India
Postal Coaching
2004

$\boxed{1}$ Let ABC and DEF be two triangles such that $A + D = 120^\circ$ and $B + E = 120^\circ$. Suppose they have the same circumradius. Prove that they have the same 'Fermat length'.

$\boxed{2}$ (a) Find all triples (x, y, z) of positive integers such that $xy \equiv 2(\text{mod} z)$, $yz \equiv 2(\text{mod} x)$ and $zx \equiv 2(\text{mod} y)$ (b) Let $n \geq 1$ be an integer. Give an algoritm to determine all triples (x, y, z) such that '2' in part (a) is replaced by 'n' in all three congruences.

$\boxed{3}$ Let $a, b, c, d,$ be real and $ad - bc = 1$. Show that $Q = a^2 + b^2 + c^2 + d^2 + ac + bd \neq 0, 1, -1$

$\boxed{4}$ In how many ways can a $2 \times n$ grid be covered by (a) 2 monominoes and $n - 1$ dominoes (b) 4 monominoes and $n - 2$ dominoes.

$\boxed{5}$ How many paths from $(0, 0)$ to (n, n) of length $2n$ are there with exactly k steps. A step is an occurence of the pair EN in the path

$\boxed{6}$ Find the number of ordered palindromic partitions of an integer n.

$\boxed{7}$ Let $ABCD$ be a square, and C the circle whose diameter is AB. Let Q be an arbitrary point on the segment CD. We know that QA meets C on E and QB meets it on F. Also CF and DE intersect in M. show that M belongs to C.

$\boxed{8}$ Solve for integers a, b, c

$$(a + b + c)^3 + \frac{1}{2}(b + c)(c + a)(a + b) = 1 - abc$$

$\boxed{9}$ Let $ABCDEF$ be a regular hexagon of side lengths 1 and O its centre, Join O cto each of the six vertices , thus getting 12 unit line segments in all. Find the number of closed paths from (i) O to O (ii) A to A each of length 2004

$\boxed{10}$ A convex quadrilateral $ABCD$ has an incircle. In each corner a circle is inscribed that also externally touches the two circles inscribed in the adjacent corners. Show that at least two circles have the same size.

$\boxed{11}$ Three circles touch each other externally and all these cirlces also touch a fixed straight line. Let A, B, C be the mutual points of contact of these circles. If ω denotes the Brocard angle of the triangle ABC, prove that $\cot \omega = 2$.

$\boxed{12}$ Suppose $z_1, z_2, \cdots z_n$ are n complex numbers such that $min_{j \neq k}|z_j - z_k| \geq max_{1 \leq j \leq n}|z_j|$. Find the maximum possible value of n. Further characterise all such maximal configurations.

$\boxed{13}$ Find all functions $f, g : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}^+$ such that

$$(\sum_{j=1}^{n} a_j b_j)^2 \leq (\sum_{j=1}^{n} f(a_j, b_j))(\sum_{j=1}^{n} g(a_j, b_j)) \leq (\sum_{j=1}^{n} (a_j)^2)(\sum_{j=1}^{n} (b_j)^2)$$

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for any two sets a_j and b_j of real numbers.

14 Find the greatest common divisor of all number in the set $(a^{41} - a | a \in \mathbb{N} and \geq 2)$. What is your guess if 41 is replaced by a natural number n

15 Show that for each integer a, there is a unique decomposition

$$a = \sum_{j=0}^{n} d_j 2^j, d_j \in (-1, 0, 1)$$

such that no two consecutive d_j's are nonzero. Show further that if f is nondecreasing function from the set of all non-negative integers in to the set of all non-negative real numbers, and if $a = \sum_{j=0}^{n} c_j 2^j$ is any other decomposition of a with $c_j \in (-1, 0, 1)$, then

$$\sum_{j=0}^{n} |d_j| f(j) \leq \sum_{j=0}^{n} |c_j| f(j)$$

16 Find all regular n-gons with the following properties: (a) a diagonal is equal to the sum of two other diagonals (b) a diagonal is equal to the sum of a side and another diagonal

17 In a system of numeration with base B , there are n one-digit numbers less than B whose cubes have $B - 1$ in the units-digits place. Determine the relation between n and B

18 Let $0 = a_1 < a_2 < a_3 < \cdots < a_n < 1$ and $0 = b_1 < b_2 < b_3 \cdots < b_m < 1$ be real numbers such that for no a_j and b_k the relation $a_j + b_k = 1$ is satisfied. Prove that if the mn numbers $a_j + b_k : 1 \leq j \leq n, 1 \leq k \leq m\}$ are reduced modulo 1, then at least $m + n - 1$ residues will be distinct.

19 Suppose a circle passes through the feet of the symmedians of a non-isosceles triangle ABC , and is tangent to one of the sides. Show that $a^2 + b^2, b^2 + c^2, c^2 + a^2$ are in geometric progression when taken in some order

20 Three numbers N, n, r are such that the digits of N, n, r taken together are formed by $1, 2, 3, 4, 5, 6, 7, 8, 9$ without repition. If $N = n^2 - r$, find all possible combinations of N, n, r