Prove RP=RQ

Source: All Russian 2014 Grade 10 Day 1 P4

mathuz
1229 posts

May 3, 2014, 8:33 pm PM #1

Given a triangle ABC with $AB > BC$, let Ω be the circumcircle. Let M, N lie on the sides AB, BC respectively, such that $AM = CN$. Let K be the intersection of MN and AC. Let P be the incentre of the triangle AMK and Q be the K-excentre of the triangle CNK. If R is midpoint of the arc ABC of Ω then prove that $RP = RQ$.

M. Kungodjin

This post has been edited 2 times. Last edited by WakeUp, Jul 20, 2014, 6:52 pm

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sedrikktl
105 posts

May 3, 2014, 9:43 pm PM #2

http://olympiads.mccme.ru/vmo/

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wiseman
211 posts

May 3, 2014, 11:21 pm PM #3

@Mathuz : Do you mean K is the intersection point of BC and MN?

War-Ham...
662 posts

May 3, 2014, 11:23 pm PM #4

K is the point of intersection MN and AC.

MrRTI
191 posts

May 4, 2014, 6:24 am PM #5

> *War-Hammer wrote:*
> K is the point of intersection MN and AC.

the intersection of MN and AC is N....

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pi37
2079 posts

May 4, 2014, 6:35 am • 1 PM #6

There is a typo in the problem statement. N should lie on BC, under the same condition $CN = AM$.

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pi37
2079 posts

May 5, 2014, 6:07 am • 1 PM #7

Solution:

Let (P) be tangent to MN, AC at D, E, and let Q be tangent to MN, AC at F, G. Let X, Y, Z be the midpoints of AC, MN, PQ, and let S, T be the midpoints of EG, DF.
Note that since PQ passes through K and $DP, FQ \perp MN, TZ \perp MN$ as well. Similarly $ZS \perp AC$. Now

$$KD = KE = \frac{1}{2}(KM + KA - MA)$$

and

$$KF = KG = \frac{1}{2}(KN + KC + NC)$$

so

$$2KT = KD + KF = \frac{1}{2}(KM + KA + KN + KC)$$

$$= \frac{1}{2}(KM + KN) + \frac{1}{2}(KA + KC) = KX + KY$$

Thus

$$KT = KS = \frac{1}{2}(KX + KY) \Rightarrow YT = SX$$

Z lies on PQ, the angle bisector of MKA, so $ZT = ZS$. Then $\triangle ZTY \cong \triangle ZSX$, yielding $ZY = ZX$. So Z, the intersection of the angle bisector of YKX and the perpendicular bisector of XY, is the midpoint of arc XY on circle (KXY).
But $RA = RC, MA = NC$, and $\angle MAR = \angle NCR$, so $\triangle MAR \cong \triangle NCR$ implying $RM = RN$. Thus $RY \perp YK$, and of course $RX \perp XK$. So $RXYK$ is cyclic with diameter RK. Then Z also lies on this circle, so $RZ \perp ZK$, and R lies on the perpendicular bisector of PQ, giving $RP = RQ$.

Dominati...
313 posts

May 5, 2014, 7:10 pm PM #8

Can anyone make all of it into the page contest?

Tangents
37 posts

May 18, 2014, 8:47 am PM #9

> *Domination1998 wrote:*
> Can anyone make all of it into the page contest?

It is here.

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nima1376
111 posts

May 18, 2014, 12:18 pm PM #10

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=587501

Dominati...
313 posts

May 18, 2014, 1:56 pm PM #11

OK thanks
for the page

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thecmd999
2874 posts

Sep 23, 2014, 1:31 am PM #12

Solution

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anantmu...
839 posts

Dec 18, 2015, 9:20 am • 1 PM #13

This probably has the following, surprisingly easy solution. It seems that its difficulty is over-rated by a long shot

But anyway, this was a very elegant problem.

Here is my solution:

It is easy to see that $RM = RN$ and so by considering the rotation $\mathcal{R}$ centered at point R that sends A to C and its inverse $\mathcal{R}^{-\infty}$ we define:

Point $Q' = \mathcal{R}^{-\infty}(Q)$ and $P' = \mathcal{R}(P)$.

Now, we see that by some trivial angle-chasing we have $PQP'Q'$ as an isosceles trapezoid and since R lies on the perpendicular bisectors of PP' and QQ' we must have that R is the center of the isosceles trapezoid $PQP'Q'$ and so $RP = RQ$.

P.S.- Sorry for the abuse of notation

This post has been edited 1 time. Last edited by anantmudgal09, Dec 18, 2015, 9:21 am

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kapilpavase
432 posts

Jan 31, 2016, 7:10 pm PM #14

Lol...rotation and done...for the foll sol we do not need angle chase at all.
Just note two things,$1)RM = RN, RC = RA$ and that $2) \odot MPA, \odot NQC$ are congruent.Now $1)$ imples R is the centre of rotation sending $\odot MPA$to$\odot NQC.2)$ implies it is equidistant from their centres,and so present on line of symmetry between the circles.Since PQ passes thru centres of those circles,,by symmetry $RP = RQ$ is obvious.

This post has been edited 2 times. Last edited by kapilpavase, Jan 31, 2016, 7:12 pm

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