India
International Mathematical Olympiad Training Camp
2009

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1. Let ABC be a triangle with $\angle A = 60$.Prove that if T is point of contact of Incircle And Nine-Point Circle, Then $AT = r$, r being inradius.

2. Let us consider a simle graph with vertex set V. All ordered pair (a, b) of integers with $gcd(a, b) = 1$, are elements of V. (a, b) is connected to $(a, b+kab)$ by an edge and to $(a+kab, b)$ by another edge for all integer k. Prove that for all $(a, b) \in V$, there exists a path fromm $(1, 1)$ to (a, b).

3. Let a, b be two distinct odd natural numbers.Define a Sequence $< a_n >_{n\geq 0}$ like following:
$a_1 = a$
$a_2 = b$
$a_n =$ largest odd divisor of $(a_{n-1} + a_{n-2})$. Prove that there exists a natural number N such that $a_n = gcd(a, b) \forall n \geq N$.

4. Let γ be circumcircle of $\triangle ABC$.Let R_a be radius of circle touching $AB, AC \gamma$ internally.Define R_b, R_c similarly. Prove That $\frac{1}{aR_a} + \frac{1}{bR_b} + \frac{1}{cR_c} = \frac{r^2}{sabc}$.

5. Let $f(x)$and $g(y)$ be two monic polynomials of degree=n having complex coefficients. We know that there exist complex numbers $a_i, b_i, c_i \forall 1 \leq i \leq n$, such that $f(x) - g(y) = \prod_{i=1}^{n} (a_i x + b_i y + c_i)$. Prove that there exists $a, b, c \in \mathbb{C}$ such that $f(x) = (x + a)^n + c$ and $g(y) = (y + b)^n + c$.

6. Prove The Following identity: $\sum_{j=0}^{n} \left(\binom{3n+2-j}{j} 2^j - \binom{3n+1-j}{j-1} 2^{j-1} \right) = 2^{3n}$. The Second term on left hand side is to be regarded zero for j=0.

7. Let P be any point in the interior of a $\triangle ABC$.Prove That $\frac{PA}{a} + \frac{PB}{b} + \frac{PC}{c} \geq \sqrt{3}$.

8. Let n be a natural number ≥ 2 which divides $3^n + 4^n$.Prove That $7 \mid n$.

9. Let $f(x) = \sum_{k=1}^{n} a_k x^k$ and $g(x) = \sum_{k=1}^{n} \frac{a_k x^k}{2^k - 1}$ be two polynomials with real coefficients. Let g(x) have $0, 2^{n+1}$ as two of its roots. Prove That $f(x)$ has a positive root less than 2^n.

10. For a certain triangle all of its altitudes are integers whose sum is less than 20. If its Inradius is also an integer Find all possible values of area of the triangle.

11. Find all integers $n \geq 2$ with the following property:

    There exists three distinct primes p, q, r such that whenever $a_1, a_2, a_3, \cdots, a_n$ are n distinct positive integers with the property that at least one of p, q, r divides $a_j - a_k$ $\forall 1 \leq j \leq k \leq n$, one of p, q, r divides all of these differences.

12. Let G be a simple graph with vertex set $V = \{0, 1, 2, 3, \cdots, n+1\}$.$jandj+1$ are connected by an edge for $0 \leq j \leq n$. Let A be a subset of V and $G(A)$ be the induced subgraph associated

with A. Let $O(G(A))$ be number of components of $G(A)$ having an odd number of vertices. Let $T(p,r) = \{A \subset V \mid 0.n+1 \notin A, |A| = p, O(G(A)) = 2r\}$ for $r \leq p \leq 2r$. Prove That $|T(p,r)| = \binom{n-r}{p-r}\binom{n-p+1}{2r-p}$.