Problems In Number Theory

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March 17, 2015

Easy [1]

1. Show that, for all positive integer n, 81 divides $10^{n+1} - 9n - 10$.

2. Find all integers m, n such that $m^2 + 3n^2$ and $m + 3n$ both are perfect cubes.

3. Prove that, the fraction $\dfrac{21n+4}{14n+3}$ is irreducible for any positive integer n.

4. For any positive integers a, n, $n|\varphi(a^n - 1)$. [2] [3]

5. Find all positive integers (m, n) such that $m^2 + 4n$ and $n^2 + 4m$ are perfect squares.

6. Show that $(36m + n)(36n + m)$ never a power of 2.

7. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that[4]

$$(f(n+2), f(n+1)) = [f(n+1), f(n)]$$

8. p is a prime. Find all positive integer k such that $\sqrt{k^2 - pk}$ is an integer.

9. Find all positive integers n such that $n|2^{n!} - 1$.

10. Let A be the sum of the digits of the number 4444^{4444} and B the sum of the digits of the number A. Find the sum of the digits of the number B.

11. Given that,

$$1 - \frac{1}{2} + \cdots - \frac{1}{1378} + \frac{1}{1379} = \frac{a}{b}$$

with $a \perp b$. Prove that, $1979|p$.

IMO 1979

[1] Problems are grouped by (but not ordered in) ascending difficulty.
[2] $a \perp b$ denotes a is co-prime to b.
[3] $a|b$ denotes a divides b.
[4] (a, b) is the greatest common divisor of a and b and $[a, b]$ is the least common multiple of a and b.

12. Determine all positive integers x, y satisfying

$$xy^2 + y + 7 | x^2y + x + y$$

13. Show that, there exists an infinite pairs of positive integers (a, b) such that $\dfrac{a^k + b^k}{a^k b^k + 1}$ is a *perfect* $k-th$ *power.*[5]

14. Find all pairs of integers (x, y) with

$$1 + 2^x + 2^{2x+1} = y^2$$

15. Let n be a positive integer and $a_1, ..., a_k (k \geq 2)$ be distinct integers in $\{1, 2, ..., n\}$ such that $n | a_i(a_{i+1} - 1)$ for $1 \leq i \leq k - 1$. Prove that, $n \not| a_k(a_1 - 1)$.

16. Find all integers x, y such that $4xy - x - y$ is a perfect square.

17. Let $r_1, r_2, ..., r_k$ be the positive integers less than or equal to n and co-prime to n. If they are in an arithmetic progression, show that n is a prime or a power of 2.

18. Find all positive integers (a, b) so that $7^a + 11^b$ is a perfect square.

19. Solve in positive integers:

$$n_1^{10} + n_2^{10} + \cdots + n_8^{10} = 19488391$$

20. Find all positive integers (a, b) such that, $7^a - 3^b$ divides $a^4 + b^2$.

IMO Shortlist 2007(N1)

21. Take $2n$ integers where $n > 2$. Consider all pair-wise differences we can have from those integers. Denote the product of these differences by S. Prove that,

$$2^{n^2-n}(2n-1)(2n-3)(2n-5)|S$$

22. Find all positive integers N whose only prime divisors are 2 and 5, such that the number $N + 25$ is a perfect square.

23. Find all primes such that $p^3 - 4p + 9$ is a square.

Turkey NMO, 2006

Medium

24. Find all positive integers n such that $n = a^2 + b^2$ where $a \perp b$ and every prime not exceeding $\sqrt{n}$ divides ab.

[5] n is a perfect $k - th$ power if it can be written as m^k for an integer m.

25. We are given three integers a, b, c such that $a, b, c, a+b—c, a+c—b, b+c—a$, and $a+b+c$ are seven distinct primes. Let d be the difference between the largest and smallest of these seven primes. Suppose that $800 \in \{a+b, b+c, c+a\}$. Determine the maximum possible value of d.

China 2001

26. Find all positive integers x, y, z, w satisfying:

$$2^x 3^y - 5^z 7^w = 1$$

27. Let $d(n)$ and $\varphi(n)$ be the number of divisors and the number of positive integers less than or equal to n which are co-prime to n. Find all c such that there is a n with

$$d(n) + \varphi(n) = n + c$$

28. Let $f(n) = \sum_{k=1}^{n} \gcd(k, n)$. Prove that,

1. For $\gcd(m, n) = 1$, $f(mn) = f(m)f(n)$.
2. For every positive integer a, there is a x such that $f(x) = ax$.
3. Find all positive integer a such that $f(x) = ax$ has a unique solution.

IMO Shortlist, 2004(N2)

29. Let p and q be two co-prime positive integers. Determine the number of integers that can be written as $ip + jq$ where $i, j \geq 0$ and $i + j \leq n$.

30. Show that, there exists a positive integer n so that the first 1992 digits of n^{1992} is 1.

Brazil, 1992(P2)

31. Find all positive integers k such that $2013 | F_{2013^k}$.[6]

32. Find all positive integers (a, b) such that

$$\left\lfloor \frac{a^2}{b} \right\rfloor + \left\lfloor \frac{b^2}{a} \right\rfloor = \left\lfloor \frac{a^2 + b^2}{ab} \right\rfloor + ab$$

IMO Shortlist, 1996(N4)

33. Let a, n be two positive integers and p be a prime such that,

$$a^p \equiv 1 \pmod{p^n}$$

Prove that, $a \equiv 1 \pmod{p^{n-1}}$.

UNESCO Competition, 1995

34. Find the number of numbers having odd sum of divisors less than or equal to n.

[6] F_n is the usual *Fibonacci number*.

35. Let a be a positive integers so that $4(a^n + 1)$ is a perfect cube for all n. Prove that, $a = 1$.

Iran Second Round, 2008

36. Let $f : \mathbb{Z} \to \mathbb{N}$ so that for any two integers m, n,

$$f(m - n)|f(m) - f(n)$$

Prove that for all $m \leq n$, $f(m)|f(n)$.

37. For every positive integer $k \geq 2$, prove that, 2^{3k} divides

$$N = \binom{2^{k+1}}{2^k} - \binom{2^k}{2^{k-1}}$$

but 2^{3k+1} does not.

38. Find all positive integers a such that $a^{a-1} - 1$ is *square-free*.[7]

39. Show that, there are an infinite primes of the form $4n + 1$.

40. Let n > 1 be an integer and $f(x) = x^n + 5x^{n-1} + 3$. Prove that f is an *irreducible polynomial.*

41. Find all positive integer n and prime p so that

$$p^n + 1|n^p + 1$$

APMO, 2012(P3)

42. Solve the equation $\dfrac{x^7 - 1}{x - 1} = y^5$ in positive integers.

IMO Shortlist, 2006(N5)

43. Define generalized Fibonacci number as

$$G_n = \begin{cases} a \text{ if } n = 0 \\ b \text{ if } n = 1 \\ G_{n-1} + G_{n-2} \text{ otherwise} \end{cases}$$

Prove that, the value of $|G_{n+1}G_{n-1} - G_n^2|$ is independent of n.

44. Show that, for any given prime p, there are integers x, y, z and $0 < w < p$ satisfying

$$x^2 + y^2 + z^2 = wp$$

45. Do there exist pairwise co-prime integers $a, b, c > 1$ such that $2^a + 1$ is divisible by b, $2^b + 1$ is divisible by c, and $2^c + 1$ is divisible by a?

[7] A positive integer is square-free if it is not divisible by any square.

Not So Easy

46. Let $rad(1) = 1$ and $rad(n)$ be the product of prime factors of n. A sequence of natural numbers a_i is defined as

$$a_{n+1} = a_n + rad(n)$$

for an arbitrary a_1. Prove that, for any n, the sequence $a_1, a_2, ...$ contains some n consecutive terms in an arithmetic progression.

Mongolia, 2000

47. Let $\varphi(5^n - 1) = 5^m - 1$. Prove that, $(m, n) > 1$.

48. Find a positive integer n with $100 \leq n \leq 1997$ such that

$$n|2^n + 2$$

APMO 1997, 2

49. If a and b are positive integers such that $a^n + n$ divides $b^n + n$ for all n, prove that $a = b$.

IMO Shortlist, 2005(N6)

50. Find all positive integers which can be represented as

$$\frac{(x+y+z)^2}{xyz}$$

Mongolia, 2000

51. Find all positive integer n and prime p with $n^{p-1}|(p-1)^n + 1$.

52. Let $a < b < c < d$ be positive integers such that $ad = bc$. If

$$a + d = 2^k, b + c = 2^m$$

for some positive integers k, m show that, $a = 1$.

53. Prove that for all odd n, $\tau(F_n) \geq \tau(n)$.

54. Find all positive integers a, b, n with n odd and prime p such that

$$a^k + b^k = p^n$$

55. Prove that, there are infinitely many positive integers n such that $n^2 + 1$ has a prime divisor greater than $2n + \sqrt{2n}$.

IMO Shortlist, 2008(N6)

56. Let $s_1, s_2, ...$ be a strictly increasing sequence of positive integers. Prove that, if

$$s_{s_1}, s_{s_2}, ..., s_{s_n} \text{ and } s_{s_1+1}, s_{s_2+1}, ..., s_{s_n+1}$$

are arithmetic sequences, then $s_1, s_2, ...$ itself is an arithmetic sequence.

57. Let $a, n > 3, d$ are positive integers so that $a, a+d, ..., a+(n-1)d$ are all primes. The number of primes strictly less than n is π_n and the number of divisors of n is $\tau(n)$. Show that,

$$\tau(2^{\lfloor \frac{d}{2} \rfloor} + 1) \geq 2^{2^{\pi_n - 2}}$$

58. For all positive integers n, show that, there exists a positive integer m such that n divides $2^m + m$.

IMO Shortlist, 2006(N7)

59. x, y are positive integers so that,

$$2^n y + 1 | x^{2^n} - 1$$

for all $n \in \mathbb{N}$. Find x.

IMO Shortlist 2012 (N6)