonal conjugate of P WRT $\triangle ABC$. AQ cuts (O) again at D. Fix AA_0 and animate P. Since P_a moves on the reflection of AA_0 on BC, then the series $Q \barwedge P \barwedge P_a \barwedge A_1$ are projective. So we'll prove that $OQ \perp AA_1$ holds for at least 3 positions of P.

When $P \in BC$, then $Q \equiv A \equiv A_1 \Longrightarrow AA_1$ becomes tangent of $(O) \Longrightarrow OQ \perp AA_1$. When $P \equiv A_0$, then Q goes to the point at infinity of AD and since $(DA_0 \parallel BC) \perp PP_a \Longrightarrow \angle DAA_1 = \angle DPA_1 = 90^\circ \Longrightarrow OQ \perp AA_1$. Finally when P is at infinity, then $Q \equiv D$ and P_a goes to the point at infinity of the reflection of AA_0 on $BC \Longrightarrow A_0Q$ bisects $\angle AA_0A_1$, i.e. Q is midpoint of the arc $AA_1 \Longrightarrow OQ \perp AA_1$. Consequently $OQ \perp AA_1$ holds for any P. Similarly OQ is perpendicular to BB_1 and $CC_1 \Longrightarrow (AA_1 \parallel BB_1 \parallel CC_1) \perp OQ$.

IDMasterz

#5 Aug 1, 2015, 1:36 pm • 1

This is very similar to the second problem here. In fact the proof of it is very similar to Telv's, so the similarity must be profound.

Quick Reply

High School Olympiads

two concentric circles

Reply

phuocdinh_vn99
#1 Jul 31, 2015, 11:19 pm

Given acute triangle ABC with orthocenter H. D is an arbitrary point on AH. (ADB) and (ADC) cut BH, CH at E, F, resp. Let X, Y, Z be circumcenter of $(ADB), (ADC), (BEC)$, resp. Prove that (DEF) and (XYZ) have common center.

This post has been edited 1 time. Last edited by phuocdinh_vn99, Jul 31, 2015, 11:19 pm

Luis González
#2 Aug 1, 2015, 12:53 am

Since $HE \cdot HB = HD \cdot HA = HF \cdot HC$, then $BEFC$ is cyclic and since $\angle HDF = \angle HCA = \angle HBA = \angle HDE \Longrightarrow DH$ is internal bisector of $\angle EDF$, cutting the arc EF of $\odot(DEF)$ at its midpoint $M \Longrightarrow XM$ is perpendicular bisector of EF passing through the circumcenter K of $\triangle DEF$.

As EF is antiparallel to $BC \parallel YZ$ WRT HB, HC and $XY \perp HC, XZ \perp HB$, we deduce that EF is also antiparallel to YZ WRT $XY, XZ \Longrightarrow XK \perp EF$ passes through the circumcenter of $\triangle XYZ$ and similarly YK, ZK pass through the circumcenter of $\triangle XYZ \Longrightarrow K$ is also circumcenter of $\triangle XYZ$.

jsphkn
#3 Aug 1, 2015, 1:01 am

Alternatively let P, Q, R be the foot of perpendicular from A, B, C respectively, and we have $DH : EH : FH = PH : QH : RH$. If we shift D, E, F at a constant rate away from H then lines XY, YZ, ZX will shift at a constant rate too, so circumcenters of (DEF) and (XYZ) will shift at constant rate. So we can just prove the question for the case when $D = H$ and $D = P$. For the first case it is quite clear AH bisects XY so H is the common center of both circles. For the second case X, Y, Z become the midpoints of sides of ABC so $DEFXYZ$ lie on the nine-point circle.

This post has been edited 1 time. Last edited by jsphkn, Aug 1, 2015, 1:01 am

Quick Reply

High School Olympiads

line passes through midpoint

Reply

andria

#1 Jul 31, 2015, 11:44 pm

Let $\mathcal{C}$ be the parabola that is tangent to the sides BC, CA, AB at A', B', C' of triangle ABC respectively. Let ℓ be the line throw A' which is parallel to the axis of $\mathcal{C}$ let $\ell \cap B'C' = S$ prove that AS passes throw the midpoint of BC.

TelvCohl

#3 Aug 1, 2015, 12:03 am

Let A_∞ be the infinity point on BC. Since $A'S$ passes through the center of $\mathcal{C}$, so $A'S$ is the polar of A_∞ WRT $\mathcal{C} \Longrightarrow AA_\infty$ is the polar of S WRT $\mathcal{C}$, hence from $A(A_\infty, S; B', C') = -1$ and $AA_\infty \parallel BC \Longrightarrow AS$ passes through the midpoint of BC.

Luis González

#4 Aug 1, 2015, 12:09 am • 1

More general: Any inconic $\mathcal{K}$ with center K touches BC, CA, AB at A', B', C'. If $A'K$ cuts $B'C'$ at S, then AS passes through the midpoint of BC.

The polar τ of S WRT $\mathcal{K}$ is the line through A with conjugate direction WRT the diameter $KA' \Longrightarrow \tau \parallel BC$. Hence $P \equiv \tau \cap B'C'$ is the pole of AS WRT $\mathcal{K} \Longrightarrow (C', B', P, S) = A(B, C, P, S) = -1$. Together with $AP \parallel BC$, it follows that AS passes through the midpoint of $\overline{BC}$.

Quick Reply

High School Olympiads

A DOUBT....

Reply

Source: Own

Viswanath

#1 Jul 31, 2015, 8:19 pm

I was brought to notice a statement "Any two quadrilaterals are projectively related....!"....which I found in one of Akopyan's book "geometry of conics"....

So in any problem if there is given a quadrilateral $ABCD$ can I consider it to be a square or any triangle to be a equilateral triangle.....

This sounds absurd I know.....but can anyone enhance the real meaning of that statement....

Luis González

#2 Jul 31, 2015, 9:01 pm

This is because there is always a projective transformation (homography) making an abritrary planar quadrilateral $ABCD$ a square. For example, letting $X \equiv AD \cap BC$ and $Y \equiv AB \cap CD,$ any projective mapping sending XY to infinity, transforms $ABCD$ into a parallelogram $A_1B_1C_1D_1$ and this can be transformed then into a square $A_2B_2C_2D_2$ through an affine mapping. Thus all quadrilaterals are said to be projectively related as they can be transformed into a same square. Same happens for triangles; any $\triangle ABC$ can be transformed into a equilateral $\triangle A^*B^*C^*$.

If a problem has a **projective** nature, then one can conveniently use projective transformations to boil it down to an easier configuration. This is certainly not valid, in general, in metric problems.

Viswanath

#3 Jul 31, 2015, 9:09 pm

Can you eleborate what is meant by "PROJECTIVE NATURE".........

Quick Reply

High School Olympiads

Another concurrent problem

Locked

Source: Mathematics and youth magazine

quangminhltv99

#1 Jul 31, 2015, 6:44 pm

Let ABC be a non-isoceles triangle. The incircle (I) touches BC, CA, AB at A_0, B_0, C_0 respectively; AI, BI, CI intersect BC, CA, AB at A_1, B_1, C_1 respectively; B_0C_0, C_0A_0, A_0B_0 meet B_1C_1, C_1A_1, A_1B_1 at A_2, B_2, C_2 respectively. Prove that the lines A_0A_2, B_0B_2, C_0C_2 concurrent at a point on the circumcircle (I)

Luis González

#2 Jul 31, 2015, 6:56 pm

Discussed before at http://www.artofproblemsolving.com/community/c6h553528.

High School Olympiads

Trilinear polar of Antigonal conjugate

Reply

Source: Own

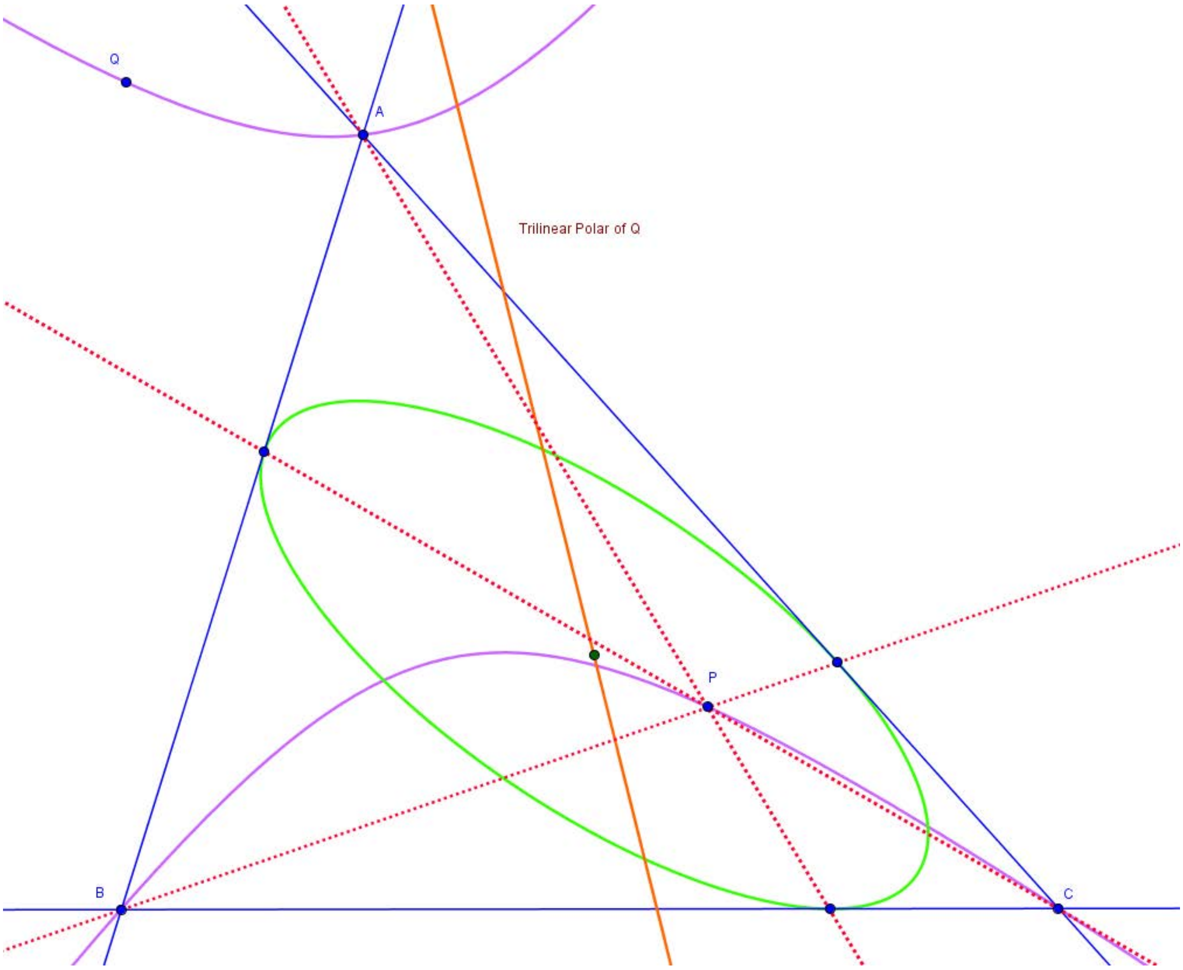
TelvCohl

#1 Jul 16, 2015, 12:08 am • 1

Let $\mathcal{C}$ be the inconic of $\triangle ABC$ with perspector P .
Let Q be the antigonal conjugate of P WRT $\triangle ABC$.

Prove that the center of $\mathcal{C}$ lie on the trilinear polar of Q WRT $\triangle ABC$

Attachments:



A-B-C

#2 Jul 18, 2015, 11:29 am

I don't have a synthetic proof for this problem. My solution uses barycentric coordinates. It is not so complicated, I think

$$A(1,0,0), B(0,1,0), C(0,0,1), P(\alpha,\beta,\gamma)$$

Then according to http://faculty.evansville.edu/ck6/encyclopedia/glossary.html, barycentric coordinates of Q is:

$$\frac{\alpha}{(b^2+c^2-a^2)\alpha(\alpha+\beta+\gamma)-(a^2\beta\gamma+b^2\gamma\alpha+c^2\alpha\beta)}$$

$$\frac{\beta}{(c^2+a^2-b^2)\beta(\alpha+\beta+\gamma)-(a^2\beta\gamma+b^2\gamma\alpha+c^2\alpha\beta)}$$

$$\frac{\gamma}{(a^2+b^2-c^2)\gamma(\alpha+\beta+\gamma)-(a^2\beta\gamma+b^2\gamma\alpha+c^2\alpha\beta)}$$

Let center of inconic of $\triangle ABC$ WRT P be I, then:

$$I(\frac{1}{\beta}+\frac{1}{\gamma},\frac{1}{\gamma}+\frac{1}{\alpha},\frac{1}{\alpha}+\frac{1}{\beta})$$

Note that equation of the inconic is:

$$x^2\alpha^2+y^2\beta^2+z^2\gamma^2-2yz\beta\gamma-2zx\gamma\alpha-2xy\alpha\beta=0$$

To show that I is center of $\mathcal{C}$, we prove that reflection of a point on $\mathcal{C}$ in I also lies on $\mathcal{C}$. This is easy to verify.
Trilinear polar ℓ of Q WRT $\triangle ABC$ has equation:

$$x\frac{(b^2+c^2-a^2)\alpha(\alpha+\beta+\gamma)-(a^2\beta\gamma+b^2\gamma\alpha+c^2\alpha\beta)}{\alpha}$$

$$+y\frac{(c^2+a^2-b^2)\beta(\alpha+\beta+\gamma)-(a^2\beta\gamma+b^2\gamma\alpha+c^2\alpha\beta)}{\beta}$$

$$+z\frac{(a^2+b^2-c^2)\gamma(\alpha+\beta+\gamma)-(a^2\beta\gamma+b^2\gamma\alpha+c^2\alpha\beta)}{\gamma}=0$$

To prove I lies on ℓ, we only need to show that:

$$(\beta+\gamma)[(b^2+c^2-a^2)\alpha(\alpha+\beta+\gamma)-(a^2\beta\gamma+b^2\gamma\alpha+c^2\alpha\beta)]$$

$$(\gamma+\alpha)[(c^2+a^2-b^2)\beta(\alpha+\beta+\gamma)-(a^2\beta\gamma+b^2\gamma\alpha+c^2\alpha\beta)]$$

$$(\alpha+\beta)[(a^2+b^2-c^2)\gamma(\alpha+\beta+\gamma)-(a^2\beta\gamma+b^2\gamma\alpha+c^2\alpha\beta)]=0$$

This equivalents to:

$$(\alpha+\beta+\gamma)[\alpha(\beta+\gamma)(b^2+c^2-a^2)+\beta(\gamma+\alpha)(c^2+a^2-b^2)+\gamma(\alpha+\beta)(a^2+b^2-c^2)]$$

$$=2(\alpha+\beta+\gamma)(a^2\beta\gamma+b^2\gamma\alpha+c^2\alpha\beta)$$

$$\Leftrightarrow \alpha(\beta+\gamma)(b^2+c^2-a^2)+\beta(\gamma+\alpha)(c^2+a^2-b^2)+\gamma(\alpha+\beta)(a^2+b^2-c^2)$$

$$=2(a^2\beta\gamma+b^2\gamma\alpha+c^2\alpha\beta)$$

The problem was solved.

This post has been edited 1 time. Last edited by A-B-C, Jul 18, 2015, 11:29 am
Reason: correct

Luis González

#3 Jul 31, 2015, 10:00 am • 5

Lemma: P is arbitrary point on the plane of $\triangle ABC$ and X, Y, Z are the reflections of P on the midpoints of BC, CA, AB. Then A, B, C, X, Y, Z and the antigonal conjugate of P WRT $\triangle ABC$ lie on a same conic.

Proof: Midpoints of BC, CA, AB, PA, PB, PC lie on a same conic; namely the 9-point conic of $ABCP$, which is the locus of the centers of all conics through A, B, C, P, particularly it contains the center of the rectangular hyberbola through A, B, C, P. Now, the homothety with center P and factor 2 yields the result. $\blacksquare$

Back to the problem, let K denote the center of $\mathcal{C}$: isotomcomplement of P WRT $\triangle ABC$. Considering an affine homology taking $\triangle ABC \cup P$ into an acute $\triangle ABC$ with orthocenter P, then K becomes its symmedian point and its known that the tripoles L of all lines through K lie on the circumcircle $\odot(ABC)$. Thus, back in the primitive figure, it follows that the tripoles L of all lines through K lie on the conic $\mathcal{K}$ through A, B, C and the reflections of P on the midpoints of BC, CA, AB. From previous lemma, this conic $\mathcal{K}$ goes through the antigonal conjugate Q of P, so it follows that the trilinear polar of Q WRT $\triangle ABC$ goes through K.

TelvCohl

#5 Jul 31, 2015, 2:14 pm

Thank you for your nice proof :) Here is my solution :

Let K be the isotomcomplement of P WRT $\triangle ABC$ and K^* be the isogonal conjugate of K WRT $\triangle ABC$. Let P_a, P_b, P_c be the reflection of P in the midpoint of BC, CA, AB, respectively. From the lemma mentioned by Luis in post #3 $\Longrightarrow A, B, C,$ P_a, P_b, P_c, Q lie on a conic $\mathcal{K}$. From Collinear Points we know $\mathcal{K}$ is the isogonal conjugate (WRT $\triangle ABC$) of the trilinear polar of K^* WRT $\triangle ABC$, so from $Q \in \mathcal{K} \Longrightarrow K$ (the center of $\mathcal{C}$) lie on the trilinear polar of Q WRT $\triangle ABC$.

P.S. My proof to the lemma :

Clearly, A, B, C, P_a, P_b, P_c lie on a conic $\mathcal{K}$ with center R where R is the common midpoint of AP_a, BP_b, CP_c (complement of P WRT $\triangle ABC$), so notice $Q \in \odot(P_aBC), \odot(P_bCA), \odot(P_cAB) \Longrightarrow Q$ is the 4th intersection of $\mathcal{K}$ and $\odot(P_aP_bP_c)$.

Quick Reply

High School Olympiads

A concurrent problem

Reply

Source: For v_Enhance

quangminhltv99

#1 Jul 30, 2015, 8:59 pm • 1

Let H denote the orthocenter of triangle ABC, and let M be the midpoint of BC. Point P is chosen on line HM. The circumcircle (K) whose diameter is AP intersects CA, AB at E and F respectively (which are differ from A). Prove that the tangent lines to the circumcircle (K) at E, F and the perpendicular bisector of BC are concurrent

Luis González

#2 Jul 30, 2015, 9:39 pm

Let X be the intersection of the tangents of (K) at $E,F.$ When P varies, the series E,F are obviously similar and since all the isosceles $\triangle XEF$ are directly similar (due to $\angle XEF=\angle XFE=\angle BAC=\text{const}$), it follows that the locus of X is a line $\ell_A.$ When $P \equiv H,$ then $X \equiv M$ (well-known) and when $P \equiv M,$ then X is on the perpendicular bisector of BC (see Another easy problem [midpoint M of BC projected], Equal segments and elsewhere). Thus we conclude that ℓ_A is the perpendicular bisector of $BC.$

A-B-C

#3 Jul 30, 2015, 9:49 pm

Let B', C' be orthogonal projections of B,C on CA,AB.
It is well-known that MB', MC' are tangent to $\odot(AH)$. Note that $\odot(XY)$is the circle that has diameter XY.
Hence the problem is equivalent to:
Pole of $BC, EF, B'C'$ WRT $\odot(AH), \odot(AP), \odot(ABC)$ are collinear.
Generalization:
$\odot(O_a), \odot(O_b), \odot(O_c)$ pass through two distinct points M,N.
Two line d_1, d_2 pass through M such thatthey are not pendiculer
d_1, d_2 intersects $\odot(O_a), \odot(O_b), \odot(O_c)$ at $A_1, A_2, B_1, B_2, C_1, C_2$.
Then poles of A_1A_2, B_1B_2, C_1C_2 WRT $\odot(O_a), \odot(O_b), \odot(O_c)$ are collinear.
Proof
Tangent lines at A_1, B_1, C_1 of $\odot(O_a), \odot(O_b), \odot(O_c)$ determine $\triangle X_1Y_1Z_1$
Tangent lines at A_2, B_2, C_2 of $\odot(O_a), \odot(O_b), \odot(O_c)$ determine $\triangle X_2Y_2Z_2$
Then we only have to prove that $\triangle X_1Y_1Z_1$ and $\triangle X_2Y_2Z_2$ are perpective.

$$(X_1B_1, X_1C_1) = (X_1B_1, MB_1) + (MC_1, X_1C_1) = (NB_1, NM) + (NM, NC_1) = (NB_1, NC_1)$$

So $\odot(X_1B_1C_1)$ passes through N. Similarly, $\odot(Y_1C_1A_1)$, $\odot(Z_1A_1B_1)$, $\odot(X_2B_2C_2)$, $\odot(Y_2C_2A_2)$, $\odot(Z_2A_2B_2)$ pass through N.
N is a spiral similarity center , then $\triangle NX_1Y_1 \sim \triangle NC_1A_1$. It is well-known that $\triangle NC_1A_1 \sim \triangle NO_aO_b$. Similarly, $\triangle NX_2Y_2 \sim \triangle NO_aO_b$ directly (*)
Then according to Miquel's theorem, $\odot(X_1Y_1Z_1), \odot(X_2Y_2Z_2)$ pass through N.
Let T be the intersection other than N of $\odot(X_1Y_1Z_1), \odot(X_2Y_2Z_2)$

$$(TX_1, TX_2) = (TX_1, TN) + (TN, TX_2) = (Y_1X_1, Y_1N) + (Y_2N, Y_2X_2) = 0(\text{from } (*))$$

Then X_1X_2 passes through T. Y_1Y_2, Z_1Z_2 pass through T, similarly.
$\Rightarrow X_1X_2, Y_1Y_2, Z_1Z_2$ are concurrent.
$\Longrightarrow \triangle X_1Y_1Z_1, \triangle X_2Y_2Z_2$ are perspective.

In the original post, the coaxial circles are $\odot(ABC), \odot(AP), \odot(AH)$ and d_1, d_2 are AB, AC.

This post has been edited 1 time. Last edited by A-B-C, Jul 30, 2015, 9:50 pm
Reason: typo

THVSH

#4 Jul 30, 2015, 9:50 pm

This problem was proposed in **Mathematics and Youth Magazine** by **Mr Tran Quang Hung (buratinogigle)**

My solution:
Let (O) be the circumcircle of $\triangle ABC$. $\odot(K)\cap\odot(O)=\{N,A\}$. The tangents at B,C of (O) intersect at I. The tangents at E,F of $\odot(K)$ intersect at G. Construct the diameter AA' of $\odot(O)$. Then $\angle ANA'=\angle ANP=90^\circ$. So $P\in NA'$. On the other hand, $BHCA'$ is a parallelogram so H,M,A' are collinear. Thus, N,H,P,M,A' are collinear.
We have $\triangle NFB\sim\triangle NEC\Longrightarrow\frac{FN}{NB}=\frac{EN}{NC}$. But
$\angle BNC=\angle BNF+\angle FNC=\angle CNE+\angle FNC=\angle FNE$
$\Longrightarrow\triangle NFE\cup G\sim\triangle NBC\cup I$ (since G,I are the intersections of the tangents at E,F and B,C of $\odot(K)$ and $\odot(O)$, respectively) $\Longrightarrow\angle NIG=\angle NBF$. On the other hand, we have $ON^2=OB^2=OM.OI\Longrightarrow\angle NIM=\angle ONM$ $=\angle NA'A=\angle NBA=\angle NBF=\angle NIG\Longrightarrow I,M,G$ are collinear. Therefore, G lies on the perpendicular bisector of BC Q.E.D

This post has been edited 1 time. Last edited by THVSH, Jul 30, 2015, 9:59 pm

TelvCohl

#5 Jul 30, 2015, 10:13 pm • 1

> *A-B-C wrote:*
> **Generalization:**
> $\odot(O_a),\odot(O_b),\odot(O_c)$ pass through two distinct points M,N.
> Two line d_1,d_2 pass through M such that they are not pendiculer
> d_1,d_2 intersects $\odot(O_a),\odot(O_b),\odot(O_c)$ at A_1,A_2,B_1,B_2,C_1,C_2.
> Then poles of A_1A_2,B_1B_2,C_1C_2 WRT $\odot(O_a),\odot(O_b),\odot(O_c)$ are collinear.

Let A be the pole of A_1A_2 WRT $\odot(O_a)$ (define B,C similarly) .

Since N is the center of the spiral similarity of $A_1A_2\mapsto B_1B_2\mapsto C_1C_2$,
so from $\triangle AA_1A_2\sim\triangle BB_1B_2\sim\triangle CC_1C_2$ we get A,B,C are collinear .

Quick Reply

High School Olympiads

Another easy problem [midpoint M of BC projected]

Reply

Source: some Kolmogorov cup (which one?)

darij grinberg

#1 Sep 16, 2004, 5:59 pm

Let M be the midpoint of the side BC of a triangle ABC, and let D and E be the orthogonal projections of this point M on the sides AB and CA. The points D and E clearly lie on the circle with diameter AM. Now, let the tangents to this circle at the points D and E meet each other at a point P. Prove that this point P lies on the perpendicular bisector of the segment BC.

Darij

grobber

#2 Sep 16, 2004, 10:39 pm

Let F be the intersection of PM with the circle (ADE). FM is the F-symmedian in FED (which, by the way, is similar to ABC). This means that $(\widehat{FM,BC})=(\widehat{AK,DE})=\frac{\pi}{2}$, where K is the Lemoine point of ABC, and the last equality holds because M lies on the isogonal cevian of AK.

darij grinberg

#3 Sep 16, 2004, 11:37 pm

> grobber wrote:
> FM is the F-symmedian in FED (which, by the way, is similar to ABC). This means that $(\widehat{FM,BC})=(\widehat{AK,DE})=\frac{\pi}{2}$,

Sorry, but I really don't understand why $(\widehat{FM,BC})=(\widehat{AK,DE})$. Could you explain please?

I have two solutions of the problem: a simple trigonometric and a nice synthetic one.

Darij

grobber

#4 Sep 16, 2004, 11:55 pm

That's because we have some similar figures over there, and $(\widehat{FM,DE})=(\widehat{AK,BC})$, and we can form a cyclic quadrilateral with these four lines (it seems Ok right now, but it could be wrong :)).

darij grinberg

#5 Sep 17, 2004, 2:53 am

Grobber, I apologize that the below sounds rather harsh, but I really don't understand your argument. I fear that what you have posted is not your solution, and your solution is not what you have posted.

You begin with writing "That's because we have some similar figures over there". Actually, could you please be more exact about which figures are similar? I know, you probably refer to your statement before that the triangles FED and ABC are similar (actually indirectly similar). But when you made that statement, you didn't prove it (you just wrote "in FED (which, by the way, is similar to ABC)"). Actually, I don't think that this statement would be so immediate to prove. (In any case, don't you see that if you can prove this statement, then the problem becomes almost trivial?)

you can prove this statement, then the problem becomes almost trivial!)

BTW, I think that the orientation of your angles is incorrect: Since the triangles FED and ABC are indirectly similar, we have < (FM; DE) = - < (AK; BC), and this, after some easy transformations, yields < (FM; BC) = - < (AK; DE). From here on, everything works as you wrote. But it remains to show that triangles FED and ABC are indirectly similar...

Darij

grobber

#6 Sep 17, 2004, 2:58 am

Sorry for this.. That is my solution, and the only reason why I'm being so vague is that I have erased my drawing (a dynamic sketch), and I didn't want to draw another one all over again :). I'll see what I can do, Ok? :)

grobber

#7 Sep 17, 2004, 3:07 am

It is obvious that they are similar, it's just a bit boring to write it out in detail:

Since $S_{ABM}=S_{ACM}$ we have $\frac{MD}{ME}=\frac{AC}{AB}$ $(*)$. On the other hand, from the similarity of PDM, PFD and PEM, PFE we get $\frac{FD}{MD}=\frac{PD}{PM}$, $\frac{FE}{ME}=\frac{PE}{PM}$, and since $PD=PE$, we have $\frac{FD}{FE}=\frac{MD}{ME}=\frac{AC}{AB}$ $(\#)$ (the last one follows from $(*)$). Now $\angle BAC=\angle EFD$, and this, together with $(\#)$ gives the inverse similarity.

This post has been edited 1 time. Last edited by grobber, Sep 17, 2004, 8:21 am

pestich

#8 Sep 17, 2004, 8:17 am

Do we really need the triangle similarity?

Simple angle chase will do the job. Angles B and C are outside the circle
and are measured by semi-difference of their arcs.

Pestich.

darij grinberg

#9 Sep 17, 2004, 8:22 pm

To Pestich:

> *pestich wrote:*
> Do we really need the triangle similarity?
>
> Simple angle chase will do the job.

I don't think so. How do you want to involve in the angle chasing the fact that M is the midpoint of the segment BC ? On the other hand, if you don't use this fact, you can't prove anything...

Sorry if I misunderstood you.

To Grobber:

> *grobber wrote:*
> It is obvious that they are similar, it's just a bit boring to write it out in detail:

Thanks a lot. Well, it is not obvious, but anyway it is rather easy (even I would have found it if you explicitely had said that you prove this using FD / FE = AC / AB; I was trying to prove it in other ways all the time).

Now, now as you know that the triangles FED and ABC are indirectly similar, it is easy to complete your solution: We have < FED = - < ABC. Now, < FMD = < FED (since the points D, E, F and M lie on one circle); thus, < FMD = - < ABC. Now,

< (MP; BC) = < (MP; MD) + < (MD; AB) + < (AB; BC)
= < FMD + < (MD; AB) + < ABC
= - < ABC + < (MD; AB) + < ABC
= < (MD; AB) = 90

(since $MD \perp AB$). Thus, $MP \perp BC$. And this means that the point P lies on the perpendicular bisector of the segment BC.

Thank you for this proof. It looks as a kind of middle between the two proofs I have.

Here is the *first proof* - alas, it is damn situation dependent:

Let U be some point on the perpendicular bisector of the segment BC. We want to prove that the point P lies on the perpendicular bisector of the segment BC; to this end, it will suffice to show that the point P lies on the line MU. In other words, it is enough to prove that the lines DP, EP and MU concur at one point.

Since the tangents from a point to a circle are equal in length, we have PD = PE, so that the triangle DPE is isosceles. Thus, < PED = < EDP. Also, we have < DMU = < ABC and < UME = < BCA since the edges of these angles are pairwisely perpendicular. We also have < MEP = < MAE and < PDM = < MAD as secant-tangent angles. Thus, with help of the Sine Law,

$$\frac{\sin\measuredangle DMU}{\sin\measuredangle UME}\cdot\frac{\sin\measuredangle MEP}{\sin\measuredangle PED}\cdot\frac{\sin\measuredangle EDP}{\sin\measuredangle PDM}$$
$$=\frac{\sin\measuredangle ABC}{\sin\measuredangle BCA}\cdot\frac{\sin\measuredangle MAE}{\sin\measuredangle EDP}\cdot\frac{\sin\measuredangle EDP}{\sin\measuredangle MAD}$$
$$=\frac{\sin\measuredangle ABC}{\sin\measuredangle BCA}\cdot\frac{\sin\measuredangle MAE}{\sin\measuredangle MAD}=\frac{\sin\measuredangle ABM}{\sin\measuredangle MCA}\cdot\frac{\sin\measuredangle MAC}{\sin\measuredangle MAB}$$
$$=\frac{\sin\measuredangle ABM}{\sin\measuredangle MAB}\cdot\frac{\sin\measuredangle MAC}{\sin\measuredangle MCA}=\frac{AM}{BM}\cdot\frac{CM}{AM}=\frac{CM}{BM}=1.$$

Hence, by the trigonometric version of the Ceva theorem, it follows that the lines DP, EP and MU concur. Thus, the problem is solved.

The *second proof* is situation independent, but it uses some theoretical preknowledge:

We will work with directed angles modulo 180.

Since the point P is a vertex of the tangential triangle of triangle MDE, the line MP is a symmedian of triangle MDE. If we call L the midpoint of the segment DE, then the line MD is a median of triangle MDE. Since a median and the corresponding symmedian of a triangle are symmetric to each other with respect to the corresponding angle bisector, we have < PME = < DML.

Let K be the symmedian point of triangle ABC, and let X, Y, Z be the orthogonal projections of this symmedian point K on the sides BC, CA, AB. It is well-known that the point K is then the centroid of triangle XYZ. Thus, the line XK passes through the midpoint X' of the side YZ of this triangle.

Now, since < AYK = 90 and < AZK = 90, the points Y and Z lie on the circle with diameter AK. Hence, < KYZ = < KAZ. In other words, < KYZ = < KAB. Similarly, < MDE = < MAC. But the line AM is a median in triangle ABC, while the line AK is a symmedian; the median and the symmedian are symmetric to each other with respect to the respective angle bisector, and hence we have < KAB = - < MAC. This, together with < KYZ = < KAB and < MDE = < MAC, gives < KYZ = - < MDE. Similarly, < KZY = - < MED. Thus, the triangles KYZ and MDE are inversely similar. Now, the point X' is the midpoint of the side YZ of the first triangle, while the point L is the midpoint of the side DE of the second triangle. Thus, the points X' and L are corresponding points in the triangles KYZ and MDE. Since corresponding points in indirectly similar triangles form oppositely equal angles, we thus have < DML = - < YKX'. In other words, < DML = - < (KY; KX) = - < (CA; BC), what is because the lines KY and KX are perpendicular to the lines CA and BC, respectively.

Since < PME = < DML, we thus have < PME = - < (CA; BC). Hence,

< (MP; BC) = < (MP; ME) + < (ME; CA) + < (CA; BC)
= < PME + 90 + < (CA; BC)
= - < (CA; BC) + 90 + < (CA; BC) = 90.

Thus, $MP \perp BC$. In other words, the point P lies on the perpendicular bisector of the segment BC. And we are done again.

So now we have three different solutions... something I really couldn't expect from this little problem!

Darij

Sailor

#10 Sep 18, 2004, 9:02 pm

#10 Sep 18, 2004, 9:02 pm

So what about the angle chasing, dear pestich? I tried to find somenthing like this but unfortunately.....

pestich

#11 Sep 19, 2004, 9:30 am

I'm still chasing those unruly angles.

Pestich.

sprmnt21

#12 Sep 20, 2004, 2:20 pm

> *darij grinberg wrote:*
>
> So now we have three different solutions... something I really couldn't expect from this little problem!
>
> Darij

here it is a fourth way. Only a sketch and a joke .

let's "complete" the traingle AKL of which M is the orthocenter, where K=AC^DM and L=AB^ME. The circle through E, D and P is the nine-point-circle of AKL. P is of course the mid-point of KL and at the same time the center of c(EDKL).
Now we can use the well-known(?) fact of the butterfly theorem applying it to the chord through C,M,B and conclude that PM is the perpendicular bisector of CB.

darij grinberg

#13 Sep 20, 2004, 4:29 pm

Wow. That was cool, Rocco. I probably should not only collect applications of the Pascal, Brianchon, Desargues and Pappos theorems, but now also of the Butterfly theorem

Darij

sprmnt21

#14 Sep 20, 2004, 7:33 pm

> *darij grinberg wrote:*
>
> I probably should not only collect applications of the Pascal, Brianchon, Desargues and Pappos theorems, but now also of the Butterfly theorem
>
> Darij

very interesting theorem indeed.

Some time ago I posted to Alexander Bogomolny a very short projective proof of this theorem, I was very proud of that results, but Alexander "kindly" answered me that the proof I posted was already known.

Now I would like to provide a specific (hopefully non already known, at least by Alexander) proof for our particular case. Or to be more precise I will prove a little bit more general case of which our question is a corollary.

Let AB be the diameter of a circle c with center O and let C a point on the plane of c. Let E=c^BC and D=c^AC, if M and N are the intersections of the orthogonal line through C to the line OC with AE and BD respectively, then CM = CN.

Proof

If H is the orthocenter of CBN, then AOC and ABH are similar and therefore AC=CH. Since AMC and HNC, as AM/HN, are similar then CM = CN.

darij grinberg

#15 Sep 20, 2004, 10:02 pm

> *sprmnt21 wrote:*
> Let AB be the diameter of a circle c with center O and let C a point on the plane of c. Let E=c^BC and D=c^AC, if M and N are the intersections of the orthogonal line through C to the line OC with AE and BD respectively, then CM = CN.
>
> Proof
>
> If H is the orthocenter of CBN, then AOC and ABH are similar and therefore AC=CH. Since AMC and HNC, as AM/HN, are similar then CM = CN.

Nice proof! Here is my proof (which I found long ago):

Since the points D and E lie on the circle with diameter AB, we have < ADB = 90 and < AEB = 90.

Let U and V be the orthogonal projections of the points A and B on the line MN. Then, the lines AU, BV and OC are parallel to each other, since they are all perpendicular to the line UV. Hence, the quadrilateral AUVB is a trapezoid, and since the line OC passes through the midpoint O of its leg AB and is parallel to its bases AU and BV, this line OC is a midparallel of the trapezoid. In other words, the point C is the midpoint of the leg UV, and it follows that CU = CV.

Now, since the triangles AUC and NDC are similar (as < ACU = < NCD and < AUC = < NDC = 90), we have CA : CU = CN : CD. Thus, $CA \cdot CD = CU \cdot CN$. Similarly, $CB \cdot CE = CV \cdot CM$. But $CA \cdot CD = CB \cdot CE$ by the intersecting chords theorem. Thus, $CU \cdot CN = CV \cdot CM$. Since CU = CV, we can conclude that CM = CN, and we are done.

Darij

hucht

#16 Feb 27, 2006, 6:29 pm

Could anybody explain me what about direct and inverse simillarity? :(I haven't found any paper about this :(

thanks

José Carlos

Virgil Nicula

#17 Feb 28, 2006, 7:31 pm

Here is a short metrical proof.

Quick Reply

High School Olympiads

Equal Segments

Reply

Source: Kolmogorov cup - 2003

stergiu

#1 Nov 24, 2009, 2:18 pm

Let D the midpoint of side BC in a triangle ABC.Let E, Z be the projections of D on AB, AC resp/ly and T the intersection of the tangents at points E, Z to the circle with diameter AD . Prove that $TB = TC$.

Babis

shoki

#2 Nov 24, 2009, 6:01 pm

hint or ...

Luis González

#3 Nov 24, 2009, 9:27 pm

Let M, N be the midpoints of AC, AB and $K \equiv MN \cap EF$. In the parallelogram $AMDN$, we have $\angle NDE = \angle MDF$, due to $\angle BND = \angle CMD$. Let L be the projection of D on MN. From the quadrilaterals $LDEN$ and $LDFM$ inscribed in the circles with diameters ND, MD, we have $\angle NLE = \angle NDE = \angle MDF = \angle MLF \Longrightarrow LD, MN$ bisects $\angle ELF$ internally and externally. If $J \equiv LD \cap EF$, then cross ratio (E, F, J, K) is harmonic $\Longrightarrow K$ is the pole of the perpendicular bisector LD of BC WRT the circle ω with diameter AD. Since T is the pole of EF WRT $\omega \Longrightarrow T \in LD$, i.e. $TB = TC$, as desired.

Quick Reply

High School Olympiads

Nice geometry 3

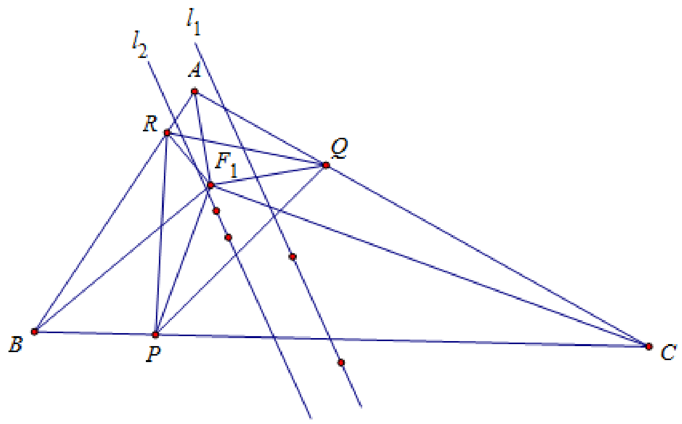
Reply

Source: Own

LeVietAn
#1 Jun 1, 2015, 10:14 am • 4

Dear Mathelinkers,
Let ABC is a triangle.
Let F_1 is 1st Fermat point.
Let $P \in BC, Q \in CA, R \in AB$ such that $\angle PF_1C = \angle QF_1A = \angle RF_1B = 90^0$.
Let l_1, l_2 are the Euler's line of the triangles ABC, PQR, reps.
Prove that $l_1 \parallel l_2 \vee l_1 \equiv l_2$.

Attachments:



TelvCohl
#3 Jun 1, 2015, 10:28 pm • 1

This nice problem can be generalized as following :

Let F_1 be the 1st Fermat point of $\triangle ABC$ $(\max\{\angle A, \angle B, \angle C\} < 120^\circ)$.
Let $P \in BC, Q \in CA, R \in AB$ be the points such that $\measuredangle AF_1P = \measuredangle BF_1Q = \measuredangle CF_1R$.

Then the Euler line of $\triangle PQR$ is parallel to the Euler line of $\triangle ABC$.

LeVietAn
#4 Jun 1, 2015, 10:52 pm

Dear TelvCohl, I think the locus of the 1st isodynamic point of $\triangle PQR$ is a line F_1J with J be the 1st isodynamic point of $\triangle ABC$. And just to prove this, the general problem will be solved. But i can't proof.

TelvCohl

#6 Jun 17, 2015, 9:17 am • 4

Problem :

Let F_1 be the 1st Fermat point of $\triangle ABC$ $(\max\{\angle A, \angle B, \angle C\} < 120^\circ)$.
Let $P \in BC, Q \in CA, R \in AB$ be the points such that $\measuredangle AF_1P = \measuredangle BF_1Q = \measuredangle CF_1R$.

Prove that the locus of the 1st Isodynamic point S of $\triangle PQR$ is a line through F_1

Proof :

Let X, Y, Z be the reflection of F_1 in QR, RP, PQ, respectively (S is the center of $\odot(XYZ)$) .
Invert the figure with center F_1 (arbitrary power) and denote T^* as the image of T under the inversion .

Now we get the new problem as following

New problem :

Let F_1 be the I st Fermat point of $\triangle A^*B^*C^*$ $(\max\{\angle B^*A^*C^*, \angle C^*B^*A^*, \angle A^*C^*B^*\} < 120^\circ)$.
Let $P^* \in \odot(B^*F_1C^*), Q^* \in \odot(C^*F_1A^*), R^* \in \odot(A^*F_1B^*)$ s.t. $\measuredangle Q^*F_1R^* = \measuredangle R^*F_1P^* = \measuredangle P^*F_1Q^* = 120^\circ$.
Let X^*, Y^*, Z^* be the center of $\odot(Q^*F_1R^*), \odot(R^*F_1P^*), \odot(P^*F_1Q^*)$, resp and S^* the image of F_1 under $\mathbf{I}_{\odot(X^*Y^*Z^*)}$.

Prove that the locus of S^* is a line through F_1

Proof of the new problem :

Let U, V, W be the center of $\odot(B^*F_1C^*), \odot(C^*F_1A^*), \odot(A^*F_1B^*)$, respectively .

Since Y^*Z^*, Z^*X^*, X^*Y^* is the perpendicular bisector of F_1P^*, F_1Q^*, F_1R^*, respectively ,
so we get $U \in Y^*Z^*, V \in Z^*X^*, W \in X^*Y^*$ and $\triangle X^*Y^*Z^*$ is an equilateral triangle .

Since $\triangle UVW$ is equilateral triangle and its center coincide with the center of $\triangle X^*Y^*Z^*$,
so the locus of S^* is a line connecting the center of $\triangle UVW$ (fixed (Centroid of $\triangle A^*B^*C^*$)) and F_1 .

Q.E.D

Luis González

#7 Jul 30, 2015, 10:18 am • 3

> *TelvCohl wrote:*
>
> **Problem:**
>
> Let F_1 be the 1st Fermat point of $\triangle ABC$ $(\max\{\angle A, \angle B, \angle C\} < 120^\circ)$.
> Let $P \in BC, Q \in CA, R \in AB$ be the points such that $\measuredangle AF_1P = \measuredangle BF_1Q = \measuredangle CF_1R.$
>
> Prove that the locus of the 1st Isodynamic point S of $\triangle PQR$ is a line through F_1

We extend this result as follows: P, Q, R are three points on the sides BC, CA, AB of $\triangle ABC,$ such that $\triangle ABC$ and $\triangle PQR$ have the same first Fermat point $F_1.$ Then the trilinear polar τ of F_1 WRT $\triangle ABC$ coincides with the trilinear polar of F_1 WRT $\triangle PQR.$ As a result, the 1st Isodymanic point S of $\triangle PQR$ is on the perpendicular from F_1 to $\tau.$

Proof: Let τ cut BC, CA, AB at $X, Y, Z,$ respectively. Since $\angle(F_1B, F_1C) = \angle(F_1C, F_1A) = \angle(F_1P, F_1Q) = 60^\circ$ and $\angle(F_1X, F_1Y) = 60^\circ$ (due to $F_1X \perp F_1A$ and $F_1Y \perp F_1B$), it follows that $(P, B, C, X) = (Q, C, A, Y)$ and similarly $(Q, C, A, Y) = (R, A, B, Z) \Longrightarrow (P, B, C, X) = (Q, C, A, Y) = (R, A, B, Z).$ Thus considering a homography taking $\triangle ABC$ into an equilateral triangle with center $F_1,$ we'll get $PB : PC = QC : QA = RA : RB \Longrightarrow$ $\triangle PQR$ is also equilateral with center $F_1 \Longrightarrow$ trilinear polar of F_1 WRT $\triangle ABC$ (line at infinity) is also trilinear polar of F_1 WRT $\triangle PQR.$ Back to the primitive figure we get the wanted result.

Quick Reply

High School Olympiads

Perky circles

Reply

Source: Own

Bandera

#1 Jul 30, 2015, 3:49 am

Given a triangle ABC having $AC \neq BC$. Its incircle touches the sides BC, CA, AB at D, E, F respectively. AD and BE meet at G. The incircle intersects CF at H, besides F. Denote the perpendicular bisector of CG by p and the line ED by q. Let I be the point where p meets the perpendicular from F to q. The circle with center at I that passes through C intersects q at K and L. If FK and HL meet at M, FL and HK meet at N, and MN is r, prove that p, q and r are concurrent.

This post has been edited 1 time. Last edited by Bandera, Jul 30, 2015, 3:50 am

Luis González

#2 Jul 30, 2015, 7:41 am

Let $U \equiv CF \cap DE$. Since $(F, U, G, C) = -1$ and $\overline{DUE} \perp FI$, it follows that DE is the polar of F WRT $\odot(CKL) \Longrightarrow FK, FL$ are tangents of $\odot(CKL)$. Moreover since $(F, H, U, C) = -1$, then we deduce that $\triangle FMN$ is the tangential triangle of $\triangle CKL$. Thus MN, KL and p concur at the pole of CF WRT $\odot(CKL)$.

Bandera

#3 Jul 31, 2015, 11:06 am

Below is an additional problem based on the same construction.

If p, q and r concur at S, V is the midpoint of CG, W is the midpoint of VS, and T is the foot of an altitude from K to CG, prove that W, T and L are collinear.

Quick Reply

High School Olympiads

Three collinear point

Reply

Source: own

jayme

#1 Jul 14, 2015, 3:49 pm

Dear Mathlinkers,

1. ABC a triangle
2. P a point
3. P* the isogonal of P wrt ABC
4. (O) the circumcircle of ABC
5. A' the second point of intersection of AP with the circumcircle of PBC
6. A'* the point of intersection of the parallel to PP* through A' with AP*
7. Y, Z the points of intersection of the perpendicular bisectors of P*A and P*A', PA and PA'*.

Prove : Y, O and Z are collinear.

Sincerely
Jean-Louis

Luis González

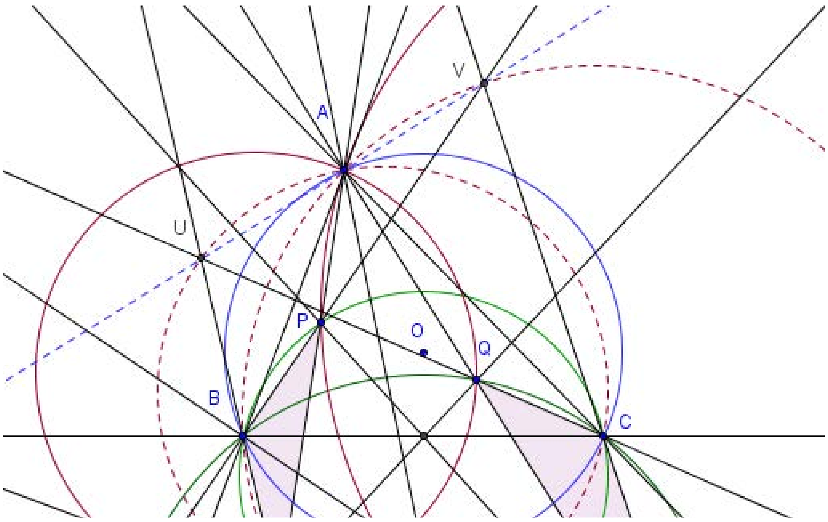
#2 Jul 15, 2015, 7:29 am

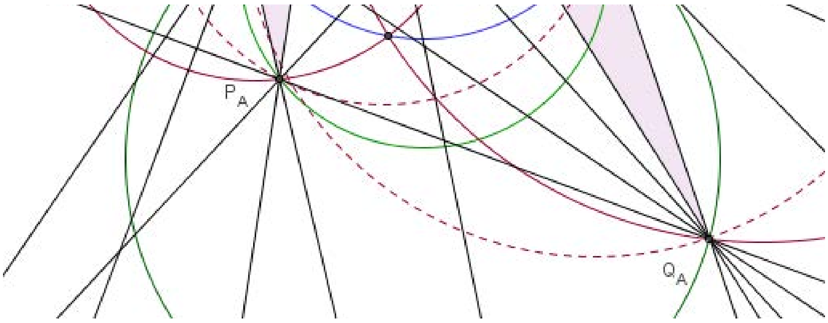
The problem can be rephrased as follows: P, Q are isogonal conjugates WRT $\triangle ABC$. AP and AQ cut $\odot(PBC)$ and $\odot(QBC)$ again at P_A, Q_A. Then $PQ \parallel P_AQ_A$ and the circumcenters of $\triangle ABC, \triangle APQ_A, \triangle AQP_A$ are collinear.

Denote $\mathbf{I}_A$ the composition of the inversion with center A and power $AB \cdot AC$ followed by symmetry on the bisector of $\widehat{BAC}$. As this swaps B and C, then BC goes to $(O) \equiv \odot(ABC)$ and since $\widehat{BPC} + \widehat{BQC} = 180^\circ + \widehat{BAC}$, we deduce that $\mathbf{I}_A : \odot(PBC) \mapsto \odot(QBC) \Longrightarrow \mathbf{I}_A : P \mapsto Q_A, Q \mapsto P_A \Longrightarrow AP \cdot AQ_A = AQ \cdot AP_A$, or $AP : AP_A = AQ : AQ_A \Longrightarrow PQ \parallel P_AQ_A$.

Let $U \equiv CQ \cap BP_A$ and $V \equiv BP \cap CQ_A$. Since $\widehat{ACQ} = \widehat{AP_AB}$, then ACP_AU is cyclic and similarly ABQ_AV is cyclic $\Longrightarrow \widehat{PAU} = \widehat{QCP_A} = \widehat{QCB} + \widehat{BPP_A} = \widehat{QCB} + \widehat{BAP} + \widehat{PBA} \Longrightarrow$ $\widehat{BAU} = \widehat{PAU} - \widehat{BAP} = \widehat{QCB} + \widehat{QBC} = \widehat{BQ_AC} = 180^\circ - \widehat{BAV} \Longrightarrow A, U, V$ are collinear $\Longrightarrow \triangle BPP_A$ and $\triangle CQ_AQ$ are perspective through UAV. Hence, by Desargues theorem PQ_A, QP_A, BC concur $\Longrightarrow$ their images $\odot(APQ_A), \odot(AQP_A), (O)$ under $\mathbf{I}_A$ are coaxal (intersecting at a second point) $\Longrightarrow$ their centers are collinear.

Attachments:





TelvCohl

#3 Jul 15, 2015, 7:32 am

My solution **:**

Let $Q \in \odot(BPC)$ s.t. $PQ \perp AP$ and $Q' \in \odot(BP'C)$ s.t. $P'Q' \perp AP'$.

Since $\measuredangle CBQ + \measuredangle CBQ' = \measuredangle CPA - 90^\circ + \measuredangle CP'A - 90^\circ = \measuredangle CBA$,
so $\measuredangle CBQ = \measuredangle Q'BA \Longrightarrow BQ$ and BQ' are isogonal conjugate WRT $\angle CBA$.
Similarly, CQ, CQ' are isogonal conjugate WRT $\angle ACB \Longrightarrow Q, Q'$ are isogonal conjugate WRT $\triangle ABC$.

Let Ψ be the composition of inversion $\mathbf{I}(A, \sqrt{AB \cdot AC})$ and reflection $\mathbf{R}(\ell_a)$ where ℓ_a is the bisector of $\angle BAC$.

From Circumcircles of BOC and BHC (post #12) $\Longrightarrow PA'^* \cap P'A' \in BC$ and $(P, P') \longleftrightarrow (A'^*, A')$ under Ψ ,
so we get $\odot(ABC), \odot(APA'^*), \odot(AP'A')$ are coaxial ($\because BC \longleftrightarrow \odot(ABC)$ under Ψ) $\Longrightarrow Y, O, Z$ are collinear .

Q.E.D

A-B-C

#4 Jul 15, 2015, 5:02 pm

Dear J.L.Ayme
Your problem was a lemma that I had found, proved, and used, to solved my generalization of a theorem here https://vn-mg61.mail.yahoo.com/neo/launch?.rand=ekfe12pddaffe#6365173662
The proof by Luis is exactly the same as my proof.
I think you shouldn't post the properties around this configuration, for my secret. Because I am writing an article about this, as I have told you here: http://www.artofproblemsolving.com/community/c6h1101842
Sincerely,
Ngo Quang Duong

Quick Reply

High School Olympiads

midpoints; generaization?

powerpi

#1 Jul 15, 2015, 3:00 am

Let $ABCD$ be a quadrilateral with P intersection of diagonals AC and BD. Let l be a line through P such that if $l\cap AD=M$ and $l\cap BC=N$, then P is midpoint of MN. Prove that if $l\cap AB=E$ and $l\cap CD=F$, then P is midpoint of EF.

Is there any generalization for example for equal ratios instead of midpoints?

Luis González

#2 Jul 15, 2015, 3:51 am

By Desargues involution theorem, l cuts the opposite sides of the quadrangle $ABCD$ at pairs of points in involution. Thus P is double in the involution defined by the pairs $\{M,N\}$ and $\{E,F\} \Longrightarrow (E,M,N,P)=(F,N,M,P) \Longrightarrow$ $\frac{EM}{EN}\cdot\frac{FM}{FN}=\frac{PM^2}{PN^2}$. Thus from this relation, it follows that P is the midpoint of $\overline{MN} \Longleftrightarrow P$ is the midpoint of $\overline{EF}$.

P.S. A solution with Menelaus' theorem is also possible. See the topic butterfly like in triangle for another formulation of the problem.

High School Olympiads

butterfly like in triangle

Reply

TripteshBiswas
#1 Dec 25, 2014, 1:07 am

In $\triangle ABC$, points D, K lie on side BC and AB respectively, point P lies on segment AD. Line DK intersects segments BP, AC, PC at E, F and L respectively. Show that if $DE = DF$, then $DK = DL$.

Arab
#2 Dec 25, 2014, 2:18 am • 2

Applying **Menelaus' Theorem** to $(\triangle ADF, \overline{PCL})$ and $(\triangle ADK, \overline{BEP})$ respectively, we obtain that

$\frac{AP}{PD}\cdot\frac{DL}{LF}\cdot\frac{FC}{CA}=1, \frac{DP}{PA}\cdot\frac{AB}{BK}\cdot\frac{KE}{ED}=1$, and hence $\frac{FC}{CA}\cdot\frac{AB}{BK}\cdot\frac{KE}{ED}\cdot\frac{DL}{LF}=1$.

Meanwhile, applying **Menelaus' Theorem** to $(\triangle AFK, \overline{BDC})$, we have $\frac{FC}{CA}\cdot\frac{AB}{BK}\cdot\frac{KD}{DF}=1$, and therefore

$\frac{KE}{ED}\cdot\frac{DL}{LF}=\frac{KD}{DF}\implies\frac{KD}{KE}=\frac{DL}{LF}\implies 1+\frac{DE}{KE}=1+\frac{DF}{LF}\implies KE=LF\implies DK=DL$, as desired.
$\blacksquare$

Luis González
#3 Dec 25, 2014, 3:14 am • 3

By Desargues involution theorem, the opposite sidelines of complete quadrilateral $PBAC$ form an involution on the line EF, namely $\{K, E, D\} \mapsto \{L, F, D\}$. Therefore

$(D, E, F, K) = (D, F, E, L) \Longrightarrow \frac{\overline{DE}}{\overline{DF}}\cdot\frac{\overline{KF}}{\overline{KE}}=\frac{\overline{DF}}{\overline{DE}}\cdot\frac{\overline{LE}}{\overline{LF}} \Longrightarrow \frac{\overline{KF}}{\overline{KE}}=\frac{\overline{LE}}{\overline{LF}} \Longrightarrow D$ is also midpoint of $\overline{KL}$.

TelvCohl
#4 Dec 25, 2014, 4:37 pm • 2

See Equality implies equality :)
(The source in the link is wrong, this problem is from 2004 China south east mathematical olympiad) :)

Quick Reply

High School Olympiads

GEOMETRY

Reply

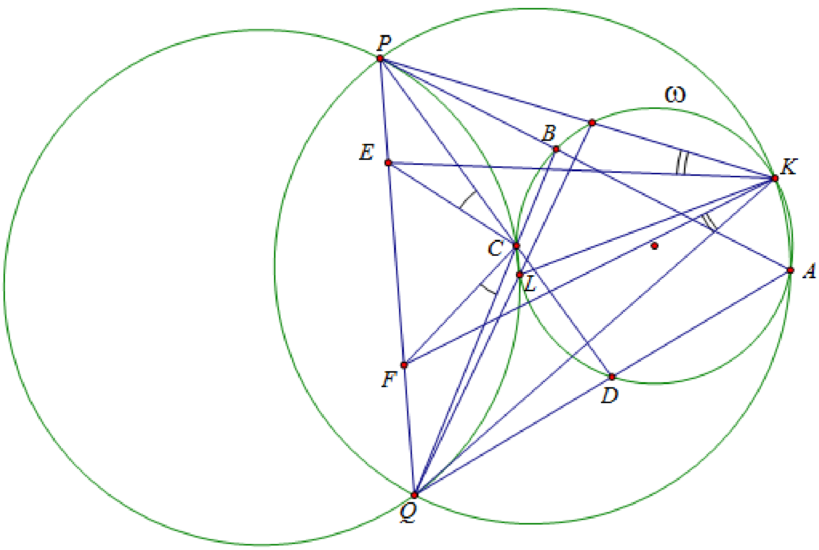
Source: OWN

LeVietAn

#1 Jul 14, 2015, 2:43 pm

Dear Mathlinkers,
Let $ABCD$ be a quadrilateral inscribed in a circle ω. The lines AB and CD meet at P, the lines AD and BC meet at Q. Let K be the second point of intersection of the circumcircle of triangle APQ and ω.
a)Suppose that the points E, F lie on the segment PQ such that P, Q, E, and F are all different. Prove that $\angle PKE = \angle QKF$ if and only if $\angle PCE = \angle QCF$.
b) Let L be the second point of intersection of the circumcircle of triangle CPQ and ω. Prove that the intersection of PK and QL is on ω .

Attachments:



TelvCohl

#2 Jul 14, 2015, 4:46 pm

My solution :

Let M be the midpoint of PQ and $K' \equiv CM \cap \odot(CPQ), L' \equiv AM \cap \odot(APQ)$.

From $\measuredangle KCP = \measuredangle KAD = \measuredangle KPQ \Longrightarrow \odot(KCP)$ is tangent to PQ at P .
Similarly, $\odot(KCQ)$ is tangent to PQ at $Q \Longrightarrow PQ$ is the common tangent of $\odot(KCP), \odot(KCQ)$,
so their radical axis CK passes through the midpoint M of PQ and K is the reflection of K' in M (well-known) .

Since $CP \cdot KQ : CQ \cdot KP = CP \cdot K'P : CQ \cdot K'Q = [CPK'] : [CQK'] = 1$,
so we get $CP : CQ = KP : KQ \Longrightarrow K$ lie on the C-apollonius circle $\mathcal{A}_C$ of $\triangle CPQ$,
hence $\angle PKE = \angle QKF \Longleftrightarrow E \longleftrightarrow F$ under the inversion $\mathbf{I}(\mathcal{A}_C) \Longleftrightarrow \angle PCE = \angle QCF$.

Similarly, we can prove $M \in AL$ and $LPL'Q$ is a parallelogram ,
so $\angle(QL, PK) = \angle LQK' = \angle LCK' = \angle LAK \Longrightarrow PK \cap QL \in \omega$

So $\angle(QE, PK) = \angle EQK = \angle ECK = \angle EAK \Longrightarrow PK \cap QE \in \omega$.

Q.E.D

Luis González

#3 Jul 15, 2015, 3:26 am

a) Let CK cut $\odot(APQ)$ again at X. $\widehat{XQP} = \widehat{CKS} = \widehat{CLQ} = \widehat{CPQ} \Longrightarrow XQ \parallel CP$ and similarly $XP \parallel CQ \Longrightarrow CPXQ$ is parallelogram $\Longrightarrow CK$ passes through the midpoint of PQ; center of the direct inversion of $\odot(APQ) \cong \odot(CPQ) \Longrightarrow$ tangents of $\odot(APQ), \odot(CPQ)$ at K, C meet at T on their radical axis BC. Hence CE, CF are isogonals WRT $\widehat{PCQ} \Longleftrightarrow \odot(CEF)$ is tangent to $\odot(CPQ) \Longleftrightarrow TE \cdot TF = TC^2 = TK^2 \Longleftrightarrow \odot(KEF)$ is tangent to $\odot(KPQ) \Longleftrightarrow KE, KF$ are isogonals WRT $\widehat{PKQ}$.

b) AK, CL, PQ concur at the radical center R of $\odot(APQ), \odot(CPQ)$ and ω. If $S \equiv PK \cap QL$, then the intersections of the opposite sides of the hexagon $AKSLCD$ are collinear, namely P, Q, R. Thus by the converse of Pascal theorem $S \in \omega$.

Quick Reply

High School Olympiads

Geometry problem

Locked

Gramos

#1 Jul 15, 2015, 1:09 am

Let ABC be a triangle, and let P be a point inside it such that $\angle PAC = \angle PBC$. The perpendiculars from P to BC and CA meet these lines at L and M, respectively, and D is the midpoint of AB. Prove that $DL = DM$.

Luis González

#2 Jul 15, 2015, 2:13 am

This problem has been posted many times before
http://artofproblemsolving.com/community/c6h341610 (Solution with congruent triangles)
http://artofproblemsolving.com/community/c6h404100 (Solution with isogonal conjugates)

Cyclic quadrilateral version:
http://artofproblemsolving.com/community/c6h316542
http://artofproblemsolving.com/community/c6h348583

High School Olympiads

Power of point 2

Reply

Aquarious
#1 Mar 29, 2010, 8:14 pm

ABC is a triangle and P is a point inside it . Angle PAC equals angle PBC . The perpendiculars form P to BC and CA

meet these side at L and M , respectively and D is the midpoint of AB . Prove that DL=DM

Luis González
#2 Mar 29, 2010, 10:43 pm

Let $\angle PBC = \angle PAC = \omega . U, V$ are the midpoints of $PB, PA.$ Then $UD = \frac{1}{2}PA = VM$ and $VD = \frac{1}{2}PB = UL.$ Since U, V are the centers of the circle with diameters $PB, PA,$ we get $\angle LUP = \angle MVP = 2\omega,$ which yields $\angle DUL = \angle DVM = \pi - \angle BPA + 2\omega \Longrightarrow \triangle DUL \cong \triangle MVD \Longrightarrow DL = DM.$

dgreenb801
#3 Mar 30, 2010, 10:42 am

Nice problem! My solution:
Make point T such that $ACBT$ is a parallelogram. Make point Q such that $APBQ$ is a parallelogram. Let the perpendiculars from Q to AT and BT be R and S, respectively. Then by symmetry $ML \parallel RS$ and $MR \parallel LS$, thus $MRSL$ is a parallelogram with center D. Also, $\angle LBS = 180 - \angle MCL = \angle MPL$ and $\frac{LB}{BS} = \frac{LP}{PM}$ (by similar triangles), so $\triangle LBS \sim \triangle LPM$, thus $\angle BLS = \angle PLM$, so $\angle MLS = 90$ and $MLSR$ is a rectangle, so $DM = DL$.

jgnr
#4 Mar 30, 2010, 1:59 pm

BP, AP produced meet AC, BC at Q, R. So $ABRQ$ is a cyclic quadrilateral and the problem becomes this:
http://www.mathlinks.ro/viewtopic.php?t=316542

saiftamboli
#5 Mar 30, 2010, 3:27 pm

what if **D-U-L** and **M-V-D?**

JRD
#6 Jan 10, 2013, 9:24 pm

let N be a point on line MD such that $ND = DM$ and $N \neq M$. so by angle chasing it's easy to see that it's enough to prove that $\triangle NBL \sim \triangle MPL$ so we have to prove that $\frac{AM}{PM} = \frac{BL}{PL}$ and it's obvious .

Quick Reply

High School Olympiads

Nice problem

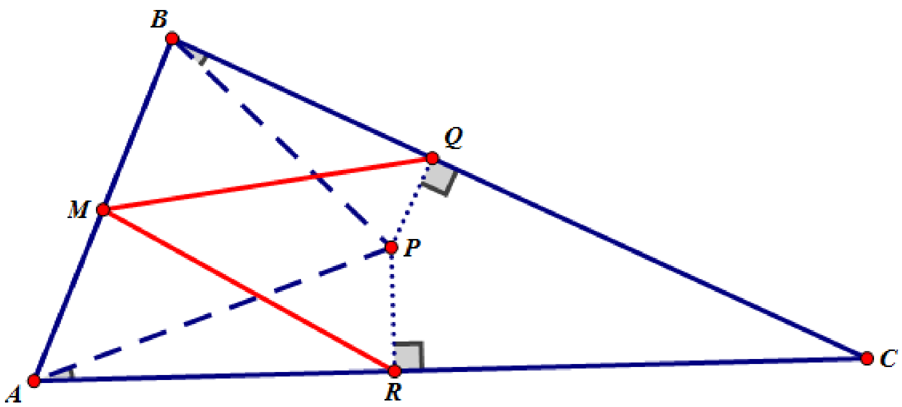
Reply

bayasaa_84

#1 Apr 27, 2011, 1:16 pm

Given a triangle ABC.
Point P is chosen inside the triangle ABC.
$\angle CBP = \angle CAP$
The foot of perpendicular from a point P to the line BC is the point Q.
The foot of perpendicular from a point P to the line AC is the point R.
Point M is midpoint of AB.
Prove that $MQ = MR$

Attachments:

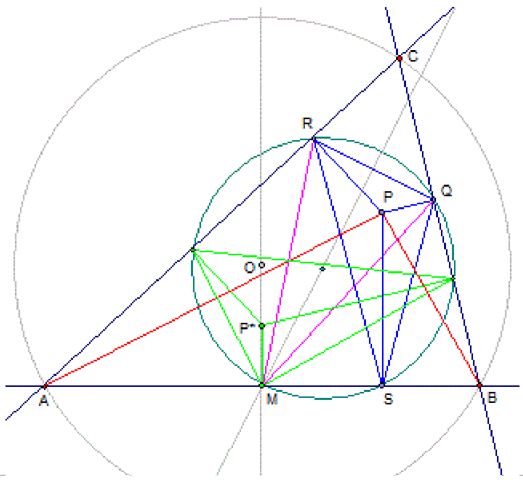


yetti

#2 May 1, 2011, 4:18 am

O is circumcenter of $\triangle ABC$. P^* is isogonal conjugate of P WRT $\triangle ABC$. $\angle ABP^* = \angle CBP = \angle CAP = \angle BAP^*$ $\Longrightarrow$ P^* is on perpendicular bisector OM of AB. S is foot of perpendicular from P to AB. Pedal triangles of P, P^* WRT $\triangle ABC$ have common circumcircle $\Longrightarrow QRSM$ is cyclic.
$\angle QSP = \angle QBP = \angle CBP = \angle CAP = \angle RAP = \angle RSP \Longrightarrow SP$ bisects $\angle QSR$ internally. $AB \perp SP$ bisects $\angle QSR$ externally, cutting the circumcircle $\odot(QRS)$ at S and again at M on the perpendicular bisector of $QR \Longrightarrow$ $MQ = MR.$

Attachments:



Virgil Nicula
#3 May 2, 2011, 12:38 am

See **here** the particular case from the proposed problem **PP3**.

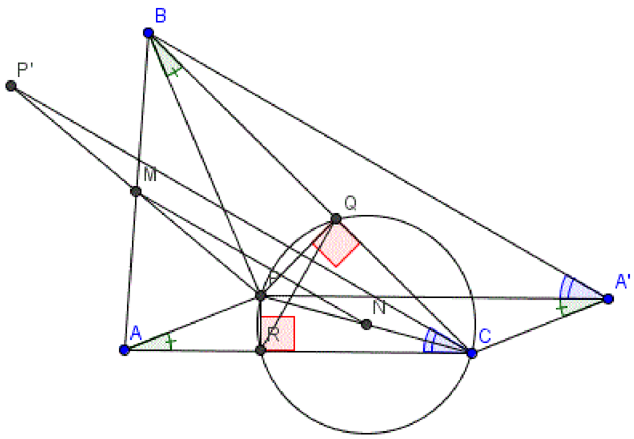
math_explorer
#4 May 2, 2011, 8:57 am

Let N be the midpoint of PC. Since $\angle PQC$ and $\angle PRC$ is right, $PQCR$ is cyclic and the center of the circle is N.

Reflect A about N to get A'; since $\angle PA'C = \angle PBC$, $PA'BC$ is cyclic and $\angle PA'B = \angle PCB$.

Reflect P about M to get P'. Now $\triangle ACP'$ is just $\triangle A'PB$ translated by $\overrightarrow{PA}$, so $\angle ACP' = \angle PA'B = \angle PCB$, so CP, CP' are isogonal wrt $\angle ACB$, so $QR \perp CP'$. M, N are midpoints of PP', PC so $MN \parallel CP' \perp QR$. Since $NQ = NR$, MN is the perpendicular bisector of QR so $MQ = MR$.

Attachments:



sunken rock
#5 May 2, 2011, 8:19 pm • 2

Take D midpoint of AP, E midpoint of BP, see that $DR = ME$ and $QE = MD$, so easily, from parallelogram $MEPD$ we get $\Delta MDR \equiv \Delta QEM$, done.

Best regards,
sunken rock

cwein3
#6 May 4, 2011, 1:22 pm

Take the isogonal conjugate of P and reflect it over AB to get point P'. Then we see a spiral similarity mapping BQP to BMP' and ARP to AMP'.
Thus, $AR/AP = MR/P'P$ and $BQ/BP = MQ/P'P$. Since $AR/AP = BQ/BP$, we see that $MQ = MR$.

Quick Reply

High School Olympiads

cyclic quadrilateral, midpoint

Reply

binaj

#1 Dec 6, 2009, 8:50 pm

Diagonals in cyclic quadrilateral $ABCD$, meet at point S, Q is midpoint of AB, P and R are projections of S on AD and BC respectively. Prove that $PQ = RQ$

sunken rock

#2 Dec 6, 2009, 9:30 pm

It's an old problem!

Call K and L the midpoint of AS, BS. Then $KQ = RL = \frac{BS}{2}, QL = KP = \frac{AS}{2}$, $\measuredangle QLR = \measuredangle QKP = m(\angle ASB) + 2m(\angle SBR)$ (see that $\angle SAD = \angle SBC$), hence $\triangle QLR = \triangle FKQ$, i.e. $QP = QR$.

Best regards,
sunken rock

Quick Reply

High School Olympiads

Perpendicular bisector

Reply

livetolove212

#1 May 9, 2010, 11:41 pm

Given a cyclic quadrilateral $ABCD$. AB intersects CD at E. Let H, K be the projections of E on BC, AD; X, Y be the midpoints of AC, BD. Prove that XY is the perpendicular bisector of the line segment HK.

yetti

#2 May 10, 2010, 4:02 am

Similar problem was posted at least 3 times, eg. Quadrilateral.
Go fly a kite.

Quick Reply

High School Olympiads

reflection of foot

powerpi

#1 Jul 14, 2015, 8:32 pm

Let ABC be a triangle, tangents at B and C to the circumcircle meet tangent at A at M and N. P is reflection of B over M and Q is reflection of C over N. D if foot of perpendicular from A on BC and E is reflection of D over A. Prove that P, E, Q are collinear.

Luis González

#2 Jul 14, 2015, 11:25 pm • 1

Redefining $E \equiv AD \cap PQ,$ then we need to prove that A is the midpoint of $DE.$

Let X,Y be the projections of P and Q on $BC.$ Since $\angle PBX=\angle QCY=\angle BAC \Longrightarrow \triangle PBX \sim \triangle QCY \Longrightarrow BX:CY=BM:CN=DB:DC \Longrightarrow DB:DC=DX:DY=EP:EQ.$ Thus by ERIQ lemma in the quadrilateral $BCQP,$ it follows that $AD:AE=MB:MP=NC:NQ=-1,$ i.e. A is midpoint of $DE.$

High School Olympiads

There exist two circles tangent to 4 circumcircles

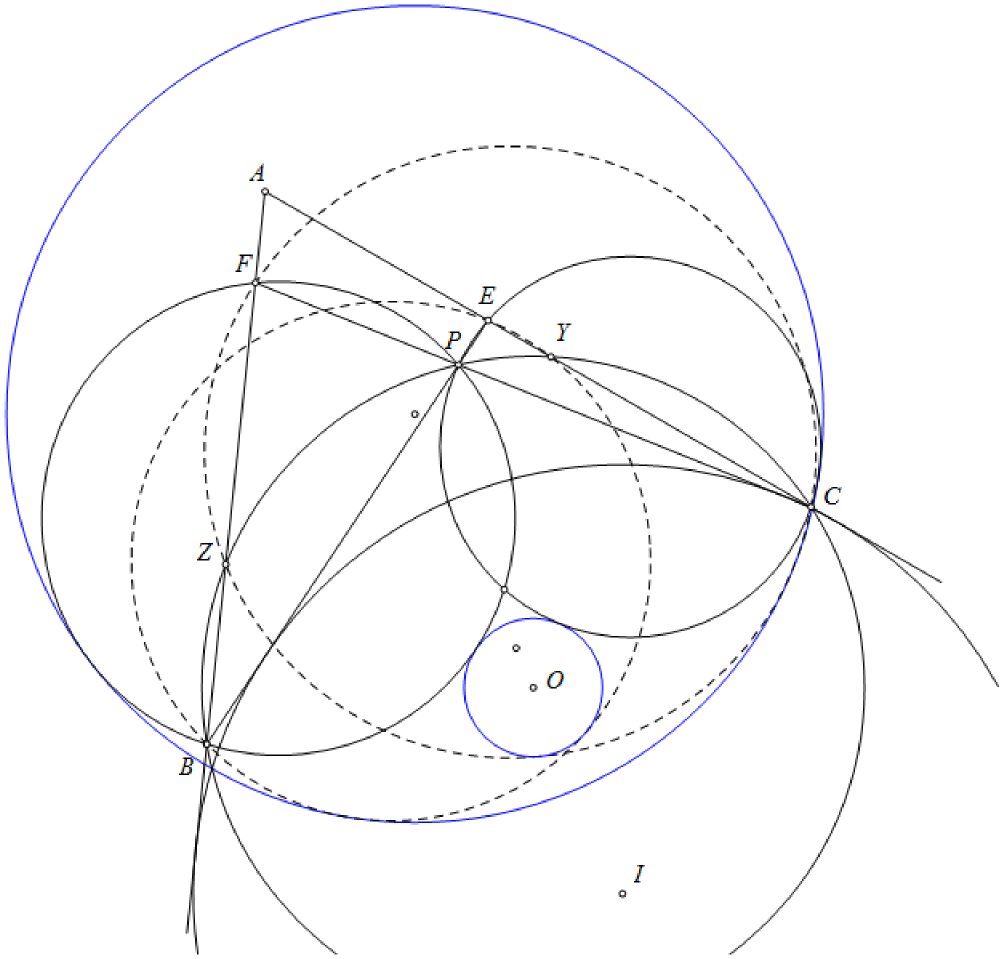
Source: Own

livetolove212

#1 Jul 10, 2015, 10:09 am

Given triangle ABC. Let P be a point such that $AB+BP=AC+CP$. BP, CP meet AC, AB at E, F, respectively. (BPC) intersects AB, AC again at Z, Y,respectively. Prove that there exist two circles tangent to 4 circumcircles of 4 triangles BPF, CPE, BEY, CFZ, including one whose center is the circumcenter of triangle BPC.

Attachments:



Luis González

#2 Jul 14, 2015, 9:15 am • 1

So far I've only proved the existence of the circle centered at the circumcenter O of $\triangle PBC$.

Since $AB+BP=AC+CP \Longrightarrow$ there is a circle (I) tangent to AB, AC and the rays $\overrightarrow{PB}, \overrightarrow{PC}$. EI is internal bisectors of $\angle BEC$ intersecting the arcs PC, BY of $\odot(EPC), \odot(EBY)$ at their midpoints U, T and FI is internal bisector of $\angle BFC$ intersecting the arcs PB, CZ of $\odot(FPB), \odot(FCZ)$ at their midpoints V, R.

Let M, N be the incenters of $\triangle PEC, \triangle PFB$. Since I is E-excenter and F-excenter of $\triangle PEC$ and $\triangle PFB$, then U, V are the midpoints of $IM, IN \Longrightarrow UV \parallel MN \Longrightarrow$
$\angle OUV=\angle(OU,MN)=90^\circ-\frac{1}{2}\angle EPC=\angle(MN,OV)=\angle OVU \Longrightarrow OU=OV$. We also have
$\angle UTO=90^\circ-\angle(BY,EM)=90^\circ-\angle(EM,PC)=\angle OUT \Longrightarrow OT=OU$ and similarly we get $OR=OV$.

Hence $OU = OV = OR = OT \Longrightarrow \odot(UVRT)$ with center O is tangent to $\odot(PEC), \odot(PFB), \odot(BEY)$, $\odot(CFZ)$.

livetolove212

#3 Jul 14, 2015, 9:50 am

I have an idea: denote J the center of the second tangent circle, if the radical axis of two pair of circles (PEC) and (PFB); (CFZ) and (BEY) meet OI again at P, Q then $(OJPQ) = -1$. Then we can use two inversions centered at P and Q to finish this problem.

Quick Reply

High School Olympiads

Reflections

Locked

dothef1

#1 Jul 14, 2015, 7:20 am

Let ABC be a triangle , and C' be the reflection of C in (AB) , and B' be the reflection of B in (AC) .
Let the tangent to $(AB'C')$ at A cut (BC) at E . Prove that the reflection of E over point A lies on (B'C')

Luis González

#2 Jul 14, 2015, 8:25 am

Discussed before at http://www.artofproblemsolving.com/community/q5h570916p3353211.

High School Olympiads

Reflections and two symmetric points

Reply

ThirdTimeLucky
#1 Jan 12, 2014, 8:51 pm • 1

Let ABC be a scalene triangle. Let B' be the reflection of B about line AC and let C' be the reflection of C about line AB. Let ω be the circumcircle of $\triangle AB'C'$. The tangent at A to ω intersects $BC, B'C'$ respectively at X, Y. Prove that $AX = AY$.

jayme
#2 Jan 12, 2014, 9:28 pm • 1

Look like an application of the butterfly theorem
Sincerely
Jean-Louis

jayme
#3 Jan 12, 2014, 9:30 pm

see also
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=568915
Sincerely
Jean-Louis

Luis González
#4 Jan 13, 2014, 1:19 am • 2

Let E, F be the midpoints of CA, AB and Q, R the feet of the altitudes on $CA, AB. O, H, N$ are the circumcenter, orthocenter and 9-point center of $\triangle ABC. U, V$ are the midpoints of EQ, FR, i.e. projections of N on CA, AB.

BQ, CR cut circumcircle (O) again at the reflections Q', R' of H on $CA, AB \Longrightarrow HB' = BQ'$, but $NU = \frac{1}{2}(OE + HQ) = \frac{1}{4}(BH + HQ') = \frac{1}{4}BQ' \Longrightarrow NU = \frac{1}{4}HB'$. Similarly, $NV = \frac{1}{4}HC'$ and it follows that $\triangle NUV$ and $\triangle HB'C'$ are homothetic with coefficient $\frac{1}{4} \Longrightarrow UV \parallel B'C' \Longrightarrow$ perpendicular τ from A to $B'C' \parallel UV$ is then the isogonal of AN WRT $\angle BAC$, but AB', AC' are isogonals WRT $\angle BAC$, hence AN, τ are isogonals WRT $\angle B'AC' \Longrightarrow AN$ passes through the circumcenter of $\triangle AB'C' \Longrightarrow$ perpendicular ℓ to AN at A is then the tangent of $\odot(AB'C')$ at A. If ℓ cut BQ, CR, QR at K, L, T, respectively, then $\angle HLK = \angle NAB = NUV = \angle HB'C' \Longrightarrow L, K, B'C'$ are concyclic $\Longrightarrow YA^2 = YB' \cdot YC' = YK \cdot YL$ $(\star)$.

Let EF cut $AX \equiv \ell$ at M. By Butterfly theorem for cyclic quadrilateral $EFRQ$, the line ℓ perpendicular to AN at A cuts its opposite sidelines EF, RQ, at points M, T equidistant from A, i.e. $\overline{AM} = -\overline{AT}$, but since the A-midline EF bisects AX, then $\overline{AX} = -2 \cdot \overline{AT}$. By Desargues involution theorem, the line ℓ cuts the opposite sidelines of $BCQR$ at pairs of points in involution $\Longrightarrow A$ is a double point of the involution $T \mapsto X, K \mapsto L$ and from $(\star)$, we deduce that Y is the center of this involution, so $(Y, A, T, X) \mapsto (\infty, A, X, T) \Longrightarrow$

$$\frac{\overline{YX}}{\overline{YA}} \cdot \frac{\overline{TA}}{\overline{TX}} = \frac{\overline{XA}}{\overline{XT}} \Longrightarrow \frac{\overline{YX}}{\overline{YA}} = -\frac{\overline{AX}}{\overline{AT}} = 2 \Longrightarrow \overline{AX} = -\overline{AY}.$$

buratinogigle
#5 Feb 8, 2014, 2:40 pm

See here the lemma

http://jcgeometry.org/Articles/Volume1/JCG2012V1pp53-56.pdf

ThirdTimeLucky

#6 Feb 9, 2014, 1:01 am • 1

@Buratinogigle:

Yes, I found it too while proving Lester's Theorem.

TelvCohl

#7 Oct 27, 2014, 4:24 pm • 2

My solution:

Let T be the center of $(AB'C')$.
Let $E = CA \cap B'C', F = AB \cap B'C', D = BE \cap CF$.
Let $E' = CA \cap (AB'C'), F' = AB \cap (AB'C')$ and D' be a point satisfy $D'E' \parallel DE, D'F' \parallel DF$.
Let $X' = DE \cap XY, Y' = DF \cap XY$.

From symmetry it's easy to see $\angle DFA = \angle AFE, \angle DEA = \angle AEF$,
so we get A is the incenter of $\triangle DEF$.
Since $\angle F'AC' = \angle CAF = \angle EAB = \angle B'AE'$,
so $B'C'E'F'$ is a isosceles trapezoid .
Since $\triangle DEF$ and $\triangle D'E'F'$ are homothety with center A ,
so we get A is also the incenter of $\triangle D'E'F'$ and A, D', D are collinear .
Since $\angle F'AT + \angle D'AF' = 90^\circ - \frac{1}{2}\angle D'E'F' + 90^\circ + \frac{1}{2}\angle D'E'F' = 180^\circ$,
so T, A, D' are collinear, ie. T, A, D', D are collinear
hence we get $DA \perp XY \equiv X'Y'$.
Since DA is the bisector of $\angle FDE$,
so A is the midpoint of $X'Y'$,
hence from **Butterfly theorem for quadrilaterals** we get $XA = AY$.

Q.E.D

Quick Reply

High School Olympiads

Concyclic circumcenters in a hexagon

Locked

Source: French Olympiad 2015 - RMM Team Selection Test #2

InCtrl

#1 Jul 10, 2015, 8:39 am

Hexagon $ABCDEF$ is convex. Diagonals $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ intersect at point M and the circumcenters of triangles MAB, MBC, MCD, MDE, MEF and MFA are concyclic. Show that quadrilaterals $ABDE$, $BCEF$ and $CDFA$ have the same area.

Luis González

#2 Jul 13, 2015, 10:33 pm

Posted before at http://www.artofproblemsolving.com/community/c6h145746.

High School Olympiads

super nice 6-th problem, an interesting result.

Reply

Source: Third Zhautykov Olympiad, Kazakhstan, 2007

pohoatza

#1 Apr 26, 2007, 4:00 am • 1

Let $ABCDEF$ be a convex hexagon and it`s diagonals have one common point M. It is known that the circumcenters of triangles $MAB, MBC, MCD, MDE, MEF, MFA$ lie on a circle.
Show that the quadrilaterals $ABDE, BCEF, CDFA$ have equal areas.

e.lopes

#2 Apr 26, 2007, 4:35 am

Nice Problem!

We know that the Area of one quadrilateral ABCD is $\frac{AC.BD.sen(k)}{2}$, where k is the angle between the diagonals!

So, we have to prove that $\frac{AD}{sen(BMC)} = \frac{BE}{sen(CMD)} = \frac{CF}{sen(AMB)}$.

Let $A_1, B_1, ..., F_1$ the circuncenters of $MAB, MBC, ..., MFA$.
We have that $A_1B_1 \parallel D_1E_1$, $B_1C_1 \parallel E_1F_1$ and $F_1A_1 \parallel C_1D_1$, and $A_1B_1C_1D_1E_1F_1$ is cyclic!
So, $A_1B_1C_1$ is congruent to $D_1E_1F_1$, $A_1C_1 = D_1F_1$ and $A_1F_1 = D_1C_1$.
This implies that $A_1C_1D_1F_1$ is one rectangle. Let A' and D' the midpoints of MA and MD. We have $AD = 2A'D'$
So, $\frac{AD}{2sen(BMC)} = \frac{A'D'}{sen(BMC)} = \frac{A_1D_1}{sen(180 - BMC)} = \frac{A_1D_1}{sen(A_1B_1C_1}$

$= 2R$, where R is the ray of $A_1B_1C_1D_1E_1F_1$ circuncircle!

Do the same for the other fractions, and we will get the same result (2R)

nayel

#3 Jun 1, 2009, 1:30 am

We denote $B = A_2, C = A_3, D = A_4, E = A_5, F = A_6$ for brevity. Let P_i denote the midpoint of MA_i and O_i the circumcenter of $\triangle MA_iA_{i+1}$, where the index is taken modulo 6. O_iO_{i-1} is the perpendicular bisector of MA_i. Therefore $O_iO_{i+1} \parallel O_{i+3}O_{i+4}$. Hence $O_iO_{i+1}O_{i+3}O_{i+4}$ is an isosceles trapezoid, implying $O_iO_{i+3} = k$ (constant).

Now consider the forces $\overrightarrow{P_1P_4}, \overrightarrow{P_5P_2}, \overrightarrow{P_3P_6}$ acting at point M. Let R denote the circumradius of circle $O_1O_2O_3O_4O_5O_6$. We have

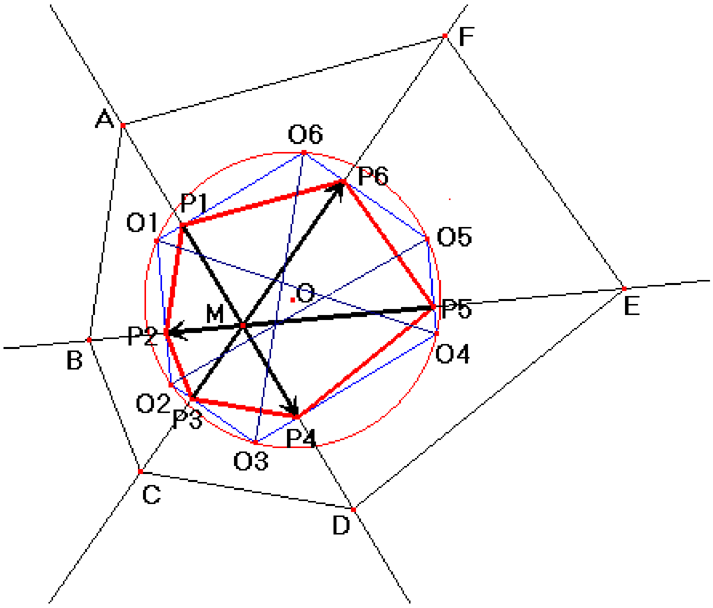
$$\frac{P_{i+2}P_{i+5}}{\sin \angle P_iMP_{i+1}} = \frac{P_{i+2}P_{i+5}}{\sin \angle P_iO_iP_{i+1}} = \frac{P_{i+2}P_{i+5} \cdot 2R}{O_{i+1}O_{i+5}} = 2R \sin \angle O_{i+1}O_{i+5}O_{i+4} = O_{i+1}O_{i+4} = k.$$

Hence the forces are in equilibrium, and so

$$\begin{aligned} \overrightarrow{P_1P_4} + \overrightarrow{P_5P_2} + \overrightarrow{P_3P_6} &= 0 \\ \Rightarrow \overrightarrow{P_2P_5} \times (\overrightarrow{P_1P_4} + \overrightarrow{P_5P_2} + \overrightarrow{P_3P_6}) &= 0 \\ \Rightarrow \overrightarrow{P_2P_5} \times (\overrightarrow{P_1P_4} + \overrightarrow{P_3P_6}) &= 0. \end{aligned}$$

Therefore $|\overline{P_2P_5'} \times \overline{P_1P_4'}| = |\overline{P_2P_5'} \times \overline{P_6P_3'}|$ implying $[P_1P_2P_4P_5] = [P_2P_3P_5P_6]$. Similarly we can show that $[P_1P_2P_4P_5] = [P_3P_4P_6P_1]$. Now the homothety with center M and ratio 2 maps P_i to A_i. Hence the conclusion.

Attachments:



This post has been edited 2 times. Last edited by nayel, Jun 1, 2009, 12:57 pm

nayel

#4 Jun 1, 2009, 1:45 am

> *e.lopes wrote:*
> This implies that $A_1C_1D_1F_1$ is one rectangle...

I think this is not true...

littletush

#5 Nov 26, 2011, 11:34 am

it's similar to an OC's problem.
let $O_1, O_2, ..., O_6$ be six centers,O_1O_2 intersects O_5O_6 at O_7
by OC's theorem,$\frac{O_7O_2}{FC} = \frac{1}{2sinBMC} = \frac{O_7O_5}{BE}$
since $O_1, ..., O_6$ are cyclic,hence
$\frac{O_7O_6}{O_7O_1} = \frac{sinAMB}{sinAMF} = \frac{FC}{BE} = \frac{O_7O_2}{O_7O_5}$
so $BEsinAMB = FCsinAMF$ yielding $(ABDE) = (CDFA)$.

littletush

#6 Nov 26, 2011, 11:36 am

by the way,I'd like to know what kind of contest Zhautykov is?
why is its 2006 identical to Kazakhstan 2006?

Blitzkrieg97

#7 Dec 17, 2014, 11:40 pm

> *littletush wrote:*
> by the way,I'd like to know what kind of contest Zhautykov is?
> why is its 2006 identical to Kazakhstan 2006?

it's international MO,held in Kazakhstan

Quick Reply

High School Olympiads

Isogonal conjugate associate with Miquel

Reply

Source: Own

A-B-C
#1 Jul 13, 2015, 9:35 pm • 1

Given $\triangle ABC, \triangle DEF, \triangle XYZ$ such that :
$A, Y, Z; A, E, F; B, Z, X; B, F, D; C, X, Y; C, D, E$ are collinear.
YZ, EF are isogonal lines WRT $\angle CAB$
ZX, FD are isogonal lines WRT $\angle ABC$
XY, DE are isogonal lines WRT $\angle BCA$
$(DBC), (ECA), (FAB)$ are concurrent at P
$(XBC), (YCA), (ZAB)$ are concurrent at Q
Prove that P, Q are isogonal conjugate WRT $\triangle ABC$

Luis González
#2 Jul 13, 2015, 10:03 pm • 1

For ease we assume the configuration where D, E, F, X, Y, Z are all outside $\triangle ABC$, as the angle chasing depends on the figure. The remaining cases are treated similarly.

Since F, Z are isogonal conjugates WRT $\triangle ABC$, we get $\angle AZB + \angle AFB = 180^\circ - \angle ACB \Longrightarrow$ $\angle APB + \angle AQB = 180^\circ + \angle ACB$ and similarly we get $\angle APC + \angle AQC = 180^\circ + \angle ABC$, which means that P and Q are isogonal conjugates WRT $\triangle ABC$.

TelvCohl
#4 Jul 13, 2015, 10:36 pm • 1

My solution :

Let Ψ be the composition of Inversion $\mathbf{I}(A, \sqrt{AB \cdot AC})$ and reflection $\mathbf{R}(\ell_a)$ where ℓ_a is the bisector of $\angle BAC$.
We use the following well-known property of the mapping Ψ which can be proved by easy angle chasing .

Property $(\star)$: If T, S are isogonal conjugate WRT $\triangle ABC$, then $\Psi(T), \Psi(S)$ are isogonal conjugate WRT $\triangle ABC$.

Since $\{E, Y\}, \{F, Z\}$ are isogonal conjugate WRT $\triangle ABC$,
so from $(\star) \Longrightarrow \{\Psi(E), \Psi(Y)\}, \{\Psi(F), \Psi(Z)\}$ are isogonal conjugate WRT $\triangle ABC$,
hence $\Psi(P) \equiv B\Psi(E) \cap C\Psi(F), \Psi(Q) \equiv B\Psi(Y) \cap C\Psi(Z)$ are isogoanl conjugate WRT $\triangle ABC$,
so from $(\star)$ we conclude P, Q are isogonal conjugate WRT $\triangle ABC$ ($\because \Psi(\Psi(P)) = P, \Psi(\Psi(Q)) = Q$) .

Q.E.D

Quick Reply

High School Olympiads

isogonal conjugates

Reply

andria

#1 Jul 13, 2015, 8:52 pm

Consider two arbitrary points P,Q in the plain of $\triangle ABC$. Let P',Q' be the isogonal conjugate of P,Q WRT $\triangle ABC$ respectively. Let $PQ\cap P'Q'=S, P'Q\cap PQ'=T$ prove that T is isogonal conjugate of S WRT $\triangle ABC$.

TelvCohl

#2 Jul 13, 2015, 9:27 pm • 1

My solution :

Let $X \equiv AQ \cap PQ', Y \equiv AQ \cap P'Q'$.

Since $A(QQ';TP)=(XQ';TP)=(YQ';P'S)=A(QQ';P'S)$,
so $A(QQ';TP)=A(Q'Q;SP') \Longrightarrow AS, AT$ are isogonal conjugate WRT $\angle BAC$.
Similarly, BS, BT are isogonal conjugate WRT $\angle CBA \Longrightarrow S, T$ are isogonal conjugate WRT $\triangle ABC$.

Q.E.D

For another approach by using conic you can see *Geometry of conics* by A.V. Akopyan and A.A. Zaslavsky (page 90) :)

Luis González

#3 Jul 13, 2015, 9:32 pm

Let T' denote the isogonal conjugate of T WRT $\triangle ABC$. Then $A(T,P,Q,Q')=A(T',P',Q',Q) \Longrightarrow AT'$ is the image of AT under the involution defined by $\{AP,AP'\},\{AQ,AQ'\}$. But by dual of Desargues involution theorem for the complete quadrilateral $PQP'Q'$, it follows that AS is the image of AT under the involution defined by $\{AP,AP'\},\{AQ,AQ\} \Longrightarrow AT' \equiv AS$ and cyclically we get $BT' \equiv BS \Longrightarrow T' \equiv S$.

Quick Reply

High School Olympiads

An old problem[2013 TWN TST]

Reply

Source: 2013 Taiwan TST

USJL

#1 Jul 12, 2015, 4:53 pm

Let P be a point in an acute triangle ABC, and d_A, d_B, d_C be the distance from P to vertices of the triangle respectively. If the distance from P to the three edges are d_1, d_2, d_3 respectively, prove that

$$d_A + d_B + d_C \geq 2(d_1 + d_2 + d_3)$$

TelvCohl

#2 Jul 12, 2015, 6:17 pm

This is known as **Erdős-Mordell Inequality**, for the proof you can see
Nikolaos Dergiades, Signed Distances and the Erdős-Mordell Inequality, Forum Geometricorum, 4 (2004) 67--68.
Claudi Alsina and Roger B. Nelsen, A visual proof of the Erdős-Mordell inequality, Forum Geometricorum, 7 (2007) 99--102.

:)

rkm0959

#3 Jul 12, 2015, 6:43 pm • 1

I will present my favorite solution of this theorem since it was not presented in the above papers.
Let D, E, F be the foot of perpendicular from P onto BC, CA, AB.
Let H, G be the foot of perpendicular from B, C onto EF.
Clearly, $BC \geq HG = HF + FE + EG$. (1)
By easy angle chasing, $\triangle BFH \sim \triangle APE$ and $\triangle CEG \sim \triangle APF$.
Therefore, we have $HF = \frac{PE \times BF}{PA}, EG = \frac{PF \times CE}{PA}$ (2)
By Ptolemy on $\square AFBE$, we have $PA \times EF = AF \times PE + AE \times PF$.
Therefore, $EF = \frac{AF \times PE + AE \times PF}{PA}$ (3)
Using (2), (3) on (1), we have $PA \geq \frac{AB}{BC} \times PE + \frac{AC}{BC} \times PF$.
Summing this cyclically and using $AM - GM$ proves the problem. ■

Luis González

#4 Jul 13, 2015, 10:46 am

Treegoner posted a stronger inequality in the topic The Erdös-Mordell Inequality (see post #3). No solutions yet.

Quick Reply

High School Olympiads

The Erdös-Mordell Inequality

Reply

Johann Peter Dirichlet

#1 May 5, 2004, 8:51 am

Hey children!

This is the most famous crazy geometric inequality.
"Erdös-Mordell Inequality:

If P is a point in the same plane of a triangle ABC, and D,E,F are the projections of P in BC, AC and AB, then
$(PA + PB + PC) \geq 2(PD + PE + PF)$,
with equality if and only if $PA = PB = PC$ and $AB = AC = BC$".

xirti

#2 May 9, 2004, 1:41 am

Do you know if this is also good in space? Of course replacing 2 by 3.

I mean for a tetrahedron.

treegoner

#3 Jul 3, 2004, 10:01 pm

This inequality can be tightened as follow :
Let ABC be a triangle. Let P be a point in it. Let D, E, F be points on BC, CA, AB such that PD, PE, PF are respectively bisectors of $\measuredangle BPC, \measuredangle CPA, \measuredangle APB$. Then
$PA + PB + PC \geq 2(PD + PE + PF)$
The proof is much easier to be thought out. Try!

Igor

#4 Jul 4, 2004, 12:56 am

Then generalize for a n side polygon
(The proof has been already posted on the forum by Pierre)

Dr Sonnhard Graubner

#5 Apr 14, 2009, 1:51 am

hello, here you can find something more about this inequality
http://www.gtsintsifas.com/uploads/erdsum.pdf
Sonnhard.

Quick Reply

High School Olympiads

The circumcircle of triangle KDE is tangent to Γ

Reply

Source: OWN

LeVietAn

#1 Jul 12, 2015, 9:21 pm

Dear Mathlinkers,
Let ABC be a triangle with $AB > AC$. Let Γ be its circumcircle. Let H be a point is inside triangle ABC such that $\angle HBA = \angle HCA$. Let K be the point on Γ such that $\angle AKH = 90^\circ$. Let D, E be the points on the segment BC such that that B, C, D, E are all diferrent and $\angle BHD = \angle CHE$. Prove that the circumcircle of triangle KDE is tangent to Γ.

This post has been edited 1 time. Last edited by math_explorer, Jul 14, 2015, 6:13 pm
Reason: more descriptive title

Luis González

#2 Jul 13, 2015, 10:03 am

Since $\angle AKH = 90^\circ$, then KH cuts Γ again at the antipode P of A. If BH, CH cut AC, AB at B', C', then $BCB'C'$ is cyclic due to $\angle HBA = \angle HCA \Longrightarrow B'C'$ is antiparallel to BC WRT AB, AC and $HB, HC \Longrightarrow$ $(AO \parallel HJ) \perp B'C'$; where O and J denote the circumcenters of $\triangle ABC$ and $\triangle HBC$. Thus from the parallel radii $OP \parallel JH$, it follows that $OJ \cap PH$ is the insimilicenter of $\Gamma \sim \odot(HBC)$, i.e. center of their direct inversion, hence the tangents of $\Gamma, \odot(HBC)$ at K, H meet at T on their radical axis BC.

Since HD, HE are isogonals WRT $\angle BHC$, then $\odot(HDE)$ is tangent to $\odot(HBC)$ at H. Hence $TD \cdot TE = TH^2 = TK^2 \Longrightarrow TK$ is tangent of $\odot(KDE) \Longrightarrow \odot(KDE)$ is tangent to Γ.

Quick Reply

High School Olympiads

IMO Shortlist 2014 G6

Reply

hajimbrak
#1 Jul 11, 2015, 2:56 pm

Let ABC be a fixed acute-angled triangle. Consider some points E and F lying on the sides AC and AB, respectively, and let M be the midpoint of EF . Let the perpendicular bisector of EF intersect the line BC at K, and let the perpendicular bisector of MK intersect the lines AC and AB at S and T , respectively. We call the pair (E, F) $interesting$, if the quadrilateral $KSAT$ is cyclic.

Suppose that the pairs (E_1, F_1) and (E_2, F_2) are interesting. Prove that $\dfrac{E_1E_2}{AB} = \dfrac{F_1F_2}{AC}$

Proposed by Ali Zamani, Iran

This post has been edited 4 times. Last edited by hajimbrak, Jul 27, 2015, 12:11 pm
Reason: Changed format

m.candales
#2 Jul 12, 2015, 5:53 am

The condition that we are being asked to prove is equivalent to proving that EF is antiparallel to BC for every (E, F) that is interesting.
So basically we need to prove that if (E, F) is interesting then $EFBC$ is cyclic.
This is an example of a geometry problem that can be solved methodically by using complex numbers.
For every point X in the plane, we denote x to be the associated complex number, and $\bar{x}$ its conjugate.

Let's first review some properties:

1- $x = e^{i\varphi}|x|$ where φ is the argument of x and $|x$ is its distance to the origin (absolute value)

2- x and $\bar{x}$ have the same absolute value and oposite argument

3- $x \in \mathbb{R}$ if and only if x = $\bar{x}$ (this follows from 1 and 2)

4- $x\bar{x} = |x|^2$ (this follows from 1 and 2)

5- $\dfrac{x}{\bar{x}} = e^{2i\varphi}$ where φ is the argument of x (this follows from 1 and 2)

6- x belongs to the unit circle if and only if $\bar{x} = \dfrac{1}{x}$ (this follows from 4)

7- $\varphi = \angle ACB$ (from A to B in positive direction) if and only if $\dfrac{b-c}{|b-c|} = e^{i\varphi}\dfrac{a-c}{|a-c|}$ (this follows from 1)

8- $\dfrac{a-b}{\bar{a}-\bar{b}} = e^{2i\varphi}\dfrac{c-d}{\bar{c}-\bar{d}}$ where φ is the angle between CD and AB (from CD to AB in positive direction) (this follows from 5)

9- $AB \parallel CD$ if and only if $\dfrac{a-b}{\bar{a}-\bar{b}} = \dfrac{c-d}{\bar{c}-\bar{d}}$ (this follows from 8)

10- $AB \perp CD$ if and only if $\dfrac{a-b}{\bar{a}-\bar{b}} = -\dfrac{c-d}{\bar{c}-\bar{d}}$ (this follows from 8)

11- A, B, C are colinear if and only if $\dfrac{a-b}{\bar{a}-\bar{b}} = \dfrac{a-c}{\bar{a}-\bar{c}}$ (this follows from 9)

12- A, B, C, D belong to a circle if and only if $\dfrac{a-c}{b-c} : \dfrac{a-d}{b-d} \in \mathbb{R}$ (this follows from 7)

13- A, B, C, D belong to a circle if and only if $\frac{a-c}{\bar{a}-\bar{c}} : \frac{b-c}{\bar{b}-\bar{c}} = \frac{a-d}{\bar{a}-\bar{d}} : \frac{b-d}{\bar{b}-\bar{d}}$ (this follows from 3 and 12. It also follows directly from 8)

14- For a chord AB of the unit circle, we have $\frac{a-b}{\bar{a}-\bar{b}} = -ab$ (this follows from 6)

15- If C belongs to the chord AB of the unit circle, then $\bar{c} = \frac{a+b-c}{ab}$ (this follows from 6 and 11)

Now, back to the problem. We can assume without loss of generality that the circumcircle of $\triangle ABC$ is the the unit circle in the complex plane.
The strategy to solve a problem like this using complex numbers is by first computing all the points(complex numbers) that we need in terms of some reduced set of points(complex numbers). The conditions of the problem will be reduced to some equation(s); what we are asked to prove will also be reduced to some equation(s); and all we need to prove is that one equation implies the other.
So, here we are going to try to express everything in terms of a, b, c, e, f
We can express $\bar{a}, \bar{b}, \bar{c}, \bar{e}, \bar{f}$ in terms of a, b, c, e, f using 6 and 15
We can express m in terms of e, f since $m = \frac{e+f}{2}$ and therefore $\bar{m}$ can be expressed in terms of a, b, c, e, f too.
We can express $\bar{k}$ in terms of b, c, k using 15
$EF \perp MK$, so, using 10 we get an equation involving k, m, e, f and their conjugates. All of those can be expressed in terms of a, b, c, e, f and k. So, we solve for k, and we will also get k expressed in terms of a, b, c, e, f and therefore $\bar{k}$
Let R be the middle point of MK. We can express r in term of m, k and therefore in terms of a, b, c, e, f. Therefore $\bar{r}$ too.
We can express $\bar{s}$ in terms of s, a, c using 15
$SR \parallel EF$, so, using 9 we get an equation involving s, r, e, f and their conjugates. All of those can be expressed in terms of a, b, c, e, f and k. So, we solve for s and we get s expressed in terms of a, b, c, e, f and therefore $\bar{s}$
In the same way we get t and $\bar{t}$ expressed in terms of a, b, c, e, f. Actually, we should be able to get the formulas for t and $\bar{t}$ from the ones from s and $\bar{s}$ interchanging b with c and e with f.
The beautiful thing up to this point, is that all the equations we get are simple linear equations, so they can be easily solved. This is because all the geometric constructions up to this point only involve straight lines.
The condition of $KSAT$ being cyclic is translated to an equation (using 13) involving k, s, a, t and their conjugates. All of them we have been able to express in terms of a, b, c, e, f. So, this is an equation that relates a, b, c, e, f. Let's call this equation (I).
We need to prove that $EFBC$ is cyclic. This is also equivalent to an equation (using 13) involving e, f, b, c and their conjugates; all of which we have expressed also in terms of a, b, c, e, f. Let's call this second equation (II). All we need to prove is that (I) implies (II)

I carried out the computations to get equation (II) and I got:
$(II) : (b-c)(fbc - ebc - fac + eab - a^2b + a^2c) = 0$
I didn't carry out the computations to get equation (I), I gave up before finishing. But I bet that it is something like:
$(I) : a^?(e-f)^?(b-c)^?(fbc - ebc - fac + eab - a^2b + a^2c) = 0$
So that (I) would clearly imply (II)

TelvCohl
#3 Jul 12, 2015, 7:58 am • 2

My solution **:**

Lemma :

Let E, F be the points on CA, AB, respectively .
Let M be the midpoint of EF and $K \in BC$ be the point s.t. $KE = KF$.
Let ℓ be the perpendicular bisector of MK and $S \equiv \ell \cap CA, T \equiv \ell \cap AB$.

If A, K, S, T are concyclic, then AK is A-symmedian of $\triangle AEF$

Proof :

Let $X \equiv AM \cap \odot(AKST)$.

From $EF \parallel ST \Longrightarrow AM$ is A-median of $\triangle AST$,
so notice M, K are symmetry WRT ST we gte M is the reflection of X in the midpoint of ST ,
hence $STKX$ is an Isosceles trapezoid $\Longrightarrow XK \parallel ST \parallel EF \Longrightarrow AK$ is A-symmedian of $\triangle AEF$.

Back to the main problem :

Let the perpendicular bisector of E_iF_i cuts BC at K_i $(i = 1, 2)$.
Let H be the orthocenter of $\triangle ABC$ and $B' \equiv BH \cap CA, C' \equiv CH \cap AB$.
Let R be the midpoint of BC and $S \equiv \odot(AH) \cap AR$ (i.e. $AB'SC'$ is a harmonic quadrilateral).

From $AS \cdot AR = AC' \cdot AB = AB' \cdot AC \Longrightarrow S$ is the Miquel point of R, B', C' WRT $\triangle ABC$. $(\star)$
From RB', RC' are the tangents of $\odot(AB'C')$ (well-knwon) $\Longrightarrow \angle RB'C' = \angle RC'B' = \angle BAC$.

From the lemma $\Longrightarrow AK_1$ is A-symmedian of $\triangle AE_1F_1$,
so combine $K_1E_1 = K_1F_1 \Longrightarrow K_1E_1, K_1F_1$ are the tangents of $\odot(AE_1F_1)$,
hence we get $\angle K_1E_1F_1 = \angle K_1F_1E_1 = \angle BAC \Longrightarrow \triangle RB'C' \sim \triangle K_1E_1F_1$.

Similarly, we can prove $\triangle RB'C' \sim \triangle K_2E_2F_2 \Longrightarrow \triangle RB'C' \sim \triangle K_1E_1F_1 \sim \triangle K_2E_2F_2$,
so from $(\star) \Longrightarrow S$ is the center of spiral similarity that maps $\triangle RB'C' \mapsto \triangle K_1E_1F_1 \mapsto \triangle K_2E_2F_2$,
hence we get $E_1E_2 : F_1F_2 = SE_1 : SF_1 = SE_2 : SF_2 = SB' : SC' = AB' : AC' = AB : AC$.

Q.E.D

Luis González

#4 Jul 13, 2015, 9:16 am

From 2015 Taiwan TST Round 3 Mock IMO Day 2 Problem 1, it follows that $K_1 \in BC$ is intersection of the tangents of $\odot(AE_1F_1)$ at E_1, F_1 and similarly $K_2 \in BC$ is intersection of the tangents of $\odot(AE_2F_2)$ at $E_2, F_2.$ This is also the lemma in Telv's previous solution.

Let P be the 2nd intersection of $\odot(AE_1F_1)$ and $\odot(AE_2F_2)$; center of the spiral similarity carrying E_1E_2 into $F_1F_2.$ Since the isosceles $\triangle K_1E_1F_1$ and $\triangle K_2E_2F_2$ are directly similar, then K_1 is the image of K_2 under the referred spiral similarity $\Longrightarrow P$ is Miquel point of $\{AB, BC, K_2F_2, K_1F_1\} \Longrightarrow BK_2PF_2$ is cyclic $\Longrightarrow \angle PBK_2 = \angle PF_2K_2 = \angle PAF_2 \Longrightarrow$ $\odot(ABP)$ touches BC and similarly $\odot(ACP)$ touches $BC \Longrightarrow AP$ is radical axis of $\odot(ABP), \odot(ACP)$ bisecting their common tangent $\overline{BC}$, i.e. AP is A-median of $\triangle ABC.$ Therefore
$AB : AC = \text{dist}(P, E_1E_2) : \text{dist}(P, F_1F_2) = E_2E_2 : F_1F_2.$

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livetolove212

#5 Jul 14, 2015, 10:19 am

I also used the same lemma as Telv Cohl did.
Let L, M be the intersections of CF_1 and BE_1, CF_2 and $BE_2.$
Since B, K_1, C are collinear then using the converse of Pascal theorem for 6 points A, F_1, F_1, L, E_1, E_1 we get L lies on (AE_1F_1). Similarly M lies on (AE_2F_2). Therefore $\angle F_2F_1C = \angle E_2E_1B, \angle F_1F_2C = \angle E_1E_2B$, which follows that $\triangle CF_1F_2 \sim \triangle BE_1E_2$. Then $\dfrac{E_1E_2}{F_1F_2} = \dfrac{dis(B, AC)}{dis(C, AB)} = \dfrac{AB}{AC}$. Thus $\dfrac{E_1E_2}{AB} = \dfrac{F_1F_2}{AC}$.

This post has been edited 1 time. Last edited by livetolove212, Jul 14, 2015, 10:20 am

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andria

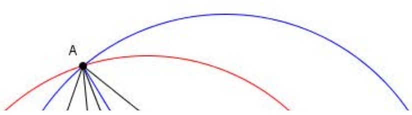
#6 Jul 26, 2015, 9:22 pm

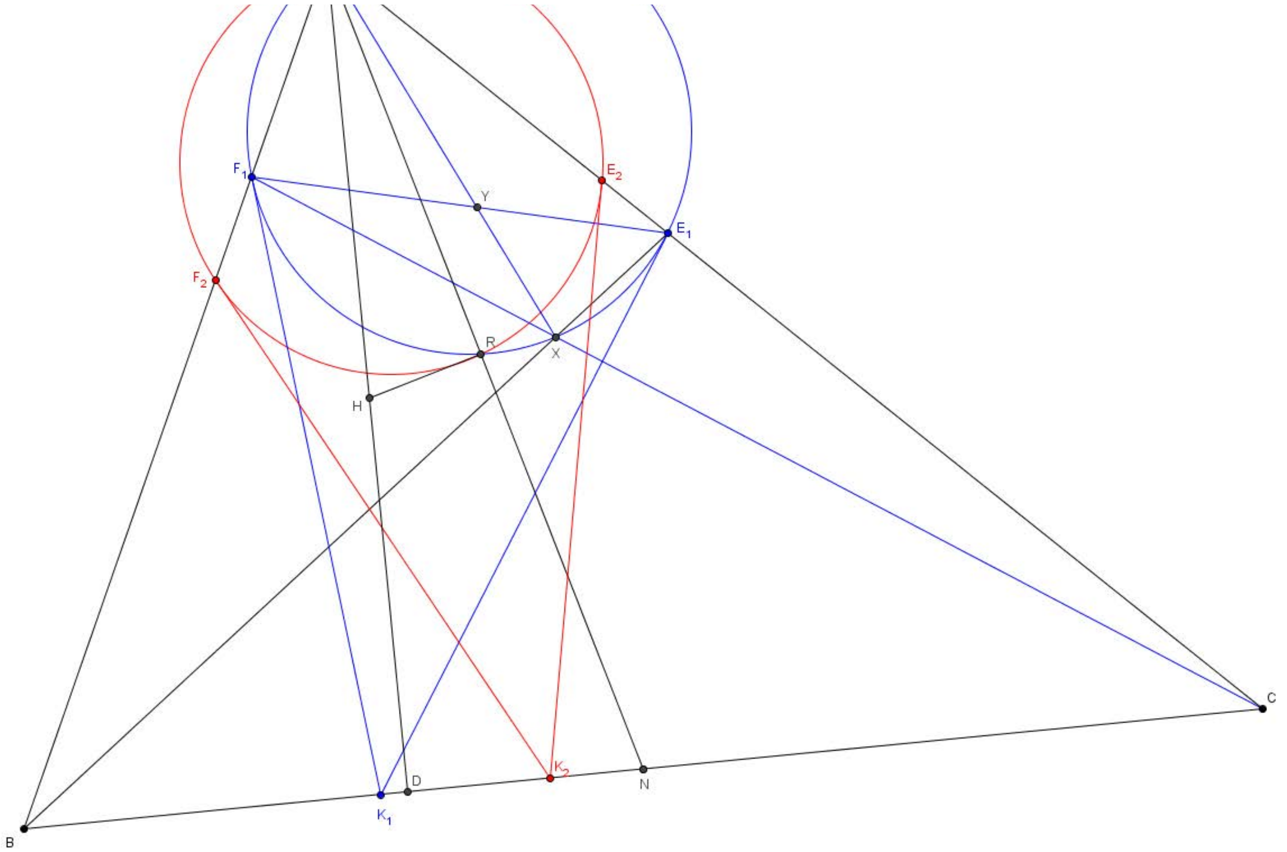
I have Another solution:
I use the following lemma that has been proved in TelvCohl's solution in the post #3#.
Lemma: the tangents from E_1, F_1 to $\odot(\triangle AE_1F_1)$ intersect each other at K_1.

Back to the main problem:
From the lemma K_1E_1, K_1F_1 are tangents to $\odot(\triangle AE_1F_1)$. So E_1F_1 is polar of K_1 WRT $\odot(\triangle AE_1F_1)$ let $BE_1 \cap \odot(\triangle AE_1F_1) = X$ and $AX \cap E_1F_1 = Y$. since B, K_1, C are collinear polar of B, K_1, C are concurrent. Notice that Y lies on polar of K_1, B so Y also belongs to the polar of C. hence C, X, F_1 are collinear.
Let H be the orthocenter of $\triangle ABC$ and $AH \cap BC = D$ and let R be the projection of H on the line AN where N is midpoint of BC. since $\odot(BC)$ is orthogonal to cyclic quadrilateral AE_1XF_1 (well known) we get that $\odot(AE_1XF_1)$ passes throw R ($\because$ the inverse of $\odot(AE_1XF_1)$ passes throw N under the inversion Ψ with center A and power $AH.AD$). Similarly $\odot(\triangle AE_2F_2)$ passes throw R. hence R is miquel point of
$E_1E_2F_1F_2 \Longrightarrow \triangle RE_1E_2 \sim \triangle RF_1F_2 \Longrightarrow \dfrac{E_1E_2}{F_1F_2} = \dfrac{\text{dis}(R, AC)}{\text{dis}(R, AB)} = \dfrac{AB}{AC}$.

DONE

Attachments:





This post has been edited 1 time. Last edited by andria, Jul 26, 2015, 9:24 pm

andria

#7 Jul 26, 2015, 10:03 pm

Remark:
In My solution in the previous post the fact that $\odot(\triangle AE_1XF_1)$ passes throw R is old. For more solutions see Turkey's TST 2010 Q5

polya78

#8 Jul 29, 2015, 11:20 pm

Let M' be the reflection of M in the midpoint of ST. Then as noted before, $\angle SM'T = \angle SMT = \angle SKT$, so M' lies on $(ASTK)$, and since $KM' \parallel ST$, $\angle SAK = \angle TAM$, which means that K is the point of intersection of the tangents at E, F of w, the circumcircle of $\triangle AEF$.

Let $X = BE \cap w$. Then applying Pascal to hexagon $AFFXEE$, we get that C, F, X are collinear. Let $Y = BC \cap (BFX)$. Then $\angle BYX = \angle AFX = \angle XEC$, so C, E, X, Y are concyclic. Then if $AE = x, AF = y$ (signed lengths), we have that $a * BY = BX * BE = c(c-y)$, $a * CY = CX * CF = b(b-x)$, which leads to $bx + cy = b^2 + c^2 - a^2$. The desired result follows easily.

Attachments:

imo shortlist 2014.pdf (393kb)

This post has been edited 1 time. Last edited by polya78, Jul 29, 2015, 11:22 pm
Reason: clean up

Gryphos

#9 Aug 4, 2015, 2:55 pm

I use the same lemma as Telv.

Let (E_1, F_1) and (E_2, F_2) be to interesting pairs and K_1, K_2 the corresponding points K. Let furthermore $X = F_1K_1 \cap F_2K_2$, $Y = E_1K_1 \cap E_2K_2$ and $Z = E_1F_1 \cap E_2F_2$. Since $\triangle E_1F_1K_1 \sim \triangle E_2F_2K_2$, it is easy to see that these lines cannot be parallel.
Obiously F_1F_2XZ and E_1E_2YZ are cyclic quadrilaterals, so sine law yields

$$\frac{F_1F_2}{XZ} = \frac{\sin\angle F_1ZF_2}{\sin\alpha}, \quad \frac{E_1E_2}{YZ} = \frac{\sin\angle E_1ZE_2}{\sin\alpha} \Longrightarrow \frac{F_1F_2}{XZ} = \frac{E_1E_2}{YZ}.$$

It remains to prove that $\frac{XZ}{YZ}=\frac{AC}{AB}$. But by simple angle chasing (XYK_1K_2 is also cyclic) we get $\angle ZYX=\beta$ and $\angle YXZ=\gamma$. This implies that the triangles XYZ and ABC are similar, and thus the desired result.

Quick Reply

High School Olympiads

2015 Taiwan TST Round 3 Mock IMO Day 2 Problem 1

Reply

Source: 2015 Taiwan TST Round 3 Mock IMO Day 2 Problem 1

wanwan4343

#1 Jul 12, 2015, 8:25 pm

Let ABC be a fixed acute-angled triangle. Consider some points E and F lying on the sides AC and AB, respectively, and let M be the midpoint of EF. Let the perpendicular bisector of EF intersect the line BC at K, and let the perpendicular bisector of MK intersect the lines AC and AB at S and T, respectively. If the quadrilateral $KSAT$ is cycle, prove that $\angle KEF = \angle KFE = \angle A$.

Luis González

#2 Jul 13, 2015, 5:44 am

Since $EF \parallel ST$, then AM cuts ST at its midpoint N. Since NK is the reflection of NA on ST, then it follows by symmetry that KN cuts $\odot(AST)$ again at D forming the isosceles trapezoid $ATSD$. Thus again, by symmetry, if AN cuts $\odot(AST)$ again at K', then $KK' \parallel ST \Longrightarrow AK$ and $AM \equiv AK'$ are isogonals WRT $\angle EAF \Longrightarrow AK$ is the A-symmedian of $\triangle AEF \Longrightarrow KE, KF$ are tangents of $\odot(AEF) \Longrightarrow \angle KEF = \angle KFE = \angle A$.

Dukejukem

#3 Aug 26, 2015, 12:00 am

Let us relabel points S, T, M as B, C, X, respectively (we can ignore the points B, C specified in the problem statement). Now, let H be the orthocenter of $\triangle ABC$ and let M be the midpoint of $\overline{BC}$. Γ_1, Γ_2 denote $\odot(ABC), \odot(BHC)$, respectively and U, V are reflections of A in BC, M, respectively.

Since $EF \parallel BC$, it follows that A, X, M are collinear. From $\triangle AFX \sim \triangle ABM$ we obtain $\frac{XF}{MB} = \frac{AX}{AM}$. From $\triangle MXK \sim \triangle MAU$ we find $\frac{XK}{AU} = \frac{MX}{AM}$. It follows that $\frac{XF}{XK} = \frac{AX}{MX} \cdot \frac{MB}{AU}$. Now, because the reflection X in BC lies on Γ_1, it follows that X lies on the reflection of Γ_1 in BC, which is just Γ_2 (well-known). Meanwhile, since Γ_1 and Γ_2 are symmetric about BC and M, it follows that U, V lie on Γ_2. By Power of a Point, we obtain $AX \cdot AV = AH \cdot AU$ and $MX \cdot MV = MB^2$. Therefore,

$$\frac{AX}{MX} \cdot \frac{MB}{AU} = \frac{AH}{MB} \cdot \frac{MV}{AV} = \frac{AH}{2MB} = \frac{OM}{MB},$$

where O denotes the center of Γ_1, and we have used the well-known fact that $AH = 2OM$. Then by side-angle-side similarity, we deduce that $\triangle KFX \sim \triangle BOM$, and hence $\angle KFE = \angle BOM = \angle A$. $\square$

pi37

#4 Aug 26, 2015, 10:00 am • 1

Suppose $GH \parallel EF$ and passes through K with G, H on AE, AF respectively. Let GF intersect HE at X. Then by parallelism

$$\angle FXE = \angle GXH = \angle TKS = 180 - \angle FAE$$

so $FAEX$ is cyclic. By Brokard's Theorem, K', the intersection of the tangents from E, F to $(AEXF)$, lies on GH. But of course K' lies on the perpendicular bisector of EF, so $K' = K$, and we're done.

This post has been edited 1 time. Last edited by pi37, Aug 27, 2015, 9:45 pm

TKDLH-99

#5 Aug 27, 2015, 7:31 pm

Dear pi37: What's X mean?

pi37

#6 Aug 27, 2015, 9:45 pm

Sorry, I edited it in.

utkarshgupta

#7 Jan 6, 2016, 12:01 am

Consider the circumcircle of $\triangle AST$.
Obviously since $ST||EF$, AM is the median of $\triangle AST$.
Now we have that a point on the arc ST not containing Awhose reflection in ST lie on the $A-$median.

Let X, Y be two such points on the arc ST whose reflections in ST lie on $A-$median.
Denote these reflections by $X'Y'$
Thus since S, T, X, Y are concyclic, $S, T, X'Y'$ are also concyclic which is not possible as they lie on the same side of ST.

Thus there exists only one such point on the arc ST.

It is well known and easy to show that this point (K in the question) lies on the symmedian.

Now the question is easy.

AdithyaBhaskar

#8 Jan 6, 2016, 3:02 pm

Never seen an uglier problem.

Quick Reply

High School Olympiads

Determine the angle[2013 TWN TST Quizzes]

Source: 2013 Taiwan TST

USJL

#1 Jul 12, 2015, 6:16 pm

Let ΔABC be a triangle with $AB=AC$ and $\angle A=\alpha$, and let O,H be its circumcenter and orthocenter, respectively. If P,Q are points on AB and AC, respectively, such that $APHQ$ forms a rhombus, determine $\angle POQ$ in terms of α.

Luis González

#2 Jul 13, 2015, 6:19 am

This is a particular case of the problem Beautiful angle relation between circumcenter & orthocenter.

High School Olympiads

Beautiful angle relation between circumcenter & orthocenter

Source: maybe own

hqdhftw

#1 Aug 15, 2014, 11:19 am

Given an acute triangle ABC and point E on AC, F on AB. Prove that $\angle EOF = 180^o - 2\angle A$ if and only if $\angle EHF = \angle A$. (there are some cases that we need to replace the assumption $\angle EHF = \angle A$ with $\angle EHF = 180^o - \angle A$ but I believe the solutions are similar).

leader

#2 Aug 18, 2014, 3:37 am

Let X, Y be symmetric points of O wrt AC, AB resp. Let $\angle BAC = x$. Now we have that X, Y are circumcenters of AHC, AHB resp. Note
$\angle XAH = \angle XHA = 90 - \angle HCA = x$(*)
Similarly $\angle YAH = \angle YHA = x$ We now know that $HYAX$ is a rhombus so $AX||HY$(1).
Now let E' be the symmetric point of E wrt AH we have that $\angle E'AY = \angle XAE = x - \angle EAH = \angle HAF$(2) ,
$\angle E'YA = \angle EXA$(3),
$\angle E'HY = \angle EHX$ (4)
The condition $\angle EOF = 180 - 2x$ is equivalent to $\angle AOE + \angle AOF = 180 - 2x$ Now adding $2x$ to both sides

$$\angle EXA + \angle XAY + \angle AYF = 2x + \angle FOA + \angle EOA = 2x + 180 - 2x = 180$$

but this is equivalent to $XE||YF$ which is using (1) equivalent to $\angle EXA = \angle FYH$ which is using (3) equivalent to $\angle FYH = \angle E'YA$ which is using (2) equivalent to E', F being isogonal conjugates wrt $\triangle AHY$ which is using (2) equivalent to $\angle E'HY = \angle FHA$ using (4) equivalent to $\angle EHX = \angle FHA$ which is equivalent to $\angle EHF = \angle AHX$ but using (*) this is equivalent to $\angle EHF = x$

Luis González

#3 Sep 2, 2014, 5:18 am

Let X, Y, Z be the feet of the altitudes on BC, CA, AB and P, Q, R the midpoints of BC, CA, AB. $F' \in AB$, such that $\widehat{EOF'} = 180^\circ - 2 \cdot \widehat{A}$. Let U be the projection of H on EF and V the projection of O on EF'. It suffices to prove that $F \equiv F'$.

From cyclic quadrilaterals $HUFZ$ and $HUEY$, we get $\widehat{AZU} = \widehat{FHU}$ and $\widehat{AYU} = \widehat{EHU} \Longrightarrow$ $\widehat{YUZ} = \widehat{YAZ} + \widehat{AZU} + \widehat{AYU} = \widehat{A} + \widehat{EHF} = 2 \cdot \widehat{A} \Longrightarrow U$ is on 9-point circle $\odot(XYZ) \Longrightarrow EF$ touches the inconic $\mathcal{C}$ with foci O, H. Likewise, from cyclic quadrilaterals $OVF'R$ and $OVEQ$, we get $\widehat{ARV} = \widehat{F'OV}$ and $\widehat{AQV} = \widehat{EOV} \Longrightarrow \widehat{QVR} = \widehat{QAR} + \widehat{ARV} + \widehat{AQV} = \widehat{A} + \widehat{EOF'} = 180^\circ - \widehat{A} \Longrightarrow V$ is on 9-point circle $\odot(PQR)$ $\Longrightarrow EF'$ also touches $\mathcal{C}$, forcing $F \equiv F'$, as desired.

TelvCohl

#4 Oct 9, 2014, 9:32 pm • 2

My solution:

I'll only prove $\angle EOF = 180 - 2\angle BAC \Longrightarrow \angle EHF = \angle BAC$.
(we can prove the converse part similarly)

Let X, Y be the intersection of (ABC) and BH, CH , respectively.
Let Y' be the reflection of Y with respect to OF .
Let P, Q be the projection of Y' on AC, AB , respectively.
Let R be the intersection of $Y'H$ and EF .

By symmetry we get Y' lie on (ABC).
Since X, Y is the reflection of H with respect to AC, AB, respectively,
so $\angle YY'X = \angle YAX = 2\angle BAC$.
Since $\angle EOF = 180^\circ - 2\angle BAC$ and $OF \perp YY'$,
so we get $OE \perp XY'$. i.e. E lie on the perpendicular bisector of XY'

Since $EH = EX = EY', FH = FY = FY'$,
so EF is the perpendicular bisector of HY',
hence we get R is the midpoint of $Y'H$.
Since PQ is the Simson line of Y' with respect to $\triangle ABC$,
so from Steiner theorem we get P, Q, R are collinear,
hence by the converse of Simson theorem we get A, E, F, Y' are concyclic,
so $\angle EHF = \angle FY'E = \angle BAC$.

Q.E.D

Remark:

we can use some properties of conic to prove this problem as following:

Let U be the intersection of OX and AC.
Let V be the intersection of OY and AB.

Notice that U, V is the tangent point of **Macbeath inellipse** with AC, AB, respectively.
so from $\angle UOV = 360^\circ - 2\angle BAC = 2\angle EOF$ we get EF is tangent to **Macbeath inellipse**,
hence $\angle EHF = \frac{1}{2}\angle UHV = \angle BAC$.

Q.E.D

From the proof above we can generalize this problem:

Let E, F be the point on AC, AB, respectively.
Let P, Q be the isogonal conjugate of $\triangle ABC$.
Then $\angle BPC + \angle EPF = 180^\circ \iff \angle BQC + \angle EQF = 180^\circ$

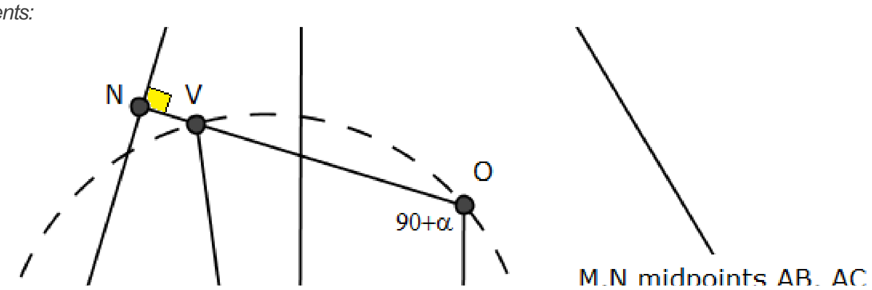
This post has been edited 1 time. Last edited by TelvCohl, May 5, 2015, 6:32 am

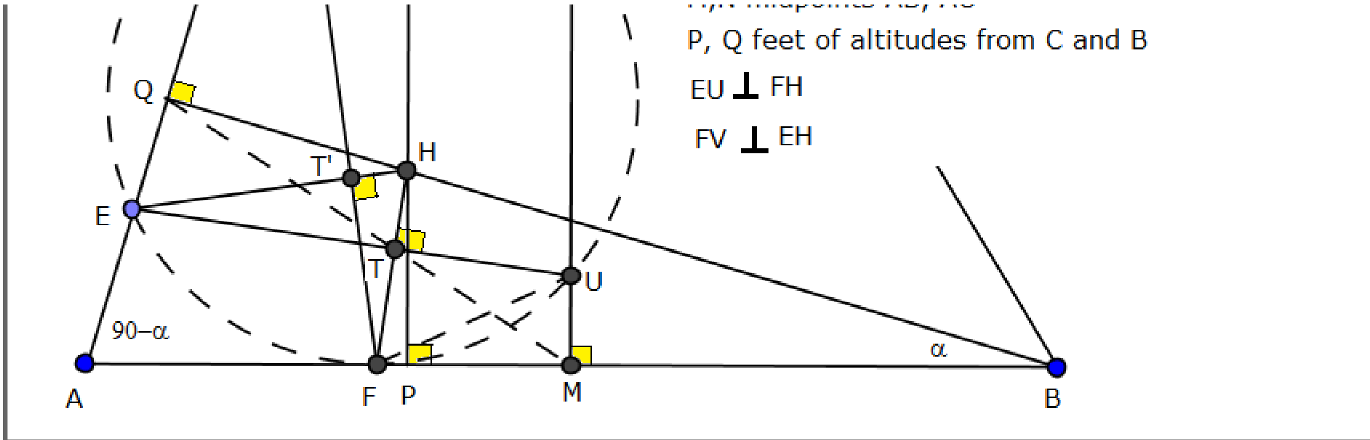
ricarlos
#5 May 8, 2015, 6:24 am

Denote $\angle ABQ = \alpha$, then problem is
"$\angle EOF = 2\alpha$ iff $\angle EHF = \angle A = 90 - \alpha$"
a)
$T = EU \cap FH, \Delta HET \sim \Delta ABQ$
$THQE$ is cyclic $\to \angle HET = \angle HQT = \alpha$,
ΔMBQ is isosceles $\to \angle HQM = \alpha \to$ Q, T and M are collinear.
$\angle QMA = 2\alpha$ (exterior angle of triangle MBQ),
$MUTF$ is cyclic $\to \angle TMF = \angle EUF = 2\alpha$.
b)
$T' = EH \cap FV$, from solution a) we have $\angle EVT' = \angle EVF = 2\alpha$
then $VEFU$ is cyclic.

$\Delta EVT' \to \angle EVT' = 2\alpha, \angle VET' = 90 - 2\alpha$
$\angle HET = \alpha$,
$\angle VEU = \angle VET' + HET = 90 - \alpha$
Then $EUOV$ is cyclic, its circumcircle is the circumcircle of $VEFU$
so $\angle EOF = 2\alpha$.

Attachments:





Quick Reply

High School Olympiads

2015 Taiwan TST Round 3 Mock IMO Day 2 Problem 1

Reply

Source: 2015 Taiwan TST Round 3 Mock IMO Day 2 Problem 1

wanwan4343

#1 Jul 12, 2015, 8:25 pm

Let ABC be a fixed acute-angled triangle. Consider some points E and F lying on the sides AC and AB, respectively, and let M be the midpoint of EF. Let the perpendicular bisector of EF intersect the line BC at K, and let the perpendicular bisector of MK intersect the lines AC and AB at S and T, respectively. If the quadrilateral $KSAT$ is cycle, prove that $\angle KEF = \angle KFE = \angle A$.

Luis González

#2 Jul 13, 2015, 5:44 am

Since $EF \parallel ST$, then AM cuts ST at its midpoint N. Since NK is the reflection of NA on ST, then it follows by symmetry that KN cuts $\odot(AST)$ again at D forming the isosceles trapezoid $ATSD$. Thus again, by symmetry, if AN cuts $\odot(AST)$ again at K', then $KK' \parallel ST \Longrightarrow AK$ and $AM \equiv AK'$ are isogonals WRT $\angle EAF \Longrightarrow AK$ is the A-symmedian of $\triangle AEF \Longrightarrow KE, KF$ are tangents of $\odot(AEF) \Longrightarrow \angle KEF = \angle KFE = \angle A$.

Dukejukem

#3 Aug 26, 2015, 12:00 am

Let us relabel points S, T, M as B, C, X, respectively (we can ignore the points B, C specified in the problem statement). Now, let H be the orthocenter of $\triangle ABC$ and let M be the midpoint of $\overline{BC}$. Γ_1, Γ_2 denote $\odot(ABC), \odot(BHC)$, respectively and U, V are reflections of A in BC, M, respectively.

Since $EF \parallel BC$, it follows that A, X, M are collinear. From $\triangle AFX \sim \triangle ABM$ we obtain $\frac{XF}{MB} = \frac{AX}{AM}$. From $\triangle MXK \sim \triangle MAU$ we find $\frac{XK}{AU} = \frac{MX}{AM}$. It follows that $\frac{XF}{XK} = \frac{AX}{MX} \cdot \frac{MB}{AU}$. Now, because the reflection X in BC lies on Γ_1, it follows that X lies on the reflection of Γ_1 in BC, which is just Γ_2 (well-known). Meanwhile, since Γ_1 and Γ_2 are symmetric about BC and M, it follows that U, V lie on Γ_2. By Power of a Point, we obtain $AX \cdot AV = AH \cdot AU$ and $MX \cdot MV = MB^2$. Therefore,

$$\frac{AX}{MX} \cdot \frac{MB}{AU} = \frac{AH}{MB} \cdot \frac{MV}{AV} = \frac{AH}{2MB} = \frac{OM}{MB},$$

where O denotes the center of Γ_1, and we have used the well-known fact that $AH = 2OM$. Then by side-angle-side similarity, we deduce that $\triangle KFX \sim \triangle BOM$, and hence $\angle KFE = \angle BOM = \angle A. \square$

pi37

#4 Aug 26, 2015, 10:00 am • 1

Suppose $GH \parallel EF$ and passes through K with G, H on AE, AF respectively. Let GF intersect HE at X. Then by parallelism

$$\angle FXE = \angle GXH = \angle TKS = 180 - \angle FAE$$

so $FAEX$ is cyclic. By Brokard's Theorem, K', the intersection of the tangents from E, F to $(AEXF)$, lies on GH. But of course K' lies on the perpendicular bisector of EF, so $K' = K$, and we're done.

This post has been edited 1 time. Last edited by pi37, Aug 27, 2015, 9:45 pm

TKDLH-99

#5 Aug 27, 2015, 7:31 pm

Dear pi37: What's X mean?

pi37

#6 Aug 27, 2015, 9:45 pm

Sorry, I edited it in.

utkarshgupta

#7 Jan 6, 2016, 12:01 am

Consider the circumcircle of $\triangle AST$.
Obviously since $ST||EF$, AM is the median of $\triangle AST$.
Now we have that a point on the arc ST not containing Awhose reflection in ST lie on the $A-$median.

Let X, Y be two such points on the arc ST whose reflections in ST lie on $A-$median.
Denote these reflections by $X'Y'$
Thus since S, T, X, Y are concyclic, $S, T, X'Y'$ are also concyclic which is not possible as they lie on the same side of ST.

Thus there exists only one such point on the arc ST.

It is well known and easy to show that this point (K in the question) lies on the symmedian.

Now the question is easy.

AdithyaBhaskar

#8 Jan 6, 2016, 3:02 pm

Never seen an uglier problem.

Quick Reply

High School Olympiads

2015 Taiwan TST Round 3 Quiz 3 Problem 2

Reply

Source: 2015 Taiwan TST Round 3 Quiz 3 Problem 2

wanwan4343

#1 Jul 12, 2015, 5:56 pm • 1

In a scalene triangle ABC with incenter I, the incircle is tangent to sides CA and AB at points E and F. The tangents to the circumcircle of triangle AEF at E and F meet at S. Lines EF and BC intersect at T. Prove that the circle with diameter ST is orthogonal to the nine-point circle of triangle BIC.

Proposed by Evan Chen

This post has been edited 1 time. Last edited by v_Enhance, Aug 12, 2015, 3:17 am
Reason: claimauthorship

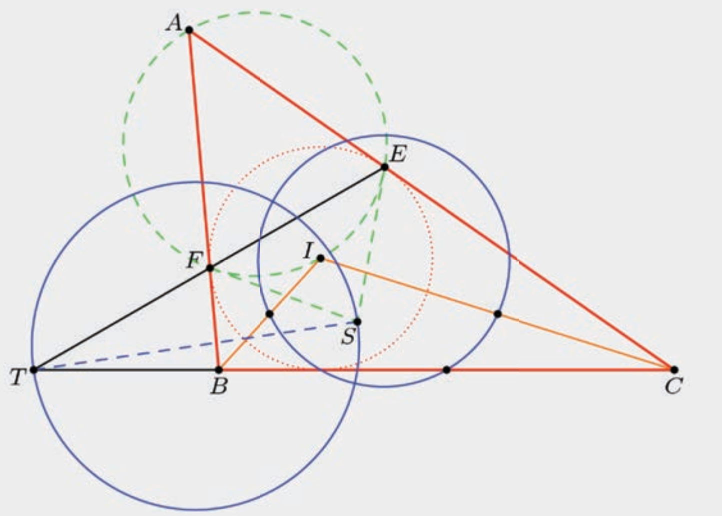
v_Enhance

#2 Jul 12, 2015, 7:01 pm • 5

This was my problem :) Hope you guys liked it!

EDIT: just for fun, here's the original I sent in, complete with picture:

> *Quote:*
> □□□□□□□ $\triangle ABC$ □□□□□□□ I□□□□□□□□□ CA, AB □□□□ E, F□ $\triangle AEF$ □□□□□ E □ F □□□□□□□□ S□
> □□ EF □ BC □□□ T□
> □□□□ ST □□□□□□□□□ $\triangle BIC$ □□□□□



This post has been edited 1 time. Last edited by v_Enhance, Jul 12, 2015, 9:19 pm
Reason: add in diagram

Luis González

#3 Jul 13, 2015, 4:44 am • 2

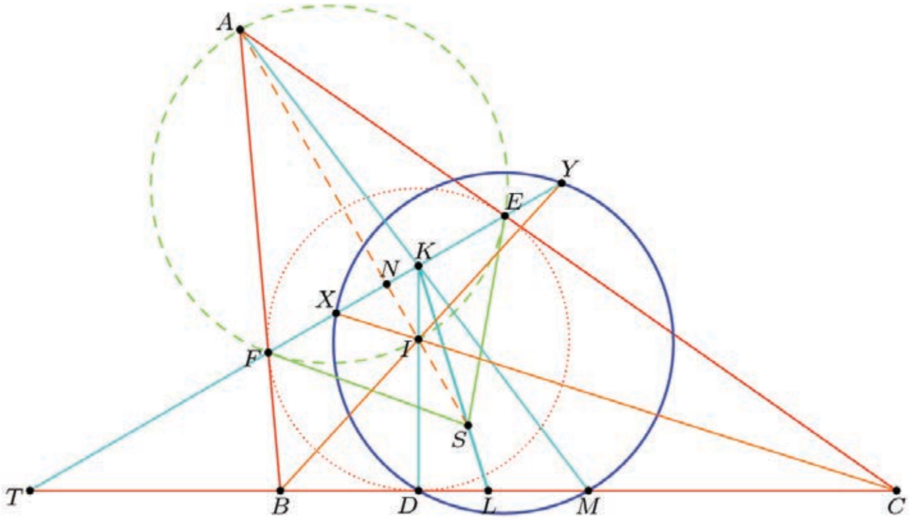
Let the incircle (I) touch BC at D. DI cuts EF at J and the perpendicular from A to DI cuts DI and EF at X, P. Since AX is the polar of J WRT (I), then $(E,F,J,P)=-1 \Longrightarrow A(E,F,J,P)=A(B,C,J,\infty)=-1 \Longrightarrow AJ$ cuts BC at its midpoint M. As SJ is the polar of P WRT $\odot(AEF)$, then $J(A,X,S,P)=-1 \Longrightarrow (M,D,L,T)=-1$, where $L \equiv JS \cap BC$.

On the other hand, if EF cuts BI, CI at Y, Z, then it's known (easy to show by angle chase) that $CY \perp BI$ and $BZ \perp CI$ $\Longrightarrow \omega \equiv \odot(DMYZ)$ is 9-point circle of $\odot(IBC)$. Now since $(Y, Z, J, T) = I(B, C, D, T) = -1$ and together with $(M, D, L, T) = -1$ previously found, we deduce that $\overline{SLJ}$ is the polar of T WRT $\omega \Longrightarrow S, T$ are conjugate points WRT ω $\Longrightarrow$ circle with diameter $\overline{ST}$ is orthogonal to ω.

v_Enhance

#4 Jul 13, 2015, 10:52 pm • 2

My solution is very similar to that of Luis.



Let D be the foot from I to BC. Let X, Y denote the feet from B, C to CI and BI. We can show that $BIFX, CIEY$ are cyclic, so that X and Y lie on EF. Now let M be the midpoint of BC, and ω the circumcircle of $DMXY$. The problem reduces to showing that T lies on the polar of S to ω.

Let K be the intersection of AM and EF. It's well known (say by SL 2005 G6) that points K, I, D are collinear. Let N be the midpoint of EF, and L the intersection of KS and BC. From

$$-1 = (A, I; N, S) \stackrel{K}{=} (T, L; M, D)$$

and

$$-1 = (T, D; B, C) \stackrel{I}{=} (T, K; Y, X)$$

we find that $T = MD \cap YX$ is the pole of line KL with respect to ω, completing the proof.

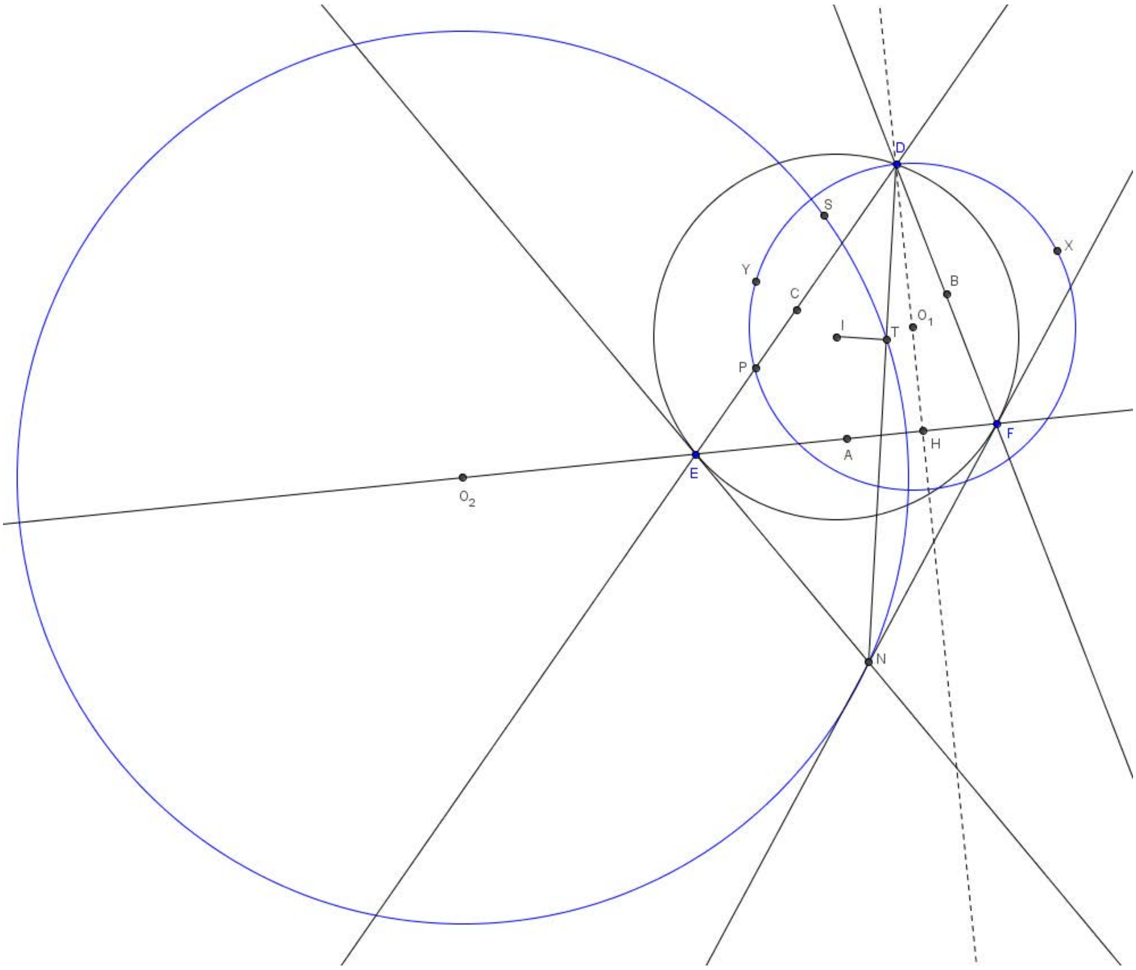
PSJL

#5 Jul 19, 2015, 8:47 pm • 2

Consider inversion with center I
we can turn the original problem into this
A triangle DEF,and its circumcenter is I
A, B, C are midpoints of EF, FD, DE
N is the intersection of the tangents to (DEF) in E, F
IT is perpendicular to DN at T
S is the reflection of N with respect to EF
X, Y are the reflections of I with respect to DF, DE
and we need to prove that (DXY) is orthogonal to (STN)
Suppose that O_1, O_2 are the circumcenters of $(DXY), (STN)$
It is obvious that O_2 lies on EF
we need to prove that $O_1D^2 + O_2S^2 = O_1O_2^2$
$\angle O_1DF = \angle O_1DX - \angle FDX = (90^o - \angle XYD) - \angle IDF = (180^o - \angle XIY) - \angle IDF = \angle IDE$
$\therefore DO_1 \perp EF$,Suppose DH is perpendicular to EF at H
$O_1O_2^2 - O_2S^2 - O_1D^2 = (O_1O_2^2 - O_2D^2) + (O_2D^2 - O_2S^2) - O_1D^2 = (O_1H^2 - DH^2) + DT \cdot DN - O_1D^2$
$= DT \cdot DN - 2 \cdot DO_1 \cdot DH$
Suppose that (DXY) meets DE at D, P
$\angle YXI = \angle YDP = \angle YXP \Rightarrow X, I, P$ are collinear

$\Rightarrow \angle DPI = 90^o - \angle EDF = \angle INE \Rightarrow I, P, E, N, F, T$ are concyclic
$\Rightarrow DT \cdot DN = DP \cdot DE = 2 \cdot DO_1 \cdot DH$
which means that $O_1D^2 + O_2S^2 = O_1O_2^2$,as desired.

Attachments:



This post has been edited 1 time. Last edited by PSJL, Jul 19, 2015, 8:51 pm

High School Olympiads

2015 Taiwan TST Round 1 Mock IMO Day 2 Problem 1

Reply

Source: 2015 Taiwan TST Round 1 Mock IMO Day 2 Problem 1

wanwan4343
#1 Jul 12, 2015, 4:51 pm

Let ABC be a triangle and M be the midpoint of BC, and let AM meet the circumcircle of ABC again at R. A line passing through R and parallel to BC meet the circumcircle of ABC again at S. Let U be the foot from R to BC, and T be the reflection of U in R. D lies in BC such that AD is an altitude. N is the midpoint of AD. Finally let AS and MN meets at K. Prove that AT bisector MK.

Luis González
#2 Jul 13, 2015, 3:32 am

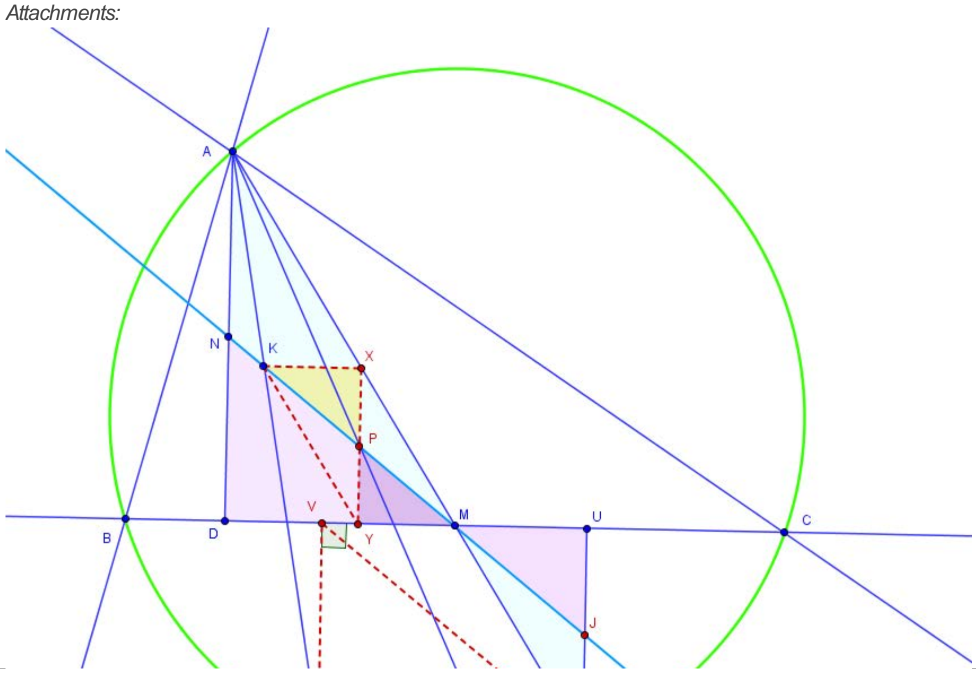
Let the tangents of the circumcircle $\odot(ABC)$ at B, C meet at Y. $AS \equiv AY$ is the A-symmedian of $\triangle ABC$. If $W \equiv AY \cap BC$, then NW cuts MY at its midpoint M', due to $AD \parallel MY$. Since $M'(D, N, A, \infty) = -1 = M'(S, W, A, Y) \Longrightarrow M', S, D$ are collinear. If $J \equiv MS \cap DY$, then $JW \parallel YM$ as S is on the D-median DM' of $\triangle DMY$. Thus from the complete $MWJY$, it follows that $(M, W, D, L) = -1$.

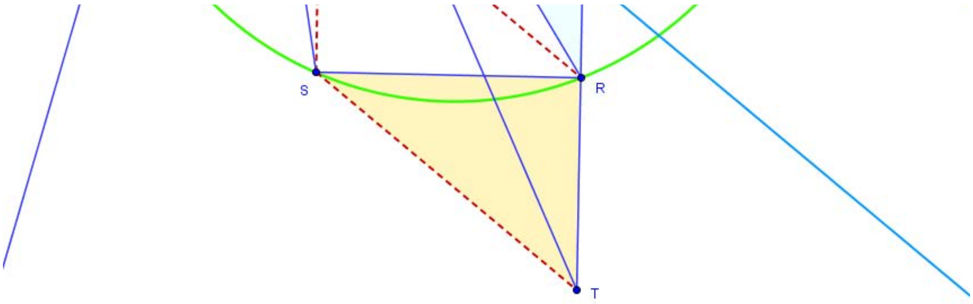
Let F be the reflection of D on M. Since M', S, D are collinear, then by symmetry M', R, F are collinear and since $UR \parallel MY$, then $T \in YF$. By symmetry, $Z \equiv MR \cap YF$ is the reflection of J across YM and therefore the projection V of Z on BC is the reflection of W on $M \Longrightarrow (M, V, F, U) = (M, W, D, L) = -1 \Longrightarrow$ $(M, V, F, U) = (Y, Z, F, T) = A(K, M, F, T) = -1$. Since $AF \parallel MN$, it follows that AT is the A-median of $\triangle AKM$, i.e. AT bisects $\overline{MK}$.

TelvCohl
#3 Aug 24, 2015, 12:24 am • 1

Let V be the projection of S on BC and $J \equiv MN \cap UR$, $P \equiv AT \cap MN$. From $VS \stackrel{\parallel}{=} UR \stackrel{\parallel}{=} RT$ we know $VRTS$ is a parallelogram. From $AD \parallel UR \Longrightarrow J$ is the midpoint of UR, so $2MJ \stackrel{\parallel}{=} VR \stackrel{\parallel}{=} ST \Longrightarrow KP \parallel ST$. Let X be the point on AM s.t. $XP \perp BC$ and $Y \equiv XP \cap BC$. Since $\triangle KPX$ and $\triangle STR$ are homothetic (with center A), so $XK \parallel RS \Longrightarrow$ $XK \parallel BC \equiv MY$, hence combine $XP = YP$ ($\because XY \parallel AD$) we get $XKYM$ is a parallelogram $\Longrightarrow KP = MP$.

Attachments:





Dukejukem

#4 Aug 25, 2015, 5:09 am

I think we can prove a more general statement:

Let $\triangle ABC$ be a triangle and let M be a variable point on BC. Let AM meet $\odot(ABC)$ for a second time at R. Let U be the projection of R onto BC and let T be the reflection of U in R. Let D be the projection of A onto BC and let N be the midpoint of $\overline{AD}$. Let τ be the line passing through R parallel to BC. Point S lies on τ and satisfies $MR = MS$. Prove that if $K \equiv AS \cap MN$, then AT bisects $\overline{MK}$.

Dukejukem

#5 Aug 25, 2015, 5:10 am

My solution to the generalization:

Let R' be the reflection of R in M and let H be the projection of AD onto τ. Denote $V \equiv AT \cap MN, X \equiv AT \cap BC, Y \equiv AS \cap BC$, and let D' be the reflection of D' in M.

Note that $-1 = A(U, T; R, \infty) = (U, X; M, D)$. Then from $\triangle MUR \sim \triangle MDA$, we infer $XM : XD = UM : UD = RM : RA$. Note that M is the circumcenter of $\triangle RR'S$, which implies that $\angle R'SR = 90^\circ$. Therefore, $\triangle RR'S \sim \triangle RAH$, implying that $\frac{RM}{RA} = \frac{RR'}{2RA} = \frac{RS}{2RH} = \frac{MY}{2MD}$. Hence, $\frac{XM}{XD} = \frac{MY}{2MD}$. It follows that

$$\frac{XY}{XM} = \frac{YM}{XM} - 1 = 1 + 2\left(\frac{MD}{XD} - 1\right) = 1 + \frac{2XM}{XD} = 1 + \frac{MY}{MD} = \frac{D'Y}{D'M}.$$

Because $AD' \parallel MN$, we obtain $-1 = A(Y, M; X, D') = (K, M; V, \infty)$. $\square$

This post has been edited 1 time. Last edited by Dukejukem, Aug 25, 2015, 5:10 am

Aiscrim

#9 Sep 5, 2015, 4:53 pm

RIP synthetic

Quick Reply

High School Olympiads

2015 Taiwan TST Round 2 Mock IMO Day 2 Problem 1

Locked

Source: 2015 Taiwan TST Round 2 Mock IMO Day 2 Problem 1

wanwan4343

#1 Jul 12, 2015, 5:15 pm • 1

Let ABC be a triangle with incircle ω, incenter I and circumcircle Γ. Let D be the tangency point of ω with BC, let M be the midpoint of ID, and let A' be the diametral opposite of A with respect to Γ. If we denote $X = A'M \cap \Gamma$, then prove that the circumcircle of triangle AXD is tangent to BC.

Luis González

#2 Jul 13, 2015, 12:54 am

Posted before at http://www.artofproblemsolving.com/community/c6h614973.

High School Olympiads

Circle tangent to BC

Reply

Source: own

MariusStanean

#1 Nov 22, 2014, 3:46 pm

Let ABC be a triangle with incircle $\omega(I)$ and circumcircle Γ. Let D be the tangency points of ω with BC, let M be the midpoint of ID, and let A' be the diametral opposite of A with respect to Γ. If we denote $X=A'M\cap\Gamma$ then prove that the circumcircle of $\triangle AXD$ is tangent to BC.

Luis González

#2 Nov 22, 2014, 10:57 pm • 4

Redefine X as the 2nd intersection of Γ with the circle passing through A tangent to BC at D. We prove that X, M and A' are collinear. (I) touches AC, AB at E, F. EF cuts BC at P and AX cuts BC at J.

Since $JB \cdot JC = JA \cdot JX = JD^2$ and $(B, C, D, P) = -1$, it follows that J is midpoint of $\overline{DP}$. But AD, being the polar of P WRT (I), is perpendicular to PI at $U \Longrightarrow$ D-midline JM of $\triangle PID$ is perpendicular bisector of $\overline{UD}$. Hence $\angle JUD = \angle JDU = \angle AXD \Longrightarrow JXUMD$ is cyclic. Hence $\angle AXM = \angle JDM = 90^\circ \Longrightarrow XM$ cuts Γ again at the antipode A' of A, i.e. X, M and A' are collinear.

shinichiman

#3 Jan 7, 2015, 11:32 am • 1

First, we will solve this problem:
Let (I) and (O) be the incircle and circumcircle of $\triangle ABC$. (I) touches BC at D. Line AL is isogonal to AD in $\angle BAC$ and $L \in (O)$. $LD \cap (O) = X$. Prove that $XM \perp AX$ with M is the midpoint of ID.

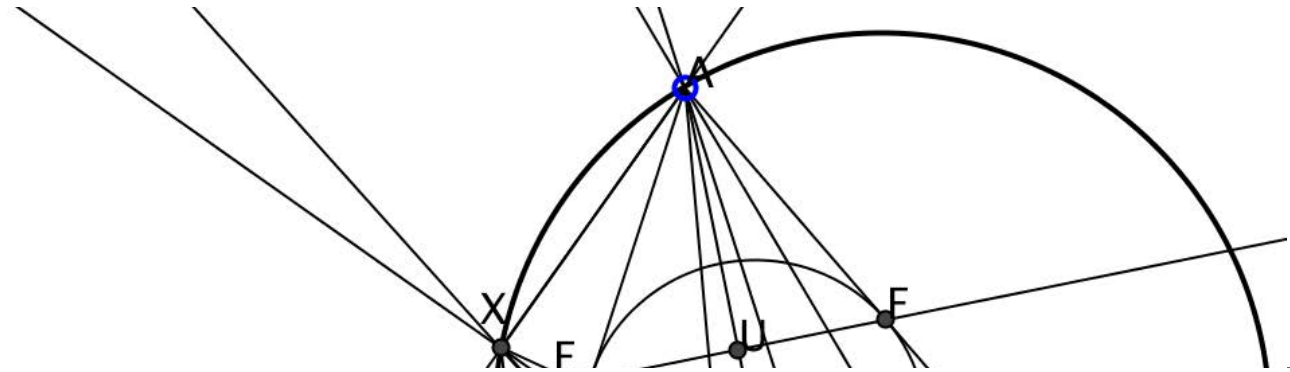
WLOG, we assume that $AB < AC$. (I) touches AB, AC at E, F, respectively. $EF \cap BC = T$, $AX \cap BC = K$. We have $(TD, BC) = -1$.

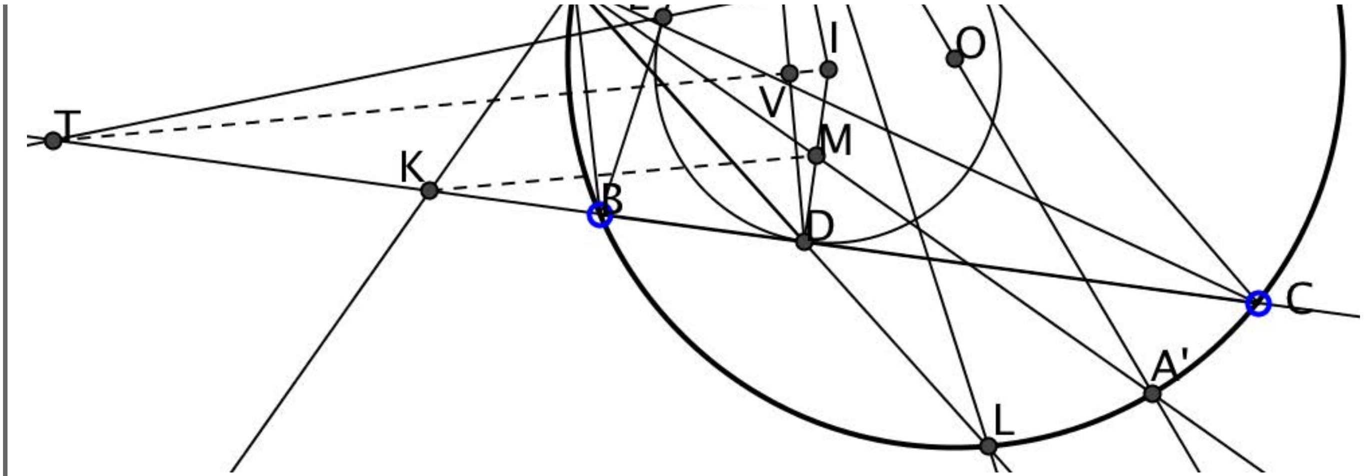
First, we will prove that $TI \perp AD$. $AI \cap EF = U$. A perpendicular line from I to AD intersects EF, AD at T', V, respectively. We have $IE^2 = IU \cdot IA = IV \cdot IT' = ID^2$. From here we obtain that $ID \perp T'D$ or $T \equiv T'$. Therefore, $TI \perp AD$.

Since AL, AD are isogonal lines in $\angle BAC$, it is easy to prove $\angle XAD = \angle XDK$ so $\triangle KDX \sim \triangle KAD$ (A.A). We get $KD^2 = KX \cdot KA = KB \cdot KC$. Also note that $(TD, BC) = -1$ so K is the midpoint of TD. Hence, $TI \parallel KM$ or $KM \perp AD$. We obtain $\angle KMD = \angle KDA$. On the other hand, we have $\angle KXD = \angle KDA$ so $\angle KXD = \angle KMD$. Thus, $KXMD$ are cyclic. Since $MD \perp KD$ so $MX \perp AX$.

Back to the problem, for convenience, let $Y = A'M \cap \Gamma$. We will prove $\triangle AYD$ is tangent to BC. We denote point X like the above problem. Then we get $X = A'M \cap \Gamma$. Thus, $X \equiv Y$. Therefore $KD^2 = KY^2 \cdot KA$ or $\triangle AYD$ is tangent to BC.

Attachments:





Aiscrim

#4 Jan 11, 2016, 9:31 pm

Let E, F be the tangency points of ω with AC, AB and let $EF \cap BC = \{R\}$. Let T be the midpoint of RD, $\{X'\} = TA \cap \Gamma$ and S the projection of I onto AD. Obviously, $I - S - R$ are collinear.

As $(R, D, B, C) = -1$, we have that $TD^2 = TB \cdot TC = TX' \cdot TA$, hence TD is tangent to $(AX'D)$. It is enough to prove that $X = X'$. Note that MT is the perpendicular bisector of DS, hence $TS^2 = TD^2 = TX' \cdot TA$, so TS is tangent to $(AX'S)$. From the two tangencies we get that $\widehat{TDX'} = \widehat{X'AD} = \widehat{X'ST}$, i.e. $X'SDT$ is cyclic. As $SMDT$ is also cyclic, we get that $TX'MD$ is cyclic, hence $\widehat{AX'M} = 90^\circ$, i.e. $X' = X$.

Quick Reply

High School Olympiads

Problem of concentric circles

Locked

equacoesdiofantinas

#1 Jul 12, 2015, 5:16 am

Let Γ_1 and Γ_2 two concentric circles, with Γ_2 within Γ_1. Starting from a point A belonging to Γ_1, it is drawn one tangent AB to $\Gamma_2 (B \in \Gamma_2)$. Let C the second point of intersection of AB with Γ_1, and D the midpoint of AB. A straight through A intersects Γ_2 in E and F such that the DE and CF bisectors intersect at a point M on AC. Determine the ratio $\frac{AM}{MC}$.

Luis González

#2 Jul 12, 2015, 5:36 am

Posted before at the topics Albanian BMO TST 2009 Question 2 and Easy Geometry.

High School Olympiads

Albanian BMO TST 2009 Question 2

Reply

ridgers

#1 Jun 5, 2010, 9:30 pm

Let C_1 and C_2 be concentric circles, with C_2 in the interior of C_1. From a point A on C_1, draw the tangent AB to C_2 $(B \in C_2)$. Let C be the second point of intersection of AB and C_1,and let D be the midpoint of AB. A line passing through A intersects C_2 at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio AM/MC.

This question is taken from Mathematical Olympiad Challenges , the 9-th exercise in 1.3 Power of a Point.

frenchy

#2 Jun 7, 2010, 4:43 pm • 2

A very nice problem.
This is an approach worked out by my friend.
From the power of the point theorem and since D is the middle of AB with $AB = AC$(because $OM \perp AC$ and $OA = OC$) we get that $AB^2 = AE \cdot AF = AD \cdot AC$.
From the above we obtain that the quadrilateral $EDCF$ is cyclic.
So M is the center of circle passing through points E, D, F, C.
So M is the middle of DC, from which $\frac{AM}{MC} = \frac{5}{3}$.
That's it. :)

littletush

#3 Nov 5, 2011, 11:49 am

> frenchy wrote:
>
> A very nice problem.
> This is an approach worked out by my friend.
> From the power of the point theorem and since D is the middle of AB with $AB = AC$(because $OM \perp AC$ and $OA = OC$) we get that $AB^2 = AE \cdot AF = AD \cdot AC$.
> From the above we obtain that the quadrilateral $EDCF$ is cyclic.
> So M is the center of circle passing through points E, D, F, C.
> So M is the middle of DC, from which $\frac{AM}{MC} = \frac{5}{3}$.
> That's it. :)

very nice solution
the key is to find that D, E, F, C are concyclic.
I have second-killed it in the exactly same way~

Cassius

#4 May 15, 2012, 11:51 pm

We have that $AD \cdot AC = AB^2 = AE \cdot AF$ since AD is one quarter of AC, so $DFEC$ is cyclic; by definition the center of its circumscribed circle is M, which is also therefore the midpoint of DC. So $\frac{AM}{MC} = \frac{5}{3}$.

Quick Reply

High School Olympiads

Easy Geometry

Reply

abhinavzandubalm

#1 Feb 10, 2011, 12:23 pm

Let ω_1, ω_2 Be Two Concentric Circles With Radii $r_1 < r_2$.
Let AB Be A Cord Of ω_2 Tangent To ω_1 At C.
Let Midpoint Of AC Be D.
Let Another Chord Of ω_2 Be AE Which Is A Secant Of ω_1 At P And Q(P Closer To A).
Let The Perpendicular Bisectors Of DP And BQ Meet At A Point On AB, Say F.
Find Ratio $\frac{AF}{FB}$.

Request

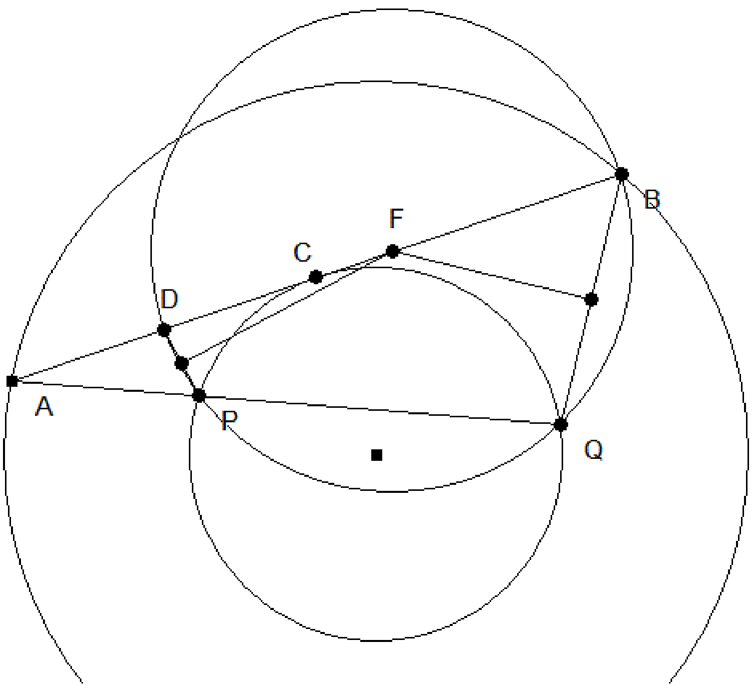
sunken rock

#2 Feb 10, 2011, 6:06 pm

$\frac{5}{3}$.
See attached drawing ($BDPQ$ - cyclic, as $AD \cdot AB = AC^2 = AP \cdot AQ$).

Best regards,
sunken rock

Attachments:



Quick Reply

High School Olympiads

Concyclic incentres and excentres

Source: Indian Team Selection Test 2015 Day 1 Problem 1

hajimbrak
#1 Jul 11, 2015, 4:43 pm

Let $ABCD$ be a convex quadrilateral and let the diagonals AC and BD intersect at O. Let I_1, I_2, I_3, I_4 be respectively the incentres of triangles AOB, BOC, COD, DOA. Let J_1, J_2, J_3, J_4 be respectively the excentres of triangles AOB, BOC, COD, DOA opposite O. Show that I_1, I_2, I_3, I_4 lie on a circle if and only if J_1, J_2, J_3, J_4 lie on a circle.

Luis González
#2 Jul 12, 2015, 5:19 am • 2

Obviously I_1, J_1, I_3, J_3 are collinear on the internal bisector of $\angle AOB$ and I_2, J_2, I_4, J_4 are collinear on the internal bisector of $\angle BOC$. In any $\triangle OAB$ we have the well-known identity $OA \cdot OB = OI_1 \cdot OJ_1$ and similarly $OC \cdot OD = OI_3 \cdot OJ_3$ $\Longrightarrow OI_1 \cdot OI_3 \cdot OJ_1 \cdot OJ_3 = OA \cdot OB \cdot OC \cdot OD$ and similarly we obtain $OI_2 \cdot OI_4 \cdot OJ_2 \cdot OJ_4 = OA \cdot OB \cdot OC \cdot OD$. Thus from these relations, we conclude that I_1, I_2, I_3, I_4 are concyclic $\Longleftrightarrow OI_1 \cdot OI_3 = OI_2 \cdot OI_4 \Longleftrightarrow OJ_1 \cdot OJ_3 = OJ_2 \cdot JI_4 \Longleftrightarrow J_1, J_2, J_3, J_4$ are concyclic.

High School Olympiads

Midpoints of arcs reflected on sides

Reply

Source: IMOTC 2015 Practice Test 2 Problem 1

hajimbrak

#1 Jul 11, 2015, 4:28 pm

Let ABC be a triangle in which $CA > BC > AB$. Let H be its orthocentre and O its circumcentre. Let D and E be respectively the midpoints of the arc AB not containing C and arc AC not containing B. Let D' and E' be respectively the reflections of D in AB and E in AC. Prove that O, H, D', E' lie on a circle if and only if A, D', E' are collinear.

This post has been edited 1 time. Last edited by hajimbrak, Jul 11, 2015, 4:29 pm
Reason: Added Source

Luis González

#2 Jul 12, 2015, 3:43 am

Since $\angle BAD' = \angle BAD = \tfrac{1}{2}\angle ACB$ and $\angle CAE' = \angle CAE = \tfrac{1}{2}\angle ABC$, we get A, D', E' are collinear $\Longleftrightarrow$ $\angle BAD' + \angle CAE' = \angle BAC \Longleftrightarrow \angle ACB + \angle ABC = 2\angle BAC \Longleftrightarrow \angle BAC = 60^\circ$.

Let I, N, Na denote the incenter, 9-point center and Nagel point of $\triangle ABC$. By Furhmann theorem, D' and E' lie on the circle with diameter $\overline{HNa}$ for any $\triangle ABC$, hence O, H, D', E' are concyclic $\Longleftrightarrow \angle HONa = 90^\circ$. As I, N are the Nagel point and circumcenter of the medial triangle, then $ONa \parallel IN$, so $\angle HONa = 90^\circ \Longleftrightarrow IN \perp OH \Longleftrightarrow IH = IO \Longleftrightarrow$ one of the angles of $\triangle ABC$ equals 60° (well-known).

From the previous discussion we get: If D', E' are collinear, then $\angle BAC = 60^\circ \Longrightarrow AH = AO \Longrightarrow IN \perp OH \Longrightarrow$ $\angle HONa = 90^\circ \Longrightarrow O, H, D', E'$ are concyclic. But the converse is not necessarily true; for instance the concyclity yields $IH = IO \Longrightarrow$ one of the angles of $\triangle ABC$ equals 60°, say $\angle ACB = 60^\circ \Longrightarrow D' \equiv O \Longrightarrow D'E'$ is perpendicular bisector of $AC \Longrightarrow A \notin D'E'$.

leeky

#3 Jul 12, 2015, 6:32 am

See http://www.artofproblemsolving.com/community/c6h567389p3324618 :)

Quick Reply

High School Olympiads

cyclic quadrilateral

Reply

Source: Own

andria

#1 Jul 11, 2015, 1:31 pm

in $\triangle ABC$ let X be the center of spiral similarity that takes B to A and A to C. let O be the circumcenter of $\triangle ABC.M, N, P$ are midpoints of BC, CA, AB respectively. let $OM \cap NP = S$ and $AS \cap \odot(\triangle BSC) = T$ and A' is antipode of A WRT $\triangle ABC$. prove that $AXA'T$ is cyclic.

Luis González

#2 Jul 12, 2015, 12:59 am • 1

Note that X is none other than the midpoint of the A-symmedian chord cut by the circumcircle. Thus using inversion with center A and power $\frac{1}{2}AB \cdot AC$ followed by symmetry WRT the angle bisector of $\angle BAC$, the problem becomes: Circle Ω passes through the midpoints M, P of AC, AB and touches the circumcircle (O) internally at S ($S \neq A$). AS cuts Ω again at T. If X, A' are the midpoints of BC and the A-altitude, then T, A', X are collinear.

The tangents of (O) at A and S and MP concur at the radical center R of $(O), \Omega, \odot(AMP)$. AO cuts MP at U and SU cuts Ω again at L. Since U is on radical axis MP of Ω and $\odot(AMOP)$, then $ASOL$ is cyclic $\Longrightarrow R, A, L, S$ are concyclic. Inversion with center R and power $RA^2 = RS^2$ swaps A', U and fixes A, S, thus AS is the inverse of $\odot(ARS)$ $\Longrightarrow L$ is the inverse of T and since S, U, L are collinear, then S, A', T, R are concyclic.

Let D be the foot of the A-altitude and DS cuts (O) again at E. According to IMO Shortlist 2011, G4, SD goes through the centroid of $\triangle ABC$, thus from the homotethy $(G, -2)$, we get $\angle XA'M = \angle DXA' = \angle DEA \equiv \angle SEA = \angle TSR = \angle TA'R \Longrightarrow T, A', X$ are collinear.

TelvCohl

#4 Jul 12, 2015, 3:08 pm • 1

Another solution (without inversion) :

Let $R \in \odot(ABC)$ be the point such that $AR \parallel BC$ and H be the projection of A on BC .
Let the tangent of $\odot(ABC)$ through A cuts BC at Y and $E \equiv AS \cap BC, F \equiv AS \cap \odot(ABC)$.
Let $Z \in \odot(ABC)$ be the point such that $ZF \parallel BC$ (i.e. AZ is the isogonal conjugate of AS WRT $\angle BAC$) .

Since X is the projection of O on A-symmedian ℓ_A of $\triangle ABC$ (well-knwon) ,
so we get OX passes through the pole Y of ℓ_A WRT $\odot(ABC)$ and $X \in \odot(OBC)$.

Since H, E are symmetry WRT the midpoint of BC ,
so $\angle BRH = \angle CAE = \angle BAZ = \angle BRZ \Longrightarrow H, R, Z$ are collinear ,
hence $\angle HZA = \angle RZA = \angle CBA - \angle ACB = \angle HYA \Longrightarrow A, H, Y, Z$ are concyclic ,
so A, H, X, Y, Z lie on a circle with diameter $AY \Longrightarrow \angle AZY = 90^\circ \Longrightarrow A', Y, Z$ are collinear .

From $YX \cdot YO = YB \cdot YC = YZ \cdot YA' \Longrightarrow A', O, X, Z$ are concyclic ,
so combine $OZ = OA' \Longrightarrow XO$ is the external bisector of $\angle ZXA'$ . ... $(\star)$

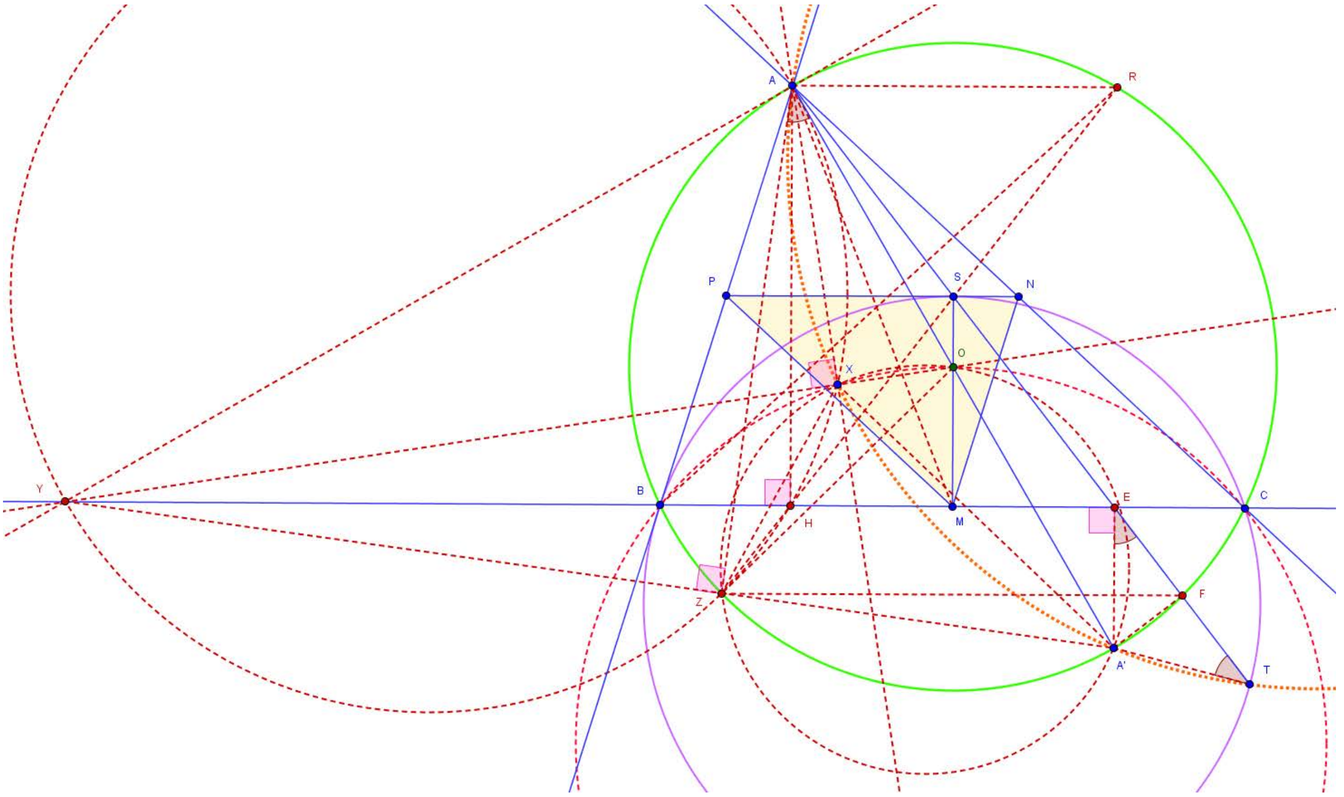
From $EA \cdot EF = EB \cdot EC = ES \cdot ET \Longrightarrow F$ is the midpoint of ET ,
so combine $A'F \perp ET \Longrightarrow A'F$ is the perpendicular bisector of $ET \Longrightarrow \angle ATA' = \angle A'ET$,
hence notice $A'E \perp BC$ and $(\star)$ we get $\angle ATA' = \angle HAE = \angle ZAA' = \frac{1}{2}\angle ZOA' = \frac{1}{2}\angle ZXA' = 180^\circ - \angle A'XA$.

i.e. A, A', T, X are concyclic

Q.E.D

Attachments:

Attachments:



Quick Reply

High School Olympiads

Geometry Problem

Locked

ThE-dArK-lOrD

#1 Jul 10, 2015, 11:56 pm

Given triangle ABC with circumcenter O and orthocenter H
Proof that there exists points D, E, F such that
$(1)\, D \in BC, E \in AC, F \in AB$
$(2)\, OD + DH = OE + EH = OF + FH$
$(3)\, AD, BE, CF$ concurrent

Luis González

#2 Jul 11, 2015, 12:10 am • 1

This is ISL 2000 G3 posted many times before. See the topics Lines AD, BE, and CF are concurrent, ellipse tangent to triangle and elsewhere.

High School Olympiads

Lines AD, BE, and CF are concurrent

Source: IMO Shortlist 2000, G3

orl

#1 Aug 10, 2008, 7:24 am

Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Show that there exist points D, E, and F on sides BC, CA, and AB respectively such that

$$OD + DH = OE + EH = OF + FH$$

and the lines AD, BE, and CF are concurrent.

mr.danh

#2 Aug 11, 2008, 3:09 pm

Solution

Valentin Vornicu

#3 Aug 11, 2009, 6:01 am • 2

We know that the orthocenter reflects over the sides of the triangle on the circumcircle. Therefore the minimal distance $OD + HD$ equals R. Obviously we can achieve this on all sides, so we assume that D, E, F are the intersection points between A', B', C' the reflections of H across BC, CA, AB respectively. All we have to prove is that AD, BE and CF are concurrent.

In order to do that we need the ratios $\frac{BD}{DC}$, $\frac{CE}{EA}$ and $\frac{AF}{FB}$, and then we can apply Ceva's theorem.

We know that the triangle ABC is acute, so $\angle BAH = 90^\circ - \angle B = \angle OAC$, therefore $\angle HAO = |\angle A - 2(90^\circ - \angle B)| = |\angle B - \angle C|$. In particular this means that $\angle OA'H = |\angle B - \angle C|$. Since $\angle BA'A = \angle C$ and $\angle AA'C = \angle B$, we have that $\angle BA'D = \angle B$ and $\angle DA'C = \angle C$.

By the Sine theorem in the triangles $BA'D$ and $DA'C$, we get

$$\frac{BD}{DC} = \frac{\sin B}{\sin C}.$$

Using the similar relationships for $\frac{CE}{EA}$ and $\frac{AF}{FB}$ we get that those three fractions multiply up to 1, and thus by Ceva's, the lines AD, BE and CF are concurrent.

Zhero

#4 Apr 27, 2010, 12:28 pm • 2

Lemma 1: If an ellipse is inscribed triangle ABC, tangent to BC, AC, and AB at D, E, and F, respectively, then AD, BE, and CF are concurrent.
Proof: Scale the ellipse about its major axis so that it becomes a circle; here, $A'D'$, $B'E'$, and $C'F'$ concur at the Gergonne point of $A'B'C'$. Scaling preserves incidence, so AD, BE, and CF concur.

Lemma 2: Let ℓ be any line, and let X and Y be any points on the same side of ℓ. Let Z be the reflection of X across ℓ, and let W be the intersection of YZ and ℓ. Then the ellipse with foci X and Y that passes through W is tangent to ℓ.
Proof: Suppose that the ellipse intersected line ℓ at another point, W'. Then $XW' + YW' = YW' + ZW' > YZ$ by the

triangle inequality, since W', Y, and Z are not collinear. On the other hand, by definition of ellipse, $XW' + YW' = XW + YW = ZW + YW = YZ$, so $YZ > YZ$, which is a contradiction.

By lemma 1, it is sufficient to show that there exists an ellipse with foci O and H that are tangent to the sides BC, AC, and AB at D, E, and F, respectively.

Reflect H across BC, CA, and AC to get A', B', and C', respectively, and let OA' hit BC at D, OB' hit CA at E, and OC' hit AB at F. . A', B', C' lie on the circumcircle of ABC, so $R = OA' = OB' = OC' = HD + DO = HE + EO = HF + FO$, where R is the circumradius of $\triangle ABC$ (since A', B', C' are the reflections of H across the sides of the triangle.)

Consider the ellipse with foci O and H with a major axis of length R. By definition, the ellipse passes through D, E, and F. However, by lemma 2, it is tangent to sides BC, CA, and AB, so by lemma 1, we are done.

Some motivation

Rofler

#5 May 2, 2010, 12:14 pm

@Zhero: Instead of scaling, it is more elegant to just use Brainchon's theorem.

Cheers,

Rofler

Zhero

#6 May 4, 2010, 6:11 pm

Probably. I believe we could also just centrally project it to a circle as well (my first approach). I chose to scale because I felt it was the most elementary solution; anyone who knows anything about ellipses should be able to understand the solution I gave.

goodar2006

#7 Aug 12, 2010, 9:03 pm • 2

since H and O are two isogonal conjugate points, there exists an ellipse which these two points are its foci and its tangent to sides of the triangle.

StefanS

#8 Jul 13, 2011, 5:57 am

> *Valentin Vornicu wrote:*
> By the Sine theorem in the triangles $BA'D$ and $DA'C$, we get
> $$\frac{BD}{DC} = \frac{\sin B}{\sin C}.$$

By the law of sines for triangles $\triangle BA'D$ and $\triangle DA'C$ we get:

$$\frac{BD}{\sin B} = \frac{A'D}{\sin \angle DBA'} \quad \wedge \quad \frac{DC}{\sin C} = \frac{A'D}{\sin \angle DCA'}$$

So for:

$$\frac{BD}{DC} = \frac{\sin B}{\sin C}.$$

to be true, the following:

$$\sin \angle DBA' = \sin \angle DCA'$$

should be true as well. I don't think that's correct.

should be true as well. I don't think that's correct.

Virgil Nicula
#9 Jul 13, 2011, 9:25 am

You are right. See **here** **PP8**.

StefanS
#10 Jul 13, 2011, 9:33 am

> *Virgil Nicula wrote:*
> You are right. See **here** **PP8**.

Wow you had a math blog!!! I unfortunately can't read the entire problems. Only half of the picture is displayed. Do you know why?

Virgil Nicula
#11 Jul 13, 2011, 9:37 am

See click on the title of message.

exmath89
#12 Jul 9, 2013, 5:06 am

Solution

IDMasterz
#13 Jul 9, 2013, 8:38 am • 1

Sol

sayantanchakraborty
#14 Apr 4, 2014, 6:51 pm

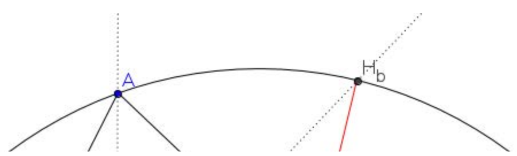
> *orl wrote:*
> Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Show that there exist points D, E, and F on sides BC, CA, and AB respectively such that
> $$OD + DH = OE + EH = OF + FH$$
> and the lines AD, BE, and CF are concurrent.

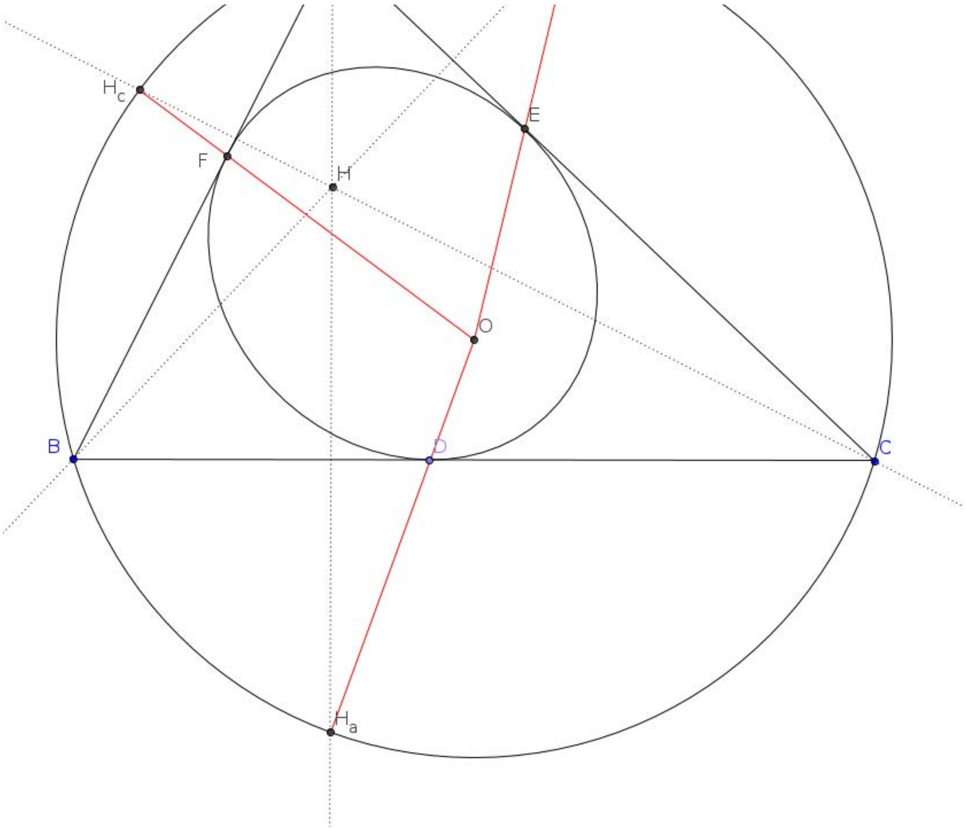
This problem also appeared in one of the Indian TSTs!!!

bonciocatciprian
#15 Jun 16, 2014, 4:51 pm

Take H_a, H_b, H_c be the mirror images of H about BC, AC, and AB respectively. They lie on the circumcircle of ΔABC. Now, take $\{D\} = OH_a \cap BC$, $\{E\} = OH_b \cap AC$ and $\{F\} = OH_c \cap AB$. Now, using Pool's Theorem, we get that $OD + DH = OE + EH = OF + FH$. Moreover, in this case the sum $OD + DH$ is minimal, so the points D, E, F are situated on an ellipse of foci O and H, having the constant sum equal to R (the ray of the circumcircle of ΔABC), and which is tangent to the edges of the triangle. Now, this is just Gergonne's theorem, under a projective transformation.

Attachments:





guldam

#16 Jan 13, 2015, 9:58 pm

Here is a bashing proof.

A', B', C' are the reflection point of H with respect to BC, CA, AB respectively
D, E, F are the intersection point of OA', OB', OC' with BC, CA, AB respectively

Let

$$S_A := (b^2 + c^2 - a^2)[b^2(c^2 + a^2 - b^2)^2 + c^2(a^2 + b^2 - c^2)^2]$$
$$S_B := (c^2 + a^2 - b^2)[c^2(a^2 + b^2 - c^2)^2 + a^2(b^2 + c^2 - a^2)^2]$$
$$S_C := (a^2 + b^2 - c^2)[a^2(b^2 + c^2 - a^2)^2 + b^2(c^2 + a^2 - b^2)^2]$$

One can calculate the barycentric cooridnates of $D, E, , F$ respectively to be

$$D = (0 : \frac{1}{S_B} : \frac{1}{S_C})$$
$$E = (\frac{1}{S_A} : 0 : \frac{1}{S_C})$$
$$F = (\frac{1}{S_A} : \frac{1}{S_B} : 0)$$

It is straightforward to see that AD, BE, CF are concurrent.

Quick Reply

High School Olympiads

ellipse tangent to triangle

Reply

Source: question

maky

#1 Apr 4, 2007, 3:52 pm

i thought about this when i saw an isl problem (2000) i think, which said
"prove that there exist points D, E, F on sides BC, CA, AB of triangle $\triangle ABC$ such that AD, BE, CF are concurrent and $OD + HD = OE + HE = OF + HF$".
well, there exists an ellipse with foci O and H that is tangent to BC, CA, AB. let the tangency points be D, E, F, the hexagon (degenerate) formed by A, B, C, D, E, F has a circle inscribed in it, so by brianchon theorem AD, BE, CF are concurrent.

this was more the "introduction". O and H are isogonal conjugates...my question is : is the question still good for P and Q isogonal conjugates ? that is, does there exists an ellipse with foci P and Q, that is tangent to the sides of the triangle ?
(another interesting case is I, which is it`s own isogonal conjugate, and the ellipse is a circle in that case - the inscribed circle).

goodar2006

#2 Aug 14, 2010, 7:35 pm

the statement is true for every two isogonal conjugate points P and Q.

Luis González

#3 Aug 14, 2010, 11:31 pm

> maky wrote:
>
> my question is: is the question still good for P and Q isogonal conjugates? that is, does there exists an ellipse with foci P and Q, that is tangent to the sides of the triangle ?

According to the location of P in $\triangle ABC$, the "shape" of the inconic varies.

We assume that point P lies inside $\triangle ABC$, so is its isogonal conjugate Q. The remaining case is treated analogously. Let P_1, P_2, P_3 and Q_1, Q_2, Q_3 be the perpendicular feet of P, Q on sidelines BC, CA, AB. It's well-known that $P_1, P_2, P_3, Q_1, Q_2, Q_3$ lie on a circle centered at the midpoint K of PQ, thus let ϱ be its radius. Let Q_0 be the reflection of Q across BC and $A_0 \equiv PQ_0 \cap BC$. Then BC is external bisector of $\angle PA_0Q$ and since K, Q_1 are midpoints of PQ, QQ_0 we get $A_0P + A_0Q = A_0P + A_0Q = PQ_0 = 2\varrho$. Similarly, there exists B_0, C_0 on CA, AB such that CA, AB are external bisectors of $\angle PB_0Q, \angle PC_0Q$, respectively and whose sum of distances to P, Q equals $2\varrho \Longrightarrow$ there exists an inellipse $\mathcal{K}$ with foci P, Q and pedal circle (K, ϱ). When P is outside $\triangle ABC, \mathcal{K}$ becomes hyperbola, or parabola if P lies on $\odot(ABC)$, since Q is at infinity. Conversely, foci of a conic $\mathcal{K}$ are isogonal conjugates with respect to any triangle bounded by three tangent lines of $\mathcal{K}$.

Quick Reply

High School Olympiads

cevian conic

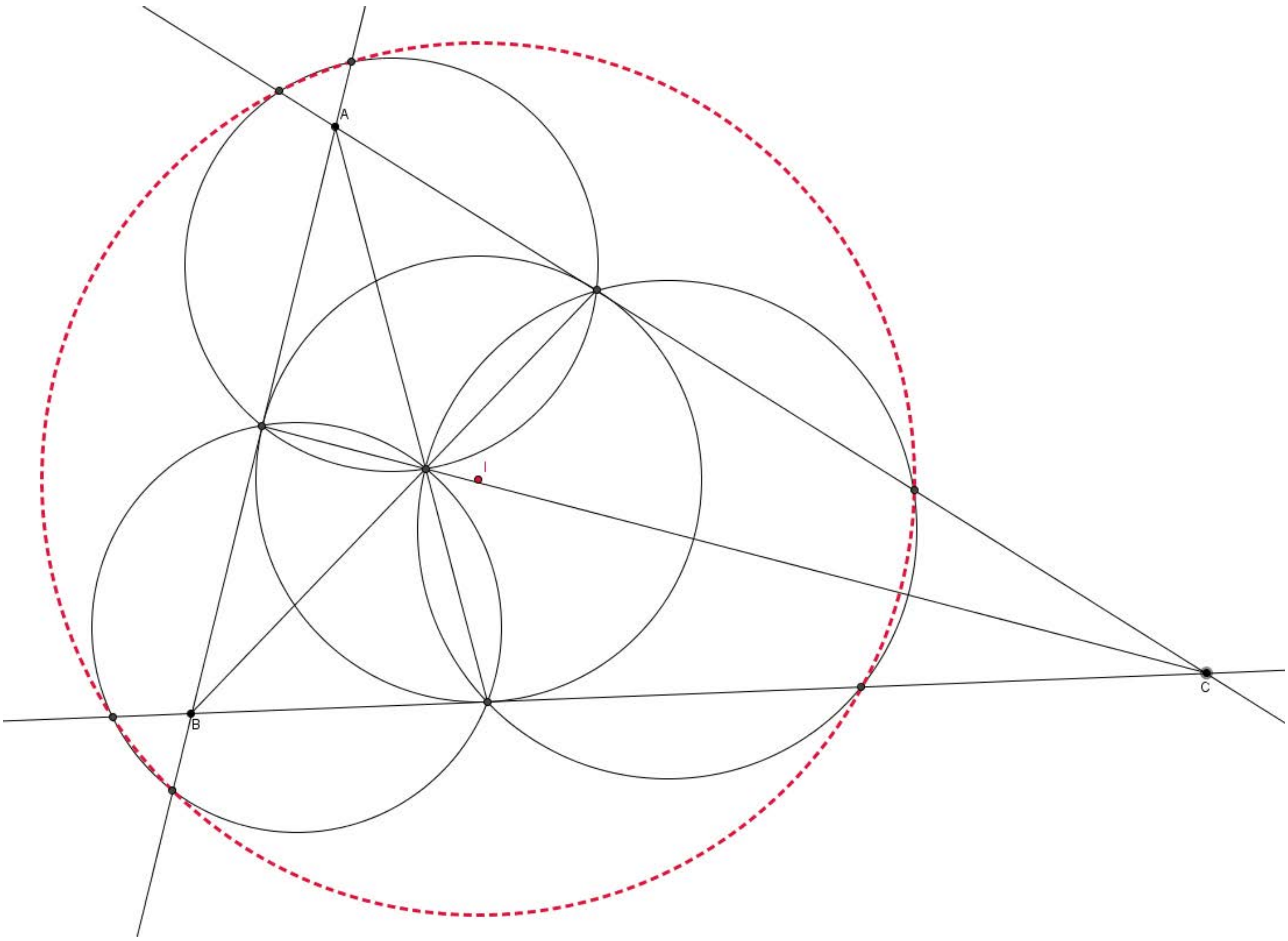
Reply

andria

#1 Jul 9, 2015, 4:30 pm

Prove that in $\triangle ABC$ the cevian conic of gergonne point is a circle with center I where I is incenter of triangle ABC.

Attachments:



This post has been edited 1 time. Last edited by andria, Jul 9, 2015, 6:41 pm
Reason: add picture

Luis González

#2 Jul 10, 2015, 12:56 am • 1

Let D,E,F denote the tangency points of the incircle (I) with BC,CA,AB and $Ge \equiv AD \cap BE \cap CF$ is the Gergonne point. $\odot(GeEF)$ cuts AC,AB again at A_b,A_c and $\{B_c,B_a\},\{C_a,C_b\}$ are defined similarly.

Since $AE \cdot AC_b = AGe \cdot AD = AF \cdot AB_c$, then EFB_cC_b is cyclic, even isosceles trapezoid with bases $EF \parallel B_cC_b$. Since EFA_cA_b is obviously an isosceles trapezoid with bases $EF \parallel A_bA_c$, then it follows that $A_bA_cB_cC_b$ is an isosceles trapezoid, i.e. A_b,A_c,B_c,C_b are concyclic. Analogously B_a,B_c,C_a,A_c and C_a,C_b,A_b,B_a are concyclic. Thus, we deduce that A_b,A_c,B_c,B_a,C_a,C_b lie on a same circle ω. As A_bA_cFE is isosceles trapezoid, the perpendicular bisector of EF is also perpendicular bisector of A_bA_c and similarly for B_cB_a and $C_aC_b \Longrightarrow I$ is the center of ω.

Quick Reply

High School Olympiads

tangential quadrilateral

Reply

Source: Own

andria

#1 Jul 9, 2015, 2:07 pm

Consider triangle ABC. An arbitrary conic with focus $\{B, C\}$ cut segments AB, AC at C_1, B_1 respectively. Let $BB_1 \cap CC_1 = T$ prove that quadrilateral AB_1TC_1 is tangential.

Luis González

#2 Jul 9, 2015, 11:02 pm

WLOG we assume that the object conic is an ellipse $\mathcal{E}$. The case when the conic is hyperbola is approached analogously. Tangents of $\mathcal{E}$ at B_1, C_1 meet at I. By conic tangent properties, it follows that IB_1, IC_1 bisect $\angle BB_1C, \angle CC_1B$ externally and BI, CI bisect $\angle B_1BC_1, \angle B_1CC_1 \Longrightarrow I$ is B-excenter of $\triangle TBC_1 \Longrightarrow TI$ bisects $\angle B_1TC_1 \Longrightarrow AB_1TC_1$ is tangential with incenter I.

Quick Reply

High School Olympiads

X57 lies on the radical axis

Reply

Source: Own

andria

#1 Jul 9, 2015, 10:07 pm

In $\triangle ABC$ let Ω_a be the circle that passes throw B, C and it is tangent to the incircle of $\triangle ABC$ at A'. The incircle touches BC at A''. We define B', B'', C', C' similarly.
1) prove that $\odot(\triangle AA'A''), \odot(\triangle BB'B''), \odot(\triangle CC'C'')$ are coaxal.
2) let ℓ be the radical axis of triangles $AA'A'', BB'B'', CC'C''$. Prove that X_{57} lies on ℓ.

Luis González

#2 Jul 9, 2015, 10:42 pm

This is a particular case of the problem pure geometry-coaxality. If τ is the radical axis of the incircle (I) and the circumcircle (O), then ℓ passes through the pole X_{57} of τ WRT (I).

Quick Reply

High School Olympiads

pure geometry-coaxality

Reply

kocabey44

#1 Jun 26, 2014, 1:38 am

Let $\triangle ABC$ be a triangle and ω be a random circle on its plane. Consider points A_1 and A_2 on ω such that circumcircles of triangles $\triangle BCA_1$ and $\triangle BCA_2$ are tangent to ω. B_1, B_2, C_1, C_2 are similarly defined. Prove that circumcircles of triangles $AA_1A_2, BB_1B_2, CC_1C_2$ are coaxal.

Luis González

#2 Jul 3, 2014, 10:25 am • 2

Tangents of ω at A_1, A_2 and BC concur at the common radical center X of $\omega, \odot(BCA_1), \odot(BCA_2)$ and the circumcircle $(O) \equiv \odot(ABC)$. Y and Z are defined similarly $\Longrightarrow \tau \equiv \overline{XYZ}$ is the radical axis of $(O), \omega$. Thus A_1A_2, B_1B_2, C_1C_2 are the polars of X, Y, Z WRT ω, concurring at the pole P of τ WRT ω.

On the other hand, let $\odot(AA_1A_2)$ cut (O) again at A_3 and B_3, C_3 are defined similarly. Then AA_3, A_1A_2, τ concur at the radical center A_0 of $(O), \omega$ and $\odot(AA_1A_2)$. B_0, C_0 are defined similarly. If $Q \equiv BB_0 \cap CC_0$, then by Desargues theorem, τ cuts the opposite sidelines of $ABQC$ at pairs of points in involution, but the application $\{X, Y, Z\} \mapsto \{A_0, B_0, C_0\}$ is indeed involutive, forcing $A_0 \in AQ$. Consequently $\odot(AA_1A_2), \odot(BB_1B_2)$ and $\odot(CC_1C_2)$ are coaxal with common radical axis PQ.

P.S. See also Triad of coaxal circles coming from an arbitrary circle.

Quick Reply

High School Olympiads

3 circle have a common radical axis

Reply

Scorpion.k48

#1 Jul 9, 2015, 8:59 am

Let triangle ABC with orthocentre H and a point P. d be a line passes through P and perpendicular to HP. d intersects BC, AH, CA, BH, AB, CH at $A_1, A_2, B_1, B_2, C_1, C_2$ respectively. Prove that $(HA_1A_2), (HB_1B_2), (HC_1C_2)$ are coaxal .

This post has been edited 2 times. Last edited by Scorpion.k48, Jul 9, 2015, 12:19 pm

Phie11

#2 Jul 9, 2015, 9:18 am

What do you mean have a common radical axis?

buratinogigle

#3 Jul 9, 2015, 9:23 am

This problem does not need orthocenter H.

Let ABC be a triangle and P, Q are two any points. The line passes through P and perpendicular to PQ which cut BC, AQ, CA, BQ, AB, CQ at $A_1, A_2, B_1, B_2, C_1, C_2$. Prove that circles $(QA_1A_2), (QB_1B_2), (QC_1C_2)$ are coaxal.

Luis González

#4 Jul 9, 2015, 9:53 am • 1

> *buratinogigle wrote:*
> This problem does not need orthocenter H.
>
> Let ABC be a triangle and P, Q are two any points. The line passes through P and perpendicular to PQ which cut BC, AQ, CA, BQ, AB, CQ at $A_1, A_2, B_1, B_2, C_1, C_2$. Prove that circles $(QA_1A_2), (QB_1B_2), (QC_1C_2)$ are coaxal.

Further, we don't need a line perpendicular to PQ as any line ℓ does the work. By Desargues involution theorem, it follows that ℓ cuts the opposite sidelines of the quadrangle $ABQC$ at pairs of points in involution, namely $(A_1, A_2), (B_1, B_2), (C_1, C_2)$. Thus $\odot(QA_1A_2), \odot(QB_1B_2), \odot(QC_1C_2)$ are coaxal for any Q in the plane.

Quick Reply

High School Olympiads

Three common tangents are concurrent

Reply

Source: Own

buratinogigle

#1 Jul 9, 2015, 8:28 am • 1

Let ABC be a triangle inscribed in circle (O). P is an any point. Circle (K_a) is tangent to segments PB,PC and tangent to (O) internally. Similarly, we have circles $(K_b),(K_c)$. Let d_a be the inner common tangent orther than PA of $(K_b),(K_c)$. Similarly, we have lines d_b,d_c. Prove that d_a,d_b,d_c are concurrent.

Phie11

#2 Jul 9, 2015, 8:35 am

Very nice problem. Did you Geogebra it? Also, can P be outside of ABC?

Luis González

#3 Jul 9, 2015, 9:33 am • 2

Let d_a,d_b,d_c cut K_bK_c,K_cK_a,K_aK_b at $X,Y,Z,$ respectively and let $Q \equiv d_b \cap d_c.$ Since the quadrangle $PYQZ$ is circumscribed to $(K_a),$ we have $YP+YQ=ZP+ZQ$ and since K_aK_b,K_aK_c bisect $\angle PZQ$ and $\angle PYQ$ externally, then there is an ellipse $\mathcal{E}$ with foci P,Q tangent to K_aK_b,K_aK_c at $Z,Y.$ As $\mathcal{E}$ is the unique inconic of $\triangle K_aK_bK_c$ with focus $P,$ then Q is isogonal conjugate of P WRT $\triangle K_aK_bK_c.$ Thus analogously we have $Q \in d_a \Longrightarrow d_a,d_b,d_c$ concur at the isogonal conjugate Q of P WRT $\triangle K_aK_bK_c.$

Quick Reply

High School Olympiads

concurrent lines

Reply

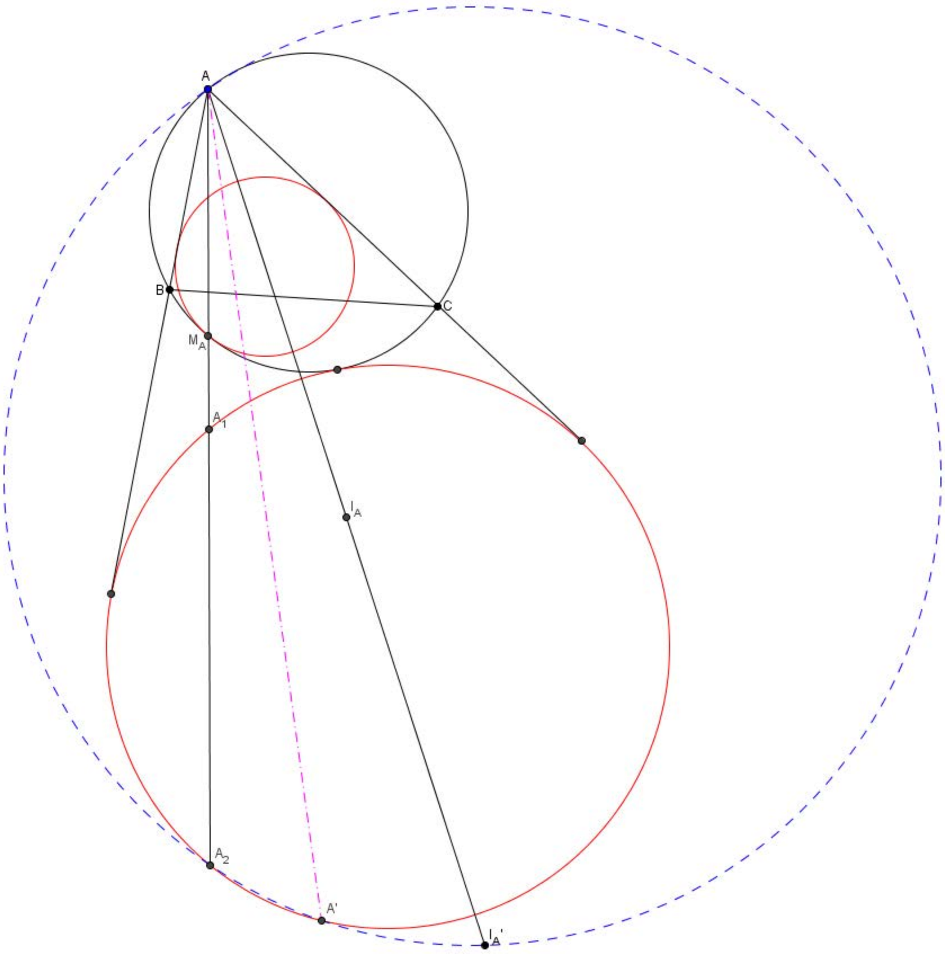
Source: Own

andria

#1 Jun 23, 2015, 2:22 pm

In $\triangle ABC$ let M_A be the tangency point of A-mixtilinear incircle with $\odot(\triangle ABC)$ and I_A is the center of A-excircle let I'_A be the reflection of A WRT I_A and let AM_A intersect A-mixtilinear excircle at A_1, A_2(A_1 is between A, A_2) let $\odot(\triangle AI'_A A_2)$ intersect A-mixtilinear excircle again at $A' \neq A_2$ we difine B', C' similarly prove that AA', BB', CC' are concurrent.

Attachments:



andria

#2 Jun 24, 2015, 12:56 pm

hint

Luis González

#3 Jul 9, 2015, 3:12 am • 1

Consider the inversion $(A, \sqrt{AB \cdot AC})$ followed by symmetry across the angle bisector of $\angle BAC$. This swaps B, C and therefore $\odot(ABC)$ and $BC \Longrightarrow$ the A-mixtilinear incircle goes to the A-excircle (I_A) and the A-mixtilinear excircle goes to the incircle (I). Thus A_2 goes to the intersection of the the A-Nagel cevian with (I) closer to A, i.e. the antipode X of the tangency point of (I) with BC. Since I_A goes to I, due to $AI \cdot AI_A = AB \cdot AC$, then it follows that $I_A{}'$ goes to the midpoint J of AI. Thus $\odot(AA_2I_A{}')$ is transformed into $XJ \Longrightarrow XJ$ cuts (I) again at the image of A', which is the Feuerbach point Fe of $\triangle ABC$ (see post #4 at Intersect on circle) $\Longrightarrow AA'$ and AFe are isogonals WRT $\triangle ABC$ and similarly BB', BFe and CC', CFe are pairs of isogonals. Hence AA', BB', CC' concur at the isogonal conjugate of Fe.

Quick Reply

High School Olympiads

Problem 7, Tuymaada, Senior league

Source: Tuymaada 2015,D. Shryyaev

Popescu

#1 Jul 8, 2015, 8:29 pm

Let ABC a triangle with OM=R-r, where M is the midpoint of AB and O is the circumcircle.The external bisctor of A meets BC in D and the external bisector of C meets AB in E.Determine the values of the angle CED.
D. Shiryaev

Luis González

#2 Jul 8, 2015, 11:27 pm

Let X,Y,Z be the tangency points of the incircle (I,r) with BC,CA,AB and let P be the midpoint of the arc BC of the circumcircle (O,R). We have $OM=R-MP \Longrightarrow MP=r$, thus since $\angle PBM=\angle IAY=\frac{1}{2}\angle BAC$, then $\triangle IAY \cong \triangle PBM \Longrightarrow IA=PB=PC=PI$, i.e. I is midpoint of $AP \Longrightarrow OI \perp AI \Longrightarrow$ $\angle CIO=\angle AIC-90^{\circ}=\frac{1}{2}\angle ABC$. Now since $CI \perp CE$ and $OI \perp DE$ (well-known for any triangle ABC), it follows that $\angle CED=\angle CIO=\frac{1}{2}\angle ABC$.

micliva

#3 Jul 13, 2015, 2:22 pm

You swap the points A and C

Aiscrim

#6 Jul 16, 2015, 4:21 pm

Computations ftw

High School Olympiads

Prove that PM=PN

Reply

Source: Own

LeVietAn
#1 Jul 8, 2015, 8:03 am

Dear Mathlinkers,
Given triangle ABC ($\angle BAC = 90°$) is inscribed into circle Γ. The mixtilinear incircle and $A-$mixtilinear excircle respectively are tangent to Γ at D and E. Let M and N respectively are the midpoints of the segments AD and AE. Prove that if P is the midpoint of the arc BAC of Γ then $PM = PN$.

TelvCohl
#2 Jul 8, 2015, 9:15 am • 1

My solution :

Let I, I_a be the incenter, A-excenter of $\triangle ABC$.
Let $X \equiv \odot(I) \cap BC, Y \equiv \odot(I_a) \cap BC, Z \equiv DX \cap \odot(ABC)$.
Let r, r' be the radius of $\odot(I)$, mixtilinear incircle of $\triangle ABC$, respectively .

From $\frac{r'}{r} = \left(\frac{1}{\cos\frac{A}{2}}\right)^2 \Longrightarrow r' = 2r \Longrightarrow M \in \odot(I) \Longrightarrow IM = IX$.

Since $AXYZ$ is an isosceles trapezoid ($AZ \parallel XY$) $\Longrightarrow DX \cdot AY = BX \cdot CX$,

so from $AY \cdot AD = bc \Longrightarrow \frac{DX}{DM} = \frac{(a+c-b)(a+b-c)}{2bc} = 1 \Longrightarrow DM = DX$,

hence DI is the perpendicular bisector of $MX \Longrightarrow PM = PX$ ($\because P \in DI$(well-known)) .

Similarly, we can prove $PN = PY \Longrightarrow PM = PX = PY = PN$ (from symmetry) .

Q.E.D

LeVietAn
#3 Jul 8, 2015, 9:48 am

Thank you **TelvCohl**, in fact we have beautiful results follows:
In triangle ABC ($\angle BAC = 90°$), the incircle touches BC at D; the $A-$excircle touches BC at X, the $B-$excircle touches CA at Y, the $C-$excircle touches AB at Z. The nine point circle is tangent to incircle, $A-$excircle at two Feuerbach point F, F_a. Then the points D, X, Y, Z, F, F_a lie on the same a circle.
see also in: http://artofproblemsolving.com/community/c6h1095833p4909913 and
http://artofproblemsolving.com/community/c6h545219p3152692

This post has been edited 1 time. Last edited by LeVietAn, Jul 8, 2015, 10:15 am

A-B-C
#4 Jul 8, 2015, 9:52 am • 1

Let the incircle (I) tangents to AB, AC, BC at Z, Y, X. IB, IC intersects (ABC) at $B', C' \neq B, C$
A-mixtilinear incircle is tangent to AB, AC at Z', Y' , then Z, Y are midpoints of AZ, AY.
O is midpoint of BC
$\angle BXC = \angle BXI + \angle CXI = \angle B'BC' + \angle B'CC' = 45^o + 45^o = 90^o$
$\angle BOC = 90^o$

Consider homothecy with center A, ratio 2
$Y, Z \rightarrow Y', Z'$
(I) passes through Y, Z and tangent to AB, AC
A-mixtilinear incircle passes through Y', Z' and tangent to AB, AC
$\Rightarrow (I) \rightarrow A$-mixtilinear incircle.
$\Rightarrow M$ lies on (I)
It is well-known that $AB'DC'$ is a harmonic quadrilateral so MA bisects $\angle B'MC'$and M, B', C', O are concyclic.
$\Longrightarrow B', C', O, M, X$ are concyclic.
Since A-mixtilinear incircle tangent to (ABC), M is also the Feuerbach point of $\triangle ABC$.
Similarly, N is tangency point of A-excircle with nine-point circle of $\triangle ABC$
Let Y_1, Z_1, X_1 be reflections of Y, Z, X in midpoints of AC, AB, BC, then P is the circumcenter of $X_1Y_1Z_1$ according to IMO 2013 Problem 3. $\Rightarrow PX_1 = PY_1 = PZ_1 = PX$
Since $X_1Y_1Z_1$ is pedal triangle of Bevan point WRT $\triangle ABC$. Incenter, O and Bevan point are collinear. According to Fontene theorem, Incircle, nine-point circle and $(X_1Y_1Z_1)$ have a common point $\Rightarrow (X_1Y_1Z_1)$ passes through Feuerbach point M
Let V_a be circumcenter of II_bI_c where I, I_b, I_c are incenter and excenter WRT B, C. Since V_a, I_a, O are collinear(O is midpoint of V_aI_a), then pedal circle of V_a WRT $\triangle ABC$ passes through N. Hence, N lies on (Y_2XZ_2)
Y_2, Z_2 are orthogonal projections of V_a on AC, AB. It is quite easy to prove that $AZ_1 = AY_2$, $AY_1 = AZ_2$, so Y_2, Z_2 are reflections of Z_1, Y_1 in I_bI_c, and $V_aI \perp BC$. By reflection, $PY_1 = PZ_1 = PY_2 = PZ_2 = PX = PN$.
$\Longrightarrow PX_1 = PY_1 = PZ_1 = PX = PY_2 = PZ_2 = PM = PN$

This post has been edited 3 times. Last edited by A-B-C, Jul 8, 2015, 9:56 am
Reason: .

TelvCohl
#5 Jul 8, 2015, 10:07 am

> *LeVietAn wrote:*
> In triangle ABC ($\angle BAC = 90^\circ$), the incircle touches BC at D; the $A-$excircle touches BC at X, the $B-$excircle touches CA at Y, the $C-$excircle touches AB at Z. The nine point circle is tangent to incircle, $A-$excircle at two Feuerbach point F, F_a. Then the points D, X, Y, Z, F, F_a lie on the same a circle.

Remark : In general, the tangency points of the incircle , the A-excircle of $\triangle ABC$ with BC and two Feuerbach points Fe, Fa lie on a circle with the center the reflection of the midpoint of arc BC of $\odot(ABC)$ in BC (see HU is perpendicular to AI) .

Luis González
#6 Jul 8, 2015, 12:02 pm

Let the incircle (I) and A-excircle (I_a) touch BC at X, Y and let the tangent of the circumcircle (O) at A cut BC at S. If the parallel from A to BC cuts (O) again at K, then by symmetry $AXYK$ is isosceles trapezoid. Under inversion with center A and power $AB \cdot AC$ followed by symmetry across AI, we have $X \mapsto E, Y \mapsto D$ and $K \mapsto S$, thus since A, X, Y, K are concyclic $\Longrightarrow D, E, S$ are collinear. So if the tangents of (O) at D, E meet at T, then AT is the polar of S WRT $(O) \Longrightarrow AT \perp BC$.

Let U be the antipode of A on (O) and let Q be the midpoint of the arc BUC. Since AD, AE pass through the homothetic centers of $(I) \sim (O)$, it follows that $A(D, E, I, O) = -1 \Longrightarrow A(D, E, Q, U) = -1 \Longrightarrow DUEQ$ is harmonic $\Longrightarrow U, Q, T$ are collinear and Q is midpoint of TU, because of $(OQ \parallel AT) \perp BC$. If TO cuts AP at R, then from $AP \parallel QU$, it follows that P is midpoint of AR. As TO is perpendicular bisector of DE, then $RD = RE$, so from homothety $(A, \frac{1}{2})$, we get $PM = PN$.

Quick Reply

High School Olympiads

Involution and constant

Reply

ricarlos

#1 Jul 8, 2015, 8:27 am

Let $P, P'; Q, Q'; R, R'$ be 3 pairs of points in involution so that R', Q, Q', P' are fixed points and P, Q are variable points between $Q - Q'$.

Prove that $\frac{2PQ\cdot RQ'}{PR} = constant$

Attachments:

Luis González

#2 Jul 8, 2015, 9:30 am • 1

$$\{P,Q,R\}\mapsto\{P',Q',R'\}\Longrightarrow(P,Q,R,Q')=(P',Q',R',Q)\Longrightarrow\frac{PQ}{PR}\cdot\frac{Q'R}{Q'Q}=\frac{P'Q'}{P'R'}\cdot\frac{QR'}{QQ'}$$

$$\Longrightarrow\frac{PQ\cdot RQ'}{PR}=\frac{P'Q'\cdot QR'}{P'R'}=\text{constant}.$$

ricarlos

#3 Jul 9, 2015, 6:49 am

> *ricarlos wrote:*
> Prove that $\frac{2PQ\cdot RQ'}{PR} = constant$

2?

Quick Reply

High School Olympiads

Prove concurrent

Reply

PhuongMath
#1 Jul 8, 2015, 8:46 am

Let triangle ABC and a circle intersect BC, CA, AB at (M;N); (P;Q); (S;T) resp; (M is located between B and N; =P is located between C and Q; S is located between A and T). $SN \cap QM = K; QM \cap TP = H; TP \cap SN = L$.
Prove AK, BH, CL are concurrent

Luis González
#2 Jul 8, 2015, 9:07 am • 1

By Pascal theorem for cyclic hexagon $TPQMNS,$ the intersections $TP \cap MN, QP \cap NS$ and $QM \cap ST$ are collinear, which means that $\triangle ABC$ and the triangle $\triangle KHL$ bounded by TP, NS, QM are perspective. Thus by Desargues theorem AK, BH, CL concur.

Quick Reply

High School Olympiads

orthocenter lie on a line

andria

#1 Jul 7, 2015, 5:24 pm

Consider two circles γ_1, γ_2 with centers O_1, O_2 such that they don't intersect each other. Let ℓ_1, ℓ_2 be common external tangents of γ_1, γ_2 and let $\ell_1 \cap \gamma_1 = A_1, \ell_1 \cap \gamma_2 = B_1, \ell_2 \cap \gamma_1 = A_2, \ell_2 \cap \gamma_2 = B_2$. Consider arbitrary point P on the segment A_1B_1 and let the tangents (other than A_1B_1) from P to γ_1, γ_2 intersect A_2B_2 at C_1, C_2 respectively (C_1 is between A_2, C_2). Let I be the P-excenter of $\triangle PC_1C_2$.
1) prove that orthocenter of $\triangle IO_1O_2$ lies on the line A_2B_2.
2) let S be the tangency point of P-excircle of $\triangle PC_1C_2$ with A_2B_2 and let M midpoint of O_1O_2. Prove that $MP = MS$

Luis González

#2 Jul 8, 2015, 3:37 am • 1

Solution to problem 1)

Let $O \equiv \ell_1 \cap \ell_2$ and WLOG we assume that γ_1 is incircle of $\triangle OPC_1$ and γ_2 is O-excircle of $\triangle OPC_2$. Since $\widehat{O_1PC_1} = \tfrac{1}{2}\angle \widehat{OPC_1}$ and $\widehat{O_2PC_2} = 90^\circ - \tfrac{1}{2}\widehat{OPC_2} \Longrightarrow$ $\widehat{O_1PO_2} = 90^\circ - \tfrac{1}{2}(\angle \widehat{OPC_2} - \widehat{OPC_1}) + \widehat{C_1PC_2} = 90^\circ + \tfrac{1}{2}\widehat{C_1PC_2}$ and since $\widehat{O_1IO_2} = 90^\circ - \tfrac{1}{2}\widehat{C_1PC_2} \Longrightarrow$ PO_1IO_2 is cyclic. If $\odot(PO_1IO_2)$ cuts ℓ_1 again at X, we get $\widehat{O_1XI} = \widehat{O_1O_2I} = \tfrac{1}{2}\widehat{OPC_2}$ and $\widehat{XO_1O_2} = \widehat{O_2PB_1} = 90^\circ - \tfrac{1}{2}\widehat{OPC_2} \Longrightarrow \widehat{O_1XI} + \widehat{XO_1O_2} = 90^\circ \Longrightarrow IX \perp O_1O_2$, i.e. IX is I-altitude of $\triangle IO_1O_2$. Moreover, by symmetry IX cuts ℓ_2 at the reflection of X on O_1O_2, thus the orthocenter of $\triangle IO_1O_2$.

Luis González

#3 Jul 8, 2015, 4:11 am

Solution to problem 2)

$SB_2 = C_2S + C_2B_2 = \tfrac{1}{2}(PC_1 + C_1C_2 - PC_2) + \tfrac{1}{2}(PO + PC_2 - OC_2) \Longrightarrow$
$SB_2 = \tfrac{1}{2}(PO + PC_1 - OC_1) = PA_1$. Similarly we have $PB_1 = SA_2 \Longrightarrow PA_1 : PB_1 = SB_2 : SA_2$.

Let the perpendicular from P to ℓ_1 cut O_1O_2 and IK at U, K and let IS cut O_1O_2 at V. Then we have $UO_1 : UO_2 = PA_1 : PB_1 = SB_2 : SA_2 = VO_2 : VO_1 \Longrightarrow U, V$ are isotomic points WRT $\overline{O_1O_2}$. But since $\widehat{KUV} = \widehat{KVU} = 90^\circ - \tfrac{1}{2}\widehat{\ell_1, \ell_2}$, then $\triangle KUV$ is K-isosceles $\Longrightarrow KM \perp O_1O_2 \Longrightarrow OPKMS$ is cyclic. Since OM bisects $\widehat{\ell_1, \ell_2} \equiv \widehat{POS}$, then $MP = MS$.

TelvCohl

#4 Jul 8, 2015, 5:53 am • 1

My solution :

Lemma :

Let P be a point on $\odot(ABC)$ and M be the midpoint of BC .
Let D be the projection of A on the Steiner line τ of P WRT $\triangle ABC$.

Then $MP = MD$

Proof :

Let $T \equiv AD \cap \odot(ABC)$ and $P_a \in \tau$ be the reflection of P in BC .

From $AT \perp \tau \implies AT$ is the isogonal conjugate of AP WRT $\angle BAC$,
so $TP \parallel BC \implies PP_a \perp PT \implies D, T, P, P_a$ lie on a circle with diameter TP_a ,
hence notice M lie on the perpendicular bisector of PP_a, PT we get $MP = MD$.

Back to the main problem :

Since the reflection of P in IO_1, IO_2, O_1O_2 all lie on A_2B_2 ,
so A_2B_2 is the Steiner line of P WRT $\triangle IO_1O_2 \implies$ the orthocenter of $\triangle IO_1O_2$ lie on A_2B_2 .
Since S is the projection of I on A_2B_2 (Steiner line of P), so from the lemma we get $MP = MS$.

Q.E.D

High School Olympiads

orthocenteric line

Source: Own

andria

#1 Jul 7, 2015, 7:04 pm

In $\triangle ABC$ points P, Q lie on AB, AC respectively such that $BP = CQ$. Prove that the orthocenteric line of complete quadrilateral $PQCB$ is a fixed line while P, Q varies on AB, AC respectively.

Luis González

#2 Jul 7, 2015, 9:15 pm

Let M, N, U, V be the midpoints of $PQ, BC, CP, BQ,$ respectively. Since $MU = \frac{1}{2}CQ = \frac{1}{2}BP = MV,$ it follows that the parallelogram $UMVN$ is a rhombus $\Longrightarrow MN$ bisects $\angle UMV$. Thus since $MU \parallel AC, MV \parallel AB,$ then MN is parallel to the internal bisector of $\angle BAC \Longrightarrow UV$ is perpendicular to the internal bisector of $\angle BAC$. As a result, the orthocentric line of $PQCB$ is the line passing through the orthocenter H of $\triangle ABC$ parallel to the internal bisector of $\angle BAC$, thus fixed.

Luis González

#3 Jul 7, 2015, 9:30 pm • 1

Another solution using the Miquel point:

Since $BP = CQ,$ then the circles $\odot(ABC)$ and $\odot(APQ)$ meet again at the center S of the rotation that swaps BP and $CQ \Longrightarrow SB = SC \Longrightarrow S$ is midpoint of the arc BAC. Now since S is Miquel point of $PQCB,$ its isogonal conjugate WRT $\triangle ABC$ is the point at infinity of its Newton line $UV \Longrightarrow UV$ is parallel to the external bisector AS of $\angle BAC \Longrightarrow$ orthocentric line of $PQCB$ is the line passing through the orthocenter H of $\triangle ABC$ parallel to the internal bisector of $\angle BAC$.

andria

#5 Jul 8, 2015, 2:42 am

Here is my solution:
It's well known that the orthocenteric line is the image of simson line of miquel point of the quadrilatreal under the hemothety with ratio 2 so we must prove that simson line of $PQCB$ WRT $PQCB$ is fixed. Since N midpoint of arc BAC is miquel point of $PQCB$ we get that its simson line WRT quadrilateral $PQCB$ is a line connecting midpoints of PQ, BC let S, T midpoints of BC and PQ respectively. Since $\overrightarrow{ST} = \frac{1}{2}(\overrightarrow{BP} + \overrightarrow{CQ})$ we get that ST is parallel to internal angle bisector of $\angle A$ which is fixed.
DONE

High School Olympiads

incenter of ABC is same as incenter of AST

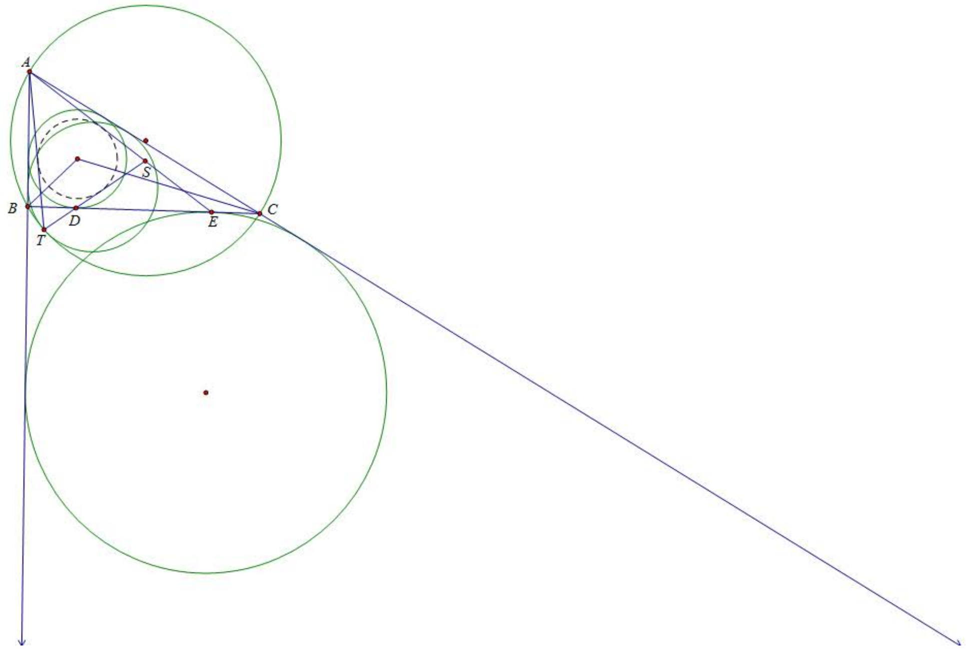
Source: Own

LeVietAn

#1 Jul 7, 2015, 8:07 am • 1

Dear Mathlinkers,
In a triangle ABC,the incircle touches segment BC at D, the $A-$excircle touches segment BC at E, the $A-$mixtilinear incircle touches circle (ABC) at T. DT intersects AE at S. Prove that the incenter of triangle ABC is the incenter of triangle ATS.

Attachments:



Luis González

#2 Jul 7, 2015, 9:04 am

Denote by I the incenter of $\triangle ABC$. The result follows from the following well-known properties of the mixtilinear incircle: AT, AE are isogonals WRT $\angle BAC \Longrightarrow AI$ also bisects $\angle EAT$ and if M is the midpoint of the arc BAC of $\odot(ABC)$ and TD cuts $\odot(ABC)$ again at X, then T, I, M are collinear and $AX \parallel BC \Longrightarrow TI$ bisects $\angle ATX \equiv \angle ATS$. Therefore I is also the incenter of $\triangle ATS$.

High School Olympiads

Bisect segment

Locked

Source: Own

buratinogigle

#1 Jul 7, 2015, 2:40 am

Let ABC be a triangle and circle (K) touches CA, AB inside triangle. E, F lie on CA, AB such that BE, CF are tangent to (K). BE cuts CF at P. Prove that radical axis of incircles of triangles PCE, PBF bisects segment EF.

Luis González

#2 Jul 7, 2015, 2:55 am • 1

Dear, you probably didn't notice you posted this problem before at Pass through midpoint. So for further discussions let us use the previous link.

High School Olympiads

Pass through midpoint

Reply

buratinogigle

#1 Feb 3, 2011, 9:30 pm

Let P be a point inside triangle ABC such that $PB+AC=PC+AB$. PB,PC intersect AC,AB at Y,Z. M is midpoint of YZ.

a) Prove that radical axis of incircles of triangles PZB and PYC pass through M.

b) Prove that radical axis of incircles of triangles YBC and ZBC pass through M.

Note that, similar problems on the post Midpoint lies on the radical axis of two incircles (2).

Luis González

#2 Feb 4, 2011, 6:55 am • 1

The proof to part a) is analogous to Midpoint lies on the radical axis of two incircles (2).

b) $PB+AC=PC+AB$ implies that $AYPZ$ has an incircle (O) touching AB,AC at M,N. Incircles $(U),(V)$ of $\triangle ZBC$ and $\triangle YBC$ touch CZ,BY at D,E, respectively. Since $AYPZ$ has incircle (O), it follows that $\triangle ZBC$ and $\triangle YBC$ have equal perimeters $\Longrightarrow ZD=YE$. From $\triangle OZM \sim \triangle ZUD$ and $\triangle OYN \sim \triangle YVE$, we have then:

$$\frac{UZ}{OZ}=\frac{ZD}{OM}, \ \frac{VY}{OY}=\frac{YE}{ON} \Longrightarrow \frac{OZ}{OY}=\frac{UZ}{VY} \ (\star)$$

If L,T,S are the orthogonal projections of O,U,V onto ZY, we have

$$\frac{OZ}{UZ}=\frac{OL}{ZT}, \ \frac{OY}{VY}=\frac{OL}{YS}.$$

Together with $(\star)$, we obtain $ZT=YS$, i.e midpoint F of YZ is also midpoint of ST.

$$FU^2-UZ^2=FV^2-VY^2=FU^2-(UD^2+ZD^2)=FV^2-(VE^2+YE^2)$$

$$\Longrightarrow FU^2-FV^2=UD^2-VE^2 \Longrightarrow F \text{ has equal power to } (U),(V).$$

Quick Reply

High School Olympiads

Newton line - fixed point

Reply

Source: Own

andria

#1 Jul 7, 2015, 12:15 am

Two circles ω_1, ω_2 intersect each other at A, B.a fixed line ℓ_1 passes throw A. Let $\ell_1 \cap \omega_1 = P, \ell_1 \cap \omega_2 = Q$ a variable line ℓ_2 passes throw A.let $\ell_2 \cap \omega_1 = T, \ell_2 \cap \omega_2 = S$ points M, N are midpoints of SP, QT respectively.
Prove that MN passes throw the fixed point while ℓ_2 varies around A.

Luis González

#2 Jul 7, 2015, 2:44 am

Let $C \equiv PT \cap QS.$ Miquel point B of $APCS$ is isogonal conjugate of the point at infinity of its Newton line MN WRT $APCS \Longrightarrow \angle(PT, MN) = \angle APB = \text{const}.$ But if D is the midpoint on $PQ,$ we have $DN \parallel PT \Longrightarrow \angle DNM = \angle(PT, MN) = \text{const}.$ Since N moves on fixed circle ω_3, image of ω_1 under homothety $(Q, \tfrac{1}{2}),$ that also passes through $D,$ then it follows that MN hits ω_3 again at a fixed point $X \Longrightarrow MN$ goes through the fixed point $X.$

Quick Reply

High School Olympiads

tangent-tangent

andria

#1 Jul 6, 2015, 10:11 pm

For every four points A, B, C, D in the plain prove that there is a point P such that the following conditions are true:
1) $\odot(\triangle PAB)$ is tangent to $\odot(\triangle PCD)$
2) $\odot(\triangle PBC)$ is tangent to $\odot(\triangle PAD)$

This post has been edited 1 time. Last edited by andria, Jul 6, 2015, 10:12 pm

TelvCohl

#5 Jul 6, 2015, 11:04 pm • 1

Let M be the Miquel point of $ABCD$. From $\triangle MBA \sim \triangle MCD \Longrightarrow \angle AMB = \angle DMC$, so $\angle AMC, \angle BMD$ share the same angle bisector ℓ and $MA \cdot MC = MB \cdot MD = \gamma^2$. Let Ψ be the composition of inversion $\mathbf{I}(\odot(M, \gamma))$ and reflection $\mathbf{R}(\ell)$. Let $P \equiv \ell \cap \odot(M, \gamma)$ be the fixed point under Ψ. Since $A \longleftrightarrow C, B \longleftrightarrow D$ under Ψ, so we get $\triangle MBP \sim \triangle MPD$ and $\triangle MAP \sim \triangle MPC$, hence $\angle DCP + \angle PBA = \angle DCP + \angle PBM + \angle MBA$ $= \angle DCP + \angle PBM + \angle MCD = \angle MCP + \angle PBM = \angle MPA + \angle DPM = \angle DPA \Longrightarrow \odot(PAB)$ and $\odot(PCD)$ are tangent at P. Similarly, $\odot(PBC)$ and $\odot(PDA)$ are tangent at P.

Luis González

#6 Jul 7, 2015, 1:08 am

In fact we can prove that, in general, there are only two points fulfilling the condition. Obviously any inversion with center P transforms $ABCD$ into a parallelogram $A'B'C'D'$, thus by inversion property, it follows that $\angle APC = \angle A'D'C' + \angle ADC$ and $\angle APC = 360^\circ - \angle ABC - \angle A'B'C'$ or the other way around $\Longrightarrow$ $\angle APC = \frac{1}{2}(\angle ABC - \angle ADC)$ mod 180 and similarly $\angle BPD = \frac{1}{2}(\angle BAD - \angle BCD)$ mod 180. These two circles seeing AC, BD under the referred angles intersect at the two desired points.

High School Olympiads

equal angles

Reply

Source: Own

andria

#1 Jul 6, 2015, 9:00 pm

In $\triangle ABC$ let ω_b be the circle that passes throw A, C and it is tangent to the incircle of $\triangle ABC$ we define ω_c similarly.let $\omega_b \cap \omega_c = T \neq A$ and let $AT \cap BC = S$. I, M are incenter of $\triangle ABC$ and midpoint of BC respectively. prove that: $\angle SIB = \angle MIC$

Luis González

#2 Jul 6, 2015, 10:12 pm

From the problem http://www.artofproblemsolving.com/community/c6h467561, we get that AT is the A-cevian of X_{57}; the orthocorrespondent of I WRT $\triangle ABC$. Thus if the perpendicular to AI at I cuts AC, AB at $Y, Z,$ we have $K \equiv BY \cap CZ \cap AT$.

Let N be the midpoint of the arc BAC of $\odot(ABC)$; intersection of the tangents of $\odot(IBC)$ at B, C and $U \equiv YZ \cap BC$. Since YZ is tangent to $\odot(BIC)$ and $(B, C, S, U) = -1,$ it follows that N, I, S are collinear on the polar of U WRT $\odot(BIC) \Longrightarrow IS$ is I-symmedian of $\triangle IBC \Longrightarrow \angle SIB = \angle MIC$.

Quick Reply

High School Olympiads

The three lines AA', BB' and CC' meet on the line IO

Reply

Source: Romanian Master Of Mathematics 2012

WakeUp

#1 Mar 4, 2012, 2:01 am • 7

Let ABC be a triangle and let I and O denote its incentre and circumcentre respectively. Let ω_A be the circle through B and C which is tangent to the incircle of the triangle ABC; the circles ω_B and ω_C are defined similarly. The circles ω_B and ω_C meet at a point A' distinct from A; the points B' and C' are defined similarly. Prove that the lines AA', BB' and CC' are concurrent at a point on the line IO.

(Russia) Fedor Ivlev

enescu

#2 Mar 4, 2012, 4:00 am • 2

http://rmm.lbi.ro/index.php?id=solutions_math

mahanmath

#3 Mar 4, 2012, 7:31 pm • 9

Congrats *F.Ivlev* ! Very very cute problem

Solution

r1234

#4 Mar 5, 2012, 2:59 pm • 3

Lemma:- Let ℓ be a line and Γ be circle.Suppose P is the pole of ℓ wrt Γ.Now let $\triangle ABC$ be inscribed in Γ.Let $\triangle A'B'C'$ be the circumcevian triangle of $\triangle ABC$ wrt P.Then ℓ is the perspective axis of $\triangle ABC$ and $\triangle A'B'C'$.

Proof:-

Let $A_1 = BC \cap B'C', B_1 = AC \cap A'C', C_1 = AB \cap A'B'$.
Now applying Pascal's theorem on the hexagon $B'C'ABCA'$ we get $A_1, C_1, C'A \cap CA'$ are collinear.
Again $\triangle AC'B$ and $\triangle A'CB'$ are perspective.So $C_1, C'A \cap CA', BC' \cap B'C$ are collinear.Hence we conclude that $A_1, C_1, BC' \cap B'C$ are collinear.But this line is nothing but the polar of P i.e ℓ.Hence we conclude that $A_1B_1C_1 \equiv \ell$.

Back to the main proof

Clearly AA' is the radical axis of ω_b, ω_c, BB' is the radical axis of ω_c, ω_a and CC' is the radical axis of ω_a, ω_b.So by radical axis theorem AA', BB', CC' are concurrent.
Let (I) be the incircle of $\triangle ABC$ and $\triangle DEF$ is its intouch triangle.D', E', F' are the touch points of (I) with $\omega_a, \omega_b, \omega_c$ respectively.Now let tangents at D', E', F' meets BC, CA, AB at X, Y, Z respectively.Then its easy to show that XYZ is the radical axis of (I) and $\odot ABC$.So X is the pole of DD' wrt (I) and similar for others.So DD', EE', FF' concur at the pole of XYZ wrt (I).Now consider the circles $(I), \omega_b, \omega_c$.Then by radical axis theorem the lines $AA', E'E', F'F'$ are concurrent ,say at X_1.Then X_1 is the pole of $E'F'$ wrt (I).Since A, X_1, A' are collinear, their polars i.e $EF, E'F', \text{Polar of A' }$ are concurrent.So $\triangle DEF$ and the triangle formed by the polars of A', B', C' are perspective wrt the perspective axis of $\triangle DEF, \triangle D'E'F'$.But according to our lemma this perspective axis is the polar of the perspective point of $\triangle DEF, \triangle D'E'F'$, i.e the radical axis of $(I), \odot ABC$.So we conclude that AA', BB', CC' concur at the pole of the radical axis of $(I), \odot ABC$ wrt (I).Since $IO \perp \text{Radical axis of (I),circumcircle of ABC}$ we conclude that the pole of the radical axis of $(I), \odot ABC$ lies on IO.Hence AA', BB', CC' concur on IO.

Swistak

#5 Mar 6, 2012, 8:19 pm • 3

Daaaamn! That problem was so incredibly similar to my solution of this problem: http://www.artofproblemsolving.com/Forum/viewtopic.php?p=2280575&sid=06651024ba8c2a2d88a8d3bf555bf086#p2280575 (I don't know why, but it's written that this is from 2010 MO, but it's from 2011 MO). During the polish final I solved that using radical axis of incenter and vertices and homothety. But during Romanian I didn't see that sixth problem is so similar :/. If I only drew one line :<... 75% of first official solution is like copy+paste from my solution from polish final :/.

Of course very nice and hard problem :).

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simplependulum

#6 Mar 8, 2012, 9:58 am • 1

Here is my solution :

By radical axis theorem , we can immediately conclude that the three lines intersect at the radical centre R of the three circles . To show that this radical centre is on IO , consider the inversion w.r.t to a circle centered at R and with radius $\sqrt{-\text{ power at R to the circles}}$ and then the reflection w.r.t R . (It is actually the same as the case that the three circles are disjoint so that we can choose an inversion at R sending these circles to themselves .) We find that the images of $\omega_A\ ,\ \omega_B\ ,\ \omega_C$ are themselves and the incircle becomes the larger circle tangent to them (internally) . From the property of this transformation (inversion + homothety) , $R\ ,\ I$ and the centre of this circle are collinear . We now show that this centre is actually O !

Let A_1 be the mid-pt of arc BC of ω_A that does not intersect the incircle . $B_1\ ,\ C_1$ are defined in similar mannar . It is well-known that the power at A_1 to the incircle is equal to A_1B^2 and A_1C^2 , so A_1 is the radical centre of the incircle , 'circle' B and 'circle' C . ($B_1\ ,\ C_1$ have similar property). Therefore , we conclude that $A_1\ ,\ B_1$ lie on the radical axis of the incircle and 'circle' C , and for other pairs , we have similar conclusion . Let $D\ ,\ E\ ,\ F$ be the mid-pts of arc $BC\ ,\ CA\ ,\ AB$ (not containing the third vertex of ΔABC .) of (ABC) respectively . Then , we deduce that $\Delta A_1B_1C_1$ is homothetic to ΔDEF and the centre of homothety is the intersection of the perpendicular bisectors of $AB\ ,\ BC\ ,\ CA$, which is O . Since O is the circumcenter of ΔDEF , so is $\Delta A_1B_1C_1$. But A_1 is an intersection of ω_A and $(A_1B_1C_1)$ and the centres of these two circles lie on the perpendicular bisector of BC which passes through A_1 , we therefore deduce that $(A_1B_1C_1)$ is tangent to ω_A . Similarly , $(A_1B_1C_1)$ is tangent to $\omega_B\ ,\ \omega_C$. In other words , $(A_1B_1C_1)$ is the larger circle we previously described , which has the centre O !

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pohoatza

#7 Mar 8, 2012, 12:48 pm

Bonus points for the one who generalizes this as much as possible. :D

Hint

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buratinogigle

#8 Mar 8, 2012, 2:34 pm

General problem

Let ABC be a triangle and a point P. $A_1B_1C_1$ is pedal triangle of P. $A_2B_2C_2$ is antipedal triangle of P. A_1A_2, B_1B_2, C_1C_2 cut pedal circle $(A_1B_1C_1)$ again at A_3, B_3, C_3. Let circumcircle $(ABC_3), (CAB_3)$ intersect again at A_4. Similarly, we have B_4, C_4. Prove that AA_4, BB_4, CC_4 are concurrent.

---

simplependulum

#9 Mar 8, 2012, 4:16 pm • 1

> *buratinogigle wrote:*
>
> General problem
>
> Let ABC be a triangle and a point P. $A_1B_1C_1$ is pedal triangle of P. $A_2B_2C_2$ is antipedal triangle of P. A_1A_2, B_1B_2, C_1C_2 cut pedal circle $(A_1B_1C_1)$ again at A_3, B_3, C_3. Let circumcircle $(ABC_3), (CAB_3)$ intersect again at A_4. Similarly, we have B_4, C_4. Prove that AA_4, BB_4, CC_4 are concurrent.

only concurrent ? but can't it be immediately proved by radical axis theorem ?

buratinogigle
#10 Mar 9, 2012, 9:55 am

Sorry, you are right, it is trivial, I will try again!

Fedyarer
#11 Apr 4, 2012, 2:42 am • 9

Thank you for all who say that like this problem. It's my
I was very wonderful that this problem was taken on this Olympiad. I didn't expecte that it's so nice, but.

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yumeidesu
#12 Apr 4, 2012, 11:26 pm

If we call P_a, P_b, P_c be the intersections of $\omega_A, \omega_B, \omega_C$ with (I), respectively, then AP_a, BP_b, CP_c are concurrent.
I find it by sketchpad but I don't know how to prove it. Who can help me?

---

mahanmath
#13 Apr 4, 2012, 11:47 pm • 1

> *yumeidesu wrote:*
> If we call P_a, P_b, P_c be the intersections of $\omega_A, \omega_B, \omega_C$ with (I), respectively, then AP_a, BP_b, CP_c are concurrent.
> I find it by sketchpad but I don't know how to prove it. Who can help me?

Keep the notations in my first post in mind , There is a celebrated theorem which asserts that , for three points A', B', C' on (I) AA', BB', CC' are concurrent if and only if $A'T_a, B'T_b, C'T_c$. It can easily proved by trigonometric *Ceva* theorem .
According to what I proved in my post we know P_aT_a, P_bT_b, P_cT_c are concurrent and above theorem implies that AP_a, BP_b, CP_c are concurrent.

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yumeidesu
#14 Apr 7, 2012, 8:58 am

I think this problem is a combine problem from:
IMO Shortlist 2002:
http://www.artofproblemsolving.com/Forum/viewtopic.php?p=118682&sid=ff0420e5b46b2284383a55c03e1bc918#p118682
Vietnam TST 2003:
http://www.artofproblemsolving.com/Forum/viewtopic.php?p=268390&sid=ff0420e5b46b2284383a55c03e1bc918#p268390

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cwein3
#15 Apr 9, 2012, 6:54 am • 1

Let the point of tangency between ω_A and the incircle be A'', and so on, and the excenters be I_A, I_B, I_C. Let the incircle be tangent to BC, AC, and AB at T_A, T_B, and T_C, respectively.

Lemma 1: $A''T_A$ passes through I_A.
Proof: This is 2002 G7.

Lemma 2: I_AT_A, I_BT_B, I_CT_C are concurrent.
Sketch of proof: Trig Ceva wrt $\triangle I_AI_BI_C$.

Lemma 3: $A''T_A$, $B''T_B$, and AA' are concurrent.
Proof: Let the tangents to the incircle at A'' and B'' intersect at N. By the Radical Axis Theorem, N lies on AA'. Let T_AT_B and $A''B''$ intersect at M, then it's easy to show that NA is the polar of M wrt to (I). Thus, $A''T_A$ and $B''T_B$ intersect on NA.

From Lemmas 2 and 3, we can conclude that AA', BB', CC', $A''T_A$, $B''T_B$, and $C''T_C$ are all concurrent. Call this point Q.
Let $A''T_A$ meet ω_A again at P_A, and define P_B and P_C the same way. Since $P_AP_BA''B''$ is cyclic, $P_AP_B \parallel T_AT_B$. The same goes for P_BP_C and P_AP_C, so $\triangle P_AP_BP_C$ is homothetic to $\triangle T_AT_BT_C$ wrt Q.

Note that the homothety from A'' takes T_A to P_A; thus, the tangent l_A to ω_A at P_A is parallel to BC; furthermore, $A''P_A$ bisects arc BC in ω_A. Define l_B and l_C similarly; then note that from the homothety from Q, $(P_AP_BP_C)$ is the incircle of the triangle formed by the intersections of l_A, l_B, and l_C. Let O' be the circumcenter of $(P_AP_BP_C)$. Then $O'P_A \perp l_A$, so $O'P_A$ is the formed by the intersections of ℓ_A, ℓ_B, and ℓ_C. Let O' be the circumcenter of $(P_A P_B P_C)$. Then $O'P_A \perp \ell_A$, so $O'P_A$ is the perpendicular bisector of BC. In addition, $O'P_B$ and $O'P_C$ bisect AC and AB, respectively. Thus, $O = O'$. But $O'I$ passes through Q by homothety, so OI passes through Q, as desired.

polya78

#16 May 23, 2012, 11:02 pm • 1

As above, let A'' and T_A be the points of tangency of (I) and w_A and BC respectively, etc. For notation's sake, for any point Z on (I), define ZZ to be the tangent to (I) at Z.

Again, as previously noted, by the radical axis theorem applied to w_A, (I) and (O), we have that $A''A''$ and $T_AT_A \equiv BC$ intersect on the radical axis of (O) and (I), which means that if M is the pole of this radical axis with respect to (I), then M (which clearly lies on OI) is on $A''T_A$.

Now let ℓ be the line through M and $T_AB'' \cap T_BA''$. Since $M = T_AA'' \cap T_BB'''$, by applying Pascal's Theorem to hexagons $T_AT_AA''T_BT_BB''$ and $A''A''T_AB''B''T_B$, we have that $C = T_AT_A \cap T_BT_B$ is on ℓ, as is $A''A'' \cap B''B''$. Now if we consider the circles w_A,w_B, and (I), and their radical axes, we see that $A''A'' \cap B''B''$, C and C' are collinear, which means that C' is on ℓ as well.

antimonyarsenide

#17 Jul 18, 2013, 5:15 am • 1

Very cool problem
My solution is not as nice as some others, but I thought it was pretty cool still.
Solution

IDMasterz

#18 Jul 19, 2013, 6:19 pm • 1

> *r1234 wrote:*
>
> **Lemma:-** Let ℓ be a line and Γ be circle.Suppose P is the pole of ℓ wrt Γ.Now let $\triangle ABC$ be inscribed in Γ.Let $\triangle A'B'C'$ be the circumcevian triangle of $\triangle ABC$ wrt P.Then ℓ is the perspective axis of $\triangle ABC$ and $\triangle A'B'C'$.
>
> **Proof:-**
>
> Let $A_1 = BC \cap B'C', B_1 = AC \cap A'C', C_1 = AB \cap A'B'$.
> Now applying Pascal's theorem on the hexagon $B'C'ABCA'$ we get $A_1, C_1, C'A \cap CA'$ are collinear.
> Again $\triangle AC'B$ and $\triangle A'CB'$ are perspective.So $C_1, C'A \cap CA', BC' \cap B'C$ are collinear.Hence we conclude that $A_1, C_1, BC' \cap B'C$ are collinear.But this line is nothing but the polar of P i.e ℓ.Hence we conclude that $A_1B_1C_1 \equiv \ell$.
>
> **Back to the main proof**
>
> Clearly AA' is the radical axis of ω_b, ω_c, BB' is the radical axis of ω_c, ω_a and CC' is the radical axis of ω_a, ω_b.So by radical axis theorem AA', BB', CC' are concurrent.
> Let (I) be the incircle of $\triangle ABC$ and $\triangle DEF$ is its intouch triangle.D', E', F' are the touch points of (I) with $\omega_a, \omega_b, \omega_c$ respectively.Now let tangents at D', E', F' meets BC, CA, AB at X, Y, Z respectively.Then its easy to show that XYZ is the radical axis of (I) and $\odot ABC$.So X is the pole of DD' wrt (I) and similar for others.So DD', EE', FF' concur at the pole of XYZ wrt (I).Now consider the circles $(I), \omega_b, \omega_c$.Then by radical axis theorem the lines $AA', E'E', F'F'$ are concurrent ,say at X_1.Then X_1 is the pole of $E'F'$ wrt (I).Since A, X_1, A' are collinear, their polars i.e $EF, E'F', \text{Polar of A'}$ are concurrent.So $\triangle DEF$ and the triangle formed by the polars of A', B', C' are perspective wrt the perspective axis of $\triangle DEF, \triangle D'E'F'$.But according to our lemma this perspective axis is the polar of the perspective point of $\triangle DEF, \triangle D'E'F'$, i.e the radical axis of $(I), \odot ABC$.So we conclude that AA', BB', CC' concur at the pole of the radical axis of $(I), \odot ABC$ wrt (I).Since $IO \perp \text{Radical axis of (I),circumcircle of ABC}$ we conclude that the pole of the radical axis of $(I), \odot ABC$ lies on IO.Hence AA', BB', CC' concur on IO.

Nice Lemma It can be generalised for any conic:

Lemma: *(Generalised)* Suppose line ℓ outside conic $\Gamma(ABC)$. Let the pole of ℓ wrt $\Gamma(ABC)$ be P and let A', B', C' be the (conic?)umcevian triangle of P wrt $\triangle ABC$.

Proof: Under a projective transformation, take the conic to a circle $\odot ABC$ and ℓ to infinity. Then, P goes to the centre of $\odot \triangle ABC$, so $\triangle A'B'C'$ and $\triangle ABC$ are homothetic, so done.

My proof goes very similar:

T_A, T_B, T_C be the tangency points of those circles with the incircle. Let the tangent $\equiv \ell_A$ through T_A intersect BC at X and define Y, Z similarly. (Note that X is the midpoint of DD' where D' is the harmonic conjugate of D wrt BC). Then, XYZ is the radical axis of $\odot ABC$ and the incircle, hence $OI \perp XYZ$. Let $P = DT_A \cap ET_B$ wrt incircle. Then, by radical axis theorem, AA' meets $\ell_B \cap \ell_C$. Then by pascals we get $T_C F \cap T_B E, \ell_B \cap \ell_C, P$ are collinear. But we get $A, P, T_C F \cap T_B E$ are also collinear by pascals, so $A, \ell_B \cap \ell_C, P$ are collinear. We conclude P is the radical centre. Note that P is the polar of XYZ of incircle, so done.

One does notice some interesting similarities. For instance, this sort of set-up was present in IMO 2002 shortlist, and 2009 Serbian MO (I think).

hqdhftw

#19 Jul 19, 2013, 11:18 pm • 2

> *mahanmath wrote:*
> Congrats *F.Ivlev* ! Very very cute problem
> Solution

Nice solution, I have another idea to prove the concurrency of $P_aT_a, P_bT_b, P_cT_c, OI$.
Let $T_bT_c \cap BC = S_a$. Similarly we define S_bS_c.
It's easy to see that P_aT_a is the angle bisector of $\angle BPC$. Hence P_aT_a is the radical axis of (I) and $(P_aT_aS_a)$, which is indeed the circle with diameter T_aS_a (because $(BCT_aS_a) = -1$). Likewise, P_bT_b is the radical axis of (I) and $(P_bS_bT_b)$. Consider 3 circles $(I), (P_aT_aS_a), (P_bT_bS_b)$, their radical axises concur at a point P. Let H be the orthocenter of $\triangle T_aT_bT_c$, it's easy to prove that HI is the common radical axis of 3 circles $(P_aS_aT_a), (P_bT_bS_c); (P_cS_cT_c)$, and from a well-known fact, we know that O, I, H lies on the Euler line of $\triangle T_aT_bT_c$. So $P_aT_a; P_bT_c$ and OI are concurrent. Similarly, we shall have all 4 lines are concurrent at P.

liberator

#20 Aug 2, 2014, 2:58 am • 1

> *WakeUp wrote:*
> Let ABC be a triangle and let I and O denote its incentre and circumcentre respectively. Let ω_A be the circle through B and C which is tangent to the incircle of the triangle ABC; the circles ω_B and ω_C are defined similarly. The circles ω_B and ω_C meet at a point A' distinct from A; the points B' and C' are defined similarly. Prove that the lines AA', BB' and CC' are concurrent at a point on the line IO.
>
> *(Russia) Fedor Ivlev*

Let Γ be the incircle of $\triangle ABC$, which is tangent to BC, CA, AB at P, Q, R respectively. Let $P \equiv \Gamma \cap \omega_A, Q \equiv \Gamma \cap \omega_B, R \equiv \Gamma \cap \omega_C$, and finally, let $P_1 \equiv P'P \cap \omega_A$, with similar definitions for Q_1, R_1.

Γ and ω_A are homothetic center P', so P_1 is the midpoint of the arc $\overset{\frown}{\mathrm{BC}}$ not containing P'. Hence

$$\angle P_1BC = \angle P_1P'C = \angle P_1P'B \implies \triangle P_1BP \sim \triangle P_1PB \implies \frac{P_1B}{P_1P'} = \frac{P_1P}{P_1B} \implies P_1B^2 = P_1P \cdot P_1P'.$$

$\therefore$ P_1 is on the radical axis of Γ and degenerate circle B. Similarly, P_1 lies on the radical axis of Γ and C, with analogous results for Q_1, R_1.

$\therefore$ $P_1Q_1 \perp CI \perp PQ$, with analogous results for Q_1R_1, R_1P_1. It follows that there exists a homothecy $\Theta : \triangle PQR \to \triangle P_1Q_1R_1$ with center X, the point of concurrency of PP_1, QQ_1, RR_1.

$PX \cdot XP' = QX \cdot XQ'$, since $PQP'Q'$ is cyclic. Noting $\dfrac{P_1X}{XP} = \dfrac{Q_1X}{XQ}$ gives $P_1X \cdot XP' = Q_1X \cdot XQ'$, so that X lies on the radical axis CC' of ω_A, ω_B. Similarly, AA', BB' pass through X.

$IP \parallel OP_1$ and $IQ \parallel OQ_1$, so it follows that O is the image of I in Θ . Hence AA', BB', CC' concur at a point on IO, as required.

Note: The point X of the required concurrency is in fact the *exsimilicenter* of $\triangle ABC$, the homothetic center of the incircle and the circumcircle, and hence the homothetic center of the intouch and circummidarc triangles. It is X_{56} in Kimberling's online encyclopedia of triangle centers encyclopedia of triangle centers.

pi37

#21 May 21, 2015, 6:01 am

Suppose $\omega_A, \omega_B, \omega_C$ are tangent to the incircle at X, Y, Z. By homothety and a well-known lemma, XD intersects ω_A again at some point M, which is the radical center of the incircle, B, and C. Define N, P similarly. Clearly MNP is homothetic to the intouch triangle. But $DI \perp BC$, so M, the center of ω_A, and the circumcenter of MNP are collinear, and (MNP) is tangent to ω_A. Note that if O is the circumcenter of MNP, $MO \perp BC$ and M lies on the perpendicular bisector of BC, so O is also the circumcenter of ABC.

We now have two circles Γ_1, the incircle and Γ_2, the circumcircle of MNP, with three circles $\omega_A, \omega_B, \omega_C$ externally tangent to both. We aim to show that the radical center of the three circles lies on the line through the centers of Γ_1 and Γ_2. In fact, we claim that this radical center is the exsimilicenter of Γ_1 and Γ_2. This fact holds in general, but we will use the points already defined in the problem to prove it.

Let S be the exsimilicenter of the two circles. Then S is the homothetic center of DEF and MNP. By Monge's theorem or by our previous results, S is collinear with X, D, M. So

$$\frac{SF}{SE} = \frac{SP}{SN}$$

and

$$SY \cdot SF = SE \cdot SZ$$

imply

$$SY \cdot SP = SZ \cdot SN$$

Therefore S has equal powers with respect to ω_B and ω_C, so it is the radical axis of all 3.

TelvCohl

#23 May 21, 2015, 9:05 pm • 2

My solution :

Let $D \equiv \odot(I) \cap BC, E \equiv \odot(I) \cap CA, F \equiv \odot(I) \cap AB$.
Let $X \equiv \omega_A \cap \odot(I), Y \equiv \omega_B \cap \odot(I), Z \equiv \omega_C \cap \odot(I)$.
Let $\mathcal{P}(P, \odot)$ be the power of a point P with respect to a circle $\odot$

From homothety with center X (maps $\odot(I) \mapsto \omega_A$) $\Longrightarrow M_A \equiv XD \cap \omega_A$ is the midpoint of arc BC in ω_A .
Similarly, $M_B \equiv YE \cap \omega_B, M_C \equiv ZF \cap \omega_C$ is the midpoint of arc CA, arc AB in ω_B, ω_C, respectively .

Since $\mathcal{P}(M_B, \odot(I)) = \mathcal{P}(M_B, C) = \mathcal{P}(M_B, A), \mathcal{P}(M_C, \odot(I)) = \mathcal{P}(M_C, B) = \mathcal{P}(M_C, A)$,
so $M_B M_C$ is the radical axis of $\{\odot(I), A\} \Longrightarrow M_B M_C$ pass through the midpoint of AE and AF .
Similarly, $M_C M_A, M_A M_B$ is B-midline, C-midline of $\triangle BFD, \triangle CDE$, resp $\Longrightarrow \triangle DEF, \triangle M_A M_B M_C$ are homothetic .

Let $T \equiv M_A D \cap M_B E \cap M_C F$ be the homothety center of $\triangle DEF$ and $\triangle M_A M_B M_C$.

From Reim theorem and $M_B M_C \parallel EF \Longrightarrow Y, Z, M_B, M_C$ are concyclic ,
so T is the radical center of $\{\odot(YZM_B M_C), \omega_B, \omega_C\} \Longrightarrow \mathcal{P}(T, \omega_B) = \mathcal{P}(T, \omega_C) \Longrightarrow T \in AA'$.
Similarly, we can prove $T \in BB'$ and $T \in CC' \Longrightarrow AA', BB', CC'$ are concurrent at T .

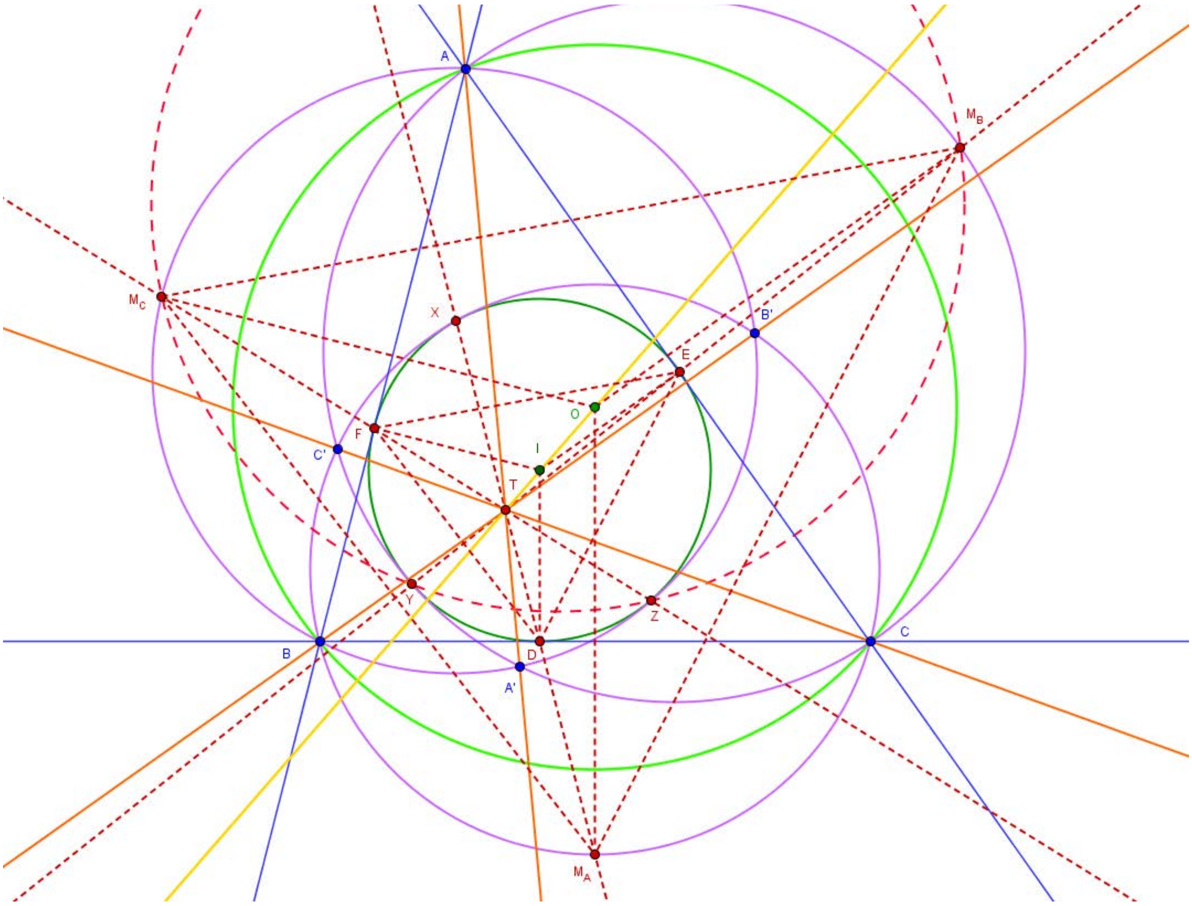
From $ID \parallel OM_A, IE \parallel OM_B, IF \parallel OM_C \Longrightarrow \triangle DEF \cup I$ and $\triangle M_A M_B M_C \cup O$ are homothetic $\Longrightarrow T \in OI$.

Q.E.D

Remark **:**

It's well-known that TD, TE, TF pass through the A-excenter I_a , B-excenter I_b , C-excenter I_c of $\triangle ABC$, respectively (see here) , so T is the homothety center of $\triangle DEF$ and $\triangle I_a I_b I_c \Longrightarrow T$ is X_{57} (in ETC) of $\triangle ABC$.

Attachments:



Luis González

#24 May 21, 2015, 10:59 pm • 1

Incircle (I) touches BC, CA, AB at D, E, F and $\omega_A, \omega_B, \omega_C$ touch (I) at T_A, T_B, T_C. Denote τ_A, τ_B, τ_c the tangents of (I) at T_A, T_B, T_C and $X \equiv \tau_b \cap \tau_C, Y \equiv \tau_C \cap \tau_A, Z \equiv \tau_A \cap \tau_B$. Since $XT_B = XT_C$, then X is on radical axis AA' of ω_B, ω_C and similarly $Y \in BB'$ and $Z \in CC' \Longrightarrow \triangle ABC$ and $\triangle XYZ$ are perspective through the radical center T of $\omega_A, \omega_B, \omega_C$ and their perspectrix is the radical axis τ of $(I), (O)$.

Since τ does not intersect (I), then there exists a homology sendind τ to infinity and taking (I) into another circle (J). Thus in this figure, $\triangle ABC$ and $\triangle XYZ$ become symmetric WRT $J \Longrightarrow AX, BY, CZ, DT_A, ET_B, FT_C$ concur at J; their common midpoint. Hence, in the original figure $AA' \equiv AX, BB' \equiv BY, CC' \equiv CZ, DT_A, ET_B, FT_C$ concur at $T \equiv X_{57}$ lying on OI.

JuanOrtiz

#25 Jun 27, 2015, 12:22 am

A nice result!

Solution

Q.E.D.

An additional challenge would be to prove that AX, BY, CZ are concurrent, which is true!

Proof of this additional result

This post has been edited 4 times. Last edited by JuanOrtiz, Jun 27, 2015, 2:13 am

drmzjoseph

#26 Jun 27, 2015, 2:17 pm • 1

Let $\triangle MNP$ be the intouch triangle of $\triangle ABC$, and let γ be the incircle. Now ω_b and ω_c touch to γ at B_1 and C_1 respectively, and the common tangents from this points cut at $A'', X \equiv B_1N \cap C_1P$, From Pascal's Theorem for fourth points in B_1, C_1, P, N we get A, X and A'' are collinear, also by radical axis on ω_b, ω_c and γ we get $A'' \in AA' \Rightarrow X \in AA'$ Moreover the common tangent between ω_b and the incircle touch AC at B_2, let ℓ be the radical axis between γ and $\odot(ABC)$, by radical axis theorem on $\gamma, \odot(ABC), \omega$ we get $B_2 \in \ell$, so the polar lines of A_2, B_2, C_2 (defined analogously) with respect to γ are concurrent at IO,(the pole of ℓ), and this point is X is a point fixed that belongs to AA' and IO as desired.

toto1234567890

#27 Jan 11, 2016, 6:13 pm

It's just a point that out-divides OI as R+r/2:r

ABCDE

#28 Feb 26, 2016, 9:00 am • 1

Let the incircle be tangent to BC and ω_A at D and D' respectively and similarly define E, F, E', and F'. Let $D'D$ intersect ω_A again at K_a, and define K_b and K_c similarly. By homothety, we have that K_a is the midpoint of arc BC on ω_A. Inversion about K_a with radius $K_aB = K_aC$ swaps line BC with ω_A, so $K_aB^2 = K_aC^2 = K_aD \cdot K_aD'$. Hence, K_a is the radical center of B, C, and the incircle.

Note that $K_aK_bK_c$ is homothetic with DEF because K_b and K_c both lie on the radical axis of A and the incircle which is parallel to EF. Let L be the center of homothety of $K_aK_bK_c$ and DEF. The perpendiculars from D to BC, E to CA, and F to AB concur at I, and the perpendiculars from K_a to BC, K_b to CA, and K_c to AB concur at O. Hence, L lies on IO. Now, note that by Power of a Point $LD \cdot LD' = LE \cdot LE' = LF \cdot LF'$ and because it's the center of homothety $LK_a \cdot LD' = LK_b \cdot LE' = LK_c \cdot LF'$, so L is the radical center of ω_A, ω_B, and ω_C, or where AA', BB', and CC' concur.

Quick Reply

High School Olympiads

Incentre of O_A O_B O_C is same as incentre of ABC

Reply

Source: All-Russian Olympiad 2012 Grade 11 Day 2

WakeUp

#1 May 31, 2012, 5:29 pm • 2

The points A_1, B_1, C_1 lie on the sides BC, CA and AB of the triangle ABC respectively. Suppose that $AB_1 - AC_1 = CA_1 - CB_1 = BC_1 - BA_1$. Let O_A, O_B and O_C be the circumcentres of triangles AB_1C_1, A_1BC_1 and A_1B_1C respectively. Prove that the incentre of triangle $O_AO_BO_C$ is the incentre of triangle ABC too.

WakeUp

#2 May 31, 2012, 7:00 pm • 6

I couldn't resist posting a solution

The weird condition $AB_1 - AC_1 = CA_1 - CB_1 = BC_1 - BA_1$ can be replaced with $AB_1 + BA_1 = AB, CA_1 + AC_1 = CA$ and $BC_1 + CB_1 = BC$.

So consider the point A_1 on BC. Let A_2, C_2 be the points on CA such that $CA_1 = CA_2$ and $AC_1 = AC_2$. Since $CA_1 + AC_1 = CA$, we have that $A_2 = C_2$.
Therefore C_1 is defined as: Take the circle with centre C and radius CA_1. Let this circle meet the segment CA at K. The circle with centre a and radius AK meets the segment AB at C_1.

Hence C_1 is uniquely determined by A_1, and the same goes for B_1.

Now, let the incentre of triangle ABC be I and consider the circle with centre I and radius IA_1. If this circle is tangent to BC then A_1 is the foot of the perpendicular from I to BC. Thus, this circle is tangent to the sides CA and AB too. In this case, O_A, O_B, O_C are the midpoints of IA, IB, IC respectively and the problem follows since $\triangle O_AO_BO_C$ is an enlargement of $\triangle ABC$ with scale factor $1/2$.

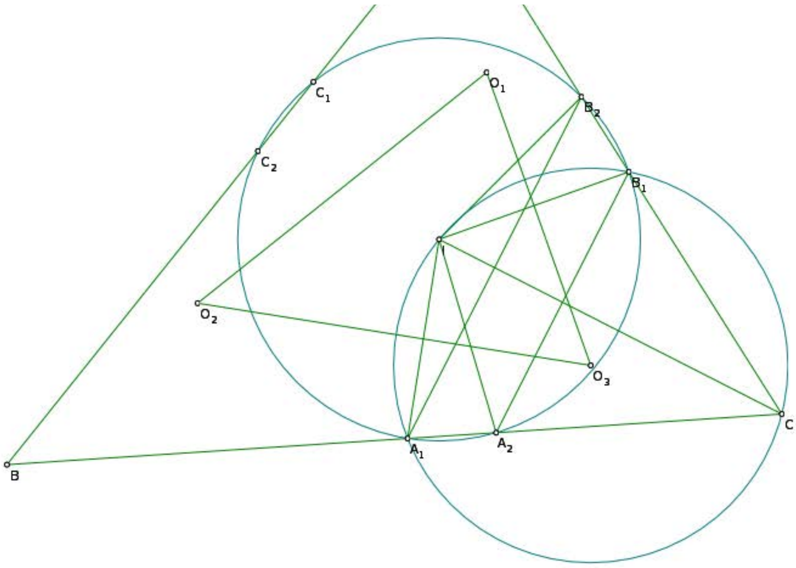
Otherwise, this circle is not tangent to any of the sides of triangle ABC. So let it cross BC, BC, CA, CA, AB, AB in $A_1, A_2, B_0, B_2, C_0, C_2$ (such that the 6 points appear in that order on the circle). Remember that A_1 uniquely determines B_1 and C_1. We will prove that $B_0 = B_1$ and $C_0 = C_1$.

Let D, E be the feet of the perpendicular from I to BC, CA respectively. Then $ID \perp A_1A_2$ so D is the midpoint of A_1A_2. Similarly E is the midpoint of B_2B_0. Clearly $CD = CE$. Let $DA_1 = DA_2 = x$ and $EB_2 = EB_0 = y$. Then the power of C with respect to the circle is $CA_2 \cdot CA_1 = CB_2 \cdot CB_0 \implies (CD - x)(CD + x) = (CE - y)(CE + y)$ $\implies CD^2 - CE^2 = x^2 - y^2$ so $x = y$. We conclude that $CA_2 = CB_0$ and $CA_1 = CB_2$. Recalling the construction of C_1 at the start, we see that B_2 is the point K, i.e. $AC_1 = AB_2$. But by a symmetric argument, we can prove that $AC_1 = AB_2$ and $AC_2 = AB_0$ and so we conclude $C_0 = C_2$. Similarly $B_0 = B_2$.

Now, note that $x = y \implies A_1A_2 = B_1B_2$ and symmetrically this length must also be equal to C_1C_2.

Note that $\angle IA_1A_2 = \angle IA_2A_1 = 180^\circ - \angle IA_2C = 180^\circ - \angle IB_1C$ due to isosceles triangles. Thus IB_1CA_1 is cyclic (with centre O_C). Similarly O_A is the centre of (AB_1IC_1). So IB_1 is the radical axis of the circles AB_1C_1 and A_1B_1C. Thus by symmetry $\angle IO_CO_A = \frac{1}{2}\angle IO_CB_1$. Now $\angle IO_CB_1 = 2\angle ICB_1 = C$. So $\angle IO_CO_A = \angle\frac{1}{2}C$. Similarly $\angle IO_3O_2 = \frac{1}{2}C$ and so IO_C bisects $\angle O_AO_CO_A$. Similarly IO_A bisects $\angle O_CO_AO_B$ and IO_B bisects $\angle O_AO_BO_C$ and therefore I is the incentre of triangle $O_AO_BO_C$.

Attachments:



jjax
#3 Jun 3, 2012, 5:05 pm • 1

Let I be the incenter of $\triangle ABC$.

We first settle the trivial case where $AB_1 - AC_1 = ... = 0$. Here, it is easily seen that A_1, B_1, C_1 are the points of tangency of the incircle to $\triangle ABC$. Then AI, BI, CI are the diameters of the three mentioned circles, so the triangle formed by the circumcentres is homothetic to the original triangle, with factor $\frac{1}{2}$ and centre of homothety I, so the conclusion follows.

Now we start for real. Construct points A_2, B_2, C_2 on the interiors of BC, CA, AB such that $AB_2 = AC_1$, $BC_2 = BA_1$, $CA_2 = CB_1$. That is, make several isosceles triangles. (Suppose such points don't exist, then WLOG $AC_1 \geq AC$, so $AC_1 - AB_1 \geq B_1C > CB_1 - CA_1$, a contradiction).

The length condition becomes $A_1A_2 = B_1B_2 = C_1C_2$. Now, let the incircle of ABC touch BC, CA, AB at X, Y, Z. Let $ZC_1 = YB_2 = d$, $XA_1 = ZC_2 = e$, $YB_1 = XA_2 = f$. Then, $B_1B_2 = C_1C_2$ gives $y = z$, and so by symmetry we have $x = y = z$. Call the incentre I, then this tells us that $\triangle IZC_1, \triangle IYB_1, \triangle IXA_1$ are congruent triangles, and in particular, $IC_1 = IB_1 = IA_1$.

The perpendicular bisector of B_1C_1 meets the angle bisector of $\angle B_1AC_1$ at a unique point, the midpoint of the minor arc B_1C_1 of the circumcircle of $\triangle AB_1C_1$. (If this point is not unique, the problem reduces to the trivial case). Since I lies on both lines, it lies on the circumcircle of $\triangle AB_1C_1$, and by symmetry, on all three circumcircles.

Clearly, I, A_1, B_1, C_1 are all distinct. Thus, the two intersection points of the circles with centres O_A, O_B are C, I, and so the perpendicular distance from I to O_AO_B is half the length IC_1. Thus, by symmetry and noting that $IA_1 = IB_1 = IC_1$, we know that I is equidistant from the three sides of $\triangle O_AO_BO_C$, so it is the incenter.

Particle
#4 Sep 19, 2012, 9:19 am • 1

Construct points A_1', B_1', C_1' are on AB, BC, CA such that $A_1B = A_1'B, B_1C = B_1'C, C_1A = C_1'A$.The given condition implies $A_1B_1' = B_1C_1' = C_1A_1'$.It's easy to prove $\angle C_1C_1'A_1' = \frac{1}{2}(\pi - \angle B) = \angle A_1A_1'B$ and so A_1, C_1, A_1', C_1' are concyclic (1).By symmetry, A_1, B_1, A_1', B_1' are concyclic (2) and B_1, C_1, B_1', C_1' are also concyclic (3).It's easy to see all of these circles have center I,the incentre of $\triangle ABC$.From (1), (2) and (3) $IA_1 = IB_1 = IC_1 = IA_1' = IB_1' = IC_1'$ and all of the six points lie on the same circle.
So $\angle A_1IC_1 = 2\angle A_1C_1'C_1 = \pi - \angle B$.Then A_1, B, C_1, I are concyclic.Now by symmetry the circumcircles of $\triangle AB_1C_1, \triangle A_1BC_1, \triangle A_1B_1C$ concur at point I.By SSS congruence theorem $\triangle O_AA_1O_B \cong \triangle O_AIO_B, \triangle O_AC_1O_C \cong \triangle O_AIO_C$ and $\triangle O_AA_1I \cong \triangle O_AC_1I$.Now it easily follows that O_AI bisects $\angle O_BO_AO_C$.By symmetry the result follows immediately.

TelvCohl
#5 Feb 9, 2015, 9:58 pm • 1

My solution:

Let I be the incenter of $\triangle ABC$.
Let D, E, F be the tangent point of $\odot(I)$ with BC, CA, AB, respectively .

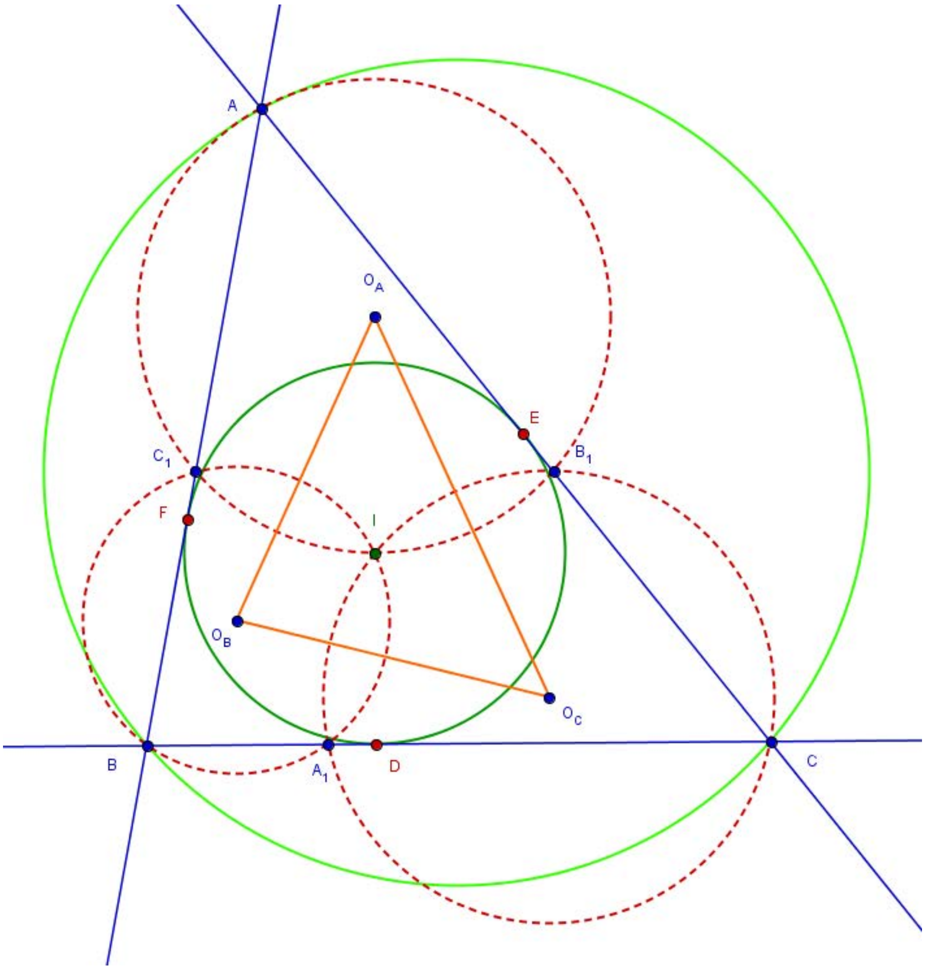
From $BC_1 - BA_1 = CA_1 - CB_1 \Longrightarrow BC_1 + CB_1 = BC$,
so we get $B_1E = C_1F \Longrightarrow Rt\triangle IEB_1 \cong Rt\triangle IFC_1 \Longrightarrow IB_1 = IC_1$.
Similarly, we can prove $IC_1 = IA_1 \Longrightarrow I$ is the circumcenter of $\triangle A_1B_1C_1$ . ... $(\star)$

Since $\angle IB_1E = \angle IC_1F$,
so A, I, B_1, C_1 are concyclic .
Similarly, we can prove $I \in \odot(BC_1A_1)$ and $I \in \odot(CA_1B_1)$.

Since O_BO_C, O_CO_A, O_AO_B is the perpendicular bisector of IA_1, IB_1, IC_1, respectively ,
so from $(\star)$ we get $\text{dist}(I, O_BO_C) = \text{dist}(I, O_CO_A) = \text{dist}(I, O_AO_B)$.
i.e. I is the incenter of $\triangle O_AO_BO_C$

Q.E.D

Attachments:



buratinogigle

#6 Feb 10, 2015, 7:05 pm

Variation problem.

Let ABC be a triangle and D, E, F lie on side BC, CA, AB such that $FB - EC = EA - DB = DC - FA$. Let X, Y, Z be circumcenters of triangles AEF, BFD, CDE. Prove that incenter of triangle XYZ is circumcenter of triangle ABC.

TelvCohl

#7 Feb 10, 2015, 8:59 pm

> *buratinogigle wrote:*
> Variation problem.

Let ABC be a triangle and D,E,F lie on side BC,CA,AB such that $FB-EC=EA-DB=DC-FA$. Let X,Y,Z be circumcenters of triangles AEF,BFD,CDE. Prove that incenter of triangle XYZ is circumcenter of triangle ABC.

My solution:

Let I,O,T be the incenter, circumcenter, Bevan point of $\triangle ABC$, respectively .
Let A^*,B^*,C^*,D^*,E^*,F^* be the projection of T on IA,IB,IC,BC,CA,AB, respectively .

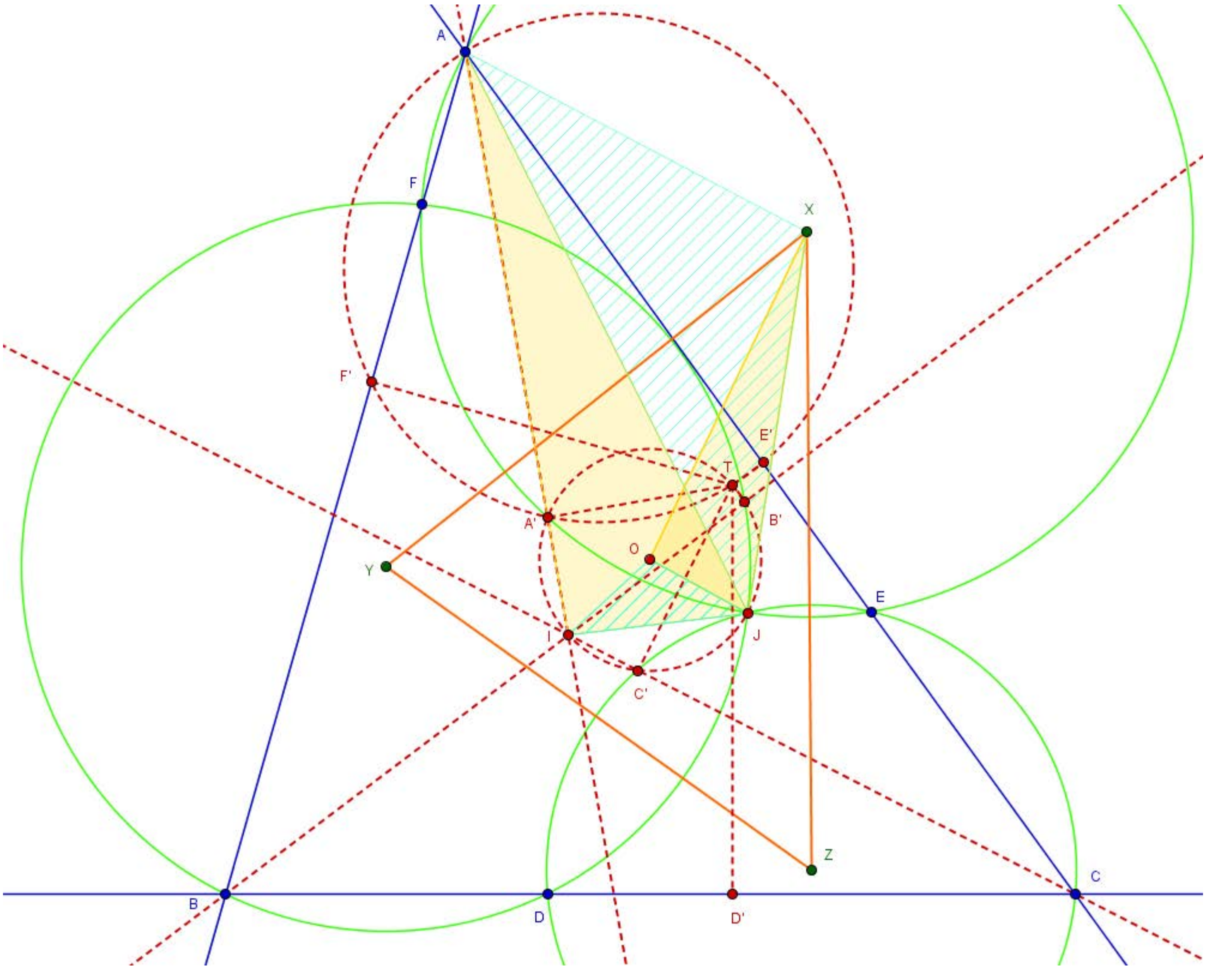
From $EA-DB=DC-FA \Longrightarrow AE+AF=BC$,
so combine with $AE^*+AF^*=BC \Longrightarrow EE^*=FF^*$.
Since A,E^*,F^*,A^*,T lie on a circle with diameter AT ,
so A^* is the midpoint of arc E^*F^* in $\odot(AT) \Longrightarrow \triangle A^*E^*E \cong \triangle A^*F^*F$,
hence from $\angle E^*EA=\angle F^*FA$ we get $A^* \in \odot(AEF)$.
Similarly we can prove $B^* \in \odot(BFD)$ and $C^* \in \odot(CDE)$.

Since AA^*,BB^*,CC^* are concurrent at I ,
so the Miquel point J of $\{D,E,F\}$ WRT $\triangle ABC$ lie on $\odot(A^*B^*C^*) \equiv \odot(IT)$.

Since $\angle IOJ=2\angle IA^*J=2\angle AEJ=\angle AXJ$,
so combine with $XA=XJ, OI=OJ \Longrightarrow \triangle IOJ \sim \triangle AXJ \Longrightarrow \triangle AIJ \sim \triangle XOJ$.
Similarly we can prove $\triangle BIJ \sim \triangle YOJ$ and $\triangle CIJ \sim \triangle ZOJ$,
so we get $\triangle ABC \cup I \sim \triangle XYZ \cup O$. i.e. O is the incenter of $\triangle XYZ$

Q.E.D

Attachments:



livetolove212

#8 Jul 5, 2015, 11:13 pm

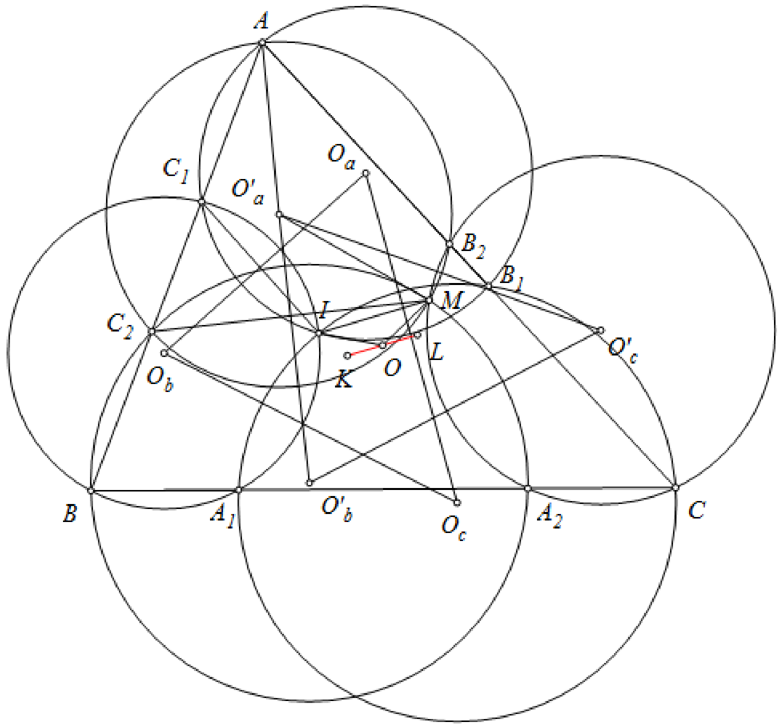
> *buratinogigle wrote:*
>
> Variation problem.
>
> Let ABC be a triangle and D,E,F lie on side BC,CA,AB such that $FB-EC=EA-DB=DC-FA$.
> Let X,Y,Z be circumcenters of triangles AEF,BFD,CDE. Prove that incenter of triangle XYZ is circumcenter of

Let X, Y, Z be circumcenters of triangles AEF, BFD, CDE. Prove that incenter of triangle XYZ is circumcenter of triangle ABC.

We need a lemma: O is a point in triangle ABC.Let P, Q, R be the feets of the perpendiculars from O to BC, CA, AB,respectively. A_1, B_1, C_1 be three points in BC, CA, AB and A_2, B_2, C_2 be the symetric of A_1, B_1, C_1 by P, Q, R.
Z_1, Z_2 be the $Miquel$ points of triangle ABC with A_1, B_1, C_1 and A_2, B_2, C_2. Prove that
$OZ_1 = OZ_2$.
See here
In my proof at this link, if we put $(Z_1A_1, BC) = (Z_1B_1, AC) = (Z_1C_1, AB) = \alpha$ then $\angle Z_1OZ_2 = 2\alpha$.
Back to this problem.
Since $FB - EC = EA - DB = DC - FA$ we get D, E, F are the reflections of A_1, B_1, C_1 wrt midpoints of BC, CA, AB, respectively. We will change the names of D, E, F to A_2, B_2, C_2. Let O'_a, O'_b, O'_c be the circumcenters of triangles $AB_2C_2, BA_2C_2, CA_2B_2$, respectively, M be Miquel point of triangle ABC wrt A_2, B_2, C_2.
We have $\angle MO'_aA = \angle MO'_bB = \angle MO'_cC = 2\alpha$ and $MO'_a = O'_aA, MO'_b = O'_bB, MO'_c = O'_cC$ then M is the center of spiral similarity which takes triangle $O'_aO'_bO'_c$ to ABC. But applying the lemma above, $MO = OI$ and $\angle MOI = 2\alpha$ then this transformation takes O to I. But I is the incenter of triangle ABC, O must be the incenter of triangle $O'_aO'_bO'_c$.

Another interesting result:
Given triangle ABC. Let A_1, A_2 on BC, B_1, B_2 on AC, C_1, C_2 on AB such that $A_1, A_2; B_1, B_2; C_1, C_2$ are symmetric wrt midpoints of BC, CA, AB, respectively. Let O_a, O_b, O_c be the circumcenters of triangles $AB_1C_1, BA_1C_1, CA_1B_1; O'_a, O'_b, O'_c$ be the circumcenters of triangles $AB_2C_2, BA_2C_2, CA_2B_2; P, Q$ be the circumcenters of triangles $O_aO_bO_c$ and $O'_aO'_bO'_c$. Prove that P and Q are symmetric through the circumcenter of triangle ABC.

Attachments:



Luis González
#9 Jul 6, 2015, 12:06 am

> *livetolove212 wrote:*
> **Another interesting result:**
> Given triangle ABC. Let A_1, A_2 on BC, B_1, B_2 on AC, C_1, C_2 on AB such that $A_1, A_2; B_1, B_2; C_1, C_2$ are symmetric wrt midpoints of BC, CA, AB, respectively. Let O_a, O_b, O_c be the circumcenters of triangles $AB_1C_1, BA_1C_1, CA_1B_1; O'_a, O'_b, O'_c$ be the circumcenters of triangles $AB_2C_2, BA_2C_2, CA_2B_2; P, Q$ be the circumcenters of triangles $O_aO_bO_c$ and $O'_aO'_bO'_c$. Prove that P and Q are symmetric through the circumcenter of triangle ABC.

Let M_1 and M_2 denote the Miquel points of $A_1B_1C_1$ and $A_2B_2C_2$ WRT $\triangle ABC$. From the solution of the problem Inequality triangle (see post #3) we obtain $\triangle OM_1M_2 \sim \triangle O_aAM_1 \sim \triangle O_a'AM_2$ and from the solution of the problem Equal triangle (see post #3), we obtain $\triangle OM_1M_2 \sim \triangle O_aAM_1 \sim \triangle O_a\ AM_2$ and from the solution of the problem Equal segments (see post #2), we get $\triangle POM_1$ is P-isosceles such that $\triangle POM_1 \sim \triangle O_aAM_1 \Longrightarrow \triangle POM_1 \sim \triangle OM_1M_2$ and likewise $\triangle QOM_2 \sim \triangle OM_1M_2 \Longrightarrow \triangle POM_1 \cong \triangle QOM_2,$ so it clearly follows that P, O, Q are collinear on a parallel to M_1M_2 and O is midpoint of $\overline{PQ}.$

va2010

#10 Jan 11, 2016, 6:23 am

This problem is made significantly easier if we've done the previous one, which can be found here: http://artofproblemsolving.com/community/c6h481933p2699668

The idea is that since IC_1AB_1 is cyclic, O_A is also the circumcenter of IC_1B_1. So now all we need to do is show that the incenter of the circumcenters of IC_1B_1, IA_1C_1, and IA_1B_1 is the point I. This is not so hard. We take a homothety about I with scale factor two, upon which each of the circumcenters O_B, O_C, and O_A map to the the poles of A_1C_1, A_1B_1, and B_1C_1. But by definition, the incenter of these points is I, so we are done.

Quick Reply

High School Olympiads

Inequality triangle

livetolove212

#1 Aug 21, 2010, 9:26 am

Problem (own): Given a triangle ABC and its circumcenter O. Let A_1, B_1, C_1 be arbitrary points on BC, CA, AB, respectively. The lines through A_1, B_1, C_1 and perpendicular to BC, CA, AB intersect each other and make triangle $A_2B_2C_2$ with its circumcenter I. Let A_3, B_3, C_3 be the reflections of A_1, B_1, C_1 wrt the midpoints of BC, CA, AB, respectively, J be the Miquel point of triangle ABC wrt (A_3, B_3, C_3). Prove that $OI \geq OJ$.

livetolove212

#2 Aug 31, 2010, 8:25 am

Dear Mathlinkers,
The idea of this problem is to show that if I' is the reflection of I wrt O then $\angle OJI' = 90^o$.

Luis González

#3 Feb 26, 2011, 9:47 pm • 1

Lemma: M, N, L are arbitrary points of the sidelines BC, CA, AB of $\triangle ABC$. P is an arbitrary point on the plane of $\triangle ABC$ and D, E, F denote its orthogonal projections onto BC, CA, AB. M', N', L' are the reflections of M, N, L about D, E, F and U, V are the Miquel points of MNL and $M'N', L'$ WRT $\triangle ABC$. Then $PU = PV$.

Let $S \equiv UL \cap PF, T \equiv UN \cap PE$ and $V' \equiv TN' \cap SL'$. Using directed angles (mod 180) we have $\angle V'N'E = \angle TNE = \angle UMC = \angle SLL' = \angle V'L'A \Longrightarrow A, L', N', V'$ are concyclic. Further, from $\angle SUT = \angle TV'S = \angle TPS = \angle BAC$, we deduce that P, S, T, U, V' are concyclic and $\angle PSV' = \angle PTU \Longrightarrow PU = PV'$. Hence if $R \equiv UM \cap PD$, then $R \in \odot(PTU)$ and RP bisects $\angle V'RU \Longrightarrow V'R$ goes through M'. Since A', N', L', V' and B, L', M', V' are concyclic, then V' is Miquel point of $\triangle ABC \cup M'N'L' \Longrightarrow V \equiv V'$.

Back to the problem, it's clear that $\triangle ABC \sim \triangle A_2B_2C_2$ are directly similar. Hence, $\angle IB_2C_2 = \angle OBC$ implies that IB_2 and OB are perpendicular at $X \in \odot(BA_1C_1)$. Likewise, IC_2 and OC are perpendicular at $Y \in \odot(CA_1B_1) \Longrightarrow X, Y, I, O$ lie on the circle $\mathcal{K}$ with diameter $\overline{OI}$. Let K be the Miquel point of $\triangle ABC \cup A_1B_1C_1$. Using the lemma for O and its pedal triangle (medial triangle of ABC) we deduce that $OK = OJ$ and $\angle XOY = 2\angle OBC = \angle YKX$ implies that $K \in \mathcal{K} \Longrightarrow \angle IKO = 90^\circ \Longrightarrow OI \geq OK = OJ$ and the proof is completed.

Quick Reply

High School Olympiads

Equal segments

Reply

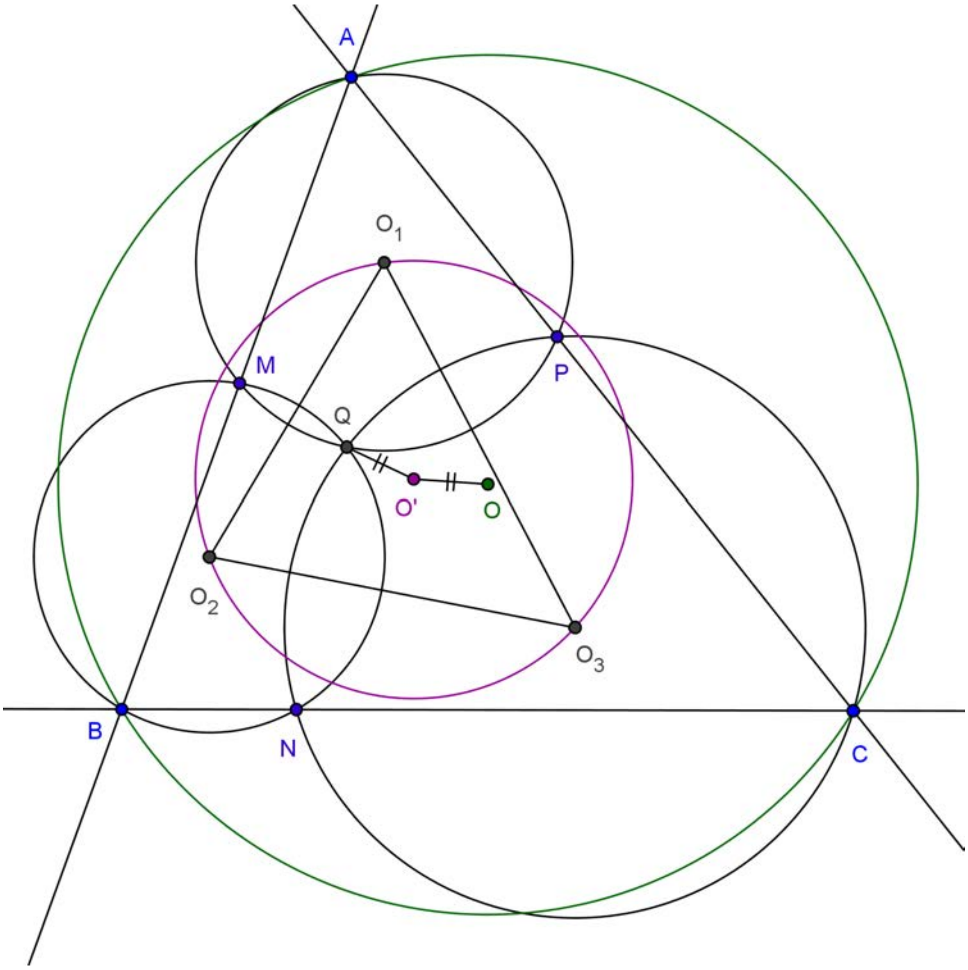
Source: mpdb

borislav_mirchev

#1 Sep 25, 2012, 12:35 am • 1

Let the triangle ABC is inscribed in a circle $k(O)$. M, N, P are three points of the sides AB, BC, CA, respectively. $k_1(O_1), k_2(O_2), k_3(O_3)$ are the circumcircles of the triangles APM, BMN, CNP, respectively. $O^$ is the circumcenter of the triangle $O_1O_2O_3$. Q is the intersection point of the circles k_1, k_2, k_3. Prove that $OO^=O^Q.$

Attachments:



This post has been edited 2 times. Last edited by borislav_mirchev, Sep 25, 2012, 1:49 am

Luis González

#2 Sep 25, 2012, 1:47 am • 3

Since $O_1O_2 \perp QM$ and $O_1O_3 \perp QP$, we have $\angle O_2O_1O_3 = \angle MQP = \angle BAC$ (mod 180º). Similarly, $\angle O_3O_2O_1 = \angle CBA \Longrightarrow \triangle ABC \sim \triangle O_1O_2O_3$ are directly similar. Let $D \equiv BC \cap O_2O_3$. Since O_2O_3 is perpendicular bisector of $\overline{QN}$, then DO_3 bisects $\angle CDQ$. Together with $O_3C = O_3Q$, we deduce that $O_3 \in \odot(DCQ)$ is the midpoint of its arc QC. Similarly, Q is on $\odot(DBO_2) \Longrightarrow Q$ is center of the spiral similarity that takes BC into O_2O_3. Thus, Q is center of the spiral similarity that carries $\triangle ABC$ into $\triangle O_1O_2O_3$. This takes A, O into $O_1, O' \Longrightarrow$ isosceles $\triangle QAO_1$ and $\triangle QOO'$ are spirally similar with center $Q \Longrightarrow \triangle QOO'$ is isosceles with $O'Q = O'O$.

borislav_mirchev

borislav_mirchev
#3 Sep 28, 2012, 5:50 pm • 1

You can see a detailed solution with simplified therms here:
http://www.math10.com/f/viewtopic.php?f=49&t=10751&p=49766#p49766

jayme
#4 Sep 28, 2012, 9:19 pm • 2

Dear Mathlinkers,
I have seen this nice result on

Die Würtzel, 12 (1998).

If some one can send me a scan of these page, welcome

Sincerely
Jean-Louis

borislav_mirchev
#5 Sep 29, 2012, 3:22 pm • 1

I haven't seen the source mentioned and it seems I rediscovered it.
What I think here is the case when $O_1O_2O_3$'s sides are parallel to the ABC's sides deserve special threatment for the solution to be correct.
I'm also interested to see the material jayme mentioned scanned here.

P.S. You can see the solution for the case when the sides mentioned are parallel on the link above. If you can see any different cases or have some remarks - you are welcome to share your opinion.

Quick Reply

High School Olympiads

beautiful geo lemma

Reply

phuocdinh_vn99
#1 Jul 5, 2015, 10:43 am • 1

Given cyclic quadrilateral $ABCD$ with circumcircle (O). AC cuts BD at I, AB cuts CD at H. HI cuts (IAD) and (IBC) at P, Q,resp. Prove that $OP = OQ$.

A-B-C
#2 Jul 5, 2015, 8:13 pm

Let M be the intersection of (OAD) and (OBC) other than I. By directed angle, we can prove that (IAB) and (ICD) pass through M.
$AD \cap BC = K$, then K, O, M are collinear, $OM \perp IH$ cause IH is polar line of K WRT (O)
Let S, T be the intersection of (OAD), (OBC) with IH other than I
We need to show that $(HIST) = -1$, hence, we prove that IK, SD, CT are concurrent.
IK is the radical axis of (IAD) and (IBC)
D, S, M, O are concyclic $\Rightarrow \widehat{ODS} = 180^o - \widehat{OMS} = 90^o$ so SD is tangent to (O), similarly, TC is tangent to (O)
$\Rightarrow SD \cap CT$ lies on KI-polar line of H WRT (O) so IK, SD, CT are concurrent $\Rightarrow (HIST) = -1$
$\overline{HA}.\overline{HB} = \overline{HC}.\overline{HD} = \overline{HK}.\overline{HM}$
$\mathbf{I}(H, \mathcal{P}_{H/(O)})$: $(ODA) \to (IBC) \Rightarrow \overline{HS}.\overline{HQ} = \overline{HT}.\overline{HP} = \mathcal{P}_{H/(O)}$
$\mathbf{I}(H, \mathcal{P}_{H/(O)})$ maps S, T, I to Q, P, M
$(HIST) = -1 \Rightarrow M$ is midpoint of PQ

Luis González
#3 Jul 5, 2015, 10:53 pm

This holds for any line through I, not necessarily IH. Just note that the circle ω with diameter $\overline{OI}$, coaxal with $\odot(IAD)$ and $\odot(IBC)$, is their midcircle $\Longrightarrow$ midpoint M of $\overline{PQ}$ is on $\omega \Longrightarrow OM \perp PQ \Longrightarrow OP = OQ$.

Gibby
#4 Jul 6, 2015, 1:44 am

Luis, could you explain why those circles are coaxal?

David_Forest
#5 Jul 6, 2015, 8:13 am • 2

As Luis says, it is true for any line through I.
Now let's rewrite this problem as following:
Given cyclic quadrilateral $ABCD$ with circumcircle O.
A line through I cuts (O) at M, N resp. and cuts (IAD) and (IBC) at P, Q resp.
Prove that $PM = QN$.

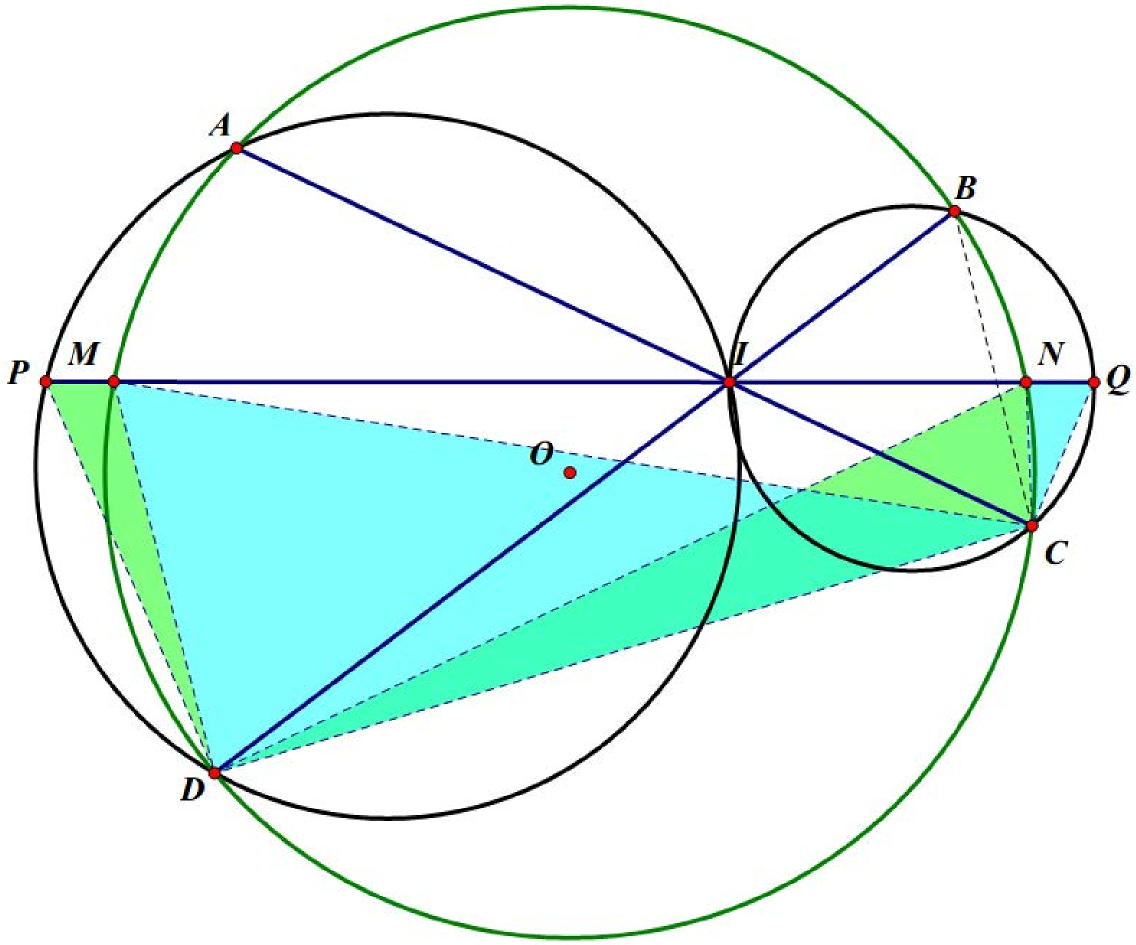
Proof:
Note that $\angle CQI = \angle CBI = \angle CMD$ and $\angle CNQ = \angle CDM$, we have $\Delta CNQ \sim \Delta CDM$. Similarly we also have $\Delta PMD \sim \Delta NCD$. Therefore

$$\frac{NQ}{MD} = \frac{NC}{CD} = \frac{PM}{MD}$$

From above we have $NQ = PM$.

Q.E.D.

Attachments:



tkhalid

#6 Jul 6, 2015, 8:57 am

Nice proof David_Forest!

TelvCohl

#7 Jul 6, 2015, 12:38 pm • 3

> *David_Forest wrote:*
> **Now let's rewrite this problem as following:**
> Given cyclic quadrilateral $ABCD$ with circumcircle O.
> A line through I cuts (O) at M, N resp. and cuts (IAD) and (IBC) at P, Q resp.
> Prove that $PM = QN$.

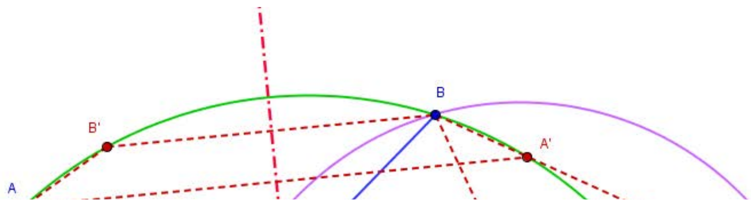
My solution **:**

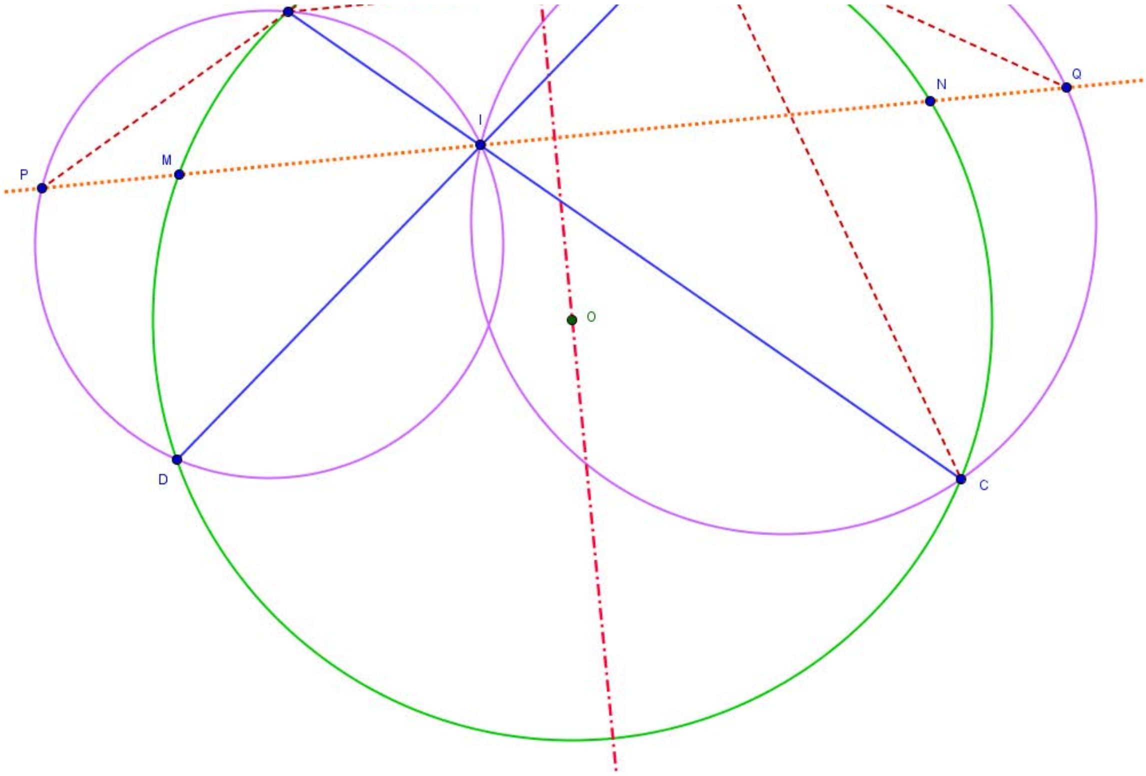
Let A', B' be the reflection of A, B in the perpendicular bisector ℓ of MN, respectively .

From $AA' \parallel MN$ and $\angle BA'A = \angle BCI = \angle BQI \Longrightarrow A' \in BQ$.
Similarly, we can prove $B' \in AP \Longrightarrow P, Q$ are symmetry WRT $\ell \Longrightarrow PM = QN$.

Q.E.D

Attachments:





David_Forest

#8 Jul 6, 2015, 7:37 pm

> *TelvCohl wrote:*
>
> > *David_Forest wrote:*
> >
> > **Now let's rewrite this problem as following:**
> > Given cyclic quadrilateral $ABCD$ with circumcircle O.
> > A line through I cuts (O) at M, N resp. and cuts (IAD) and (IBC) at P, Q resp.
> > Prove that $PM = QN$.
>
> My solution :
>
> Let A', B' be the reflection of A, B in the perpendicular bisector ℓ of MN, respectively .
>
> From $AA' \parallel MN$ and $\angle BA'A = \angle BCI = \angle BQI \Longrightarrow A' \in BQ$.
> Similarly, we can prove $B' \in AP \Longrightarrow P, Q$ are symmetry WRT $\ell \Longrightarrow PM = QN$.
>
> Q.E.D

Another excellent proof! Your proof reminded me of the so-called "Butterfly" theorem.
We may also use similar method of symmetry on the "Butterfly" theorem.

This post has been edited 1 time. Last edited by David_Forest, Jul 6, 2015, 7:38 pm
Reason: Make addition.

muuratjann

#9 Jul 6, 2015, 8:03 pm

> *David_Forest wrote:*
>
> As Luis says, it is true for any line through I.
> **Now let's rewrite this problem as following:**
> Given cyclic quadrilateral $ABCD$ with circumcircle O.
> A line through I cuts (O) at M, N resp. and cuts (IAD) and (IBC) at P, Q resp.
> Prove that $PM = QN$.
>
> **Proof:**
> Note that $\angle CQI = \angle CBI = \angle CMD$ and $\angle CNQ = \angle CDM$,we have $\Delta CNQ \sim \Delta CDM$ .Similarly we also

Note that $\angle CQP = \angle CBP = \angle CMD$ and $\angle CNQ = \angle CDM$, we have $\Delta CNQ \sim \Delta CDM$. Similarly we also have $\Delta PMD \sim \Delta NCD$. Therefore

$$\frac{NQ}{MD} = \frac{NC}{CD} = \frac{PM}{MD}$$

From above we have $NQ = PM$.

Q.E.D.

Very niceeee)))

High School Olympiads

geo. not easy

Locked

JSJ20142014

#1 Jul 5, 2015, 8:14 am

In triangle ABC,point M is middle point of BC,point N is on AM,$AN=BN$,point Q is on NM,$AQ=CQ$,extend CQ intersects BN at X,prove $<BAX=<CAM$.

Luis González

#2 Jul 5, 2015, 9:03 am • 1

Posted before at http://www.artofproblemsolving.com/community/c6h396352.

trunglqd91

#3 Jul 5, 2015, 6:40 pm

I think it is a beautiful lemma. Thank you so much.

My solution:

From the problem we get $\triangle ABN$ and $\triangle AQC$ are isosceles.

$\Longrightarrow \angle NBA=\angle NAB$ and $\angle QCA=\angle QAC$.

By using **Menelaus** theorem for $\triangle BXC$ with $\overline{N,Q,M}$:

We get $\frac{NB}{NX}.\frac{QX}{QC}.\frac{MC}{MB}=1.$

$\Longrightarrow \frac{NB}{NX}=\frac{QC}{QX}$ (because M is the midpoint of BC)

$\Longrightarrow \frac{NA}{QA}=\frac{NX}{QX}$

$\Longrightarrow AX$ is external bisector of $\angle NXQ$

Let $CQ\cap AB=H$

$NB\cap AC=K$

So we get $\angle AXH=\angle AXK$

And $\angle AXK=\angle BAX+\angle XBA=\angle BAX+\angle NBA=\angle BAX+\angle BAN=2\angle BAX+\angle XAN$

$\angle AXH=\angle XCA+\angle XAC=\angle XAC+\angle QCA=\angle XAC+\angle QAC=2\angle QAC+\angle XAN$

$\Longrightarrow \angle BAX=\angle QAC=\angle MAC$

Q.E.D.

High School Olympiads

congruent quadrilaterals

Reply

andria

#1 Jun 30, 2015, 5:32 pm

$ABCD$ is a cyclic quadrilateral with circumcircle (O).$(O_1), (O_2), (O_3), (O_4)$ are symmetric circles to (O) WRT AB, BC, CD, DA respectively.let $(O_1) \cap (O_4) = A', (O_1) \cap (O_2) = B', (O_2) \cap (O_3) = C', (O_3) \cap (O_4) = D'$ prove that quadrilateral $ABCD$ is congruent to $A'B'C'D'$.

Luis González

#2 Jul 5, 2015, 6:01 am

This is another formulation of an old problem. A', B', C', D' are clearly the orthocenters of $\triangle DAB, \triangle ABC, \triangle BCD, \triangle CDA$. Thus $A'B'C'D'$ and $ABCD$ are symmetric about the anticenter of $ABCD$. See Orthocenters and Cyclic Quadrilaterals, Well known? [cyclic ABCD, orthocenters of ABC, BCD, ...], congruence of quadrilaterals and elsewhere.

Quick Reply

High School Olympiads

Orthocenters and Cyclic Quadrilaterals

Reply

Source: Chinese Competition, 1992

ThAzN1

#1 Oct 23, 2004, 6:43 am

Let $A_1A_2A_3A_4$ be a cyclic quadrilateral. Let the orthocenters of triangles $A_2A_3A_4, A_3A_4A_1, A_4A_1A_2, A_1A_2A_3$ be $H_1, H_2, H_3, H_4,$ respectively. Prove that $H_1H_2H_3H_4$ is a cyclic quadrilateral.

grobber

#2 Oct 23, 2004, 8:48 am

A standard complex numbers solution: let the circumcenter of $A_1A_2A_3A_4$ be the origin of the plane, and let a_1, a_2, a_3, a_4 be the complex numbers representing A_1, A_2, A_3, A_4 respectively. In this case, H_1, H_2, H_3, H_4 are represented by $a_2+a_3+a_4, a_3+a_4+a_1, a_4+a_1+a_2, a_1+a_2+a_3$ respectively. This means that the midpoints of A_iH_i coincide, so $H_1H_2H_3H_4$ is obtained from $A_1A_2A_3A_4$ through a homothety of ratio -1, so they're congruent (this is even more than we wanted).

Soarer

#3 Oct 23, 2004, 12:45 pm

sorry, i'm not familiar with complex no. at all, could you briefly explain this?

> *grobber wrote:*
> This means that the midpoints of A_iH_i coincide, so $H_1H_2H_3H_4$ is obtained from $A_1A_2A_3A_4$ through a homothety of ratio -1, so they're congruent (this is even more than we wanted).

grobber

#4 Oct 23, 2004, 12:58 pm

Since A_1 has corresponding complex number a_1 and H_1 has complex number (the corresponding complex number is called affix; in Romanian at least :)) $a_2+a_3+a_4$, the midpoint of A_1H_1 has affix $\frac{a_1+a_2+a_3+a_4}{2}$, which is a symmetric expression, so it's the same for the midpoints of A_2H_2, A_3H_3, A_4H_4.

Soarer

#5 Oct 23, 2004, 2:02 pm

ah ic, thx!

darij grinberg

#6 Oct 23, 2004, 2:35 pm

> *grobber wrote:*
> (the corresponding complex number is called affix; in Romanian at least :))

Yes, "affix" is "affix" in English, too.

Anyway, a proof without complex numbers is given in the note "The Euler point of a cyclic quadrilateral" on my website. The most important thing is that the quadrilaterals $A_1A_2A_3A_4$ and $H_1H_2H_3H_4$ are symmetric to each other with respect to a point. (Theorem 2 in my note.)

Darij

Quick Reply

High School Olympiads

Well known? [cyclic ABCD, orthocenters of ABC, BCD, ...]

Reply

Source: Does anyone know the source?

Yimin Ge

#1 Sep 19, 2006, 4:38 am

Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R, S be the orthocenters of the triangles BCD, CDA, DAB, ABC respectively. Prove that the quadrilaterals $ABCD$ and $PQRS$ are congruent.

Negative_3

#2 Sep 20, 2006, 3:56 am

Lets represent the points A, B, C, D with the complex numbers a, b, c, d. W.L.O.G. we may assume a, b, c, d lie on the complex unit circle. Using useful formula #40 at

http://mathcircle.berkeley.edu/complexBMC2.pdf

we have

$$p = b + c + d = (a + b + c + d) - a$$
$$q = c + d + a = (a + b + c + d) - b$$
$$r = d + a + b = (a + b + c + d) - c$$
$$s = a + b + c = (a + b + c + d) - d$$

Thus $pqrs$ corresponds to rotating $abcd$ by 180 degrees, and then translating it by ammount $a + b + c + d$. Since rotation and translation are linear transformations, we have $pqrs$ is congruent to $abcd$.

Albanian Eagle

#3 Sep 28, 2006, 7:50 am

the source of the problem is Balkan Mathematical Olympiad 1984.

shobber

#4 Sep 30, 2006, 6:43 pm

Let $G_1G_2G_3G_4$ be the quadrilateral formed by the four centroids of the triangles, then by Euler's theorem we can prove that $G_1G_2G_3G_4 \sim ABCD$ with ratio $\frac{1}{3}$ and $G_1G_2G_3G_4 \sim PQRS$ with ratio $\frac{1}{3}$. Hence $ABCD \cong PQRS$.

This problem is also used at a Chinese Math League in 1992. The problem asked students to show that $PQRS$ is cyclic and find the location of its circumcentre.

Quick Reply

High School Olympiads

congruence of quadrilaterals

Reply

Source: old problem from balkan olymp.

v235711

#1 Mar 29, 2009, 8:46 am

Let ABCD be a cyclic quadrilateral.Let the orthocenters of BCD,CDA,DAB,ABC be A1,B1,C1,D1, respectively.prove that quadrilateral A1B1C1D1 is congruent to ABCD.

v235711

#2 Mar 29, 2009, 9:04 am

any ideas?

Luis González

#3 Mar 29, 2009, 9:17 am

Using the fact that in any triangle the distance from the orthocenter to a vertex is twice the distance from the circumcenter to the opposite side, it follows that $\overline{BA_1}$ and $\overline{B_1A}$ are parallel and equal to twice the distance from O to DC (O is the circumcenter of ABCD). Hence, B_1BA_1A is a parallelogram $\Longrightarrow \overline{A_1B_1} = \overline{AB}$. By similar reasoning, we have that $\overline{B_1C_1} = \overline{BC}$, $\overline{C_1D_1} = \overline{CD}$ and $\overline{D_1A_1} = \overline{DA}$. Likewise, C_1ACA_1 is parallelogram $\Longrightarrow \overline{AC} = \overline{A_1C_1}$ and $\overline{BD} = \overline{B_1C_1} \Longrightarrow$ Quadrilaterals $ABCD$ and $A_1B_1C_1D_1$ have congruent sides and diagonals.

Quick Reply

High School Olympiads

Radical center and concurrent

Reply

Source: Own

LeVietAn
#1 Jul 4, 2015, 8:42 am

Dear Mathlinkers,
Let ABC be a triangle inscribed in the circle Ω. Let (A) be the circle centered A and touches the line BC, and let ω_a be the circle reflection of (A) in the line BC. The circles ω_b, ω_c are determined similarly to the way we construct the circle ω_a. Let A_r, B_r and C_r respectively be the radical centers of $\{\Omega, \omega_b, \omega_c\}$, $\{\Omega, \omega_c, \omega_a\}$ and $\{\Omega, \omega_a, \omega_b\}$. Prove that the lines AA_r, BB_r and CC_r are concurrent.

Luis González
#2 Jul 4, 2015, 9:04 am • 1

Let ℓ_A, ℓ_B, ℓ_C be the radical axis of $\{\Omega, \omega_a\}, \{\Omega, \omega_b\}, \{\Omega, \omega_c\}$, resp. They cut BC, CA, AB at X, Y, Z, resp. If D, E, F are the feet of the altitudes on BC, CA, AC, we have then $XD^2 = XB \cdot XC \Longrightarrow X$ is the midpoint between D and its harmonic conjugate WRT B, C and similarly for Y, Z. Thus $\overline{XYZ}$ is Newton line of the quadrilateral bounded by the sides of $\triangle DEF$ and the trilinar polar of the orthocenter $\Longrightarrow \triangle ABC$ and $\triangle(\ell_A, \ell_B, \ell_C) \equiv \triangle A_rB_rC_r$ are perspective $\Longrightarrow AA_r$, BB_r, CC_r concur.

Luis González
#3 Jul 5, 2015, 1:43 am

In fact the concurrency point $AA_r \cap BB_r \cap CC_r$ is the triangle center X_{2963}; the barycentric product of the two Napoleon points X_{17}, X_{18}.

We use barycentric coordinates WRT $\triangle ABC$. Equation of ω_a centered at the reflection $(-a^2 : 2S_C : 2S_B)$ of A across BC and passing through $D \equiv (0 : S_C : S_B)$ is $a^2yz + b^2zx + c^2xy - (x+y+z)(3S^2x + {S_B}^2y + {S_C}^2z) = 0 \Longrightarrow$ $\ell_A \equiv 3S^2x + {S_B}^2y + {S_C}^2z$ and cyclically we have $\ell_B \equiv {S_A}^2x + 3S^2y + {S_C}^2z = 0$ and $\ell_C \equiv {S_A}^2x + {S_B}^2y + 3S^2z = 0$. Hence

$A_r \equiv \ell_B \cap \ell_C \equiv \left(9S^4 - {S_B}^2{S_C}^2 : {S_A}^2({S_C}^2 - 3S^2) : {S_A}^2({S_B}^2 - 3S^2)\right)$ and cyclic expressions for B_r and C_r. Thus, we conclude that AA_r, BB_r, CC_r concur at

$$X_{2963} \equiv X_{17} \cdot X_{18} \equiv \left(\frac{1}{{S_A}^2 - 3S^2} : \frac{1}{{S_B}^2 - 3S^2} : \frac{1}{{S_C}^2 - 3S^2}\right).$$

Quick Reply

High School Olympiads

Some properties of orthocorrespondent

Reply

TelvCohl

#1 Jun 9, 2015, 12:06 pm • 7

Some properties of orthocorrespondent

Theorem :

Let H be the orthocenter of $\triangle ABC$.
Let P, Q be the isogonal conjugate of $\triangle ABC$.
Let $\mathcal{L}_P$ be the orthotransversal of P WRT $\triangle ABC$.
Let $\mathcal{O}_P$ be the orthocorrespondent of P WRT $\triangle ABC$.
Let (H/P) be the cevian quotient of H and P WRT $\triangle ABC$.
Let R be the isogonal conjugate of P WRT the anticevian triangle of P WRT $\triangle ABC$.

Then
(1) $P(H/P) \perp \mathcal{L}_P$
(2) $Q, R, (H/P), \mathcal{O}_P$ are collinear
(3) $(Q, (H/P); R, \mathcal{O}_P) = -1$

Proof :

Let $\triangle DEF$ be the orthic triangle of $\triangle ABC$.
Let $\triangle A_1B_1C_1$ be the anticevian triangle of P WRT $\triangle ABC$.
Let $\triangle A_2B_2C_2$ be the cevian triangle of P WRT $\triangle ABC$.
Let A_3, B_3, C_3 be the reflection of P in BC, CA, AB, respectively .
Let $\triangle A_4B_4C_4$ be the pedal triangle of P WRT $\triangle A_1B_1C_1$.
Let $A_5 \equiv PA_3 \cap BC, B_5 \equiv PB_3 \cap CA, C_5 \equiv PC_3 \cap AB$.
Let A_6, B_6, C_6 be the midpoint of PA_1, PB_1, PC_1, respectively .
Let $A_7 \equiv \mathcal{L}_P \cap BC, B_7 \equiv \mathcal{L}_P \cap CA, C_7 \equiv \mathcal{L}_P \cap AB$.
Let $A_8 \equiv A\mathcal{O}_P \cap BC, B_8 \equiv B\mathcal{O}_P \cap CA, C_8 \equiv C\mathcal{O}_P \cap AB$.

Let T^* be the image of T under the inversion $\mathbf{I}_P$ with arbitrary power .

(1)

From $D(A, A_2; P, A_3) = -1 = D(A, A_2; P, A_1) \Longrightarrow A_3 \in DA_1$.
Similarly, we can prove E, B_3, B_1 are collinear and F, C_3, C_1 are collinear ,
so (H/P) is the isogonal conjugate of P WRT $\triangle DEF$ ($\because$ H is the incenter of $\triangle DEF$) .

Since A, B, C is the D-excenter, E-excenter, F-excenter of $\triangle DEF$, respectively ,
so the polar of P WRT the conic $\mathcal{H}$ passing through A, B, C, H, P passes through (H/P) ,
hence the tangent of $\mathcal{H}$ through P passes through $(H/P) \Longrightarrow P(H/P) \perp \mathcal{L}_P$ (see here (post #3)) .

---

(2)

From $P, B_4, C_4, A_1 \in \odot(A_1P) \Longrightarrow A_1^*$ is the projection of P on $B_4^*C_4^*$.
Similarly, we can prove B_1^*, C_1^* is the projection of P on $C_4^*A_4^*, A_4^*B_4^*$, respectively .

Since A, P, B_5, C_5, A_4 all lie on $\odot(AP)$ and $A(A_4, P; B_5, C_5) = -1$,
so we get A_4^* is the midpoint of $B_5^*C_5^*$ and A^* is the projection of P on $B_5^*C_5^*$.
Similarly, B_4^*, C_4^* is the midpoint of $C_5^*A_5^*, A_5^*B_5^*$ and B^*, C^* is the projection of P on $C_5^*A_5^*, A_5^*B_5^*$.

Since the center O_5 of $\odot(A_5B_5C_5)$ is the midpoint of PQ ,
so Q^* is the midpoint of PO_5^* (O_5^* is the image of P under the inversion $\mathbf{I}(\odot(A_5^*B_5^*C_5^*))$) .

Since the center O_4 of $\odot(A_4B_4C_4)$ is the midpoint of PR ,
so R^* is the midpoint of PO_4^* (O_4^* is the image of P under the inversion $\mathbf{I}(\odot(A_4^*B_4^*C_4^*))$) .

Since $(H/P) \equiv A_1A_3 \cap B_1B_3 \cap C_1C_3$,
so $(H/P)^*$ is the midpoint of PS^* where $S^* \equiv \odot(PA_5^*A_6^*) \cap \odot(PB_5^*B_6^*) \cap \odot(PC_5^*C_6^*)$.

From symmetry $\Longrightarrow$ $\odot(PA_5^*A_6^*)$ passes through the projection A_8^* of A_5^* on $B_5^*C_5^*$.
Similarly, we can prove $B_9^* \in \odot(PB_5^*B_6^*)$ and $C_9^* \in \odot(PC_5^*C_6^*)$ (B_9^*, C_9^* defined in the same way) .

Let V^* be the orthocenter of $\triangle A_5^*B_5^*C_5^*$.
Let Ψ be the inversion with center V^* that maps $A_5^*, B_5^*, C_5^* \longleftrightarrow A_9^*, B_9^*, C_9^*$.

Since $\odot(PA_5^*A_6^*), \odot(PB_5^*B_6^*), \odot(PC_5^*C_6^*)$ are fixed under Ψ ,
so $S^* = \Psi(P) \Longrightarrow P, S^*, O_4^*, O_5^*$ are concyclic (circle through P and $\perp \{\odot(A_4^*B_4^*C_4^*), \odot(A_5^*B_5^*C_5^*)\}$) .

Since $\odot(A_4^*B_4^*C_4^*)$ is the 9-point circle of $\triangle A_5^*B_5^*C_5^*$,
so the center of $\Omega_P \equiv \odot(PS^*O_4^*O_5^*)$ lie on the orthic axis τ of $\triangle A_5^*B_5^*C_5^*$ . ... $(\star)$

From $A_7^*P \parallel B_5^*C_5^*$ and $A_7^* \in (PB^*C^*) \Longrightarrow A_7^*$ is the projection of P on $A_5^*A_9^*$.
Similarly, we can prove B_7^*, C_7^* is the projection of P on $B_5^*B_9^*, C_5^*C_9^*$, respectively .

Since $A_5^*(A_7^*, A_8^*; B^*, C^*) = P^*(A_7^*, A_8^*; B^*, C^*) = (A_7, A_8; B, C) = -1$,
so $\tau \cap B_5^*C_5^* \in A_5^*A_8^* \Longrightarrow \odot(PA^*A_8^*)$ passes through the projection of P on τ .
Similarly, we can prove the projection of P on τ lie on $\odot(PB^*B_8^*)$ and $\odot(PC^*C_8^*)$,
so $\mathcal{O}_P^*$ is the projection of P on $\tau \Longrightarrow$ the reflection of P in $\mathcal{O}_P^*$ lie on Ω_P (from $(\star)$) ,
hence we get $P, Q^*, R^*, (H/P)^*, \mathcal{O}_P^*$ are concyclic $\Longrightarrow Q, R, (H/P), \mathcal{O}_P$ are collinear .

---

(3)

Since PQ^*, PR^* passes through the circumcenter, 9-point center of $\triangle A_5^*B_5^*C_5^*$, respectively ,
so from $V^* \in P(H/P)^*, P\mathcal{O}_P^* \perp \tau \Longrightarrow (Q, (H/P); R, \mathcal{O}_P) = P^*(Q^*, (H/P)^*; R^*, \mathcal{O}_P^*) = -1$.

Done

Luis González

#2 Jul 4, 2015, 9:22 pm • 2

I found another general property. Keeping the same initial notations, the trilinear polar of the isotomic conjugate of $\mathcal{O}_P$ is perpendicular to PH.

Let the perpendiculars to PA, PB, PC at P cut BC, CA, AB at $A_1, B_1, C_1 \Longrightarrow \overline{A_1B_1C_1}$ is the orthotransversal of P and the reflections A_2, B_2, C_2 of A_1, B_1, C_1 on the midpoints M_A, M_B, M_C of BC, CA, AB are then collinear on the trilinear polar of the isotomic conjugate of $\mathcal{O}_P$. It's known that $\odot(PAA_1), \odot(PBB_1), \odot(PCC_1)$ with centers U_A, U_B, U_C are coaxal with radical axis $PH \Longrightarrow PH \perp \overline{U_AU_BU_C}$.

On the other hand, note that AM_A and A_2U_A are medians of $\triangle AA_1A_2$, metting at its centroid $G \Longrightarrow \overline{GM_A} : \overline{GA} = -1 : 2 \Longrightarrow G$ is also the centroid of $\triangle ABC$. Therefore $\overline{A_2B_2C_2}$ is the image of $\overline{U_AU_BU_C}$ under homothety $(G, -2) \Longrightarrow \overline{A_2B_2C_2} \parallel \overline{U_AU_BU_C} \Longrightarrow PH \perp \overline{A_2B_2C_2}$.

Quick Reply

High School Olympiads

Source: Kürschák 2012, problem 1

randomusername

#1 Jul 4, 2015, 3:51 pm

Let J_A and J_B be the A-excenter and B-excenter of $\triangle ABC$. Consider a chord $\overline{PQ}$ of circle ABC which is parallel to AB and intersects segments $\overline{AC}$ and $\overline{BC}$. If lines AB and CP intersect at R, prove that

$$\angle J_A Q J_B + \angle J_A R J_B = 180^\circ.$$

wiseman

#2 Jul 4, 2015, 8:21 pm

$\rightarrow$ Q' =The reflection of Q WRT J_AJ_B.
$\rightarrow$ I =The incenter of $\triangle ABC$.
$\rightarrow$ Denote by a, b, c, x, y, z the lengths of BC, AC, AB, AE, CD, BF respectively where D, E, F are the touch points of the incircle of $\triangle ABC$ with BC, CA, AB respectively.

Lemma: In any $\triangle ABC$ we have: $\sin(\frac{\widehat{ACB}}{2})^2 = \frac{xy}{ab}$.
Proof of the lemma: The proof is really not hard. Just note that $CI.CJ_C = ab$ and $xy = rr_c$.

Back to the main problem:
$\rightarrow$ Q' is the reflection of Q WRT $J_AJ_B \Rightarrow \widehat{J_AQJ_B} = \widehat{J_AQ'J_B}$.
$\rightarrow 2\widehat{QCJ_B} = \frac{\text{arc}AB}{2} - \text{arc}BQ = 180 - J_BCP \Rightarrow Q' \in CP$.
$\rightarrow AB \parallel PQ \Rightarrow \widehat{BCQ} = \widehat{PCA}$. (a)
$\rightarrow \widehat{AQC} = \widehat{RBC}$. (b)
Combining (a) and (b) yields that $\triangle ACQ \sim \triangle RCB \Rightarrow CR.CQ = ab$.
$\Rightarrow$ By the lemma we have: $ab = \frac{xy}{\sin(\frac{\widehat{ACB}}{2})^2} = CJ_A.CJ_B$. But it's obvious that
$CQ = CQ' \Rightarrow CQ'.CR = CQ.CR = ab = CJ_A.CJ_B \Rightarrow RJ_AQ'J_B$ is cyclic
$\Rightarrow \widehat{J_AQ'J_B} + \widehat{J_ARJ_B} = \widehat{J_AQJ_B} + \widehat{J_ARJ_B} = 180\blacksquare$.

This post has been edited 2 times. Last edited by wiseman, Jul 4, 2015, 8:22 pm
Reason: typo

TelvCohl

#3 Jul 4, 2015, 8:44 pm

My solution :

Let Q^* be the reflection of Q in J_AJ_B (easy to see Q^* lie on CP) .
Let Ψ be the composition of inversion $\mathbf{I}(C, \sqrt{CA \cdot CB})$ and reflection $\mathbf{R}(J_AJ_B)$.

Since $CP \longleftrightarrow CQ, \odot(ABC) \longleftrightarrow AB$ under Ψ ,
so $Q \longleftrightarrow R$ under $\Psi \Longrightarrow CR \cdot CQ = CA \cdot CB$,
hence combine $\triangle CJ_AB \sim \triangle CAJ_B$ we get $CR \cdot CQ^* = CJ_A \cdot CJ_B$,
so J_A, J_B, Q^*, R are concyclic $\Longrightarrow \angle J_AQJ_B + \angle J_ARJ_B = \angle J_AQ^*J_B + \angle J_ARJ_B = 180^\circ$.

Q.E.D

Luis González

#4 Jul 4, 2015, 8:57 pm • 1

Didn't you post it before at http://www.artofproblemsolving.com/community/c6h553615 ?. Please do not double post.

Quick Reply