2-nd Czech–Slovak Match 1996

Žilina, June 2–5, 1996

1. Show that an integer $p > 3$ is a prime if and only if for every two nonzero integers a, b exactly one of the numbers

$$N_1 = a + b - 6ab + \frac{p-1}{6}, \quad N_2 = a + b + 6ab + \frac{p+1}{6}$$

is a nonzero integer.

2. Let $\star$ be a binary operation on a nonempty set M. That is, every pair $(a, b) \in M$ is assigned an element $a \star b$ in M. Suppose that $\star$ has the additional property that

$$(a \star b) \star b = a \text{ and } a \star (a \star b) = b \quad \text{for all } a, b \in M.$$

   (a) Show that $a \star b = b \star a$ for all $a, b \in M$.

   (b) On which finite sets M does such a binary operation exist?

3. The base of a regular quadrilateral pyramid π is a square with side length $2a$ and its lateral edge has length $a\sqrt{17}$. Let M be a point inside the pyramid. Consider the five pyramids which are similar to π, whose top vertex is at M and whose bases lie in the planes of the faces of π. Show that the sum of the surface areas of these five pyramids is greater or equal to one fifth the surface of π, and find for which M equality holds.

4. Decide whether there exists a function $f : \mathbb{Z} \to \mathbb{Z}$ such that for each $k = 0, 1, \dots, 1996$ and for any integer m the equation

$$f(x) + kx = m$$

has at least one integral solution x.

5. Two sets of intervals $\mathcal{A}, \mathcal{B}$ on the line are given. The set $\mathcal{A}$ contains $2m-1$ intervals, every two of which have an interior point in common. Moreover, every interval from $\mathcal{A}$ contains at least two disjoint intervals from $\mathcal{B}$. Show that there exists an interval in $\mathcal{B}$ which belongs to at least m intervals from $\mathcal{A}$.

6. The points E and D are taken on the sides AC and BC respectively of a triangle ABC. The lines AD and BE intersect at F. Show that the areas of the triangles ABC and ABF satisfy

$$\frac{S_{ABC}}{S_{ABF}} = \frac{AC}{AE} + \frac{BC}{BD} - 1.$$

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