China Girls Math Olympiad 2007

Day 1

1 A positive integer m is called *good* if there is a positive integer n such that m is the quotient of n by the number of positive integer divisors of n (including 1 and n itself). Prove that $1, 2, \ldots, 17$ are good numbers and that 18 is not a good number.

2 Let ABC be an acute triangle. Points D, E, and F lie on segments BC, CA, and AB, respectively, and each of the three segments AD, BE, and CF contains the circumcenter of ABC. Prove that if any two of the ratios $\frac{BD}{DC}$, $\frac{CE}{EA}$, $\frac{AF}{FB}$, $\frac{BF}{FA}$, $\frac{AE}{EC}$, $\frac{CD}{DB}$ are integers, then triangle ABC is isosceles.

3 Let n be an integer greater than 3, and let $a_1, a_2, \cdots, a_n$ be non-negative real numbers with $a_1 + a_2 + \cdots + a_n = 2$. Determine the minimum value of

$$\frac{a_1}{a_2^2+1} + \frac{a_2}{a_3^2+1} + \cdots + \frac{a_n}{a_1^2+1}.$$

4 The set S consists of $n > 2$ points in the plane. The set P consists of m lines in the plane such that every line in P is an axis of symmetry for S. Prove that $m \le n$, and determine when equality holds.

Day 2

5 Point D lies inside triangle ABC such that $\angle DAC = \angle DCA = 30^\circ$ and $\angle DBA = 60^\circ$. Point E is the midpoint of segment BC. Point F lies on segment AC with $AF = 2FC$. Prove that $DE \perp EF$.

6 For $a, b, c \ge 0$ with $a + b + c = 1$, prove that
$\sqrt{a + \frac{(b-c)^2}{4}} + \sqrt{b} + \sqrt{c} \le \sqrt{3}$

7 Let a, b, c be integers each with absolute value less than or equal to 10. The cubic polynomial $f(x) = x^3 + ax^2 + bx + c$ satisfies the property

$$\left| f\left(2+\sqrt{3}\right) \right| < 0.0001.$$

Determine if $2 + \sqrt{3}$ is a root of f.

8 In a round robin chess tournament each player plays every other player exactly once. The winner of each game gets 1 point and the loser gets 0 points. If the game is tied, each player gets 0.5 points. Given a positive integer m, a tournament is said to have property $P(m)$ if the following holds: among every set S of m players, there is one player who won all her games against the other $m-1$ players in S and one player who lost all her games against the other $m-1$ players in S. For a given integer $m \geq 4$, determine the minimum value of n (as a function of m) such that the following holds: in every n-player round robin chess tournament with property $P(m)$, the final scores of the n players are all distinct.