IMO Shortlist 2007

– Algebra

1 Real numbers a_1, a_2, $\ldots$, a_n are given. For each i, $(1 \leq i \leq n)$, define

$$d_i = \max\{a_j \mid 1 \leq j \leq i\} - \min\{a_j \mid i \leq j \leq n\}$$

and let $d = \max\{d_i \mid 1 \leq i \leq n\}$.

(a) Prove that, for any real numbers $x_1 \leq x_2 \leq \cdots \leq x_n$,

$$\max\{|x_i - a_i| \mid 1 \leq i \leq n\} \geq \frac{d}{2}. \qquad (*)$$

(b) Show that there are real numbers $x_1 \leq x_2 \leq \cdots \leq x_n$ such that the equality holds in (*).

Author: Michael Albert, New Zealand

2 Consider those functions $f : \mathbb{N} \mapsto \mathbb{N}$ which satisfy the condition

$$f(m+n) \geq f(m) + f(f(n)) - 1$$

for all $m, n \in \mathbb{N}$. Find all possible values of $f(2007)$.

Author: Nikolai Nikolov, Bulgaria

3 Let n be a positive integer, and let x and y be a positive real number such that $x^n + y^n = 1$. Prove that

$$\left(\sum_{k=1}^{n} \frac{1+x^{2k}}{1+x^{4k}}\right) \cdot \left(\sum_{k=1}^{n} \frac{1+y^{2k}}{1+y^{4k}}\right) < \frac{1}{(1-x)\cdot(1-y)}.$$

Author: Juhan Aru, Estonia

4 Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $f\left(x + f\left(y\right)\right) = f\left(x + y\right) + f\left(y\right)$ for all pairs of positive reals x and y. Here, $\mathbb{R}^+$ denotes the set of all positive reals.

Proposed by Paisan Nakmahachalasint, Thailand

5 Let $c > 2$, and let $a(1), a(2), \ldots$ be a sequence of nonnegative real numbers such that

$$a(m+n) \leq 2 \cdot a(m) + 2 \cdot a(n) \text{ for all } m, n \geq 1,$$

and $a\left(2^k\right) \leq \frac{1}{(k+1)^c}$ for all $k \geq 0$. Prove that the sequence $a(n)$ is bounded.

Author: Vjekoslav Kova, Croatia

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6 Let $a_1, a_2, \ldots, a_{100}$ be nonnegative real numbers such that $a_1^2 + a_2^2 + \ldots + a_{100}^2 = 1$. Prove that

$$a_1^2 \cdot a_2 + a_2^2 \cdot a_3 + \ldots + a_{100}^2 \cdot a_1 < \frac{12}{25}.$$

Author: Marcin Kuzma, Poland

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7 Let n be a positive integer. Consider

$$S = \{(x, y, z) \mid x, y, z \in \{0, 1, \ldots, n\}, x + y + z > 0\}$$

as a set of $(n+1)^3 - 1$ points in the three-dimensional space. Determine the smallest possible number of planes, the union of which contains S but does not include $(0, 0, 0)$.

Author: Gerhard Wginger, Netherlands

– Combinatorics

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1 Let $n > 1$ be an integer. Find all sequences $a_1, a_2, \ldots a_{n^2+n}$ satisfying the following conditions:

$$\text{(a) } a_i \in \{0, 1\} \text{ for all } 1 \leq i \leq n^2 + n;$$

$$\text{(b) } a_{i+1} + a_{i+2} + \ldots + a_{i+n} < a_{i+n+1} + a_{i+n+2} + \ldots + a_{i+2n} \text{ for all } 0 \leq i \leq n^2 - n.$$

Author: Dusan Dukic, Serbia

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2 A rectangle D is partitioned in several (≥ 2) rectangles with sides parallel to those of D. Given that any line parallel to one of the sides of D, and having common points with the interior of D, also has common interior points with the interior of at least one rectangle of the partition; prove that there is at least one rectangle of the partition having no common points with D's boundary.

Author: Kei Irie, Japan

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3 Find all positive integers n for which the numbers in the set $S = \{1, 2, \ldots, n\}$ can be colored red and blue, with the following condition being satisfied: The set $S \times S \times S$ contains exactly 2007 ordered triples (x, y, z) such that:

(i) the numbers x, y, z are of the same color,
and
(ii) the number $x + y + z$ is divisible by n.

Author: Gerhard Wginger, Netherlands

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4 Let $A_0 = (a_1, \ldots, a_n)$ be a finite sequence of real numbers. For each $k \geq 0$, from the sequence $A_k = (x_1, \ldots, x_k)$ we construct a new sequence A_{k+1} in the following way.
1. We choose a partition $\{1, \ldots, n\} = I \cup J$, where I and J are two disjoint sets, such that the expression

$$\left| \sum_{i \in I} x_i - \sum_{j \in J} x_j \right|$$

attains the smallest value. (We allow I or J to be empty; in this case the corresponding sum is 0.) If there are several such partitions, one is chosen arbitrarily.
2. We set $A_{k+1} = (y_1, \ldots, y_n)$ where $y_i = x_i + 1$ if $i \in I$, and $y_i = x_i - 1$ if $i \in J$.
Prove that for some k, the sequence A_k contains an element x such that $|x| \geq \frac{n}{2}$.

Author: Omid Hatami, Iran

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5 In the Cartesian coordinate plane define the strips $S_n = \{(x, y) | n \leq x < n + 1\}$, $n \in \mathbb{Z}$ and color each strip black or white. Prove that any rectangle which is not a square can be placed in the plane so that its vertices have the same color.

IMO Shortlist 2007 Problem C5 as it appears in the official booklet:
In the Cartesian coordinate plane define the strips $S_n = \{(x, y) | n \leq x < n + 1\}$ for every integer n. Assume each strip S_n is colored either red or blue, and let a and b be two distinct positive integers. Prove that there exists a rectangle with side length a and b such that its vertices have the same color.
(*Edited by Orlando Dhring*)

Author: Radu Gologan and Dan Schwarz, Romania

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6 In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a *clique* if each two of them are friends. (In particular, any group of fewer than two competitiors is a clique.) The number of members of a clique is called its *size*.

Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged into two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

Author: Vasily Astakhov, Russia

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7 Let $\alpha < \frac{3-\sqrt{5}}{2}$ be a positive real number. Prove that there exist positive integers n and $p > \alpha \cdot 2^n$ for which one can select $2 \cdot p$ pairwise distinct subsets $S_1, \ldots, S_p, T_1, \ldots, T_p$ of the set $\{1, 2, \ldots, n\}$ such that $S_i \cap T_j \neq \emptyset$ for all $1 \leq i, j \leq p$

Author: Gerhard Wginger, Austria

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8 Given is a convex polygon P with n vertices. Triangle whose vertices lie on vertices of P is called *good* if all its sides are equal in length. Prove that there are at most $\frac{2n}{3}$ *good* triangles.

Author: Vyacheslav Yasinskiy, Ukraine

– Geometry

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1 In triangle ABC the bisector of angle BCA intersects the circumcircle again at R, the perpendicular bisector of BC at P, and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoint of AC is L. Prove that the triangles RPK and RQL have the same area.

Author: Marek Pechal, Czech Republic

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2 Denote by M midpoint of side BC in an isosceles triangle $\triangle ABC$ with $AC = AB$. Take a point X on a smaller arc $\overset{\frown}{MA}$ of circumcircle of triangle $\triangle ABM$. Denote by T point inside of angle BMA such that $\angle TMX = 90$ and $TX = BX$.

Prove that $\angle MTB - \angle CTM$ does not depend on choice of X.

Author: Farzan Barekat, Canada

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3 The diagonals of a trapezoid $ABCD$ intersect at point P. Point Q lies between the parallel lines BC and AD such that $\angle AQD = \angle CQB$, and line CD separates points P and Q. Prove that $\angle BQP = \angle DAQ$.

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Author: Vyacheslav Yasinskiy, Ukraine

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4 Consider five points A, B, C, D and E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line passing through A. Suppose that ℓ intersects the interior of the segment DC at F and intersects line BC at G. Suppose also that $EF = EG = EC$. Prove that ℓ is the bisector of angle DAB.

Author: Charles Leytem, Luxembourg

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5 Let ABC be a fixed triangle, and let A_1, B_1, C_1 be the midpoints of sides BC, CA, AB, respectively. Let P be a variable point on the circumcircle. Let lines PA_1, PB_1, PC_1 meet the circumcircle again at A', B', C', respectively. Assume that the points A, B, C, A', B', C' are distinct, and lines AA', BB', CC' form a triangle. Prove that the area of this triangle does not depend on P.

Author: Christopher Bradley, United Kingdom

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6 Determine the smallest positive real number k with the following property. Let $ABCD$ be a convex quadrilateral, and let points A_1, B_1, C_1, and D_1 lie on sides AB, BC, CD, and DA, respectively. Consider the areas of triangles AA_1D_1, BB_1A_1, CC_1B_1 and DD_1C_1; let S be the sum of the two smallest ones, and let S_1 be the area of quadrilateral $A_1B_1C_1D_1$. Then we always have $kS_1 \geq S$.

Author: Zuming Feng and Oleg Golberg, USA

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7 Given an acute triangle ABC with $\angle B > \angle C$. Point I is the incenter, and R the circumradius. Point D is the foot of the altitude from vertex A. Point K lies on line AD such that $AK = 2R$, and D separates A and K. Lines DI and KI meet sides AC and BC at E, F respectively. Let $IE = IF$.

Prove that $\angle B \leq 3\angle C$.

Author: Davoud Vakili, Iran

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8 Point P lies on side AB of a convex quadrilateral $ABCD$. Let ω be the incircle of triangle CPD, and let I be its incenter. Suppose that ω is tangent to the incircles of triangles APD and BPC at points K and L, respectively. Let lines AC and BD meet at E, and let lines AK and BL meet at F. Prove that points E, I, and F are collinear.

Author: Waldemar Pompe, Poland

–	Number Theory
1	Find all pairs of natural numbers (a, b) such that $7^a - 3^b$ divides $a^4 + b^2$. *Author: Stephan Wagner, Austria*
2	Let $b, n > 1$ be integers. Suppose that for each $k > 1$ there exists an integer a_k such that $b - a_k^n$ is divisible by k. Prove that $b = A^n$ for some integer A. *Author: Dan Brown, Canada*
3	Let X be a set of 10,000 integers, none of them is divisible by 47. Prove that there exists a 2007-element subset Y of X such that $a - b + c - d + e$ is not divisible by 47 for any $a, b, c, d, e \in Y$. *Author: Gerhard Wginger, Netherlands*
4	For every integer $k \geq 2$, prove that 2^{3k} divides the number $$\binom{2^{k+1}}{2^k} - \binom{2^k}{2^{k-1}}$$ but 2^{3k+1} does not. *Author: Waldemar Pompe, Poland*
5	Find all surjective functions $f : \mathbb{N} \to \mathbb{N}$ such that for every $m, n \in \mathbb{N}$ and every prime p, the number $f(m + n)$ is divisible by p if and only if $f(m) + f(n)$ is divisible by p. *Author: Mohsen Jamaali and Nima Ahmadi Pour Anari, Iran*
6	Let k be a positive integer. Prove that the number $(4 \cdot k^2 - 1)^2$ has a positive divisor of the form $8kn - 1$ if and only if k is even. Actual IMO 2007 Problem, posed as question 5 in the contest, which was used as a lemma in the official solutions for problem N6 as shown above. (`http://www.mathlinks.ro/viewtopic.php?p=89465#894656`) *Author: Kevin Buzzard and Edward Crane, United Kingdom*

7 For a prime p and a given integer n let $\nu_p(n)$ denote the exponent of p in the prime factorisation of $n!$. Given $d \in \mathbb{N}$ and $\{p_1, p_2, \dots, p_k\}$ a set of k primes, show that there are infinitely many positive integers n such that $d \mid \nu_{p_i}(n)$ for all $1 \leq i \leq k$.

Author: Tejaswi Navilarekkallu, India