India National Olympiad 2016

– Problems

P1 Let ABC be a triangle in which $AB = AC$. Suppose the orthocentre of the triangle lies on the incircle. Find the ratio $\frac{AB}{BC}$.

P2 For positive real numbers a, b, c which of the following statements necessarily implies $a = b = c$: (I) $a(b^3 + c^3) = b(c^3 + a^3) = c(a^3 + b^3)$, (II) $a(a^3 + b^3) = b(b^3 + c^3) = c(c^3 + a^3)$? Justify your answer.

P3 Let $\mathbb{N}$ denote the set of natural numbers. Define a function $T : \mathbb{N} \to \mathbb{N}$ by $T(2k) = k$ and $T(2k + 1) = 2k + 2$. We write $T^2(n) = T(T(n))$ and in general $T^k(n) = T^{k-1}(T(n))$ for any $k > 1$.

(i) Show that for each $n \in \mathbb{N}$, there exists k such that $T^k(n) = 1$.

(ii) For $k \in \mathbb{N}$, let c_k denote the number of elements in the set $\{n : T^k(n) = 1\}$. Prove that $c_{k+2} = c_{k+1} + c_k$, for $k \geq 1$.

P4 Suppose 2016 points of the circumference of a circle are colored red and the remaining points are colored blue . Given any natural number $n \geq 3$, prove that there is a regular n-sided polygon all of whose vertices are blue.

P5 Let ABC be a right-angle triangle with $\angle B = 90°$. Let D be a point on AC such that the inradii of the triangles ABD and CBD are equal. If this common value is r' and if r is the inradius of triangle ABC, prove that

$$\frac{1}{r'} = \frac{1}{r} + \frac{1}{BD}.$$

P6 Consider a nonconstant arithmetic progression $a_1, a_2, \cdots, a_n, \cdots$. Suppose there exist relatively prime positive integers $p > 1$ and $q > 1$ such that a_1^2, a_{p+1}^2 and a_{q+1}^2 are also the terms of the same arithmetic progression. Prove that the terms of the arithmetic progression are all integers.