USA Team Selection Test 2017

TST#1 December 8th, 2016

1 In a sports league, each team uses a set of at most t signature colors. A set S of teams is *color-identifiable* if one can assign each team in S one of their signature colors, such that no team in S is assigned any signature color of a different team in S.

For all positive integers n and t, determine the maximum integer $g(n,t)$ such that: In
any sports league with exactly n distinct colors present over all teams, one can always
find a color-identifiable set of size at least $g(n,t)$.

2 Let ABC be an acute scalene triangle with circumcenter O, and let T be on line BC such that $\angle TAO = 90^\circ$. The circle with diameter $\overline{AT}$ intersects the circumcircle of $\triangle BOC$ at two points A_1 and A_2, where $OA_1 < OA_2$. Points B_1, B_2, C_1, C_2 are defined analogously.
- Prove that $\overline{AA_1}$, $\overline{BB_1}$, $\overline{CC_1}$ are concurrent.
- Prove that $\overline{AA_2}$, $\overline{BB_2}$, $\overline{CC_2}$ are concurrent on the Euler line of triangle ABC.
Evan Chen

3 Let $P, Q \in \mathbb{R}[x]$ be relatively prime nonconstant polynomials. Show that there can be at most three real numbers λ such that $P + \lambda Q$ is the square of a polynomial.

Alison Miller

TST#2 January 19th, 2017

1 You are cheating at a trivia contest. For each question, you can peek at each of the $n > 1$ other contestants' guesses before writing down your own. For each question, after all guesses are submitted, the emcee announces the correct answer. A correct guess is worth 0 points. An incorrect guess is worth -2 points for other contestants, but only -1 point for you, since you hacked the scoring system. After announcing the correct answer, the emcee proceeds to read the next question. Show that if you are leading by 2^{n-1} points at any time, then you can surely win first place.

Linus Hamilton

2 Let ABC be a triangle with altitude $\overline{AE}$. The A-excircle touches $\overline{BC}$ at D, and intersects the circumcircle at two points F and G. Prove that one can select points V and N on lines DG and DF such that quadrilateral $EVAN$ is a rhombus.

Danielle Wang

3 Prove that there are infinitely many triples (a, b, p) of positive integers with p prime, $a < p$, and $b < p$, such that $(a+b)^p - a^p - b^p$ is a multiple of p^3.