Romania National Olympiad 2013

–	IX
1	A series of numbers is called complete if it has non-zero natural terms and any nonzero integer has at least one among multiple series. Show that the arithmetic progression is a complete sequence if and only if it divides the first term relationship.
2	Given $f : \mathbb{R} \to \mathbb{R}$ an arbitrary function and $g : \mathbb{R} \to \mathbb{R}$ a function of the second degree, with the property: for any real numbers m and n equation $f(x) = mx + n$ has solutions if and only if the equation $g(x) = mx + n$ has solutions Show that the functions f and g are equal.
3	Given P a point m inside a triangle acute-angled ABC and DEF intersections of lines with that AP, BP, CP with$[BC]$, $[CA]$,respective $[AB]$ a) Show that the area of the triangle DEF is at most a quarter of the area of the triangle ABC b) Show that the radius of the circle inscribed in the triangle DEF is at most a quarter of the radius of the circle circumscribed on triangle $4ABC$.
4	Consider a nonzero integer number n and the function $f : \mathbb{N} \to \mathbb{N}$ by $$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{x-1}{2} + 2^{n-1} & \text{if } x \text{ is odd} \end{cases}.$$ Determine the set: $$A = \{x \in \mathbb{N} \mid \underbrace{(f \circ f \circ .... \circ f)}_{n\ f\text{'s}}(x) = x\}.$$
–	X
1	Solve the following equation $2^{\sin^4 x - \cos^2 x} - 2^{\cos^4 x - \sin^2 x} = \cos 2x$
2	To be considered the following complex and distinct a, b, c, d. Prove that the following affirmations are equivalent: i)For every $z \in \mathbb{C}$ the inequality takes place :$\lvert z-a\rvert + \lvert z-b\rvert \geq \lvert z-c\rvert + \lvert z-d\rvert$; ii)There is $t \in (0,1)$ so that $c = ta + (1-t)b$ si $d = (1-t)a + tb$

3 Find all injective functions$f:\mathbb{Z}\to \mathbb{Z}$ that satisfy: $\left| f\left(x \right)-f\left(y \right) \right|\le \left| x-y \right|$,for any $x,y\in \mathbb{Z}$.

4 a)Prove that $\frac{1}{2}+\frac{1}{3}+...+\frac{1}{{{2}^{m}}}< m$, for any $m\in {{\mathbb{N}}^{*}}$.
b)Let ${{p}_{1}},{{p}_{2}},...,{{p}_{n}}$ be the prime numbers less than ${{2}^{100}}$. Prove that $\frac{1}{{{p}_{1}}}+\frac{1}{{{p}_{2}}}+...+\frac{1}{{{p}_{n}}}<10$

– XI

1 Given A, non-inverted matrices of order n with real elements, $n\ge 2$ and given ${{A}^{*}}$adjoin matrix A. Prove that $tr({{A}^{*}})\ne -1$ if and only if the matrix ${{I}_{n}}+{{A}^{*}}$ is invertible.

2 Whether m and n natural numbers, $m,n\ge 2$. Consider matrices, ${{A}_{1}},{{A}_{2}},...,{{A}_{m}}\in {{M}_{n}}(R)$ not all nilpotent. Demonstrate that there is an integer number $k>0$ such that ${{A}^{k}}_{1}+{{A}^{k}}_{2}+.....+{{A}^{k}}_{m}\ne {{O}_{n}}$

3 A function

$$\text{f:}\left(0\text{,}\infty \right)\text{ }\to \left(0\text{,}\infty \right)$$

is called contract if, for every numbers $x,y\in \text{(0,}\infty \text{)}$ we have, $\underset{n\to \infty }{\mathop{\lim }}\,\left({{f}^{n}}\left(x \right)-{{f}^{n}}\left(y \right) \right)=0$ where ${{f}^{n}}=\underbrace{f\circ f\circ ...\circ f}_{n\text{ }f\text{'s}}$
a) Consider

$$f:\left(0\text{,}\infty \right)\text{ }\to \left(0\text{,}\infty \right)$$

a function contract, continue with the property that has a fixed point, that existing ${{x}_{0}}\in \left(0\text{,}\infty \right)\text{ }$ there so that $f\left({{x}_{0}} \right)={{x}_{0}}$. Show that $f\left(x \right)>x$, for every $x\in \left(0,{{x}_{0}} \right)\text{ }$ and $f\left(x \right)<x$, for every $x\in \left({{x}_{0}},\infty \right)\text{ }$.
b) Show that the given function

$$f\text{:}\left(0\text{,}\infty \right)\text{ }\to \left(0\text{,}\infty \right)$$

given by $f\left(x \right)=x+\frac{1}{x}$ is contracted but has no fix number.

4 a) Consider

$$f\text{: }\!\![\!\!\text{ 0,}\infty)\to [0,\infty)$$

a differentiable and convex function .Show that $f\left(x \right)\le x$, for every $x\ge 0$, than ${f}'\left(x \right)\le 1$,for every $x\ge 0$
b) Determine

$$f\text{: }\!\![\!\!\text{ 0,}\infty)\to [0,\infty)$$

differentiable and convex functions which have the property that $f\left(0\right)=0$, and $f'\left(x\right)f\left(f\left(x\right)\right)=x$, for every $x\geq 0$

– XII

1 Determine continuous functions $f:\mathbb{R}\to \mathbb{R}$ such that $\left({{a}^{2}}+ab+{{b}^{2}} \right)\int\limits_{a}^{b}{f\left(x \right)dx=3\int\limits_{a}^{b}{{{x}^{2}}f\left(x \right)dx,}}$ for every $a,b\in \mathbb{R}$.

2 Given a ring $(A,+,\cdot)$ that meets both of the following conditions:
(1) A is not a field, and
(2) For every non-invertible element x of A, there is an integer $m>1$ (depending on x) such that $x=x^2+x^3+\ldots+x^{2^m}$.
Show that
(a) $x+x=0$ for every $x\in A$, and
(b) $x^2=x$ for every non-invertible $x\in A$.

3 Given $a\in (0,1)$ and C the set of increasing functions $f:[0,1]\to [0,\infty)$ such that $\int\limits_{0}^{1}{f(x)}dx=1$. Determine: $(a)\underset{f\in C}{\max}\,\int\limits_{0}^{a}{f(x)dx}$ $(b)\underset{f\in C}{\max}\,\int\limits_{0}^{a}{{{f}^{2}}(x)dx}$

4 Given $n\ge 2$ a natural number, $(K,+,\cdot)$ a body with commutative property that $\underbrace{1+...+1}_{m}\ne 0,m=2,...,n,f\in K[X]$ a polynomial of degree n and G a subgroup of the additive group $(K,+,\cdot)$, $G\ne K$.Show that there is $a\in K$ so$f(a)\notin G$.