India
National Olympiad
1990

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1 Given the equation

$$x^4 + px^3 + qx^2 + rx + s = 0$$

has four real, positive roots, prove that (a) $pr - 16s \geq 0$ (b) $q^2 - 36s \geq 0$ with equality in each case holding if and only if the four roots are equal.

2 Determine all non-negative integral pairs (x, y) for which

$$(xy - 7)^2 = x^2 + y^2.$$

3 Let f be a function defined on the set of non-negative integers and taking values in the same set. Given that

(a) $x - f(x) = 19\left[\frac{x}{19}\right] - 90\left[\frac{f(x)}{90}\right]$ for all non-negative integers x;

(b) $1900 < f(1990) < 2000$,

find the possible values that $f(1990)$ can take. (Notation : here $[z]$ refers to largest integer that is $\leq z$, e.g. $[3.1415] = 3$).

4 Consider the collection of all three-element subsets drawn from the set $\{1, 2, 3, 4, \ldots, 299, 300\}$. Determine the number of those subsets for which the sum of the elements is a multiple of 3.

5 Let a, b, c denote the sides of a triangle. Show that the quantity

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$$

must lie between the limits $3/2$ and 2. Can equality hold at either limits?

6 Triangle ABC is scalene with angle A having a measure greater than 90 degrees. Determine the set of points D that lie on the extended line BC, for which

$$|AD| = \sqrt{|BD| \cdot |CD|}$$

where $|BD|$ refers to the (positive) distance between B and D.

7 Let ABC be an arbitrary acute angled triangle. For any point P lying within the triangle, let D, E, F denote the feet of the perpendiculars from P onto the sides AB, BC, CA respectively. Determine the set of all possible positions of the point P for which the triangle DEF is isosceles. For which position of P will the triangle DEF become equilateral?