IMO 1961

Veszprem, Hungary

Day 1

1. Solve the system of equations:

$$x + y + z = a$$
$$x^2 + y^2 + z^2 = b^2$$
$$xy = z^2$$

where a and b are constants. Give the conditions that a and b must satisfy so that x, y, z are distinct positive numbers.

2. Let a, b, c be the sides of a triangle, and S its area. Prove:

$$a^2 + b^2 + c^2 \geq 4S\sqrt{3}$$

In what case does equality hold?

3. Solve the equation $\cos^n x - \sin^n x = 1$ where n is a natural number.

Day 2

4. Consider triangle $P_1P_2P_3$ and a point p within the triangle. Lines P_1P, P_2P, P_3P intersect the opposite sides in points Q_1, Q_2, Q_3 respectively. Prove that, of the numbers

$$\frac{P_1P}{PQ_1}, \frac{P_2P}{PQ_2}, \frac{P_3P}{PQ_3}$$

at least one is ≤ 2 and at least one is ≥ 2

5. Construct a triangle ABC if $AC = b$, $AB = c$ and $\angle AMB = w$, where M is the midpoint of the segment BC and $w < 90$. Prove that a solution exists if and only if

$$b \tan \frac{w}{2} \leq c < b$$

In what case does the equality hold?

6. Consider a plane ϵ and three non-collinear points A, B, C on the same side of ϵ; suppose the plane determined by these three points is not parallel to ϵ. In plane ϵ take three arbitrary points A', B', C'. Let L, M, N be the midpoints of segments AA', BB', CC'; Let G be the centroid of the triangle LMN. (We will not consider positions of the points A', B', C' such that the points L, M, N do not form a triangle.) What is the locus of point G as A', B', C' range independently over the plane ϵ?