Art of Problem Solving
1996 APMO

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APMO 1996

1 Let $ABCD$ be a quadrilateral $AB = BC = CD = DA$. Let MN and PQ be two segments perpendicular to the diagonal BD and such that the distance between them is $d > \frac{BD}{2}$, with $M \in AD$, $N \in DC$, $P \in AB$, and $Q \in BC$. Show that the perimeter of hexagon $AMNCQP$ does not depend on the position of MN and PQ so long as the distance between them remains constant.

2 Let m and n be positive integers such that $n \leq m$. Prove that

$$2^n n! \leq \frac{(m+n)!}{(m-n)!} \leq (m^2+m)^n$$

3 If $ABCD$ is a cyclic quadrilateral, then prove that the incenters of the triangles ABC, BCD, CDA, DAB are the vertices of a rectangle.

4 The National Marriage Council wishes to invite n couples to form 17 discussion groups under the following conditions:

(1) All members of a group must be of the same sex; i.e. they are either all male or all female.

(2) The difference in the size of any two groups is 0 or 1.

(3) All groups have at least 1 member.

(4) Each person must belong to one and only one group.

Find all values of n, $n \leq 1996$, for which this is possible. Justify your answer.

5 Let a, b, c be the lengths of the sides of a triangle. Prove that

$$\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} \leq \sqrt{a} + \sqrt{b} + \sqrt{c}$$

and determine when equality occurs.