High School Olympiads

4 points are concyclic

Reply

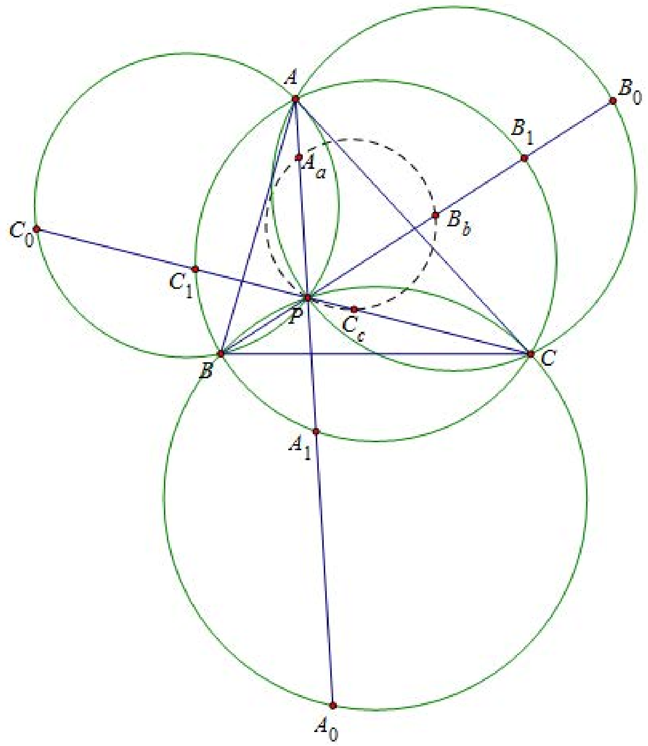
Source: OWN

LeVietAn

#1 Sep 23, 2015, 1:29 pm

Dear Mathlinkers,
Let P is a point which is not on the lines through BC, CA, AB and circumcirle of triangle ABC. The lines PA, PB, PC respectively intersect the circles $(PBC), (PCA), (PAB)$ again at A_0, B_0, C_0 . And the lines PA, PB, PC respectively intersect the circle (ABC) again at A_1, B_1, C_1 . Let A_a, B_b, C_c respectively be the reflections of A_0, B_0, C_0 across A_1, B_1, C_1 . Prove that A_a, B_b, C_c and P are concyclic.

Attachments:



TelvCohl

#2 Sep 23, 2015, 4:30 pm • 1

Let $\triangle DEF$ be the antipedal triangle of P WRT $\triangle ABC$. Let O be the circumcenter of $\triangle ABC$ and let A^*, B^*, C^* be the antipode of A, B, C in $\odot(O)$, respectively. Let D_1 be the reflection of D in A^* and define E_1, F_1 in similar way. Let $D_d \equiv D_1O \cap AD, E_e \equiv E_1O \cap BE, F_f \equiv F_1O \cap CF$ and let G be the Centroid of $\triangle DEF$.

From Menelaus theorem ($\triangle ADA^*$ and $\overline{D_1OD_d}$) we get $DD_d : D_dA = 2 : 1$, so $GD_d \parallel EF \Longrightarrow GD_d \perp AP$. On the other hand, from Menelaus theorem ($\triangle D_1A^*O$ and $\overline{AD_dD}$) $\Longrightarrow OD_1 : D_dO = 3 : 1$, so $D_1G^* \perp AP$ where G^* is the image of G under the homothety $\mathbf{H}(O, -3) \Longrightarrow A_aG^* \perp AP$ (notice $A_1A^* \perp AP$ and $A_0D \perp AP$), hence we get A_a lie on the circle ω with diameter PG^*. Similarly, we can prove B_b, C_c lie on $\omega \Longrightarrow A_a, B_b, C_c, P$ are concyclic.

Luis González

#3 Sep 24, 2015, 3:03 am • 1

Inverting with center P we get the following projective problem: P is arbitrary point on the plane of $\triangle ABC$. $\triangle A_0B_0C_0$ and

$\triangle A_1B_1C_1$ are the cevian and circumcevian triangle of P, resp. A_a lies on PA such that $(A_0, P, A_1, A_a) = -1$ and B_b, C_c are defined similarly. Then A_a, B_b, C_c are collinear and moreover $\overline{A_aB_bC_c}$ is the perspectrix of $\triangle A_1B_1C_1$ and $\triangle A_0B_0C_0$.

Let $D \equiv A_1C_1 \cap BC, E \equiv A_1B_1 \cap BC, F \equiv B_1A_1 \cap CA, G \equiv B_1C_1 \cap CA, H \equiv C_1B_1 \cap AB,$ $I \equiv C_1A_1 \cap AB.$ Considering a homology that sends P into the center of the conic image of $\odot(ABC)$, we get by symmetry that $P \equiv DG \cap EH \cap IF$ is the symmetry center of $\triangle ABC$ and $\triangle A_1B_1C_1$. Further $\triangle B_0B_1F, \triangle C_0C_1I$ are perspective through $\overline{APA_1}$, therefore $X \equiv B_0C_0 \cap B_1C_1 \cap IF$ and similarly $Y \equiv C_0A_0 \cap C_1A_1 \cap EH$ and $Z \equiv A_0B_0 \cap A_1B_1 \cap DG.$ Now from the complete $IHYX$, we get $X(C_0, P, C_1, Y) = -1 \Longrightarrow C_c \in \overline{XYZ}$ and likewise A_a and B_b fall on $\overline{XYZ}.\blacksquare$

Remark: Inverting back we get that the circle through P, A_a, B_b, C_c also passes through $A_3 \equiv \odot(PB_0C_0) \cap \odot(PB_1C_1),$ $B_3 \equiv \odot(PC_0A_0) \cap \odot(PC_1A_1)$ and $C_3 \equiv PA_0B_0 \cap \odot(PA_1B_1)$, all distinct from P.

Quick Reply

High School Olympiads

Euler Line and the Feuerbach Point

Reply

JuanOrtiz

#1 Aug 7, 2013, 6:47 am

Let ABC be a triangle. Consider L the line parallel to BC through A. The tangency points of the incircle of triangle ABC with the sides BC, CA and AB are D, E and F, respectively. Let P and Q be the intersections of DE and DF with L. Prove that the Euler line of triangle DPQ passes through the Feuerbach point of triangle ABC.

NOTE: It is known that the nine-point circle and the incircle of a triangle are tangent. This point of tangency is called the Feuerbach point. It is also known that the orthocenter, nine-point center, circumcenter and the centroid of a triangle are collinear. This line is called the Euler line of the triangle.

XmL

#2 Aug 7, 2013, 11:10 am

Here's my solution to this nice problem:
First of all let H, O be the orthocenter,circumcenter of DPQ. Thus OH is the Euler line of DPQ. Note that since $\angle AQD = \angle BDF = \angle BFD = \angle QFA$, we have $AQ = AF$. Similarly we can obtain $AE = AP$, thus $AQ = AF = AE = AP$ and $\angle QEP = \angle PFQ = 90$. Hence QE, PF intersect at H, which is on incircle (I)(since D, E, H, F are concyclic). Since $DH \perp BC$, thus I is the midpoint of HD. Now let K be the Feuerbach point, thus K is on both the incircle and the nine point circle of ABC, now we are sufficed to prove K, O, H are collinear.

Since $AO \perp BC$, let it meet BC at L. Denote M as the midpoint of BC, thus L, M are on the nine-point circle. Since K is the tangency of both circles. thus by their homothety transformation we have KD bisects $\angle LKM$. After some angle chasing we can reveal that $\angle KLM = \angle B - \angle C = 2\angle OAI$, which means that $\angle DKL = \angle OAI$. In $\triangle PQD$, it's well known that $AODI$ is a parallelogram, thus $\angle KDL = \angle LAI = \angle LOI$, which means that K, L, D, O are concyclic $\Rightarrow$ $\angle OKD = \angle OLD = 90$. Since $\angle DKH = 90$, hence K, O, H are collinear. Q.E.D

Btw: It's an honor to answer a question for an IMO silver medalist!

jayme

#3 Aug 7, 2013, 2:30 pm

Dear Mathlinkers,
for history and more, you can have a look to
http://forumgeom.fau.edu/FG2006volume6/FG200621.pdf
incerely
Jean-Louis

Luis González

#4 Sep 23, 2015, 11:27 am

The antipode X of D WRT (I) is clearly orthocenter of $\triangle DPQ$ and A is midpoint of $PQ.$ Therefore the midpoint M of $\overline{IA}$ is 9-point center of $\triangle DPQ \Longrightarrow XM$ is Euler of $\triangle DPQ.$ Now according to problem Intersect on circle (post #4), XM cuts (I) again at the Poncelet point of $ABCI,$ i.e. the Feuerbach point of $\triangle ABC.$

Quick Reply

High School Olympiads

MacBeath circumconic of X2X13X14 pass through X110

Reply

Source: discovered by rodinos

TelvCohl

#1 Mar 13, 2015, 1:35 am • 1

Let G be the Centroid of $\triangle ABC$.
Let T be the Kiepert focus of $\triangle ABC$.
Let F_1 be the 1st Fermat point of $\triangle ABC$.
Let F_2 be the 2nd Fermat point of $\triangle ABC$.

Prove that T lie on the MacBeath circumconic of $\triangle GF_1F_2$

Luis González

#2 Sep 23, 2015, 9:15 am • 2

Let H and N be the orthocenter and 9-point center of $\triangle ABC$. F_1F_2 cuts $\overline{HG}$ at its midpoint $D,$ thus N is midpoint of $\overline{GD}.$ Recalling the previous problem X2, X5, X13, X14, X110, N,T are isogonal conjugates WRT $\triangle GF_1F_2$ and from Collinear Points (Hyacinthos #22972), we deduce that the MacBeath circumconic of $\triangle GF_1F_2,$ centered at its symmedian point, is the isogonal of the trilinear polar of its orthocenter, i.e. isogonal of its orthic axis. Thus it's enough to show that N is on the orthic axis of $\triangle GF_1F_2.$

Let G_1,G_2 be the midpoints of $GF_1,GF_2 \Longrightarrow G_1G_2$ is G-midline of $\triangle GF_1F_2$ cutting the cevian $\overline{GD}$ at its midpoint $N.$ Since $G_1G_2 \parallel F_1F_2,$ then $\odot(GG_1G_2)$ and $\odot(GF_1F_2)$ are tangent and since GN is tangent to $\odot(GF_1F_2) \Longrightarrow$ $NG^2=NG_1 \cdot NG_2 \Longrightarrow N$ has equal power WRT the circumcircle of $\triangle GF_1F_2$ and its 9-point circle, in other words, N is on orthic axis of $\triangle GF_1F_2,$ as desired.

Quick Reply

High School Olympiads

Collinear Points

Reply

Source: Hyacinthos #22972

rodinos
#1 Jan 5, 2015, 3:52 am

Let ABC be a triangle and P a point.

The isogonal conjugates of the reflections of P in the vertices
of the medial triangle are collinear.

(The line is the trilinear polar of gctP = isogonal conjugate of the complement of the isotomic
conjugate of P)

Synthetic proof?

APH

TelvCohl
#2 Jan 5, 2015, 5:02 am • 2

My solution:

Let M_a, M_b, M_c be the midpoint of BC, CA, AB, respectively .
Let P_a, P_b, P_c be the reflection of P in M_a, M_b, M_c, respectively .

Since CP_bAP, BP_cAP are parallelogram ,
so $CP_b \parallel AP \parallel BP_c$ (similarly, $AP_c \parallel CP_a, BP_a \parallel AP_b$) .

From Pascal theorem (for $AP_cBP_aCP_b$) we get P_a, P_b, P_c lie on a circumconic of $\triangle ABC$,
so the isogonal conjugate P^*_a, P^*_b, P^*_c of P_a, P_b, P_c WRT $\triangle ABC$ are collinear .

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, Jul 18, 2015, 6:42 pm

TelvCohl
#3 Jan 5, 2015, 6:41 am • 2

Theorem :

Let M_a, M_b, M_c be the midpoint of BC, CA, AB, respectively .
Let P_a, P_b, P_c be the reflection of P in M_a, M_b, M_c, respectively .
Let P^*_a, P^*_b, P^*_c be the isogonal conjugate of P_a, P_b, P_c WRT $\triangle ABC$, respectively .

Then P^*_a, P^*_b, P^*_c are collinear at the trilinear polar $\mathcal{T}$ of the isogonal conjugate of isotomcomplement of P

Proof :

Let $\triangle DEF$ be the cevian triangle of P .
Let $\triangle A'B'C'$ be the anticevian triangle of the isotomcomplement of P .
Let $B'' \in CA, C'' \in AB$ be the point satisfy $BB'' \parallel A'B', CC'' \parallel C'A'$.
Let X, Y, Z be the intersection of $\mathcal{T}$ with BC, CA, AB, respectively .

It's well-known that $\triangle DEF$ and $\triangle A'B'C'$ are homothetic .

Since $B(A,P_a;A',C)=(BA,CP;CC'',CB)=(\infty,F;C'',B)$
$=\frac{BD}{DC}=(\infty,E;C,B'')=(CA,BP;BC,BB'')=C(A,P_a;B,A')$,
so reflect in the corresponding angle bisector we get $B(C;P_a^*;Y,A)=C(B;P_a^*;A,Z)$,
hence P_a^*,Y,Z are collinear . i.e. $P_a^*\in\mathcal{T}$

Similarly, we can prove $P_b^*\in\mathcal{T}$ and $P_c^*\in\mathcal{T}$, so P_a^*,P_b^*,P_c^* are collinear at $\mathcal{T}$.

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, Jul 31, 2015, 6:12 pm

Luis González

#4 Jan 5, 2015, 11:49 am • 1

Let $\triangle P_1P_2P_3$ be the cevian triangle of P WRT $\triangle ABC$. B-symmedian of $\triangle BP_1P_3$ passes through the isogonal of the isotomcomplement of P, thus its harmonic conjugate WRT BA, BC (tangent of $\odot(BP_1P_2)$ at B) cuts AC at a point B_1 on its trilinear polar τ. Likewise τ goes through the intersection of AB with the tangent of $\odot(CP_1P_2)$ at C.

Let $BPCP_A$ be a parallelogram and let Q_A be the isogonal of P_A WRT $\triangle ABC$. Let $A_\infty, U_\infty, V_\infty$ denote the points at infinity of $BC, P_1P_3, P_1P_2,$ resp. The isogonal of BB_1 WRT $\angle ABC$ is the parallel from B to P_1P_3, thus $B(A,B_1,Q_A,C)=B(C,U_\infty,P_A,A)=P_3(A_\infty,P_1,C,A)$ and similarly we have $C(C_1,A,Q_A,B)=P_2(A_\infty,P_1,A,B)$. But $P_3(A_\infty,P_1,C,A)$ and $P_2(A_\infty,P_1,A,B)$ are perspective $\Longrightarrow$ $B(A,B_1,Q_A,C)=C(C_1,A,Q_A,B)\Longrightarrow Q_A,B_1,C_1$ are collinear, i.e. $Q_A\in\tau$.

IDMasterz

#5 Jan 7, 2015, 6:24 pm

For first assertion. This is pretty well-known; note that under an affine transformation we can take $P=H$, the orthocentre and its equivalent to the circumcircle, or really it is just pascal's theorem.

For the second assertion, let the conic through $ABCP_aP_bP_c$ be $\mathcal{C}$. Let the isotomic conjugate of P w.r.t. ABC be Q and let the reflection of Q over M_a be Q_a; define Q_b,Q_c similarly. Let ℓ_A be the point at infinity on the tangent at A to $\mathcal{C}$.

Of course, the isotomic conjugate of the orthocentre of ABC is the symmedian point of the anti-complementary triangle. Hence, under the aforementioned affine transformation, we have $A(\ell_A,Q_a;B,C)=-1$. The result follows.

Note: I used the following facts;

1. For a point P, the perspector of ABC and $P_aP_bP_c$ is the complement of P w.r.t. ABC.

2. For two isogonal conjugates P,Q, let the isogonal conjugate of the trilinear polar of P be $\mathcal{C}$. Then the Pascal line of hexagon $AABBCC$ w.r.t. $\mathcal{C}$ is the trilinear polar of Q w.r.t. ABC.

rodinos

#6 Jan 8, 2015, 1:59 am

IDMasterz: "This is pretty well-known" References ?

IDMasterz

#7 Jan 8, 2015, 1:56 pm

Well, I don't really know any references; RSM used to use this conic a lot (Sorry, I guess this just has been known to me for a long time). $ABCP_aP_bP_c$ lie on an affine equivalent of the circumcircle. Also, the antigonal conjugate of P lies on the aforementioned conic; this is just a corollary of the three conics theorem.

Quick Reply

High School Olympiads

Midpoint of segment

Source: Own, HSGS TST 2015

buratinogigle

#1 Sep 22, 2015, 11:27 am • 1

Let ABC be a triangle. Circle (K) passes through B,C which intersects CA,AB again at E,F, reps. Let M,N be symmetric points of B,C through E,F, resp. The tangent at A of circumcircle of triangle AMN which intersects BC,MN at P,Q, resp. Prove that A is midpoint of PQ.

Note that, this is generalization of lemma of Lester's circle here.

Luis González

#2 Sep 22, 2015, 1:55 pm • 1

More general: Take points M,N on BE,CF such that $\tfrac{EB}{EM}=\tfrac{FC}{FN}=k.$ Tangent of $\odot(AMN)$ at A cuts BC,MN at $P,Q,$ respectively. Then we have $\tfrac{AP}{AQ}=k.$

Proof: Take points X,Y on $AB,AC,$ such that $\tfrac{AB}{AX}=\tfrac{AC}{AY}=k$ and let $Z \equiv MX \cap NY.$ Then $XY \parallel BC, XZ \parallel AC$ and $YZ \parallel AB.$ Since $\tfrac{AX}{AY}=\tfrac{AB}{AC}=\tfrac{AE}{AF}=\tfrac{XM}{YN}$ and $\angle AXM=\angle AYN,$ it follows that $\triangle AXM \sim \triangle AYN \Longrightarrow \tfrac{AM}{AN}=\tfrac{AX}{AY}.$ Thus if $Q' \equiv MN \cap XY,$ then by Menelaus' theorem for $\triangle ZMN$ cut by $Q'XY,$ we get

$$\frac{Q'M}{Q'N}=\frac{YZ}{YN} \cdot \frac{XM}{XZ}=\frac{AX}{YN} \cdot \frac{XM}{AY}=\frac{AX^2}{AY^2}=\frac{AM^2}{AN^2},$$

which means that AQ' is tangent of $\odot(AMN) \Longrightarrow Q \equiv Q',$ thus from $XYQ \parallel BCP,$ we get $\tfrac{AP}{AQ}=\tfrac{AB}{AX}=\tfrac{EB}{EM}=k.$

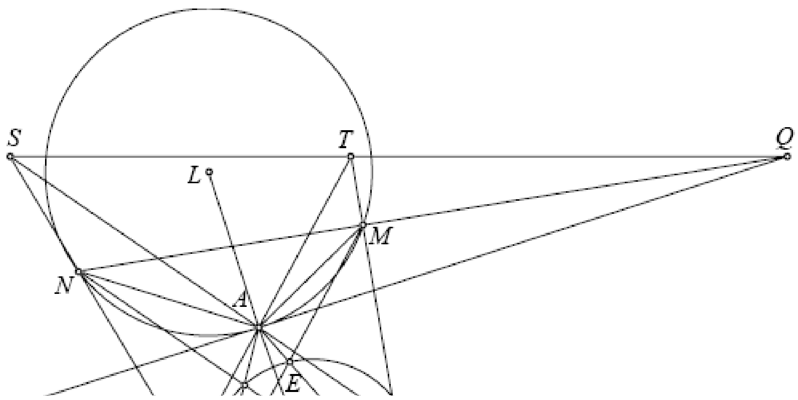
buratinogigle

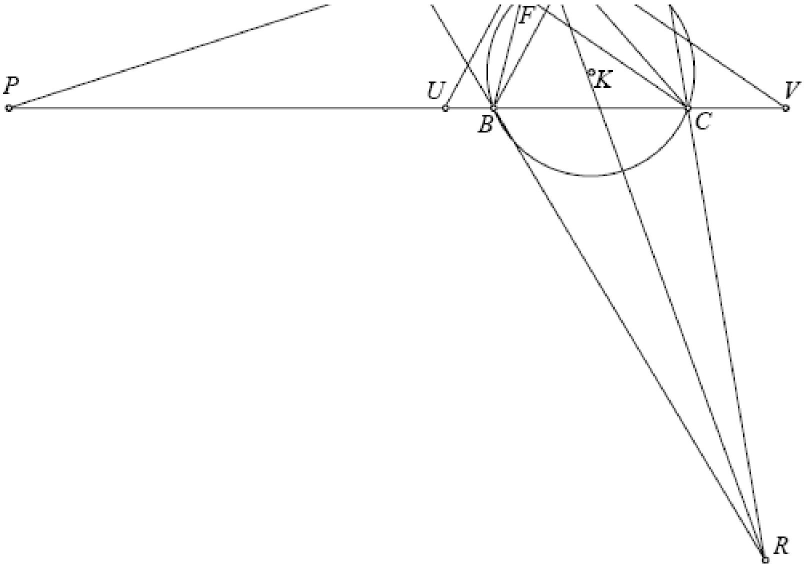
#3 Sep 22, 2015, 2:18 pm

Thank you Luis for nice generalization and solution, here is my solution to original problem, I think it works for your generalization

The line passes through A which is parallel to BM,CN cut BC at U,V, reps and cut BN,CM at S,T, reps. From E,F are midpoint of BM,CN so A is midpoint of UT,SV. Thus line ST is symmetric of BC through A. We will prove that S,T,Q are collinear, indeed. Let SB cut TC at R. Note that, $BCEF$ is cyclic so $\dfrac{BM}{BA}=\dfrac{2BE}{BA}=\dfrac{2CF}{CA}=\dfrac{CN}{CA}$. Hence, triangles ACN and ABN are similar. Follow, Thales theorem $\dfrac{TM}{TR}.\dfrac{SR}{SN}=\dfrac{TM}{SN}.\dfrac{BS}{CT}=\dfrac{TM}{TC}.\dfrac{SB}{SN}=\dfrac{AE}{AC}.\dfrac{AB}{AF}=\dfrac{AB^2}{AC^2}=\dfrac{AM^2}{AN^2}=\dfrac{QM}{QN}$. Thus, $\dfrac{TM}{TR}.\dfrac{SR}{SN}.\dfrac{QN}{QM}=1$ we deduce Q,S,T are collinear. We are done.

Attachments:





Quick Reply

High School Olympiads

Fixed point

Locked

Scorpion.k48

#1 Sep 22, 2015, 10:55 am

Let $\triangle ABC$ with M lies on A - Apollonius circle of $\triangle ABC$. Let I_1, I_2 be incenter of $\triangle AMB, \triangle AMC$. Prove that I_1I_2 passes through fixed point.

Luis González

#2 Sep 22, 2015, 11:00 am

Posted several times before; the fixed point is the foot of the external bisector of <BAC. See http://www.artofproblemsolving.com/community/c6h410908 and elsewhere.

High School Olympiads

Apolonius circle

mastergeo

#1 Jun 10, 2011, 9:00 am

Let ABC be a triangle with Apollonius circle ω. M is a point on the circle ω and inside triangle ABC. The incenter of triangle ABM and ACM is I_1, I_2 respectively. AE is the external bisector of angle $\angle A$ with $E \in BC$. Prove that I_1, I_2, E is collinear.

Luis González

#2 Jun 10, 2011, 10:12 am • 1

Assuming that ω is the A-Apollonius circle, then $M \in \omega \Longleftrightarrow \frac{BA}{BM} = \frac{CA}{CM}$. Thus, by angle bisector theorem, it follows that bisectors BI_1, CI_2 of $\angle ABM, \angle ACM$ cut AM at the same point D. Let I_1I_2 cut BC at E'. By Menelaus' theorem for $\triangle DBC$ cut by $\overline{E'I_2I_1}$, we get

$$\frac{\overline{E'C}}{\overline{E'B}} = \frac{\overline{DI_1}}{\overline{BI_1}} \cdot \frac{\overline{CI_2}}{\overline{DI_2}} = \frac{AD}{AB} \cdot \frac{AC}{AD} = \frac{AC}{AB} \Longrightarrow E \equiv E'.$$

High School Olympiads

Incenter, Excenter, Nine Point Center

Reply

Headhunter
#1 Sep 14, 2010, 12:01 am

Hello.

I=incenter, I_k=excenter (k=1,2,3), N=nine point center, R=radius of circumcircle
Show that $NI+NI_1+NI_2+NI_3=6R$
I have no solution, and failed to prove it.

Luis González
#2 Sep 17, 2010, 3:19 am • 1

According to Feuerbach theorem, incircle (I,r) is internally tangent to the 9-point circle $(N,\frac{R}{2})$ and excircles $(I_a,r_a),(I_b,r_b),(I_c,r_c)$ are externally tangent to $(N,\frac{R}{2})$. Thus

$$\overline{NI}=\frac{R}{2}-r\ ,\ \overline{NI_a}=\frac{R}{2}+r_a\ ,\ \overline{NI_b}=\frac{R}{2}+r_b\ ,\ \overline{NI_c}=\frac{R}{2}+r_c$$

$$\overline{NI}+\overline{NI_a}+\overline{NI_b}+\overline{NI_c}=2R-r+(r_a+r_b+r_c)=2R-r+(4R+r)=6R$$

Quick Reply

High School Olympiads

An easy problem!

Reply

Source: own

MRF2017

#1 Sep 21, 2015, 11:25 pm

Point G is centroid of scalene triangle ABC.We know reflection of G wrt BC lies on circumcircle of ABC.
Prove that $\angle BAC < 60$.

Luis González

#2 Sep 21, 2015, 11:55 pm

Let O and H be the circumcenter and orthocenter of $\triangle ABC$. As $\odot(HBC)$ is the reflection of the circumcircle (O) on BC, then $G \in \odot(HBC)$. Now since G is always between O and H, it follows that O is outside of $\odot(HBC) \Longrightarrow$ $\angle BHC > \angle BOC \Longrightarrow 180^\circ - \angle BAC > 2\angle BAC \Longrightarrow \angle BAC < 60^\circ$.

Quick Reply

High School Olympiads

Radical axis

Reply

Scorpion.k48

#1 Sep 21, 2015, 10:04 pm

Let $\triangle ABC$ with circumcenter O. A_1, B_1, C_1 be the reflection of A, B, C to BC, CA, AB. Let O_1 is circumcenter of $\triangle A_1B_1C_1$. Prove that O_1 lies on radical axis of $\odot OAA_1, \odot OBB_1, \odot OCC_1$

This post has been edited 1 time. Last edited by Scorpion.k48, Sep 21, 2015, 10:48 pm

Luis González

#2 Sep 21, 2015, 11:05 pm

Let O, G, N be the circumcenter, centroid and 9-point center of $\triangle ABC$. $\triangle N_AN_BN_C$ is pedal circle of N with center U. Reflection K of N on U is isogonal conjugate of N; Kosnita point. Redefine O_1 as the intersection of UG and OK. By Menelaus' theorem for $\triangle OKN$ cut by $\overline{GUO_1}$, we get $\frac{O_1K}{O_1O} = \frac{UK}{UN} \cdot \frac{GN}{GO} = \frac{1}{2} \Longrightarrow K$ is midpoint of O_1. Now by Meneleaus' theorem for $\triangle NGU$ cut by $\overline{OKO_1}$, we'll obtain $\frac{GO_1}{GU} = 4$. Since the homothety $\mathcal{H}(G, 4)$ takes $\triangle N_AN_BN_C$ into $\triangle A_1B_1C_1$ (well-known), then it follows that O_1 is the image of $U \Longrightarrow O_1$ is the circumcenter of $\triangle A_1B_1C_1$.

Let O_a, O_b, O_c be the centers of $\odot(OBC), \odot(OCA), \odot(OAB)$ and let O_1, O_2, O_3 be the reflections of O on BC, CA, AB lying on $\odot(OAA_1), \odot(OBB_1), \odot(OCC_1)$, resp. Inversion on (O) takes $O_a \mapsto O_1$, etc $\Longrightarrow AO_a, BO_b, CO_c$, concurring at K, are the inverses of $\odot(OAA_1), \odot(OBB_1), \odot(OCC_1) \Longrightarrow \odot(OAA_1), \odot(OBB_1), \odot(OCC_1)$ are coaxal, i.e. they meet at O and the inverse of K on $(O) \Longrightarrow \overline{OKO_1}$ is their common radical axis.

Quick Reply

High School Math

Leibniz's theorem

Reply

AndrewTom
#1 Sep 21, 2015, 3:49 am

If P is any point in the plane of triangle ABC and G is the centroid, prove that

$$PA^2+PB^2+PC^2=\frac{1}{3}(a^2+b^2+c^2)+3PG^2.$$

Luis González
#2 Sep 21, 2015, 7:56 am • 1

For a more general expression of the Leibniz theorem see the topic Computing distances with barycentric coordinates.

AndrewTom
#3 Sep 21, 2015, 2:07 pm

Thanks Luis. I'm afraid I'm not familiar with barycentric coordinates.

Could we have a proof using another method such as vectors?

nikolapavlovic
#4 Sep 21, 2015, 6:32 pm • 1

we can prove it by complex numbers if you want

nikolapavlovic
#5 Sep 21, 2015, 9:38 pm • 1

let p be the origin of the complex plane
by using complex product $(a\cdot b=\frac{a\bar{b}+b\bar{a}}{2})$
$PA^2=a^2$
$PB^2=b^2$
$PC^2=c^2$
$g=\frac{a+b+c}{3}\implies$
$PG^2=\frac{(a+b+c)^2}{9}$
$AB^2=(a-b)^2$ and others(AC,BC).
Now by expanding we get the desired result

This post has been edited 2 times. Last edited by nikolapavlovic, Sep 21, 2015, 9:41 pm

Quick Reply

High School Olympiads

Computing distances with barycentric coordinates

Reply

Source: 0

Luis González

#1 May 10, 2009, 3:36 am • 1

$P(x_1:y_1:z_1)$ and $Q(x_2:y_2:z_2)$ with $x_1+y_1+z_1=x_2+y_2+z_2$ represent the barycentric coordinates of P,Q WRT $\triangle ABC.$ The distance betwen P,Q is given by

$$PQ^2=S_A(x_2-x_1)^2+S_B(y_2-y_1)^2+S_C(z_2-z_1)^2$$

With the usual Conway notation

$$S_A=\frac{b^2+c^2-a^2}{2} \ , \ S_B=\frac{c^2+a^2-b^2}{2} \ , \ S_C=\frac{a^2+b^2-c^2}{2}$$

Leibniz formula: Let Q be a point with homogeneus barycentric coordinates $(u:v:w)$ with respect to $\triangle ABC.$ For any point P on the plane $ABC,$ the following relation holds:

$$uPA^2+vPB^2+wPC^2=(u+v+w)PQ^2+uQA^2+vQB^2+wQC^2$$

April

#2 May 10, 2009, 5:46 am

Thank you very much for your message, dear Luis! I am not familiar with algebraic geometry (especially about barycentric coordinates) so I found your post is really helpful. Thank you!

Luis González

#3 May 22, 2009, 7:22 am • 1

Due to a request via PM, I will write the proof of Leibniz theorem

By definition $(u:v:w)$ are three scalars that satisfy $u\overrightarrow{QA}+v\overrightarrow{QB}+w\overrightarrow{QC}=\overrightarrow{0} \ (\star)$

For any point P in the plane $ABC,$ we'll have

$$u\overrightarrow{PA}^2=uPQ^2+uQA^2+2u\cdot\overrightarrow{PQ}\cdot\overrightarrow{QA}$$

$$v\overrightarrow{PB}^2=vPQ^2+vQB^2+2v\cdot\overrightarrow{PQ}\cdot\overrightarrow{QB}$$

$$w\overrightarrow{PC}^2=wPQ^2+wQC^2+2w\cdot\overrightarrow{PQ}\cdot\overrightarrow{QC}$$

Now, summing up the 3 latter equations keeping in mind $(\star),$ we get then

$$uPA^2+vPB^2+wPC^2=(u+v+w)PQ^2+uQA^2+vQB^2+wQC^2$$

Quick Reply

High School Olympiads

Blacklord
#1 Sep 21, 2015, 12:29 am

Let O be the circumcenter of the triangle ABC.L is a fixed point passing throw the O.P is the intersection of the line tangent with (O) at B and the line tangent with (O) at C.
A1 is the feet of prependicular from P to L.B1 and C1 defines similarly.
Prove that AA1,BB1 and CC1 concur.

Luis González
#2 Sep 21, 2015, 2:47 am • 1

We can rephrase the problem as follows: ℓ is abritrary line through the circumcenter O of $\triangle ABC$. If ℓ cuts the circles $\odot(OBC), \odot(OCA), \odot(OAB)$ again at A_1, B_1, C_1, then AA_1, BB_1, CC_1 concur at a point S. Moreover S is the isogonal conjugate of the orthopole of ℓ WRT $\triangle ABC$.

Proof: Denote Y, Z the midpoint of AC, AB and U the projection of A on ℓ. It's well-known that the orthopole T of ℓ WRT $\triangle ABC$ is the reflection of U on YZ. If OY cuts AB at L, we have $\angle BLC = 2 \cdot \angle ALO = 2 \cdot \angle CAB = \angle BOC$ (mod 180º) $\Longrightarrow L \in \odot(OBC) \Longrightarrow \angle YZT = \angle YZU = \angle YOU = \angle LOA_1 = \angle ABA_1$. Hence if A_2 is the midpoint of AA_1, keeping in mind that $ZA_2 \parallel BA_1$, we get $\angle AZA_2 = \angle ABA_1 = \angle YZT \Longrightarrow ZT, ZA_2$ are isogonals WRT $\angle AZY$ and similarly YT, YA_2 are isogonals WRT $\angle AYZ \Longrightarrow T, A_2$ are isogonal conjugates WRT $\triangle AYZ \Longrightarrow$ $AA_2 \equiv AA_1$ and AT are isogonals WRT $\angle BAC$ and similarly BB_1 and BT are isogonals WRT $\angle CBA \Longrightarrow$ AA_1, BB_1, CC_1 concur at the isogonal conjugate S of T WRT $\triangle ABC$.

Dukejukem
#4 Sep 22, 2015, 3:53 am

> *Luis González wrote:*
> We can rephrase the problem as follows: ℓ is abritrary line through the circumcenter O of $\triangle ABC$. If ℓ cuts the circles $\odot(OBC), \odot(OCA), \odot(OAB)$ again at A_1, B_1, C_1, then AA_1, BB_1, CC_1 concur.

Under inversion with pole O, we obtain the following equivalent problem: An arbitrary line ℓ passing through the circumcenter O of $\triangle ABC$ cuts BC, CA, AB at A_1, B_1, C_1, respectively. Prove that $\odot(OAA_1), \odot(OBB_1), \odot(OCC_1)$ are coaxial.

Proof: For a point X and a circle ω, let $\mathcal{P}(X, \omega)$ be the power of X WRT ω. Define $\Gamma \equiv \odot(ABC), \omega_A \equiv \odot(OAA_1), \omega_B \equiv \odot(OBB_1), \omega_C \equiv \odot(OCC_1)$. Let $\lambda := \frac{CA}{CB_1}, \mu := \frac{BA}{BC_1}$.

Define the function $f : \mathbb{R}^2 \mapsto \mathbb{R}$ by $f(X) := \mathcal{P}(X, \omega_C) - \mathcal{P}(X, \Gamma)$. It is straightforward to check that f is linear. Then since $A = \lambda B_1 + (1 - \lambda)C$, we obtain

$$\mathcal{P}(A, \omega_C) = f(A) = \lambda f(B_1) + (1-\lambda) f(C) = \lambda \left(B_1O \cdot B_1C_1 - B_1O^2 + R^2\right).$$

Similarly, let us define $g(X) := \mathcal{P}(X, \omega_B) - \mathcal{P}(X, \Gamma)$. Then note that

$$\mathcal{P}(A, \omega_B) = g(A) = \mu g(C_1) + (1-\mu) g(B) = \mu \left(C_1O \cdot C_1B_1 - C_1O^2 + R^2\right).$$

Consequently, if AO cuts ω_B, ω_C for a second time at Y, Z, respectively, we obtain

$$\frac{AY}{AZ} = \frac{\mathcal{P}(A, \omega_B)}{\mathcal{P}(A, \omega_C)} = \frac{\mu}{\lambda} \cdot \frac{B_1O(B_1C_1 - B_1O) + R^2}{C_1O(C_1B_1 - C_1O) + R^2} = \frac{\mu}{\lambda} \cdot \frac{B_1O \cdot OC_1 + R^2}{C_1O \cdot OB_1 + R^2} = \frac{\mu}{\lambda} = \frac{A_1B_1}{A_1C_1},$$

where the last step follows from Menelaus Theorem applied to $\triangle AB_1C_1$ for transversal $\overline{A_1BC}$. Consequently, there exists a spiral similarity that sends $\overline{A_1B_1C_1} \mapsto \overline{AYZ}$. Since $O \equiv \overline{A_1B_1C_1} \cap \overline{AYZ}$, it is well-known that the center S of this spiral similarity lies on $\odot(OAA_1), \odot(OYB_1), \odot(OZC_1)$. Thus, OS is the common chord of $\omega_A, \omega_B, \omega_C$. $\square$

This post has been edited 2 times. Last edited by Dukejukem Sep 22, 2015, 6:56 am

Quick Reply

High School Olympiads

Feuerbach point

Reply

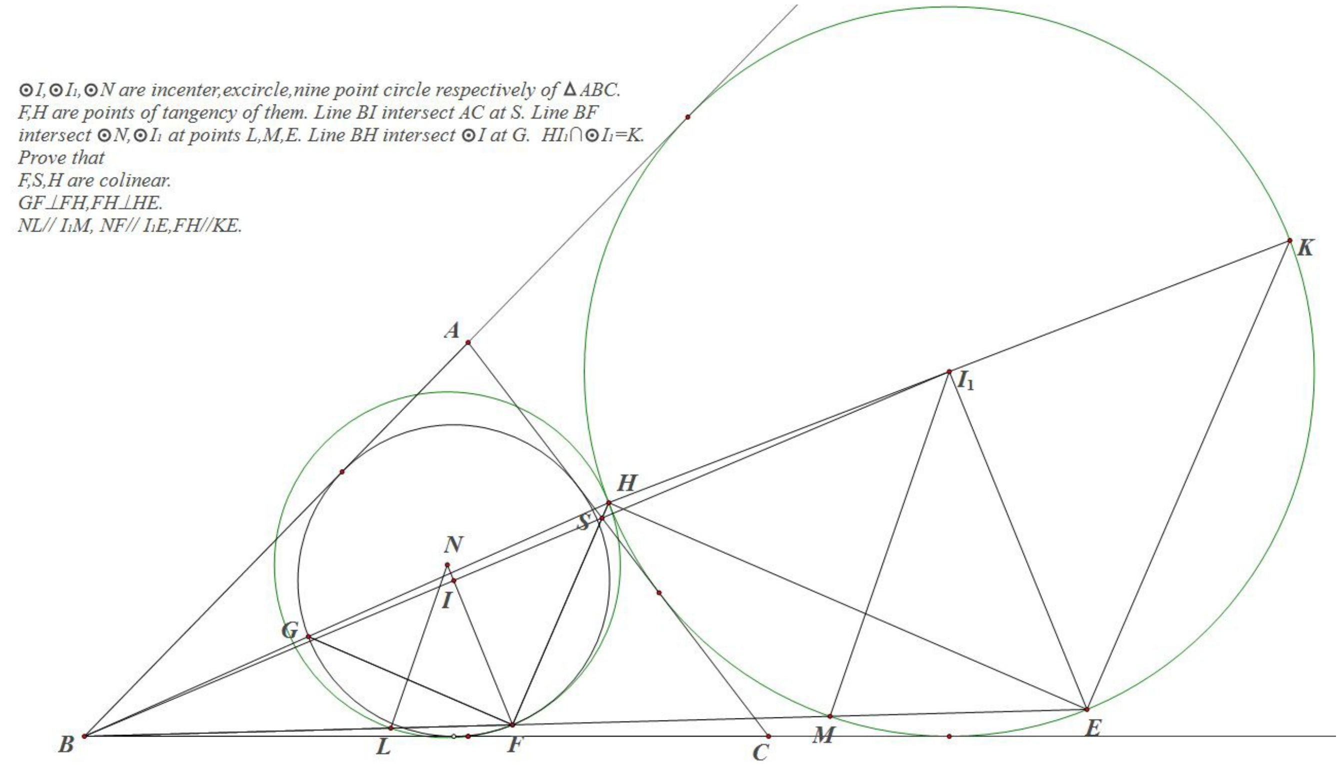
Source: own

magrv

#1 Sep 20, 2015, 5:54 pm

It's about incenter,excircle,nine point circle tangency respectively of a triangle.

Attachments:



Luis González

#2 Sep 20, 2015, 11:16 pm • 1

Since F is the exsimilicenter of $(I) \sim (N)$, H is the insimilicenter of $(N) \sim (I_1)$ and S is the insimilicenter of $(I) \sim (I_1)$, then by Monge & d'Alembert theorem, it follows that F, S, H are collinear.

Let IG cut (I) again at U. Since B is the exsimilicenter of $(I) \sim (I_1)$, it follows that $IU \equiv IG \parallel I_1H \equiv NH$, thus as F is the exsimilicenter of $(I) \sim (N)$, then F, U, H are collinear $\Longrightarrow \angle HFG \equiv \angle UFG = 90^\circ$, i.e. $GF \perp FH$. By similar reasoning we obtain $FH \perp HE$.

Let FB cut (I) again at V. As B is the exsimilicenter of $(I) \sim (I_1)$ and F is the exsimilicenter of $(I) \sim (N)$, we get $\overline{NIF} \parallel I_1E, IV \parallel I_1M$ and $NL \parallel IV \Longrightarrow NL \parallel I_1M$. Finally, since $FG \parallel EH$ and $HF \perp FG, KE \perp EH \Longrightarrow FH \parallel KE$.

Quick Reply

High School Olympiads

Equal angle and fixed point

Source: Own, HSGS TST 2015

buratinogigle

#1 Sep 20, 2015, 11:35 am • 1

Let ABC be an acute triangle with E, F lie on side CA, AB such that $EF \parallel BC$. The tangents at E, F of circumcircle of triangle AEF intersect BC at M, N, reps. Assume that BE, CF cut FN, EM at K, L, reps.

a) Prove that $\angle KAB = \angle LAC$.

b) Let BE cut CF at X. FM cut EN at Y. Prove that line XY always passes through fixed point when E, F move.

Luis González

#2 Sep 20, 2015, 12:44 pm • 2

a) Let $Z \equiv EM \cap FN.$ Since $EF \parallel BC,$ then clearly X is on the A-median of $\triangle AEF,$ thus AZ being the A-symmedian of $\triangle AEF$ is the isogonal of AX WRT $\angle BAC.$ But by dual Desargues involution theorem for $XLZK,$ we deduce that $AE \mapsto AF, AZ \mapsto AX, AK \mapsto AL$ is an involution, which coincides then with the reflection across the angle bisector of $\angle BAC \Longrightarrow AK, AL$ are isogonals WRT $\angle BAC.$

b) The homography sending $NMEF \mapsto BCEF$ fixes the line $EF,$ so it is a homology fixing a line pencil. This carries $Y \mapsto X$ and $Z \mapsto A \Longrightarrow S \equiv XY \cap AZ \cap BN$ is the vertex of this pencil $\Longrightarrow$ all lines XY go through the intersection S of BC with the A-smmedian of $\triangle ABC,$ which is fixed.

huricane

#3 Sep 20, 2015, 9:23 pm • 2

Here is my solution for b):

Let $S \equiv FN \cap EM, P \equiv AS \cap EF, S' \equiv AS \cap BC, M' \equiv AX \cap EF, T \equiv EN \cap CF, R \equiv FM \cap BE$ and let Q be the symmetric point of P wrt point M'.
It's clear that $\overline{APSS'}$ is the A-symmedian of triangles $\triangle AEF$ and $\triangle ABC$.Since $\triangle FSE \sim \triangle NSM$,it follows that $\frac{MS'}{NS'} = \frac{EP}{FP} = \left(\frac{AE}{AF}\right)^2 = \left(\frac{AC}{AB}\right)^2 = \frac{CS'}{BS'}$.Thus $BS' \cdot MS' = CS' \cdot NS'$,so $BS' \cdot MS' + BS' \cdot CS' = CS' \cdot NS' + BS' \cdot CS'$,which is equivalent to $BS' \cdot MC = CS' \cdot NB$ or $\frac{MC}{BN} = \frac{CS'}{BS'} = \left(\frac{AC}{AB}\right)^2$ $(\bigstar)$.
Now,from Sine Rule in triangles $\triangle FER, \triangle YER, \triangle YFT, \triangle EFT$ we deduce that

$$FR = \frac{RE \cdot \sin \measuredangle FER}{\sin \measuredangle RFE},$$

$$YR = \frac{RE \cdot \sin \measuredangle YER}{\sin \measuredangle RYE},$$

$$YT = \frac{FT \cdot \sin \measuredangle TFY}{\sin \measuredangle FYT},$$

$$ET = \frac{FT \cdot \sin \measuredangle EFT}{\sin \measuredangle FET}$$

,so

$$\frac{FR}{YR}\cdot\frac{YT}{ET}=\frac{\sin\measuredangle FER}{\sin\measuredangle YER}\cdot\frac{\sin\measuredangle YFT}{\sin\measuredangle EFT}(\bigstar\bigstar).$$

But $\measuredangle FER\equiv\measuredangle EBN$ and $\measuredangle EFT\equiv\measuredangle FCM$,implying that

$$\frac{\sin\measuredangle FER}{\sin\measuredangle YER}\cdot\frac{\sin\measuredangle YFT}{\sin\measuredangle EFT}=\frac{\sin\measuredangle EBN}{\sin\measuredangle BEN}\cdot\frac{\sin\measuredangle CFM}{\sin\measuredangle FCM}=\frac{EN}{BN}\cdot\frac{MC}{FM}=\frac{MC}{BN}(\bigstar\bigstar\bigstar).$$

Now,by letting $Q'\equiv XY\cap EF$ and combining $(\bigstar),(\bigstar\bigstar)$ and $(\bigstar\bigstar\bigstar)$ with Ceva's Theorem in triangle $\triangle YEF$ we get $\frac{FQ'}{EQ'}=\frac{FR}{YR}\cdot\frac{YT}{ET}=\frac{MC}{BN}=\left(\frac{AC}{AB}\right)^2=\left(\frac{AE}{AF}\right)^2=\frac{EP}{FP}=\frac{FQ}{EQ}\implies Q\equiv Q'.$

If $W\equiv\overline{XYQ}\cap MN$,then $\frac{MW}{NW}=\frac{FQ}{EQ}=\frac{EP}{FP}=\left(\frac{AE}{AF}\right)^2=\left(\frac{AC}{AB}\right)^2=\frac{MS'}{NS'}\implies W\equiv S'$ i.e. the line XY passes through point S',which is obviously fixed.

This post has been edited 1 time. Last edited by huricane, Sep 22, 2015, 1:21 am
Reason: Ceva ,not Menelaus...

buratinogigle

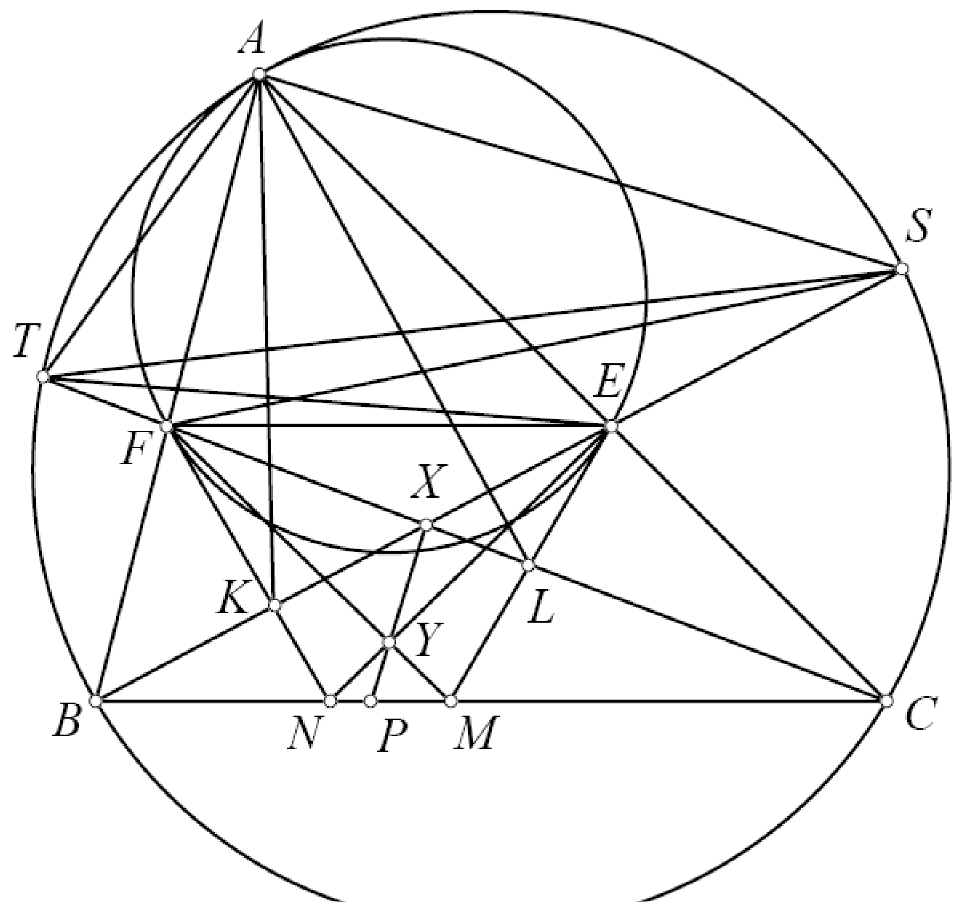
#4 Sep 22, 2015, 10:59 am

Thank you for your interest, here is my solution

a) Let BE,CF cut (ABC) again at S,T. We have $\angle BFN=180^\circ-\angle NFE-\angle EFA=180^\circ-\angle BAC-\angle ABC=\angle ACB=\angle ASB$. Hence, $AFKS$ is cyclic, we deduce $\angle KAB=\angle FSE$. Similarly, $\angle LAC=\angle FTE$. We have $\angle FEB=\angle EBC=\angle STF$ so $EFTS$ is cyclic. Now, $\angle KAB=\angle FSE=\angle FTE=\angle LAC$. We are done.

b) FN,EM are tangent to AEF so $\angle BFN=\angle ECM,\angle CEM=\angle FBN$. Thus, triangles FBN and CEM are similar. Note, $FN=EM$ so $\frac{BN}{CM}=\frac{BN}{ME}.\frac{FN}{CM}=\frac{FB}{EC}.\frac{FB}{EC}=\frac{FB^2}{EC^2}=\frac{AB^2}{AC^2}$.Let XY cut BC at P. Apply Menelaus theorem for triangle FMC with X,Y,P are collinear $\frac{PM}{PC}.\frac{XC}{XF}.\frac{YF}{YM}=1$. Apply Menelaus theorem for triangle ENB with X,Y,P are collinear, we have $\frac{PN}{PB}.\frac{XB}{XE}.\frac{YE}{YN}=1$.From, $EF\parallel BC$ we get $\frac{XC}{XF}=\frac{XB}{XE}$ and $\frac{YF}{YM}=\frac{YE}{YN}$. Hence, $\frac{PM}{PC}=\frac{PN}{PB}$ or $\frac{PM}{PN}=\frac{PC}{PB}=\frac{PC-PM}{PB-PN}=\frac{CM}{BN}=\frac{AC^2}{AB^2}$ is constant, so P is fixed.

Attachments:



huricane

#5 Sep 22, 2015, 11:37 pm

Finally,here is my solution(which uses only Menelaus' Theorem) to a):

Let's define the points $K'\equiv AK\cap BC, L'\equiv AL\cap BC, U\equiv KL\cap BC$(in my picture,the order of the points on line BC is U,B,M,N,C).
By applying Menelaus' Theorem in triangle $\triangle BEC$ and $\triangle BFC$ for transversals $\overline{AKK'}$ and respectively $\overline{ALL'}$ we obtain

$$\frac{BK'}{CK'}\cdot\frac{CA}{EA}\cdot\frac{EK}{BK}=1,$$

$$\frac{BL'}{CL'}\cdot\frac{CL}{FL}\cdot\frac{FA}{AB}=1.$$

By multiplying the above two relations we obtain

$$\left(\frac{BK'}{CK'}\cdot\frac{BL'}{CL'}\right)\cdot\left(\frac{EK}{BK}\cdot\frac{CL}{FL}\right)=1(\bigstar).$$

From Menelaus' Theorem applied in triangles $\triangle BEM$ and $\triangle CFN$ for the transversal $\overline{LKU}$ give us

$$\frac{EK}{BK}\cdot\frac{UB}{UM}\cdot\frac{ML}{LE}=1,$$

$$\frac{FK}{KN}\cdot\frac{UN}{UC}\cdot\frac{CL}{FL}=1.$$

But $\frac{EK}{BK}\cdot\frac{ML}{LE}=\frac{EK}{BK}\cdot\frac{CL}{FL}=\frac{FK}{NK}\cdot\frac{CL}{FL}$,so from the above two equations we deduce that $\frac{UB}{UM}=\frac{UN}{UC}$.
Thus,

$$\begin{aligned}\frac{UB}{UM}=\frac{UN}{UC}&\implies UB\cdot UC=UN\cdot UM\\&\implies UB\cdot UN+UB\cdot NC=UN\cdot UB+UN\cdot BM\\&\implies UB\cdot NC=UN\cdot BM\\&\implies UB\cdot MC-UB\cdot MN=UN\cdot BN-UN\cdot MN\\&\implies UB\cdot MC+MN\cdot BN=UN\cdot BN\\&\implies UB\cdot MC=BN\cdot UM\\&\implies\frac{UB}{UM}=\frac{BN}{MC}=\left(\frac{AB}{AC}\right)^2.\end{aligned}$$

Now,from $\frac{EK}{BK}\cdot\frac{UB}{UM}\cdot\frac{ML}{LE}=1$ and $\frac{UB}{UM}=\left(\frac{AB}{AC}\right)^2$ it follows that $\frac{EK}{BK}\cdot\frac{ML}{LE}=\left(\frac{AC}{AB}\right)^2$ i.e.
$\frac{EK}{BK}\cdot\frac{CL}{FL}=\left(\frac{AC}{AB}\right)^2(\bigstar\bigstar)$.

Finally,by combining $(\bigstar)$ with $(\bigstar\bigstar)$ we conclude that $\frac{BK'}{CK'}\cdot\frac{BL'}{CL'}=\left(\frac{AB}{AC}\right)^2$ i.e. the lines AK and AL are isogonal in $\triangle ABC$.

Vanescralet

#6 Sep 26, 2015, 3:08 pm

I have an extension of this problem:

c) FN intersects EM at Z. Prove that line XZ always passes through fixed point when E,F move.

Quick Reply

High School Olympiads

Isogonal conjugate

Reply

Scorpion.k48
#1 Sep 20, 2015, 11:42 am

Let $\triangle ABC$ with Lemoine L, median triangle $A_1B_1C_1$ of $\triangle ABC$. Let K be isogonal conjugate of L WRT $\triangle A_1B_1C_1$. Prove that K lies on Euler line of $\triangle ABC$.

Luis González
#2 Sep 20, 2015, 12:05 pm • 1

Let O and H be the circumcenter and orthocenter of $\triangle ABC$. AH, BH, CH cut B_1C_1, C_1A_1, A_1B_1 at A_0, B_0, C_0 (midpoints of corresponding altitudes of $\triangle ABC$). A_1A_0, B_1B_0, C_1C_0 are Schwatt lines of $\triangle ABC$ concurring at L (well-known), thus L becomes isotomic conjugate of orthocenter O of $\triangle A_1B_1C_1$ WRT it. According to Isotomic Conjugate's Property, we conclude that the isogonal conjugate of L WRT $\triangle A_1B_1C_1$ is on Euler line of $\triangle DEF, \triangle ABC$.

Quick Reply

High School Olympiads

Isotomic Conjugate's Property

Reply

Arab

#1 Jul 1, 2013, 5:23 pm

Denote by O' the isotomic conjugate of O the circumcenter of $\triangle ABC$ and H' the isotomic conjugate of H the orthocenter. Prove that,O', H', H are collinear.

Has this been proved before?

Wizzy

#2 Jul 1, 2013, 6:24 pm • 1

Well, yes, it's a reformulation of a well-known fact. Let L be the point isogonal conjugate to H'. Then O', H', H are collinear $\Leftarrow O, H, H', A, B, C$ lie on the conic $\Leftarrow H, O, L$ are collinear - and this fact is pretty well-known. The geometric proof of this fact is the following.
Let D be the point symmetric to B w.r.t. perpendicular bisector of AC, BB_1 be the altitude of ABC, A_0, B_0, C_0 - the midpoints of the corresponding sides of ABC, Ω and ω - the circumcircles of ABC and $A_0B_0C_0$ respectively. Let B_2 be the second point of intersection of DB_1 and Ω. It is clear that L lies on BB_2 (some trivial angle calculation).
WLOG $\angle C > \angle A$. Let B_3 be the point on the ray AC such that B_3B is tangent to Ω. Let B_4 be the intersection of BB_3 and A_0C_0.
We have $\angle CB_2B_1 = \angle CB_2D = \angle C$ so $\angle BB_2B_1 = \angle CB_2D - \angle CB_2B = \angle C - \angle A = \angle BB_3A$ so the points B, B_3, B_2, B_1 are concyclic. But $\angle BB_1B_3 = 90^\circ$, so $\angle BB_2B_3 = 90^\circ$. We also have $BB_4 = B_4B_3$, so $B_4B = B_4B_1 = B_4B_3 = B_4B_2 \Rightarrow B_4B_2$ is tangent to Ω. But also $B_4B_2^2 = B_4B^2 = B_4A_0 \cdot B_4C_0$ so B_4 lies on the radical axis of Ω and ω. Similarly, the analogous points A_4 and C_4 also lie on the radical axis of Ω and ω. But A_4, B_4, C_4 also lie on the polar of the point L w.r.t. circle Ω, so [radical axis of Ω and ω] $\perp OL$, but also obviously [radical axis of Ω and ω] $\perp OH$, so $L \in OH$. Q.E.D.

TelvCohl

#3 Feb 5, 2015, 7:44 am • 4

Another way to prove the theorem mentioned by **Wizzy** :

Theorem:

Let O, H be the circumcenter, orthocenter of $\triangle ABC$, respectively .
Let H^* be the isogonal conjugate of the isotomic conjugate of H WRT $\triangle ABC$.

Then $H^* \in OH$.

Proof:

Let $\triangle DEF$ be the orthic triangle of $\triangle ABC$.
Let $\triangle A^*B^*C^*$ be the tangential triangle of $\triangle ABC$.
Let X, Y, Z be the projection of H on EF, FD, DE, respectively .
Let V be the projection of A on EF .

Since H, A is the incenter, $D-$ excenter of $\triangle DEF$, respectively ,
so X, V are symmetry WRT the midpoint of $EF \Longrightarrow H^* \in AX$.
Similarly, we can prove $H^* \in BY$ and $H^* \in CZ$.

Since figure $\triangle A^*B^*C^* \cup \triangle ABC$ and figure $\triangle DEF \cup \triangle XYZ$ are homothetic ,
so $H^* \equiv AX \cap BY \cap CZ$ is the homothety center of $\triangle A^*B^*C^*$ and $\triangle DEF$,
hence H^* lie on the line connecting the incenter O of $\triangle A^*B^*C^*$ and the incenter H of $\triangle DEF$.

Done 🙂

This post has been edited 1 time. Last edited by TelvCohl, Feb 8, 2015, 5:02 pm

drmzjoseph

#4 Feb 5, 2015, 8:10 am • 1

Barycentric coordinates and Conway notation

$$H=\frac{1}{S_A}:\frac{1}{S_B}:\frac{1}{S_C}$$

$$O'=\frac{1}{a^2S_A}:\frac{1}{b^2S_B}:\frac{1}{c^2S_C}$$

$$H'=S_A:S_B:S_C$$

O',H',H are collinear $\iff \begin{pmatrix} \frac{1}{S_A} & \frac{1}{S_B} & \frac{1}{S_C} \\ S_A & S_B & S_C \\ \frac{1}{a^2S_A} & \frac{1}{b^2S_B} & \frac{1}{c^2S_C} \end{pmatrix}=0$

$$\sum_{cyc}\frac{c^2a^2S_B^2-a^2b^2S_C^2}{a^2b^2c^2S_AS_BS_c}=0$$

Q.E.D

This post has been edited 2 times. Last edited by drmzjoseph, Feb 5, 2015, 11:04 am

Luis González

#5 Feb 5, 2015, 10:37 am • 2

The problem is equivalent to prove that the isotomic conjugate H' of H is on Jerabek hyperbola $\mathcal{J}$ of $\triangle ABC$. For any point P on Euler line OH, we have that $\triangle ABC$ and the triangle formed by the reflections of P on the perpendicular bisectors of BC,CA,AB are perspective with perspector Q on $\mathcal{J}$ (for a proof see with the Euler's line). When P is the centroid of $\triangle ABC$, then clearly $Q\equiv H'\Longrightarrow H'\in\mathcal{J}$.

Quick Reply

High School Olympiads

A line bisects segment

Reply

Scorpion.k48

#1 Sep 20, 2015, 11:19 am

Let $\triangle ABC$ with circumcircle $\odot(O)$. K is a point such that AK is an internal angle bisector of $\triangle ABC$. P lies on AK. BP cuts AC and $\odot(O)$ at X, Y. CP cuts AB and $\odot(O)$ at Z, T. Q is intersection of YZ and XT. Prove that PQ bisects BC.

Luis González

#2 Sep 20, 2015, 11:27 am • 1

This configuration was discussed before at Line KI passes through midpoint of AC (see post #4).

jayme

#3 Sep 20, 2015, 3:17 pm

Dear Mathlinkers,
see also

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=487265

Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Line KI passes through midpoint of AC

math707
#1 Feb 8, 2012, 11:32 pm

w-circumcircle of triangle ABC. Let bisector of angle BAC meet BC and w at A_1 and A_2 respectively. Let bisector of angle BCA meet AB and w at C_1 and C_2 respectively. $K = A_1C_2 \cap A_2C_1$. Prove that KI passes through a midpoint of AC (I-incenter)

yetti
#2 Feb 9, 2012, 7:10 am • 1

I_a, I_c are A- and C-excenters of $\triangle ABC$. Cross ratio $(A, A_1, I, I_a) = -1$ is harmonic and A_2 is midpoint of $II_a \Longrightarrow$
$\frac{\overline{A_1A_2}}{\overline{A_1I}} = \frac{\overline{IA_2}}{\overline{IA}}$. Similarly, $\frac{\overline{C_1C_2}}{\overline{C_1I}} = \frac{\overline{IC_2}}{\overline{IC}}$.
IK cuts A_2C_2 at L. By Ceva theorem for $\triangle IA_2C_2$ with lines A_2C_1, C_2A_1, IL concurrent at $K \Longrightarrow$
$\frac{\overline{LA_2}}{\overline{LC_2}} = -\frac{\overline{A_1A_2}}{\overline{A_1I}} \cdot \frac{\overline{C_1I}}{\overline{C_1C_2}} = -\frac{\overline{IA_2}}{\overline{IA}} \cdot \frac{\overline{IC}}{\overline{IC_2}} = -\left(\frac{\overline{IA_2}}{\overline{IC_2}}\right)^2 \Longrightarrow$
IKL is I-symmedian of $\triangle IA_2C_2 \Longrightarrow IKL$ is I-median of $\triangle ICA$, oppositely similar to $\triangle IA_2C_2$, cutting CA at its midpoint.

r1234
#3 Feb 10, 2012, 3:29 pm

I will also prove that IK is the I-symmedian of ΔIA_2C_2 but in just a different way...

Let us draw a parallel line l to BC through I.Note that l is tangent to $\odot IA_2C_2$.Let I_a, I_c be the ex-centres of ABC opposite to A, C respectively.Note that ΔIA_2C_2 is homothetic to ΔII_aI_c. So l is also tangent to $\odot II_aI_c$ at I.Now its well-known that A_1C_1 is the radical axis of $\odot II_aI_c$.Let $X = A_2C_2 \cap l$.So X, A_1, C_1 are collinear.Hence we get
$(IX, IK; IC_2, IA_2) = -1$.
Hence IK is the I-symmedian of ΔIA_2C_2.Hence IK passes trough the midpoint of BC.

Luis González
#4 Feb 11, 2012, 9:46 am • 1

Let I be an arbitrary point on the angle bisector of $\angle ABC$ (not necessarily the incenter of ABC as the problem states). IA, IC cut CB, BA at A_1, C_1 and cut the circumcircle ω of $\triangle ABC$ again at A_2, C_2. $K \equiv A_1C_2 \cap A_2C_1$.

BI cuts ω again at the midpoint M of the arc AC of ω. MA_2, MC_2 cut CB, BA at U, V. Then
$\angle UA_2I = \angle MBA = \angle UBI \Longrightarrow B, I, U, A_2$ are concyclic $\Longrightarrow \angle BUI = \angle BA_2A = \angle BCA \Longrightarrow IU \parallel AC$.
Similarly, $IV \parallel AC$, i.e. $\overline{UIV} \parallel AC$ $(\star)$. On the other hand, let $P \equiv UV \cap A_2C_2$. Then
$\angle PVC_2 = \angle IBC_2 = \angle UA_2C_2 \Longrightarrow U, V, A_2, C_2$ are concyclic. Now, since P, A_1, C_1 have equal power WRT $\odot(BUV)$ and $\odot(IA_2C_2)$, then P, A_1, C_1 lie on radical axis of $\odot(BUV), \odot(IA_2C_2)$, i.e. $P \equiv UV \cap A_1C_1 \cap A_2C_2 \Longrightarrow$
$I(A, C, K, P) = -1$. Together with $(\star)$, we conclude that IK is the I-median of $\triangle IAC$.

High School Olympiads

Feorbakh

Locked

Source: I don't know

hayoola

#1 Sep 20, 2015, 10:15 am

the incircle of triangle ABC touches BC, AB, AC at D, P, T let L be a parrale line to BS passes from A and let Y, X be the interestion DT, DP with line L prove that the ouler line(passes from O,G,H) of DXY passes from the feorbakh of ABC

Luis González

#2 Sep 20, 2015, 11:16 am

Please proofread before submitting; you have a typo and the correct spelling is *Feuerbach*. The problem has been discussed before at Feuerbach point and Euler Line and the Feuerbach Point.

High School Olympiads

Feuerbach point

Locked

Source: Iranian National Olympiad (3rd Round) 2008

Omid Hatami

#1 Sep 12, 2008, 5:05 pm

Let D,E,F be tangency point of incircle of triangle ABC with sides BC,AC,AB. DE and DF intersect the line from A parallel to BC at K and L. Prove that the Euler line of triangle DKL passes through Feuerbach point of triangle ABC.

pohoatza

#2 Sep 12, 2008, 8:40 pm

See Theorem 1 from Bogdan Suceava and Paul Yiu, The Feuerbach point and Euler lines, *Forum Geom.*, 6 (2006) 191-197.

High School Olympiads

Old VMO Geometry 2

Reply

Vietnamisalwaysinmyheart
#1 Sep 19, 2015, 1:48 pm

VMO 2003.B
Let ΔABC be an acute triangle, (O) is the circumcircle. M, N be points of AC : $\overline{MN} = \overline{AC}$.
Let D, E be a point: $MD \perp BC$; $NE \perp AB$
H is orthocenter of ΔABC.
O' be the center of circumcircle (BED)
a/ Prove that: $H \epsilon (BED)$
b/ Prove that midpoint of AN and B are symetric through midpoint of OO'.

This post has been edited 1 time. Last edited by Vietnamisalwaysinmyheart, Sep 19, 2015, 6:48 pm

Luis González
#2 Sep 20, 2015, 4:41 am • 1

a) Let $P \in MD \cap NE.$ Since $(MP \parallel AH) \perp BC$ and $(NP \parallel CH) \perp BA,$ it follows that P is the image of H under the translation determined by $\overrightarrow{MN} \Longrightarrow (PH \parallel AC) \perp BH \Longrightarrow H$ is on circle $\odot(BED)$ with diameter $\overline{PB}.$

b) Denote by U the midpoint of $\overline{AN},$ which is clearly midpoint of $\overline{CM}$ as well. If CH cuts $\odot(BED)$ again at $X,$ then since $HX \parallel PE,$ we have $\angle MDX = \angle EBH = \angle MCX \Longrightarrow X$ is on circle (U) with diameter $\overline{CM} \Longrightarrow UO' \perp XD.$ But since $\angle XDP = \angle EBH = \angle DBO$ and $\angle BDP = 90^\circ,$ it follows that $BO \perp XD,$ i.e. XD is antiparallel to AC WRT $BA, BC \Longrightarrow BO \parallel UO'.$ If AX cuts (O) again at $Y,$ we have $\angle BYC = \angle BAC = \angle BDX \Longrightarrow Y \in (U)$ and CY is antiparallel to XD WRT $BD, BX \Longrightarrow BO' \perp CY$ and $CY \perp OU \Longrightarrow OU \parallel BO'.$ As as result, $BOUO'$ is a parallelogram $\Longrightarrow U$ is reflection of B on the midpoint of $\overline{OO'}.$

jayme
#3 Oct 14, 2015, 6:22 pm

Dear Mathlinkers,

for (a)

1. the triangles PMN and HAC being homothetic and having MN = AC are equal; then AH = MP
2. the quadrilateral AHPM being a parallelogram, HP is perpendicular to HB
and we are done...

Sincerely
Jean-Louis

Quick Reply

High School Olympiads

areas in square

Locked

zukazuk

#1 Sep 19, 2015, 4:24 pm

Given $ABCD$ square which is inscribed in a circle. M is a point on smaller arc AB. H and G are points of intersection of MC and MD with the diagonals respectively. Prove that area of ABH equals area of HGD.

Luis González

#2 Sep 20, 2015, 2:23 am

Baltic Way 2014 (P13). See http://www.artofproblemsolving.com/community/c6h613451.

High School Olympiads

Triangles with equal areas

Reply

Source: Baltic Way 2014, Problem 13

socrates
#1 Nov 11, 2014, 8:42 pm

Let $ABCD$ be a square inscribed in a circle ω and let P be a point on the shorter arc AB of ω. Let $CP\cap BD = R$ and $DP \cap AC = S.$
Show that triangles ARB and DSR have equal areas.

Bandera
#2 Nov 13, 2014, 2:29 am

Let O be the intersection point of AC and BD. As $\angle SOD=90^{\circ}=\angle BPD$, $\angle OSD=90^{\circ}-\angle SDO=\angle PBD$. Furthermore, $\angle SAD=45^{\circ}=\angle BAC=\angle BPC$. By the law of sines:

$$\frac{DA}{DS}=\frac{\sin\angle ASD}{\sin\angle SAD}=\frac{\sin\angle OSD}{\sin\angle SAD}=\frac{\sin\angle PBR}{\sin\angle BPR}=\frac{RP}{RB}.$$

Calculating the power of R WRT ω, we get: $DR\cdot RB=CR\cdot RP$. Thus, $\frac{DA}{DS}=\frac{RP}{RB}=\frac{DR}{CR}$ and $CB\cdot CR=DA\cdot CR=DS\cdot DR$. Noting that $\angle BCR=\angle BCP=\angle BDP=\angle RDS$, we obtain that $\triangle CBR$ and $\triangle DSR$ have equal areas. But $\triangle CRB\cong\triangle ARB$.

sunken rock
#3 Nov 21, 2014, 9:50 pm

I met this problem few years ago and I have used a small trick to avoid trigo; it took me some time to recall it, but here it is:
Clearly $\triangle ARB\cong\triangle CBR\ (\ 1\ )$. Adding to its area the area of $\triangle CDR$ we get:
$[CRB]+[CRD]=[BCD]=R^2\ (\ 2\ )$. Adding to $[RDS]$ the same $[CDR]$ we get $[RDS]+[CDR]=[CDSR]$.
$CDSR$ having perpendicular diagonals, its area is $[CDSR]=\frac{CS\cdot DR}{2}\ (\ 3\ )$, so we need $CS\cdot DR=2R^2$. The last relation we get from $\triangle CDR\sim\triangle SCD\implies\frac{DR}{CD}=\frac{CD}{CS}$, done.
Here R is the circumradius the square.

Best regards,
sunken rock

Luis González
#4 Nov 22, 2014, 1:45 am

Let O be the center of $ABCD$ and let D_{∞} be the point at infinity of AC (also point at infinity of the tangent of ω at D).

$$C(O,B,R,D)\equiv C(A,B,P,D)=D(A,B,P,D_{\infty})\equiv D(A,O,S,D_{\infty})\Longrightarrow$$

$$\frac{OB}{OD}\cdot\frac{RD}{RB}=\frac{OA}{OS}\Longrightarrow OS\cdot RD=OA\cdot RB\Longrightarrow[DSR]=[ARB].$$

jayme
#5 Nov 7, 2015, 8:06 pm

Dear Mathlinkers,
there is a typo in this problem: P is the midpoint of the shorter arc AB...

Sincerely
Jean-Louis

IstekOlympiadTeam
#6 Nov 7, 2015, 8:29 pm

> *jayme wrote:*
> Dear Mathlinkers,
> there is a typo in this problem: P is the midpoint of the shorter arc AB...
>
> Sincerely
> Jean-Louis

Special Case

jayme
#7 Nov 7, 2015, 8:40 pm

Dear Mathlinkers,
my figure after toying desn't give the result...
Make a figure and verify again... Perhaps I am wrong but make a tentative...

Sincerely
Jean-Louis

Quick Reply

High School Olympiads

parallel

Reply

Source: Created by me

ferma2000

#1 Sep 19, 2015, 1:30 am

Dear mathlinkers;
1) ABC is a triangle.
2) $(P,Q),(R,S)$ are two pairs of isogonal conjugates such that (B,P,Q,C) and (B,R,S,C) lie on a circle.
Claim:
$PQ \parallel RS$
Best regards;

Luis González

#2 Sep 19, 2015, 2:48 am • 1

In general, we have $\angle BPC+\angle BQC=180^\circ+\angle BAC$. Thus if B,P,Q,C are concyclic, then $\angle BPC=\angle BQC \Longrightarrow 2\cdot\angle BPC=180^\circ+\angle BAC \Longrightarrow \angle BPC=\angle BIC=90^\circ+\frac{1}{2}\angle BAC$ where I is the incenter of $\triangle ABC$, i.e. $P\in\odot(IBC)$ and likewise $Q\in\odot(IBC)$. Thus by the symmetry across AI, it follows that $PQ\perp AI$ and similarly $RS\perp AI \Longrightarrow (PQ \parallel RS)\perp AI$.

Quick Reply

High School Olympiads

Incircle/excircle tangency points

Reply

Source: own, nice, simple

yetti
#1 Jun 22, 2008, 12:36 am

(O) is circumcircle of a $\triangle ABC, AA'$ the the circumcircle diameter. $D \in BC$ is foot of the A-altitude, $AD \perp BC.$ Circle (A) with center A and radius AD cuts the ray $(AA'$ at $Z.$ Circle (A') with center A' and radius $A'Z$ cuts BC at $X, Y.$ Show that the triangle incircle and A-excircle in the angle $\angle A$ are tangent to BC at the midpoints of the segments $DX, DY.$

Little Gauss
#2 Jun 24, 2008, 6:44 pm

Let D' be a foot of a altitude from A' to BC. $BD = CD'$.
From $XD' = YD'$, we get 2 midpoints are symmetric wrt midpoint of BC.
Let $XY = 2x$. Only remains to show is $x = b - c$
If E is on the BC and $\angle BAE = CAE$, $DE^2 = EZ^2 = D'E^2 - x^2$
So $x^2 = ED'^2 - ED^2 = DD'(ED' - ED) = DD'(CE - BE)$
$= 2R\sin(B - C)\dfrac{(b - c)a}{b + c} = \dfrac{b^2 - c^2}{a}\dfrac{(b - c)a}{b + c} = (b - c)^2.$
Done

28121941
#3 Jun 24, 2008, 8:56 pm

you wrote
From XD'=YD' , we get 2 midpoints are symmetric wrt midpoint of BC.
From what segments are the mid points you are speaking? I suppose these are the mid points of the segments of the statement of the problem, I am right?
Thanks in advance

Little Gauss
#4 Jun 25, 2008, 7:47 pm

DX, DY...

Luis González
#5 Sep 18, 2015, 9:59 am

Incircle (I) touches BC at $P,$ AI cuts BC at L and cuts (O) again at $M.$ Circle $\odot(I, ID)$ cuts BC again at X'; reflection of D on P and by symmetry across $AI,$ we have $Z \in \odot(I, ID).$

Let J be the projection of I on AD and $S \equiv MA' \cap AD.$ Since M, P, J are collinear (well-known), then $\angle IPM = \angle IDS$ and from cyclic $IJSM,$ we have $\angle IMP = \angle ISD,$ yielding $\angle PIM = \angle DIS = \angle ZIA'$ (due to symmetry across AI). Thus from cyclic $IZX'L$ (due to $\angle IX'L = \angle IDL = \angle IZL$), we get $\angle IZX' = \angle ILP \Longrightarrow$ $\angle(IA', X'Z) = \angle IPL = 90^\circ,$ i.e. $IA' \perp ZX'$ is perpendicular bisector of $ZX' \Longrightarrow A'Z = A'X' \Longrightarrow X \equiv X',$ i.e. P is midpoint of $DX.$ The proof for the excircle is exactly the same.

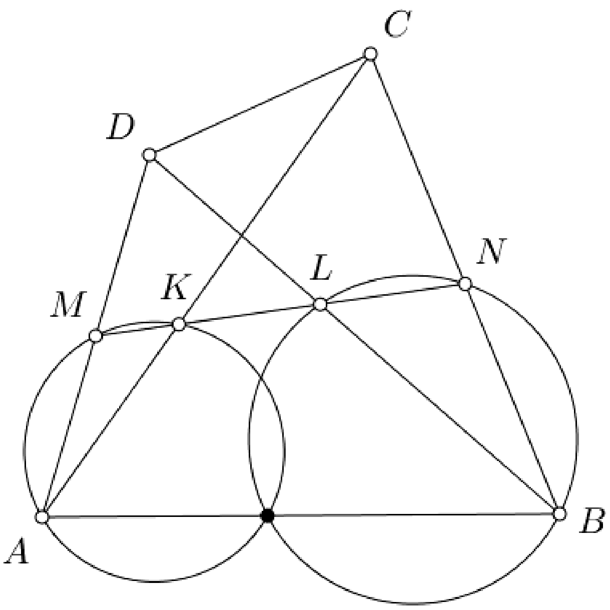
Quick Reply

High School Olympiads

2015 Bulgaria Team Selection Test Round 1, Problem 2

estoyanovvd
#1 Sep 17, 2015, 7:38 pm

Quadrilateral $ABCD$ is given with $AD \nparallel BC$. The midpoints of AD and BC are denoted by M and N, respectively. The line MN intersects the diagonals AC and BD in points K and L, respectively. Prove that the circumcircles of the triangles AKM and BNL have common point on the line AB.(Proposed by Emil Stoyanov)



This post has been edited 2 times. Last edited by estoyanovvd, Sep 17, 2015, 7:41 pm
Reason: edit

Luis González
#2 Sep 17, 2015, 9:47 pm • 1

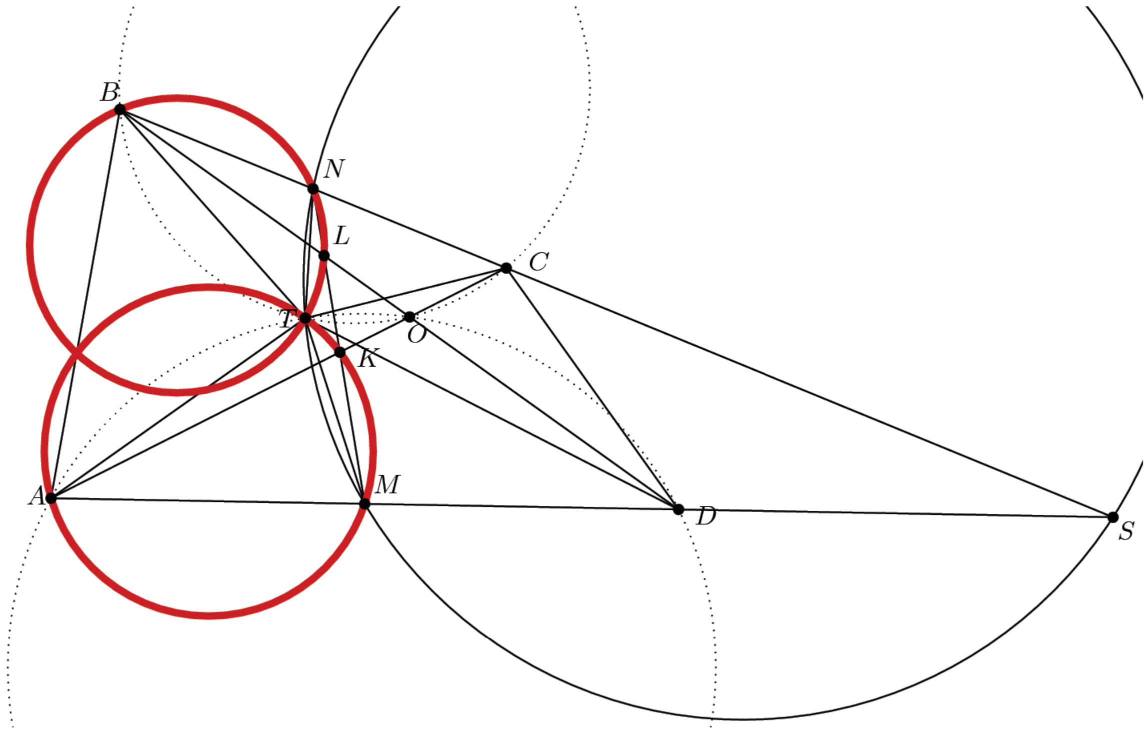
Let $X \equiv MN \cap CD$ and $P \equiv AC \cap BD$. By Menelaus' theorem for $\triangle ACD$ and $\triangle BCD$ cut by MN, we obtain $\frac{KA}{KC} = \frac{LD}{LB} = -\frac{XD}{XC} \Longrightarrow \frac{KA \cdot KP}{KP \cdot KC} = \frac{LD \cdot LP}{LP \cdot LB} \Longrightarrow$ powers of K and L WRT $\odot(PAD)$ and $\odot(PBC)$ are in the same ratio $\Longrightarrow \odot(PAD), \odot(PBC)$ and $\odot(PKL)$ are coaxal, i.e. if Q is the second intersection of $\odot(PAD)$ and $\odot(PBC)$, then $PKQL$ is cyclic. Therefore Q is the Miquel point of MN WRT $\triangle PAD \Longrightarrow Q \in \odot(AMK)$ and likewise $Q \in \odot(BNL)$. Now by Miquel theorem in $\triangle PAB$, it follows that the 2nd intersection of $\odot(AMKQ)$ and $\odot(BNLQ)$ lies on AB.

P.S. The property still holds for all points M, N on AD, BC verifying $\frac{MA}{MD} = \frac{NC}{NB}$. The proof is exactly the same.

silouan
#3 Sep 18, 2015, 12:56 am • 1

Let me add another quick solution I think which also works on the general case posted by Luis! Note also that the problem is similar to:
http://artofproblemsolving.com/community/c6h84559p490691
Let the $AC \cap BD \equiv O$ and $BC \cap AD \equiv S$
Then the center T of spiral similarity sending $A \mapsto C$ and $D \mapsto B$ is the second point of intersection of $\odot(AOD), \odot(BOC)$
But this is the same sending $A \mapsto C$ and $M \mapsto N$ and the same sending $M \mapsto N$ and $D \mapsto B$ so
T, K, M, A and T, L, N, B are cocyclic and this means that T is the Miquel point of the complete quadrilateral $SCANM$, so $SNTM$ is cyclic

so $SNTM$ is cyclic,
so now by Miquel's pivot theorem in triangle SAB we have the desired.



This post has been edited 2 times. Last edited by silouan, Sep 18, 2015, 1:08 am
Reason: edit

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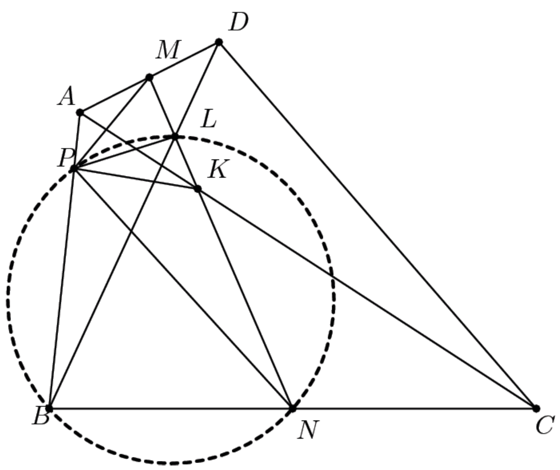
gavrilos

#4 Oct 25, 2015, 7:50 pm • 1

Hello!

My solution.



Let $P \equiv AB \cap \odot(BLN)$.It suffices to show that $APKM$ is cyclic.

We will show that $\triangle MLP \simeq \triangle ABC$.Since $\angle MLP = \angle ABC$ it suffices to show that $\dfrac{AB}{CB} = \dfrac{ML}{PL}$ (1).

● The sine law gives $\dfrac{PL}{\sin P\hat{N}L} = \dfrac{NL}{\sin N\hat{P}L} \Rightarrow PL = \dfrac{NL \cdot \sin A\hat{B}D}{\sin C\hat{B}D}$.

Thus,(1) becomes $\dfrac{AB}{CB} = \dfrac{ML \cdot \sin C\hat{B}D}{NL \cdot \sin A\hat{B}D}$ (2).

● The sine law gives $\dfrac{ML}{\ldots} = \dfrac{MD}{\ldots} \Rightarrow ML = \dfrac{MD \cdot \sin A\hat{D}B}{\ldots}$

- The sine law gives $\dfrac{}{\sin M\hat{D}L} = \dfrac{}{\sin M\hat{L}D} \Rightarrow ML = \dfrac{}{\sin M\hat{L}D}$

and $\dfrac{NL}{\sin N\hat{B}L} = \dfrac{BN}{\sin B\hat{L}N} \Rightarrow NL = \dfrac{BN \cdot \sin C\hat{B}D}{\sin B\hat{L}N}$.

Thus,since $\angle MLD = \angle BLN$,(2) becomes $\dfrac{AB}{CB} = \dfrac{MD \cdot \sin A\hat{D}B}{BN \cdot \sin C\hat{B}D} \cdot \dfrac{\sin C\hat{B}D}{\sin A\hat{B}D} \Leftrightarrow$

$\Leftrightarrow \dfrac{AB}{CB} = \dfrac{AD \cdot \sin A\hat{D}B}{BC \cdot \sin A\hat{B}D} \Leftrightarrow \dfrac{AB}{AD} = \dfrac{\sin A\hat{D}B}{\sin A\hat{B}D}$which is true from the sine law.

Thus $\triangle MLP \simeq \triangle ABC$ as we wanted.This gives $\angle PML = \angle BAC \Leftrightarrow \angle PMK = \angle PAK$

which gives that $APMK$ is cyclic and we are done.

This post has been edited 1 time. Last edited by gavrilos, Oct 25, 2015, 7:51 pm

Misha57

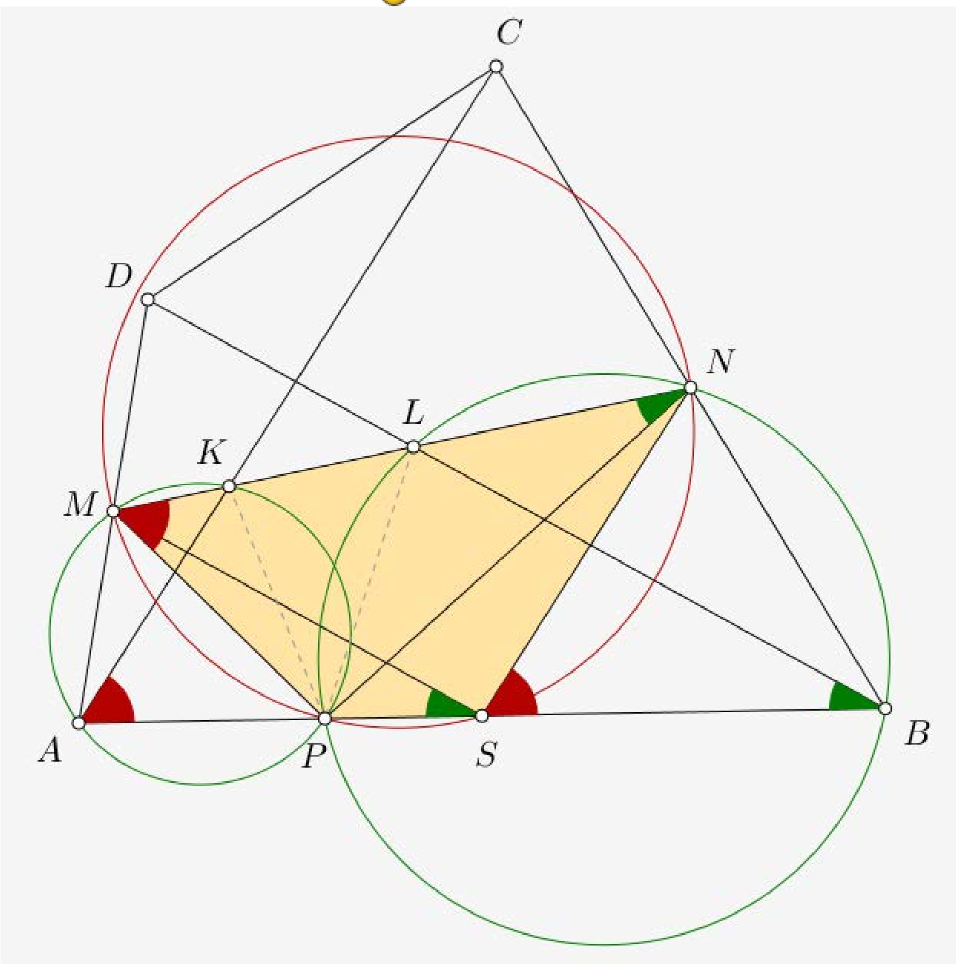
#5 Oct 25, 2015, 9:02 pm • 1

Let S be midpoint of AB.Circle MNS intersects AB second time in point W.Actually this point is common point of our circles(which can be proved by counting angles).

estoyanovvd

#8 Oct 25, 2015, 9:48 pm

Misha57:It is an incrediblle solution!



Quick Reply

High School Olympiads

maybe easy but nice geometry

phuocdinh_vn99
#1 Sep 17, 2015, 5:39 am

Given triangle ABC with incenter I and circumcircle (O). AO cuts (O) at D. DI cuts (O) at E. AE cuts BC at F. Prove that $FI \perp AI$

Luis González
#2 Sep 17, 2015, 6:11 am • 1

Let Y,Z be the projections of I on $AC,AB.$ Since $\angle AEI \equiv \angle AED = 90^\circ$, then E is on circle $\odot(AYIZ)$ with diameter $\overline{IA}$. AE, BC are radical axes of $(O), \odot(AYIZ)$ and $(O), \odot(BIC)$ meeting at the radical center F of $(O), \odot(IBC)$, $\odot(AYIZ) \Longrightarrow F$ is on the radical axis of $\odot(BIC), \odot(AYIZ)$. But since IA is the I-circumdiameter of $\triangle IBC$ (well-known), then $\odot(BIC), \odot(AYIZ)$ are tangent at $I \Longrightarrow FI \perp AI.$

Dukejukem
#3 Sep 17, 2015, 9:25 am • 3

Let BI cut (O) for a second time at M, the midpoint of arc $\widehat{AC}$, and denote $X \equiv CD \cap AM.$ Since AD is a diameter of (O), it follows that $\triangle ACX$ is right-angled at $C.$ Therefore, $\angle AXC = 90^\circ - \angle MAC = 90^\circ - \frac{B}{2}.$ Meanwhile, $\angle AIC = 90^\circ + \frac{B}{2}$, whence it follows that $AICX$ is cyclic. Hence, $\angle AIX = \angle AXC = 90^\circ.$ But from Pascal's Theorem applied to $AEDCBM$, we find that F, I, X are collinear. The desired result follows. $\square$

jayme
#4 Sep 17, 2015, 4:18 pm

Dear Mathlinkers,

1. like Luis, the circle with diameter AI goes through E
2. A+, A- the midpoints of the arcs BAC and BC which doesn't contain A
3. A* the second point of intersection of A+I with (O) (note that A* is the contact point of the A-mixtilinear incircle with (O))
4. the circle with diameter IA-

and we finish with the three chords theorem...

sincerely
Jean-Louis

High School Olympiads

Geometry

Locked

Blacklord

#1 Sep 17, 2015, 12:41 am

P is an interior point of the triangle ABC.H is the feet of altitude from A.A1 and A2 are the points on AH and BC such that angles PA1A=90 and PA2C=90.
BBCC defines similarly. Prove that A1A2,B1B2,C1C2 are concurent.

Luis González

#2 Sep 17, 2015, 12:52 am

Discussed before at http://www.artofproblemsolving.com/community/c6h378378.

High School Olympiads

4 points of intersection of 4 circles are concyclic

Reply

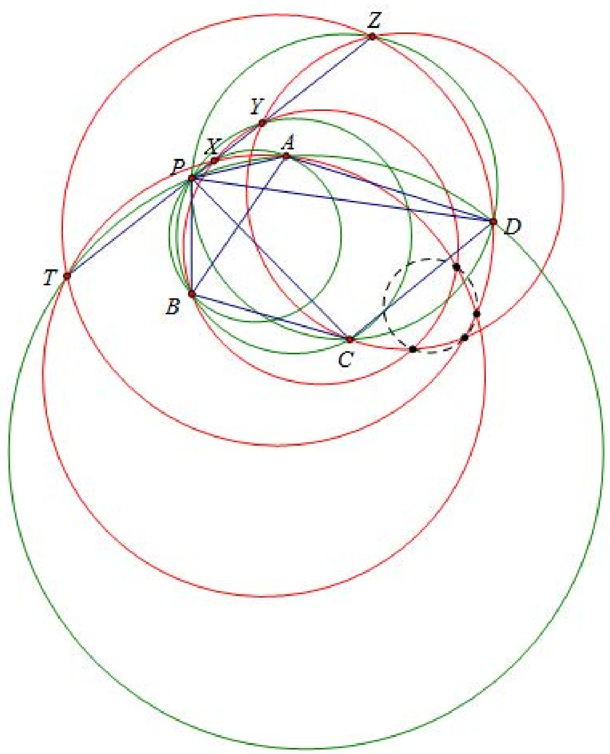
Source: OWN

LeVietAn

#1 Sep 14, 2015, 4:36 pm

Dear Mathlinkers,
Assume that $ABCD$ is a quadrilateral (convex or concave) and P is a point which is not on circumcircle and lines through the side of triangles ABC, BCD, CDA, DAB.
An arbitrary line passing through P intersects $\odot(PAB), \odot(PBC), \odot(PCD), \odot(PDA)$ at X, Y, Z, T, resp.
Prove that the four points of intersection (other than X, Y, Z, T) of $\odot(BXY)$ and $\odot(CYZ)$, of $\odot(CYZ)$ and $\odot(DZT)$, of $\odot(DZT)$ and $\odot(ATX)$, of $\odot(ATX)$ and $\odot(BXY)$ are concyclic.

Attachments:



Luis González

#2 Sep 14, 2015, 10:05 pm

Inverting with center P we get a well-known problem: $ABCD$ is a quadrilateral and ℓ is an arbitrary line on its plane. The Miquel points of the quadrilaterals formed by ℓ and each three sides of $ABCD$ are concyclic and this circle also contains the Miquel point of $ABCD$.

Luis González

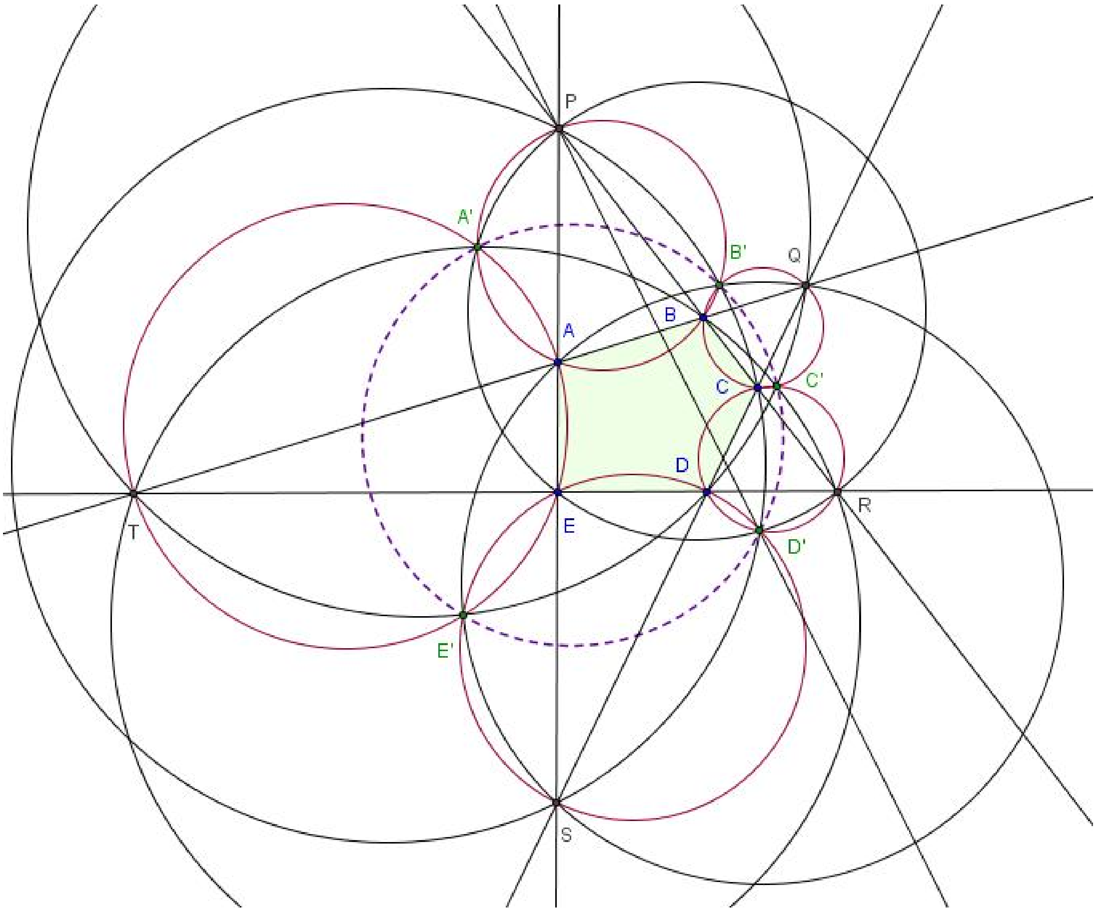
#3 Sep 15, 2015, 3:51 am

The previous statement is equivalent to the Miquel-Morley pentagram problem:

$ABCDE$ is a pentagon and let $P \equiv EA \cap BC, Q \equiv AB \cap CD, R \equiv BC \cap DE, S \equiv CD \cap AE, T \equiv DE \cap AB$. Circles $\odot(EAT)$ and $\odot(ABP)$ meet again at A' and define B', C', D', E' in the same way. Then A', B', C', D', E' are concyclic.

Proof: As A' is the Miquel point of $AERB$, it follows that $A' \equiv \odot(TBR) \cap \odot(PER)$. Likewise we have $B' \equiv \odot(QAS) \cap \odot(PCS), C' \equiv \odot(QDT) \cap \odot(RBT), D' \equiv \odot(SCP) \cap \odot(REP)$ and $E' \equiv \odot(SAQ) \cap \odot(TDQ)$. Therefore $\angle B'D'P = \angle B'CP = \angle B'C'B$ and $\angle A'D'P = \angle A'RB = \angle A'C'B \Longrightarrow \angle B'D'A' = \angle B'D'P + \angle A'D'P = \angle B'C'B + \angle A'C'B = \angle B'C'A' \Longrightarrow A', B', C', D'$ are concyclic and in exactly the same way B', C', D', E' are concyclic $\Longrightarrow A', B', C', D', E'$ are concyclic.

Attachments:



Quick Reply

High School Olympiads

Line bisects segment

Source: Own, HSGS TST 2015

buratinogigle

#1 Sep 13, 2015, 3:25 pm • 3

Let ABC be a triangle with bisector AD, D lies on segment BC. E, F lie on segments CA, AB, reps, such that $EF \parallel BC$. M, N are projections of C, B on line DE, DF, reps. Circumcircle of triangle AFN, AEM intersect again at P. Prove that AP bisects segment EF.

TelvCohl

#2 Sep 13, 2015, 7:20 pm • 4

Let $U \equiv \odot(BD) \cap \odot(AFN), V \equiv \odot(CD) \cap \odot(AEM), Y \equiv EF \cap \odot(AFN), Z \equiv EF \cap \odot(AEM)$.

From Reim's theorem we get $U \in BY$ and $V \in CZ$. Since $\measuredangle DBU = \measuredangle DNU = \measuredangle BAU$, so $\odot(ABU)$ is tangent to BC at B. Similarly, we can prove $\odot(ACV)$ is tangent to BC at C, so the radical axis of $\odot(ABU)$ and $\odot(ACV)$ is the A-median of $\triangle ABC$. Let $J \equiv AD \cap \odot(ABC)$ be the midpoint of arc BC of $\odot(ABC)$ and Φ_d be the Inversion with center D that swaps B, C. Since $\Phi_d(\odot(BD)), \Phi_d(\odot(CD))$ is the perpendicular from C, B to BC, resp and $\Phi_d(\odot(ABU))$, $\Phi_d(\odot(ACV))$ is the circle passing through J and tangent to BC at C, B, resp, so we get $B, C, \Phi_d(U), \Phi_d(V)$ lie on a circle (from symmetry) $\Longrightarrow B, C, U, V$ are concyclic, hence $X \equiv BU \cap CV$ lies on the A-median of $\triangle ABC$ (radical axis of $\odot(ABU), \odot(ACV)$). From Reim's theorem $\Longrightarrow U, V, Y, Z$ are concyclic, so X lies on the radical axis AP of $\odot(AEM)$ and $\odot(AFN)$, hence AP is the A-median of $\triangle ABC \Longrightarrow AP$ passes through the midpoint of EF ($\because EF \parallel BC$).

This post has been edited 2 times. Last edited by TelvCohl, Sep 19, 2015, 12:06 am

buratinogigle

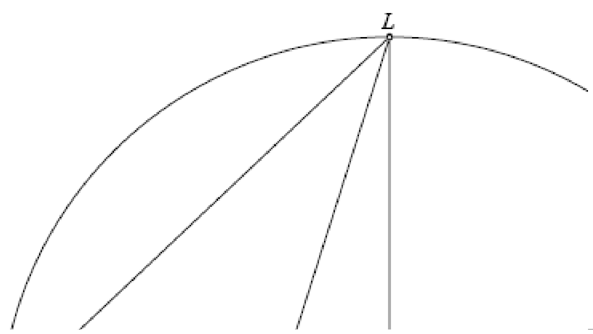
#3 Sep 14, 2015, 1:19 am • 1

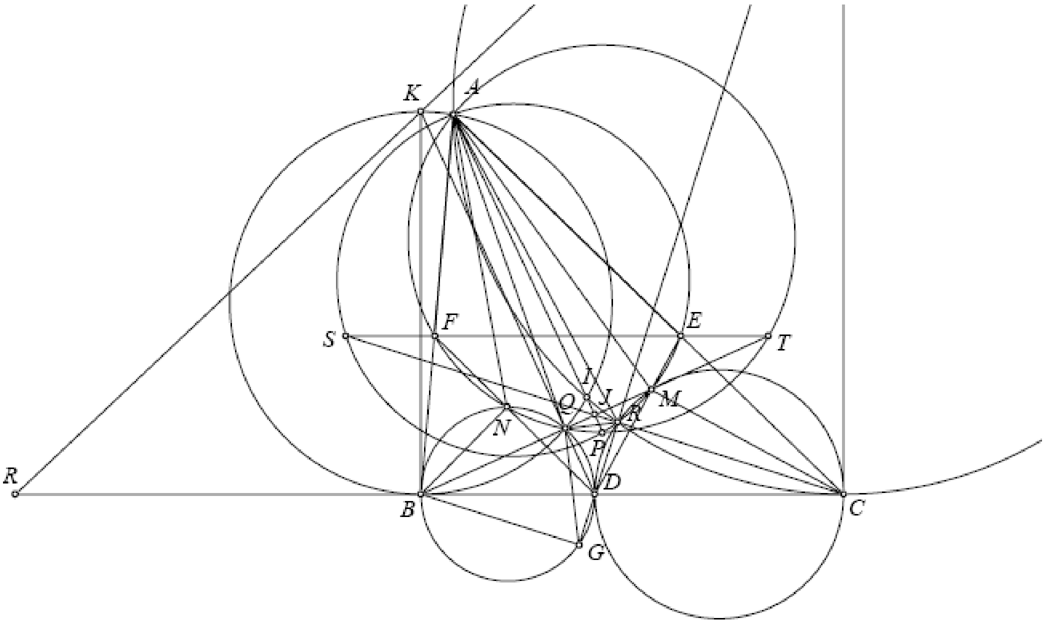
Thank Telv for your interest, my solution is the same idea with you.

My solution. Let EF cut $(AEM), (AFN)$ again at S, T. Let CS, BT cut $(CDM), (BDN)$ again at R, Q. Since $EF \parallel BC$, we have $\angle FAQ = \angle FTQ = \angle QBD$ hence (ABQ) is tangent to BC. Similarly, (ACR) is tangent to BC. So (ABQ) incetersect (ACR) again at I then AI bisects BC. We will prove that A, P, I are collinear, indeed.

Let BK, CL be diameters of $(ABQ), (ACR)$ then KL cuts BC at X, easily seen $XA^2 = XB.XC$, this implies $\frac{BK}{CL} = \frac{XB}{XC} = \frac{AB^2}{AC^2} = \frac{DB^2}{DC^2} = \frac{DQ.DK}{DR.DL}$ or $\frac{DQ}{DR} = \frac{BK}{DK}.\frac{DL}{BL} = \frac{BQ}{BD}.\frac{CD}{CR}$. Let RD cuts (BD) again at G then $\angle QBG = \angle RDQ$ and $\frac{BQ}{BG} = \frac{BQ}{CR}.\frac{CR}{BG} = \frac{BQ}{CR}.\frac{DC}{DR} = \frac{DQ}{DR}$. From this, $\triangle DQR \sim \triangle BGQ$ deduce $\angle DRQ = \angle BGQ = \angle BDQ$, hence $\angle QRC + \angle QBD = 180^\circ$ or $BQRC$ is cyclic. Easily seen, $STRQ$ is cyclic, too. Let BT cut CS at J then $JQ.JB = JR.JC$ thus J lies on AI which is radical axis of $(ABQ), (ACR)$. From $STRQ$ is cyclic then $JQ.JT = JS.JR$ this means J lies on AP which is radical axis of $(AEM), (AFN)$. Now we see P, I lie on AJ or A, P, I are collinear, we are done.

Attachments:





Luis González

#4 Sep 14, 2015, 4:41 am • 3

Let X be the 2nd intersection of $\odot(AFN)$ with the circle with diameter $\overline{DB}$ and let BX cut EF at U.
$\angle DNX = \angle DBX = \angle XUF \Longrightarrow U \in \odot(AFN) \Longrightarrow \angle BAX = \angle FUX = \angle DBX \Longrightarrow \odot(ABX)$ is tangent to BC. Similarly $\odot(AEM)$ goes to the 2nd intersection Y of the circle with diameter $\overline{DC}$ with the circle passing through A, C and tangent to BC. Thus after inverting with center A, the problem becomes:

Tangents to the circumcircle (O) of $\triangle ABC$ at B, C meet the tangent to (O) through the midpoint D of its arc BC at $B,' C'$. X, Y are the reflections of B, C on B', C' and E, F are points on AC, AB, such that $EF \parallel BC$. Then $P \equiv EY \cap FX$ is on the A-symmedian of $\triangle ABC$.

Let XY cut AB, AC at U, V and let the tangents of (O) at B, C meet at S. Since $EF \parallel BC \parallel XY \equiv UV \Longrightarrow$ $X(F, U, B, A) = Y(E, V, C, A) \Longrightarrow P \equiv XF \cap YE, S \equiv XB \cap YC$ and A are collinear, i.e. P is on the A-symmedian AS of $\triangle ABC$, as required.

buratinogigle

#5 Sep 14, 2015, 8:55 am • 1

I have seen general problem

Let ABC be a triangle and E, F lie on CA, AB such that $EF \parallel BC$. K, L lie on BC such that $\angle KAB = \angle LAC$. Let M, N be projections of C, B on LE, KF. Circumcircles of triangles AEM, AFN intersect again at P. Prove that AP bisects EF.

Luis González

#6 Sep 14, 2015, 9:20 am • 2

> *buratinogigle wrote:*
> I have seen general problem
>
> Let ABC be a triangle and E, F lie on CA, AB such that $EF \parallel BC. K, L$ lie on BC such that $\angle KAB = \angle LAC$. Let M, N be projections of C, B on LE, KF. Circumcircles of triangles AEM, AFN intersect again at P. Prove that AP bisects EF.

My previous solution still works. Inverting with center A we get the equivalent problem:

K, L are two points on the circumcircle (O) of $\triangle ABC$ such that AK, AL are isogonals WRT $\angle BAC$. Tangents of (O) through K, L cut the tangents of (O) through B, C at B', C', resp. X, Y are the reflections of B, C on B', C'. E, F are points on AC, AB, such that $EF \parallel BC$. Then $P \equiv EY \cap FX$ is on the A-symmedian of $\triangle ABC$.

Again let XY cut AB, AC at U, V and let the tangents of (O) at B, C meet at S. By obvious symmetry we have $EF \parallel BC \parallel XY \equiv UV \Longrightarrow X(F,U,B,A) = Y(E,V,C,A) \Longrightarrow P, S$ and A are collinear.

buratinogigle

#7 Sep 14, 2015, 3:17 pm

Thank Luis so much for your interest, I proposed other general problem as following

Let ABC be a triangle with bisector AD. Perpendicular bisector of AD cuts BC at R. Circle (K) passes through B, D and circle (L) passes through C, D such that KL passes through R. E, F lie on CA, AB such that $EF \parallel BC$. DE, DF cuts $(L), (K)$ again at M, N, reps. Circumcircle of triangles AEM, AFN intersect again at P. Prove that AP bisects EF.

I think my solution and of Telv are avaliable with the inversion center R.

THVSH

#8 Sep 14, 2015, 8:51 pm • 1

> *buratinogigle wrote:*
>
> I have seen general problem
>
> Let ABC be a triangle and E, F lie on CA, AB such that $EF \parallel BC$. K, L lie on BC such that $\angle KAB = \angle LAC$. Let M, N be projections of C, B on LE, KF. Circumcircles of triangles AEM, AFN intersect again at P. Prove that AP bisects EF.

Let $X = EF \cap \odot(AEM); Y = EF \cap \odot(AFN)$. Let G, H, E_1, F_1 be the projection of X, A, E, F on BC, respectively. Let R be the midpoint of EF. I is antipode of E in $\odot(AEM)$.

It's easy to see that C, M, I are collinear. So since $\angle CML = \angle IGC = 90^\circ$, $MLGI$ are concyclic. So $CL.CG = CM.CI = CE.CA. \Longrightarrow XE = GE_1 = CG - CE_1 = \frac{CA}{CL}.CE - CE_1$. Similarly, we get $YF = \frac{BA}{BK}.BF - BF_1$.
Now we have

$$BH^2 - CH^2 = AB^2 - AC^2 \Longrightarrow BH - CH = (AB - AC).\frac{AB + AC}{BC} \Longrightarrow BH - CH = \frac{BA}{BD}.BA - \frac{CA}{CD}.CA$$

$$\Longrightarrow BF_1 - CE_1 = \frac{BA}{BD}.BF - \frac{CA}{CD}.CE \Longrightarrow \frac{CA}{CD}.CE - CE_1 = \frac{BA}{BD}.BF - BF_1$$

On the other hand, we have $\frac{BK}{CL} = \frac{AB}{AC}.\frac{AK}{AL} = \frac{AB}{AC}.\frac{DK}{DL} \Longrightarrow \frac{CA^2.DL}{CL.CD} = \frac{BA^2.DK}{BK.BD} \Longrightarrow \frac{CA^2}{CL} - \frac{CA^2}{CD} = \frac{BA^2}{BK} - \frac{BA^2}{BD} \Longrightarrow \frac{CA}{CL}.CE - \frac{CA}{CD}.CE = \frac{BA}{BK}.BF - \frac{BA}{BD}.BF$
So now we get $\frac{CA}{CL}.CE - CE_1 = \frac{BA}{BK}.BF - BF_1. \Longrightarrow XE = YF$. Hence, $RE.RX = RF.RY$ It means: R lies on the radical axis of $\odot(AEM); \odot(AFN)$. i.e $R \in AP$ Q.E.D

THVSH

#11 Sep 14, 2015, 10:36 pm • 1

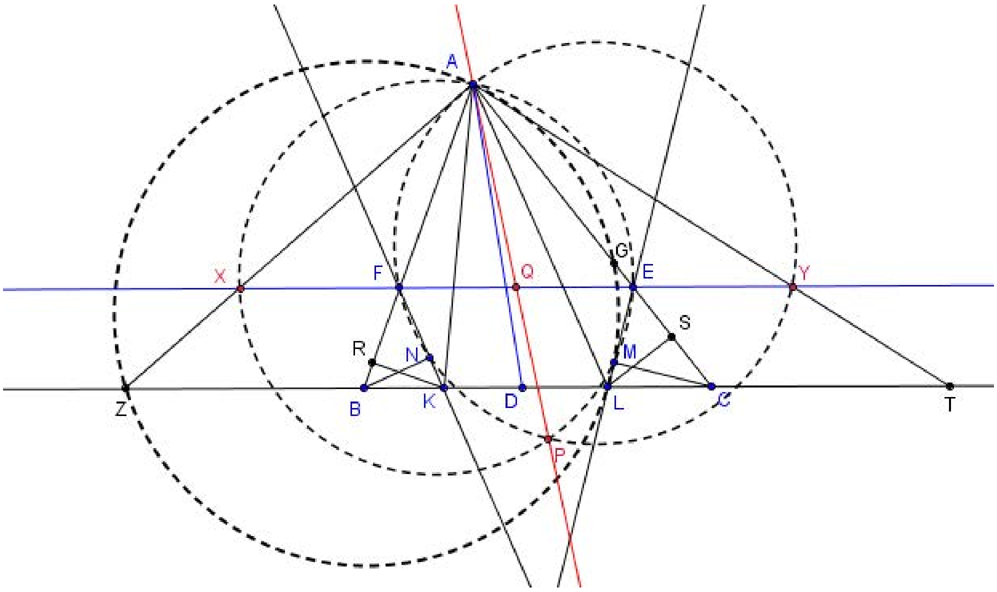
> *buratinogigle wrote:*
>
> I have seen general problem
>
> Let ABC be a triangle and E, F lie on CA, AB such that $EF \parallel BC$. K, L lie on BC such that $\angle KAB = \angle LAC$. Let M, N be projections of C, B on LE, KF. Circumcircles of triangles AEM, AFN intersect again at P. Prove that AP bisects EF.

Another proof: Let $X = EF \cap \odot(AEM); Y = EF \cap \odot(AFN); Z = AX \cap BC; T = AY \cap BC.$. Let $G = \odot(AML) \cap AC$. Let Q be the midpoint of EF. R, S are the projection of K, L on AB, AC,respectively.

Since $\angle AZL = \angle AXE = \angle AME$, we get $AZLM$ are concyclic i.e $Z \in \odot(A, L, M, G). \Longrightarrow CL.CZ = CG.CA$.
On the other hand, we have $EG.EA = EM.EL = ES.EC \Longrightarrow \frac{EG}{ES} = \frac{EC}{EA} = \frac{EG+EC}{ES+EA} = \frac{CG}{AS} \Longrightarrow CZ = \frac{EC}{EA}.AS.\frac{CA}{CL}$.

Similarly, we have: $BT = \frac{FB}{FA}.AR.\frac{BA}{BK}$.
On the other hand, we have $\frac{BK}{CL} = \frac{AB}{AC}.\frac{AK}{AL} = \frac{AB}{AC}.\frac{AR}{AS} \Longrightarrow \frac{AS.AC}{CL} = \frac{AR.AB}{BK}$, combine $\frac{EC}{EA} = \frac{FB}{FA}$ (since $EF \parallel BC$) $\Longrightarrow CZ = BT$ i.e $EX = FY$. Thus, $QE.QX = QF.QY$, so Q lies on the radical axis of $\odot(AEM); \odot(AFN)$. Hence, $Q \in AP$. Q.E.D

Attachments:



THVSH

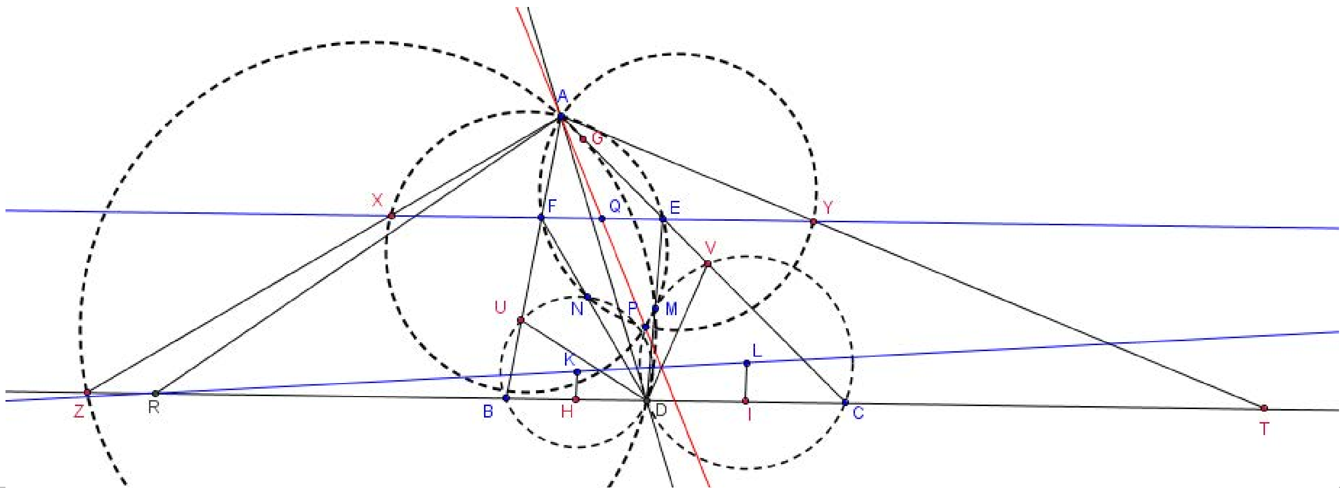
#12 Sep 14, 2015, 11:23 pm

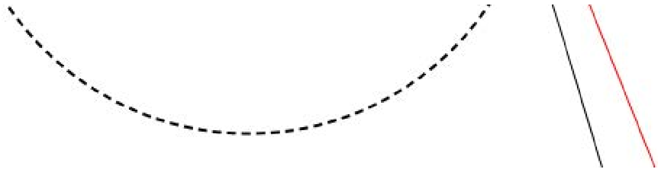
> *buratinogigle wrote:*
> Thank Luis so much for your interest, I proposed other general problem as following
>
> Let ABC be a triangle with bisector AD. Perpendicular bisector of AD cuts BC at R. Circle (K) passes through B, D and circle (L) passes through C, D such that KL passes through R. E, F lie on CA, AB such that $EF \parallel BC$. DE, DF cuts $(L), (K)$ again at M, N, reps. Circumcircle of triangles AEM, AFN intersect again at P. Prove that AP bisects EF.
>
> I think my solution and of Telv are avaliable with the inversion center R.

Let $X = EF \cap \odot(AEM); Y = EF \cap \odot(AFN); Z = AX \cap BC; T = AY \cap BC$.. Let $G = \odot(AMD) \cap AC$. Let Q, H, I be the midpoint of EF, BD, CD, respectively. Let $U = \odot(K) \cap AB; V = \odot(L) \cap AC$.

It is known that AR is tangent to $\odot(ABC)$. Then $RD^2 = RA^2 = RB.RC \Longrightarrow \frac{RB}{RD} = \frac{RD}{RC} = \frac{RD-RB}{RC-RD} = \frac{BD}{CD} = \frac{DH}{CI} = \frac{RD-DH}{RC-CI} = \frac{RH}{RI} = \frac{RK}{RL} \Longrightarrow DK \parallel CL \Longrightarrow \angle KDB = \angle LCD$ i.e $\angle BKD = \angle CLD$. So we get $\angle BUD = \angle CVD$. Thus, $\triangle AUD = \triangle AVD. \Longrightarrow AU = AV$.
Since $\angle AZD = \angle AXE = \angle AME$, we get $AZDM$ are concyclic i.e $Z \in \odot(A, D, M, G) \Longrightarrow CD.CZ = CG.CA$.
On the other hand, we have $EG.EA = EM.ED = EV.EC \Longrightarrow \frac{EG}{EV} = \frac{EC}{EA} = \frac{EG+EC}{EV+EA} = \frac{CG}{AV} \Longrightarrow CZ = \frac{EC}{EA}.AV.\frac{CA}{CD}$. Similarly, $BT = \frac{FB}{FA}.AU.\frac{BA}{BD}$. Hence, $CZ = BT$ i.e $EX = FY$. So $QE.QX = QF.QY$, then Q lies on the radical axis of $\odot(AEM); \odot(AFN)$, so $Q \in AP$. Q.E.D

Attachments:





Luis González

#13 Sep 14, 2015, 11:50 pm • 1

> buratinogigle wrote:
> Let ABC be a triangle with bisector AD. Perpendicular bisector of AD cuts BC at R. Circle (K) passes through B, D and circle (L) passes through C, D such that KL passes through R. E, F lie on CA, AB such that $EF \parallel BC$. DE, DF cuts $(L), (K)$ again at M, N, reps. Circumcircle of triangles AEM, AFN intersect again at P. Prove that AP bisects EF.

Invertion with center A still works nicely. Note that $(L), (K)$ and the A-Apollonius circle $\odot(R, RA)$ are coaxal, thus after inverting with center A, we get the following problem:

In a $\triangle ABC$, let D be the midpoint of the arc BC of its circumcircle (O). T is abritrary point on the perpendicular bisector of BC. Circles $\odot(TDB)$ and $\odot(TDC)$ cut the tangents of (O) at B, C again at X, Y, respectively. E, F are points on AC, AB, such that $EF \parallel BC$. Then $P \equiv EY \cap FX$ is on the A-symmedian.

Again by obvious symmetry, we have $EF \parallel BC \parallel XY$ and the proof is the same.

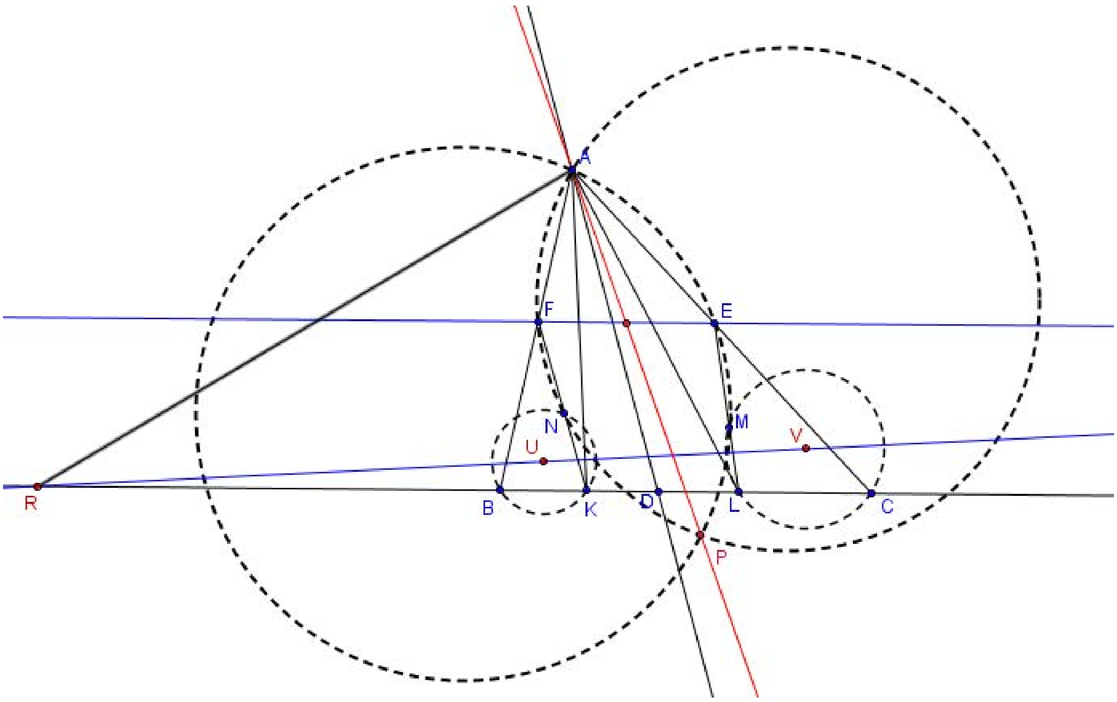
THVSH

#14 Sep 15, 2015, 12:00 am • 1

Another generalization: Let ABC be a triangle and E, F lie on CA, AB such that $EF \parallel BC$. K, L lie on BC such that $\angle KAB = \angle LAC$. The tangent at A of $\odot(ABC)$ intersects BC at R. A circle $\odot(U)$ passes through B, K and a circle $\odot(V)$ passes through C, L such that UV passes through R. LE, KF intersect $\odot(V)$; $\odot(U)$ again at M, N, respectively. $\odot(AEM)$; $\odot(AFN)$ intersect again at P. Prove that AP bisects EF.

This problem can be solved similar to my two previous posts.

Attachments:



Quick Reply

High School Olympiads

Angle bisectors and isogonal lines

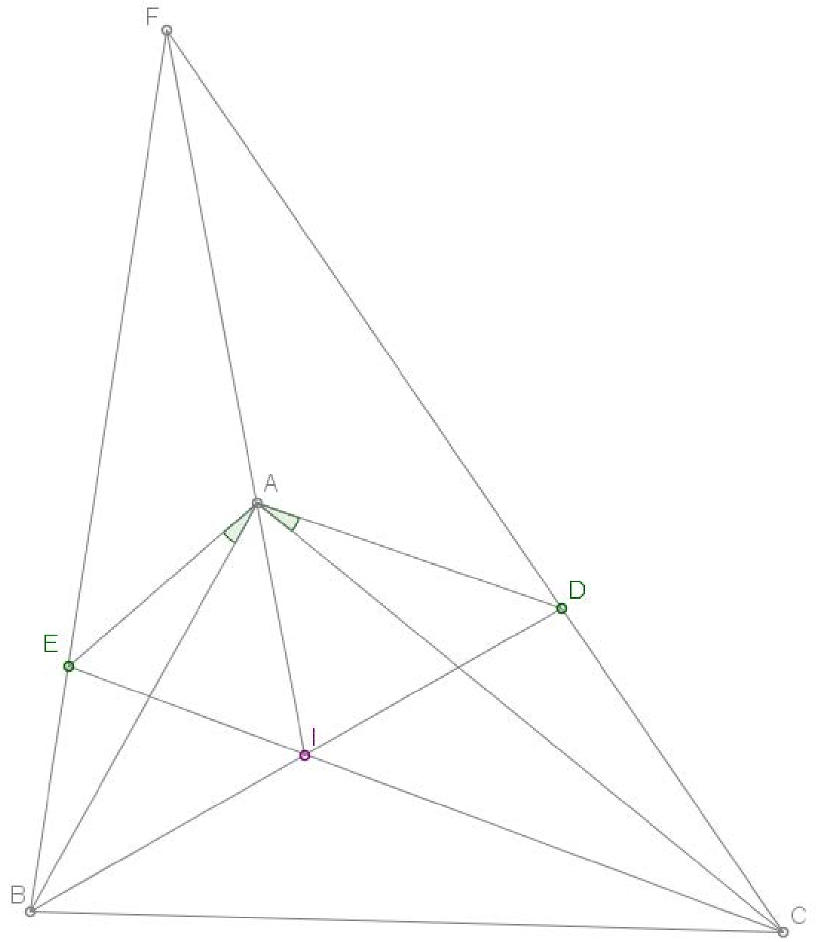
Source: A problem from Geometry of Figures by A. Akopyan.

Cezar

#1 Sep 14, 2015, 4:28 am

Let I be the incenter of $\triangle ABC$. Let D, E be points on BI,CI such that $\angle EAB = \angle DAC$.
Prove that I, A and the intersection of BE and CD are collinear.

Attachments:



Luis González

#2 Sep 14, 2015, 5:00 am • 1

This holds for any I on the angle bisector of $\angle BAC$. Let AI, BI, CI cut BC, CA, AB at X, Y, Z. Since $\angle BAI = \angle CAI$ and $\angle BAE = \angle CAD \Longrightarrow (C, I, Z, E) = (B, I, Y, D)$ or $B(C, I, Z, E) = C(B, I, Y, D) \Longrightarrow I$, $A \equiv BZ \cap CY$ and $F \equiv BE \cap CD$ are collinear.

High School Olympiads

Four circles touching a circle and a line

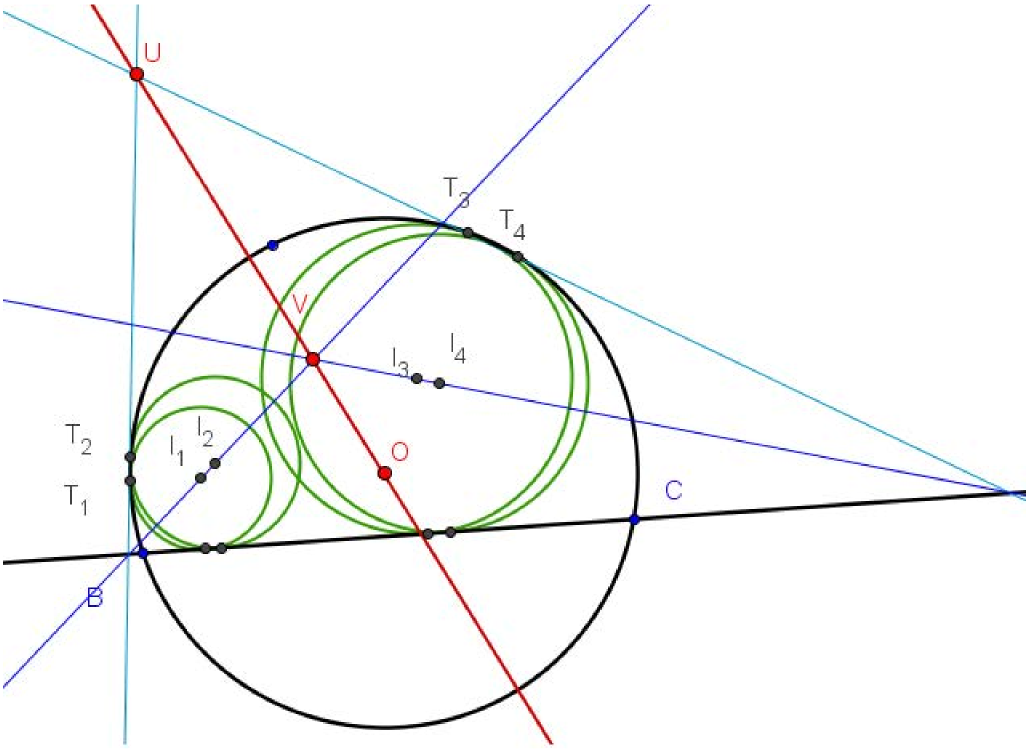
Reply

daothanhoai

#1 Sep 13, 2015, 5:39 pm

Let (O) be a circle through two points B, C. Let four circles with centers I_1, I_2, I_3, I_4 such that they are touch BC, and touch (O) at T_1, T_2, T_3, T_4 respectively. Let $U = T_1T_2 \cap T_3T_4$, $V = I_1I_2 \cap I_3I_4$. Show that U, V, O are collinear.

Attachments:



Luis González

#2 Sep 14, 2015, 12:53 am

Since T_1 and T_2 are the exsimilicenters of $(O) \sim (I_1)$ and $(O) \sim (I_2)$, then by Monge & d'Alembert theorem, it follows that $X \equiv T_1T_2 \cap BC$ is the exsimilicenter of $(I_1) \sim (I_2) \Longrightarrow X \in I_1I_2$ and likewise we have $Y \equiv T_3T_4 \cap BC \cap I_3I_4$. Again by Monge & d'Alembert theorem, it follows that $Z \equiv T_1T_3 \cap BC \cap I_1I_3$ is the exsimilicenter of $(I_1) \sim (I_3) \Longrightarrow \triangle XI_1T_1$ and $\triangle YI_3T_3$ are perspective through Z. Thus by Desargues theorem, the intersections $V \equiv XI_1 \cap YI_3, U \equiv XT_1 \cap YT_3$ and $O \equiv T_1I_1 \cap T_3I_3$ are collinear, as desired.

Quick Reply

High School Olympiads

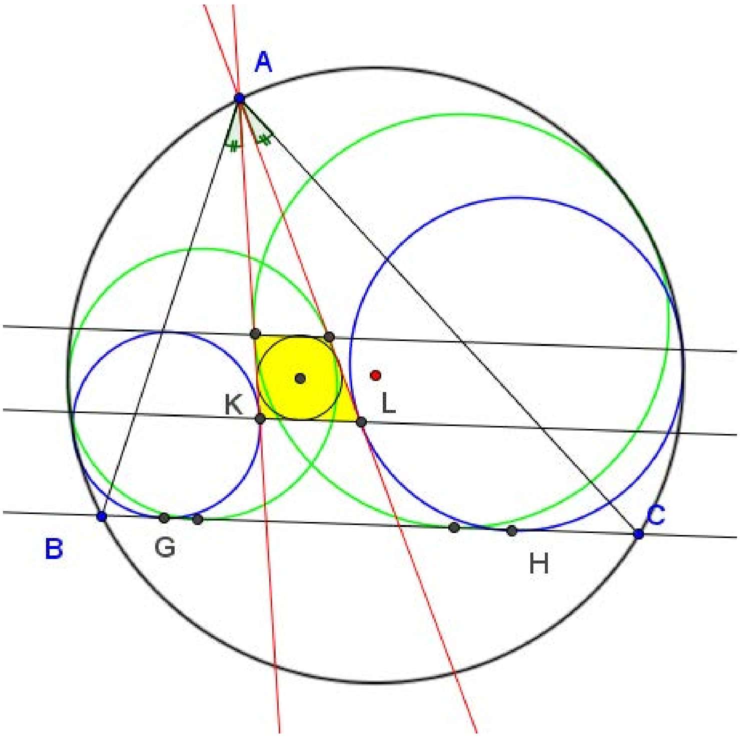
Two pairs of Thebault circles with reflection line

daothanhoai

#1 Sep 13, 2015, 6:00 pm

Let ABC be a triangle, let two cevian lines AK, AL such that let $\angle BAK = \angle LAC$. Let a Thebault circle touch AK at K, and touch BC at G; and Let another Thebault circle tough AL at L
and touch BC at H (see in the figure). Then show that:
1, KL and BC are parallel
2, Incenter of AKL and ABC is the same points

Attachments:



This post has been edited 1 time. Last edited by daothanhoai, Sep 13, 2015, 6:01 pm

Luis González

#2 Sep 14, 2015, 12:05 am

1. See http://www.artofproblemsolving.com/community/c6h603703.
2. Let AK, AL cut BC at K', L' and let J be the incenter of $\triangle AK'L'$. By Thebault-Sawayamma theorem $I \equiv GK \cap HL$ is the incenter of $\triangle ABC$ and since $IKG \parallel K'J, ILH \parallel L'J$ (due to $KL \parallel BC$), it follows that KI, LI are internal bisectors of $\angle AKL$ and $\angle ALK \Longrightarrow I$ is also incenter of $\triangle AKL$.

High School Olympiads

a famus problem

Reply

fandogh

#1 Aug 24, 2014, 3:56 pm

C_1 and C_2 are circles outside from triangle ABC, tangent to the circumcircle and tangent to AB at D and to AC at E respectively. Line ℓ is tangent to C_1, C_2 such that C_1, C_2 are in a same side and segment BC in another side of ℓ. Prove that DE is parallel to BC iff ℓ is parallel to BC.

Luis González

#2 Aug 29, 2014, 10:37 am

Let ℓ cut the arcs AB, AC of the circumcircle (O) of $\triangle ABC$ at U, V, resp. $\mathcal{C}_1, \mathcal{C}_2$ touch ℓ at Y, Z, resp. $\mathcal{C}_1$ becomes a Thebault circle of the cevian AB of $\triangle AUV$ internally tangent to the arc AU of its circumcircle (O), thus by Sawayama lemma, DY goes through the incenter J of $\triangle AUV$ and similarly EZ goes through J.

Assume that $\ell \parallel BC \Longrightarrow \angle BAU = \angle CAV \Longrightarrow AJ$ also bisects $\angle BAC$. If $I \in AJ$ is the incenter of $\triangle ABC$, then it's clear that $JDY \parallel IB$ and $JEZ \parallel IC \Longrightarrow \triangle JDE$ and $\triangle IBC$ are homothethic with center $A \Longrightarrow DE \parallel BC$.

Now conversely assume that $DE \parallel BC$. Let ℓ' be the tangent of $\mathcal{C}_1$ parallel to BC, leaving $\mathcal{C}_1$ and BC in different sides. This cuts the arcs AB, AC of (O) at U', V' and let $\mathcal{C}_2{}'$ be the Thebault circle of the cevian AC of $\triangle AU'V'$ internally tangent to the arc AV' of (O) and tangent to AC at E'. According to the previous result, we have $DE' \parallel BC \Longrightarrow E \equiv E'$, which implies that $\mathcal{C}_2 \equiv \mathcal{C}_2{}'$ and therefore $\ell \equiv \ell' \Longrightarrow \ell \parallel BC$.

Quick Reply

High School Olympiads

Equilateral Hyperbolas

Reply

ferma2000

#1 Sep 13, 2015, 10:15 pm

Dear mathlinkers;

1) Denote by $\mathcal{H}$ an equilateral hyperbola that pass throw the vertex of cyclic quadrilateral $ABCD$.
2) Let $\mathcal{L}$ be the center of $\mathcal{H}$.
3) Let O be the center of circumcircle of $ABCD$.
4) Denote by G the centroid of $ABCD$.

Claim:

$\mathcal{L}, O, G$ are collinear.

Luis González

#2 Sep 13, 2015, 11:17 pm

The center $\mathcal{L}$ of $\mathcal{H}$ is the Poncelet point of $ABCD$, which coincides with the anticenter of $ABCD$ when the quadrilateral is cyclic. Now, it's well-known that $\mathcal{L}, O$ and G are collinear, being G the midpoint of $O\mathcal{L}$.

Quick Reply

High School Olympiads

4 points are concyclic

Reply

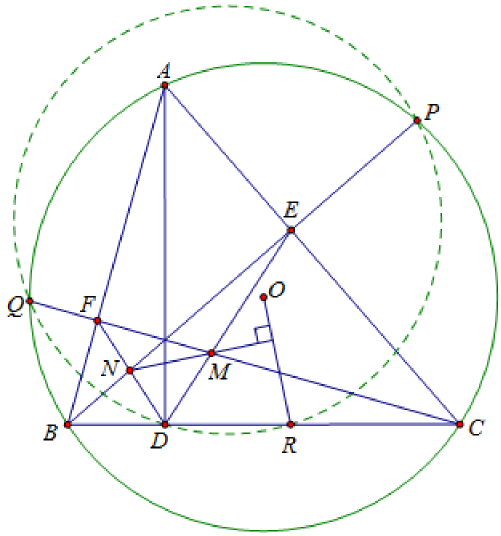
Source: OWN

LeVietAn

#1 Sep 13, 2015, 11:51 am

Dear Mathlinkers,
Given an acute triangle ABC inscribed in circle (O) and AD, BE, CF be the altitudes. BE, CF intersect (O) again at P, Q, resp. Let M, N be respectively the points of intersection of DE and CF, of DF and BE. Choose point R on side BC such that OR perpendicular to MN. Prove that the four points D, P, Q, R are concyclic.

Attachments:



Luis González

#2 Sep 13, 2015, 12:56 pm

H, N_9 are the orthocenter and 9-point center of $\triangle ABC, X \equiv PQ \cap BC$ and L is the 2nd intersection of $(U) \equiv \odot(HBC)$ and $\odot(PHQ)$ whose center is A. As HX is radical axis of $\odot(HBC)$ and $\odot(PHQ)$, then $L \in XH$. Since A, U, N_9 are collinear, it follows that AN_9U is the perpendicular bisector of $HL \Longrightarrow (OL \parallel AN_9) \perp HL$. Since $AN_9 \perp MN$ (well-known), then $OL \perp MN \Longrightarrow L \in OR$. Now, from cyclic $HDRL$, we get $XD \cdot XR = XH \cdot XL = XP \cdot XQ \Longrightarrow D, P, Q, R$ are concyclic.

Quick Reply

High School Math

Euler line parallel to a side

Reply

Kezer

#1 Sep 13, 2015, 4:29 am

If $\triangle ABC$ has the special property that its Euler line is parallel to its side BC, then prove

$$\tan\angle CBA\cdot\tan\angle ACB=3.$$

tkhalid

#2 Sep 13, 2015, 5:18 am • 1

Solution

Luis González

#3 Sep 13, 2015, 11:19 am • 2

More general: If θ denotes the angle formed by the Euler line of $\triangle ABC$ with BC, then

$$\tan\theta=\frac{\tan B\cdot\tan C-3}{\tan B-\tan C}.$$

Let O, H be the circumcenter and orthocenter of $\triangle ABC$. M is the midpoint of BC (projection of O on BC), D is the foot of the A-altitude and X is the projection of H on OM. We assume the configuration where $\triangle ABC$ is acute with $AC > AB$ and X is between O, M. It's well-known that $AH=2\cdot OM$ and $\angle HAO=\angle B-\angle C$, thus

$$\tan\theta=\frac{OX}{HX}=\frac{OM-HD}{DM}=\frac{OM-(AD-2\cdot OM)}{DM}=\frac{3\cdot OM-AD}{DM}=$$

$$=\frac{3R\cos A-2R\sin A\cdot\sin B}{R\sin(B-C)}=\frac{3(\sin B\cdot\sin C-\cos B\cdot\cos C)-2\sin B\cdot\sin C}{\sin B\cdot\cos C-\cos B\cdot\sin C}=$$

$$=\frac{\sin B\cdot\sin C-3\cos B\cdot\cos C}{\sin B\cdot\cos C-\cos B\cdot\sin C}=\frac{\tan B\cdot\tan C-3}{\tan B-\tan C}.$$

Kezer

#4 Sep 13, 2015, 6:36 pm

> tkhalid wrote:
> Solution

Nice solution! There is 1 (or 2?) step that I don't really get, though. How do you show $AD=3R\cos A$ in the first line? And how do you prove $AH=2R\cos A$?

rkm0959

#5 Sep 13, 2015, 9:04 pm • 1

Try proving $AH=2OM=2R\cos A$. Then we have $AD=AH+OM=3OM=3R\cos A$.
To prove that $AH=2OM$, let the antipode of B wrt the circumcircle be B' and prove that $AHCB'$ is a parallelogram.

Quick Reply

High School Olympiads

Prove that FL,QR,GI are concurrent

Reply

Source: HSGS TST 2016, Round 1, Day 1, P3

A_Gappus
#1 Sep 12, 2015, 7:01 pm

Given scalene triangle ABC inscribed fixed circle (O). B, C are fixed and A be variable, I be incenter of $\triangle ABC$.
$AI \cap O \equiv M$. F is projection of I on AB. IF intersects BC at S. SM cuts (O) at T.
a) Prove that TI passes through a fixed point - G
b) Denote that H is orthocentre of $\triangle ABC$. Q is reflection of H through F. L is projection of F on IC, R is reflection of I through L. Prove that FL, QR, GI are concurrent

Luis González
#2 Sep 12, 2015, 11:40 pm

a) Let SI cut $\odot(IBC)$ again at X. As S is on radical axis BC of $(O), \odot(IBC)$, it follows that $IMTX$ is cyclic $\Longrightarrow$ $\angle MTI = \angle MXI$. But $\triangle MXI$ is M-isosceles $\Longrightarrow \angle MTI = \angle MIX = \angle AIF = 90^\circ - \frac{1}{2}\angle BAC$. Therefore if TI cuts (O) again at G, we have $\angle GCB = \angle ITM - \angle BTM = 90^\circ - \frac{1}{2}\angle BAC - \frac{1}{2}\angle BAC = 90^\circ - \angle BAC \Longrightarrow G$ coincides with the antipode of C on (O), obviously fixed.

Dukejukem
#3 Sep 13, 2015, 12:57 am

a) Let N be the midpoint of $\widehat{AB}$ on (O) and let D be the projection of I onto $\overline{BC}$. Denote $X \equiv IF \cap MN$. It is well-known that M, N are the circumcenters of $\triangle BIC, \triangle AIB$, respectively. Hence, MN is the perpendicular bisector of $\overline{BI}$, implying that $\angle XBI = \angle XIB = \angle FIB$. But from symmetry in the bisector of $\angle ABC$, we get $\angle FIB = \angle DIB = 90^\circ - \angle IBD$. It follows that $\angle XBD = 90^\circ$.

Then if TI cuts (O) for a second time at G, Pascal's Theorem on $MTGBCN$ yields that $S, I, GB \cap MN$ are collinear. Therefore, $GB \cap MN \equiv X \implies \angle GBC = \angle XBD = 90^\circ$. Hence, G is the antipode of C w.r.t. (O), which is clearly a fixed point.

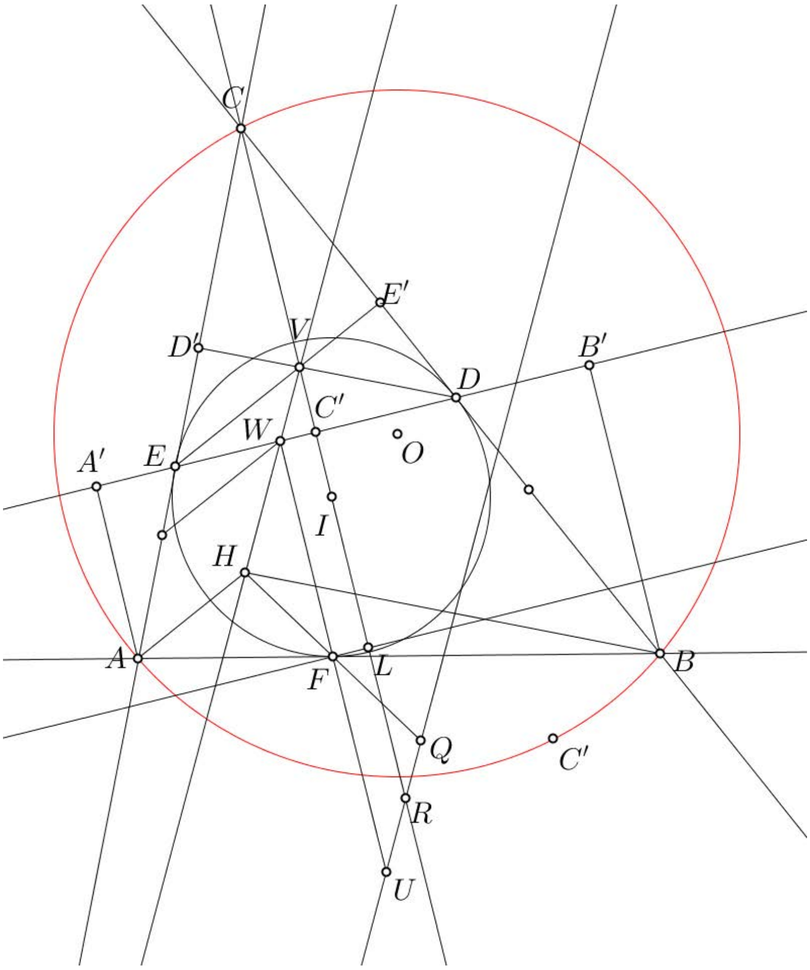
This post has been edited 1 time. Last edited by Dukejukem, Sep 13, 2015, 12:58 am

A-B-C
#4 Sep 13, 2015, 7:54 am

My solution for b.
BB_1, CC_1 are the altitudes of $\triangle ABC$
$\odot(I)$ touches CA, CB at E, D. Let A', B', C' be orthogonal projections of A, B, C on DE.
V is the reflection of I in DE. W is the orthogonal projection of F on DE. U is the reflection of W in F.
$WD/WE = \cot\angle EDF/\cot\angle FED = \tan A/2/\tan B/2 = FB/FA = WB'/WA'$
$\Rightarrow WA'.WD = WB'.WE$ so W lies on radical axis of $\odot(AD), \odot(BE)$.
$\mathcal{P}_{H/\odot(AD)} = \mathcal{P}_{H/\odot(BE)} = \overline{HB}.\overline{HB_1} = \overline{HC}.\overline{HC_1}$
I is antipode of C in $\odot(IC)$ so V is orthocenter of $\triangle CDE$. DD', EE' are the altitudes of $\triangle CDE$
$\Rightarrow \overline{VD}.\overline{VD'} = \overline{VE}.\overline{VE'}$
$\Longrightarrow H, V, W$ lie on radical axis of $\odot(AD), \odot(BE)$.
$\overrightarrow{VR} = 2\overrightarrow{C'L} = 2\overrightarrow{WF} = \overrightarrow{WU}$
$\Rightarrow VRUW$ is a parallelogram $\Rightarrow VW \parallel UR$
Q, U are reflections of H, W in $F \Rightarrow HW \parallel QU$
$\Longrightarrow HW, QU$ are parallel and HW is reflection of QU in F.
Considering the inversion $\mathbf{I}_I^{r^2}$:
$\odot(ABC) \rightarrow$ nine-point circle of $\triangle DEF$
$\odot(CDE) \rightarrow DE$

$\grave{W} \mapsto \odot(\acute{A}BC) \cap \odot(CDE) = T$
$\Rightarrow I, W, T$ are collinear.
$\angle(QR, FL) = \angle(HW, DE) = \angle(DE, IW)$ (IW is the reflection of HW in DE)
$= \angle(FL, IW)$
Furthermore, W is reflection of U in $FL \Rightarrow \overline{I, W, T, G}$ is reflection of $\overline{Q, U, R}$ in FL.
Hence, FL, QR, GI are concurrent.

Attachments:



tranquanghuy7198

#5 Sep 25, 2015, 6:27 pm

My solution:

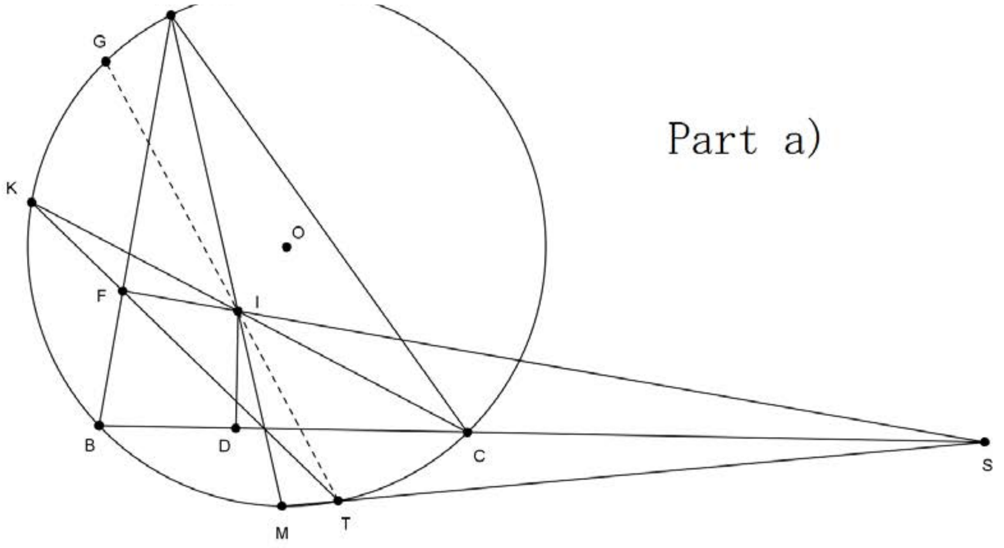
a) Let G be the antipode of C in (O), $CI \cap (O) = \{C, K\}$
We apply Pascal theorem for $\begin{pmatrix} M & B & K \\ C & T & A \end{pmatrix}$ to see that: $\overline{T, F, K}$
We have: $\triangle KTI \sim \triangle KIF \Rightarrow \angle KTI = \angle KIF = \angle KCG = \angle KTG$
$\Rightarrow TI$ passes through the fixed point G.

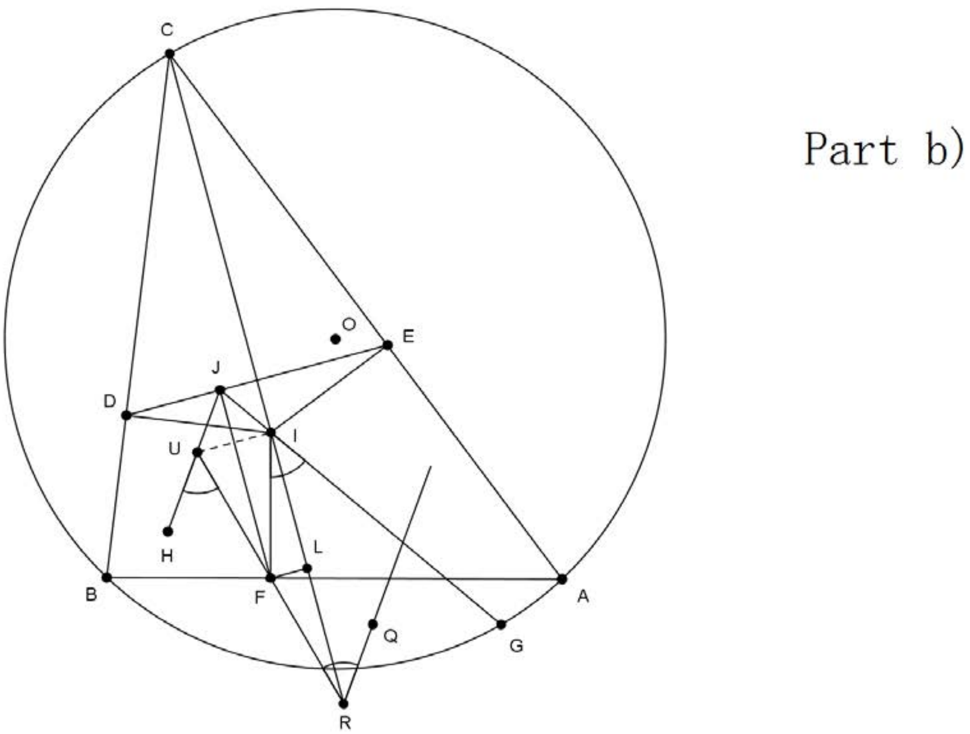
b) Let D, E be the projections of I on BC, CA, $GI \cap DE = J, U = \mathcal{S}_F(R)$
From here (post #1) we get: $FJ \perp DE$
From here (post #4) we get: JF bisects $\angle IJH$

On the other hand, notice that: $FL \perp FJ, IR$ and $U = \mathcal{S}_F(R) = \mathcal{S}_F(\mathcal{S}_L(I))$
$\Rightarrow U = \mathcal{R}_{FJ}(I) \Rightarrow U \in JH$

Now we have: $\angle FIG = \angle FUH = \angle FRQ; R = \mathcal{R}_{FL}(I)$
$\Rightarrow RQ = \mathcal{R}_{FL}(IG) \Rightarrow RQ, IG, FL$ concur.
Q.E.D

Attachments:





Quick Reply

High School Olympiads

symmedian line

Source: Own

andria

#1 Sep 12, 2015, 10:33 pm

Let ABC be a triangle with incenter I. let P be a point such that the midpoint of a segment joining the projections of P on the external bisectors of B,C lies on perpendicular bisector of BC. Prove that PI is symmedian line of $\triangle IBC$.

Luis González

#2 Sep 12, 2015, 11:04 pm

$\triangle I_aI_bI_c$ is the excentral triangle of $\triangle ABC$ and X is the midpoint of I_bI_c. Since XB, XC are tangents of $\odot(BICI_a)$, then IX is the I-symmedian of $\triangle IBC$. S,T are the projections of P on I_aI_b, I_aI_c and M is the midpoint of ST.

In general, the application $P \mapsto M$ is an affine homography for arbitrary P on the plane, thus if M moves on a line, then P moves a line as well. When M is midpoint of BC, then $S \equiv C, T \equiv B \Longrightarrow I \equiv P$ and when $M \equiv X$, then $S \equiv I_b, T \equiv I_c \Longrightarrow P$ is the antipode of I_a on $\odot(I_aI_bI_c)$ lying on IX (well-known). As a result $P \in IX$ for any M on perpendicular bisector of $BC \Longrightarrow IP$ is I-symmedian of $\triangle IBC$.

Dukejukem

#3 Sep 13, 2015, 12:24 am

We can also use complex numbers. Let I_a be the A-excenter of $\triangle ABC$. It is well-known that B, C, I, I_a are inscribed in the circle of diameter $\overline{II_a}$. Therefore, we may WLOG set $\odot(BCII_a)$ to be the unit circle with $i = -1, i_a = 1$. From the projection formula, it follows that if U, V are the projections of P onto I_aB, I_aC, respectively, then the midpoint M of $\overline{UV}$ satisfies

$$2m = u + v = \frac{1 + b + p - b\overline{p}}{2} + \frac{1 + c + p - c\overline{p}}{2} = \frac{b+c}{2} - \frac{(b+c)\overline{p}}{2} + 1 + p.$$

But $\frac{m}{b+c} \in \mathbb{R}$ since M lies on the perpendicular bisector of $\overline{BC}$. Therefore,

$$-\frac{\overline{p}}{2} + \frac{1+p}{b+c} \in \mathbb{R} \implies -\frac{\overline{p}}{2} + \frac{1+p}{b+c} = -\frac{p}{2} + \frac{bc\left(1+\overline{p}\right)}{b+c} \qquad (\star)$$

where the last step follows from conjugation.

Now, let the tangents to $\odot(BCII_a)$ at B and C meet at K. It is well-known (Lemma 1) that IK is the I-symmedian in $\triangle IBC$. Thus, it is sufficient to show that I, K, P are collinear. Since $k = \frac{2bc}{b+c}$, we compute

$$\frac{k-i}{p-i} = \frac{k+1}{p+1} = \frac{2bc+b+c}{(b+c)(p+1)} \implies \overline{\left(\frac{k-i}{p-i}\right)} = \frac{2+b+c}{(b+c)\left(\overline{p}+1\right)}.$$

Then

$$\frac{k-i}{p-i} = \overline{\left(\frac{k-i}{p-i}\right)} \iff \left(2bc+b+c\right)\left(\overline{p}+1\right) = (2+b+c)(1+p)$$
$$\iff 2bc\left(\overline{p}+1\right) + (b+c)\overline{p} = 2(1+p) + (b+c)p,$$

which just reduces to $(\star)$ upon division by $2(b+c)$. The desired result follows. $\square$

High School Olympiads

angle conditions with angle bisector, prove collinearity

Reply

SMOJ
#1 Sep 12, 2015, 8:12 pm

Let the angle bisector at $\angle B$ of triangle ABC meet AC at D and the circumcircle of ABC at P. Let K be a point on the circumcircle of ABC such that $DK \perp BC$. Let L be a point on AC such that $AK \perp BL$. Prove that K, L, P are collinear

Luis González
#2 Sep 12, 2015, 9:38 pm • 1

Let PK cut AC at L'. $\angle BKL' = \angle PCB = \angle DBC + \angle BCA = \angle BDA \Longrightarrow BDKL'$ is cyclic $\Longrightarrow \angle KBL' = \angle KDL' = 90^\circ - \angle ACB \Longrightarrow \angle KBL' + \angle BKA = 90^\circ - \angle ACB + \angle ACB = 90^\circ \Longrightarrow AK \perp BL'$ $\Longrightarrow L \equiv L' \Longrightarrow K, L, P$ are collinear.

hayoola
#3 Sep 13, 2015, 2:21 am

let AK and BL meet each other at X and let KD and BC meet each other at F so we know that XKFB is cyclic we find that the angels ACK=AKB=XFB so XF is paralled LC so we have BXF=BKF=BLC so we find that LKDB is cyclic so we have KLD=KBD we have KBP=PBC-ABK so we are done

jayme
#4 Sep 14, 2015, 5:56 pm

Dear Mathlinkers,
this last and nice proof can be solved by using uniquely the Reim's theorem...
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

equal angles

Reply

Source: Iranian third round 2015 geometry problem 3

andria

#1 Sep 10, 2015, 5:52 pm

Let ABC be a triangle. consider an arbitrary point P on the plain of $\triangle ABC$. Let R,Q be the reflections of P wrt AB, AC respectively. Let $RQ\cap BC=T$. Prove that $\angle APB=\angle APC$ if and if only $\angle APT=90^{\circ}$.

This post has been edited 1 time. Last edited by andria, Sep 11, 2015, 3:12 pm

Luis González

#2 Sep 11, 2015, 3:44 am • 1

Let the perpendicular to PA at P cut AC, AB, BC at Y, Z, T^*, resp and let $X\equiv RZ\cap QY$. Since AC, AB are perpendicular bisectors of PQ, PR, then A is the center of $\odot(PQR) \Longrightarrow YZ, QY, RZ$ are tangents at $P, Q, R \Longrightarrow XP$ is the polar of T^* WRT $\odot(PQR) \Longrightarrow X(Z,Y,P,T^*)=-1 \Longrightarrow A(B,C,P,T^*)=-1$ or $P(B,C,A,T^*)=-1$. As a result, $\angle APT=90^{\circ} \Longleftrightarrow T\equiv T^* \Longleftrightarrow PA$ bisect $\angle BPC \Longleftrightarrow \angle APB=\angle APC$.

Luis González

#3 Sep 11, 2015, 9:43 pm

Sorry there is a flaw in the previous resolution. T^* is the pole of XP WRT $\odot(PQR)$ only when $T\equiv T^*$, though this can be easily fixed keeping the same notations.

If $\angle APT=90^{\circ} \Longrightarrow T$ is the pole of XP WRT $\odot(PQR) \Longrightarrow X(Z,Y,P,T)=-1 \Longrightarrow P(B,C,A,T)=-1 \Longrightarrow AP$ bisects $\angle BPC$. Conversely if AP bisects $\angle BPC \Longrightarrow P(B,C,A,T^*)=-1$ or $A(B,C,P,T^*)=-1 \Longrightarrow$ perpendiculars from P to AB, AC, AP, AT^* form a harmonic pencil as well. If $D\in\odot(PQR)$ is the reflection of P on AT^*, then $P(Q,R,D,T^*)=-1 \Longrightarrow PQDR$ is harmonic $\Longrightarrow Q,R,T^*$ are collinear $\Longrightarrow T\equiv T^* \Longrightarrow \angle APT=90^{\circ}$.

TelvCohl

#4 Sep 12, 2015, 12:34 am

After performing the Inversion with center A we get the following equivalent problem :

Given a $\triangle ABC$ and an arbitrary point P. Let Q, R be the reflection of P in CA, AB, respectively. Let T be the second intersection of $\odot(ABC)$ and $\odot(AP)$. Prove that A, Q, R, T are concyclic if and only if $\angle PBA=\angle PCA$.

Proof :

Let Y, Z be the projection of P on CA, AB, respectively. Since A lies on the perpendicular bisector CA, AB of PQ, PR, so A is the circumcenter of $\triangle PQR \Longrightarrow \angle AQR=\angle ARQ=90^{\circ}-\angle BAC$, hence T lies on $\odot(AQR)$ iff $\angle QTP=\angle RTP=\angle BAC$. On the other hand, since T is the Miquel point of the complete quadrilateral $\{BC, CA, AB, YZ\}$, so $\triangle TYC\sim\triangle TZB \Longrightarrow \angle QTP=\angle RTP=\angle BAC$ iff $\triangle TYC\cup(P,Q)\sim\triangle TZB\cup(R,P)$ (notice $PQ\perp CY, RP\perp BZ$ and Y, Z is the midpoint of PQ, RP, respectively.) iff $\angle PBA=\angle PBZ=\angle RBZ=\angle QCY=\angle PCY=\angle PCA$.

andria

#5 Sep 12, 2015, 1:21 am

Different solution by inversion:
Consider an inversion Ψ with center P. After performing Ψ we get the following problem:
Problem:
Let PBC be a triangle. Consider an arbitrary point A. Let R, Q be the circumcenters of $\triangle PAB, \triangle PAC$ respectively. Let T

be a point on circumcircle of $\triangle ABC$ such that $PT \perp PA$ then $RPTQ$ is cyclic if and if only $\angle APB = \angle APC$.
Proof:
Let O be the circumcenter of $\triangle PBC$ then RO, OQ are perpendicular bisectors of PB, PC respectively. since RQ is perpendicular bisector of PA so $PT \parallel RQ$. also $OP = OT$. Now if:

1) $RPTQ$ is cyclic then it is isosceles trapezoid so $PR = QT$ and $\angle OPR = \angle OTQ$ hence $\triangle OTQ = \triangle OPR \implies OR = OQ \implies \angle APC = \angle APB$.

2) $\angle APC = \angle APB$ then $\angle OQR = \angle ORQ \implies OR = OQ$ Also since T is midpoint of arc BPC of $\odot(PBC)$ we get $\angle TOQ = \angle POR = \angle C$ hence $\triangle OTQ = \triangle OPR \implies TQ = PR \implies RPTQ$ is isosceles trapezoid so it is cyclic.
DONE

Quick Reply

High School Olympiads

Three collinear points

Reply

Source: Iranian third round geometry problem 4

andria

#1 Sep 10, 2015, 6:27 pm

Let ABC be a triangle with incenter I. Let K be the midpoint of AI and $BI\cap \odot(\triangle ABC)=M, CI\cap \odot(\triangle ABC)=N$. points P,Q lie on AM,AN respectively such that $\angle ABK=\angle PBC,\angle ACK=\angle QCB$. Prove that P,Q,I are collinear.

Luis González

#2 Sep 11, 2015, 4:56 am • 2

Let $L \equiv AI \cap BC, D \equiv MN \cap BC, Y \equiv AM \cap BC$ and $U \equiv BP \cap AI.$ As MN is the perpendicular bisector of AI (well-known), then $K \in MN.$ Since BU,BL are the reflections of BK,BA across $BI,$ we deduce that $(L,I,A,K)=(A,K,L,I)=(L,U,A,I)=B(L,U,A,I)=(Y,P,A,M) \Longrightarrow D(L,I,A,K)=D(Y,P,A,M)$ $\Longrightarrow P \in DI.$ By similar reasoning $Q \in DI \Longrightarrow P,Q,I$ are collinear.

buratinogigle

#3 Sep 11, 2015, 9:15 am

I have seen general problem

Let ABC be a triangle inscribed in circle (O) and P,Q are two isogonal conjugate points. PB,PC cut (O) again at M,N. QA cuts MN at K. L is isogonal conjugate of K. LB,LC cut AM,AN at S,T, resp. Prove that S,Q,T are collinear.

Luis González

#4 Sep 11, 2015, 9:38 am • 1

> *buratinogigle wrote:*
>
> I have seen general problem
>
> Let ABC be a triangle inscribed in circle (O) and P,Q are two isogonal conjugate points. PB,PC cut (O) again at $M,N.QA$ cuts MN at $K.L$ is isogonal conjugate of $K.LB,LC$ cut AM,AN at $S,T,$ resp. Prove that S,Q,T are collinear.

The proof to this generalization is very similar to what I did in my previous solution. Letting $D \equiv MN \cap BC,$ $Y \equiv AM \cap BC, U \equiv AQ \cap BC$ and $V \equiv AP \cap BC,$ we get $B(Y,S,A,M)=B(V,L,A,P)=(A,K,U,Q)=(U,Q,A,K) \Longrightarrow D(Y,S,A,M)=D(U,Q,A,K) \Longrightarrow$ $S \in DQ$ and likewise $T \in DQ \Longrightarrow S,Q,T$ are collinear.

Quick Reply

High School Olympiads

Generalization of Newton line

Source: Own

buratinogigle

#1 Sep 11, 2015, 12:47 am

Let $ABCD$ be cyclic quadrilateral inscribed in circle (O) and P is any point. X lies on circle (PAB). XB cuts circle (PBC) again at Y. YC cuts circle (PCD) again at Z. ZD cuts circle (PDA) again at T. Let M,N be midpoints of XZ,YT.

a) Prove that line MN always passes through fixed point Q on circle diameter OP when X moves.

b) Prove that $\angle(MN,PQ)=\angle(BX,BP)$.

Luis González

#2 Sep 11, 2015, 7:59 am • 1

Let I be the 2nd intersection of $\odot(PAB),\odot(PCD)$ and J the 2nd intersection of $\odot(PBC),\odot(PDA)$. By Miquel theorem repeatedly, we easily deduce that $A\in TX, I\in XZ$ and $J\in YT$. Thus M moves on midcircle ω_I of $\odot(PAB),\odot(PCD)$ through P,I and N moves on midcircle ω_J of $\odot(PBC),\odot(PDA)$. As all $XYZT$ are spirally similar with center P, it follows that all $\triangle PMN$ are similar, thus since M,N run on ω_I,ω_J, then we deduce that all lines MN go through the 2nd intersection Q of ω_I,ω_J.

Let E,F,G,H be the 2nd intersections of PQ with $\odot(PAD),\odot(PBC),\odot(PAB),\odot(PCD)$ and let PQ cut (O) at U,V. Invert with center P and arbitrary power, denoting inverse images with primes. PQ cuts $A'D',B'C',A'B',C'D'$ at E',F',G',H' and (O) goes to $\odot(A'B'C'D')$. By Desargues involution theorem for $A'B'C'D'$ cut by PQ, it follows that $E'\mapsto F',G'\mapsto H',U'\mapsto V'$ is an involution on $PQ\Longrightarrow E\mapsto F,G\mapsto H,U\mapsto V$ is an involution on PQ as well. But by midcircle property E,F and G,H are symmetric WRT $Q\Longrightarrow$ the latter involution coincides with the reflection on $Q\Longrightarrow Q$ is also midpoint of $UV\Longrightarrow OQ$ is common perpendicular bisector of $EF,GH,UV\Longrightarrow OQ\perp PQ\Longrightarrow Q$ is on circle with diameter OP.

From cyclic $PQMI$ and $PIXB$, we get $\angle(MN,PQ)=\angle(IM,IP)=\angle(BX,BP)$.

Quick Reply

High School Olympiads

Tangent 4

Reply

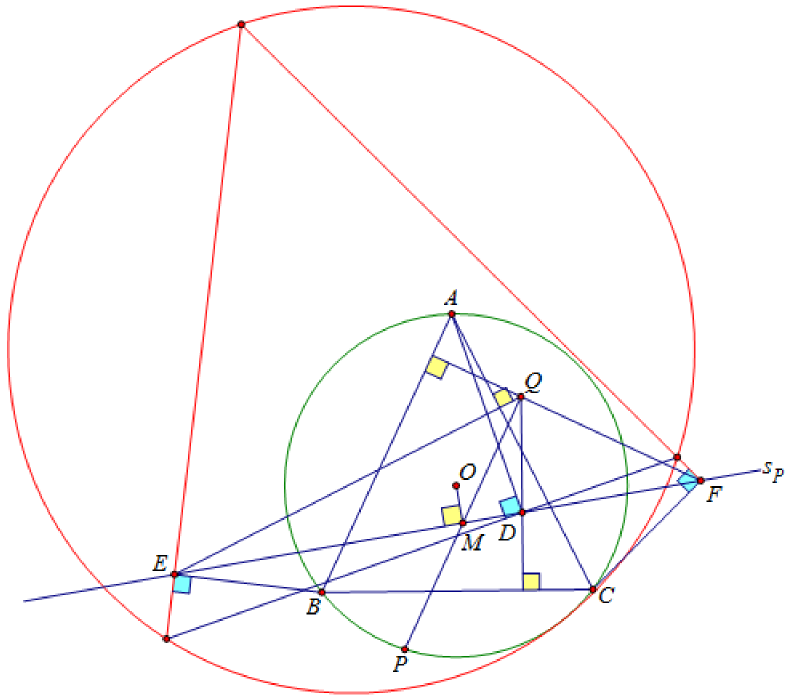
Source: OWN

LeVietAn

#1 Sep 10, 2015, 1:44 am

Dear Mathlinkers,
Given triangle ABC inscribed circle (O) and P is a point on (O). Let s_P is the Steiner line of P WRT triangle ABC. Draw OM perpendicular to s_P at M. Let Q be the reflection of P in M. The lines through Q and perpendicular to BC, CA, AB respectively intersect s_P at D, E, F. Prove that the circumcircle of triangle determined by the lines perpendicular to AD, BE, CF respectively at D, E, F is tangent to the (O).

Attachments:



Luis González

#2 Sep 10, 2015, 5:46 am • 1

Let H be the orthocenter of $\triangle ABC$ and AH, BH, CH cut (O) again at X, Y, Z. If the parallel from P to s_P cuts (O) again at R and AR cuts s_P at D', we clearly have $QD' \perp BC \Longrightarrow D \equiv D' \Longrightarrow R \equiv AD \cap BE \cap CF$. If s_P cuts AC, AB at S, T, we have $S \in PY$ and $T \in PZ$, thus $\angle CYS \equiv \angle CYP = \angle(RP, RC) = \angle CFS \Longrightarrow CFYS$ is cyclic and likewise $BEZT$ is cyclic. If $O_a \equiv FY \cap EZ$, we have then $\angle O_aFS = \angle YCA = 90^\circ - \angle BAC = 90^\circ - \angle(RB, RC) \Longrightarrow FO_a$ is F-circumdiameter of $\triangle REF$ and likewise EO_a is E-circumdiameter of $\triangle REF \Longrightarrow O_a$ is circumcenter of $\triangle REF$. Furthermore, $\angle O_aYP = \angle ACF = \angle ABR = \angle EZP \Longrightarrow PYO_aZ$ is cyclic, i.e. $O_a \in (O)$. By similar reasoning, circumcenters O_b and O_c of $\triangle RDF$ and $\triangle RDE$ lie on (O). The circumcircle of the triangle bounded by perpendiculars to AD, BE, CF at D, E, F is image of $\odot(O_aO_bO_c)$ under homothety $\mathcal{H}(R, 2)$, so it is tangent to (O) at R.

TelvCohl

#4 Sep 10, 2015, 12:12 pm • 1

Let $\triangle XYZ$ be the triangle formed by the perpendicular from D, E, F to AD, BE, CF, respectively. Let R be the reflection of Q in s_P. Obviously, R lies on $\odot(ABC)$ and $PR \parallel s_P$, so notice the Simson line of A WRT $\triangle ABC$ is perpendicular to BC

of Q in s_P. Obviously, R lies on $\odot(ABC)$ and $PR \parallel s_P$, so notice the Simson line of R wrt $\triangle ABC$ is perpendicular to BC we get $\measuredangle ARP = \measuredangle(s_P, QD) = \measuredangle(RD, s_P) = \measuredangle DRP \Longrightarrow R \in AD$. Similarly, we can prove R lies on BE and CF.

Since R lies on $\odot(XEF), \odot(YFD), \odot(ZDE)$, so R is the Miquel point of the complete quadrilateral $\{\triangle XYZ, s_P\} \Longrightarrow R \in \odot(XYZ)$. Let τ be the line through R and tangent to $\odot(XYZ)$. Since $\measuredangle(AR, \tau) = \measuredangle DRZ + \measuredangle(ZR, \tau) = \measuredangle DEZ + \measuredangle ZYR = \measuredangle(CA, s_P) + \measuredangle(s_P, CR) = \measuredangle ACR$, so τ is tangent to $\odot(ABC) \Longrightarrow \odot(ABC)$ is tangent to $\odot(XYZ)$.

livetolove212

#5 Sep 11, 2015, 10:00 am • 1

Generalization: Given triangle ABC inscribed in (O). Let d be an arbitrary line, d_a, d_b, d_c be the reflections of d wrt BC, CA, AB. d_a, d_b, d_c intersect each other and form triangle XYZ. Let I be the incenter of triangle XYZ, M be the projection of O on d, Q be the reflection of I wrt M. The lines through Q and perpendicular to BC, CA, AB intersect d at D, E, F. Then the circumcircle of triangle formed by the lines through D, E, F and perpendicular to AD, BE, CF, respectively, is tangent to (O).

TelvCohl

#6 Sep 11, 2015, 10:54 am • 1

> *livetolove212 wrote:*
>
> **Generalization:** Given triangle ABC inscribed in (O). Let d be an arbitrary line, d_a, d_b, d_c be the reflections of d wrt BC, CA, AB. d_a, d_b, d_c intersect each other and form triangle XYZ. Let I be the incenter of triangle XYZ, M be the projection of O on d, Q be the reflection of I wrt M. The lines through Q and perpendicular to BC, CA, AB intersect d at D, E, F. Then the circumcircle of triangle formed by the lines through D, E, F and perpendicular to AD, BE, CF, respectively, is tangent to (O).

My proof still works for this generalization (notice I is the pole of the Simson line of $\triangle ABC$ with direction $\parallel d$) :).

livetolove212

#7 Sep 11, 2015, 11:18 am

I have more generalization: Given triangle ABC inscribed in (O). Let d be an arbitrary line and P be an arbitrary point on (O). Q is the reflection of P wrt d. The lines through Q and perpendicular to BC, CA, AB intersect d at D, E, F, respectively. Let I is the pole of the Simson line of $\triangle ABC$ with direction $\parallel d$. The line through I and parallel to d cuts (O) again at P'. Let XYZ be triangle formed by the lines through D, E, F and perpendicular to PA, PB, PC, respectively. Then (XYZ) passes through P and P'.

When $P \equiv P'$, (XYZ) is tangent to (O).

Quick Reply

High School Olympiads

A Cyclic Quadrilateral

Reply

Source: Spring 2006 Tournament of Towns Senior O-Level #4

bluecarneal

#1 Sep 10, 2015, 1:30 am

Quadrilateral $ABCD$ is a cyclic, $AB = AD$. Points M and N are chosen on sides BC and CD respectfully so that $\angle MAN = 1/2(\angle BAD)$. Prove that $MN = BM + ND$.

(5 points)

Luis González

#2 Sep 10, 2015, 2:27 am

WLOG assume that $CD > CB$. Let U, V be the projections of A on CD, CB (U lying on segment CD and V lying on ray CB beyond B). Clearly $\triangle AVB \cong \triangle AUD$ are directly congruent $\Longrightarrow BV = UD$. Since $\angle MAN = \frac{1}{2}\angle BAD = 90^\circ - \frac{1}{2}\angle MCN$ and CA bisects $\angle MCN$, we deduce that A is the C-excenter of $\triangle CMN$ $\Longrightarrow \odot(A, AU)$ is C-excircle of $\triangle CMN$ touching MN at $X \Longrightarrow MX = MV$ and $NX = NU \Longrightarrow$ $MN = MV + NU = BM + BV + ND - UD = BM + ND.$

Gryphos

#3 Sep 10, 2015, 1:10 pm

My solution:
Let K be the point on the ray AB beyond B, such that $BK = DN$. Then we have $\angle ABK = \angle ADN$ and $AB = AD$, so the triangles $\triangle AND$ and $\triangle AKB$ are congruent. In particular, $AK = AN$. Moreover, we have $\angle KAM = \angle BAM + \angle NAD = \angle BAD - \angle MAN = \angle MAN$, hence the triangles $\triangle AMN$ and $\triangle AKM$ are also congruent.
From this we infer $MN = KM = BM + BK = BM + DN$.

henderson

#4 Oct 5, 2015, 6:31 pm

Let's construct the triangle AED such that is congruent to the triangle AMB ($AB = AD, BM = DE$ and $MA = EA$). Since $MA = EA$, $\angle MAN = \angle EAN$ and AN is common, $\triangle MAN$ is congruent to $\triangle EAN$. Then $MN = NE = ND + DE = ND + BM$, as desired.

This post has been edited 1 time. Last edited by henderson, Oct 5, 2015, 6:36 pm

hayoola

#5 Oct 5, 2015, 9:49 pm

Continiue segment BC Through B to make point T such that $TB = DN$ it is easy to find that the triangle $TBA$is eqal to triangle ADN so angles $TAB, DAN$are equal so $AT = AN$ and angles $TAM = MAN$ so AM is the perpendicular bisector of TN so $TM = MN$ So $DN + BM = MN$

Quick Reply

High School Olympiads

All lines are parallel

Reply

Source: created by me

ferma2000

#1 Sep 9, 2015, 11:35 pm

Dear mathlinkers;
1) $ABCD$ is a cyclic quadrilateral with circumcenter O.
2) ℓ denotes an arbitrary line passing throw O.
3) P denotes an arbitrary point on the line ℓ.
4) $P_{ab}, P_{cd}, P_{ac}, P_{ad}, P_{bc}, P_{bd}$ are projections of P on AB, CD, AC, AD, BC, BD res.
5) M_1, M_2, M_3 are midpoint of $P_{ab}P_{cd}, P_{ad}P_{bc}, P_{ac}Pbd$ res.
Claim:
a) M_1, M_2, M_3 are collinear.
b) the lines $\overline{M_1M_2M_3}$ are parallel when P vary on ℓ.
Best regards;

Luis González

#2 Sep 10, 2015, 12:05 am • 1

a) See the problem A wonderful exercise [midpoints between projections collin.]
b) The point P can run on any line not necessarily through O. The projections of P on AB, CD, AC, AD, BC, BD describe similar series, so M_1, M_2, M_3 run on 3 different lines and describe similar series as well $\Longrightarrow \overline{M_1M_2M_3}$ has fixed direction.

ferma2000

#3 Sep 10, 2015, 12:10 am

> Luis González wrote:
> describe similar series as well

Hello i'm sorry could you explain this part.

Luis González

#5 Sep 10, 2015, 12:40 am • 1

@ferma2000. Let $A_1, A_2, A_3, \ldots, A_n$ be n points on a line a and $B_1, B_2, B_3, \ldots, B_n$ be n poins on a line b. The series of points $\{A_1, A_2, A_3, \ldots, A_n\}$ and $\{B_1, B_2, B_3, \ldots, B_n\}$ are said to be projective iff $(A_1, A_2, A_3, A_i) = (B_1, B_2, B_3, B_i)$ (cross ratio). When the point at infinity A_∞ of a corresponds to the point at infinity B_∞ of b then we say that these series are similar.

Actually for part b) we did need the line ℓ through O. otherwise $\overline{M_1M_2M_3}$ envelopes a parabola. When $P \equiv M$ then $M_1 \equiv M_2 \equiv M_3 \equiv M$ coincides with the centroid of $\{A, B, C, D\}$, so we now can conclude $\overline{M_1M_2M_3}$ are all parallel.

TelvCohl

#8 Sep 10, 2015, 1:36 am • 1

Remark :

In general, for arbitrary quadrilateral $ABCD$ and an arbitrary point P. If $H_{ab}, H_{ac}, H_{ad}, H_{bc}, H_{bd}, H_{cd}$ is the projection of P on AB, AC, AD, BC, BD, CD, resp, then the $\triangle XYZ$ whose vertexes are the midpoints of $H_{ab}H_{cd}, H_{ac}H_{bd}, H_{ad}H_{bc}$, resp is similar to the pedal triangle of D WRT $\triangle ABC$. Let Q be the Isogonal center of $ABCD$. If P varies on a line passing through Q, then all $\triangle XYZ$ are homothetic (notice Q is the unique point (in general) s.t. $X \equiv Y \equiv Z$). Furthermore, the area of $\triangle XYZ$ only depend on the distance between P and Q. Here is a simple way to construct Q : Q is the isogonal conjugate (WRT $\triangle ABC$) of the antigonal conjugate of D WRT $\triangle ABC$.

P.S. I got this generalization when I tried to solve the problem similar midpoint triangles.

Quick Reply

High School Olympiads

A wonderful exercise [midpoints between projections collin.]

Reply

Source: a seminar of Nguyen Minh Ha

treegoner
#1 Mar 20, 2004, 10:59 am

Let A, B, C, D be four points on a circle, and let M be a point in the same plane. Let E, F, G, H, K, L be the orthogonal projections of the point M on the lines AB, BC, CD, DA, AC, BD, respectively. Prove that the three midpoints of the segments EG, FH, KL are collinear.

grobber
#2 Mar 20, 2004, 6:04 pm

I don't have the actual solution, but I think I've got the main idea:

If we denote by X and Y the midpts of EG and FH respectively and by O the circumcenter of ABCD, then the lines OM and XY are antiparallelles with respect to any of the angles formed by 2 opposite sides of the quadrilateral (including the angle formed by the 2 diagonals). If we show thif for XY then it applies to YZ and XZ as well, with Z being the midpt of KL, and from here we could derive that lines XY and XZ are parallel, so they're actually one and the same because they both pass through X.

I'm trying to prove my first statement.

darij grinberg
#3 Mar 20, 2004, 9:42 pm

Any complete quadrilaterals with Gauss lines out there? Could be of help.

dg

grobber
#4 Mar 20, 2004, 10:06 pm

I tried to look for one but didn't find any.

One thing isn't hard to show, and that is that if we draw a line d through O and move M along that fixed line, then the line XY will also have a fixed direction, meaning that all the lines XY (and YZ, and XZ, of course) which are obtained for various positions of M on d are parallel to each other.

This means that it's enough to prove everything for a certain position of M, like M being on the circle, which would make some of the points E, F, G, H, L, K collinear (because of Simson's thm).

Myth
#5 Mar 20, 2004, 10:20 pm

Since we can construct line M₁M₂ where M \in M₁M₂, M₁ and M₂ lie on circle, it is sufficient to prove statement for points on circle (due to linearity of construction).

For point M on circle we have on figure 4 Simson's lines and statement in this case transform to following one:
Let ABC is triangle, D \in AB, E \in AC, F is point of intersection of CD and BE, then midpoints of AF, DE and BC are collinear.

This statement can be easely proved using center of mass technique.

grobber

#6 Mar 20, 2004, 10:31 pm

Oh, so you did manage to find a quadrilateral (ADFE) and its Gauss line .

Myth

#7 Mar 20, 2004, 10:46 pm

Now I know what Gauss's line is.

darij grinberg

#8 Mar 21, 2004, 1:49 am

Thanks to Grobber and Myth for the proof. Notably, in the special case where M is the point of intersection of the diagonals AC and BD, we get quite a nice theorem (the midpoints of EG and FH are collinear with M). The only - more or less - trivial special case is when M is the circumcenter of our quadrilateral ABCD.

Darij

treegoner

#9 Mar 24, 2004, 2:07 pm

Your solution is right. The point of this problem is to prove when M is O, in (O), and based on the second case when M is in (O) to prove for every M.

Luis González

#10 Dec 6, 2013, 10:46 pm

Clearly E,H,K lie on the circle with diameter $\overline{MA}$ and E,F,L lie on the circle with diameter $\overline{MB}.$ Hence $\angle HEK=\angle HAK=\angle LBF=\angle LEF \Longrightarrow EK,EL$ are isogonals WRT $\angle HEF.$ By similar reasoning, the pairs of lines $(FK,FL),(GK,GL)$ and (HK,HL) are isogonals WRT $\angle EFG,\angle FGH$ and $\angle GHE,$ respectively. Therefore, K,L are isogonal conjugates WRT $EFGH \Longrightarrow$ there is a conic $\mathcal{C}$ with foci K,L inscribed in the quadrangle $EFGH.$ Thus, by Newton's theorem its center (midpoint of LK) is on the Newton line of $EFGH,$ i.e. midpoints of LK,EG,FH are collinear.

Ligouras

#11 Dec 6, 2013, 11:19 pm

Very nice solution Luis!

Quick Reply

High School Olympiads

Collinear 2

Reply

buratinogigle
#1 Sep 25, 2009, 10:19 pm

Let ABC be a triangle and $A'B'C'$ is pedal triangle of an arbitrary point P, let PB', PC' intersect $C'A'$, $A'B'$ ay Y, Z, resp. BY, CZ intersect CA, AB at E, F, resp. PB intersects EF at D. Prove that A', B', D are collinear.

Note that, this is generalization of the problem ABOUT collinear.

Luis González
#2 Feb 10, 2011, 9:56 am

Let $(u:v:w)$ be the barycentric coordinates of P with respect to $\triangle ABC$. Then

$A' \ (0 : a^2v+uS_C : a^2w+uS_B)$

$B' \ (b^2u+vS_C : 0 : b^2w+vS_A)$

$C' \ (c^2u+wS_B : c^2v+wS_A : 0)$

$$A'B' \equiv \begin{bmatrix} 0 & a^2v+uS_C & a^2w+uS_B \\ b^2u+vS_C & 0 & b^2w+vS_A \\ x & y & z \end{bmatrix} = 0$$

$E \ (u(c^2u+wS_B) : 0 : w(a^2w+uS_B))$, $F \ (u(b^2u+vS_C) : v(a^2v+uS_C) : 0)$

$$EF \equiv \begin{bmatrix} u(c^2u+wS_B) & 0 & w(a^2w+uS_B) \\ u(b^2u+vS_C) & v(a^2v+uS_C) & 0 \\ x & y & z \end{bmatrix} = 0$$

Therefore, $A'B', EF, BP$ concur at a point $D \ (x_0 : y_0 : z_0)$ with barycentric coordinates:

$x_0 = u(a^2w+uS_B)(b^2u+vS_C)$

$y_0 = v(a^2v+uS_C)(wS_C-uS_A)$

$z_0 = w(a^2w+uS_B)(b^2u+vS_C)$

Luis González
#3 Sep 9, 2015, 11:52 pm • 2

Actually the problem is merely projective. It can be generalized as follows: P is arbitrary point on the plane and A', B', C' are abitrary points on BC, CA, AB, resp. PB', PC' cut $A'C', A'B'$ at Y, Z, resp and BY, CZ cut AC, AB at E, F. Then the lines $A'B', EF, BP$ concur.

Proof: Let PB, PC cut $A'B', A'C'$ at M, N, resp and MN cuts AC, AB at E', F'. By Pappus theorem for $A'NCE'BM$, the intersections $Y' \equiv A'N \cap E'B$, $P \equiv NC \cap BM$ and $B' \equiv CE' \cap MA'$ are collinear $\Longrightarrow Y' \equiv Y \Longrightarrow E' \equiv E$. By similar reasoning $F' \equiv F \equiv MN \cap AB \cap CZ \Longrightarrow A'B', EF, BP$ concur at M.

Quick Reply

High School Olympiads

Concurrency

Reply

Scorpion.k48

#1 Sep 9, 2015, 4:50 pm

Let $\triangle ABC$ with centroid G. $\triangle A_1B_1C_1$ is pedal triangle of G WRT $\triangle ABC$. A_2, B_2, C_2 be reflection of A_1, B_1, C_1 to G. Prove that AA_2, BB_2, CC_2 are concurrent.

TelvCohl

#3 Sep 9, 2015, 5:24 pm • 1

Let D, E, F be the midpoint of BC, CA, AB, respectively. Let X, Y be the projection of A on BC, A_1A_2, respectively. Obviously, A_2 is the reflection of G in $EF \Longrightarrow A_2G = YA_2$, so if A_3 is the intersection of AA_2 and BC then we get

$$\frac{A_1A_3}{A_1X} = \frac{A_1A_3}{YA} = \frac{A_2A_1}{A_2Y} = -2 \Longrightarrow D \text{ is the midpoint of } A_3X,$$

hence AA_2 passes through the isotomic conjugate T of the orthocenter of $\triangle ABC$ WRT $\triangle ABC$. Analogously, we can prove $T \in BB_2, CC_2 \Longrightarrow AA_2, BB_2, CC_2$ are concurrent at the isotomic conjugate of the orthocenter of $\triangle ABC$ WRT $\triangle ABC$.

Luis González

#4 Sep 9, 2015, 9:06 pm

Posted before at http://www.artofproblemsolving.com/community/c6h463306.

Quick Reply

High School Olympiads

AA", BB" and CC" are concurrent

Reply

Source: Donova Mathmatical Olympiad 2010

littletush
#1 Feb 11, 2012, 11:06 am

Given a triangle ABC, let A', B', C' be the perpendicular feet dropped from the centroid G of the triangle ABC onto the sides BC, CA, AB respectively. Reflect A', B', C' through G to A'', B'', C'' respectively. Prove that the lines AA'', BB'', CC'' are concurrent.

Luis González
#2 Feb 11, 2012, 12:23 pm

Let $\triangle A_0B_0C_0$ be the antimedial triangle of $\triangle ABC$. P is the projection of A_G on B_0C_0, $M \equiv AG \cap BC$ is the midpoint of BC, $U \equiv PA_0 \cap BC$ is the midpoint of PA_0 and $D \equiv GA' \cap B_0C_0$ is clearly the reflection of G about A'', since $\frac{\overline{GA'}}{\overline{GD}} = \frac{\overline{GM}}{\overline{GA}} = -\frac{1}{2}$. Hence A, A'', U are collinear, i.e. A'' lies on the A-cevian AU of the isotomic conjugate X_{69} of the orthocenter of $\triangle ABC$. Likewise, BB'' and CC'' pass through X_{69}.

P.S. This property is valid for all the points lying on the Thomson cubic of ABC.

r1234
#3 Feb 12, 2012, 4:08 pm

Let D be the foot of the A-altitude.Let AA'' meets the the side BC at D'.Now clearly $A'A'' \parallel AD$.So clearly $D'G$ bisects AD.Let A_1 be the midpoint of AD.Note that A_1, G, D' are collinear.Let A_2 be the reflection of A wrt G.Then $A_1G \parallel A_2D \implies A_2D \parallel D'G$.Now BC bisects $G'A_2$.Hence GDA_2D' is a parallelogram and so D' is the isotomic point of D wrt BC.Hence AA'', BB'', CC'' concur at the isotomic conjugate of the orthocenter of ABC which is again the symmedian point of the anti-complementary triangle of $\triangle ABC$.

Potla
#4 Feb 24, 2012, 12:19 am

Here is my solution, though a bit long. :)

Firstly, I will use a lemma.
Lemma.
If A_1, B_1, C_1 are the midpoints of BC, CA, AB and if A_2, B_2, C_2 are the feet of perpendiculars from G onto BC, CA, AB such that X, Y, Z are the midpoints of GA_2, GB_2, GC_2; then the lines A_1X, B_1Y, C_1Z concur at the symmedian point of ABC.
Proof.
Let AA_3 be the A-altitude, and let A_4 be the midpoint of AA_3. Then if K is the symmedian point, it's well-known that KA_1 passes through A_4. Homothety about A_1 that maps AA_3 to GA_1 maps the point A_4 to X. So, A_1A_4 passes through X. So, we are done.$\square$

Coming back to our original problem, we see that a homothety about G with ratio -2 maps $A_1B_1C_1$ to ABC. This also maps the point X to the point A''.So we finally get that, AA'', BB'', CC'' concur at the point K' where K' is the image obtained from homothety about G with a ratio -2 of K.
We are done. $\square$
:)

Babai
#5 May 7, 2012, 2:04 pm

Use barycentric coordinate.
We get $A' = (0, 3a^2+b^2-c^2, 3a^2+c^2-b^2)$ and like wise.
Reflection in G we get $A" = (4a^2, a^2+c^2-b^2, a^2+b^2-c^2)$ and like wise.
So The equation of $AA"$ is $z(a^2+c^2-b^2) = y(a^2+b^2-c^2)$ and like wise.

We can now easily use determinant method to show $AA", BB", CC"$ are concurrent.
The point of concurrence is $(b^2+c^2-a^2, a^2+c^2-b^2, a^2+b^2-c^2)$ which is nothing but the isotomic conjugate of orthocentre.

Quick Reply

High School Olympiads

Reflections of Simson lines

Reply

Source: Own

buratinogigle
#1 Sep 8, 2015, 10:41 pm • 1

Prove that the line connecting a point on circumcircle of a triangle with circumcenter of the triangle which is created by reflecting the Simson lines of that point with original triangle through side line of this triangle, always passes through a fixed point.

andria
#2 Sep 8, 2015, 11:03 pm • 1

it has been posted before see http://www.artofproblemsolving.com/community/c6h421236

TelvCohl
#3 Sep 8, 2015, 11:05 pm • 1

In general, the incenter of the triangle formed by the reflection of ℓ in BC, CA, AB is the pole of the Simson line of $\triangle ABC$ with direction $\parallel \ell$ (well-known), so if $\triangle DEF$ is the triangle formed by the reflection of the Simson line of P WRT $\triangle ABC$ in BC, CA, AB then P is the inceter of $\triangle DEF$, hence from On the reflections of a line wrt the sidelines of a triangle (b) $\Longrightarrow$ the line connecting P and the circumcenter of $\triangle DEF$ passes through the Euler reflection point of $\triangle ABC$ (fixed).

buratinogigle
#4 Sep 8, 2015, 11:23 pm • 1

Thank you dear friends for point out me the links, I have seen another problem with reflection of Simson lines, is it known before ?

Let ABC be a triangle inscribed in circle (O) and P is a point on (O). Let d is Simson line of P. d_a is reflection of d through BC. d_b is reflection of d_a through CA. d_c is reflection of d_b through AB. Then the line passing through P and are perpendicular to d_c always passes through a fixed point.

TelvCohl
#6 Sep 9, 2015, 12:55 am • 1

> buratinogigle wrote:
> I have seen another problem with reflection of Simson lines, is it known before ?
>
> Let ABC be a triangle inscribed in circle (O) and P is a point on (O). Let d is Simson line of P. d_a is reflection of d through BC. d_b is reflection of d_a through CA. d_c is reflection of d_b through AB. Then the line passing through P and are perpendicular to d_c always passes through a fixed point.

Let X be the point on $\odot(ABC)$ such that $PX \parallel d_c$. Since d_b is the image of d under $\mathbf{R}(BC) \circ \mathbf{R}(CA)$ (composition of the reflection $\mathbf{R}(BC)$ and the reflection $\mathbf{R}(CA)$), so $\measuredangle(d, d_b) = 2\measuredangle(BC, CA)$. Since $\measuredangle XPC = \measuredangle(d_c, PC) = \measuredangle(d_c, AB) + \measuredangle(AB, PC) = \measuredangle(AB, d_b) + \measuredangle(AB, BC) + \measuredangle(BC, PC) = \measuredangle(AB, d_b) + \measuredangle(AB, BC) + \measuredangle(d, CA) - 90^\circ = \measuredangle BAC + \measuredangle ABC - 90^\circ - \measuredangle(d_b, d) = \measuredangle BAC + \measuredangle ABC - 90^\circ + 2\measuredangle(BC, CA) = \text{Constant}$, so X is fixed as P varies $\Longrightarrow$ the perpendicular from P to d_c passes through a fixed point (antipode of X in $\odot(ABC)$) when P moves on $\odot(ABC)$.

I've never seen this problem before .

Luis González

#7 Sep 9, 2015, 2:05 am • 2

> buratinogigle wrote:
>
> Let ABC be a triangle inscribed in circle (O) and P is a point on (O). Let d is Simson line of P. d_a is reflection of d through BC. d_b is reflection of d_a through CA. d_c is reflection of d_b through AB. Then the line passing through P and are perpendicular to d_c always passes through a fixed point.

The lines d_a, d_b are not necessary in this configuration. Let the isogonal of CP cut (O) again at $R \Longrightarrow PR \parallel AB$ and $CR \perp d$. If the perpendicular from P to d_c cuts (O) again at Q, we have $\angle QPR = \angle(AB, PQ) = 90^\circ - \angle(AB, d_c)$ and $\angle CRP = \angle(CR, AB) = 90^\circ - \angle(AB, d) \Longrightarrow \angle QPR = \angle CRP \Longrightarrow CPRQ$ is isosceles trapezoid with $CQ \parallel PR \parallel AB \Longrightarrow Q$ is fixed.

Quick Reply

High School Olympiads

Hard Geometry

Reply

Source: Myself

comboisreallyhard

#1 Sep 8, 2015, 6:48 am

Let ABC be a triangle and let Γ be the circle externally tangent to the circumcircle of ABC and the extensions of lines AB and AC. Let F be a randomly selected point on segment BC. Construct the two circles through A and F tangent to Γ. Suppose the two aforementioned circles are tangent to Γ at G and H; prove that F, G, H and I_a are concyclic, where I_a is the A-excenter.

Luis González

#2 Sep 8, 2015, 7:52 am

Let Γ touch AC, AB at Y, Z. It's well-known that I_a is the midpoint of $\overline{YZ}$ (extraversion of Mannheim's theorem). AF and the common internal tangents of $\odot(AFG), \Gamma$ and $\odot(AFH), \Gamma$ concur at the radical center R of $\Gamma, \odot(AFG), \odot(AFH)$.

Inversion with center R and power $RG^2 = RH^2 = RA \cdot RF$ leaves Γ fixed and swaps $\odot(FGH)$ and $\odot(AGH)$. Thus by conformity Γ forms equal angles with $\odot(FGH)$ and $\odot(AGH) \Longrightarrow$ by conformity $\odot(AGH)$ and $\odot(FGH)$ are images under inversion WRT Γ. As this inversion clearly swaps A and $I_a \Longrightarrow I_a \in \odot(FGH)$.

comboisreallyhard

#3 Sep 8, 2015, 9:15 am

Is there a way to do this without inversion? A related problem: suppose the two circles through A and F tangent to Γ intersect the circumcircle of ABC again at P and Q. Let G be the projection of I_a onto BC; prove that F, P, Q and G are concyclic.

This post has been edited 1 time. Last edited by comboisreallyhard, Sep 8, 2015, 9:20 am
Reason: added problem

Luis González

#4 Sep 8, 2015, 11:11 pm

Here is a proof to the previous related problem:

Perform inversion with center A and power $AB \cdot AC$ followed by symmetry on the internal bisector of $\angle BAC$. This swaps $\odot(ABC)$ with BC, Γ with the incircle the incircle (I) of $\triangle ABC$ and the A-excircle (I_a) with the A-mixtilinear incircle γ. F goes to the 2nd intersection F' of $\odot(ABC)$ with the isogonal of AF and $\odot(AFG), \odot(AFH)$ go to the tangents from F' to (I) cutting BC at the images P', Q' of $P, Q \Longrightarrow \odot(FPQ)$ goes to $\odot(F'P'Q')$. This circle $\odot(F'P'Q')$ goes to the tangency point G' of γ with $\odot(ABC)$ (see Mixtilinear incircles and somehow Poncelet's porism or Cevian and mixtilinear incircle), thus it follows that $\odot(FPQ)$ goes to the image G of G', i.e. F, P, Q, G are concyclic.

Quick Reply

High School Olympiads

hard geometry

Reply

Source: created by my teacher

ferma2000

#1 Sep 8, 2015, 9:45 pm • 1

Dear mathlinkers;
1) ABC is a triangle
2) denote by ℓ the perpendicular bisector of BC
3) ϱ is arbitrary circle passing throw B, C.
4) ℓ cut ϱ at two points P, Q.
5) AP, AQ cut again ϱ at X, Y.
Claim:
XY passes throw fixed point.
Best regards;

Luis González

#2 Sep 8, 2015, 9:52 pm • 1

See http://www.artofproblemsolving.com/community/c6h612794 (post #2) and for a generalization see post #4.

andria

#4 Sep 8, 2015, 9:57 pm • 1

Also the generalization has been proved here.

Quick Reply

High School Olympiads

Collinear

Reply

Source: Own

TelvCohl

#1 Nov 6, 2014, 6:26 am • 1

Let Ω be a circle passing through B and C .
Let M be the midpoint of BC and H be the projection of A on BC .
Let P, Q be the intersection of Ω and the perpendicular bisector of BC .
Let X be a point satisfy AX is tangent to (ABC) and $HX \parallel AM$.
Let $P' = AP \cap \Omega$ and $Q' = AQ \cap \Omega$.

Prove that P', Q', X are collinear

This post has been edited 1 time. Last edited by TelvCohl, Aug 21, 2015, 1:54 am

Luis González

#2 Nov 26, 2014, 10:12 am • 1

Let $R \equiv PQ \cap P'Q'$, $S \equiv PQ' \cap QP'$, $J \equiv PQ \cap AS$ and $K \equiv P'Q' \cap AS$. P is obviously orthocenter of $\triangle QAS$ $\Longrightarrow AS \perp PQ \Longrightarrow$ as Ω varies, S moves on parallel to BC through A.

Let A' be the conjugate of A WRT Ω, lying on the polar RS of A. Since the circle with diameter $\overline{AA'}$ is orthogonal to Ω, the perpendicular bisector τ of $\overline{AA'}$ is then radical axis of A and the pencil Ω through $B, C \Longrightarrow \tau$ goes through a fixed point Y; common radical center of A and the circles $\Omega \Longrightarrow RS$ goes through the reflection of A on $Y \Longrightarrow R \mapsto S$ is a perspectivity between PQ and AJ. But as $(A, S, J, K) = -1$ and A, J are fixed, then $S \mapsto K$ is a projectivity. Consequently $R \mapsto K$ is also a perspectivity between PQ and AJ, because it clearly fixes $J \Longrightarrow RK \equiv P'Q'$ goes through a fixed point.

Considering the cases when $\Omega \equiv \odot(ABC)$ and when Ω degenerates into the line BC, we deduce that the fixed point is the described X, i.e. P', Q', X are collinear.

XmL

#3 Nov 27, 2014, 7:19 am • 5

Construct Q'' on AH such that $XQ'' \parallel AQ$, thus $AQM, XQ''H$ are homothetic$\Rightarrow$ AX, HM, QQ'' concur at a point, denote it K. Since K lies on the radical axis of $(ABC), \Omega$, therefore $\angle XQ''K = \angle AQK = \angle KAD \Rightarrow A, X, D, Q''$ are concyclic, where $D = KQ \cap \omega$.

On the other hand, note that $\angle P'AQ'' = \angle P'PQ = \angle P'DQ$, therefore A, D, Q'', P' are concyclic.

Together, A, X, D, Q'', P' are concyclic, which means $\angle AP'X = \angle AQ''X = \angle AQP = \angle AP'Q' \Rightarrow X, P', Q'$ are collinear.

This post has been edited 1 time. Last edited by XmL, Nov 28, 2014, 3:03 pm

Luis González

#4 Nov 27, 2014, 9:17 am • 1

Here is a generalization of the problem:

Let A, B, C, D, E be 5 distinct points on a conic $\mathcal{C}$ and let $\mathcal{K}$ be another conic through B, C, D, E. A line ℓ cuts BC, DE at M, N and cuts $\mathcal{K}$ at P, Q. AM, AN cut DE, BC at U, V, respectively and UV cuts the tangent of $\mathcal{C}$ at A at X. If AP, AQ cut $\mathcal{K}$ again at P', Q', then X, P', Q' are collinear.

Proof: Let $R \equiv PQ \cap P'Q'$, $S \equiv PQ' \cap QP'$ and $T \equiv RS \cap QQ'$. Polars RS of A WRT $\mathcal{K}$ go through a fixed point Y (for this just consider a homography with complex coefficients sending any pair of the quadruplet $\{B, C, E, D\}$ to the umbilics. RS will pass through the reflection of A on the common radical center Y of $A, \mathcal{K}$). Since $(Q, Q', A, T) = -1$, then the application sending the line $RT \equiv RS$ to the line $RQ' \equiv P'Q'$ is an involutive homography fixing the pencil A and the line ℓ $\Longrightarrow$ lines $P'Q'$ also go through a fixed point.

Considering the cases when $\mathcal{K} \equiv \mathcal{C}$ and when $\mathcal{K} \equiv BC \cap ED$, we deduce that the fixed point is the described X, i.e. X, P', Q' are collinear.

Quick Reply

High School Olympiads

quadrilateral, circumcenter, angle equal and perpendicular

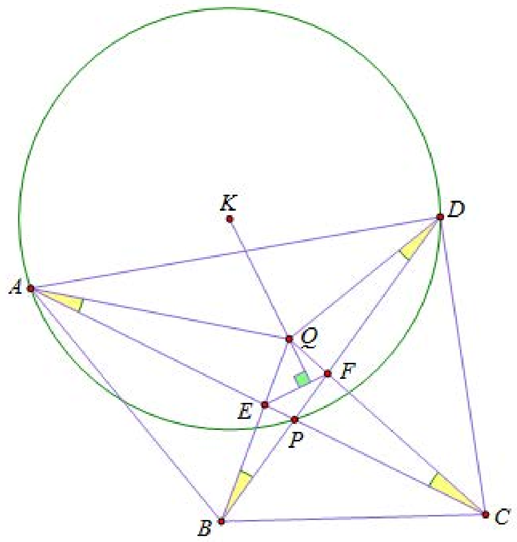
Source: OWN

LeVietAn

#1 Sep 7, 2015, 8:39 pm

Dear Mathlinkers,
Let convex quadrilateral $ABCD$ has $AC \cap BD = P$.
Let point Q lies inside $\triangle APD$ and $\measuredangle QAC = \measuredangle QCA = \measuredangle QBD = \measuredangle QDB$.
Let $QB \cap AC = E, QC \cap BD = F$.
Let K is circumcenter of $\triangle APD$.
Prove that KQ is perpendicular to EF.

Attachments:



Luis González

#2 Sep 8, 2015, 5:34 am • 1

Lemma: P is a point on the angle bisector $\angle BAC$ of $\triangle ABC$. PB, PC cut AC, AB at Y, Z and S is the reflection of P on YZ. Then the circle $\odot(PAS)$ is orthogonal to $\odot(ABC)$.

Let PA, PB, PC cut $\odot(ABC)$ again at M, N, L. $\angle BNM = \angle BAM = \angle CAM \Longrightarrow APEN$ is cyclic $\Longrightarrow \angle AEP = \angle ANB = \angle ACB \Longrightarrow PE \parallel BC$. Similarly $APFL$ is cyclic and $PF \parallel BC \Longrightarrow \overline{EPF} \parallel BC \Longrightarrow \odot(AEF)$ is tangent to $\odot(ABC)$. Since $\angle PLN = \angle PBC = \angle NPE \Longrightarrow \odot(PNL)$ is tangent to EF, thus if $U \equiv EF \cap NL$, then $UP^2 = UN \cdot UL = UE \cdot EF \Longrightarrow UA$ is tangent of $\odot(ABC)$. Since $YA \cdot YE = YP \cdot YN$ and $ZA \cdot ZF = ZP \cdot ZL$, it follows that YZ is radical axis of $\odot(PNL)$ and $\odot(AEF)$ passing through the radical center U of $\odot(ABC), \odot(PNL), \odot(AEF) \Longrightarrow U$ coincides with the center of $\odot(PAS)$. As a result, $\odot(PAS)$ is the A-Apollonius circle of $\triangle AEF$ orthogonal to its circumcircle $\odot(AEF) \Longrightarrow \odot(PAS)$ is also orthogonal to $\odot(ABC)$.

Back to the problem. Invert with center P and power $PB \cdot PF = PC \cdot PE$. Q goes to the 2nd intersection Q' of $\odot(PEB)$, $\odot(PFC)$ and A and D go to $A' \equiv CE \cap Q'F$ and $D' \equiv BF \cap Q'E \Longrightarrow K$ goes to the reflection K' of P on $A'D' \Longrightarrow$ QK goes to $\odot(PQ'K')$ and EF goes to $\odot(PBC)$. By conformity, it suffices to prove that $\odot(PQ'K')$ and $\odot(PBC)$ are

QK goes to $\odot(PQ'K')$ and EF goes to $\odot(PBC)$. By conformity, it suffices to prove that $\odot(PQ'K')$ and $\odot(PBC)$ are orthogonal.

Since $\angle PQ'E = \angle PBE = \angle PCF = \angle PQ'F$, then $Q'P$ bisects $\angle EQ'F$. Using the previous lemma for $\triangle Q'EF$, we get that $(J) \equiv \odot(PQ'K')$ and $\odot(Q'EF)$ are orthogonal. But if $A'D'$ cuts $\odot(Q'EF)$ at U, V, then $UV \equiv A'D'$ is the radical axis of $\odot(PBC)$ and $\odot(Q'EF)$ because of $A'Q \cdot A'F = A'P \cdot A'C$ and $D'Q \cdot D'E = D'P \cdot D'B \Longrightarrow$ $\{U, V\} \in \odot(PBC) \Longrightarrow \odot(PQ'K')$ and $\odot(PBC) \equiv \odot(PUV)$ are orthogonal, as desired.

Dukejukem

#3 Sep 8, 2015, 1:19 pm

Let AQ, DQ cut $\odot(APD)$ for a second time at R, S, respectively. Let M be the midpoint of $\overline{AD}$ and let U, V be the projections of K onto AQ, DQ, respectively.

Note that $\angle AQC = \angle DQB \implies QB, QC$ are isogonal w.r.t. $\angle AQD$. Meanwhile, B, C, E, F are concyclic because $\angle EBF = \angle ECF$. Hence, BC and EF are antiparallel w.r.t. $\angle BQC$, so a reflection in the bisector l of $\angle BQC$ sends EF to a line parallel to BC. But since l is also the bisector of $\angle AQD$, it is a well-known property of pedal triangles that the reflection of KQ in l is perpendicular to UV. Therefore, it is sufficient to show that $BC \parallel UV$.

From $\triangle QAS \sim \triangle QDR$, we obtain $\frac{QB}{QC} = \frac{QD}{QA} = \frac{DR}{AS} = \frac{MU}{MV}$. Meanwhile, note that $\angle PDR = \angle PAR = \angle PBQ \implies DR \parallel QB$. Since M, U are the midpoints of $\overline{AD}, \overline{AR}$, it follows that $MU \parallel QB$. Similarly, $MV \parallel QC$. Therefore, $\triangle MUV \sim \triangle QBC$ and it follows that $UV \parallel BC$ as desired. $\square$

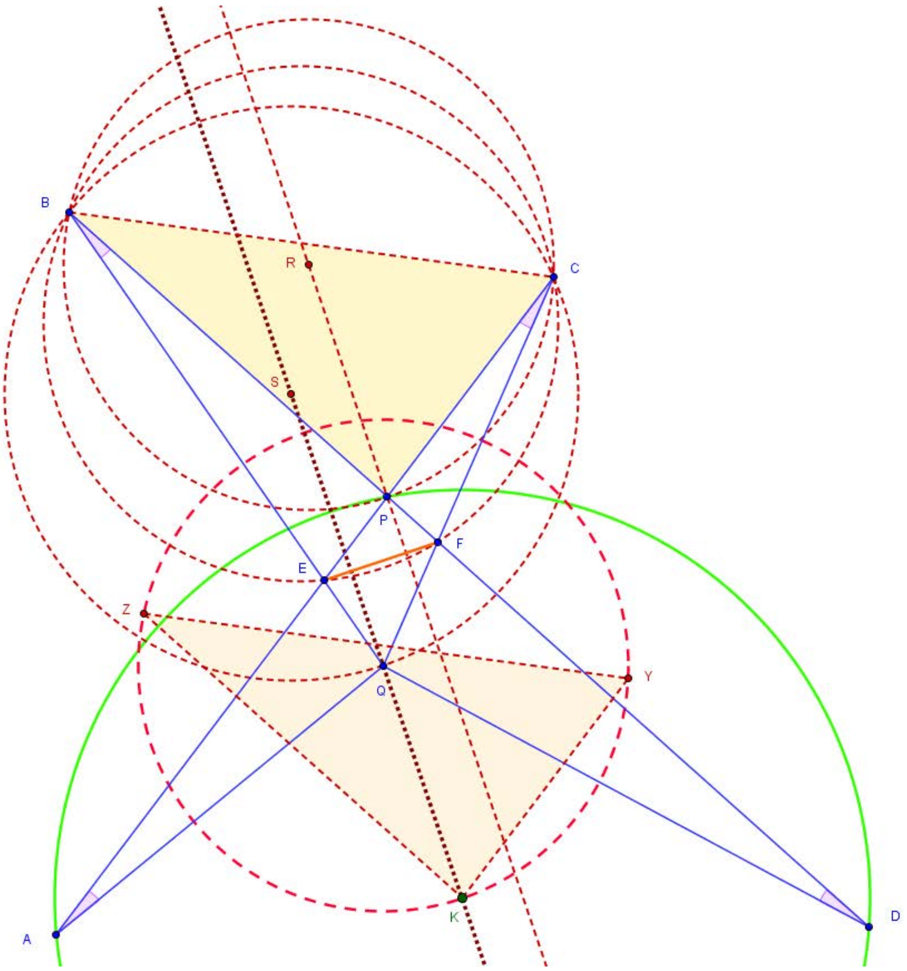
TelvCohl

#5 Sep 8, 2015, 2:06 pm

Let Y, Z be the points on $\odot(Q, QK)$ such that $KY \parallel AC, KZ \parallel BD$. Since the projection of Q, K on AC is the midpoint of AC, AP, resp, so $KY = PC$. Similarly, we can prove $KZ = PB$, so $\triangle YKZ$ and $\triangle CPB$ are congruent and homothetic, hence if R is the circumcenter of $\triangle CPB$ then $RP \parallel QK$ (and $RP = QK$).

Since the bisectors of $\angle CPB$ and $\angle CQB$ are parallel, so if S is the circumcenter of $\triangle CQB$ then $SQ \parallel RP \Longrightarrow S \in QK$. Finally, notice B, C, E, F are concyclic we get EF is anti-parallel to BC WRT $\angle CQB$, so we conclude that $QK \perp EF$.

Attachments:



Quick Reply

High School Math

Triangle - prove

Reply

soulforged

#1 Sep 7, 2015, 1:02 am

We have triangle ABC, in which $|AB|^2 = |AC| \cdot |BC|$. Point D lies on BC and point E lies on AB and $|BD| = |AC|$, $|AD| = |AE|$. Prove, that $|AB| = |CE|$.

Luis González

#2 Sep 7, 2015, 11:14 am

$BA^2 = BC \cdot CA = BC \cdot BD \Longrightarrow BA$ touches $\odot(ACD) \Longrightarrow \angle ACB = \angle BAD \Longrightarrow \triangle BAD \sim \triangle BCA \Longrightarrow$ $\frac{AB}{BC} = \frac{AD}{AC} \Longrightarrow AE = AD = \frac{AB \cdot AC}{BC} \Longrightarrow AE \cdot AB = \frac{AB^2 \cdot AC}{BC} = AC^2 \Longrightarrow AC$ touches $\odot(BCE) \Longrightarrow$ $\angle ACE = \angle ABC \Longrightarrow \triangle ACE \sim \triangle ABC \Longrightarrow \frac{CE}{BC} = \frac{AC}{AB} \Longrightarrow CE = \frac{AC \cdot BC}{AB} = \frac{AB^2}{AB} = AB.$

Quick Reply

High School Olympiads

Generalization of Gossard's problem

Reply

Source: Own

buratinogigle

#1 Sep 5, 2015, 6:26 pm • 1

Let ABC be a triangle and P is an any point. Line ℓ cuts BC,CA,AB at D,E,F. Let X be the point such that $EX\parallel PC,FX\parallel PB$. Let Y be the point such that $DY\parallel PC,FY\parallel PA$. Let Z be the point such that $EZ\parallel PA,DZ\parallel PB$.

a) Prove that the lines pass through X,Y,Z and are parallel to BC,CA,AB, reps, bound a triangle which is congruent to ABC.

b) The homothety center of this triangle and ABC lies on ℓ iff ℓ passes through P.

Luis González

#2 Sep 6, 2015, 1:58 pm • 2

Lemma (well-known): $ABCD$ is a quadrilateral. Parallel from A to BC cuts BD at S and the parallel from B to AD cuts AC at T. Then $ST\parallel DC. \blacksquare$

Label $\triangle A'B'C'$ the triangle bounded by the parallels from X,Y,Z to BC,CA,AB. Parallel ℓ' from P to ℓ cuts BC,CA,AB at U,V,W and ℓ cuts $B'C',C'A',A'B'$ at U',V',W'. Animate ℓ so that it has fixed direction, thus the series D,E,F are similar. Since the directions $EX\parallel PC,FX\parallel PB$ are fixed, then X runs on a line and similarly Y,Z. Moreover, the series X,Y,Z are clearly similar, thus A',B',C',U',V',W' move each on different lines.

Assume the case when $D\equiv B$. Then B,P,X,Z are collinear. Using the lemma for $ABCZ$, we get $B'\in AC$ and again using the lemma for $ABEZ$, we get $PB'\parallel BE\Longrightarrow B'\equiv V$. From parallelograms $ABA'B'$ and $UBU'B'$, we deduce that $\triangle BUA'\cong\triangle B'U'A$ are centrally congruent $\Longrightarrow UA'\parallel AU$. From parallelogram $BCB'C'$ we get $BC=B'C'$ and by analogous reasoning the same holds when $D\equiv C$, thus $BC=B'C'$ holds for all ℓ and similarly we get $CA=C'A'$ and $AB=A'B'\Longrightarrow\triangle ABC\cong\triangle A'B'C'$ are centrally congruent with center J. As a result, for any ℓ we have $UA'\parallel VB'\parallel WC'\parallel AU'\parallel BV'\parallel CW'$. Hence, by symmetry it follows that U',V',W' and U,V,W are homologous under the symmetry with center $J\Longrightarrow\ell'$ and ℓ are symmetric about $J\Longrightarrow$ the point P' veryfing $\triangle ABC\cup P\cong\triangle A'B'C'\cup P'$ lies on ℓ and J is midpoint of PP'. Hence, we deduce that $J\in\ell\Longleftrightarrow P\in\ell$.

Luis González

#3 Sep 7, 2015, 2:02 am • 2

Remark: The homothetic center J of $\triangle ABC$ and $\triangle A'B'C'$ lies on the Newton line of the complete quadrangle $\{BC,CA,AB,\ell\}$.

This time we fix the line ℓ and animate P on the plane. Denote A_∞ and L_∞ the points at infinity of BC and ℓ, respectively. Since $\overline{XF}\parallel\overline{PB},\overline{XE}\parallel\overline{PC},\overline{PU}\parallel\ell$ and $XU'\parallel BC$, it follows that $P(B,C,U,A_\infty)=P(F,E,L_\infty,U')\Longrightarrow\overline{UB}:\overline{UC}=\overline{U'E}:\overline{U'F}\Longrightarrow$ the series $\{U,B,C\}$ and $\{U',E,F\}$ are similar $\Longrightarrow$ midpoint J of $\overline{UU'}$ describes a line τ. When $U\equiv B$, then $U'\equiv E$ and when $U\equiv C$, then $U'\equiv F\Longrightarrow\tau$ is the line through the midpoints of $\overline{BE},\overline{CF}$, i.e. the Newton line of $\{BC,CA,AB,\ell\}\Longrightarrow J\in\tau$.

Quick Reply

High School Olympiads

A generalization of the Simson line again

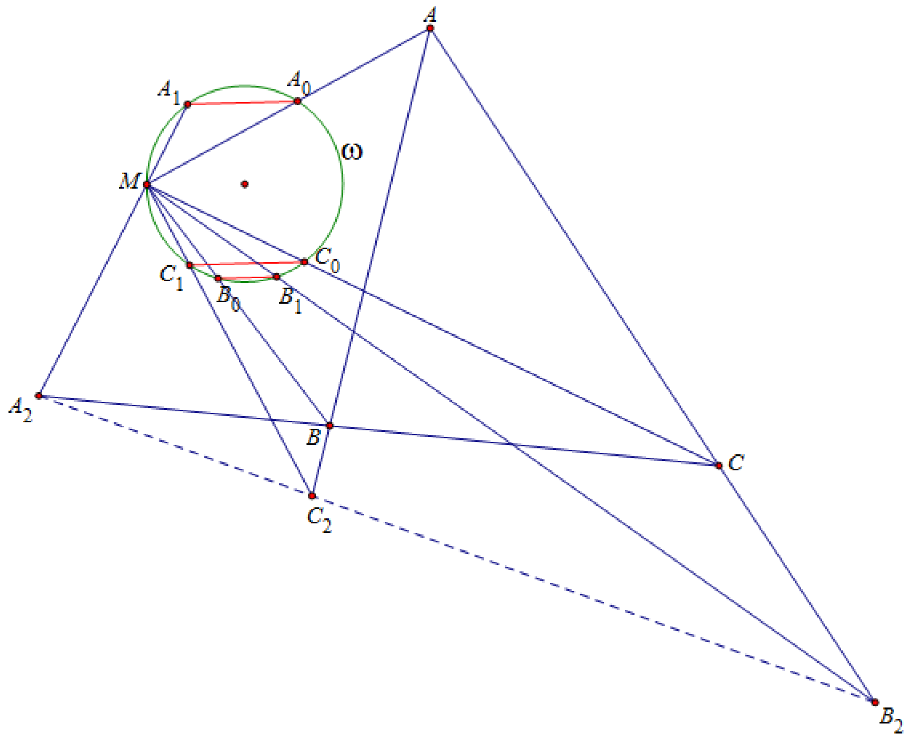
Reply

Source: OWN

LeVietAn
#1 Sep 6, 2015, 10:58 pm

Dear Mathlinkers,
Let ABC be a triangle and M is a point. A circle ω passing through M and respectively intersect MA, MB, MC again at A_0, B_0, C_0. Choose three points A_1, B_1, C_1 lie on ω such that $A_0A_1 \parallel B_0B_1 \parallel C_0C_1$. Suppose that the lines MA_1, MB_1, MC_1 respectively intersect the lines BC, CA, AB at A_2, B_2, C_2. Prove that A_2, B_2, C_2 are conlinear.
Note: When $\omega \equiv \odot(ABC)$ and $MA_1 \perp BC$ we get line $\overline{A_2B_2C_2}$ be the Simson line of M WRT $\triangle ABC$.

Attachments:



Luis González
#2 Sep 6, 2015, 11:08 pm

Since $A_0 \mapsto A_1, B_0 \mapsto B_1, C_0 \mapsto C_1$ is an involution on ω (A_0A_1, B_0B_1, C_0C_1 pass through a point at infinity), then it follows that the pencil $MA \mapsto MA_2, MB \mapsto MB_2, MC \mapsto MC_2$ is involutive. Thus by dual of Desargues involution theorem for the quadrangle BCB_2C_2, this forces $A_2 \in B_2C_2$.

Quick Reply

High School Olympiads

Tangent Circles

Locked

Source: Own

Mentalist

#1 Sep 6, 2015, 4:34 pm

In an acute triangle ABC lets take a point I on the perpendicular from A to BC. A circle w with the circumcenter I cuts the sides AB and AC at points P and Q respectively. If $|BP|\cdot|CQ|=|AP|\cdot|AQ|$ then prove that the circle w is tangent to the circumcircle of $\triangle BOC$.

Luis González

#2 Sep 6, 2015, 10:59 pm

The problem comes from the CMO 2015. See circumcircle and altitudes! CMO 2015 P4.

High School Olympiads

Equilateral triangle and angles

Reply

Source: M,C-19

stergiu

#1 Dec 14, 2009, 9:56 pm

In an equilateral triangle ABC let D the midpoint of BC and E a point on DB extended, such that $BE = BD$.

The point X is on side AB and Y is a point on the ray BA (on the extention of BA to A), such that XY= AB. The lines EX , DY meet

each other at Z.Prove that $\angle ZCE = 2\angle ZEC$.

Babis

Ulanbek_Kyzylorda KTL

#2 Dec 15, 2009, 7:59 am • 1

anybody solved? i'm thinking of this problem...

vittasko

#3 Dec 19, 2009, 1:22 am

LEMMA 1. - A triangle $\triangle ABC$ is given, with $\angle C = 2\angle B$ and let D be, the trace on AB, of the angle bisector of $\angle C$. We draw the circumcircle (K) of the triangle $\triangle DBC$ and we denote as D', the antiadiametric (= diametric opposite) point of D. Prove that $BP = 2\cdot PC$, where $P \equiv BC\cap AD'$.

PROOF OF THE LEMMA 1. - We draw the line through the point A and parallel to BC, which intersects the line segments $D'B,\ D'C$, at points $E,\ F$ respectively and let be the point $A'\equiv EF\cap CD$, as the symmetric point of A with respect to DD' as well.

It is easy to show that $AA' = AF$, (1) from the right triangle $\triangle CA'F$ with $\angle A'CF = 90^o$ and $\angle AA'C = \angle A'CB = \angle A'CA$.

Similarly, we have that $A'A = A'E$, (2) from the right triangle $\triangle BAE$.

From $(1),\ (2) \Longrightarrow EA' = A'A = AF \Longrightarrow EA = 2\cdot AF$, (3)

From (3) and because of $EF \parallel BC$, based on the **Thales theorem**, we conclude that $BP = 2\cdot PC$ and the proof of the **Lemma 1** is completed.

LEMMA 2. - A right triangle $\triangle ABC$ with $\angle A = 90^o$ and $\angle B = 60^o$ is given and let P be, an arbitrary point on the midperpendicular of its side-segment AB, outwardly to it. Through the point D, as the symmetric point of A with respect to B, we draw the line perpendicular to DP, which intersects BC at point so be it E. Prove that $BE = CF$, where $F\equiv BC\cap AP$.

PROOF OF THE LEMMA 2. - We define the point Q between $M,\ P$, where M is the midpoint of the side-segment AB, such that $\angle ABQ = 60^o$.

It is easy to show that $BC\perp DQ$ and also $AQ\perp DQ$.

We draw the lines through the points $D,\ A$ and parallel to MP and also the line through the point D and parallel to BC.

Because of the perpendicularities of $DX_{\infty}\perp DM$ and $DV_{\infty}\perp DQ$ and $DE\perp DP$ and $DB\perp DZ$, where $Z = DX_{\infty}\cap BC$, we conclude that the pencils $D.X,\ V,\ EB$ and $D.MQPZ$ have equal **Double ratios** (= **Cross ratios**

$Z \equiv DX_\infty \cap BC$, we conclude that the pencils $D.X_\infty V_\infty EB$ and $D.MQPZ$ have equal **Double ratios** (**Cross ratios**
).

So, we have that $(D.X_\infty V_\infty EB) = (D.MQPZ), (1)$

We have also that $(A.Y_\infty U_\infty FC) = (A.MQPT_\infty), (2)$

But, $(D.MQPZ) = (A.MQPT_\infty), (3)$

From $(1),\ (2),\ (3) \Longrightarrow (D.X_\infty V_\infty EB) = (A.Y_\infty U_\infty FC), (4)$

$(D.X_\infty V_\infty EB) = (D.ZV_\infty EB) = (E, B, Z) = \dfrac{ZE}{ZB}, (5)$

$(A.Y_\infty U_\infty FC) = (A.BU_\infty FC) = (F, C, B) = \dfrac{BF}{BC}, (6)$

From $(4),\ (5),\ (6) \Longrightarrow \dfrac{ZE}{ZB} = \dfrac{BF}{BC} \Longrightarrow ZE = BF\ , (7)$ because of $ZB = BC$

From $(7) \Longrightarrow BE = CF$ and the proof of the **Lemma 2** is completed.

● Return now in to the proposed problem, we denote as $Z,$ the point on the line segment $EX,$ where X is an arbitrary point on the side-segment AB of the given equilateral triangle $\triangle ABC,$ such that $\angle ZCE = 2 \cdot \angle ZEC$ and it is enough to prove that $AY = BX,$ where $Y \equiv AB \cap DZ$ and D is the midpoint of the side-segment $BC.$

Based on the above **Lemma 1**, we have that the line segment DZ passes through the point $F',$ as the antidiametric point of $F,$ with respect to the circumcircle (K) of the isosceles triangle $\triangle FEC,$ where F is the trace on EZ of the angle bisector of $\angle ZCE.$

We have now, the configuration of the right triangle $\triangle DAB$ with $\angle ADB = 90^o$ and $\angle ABD = 60^o$ and the point F' on the midperpendicular of its side-segment BD outwardly to it.

We have also that $EX \perp EF'$ and based on the **Lemma 2**, we conclude that $BX = AY\ , (8)$

From (8) we conclude that $XY = AB$ and the proof of the proposed problem is completed.

● This proof is dedicated to **Seraphim Tsipelis**.

Kostas Vittas.

Attachments:
t=318425(b).pdf (5kb)
t=318425(a).pdf (6kb)
t=318425.pdf (5kb)

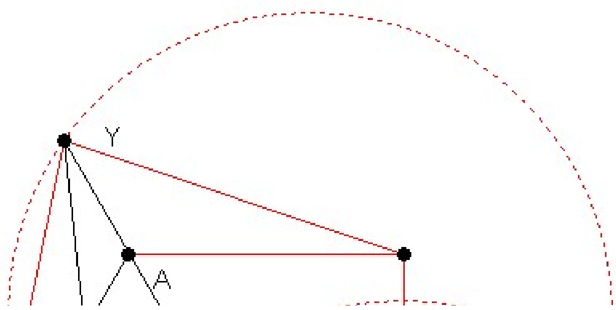
sunken rock
#4 Dec 20, 2009, 9:44 pm

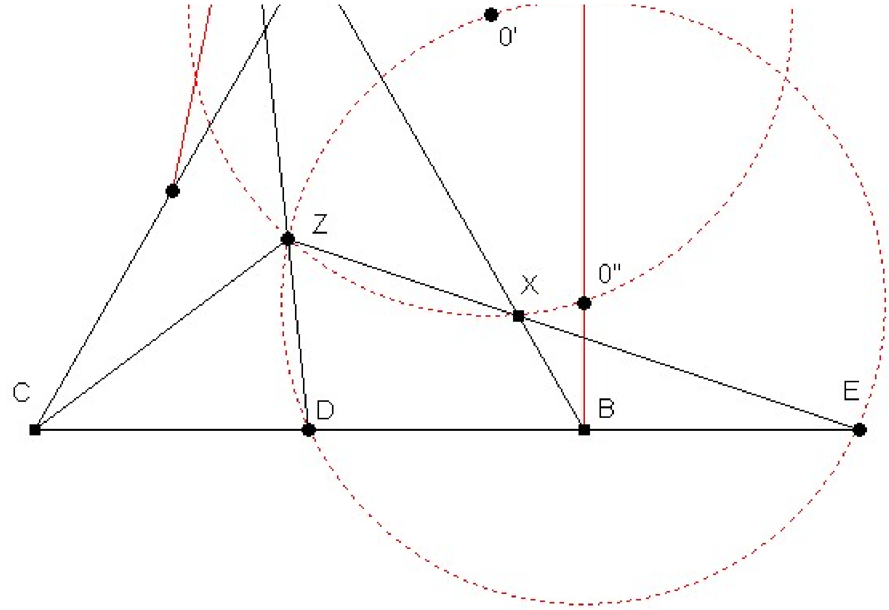
Under the conditions of the general problem: the circles (XYZ) and (DEZ) are equal, easily to be seen. However there is the following problem: the center of each of these circles lies on the other circle.
I did not prove it as yet.

N.B. Good proof, **Kostas**!

Best regards,
sunken rock

Attachments:





sunken rock

#5 Dec 21, 2009, 12:05 am

In fact, as my son proved, it's a piece of cake!

Best regards,
sunken rock

Luis González

#6 Sep 6, 2015, 4:24 am

As X, Y vary they describe congruent series since $\overrightarrow{XY}$ is a translation, so the pencils EX and DY are homographic $\Longrightarrow$ $Z \equiv EX \cap DY$ moves on a fixed conic $\mathcal{H}$ through $E, D.$ When $X \equiv B,$ then $A \equiv Y \Longrightarrow AD$ is tangent of $\mathcal{H}$ and when X is the reflection of A on $B,$ then $Y \equiv B \Longrightarrow$ perpendicular to BC at E is tangent of $\mathcal{H} \Longrightarrow B$ is the center of $\mathcal{H}.$ When X is at infinity, then Y is also at infinity $\Longrightarrow Z$ is the point at infinity of AB and when B is the midpoint of $XY,$ then $EX \parallel DY \parallel AC \Longrightarrow Z$ is the point at infinity of $AC.$ Thus $\mathcal{H}$ is a hyperbola with vertices D, E and asymptotes BA and its reflection across $BC.$

Label $2a, 2b, 2c$ the lenghts of its major axis, minor axis and focal distance. As the angle between its asymptotes is $60^\circ,$ it follows that $\frac{b}{a} = \tan 60^\circ = \sqrt{3} \Longrightarrow c = \sqrt{a^2 + (\sqrt{3}a)^2} = 2a = BC \Longrightarrow C$ is a focus of $\mathcal{H}.$ Since its eccentricity is $\epsilon = \frac{c}{a} = 2,$ then its directrix d is the perpendicular bisector of $BD, EC.$ If M is the midpoint of EC and EZ cuts d at $P,$ we get

$$\frac{PZ}{PE} = \frac{\text{dist}(Z,d)}{ME} = \frac{\text{dist}(Z,d)}{\frac{1}{2}CE} = \frac{ZC}{\epsilon \cdot \frac{1}{2}CE} = \frac{ZC}{CE},$$

which means that CP bisects $\angle ECZ \Longrightarrow \angle ZEC = \angle PCE = \frac{1}{2}\angle ZCE.$

Quick Reply

High School Olympiads

Sum of distances to sides => perpendicularity

Reply

Source: Bundeswettbewerb Mathematik 2015 - Round 2 - #4

Kezer

#1 Sep 5, 2015, 4:28 am

Let ABC be a triangle, such that its incenter I and circumcenter U are distinct. For all points X in the interior of the triangle let $d(X)$ be the sum of distances from X to the three (possibly extended) sides of the triangle.
Prove: If two distinct points P, Q in the interior of the triangle ABC satisfy $d(P) = d(Q)$, then PQ is perpendicular to UI.

Luis González

#2 Sep 5, 2015, 5:24 am

See http://www.artofproblemsolving.com/community/c6h209657 (Lemma at post #4) and for a synthetic proof see the subsequent post #5.

Kezer

#3 Sep 5, 2015, 5:30 am

The synthetic proof is exactly my solution in the contest!

Quick Reply

High School Olympiads

RATIO IN GEOMETRY

Locked

Source: JMO

ind
#1 Sep 4, 2015, 10:29 pm

ABC is atriangle in which A=60 ,AB>AC ,Point O is the cicumcentre and H is the Orthocentre .Points M & N are taken on altitudes BE and CF such that BM =CN.Determine the value of (MH+NH)/OH

Luis González
#2 Sep 4, 2015, 11:03 pm • 1

See http://www.artofproblemsolving.com/community/c6h604436.

High School Olympiads

Triangle altitudes

Reply

Source: 2002 China Second Round Olympiad

tau172

#1 Aug 30, 2014, 7:11 am

In $\triangle ABC$, $\angle A = 60$, $AB > AC$, point O is the circumcenter and H is the intersection point of two altitudes BE and CF. Points M and N are on the line segments BH and HF respectively, and satisfy $BM = CN$. Determine the value of $\dfrac{MH+NH}{OH}$.

Luis González

#2 Aug 30, 2014, 10:08 am • 1

Circles $\odot(HBC)$ and $\odot(HMN)$ meet at H and the center of the rotation that swaps the congruent segments $\overline{BM}$ and $\overline{CN}$. Since $O \in \odot(HBC)$, because of $\angle BOC = \angle BHC = 120^\circ$, and $OB = OC$, then it follows that O is the center of the aforementioned rotation $\Longrightarrow O \in \odot(HMN)$ such that $OM = ON$, i.e. $\triangle OMN$ is isosceles with apex angle $\angle MON = \angle BHC = 120^\circ$. Hence Ptolemy's theorem for the cyclic $OMHN$ gives

$$OH \cdot MN = OM \cdot (HM + HN) \Longrightarrow \frac{HM+HN}{OH} = \frac{MN}{OM} = 2 \cdot \sin 60^\circ = \sqrt{3}.$$

ind

#3 Sep 4, 2015, 11:16 pm

could you please explain this solution in simpler manner
I could not get
that O is the center of the aforementioned rotation $\Longrightarrow O \in \odot(HMN)$

Quick Reply

High School Olympiads

Prove that $AW \parallel BC$

Reply

Source: BYaas

variabestjako

#1 Sep 3, 2015, 3:41 pm • 1

Hard Problem: Given a triangle ABC inscribed circle (O). D is the midpoint of BC. E is the intersection of the tangent at A to the circle (O) with BC. F is the second intersection of (EAB) with (DAC). The line through F cut AB, AC respectively in M, N. P is the intersection BN with CM. X, Y is the intersection of PF with AD, AE. W is the intersection NY with MX.

This post has been edited 2 times. Last edited by variabestjako, Jan 9, 2016, 4:04 pm
Reason: no

Luis González

#3 Sep 4, 2015, 12:11 pm • 1

More general: F is arbitrary point on the plane of $\triangle ABC$. $\odot(FAC)$ and $\odot(FAB)$ cut BC again at D, E. Arbitrary line through F cuts AB, AC at M, N. $P \equiv BN \cap CM$ and PF cuts AD, AE at X, Y. If $W \equiv NY \cap MX$, then $AW \parallel BC$.

Radical axis AF of $\odot(FAC), \odot(FAB)$ cuts BC at T. $\Longrightarrow TD \cdot TC = TE \cdot TB \Longrightarrow T$ is center of the involution that swaps B, E and C, D. As T gets mapped to the point at infinity A_∞ of BC, then the pencil $AM \mapsto AY, AX \mapsto AN$, $AF \mapsto AA_\infty$ is involutive. But by dual of Desargues involution theorem for the complete quadrangle $FXWN$, it follows that $AM \mapsto AY, AX \mapsto AN, AF \mapsto AW$ is an involution, which forces $AW \equiv AA_\infty \Longrightarrow AW \parallel BC$.

Dukejukem

#4 Sep 5, 2015, 2:19 pm

> *Luis González wrote:*
> **More general:** F is arbitrary point on the plane of $\triangle ABC$. $\odot(FAC)$ and $\odot(FAB)$ cut BC again at D, E. Arbitrary line through F cuts AB, AC at M, N. $P \equiv BN \cap CM$ and PF cuts AD, AE at X, Y. If $W \equiv NY \cap MX$, then $AW \parallel BC$.

Here's another solution: Let R, S lie on AD, AE, respectively, and satisfy $MR \parallel NS \parallel BC$. Let AF cut BC at T. Since T lies on the radical axis of $\odot(FAC)$ and $\odot(FAB)$, we have

$$TB(TC + TE) = TC(TB + TD) \implies \frac{TB}{TC} = \frac{BD}{CE}.$$

Meanwhile, from $\triangle AMR \sim \triangle ABD$ and $\triangle ANS \sim \triangle ACE$, we obtain $\frac{MR}{AM} = \frac{BD}{AB}$ and $\frac{NS}{AN} = \frac{CE}{AC}$. Using $\frac{BD}{CE} = \frac{TB}{TC}$, it follows that

$$\frac{MR}{NS} = \frac{AM}{AN} \cdot \frac{AC}{AB} \cdot \frac{TB}{TC} = \frac{AM}{AN} \cdot \frac{\sin \angle BAT}{\sin \angle CAT},$$

where the last step follows from the Ratio Lemma applied to $\triangle ABC$. Finally, from the Ratio Lemma applied to $\triangle AMN$, we obtain $\frac{MR}{NS} = \frac{FM}{FN}$. Combining this relation with $\angle FMR = \angle FNS$, it follows that $\triangle FMR \sim \triangle FNS$. Therefore, F, R, S are collinear, and it follows that $\triangle MXR$ and $\triangle NYS$ are perspective with perspector F. By Desargue's Theorem, $W \equiv MX \cap NY, A \equiv XR \cap YS, \infty \equiv RM \cap SN$ are collinear. Hence, $AW \parallel BC$ as desired. $\square$

This post has been edited 1 time. Last edited by Dukejukem Sep 5, 2015, 2:20 pm
Reason: Typo

Quick Reply

High School Olympiads

Perpendicular if cyclic

Reply

Scorpion.k48

#1 Sep 3, 2015, 11:13 pm

Let 2 circles $\odot(O_1;R_1)$ and $\odot(O_2;R_2)$ with $R_1 \neq R_2$, Δ_1,Δ_2 be two internal common tangents of $\odot(O_1),\odot(O_2)$. Δ_3 be the external common tangent of $\odot(O_1),\odot(O_2)$ and touches $\odot(O_1),\odot(O_2)$ at M,N. $A=\Delta_1\cap\Delta_3, B=\Delta_2\cap\Delta_3$. Let $\odot(O_3)$ is incircle of $\triangle ABI$ and touches Δ_1,Δ_2 at P,Q. Assume that M,N,P,Q are cyclic, let $\odot(O_4)=\odot(MNPQ)$. Prove that I,A,B,O_4 are cyclic and $\Delta_1\perp\Delta_2$

Luis González

#2 Sep 4, 2015, 5:08 am

We can rephrase the problem as follows: The incircle (I) of $\triangle ABC$ touches BC,CA,AB at D,E,F and the B- and C-excircles touch BC at $Y,Z.$ If $EFZY$ is cyclic, then $\angle BAC=90^{\circ}$ and the center of this circle is the midpoint of the arc BC of $\odot(ABC).$

Let M be the common midpoint of BC,YZ and let EF cut BC at $U.$ V is the reflection of D on $U.$ Since $UY\cdot UZ=UE\cdot UF=UD^2=UV^2 \Longrightarrow (Y,Z,D,V)=-1 \Longrightarrow DY\cdot DZ=DM\cdot DV=2\cdot DM\cdot DU \Longrightarrow$ $(b-c)\cdot\dfrac{2(s-b)(s-c)}{b-c}=bc \Longrightarrow 2(s-b)(s-c)=bc \Longrightarrow b^2+c^2=a^2 \Longrightarrow \angle BAC=90^{\circ}.$

The center J of $\odot(EFZY)$ is found intersecting the perpendicular bisector of EF (internal bisector of $\angle BAC$) and the perpendicular bisector of YZ (perpendicular bisector of BC as well) $\Longrightarrow J$ is the midpoint of the arc BC of $\odot(ABC).$

Quick Reply

High School Olympiads

Triangle bounded by perpendiculars

Reply

Source: Hyacinthos #22376

rodinos

#1 Jun 10, 2014, 2:06 am

Let ABC be a triangle and P a point. The line OP intersects the circumcircle of BPC at A', other than P. Similarly the points B',C'. The perpendiculars from A',B',C' to BC, CA, AB, resp. bound a triangle A*B*C*. For which points P's the triangles ABC and A*B*C* are perspective? For all?

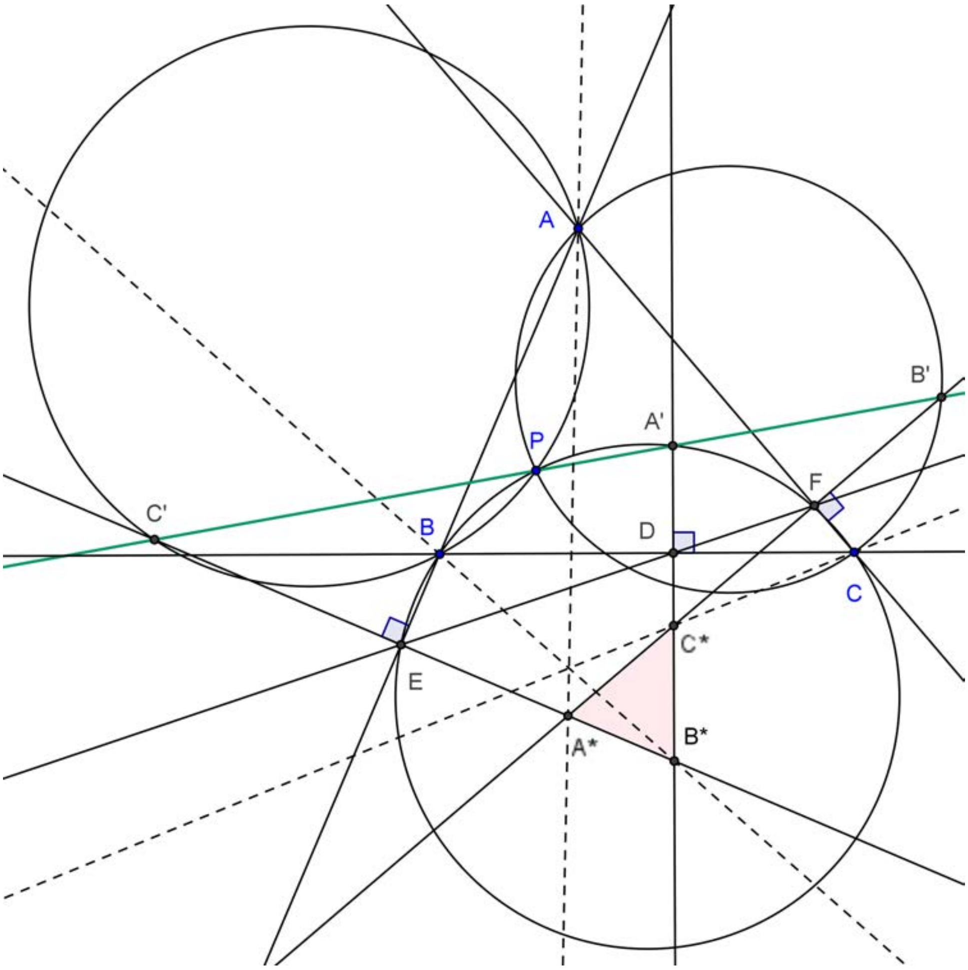
PS: If this problem is already known, I would appreciate references to.

Luis González

#2 Jun 10, 2014, 8:26 am • 2

Is O specifically the circumcenter of ABC?, because according to my calculations, ABC and A*B*C* are perspective for any line through P. If D,E,F denote the projections of A',B',C' on BC,CA,AB, then it suffices to prove that D,E,F are collinear.

Attachments:



rodinos

#3 Jun 10, 2014, 4:25 pm

I seem to recall that the collinearity for any P was discussed in Hyacinthos.

Now, I am interested in mappings P --> P' and loci.
An example: Here:
https://groups.yahoo.com/neo/groups/Hyacinthos/conversations/messages/22410

A-B-C

#4 Sep 3, 2015, 8:58 pm • 1

> *Luis González wrote:*
> it suffices to prove that D,E,F are collinear.

My solution:
Let $\triangle A_1B_1C_1$ be antipedal triangle of P WRT $\triangle ABC$.
Then D is orthopole of $\overline{A'B'C'}$ WRT $\triangle A_1BC$.
From Three Collinear Orthopoles, D, E, F are collinear.
Thanks to Telv for giving me the link.

Luis González

#5 Sep 4, 2015, 1:55 am

Here is the proof I had in mind: Denote $\ell \equiv \overline{A'B'C'}$ and $\angle(PA, \ell) = \lambda_A, \angle(PB, \ell) = \lambda_B, \angle(PC, \ell) = \lambda_C$. Then we have

$$\frac{\overline{DB}}{\overline{DC}} = \frac{\cot\angle A'BC}{\cot\angle A'CB} = \frac{\cot\angle A'PC}{\cot\angle A'PB} = \frac{\cot\lambda_C}{\cot\lambda_B}.$$

Multiplying the cyclic expressions together, we conclude that D, E, F are collinear by Menelaus' theorem.

Quick Reply

High School Olympiads

I need ur help :D

Locked

Source: British Math Olympiad 1993

Giovanni98

#1 Sep 3, 2015, 4:32 pm

Let P be an internal point of a triangle ABC. Let's define $\alpha = \angle BPC - \angle BAC$, $\beta = \angle APC - \angle ABC$, $\gamma = \angle APB - \angle ACB$.

Prove that $PA\frac{\sin \angle BAC}{\sin \alpha} = PB\frac{\sin \angle ABC}{\sin \beta} = PC\frac{\sin \angle ACB}{\sin \gamma}$

Giovanni98

#2 Sep 3, 2015, 5:30 pm

Up......

Luis González

#3 Sep 3, 2015, 8:39 pm

Please give your topics meaningful subjects and use the search before posting contest problems. See http://www.artofproblemsolving.com/community/c6h312075.

High School Olympiads

trigonometry trivia

Reply

Source: Indonesia IMO 2007 TST, Stage 2, Test 1, Problem 1

Raja Oktovin
#1 Nov 15, 2009, 12:26 pm

Let P be a point in triangle ABC, and define α, β, γ as follows:

$$\alpha = \angle BPC - \angle BAC, \quad \beta = \angle CPA - \angle\angle CBA, \quad \gamma = \angle APB - \angle ACB.$$

Prove that

$$PA \frac{\sin \angle BAC}{\sin \alpha} = PB \frac{\sin \angle CBA}{\sin \beta} = PC \frac{\sin \angle ACB}{\sin \gamma}.$$

Luis González
#2 Nov 17, 2009, 11:30 pm

Let A', B', C' be the second intersections of rays AP, BP, CP with the circumcircle (O) of $\triangle ABC$. Angle chase gives

$\angle B'CP = \angle BPC - \angle BB'C = \angle BPC - \angle BAC = \alpha$

Similarly, $\angle C'AP = \beta$ and $\angle A'BP = \gamma$.

By sine law in the triangles $\triangle PCB', \triangle PAC'$ and $\triangle PBA'$ we obtain

$\frac{\sin \widehat{BAC}}{\sin \alpha} = \frac{PC}{PA}, \ \frac{\sin \widehat{CBA}}{\sin \beta} = \frac{PA}{PC'}, \ \frac{\sin \widehat{ACB}}{\sin \gamma} = \frac{PB}{PA'}$

From power of P WRT (O) , $PA \cdot PA' = PB \cdot PB' = PC \cdot PC'$ we get

$PA \cdot \frac{\sin \widehat{BAC}}{\sin \alpha} = PB \cdot \frac{\sin \widehat{CBA}}{\sin \beta} = PC \cdot \frac{\sin \widehat{ACB}}{\sin \gamma}$

Quick Reply

High School Olympiads

Non-Projective Geometry

Locked

droid347

#1 Sep 3, 2015, 12:52 am

Let ABC be a right triangle with $\angle A = 90^\circ$, and let D be a point lying on the side AC. Denote by E the reflection of A into the line BD, and by F the intersection point of CE with the perpendicular in D to the line BC. Prove that AF, DE, and BC are concurrent.

I was able to solve this problem with projective tools, but was not able to find an elementary solution. Could anyone help me? Thank you!

Luis González

#2 Sep 3, 2015, 1:31 am

Discussed at http://www.artofproblemsolving.com/community/c6h152838. So for further discussions use the previous thread.

High School Olympiads

Concurrent lines in a right triangle

Reply

Source: Romanian JBTST VI 2007, problem 3

PhilAndrew

#1 Jun 8, 2007, 12:59 pm • 5

Let ABC be a right triangle with $A = 90^\circ$ and $D \in (AC)$. Denote by E the reflection of A in the line BD and F the intersection point of CE with the perpendicular in D to BC. Prove that AF, DE and BC are concurrent.

pohoatza

#2 Jun 9, 2007, 1:30 am • 4

Denote the points $X \in AE \cap BD$, $Y \in AE \cap BC$, $Z \in AE \cap DF$ and $T \in DF \cap BC$.

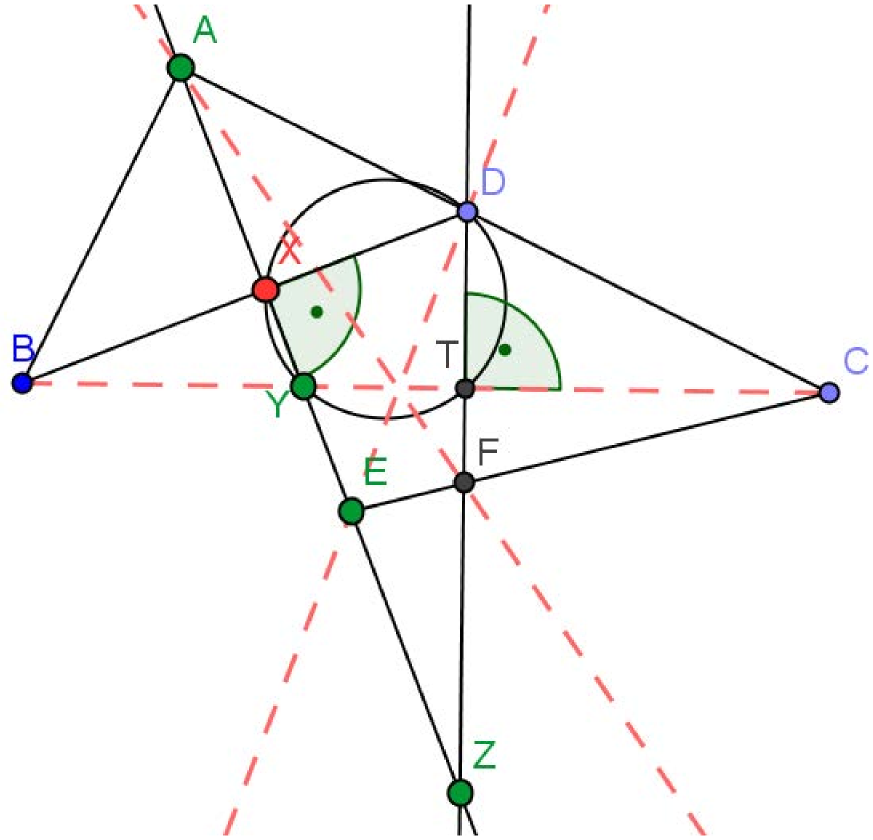
In $\triangle AEC$, we observe that the lines AF, DE and BC are concurrent, if and only if, the division $(AYEZ)$ is harmonic.

Since the quadrilateral $XYTD$ is cyclic, $\tan XYB = \tan XDZ$, which is equivalent to

$$\frac{XB}{XY} = \frac{XZ}{XD} \Longleftrightarrow XB \cdot XD = XY \cdot XZ.$$

Due to the similarity of the triangles $\triangle XAB$ and $\triangle XDA$, we have that $XA^2 = XB \cdot XD$, so $XA^2 = XY \cdot XZ$, which by using $XA = XF$, it is equivalent with $\frac{YA}{YE} = \frac{ZA}{ZE}$, i.e. the division $(AYEZ)$ is harmonic.

Attachments:



This post has been edited 2 times. Last edited by pohoatza, Jul 14, 2007, 10:38 am vittasko

#3 Jun 10, 2007, 1:43 am • 4

We denote as $K, L,$ the intersection points of $BC,$ from DF, AE respectively and also the points $X\equiv AC\cap BE,$ $Y\equiv AB\cap CE, Z\equiv AE\cap XY.$

From the complete quadrilateral $ABECXY,$ we conclude that the points $L, Z,$ are harmonic conjugates, with respect to the points $A, E.$

Because of $AD\perp AB, DE\perp BE, DK\perp KB,$ we conclude that the pentagon $ABEKD$ is cyclic (taken as diameter the segment BD) and so, we have that the segment line $KB,$ is the angle bisector of the angle $\angle AKE,$ because of $AB = BE.$

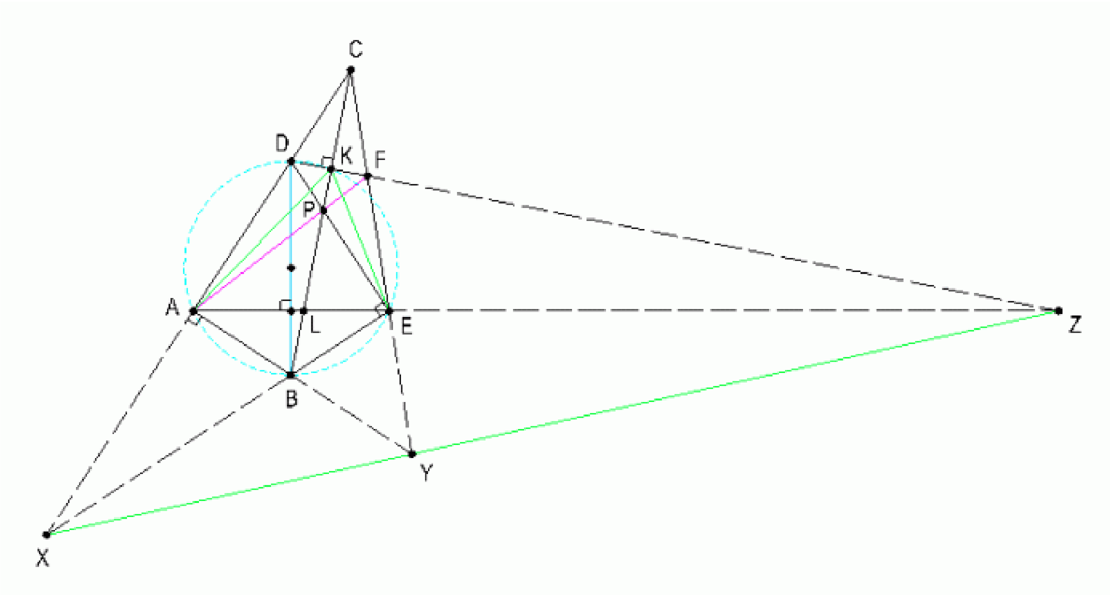
So, because of $DK\perp KB,$ we conclude that the segment line $DF,$ passes through the point $Z,$ as the external angle bisector of the triangle $\triangle KAE,$ through vertex $K.$

Because of now, the points X, Y, Z are collinear, based on the **Desarques's theorem**, we conclude that the triangles $\triangle BEA, \triangle CDF,$ are perspective ($X\equiv BE, Y\equiv BA\cap CF, Z\equiv AE\cap DF$).

Hence, the segment lines AF, DE, BC are concurrent at one point and the proof is completed.

Kostas Vittas.

Attachments:



andyciup

#4 Jun 18, 2007, 3:27 pm • 5

Here is another solution to this nice problem:
Let $DF\cap BC = O, AE\cap BD = M, AE\cap BC = N$ and assume $AD\geq DC.$
Let $\measuredangle CBD = x, \measuredangle DBA = y,$ and $\measuredangle BCA = c.$ It`s clear that $x + y + c = 90^\circ$
Because triangle BAM is right-angled, we get $\measuredangle BAM = c + x.$
Because the quadrilateral $ABDO$ is cyclic, we have $\measuredangle OAD = \measuredangle OBD = x,$
thus $\measuredangle OAE = y - x.$
In isosceles triangle $ABE,$ we have $\measuredangle BAE = c + x,$ therefore $\measuredangle ABE = 2y.$
But $\measuredangle ABN = x + y,$ therefore $\measuredangle EBO = y - x = \measuredangle OAE.$
This means that quadrilaterel $OABE$ is cyclic, and thus the pentagon $ADOEB$ is cyclic.
This means that $\measuredangle EDB = \measuredangle EAB = c + x \Longrightarrow \measuredangle ADE = 2(c + x) \Longrightarrow \measuredangle EDF = y - x.$
In triangle $ABE,$ $\ds N\in (AE)\Longrightarrow \frac{AN}{NE}=\frac{AB}{BE}\cdot\frac{\sin \measuredangle{ABN}}{\sin\measuredangle{EBN}}\Longrightarrow \frac{AN}{NE}=\frac{\sin(x+y)}{\sin(y-x)}$
In triangle $EDC,$ $\ds F\in (EC)\Longrightarrow \frac{EF}{FC}=\frac{ED}{DC}\cdot\frac{\sin\measuredangle{EDF}}{\sin\measuredangle{FDC}}\Longrightarrow \frac{EF}{FC}=\frac{ED}{DC}\cdot\frac{\sin(y-x)}{\sin(x+y)}$
By multiplying the last two relations we obtain $\ds \frac{AN}{NE}\cdot\frac{EF}{FC}\cdot\frac{CD}{DA}=\frac{ED}{DA}=1$,
therefore, by the converse of Ceva`s theorem, the lines AF, CN, ED are concurent, QED.

Sergey

#5 May 5, 2009, 12:27 am

> pohoatza wrote:
> Due to the similarity of the triangles $\triangle XAB$ and $\triangle XDA$, we have that $XA^2 = XB \cdot XD$, so $XA^2 = XY \cdot XZ$, which by using $XA = XF$, it is equivalent with $\frac{YA}{YE} = \frac{ZA}{ZE}$, i.e. the division $(AYEZ)$ is harmonic.

Please could you tell how exactly using $XA = XF$ we get $\frac{YA}{YE} = \frac{ZA}{ZE}$?

Virgil Nicula
#6 May 6, 2009, 10:17 pm

Remark (a short commentary). It is a very nice application of the following **remarkable harmonical division** (see the nice **Pohoatza**'s proof !) :

> Quote:
> **Lemma.** Let $w(O)$, w_o be two secant circles in $\{A, B\}$ so that $O \in w_o$. For $P \in (AB)$ (segment !) denote $\{M, N, R\}$
>
> such that $\{M, N\} \subset w$, $R \in w_o$ and N separates P , R . Then the division $\{\ M\ ,\ N\ ;\ P\ ,\ R\ \}$ is harmonically.
>
> **Particular case.** Let ABC be a triangle with the orthocenter H . Denote $D \in BC \cap AH$ and the intersections N , S
>
> between the line AH and the circle with the diameter $[BC]$. Then the division $\{\ A\ ,\ H\ ,\ N\ ,\ S\ \}$ is harmonically.

Proof. $PO \cdot PR = PA \cdot PB = PM \cdot PN \quad \Longrightarrow \quad PO \cdot PR = PM \cdot PN$ $\Longleftrightarrow$ the division $\{\ M\ ,\ N\ ;\ P\ ,\ R\ \}$ is harmonically.

The following interesting problem is a nice consequence of the harmonical division which was mentioned in the above lemma :

> Quote:
> Let $w(O)$, w_o be two secant circles so that $O \in w_o$. Consider $A \in w \cap w_o$. For $P \in (AB)$ (segment !) denote $\{M, N, R\}$ such that
>
> $\{M, N\} \subset w$, $R \in w_o$ and N separates P , R . For $C \in AB$ (line !) denote $\left\| \begin{array}{c} X \in MC \cap RA \\ Y \in NC \cap RA \end{array} \right\|$ .Then $MY \cap NX \cap CP \ne \emptyset$.

Proof. From the upper lemma obtain that the division $\{\ M\ ,\ N\ ;\ P\ ,\ R\ \}$ is harmonically, i.e. $\frac{\overline{PM}}{\overline{PN}} = -\frac{\overline{RM}}{\overline{RN}}$. Apply the **Menelaus' theorem**

to the transversal $\overline{RXY}$ and the triangle CMN : $\frac{\overline{RM}}{\overline{RN}} \cdot \frac{\overline{YN}}{\overline{YC}} \cdot \frac{\overline{XC}}{\overline{XM}} = +1 \Longrightarrow \frac{\overline{PM}}{\overline{PN}} \cdot \frac{\overline{YN}}{\overline{YC}} \cdot \frac{\overline{XC}}{\overline{XM}} = -1$, i.e. $MY \cap NX \cap CP \ne \emptyset$.

vittasko
#7 May 7, 2009, 12:27 pm

> Sergey wrote:
>> pohoatza wrote:
>> Due to the similarity of the triangles $\triangle XAB$ and $\triangle XDA$, we have that $XA^2 = XB \cdot XD$, so $XA^2 = XY \cdot XZ$, which by using $XA = XF$, it is equivalent with $\frac{YA}{YE} = \frac{ZA}{ZE}$, i.e. the division $(AYEZ)$ is harmonic.
>
> Please could you tell how exactly using $XA = XF$ we get $\frac{YA}{YE} = \frac{ZA}{ZE}$?

There is a typo. He means $XA = XE$, instead of $XA = XF$.

Kostas Vittas.

Petry

#8 May 7, 2009, 11:24 pm

Hello!

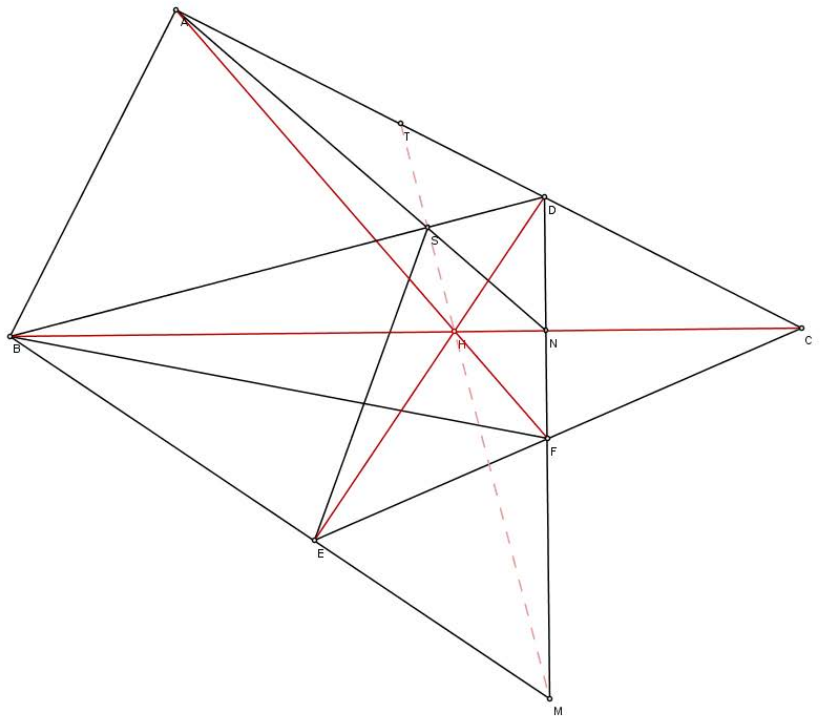
$\{M\} = BE \cap DF$, $\{N\} = BC \cap DF$, $\{H\} = DE \cap BN$,
$\{S\} = MH \cap BD$, $\{T\} = MH \cap AC$.

$DE \perp MB$ and $BN \perp MD \Rightarrow$ the point H is the orthocenter of the triangle $\Delta MBD \Rightarrow$
$\Rightarrow MS \perp BD$ and $\angle MSN = \angle MSE$ (1)
$\Delta ABS \equiv \Delta EBS \Rightarrow \angle ASB = \angle ESB \Rightarrow \angle AST = \angle MSE$ (2)
(1), (2) $\Rightarrow \angle MSN = \angle AST \Rightarrow$ the points A, S, N are collinear.

Let's consider the triangles ΔSDA and ΔMEF.
$\{B\} = SD \cap ME$, $\{C\} = DA \cap EF$, $\{N\} = AS \cap FM$ and the points B, C, N are collinear.
Based on the Desargues's theorem, we conclude that the triangles ΔSDA and ΔMEF are perspective (the lines SM, DE and AF are concurrent at the point H).
Hence, the lines AF, DE and BC are concurrent at the point H.

Best regards, Petrisor Neagoe

Attachments:



mathVNpro

#9 May 7, 2009, 11:31 pm

> *PhilAndrew wrote:*
> Let ABC be a right triangle with $A = 90^\circ$ and $D \in (AC)$. Denote by E the reflection of A in the line BD and F the intersection point of CE with the perpendicular in D to BC. Prove that AF, DE and BC are concurrent.

Here is another approach:
Let $T \equiv DF \cap AE$, $S \equiv AE \cap BC$, $K \equiv AE \cap BD$, $H \equiv DT \cap BC$. In order to prove AF, DE, BC are concurrent, we need to prove $(A, E, S, T) = -1$. Define (BD) is the circle with diameter BD, (DS) is the circle with diameter DS. It is easy to notice that $\{A, B, D, E, H\} \subset (BD)$ and $\{K, D, H, S\} \subset (DS) \Longrightarrow DH$ is the radical axis wrt (BD), (DS)

easy to notice that $\{A, B, D, E, H\} \in (BD)$ and $\{K, D, H, S\} \in (DS) \Longrightarrow DH$ is the radical axis wrt $(BD), (DS)$.
Thus, $\mathcal{P}_{T/(BD)} = \mathcal{P}_{T/(DS)}$
$\Longrightarrow \overline{TS}.\overline{TK} = \overline{TE}.\overline{TA}$, but K is the midpoint of AE, hence $(AEST) = -1$, which implies to the result of the problem.
Our proof is completed $\blacksquare$.

fmasroor

#10 Dec 2, 2013, 5:56 am

> *andyciup wrote:*
>
> Here is another solution to this nice problem:
> Let $DF \cap BC = O$, $AE \cap BD = M$, $AE \cap BC = N$ and assume $AD \geq DC$.
> Let $\measuredangle CBD = x$, $\measuredangle DBA = y$, and $\measuredangle BCA = c$. It`s clear that $x + y + c = 90^\circ$
> Because triangle BAM is right-angled, we get $\measuredangle BAM = c + x$.
> Because the quadrilateral $ABDO$ is cyclic, we have $\measuredangle OAD = \measuredangle OBD = x$,
> thus $\measuredangle OAE = y - x$.
> In isosceles triangle ABE, we have $\measuredangle BAE = c + x$, therefore $\measuredangle ABE = 2y$.
> But $\measuredangle ABN = x + y$, therefore $\measuredangle EBO = y - x = \measuredangle OAE$.
> This means that quadrilaterel $OABE$ is cyclic, and thus the pentagon $ADOEB$ is cyclic.
> This means that $\measuredangle EDB = \measuredangle EAB = c + x \Longrightarrow \measuredangle ADE = 2(c + x) \Longrightarrow \measuredangle EDF = y - x$.
> In triangle ABE, $\ds N\in (AE)\Longrightarrow \frac{AN}{NE}=\frac{AB}{BE}\cdot\frac{\sin \measuredangle{ABN}}{\sin\measuredangle{EBN}}\Longrightarrow \frac{AN}{NE}=\frac{\sin(x+y)}{\sin(y-x)}$
> In triangle EDC, $\ds F\in (EC)\Longrightarrow \frac{EF}{FC}=\frac{ED}{DC}\cdot\frac{\sin\measuredangle{EDF}}{\sin\measuredangle{FDC}}\Longrightarrow \frac{EF}{FC}=\frac{ED}{DC}\cdot\frac{\sin(y-x)}{\sin(x+y)}$
> By multiplying the last two relations we obtain $\ds \frac{AN}{NE}\cdot\frac{EF}{FC}\cdot\frac{CD}{DA}=\frac{ED}{DA}=1$,
> therefore, by the converse of Ceva`s theorem, the lines AF, CN, ED are concurent, QED.

Darn it this was exactly the same proof I just rediscovered a few minutes ago, which I originally wrote down a while ago but threw away the paper.

Dukejukem

#11 Sep 9, 2014, 11:15 am

> *PhilAndrew wrote:*
>
> Let ABC be a right triangle with $A = 90^\circ$ and $D \in (AC)$. Denote by E the reflection of A in the line BD and F the intersection point of CE with the perpendicular in D to BC. Prove that AF, DE and BC are concurrent.

The inspiration for this solution comes from Cosmin (pohoatza) in https://www.google.com/url?sa=t&rct=j&q=&esrc=s&source=web&cd=1&cad=rja&uact=8&ved=0CCAQFjAA&url=http%3A%2F%2Fdiendantoanhoc.net%2Ff
(Harmonic Divisions and its Applications).

Solution

Quick Reply

High School Olympiads

Coaxial circles associate with X13,X14

Reply

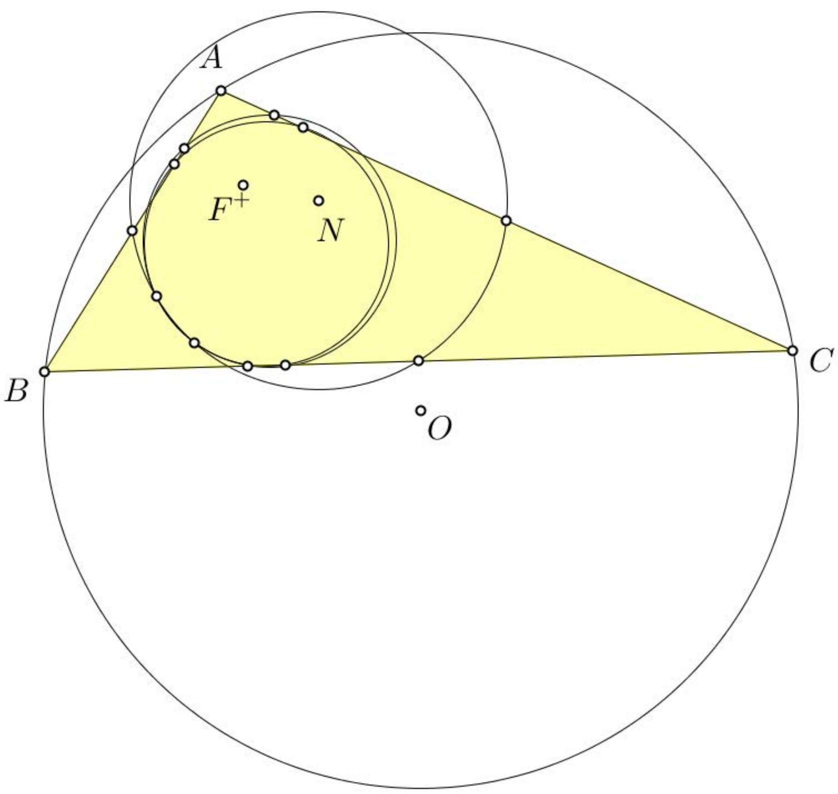
Source: Own

A-B-C

#1 Sep 2, 2015, 7:48 pm • 4

Nine-point circle, Pedal circle and Cevian circle of Fermat point are coaxial.

Attachments:



TelvCohl

#3 Sep 2, 2015, 10:13 pm • 2

Lemma :

Let T be the Fermat point of $\triangle ABC$ and $\triangle DEF$ be the cevian triangle of T WRT $\triangle ABC$. Let O be the circumcenter of $\triangle ABC$ and H be the orthocenter of $\triangle DEF$. Then O, H, T are collinear.

Proof :

Let the orthotransversal of T WRT $\triangle ABC$ cuts BC, CA, AB at X, Y, Z, respectively. From $\measuredangle BTD = \measuredangle DTC = 60^\circ$ and $AT \perp TX \Longrightarrow (BC;DX) = -1$, so X lie on the trilinear polar τ of T WRT $\triangle ABC$. Analogously, we can prove Y, Z lie on τ, so $\overline{XYZ}$ coincide with the trilinear polar of T WRT $\triangle ABC$. From $(BC;DX) = -1 \Longrightarrow \odot(DX) \perp \odot(O)$. Analogously, we can prove $\odot(EY) \perp \odot(O)$ and $\odot(FZ) \perp \odot(O)$, so OT is the common radical axis Ω of $\{\odot(DX), \odot(EY), \odot(FZ)\}$, hence combine with $H \in \Omega$ (well-known) we conclude that O, H, T are collinear.

Back to the main problem :

Let T be the Fermat point of $\triangle ABC$ and S be the isogonal conjugate of T WRT $\triangle ABC$. Let P, Q be the intersection of the 9-point circle $\odot(N)$ of $\triangle ABC$ and the circumcircle of the pedal triangle of T WRT $\triangle ABC$ where P is the Poncelet point of $ABCT$ and Q is the Ponclete point of $ABCS$. From Poncelet points $\Longrightarrow P$ also lie on the circumcircle of the cevian triangle $\triangle XYZ$ of T WRT $\triangle ABC$, so we only have to prove the other intersection $R \neq P$ of $\odot(N)$ and $\odot(XYZ)$ coincide with Q.

Let O be the circumcenter of $\triangle ABC$ and H be the orthocenter of $\triangle XYZ$. From the lemma $\Longrightarrow O, T, H$ are collinear, so notice OT, OH is the Steiner line of Q, R WRT the medial triangle of $\triangle ABC$, respectively (well-known) we get $Q \equiv R$.

Luis González

#4 Sep 2, 2015, 11:08 pm • 1

Label F, S the 1st Fermat point and it's isogonal conjugate; 1st isodynamic point. $\triangle F_aF_bF_c$ is cevian triangle of F and $\triangle S_aS_bS_c$ is pedal triangle of S. $\odot(S_aS_bS_c)$ is pedal circle of F, S cutting BC again at the projection U_a of F on BC. $\triangle H_aH_bH_c$ is orthic triangle of $\triangle ABC$ and A_c is the apex of the equilateral triangle erected outside $\triangle ABC$. It's well-known that the reflection X of S on BC is on AF_a. Hence

$$\frac{F_aS_a}{F_aH_a} = \frac{S_aX}{AH_a} = \frac{SS_a}{AH_a} \ , \ \frac{F_aU_a}{F_aM_a} = \frac{FU_a}{A_0M_a} = \frac{FU_a}{\frac{\sqrt{3}}{2}BC} \Longrightarrow$$

$$\frac{F_aU_a \cdot F_aS_a}{F_aH_a \cdot F_aM_a} = \frac{FU_a \cdot SS_a}{\frac{\sqrt{3}}{2}BC \cdot AH_a} = \frac{FU_a \cdot SS_a}{\sqrt{3}[ABC]}.$$

As F, S are isogonal conjugates, the product of their distaces to BC, CA, AB is constant. Thus the latter expression reveals that the ratio of the powers of F_a WRT the 9-point circle $\odot(H_aH_bH_c)$ and $\odot(S_aS_bS_c)$ is a constant and similarly for F_b and F_c $\Longrightarrow \odot(F_aF_bF_c), \odot(S_aS_bS_c)$ and $\odot(H_aH_bH_c)$ are coaxal.

High School Olympiads

a propertyof the line through circumcenter 5

Reply

Source: OWN

LeVietAn

#1 Sep 2, 2015, 10:04 am

Dear Mathelinkers,
Given a triangle ABC inscribed in a circle (O) and ℓ is a line passing through O. Let P vary on (O). Let PA, PB, PC intersect ℓ at D, E, F, resp. Let XYZ be the triangle bounded by the line perpendicular to BC through D, the line perpendicular to CA through E, and the line perpendicular to AB through F. Let H and K be the orthocenter of the triangles ABC and XYZ, resp. Let M and N be the midpoints of the segments PH and HK. Prove that the line MN always goes through a fixed point when P moves.

Luis González

#2 Sep 2, 2015, 10:22 am • 2

This follows from the problem A generalization of the Simson line theorem and Sondat's theorem (see An old result but very hard). All lines MN go through the orthopole of ℓ WRT $\triangle ABC$.

TelvCohl

#3 Sep 2, 2015, 12:24 pm • 1

Let Q be the reflection of P in ℓ and $T \equiv PK \cap \odot(ABC)$. From $\measuredangle EQF = \measuredangle FPE = \measuredangle CPB = \measuredangle CAB = \measuredangle EXF \Longrightarrow Q \in \odot(XEF)$. Similarly, we can prove Q lie on $\odot(YFD), \odot(ZDE)$, so Q is the Miquel point of the complete quadrilateral $\{\triangle XYZ, \ell\}$, hence PT is the Steiner line of the complete quadrilateral $\{\triangle XYZ, \ell\}$. Since the isogonal conjugate Q^* of Q WRT $\triangle XYZ$ is the infinity point with direction $\perp PT$, so $\measuredangle(PT, BC) = \measuredangle Q^*ZY = \measuredangle XZQ = \measuredangle(\ell, QD) = \measuredangle(PA, \ell) = \measuredangle(PA, BC) + \measuredangle(BC, \ell) \Longrightarrow \measuredangle(PT, PA) = \measuredangle(BC, \ell)$, hence T is the pole of the Simson line of $\triangle ABC$ with direction $\perp \ell \Longrightarrow T$ is a fixed point when P varies on $\odot(O) \Longrightarrow MN$ passes through the midpoint of HT (fixed) as P varies on $\odot(O)$.

Quick Reply

High School Olympiads

An old result but very hard

Source: Come on!

Nbach

#1 Jul 29, 2007, 7:52 pm • 1

Let be $\triangle ABC$.A line p intersects AB, BC, CA at poits C_0, A_0, B_0 (respect) .The lines which through C_0, B_0, A_0 perpendicular to AB, AC, BC intersect at poits D, E, F. H_1, H_2 are orthocenters of triangles ABC, DEF .Denote $f(p)$ be the ratio which p intersect $[H_1H_2]$. Prove that $f(p)$ is a constant number for all line p in the plane.

yetti

#2 Nov 14, 2007, 9:14 am

If the ratio, in which p cuts H_1H_2, is constant, it has to be $\frac{1}{2}$. If the normals to $BC,\ CA,\ AB$ at $A_0,\ B_0,\ C_0$ happen to concur at a point, the $\triangle DEF$, including its orthocenter H_2, degenerates to the point H_2, which is then pole of the Simson line p of the $\triangle ABC$. Any Simson line cuts the segment H_1H_2 between its pole H_2 and the triangle orthocenter H_1 in half. If the normals do not concur and if $\mathcal{T}$ is the defined transformation $\triangle DEF = \mathcal{T}(\triangle ABC,\ p)$, then $\triangle ABC = \mathcal{T}(\triangle DEF,\ p)$, so that $\mathcal{T}$ is an involution (its own inverse) and again, if the ratio is constant, it has to be 1/2. Changing notation:

Let an arbitrary line k intersect the triangle sidelines $BC,\ CA,\ AB$ at points $X,\ Y,\ Z$. Let normals $x,\ y,\ z$ to $BC,\ CA,\ AB$ at the points $X,\ Y,\ Z$ meet at $A' \equiv y \cap z,\ B' \equiv z \cap x,\ C' \equiv x \cap y$. The line k cuts the segment HH' between the orthocenters $H,\ H'$ of the $\triangle ABC,\ \triangle A'B'C'$ in half.

The $\triangle ABC \sim \triangle A'B'C'$ are similar, having perpendicular their corresponding sides $BC \perp x \equiv B'C'$, $CA \perp y \equiv C'A'$, $AB \perp z \equiv A'B'$. Let $(O),\ (O')$ be the respective circumcircles of these two triangles. Let $P \in (O)$ be pole of the Simson line p of the original $\triangle ABC$ parallel to the line $k \equiv XYZ$. Let $F_a,\ F_b,\ F_c$ be feet of normals from P to $BC,\ CA,\ AB$, forming the Simson line $p \equiv F_aF_bF_c$ of the $\triangle ABC$ with the pole P. The quadrilaterals $AF_bPF_c \sim AYA'Z$ are centrally similar with similarity center A, having parallel sides $AF_b \equiv AY,\ AF_c \equiv AZ, PF_b \parallel A'Y$ (both $\perp CA$), $PF_c \parallel A'Z$ (both $\perp AB$) and diagonals $F_bF_c \equiv p \parallel k \equiv YZ$. Consequently, PA' passes through the similarity center A, $A' \in PA$, and similarly, $B' \in PB,\ C' \in PC$. Let $Q \in (O)$ be the diametrically opposite point of P, the pole of the Simson line q of the $\triangle ABC$ perpendicular to the line $k \equiv XYZ$. Since PQ is a diameter of (O), $A'P \equiv AP \perp AQ, B'P \equiv BP \perp BQ$ and since the $\triangle ABC \sim \triangle A'B'C'$ have the corresponding sides perpendicular, $A'B' \perp AB$. As a result, the $\triangle A'PB' \sim \triangle AQB$ are similar, which means that $P \equiv Q' \in (O')$ is also pole of the Simson line $q' \perp q$ of the $\triangle A'B'C'$, hence $q' \parallel p \parallel k$.

Let $F'_a,\ F'_b,\ F'_c$ be feet of normals from P to $B'C',\ C'A',\ A'B'$, forming the Simson line $q' \equiv F'_aF'_bF'_c$ of the $\triangle A'B'C'$ with the pole P. Furthermore, let $P_a,\ P_b,\ P_c$ be reflections of P in $BC,\ CA,\ AB$ and $P'_a,\ P'_b,\ P'_c$ reflections of P in $B'C',\ C'A',\ A'B'$, forming Steiner lines $s \equiv P_aP_bP_c \parallel F_aF_bF_c \equiv p, s' \equiv P'_aP'_bP'_c \parallel F'_aF'_bF'_c \equiv q'$ of the $\triangle ABC,\ \triangle A'B'C'$, respectively, $s \parallel s' \parallel k \equiv XYZ$. The Simson lines $p,\ q'$ cut in half the segments $PH,\ PH'$ from their common pole P to the orthocenters $H,\ H'$, hence $H \in s,\ H' \in s'$. The quadrilateral $PF_cZF'_c$ is a rectangle, because $PF_c \parallel ZF'_c \equiv A'B'$ (both $\perp AB \equiv ZF_c$), $PF'_c \parallel ZF_c \equiv AB$ (both $\perp A'B'$, hence $ZF_c = PF'_c$. But F'_c is the midpoint of PP'_c, so that $ZF_c = F'_cP'_c$. Since $Z \in k,\ F_c \in p,\ F'_c \in q',\ P'_c \in s'$, the distance between the parallels $k \parallel p$ is equal to the distance between the parallels $q' \parallel s'$. It follows that the distance between the parallels $k \parallel s'$ (the given line and the Steiner line of the $\triangle A'B'C'$) is equal to the distance between the parallels $p \parallel q'$ (the Simson lines of the $\triangle ABC,\ \triangle A'B'C'$). In exactly the same way, we can show that the distance between the parallels $k \parallel s$ (the given line and the Steiner line of the $\triangle ABC$) is equal to the distance between the Simson lines $p \parallel q'$ of the two triangles. Even simpler, the distance between the Steiner lines $s \parallel s'$ is obviously twice the distance between the Simson lines $p \parallel q'$ of the two triangles. Thus k is the midparallel of the parallels $s \parallel s'$, cutting in half the segment HH', where $H \in s,\ H' \in s'$.

See Paralogic Triangles, Sondat's Theorem.

Luis González

#3 Mar 8, 2009, 3:24 am

Let us define the rectangular coordinates $A:(0,a)$, $B:(b,0)$, $C:(c,0)$ and τ is an arbitrary line with equation $\tau : px+qy+r=0$. Under this reference the orthocenter $H(x_0,y_0)$ of $\triangle ABC$ and the sidelines AB, AC are

$$H\left(0\ ,\ -\frac{bc}{a}\right)\ ,\ AB\equiv ax+by-ab=0\ ,\ AC\equiv ax+cy-ac=0.$$

Line τ intersects AB, AC, BC at points B_0, C_0, A_0 with coordinates:

$$B_0\left(b\cdot\frac{r+aq}{aq-bp},\ a\cdot\frac{r+pb}{bp-aq}\right)\ ,\ C_0\left(c\cdot\frac{r+aq}{aq-cp},\ a\cdot\frac{r+pc}{cp-aq}\right),\ A_0\left(-\frac{r}{p}\ ,\ 0\right)$$

Equations of the perpendiculars β, γ to AB, AC through B_0, C_0 are

$$\beta\equiv y-a\cdot\frac{r+pb}{bp-aq}-\frac{b}{a}\left(x-b\cdot\frac{r+aq}{aq-bp}\right)=0$$

$$\gamma\equiv y-a\cdot\frac{r+pc}{cp-aq}-\frac{c}{a}\left(x-c\cdot\frac{r+aq}{aq-cp}\right)=0$$

$$E\left[-\frac{r}{p}\ ,\ \beta\left(-\frac{r}{p}\right)\right]\Longrightarrow E\left[-\frac{r}{p}\ ,\ \frac{(r+pb)(bq+ap)}{p(bp-aq)}\right]$$

$$D\left[-\frac{r}{p}\ ,\ \gamma\left(-\frac{r}{p}\right)\right]\Longrightarrow D\left[-\frac{r}{p}\ ,\ \frac{(r+pc)(cq+ap)}{p(cp-aq)}\right]$$

Parallels through D, E to AB, AC meet at the orthocenter $H:(x_1,y_1)$ of $\triangle DEF$.

$$H_1\equiv\left(\frac{bc}{a(c-b)}\cdot(n-m)-\frac{r}{p}\ ,\ \frac{cm-bn}{c-b}\right)$$

$$m=\frac{(r+pb)(bq+ap)}{p(bp-aq)}\quad ,\ n=\frac{(r+pc)(cq+ap)}{p(cp-aq)}$$

$$\Longrightarrow\ |px_1+qy_1+r|=\left|\frac{bcq}{a}-r\right|\quad ,\ |px_0+qy_0+r|=\left|\frac{bcq}{a}-r\right|$$

$$\Longrightarrow\ |px_1+qy_1+r|=|px_0+qy_0+r|\Longrightarrow H, H_1 \text{ are equidistant from } \tau.$$

jayme

#4 Mar 9, 2009, 2:41 pm

Dear Mathlinkers,
for me it seems that this problem come from Sondat?
Any reference?
Sincerely
Jean-Louis

yetti

#5 Mar 11, 2009, 8:43 am

Links to references are at the end of the 1st reply ?

jayme

#6 Aug 30, 2009, 3:24 pm

Dear Mathlinkers,
an article concerning another Sondat's theorem can be found on
http://perso.orange.fr/jl.ayme/ vol. 5
Sincerely
Jean-Louis

TelvCohl

#7 Dec 5, 2014, 11:06 pm • 1

I'll prove the midpoint of H_1H_2 lie on $\overline{A_0B_0C_0}$.

My solution:

Let X, Y, Z be the midpoint of AD, BE, CF, respectively .
Let M be the Miquel point of complete quadrilateral $AC_0BCA_0B_0$.
Let H be the orthocenter of $\triangle XYZ$.

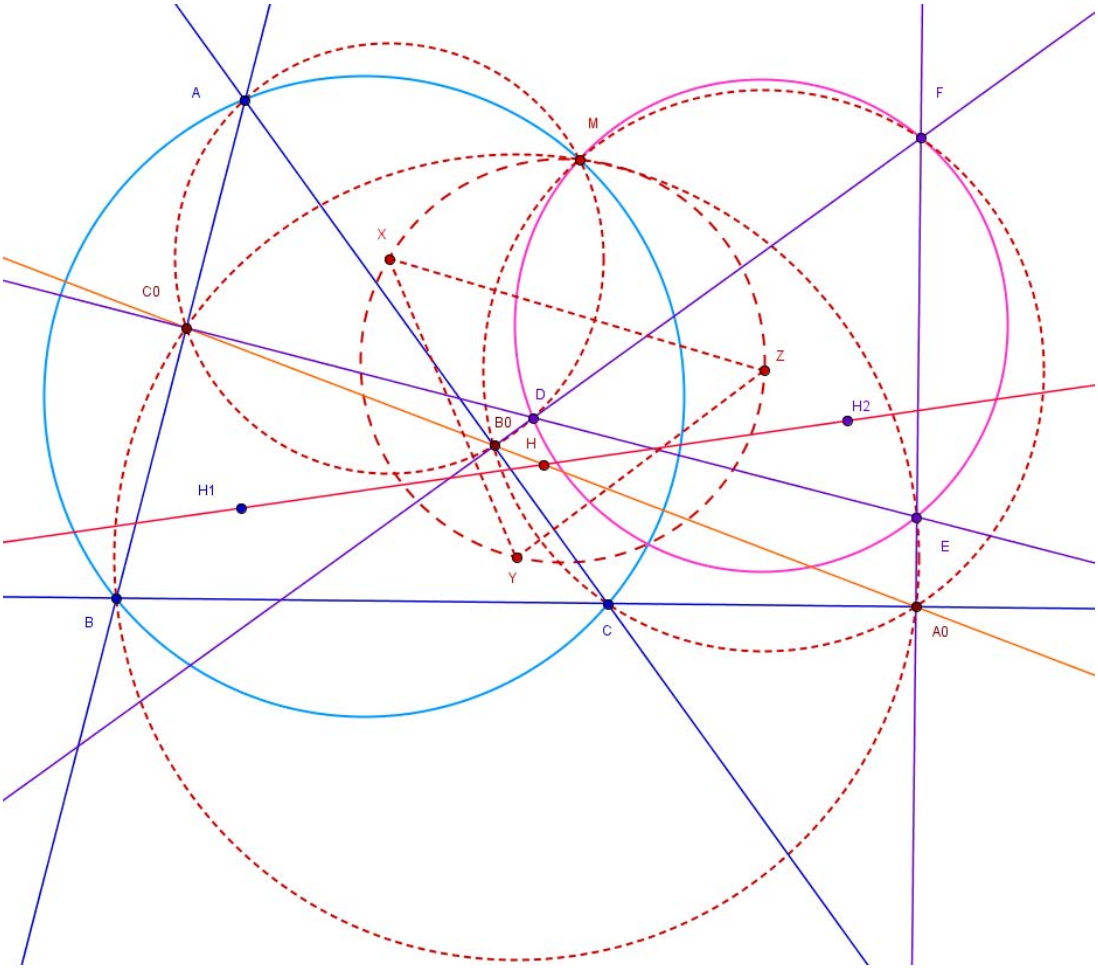
Easy to see $\triangle ABC \sim \triangle DEF$.

Since $\angle C_0DB_0 = \angle C_0AB_0$,
so we get $D \in (AB_0C_0)$. ie. A, D, B_0, C_0, M are concyclic
Similarly, we can prove $E \in (BC_0A_0M)$ and $F \in (CA_0B_0M)$,
so M is also the Miquel point of complete quadrilateral $DC_0B_0A_0EF$,
hence M lie on the circumcircle of $\triangle DEF$.

From Peterson-Schoute theorem
we get $\triangle ABC \sim \triangle XYZ \sim \triangle DEF$ and H is the midpoint of H_1H_2 .
Since M is the spiral similar center of $\triangle ABC$ and $\triangle DEF$,
so M also lie on the circumcircle of $\triangle XYZ$.
Since X, Y, Z is the center of $(MB_0C_0), (MC_0A_0), (MA_0B_0)$, respectively ,
so we get YZ, ZX, XY is the perpendicular bisector of MA_0, MB_0, MC_0, respectively ,
hence $\overline{A_0B_0C_0}$ is the Steiner line of M WRT $\triangle XYZ$.
ie. the midpoint H of H_1H_2 lie on $\overline{A_0B_0C_0}$.

Q.E.D

Attachments:



Quick Reply

High School Olympiads

Geometry

Reply

Source: Kazakhstan national olympiad 10th grade

muuratjann

#1 Sep 2, 2015, 12:15 am

Given circles ω_1, ω_2. AB, CD are external tangents to this circles (A, C are on ω_1 and B, D on ω_2). Let AD intersects ω_1, ω_2 at P, Q respectively. Let tangent line to ω_1 at P intersects AB at R. Let tangent line to ω_2 at Q intersects CD at S. Prove that $MP = MQ$ if M is midpoint of RS

Luis González

#2 Sep 2, 2015, 1:32 am

Let U, V be the midpoints of PA, QD, respectively and let T be the midpoint of PQ. By power of point, we have $DP \cdot DA = DC^2 = AB^2 = AQ \cdot AD \Longrightarrow AQ = DP \Longrightarrow AP = DQ \Longrightarrow PU = QV \Longrightarrow T$ is also midpoint of $UV \Longrightarrow MT$ is midline of the trapezoid $URVS \Longrightarrow (MT \parallel UR \parallel VS) \perp UV \Longrightarrow MT$ is perpendicular bisector of PQ $\Longrightarrow MP = MQ$.

Quick Reply

High School Olympiads

Equal angles

Reply

andria

#1 Sep 1, 2015, 4:35 pm

In $\triangle ABC$ with orthocenter H and circumcenter O. let B', C' be the reflections of B, C WRT AC, AB respectively. let O' be the circumcenter of $\triangle HB'C'$. let K be the midpoint of AH. Let $KO' \cap BC = T$.
Prove that $\angle OTC = \angle O'TB$.

gavrilos

#2 Sep 1, 2015, 7:09 pm • 1

Hello!

Let O'' be the circumcentre of $\triangle HBC$.Also,let c_1 be the circle with diameter AH and c_2 be the circle through H, B', C'.Finally,

let $L \equiv c_1 \cap c_2$ with $L \neq H$ and BE, CF be altitudes of the triangle ABC.Obviously $E, F \in c_1$.

Lemma: *The quadrilateral $HBCL$ is cyclic.*

Proof: We have $\angle HC'L = \angle HB'L$.Also $\angle HFL = \angle HEL \Leftrightarrow \angle C'FL = \angle B'EL$.

From these two relations we get $\triangle C'FL \simeq \triangle B'EL$.Thus $\frac{C'L}{B'L} = \frac{C'F}{B'E} = \frac{C'C}{B'B}$.

The latter,combined with the relation $\angle CC'L = \angle BB'L$ gives $\triangle CC'L \simeq \triangle BB'L$.

Thus,$\angle B'BL = \angle C'CL \Leftrightarrow \angle HBL = \angle HCL$ which is the desired result.

According to the lemma,the circles c_1, c_2 and the circumcircle of $\triangle HBC$ (name it c_3) have a common segment,that is,HL.

Thus,their centers are collinear.More specifically,O', K, O'' are collinear.

It is a well known fact that O'' is the symmetric of O with respect to BC.However,we will give a proof of this preposition.

Let R_1, R_2 be the radii of c_3 and the circumcircle of $\triangle ABC$.

We have $\angle BHC = \angle FHE = 180^\circ - \angle BAC$.

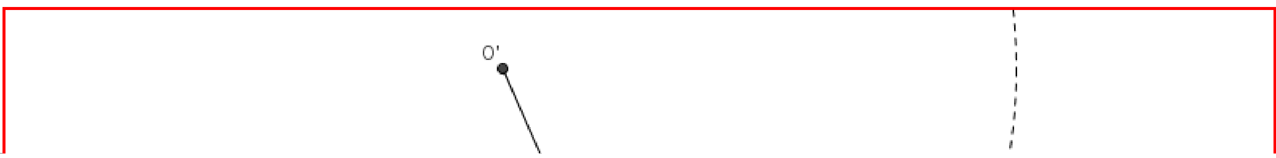
Sine law gives $2R_1 = \frac{BC}{\sin \angle BHC} = \frac{BC}{\sin(180^\circ - \angle BAC)} = \frac{BC}{\sin \angle BAC} = 2R_2$.

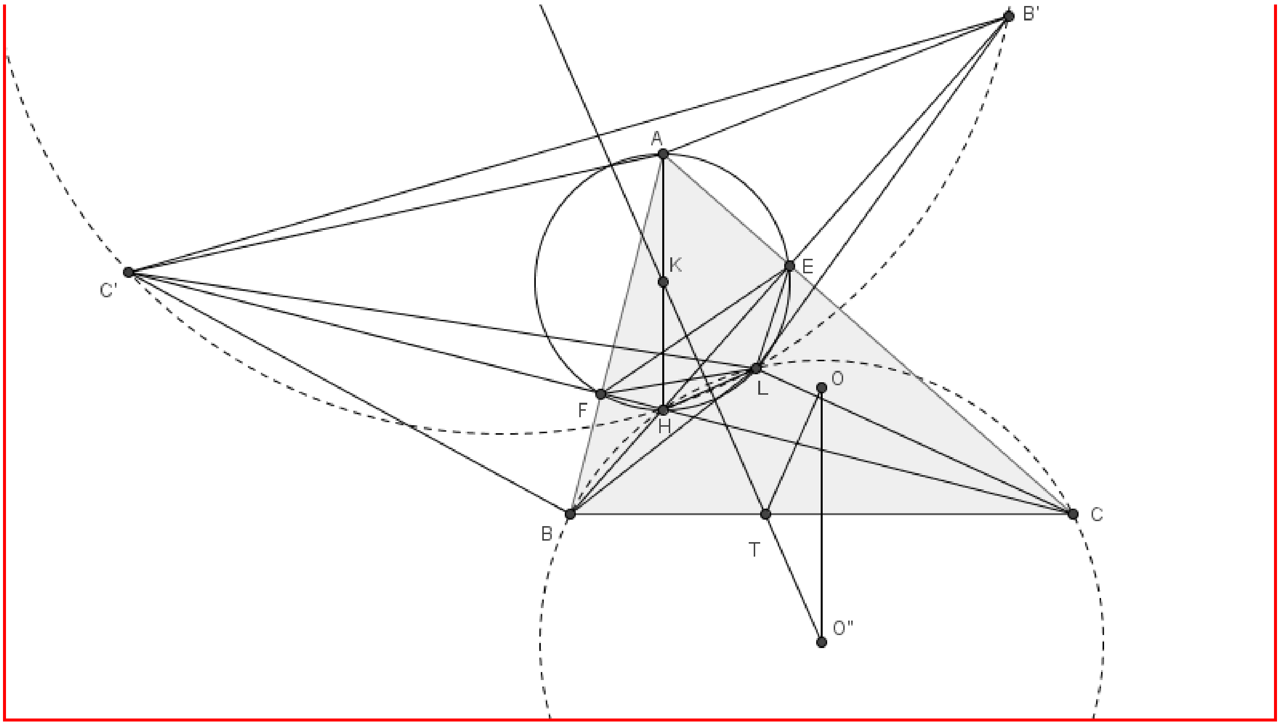
Thus $R_1 = R_2$.Thus,$O''B = O''C = OB = OC$,that is,$OBO''C$ is a rhombus,whence we get the desired result.

From this symmetry and the collinearity we proved earlier we obtain $\angle O'TB = \angle O''TC = \angle OTC$ q.e.d.

Nice problem!

Attachments:





Cezar

#3 Sep 1, 2015, 8:14 pm

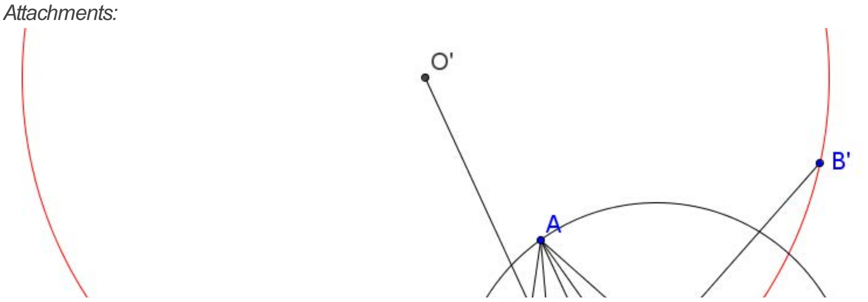
Let M, N reflections of O wrt AB, AC, and D the projection of A on BC. Let F, E, R midpoints of AB, AC, BC.
Then $O'MON$ is a homothety of $AFOE$ with center O with ratio 2. So A is the middle of OO'.
And since $AH = 2 \cdot OR \longrightarrow \triangle O'AK = \triangle AOR \longrightarrow O'K \parallel AR$.
So $\dfrac{DK}{OR} = \dfrac{DK}{KA} = \dfrac{DT}{TR} \longrightarrow \triangle KTD = \triangle OTR$. So $\angle OTC = \angle O'TB$.

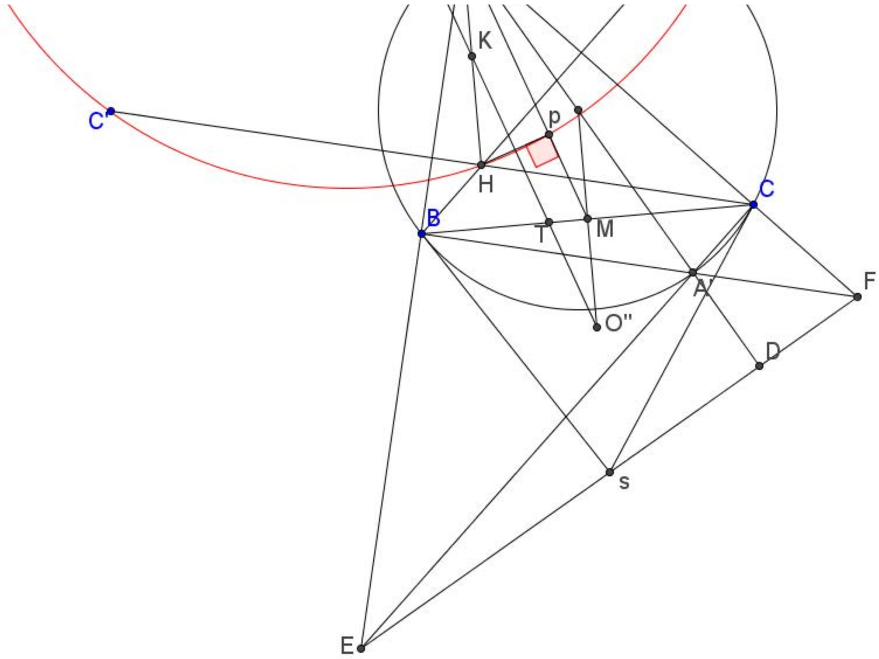
andria

#4 Sep 1, 2015, 8:33 pm

I Present two different solutions:
First solution:
Let Ω be the circumcircle of $\triangle ABC$. Let the tangents from B, C to Ω intersect at S. Let P be the projection of H on AM where M is midpoint of BC.
Since $\odot(\triangle APB), \odot(\triangle APC)$ are tangent to BC at B, C respectively and $HPBC$ is cyclic (well known result) we deduce that S is image of P under the combination of inversion $\Psi(A, bc)$ and reflecting throw the internal angle bisector of A. Let A' be the antipode of A wrt Ω. Let $BA' \cap AC = F, CA' \cap AB = E, AA' \cap EF = D$. it's easy to see that E, F, S are collinear and S is midpoint of EF. But from cyclic quadrilateral we get that $\odot(\triangle B'C'D)$ passes throw S. but
$\Psi(B') = C', \Psi(C') = B', \Psi(\odot(\triangle HB'C')) = \odot(\triangle B'C'D), \Psi(S) = P$. Hence $\odot(\triangle HB'C')$ passes throw P.
Since HP is radical axis of $\odot(\triangle HPBC), \odot(\triangle HPB'C')$ we get that KO' is perpendicular bisector of PH hence it passes throw O'' the circumcenter of $\triangle HBC$ in BC which is reflection of O in BC as desired.
DONE
Second solution:
Let O'' be the reflection of O in BC and P, Q be the circumcenters of $\odot(AHB), \odot(AHC)$ and M be the midpoint of BC.
since $R_{\odot(HAB)} = R_{\odot(HAC)} = R_{\odot(ABC)}$ we deduce that $CQPB$ is parallelogram and $PQ = BC$ but since
$O'Q \perp BH, O'B \perp CH, PQ \perp AH$ we deduce that $\triangle O'PQ = \triangle ABC$ But since K is midpoint BC we get
$AM \parallel AM$.
$AK = MO', AK \parallel MO' \implies KO' \parallel AM$(2)
From (1),(2) we get $O'K$ passes throw O''.
DONE

Attachments:





Luis González

#5 Sep 1, 2015, 11:16 pm

Let $E \equiv BH \cap CA$ and $F \equiv CH \cap AB.$ Since E, F are the midpoints of BB', CC', then it follows that $(K) \equiv \odot(HEF)$ is the midcircle of $\odot(HBC) \equiv (X)$ and $\odot(HB'C') \equiv (O)'$, i.e. $(K), (O'), (X)$ are coaxal $\Longrightarrow$ O', K, X are collinear. But X is the reflection of O across $BC \Longrightarrow BC$ bisects $\angle OTO'$ or $\angle OTC = \angle O'TB.$

drmzjoseph

#6 Sep 2, 2015, 12:21 am

Let N, P be the foots of altitudes from B and C respectively, now let M be the mipoint of BC, and R the projection of H at AM well-known that PN, BC, HR are concurrent (at Z) and $BHRC$ is cyclical.
Consider $\Psi: X \to X_1$ the inversion with center H that interchanges $\{B; N\}, \{C; P\}$

Easy notice that $\frac{HB_1'}{B_1'B} = \frac{HN}{NB'} = \frac{AN}{NC}$, analogously $\frac{HC_1'}{C_1'C} = \frac{AP}{PB}$, by two Menelaus's theorem at $\triangle ABC - P - N - Z$ and $\triangle HBC - C_1 - B_1 - Z$ we get $Z \in C_1B_1$ i.e. $HC'B'R$ are cyclical, because $\Psi(Z) = R$

Let Q the reflection of O at BC, and Q is the center of $\odot(BHRC)$ then $O'Q \perp HO$, also $QK \parallel AM \Rightarrow QK \perp HR$ i.e. O', K, T, Q are collinear, this is sufficient.

Quick Reply

High School Olympiads

Nice problem

Reply

DesertEagle735
#1 Sep 1, 2015, 10:34 am

Let circle (I) be inscribed ΔABC. (I) touches CA, AB at E,F. A variable line d which is through C, intersect AB, EF at M, N. ME intersect CF at J.
Prove that $AJ \perp IN$

Luis González
#2 Sep 1, 2015, 11:47 am

Let $X \equiv AJ \cap EF.$ From the complete quadrilateral $EFMC,$ we have $(E,F,X,N)=-1 \Longrightarrow AXJ$ is the polar of N WRT $(I) \Longrightarrow AJ \perp IN.$

jayme
#3 Sep 1, 2015, 4:22 pm

Dear Mathlinkers,
see also

http://www.artofproblemsolving.com/community/c6t48f6h1082931_nice_geo

Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Tangent 1

Reply

Source: OWN

LeVietAn
#1 Sep 1, 2015, 8:00 am

Dear Mathlinkers,
Let $ABCD$ be a trapezium inscribed in a circle (O) with $AD < BC$. The reflection of OD in the line BC intersects AB, AC at E, F, resp. Let (O') be the circumcircle of triangle DEF. The tangent line at A of (O) intersects (O') at P, Q. Prove that circumcircle of triangle $O'PQ$ is tangent to (O).

Luis González
#2 Sep 1, 2015, 11:31 am

Let $S \equiv OD \cap BC$ and $X \equiv CD \cap BA,$ respectively. Since $\angle FCD \equiv \angle ACD = 90^\circ - \angle DSC = \tfrac{1}{2}\angle FSD$ and SC bisects $\angle FSD$ externally, it follows that C is the F-excenter of $\triangle DFS \Longrightarrow FA$ bisects $\angle DFE$ externally and $\tfrac{1}{2}\angle DFE = \angle DCS = \angle EAB \Longrightarrow A$ is the D-excenter of $\triangle DEF \Longrightarrow \angle XED = \angle FEA = \angle SDC = \angle FDX$ $\Longrightarrow X \in (O')$ and SD is tangent of $(O'),$ i.e. (O) and $(O)'$ are orthogonal. Thus if $O'A$ cuts (O) again at $M,$ we have $O'P^2 = O'Q^2 = O'M \cdot O'A,$ i.e. the inversion WRT $(O)'$ fixes (O) and carries $\odot(O'PQ)$ into the tangent of (O) at A $\Longrightarrow \odot(O'PQ)$ is tangent to (O) at $M.$

Quick Reply

High School Olympiads

Tangent 3

Reply

Source: OWN

LeVietAn

#1 Sep 1, 2015, 8:05 am

Dear Mathlinkers,
Given a trapezoid $ABCD$ with $BC||AD$ and $BC \neq AD$. Choose point K such that $AK \perp BC, BK \perp CD$. Choose point R on AB such that $\angle CKR = 90^\circ$. The line DK meets the line BC at L. Prove that the circle containing C, K and L is tangent to the line going through RC.

Luis González

#2 Sep 1, 2015, 9:56 am • 1

Let $E \equiv KA \cap CD, X \equiv RC \cap AD$ and $Y \equiv CK \cap BE. K$ is orthocenter of $\triangle EBC \Longrightarrow (RK \parallel EB) \perp CK \Longrightarrow \frac{XA}{BC} = \frac{RA}{RB} = \frac{KA}{KE} \Longrightarrow \frac{XA}{KA} = \frac{BC}{KE} = \frac{CY}{EY} \Longrightarrow$ right triangles $\triangle KAX$ and $\triangle EYC$ are similar by SAS $\Longrightarrow$ $\angle KXA = \angle ECY \Longrightarrow KCDX$ is cyclic $\Longrightarrow \angle KCR = \angle KDX = \angle KLC \Longrightarrow RC$ is tangent of $\odot(CKL)$.

Quick Reply

High School Olympiads

Tangent 2

Reply

Source: OWN

LeVietAn

#1 Sep 1, 2015, 8:02 am

Dear Mathlinkers,
Let ABC is an acute triangle inscribed a circle (O). AO intersects BC at D. Choose the points E and F respectively on the lines CA and AB such that D is orthocenter of triangle AEF. Let XYZ be the triangle bounded by the line BC, tangent line at A of (O), and the reflection of BC in EF. Prove that the circumcircle of triangle XYZ is tangent to (O).

Luis González

#2 Sep 1, 2015, 8:51 am • 1

Denote $X \equiv BC \cap EF$ and Y the intersection of BC with the tangent of (O) at A. AD cuts (O) again at U and the tangent of (O) at U cuts BC at M. Since $\angle EDC = \angle EFD = 90^\circ - \angle ABC \Longrightarrow BC$ is tangent of $\odot(DEF) \Longrightarrow$ $XD^2 = XE \cdot XF = XB \cdot XC \Longrightarrow X$ is center of the involution on BC that fixes D and swaps B, C. But by Desargues involution theorem for $AAUU$ cut by BC, it follows that M, Y are homologous points in this involution $\Longrightarrow$ $XM \cdot XY = XB \cdot XC$.

Clearly $YZ \parallel EF$, thus $\triangle XYZ$ is X-isosceles $\Longrightarrow \odot(XYZ)$ is tangent to EF. So the inversion with center X and power $XM \cdot XY = XB \cdot XC$ leaves (O) fixed and carries $\odot(XYZ)$ into the parallel UM from M to EF (tangent of (O) through U) $\Longrightarrow \odot(XYZ)$ is tangent to (O).

TelvCohl

#4 Sep 1, 2015, 9:32 pm • 1

Let X be the intersection of BC with the tangent τ of $\odot(ABC)$ through A and Y be the intersection of τ with the reflection of BC in EF. Let $\triangle A^*E^*F^*$ be the orthic triangle of $\triangle AEF$ and $T \equiv \odot(ABC) \cap \odot(ADX)$, $R \equiv AT \cap EF, V \equiv AO \cap \odot(ABC)$. From $D \in AO \Longrightarrow E^*F^* \parallel BC$, so notice E, F, E^*, F^* are concyclic we get BC is tangent to $\odot(DEF)$ at D (Reim theorem). Since $\measuredangle DTA = \measuredangle DXA$, so AS is parallel to BC where $S \equiv DT \cap \odot(ABC)$. hence if D^* is the isotomic conjugate of D on BC then $\measuredangle BAT = \measuredangle BST = \measuredangle BSD = \measuredangle D^*AC \Longrightarrow AT$ is the isogonal conjugate of AD^* WRT $\angle A$.

From $\triangle ABC \overset{-}{\sim} \triangle AEF \Longrightarrow \triangle ABC \cup D^* \overset{-}{\sim} \triangle AEF \cup R$, so DR, DA^* are isogonal conjugate WRT $\angle FDE$ (notice the center of $\odot(AEF), \odot(DEF)$ are symmetry WRT the midpoint of EF) $\Longrightarrow \odot(DA^*R)$ is tangent to $\odot(DEF)$ at D. From $BC \parallel E^*F^* \Longrightarrow B, C, E, F$ are concyclic, so $ZA^* \cdot ZR = ZD^2 = ZE \cdot ZF = ZB \cdot ZC \Longrightarrow Z$ lie on the radical axis VT of $\odot(ABC)$ and $\odot(VTRA^*)$, hence A, A^*, T, Z lie on a circle with diameter AZ.

From $XY \parallel EF$ (both anti-parallel to BC WRT $\angle A$) $\Longrightarrow \triangle XYZ$ is an isosceles triangle, so $\measuredangle ZTX = \measuredangle ZTA + \measuredangle ATX$ $= 90^\circ + \measuredangle ATX = 90^\circ + \measuredangle ADX = \measuredangle AXD = \measuredangle ZYX \Longrightarrow T \in \odot(XYZ)$. Finally, from $\measuredangle ZTC = \measuredangle DAC = \measuredangle ADB$ $- \measuredangle ACB = \measuredangle ATX - \measuredangle ATB = \measuredangle BTX$ we conclude that $\odot(ABC)$ is tangent to $\odot(XYZ)$ at T.

livetolove212

#5 Sep 2, 2015, 12:22 am • 2

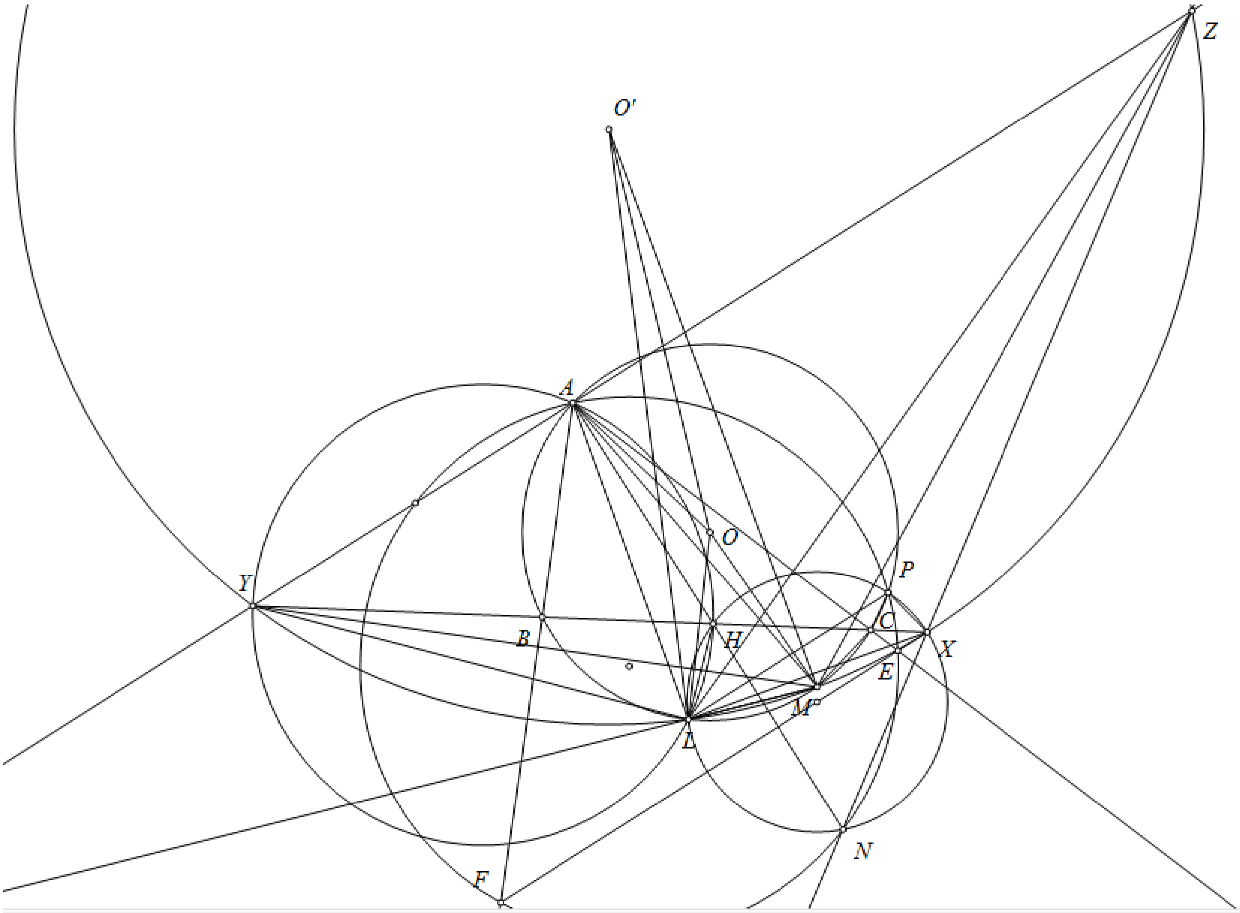
Generalization. Given triangle ABC inscribed in (O). Let H be an arbitrary point on BC. Choose points E, F on the lines CA, AB, respectively, such that H is the orthocenter of triangle AEF. Let XYZ be the triangle formed by line BC, the reflection of BC wrt EF and the line through A and perpendicular to AH. Let (O') be the circumcircle of triangle XYZ. Then (O') intersects (O) at two points L and M such that $\angle LOM - \angle LO'M = 2\angle HAO$.

Proof.
Let N be the second intersection of AH and (AEF), P be the second intersection of (O) and (AEF). (AHY) cuts (HNX) at H and L'.
We have L' is the Miquel point of completed quadrilateral $AHXZYN$ then L' lies on (O'). On the other side, P is the Miquel

point of completed quadrilateral $FBCEAX$ then P lies on (CEX). We get $\angle PXH = \angle PXC = \angle PEC = \angle PNH$ then H, P, N, X are concyclic. From this,
$\angle AL'P = \angle AL'H + \angle HL'P = \angle AYH + \angle HXP = \angle HXE + \angle HXP = \angle PXE = \angle ACP$. This means L' lies on (O). Therefore $L' \equiv L$.
Let M be the second intersection of (O) and (O').
We have $\angle HAO = \angle LAO - \angle LAH = \angle LAO - \angle LYH = \angle LAO - \angle LZX$.
$\frac{1}{2}(\angle LOM - \angle LO'M) = \angle LAM - \angle LZM$.
We need to prove that $\angle LAO - \angle LZX = \angle LAM - \angle LZM$, which is equivalent to $\angle LAO - \angle MZX = \angle LAM$ or $\angle OAM = \angle MZX$.
But $\angle OAM = \angle ALM - 90^\circ$ and $\angle MZX = \angle MLX$ hence we need to prove that $\angle ALX = 90^\circ$.
The last part is easy since $\angle ALX = \angle ALH + \angle HLX = \angle AYH + \angle HNX = \angle AYH + \angle AHY = 90^\circ$. This problem is solved.

Remark: When A, O, H are collinear, $\angle HAO = 0^o$, then $L \equiv M$ and we get (XYZ) is tangent to (O).

Attachments:



This post has been edited 2 times. Last edited by livetolove212, Sep 2, 2015, 12:25 am

Quick Reply

High School Olympiads

Intersection of circumcircles

Reply

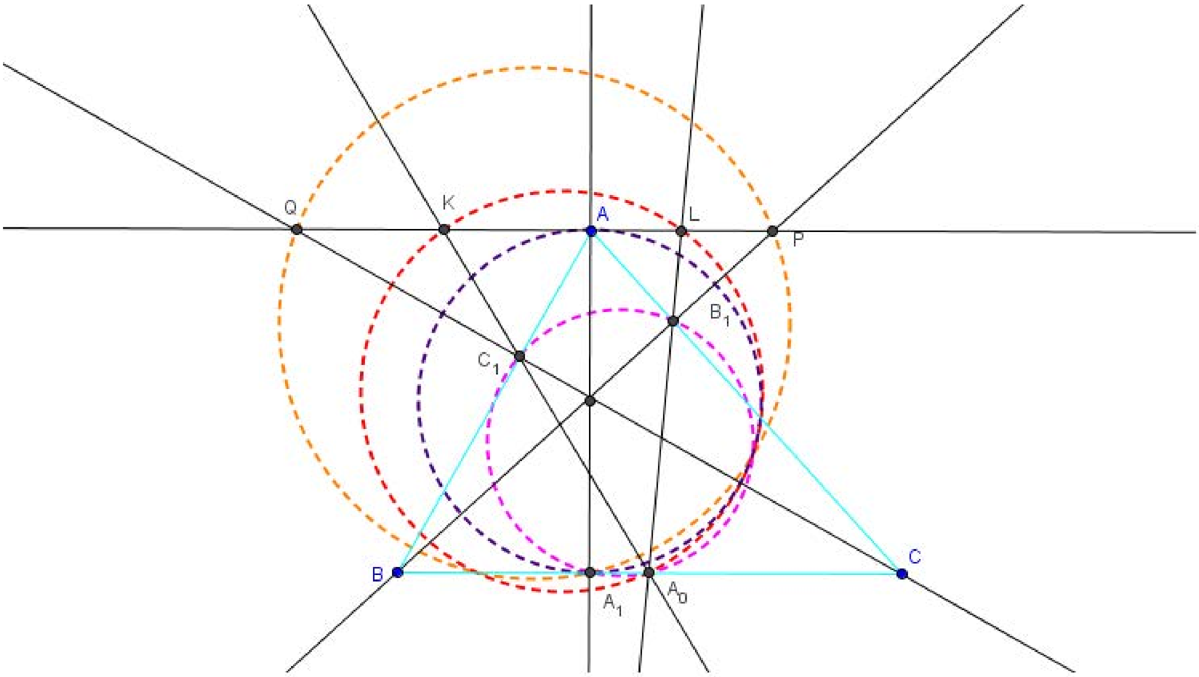
Source: own

MRF2017

#1 Sep 1, 2015, 3:13 am

AA_1, BB_1, CC_1 are altitudes of an acute scalene triangle ABC.BB_1, CC_1 intersect with the line passes through A and parallel to BC at P, Q,respectively.A_0 is midpoint of BC.Lines A_0C_1, A_0B_1 intersect with PQ at K, L,respectively.prove that circumcircle of triangles PQA_1,KLA_0,$A_1B_1C_1$ and a circle with diameter AA_1 intersect at one point.

Attachments:



Luis González

#2 Sep 1, 2015, 4:50 am

Let T be the second intersection of $\odot(A_1B_1C_1)$ and the circle ω_A with diameter AA_1. Since PQ is common tangent of ω_A and $\odot(AB_1C_1)$, then PQ, B_1C_1, A_1T are pairwise radical axes of $\omega_A, \odot(A_1B_1C_1), \odot(AB_1C_1)$ concurring at their radical center X. Since $\angle APB = \angle B_1BC = \angle B_1C_1C \Longrightarrow B_1C_1QP$ is cyclic $\Longrightarrow$ $XP \cdot XQ = XB_1 \cdot XC_1 = XA_1 \cdot XT \Longrightarrow T \in \odot(PQA_1)$.

It's well-known that A_0B_1, A_0C_1 are tangents of $\odot(AB_1C_1) \Longrightarrow \odot(AB_1C_1)$ becomes incircle of $\triangle A_0KL$. Second intersection T^* of $\odot(A_1B_1C_1)$ and $\odot(A_0KL)$ is center of the spiral similarity that swaps B_1L and $C_1K \Longrightarrow$ $T^*K : T^*L = KC_1 : LB_1 = KA : LA \Longrightarrow T^*A$ bisects $\angle KT^*L$ and together with $(K, L, A, X) = -1$, it follows that $\angle AT^*A_1 = 90^\circ \Longrightarrow T^* \in \omega_A \Longrightarrow T \equiv T^*$. Hence, $\odot(PQA_1), \odot(KLA_0), \odot(A_1B_1C_1)$ and ω_A concur at T.

Quick Reply

High School Olympiads

Locus of common internal tangents

Locked

Echoz

#1 Aug 31, 2015, 8:34 pm

Triangle ABC is inscribed in circle ω. A variable line l is chosen parallel to BC. Let $l \cap AB = D, l \cap AC = E, l \cap \omega = K, L$ (with D between K and E). Circle γ_1 is tangent to segments KD, BD and also tangent to ω, while circle γ_2 is tangent to segments LE, CE and ω. Determine the locus of the common internal tangents to γ_1, γ_2 as l varies.

Luis González

#2 Aug 31, 2015, 8:47 pm

Posted before at http://www.artofproblemsolving.com/community/c6h393650.

High School Olympiads

RMM2011, P 3, Day 1 - Determine the locus as line varies

Reply

Source: 0

mavropnevma

#1 Feb 25, 2011, 10:54 pm • 2

A triangle ABC is inscribed in a circle ω.
A variable line ℓ chosen parallel to BC meets segments AB, AC at points D, E respectively, and meets ω at points K, L (where D lies between K and E).
Circle γ_1 is tangent to the segments KD and BD and also tangent to ω, while circle γ_2 is tangent to the segments LE and CE and also tangent to ω.
Determine the locus, as ℓ varies, of the meeting point of the common inner tangents to γ_1 and γ_2.

(Russia) Vasily Mokin and Fedor Ivlev

Luis González

#2 Mar 3, 2011, 11:18 pm • 11

A-angle bisector cuts DE at T and the circumcircle (O) of $\triangle ABC$ again at F. Since $KL \parallel BC$, then arcs BK, CL are equal $\Longrightarrow AT$ also bisects $\angle KAL$. P lies on the ray AK such that $AP = AL$. Since AT is self isogonal with respect to $\angle KAL$, it follows that $AK \cdot AL = AT \cdot AF \Longrightarrow AK \cdot AP = AT \cdot AF \Longrightarrow$ inversion with center A and power $AK \cdot AP$ takes PT into (O) and $\ell \equiv DE$ into the circle $\odot(APF)$, congruent to (O). Now, due to conformity, $(I_1) \equiv \gamma_1$ is taken into the circle (U_1) tangent to AD, PT and internally tangent to $\odot(APF)$. Consequently, $(I_2) \equiv \gamma_2 \cong (U_1) \Longrightarrow AI_1 \equiv AU_1, AI_2$ are isogonals with respect to $\angle BAC$.

If $(I_1), (I_2)$ touch AB, AC at M, N, then the right triangles $\triangle AMI_1$ and $\triangle ANI_2$ are similar. Let line I_1I_2 cut AF at S. By angle bisector theorem we have then $\frac{SI_1}{SI_2} = \frac{AI_1}{AI_2} = \frac{I_1M}{I_2N} \Longrightarrow S$ is the insimilicenter of $(I_1) \sim (I_2)$, i.e common internal tangents of $(I_1), (I_2)$ intersect on the A-internal bisector.

Remark: Likewise, common external tangents of $(I_1), (I_2)$ intersect on the A-external bisector.

$\frac{d}{dt}\text{Fail} > t$

dnkywin

#3 Mar 4, 2011, 3:44 am • 5

A more computation intensive solution
For people who are not tricky enough to find solutions like the one above.

This post has been edited 1 time. Last edited by dnkywin, Mar 6, 2011, 3:05 am

math154

#4 Mar 4, 2011, 8:48 am • 6

dnkywin: Shouldn't $1 - \cos(2x) = 2\sin^2(x)$? Well, it doesn't seem like a real issue anyway.

In the same spirit, but with absolutely zero ingenuity... (maybe I should've thought of this during the test darn).

Let $\triangle L'MN$ be the medial triangle of $\triangle ABC$, with $R_1 = (I_1) \cap (O)$ and $S_1 = (I_1) \cap AB$ (define R_2, S_2 similarly). Using directed angles modulo 180°, define $\angle L'OR_i = \theta_i$. Then $\angle OI_1S_1 = \angle I_1ON = -(B + \theta_1)$, so by a simple trigonometric calculation, we have

$$r_1 = R\frac{\cos C + \cos(B+\theta_1)}{\cos(B+\theta_1) - 1} \implies R - r_1 = R\frac{1+\cos C}{1-\cos(B+\theta_1)},$$

so

$$AS_1 = R\sin C + (R - r_1)\sin(B+\theta_1) = R\frac{\sin C + \sin(B+\theta_1) + \sin(B+\theta_1 - C)}{1 - \cos(B+\theta_1)}$$

and

$$\frac{r_1}{AS_1} = -\frac{2\cos\frac{C}{2}\sin\frac{B+\theta_1}{2}}{\cos\frac{B+C+\theta_1}{2}},$$

with (distance defined up to some constant sign)

$$\begin{aligned} d(O, KL) &= r_1 + (R - r_1)\sin(\theta_1 - 90^\circ) \\ &= R\frac{\cos C + \cos(B+\theta_1) + \cos\theta_1 + \cos C\cos\theta_1}{\cos(B+\theta_1) - 1} \\ &= R + R\frac{(1+\cos C)(1+\cos\theta_1)}{\cos(B+\theta_1) - 1} \\ &= R - 2R\frac{\cos^2\frac{C}{2}\cos^2\frac{\theta_1}{2}}{\sin^2\frac{B+\theta_1}{2}}. \end{aligned}$$

By symmetry,

$$\frac{\cos^2\frac{C}{2}\cos^2\frac{\theta_1}{2}}{\sin^2\frac{B+\theta_1}{2}} = \frac{\cos^2\frac{B}{2}\cos^2\frac{\theta_2}{2}}{\sin^2\frac{C+\theta_2}{2}} \implies \frac{\cos\frac{C}{2}\cos\frac{\theta_1}{2}}{\sin\frac{B+\theta_1}{2}} = \frac{\cos\frac{B}{2}\cos\frac{\theta_2}{2}}{\sin\frac{C+\theta_2}{2}}$$

(it's not difficult to see each individual thing is positive by the definitions of the angles).

As in the previous two solutions, it suffices to show that $r_1/AS_1 = r_2/AS_2$, or

$$\frac{\cos\frac{C}{2}\sin\frac{B+\theta_1}{2}}{\cos\frac{B+C+\theta_1}{2}} = \frac{\cos\frac{B}{2}\sin\frac{C+\theta_2}{2}}{\cos\frac{B+C+\theta_2}{2}}.$$

Let $x = B/2$, $y = C/2$, $z = \theta_1/2$, and $w = \theta_2/2$. We can easily verify that

$$\begin{aligned} &\cos y\cos z\sin(y+w) - \cos x\cos w\sin(x+z) \\ &= \cos y\sin(x+z)\cos(x+y+w) - \cos x\sin(y+w)\cos(x+y+z), \end{aligned}$$

so we're done. 🙂

Edit: Accidentally used L twice.

This post has been edited 1 time. Last edited by math154, Mar 5, 2011, 9:37 pm

Zhero

#5 Mar 5, 2011, 5:05 am

Here's another problem which uses the inversion idea used to solve this (i.e., inverting and invoking symmetry): http://www.artofproblemsolving.com/Forum/viewtopic.php?t=318917.

polya78

#6 Mar 30, 2015, 2:03 am • 1 👍

Let f be the transformation consisting of an inversion about A with radius $\sqrt{AK*AL}$ followed by a reflection about l, the common angle bisector of $\angle BAC$ and $\angle KAL$. Then $f(KL) = w, f(AB) = AC, f(AC) = AB$, and so $f(\gamma_1) = \gamma_2$ and visa versa. Let N be the intermal center of similarity of γ_1, γ_2 and for notation's sake, define $X' = f(X)$ for any point X. Let R, S, T, U be the points of tangency of the two common inner tangents of γ_1, γ_2.

Then $(AR'T')$ is internally tangent to γ_2 and externally to γ_1 at $R,' T'$ respectively, so R', T', N are collinear, as are S', U', N. This means that N' is the second intersection of (ART) and (ASU). But clearly N is on the radical axis of (ART) and (ASU), so A, N, N' are collinear, which means that N lies on l.

This post has been edited 1 time. Last edited by polya78, Mar 30, 2015, 9:06 pm
Reason: Clean-up

drmzjoseph

#7 Oct 28, 2015, 2:05 pm

Let X be the point of the common inner tangents to γ_1 and γ_2.

γ_1 (center O_1 and radius r_1) touch KL at F and ω at G, γ_2 (center O_2 and radius r_2) touch KL at P and ω at Q, if $M \equiv PQ \cap FG$, then M is the midpoint of the arc KAL. Now $KC \cap BL \equiv Y$, since $KY = LY$ we get the circle tangent to the segments KY, LY and ω is tangent to ω at M. Using A concyclic problem we obtain A, M, P, F are concyclic, Now AM, KL, GQ are concurrent at Z, from Radical Axis Theorem $\odot(GFPQ), \odot(AMPF), \omega$, The parallel to AB and BC cut AM at B' and C' respectively. Since $\angle O_1B'A = \angle O_2C'A \Rightarrow \frac{r_1}{r_2} = \frac{B'A}{C'A}$ and $\frac{B'O_1}{C'O_2} = \frac{ZO_1}{ZO_2} = \frac{r_1}{r_2} \Rightarrow \triangle O_1B'A \sim \triangle O_2C'A$ using $(Z, X, O_1, O_2) = -1 \Rightarrow AX$ is bisector angle of $\angle O_1AO_2 \Rightarrow AM$ is bisector angle of $\angle BAC$

Quick Reply

High School Olympiads

property for triangle

Reply

Source: creted by me

ferma2000

#1 Aug 31, 2015, 1:04 am

Dear mathlinkers;
1) ABC is triangle.
2) I, H are incenter and orthocenter of ABC.
3) ϱ is mixtilinear incircle with center S in front of A
3) assume that S lies on BC.
4) The incircle of ABC touches AB, AC at F, E.
5) N is nagell point of ABC.
Claim:
A) $IH \parallel AN$
B) H lies on EF.
Best regards;

Luis González

#2 Aug 31, 2015, 5:13 am • 1

Let $\triangle I_aI_bI_c$ be the excentral triangle and let O be the circumcenter. The incircle (I,r) and A-excircle (I_a,r_a) touch BC at $D, U,$ respectively and M is the common midpoint of BC and DU.

According to problem center of mixtilinear incircle, $S \in BC \Longleftrightarrow$ midpoint X of II_a lies on $(O) \Longleftrightarrow R = \frac{1}{2}r_a$. Thus since $OX \parallel I_aU,$ then U is reflection of I on O, i.e. U is the circumcenter of $\triangle I_aI_bI_c$ (Bevan point of $\triangle ABC$). Furthermore, $ID = 2 \cdot OM = AH \Longrightarrow AHDI$ is parallellogram $\Longrightarrow HD \parallel AI \Longrightarrow HD$ bisects $\angle BHC$ internally. Thus if the external bisector of $\angle BHC$ cuts BC at D', then $(B,C,D,D') = -1 \Longrightarrow D' \in EF.$ Together with $(D'FE \parallel D'H) \perp AI \Longrightarrow H \in EF.$

On the other hand, since H coincides with the projection of D on $EF,$ then by the homothety carrying $\triangle DEF$ into $\triangle I_aI_bI_c,$ we get $IH \parallel UA \equiv AN.$

ferma2000

#3 Sep 7, 2015, 4:22 pm

Dear mathlinkers;
two more claims:
1) the B excircle touch AC at E'.
2) the C excircle touch AB at F'.
Claim 1:
I lies on $E'F'$.
Claim 2:
the A excircle is orthogonal to circumcircle of ABC.
Best regards;

This post has been edited 1 time. Last edited by ferma2000, Sep 7, 2015, 4:32 pm

TelvCohl

#5 Sep 7, 2015, 6:35 pm • 1

> *ferma2000 wrote:*
> **Claim 1:**

I lies on $E''F''$.

Since the projection D' of the A-excenter I_a of $\triangle ABC$ on BC coincide with the Bevan point of $\triangle ABC$, so if I_b, I_c is the B-excenter, C-excenter of $\triangle ABC$, resp then $E' \in I_bD'$, $F' \in I_cD'$, hence from Pappus theorem (for A-I_b-I_c and D'-C-B) we get $I \in E'F'$.

> *ferma2000 wrote:*
> **Claim 2:**
> the A excircle is orthogonal to circumcircle of ABC.

See center of mixtilinear incircle (post #2).

Quick Reply

High School Olympiads

center of mixtilinear incircle

Reply

andria

#1 Jun 15, 2015, 12:05 am • 1

In $\triangle ABC$ let R, r_a radius of circumcircle and A_excircle.assume that $r_a = 2R$
a) prove that center of A_mixtilinear incircle lies on BC.
b) let the A_excircle touch AB, BC, CA at D, E, F prove that circumcircle of $\triangle ABC$ is nine point circle of $\triangle DEF$.

Luis González

#2 Jun 15, 2015, 1:52 am • 1

For convenience we first solve b) anf then a)

b) Let X, Y, Z be the midpoints of FD, DE, EF. Inversion on A-excircle (I_a) swap A, B, C and $X, Y, Z \Longrightarrow$ it swaps (O) and the 9-point circle $\odot(XYZ)$ of $\triangle DEF \Longrightarrow \odot(XYZ)$ goes through $(I_a) \cap (O)$. $R = \frac{1}{2}r_a \Longleftrightarrow (O) \cong \odot(XYZ)$ $\Longleftrightarrow$ either (O) and $\odot(XYZ)$ coincide or they are distinct with homothety center I_a at infinity, which is a contradiction, so (O) is 9-point circle of $\triangle DEF$.

a) Consider the inversion with center A that swaps the incircle (I) and the A-mixtilinear incircle (O_a). (O) goes to a tangent $B'C'$ of (I) leaving A, I on different sides and therefore BC goes to $\odot(AB'C')$. From b), it follows that (I_a) is orthogonal to (O), hence since $\triangle AC'B' \sim \triangle ABC$ ($B'C'$ is antiparallel to BC), then $\odot(AB'C')$ is orthogonal to (I). Thus by conformity, we get $BC \perp (O_a)$, or $O_a \in BC$.

Quick Reply

High School Olympiads

Line passes through centroid

Reply

Source: Own

buratinogigle
#1 Aug 30, 2015, 5:21 pm

Let ABC be a triangle and K, L are excenters with respect to vexter B, C. Euler lines of triangle KCA and LAB cut CA, AB at M, N, reps. Prove that MN passes through centroid of triangle ABC.

TelvCohl
#3 Aug 30, 2015, 6:11 pm • 1

See Excentral triangle, Euler line, Nagel line, M and N lie on Nagel line of $\triangle ABC$:).

buratinogigle
#4 Aug 30, 2015, 6:59 pm

Thank you Telv :) I have seen a genralizaion for your problem

Let ABC be a triangle and DEF is cevian triangle point P. Line passes through midpoint of PA and centroid of triangle AEF intersects EF at X. Similarly, we have Y, Z. Q is symmetric of P through midpoint of PG with G is centroid of triangle ABC. QA, QB, QC cut EF, FD, DE at U, V, W. Prove that X, Y, Z are collinear on a line passing through centroid of triangle DEF and this line is concurrent with DU, EV, FW.

Luis González
#5 Aug 31, 2015, 12:37 am • 1

> buratinogigle wrote:
> Let ABC be a triangle and DEF is cevian triangle point P. Line passes through midpoint of PA and centroid of triangle AEF intersects EF at X. Similarly, we have Y, Z. Q is symmetric of P through midpoint of PG with G is centroid of triangle ABC. QA, QB, QC cut EF, FD, DE at U, V, W. Prove that X, Y, Z are collinear on a line passing through centroid of triangle DEF and this line is concurrent with DU, EV, FW.

I think you mean Q is the reflection of the midpoint of PG through G. If P is inside of $\triangle ABC$, then $\triangle ABC \cup P$ can be parallel projected into an acute $\triangle ABC$ with orthocenter P and the problem is the same as Excentral triangle, Euler line, Nagel line.

buratinogigle
#6 Aug 31, 2015, 12:40 am

Thank you very much, I mean Q is the reflection of the midpoint of PG through G. I get this problem exactly from Telv's problem by parallel projection :)!

Quick Reply

High School Olympiads

Excentral triangle, Euler line, Nagel line

Reply

Source: drmzjoseph asked me to post it

TelvCohl

#1 Mar 15, 2015, 10:04 pm • 3

Let $\triangle I_aI_bI_c$ be the excentral triangle of $\triangle ABC$.
Let A^* be the intersection of BC with the Euler line of $\triangle I_aBC$.
Let B^* be the intersection of CA with the Euler line of $\triangle I_bCA$.
Let C^* be the intersection of AB with the Euler line of $\triangle I_cAB$.

Prove that A^* , B^* , C^* are collinear at the Nagel line of $\triangle ABC$

Luis González

#2 Mar 16, 2015, 12:31 am • 1

Let I, Na be the incenter and Nagel point of $\triangle ABC$ and let D, M be the midpoints of $\overline{BC}$ and the arc BC of $\odot(ABC)$. A-excircle (I_a) touches BC at X. It's known that M is the circumcenter of $\triangle I_aBC$ and its orthocenter T is the reflection of I on $D \Longrightarrow MT$ is Euler line of $\triangle I_aBC$.

Since Na is the incenter of the antimedial triangle of $\triangle ABC$, we deduce that $MI \parallel TNa$ and $ID \parallel XN_a \Longrightarrow \triangle MDI$ and $\triangle TXNa$ are homothetic $\Longrightarrow INa, DX \equiv BC$ and MT concur at their homothetic center A^*, i.e. A^* is on Nagel line INa of $\triangle ABC$ and likewise B^*, C^* lie on INa.

Quick Reply

High School Olympiads

tangent circles

Locked

ferma2000

#1 Aug 31, 2015, 12:13 am

Dear mathlinkers;
1) ABC is a triangle.
2) $AB + AC = 2BC$.
3) H is orthocenter of ABC and M is midpoint of side BC.
4) O is circumcenter of ABC.
Claim:
circles with diameters HM, AO are tangent.
Best regards;

Luis González

#2 Aug 31, 2015, 12:24 am

Posted before at (HM) is tangent to (AO) and for a generalization see Tangent circles.

High School Olympiads

(HM) is tangent to (AO)

Source: Vietnam IMO training 2015- Own

livetolove212

#1 May 8, 2015, 9:57 am • 1

Given triangle ABC inscribed in (O) with $AB + AC = 2BC$. Let H be the orthocenter of triangle ABC, M be the midpoint of BC. Prove that (HM) is tangent to (AO).

Luis González

#2 May 8, 2015, 12:07 pm • 4

WLOG we assume that $c > a > b$. Let E, F be the midpoints of AC, AB and let P, Q, R be the tangency points of the incircle (I, r) with BC, CA, AB. $b + c = 2a \Longrightarrow QE = \frac{1}{2}(c - a) = \frac{1}{2}(a - b) = RF \Longrightarrow \triangle IQE \cong \triangle IRF \Longrightarrow IE = IF \Longrightarrow I$ is the midpoint of the arc EOF of $\odot(AEOF)$. Thus if K is the midpoint of AO, then IK is perpendicular bisector of $EF \Longrightarrow IK$ is median of the trapezoid $AHMO$, cutting MH at its midpoint S.

By Pythagorean theorem $IM^2 = r^2 + PM^2 = r^2 + \frac{1}{4}(b - c)^2$, but using $b + c = 2a$ in the inradius formula yields $3r^2 = \frac{1}{4}a^2 - \frac{1}{4}(b - c)^2 \Longrightarrow IM^2 = \frac{1}{4}a^2 - 2r^2$. From the problem A useful equality, we have $HM^2 - HI^2 = \frac{1}{4}a^2 - 2r^2 \Longrightarrow IM^2 = HM^2 - HI^2 \Longrightarrow \angle MIH = 90^\circ \Longrightarrow$ circle (S) with diameter MH is tangent to the circle (K) with diameter AO at I.

TelvCohl

#3 May 10, 2015, 4:13 pm • 2

My solution :

Let I, G be the incenter, centroid of $\triangle ABC$, respectively .
Let $T = AI \cap \odot(ABC)$ ($T \neq A$) and $N = IG \cap OM$.

From Ptolemy theorem $\Longrightarrow AB \cdot CT + AC \cdot BT = AT \cdot BC$,
so combine $AB + AC = 2BC$ and $TB = TC = TI \Longrightarrow I$ is the midpoint of AT (or $I \in \odot(AO)$) .

Since $(AB + AC + BC) \cdot \text{dist}(I, BC) = 2[ABC] = BC \cdot \text{dist}(A, BC)$,
so combine $AB + AC = 2BC \Longrightarrow \text{dist}(G, BC) = \frac{1}{3}\text{dist}(A, BC) = \text{dist}(I, BC) \Longrightarrow IG \parallel BC$.

Since $IN : GN = \frac{1}{2}\text{dist}(A, OM) : \frac{1}{3}\text{dist}(A, OM) = 3 : 2$,
so $IG : GN = 1 : 2 \Longrightarrow OI \parallel HN \Longrightarrow AI \perp HN \Longrightarrow I$ is the orthocenter of $\triangle AHN$,
hence we get $HI \perp AN \Longrightarrow \angle HIM = 90^\circ$ ($\because AG : GM = 2 : 1$) $\Longrightarrow I$ lie on $\odot(HM)$.

Since $\text{dist}(I, AH) = \text{dist}(I, OM)$,
so I lie on the line connecting the center of $\odot(AO)$ and $\odot(HM)$,
hence we get $\odot(AO)$ and $\odot(HM)$ are tangent to each other at I .

Q.E.D

drmzjoseph

#4 May 11, 2015, 6:51 pm • 1

Denote r the inradius of $\triangle ABC, \quad E \equiv AI \cap BC, \quad D = AI \cap \odot(ABC), \quad D \neq A$, is well-known $DI = IA$, and
$2 = \dfrac{AC + AB}{BC} = \dfrac{AI}{IE} \Rightarrow MD = r$ and if $F \equiv BC \cap AH \Rightarrow AF = 3r$ and
$HF = m, \Rightarrow AH = 3r - m \Rightarrow MO = \dfrac{3r - m}{\ldots}$

$T \equiv AH \cap MI \Rightarrow MI = IT \wedge TA = r \wedge TH = 2r - m$, the ray HM cut $\odot(ABC)$ at $X \Rightarrow XM = MH$, the ray MH cut $\odot(ABC)$ at $Y \Rightarrow XH.HY = AH.2(HF) = 2AH.m \Rightarrow HY = \frac{AH.m}{MH}$
$OD^2 - OM^2 = XM.MY = MH.(MH + HY) = MH^2 + AH.m = MH^2 + (3r - m)m$, since $MO = \frac{3r - m}{2}$ and $OD = \frac{5r - m}{2}$
$OD^2 - OM^2 = (\frac{5r - m}{2})^2 - (\frac{3r - m}{2})^2 = MH^2 + (3r - m)m \Rightarrow MH = 2r - m = HT \Rightarrow \angle MIH = 90^\circ$,
Since $\angle OIA = 90^\circ \Rightarrow \angle MIO = \angle IAH + \angle IHA \Rightarrow \odot(OIA)$ is tangent to $\odot(MIH)$.

Quick Reply

High School Olympiads

Tangent circles

Reply

Source: Own

buratinogigle

#1 Jul 9, 2015, 10:49 pm

Let ABC be a triangle inscribed in circle (O). AD is diameter of (O). AL is bisector with L is on BC. E,F lie on CA,AB such that $CE=CL,BF=BL$. AD cuts circumcircle of triangle AEF again at K. H is projection K on BC. DH cuts altitude from A at N. Prove that circle diameter HN is tangent to circumcircle of triangle AEF.

Note that, when $AB+AC=2BC$ then E,F are midpoints of CA,AB. We get problem in the post http://www.artofproblemsolving.com/community/q1h1086462p4806122

buratinogigle

#2 Jul 9, 2015, 11:14 pm

Let ABC be a triangle inscribed in circle (O). AD is diameter of (O). AL is bisector with L is on BC. E,F lie on CA,AB such that $CE=CL,BF=BL$. AD cuts circumcircle of triangle AEF again at K. H is projection K on BC. AX is altitude of triangle ABC and $(Y),(Z)$ are Thebault circle wrt AX. Prove that DH is common inner tangent of $(Y),(Z)$.

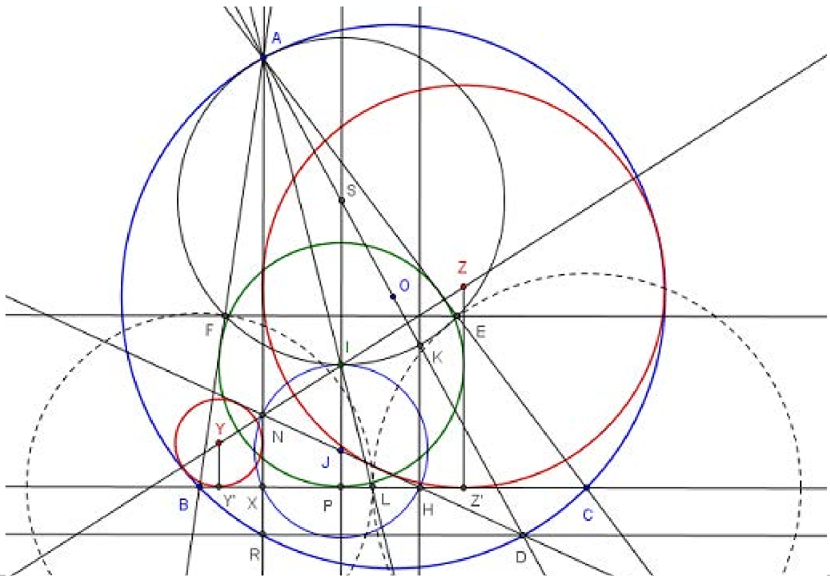
Luis González

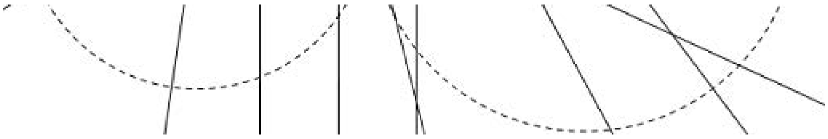
#4 Jul 10, 2015, 4:13 am • 2

Let (I) be the incircle of $\triangle ABC$ tangent to BC at P. By symmetry, we have $IE=IL$ and $IF=IL \Longrightarrow IE=IF$ and together with $\angle IAE=\angle IAF$, it follows that I is the midpoint of the arc EF of $\odot(AEF)$. Moreover, $\angle AIF=180^\circ-2\angle BIL=180^\circ-2(90^\circ-\tfrac{1}{2}\angle ACB)=\angle ACB \Longrightarrow \angle AEF=\angle AIF=\angle ACB \Longrightarrow$ $EF \parallel BC \Longrightarrow \odot(AEF)$ is tangent to (O), thus its center S is the midpoint of $AK \Longrightarrow$ perpendicular bisector $\overline{SIP}$ of EF is midparalell of $KH \parallel AX$ cutting NH at its midpoint J.

Let R be the 2nd intersection of AX and (O). Since $DR \parallel BC$, then by parallel tangent theorem for the Thebault circles $(Y),(Z)$ of AX, it follows that their common internal tangent ℓ, other than AX, passes through D. Now let ℓ cut BC,AX at H',N' and let $(Y),(Z)$ touch BC at Y',Z'. From Thebault's theorem we deduce that I is midpoint of $YZ \Longrightarrow P$ is midpoint of $Y'Z'$ and since Y',Z' are isotomic points WRT XH', then H' is reflection of X on $P \Longrightarrow H\equiv H'$ and $N\equiv N' \Longrightarrow NI$ is external bisector of $\angle XNH$ cutting the perpendicular bisector of XH at the midpoint I of the arc XNH, i.e. I is on the circle (J) with diameter $HN \Longrightarrow (J)$ is tangent to $\odot(AEF)$ at I.

Attachments:





livetolove212

#5 Jul 10, 2015, 10:20 am

I have another generalization, which is similar to this problem.
Given triangle ABC with $AB+AC=kBC$ and its circumcircle (O), orthocenter H. Let $AK\perp BC$, L be a point on AK such that $\frac{\overline{KH}}{\overline{KL}}=\frac{1}{k-1}$. Let G be the antipode of A, LG meets BC at M. A line through M and perpendicular to BC intersects AO at N. Prove that (LM) is tangent to (AN).

Quick Reply

High School Olympiads

Two fixed circle

Reply

Source: Own

buratinogigle

#1 Aug 30, 2015, 2:12 pm

Let ABC be a triangle and P is a point. ℓ is a line passing through P. ℓ cuts CA, AB at E, F, reps. Let M, N be on BC such that $EM \parallel PB, FN \parallel PC$. EM cuts FN at Q. Prove that circumcircle of triangle QMN is always tangent to two fixed circles when ℓ moves.

Luis González

#2 Aug 30, 2015, 10:54 pm • 3

As the line ℓ spins around P, it induces a perspectivity $E \mapsto F$ between AC, AB and since the directions $EM \parallel PB$ and $FN \parallel PC$ are fixed, then $\mathbf{P}: M \mapsto N$ is a proyectivity on BC whose fixed points are clearly B and C. Now invert with center B and arbitrary power, denoting inverse points with primes. By conformity the inverse of $\omega \equiv \odot(QMN)$ is the circle ω' through M', N' such that $\angle(BC, \omega) = \angle(BC, \omega') = \angle BPC$. Further $(C, M, N, B) = (C', M', N', \infty) \Longrightarrow \frac{CM'}{CN'} = \frac{CM}{CN} \cdot \frac{BN}{BM} = \text{const}$. The RHS being constant, as (C, M, N, B) is the characteristic of $\mathbf{P}$, thus we deduce that ω' are all homothetic with center C'. Since $\mathbf{P}$ is concordant, then $M' \mapsto N'$ is concordant as well $\Longrightarrow \omega'$ have common external tangents τ_1, τ_2 meeting at $C' \Longrightarrow \omega$ touches two fixed circles through B, C; inverse images of τ_1, τ_2.

Quick Reply

High School Olympiads

With the Feuerbach's point

Reply

jayme
#1 Oct 5, 2010, 9:58 pm

Dear Mathlinkers,
Let ABC a triangle, G the centroid, Fe the Feuerbach's point, A'B'C' the orthic triangle, Ea, Eb, Ec the Euler's lines wrt AB'C', BC'A', CA'B'. It is know that these three lines are concurrent in a point M. Prove that M, G and Fe are collinear.
Sincerely
Jean-Louis

jayme
#2 Oct 6, 2010, 6:37 pm

Dear Mathlinkers,
any ideas?
Sincerely
Jean-Louis

Luis González
#3 Oct 6, 2010, 8:08 pm

> jayme wrote:
> Let ABC a triangle, G the centroid, Fe the Feuerbach's point, A'B'C' the orthic triangle, Ea, Eb, Ec the Euler's lines wrt AB'C', BC'A', CA'B'. It is know that these three lines are concurrent in a point M. Prove that M, G and Fe are collinear.

Dear Jean Louis, unless I'm missunderstanding your wording, this proposition is false. In general, Euler lines of $\triangle AB'C', \triangle BC'A'$ and $\triangle CA'B'$ concur at the anti-Steiner point of the Euler line of $\triangle ABC$ WRT its medial triangle, namely X_{125}. But centroid X_2, Feuerbach point X_{11} and X_{125} are not collinear.

jayme
#4 Oct 6, 2010, 8:26 pm

Dear Luis and Mathlinkers, yes, I made a confusion with the lot of triangles that I have on my figure. If I am not wrong, Fe and G concern the orthic triangle of ABC. Is now my proposition OK?
Sincerely
Jean-Louis

Luis González
#5 Oct 6, 2010, 8:48 pm

Now, It's indeed correct. $M \equiv X_{125}$ becomes X_{100} of the orthic triangle $\triangle ABC$, i.e the anticomplement of its Feuerbach point $\Longrightarrow$ points F_e, G, M are collinear such that $\overline{GF_e} : \overline{GM} = -1 : 2$, but I don't have a synthetic proof in mind.

References:
http://www.xtec.cat/~qcastell/ttw/ttweng/resultats/r125.html
http://www.xtec.cat/~qcastell/ttw/ttweng/resultats/r128.html

jayme
#6 Oct 6, 2010, 8:55 pm

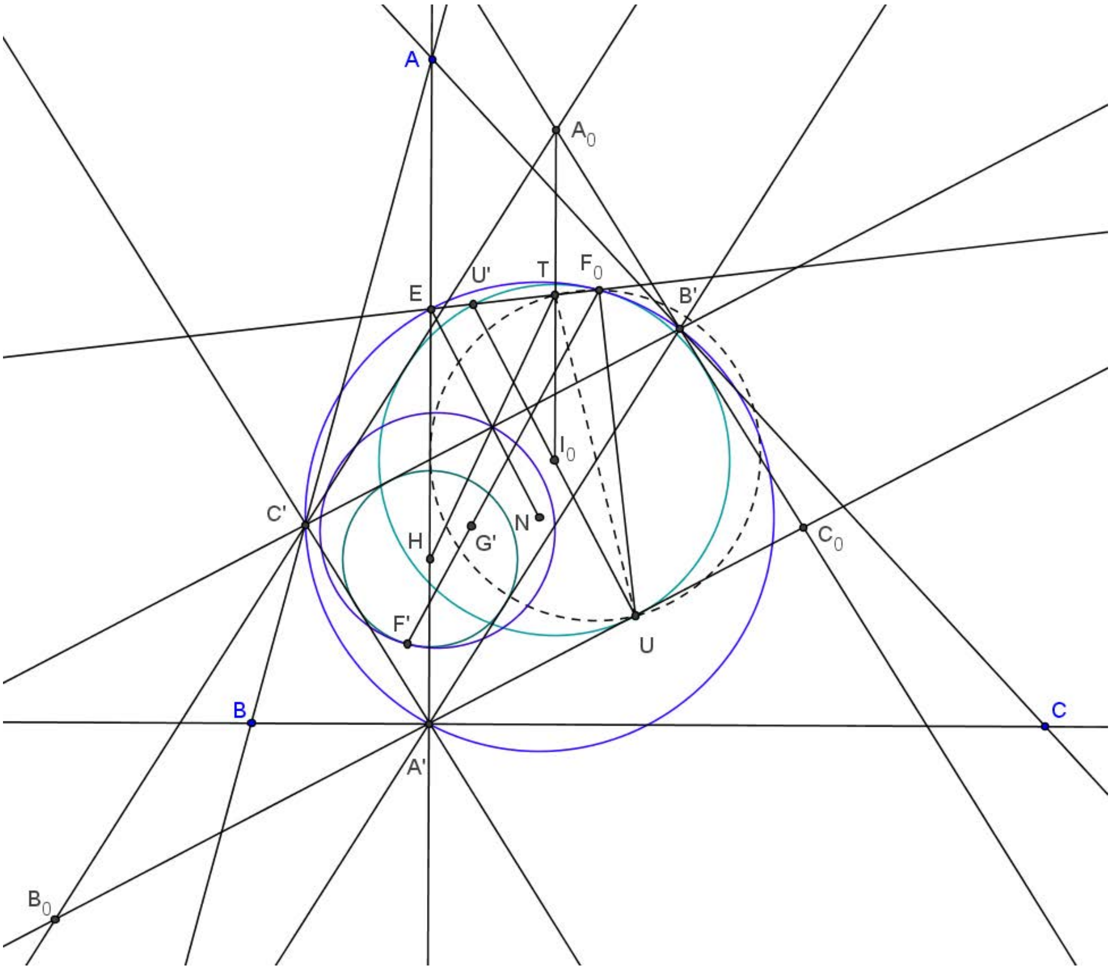
Dear Luis and Mathlinkers,
thanks for your quick answer.
As you have perhaps understand, the different questions I have send concern a little study that lead synthetically on this unexpect result.
I have to write an article...
Sincerely
Jean-Louis

Luis González
#7 Oct 6, 2010, 10:53 pm

For conveniece let G', F' denote the centroid and Feuerbach point of the orthic triangle $\triangle A'B'C'$. If $\triangle ABC$ is acute, then H becomes incenter of $\triangle A'B'C'$. Since AH is the A-circumdiameter of $\triangle AB'C'$, then its circumcenter E is the midpoint of AH, i.e. midpoint of the arc $B'C'$ of the 9-point circle (N) of $\triangle ABC$. Thus, orthocenter T of $\triangle AB'C'$ is the reflection of H about the midpoint of $B'C' \Longrightarrow e_a \equiv ET$ is the Euler line of $\triangle AB'C'$. Let $\triangle A_0B_0C_0$ be the antimedial triangle of $\triangle A'B'C'$ and I_0 be its incenter. Then T is the midpoint of A_0I_0.

On the other hand, incircle (I_0) of $\triangle A_0B_0C_0$ touches B_0C_0 at U and U' is the antipode of U WRT (I_0). Circle with diameter UT is the A_C- Garitte's circle passing through the Feuerbach point F_0 of $\triangle A_0B_0C_0$, i.e. $\angle TF_0U = 90^\circ$, thus F_0, T, U are collinear. But since radii NE and I_0U' of 9-point circle and incircle of $\triangle A_0B_0C_0$ are parallel, EU' passes through their exsimilicenter $F_0 \Longrightarrow F_0 \in e_a$. Likewise, Euler lines e_b, e_c of $\triangle BC'A'$ and $\triangle CA'B'$ pass through the Feuerbach point F_0 of $\triangle A_0B_0C_0$, i.e. the anticomplement of F' WRT $\triangle A'B'C'$.

Attachments:



Luis González
#8 Aug 30, 2015, 6:42 am • 1

The problem can be generalized as follows:

P is an arbitrary point on the plane of $\triangle ABC$ and $\triangle XYZ$ is the antipedal triangle of P WRT $\triangle ABC$. Q is the isogonal conjugate of P WRT $\triangle XYZ$. Then the Poncelet point of $XYZQ$ is the anticomplement of the Poncelet point of $ABCP$ WRT $\triangle ABC$.

For a proof see An easy problem on isogonal conjugate (combination of the results found in posts #2 and #3).

TelvCohl

#9 Aug 30, 2015, 1:40 pm • 1

> *Luis González wrote:*
> The problem can be generalized as follows:
>
> P is an arbitrary point on the plane of $\triangle ABC$ and $\triangle XYZ$ is the antipedal triangle of P WRT $\triangle ABC.$ Q is the isogonal conjugate of P WRT $\triangle XYZ.$ Then the Poncelet point of $XYZQ$ is the anticomplement of the Poncelet point of $ABCP$ WRT $\triangle ABC.$
>
> For a proof see An easy problem on isogonal conjugate (combination of the results found in posts #2 and #3).

Equivalence Proposition :

Given a $\triangle ABC$ and a point P (arbitrary point). Let $\triangle P_AP_BP_C$ be the pedal triangle of P WRT $\triangle ABC$ and let O, V be the circumcenter of $\triangle ABC, \triangle P_AP_BP_C$, resp. Let T be the orthopole of OP WRT $\triangle ABC$ and let R be the isogonal conjugate of P WRT $\triangle P_AP_BP_C$. Then the Simson line of T WRT $\triangle P_AP_BP_C$ is parallel to VR.

For the proof see also here **:** A Steiner line parallel to an orthotransversal (post #4) :)

Quick Reply

High School Olympiads

An easy problem on isogonal conjugate

Reply

Source: Own

A-B-C

#1 Jul 6, 2015, 8:17 am • 1

$\triangle ABC$ and P, Q are isogonal conjugate points WRT $\triangle ABC$
$\triangle DEF$ is circumcevian triangle of P WRT $\triangle ABC$
Let X, Y, Z be the reflections of Q in midpoints of BC, CA, AB
Prove that: DX, EY, FZ are concurrent at a point on (ABC)

Luis González

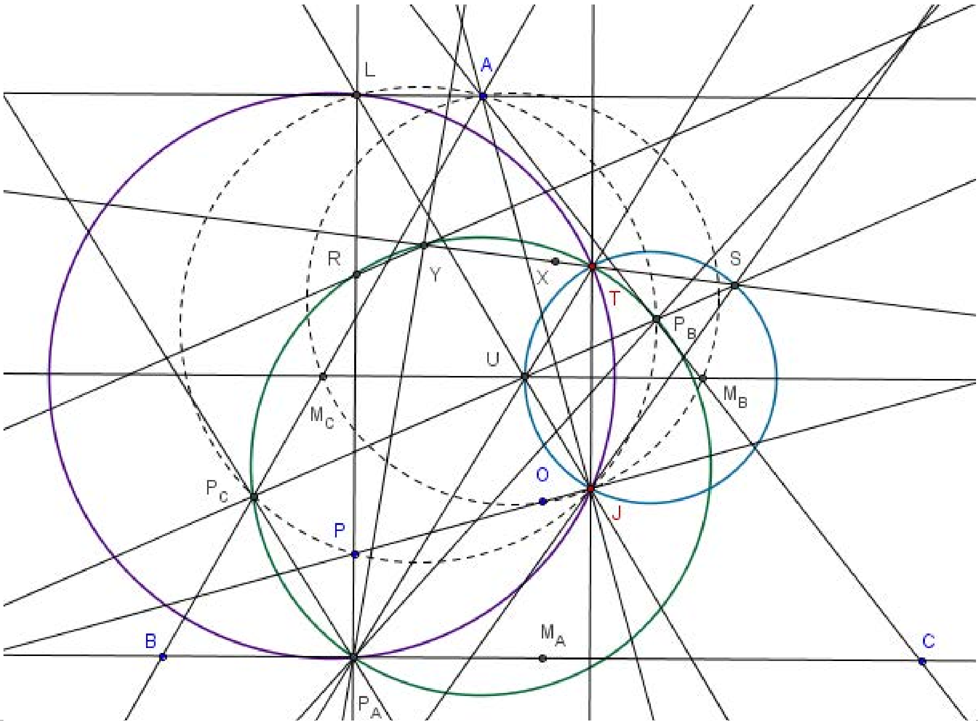
#2 Jul 7, 2015, 1:03 pm • 3

Lemma: P is arbitrary point in the plane of $\triangle ABC$ with circumcenter O. $\triangle P_AP_BP_C$ is the pedal triangle of P WRT $\triangle ABC$ and T is the orthopole of OP WRT $\triangle ABC$, lying on $\odot(P_AP_BP_C)$. X is the orthocenter of $\triangle AP_BP_C$ and TX cuts $\odot(P_AP_BP_C)$ again at Y. Then P_AP, P_AY are isogonals WRT $\angle P_BP_AP_C$.

Let J, L denote the projections of A on PO, PP_A, resp. From the Fontené configuration discussed at Two Yango's problem (see the solution at post #2), we get that J is the Miquel point of the quadrangle $\{AB, AC, M_BM_C, P_BP_C\}$, T is reflection of J on M_BM_C and $JTLP_A$ is an isosceles trapezoid with diagonal intersection $U \equiv JL \cap TP_A \equiv M_BM_C \cap P_BP_C$. As TX is the Steiner line of J WRT $\{AB, AC, M_BM_C, P_BP_C\}$, then J is anti-Steiner point of TX WRT $\triangle AP_BP_C$. Hence if $S \equiv TX \cap P_BP_c$, then SU bisects $\angle TSJ$. Together with $UT = UJ$, then $STUJ$ is cyclic $\Longrightarrow$ $\angle YSU = \angle USJ = \angle UTJ = \angle TP_AL$. So if PP_A cuts $\odot(P_AP_BP_C)$ again at R, we have $\angle(YT, YR) = \angle TP_AL = \angle YSU \Longrightarrow RY \parallel P_BP_C \Longrightarrow P_AP$ and P_AY are isogonals WRT $\angle P_BP_AP_C$.

Back to the proposed problem. Let $\triangle A^*B^*C^*$ be the antipedal triangle of Q WRT $\triangle ABC$ and let O^* denote the circumcenter of $\triangle A^*B^*C^*$. Since X becomes the orthocenter of $\triangle BCA^*$, then using the previous lemma for $\triangle A^*B^*C^*$, it follows that DX cuts $\odot(ABC)$ again at the orthopole T of QO^* WRT $\triangle A^*B^*C^*$. Similarly EY and FZ hit $\odot(ABC)$ at T.

Attachments:



TelvCohl

#3 Jul 7, 2015, 3:01 pm • 2

My solution :

Let R be the anticomplement of Q WRT $\triangle ABC$.
Let $\triangle A'B'C'$ be the anticomplementary triangle of $\triangle ABC$.

Since $\odot(AYZ), \odot(BZX), \odot(CXY)$ is the 9-point circle of $\triangle RB'C', \triangle RC'A', \triangle RA'B'$, resp,
so $\odot(ABC), \odot(AYZ), \odot(BZX), \odot(CXY)$ are concurrent at the Poncelet point T of $A'B'C'R$.

From $\angle XTC = \angle XYC = \angle BAQ = \angle DAC = \angle DTC \Longrightarrow T \in DX.$
Similarly, we can prove $T \in EY, T \in FZ \Longrightarrow DX, EY, FZ$ are concurrent on $\odot(ABC)$.

Q.E.D

Quick Reply

High School Olympiads

Nice harmonic

Locked

ronaldo58

#1 Aug 29, 2015, 10:16 pm

Let ABC be a triangle inscribed (O). AA_0, BB_0, CC_0 are altitudes. AA_0 intersect (O) at second point D. E is a point on A_0B_0 such that BE OA. DB_0 intersect (O) at second point F. BF intersects AE at K. Prove that K is midpoint of B_0C_0.

Luis González

#2 Aug 30, 2015, 3:05 am

Discussed before at http://www.artofproblemsolving.com/community/c6h389720.

High School Olympiads

Midpoint

Reply

buratinogigle

#1 Feb 3, 2011, 9:15 pm

Let ABC be a triangle inscribed (O). AA', BB', CC' are altitudes. AA' intersect (O) at second point D. E is a point on $A'B'$ such that $BE \perp OA$. DB' intersect (O) at second point F. BF intersects AE at K. Prove that K is midpoint of $B'C'$.

Luis González

#2 Feb 4, 2011, 12:38 am

This configuration is the same as Two parallels.

From that exercise, we know that BF passes through the midpoint K of $B'C'$. Now, it suffices to show that A, K, E are collinear. Let BE cut AC at M. Since $\triangle EBB'$ is isosceles with apex E, it follows that E is the midpoint of BM. Now $BEM \parallel B'C'$ implies that A, K, E are collinear on the A-median line of $\triangle AB'C'$, i.e. the A-symmedian of $\triangle ABC$.

buratinogigle

#3 Feb 5, 2011, 8:14 pm

According problem on the post Divide in two eqal segments. BF passes through midpoint K of $B'C'$. We will prove that A, K, E are collinear.

Easily seen $B'C' \perp OA$ thus $AE \parallel B'C'$. BF passes through midpoint K of $B'C'$. We deduce $B(B'C'KE) = -1$ (1).

Let H be orthocenter, we have $B'(BAC'A') = -1$ (2).

From (1) and (2) we get $B(B'C'KE) = B'(BAC'A')$, therefore the intersections A, K, E are collinear. We are done.

Attachments:

Figure373.pdf (6kb)

Quick Reply

High School Olympiads

Two isogonals in incenter diagram

Source: MEMO 2015, problem T-6

randomusername

#1 Aug 29, 2015, 1:32 am

Let I be the incentre of triangle ABC with $AB > AC$ and let the line AI intersect the side BC at D. Suppose that point P lies on the segment BC and satisfies $PI = PD$. Further, let J be the point obtained by reflecting I over the perpendicular bisector of BC, and let Q be the other intersection of the circumcircles of the triangles ABC and APD. Prove that $\angle BAQ = \angle CAJ$.

Luis González

#2 Aug 29, 2015, 2:27 am

Let M be the midpoint of the arc BC of $\odot(ABC)$. $\angle AQP = \angle ADP = \angle ACM \Longrightarrow M, Q, P$ are collinear. Since $MB^2 = MC^2 = MI^2 = MQ \cdot MP \Longrightarrow MI$ is tangent of $\odot(PIQ) \Longrightarrow \angle IQM = \angle PIM = \angle IDP = \angle ACM$. Thus if QI cuts $\odot(ABC)$ again at E, then $AE \parallel BC \Longrightarrow E$ is reflection of A across the perpendicular bisector of BC. Hence, by obvious symmetry, $J \in EM$ and $AEJI$ is isosceles trapezoid $\Longrightarrow \angle MAQ = \angle MEQ = \angle MAJ \Longrightarrow \angle BAQ = \angle CAJ$.

andria

#3 Aug 29, 2015, 2:28 am

Let $AI \cap \odot(\triangle ABC) = M$.

$IJ \parallel BC \Longrightarrow \angle IDP = \angle MIJ \Longrightarrow \Delta MIJ \sim \Delta PID \Longrightarrow \frac{IJ}{ID} = \frac{MI}{PD} \Longrightarrow IJ.PD = MI.ID = AI.MD \Longrightarrow \frac{AI}{IJ} = \frac{PD}{DM} \bigstar$

$(\because \frac{AI}{ID} = \frac{MI}{MD} = \frac{b+c}{a})$

$\angle BPQ = \angle MAQ = \frac{\angle A - \angle CAQ}{2} \Longrightarrow P, Q, M$ are collinear.

From $\angle IDP = \angle DIJ$ and $\bigstar \Longrightarrow \Delta IAJ \sim \Delta DPM \Longrightarrow \angle MAJ = \angle BPM = \angle QAM$.

DONE

High School Olympiads

Property of Orthodiagonal quadrilateral

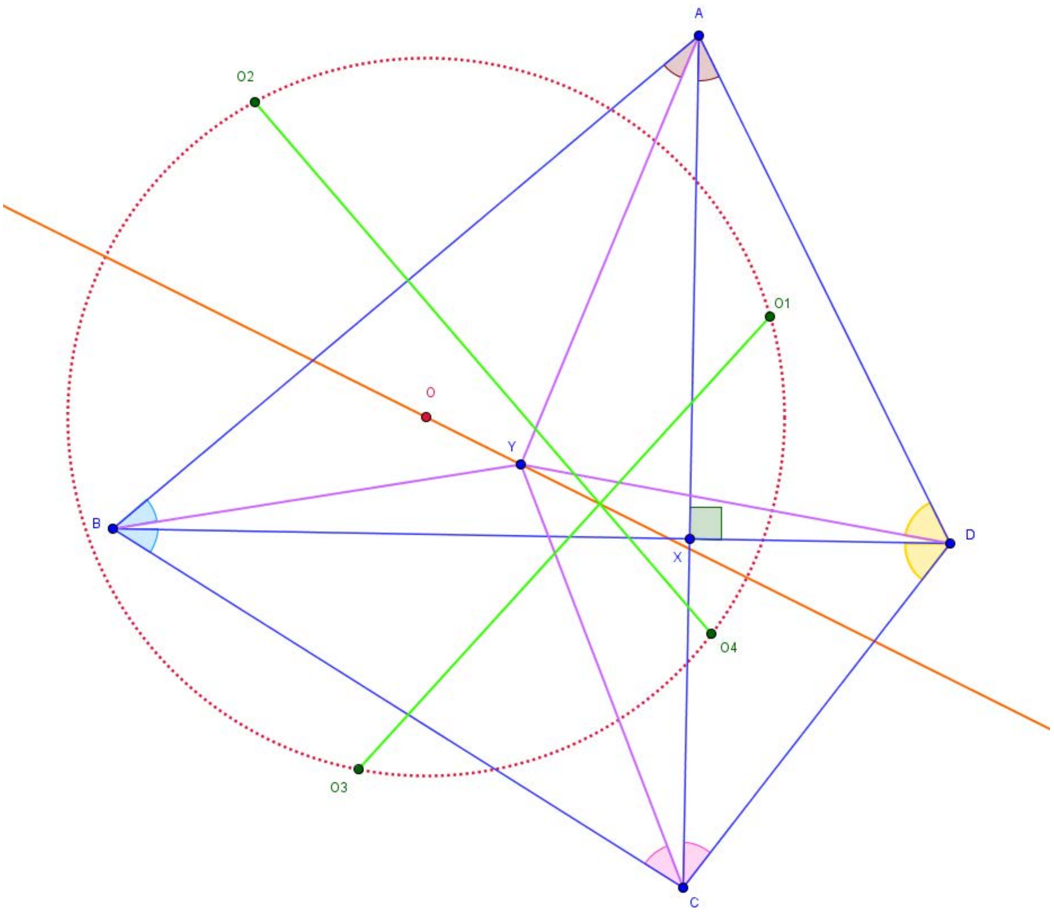
Source: Own

TelvCohl
#1 Aug 24, 2015, 2:49 am • 5

Given an Orthodiagonal quadrilateral $ABCD$ (i.e. $AC \perp BD$) .
Let $X \equiv AC \cap BD$ and Y be the isogonal conjugate of X WRT $ABCD$.
Let O_1, O_2, O_3, O_4 be the circumcenter of $\triangle YDA, \triangle YAB, \triangle YBC, \triangle YCD$, respectively .
Let O be the center of the circle passing through O_1, O_2, O_3, O_4 .

Prove that OY, O_1O_3, O_2O_4 are concurrent

Attachments:



pi37
#2 Aug 24, 2015, 6:16 am • 2

Invert about Y; the fact that Y is isogonal to the intersection of the diagonals of an orthodiagonal quadrilateral is equivalent to the fact that $\angle YAB + \angle YDC = 90$, and cyclic permutations. But after inversion, this is $\angle YB'A' + \angle YC'D' = 90$, which is equivalent. So the isogonal conjugate of Y in the new quadrilateral $ABCD$ (it exists because $\angle YA'B' = 180 - \angle YC'D'$) is Z', the intersection of the perpendicular diagonals of $A'B'C'D'$. O_1' is now the reflection of Y across $A'D'$, so $O_1'O_2'O_3'O_4'$ is cyclic on a circle centered at Z'. Now

$$\angle O_2'O_1'O_4' = \angle O_2'O_1'Y + \angle YO_1'O_4' = \angle YA'B' + \angle YD'C' = 90$$

implying $O_1'O_2'O_3'O_4'$ is a rectangle. Thus $O_1'O_3'$ and $O_2'O_4'$ pass through Z'. But $O_1'O_2'O_3'O_4'$ is cyclic, so YZ' is the radical axis of $(YO_1'O_3')$ and $(YO_2'O_4')$, giving that YZ' is collinear with their second intersection, implying the original concurrence.

mineiraojose

#3 Aug 24, 2015, 9:57 pm

Remark: If $\odot(O)$ is the circle passing through O_1, O_2, O_3, O_4 and M is the Miquel point of $ABCD$ then M and Y are inverses to each wrt. $\odot(O)$.

This post has been edited 1 time. Last edited by mineiraojose, Aug 24, 2015, 9:57 pm

THVSH

#4 Aug 25, 2015, 8:54 pm • 2

> TelvCohl wrote:
>
> Given an Orthodiagonal quadrilateral $ABCD$ (i.e. $AC \perp BD$) .
> Let $X \equiv AC \cap BD$ and Y be the isogonal conjugate of X WRT $ABCD$.
> Let O_1, O_2, O_3, O_4 be the circumcenter of $\triangle YDA, \triangle YAB, \triangle YBC, \triangle YCD$, respectively .
> Let O be the center of the circle passing through O_1, O_2, O_3, O_4 .
>
> Prove that OY, O_1O_3, O_2O_4 are concurrent

My solution: (without inversion)
Let $O_1O_3 \cap O_2O_4 = E$. F, G are the projection of E on O_1O_2, O_3O_4, respectively. H, I are the midpoints of O_1O_2, O_3O_4, respectively. M, N are the midpoints of YA, YC,respectively. Since $O_1O_2; O_3O_4$ are the perpendicular bisector of YA, YC, we get $M \in O_1O_2$; $N \in O_3O_4$. On the other hand, we have $\angle YO_3O_4 = \angle YBC = \angle ABD$, similarly, we have $\angle YO_4O_3 = \angle ADB \Longrightarrow \triangle YO_3O_4 \cup N \sim \triangle ABD \cup X \Longrightarrow \frac{NO_3}{NO_4} = \frac{XB}{XD}$. Similarly, $\frac{MO_2}{MO_1} = \frac{XB}{XD}$. Hence, $\frac{MO_2}{MO_1} = \frac{NO_3}{NO_4}$. So $\triangle EO_2O_1 \cup M \cup F \cup H \sim \triangle EO_3O_4 \cup N \cup G \cup I. \Longrightarrow \frac{MF}{MH} = \frac{NG}{NI} = k$.
Let $Y^* \in OE$ such that $\frac{Y^*E}{Y^*O} = k$. Now we get $Y^*M \parallel EF \perp O_1O_2$ and $Y^*N \parallel EG \perp O_3O_4 \Longrightarrow Y^* \equiv Y$. So $Y \in OE$.
Therefore, O_1O_3, O_2O_4, OY are concurrent at E. Q.E.D

Petry

#6 Aug 26, 2015, 4:30 pm • 1

My solution:

$\{V\} = O_1O_3 \cap O_2O_4$ and I, J are the circumcenters of ΔO_1YO_3, ΔO_2YO_4, respectively.
$\{Y, Z\} = \odot(I) \cap \odot(J) \Rightarrow YZ$ is the radical axis of $\odot(I)$ and $\odot(J)$.
O_1O_3 is the radical axis of $\odot(O)$ and $\odot(I)$,
O_2O_4 is the radical axis of $\odot(O)$ and $\odot(J)$.
So, V is the radical center of $\odot(O)$, $\odot(I)$ and $\odot(J) \Rightarrow V \in YZ$. (*)
Let's prove that the points O, Y, Z are collinear.

O_1, O_2 are the circumcenters of $\Delta AYD, \Delta AYB$, respectively $\Rightarrow O_1O_2$ is the perpendicular bisector of AY.
Similarly, O_2O_3, O_3O_4, O_4O_1 are the perpendicular bisectors of BY, CY, DY, respectively.
$\{M\} = AY \cap O_1O_2$, $\{N\} = BY \cap O_2O_3$, $\{P\} = CY \cap O_3O_4$, $\{Q\} = DY \cap O_4O_1$.

$$\angle YO_1Q = \frac{\angle YO_1D}{2} = \angle YAD = \angle BAX$$
$$\angle O_3YP = 90^\circ - \angle YO_3P = 90^\circ - \frac{YO_3C}{2} = 90^\circ - \angle YBC = 90^\circ - \angle ABX = \angle BAX$$
$$\Rightarrow \angle YO_1Q = \angle BAX = \angle O_3YP$$

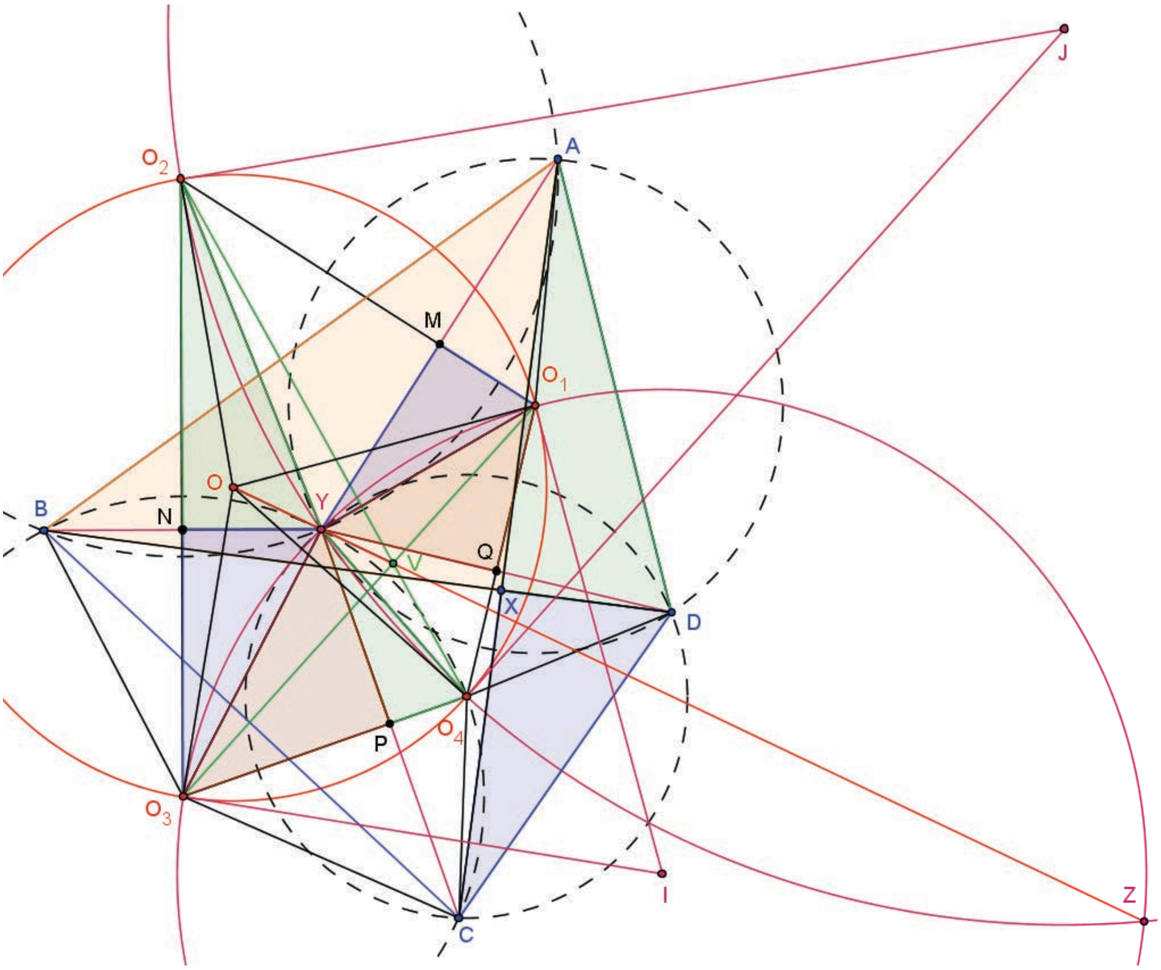
So, $\Delta YO_1Q \sim \Delta BAX \sim \Delta O_3YP$.
Similarly, $\Delta YO_1M \sim \Delta CDX \sim \Delta O_3YN$, $\Delta YO_2M \sim \Delta CBX \sim \Delta O_4YQ$ and $\Delta YO_2N \sim \Delta DAX \sim \Delta O_4YP$.
$\angle O_1O_2O_3 = \angle YO_2M + \angle YO_2N = \angle O_4YQ + \angle O_4YP = \angle DYC$.

OY intersects again $\odot(I)$, $\odot(J)$ at Z_1, Z_2, respectively.
$$\angle OO_1I = \angle OO_1O_3 + \angle IO_1O_3 = \frac{180^\circ - \angle O_1OO_3}{2} + \frac{180^\circ - \angle O_1IO_3}{2} =$$
$$= 180^\circ - \angle O_1Z_1O_3 - \angle O_1O_2O_3 = \angle O_1YO_3 - \angle O_1O_2O_3 = 90^\circ + \angle DYC - \angle O_1O_2O_3 = 90^\circ$$
So, $\angle OO_1I = 90^\circ \Rightarrow OO_1 \perp IO_1$

OO_1 is tangent to $\odot(I)$ at $O_1 \Rightarrow OY \cdot OZ_1 = OO_1^2$
Similarly, OO_2 is tangent to $\odot(J)$ at $O_2 \Rightarrow OY \cdot OZ_2 = OO_2^2$
$OO_1 = OO_2 \Rightarrow OZ_1 = OZ_2 \Rightarrow Z_1 = Z_2 = Z \Rightarrow Z \in OY$ (**)

(*) and (**) $\Rightarrow$ the points O, Y and V are collinear.
So, OY, O_1O_3 and O_2O_4 are concurrent.

Attachments:



buratinogigle

#9 Aug 26, 2015, 5:53 pm

Nice configuration, I have an idea from this, we can start with original cyclic quadrilateral $O_1O_2O_3O_4$.

Let $ABCD$ be a cyclic quadrilateral inscribed in circle (O) with AB cuts CD at E, AD cuts BC at F. Let P be an arbitrary point. Let X, Y, Z, T be reflections of P though AB, BC, CD, DA.

a) Prove that $XZ \perp YT$ iff $PE \perp PF$.

b) Prove that $\angle(XZ, YT) = \angle(PE, PF)$.

Note that, when $PE \perp PF$ and P lies on line OG with AC cuts BD at G then quadrilateral $XYZT$ is the same as $ABCD$ in original problem.

Petry

#11 Aug 26, 2015, 9:16 pm • 1

My solution for buratinogigle's problem:

b)$\{M\} = PX \cap AB$, $\{N\} = PY \cap BC$, $\{Q\} = PZ \cap CD$ and $\{S\} = PT \cap AD$.
M, N, Q, S are the midpoints of PX, PY, PZ, PT, respectively.
$MQ || XZ$ and $NS || YT \Rightarrow \angle(MQ, NS) = \angle(XZ, YT)$. (1)

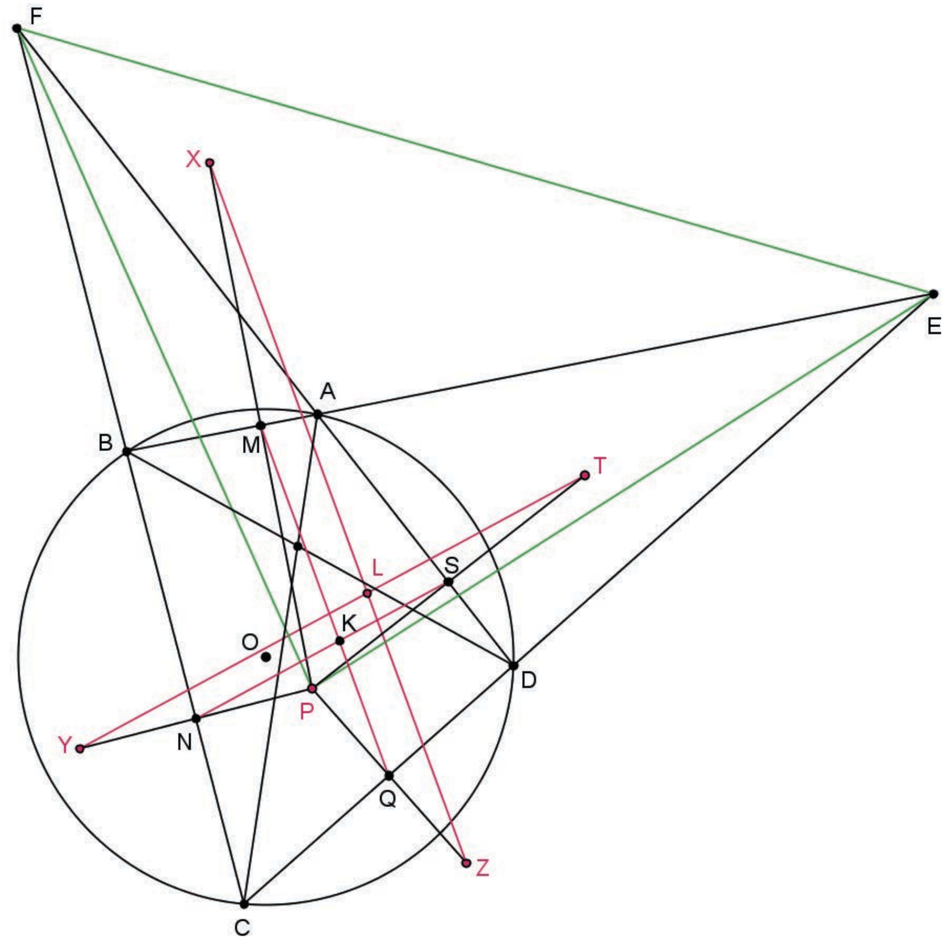
$PN \perp BC$ and $PS \perp AD \Rightarrow FNPS$ is cyclic $\Rightarrow \angle FPS = \angle FNS$
$PM \perp AB$ and $PQ \perp CD \Rightarrow EMPQ$ is cyclic $\Rightarrow \angle EPQ = \angle EMQ$
$PQ \perp CD$ and $PS \perp AD \Rightarrow PQDS$ is cyclic $\Rightarrow \angle SPQ = 180° - \angle SDQ$

$\{K\} = MQ \cap NS$
$\angle EPF = \angle EPS + \angle FPS = \angle SPQ - \angle EPQ + \angle FNS = 180^\circ - \angle D - \angle EMQ + \angle FNS =$
$= \angle B - 180^\circ + \angle BMK + \angle BNK = 360^\circ - \angle MKN - 180^\circ = \angle NKQ$.
So, $\angle(PE, PF) = \angle(MQ, NS)$. (2)

(1), (2) $\Rightarrow \angle(XZ, YT) = \angle(PE, PF)$.

a) $\angle(XZ, YT) = \angle(PE, PF) \Rightarrow XZ \perp YT$ iff $PE \perp PF$.

Attachments:



TelvCohl
#13 Aug 26, 2015, 9:46 pm • 1

> *buratinogigle wrote:*
> Let $ABCD$ be a cyclic quadrilateral inscribed in circle (O) with AB cuts CD at E, AD cuts BC at F. Let P be an arbitrary point. Let X, Y, Z, T be reflections of P though AB, BC, CD, DA.
>
> b) Prove that $\angle(XZ, YT) = \angle(PE, PF)$.

Let ℓ_1, ℓ_2 be the bisector of $\measuredangle(CD, AB), \measuredangle(DA, BC)$, resp. Let τ_E, τ_F be the reflection of EP, FP in ℓ_1, ℓ_2, resp. Notice $\tau_E \perp XZ$ and $\tau_F \perp YT$ we get $\measuredangle(XZ, YT) = \measuredangle(\tau_E, \tau_F) = \measuredangle(\tau_E, AB) + \measuredangle(AB, DA) + \measuredangle(DA, \tau_F) = \measuredangle(CD, EP) + \measuredangle(BC, CD) + \measuredangle(FP, BC) = \measuredangle(FP, EP)$.

Luis González
#14 Aug 28, 2015, 7:06 am • 1

Let $PQRS$ be the antipedal quadrilateral of Y WRT $ABCD$. If K is the circumcenter of $PQRS$, then by homothety $(Y, \tfrac{1}{2})$, it's enough to prove that PR, QS, KY concur.

From cyclic $YAPD$, we get $\angle YPA = \angle YDA = \angle CDB$ and likewise $\angle YQA = \angle CBD \Longrightarrow \triangle YPQ$ and $\triangle CDB$ are similar with corresponding altitudes YA and $CX \Longrightarrow \overline{AP} : \overline{AQ} = \overline{XD} : \overline{XB}$ and similarly we have

$CS : CR = XD : XB \Longrightarrow AP : AQ = CS : CR.$ Now from the problem nice lemma on cyclic quadrilateral, it follows that Y is on the line through K and $PR \cap QS,$ i.e. PR, QS, KY concur, as desired.

Quick Reply

High School Olympiads

Centroid on the inscribed circle of a right triangle

Reply

mdelyakova

#1 Aug 28, 2015, 1:13 am

The centroid of a right triangle lies on the inscribed circle for the triangle. Find the angles of the triangle.

Luis González

#2 Aug 28, 2015, 5:52 am

WLOG assume that $\triangle ABC$ is right angled at A. Label $BC = a, CA = b, AB = c$ and $\angle ABC = \beta$. In general, the centroid of an scalene $\triangle ABC$ lies on its incircle $\Longleftrightarrow 5(a^2 + b^2 + c^2) = 6(bc + ca + ab)$ (for various proofs see the topic Barycenter and incircle). Thus $10a^2 = 6(bc + ca + ab) \Longrightarrow \sin\beta + \cos\beta + \sin\beta \cdot \cos\beta = \frac{5}{3}$. Solving this trigonometic equation for $0 \leq \beta \leq \frac{\pi}{2} \in \mathbb{R}$ gives

$$\beta = 2 \cdot \arctan\left[\frac{1}{4}\left(\sqrt{3} \pm 2\sqrt{2\sqrt{3} - \frac{13}{4}}\right)\right]$$

Quick Reply

High School Olympiads

nice lemma on cyclic quadrilateral

Reply

jgnr
#1 Apr 30, 2011, 6:45 am

Let $ABCD$ be a cyclic quadrilateral with circumcenter O. The diagonals AC and BD meet at P. Prove that line OP is the locus of points X satisfying the following property: if Y and Z are the projections from X to AB and CD respectively, then $\frac{AY}{BY}=\frac{DZ}{CZ}$ (we use directed length).

[edited]

This post has been edited 2 times. Last edited by jgnr, Apr 30, 2011, 11:25 am

Tales749
#2 Apr 30, 2011, 7:55 am

> *Johan Gunardi wrote:*
> then $\frac{AY}{BY}=\frac{CZ}{DZ}$ (we use directed length).

are you sure that is $\frac{AY}{BY}=\frac{CZ}{DZ}$? I think that is $\frac{AY}{BY}=\frac{DZ}{CZ}$

Luis González
#3 Apr 30, 2011, 10:19 am

LeandroRemolina is right. The relation should be $\frac{AY}{BY}=\frac{DZ}{CZ}$, otherwise, the locus of X is a line that does not pass through P.

Let $E \equiv AB \cap DC$. Circles $\odot(EAD)$ and $\odot(EBC)$ intersect at E and the center H of the spiral similarity taking the oriented segments $\overline{AB}$ and $\overline{DC}$ into each other. Since $\frac{AY}{BY}=\frac{DZ}{CZ}$, it follows that Y, Z are homologous points under such spiral similarity $\Longrightarrow \angle YHX=\angle YEZ$, i.e. All $\odot(EYZ)$ pass through the fixed points $E, H \Longrightarrow$ Locus of the center O' of $\odot(EYZ)$ is the perpendicular bisector ℓ of $\overline{EH}$. Since X is the antipode of E WRT $\odot(EYZ)$, then the locus of X is the image τ of ℓ under the homothety with center E and factor 2. This is true for any quadrilateral $ABCD$ but τ passes through $P \equiv AC \cap BD$ when $ABCD$ is cyclic.

jgnr
#4 Apr 30, 2011, 11:39 am

Thank you for the correction, you are both correct.

A direct consequence of this lemma is Russia 2000.

Quick Reply

High School Olympiads

Barycenter and incircle.

Reply

Vladislao
#1 May 24, 2011, 3:51 am

Show that if the barycenter of a triangle $\triangle ABC$ lies in the incircle (I mean the circunference), then $5(a^2+b^2+c^2)=6(ab+bc+ca)$; $(a,b,c$ are the sides of the triangle).

yetti
#2 May 24, 2011, 6:29 am • 1

r is inradius of $\triangle ABC$.

Moment of inertia of A,B,C WRT centroid G is:
$\mathcal{J}_G=GA^2+GB^2+GC^2=\frac{1}{3}(a^2+b^2+c^2)$.

Moment of inertia of A,B,C WRT incenter I is:
$\mathcal{J}_I=IA^2+IB^2+IC^2=\frac{1}{4}\left[(b+c-a)^2+(c+a-b)^2+(a+b-c)^2\right]+3r^2=$
$=\frac{3}{4}(a^2+b^2+c^2)-\frac{1}{2}(bc+ca+ab)+3r^2$.

By Huygens-Steiner theorem, $\mathcal{J}_I=\mathcal{J}_G+3\cdot GI^2 \Longrightarrow 5(a^2+b^2+c^2)=6(bc+ca+ab)+36\cdot(GI^2-r^2)$.
If $GI=r$, the conclusion follows.

Virgil Nicula
#3 May 24, 2011, 7:52 am

Very nice the **Yetti**'s proof ! **Thank you**.

Classical proof.

$\boxed{IG=r} \iff IN^2=9\cdot IG^2=9r^2 \iff s^2+5r^2-16Rr=9r^2 \iff \boxed{s^2=4r(4R+r)}$.

$\boxed{5\cdot\left(a^2+b^2+c^2\right)=6\cdot(ab+bc+ca)} \iff 5\left[s^2-r(4R+r)\right]=3\left[s^2+r(4R+r)\right] \iff$
$\boxed{s^2=4r(4R+r)}$.

Luis González
#4 May 24, 2011, 8:32 am

Barycentric equation of the incircle (I) with respect to $\triangle ABC$ is

$(I)\equiv a^2yz+b^2zx+c^2xy-(x+y+z)((s-a)^2x+(s-b)^2y+(s-c)^2z)=0$

Centroid G of $\triangle ABC$ lies on $(I) \iff G\equiv(1:1:1)$ satisfies its equation

$\iff \ a^2+b^2+c^2-3[(s-a)^2+(s-b)^2+(s-c)^2]=0$

$\iff \ a^2+b^2+c^2-3(a^2+b^2+c^2-s^2)=0$

$\iff \ 3(a+b+c)^2-8(a^2+b^2+c^2)=0$

$\iff \ 5(a^2+b^2+c^2)=6(bc+ca+ab)$

Vladislao
#5 May 25, 2011, 7:52 am

> *Virgil Nicula wrote:*
> Very nice the **Yetti**'s proof ! **Thank you**.
>
> **Classical proof.**
>
> $\boxed{IG=r}\iff IN^2=9\cdot IG^2=9r^2\iff s^2+5r^2-16Rr=9r^2\iff\boxed{s^2=4r(4R+r)}$.
>
> $\boxed{5\cdot\left(a^2+b^2+c^2\right)=6\cdot(ab+bc+ca)}\iff 5\left[s^2-r(4R+r)\right]=3\left[s^2+r(4R+r)\right]\iff$ $\boxed{s^2=4r(4R+r)}$.

Hmm... I think you didn't say what's N. (You use that $IN^2=9\cdot IG^2=9r^2$ and then go on with the proof).

Quick Reply

High School Olympiads

anti-steiner point of Nagel point

Locked

powerpi

#1 Aug 27, 2015, 10:21 pm

In $\triangle ABC$ let N be the Nagel point, O the circumcenter and T the anti-steiner point of N with respect to $\triangle ABC$. Prove that T lies on ON.

Luis González

#2 Aug 27, 2015, 10:28 pm

Well-known and posted many times before. N is the incenter of the antimedial triangle $\triangle A^*B^*C^*$ of $\triangle ABC,$ thus T is the Feuerbach point of $\triangle A^*B^*C^*$ lying on the line joining N and its 9-point center $O.$ See the topic two Yango's problems (post #2) for the general case

High School Olympiads

Lots of cyclic quadrilaterals

Reply

Source: MEMO 2015, problem I-3.

randomusername

#1 Aug 27, 2015, 9:18 pm

Let $ABCD$ be a cyclic quadrilateral. Let E be the intersection of lines parallel to AC and BD passing through points B and A, respectively. The lines EC and ED intersect the circumcircle of AEB again at F and G, respectively. Prove that points C, D, F, and G lie on a circle.

raf616

#2 Aug 27, 2015, 9:40 pm • 1

My solution:

Let $O \equiv AC \cap BD$. Then we get that $EAOB$ is a parallelogram and so $\angle EAB = \angle ABO$. Then, we get:

$\angle EGF = \angle EAF = \angle EAB + \angle BAF = \angle ABO + \angle BEF = \angle ABD + \angle BEC = \angle ACD + \angle ECA = \angle FCD$
which gives the desired result.

Luis González

#3 Aug 27, 2015, 10:06 pm • 1

Let EA, EB cut DC at U, V. Then $\angle EAB = \angle ABD = \angle ACD = \angle EVU \Longrightarrow ABVU$ is cyclic $\Longrightarrow CD$ is the image of $\odot(EAB)$ under inversion with center E and power $EA \cdot EU = EB \cdot EV \Longrightarrow D, G$ and C, F are pairs of inverse points under this inversion $\Longrightarrow CDGF$ is cyclic.

hayoola

#4 Aug 28, 2015, 8:17 am

We have to prove that the Angels EGF and ECD are equal let p be a point on the line EB out of the segment EB near B we know that EGF=FBP so we have to prove FBP=ECD FBP =BEF+BFE we know that EB is parallede AC so BEF =ECA and we know that EBFGA is cyclic so EFB=EAB we know AE is parallede BD so EAB=ABD finally ABCD is cyclic so ABD=ACD and we are done

Math_CYCR

#5 Aug 28, 2015, 9:28 am

$\angle BDC = \angle BAC = \angle EBA$ and $\angle EDB = \angle AED$
$\longrightarrow \angle EFG = \angle AED + \angle EBA = \angle BDC + \angle EDB = \angle EDC$

$Q.E.D.$

Gryphos

#6 Aug 28, 2015, 8:32 pm

Here is another way of doing the angle chasing:
Let t be the tangent to the circumcircle of ABE at E. If we can prove that $t \parallel CD$, we are done, because then $\angle GFE = \angle(DE, t) = \angle EDC$, proving the assertion.
But because $AEBO$ is a parallelogramm, t is parallel to the tangent to the circumcircle of AOB at O ($O = AC \cap BD$ as in the first solution), and this is parallel to CD because of Reim´s theorem in a limit case, which ends the proof.

Quick Reply

High School Olympiads

BM= MI+ IK

Locked

Albert.V

#1 Aug 27, 2015, 9:40 pm

Let triangle ABC, D,E,F are foot perpendicual from A,B, C respectiely. On DE take K such that DH = DK (H is orthocentre). Through K guys perpendicular lines cut AD at I. M is the mid point of BC. Prove: BM= MI+ IK

Luis González

#2 Aug 27, 2015, 9:56 pm

Posted before at http://www.artofproblemsolving.com/community/c6h503957.

High School Olympiads

Prove that

Reply

robinson123

#1 Oct 25, 2012, 10:57 am

In $\triangle ABC$ with orthecenter H, perpendiculars AD, BE. K is the point lying on DE such that $DK = DH$. The perpendiicular to DE at K meets AD at I. M is the midpoint of BC prove that $BM = MI + IK$

yetti

#2 Oct 27, 2012, 3:49 pm • 2

$BM = MI + IK \Longleftrightarrow$
$(BM - IK)^2 = MI^2 = MD^2 + ID^2 = (BM - BD)^2 + ID^2 \Longleftrightarrow$
$BC \cdot (BD - IK) = BD^2 + ID^2 - IK^2 = BD^2 + KD^2 = BD^2 + HD^2 = BH^2 \Longleftrightarrow$
$BH^2 = BC \cdot (BH \cdot \sin\widehat{C} - KD \cdot \cot\widehat{A}) = BC \cdot (BH \cdot \sin\widehat{C} - HD \cdot \cot\widehat{A}) =$
$= BC \cdot BH \cdot (\sin\widehat{C} - \cos\widehat{C} \cdot \cot\widehat{A}) = BC \cdot BH \cdot \frac{\cos\widehat{B}}{\sin\widehat{A}} \Longleftrightarrow$
$\frac{BH}{BC} = \frac{\cos\widehat{B}}{\sin\widehat{A}} = \frac{\sin\widehat{BCH}}{\sin\widehat{CHB}}$, which is true by sine theorem for $\triangle HBC$.

Luis González

#3 Nov 6, 2012, 12:24 am • 1

We assume that $\triangle ABC$ is acute. F is the foot of the C-altitude. H becomes incenter of orthic $\triangle DEF$. L is on the ray DF, such that $DL = DK \Longrightarrow$ circle (I) with center I is tangent to DE, DF at K, L. Obviously $LK \parallel BC$, thus if $J \equiv FH \cap LK$, we have $\angle FJL = \angle FCB = \angle FEH = \angle HEK \Longrightarrow EHJK$ is cyclic. Since $\triangle DHK$ is D-isosceles, we deduce that $\angle FJE = \angle HKE = 90^\circ + \frac{1}{2}\angle HDE$. If FD cuts the circle (M) with diameter $\overline{BC}$ again at P, then $\angle FJE = 90^\circ + \frac{1}{2}\angle FPE \Longrightarrow J$ is incenter of $\triangle PEF \Longrightarrow (I)$ coincides with a Thebault circle of the cevian ED of $\triangle PEF$ internally tangent to (M) at T (Sawayama's lemma) $\Longrightarrow BM = MT = MI + IK$.

Quick Reply

High School Olympiads

nine point center

Locked

ferma2000

#1 Aug 11, 2015, 11:38 pm

Let $\triangle ABC$ be a triangle with nine point center N and centroid G and circumcircle (O). Let $BG \cap \odot(O) = E, CG \cap (O) = F$.
ω_1 denotes a circle throw B, F and tangent to AB at B.
ω_2 denotes a circle throw C, E and tangent to AC at C.
O_1, O_2 are centers of ω_1, ω_2 respectively.
Claim: prove that O_1, O_2, N are collinear.
scinearly,

Luis González

#2 Aug 11, 2015, 11:50 pm

Posted before at http://www.artofproblemsolving.com/community/c6h545088.

High School Olympiads

O_1O_2 passes through the nine-point center

Source: ELMO Shortlist 2013: Problem G7, by Michael Kural

v_Enhance
#1 Jul 23, 2013, 7:32 am

Let ABC be a triangle inscribed in circle ω, and let the medians from B and C intersect ω at D and E respectively. Let O_1 be the center of the circle through D tangent to AC at C, and let O_2 be the center of the circle through E tangent to AB at B. Prove that O_1, O_2, and the nine-point center of ABC are collinear.

Proposed by Michael Kural

Luis González
#2 Jul 23, 2013, 8:39 am

Let Y,Z be the midpoints of $AC,AB.G \equiv BY \cap CZ$ is the centroid of $\triangle ABC.$ If CE cuts (O_2) again at $L,$ then $\angle ELB=\angle EBA=\angle ECA \Longrightarrow AC \parallel BL \Longrightarrow GC:GL=GY:GB=-1:2 \Longrightarrow \frac{GC \cdot GE}{GE \cdot GL}=-\frac{1}{2},$ i.e. the ratio of the powers of G WRT $(O) \equiv \omega$ and (O_2) equals $-\frac{1}{2},$ thus the center U of $\odot(GBE)$ coaxal with $(O),(O_2)$ verifies $UO:UO_2=-1:2=GO:GH$ (H is the orthocenter of ABC) $\Longrightarrow HO_2 \parallel GU,$ but $\angle UGE=90^\circ-\angle EBG=90^\circ-\angle DCG \Longrightarrow GU \perp CD \Longrightarrow HO_2 \perp CD \Longrightarrow HO_2 \parallel OO_1.$ Similarly $HO_1 \parallel OO_2 \Longrightarrow OO_1HO_2$ is a parallelogram $\Longrightarrow \overline{O_1O_2}$ and $\overline{OH}$ bisect each other through the 9-point center.

thecmd999
#3 Apr 27, 2014, 1:20 am

Computational

Wolstenholme
#4 Jul 29, 2014, 1:25 am

Let H be the orthocenter of $\triangle ABC$. I shall show that O_1HO_2O is a parallelogram, which will immediately imply the desired result. We proceed with a complex number bash. Assume WLOG that the circumcircle of $\triangle ABC$ is the unit circle and denote the complex coordinates of A,B,C by a,b,c respectively.

Denote the complex coordinate of O_2 by x. Then since $O_2B \perp AB$ we have that $\frac{x-b}{\overline{x}-\overline{b}}=-\frac{b-a}{\overline{b}-\overline{a}}=ab$ so x satisfies the equation $x=ab\overline{x}+b-a$.

Now, denote the complex coordinates of E by e. Since E is on the C-median of $\triangle ABC$ we have that $\frac{e-c}{\overline{e}-\overline{c}}=\frac{a+b-2c}{\overline{a}+\overline{b}-2\overline{c}}=\frac{(a+b-2c)abc}{bc+ac-2ab}$ so e satisfies the equation $e=\frac{(a+b-2c)abc}{bc+ac-2ab}\overline{e}+\frac{(a+b)(c^2-ab)}{bc+ac-2ab}$

But e lies on the unit circle so $\overline{e}=\frac{1}{e}$ and so, after plugging in, we find that e satisfies: $e^2-\frac{(a+b)(c^2-ab)}{bc+ac-2ab}e-\frac{(a+b-2c)abc}{bc+ac-2ab}=0$. Since c also is a root of this quadratic, by Vieta's formulas we find that $e=\frac{(2c-a-b)ab}{bc+ac-2ab}$.

Now let M denote the midpoint of segment BE and let m be its complex coordinate. Then we easily find that $m=\frac{b+e}{2}=\frac{b(3ac+bc-3ab-a^2)}{2(bc+ac-2ab)}$. Since $O_2M \perp BE$ we have that $\frac{x-m}{\overline{x}-\overline{m}}=-\frac{b-e}{\overline{b}-\overline{e}}$. But since $E \in \omega$ we have that $\overline{e}=\frac{1}{e}$ so $\frac{x-m}{\overline{x}-\overline{m}}=be=\frac{(2c-a-b)ab^2}{}$

Therefore $x = \frac{(2c-a-b)ab^2}{bc+ac-2ab}\overline{x} + m - \overline{m}\cdot\frac{(2c-a-b)ab^2}{bc+ac-2ab}$. But we can compute that $\overline{m} = \frac{3ab+a^2-3ac-bc}{2ab(a+b-2c)}$ so $\overline{m}\cdot\frac{(2c-a-b)ab^2}{bc+ac-2ab} = -\frac{b(3ab+a^2-3ac-bc)}{2(bc+ac-2ab)} = m$ and so $x = \frac{(2c-a-b)ab^2}{bc+ac-2ab}\overline{x}$.

Plugging this into our original equation $x = ab\overline{x}+b-a$, we find that $\frac{(2c-a-b)ab^2}{bc+ac-2ab}\overline{x} = ab\overline{x}+b-a$ and solving for $\overline{x}$ we obtain $\overline{x} = \frac{ac+bc-2ab}{ab(c-b)}$. Letting O_1 have complex coordinate y we similarly find that $\overline{y} = \frac{ab+cb-2ac}{ac(b-c)}$.

Now since O has complex coordinate 0 and H has complex coordinate $a+b+c$ it suffices to show that $\overline{x}+\overline{y} = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$. But $\overline{x}+\overline{y} = \frac{ac+bc-2ab}{ab(c-b)} + \frac{ab+cb-2ac}{ac(b-c)} = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ as desired so we are done.

Dukejukem

#5 Sep 25, 2015, 7:18 am

Let r_1, r_2 be the radii of $(O_1), (O_2)$, respectively. Let O be the circumcenter of $\triangle ABC$ and let M be the midpoint of $\overline{AC}$. If BM cuts (O_1) for a second time at U, then by Power of a Point we have

$$MD\cdot MU = MC^2 = -MC\cdot MA = -MD\cdot MB,$$

implying that M is the midpoint of $\overline{BU}$. Then since $\triangle AUC$ is the image of $\triangle ABC$ under the homothety $\mathcal{H}(M,-1)$, it follows that the circumcenter O' of $\triangle AUC$ is the reflection of O in M. Thus, it is well-known that the midpoint of $\overline{BO'}$ is the nine-point center of $\triangle ABC$. Meanwhile, since $O'O_1$ is the perpendicular bisector of $\overline{CU}$, we have

$$\measuredangle UO_1O' = \measuredangle UDC = \measuredangle BDC = \measuredangle BAC \quad \text{and} \quad \measuredangle UO'O_1 = \measuredangle UAC = \measuredangle BCA,$$

where the last step follows since $ABCU$ is a parallelogram. Therefore, $\triangle ABC \sim \triangle O_1UO'$. Then since $O'U = R$ (which follows from the aforementioned homothety), we obtain $\frac{r_1}{AB} = \frac{O_1O'}{AC} = \frac{R}{BC}$. Analagously, we find that $\frac{r_2}{AC} = \frac{R}{BC}$, and hence $O_1O' = r_2$. Consequently, $O_1O' = O_2B$ and meanwhile $O_1O' \parallel O_2B$, because both lines are perpendicular to AB. Therefore, $O_1O'O_2B$ is a parallelogram, and thus the common midpoint of $\overline{O_1O_2}$ and $\overline{BO'}$ is the nine-point center of $\triangle ABC$. $\square$

Quick Reply

High School Olympiads

fixed point

Reply

PhuongMath

#1 Aug 11, 2015, 10:13 pm

Let a triangle ABC, the point M is moving on [BC]. Two point I,J are center of two triangle MAB, MAC, PQ is common external tangents of two circle $\odot(I), \odot(J)$ $(P \in \odot(I), Q \in \odot(J))$.
Prove that:
1. The circle (MIJ) always passes through a fixed point.
2. The intersection of BP and CQ are moving on a fixed circle when M moves on the side BC.

Luis González

#2 Aug 11, 2015, 10:34 pm • 1

Let (K) be the incircle of $\triangle ABC$ tangent to BC at $D.$ All circles $\odot(MIJ)$ go through the fixed point D (see two problems about cyclic quadrilateral (problem 1), incenters and cyclic, etc.)

Letting X be a point of (K) such that the radius KX is parallel to the radii $IP \parallel JQ,$ it follows that XP and XQ go through the exsimilicenters B and C of $(I) \sim (K)$ and $(J) \sim (K) \Longrightarrow X \equiv BP \cap CQ$ moves on $(K).$

Quick Reply

High School Olympiads

Symmedian again

Reply

Source: Own

Bandera

#1 Aug 11, 2015, 9:38 am

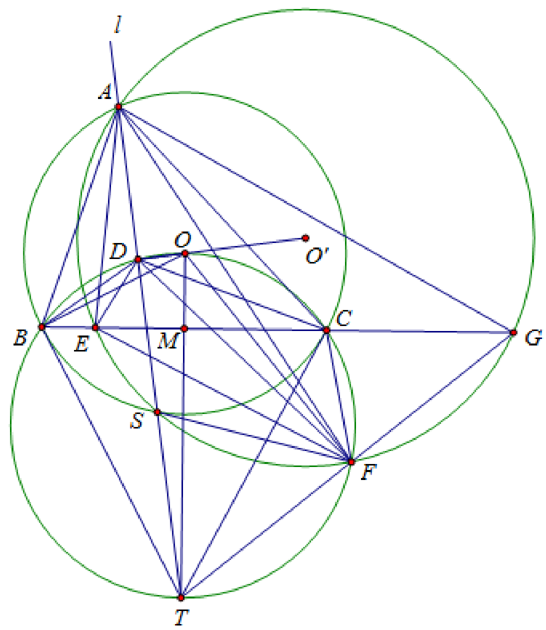
ABC is a triangle with an acute $\angle BAC$. Line ℓ is the symmedian at A. A point D is taken on ℓ inside $\triangle ABC$ such that $\angle BDC = 2\,\angle BAC$. Let E be an arbitrary point on the line BC not belonging to ℓ. Two circles are drawn: one that passes through B, C, D and the other that passes through A, E and whose center lies on the perpendicular to ℓ at D. Let F be that point of intersection of these circles that lies at the other side of ℓ against E. Prove that ℓ is also the symmedian of $\triangle AEF$ at A.

LeVietAn

#2 Aug 11, 2015, 11:08 am • 1

My solution:
Let $(O), (O')$ be the circumcircle of $\triangle ABC, \triangle AEF$, reps. The tangent lines at B, C of (O) intersect at T.
We have $\ell \equiv AT$ and $\angle BDC = 2\angle BAC = \angle BOC \Rightarrow D \in \odot(BOCT) \Rightarrow \angle ODT = \angle OBT = 90^\circ$
$\Rightarrow O' \in OD \Rightarrow AT \perp OO' \Rightarrow AT$ be the radical axis of (O) and (O'). Let $S = (O) \cap (O'), S \neq A \Rightarrow S \in AT$. Let $TF \cap BC = G, OT \cap BC = M$.
We have $\angle OMG = \angle OFG = 90^\circ \Rightarrow OMFG$ is cyclic $\Rightarrow TF.TG = TM.TO = TB^2 = TS.TA \Rightarrow AEFG$ is cyclic $\Rightarrow \angle EAF = \angle EGF = \angle MGF = \angle MOF = \angle TOF = \angle TDF = \angle SDF \Rightarrow \Delta SDF \sim \Delta EAF$
$\Rightarrow \angle DFS = \angle AFE \Rightarrow EF$ is F-symmedian of $\Delta AFS(\because D$ is midpoint of $AS) \Rightarrow AS$ is A-symmedian of ΔAEF.
DONE

Attachments:



Luis González

#3 Aug 11, 2015, 11:14 am

Let (O) denote the circumcircle of $\triangle ABC$. Tangents of (O) at B, C meet at X and AX cuts (O) again at L. Since $\angle BDC = \angle BOC = 2\angle BAC \Longrightarrow D \in \odot(OBXC) \Longrightarrow OD \perp AL \Longrightarrow$ the circle ω passing through A, E centered at OD passes through $L.$ Let M and N be the second intersections of ω with $\odot(OBC)$ and $BC,$ resp.

Inversion WRT $\odot(X,XB)$ fixes $(O),\omega$ and swaps BC and $\odot(OBC) \Longrightarrow E \in XM$ and $F \in XN.$ Moreover AX,MF,NE concur at the radical center R of $(O),\omega,\odot(OBC).$ Thus if $P \equiv MN \cap EF,$ from the complete cyclic $EFNM,$ we deduce that $\overline{ARX}$ is the polar of P WRT $\omega \Longrightarrow PA,PL$ are tangents of $\omega \Longrightarrow AELF$ is harmonic, i.e. AX is A-symmedian of $\triangle AEF.$

livetolove212

#4 Nov 17, 2015, 9:53 am • 1

Using LeVietAn's figure.
Let AS intersect BC at P. Just note that $(ASPT)=-1$ then $G(ASPT)=-1$. Therefore $(ASEF)=-1.$

Quick Reply

High School Olympiads

The intersection lies on the Euler-circle.

Locked

duyptnk0

#1 Aug 11, 2015, 2:27 am

Let ABC be triangle inscribed circle (O). Let M,N are points on AB,AC respectively such that O lies on MN. Prove that one of the two intersections of $\odot(BN)$ and $\odot(CM)$ lies on the Euler-circle of ABC.

Luis González

#2 Aug 11, 2015, 2:44 am

Discussed many times before, e.g. http://www.artofproblemsolving.com/community/c6h186883. One intersection is the orthopole of MN WRT ABC lying on its 9-point circle.

High School Olympiads

Orthopole

Reply

Source: own

jayme
#1 Feb 5, 2008, 6:42 pm

Dear all,
Let ABC be a triangle, 0 the circumcircle of ABC, O the center of 0, P and Q two points on 0, Q1 the symmetric of Q wrt BC, Q2 the symmetric of Q wrt CA, S1 the point of intersection of PQ1 and BC, S2 the point of intersection of PQ2 and CA.
Prouve that the orthopole of ABC wrt the mediator of PQ, is on S1S2.
Any synthetic proof?
Sorry, I am not good with latex.
Jean-Louis

yetti
#2 Apr 5, 2008, 5:18 pm

Lemma: Let a line through the circumcenter O cut the triangle sidelines BC, CA, AB at $X, Y, Z.$ Circles $(P_a), (P_b), (P_c)$ with diameters AX, BY, CZ are coaxal with 2 common points $K, L.$ The 1st point K lies on the triangle circumcircle (O) and the 2nd point L is orthopole of the line XYZ through $O.$

The circle centers P_a, P_b, P_c lie on the triangle midlines $B'C', C'A', A'B'$, where A', B', C' are midpoints of $(BC), (CA), (AB).$ By Menelaus theorem for the $\triangle ABC$ and its medial $\triangle A'B'C'$,

$$\frac{\overline{P_aB'}}{\overline{P_aC'}} \cdot \frac{\overline{P_bC'}}{\overline{P_bA'}} \cdot \frac{\overline{P_cA'}}{\overline{P_cB'}} = \frac{\overline{XC}}{\overline{XB}} \cdot \frac{\overline{YA}}{\overline{YC}} \cdot \frac{\overline{ZB}}{\overline{ZA}} = 1,$$

which means that P_a, P_b, P_c are collinear. Since AX, BY, CZ are diameters of $(P_a), (P_b), (P_c)$, these circles pass through the altitude feet $D \in BC, E \in CA, F \in AB$ of the $\triangle ABC.$ Therefore, powers of the orthocenter H to the circles $(P_a), (P_b), (P_c)$ are all equal to half the power of H to the circumcircle (O):

$$\overline{HA} \cdot \overline{HD} = \overline{HB} \cdot \overline{HE} = \overline{HC} \cdot \overline{HF} = \frac{1}{2}\ p(H, (O)).$$

Together with the collinear centers, this means that the circles $(P_a), (P_b), (P_c)$ are coaxal. (The coaxality does not depend on the line XYZ passing through the circumcenter O.) Let $(P_a), (P_b), (P_c)$ meet XYZ again at $X', Y', Z'.$ The reflection L of X', Y', Z' in $B'C', C'A', A'B'$ is the orthopole of XYZ through O and it lies on $(P_a), (P_b), (P_c)$ (well known, due to O being the orthocenter of the medial $\triangle A'B'C'$, for example, see the last paragraph of http://www.mathlinks.ro/viewtopic.php?t=173756, or http://www.mathlinks.ro/viewtopic.php?t=184885, etc.) Let the ray $(HL$ of the common radical axis KHL of $(P_a), (P_b), (P_c)$ cut (O) at $M.$ Since the orthopole L lies on the 9-point circle of the $\triangle ABC$, similar to the circumcircle (O) with center H and coefficient $\frac{1}{2}$, it is the midpoint of $(HM).$ Power of H to $(P_a), (P_b), (P_c)$ is

$$\frac{1}{2}\ p(H, (O)) = \overline{HK} \cdot \overline{HL} = \frac{1}{2} \overline{HK} \cdot \overline{HM}.$$

Since $M \in (O)$, it follows that $K \in (O).$ (This last fact will not be used any more.) This proves the lemma.

Let (O_a) be an arbitrary circle centered of the line XYZ through O and passing through X (intersecting the circumcircle (O) or not). The circle (O_a) meets the lines BC, XYZ and the circle (P_a) at $X.$ Let (O_a) meet the line BC again at S_a, the line XYZ again at M_a and the circle (P_a) again at $U.$ Since AX, M_aX are diameters of $(P_a), (O_a), XU \perp UA, UM_a$ and A, U, M_a are collinear. From the cyclic hexagon $AULX'XD$ and cyclic (isosceles) trapezoid $ALX'D$ (because $LX' \parallel AD$, both perpendicular to $B'C' \parallel BC$),

$$\measuredangle LUM_a = \measuredangle LUA = \measuredangle LX'A = \measuredangle LDA = \measuredangle DAX' \mod 180°.$$

Since $AD \perp (BC \equiv X_aD)$ and $AX' \perp (M_aX' \equiv XYZ)$,

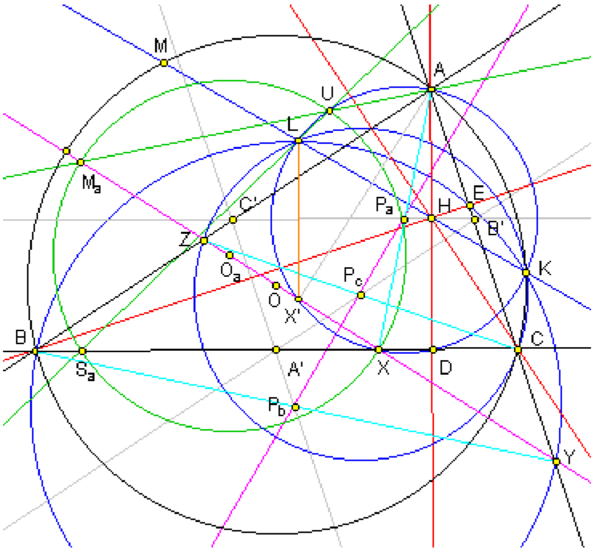
$$\measuredangle DAX' = \measuredangle S_aXM_a \mod 180^\circ,$$

and from the cyclic quadrilateral XS_aM_aU,

$$\measuredangle S_aXM_a = \measuredangle S_aUM_a \mod 180^\circ.$$

Combining, $\measuredangle LUM_a = \measuredangle S_aUM_a \mod 180^\circ$, which means that the points S_a, L, U are collinear. Using the result of http://www.mathlinks.ro/viewtopic.php?t=186477, the posted problem is a particular case of the circle (O_a) intersecting the circumcircle (O) at two arbitrary points P, Q and passing through the intersection X of the perpendicular bisector XYZ of PQ with BC.

Attachments:



yetti

#3 Apr 7, 2008, 12:07 pm

The lemma used in the proof has the following generalization:

Let H be ortocenter of a $\triangle ABC$ and P an arbitrary point in its plane. Let XYZ be the orthotransversal of P with respect to the $\triangle ABC$. The orthopole L of the line XYZ lies on the line PH.

Any synthetic proof ?

darij grinberg

#4 Apr 7, 2008, 3:40 pm

> *yetti wrote:*
> The lemma used in the proof has the following generalization:
>
> Let H be ortocenter of a $\triangle ABC$ and P an arbitrary point in its plane. Let XYZ be the orthotransversal of P with respect to the $\triangle ABC$. The orthopole L of the line XYZ lies on the line PH.

I am not feeling quite well right now, so please excuse me if the following has some mistakes:

Theorem 1. Let P be a point in the plane of a triangle ABC, and let the perpendiculars to the lines AP, BP, CP at the point P meet the lines BC, CA, AB at the points X, Y, Z. Then, the point P, the orthocenter of triangle ABC, and the orthopole of the line XYZ with respect to triangle ABC are collinear.

Proof. This turns out to be pretty easy when considered from the right viewpoint, namely that of complete quadrilaterals:

It is known that the points X, Y, Z lie on one line. Consider the complete quadrilateral q formed by the lines XYZ, BC, CA, AB. After Theorem 2 of [1], the orthocenters of triangles ABC, AYZ, BZX, CXY lie on the Steiner line of this complete quadrilateral q, and after Corollary 4 of [1], the orthopoles of the lines XYZ, BC, CA, AB with respect to the triangles ABC, AYZ, BZX, CXY also lie on the Steiner line of this complete quadrilateral q.

According to the proofs at http://www.mathlinks.ro/Forum/viewtopic.php?t=842 , the Steiner line of a complete quadrilateral coincides with the pairwise radical axis of the circles with the diagonals of the complete quadrilateral as diameters. Hence, the Steiner line of our complete quadrilateral q also happens to be the pairwise radical axis of the circles with diameters AX, BY, CZ (because AX, BY, CZ are the diagonals of the complete quadrilateral q), so that the point P must lie on this Steiner line (because P has equal powers with respect to the circles with diameters AX, BY, CZ (in fact, P has the power 0 with respect to all of these circles, because P lies on each of these circles, since $\measuredangle APX = 90^\circ$, $\measuredangle BPY = 90^\circ$ and $\measuredangle CPZ = 90^\circ$)).

Altogether, we have seen that the Steiner line of the complete quadrilateral q passes through:
- the orthocenters of triangles ABC, AYZ, BZX, CXY;
- the orthopoles of the lines XYZ, BC, CA, AB with respect to the triangles ABC, AYZ, BZX, CXY;
- the point P.

This proves even more than Theorem 1 asked us to show.

References

[1] Atul Dixit and Darij Grinberg, *Orthopoles and the Pappus Theorem*, Forum Geometricorum, 4 (2004) pp. 53-59

darij

Quick Reply

High School Olympiads

Difficult Geometry

Locked

droid347

#1 Aug 11, 2015, 2:22 am

Let ABC be a triangle with incircle Γ, and let D, E, F be the tangency points of Γ with sides BC, CA, AB, respectively. Let K be the orthocenter of triangle ABC. Furthermore, let $\Gamma_A, \Gamma_B, \Gamma_C$ be the circles centered at A, B, C with radii AD, BE, CF, respectively. Prove that K is the radical center of $\Gamma_A, \Gamma_B, \Gamma_C$.

Luis González

#2 Aug 11, 2015, 2:35 am

I'm sure you mean K is the orthocenter of DEF and not ABC. This was posted before at Radical center and line OI (first problem)

High School Olympiads

equilateral triangle and equal angles

Reply

linqaszayi
#1 Aug 10, 2015, 7:53 pm

given a equilateral triangle ABC. let D be a point on edge AB. consider a circle with center D and radius CD. let this circle cut the line CB ,CA respectfully at E, F. let M be the midpoint of CD.
prove that $\angle AEC + \angle BFC = \angle EMF$.

Luis González
#2 Aug 11, 2015, 12:33 am • 2

Let U, V be the midpoints of $CA, CB.$ Since $\triangle CEF$ has $\angle ECF = 60^\circ$, the reflection X of its circumcenter D on EF is clearly midpoint of its circumcircle arc $EF.$ Since $\triangle XDF$ is obviously equilateral and $\angle DAF = 120^\circ$, then $XDAF$ is cyclic $\Longrightarrow \angle DAX = \angle DFX = 60^\circ$ and likewise $\angle DBX = 60^\circ \Longrightarrow \triangle XAB$ is equilateral. Thus $AC = AB = XB,$ $CD = XD = XE$ and since
$\angle CDA = 60^\circ + \angle BCD = 60^\circ + 90^\circ - \angle CFE = 150^\circ - \angle CFE = 30^\circ + \angle CEF = \angle XEB,$ it follows that $\triangle CAD \cong \triangle XBE \Longrightarrow BE = AD = 2 \cdot MU.$ Together with $BA = 2 \cdot UA$ and $\angle ABE = \angle AUM = 120^\circ \Longrightarrow$ $\triangle ABE \sim \triangle AUM$ are spirally similar $\Longrightarrow \angle AEC = \angle AMU$ and $\triangle AEM \sim \triangle ABU$ are spirally similar $\Longrightarrow$ $\angle AME = \angle AUB = 90^\circ.$ In the same way, we have $\angle BFC = \angle BMV$ and $\angle BMF = 90^\circ.$ Therefore $\angle EMF = 180^\circ - \angle AMB = \angle AMU + \angle BMV = \angle AEC + \angle BFC.$

Gryphos
#3 Aug 11, 2015, 3:09 am • 1

Very nice solution! I also tried this problem, but could only prove that $\angle AEC + \angle BFC = 180^\circ - \angle AMB$.
My proof for this uses that ED and EX are both tangent to the circumcircle of cyclic quadrilateral $AFXD$, and hence AE is the A-symmedian of $\triangle AXD$.
If N is the midpoint of DX, this implies $\angle EAD = \angle XAN = \angle MAC$ because C and X are symmetric wrt AB and analogously $\angle DBF = \angle CBM$. Thus,

$$\angle AEC + \angle BFC = 120^\circ - \angle EAD - \angle DBF = 120^\circ - \angle MAC - \angle CBM = \angle BAM + \angle MBA = 180^\circ - \angle AMB.$$

Quick Reply

High School Olympiads

Triangle Center Geometry

Reply

tkhalid

#1 Aug 10, 2015, 11:16 pm

Prove that $X_3X_8 \parallel X_5X_{355}$.

$X_3 = \text{circumcenter}$
$X_5 = \text{Nine Point Center}$
$X_8 = \text{Nagel Point}$
$X_{355} = \text{Fuhrmann Center}$

Luis González

#2 Aug 10, 2015, 11:33 pm

It's known (posted before) that X_{355} is the midpoint between the orthocenter X_4 and X_8. Thus since X_5 is midpoint of X_4X_3, then it follows that $X_3X_8 \parallel X_5X_{355}$.

jayme

#3 Aug 11, 2015, 10:33 am

Dear Mathlinkers,
you can have a look on

http://jl.ayme.pagesperso-orange.fr/Docs/Le%20cercle%20de%20Fuhrmann.pdf p. 8

Sincerely
Jean-Louis

Quick Reply

High School Olympiads

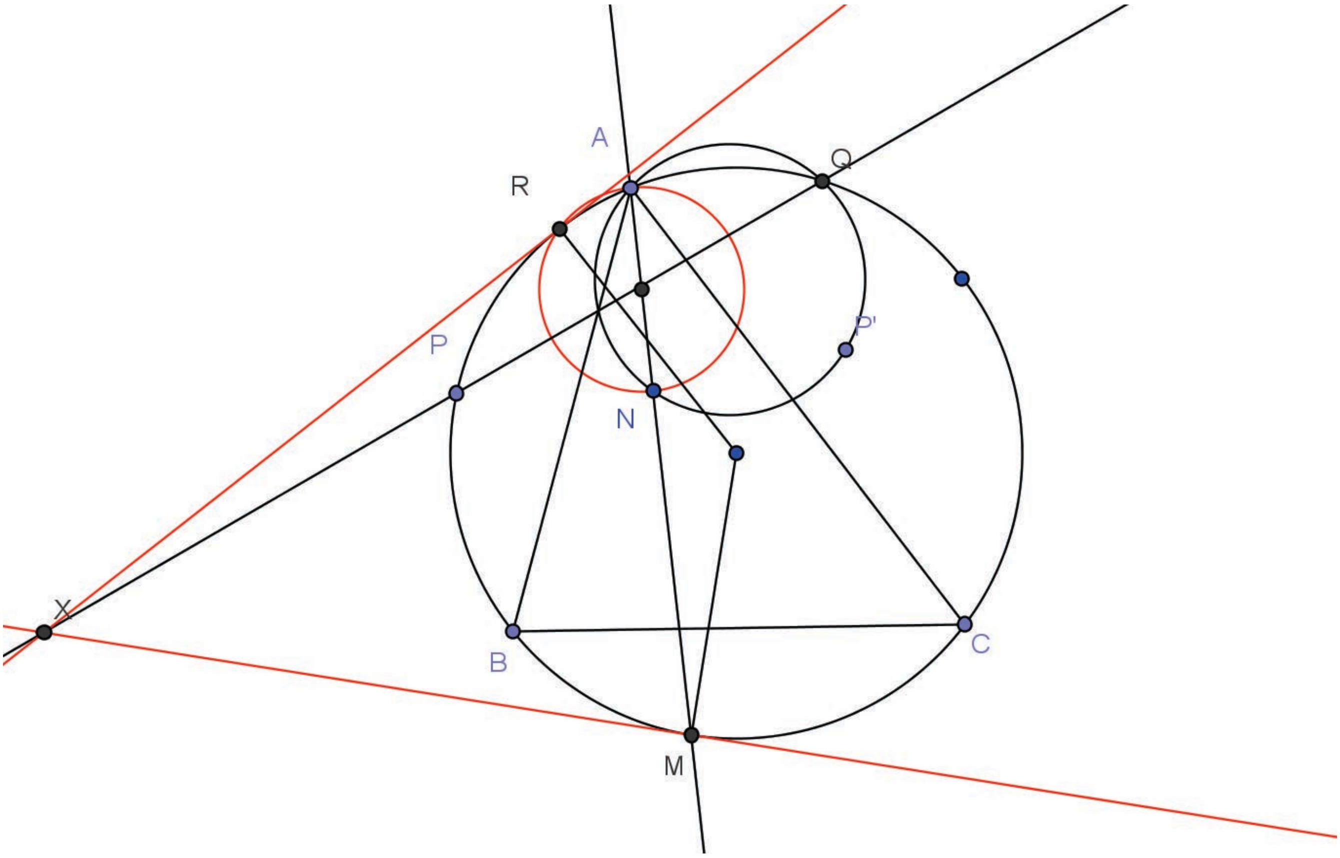
Proof PQ always pass through the fixed point

Reply

kingmathvn
#1 Aug 10, 2015, 6:53 pm

Give triangle ABC, N is a fixed point in triangle ABC, AN cut (ABC) at M (fixed point), P move in (ABC), P' symmetric to P through AN, (ANP') cut (ABC) at Q. Proof PQ always pass through the fixed point.

Attachments:



This post has been edited 1 time. Last edited by kingmathvn, Aug 10, 2015, 7:14 pm
Reason: picture

Luis González
#2 Aug 10, 2015, 9:28 pm

Invert with center A and arbitrary power. P,Q,N go to $P',Q',N' \Longrightarrow \odot(APN)$ and $\odot(AQN)$ go to lines $N'P'$ and $N'Q'$ equidistant from the inversion pole A, because $\odot(APN) \cong \odot(AQN) \Longrightarrow N'A$ bisects $\angle P'N'Q'$. Hence as P',Q' vary, $P' \mapsto Q'$ is an involution on the fixed line $P'Q' \Longrightarrow AP \mapsto AQ$ is an involution $\Longrightarrow P \mapsto Q$ is an involuton on $\odot(ABC)$ $\Longrightarrow PQ$ goes through a fixed point; namely the pole of this involution.

kingmathvn
#3 Aug 10, 2015, 10:01 pm

Thanks Luis Gonzalez, any basic solution???

Quick Reply

High School Olympiads

ellipse-concurrent

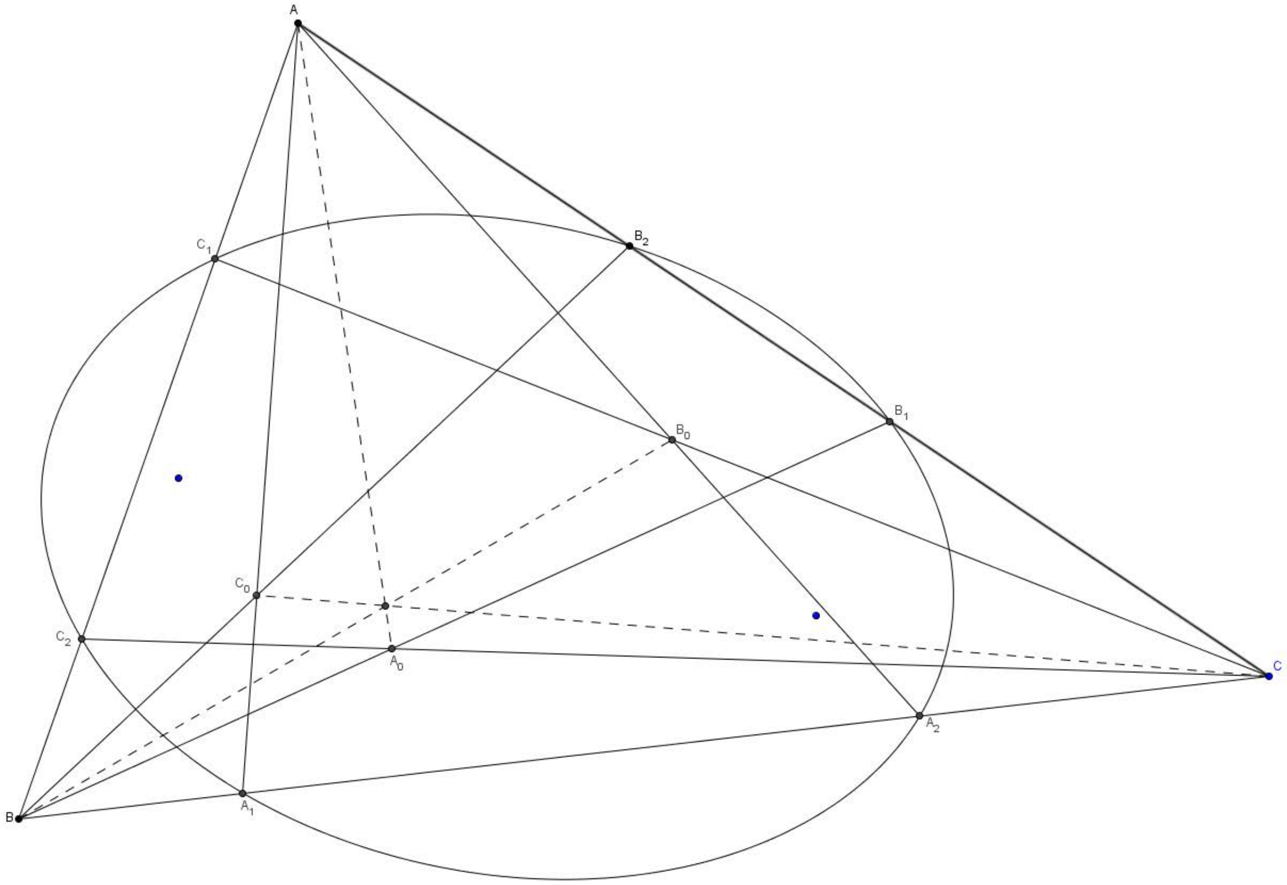
Reply

ferma2000

#1 Aug 10, 2015, 12:07 pm

an ellipse $\mathcal{C}$ intersect sides of $\triangle ABC$ (see picture) in the picture prove AA_0, BB_0, CC_0 are concurrent.

Attachments:



Luis González

#2 Aug 10, 2015, 12:36 pm • 1

AA_0, BB_0, CC_0 cut BC, CA, AB at X, Y, Z and B_1C_2, C_1A_2, A_1B_2 cut BC, CA, AB at X', Y', Z'. From the complete quadrangle BC_2B_1C, it follows that $(B, C, X, X') = -1$ and similarly $(C, A, Y, Y') = -1$, $(A, B, Z, Z') = -1$. But by Pascal theorem for $A_1A_2C_1C_2B_1B_2$, it follows that X', Y', Z' are collinear $\Longrightarrow AA_0 \equiv AX$, $BB_0 \equiv BY, CC_0 \equiv CZ$ concur at the tripole of $\overline{X'Y'Z'}$ WRT $\triangle ABC$.

Quick Reply

High School Olympiads

Two surprising perpendiculars

Reply

jayme
#1 Jun 13, 2013, 2:42 pm • 1

Dear Mathlinkers,

1. ABC a triangle
2. (e) the Euler line of ABC
3. H the orthocenter of ABC
4. P a point on (e)
5. DEF the median triangle of ABC
6. A'B'C' the P-cevian triangle of ABC
7. U the point of intersection of EF and B'C'
8. P* the isogonal of P wrt ABC

Prove that HP* is perpendicular to AU.

Sincerely
Jean-Louis

Luis González
#2 Aug 10, 2014, 9:48 am • 6

Animate the point P on the Euler line e. The perspective pencils BP, CP induce a homography $B' \mapsto C'$ between AC and $AB \Longrightarrow B'C'$ envelopes a fixed conic $\mathcal{C}$ touching AC, AB and EF when P coincides with the centroid $G \equiv BE \cap CF$ $\Longrightarrow P \mapsto U$ is a homography $\Longrightarrow$ the application sending HP to the perpendicular fom H to AU is a homography $\mathbb{H}_1$. Now, since the application $\mathbb{H}_2$ sending P to its isogonal conjugate P^* on a circum-conic is also homographic, then all we have to show is that $\mathbb{H}_1$ and $\mathbb{H}_2$ are identical, in other words that $HP^* \perp AU$ holds for at least three positions of P.

When $P \equiv e \cap AB$, then P^* coincides with C and $AU \equiv AB \Longrightarrow HC \equiv HP^* \perp AB$ and the same happens when $P \equiv e \cap AC$. Finally when $P \equiv e \cap BC$, then P^* coincides with A and U goes to infinity, i.e. $AU \parallel BC \Longrightarrow$ $HA \equiv HP^* \perp AU$. Hence, $\mathbb{H}_1 \equiv \mathbb{H}_2$ as desired. $\blacksquare$

Remark: Since the trilinear polar of P WRT $\triangle ABC$ is parallel to AU (this is easy to prove), then we just got a proof of the following result: If P is a point on the Euler line of $\triangle ABC$, then the line connecting its orthocenter H with the isogonal conjugate of P is perpendicular to the trilinear polar of P.

IDMasterz
#3 Jan 21, 2015, 10:01 pm • 1

Here is my solution to the trilinear polar version that is practically equivalent to Luis' solution.

Let XYZ be the trilinear polar of P w.r.t. ABC. Let the cevian triangle of P w.r.t ABC be $P_AP_BP_C$. Let $OH \cap BC = T$ and let the harmonic conjugate of T w.r.t. BC be T'.

Note that the cross ratio as $P \mapsto P_A \mapsto X$ preserves cross ratio (an involution on BC, or just inversion about the diametre circle of BC). Similarly, $P \mapsto Y$ preserves cross ratio, so $X \mapsto Y$ preserves cross ratio and is thus must be tangent to a fixed conic $\mathcal{C}$. Since the trilinear polar of G, the centroid, is the line at infinity, it follows that $\mathcal{C}$ is also tangent to the line at infinity. Hence, $\mathcal{C}$ is a parabola.

Let the focus of $\mathcal{C}$ be F, and the reflection of F over XY be F_{XY}. Since all circles CXY pass through F, there is a spiral similarity mapping $F_{XY} \mapsto X$ (X moves on the directrix of the parabola). Therefore;

$$F(F_{\ldots} F_{\ldots}; F_{\ldots}, F_{\ldots}) = F(T', C; B, X)$$

$$F(F_{BC}, F_{AC}; F_{AB}, F_{XY}) = F(T', C; B, X)$$

After inversion about circle with diametre BC;

$$F(T', C; B, X) = F(T, C; B, P_A) = A(T, C; B, P)$$

$$\implies F(F_{BC}, F_{AC}; F_{AB}, F_{XY}) = H(A, B; C, P^*)$$

So, $HP^* \parallel F_{XY} \perp XYZ$.

rodinos

#4 Jan 22, 2015, 1:48 am

Rephrasing Luis's remark:

Let P, P* be two isogonal conjugate points. If P moves on the Euler line, then the envelope of the perpendicular from P* to trilinear polar of P is the circle (H, 0) = H

In general, which is the envelope, if P moves on a line L (passing or not passing through H ?

APH

IDMasterz

#5 Jan 22, 2015, 4:55 pm

> *rodinos wrote:*
>
> Rephrasing Luis's remark:
>
> Let P, P* be two isogonal conjugate points. If P moves on the Euler line, then the envelope of the perpendicular from P* to trilinear polar of P is the circle (H, 0) = H
>
> In general, which is the envelope, if P moves on a line L (passing or not passing through H ?
>
> APH

Do you have any idea for the locus? I think it is nice when $G \in \ell$, because then the proof I provided works and there is a projectivity between the line at infinity to P^*.

rodinos

#6 Jan 22, 2015, 5:44 pm

No, I have no idea... but when we do not have idea, we construct as many as possible points (in the case of locus) or lines/curves (in the case of envelope) to get an idea about, thereafter make a conjecture and finally we try a proof.

jayme

#7 Jan 22, 2015, 6:52 pm

Dear Mathlinkers,
I have spend a lot of hours on this problem by searching an elementary proof without success.
Is there a such proof?
Sincerely
Jean-Louis

IDMasterz

#8 Jan 22, 2015, 10:35 pm

> jayme wrote:
> Dear Mathlinkers,
> I have spend a lot of hours on this problem by searching an elementary proof without success.
> Is there a such proof?
> Sincerely
> Jean-Louis

I know of a synthetic proof, but it uses a lot of cross ratio. The two solutions already presented are much cleaner :)

IDMasterz
#9 Jan 22, 2015, 10:36 pm

> rodinos wrote:
> No, I have no idea... but when we do not have idea, we construct as many as possible points (in the case of locus) or lines/curves (in the case of envelope) to get an idea about, thereafter make a conjecture and finally we try a proof. :)

:) Unfortunately, I can't test this because I don't really have a program to make the envelope, but I believe the nicest locus will be when the line contains the centroid.

rodinos
#10 Jan 22, 2015, 10:57 pm

I will ask Telv in FB for a figure :)

TelvCohl
#11 Jan 9, 2016, 4:05 am • 2

> Luis González wrote:
> Remark: Since the trilinear polar of P WRT $\triangle ABC$ is parallel to AU (this is easy to prove), then we just got a proof of the following result: If P is a point on the Euler line of $\triangle ABC$, then the line connecting its orthocenter H with the isogonal conjugate of P is perpendicular to the trilinear polar of P.

Generalization : Given a $\triangle ABC$ and a fixed point M. Let (P, P^*) be the isogonal conjugate WRT $\triangle ABC$ and let τ be the trilinear polar of P^* WRT $\triangle ABC$. If $\measuredangle(\tau, PM) = \theta$ (fixed), then P lies on a fixed conic $\Gamma_{(M,\theta)}$ which passes through M and the symmedian point K of $\triangle ABC$.

Proof : Actually, this simplified follows from the construction of P for given M and θ. Let ℓ be a line passing through M and let ℓ^* be the line passing through M such that $\measuredangle(\ell^*, \ell) = \theta$. If P is the point on ℓ such that $\tau \parallel \ell^*$, then P^*, the trilinear pole Q^* of ℓ^* WRT $\triangle ABC$ and the Centroid G of $\triangle ABC$ lie on a circumconic of $\triangle ABC$, so P, K and the isogonal conjugate Q of Q^* WRT $\triangle ABC$ are collinear. Conversely, if P is the intersection of ℓ and KQ, then we can prove $\tau \parallel \ell^*$ similarly.

When ℓ varies around M, it's well-known that Q^* varies on the circumconic $\mathcal{C}$ of $\triangle ABC$ with perspector $M \Longrightarrow Q$ varies on the trilinear polar of the isogonal conjugate of M WRT $\triangle ABC$ WRT $\triangle ABC$ (isogonal conjugate of $\mathcal{C}$ WRT $\triangle ABC$). Since pencil $\ell^* \mapsto$ pencil AQ^* and pencil $AQ^* \mapsto$ pencil AQ are homography, so pencil $\ell \mapsto$ pencil KQ is a homography $\Longrightarrow$ their intersection P varies on a fixed conic $\Gamma_{(M,\theta)}$ passing through M and K.

Remark :

Property of $\Gamma_{(M,90^\circ)}$: $\Gamma_{(M,90^\circ)}$ and the Jerabek hyperbola $\mathcal{J}$ of $\triangle ABC$ are homothetic.

Proof : Let U, V be the intersection of the Euler line of $\triangle ABC$ and $\odot(ABC)$. It suffices to prove the direction of the isogonal conjugate of U (at infinity) WRT $\triangle ABC$ is perpendicular to the trilinear polar $\mathcal{T}_U$ of U WRT $\triangle ABC \Longleftrightarrow \mathcal{T}_U$ is parallel to the Steiner line $\mathcal{S}_U$ of U WRT $\triangle ABC$.

From Nice concurrent problem $\Longrightarrow \mathcal{T}_U \cap \mathcal{S}_U$ lies on the orthotransversal of U WRT $\triangle ABC$, so combine the problem Intersect on Jerabek hyperbola we get $\mathcal{T}_U \cap \mathcal{S}_U$ lies on $\mathcal{J}$, hence notice $\mathcal{S}_U$ is parallel to the asymptote of $\mathcal{J}$ (the simson line of U WRT $\triangle ABC$) we conclude that $\mathcal{T}_U \parallel \mathcal{S}_U$.

Some special cases :

(1) When M coincide with the circumcenter O of $\triangle ABC$ and $\theta = 90^\circ$:

Given a $\triangle ABC$ with circumcenter O. Let I, I_a, I_b, I_c be the incenter, A-excenter, B-excenter, C-excenter of $\triangle ABC$, resp. and let $\mathcal{H}$ be the conic passing through O and these four points. Let P be a point on $\mathcal{H}$ and let Q be the isogonal conjugate of P WRT $\triangle ABC$. Then OP is perpendicular to the trilinear polar of Q WRT $\triangle ABC$.

(2) When M coincide with the orthocenter H of $\triangle ABC$ and $\theta = 90^\circ$:

Given a $\triangle ABC$ with orthocenter H. Let P be a on the Euler line of $\triangle ABC$ and let Q be the isogonal conjugate of P WRT $\triangle ABC$. Then HQ is perpendicular to the trilinear polar of P WRT $\triangle ABC$.

Quick Reply

High School Olympiads

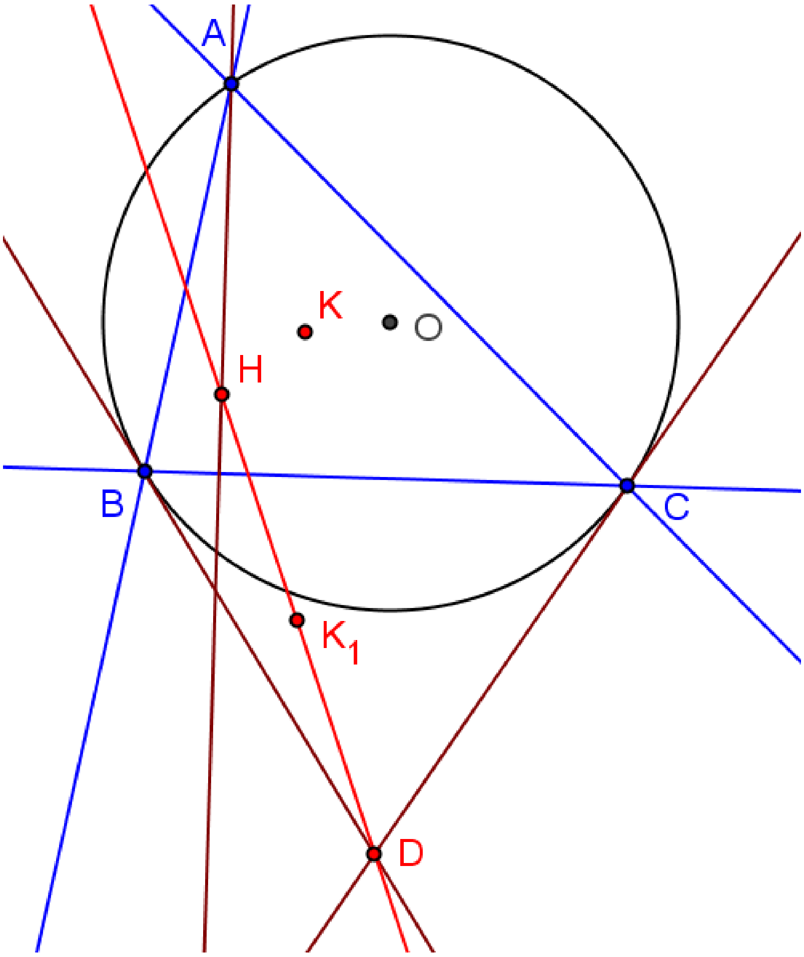
H,K_1,D are collinear.

THVSH

#1 Aug 9, 2015, 11:11 pm • 3

Let ABC be a triangle with circumcircle $\odot(O)$, orthocenter H and the **Kosnita** point K. K_1 is reflection of K in BC. The tangents of $\odot(O)$ at B,C intersect at D.
Prove that H,K_1,D are collinear.

Attachments:



Luis González

#2 Aug 10, 2015, 6:59 am • 1

Let N be the 9-point center of $\triangle ABC$. If S is the reflection of O on BC and U is the midpoint of OD (circumcenter of $\triangle OBC$), then it's well-known that $N\in AS$ and $K\in AU$. If X,Y,Z be the projections of K on BC,CA,AB and T is the orthocenter of $\triangle XYZ$, then O,K,T and $\overline{KT}:\overline{KO}=-1:2$ (see the thread HK passes through the circumcenter).
Hence, if M is the midpoint of BC, $E\equiv XT\cap OM$ and $J\equiv KX\cap AS$, we obtain

$$XK=\tfrac{1}{3}OE=\tfrac{1}{3}(OS+SE)=\tfrac{1}{3}(AH+XJ)\Longrightarrow$$

$$XK=\tfrac{1}{3}AH+\tfrac{1}{3}(XK+KJ)\Longrightarrow AH+KJ=2\cdot XK=KK_1.$$

But $KJ=\frac{AK}{AU}\cdot US$ (due to $KJ\parallel SU$) $\Longrightarrow KK_1=AH+US\cdot\frac{AK}{AU}$.

Combining with $DU-US=OU-US=OS=AH$ the latter expression becomes $KK_1=\frac{AH}{AU}\cdot KU+\frac{DU}{AU}\cdot AK,$ which means that in the trapezoid $AHDU,$ the points H,D,K_1 are collinear.

TelvCohl

#3 Apr 13, 2016, 3:19 am • 1

Let N be the 9-point center of $\triangle ABC$ and let $E\equiv BK\cap\odot(O), F\equiv CK\cap\odot(O).$ Obviously, $\triangle BNC$ and $\triangle FAE$ are directly similar, so if T is the second intersection of AH with $\odot(O),$ then we get $\measuredangle FAT=\measuredangle KCT=\measuredangle(AO,CN),$ hence AT is the A-symmedian of $\triangle FAE$ ($\because$ the N-median of $\triangle BNC$ is parallel to AO) **......** $(\star).$

Let $U\equiv TK\cap\odot(O)$ and let $V\in\odot(O)$ be the point such that $UV\parallel BC.$ Since

$$A(V,N;B,C)=A(U,K;C,B)=(T,A;F,E)\overset{(\star)}{=}-1,$$

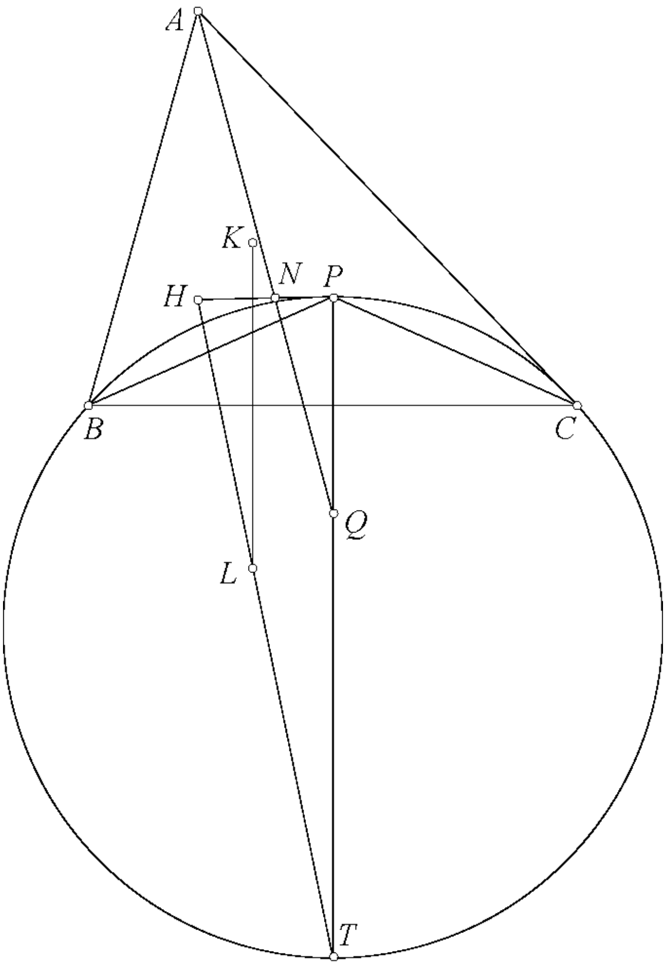
so notice $\measuredangle(AH,HK_1)=\measuredangle UTA=\measuredangle UVA\Longrightarrow AV\perp HK_1$ we get $(\perp HK_1,AN;AB,AC)=-1$ **......** $(\spadesuit).$ On the other hand, since AN is the Euler line of $\triangle BHC,$ so from AF perpendicular to Euler line we get $(\perp HD,AN;AB,AC)=(HD,\perp AN;HC,HB)=-1,$ hence combine $(\spadesuit)$ we conclude that D,H,K_1 are collinear.

buratinogigle

#4 Apr 13, 2016, 2:07 pm • 1

General problem. Let ABC be a triangle with orthocenter H. P is a point on perpendicular bisector of BC. Q is reflection of P through BC. AQ cuts HP at N. K is isogonal conjugate of N with respect to triangle ABC. L is reflection of K through BC. Prove that HL and PQ intersect on (PBC).

Attachments:



TelvCohl

#5 Apr 13, 2016, 6:47 pm • 2

> *buratinogigle wrote:*
>
> **General problem.** Let ABC be a triangle with orthocenter H. P is a point on perpendicular bisector of BC. Q is reflection of P through BC. AQ cuts HP at N. K is isogonal conjugate of N with respect to triangle ABC. L is

reflection of K through BC. Prove that HL and PQ intersect on (PBC).

From my proof at post #3 we get $(\perp HL, AN; AB, AC) = -1,$ so it suffices to prove $(\perp HT, AN; AB, AC) = -1.$

Let D be the antipode of H in $\odot(BHC).$ Since $P \mapsto T$ is a homography when P varies on the perpendicular bisector of $BC,$ so notice $AN \parallel DP \Longrightarrow$ it suffices to prove $(HT, \perp DP; HC, HB) = -1$ ... $(\star)$ for three positions of $P.$

When P coincide with the midpoint of BC, T coincide with the infinity point on the H-altitude of $\triangle BHC \Longrightarrow (\star)$ holds. When P coincide with the midpoint of arc BC in $\odot(BHC), DP$ is parallel to the bisector HT of $\angle BHC \Longrightarrow (\star)$ holds.

Quick Reply

High School Olympiads

Dual of the Maxwell theorem

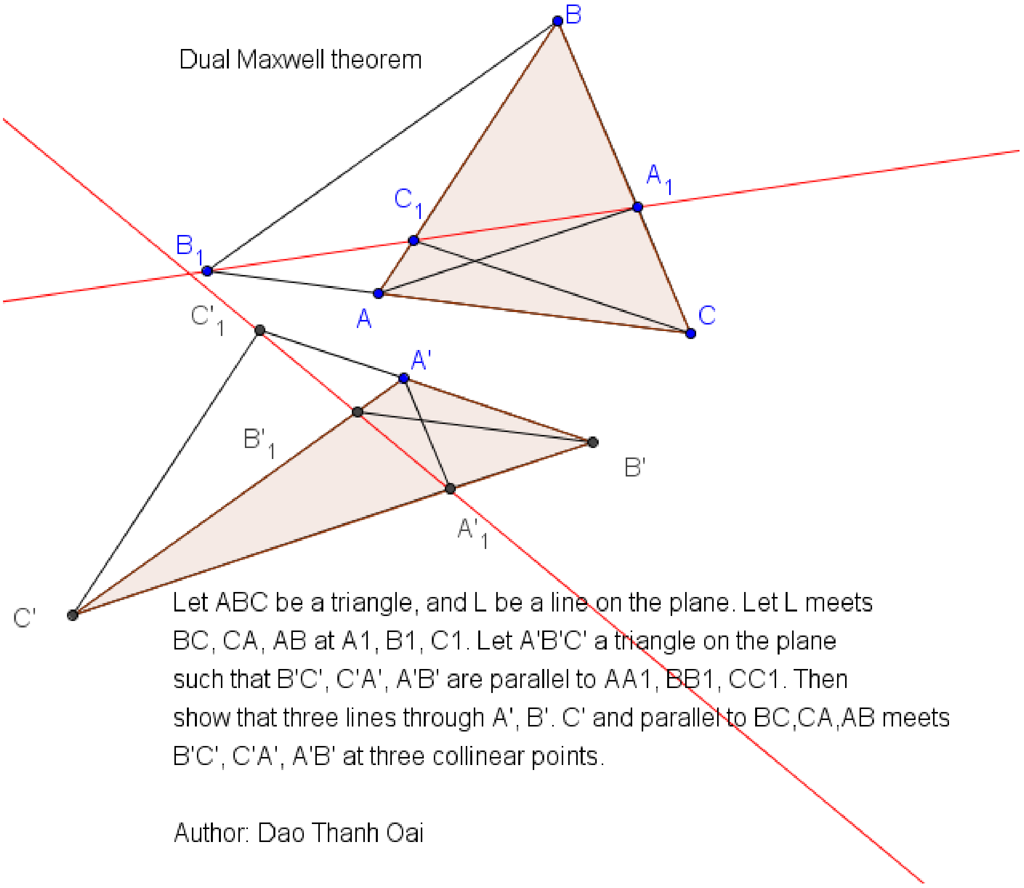
Reply

daothanhoai

#1 Aug 9, 2015, 10:24 pm

Let ABC be a triangle, and L be a line on the plane. Let L meets BC, CA, AB at A_1, B_1, C_1. Let A'B'C' a triangle on the plane such that $B'C', C'A', A'B'$ are parallel to AA_1, BB_1, CC_1. Then show that three lines through A', B', C' and parallel to BC, CA, AB respectively meets $B'C', C'A', A'B'$ at three collinear points.

Attachments:



Luis González

#2 Aug 10, 2015, 4:30 am • 1

$$\frac{A_1'B'}{A_1'C'} = \frac{\sin \widehat{B'A'A_1'}}{\sin \widehat{C'A'A_1'}} \cdot \frac{A'B'}{C'A'} = \frac{\sin \widehat{BCC_1}}{\sin \widehat{CBB_1}} \cdot \frac{A'B'}{C'A'} \quad (1).$$

Similarly we have the expressions

$$\frac{B_1'C'}{B_1'A'} = \frac{\sin \widehat{CAA_1}}{\sin \widehat{ACC_1}} \cdot \frac{B'C'}{A'B'} \quad (2)\ , \quad \frac{C_1'A'}{C_1'B'} = \frac{\sin \widehat{ABB_1}}{\sin \widehat{BAA_1}} \cdot \frac{C'A'}{B'C'} \quad (3).$$

Multiplying (1), (2) and (3) gives

$$\frac{A_1'B'}{A_1'C'} \cdot \frac{B_1'C'}{B_1'A'} \cdot \frac{C_1'A'}{C_1'B'} = \frac{\sin \widehat{CAA_1}}{\sin \widehat{BAA_1}} \cdot \frac{\sin \widehat{BCC_1}}{\sin \widehat{ACC_1}} \cdot \frac{\sin \widehat{ABB_1}}{\sin \widehat{CBB_1}} = \frac{A_1C}{A_1B} \cdot \frac{B_1A}{B_1C} \cdot \frac{C_1B}{C_1A}.$$

Hence, by Menelaus' theorem we conclude that A_1, B_1, C_1 are collinear $\Longleftrightarrow A_1', B_1', C_1'$ are collinear.

Quick Reply

High School Olympiads

I is incenter of AQR

Reply

Source: Own

buratinogigle

#1 Aug 10, 2015, 12:55 am

Let ABC be a triangle inscribed in circle (O) with incenter I. P is a point such that $PI\perp IA$ and Q is isogonal conjugate of P. AP cuts (O) again at R. Prove that I is incenter of triangle AQR.

Luis González

#2 Aug 10, 2015, 2:04 am

If QR cuts BC at D, then $PD \parallel AQ$ (see Isogonal points and parallelism, An extension of a property from mixtilinear incircle and elsewhere) $\Longrightarrow PD$ is reflection of PA on $PI \Longrightarrow PD$ touches the circle (I) centered at I and tangent to AP, AQ. But from Goormaghtigh theorem, the reflections of PA, QA on IP, IQ, resp, concur on BC for any pair of isogonal conjugates $P, Q \Longrightarrow QD$ touches $(I) \Longrightarrow (I)$ is incircle of $\triangle AQR$.

Quick Reply

High School Olympiads

Isogonal points and parallelism

Reply

Source: very very difficult

jayme

#1 Dec 5, 2007, 3:26 pm

Let ABC be a triangle, 1 the circumcircle of ABC, P a point, P* the isogonal of P, A' the second intersection of AP* with 1, and M the intersection of A'P and BC.
Prouve: P*M is parallel to AP.

If this problem has being posted, I ask the moderator to put it in his right place. Thank you.
Jean-Louis

darij grinberg

#2 Dec 5, 2007, 3:34 pm

It was posted at http://www.mathlinks.ro/viewtopic.php?t=46718 , but the shortness of Grobber's solution doesn't make the problem easier. I will try to look for something synthetic when I have got time.

darij

Leonhard Euler

#3 Dec 5, 2007, 8:55 pm

Equivalent problem: Let M be the point on BC such that $P^{'}M \parallel AP$.Show $P, M, A^{'}$ are collinear.

Denot X as intersection of $A^{'}B$ and line that pass $P^{'}$ and parrallel to AB and denote Y as intersection of $A^{'}C$ and line that pass $P^{'}$ and parrallel to AC. Since $P^{'}X \parallel AB, P^{'}M \parallel AP$, we have $\angle XP^{'}M = \angle BAP = \angle P^{'}AC = \angle A^{'}BC = \angle XBM$. Hence, $P^{'}, M, X, B$ are cyclic. Then $\angle XMB = \angle XP^{'}B = \angle ABP^{'} = \angle PBM$. Therefore, $MX \parallel PB$ and similarly $MY \parallel PC$. Since $P^{'}X \parallel AB$, we have $\frac{XB}{A^{'}X} = \frac{P^{'}A}{A^{'}P^{'}} = \frac{YC}{A^{'}Y}$.So $XY \parallel BC$. Therefore, $\triangle MXY$ and $\triangle PBC$ are homothetic. Since BX and CY meet at $A^{'}$, PM pass $A^{'}$.ie, $P, M, A^{'}$ are collinear.

jayme

#4 Dec 5, 2007, 11:52 pm

Very nice proof for your proposition.
I reformulate the problem :
Let ABC be a triangle, 1 the circumcircle of ABC, P a point, P* the isogonal of P, A' the second intersection of AP* and 1, and finally M a point on BC.
[P, M and A' are colinear] if, and only if, [P*M is parallel to AP].
You prouve (only if) and I suppose you prouve (if) with reductio ad absurdum i.e. with an indirect reasoning.
I omit to ask for a direct synthetic proof of (if).
I am sure that this proof will be more complicated than yours, but I hope that it will perhaps teach some point of view...
Thanks again.
Jean-Louis.

armpist

#5 Dec 6, 2007, 2:42 am

When we involve a point on the circumcircle, like A' here, it's conjugate is at infinity (the favorite place of mine). I will give you directions, so that when you send there a line, or whatever you want, you know where to look for it later.

The direction is orthogonal to the Simson line of A'.

Thank you.
M.T.

jayme

#6 Feb 3, 2010, 4:27 pm

Dear Mathlinkers,
you can see for example
http://perso.orange.fr/jl.ayme/ vol. 5 Le P-cercle de Hagge p. 55.
Sincerely
Jean-Louis

vittasko

#7 Feb 7, 2010, 2:56 am • 1

Let $D,\ E$ be, the points of intersections of the angle bisector of $\angle B,$ from the line segments $AP,\ AP'$ respectively and also let F be, the point of intersection of $BC,$ from the line trhough the point D and parallel to $AP'.$

Based on the below **Lemma**, we have that the points $A',\ F,\ E,$ are collinear.

We denote the points $K\equiv AP'\cap BP$ and $L\equiv BC\cap AP'$ and because of the line segments $BP,\ BP'$ are isogonal conjugates with respect to the angle $\angle B,$ it is easy to show that $(A,E,P',L) = (L,E,K,A)\ ,(1)$

(The pencils $B.AEP'L,\ B.LEKA$ have equal **Double Ratios**, because of the angles formed by their homologous rays are equal).

But, $(A,E,P',L) = (Z,F,M,L)\ ,(2)$ and $(L,E,K,A) = (Z,D,P,A\ ,(3)$ where $Z\equiv BC\cap AP.$

From $(1),\ (2),\ (3)\Longrightarrow (Z,F,M,L) = (Z,D,P,A)\ ,(4)$

From (4) we conclude that the line segments $DF,\ PM,\ AL$ are parallels because of $DF\parallel AL$ and the proof is completed.

LEMMA - A triangle $\triangle ABC$ with circumcircle (O) is given and let $D,\ E$ be, the points of intersection of its B-angle bisector, from two arbitrary lines through vertex A and isogonal conjugates with respect to $\angle A.$ The line through the point D and parallel to $AE,$ intersects BC at point so be it F and let be the point $Z\equiv (O)\cap AD.$ Prove that the points $E,\ F,\ Z,$ are collinear.

Kostas Vittas.

PS. I will post here next time, the proof of the above **Lemma** I have in mind.

Attachments:
t=177608.pdf (5kb)

vittasko

#8 Feb 10, 2010, 1:33 am • 1

> *vittasko wrote:*
> **LEMMA - A triangle $\triangle ABC$ with circumcircle (O) is given and let $D,\ E$ be, the points of intersection of its B-angle bisector, from two arbitrary lines through vertex A and isogonal conjugates with respect to $\angle A.$ The line through the point D and parallel to $AE,$ intersects BC at point so be it F and let be the point $Z\equiv (O)\cap AD.$ Prove that the points $E,\ F,\ Z,$ are collinear.**

PROOF OF THE LEMMA - It is enough to prove that $\frac{DF}{AE} = \frac{ZD}{ZA}\ ,(1)$ and let be the point $P\equiv BC\cap AD.$

Through the vertex A of $\triangle ABC,$ we draw the line parallel to$BZ,$ which intersects $BE,$ at point so be it $Q.$

It is easy to show that $AE = AQ\ ,(2)$ from

$\angle AQE = \angle QBZ = \dfrac{\angle B}{2} + \angle ZBC = \dfrac{\angle B}{2} + \angle ZAC = \dfrac{\angle B}{2} + \angle BAE = \angle AEQ \Longrightarrow \angle AQE = \angle AEQ.$

From $AQ \parallel BZ \Longrightarrow \dfrac{ZD}{ZA} = \dfrac{BD}{BQ}, (3)$

From the similar triangles $\triangle BDF,\ \triangle BQA \Longrightarrow \dfrac{DF}{AQ} = \dfrac{BD}{BQ}, (4)$

From $(3),\ (4) \Longrightarrow \dfrac{DF}{AQ} = \dfrac{ZD}{ZA}, (5)$

From $(2),\ (5) \Longrightarrow , (1)$ and the proof of the **Lemma** is completed.

Kostas Vittas.

Attachments:
t=177608(a).pdf (5kb)

vittasko

#9 Feb 10, 2010, 8:52 pm

> *vittasko wrote:*
> **LEMMA. - A triangle $\triangle ABC$ with circumcircle (O) is given and let $D,\ E$ be, the points of intersection of its B-angle bisector, from two arbitrary lines through vertex A and isogonal conjugates with respect to $\angle A$. The line through the point D and parallel to AE, intersects BC at point so be it F and let be the point $Z \equiv (O) \cap AD$. Prove that the points $E,\ F,\ Z$, are collinear.**

PROOF OF THE LEMMA. - (By a friend of mine **Andreas Varverakis**).

● Let be the points $Z \equiv (O) \cap AD$ and $F \equiv BC \cap EZ$ and we will prove that $DF \parallel AE.$

We draw the line through the point D and parallel to AB, which intersects BZ at point so be it N and we denote the points $K \equiv BC \cap AE$ and $H \equiv (O) \cap AE.$

It is easy to show that $BK,\ DN$, are two homologous line segments of the similar triangles $\triangle BEH,\ \triangle DBZ$ respectively and so, we have that $\dfrac{EK}{KH} = \dfrac{BN}{NZ}, (1)$

But, from $FK \parallel ZH$ and $DN \parallel AB$ we have $\dfrac{EK}{EH} = \dfrac{EF}{FZ}, (2)$ and $\dfrac{BN}{NZ} = \dfrac{AD}{DZ}, (3)$

From $(1),\ (2),\ (3) \Longrightarrow \dfrac{EF}{FZ} = \dfrac{AD}{DZ} \Longrightarrow DF \parallel AE$ and the proof is completed.

● Many thanks to **Andreas.**

Kostas Vittas.

Attachments:
t=177608(b).pdf (5kb)

Mashimaru

#10 Feb 18, 2010, 8:51 pm

I have a simpler solution, or at least, a shorter one. Since a button on my keyboard is not good, I will replace P^* by P'.

Let $B' \equiv BP' \cap (O)$. We have $\angle B'A'C = \angle B'BC = \angle ABP$ and $\angle A'B'C = \angle A'AC = \angle PAB$ so $\triangle ABP$ and $\triangle B'A'C$ are similar in the same direction. Thus $\dfrac{BP}{BA} = \dfrac{A'C}{A'B'} = \dfrac{\sin \angle A'BC}{\sin \angle A'BP'}$. From this, multiply the same thing to both sides, we deduce that $\dfrac{BP \cdot \sin \angle PBC}{BA' \cdot \sin \angle A'BC} = \dfrac{BA \cdot \sin \angle ABP'}{BA' \cdot \sin \angle A'BP'}$, which is equivalent to $\dfrac{S_{BMP}}{S_{BMA'}} = \dfrac{S_{BP'A}}{S_{BP'A'}}$ or $\dfrac{MP}{MA'} = \dfrac{P'A}{P'A'}$, from which we directly have $AP \parallel P'M$.

Quick Reply

High School Olympiads

An extension of a property from mixtilinear incircle

Reply

Source: Own

buratinogigle

#1 Dec 28, 2014, 12:40 am

Let ABC be a triangle inscribed circle (O) and P, Q are two isogonal conjugate points. AP cuts (O) again at D. DE is diameter of (O). EQ cuts (O) again at F. Prove that BC, DF and the line passing through P and perpendicular to AQ are concurrent.

Luis González

#2 Dec 28, 2014, 4:41 am • 3

Let AQ cut (O) again at D' and let $X \equiv DF \cap BC$. If we fix AD, AD', the relation between P and the intersection of BC with the perpendicular from P to AD' is obviously a perspectivity and $EP \barwedge EQ \equiv EF \barwedge DF \equiv DX$. So it suffices to show that the desired concurrency holds for at least 3 positions of P.

When $P \equiv AD \cap BC$, then $X \equiv P$ and $Q \equiv A \equiv F \Longrightarrow$ the concurrency trivially holds. When P is at infinity, then $Q \equiv F \equiv D' \Longrightarrow DF \parallel BC \Longrightarrow X$ is at infinity $\Longrightarrow$ the concurrency holds. Finally, when $P \equiv A$, then $Q \equiv AD' \cap BC$ and we have $\angle FAD' = \angle FDD' = \angle FXQ \Longrightarrow AXFQ$ is cyclic $\Longrightarrow \angle XAQ = \angle DFQ = 90^\circ$, i.e. $AX \perp AQ \Longrightarrow$ the concurrency holds.

TelvCohl

#3 Dec 28, 2014, 7:54 am • 3

My solution:

Let X be the projection of P on AQ .
Let Y be the projection of Q on BC .
Let $Z = AQ \cap \odot(ABC), T = YZ \cap \odot(ABC), U = AP \cap BC, V = DF \cap BC$.

It's suffices to prove $V \in PX$

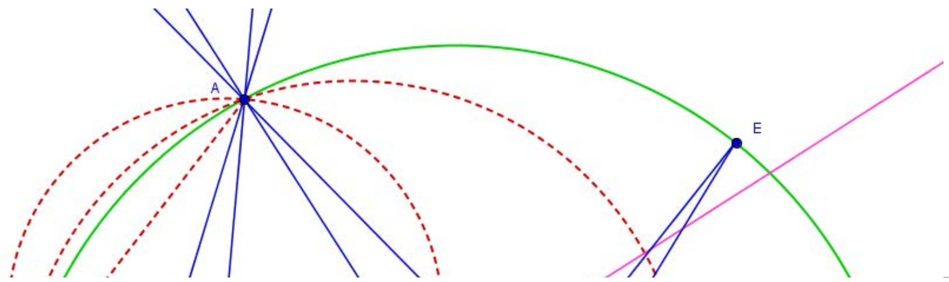
Easy to see F, Q, V, Y lie on a circle with diameter VQ .

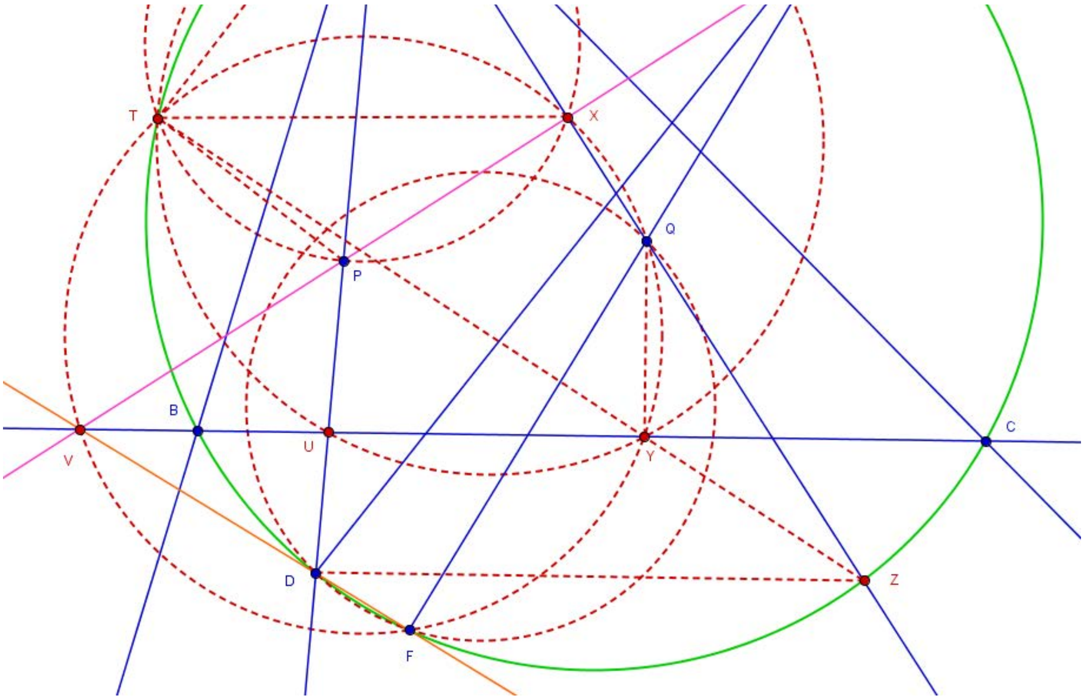
From Reim theorem (for $T-Y-Z$ and $A-U-D$) we get A, T, U, Y are concyclic .
From my post at Concurrent with PQ line (lemma 3) we get A, T, P, X all lie on $\odot(AP)$.
From Reim theorem (for $F-D-V$ and $T-Z-Y$) we get $T \in \odot(VQ)$ . ... $(\star)$

Since $\angle AXT = 90^\circ - \angle TAP = 90^\circ - \angle TYU = \angle QYT$,
so we get T, Q, X, Y are concyclic ,
hence combine with $(\star)$ we get F, T, Q, V, X, Y all lie on $\odot(VQ)$,
so we get $\angle AXV = 90^\circ = \angle AXP$. ie. X, P, V are collinear

Q.E.D

Attachments:





buratinogigle

#4 Feb 4, 2015, 2:18 pm • 2

Lemma. Let ABC be a triangle inscribed circle (O). P, Q are isogonal conjugate points. AP cuts (O) agian at M. QM cuts BC at E then $PE \parallel AQ$.

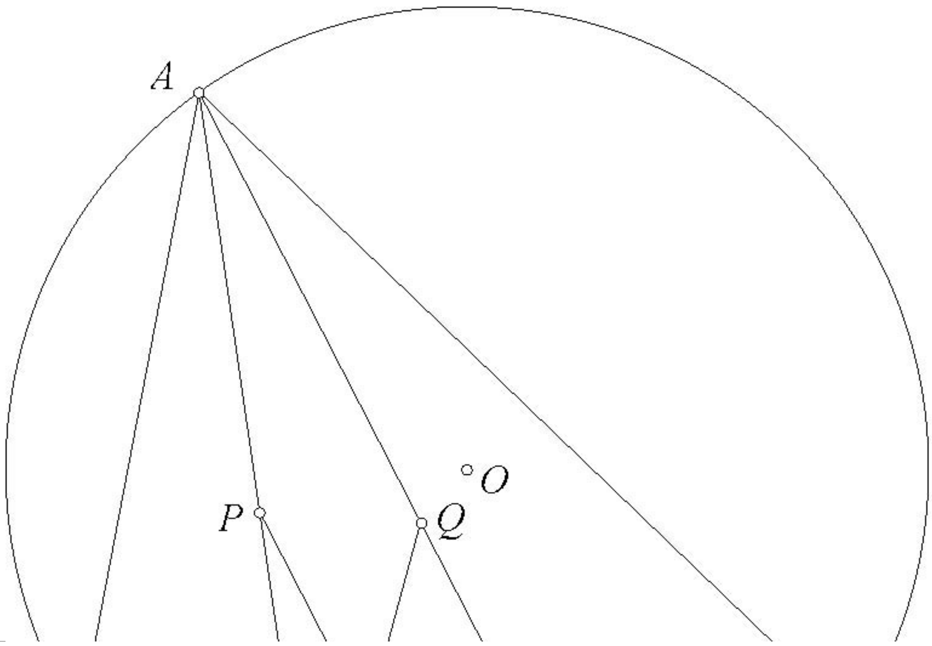
Proof by my pupil **Phan Anh Quan**. AQ cuts (O) again at N and cut BC at H. From P, Q are isogonal conjugate we have $\triangle CHN \sim \triangle ACM$ and $\triangle CPM \sim \triangle QCN$ (g.g) deduce $HN.AM = CM.CN = QN.PM$, therefore $\frac{MP}{MA} = \frac{NH}{NQ} = \frac{ME}{MQ}$, thus $PE \parallel AQ$. We are done.

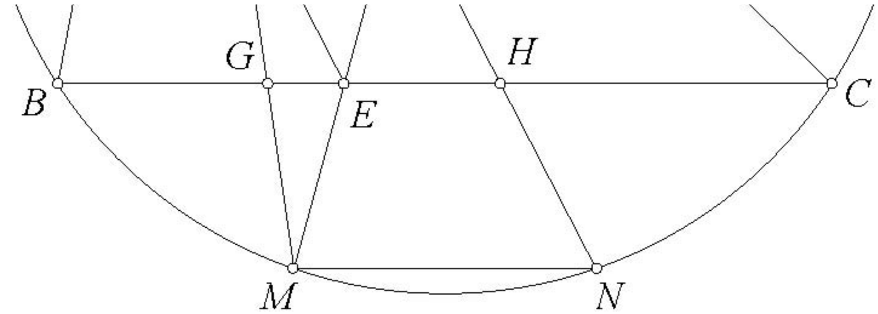
Proof of the problem by my pupil **Trinh Huy Vu**. Let FD cut BC at S, we will prove that $PS \perp AQ$, ineed. Let QD cuts BC at R. T is projection of R on FD. From the lemma, $PR \parallel AQ$. From this, $\frac{DT}{DF} = \frac{DR}{DQ} = \frac{DP}{DA}$ deduce $PT \parallel AF$. We have angle charsing,

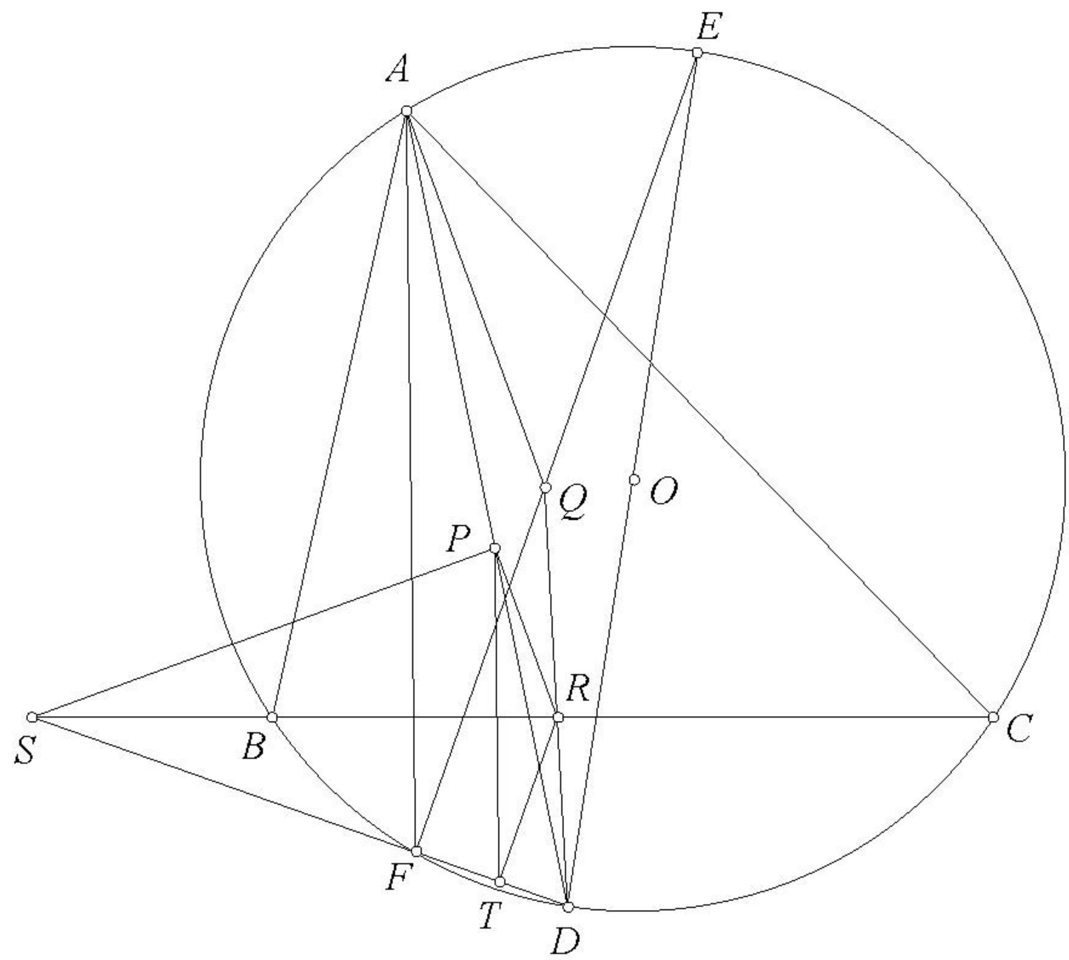
$$\angle TPR = \angle FAQ = \angle BAQ - \angle BAF = \angle PAC - \angle BDF = \angle DBC - \angle BDF = \angle TSR.$$

Therfore, $STRP$ is cyclic or $SP \perp PR \parallel AQ$. We are done.

Attachments:







shinichiman

#5 Feb 9, 2015, 9:44 am • 2

> *buratinogigle wrote:*
>
> Let ABC be a triangle inscribed circle (O) and P, Q are two isogonal conjugate points. AP cuts (O) again at D. DE is diameter of (O). EQ cuts (O) again at F. Prove that BC, DF and the line passing through P and perpendicular to AQ are concurrent.

AQ cuts (O) again at M, DF cuts BC at T. Since P, Q are isogonal conjugates wrt $\triangle ABC$ so $\triangle DBP \sim \triangle MQB$ (A.A). Hence $MQ \cdot DP = MB \cdot DB$. We can also prove that $\triangle MFB \sim \triangle DBT$ (A.A) so $MB \cdot DB = MF \cdot DT$. Thus, $MF \cdot DT = MQ \cdot DP$ or $\dfrac{MQ}{MF} = \dfrac{DT}{DP}$. From here we get $\triangle MQF \sim \triangle DTP$ (S.A.S). Therefore $\angle DPT = \angle MFQ = \angle MAE = 90^\circ - \angle PAQ$. This follows $TP \perp AQ$.

buratinogigle

#6 May 10, 2015, 9:40 pm • 2

A better extension

Let ABC be a triangle inscribed circle (O) and P, Q are two isogonal conjugate point. AP cuts (O) again at D. E is a point on (O). EQ cuts (O) again at F. DF cuts BC at M. AQ cuts (FMQ) again at N. Prove that M, N, P are colliear.

The solution is the same as in #4 or in #5 of shinichiman.

Luis González

#7 May 11, 2015, 4:18 am • 1

We can also solve the extension with the same method I gave in my previous post.

Fix D and E and animate P. If we redefine $N \in AQ$, such that $\angle(NA, NP) = \angle(FE, FD) = \text{const}$, then all lines NP are parallel to each other $\Longrightarrow$ series P, N are similar, but clearly $P \barwedge Q \barwedge F \barwedge M$. So it suffices to prove that M, N, P are collinear for at least 3 positions of P.

When P is at infinity and $P \equiv D$ the collinearity trivially holds. When $P \equiv A$, then $Q \in BC$ and we have $\angle AQB = \angle ACD = \angle AFD$ making $AQFM$ cyclic $\Longrightarrow \angle(AQ, AM) = \angle(FE, FD) \Longrightarrow$ the collinearity holds. Thus M, N, P are collinear for any P.

Quick Reply

Site Support

Why is my question getting deleted?

Reply

XA-GTHO-PIV

#1 Aug 9, 2015, 11:53 am

Trying to propose a question to High School Olympiads but it keeps getting deleted? Why?

Luis González

#2 Aug 9, 2015, 12:45 pm

Your problem was moved to High School Math since it is too easy for the olympiad forum.

Quick Reply

High School Math

40

Functions

Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be a function such that $f(a+b)=f(a)+f(b)$ and that $f(2008)=3012$. What is $f(2009)$?

5

MBMT Lobachevsky Counting & Probability #8

Alice picks a random number between 0 and 0.75. Charlie picks a random number between 0.25 and 1. Bob picks a random number between 0 and 1. What is the probability that Bob's number is between Charlie and Alice's numbers?

Problem created by person361

5

MBMT Lobachevsky Algebra #8

Let $f(x)$ be a function such that $f(x)f(y)-f(xy)=xy$ for all real x and y. Let M and m be the maximum possible value and minimum possible value, respectively, of $f(2016)$. Find $M-m$.

Problem created by blawho12

5

st05__#12

Let $A_1, A_2, ..., A_n$ be the interior angles of an n-sided convex polygon.Then the value of

$$\frac{\cos(A_1+A_2+...+A_k)}{\cos(A_{k+1}+A_{k+2}+...+A_n)}$$

5

Factorial

How many zeroes does end in when 20! written in base 11?

7

Fermat and Euler's theorems

Prove Euler's theorem, that for positive integers $\gcd(a,m)=1, a^{\phi(m)}\equiv 1 \mod m$, by only using Fermat's Little Theorem.

1

Counting

Can someone explain to me (or link me to) how to count the number of possible ways to have $a+2b+3c=2015$ with a,b,c being integers?

2

arrangement with repetition

How many arrangements of a,a,a,b,b,b,c,c,c are there such that no two adjacent letters are the same?

7

numbers of non-repeating digits.

n is a positive integer such that n^2 and n^3 contain digits 1, 2, 3, 4, 5, 6, 7, 8 with non-repeating digits. Please give me some guidance on how to find n.

4

Fun Area Probem

10

A point P is chosen in the interior of $\triangle ABC$ so that when lines are drawn through P parallel to the sides of $\triangle ABC$, the resulting smaller triangles, t_1, t_2, and t_3 in the figure, have areas 4, 9, and 49, respectively. Find the area of $\triangle ABC$.
IMAGE

Function of conditions

1

$$\delta(x,Q) = \begin{cases} 0 \text{if} x \neq Q \\ 1 & \text{if} x = Q \end{cases}$$

then

Range of a function

3

Range of f(x) = |x + 1| + |x| + |x - 1| +|x - 2| + |x - 3| is (a,b) then what is the value of a?

Please help me do this question.

K.Bala

Help with trigonometry please

3

I have this trig problem, which is part of a geometry question. Please help

$a^2 = r^2 + r^2 - 2r^2 Cos\alpha$
$b^2 = r^2 + r^2 - 2r^2 Cos\beta$
Where $\alpha = 180° - \beta$

show the m1m2m3m4

0

If(mi ‹1/mi) where i=1 and 2 and 3 and 4
are the coordinates of four concyclic points, then what is the value of ?
m1m2m3m4