India
Postal Coaching
2005

$\boxed{1}$ Consider the sequence $< a_n >$ of natural numbers such that i a_n is a square numver for all n ; (ii) $a_{n+1} - a_n$ is either a prime or a square of a prime for each n. Show that $< a_n >$ is a finite sequence. Determine the longest such sequence.

$\boxed{2}$ Let $< \Gamma_j >$ be a sequnce of concentric circles such that the sequence $< R_j >$, where R_j denotes the radius of Γ_j, is increasing and $R_j \longrightarrow \infty$ as $j \longrightarrow \infty$. Let $A_1B_1C_1$ be a triangle inscribed in Γ_1. extend the rays $A_i\vec{B}_1, B_1\vec{C}_1, C_1\vec{A}_1$ to meet Γ_2 in $B_2, C_2$and A_2 respectively and form the triangle $A_2B_2C_2$. Continue this process. Show that the sequence of triangles $< A_nB_nC_n >$ tends to an equilateral triangle as $n \longrightarrow \infty$

$\boxed{3}$ Find all real α s.t.

$$[\sqrt{n+\alpha} + \sqrt{n}] = [\sqrt{4n+1}]$$

holds for all natural numbers n

$\boxed{4}$ Let m, n be natural numbers and let $d = gcd(m, n)$. Let $x = 2^m - 1$ and $y = 2^n + 1$ (a) If $\frac{m}{d}$ is odd, prove that $gcd(x, y) = 1$ (b) If $\frac{m}{d}$ is even, Find $gcd(x, y)$

$\boxed{5}$ Characterize all triangles ABC s.t.

$$AI_a : BI_b : CI_c = BC : CA : AB$$

where I_a etc. are the corresponding excentres to the vertices A, B, C

$\boxed{6}$ Let $ABCD$ be a trapezoid such that AB is parallel to CD, and let E be the midpoint of its side BC. Suppose we can inscribe a circle into the quadrilateral $ABED$, and that we can inscribe a circle into the quadrilateral $AECD$. Denote $|AB| = a$, $|BC| = b$, $|CD| = c$, $|DA| = d$. Prove that

$$a + c = \frac{b}{3} + d;$$

$$\frac{1}{a} + \frac{1}{c} = \frac{3}{b}$$

$\boxed{7}$ Fins all ordered triples (a, b, c) of positive integers such that $abc + ab + c = a^3$.

$\boxed{8}$ Prove that For all positive integers m and n , one has $|n\sqrt{2005} - m| > \frac{1}{90n}$

$\boxed{9}$ In how many ways can n identical balls be distributed to nine persons $A, B, C, D, E, F, G, H, I$ so that the number of balls recieved by A is the same as the total number of balls recieved by B, C, D, E together,.

$\boxed{10}$ On the sides AB and BC of triangle ABC, points K and M are chosen such that the quadrilaterals $AKMC$ and $KBMN$ are cyclic , where $N = AM \cap CK$. If these quads have the same circumradii, find $\angle ABC$

$\boxed{11}$ (a) Prove that the set $X = (1, 2, ...100)$ cannot be partitoned into THREE subsets such that two numbers differing by a square belong to different subsets. (b) Prove that X can so be partitioned into 5 subsets.

$\boxed{12}$ Let ABC be a triangle with vertices at lattice points. Suppose one of its sides in $\sqrt{n}$, where n is square-free. Prove that $\frac{R}{r}$ is irraational . The symbols have usual meanings.

$\boxed{13}$ Let $a_1 < a_2 < .... < a_n < 2n$ ne n positive integers such that a_j does not divide a_k or $j \neq k$. Prove that $a_1 \geq 2^k$ where k is defined by the condition $3^k < 2n < 3^{k+1}$ and show that it is the best estimate for a_1

$\boxed{14}$ Let $f(z) = a_m z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree $n \geq 3$ with real coefficients.Suppose all roots of $f(z) = 0$ lie in the half plane $z \in \mathbb{C} : Re(z) < 0\}$. Prove that $a_k a_{k+3} < a_{k+1} a_{k+2}$ for $k = 0, 1, 2, 3, ...n - 3$

$\boxed{15}$ Let X be a set with $|X| = n$, and let $X_1, X_2, ...X_n$ be the $n$subsets eith $|X_j| \geq 2$, for $1 \leq j \leq n$. Suppose for each 2 element subset Y of X, there is a unique j in the set $1, 2, 3...., n$ such that $Y \subset X_j$. Prove that $X_j \cap X_k \neq \Phi$ for all $1 \leq j < k \leq n$

$\boxed{16}$ The diagonals AC and BD of a cyclic ABCD intersect at E. Let O be circumcentre of ABCD. If midpoints of AB, CD, OE are collinear prove that AD=BC.

Bomb

[Moderator edit: The problem is wrong. See also ¡!– m –¿¡a class="postlink" href="http://www.mathlinks.ro/
m –¿ .]

$\boxed{17}$ Let A', B', C' be points, in which excircles touch corresponding sides of triangle ABC. Circumcircles of triangles $A'B'C$, $AB'C'$, $A'BC'$ intersect a circumcircle of ABC in points $C_1 \neq C$, $A_1 \neq A$, $B_1 \neq B$ respectively. Prove that a triangle $A_1B_1C_1$ is similar to a triangle, formed by points, in which incircle of ABC touches its sides.

$\boxed{18}$ Find the least positive integer, which may not be represented as $\frac{2^a - 2^b}{2^c - 2^d}$, where a, b, c, d are positive integers.

$\boxed{19}$ Find all functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that $f(xy + f(x)) = xf(y) + f(x)$ for all $x, y \in \mathbb{R}$.

$\boxed{20}$ In the following, the point of intersection of two lines g and h will be abbreviated as $g \cap h$.

Suppose ABC is a triangle in which $\angle A = 90^\circ$ and $\angle B > \angle C$. Let O be the circumcircle of the triangle ABC. Let l_A and l_B be the tangents to the circle O at A and B, respectively.

Let $BC \cap l_A = S$ and $AC \cap l_B = D$. Furthermore, let $AB \cap DS = E$, and let $CE \cap l_A = T$. Denote by P the foot of the perpendicular from E on l_A. Denote by Q the point of intersection of the line CP with the circle O (different from C). Denote by R be the point of intersection of the line QT with the circle O (different from Q). Finally, define $U = BR \cap l_A$. Prove that

$$\frac{SU \cdot SP}{TU \cdot TP} = \frac{SA^2}{TA^2}.$$

21 Find all positive integers n that can be *uniquely* expressed as a sum of five or fewer squares.

22 Consider the points P =$(0,0)$,$Q = (1,0)$, R= $(2,0)$, S =$(3,0)$ in the xy-plane. Let A, B, C, D be four finite sets of colours(not necessarily distinct nor disjoint). In how many ways can P, Q, R be coloured bu colours in A, B, C respectively if adjacent points have to get different colours? In how many ways can P, Q, R, S be coloured by colours in A, B, C, D respectively if adjacent points have to get different colors?

23 Let Γ be the incircle of an equilateral triangle ABC of side length 2 units. (a) Show that for all points P on Γ, $PA^2 + PB^2 + PC^2 = 5$. (b) Show that for all points P on Γ, it is possible to construct a triangle of sides equal to PA, PB, PC and whose area is equal to $\frac{\sqrt{3}}{4}$ units.

24 Find all nonnegative integers x, y such that

$$2 \cdot 3^x + 1 = 7 \cdot 5^y.$$

25 Find all pairs of cubic equations $x^3 + ax^2 + bx + c = 0$ and $x^3 + bx^2 + ax + c = 0$ where a, b, c are integers, such that each equation has three integer roots and both the equations have exactly one common root.

26 Let $a_1, a_2, \dots a_n$ be real numbers such that their sum is equal to zero. Find the value of

$$\sum_{j=1}^{n} \frac{1}{a_j(a_j + a_{j+1})(a_j + a_{j+1} + a_{j+2}) \dots (a_j + \dots a_{j+n-2})}.$$

where the subscripts are taken modulo n assuming none of the denominators is zero.

27 Let k be an even positive integer and define a sequence $< x_n >$ by

$$x_1 = 1, x_{n+1} = k^{x_n} + 1.$$

Show that x_n^2 divides $x_{n-1}x_{n+1}$ for each $n \geq 2$.