

USAMO 2015

### Day 1

- 1 Solve in integers the equation

$$x^2 + xy + y^2 = \left( \frac{x+y}{3} + 1 \right)^3.$$

- 2 Quadrilateral  $APBQ$  is inscribed in circle  $\omega$  with  $\angle P = \angle Q = 90^\circ$  and  $AP = AQ < BP$ . Let  $X$  be a variable point on segment  $\overline{PQ}$ . Line  $AX$  meets  $\omega$  again at  $S$  (other than  $A$ ). Point  $T$  lies on arc  $AQB$  of  $\omega$  such that  $\overline{XT}$  is perpendicular to  $\overline{AX}$ . Let  $M$  denote the midpoint of chord  $\overline{ST}$ . As  $X$  varies on segment  $\overline{PQ}$ , show that  $M$  moves along a circle.

- 3 Let  $S = \{1, 2, \dots, n\}$ , where  $n \geq 1$ . Each of the  $2^n$  subsets of  $S$  is to be colored red or blue. (The subset itself is assigned a color and not its individual elements.) For any set  $T \subseteq S$ , we then write  $f(T)$  for the number of subsets of  $T$  that are blue.
- Determine the number of colorings that satisfy the following condition: for any subsets  $T_1$  and  $T_2$  of  $S$ ,

$$f(T_1)f(T_2) = f(T_1 \cup T_2)f(T_1 \cap T_2).$$

### Day 2 [b]

- 4 Steve is piling  $m \geq 1$  indistinguishable stones on the squares of an  $n \times n$  grid. Each square can have an arbitrarily high pile of stones. After he finished piling his stones in some manner, he can then perform *stone moves*, defined as follows. Consider any four grid squares, which are corners of a rectangle, i.e. in positions  $(i, k), (i, l), (j, k), (j, l)$  for some  $1 \leq i, j, k, l \leq n$ , such that  $i < j$  and  $k < l$ . A stone move consists of either removing one stone from each of  $(i, k)$  and  $(j, l)$  and moving them to  $(i, l)$  and  $(j, k)$  respectively, or removing one stone from each of  $(i, l)$  and  $(j, k)$  and moving them to  $(i, k)$  and  $(j, l)$  respectively.
- Two ways of piling the stones are equivalent if they can be obtained from one another by a sequence of stone moves.
- How many different non-equivalent ways can Steve pile the stones on the grid?



# Art of Problem Solving

## 2015 USAMO

- 
- 5 Let  $a, b, c, d, e$  be distinct positive integers such that  $a^4 + b^4 = c^4 + d^4 = e^5$ . Show that  $ac + bd$  is a composite number.
- 
- 6 Consider  $0 < \lambda < 1$ , and let  $A$  be a multiset of positive integers. Let  $A_n = \{a \in A : a \leq n\}$ . Assume that for every  $n \in \mathbb{N}$ , the set  $A_n$  contains at most  $n\lambda$  numbers. Show that there are infinitely many  $n \in \mathbb{N}$  for which the sum of the elements in  $A_n$  is at most  $\frac{n(n+1)}{2}\lambda$ . (A multiset is a set-like collection of elements in which order is ignored, but repetition of elements is allowed and multiplicity of elements is significant. For example, multisets  $\{1, 2, 3\}$  and  $\{2, 1, 3\}$  are equivalent, but  $\{1, 1, 2, 3\}$  and  $\{1, 2, 3\}$  differ.)
- 



— These problems are copyright © Mathematical Association of America (<http://maa.org>).

---