All-Russian Olympiad 2016

Grade 11	Day 1	
1	There are 30 teams in NBA and every team play 82 games in the year. Bosses of NBA want to divide all teams on Western and Eastern Conferences (not necessary equally), such that number of games between teams from different conferences is half of number of all games. Can they do it?	
2	In the space given three segments A_1A_2, B_1B_2 and C_1C_2, do not lie in one plane and intersect at a point P. Let O_{ijk} be center of sphere that passes through the points A_i, B_j, C_k and P. Prove that $O_{111}O_{222}, O_{112}O_{221}, O_{121}O_{212}$ and$O_{211}O_{122}$ intersect at one point. (P.Kozhevnikov)	
3	We have sheet of paper, divided on 100×100 unit squares. In some squares we put rightangled isosceles triangles with leg =1 (Every triangle lies in one unit square and is half of this square). Every unit grid segment(boundary too) is under one leg of triangle. Find maximal number of unit squares, that don't contains triangles.	
4	There is three-dimensional space. For every integer n we build planes $x \pm y \pm z = n$. All space is divided on octahedrons and tetrahedrons. Point (x_0, y_0, z_0) has rational coordinates but not lies on any plane. Prove, that there is such natural k , that point (kx_0, ky_0, kz_0) lies strictly inside the octahedron of partition.	
Grade 11	Day 2	
5	Let $n \in \mathbb{N}$. $k_0, k_1, ..., k_{2n}$ - nonzero integers, and $k_0 + ... + k_{2n} \neq 0$. Can we always find such permutation $(a_0, ...a_{2n})$ of $(k_0, k_1, ..., k_{2n})$, that equation $a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + ... + a_0 = 0$ has not integer roots?	
6	There are $n > 1$ cities in the country, some pairs of cities linked two-way through straight flight. For every pair of cities there is exactly one aviaroute (can have interchanges). Major of every city X counted amount of such numberings of all cities from 1 to n , such that on every aviaroute with the beginning in X, numbers of cities are in ascending order. Every major, except one, noticed that results of counting are multiple of 2016. Prove, that result of last major is multiple of 2016 too.	

7 All russian olympiad 2016,Day 2 ,grade 9,P8 :
Let a, b, c, d be are positive numbers such that $a + b + c + d = 3$ .Prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \le \frac{1}{a^2b^2c^2d^2}$$

All russian olympiad 2016,Day 2,grade 11,P7 :
Let a, b, c, d be are positive numbers such that $a + b + c + d = 3$ .Prove that

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3} \le \frac{1}{a^3b^3c^3d^3}$$

Russia national 2016

8 Medians AM_A, BM_B, CM_C of triangle ABC intersect at M.Let Ω_A be circumcircle of triangle passes through midpoint of AM and M_B, M_C.Define Ω_B and Ω_C analogusly.Prove that Ω_A, Ω_B and Ω_C intersect at one point.(A.Yakubov)

All-Russian Olympiad 2015

– Grade 9

– Day 1

1 Real numbers a and b are chosen so that each of two quadratic trinomials $x^2 + ax + b$ and $x^2 + bx + a$ has two distinct real roots,and the product of these trinomials has exactly three distinct real roots.Determine all possible values of the sum of these three roots. *(S.Berlov)*

3 Let a, x, y be positive integer such that $a > 100, x > 100, y > 100$ and $y^2 - 1 = a^2(x^2 - 1)$. Find the minimum value of $\frac{a}{x}$.

– Day 2

5 100 integers are arranged in a circle. Each number is greater than the sum of the two subsequent numbers (in a clockwise order). Determine the maximal possible number of positive numbers in such circle. *(S.Berlov)*

6 A field has a shape of checkboard 41x41 square. A tank concealed in one of the cells of the field. By one shot, a fighter airplane fires one of the cells. If a shot hits the tank, then the tank moves to a neighboring cell of the field, otherwise it stays in its cell (the cells are neighbours if they share a side). A pilot has no information about the tank , one needs to hit it twice. Find the least number of shots sufficient to destroy the tank for sure. *(S.Berlov,A.Magazinov)*

7 An acute-angled ABC ($AB < AC$) is inscribed into a circle ω. Let M be the centroid of ABC, and let AH be an altitude of this triangle. A ray MH meets ω at A'. Prove that the circumcircle of the triangle $A'HB$ is tangent to AB. *(A.I. Golovanov , A.Yakubov)*

8 $N \geq 9$ distinct real numbers are written on a blackboard. All these numbers are nonnegative, and all are less than 1. It happens that for very 8 distinct numbers on the board, the board contains the ninth number distinct from eight such that the sum of all these nine numbers is integer. Find all values N for which this is possible. *(F. Nilov)*

– Grade 10

– Day 1

1 We say that a positive integer is an *almost square*, if it is equal to the product of two consecutive positive integers. Prove that every almost square can be expressed as a quotient of two almost squares.
V. Senderov

2 Given is a parallelogram $ABCD$, with $AB < AC < BC$. Points E and F are selected on the circumcircle ω of ABC so that the tangenst to ω at these points pass through point D and the segments AD and CE intersect.
It turned out that $\angle ABF = \angle DCE$. Find the angle $\angle ABC$.
A. Yakubov, S. Berlov

4 We denote by $S(k)$ the sum of digits of a positive integer number k. We say that the positive integer a is n-good, if there is a sequence of positive integers $a_0, a_1, \dots, a_n$, so that $a_n = a$ and $a_{i+1} = a_i - S(a_i)$ for all $i = 0, 1, ..., n-1$.
Is it true that for any positive integer n there exists a positive integer b, which is n-good, but not $(n+1)$-good?
A. Antropov

– Day 2

5 It is known that a cells square can be cut into n equal figures of k cells.
Prove that it is possible to cut it into k equal figures of n cells.

7 In an acute-angled and not isosceles triangle ABC, we draw the median AM and the height AH.
Points Q and P are marked on the lines AB and AC, respectively, so that the $QM \perp AC$ and $PM \perp AB$.
The circumcircle of PMQ intersects the line BC for second time at point X. Prove that $BH = CX$.
M. Didin

– Grade 11

– Day 1

1 Parallelogram $ABCD$ is such that angle $B < 90$ and $AB < BC$. Points E and F are on the circumference of ω inscribing triangle ABC, such that tangents to ω in those points pass through D. If $\angle EDA = \angle FDC$, find $\angle ABC$.

2 Let $n > 1$ be a natural number. We write out the fractions $\frac{1}{n}, \frac{2}{n}, \dots , \dfrac{n-1}{n}$ such that they are all in their simplest form. Let the sum of the numerators be $f(n)$. For what $n > 1$ is one of $f(n)$ and $f(2015n)$ odd, but the other is even?

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3 110 teams participate in a volleyball tournament. Every team has played every other team exactly once (there are no ties in volleyball). Turns out that in any set of 55 teams, there is one which has lost to no more than 4 of the remaining 54 teams. Prove that in the entire tournament, there is a team that has lost to no more than 4 of the remaining 109 teams.

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4 You are given N such that $n \geq 3$. We call a set of N points on a plane acceptable if their abscissae are unique, and each of the points is coloured either red or blue. Let's say that a polynomial $P(x)$ divides a set of acceptable points either if there are no red dots above the graph of $P(x)$, and below, there are no blue dots, or if there are no blue dots above the graph of $P(x)$ and there are no red dots below. Keep in mind, dots of both colors can be present on the graph of $P(x)$ itself. For what least value of k is an arbitrary set of N points divisible by a polynomial of degree k?

– Day 2

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5 An immortal flea jumps on whole points of the number line, beginning with 0. The length of the first jump is 3, the second 5, the third 9, and so on. The length of k^{th} jump is equal to $2^k + 1$. The flea decides whether to jump left or right on its own. Is it possible that sooner or later the flee will have been on every natural point, perhaps having visited some of the points more than once?

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6 Let a,b,c,d be real numbers satisfying $|a|, |b|, |c|, |d| > 1$ and $abc + abd + acd + bcd + a + b + c + d = 0$. Prove that $\frac{1}{a-1} + \frac{1}{b-1} + \frac{1}{c-1} + \frac{1}{d-1} > 0$

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7 A scalene triangle ABC is inscribed within circle ω. The tangent to the circle at point C intersects line AB at point D. Let I be the center of the circle inscribed within $\triangle ABC$. Lines AI and BI intersect the bisector of $\angle CDB$ in points Q and P, respectively. Let M be the midpoint of QP. Prove that MI passes through the middle of arc ACB of circle ω.

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8 Given natural numbers a and b, such that $a < b < 2a$. Some cells on a graph are colored such that in every rectangle with dimensions $A \times B$ or $B \times A$, at least one cell is colored. For which greatest α can you say that for every natural number N you can find a square $N \times N$ in which at least $\alpha \cdot N^2$ cells are colored?

All-Russian Olympiad 2014

– Grade level 9

Day 1

1 On a circle there are 99 natural numbers. If a, b are any two neighbouring numbers on the circle, then $a-b$ is equal to 1 or 2 or $\frac{a}{b}=2$. Prove that there exists a natural number on the circle that is divisible by 3.

S. Berlov

2 Sergei chooses two different natural numbers a and b. He writes four numbers in a notebook: a, $a+2$, b and $b+2$. He then writes all six pairwise products of the numbers of notebook on the blackboard. Let S be the number of perfect squares on the blackboard. Find the maximum value of S.

S. Berlov

3 In a convex n-gon, several diagonals are drawn. Among these diagonals, a diagonal is called *good* if it intersects exactly one other diagonal drawn (in the interior of the n-gon). Find the maximum number of good diagonals.

4 Let M be the midpoint of the side AC of acute-angled triangle ABC with $AB > BC$. Let Ω be the circumcircle of ABC. The tangents to Ω at the points A and C meet at P, and BP and AC intersect at S. Let AD be the altitude of the triangle ABP and ω the circumcircle of the triangle CSD. Suppose ω and Ω intersect at $K \neq C$. Prove that $\angle CKM = 90^\circ$.

V. Shmarov

Day 2

1 Define $m(n)$ to be the greatest proper natural divisor of $n \in \mathbb{N}$. Find all $n \in \mathbb{N}$ such that $n + m(n)$ is a power of 10.

N. Agakhanov

2 Let $ABCD$ be a trapezoid with $AB \parallel CD$ and Ω is a circle passing through A, B, C, D. Let ω be the circle passing through C, D and intersecting with CA, CB at A_1, B_1 respectively. A_2 and B_2 are the points symmetric to A_1 and B_1 respectively, with respect to the midpoints of CA and CB. Prove that the points A, B, A_2, B_2 are concyclic.

I. Bogdanov

3 In a country, mathematicians chose an $\alpha > 2$ and issued coins in denominations of 1 ruble, as well as α^k rubles for each positive integer k. α was chosen so that the value of each coins, except the smallest, was irrational. Is it possible that any natural number of rubles can be formed with at most 6 of each denomination of coins?

4 In a country of n cities, an express train runs both ways between any two cities. For any train, ticket prices either direction are equal, but for any different routes these prices are different. Prove that the traveler can select the starting city, leave it and go on, successively, $n-1$ trains, such that each fare is smaller than that of the previous fare. (A traveler can enter the same city several times.)

– Grade level 10

Day 1

1 Let a be *good* if the number of prime divisors of a is equal to 2. Do there exist 18 consecutive good natural numbers?

2 Given a function $f\colon \mathbb{R} \to \mathbb{R}$ with $f(x)^2 \le f(y)$ for all $x, y \in \mathbb{R}$, $x > y$, prove that $f(x) \in [0, 1]$ for all $x \in \mathbb{R}$.

3 There are n cells with indices from 1 to n. Originally, in each cell, there is a card with the corresponding index on it. Vasya shifts the card such that in the i-th cell is now a card with the number a_i. Petya can swap any two cards with the numbers x and y, but he must pay $2|x-y|$ coins. Show that Petya can return all the cards to their original position, not paying more than $|a_1 - 1| + |a_2 - 2| + \ldots + |a_n - n|$ coins.

4 Given a triangle ABC with $AB > BC$, let Ω be the circumcircle. Let M, N lie on the sides AB, BC respectively, such that $AM = CN$. Let K be the intersection of MN and AC. Let P be the incentre of the triangle AMK and Q be the K-excentre of the triangle CNK. If R is midpoint of the arc ABC of Ω then prove that $RP = RQ$.

M. Kungodjin

Day 2

1 Define $m(n)$ to be the greatest proper natural divisor of $n \in \mathbb{N}$. Find all $n \in \mathbb{N}$ such that $n + m(n)$ is a power of 10.

N. Agakhanov

2 Let M be the midpoint of the side AC of $\triangle ABC$. Let $P \in AM$ and $Q \in CM$ be such that $PQ = \frac{AC}{2}$. Let (ABQ) intersect with BC at $X \neq B$ and (BCP) intersect with BA at $Y \neq B$. Prove that the quadrilateral $BXMY$ is cyclic.

F. Ivlev, F. Nilov

3 In a country, mathematicians chose an $\alpha > 2$ and issued coins in denominations of 1 ruble, as well as α^k rubles for each positive integer k. α was chosen so that the value of each coins, except the smallest, was irrational. Is it possible that any natural number of rubles can be formed with at most 6 of each denomination of coins?

4 Given are n pairwise intersecting convex k-gons on the plane. Any of them can be transferred to any other by a homothety with a positive coefficient. Prove that there is a point in a plane belonging to at least $1 + \frac{n-1}{2k}$ of these k-gons.

– Grade level 11

Day 1

1 Does there exist positive $a \in \mathbb{R}$, such that

$$|\cos x| + |\cos ax| > \sin x + \sin ax$$

for all $x \in \mathbb{R}$?

N. Agakhanov

2 Peter and Bob play a game on a $n \times n$ chessboard. At the beginning, all squares are white apart from one black corner square containing a rook. Players take turns to move the rook to a white square and recolour the square black. The player who can not move loses. Peter goes first. Who has a winning strategy?

3 Positive rational numbers a and b are written as decimal fractions and each consists of a minimum period of 30 digits. In the decimal representation of $a - b$, the period is at least 15. Find the minimum value of $k \in \mathbb{N}$ such that, in the decimal representation of $a + kb$, the length of period is at least 15.

A. Golovanov

4 Given a triangle ABC with $AB > BC$, Ω is circumcircle. Let M, N are lie on the sides AB, BC respectively, such that $AM = CN$. $K(.) = MN \cap AC$ and P is incenter of the triangle AMK, Q is K-excenter of the triangle CNK (opposite to K and tangents to CN). If R is midpoint of the arc ABC of Ω then prove that $RP = RQ$.

M. Kungodjin

Day 2

1 Call a natural number n *good* if for any natural divisor a of n, we have that $a+1$ is also divisor of $n+1$. Find all good natural numbers.

S. Berlov

2 The sphere ω passes through the vertex S of the pyramid $SABC$ and intersects with the edges SA, SB, SC at A_1, B_1, C_1 other than S. The sphere Ω is the circumsphere of the pyramid $SABC$ and intersects with ω circumferential, lies on a plane which parallel to the plane (ABC).
Points A_2, B_2, C_2 are symmetry points of the points A_1, B_1, C_1 respect to midpoints of the edges SA, SB, SC respectively. Prove that the points A, B, C, A_2, B_2, and C_2 lie on a sphere.

3 If the polynomials $f(x)$ and $g(x)$ are written on a blackboard then we can also write down the polynomials $f(x) \pm g(x)$, $f(x)g(x)$, $f(g(x))$ and $cf(x)$, where c is an arbitrary real constant. The polynomials $x^3 - 3x^2 + 5$ and $x^2 - 4x$ are written on the blackboard. Can we write a nonzero polynomial of form $x^n - 1$ after a finite number of steps?

4 Two players play a card game. They have a deck of n distinct cards. About any two cards from the deck know which of them has a different (in this case, if A beats B, and B beats C, then it may be that C beats A). The deck is split between players in an arbitrary manner. In each turn the players over the top card from his deck and one whose card has a card from another player takes both cards and puts them to the bottom of your deck in any order of their discretion. Prove that for any initial distribution of cards, the players can with knowing the location agree and act so that one of the players left without a card.

E. Lakschmanov

All-Russian Olympiad 2013

– Grade level 9

Day 1

1 Given three distinct real numbers a, b, and c, show that at least two of the three following equations

$$(x-a)(x-b)=x-c$$

$$(x-c)(x-b)=x-a$$

$$(x-c)(x-a)=x-b$$

have real solutions.

2 Acute-angled triangle ABC is inscribed into circle Ω. Lines tangent to Ω at B and C intersect at P. Points D and E are on AB and AC such that PD and PE are perpendicular to AB and AC respectively. Prove that the orthocentre of triangle ADE is the midpoint of BC.

3 100 distinct natural numbers $a_1, a_2, a_3, \ldots, a_{100}$ are written on the board. Then, under each number a_i, someone wrote a number b_i, such that b_i is the sum of a_i and the greatest common factor of the other 99 numbers. What is the least possible number of distinct natural numbers that can be among $b_1, b_2, b_3, \ldots, b_{100}$?

4 N lines lie on a plane, no two of which are parallel and no three of which are concurrent. Prove that there exists a non-self-intersecting broken line $A_0A_1A_2A_3...A_N$ with N parts, such that on each of the N lines lies exactly one of the N segments of the line.

Day 2

1 $2n$ real numbers with a positive sum are aligned in a circle. For each of the numbers, we can see there are two sets of n numbers such that this number is on the end. Prove that at least one of the numbers has a positive sum for both of these two sets.

2 Peter and Basil together thought of ten quadratic trinomials. Then, Basil began calling consecutive natural numbers starting with some natural number. After each called number, Peter chose one of the ten polynomials at random and plugged in the called number. The results were recorded on the board. They eventually form a sequence. After they finished, their sequence was arithmetic. What is the greatest number of numbers that Basil could have called out?

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3 Squares $CAKL$ and $CBMN$ are constructed on the sides of acute-angled triangle ABC, outside of the triangle. Line CN intersects line segment AK at X, while line CL intersects line segment BM at Y. Point P, lying inside triangle ABC, is an intersection of the circumcircles of triangles KXN and LYM. Point S is the midpoint of AB. Prove that angle $\angle ACS = \angle BCP$.

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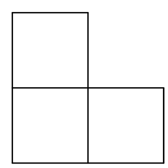
4 On a 55×55 square grid, 500 unit squares were cut out as well as 400 L-shaped pieces consisting of 3 unit squares (each piece can be oriented in any way) [refer to the figure]. Prove that at least two of the cut out pieces bordered each other before they were cut out.



– Grade level 10

Day 1

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1 Given three distinct real numbers a, b, and c, show that at least two of the three following equations

$$(x-a)(x-b) = x-c$$

$$(x-c)(x-b) = x-a$$

$$(x-c)(x-a) = x-b$$

have real solutions.

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2 Circle is divided into n arcs by n marked points on the circle. After that circle rotate an angle $2\pi k/n$ (for some positive integer k), marked points moved to n *new points* , dividing the circle into n *new arcs*. Prove that there is a new arc that lies entirely in the one of the old ars.
(It is believed that the endpoints of arcs belong to it.)

I. Mitrophanov

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3 Find all positive integers k such that for the first k prime numbers $2, 3, \ldots, p_k$ there exist positive integers a and $n > 1$, such that $2 \cdot 3 \cdot \ldots \cdot p_k - 1 = a^n$.

V. Senderov

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4 Inside the inscribed quadrilateral $ABCD$ are marked points P and Q, such that $\angle PDC + \angle PCB$, $\angle PAB + \angle PBC$, $\angle QCD + \angle QDA$ and $\angle QBA + \angle QAD$ are all equal to 90°. Prove that the line PQ has equal angles with lines AD and BC.

A. Pastor

Day 2

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1 Does exist natural n, such that for any non-zero digits a and b

$$\overline{ab} \mid \overline{anb} \text{ ?}$$

(Here by $\overline{x\ldots y}$ denotes the number obtained by concatenation decimal digits x, $\ldots$, y.)

V. Senderov

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2 Peter and Vasil together thought of ten 5-degree polynomials. Then, Vasil began calling consecutive natural numbers starting with some natural number. After each called number, Peter chose one of the ten polynomials at random and plugged in the called number. The results were recorded on the board. They eventually form a sequence. After they finished, their sequence was arithmetic. What is the greatest number of numbers that Vasil could have called out?

A. Golovanov

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3 The incircle of triangle ABC has centre I and touches the sides BC, CA, AB at points A_1, B_1, C_1, respectively. Let I_a, I_b, I_c be excentres of triangle ABC, touching the sides BC, CA, AB respectively. The segments I_aB_1 and I_bA_1 intersect at C_2. Similarly, segments I_bC_1 and I_cB_1 intersect at A_2, and the segments I_cA_1 and I_aC_1 at B_2. Prove that I is the center of the circumcircle of the triangle $A_2B_2C_2$.

L. Emelyanov, A. Polyansky

4 A square with horizontal and vertical sides is drawn on the plane. It held several segments parallel to the sides, and there are no two segments which lie on one line or intersect at an interior point for both segments. It turned out that the segments cuts square into rectangles, and any vertical line intersecting the square and not containing segments of the partition intersects exactly k rectangles of the partition, and any horizontal line intersecting the square and not containing segments of the partition intersects exactly ℓ rectangles. How much the number of rectangles can be?

I. Bogdanov, D. Fon-Der-Flaass

– Grade level 11

Day 1

1 Let $P(x)$ and $Q(x)$ be (monic) polynomials with real coefficients (the first coefficient being equal to 1), and $\deg P(x) = \deg Q(x) = 10$. Prove that if the equation $P(x) = Q(x)$ has no real solutions, then $P(x+1) = Q(x-1)$ has a real solution.

2 The inscribed and exscribed sphere of a triangular pyramid $ABCD$ touch her face BCD at different points X and Y. Prove that the triangle AXY is obtuse triangle.

3 Find all positive k such that product of the first k odd prime numbers, reduced by 1 is exactly degree of natural number (which more than one).

4 On each of the cards written in 2013 by number, all of these 2013 numbers are different. The cards are turned down by numbers. In a single move is allowed to point out the ten cards and in return will report one of the numbers written on them (do not know what). For what most w guaranteed to be able to find w cards for which we know what numbers are written on each of them?

Day 2

1 101 distinct numbers are chosen among the integers between 0 and 1000. Prove that, among the absolute values of their pairwise differences, there are ten different numbers not exceeding 100.

2 Let a, b, c, d be positive real numbers such that $2(a+b+c+d) \geq abcd$. Prove that

$$a^2+b^2+c^2+d^2 \geq abcd.$$

3 The head of the Mint wants to release 12 coins denominations (each - a natural number rubles) so that any amount from 1 to 6543 rubles could be paid without having to pass, using no more than 8 coins. Can he do it? (If the payment amount you can use a few coins of the same denomination.)

4 Let ω be the incircle of the triangle ABC and with centre I. Let Γ be the circumcircle of the triangle AIB. Circles ω and Γ intersect at the point X and Y. Let Z be the intersection of the common tangents of the circles ω and Γ. Show that the circumcircle of the triangle XYZ is tangent to the circumcircle of the triangle ABC.

All-Russian Olympiad 2012

– Grade level 9

Day 1

1 Let $a_1, \dots a_{11}$ be distinct positive integers, all at least 2 and with sum 407. Does there exist an integer n such that the sum of the 22 remainders after the division of n by $a_1, a_2, \dots, a_{11}, 4a_1, 4a_2, \dots, 4a_{11}$ is 2012?

2 A regular 2012-gon is inscribed in a circle. Find the maximal k such that we can choose k vertices from given 2012 and construct a convex k-gon without parallel sides.

3 Consider the parallelogram $ABCD$ with obtuse angle A. Let H be the feet of perpendicular from A to the side BC. The median from C in triangle ABC meets the circumcircle of triangle ABC at the point K. Prove that points K, H, C, D lie on the same circle.

4 The positive real numbers $a_1, \dots, a_n$ and k are such that $a_1 + \dots + a_n = 3k$, $a_1^2 + \dots + a_n^2 = 3k^2$ and $a_1^3 + \dots + a_n^3 > 3k^3 + k$. Prove that the difference between some two of $a_1, \dots, a_n$ is greater than 1.

Day 2

1 101 wise men stand in a circle. Each of them either thinks that the Earth orbits Jupiter or that Jupiter orbits the Earth. Once a minute, all the wise men express their opinion at the same time. Right after that, every wise man who stands between two people with a different opinion from him changes his opinion himself. The rest do not change. Prove that at one point they will all stop changing opinions.

2 The points A_1, B_1, C_1 lie on the sides sides BC, AC and AB of the triangle ABC respectively. Suppose that $AB_1 - AC_1 = CA_1 - CB_1 = BC_1 - BA_1$. Let I_A, I_B, I_C be the incentres of triangles AB_1C_1, A_1BC_1 and A_1B_1C respectively. Prove that the circumcentre of triangle $I_AI_BI_C$ is the incentre of triangle ABC.

3 Initially, ten consecutive natural numbers are written on the board. In one turn, you may pick any two numbers from the board (call them a and b) and replace them with the numbers $a^2 - 2011b^2$ and ab. After several turns, there were no initial numbers left on the board. Could there, at this point, be again, ten consecutive natural numbers?

4 In a city's bus route system, any two routes share exactly one stop, and every route includes at least four stops. Prove that the stops can be classified into two groups such that each route includes stops from each group.

– Grade level 10

Day 1

1 Let $a_1, \dots, a_{10}$ be distinct positive integers, all at least 3 and with sum 678. Does there exist a positive integer n such that the sum of the 20 remainders of n after division by $a_1, a_2, \dots, a_{10}, 2a_1, 2a_2, \dots, 2a_{10}$ is 2012?

2 The inscribed circle ω of the non-isosceles acute-angled triangle ABC touches the side BC at the point D. Suppose that I and O are the centres of inscribed circle and circumcircle of triangle ABC respectively. The circumcircle of triangle ADI intersects AO at the points A and E. Prove that AE is equal to the radius r of ω.

3 Any two of the real numbers a_1, a_2, a_3, a_4, a_5 differ by no less than 1. There exists some real number k satisfying

$$a_1 + a_2 + a_3 + a_4 + a_5 = 2k$$

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 = 2k^2$$

Prove that $k^2 \geq 25/3$.

4 Initially there are $n+1$ monomials on the blackboard: $1, x, x^2, \dots, x^n$. Every minute each of k boys simultaneously write on the blackboard the sum of some two polynomials that were written before. After m minutes among others there are the polynomials $S_1 = 1 + x, S_2 = 1 + x + x^2, S_3 = 1 + x + x^2 + x^3, \dots, S_n = 1 + x + x^2 + \dots + x^n$ on the blackboard. Prove that $m \geq \frac{2n}{k+1}$.

Day 2

1 101 wise men stand in a circle. Each of them either thinks that the Earth orbits Jupiter or that Jupiter orbits the Earth. Once a minute, all the wise men express their opinion at the same time. Right after that, every wise man who stands between two people with a different opinion from him changes his opinion himself. The rest do not change. Prove that at one point they will all stop changing opinions.

2 Does there exist natural numbers a, b, c all greater than 10^{10} such that their product is divisible by each of these numbers increased by 2012?

3 On a Cartesian plane, n parabolas are drawn, all of which are graphs of quadratic trinomials. No two of them are tangent. They divide the plane into many areas, one of which is above all the parabolas. Prove that the border of this area has no more than $2(n-1)$ corners (i.e. the intersections of a pair of parabolas).

4 The point E is the midpoint of the segment connecting the orthocentre of the scalene triangle ABC and the point A. The incircle of triangle ABC incircle is tangent to AB and AC at points C' and B' respectively. Prove that point F, the point symmetric to point E with respect to line $B'C'$, lies on the line that passes through both the circumcentre and the incentre of triangle ABC.

– Grade level 11

Day 1

1 Initially, there are 111 pieces of clay on the table of equal mass. In one turn, you can choose several groups of an equal number of pieces and push the pieces into one big piece for each group. What is the least number of turns after which you can end up with 11 pieces no two of which have the same mass?

2 Any two of the real numbers a_1, a_2, a_3, a_4, a_5 differ by no less than 1. There exists some real number k satisfying

$$a_1 + a_2 + a_3 + a_4 + a_5 = 2k$$

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 = 2k^2$$

Prove that $k^2 \geq 25/3$.

3 A plane is coloured into black and white squares in a chessboard pattern. Then, all the white squares are coloured red and blue such that any two initially white squares that share a corner are different colours. (One is red and the other is blue.) Let ℓ be a line not parallel to the sides of any squares. For every line segment I that is parallel to ℓ, we can count the difference between the length of its red and its blue areas. Prove that for every such line ℓ there exists a number C that exceeds all those differences that we can calculate.

4 Given is a pyramid $SA_1A_2A_3\dots A_n$ whose base is convex polygon $A_1A_2A_3\dots A_n$. For every $i = 1, 2, 3, \dots, n$ there is a triangle $X_iA_iA_{i+1}$ congruent to triangle SA_iA_{i+1} that lies on the same side from A_iA_{i+1} as the base of that pyramid. (You can assume a_1 is the same as a_{n+1}.) Prove that these triangles together cover the entire base.

Day 2

1 Given is the polynomial $P(x)$ and the numbers $a_1, a_2, a_3, b_1, b_2, b_3$ such that $a_1a_2a_3 \neq 0$. Suppose that for every x, we have

$$P(a_1x + b_1) + P(a_2x + b_2) = P(a_3x + b_3)$$

Prove that the polynomial $P(x)$ has at least one real root.

2 The points A_1, B_1, C_1 lie on the sides BC, CA and AB of the triangle ABC respectively. Suppose that $AB_1 - AC_1 = CA_1 - CB_1 = BC_1 - BA_1$. Let O_A, O_B and O_C be the circumcentres of triangles AB_1C_1, A_1BC_1 and A_1B_1C respectively. Prove that the incentre of triangle $O_AO_BO_C$ is the incentre of triangle ABC too.

3 On a circle there are $2n + 1$ points, dividing it into equal arcs ($n \geq 2$). Two players take turns to erase one point. If after one player's turn, it turned out that all the triangles formed by the remaining points on the circle were obtuse, then the player wins and the game ends.
Who has a winning strategy: the starting player or his opponent?

4 For a positive integer n define $S_n = 1! + 2! + \ldots + n!$. Prove that there exists an integer n such that S_n has a prime divisor greater than 10^{2012}.

All-Russian Olympiad 2011

– Grade 9

Day 1

1 A quadratic trinomial $P(x)$ with the x^2 coefficient of one is such, that $P(x)$ and $P(P(P(x)))$ share a root. Prove that $P(0) * P(1) = 0$.

2 Given is an acute angled triangle ABC. A circle going through B and the triangle's circumcenter, O, intersects BC and BA at points P and Q respectively. Prove that the intersection of the heights of the triangle POQ lies on line AC.

3 A convex 2011-gon is drawn on the board. Peter keeps drawing its diagonals in such a way, that each newly drawn diagonal intersected no more than one of the already drawn diagonals. What is the greatest number of diagonals that Peter can draw?

4 Do there exist any three relatively prime natural numbers so that the square of each of them is divisible by the sum of the two remaining numbers?

Day 2

1 For some 2011 natural numbers, all the $\frac{2010 \cdot 2011}{2}$ possible sums were written out on a board. Could it have happened that exactly one third of the written numbers were divisible by three and also exactly one third of them give a remainder of one when divided by three?

2 In the notebooks of Peter and Nick, two numbers are written. Initially, these two numbers are 1 and 2 for Peter and 3 and 4 for Nick. Once a minute, Peter writes a quadratic trinomial $f(x)$, the roots of which are the two numbers in his notebook, while Nick writes a quadratic trinomial $g(x)$ the roots of which are the numbers in *his* notebook. If the equation $f(x) = g(x)$ has two distinct roots, one of the two boys replaces the numbers in his notebook by those two roots. Otherwise, nothing happens. If Peter once made one of his numbers 5, what did the other one of his numbers become?

3 Let ABC be an equilateral triangle. A point T is chosen on AC and on arcs AB and BC of the circumcircle of ABC, M and N are chosen respectively, so that MT is parallel to BC and NT is parallel to AB. Segments AN and MT

intersect at point X, while CM and NT intersect in point Y. Prove that the perimeters of the polygons $AXYC$ and $XMBNY$ are the same.

4 There are some counters in some cells of 100×100 board. Call a cell *nice* if there are an even number of counters in adjacent cells. Can exactly one cell be *nice*?

K. Knop

– Grade 10

Day 1

1 In every cell of a table with n rows and ten columns, a digit is written. It is known that for every row A and any two columns, you can always find a row that has different digits from A only when it intersects with two columns. Prove that $n \geq 512$.

2 Nine quadratics, $x^2+a_1x+b_1, x^2+a_2x+b_2, ..., x^2+a_9x+b_9$ are written on the board. The sequences $a_1, a_2, ..., a_9$ and $b_1, b_2, ..., b_9$ are arithmetic. The sum of all nine quadratics has at least one real root. What is the the greatest possible number of original quadratics that can have no real roots?

3 The graph G is not 3-coloured. Prove that G can be divided into two graphs M and N such that M is not 2-coloured and N is not 1-coloured.

V. Dolnikov

4 Perimeter of triangle ABC is 4. Point X is marked at ray AB and point Y is marked at ray AC such that AX=AY=1. BC intersects XY at point M. Prove that perimeter of one of triangles ABM or ACM is 2.
(V. Shmarov).

Day 2

1 Given are 10 distinct real numbers. Kyle wrote down the square of the difference for each pair of those numbers in his notebook, while Peter wrote in his notebook the absolute value of the differences of the squares of these numbers. Is it possible for the two boys to have the same set of 45 numbers in their notebooks?

2 Given is an acute triangle ABC. Its heights BB_1 and CC_1 are extended past points B_1 and C_1. On these extensions, points P and Q are chosen, such that angle PAQ is right. Let AF be a height of triangle APQ. Prove that angle BFC is a right angle.

3 For positive integers $a > b > 1$, define

$$x_n = \frac{a^n - 1}{b^n - 1}$$

Find the least d such that for any a, b, the sequence x_n does not contain d consecutive prime numbers.

V. Senderov

4 A 2010×2010 board is divided into corner-shaped figures of three cells. Prove that it is possible to mark one cell in each figure such that each row and each column will have the same number of marked cells.

I. Bogdanov & O. Podlipsky

– Grade 11

Day 1

1 Two natural numbers d and d', where $d' > d$, are both divisors of n. Prove that $d' > d + \frac{d^2}{n}$.

2 On side BC of parallelogram $ABCD$ (A is acute) lies point T so that triangle ATD is an acute triangle. Let O_1, O_2, and O_3 be the circumcenters of triangles ABT, DAT, and CDT respectively. Prove that the orthocenter of triangle $O_1O_2O_3$ lies on line AD.

3 There are 999 scientists. Every 2 scientists are both interested in exactly 1 topic and for each topic there are exactly 3 scientists that are interested in that topic. Prove that it is possible to choose 250 topics such that every scientist is interested in at most 1 theme.

A. Magazinov

4 Ten cars are moving at the road. There are some cities at the road. Each car is moving with some constant speed through cities and with some different constant speed outside the cities (different cars may move with different speed).

There are 2011 points at the road. Cars don't overtake at the points. Prove that there are 2 points such that cars pass through these points in the same order.

S. Berlov

Day 2

1 Given are two distinct monic cubics $F(x)$ and $G(x)$. All roots of the equations $F(x) = 0$, $G(x) = 0$ and $F(x) = G(x)$ are written down. There are eight numbers written. Prove that the greatest of them and the least of them cannot be both roots of the polynomial $F(x)$.

2 There are more than n^2 stones on the table. Peter and Vasya play a game, Peter starts. Each turn, a player can take any prime number less than n stones, or any multiple of n stones, or 1 stone. Prove that Peter always can take the last stone (regardless of Vasya's strategy).

S Berlov

3 Let $P(a)$ be the largest prime positive divisor of $a^2 + 1$. Prove that exist infinitely many positive integers a, b, c such that $P(a) = P(b) = P(c)$.

A. Golovanov

4 Let N be the midpoint of arc ABC of the circumcircle of triangle ABC, let M be the midpoint of AC and let I_1, I_2 be the incentres of triangles ABM and CBM. Prove that points I_1, I_2, B, N lie on a circle.

M. Kungojin

All-Russian Olympiad 2010

– Grade level 9

Day 1

1 There are 24 different pencils, 4 different colors, and 6 pencils of each color. They were given to 6 children in such a way that each got 4 pencils. What is the least number of children that you can randomly choose so that you can guarantee that you have pencils of all colors.

P.S. for 10 grade gives same problem with 40 pencils, 10 of each color and 10 children.

2 There are 100 random, distinct real numbers corresponding to 100 points on a circle. Prove that you can always choose 4 consecutive points in such a way that the sum of the two numbers corresponding to the points on the outside is always greater than the sum of the two numbers corresponding to the two points on the inside.

3 Lines tangent to circle O in points A and B, intersect in point P. Point Z is the center of O. On the minor arc AB, point C is chosen not on the midpoint of the arc. Lines AC and PB intersect at point D. Lines BC and AP intersect at point E. Prove that the circumcentres of triangles ACE, BCD, and PCZ are collinear.

4 There are 100 apples on the table with total weight of 10 kg. Each apple weighs no less than 25 grams. The apples need to be cut for 100 children so that each of the children gets 100 grams. Prove that you can do it in such a way that each piece weighs no less than 25 grams.

Day 2

1 Let $a \neq ba, b \in \mathbb{R}$ such that $(x^2 + 20ax + 10b)(x^2 + 20bx + 10a) = 0$ has no roots for x. Prove that $20(b - a)$ is not an integer.

2 Each of 1000 elves has a hat, red on the inside and blue on the outside or vise versa. An elf with a hat that is red outside can only lie, and an elf with a hat that is blue outside can only tell the truth. One day every elf tells every other elf, Your hat is red on the outside. During that day, some of the elves turn their hats inside out at any time during the day. (An elf can do that more than once per day.) Find the smallest possible number of times any hat is turned inside out.

3 Let us call a natural number *unlucky* if it cannot be expressed as $\frac{x^2-1}{y^2-1}$ with natural numbers $x, y > 1$. Is the number of *unlucky* numbers finite or infinite?

4 In a acute triangle ABC, the median, AM, is longer than side AB. Prove that you can cut triangle ABC into 3 parts out of which you can construct a rhombus.

– Grade level 10

Day 1

1 There are 24 different pencils, 4 different colors, and 6 pencils of each color. They were given to 6 children in such a way that each got 4 pencils. What is the least number of children that you can randomly choose so that you can guarantee that you have pencils of all colors.

P.S. for 10 grade gives same problem with 40 pencils, 10 of each color and 10 children.

2 There are 100 random, distinct real numbers corresponding to 100 points on a circle. Prove that you can always choose 4 consecutive points in such a way that the sum of the two numbers corresponding to the points on the outside is always greater than the sum of the two numbers corresponding to the two points on the inside.

3 Let O be the circumcentre of the acute non-isosceles triangle ABC. Let P and Q be points on the altitude AD such that OP and OQ are perpendicular to AB and AC respectively. Let M be the midpoint of BC and S be the circumcentre of triangle OPQ. Prove that $\angle BAS = \angle CAM$.

4 In each unit square of square $100 * 100$ write any natural number. Called rectangle with sides parallel sides of square *good* if sum of number inside rectangle divided by 17. We can painted all unit squares in *good* rectangle. One unit square cannot painted twice or more.
Find maximum d for which we can guaranteed paint at least d points.

Day 2

1 Let $a \neq ba, b \in \mathbb{R}$ such that $(x^2 + 20ax + 10b)(x^2 + 20bx + 10a) = 0$ has no roots for x. Prove that $20(b - a)$ is not an integer.

2 Into triangle ABC gives point K lies on bisector of $\angle BAC$. Line CK intersect circumcircle ω of triangle ABC at $M \neq C$. Circle Ω passes through A, touch CM at K and intersect segment AB at $P \neq A$ and ω at $Q \neq A$.
Prove, that P, Q, M lies at one line.

3 Given $n \geq 3$ pairwise different prime numbers $p_1, p_2, ..., p_n$. Given, that for any $k \in \{1, 2, ..., n\}$ residue by division of $\prod_{i \neq k} p_i$ by p_k equals one number r. Prove, that $r \leq n - 2$.

4 In the county some pairs of towns connected by two-way non-stop flight. From any town we can flight to any other (may be not on one flight). Gives, that if we consider any cyclic (i.e. beginning and finish towns match) route, consisting odd number of flights, and close all flights of this route, then we can found two towns, such that we can't fly from one to other.
Proved, that we can divided all country on 4 regions, such that any flight connected towns from other regions.

– Grade level 11

Day 1

1 Do there exist non-zero reals numbers $a_1, a_2, ..., a_{10}$ for which

$$(a_1 + \frac{1}{a_1})(a_2 + \frac{1}{a_2}) \cdots (a_{10} + \frac{1}{a_{10}}) = (a_1 - \frac{1}{a_1})(a_2 - \frac{1}{a_2}) \cdots (a_{10} - \frac{1}{a_{10}}) \ ?$$

2 On an $n \times n$ chart, where $n \geq 4$, stand "+" signs in the cells of the main diagonal and "$-$" signs in all the other cells. You can change all the signs in one row or in one column, from $-$ to $+$ or from $+$ to $-$. Prove that you will always have n or more $+$ signs after finitely many operations.

3 Quadrilateral $ABCD$ is inscribed into circle ω, AC intersect BD in point K. Points M_1, M_2, M_3, M_4-midpoints of arcs AB, BC, CD, and DA respectively. Points I_1, I_2, I_3, I_4-incenters of triangles ABK, BCK, CDK, and DAK respectively. Prove that lines M_1I_1, M_2I_2, M_3I_3, and M_4I_4 all intersect in one point.

4 Given is a natural number $n \geq 3$. What is the smallest possible value of k if the following statements are true?
For every n points $A_i = (x_i, y_i)$ on a plane, where no three points are collinear, and for any real numbers c_i $(1 \leq i \leq n)$ there exists such polynomial $P(x, y)$, the degree of which is no more than k, where $P(x_i, y_i) = c_i$ for every $i = 1, \dots, n$.

(The degree of a nonzero monomial $a_{i,j}x^iy^j$ is $i + j$, while the degree of polynomial $P(x, y)$ is the greatest degree of the degrees of its monomials.)

Day 2

1 If $n \in \mathbb{N}n > 1$ prove that for every n you can find n consecutive natural numbers the product of which is divisible by all primes not exceeding $2n + 1$, but is not divisible by any other primes.

2 Could the four centers of the circles inscribed into the faces of a tetrahedron be coplanar?

(vertexes of tetrahedron not coplanar)

3 Polynomial $P(x)$ with degree $n \geq 3$ has n real roots $x_1 < x_2 < x_3 < ... < x_n$, such that $x_2 - x_1 < x_3 - x_2 < .... < x_n - x_{n-1}$. Prove that the maximum of the function $y = |P(x)|$ where x is on the interval $[x_1, x_n]$, is in the interval $[x_n - 1, x_n]$.

4 In a board school, there are 9 subjects, 512 students, and 256 rooms (two people in each room.) For every student there is a set (a subset of the 9 subjects) of subjects the student is interested in. Each student has a different set of subjects, (s)he is interested in, from all other students. (Exactly one student has no subjects (s)he is interested in.)
Prove that the whole school can line up in a circle in such a way that every pair of the roommates has the two people standing next to each other, and those pairs of students standing next to each other that are not roommates, have the following properties. One of the two students is interested in all the subjects that the other student is interested in, and also exactly one more subject.

All-Russian Olympiad 2009

– Grade level 9

1 The denominators of two irreducible fractions are 600 and 700. Find the minimum value of the denominator of their sum (written as an irreducible fraction).

2 Let be given a triangle ABC and its internal angle bisector BD ($D \in BC$). The line BD intersects the circumcircle Ω of triangle ABC at B and E. Circle ω with diameter DE cuts Ω again at F. Prove that BF is the symmedian line of triangle ABC.

3 Given are positive integers $n > 1$ and a so that $a > n^2$, and among the integers $a+1, a+2, \ldots, a+n$ one can find a multiple of each of the numbers $n^2+1, n^2+2, \ldots, n^2+n$. Prove that $a > n^4 - n^3$.

4 There are n cups arranged on the circle. Under one of cups is hiden a coin. For every move, it is allowed to choose 4 cups and verify if the coin lies under these cups. After that, the cups are returned into its former places and the coin moves to one of two neigbor cups. What is the minimal number of moves we need in order to eventually find where the coin is?

5 Let a, b, c be three real numbers satisfying that

$$\begin{cases} (a+b)(b+c)(c+a) & = & abc \\ \left(a^3+b^3\right)\left(b^3+c^3\right)\left(c^3+a^3\right) & = & a^3b^3c^3 \end{cases}$$

Prove that $abc = 0$.

6 Can be colored the positive integers with 2009 colors if we know that each color paints infinitive integers and that we can not find three numbers colored by three different colors for which the product of two numbers equal to the third one?

7 We call any eight squares in a diagonal of a chessboard as a fence. The rook is moved on the chessboard in such way that he stands neither on each square over one time nor on the squares of the fences (the squares which the rook passes is not considered ones it has stood on). Then what is the maximum number of times which the rook jumped over the fence?

8 Triangles ABC and $A_1B_1C_1$ have the same area. Using compass and ruler, can we always construct triangle $A_2B_2C_2$ equal to triangle $A_1B_1C_1$ so that the lines AA_2, BB_2, and CC_2 are parallel?

– Grade level 10

1 Find all value of n for which there are nonzero real numbers a, b, c, d such that after expanding and collecting similar terms, the polynomial $(ax+b)^{100}-(cx+d)^{100}$ has exactly n nonzero coefficients.

2 Let be given a triangle ABC and its internal angle bisector BD ($D \in BC$). The line BD intersects the circumcircle Ω of triangle ABC at B and E. Circle ω with diameter DE cuts Ω again at F. Prove that BF is the symmedian line of triangle ABC.

3 How many times changes the sign of the function

$$f(x) = \cos x \cos \frac{x}{2} \cos \frac{x}{3} \cdots \cos \frac{x}{2009}$$

at the interval $\left[0, \frac{2009\pi}{2}\right]$?

4 On a circle there are 2009 nonnegative integers not greater than 100. If two numbers sit next to each other, we can increase both of them by 1. We can do this at most k times. What is the minimum k so that we can make all the numbers on the circle equal?

5 Given strictly increasing sequence $a_1 < a_2 < \dots$ of positive integers such that each its term a_k is divisible either by 1005 or 1006, but neither term is divisible by 97. Find the least possible value of maximal difference of consecutive terms $a_{i+1} - a_i$.

6 Given a finite tree T and isomorphism $f : T \to T$. Prove that either there exist a vertex a such that $f(a) = a$ or there exist two neighbor vertices a, b such that $f(a) = b$, $f(b) = a$.

7 The incircle (I) of a given scalene triangle ABC touches its sides BC, CA, AB at A_1, B_1, C_1, respectively. Denote ω_B, ω_C the incircles of quadrilaterals BA_1IC_1 and CA_1IB_1, respectively. Prove that the internal common tangent of ω_B and ω_C different from IA_1 passes through A.

8 Let x, y be two integers with $2 \le x, y \le 100$. Prove that $x^{2^n} + y^{2^n}$ is not a prime for some positive integer n.

– Grade level 11

1 In a country, there are some cities linked together by roads. The roads just meet each other inside the cities. In each city, there is a board which showing the shortest length of the road originating in that city and going through all other cities (the way can go through some cities more than one times and is not necessary to turn back to the originated city). Prove that 2 random numbers in the boards can't be greater or lesser than 1.5 times than each other.

2 Consider the sequence of numbers (a_n) $(n = 1, 2, \ldots)$ defined as follows: $a_1 \in (1, 2)$, $a_{k+1} = a_k + \frac{k}{a_k}$ $(k = 1, 2, \ldots)$. Prove that there exists at most one pair of distinct positive integers (i, j) such that $a_i + a_j$ is an integer.

3 Let $ABCD$ be a triangular pyramid such that no face of the pyramid is a right triangle and the orthocenters of triangles ABC, ABD, and ACD are collinear. Prove that the center of the sphere circumscribed to the pyramid lies on the plane passing through the midpoints of AB, AC and AD.

4 Given a set M of points (x, y) with integral coordinates satisfying $x^2+y^2 \le 10^{10}$. Two players play a game, making rather sophisticated moves in turn. One of them marks a point on his first move. After this, on each move the moving player marks a point, which is not yet marked and joins it with the previous marked point. So, they draw a broken line. The requirement is that lengths of edges of this broken line must strictly increase. The player, which can not make a move, loses. Who have a winning strategy?

5 Prove that
$$\log_a b + \log_b c + \log_c a \le \log_b a + \log_c b + \log_a c$$
for all $1 < a \le b \le c$.

6 There are k rooks on a 10×10 chessboard. We mark all the squares that at least one rook can capture (we consider the square where the rook stands as captured by the rook). What is the maximum value of k so that the following holds for some arrangement of k rooks: after removing any rook from the chessboard, there is at least one marked square not captured by any of the remaining rooks.

7 Let be given a parallelogram $ABCD$ and two points A_1, C_1 on its sides AB, BC, respectively. Lines AC_1 and CA_1 meet at P. Assume that the circumcircles of triangles AA_1P and CC_1P intersect at the second point Q inside triangle ACD. Prove that $\angle PDA = \angle QBA$.

8 Let x, y be two integers with $2 \leq x, y \leq 100$. Prove that $x^{2^n} + y^{2^n}$ is not a prime for some positive integer n.

All-Russian Olympiad 2008

– Grade level 9

1 Do there exist 14 positive integers, upon increasing each of them by 1,their product increases exactly 2008 times?

2 Numbers a, b, c are such that the equation $x^3 + ax^2 + bx + c$ has three real roots.Prove that if $-2 \leq a + b + c \leq 0$,then at least one of these roots belongs to the segment $[0, 2]$

3 In a scalene triangle ABC, H and M are the orthocenter an centroid respectively. Consider the triangle formed by the lines through A, B and C perpendicular to AM, BM and CM respectively. Prove that the centroid of this triangle lies on the line MH.

4 There are several scientists collaborating in Niichavo. During an 8-hour working day, the scientists went to cafeteria, possibly several times.It is known that for every two scientist, the total time in which exactly one of them was in cafeteria is at least x hours ($x > 4$).
What is the largest possible number of scientist that could work in Niichavo that day,in terms of x?

5 The distance between two cells of an infinite chessboard is defined as the minimum nuber to moves needed for a king for move from one to the other.One the board are chosen three cells on a pairwise distances equal to 100. How many cells are there that are on the distance 50 from each of the three cells?

6 The incircle of a triangle ABC touches the side AB and AC at respectively at X and Y. Let K be the midpoint of the arc $\widehat{AB}$ on the circumcircle of ABC. Assume that XY bisects the segment AK. What are the possible measures of angle BAC?

7 A natural number is written on the blackboard. Whenever number x is written, one can write any of the numbers $2x + 1$ and $\frac{x}{x+2}$. At some moment the number 2008 appears on the blackboard. Show that it was there from the very beginning.

8 We are given 3^{2k} apparently identical coins,one of which is fake,being lighter than the others. We also dispose of three apparently identical balances without weights, one of which is broken (and yields outcomes unrelated to the actual situations). How can we find the fake coin in $3k+1$ weighings?

– Grade level 10

1 Do there exist 14 positive integers, upon increasing each of them by 1,their product increases exactly 2008 times?

2 The columns of an $n \times n$ board are labeled 1 to n. The numbers $1, 2, ..., n$ are arranged in the board so that the numbers in each row and column are pairwise different. We call a cell "good" if the number in it is greater than the label of its column. For which n is there an arrangement in which each row contains equally many good cells?

3 A circle ω with center O is tangent to the rays of an angle BAC at B and C. Point Q is taken inside the angle BAC. Assume that point P on the segment AQ is such that $AQ \perp OP$. The line OP intersects the circumcircles ω_1 and ω_2 of triangles BPQ and CPQ again at points M and N. Prove that $OM = ON$.

4 The sequences $(a_n), (b_n)$ are defined by $a_1 = 1, b_1 = 2$ and

$$a_{n+1} = \frac{1 + a_n + a_n b_n}{b_n}, \quad b_{n+1} = \frac{1 + b_n + a_n b_n}{a_n}.$$

Show that $a_{2008} < 5$.

5 Determine all triplets of real numbers x, y, z satisfying

$$1 + x^4 \leq 2(y-z)^2, \quad 1 + y^4 \leq 2(x-z)^2, \quad 1 + z^4 \leq 2(x-y)^2.$$

6 In a scalene triangle ABC the altitudes AA_1 and CC_1 intersect at H, O is the circumcenter, and B_0 the midpoint of side AC. The line BO intersects side AC at P, while the lines BH and A_1C_1 meet at Q. Prove that the lines HB_0 and PQ are parallel.

7 For which integers $n > 1$ do there exist natural numbers $b_1, b_2, \ldots, b_n$ not all equal such that the number $(b_1 + k)(b_2 + k) \cdots (b_n + k)$ is a power of an integer for each natural number k? (The exponents may depend on k, but must be greater than 1)

8 On the cartesian plane are drawn several rectangles with the sides parallel to the coordinate axes. Assume that any two rectangles can be cut by a vertical or a horizontal line. Show that it's possible to draw one horizontal and one vertical line such that each rectangle is cut by at least one of these two lines.

– Grade level 11

1 Numbers a, b, c are such that the equation $x^3 + ax^2 + bx + c$ has three real roots.Prove that if $-2 \leq a + b + c \leq 0$,then at least one of these roots belongs to the segment $[0, 2]$

2 Petya and Vasya are given equal sets of N weights, in which the masses of any two weights are in ratio at most 1.25. Petya succeeded to divide his set into 10 groups of equal masses, while Vasya succeeded to divide his set into 11 groups of equal masses. Find the smallest possible N.

3 Given a finite set P of prime numbers, prove that there exists a positive integer x such that it can be written in the form $a^p + b^p$ (a, b are positive integers), for each $p \in P$, and cannot be written in that form for each p not in P.

4 Each face of a tetrahedron can be placed in a circle of radius 1. Show that the tetrahedron can be placed in a sphere of radius $\frac{3}{2\sqrt{2}}$.

5 The numbers from 51 to 150 are arranged in a 10×10 array. Can this be done in such a way that, for any two horizontally or vertically adjacent numbers a and b, at least one of the equations $x^2 - ax + b = 0$ and $x^2 - bx + a = 0$ has two integral roots?

6 A magician should determine the area of a hidden convex 2008-gon $A_1A_2 \cdots A_{2008}$. In each step he chooses two points on the perimeter, whereas the chosen points can be vertices or points dividing selected sides in selected ratios. Then his helper divides the polygon into two parts by the line through these two points and announces the area of the smaller of the two parts. Show that the magician can find the area of the polygon in 2006 steps.

7 In convex quadrilateral $ABCD$, the rays BA, CD meet at P, and the rays BC, AD meet at Q. H is the projection of D on PQ. Prove that there is a circle inscribed in $ABCD$ if and only if the incircles of triangles ADP, CDQ are visible from H under the same angle.

8 In a chess tournament $2n+3$ players take part. Every two play exactly one match. The schedule is such that no two matches are played at the same time, and each player, after taking part in a match, is free in at least n next (consecutive) matches. Prove that one of the players who play in the opening match will also play in the closing match.

All-Russian Olympiad 2007

– Grade level 8

Day 1

1 Given reals numbers a, b, c. Prove that at least one of three equations $x^2 + (a-b)x + (b-c) = 0$, $x^2 + (b-c)x + (c-a) = 0$, $x^2 + (c-a)x + (a-b) = 0$ has a real root.
O. Podlipsky

2 The numbers $1, 2, \dots, 100$ are written in the cells of a 10×10 table, each number is written once. In one move, Nazar may interchange numbers in any two cells. Prove that he may get a table where the sum of the numbers in every two adjacent (by side) cells is composite after at most 35 such moves.
N. Agakhanov

3 Given a rhombus $ABCD$. A point M is chosen on its side BC. The lines, which pass through M and are perpendicular to BD and AC, meet line AD in points P and Q respectively. Suppose that the lines PB, QC, AM have a common point. Find all possible values of a ratio $\frac{BM}{MC}$.
S. Berlov, F. Petrov, A. Akopyan

4 *A. Akopyan, A. Akopyan, A. Akopyan, I. Bogdanov*

A conjurer Arutyun and his assistant Amayak are going to show following super-trick. A circle is drawn on the board in the room. Spectators mark 2007 points on this circle, after that Amayak
removes one of them. Then Arutyun comes to the room and shows a semicircle, to which the removed point belonged. Explain, how Arutyun and Amayak may show this super-trick.

Day 2

5 The distance between Maykop and Belorechensk is 24 km. Two of three friends need to reach Belorechensk from Maykop and another friend wants to reach Maykop from Belorechensk. They have only one bike, which is initially in Maykop. Each guy may go on foot (with velocity at most 6 kmph) or on a bike (with velocity at most 18 kmph). It is forbidden to leave a bike on a road.

Prove that all of them may achieve their goals after 2 hours 40 minutes. (Only one guy may seat on the bike simultaneously).

Folclore

6 A line, which passes through the incentre I of the triangle ABC, meets its sides AB and BC at the points M and N respectively. The triangle BMN is acute. The points K, L are chosen on the side AC such that $\angle ILA = \angle IMB$ and $\angle KC = \angle INB$. Prove that $AM + KL + CN = AC$.

S. Berlov

7 For an integer $n > 3$ denote by $n?$ the product of all primes less than n. Solve the equation $n? = 2n + 16$.

V. Senderov

8 Given a matrix $\{a_{ij}\}_{i,j=0}^{9}$, $a_{ij} = 10i + j + 1$. Andrei is going to cover its entries by 50 rectangles 1×2 (each such rectangle contains two adjacent entries) so that the sum of 50 products in these rectangles is minimal possible. Help him.

A. Badzyan

– Grade level 9

Day 1

1 Unitary quadratic trinomials $f(x)$ and $g(x)$ satisfy the following interesting condition: $f(g(x)) = 0$ and $g(f(x)) = 0$ do not have real roots. Prove that at least one of equations $f(f(x)) = 0$ and $g(g(x)) = 0$ does not have real roots too.

S. Berlov

2 100 fractions are written on a board, their numerators are numbers from 1 to 100 (each once) and denominators are also numbers from 1 to 100 (also each once). It appears that the sum of these fractions equals to $a/2$ for some odd a. Prove that it is possible to interchange numerators of two fractions so that sum becomes a fraction with odd denominator.

N. Agakhanov, I. Bogdanov

3 Two players by turns draw diagonals in a regular $(2n+1)$-gon $(n > 1)$. It is forbidden to draw a diagonal, which was already drawn, or intersects an odd number of already drawn diagonals. The player, who has no legal move, loses. Who has a winning strategy?

K. Sukhov

4 BB_1 is a bisector of an acute triangle ABC. A perpendicular from B_1 to BC meets a smaller arc BC of a circumcircle of ABC in a point K. A perpendicular from B to AK meets AC in a point L. BB_1 meets arc AC in T. Prove that K, L, T are collinear.
V. Astakhov

Day 2

5 Two numbers are written on each vertex of a convex 100-gon. Prove that it is possible to remove a number from each vertex so that the remaining numbers on any two adjacent vertices are different.
F. Petrov

6 Let ABC be an acute triangle. The points M and N are midpoints of AB and BC respectively, and BH is an altitude of ABC. The circumcircles of AHN and CHM meet in P where $P \neq H$. Prove that PH passes through the midpoint of MN.
V. Filimonov

7 Given a matrix $\{a_{ij}\}_{i,j=0}^{9}$, $a_{ij} = 10i + j + 1$. Andrei is going to cover its entries by 50 rectangles 1×2 (each such rectangle contains two adjacent entries) so that the sum of 50 products in these rectangles is minimal possible. Help him.
A. Badzyan

8 Dima has written number $1/80!$, $1/81!$, $\ldots$, $1/99!$ on 20 infinite pieces of papers as decimal fractions (the following is written on the last piece: $\frac{1}{99!} = 0{,}00\ldots0010715\ldots$, 155 0-s before 1). Sasha wants to cut a fragment of N consecutive digits from one of pieces without the comma. For which maximal N he may do it so that Dima may not guess, from which piece Sasha has cut his fragment?

A. Golovanov

– Grade level 10

Day 1

1 Faces of a cube $9 \times 9 \times 9$ are partitioned onto unit squares. The surface of a cube is pasted over by 243 strips 2×1 without overlapping. Prove that the number of bent strips is odd.

A. Poliansky

2 Given polynomial $P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$. Put $m = \min\{a_0, a_0 + a_1, \ldots, a_0 + a_1 + \cdots + a_n\}$. Prove that $P(x) \geq mx^n$ for $x \geq 1$.
A. Khrabrov

3 BB_1 is a bisector of an acute triangle ABC. A perpendicular from B_1 to BC meets a smaller arc BC of a circumcircle of ABC in a point K. A perpendicular from B to AK meets AC in a point L. BB_1 meets arc AC in T. Prove that K, L, T are collinear.
V. Astakhov

4 Arutyun and Amayak show another effective trick. A spectator writes down on a board a sequence of N (decimal) digits. Amayak closes two adjacent digits by a black disc. Then Arutyun comes and says both closed digits (and their order). For which minimal N they may show such a trick?
K. Knop, O. Leontieva

Day 2

5 Given a set of $n > 2$ planar vectors. A vector from this set is called *long*, if its length is not less than the length of the sum of other vectors in this set. Prove that if each vector is long, then the sum of all vectors equals to zero.
N. Agakhanov

6 Two circles ω_1 and ω_2 intersect in points A and B. Let PQ and RS be segments of common tangents to these circles (points P and R lie on ω_1, points Q and S lie on ω_2). It appears that $RB \parallel PQ$. Ray RB intersects ω_2 in a point $W \neq B$. Find RB/BW.

S. Berlov

7 Given a convex polyhedron F. Its vertex A has degree 5, other vertices have degree 3. A colouring of edges of F is called nice, if for any vertex except A all three edges from it have different colours. It appears that the number of nice colourings is not divisible by 5. Prove that there is a nice colouring, in which some three consecutive edges from A are coloured the same way.
D. Karpov

8 Dima has written number $1/80!$, $1/81!$, $\ldots$, $1/99!$ on 20 infinite pieces of papers as decimal fractions (the following is written on the last piece: $\frac{1}{99!} = 0{,}00\ldots0010715\ldots$, 155 0-s before 1). Sasha wants to cut a fragment of N

consecutive digits from one of pieces without the comma. For which maximal N he may do it so that Dima may not guess, from which piece Sasha has cut his fragment?

A. Golovanov

– Grade level 11

Day 1

1 Prove that for $k > 10$ Nazar may replace in the following product some one cos by sin so that the new function $f_1(x)$ would satisfy inequality $|f_1(x)| \leq 3 \cdot 2^{-1-k}$ for all real x.

$$f(x) = \cos x \cos 2x \cos 3x \dots \cos 2^k x$$

N. Agakhanov

2 The incircle of triangle ABC touches its sides BC, AC, AB at the points A_1, B_1, C_1 respectively. A segment AA_1 intersects the incircle at the point $Q \neq A_1$. A line ℓ through A is parallel to BC. Lines A_1C_1 and A_1B_1 intersect ℓ at the points P and R respectively. Prove that $\angle PQR = \angle B_1QC_1$.
A. Polyansky

3 Arutyun and Amayak show another effective trick. A spectator writes down on a board a sequence of N (decimal) digits. Amayak closes two adjacent digits by a black disc. Then Arutyun comes and says both closed digits (and their order). For which minimal N they may show such a trick?
K. Knop, O. Leontieva

4 An infinite sequence (x_n) is defined by its first term $x_1 > 1$, which is a rational number, and the relation $x_{n+1} = x_n + \frac{1}{[x_n]}$ for all positive integers n. Prove that this sequence contains an integer.
A. Golovanov

Day 2

5 Two numbers are written on each vertex of a convex 100-gon. Prove that it is possible to remove a number from each vertex so that the remaining numbers on any two adjacent vertices are different.
F. Petrov

6 Do there exist non-zero reals a, b, c such that, for any $n > 3$, there exists a polynomial $P_n(x) = x^n + \cdots + ax^2 + bx + c$, which has exactly n (not necessary distinct) integral roots?
N. Agakhanov, I. Bogdanov

7 Given a tetrahedron T. Valentin wants to find two its edges a, b with no common vertices so that T is covered by balls with diameters a, b. Can he always find such a pair?

A. Zaslavsky

8 Given an undirected graph with N vertices. For any set of k vertices, where $1 \leq k \leq N$, there are at most $2k-2$ edges, which join vertices of this set. Prove that the edges may be coloured in two colours so that each cycle contains edges of both colours. (Graph may contain multiple edges).
I. Bogdanov, G. Chelnokov

All-Russian Olympiad 2006

– Grade level 9

1 Given a 15×15 chessboard. We draw a closed broken line without self-intersections such that every edge of the broken line is a segment joining the centers of two adjacent cells of the chessboard. If this broken line is symmetric with respect to a diagonal of the chessboard, then show that the length of the broken line is ≤ 200.

2 Show that there exist four integers a, b, c, d whose absolute values are all > 1000000 and which satisfy $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{1}{abcd}$.

3 Given a circle and 2006 points lying on this circle. Albatross colors these 2006 points in 17 colors. After that, Frankinfueter joins some of the points by chords such that the endpoints of each chord have the same color and two different chords have no common points (not even a common endpoint). Hereby, Frankinfueter intends to draw as many chords as possible, while Albatross is trying to hinder him as much as he can. What is the maximal number of chords Frankinfueter will always be able to draw?

4 Given a triangle ABC. Let a circle ω touch the circumcircle of triangle ABC at the point A, intersect the side AB at a point K, and intersect the side BC. Let CL be a tangent to the circle ω, where the point L lies on ω and the segment KL intersects the side BC at a point T. Show that the segment BT has the same length as the tangent from the point B to the circle ω.

5 Let a_1, a_2, ..., a_{10} be positive integers such that $a_1 < a_2 < ... < a_{10}$. For every k, denote by b_k the greatest divisor of a_k such that $b_k < a_k$. Assume that $b_1 > b_2 > ... > b_{10}$. Show that $a_{10} > 500$.

6 Let P, Q, R be points on the sides AB, BC, CA of a triangle ABC such that $AP = CQ$ and the quadrilateral $RPBQ$ is cyclic. The tangents to the circumcircle of triangle ABC at the points C and A intersect the lines RQ and RP at the points X and Y, respectively. Prove that $RX = RY$.

7 A 100×100 chessboard is cut into dominoes (1×2 rectangles). Two persons play the following game: At each turn, a player glues together two adjacent cells (which were formerly separated by a cut-edge). A player loses if, after his turn, the 100×100 chessboard becomes connected, i. e. between any two cells there exists a way which doesn't intersect any cut-edge. Which player has a winning strategy - the starting player or his opponent?

8 Given a quadratic trinomial $f(x) = x^2 + ax + b$. Assume that the equation $f(f(x)) = 0$ has four different real solutions, and that the sum of two of these solutions is -1. Prove that $b \leq -\frac{1}{4}$.

– Grade level 10

1 Given a 15×15 chessboard. We draw a closed broken line without self-intersections such that every edge of the broken line is a segment joining the centers of two adjacent cells of the chessboard. If this broken line is symmetric with respect to a diagonal of the chessboard, then show that the length of the broken line is ≤ 200.

2 If an integer $a > 1$ is given such that $(a-1)^3 + a^3 + (a+1)^3$ is the cube of an integer, then show that $4 \mid a$.

3 Given a circle and 2006 points lying on this circle. Albatross colors these 2006 points in 17 colors. After that, Frankinfueter joins some of the points by chords such that the endpoints of each chord have the same color and two different chords have no common points (not even a common endpoint). Hereby, Frankinfueter intends to draw as many chords as possible, while Albatross is trying to hinder him as much as he can. What is the maximal number of chords Frankinfueter will always be able to draw?

4 Consider an isosceles triangle ABC with $AB = AC$, and a circle ω which is tangent to the sides AB and AC of this triangle and intersects the side BC at the points K and L. The segment AK intersects the circle ω at a point M (apart from K). Let P and Q be the reflections of the point K in the points B and C, respectively. Show that the circumcircle of triangle PMQ is tangent to the circle ω.

5 Let a_1, a_2, ..., a_{10} be positive integers such that $a_1 < a_2 < ... < a_{10}$. For every k, denote by b_k the greatest divisor of a_k such that $b_k < a_k$. Assume that $b_1 > b_2 > ... > b_{10}$. Show that $a_{10} > 500$.

6 Let K and L be two points on the arcs AB and BC of the circumcircle of a triangle ABC, respectively, such that $KL \parallel AC$. Show that the incenters of triangles ABK and CBL are equidistant from the midpoint of the arc ABC of the circumcircle of triangle ABC.

7 Given a quadratic trinomial $f(x) = x^2 + ax + b$. Assume that the equation $f(f(x)) = 0$ has four different real solutions, and that the sum of two of these solutions is -1. Prove that $b \leq -\frac{1}{4}$.

8 A 3000×3000 square is tiled by dominoes (i. e. 1×2 rectangles) in an arbitrary way. Show that one can color the dominoes in three colors such that the number of the dominoes of each color is the same, and each dominoe d has at most two neighbours of the same color as d. (Two dominoes are said to be *neighbours* if a cell of one domino has a common edge with a cell of the other one.)

– Grade level 11

1 Prove that $\sin\sqrt{x} < \sqrt{\sin x}$ for every real x such that $0 < x < \frac{\pi}{2}$.

2 The sum and the product of two purely periodic decimal fractions a and b are purely periodic decimal fractions of period length T. Show that the lengths of the periods of the fractions a and b are not greater than T.
Note. A *purely periodic decimal fraction* is a periodic decimal fraction without a non-periodic starting part.

3 On a 49×69 rectangle formed by a grid of lattice squares, all $50 \cdot 70$ lattice points are colored blue. Two persons play the following game: In each step, a player colors two blue points red, and draws a segment between these two points. (Different segments can intersect in their interior.) Segments are drawn this way until all formerly blue points are colored red. At this moment, the first player directs all segments drawn - i. e., he takes every segment AB, and replaces it either by the vector $\overrightarrow{AB}$, or by the vector $\overrightarrow{BA}$. If the first player succeeds to direct all the segments drawn in such a way that the sum of the resulting vectors is $\overrightarrow{0}$, then he wins; else, the second player wins.
Which player has a winning strategy?

4 Given a triangle ABC. The angle bisectors of the angles ABC and BCA intersect the sides CA and AB at the points B_1 and C_1, and intersect each other at the point I. The line B_1C_1 intersects the circumcircle of triangle ABC at the points M and N. Prove that the circumradius of triangle MIN is twice as long as the circumradius of triangle ABC.

5 Two sequences of positive reals, (x_n) and (y_n), satisfy the relations $x_{n+2} = x_n + x_{n+1}^2$ and $y_{n+2} = y_n^2 + y_{n+1}$ for all natural numbers n. Prove that, if the numbers x_1, x_2, y_1, y_2 are all greater than 1, then there exists a natural number k such that $x_k > y_k$.

6 Consider a tetrahedron $SABC$. The incircle of the triangle ABC has the center I and touches its sides BC, CA, AB at the points E, F, D, respectively. Let A', B', C' be the points on the segments SA, SB, SC such that $AA' = AD$, $BB' = BE$, $CC' = CF$, and let S' be the point diametrically opposite to the point S on the circumsphere of the tetrahedron $SABC$. Assume that the line SI is an altitude of the tetrahedron $SABC$. Show that $S'A' = S'B' = S'C'$.

7 Assume that the polynomial $(x+1)^n - 1$ is divisible by some polynomial $P(x) = x^k + c_{k-1}x^{k-1} + c_{k-2}x^{k-2} + ... + c_1 x + c_0$, whose degree k is even and whose coefficients c_{k-1}, c_{k-2}, ..., c_1, c_0 all are odd integers. Show that $k+1 \mid n$.

8 At a tourist camp, each person has at least 50 and at most 100 friends among the other persons at the camp. Show that one can hand out a t-shirt to every person such that the t-shirts have (at most) 1331 different colors, and any person has 20 friends whose t-shirts all have pairwisely different colors.

All-Russian Olympiad 2005

–	Grade level 9
Day 1	
1	Given a parallelogram $ABCD$ with $AB < BC$, show that the circumcircles of the triangles APQ share a second common point (apart from A) as P, Q move on the sides BC, CD respectively s.t. $CP = CQ$.
2	Lesha put numbers from 1 to 22^2 into cells of 22×22 board. Can Oleg always choose two cells, adjacent by the side or by vertex, the sum of numbers in which is divisible by 4?
3	Given three reals $a_1,\ a_2,\ a_3 > 1$, $S = a_1 + a_2 + a_3$. Provided $\frac{a_i^2}{a_i - 1} > S$ for every $i = 1,\ 2,\ 3$ prove that $$\frac{1}{a_1 + a_2} + \frac{1}{a_2 + a_3} + \frac{1}{a_3 + a_1} > 1.$$
4	Given 365 cards, in which distinct numbers are written. We may ask for any three cards, the order of numbers written in them. Is it always possible to find out the order of all 365 cards by 2000 such questions?
Day 2	
1	Ten mutually distinct non-zero reals are given such that for any two, either their sum or their product is rational. Prove that squares of all these numbers are rational.
2	Find the number of subsets $A \subset M = \{2^0,\ 2^1,\ 2^2, \dots, 2^{2005}\}$ such that equation $x^2 - S(A)x + S(B) = 0$ has integral roots, where $S(M)$ is the sum of all elements of M, and $B = M \setminus A$ (A and B are not empty).
3	We have an acute-angled triangle ABC, and AA', BB' are its altitudes. A point D is chosen on the arc ACB of the circumcircle of ABC. If $P = AA' \cap BD, Q = BB' \cap AD$, show that the midpoint of PQ lies on $A'B'$.

4	100 people from 50 countries, two from each countries, stay on a circle. Prove that one may partition them onto 2 groups in such way that neither no two countrymen, nor three consecutive people on a circle, are in the same group.
–	Grade level 10
Day 1	
1	Find the least positive integer, which may not be represented as $\frac{2^a-2^b}{2^c-2^d}$, where a, b, c, d are positive integers.
2	In a $2 \times n$ array we have positive reals s.t. the sum of the numbers in each of the n columns is 1. Show that we can select a number in each column s.t. the sum of the selected numbers in each row is at most $\frac{n+1}{4}$.
3	Given 2005 distinct numbers $a_1, a_2, \dots, a_{2005}$. By one question, we may take three different indices $1 \le i < j < k \le 2005$ and find out the set of numbers $\{a_i, a_j, a_k\}$ (unordered, of course). Find the minimal number of questions, which are necessary to find out all numbers a_i.
4	w_B and w_C are excircles of a triangle ABC. The circle w_B' is symmetric to w_B with respect to the midpoint of AC, the circle w_C' is symmetric to w_C with respect to the midpoint of AB. Prove that the radical axis of w_B' and w_C' halves the perimeter of ABC.
Day 2	
1	We select 16 cells on an 8×8 chessboard. What is the minimal number of pairs of selected cells in the same row or column?
2	We have an acute-angled triangle ABC, and AA', BB' are its altitudes. A point D is chosen on the arc ACB of the circumcircle of ABC. If $P = AA' \cap BD, Q = BB' \cap AD$, show that the midpoint of PQ lies on $A'B'$.
3	Positive integers $x > 1$ and y satisfy an equation $2x^2 - 1 = y^{15}$. Prove that 5 divides x.
4	A white plane is partitioned onto cells (in a usual way). A finite number of cells are coloured black. Each black cell has an even (0, 2 or 4) adjacent (by the side) white cells. Prove that one may colour each white cell in green or red such that every black cell will have equal number of red and green adjacent cells.

– Grade level 11

Day 1

1 Find the maximal possible finite number of roots of the equation $|x-a_1|+\dots+|x-a_{50}|=|x-b_1|+\dots+|x-b_{50}|$, where $a_1,\ a_2,\ \dots,a_{50},\ b_1,\dots,\ b_{50}$ are distinct reals.

2 Given 2005 distinct numbers $a_1,\ a_2,\dots,a_{2005}$. By one question, we may take three different indices $1\le i<j<k\le 2005$ and find out the set of numbers $\{a_i,\ a_j,\ a_k\}$ (unordered, of course). Find the minimal number of questions, which are necessary to find out all numbers a_i.

3 Let A', B', C' be points, in which excircles touch corresponding sides of triangle ABC. Circumcircles of triangles $A'B'C$, $AB'C'$, $A'BC'$ intersect a circumcircle of ABC in points $C_1\ne C$, $A_1\ne A$, $B_1\ne B$ respectively. Prove that a triangle $A_1B_1C_1$ is similar to a triangle, formed by points, in which incircle of ABC touches its sides.

4 Integers $x>2$, $y>1$, $z>0$ satisfy an equation $x^y+1=z^2$. Let p be a number of different prime divisors of x, q be a number of different prime divisors of y. Prove that $p\geq q+2$.

Day 2

1 Do there exist a bounded function $f:\mathbb{R}\to\mathbb{R}$ such that $f(1)>0$ and $f(x)$ satisfies an inequality $f^2(x+y)\ge f^2(x)+2f(xy)+f^2(y)$?

2 Do there exist 12 rectangular parallelepipeds $P_1,\ P_2,\dots,P_{12}$ with edges parallel to coordinate axes OX, OY, OZ such that P_i and P_j have a common point iff $i\ne j\pm 1$ modulo 12?

3 A quadrilateral $ABCD$ without parallel sides is circumscribed around a circle with centre O. Prove that O is a point of intersection of middle lines of quadrilateral $ABCD$ (i.e. barycentre of points A, B, C, D) iff $OA\cdot OC=OB\cdot OD$.

4 100 people from 25 countries, four from each countries, stay on a circle. Prove that one may partition them onto 4 groups in such way that neither no two countrymans, nor two neighbours will be in the same group.

All-Russian Olympiad 2004

– Grade level 9

Day 1

1 Each grid point of a cartesian plane is colored with one of three colors, whereby all three colors are used. Show that one can always find a right-angled triangle, whose three vertices have pairwise different colors.

2 Let $ABCD$ be a circumscribed quadrilateral (i. e. a quadrilateral which has an incircle). The exterior angle bisectors of the angles DAB and ABC intersect each other at K; the exterior angle bisectors of the angles ABC and BCD intersect each other at L; the exterior angle bisectors of the angles BCD and CDA intersect each other at M; the exterior angle bisectors of the angles CDA and DAB intersect each other at N. Let K_1, L_1, M_1 and N_1 be the orthocenters of the triangles ABK, BCL, CDM and DAN, respectively. Show that the quadrilateral $K_1L_1M_1N_1$ is a parallelogram.

3 On a table there are 2004 boxes, and in each box a ball lies. I know that some the balls are white and that the number of white balls is even. In each case I may point to two arbitrary boxes and ask whether in the box contains at least a white ball lies. After which minimum number of questions I can indicate two boxes for sure, in which white balls lie?

4 Let $n > 3$ be a natural number, and let x_1, x_2, ..., x_n be n positive real numbers whose product is 1. Prove the inequality

$$\frac{1}{1+x_1+x_1\cdot x_2}+\frac{1}{1+x_2+x_2\cdot x_3}+...+\frac{1}{1+x_n+x_n\cdot x_1}>1.$$

Day 2

1 Are there such pairwise distinct natural numbers m, n, p, q satisfying $m+n=p+q$ and $\sqrt{m}+\sqrt[3]{n}=\sqrt{p}+\sqrt[3]{q}>2004$?

2 In the cabinet 2004 telephones are located; each two of these telephones are connected by a cable, which is colored in one of four colors. From each color there is one cable at least. Can one always select several telephones in such a way that among their pairwise cable connections exactly 3 different colors occur?

3 The natural numbers from 1 to 100 are arranged on a circle with the characteristic that each number is either larger as their two neighbours or smaller than their two neighbours. A pair of neighbouring numbers is called "good", if you cancel such a pair, the above property remains still valid. What is the smallest possible number of good pairs?

4 Let O be the circumcenter of an acute-angled triangle ABC, let T be the circumcenter of the triangle AOC, and let M be the midpoint of the segment AC. We take a point D on the side AB and a point E on the side BC that satisfy $\angle BDM = \angle BEM = \angle ABC$. Show that the straight lines BT and DE are perpendicular.

– Grade level 10

Day 1

1 Each grid point of a cartesian plane is colored with one of three colors, whereby all three colors are used. Show that one can always find a right-angled triangle, whose three vertices have pairwise different colors.

2 Let $ABCD$ be a circumscribed quadrilateral (i. e. a quadrilateral which has an incircle). The exterior angle bisectors of the angles DAB and ABC intersect each other at K; the exterior angle bisectors of the angles ABC and BCD intersect each other at L; the exterior angle bisectors of the angles BCD and CDA intersect each other at M; the exterior angle bisectors of the angles CDA and DAB intersect each other at N. Let K_1, L_1, M_1 and N_1 be the orthocenters of the triangles ABK, BCL, CDM and DAN, respectively. Show that the quadrilateral $K_1L_1M_1N_1$ is a parallelogram.

3 Let $ABCD$ be a quadrilateral which is a cyclic quadrilateral and a tangent quadrilateral simultaneously. (By a *tangent quadrilateral*, we mean a quadrilateral that has an incircle.)

Let the incircle of the quadrilateral $ABCD$ touch its sides AB, BC, CD, and DA in the points K, L, M, and N, respectively. The exterior angle bisectors of the angles DAB and ABC intersect each other at a point K'. The exterior angle bisectors of the angles ABC and BCD intersect each other at a point L'. The exterior angle bisectors of the angles BCD and CDA intersect each other at a point M'. The exterior angle bisectors of the angles CDA and DAB intersect each other at a point N'. Prove that the straight lines KK', LL', MM', and NN' are concurrent.

Day 2

1 A sequence of non-negative rational numbers $a(1), a(2), a(3), \ldots$ satisfies $a(m)+a(n) = a(mn)$ for arbitrary natural m and n. Show that not all elements of the sequence can be distinct.

2 A country has 1001 cities, and each two cities are connected by a one-way street. From each city exactly 500 roads begin, and in each city 500 roads end. Now an independent republic splits itself off the country, which contains 668 of the 1001 cities. Prove that one can reach every other city of the republic from each city of this republic without being forced to leave the republic.

3 A triangle T is contained inside a point-symmetrical polygon M. The triangle T' is the mirror image of the triangle T with the reflection at one point P, which inside the triangle T lies. Prove that at least one of the vertices of the triangle T' lies in inside or on the boundary of the polygon M.

4 Is there a natural number $n > 10^{1000}$ which is not divisible by 10 and which satisfies: in its decimal representation one can exchange two distinct non-zero digits such that the set of prime divisors does not change.

– Grade level 11

Day 1

1 Each grid point of a cartesian plane is colored with one of three colors, whereby all three colors are used. Show that one can always find a right-angled triangle, whose three vertices have pairwise different colors.

2 Let $I(A)$ and $I(B)$ be the centers of the excircles of a triangle ABC, which touches the sides BC and CA in its interior. Furthermore let P a point on the circumcircle ω of the triangle ABC. Show that the center of the segment which connects the circumcenters of the triangles $I(A)CP$ and $I(B)CP$ coincides with the center of the circle ω.

3 The polynomials $P(x)$ and $Q(x)$ are given. It is known that for a certain polynomial $R(x,y)$ the identity $P(x)-P(y)=R(x,y)(Q(x)-Q(y))$ applies. Prove that there is a polynomial $S(x)$ so that $P(x)=S(Q(x)) \quad \forall x.$

4 A rectangular array has 9 rows and 2004 columns. In the 9 * 2004 cells of the table we place the numbers from 1 to 2004, each 9 times. And we do this in such a way that two numbers, which stand in exactly the same column in and differ around at most by 3. Find the smallest possible sum of all numbers in the first row.

Day 2

1 Let $M=\{x_1 ..., x_{30}\}$ a set which consists of 30 distinct positive numbers, let A_n, $1 \leq n \leq 30$, the sum of all possible products with n elements each of the set M. Prove if $A_{15} > A_{10}$, then $A_1 > 1$.

2 Prove that there is no finite set which contains more than $2N$, with $N>3$, pairwise non-collinear vectors of the plane, and to which the following two characteristics apply:

1) for N arbitrary vectors from this set there are always further $N-1$ vectors from this set so that the sum of these is $2N-1$ vectors is equal to the zero-vector;

2) for N arbitrary vectors from this set there are always further N vectors from this set so that the sum of these is $2N$ vectors is equal to the zero-vector.

3 In a country there are several cities; some of these cities are connected by airlines, so that an airline connects exactly two cities in each case and both flight directions are possible. Each airline belongs to one of k flight companies; two airlines of the same flight company have always a common final point. Show that one can partition all cities in $k+2$ groups in such a way that two cities from exactly the same group are never connected by an airline with each other.

4 A parallelepiped is cut by a plane along a 6-gon. Supposed this 6-gon can be put into a certain rectangle π (which means one can put the rectangle π on the parallelepiped's plane such that the 6-gon is completely covered by the rectangle). Show that one also can put one of the parallelepiped' faces into the rectangle π.

All-Russian Olympiad 2003

– Grade level 9

Day 1

1 Suppose that M is a set of 2003 numbers such that, for any distinct $a, b \in M$, the number $a^2 + b\sqrt{2}$ is rational. Prove that $a\sqrt{2}$ is rational for all $a \in M$.

2 Two circles S_1 and S_2 with centers O_1 and O_2 respectively intersect at A and B. The tangents at A to S_1 and S_2 meet segments BO_2 and BO_1 at K and L respectively. Show that $KL \parallel O_1O_2$.

3 On a line are given $2k-1$ white segments and $2k-1$ black ones. Assume that each white segment intersects at least k black segments, and each black segment intersects at least k white ones. Prove that there are a black segment intersecting all the white ones, and a white segment intersecting all the black ones.

4 A sequence (a_n) is defined as follows: $a_1 = p$ is a prime number with exactly 300 nonzero digits, and for each $n \geq 1$, a_{n+1} is the decimal period of $1/a_n$ multiplies by 2. Determine a_{2003}.

Day 2

1 There are N cities in a country. Any two of them are connected either by a road or by an airway. A tourist wants to visit every city exactly once and return to the city at which he started the trip. Prove that he can choose a starting city and make a path, changing means of transportation at most once.

2 Let a, b, c be positive numbers with the sum 1. Prove the inequality

$$\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} \geq \frac{2}{1+a} + \frac{2}{1+b} + \frac{2}{1+c}.$$

3 Is it possible to write a natural number in every cell of an infinite chessboard in such a manner that for all integers $m, n > 100$, the sum of numbers in every $m \times n$ rectangle is divisible by $m + n$?

Contributors: Amir Hossein, vladimir92, v_Enhance, math10, ehsan2004, dragon96, specialone

4 Let B and C be arbitrary points on sides AP and PD respectively of an acute triangle APD. The diagonals of the quadrilateral $ABCD$ meet at Q, and H_1, H_2 are the orthocenters of triangles APD and BPC, respectively. Prove that if the line H_1H_2 passes through the intersection point X $(X \neq Q)$ of the circumcircles of triangles ABQ and CDQ, then it also passes through the intersection point Y $(Y \neq Q)$ of the circumcircles of triangles BCQ and ADQ.

– Grade level 10

Day 1

1 Suppose that M is a set of 2003 numbers such that, for any distinct $a, b, c \in M$, the number $a^2 + bc$ is rational. Prove that there is a positive integer n such that $a\sqrt{n}$ is rational for all $a \in M$.

2 The diagonals of a cyclic quadrilateral $ABCD$ meet at O. Let S_1, S_2 be the circumcircles of triangles ABO and CDO respectively, and O, K their intersection points. The lines through O parallel to AB and CD meet S_1 and S_2 again at L and M, respectively. Points P and Q on segments OL and OM respectively are taken such that $OP : PL = MQ : QO$. Prove that O, K, P, Q lie on a circle.

3 Consider a tree (i.e. a connected graph with no cycles) with n vertices. Its vertices are assigned numbers $x_1, x_2, \cdots x_n$ and each edge is assigned the product of the numbers at its endpoints. Let S denote the sum of the numbers at the deges. Prove that

$$2S \leq \sqrt{n-1}\left(x_1^2 + x_2^2 + \cdots + x_n^2\right).$$

4 A finite set of points X and an equilateral triangle T are given on a plane. Suppose that every subset X' of X with no more than 9 elements can be covered by two images of T under translations. Prove that the whole set X can be covered by two images of T under translations.

Day 2

1 There are N cities in a country. Any two of them are connected either by a road or by an airway. A tourist wants to visit every city exactly once and return to the city at which he started the trip. Prove that he can choose a starting city and make a path, changing means of transportation at most once.

2 Let a_0 be a natural number. The sequence (a_n) is dened by $a_{n+1} = \frac{a_n}{5}$ if a_n is divisible by 5
and $a_{n+1} = [a_n\sqrt{5}]$ otherwise . Show that the sequence a_n is increasing starting from some term.

3 In a triangle ABC, O is the circumcenter and I the incenter. The excircle ω_a touches rays AB, AC and side BC at K, M, N, respectively. Prove that if the midpoint P of KM lies on the circumcircle of $\triangle ABC$, then points O, N, I lie on a line.

4 Find the greatest natural number N such that, for any arrangement of the numbers $1, 2, \ldots, 400$ in a chessboard 20×20, there exist two numbers in the same row or column, which differ by at least N.

– Grade level 11

Day 1

1 Let $\alpha, \beta, \gamma, \delta$ be positive numbers such that for all x, $\sin \alpha x + \sin \beta x = \sin \gamma x + \sin \delta x$. Prove that $\alpha = \gamma$ or $\alpha = \delta$.

2 The diagonals of a cyclic quadrilateral $ABCD$ meet at O. Let S_1, S_2 be the circumcircles of triangles ABO and CDO respectively, and O, K their intersection points. The lines through O parallel to AB and CD meet S_1 and S_2 again at L and M, respectively. Points P and Q on segments OL and OM respectively are taken such that $OP : PL = MQ : QO$. Prove that O, K, P, Q lie on a circle.

3 Let $f(x)$ and $g(x)$ be polynomials with non-negative integer coefficients, and let m be the largest coefficient of f. Suppose that there exist natural numbers $a < b$ such that $f(a) = g(a)$ and $f(b) = g(b)$. Show that if $b > m$, then $f = g$.

4 Ana and Bora are each given a sufficiently long paper strip, one with letter A written , and the other with letter B. Every minute, one of them (not necessarily one after another) writes either on the left or on the right to the word on his/her strip the word written on the other strip. Prove that the day after, one will be able to cut word on Ana's strip into two words and exchange their places, obtaining a palindromic word.

Day 2

1 The side lengths of a triangle are the roots of a cubic polynomial with rational coefficients. Prove that the altitudes of this triangle are roots of a polynomial of sixth degree with rational coefficients.

2 Is it possible to write a positive integer in every cell of an infinite chessboard, in such a manner that, for all positive integers m, n, the sum of numbers in every $m \times n$ rectangle is divisible by $m + n$?

3 There are 100 cities in a country, some of them being joined by roads. Any four cities are connected to each other by at least two roads. Assume that there is no path passing through every city exactly once. Prove that there are two cities such that every other city is connected to at least one of them.

4 The inscribed sphere of a tetrahedron $ABCD$ touches ABC, ABD, ACD and BCD at D_1, C_1, B_1 and A_1 respectively. Consider the plane equidistant from A and plane $B_1C_1D_1$ (parallel to $B_1C_1D_1$) and the three planes defined analogously for the vertices B, C, D. Prove that the circumcenter of the tetrahedron formed by these four planes coincides with the circumcenter of tetrahedron of $ABCD$.

All-Russian Olympiad 2002

– Grade level 9

Day 1

1 Can the cells of a 2002×2002 table be filled with the numbers from 1 to 2002^2 (one per cell) so that for any cell we can find three numbers a, b, c in the same row or column (or the cell itself) with $a = bc$?

2 Point A lies on one ray and points B, C lie on the other ray of an angle with the vertex at O such that B lies between O and C. Let O_1 be the incenter of $\triangle OAB$ and O_2 be the center of the excircle of $\triangle OAC$ touching side AC. Prove that if $O_1A = O_2A$, then the triangle ABC is isosceles.

3 On a plane are given 6 red, 6 blue, and 6 green points, such that no three of the given points lie on a line. Prove that the sum of the areas of the triangles whose vertices are of the same color does not exceed quarter the sum of the areas of all triangles with vertices in the given points.

4 A hydra consists of several heads and several necks, where each neck joins two heads. When a hydra's head A is hit by a sword, all the necks from head A disappear, but new necks grow up to connect head A to all the heads which weren't connected to A. Heracle defeats a hydra by cutting it into two parts which are no joined. Find the minimum N for which Heracle can defeat any hydra with 100 necks by no more than N hits.

Day 2

1 There are eight rooks on a chessboard, no two attacking each other. Prove that some two of the pairwise distances between the rooks are equal. (The distance between two rooks is the distance between the centers of their cell.)

2 We are given one red and $k > 1$ blue cells, and a pack of $2n$ cards, enumerated by the numbers from 1 to $2n$. Initially, the pack is situated on the red cell and arranged in an arbitrary order. In each move, we are allowed to take the top card from one of the cells and place it either onto the top of another cell on which the number on the top card is greater by 1, or onto an empty cell. Given k, what is the maximal n for which it is always possible to move all the cards onto a blue cell?

3 Let O be the circumcenter of a triangle ABC. Points M and N are choosen on the sides AB and BC respectively so that the angle AOC is two times greater than angle MON. Prove that the perimeter of triangle MBN is not less than the lenght of side AC

4 From the interval $(2^{2n}, 2^{3n})$ are selected $2^{2n-1}+1$ odd numbers. Prove that there are two among the selected numbers, none of which divides the square of the other.

– Grade level 10

Day 1

1 The polynomials P, Q, R with real coefficients, one of which is degree 2 and two of degree 3, satisfy the equality $P^2+Q^2=R^2$. Prove that one of the polynomials of degree 3 has three real roots.

2 A quadrilateral $ABCD$ is inscribed in a circle ω. The tangent to ω at A intersects the ray CB at K, and the tangent to ω at B intersects the ray DA at M. Prove that if $AM=AD$ and $BK=BC$, then $ABCD$ is a trapezoid.

3 Prove that for every integer $n>10000$ there exists an integer m such that it can be written as the sum of two squares, and $0<m-n<3\sqrt[4]{n}$.

4 There are 2002 towns in a kingdom. Some of the towns are connected by roads in such a manner that, if all roads from one city closed, one can still travel between any two cities. Every year, the kingdom chooses a non-self-intersecting cycle of roads, founds a new town, connects it by roads with each city from the chosen cycle, and closes all the roads from the original cycle. After several years, no non-self-intersecting cycles remained. Prove that at that moment there are at least 2002 towns, exactly one road going out from each of them.

Day 2

1 For positive real numbers a, b, c such that $a+b+c=3$, show that:

$$\sqrt{a}+\sqrt{b}+\sqrt{c} \geq ab+bc+ca.$$

2 We are given one red and $k > 1$ blue cells, and a pack of $2n$ cards, enumerated by the numbers from 1 to $2n$. Initially, the pack is situated on the red cell and arranged in an arbitrary order. In each move, we are allowed to take the top card from one of the cells and place it either onto the top of another cell on which the number on the top card is greater by 1, or onto an empty cell. Given k, what is the maximal n for which it is always possible to move all the cards onto a blue cell?

3 Let A' be the point of tangency of the excircle of a triangle ABC (corrsponding to A) with the side BC. The line a through A' is parallel to the bisector of $\angle BAC$. Lines b and c are analogously dened. Prove that a, b, c have a common point.

4 On a plane are given finitely many red and blue lines, no two parallel, such that any intersection point of two lines of the same color also lies on another line of the other color. Prove that all the lines pass through a single point.

– Grade level 11

Day 1

1 The polynomials P, Q, R with real coefficients, one of which is degree 2 and two of degree 3, satisfy the equality $P^2 + Q^2 = R^2$. Prove that one of the polynomials of degree 3 has three real roots.

2 Several points are given in the plane. Suppose that for any three of them, there exists an orthogonal coordinate system (determined by the two axes and the unit length) in which these three points have integer coordinates. Prove that there exists an orthogonal coordinate system in which all the given points have integer coordinates.

3 Prove that if $0 < x < \frac{\pi}{2}$ and $n > m$, where n,m are natural numbers,

$$2\left|\sin^n x - \cos^n x\right| \le 3\left|\sin^m x - \cos^m x\right|.$$

4 There are some markets in a city. Some of them are joined by one-way streets, such that for any market there are exactly two streets to leave it. Prove that the city may be partitioned into 1014 districts such that streets join only markets from different districts, and by the same one-way for any two districts (either only from first to second, or vice-versa).

Day 2

1 Determine the smallest natural number which can be represented both as the sum of 2002 positive integers with the same sum of decimal digits, and as the sum of 2003 integers with the same sum of decimal digits.

2 The diagonals AC and BD of a cyclic quadrilateral $ABCD$ meet at O. The circumcircles of triangles AOB and COD intersect again at K. Point L is such that the triangles BLC and AKD are similar and equally oriented. Prove that if the quadrilateral $BLCK$ is convex, then it is tangent [has an incircle].

3 On a plane are given finitely many red and blue lines, no two parallel, such that any intersection point of two lines of the same color also lies on another line of the other color. Prove that all the lines pass through a single point.

4 Prove that there exist infinitely many natural numbers n such that the numerator of $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n}$ in the lowest terms is not a power of a prime number.

All-Russian Olympiad 2001

– Grade level 9

Day 1

1 The integers from 1 to 999999 are partitioned into two groups: the first group consists of those integers for which the closest perfect square is odd, whereas the second group consists of those for which the closest perfect square is even. In which group is the sum of the elements greater?

2 The two polynomials $(x) = x^4 + ax^3 + bx^2 + cx + d$ and $Q(x) = x^2 + px + q$ take negative values on an interval I of length greater than 2, and nonnegative values outside of I. Prove that there exists $x_0 \in \mathbb{R}$ such that $P(x_0) < Q(x_0)$.

3 A point K is taken inside parallelogram $ABCD$ so that the midpoint of AD is equidistant from K and C, and the midpoint of CD is equidistant form K and A. Let N be the midpoint of BK. Prove that the angles NAK and NCK are equal.

4 Consider a convex 2000-gon, no three of whose diagonals have a common point. Each of its diagonals is colored in one of 999 colors. Prove that there exists a triangle all of whose sides lie on diagonals of the same color. (Vertices of the triangle need not be vertices of the original polygon.)

Day 2

1 Yura put 2001 coins of 1, 2 or 3 kopeykas in a row. It turned out that between any two 1-kopeyka coins there is at least one coin; between any two 2-kopeykas coins there are at least two coins; and between any two 3-kopeykas coins there are at least 3 coins. How many 3-koyepkas coins could Yura put?

2 In a party, there are $2n + 1$ people. It's well known that for every group of n people, there exist a person(out of the group) who knows all them(the n people of the group). Show that there exist a person who knows all the people in the party.

3 Let N be a point on the longest side AC of a triangle ABC. The perpendicular bisectors of AN and NC intersect AB and BC respectively in K and M. Prove that the circumcenter O of $\triangle ABC$ lies on the circumcircle of triangle KBM.

Contributors: v_Enhance, manlio, Pascual2005, hvaz, iandrei, sisioyus, Superficial, Moonmathpi496, EQSon, towersfreak2006, elegant, Mathx, mathVNpro

4 Find all odd positive integers $n > 1$ such that if a and b are relatively prime divisors of n, then $a + b - 1$ divides n.

– Grade level 10

Day 1

1 The integers from 1 to 999999 are partitioned into two groups: the first group consists of those integers for which the closest perfect square is odd, whereas the second group consists of those for which the closest perfect square is even. In which group is the sum of the elements greater?

2 Let $A_1, A_2, ..., A_{100}$ be subsets of a line, each a union of 100 pairwise disjoint closed segments. Prove that the intersection of all hundred sets is a union of at most 9901 disjoint closed segments.

3 Let the circle ω_1 be internally tangent to another circle ω_2 at N.Take a point K on ω_1 and draw a tangent AB which intersects ω_2 at A and B. Let M be the midpoint of the arc AB which is on the opposite side of N. Prove that, the circumradius of the $\triangle KBM$ doesnt depend on the choice of K.

4 Some towns in a country are connected by twoway roads, so that for any two towns there is a unique path along the roads connecting them. It is known that there is exactly 100 towns which are directly connected to only one town. Prove that we can construct 50 new roads in order to obtain a net in which every two towns will be connected even if one road gets closed.

Day 2

1 The polynomial $P(x) = x^3 + ax^2 + bx + d$ has three distinct real roots. The polynomial $P(Q(x))$, where $Q(x) = x^2 + x + 2001$, has no real roots. Prove that $P(2001) > \frac{1}{64}$.

2 In a magic square $n \times n$ composed from the numbers $1, 2, \cdots, n^2$, the centers of any two squares are joined by a vector going from the smaller number to the bigger one. Prove that the sum of all these vectors is zero. (A magic square is a square matrix such that the sums of entries in all its rows and columns are equal.)

3 Points A_1, B_1, C_1 inside an acute-angled triangle ABC are selected on the altitudes from A, B, C respectively so that the sum of the areas of triangles ABC_1, BCA_1, and CAB_1 is equal to the area of triangle ABC. Prove that the circumcircle of triangle $A_1B_1C_1$ passes through the orthocenter H of triangle ABC.

4 Find all odd positive integers $n > 1$ such that if a and b are relatively prime divisors of n, then $a + b - 1$ divides n.

– Grade level 11

Day 1

1 The total mass of 100 given weights with positive masses equals $2S$. A natural number k is called *middle* if some k of the given weights have the total mass S. Find the maximum possible number of middle numbers.

2 Let the circle ω_1 be internally tangent to another circle ω_2 at N.Take a point K on ω_1 and draw a tangent AB which intersects ω_2 at A and B. Let M be the midpoint of the arc AB which is on the opposite side of N. Prove that, the circumradius of the $\triangle KBM$ doesnt depend on the choice of K.

3 There are two families of convex polygons in the plane. Each family has a pair of disjoint polygons. Any polygon from one family intersects any polygon from the other family. Show that there is a line which intersects all the polygons.

4 Participants to an olympiad worked on n problems. Each problem was worth a positive integer number of points, determined by the jury. A contestant gets 0 points for a wrong answer, and all points for a correct answer to a problem. It turned out after the olympiad that the jury could impose worths of the problems, so as to obtain any (strict) final ranking of the contestants. Find the greatest possible number of contestants.

Day 2

1 Two monic quadratic trinomials $f(x)$ and $g(x)$ take negative values on disjoint intervals. Prove that there exist positive numbers α and β such that $\alpha f(x) + \beta g(x) > 0$ for all real x.

2 Let a, b be 2 distinct positive interger number such that $(a^2+ab+b^2)|ab(a+b)$. Prove that: $|a-b| > \sqrt[3]{ab}$.

3 The 2001 towns in a country are connected by some roads, at least one road from each town, so that no town is connected by a road to every other city. We call a set D of towns *dominant* if every town not in D is connected by a road to a town in D. Suppose that each dominant set consists of at least k towns. Prove that the country can be partitioned into $2001-k$ republics in such a way that no two towns in the same republic are connected by a road.

4 A sphere with center on the plane of the face ABC of a tetrahedron $SABC$ passes through A, B and C, and meets the edges SA, SB, SC again at A_1, B_1, C_1, respectively. The planes through A_1, B_1, C_1 tangent to the sphere meet at O. Prove that O is the circumcenter of the tetrahedron $SA_1B_1C_1$.

Contributors: v_Enhance, manlio, Pascual2005, hvaz, iandrei, sisioyus, Superficial, Moonmathpi496, EQSon, towersfreak2006, elegant, Mathx, mathVNpro

All-Russian Olympiad 2000

– Grade level 9

Day 1

1 Let a, b, c be distinct numbers such that the equations $x^2 + ax + 1 = 0$ and $x^2 + bx + c = 0$ have a common real root, and the equations $x^2 + x + a = 0$ and $x^2 + cx + b$ also have a common real root. Compute the sum $a + b + c$.

2 Tanya chose a natural number $X \leq 100$, and Sasha is trying to guess this number. He can select two natural numbers M and N less than 100 and ask about $\gcd(X + M, N)$. Show that Sasha can determine Tanya's number with at most seven questions.

3 Let O be the center of the circumcircle ω of an acute-angle triangle ABC. A circle ω_1 with center K passes through A, O, C and intersects AB at M and BC at N. Point L is symmetric to K with respect to line NM. Prove that $BL \perp AC$.

4 Some pairs of cities in a certain country are connected by roads, at least three roads going out of each city. Prove that there exists a round path consisting of roads whose number is not divisible by 3.

Day 2

5 The sequence $a_1 = 1, a_2, a_3, \cdots$ is defined as follows: if $a_n - 2$ is a natural number not already occurring on the board, then $a_{n+1} = a_n - 2$; otherwise, $a_{n+1} = a_n + 3$. Prove that every nonzero perfect square occurs in the sequence as the previous term increased by 3.

6 On some cells of a $2n \times 2n$ board are placed white and black markers (at most one marker on every cell). We first remove all black markers which are in the same column with a white marker, then remove all white markers which are in the same row with a black one. Prove that either the number of remaining white markers or that of remaining black markers does not exceed n^2.

7 Let E be a point on the median CD of a triangle ABC. The circle $\mathcal{S}_1$ passing through E and touching AB at A meets the side AC again at M. The circle

S_2 passing through E and touching AB at B meets the side BC at N. Prove that the circumcircle of $\triangle CMN$ is tangent to both $\mathcal{S}_1$ and $\mathcal{S}_2$.

8 One hundred natural numbers whose greatest common divisor is 1 are arranged around a circle. An allowed operation is to add to a number the greatest common divisor of its two neighhbors. Prove that we can make all the numbers pairwise copirme in a finite number of moves.

– Grade level 10

Day 1

1 Evaluate the sum

$$\left\lfloor \frac{2^0}{3} \right\rfloor + \left\lfloor \frac{2^1}{3} \right\rfloor + \left\lfloor \frac{2^2}{3} \right\rfloor + \cdots + \left\lfloor \frac{2^{1000}}{3} \right\rfloor.$$

2 Let $-1 < x_1 < x_2, \cdots < x_n < 1$ and $x_1^{13} + x_2^{13} + \cdots + x_n^{13} = x_1 + x_2 + \cdots + x_n$. Prove that if $y_1 < y_2 < \cdots < y_n$, then

$$x_1^{13}y_1 + \cdots + x_n^{13}y_n < x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

3 In an acute scalene triangle ABC the bisector of the acute angle between the altitudes AA_1 and CC_1 meets the sides AB and BC at P and Q respectively. The bisector of the angle B intersects the segment joining the orthocenter of ABC and the midpoint of AC at point R. Prove that P, B, Q, R lie on a circle.

4 We are given five equal-looking weights of pairwise distinct masses. For any three weights A, B, C, we can check by a measuring if $m(A) < m(B) < m(C)$, where $M(X)$ denotes the mass of a weight X (the answer is *yes* or *no*.) Can we always arrange the masses of the weights in the increasing order with at most nine measurings?

Day 2

5 Let M be a finite sum of numbers, such that among any three of its elements there are two whose sum belongs to M. Find the greatest possible number of elements of M.

6 A perfect number, greater than 6, is divisible by 3. Prove that it is also divisible by 9.

7 Two circles are internally tangent at N. The chords BA and BC of the larger circle are tangent to the smaller circle at K and M respectively. Q and P are midpoint of arcs AB and BC respectively. Circumcircles of triangles BQK and BPM are intersect at L. Show that $BPLQ$ is a parallelogram.

8 Some paper squares of k distinct colors are placed on a rectangular table, with sides parallel to the sides of the table. Suppose that for any k squares of distinct colors, some two of them can be nailed on the table with only one nail. Prove that there is a color such that all squares of that color can be nailed with $2k-2$ nails.

– Grade level 11

Day 1

1 Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(x+y)+f(y+z)+f(z+x)\geq 3f(x+2y+3z)$$

for all $x, y, z \in \mathbb{R}$.

2 Prove that one can partition the set of natural numbers into 100 nonempty subsets such that among any three natural numbers a, b, c satisfying $a+99b=c$, there are two that belong to the same subset.

3 A convex pentagon $ABCDE$ is given in the coordinate plane with all vertices in lattice points. Prove that there must be at least one lattice point in the pentagon determined by the diagonals AC, BD, CE, DA, EB or on its boundary.

4 Let $a_1, a_2, \cdots, a_n$ be a sequence of nonnegative integers. For $k = 1, 2, \cdots, n$ denote

$$m_k = \max_{1 \le l \le k} \frac{a_{k-l+1} + a_{k-l+2} + \cdots + a_k}{l}.$$

Prove that for every $\alpha > 0$ the number of values of k for which $m_k > \alpha$ is less than $\frac{a_1+a_2+\cdots+a_n}{\alpha}$.

Day 2

5 Prove the inequality

$$\sin^n(2x) + (\sin^n x - \cos^n x)^2 \leq 1.$$

6 A perfect number, gerater than 28 is divisible by 7. Prove that it is also divisible by 49.

7 A quadrilateral $ABCD$ is circumscribed about a circle ω. The lines AB and CD meet at O. A circle ω_1 is tangent to side BC at K and to the extensions of sides AB and CD, and a circle ω_2 is tangent to side AD at L and to the extensions of sides AB and CD. Suppose that points O, K, L lie on a line. Prove that the midpoints of BC and AD and the center of ω also lie on a line.

8 All points in a 100×100 array are colored in one of four colors red, green, blue or yellow in such a way that there are 25 points of each color in each row and in any column. Prove that there are two rows and two columns such that their four intersection points are all in different colors.

All-Russian Olympiad 1999

– Grade level 9

Day 1

1 The decimal digits of a natural number A form an increasing sequence (from left to right). Find the sum of the digits of $9A$.

2 There are several cities in a country. Some pairs of the cities are connected by a two-way airline of one of the N companies, so that each company serves exactly one airline from each city, and one can travel between any two cities, possibly with transfers. During a financial crisis, $N-1$ airlines have been canceled, all from different companies. Prove that it is still possible to travel between any two cities.

3 A triangle ABC is inscribed in a circle S. Let A_0 and C_0 be the midpoints of the arcs BC and AB on S, not containing the opposite vertex, respectively. The circle S_1 centered at A_0 is tangent to BC, and the circle S_2 centered at C_0 is tangent to AB. Prove that the incenter I of $\triangle ABC$ lies on a common tangent to S_1 and S_2.

4 Initially numbers from 1 to 1000000 are all colored black. A move consists of picking one number, then change the color (black to white or white to black) of itself and all other numbers NOT coprime with the chosen number. Can all numbers become white after finite numbers of moves?

Edited by pbornsztein

Day 2

5 An equilateral triangle of side n is divided into equilateral triangles of side 1. Find the greatest possible number of unit segments with endpoints at vertices of the small triangles that can be chosen so that no three of them are sides of a single triangle.

6 Prove that for all natural numbers n,

$$\sum_{k=1}^{n^2} \left\{\sqrt{k}\right\} \le \frac{n^2-1}{2}.$$

Here, $\{x\}$ denotes the fractional part of x.

7 A circle through vertices A and B of triangle ABC meets side BC again at D. A circle through B and C meets side AB at E and the first circle again at F. Prove that if points A, E, D, C lie on a circle with center O then $\angle BFO$ is right.

8 There are 2000 components in a circuit, every two of which were initially joined by a wire. The hooligans Vasya and Petya cut the wires one after another. Vasya, who starts, cuts one wire on his turn, while Petya cuts one or three. The hooligan who cuts the last wire from some component loses. Who has the winning strategy?

– Grade level 10

Day 1

1 There are three empty jugs on a table. Winnie the Pooh, Rabbit, and Piglet put walnuts in the jugs one by one. They play successively, with the initial determined by a draw. Thereby Winnie the Pooh plays either in the first or second jug, Rabbit in the second or third, and Piglet in the first or third. The player after whose move there are exactly 1999 walnuts loses the games. Show that Winnie the Pooh and Piglet can cooperate so as to make Rabbit lose.

2 Find all bounded sequences $(a_n)_{n=1}^\infty$ of natural numbers such that for all $n \geq 3$,

$$a_n = \frac{a_{n-1} + a_{n-2}}{\gcd(a_{n-1}, a_{n-2})}.$$

3 The incircle of $\triangle ABC$ touch AB,BC,CA at K,L,M. The common external tangents to the incircles of $\triangle AMK$,$\triangle BKL$,$\triangle CLM$, distinct from the sides of $\triangle ABC$, are drawn. Show that these three lines are concurrent.

4 A frog is placed on each cell of a $n \times n$ square inside an infinite chessboard (so initially there are a total of $n \times n$ frogs). Each move consists of a frog A jumping over a frog B adjacent to it with A landing in the next cell and B disappearing (adjacent means two cells sharing a side). Prove that at least $\left[\frac{n^2}{3}\right]$ moves are needed to reach a configuration where no more moves are possible.

Day 2

5 The sum of the (decimal) digits of a natural number n equals 100, and the sum of digits of $44n$ equals 800. Determine the sum of digits of $3n$.

6 In triangle ABC, a circle passes through A and B and is tangent to BC. Also, a circle that passes through B and C is tangent to AB. These two circles intersect at a point K other than B. If O is the circumcenter of ABC, prove that $\angle BKO = 90^\circ$.

7 Positive numbers x, y satisfy $x^2 + y^2 \geq x^3 + y^4$. Prove that $x^3 + y^3 \leq 2$.

8 In a group of 12 persons, among any 9 there are 5 which know each other. Prove
that there are 6 persons in this group which know each other

– Grade level 11

Day 1

1 Do there exist 19 distinct natural numbers with equal sums of digits, whose sum equals 1999?

2 Each rational point on a real line is assigned an integer. Prove that there is a segment such that the sum of the numbers at its endpoints does not exceed twice the number at its midpoint.

3 A circle touches sides DA, AB, BC, CD of a quadrilateral $ABCD$ at points K, L, M, N, respectively. Let S_1, S_2, S_3, S_4 respectively be the incircles of triangles AKL, BLM, CMN, DNK. The external common tangents distinct from the sides of $ABCD$ are drawn to S_1 and S_2, S_2 and S_3, S_3 and S_4, S_4 and S_1. Prove that these four tangents determine a rhombus.

4 A frog is placed on each cell of a $n \times n$ square inside an infinite chessboard (so initially there are a total of $n \times n$ frogs). Each move consists of a frog A jumping over a frog B adjacent to it with A landing in the next cell and B disappearing (adjacent means two cells sharing a side). Prove that at least $\left[\frac{n^2}{3}\right]$ moves are needed to reach a configuration where no more moves are possible.

Day 2

5 Four natural numbers are such that the square of the sum of any two of them is divisible by the product of the other two numbers. Prove that at least three of these numbers are equal.

6 Three convex polygons are given on a plane. Prove that there is no line cutting all the polygons if and only if each of the polygons can be separated from the other two by a line.

7 Through vertex A of a tetrahedron $ABCD$ passes a plane tangent to the circumscribed sphere of the tetrahedron. Show that the lines of intersection of the plane with the planes ABC, ABD, ACD, form six equal angles if and only if:

$$AB \cdot CD = AC \cdot BD = AD \cdot BC$$

8 There are 2000 components in a circuit, every two of which were initially joined by a wire. The hooligans Vasya and Petya cut the wires one after another. Vasya, who starts, cuts one wire on his turn, while Petya cuts two or three. The hooligan who cuts the last wire from some component loses. Who has the winning strategy?

All-Russian Olympiad 1998

– Grade level 9

Day 1

1 The angle formed by the rays $y = x$ and $y = 2x$ $(x \geq 0)$ cuts off two arcs from a given parabola $y = x^2 + px + q$. Prove that the projection of one arc onto the x-axis is shorter by 1 than that of the second arc.

2 A convex polygon is partitioned into parallelograms. A vertex of the polygon is called *good* if it belongs to exactly one parallelogram. Prove that there are more than two good vertices.

3 Let $S(x)$ denote the sum of the decimal digits of x. Do there exist natural numbers a, b, c such that

$$S(a+b) < 5, \quad S(b+c) < 5, \quad S(c+a) < 5, \quad S(a+b+c) > 50?$$

4 A maze is an 8×8 board with some adjacent squares separated by walls, so that any two squares can be connected by a path not meeting any wall. Given a command LEFT, RIGHT, UP, DOWN, a pawn makes a step in the corresponding direction unless it encounters a wall or an edge of the chessboard. God writes a program consisting of a finite sequence of commands and gives it to the Devil, who then constructs a maze and places the pawn on one of the squares. Can God write a program which guarantees the pawn will visit every square despite the Devil's efforts?

Day 2

5 We are given five watches which can be winded forward. What is the smallest sum of winding intervals which allows us to set them to the same time, no matter how they were set initially?

6 In triangle ABC with $AB > BC$, BM is a median and BL is an angle bisector. The line through M and parallel to AB intersects BL at point D, and the line through L and parallel to BC intersects BM at point E. Prove that ED is perpendicular to BL.

7 A jeweller makes a chain consisting of $N > 3$ numbered links. A querulous customer then asks him to change the order of the links, in such a way that the number of links the jeweller must open is maximized. What is the maximum number?

8 Two distinct positive integers a, b are written on the board. The smaller of them is erased and replaced with the number $\frac{ab}{|a-b|}$. This process is repeated as long as the two numbers are not equal. Prove that eventually the two numbers on the board will be equal.

– Grade level 10

Day 1

1 Two lines parallel to the x-axis cut the graph of $y = ax^3 + bx^2 + cx + d$ in points A, C, E and B, D, F respectively, in that order from left to right. Prove that the length of the projection of the segment CD onto the x-axis equals the sum of the lengths of the projections of AB and EF.

2 Two polygons are given on the plane. Assume that the distance between any two vertices of the same polygon is at most 1, and that the distance between any two vertices of different polygons is at least $1/\sqrt{2}$. Prove that these two polygons have no common interior points.

By the way, can two sides of a polygon intersect?

3 In scalene $\triangle ABC$, the tangent from the foot of the bisector of $\angle A$ to the incircle of $\triangle ABC$, other than the line BC, meets the incircle at point K_a. Points K_b and K_c are analogously defined. Prove that the lines connecting K_a, K_b, K_c with the midpoints of BC, CA, AB, respectively, have a common point on the incircle.

4 Let k be a positive integer. Some of the $2k$-element subsets of a given set are marked. Suppose that for any subset of cardinality less than or equal to $(k+1)^2$ all the marked subsets contained in it (if any) have a common element. Show that all the marked subsets have a common element.

Day 2

5 Initially the numbers 19 and 98 are written on a board. Every minute, each of the two numbers is either squared or increased by 1. Is it possible to obtain two equal numbers at some time?

6 A binary operation $*$ on real numbers has the property that $(a*b)*c = a+b+c$ for all a, b, c. Prove that $a*b = a+b$.

7 Let n be an integer at least 4. In a convex n-gon, there is NO four vertices lie on a same circle. A circle is called circumscribed if it passes through 3 vertices of the n-gon and contains all other vertices. A circumscribed circle is called boundary if it passes through 3 consecutive vertices, a circumscribed circle is called inner if it passes through 3 pairwise non-consecutive points. Prove the number of boundary circles is 2 more than the number of inner circles.

8 Each square of a $(2^n-1)\times(2^n-1)$ board contains either 1 or -1. Such an arrangement is called *successful* if each number is the product of its neighbors. Find the number of successful arrangements.

– Grade level 11

Day 1

1 Two lines parallel to the x-axis cut the graph of $y = ax^3+bx^2+cx+d$ in points A, C, E and B, D, F respectively, in that order from left to right. Prove that the length of the projection of the segment CD onto the x-axis equals the sum of the lengths of the projections of AB and EF.

2 Let ABC be a triangle with circumcircle w. Let D be the midpoint of arc BC that contains A. Define E and F similarly. Let the incircle of ABC touches BC, CA, AB at K, L, M respectively. Prove that DK, EL, FM are concurrent.

3 A set $\mathcal{S}$ of translates of an equilateral triangle is given in the plane, and any two have nonempty intersection. Prove that there exist three points such that every triangle in $\mathcal{S}$ contains one of these points.

4 yeah you're right,the official problem is the following one:

there are 1998 cities in Russia, each being connected (in both directions) by flights to three other cities. any city can be reached by any other city by a sequence of flights. the KGB plans to close off 200 cities, no two joined by a single flight. show that this can be done so that any open city can be reached from any other open city by a sequence of flights only passing through open cities.

we begin with some terminology. define a **trigraph** to be a connected undirected graph in which every vertex has degree at most 3. a **trivalent** vertex of such a graph is a vertex of degree 3. in this wording, the problem becomes: we have a trigraph G with 1998 vertices, all of which are trivalent. we want to remove 200 vertices, no two of which are adjacent, such that the remaining vertices stay connected.
we remove the vertices one at a time. suppose we have deleted k of the 1998 vertices, no two of which are adjacent, such the trigraph G' induced by the remaining vertices is connected. we will show that if $K < 200$, we can always delete a trivalent vertex of G' such that the graph remains connected. this vertex cannot be adjacent in G to any of the other k deleted vertices, because then its degree in G' would be less than 3. hence repeating this 200 times gives us the desired set of vertices.

Lemma. let G be a trigraph such that the removal of any trivalent vertex disconnects G. then G is planer. moreover G can be drawn in such a way that every vertex lies on the "outside" face; in other words, for any point P outside some bounded set, each vertex v of G can be joined to P by a curve which does not intersect any edges of G (except at v).

Proof. we induct on the number of trivalent vertices of G. if G does not contain any trivalent vertices, then G must be a path or a cycle and the claim is obvious. so suppose G contains $n \geq 1$ trivalent vertices and that every trigraph with fewer trivalent vertices can be drawn as described. if G is a tree the claim is obvious, so suppose G contains a cycle; let $v_1, \ldots, v_k$ ($k \geq 3$) be a minimal cycle. let $S = \{v_1, \ldots, v_k\}$, and let $T = \{i \mid v_i \text{ is trivalent }\}$. ($T$ cannot be empty, because then no v_i would be connected to a vertex of degree 3.) for each $i \in T$, let w_i denote the third vertex which is adjacent to v_i (other than v_{i-1} and v_{i+1}), and let S_i be the set of vertices in G which can be reached from w_i without passing though S. (for $i \notin T$, let $S_i = \emptyset$.) we claim that the sets $S, S_1, \ldots, S_k$ partition the vertices of G. first, note that if v is a vertex of G not in S then there is a shortest path joining v to a vertex v_i of S; the penultimate vertex on this path must be w_i, so $v \in S_i$. now suppose that $v \in S_i \cap S_j$ for some $i \neq j$; then v_i and v_j are trivalent and there exist paths $w_i \to v$, $w_j \to v$ which do not pass though S. we will show that there exists a path from every vertex of $G - \{v_i\}$ to v which does not pass though v_i. for $k \neq i$ there is a path $v_k \to v_j \to w_j \to v$; if $w \in S_k$ for $k \neq i$, then there is a path $w \to w_k \to v_k \to v$; if $w \in S_i$, there is a path $w \to w_i \to v$. since $S \cup S_1 \cup \ldots \cup S_k = G$, we have shown that the graph obtained from G by deleting v_i is connected,a contradiction, as v_i is trivalent. therefore $S_i \cap S_j = \emptyset$ for $i \neq j$. obviously $S_i \cap S$ is empty for all i;

hence $S, S_1, \dots, S_k$ partition the vertices of G. let $G', G_1, \dots, G_k$ be the induced subgraphs of $S, S_1, \dots, S_k$ in G, respectively. by construction, the only edges in G which are not in one of the graphs $G', G_1, \dots, G_k$ are the edges $v_i w_i$ for $i \in T$. now G_i is a trigraph with fewer than n trivalent vertices, since at least one of the n trivalent vertices in G is in S. hence by the inductive hypothesis, we can draw each G_i in the plane in such a way that every vertex lies on the outside face. since $v_1, \dots, v_k$ was a minimal cycle, there are no "extra" edges between these vertices, so the graph G' is a k-cycle. now place the vertices of S at the vertices of a small regular k-gon far from all the graphs G_i; then we can draw a curve joining each pair v_i, w_i. it is east to check that this gives us a drawing of G with the desired properties. $\square$

now suppose we have removed k vertices from G, no two of which are adjacent, such that the trigraph G' induced by the remaining vertices is connected, and suppose that removing any trivalent vertex of G' disconnects the graph; we must show $k \geq 200$. by the lemma, G' is planner. we will call a dace other than the outside one a "proper face". let F be the number of proper faces of G'; since G' has $1998 - k$ vertices and $2997 - 3k$ edges,

$F \geq 1 - (1998 - k) + (2997 - 3k) = 1000 - 2k.$

we now show that no two proper faces can share a vertex. observe that each vertex belongs to at most as many faces as its degree; thus vertices of degree 1 lie only on the outside face. no two proper faces can intersect in a vertex of degree 2, or that vertex would not lie on the outside face, contracting the lemma. if two proper daces intersected in a trivalent vertex v, each face would give a path between two of v's neighbors, so removing v would not disconnect the graph, by an argument similar to that of the lemma.
since each proper face contains at least 3 vertices and no two share a vertex, we have $3F \leq 1998 - k$. combining this with the previous inequality gives

$3000 - 6k \leq 3F \leq 1998 - k$

so $1002 \leq 5k$ and $k \geq 200$, as desired.

Day 2

5 A sequence of distinct circles $\omega_1, \omega_2, \cdots$ is inscribed in the parabola $y = x^2$ so that ω_n and ω_{n+1} are tangent for all n. If ω_1 has diameter 1 and touches the parabola at $(0,0)$, find the diameter of ω_{1998}.

6 Are there 1998 different positive integers, the product of any two being divisible by the square of their difference?

7 A tetrahedron $ABCD$ has all edges of length less than 100, and contains two nonintersecting spheres of diameter 1. Prove that it contains a sphere of diameter 1.01.

8 A figure Φ composed of unit squares has the following property: if the squares of an $m \times n$ rectangle (m, n are fixed) are filled with numbers whose sum is positive, the figure Φ can be placed within the rectangle (possibly after being rotated) so that the sum of the covered numbers is also positive. Prove that a number of such figures can be put on the $m \times n$ rectangle so that each square is covered by the same number of figures.

All-Russian Olympiad 1997

– Grade level 9

Day 1

1 Let $P(x)$ be a quadratic polynomial with nonnegative coeficients. Show that for any real numbers x and y, we have the inequality $P(xy)^2 \leqslant P(x^2)P(y^2)$.
E. Malinnikova

2 Given a convex polygon M invariant under a 90° rotation, show that there exist two circles, the ratio of whose radii is $\sqrt{2}$, one containing M and the other contained in M.
A. Khrabrov

3 The lateral sides of a box with base $a \times b$ and height c (where a; b;c are natural numbers) are completely covered without overlap by rectangles whose edges are parallel to the edges of the box, each containing an even number of unit squares. (Rectangles may cross the lateral edges of the box.) Prove that if c is odd, then
the number of possible coverings is even.
D. Karpov, C. Gukshin, D. Fon-der-Flaas

4 The Judgment of the Council of Sages proceeds as follows: the king arranges the sages in a line and places either a white hat or a black hat on each sage's head. Each sage can see the color of the hats of the sages in front of him, but not of his own hat or of the hats of the sages behind him. Then one by one (in an order of their choosing), each sage guesses a color. Afterward, the king executes those sages who did not correctly guess the color of their own hat. The day before, the Council meets and decides to minimize the number of executions. What is the smallest number of sages guaranteed to survive in this case?

See also `http://www.artofproblemsolving.com/Forum/viewtopic.php?f=42&t=530553`

Day 2

1 Do there exist real numbers b and c such that each of the equations $x^2+bx+c=0$ and $2x^2+(b+1)x+c+1=0$ have two integer roots?
N. Agakhanov

2 A class consists of 33 students. Each student is asked how many other students in the class have his first name, and how many have his last name. It turns out that each number from 0 to 10 occurs among the answers. Show that there are two students in the class with the same first and last name.
A. Shapovalov

3 The incircle of triangle ABC touches sides AB;BC;CA at M;N;K, respectively. The line through A parallel to NK meets MN at D.
The line through A parallel to MN meets NK at E.
Show that the line DE bisects sides AB and AC of triangle ABC.
M. Sonkin

4 The numbers from 1 to 100 are arranged in a 10×10 table so that any two adjacent numbers have sum no larger than S. Find the least value of S for which this is possible.
D. Hramtsov

– Grade level 10

Day 1

1 Find all integer solutions of the equation $(x^2 - y^2)^2 = 1 + 16y$.
M. Sonkin

2 An $n \times n$ square grid ($n \geqslant 3$) is rolled into a cylinder. Some of the cells are then colored black. Show that there exist two parallel lines (horizontal, vertical or diagonal) of cells containing the same number of black cells.
E. Poroshenko

3 Two circles intersect at A and B. A line through A meets the first circle again at C and the second circle again at D. Let M and N be the midpoints of the arcs BC and BD not containing A, and let K be the midpoint of the segment CD. Show that $\angle MKN = \pi/2$.
(You may assume that C and D lie on opposite sides of A.)
D. Tereshin

4 A polygon can be divided into 100 rectangles, but not into 99. Prove that it cannot be divided into 100 triangles.
A. Shapovalov

Day 2

1 Do there exist two quadratic trinomials ax^2+bx+c and $(a+1)x^2+(b+1)x+(c+1)$ with integer coeficients, both of which have two integer roots?
N. Agakhanov

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2 A circle centered at O and inscribed in triangle ABC meets sides $AC;AB;BC$ at $K;M;N$, respectively. The median BB_1 of the triangle meets MN at D. Show that $O;D;K$ are collinear.
M. Sonkin

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3 Find all triples $m;\ n;\ l$ of natural numbers such that $m+n = gcd(m;n)^2$; $m+l = gcd(m;l)^2$; $n+l = gcd(n;l)^2$:
S. Tokarev

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4 On an infinite (in both directions) strip of squares, indexed by the integers, are placed several stones (more than one may be placed on a single square). We perform a sequence of moves of one of the following types:
(a) Remove one stone from each of the squares $n-1$ and n and place one stone on square $n+1$.
(b) Remove two stones from square n and place one stone on each of the squares $n+1$, $n-2$.
Prove that any sequence of such moves will lead to a position in which no further moves can be made, and moreover that this position is independent of the sequence of moves.
D. Fon-der-Flaas

– Grade level 11

Day 1

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1 Find all integer solutions of the equation $(x^2-y^2)^2 = 1+16y$.
M. Sonkin

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2 The Judgment of the Council of Sages proceeds as follows: the king arranges the sages in a line and places either a white hat, black hat or a red hat on each sage's head. Each sage can see the color of the hats of the sages in front of him, but not of his own hat or of the hats of the sages behind him. Then one by one (in an order of their choosing), each sage guesses a color. Afterward, the king executes those sages who did not correctly guess the color of their own hat. The day before, the Council meets and decides to minimize the number of executions. What is the smallest number of sages guaranteed to survive in this case?
K. Knop

P.S. Of course, the sages hear the previous guesses.

See also http://www.artofproblemsolving.com/Forum/viewtopic.php?f=42&t=530552

3 Two circles intersect at A and B. A line through A meets the first circle again at C and the second circle again at D. Let M and N be the midpoints of the arcs BC and BD not containing A, and let K be the midpoint of the segment CD. Show that $\angle MKN = \pi/2$.
(You may assume that C and D lie on opposite sides of A.)
D. Tereshin

4 An $n \times n \times n$ cube is divided into unit cubes. We are given a closed non-self-intersecting polygon (in space), each of whose sides joins the centers of two unit cubes sharing a common face. The faces of unit cubes which intersect the polygon are said to be distinguished. Prove that the edges of the unit cubes may be colored in two colors so that each distinguished face has an odd number of edges of each color, while each nondistinguished face has an even number of edges of each color.
M. Smurov

Day 2

1 Of the quadratic trinomials $x^2 + px + q$ where p; q are integers and $1 \leqslant p, q \leqslant 1997$, which are there more of: those having integer roots or those not having real roots?
M. Evdokimov

2 We are given a polygon, a line l and a point P on l in general position: all lines containing a side of the polygon meet l at distinct points diering from P.
We mark each vertex of the polygon the sides meeting which, extended away from the vertex, meet the
line l on opposite sides of P. Show that P lies inside the polygon if and only if on each side of l there are an odd number of marked vertices.
O. Musin

3 A sphere inscribed in a tetrahedron touches one face at the intersection of its angle bisectors, a second face at the intersection of its altitudes, and a third face at the intersection of its medians. Show that the tetrahedron is regular.
N. Agakhanov

4 In an $m \times n$ rectangular grid, where m and n are odd integers, 1×2 dominoes are initially placed so as to exactly cover all but one of the 1×1 squares at one

corner of the grid.
It is permitted to slide a domino towards the empty square, thus exposing another square.
Show that by a sequence of such moves, we can move the empty square to any corner of the rectangle.
A. Shapovalov

All-Russian Olympiad 1996

– Grade level 9

Day 1

1 Which are there more of among the natural numbers from 1 to 1000000, inclusive: numbers that can be represented as the sum of a perfect square and a (positive) perfect cube, or numbers that cannot be?

A. Golovanov

2 The centers O_1; O_2; O_3 of three nonintersecting circles of equal radius are positioned at the vertices of a triangle. From each of the points O_1; O_2; O_3 one draws tangents to the other two given circles. It is
known that the intersection of these tangents form a convex hexagon. The sides of the hexagon are alternately colored red and blue. Prove that the sum of the lengths of the red sides equals the sum of the lengths of the blue sides.

D. Tereshin

3 Let x, y, p, n, and k be positive integers such that $x^n + y^n = p^k$. Prove that if $n > 1$ is odd, and p is an odd prime, then n is a power of p.

A. Kovaldji, V. Senderov

4 In the Duma there are 1600 delegates, who have formed 16000 committees of 80 persons each. Prove that one can find two committees having no fewer than four common members.

A. Skopenkov

Day 2

5 Show that in the arithmetic progression with first term 1 and ratio 729, there are infinitely many powers of 10.

L. Kuptsov

6 In the isosceles triangle ABC ($AC = BC$) point O is the circumcenter, I the incenter, and D lies on BC so that lines OD and BI are perpendicular. Prove that ID and AC are parallel.

M. Sonkin

7 Two piles of coins lie on a table. It is known that the sum of the weights of the coins in the two piles are equal, and for any natural number k, not exceeding the number of coins in either pile, the sum of the weights of the k heaviest coins in the first pile is not more than that of the second pile. Show that for any natural number x, if each coin (in either pile) of weight not less than x is replaced by a coin of weight x, the first pile will not be lighter than the second.

D. Fon-der-Flaas

8 Can a 5×7 checkerboard be covered by L's (figures formed from a 2×2 square by removing one of its four 1×1 corners), not crossing its borders, in several layers so that each square of the board is covered by the same number of L's?

M. Evdokimov

– Grade level 10

Day 1

1 Points E and F are given on side BC of convex quadrilateral $ABCD$ (with E closer than F to B). It is known that $\angle BAE = \angle CDF$ and $\angle EAF = \angle FDE$. Prove that $\angle FAC = \angle EDB$.

M. Smurov

2 On a coordinate plane are placed four counters, each of whose centers has integer coordinates. One can displace any counter by the vector joining the centers of two of the other counters. Prove that any two preselected counters can be made to coincide by a finite sequence of moves.

. Sadykov

3 Find all natural numbers n, such that there exist relatively prime integers x and y and an integer $k > 1$ satisfying the equation $3^n = x^k + y^k$.

A. Kovaldji, V. Senderov

4 Show that if the integers $a_1; \dots\ a_m$ are nonzero and for each $k = 0; 1; \dots; n$ $(n < m - 1)$, $a_1 + a_2 2^k + a_3 3^k + \cdots + a_m m^k = 0$; then the sequence $a_1, \dots, a_m$ contains at least $n + 1$ pairs of consecutive terms having opposite signs.

O. Musin

Day 2

5 At the vertices of a cube are written eight pairwise distinct natural numbers, and on each of its edges is written the greatest common divisor of the numbers at the endpoints of the edge. Can the sum of the numbers written at the vertices be the same as the sum of the numbers written at the edges?

A. Shapovalov

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6 Three sergeants and several solders serve in a platoon. The sergeants take turns on duty. The commander has given the following orders:
(a) Each day, at least one task must be issued to a soldier.
(b) No soldier may have more than two task or receive more than one tasks in a single day.
(c) The lists of soldiers receiving tasks for two different days must not be the same.
(d) The first sergeant violating any of these orders will be jailed.
Can at least one of the sergeants, without conspiring with the others, give tasks according to these rules and avoid being jailed?

M. Kulikov

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7 A convex polygon is given, no two of whose sides are parallel. For each side we consider the angle the side subtends at the vertex farthest from the side. Show that the sum of these angles equals 180°.

M. Smurov

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8 Goodnik writes 10 numbers on the board, then Nogoodnik writes 10 more numbers, all 20 of the numbers being positive and distinct. Can Goodnik choose his 10 numbers so that no matter what Nogoodnik writes, he can form 10 quadratic trinomials of the form $x^2 + px + q$, whose coeficients p and q run through all of the numbers written, such that the real roots of these trinomials comprise exactly 11 values?

I. Rubanov

– Grade level 11

Day 1

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1 Can the number obtained by writing the numbers from 1 to n in order $(n > 1)$ be the same when read left-to-right and right-to-left?

N. Agakhanov

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2 Several hikers travel at fixed speeds along a straight road. It is known that over some period of time, the sum of their pairwise distances is monotonically decreasing. Show that there is a hiker, the sum of whose distances to the other hikers is monotonically decreasing over the same period.

A. Shapovalov

3 Show that for $n \geq 5$, a cross-section of a pyramid whose base is a regular n-gon cannot be a regular $(n+1)$-gon.

N. Agakhanov, N. Tereshin

4 Show that if the integers $a_1; \dots\ a_m$ are nonzero and for each $k = 0; 1; \dots; n$ $(n < m-1)$, $a_1 + a_2 2^k + a_3 3^k + \dots + a_m m^k = 0$; then the sequence $a_1, \dots, a_m$ contains at least $n+1$ pairs of consecutive terms having opposite signs.

O. Musin

Day 2

5 Do there exist three natural numbers greater than 1, such that the square of each, minus one, is divisible by each of the others?

A. Golovanov

6 In isosceles triangle ABC $(AB = BC)$ one draws the angle bisector CD. The perpendicular to CD through the center of the circumcircle of ABC intersects BC at E. The parallel to CD through E meets AB at F. Show that $BE = FD$.

M. Sonkin

7 Does there exist a finite set M of nonzero real numbers, such that for any natural number n a polynomial of degree no less than n with coeficients in M, all of whose roots are real and belong to M?

E. Malinnikova

8 The numbers from 1 to 100 are written in an unknown order. One may ask about any 50 numbers and find out their relative order. What is the fewest questions needed to find the order of all 100 numbers?

S. Tokarev

All-Russian Olympiad 1995

– Grade level 9

Day 1

1 A freight train departed from Moscow at x hours and y minutes and arrived at Saratov at y hours and z minutes. The length of its trip was z hours and x minutes. Find all possible values of x.

S. Tokarev

2 A chord CD of a circle with center O is perpendicular to a diameter AB. A chord AE bisects the radius OC. Show that the line DE bisects the chord BC
V. Gordon

3 Can the equation $f(g(h(x))) = 0$, where f, g, h are quadratic polynomials, have the solutions $1, 2, 3, 4, 5, 6, 7, 8$?

S. Tokarev

4 Can the numbers from 1 to 81 be written in a 99 board, so that the sum of numbers in each 33 square is the same?
S. Tokarev

Day 2

5 We call natural numbers *similar* if they are written with the same (decimal) digits. For example, 112, 121, 211 are similar numbers having the digits 1, 1, 2. Show that there exist three similar 1995-digit numbers with no zero digits, such that the sum of two of them equals the third.
S. Dvoryaninov

6 In an acute-angled triangle ABC, points A_2, B_2, C_2 are the midpoints of the altitudes AA_1, BB_1, CC_1, respectively. Compute the sum of angles $B_2A_1C_2$, $C_2B_1A_2$ and $A_2C_1B_2$.
D. Tereshin

7 There are three boxes of stones. Sisyphus moves stones one by one between the boxes. Whenever he moves a stone, Zeus gives him the number of coins that is equal to the difference between the number of stones in the box the stone was put in, and that in the box the stone was taken from (the moved stone does not

count). If this difference is negative, then Sisyphus returns the corresponding amount to Zeus (if Sisyphus cannot pay, generous Zeus allows him to make the move and pay later).
After some time all the stones lie in their initial boxes. What is the greatest possible earning of Sisyphus at that moment?
I. Izmestev

8 Numbers 1 and 1 are written in the cells of a board 20002000. It is known that the sum of all the numbers in the board is positive. Show that one can select 1000 rows and 1000 columns such that the sum of numbers written in their intersection cells is at least 1000.
D. Karpov

– Grade level 10

Day 1

1 A freight train departed from Moscow at x hours and y minutes and arrived at Saratov at y hours and z minutes. The length of its trip was z hours and x minutes. Find all possible values of x.

S. Tokarev

2 A chord CD of a circle with center O is perpendicular to a diameter AB. A chord AE bisects the radius OC. Show that the line DE bisects the chord BC
V. Gordon

3 Does there exist a sequence of natural numbers in which every natural number occurs exactly once, such that for each $k = 1, 2, 3, \dots$ the sum of the first k terms of the sequence is divisible by k?
A. Shapovalov

4 Prove that if all angles of a convex n-gon are equal, then there are at least two of its sides that are not longer than their adjacent sides.
A. Berzinsh, O. Musin

Day 2

5 The sequence $a_1, a_2, ...$ of natural numbers satisfies $GCD(a_i, a_j) = GCD(i, j)$ for all $i \neq j$. Prove that $a_i = i$ for all i.

6 Let be given a semicircle with diameter AB and center O, and a line intersecting the semicircle at C and D and the line AB at M ($MB < MA$, $MD < MC$). The circumcircles of the triangles AOC and DOB meet again at L. Prove that $\angle MKO$ is right.
L. Kuptsov

7 Numbers 1 and 1 are written in the cells of a board 20002000. It is known that the sum of all the numbers in the board is positive. Show that one can select 1000 rows and 1000 columns such that the sum of numbers written in their intersection cells is at least 1000.
D. Karpov

8 Let $P(x)$ and $Q(x)$ be monic polynomials. Prove that the sum of the squares of the coeficients of the polynomial $P(x)Q(x)$ is not smaller than the sum of the squares of the free coefficients of $P(x)$ and $Q(x)$.
A. Galochkin, O. Ljashko

– Grade level 11

Day 1

1 Can the numbers $1, 2, 3, \dots, 100$ be covered with 12 geometric progressions?
A. Golovanov

2 Prove that every real function, defined on all of $\mathbb{R}$, can be represented as a sum of two functions whose graphs both have an axis of symmetry.
D. Tereshin

3 Two points on the distance 1 are given in a plane. It is allowed to draw a line through two marked points, as well as a circle centered in a marked point with radius equal to the distance between some two marked points. By marked points we mean the two initial points and intersection points of two lines, two circles, or a line and a circle constructed so far. Let $C(n)$ be the minimum number of circles needed to construct two points on the distance n if only a compass is used, and let $LC(n)$ be the minimum total number of circles and lines needed to do so if a ruler and a compass are used, where n is a natural number. Prove that the sequence $C(n)/LC(n)$ is not bounded.
A. Belov

4 Prove that if all angles of a convex n-gon are equal, then there are at least two of its sides that are not longer than their adjacent sides.
A. Berzinsh, O. Musin

Day 2

5 Prove that for every natural number $a_1 > 1$ there exists an increasing sequence of natural numbers a_n such that $a_1^2+a_2^2+\cdots+a_k^2$ is divisible by $a_1+a_2+\cdots+a_k$ for all $k \geq 1$.

A. Golovanov

6 A boy goes n times at a merry-go-round with n seats. After every time he moves in the clockwise direction and takes another seat, not making a full circle. The number of seats he passes by at each move is called the length of the move. For which n can he sit at every seat, if the lengths of all the $n-1$ moves he makes have different lengths?
V. New

7 The altitudes of a tetrahedron intersect in a point. Prove that this point, the foot of one of the altitudes, and the points dividing the other three altitudes in the ratio $2:1$ (measuring from the vertices) lie on a sphere.
D. Tereshin

8 Let $P(x)$ and $Q(x)$ be monic polynomials. Prove that the sum of the squares of the coeficients of the polynomial $P(x)Q(x)$ is not smaller than the sum of the squares of the free coefficients of $P(x)$ and $Q(x)$.
A. Galochkin, O. Ljashko