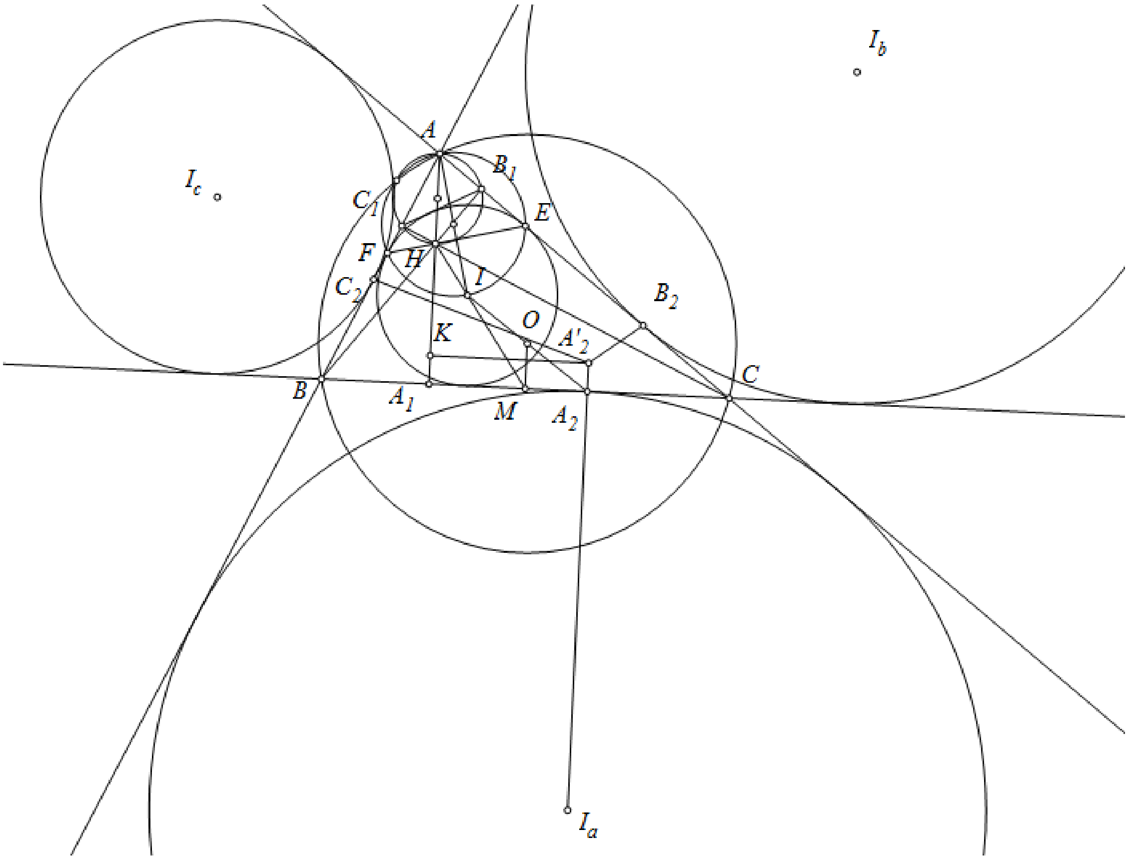
hence $IE \perp AC, IF \perp AB$. We get B_1C_1BC is a bicentric quadrilateral or B_1C_1 is tangent to (I).

Attachments:



livetolove212

#6 Jan 21, 2016, 11:36 pm

After proving that H, I, M are collinear we note that (I) is a circle which is tangent to BC and tangent to 9-point circle of triangle ABC at Feuerbach point, then applying my last post at here, we get (I) is tangent to B_1C_1.

Quick Reply

High School Olympiads

Intersect on circumcircle

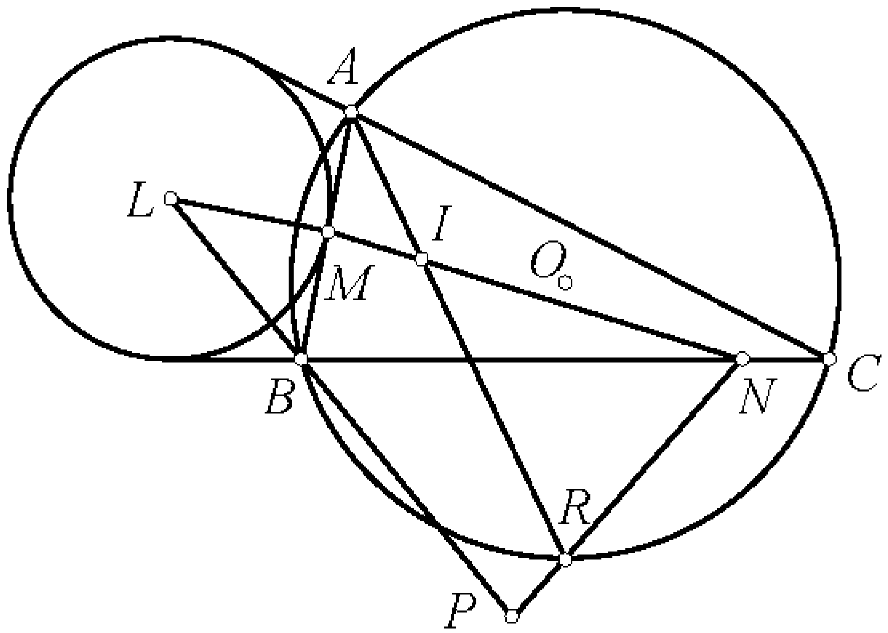
Source: Problem weekly, forth week, November 2015

buratinogigle

#1 Feb 8, 2016, 8:54 am • 1

Let ABC be a triangle inscribed in circle (O) and incenter I. C-excircle (L) of ABC touches AB at M. MI cuts BC at N. P is projection of C on LB. Prove that AI and PN intersect on (O).

Attachments:



IsoLyS

#2 Feb 8, 2016, 9:55 am • 2

Say $AI \cap (O) = R, CI \cap AB = X, CR \cap BL = Y$.
Since $\triangle BLM \sim \triangle BCP$, we get $\triangle BLC \sim \triangle BMP$, hence $\angle BMP = \angle BLC = \angle BAI$, which means $MP \| IR$.
Since $\angle ABL = \angle ICR$, $BXCY$ is cyclic, hence $\angle BXY = \angle BCR = \angle BAI$, which means $XY \| IR$.
Now applying Desargues's theorem on MIX and PRY, we get MI, PR, BC are concurrent, as desired.

Luis González

#3 Feb 8, 2016, 10:13 am • 1

Let J be the A-excenter of $\triangle ABC. U \equiv AI \cap BC, W \equiv CI \cap AB, Z \equiv CD \cap BL$ and let $D \equiv AI \cap (O)$, other than A. We need to prove that P, D, N are collinear.

Notice that $\triangle JBC \cup Z, P \sim \triangle ABL \sim W, M \Longrightarrow (W, B, A, M) = (Z, B, J, P)$. But
$I(W, B, A, M) = I(C, B, U, N) \Longrightarrow (C, B, U, N) = D(Z, B, J, P) \equiv D(C, B, U, P) \Longrightarrow P, D, N$ are collinear.

buratinogigle

#5 Feb 8, 2016, 12:44 pm

Thank for your ineterest, here is my solution

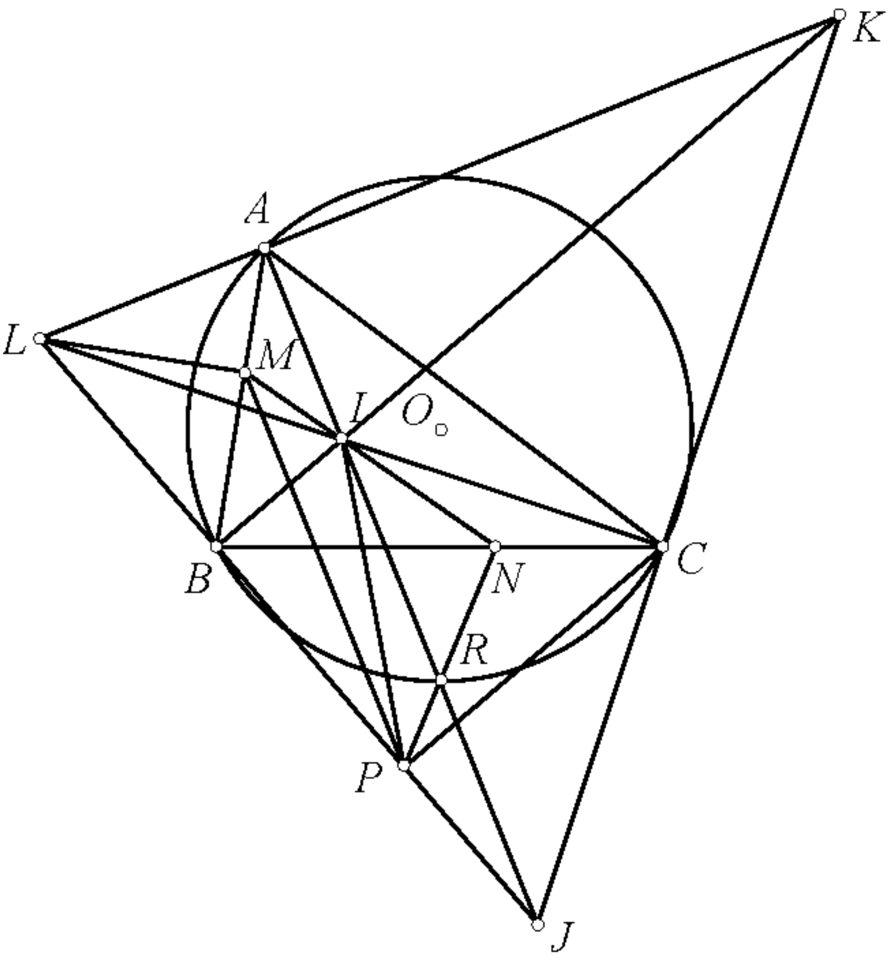
Let J, K be the excenters at vertex A, B of triangle ABC. Because M, I, N are collinear so

Let J, K be the excenters at vertex A, B of triangle ABC. Because M, I, N are collinear so $P(BI, MN) = B(PI, MN) = B(JK, AC) = -1$. We easily seen triangles BAL and BJC are similar, with altitudes LM and CP so $\frac{BM}{MA} = \frac{BP}{PJ}$, hence $PM \parallel AJ$. Combine with $P(BI, MN) = -1$ we get PN bisecets IJ. Follow the fundamental result, midpoint of IJ lies on (O). We are done.

Here is a general problem

Let ABC be a triangle inscribed in circle (O) and P, Q are two isogonal conjugate point. CP, CQ cut circles $(PAB), (QAB)$ again at M, N. K, L are projections of M, C on AB, NB, reps. KP cuts BC at R. Prove that LR and AP intersect on (O).

Attachments:



Luis González

#6 Feb 8, 2016, 1:13 pm • 1

> *buratinogigle wrote:*
> Here is a general problem
>
> Let ABC be a triangle inscribed in circle (O) and P, Q are two isogonal conjugate point. CP, CQ cut circles $(PAB), (QAB)$ again at M, N. K, L are projections of M, C on AB, NB, reps. KP cuts BC at R. Prove that LR and AP intersect on (O).

This can be done similarly as I did in my previous post. Let $J \equiv AP \cap \odot(PBC)$ (other than P), $D \equiv AP \cap (O)$ (other than A), $W \equiv CM \cap AB$ and $Z \equiv CD \cap BN$. Straightforward angle chase reveals that $\triangle JBC \cup Z, L \sim \triangle ABM \cup W, K \Longrightarrow (W, B, A, K) = (Z, B, J, L)$ and from here we get the conclusion.

IsoLyS

#7 Feb 8, 2016, 1:38 pm • 1

> *buratinogigle wrote:*
> Let ABC be a triangle inscribed in circle (O) and P, Q are two isogonal conjugate point. CP, CQ cut circles $(PAB), (QAB)$ again at M, N. K, L are projections of M, C on AB, NB, reps. KP cuts BC at R. Prove that

$(PAB), (QAB)$ again at M, N. K, L are projections of M, C on AB, NB, resp. KP cuts BC at R. Prove that LR and AP intersect on (O).

Same way, too. Similarly, say $AP \cap (O) = D, CP \cap AB = X, CD \cap BL = Y$.
Since $\triangle BMK \sim \triangle BCL$, so $\triangle BMC \sim \triangle BKL$, hence $\angle BKL = \angle BMC = \angle BAP$, which means $KL \| PD$.
Since $\angle PCD = \pi - \angle AQC = \angle ABN$, $BXCY$ is cyclic, hence $\angle BXY = \angle BCY = \angle BAD$, which means $XY \| PD$.
Now applying Desargues's theorem on XKP and YLD, we get KP, DL, BC are concurrent, as desired.

FabrizioFelen

#8 Feb 8, 2016, 3:19 pm • 1

My solution:(it's not fine)
Let $N' = PR \cap BC$ and let $\angle BAC = 2\alpha$ and $\angle BCA = 2\theta \Longrightarrow$ By angle-chasing $\angle RBC = \angle BCR = \angle ILB = \alpha$ and $\angle PBR = \theta$ and $\angle ILM = \angle PCR = 90 - 2\alpha - \theta \Longrightarrow$ By law of sines in $\triangle PBC$ we get
$\frac{BN'}{N'C} = \frac{sin\theta.sin(90-\alpha-\theta)}{sin(\alpha+\theta).sin(90-2\alpha-\theta)}$ in the other hand by law of sines $\frac{MB}{MI} = \frac{sin\angle BIN}{sin(90-\alpha-\theta)}$ and
$\frac{MI}{ML} = \frac{sin(90-2\alpha-\theta)}{sin\angle NIC}$ and $\frac{ML}{MB} = \frac{sin(\alpha+\theta)}{sin(90-\alpha-\theta)} \Longrightarrow$
$1 = \frac{MB.MI.ML}{MI.ML.MB} = \frac{sin\angle BIN.sin(90-2\alpha-\theta).sin(\alpha+\theta)}{sin(90-\alpha-\theta).sin\angle NIC.sin(90-\alpha-\theta)} \Longrightarrow$
$\frac{sin\angle BIN}{sin\angle NIC} = \frac{sin(90-\alpha-\theta).sin(90-\alpha-\theta)}{sin(90-2\alpha-\theta).sin(\alpha+\theta)} \Longrightarrow$ By law of sines in $\triangle BIC$ we get
$\frac{BN}{NC} = \frac{sin\angle BIN.sin\theta}{sin\angle NIC.sin(90-\alpha-\theta)} = \frac{sin\theta.sin(90-\alpha-\theta)}{sin(\alpha+\theta).sin(90-2\alpha-\theta)} = \frac{BN'}{N'C} \Longrightarrow N' = N$ hence AI and PN intersect on (O)...

This post has been edited 1 time. Last edited by FabrizioFelen, Feb 13, 2016, 7:39 am

Quick Reply

Spain

Problema de Geometría

Reply

Axlrose

#1 Feb 7, 2016, 3:24 am

Aquí otro problema creado por mi saludos desde Perú
Sea $ABCD$ un cuadrilátero inscrito en una circunferencia Ω tal que AB es diámetro de Ω y sean P, Q puntos en los segmentos CA y BD tal que $PQ \parallel AB$. Sea $P' = BP \cap \Omega$ y $Q' = AQ \cap \Omega$ y $X = DQ' \cap CP'$ y sea $Y = \odot(AXC) \cap AB$ y $Z = \odot(BXD) \cap AB$. Probar que: $AY = BZ$

Luis González

#2 Feb 8, 2016, 12:13 pm

Sea O el punto medio de $\overline{AB}, M \equiv AQ \cap BP$ y $T \equiv AC \cap BD.$ Como $PQ \parallel AB,$ entonces M yace en la T-mediana TO de $\triangle TAB.$ Por teorema de Pascal para el hexágono $ACP'\underline{BDQ'},$ se sigue que X, T, M están alineados $\Longrightarrow$ X, T, O están alineados. Como $TA \cdot TC = TB \cdot TD,$ entonces $\overline{XTO}$ es el eje radical de $\odot(AXC)$ y $\odot(BXD) \Longrightarrow$ $OA \cdot OY = OB \cdot OZ \Longrightarrow OY = OZ,$ que implica por simetría que $AY = BZ.$

Quick Reply

High School Olympiads

A nice concurrecy!!

Reply

Source: 10.6 Final Round of Sharygin geometry Olympiad 2015

MRF2017

#1 Feb 7, 2016, 10:10 pm

Let H and O be the orthocenter and the circumcenter of triangle ABC. The circumcircle of triangle AOH meets the perpendicular bisector of BC at point $A_1 \neq O$. Points B_1 and C_1 are defined similarly. Prove that lines AA_1, BB_1 and CC_1 concur.

Luis González

#2 Feb 7, 2016, 11:26 pm

Let $\odot(AOH)$ cut the circumcircle $(O) \equiv \odot(ABC)$ again at A_2 and let AA_2 cut OH at L. If OH cuts (O) at U, V, we have then $LU \cdot LV = LA \cdot LA_2 = LO \cdot LH \Longrightarrow (H, U, V, L) = -1 \Longrightarrow L$ is the inverse of H on (O). Now from isosceles trapezoid $AHOA_1$ and the O-isosceles $\triangle OAA_2$, we get $\angle(AA_1, AH) = \angle OA_1A = \angle OA_2A = \angle OAA_2$, which means that AA_1 is the isogonal of AL WRT $\angle BAC$, passing through the isogonal conjugate L^* of L WRT $\triangle ABC$, i.e. the antigonal conjugate of O. Similarly BB_1 and CC_1 go through the antigonal conjugate of O.

mjuk

#3 Feb 7, 2016, 11:30 pm

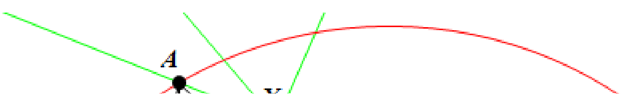
I didn't use directed angles, so here's my figure.
Let $D = AH \cap BC, E = BH \cap AC, F = CH \cap AB$.
$BH \parallel OB_1$, hence BB_1OH is isosceles trapezoid, similarly CC_1OH and AA_1OH are isosceles trapezoids.
$HB_1 = OB, HC_1 = OC, HA_1 = OA \rightarrow HA_1 = HB_1 = HC_1 \rightarrow H$ is circumcenter of $\odot A_1B_1C_1$.
Let $X = BB_1 \cap CC_1$

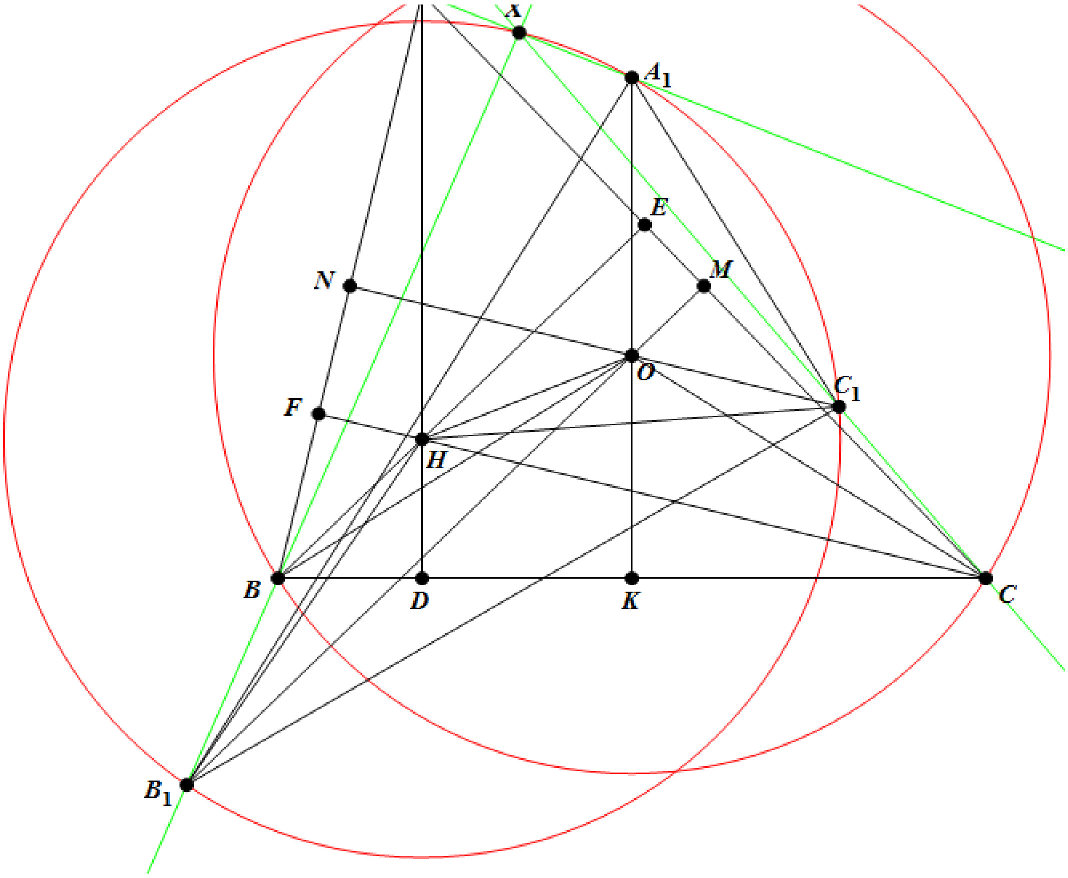
$$\begin{aligned}
\angle B_1XC_1 &= \angle BXC \\
&= \angle BHC - \angle HBX - \angle HCX \\
&= \angle B + \angle C - \angle OB_1X - \angle OHC \\
&= \angle B + \angle C - \angle OHE - \angle OHC \\
&= \angle B + \angle C - \angle CHE \\
&= \angle B + \angle C - \angle A \\
&= 180 - 2\angle A \\
\angle B_1HC_1 &= \angle B_1XC_1 + \angle XB_1H + \angle HC_1X \\
&= 180 - 2\angle A + \angle BOH + 180 - \angle HOC \\
&= 360 - 2\angle A - \angle BOC \\
&= 360 - 4\angle A
\end{aligned}$$

$\angle B_1A_1C_1 = \angle B_1HC_1/2 = 180 - 2\angle A = \angle B_1XC_1$
So $X \in \odot A_1B_1C_1$.
Similarly $AA_1 \cap BB_1 \in \odot A_1B_1C_1$, so we conclude that AA_1, BB_1, CC_1 concur at $X \in \odot A_1B_1C_1$.

Some other interesting results:
$\triangle A_1B_1C_1 \sim \triangle DEF$
A_1D, B_1E, C_1F intersect at Y on OH, such that $\dfrac{HY}{OY} = \dfrac{1}{2}$
O is incenter of $\triangle A_1B_1C_1$

Attachments:



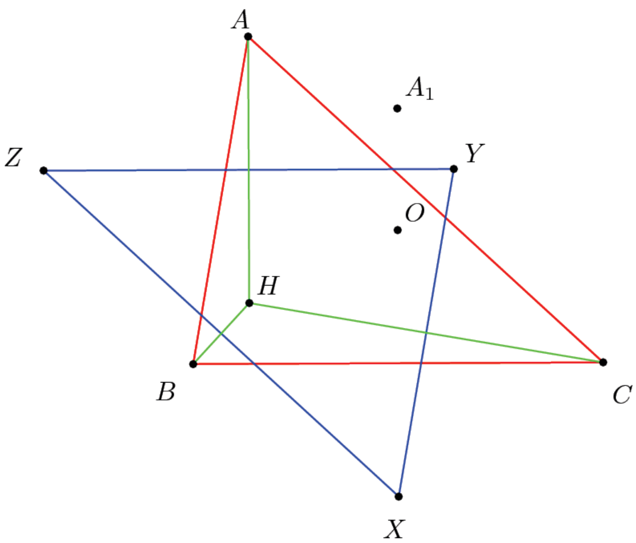


This post has been edited 4 times. Last edited by mjuk, Feb 8, 2016, 1:34 am
Reason: edit

Dukejukem

#4 Feb 7, 2016, 11:56 pm • 1

Let X, Y, Z be the circumcenters of $\triangle BHC, \triangle CHA, \triangle AHB$ respectively. Note that YZ is the perpendicular bisector of $\overline{AH}$ and OX is the perpendicular bisector of $\overline{BC}$. Hence $OX \perp YZ$. Analagous relations imply that O is the orthocenter of $\triangle XYZ$.



On the other hand, as $AH \parallel OA_1$, we deduce that $AHOA_1$ is an isoceles trapezoid. Hence, AA_1 is the reflection of OH in YZ. Similarly, BB_1, CC_1 are the reflections of OH in ZX, XY respectively. It follows that AA_1, BB_1, CC_1 are concurrent at the Anti-Steiner point of OH WRT $\triangle XYZ$.

This post has been edited 2 times. Last edited by Dukejukem, Feb 8, 2016, 12:04 am
Reason: Diagram

Quick Reply

High School Olympiads

Concurrent

Reply

Source: Iran TST 2008

Omid Hatami

#1 Jul 6, 2008, 4:05 am

I_a is the excenter of the triangle ABC with respect to A, and AI_a intersects the circumcircle of ABC at T. Let X be a point on TI_a such that $XI_a^2 = XA.XT$. Draw a perpendicular line from X to BC so that it intersects BC in A'. Define B' and C' in the same way. Prove that AA', BB' and CC' are concurrent.

Zagros

#2 Jul 7, 2008, 6:58 pm

Let $AB \leq AC \leq BC$. Denote the incenter by I. Let $BC = a$, $AC = b$, $AB = c$, $I_aT = IT = IB = IC = i$, $AT = t$ and $XT = x$.

$XI_a^2 = XA.XT \Longrightarrow (i-x)^2 = x(x+t) \Longrightarrow x = \frac{i^2}{2i+t}$. (1)

Let lines passing from I and T and perpendicular to BC, intersect BC in points D and M, respectively.
Let line passing from I and parallel to BC, intersect TM and XA' in points F and G, respectively.
We have $TF \parallel XG$, $IF = DM$ and $FG = MA'$.

$TF \parallel XG \Longrightarrow \frac{IF}{FG} = \frac{IT}{TX} \Longrightarrow \frac{DM}{MA'} = \frac{i}{x}$. Using (1) we have $\frac{DM}{MA'} = \frac{2i+t}{i} = 2 + \frac{t}{i}$. (2)

$ABTC$ is a cyclic quadrilateral. By using Ptolemy's theorem, we have
$TB.AC + TC.AB = BC.AT \Longrightarrow i(b+c) = at \Longrightarrow \frac{t}{i} = \frac{b+c}{a}$. (3)

$AB \leq AC \Longrightarrow DM = BM - BD = \frac{a}{2} - \frac{a+c-b}{2} = \frac{b-c}{2}$. (4)

From (2), (3) and (4), it is concluded that $MA' = \frac{a(b-c)}{2(2a+b+c)}$. (5)

We have $\frac{BA'}{A'C} = \frac{BM+MA'}{CM-MA'}$. (6)

From (5) and (6), it is concluded that

$$\frac{BA'}{A'C} = \frac{\frac{a}{2} + (\frac{a}{2})(\frac{b-c}{2a+b+c})}{\frac{a}{2} - (\frac{a}{2})(\frac{b-c}{2a+b+c})} = \frac{(\frac{a}{2})(1+\frac{b-c}{2a+b+c})}{(\frac{a}{2})(1-\frac{b-c}{2a+b+c})} = \frac{a+b}{a+c}.$$

Similarly, we can find that $\frac{CB'}{B'A} = \frac{b+c}{b+a}$ and $\frac{AC'}{C'B} = \frac{c+a}{c+b}$.

$(\frac{BA'}{A'C})(\frac{CB'}{B'A})(\frac{AC'}{C'B}) = (\frac{a+b}{a+c})(\frac{b+c}{b+a})(\frac{c+a}{c+b}) = 1$.
By using Ceva theorem, it is concluded that AA', BB' and CC' are concurrent.

This post has been edited 1 time. Last edited by rusczyk, Mar 7, 2015, 10:02 am
Reason: Patch LaTeX

yetti

#3 Jul 9, 2008, 8:35 am

D, E, F are feet of the internal bisectors of the angles $\angle A, \angle B, \angle C$. EF cuts AI and X' and BC at D'. BC, AI, EF are

D, E, F are feet of the internal bisectors of the angles $\angle A, \angle B, \angle C$. EF cuts AI and X' and BC at D'. BC, AI, EF, are diagonals of the complete quadrilateral $(AE, AF, IE, IF) \Longrightarrow$ the cross ratios $(D', D, B, C) = -1$, $(A, I, X', D) = -1$ are harmonic. From the 1st one, D' is foot of the external bisector of the angle $\angle A$ and $D'A \perp AI_a$. From the 2nd one,

$$\frac{\overline{DX'}}{\overline{DI}} = \frac{2\ \overline{DA}}{\overline{DA} + \overline{DI}} = \frac{2(a+b+c)}{2a+b+c}.$$

(P) is circumcircle of the $\triangle II_bI_c$, (O) is its 9-point circle, I_a its orthocenter, EF is the radical axis of $(P), (O)$, $D'X' \equiv EF \perp OI_a$. U is tangency point of the incircle (I) with BC and $A'' \in BC$ foot of perpendicular from X' to BC.

$$\frac{\overline{DA''}}{\overline{DU}} = \frac{\overline{DX'}}{\overline{DI}}$$

$M \in (O)$ is the midpoint of II_a, (M) is the circumcircle of $BICI_a$. $\overline{X}A \cdot XT = XI_a^2 \Longrightarrow X$ is on radical axis $s \perp OI_a$ of (O) and the point I_a. Tangent $t \perp MI_a \equiv AI_a$ to (M) at I_a is radical axis of $(M), I_a$ and BC is radical axis of $(O), (M)$. They meet at the radical center $R_a \in BC$ of $(O), (M), I_a$. $R_aI_a \parallel D'A$ (both $\perp AI_a$) and $R_aX \parallel D'X'$ (both $\perp OI_a$). Consequently,

$$\frac{\overline{DA'}}{\overline{DA''}} = \frac{\overline{DX}}{\overline{DX'}} = \frac{\overline{DI_a}}{\overline{DA}}$$

Substituting

$$\frac{\overline{DA}}{\overline{DI_a}} = 1 + \frac{\overline{I_aA}}{\overline{DI_a}} = 1 - \frac{b+c}{a}$$

$$\overline{DU} = \overline{DC} - \overline{UC} = \frac{ab}{c+b} - \frac{a+b-c}{2} = -\frac{(b-c)(b+c-a)}{2(b+c)}$$

$$\overline{DA'} = -\frac{a}{b+c-a}\ \overline{DA''} = -\frac{2a(a+b+c)}{(b+c-a)(2a+b+c)}\ \overline{DU} = \frac{a(b-c)(a+b+c)}{(b+c)(2a+b+c)}$$

$$\overline{A'B} = \overline{DB} - \overline{DA'} = -\frac{ca}{b+c} - \frac{a(b-c)(a+b+c)}{(b+c)(2a+b+c)} = -\frac{a(b-c)(a+b)}{2a+b+c}$$

$$\overline{A'C} = \overline{DC} - \overline{DA'} = +\frac{ab}{b+c} - \frac{a(b-c)(a+b+c)}{(b+c)(2a+b+c)} = +\frac{a(b-c)(c+a)}{2a+b+c}$$

$$\frac{\overline{A'B}}{\overline{A'C}} = -\frac{a+b}{c+a}$$

By cyclic exchange, $\dfrac{\overline{A'B}}{\overline{A'C}} \cdot \dfrac{\overline{B'C}}{\overline{B'A}} \cdot \dfrac{\overline{C'A}}{\overline{C'B}} = -\dfrac{a+b}{c+a} \cdot \dfrac{b+c}{a+b} \cdot \dfrac{c+a}{b+c} = -1$ and AA', BB', CC' are concurrent by Ceva theorem.

Remark: $X' \equiv AI \cap EF$ and $A'' \in BC$ was defined as foot of perpendicular from X' to BC. If $Y' \equiv BI \cap FD, Z' \equiv CI \cap DE$ and B'', C'' are feet of perpendiculars from Y', Z' to CA, AB, then AA'', BB'', CC'' are also concurrent.

Aiscrim

#5 Feb 6, 2016, 9:40 pm

Solution

Luis González

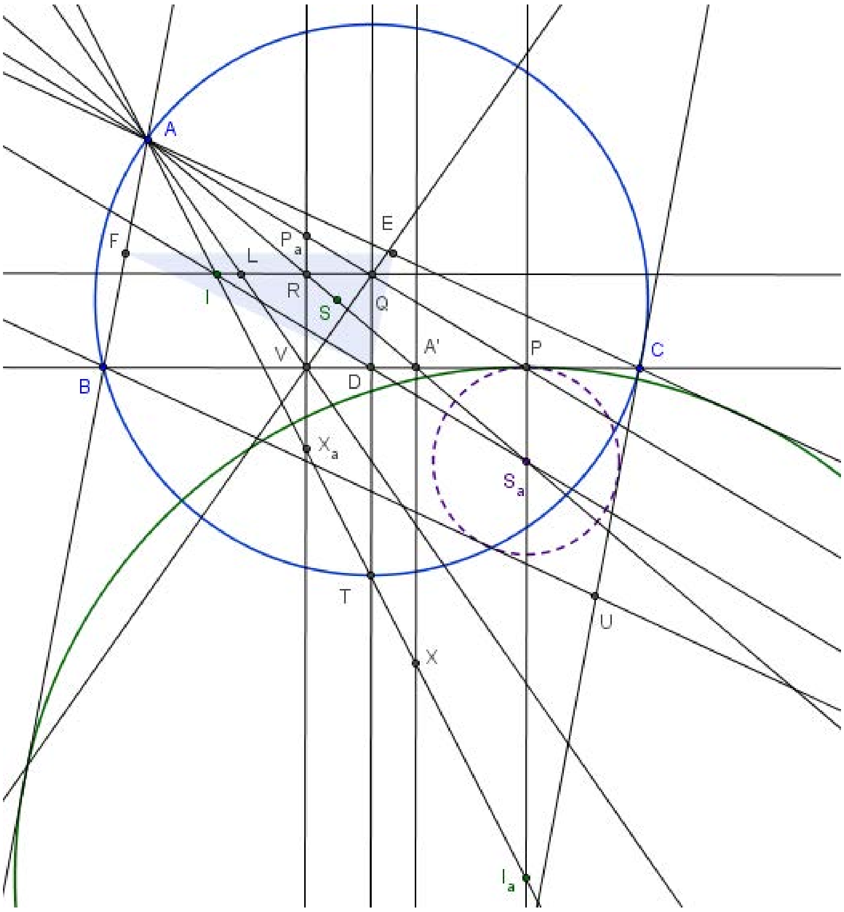
#6 Feb 7, 2016, 10:51 am • 1

Let D, E, F be the midpoints of BC, CA, AB. I and S are the incenters of $\triangle ABC$ and $\triangle DEF$ (Spieker point) and S_a is the reflection of A on S, which is incenter of $\triangle UBC$, being U the reflection of A on D. If the A-excircle (I_a) touches BC at P, then by the symmetry WRT D, we deduce that P is the tangency point of the incircle of $\triangle UBC$ with BC, i.e. $S_a \in I_aP$.

Redefine $A' \equiv AS \cap BC$ and let V be the reflection of P on A'. Parallel from I to BC cuts AP, AA', AV at Q, R, L, resp. Since D is clearly midpoint of IS_a, then A' is midpoint of $RS_a \Longrightarrow (PS_a \parallel RV) \perp BC$ and since $DI \parallel AP$ (well-known),

$PQDS_a$ is a parallelogram $\Longrightarrow (DQ \parallel PS_a) \perp BC$. As a result $Q \in DT$ and VR is perpendicular bisector of $LQ \Longrightarrow$ $V(R, Q, L, P) = -1$. If RV cuts AI_a, AP at X_a, P_a, we get then $V(R, Q, L, P) = (P_a, Q, A, P) = (X_a, T, A, I_a) = -1 \Longrightarrow X$ is midpoint of $I_aX_a \Longrightarrow (XA' \parallel I_aP \parallel X_aV) \perp BC$ and similarly for B' and C'. So we conclude that AA', BC', CC' concur at the Spieker point S of $\triangle ABC$.

Attachments:



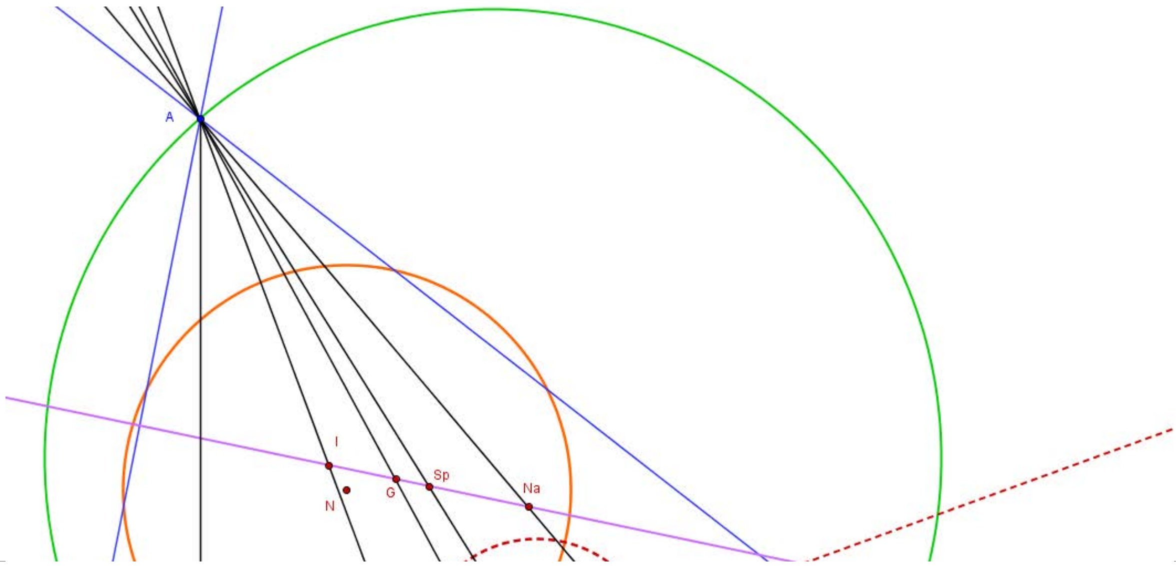
TelvCohl

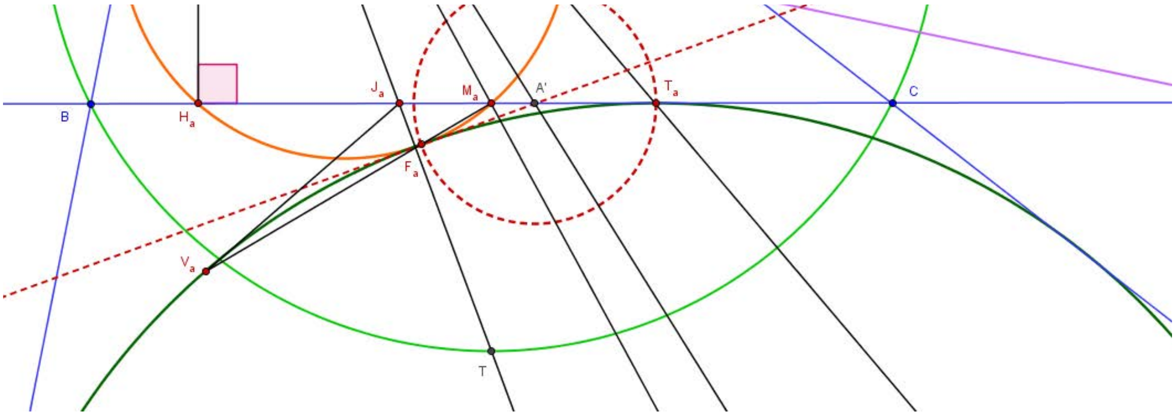
#8 Feb 7, 2016, 2:09 pm

Let G, I, N_a be the centroid, incenter, Nagel point of $\triangle ABC$, respectively. Let H_a, M_a, T_a be the projection of A, T, I_a on BC, respectively. Obviously, M_a is the midpoint of BC, so from $XT \cdot XA = XI_a{}^2 \Longrightarrow A'M_a \cdot A'H_a = A'T_a{}^2$ we get the circle with center A' passing through T_a is orthogonal to $\odot(I_a)$ and the 9-point circle $\odot(N)$ of $\triangle ABC$, hence A' lies on the radical axis of $\odot(N)$ and $\odot(I_a) \Longrightarrow A'$ lies on the common tangent of $\odot(N), \odot(I_a)$.

On the other hand, if AI cuts BC at J_a and the tangent from J_a to $\odot(I_a)$ ($\neq BC$) touches $\odot(I_a)$ at V_a, then M_aV_a passes through the A-Ex-Feuerbach point F_a of $\triangle ABC$ (well-known), so from $(A', J_a; M_a, T_a) = F_a(F_a, J_a; V_a, T_a) = -1 \Longrightarrow$ $A(A', I; G, N_a) = A(A', J_a; M_a, T_a) = -1$, hence we get AA' passes through the Spieker point S_p of $\triangle ABC$. Similarly, we can prove S_p lies on BB' and CC', so we conclude that AA', BB', CC' are concurrent at S_p.

Attachments:





buratinogigle

#9 Feb 8, 2016, 9:58 pm

I think there is a projective solution for parallel model

Let ABC be a triangle and DEF is cevian triangle of P. K, L, M are midpoint of PA, PB, PC. Let X, Y, Z lie on PA, PB, PC such that $XA^2 = XK.XD, YB^2 = YL.YE, ZC^2 = ZM.ZF$. G is centroid of triangle ABC and Q is symmetric of midpoint PG through G. U, V, W lie on EF, FD, DE such that $XU \parallel QA, YV \parallel QB, ZW \parallel QC$. Prove that DU, EV, FW are concurrent.

When P is orthocenter, we get original problem, but using parallel projection from original problem, we get this problem, also.

Quick Reply

High School Olympiads

the lines intersect on 9-point cirlce

Source: OWN

LeVietAn
#1 Feb 6, 2016, 6:15 pm

Dear Mathlinkers,
Let ABC be an acute triangle with $AB \neq AC$. Let (O), H and G respectively be the circumcircle, orthorcenter, centroid of it. Choose the point K on the (O) such that $\angle AKH = 90^\circ$. The line AG intersects (O) again at L. Prove that the lines GK and HL intersect on nine-point circle of triangle ABC.

Luis González
#2 Feb 7, 2016, 1:25 am • 2

Let D be the midpoint of BC and let P be the antipode of A on (O). It's well-known that P is the reflection of H on D and $K \in PH$. Since G is the insimilicenter of (O) and the 9-point circle (N), then it is also center of their direct inversion, thus if the ray GK cuts (N) at X, we have $GD \cdot GL = GX \cdot GK \Longrightarrow KXDL$ is cyclic $\Longrightarrow \angle GKD = \angle XLD$.

Take $T \in HP$, such that $\frac{TD}{TP} = -\frac{1}{2}$. Then $\frac{TD}{TP} \cdot \frac{HP}{HD} = -\frac{1}{2} \cdot 2 = -1 \Longrightarrow (T, D, P, H) = -1$ and since $\angle DLP = 90^\circ$, then LD, LP bisect $\angle HLT \Longrightarrow \angle HLD = \angle DLT$. But from $\triangle AKD \cup G \sim \triangle PLD \cup T$, we get $\angle DLT = \angle GKD \Longrightarrow \angle HLD = \angle GKD = \angle XLD \Longrightarrow X \in HL$ and the conclusion follows.

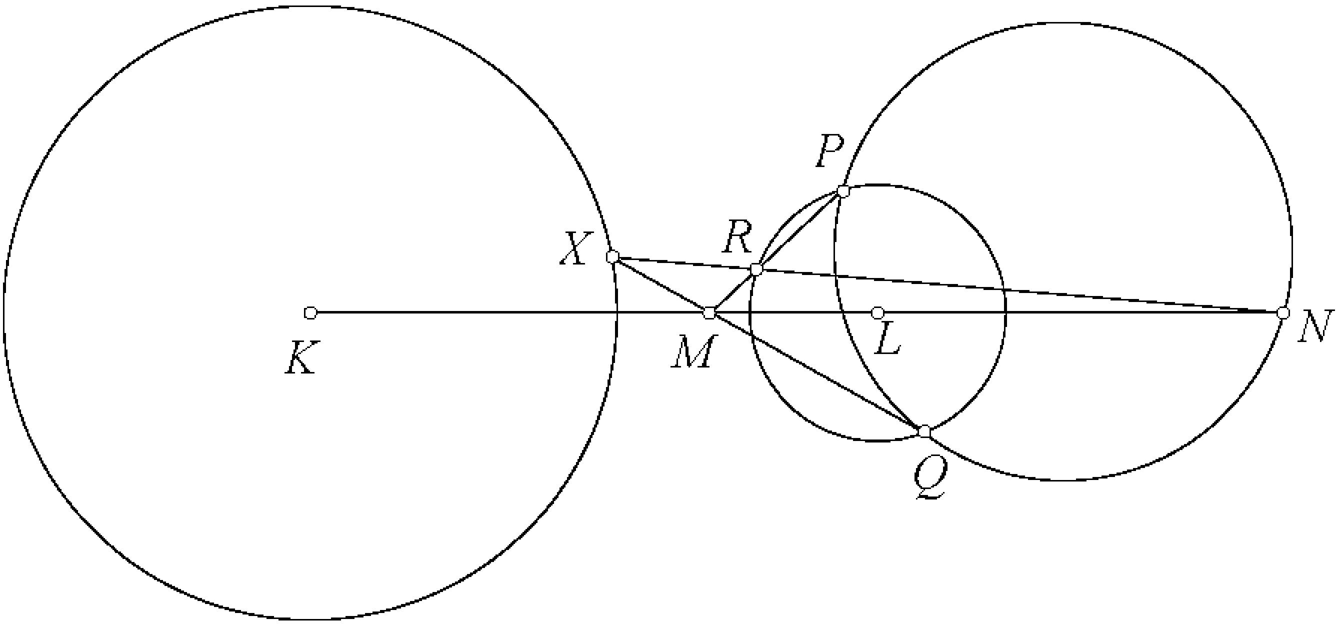
buratinogigle
#3 Feb 8, 2016, 8:48 am • 1

This configuration is actually true for two circles as following

Let (K), (L) be two circles with insimilicenter and exsimilicenter are M, N ,reps. P is a point on (L). Circle diameter PN cuts (L) again at Q. PM cuts (L) again at R. Prove that RN and MQ intersect on (K).

I think solution of Luis is available for this problem.

Attachments:



Spain

Problema de Geometría

Reply

Axlrose

#1 Feb 5, 2016, 2:26 pm

Bueno ahí va un problema creado por mi espero que les agrade saludos desde Perú.
Sea $ABCD$ un trapecio isósceles con $AB = CD$ y $AD \parallel BC$. Sea l una linea que pasa por D además l corta a AC y AB en E y F respectivamente tal que $EFBC$ es cíclico y sea M el punto medio de BC y $l \cap AM = G$. Sea N el punto medio de DG y sea $H = \odot(AFE) \cap NA$. Probar que $\odot(AFE)$ es tangente a $\odot(NHD)$.

This post has been edited 1 time. Last edited by Axlrose, Feb 6, 2016, 1:06 pm

Luis González

#2 Feb 6, 2016, 11:16 am • 1

El resultado se cumple para todo trapecio $ABCD$ no necesariamente isósceles.

Note que AD es obviamente tangente a $\odot(AEF)$ y como EF es antiparalela a BC respecto a AC, AB, entonces la A-mediana de AM de $\triangle ABC$ es la A-simediana AG de $\triangle AEF$. Por ende resulta $(E, F, G, D) = -1 \Longrightarrow NG^2 = ND^2 = NE \cdot NF = NH \cdot NA$. Esto quiere decir que la inversión con centro N y radio $NG = ND$ fija a $\odot(AEF)$ y transforma a $\odot(NHD)$ en AD. Como AD es tangente a $\odot(AEF)$, entonces por conformidad $\odot(NHD)$ es tangente a $\odot(AEF)$.

Quick Reply

High School Olympiads

Tripole of the Orthotransversal of I WRT DEF WRT ABC

Reply

Source: Own

TelvCohl

#1 Feb 5, 2016, 8:51 pm • 1

Let I, H be the incenter, orthocenter of $\triangle ABC$, respectively. Let $\triangle DEF$ be the intouch triangle of $\triangle ABC$ and let $\triangle XYZ$ be the cevian triangle of H WRT $\triangle DEF$. Prove that the trilinear polar (WRT $\triangle ABC$) of the perspector of $\triangle ABC, \triangle XYZ$ is the orthotransversal of I WRT $\triangle DEF$.

Luis González

#2 Feb 5, 2016, 10:57 pm • 2

Lemma: Incircle (I) of $\triangle ABC$ touches BC, CA, AB at $D, E, F.$ P is an arbitrary point and $\triangle XYZ$ is the cevian triangle of P WRT $\triangle DEF.$ Then the polar of P WRT (I) coincides with the trilinear polar of the perspector of $\triangle ABC$ and $\triangle XYZ$ WRT $\triangle ABC.$

Proof: Let $\triangle MNL$ be the cevian triangle of the perspector Q of $\triangle ABC, \triangle XYZ$ WRT $\triangle ABC.$ Consider a homology sending P to the center of the conic $\mathcal{C}$ image of $(I).$ Since the parallel ℓ_A from A to BC and DP have conjugate directions WRT $\mathcal{C},$ then it follows that ℓ_A is the polar of $X \equiv DP \cap EF$ WRT $\mathcal{C}.$ Hence if $X' \equiv \ell_A \cap EF,$ then $A(E, F, X, X') = -1 \Longrightarrow A(C, B, M, X') = -1 \Longrightarrow M$ is midpoint of BC and likewise N, L are midpoints of $CA, AB \Longrightarrow Q$ is centroid of $\triangle ABC,$ so its trilinear polar WRT $\triangle ABC$ is the line at infinity, which is polar of P WRT $\mathcal{C}.$ So back in the primitive figure, we get the desired result.

Back to the proposed problem. Let U, V, W be the intersections of EF, FD, DE with the perpendiculars to DI, EI, FI at $I,$ resp. A-altitude of $\triangle ABC,$ passing through the pole A of EF WRT (I) perpendicular to UI is then the polar of U WRT (I) and similarly the B- and C- altitude are the polars of V and W WRT $(I) \Longrightarrow$ orthotransversal $\overline{UVW}$ of I WRT $\triangle DEF$ is the polar of H WRT $(I).$ So using the previous lemma for $P \equiv H,$ we get the conclusion.

A-B-C

#3 Feb 6, 2016, 7:24 pm

Remarks

1) P lies on Darboux cubic. $\triangle DEF$ is pedal triangle of P WRT $\triangle ABC,$ H is orthocenter of $\triangle ABC.$ $\triangle XYZ$ is cevian triangle of H WRT $\triangle DEF.$ Then trilinear polar of the perspector of $\triangle ABC$ and $\triangle XYZ$ is parallel to orthotransversal of P WRT $\triangle DEF.$

2)(This is true for all P)Orthotransversal of P WRT $\triangle DEF$ is pendicular to HP^*, where P^* is isogonal conjugate of P WRT $\triangle ABC.$

For the 2nd remark, see complement

Quick Reply

High School Olympiads

A short and nice geometry problem

Reply

phuocdinh_vn99
#1 Feb 5, 2016, 7:35 pm • 1

Given acute triangle ABC with circumcircle (O). Let N be midpoint of arc BAC and M be midpoint of AN. Internal bisector of $\angle BAC$ cuts BC at D. OD cuts (BOC) at E. Prove that (OME) goes through midpoint of BC.

Luis González
#2 Feb 5, 2016, 9:06 pm

Let X and L be the midpoints of $\overline{BC}$ and let arc BC of (O). Tangents of (O) at B, C meet at U and tangents of (O) at A, N meet at J. Inversion WRT (O) takes X, E, M into U, D, J, respectively, thus it suffices to prove that U, D, J are collinear.

Let ND cut (O) again at S. Since BC is the polar of U WRT (O), it follows that $U \in AS$. Now from the complete cyclic $ANLS$, we deduce that U, D, J are collinear on the polar of $AN \cap LS$ WRT (O), as desired.

dothef1
#5 Feb 5, 2016, 9:58 pm

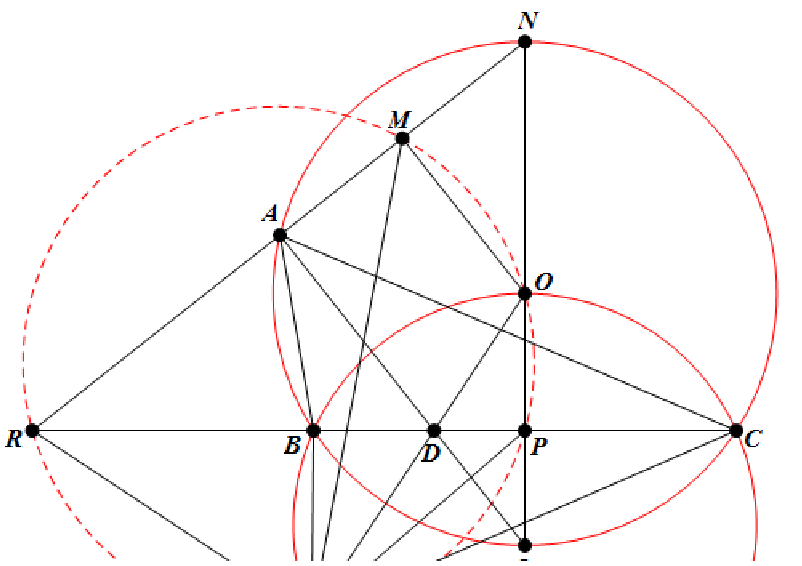
Let X be the midpoint of $[BC]$, and let the external angle bisector of A (that is (AM)) intersect (BC) at Y, since $OMYX$ is cyclic , it suffice to show that $OEY = 90$, or simply $DEY = 90$, wich is equivalent to E lying on the Apollonius circle of ABC.
Now invert wrt the circumcenter , obviously the image of D is E , and since the appolonius circle is fixed under this inversion and D lies on that circle , we're done .

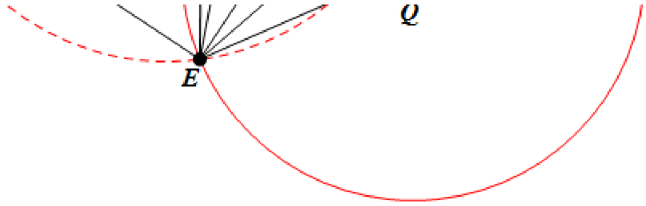
This post has been edited 1 time. Last edited by dothef1, Feb 5, 2016, 9:59 pm
Reason: typos :(

mjuk
#6 Feb 6, 2016, 12:03 am

Let P be midpoint of BC, let Q be midpoint of arc BC, then $\angle QAN = 90^\circ \to AN \perp AD \to AN$ is external angle bisector of $\angle A$, Let $AN \cap BC = R$. O is midpoint of arc BOC in $\odot BEC$, hence $\angle BED = \angle DEC$, but $(BCRP) = -1$, hence $\angle DER = \angle OER = 90^\circ$. But $RMO = RPO = 90^\circ \to REPOM$ is cyclic, hence $\odot OME$ goes through P.

Attachments:





This post has been edited 3 times. Last edited by mjuk, Feb 6, 2016, 12:06 am
Reason: ga

PROF65

#7 Feb 6, 2016, 1:14 am

Let I be midpoint of BC and F be the point where AN (the A-exterior bisector) hit BC then
$DE \cdot DO = DB \cdot DC = DI \cdot DF$ thus $FEIO$ is cyclic but $\widehat{OIF} = \widehat{ONF} = 90$ therefore the result follows.
WCP

Quick Reply

High School Olympiads

Nice geometry from Serbia

mihajlon
#1 Jan 26, 2016, 1:08 am

Let AD and BE be altitudes, H orthocenter and O circumcenter of acute-angled triangle ABC. If K is the orthocenter of $\triangle AOB$ and M is midpoint of DE then prove that points H, K and M are colinear.

FabrizioFelen
#2 Jan 26, 2016, 2:24 am

My solution:
Let N and P the midpoints of AB and $CH \Longrightarrow M, N, P$ are collinear(the Gauss-Newton line) and $AN^2 = NE^2 = NM.NP$ and $PH^2 = PE^2 = PM.PN$ since $\angle PEN = 90^\circ$ and $\angle EMP = 90^\circ \Longrightarrow \angle PHM = \angle HNP$ since $PH^2 = PM.PN$ and $\angle HNP = \angle OPN$ since $PHNO$ is a parallelogram. Since K is the orthocenter of $AOB \Longrightarrow NO.NK = AN^2 \Longrightarrow AN^2 = NO.NK = NM.NP \Longrightarrow MOKP$ is cyclic $\Longrightarrow \angle MKO = \angle OPM$ but $\angle OPM = \angle PHM = \angle PNH \Longrightarrow \angle MKN = \angle MHP$ and $HP \parallel KN \Longrightarrow M, K, H$ are collinear...

drmzjoseph
#3 Jan 26, 2016, 2:40 am

Easy notice $\odot(BHC)$ is tangent to BK and $CK \Rightarrow HK$ is H-symmedian of $\triangle BHC$
Denote F the foot of the C-altitude, so $\triangle FEH \sim \triangle BCH \Rightarrow HM$ is H-symmedian of $\triangle BHC \Rightarrow H, K, M$ are collinear.

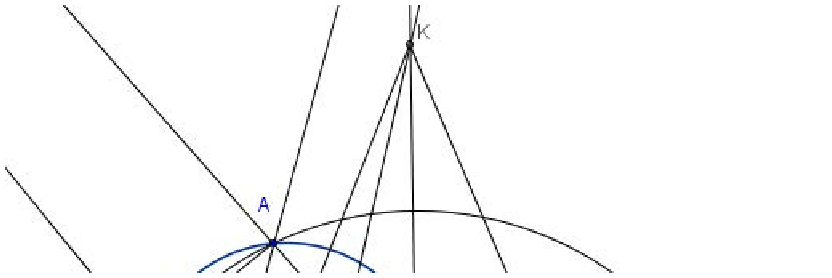
Luis González
#4 Jan 26, 2016, 4:49 am • 1

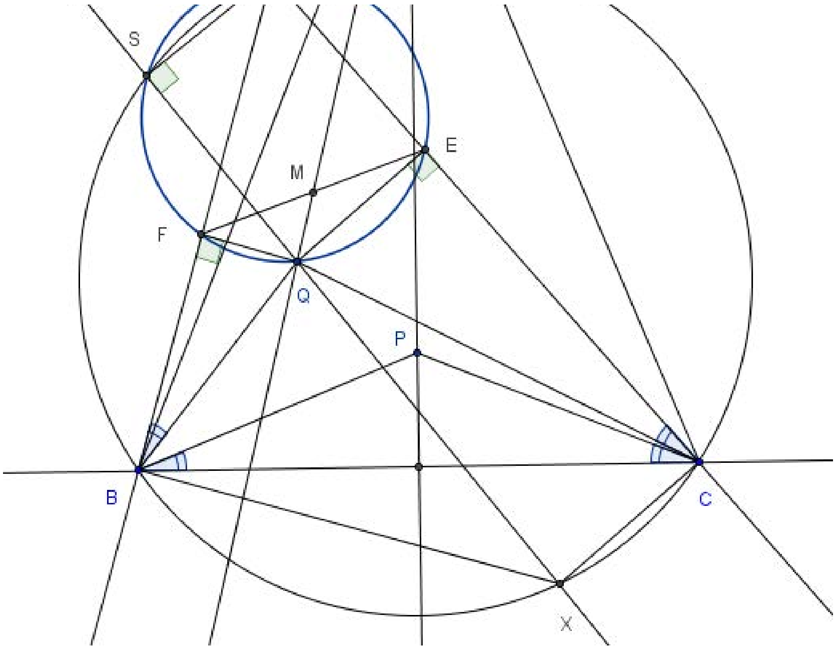
Generalization: In a $\triangle ABC$, let P be a point on the perpendicular bisector of $\overline{BC}$. Q is isogonal conjugate of P WRT $\triangle ABC$ and E, F are the projections of Q on AC, AB. M is the midpoint of $\overline{EF}$ and K is the orthocenter of $\triangle PBC$. Then Q, K, M are collinear.

Proof: Let X be the antipode of A on the circumcircle $(O) \equiv \odot(ABC)$. Then $\angle KCB = 90^\circ - \angle PCB = 90^\circ - \angle QCA = \angle XCQ \Longrightarrow CX, CK$ are isogonals WRT $\angle QCB$ and likewise BX, BK are isogonals WRT $\angle QBC \Longrightarrow X, K$ are isogonal conjugates WRT $\triangle QBC \Longrightarrow QX$ and QK are isogonals WRT $\angle BQC$ $(\star)$.

QX cuts (O) again at second intersection S of (O) with the circle $\odot(AEF)$ with diameter $\overline{AQ}$, which is the center of the spiral similarity that swaps $\overline{BF}$ and $\overline{CE} \Longrightarrow SE : SF = CE : BF$. But $\angle FBQ = \angle PBC = \angle PCB = \angle ECQ$ yields $\triangle BFQ \sim \triangle CEQ \Longrightarrow QE : QF = CE : BF = SE : SF \Longrightarrow QESF$ is harmonic $\Longrightarrow QS$ is the Q-symmedian of $\triangle QEF$ isogonal of its Q-median QM WRT $\angle EQF$. Since QE, QF are clearly isogonals WRT $\angle BQC$, then it follows that $QS \equiv QX$ and QM are isogonals WRT $\angle BQC$. Together with $(\star)$, we conclude that $QM \equiv QK$, i.e. Q, K, M are collinear.

Attachments:





drmzjoseph

#5 Jan 26, 2016, 9:23 am

> *Luis González wrote:*
>
> **Generalization:** In a $\triangle ABC$, let P be a point on the perpendicular bisector of $\overline{BC}$. Q is isogonal conjugate of P WRT $\triangle ABC$ and E, F are the projections of Q on AC, AB. M is the midpoint of $\overline{EF}$ and K is the orthocenter of $\triangle PBC$. Then Q, K, M are collinear.

Stronger:

Let P and Q isogonal conjugate WRT $\triangle ABC$
Let ω be the pedal circle of P and Q
Let $\triangle DEF$ be the pedal triangle of Q WRT $\triangle ABC$
QD cut again to ω at J, and let T be the projection of P at BC.
$M \equiv JT \cap EF$
K the orthocenter of $\triangle PBC$. Then Q, K, M are collinear.

Proof:

Let S be the reflection of P at BC
Let R be the projection of S at KC

$\measuredangle TRS = \measuredangle TCS = \measuredangle ECQ = \measuredangle EDQ = \measuredangle EFJ$
$\measuredangle TKR = \measuredangle QBF = \measuredangle QDF = \measuredangle JTF \Rightarrow \Delta KRS \cup T \sim \Delta TFJ \cup M \Rightarrow \frac{ST}{TK} = \frac{JM}{MD} = \frac{JQ}{TK} \Rightarrow Q, M, K$
are collinear.

sunken rock

#6 Jan 26, 2016, 10:50 am

Let N, P be the midpoints of AB, CH; $P - M - N$ are collinear, Newton-Gauss line of $CEHD$. Let T the symmetrical of of K about N, T is the intersection of the tangents at A, B to circle (ABC), thus CT is symmedian and it passes through M, since $ABDE$ is cyclic. Consequently we get $C - M - T$ and $P - M - N$ collinear, but P, N are midpoints of CH, KT and we are done.

Best regards,
sunken rock

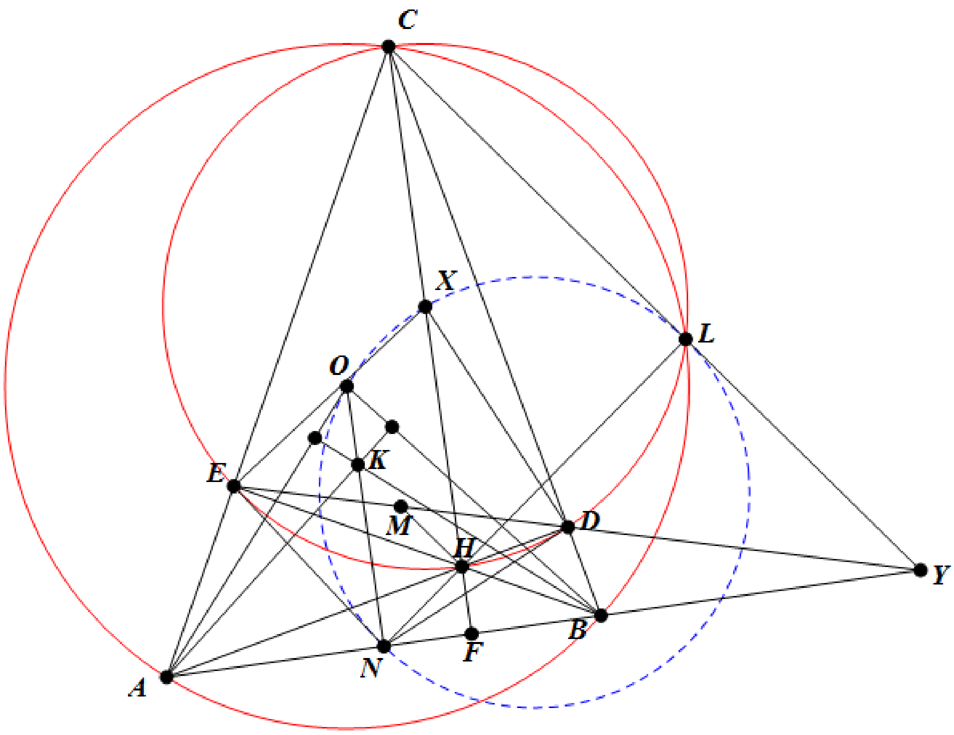
mjuk

#7 Jan 26, 2016, 10:48 pm

Inversion solution:

Inversion solution:
Let X be the midpoint of CH, F foot of altitude from C to AB, N midpoint of AB.
X is center of circle $CEHD$.
(1) DE is polar of N wrt. $\odot CED$ (well-known).
(2) $NB^2 = NK \cdot NO$ (this follows from $\triangle NKB \sim \triangle NBO$)
Apply inversion Ψ with center N and radius AN.
(1)$\rightarrow$ $\odot CED$ is fixed (3), and $\Psi(M) = X$.
(2)$\rightarrow$ $\Psi(K) = O$
Let L be second intersection of $\odot CED$ and $\odot ABC$.
CL, DE, AB intersect at radical center Y, which is also harmonic conjugate of F wrt. AB, so $\Psi(F) = Y$.
Let $\Psi(H) = H'$, (3)$\rightarrow$ $H' \in \odot CED$. We know that $\angle NH'Y = \angle NFH = 90$, but $\angle NLY = 180 - \angle HLC = 90$, so $H' = L \rightarrow \Psi(H) = L$
Suffices to prove that $NLXO$ is cyclic, but $NHXO$ is a parallelogram, so $NL \parallel OX$, and $ON = XH = XL \rightarrow NLXO$ is an isosceles trapezoid $\rightarrow NLXO$ is cyclic $\rightarrow H, K, M$ are collinear.

Attachments:



Quick Reply

High School Olympiads

A is incenter of triangle KLN

Reply

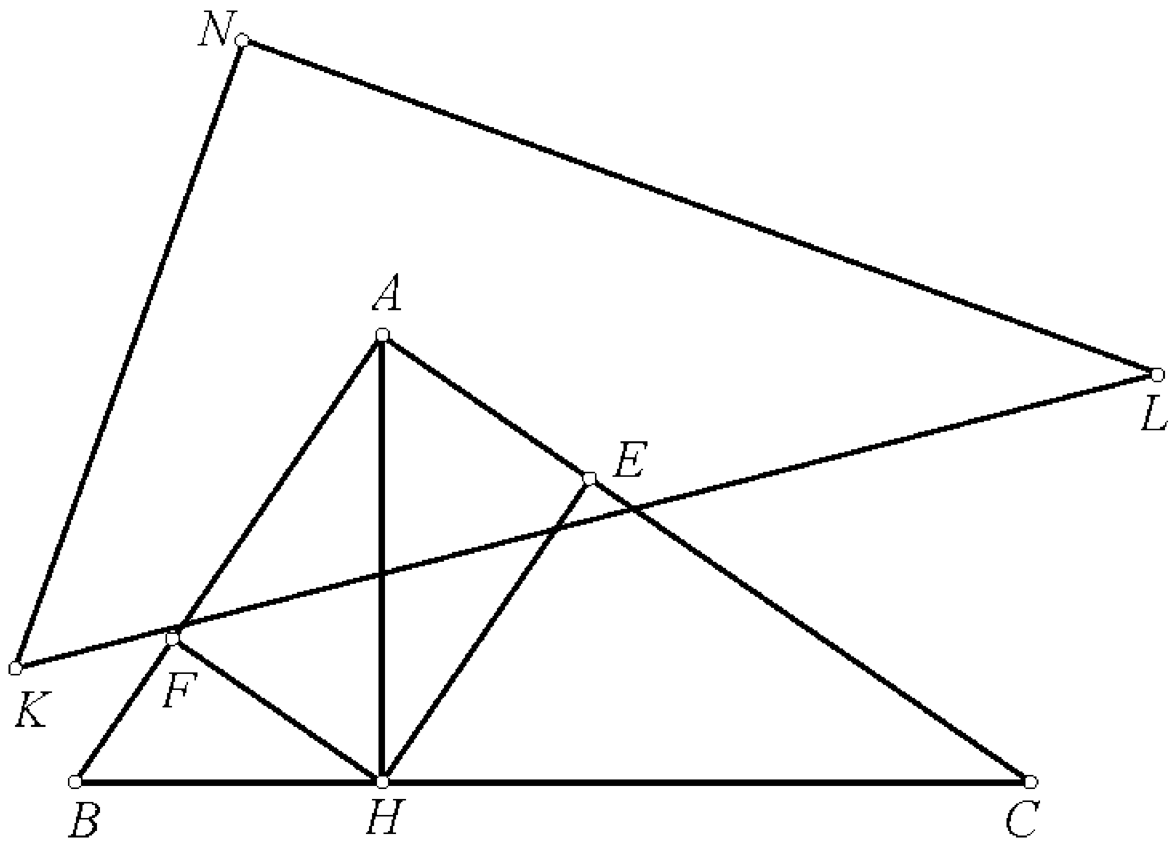
Source: Problem weekly, first week, September 2015

buratinogigle

#1 Jan 25, 2016, 5:15 pm • 1

Let ABC be triangle with $\angle A = 90^\circ$ and altitude AH. E, F are projections of H on CA, AB. Let K, L, N be the H-excenters of triangles HBF, HCE, HEF. Prove that A is incenter of triangle KLN.

Attachments:



Luis González

#2 Jan 25, 2016, 7:59 pm • 2

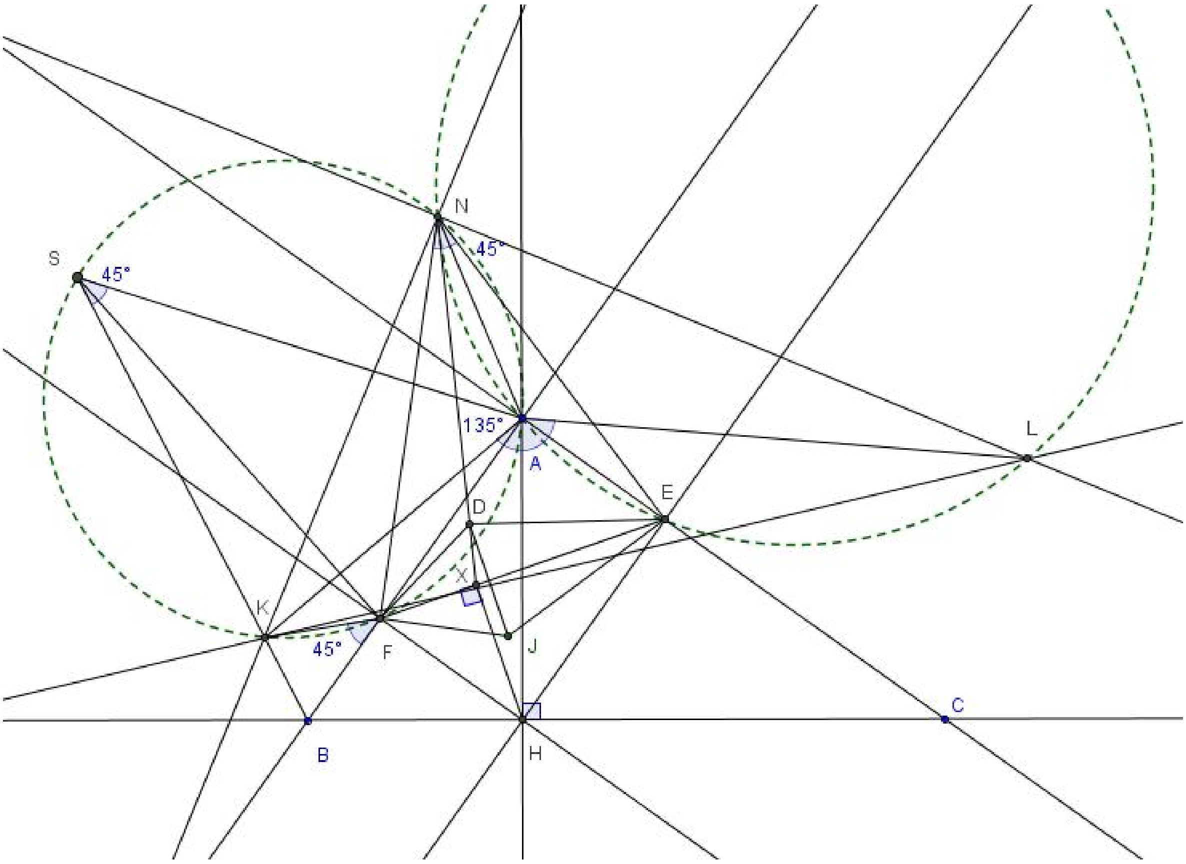
Let J be the incenter of $\triangle HEF$ and let $D \in NX$ be the reflection of J on EF.
$\angle NED = 90^\circ - \tfrac{1}{2}\angle FEH - \tfrac{1}{2}FEH = 90^\circ - \angle FEH = \angle AEF \Longrightarrow ED, EA$ are isogonals WRT $\angle NEF$ and likewise FD, FA are isogonals WRT $\angle NFE \Longrightarrow A, D$ are isogonal conjugates WRT $\triangle NEF \Longrightarrow$ $\angle FNA = \angle END \equiv \angle ENX.$ If S is the H-excenter of $\triangle HAB,$ then from $\triangle HEF \cup X \cup A \sim \triangle HAB \cup F \cup S,$ it follows that $\angle ASF = \angle ENX = \angle FNA \Longrightarrow ANSF$ is cyclic. But $\angle BFK = \angle ASB = 45^\circ \Longrightarrow ASKF$ is cyclic $\Longrightarrow ANKF$ is cyclic and similarly $ANLE$ is cyclic. Thus $\angle KNA = \angle BFK = 45^\circ$ and likewise $\angle ANL = 45^\circ \Longrightarrow$ NA is internal bisector of right $\angle KNL.$ Moreover
$\angle KAL = 90^\circ + \angle KAF + \angle LAE = 90^\circ + \angle KNF + \angle LNE = 90^\circ + 90^\circ - \angle ENF = 135^\circ,$ which means that A coincides with the incenter of $\triangle KLN.$

Luis González

#3 Jan 25, 2016, 8:14 pm • 1

Here is a diagram for the previous solution. I forgot to mention that X is the projection of H on EF.

Attachments:



buratinogigle

#4 Jan 25, 2016, 8:22 pm

Thank Luis for nice solution.

Here is my solution. Let LE, KF cut HN at X, Y, reps. Let Z, T be projections of N, L on FH, CE, resp. Note the isoceleses triangles, so we have $\frac{FY}{FY+NY} = \frac{YH}{YH+NY} = \frac{YH}{HN} = \frac{HF}{2HZ} = \frac{HF}{HE+HF+EF} = \frac{AB}{AB+AC+BC} \quad (1)$.

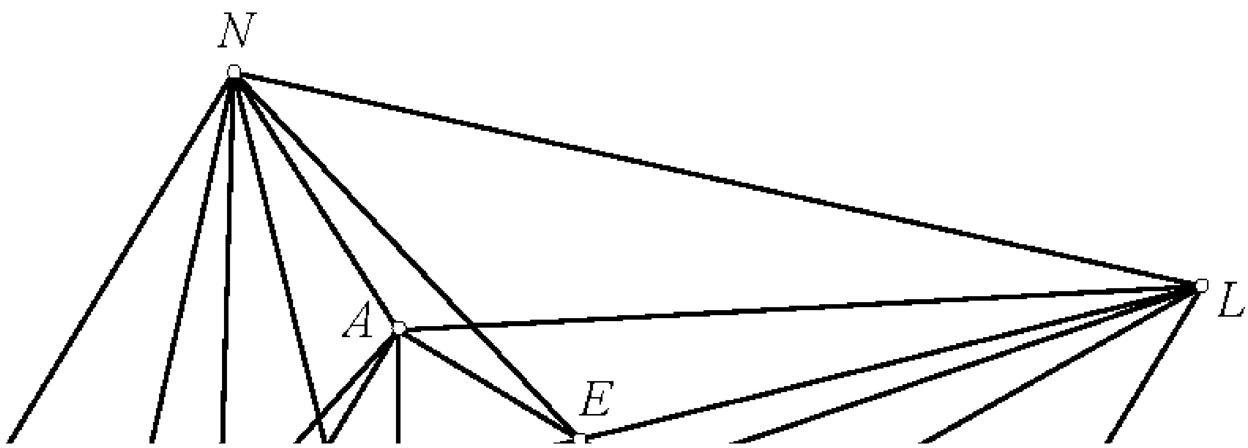
We have $\frac{XH}{XH+LX} = \frac{XE}{2XE+LE} = \frac{HE}{2HE+ET} = \frac{HE}{2HE+(CE+CH-HE)} = \frac{AB}{AB+AC+BC} \quad (2)$.

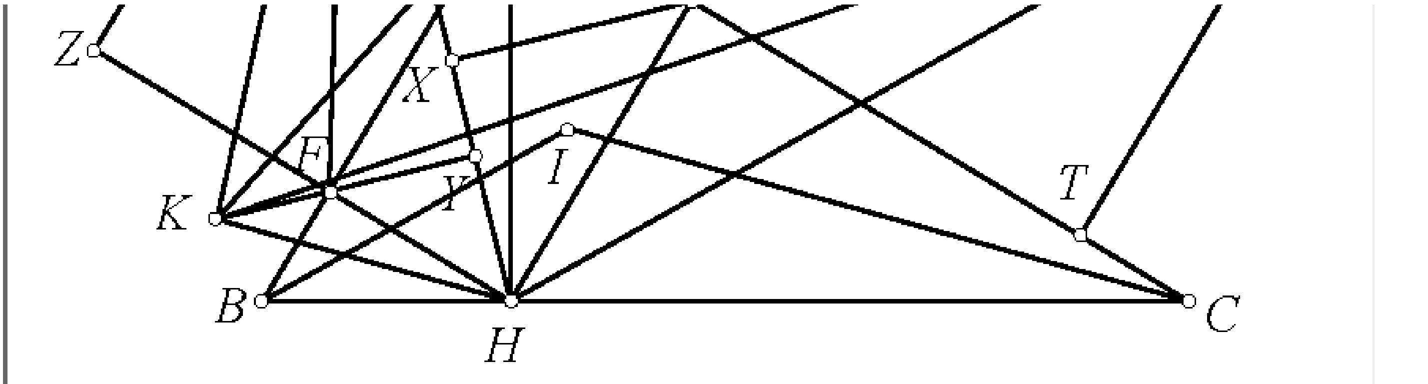
From (1),(2) we deduce $\frac{YF}{YN} = \frac{XH}{XL}$. Similarly, $\frac{XE}{XN} = \frac{YH}{YK}$. Now multiply the ratio with notice that $YF = YH, XE = XH$ we get $\frac{1}{YN.XN} = \frac{1}{XL.YK}$. Hence right triangles YNK and XLN are similar, so $\angle KNL = 90°$.

We see $\frac{NK}{NL} = \frac{KY}{XN} = \frac{YH}{XE} = \frac{HF}{HE} = \frac{AB}{AC}$. Therefore, NKL and ABC are similar.

Let I be incenter of triangle ABC. We have HFK and LEH are similar. Thus, $AE.AF = HE.HF = FK.EL$. But $\angle AEL = \angle KFA = 135°$. From this, triangles AEL and KAF are similar. So $\frac{AK}{AL} = \frac{KF}{AE} = \frac{KF}{FH} = \frac{IB}{IC}$. Note that the last equality because triangles FKH and IBC are similar. We have $\angle KAL = 90° + \angle KAF + \angle LAE = 90° + 45° = 135° = \angle BIC$. So AKL and IBC are similar. We get A is incenter of triangle NKL.

Attachments:





Quick Reply

High School Olympiads

Perpendicular segments

Source: Problem weekly, first week, August 2015

buratinogigle

#1 Jan 25, 2016, 9:24 am • 1

Let ABC be a triangle inscribed in circle (O) with altitudes AD, CF. Let AD cut (O) again at K. Let KF cut (O) again at L. Perpendicular line from A to OC cuts CL at N. Prove $FN \perp FO$.

Luis González

#2 Jan 25, 2016, 12:34 pm • 2

Let H be the orthocenter of $\triangle ABC$. E is the foot of the B-altitude and CF cuts (O) again at T. If the parallel from C to EF cuts (O) again at S, we have $\angle SKA = \angle SCA = \angle FEA = \angle DHC \Longrightarrow KS \parallel CH$ and since F is midpoint of HT, then $K(F, H, T, S) = K(L, A, T, S) = C(L, A, T, S) = -1 \Longrightarrow CL$ passes through the midpoint M of EF.

Since EF is antiparallel to HA WRT CH, CA, then the C-median CN of $\triangle CEF$ is the C-symmedian of $\triangle CHA$. But since $\angle NAC = \angle FHA$, NA is tangent of $\triangle HAC \Longrightarrow N$ is on perpendicular bisector of AH, i.e. $\triangle NAH$ is N-isosceles with $\angle NAH = \angle NHA = \angle HCA = \angle OCB \Longrightarrow \triangle AFH \cup N \sim \triangle CFB \cup O \Longrightarrow \angle NFA = \angle OFC \Longrightarrow \angle OFN = \angle CFA = 90^\circ$.

buratinogigle

#3 Jan 25, 2016, 1:31 pm

Here is my solution. We will prove $\triangle FNA \sim \triangle FOC$ to deduce $\triangle FNO \sim \triangle FAC$ then $\angle NFO = 90^\circ$, indeed. We see $\angle NAC = \angle ABC = \angle ALC$. Hence $\angle NAF = \angle NAC - \angle BAC = \angle ABC - \angle BAC = \angle FCO$. Let AD cut CF at H and P is projection of F on AD, M is symmetric of H through P. Easily seen $\angle FMH = \angle FHM = \angle ABC = \angle NAC$ and $\angle FKM = \angle ACN$ therefore $\triangle KFM \sim \triangle CNA$. We also have $\triangle FAL \sim \triangle FKB$ and $\triangle FDP \sim \triangle ACF$. From this, $\frac{NA}{FA} = \frac{NA}{LA} \cdot \frac{LA}{FA} = \frac{NC}{AC} \cdot \frac{KB}{FK} = \frac{HB}{AC} \cdot \frac{NC}{FK} = \frac{2OQ}{AC} \cdot \frac{AC}{MK} = \frac{2OQ}{2DP} = \frac{OQ}{OC} \cdot \frac{OC}{CF} \cdot \frac{CF}{DP} = \frac{BF}{BC} \cdot \frac{OC}{CF} \cdot \frac{AC}{FD} = \frac{OC}{CF}$. We deduce $\triangle FNA \sim \triangle FOC$ so we are done.

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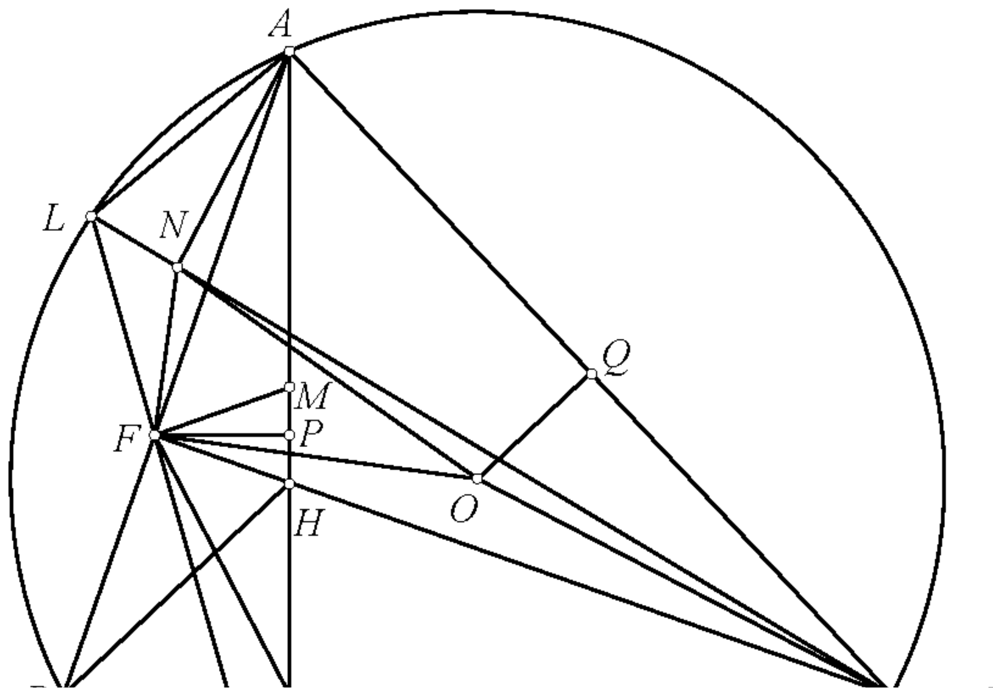
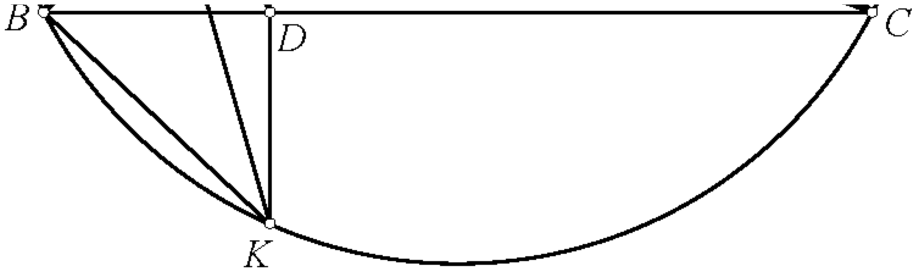



B
D
C
K

High School Olympiads

Radical center is the center of pedal circle

Reply

Source: Own

livetolove212
#1 Jan 20, 2016, 9:31 pm • 1

Given triangle ABC and an arbitrary point P in plane. Let A_1, A_2; B_1, B_2; C_1, C_2 be three pairs of point which lie on BC, CA, AB, respectively such that 3 triangles $PA_1A_2, PB_1B_2, PC_1C_2$ are isosceles at P and they are similar. Let (O_a) be circle tangent to PA_1, PA_2 at A_1, A_2, respectively. Similarly with $(O_b), (O_c)$. Prove that the radical center of $(O_a), (O_b), (O_c)$ is the center of pedal circle of P wrt triangle ABC.
This problem is not hard but nice

Luis González
#2 Jan 20, 2016, 10:57 pm • 1

Let X, Y, Z be the projections of P on BC, CA, AB and let U be the center of $\odot(PYAZ)$. Since $O_bB_1{}^2 = O_bY \cdot O_bP$, then (O_b) and $(U) \equiv \odot(PYAZ)$ are orthogonal and likewise (U) is orthogonal to $(O_c) \Longrightarrow U$ is on the radical axis τ_A of $(O_b), (O_c)$. Since $YZ \parallel O_bO_c$ (due to $PY : PO_b = PZ : PO_c$), then it follows that τ_A is the line through U perpendicular to YZ, i.e. the perpendicular bisector of $\overline{YZ}$. Similarly the other radical axes τ_B, τ_C are the perpendicular bisectors of $\overline{ZX}, \overline{XY}$, concurring at the circumcenter of $\triangle XYZ$; the radical center of $(O_a), (O_b), (O_c)$.

buratinogigle
#4 Jan 25, 2016, 10:01 am

Extension of problem. Let P be the point see three circle $(O_a), (O_b), (O_c)$ in three equal angles. D, E, F divide PO_a, PO_b, PO_c in the same ratios, reps. X, Y, Z are inversions of D, E, F through $(O_a), (O_b), (O_c)$, reps. Let K be circumcenter of triangle XYZ. Prove that radical center of $(O_a), (O_b), (O_c)$ lies on line PK.

Luis González
#5 Jan 25, 2016, 10:23 am • 1

> *buratinogigle wrote:*
> Extension problem. Let P be the point see three circle $(O_a), (O_b), (O_c)$ in three equal angles. D, E, F divide PO_a, PO_b, PO_c in the same ratios, reps. X, Y, Z are inversions of D, E, F through $(O_a), (O_b), (O_c)$, reps. Let K be circumcenter of triangle DEF. Prove that radical center of $(O_a), (O_b), (O_c)$ lies on line PK.

I believe you mean K is the circumcenter of $\triangle XYZ$, otherwise X, Y, Z play no role in the conclusion.

As before, denote A_1, A_2 the points where the tangents from P to (O_a) touch it and denote B_1, B_2 and C_1, C_2 cyclically. Let M, N, L be the projections of P on A_1A_2, B_1B_2, C_1C_2. From the original problem the radical center of $(O_a), (O_b), (O_c)$ is the circumcenter R of $\triangle MNL$ and since $\triangle PA_1A_2 \sim \triangle PB_1C_2 \sim \triangle PC_1C_2$, we deduce that $PX : XM = PY : YN = PZ : ZL \Longrightarrow P$ is homothety center of $\triangle XYZ$ and $\triangle MNL \Longrightarrow P, K, R$ are collinear.

Quick Reply

High School Olympiads

Fixed circle 2

Reply

Source: Own

livetolove212

#1 Jan 20, 2016, 9:50 pm

Given triangle ABC inscribed in (O) (B, C are fixed, A moves on (O)). Let D be a fixed point in the plane, X, Y, Z be the projections of P onto BC, CA, AB. Let T be the intersection of the tangents through Y, Z of (XYZ). Prove that T lies on a fixed circle.

Luis González

#2 Jan 21, 2016, 1:20 am • 1

Straightforward angle chase yields $\angle YZX = \angle BPC - \angle BAC \Longrightarrow \triangle TYZ$ is T-isosceles with constant base angle $\angle TYZ = \angle TZY = \angle BPC - \angle BAC,$ i.e. all $\triangle TYZ$ are directly similar. Z, Y move on fixed circles ω_B, ω_C with diameters $\overline{PB}, \overline{PC}$ and since $\angle(PZ, PY) = \angle(AB, AC) = \text{const},$ we deduce that Z, Y move on ω_B, ω_C with the same speed and direction. Thus from the general configuration discussed at All Russian olympiad 1961, we conclude that T runs on a fixed circle.

buratinogigle

#3 Jan 25, 2016, 2:02 am

General problem. Let ABC be a triangle inscribed a fixed circle (O) with fixed chord BC. P is a fixed point on plane. DEF is pedal triangle of P. Q is the point which devides EF in constant ratio. R is inversion of Q through circle (DEF). Prove that R lies on a fixed circle when A moves.

When Q is midpoint of EF we have a bove problem.

Luis González

#4 Jan 25, 2016, 2:53 am • 1

> buratinogigle wrote:
>
> General problem. Let ABC be a triangle inscribed a fixed circle (O) with fixed chord $BC.$ P is a fixed point on plane. DEF is pedal triangle of $P.$ Q is the point which devides EF in constant ratio. R is inversion of Q through circle (DEF). Prove that R lies on a fixed circle when A moves.
>
> When Q is midpoint of EF we have a bove problem.

If K denotes the center of $\odot(DEF)$, then R is clearly the second intersection of KQ with $\odot(KEF).$ Since $\angle EKF = 2\angle EDF = 2(\angle BPC - \angle BAC) = \text{const}$ and $QE : QF = \text{const},$ then all $\triangle KEF \cup Q$ are directly similar. Hence $\angle EFR = \angle EKQ = \text{const}$ and $\angle FER = \angle FKQ = \text{const} \Longrightarrow$ all $\triangle REF$ are directly similar. Hence as before, R moves on a fixed circle.

Quick Reply

High School Olympiads

Fixed circle 1

Reply

Source: Own

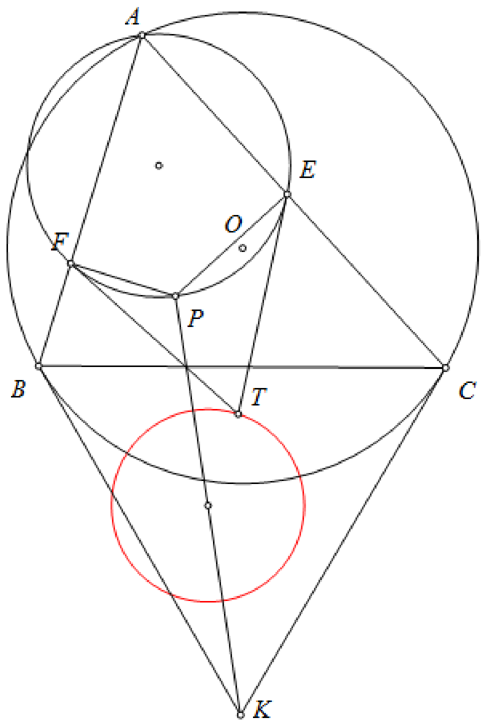
livetolove212

#1 Jan 23, 2016, 5:49 pm • 1

Given triangle ABC inscribed in (O) with B, C are fixed, A moves on (O). Let P be an arbitrary point in the plane, E, F be the projections of P onto AC, AB, respectively. Let T be the intersection of the tangents through E and F of triangle AEF. Prove that T moves on fixed circle whose center is the midpoint of segment joining P and the intersection of tangents through B and C of (O).

I don't know why my previous post was deleted, so I attach a diagram.

Attachments:



IsoLyS

#2 Jan 23, 2016, 6:30 pm

Using complex numbers, let B_1 be midpoint of BP and C_1 be midpoint of CP.
Since $EFT \sim CBK \sim C_1B_1M$, we get

$$\frac{m-c_1}{m-b_1} = \frac{t-e}{t-f}$$

and by freshman sum

$$\frac{m-c_1}{m-b_1} = \frac{t-e}{t-f} = \frac{(t-m)-(e-c_1)}{(t-m)-(f-b_1)}$$

Now if we move A, let E, F, T moves to E', F', T'. Since $\angle EC_1E_1 = \angle FB_1F_1$ and E, F moves on circle which has center on C_1, B_1, respectively, there exists a complex number $|\omega| = 1$ satisfying

$$e' = c_1 + (e-c_1)\omega, f' = b_1 + (f-b_1)\omega$$

Let t'' be a complex such that

$$t'' = m + (t - m)\omega$$

We'll prove $t' = t''$. Indeed, we know $E'F'T' \sim EFT$, so

$$\frac{t'-e'}{t'-f'} = \frac{t-e}{t-f} = \frac{m-c_1}{m-b_1} = \frac{(t-m)-(e-c_1)}{(t-m)-(f-b_1)} = \frac{t''-e'}{t''-f'}$$

Therefore $t' = t''$ holds, which means absolute value of $\dfrac{t'-m}{t-m}$ is 1, hence T' lies on a fixed circle whose center is M, as desired.

---

tranquanghuy7198

#3 Jan 24, 2016, 4:07 pm • 1

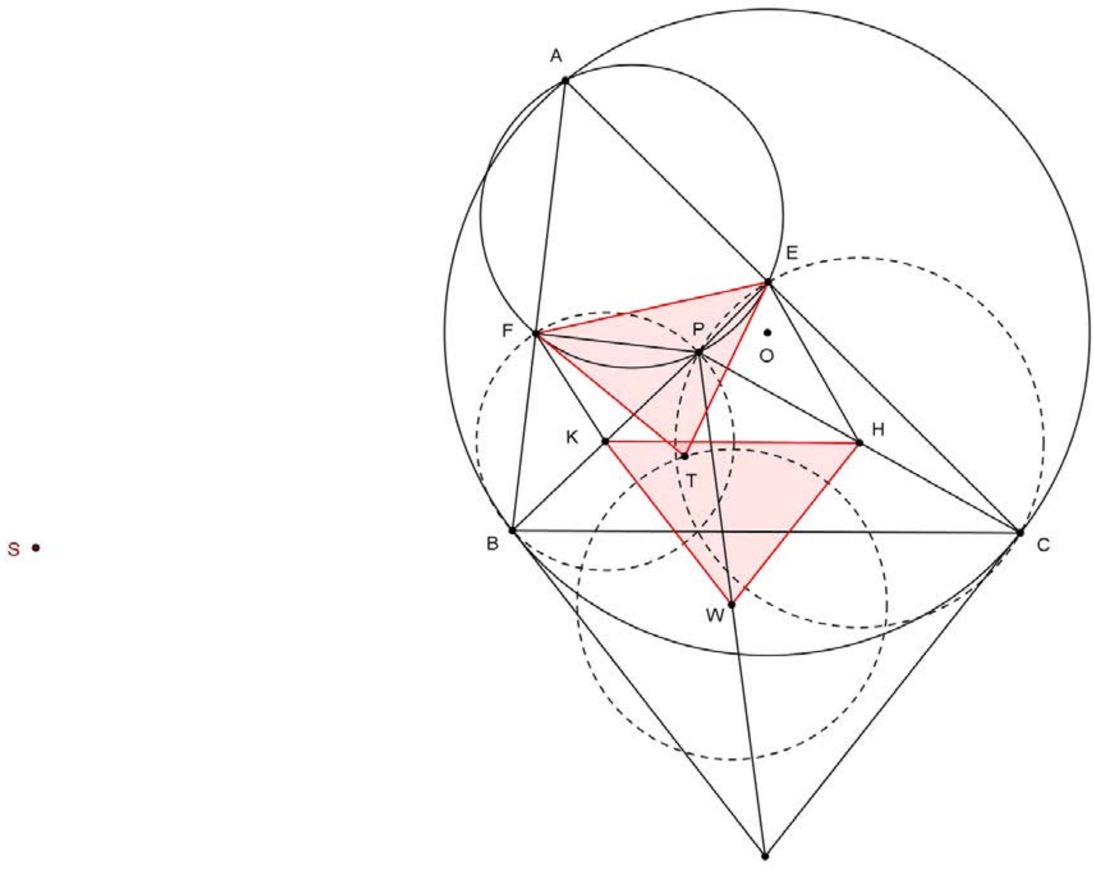
My solution.

Let R be the intersection of the tangents of (O) at B, C. W, K, H are the midpoints of PR, PB, PC, resp.
It's clear that: $\angle TEF = \angle TFE = \angle A \Rightarrow \triangle TEF \sim \triangle WHK$, which is fixed.
We have:
$(\overrightarrow{KF}, \overrightarrow{HE}) = (\overrightarrow{KF}, \overrightarrow{KP}) + (\overrightarrow{KP}, \overrightarrow{HP}) + (\overrightarrow{HP}, \overrightarrow{HE})$
$= 2(BA, BP) + (\overrightarrow{KP}, \overrightarrow{HP}) + 2(CP, CA)$
$= 2(AB, AC) + 2(PC, PB) + (\overrightarrow{KP}, \overrightarrow{HP})$
$= \varphi = \text{const}$ (mod 2π)
It means that there exists a fixed spiral similarity $\mathcal{S}_{(S,k,\varphi)}$ which maps:

$$\mathcal{S}_{(S,k,\varphi)} : K \mapsto H, (K) \mapsto (H), F \mapsto E$$

$\Rightarrow \triangle SFE \overset{+}{\sim} \triangle SKH$
$\Rightarrow (\triangle SFE \cup T) \overset{+}{\sim} (\triangle SKH \cup W)$
$\Rightarrow \triangle SFK \overset{+}{\sim} \triangle STW$
$\Rightarrow \dfrac{TW}{FK} = \dfrac{SW}{SK} = \text{const}$ ($\because S, K, H, W$ are fixed)
$\Rightarrow TW = \text{const}$ and the conclusion follows.
Q.E.D

Attachments:



R

This post has been edited 1 time. Last edited by tranquanghuy7198, Jan 24, 2016, 9:28 pm

Luis González

#4 Jan 25, 2016, 12:31 am

This can be solved exactly in the same way as the problem Fixed circle 2.

F, E run on fixed circles $(O_1), (O_2)$ with diameters $\overline{PB}, \overline{PC}$ with the same direction and speed due to $\angle(PF, PE) = \angle(AB, AC) = \text{const}.$ Since all the isosceles $\triangle TEF$ are directly similar ($\angle TEF = \angle TFE = \angle BAC$), then from the general configuration discussed at All Russian olympiad 1961, we conclude that T runs on a fixed circle whose center is the point U verifying that $\triangle UO_1O_2 \sim \triangle TFE$ are directy similar. By homothety $\mathcal{H}(P, 2)$, it's clear that U is the midpoint of $\overline{PK}$.

buratinogigle

#5 Jan 25, 2016, 2:04 am

General problem. Let ABC be a triangle inscribed a fixed circle (O) with fixed chord BC. P is a fixed point on plane. E, F are projections of P on line CA, AB, reps. Q is the point which devides EF in a constant ratio. R is the inversion of Q through circle (AEF). Prove that R lies on a fixed circle when A moves.

When Q is midpoint of EF we have a bove problem.

Luis González

#6 Jan 25, 2016, 2:38 am • 1

> *buratinogigle wrote:*
>
> General problem. Let ABC be a triangle inscribed a fixed circle (O) with fixed chord BC. P is a fixed point on plane. E, F are projections of P on line CA, AB, reps. Q is the point which devides EF in a constant ratio. R is the inversion of Q through circle (AEF). Prove that R lies on a fixed circle when A moves.
>
> When Q is midpoint of EF we have a bove problem.

If K denotes the center of $\odot(AEF)$, then R is clearly the second intersection of KQ with $\odot(KEF)$. Since $\angle EKF = 2\angle BAC = \text{const}$ and $QE : QF = \text{const}$, then all $\triangle KEF \cup Q$ are directly similar $\Longrightarrow$ $\angle FER = \angle FKQ = \text{const}$ and $\angle EFR = \angle EKQ = \text{const} \Longrightarrow$ all $\triangle REF$ are directly similar. Hence as before, R moves on a fixed circle whose center is the midpoint of $\overline{PS}$, where S is the point that fulfills $\triangle SBC \sim \triangle RFE$ are directly similar.

Quick Reply

High School Olympiads

4 Squares

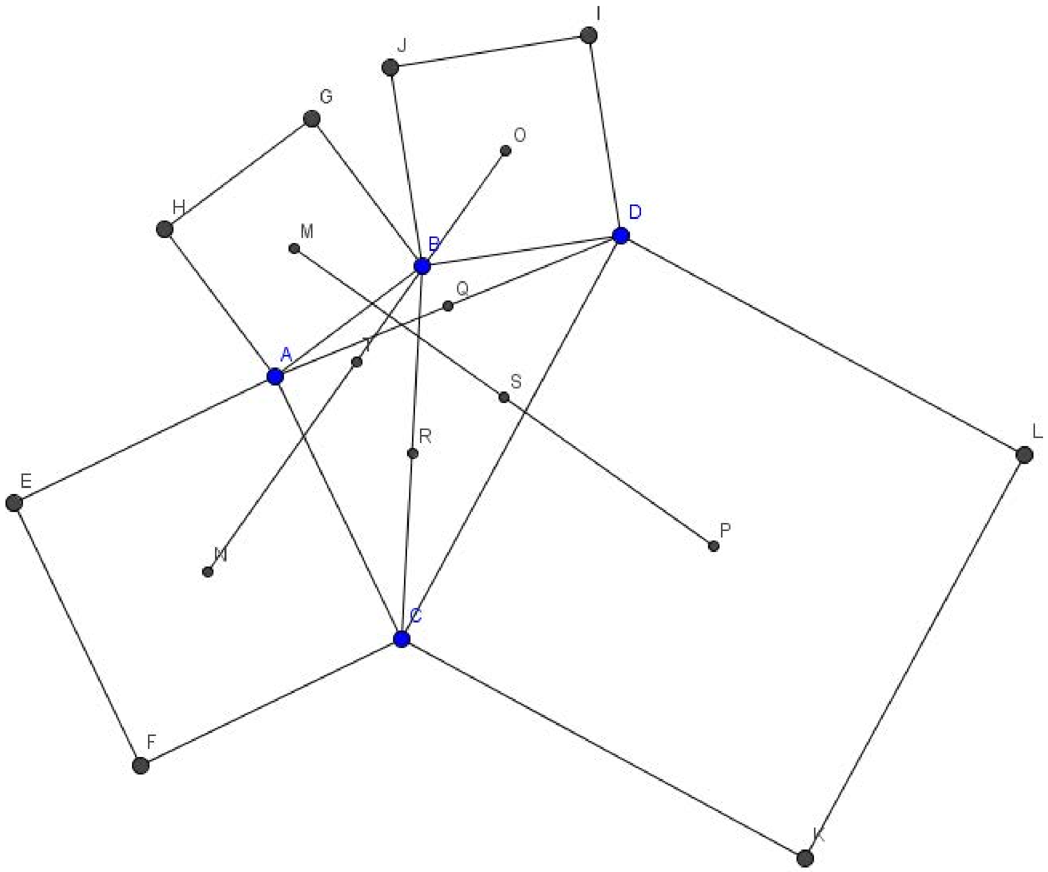
Reply

Emirhan

#1 Jan 24, 2016, 10:04 pm

Let $ABCD$ a convex quadrangle. Construct a square each side. Merge opposite square's centers.Prove that these line segment's midpoints and midpoint of $ABCD$'s diagonals specify a square.

Attachments:



Imgoodinmaths_dude

#2 Jan 24, 2016, 11:16 pm

Seems a bit hard

Luis González

#3 Jan 24, 2016, 11:25 pm

See http://www.artofproblemsolving.com/community/c6h1062591 for a proof of this problem and its generalization.

mjuk

#4 Jan 24, 2016, 11:40 pm

> *Imgoodinmaths_dude wrote:*
> Seems a bit hard

This can be bashed with complex numbers using only basic ideas like vector rotation.

mjuk

#5 Jan 25, 2016, 12:13 am

Complex numbers bash:
I will use notation as in diagram.
Assign complex number x to point X.
We get L by rotating C around D counterclockwise by $\pi/2$, so:
$l-d=i(c-d)$, and similar for other points.
So we get:
$l=ic-id+d$
$j=id-ib+b$
$h=ib-ia+a$
$f=ia-ic+c$

$p=(c+l)/2=(c+ic-id+d)/2$
$o=(d+j)/2=(d+id-ib+b)/2$
$m=(b+h)/2=(b+ib-ia+a)/2$
$n=(a+f)/2=(a+ia-ic+c)/2$

$t=(o+n)/2=(a+b+c+d+ia+id-ib-ic)/4$
$s=(p+m)/2=(a+b+c+d+ib+ic-ia-id)/4$
$q=(a+d)/2$
$r=(b+c)/2$

$t-s=i(a+d-b-c)/2=i(q-r)\rightarrow QR=ST$ and $QR\perp ST$
$t-q=(-a-d+b+c+ia+id-ib-ic)/4=r-s\rightarrow TQ=RS$ and $TQ\parallel RS$
$t-r=(a+d-b-c+ia+id-ib-ic)/4=q-s\rightarrow TR=QS$ and $TR\parallel QS$
From this we conclude that $RSQT$ is a parallelogramm with equal perpendicular diagonals, so it has to be a square.

This post has been edited 2 times. Last edited by mjuk, Jan 25, 2016, 1:15 am
Reason: uz

Quick Reply

High School Olympiads

Isosceles,similar triangle play the midpoint make an Rhombus

Reply

Nimplesy

#1 Mar 13, 2015, 12:11 pm

Let $ABCD$ be a convex quadrilateral. Let ABE,BCF,CDG,DAH are isoceles and similar triangles where AB, BC, CD, DA are the base of all the triangles. Let W, X, Y, Z be midpoint of AC, BD, EG, FH respectively. Prove that $WXYZ$ is a rhombus

Luis González

#2 Mar 13, 2015, 12:36 pm

Assuming that all triangles are constructed outwardly or inwardly, then W, X, Y, Z are in fact vertices of a square. See Squares Constructed on Sides of Quadrilateral, $\triangle WEG$ is the image of $\triangle WFH$ under rotation with center W and rotational angle 90°. This rotation takes Z into Y, thus $\triangle WYZ$ is isosceles right at W and by similar reasoning $\triangle XYZ$ is isosceles right at $X \Longrightarrow XYWZ$ is a square.

Nimplesy

#3 Mar 13, 2015, 6:19 pm

@Luis González this hold only if the triangle drawn is isosceles and RIGHT similar triangle but at my problem only hold for isosceles and similar triangle and i think it doesnot hold can you give explanation for me? Thank you

TelvCohl

#4 Mar 15, 2015, 8:32 pm • 1

My solution:

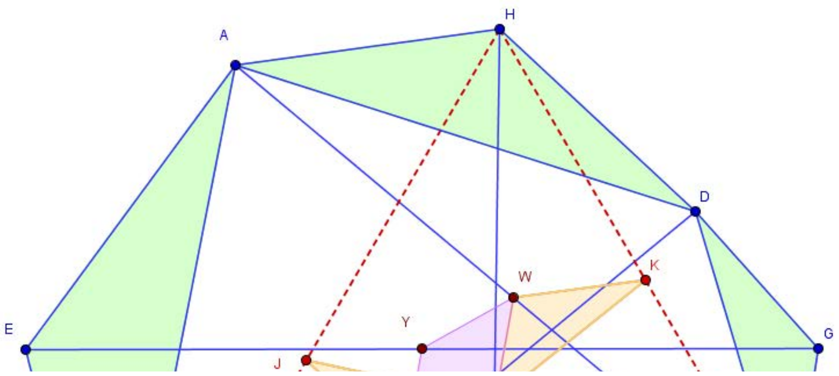
Let J, K be the midpoint of HB, HC, respectively .
Let $\angle BEA = \angle CFB = \angle DGC = \angle AHD = \theta$.

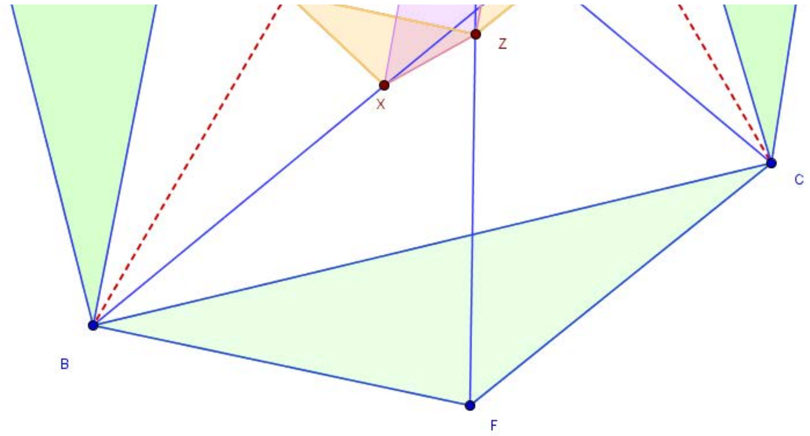
Since $JX = \frac{1}{2}HD = \frac{1}{2}HA = KW, JZ = \frac{1}{2}FB = \frac{1}{2}FC = KZ$,
so combine with $\angle XJZ = \angle(HD, FB) = \angle(HA, FC) = \angle WKZ \Longrightarrow \triangle JXZ \cong \triangle KWZ$,
hence we get $ZX = ZW$ and $\angle WZX = \angle KZJ = \angle CFB = \theta$.

Similarly we can prove $YX = YW$ and $\angle XYW = \theta$,
so we get $WXYZ$ is a rhombus and $\angle XYW = \angle WZX = \theta$.

Q.E.D

Attachments:





Quick Reply

High School Olympiads

Passes through tangent point

Source: Own

buratinogigle
#1 Jan 1, 2016, 1:14 pm • 7

Let ABC be a triangle inscribed in circle (O) with A-excircle (J). Circle passing through A,B touches (J) at M. Circle passing through A,C touches (J) at N. BM cuts CN at P. Prove that AP passes through tangent point of A-mixtilinear incircle with (O).

Happy new year 2016!

huynguyen
#2 Jan 2, 2016, 2:55 pm

Up.I hope to see a solution to this nice problem.

buratinogigle
#5 Jan 3, 2016, 2:03 am • 3

Using variation problem for incircle as following

Let ABC be a triangle with incircle (I) touches BC at D. Circle passing through A,B touches (I) at M. Circle passing through A,C touches (I) at N. BM cuts CN at P. Prove that $\angle PAB=\angle DAC$.

Lemma. Let ABC be a triangle and incircle (I,r) touches BC,CA,AB at D,E,F, resp. Circle passes through B,C and touches (I) at X then

i) $\dfrac{XE.XF}{XD^2}=\dfrac{r^2}{IB.IC}$.

ii) $\dfrac{XE}{XF}=\dfrac{IB.DE^2}{IC.DF^2}$

Consequence. $\dfrac{XE^2}{XD^2}=\dfrac{r^2.DE^2}{IB^2.DF^2}$ and $\dfrac{XF^2}{XD^2}=\dfrac{r^2.DF^2}{IC^2.DE^2}$.

Proof of problem. Let (I) touches CA,AB at E,F and X,Y,Z is cevian triangle of P. Let BY cuts DF at K. We have

$$\frac{YC}{YA}=\frac{[YBC]}{[YBA]}=\frac{[YBC]}{[KBD]}\cdot\frac{[KBD]}{[KBF]}\cdot\frac{[KBF]}{[YBA]}=\frac{BC.BY}{BD.BK}\cdot\frac{KD}{KF}\cdot\frac{BF.BK}{BY.BA}=\frac{BC}{BA}\frac{MD^2}{MF^2}=\frac{BC}{BA}\cdot\frac{r^2.DF^2}{EF^2.IA^2}.$$

Similarly, $\dfrac{ZB}{ZA}=\dfrac{BC}{CA}\cdot\dfrac{r^2.DE^2}{EF^2.IA^2}$.

Now by Ceva's theorem then $\dfrac{XB}{XC}=\dfrac{YA}{YC}\cdot\dfrac{ZB}{ZA}=\dfrac{AB.DE^2}{AC.DF^2}$.

So $\dfrac{DB}{DC}\cdot\dfrac{XB}{XC}=\dfrac{p-b}{p-c}\cdot\dfrac{AB.DE^2}{AC.DF^2}=\dfrac{DF.IB}{DE.IC}\cdot\dfrac{AB.DE^2}{AC.DF^2}=\dfrac{AB^2}{AC^2}$. Thus AD, AX are isogonal.

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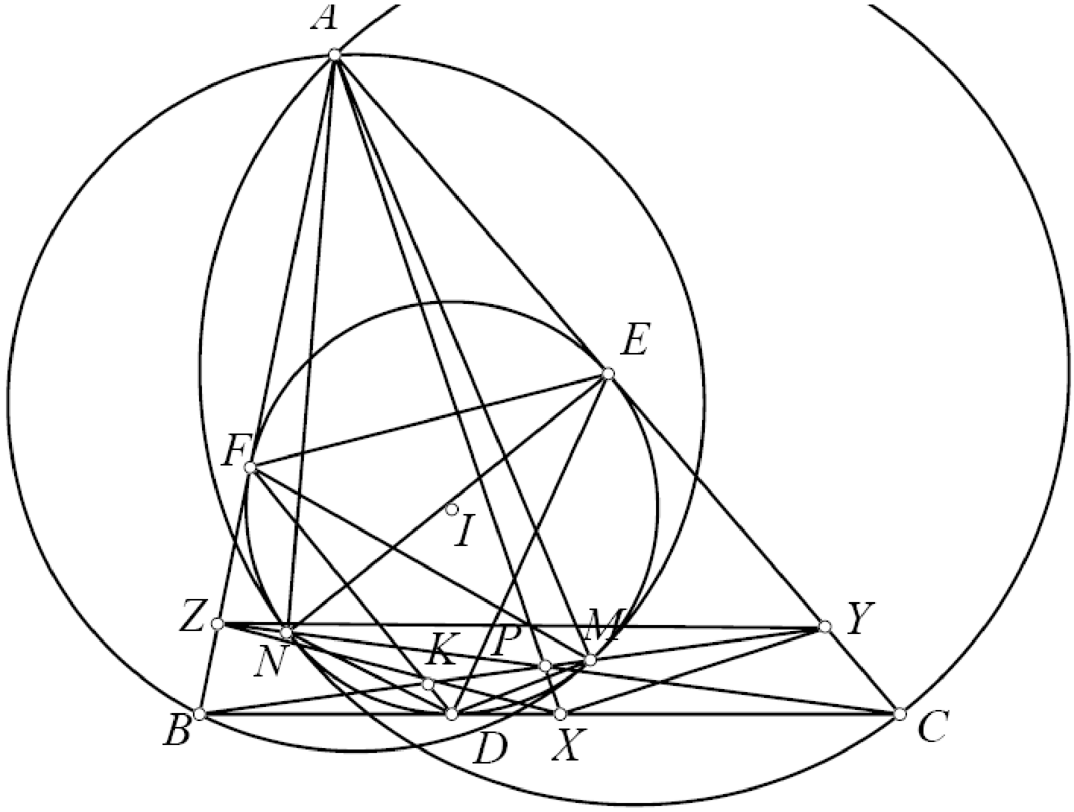


buratinogigle

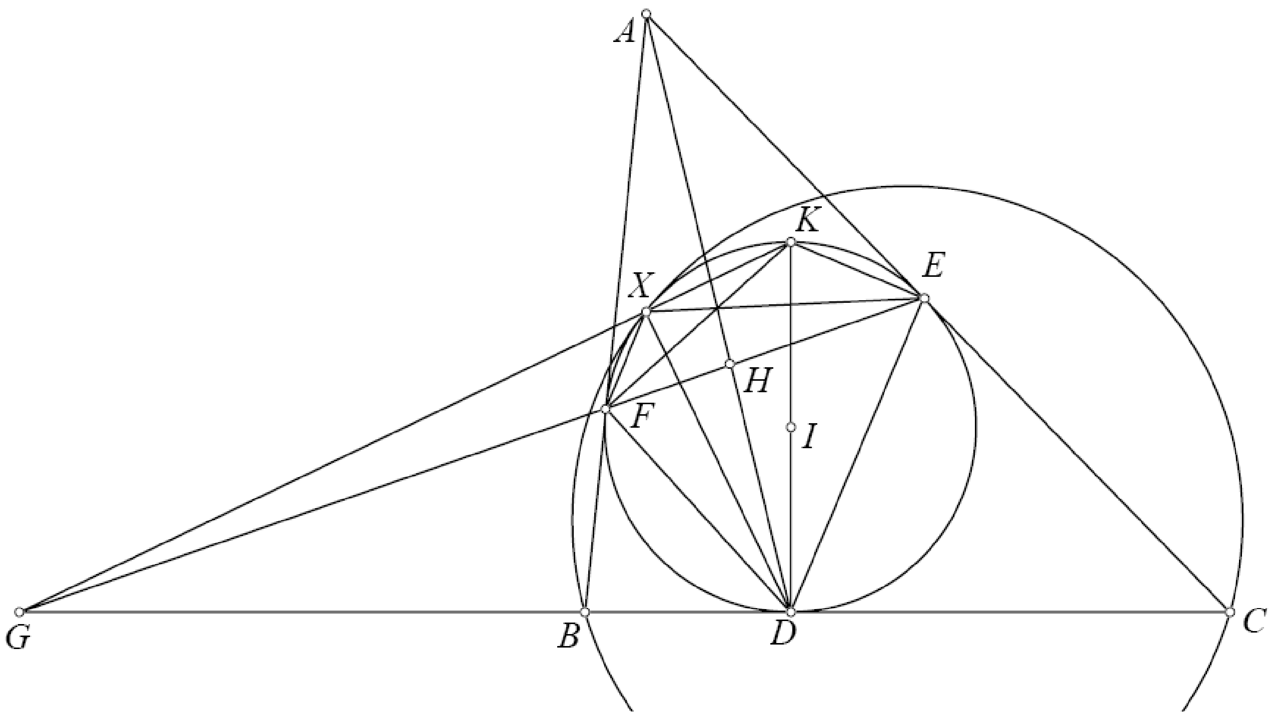
#6 Jan 3, 2016, 8:33 am • 2

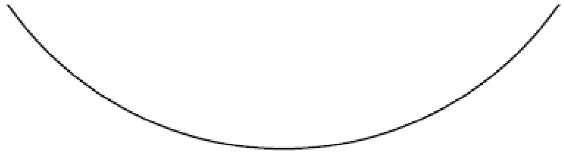
Proof of lemma. Follow topic IMO ShortList 2002, geometry problem 7, let DK is diameter of (I) then XK, EF, BC are concurrent at G. AD cuts EF at H.

We have $\frac{XE}{XF}.\frac{KE}{KF}=\frac{GE}{GF}=\frac{HE}{HF}=\frac{[AED]}{[AFD]}=\frac{DE}{DF}.\frac{DB}{IC}.\frac{IB}{DC}$ and note that $KE.IC=2r^2=KF.IB$. So $\frac{XE}{XF}=\frac{IB.DE^2}{IC.DF^2}$.

And $\frac{FX}{FK}.\frac{EX}{EK}=\frac{GX}{GK}=\frac{GD^2}{GK^2}=\frac{XD^2}{4r^2}$ and note that $KE.IC=2r^2=KF.IB$. So $\frac{XE.XF}{XD^2}=\frac{r^2}{IB.IC}$.

Attachments:





buratinogigle

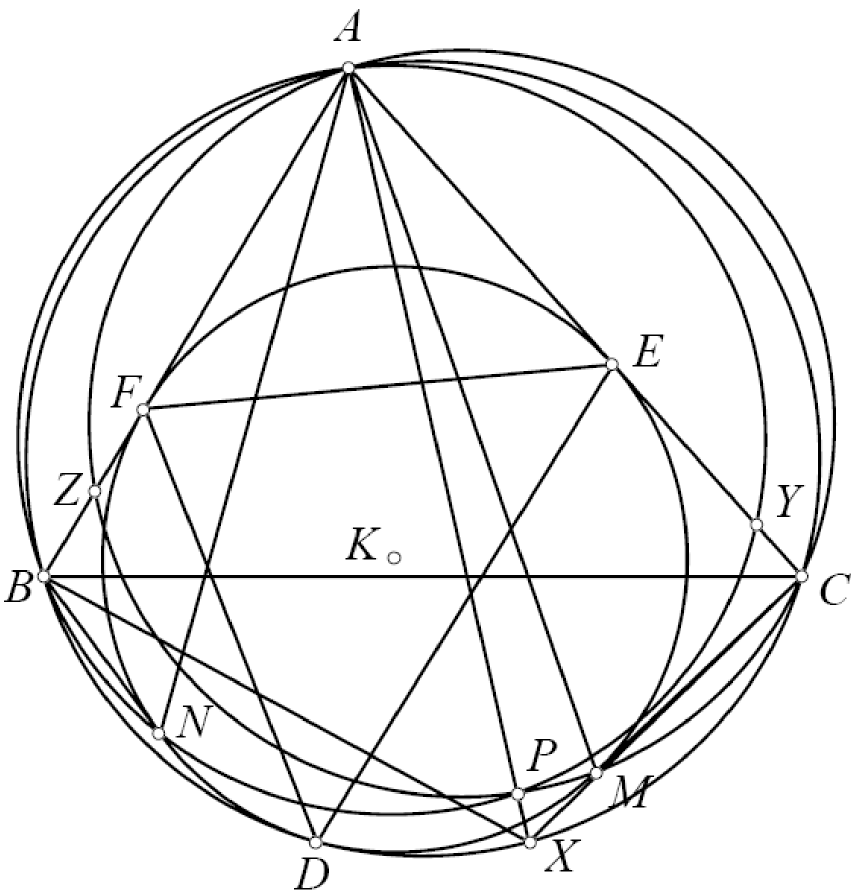
#7 Jan 3, 2016, 8:42 am • 3

Solution of original problem by inversion. Using $\sqrt{bc}$ inversion we need to prove the problem

Let ABC be a triangle inscriebd in circle (O). Circle (K) touches CA, AB and (O) internally. Draw tangent line CM, BN to (K) with M, N lie on (K). Circles $(ABN), (ACM)$ intersect again at P. Prove that AP passes through Nagel point.

Proof. Let (ABN) cuts CA again at Y and (ACM) cuts AB again at Z. AP cuts (O) again at X. We easily seen $\frac{XB}{XC} = \frac{ZB}{YC}$. But by inversion, we can prove $\frac{ZB}{YC} = \frac{DE^2}{DF^2}$. So $\frac{XB}{XC} = \frac{DE^2}{DF^2}$. From this, we can prove that AP passes through Nagel point.

Attachments:



huynguyen

#8 Jan 3, 2016, 8:46 am

Thank you, mr.buratinogiggle, for the lemma and the solution. 🙂

mihaith

#9 Jan 3, 2016, 1:16 pm • 1

And how we get those ratios in the inversion proof, dear **buratinogigle** ?

This post has been edited 5 times. Last edited by mihaith, Jan 26, 2016, 11:34 pm buratinogigle

#10 Jan 3, 2016, 2:06 pm • 2

Dear mihaith, actually I get it from the above lemmas before I use inversion.

mihaith

#11 Jan 4, 2016, 7:27 pm

> buratinogigle wrote:
> Dear mihaith, actually I get it from the above lemmas before I use inversion.

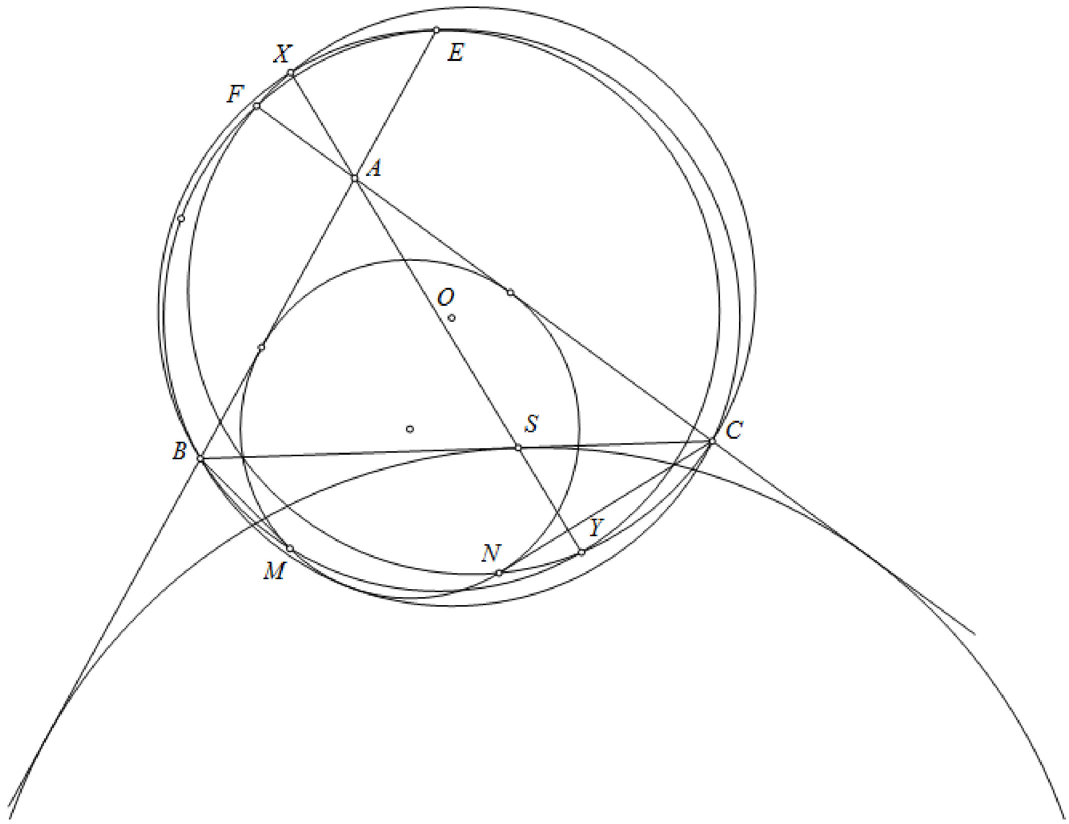
I understand, thank you.

livetolove212

#12 Jan 5, 2016, 10:45 pm

Generalization of the inversion version:
Given triangle ABC. Let (O) be an arbitrary circle passing through B, C and cuts AB, AC at E and F, respectively. Let ω be circle internally tangent to (O) and tangent to rays AB and AC. Construct tangents BM and CN to ω. The A-excircle of triangle ABC touches BC at S. Then AS is the radical axis of (EBM) and (FCN).

Attachments:



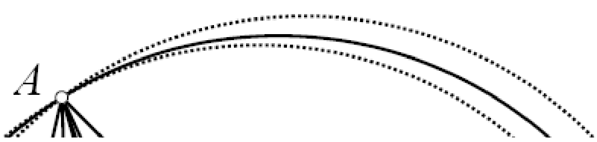
buratinogigle

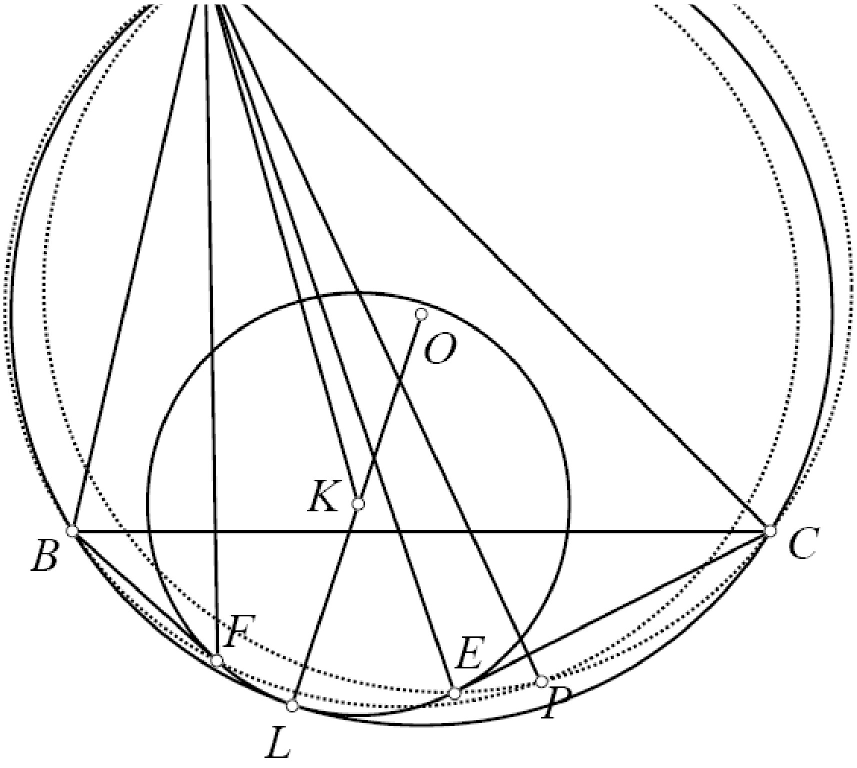
#16 Jan 6, 2016, 7:08 pm

Another way to generalize the version created by inversion

Let ABC be a triangle inscribed in circle (O). K is a point on bisector of angle A and circle (K) touches (O) internally at L. E, F lie on (K) such that CE, DF are tangents of (K) with E, F and A are not the same side with respect to BC. Circumcircle of triangle ACE and ABF intersect again at P. Prove that $\angle PAC = \angle LAB$.

Attachments:





TelvCohl

#17 Jan 19, 2016, 5:43 pm • 2

> buratinogigle wrote:
> Using variation problem for incircle as following
>
> Let ABC be a triangle with incircle (I) touches BC at D. Circle passing through A, B touches (I) at M. Circle passing through A, C touches (I) at N. BM cuts CN at P. Prove that $\angle PAB = \angle DAC$.

Let $\odot(I)$ touches CA, AB at E, F, respectively and let $R \equiv EF \cap BC$. Let $\odot(BTC)$ be the circle passing through B, C and tangent to $\odot(I)$ at T. Let $J \equiv DT \cap EN \cap FM$ be the perspector of the intouch triangle of $\triangle ABC$ and the excentral triangle of $\triangle ABC$ (X_{57} of $\triangle ABC$ (well-known)). From Steinbart's theorem we get AT, BN, CM are concurrent at K, so from the dual of Desargue involution theorem (for $BMCN$) $\Longrightarrow$ there is an involution Φ that swaps $(AB, AC), (AK, AP), (AM, AN)$. On the other hand, it's easy to see that the pole L of MN WRT $\odot(I)$ lies on AJ, so from the dual of Desargue involution theorem (for $LMLN$) $\Longrightarrow AL$ is fixed under Φ.

Let AL, AP, AK cuts $\odot(ABC)$ again at X, Y, Z, respectively. From the discussion above $\Longrightarrow XS$ is tangent to $\odot(ABC)$ at X where $S \equiv BC \cap YZ$, so S is the midpoint of DR (well-known property of X_{57}). From $(B, C; D, R) = -1$ we get $SD^2 = SB \cdot SC$, so S lies on the radical axis of $\odot(I)$ and $\odot(BTC) \Longrightarrow ST$ is tangent to $\odot(BTC)$ at T.

Let $V \equiv AT \cap BC$. Since the isogonal conjugate of TV WRT $\angle BTC$ passes through the tangency point of the A-excircle of $\triangle ABC$ with BC which is the isotomic conjugate of D WRT B, C (well-known), so $BV : CV = BD^3 : CD^3 \Longrightarrow$

$$Y(\infty, S; B, C) = \frac{CS}{BS} = \left(\frac{CT}{BT}\right)^2 = \left(\frac{CD}{BD}\right)^2 = \frac{BD}{CD} \cdot \left(\frac{CD}{BD}\right)^3 = \frac{BD}{CD} \cdot \frac{CV}{BV} = (D, V; B, C),$$

hence if AD cuts $\odot(ABC)$ again at U then $YU \parallel BC \Longrightarrow \angle PAB = \angle DAC$.

TelvCohl

#18 Jan 20, 2016, 10:08 am • 1

> buratinogigle wrote:
> Using variation problem for incircle as following
>
> Let ABC be a triangle with incircle (I) touches BC at D. Circle passing through A, B touches (I) at M. Circle passing through A, C touches (I) at N. BM cuts CN at P. Prove that $\angle PAB = \angle DAC$.

Another way to finish the proof (without calculation) **:**

Let $\odot(I)$ touches CA, AB at E, F, respectively and let $R \equiv EF \cap BC$. Let $\odot(BTC)$ be the circle passing through B, C and tangent to $\odot(I)$ at T. Let $J \equiv DT \cap EN \cap FM$ be the perspector of the intouch triangle of $\triangle ABC$ and the excentral triangle of $\triangle ABC$ (X_{57} of $\triangle ABC$ (well-known)). From Steinbart's theorem we get AT, BN, CM are concurrent at K, so from the dual of Desargue involution theorem (for $BMCN$) $\Longrightarrow$ there is an involution Φ that swaps $(AB, AC), (AK, AP), (AM, AN)$. On the other hand, it's easy to see that the pole L of MN WRT $\odot(I)$ lies on AJ, so from the dual of Desargue involution theorem (for $LMLN$) $\Longrightarrow AL$ is fixed under Φ.

Let AL, AP, AK cuts $\odot(ABC)$ again at X, Y, Z, respectively. From the discussion above $\Longrightarrow XS$ is tangent to $\odot(ABC)$ at X where $S \equiv BC \cap YZ$, so S is the midpoint of DR (well-known property of X_{57}). From $(B, C; D, R) = -1$ we get $SD^2 = SB \cdot SC$, so S lies on the radical axis of $\odot(I)$ and $\odot(BTC) \Longrightarrow ST$ is tangent to $\odot(BTC)$ at T.

Let $Q \equiv AT \cap \odot(BTC), V \equiv AT \cap BC$. It's well-known that there is a circle tangent to AB, AC and tangent to $\odot(BTC)$ at Q, so QI is the bisector of $\angle CQB$ (see incenter of triangle) $\Longrightarrow QI$ passes through the midpoint of arc BC in $\odot(BTC)$, hence notice TD is the bisector of $\angle BTC$ we get D, I, Q, S, T are concyclic (Reim theorem). Since $VA \cdot VZ = VB \cdot VC = VT \cdot VQ = VD \cdot VS$, so A, D, S, Z are concyclic, hence we conclude that

$$\measuredangle BAP = \measuredangle BZS = \measuredangle AZS - \measuredangle AZB = \measuredangle ADB - \measuredangle ACB = \measuredangle DAC.$$

Luis González

#19 Jan 24, 2016, 9:49 am • 3

Generalization: The incircle (I) of $\triangle ABC$ touches BC, CA, AB at D, E, F and let $\mathcal{H}_A$ be the conic through A, D, E, F and the isogonal $T \equiv X_{57}$ of the Mittenpunkt of $\triangle ABC$. J is an arbitrary point on $\mathcal{H}_A$ and JF, JE cut (I) again at N, M. If $P \equiv BM \cap CN$, then AP, AD are isogonals WRT $\angle BAC$.

Lemma: $\triangle ABC$ is acute and $\triangle A'B'C'$ is its tangential triangle. ω_A is the circle pasing through B', C' tangent to $\odot(ABC)$ at V. Then AV is the isogonal of the isotomic of the A-altitude of $\triangle ABC$.

Let M, N, L be the midpoints of BC, CA, AB and let D be the foot of the A-altitude. Inversion WRT $\odot(ABC)$ takes B', C' into N, L, hence by conformity ω_A is transformed into the circle through N, L and tangent to $\odot(ABC)$ at V. From IMO Shortlist 2011 (G4), VD passes through the centroid G of $\triangle ABC$. Thus if $U \equiv VDG \cap NL$, then $(UM \parallel AD) \perp BC$, so by symmetry AU is the isotomic of AD; reflection of UD WRT UM. Thus by obvious symmetry AU cuts $\odot(ABC)$ again at the reflection of V across $UM \Longrightarrow AU$ is the isogonal of AV.

Back to the main problem. Let $\triangle D'E'F'$ be the antimedial triangle of $\triangle DEF$. The circle through B, C tangent to (I) touches this at $T_A \Longrightarrow T \in DT_A$ (well-known), so applying the previous lemma in the acute $\triangle DEF$, we deduce that T is the isogonal conjugate of the isotomic conjugate of the orthocenter of $\triangle DEF$, i.e. isogonal conjugate of the symmedian point K' of $\triangle D'E'F'$ WRT $\triangle DEF$. Thus since D' is the isogonal conjugate of A WRT $\triangle DEF$, then it follows that $D'K'$ is the isogonal of $\mathcal{H}_A$ WRT $\triangle DEF$. As $D'K'$ is parallel to the D-symmedian DA of $\triangle DEF$, then $\mathcal{H}_A$ cuts (I) again at X, such that DA, DX are isogonals WRT $\triangle DEF$, i.e. DX is the D-median of $\triangle DEF \Longrightarrow AX$ is the isogonal of AD WRT $\angle BAC$.

On the other hand let $Q \equiv FN \cap EM$. Then $F(E, D, X, N) = E(A, D, X, N)$ and similarly $E(F, D, X, M) = F(A, D, X, M)$. But $E(A, D, X, N) = E(A, D, X, J) = F(A, D, X, J) = F(A, D, X, M) \Longrightarrow F(E, D, X, N) = E(F, D, X, M)$ $\Longrightarrow D, X, Q$ are collinear, i.e. $\triangle XMN$ is the circumcevian triangle of Q WRT $\triangle DEF$, so by Steinbart theorem $P \in AX$ $\Longrightarrow AP, AD$ are isogonals WRT $\angle BAC$.

Quick Reply

High School Olympiads

IMO Shortlist 2011, G4

Reply

Source: IMO Shortlist 2011, G4

WakeUp

#1 Jul 13, 2012, 5:41 pm • 3

Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G and X are collinear.

Proposed by Ismail Isaev and Mikhail Isaev, Russia

malcolm

#2 Jul 13, 2012, 5:47 pm • 6

Let Γ, O be the circumcircle and circumcenter of $\triangle ABC$, and let A_C be the midpoint of BC. Let XB_0, XC_0 meet Γ again at B', C', and let $E = BB' \cap CC'$. Pascal's Theorem on hexagon $ABB'XC'C$ gives $E \in B_0C_0$. Since the dilation carrying ω to Γ carries B_OC_0 to $B'C'$, $B'C' \parallel B_0C_0 \parallel BC$. Then $B'C'BC$ is an isosceles trapezium, so E is the foot of the perpendicular of A_C on B_0C_0. The dilation centered at G taking $\triangle ABC$ to $\triangle A_0B_0C_0$ takes D to E, so D, G, E are collinear. It suffices to show $B'B_0, C'C_0, ED$ are concurrent. Since lines BB_0, CC_0, ED concur at G, $\triangle BCE$ and $\triangle B_0C_0D$ are perspective. Let $U = BE \cap B_0D$ and $V = CE \cap C_0D$. Desargue's Theorem implies $UV \parallel B_0C_0$. By Desargue's Theorem again, this implies $\triangle B'C'E$ and $\triangle B_0C_0D$ are perspective, so $B'B_0, C'C_0, ED$ are concurrent as desired.

daniel73

#3 Jul 13, 2012, 5:56 pm • 3

Alternative solution:

Click to reveal hidden text

prime04

#4 Jul 13, 2012, 8:42 pm • 8

solution by inversion

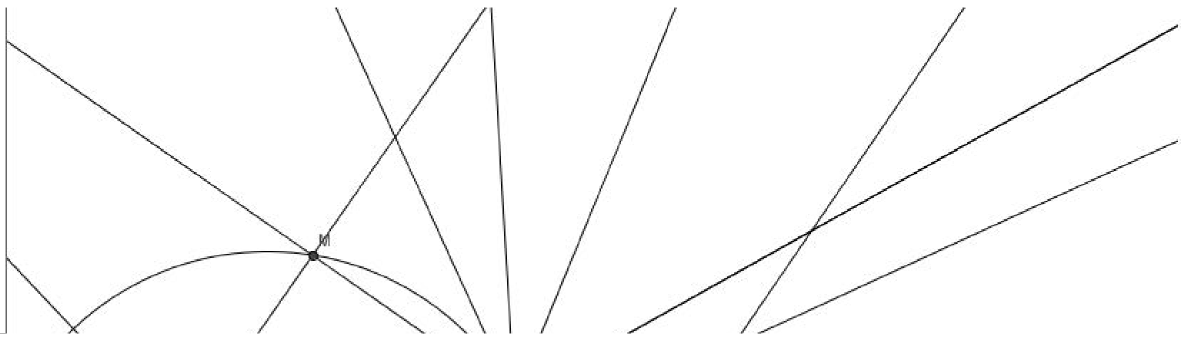
This post has been edited 6 times. Last edited by prime04, Jul 17, 2012, 8:53 pm

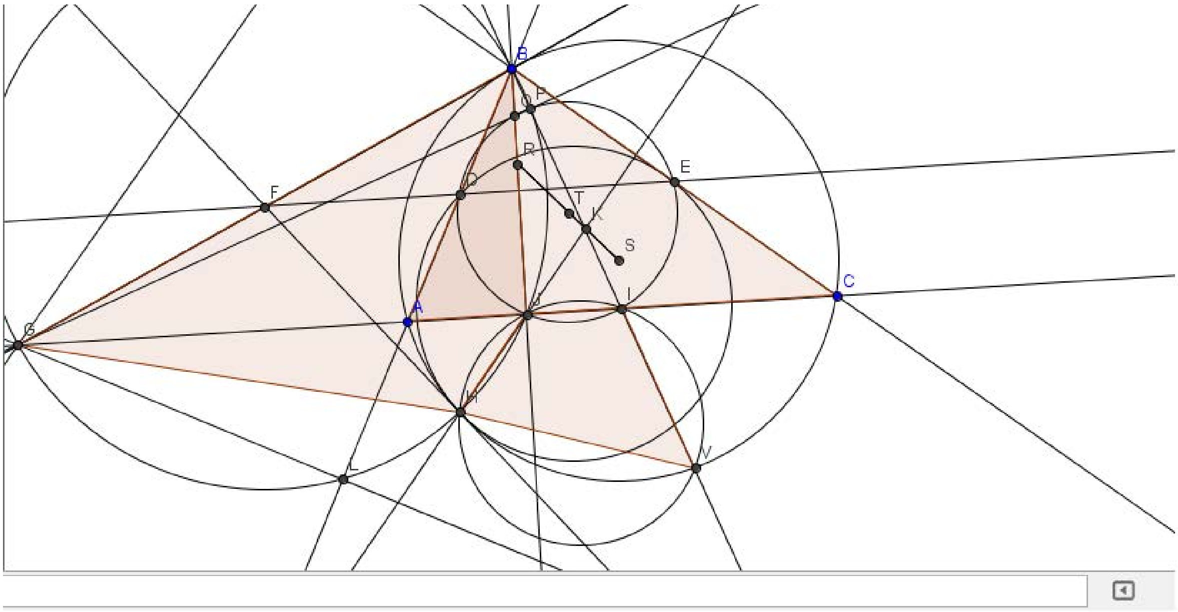
paul1703

#5 Jul 13, 2012, 10:23 pm • 1

My solution interprets DX as a radical axis, the power of G is easy to calculate (no trigo).

Attachments:





Goutham

#6 Jul 14, 2012, 12:23 am • 2

I believe my solution is along similar lines to daniel's solution but I didn't read that fully...

Let T be on BC such that AT tangents cirumcircle of ABC. Now by radical axis theorem on ABC, AB_0C_0, B_0C_0X, their radical centre is midpoint of AT, call it as S. Let γ be circle TAD with centre S. Since $SA \perp AO$ where O is cirumcentre of ABC, ω and γ are orthogonal. Also, SX is tangent to ω and so, γ passes through X. Now by Reim's theorem on circles γ, ω with lines XD and AP, we have that if XD intersects ω at U, then $AU \parallel BC$. The foot of the altitude on the antimedial triangle from vertex corresponding to A lies on the nine point circle ω and is also the image of the point D under a homothety with centre G and so, we see that that point is U. This proves the result.

Zhero

#7 Jul 14, 2012, 1:06 am • 2

Let G' and X' be the reflections of G and X across B_0C_0, respectively, let $T = AA \cap B_0C_0 \cap XX$ be the radical center of (ABC), (AB_0C_0), and (B_0C_0X), let AG' meet B_0C_0 at P, let O be the center of (ABC), let D' be where AD meets B_0C_0, let $A_1B_1C_1$ be the medial triangle of $\triangle A_0B_0C_0$, let A' be the reflection of A across the perpendicular bisector of B_0C_0, and let G_0 be the centroid of AB_0C_0.

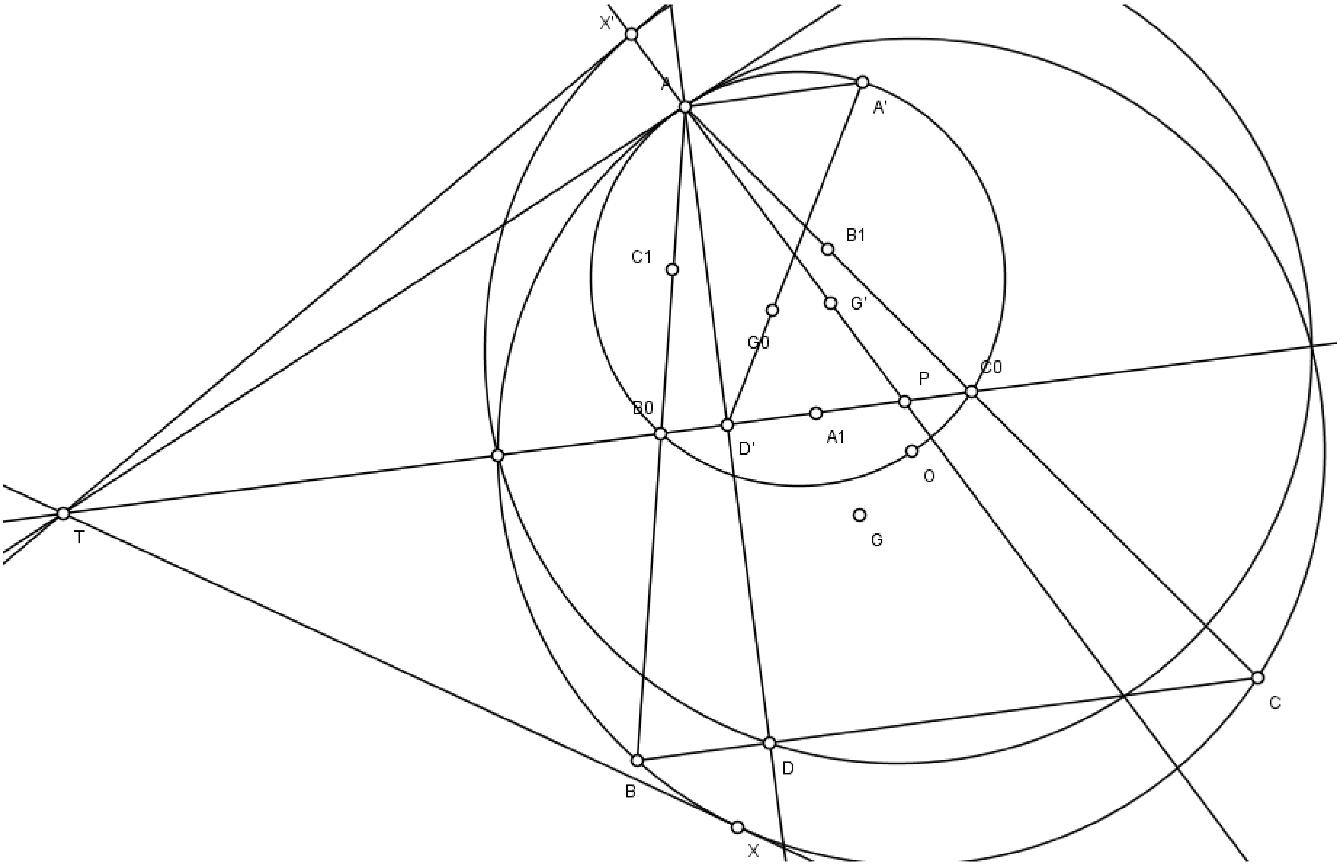
It is easy to see that G' is the reflection of G_0 across the perpendicular bisector of B_0C_0. We first claim that $OP \perp B_0C_0$. Now, since D' is the foot of the altitude from A to B_0C_0, $D'A_1B_1C_1$ is an isosceles trapezoid. Since a homothety centered at G_0 with factor -2 sends $A_1B_1C_1$ to AB_0C_0, and $A'AB_0C_0$ is an isosceles trapezoid, this homothety must send D' to A' as well, so A', G_0, and D' are collinear. Reflecting these points across the perpendicular bisector of B_0C_0 shows that $B_0D' = PC_0$. Since A and O are antipodes with respect to (AB_0C_0), $AD' \perp B_0C_0$, and $B_0D' = PC_0$, $OP \perp B_0C_0$. Since $OA \perp B_0C_0$, quadrilateral $OPAT$ must be cyclic. Thus, $\angle APT = \angle AOT$. We thus have

$$\begin{aligned}
\angle G'AT &= \angle B_0AT + \angle G'AB_0 = \angle B_0C_0A + \angle PAB_0 \\
&= \angle C + (180^\circ - \angle AB_0P - \angle B_0PA) \\
&= 180^\circ + \angle C - \angle ABC - \angle TPA = 180^\circ + \angle C - \angle B - \angle TOA \\
&= 90^\circ + \angle C - \angle B + \angle ATO = 90^\circ + \angle C - \angle B + \frac{\angle ATX}{2} \\
&= 90^\circ + \angle C - \angle B + \frac{\angle ATB_0 + \angle B_0TX}{2} \\
&= 90^\circ + \angle C - \angle B + \frac{\angle ATB_0 + \angle X'TB_0}{2} \\
&= 90^\circ + \angle C - \angle B + \frac{\angle ATB_0 + \angle X'AT + \angle ATB_0}{2} \\
&= 90^\circ + \angle C - \angle B + \angle ATB_0 + \frac{\angle X'TA}{2} \\
&= 90^\circ + \angle C - \angle B + 180^\circ - \angle TB_0A - \angle B_0AT) + 90^\circ - \angle TAX'
\end{aligned}$$

$$= 180^\circ - \angle TAX' + \angle C - \angle B + \angle AB_0C_0 - \angle AC_0B_0$$
$$= 180^\circ - \angle TAX' + \angle C - \angle B + \angle B - \angle C = 180^\circ - \angle TAX',$$

so X', A, and G' are collinear. Reflecting these points across B_0C_0 shows that X, D, and G are collinear, as desired.

Attachments:



v_Enhance

#8 Jul 14, 2012, 4:14 am • 7

This very short solution is from Evan o'Dorney:

Let R be intersection of B_0C_0 and tangents at A and X. Take E s.t. $AEBC$ is an isosceles trapezoid; well-known D, G and E are collinear. $RA = RD = RX \implies \angle AXD = \frac{1}{2}\angle ARD = \angle ARB_0 = \pi - \angle RAE = \angle AXE$, done.

r1234

#9 Jul 15, 2012, 8:39 pm • 2

Here are some facts I want to say about this problems and these will also lead to the solution.....

1. A_0, B_0, C_0 are the midpoints of the sides BC, CA, AB. Now we define the points Y, Z such that $\odot YA_0C_0$ us tangent to $\odot ABC$ at Y and similar for Z. Then the lines AX, BY, CZ concur at the isogonal conjugate of the isotomic conjugate of orthocente rof $\triangle ABC$.

2. Let D' be the isotomic point of D wrt BC. Then GD bisects AD'.

Proof of (1):- Inverting with respect to A with some power, B_0', C_0' becomes the reflection of A wrt B', C' respectively. The point X' is such that the line $B'C'$ is tangent to $\odot X'B_0'C_0'$ at X' Hence X' is the midpoint of the arc $B_0'C_0'$. So X' lies on the perpendicular bisector of $B_0'C_0'$. Hence X' is the reflection of the foot of A-altitude of the triangle $\triangle AB'C'$ wrt the midpoint of $B'C'$. So AX, BY, CZ concur at the isogonal conjugate of isotomic conjugate of the orthocenter of $\triangle ABC$.

Proof of (2):- Reflect G wrt the midpoint of BC. Let it be G'.Then clearly $G'D' \parallel GD$. Now G is the midpoint of AG'. So GD bisects AD'.

Main Proof:- Suppose the isogonal ray of AD' cuts $\odot ABC$ at D_2. And AD' cuts the circumcircle at D_1. Then clearly $D_1D_2 \parallel BC$.So the perpendicular bisector of D_1D_2 passes through the midpoint of BC which again passes through the midpoint of AD' (easy to see). So $D_2, D, \text{midpoint of} AD'$ are collinear. So D_1, D, G are collinear. Hence proven.

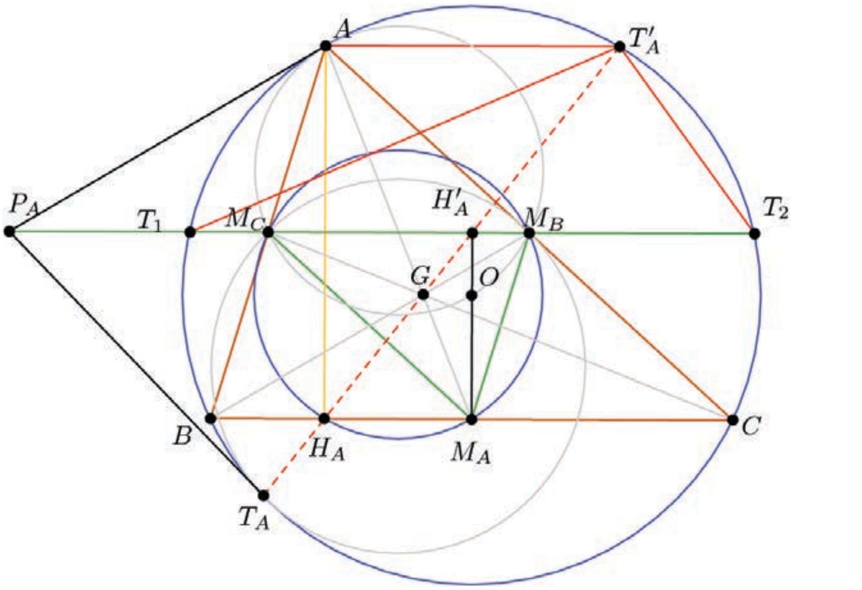
proglote

#10 Jul 16, 2012, 5:22 am • 5

Posted on my blog a while ago, didn't know it was from the shortlist.

http://www.artofproblemsolving.com/Forum/blog.php?u=112643&b=72551

$3.$ ► (ISL 2011 G4) Let $\mathcal{C}$ denote the circle passing through M_B, M_C tangent to the circumcircle ω of ABC at a point $T_A \neq A$. Then T_A, H_A, G are collinear.



Proof: Let $\mathcal{C}_2$ denote the circumcircle of AM_CM_B, which is also tangent to ω. Let P_A be the radical center of $\mathcal{C}, \mathcal{C}_2, \omega$, and let T_1, T_2 be the intersections of M_BM_C and ω, as above. Then since P_AA and P_AT_A are tangent to ω, $AT_1T_AT_2$ is a harmonic quadrilateral. Let T_A^* denote the second intersection of T'_AG and ω, and H'_A denote the image of H_A under the homothety $\mathcal{H}$ which sends $\triangle ABC$ to $\triangle M_AM_BM_C$. Then $M_AH'_A \perp T_1T_2$, so $OH'_A \perp T_1T_2$ and H'_A is the midpoint of T_1T_2. We have shown in (2) that $AT'_A \parallel T_1T_2$, so $T'_A(T_2, T_1; H'_A, A)$ is a harmonic pencil, which implies that $T'_A(T_2, T_1; T_A^*, A)$ is a harmonic pencil, and $AT_1T_A^*T_2$ is a harmonic quadrilateral $\implies T_A^* \equiv T_A$, as desired.

Orin

#11 Jul 25, 2012, 12:50 am • 4

Let $\odot AB_0C_0 = \alpha, \odot XB_0C_0 = \omega, M = AD \cap B_0C_0$.
Let $AD' \perp B_0C_0, D' \in B_0C_0$, O be the centre of Ω.
Let K be the intersection of the tangents to Ω from A, X.
AK is the radical axis of Ω, α; KX is the radical axis of Ω, ω and B_0, C_0 is the radical axis of α, ω.
So $K \in B_0C_0$.
$\angle OAK + \angle OXK = 90^\circ + 90^\circ = 180^\circ \to AKXO$ are cyclic.
A homothety of ratio -2 and centre G sends O to H, the orthocentre of ABC and $\triangle A_0B_0C_0$ to $\triangle ABC$.
So O is the orthocentre of $\triangle A_0B_0C_0$.
$\angle OD'K = \angle OAK = 90^\circ \to AKXOD'$ is cyclic.
$AB_0C_0 \cong A_0B_0C_0 \to AM = A_0D'$
$AM \perp B_0C_0, A_0D' \perp B_0C_0 \to AM \parallel A_0D'$
which implies AMA_0D' is a parallelogram.
Note that a homothety of ratio -2 and centre G sends D' to D.
Hence D, G, D' are collinear.
So it suffices to prove that X, D, D' are collinear.
$\angle KD'X = \angle KAX = \angle KXA = \angle KD'A = \angle AD'M$
$= \angle A_0MD' = \angle MA_0D = \angle MD'D$[The last arguement comes from the fact that MDA_0D' is a rectangle.]
which implies X, D, D' are collinear.

sjaelee

#12 Sep 8, 2012, 10:44 am • 1

Lemma: If DG intersects the circumcircle at E, then $ABCE$ is an isosceles trapezoid.

Proof

Now we note that the tangent from X, C_0B_0, and the tangent from A are concurrent at the radical center M of circles AB_0C_0, B_0C_0X, ABC. Then we note that $MX = MA$ from PoP and $MD = MA$ since C_0B_0 is the perpendicular bisector of AD. Then M is the center of circle ADX and $2\angle AXE = \angle DMA = 2\angle AXD$, thus XD and XE are equivalent lines and X, D, E are collinear, implying X, G, D are collinear.

MBGO

#13 Feb 27, 2013, 12:28 am

new elementary proof :

Let the circumcircles of XC_0B and XB_0C meet at point K different from X,because circle ω and circumcircle of ABC are tangent,it follows that K is on B_0C_0. Let XB_0 and XC_0 intersect the circumcircle of ABC again at M,N. We have B, K, M and C, K, N collinear, all else we should prove is K, G, X collinear, equivalent to proving that G lies on Radical axes of circumcircles of XB_0C and XC_0B. Let XB_0C and XC_0B meet lines BB_0 and CC_0 again at B_1,C_1 respectively,hence we should prove that BB_1C_1C is a cyclic quadrilateral,which is obvious as we have $\angle BB_1C = \angle CC_1B = 180 - \angle CBM = 180 - \angle BCN$.

and it's done

leader

#14 May 4, 2013, 11:23 pm

another way $E \in BC$ $CE = BD$ $R \in \odot ABC$ $\angle BAR = \angle CAE(R, A$ are on opposite sides of $BC)$ Y-midpoint of AE.Since $ARC \sim ABE$,$ARB \sim ACE$ and Y is on perp bisector of ED(and BC) then if RB_0, RC_0 meet $\odot ABC$ in B', C' then $\angle C'RB = \angle YCB = \angle YBC = \angle B'RC$ so $B'C'||BC||B_0C_0$ and $\odot RB_0C_0$ and $\odot ABC$ touch so $X = R$ but if AE meets $\odot ABC$ again at P then $XB = CP$ $BD = CE$ $\angle XBD = \angle PCE$ so $PCE \cong XBD$ and $\angle XDB = \angle PEC = \angle AED$ but G is centroid of ADE so $\angle AED = \angle GDE$ and G, D, X are collinear.

mathocean97

#15 May 23, 2013, 7:52 pm • 1

A less ingenious solution.

thecmd999

#16 Dec 21, 2013, 12:17 pm

Solution

sayantanchakraborty

#17 Jul 21, 2014, 9:51 pm

Let B', C' denote the points of intersections of ω with DB_0, DC_0.Then by Pascal's theorem on hexagon $ABB'XC'C$ we see that $P = BB' \cap CC' \in B_0C_0$.Also $B'C' \parallel BC \parallel B_0C_0$ since $\angle B_0C_0X = \angle B'C'X = \angle \omega, l$ where l is the tangent line through X.Thus $BC'B'C$ is an isosceles trapezoid $\Rightarrow A_0P \perp B_0C_0$.So the homothety with center G that carries Ω to ω takes D to $P \Rightarrow D, G, P$ are collinear.Now it is to note that $B_0C_0 \cap B'C' = \infty, DC_0 \cap PC', DB_0 \cap PB'$ are collinear so $\triangle DB_0C_0, \triangle PB'C'$ are perspective.So by Desargue's theorem $DP, C'C_0, B'B_0$ are concurrent,or,in other words D, G, X are collinear,as desired.

junioragd

#18 Aug 1, 2014, 3:21 am

I have a solution with appolonius circles.WLOG AC>BC.Now,let XBo and XCo meet the circle of ABC at B' and C'.Now,since circumsircles of XB'C' and XBoCo are tangent and X,Bo,B' and X,Co,C' are respectively colinear,we have BoCo//B'C' and BoB'*BoX=BBo*BBo and C'Co*CoX=CCo*CCo and XCo/XC'=XBo/XB',so we have XCo/XBo=DCo/DBo=AB/AC=ABo/ACo,so we have that X is on the appolonius circle of DBoCo.Now,let interal angle bisector of <BAC intersects BoCo at S.Now,we have that XDSA is a cyclic,and we have that <ASD = 2<BCA+<BAC(Angle chase and use the fact that D is the reflection of A over BoCo),so we obtain <DXA=180-<ASD=<ABC-<BCA.Now,let XD intersects the circumcircle of ABC at A'.Since <AXA'=<ACA',we obtain <BCA'A is an iscolles trapezoid.Now,we want to prove that A',D and G are collinear.Let Ao be the midpoint of BC.Since A,G and Ao are collinear,it will be enough to prove that AA'=2*DAo,but this is obvious,let A1 be the projection of A' on BC.It is easy

to see that DA1A'A is a rectangle and AAo=AA',(since BCA'A is an iscolles trapezoid),so we are done.

utkarshgupta

#19 Apr 5, 2015, 5:24 pm

> *thecmd999 wrote:*
> Solution

Can you elaborate how AX is the symmedian ???
Coz it doesn't look like it is

This post has been edited 1 time. Last edited by utkarshgupta, Apr 6, 2015, 7:56 pm

hayoola

#20 Aug 22, 2015, 12:09 pm

let the line GD intersect arc BC not contains A at p and arc BAC at L we khnow that G is the homogenous center so GL=2DG we know that AG=2GM so AL is parraled to DM and we find that angle BPD=angle B now we find that P is the out homogenous point of B0C0A0 and we find that the circle who passed from C0 and B0 and is tangant to the circume circele of ABC passed from P

EulerMacaroni

#21 Oct 2, 2015, 10:09 am

First we claim that DG intersects the circumcircle at $F \neq X$, and that F is the reflection of A over the perpendicular bisector of BC. Indeed, note that the reflection of F over the midpoint of BC is the reflection of A over D (call this point A'), so G is the centroid in $\triangle AFA'$.

Now, invert about A with radius $\sqrt{AB \cdot AC}$, composed with an additional reflection about the A-angle bisector. Since $AF \parallel BC$, F' is the intersection of the tangent from A in the circumcircle to BC. D' is the antipode of A in the circumcircle, and we claim that X' is the reflection of D about the perpendicular bisector of BC. It suffices to prove that $AX'D'F'$ is cyclic, but this is obvious, as $\angle F'AD' = \angle F'X'D'$. B_0' is the reflection of A over C and C_0' is the reflection of A over B, so we want to show that $(X'B_0'C_0')$ is tangent to BC, or that X' lies on the perpendicular bisector of $B_0'C_0'$. Taking a homothety with ratio $\frac{1}{2}$ at A, we want to show that the midpoint of AX' lies on the perpendicular bisector of BC , but this is obvious, as $AFDX'$ is a rectangle, hence the image of X' lies on the perpendicular bisector of DX', which is the same perpendicular bisector of BC.

This post has been edited 1 time. Last edited by EulerMacaroni, Oct 4, 2015, 8:01 am
Reason: forgot reflection

dothef1

#22 Oct 24, 2015, 5:57 pm

Let R be on (ABC) such that (AR) is parallel to (BC) ,
First note that :by homothety (centered at G taking B_0 to B) , D, G, R are collinear , and (AB_0C_0) is tangent to (ABC) .
Then , the weird tangeancy gives a radical axis approach , so we have that $(XX), (AA)$ and (B_0C_0) concur , so our new aim would be to prove that , If (DG) intersect (ABC) at M , and (B_0C_0) intersect (ABC) at P and Q (P closer to B_0 , and Q closer to C_0) then $APMQ$ is harmonic , in other words : (MA) is the $M-$symmedian of MPQ.
Then notice that : (AR) is parallel to (BC) , implies that (MA) and (MR) are isogonals wrt MPQ , so it is enough to prove that (MR) is the $M-$median of MPQ , but wait; (MR) is just (DG) , now take nother homothety centered at G , taking B to B_0 and we're done.

This post has been edited 1 time. Last edited by dothef1, Oct 24, 2015, 6:48 pm
Reason: edit

kapilpavase

#23 Mar 17, 2016, 1:42 pm

Let DG intersect B_0C_0 at Q and M be midpt of BC. By homothethy we see that $MQ \perp B_0C_0$. So
$(QD, QA; QB_0, QM) = -1$. Hence if $QD \cap \Omega = X', AX' \cap QM = R, AX' \cap B_0C_0 = S$ then
$(A, X'; S, R) = -1$ implying S, R are conjugate pts. So that if P is the intersection of tangent at A and B_0C_0 then as PS is perp to QR ie line passing through centre of Ω, it follows that P, R are conjugate too. Hence RA is polar line of P and hence
$X = X'$ by radical axis
Thm.

This post has been edited 2 times. Last edited by kapilpavase, Mar 17, 2016, 2:32 pm

kapilpavase

#24 Mar 17, 2016, 2:44 pm

Another fantanstic approach...
We have by radical axis thm that tangent to Ω at A, X and line B_0C_0 concur at say P. Also $PD=PA=PE$. This implies the existence of a circle ω tangent to Ω, PE at E and at D to line PD. Also we easily see that $\odot B_0C_0D$ is tangent to line PD and hence tangent to ω too at D. Now we are done by de monge, as G is the centre of negative homothethy mapping $\odot B_0C_0D$ to Ω, D is centre of negative homothethy maping $\odot B_0C_0D$ to ω and E maps ω to Ω by positive homothethy.
Done!!!

This post has been edited 1 time. Last edited by kapilpavase, Mar 17, 2016, 2:46 pm

Quick Reply

High School Olympiads

geometry

Reply

Source: Own

Dadgarnia

#1 Jan 23, 2016, 6:53 pm

Let BE and CF be altitudes of triangle ABC.A circle ω_1 passes through A and B in such a way that BC touches ω_1 and a circle ω_2 passes through A and C in such a way that BC touches ω_2. Let $P \equiv \omega_1 \cap AC$ and $Q \equiv \omega_2 \cap AB$. If $R \equiv PQ \cap BC$ and A' be the reflection of A with respect to the EF prove that $EF \parallel A'R$.

Luis González

#2 Jan 23, 2016, 7:48 pm • 2

Upon inverting the figure with center A and power $AE \cdot AC = AF \cdot AB$, the result follows from the problem Concyclic Quadrilateral. Notice that A' goes to the circumcenter O of $\triangle ABC$ and ω_1, ω_2 go to the lines joining the midpoint M of BC with F, E, resp. MF, ME cut then AC, AB at the inverses P', Q' of $P, Q \Longrightarrow$ inverse R' of R is the 2nd intersection of $\odot(AP'Q')$ and $\odot(AEF)$, which according to the given reference is on the circle ω_A with diameter OA. Since ω_A and (O) are obviously tangent, then their inverse lines RA' and EF are parallel.

Quick Reply

High School Olympiads

Concyclic Quadrilateral

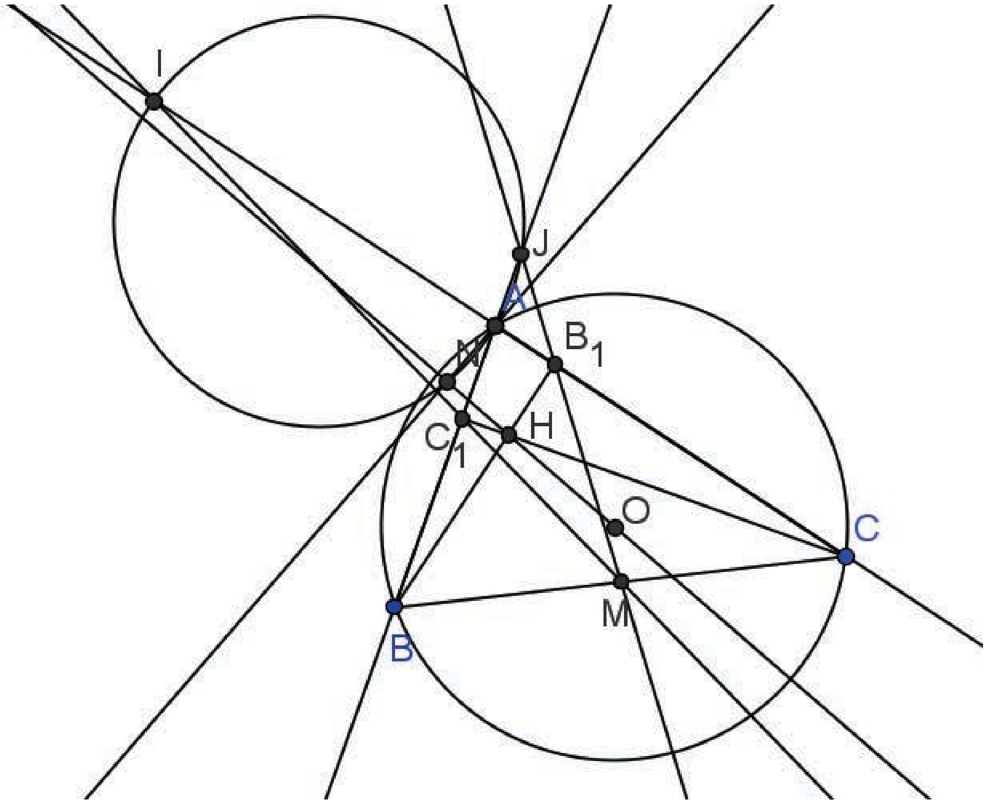
Reply

CTK9CQT

#1 Nov 3, 2012, 10:30 am

Given a triangle ABC and (O) be its circumcircle. B_1 and C_1 are feets of altitude from B and C, respectively. H is the orthocenter. Let M be the midpoint of BC. J is the intersection point of MB_1 and AB. I is the intersection point of MC_1 and AC. N is feet of A on OH. Prove that $ANIJ$ are cyclic.

Attachments:



Luis González

#2 Nov 4, 2012, 1:40 pm

If E,F are the midpoints of CA, AB, then N is clearly the 2nd intersection of $\odot(AEOF)$ and $\odot(AB_1HC_1)$. E,F,C_1,B_1 are on 9-point circle of $\triangle ABC$ and easy angle chase reveals that $\angle B_1IC_1=\angle FEC_1 \Longrightarrow \triangle IB_1C_1 \sim \triangle EFC_1$. Likewise, $\triangle JC_1B_1 \sim \triangle FEB_1$. Thus

$$\frac{IB_1}{EF}=\frac{B_1C_1}{FC_1}, \ \frac{JC_1}{FE}=\frac{C_1B_1}{EB_1} \Longrightarrow \frac{JC_1}{FC_1}=\frac{IB_1}{EB_1} \Longrightarrow$$

$$\frac{JC_1}{JF}=\frac{IB_1}{IE} \Longrightarrow \frac{JC_1\cdot JA}{JF\cdot JA}=\frac{IB_1\cdot IA}{IE\cdot IA}.$$

The latter expression means that the ratio of the powers of I and J WRT $\odot(AB_1C_1)$ and $\odot(AEF)$ are equal $\Longrightarrow I,J$ lie on a circle coaxal with $\odot(AB_1C_1)$ and $\odot(AEF) \Longrightarrow A,N,I,J$ are concyclic.

Quick Reply

High School Olympiads

Nice geometry

Reply

Re1gnover
#1 Jan 23, 2016, 12:31 am

Let ABC be a triangle which $AB+AC=3BC$. I is incenter of the triangle. (I) is tangent to BC, CA, AB at D, E, F, resp. K, L is symmetric of E, F through I, resp. The circle with diameter AI intersect (O) again at T. Prove that (LKT) is tangent to (O).

Luis González
#2 Jan 23, 2016, 1:25 am • 3

Let M be the midpoint of the arc BC of (O) and let EF cut BC at X; harmonic conjugate of D WRT B, C. In any $\triangle ABC$, the points M, D, T are collinear (see for instance incenter I and touches BC side with D for some proofs). Since TDM bisects $\angle BTC$ and $(B, C, D, X) = -1$, it follows that TX is external bisector of $\angle BTC \Longrightarrow \angle DTX = 90^\circ$. Since the circle (S) with diameter $\overline{DX}$ is orthogonal to (O), then the tangent of (O) at T hits DX at S.

Let $\odot(I_a, r_a)$ be the A-excircle of $\triangle ABC$. Then $\frac{AI}{AI_a} = \frac{r}{r_a} = \frac{s-a}{s} = \frac{1}{2} \Longrightarrow I$ is midpoint of $AI_a \Longrightarrow$ circles $\odot(IEAF)$ and $\odot(BICI_a)$ with diameters AI and II_a are symmetric WRT $I \Longrightarrow K$ and L lie on $\odot(BIC)$. Since (S) is orthogonal to both (I) and $\odot(BIC)$, then S is on their radical axis KL. Therefore $SK \cdot SL = SB \cdot SC = ST^2 \Longrightarrow \odot(LKT)$ is tangent to (O).

TelvCohl
#4 Jan 23, 2016, 5:39 pm • 1

Let $M \in AI$ be the midpoint of arc BC in $\odot(ABC)$ and let I_a be A-excenter of $\triangle ABC$. Since $\frac{AI}{AI_a} = \frac{s-a}{s} = \frac{1}{2}$, so K, L lie on $\odot(II_a) \equiv \odot(BIC)$. Since M, D, T are collinear (well-known), so $D \mapsto T$ under the inversion $\mathbf{I}(\odot(BIC))$, hence $\odot(I) \mapsto \odot(TKL)$ under $\mathbf{I}(\odot(BIC)) \Longrightarrow \odot(TKL)$ is tangent to $\odot(ABC)$ (image of BC under $\mathbf{I}(\odot(BIC))$) at T.

livetolove212
#5 Jan 23, 2016, 5:55 pm

This problem was proposed by me in my article (in Vietnamese) $AB+AC=kBC$ (see page 11).
https://nguyenvanlinh.wordpress.com/2015/07/27/abackbc/

Quick Reply

High School Olympiads

Prove tangent(HARD)

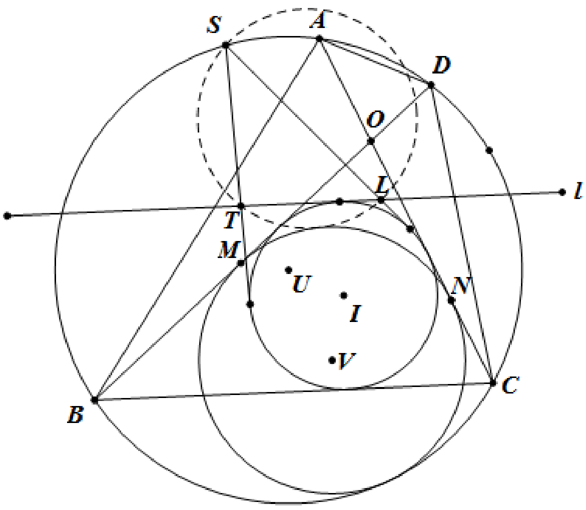
Lin_yangyuan

#1 Jan 10, 2016, 3:39 pm

Very difficult problem

Attachments:

Given ΔOBC and incenter I,$\odot U$ passing through B,C.Let $\odot V$ be the circle tangent toOB,OC and tangent to $\odot U$.
Let l be a line tangent to $\odot I$ and $l \parallel BC$ as well.S is on $\odot U$, T,L are on l and ST,SL are tangent to $\odot I$.
Prove that: $\odot STL$ is tangent to $\odot V$.



TelvCohl

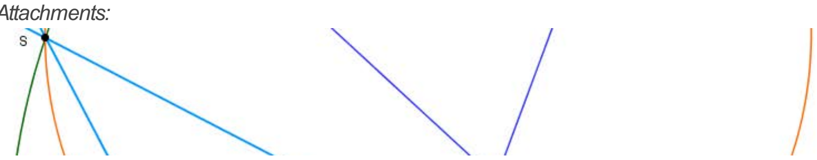
#2 Jan 22, 2016, 3:29 am • 4

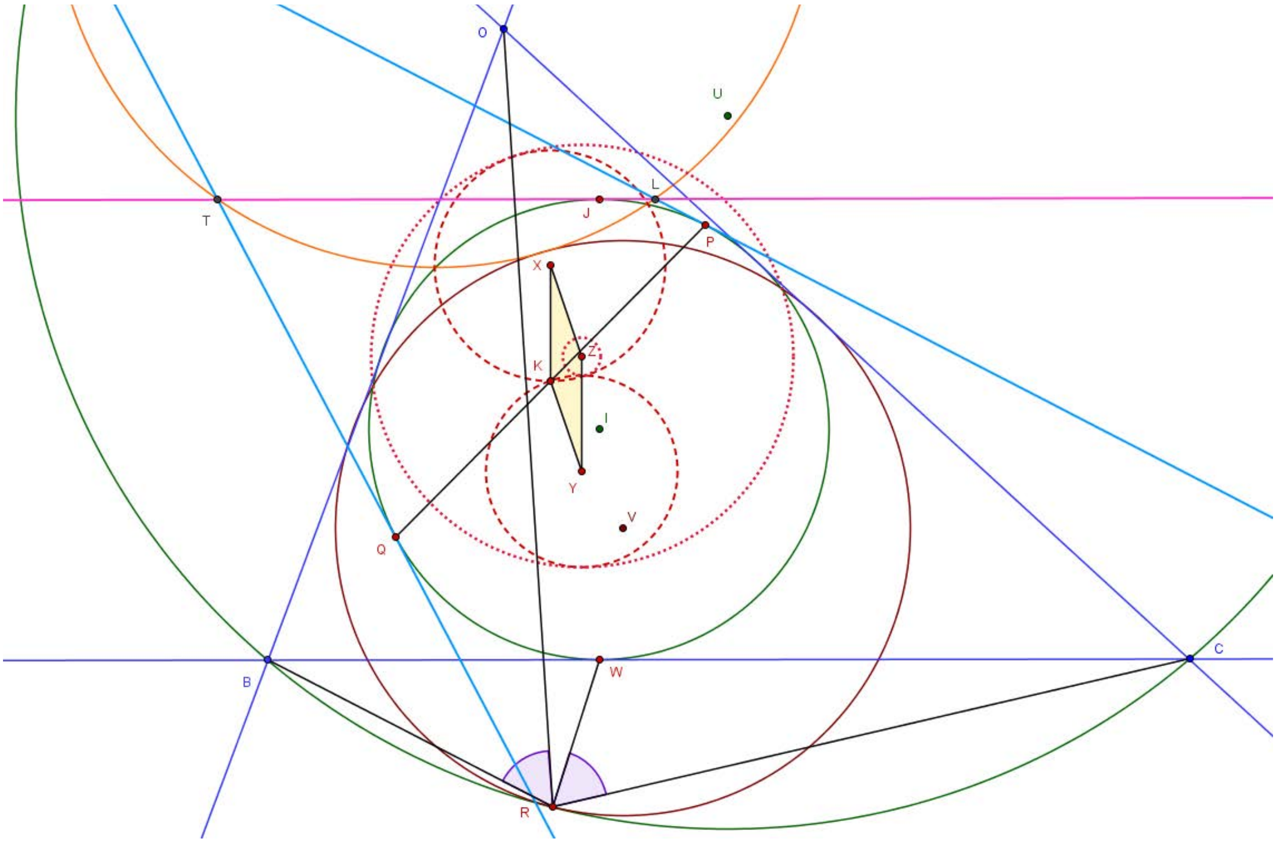
Let $\odot(I)$ touches ℓ, SL, ST at J, P, Q, respectively. Let $\odot(X), \odot(Y)$ be the image of $\odot(STL), \odot(U)$ under the inversion $\mathbf{I}(\odot(I))$, respectively. Let K be the midpoint of PQ and let Z be the point s.t. $XKYZ$ is a parallelogram. Since $\odot(X)$ is the 9-point circle of $\triangle JPQ$, so $YZ = XK = \frac{1}{2}r$ where r is the radius of $\odot(I)$, hence combine $YZ \parallel KX \stackrel{\parallel}{=} \frac{1}{2}IJ$ we get Z is fixed when S varies on $\odot(U)$. Since $ZX = KY = \varrho$ is fixed when S varies on $\odot(U)$, so $\odot(X)$ is tangent to $\odot(Z, \frac{1}{2}r - \varrho)$, $\odot(Z, \frac{1}{2}r + \varrho)$ which are fixed when S varies on $\odot(U)$. since $\odot(Z, \frac{1}{2}r - \varrho), \odot(Z, \frac{1}{2}r + \varrho)$ are also tangent to $\odot(Y)$, so we conclude that $\odot(STL)$ is tangent to two fixed circles Ω_1, Ω_2 which are tangent to $\odot(U)$ as S varies on $\odot(U)$.

Let $R \equiv \odot(U) \cap \odot(V), W \equiv \odot(I) \cap BC$. If S coincide with B, then $\odot(STL)$ coincide with $OB \Longrightarrow OB$ is tangent to Ω_1, Ω_2. Analogously, we can prove OC is tangent to Ω_1 and Ω_2. From the dual of Desargue involution theorem (for $OBWC$) $\Longrightarrow$ there is an involution that swaps $(RB, RC), (RO, RW)$ and the tangents from R to $\odot(I)$, so notice RO, RW are isogonal conjugate WRT $\angle CRB$ (well-known) we get the tangents from R to $\odot(I)$ are isogonal conjugate WRT $\angle CRB \Longrightarrow R_1R_2$ is parallel to BC where R_1, R_2 is the intersection of $\odot(U)$ and the tangents from R to $\odot(I)$.

When S coincide with R, from $R_1R_2 \parallel \ell$ we get $\odot(U), \odot(V), \odot(STL)$ are tangent to each other at R, so we conclude that $\odot(V)$ is one of Ω_1, Ω_2. i.e. $\odot(STL)$ always tangent to $\odot(V)$ when S varies on $\odot(U)$

Attachments:





This post has been edited 1 time. Last edited by TelvCohl, Jan 22, 2016, 8:41 am

Luis González

#3 Jan 22, 2016, 4:29 am • 1

This immediately follows from the configuration discussed at Two fixed circles. 9-point circle $\odot(DEF)$ of $\triangle ABC$ always touches $\odot(V, \frac{1}{2}\varrho \pm \delta)$ both touching the inverse $\odot(U, \delta)$ of (L) WRT (K, ϱ). Thus by conformity $\odot(PQR)$ touch two circles internally tangent to (L).

Back to the proposed problem. Using the previous result for (U), (I) and l tangent to (I), it follows that $\odot(STL)$ touch two fixed circles internally tangent o (U). When $S \equiv B$ or $S \equiv C$, then $\odot(STL)$ degerates into the lines BD or AC, so $\odot(STL)$ touches (V) for any S selected.

Quick Reply

High School Olympiads

Two fixed circles

Reply

Source: Own

buratinogigle

#1 Jul 26, 2014, 2:05 pm

Let (K) be a circle which is contained in circle (L) and d is a fixed tangent of (K). P is a point on (L). Tangent to (K) from P cuts d at Q, R. Prove that circumcircle of triangle PQR always touches two fixed circle when P moves on (L).

Reference

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=275799

Luis González

#2 Aug 6, 2014, 6:08 am • 1

Let QR, RP, PQ touch $\odot(K, \varrho)$ at A, B, C, respectively. Inversion WRT $\odot(K, \varrho)$ takes P, Q, R into the midpoints D, E, F of $BC, CA, AB \Longrightarrow \odot(PQR)$ goes tho the 9-point circle (N) of $\triangle ABC$. So by conformity, it suffices to prove that (N) touches two fixed circles.

Since $\triangle DEF \cup N$ and $\triangle ABC \cup K$ are homothetic, then $(ND \parallel KA) \perp d \Longrightarrow \overline{ND}$ has constant direction and constant length $\frac{1}{2}\varrho \Longrightarrow D \mapsto N$ is a translation. Now since the locus of D is the inverse circle $\odot(U, \delta)$ of (L) WRT (K, ϱ), then N describes the image $\odot(V, \delta)$ of $\odot(U, \delta)$ under the aforementioned translation $\Longrightarrow V$ is fixed and $VN = \delta = \text{const}$ $\Longrightarrow$ all (N) envelope the circles $\odot(V, \frac{1}{2}\varrho \pm \delta)$, as desired.

Quick Reply

High School Olympiads

A cute problem

Reply

Source: OWN

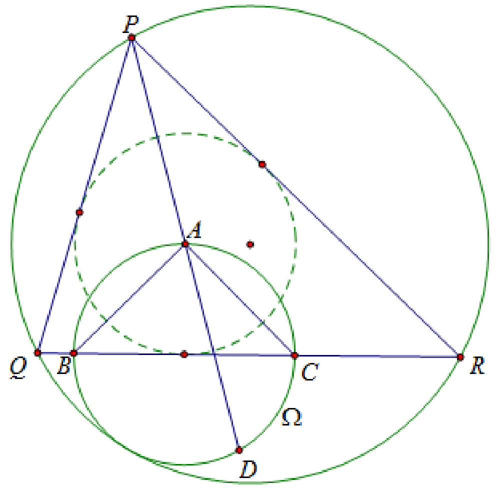
LeVietAn

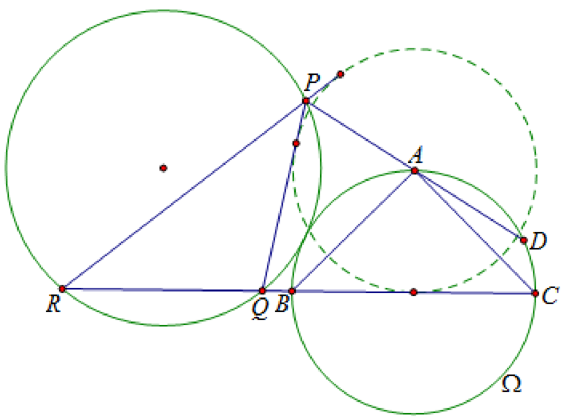
#1 Jan 20, 2016, 5:27 pm • 1

Dear Mathlinkers,

Let ABC be a right triangle with $AB = AC$ inscribed a circle Ω. Let D be a point on Ω. Let P be the reflection point of D through A. Suppose that there exists two points Q and R on BC such that A is the incenter or excenter of triangle PQR. Prove that the circumcircle of triangle PQR is tangent to Ω.

Attachments:





livetolove212

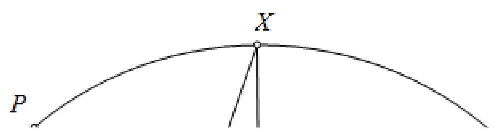
#2 Jan 20, 2016, 9:16 pm • 1

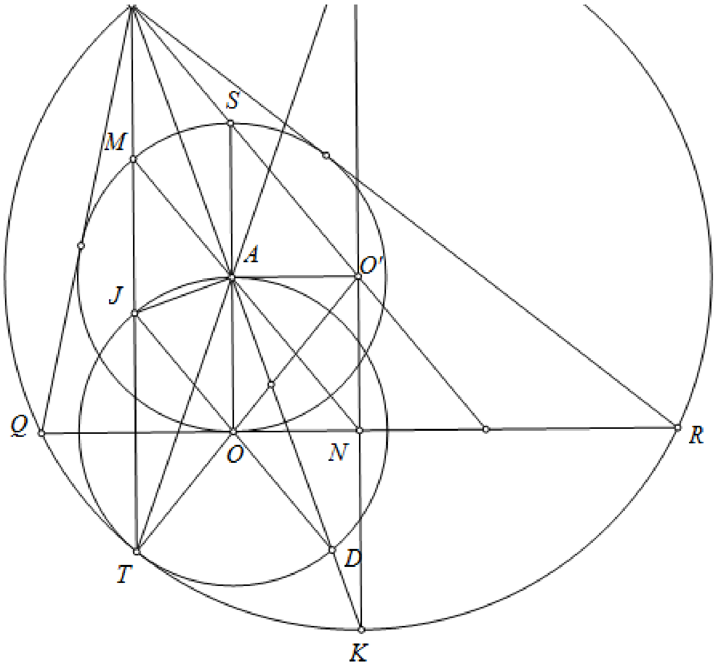
Very nice problem dear friend!

My solution.

Let J be the reflection of D wrt O. We have AO is the midline of triangle PJD then $PJ \parallel AO$ and $PJ = 2AO = 2r$.
Let M be the midpoint of PJ then $MP = AO = r$, which follows that MA passes through midpoint N of QR. Let S be the reflection of O wrt A. We get $PMAS$ is a parallelogram. But $MA = MP$, we deduce that $PMAS$ is a rhombus. This means PS and PJ are isogonals wrt $\angle QPR$. But $PJ \perp QR$ we get PS passes through circumcenter O' of triangle PQR.
Hence $ANO'S$ is a parallelogram or $O'N = AS = AO$. Then $AO' \parallel QR$.
Let X be the midpoint of arc QPR, K be the midpoint of arc QR, XA meets (O') again at T. We have
$\frac{TA}{AX} = \frac{PA}{AK} = \frac{AS}{O'K} = \frac{AO}{O'K}$. Therefore T, O, O' are collinear. We get $\frac{TO}{TO'} = \frac{AO}{XO'}$ then $OA = OT$.
Therefore (O) is tangent to (O') at the tangency of P-mixtilinear incircle of triangle PQR with (O').

Attachments:





This post has been edited 1 time. Last edited by livetolove212, Jan 20, 2016, 9:16 pm

Luis González

#3 Jan 20, 2016, 10:17 pm • 2

Let X and T be the projections of A and P on BC and let X' be the antipode of X on $\odot(A, AX)$. By obvious symmetry, we have $\triangle X'AP \cong \triangle XAD$, i.e. $\triangle X'AP$ is X'- isosceles. Moreover $\angle X'PA = \angle X'AP \equiv \angle XAD = \angle TPA \Longrightarrow$ P-Nagel cevian PX' of $\triangle PQR$ passes through the circumcenter O of $\triangle PQR$. Thus if PX' cuts QR at Y (tangency point of the P-excircle of $\triangle PQR$ with QR), then O is the intersection of PX' with the perpendicular bisector of XY, i.e. midpoint of $YX' \Longrightarrow OA \parallel BC$. Since $\odot(X', X'P)$ is tangent to $(O) \equiv \odot(PQR)$, then its reflection $\odot(ABC)$ on OA also touches (O).

Quick Reply

High School Olympiads

r_1 , r_2 , r

Reply

sororak

#1 Oct 22, 2010, 1:23 am

Let ABC be a triangle and r be the radius of its incircle. D is an arbitrary point on side BC. Let r_1, r_2 be the radiuses of the incircles of triangles ADB, ADC, respectively. Let h_a be the length of perpendicular from A to BC. Prove that:

$$\frac{1}{r_1}+\frac{1}{r_2}-\frac{r}{r_1r_2}=\frac{2}{h_a}.$$

Luis González

#2 Oct 22, 2010, 10:04 am

Incircle (I,r) and C-excircle (I_c,r_c) of $\triangle ABC$ are tangent to BC at $X,Y,$ respectively. If $\angle ABY=\omega$, then

$$\frac{\tan\frac{\gamma}{2}}{\tan\frac{\omega}{2}}=\frac{r}{s-c}\cdot\frac{s-a}{r_c}=\frac{s-a}{s}=1-\frac{2r}{h_a} \quad (*)$$

Label $\angle ADB=\lambda$. Using $(*)$ for $\triangle ADB$ and $\triangle ADC$ with common A-altitude h_a

$$1-\frac{2r_1}{h_a}=\frac{\tan\frac{\lambda}{2}}{\tan\frac{\omega}{2}} \ \ , \ \ 1-\frac{2r_1}{h_a}=\frac{\tan\frac{\gamma}{2}}{\tan\frac{\lambda}{2}}$$

$$\Longrightarrow \left(1-\frac{2r_1}{h_a}\right)\cdot\left(1-\frac{2r_2}{h_a}\right)=\frac{\tan\frac{\lambda}{2}}{\tan\frac{\omega}{2}}\cdot\frac{\tan\frac{\gamma}{2}}{\tan\frac{\lambda}{2}}=\frac{\tan\frac{\gamma}{2}}{\tan\frac{\omega}{2}}=1-\frac{2r}{h_a}$$

$$\Longrightarrow \frac{1}{r_1}+\frac{1}{r_2}-\frac{r}{r_1\cdot r_2}=\frac{2}{h_a}.$$

yetti

#3 Oct 22, 2010, 10:08 am

Ha! The infamous inradii problem again.

Click to reveal hidden text

Quick Reply

High School Olympiads

Point on hypotenuse making the two inradii equal

Source: INMO 2016 Problem 5

YESMAths

#1 Jan 17, 2016, 7:08 pm

Let ABC be a right-angle triangle with $\angle B = 90^\circ$. Let D be a point on AC such that the inradii of the triangles ABD and CBD are equal. If this common value is r' and if r is the inradius of triangle ABC, prove that

$$\frac{1}{r'} = \frac{1}{r} + \frac{1}{BD}.$$

Luis González

#2 Jan 17, 2016, 10:23 pm • 8

We use standard notation $BC = a, CA = b, AB = c$ and $s = \tfrac{1}{2}(a+b+c)$. Thus letting I_1 and I_2 be the incenters of $\triangle ABD$ and $\triangle CBD$, we get:

$$[ABC] = [BI_1D] + [BI_2D] + [ABI_1] + [CBI_2] + [AI_1D] + [CI_2D] =$$

$$= r' \cdot BD + \tfrac{1}{2} r' \cdot (a+b+c) = r' \cdot (BD + s) \Longrightarrow$$

$$BD = \frac{[ABC]}{r'} - s = \frac{[ABC]}{r'} - \frac{[ABC]}{r} = [ABC] \cdot \left(\frac{1}{r'} - \frac{1}{r}\right) \ (\star).$$

But from Cono Sur 1994 (P6), we have $[ABC] = BD^2$. Thus combining with $(\star)$ gives

$$BD = BD^2 \cdot \left(\frac{1}{r'} - \frac{1}{r}\right) \Longrightarrow \frac{1}{r'} = \frac{1}{r} + \frac{1}{BD}.$$

TheOneYouWant

#3 Jan 17, 2016, 10:54 pm

Thanks Luis Gonzalez, almost everyone were stuck with the Cono Sur problem lemma...

Luis González

#4 Jan 18, 2016, 5:29 am • 2

Generalization: D is a point on the side $\overline{BC}$ of $\triangle ABC$, such that the inradii of $\triangle ABD$ and $\triangle ACD$ are equal to ϱ. If r is the inradius of $\triangle ABC$, then we have the relation:

$$\frac{1}{\varrho} = \frac{1}{r} + \cot\frac{A}{2} \cdot \frac{1}{AD}$$

As before, we have $[ABC] = r \cdot s = \varrho \cdot (AD + s) \ (\star) \Longrightarrow \frac{2\varrho}{h_a} = \frac{a}{s+AD}$. But from the infamous inradii problem, we have $1 - \frac{a}{s} = \left(1 - \frac{2\varrho}{h_a}\right)^2 \Longrightarrow$

$$\frac{s-a}{s} = \left(1 - \frac{a}{s+AD}\right)^2 \Longrightarrow AD = \frac{a\sqrt{s}}{\sqrt{s} - \sqrt{s-a}} - s = \sqrt{s(s-a)} = \sqrt{[ABC]\cot\frac{A}{2}}.$$

Combining this latter relation with $(\star)$ yields

$$\frac{1}{\varrho}-\frac{1}{r}=\frac{AD}{[ABC]}=\frac{AD}{AD^2}\cdot\cot\frac{A}{2}=\cot\frac{A}{2}\cdot\frac{1}{AD}\Longrightarrow\frac{1}{\varrho}=\frac{1}{r}+\cot\frac{A}{2}\cdot\frac{1}{AD}.$$

anantmudgal09

#5 Jan 18, 2016, 6:07 am • 1

Here is my solution using some length chasing sort of arguments. This is quite hard for an average INMO problem but nevertheless, it can be done in an hour at the most at the contest.

Let $AB=1$ and $BC=x$. Let D be the point on AC and let $\frac{AD}{DC}=k$. Let $[ABD]=\Delta_1$ and $[CBD]=\Delta_2$ and let $2s_1=AB+BD+DA$ and $2s_2=CB+BD+DC$.

Now, we know that $\frac{r_1}{r_2}=1=\frac{\Delta_1(2s_2)}{\Delta_2(2s_1)}=\frac{AD}{DC}\frac{x+BD+DC}{1+BD+DA}$ which is equivalent to

$$\frac{AD}{DC}=k=\frac{1+BD}{x+BD}$$

Now, this give that $BD=\frac{kx-1}{1-k}$

However, by Stewart's theorem $AC.BD^2=CD.AB^2+AD.BC^2$ and so computing this gives that $k=\frac{x+3}{3x+1}$. and similarly we get that $BD=\frac{(x+1)}{2}$.

Now, $r=\frac{x+1-\sqrt{x^2+1}}{2}$ and that $r'=\frac{\Delta_1}{s_1}$.

Now, $[ABC]=\frac{x}{2}$ and so we have that $\Delta_1(1+k)=[ABC]$ and so $\Delta_1=\frac{x(x+3)}{8(x+1)}$.

Also, we know that $AD(1+k)=AC=\sqrt{x^2+1}$ and so $2AD=\frac{(x+3)\sqrt{x^2+1}}{2(x+1)}$ and hence, we have that

$$r_1=r'=\frac{\frac{x(x+3)}{8(x+1)}}{\frac{1+\frac{x+1}{2}+AD}{2}}=\frac{2x(x+1)}{(x+1)^2+2x+(x+1)\sqrt{x^2+1}}$$

Now, we have that $\frac{1}{r}+\frac{1}{BD}=\frac{2}{x+1}+\frac{x}{x+1-\sqrt{x^2+1}}=\frac{2x(x+1)}{(x+1)^2+2x+(x+1)\sqrt{x^2+1}}$.

This completes the proof $\blacksquare$
(This was just a sketch of what the expressions come out to be and is done by hand. It's not really hard to do all the intermediate steps, just that one needs a bit of computational fortitude)

AdithyaBhaskar

#6 Jan 18, 2016, 1:14 pm

> *anantmudgal09 wrote:*
> ... just that one needs a bit of computational fortitude

Unfortunatley, I don't have it. But anyway, I mentioned Stewart's theorem, and wrote down $r'=\frac{\triangle_1}{s_1}=\frac{\triangle_2}{s_2}=\frac{\triangle_1+\triangle_2}{s_1+s_2}=\frac{\triangle}{s+BD}$ and thus it is enough to prove $\triangle=BD^2$ (as $\frac{1}{r}=\frac{s}{\triangle}$). And I fooled around a bit more with this, to add up to about 2 pages in total. How much, according to you, should that fetch me?

AdithyaBhaskar
#8 Jan 18, 2016, 3:37 pm

> *anantmudgal09 wrote:*
>
> > *AdithyaBhaskar wrote:*
> >
> > > *anantmudgal09 wrote:*
> > > ... just that one needs a bit of computational fortitude
> >
> > Unfortunatley, I don't have it. But anyway, I mentioned Stewart's theorem, and wrote down $r' = \frac{\triangle_1}{s_1} = \frac{\triangle_2}{s_2} = \frac{\triangle_1 + \triangle_2}{s_1 + s_2} = \frac{\triangle}{s + BD}$ and thus it is enough to prove $\triangle = BD^2$ (as $\frac{1}{r} = \frac{s}{\triangle}$). And I fooled around a bit more with this, to add up to about 2 pages in total. How much, according to you, should that fetch me?
>
> I don't really think that saying: "To proving X it shall suffice to proving Y" and then not proving Y is worth anything reasonable, at least not at INMO and certainly not at TST, not sure for RMO.

Hey just observed: The result that I 'had to prove' is nothing but Cono Sur 1994/6

kapilpavase
#9 Jan 18, 2016, 5:38 pm • 2

I think i finally have a 'not so bash ' sol to this problem
Denote $AD = x, CD = y, BD = z$
Indeed,it suffices to show $z^2 = ac/2$
Drop perps from D on AB, BC and use pyth to get

$$z^2 = (\frac{yc}{b})^2 + (\frac{xa}{b})^2$$

,that is,

$$z^2 = \frac{y^2c^2 + x^2a^2}{x^2 + y^2 + 2xy}$$

.
Now compare areas of ABD, BDC by using formula $\Delta = rs$ and ratios of areas of triangle with same height.We get

$$\frac{x}{y} = \frac{z + c + x}{z + a + y} = \frac{z + c}{z + a}$$

.
So

$$z^2 = \frac{(yc - xa)^2}{(x - y)^2} = \frac{y^2c^2 + x^2a^2 - 2xayc}{x^2 + y^2 - 2xy}$$

Now compare this two equations for z^2, we get by componendo(or whatever you call that}

$$z^2 = 2xayc/4xy = ac/2$$

And done

aditya21
#10 Jan 18, 2016, 10:45 pm

though this question is less bashy than INMO 2015 P5, still ....

my solution sketch =

we easily get by $[ABC] = rs$ where r is inradii.

$\frac{1}{}\ \frac{s + BD}{}$

we get $\frac{1}{r'} = \frac{s+BD}{[ABC]}$ where s is semi-perimeter of triangle ABC

and hence $\frac{1}{r'} = \frac{1}{r} + \frac{1}{BD}$ is equivalent to prove that

$[ABC] = BD^2$

now since triangles ABD, BCD have same inradii. than again using $[ABC] = rs$

we get $\frac{AB+BD}{AD} = \frac{BD+BC}{CD}$

which can than be bashed to get $2BD^2 = AB.AC = 2[ABC]$ as desired.

so we are done.

This post has been edited 1 time. Last edited by aditya21, Jan 18, 2016, 10:46 pm
Reason: e

kapilpavase
#11 Jan 19, 2016, 2:40 pm • 8

'To bash or not to bash,that is the question' describes the geos of 2016 Inmo

TheOneYouWant
#12 Jan 19, 2016, 2:46 pm

Exactly @ above. Infact, on the overall the paper was so bashy, in the first 2 hours i got only 0.5 problems () since i was thinking up creative ideas for the others. However, in the last 2 hours reality struck and thankfully i was able to solve enough problems to pass(hopefully)...

anantmudgal09
#16 Mar 18, 2016, 12:04 am

Amusingly, it seems that something related to this was given in An Indian Practice test

Quick Reply

High School Olympiads

Prove area

Reply

Source: Cono Sur 1994-problem 6

José
#1 Jun 2, 2006, 4:56 am

Consider a $\triangle ABC$, with $AC \perp BC$. Consider a point D on AB such that $CD = k$, and the radius of the inscribe circles on $\triangle ADC$ and $\triangle CDB$ are equals. Prove that the area of $\triangle ABC$ is equal to k^2.

Bourne
#2 Jun 3, 2006, 8:25 pm

My solution :
Let $AB = c, AC = b, BC = a, AD = u, BD = v$
$u + v = \sqrt{a^2 + b^2}$
radius of the inscribe circles on $\triangle ADC$ and $\triangle CDB$ are equals

$$\frac{AC + CD}{AD} = \frac{BC + CD}{BD}$$
$$\frac{b + k}{u} = \frac{a + k}{v}$$
$$\frac{b^2 v + a^2 u}{u + v} - uv = k^2$$
$$b^2 \frac{a + b}{a + b + 2k} + a^2 \frac{b + k}{a + b + 2k} = k^2 + \frac{(a^2 + b^2)(a + k)(b + k)}{(a + b + 2k)^2}$$
$$4k^4 + 4ak^3 + 4bk^3 + 2k^2 ab - 2a^2 b^2 - 2kab^2 - 2kba^2 = 0$$
$$2(2k^2 - ab)(a + k)(b + k) = 0$$
$$\frac{ab}{2} = k^2$$

Satyaprakash2009rta
#3 Jan 17, 2016, 11:54 pm

> Bourne wrote:
> radius of the inscribe circles on $\triangle ADC$ and $\triangle CDB$ are equals
> $$\frac{AC + CD}{AD} = \frac{BC + CD}{BD}$$

How did you find that??

This post has been edited 1 time. Last edited by Satyaprakash2009rta, Jan 18, 2016, 9:37 am

tarzanjunior
#4 Jan 18, 2016, 7:38 pm

Any other solution?

aditya21
#5 Jan 18, 2016, 10:34 pm

> Satyaprakash2009rta wrote:

> *Bourne wrote:*
> radius of the inscribe circles on $\triangle ADC$ and $\triangle CDB$ are equals
> $\frac{AC+CD}{AD}=\frac{BC+CD}{BD}$
>
> How did you find that??

this follows as $r=\frac{[ABC]}{s}$ where s is semiperimeter and r is inradius.
apply this for both triangles and put them equal to get above relation

tarzanjunior
#6 May 23, 2016, 3:09 pm
Is there any non-computational proof?

Quick Reply

High School Math

ellipse and slopes

Reply

AndrewTom

#1 Oct 12, 2015, 1:55 pm

Let A and B be the end-points of the major axis of an ellipse. Let P be any point on the ellipse, distinct from A and B. Show that the product of the slopes of the lines PA and PB are constant.

AndrewTom

#2 Oct 17, 2015, 1:50 am

Any ideas on this one?

talkinaway

#3 Oct 17, 2015, 2:21 am

I assume this problem refers to ellipses of the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ only, and not tilted ellipses such as $x^2+xy+y^2=4$.

I can make this assumption because in a tilted ellipse whose major/minor axes aren't along the x and y axis, there exists a point P on the ellipse that is either directly below or directly above A, making the slope of PA indeterminate.

Additionally, you can translate the generic "lined-up" ellipse to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ without changing the slope of PA or the slope of PB, so I'll just deal with that simpler form.

Coordinate solution

Proofread my proof: I left out something kind of major; what did I leave out? I actually proved something slightly more general...can you figure out what?

This post has been edited 1 time. Last edited by talkinaway, Oct 17, 2015, 2:22 am

Luis González

#4 Jan 17, 2016, 4:52 am • 1

Label a and b the lengths of the semi-major axis and semi-minor axis of the given ellipse $\mathcal{E}$. Circle (O) with diameter $\overline{AB}$ is the pedal circle of $\mathcal{E}$. X is the projection of P on AB and the ray XP cuts (O) at Q. Since $P \mapsto Q$ is an affine homology fixing AB that takes $\mathcal{E}$ into (O), then $\frac{PX}{QX}=\text{const}=\frac{b}{a}$. Therefore

$$\tan\measuredangle PAB\cdot\tan\measuredangle PBA=-\frac{PX}{XA}\cdot\frac{PX}{XB}=-\frac{PX^2}{XA\cdot XB}=-\frac{PX^2}{QX^2}=-\frac{b^2}{a^2}.$$

Quick Reply

High School Olympiads

Tetrahedron 3d geometry

Reply

Source: Competition

niti

#1 Jan 16, 2016, 6:41 pm

Let A, B, C, D be points in space. A spehere touches the sides AB, BC, CD, DA in E, F, G, H. Prove, that E, F, G, H lie in one plane.

huricane

#2 Jan 16, 2016, 7:45 pm

We have $AE = AH$,$BE = BF$,$CF = CG$ and $DG = DH$,so $\frac{AE}{BE} \cdot \frac{BF}{CF} \cdot \frac{CG}{DG} \cdot \frac{DH}{AH} = 1$.
Now by Menelaus' Theorem in space we deduce that points E, F, G, H are coplanar.

Luis González

#3 Jan 17, 2016, 1:11 am

Posted before at http://www.artofproblemsolving.com/community/c6h423195.

Quick Reply

High School Olympiads

Tangent points of a quadrilateral to a sphere are coplanar

Source: Spanish Mathematical Olympiad, National Stage, 2011

Shu

#1 Aug 9, 2011, 9:43 pm • 1

Let A, B, C, D be four points in space not all lying on the same plane. The segments AB, BC, CD, and DA are tangent to the same sphere. Prove that their four points of tangency are coplanar.

Luis González

#2 Aug 9, 2011, 10:41 pm • 1

Let P,Q,R,S be the tangency points of the given sphere with $AB,BC,CD,DA,$ respectively. Assume that PS and QR cut BD at X,X', respectively. By Menelaus' theorem for $\triangle ABD$ and $\triangle CBD$ cut by the transversals $\overline{PSX}$ and $\overline{QRX'}$, we get

$$\frac{\overline{XD}}{\overline{XB}}=\frac{\overline{PA}}{\overline{BP}}\cdot\frac{\overline{SD}}{\overline{AS}}\ \ ,\ \ \frac{\overline{X'D}}{\overline{X'B}}=\frac{\overline{QC}}{\overline{BQ}}\cdot\frac{\overline{RD}}{\overline{CR}}$$

Since $PA=AS$, $QC=CR$, $BP=BQ$ and $RD=SD$, then we obtain $\frac{\overline{XD}}{\overline{XB}}=\frac{\overline{X'D}}{\overline{X'B}} \Longrightarrow X\equiv X'$ i.e. $PS\cap QR\neq\emptyset \Longrightarrow P,Q,R,S$ are coplanar.

High School Olympiads

X(68) and X(389) as radical centers

Reply

Source: Own

Luis González

#1 Jan 16, 2016, 1:37 pm • 2

$\triangle ABC$ is acute and $\triangle DEF$ is its orthic triangle. Show that:
a) Radical center of $\odot(A, AD), \odot(B, BE), \odot(C, CF)$ is Spieker point of $\triangle DEF$.
b) Radical center of $\odot(D, DA), \odot(E, EB), \odot(F, FC)$ is Prasolov point of $\triangle ABC$.

leonardg

#2 Jan 16, 2016, 4:03 pm

Demonstracion a) :

Attachments:

Luis

Let $A(0,2)$, $B(-2b,0)$ *and* $C(2c,0)$ *,where* $b,c > 0$ *and* $bc < 1$ *.Then* $S(x_S, y_S)$ *,where*

$$x_S = \frac{c\left(\sqrt{c^2+1}+c\right)-b\left(\sqrt{b^2+1}+b\right)}{b+c+\sqrt{b^2+1}+\sqrt{c^2+1}} \text{ and } y_S = \frac{\sqrt{b^2+1}+\sqrt{c^2+1}}{b+c+\sqrt{b^2+1}+\sqrt{c^2+1}} \text{ .Also ,} AD^2 = 4 \text{ ,}$$

$$BE^2 = \frac{4(b+c)^2}{c^2+1} \text{ and } CE^2 = \frac{4(b+c)^2}{b^2+1} \text{ . Now redenote } b = \frac{x^2-1}{2x} \text{ and } c = \frac{y^2-1}{2y} \text{ ,where } x,y > 1 \text{ .}$$

$$\text{Hence } x_S = \frac{y-x}{2} \text{ and } y_S = \frac{xy+1}{2xy}$$

.From these ,we can check immediately that S *have the same power relative to the 3 circles .*

Leo

Quick Reply

High School Olympiads

Concyclic points

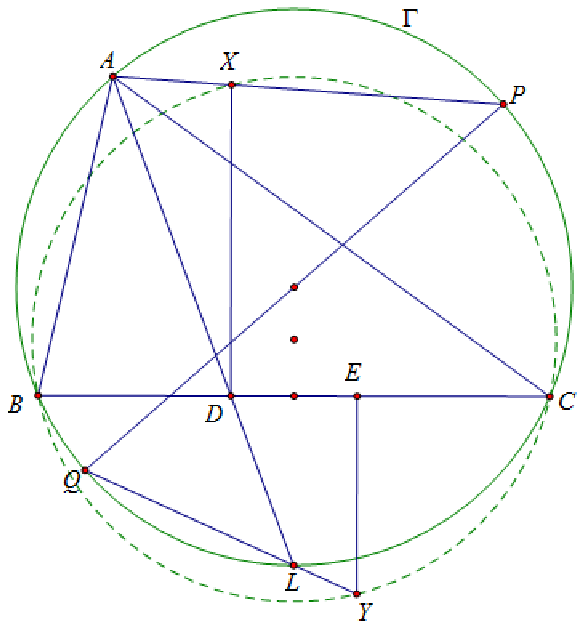
Reply

Source: OWN

LeVietAn
#1 Jan 4, 2016, 3:08 pm

Dear Mathlinkers,
Let ABC be a triangle with $AB \neq AC$ and circumcircle Γ. The bisector of $\angle BAC$ intersects BC at D and Γ again at L. Let E be the reflection of D with respect to the midpoint of BC. Let PQ be a diameter of Γ. The lines through D and E perpendicular to BC intersect the lines AP and QL at X and Y respectively. Prove that the quadrilateral $BXCY$ is cyclic.
This problem is a generalization of problem here: http://www.artofproblemsolving.com/community/c6h546178p3160582

Attachments:



This post has been edited 1 time. Last edited by LeVietAn, Jan 11, 2016, 8:41 pm

Luis González
#2 Jan 16, 2016, 5:40 am • 1

Redefine X,Y as the intersections of a circle ω through B,C with the perpendiculars to BC at D,E. Hence if AX and LY cut $(O) \equiv \Gamma$ again at P,Q, then we need to show that PQ is a diameter of (O).

If XD cuts ω again at Z, then by symmetry $BCYZ$ is an isosceles trapezoid with $YZ \parallel BC$. Thus when ω varies, the series Y,Z are congruent and moreover $DX \cdot DZ = DB \cdot DC = \text{const} \Longrightarrow X \mapsto Z$ is an involution $\Longrightarrow$ series X,Y are projective $\Longrightarrow$ pencils AX and LY are projective, inducing a projectivity $P \mapsto Q$ on (O). Hence it is enough to prove that PQ goes through O for three positions of P conveniently taken.

When $P \equiv X$ (two cases), it follows by trivial symmetry that PQ is a diameter of (O) and finally when $P \equiv L$, then $X \equiv D$ and Y goes to the point at infinity of $\perp BC \Longrightarrow Q$ becomes midpoint of the arc BAC, i.e. PQ is a diameter of (O). Therefore PQ is a diameter of (O) for any X, as desired.

hayoola

#3 Jan 18, 2016, 6:58 pm

I use function to solve it Let the line that is perpendicular to BC from E is R and let the line that is perpendicular to BC from D is s

Function:for any point Y on R we project it a bout point L on the circumcircle to make point Y_1

Then we project point Y_1 about the circumcenter on the circumcircle to make point Y_2

Then we project point Y_2 about point A on the line s to make point Y_3 .

The question wants to prove that YBY_3C is cyclice

We know that for any for points $H, I, J, K. (HIJK) = (H_3, I_3, J_3, K_3)$

I find that for three points like B, C, L it is cyclice

So by the function and three fixed point of the function we can say that for any point $K. kBK_3C$ is cyclice

Quick Reply

High School Olympiads

Perspective triangles

Source: own

quangMavis1999

#1 Jan 15, 2016, 10:30 am

Problem Let $\triangle ABC$ with incenter I, inscribed in circle (O). AD, BE, CF are diameter of circle (O), resp. DI, EI, FI intersect circle (O) again at X, Y, Z, resp. Let $C' = BY \cap AX, B' = AX \cap CZ, A' = BY \cap CZ$. Prove that $\triangle A'B'C'$ & $\triangle ABC$ are perspective. Moreover perpector P of $\triangle A'B'C'$ and $\triangle ABC$ lies on OI line

Luis González

#2 Jan 15, 2016, 11:23 am

If the incircle (I) touches BC, CA, AB at U, V, W, then X, Y, Z are clearly the second intersections of (O) with $\odot(AVW), \odot(BWU), \odot(CUV)$, respectively. Thus according to The isogonal of the complement of a point, we deduce that $\triangle ABC$ and $\triangle A'B'C'$ are perspective with perspector the isogonal of the complement of the Gergonne point, i.e. the isogonal of the Mittenpunkt (X_{57} of $\triangle ABC$), which lies on OI (well-known).

High School Olympiads

The isogonal of the complement of a point

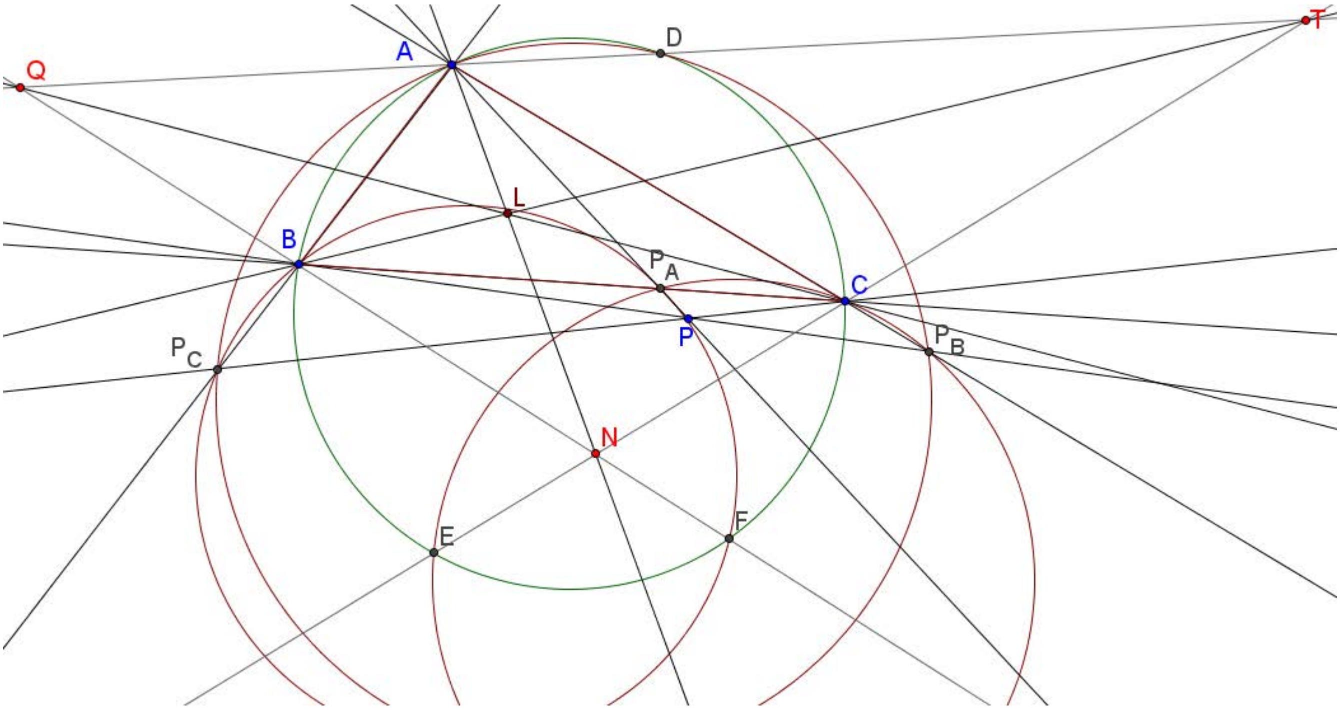
Reply

Cezar

#1 Apr 13, 2015, 4:52 am

Let a point P and $\triangle ABC$.
Let the cevian triangle of P wrt $\triangle ABC$ be $\triangle P_AP_BP_C$.
The radical axes of $\odot ABC$ with $\odot P_AP_BC$, $\odot P_CP_BA$, $\odot P_AP_CB$ intersect in points Q, N, T.
Prove that lines AN, BT, QC are concurrent in the isogonal of the complement of P wrt $\triangle ABC$.

Attachments:



Luis González

#2 Apr 13, 2015, 5:53 am • 1

Let Q be the isotomic conjugate of P and let $\triangle Q_AQ_BQ_C$ be the cevian triangle of Q. From the lemma discussed at Concurrency in harmonic quadrilateral (see hidden text at post #2), it follows that $\odot(AQ_BQ_C), \odot(BQ_CQ_A), \odot(CQ_AQ_B)$ are tangent to TQ, QN, NT. Now from the problem Perspective triangles from tangents, AN, BT, CQ concur then at the isogonal of the isotomcomplement of the isotomic of P, i.e. the isogonal of the complement of P.

Quick Reply

High School Olympiads

Prove BK=CL

Reply

Scorpion.k48
#1 Jan 13, 2016, 10:20 pm

Let $\triangle ABC$ with incenter I. P, Q is midpoint of IB, IC, res. $\odot(APB)$ and $\odot(AQC)$ cuts BC again at K, L. Prove that $BK = CL$.

ind
#2 Jan 13, 2016, 10:30 pm

ANOTHER QUESTION with same feeling will be
CHINESE Girls math olympiad
n acute triangle ABC, $AB > AC$.
D and E are the midpoints of AB, AC respectively.
The circumcircle of ADE intersects the circumcircle of BCE again at P.
The circumcircle of ADE intersects the circumcircle BCD again at Q.
Prove that $AP = AQ$.

Luis González
#3 Jan 14, 2016, 5:35 am • 1

Let M be the midpoint of BC and let T be the second intersection of $\odot(APB)$ and $\odot(AQC)$. $\angle(TP, TA) = \angle(BI, BA)$ and similarly $\angle(TA, TQ) = \angle(CA, CI) \Longrightarrow$ $\angle(TP, TQ) = \angle(BI, BA) + \angle(CA, CI) = \angle(IC, IB) = \angle(MP, MQ) \Longrightarrow T$ is on 9-point circle $\odot(MPQ)$ of $\triangle IBC$. Hence if $X \in \odot(MPQ)$ is the projection of I on BC, we have $\angle(TP, TM) = \angle(XP, XM) = \angle(BC, BI) = \angle(BP, BA) \Longrightarrow A, T, M$ are collinear $\Longrightarrow AM$ is radical axis of $\odot(APB)$ and $\odot(AQC) \Longrightarrow MB \cdot MK = MC \cdot ML \Longrightarrow MK = ML \Longrightarrow BK = CL.$

buratinogigle
#4 Jan 14, 2016, 3:27 pm • 1

Thank Luis for your nice solution but Scorpion.k48 you shouldn't post problem weekly on my blog here. However there are some extensions for the problem, here is an example

Let ABC be a triangle with incenter I. P is a point on BC. M, N are on IB, IC, reps, such that $PM \parallel IC$ and $PN \parallel IB$. BC cuts circumcircles of triangles ABM and ACN again at S, T. Prove that P is midpoint of ST.

TelvCohl
#5 Jan 14, 2016, 4:03 pm • 2

> *buratinogigle wrote:*
> Let ABC be a triangle with incenter I. P is a point on BC. M, N are on IB, IC, reps, such that $PM \parallel IC$ and $PN \parallel IB$. BC cuts circumcircles of triangles ABM and ACN again at S, T. Prove that P is midpoint of ST.

Let $E \equiv PN \cap \odot(ACN), F \equiv PM \cap \odot(ABM)$. Since $\measuredangle CAE = \measuredangle CNE = \measuredangle CIB = 90^\circ - \tfrac{1}{2}\measuredangle BAC$ and similarly we get $\measuredangle FAB = 90^\circ - \tfrac{1}{2}\measuredangle BAC$, so E, A, F are collinear. From $\measuredangle PEF = \measuredangle BCI$ and $\measuredangle PFE = \measuredangle CBI$ we get $\triangle EPF \sim \triangle CIB$, so combine $PN = \frac{IB \cdot PC}{BC}$ and $PM = \frac{IC \cdot PB}{BC}$ we conclude that

$$\frac{PT}{PS} = \frac{\frac{PN \cdot PE}{PC}}{\frac{PM \cdot PF}{PB}} = \frac{IB \cdot PE}{IC \cdot PF} = 1 \Longrightarrow PT = PS.$$

ljiljak1707

#6 Jan 14, 2016, 4:22 pm

Let D be the midpoint of BC. And let R be the second point of intersection of $\odot(BPA)$ and $\odot(ACQ)$. Since $DP \parallel IC$ and $DQ \parallel IB \implies \angle PDQ = \alpha + \beta/2 + \gamma/2$. It's easy to see that $\angle PRQ = \beta/2 + \gamma/2$, so $PRQD$ is a cyclic quadrilateral. Since $\angle PRD = \angle PQD = \beta/2$ and $\angle ARP = 180^\circ - \beta/2 \implies A, R, D$ are collinear.Since AR is a radical axis of $\odot(BPA)$ and $\odot(ACQ)$, $DB \cdot DK = DC \cdot DL$ and the conclusion follows

Quick Reply

High School Olympiads

Cyclic quadrilateral + Tangent circles.

Reply

Source: 0

Luis González

#1 Jul 31, 2010, 5:27 am • 3

Quadrilateral $ABCD$ is cyclic with circumcircle (O). Tangents of (O) through vertices B, C cut line AD at M, N, respectively. Define the points $E \equiv BN \cap CM$ and $F \equiv AE \cap BC$. If L denotes the midpoint of segment $\overline{BC}$, show that the circumcircle of $\triangle DLF$ is tangent to (O) through D.

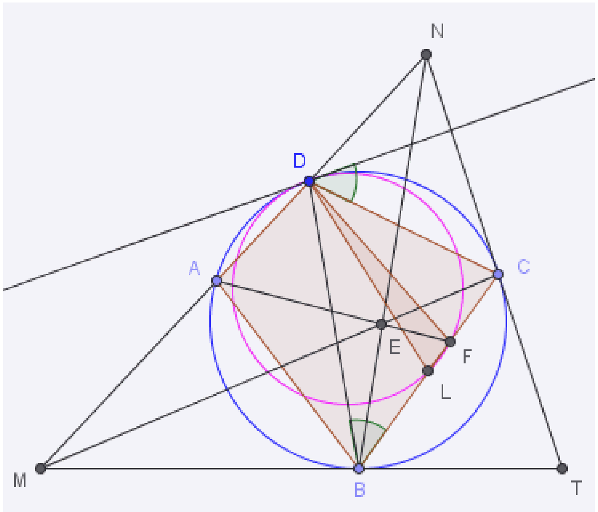
77ant

#2 Aug 1, 2010, 12:27 am

I guess the collinearity of (D,F,T) is crucial, but no idea.
Assuming it, DF is symmedian of $\triangle DBC$. $\angle FDD = \angle DLF$, and we're done.
May I ask you some hint?

p.s. I edited, and will try it

Attachments:

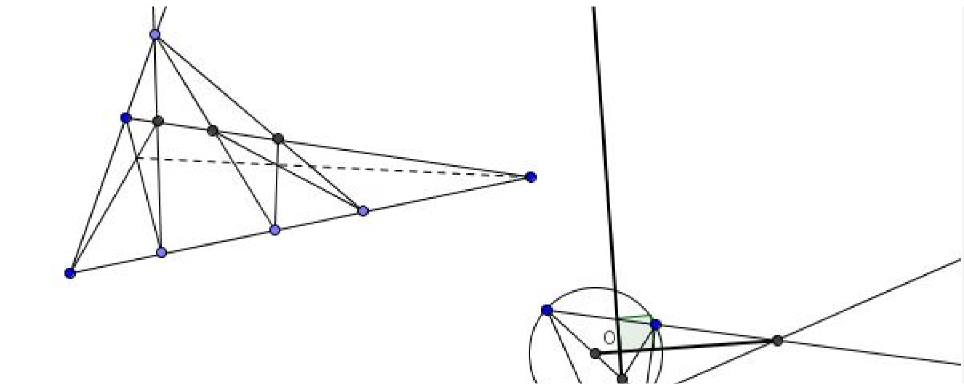


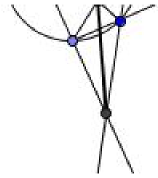
skytin

#3 Aug 1, 2010, 1:32 am

two important lemma's

Attachments:



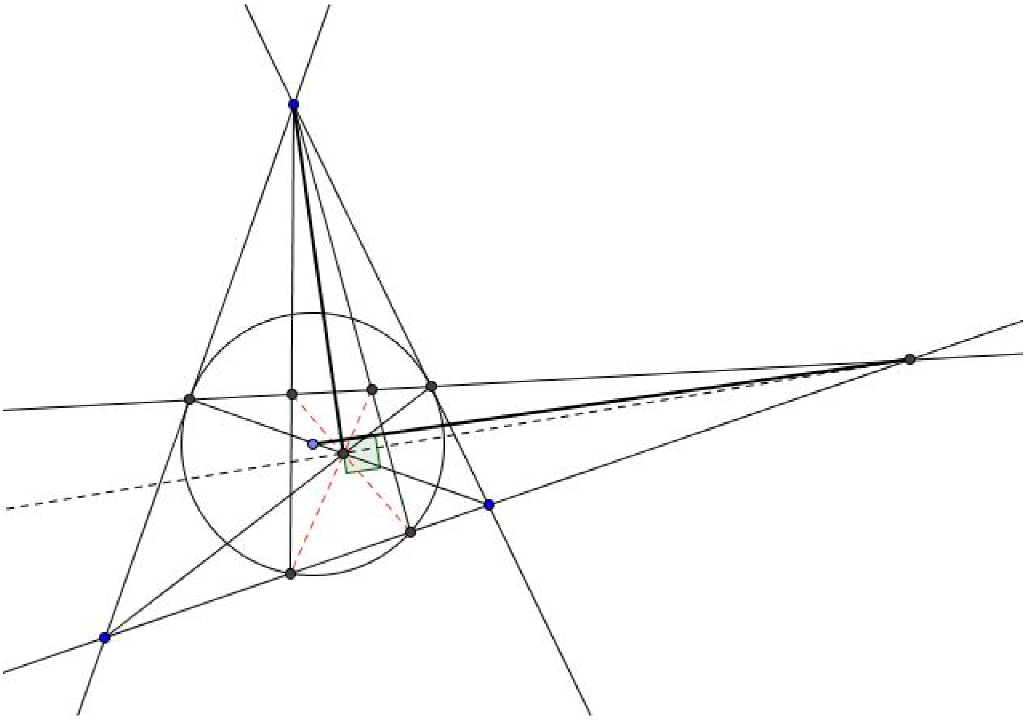


skytin

#4 Aug 1, 2010, 1:39 am

jpg to my solution

Attachments:



Vikernes

#5 Aug 1, 2010, 2:12 am

> *skytin wrote:*
> jpg to my solution

Can you write your solution, please?

mathVNpro

#6 Aug 1, 2010, 11:25 pm

> *Quote:*
> Quadrilateral $ABCD$ is cyclic with circumcircle (O). The tangent lines to (O) through vertices B, C cut line AD at M, N respectively. Define the points $E \equiv BN \cap CM$ and $F \equiv AE \cap BC$. If L denotes the midpoint of segment $\overline{BC}$, show that circumcircle of $\triangle DLF$ is tangent to (O) through D.

Nice one, my dear Luis! Here is my solution:

Let P_a be the intersection of the tangents from B and C of (O) and P'_a be the intersection of P_aE with BC. Denote $P \equiv AB \cap CD$ and $Q \equiv AD \cap BC$. Since (P_aM, P_aN, BC, MN) is a complete quadrilateral, we deduce that P_aE is the *polar of* Q *with respect to* P_aM and P_aN. Hence, P'_a and Q are *conjugated with respect to* P_aM and P_aN, which implies that $(QP'_aCB) = -1$. Therefore, P'_a also lies on the polar of Q with respect to $(O) \Longrightarrow P'_a \in PP_a$. As a result, P, E and P_a are collinear. Now, let consider the two triples (P, D, C) and (M, B, P_a) of collinear points. Denote F' by the intersection of DP_a and BC. Then by *Pappus* theorem, we conclude that A, E and F' are collinear, which implies that F' must $\equiv F$. Thus, D, F and P_a are collinear.

Consider the inversion through pole P_a, power $k = \mathcal{P}_{P_a/(O)}$, we have $\mathcal{I}(P_a, k)$ maps D into D', then $D' \equiv P_aD \cap (O)$; maps

L into O and F into F', which is the midpoint of DD'. Hence, $\mathcal{I}(P_a, k) : (DLF) \mapsto (OF'D')$. However, since $\triangle OF'D'$ is a $F'-$ right triangle; thus the circumcenter of $(OF'D')$ must be the midpoint of OD'. Therefore, we deduce that $(OF'D')$ is tangent to (O) at D'. (O) is preserved by the inversion $\mathcal{I}(P_a, k)$. As a result, we obtain that (DLF) is tangent to (O) at D. Our proof is completed then. $\square$

Luis González

#7 Aug 2, 2010, 3:58 am • 5

Let P be the intersection of the tangents of (O) through B, C and Q be the intersection of the tangents of (O) through $A, D.$ Points B, C, Q are the poles of the sidelines MP, PN, NM of $\triangle MNP$ WRT $(O) \Longrightarrow E \equiv PQ \cap CM \cap BN.$ Thereby if PQ cuts AD, BC at $X, Y,$ then $(P, E, Y, X) = -1.$

Let $F' \equiv BC \cap DP, U \equiv AD \cap BC, V \equiv AP \cap BC$ and $K \equiv DP \cap (O).$ Since BC, PQ are the polars of P, U WRT $(O),$ it follows that $P(A, D, X, U) = P(V, F', Y, U) = -1,$ thus $A(P, F', K, D) = A(V, F', Y, U) = -1 \Longrightarrow K \in AY.$ Hence, $A(P, F', K, D) = A(P, E, Y, X) = -1$ implies that $F' \in AE \Longrightarrow F \equiv F'.$ If $R \equiv QD \cap BC,$ then from $(B, C, F, R) = -1,$ it follows that $RB \cdot RC = RD^2 = RF \cdot RL,$ i.e. $\odot(DLF)$ is tangent to (O) through $D.$

shinichiman

#8 Dec 24, 2014, 4:36 pm • 1

Let $BM \cap CN = P, BC \cap AD = K$. We get $PB = PC$. First, we will prove D, K, P are collinear. Applying Menelaus theorem to K, B, C and $\triangle PMN$ we have $\frac{BM}{BP} \cdot \frac{CP}{CN} \cdot \frac{KN}{KM} = 1$. Since $PB = PC$ so $\frac{BM}{CN} = \frac{KM}{KN}$. On the other hand, we also have $MB^2 = MA \cdot MD, NC^2 = ND \cdot NA$. Therefore we have

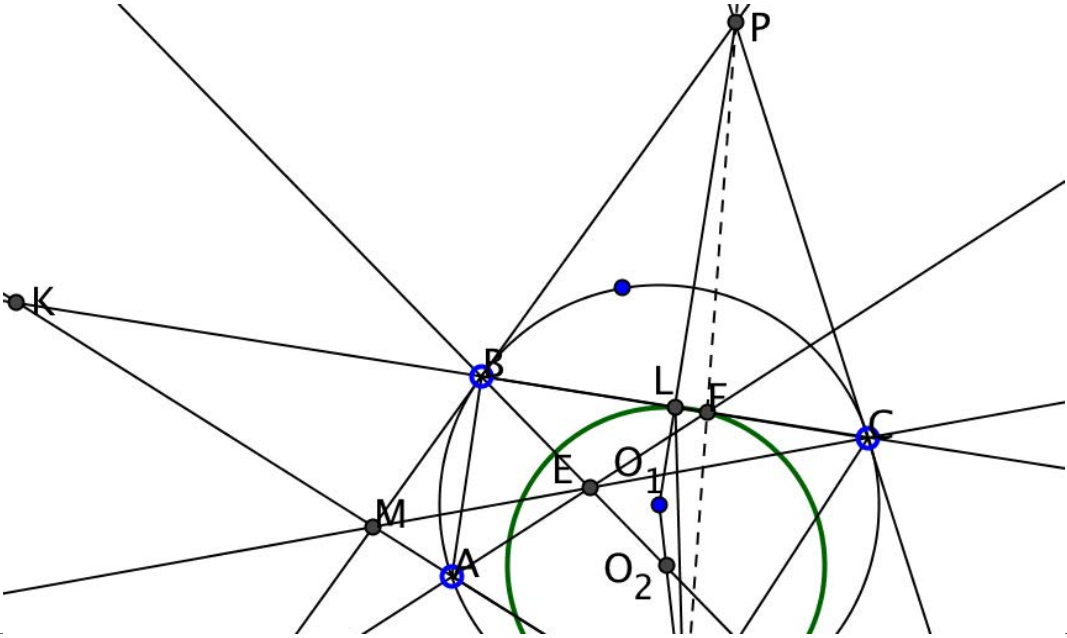
$$\frac{KM^2}{KN^2} = \frac{AM}{AN} \cdot \frac{DM}{DN} \left(= \frac{BM^2}{CN^2}\right).$$

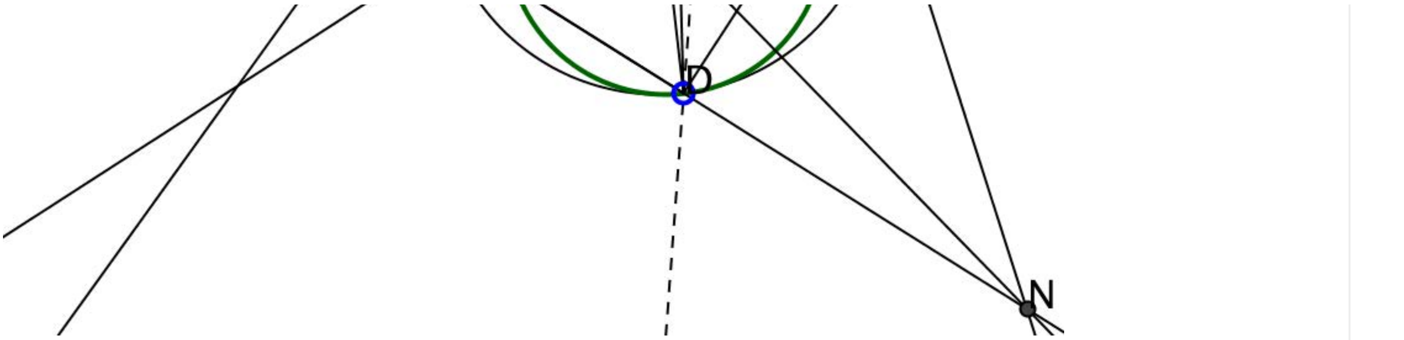
From here and since its is obvious that A, D are between M, N and K is not between M, N so $(KA, MN) = (DK, MN)$. We also have $E(KA, MN) = E(KF, CB)$ so $(KA, MN) = (KF, CB) = (DK, MN)$. Since $(KF, CB) = (NM, KD)$ so $(KC, FB) = (NK, MD)$ or $\frac{\overline{FK}}{\overline{FC}} \cdot \frac{\overline{DN}}{\overline{DK}} = \frac{\overline{BK}}{\overline{BC}} \cdot \frac{\overline{MN}}{\overline{ML}}$.
Applying Menelaus theorem to $\triangle KCN$, we see that

$$\overline{D, F, P} \iff \frac{\overline{FK}}{\overline{FC}} \cdot \frac{\overline{DN}}{\overline{DK}} \cdot \frac{\overline{PC}}{\overline{PN}} = 1 \iff \frac{\overline{BK}}{\overline{BC}} \cdot \frac{\overline{MN}}{\overline{ML}} \cdot \frac{\overline{PC}}{\overline{PN}} = 1 \iff \overline{P, M, B}.$$

Thus, D, F, P are collinear. Let O_1, O_2 be the circumcenter of triangle ABC, DLF, respectively. Since $O_1D^2 = O_1L \cdot O_1P$ so $\triangle O_1LD \sim \triangle O_1DP$ (S.A.S). We get $\angle O_1DL = \angle O_1PD = 90^\circ - \angle PFL = \angle LFD - 90^\circ$ (since F is between D and P). From here we also get that $\angle LFD \geq 90^\circ$. This means $\angle O_2DL = \angle LFD - 90^\circ$ or $\angle O_1DL = \angle O_2DL$. Also note that O_1, O_2 lie on the same side of LD. Thus, O_1, O_2, D are collinear or (DLF) is tangent to (O_1) through D.

Attachments:





TelvCohl

#9 Dec 24, 2014, 5:35 pm • 5

My solution:

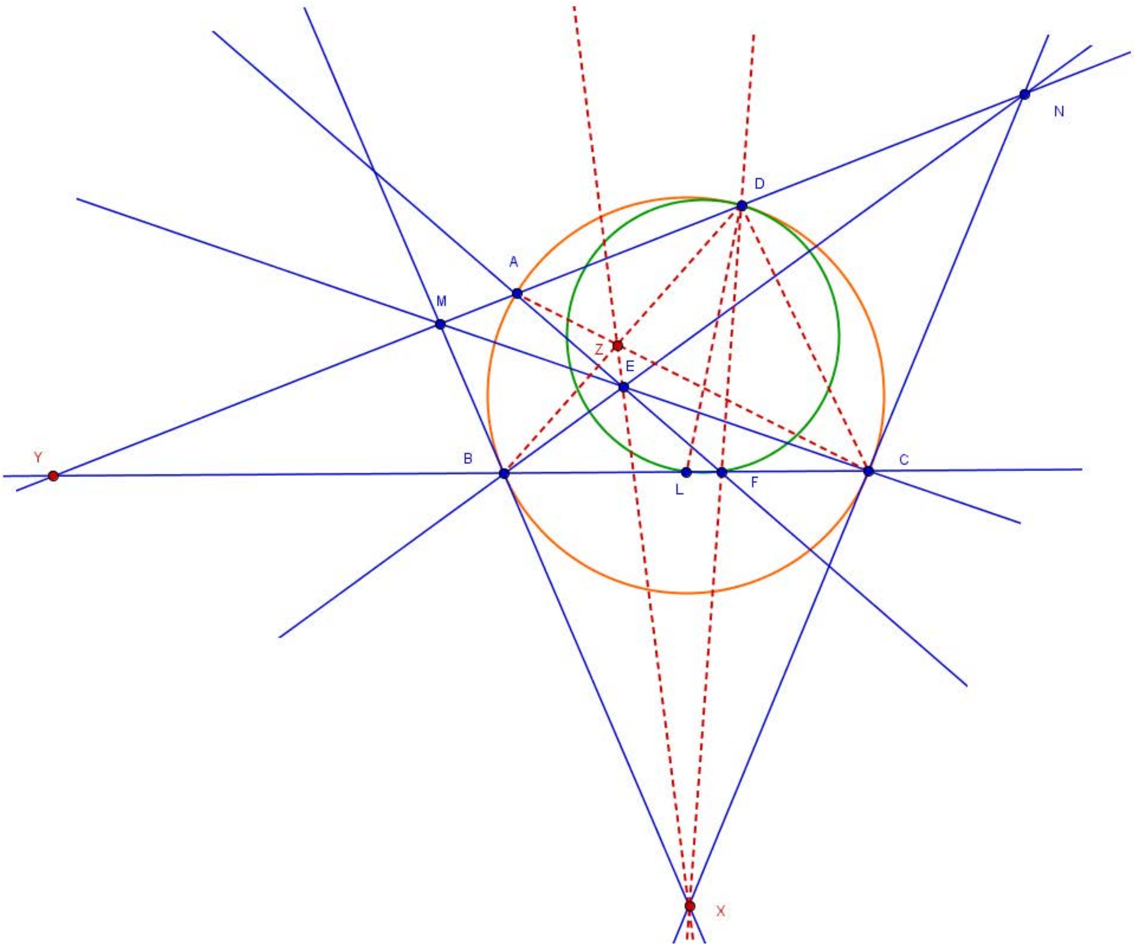
Let $X = BM \cap CN, Y = AD \cap BC, Z = AC \cap BD$.

Since BC is the polar of X WRT (O) ,
so from $X(E, Y; C, B) = -1$ we get XE is the polar of Y WRT (O) .
Since XZ is the polar of Y WRT (O) ,
so we get X, E, Z are collinear and $C(B, A; N, M) = B(C, D; M, N)$,
hence $(Y, F; B, C) = (Y, A; N, M) = C(B, A; N, M) = B(C, D; M, N) = (Y, D; M, N)$.
ie. D, F, X are collinear

Since DF is $D-$ symmedian of $\triangle DBC$,
so we get $\angle BDL = \angle FDC$ and (DLF) is tangent to (O) at D .

Q.E.D

Attachments:



buratinogigle

#10 Jan 1, 2015, 1:00 am

Rewrite this problem as following

Let ABC be a triangle inscribed (O). Tangent at B, C of (O) intersect at T. P is a point on (O). AP cuts TB, TC at E, F. CE cuts BF at D. AD cuts BC at G. M is midpoint of BC. Prove that circle (PDM) is tangent to (O).

Note that TP passes through G. We have general problem

Let ABC be a triangle inscribed (O). M, N lie on (O) such that $MN \parallel BC$ and AB is between AM, AC. CM cuts BN at S. BM cuts CN at T. P is a point on (O). PT cuts the line passing through S and parallel to BC at Q. Prove that circle (QPS) is tangent to (O).

Luis González

#11 Jan 1, 2015, 2:22 am • 1

A proof to buratinogigle's problem:

PT cuts (O) again at D and SD, SP cut (O) again at E, F, resp. From the complete quadrilateral $DEPF$, it follows that $EP \cap DF$ is the pole of ST WRT (O), but from the complete quadrilateral $BMNC$, this pole is the point at infinity of $BC \parallel MN \Longrightarrow DEPF$ is isosceles trapezoid with bases EP, DF parallel to BC. Thus, if τ denotes the tangent of (O) at P, we have $\angle(PS, \tau) = \angle PET = \angle EPT = \angle PQS \Longrightarrow (O)$ and $\odot(QPS)$ are tangent at P.

TelvCohl

#12 Jan 1, 2015, 6:39 am • 1

> *buratinogigle wrote:*
>
> Note that TP passes through G. We have general problem
>
> Let ABC be a triangle inscribed (O). M, N lie on (O) such that $MN \parallel BC$ and AB is between AM, AC. CM cuts BN at S. BM cuts CN at T. P is a point on (O). PT cuts the line passing through S and parallel to BC at Q. Prove that circle (QPS) is tangent to (O).

My solution:

Let $T' = PT \cap \odot(O)$ and $S' = PS \cap \odot(O)$.
Let $\{X, Y\} = SQ \cap \odot(O)$ and $\{U, V\} = TS \cap \odot(O)$.

Since SQ is the polar of T WRT $\odot(O)$,
so we get $P(T', S'; U, V) = (T, S; U, V) = -1$,
hence combine with $\angle UPV = 90^\circ$ we get $\angle BPQ = \angle SPC$ . ... $(\star)$
Since $\angle XPB = \angle CPY$ ($\because XY \parallel BC$) ,
so combine with $(\star)$ we get $\angle XPQ = \angle SPY$,
hence $\odot(PQS)$ is tangent to $\odot(O)$ at P .

Q.E.D

Quick Reply

High School Olympiads

Small problem about radical center lies on Euler line

Reply

Source: own

quangMavis1999

#1 Jan 13, 2016, 8:24 pm • 2

Problem Let $\triangle ABC$ with circumcenter O, orthocenter H, centroid G.. Let $\triangle DEF$ is pedal triangle of H WRT $\triangle ABC$, $\triangle XYZ$ is circumcevian triangle of a point G WRT $\triangle ABC$. Prove that radical center of three circles $(DYZ),(EXZ),(FXY)$ lies on Euler line of $\triangle ABC$

This post has been edited 2 times. Last edited by quangMavis1999, Jan 14, 2016, 11:50 pm

Luis González

#2 Jan 14, 2016, 2:25 am • 3

Let U,V,W be the midpoints of $BC,CA,AB.$ Since G is the insimilicenter of the circumcircle $(O)\equiv\odot(ABC)$ and 9-point circle $(N)\equiv\odot(DEF),$ then it is also center of its direct inversion $\Longrightarrow GV\cdot GY=GW\cdot GZ\Longrightarrow VWZY$ is cyclic. Thus YZ,VW and the common tangent ℓ_A of $\odot(ABC),\odot(AVW)$ concur at the radical center J of $\odot(AVW),(O)$ and $\odot(VWZY).$ Since D is the reflection of A across $VW,$ we have $JD^2=JA^2=JY\cdot JZ=JV\cdot JW\Longrightarrow\odot(DYZ)$ and (N) are tangent at D and likewise $\odot(EXZ)$ and $\odot(FXY)$ are tangent to $(N).$ Thus the radical axis of $\odot(EXZ)$ and $\odot(FXY)$ is the line τ_A connecting X and the intersection T of the tangents of (N) at $E,F.$ τ_B and τ_C are characterized analogously.

Let $\triangle A'B'C'$ be the tangential triangle of $\triangle ABC$ (A',B',C' againts A,B,C). Inversion with center H and power $HA\cdot HD=HB\cdot HE=HC\cdot HF$ swaps (O) and $(N),$ thus by conformity $\odot(T,TE)$ orthogonal to (N) goes to $\odot(A',A'B)$ orthogonal to $(O),$ so their inversion center H is also their insimilicenter.

On the other hand, let $\overline{AU}$ cut $\odot(T,TE)$ and $\odot(A',A'B)$ at $S,L,$ resp. Thus defining P such that $\triangle ABC\cup P\sim\triangle AEF\cup T$ and letting $\overline{AA'}$ cut $\odot(P,PB)$ at $M,$ then clearly $\triangle ABC\cup M\sim\triangle AEF\cup S.$ Notice that $\odot(P,PB)$ is the inverse of the circle (U) with diameter BC WRT $\odot(A',AB),$ thus if $\overline{AM}$ cuts (U) at $L',$ then M,L' are inverse points under this inversion $\Longrightarrow$ tangents of (U) and $\odot(P,PB)$ at L',M meet at a point A'' on their radical axis $BC.$ But L' is precisely the image of L under the inversion with center A and power $AB\cdot AC$ followed by reflection on the angle bisector of $\angle BAC$ (this swaps (U) and $\odot(A',A'B)$ due to conformity). Consequently, if the tangents of $\odot(T,TE)$ and $\odot(A',A'B)$ at S,L meet at $Q,$ we get $\angle QSL=\angle A''ML'=\angle A''L'M=\angle QLS\Longrightarrow\triangle QSL$ is Q-isosceles, i.e. $QS=QL\Longrightarrow Q$ is on the radical axis of $\odot(T,TE)$ and $\odot(A',A'B)\Longrightarrow S,L$ are inverse points under the direct inversion that swaps $\odot(T,TE)$ and $\odot(A',A'B)\Longrightarrow SL$ goes through their exsimilicenter $K.$

It's known that $A'X,B'Y,C'Z$ concur at a point on $OH;$ the Exeter point of $\triangle ABC$ (X_{22} of $\triangle ABC$). Thus if $TX\equiv\tau_A$ cuts OH at $R,$ we get $(R,H,G,X_{22})=X(R,H,G,X_{22})=X(T,H,K,A')=(T,H,K,A')=-1\Longrightarrow R$ is the harmonic conjugate of X_{22} WRT $H,G.$ Similarly τ_B and τ_C hit OH at the same point $R\Longrightarrow R$ is the radical center of $\odot(DYZ),\odot(EXZ),\odot(FXY)$ lying on the Euler line $OH.$

P.S. This radical center $R,$ being the harmonic conjugate of the Exeter point WRT the orthocenter and the centroid, turns out to be the triangle center $X_{5133}.$

buratinogigle

#3 Jan 14, 2016, 10:22 am • 1

That's strange problem, I see a similar problem

Let ABC be a triangle with circumcircle (O) and Lemoine point L. D,E,F are midpoints of BC,CA,AB. XYZ is circumcevian triangle of L. Prove that radical center of circles $(DYZ),(EZX),(FXY)$ lies on line OL.

TelvCohl

#4 Jan 14, 2016, 12:38 pm • 4

Let $\triangle T_dT_eT_f$ be the tangential triangle of $\triangle DEF$ and let N be the 9-point center of $\triangle ABC$. Since G is the insimilicenter of $\odot(N) \sim \odot(O)$, so there exist an inversion $\mathbf{I}_G$ with center G that swaps $\odot(N)$ and $\odot(O)$. Let P^* be the image of P under $\mathbf{I}_G$ (Clearly, X^*, Y^*, Z^* is the midpoint of BC, CA, AB, respectively.) and let $L \equiv T_eT_f \cap Y^*Z^*, M\ (\neq D^*) \equiv DG \cap \odot(O)$.

From symmetry we get LA is tangent to $\odot(O)$ at A and $LA = LD$, so combine $\measuredangle DD^*A = \measuredangle(T_eT_f, Y^*Z^*)$ (notice AM is parallel to BC) $= \frac{1}{2}\measuredangle DLA \Longrightarrow L$ is the circumcenter of $\triangle ADD^*$, hence LD^* is tangent to $\odot(O)$ at D^*. On the other hand, from $LD^{*2} = LD^2 = LY^* \cdot LZ^*$ we get LD^* is tangent to $\odot(D^*Y^*Z^*)$ at D^*, so $\odot(D^*Y^*Z^*)$ is tangent $\odot(O)$ at $D^* \Longrightarrow \odot(DYZ)$ is tangent to $\odot(N)$ at D. Similarly, we can prove $\odot(EZX), \odot(FXY)$ are tangent to $\odot(N)$.

Since Y, Z, Y^*, Z^* are concyclic, so L is the radical center of $\odot(O), \odot(N)$ and $\odot(YZY^*Z^*)$, hence $L \in YZ$. Analogously, we can prove ZX, Z^*X^*, T_fT_d are concurrent and XY, X^*Y^*, T_dT_e are concurrent, so $\triangle XYZ$, $\triangle X^*Y^*Z^*, \triangle T_dT_eT_f$ share the same perspectrix, hence G (perspector of $\triangle XYZ, \triangle X^*Y^*Z^*$), N (perspector of $\odot X^*Y^*Z^*, \triangle T_dT_eT_f$) and the perspector K of $\triangle T_dT_eT_f, \triangle XYZ$ are collinear $\Longrightarrow K$ lies on the Euler line of $\triangle ABC$.

On the other hand, from $T_dE^2 = T_dF^2 \Longrightarrow XT_d$ is the radical axis of $\odot(EZX)$ and $\odot(FXY)$. Similarly, we can prove YT_e is the radical axis of $(\odot(FXY), \odot(DYZ))$ and ZT_f is the radical axis of $(\odot(DYZ), \odot(EZX))$, so we conclude that K is the radical center of $\odot(DYZ), \odot(EZX), \odot(FXY)$.

quangMavis1999

#5 Jan 14, 2016, 4:06 pm • 1

Thanks for yours interest.

Summary, my bad solution : Let X', Y', Z' is midpoint of BC, CA, AB,resp. Invert $\mathbf{I}_G^k (k = GX'.GX)$ that swaps : $D \to D', Y \to Y', Z \to Z'$. Let U is projection of O to $Y'Z'$, GU intersect (O) at a point P ($U \in DG$, $AP||BC$ (well-known))).

Let Q is intersection of EF and AA. The tangents from Q intersect (O) at a point D^*. We get A, Q, D^*, U, O are concylic $\Rightarrow \angle AD^*U = \angle AQU = \dfrac{1}{2}$ arc $AP = \angle AD'U. \Rightarrow D' \equiv D^* \Rightarrow (Y'Z'D')$ are tangent to (O)

Invert $\mathbf{I}_O^l (l = OA^2)$: $A \to A, B \to B, C \to C, X' \to X^*, Y' \to Y^*, Z' \to Z^*, D' \to D'$ and (Y^*Z^*D') are tangent to (O). Similarly, we get the links: The three lines AA', BB' and CC' meet on the line IO

This post has been edited 3 times. Last edited by quangMavis1999, Jan 15, 2016, 12:04 am

toto1234567890

#7 Jan 14, 2016, 9:05 pm

My solution is shorter using Desargue(not involution)

Let U, V, W be the midpoints of BC, CA, AB. Since G is the insimilicenter of the circumcircle $(O) \equiv \odot(ABC)$ and 9-point circle $(N) \equiv \odot(DEF)$, then it is also center of its direct inversion $\Longrightarrow GV \cdot GY = GW \cdot GZ \Longrightarrow VWZY$ is cyclic. Thus YZ, VW and the common tangent ℓ_A of $\odot(ABC), \odot(AVW)$ concur at the radical center J of $\odot(AVW), (O)$ and $\odot(VWZY)$. Since D is the reflection of A across VW, we have $JD^2 = JA^2 = JY \cdot JZ = JV \cdot JW \Longrightarrow \odot(DYZ)$ and (N) are tangent at D and likewise $\odot(EXZ)$ and $\odot(FXY)$ are tangent to (N). Thus the radical axis of $\odot(EXZ)$ and $\odot(FXY)$ is the line τ_A connecting X and the intersection T_A of the tangents of (N) at E, F. τ_B and τ_C are characterized analogously.

Let's use desargue in $\triangle UXT_A, \triangle VYT_B$. Then we can know that XT_A and YT_B meet on NG which is the Euler Line.

This post has been edited 1 time. Last edited by toto1234567890, Jan 14, 2016, 9:32 pm

TelvCohl

#9 Jan 15, 2016, 7:05 pm • 1

Small remark on Luis's proof (keep the same notation (A', P, R, T, U, X_{22}) in Luis's proof) **:** After proving the collinearly of A', H and T, we can finish the proof as following **:** Notice BC is the polar of P WRT $\odot(BOC)$ we get

$$(G, H; R, X_{22}) \stackrel{X}{=} (AX \cap A'T, H; T, A') = A(U, H; T, A') = A(A', O; P, U) = (A', O; P, U) = -1.$$

Quick Reply

High School Olympiads

Two perspective and inversely similar triangles

Source: Own

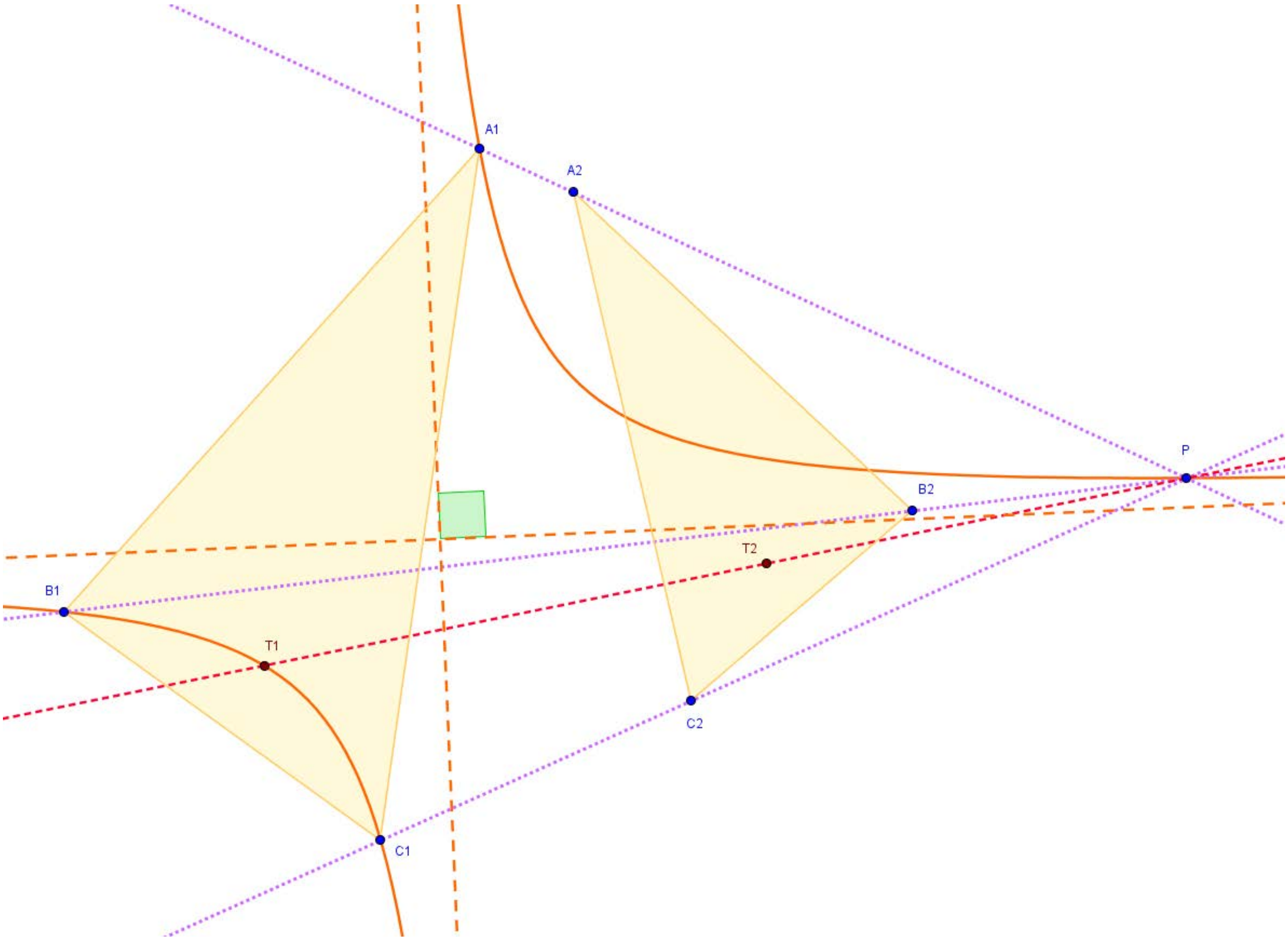
TelvCohl

#1 Jun 6, 2015, 3:38 pm • 2

Let $\triangle A_1B_1C_1, \triangle A_2B_2C_2$ be 2 perspective and inversely similar triangles .
Let $P \equiv A_1A_2 \cap B_1B_2 \cap C_1C_2$ and T_1, T_2 be the points s.t. $\triangle A_1B_1C_1 \cup T_1 \sim \triangle A_2B_2C_2 \cup T_2$.

Prove that T_1, T_2, P are collinear $\Longleftrightarrow$ A_1, B_1, C_1, T_1, P lie on a rectangular hyperbola

Attachments:



Luis González

#3 Jan 13, 2016, 4:09 am • 1

Let the perpendiculars from B_1 and C_1 to C_2A_2 and A_2B_2, resp, intersect at O_1. Then $\angle(O_1B_1, O_1C_1) = \angle(A_2C_2, A_2B_2) = \angle(A_1B_1, A_1C_1) \Longrightarrow O_1 \in \odot(A_1B_1C_1) \Longrightarrow$ $\angle(O_1C_1, O_1A_1) = \angle(B_1C_1, B_1A_1) = \angle(B_2A_2, B_2C_2) \Longrightarrow A_1O_1 \perp B_2C_2 \Longrightarrow \triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ are orthologic, so let O_2 be the orthology center of $\triangle A_2B_2C_2$ WRT $\triangle A_1B_1C_1$. Since these triangles are also perspective with perspector T, then by Sondat's theorem T, O_1, O_2 are collinear and the conics through A_1, B_1, C_1, T, O_1 and A_2, B_2, C_2, T, O_2 are rectangular hyperbolae $\mathcal{H}_1$ and $\mathcal{H}_2$, resp (well-known for orthologic triangles).

Let $H_1 \in \mathcal{H}_1$ and $H_2 \in \mathcal{H}_2$ be the orthocenters of $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$, resp. Redefining T_1 as a point on $\mathcal{H}_1$ and T_2 the second intersection of TT_1 with $\mathcal{H}_2$, then it's enough to prove that $\triangle A_1B_1C_1 \cup T_1 \sim \triangle A_2B_2C_2 \cup T_2$.

We know that H_1, H_2 and T are collinear (see for instance the topic A Nice Result. post #7). Thus, using cross ratios on $\mathcal{H}_1$ and $\mathcal{H}_2$, we obtain $C_1(A_1, B_1, O_1, H_1) = T(A_1, B_1, O_1, H_1) \equiv T(A_2, B_2, O_2, H_2) = C_2(A_2, B_2, O_2, H_2)$. But clearly $\triangle A_1B_1C_1 \cup \mathcal{H}_1, H_1, O_1 \sim \triangle A_2B_2C_2 \cup \mathcal{H}_2, H_2, O_2$, thus we deduce that $\triangle A_1B_1C_1 \cup T_1 \sim \triangle A_2B_2C_2 \cup T_2$, as desired.

Luis González

#4 Jan 13, 2016, 9:12 am

Actually the previous approach can be even simpler. Either the collinearity of T, O_1, O_2 or T, H_1, H_2 is sufficient to yield the result. So for example using the collinearity of T, O_1, O_2 found in the first paragraph, we have $C_1(A_1, B_1, O_1, T_1) = T(A_1, B_1, O_1, T_1) \equiv T(A_2, B_2, O_2, T_2) = C_2(A_2, B_2, O_2, T_2)$ and the conclusion follows.

Quick Reply

High School Olympiads

A Nice Result.

Reply

Source: Mathley

doxuanlong15052000

#1 Jan 7, 2016, 11:05 am

Let ABC be a triangle. Points E and F are outside the triangle such that EA=EC and FA=FB and angle AEC=angle AFB. Let H and K be the orthocenters of the triangles AEC and AFB respectively. Prove that EF, HK, BC are concurrent.

ThE-dArK-lOrD

#2 Jan 7, 2016, 7:43 pm

Let $P = BK \cap CH, Q = BF \cap CE$ and O is circumcenter of triangle ABC
By Desargue's Theorem on $\triangle BFK, \triangle CHE$
We need to proof P, O, Q collinear
Let BO cut PQ at T
We will proof that $\angle TCB = 90 - \angle BAC$
Let $\angle CAE = \angle ACE = \angle BAF = \angle ABF = m$
We get $\angle BFA = \angle CAE = 180 - 2m$
Simple angle-chasing gives us $\angle QBC = 180 - m - \angle B$ and $\angle QCB = 180 - m - \angle C$
By Ceva's Theorem with cevian PQ, BQ, CQ of triangle PBC
We get $\frac{sin(\angle BPT)}{sin(\angle CPT)} \frac{sin(270 - m)}{sin(m + \angle C)} \frac{sin(m + \angle B)}{sin(270 - m)} = 1$
So $\frac{sin(\angle BPT)}{sin(\angle CPT)} = \frac{sin(m + \angle C)}{sin(m + \angle B)}$
By Ceva's Theorem with cevian PT, BT, CT of triangle PBC
Let $\angle BCT = t$ give $\angle ACT = \angle C - t$
We get $\frac{sin(\angle BPT)}{sin(\angle CPT)} \frac{sin(90 - m + \angle C - t)}{sin(t)} \frac{sin(90 - \angle A)}{sin(m + \angle C)} = 1$
Plugging value of $\frac{sin(\angle BPT)}{sin(\angle CPT)}$
Give us $\frac{sin(90 - \angle A)}{sin(m + \angle B)} = \frac{sin(t)}{sin(90 - m + \angle C - t)}$
We easily get that $t = 90 - \angle A$ is uniquely satisfy the equation.
So $\angle TCB = 90 - \angle A$
Which mean $T = O$

Another result
Let H is orthocenter of triangle ABC
Proof that P, Q, A, B, C, H lie on the same hyperbola.

This post has been edited 4 times. Last edited by ThE-dArK-lOrD, Jan 7, 2016, 8:05 pm

doxuanlong15052000

#3 Jan 7, 2016, 8:18 pm

Thank you !!

ThE-dArK-lOrD

#4 Jan 7, 2016, 9:08 pm

Try to proof that A, B, C, H, P, Q lie on the same hyperbola or
Locus of point Q when we varied point E on perpendicular bisector of AC is hyperbola pass through A, B, C, H

PROF65

#5 Jan 8, 2016, 5:22 am

First the condition of E and F isn't sufficient . Either BFK and CEH overlap ABC at the same time or they don't overlap it.
Let $S = BF \cap CE$ and $FK \cap EH = O, T = BK \cap OS, T' = CH \cap OS, U = BC \cap OS$ (O circumcenter of triangle ABC).
WLG consider BFA and ACH acute triangle ,$\widehat{ABF} = \widehat{ACE} = \alpha, \widehat{KBF} = \widehat{HCE} = \frac{\pi}{2} - \alpha$.

$$\frac{\sin \widehat{UBS}}{\sin \widehat{UBT}} : \frac{\sin \widehat{OBS}}{\sin \widehat{OBT}} = \frac{\sin(\hat{B}+\alpha)}{\sin(\hat{B}+\frac{\pi}{2}-\alpha)} : \frac{\sin(\frac{\pi}{2}-\hat{C}+\alpha)}{\sin(\frac{\pi}{2}-\hat{C}+\frac{\pi}{2}-\alpha)} = \frac{\sin(\hat{B}+\alpha)}{\cos(\hat{B}-\alpha)} : \frac{\cos(\hat{C}-\alpha)}{\sin(\hat{C}+\alpha)}$$

similarly we got $\frac{\sin \widehat{UCS}}{\sin \widehat{UCT'}} : \frac{\sin \widehat{OCS}}{\sin \widehat{OCT'}} = \frac{\sin(\hat{B}+\alpha)}{\cos(\hat{B}-\alpha)} : \frac{\cos(\hat{C}-\alpha)}{\sin(\hat{C}+\alpha)}$
which means $B(U, O; S, T) = C(U, O; S, T')$ then $T = T'$ finaly by desargues we conclude
WCP

TelvCohl

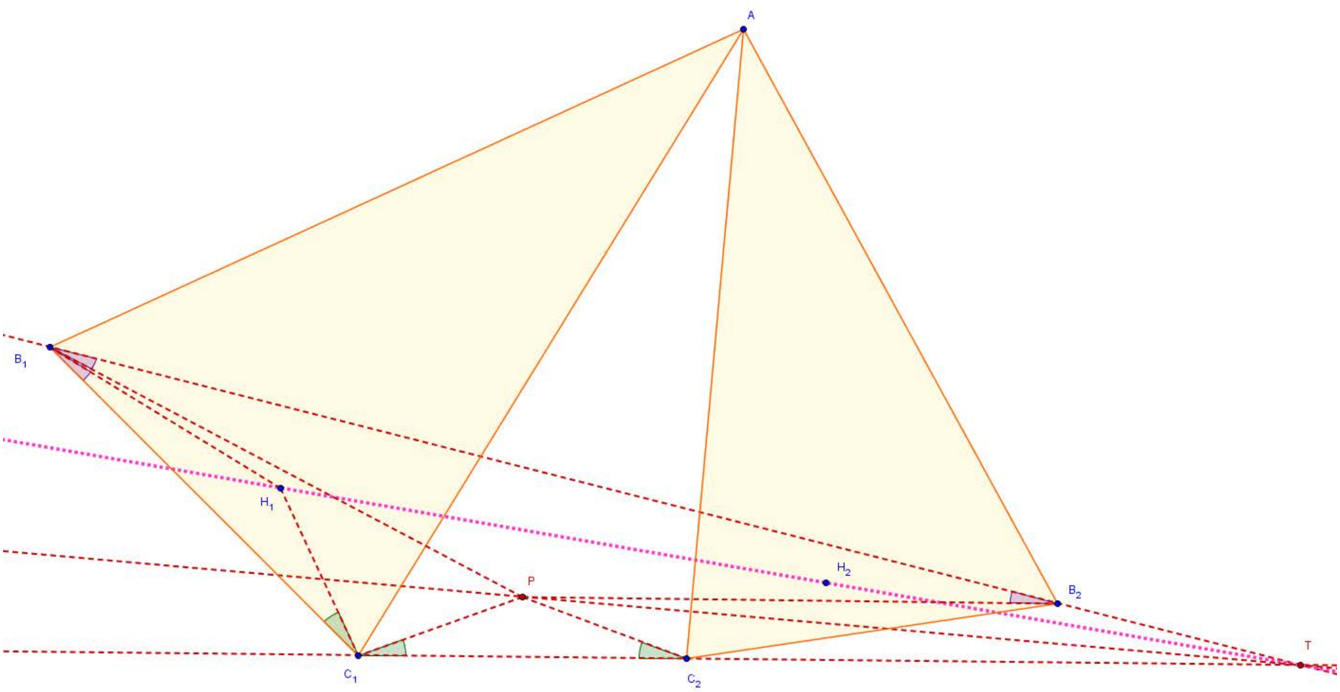
#7 Jan 8, 2016, 7:22 am • 1

Generalization : Given two inversely similar triangles $\triangle AB_1C_1$ and $\triangle AB_2C_2$. Let H_1, H_2 be the orthocenter of $\triangle AB_1C_1$, $\triangle AB_2C_2$, respectively. Then B_1B_2, C_1C_2, H_1H_2 are concurrent.

Proof : Let P be the point such that $PB_1 = PB_2, PC_1 = PC_2$ and let $T \equiv B_1B_2 \cap C_1C_2$. From 2000 IMO Shortlist G6 $\Longrightarrow$ $\measuredangle C_1PC_2 = 2\measuredangle C_1B_1A$, so $\measuredangle TC_1P = 90^\circ - \measuredangle C_1B_1A = \measuredangle H_1C_1B_1$. Analogously, we can prove $\measuredangle TB_1P = \measuredangle H_1B_1C_1$, so H_1 and P are isogonal conjugate WRT $\triangle TB_1C_1 \Longrightarrow H_1$ lies on the isogonal conjugate τ of TP WRT $\angle(B_1B_2, C_1C_2)$. Similarly, we can prove H_2 lies on τ, so T, H_1, H_2 are collinear. i.e. B_1B_2, C_1C_2, H_1H_2 are concurrent

Remark : See also here (another generalization) and here (generalization of **Generalization**).

Attachments:



Quick Reply

High School Olympiads

Harmonic pencil

Locked

Source: Vietnam national math Olympiad 2003

amrit1968

#1 Jan 11, 2016, 11:39 pm

(Vietnam National Olympiad 2003, Problem 2) The circles C1 and C2 touch externally at M and the radius of C2 is larger than that of C1. Let A be a point on C2 which does not lie on the line joining the centers of the circles, B and C points on C1 such that AB and AC are tangent to C1. The lines BM, CM intersect C2 again at E, F respectively. Let D be the intersection of the tangent at A and the line EF. Show that the locus of D as A varies is a straight line.

Luis González

#2 Jan 11, 2016, 11:52 pm

Please check the Contests Collections before posting problems you know the source. This was already posted at http://www.artofproblemsolving.com/community/c6h16007p112198.

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- IMO Shortlist
- IMO Longlists
- APMO
- Austrian-Polish
- Balkan MO
- Baltic Way
- Benelux
- CentroAmerican
- Cono Sur Olympiad

National and Regional Conte…
National and regional competitions

- USA Contests — Contests from USA
- Albania Contests — Contests from Albania
- Argentina Contests — Contests from Argentina
- Australia Contests — Contests from Australia
- Austria Contests — Contests from Austria
- Azerbaijan Contes… — Contests from Azerbaijan
- Belarus Contests — Contests from Belarus
- Belgium Contests — Contests from Belgium
- Bosnia Herzegovin.. — Contests from Bosnia Her..
- Brazil Contests — Contests from Brazil

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Collegiate competitions

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- Brazil Undergrad M..
- IberoAmerican Olym.
- IMC
- IMS
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- Nanyang Technolo…
- Paenza
- Putnam
- Ural State Universi…

National Olympiads
National math olympiads from around the world

- Albania National O…
- All-Russian Olymp…
- Austria Federal Co..
- Austria Regional C..
- Azerbaijan Nationa..
- Brazil National Oly…
- Bulgaria National O.
- Canada National O..
- China Girls Math O..
- China National Oly…

Team Selection Tests
IMO Team Selection Tests

- Albania Team Sele…
- Argentina Team Se..
- Austrian-Polish
- Balkan MO TST
- Belarus Team Sele..
- Bosnia Herzegovin.
- Bulgaria Team Sel…
- China Team Select..
- Croatia Team Sele…
- Estonia Team Sele…

Junior Olympiads
Junior olympiads from around the world

- Flanders Junior Ol…
- JBMO ShortLists
- JBMO TST Azerbaij…
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- Junior Olympiad of…
- Turkey Junior Natio..
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2016 Contests
Contests in the 2015-16 school year

- 2016 AIME Problem. — AIME Problems 2016
- 2016 All-Russian O.. — All-Russian Olympiad 2016
- 2016 AMC 10 — AMC 10 2016
- 2016 AMC 12/AHSM. — AMC 12/AHSME 2016
- 2016 Azerbaijan Ba.. — Azerbaijan Balkan MO TS…
- 2016 Azerbaijan Ju.. — Azerbaijan Junior Mathem..
- 2016 Balkan MO — Balkan MO 2016
- 2016 BAMO — 2016, Bay Area Mathema…
- 2016 Benelux — Benelux 2016
- 2016 Bosnia and H.. — Bosnia and Herzegovina …

2015 Contests
Contests in the 2014-15 school year

- 2015 AIME Problem. — AIME Problems 2015
- 2015 All-Russian O.. — All-Russian Olympiad 2015
- 2015 AMC 10 — AMC 10 2015
- 2015 AMC 12/AHSM. — AMC 12/AHSME 2015
- 2015 AMC 8 — AMC 8 2015
- 2015 AoPS Mathem..
- 2015 APMO — APMO 2015
- 2015 Azerbaijan IM.. — Azerbaijan IMO TST 2015
- 2015 Azerbaijan JB.. — 2015 Azerbaijan JBMO TST
- 2015 Azerbaijan Na.. — Azerbaijan National Olymp..

2014 Contests

Contests in the 2013-14 school year

- **2014 AIME Problem.** AIME Problems 2014
- **2014 All-Russian O..** All-Russian Olympiad 2014
- **2014 AMC 10** AMC 10 2014
- **2014 AMC 12/AHSM.** AMC 12/AHSME 2014
- **2014 AMC 8** AMC 8 2014
- **2014 APMO** APMO 2014
- **2014 Balkan MO** Balkan MO 2014
- **2014 Baltic Way** Baltic Way 2014
- **2014 BAMO** 2014, Bay Area Mathema...
- **2014 Benelux** Benelux 2014

2013 Contests

Contests in the 2012-13 school year

- **2013 AIME Problem.** AIME Problems 2013
- **2013 Albania Team..** Albania Team Selection T...
- **2013 All-Russian O..** All-Russian Olympiad 2013
- **2013 AMC 10** AMC 10 2013
- **2013 AMC 12/AHSM.** AMC 12/AHSME 2013
- **2013 AMC 8** AMC 8 2013
- **2013 APMO** APMO 2013
- **2013 Balkan MO** Balkan MO 2013
- **2013 Baltic Way** Baltic Way 2013
- **2013 BAMO** 2013, Bay Area Mathema...

New Post Collections

New contests and other new collections of posts

- **2005 Greece Team..** Greece Team Selection T...
- **2008 Greece Team..** Greece Team Selection T...
- **2009 Greece Team..** Greece Team Selection T...
- **2011 Greece Team..** Greece Team Selection T...
- **2012 Greece Team..** Greece Team Selection T...
- **2013 China Second.** Second Round Olympiad ...
- **2013 Greece Team..** Greece Team Selection T...
- **2014 China Second.** Second Round Olympiad ...
- **2014-2015 Turkme...**
- **2015 Benelux** Benelux 2015

High School Olympiads

Straight line

Reply

Source: 41-st Vietnamese Mathematical Olympiad 2003

uTOPi_a

#1 Aug 28, 2004, 10:18 am • 1

The circles C_1 and C_2 touch externally at M and the radius of C_2 is larger than that of C_1. A is any point on C_2 which does not lie on the line joining the centers of the circles. B and C are points on C_1 such that AB and AC are tangent to C_1. The lines BM, CM intersect C_2 again at E, F respectively. D is the intersection of the tangent at A and the line EF. Show that the locus of D as A varies is a straight line.

grobber

#2 Aug 28, 2004, 7:54 pm • 1

And that line is the perpendicular to the line of the centers from M. EF is the image of BC through the homothety of center M turning C_1 into C_2, which means that the pole of EF wrt C_2 is the image of A through this homothety. Call this point A'. Since A, M, A' are collinear, we get that AA' passes through M, so the intersection of the polars of A, A' wrt C_2 must lie on a line, which is the polar of M wrt C_2 (duality principle), and this line is the tangnt from M to C_2.

darij grinberg

#3 Aug 28, 2004, 9:29 pm

Okay, I have the same proof but in a different form and a bit more detailed, so I'm posting it...

In fact, the locus of the point D is the common tangent of the circles C_1 and C_2 at the point M. In order to prove this, I will show that the point D always lies on the common tangent of the circles C_1 and C_2 at the point M.

Here is how I prove it (sorry, the proof uses pole-polar relationship): Let m be the common tangent of the circles C_1 and C_2 at the point M. Then we have to show that the point D lies on the line m.

The polar of the point A with respect to the circle C_1 is the line BC, since the points B and C are the points where the tangents from A to C_1 touch C_1. Now, since the line MA passes through the point A, the pole of the line MA with respect to the circle C_1 must lie on the polar of the point A with respect to C_1. Hence we have obtained the fact that the pole of the line MA with respect to the circle C_1 lies on the line BC.

Now, since M is the point of tangency of the circles C_1 and C_2, there exists a homothety h with center M mapping the circle C_1 to the circle C_2. This homothety h must take the points B and C to the points E and F, respectively (since the points E and F lie on the circle C_2 and on the lines BM and CM, respectively). Hence, this homothety h takes the line BC to the line EF. Also, this homothety h leaves the line MA invariant (since this line passes through the center M of the homothety). Finally, polar relation is clearly invariant under homotheties. Hence, from the fact that the pole of the line MA with respect to the circle C_1 lies on the line BC, we can derive using our homothety h that the pole of the line MA with respect to the circle C_2 lies on the line EF. But the pole of the line MA with respect to the circle C_2 is the point of intersection of the tangents to C_2 at the points M and A. The tangent to C_2 at M is the line m that we have met before. Hence, we see that the point of intersection of the line m with the tangent to C_2 at A lies on the line EF. In other words, the point of intersection D of the tangent to C_2 at A with the line EF must lie on the line m. And this is exactly what we wanted to prove.

Darij

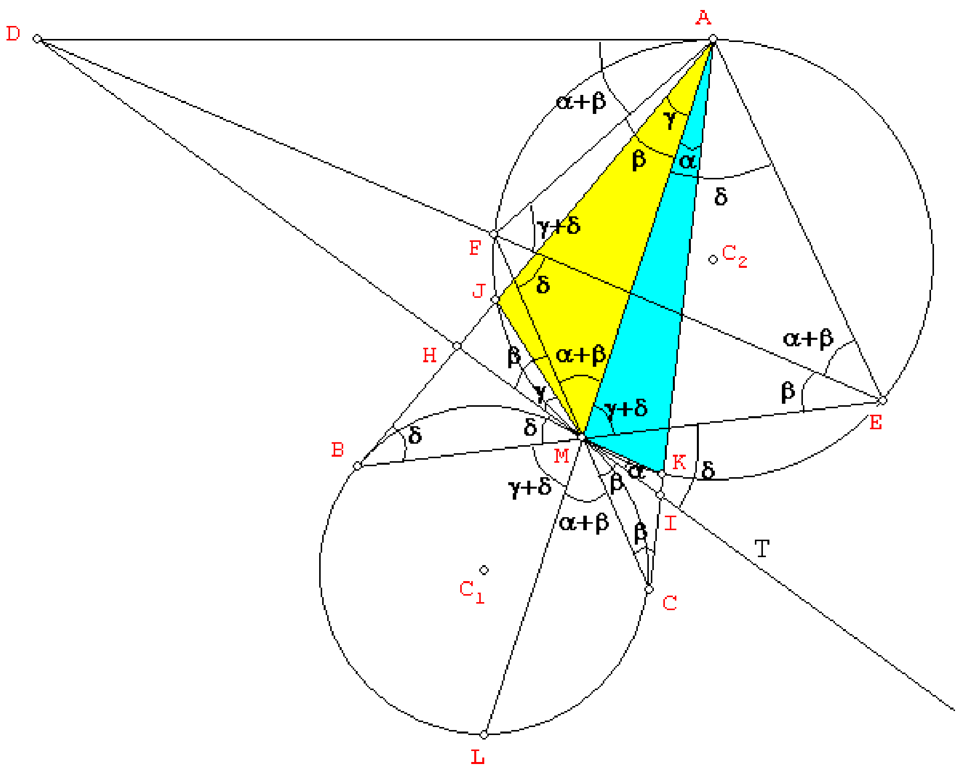
juancarlos

#4 Aug 30, 2004, 2:36 am

The condition:...the radius of C_2 is larger than that of C_1...
It is not necessary.
$AFME$ performed in this way is harmonic quadrilateral, also: $AE.MF = AF.ME$

From the another view point: AB, AC cut at J, K to circle C_2, we know:
MB and MC are external bisector of AMJ and AMK triangles.
Here the proof:
The line AM cut at L to circle C_1.
Draw the line T common tangent at M for two circles C_1 and C_2.
The tangent line T cut at H, I to AB, AC ,then:
$< HMJ =< MAJ = \gamma, < LMB =< MAB + < MBA = \gamma + \delta$,
$< MBA =< MBH =< HMB = \delta$, so $< BMJ =< BML = \gamma + \delta$ then MB is external bisector of AMJtriangle.
$< KMI =< KAM = \alpha, < CML =< CAM + < ACM = \alpha + \beta$,
$< ACM =< ICM =< IMC = \beta$, so MC is external bisector of AMK triangle.
The another angular relations in the figure.
It is not hard prove that the tangent line T at M and the tangent at A with EF are concur at D.
Therefore the locus is the tangent line T at M.

Attachments:



yetti
#5 Dec 17, 2006, 2:21 pm

Let $O_1,\ O_2$ be centers and $r_1,\ r_2$ radii of the circles $\mathcal{C}_1,\ \mathcal{C}_1$ touching at M. These 2 circles can touch either externally or internally. Only in the case of internal tangency we need $r_2 > r_1$, otherwise the tangents to $\mathcal{C}_1$ from $A \in \mathcal{C}_2$ would not exist. Assuming external tangency, M is the internal similarity center of the circles $\mathcal{C}_1,\ \mathcal{C}_2$, hence $\mathcal{C}_1 \sim \mathcal{C}_2$ are centrally similar with similarity center M and similarity coefficient $-\frac{r_1}{r_2}$. The triangles $\triangle MBC \sim \triangle MEF$ are centrally similar with the same similarity center and coefficient, hence their corresponding sides $BC \parallel EF$ are parallel. Since BC is a polar of A with respect to $\mathcal{C}_1$, $BC \perp AO_1$ and consequently, $EF \perp AO_1$ as well. Let AM meet $\mathcal{C}_1$ at Z. Since AB is a tangent of $\mathcal{C}_1$ at B, $\angle ABM = \angle MZB = \angle MAE$. Using the sine theorem for the triangles $\triangle ABM,\ \triangle AEM$ then yields

$$\frac{EA}{EM} = \frac{\sin \widehat{AME}}{\sin \widehat{MAE}} = \frac{\sin \widehat{AMB}}{\sin \widehat{ABM}} = \frac{AB}{AM}, \quad EA^2 = EM^2 \cdot \frac{AB^2}{AM^2}$$

Because of the central similarity of $\mathcal{C}_1,\ \mathcal{C}_2$ with center M, $AZ = AM + MZ = AM\left(1 + \frac{r_1}{r_2}\right)$ and $EB = EM + MB = EM\left(1 + \frac{r_1}{r_2}\right)$. Power of A to $\mathcal{C}_1$ is then $AB^2 = AM \cdot AZ = AM^2\left(1 + \frac{r_1}{r_2}\right)$. Substituting this to the above equation yields $EA^2 = EM \cdot EB$, which means that E lies on the radical axis of the point A and the circle $\mathcal{C}_1$. Since $EF \perp AO_1$, EF is their radical axis. On the other hand, the tangent t_A of $\mathcal{C}_2$ at A is the radical axis of the point A and the circle $\mathcal{C}_2$. The radical axes $EF,\ t_A$ meet at the radical center D of the point A and the circles $\mathcal{C}_1,\ \mathcal{C}_2$, which lies on the radical axis t_M of the circles $\mathcal{C}_1,\ \mathcal{C}_2$, their single common internal tangent at M.

yetti

#6 Dec 19, 2006, 2:23 pm

And I could have made this simpler, too:

... Since AB is a tangent of $\mathcal{C}_1$ at B, $\angle ABM = \angle MZB = \angle MAE.$ The triangles $\triangle ABE \sim \triangle MAE$ with a common angle at the vertex E are then similar, having equal angles, hence

$$\frac{EA}{AB} = \frac{EM}{AM}, \quad EA^2 = EM^2 \cdot \frac{AB^2}{AM^2}$$

etc.

Virgil Nicula

#7 Dec 19, 2006, 8:40 pm

Very nice both the proposed problem and the Yetti proof !

Lemma. If the circles c , c' are tangent in the point T and are given the points $\{A, B\} \subset c$, $\{A', B'\} \subset c'$ so that $T \in AA' \cap BB'$, then $AB \parallel A'B'$.

Another proof of the proposed problem (similarly with the Yetti's).

Denote the second intersection L of the line AM with the circle C_1 . Therefore,

$$\left.\begin{array}{ccccccc} BL \parallel AE & \Longrightarrow & \widehat{ABE} \equiv \widehat{BLA} \equiv \widehat{EAM} & \Longrightarrow & EAM \sim EBA & \Longrightarrow & EA^2 = EM \cdot EB \\ CL \parallel AF & \Longrightarrow & \widehat{ACF} \equiv \widehat{CLA} \equiv \widehat{FAM} & \Longrightarrow & FAM \sim FCA & \Longrightarrow & FA^2 = FM \cdot FC \end{array}\right\|$$

$\Longrightarrow$ the line EF is the radical axis between the circle C_1 and the point A (null circle).

$D \in EF \cap AA$ and AA is the radical axis of the circles C_2 and A $\Longrightarrow D \in MM$.

This post has been edited 5 times. Last edited by Virgil Nicula, Dec 20, 2006, 1:26 am

silouan

#8 Dec 19, 2006, 10:21 pm

Thank you very much. Very nice solution mr Virgil .

I remember I friend Nick Rapanos solved it by homothety ,but I don't remember the full solution .Could anyone find such one ?

vittasko

#9 Dec 23, 2006, 3:34 am

We denote as $L,$ the intersection point of the circle $(O_1),$ from the segment line $AM.$ Also we denote as $D, H,$ the intersection point of the common tangent line at M (radical axis), of $(O_1), (O_2),$ at points $A, L,$ respectively.

By applying the polar theory, because of the segment line $LM,$ as the polar of $H,$ with respect to the circle $(O_1),$ passes through the point $A,$ we have that the segment line $BC,$ as the polar of $A,$ with respect also to $(O_1),$ passes through the point $H.$

(We can also prove this result, without polar theory. If we denote as $K, N,$ the intersection points of the segment line $LH,$ from the tangent lines AB, AC respectively, we have the configuration of the triangle $\triangle AKN,$ taken the circle (O_1) as it's incircle and so, based on a well known (at least to me) Lemma, we have that the segment lines KN, BC and the tangent line of (O_1) at $M,$ are concurrent. An elementary proof can be found, by Newton's, Menelaus's and Ceva's theorems).

We will prove now, that the segment line $EF,$ passes through the point $D.$ From similar isosceles triangles $\triangle DAM \sim \triangle HLM, \Longrightarrow \frac{MD}{MH} = \frac{MA}{ML} = \frac{R_2}{R_1}, (1)$ (because of $\triangle O_1LM \sim \triangle O_2AM$).

It is easy to prove that $BC \parallel EF$ (from $\angle CBM = \angle CMH = \angle FMD = \angle FEM$).

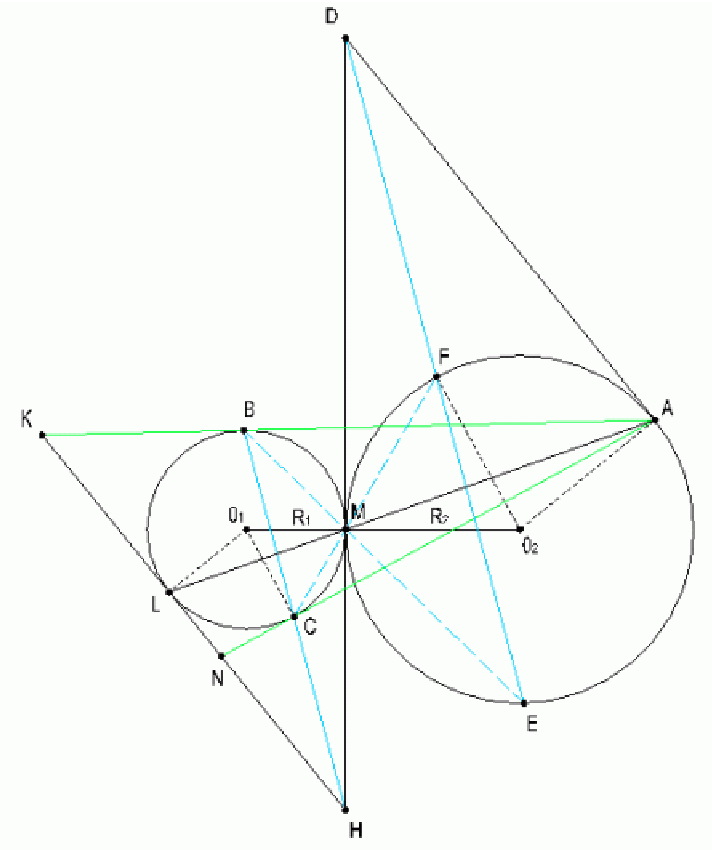
We denote as $D',$ the intersection point of $EF, HM.$ From similar triangles $\triangle CMH \sim \triangle FMD' \Longrightarrow \frac{MD'}{MH} = \frac{MF}{MC} = \frac{R_2}{R_1}, (2)$ (because of $\triangle O_1MC \sim \triangle O_2MF$).

From $(1), (2) \Longrightarrow MD' = MD \Longrightarrow D' \equiv D, (3)$

From $(3),$ we conclude that the concurrency point of the segment line $EF,$ with the tangent line of $(O_1),$ at point $A,$ lies on the radical axis of $(O_1), (O_2)$ and the proof is completed.

Kostas Vittas.

Attachments:



mathVNpro

#10 Apr 22, 2009, 6:17 pm

Let me restate the problem so that I can fit my solution:
"The circles $(C_1), (C_2)$ externally touches at A ($(C_2) > (C_1)$). Let B be an any point on (C_2). From B, let BM, BN be the tangents wrt (C_1). AM, AN, respectively, intersects (C_2) at P, Q. PQ intersects the tangents from B of (C_2) by the point K. Prove that K blongs to a fixed line."
Proof:
Consider the homothety with center A, ratio $\frac{-r_1}{r_2} = k$, we get:
$H(A, k) : P \mapsto M, Q \mapsto N$. Hence, $H(A, k) : PQ \mapsto MN$. Also, the tagents from B also maps to the tangents from C. Let H be the intersection of MN to the tangent from C. It is easy to see that $H(A, k) : H \mapsto K$. Now, we need to prove that H belongs to the fixed line. But this is obviously true because the fact that $AMCN$ is the harmonic quadrilateral, therefore, H belongs the the common internal tangent of $(C_1), (C_2)$. The homothety center A, ratio k, which turns the tangent at A of $(C_1), (C_2)$ into itself. Therefore, K is also a member of the common internal tangent of $(C_1), (C_2)$, which is fixed.
Our proof is completed

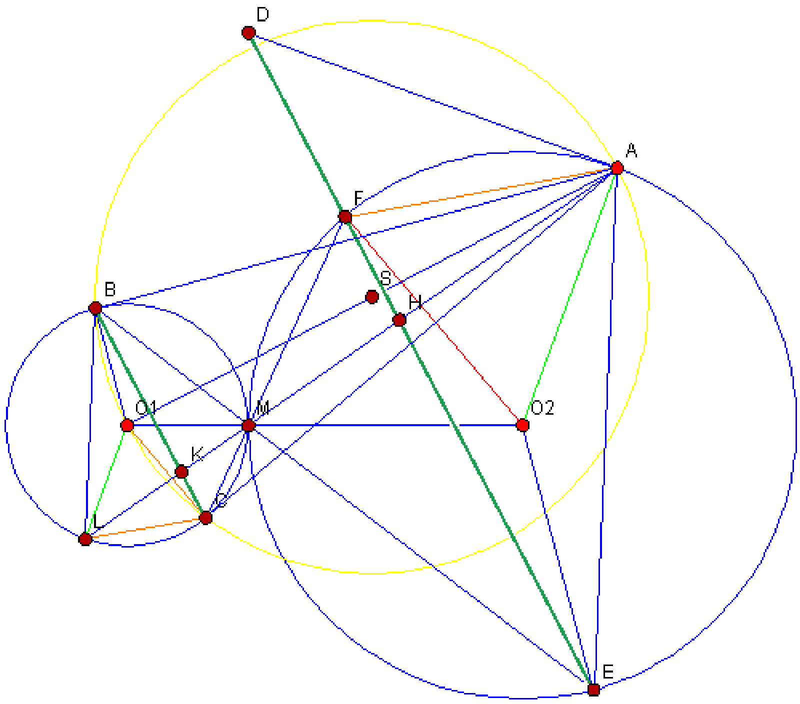
p/s: Sorry if my solution is the same to anyone's

vslmat

#11 Aug 8, 2012, 8:16 pm

Another solution using properties of symmedian:
Easy to show that $LO_1//AO_2$ and $CO_1//FO_2$ and $LC//FA$, also $LB//AE$, therefore $BC//EF$.
$\Delta BLC \sim \Delta EAF$. Since $LC//AF$, point K in ΔBLC corresponds to point H in ΔEAF.
As AB and AC are two tangents to the circle C_1, AL, resp. KL is the symmedian in ΔBLC, so AH is the symmedian in ΔEAF.
Furthermore, HE, resp. DE is the symmedian in ΔMEA, so D must be the intersection of the tangents at A and M to the circle C_2.
With the solution, other interesting facts can be seen, e.g. the two tangents at E and F to the circle C_2 must intersect at a point on line AL.

Attachments:



Ankoganit
#12 Feb 22, 2016, 10:33 am

Sorry to revive an old thread, but can someone please tell me if the following proof is correct or not?

Suppose line AM meets C_1 again at G. Evidently, $MCGB$ is a harmonic quadrilateral, and so $M(MCGB)$ is a harmonic pencil. Intersecting this with C_2, we conclude that $MFAE$ is a harmonic quadrilateral. If the tangent to C_2 from D touches C_2 at D', then $D'FAE$ is harmonic as well. Thus $D' \equiv M \implies D \in$ the common tangent of C_1 and C_2, which is a straight line.

Please let me know if it's wrong, since it appears to be too short to be true.

Ankoganit
#13 Feb 24, 2016, 9:38 am

Hello, can anyone please check the above proof?

bonciocatciprian
#14 Feb 24, 2016, 9:42 pm • 2

> *Ankoganit wrote:*
> Sorry to revive an old thread, but can someone please tell me if the following proof is correct or not?
>
> Suppose line AM meets C_1 again at G. Evidently, $MCGB$ is a harmonic quadrilateral, and so $M(MCGB)$ is a harmonic pencil. Intersecting this with C_2, we conclude that $MFAE$ is a harmonic quadrilateral. If the tangent to C_2 from D touches C_2 at D', then $D'FAE$ is harmonic as well. Thus $D' \equiv M \implies D \in$ the common tangent of C_1 and C_2, which is a straight line.
>
> Please let me know if it's wrong, since it appears to be too short to be true.

It seems okay. Also, you should note that for a specific $\triangle ABC$ inscribed in (O) (unrelated to our problem) there are three distinct points on the circle, each completing $\triangle ABC$ to a harmonic quadrilateral. What makes $D' \equiv M$ in your case is that they are both on the same arc determined by E and F.

Ankoganit

Arkoganit

#15 Feb 25, 2016, 9:40 am

@bonciocatciprian Thank you for your comments.

Quick Reply

High School Olympiads

Source: Estonia IMO TST Day 2 P2

IstekOlympiadTeam

#1 Nov 28, 2015, 2:40 am

A quadrilateral $ABCD$ is inscribed in a circle. On each of the sides AB, BC, CD, DA one erects a rectangle towards the interior of the quadrilateral, the other side of the rectangle being equal to CD, DA, AB, BC, respectively. Prove that the centers of these four rectangles are vertices of a rectangle.

Luis González

#2 Jan 11, 2016, 11:03 pm

Construct right triangles $\triangle ABP, \triangle ADQ, \triangle CDR, \triangle CBS$ towards the interior of $ABCD$, such that $\angle ABP = \angle ADQ = \angle CDR = \angle CBS = 90^\circ$ and $BP = CD, DR = AB, DQ = BC, BS = DA \Longrightarrow$ midpoints O_1, O_2, O_3, O_4 of AP, CS, CR, AQ, resp, are the centers of the object rectangles.

Since $\angle PBS = 180^\circ - \angle ABC = \angle ACD$ and $BS = DA, BP = CD$, then $\triangle ADC \cong \triangle SBP$ are inversely congruent by SAS $\Longrightarrow$ midpoints M, N, L of DB, AS, CP are collinear (well-known). Moreover $LO_2 = \frac{1}{2}PS = \frac{1}{2}AC = NO_2$ and likewise $LO_1 = NO_1 = \frac{1}{2}AC \Longrightarrow LO_1NO_2$ is a rhombus $\Longrightarrow O_1O_2 \perp MN$. By similar reasoning if K, H are the midpoints of AB, DS, then $MKNH$ is a rhombus $\Longrightarrow NM$ is internal bisector of $\angle KNH$ whose sides are parallel to those of $\angle BTD \Longrightarrow MN$ is parallel to the internal bisector τ of $\angle BTD \Longrightarrow$ $O_1O_2 \perp \tau$ $(\star)$.

Let $X \equiv DR \cap BS$ be the antipode of C on $\odot(ABCD)$ and $T \equiv DA \cap BS$. If we define U, such that $BSRU$ is a parallelogram, we get $\angle DRU = \angle DXB = \angle DAB$ and since $RU = BS = DA, DR = AB$, then $\triangle BAD \cong \triangle DRU$ by SAS $\Longrightarrow \angle RDU \equiv \angle XDU = \angle ABD \Longrightarrow$ $\angle UDB = \angle XDB - \angle ABD = 90^\circ - \angle BDC - \angle ABD$. Since $DB = DU$, then $\triangle DUB$ is D-isosceles, hence $\angle UBD = 90^\circ - \frac{1}{2}\angle UDB = \frac{1}{2}(\angle BDC + \angle ABD) \Longrightarrow \angle RST = \angle UBT = \frac{1}{2}(\angle BDC + \angle ABD) - \angle TBD$. But $\angle BDC = 90^\circ - \angle XDB$ and $\angle ABD = \angle TBD - \angle ADX \Longrightarrow$ $\angle UBT = \frac{1}{2}(90^\circ + \angle TBD - \angle TDB) - \angle TBD = \frac{1}{2}\angle BTD \Longrightarrow RS \parallel \tau \Longrightarrow O_2O_3 \parallel RS \parallel \tau$. Together with $(\star)$, it follows that $O_2O_3 \perp O_2O_1$, i.e. $\angle O_1O_2O_3 = 90^\circ$. By similar reasoning we prove that the remaining angles of $O_1O_2O_3O_4$ are right $\Longrightarrow O_1O_2O_3O_4$ is a rectangle.

PROF65

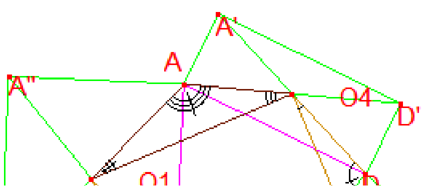
#3 Jan 12, 2016, 4:17 pm

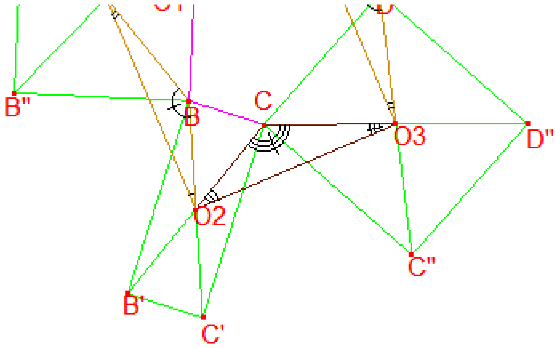
Let $BB''A''A, BB'C'C, DD''C''C$ and $AA'D'D$ be the erected rectangles and O_1, O_2, O_3, O_4 respective centers it's clear that

$O_1A = O_1B = O_3C = O_3D; O_2B = O_2C = O_4A = O_4D; \widehat{B''BA''} = \widehat{CDD''} \equiv \alpha; \widehat{A'DA} = \widehat{B'BC} \equiv \beta; \gamma \equiv \widehat{ADC} = \pi - \widehat{ABC}$ then $\widehat{O_1BO_2} = \alpha + \gamma + \beta; \widehat{O_3DO_4} = \alpha + \gamma + \beta$ thus $O_4DO_3 \cong O_2BO_1(*) \implies O_1O_2 = O_3O_4$ simlarly we deduce $O_1O_4 = O_3O_2; O_4AO_1 \cong O_2CO_3(**)$ which means that $O_1O_2O_3O_4$ is parallelogram but $\widehat{AO_1B} + \widehat{DO_3C} = \pi$ besides $(*)$ and $(**)$ yields $\widehat{DO_3O_4} = \widehat{BO_1O_2}, \widehat{AO_1O_4} = \widehat{CO_3O_2}$ hence $\widehat{O_4O_1O_2} + \widehat{O_4O_3O_2} = \pi$ therefore $\widehat{O_4O_1O_2} = \widehat{O_4O_3O_2} = \frac{\pi}{2}$.

WCP

Attachments:





This post has been edited 1 time. Last edited by PROF65, Jan 12, 2016, 10:29 pm
Reason: typo

Quick Reply

High School Olympiads

Four concurrent lines with two cevian triangles

Source: Own

buratinogigle

#1 Dec 28, 2015, 6:19 pm

Let ABC be a triangle. DEF is cevian triangle of P and XYZ is cevian triangle of Q. PX,PY,PZ cut EF,FD,DE at U,V,W, reps. Prove that DU,EV,FW and PQ are concurrent.

A-B-C

#3 Jan 9, 2016, 12:45 pm • 1

My solution: barycentric coordinates
Let $P=(x_1,y_1,z_1)$ and $Q=(x_2,y_2,z_2)$
EF intersects BC at $D'\Rightarrow (D'UEF)=(D'XBC)$

$$\frac{\overline{D'E}}{\overline{D'F}}.\frac{\overline{UF}}{\overline{UE}}=\frac{\overline{D'B}}{\overline{D'C}}.\frac{\overline{XC}}{\overline{XB}}$$

$$\Rightarrow \overline{UE}.\frac{y_2(z_1+x_1)}{y_1}+\overline{UF}.\frac{z_2(x_1+y_1)}{z_1}=0$$

DU,EV,FW are concurrent at R such that:

$$R(\frac{x_2(y_1+z_1)}{x_1}+\frac{y_2(z_1+x_1)}{y_1}+\frac{z_2(x_1+y_1)}{z_1})=D.\frac{x_2(y_1+z_1)}{x_1}+E\frac{y_2(z_1+x_1)}{y_1}+F\frac{z_2(x_1+y_1)}{z_1}$$

$$R(\frac{x_2(y_1+z_1)}{x_1}+\frac{y_2(z_1+x_1)}{y_1}+\frac{z_2(x_1+y_1)}{z_1})=Ax_1(\frac{y_2}{y_1}+\frac{z_2}{z_1})+By_1(\frac{z_2}{z_1}+\frac{x_2}{x_1})+Cz_1(\frac{x_2}{x_1}+\frac{y_2}{y_1})$$

It is easy to verify that R, P, Q are collinear.
After I see this post, I check Point mapping. It seems you have found a new transformation :)

This post has been edited 1 time. Last edited by A-B-C, Jan 9, 2016, 12:47 pm
Reason: correct

tranquanghuy7198

#4 Jan 9, 2016, 9:37 pm • 1

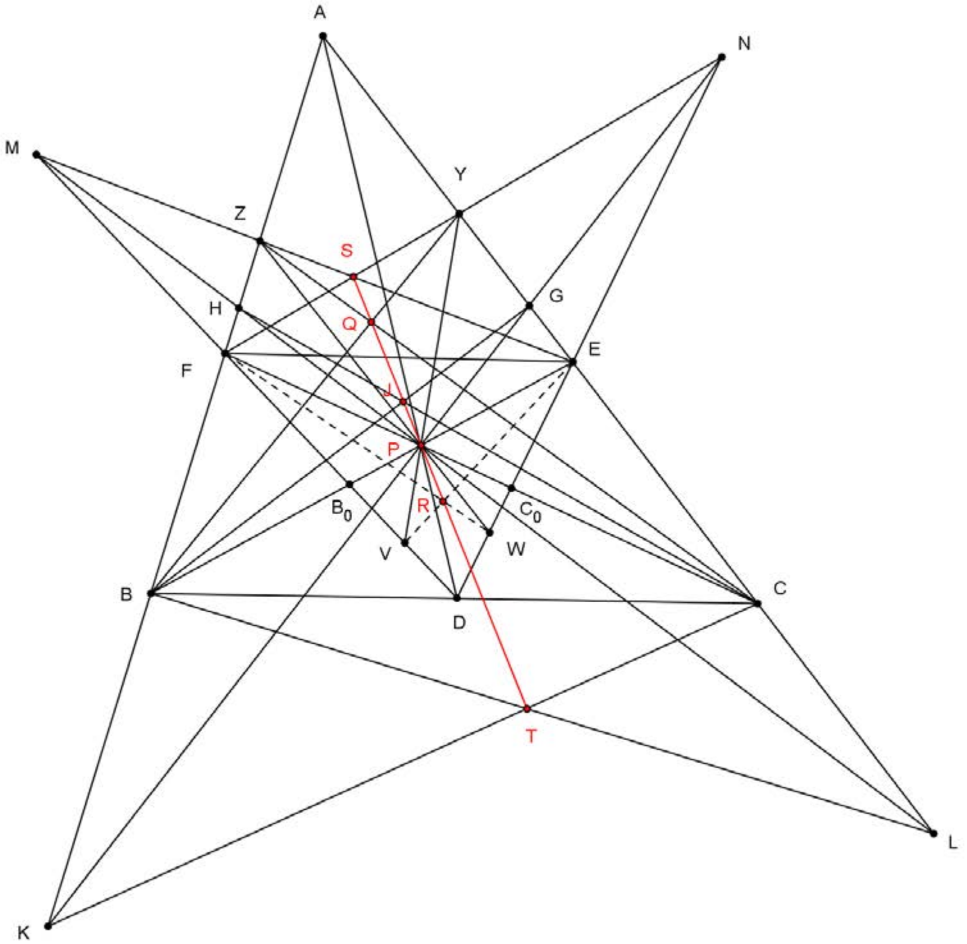
My solution (using Pappus and double ratio):

$EV\cap FW=R, EZ\cap DF=M, FY\cap DE=N$
$PN\cap AB,AC=K,G; PM\cap AB,AC=H,L$
$EZ\cap FY=S, BG\cap CH=J, BL\cap CK=T$

First: $B(PQCJ)=B(EYCG)=N(EYCG)=N(C_0FCP)=-1$ and similarly, $C(PQBJ)=-1$
$\Rightarrow B(PQCJ)=C(PQBJ)\Rightarrow \boxed{\overline{P,Q,J}}$
Pappus: $\begin{pmatrix} B & Z & F \\ C & Y & E \end{pmatrix}\Rightarrow \boxed{\overline{P,Q,S}}$
$\begin{pmatrix} K & B & H \\ L & C & G \end{pmatrix}\Rightarrow \boxed{\overline{T,P,J}}$
So we have: $\overline{P,Q,S,T,J}$
On the other hand: $E(FPSR)=E(FPMV)=P(FEMV)=P(CELY)$
$=B(CELY)=B(CPTQ)=C(BPTQ)=C(BFKZ)$

$= P(BFKZ) = P(EFNW) = F(EPNW) = F(EPSR)$
$\Rightarrow \overline{P,S,R} \Rightarrow \overline{P,Q,R} \Rightarrow (EV \cap FW) \in PQ$
Similarly, we have: $(DU \cap EV) \in PQ \Rightarrow DU, EV, FW, PQ$ concur.
Q.E.D

Attachments:



Luis González

#5 Jan 11, 2016, 4:17 am • 1

Consider a homology taking the trilinear polar of P WRT $\triangle ABC$ to infinity, i.e. P becomes centroid G of $\triangle ABC \Longrightarrow \triangle DEF$ becomes medial triangle $\Longrightarrow \frac{GU}{GX} = \frac{GF}{GC} = -\frac{1}{2} = \frac{GD}{GA} \Longrightarrow DU \parallel AQX \Longrightarrow DU$ goes through the complement R of Q WRT $\triangle ABC$. Likewise EV and FW pass through $R,$ which clearly lies on $GQ,$ such that $\overline{GR} : \overline{GQ} = -1 : 2.$ So back in the primitive figure, it follows that DU, EV, FW meet at a point R on $PQ.$

Quick Reply

High School Olympiads

Tangents to 3 mixtilinear incircle

Reply

Source: own?

quangMavis1999

#1 Jan 8, 2016, 10:44 pm • 3

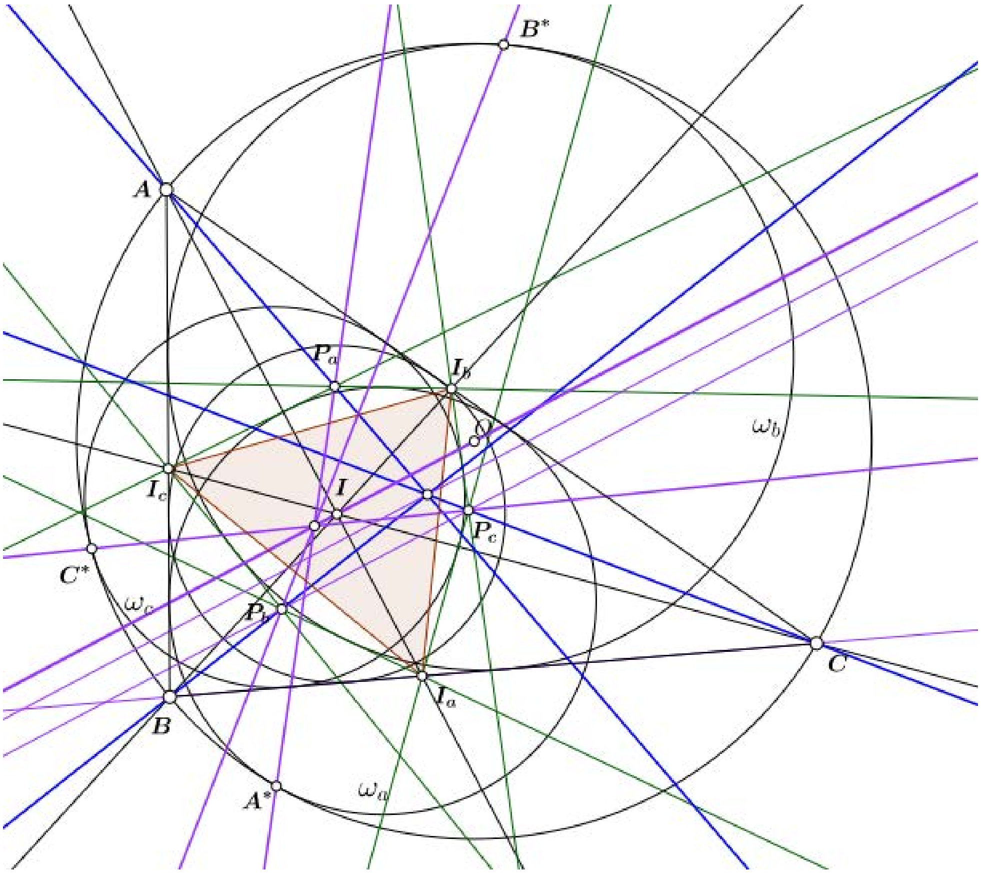
Problem Let ABC be a triangle is incribed (O), incenter I. $\triangle I_aI_bI_c$ is cevian triangle of I WRT $\triangle ABC$. Let $\omega_a, \omega_b, \omega_c$ are the A, B, C-mixtilinear incricle of $\triangle ABC$,respectively. The tangents from I_b, I_c $(\neq AC, AB)$ to ω_a intersect at P_a. Similary, we define for P_b, P_c.

a, Prove that AP_a, BP_b, CP_c are concurrent

b, A^*, B^*, C^* are intersection points of $\omega_a, \omega_b, \omega_c$ with (O),respectively.
Prove that A^*P_a, B^*P_b, C^*P_c, are concurrent on OI

Proposed by Nguyen Minh Quang

Attachments:



This post has been edited 5 times. Last edited by quangMavis1999, Jan 9, 2016, 12:10 am

Luis González

#2 Jan 9, 2016, 5:22 am • 2

Here is a proof for problem a). We'll show that the aforementioned lines concur at the Mittenpunkt X_9 of $\triangle ABC$.

Lemma: Arbitrary circle (J) touches the sides AC, AB of $\triangle ABC$ at E, F and cuts BC at A_1, A_2. P is a point on $\overline{EF}$ and BP, CP cut AC, AB at P_b, P_c. Second tangents from P_b, P_c to (J) cut the tangents of (J) through A_1, A_2 at U, V, resp. Then AP is fixed under the involution $AB \mapsto AC, AU \mapsto AV$.

The proof is straightforward considering the homology that takes (J) into another circle whose center is the image of P and taking the poles of the referred rays AB, AC, AP, AU, AV WRT (J). $\blacksquare$

Back to the problem. Using the previous lemma for $(J) \equiv \omega_a$ and $P \equiv I$, it follows that the involution $AB \mapsto AC$, $AU \mapsto AV$ is the reflection across the angle bisector of $\angle BAC$, i.e. AU, AV are isogonals WRT $\angle BAC$. Now denoting X_a the intersection of the tangents of ω_a at A_1, A_2, by dual of Desargues involution theorem for X_aUP_aV, it follows that $AP_a \mapsto AX_a$ are involutive under the referred involution $\Longrightarrow AP_a, AX_a$ are isogonals WRT $\angle BAC$.

On the other hand, as X_a, E, F are the poles of BC, CA, AB WRT ω_a, then BE, CF, AX_a concur at the perspector L_a of ω_a WRT $\triangle ABC$. But from the complete $BCEF$, it follows that $AL_a \equiv AX_a$ passes through the tripole of the orthotransversal of I WRT $\triangle ABC$; the isogonal of X_9 (well-known) $\Longrightarrow AP_a$ passes through X_9. Similarly BP_b and CP_c pass through X_9.

TelvCohl

#3 Jan 9, 2016, 7:29 pm • 5

Lemma : Given a $\triangle ABC$ and a point P on BC. Let Q be the isotomic conjugate of P WRT B, C (i.e. PQ and BC have the same midpoint) and let the isogonal conjugate of AP, AQ WRT $\angle A$, respectively cuts $\odot(ABC)$ at P^*, Q^*, respectively. Then the tangent of $\odot(ABC)$ passing through A passes through the intersection V of BC and P^*Q^*.

Proof : Let ℓ be the line passing through A and parallel to BC. Since ℓ is fixed under the involution that swaps (AB, AC) and (AP, AQ), so the isogonal conjugate of ℓ WRT $\angle A$ is fixed under the involution that swaps (AB, AC) and $(AP^*, AQ^*) \Longrightarrow BC, P^*Q^*$ and the tangent of $\odot(ABC)$ passing through A are concurrent. i.e. AV is tangent to $\odot(ABC)$ at A

Back to the main problem :

(a)

Let the perpendicular from I to AI cuts BC, I_bI_c at D, D^*, respectively. From Mannheim theorem we get DD^* is the polar of A WRT ω_a, so AP_a is the polar of D^* WRT ω_a (well-known property of tangential quadrilateral) $\Longrightarrow A(B, C; D^*, P_a) = -1$.

On the other hand, from $A(D, I; B, C) = A(D^*, I; B, C)$ we know AD^* and AD are isogonal conjugate WRT $\angle A$, so AP_a passes through the isogonal conjugate (WRT $\triangle ABC$) of the orthocorrespondent of I WRT $\triangle ABC \Longrightarrow AP_a$ passes through the Mittenpunkt M_t of $\triangle ABC$. Similarly, we can prove M_t lies on BP_b and CP_c, so AP_a, BP_b, CP_c are concurrent at M_t.

(b)

Let J_a, J_b, J_c be the A-excenter, B-excenter, C-excenter of $\triangle ABC$, respectively. Let $M_a \in J_bJ_c, M_b \in J_cJ_a, M_c \in J_aJ_b$ be the midpoint of arc BC, arc CA, arc AB, respectively. Let O be the circumcenter of $\triangle ABC$ and let K_a be the exsimilicenter of $\odot(O) \sim \odot(J_a)$ ($K_a \in AA^*$ (D'Alembert theorem)).

Since $\triangle J_aJ_bJ_c$ and $\triangle M_aM_bM_c$ are homothetic, so notice the circumcenter B_e of $\triangle J_aJ_bJ_c$ is the reflection of I in O we get J_aM_a, J_bM_b, J_cM_c are concurrent at $R \in OI$. Since the second intersection T_a of J_aM_a and $\odot(O)$ is the tangency point of $\odot(O)$ and the A-mixtilinear excircle Ω_a of $\triangle ABC$ (well-known), so from D'Alembert theorem we get K_aT_a passes through the insimilicenter of $\odot(J_a) \sim \Omega_a \Longrightarrow T_a(V, K_a; O, R) = T_a(A, K_a; O, J_a) = -1$ where $V \equiv AT_a \cap OI$ is the insimilicenter of $\odot(I) \sim \odot(O)$, hence $S \equiv K_aT_a \cap OI$ is the harmonic conjugate of V WRT O and R.

Let $L \equiv A^*P_a \cap I_bI_c$. Since I_bI_c is the polar of K_a WRT $\odot(O)$ (well-known), so the pole G of AA^* WRT $\odot(O)$ lies on I_bI_c. From the dual of Desargue involution theorem $\Longrightarrow A^*G$ is fixed under the involution that swaps $(A^*I_b, A^*I_c), (A^*A, A^*P_a)$, so AG is fixed under the involution that swaps $(AB \equiv AI_c, AC \equiv AI_b)$ and (AA^*, AL), hence if $Q \equiv AG \cap BC$ then the intersection of QA^* and AL lies on $\odot(O) \Longrightarrow A, L, T_a$ are collinear (from **Lemma**).

Let $U \equiv AA^* \cap OI$ be the exsimilicenter of $\odot(I) \sim \odot(O)$. Let X_1 and X_2 be the intersection of $\odot(O)$ with OI. Since L lies on the polar I_bI_c of K_a WRT $\odot(O)$, so the intersection Z of K_aT_a and A^*P_a lies on $\odot(O)$, hence from Desargue involution theorem (for AA^*ZT_a) we get $W \equiv A^*P_a \cap OI$ is the image of V under the involution that swaps $(S, U), (X_1, X_2)$.

Similarly, we can prove W lies on B^*P_b and C^*P_c, so we conclude that A^*P_a, B^*P_b, C^*P_c are concurrent on OI.

Quick Reply

High School Olympiads

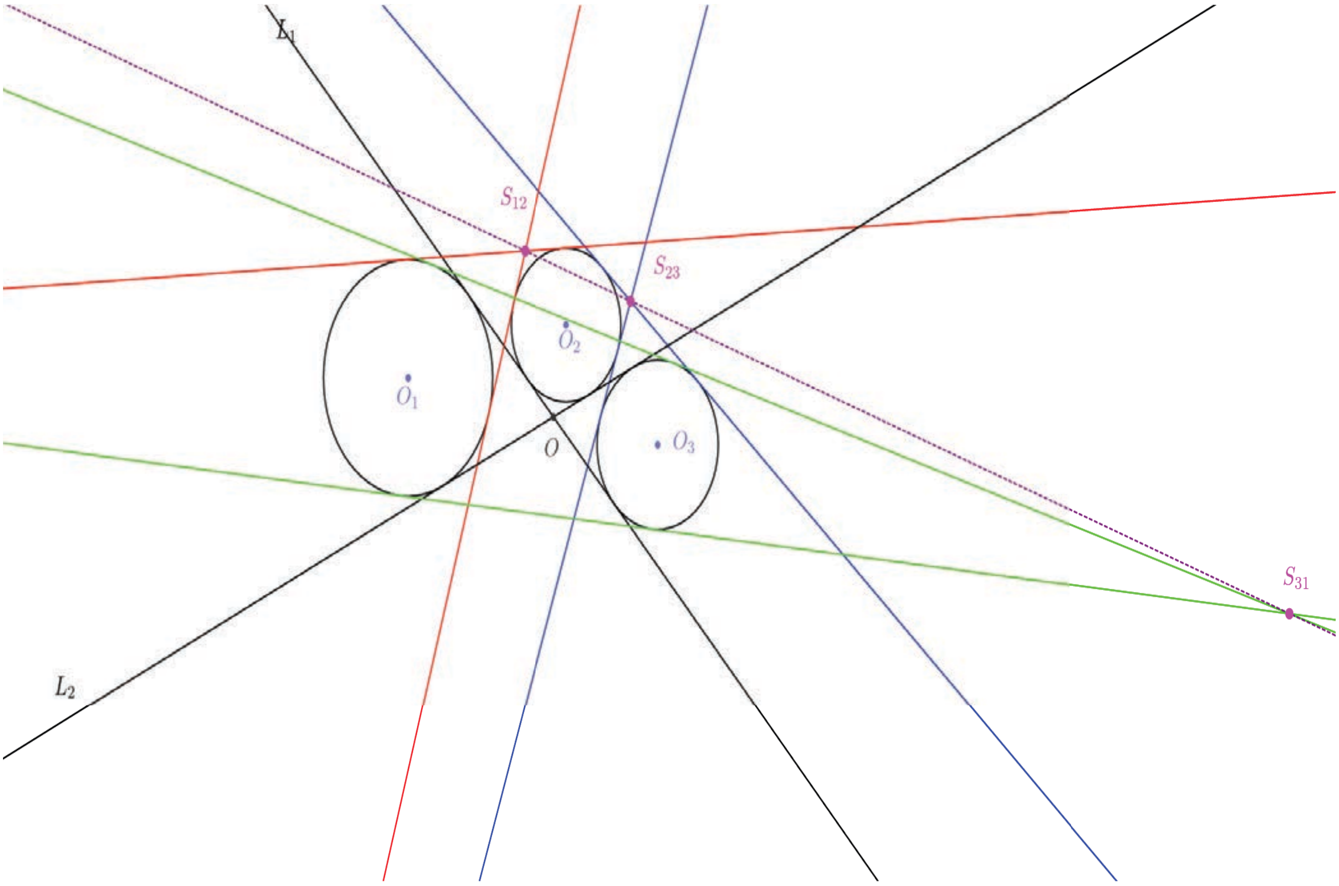
Common tangents

Reply

andria

#1 Jan 7, 2016, 9:00 pm

Two arbitrary lines L_1, L_2 intersect at O. Three circles $(O_1), (O_2), (O_3)$ are tangent to lines L_1, L_2 (see picture). circles $(O_1), (O_2)$ have two other common tangents different from L_1, L_2 let S_{12} be their intersection point. We get S_{23}, S_{31} similarly. Prove that S_{12}, S_{23}, S_{31} are collinear.



Luis González

#2 Jan 8, 2016, 6:39 am

This is a particular case of the dual of the three conics theorem, which states that: If three conics are tangent to two given lines, then the intersections of the common tangents to each pair of conics, different from the given lines, lie on a straight line.

For a proof of the three conics theorem see Geometry of conics by A.V. Akopyan and A.A. Zaslavsky (page 86).

fractals

#3 Jan 8, 2016, 6:55 am

That's the general case, but I think it's also good to mention the name of this specific result: Monge's Theorem.

A simple proof notices that S_{23} is on line O_2O_3 and similar for the others. Then you can do Menelaus on triangle $O_1O_2O_3$, since e.g. $\dfrac{S_{23}O_2}{S_{23}O_3} = \dfrac{r_2}{r_3}$ and cyclic multiplication gives the result (once you use directed segments or orientation considerations, of course).

Quick Reply

High School Olympiads

TS is the bisector of <ETF (own)

Reply

Petry

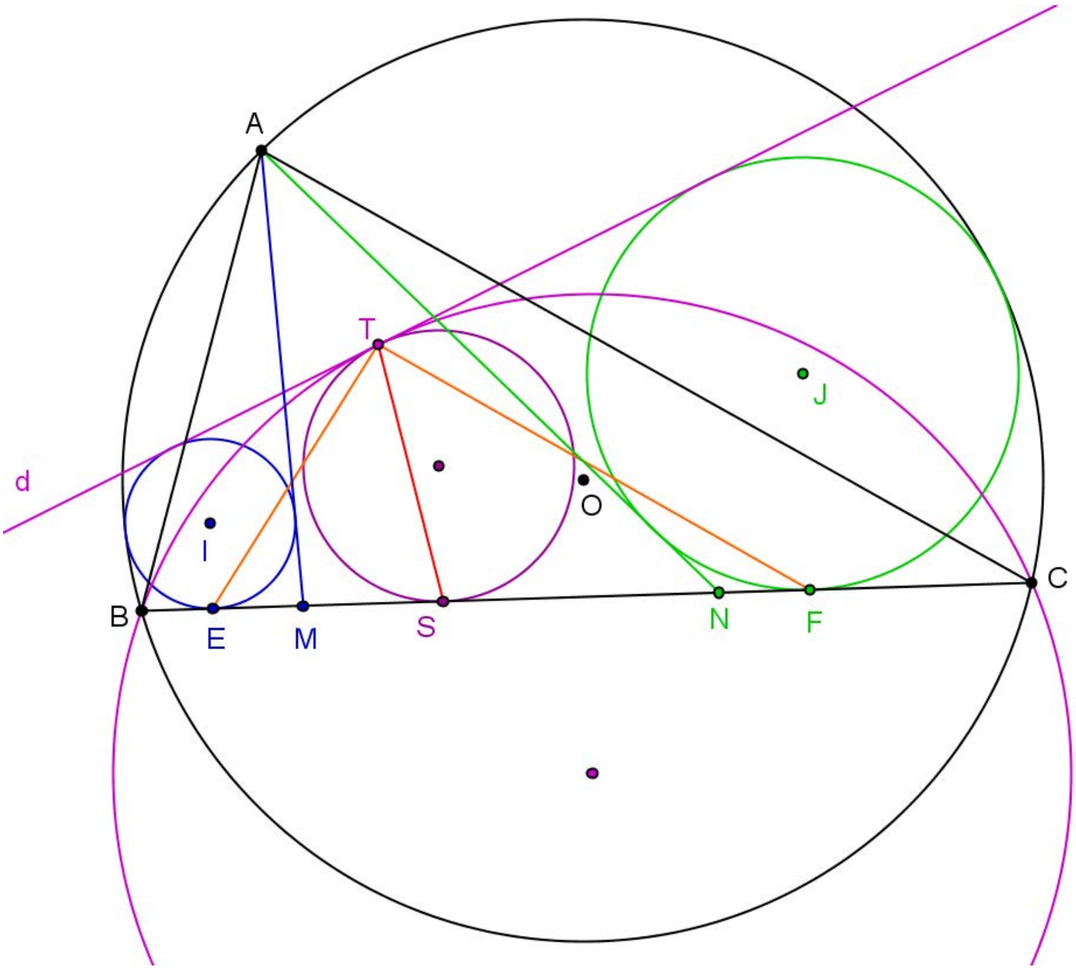
#1 Dec 29, 2010, 4:04 pm • 1

Hello!

Let ABC be a triangle, (O) is the circumcircle and $M, N \in (BC)$.
(I) is the circle tangent to AM, BM (at E) and intouches (O) simultanously.
(J) is the circle tangent to AN, CN (at F) and intouches (O) simultanously.
d is the common external tangent to the circles (I) and (J), $d \neq BC$.
$T \in d$ such that d is tangent to the circumcircle of ΔBTC.
A circle is tangent to d at T and to BC (at S).
Prove that TS is the bisector of $\angle ETF$.

Best regards,
Petrisor Neagoe

Attachments:



Luis González

#2 Dec 31, 2010, 6:35 am • 1

Points M,N can be disregarded. Let X, Y be the tangency points of $(I), (J)$ with (O). X is the exsimilicenter of $(I) \sim (O), Y$ is the exsimilicenter of $(J) \sim (O)$ and $P \equiv d \cap BC$ is the exsimilicenter of $(I) \sim (J)$. By Monge & d'alembert theorem, it follows that P, X, Y are collinear. Since P is also center of the inversion taking $(I), (J)$ into each other, we have $PX \cdot PY = PB \cdot PC = PE \cdot PF = PT^2 \Longrightarrow \odot(ETF)$ is tangent to d at T, i.e. $\odot(EDF)$ and the circle ω, tangent to d, BC through T, S, are internally tangent. T is the exsimilicenter of $\odot(EDF) \sim \omega$. If rays TE, TF cut ω at U, V, then $UV \parallel EF \Longrightarrow$ Arcs $SU = SV$ are congruent $\Longrightarrow TS$ bisects $\angle ETF$.

Petry

#3 Dec 31, 2010, 2:37 pm • 1

Thank you, dear Luis.

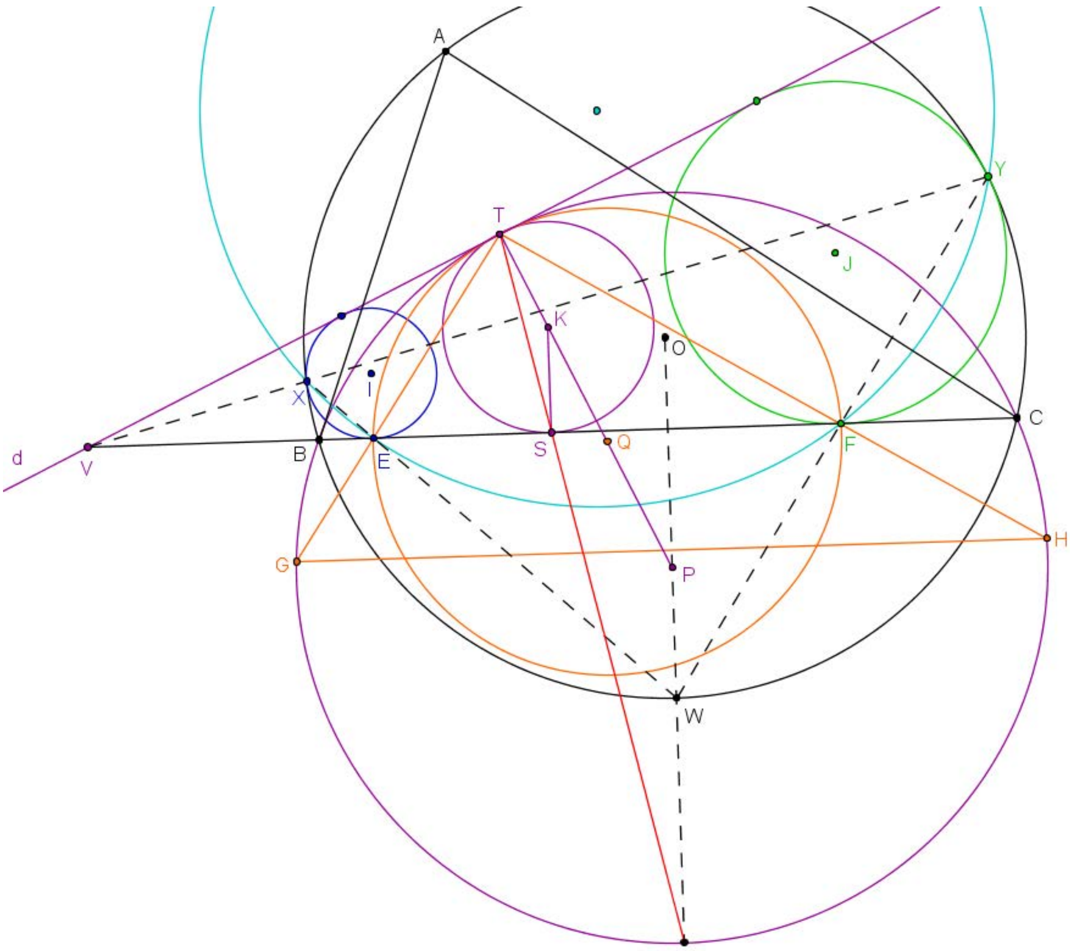
My solution:
(P) is the circumcircle of ΔBTC.
(K) is the circle tangent to d and BC at the points T and S respectively.
The points P, K and T are collinear.
$\{V\} = d \cap BC$. It's known that $V \in XY$.
Let W be the midpoint of $\widehat{BC}$, $W \notin \widehat{BAC}$.
It's known that $EX \cap FY = \{W\}$ and $WE \cdot WX = WB^2 = WC^2 = WF \cdot WY \Rightarrow$
$\Rightarrow$ the quadrilateral $XEFY$ is cyclic $\Rightarrow VE \cdot VF = VX \cdot VY$ (1)
the quadrilateral $XBCY$ is cyclic $\Rightarrow VX \cdot VY = VB \cdot VC$ (2)
d is tangent to (P) at $T \Rightarrow VB \cdot VC = VT^2$ (3)
(1),(2),(3)$\Rightarrow VE \cdot VF = VT^2$

(Q) is the circumcircle of ΔETF
$VE \cdot VF = VT^2 \Rightarrow d$ is tangent to (Q) at T
So, the points P, Q, K, T are collinear.

The circles (P) and (Q) are internally tangent at T.
$BC \cap (Q) = \{E, F\}$, $\{T, G\} = TE \cap (P)$ and $\{T, H\} = TF \cap (P)$.
So, $GH || BC \Rightarrow$ the arcs $\widehat{BG}$ and $\widehat{CH}$ are congruent $\Rightarrow \angle BTE = \angle CTF$ (4)
It's easy to prove that TS is the bisector of $\angle BTC$ (5)
(4),(5)$\Rightarrow TS$ is the bisector of $\angle ETF$.

Best regards,
Petrisor Neagoe

Attachments:



vittasko

#4 Jan 1, 2011, 3:03 am

Let be the point $V \equiv BC \cap IJ,$ as the external homothetic center of the circles $(I),\ (J)$ and it is easy to show that the line segment XY passes through the point $V,$ from $\angle OXY = \angle OYX = \angle YUJ \Longrightarrow IX \parallel JU,$ where $U \equiv (J) \cap XY$ (we use here the collinearities of the points $O,\ I,\ X$ and $O,\ J,\ Y$ as well).

It is also easy to show that the points $X,\ E,\ F,\ Y$ are concyclic, from $\angle XYF = \angle UFE = \angle XEV$ (because of $EX \parallel FU$).

So, we have that $(VE) \cdot (VF) = (VX) \cdot (VY) = (VT)^2 = (VS^2 \Longrightarrow (VE) \cdot (VF) = (VS)^2$, (1)

From $(1),$ based on the **Newton theorem**, we conclude that the points $Z,\ E,\ S,\ F,$ where $Z \equiv (V) \cap BC$ and where (V) is the circle centered at point V with radius $VT = VS$, are in harmonic conjugation.

Because of now the harmonic pencil $S.ZESF$ and $ZS \perp TS$ (from the diameter ZS of (V)), we conclude that the line segment TS bisects the angle $\angle ETF$ and the proof is completed.

● All my best wishes to my friends in Geometry here, in the coming next year.

Kostas Vittas.

Attachments:

t=384112.pdf (10kb)

Quick Reply

High School Olympiads

Just too many curvilinear incircles

Locked

Source: India IMO training camp practice problems

anantmudgal09

#1 Dec 27, 2015, 9:23 pm

In $\triangle ABC$, let Ω be its circumcircle and let M, N be points on BC with B, M, NC, col-linear in this order. Let ω_1 and ω_2 be circles tangent to BM, MA, Ω and CM, MA, Ω (internally) respectively. Let d be the other external common tangent of ω_1, ω_2 respectively and let ω be a circle tangent to d and segment BC at a points T, S such that the line d is tangent to the circumcircle of $\triangle BTC$. Prove that if ω_1, ω_2 touch line BC at E, F respectively then TS bisects $\angle ETF$.

Luis González

#2 Dec 27, 2015, 9:45 pm

Posted before at TS is the bisector of <ETF (own). In fact the points M and N are not necessary

High School Olympiads

circle through fixed point

Reply

Source: OWN

LeVietAn
#1 Dec 25, 2015, 11:08 pm

Dear Mathlinkers,
Given a triangle ABC has and Γ and incenter I. A circle Ω through A and touches the side BC at D. Let P be the second point of intersection of Γ and Ω. Let Q be the second point of intersection of Γ and the line DP. Let M be the midpoint of ID. Suppose that the lines PM and AI are different and intersect at the point R. Prove that the circumcircle of the triangle PQR always goes through a fixed point when Ω change.

Luis González
#2 Dec 26, 2015, 1:50 am

Since $\angle ACQ=\angle APQ=\angle ADB$ and $\angle ABD=\angle AQC \Longrightarrow \angle BAD=\angle CAQ \Longrightarrow AD,AQ$ are isogonals WRT $\angle BAC$. If AI cuts Γ again at X, then from IMO 2010 Problem 2, we get that $U \equiv QI \cap XM$ lies on Γ. Thus if XD cuts Γ again at S and SI cuts Γ again at T, then by Pascal theorem for $STPQUX$, it follows that P,M,T are collinear. Moreover, if AX cuts BC at L and AD cuts Γ again at E, we get $XB^2=XC^2=XI^2=XL \cdot XA=XD \cdot SX$, yielding $\angle ADI=\angle LSI$ and $ALDS$ is cyclic. Since $QE \parallel DL$, then by Reim's theorem Q,L,S are collinear $\Longrightarrow$ $\angle QPR \equiv \angle QPT=\angle QST \equiv \angle LSI=\angle ADI$.

Let I_a be the A-excenter of $\triangle ABC$. From $\triangle ABD \sim \triangle AQC$, we get $AB \cdot AC=AQ \cdot AD$ and from $\triangle ABI \sim \triangle AI_aC$, we get $AB \cdot AC=AI \cdot AI_a \Longrightarrow \frac{AQ}{AI}=\frac{AI_a}{AD}$, which means that $\triangle AQI_a \sim \triangle AID \Longrightarrow$ $\angle QI_aR \equiv \angle QI_aA=\angle ADI=\angle QPR \Longrightarrow P,Q,R,I_a$ are concyclic, i.e. $\odot(PQR)$ always passes through I_a.

Quick Reply

High School Olympiads

IMO 2010 Problem 2

Reply

orl

#1 Jul 7, 2010, 11:33 pm • 4

Given a triangle ABC, with I as its incenter and Γ as its circumcircle, AI intersects Γ again at D. Let E be a point on the arc BDC, and F a point on the segment BC, such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. If G is the midpoint of IF, prove that the meeting point of the lines EI and DG lies on Γ.

Proposed by Tai Wai Ming and Wang Chongli, Hong Kong

silouan

#2 Jul 8, 2010, 12:16 am • 11

It suffices to prove that $\angle IDG = \angle AEI$. Taking the excenter we have to prove that the triangles AFI_a and AIE are similar. But this is easy because it is enough to show that $\frac{AF}{AI_a} = \frac{AI}{AE}$. But from the similarity of ABF,AEC we have that $\frac{AE}{AC} = \frac{AB}{AF}$. So we have to prove that $AI \cdot AI_a = AB \cdot AC$ which is clearly true.

kalantzis

#3 Jul 8, 2010, 1:22 am • 3

I have a different aproach..
Suppose EI cuts Γ at K. Let the parallel from I to BC cut AF to P. Then $AKPI$ is cyclic (because AK is antiparallel to IP or by simple angle chase).
Now we prove that D,P,K are collinear:From the cyclic $AKPI$ $\widehat{AKP} = \widehat{PID} = \widehat{PIB} + \widehat{BID} = \widehat{B} + \widehat{A}/2$. But $\widehat{AKD} = \widehat{ABD} = \widehat{ABC} + \widehat{CBD} = \widehat{B} + \widehat{A}/2$.

If line DPK cuts BC at Q, it suffices to prove that $IQ \parallel AF$ since then, $PIQF$ will be parallelogram and G the intersection of DK and IE.
Since $\widehat{IAP} = \widehat{IKP}$ and we want to show $\widehat{IAP} = \widehat{DIQ}$ it is enough $DI^2 = DQ \cdot DK$
But DI=DB (well-known fact) so we have to prove $DB^2 = DQ \cdot DK$ which is obvious from similar triangles DBQ, DKB since arcs BD and DC are equal.
QED
Image not found

Note: the condition $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$ it is not necessary.

m.candales

#4 Jul 8, 2010, 3:54 am

Let L be the second intersection of IE and Γ. I will prove that G, D, L are collinear

Let M be a point on the prolongation of AD (D is between A and M) such that $\angle IBM = 90$
$\angle IBC = \frac{1}{2}\angle B$ and $\angle CBD = \frac{1}{2}\angle A$. Then $\angle DBM = \frac{1}{2}\angle C$,
$\angle BIM = \frac{1}{2}\angle A + \frac{1}{2}\angle B$. Then $\angle BMI = \frac{1}{2}\angle C$.
Then $BD = DM$. Then BD bisects IM because $\angle IBM = 90$. Then $ID = DM$
(I believe all this results are well-known)

AM AB

$\triangle ABM$ is similar to $\triangle AIC$. Then $\frac{}{AC} = \frac{}{AI}$

$\triangle AEC$ is similar to $\triangle ABF$. Then $\frac{AE}{AC} = \frac{AB}{AF}$

Then we have $\frac{AM}{AE} = \frac{AF}{AI}$. But $\angle FAM = \angle IAE$.
Then $\triangle AFM$ and $\triangle AIE$ are similar. Then $\angle FMA = \angle IEA = \angle LDA$
Then LD and FM are parallel. But $ID = DM$ then LD bisects FI. Then G, D, L are collinear.

m.candales

#5 Jul 8, 2010, 4:21 am

I just want to add that there is another way of continuing my solution after we got that $ID = DM$ which I believe is well-known.
This is the continuation:
Let $AI = i, AD = l, AB = c, AC = b$. Then $AM = 2l - i$
$\triangle ABM$ is similar to $\triangle AIC$. Then $\frac{AM}{AB} = \frac{AC}{AI}$. Then $\frac{2l-i}{c} = \frac{b}{i}$
Then $(2l - i)i = bc$. Then $il - i^2 = bc - il$ (*)

Let N the intersection of LD and AF.
$\frac{AN}{AD} = \frac{AI}{AE}$ because $\triangle AND$ and $\triangle AIE$ are similar.
Then $AN = \frac{il}{AE}$
$\frac{AF}{AB} = \frac{AC}{AE}$ because $\triangle ABF$ and $\triangle AEC$ are similar. Then $AF = \frac{bc}{AE}$
Then $FN = AF - AN = \frac{bc - il}{AE}$

Let G' the intersection of LD and AF. Then $\frac{IG'}{G'F}\frac{FN}{AN}\frac{AD}{ID} = 1$ by Menelaus

Then $\frac{IG'}{G'F} = \frac{ID}{AD}\frac{AN}{FN} = \frac{il(l-i)}{(bc-il)l} = \frac{il - i^2}{bc - il} = 1$ by (*)
Then $G' = G$, and then D, G, L are collinear.

This problem can also be solved automatically using complex numbers. The solution is long and painful to write, but I will try to post it soon

NickNafplio

#6 Jul 8, 2010, 5:59 am • 1

Another solution:

We need to prove that <GMI = <IEA. It is well known (and it can be proved easily) that the midpoint D of the arc BC is the center of the circumcircle U of the triangle BIC. Let T be the symmetric of I with respect to D, which is the intersection of the line AD and the circle U, then FT//MG and <FTI = <GMI, so we need <FTI = <IEA, or its enough to prove that triangles AFT and AIE are simillar, or equivalently AI/AE = AF/AT <=> AI*AT = AE*AF (1). From the simillar triangles ABF and AEC we have AE/AC = AB/AF <=> AE*AF = AB*AC. So we need AI*AT = AB*AC <=> AI/AB = AC/AT, which is true because the triangles ABI, ATC are simillar (<BAI = <TAC = A/2, <ABI = <ABC/2 = <ADC/2 = <ATC)!

armpist

#7 Jul 8, 2010, 6:25 am • 1

Problem with a lot of points on a circumference calls for
Pascal theorem.

Historical note:
he was 13 y.o. when he discovered it, probably solving something similar
to this Problem #2 at French National math olympiad years ago.

Mr. T

abacadaea

#8 Jul 8, 2010, 7:27 am

Click to reveal hidden text

Sung-yoon Kim

#9 Jul 8, 2010, 4:38 pm • 1

We show that $\triangle AFI_a$ and $\triangle AIE$ are similar. Then we have $\angle AEI = \angle AI_aF = \angle ADG$ and we're done. To show that, inverse the plane with regard to A with the radius $\sqrt{bc}$ where $b = AC, c = AB$. Then we have another figure which can be also obtained by reflecting the original figure. Note that E, F are mapped to F, E resp. Hence $AE \cdot AF = bc = AI \cdot AI_a$, which implies directly that $\triangle AFI_a$ and $\triangle AIE$ are similar, as desired.

April

#10 Jul 8, 2010, 6:02 pm

> orl wrote:
>
> Given a triangle ABC, with I as its incenter and Γ as its circumcircle, AI intersects Γ again at D. Let E be a point on the arc BDC, and F a point on the segment BC, such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. If G is the midpoint of IF, prove that the meeting point of the lines EI and DG lies on Γ.

Let D' be a point on AI such that $DI = DD'$. Notice that $\angle IDG = \angle AD'F$. So we only have to prove that $\angle AD'F = \angle AEI$.

We have $\triangle ABF \sim \triangle AEC$, therefore $AB \cdot AC = AE \cdot AF \quad (1)$.
$\triangle ABI \sim \triangle AD'C \Longrightarrow AB \cdot AC = AI \cdot AD' \quad (2)$.
Combine (1) and (2), we have $AE \cdot AF = AI \cdot AD'$, i.e. $\frac{AF}{AD'} = \frac{AI}{AE}$. On the other hand, $\angle FAD' = \angle IAE$, so $\triangle FAD' \sim \triangle IAE$. It follows $\angle AD'F = \angle AEI$, which completes our solution.

feliz

#11 Jul 9, 2010, 2:01 am

To add a little algebraic point of view...

The points where EI and DG meet Γ are functions of $\angle AEI$ and $\angle ADG$, which we must prove are equal. Let DG meet AF at T. Since $\angle TAD = \angle IAE$, we must prove that TAD and IAE are congruent. To do this, we need a spiral similarity between ATI and ADE. On the other hand, naming $\theta = \angle BAF$, we see that $\angle AFC = \angle ADE = \angle ABC + \theta$. Thus, if $\{P\} = FC \cap AD$, we have AFP similar to ADE, and it remains to prove AFP is similar to ATI, or $TI \parallel FP$. If this happens, let $Q = BC \cap DG$. Since $G = \frac{F+I}{2}$, $TIQF$ must be a parallelogram. Reversely, if $QI \parallel AF$, we are going to have that parallelogram. In other words, we need to prove QIP is similar to FAP.

We prove this happens in the degenerate cases $F = B$ and $F = C$. When F varies on (in? at?) BC, G varies linearly, and so does P, because it is a mean of G and D, with weights that depend only on the ratio IP/PD. So our linearity ends the argument. If $F = B$ (draw another figure!), however, BID is isosceles, so that DG is a simmetry axis, which leads to $\angle PIQ = \angle DIQ = \angle DBQ = \frac{1}{2}\angle BAC = \angle PAB$. And here is the similarity!

Zhero

#12 Jul 9, 2010, 3:04 am

Let M be the midpoint of AI. We want to show that $\angle GDM = \angle IEA$, that is, $\triangle MGD \sim \triangle AIE$.

MG is parallel to AF, so $\angle GMD = \angle FAD = \angle IAE$. Hence, $\triangle MGD \sim \triangle AIE$ if and only if $\frac{MG}{MD} = \frac{AI}{AE}$.
$MG = \frac{AF}{2}$, so this is equivalent to $2AI \cdot MD = AE \cdot AF$.

We claim that $AE \cdot AF$ is fixed. It is then sufficient to show that the result is true for some choice of E and F (namely, when $E = C$ and $F = B$), as it would imply that $2AI \cdot MD = AE \cdot AF$ for some choices of E and F, and thus for all choices of E and F.

Showing that $AE \cdot AF$ is not difficult. $\angle ABF = \angle AEC$ and $\angle BAF = \angle CAE$, so $\triangle ABF \sim \triangle AEC$, so $AE \cdot AF = AB \cdot AC$.

That the result is true when $F = B$ and $E = C$ is not difficult to show either. In this case, G is the midpoint of the base of isosceles $\triangle IBD$, so DG is the bisector of $\angle BDA$, so it meets Γ on the midpoint of minor arc AB. On the other hand, CI trivially meets Γ on the midpoint of minor arc AB as well, so our proof is complete.

Mithril

#13 Jul 9, 2010, 8:57 pm

Let EI intersect Γ again at P. We'll prove that DP meets FI at its midpoint.

Let DP cut AF and BC at X and Y, resp. Notice that $APXI$ is cyclic, because $\angle XAI = \angle DAE = \angle XPI$. We'll see that $XFYI$ is a parallelogram.

Lemma 1: XI is parallel to FY.
Proof: As $APXI$ is cyclic, we have $\angle PAI = \angle DXI$. But $\angle PAI$ is equal to the angle between DP and the tangent to Γ through D. Then, it follows that XI and that tangent are parallel. But as D is the midpoint of arc BC, XI must be parallel to BC too, and the lemma follows.

Lemma 2: D is the circumcenter of BIC.
Proof: As D is the midpoint of arc BC, we have $DB = DC$. Now, let the circle with center D which passes through B and C meet AD at I'.
We know that $\angle I'CB = \angle BDI'/2 = \angle BCA/2$. Thus I' is also on the bisector of $\angle C$, so I' is the incenter, and the lemma follows.

Lemma 3: FX is parallel to YI.
Proof: Consider the inversion with center D and radius DB. It maps BC into Γ, and, by lemma 2, fixes I.
Now, as Y is mapped to P and I is fixed, we have $\angle YID = \angle DPI$. But we know that $XPAI$ is cyclic, thus $\angle YID = \angle XAI$, and the result follows.

Now, by lemmas 1 and 3, we get that $XFYI$ is a parallelogram. Thus DP meets FI at its midpoint, which is what we wanted to prove.

wenxin

#14 Jul 10, 2010, 7:11 pm

> *orl wrote:*
> Given a triangle ABC, with I as its incenter and Γ as its circumcircle, AI intersects Γ again at D. Let E be a point on the arc BDC, and F a point on the segment BC, such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. If G is the midpoint of IF, prove that the meeting point of the lines EI and DG lies on Γ.

2010 IMO Problem 2 was proposed by Tai Wai Ming (2008 Hong Kong IMO team member) and Wang Chongli

lchserious

#15 Jul 10, 2010, 9:02 pm

> *wenxin wrote:*
>> *orl wrote:*
>> Given a triangle ABC, with I as its incenter and Γ as its circumcircle, AI intersects Γ again at D. Let E be a point on the arc BDC, and F a point on the segment BC, such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. If G is the midpoint of IF, prove that the meeting point of the lines EI and DG lies on Γ.
>
> 2010 IMO Problem 2 was proposed by Tai Wai Ming (2008 Hong Kong IMO team member) and Wang Chongli

Oh really !?

Tai Wai Ming was my teammate but we have not kept in contact since IMO 2008 :(

pavel kozlov
#16 Jul 11, 2010, 2:39 pm • 1

> silouan wrote:
> It suffices to prove that $\angle IDG = \angle AEI$. Taking the excenter we have to prove that the triangles AFI_a and AIE are similar. But this is easy because it is enough to show that $\frac{AF}{AI_a} = \frac{AI}{AE}$. But from the similarity of ABF,AEC we have that $\frac{AE}{AC} = \frac{AB}{AF}$. So we have to prove that $AI \cdot AI_a = AB \cdot AC$ which is clearly true.

The most short and nice solution of this problem! I have the same, but it took of me about 2,5 hours to get it.

Lepuslapis
#17 Jul 11, 2010, 9:37 pm

I do not think anyone has used Menelaos yet (probably because it does not yield a very short solution)

Let DG intersect AF and BC at M and N, respectively. Applying Menelaos' Theorem to AFI, we obtain

$$\frac{AM}{MF}\frac{FG}{GI}\frac{ID}{DA} = 1 \Longleftrightarrow \frac{AM}{MF} = \frac{AD}{DI}$$

since G is the midpoint of $[IF]$. Now, straightforward computation on the line containing A, I, D and J, the foot of the angular bisector through A, yields $AD \cdot IJ = DI \cdot AI$ - it is basically the power-of-a-point condition $AJ \cdot DJ = IJ \cdot (2DI - IJ)$ rearranged, so

$$\frac{AM}{MF} = \frac{AI}{IJ}$$

Thus $MI \parallel BC$. Now let H and H' the second points of intersection of EI and DG, respectively, with Γ. Observe that $\widehat{BJA} = \widehat{DCA} = \widehat{DEA}$, so AFJ and ADE are similar, whence $\widehat{ADE} = \widehat{AFJ} = \widehat{AMI}$.
Now $\widehat{AHI} = \widehat{ADE}$, so $\widehat{AHI} = \widehat{AMI}$, so H lies on the circumcircle of AMI. Also,
$\widehat{AH'I} = 180^\circ - \widehat{AED} = \widehat{FJA} = \widehat{MIA}$, so H' lies on the circumcircle of AMI as well.
Thus H and H' both lie on the circumcircle of AMI and on Γ. But neither of these points can coincide with A (since $D \neq E$), so $H = H'$, and we are done.

CatalystOfNostalgia
#18 Jul 12, 2010, 7:10 am

Here's an outline:

Lemma: $AE * AF - AI * AD = 2Rr$.
Proof: We can compute all four side-lengths in terms of parameters of the triangle and $\angle BAF = \angle CAE$. Specifically, AE and AD are chords, so we can get both them in terms of R, and we can get AI in terms of r. Then, compute all of the trig expressions (really, it's not too bad), using the fact that $2[ABC] = ac\sin B = r(a+b+c)$, the law of sines, and in the end the sum-to-product formula.

From here, we want to let CI meet the circumcircle at P, then let PD meet AF and FI at X and G', respectively. Note that AIE ~ AXD. Menelaus on AIF reduces the problem to AX/XF=AD/DI, which is equivalent to AI/AE=XF/DI, and AI*DI=AE*XF. But AI*DI is the (negative) power of I with respect to the circumcircle, which is 2Rr, and we can find that AE*XF=AE*AF-AI*AD by the fact that AX*AE=AI*AD. Then, we're done by the lemma.

meletvas
#19 Jul 15, 2010, 3:30 pm • 1

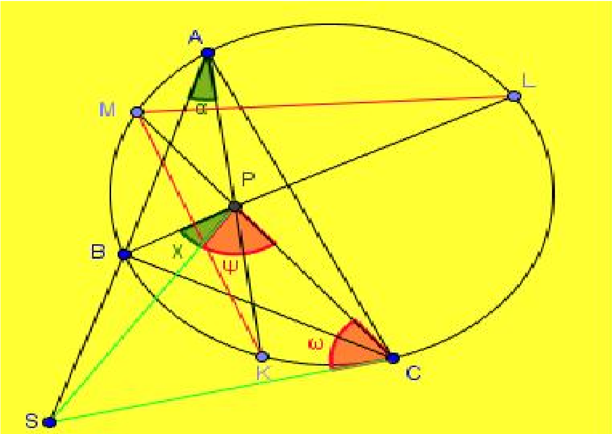
Is valid that SP.SP=SC.SC=SA.SB , so the triangles ABP , ASP are similar.

Hence χ=α=mod arc BK/2 =arc BK/2(more simple) (1)

Since angl.SPC=angl.SCP or (χ+ψ)-χ=ω ,we take arc(ML/2+ BC/2-BK/2)= arc (BC/2+MB/2)

Hence arc ML/2=arcMK/2 or MK=ML

Attachments:



simon89889

#20 Jul 15, 2010, 4:06 pm

why triangles AFI_a and AIE are similar??

Vikernes

#21 Jul 31, 2010, 7:11 am • 2

> *simon89889 wrote:*
> why triangles AFI_a and AIE are similar??

See silouan's solution. Note that D is midpoint of II_a (well known fact, instead point D is the center of the cyclic cuadrilateral IBI_aC, in April's notation, $D\equiv I_a$), so $\angle GDI=\angle IEA \iff \triangle AFI_a\sim\triangle AIE$.

kalantzis

#22 Aug 5, 2010, 5:20 am • 1

At #8 :

> *abacadaea wrote:*
> Click to reveal hidden text

I think this is not sufficient because from this relation you can't get $AEI=ADG$ since you only have $AEI+IEM=IDG+GDF$. Please someone correct me if i am wrong

dgreenb801

#23 Aug 11, 2010, 12:38 am • 2

My solution:
We wish to show $\angle AEI=\angle ADG$, because then if EI intersected DG at a point X, then $AEDX$ would be cyclic, which would mean that X lies on the circumcircle of $\triangle ABC$.

Let DG meet AF at H. To show that $\angle AEI=\angle ADH$, we try to show $\triangle AHD\sim\triangle AIE$, or since $\angle HAD=\angle IAE$, we wish to show $\frac{AH}{AD}=\frac{AI}{AE}$.

By Menelaus,
$\frac{DI}{DA}\cdot\frac{AH}{HF}=1$, or $\frac{HF}{AH}=\frac{DI}{DA}$. Adding 1 to both sides,

$\frac{AF}{AH}=\frac{DA+DI}{DA}$, or $\frac{AH}{\ldots}=\frac{AF}{\ldots}$

AH DA AD $DA+DI$

Thus, we wish to show $\frac{AF}{DA+DI} = \frac{AI}{AE}$.

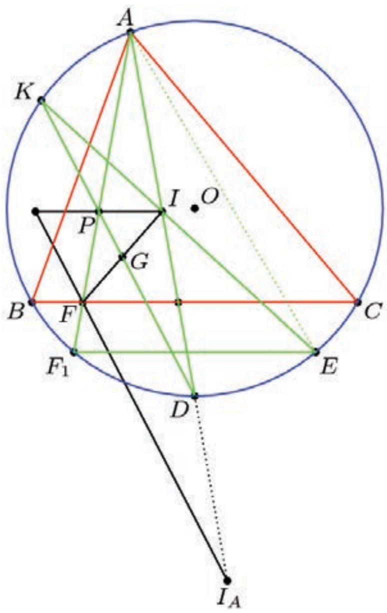
Note that $\angle BAF = \angle CAE$ and $\angle ABF = \angle ABC = \angle AEC$, thus $\triangle ABF \sim \triangle AEC$. Thus, $\frac{AE}{AC} = \frac{AB}{AF}$, or $AE = \frac{AB \cdot AC}{AF}$. Substituting this into our previous equation, we wish to show $AB \cdot AC = AI(DA + DI)$.

It is well-known that D is the circumcenter of $\triangle BIC$. Since D lies on the bisector of $\angle BAC$, it follows by symmetry that line AB must meet the circumcircle of $\triangle BAC$ at a point Y such that $AY = AC$. Then by power of a point, we find $AB \cdot AC = AI \cdot (AI + 2ID) = AI \cdot (AD + ID)$, which is what we wanted to show.

v_Enhance

#24 Jun 8, 2014, 5:15 am • 2

Evidently I remember projective geometry but not similar triangles...



Let line $K = \overline{EI} \cap \Gamma \neq E$, $P = \overline{DK} \cap \overline{AF}$, $F_1 = \overline{AF} \cap \Gamma \neq A$. By Pascal's Theorem on AF_1EKDD, we see that $\overline{IP} \parallel \overline{BC}$. Setting I_A as the A-excenter:

$$-1 = (I, I_A; A, \overline{AI} \cap \overline{BC}) \stackrel{F}{=} (I, \overline{I_AF} \cap \overline{IP}, P, \overline{BC} \cap \overline{IP})$$

implying that $\overline{I_AF} \cap \overline{IP}$ is the reflection of I over P, whence homothety at I finishes the problem.

Zeref

#25 Jul 29, 2014, 3:35 am • 3

Actually this problem is very easy with complex numbers.
Let the circumcircle of ABC be the unitary circle, let $a, b, c = u^2, v^2, w^2$; the midpoints of arcs AB, BC, CA are $-uv, -vw, -wu$. Assume wlog $w = -1/v$ (iff $d = 1$), and let $x = v - 1/v$. Then $i = -ux + 1$.
Let E'=$AF \cap \Gamma$. Since $\angle DAE = \angle E'AD$ and $d = 1$, $e' = 1/e$.
The intersection of cords ab and cd is $\frac{ab(c+d) - cd(a+b)}{ab - cd}$.
Then f=$bc \cap ae'$ implies $f = \frac{x^2u^2 - u^2e - 1 + 2u^2}{u^2 - e}$.
Let $p' = ei \cap \Gamma$. Since p' lies on cord ei, $p' + e - i - p'e\bar{i} = 0$, which implies
$p' = \frac{u(xu - 1 + e)}{xe + ue - u}$.
$g = \frac{i + f}{2} = \frac{x^2u^2 + x(-u^3 + ue) + 3u^2 - u^2e - 1 - e}{2(u^2 - e)}$.
Now it suffice to show that $1, p', g$ are collinear, but
$\frac{g - 1}{p - 1} = \frac{(xu - u + 1)(xu + e - 1)(xe + ue - u)}{2(u^2 - e)^2x} \in \mathbb{R}$

junioragd

#26 Aug 2, 2014, 12:53 am • 4

First,note triangles ABF and ACE are simmilar from which we obtain AB*AC=AE*AF.Now,let Ia be the excenter corresponding to A.Now,we have DI=DIa and A,I,D and Ia are collinear,from which we obtain IaF//DG.We have to prove <FIaI=<GDI(GD//FIa)=<IEA and this is true because triangles AIE and AFIa are simmilar(because AI*AIa=AB*AC=AE*AF(because ABIa and ACI are simmilar) and <FAD=<DAE),so we are finished.

Wolstenholme

#27 Nov 1, 2014, 9:03 am • 1

Let AF meet Γ again at F' and let EI meet Γ again at K. It suffices to show that points D, G, X are collinear. Now by Pascal's Theorem on degenerate cyclic hexagon $F'EKDDA$ we have that if $P = DK \cap AF'$ then $IP \parallel BC$. Therefore it suffices to show that points D, G, P are collinear.

Now we proceed with barycentric coordinates. Let $A = (1, 0, 0)$ and $B = (0, 1, 0)$ and $C = (0, 0, 1)$. Moreover let $a = BC$ and $b = CA$ and $c = AB$. Let $F = (0 : f : a + b + c - f)$ for some $f \in \mathbb{R}$. It is well-known that $D = (-a^2 : b(b + c) : c(b + c))$ and that $I = (a : b : c)$. Therefore we easily find $G = (a : b + f : a + b + 2c - f)$. Now denote the point at infinity on line BC as P_∞. Clearly $P_\infty = (0 : 1 : -1)$. Therefore line IP has equation $x(b + c) = a(y + z)$. Since line AF has equation $fz = (a + b + c - f)y$ we find that $P = (a(a + b + c) : (b + c)f : (b + c)(a + b + c - f))$. Now it suffices to show that the determinant $\begin{vmatrix} -a^2 & b(b+c) & c(b+c) \\ a & b+f & a+b+2c-f \\ a(a+b+c) & (b+c)f & (b+c)(a+b+c-f) \end{vmatrix} = 0$ which is a trivial computation (made even easier by the fact that a and $b + c$ cancel out).

EulerMacaroni

#28 Aug 10, 2015, 6:04 am

Seeing the construction of the midpoint G, we add in the A-excenter I', making DG the midline in $\triangle II'F$.

Lemma:
$\triangle AEI \sim \triangle AI'F$

Proof:
Consider the composition of inversion about A with radius $\sqrt{AB \cdot AC}$ and reflection over the A-angle bisector; remark that E is sent to F and I is sent to I'. Therefore, $AE \cdot AF = AB \cdot AC = AI \cdot AI'$ whence $\frac{AE}{AI} = \frac{AI'}{AF}$ as desired.

Extend EI to hit the circumcircle again at P and define P' for DG similarly. Then by the above similarity, we have $\angle AI'F = \angle ADG = \angle ADP'$, but also $\angle AI'F = \angle AEI = \angle AEP$, so $P \equiv P'$ by cyclicity, and we're done.

math2468

#29 Sep 15, 2015, 1:21 am

There is a short solution with cross ratio chasing. Let M, N be the midpoints of BI, CI respectively. Let $K' = DG \cap \Gamma, K = EI \cap \Gamma, D_1 = AD \cap \Gamma, P = BI \cap \Gamma, Q = CI \cap \Gamma$ and let D_2 be the midpoint of ID_1. It follows from easy angle chasing that D, P, N are collinear, and D, Q, M are collinear. Then,we have that

$$(B, D_1; F, C) \stackrel{I}{=} (M, D_2; G, N) \stackrel{U}{=} (Q, A; K', P)_\Gamma$$

But we also have that

$$(B, D_1; F, C) = (AB, AD; AF, AC) = (AC; AD; AE, AB) \stackrel{A}{=} (C, D; E, B)_\Gamma \stackrel{I}{=} (Q, A; K, P)_\Gamma$$

so $K = K'$ as desired.

This post has been edited 1 time. Last edited by math2468, Sep 15, 2015, 1:22 am
Reason: Typo

Dukejukem

#30 Sep 16, 2015, 9:41 am

Let K, H be the second intersections of EI, AF with Γ, respectively. Denote $Q \equiv AD \cap BC$ and $P \equiv DK \cap AH$. We will show that P, D, G are collinear.

Because $\angle BAH = \angle CAE$, it follows that D is the midpoint of arc $\widehat{HDE}$. Then from Pascal's Theorem applied to $EHADDK$, we find that $PI \parallel DD$. In particular, $PI \parallel BC$, implying that $\frac{PA}{PF} = \frac{IA}{IQ}$. Now, recall that D is the circumcenter of $\triangle BIC$. Thus, the inversion with center D and radius $r = DB = DC = DI$ fixes I and swaps Γ and BC. In particular, A and Q are swapped, so the inversive distance formula implies that

$$IQ = IA \cdot \frac{r^2}{DI \cdot DA} \implies \frac{IA}{IQ} = \frac{DA}{DI}.$$

Then from Menelaus' Theorem on $\triangle AFI$ cut by PD, we deduce that PD bisects $\overline{IF}$, i.e. P, D, G are collinear. $\square$

AMN300

#31 Dec 28, 2015, 3:51 am

Set the A-excenter as I_a. Let $EI \cap \Gamma \equiv X$ and $DG \cap \Gamma \equiv X'$. We will show $X \equiv X'$.

Lemma: $AI \cdot AI_a = bc$

Proof: Define $\alpha = \frac{A}{2}$, $[ABC] = K$, the inradius of ABC be r, and the radius of the A-excircle be r_a. Then $AI \cdot AI_a = \frac{r}{\sin \alpha} \cdot \frac{r_a}{\sin \alpha}$. Using $r = \frac{K}{s}$ and $r_a = \frac{K}{s-a}$, it remains to show that

$$K^2 = s(s-a) \cdot bc \cdot \sin^2 \alpha$$

That is,

$$K^2 = s(s-a)(bc)(\frac{1-\cos A}{2}) = s(s-a)\frac{2bc - b^2 - c^2 + a^2}{4} = s(s-a)(s-b)(s-c)$$

which is Heron's formula.

Since F and E switch roles under $\sqrt{bc}$ inversion, we have $AF \cdot AE = bc = AI \cdot AI_a$, so $\frac{AF}{AI_a} = \frac{AI}{AE}$. Combined with $\angle FAI_a = \angle IAE$, we have $\triangle FAI_a \sim IAE$, so

$$\angle AEX = \angle FI_aA = \angle GDX'$$

Thus $X \equiv X'$ and we're done.

navi_09220114

#32 Mar 29, 2016, 4:46 pm

A good exercise to apply the Shooting Lemma :)

Let $EI \cap \Gamma = X$, $DX \cap BC = M$, $DX \cap AF = N$ and $AI \cap BC = P$. Clearly it suffices to show $MINF$ is a parallelogram, then MN bisects IF and we are done.

Apply Shooting Lemma, $DI^2 = DB^2 = DP \cdot DA \Rightarrow \frac{DP}{DI} = \frac{DI}{DA}$. Apply it again then $DM \cdot DX = DB^2 = DP \cdot DA$. From the given angle condition, $\angle NAI = \angle DAE = \angle NXI$, so $DN \cdot DX = DI \cdot DA$, divide both equations gives $\frac{DP}{DI} = \frac{DM}{DN} = \frac{DI}{DA}$. First equality gives $PM \parallel NI$, while the second equality gives $IM \parallel AF$, as desired.

DrMath

#33 Apr 16, 2016, 4:33 am

Note E and F are inverses w.r.t. a $\sqrt{bc}$ inversion about A. But so are I and I_A, the $A-$excenter. Thus $\angle FI_AA = \angle IEA$. But note GD is a midline of $\triangle FII_A$ w.r.t. side FI_A so $\angle IEA = \angle FI_AA = \angle GDA$. Thus EI and DG concur on Γ as desired.

Kirilbangachev

#34 May 1, 2016, 2:35 pm

A very simple way is to use some sine theorems and Menelaus theorem for $\triangle AFI$ and the line $D-G-S$. This will imply that $\frac{AD}{AS}=\frac{AE}{AI} \Rightarrow \triangle AEI \sim \triangle ADS \Rightarrow \angle AEI = \angle ADS$. This implies the desired result.

Quick Reply

High School Olympiads

a Geometry Proposition

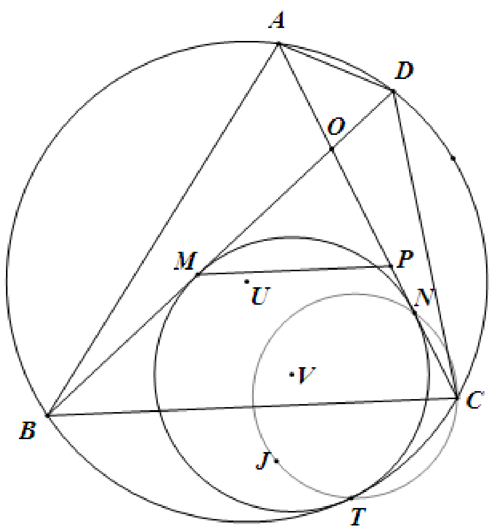
Lin_yangyuan

#1 Dec 25, 2015, 12:23 pm

The picture contains many other conclusions.

Attachments:

Given $\triangle OBC$ and $\odot U$ passing through B,C.Let $\odot V$ be the circle tangent toOB,OC and tangent to $\odot U$ at T.$MP \parallel BC$， and J is the O-excenter of OMP.
Prove that： $JNCT$ are on the same circle.



Math_tricks

#2 Dec 25, 2015, 3:51 pm

What is point M? It is a random point on the side OB or it is the tangent point of circle of center V with OB?

Lin_yangyuan

#3 Dec 25, 2015, 8:24 pm

> *Math_tricks wrote:*
> What is point M? It is a random point on the side OB or it is the tangent point of circle of center V with OB?

the tangent point of circle of center V with OB□N is the same.

Luis González

#4 Dec 25, 2015, 10:36 pm

Let I be the incenter of $\triangle OBC$. Since $\angle MOJ = \angle IOC$ and $\angle OJM = \frac{1}{2}\angle OPM = \frac{1}{2}\angle OCB = \angle OCI \Longrightarrow \triangle OMJ \sim \triangle OIC \Longrightarrow \frac{OM}{OJ} = \frac{OI}{OC} \Longrightarrow OI \cdot OJ = OC \cdot OM = OC \cdot ON \Longrightarrow J, I, N, C$ are concyclic. But according to the problem incenter of triangle, T, I, N, C are concyclic $\Longrightarrow J, N, C, T$ are concyclic.

PROF65

#5 Dec 25, 2015, 10:38 pm

Let I,I' incenter and excenter of OBC .OBC and OMP are homothetic then
$\frac{OI'}{OJ}=\frac{OB}{OM}\implies\frac{OB}{OM}=\frac{OI'\cdot OI}{OJ\cdot OI}=\frac{OB\cdot OC}{OI\cdot OJ}\implies OI\cdot OJ=OM\cdot OC=ON\cdot OC$ but it 's known that $NCTI$ is cyclic therefore the result follows.

WCP

Quick Reply

High School Olympiads

Isotomic conjugate of Isogonal conjugate of Inconic

Reply

Source: Own

TelvCohl

#1 Dec 23, 2015, 2:25 am • 1

Given a $\triangle ABC$ and a point P. Let Q be the isotomic conjugate of the isogonal conjugate of P WRT $\triangle ABC$ WRT $\triangle ABC$. Let $\mathcal{C}_P, \mathcal{C}_Q$ be the inconic of $\triangle ABC$ with perspector P, Q, respectively. Prove that $\mathcal{C}_Q$ is the isotomic conjugate of the isogonal conjugate of $\mathcal{C}_P$ WRT $\triangle ABC$ WRT $\triangle ABC$.

Luis González

#2 Dec 24, 2015, 9:08 am • 3

Let X be an arbitrary point on $\mathcal{C}_P$. P', X' denote the isogonal conjugates of P, X WRT $\triangle ABC$ and Y denotes the isotomic conjugate of X' WRT $\triangle ABC$. Therefore $A(B,C,P,X) = A(C,B,P',X') = A(B,C,Q,Y)$ and similarly we have the $B(C,A,P,X) = B(C,A,Q,Y), C(A,B,P,X) = C(A,B,Q,Y) \Longrightarrow \mathbf{H} : X \mapsto Y$ is the homography fixing A, B, C and carrying P into Q. $\mathbf{H}$ clearly maps $\mathcal{C}_P$ into $\mathcal{C}_Q$, i.e. $Y \in \mathcal{C}_Q$ and the conclusion follows.

Quick Reply

High School Olympiads

acute triangle

Locked

tyuighp

#1 Dec 24, 2015, 4:06 am

Point P lies in the interior of the acute triangle ABC. The circumcenters of triangles ABC, BCP, CAP and ABP are $O, \Delta A_1B_1$ and $\Delta C_1,$ respectively. Prove that

$$\frac{\text{area}\,(\Delta A_1B_1O)}{\text{area}\,(\Delta ABP)}=\frac{\text{area}\,(\Delta B_1C_1O)}{\text{area}\,(\Delta BCP)}=\frac{\text{area}\,(\Delta C_1A_1O)}{\text{area}\,(\Delta CAP)}.$$

Luis González

#2 Dec 24, 2015, 4:16 am

Notice that A_1,B_1,C_1 lie on perpendicular bisectors of $BC,CA,AB,$ respectively, thus $\triangle ABC$ and $\triangle A_1B_1C_1$ are orthologic with orthology centers $P,O.$ Now the problem is just a particular case of http://www.artofproblemsolving.com/community/c6h441141 (see post #4).

High School Olympiads

pedal:)

Reply

erfan_Ashorion

#1 Oct 27, 2011, 12:03 am • 1

proof The symmedian point of a triangle is the triangle centroid of its pedal triangle

unt

#2 Oct 27, 2011, 1:22 am • 1

This is a famous result. See http://www.cip.ifi.lmu.de/~grinberg/SymmedianPedalPDF.zip

yetti

#3 Oct 27, 2011, 4:17 am • 1

See also http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=209057.

Luis González

#4 Oct 27, 2011, 8:17 am • 1

There's a generalization due to Eric Danneels and Nikolaos Dergiades.

Theorem: If $\triangle XYZ$ and $\triangle ABC$ are orthologic with orthology centers U and V, then the barycentric coordinates of U with respect to $\triangle XYZ$ are equal to the barycentric coordinates of V with respect to $\triangle ABC$.

Proof: Let $M \equiv XU \cap YZ$ and $N \equiv AV \cap BC$. From $UX \perp BC, UY \perp CA, UZ \perp AB$ and $VA \perp YZ$, $VB \perp ZX, VC \perp XY$, we get $\angle XYM = \angle CVN$ and $\angle MXY = \angle NCV \Longrightarrow \triangle MXY \sim \triangle NCV$. Likewise, $\triangle MXZ \sim \triangle NBV$. Thus

$$\frac{MY}{VN} = \frac{MX}{NC}, \ \frac{MZ}{VN} = \frac{MX}{NB} \Longrightarrow \frac{MZ}{MY} = \frac{|\triangle UZX|}{|\triangle UXY|} = \frac{NC}{NB} = \frac{|\triangle VCA|}{|\triangle VAB|}$$

Similarly, we have $\dfrac{|\triangle UXY|}{|\triangle UYZ|} = \dfrac{|\triangle VAB|}{|\triangle VBC|}$ and $\dfrac{|\triangle UYZ|}{|\triangle UZX|} = \dfrac{|\triangle VBC|}{|\triangle VCA|}$

Thus, barycentrics of U WRT $\triangle XYZ$ are then $(|\triangle VBC| : |\triangle VCA| : |\triangle VAB|)$.

yetti

#5 Oct 27, 2011, 12:28 pm • 2

Another generalization:

$\triangle DEF$ is pedal triangle and $\triangle XYZ$ is circumcevian triangle of a point P WRT reference triangle $\triangle ABC \Longrightarrow$
$\triangle DEF \sim \triangle XYZ$ are similar, having equal angles:
$\angle ZXY = \angle ZXA + \angle AXY = \angle ZCA + \angle ABY = \angle PCE + \angle FBP = \angle PDE + \angle FDP = \angle FDE$, etc.
Let P^* be isogonal conjugate of P WRT circumcevian triangle $\triangle XYZ$. Then P, P^* are corresponding points of similar triangles $\triangle DEF \sim \triangle XYZ$, because
$\angle P^*XY = \angle ZXP = \angle ZXA = \angle ZCA = \angle PCE = \angle PDE$, etc.

Symmedians are common chords of the circumcircle (O) of $\triangle ABC$ and its three Apollonius circles $\perp (O)$. Centroid is isogonal conjugate of symmedian point.
P is symmedian point of $\triangle ABC \Longleftrightarrow$ Apollonius circles of $\triangle ABC$ are also Apollonius circles of the circumcevian triangle $\triangle XYZ$ of P WRT $\triangle ABC \Longleftrightarrow$
P is symmedian point of $\triangle XYZ \Longleftrightarrow P$ is centroid the pedal triangle $\triangle DEF$ of P WRT $\triangle ABC$.

Quick Reply

High School Olympiads

Equal distances between pairs of orthocenters in cyclic quad

Source: XVII Tuymaada Mathematical Olympiad (2010), Senior Level

Shu

#1 Aug 1, 2011, 3:42 am • 1

In a cyclic quadrilateral $ABCD$, the extensions of sides AB and CD meet at point P, and the extensions of sides AD and BC meet at point Q. Prove that the distance between the orthocenters of triangles APD and AQB is equal to the distance between the orthocenters of triangles CQD and BPC.

Luis González

#2 Aug 1, 2011, 4:47 am • 1

Let H_1, H_2, H_3, H_4 be the orthocenters of $\triangle APD, \triangle AQB, \triangle CQD, \triangle BPC$. H_1, H_2, H_3, H_4 are collinear on the Steiner line of the complete quadrangle $ABCD$, thus it suffices to prove that $\overline{H_1H_4}$ and $\overline{H_2H_3}$ have the same midpoint. Since AB, CD are antiparallel with respect to QD, QC, then $QH_2 \perp AB$ is the Q-circumdiameter of $\triangle QDC$, which cuts its circumcircle again at the reflection F of its orthocenter H_3 about the midpoint M of CD. Hence, if T is the midpoint of $\overline{H_1H_2}$, then $MT \parallel FH_2$, i.e. $MT \perp AB \Longrightarrow$ CD-maltitude of $ABCD$ passes through the midpoint T of $\overline{H_2H_3}$. Similarly, AB-maltitude of $ABCD$ passes through $T \Longrightarrow T$ is the anticenter of $ABCD$. Likewise, T is the midpoint of $\overline{H_1H_4} \Longrightarrow$ $H_1H_2 = H_3H_4$.

High School Olympiads

Adventure on complete quadrilateral

Reply

ThE-dArK-lOrD

#1 Dec 23, 2015, 9:41 pm

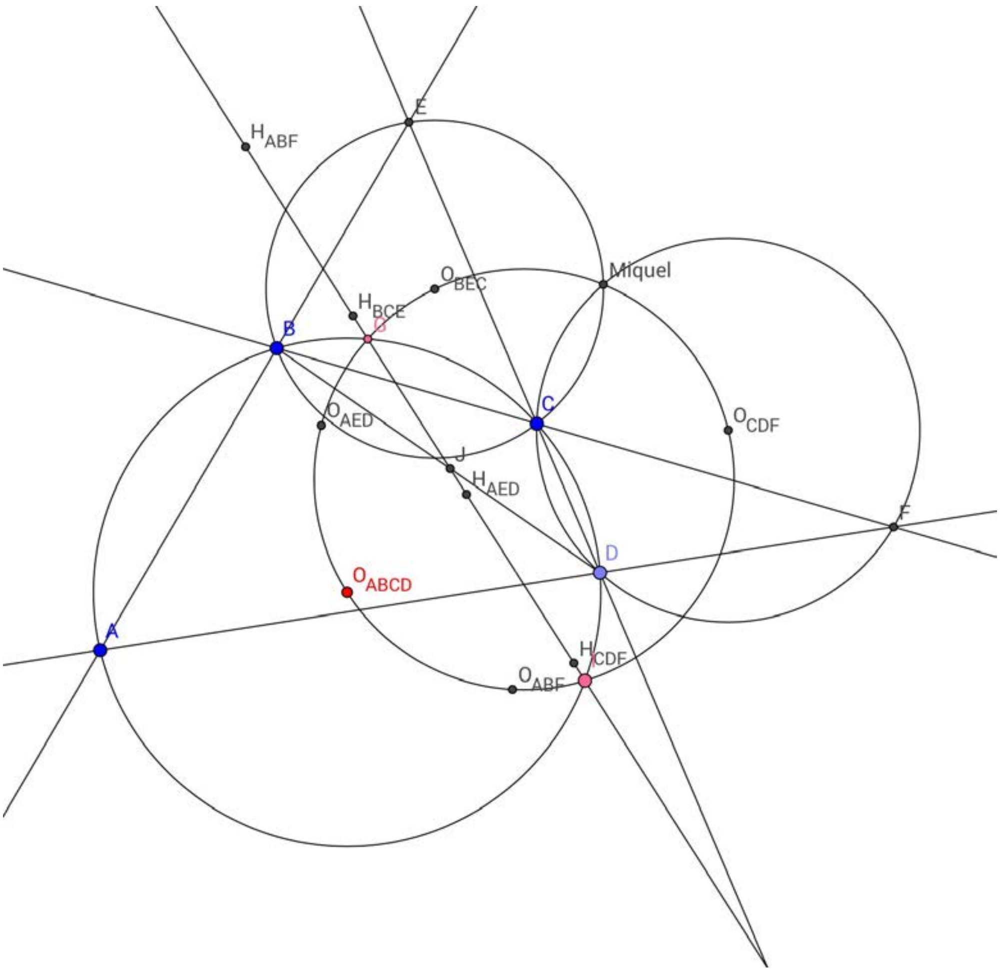
Given cyclic quadrilateral $ABCD$ with circumcenter O
Let AB intersect CD at E and AD intersect BC at F
Let circumcircle of BEC intersect CDF at Miquel point M
Denote H_{XYZ} represent orthocenter of triangle XYZ
And O_{XYZ} represent circumcenter of triangle XYZ
It is known as Steiner-Ortholine that $H_{ABF},H_{BCE},H_{AED},H_{CDF}$ are on the same line called l
More over we have J,intersection of AC and BD is on that line to.
Proof that
(1) $O,O_{ABF},O_{BCE},O_{AED},O_{CDF},M$ are on the same circle called w
(2) Let line l intersect circle w at G and I, show that there are on circumcircle of $ABCD$

ThE-dArK-lOrD

#2 Dec 23, 2015, 9:43 pm

Figure@below

Attachments:



Luis González

#3 Dec 23, 2015, 11:03 pm • 1

1) It's the well-known Miquel circle of $ABCD,$ which actually exists in any complete quadrilateral. The proof is rather easy with angle chase so I'm not posting it.

2) It suffices to show that the Steiner line of M WRT $ABCD$ coincides with the radical axis of $(O) \equiv \odot(ABCD)$ and the Miquel circle ω of $ABCD.$ It's well-known that $M \in EF$ is the inverse of $J \equiv AC \cap BD$ WRT $(O) \Longrightarrow A, C, M, O$ are concyclic and since $OA = OC,$ then MO bisects $\angle AMC.$ Thus from $\triangle MCO \sim \triangle MJA,$ we get $MA \cdot MC = MO \cdot MJ = \varrho^2$ and analogously MO bisects $\angle BMD$ and $MB \cdot MD = \varrho^2.$ Thus inversion WRT $\odot(M, \varrho)$ followed by reflection on MO swaps A, C and $B, D.$ Hence it fixes (O) and clearly swaps the intersections of (O) with any circle through $M, O,$ particularly it swaps $\{G, I\} \equiv (O) \cap \omega \Longrightarrow \omega$ goes to GI and notice that the centers of $\odot(EBC)$ and $\odot(FCD)$ go to the reflections of M on AD and $AB,$ lying on the Steiner line of M WRT $ABCD \Longrightarrow GI$ coincides with the Steiner line of M WRT $ABCD,$ as desired.

ThE-dArK-lOrD

#4 Dec 23, 2015, 11:21 pm

... Oops! I discovered another fact. COOL!

(3) Proof that $H_{ABF}H_{BCE} = H_{AED}H_{CDF}$

Luis González

#5 Dec 23, 2015, 11:44 pm

3) See the topic Equal distances between pairs of orthocenters in cyclic quad.

Quick Reply

High School Olympiads

Isogonal conjugate (P,Q), (R,S) such that PQ||RS

Reply

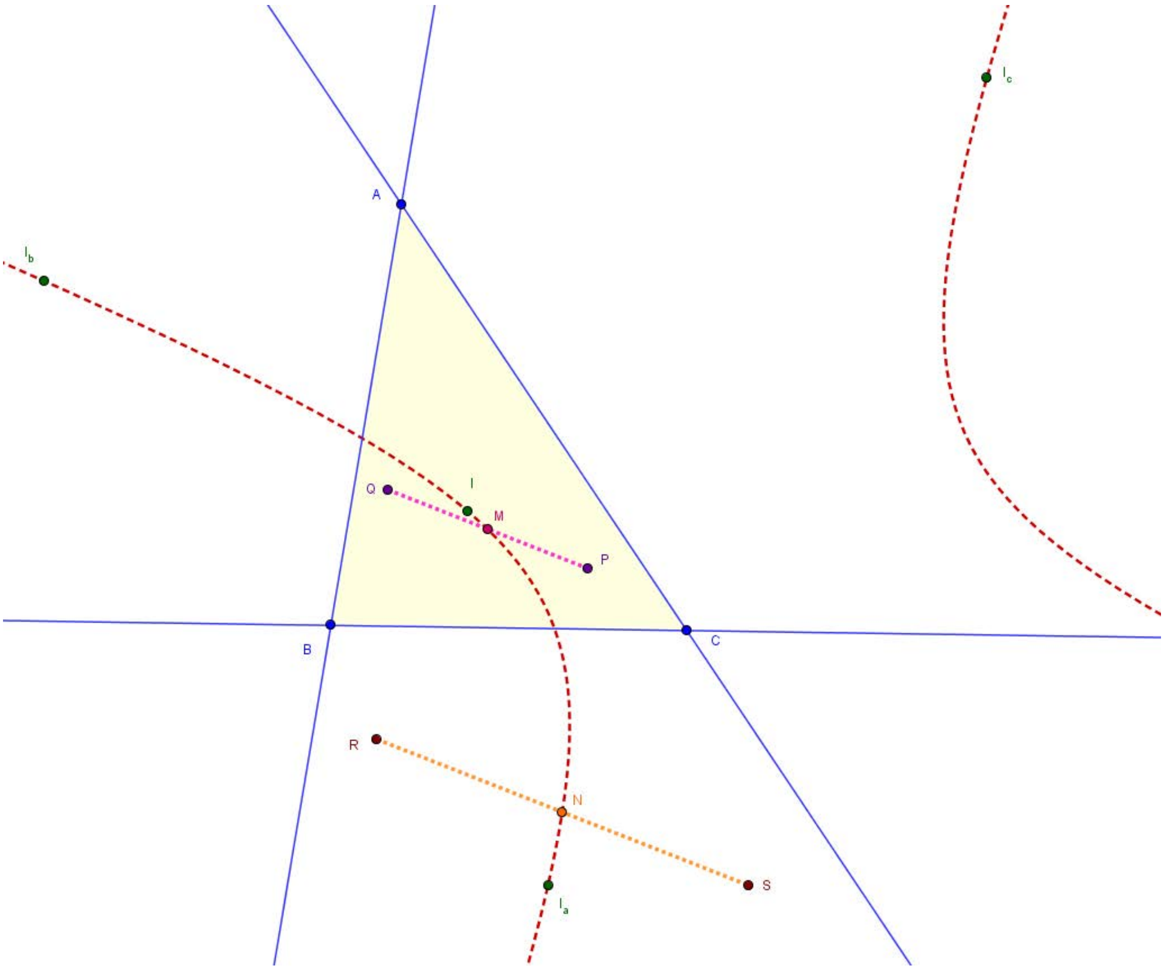
Source: Own

TelvCohl

#1 Nov 27, 2015, 11:14 am • 1

Let $(P,Q),(R,S)$ be two pairs of isogonal conjugate of $\triangle ABC$ s.t. $PQ \parallel RS$. Let I, I_a, I_b, I_c be the incenter, A-excenter, B-excenter, C-excenter of $\triangle ABC$, respectively. Let M,N be the midpoint of PQ, RS, respectively. Prove that I, I_a, I_b, I_c, M, N lie on a conic.

Attachments:



Luis González

#2 Dec 23, 2015, 9:44 pm • 3

Recalling that a pair of isogonal conjugates are foci of an inconic, we can rephrase the problem as follows: The locus of the centers J of all the inconics $\mathcal{C}$ of $\triangle ABC$ whose main axes are parallel is a rectangular hyperbola through the incenter and 3 excenters of $\triangle ABC$.

Label e, f the main axes of $\mathcal{C}$. (O) is the circumcircle of $\triangle ABC$ and $\triangle XYZ$ is its tangential triangle. The dual of $\mathcal{C}$ WRT (O) is a conic $\mathcal{H}$ through X, Y, Z such that the fixed lines ℓ, τ through O perpendicular to e, f are conjugates WRT $\mathcal{H}$, in other words, the pole E of τ WRT $\mathcal{H}$ lies on $\ell \Longleftrightarrow$ the pole F of ℓ WRT $\mathcal{H}$ lies on τ.

Consider a homography with complex coefficient sending Y, Z to the cyclic points at infinity. Thus $\mathcal{H}$ becomes a circle (K) passing through X such that ℓ, τ are conjugate lines WRT (K). Thus letting S be the intersections of KE with τ, we get from the condition that $E \in \ell$ is the pole of τ WRT $(K) \Longrightarrow KX^2 = KS \cdot KE = \text{dist}(K, \ell) \cdot \text{dist}(K, \tau) \cdot \sec \angle(\ell, \tau)$ $(\star)$. Now setting this in a rectangular reference, for example, it's clear that the locus $\mathcal{K}$ of $K \equiv (x, y)$ is certain conic, as the LHS is a second degree expression and the RHS is a product of two linear expresions. In addition, it's clear there exists two points E, S fulfilling that the perpendicular bisector of $\overline{ES}$ goes through X. This is precisely case where (K) degenerates into a line, thus $\mathcal{K}$ goes through the point at infinity of $\perp \tau$.

Note that the locus of K such that $F \in \tau$ is the pole of ℓ WRT (K) again fulfills the expression $(\star)$, thus it's the same conic $\mathcal{K}$ and then it similarly passes though the point at infinity of $\perp \ell \Longrightarrow \mathcal{H}$ has asymptotes perpendicular to ℓ, τ. Since $KE \perp \tau, KF \perp \ell$, then KE, KF are proyective pencils inducing a proyectivity $E \mapsto F$ between τ and ℓ. So back in the primitive figure $E \mapsto F$ is a proyectivity as well $\Longrightarrow$ their polars e, f WRT (O) are proyective pencils $\Longrightarrow J$ moves on a conic $\mathcal{J}$ with asymptotes parallel to e, f. Since $\mathcal{C}$ eventually becomes incircle or an excircle of $\triangle ABC$, it follows that $\mathcal{J}$ passes through the incenter and three excenters of $\triangle ABC$, as desired. $\blacksquare$

P.S. From this solution we are obtaining the following nice property: The main axes of an inconic of $\triangle ABC$ with center J are parallel to the asymptotes of the rectangular hyperbola that passes though J and the incenter and 3 excenters of $\triangle ABC$.

A-B-C

#3 Dec 24, 2015, 10:10 pm • 1

More general: $\mathcal{C}$ is a pivotal isocubic WRT $\triangle ABC$. P is a point on $\mathcal{C}$. A line ℓ passes through P, intersects $\mathcal{C}$ at P_1, P_2. Q satisfies $(P, Q, P_1, P_2) = -1$ then Q lies on a conic called *polar conic* of P WRT $\mathcal{C}$.
See special isocubics in triangle plane.
Back to the problem. Let W be infinity point of PQ, RS. Consider the circular isogonal cubic, pivot W. WI, WI_a, WI_b, WI_c are tangent to the cubic at I, I_a, I_b, I_c, respectively. Notice that $(W, M, P, Q) = (W, N, R, S) = -1$, hence I, I_a, I_b, I_c, M, N lie on a conic.
Remark. W^* is isogonal conjugate of W. Center of the conic is the intersection other than W^* of WW^* and circumcircle of $\triangle ABC$.

TelvCohl

#4 Dec 26, 2015, 5:24 am • 1

Thank you all for your nice solutions. Here is mine :

Problem :

Given a $\triangle ABC$ and a point L (at infinity). Let J, K be the isogonal conjugate of $\triangle ABC$ such that $L \in JK$ and let $\mathcal{C}$ be the inconic of $\triangle ABC$ with focus J, K. Prove that the center T of $\mathcal{C}$ lies on a fixed conic as J, K varies.

Proof :

Lemma 1 : Given a $\triangle ABC$ with the points $E \in CA, F \in AB$. Let $\mathcal{P}$ be the parabola tangent to BC, CA, AB, EF. Let Y be the reflection of E in the midpoint of CA and let Z be the reflection of F in the midpoint of AB. Then $T \equiv \mathcal{P} \cap EF$ lies on the line connecting the anticomplement Y_0, Z_0 of Y, Z WRT $\triangle ABC$, respectively.

Proof : Let $\triangle A_0B_0C_0$ be the anticomplementary triangle of $\triangle ABC$ and let O be the circumcenter of $\triangle ABC$. Let M be the Miquel point of the complete quadrilateral $\mathcal{Q} \equiv \{\triangle ABC, EF\}$ (focus of $\mathcal{P}$) and let M^* be the reflection of M in EF. Let M_0 be the midpoint of Y_0Z_0 and M_1 be the projection of O on Y_0Z_0. Since M^*T passes through the center of $\mathcal{P}$ (well-known), so M^*T is parallel to the Newton line of $\mathcal{Q} \Longrightarrow M^*T \parallel YZ \parallel Y_0Z_0$, hence it suffices to prove $M^* \in Y_0Z_0$.

We'll prove $M^* \equiv M_1$. Since M is the center of the spiral similarity that swaps BC and FE, so we get $\frac{ME}{MF} = \frac{EC}{FB} = \frac{AY}{AZ} \Longrightarrow \triangle AYZ \sim \triangle MEF \sim \triangle MCB$. On the other hand, from the generalization of 2009 IMO P2 (see Hard Geometry) we get E, F, M_0, M_1 are concyclic, so notice $M_0E \parallel AB$ and $M_0F \parallel CA$ we get $\measuredangle M_1EF = \measuredangle M_1M_0F = \measuredangle ZYA = \measuredangle FEM$ and similarly $\measuredangle M_1FE = \measuredangle EFM$, hence M_1 and M are symmetry WRT $EF \Longrightarrow M^* \equiv M_1$.

EDIT : O is the circumcenter of $\triangle A_0B_0C_0$.

Lemma 2 : Given a fixed conic $\mathcal{C}$ and a fixed point $T \in \mathcal{C}$. Let V be a fixed point such that VT is tangent to $\mathcal{C}$ at T. Let ℓ be a fixed line and let R be a point varies on ℓ. Let $J \equiv \ell \cap VT, S \equiv TR \cap \mathcal{C}$ and let VR cuts the tangent of $\mathcal{C}$ passing through S at K. Then K lies on a fixed conic passing through V and the harmonic conjugate U of V WRT J and T.

Proof : Let τ be the polar of R WRT $\mathcal{C}$ and let $L \in \tau$ be the pole of ℓ WRT $\mathcal{C}$. Let $P \equiv VR \cap \tau, Q \equiv ST \cap \tau, X \equiv TP \cap \ell$. Since pencil $VR \mapsto$ pencil LP is a homography, so P lies on a fixed conic $\mathcal{H}$ passing through L, V and T (when $R \equiv J$) as R varies on ℓ. Since $X(P, R; K, V) = (Q, R; S, T) = -1 = X(T, J; U, V)$, so K, U, X are collinear.

Since pencil $TP \mapsto$ pencil VP is a homography, so $X \mapsto R$ is a homography $\Longrightarrow$ pencil $UX \mapsto$ pencil VR is a homography, hence we conclude that $K \equiv UX \cap VR$ lies on a fixed conic passing through U and V.

Back to the main problem :

Let $V \equiv JK \cap BC$ and let the 2nd tangent of $\mathcal{C}$ passing through V cuts CA, AB at E, F, respectively. Let A^*, B^*, C^*, E^*, F^* be the midpoint of BC, CA, AB, BE, CF, respectively. From Newton-Gauss Line $\Longrightarrow T$ lies on the Newton line E^*F^* of the complete quadrilateral $\{BC, CA, AB, EF\}$. From **Lemma 1** we get the inscribed parabola $\mathcal{P}$ (notice the center of $\mathcal{P}$ is the infinity point on EF, so $\mathcal{P}$ is fixed) of the complete quadrilateral $\mathcal{Q} \equiv \{B^*C^*, C^*A^*, A^*B^*, E^*F^*\}$ is tangent to E^*F^* at the intersection G of EF and E^*F^*.

Since $VG \equiv EF$ is parallel to the Newton line of $\mathcal{Q}$, so VG passes through the intersection of $\mathcal{P}$ and the line at infinity, hence from **Lemma 2** we get T lies on a fixed conic passing through the infinity point with direction $\parallel JK, \perp JK$, resp. (notice JK is the bisector of $\angle(EF, BC)$) when V varies on BC.

P.S. Obviously, this fixed conic passes through the incenter and the excenters of $\triangle ABC$.

This post has been edited 1 time. Last edited by TelvCohl, Apr 22, 2016, 3:55 pm

Quick Reply

High School Olympiads

2 triangles are similar

Reply

Source: OWN

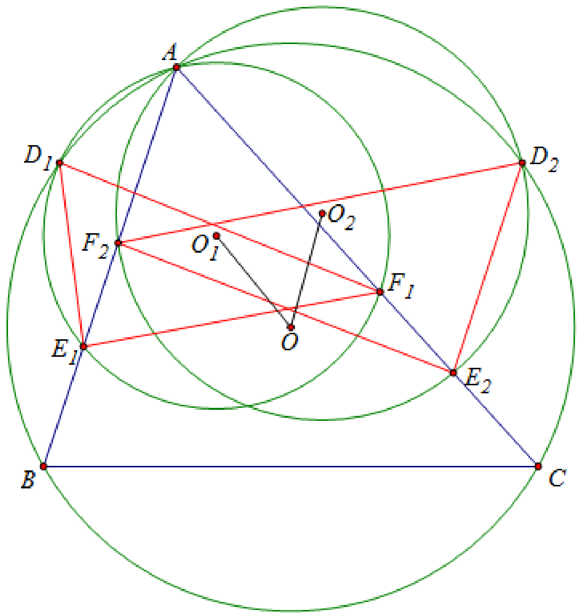
LeVietAn

#1 Dec 21, 2015, 12:57 pm

Dear Mathlinkers,

Let ABC be a triangle with circumcircle (O). Choose the points E_1 and F_1 respectively on the sides AB and AC such that $BE_1 < CF_1$. Choose the points E_2 and F_2 respectively on the sides CA and AB such that $CE_2 = BE_1$ and $BF_2 = CF_1$. Let (O_1) be the circumcircle of the triangle AE_1F_1, and let D_1 be the the second point of intersection of the circles (O) and (O_1). Let (O_2) be the circumcircle of the triangle AE_2F_2, and let D_2 be the the second point of intersection of the circles (O) and (O_2). Prove that the triangles $D_1E_1F_1$ and $D_2E_2F_2$ are similar to each other and $OO_1 = OO_2$.

Attachments:



Luis González

#2 Dec 22, 2015, 2:37 am

Since D_1 and D_2 are the centers of the spiral similarities that swap $\overline{BE_1}, \overline{CF_1}$ and $\overline{BF_2}, \overline{CE_2}$, resp, we get $\frac{D_1B}{D_1C} = \frac{BE_1}{CF_1} = \frac{CE_2}{BF_2} = \frac{D_2C}{D_2B}$. Since D_1, D_2 lie on the same side of the line BC, then by symmetry it follows that $BC \parallel D_1D_2 \Longrightarrow \triangle D_1BC \cong \triangle D_2CB$ are symmetric WRT the perpendicular bisector of $\overline{BC}$. As D_1 and D_2 are also the centers of the spiral similarites that swap $\overline{BC}, \overline{E_1F_1}$ and $\overline{BC}, \overline{F_2E_2}$, we get $\triangle D_1E_1F_1 \sim \triangle D_1BC$ and $\triangle D_2E_2F_2 \sim \triangle D_2CB \Longrightarrow \triangle D_1E_1F_1 \sim \triangle D_2E_2F_2$.

Second intersection J of $(O_1), (O_2)$ is the center of the rotation that swaps the congruent segments $\overline{E_1F_2}$ and $\overline{F_1E_2} \Longrightarrow JE_1 = JF_1 \Longrightarrow J$ is midpoint of the arc E_1F_1 of $(O_1) \Longrightarrow AJ$ bisects $\angle BAC$, but as $D_1D_2 \parallel BC$, then AJ also bisects $\angle D_1AD_2$. Since $OO_1 \perp AD_1$ and $O_1O_2 \perp AJ$, then $\angle OO_1O_2 = \angle JAD_1 = \frac{1}{2}\angle D_1AD_2$ and similarly $\angle OO_2O_1 = \frac{1}{2}\angle D_1AD_2 \Longrightarrow \triangle OO_1O_2$ is isosceles with apex $O \Longrightarrow OO_1 = OO_2$.

Quick Reply

High School Olympiads

Intersection of three lines

Locked

emil2000

#1 Dec 22, 2015, 12:50 am

Let ABC be an acute triangle with orthocenter H. Let P be a point on plane ABC and let points $A_1, A_2, B_1, B_2, C_1, C_2$ be projections of point P on lines BC, AH, AC, BH, AB, CH respectively. Prove that lines A_1A_2, B_1B_2, C_1C_2 intersect at one point.

Luis González

#2 Dec 22, 2015, 1:42 am

Posted many times before, e.g. see the topics perpendicular feet and Concurrent lines formed by projections.

High School Olympiads

perpendicular feet

Reply

barasawala

#1 Dec 29, 2006, 1:51 pm

Let AA_1, BB_1, CC_1 be the altitudes in acute triangle ABC, and let X be an arbitrary point. Let M, N, P, Q, R, S be the feet of the perpendiculars from X to the lines $AA_1, BC, BB_1, CA, CC_1, AB$. Prove that MN, PQ, RS are concurrent.

darij grinberg

#2 Dec 29, 2006, 4:35 pm

> *barasawala wrote:*
> Let AA_1, BB_1, CC_1 be the altitudes in acute triangle ABC, and let X be an arbitrary point. Let M, N, P, Q, R, S be the feet of the perpendiculars from X to the lines $AA_1, BC, BB_1, CA, CC_1, AB$. Prove that MN, PQ, RS are concurrent.

This is Theorem 2 from the note "On the paracevian perspector" on my website (which currently lives at http://www.cip.ifi.lmu.de/~grinberg/ or, archived, at http://sites.google.com/site/darijgrinberg/website). In that note, I also prove a generalization of your problem (see also http://www.mathlinks.ro/Forum/viewtopic.php?t=14692 for a discussion of that generalization).

I will leave to you the fun of finding the short solution of the above problem which does *not* apply to the generalization. :)

Darij

This post has been edited 3 times. Last edited by darij grinberg, Sep 13, 2009, 5:10 pm

vittasko

#3 Dec 29, 2006, 11:50 pm

> *barasawala wrote:*
> Let AA_1, BB_1, CC_1 be the altitudes in acute triangle ABC, and let X be an arbitrary point. Let M, N, P, Q, R, S be the feet of the perpendiculars from X to the lines $AA_1, BC, BB_1, CA, CC_1, AB$. Prove that MN, PQ, RS are concurrent.

This problem also has been presented last time in Hyacinthos Forum on Aug 8, 2006, by Quang Tuan Bui and there was an extensive discussion, about also the generalization of the problem, as it has been presented first time by Eric Daneels ("**paracevian perspector** ?"), on Jul 23,2004.

You can see at: http://groups.yahoo.com/group/Hyacinthos/message/10135 and http://groups.yahoo.com/group/Hyacinthos/message/13907

An alternative (than Darij's in his website) proof, it has already been posted also in Hyacinthos Forum at http://groups.yahoo.com/group/Hyacinthos/message/14034 and some notes at http://groups.yahoo.com/group/Hyacinthos/message/14088 for a simpler yet proof.

I will post here next time the two alternative synthetic proofs I have in mind of the "paracevian perspector " theorem.

Kostas Vittas.

This post has been edited 1 time. Last edited by vittasko, Jun 17, 2011, 2:34 am

probability1.01

#4 Dec 30, 2006, 7:16 am

I did not check the above links, but here is a way to prove it:

Dilate with a factor of 2 about X. The image of MN becomes the line A_1X reflected about AA_1, and analogous results hold for the other two sides. Since the altitudes of ABC are the angle bisectors of its orthic triangle, the images of MN, PQ, and RS concur at the isogonal conjugate of X in the orthic triangle. Thus MN, PQ, and RS themselves must also concur.

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darij grinberg

#5 Dec 30, 2006, 4:21 pm

> *probability1.01 wrote:*
>
> I did not check the above links, but here is a way to prove it:
>
> Dilate with a factor of 2 about X. The image of MN becomes the line A_1X reflected about AA_1, and analogous results hold for the other two sides. Since the altitudes of ABC are the angle bisectors of its orthic triangle, the images of MN, PQ, and RS concur at the isogonal conjugate of X in the orthic triangle. Thus MN, PQ, and RS themselves must also concur.

This is the simple proof I spoke of .

The problem also appears as problem 5.81 in Prasolov's Planimetry problems book (see http://www.math.su.se/~mleites/ and http://michaj.home.staszic.waw.pl/prasolov.html for English and http://www.mccme.ru/free-books/ for Russian versions), with the same proof.

Darij

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vittasko

#6 Dec 31, 2006, 4:22 am • 1

THE PARACEVIAN PERSPECTOR THEOREM.

– A triangle $\triangle ABC$ is given and let H be, an arbitrary fixed point, and let $\triangle HaHbHc$ be, it's cevian triangle.

- Let P be, an arbitrary also point and $Pa, Pb, Pc,$ the traces on sidelines $BC, CA, AB,$ from the lines through P and parallel to $AHa, BHb, CHc,$ respectively.

- Let Ap, Bp, Cp be, the traces on segment lines $AHa, BHb, CHc,$ from the lines through P and parallel to the sidelines $BC,$ $CA, AB,$ respectively.

- Prove that the lines $ApPa, BpPb, CpPc,$ are concurrent at one point so be it, $Q,$ as the midpoint of the segment between the point P and the $H-Ceva$ conjugate of $P.$

PROOF 1. - (In my drawing $AB = 15.3,AC = 14.2, BC = 11.2, AH = 12.8, BH = 4.2, AP = 8.2, CP = 6.8$).

$1)$ – We denote the intersection points of sidelines of $\triangle ABC,$ from the segment lines $PAp, PBp, PCp,$ as follows:

- As $D, D',$ the intersection points of AC, AB respectively, from the segment line $PAp.$

- As $E, E',$ the intersection points of AB, BC respectively, from the segment line $PBp.$

- As $F, F',$ the intersection points of BC, AC respectively, from the segment line $PCp.$

Also we denote:

- As $A',$ the intersection point of $DE', D'F.$

- As $B',$ the intersection point of $DE', EF'.$

- As $C',$ the intersection point of $EF', D'F.$

Based on the below proposition, we will prove that the lines AP, BP, CP, pass through the points A', B', C', respectively.

$2)$ – **PROPOSITION 1.** - A triangle $\triangle ABC$ is given and let P be, an arbitrary point inwardly to it. We denote as D, D', the intersection points of AC, AB respectively, from the line through P and parallel to BC. Also we denote as E', F, the intersection points of BC, from the lines through P and parallel to AC, AB respectively. Prove that the lines $DE', D'F, AP$, are concurrent.

$2a)$ - **PROOF.** - We denote as A', the intersection point of $DE', D'F$ and as P', the one of BC, from the segment line AP.

From $DD'//BC, \Longrightarrow \dfrac{PD}{P'C} = \dfrac{AP}{AP'} = \dfrac{PD'}{P'B}$

$\Longrightarrow \dfrac{PD}{(P'C)-(PD)} = \dfrac{PD'}{(P'B)-(PD')} \Longrightarrow \dfrac{PD}{P'E'} = \dfrac{PD'}{P'F}, (1)$ (because of $PD = E'C$ and $PD' = FB$).

From $(1) \Longrightarrow \dfrac{PD}{PD'} = \dfrac{P'E'}{P'F}$ and so, the points P, P', A', are collinear. Hence the lines $DE', D'F, AP$, are concurrent and the proof of **proposition 1,** is completed.

$3)$ – In our configuration now, we conclude that the line AP, passes through the point A' (intersection point of $DE', D'F$) and similarly the lines BP, CP, pass through the points B', C', respectively.

By the same way, from the triangle $\triangle AHaC$ and the point P inwardly to it, we conclude that the lines $ApPa, DE', AP$, are concurrent. Hence, the line $ApPa$ passes through the point A' and similarly the lines $BpPb, CpPc$, pass through the points B', C', respectively.

$4)$ – We denote as K, L, M, the intersection points of the line CC', from the segment lines $AB, A'Ap, AHa$ respectively and as N, the intersection point of EE', from the line $B'L$. We will prove that the points B, M, N, are collinear.

It is enough to prove that $(E, N, P, E') = (K, M, P, C)$.

We consider the pencils $B'.C'LPT$ and $A'.C'LPT$, where T, is the intersection point of $CC', A'B'$ and then we have that

$(B'.C'LPT) = (A'.C'LPT), (2)$ (congruent double ratios of these bundles of lines).

$(B'.C'LPT) = (E, N, P, E'), (3)$ (intersection of $B'.C'LPT$, from the line EE').

$(A'.C'LPT) = (D', Ap, P, D), (4)$ (intersection of $A'.C'LPT$, from the line DD').

from $(2), (3), (4), \Longrightarrow (E, N, P, E') = (D', Ap, P, D), (5)$.

The lines CC', DD', intersect the pencil $A.BHaPC$ and so, we have that $(K, M, P, C) = (D', Ap, P, D), (6)$.

From $(5), (6), \Longrightarrow (E, N, P, E') = (K, M, P, C), (7)$.

From (7), we conclude that the lines EK, MN, CE', are concurrent. So, the line MN passes through the vertex B of $\triangle ABC$ (as the intersection point of EK, CE'). Hence, the points B, M, N, are collinear.

$5)$ – We will prove now, that the lines $ApPa, BpPb, CpPc$, are concurrent. Let Q be, the intersection point of $BpPb, CpPc$ and it is enough to prove that this point lies on $A'L$ (because of it has already been proved that the line $ApPa$, passes through the point A'). We will prove that the pencils $B'.C'LBpA', C'.B'LCpA'$, have congruent double ratios. That is, we will prove that

$(B'.C'LBpA') = (C'.B'LCpA')$.

$(B'.C'LBpA') = (E, N, Bp, E'), (8)$.

$(C'.B'LCpA') = (F', P, Cp, F), (9)$.

We consider the bundles of lines (= pencils) $B.ENBpE'$ and $C.F'PCpF$. They have the sideline BC of $\triangle ABC$, as their common ray and the points A (as the intersection point of BE, CF'), M (as the intersection point of BN, CP) and H (as the intersection point of BBp, CCp), lie on the same line AHa. So, we have that

$(B.ENBpE') = (C.F'PCpF), (10)$.

From $(10) \Longrightarrow (E, N, Bp, E') = (F', P, Cp, F), (11)$.

From $(8),(9),(11),\Longrightarrow (B'.C'LBpA') = (C'.B'LCpA'),(12).$

From (12) and because of $B'C'$ is the common ray of pencils $B'.C'LBpA'$ and $C'.B'LCpA'$, we conclude that the points L (as the intersection point of $B'L, C'L$), Q (as the intersection point of $B'Bp, C'Cp$) and A' (as the intersection point of $B'A', C'A'$), are collinear. Hence, the lines $ApPa, BpPb, CpPc$, are concurrent at one point, so be it Q.

$6)$ – We will prove now, that the concurrency point Q, of the lines $ApPa, BpPb, CpPc$, is the midpoint of the segment between the point P and the $H - Ceva$ conjugate of P.

This result has been mentioned by Quang Tuan Bui, at last discussion in Hyacinthos Forum http://groups.yahoo.com/group/Hyacinthos/message/13938 and first time by Bernard Gibert http://groups.yahoo.com/group/Hyacinthos/message/10136

Let S be, the intersection point of $AA', B'C'$. So, from the parallelogram $AEPF'$, we have that $PS = SA$. Hence, the line through vertex A of $\triangle ABC$ and parallel to $B'C'$, intersects the line PB' at a point so be it B'', such that $PB' = B'B''$.

Similarly the line through vertex C of $\triangle ABC$ and parallel to $A'B'$, intersects the line PB', at the same point B'', because of $PT = TC$, from the parallelogram $CDPE'$.

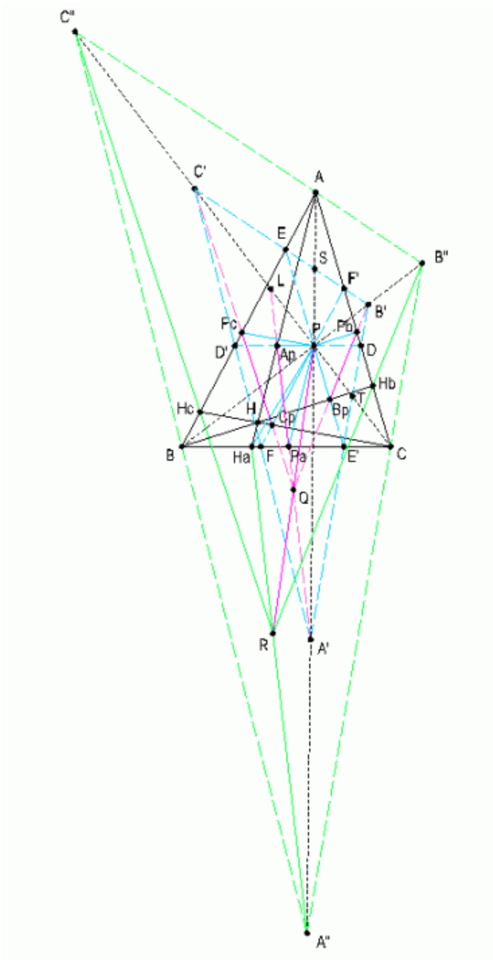
Hence, the lines through vertices A, B, C, of $\triangle ABC$ and parallel to $B'C', A'C, A'B'$ respectively, intersect each other at points A'', B'', C'', lie on the lines AA', BB', CC' respectively. The triangle $\triangle A''B''C''$, is the anticevian triangle of $\triangle ABC$, with respect to the point P.

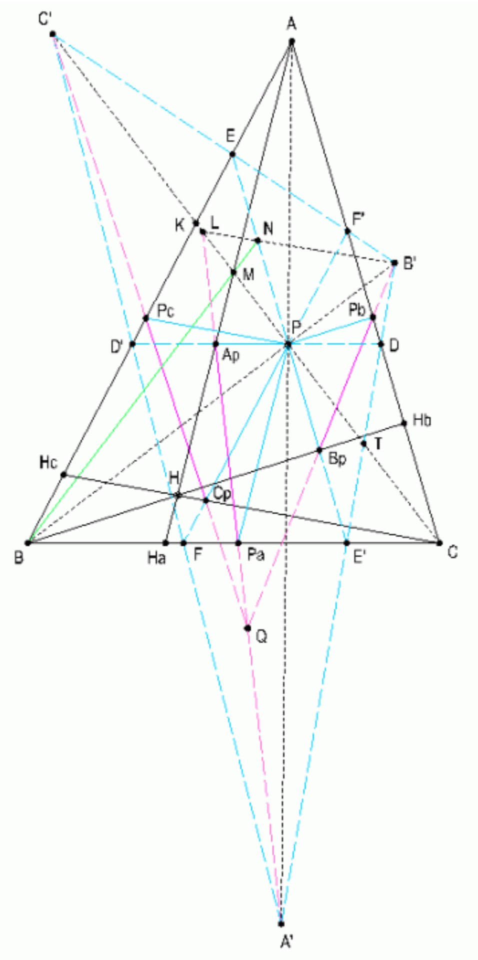
$6a)$ – We will prove now, that $B''Hb \parallel BpPb$. In triangle $\triangle B'BBp$ and from $PPb \parallel BBp$ we have that

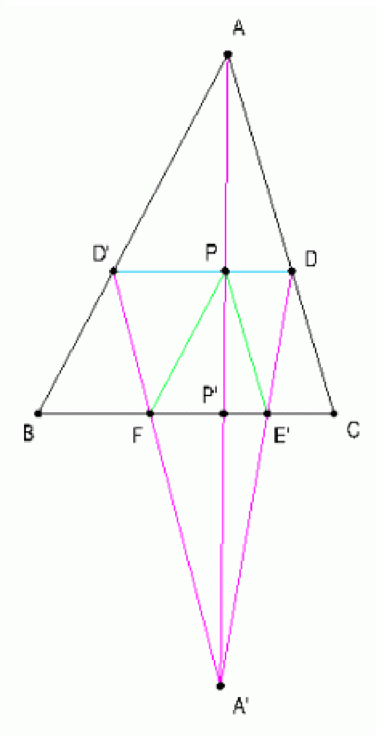
$$\frac{PPb}{BBp} = \frac{B'P}{B'B} \Longrightarrow \frac{B'B''}{B'B} = \frac{BpHp}{BBp}, (13)$$ (because of $B'P = B'B''$ and $PPb = BpHb$).

From $(13), \Longrightarrow B''Hb \parallel B'Bp$ and so, we conclude that the line $B''Hb$ connecting the vertices B'', Hb, of the triangles $\triangle A''B''C'', \triangle HaHbHc$ respectively, intersects the segment line PQ,

Attachments:







This post has been edited 1 time. Last edited by vittasko, Jun 17, 2011, 2:32 am

vittasko

#7 Dec 31, 2006, 10:57 pm • 1

PROOF 2. - (see the figure t=126138(c)).

It has already been proved that the segment lines $PAp, PBp, PCp,$ as parallel to the sidelines BC, AC, AB of $\triangle ABC$ respectively, intersect them at pairs of points D, D' and E, E' and $F, F',$ such that the intersection points $A' \equiv DE' \cap D'F,$ $B' \equiv DE' \cap EF', C' \equiv EF' \cap D'F,$ lie on the segment lines $AP, BP, CP,$ respectively.

The lines through vertices $A, B, C,$ of $\triangle ABC$ and parallel to $B'C', A'C', A'B'$ respectively, intersect each other at the points $A'', B'', C'',$ also lie on the segment lines $AP, BP, CP,$ respectively.

So, we have the configuration of the triangle $\triangle ABC,$ as the Cevian triangle of $\triangle A''B''C'',$ with respect to the point P (Hence the triangle $\triangle A''B''C'',$ is the Anticevian triangle of $\triangle ABC,$ wrt P).

Because of the segment lines AHa, BHb, CHc are concurrent at one point (here the point H) and also the segment lines $A''A, B''B, C''C$ are concurrent at one point (here the point P), based on the well known **proposition 2,** we conclude that the segment lines $A''Ha, B''Hb, C''Hc,$ are concurrent at one point, so be it $R,$ which in a such our configuration, is called the $H-Ceva$ conjugate of $P,$ with respect to the triangle $\triangle ABC.$

It has already been proved (see the proof 1), that the segment lines $ApPa, BpPb, CpPc,$ pass through the points $A', B',$ $C',$ as the midpoints of the segments PA'', PB'', PC'' respectively.

Because of $ApPa \equiv A'Ap \parallel A''Ha$ and $BpPb \equiv B'Bp \parallel B''Hb,$ and $CpPc \equiv C'Cp \parallel C''Hc,$ we conclude that the segment lines $ApPa, BpPb, CpPc,$ intersect the segment $PR,$ at the same point, so be it $Q,$ as it's midpoint. That is the segment lines $ApPa, BpPb, CpPc,$ are concurrent at the point $Q,$ as the midpoint of the segment between P and R as the $H-Ceva$ conjugate of P with respect to the triangle $\triangle ABC$ and the proof is completed.

● This proof is dedicated to **Vladimir Zajic.**

Kostas Vittas.

PS. **PROPOSITION 2.** - (well known) - A triangle $\triangle ABC$ is given and let $\triangle DEF$ be, the Cevian triangle of an arbitrary point $P,$ in the plane. If $D', E', F',$ are three points on the sidelines EF, DF, DE of $\triangle DEF$ respectively, such that the segment lines $DD', EE', FF',$ to be concurrent at one point, prove that the segment lines $AD', BE', CF',$ are also concurrent at one point.

This post has been edited 1 time. Last edited by vittasko, Jun 17, 2011, 2:31 am

jayme

#8 Dec 25, 2008, 7:24 pm

Dear Mathlinkers,
a new proof of "The paracevian perspector" has been put on my website
http://perso.orange.fr/jl.ayme vol. 4
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Concurrent lines formed by projections

Reply

hatchguy
#1 Nov 19, 2010, 4:00 am • 1

Let ABC be an non-isosceles triangle. Let A_1, B_1 and C_1 be the feet of its altitudes. Let O be an arbitrary point inside triangle $A_1B_1C_1$. Let N, Q and S be the feet of the altitudes from O to AA_1, BB_1 and CC_1 respectively. Let M, P and R be the feet of the altitudes from O to BC, AC and AB, respectively. Show that MN, PQ and RS concurr.

Luis González
#2 Nov 19, 2010, 5:13 am • 1

Let O_1, O_2, O_3 be the reflections of O on $BC, CA, AB.$ Then, A_1O_1, B_1O_2, C_1O_3 are the reflections of A_1O, B_1O, C_1O on angle bisectors AA_1, BB_1 and CC_1 of the orthic triangle $\triangle A_1B_1C_1 \Longrightarrow A_1O_1, B_1O_2, C_1O_3$ concur at the isogonal conjugate O^* of O WRT $\triangle A_1B_1C_1,$ i.e. the cevian quotient $H/O,$ where H is the orthocenter of $\triangle ABC.$ But since diagonals OA_1, MN of the rectangle $NOMA_1$ bisect each other, it follows that MN is the image of O_1A_1 under the homothety with center O and factor $\frac{1}{2}.$ Likewise, PQ and RS are the images of O_2B_1 and O_3C_1 under the same homothety $\Longrightarrow MN, PQ, RS$ concur at the midpoint of $OO^*.$

jayme
#3 Nov 19, 2010, 11:23 am

Dear Mathlinkers,
a generalization of this result is known under "the para cevian perspector"...
See:
http://perso.orange.fr/jl.ayme vol. 4 the paracevian perspector p. 9.
Sincerely
Jean-Louis

jayme
#4 Feb 6, 2011, 9:12 pm

Dear Mathlinkers,
see also
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=389908
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Tangent line bisects segment

Reply

Source: Own

buratinogigle

#1 Dec 14, 2015, 10:49 pm

Let ABC be a triangle inscribed in circle (O). P is a point on AO. Let M,N be two points inside ABC such that $\angle MAB=\angle NAC$ and $PM\perp AB,PN\perp AC$. K,L are symmetric of M,N through midpoints of AB,AC. MN,KL cut BC at S,T, reps. Prove that the tangent at A of (O) bisects ST.

Luis González

#2 Dec 18, 2015, 12:43 pm • 1

See the lemma at http://www.artofproblemsolving.com/community/c6h1175442.

Quick Reply

High School Olympiads

Harmonic pencil

Reply

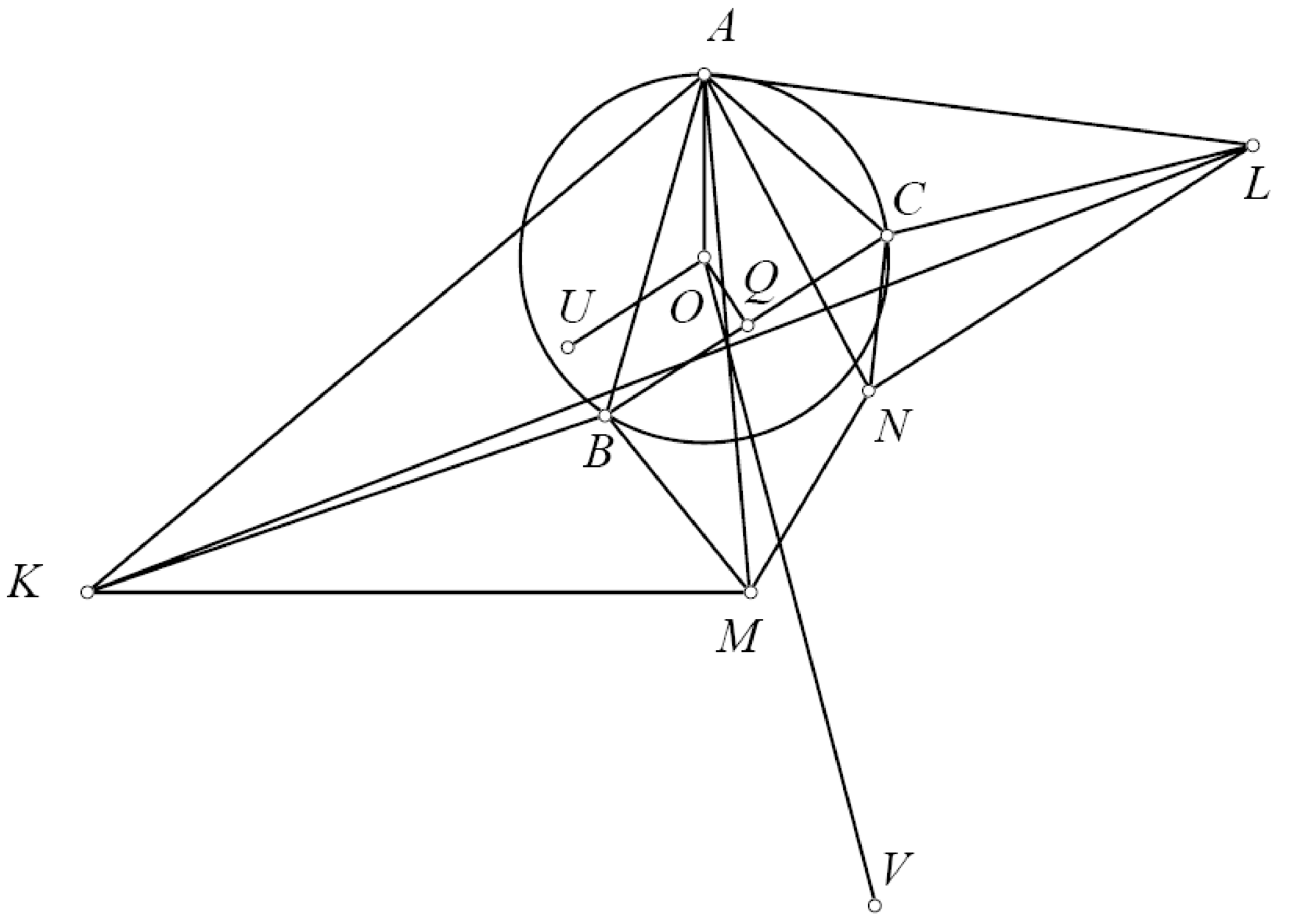
Source: Own

buratinogigle

#1 Dec 18, 2015, 10:08 am

Let ABC be a triangle inscribed in circle (O). Construct inside two similar triangles $\triangle BAM \sim \triangle CAN$. Construct outside triangles $\triangle BKA \sim \triangle BAM$ and $\triangle CLA \sim \triangle CAN$. Let U, V be circumcenters of triangles AMN và AKL. Q is midpoint of BC. Prove that pencil $O(UV, AQ)$ is harmonic.

Attachments:



Luis González

#2 Dec 18, 2015, 12:38 pm • 2

Lemma: In the plane of $\triangle ABC$, construct triangles $\triangle BAM \sim \triangle CAN$ either inwardly or outwardly. Define K, L such that $AMBK$ and $ANCL$ are paralellograms. MN and KL cut BC at X, Y. If the tangent of (O) at A cuts BC at S, then S is midpoint of $\overline{XY}$.

Proof: Let AM, AN cut BC at M', N' and let $D \equiv BM \cap CN$. Fix A, B, C, D and animate M', N' on BC. K, L run on fixed lines through A parallel to DB, DC and since $BM : CN = AK : AL = AB : AC = \text{const}$, then the series K, L, M, N are similar $\Longrightarrow$ series K, L, M, N, X, Y are all similar. Thus it suffices to show that S is the midpoint of $\overline{XY}$ for two positions of M'. When $M' \equiv N' \equiv X$ coincide with the foot of the internal bisector of $\angle BAC$, then $BX \parallel CL \parallel AX$ and $YB : YC = BK : CL = AB : AC \Longrightarrow Y$ is the foot of the external bisector of $\angle BAC \Longrightarrow S$ is the midpoint of XY and the same happens when $M' \equiv N' \equiv Y$ coincide with the foot of the external bisector of $\angle BAC$.

Thus we conclude that S is the midpoint of XY for any M'.

Back to the problem. Let X and Y be the second intersections of (O) with $\odot(AMN)$ and $\odot(AKL)$, resp and let the parallel to BC through A cut (O) again at S. Inverting with center A and arbitrary power, denoting inverse point with primes, we get $\triangle B'AM' \sim \triangle C'AN'$, $AM'B'K'$ and $AN'C'L'$ are parallelograms and $M'N'$ and $K'L'$ cut $B'C'$ at X', Y'. Moreover the tangent of $\odot(AB'C')$ at A cuts $B'C'$ at S'. Using the previous lemma in $\triangle AB'C'$, we get that S' is the midpoint of $\overline{X'Y'}$, thus inverting back $AYSX$ is harmonic $\Longrightarrow$ the pencil $O(UV, AQ)$ formed by the perpendiculars from O to AX, AY, AS, AA is hamonic.

Quick Reply

High School Olympiads

geometry

Locked

quantummath

#1 Dec 16, 2015, 11:03 am

In a acute-angled triangle ABC, a point D lies on the segment BC. Let O_1, O_2 denote the circumcentres of triangles ABD and ACD respectively.If H and H' denote the orthocentres of trianglesO_1O_2D and ABC and U the intersection of perpendicular from H to BC,then prove that $AH' = 2HU$.

baopbc

#2 Dec 16, 2015, 11:44 am

It not a correct problem

quantummath

#3 Dec 16, 2015, 12:00 pm

It is a part of INMO 2014 problem 5.

M-A-K

#4 Dec 17, 2015, 5:12 pm

> *quantummath wrote:*
> It is a part of INMO 2014 problem 5.

yes!

Luis González

#5 Dec 17, 2015, 9:47 pm

The problem is indeed correct. It follows from the fact that the line connecting the orthocenter H of $\triangle DO_1O_2$ and the circumcenter of $\triangle ABC$ is parallel to BC. See the topic 3 circumcenters.

High School Olympiads

3 circumcenters

Reply

dragonx111

#1 Apr 23, 2015, 10:33 pm

Let $D \in BC$, BC is the side of an acute angled triangle ABC, O is its circumcenter. Now let O_1, O_2 be the circumcenters of the triangles ABD, ACD, respectively.Prove that the line which contains the orthocenter of the triangle O_1O_2D and the point O is parallel with BC.

Luis González

#2 Apr 23, 2015, 11:59 pm

Since $\angle(O_1B, O_1A) = 2\angle(DB, DA) = \angle(O_2C, O_2A)$, the isosceles $\triangle ABO_1$ and $\triangle ACO_2$ are spirally similar $\Longrightarrow$ $\triangle ABC$ and $\triangle AO_1O_2$ are spirally similar. Since $OO_1 \perp AB$ and $OO_2 \perp AC$, then $\angle(OO_1, OO_2) = \angle(AB, AC) = \angle(AO_1, AO_2) \Longrightarrow O \in \odot(AO_1O_2)$.

Let H be the orthocenter of $\triangle DO_1O_2$ lying on its D-altitude DA. Then $\angle(HO_1, HO_2) = \angle(DO_2, DO_1) = \angle(AO_1, AO_2) \Longrightarrow H \in \odot(AO_1O_2)$. Hence $\angle(HA, HO) = \angle(O_1A, O_1O) = \angle(O_2A, O_2O) = \angle(DA, DB) \Longrightarrow OH \parallel BC$, as desired.

Quick Reply

High School Olympiads

Perpendicular

Reply

Source: Unknown

mamavuabo

#1 Dec 16, 2015, 6:31 pm

Given an acute triangle ABC inscribed in (O).D, E are respectively the midpoints of AB, AC.DE cuts the minor arc AB, AC at M, N respectively.(DON) and (EOM) cuts (O) respectively at I, J. (I is different from N, J is different from M).Prove that IJ is perpendicular to OA.

This post has been edited 1 time. Last edited by mamavuabo, Dec 16, 2015, 6:31 pm

Luis González

#2 Dec 16, 2015, 7:56 pm

Let the tangents of (O) at C, B cut the tangent of (O) through A at $Y, Z,$ resp. Inversion WRT (O) takes E, D into $Y, Z,$ thus DE goes to $\odot(OYZ)$ and $\odot(EOM)$ and $\odot(DON)$ go to MY and $NZ.$ Since M, N, I, J are fixed in this inversion, we get $M, N \in \odot(OYZ), I \in YM$ and $J \in ZN,$ thus $\angle MIJ = \angle MNJ \equiv \angle MNZ = \angle MYZ \Longrightarrow IJ \parallel YZ,$ i.e. $IJ \perp OA.$

FabrizioFelen

#4 Dec 18, 2015, 1:48 pm

My solution:
Since $MOEJ$ and $NODI$ are concyclic: $\angle DIO = \angle DMO = \angle EJO = \angle ENO \Longrightarrow \triangle DIO \cong \triangle DMO$ since $OI = OM$ and $\angle DIO = \angle DMO$ and $\angle IDO$ is obtuse similarly $\triangle OJE \cong \triangle ONE \Longrightarrow OD \perp IM$ and $OE \perp JN$ $\Longrightarrow AJNC$ and $AIMB$ are isosceles trapezoid $\Longrightarrow AJ = NC$ and $AI = MB \Longrightarrow AI = AJ$ since $MB = NC \Longrightarrow$ $AI = AJ$ and $OI = OJ \Longrightarrow OA \perp IJ$...

sunken rock

#5 Dec 19, 2015, 1:25 am • 1

Using an inversion of pole A and power AO^2 we get a nice solution, the points onto the circle going to points onto the diameter perpendicular to AO, while the midpoints of AB, AC go onto the tangent at A', diametrically opposite to A, thus the two circles go to other two circles, intersecting the two parallels at the vertices of two isosceles trapezoids, a.s.o.

Best regards,
sunken rock

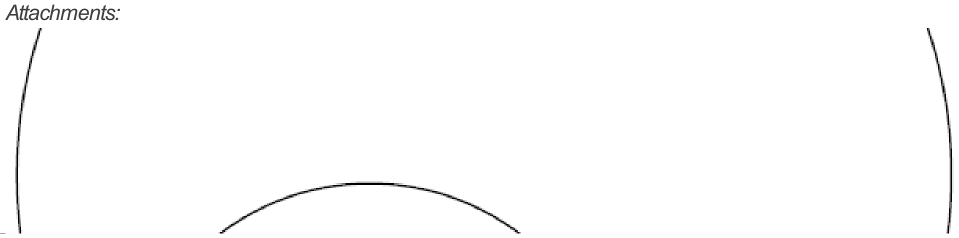
buratinogigle

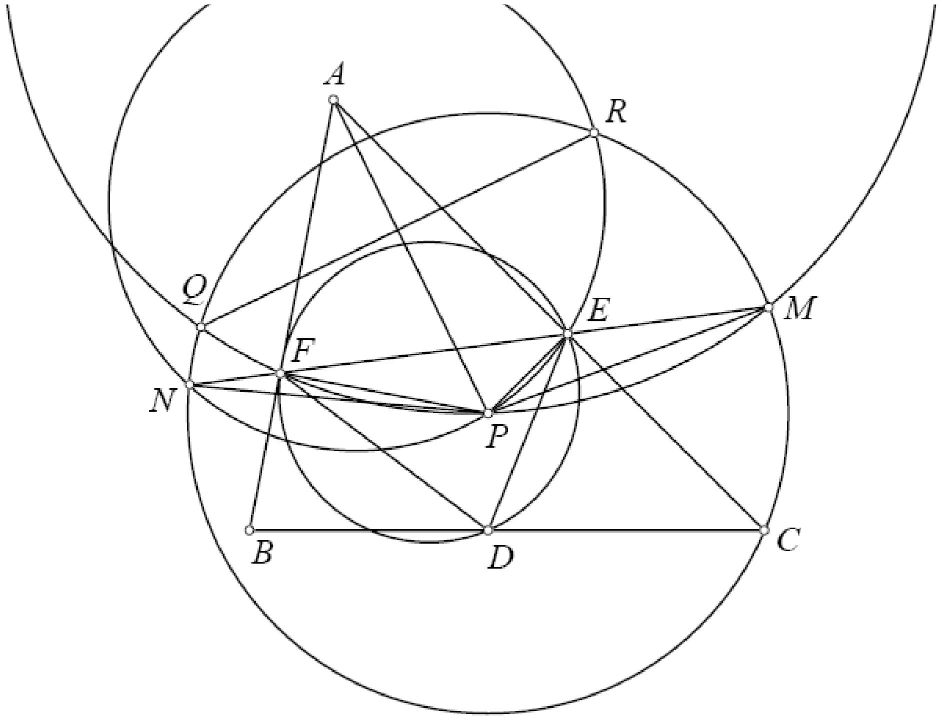
#6 Dec 19, 2015, 8:28 pm

General problem

Let ABC be a triangle and P is an any point. DEF is pedal triangle of P with circumradii is ρ. Let (ω) is circle center P with radius 2ρ. EF cuts (ω) at M, N. Circumcircle of triangle PMF, PNE cut (ω) again at Q, R. Prove that $AP \perp QR$.

Attachments:





TelvCohl

#7 Dec 19, 2015, 9:24 pm • 1

> *buratinogigle wrote:*
> General problem
>
> Let ABC be a triangle and P is an any point. DEF is pedal triangle of P with circumradii is ρ. Let (ω) is circle center P with radius 2ρ. EF cuts (ω) at M, N. Circumcircle of triangle PMF, PNE cut (ω) again at Q, R. Prove that $AP \perp QR$.

From $\measuredangle PME = \measuredangle FNP = \measuredangle ERP$ and $PM = PR \Longrightarrow M, R$ are symmetry WRT PE. Similarly, we can prove N, Q are symmetry WRT PF, so notice the P-altitude τ_p of $\triangle EPF$ is the bisector of $\angle MPN$ we get the isogonal conjugate PA of τ_p WRT $\angle EPF$ is the bisector of $\angle RPQ$. i.e. AP is the perpendicular bisector of QR

Remark : The circle $\odot(P, 2\rho)$ can be replaced by any circle with center P.

Quick Reply

High School Olympiads

Set of incenters and excenters

Reply

Source: Romanian TST 1996

madmathlover

#1 Dec 14, 2015, 9:48 pm

Let $ABCD$ be a cyclic quadrilateral and let M be the set of incenters and excenters of the triangles BCD, CDA, DAB, ABC (16 points in total).Prove that there are two sets K and L of four parallel lines each,such that every line in $K \cup L$ contains exactly four points of M.

Luis González

#2 Dec 15, 2015, 3:34 am

Label I, J, K, L the incenters of $\triangle ABC, \triangle BCD, \triangle CDA, \triangle DAB,$ resp. Label I_A, I_B, I_C the excenters of $\triangle ABC$ againts A, B, C and $\{J_B, J_C, J_D\}, \{K_C, K_D, K_A\}, \{L_D, L_A, L_B\}$ are defined similarly.

Let ω be the circle tangent to BC, DA at X, Y and internally tangent to $\odot(ABCD)$ through its arc $CD.$ ω becomes a Thebault circle of the cevian CB of $\triangle CDA,$ thus by Sawayama's lemma, it follows that $K \in XY$ and similarly $J \in XY.$ Likewise, by Sawayama's lemma for $\triangle DAB,$ it follows that $L_B \in XY$ and similarly $I_A \in XY \Longrightarrow J, K, I_A, L_B$ are collinear on $XY,$ which is obviously perpendicular to the internal bisector ℓ of $\angle(AD, BC).$ Similarly I, L, J_D, K_C are collinear on a perpendicular to $\ell.$ By extraversion of the Sawayama's lemma, we also deduce that I_B, J_B, K_A, L_A and I_C, J_C, K_D, L_D are respectively collinear on two lines parallel to $\ell \Longrightarrow$ $(JKI_AL_B \parallel ILJ_DK_C \parallel I_BJ_BK_AL_A \parallel I_CJ_CK_DL_D) \perp \ell.$ In the same way, we prove that $IJL_DK_A \parallel LKI_CJ_B \parallel I_BJ_CK_CL_B \parallel I_AJ_DK_DL_A.$

MathPanda1

#3 Dec 15, 2015, 4:06 am

Already posted: http://artofproblemsolving.com/community/c6h53558p335762.

Quick Reply

High School Olympiads

Pure geometry

Reply

Pirkuliyev Rovsen
#1 Dec 13, 2015, 12:25 am

The medians of the triangle ABC intersect at point O.Some line intersects segments AC, AO, BO and BC at points K, L,M, N respectively.It is known that $KL=LM$.Prove that: $LM=MN$

TelvCohl
#2 Dec 14, 2015, 9:53 pm • 2

Recall a well-known property about conic : Let $\mathcal{C}$ be a conic which is tangent to $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5$. Let ℓ_5 cuts $\ell_1, \ell_2, \ell_3, \ell_4$ at P_1, P_2, P_3, P_4, respectively. Let τ be a line cuts $\ell_1, \ell_2, \ell_3, \ell_4$ at Q_1, Q_2, Q_3, Q_4, respectively. Then τ is tangent to $\mathcal{C}$ if and only if $(P_1, P_2; P_3, P_4) = (Q_1, Q_2; Q_3, Q_4)$. When $\mathcal{C}$ is a parabola and ℓ_4 coincide with the line at infinity we get $(\bigstar)$:

Let $\mathcal{P}$ be a parabola which is tangent to a, b, c and d. Let d cuts a, b, c at A, B, C, respectively and let τ be a line cuts a, b, c at A_1, B_1, C_1, respectively. Then τ is tangent to $\mathcal{P}$ if and only if $\frac{AB}{AC} = \frac{A_1B_1}{A_1C_1}$.

Back to the main problem :

Let $\mathcal{P}$ be the parabola which is tangent to BC, CA, AO, BO. Let D, E be the midpoint of BC, CA, respectively. From $(\bigstar)$ and $\frac{KL}{LM} = \frac{CD}{DB} = 1$ we get $KLMN$ is tangent to $\mathcal{P}$, so from $(\bigstar)$ again we conclude that $\frac{NM}{ML} = \frac{CE}{EA} = 1$.

Luis González
#3 Dec 14, 2015, 11:15 pm • 2

Denote D, E the midpoints of BC, CA. Clearly for each $L \in AD$, the points K, M, N are unique. Thus redefining $M \in EB$ such that $ME : MB = LA : LD$ and letting K, N be the intersections of LM with AC, BC, it suffices to show that $KL = LM = MN$.

$\odot(OEA), \odot(OBD)$ and $\odot(OML)$ meet again at the center J of the spiral similarity that swaps $\overline{BME}$ and $\overline{DLA} \Longrightarrow J$ is the Miquel point of the complete $EOLK$ and $DOMN \Longrightarrow J$ is also center of the spiral similarity that swaps $\overline{AOD}$ and $\overline{KMN} \Longrightarrow MN : MK = OD : OA = -\frac{1}{2}$ and similarly $LK : LN = -\frac{1}{2} \Longrightarrow KL = LM = MN$, as desired.

Quick Reply

High School Olympiads

Ratio of lengths in kite

Reply

jlammy

#1 Dec 14, 2015, 1:34 am

Let $ABCD$ be a cyclic quadrilateral with $\angle B = \angle D = 90^\circ$ and $AB = AD$ and circumcircle Γ. Let the tangent to Γ at D meet AB at P and BC at Q. Let E be on Γ distinct from D such that PE is tangent to Γ, and K be on line AE such that QA is tangent to Γ.

Prove that $BD = 3BK$.

Luis González

#2 Dec 14, 2015, 2:05 am

This configuration has been discussed before. See A nice property of the incircle for a right triangle for another formulation of the problem.

Quick Reply

High School Olympiads

A nice property of the incircle for a right triangle.

Reply

Source: Miquel Ochoa

Virgil Nicula

#1 Aug 9, 2013, 8:16 am

PP16 (Miguel Ochoa). Let ABC be an B-right-angled triangle with the incircle $w = C(I,r)$. Denote

$\left\|\begin{array}{c} D\in BC\cap w \\ E\in AC\cap w \\ F\in AB\cap w \end{array}\right\|$ and $\left\|\begin{array}{c} \{P,D\}=AD\cap w \\ \{P,X\}=PC\cap w \\ \{P,Y\}=PB\cap w \end{array}\right\|$. Prove that $PB\perp PC \iff PD=3\cdot PA$.

yetti

#2 Aug 11, 2013, 12:26 am

Condition that the $\triangle ABC$ is B-right is irrelevant.
See http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=147900&p=838532#p838532.

Luis González

#3 Aug 11, 2013, 1:08 am

See also the problem Proof Of It. All steps are reversible.

Assume that $PD=3\cdot PA \Longrightarrow AD=4\cdot AP \Longrightarrow \frac{ED}{EP}=\sqrt{\frac{AD}{AP}}=2$. Thus if M is the midpoint of ED, then $\triangle EMP$ is E-isosceles. Since PX is P-symmedian of $\triangle PED$, it follows that $\triangle PXD\sim\triangle PEM \Longrightarrow \triangle PXD$ is X-isosceles. Similarly, $\triangle PYD$ is Y-isosceles $\Longrightarrow XY$ is perpendicular bisector of $PD \Longrightarrow I\in XY \Longrightarrow \angle YPX=90^\circ$.

Quick Reply

High School Olympiads

Angle relation related to Seven 9-point centers

Reply

Source: Own

TelvCohl

#1 Dec 10, 2015, 8:47 am • 1

Given a $\triangle ABC$ with circumcenter O. Let P, Q be the isogonal conjugate WRT $\triangle ABC$. Let $N, A_P, B_P, C_P, A_Q, B_Q, C_Q$ be the 9-point center of $\triangle ABC, \triangle BPC, \triangle CPA, \triangle APB, \triangle BQC, \triangle CQA, \triangle AQB$, respectively. Let O_P, O_Q be the circumcenter of $\triangle A_P B_P C_P, \triangle A_Q B_Q C_Q$, respectively. Prove that $\measuredangle POQ = \measuredangle O_P N O_Q$.

Luis González

#2 Dec 10, 2015, 12:42 pm • 1

Let $\triangle P_A P_B P_C$ and $\triangle Q_A Q_B Q_C$ be the pedal triangles of P, Q WRT $\triangle ABC$ and let M be the midpoint of PQ; center of the pedal circle of P, Q. If P^*, Q^* are the isogonal conjugates of P, Q WRT $\triangle P_A P_B PC$ and $\triangle Q_A Q_B Q_C$, respectively, then we know that $\triangle A_P B_P C_P \cup N \cup M \sim \triangle Q_A Q_B Q_C \cup Q \cup Q^*$ and likewise $\triangle A_Q B_Q C_Q \cup N \cup M \sim \triangle P_A P_B P_C \cup P \cup P^*$ (this has been discussed before in other topics). As a result, $\measuredangle MNO_P = \measuredangle MQQ^* \equiv \measuredangle PQQ^*$ and similarly we have $\measuredangle MNO_Q = \measuredangle QPP^*$. Thus if $R \equiv PP^* \cap QQ^*$, it follows that $\measuredangle QRP = \measuredangle O_P N O_Q$.

Let $\triangle A^* B^* C^*$ be the pedal triangle of Q^* WRT $\triangle Q_A Q_B Q_C$ and $\triangle X_A X_B X_C$ the antipedal triangle of P WRT $\triangle ABC$. Let X be the isogonal conjugate of P WRT $\triangle X_A X_B X_C$ (reflection of P on O). Clearly $\triangle A^* B^* C^* \cup \triangle Q_A Q_B Q_C$ and $\triangle ABC \cup \triangle X_A X_B X_C$ are homothetic and $\{P, X\}$ and $\{Q^*, Q\}$ are pairs of homologous point $\Longrightarrow QQ^* \parallel PXO$ and similarly we have $PP^* \parallel OQ \Longrightarrow OPQR$ is a parallelogram $\Longrightarrow \measuredangle QRP = \measuredangle POQ \Longrightarrow \measuredangle POQ = \measuredangle O_P N O_Q$.

TelvCohl

#3 Dec 10, 2015, 5:19 pm • 1

Remark : More properties in this configuration :

(1) Let K_P be the point on $\odot(ABC)$ such that $OK_P \parallel NO_P$ and define K_Q similarly. Then the Simson line of K_P, K_Q WRT $\triangle ABC$ is parallel to the bisector of $\angle OQP, \angle OPQ$, respectively.

(2) Let $\triangle P_a P_b P_c, \triangle Q_a Q_b Q_c$ be the antipedal triangle of P, Q WRT $\triangle ABC$, respectively. Then O_P, O_Q is the complement of the 9-point center of $\triangle P_a P_b P_c, \triangle Q_a Q_b Q_c$ WRT $\triangle ABC$, respectively.

Quick Reply

High School Olympiads

A property of the triangle ABC with A=60 (degrees).

Reply

Source: own

Virgil Nicula

#1 Dec 10, 2015, 12:49 am

PP. Let $\triangle ABC$ with the incenter I and $A=60^{\circ}$. Denote the intersections $E\in BI\cap AC$, $F\in CI\cap AB$ and the areas $[BIF]=m, [CIE]=n, [EIF]=p$. Prove that $\frac{1}{m}+\frac{1}{n}=\frac{1}{p}$.

This post has been edited 4 times. Last edited by Virgil Nicula, Dec 10, 2015, 12:53 am

Luis González

#2 Dec 10, 2015, 9:27 am

$$\frac{p}{m}=\frac{IE}{IB}=\frac{AE}{c}=\frac{\frac{bc}{a+c}}{c}=\frac{b}{a+c}\ ,\ \frac{p}{n}=\frac{IF}{IC}=\frac{AF}{b}=\frac{\frac{bc}{a+b}}{b}=\frac{c}{a+b}\Longrightarrow$$

$$\frac{p}{m}+\frac{p}{n}=1\Longleftrightarrow\frac{b}{a+c}+\frac{c}{a+b}=1\Longleftrightarrow a^2=b^2+c^2-bc\Longleftrightarrow\widehat{A}=60^{\circ}.$$

Quick Reply

High School Olympiads

&&&&&&&&&&&&&

Locked

izat

#1 Dec 10, 2015, 8:40 am

CC_0 is median of triangle ABC. Perpendicular bisector of BC intersects CC_0 at B_1 and perpendicular bisector of AC intersects CC_0 at A_1. Let AA_1 and BB_1 intersect at C_1. Prove that CC_1 is symedian

Luis González

#2 Dec 10, 2015, 8:54 am

Please give your threads meaningful subjects and try to use the search before posting. This was posted before at http://www.artofproblemsolving.com/community/c6h396352.

High School Olympiads

Common tangents

Reply

Source: Own

livetolove212

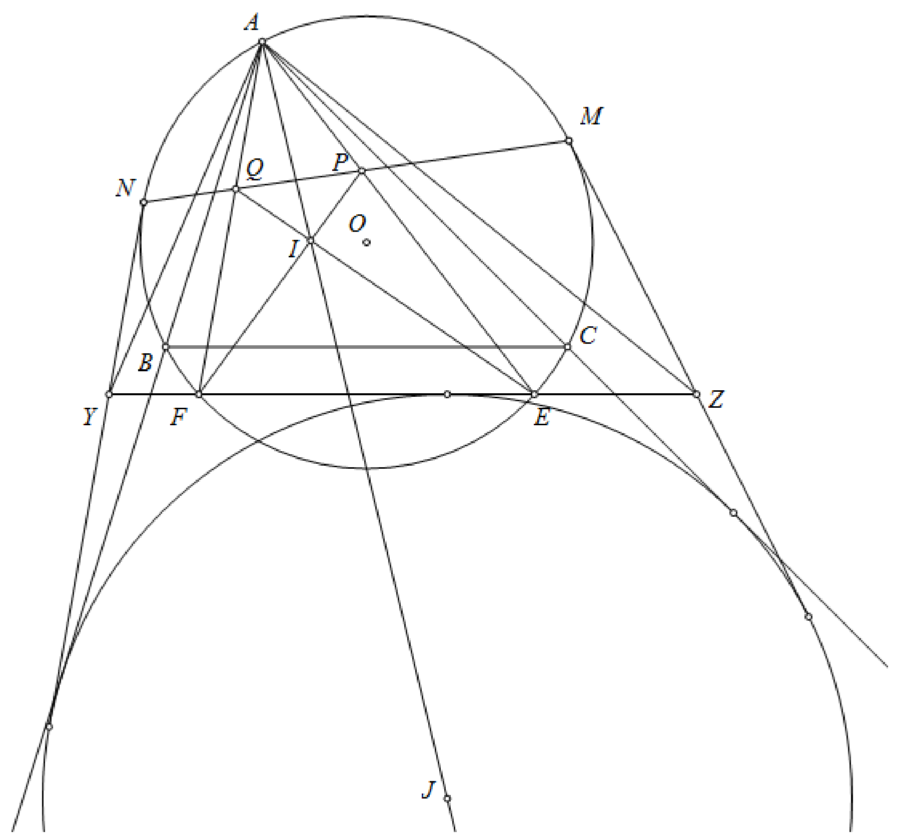
#1 Dec 5, 2015, 1:59 pm • 2

Given triangle ABC and its circumcircle (O), incenter I. Let (J) be a circle tangent to two sides AB and AC. Tangent of (J) which is parallel to BC intersects (O) at E and F. EI cuts AE at Q, FI cuts AE at P. PQ cuts (O) at M and N. Prove that M and N lie on two common external tangents of (O) and (J).

Moreover, let the common tangents through M and N intersect EF at Y and Z. Then prove that AY and AZ are isogonal wrt $\angle BAC$.

Note that this is the generalization of problem at http://www.artofproblemsolving.com/community/c6h385175

Attachments:



Luis González

#2 Dec 10, 2015, 5:27 am • 1

Let L and D be the midpoints of the arc BC and BAC of (O). Parallel through I to $BC \parallel EF$ cuts the tangent of (O) through A at S and cuts (O) at U, V. If A_∞ is the point at infinity of $BC \parallel EF$, then by Desargues involution theorem for the degenerate $AALL$ cut by the line UV, it follows that I is fixed point in the involution $U \mapsto V, A_\infty \mapsto S$. Moreover $U \mapsto V$, $E \mapsto F$ is an involution on (O) with pole A_∞ and a fixed point L. So $AI \equiv AL$ is fixed under the involution $AU \mapsto AV$, $AE \mapsto AF, AS \mapsto AA_\infty$. But by Desargues involution theorem for $EFQP$ cut by the line UV, it follows that $UV \cap PQ$ is the image of A_∞ in this involution, i.e. $S \in PQ$.

Fix A, B, C and move (J). From the previous result S is fixed, thus if the tangents of (O) at M, N meet at T, the $AT \equiv \tau_A$ is the polar of S WRT $(O) \Longrightarrow T$ moves on the fixed line τ_A. If X is the tangency point of the incircle (I) with BC and X' is the antipode of X on (I), then AX' passes through the tangency point K of (J) with EF, because A is the exsimilicenter of

$(I) \sim (J) \Longrightarrow K$ moves on the fixed line $AX' \equiv \lambda_A$. Since UV is the polar of T WRT $(O) \Longrightarrow OT \perp SUV \Longrightarrow$ pencils SUV and OT are projective. But if AI cuts EF, PQ at A_1, A_2, then from the complete $EFQP$, we have $(A_1, I, A, A_2) = -1 \Longrightarrow A_1 \mapsto A_2$ is an involution on AI, thus the pencils $AA_1 \equiv AK$ and $OT \barwedge SA_2$ are projective inducing a perspective between τ_A and $\lambda_A \Longrightarrow$ all lines TK go through a fixed point.

When $\{E \equiv C, F \equiv B\}$, then (J) becomes A-excircle of $\triangle ABC$ and according to the problem http://www.artofproblemsolving.com/community/c6h385175, T, D, K are collinear due to the fact that T is the exsimilicenter of $(O) \sim (J)$ and their radii $OD \parallel JK$ are parallel. When $E \equiv F \equiv L$, then (O) becomes incircle of the triangle bounded by TM, TN, the tangent of (O) at L and (J) is its T-excircle. Since T is exsimilicenter of $(O) \sim (J)$ and $JK \parallel OD$, then again T, D, K are collinear $\Longrightarrow D$ is the fixed point of the referred perspectivity, i.e. T, D, K are collinear for any (K). Since $OD \parallel JK$ are parallel radii of $(O), (J)$, then T is the exsimilicenter of $(O) \sim (J) \Longrightarrow TM, TN$ are common external tangents of $(O), (J)$.

Luis González

#3 Dec 10, 2015, 5:30 am • 1

For the second part of the problem, denote $R \equiv PQ \cap EF$. From the complete $EFQP$, it follows that $A(E, F, I, R) = -1 \Longrightarrow AR$ is external bisector of $\angle EAF$. Now by Desargues involution theorem for the degenerate $MMNN$ cut by EF, it follows that R is fixed in the involution $Y \mapsto Z, E \mapsto F \Longrightarrow AY, AZ$ are isogonals WRT $\angle EAF$, i.e. AY, AZ are isogonals WRT $\angle BAC$.

livetolove212

#4 Dec 12, 2015, 3:45 pm

See http://www.artofproblemsolving.com/community/c6t48f6h1172663_concyclic_points for another proof of this problem.

TelvCohl

#6 Dec 15, 2015, 9:27 pm

I'll use some result proved at Concyclic points (post #5) ... $(\bigstar)$.

Let I_a be the A-excenter of $\triangle ABC$ and let $B_1 \equiv AB \cap EF, C_1 \equiv CA \cap EF$. Since $\triangle I_aBC$ and $\triangle JB_1C_1$ are homothetic with center A, so $AB : AB_1 = AC : AC_1 = AI_a : AJ$, hence combine $\triangle ABF \sim \triangle AEC_1$ and $AI \cdot AI_a = AB \cdot AC \Longrightarrow AE \cdot AF = AI \cdot AJ \Longrightarrow$ the isogonal conjugate of I WRT $\triangle AEF$ is the second intersection of AJ with $\odot(JEF)$.

Let $I_B \equiv FI \cap \odot(AIE), I_C \equiv EI \cap \odot(AIF), I'_B \equiv JI_B \cap \odot(II_BI_C), I'_C \equiv JI_C \cap \odot(II_BI_C)$. From $(\bigstar)$ we know the intersection X of EF with the external bisector of $\angle EAF$ lies on I_BI_C and $E \in JI_B, F \in JI_C$. From $PA \cdot PE = PI \cdot PI_B$ we get I_B lies on $\odot(IMN)$. Similarly, we can prove $I_C \in \odot(IMN)$, so notice E, F, I_B, I_C are concyclic (from $(\bigstar)$) we get X lies on radical axis of $\odot(O)$ and $\odot(II_BI_C)$. i.e. $X \in MN$

Let T be the center of $\odot(IMN)$. Since $\measuredangle I'_BTI'_C = 2\measuredangle I'_BI_BI + 2\measuredangle II'_CI_C = 2\measuredangle EAF = \measuredangle EOF$, so combine $I'_BI'_C \parallel EF$ (Reim theorem) we get $\triangle OEF$ and $\triangle TI'_BI'_C$ are homothetic with center $J \Longrightarrow J, O, T$ are collinear, hence $MN \perp OJ$. If M^*, N^* is the intersection of $\odot(O)$ with the common external tangents of $\odot(O)$ and $\odot(J)$, then $M^*N^* \perp OJ$ and from $(\bigstar)$ we get M^*N^* passes through X, so we conclude that $M^* \equiv M$ and $N^* \equiv N$.

Quick Reply

High School Olympiads

Very hard

Reply

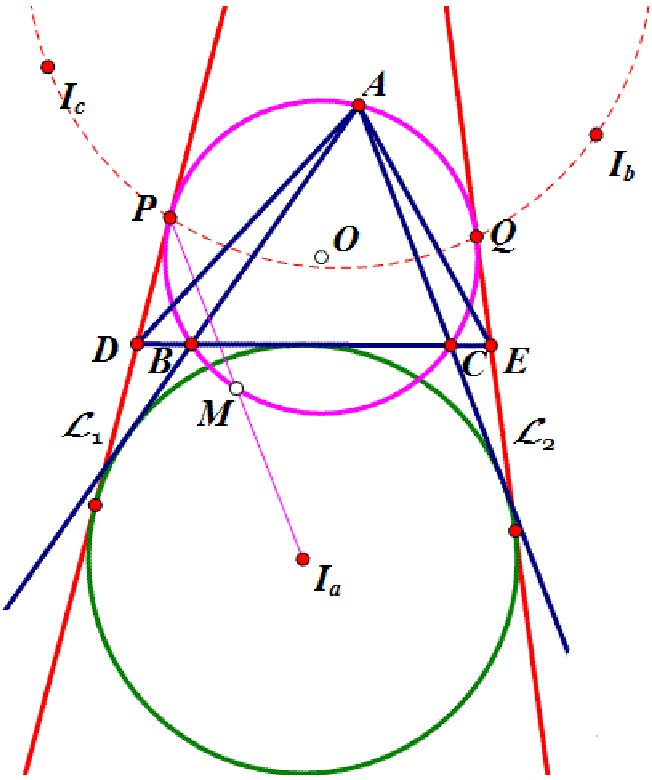
lym

#1 Jan 5, 2011, 6:55 pm • 1

See the picture□given $\triangle ABC$ with its excircle $\odot I_a$□ℓ_1□ℓ_2 the common tangents of $\odot O(ABC)$ with $\odot I_a$ cut BC at D□E . P□Q on $\odot O$ are the tangent poins□and I_b□I_c are the other excenters of $\triangle ABC$. Prove that□

□1□□P□Q□I_b□I_c are on a circle (oWn) .
□2□□$\angle BAD = \angle CAE$.

Attachments:



TheIronChancellor

#2 Jan 7, 2011, 8:47 pm

Could you post a solution to this beautiful and teachful problem.

I have an idea for the first question :

We start with : to be those points in a circle it is enough to prove the relation to the chords.Power of a point...

For the second can we think for isogonal lines ...

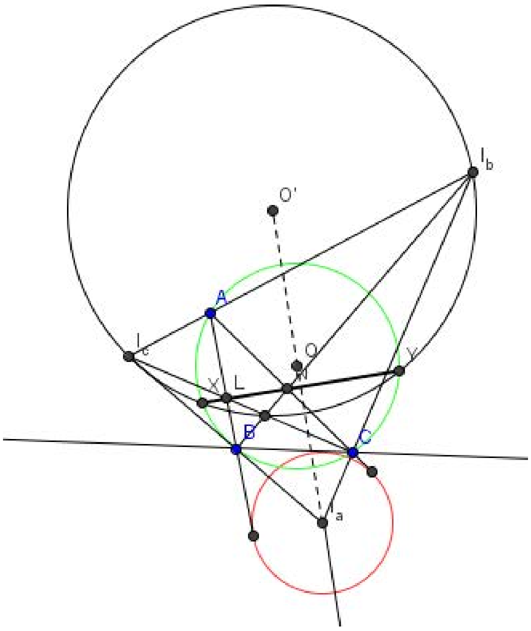
skytin

#3 Jan 7, 2011, 9:33 pm

(a)
Idea for algebraic solution :

Use Euler Formula for Excircle and segment I_aO
Look on picture
I_aO = OO'
O'I_b = 2*R
I_aO intersect XY at point P
count OP (Use R and R_a) , construct point H on I_aO such that OP*OH = R*R , and get that HI_a/HO = R_a/R , so X = P and Y = Q

Attachments:



Luis González

#4 Jan 8, 2011, 3:16 am • 5

$\triangle ABC$ and its circumcircle (O) become orthic triangle and 9-point circle of the excentral $\triangle I_aI_bI_c$. I is incenter of $\triangle ABC$ and BI, CI cut CA, AB at X, Y. Line XY cuts (O) at P_0, Q_0 and assume that P_0, Q_0 lie on the arcs AB, CA, respectively. P_0Q_0, BC, I_bI_c concur at the foot V of the A-external bisector. I_bI_c is radical axis of $\odot(IB_bI_c)$ and $\odot(BCI_bI_c)$ and BC is radical axis of $\odot(BCI_bI_c), (O) \Longrightarrow V$ is radical center of $(O), \odot(II_bI_c)$ and $\odot(BCI_bI_c) \Longrightarrow$ Points P_0, Q_0, I, I_b, I_c are concyclic. Now, we need to prove that $P \equiv P_0, Q \equiv Q_0$.

Homothety $(I_a, 2)$ takes (O) into $\odot(I_bI_bI_c)$, thus (O) cuts $\overline{I_aQ_0}$ at its midpoint M.

$$\Longrightarrow I_aQ_0{}^2 = 2 \cdot I_aM \cdot I_aQ_0 = 2 \cdot p(I_a, (O)) = 4R \cdot r_a \quad (1)$$

Let $\delta(A), \delta(B), \delta(C)$ denote the distances from A, B, C to the common external tangent $\mathcal{L}_2$ of $(O), (I_a)$. By Harcourt's theorem for the excircle (I_a) and its tangent $\mathcal{L}_2$ (see page #119) we get

$$2|\triangle ABC| = CA \cdot \delta(B) + AB \cdot \delta(C) - BC \cdot \delta(A)$$

Now, we substitute in the latter identity the expressions coming from $Q \in (O) \cap \mathcal{L}_2$

$$\delta(A) = \frac{QA^2}{2R} \ \ , \ \ \delta(B) = \frac{QB^2}{2R} \ \ , \ \ \delta(C) = \frac{QC^2}{2R}$$

$$\Longrightarrow BC \cdot CA \cdot AB = CA \cdot QB^2 + AB \cdot QC^2 - BC \cdot QA^2 \quad (2)$$

On the other hand, from Lagrange theorem for Q, I_a, we obtain

$$CA \cdot QB^2 + AB \cdot QC^2 - BC \cdot QA^2 = (CA + AB - BC)I_aQ^2 - BC \cdot CA \cdot AB \ (3)$$

Combining (2) and (3) yields: $I_aQ^2 = \dfrac{2 \cdot BC \cdot CA \cdot AB}{CA + AB - BC} = 4R \cdot r_a \quad (4)$

From $(1), (4)$ we deduce $I_aQ = I_aQ_0$, i.e. $Q \equiv Q_0$. Similarly, $P \equiv P_0$ and conclusion follows.

lym

#5 Jan 8, 2011, 4:08 am

Thanks for your joining□luis's way is a little complicated but worth to learn.

This problems have pure geometrical way to prove.
For question □1□□□2□□we need to prove I_bI_c□PQ□BC are concurrent.

You guys can continue to consider□later I will post a proof.

lym

#6 Jan 9, 2011, 12:08 am • 3

My solution□

□1□□See figure a□$J=\ell_1\cap\ell_2$□$X=I_bI_c\cap BC$□$L=PQ\cap OJ$□then $XA\perp AI$. Let $\odot I_a$ tangent BC at F□JA intersect (O) at T□$S=OI\cap JA$□then A□F□X□I_a and A□T□O□L are concylic respectively□accroding to **Monge & d'alembert theorem**□S is the similar center of (O) and incircle $\odot I$.

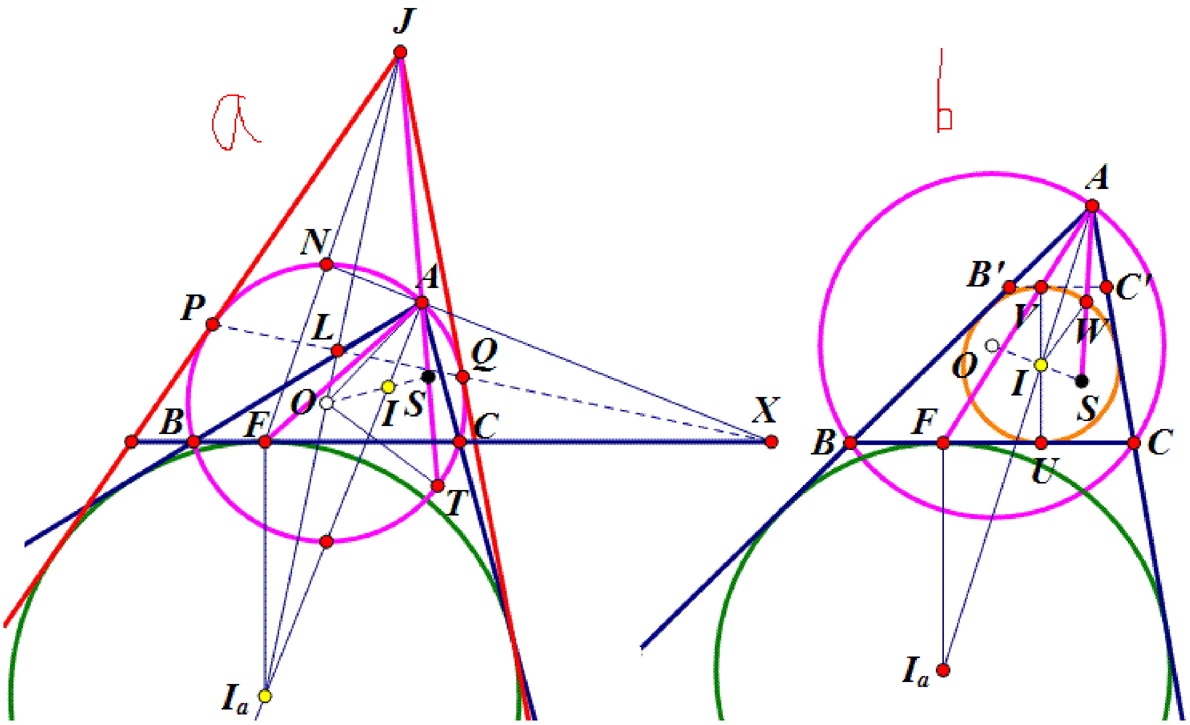
See figure b□let $\odot I$ tangent BC at U□UI intersect $\odot I$ at V□$B'C'\parallel BC$ and $V\in B'C'$. Then $\triangle AB'C'$ and $\triangle ABC$ are homothetic□so A□V□F are collinear. Let $\overline{SA}$ intersect $\odot I$ at W□becase of S is the similar center of (O) and incircle $\odot I$□hence $IW\parallel AO$□so we can easy to found V and W are symmetric about AI□so we get $\angle BAF=\angle CAS$.

Now see figure a□since $\angle BAF=\angle CAS$□therefor $\angle AFC=\dfrac{\angle AOT}{2}$□so $\angle AFC+\angle ATO=90$□i.e. $\angle AFC+\angle ALJ=90$□so $\angle ALI_a=\angle AFI_a$□so A□L□F□I_a are concylic□i.e. A□L□F□X□I_a are concylic□so we get $XL\perp I_aL$□but $PQ\perp I_aL$ and L is on PQ□so X□P□Q are collinear . So $XP\cdot XQ=XB\cdot XC=XI_b\cdot XI_c$□hence we get P□Q□I_b□I_c are concylic. If the radius of (O) and $\odot I_a$ are equal□then $O\equiv L$□$AT\parallel OI_a$□and we can also prove A□O□F□X□I_a are concylic□i.e. X□P□Q are collinear□hence P□Q□I_b□I_c are concylic. Done.

PS□Since X is on the radical axis of $\triangle ABC$ and $\triangle II_bI_c$□so we can found PQ is the radical axis further□so P□Q□I□I_b□I_c are on a circle .

□2□□use □1□□we have X□P□Q are collinear□and $\dfrac{XC}{XB}=\dfrac{AC}{AB}$□$\dfrac{XE^2}{XD^2}=\dfrac{EQ^2}{DP^2}=\dfrac{EC\cdot EB}{DB\cdot DC}$□then we can infer $\angle BAD=\angle CAE$.

Attachments:



Zhang Fangyu

#7 Jan 9, 2011, 4:29 pm

dear lym!
ver nice problem!very nice solution!

Luis González

#8 Jan 10, 2011, 11:24 am • 2

Okay, here is my solution to question (2).

Using the result of A metrical relation in a circle for the cyclic quadrilateral $BCQP$, it follows that the tangents $\mathcal{L}_1, \mathcal{L}_2$ of its circumcircle (O) through P, Q cut BC at D, E, such that

$$\left(\frac{VB}{VC}\right)^2 = \left(\frac{AB}{AC}\right)^2 = \frac{BD}{DC} \cdot \frac{BE}{EC}$$

By the converse of Steiner theorem, we get that AD, AE are isogonals WRT $\angle BAC$.

Petry

#9 Jan 11, 2011, 4:35 am

My solution for (1):

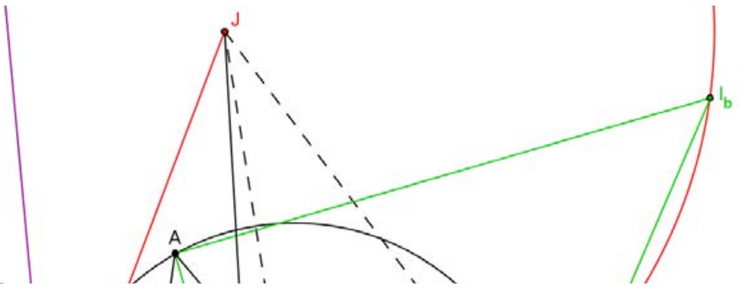
Let R, r_a be the radii of the circles (O) and (I_a) respectively.
$F \in (O) \cap (I_a)$ and $\{F, G\} = I_aF \cap (O)$.
Let's prove that $I_aG = 2R$.
$\{L\} = (I_a) \cap BC$, $\{K\} = AI_a \cap BC$,
$\{A, M\} = AI_a \cap (O)$ and N is the midpoint of $[AM]$.
It's easy to prove that $\Delta I_aLK \sim \Delta MNO \Rightarrow \frac{I_aL}{MN} = \frac{I_aK}{MO} \Rightarrow \frac{2r_a}{AM} = \frac{I_aK}{R} \Rightarrow$
$\Rightarrow AM \cdot I_aK = 2Rr_a$ (1)
It's known that M is the midpoint of $[I_aI]$.
$\Delta AMB \sim \Delta BMK \Rightarrow \frac{AM}{BM} = \frac{BM}{KM} \Rightarrow \frac{AM}{I_aM} = \frac{I_aM}{KM} \Rightarrow$
$\Rightarrow \frac{AM}{I_aA} = \frac{I_aM}{I_aK} \Rightarrow AM \cdot I_aK = I_aM \cdot I_aA$ (2)
$I_aM \cdot I_aA = I_aF \cdot I_aG \Rightarrow I_aM \cdot I_aA = r_a \cdot I_aG$ (3)
(1),(2),(3)$\Rightarrow r_a \cdot I_aG = 2Rr_a \Rightarrow I_aG = 2R$.

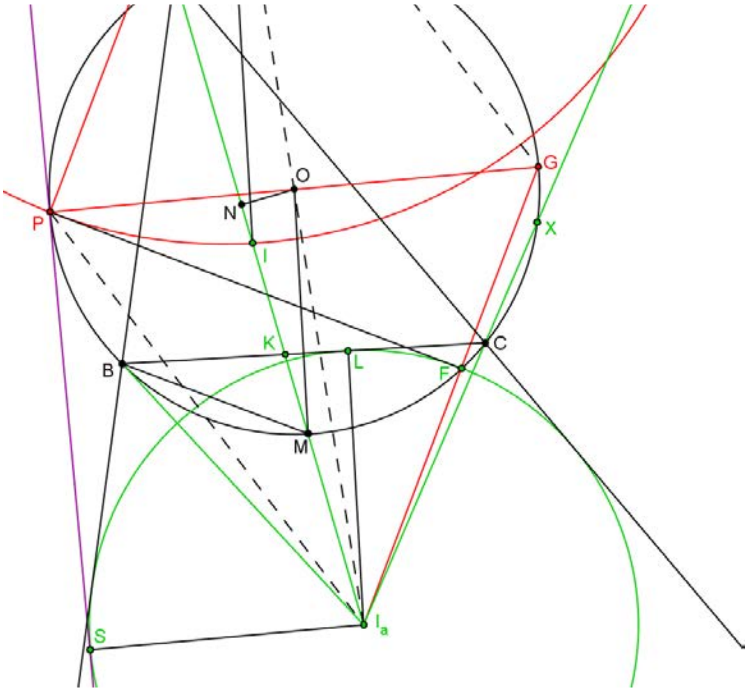
$\{G, P'\} = GO \cap (O)$ and
the tangent to (O) at P' intersects the parallel to $P'G$ through I_a at S.
$\angle P'FG = 90° \Rightarrow P'F$ is tangent to (I_a) at F.
$I_aS || GP' \Rightarrow \angle P'I_aS = \angle I_aP'G$ (4)
$I_aG = P'G = 2R \Rightarrow \angle P'I_aG = \angle I_aP'G$ (5)
(4),(5)$\Rightarrow \angle P'I_aS = \angle P'I_aF$
It's easy to prove that $\Delta P'SI_a \cong \Delta P'FI_a \Rightarrow I_aS = I_aF = r_a$.
So, $P'S$ is the common tangent to the circles (O) and $(I_a) \Rightarrow P' = P$.

$J \in I_aO$ such that O is the midpoint of $[I_aJ]$.
The quadrilateral JPI_aG is parallelogram and $JP = I_aG = PG = 2R$ (6)
O, M are the midpoints of $[I_aJ]$ and $[I_aI]$ respectively $\Rightarrow JI = 2 \cdot OM = 2R$ (7)
Let X be the midpoint of $[I_aI_b]$. It's known that $X \in (O)$.
O, X are the midpoints of $[I_aJ]$ and $[I_aI_b]$ respectively $\Rightarrow JI_b = 2 \cdot OX = 2R$ (8)
Let (J) be the circle with center J and radius $2R$.
(6),(7),(8)$\Rightarrow P, I, I_b \in (J)$.
Similarly $\Rightarrow Q, I, I_c \in (J)$.
So, $I, I_b, I_c, P, Q \in (J)$.

Best regards,
Petrisor Neagoe :)

Attachments:





buratinogigle

#10 Feb 12, 2011, 4:52 pm

Is this post heplfull ?

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=296809

jayme

#11 Dec 15, 2012, 9:12 pm

Dear Lym and Mathlinkers,

this is really a very nice problem...
To make the proof more geometrical alive and simple , we can make a link with a mixtilinear incircle and the results that follow

see for example

http://perso.orange.fr/jl.ayme vol. 4 A new mixtilinear incircle adventure I p. 17, 20, 23

In this point of view, the angle chasing can be much more simple.

Sincerely
Jean-Louis

Lyub4o

#12 Oct 24, 2013, 1:26 pm

> *Luis González wrote:*
> Homothety $(I_a, 2)$ takes (O) into $\odot(I_bI_bI_c)$, thus (O) cuts $\overline{I_aQ_0}$ at its midpoint $

Can someone explain again this part of Luis González's solution?

oneplusone

#13 Dec 9, 2013, 9:32 pm • 3

A very simple proof for (2):

Let N be the midpoint of the minor arc BC. Then $\angle PAI_a = \frac{1}{2}\angle PON = \frac{1}{2}(180 - \angle PDE) = 180 - \angle PDI_a$.
Therefore PAI_aD is cyclic and similarly QAI_aE is cyclic. So $\angle DAI_a = \angle DPI_a = \angle EQI_a = \angle EAI_a$ done.

TelvCohl

#14 Feb 18, 2015, 12:26 am • 1

According to the solution of (2) by **Luis** we can generalize (2) as following **:**

Let X be a point on the A-internal bisector of $\triangle ABC$.
Let $E = BP \cap AC, F = CP \cap AB, \{P, Q\} = EF \cap \odot(ABC)$.
Let the tangent of $\odot(ABC)$ through P, Q cut BC at Y, Z, respectively .

Then AY, AZ are isogonal conjugate WRT $\angle BAC$.

This post has been edited 1 time. Last edited by TelvCohl, Dec 11, 2015, 4:27 pm

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drmzjoseph

#15 Feb 20, 2015, 6:38 am • 2

Solution part (1)

L_1 is tangent at P and X
$\odot(O)$ is the circumcircle of ΔABC

Using $\mathcal{P}(O)I_a = 2r_aR \quad (1)$ (Well Known- Where $\mathcal{P}$ is power)
$\Rightarrow OI_a^2 = R^2 + 2r_aR$ and $PX^2 = 4Rr_a - r_a^2 \Rightarrow PI_a^2 = 4r_aR$ using $(1) \quad \Rightarrow M$ is midpoint of PI_a

$\triangle ABC$and it's circumcircle $\odot(O)$ become orthic triangle and 9-point circle of the $\triangle I_bII_c$.
(I is incenter of $\triangle ABC$)
$\odot I_a$ exsimilicenter of $\odot(O)$ and $\odot(I_bII_c)$ ratio $1:2$ i.e. $P \in \odot(I_bII_c)$
Analogously $Q \in \odot(I_bII_c)$

(2)
I prefer solving Luis Gonzáles

This post has been edited 1 time. Last edited by drmzjoseph, Apr 21, 2015, 8:13 am

Quick Reply

High School Olympiads

Cevian triangle directly similar to reference triangle

Reply

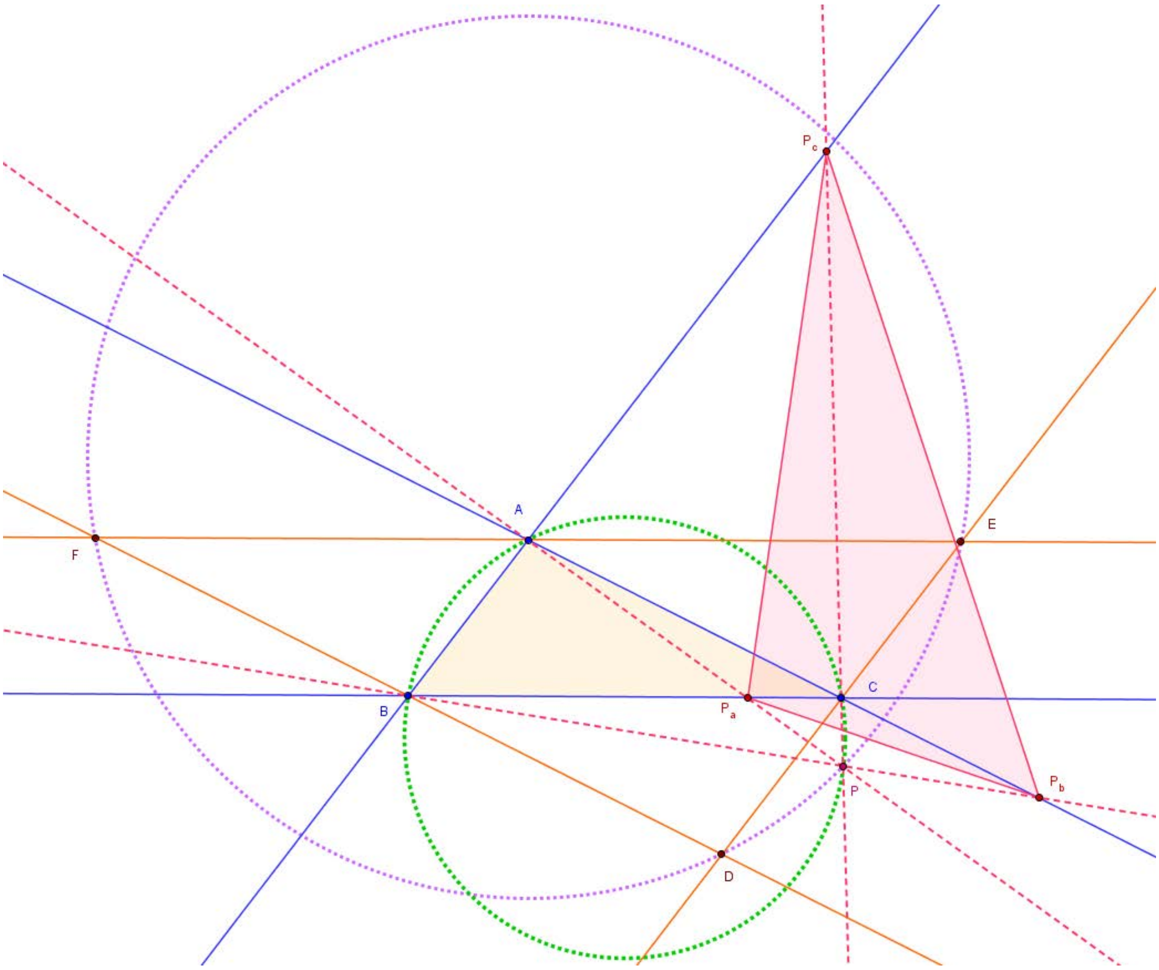
Source: Own

TelvCohl

#1 Dec 3, 2015, 9:54 pm • 2

Given an obtuse triangle $\triangle ABC$. Let $\triangle DEF$ be the anticomplementary triangle of $\triangle ABC$. Prove that the cevian triangle $\triangle P_aP_bP_c$ of a point P WRT $\triangle ABC$ is directly similar to $\triangle ABC$ if and only if P is the Centroid of $\triangle ABC$ or P is one of the intersection of $\odot(ABC)$ and $\odot(DEF)$.

Attachments:



SalaF

#2 Dec 4, 2015, 2:43 am • 2

I will prove just the second implication, that is $P_aP_bP_c$ is directly similar to ABC if P lies on both $\odot(ABC)$ and $\odot(DEF)$.

Let A_0, B_0, C_0 be the midpoints of BC, CA, AB. Let the circles $\odot(ABC), \odot(A_0B_0C_0)$ intersect at the points X, Y and let Z be the anticomplement of X; i will prove that the cevian triangle of Z is similar to ABC.

The main step of the proof is the following

Lemma: Let be given a triangle ABC with orthocenter H and centroid G. If A_1, B_1, C_1 are points on the sides BC, CA, AB respectively, such that the triangles $\triangle ABC$ and $\mathcal{T} = \triangle A_1B_1C_1$ are directly similar, let $\mathcal{R}$ be the spiral similarity which sends the first to the second.

Then the locus of the center of $\mathcal{K}$ as $\mathcal{T}$ varies is the circle with diameter GH.

Proof: Let O be the Miquel point of (A_1, B_1, C_1). Clearly $\angle(B_1O, OC_1) = \angle(B_1A, AC_1) = \angle(C_1A_1, A_1B_1)$ and analogous relations for the other vertices, and we trivially conclude that O is the orthocenter of $\mathcal{T}$. Moreover $\angle(BO, OC) = \angle(BO, OA_1) + \angle(OA_1, OC) = \angle(C_1B, C_1A_1) + \angle(B_1A_1, B_1C) = \angle(BA, AC) + \angle(B_1A_1, A_1C_1) = 2\angle(BA, AC)$ and analogous relations; we similarly conclude that O is the circumcenter of ABC.
Then, if X is the intersection point of AA_1 and OH, the center of $\mathcal{R}$ (we'll denote it K) must be the second intersection point of $\odot(AXH)$ e $\odot(A_1XO)$ (because $\mathcal{R}(A) = A_1$ and $\mathcal{R}(H) = O$. Let V be the point on $\odot(AXH)$ such that $AV//BC$ and U be the second intersection point of $\odot(A_1XO)$ and BC. We have $\angle(VK, KX) + \angle(XK, KU) = \angle(AV, AA_1) + \angle(AA_1, A_1U) = 0$, so that U, V, K are on a line ℓ, which intersects OH at a point G_1. We see that $\angle(VH, OU) = \angle(VH, HO) + \angle(HO, OU) = \angle(AV, AX) + \angle(A_1X, A_1U) = 0$ which implies $VH//OU$; we conclude that $\frac{G_1H}{G_1O} = \frac{VH}{OU}$. If M is the midpoint of BC then $\triangle AVH \simeq \triangle MUO$ (right triangles with a common acute angle from the above parallelism) and consequently $\frac{G_1H}{G_1O} = \frac{VH}{OU} = \frac{AH}{OM} = 2$ (well-known). We immediately conclude that G_1 is exactly the centroid G.
Then $\angle(HK, KG) = \angle(HK, KV) = \angle(AH, AV) = 90^\circ$ and we have our Lemma. (Note: as the angle α of the similitude varies so does $\angle(KHO) = \alpha$, from which we conclude that as $\mathcal{R}$ varies K spans the entire circle).

Now we cann apply our **Lemma** to this problem. Indeed H, G are the two homothety centers of $\odot(ABC)$ and $\odot(A_0B_0C_0)$, so that HG is coaxal to them and passes through X, Y.
By our **lemma** we can take A_1, B_1, C_1 on BC, CA, AB such that a spiral similarity of center Y sends $\triangle ABC$ to $\triangle A_1B_1C_1$. But Y is a fixed point of the transformation and $Y \in \odot(ABC)$; then $Y \in \odot(A_1B_1C_1)$. Now, if X_1 is the second intersection of the circles $\odot(ABC)$ and $\odot(A_1B_1C_1)$ we have $\angle(X_1A_1, X_1Y) = \angle(X_1A, XY)$ (correspondent angles on the circles under the similitude). We conclude that A, A_1 and X_1 are collinear, and analogous relations for the other vertices.
Finally we note that $\angle(XX_1, XY) = \angle(AX_1, AY) = \angle(AA_1, AY) =$ angle of similitude $= \angle(HO, HY) = \angle(XG, XY)$ (because O is the image of H). It follows that XX_1 passes through G, and X_1 is the anticomplement of X, that is Z. Then $A_1B_1C_1$ is the cevian triangle of Z and is directly similar to ABC by construction.

This problem gives a nice application of the **Lemma** and is indeed very nice.

This post has been edited 1 time. Last edited by SalaF, Dec 4, 2015, 3:10 pm

Luis González

#3 Dec 8, 2015, 11:46 am • 2

Let P_a, P_b, P_c be points on BC, CA, AB such that $\triangle ABC \sim \triangle P_aP_bP_c$ are directly similar but not necessarily perspective. Clearly the center of the spiral similarity of all $\triangle P_aP_bP_c$ is fixed, being in fact the circumcenter O of $\triangle ABC$. Thus, the series P_a, P_b, P_c are all similar (projective) $\Longrightarrow$ pencils AP_a, BP_b, CP_c are projective $\Longrightarrow P_1 \equiv AP_a \cap CP_c$ moves on a conic $\mathcal{H}_B$ through A, C that obviously passes through the centroid G of $\triangle ABC$ when $\triangle P_aP_bP_c$ coincides with the medial triangle. Similarly $P_2 \equiv AP_a \cap CP_b$ describes a conic $\mathcal{H}_C$ though A, B, G. Thus $\triangle P_aP_bP_c$ is a cevian triangle of $\triangle ABC \Longleftrightarrow P_1 \equiv P_2$, in other words there are only at most 3 points P verifying that its cevian triangle WRT $\triangle ABC$ is directly similar to it, namely the 3 intersections of $\mathcal{H}_B$ and $\mathcal{H}_C$ excluding A. As one of them is trivally G, then it's enough to show that the intersections $\odot(ABC) \cap \odot(DEF)$ fulfill the property.

Label P, Q the intersections of $\odot(ABC)$ and $\odot(DEF)$ and let $\triangle P_aP_bP_c$ be the cevian triangle of P WRT $\triangle ABC$. It's known that the isotomic conjugate of $\odot(ABC)$ WRT $\triangle ABC$ is the orthic axis of $\triangle DEF$, i.e. radical axis PQ of $\odot(ABC)$ and $\odot(DEF)$, thus it follows that P, Q are isotomic conjugates WRT $\triangle ABC$. Therefore if AQ cuts BC at Q_a, then by symmetry AP_aDQ_a is a parallelogram $\Longrightarrow DP_a$ is the anticomplement of AQ WRT $\triangle ABC$ passing through the anticomplement of Q; 2nd intersection U of GQ with $\odot(DEF)$. Thus if QG cuts $\odot(ABC)$ again at J, we have $\angle JUP_a \equiv \angle JUD = \angle JQA = \angle JPP_a \Longrightarrow P_a \in \odot(PJU)$ and by similar reasoning P_b and P_c lie on $\odot(PJU)$, i.e. $P \in \odot(P_aP_bP_c)$. Therefore $\angle P_bP_aP_c = \angle P_bPP_c = \angle BAC$ and similarly $\angle P_aP_bP_c = \angle ABC \Longrightarrow$ $\triangle ABC \sim \triangle P_aP_bP_c$. Similarly the cevian triangle of Q is directly similar to $\triangle ABC$, as desired. So we have proved the uniqueness of the three points verifying that their cevian triangles WRT $\triangle ABC$ are directly similar to it.

Quick Reply

High School Olympiads

Tangent circles

Source: Own

buratinogigle

#1 Dec 5, 2015, 11:15 pm • 2

Let ABC be a triangle with incircle (I) and A-excircle (J). Circle (K) passes through B,C cuts CA,AB again at E,F.

a) Prove that there is a circle (X) passes through E,F and it is tangent to the circles $(I),(J)$ internally, externally, reps.

b) Let AI cut BC at D. L is on AK such that $DL\perp BC$. Prove that circle (L,LD) is tangent to (X).

c) Assume circumcircle (O) of triangle ABC, BC and (K) are fixed. Prove that circle passing through B,C and tangent to (L,LD) is fixed when A moves.

livetolove212

#2 Dec 6, 2015, 3:38 pm

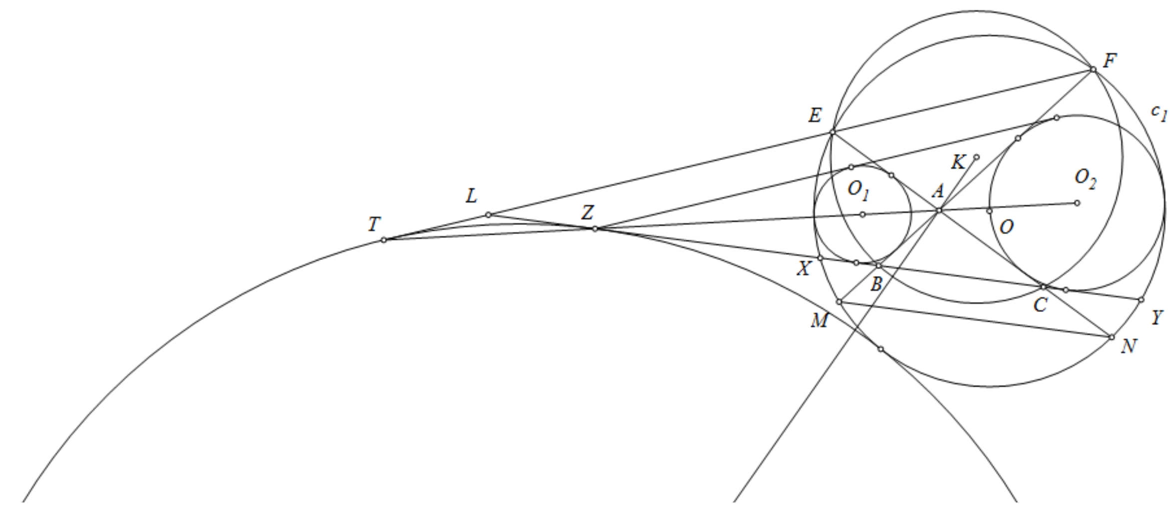
My solution for part a) and b)
I will change the version of this problem to another way.
Lemma 1. Given (O) and two circles (O_1) and (O_2) are internally tangent to (O). Let EN and FM be two internal common tangents of (O_1) and (O_2) $(M,N,E,F\in (O))$, XY be the external common tangent of (O_1) and (O_2) such that XY and MN lie on a half-plane wrt O_1O_2. Then $XY\parallel MN$.

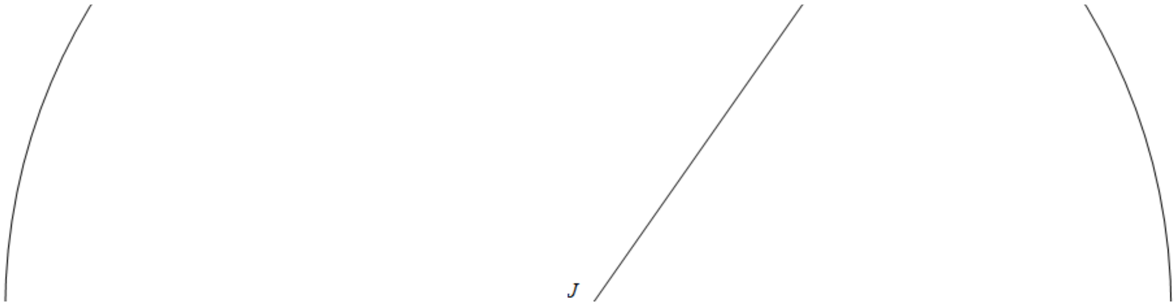
This lemma is well-known. We note that if XY intersects AM and AN at B,C then $BC\parallel MN$ and $BCFE$ is cyclic. We get (K) is $(BCFE)$. Part a) is done.
Let Z,T be the intersections of O_1O_2 and BC. EF intersects BC at L. The second external common tangent t of (O_1) and (O_2) passes through Z then $t\parallel EF$. We get $LZ=LT$. Let (J) be circle tangent to BC and EF at Z,T, respectively.

We need the second lemma.
Lemma 2. Given (O) and (I). Line d intersects (O) and (I) at X,Y and Z,T. Then the quadrilateral formed by intersections of the tangents of (O) through X,Y and of (I) through Z,T is inscibed in circle ω and $\omega,(O),(I)$ are coaxal.
This lemma can be proved by using ratio of power of a point wrt 2 circles.

Applying this lemma for (J) and $(A,0)$ with line TZ we get (K) is coaxal with (J) and $(A,0)$. This means J,A,K are collinear.
Since (O_1) is Thebault circle of triangle EXY then the incenter, excenter of triangle EXY lie on O_1O_2, then applying Sawayama-Thebault theorem we get (J) is tangent to (O).

Attachments:





This post has been edited 2 times. Last edited by livetolove212, Dec 8, 2015, 8:34 am

livetolove212

#3 Dec 6, 2015, 6:43 pm

Part c can be solved similarly as Telv did. Note that the image of J under the inversion center A followed by the symmetry wrt AD is the intersection of the perpendicular bisector of BC and the isogonal of AK, which is fixed.

TelvCohl

#4 Dec 8, 2015, 5:40 am • 1

Another proof to (a) :

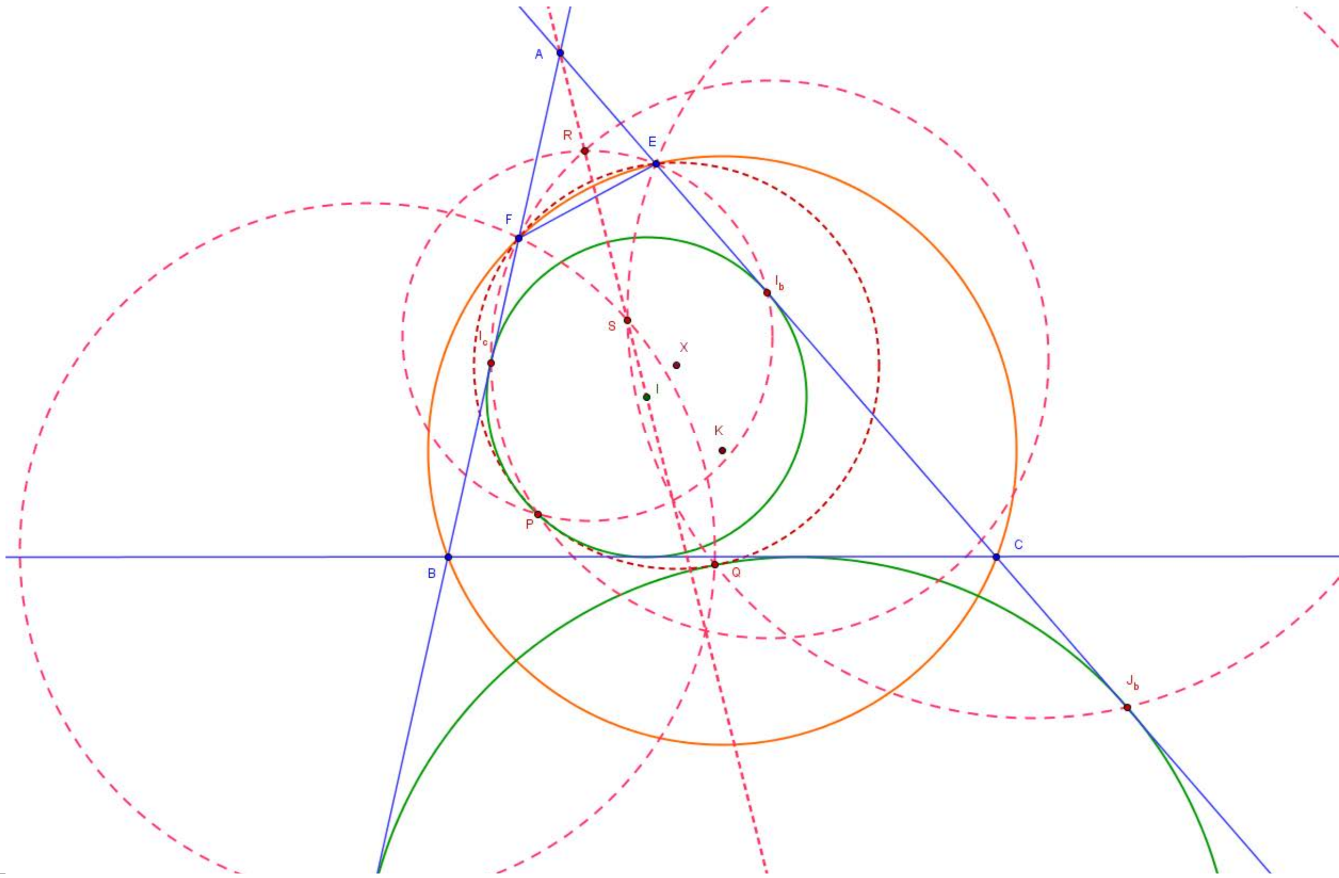
Lemma : Given a $\triangle ABC$ with A-excenter J. Let $\odot(I)$ be a circle which is tangent to CA, AB at E, F, respectively. Then $T \equiv \odot(BFJ) \cap \odot(CEJ)$ lies on $\odot(I)$ and $\odot(BTC)$ is tangent to $\odot(I)$ at T.

Proof : Let the bisector of $\angle BTC$ cuts $\odot(I)$ again at M. From $\measuredangle FTE = \measuredangle FTJ + \measuredangle JTE = \measuredangle FBJ + \measuredangle JCE = 90^\circ + \frac{1}{2}\measuredangle BAC$ we get $T \in \odot(I)$. Since E and F are symmetry WRT IJ, so $\measuredangle BTF + \measuredangle CTE = \measuredangle BJF + \measuredangle CJE = \measuredangle JBA + \measuredangle JCA \Longrightarrow$ the tangent of $\odot(I)$ passing through M is parallel to BC, hence $\odot(BTC)$ is tangent to $\odot(I)$ at T.

Back to the main problem :

Let R, S be the incenter, A-excenter of $\triangle AEF$, respectively. Let $I_b \equiv \odot(I) \cap CA, J_b \equiv \odot(J) \cap CA, I_c \equiv \odot(I) \cap AB, J_c \equiv \odot(J) \cap AB, P \equiv \odot(ERI_b) \cap \odot(FRI_c), Q \equiv \odot(ESJ_b) \cap \odot(FSJ_c)$. Since $\triangle AEF \cup R \cup S \sim \triangle ABC \cup I \cup J$, so $AR : AI = AS : AJ \Longrightarrow \triangle AI_bI_c \cup R \cup I$ and $\triangle AJ_bJ_c \cup S \cup J$ are homothetic, hence $\measuredangle EPF = \measuredangle EPR + \measuredangle RPF = \measuredangle AI_bR + \measuredangle RI_cA = \measuredangle AJ_bS + \measuredangle SJ_cA = \measuredangle EQS + \measuredangle SQF = \measuredangle EQF \Longrightarrow E, F, P, Q$ are concyclic $\Longrightarrow$ there exist a circle $\odot(X) \equiv \odot(EFPQ)$ tangent to $\odot(I), \odot(J)$ at P, Q, respectively (**Lemma**).

Attachments:



Luis González

#5 Dec 8, 2015, 7:23 am • 1

Problem a) is an extraversion of the celebrated "Parallel tangent theorem". If a circle, for example internally tangent to (I) and externally tangent to (J), cuts $\overline{CA}, \overline{AB}$ at E, F, then by the extraversion of the Parallel tangent theorem, it follows that EF is parallel to the internal common tangent of $(I), (J)$, other than BC, which is obviously an antiparallel WRT $AB, AC \Longrightarrow BCEF$ is cyclic.

For various proofs of the Parallel tangent theorem you can see the topics Parallel tangent (post #8) and XY//BD,tangent.

Quick Reply

High School Olympiads

Parallel tangent

Reply

Source: Hyacinthos message #9140

juancarlos

#1 Aug 27, 2004, 3:13 am • 2

H

is orthocenter of

ABC

triangle acutangle. Prove that for the incircles of

AHB

and

AHC

the external tangent line that is farther from the vertex

A

is parallel to

BC

side.

grobber

#2 Aug 27, 2004, 4:32 am • 1

Let D be the projection of A on BC. The problem is equivalent to showing that $\frac{r_{AHB}}{r_{AHC}}=\frac{r_{ADB}}{r_{ADC}}$ where, in general, r_{XYZ} is the inradius of XYZ. Since in a triangle $r=\frac{S}{p}$, we apply this for our four triangles, and after noticing that $\frac{S_{AHB}}{S_{AHC}}=\frac{S_{ADB}}{S_{ADC}}$, the only thing we have to do is show that $\frac{HA+HB+AB}{HA+HC+CA}=\frac{DA+DB+AB}{DA+DC+CA}$ or, in other words, $\frac{\cos A+\cos B+\sin C}{\cos A+\cos C+\sin B}=\frac{1+\sin B+\cos B}{1+\sin C+\cos C}$. This is trivial to check.

I know it's not exactly nice, but... :)

darij grinberg

#3 Aug 27, 2004, 4:41 pm • 2

Grobber, your idea with proving $\frac{r_{AHB}}{r_{AHC}}=\frac{r_{ADB}}{r_{ADC}}$ is great!

In fact, here is a simpler proof of this: For every triangle ABC, we have $r=\frac{b\sin\frac{C}{2}\sin\frac{A}{2}}{\cos\frac{B}{2}}$. Thus,

$$r_{AHB}:r_{ADB}=\frac{AB\sin\frac{\measuredangle ABH}{2}\sin\frac{\measuredangle BAH}{2}}{\cos\frac{\measuredangle AHB}{2}}:\frac{AB\sin\frac{\measuredangle ABD}{2}\sin\frac{\measuredangle BAD}{2}}{\cos\frac{\measuredangle ADB}{2}}$$

$$= \frac{\sin \frac{\text{[illegible]}}{2} \sin \frac{\text{[illegible]}}{2}}{\cos \frac{\measuredangle AHB}{2}} : \frac{\sin \frac{\text{[illegible]}}{2} \sin \frac{\text{[illegible]}}{2}}{\cos \frac{\measuredangle ADB}{2}}$$
$$= \frac{\sin \frac{90^\circ - A}{2} \sin \frac{90^\circ - B}{2}}{\cos \frac{180^\circ - C}{2}} : \frac{\sin \frac{B}{2} \sin \frac{90^\circ - B}{2}}{\cos \frac{90^\circ}{2}}$$
$$= \frac{\sin \frac{90^\circ - A}{2}}{\cos \frac{180^\circ - C}{2}} : \frac{\sin \frac{B}{2}}{\cos \frac{90^\circ}{2}} = \frac{\cos \frac{90^\circ - A}{2} \cos \frac{90^\circ}{2}}{\cos \frac{180^\circ - C}{2} \sin \frac{B}{2}}$$
$$= \frac{\cos\left(45^\circ - \frac{A}{2}\right) \cos 45^\circ}{\cos\left(90^\circ - \frac{C}{2}\right) \sin \frac{B}{2}} = \frac{\cos\left(45^\circ - \frac{A}{2}\right) \cos 45^\circ}{\sin \frac{B}{2} \sin \frac{C}{2}}.$$

This is symmetric in B and C, hence $r_{AHC} : r_{ADC}$ yields the same, and thus we have $r_{AHB} : r_{ADB} = r_{AHC} : r_{ADC}$, or, in other words, $\frac{r_{AHB}}{r_{AHC}} = \frac{r_{ADB}}{r_{ADC}}$.

This looks very much like a trigonometric proof which can be easily transformed into a synthetic one by using similar triangles instead of trig. Anyway, I am already satisfied!

Juan Carlos, thanks a lot for bringing this question up on MathLinks. An additional property (by Paul Yiu) I have not succeeded to prove yet: The internal common tangent of the incircles of triangles AHB and AHC different from the line AH passes through the midpoint of the side BC of triangle ABC.

Darij

jhaussmann5

#4 Aug 27, 2004, 6:20 pm

> *grobber wrote:*
> Let D be the projection of A on BC. The problem is equivalent to showing that $\frac{r_{AHB}}{r_{AHC}} = \frac{r_{ADB}}{r_{ADC}}$ where, in general, r_{XYZ} is the inradius of XYZ.

I don't see this. Grobber, could you explain?

juancarlos

#5 Aug 27, 2004, 10:09 pm • 2

You are welcome Darij.
I thing that, if we consider the condition of external tangent of two incircles as parallel line to

BC

side,now we can use the converse of the parallel tangent theorem wrt the

NPC

of

ABC

, two incircles tangents to

NPC

, the altitude

AD

(internal tangent to two incircles).
Therefore the another internal tangent pass through the midpoint of

BC

side.QED.

isotomion

#6 Aug 27, 2004, 10:51 pm • 1

Juan Carlos, thanks but I don't know what you mean by "parallel tangent theorem"!

To answer Jhaussmann5's question: Once we have proven that $\frac{r_{AHB}}{r_{AHC}} = \frac{r_{ADB}}{r_{ADC}}$, then we also have $\frac{r_{AHB}}{r_{ADB}} = \frac{r_{AHC}}{r_{ADC}}$. Now if we denote $\frac{r_{AHB}}{r_{ADB}} = \frac{r_{AHC}}{r_{ADC}} = k$, then the homothety with center A and factor k maps the incircles of triangles ADB and ADC to the incircles of triangles AHB and AHC, respectively. Hence, this homothety maps the line BC, which is an external common tangent of the incircles of triangles ADB and ADC, to a line x which therefore must be an external common tangent of the incircles of triangles AHB and AHC. But since the image of any line under a homothety is parallel to that line, we must have x || BC, and thus we have found an external tangent of the incircles of triangles AHB and AHC that is parallel to BC.

Darij

jhaussmann5

#7 Aug 28, 2004, 10:36 pm

Darij, thank you for your explanation.

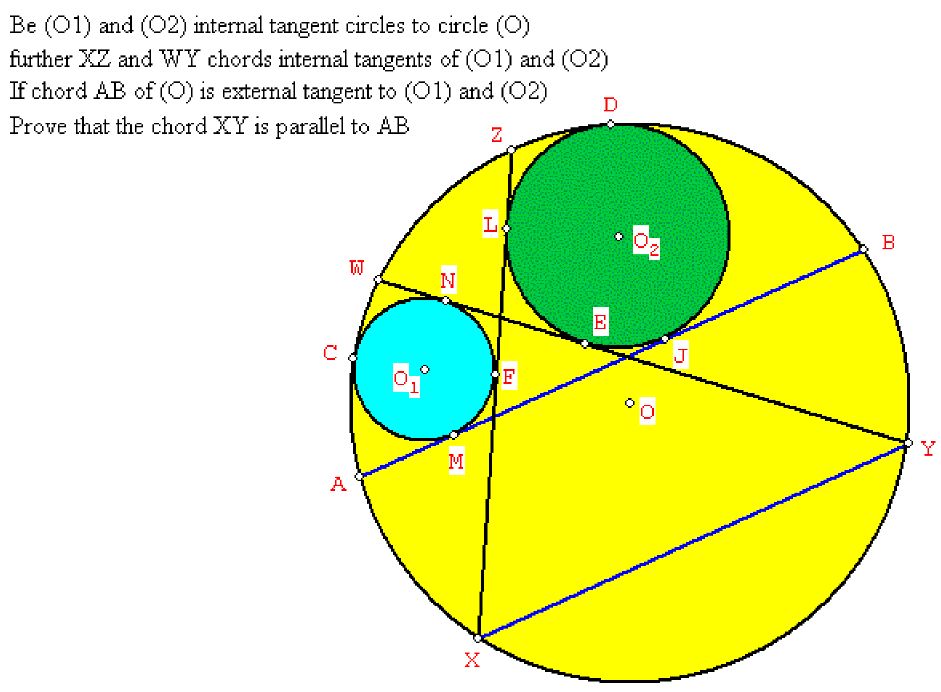
I am curious about this parallel tangent theorem as well, what is it?

juancarlos

#8 Aug 29, 2004, 12:25 am • 1

I talk about this case of the theorem of the parallel tangent, please
to see the annexed figure in the GIF file.

Attachments:



jhaussmann5

#9 Aug 29, 2004, 10:42 pm

juancarlos, that is truly a beatiful result. Do you have a proof?

darij grinberg

#10 Aug 29, 2004, 10:51 pm • 1

Actually, Juan Carlos has sent me a copy of a paper where this theorem is proved:

Shay Gueron, *Two Applications of the Generalized Ptolemy Theorem*, American Mathematical Monthly 2002, p. 362-370.

But I don't like the proof, it uses too much irrational points (points of intersection of lines with circles. I don't like such things)... I'll

But I don't like the proof, it uses too much irrational points (points of intersection of lines with circles, I don't like such things)... I'll try to find a simpler one...

Darij

grobber

#11 Aug 31, 2004, 12:31 am • 1

This is equivalent to so many nice facts (and I can't prove any of them :)). For example:

The midpoints of arcs XY, ZW are on the radical axis of the two smaller circles

If you fix the smaller circles and move the big circle, then the incircles of WTX, ZTY are constant ($T = WY \cap ZX$).

There are many more.

jhaussmann5

#12 Sep 1, 2004, 8:48 am

> *juancarlos wrote:*
>
> You are welcome Darij.
> I thing that, if we consider the condition of external tangent of two incircles as parallel line to BC side,now we can use the converse of the parallel tangent theorem wrt the NPC of ABC, two incircles tangents to NPC, the altitude AD(internal tangent to two incircles).
> Therefore the another internal tangent pass through the midpoint of BCside.QED.

This is really remarkable, I'm really getting into the problem. juancarlos, can you tell me why the circles are tangent to the nine-point circle?

darij grinberg

#13 Sep 1, 2004, 1:57 pm • 1

> *jhaussmann5 wrote:*
>
> juancarlos, can you tell me why the circles are tangent to the nine-point circle?

Well, as you probably know, the great Feuerbach theorem states that the nine-point circle of a triangle touches the incircle and the excircles. Now, apply this theorem not to triangle ABC, but to triangle AHB, for instance; notably, the triangle AHB has the same nine-point circle as the triangle ABC (this is easy to see; actually, the nine-point circle of a triangle is the circle through the feet of the altitudes, and the feet of the altitudes of triangle AHB exactly coincide with the feet of the altitudes of triangle ABC), so we see that the nine-point circle of triangle ABC touches the incircle of triangle AHB. Similarly, the same nine-point circle touches the incircle of triangle AHC.

Darij

pestich

#14 Sep 2, 2004, 6:50 am

We know that if in triangle ABC the line AD is a common internal tangent
to the incircles of ABD and ACD, then the distance between touch points
on AD is equal to abs(DT-DU) where T and U are the touch points on BC.

In the case of AD being the altitude, both distances equal to the difference
of the incircle radii, and this makes it a solution.

So, the common external tangents of the incircles of the 3 triangles
made by orthocenter in the original problem are parallel to the opposite sides.

Pestich.

Fang-jh

#15 Dec 26, 2006, 5:47 pm • 1

let CM∩DJ∩Circle O at P. Four points C.M.J.D are concyclic. P is on the radical axis of circles O1 and O2. P is midpoint arc AXYB. XF/XC=YN/YC=m/PC=invariable (m is the touch lines segments from P to circles O1 and O2). □(XF+YN)/(XC+YC)=m/PC XL/XD=YE/YD=m/PD
□(XL+YE)/(XD+YD)=m/PD □XL-XF=FL=NE=YN-YE □XL+YE=YN+XF □(XC+YC)/(XD+YD)=PC/PD=(PC*XY)/(PD*XY)=(PY*XC+PX*YC)/(PY*XD+PX*YD)(Ptolemy's theorem)(*=×)□(PX-PY)(XD*YC-XC*YD)=0
(PX-PY)*XY*CD=0(Ptolemy's theorem)□□PX=PY
□P is midpoint arc XY □XY□AB

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yetti

#16 Dec 26, 2006, 8:09 pm • 1

Excellent Thank you

$\LaTeX$ing Fang-jh solution (and some more details in blue):

Let $CM \cap DJ \cap (O)$ at P. (From similarities $(O_1) \sim (O)$ with center at the tangency point C and $(O_2) \sim (O)$ with center at the tangency point D.) The four points C, M, J, D are concyclic. P is on the radical axis of circles (O_1) and (O_2). P is midpoint of the arc AXYB. (Inversion in (P) with radius PA = PB carries (O) into AB, AB into (O) and $(O_1),\ (O_2)$ into themselves, hence $(O_1),\ (O_2) \perp (P)$ and PA = PB is the tangency length from P to $(O_1),\ (O_2)$.)

$$\frac{XF}{XC} = \frac{YN}{YC} = \frac{m}{PC} = \text{const}$$

where m = PA = PB is the tangent length segment from P to the circles (O_1) and (O_2). (Point C and circles $(O_1),\ (O)$ form a pencil of tangent circles, hence $\frac{XF}{XC} = \frac{YN}{YC} = \frac{m}{PC} = \sqrt{\frac{O_1O}{CO}}$, see the problem Constant value.)

$$\frac{XF+YN}{XC+YC} = \frac{m}{PC}$$

Similarly,

$$\frac{XL}{XD} = \frac{YE}{XD} = \frac{m}{PD}, \quad \frac{XL+YE}{XD+YD} = \frac{m}{PD}.$$

$$XL - XF = FL = NE = YN - YE, \quad XL + YE = XF + YN$$

$$\frac{XC+YC}{XD+YD} = \frac{PC}{PD} = \frac{PC \cdot XY}{PD \cdot XY} = \frac{PY \cdot XC + PX \cdot YC}{PY \cdot XD + PX \cdot YD}$$

by Ptolemy's theorem. Multiplying out:

$$(PX - PY) \cdot (XD \cdot YC - XC \cdot YD) = 0$$

$$(PX - PY) \cdot XY \cdot CD = 0$$

by Ptolemy's theorem. As a result, $PX = PY$, P is midpoint of the arc XY, $XY \parallel AB$.

This post has been edited 2 times. Last edited by yetti, Dec 27, 2006, 6:43 am

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conejita

#17 Dec 27, 2006, 1:06 am

Hello, i have a solution using the Theubalt Teorem.... but i dont speak english very good... do you like post my solution in spanish??

Huyền Vũ

#18 Jan 7, 2007, 1:46 pm

to juancarlos:
Can you send me
Shay Gueron, Two Applications of the Generalized Ptolemy Theorem, American Mathematical Monthly 2002, p. 362-370.
Thanks

Thanks
my e-mal:greenswan3@yahoo.co.uk
Thanks!

Huyền Vũ
#19 Jan 12, 2007, 10:40 pm • 1

Parallel_tangent_theorem
Let (O1) (O2) touch the circle (O) internally at M,N.The internal common tangent of (O1) and (O2) cut (O) at E,F,S,R. The external common tangent of (O1) and (O2) cut (O) at A,B.Prove that AB//EF or AB//SR
Proof:
I have a simple solution for this problem
K is the midpoint of arc AB that doesn't contain S and R
I1 is the intersection of O1O2 and SK
I2 is the intersection of O1O2 and RK
By using Thebault theorem we have I1and I2 are the incircle centers of Triangle ASB and Triangle ARB
Otherwise KA=KI1 and KB=KI2
So KI1=KI2 --> Triangle I1I2K is isosceles -->Angle I1I2K= Angle I2I1K --> Angle I1I2R = Angle I2I1S --> Angle I1SI=Angle I2SI --> K is the midpoint of the axis EF
But K is also the midpoint of the arc AB so AB // EF. :)

> Quote:
> This is equivalent to so many nice facts (and I can't prove any of them). For example:
> The midpoints of arcs XY,ZW are on the radical axis of the two smaller circles

to grobber: your problem is very interesting. I call it the problem 2
Problem 2
Let (O1) (O2) touch the circle (O) internally at M,N.The internal common tangent AC,BD of (O1) ,(O2) cut (O) at S,F,R,E. Prove PQ is the radical axis of (O1), (O2) where Q,P is the midpoint of arc EF,RS
Proof
Let (O3) be the circle which touches (O) and SE, SF at X,Y
Let I be the intersection of SQ and AD
Let J be the intersection of RQ and BC
By using Thebault theorem I,J are the incircle centers of Triangle SEF and Triangle RFE.
So X,I,J,Y are collinear
O1O2 // XY (Both are perpendicular to the bisector of Angle EKF) or IJ // O1O2 (1)
Let L and T be the intersection of SQ, RQ and O1O2
We have Angle KRT=Angle KSL and Angle SKL=Angle RKT so Angle TLQ=Angle LTQ -->Triangle QLT is isosceles --> QL=QT (2)
Similarly we have PL=PT (3)
From (2) and (3) we have PQ is the perpendicular bisector of O1O2
Because of (1) PQ is also the perpendicular bisector of IJ
Otherwise AD // BC so PQ passes through the midpoint of BD and AC
--> PQ is the radical axis of (O1), (O2)
Can you post more interesting result about this problem.Thanks

This is my solution in Geometer's Sketchpad 4.06. It is easier to understand
Parallel_tangent_theorem.gsp

Fang-jh
#20 Jan 15, 2007, 5:35 pm • 1

By using Parallel tangent theorem we can also prove Thebault theorem.

Huyền Vũ
#21 Jan 15, 2007, 9:13 pm • 2

Fang-jh is right
I've used two independent proofs for the theorem and problem two. But we also can use theorem to prove the problem 2 or use problem 2 to prove theorem(Maybe the proofs are longer)

Using theorem to prove the problem 2
Q is the midpoint of arc EF
But AB // EF so Q is also the midpoint of arc AB

So that Q is on the radical axis of (O1) and (O2) (Fang-jh and yetti proved above)
Similarly P is on the radical axis of (O1) and (O2)
This is result.

Using problem 2 to prove theorem
Q is on the radical axis of (O1) and (O2)(1)
But Q' which is the midpoint of arc EF is also on the radical axis of (O1) and (O2) (Fang-jh and yetti proved above) (2)
From (1) and (2) we have Q' is Q
Q is the midpoint of both arc EF and arc AB so AB // EF.

to Juan Carlos: I've received that file. Thanks a lot

Fang-jh
#22 Dec 28, 2008, 12:57 pm

> *Quote:*
> An additional property (by Paul Yiu) I have not succeeded to prove yet: The internal common tangent of the incircles of triangles AHB and AHC different from the line AH passes through the midpoint of the side BC of triangle ABC.
>
> Darij

it follows immediately by http://www.mathlinks.ro/Forum/viewtopic.php?t=246658

yetti
#23 Jan 11, 2009, 4:04 am

AB, ZX are 2 chords of a circle (O) intersecting at C. Circles $(O_1), (O_2)$ are tangent to the rays CA, CZ resp. CB, CZ and internally to the circle (O).

R, r_1, r_2 are radii of $(O), (O_1), (O_2)$. o is a parallel to AB on the same side as Z, such that the distance $d(AB, o) = R$. Then $OA = R = d(A, o), OB = R = d(B, o), OO_1 = R - r_1 = d(O_1, o), OO_2 = R - r_2 = d(O_2, o) \Longrightarrow$ the centers O_1, O_2 are on a parabola $\mathcal P$ through A, B with focus O. Perpendicular bisector p of AB is its axis. Let a parallel to AB through X cut (O) again at Y. Let $\mathcal E$ be ellipse through A, B with foci X, Y. The midpoint K of XY is the ellipse center. Let U_1U_2 be its major axis. External bisectors of angles $\angle XAY, \angle XBY$ are the ellipse tangents at A, B. Reflections of OA, OB in these external angle bisectors are parallel to the parabola axis p, hence these external angle bisectors are also the parabola tangents $\Longrightarrow$ the parabola $\mathcal P$ and the ellipse $\mathcal E$ are tangent at A, B. Let ZX cut the ellipse minor arc AB at S and let SY cut AB at D.

Keeping the points A, B in place, project the ellipse $\mathcal E$ into a circle (K'). $S', U'_1, U'_2 \in (K')$ are images of $S, U_1, U_2 \in \mathcal E$. Parabola $\mathcal P$ goes to a parabola $\mathcal P'$ through A, B with the same axis p and tangent to the circle (K') at A, B. It follows that its focus is circumcenter Q of $\triangle ABK'$. Since inversion in (K') takes (Q) into AB and the other way around, circles $(Q_1), (Q_2)$ tangent to AB and passing through $S' \in (K')$ are also internally tangent to (Q), therefore centered of the projected parabola $\mathcal P'$. Since they are perpendicular to (K'), they are also centered on a tangent of (K') at S', tangent to each other at S', $S'K' \perp Q_1Q_2$ being their single common internal tangent and radical axis. Let V_1, V_2 be their tangency points with AB. Let $L \equiv S'K' \cap AB$ be the midpoint of V_1V_2. Circumcircle (L) of the right $\triangle S'V_1V_2$ is tangent to the circumcircle (K') of the right $\triangle S'U'_1U'_2$ at S'. Since the hypotenuses $V_1V_2 \parallel U'_1U'_2$ are parallel, it follows that $V_1 \in S'U'_1, V_2 \in S'U'_2$, and consequently, $V_1 \in SU_1, V_2 \in SU_2$. As V_1, V_2 are feet of perpendiculars to AB from the intersections Q_1, Q_2 of the projected parabola $\mathcal P'$ with the circle (K') tangent at S', they are also feet of perpendiculars to AB from the intersections of the original parabola $\mathcal P$ with the ellipse $\mathcal E$ tangent at S.

U_1, U_2 are tangency points of the Y-, X-excircles of the $\triangle SXY$ with XY, because $XS + YS = U_1U_2 \Longrightarrow YU_1 = XU_2 = \frac{1}{2}(XS + YS + XY)$. Consequently, V_1, V_2 are tangency points of the D-, C-excircles of the $\triangle SCD$ centrally similar to the $\triangle SXY$ with similarity center S. Since these excircles are centered on the ellipse tangent at S externally bisecting the angle $\angle XSY = \angle CSD$, they are also centered on the parabola $\mathcal P$, Consequently, they are both internally tangent to (O) and to the lines AB, ZX and therefore identical with the original circles $(O_1), (O_2)$.

This proves the **parallel tangent theorem** by just parallel projection and using definitions of a parabola and an ellipse. In addition, it shows that the center line O_1O_2 is tangent to the ellipse $\mathcal E$ with foci X, Y and passing through A, B.

Leonhard Euler
#24 Jan 18, 2009, 6:13 pm

Let B' be the reflection of B on AH. Since $\angle ACH = \angle ABH = \angle AB'H$, A, C, B', H are cyclic. Let I, J be incenter of

Let B' be the reflection of B on AH.Since $\angle ACH = \angle ABH = \angle AB'H$, A, C, B', H are cyclic. Let I, J be incenter of triangle $ACH, AB'H$. By angle chasing with using fact that A, I, I', H are cyclic, it is not hard to show that angle of intersection of AH, II' is 45. So external common tangent of incircle of triangle ACH and $AB'H$ farther from the vertex A, which is reflection of AH on II', is parallel to BC. Also, Since triangle ABH and $AB'H$ are symmetric about AH, external common tangent of incircle of triangle ABH and $AB'H$ farther from the vertex A is parallel to BC. Hence we conclude that external common tangent of incircle of triangle ABH and ACH farther from the vertex A is parallel to BC.

jayme

#25 Feb 26, 2011, 3:23 pm

Dear Mathlinkers,
the Grobber result has been rediscovered by J.P. Ehrmann in 2008
http://tech.groups.yahoo.com/group/Hyacinthos/message/15979.
A synthetic proof of this result can be seen on
http://perso.orange.fr/jl.ayme vol. 4 A new mixtilinear incircle adventure III p. 42
with a following nice result.
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

XY//BD,tangent

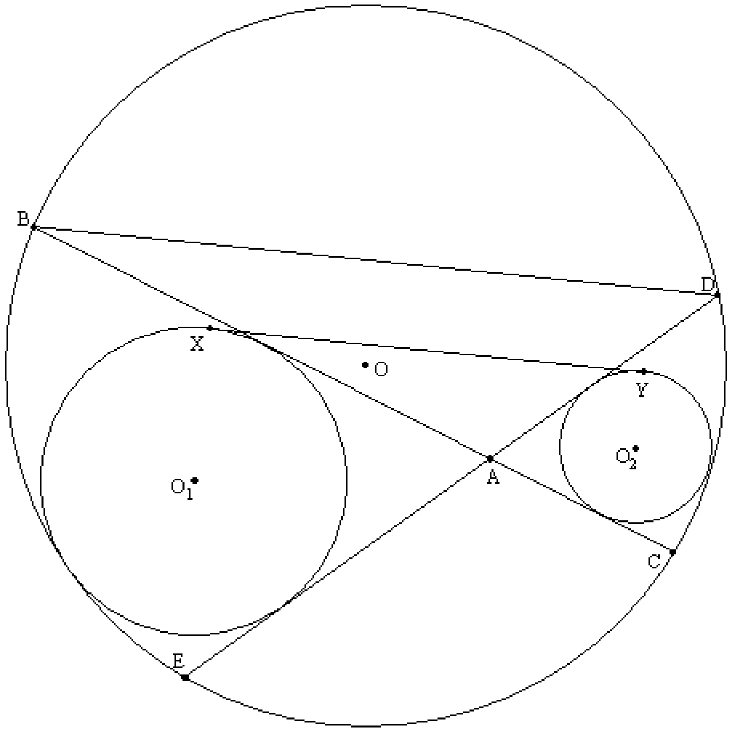
Reply

4dspace

#1 Sep 11, 2011, 11:11 pm

See the picture. Prove that XY//BD.

Attachments:



Luis González

#2 Sep 12, 2011, 9:48 am • 3

Let (O_2) touch (O) and AC at P and $Q.$ Common internal tangent XY cuts (O) at M, N and AC cuts XY at $R.$ Thus, $(O_1), (O_2)$ become the Thebault circles of the cevian CR of $\triangle CMN.$ By Sawayama's lemma, $\odot(CPQ)$ passes through the incenter I of $\triangle CMN$ and $I \in O_1O_2.$ Thus, using the result of the problem incenter of triangle for $\triangle CAD,$ the circle (O) passing through C, D and (O_2) touching $\overline{AC}, \overline{AD}, (O),$ we deduce that I is also the incenter of $\triangle CAD \Longrightarrow$ bisectors of $\angle ACD$ and $\angle MCN$ coincide, i.e. lines CB, CD are isogonals WRT $\angle MCN \Longrightarrow MN \equiv XY \parallel BD.$

4dspace

#3 Sep 12, 2011, 6:30 pm

Thank you very much.

Quick Reply

High School Olympiads

property of incircle

Locked

phuong

#1 Dec 8, 2015, 4:57 am

Let ABC be a triangle and (I) is an incircle which is tangent with BC, CA, AB at D, E, F respectively. K is perpendicular projection of D on EF and H is a orthocentre of triangle. Prove that $\widehat{IKD} = \widehat{DKH}$

Luis González

#2 Dec 8, 2015, 6:19 am

Already discussed at http://www.artofproblemsolving.com/community/c6h614584 (see the remark at post #4 and a solution at post #7).

High School Olympiads

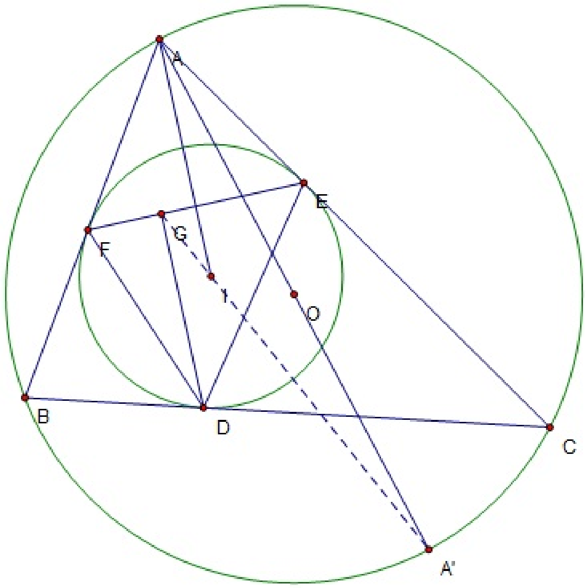
interesting problem

Reply

mathandyou
#1 Nov 19, 2014, 11:00 pm

Given $\triangle ABC$, circumcircle (O), incircle (I). (I) tangents BC, CA, AB at D, E, F. $DG \perp EF$. AO cuts (O) at A'.
Prove that A', I, G is collinear.

Attachments:



TelvCohl
#2 Nov 19, 2014, 11:10 pm • 4

My solution:

Let I' be the reflection of I in O .
Let I_a, I_b, I_c be three excenters of $\triangle ABC$.

Easy to see $\triangle DEF$ and $\triangle I_aI_bI_c$ are homothetic .

Since I, O is the orthocenter, nine point center of $\triangle I_aI_bI_c$, respectively .
so we get I' is the circumcenter of $\triangle I_aI_bI_c$,
hence $\triangle DEF \cap G \cap I$ and $\triangle I_aI_bI_c \cap A \cap I'$ are homothetic ,
so we get $GI \parallel AI'$ . ... (1)
Since $AIA'I'$ is a parallelogram ,
so we get $IA' \parallel AI'$ . ... (2)

From (1) and (2) we get G, I, A' are collinear .

Q.E.D

jayme
#3 Nov 20, 2014, 11:04 am

Dear Mathlinkers,
you can look at

you can look at
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=499885
and we have more.

Sincerely
Jean-Louis

TelvCohl

#4 Nov 21, 2014, 2:24 pm • 2

Remark

From this probelm we can get the interesting property of incenter and orthocenter :

Let I, H be the incenter, orthocenter of $\triangle ABC$, respectively .
Let D, E, F be the tangent point of (I) with BC, CA, AB, respectively .
Let T be the projection of D on EF .

Then DT bisect $\angle HTI$.

Luis González

#5 Nov 21, 2014, 8:06 pm • 2

Let $M \equiv AI \cap EF$ be the midpoint of $\overline{EF}$. Inversion WRT (I) takes 9-point circe of $\triangle DEF$, passing through M, G, into (O) and takes EF into the circle $\odot(IEAF)$ with diameter $\overline{AI} \Longrightarrow G$ goes to the 2nd intersection X of $\odot(IEAF)$ with (O), i.e. I, G, X are collinear and $\angle AXI = 90^\circ \Longrightarrow IGX$ cuts (O) again at the antipode A' of A.

TelvCohl

#6 Nov 24, 2014, 8:42 pm • 2

Another solution:

Let $T = (AI) \cap (ABC), X = EF \cap BC$.

Since $(X, D; B, C) = -1$,
so DG bisect $\angle BGC$,
hence we get $\triangle GFB \sim \triangle GEC$ . ... $(*)$
Since T is the Miquel point of $BCEF$,
so we get $\triangle TBF \sim \triangle TCE$,
hence combine with $(*)$ we get $\frac{TE}{TF} = \frac{EC}{FB} = \frac{EG}{FG}$. ie. TG bisect $\angle FTE$

Since I is the midpoint of arc EF in (TEF) ,
so we get T, G, I are collinear and $\angle ITA = \angle GTA = 90^\circ$,
hence GI pass through the antipode A' of A in (ABC) .

Q.E.D

andria

#7 Sep 9, 2015, 12:37 am

> *mathandyou wrote:*
> Given $\triangle ABC$, circumcircle (O), incircle (I). (I) tangents BC, CA, AB at D, E, F. $DG \perp EF$. AO cuts (O) at A'. Prove that A', I, G is collinear.

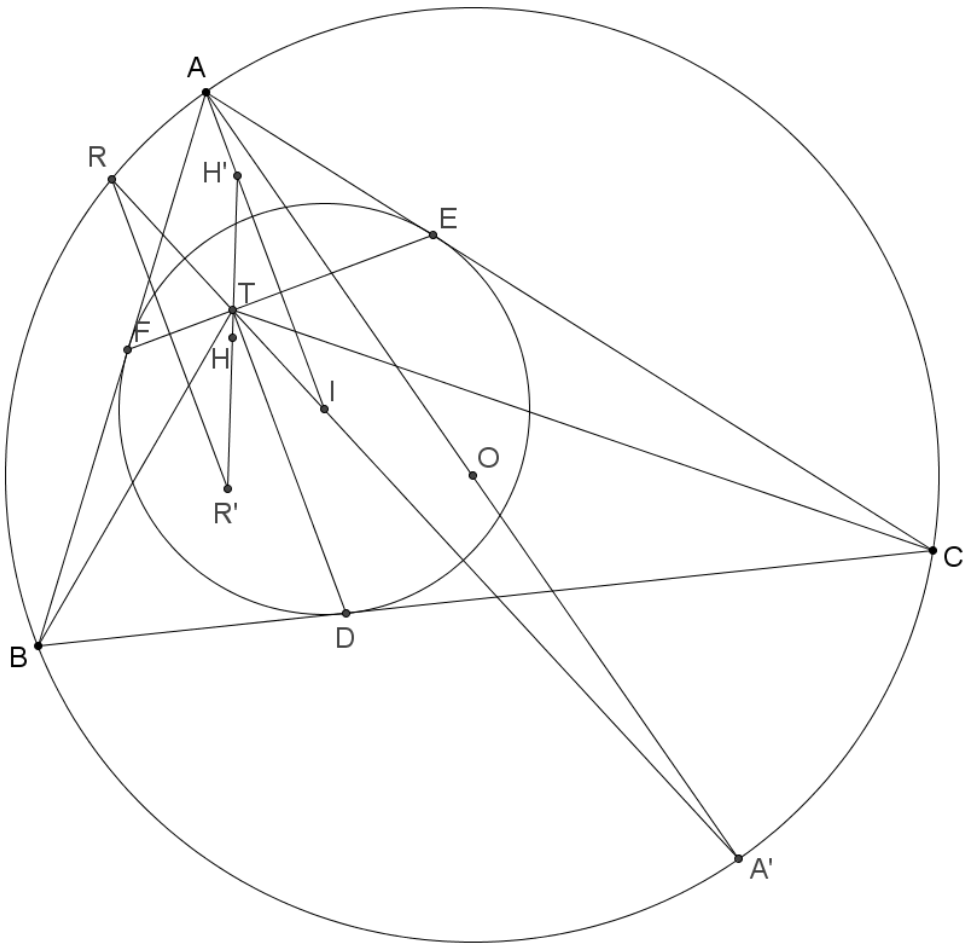
Since
$G(FDBC) = -1, \angle DGF = 90 \Longrightarrow \angle BGF = \angle CGE \Longrightarrow \triangle BGF \sim \triangle CGE \Longrightarrow \frac{FG}{EG} = \frac{BF}{CE} = \frac{BD}{CD}\bigstar$. Let $\odot(AI) \cap \odot(\triangle ABC) = R$. since R is the center of spiral similarity that sends BF to CE we have that $\frac{RF}{RE} = \frac{BF}{CE} = \frac{BD}{CD}$ Hence RD is bisector of $\angle BRC$ so combine with $\bigstar$ and $\triangle REF \sim \triangle RCB$ we get that RG is bisector of $\triangle ABC \Longrightarrow R, G, I$ are collinear $\Longrightarrow \angle GRA = 90$.

DONE

> TelvCohl wrote:
> **Remark**
>
> From this probelm we can get the interesting property of incenter and orthocenter :
>
> Let I, H be the incenter, orthocenter of $\triangle ABC$, respectively .
> Let D, E, F be the tangent point of (I) with BC, CA, AB, respectively .
> Let T be the projection of D on EF .
>
> Then DT bisect $\angle HTI$.

Let H' be the reflection of I in EF obviously H' is orthocenter of $\triangle AEF$ so HH' is steiner line of complete quadrilateral $CEFB$ hence since R is miquel point of $CEFB$ the reflection R' of R in EF lies on HH'. On the other hand since $RI \cap EF = T \Longrightarrow R', T, H'$ are collinear thus H', T, H are collinear $\Longrightarrow \angle HTF = \angle ITE \Longrightarrow TD$ is angle bisector of $\angle ITH$.
DONE

Attachments:



Dukejukem

#8 Sep 9, 2015, 1:21 am

Let $\triangle D'E'F'$ be the orthic triangle of $\triangle DEF$ (here we are relabeling the point G). Let OI cut DD' at H and let H' be the reflection of H in EF.

The inversion w.r.t. (I) sends A, B, C to the midpoints of $\overline{EF}, \overline{FD}, \overline{DE}$, respectively. Consequently, this inversion swaps (O) and the nine-point circle of $\triangle DEF$. Thus if N is the nine-point center of $\triangle DEF$, then I, O, N are collinear, implying that $H \equiv OI \cap DD'$ is the orthocenter of $\triangle DEF$. Then a homothety with center H and ratio $\frac{1}{2}$ shows that $IH' \parallel ND'$. But notice that $E'F'$ and BC are both antiparallel to EF w.r.t. $\angle FDE$. Therefore, $E'F' \parallel BC$ and it follows by symmetry that $\triangle D'E'F'$ and $\triangle ABC$ have parallel sides. Thus they are homothetic, and there exists a homothety that swaps ND' and OA. It follows that $IH' \parallel OA$.

Let P_∞, Q_∞ denote the points at infinity on DD', OA, respectively. Then from $IH' \parallel OA$ and $IP_\infty \parallel IA$, we have

$-1 = I(H', H; P_\infty, D') = (Q_\infty, O; A, ID' \cap OA)$. Therefore, $ID' \cap OA$ is the reflection of A in O, as desired. $\square$

Rmasters

#9 Sep 14, 2015, 6:32 pm

> TelvCohl wrote:
>
> **Remark**
>
> From this probelm we can get the interesting property of incenter and orthocenter :
>
> Let I, H be the incenter, orthocenter of $\triangle ABC$, respectively .
> Let D, E, F be the tangent point of (I) with BC, CA, AB, respectively .
> Let T be the projection of D on EF .
>
> Then DT bisect $\angle HTI$.

Hi. Can i see your solution for that?

Quick Reply

High School Olympiads

Source: OWN

LeVietAn

#1 Dec 7, 2015, 10:34 am • 1

Dear Mathlinkers,

Let ABC be a triangle with $AB > AC$ and Γ be its circumcircle. Let P, Q be the points respectively on the rays CA, BA such that $CP = BQ$. Let K, L and M be the midpoints of the segments BP, CQ and PQ, respectively. The line AM intersects the line KL at the point R and intersects circle Γ at the second point T. The tangent line at T of Γ intersects BC at S. Prove that RS is paralell to PQ.

Luis González

#3 Dec 8, 2015, 6:07 am • 1

Let O be the circumcenter of $\triangle ABC$ and let D be the projection of O on PQ. It's known that K, L, M, D are concyclic (for proofs see concylic midpoints, Cyclic and the symmetry point is in Euler circle and elsewhere), moreover M is midpoint of the arc KDL as $MK = \tfrac{1}{2}BQ = \tfrac{1}{2}CP = ML \Longrightarrow DO, DM$ bisect $\angle KDL$. If $F \equiv PQ \cap KL$ and A_∞ denotes the point at infinity of PQ, we have from $MK \parallel AB$ and $ML \parallel AC$ that $M(R, K, L, F) = A(M, Q, P, A_\infty) = -1 \Longrightarrow (R, K, L, F) = -1 \Longrightarrow R \in OD$.

Let $X \equiv PQ \cap BC$ and let $J \equiv \odot(ABC) \cap \odot(APQ)$ be the Miquel point of $BCPQ$. J is the center of the rotation that swaps $\overline{CP}$ and $\overline{BQ} \Longrightarrow JB = JC \Longrightarrow J$ is midpoint of the arc BAC of (O). Also J is the center of the spiral similarity that swaps PMQ and CNB, where N is the midpoint of $BC \Longrightarrow J, M, N, X$ are concyclic. Since $MKNL$ is clearly a rhombus, then MN is internal bisector of $\angle KML$ whose sides are parallel to $AB, AC \Longrightarrow MN \parallel AU$, where U is midpoint of the arc BC of (O), so $\angle TMN = \angle TAU = \angle TJU \equiv \angle TJN \Longrightarrow T \in \odot(JMNX) \Longrightarrow \angle MXN = \angle MTN$. But from $ROD \perp PQ$ and $ON \perp BC$, we have $\angle RON = \angle MXN = \angle MTN \equiv \angle RTN \Longrightarrow ORNT$ is cyclic, but $ONTS$ is cyclic on account of the right angles at N and $T \Longrightarrow ORNS$ is cyclic $\Longrightarrow \angle ORS = \angle ONS = 90^\circ \Longrightarrow (RS \parallel PQ) \perp OR$.

XmL

#4 Dec 10, 2015, 4:49 am

Lemma1: Let $CQ \cap BP = X$, let D denote the miquel point of the complete quad. of $AQXP$. Prove that $DO \perp QP$ where O is the circumcenter of ABC.

By definition, $D = (AQC) \cap (APB)$, whose centers are denoted O_C, O_B resp. Also, $DPC \cong DBQ \implies DQ = DC \implies AD$ bisects $\angle A$. Let I be the circumcenter of AQP; it is not hard to show that $IO \parallel AD$. Since IO_COO_B is a parallelogram and $O_CO_B \perp (AD \parallel IO)$, thus AI, DO are symmetric over the perpendicular bisector of AD. Hence OD is parallel to the reflection of AI over AD, which is the A-altitude of QP, and we are done.

Lemma2: $R \in OD$.

Let $(I) \cap (O) = X$ the midpoint of arcs BAC, QAP. Let N, N' denote the antipode of X wrt $(I), (O)$. Then A, N', N, D are collinear. Since $DO \parallel N'X$, it suffices to show $\frac{AM}{RM} = \frac{AN'}{DN'}$.

Define Y the midpoint of BC. Note that D is the midpoint of NN' and KL perpendicularly bisects MY,which is parallel to AN. Let $MY \cap AX = Z$, then $\frac{AM}{2RM} = \frac{ZM}{MY} = \frac{AN'}{2DN'}$, and we are done.

Main proof: From the two lemmas,$RS \parallel QP \iff SR \perp OD \iff R, Y, O, T$ are concyclic. This can be proven easily as $\angle TON = 2\angle RAN = \angle TRY$. $\square$

Quick Reply

High School Olympiads

concyclic midpoints

Reply

Source: generalization of IMO 2009.2

CatalystOfNostalgia
#1 Jul 24, 2009, 11:29 pm • 1

Let ABC be a triangle, with P and Q arbitrary points on CA, AB, respectively. Let PQ meet the circumcircle of ABC at X and Y. Prove that the midpoints of BP, CQ, PQ, and XY are concyclic.

kaka__2004
#2 Jul 25, 2009, 4:24 pm

could you post your solution?
thank you **CatalystOfNostalgia**

does anyone have idea?

math10
#3 Jul 26, 2009, 8:38 am

> CatalystOfNostalgia wrote:
> Let ABC be a triangle, with P and Q arbitrary points on CA, AB, respectively. Let PQ meet the circumcircle of ABC at X and Y. Prove that the midpoints of BP, CQ, PQ, and XY are concyclic.

very nice,but hard.
if this problem was problem IMO 2009, I think the exam would be very interseting
can you post solution?

yetti
#4 Jul 26, 2009, 8:52 am • 1

K, L, M are midpoints of BP, CQ, PQ. A', B', C' are midpoints of BC, CA, AB. Z is midpoint of XY.
$MK \parallel (QB \equiv AB)$ and $LM \parallel (CP \equiv CA) \Longrightarrow \angle LMK = \angle CAB$. $OZB'P, OZC'Q$ are cyclic on account of right angles at B', C', Z. Let $(O_P), (O_Q)$ be their circumcircles. Midlines $A'B', C'A'$ cut $(O_P), (O_Q)$ again at E, F. Perpendicular bisectors OC', OB' of AB, CA cut $(O_P), (O_Q)$ again at S, T. ES, FT cut AB, CA at U, V. $OS \perp PS \Longrightarrow PS \parallel AB \parallel A'B' \Longrightarrow B'PSE, APSU$ are isosceles trapezoids. $\triangle ABS$ is isoscelses with $AS = BS \Longrightarrow BSPU$ is parallelogram, $K \in (SU \equiv SE)$ and similarly, $L \in (TV \equiv TF)$. Then $MKEL$ is iscosceles trapezoid, cyclic, $E \in \odot(MKL)$ and similarly, $F \in \odot(MKL)$.

From $(O_P), (O_Q), \angle PZE + \angle FZQ = \angle CPS + \angle TQB = 2\angle CAB$
From $\odot(MFE), \angle FEM + \angle MFE = \angle FKM + \angle MKE = 2\angle LMK = 2\angle CAB$
$\Longrightarrow \angle EZF = \angle EMF \Longrightarrow Z \in \odot(MFE) \equiv \odot(MKL)$.

plane geometry
#5 Jul 26, 2009, 7:04 pm

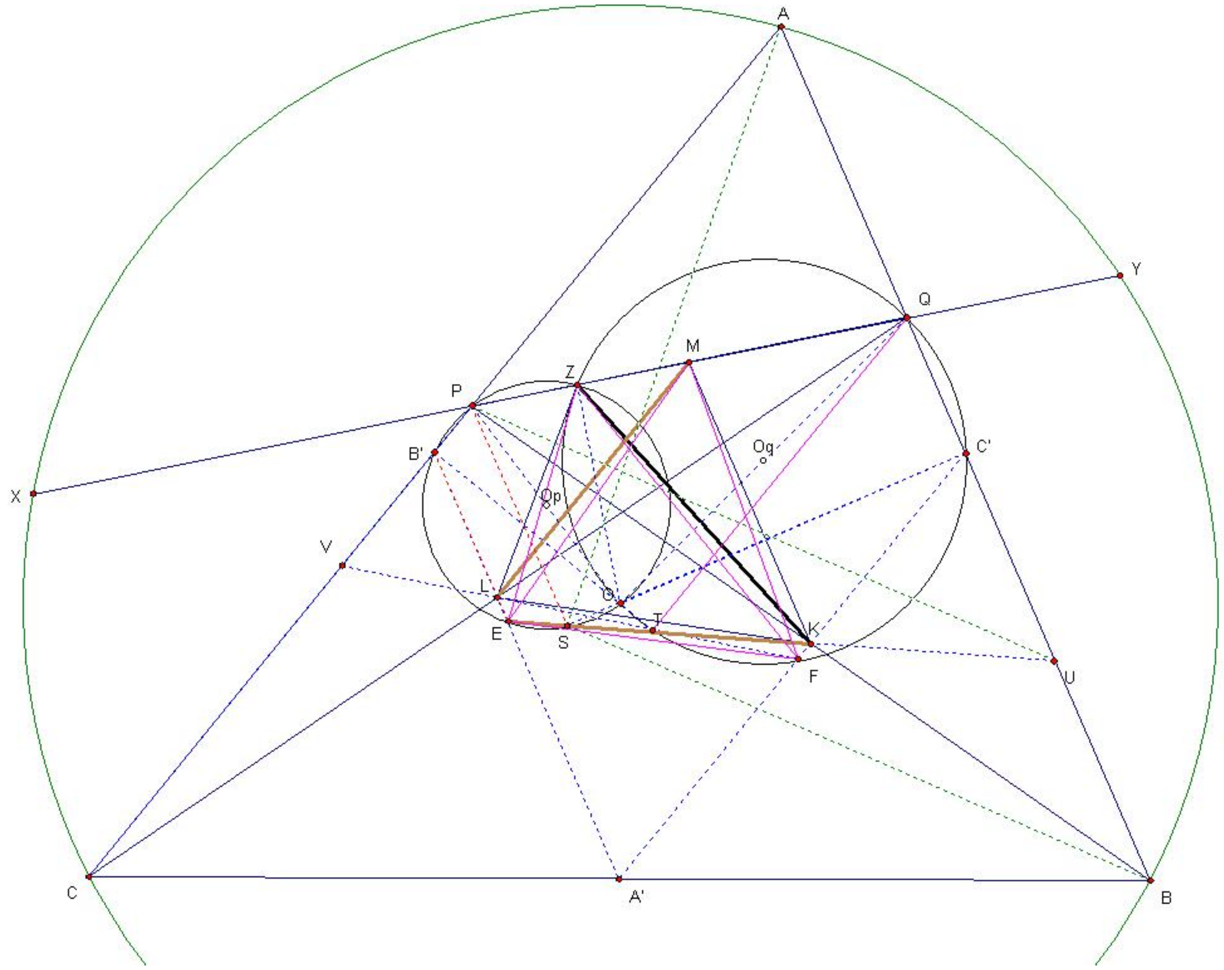
> yetti wrote:
> K, L, M are midpoints of BP, CQ, PQ. A', B', C' are midpoints of BC, CA, AB. Z is midpoint of XY.
> $MK \parallel (QB \equiv AB)$ and $LM \parallel (CP \equiv CA) \Longrightarrow \angle LMK = \angle CAB$. $OZB'P, OZC'Q$ are cyclic on account of right angles at B', C', Z. Let $(O_P), (O_Q)$ be their circumcircles. Midlines $A'B', C'A'$ cut $(O_P), (O_Q)$ again at

E, F. Perpendicular bisectors OC', OB' of AB, CA cut $(O_P), (O_Q)$ again at S, T. ES, FT cut AB, CA at U, V. $OS \perp PS \Longrightarrow PS \parallel AB \parallel A'B' \Longrightarrow B'PSE, APSU$ are isosceles trapezoids. $\triangle ABS$ is isoscelses with $AS = BS \Longrightarrow BSPU$ is parallelogram, $K \in (SU \equiv SE)$ and similarly, $L \in (TV \equiv TF)$. Then $MKEL$ is iscosceles trapezoid, cyclic, $E \in \odot(MKL)$ and similarly, $F \in \odot(MKL)$.

From $(O_P), (O_Q), \angle PZE + \angle FZQ = \angle CPS + \angle TQB = 2\angle CAB$
From $\odot(MFE), \angle FEM + \angle MFE = \angle FKM + \angle MKE = 2\angle LMK = 2\angle CAB$
$\Longrightarrow \angle EZF = \angle EMF \Longrightarrow Z \in \odot(MFE) \equiv \odot(MKL)$.

dear **yetti**, here is a picture for your powerful solution, your solution is very very impressive to me

Attachments:



Jorge Miranda

#6 Jul 27, 2009, 9:06 pm

Lemma: Let R, M, N be the midpoints of XY, CQ, BP. Then $\triangle APQ$ and $\triangle RNM$ are inversely similar.

Proof of the Lemma:
Complex number bash

Let S be the midpoint of PQ. Suppose WLOG that S lies between R and Q. Then we have that $\angle RSN = \angle RQB = \pi - \angle AQP = \pi - \angle RMN$, so $RSNM$ is cyclic, as desired ($SN \parallel QB$ since S, N are the midpoints of PQ, PB).

Altheman

#7 Aug 6, 2009, 12:49 pm

Man this is awesome!!

Anyway, this problem is also a generalization of this classic problem (perhaps old IMO?)

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=218486

Petry

#8 Aug 7, 2009, 4:13 am

Hello!
Nice problem. Here is my solution.

O is the circumcenter of ABC.
K, L, M are the midpoints of BP, CQ, PQ.
Z is the midpoint of XY.
$MK||QB \Rightarrow MK||AB$ (1)
$ML||PC \Rightarrow ML||AC$ (2)
(1), (2) $\Rightarrow \angle KML = \angle BAC$ (3)

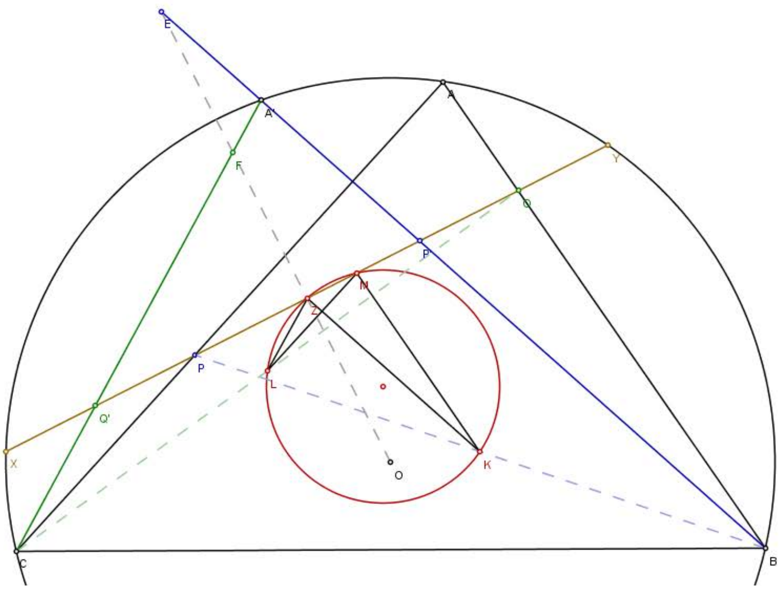
$P', Q' \in XY$ such that $ZP' = ZP$ and $ZQ' = ZQ \Rightarrow P'X = PY, P'Y = PX, Q'X = QY, Q'Y = QX$.
$\{A'\} = BP' \cap (O)$ and $\{Q''\} = A'C \cap XY$.
$(XYPQ) = A(XYPQ) = (XYCB) = A'(XYQ''P') = (XYQ''P') \Rightarrow$
$\Rightarrow (XYPQ) = (XYQ''P') \Rightarrow$
$\Rightarrow \dfrac{PX}{PY} : \dfrac{QX}{QY} = \dfrac{Q''X}{Q''Y} : \dfrac{P'X}{P'Y} \Rightarrow \dfrac{PX}{PY} \cdot \dfrac{P'X}{P'Y} = \dfrac{Q''X}{Q''Y} \cdot \dfrac{QX}{QY} \Rightarrow$
$\Rightarrow \dfrac{Q''X}{Q''Y} \cdot \dfrac{Q'Y}{Q'X} = 1 \Rightarrow \dfrac{Q''X}{Q''Y} = \dfrac{Q'X}{Q'Y} \Rightarrow Q'' = Q'$

$\{E\} = OZ \cap A'B$ and $\{F\} = OZ \cap A'C$
$ZK||P'B \Rightarrow ZK||EB \Rightarrow \angle OZK = \angle A'EF$ (4)
$ZL||Q'C \Rightarrow ZL||FC \Rightarrow \angle OZL = \angle A'FE$ (5)
(4), (5) $\Rightarrow \angle KZL = \angle BA'C \Rightarrow \angle KZL = \angle BAC$ (6)

(3), (6) $\Rightarrow \angle KML = \angle KZL \Rightarrow$ the points K, L, M, Z are concyclic.

Best regards, Petrisor Neagoe

Attachments:



Virgil Nicula

#9 Aug 9, 2009, 2:31 am

A simple equivalent enunciation. Let ABC be a triangle with the circumcentre O. For two points $P \in AC$, $Q \in AB$ consider

the midpoints M, N of the segments $[BP]$, $[CQ]$ respectively and the projection R of O on PQ. Prove that $\widehat{MRN} \equiv \widehat{BAC}$.

SnowEverywhere

#10 Feb 25, 2012, 2:17 am

Let M, M_b, M_c and M_a be the midpoints of XY, BP, CQ and PQ, respectively. Note that $M_aM_b||AB$ and $M_aM_c||AC$.

Therefore $\angle M_bM_aM_c = \angle BAC$ and it suffices to show that $\angle M_bMM_c = \angle BAC$. Now let P' be the reflection of P in M and Q' be the reflection of Q in M. Note that since $BP' \| MM_b$ and $CQ' \| MM_c$, it follows, if BP' and CQ' intersect on the circumcircle Γ of $\triangle ABC$, then by arc-angle theorem $\angle M_bMM_c = \angle BAC$. Therefore it suffices to show that BP' and CQ' meet at a point on Γ.

Let CM and BM meet Γ at R and S, respectively. Now let RQ intersect Γ at T and ST meet AC at P''. It follows by Pascal's theorem that Q, M and P'' are collinear which implies that P'' lies on XY and hence that $P'' = P$. Hence S, P and T are collinear. Now let TM meet Γ at Z and let CZ intersect XY at Q''. By the Butterfly theorem, it follows that $MQ'' = MQ$ since M is the midpoint of XY. Hence $Q' = Q''$ and Z, Q' and C are collinear. By the same argument, B, P' and Z are collinear. Hence BP' and CQ' meet at the point Z on Γ, as desired.

jayme

#11 Feb 25, 2012, 2:23 pm

Dear Mathlinkers,
you can also see my own synthetic proof in English on

http://perso.orange.fr/jl.ayme vol. 10 A generalization of problem 2....

Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Cyclic and the symmetry point is in Euler circle

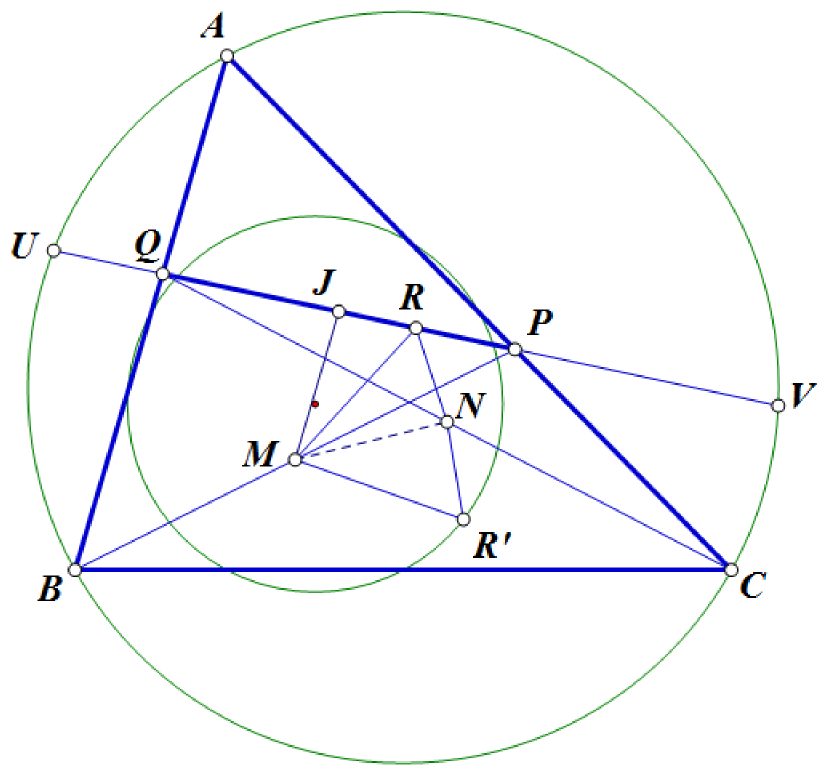
Reply

Love_Math1994

#1 Oct 21, 2011, 12:02 am

Let triangle ABC, circrumcircle (O). UV is a chord of (O), UV cuts AB,AC at Q,P. Denote M,N,J,R is midpoint of BP,CQ,PQ,UV.
a. Prove MNJR is cyclic
b. Prove that the symmetry point of R with mirror MN is lies on Euler circle of ABC
(a appear in a contest in Vietnam and b is i propose when solve a)

Attachments:



Luis González

#2 Oct 21, 2011, 11:51 am • 1

a) X is the reflection of Q about R. CX cuts (O) again at S and BS cuts UV at Y. By Klamkin's theorem (Butterfly theorem extended), R is also midpoint of $PY \Longrightarrow RM \parallel SB$ and $RN \parallel SC \Longrightarrow \angle MRN = \angle BAC = \angle MJN \Longrightarrow$ M, N, R, J are concyclic.

b) D, E, F are the midpoints of BC, CA, AB and P', Q' are the reflections of P, Q about E, F. $(O_1), (O_2)$ are circumcircles of $\triangle APQ, \triangle AP'Q'$ and $(O_0) \equiv \odot(AFOE)$ is the midcircle of $(O_1), (O_2)$. Thus, if (O_1) cuts (O) again at T and $\overline{AT}$ cuts (O_0) at H, then OO_1H is the perpendicular bisector of $\overline{AT}$. Since AO_1OO_2 is a parallelogram with diagonal intersection O_0, then $O_2A \perp TA$, i.e. TA is tangent to (O_2) at $A \Longrightarrow \angle PAT = \angle PQA$. Now, the complement K of T is clearly the 2nd intersection of $\odot(DMN)$ and $\odot(DEF) \Longrightarrow \angle KDM = \angle PQA = \angle QJM = \angle RNM \Longrightarrow K$ is reflection of R about MN, due to $\odot(DMN) \cong \odot(RMN)$.

muathuhanoi

#3 Oct 21, 2011, 4:34 pm • 1

Here is my proof for a) .

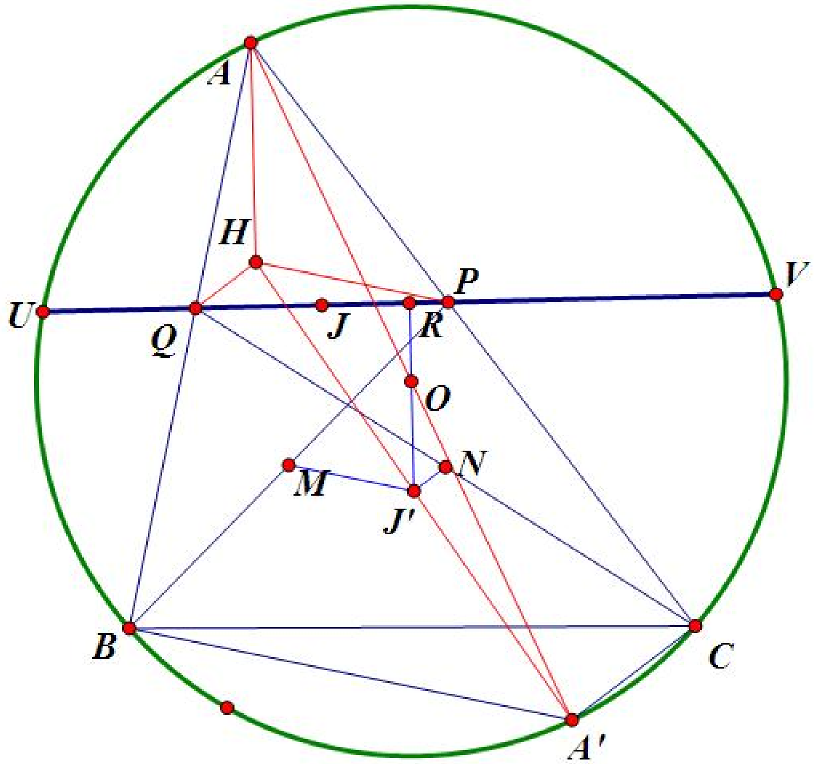
H is the orthocenter of triangle AQP .(O) is circumcircle of ABC
A' is the antipole of A wrt (O) ; J' is the midpoint of HA'

-$HP||BA' \to MJ'||HP$ and because of having $HP \perp AB; JM||AB$, we can imply that :$\angle JMJ' = 90^o$(1)
-Simlaritily , we have :$\angle JNJ' = 90^o$(2)
-O, J' are midpoints of $AA', A'H$ respectively hence :$OJ'||AH \to OJ' \perp QP(Cause AH \perp AQ)$
$\Rightarrow J'O$ passes through $R \to \angle JRJ' = 90^o$ (3)

From (1);(2);(3) we get what we need

b) is pretty obvious when we consider the median triangle of ABC

Attachments:



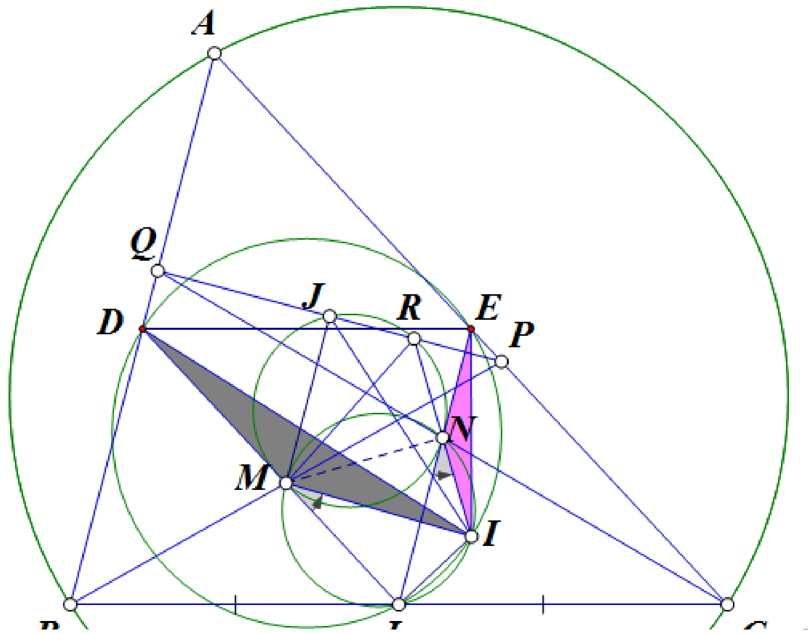
This post has been edited 1 time. Last edited by muathuhanoi, Oct 21, 2011, 5:08 pm

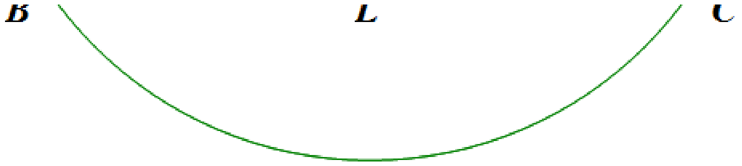
Love_Math1994

#4 Oct 21, 2011, 4:36 pm

Solution of me for b)
You can also see an another proof of a in attachment

Attachments:





IMO P2 generalization(livetolove).PDF (59kb)

skytin

#5 Oct 21, 2011, 10:43 pm • 1

(b) reflect A wrt R' and get point A'
Not hard to prove that PQA ~ BCA' ~ MNR' , so after homotety of (CA'B) with center at A and k = 1/2 we get Euler circle of ABC .
done

TelvCohl

#6 Nov 7, 2014, 9:11 pm

My solution for part (b) :

Let D, E, F be the midpoint of BC, CA, AB, respectively.
Let P', Q' be the reflection of P, Q in E, F, respectively.
Let T be the second intersection of $(AP'Q')$ and (ABC) .
Redefine R' be the second intersection of (DMN) and (DEF).

Let's prove R' is the reflection of R in MN .

Since $\frac{DN}{NE}=\frac{AQ'}{Q'B}, \frac{DM}{MF}=\frac{AP'}{P'C}$,
so we get $\triangle ABC \cap T \cap P' \cap Q' \sim \triangle DEF \cap R' \cap M \cap N$.
Since $\angle TBQ'=\angle TCP', \angle TQ'A=\angle TP'A$,
so we get $\triangle TBQ' \sim \triangle TCP'$ and $\frac{AP}{AQ}=\frac{CP'}{BQ'}=\frac{TP'}{TQ'}$,
combine with $\angle Q'TP'=\angle QAP$ we get $\triangle TP'Q' \sim \triangle APQ$.

From part (a) we get M, N, R, J are concyclic ,
so we get $\angle NMR=\angle NJR=\angle APQ=\angle TP'Q'=\angle R'MN$ . ... (1)
Since $\angle MRN=\angle BAC=\angle NDM$,
so (MNR) and (MNR') are congruent . ... (2)

From (1) and (2) we get R' is the reflection of R in MN .

Q.E.D

Quick Reply

High School Olympiads

Tangent to fixed circle

Reply

Source: Own

livetolove212
#1 Dec 5, 2015, 5:44 pm

Given triangle ABC inscribed in (O). Let AD be the A-bisector, M be the midpoint of BC. Line through D and perpendicular to BC intersects AM at J. Prove that when A moves on arc BC of (O), circle (J, JD) is always tangent to a fixed circle.

Inspired from topic http://www.artofproblemsolving.com/community/c6t48f6h1155918_geometry

TelvCohl
#2 Dec 5, 2015, 7:44 pm

Let $\odot(L)$ be the circle passing through B, C and tangent to $\odot(J)$. Let K be the intersection of the tangent of $\odot(O)$ passing through B, C, respectively. Let Φ be Inversion $\mathbf{I}(A, \sqrt{AB \cdot AC})$ followed by reflection in AD and denote P^* as the image of P (arbitrary point) under Φ. Since JD is tangent to the A-apollonius circle $\mathcal{A}_a$ of $\triangle ABC$ at D, so $\odot(J) \perp \mathcal{A}_a \Longrightarrow$ the center of the image of $\odot(J)$ under Φ lies on the A-symmedian AK of $\triangle ABC$ and perpendicular bisector of BC (image of $\mathcal{A}_a$ under Φ), hence the image of $\odot(J)$ under Φ is the circle with center K passing through the midpoint of arc BC in $\odot(O)$.

Since the image of $\odot(L)$ under Φ is the circle (different from $\odot(O)$) passing through B, C and tangent to $\odot(K)$ (fixed), so its center is a fixed point on the perpendicular bisector of $BC \Longrightarrow L$ is fixed, hence $\odot(J)$ is tangent to a fixed circle $\odot(L)$.

Luis González
#3 Dec 7, 2015, 4:28 am

Let U and V be the midpoints of the arcs BC and BAC of (O), resp and let $P \in (J)$, such that $\odot(PBC)$ is tangent to (J). AV cuts BC at E and F is the antipode of D on (J). Clearly AF cuts UM at the reflection K of U on M.

Since $PD \perp PF$ bisect $\angle BPC$ internally and externally (well-known), it follows that P is on the A-Apollonius circle of $\triangle ABC$ with diameter $\overline{DE}$, so $\overline{EPF}$ passes through the midpoint S of the arc BPC, obviously on UV. Consequently if $T \equiv FD \cap AV$, we have $\frac{SM}{SV} = \frac{FD}{FT} = \frac{KU}{KV} = \text{const} \Longrightarrow S$ is fixed $\Longrightarrow \odot(PBCS)$ is fixed and the conclusion follows.

XmL
#4 Dec 8, 2015, 7:59 am

This solution provides more info about the fixed circle.

Define N the midpoint of arc BC that doesn't contain A. Let the A-apollonius circle (K, KD) intersect (J, JD) at P, Let $PD \cap MN = X$. Since $(PXBC)$ is tangent to (J), it suffices to show that X is fixed.

Let D' is the antipode of D in (K). Since $MDX \sim DJK, DMJ \sim AND'$, then
$MX = \frac{DD'}{2}\frac{DM}{JD} = \frac{DD'}{2}\frac{AN}{AD'} = \frac{AN \cdot DN}{2MN} = R$, the circumradius of (ABC), thus X is fixed.

Quick Reply

High School Olympiads

Isogonal conjugate and Cevian quotient

Source: Own?

A-B-C

#1 Nov 28, 2015, 7:23 pm • 2

Denote P^* is isogonal conjugate of P WRT $\triangle ABC$.
P/Q is cevian quotient of P, Q.
Prove that $P, (P/P^*), (P/P^*)^*$ are collinear.

TelvCohl

#2 Dec 4, 2015, 1:13 am • 1

Lemma : Given a $\triangle ABC$ with incenter I. Let P, Q be the isogonal conjugate WRT $\triangle ABC$. Then the Cevapoint R of I and P WRT $\triangle ABC$ (i.e. Cevian product of I, P WRT $\triangle ABC$) lies on PQ.

Proof : Let I_a, I_b, I_c be the A-excenter, B-excenter, C-excenter of $\triangle ABC$, respectively. Since R is the Crosspoint of I and P WRT $\triangle I_aI_bI_c$, so R is the pole of the circumconic $\mathcal{H}$ of $\triangle I_aI_bI_c$ passing through I and P (See the generalization discussed in the topic Schwatt's lines), hence PR is tangent to $\mathcal{H}$ at P. Since the polar of P WRT any conic passing through I, I_a, I_b, I_c passes through Q (well-known), so we conclude that P, Q, R lie on a line which is tangent to $\mathcal{H}$.

Back to the main problem :

Let I, I_a, I_b, I_c be the incenter, A-excenter, B-excenter, C-excenter of $\triangle ABC$, respectively. Let J be the Cevapoint of I and P^* WRT $\triangle ABC$. Let $\triangle P_a^* P_b^* P_c^*$ be the anticevian triangle of P^* WRT $\triangle ABC$. Since P, J and P^* are collinear (**Lemma**), so $P^*, (J/P^*) \equiv I, (P/P^*)$ lie on a circumconic $\mathcal{C}$ of $\triangle P_a^* P_b^* P_c^*$, hence notice $\mathcal{C}$ also passes through the vertices I_a, I_b, I_c of the anticevian triangle of I WRT $\triangle ABC$ we get $(P/P^*)(P/P^*)^*$ is tangent to $\mathcal{C}$ at (P/P^*).

On the other hand, Since P is the Crosspoint of $P^*, (P/P^*)$ WRT $\triangle P_a^* P_b^* P_c^*$, so we get P is the pole of $P^*(P/P^*)$ WRT $\mathcal{C}$ $\Longrightarrow P(P/P^*)$ is tangent to $\mathcal{C}$ at (P/P^*), hence we conclude that $P, (P/P^*), (P/P^*)^*$ are collinear.

Luis González

#3 Dec 4, 2015, 2:26 am

> TelvCohl wrote:
> **Lemma :** Given a $\triangle ABC$ with incenter I. Let P, Q be the isogonal conjugate WRT $\triangle ABC$. Then the Cevapoint R of I and P WRT $\triangle ABC$ (i.e. Cevian product of I, P WRT $\triangle ABC$) lies on PQ.

For a generalization of this lemma see Concurrent on Euler line (post #4). So using that generalization and then proceeding exactly as Telv did, we have the following generalization of the original problem:

P, Q are arbitrary points on the plane of $\triangle ABC$. $\triangle Q_AQ_BQ_C$ is the anticevian triangle of Q WRT $\triangle ABC$ and P^* is the cevian quotient Q/P WRT $\triangle Q_AQ_BQ_C$. Then the cevian quotient $R \equiv P/P^*$ WRT $\triangle ABC$, the cevian quotient $S \equiv Q/R$ WRT $\triangle Q_AQ_BQ_C$ and P are collinear.

TelvCohl

#4 Apr 17, 2016, 1:33 pm • 2

We can prove the stronger result as following :

Given a $\triangle ABC$ and a point P. Then a point U lies on the Pivotal Isogonal cubic with pivot P if and only if the cevian quotient P/U of P and U WRT $\triangle ABC$ lies on it.

P/U of P and U WRT $\triangle ABC$ lies on it.

Proof : Actually, this simplified follows from Property of Cevian quotient (post #4) ... $(\star)$. Let $P^*, U^*, (P/U)^*$ be the isogonal conjugate of $P, U, P/U$ WRT $\triangle ABC$, respectively. Then

$$P, U, U^* \text{ are collinear } . \stackrel{(\star)}{\Longleftrightarrow} U, P^*, P/U \text{ are collinear } . \stackrel{(\star)}{\Longleftrightarrow} P, P/U, (P/U)^* \text{ are collinear } .$$

This post has been edited 1 time. Last edited by TelvCohl, Apr 17, 2016, 6:04 pm

Quick Reply

High School Olympiads

Concurrent on Euler line

Reply

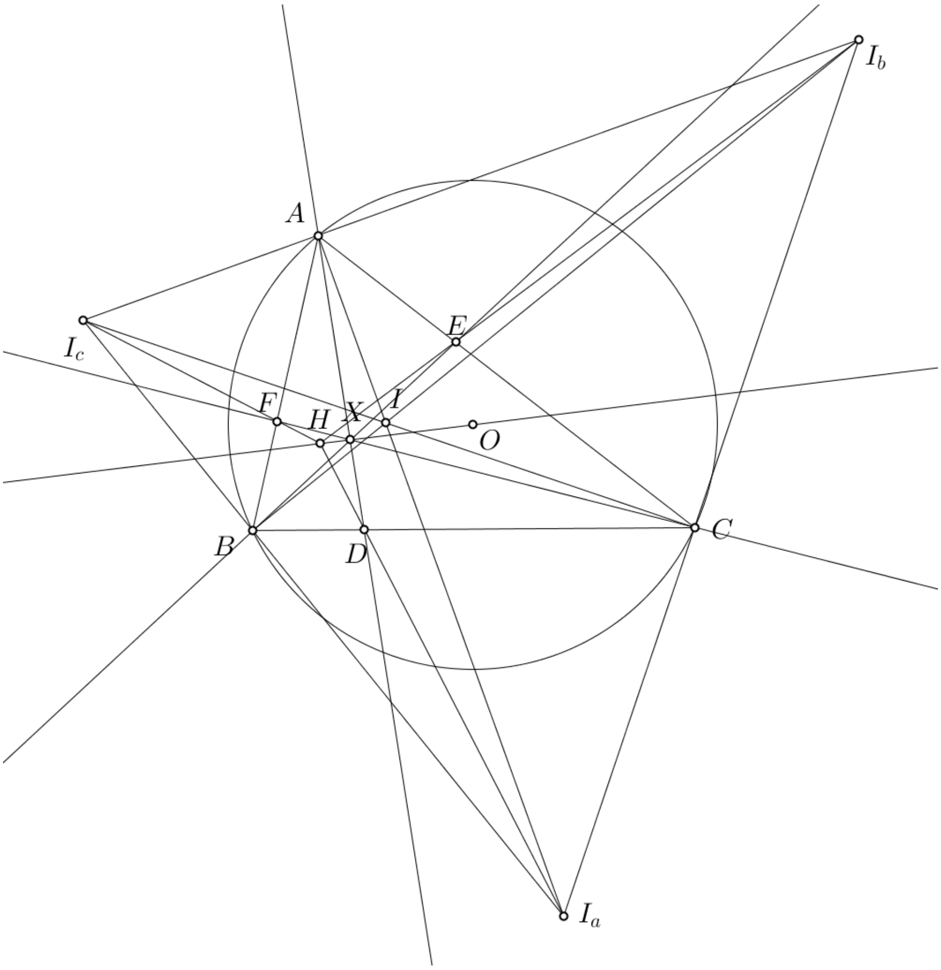
Source: Own???

A-B-C

#1 Sep 18, 2015, 7:15 pm • 1

$\triangle ABC$. I, I_a, I_b, I_c are incenter and excenters.
O, H are circumcenter and orthocenter.
HI_a, HI_b, HI_c intersect BC, CA, AB at D, E, F, resp.
Prove that AD, BE, CF, OH are concurrent.

Attachments:



Luis González

#2 Sep 19, 2015, 5:07 am • 1

We can generalized the problem as follows:

Let I, O, H be the incenter, circumcenter and orthocenter of $\triangle ABC$. $\triangle I_aI_bI_c$ is the excentral triangle and P is a point on the plane. PI_a, PI_b, PI_c cut BC, CA, AB at D, E, F. It's well-known that AD, BE, BF concur at Q; namely the Ceva-point of I and P. Now Q lies on the Euler line of $\triangle ABC \iff P$ is on the conic through I_a, I_b, I_c, O, H.

This is a consequence of the following known facts:

1) If $\triangle A'B'C'$ is anticevian triangle of X WRT $\triangle ABC$ and P is a variable point on a fixed conic though A', B', C', the locus

of the Ceva-point Q of X and P WT $\triangle ABC$ is a line.

2) I_a, I_b, I_c, O, H and the Mittenpunkt M of $\triangle ABC$ lie on a same conic $\mathcal{C}$ (to be proved). When $P \equiv M$, then Q is the centroid G of $\triangle ABC$ and when $P \equiv O$, then Q is the Schiffler point S of $\triangle ABC$ (well-known). Thus Q is on Euler line GS for any P on $\mathcal{C}$.

buratinogigle

#3 Sep 19, 2015, 10:33 am • 1

Actually, this problem is true for two isogonal conjugate points

Let ABC be a triangle with P, Q are two isogonal conjugate points. $\triangle I_aI_bI_c$ is the excentral triangle and R is a point on the plane. RI_a, RI_b, RI_c cut BC, CA, AB at D, E, F. Prove that AD, BE, CF, PQ are concurrent iff R lie on the conic through I_a, I_b, I_c, P, Q.

Luis González

#4 Sep 19, 2015, 10:47 am • 5

Here is a stronger projective generalization:

P, Q are arbitrary points on the plane of $\triangle ABC$. $\triangle P_AP_BP_C$ is the cevian triangle of P and $\triangle Q_AQ_BQ_C$ is the anticevian triangle of Q. X is the cevian quotient P/Q WRT $\triangle ABC$ and Y is the Ceva-point of P, Q WRT $\triangle P_AP_BP_C$. Then X, Y, Q are collinear.

Proof: Let AQ, BQ, CQ cut P_BP_C, P_CP_A, P_AP_B at A', B', C', resp. $U \equiv P_BP_C \cap Q_BQ_C, V \equiv P_CP_A \cap Q_CQ_A$, $W \equiv P_AP_B \cap Q_AQ_B$ are collinear on the perspectrix of $\triangle P_AP_BP_C$ and $\triangle Q_AQ_BQ_C$.
$A(B, C, Q, Q_B) = A(P_C, P_B, A', Q_B) = (P_C, P_B, A', U) = -1$, thus since $\triangle A'B'C'$ is cevian triangle of Y WRT $\triangle P_AP_BP_C$, it follows that $U \in B'C'$ and similarly we have $V \in C'A'$ and $W \in A'B' \Longrightarrow \triangle P_AP_BP_C, \triangle Q_AQ_BQ_C$ and $\triangle A'B'C'$ are perspective through the same perspectrix $\overline{UVW}$. Thus by the 3 homologies theorem, their perspectors X, Y, Q are collinear. $\blacksquare$

When P coincides with the orthocenter of $\triangle ABC$, then Q and X become isogonal conjugates WRT $\triangle P_AP_BP_C$ and we get the following corollary:

I and I_a, I_b, I_c are the incenter and three excenters of $\triangle ABC$. If P, Q are isogonal conjugates WRT $\triangle ABC$, then the Ceva-points of I, P and I, Q lie on PQ.

Quick Reply

High School Olympiads

Legendary Four Circles Problem

amplreneo

#1 Dec 2, 2015, 6:17 am

Circles k_1, k_2, and k_3 are tangent pairwise, and each is tangent to a line l. A fourth circle k is tangent to k_1, k_2, k_3, so that k and l do not intersect. Find the distance d from the center of k to l if the radius of k equals 1.

Luis González

#2 Dec 3, 2015, 10:45 pm • 2

Let O_i, r_i denote the center and radius of k_i and let O, r denote the center and radius of k. k_1, k_2, k_3 touch l at A, B, C (C between A and B), k_3 touches k_1, k_2 externally at X, Y and k_1, k_2 are externally tangent at Z. We shall evaluate the ratio d/r.

Let R be the radical center of $k_1, k_2, k_3 \Longrightarrow \odot(XYZ) \equiv \odot(R, \varrho)$ is their radical circle. RY is the radical axis of k_2, k_3 cutting BC at its midpoint M. Clearly M is the center of $\odot(BCY)$ and O_2O_3 is tangent of this circle cutting BC at the exsimilicenter H of $k_2 \sim k_3$. Hence we have

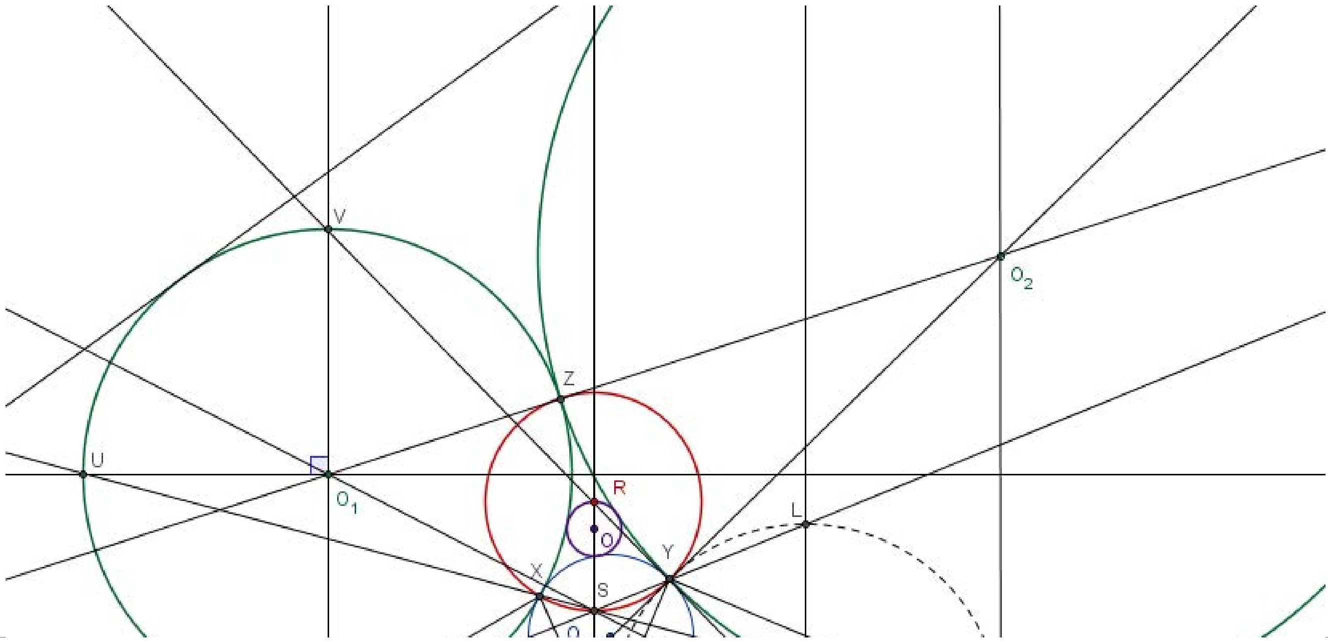
$$\frac{r_2}{r_3} = \frac{HB}{HC} = \frac{YB^2}{YC^2} \Longrightarrow \frac{YB}{YC} = \frac{\sqrt{r_2}}{\sqrt{r_3}} = \frac{4\sqrt{r_1 \cdot r_2}}{4\sqrt{r_1 \cdot r_3}} = \frac{AB}{AC},$$

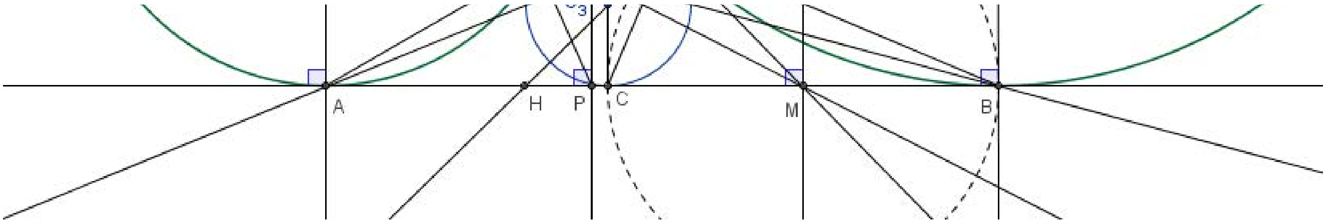
which means that YA is external bisector of $\angle BYC$, thus passing through the midpoint L of the arc BYC. Since Y is the insimilicenter of $\odot(XYZ)$ and $\odot(BYC)$, then $\overline{AYL}$ cuts $\odot(XYZ)$ again at S, such that $RS \parallel ML \Longrightarrow RS \perp AB$ at P. Analogously XB bisects $\angle AXP$ externally and $S \in BX$. Thus if BX cuts k_1 again at U, then $\angle AO_1U = 2 \cdot \angle AXU = 2 \cdot 45^\circ = 90^\circ \Longrightarrow O_1U \parallel AB$. Since $O_1U \parallel MB$ and $O_1A \parallel ML$, then we deduce that $S \equiv BU \cap AL$ is the insimilicenter of $k_1 \sim \odot(BYC) \Longrightarrow M, S, O_1$ are collinear.

On the other hand, let V be the farthest intersection $RY \cap k_1$ from AB. Inversion with center V and radius VY fixes k_2, k_3 and by conformity it swaps k_1 and $AB \Longrightarrow VO_1 \perp AB$, i.e. V is antipode of A on k_1. Thus since O_1 is midpoint of AV, then S is midpoint of $RP \Longrightarrow RP = 2\varrho$. Now, Inversion WRT $\odot(R, \varrho)$ fixes k_1, k_2, k_3 and by conformity it swaps k and $l \Longrightarrow R \in k$ and $RO \perp l$, i.e. $O \in RP$. Thus by inversion property we obtain

$$\varrho^2 = 2r \cdot RP = 2r \cdot 2\varrho \Longrightarrow \varrho = 4r \Longrightarrow d = RP - r = 8r - r = 7r \Longrightarrow \frac{d}{r} = 7.$$

Attachments:





amplreneo

#3 Dec 26, 2015, 10:53 am • 1

Solution

Quick Reply

High School Olympiads

concurrent lines

Reply

henderson
#1 Dec 2, 2015, 11:43 pm

Let ABC be a triangle with orthocenter H and circumcenter O. The parallels through B and C to AO intersect the external angle bisector of $\angle BAC$ at D and E, respectively. Prove that BE, CD, AH are concurrent.

Luis González
#2 Dec 3, 2015, 2:07 am

Let AH cut $\odot(ABC)$ again at X. $\angle ECA=\angle OAC=\angle BAX=\angle BCX$ and similarly $\angle DBA=\angle CBX$. Since $\angle CAE=\angle BAD$, then by Jacobi's theorem, it follows that BE, CD and $AX\equiv AH$ concur.

PROF65
#3 Dec 3, 2015, 3:50 am

Let Z ,H' and O' be the feet of the external angle bisector, the feet of the A-altitude and the intersection of BC with AO resp.
$(D,E;A,Z)=(B,C;O',Z)=A(B,C;O',Z)\overset{(1)}{=}A(B,C;H',Z)=(B,C;H',Z)\overset{(2)}{=}(E,D;H'',Z)$ then $H''=A$.
(1) by using the reflection in the internal bisector .
(2) by using the P-projection where P is the intersection of BE and CD
WCP

FabrizioFelen
#4 Dec 3, 2015, 6:16 am

My solution:
Let $X=BE\cap CD$ and $\angle ABC=2\alpha$, $\angle ACB=2\beta\Longrightarrow\angle ACE=90^{\circ}-2\alpha$ and $\angle ABD=90^{\circ}-2\beta\Longrightarrow$ By law of sines in $\triangle ABE$ and $\triangle BEC$ we get: $\frac{\sin\angle ABX}{\sin\angle CBX}=\frac{\cos 2(\alpha-\beta)}{\cos 2\alpha}...(1)$ similarly by law of sines in $\triangle ADC$ and $\triangle DBC$ we get: $\frac{\sin\angle XCB}{\sin\angle XCA}=\frac{\cos 2\beta}{\cos 2(\alpha-\beta)}...(2)\Longrightarrow$ By Ceva's theorem in $\triangle ABC$ and $...(1)$ and $...(2)$ we get:
$\frac{\sin\angle XAB}{\sin\angle XAC}=\frac{\cos 2\alpha}{\cos 2\beta}$ and $\angle XAB+\angle XAC=180^{\circ}-2\alpha-2\beta\Longrightarrow\angle XAB=90^{\circ}-2\alpha$ and $\angle XAC=90^{\circ}-2\beta\Longrightarrow X\in AH\Longrightarrow BE, CD, AH$ are concurrent...

This post has been edited 1 time. Last edited by FabrizioFelen, Dec 3, 2015, 6:20 am
Reason: ops

Quick Reply

High School Olympiads

Thebault Circle□Tangent

Reply

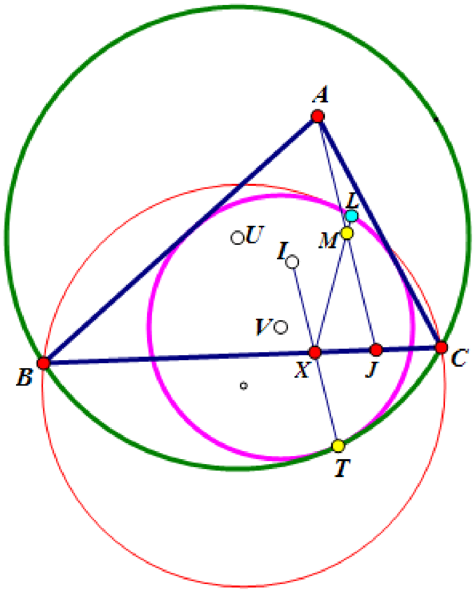
Lin_yangyuan

#1 Dec 2, 2015, 11:30 pm

Thebault Circle

Attachments:

Given ΔABC and $\odot U$ passing through B,C.Let $\odot V$ be the circle tangent to AB,AC and tangent to $\odot U$ at T.I is the incentre of ΔABC, $AJ \parallel IT$, M is the midpoint of AJ, $IT \cap BC = X$, $XM \cap \odot V = L$. Prove that $\odot(BLC)$ is the circle tangent to $\odot V$.



Luis González

#2 Dec 3, 2015, 1:15 am

Redefine L such that $\odot(LBC)$ is tangent to (V). Thus we'll prove that X, M, L are collinear.

Common tangent of $(U), (V)$, common tangent of $(V), \odot(LBC)$ and BC concur at the radical center R of $\odot(LBC)$, $(U), (V)$. Since TXI bisects $\angle BTC$ (well-known), then $(R) \equiv \odot(R, RX)$ is the T-Apollonius circle of $\triangle TBC$ and $L \in (R)$. If Y is the antipode of X on (R) and (V) touches AC, AB at B', C', then it's known that $Y \in B'C'$ (see Internally tangent circles and lines and concurrency). Thus since $A(B, C, X, Y) = -1$, it follows that AX is the polar of Y WRT (V) $\Longrightarrow AX \perp YV$ at Z and $Z \in (R)$. Now since $TZLY$ is hamonic (as the tangents of (R) at T, L meet at V), we get $X(Z, Y, T, L) \equiv X(A, J, I, L) = -1 \Longrightarrow M \in XL$, as desired.

TelvCohl

#3 Dec 3, 2015, 1:29 am

Redefine L as the point such that $\odot(BLC)$ is tangent to $\odot(V)$ at L. We prove that M lies on LX. Since $\odot(V)$ is tangent to $\odot(U), \odot(BLC)$ at T, L, resp, so the intersection K of the tangent of $\odot(U)$ passing through T and the tangent of $\odot(BLC)$ passing through L lies on BC. Since TI is the bisector of $\angle CTB$ (see incenter of triangle), so the circle with center K and radius $KX = KT = KL$ is the T-apollonius circle of $\triangle TBC \Longrightarrow LX$ is the bisector of $\angle BLC$, hence LX passes through the A-excenter I_a of $\triangle ABC \Longrightarrow X(J, A; L, T) = -1 \Longrightarrow M \in LX$.

Quick Reply

High School Olympiads

Internally tangent circles and lines and concurrency

Reply

swaqar

#1 Mar 17, 2011, 7:53 pm

Given a triangle ABC with cirumcirlce Ω . Let D be a point on segment BC and let ω be a circle tangent to the rays DA and DC at points E and F and internally tangent to Ω at K . Let M be the midpoint of arc CA which doesn't contain B . Show that EF , AC and MK are concurrent .

Luis González

#2 Mar 18, 2011, 1:11 am

By Sawayama's lemma, we know that EF passes through the incenter I of $\triangle ABC.$ Since KF bisects $\angle BKC$ internally, it follows that KF passes through the midpoint N of the arc BC of $\Omega.$ By Pascal theorem for the nonconvex cyclic hexagon $MKNACB,$ the intersections $P \equiv MK \cap AC, F \equiv KN \cap BC$ and $I \equiv NA \cap BM$ are collinear $\Longrightarrow$ Lines $IF \equiv EF, AC$ and MK concur at $P.$

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High School Olympiads

Geometry problem

Locked

Source: Hong Kong TST 2 IMO 2016 Problem 2

Math_CYCR

#1 Dec 2, 2015, 9:44 pm

Let Γ be a circle and AB a diameter. Let l be a line outside the circle, and is perpendicular to AB. Let X, Y be two points on l. If X' and Y' are two points on l such that AX and BX' intersect on Γ and such that AY and BY' intersect on Γ, prove that the circumcircles of the triangles AXY and $AX'Y'$ intersect at a point on Γ other than A, or the three circles are tangent to A.

Luis González

#2 Dec 2, 2015, 10:55 pm

Please use the search before posting contest problems. This was recently posted at Hong Kong TST2 2016 P2.