

China National Olympiad 2005

### Day 1

- 1 Suppose  $\theta_i \in (-\frac{\pi}{2}, \frac{\pi}{2}), i = 1, 2, 3, 4$ . Prove that, there exist  $x \in \mathbb{R}$ , satisfying two inequalities

$$\begin{aligned}\cos^2 \theta_1 \cos^2 \theta_2 - (\sin \theta_1 \sin \theta_2 - x)^2 &\geq 0, \\ \cos^2 \theta_3 \cos^2 \theta_4 - (\sin \theta_3 \sin \theta_4 - x)^2 &\geq 0\end{aligned}$$

if and only if

$$\sum_{i=1}^4 \sin^2 \theta_i \leq 2(1 + \prod_{i=1}^4 \sin \theta_i + \prod_{i=1}^4 \cos \theta_i).$$

- 2 A circle meets the three sides  $BC, CA, AB$  of a triangle  $ABC$  at points  $D_1, D_2; E_1, E_2; F_1, F_2$  respectively. Furthermore, line segments  $D_1E_1$  and  $D_2F_2$  intersect at point  $L$ , line segments  $E_1F_1$  and  $E_2D_2$  intersect at point  $M$ , line segments  $F_1D_1$  and  $F_2E_2$  intersect at point  $N$ . Prove that the lines  $AL, BM, CN$  are concurrent.

- 3 As the graph, a pond is divided into  $2n$  ( $n \geq 5$ ) parts. Two parts are called neighborhood if they have a common side or arc. Thus every part has three neighborhoods. Now there are  $4n+1$  frogs at the pond. If there are three or more frogs at one part, then three of the frogs of the part will jump to the three neighborhoods respectively. Prove that for some time later, the frogs at the pond will uniformly distribute. That is, for any part either there are frogs at the part or there are frogs at the each of its neighborhoods.

[http://www.mathlinks.ro/Forum/files/china2005\\_2\\_214.gif](http://www.mathlinks.ro/Forum/files/china2005_2_214.gif)

### Day 2

- 4 The sequence  $\{a_n\}$  is defined by:  $a_1 = \frac{21}{16}$ , and for  $n \geq 2$ ,

$$2a_n - 3a_{n-1} = \frac{3}{2^{n+1}}.$$

Let  $m$  be an integer with  $m \geq 2$ . Prove that: for  $n \leq m$ , we have

$$\left(a_n + \frac{3}{2^{n+3}}\right)^{\frac{1}{m}} \left(m - \left(\frac{2}{3}\right)^{\frac{n(m-1)}{m}}\right) < \frac{m^2 - 1}{m - n + 1}.$$



# Art of Problem Solving

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5                      There are 5 points in a rectangle (including its boundary) with area 1, no three of them are in the same line. Find the minimum number of triangles with the area not more than  $\frac{1}{4}$ , vertex of which are three of the five points.

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6                      Find all nonnegative integer solutions  $(x, y, z, w)$  of the equation

$$2^x \cdot 3^y - 5^z \cdot 7^w = 1.$$

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