China National Olympiad 2016

Day 1 December 16th

1 Let $a_1, a_2, \cdots, a_{31}; b_1, b_2, \cdots, b_{31}$ be positive integers such that $a_1 < a_2 < \cdots < a_{31} \leq 2015$, $b_1 < b_2 < \cdots < b_{31} \leq 2015$ and $a_1+a_2+\cdots+a_{31} = b_1+b_2+\cdots+b_{31}$. Find the maximum value of $S = |a_1 - b_1| + |a_2 - b_2| + \cdots + |a_{31} - b_{31}|$.

2 In $\triangle AEF$, let B and D be on segments AE and AF respectively, and let ED and FB intersect at C. Define K, L, M, N on segments AB, BC, CD, DA such that $\frac{AK}{KB} = \frac{AD}{BC}$ and its cyclic equivalents. Let the incircle of $\triangle AEF$ touch AE, AF at S, T respectively; let the incircle of $\triangle CEF$ touch CE, CF at U, V respectively.
Prove that K, L, M, N concyclic implies S, T, U, V concyclic.

3 Let p be an odd prime and $a_1, a_2, ..., a_p$ be integers. Prove that the following two conditions are equivalent:

1) There exists a polynomial $P(x)$ with degree $\leq \frac{p-1}{2}$ such that $P(i) \equiv a_i \pmod{p}$ for all $1 \leq i \leq p$

2) For any natural $d \leq \frac{p-1}{2}$,

$$\sum_{i=1}^{p} (a_{i+d} - a_i)^2 \equiv 0 \pmod{p}$$

where indices are taken $\pmod{p}$

Day 2 December 17th

4 Let $n \geq 2$ be a positive integer and define k to be the number of primes $\leq n$. Let A be a subset of $S = \{2, ..., n\}$ such that $|A| \leq k$ and no two elements in A divide each other. Show that one can find a set B such that $|B| = k$, $A \subseteq B \subseteq S$ and no two elements in B divide each other.

5 Let $ABCD$ be a convex quadrilateral. Show that there exists a square $A'B'C'D'$ (Vertices maybe ordered clockwise or counter-clockwise) such that $A \neq A', B \neq B', C \neq C', D \neq D'$ and AA', BB', CC', DD' are all concurrent.

6 Let G be a complete directed graph with 100 vertices such that for any two vertices x, y one can find a directed path from x to y.

a) Show that for any such G, one can find a m such that for any two vertices x, y one can find a directed path of length m from x to y (Vertices can be repeated in the path)

b) For any graph G with the properties above, define $m(G)$ to be smallest possible m as defined in part a). Find the minimim value of $m(G)$ over all such possible G's.

China National Olympiad 2015

Day 1

1 Let $z_1, z_2, ..., z_n$ be complex numbers satisfying $|z_i - 1| \le r$ for some r in $(0,1)$. Show that

$$\left|\sum_{i=1}^{n} z_i\right| \cdot \left|\sum_{i=1}^{n} \frac{1}{z_i}\right| \ge n^2(1-r^2).$$

2 Let A, B, D, E, F, C be six points lie on a circle (in order) satisfy $AB = AC$. Let $P = AD \cap BE, R = AF \cap CE, Q = BF \cap CD, S = AD \cap BF, T = AF \cap CD$.

Let K be a point lie on ST satisfy $\angle QKS = \angle ECA$.

Prove that $\frac{SK}{KT} = \frac{PQ}{QR}$

3 Let $n \ge 5$ be a positive integer and let A and B be sets of integers satisfying the following conditions:

i) $|A| = n$, $|B| = m$ and A is a subset of B
ii) For any distinct $x, y \in B$, $x + y \in B$ iff $x, y \in A$

Determine the minimum value of m.

Day 2

1 Determine all integers k such that there exists infinitely many positive integers n satisfying

$$n + k \nmid \binom{2n}{n}$$

2 Given 30 students such that each student has at most 5 friends and for every 5 students there is a pair of students that are not friends, determine the maximum k such that for all such possible configurations, there exists k students who are all not friends.

3 Let $a_1, a_2, ...$ be a sequence of non-negative integers such that for any m, n

$$\sum_{i=1}^{2m} a_{in} \le m.$$

Show that there exist k, d such that

$$\sum_{i=1}^{2k} a_{id} = k - 2014.$$

China National Olympiad 2014

Day 1

1 Let ABC be a triangle with $AB > AC$. Let D be the foot of the internal angle bisector of A. Points F and E are on AC, AB respectively such that B, C, F, E are concyclic. Prove that the circumcentre of DEF is the incentre of ABC if and only if $BE + CF = BC$.

2 For the integer $n > 1$, define $D(n) = \{a - b \mid ab = n, a > b > 0, a, b \in \mathbb{N}\}$. Prove that for any integer $k > 1$, there exists pairwise distinct positive integers $n_1, n_2, \ldots, n_k$ such that $n_1, \ldots, n_k > 1$ and $|D(n_1) \cap D(n_2) \cap \cdots \cap D(n_k)| \geq 2$.

3 Prove that: there exists only one function $f : \mathbb{N}^* \to \mathbb{N}^*$ satisfying:
i) $f(1) = f(2) = 1$;
ii)$f(n) = f(f(n-1)) + f(n - f(n-1))$ for $n \geq 3$.
For each integer $m \geq 2$, find the value of $f(2^m)$.

Day 2

1 Let $n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}$ be the prime factorisation of n. Define $\omega(n) = t$ and $\Omega(n) = a_1 + a_2 + \ldots + a_t$. Prove or disprove:
For any fixed positive integer k and positive reals α, β, there exists a positive integer $n > 1$ such that
i) $\frac{\omega(n+k)}{\omega(n)} > \alpha$
ii) $\frac{\Omega(n+k)}{\Omega(n)} < \beta$.

2 Let $f : X \to X$, where $X = \{1, 2, \ldots, 100\}$, be a function satisfying:
1) $f(x) \neq x$ for all $x = 1, 2, \ldots, 100$;
2) for any subset A of X such that $|A| = 40$, we have $A \cap f(A) \neq \emptyset$.
Find the minimum k such that for any such function f, there exist a subset B of X, where $|B| = k$, such that $B \cup f(B) = X$.

3 For non-empty number sets S, T, define the sets $S + T = \{s + t \mid s \in S, t \in T\}$ and $2S = \{2s \mid s \in S\}$.
Let n be a positive integer, and A, B be two non-empty subsets of $\{1, 2 \ldots, n\}$. Show that there exists a subset D of $A + B$ such that
1) $D + D \subseteq 2(A + B)$,

2) $|D| \geq \frac{|A|\cdot|B|}{2n}$,
where $|X|$ is the number of elements of the finite set X.

China National Olympiad 2013

Day 1

1 Two circles K_1 and K_2 of different radii intersect at two points A and B, let C and D be two points on K_1 and K_2, respectively, such that A is the midpoint of the segment CD. The extension of DB meets K_1 at another point E, the extension of CB meets K_2 at another point F. Let l_1 and l_2 be the perpendicular bisectors of CD and EF, respectively.
i) Show that l_1 and l_2 have a unique common point (denoted by P).
ii) Prove that the lengths of CA, AP and PE are the side lengths of a right triangle.

2 Find all nonempty sets S of integers such that $3m-2n \in S$ for all (not necessarily distinct) $m, n \in S$.

3 Find all positive real numbers t with the following property: there exists an infinite set X of real numbers such that the inequality

$$\max\{|x-(a-d)|, |y-a|, |z-(a+d)|\} > td$$

holds for all (not necessarily distinct) $x, y, z \in X$, all real numbers a and all positive real numbers d.

Day 2

1 Let $n \geqslant 2$ be an integer. There are n finite sets $A_1, A_2, \ldots, A_n$ which satisfy the condition

$$|A_i \Delta A_j| = |i-j| \quad \forall i, j \in \{1, 2, ..., n\}.$$

Find the minimum of $\sum\limits_{i=1}^{n} |A_i|$.

2 For any positive integer n and $0 \leqslant i \leqslant n$, denote $C_n^i \equiv c(n,i) \pmod 2$, where $c(n,i) \in \{0,1\}$. Define

$$f(n,q) = \sum_{i=0}^{n} c(n,i) q^i$$

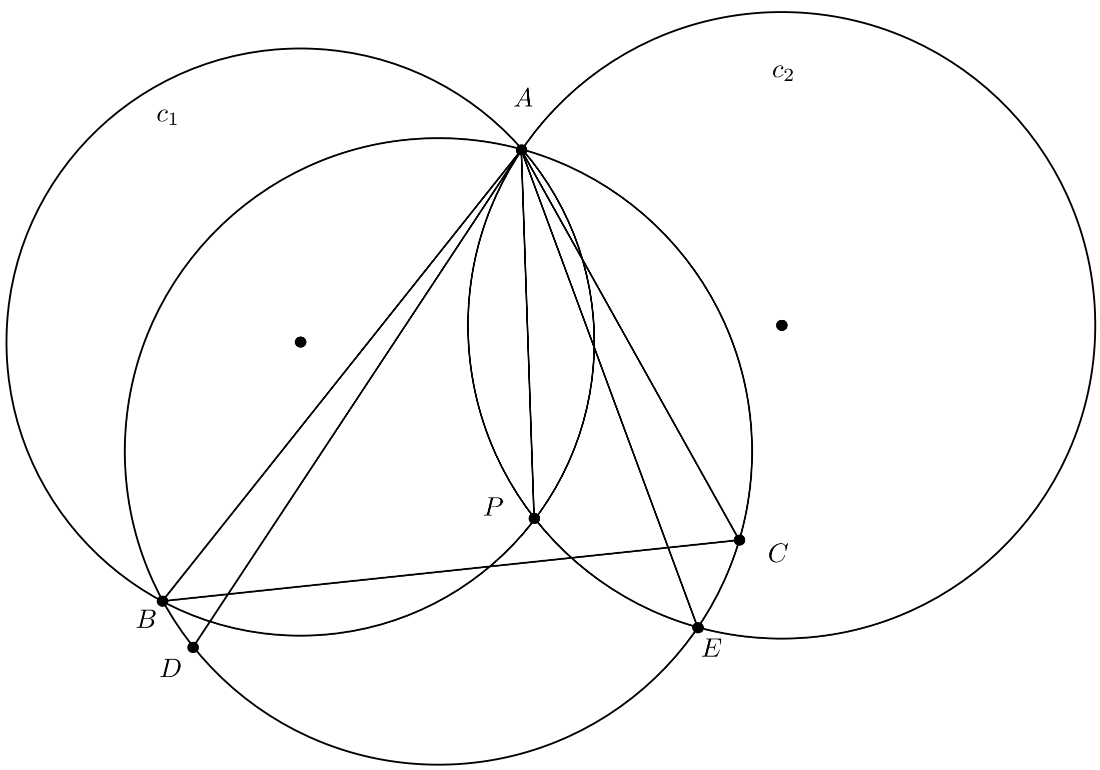
where m, n, q are positive integers and $q+1 \neq 2^\alpha$ for any $\alpha \in \mathbb{N}$. Prove that if $f(m,q) \,|\, f(n,q)$, then $f(m,r) \,|\, f(n,r)$ for any positive integer r.

3 Let m, n be positive integers. Find the minimum positive integer N which satisfies the following condition. If there exists a set S of integers that contains a complete residue system module m such that $|S| = N$, then there exists a nonempty set $A \subseteq S$ so that $n \mid \sum\limits_{x\in A} x$.

China National Olympiad 2012

Day 1

1 In the triangle ABC, $\angle A$ is biggest. On the circumcircle of $\triangle ABC$, let D be the midpoint of $\widehat{ABC}$ and E be the midpoint of $\widehat{ACB}$. The circle c_1 passes through A, B and is tangent to AC at A, the circle c_2 passes through A, E and is tangent AD at A. c_1 and c_2 intersect at A and P. Prove that AP bisects $\angle BAC$.



2 Let p be a prime. We arrange the numbers in $\{1, 2, \ldots, p^2\}$ as a $p \times p$ matrix $A = (a_{ij})$. Next we can select any row or column and add 1 to every number in it, or subtract 1 from every number in it. We call the arrangement *good* if we can change every number of the matrix to 0 in a finite number of such moves. How many good arrangements are there?

3 Prove for any $M > 2$, there exists an increasing sequence of positive integers $a_1 < a_2 < \ldots$ satisfying:
1) $a_i > M^i$ for any i;
2) There exists a positive integer m and $b_1, b_2, \ldots, b_m \in \{-1, 1\}$, satisfying $n = a_1b_1 + a_2b_2 + \ldots + a_mb_m$ if and only if $n \in \mathbb{Z}/\{0\}$.

Day 2

1 Let $f(x) = (x+a)(x+b)$ where $a, b > 0$. For any reals $x_1, x_2, \ldots, x_n \geqslant 0$ satisfying $x_1 + x_2 + \ldots + x_n = 1$, find the maximum of $F = \sum_{1 \leqslant i < j \leqslant n} \min\{f(x_i), f(x_j)\}$.

2 Consider a square-free even integer n and a prime p, such that
1) $(n, p) = 1$;
2) $p \leq 2\sqrt{n}$;
3) There exists an integer k such that $p|n + k^2$.
Prove that there exists pairwise distinct positive integers a, b, c such that $n = ab + bc + ca$.

Proposed by Hongbing Yu

3 Find the smallest positive integer k such that, for any subset A of $S = \{1, 2, \ldots, 2012\}$ with $|A| = k$, there exist three elements x, y, z in A such that $x = a+b$, $y = b+c$, $z = c+a$, where a, b, c are in S and are distinct integers.

Proposed by Huawei Zhu

China National Olympiad 2011

Day 1

1 Let $a_1, a_2, \ldots, a_n$ are real numbers, prove that;

$$\sum_{i=1}^{n} a_i^2 - \sum_{i=1}^{n} a_i a_{i+1} \le \left\lfloor \frac{n}{2} \right\rfloor (M-m)^2.$$

where $a_{n+1} = a_1, M = \max_{1 \le i \le n} a_i, m = \min_{1 \le i \le n} a_i$.

2 On the circumcircle of the acute triangle ABC, D is the midpoint of $\widehat{BC}$. Let X be a point on $\widehat{BD}$, E the midpoint of $\widehat{AX}$, and let S lie on $\widehat{AC}$. The lines SD and BC have intersection R, and the lines SE and AX have intersection T. If $RT \parallel DE$, prove that the incenter of the triangle ABC is on the line RT.

3 Let A be a set consist of finite real numbers,$A_1, A_2, \cdots, A_n$ be nonempty sets of A, such that

(a) The sum of the elements of A is 0,

(b) For all $x_i \in A_i (i = 1, 2, \cdots, n)$,we have $x_1 + x_2 + \cdots + x_n > 0$.

Prove that there exist $1 \le k \le n$, and $1 \le i_1 < i_2 < \cdots < i_k \le n$, such that

$$|A_{i_1} \bigcup A_{i_2} \bigcup \cdots \bigcup A_{i_k}| < \frac{k}{n}|A|.$$

Where $|X|$ denote the numbers of the elements in set X.

Day 2

1 Let n be an given positive integer, the set $S = \{1, 2, \cdots, n\}$.For any nonempty set A and B, find the minimum of $|A \Delta S| + |B \Delta S| + |C \Delta S|$, where $C = \{a+b | a \in A, b \in B\}, X \Delta Y = X \cup Y - X \cap Y$.

2 Let $a_i, b_i, i = 1, \cdots, n$ are nonnegitive numbers,and $n \ge 4$,such that $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n > 0$.

Find the maximum of $\frac{\sum_{i=1}^{n} a_i(a_i+b_i)}{\sum_{i=1}^{n} b_i(a_i+b_i)}$

3 Let m, n be positive integer numbers. Prove that there exist infinite many couples of positive integer nubmers (a, b) such that

$$a+b|am^a+bn^b, \quad \gcd(a,b)=1.$$

China National Olympiad 2010

Day 1

1 Two circles Γ_1 and Γ_2 meet at A and B. A line through B meets Γ_1 and Γ_2 again at C and D repsectively. Another line through B meets Γ_1 and Γ_2 again at E and F repsectively. Line CF meets Γ_1 and Γ_2 again at P and Q respectively. M and N are midpoints of arc PB and arc QB repsectively. Show that if $CD = EF$, then C, F, M, N are concyclic.

2 Let k be an integer ≥ 3. Sequence $\{a_n\}$ satisfies that $a_k = 2k$ and for all $n > k$, we have

$$a_n = \begin{cases} a_{n-1} + 1 & \text{if } (a_{n-1}, n) = 1 \\ 2n & \text{if } (a_{n-1}, n) > 1 \end{cases}$$

Prove that there are infinitely many primes in the sequence $\{a_n - a_{n-1}\}$.

3 Given complex numbers a, b, c, we have that $|az^2 + bz + c| \leq 1$ holds true for any complex number $z, |z| \leq 1$. Find the maximum value of $|bc|$.

Day 2

1 Let $m, n \geq 1$ and $a_1 < a_2 < \ldots < a_n$ be integers. Prove that there exists a subset T of $\mathbb{N}$ such that

$$|T| \leq 1 + \frac{a_n - a_1}{2n+1}$$

and for every $i \in \{1, 2, \ldots, m\}$, there exists $t \in T$ and $s \in [-n, n]$, such that $a_i = t + s$.

2 There is a deck of cards placed at every points $A_1, A_2, \ldots, A_n$ and O, where $n \geq 3$. We can do one of the following two operations at each step: 1) If there are more than 2 cards at some points A_i, we can withdraw three cards from that deck and place one each at A_{i-1}, A_{i+1} and O. (Here $A_0 = A_n$ and $A_{n+1} = A_1$); 2) If there are more than or equal to n cards at point O, we can withdraw n cards from that deck and place one each at $A_1, A_2, \ldots, A_n$.
Show that if the total number of cards is more than or equal to $n^2 + 3n + 1$, we can make the number of cards at every points more than or equal to $n + 1$ after finitely many steps.

3 Suppose $a_1, a_2, a_3, b_1, b_2, b_3$ are distinct positive integers such that

$$(n+1)a_1^n + na_2^n + (n-1)a_3^n | (n+1)b_1^n + nb_2^n + (n-1)b_3^n$$

holds for all positive integers n. Prove that there exists $k \in N$ such that $b_i = ka_i$ for $i = 1, 2, 3$.

China National Olympiad 2009

Day 1

1 Given an acute triangle PBC with $PB \neq PC$. Points A, D lie on PB, PC, respectively. AC intersects BD at point O. Let E, F be the feet of perpendiculars from O to AB, CD, respectively. Denote by M, N the midpoints of BC, AD. (1): If four points A, B, C, D lie on one circle, then $EM \cdot FN = EN \cdot FM$. (2): Determine whether the converse of (1) is true or not, justify your answer.

2 Find all the pairs of prime numbers (p, q) such that $pq|5^p + 5^q$.

3 Given two integers m, n satisfying $4 < m < n$. Let $A_1A_2 \cdots A_{2n+1}$ be a regular $2n+1$ polygon. Denote by P the set of its vertices. Find the number of convex m polygon whose vertices belongs to P and exactly has two acute angles.

Day 2

1 Given an integer $n > 3$. Let $a_1, a_2, \cdots, a_n$ be real numbers satisfying $min|a_i - a_j| = 1, 1 \leq i \leq j \leq n$. Find the minimum value of $\sum_{k=1}^{n} |a_k|^3$.

2 Let P be a convex n polygon each of which sides and diagnoals is colored with one of n distinct colors. For which n does: there exists a coloring method such that for any three of n colors, we can always find one triangle whose vertices is of P' and whose sides is colored by the three colors respectively.

3 Given an integer $n > 3$. Prove that there exists a set S consisting of n pairwisely distinct positive integers such that for any two different non-empty subset of S:A, B, $\frac{\sum_{x\in A} x}{|A|}$ and $\frac{\sum_{x\in B} x}{|B|}$ are two composites which share no common divisors.

China National Olympiad 2008

Day 1

1 Suppose $\triangle ABC$ is scalene. O is the circumcenter and A' is a point on the extension of segment AO such that $\angle BA'A = \angle CA'A$. Let point A_1 and A_2 be foot of perpendicular from A' onto AB and AC. H_A is the foot of perpendicular from A onto BC. Denote R_A to be the radius of circumcircle of $\triangle H_A A_1 A_2$. Similiarly we can define R_B and R_C. Show that:

$$\frac{1}{R_A} + \frac{1}{R_B} + \frac{1}{R_C} = \frac{2}{R}$$

where R is the radius of circumcircle of $\triangle ABC$.

2 Given an integer $n \geq 3$, prove that the set $X = \{1, 2, 3, \ldots, n^2 - n\}$ can be divided into two non-intersecting subsets such that neither of them contains n elements $a_1, a_2, \ldots, a_n$ with $a_1 < a_2 < \ldots < a_n$ and $a_k \leq \frac{a_{k-1} + a_{k+1}}{2}$ for all $k = 2, \ldots, n-1$.

3 Given a positive integer n and $x_1 \leq x_2 \leq \ldots \leq x_n, y_1 \geq y_2 \geq \ldots \geq y_n$, satisfying

$$\sum_{i=1}^{n} i x_i = \sum_{i=1}^{n} i y_i$$

Show that for any real number α, we have

$$\sum_{i=1}^{n} x_i [i\alpha] \geq \sum_{i=1}^{n} y_i [i\alpha]$$

Here $[\beta]$ denotes the greastest integer not larger than β.

Day 2

1 Let A be an infinite subset of $\mathbb{N}$, and n a fixed integer. For any prime p not dividing n, There are infinitely many elements of A not divisible by p. Show that for any integer $m > 1, (m, n) = 1$, There exist finitely many elements of A, such that their sum is congruent to 1 modulo m and congruent to 0 modulo n.

2 Find the smallest integer n satisfying the following condition: regardless of how one colour the vertices of a regular n-gon with either red, yellow or blue, one can always find an isosceles trapezoid whose vertices are of the same colour.

3 Find all triples (p, q, n) that satisfy

$$q^{n+2} \equiv 3^{n+2} (\mod p^n), \quad p^{n+2} \equiv 3^{n+2} (\mod q^n)$$

where p, q are odd primes and n is an positive integer.

China National Olympiad 2007

Day 1

1 Given complex numbers a, b, c, let $|a+b| = m, |a-b| = n$. If $mn \neq 0$, Show that

$$\max\{|ac+b|, |a+bc|\} \geq \frac{mn}{\sqrt{m^2+n^2}}$$

2 Show that:
1) If $2n-1$ is a prime number, then for any n pairwise distinct positive integers $a_1, a_2, \ldots, a_n$, there exists $i, j \in \{1, 2, \ldots, n\}$ such that

$$\frac{a_i + a_j}{(a_i, a_j)} \geq 2n - 1$$

2) If $2n-1$ is a composite number, then there exists n pairwise distinct positive integers $a_1, a_2, \ldots, a_n$, such that for any $i, j \in \{1, 2, \ldots, n\}$ we have

$$\frac{a_i + a_j}{(a_i, a_j)} < 2n - 1$$

Here (x, y) denotes the greatest common divisor of x, y.

3 Let $a_1, a_2, \ldots, a_{11}$ be 11 pairwise distinct positive integer with sum less than 2007. Let S be the sequence of $1, 2, \ldots, 2007$. Define an **operation** to be 22 consecutive applications of the following steps on the sequence S: on i-th step, choose a number from the sequense S at random, say x. If $1 \leq i \leq 11$, replace x with $x + a_i$; if $12 \leq i \leq 22$, replace x with $x - a_{i-11}$. If the result of **operation** on the sequence S is an odd permutation of $\{1, 2, \ldots, 2007\}$, it is an **odd operation**; if the result of **operation** on the sequence S is an even permutation of $\{1, 2, \ldots, 2007\}$, it is an **even operation**. Which is larger, the number of odd operation or the number of even permutation? And by how many?

Here $\{x_1, x_2, \ldots, x_{2007}\}$ is an even permutation of $\{1, 2, \ldots, 2007\}$ if the product $\prod_{i>j}(x_i - x_j)$ is positive, and an odd one otherwise.

Day 2

1 Let O, I be the circumcenter and incenter of triangle ABC. The incircle of $\triangle ABC$ touches BC, CA, AB at points D, E, F repsectively. FD meets CA at P, ED meets AB at Q. M and N are midpoints of PE and QF respectively. Show that $OI \perp MN$.

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2 Let $\{a_n\}_{n\ge 1}$ be a bounded sequence satisfying

$$a_n < \sum_{k=a}^{2n+2006} \frac{a_k}{k+1} + \frac{1}{2n+2007} \quad \forall \quad n = 1, 2, 3, \ldots$$

Show that

$$a_n < \frac{1}{n} \quad \forall \quad n = 1, 2, 3, \ldots$$

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3 Find a number $n \ge 9$ such that for any n numbers, not necessarily distinct, $a_1, a_2, \ldots, a_n$, there exists 9 numbers $a_{i_1}, a_{i_2}, \ldots, a_{i_9}, (1 \le i_1 < i_2 < \ldots < i_9 \le n)$ and $b_i \in 4, 7, i = 1, 2, \ldots, 9$ such that $b_1 a_{i_1} + b_2 a_{i_2} + \ldots + b_9 a_{i_9}$ is a multiple of 9.

China National Olympiad 2006

Day 1 January 12th

1 Let $a_1, a_2, \ldots, a_k$ be real numbers and $a_1 + a_2 + \ldots + a_k = 0$. Prove that

$$\max_{1 \le i \le k} a_i^2 \le \frac{k}{3}\left((a_1-a_2)^2 + (a_2-a_3)^2 + \cdots + (a_{k-1}-a_k)^2\right).$$

2 For positive integers $a_1, a_2, \ldots, a_{2006}$ such that $\frac{a_1}{a_2}, \frac{a_2}{a_3}, \ldots, \frac{a_{2005}}{a_{2006}}$ are pairwise distinct, find the minimum possible amount of distinct positive integers in the set$\{a_1, a_2, ..., a_{2006}\}$.

3 Positive integers k, m, n satisfy $mn = k^2 + k + 3$, prove that at least one of the equations $x^2 + 11y^2 = 4m$ and $x^2 + 11y^2 = 4n$ has an odd solution.

Day 2 January 13th

4 In a right angled-triangle ABC, $\angle ACB = 90^o$. Its incircle O meets BC, AC, AB at D,E,F respectively. AD cuts O at P. If $\angle BPC = 90^o$, prove $AE + AP = PD$.

5 Let $\{a_n\}$ be a sequence such that: $a_1 = \frac{1}{2}$, $a_{k+1} = -a_k + \frac{1}{2-a_k}$ for all $k = 1, 2, \ldots$. Prove that

$$\left(\frac{n}{2(a_1+a_2+\cdots+a_n)} - 1\right)^n \le \left(\frac{a_1+a_2+\cdots+a_n}{n}\right)^n \left(\frac{1}{a_1}-1\right)\left(\frac{1}{a_2}-1\right)\cdots\left(\frac{1}{a_n}-1\right).$$

6 Suppose X is a set with $|X| = 56$. Find the minimum value of n, so that for any 15 subsets of X, if the cardinality of the union of any 7 of them is greater or equal to n, then there exists 3 of them whose intersection is nonempty.

China National Olympiad 2005

Day 1

1 Suppose $\theta_i \in (-\frac{\pi}{2}, \frac{\pi}{2}), i = 1, 2, 3, 4$. Prove that, there exist $x \in \mathbb{R}$, satisfying two inequalities

$$\begin{aligned} \cos^2\theta_1\cos^2\theta_2 - (\sin\theta\sin\theta_2 - x)^2 &\geq 0, \\ \cos^2\theta_3\cos^2\theta_4 - (\sin\theta_3\sin\theta_4 - x)^2 &\geq 0 \end{aligned}$$

if and only if

$$\sum_{i=1}^{4}\sin^2\theta_i \leq 2(1 + \prod_{i=1}^{4}\sin\theta_i + \prod_{i=1}^{4}\cos\theta_i).$$

2 A circle meets the three sides BC, CA, AB of a triangle ABC at points $D_1, D_2; E_1, E_2; F_1, F_2$ respectively. Furthermore, line segments D_1E_1 and D_2F_2 intersect at point L, line segments E_1F_1 and E_2D_2 intersect at point M, line segments F_1D_1 and F_2E_2 intersect at point N. Prove that the lines AL, BM, CN are concurrent.

3 As the graph, a pond is divided into 2n (n $\geq$ 5) parts. Two parts are called neighborhood if they have a common side or arc. Thus every part has three neighborhoods. Now there are 4n+1 frogs at the pond. If there are three or more frogs at one part, then three of the frogs of the part will jump to the three neighborhoods repsectively. Prove that for some time later, the frogs at the pond will uniformily distribute. That is, for any part either there are frogs at the part or there are frogs at the each of its neighborhoods.

`http://www.mathlinks.ro/Forum/files/china2005_2_214.gif`

Day 2

4 The sequence $\{a_n\}$ is defined by: $a_1 = \frac{21}{16}$, and for $n \geq 2$,

$$2a_n - 3a_{n-1} = \frac{3}{2^{n+1}}.$$

Let m be an integer with $m \geq 2$. Prove that: for $n \leq m$, we have

$$\left(a_n + \frac{3}{2^{n+3}}\right)^{\frac{1}{m}}\left(m - \left(\frac{2}{3}\right)^{\frac{n(m-1)}{m}}\right) < \frac{m^2-1}{m-n+1}.$$

5 There are 5 points in a rectangle (including its boundary) with area 1, no three of them are in the same line. Find the minimum number of triangles with the area not more than $\frac{1}{4}$, vertex of which are three of the five points.

6 Find all nonnegative integer solutions (x, y, z, w) of the equation

$$2^x \cdot 3^y - 5^z \cdot 7^w = 1.$$

China National Olympiad 2004

Day 1

1 Let $EFGH, ABCD$ and $E_1F_1G_1H_1$ be three convex quadrilaterals satisfying:

i) The points E, F, G and H lie on the sides AB, BC, CD and DA respectively, and $\frac{AE}{EB} \cdot \frac{BF}{FC} \cdot \frac{CG}{GD} \cdot \frac{DH}{HA} = 1$;
ii) The points A, B, C and D lie on sides H_1E_1, E_1F_1, F_1, G_1 and G_1H_1 respectively, and $E_1F_1||EF, F_1G_1||FG, G_1H_1||GH, H_1E_1||HE$.

Suppose that $\frac{E_1A}{AH_1} = \lambda$. Find an expression for $\frac{F_1C}{CG_1}$ in terms of λ.

Xiong Bin

2 Let c be a positive integer. Consider the sequence $x_1, x_2, \dots$ which satisfies $x_1 = c$ and, for $n \geq 2$,

$$x_n = x_{n-1} + \left\lfloor \frac{2x_{n-1} - (n+2)}{n} \right\rfloor + 1$$

where $\lfloor x \rfloor$ denotes the largest integer not greater than x. Determine an expression for x_n in terms of n and c.

Huang Yumin

3 Let M be a set consisting of n points in the plane, satisfying:
i) there exist 7 points in M which constitute the vertices of a convex heptagon;
ii) if for any 5 points in M which constitute the vertices of a convex pentagon, then there is a point in M which lies in the interior of the pentagon.
Find the minimum value of n.

Leng Gangsong

Day 2

1 For a given real number a and a positive integer n, prove that:
i) there exists exactly one sequence of real numbers $x_0, x_1, \dots, x_n, x_{n+1}$ such that

$$\begin{cases} x_0 = x_{n+1} = 0, \\ \frac{1}{2}(x_i + x_{i+1}) = x_i + x_i^3 - a^3, \ i = 1, 2, \dots, n. \end{cases}$$

ii) the sequence $x_0, x_1, \dots, x_n, x_{n+1}$ in i) satisfies $|x_i| \leq |a|$ where $i = 0, 1, \dots, n+1$.

Liang Yengde

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2 For a given positive integer $n \geq 2$, suppose positive integers a_i where $1 \leq i \leq n$ satisfy $a_1 < a_2 < \ldots < a_n$ and $\sum_{i=1}^{n} \frac{1}{a_i} \leq 1$. Prove that, for any real number x, the following inequality holds

$$\left(\sum_{i=1}^{n} \frac{1}{a_i^2 + x^2} \right)^2 \leq \frac{1}{2} \cdot \frac{1}{a_1(a_1 - 1) + x^2}$$

Li Shenghong

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3 Prove that every positive integer n, except a finite number of them, can be represented as a sum of 2004 positive integers: $n = a_1 + a_2 + \cdots + a_{2004}$, where $1 \leq a_1 < a_2 < \cdots < a_{2004}$, and $a_i \mid a_{i+1}$ for all $1 \leq i \leq 2003$.

Chen Yonggao

China National Olympiad 2003

Day 1 January 15th

1 Let I and H be the incentre and orthocentre of triangle ABC respectively. Let P, Q be the midpoints of AB, AC. The rays PI, QI intersect AC, AB at R, S respectively. Suppose that T is the circumcentre of triangle BHC. Let RS intersect BC at K. Prove that A, I and T are collinear if and only if $[BKS] = [CKR]$.

Shen Wunxuan

2 Determine the maximal size of the set S such that:
i) all elements of S are natural numbers not exceeding 100;
ii) for any two elements a, b in S, there exists c in S such that $(a, c) = (b, c) = 1$;
iii) for any two elements a, b in S, there exists d in S such that $(a, d) > 1, (b, d) > 1$.

Yao Jiangang

3 Given a positive integer n, find the least $\lambda > 0$ such that for any $x_1, \dots x_n \in \left(0, \frac{\pi}{2}\right)$, the condition $\prod_{i=1}^{n} \tan x_i = 2^{\frac{n}{2}}$ implies $\sum_{i=1}^{n} \cos x_i \leq \lambda$.

Huang Yumin

Day 2 January 16th

1 Find all integer triples (a, m, n) such that $a^m + 1 | a^n + 203$ where $a, m > 1$.

Chen Yonggao

2 Ten people apply for a job. The manager decides to interview the candidates one by one according to the following conditions:
i) the first three candidates will not be employed;
ii) from the fourth candidates onwards, if a candidate's comptence surpasses the competence of all those who preceded him, then that candidate is employed;
iii) if the first nine candidates are not employed, then the tenth candidate will be employed.
We assume that none of the 10 applicants have the same competence, and these competences can be ranked from the first to tenth. Let P_k represent the probability that the kth-ranked applicant in competence is employed. Prove that:
i) $P_1 > P_2 > \dots > P_8 = P_9 = P_{10}$;

ii) $P_1 + P_2 + P_3 > 0.7$
iii) $P_8 + P_9 + P_{10} \leq 0.1$.

Su Chun

3 Suppose a, b, c, d are positive reals such that $ab + cd = 1$ and x_i, y_i are real numbers such that $x_i^2 + y_i^2 = 1$ for $i = 1, 2, 3, 4$. Prove that

$$(ax_1 + bx_2 + cx_3 + dx_4)^2 + (ay_4 + by_3 + cy_2 + dy_1)^2 \leq 2\left(\frac{a^2+b^2}{ab} + \frac{c^2+d^2}{cd}\right).$$

Li Shenghong

China National Olympiad 2002

Day 1

1 the edges of triangle ABC are a, b, c respectively,$b < c$,AD is the bisector of $\angle A$,point D is on segment BC.
(1)find the property $\angle A$,$\angle B$,$\angle C$ have,so that there exists point E, F on AB, AC satisfy $BE = CF$,and $\angle NDE = \angle CDF$
(2)when such E, F exist,express BE with a, b, c

2 Given the polynomial sequence $(p_n(x))$ satisfying $p_1(x) = x^2 - 1$, $p_2(x) = 2x(x^2 - 1)$, and $p_{n+1}(x)p_{n-1}(x) = (p_n(x))^2 - (x^2 - 1)^2$, for $n \geq 2$, let s_n be the sum of the absolute values of the coefficients of $p_n(x)$. For each n, find a non-negative integer k_n such that $2^{-k_n} s_n$ is odd.

3 In a competition there are 18 teams and in each round 18 teams are divided into 9 pairs where the 9 matches are played coincidentally. There are 17 rounds, so that each pair of teams play each other exactly once. After n rounds, there always exists 4 teams such that there was exactly one match played between these teams in those n rounds. Find the maximum value of n.

Day 2

1 For every four points P_1, P_2, P_3, P_4 on the plane, find the minimum value of $\frac{\sum_{1 \leq\ i<j \leq\ 4} P_i P_j}{\min_{1 \leq\ i<j \leq\ 4}(P_i P_j)}$.

2 Suppose that a point in the plane is called *good* if it has rational coordinates. Prove that all good points can be divided into three sets satisfying:
1) If the centre of the circle is good, then there are three points in the circle from each of the three sets.
2) There are no three collinear points that are from each of the three sets.

3 Suppose that $c \in \left(\frac{1}{2}, 1\right)$. Find the least M such that for every integer $n \geq 2$ and real numbers $0 < a_1 \leq a_2 \leq \ldots \leq a_n$, if $\frac{1}{n} \sum_{k=1}^{n} k a_k = c \sum_{k=1}^{n} a_k$, then we always have that $\sum_{k=1}^{n} a_k \leq M \sum_{k=1}^{m} a_k$ where $m = [cn]$

China National Olympiad 2001

Day 1

1 Let a be real number with $\sqrt{2} < a < 2$, and let $ABCD$ be a convex cyclic quadrilateral whose circumcentre O lies in its interior. The quadrilateral's circumcircle ω has radius 1, and the longest and shortest sides of the quadrilateral have length a and $\sqrt{4-a^2}$, respectively. Lines L_A, L_B, L_C, L_D are tangent to ω at A, B, C, D, respectively.

Let lines L_A and L_B, L_B and L_C,L_C and L_D,L_D and L_A intersect at A', B', C', D' respectively. Determine the minimum value of $\frac{S_{A'B'C'D'}}{S_{ABCD}}$.

2 Let $X = \{1, 2, \dots, 2001\}$. Find the least positive integer m such that for each subset $W \subset X$ with m elements, there exist $u, v \in W$ (not necessarily distinct) such that $u + v$ is of the form 2^k, where k is a positive integer.

3 Let P be a regular n-gon $A_1A_2 \dots A_n$. Find all positive integers n such that for each permutation $\sigma(1), \sigma(2), \dots, \sigma(n)$ there exists $1 \le i, j, k \le n$ such that the triangles $A_iA_jA_k$ and $A_{\sigma(i)}A_{\sigma(j)}A_{\sigma(k)}$ are both acute, both right or both obtuse.

Day 2

1 Let a, b, c be positive integers such that $a, b, c, a+b-c, a+c-b, b+c-a, a+b+c$ are 7 distinct primes. The sum of two of a, b, c is 800. If d be the difference of the largest prime and the least prime among those 7 primes, find the maximum value of d.

2 Let $P_1P_2 \dots P_{24}$ be a regular 24-sided polygon inscribed in a circle ω with circumference 24. Determine the number of ways to choose sets of eight distinct vertices from these 24 such that none of the arcs has length 3 or 8.

3 Let $a = 2001$. Consider the set A of all pairs of integers (m, n) with $n \neq 0$ such that
(i) $m < 2a$;
(ii) $2n|(2am - m^2 + n^2)$;
(iii) $n^2 - m^2 + 2mn \le 2a(n - m)$.
For $(m, n) \in A$, let

$$f(m, n) = \frac{2am - m^2 - mn}{n}.$$

Determine the maximum and minimum values of f.