China National Olympiad 2011

Day 1

1 Let $a_1, a_2, \ldots, a_n$ are real numbers, prove that;

$$\sum_{i=1}^{n} a_i^2 - \sum_{i=1}^{n} a_i a_{i+1} \le \left\lfloor \frac{n}{2} \right\rfloor (M-m)^2.$$

where $a_{n+1} = a_1, M = \max_{1\le i\le n} a_i, m = \min_{1\le i\le n} a_i$.

2 On the circumcircle of the acute triangle ABC, D is the midpoint of $\overset{\frown}{BC}$. Let X be a point on $\overset{\frown}{BD}$, E the midpoint of $\overset{\frown}{AX}$, and let S lie on $\overset{\frown}{AC}$. The lines SD and BC have intersection R, and the lines SE and AX have intersection T. If $RT \parallel DE$, prove that the incenter of the triangle ABC is on the line RT.

3 Let A be a set consist of finite real numbers,$A_1, A_2, \cdots, A_n$ be nonempty sets of A, such that

(a) The sum of the elements of A is 0,

(b) For all $x_i \in A_i (i = 1, 2, \cdots, n)$,we have $x_1 + x_2 + \cdots + x_n > 0$.

Prove that there exist $1 \le k \le n$, and $1 \le i_1 < i_2 < \cdots < i_k \le n$, such that

$$|A_{i_1} \bigcup A_{i_2} \bigcup \cdots \bigcup A_{i_k}| < \frac{k}{n}|A|.$$

Where $|X|$ denote the numbers of the elements in set X.

Day 2

1 Let n be an given positive integer, the set $S = \{1, 2, \cdots, n\}$.For any nonempty set A and B, find the minimum of $|A\Delta S| + |B\Delta S| + |C\Delta S|$, where $C = \{a+b|a \in A, b \in B\}, X\Delta Y = X \cup Y - X \cap Y$.

2 Let $a_i, b_i, i = 1, \cdots, n$ are nonnegitive numbers,and $n \ge 4$,such that $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n > 0$.

Find the maximum of $\frac{\sum_{i=1}^{n} a_i(a_i+b_i)}{\sum_{i=1}^{n} b_i(a_i+b_i)}$

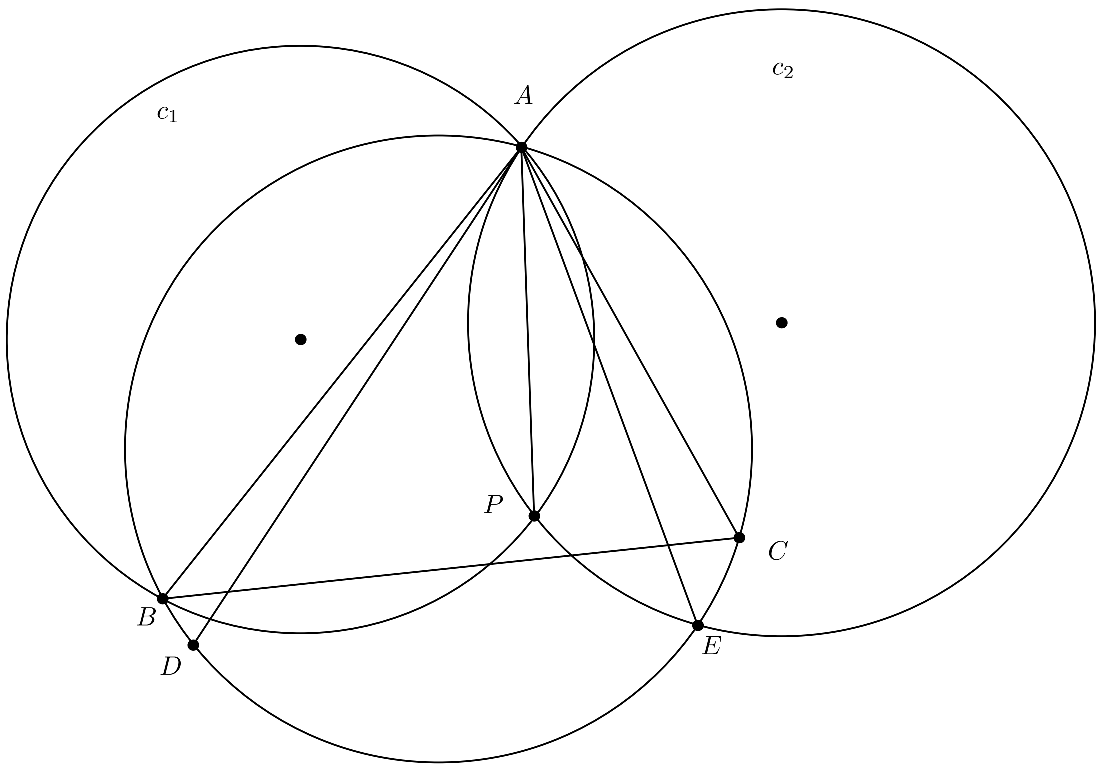
3 Let m, n be positive integer numbers. Prove that there exist infinite many couples of positive integer nubmers (a, b) such that

$$a + b | am^a + bn^b, \quad \gcd(a, b) = 1.$$

China National Olympiad 2012

Day 1

1 In the triangle ABC, $\angle A$ is biggest. On the circumcircle of $\triangle ABC$, let D be the midpoint of $\widehat{ABC}$ and E be the midpoint of $\widehat{ACB}$. The circle c_1 passes through A, B and is tangent to AC at A, the circle c_2 passes through A, E and is tangent AD at A. c_1 and c_2 intersect at A and P. Prove that AP bisects $\angle BAC$.



2 Let p be a prime. We arrange the numbers in $\{1, 2, \ldots, p^2\}$ as a $p \times p$ matrix $A = (a_{ij})$. Next we can select any row or column and add 1 to every number in it, or subtract 1 from every number in it. We call the arrangement *good* if we can change every number of the matrix to 0 in a finite number of such moves. How many good arrangements are there?

3 Prove for any $M > 2$, there exists an increasing sequence of positive integers $a_1 < a_2 < \ldots$ satisfying:
1) $a_i > M^i$ for any i;
2) There exists a positive integer m and $b_1, b_2, \ldots, b_m \in \{-1, 1\}$, satisfying $n = a_1b_1 + a_2b_2 + \ldots + a_mb_m$ if and only if $n \in \mathbb{Z}/\{0\}$.

Day 2

1 Let $f(x) = (x+a)(x+b)$ where $a, b > 0$. For any reals $x_1, x_2, \ldots, x_n \geqslant 0$ satisfying $x_1 + x_2 + \ldots + x_n = 1$, find the maximum of $F = \sum_{1 \leqslant i < j \leqslant n} \min\{f(x_i), f(x_j)\}$.

2 Consider a square-free even integer n and a prime p, such that
1) $(n, p) = 1$;
2) $p \leq 2\sqrt{n}$;
3) There exists an integer k such that $p | n + k^2$.
Prove that there exists pairwise distinct positive integers a, b, c such that $n = ab + bc + ca$.

Proposed by Hongbing Yu

3 Find the smallest positive integer k such that, for any subset A of $S = \{1, 2, \ldots, 2012\}$ with $|A| = k$, there exist three elements x, y, z in A such that $x = a+b$, $y = b+c$, $z = c+a$, where a, b, c are in S and are distinct integers.

Proposed by Huawei Zhu

China National Olympiad 2013

Day 1

1 Two circles K_1 and K_2 of different radii intersect at two points A and B, let C and D be two points on K_1 and K_2, respectively, such that A is the midpoint of the segment CD. The extension of DB meets K_1 at another point E, the extension of CB meets K_2 at another point F. Let l_1 and l_2 be the perpendicular bisectors of CD and EF, respectively.
i) Show that l_1 and l_2 have a unique common point (denoted by P).
ii) Prove that the lengths of CA, AP and PE are the side lengths of a right triangle.

2 Find all nonempty sets S of integers such that $3m - 2n \in S$ for all (not necessarily distinct) $m, n \in S$.

3 Find all positive real numbers t with the following property: there exists an infinite set X of real numbers such that the inequality

$$\max\{|x-(a-d)|,|y-a|,|z-(a+d)|\} > td$$

holds for all (not necessarily distinct) $x, y, z \in X$, all real numbers a and all positive real numbers d.

Day 2

1 Let $n \geqslant 2$ be an integer. There are n finite sets $A_1, A_2, \ldots, A_n$ which satisfy the condition

$$|A_i \Delta A_j| = |i-j| \quad \forall i,j \in \{1,2,...,n\}.$$

Find the minimum of $\sum\limits_{i=1}^{n} |A_i|$.

2 For any positive integer n and $0 \leqslant i \leqslant n$, denote $C_n^i \equiv c(n,i) \pmod 2$, where $c(n,i) \in \{0,1\}$. Define

$$f(n,q) = \sum_{i=0}^{n} c(n,i) q^i$$

where m, n, q are positive integers and $q+1 \neq 2^\alpha$ for any $\alpha \in \mathbb{N}$. Prove that if $f(m,q) \,|\, f(n,q)$, then $f(m,r) \,|\, f(n,r)$ for any positive integer r.

3 Let m, n be positive integers. Find the minimum positive integer N which satisfies the following condition. If there exists a set S of integers that contains a complete residue system module m such that $|S| = N$, then there exists a nonempty set $A \subseteq S$ so that $n \mid \sum\limits_{x\in A} x$.

China National Olympiad 2014

Day 1

1 Let ABC be a triangle with $AB > AC$. Let D be the foot of the internal angle bisector of A. Points F and E are on AC, AB respectively such that B, C, F, E are concyclic. Prove that the circumcentre of DEF is the incentre of ABC if and only if $BE + CF = BC$.

2 For the integer $n > 1$, define $D(n) = \{a - b \mid ab = n, a > b > 0, a, b \in \mathbb{N}\}$. Prove that for any integer $k > 1$, there exists pairwise distinct positive integers $n_1, n_2, \ldots, n_k$ such that $n_1, \ldots, n_k > 1$ and $|D(n_1) \cap D(n_2) \cap \cdots \cap D(n_k)| \geq 2$.

3 Prove that: there exists only one function $f : \mathbb{N}^* \to \mathbb{N}^*$ satisfying:
i) $f(1) = f(2) = 1$;
ii)$f(n) = f(f(n-1)) + f(n - f(n-1))$ for $n \geq 3$.
For each integer $m \geq 2$, find the value of $f(2^m)$.

Day 2

1 Let $n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}$ be the prime factorisation of n. Define $\omega(n) = t$ and $\Omega(n) = a_1 + a_2 + \ldots + a_t$. Prove or disprove:
For any fixed positive integer k and positive reals α, β, there exists a positive integer $n > 1$ such that
i) $\frac{\omega(n+k)}{\omega(n)} > \alpha$
ii) $\frac{\Omega(n+k)}{\Omega(n)} < \beta$.

2 Let $f : X \to X$, where $X = \{1, 2, \ldots, 100\}$, be a function satisfying:
1) $f(x) \neq x$ for all $x = 1, 2, \ldots, 100$;
2) for any subset A of X such that $|A| = 40$, we have $A \cap f(A) \neq \emptyset$.
Find the minimum k such that for any such function f, there exist a subset B of X, where $|B| = k$, such that $B \cup f(B) = X$.

3 For non-empty number sets S, T, define the sets $S + T = \{s + t \mid s \in S, t \in T\}$ and $2S = \{2s \mid s \in S\}$.
Let n be a positive integer, and A, B be two non-empty subsets of $\{1, 2 \ldots, n\}$. Show that there exists a subset D of $A + B$ such that
1) $D + D \subseteq 2(A + B)$,

2) $|D| \geq \frac{|A|\cdot|B|}{2n}$,
where $|X|$ is the number of elements of the finite set X.

China National Olympiad 2015

Day 1

1 Let $z_1, z_2, ..., z_n$ be complex numbers satisfying $|z_i - 1| \leq r$ for some r in $(0, 1)$. Show that

$$\left|\sum_{i=1}^{n} z_i\right| \cdot \left|\sum_{i=1}^{n} \frac{1}{z_i}\right| \geq n^2(1 - r^2).$$

2 Let A, B, D, E, F, C be six points lie on a circle (in order) satisfy $AB = AC$. Let $P = AD \cap BE, R = AF \cap CE, Q = BF \cap CD, S = AD \cap BF, T = AF \cap CD$.

Let K be a point lie on ST satisfy $\angle QKS = \angle ECA$.

Prove that $\frac{SK}{KT} = \frac{PQ}{QR}$

3 Let $n \geq 5$ be a positive integer and let A and B be sets of integers satisfying the following conditions:

i) $|A| = n$, $|B| = m$ and A is a subset of B
ii) For any distinct $x, y \in B$, $x + y \in B$ iff $x, y \in A$

Determine the minimum value of m.

Day 2

1 Determine all integers k such that there exists infinitely many positive integers n satisfying

$$n + k \nmid \binom{2n}{n}$$

2 Given 30 students such that each student has at most 5 friends and for every 5 students there is a pair of students that are not friends, determine the maximum k such that for all such possible configurations, there exists k students who are all not friends.

3 Let $a_1, a_2, ...$ be a sequence of non-negative integers such that for any m, n

$$\sum_{i=1}^{2m} a_{in} \leq m.$$

Show that there exist k, d such that

$$\sum_{i=1}^{2k} a_{id} = k - 2014.$$

China National Olympiad 2016

Day 1 December 16th

1 Let $a_1, a_2, \cdots, a_{31}; b_1, b_2, \cdots, b_{31}$ be positive integers such that $a_1 < a_2 < \cdots < a_{31} \leq 2015$, $b_1 < b_2 < \cdots < b_{31} \leq 2015$ and $a_1 + a_2 + \cdots + a_{31} = b_1 + b_2 + \cdots + b_{31}$. Find the maximum value of $S = |a_1 - b_1| + |a_2 - b_2| + \cdots + |a_{31} - b_{31}|$.

2 In $\triangle AEF$, let B and D be on segments AE and AF respectively, and let ED and FB intersect at C. Define K, L, M, N on segments AB, BC, CD, DA such that $\frac{AK}{KB} = \frac{AD}{BC}$ and its cyclic equivalents. Let the incircle of $\triangle AEF$ touch AE, AF at S, T respectively; let the incircle of $\triangle CEF$ touch CE, CF at U, V respectively.
Prove that K, L, M, N concyclic implies S, T, U, V concyclic.

3 Let p be an odd prime and $a_1, a_2, ..., a_p$ be integers. Prove that the following two conditions are equivalent:

1) There exists a polynomial $P(x)$ with degree $\leq \frac{p-1}{2}$ such that $P(i) \equiv a_i \pmod{p}$ for all $1 \leq i \leq p$

2) For any natural $d \leq \frac{p-1}{2}$,

$$\sum_{i=1}^{p} (a_{i+d} - a_i)^2 \equiv 0 \pmod{p}$$

where indices are taken $\pmod{p}$

Day 2 December 17th

4 Let $n \geq 2$ be a positive integer and define k to be the number of primes $\leq n$. Let A be a subset of $S = \{2, ..., n\}$ such that $|A| \leq k$ and no two elements in A divide each other. Show that one can find a set B such that $|B| = k$, $A \subseteq B \subseteq S$ and no two elements in B divide each other.

5 Let $ABCD$ be a convex quadrilateral. Show that there exists a square $A'B'C'D'$ (Vertices maybe ordered clockwise or counter-clockwise) such that $A \neq A', B \neq B', C \neq C', D \neq D'$ and AA', BB', CC', DD' are all concurrent.

6 Let G be a complete directed graph with 100 vertices such that for any two vertices x, y one can find a directed path from x to y.

a) Show that for any such G, one can find a m such that for any two vertices x, y one can find a directed path of length m from x to y (Vertices can be repeated in the path)

b) For any graph G with the properties above, define $m(G)$ to be smallest possible m as defined in part a). Find the minimim value of $m(G)$ over all such possible G's.

All-Russian Olympiad 2014

– Grade level 9

Day 1

1 On a circle there are 99 natural numbers. If a, b are any two neighbouring numbers on the circle, then $a - b$ is equal to 1 or 2 or $\frac{a}{b} = 2$. Prove that there exists a natural number on the circle that is divisible by 3.

S. Berlov

2 Sergei chooses two different natural numbers a and b. He writes four numbers in a notebook: a, $a + 2$, b and $b + 2$. He then writes all six pairwise products of the numbers of notebook on the blackboard. Let S be the number of perfect squares on the blackboard. Find the maximum value of S.

S. Berlov

3 In a convex n-gon, several diagonals are drawn. Among these diagonals, a diagonal is called *good* if it intersects exactly one other diagonal drawn (in the interior of the n-gon). Find the maximum number of good diagonals.

4 Let M be the midpoint of the side AC of acute-angled triangle ABC with $AB > BC$. Let Ω be the circumcircle of ABC. The tangents to Ω at the points A and C meet at P, and BP and AC intersect at S. Let AD be the altitude of the triangle ABP and ω the circumcircle of the triangle CSD. Suppose ω and Ω intersect at $K \neq C$. Prove that $\angle CKM = 90^\circ$.

V. Shmarov

Day 2

1 Define $m(n)$ to be the greatest proper natural divisor of $n \in \mathbb{N}$. Find all $n \in \mathbb{N}$ such that $n + m(n)$ is a power of 10.

N. Agakhanov

2 Let $ABCD$ be a trapezoid with $AB \parallel CD$ and Ω is a circle passing through A, B, C, D. Let ω be the circle passing through C, D and intersecting with CA, CB at A_1, B_1 respectively. A_2 and B_2 are the points symmetric to A_1 and B_1 respectively, with respect to the midpoints of CA and CB. Prove that the points A, B, A_2, B_2 are concyclic.

I. Bogdanov

3 In a country, mathematicians chose an $\alpha > 2$ and issued coins in denominations of 1 ruble, as well as α^k rubles for each positive integer k. α was chosen so that the value of each coins, except the smallest, was irrational. Is it possible that any natural number of rubles can be formed with at most 6 of each denomination of coins?

4 In a country of n cities, an express train runs both ways between any two cities. For any train, ticket prices either direction are equal, but for any different routes these prices are different. Prove that the traveler can select the starting city, leave it and go on, successively, $n-1$ trains, such that each fare is smaller than that of the previous fare. (A traveler can enter the same city several times.)

– Grade level 10

Day 1

1 Let a be *good* if the number of prime divisors of a is equal to 2. Do there exist 18 consecutive good natural numbers?

2 Given a function $f\colon \mathbb{R} \to \mathbb{R}$ with $f(x)^2 \le f(y)$ for all $x, y \in \mathbb{R}$, $x > y$, prove that $f(x) \in [0, 1]$ for all $x \in \mathbb{R}$.

3 There are n cells with indices from 1 to n. Originally, in each cell, there is a card with the corresponding index on it. Vasya shifts the card such that in the i-th cell is now a card with the number a_i. Petya can swap any two cards with the numbers x and y, but he must pay $2|x-y|$ coins. Show that Petya can return all the cards to their original position, not paying more than $|a_1 - 1| + |a_2 - 2| + \ldots + |a_n - n|$ coins.

4 Given a triangle ABC with $AB > BC$, let Ω be the circumcircle. Let M, N lie on the sides AB, BC respectively, such that $AM = CN$. Let K be the intersection of MN and AC. Let P be the incentre of the triangle AMK and Q be the K-excentre of the triangle CNK. If R is midpoint of the arc ABC of Ω then prove that $RP = RQ$.

M. Kungodjin

Day 2

1 Define $m(n)$ to be the greatest proper natural divisor of $n \in \mathbb{N}$. Find all $n \in \mathbb{N}$ such that $n + m(n)$ is a power of 10.

N. Agakhanov

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2 Let M be the midpoint of the side AC of $\triangle ABC$. Let $P \in AM$ and $Q \in CM$ be such that $PQ = \frac{AC}{2}$. Let (ABQ) intersect with BC at $X \neq B$ and (BCP) intersect with BA at $Y \neq B$. Prove that the quadrilateral $BXMY$ is cyclic.

F. Ivlev, F. Nilov

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3 In a country, mathematicians chose an $\alpha > 2$ and issued coins in denominations of 1 ruble, as well as α^k rubles for each positive integer k. α was chosen so that the value of each coins, except the smallest, was irrational. Is it possible that any natural number of rubles can be formed with at most 6 of each denomination of coins?

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4 Given are n pairwise intersecting convex k-gons on the plane. Any of them can be transferred to any other by a homothety with a positive coefficient. Prove that there is a point in a plane belonging to at least $1 + \frac{n-1}{2k}$ of these k-gons.

– Grade level 11

Day 1

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1 Does there exist positive $a \in \mathbb{R}$, such that

$$|\cos x| + |\cos ax| > \sin x + \sin ax$$

for all $x \in \mathbb{R}$?

N. Agakhanov

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2 Peter and Bob play a game on a $n \times n$ chessboard. At the beginning, all squares are white apart from one black corner square containing a rook. Players take turns to move the rook to a white square and recolour the square black. The player who can not move loses. Peter goes first. Who has a winning strategy?

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3 Positive rational numbers a and b are written as decimal fractions and each consists of a minimum period of 30 digits. In the decimal representation of $a - b$, the period is at least 15. Find the minimum value of $k \in \mathbb{N}$ such that, in the decimal representation of $a + kb$, the length of period is at least 15.

A. Golovanov

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4 Given a triangle ABC with $AB > BC$, Ω is circumcircle. Let M, N are lie on the sides AB, BC respectively, such that $AM = CN$. $K(.) = MN \cap AC$ and P is incenter of the triangle AMK, Q is K-excenter of the triangle CNK (opposite to K and tangents to CN). If R is midpoint of the arc ABC of Ω then prove that $RP = RQ$.

M. Kungodjin

Day 2

1 Call a natural number n *good* if for any natural divisor a of n, we have that $a + 1$ is also divisor of $n + 1$. Find all good natural numbers.

S. Berlov

2 The sphere ω passes through the vertex S of the pyramid $SABC$ and intersects with the edges SA, SB, SC at A_1, B_1, C_1 other than S. The sphere Ω is the circumsphere of the pyramid $SABC$ and intersects with ω circumferential, lies on a plane which parallel to the plane (ABC).
Points A_2, B_2, C_2 are symmetry points of the points A_1, B_1, C_1 respect to midpoints of the edges SA, SB, SC respectively. Prove that the points A, B, C, A_2, B_2, and C_2 lie on a sphere.

3 If the polynomials $f(x)$ and $g(x)$ are written on a blackboard then we can also write down the polynomials $f(x) \pm g(x)$, $f(x)g(x)$, $f(g(x))$ and $cf(x)$, where c is an arbitrary real constant. The polynomials $x^3 - 3x^2 + 5$ and $x^2 - 4x$ are written on the blackboard. Can we write a nonzero polynomial of form $x^n - 1$ after a finite number of steps?

4 Two players play a card game. They have a deck of n distinct cards. About any two cards from the deck know which of them has a different (in this case, if A beats B, and B beats C, then it may be that C beats A). The deck is split between players in an arbitrary manner. In each turn the players over the top card from his deck and one whose card has a card from another player takes both cards and puts them to the bottom of your deck in any order of their discretion. Prove that for any initial distribution of cards, the players can with knowing the location agree and act so that one of the players left without a card.

E. Lakschmanov

All-Russian Olympiad 2015

–	Grade 9	
–	Day 1	
1	Real numbers a and b are chosen so that each of two quadratic trinomials $x^2 + ax + b$ and $x^2 + bx + a$ has two distinct real roots,and the product of these trinomials has exactly three distinct real roots.Determine all possible values of the sum of these three roots. *(S.Berlov)*	
3	Let a, x, y be positive integer such that $a > 100, x > 100, y > 100$ and $y^2 - 1 = a^2(x^2 - 1)$. Find the minimum value of $\frac{a}{x}$.	
–	Day 2	
5	100 integers are arranged in a circle. Each number is greater than the sum of the two subsequent numbers (in a clockwise order). Determine the maximal possible number of positive numbers in such circle. *(S.Berlov)*	
6	A field has a shape of checkboard 41x41 square. A tank concealed in one of the cells of the field. By one shot, a fighter airplane fires one of the cells. If a shot hits the tank, then the tank moves to a neighboring cell of the field, otherwise it stays in its cell (the cells are neighbours if they share a side). A pilot has no information about the tank , one needs to hit it twice. Find the least number of shots sufficient to destroy the tank for sure. *(S.Berlov,A.Magazinov)*	
7	An acute-angled ABC $(AB < AC)$ is inscribed into a circle ω. Let M be the centroid of ABC, and let AH be an altitude of this triangle. A ray MH meets ω at A'. Prove that the circumcircle of the triangle $A'HB$ is tangent to AB. *(A.I. Golovanov , A.Yakubov)*	
8	$N \geq 9$ distinct real numbers are written on a blackboard. All these numbers are nonnegative, and all are less than 1. It happens that for very 8 distinct numbers on the board, the board contains the ninth number distinct from eight such that the sum of all these nine numbers is integer. Find all values N for which this is possible. *(F. Nilov)*	
–	Grade 10	

–	Day 1
1	We say that a positive integer is an *almost square*, if it is equal to the product of two consecutive positive integers. Prove that every almost square can be expressed as a quotient of two almost squares. V. Senderov
2	Given is a parallelogram $ABCD$, with $AB < AC < BC$. Points E and F are selected on the circumcircle ω of ABC so that the tangenst to ω at these points pass through point D and the segments AD and CE intersect. It turned out that $\angle ABF = \angle DCE$. Find the angle $\angle ABC$. A. Yakubov, S. Berlov
4	We denote by $S(k)$ the sum of digits of a positive integer number k. We say that the positive integer a is n-good, if there is a sequence of positive integers $a_0, a_1, \ldots, a_n$, so that $a_n = a$ and $a_{i+1} = a_i - S(a_i)$ for all $i = 0, 1, ..., n-1$. Is it true that for any positive integer n there exists a positive integer b, which is n-good, but not $(n+1)$-good? A. Antropov
–	Day 2
5	It is known that a cells square can be cut into n equal figures of k cells. Prove that it is possible to cut it into k equal figures of n cells.
7	In an acute-angled and not isosceles triangle ABC, we draw the median AM and the height AH. Points Q and P are marked on the lines AB and AC, respectively, so that the $QM \perp AC$ and $PM \perp AB$. The circumcircle of PMQ intersects the line BC for second time at point X. Prove that $BH = CX$. M. Didin
–	Grade 11
–	Day 1
1	Parallelogram $ABCD$ is such that angle $B < 90$ and $AB < BC$. Points E and F are on the circumference of ω inscribing triangle ABC, such that tangents to ω in those points pass through D. If $\angle EDA = \angle FDC$, find $\angle ABC$.

2 Let $n > 1$ be a natural number. We write out the fractions $\frac{1}{n}, \frac{2}{n}, \dots, \dfrac{n-1}{n}$ such that they are all in their simplest form. Let the sum of the numerators be $f(n)$. For what $n > 1$ is one of $f(n)$ and $f(2015n)$ odd, but the other is even?

3 110 teams participate in a volleyball tournament. Every team has played every other team exactly once (there are no ties in volleyball). Turns out that in any set of 55 teams, there is one which has lost to no more than 4 of the remaining 54 teams. Prove that in the entire tournament, there is a team that has lost to no more than 4 of the remaining 109 teams.

4 You are given N such that $n \geq 3$. We call a set of N points on a plane acceptable if their abscissae are unique, and each of the points is coloured either red or blue. Let's say that a polynomial $P(x)$ divides a set of acceptable points either if there are no red dots above the graph of $P(x)$, and below, there are no blue dots, or if there are no blue dots above the graph of $P(x)$ and there are no red dots below. Keep in mind, dots of both colors can be present on the graph of $P(x)$ itself. For what least value of k is an arbitrary set of N points divisible by a polynomial of degree k?

– Day 2

5 An immortal flea jumps on whole points of the number line, beginning with 0. The length of the first jump is 3, the second 5, the third 9, and so on. The length of k^{th} jump is equal to $2^k + 1$. The flea decides whether to jump left or right on its own. Is it possible that sooner or later the flee will have been on every natural point, perhaps having visited some of the points more than once?

6 Let a,b,c,d be real numbers satisfying $|a|, |b|, |c|, |d| > 1$ and $abc + abd + acd + bcd + a + b + c + d = 0$. Prove that $\frac{1}{a-1} + \frac{1}{b-1} + \frac{1}{c-1} + \frac{1}{d-1} > 0$

7 A scalene triangle ABC is inscribed within circle ω. The tangent to the circle at point C intersects line AB at point D. Let I be the center of the circle inscribed within $\triangle ABC$. Lines AI and BI intersect the bisector of $\angle CDB$ in points Q and P, respectively. Let M be the midpoint of QP. Prove that MI passes through the middle of arc ACB of circle ω.

8 Given natural numbers a and b, such that $a < b < 2a$. Some cells on a graph are colored such that in every rectangle with dimensions $A \times B$ or $B \times A$, at least one cell is colored. For which greatest α can you say that for every natural number N you can find a square $N \times N$ in which at least $\alpha \cdot N^2$ cells are colored?

All-Russian Olympiad 2016

Grade 11 Day 1

1 There are 30 teams in NBA and every team play 82 games in the year. Bosses of NBA want to divide all teams on Western and Eastern Conferences (not necessary equally), such that number of games between teams from different conferences is half of number of all games. Can they do it?

2 In the space given three segments A_1A_2, B_1B_2 and C_1C_2, do not lie in one plane and intersect at a point P. Let O_{ijk} be center of sphere that passes through the points A_i, B_j, C_k and P. Prove that $O_{111}O_{222}, O_{112}O_{221}, O_{121}O_{212}$ and$O_{211}O_{122}$ intersect at one point. (P.Kozhevnikov)

3 We have sheet of paper, divided on 100×100 unit squares. In some squares we put rightangled isosceles triangles with leg =1 (Every triangle lies in one unit square and is half of this square). Every unit grid segment(boundary too) is under one leg of triangle. Find maximal number of unit squares, that don't contains triangles.

4 There is three-dimensional space. For every integer n we build planes $x \pm y \pm z = n$. All space is divided on octahedrons and tetrahedrons.
Point (x_0, y_0, z_0) has rational coordinates but not lies on any plane. Prove, that there is such natural k , that point (kx_0, ky_0, kz_0) lies strictly inside the octahedron of partition.

Grade 11 Day 2

5 Let $n \in \mathbb{N}$. $k_0, k_1, ..., k_{2n}$ - nonzero integers, and $k_0 + ... + k_{2n} \neq 0$.

Can we always find such permutation $(a_0, ...a_{2n})$ of $(k_0, k_1, ..., k_{2n})$, that equation $a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + ... + a_0 = 0$ has not integer roots?

6 There are $n > 1$ cities in the country, some pairs of cities linked two-way through straight flight. For every pair of cities there is exactly one aviaroute (can have interchanges).
Major of every city X counted amount of such numberings of all cities from 1 to n , such that on every aviaroute with the beginning in X, numbers of cities are in ascending order. Every major, except one, noticed that results of counting are multiple of 2016.
Prove, that result of last major is multiple of 2016 too.

7 All russian olympiad 2016,Day 2 ,grade 9,P8 :
Let a,b,c,d be are positive numbers such that $a+b+c+d=3$ .Prove that

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\le\frac{1}{a^2b^2c^2d^2}$$

All russian olympiad 2016,Day 2,grade 11,P7 :
Let a,b,c,d be are positive numbers such that $a+b+c+d=3$ .Prove that

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+\frac{1}{d^3}\le\frac{1}{a^3b^3c^3d^3}$$

Russia national 2016

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8 Medians AM_A, BM_B, CM_C of triangle ABC intersect at M.Let Ω_A be circumcircle of triangle passes through midpoint of AM and M_B, M_C.Define Ω_B and Ω_C analogusly.Prove that Ω_A, Ω_B and Ω_C intersect at one point.(A.Yakubov)

All-Russian Olympiad

–	grade 9 day 1
1	In country some cities are connected by oneway flights(There are no more then one flight between two cities). City A called "available" for city B, if there is flight from B to A, maybe with some transfers. It is known, that for every 2 cities P and Q exist city R, such that P and Q are available from R. Prove, that exist city A, such that every city is available for A.
2	$ABCD$ is an isosceles trapezoid with $BC\|\|AD$. A circle ω passing through B and C intersects the side AB and the diagonal BD at points X and Y respectively. Tangent to ω at C intersects the line AD at Z. Prove that the points X, Y, and Z are collinear.
3	There are 100 dwarfes with weight $1, 2, ..., 100$. They sit on the left riverside. They can not swim, but they have one boat with capacity 100. River has strong river flow, so every dwarf has power only for one passage from right side to left as oarsman. On every passage can be only one oarsman. Can all dwarfes get to right riverside?
4	Are there infinite increasing sequence of natural numbers, such that sum of every 2 different numbers are relatively prime with sum of every 3 different numbers?
–	grade9 day 2
1	There are $n > 3$ different natural numbers, less than $(n-1)!$ For every pair of numbers Ivan divides bigest on lowest and write integer quotient (for example, 100 divides $7 = 14$) and write result on the paper. Prove, that not all numbers on paper are different.
2	a, b, c - different natural numbers. Can we build quadratic polynomial $P(x) = kx^2 + lx + m$, with k, l, m are integer, $k > 0$ that for some integer points it get values a^3, b^3, c^3 ?
3	In the scalene triangle ABC,$\angle ACB = 60$ and Ω is its cirumcirle.On the bisectors of the angles BAC and CBA points A',B' are chosen respectively such that $AB' \parallel BC$ and $BA' \parallel AC$.$A'B'$ intersects with Ω at D, E.Prove that triangle CDE is isosceles.(A. Kuznetsov)

4	Every cell of 100×100 table is colored black or white. Every cell on table border is black. It is known, that in every 2×2 square there are cells of two colors. Prove, that exist 2×2 square that is colored in chess order.
–	Grade 10
1	$f_1(x) = x^2 + p_1x + q_1$, $f_2(x) = x^2 + p_2x + q_2$ are two parabolas. l_1 and l_2 are two not parallel lines. It is knows, that segments, that cuted on the l_1 by parabolas are equals, and segments, that cuted on the l_2 by parabolas are equals too. Prove, that parabolas are equals.
2	Let ABC be an acute angled isosceles triangle with $AB = AC$ and circumcentre O. Lines BO and CO intersect AC, AB respectively at B', C'. A straight line l is drawn through C' parallel to AC. Prove that the line l is tangent to the circumcircle of $\triangle B'OC$.
3	There are 3 heaps with $100, 101, 102$ stones. Ilya and Kostya play next game. Every step they take one stone from some heap, but not from same, that was on previous step. They make his steps in turn, Ilya make first step. Player loses if can not make step. Who has winning strategy?
5	n is composite. $1 < a_1 < a_2 < ... < a_k < n$ - all divisors of n. It is known, that $a_1 + 1, ..., a_k + 1$ are all divisors for some m (except $1, m$). Find all such n.
8	In a non-isosceles triangle ABC,O and I are circumcenter and incenter,respectively.B' is reflection of B with respect to OI and lies inside the angle ABI.Prove that the tangents to circumcirle of $\triangle BB'I$ at B',I intersect on AC. (A. Kuznetsov)
–	Grade 11
1	$S = \sin 64x + \sin 65x$ and $C = \cos 64x + \cos 65x$ are both rational for some x. Prove, that for one of these sums both summands are rational too.
2	Same as Grade 10 P2
3	There are n positive real numbers on the board $a_1, \ldots, a_n$. Someone wants to write n real numbers $b_1, \ldots, b_n$,such that: $b_i \geq a_i$ If $b_i \geq b_j$ then $\frac{b_i}{b_j}$ is integer. Prove that it is possible to write such numbers with the condition $$b_1 \cdots b_n \leq 2^{\frac{n-1}{2}} a_1 \cdots a_n.$$

4 Magicman and his helper want to do some magic trick. They have special card desk. Back of all cards is common color and face is one of 2017 colors.
Magic trick: magicman go away from scene. Then viewers should put on the table $n > 1$ cards in the row face up. Helper looks at these cards, then he turn all cards face down, except one, without changing order in row. Then magicman returns on the scene, looks at cards, then show on the one card, that lays face down and names it face color.
What is minimal n such that magicman and his helper can has strategy to make magic trick successfully?

5 $P(x)$ is polynomial with degree $n \geq 2$ and nonnegative coefficients. a, b, c - sides for some triangle. Prove, that $\sqrt[n]{P(a)}$, $\sqrt[n]{P(b)}$, $\sqrt[n]{P(c)}$ are sides for some triangle too.

6 In the 200×200 table in some cells lays red or blue chip. Every chip "see" other chip, if they lay in same row or column. Every chip "see" exactly 5 chips of other color. Find maximum number of chips in the table.

7 There is number N on the board. Every minute Ivan makes next operation: takes any number a written on the board,
erases it, then writes all divisors of a except a(Can be same numbers on the board). After some time on the board there are N^2 numbers.
For which N is it possible?

8 Given a convex quadrilateral $ABCD$. We denote I_A, I_B, I_C and I_D centers of $\omega_A, \omega_B, \omega_C$and ω_D,inscribed In the triangles DAB, ABC, BCD and CDA, respectively.It turned out that $\angle BI_AA + \angle I_CI_AI_D = 180^\circ$. Prove that $\angle BI_BA + \angle I_CI_BI_D = 180^\circ$. (A. Kuznetsov)

All-Russian Olympiad 2018

– Grade 9

1 Suppose $a_1, a_2, \dots$ is an infinite strictly increasing sequence of positive integers and $p_1, p_2, \dots$ is a sequence of distinct primes such that $p_n \mid a_n$ for all $n \geq 1$. It turned out that $a_n - a_k = p_n - p_k$ for all $n, k \geq 1$. Prove that the sequence $(a_n)_n$ consists only of prime numbers.

2 Circle ω is tangent to sides AB, AC of triangle ABC. A circle Ω touches the side AC and line AB (produced beyond B), and touches ω at a point L on side BC. Line AL meets ω, Ω again at K, M. It turned out that $KB \parallel CM$. Prove that $\triangle LCM$ is isosceles.

3 Suppose that $a_1, \cdots, a_{25}$ are non-negative integers, and k is the smallest of them. Prove that
the smallest of them. Prove that

$$\left[\sqrt{a_1}\right] + \left[\sqrt{a_2}\right] + \cdots + \left[\sqrt{a_{25}}\right] \geq \left[\sqrt{a_1 + a_2 + \cdots + a_{25} + 200k}\right].$$

(As usual, $[x]$ denotes the integer part of the number x , that is, the largest integer not exceeding x.)

4 On the $n \times n$ checker board, several cells were marked in such a way that lower lefft (L) and upper right(R) cells are not marked and that for any knight-tour from L to R, there is at least one marked cell. For which $n > 3$, is it possible that there always exists three consective cells going through diagonal for which at least two of them are marked?

5 On the circle, 99 points are marked, dividing this circle into 99 equal arcs. Petya and Vasya play the game, taking turns. Petya goes first; on his first move, he paints in red or blue any marked point. Then each player can paint on his own turn, in red or blue, any uncolored marked point adjacent to the already painted one. Vasya wins, if after painting all points there is an equilateral triangle, all three vertices of which are colored in the same color. Could Petya prevent him?

6 a and b are given positive integers. Prove that there are infinitely many positive integers n such that $n^b + 1$ doesn't divide $a^n + 1$.

Contributors: lifeisgood03, anantmudgal09, sqing, toshihiro shimizu, TwoTimes3TimesSeven, ThE-dArK-lOrD, Davrbek

7 In a card game, each card is associated with a numerical value from 1 to 100, with each card beating less, with one exception: 1 beats 100. The player knows that 100 cards with different values lie in front of him. The dealer who knows the order of these cards can tell the player which card beats the other for any pair of cards he draws. Prove that the dealer can make one hundred such messages, so that after that the player can accurately determine the value of each card.

8 $ABCD$ is a convex quadrilateral. Angles A and C are equal. Points M and N are on the sides AB and BC such that $MN||AD$ and $MN = 2AD$. Let K be the midpoint of MN and H be the orthocenter of $\triangle ABC$. Prove that HK is perpendicular to CD.

– Grade 10

1 Determine the number of real roots of the equation

$$|x| + |x+1| + \cdots + |x+2018| = x^2 + 2018x - 2019$$

2 Let $\triangle ABC$ be an acute-angled triangle with $AB < AC$. Let M and N be the midpoints of AB and AC, respectively; let AD be an altitude in this triangle. A point K is chosen on the segment MN so that $BK = CK$. The ray KD meets the circumcircle Ω of ABC at Q. Prove that C, N, K, Q are concyclic.

3 A positive integer k is given. Initially, N cells are marked on an infinite checkered plane. We say that the cross of a cell A is the set of all cells lying in the same row or in the same column as A. By a turn, it is allowed to mark an unmarked cell A if the cross of A contains at least k marked cells. It appears that every cell can be marked in a sequence of such turns. Determine the smallest possible value of N.

4 Initially, a positive integer is written on the blackboard. Every second, one adds to the number on the board the product of all its nonzero digits, writes down the results on the board, and erases the previous number. Prove that there exists a positive integer which will be added inifinitely many times.

5 In a 10×10 table, positive numbers are written. It is known that, looking left-right, the numbers in each row form an arithmetic progression and, looking up-down, the numbers is each column form a geometric progression. Prove that all the ratios of the geometric progressions are equal.

6	Same as Grade 9 P6
7	Same as Grade 9 P8
8	The board used for playing a game consists of the left and right parts. In each part there are several fields and therere several segments connecting two fields from different parts (all the fields are connected.) Initially, there is a violet counter on a field in the left part, and a purple counter on a field in the right part. Lyosha and Pasha alternatively play their turn, starting from Pasha, by moving their chip (Lyosha-violet, and Pasha-purple) over a segment to other field that has no chip. Its prohibited to repeat a position twice, i.e. cant move to position that already been occupied by some earlier turns in the game. A player losses if he cant make a move. Is there a board and an initial positions of counters that Pasha has a winning strategy?
–	Grade 11
1	The polynomial $P(x)$ is such that the polynomials $P(P(x))$ and $P(P(P(x)))$ are strictly monotone on the whole real axis. Prove that $P(x)$ is also strictly monotone on the whole thing axis.
2	Let $n \geq 2$ and $x_1, x_2, ..., x_n$ positive real numbers.Prove that $\frac{1+x_1^2}{1+x_1x_2} + \frac{1+x_2^2}{1+x_2x_3} + ... + \frac{1+x_n^2}{1+x_nx_1} \geq n$
3	Same as Grade 10 P3
4	On the sides AB and AC of the triangle ABC, the points P and Q are chosen, respectively, so that $PQ \vert\vert BC$. Segments of BQ and CP intersect at point O. Point A' is symmetric to point A relative to lineBC. The segment $A'O$ intersects circle w circumcircle of the triangle APQ, at the point S. Prove that circumcircle of BSC is tangent to the circle w.
5	On the table, there're 1000 cards arranged on a circle. On each card, a positive integer was written so that all 1000 numbers are distinct. First, Vasya selects one of the card, remove it from the circle, and do the following operation: If on the last card taken out was written positive integer k, count the k^{th} clockwise card not removed, from that position, then remove it and repeat the operation. This continues until only one card left on the table. Is it possible that, initially,

there's a card A such that, no matter what other card Vasya selects as first card, the one that left is always card A?

6 Three diagonals of a regular n-gon prism intersect at an interior point O. Show that O is the center of the prism.
(The diagonal of the prism is a segment joining two vertices not lying on the same face of the prism.)

7 Given a sequence of positive integers $a_1, a_2, a_3, ...$ defined by $a_n = \lfloor n^{\frac{2018}{2017}} \rfloor$. Show that there exists a positive integer N such that among any N consecutive terms in the sequence, there exists a term whose decimal representation contain digit 5.

8 Initially, on the lower left and right corner of a 2018×2018 board, there're two horses, red and blue, respectively. A and B alternatively play their turn, A start first. Each turn consist of moving their horse (A-red, and B-blue) by, simultaneously, 20 cells respect to one coordinate, and 17 cells respect to the other; while preserving the rule that the horse can't occupied the cell that ever occupied by any horses in the game. The player who can't make the move loss, who has the winning strategy?

Canada National Olympiad 2014

– April 2nd

1 Let $a_1, a_2, \dots, a_n$ be positive real numbers whose product is 1. Show that the sum

$$\frac{a_1}{1+a_1} + \frac{a_2}{(1+a_1)(1+a_2)} + \frac{a_3}{(1+a_1)(1+a_2)(1+a_3)} + \cdots + \frac{a_n}{(1+a_1)(1+a_2)\cdots(1+a_n)}$$

is greater than or equal to $\frac{2^n-1}{2^n}$.

2 Let m and n be odd positive integers. Each square of an m by n board is coloured red or blue. A row is said to be red-dominated if there are more red squares than blue squares in the row. A column is said to be blue-dominated if there are more blue squares than red squares in the column. Determine the maximum possible value of the number of red-dominated rows plus the number of blue-dominated columns. Express your answer in terms of m and n.

3 Let p be a fixed odd prime. A p-tuple $(a_1, a_2, a_3, \dots, a_p)$ of integers is said to be *good* if

- **(i)** $0 \le a_i \le p-1$ for all i, and
- **(ii)** $a_1 + a_2 + a_3 + \cdots + a_p$ is not divisible by p, and
- **(iii)** $a_1a_2 + a_2a_3 + a_3a_4 + \cdots + a_pa_1$ is divisible by p.

Determine the number of good p-tuples.

4 The quadrilateral $ABCD$ is inscribed in a circle. The point P lies in the interior of $ABCD$, and $\angle PAB = \angle PBC = \angle PCD = \angle PDA$. The lines AD and BC meet at Q, and the lines AB and CD meet at R. Prove that the lines PQ and PR form the same angle as the diagonals of $ABCD$.

5 Fix positive integers n and $k \ge 2$. A list of n integers is written in a row on a blackboard. You can choose a contiguous block of integers, and I will either add 1 to all of them or subtract 1 from all of them. You can repeat this step as often as you like, possibly adapting your selections based on what I do. Prove that after a finite number of steps, you can reach a state where at least $n-k+2$ of the numbers on the blackboard are all simultaneously divisible by k.

Canada National Olympiad 2015

1 Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of positive integers. Find all functions f, defined on $\mathbb{N}$ and taking values in $\mathbb{N}$, such that $(n-1)^2 < f(n)f(f(n)) < n^2 + n$ for every positive integer n.

2 Let ABC be an acute-angled triangle with altitudes AD, BE, and CF. Let H be the orthocentre, that is, the point where the altitudes meet. Prove that

$$\frac{AB \cdot AC + BC \cdot CA + CA \cdot CB}{AH \cdot AD + BH \cdot BE + CH \cdot CF} \leq 2.$$

3 On a $(4n+2) \times (4n+2)$ square grid, a turtle can move between squares sharing a side.The turtle begins in a corner square of the grid and enters each square exactly once, ending in the square where she started. In terms of n, what is the largest positive integer k such that there must be a row or column that the turtle has entered at least k distinct times?

4 Let ABC be an acute-angled triangle with circumcenter O. Let I be a circle with centre on the altitude from A in ABC, passing through vertex A and points P and Q on sides AB and AC. Assume that

$$BP.CQ = AP.AQ$$

Prove that I is tangent to the circumcircle of triangle BOC

5 Let p be a prime number for which $\frac{p-1}{2}$ is also prime, and let a, b, c be integers not divisible by p. Prove that there are at most $1 + \sqrt{2p}$ positive integers n such that $n < p$ and p divides $a^n + b^n + c^n$.

Canada National Olympiad 2016

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1 The integers $1, 2, 3, \ldots, 2016$ are written on a board. You can choose any two numbers on the board and replace them with their average. For example, you can replace 1 and 2 with 1.5, or you can replace 1 and 3 with a second copy of 2. After 2015 replacements of this kind, the board will have only one number left on it.

(a) Prove that there is a sequence of replacements that will make the final number equal to 2.

(b) Prove that there is a sequence of replacements that will make the final number equal to 1000.

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2 Consider the following system of 10 equations in 10 real variables $v_1, \ldots, v_{10}$:

$$v_i = 1 + \frac{6v_i^2}{v_1^2 + v_2^2 + \cdots + v_{10}^2} \qquad (i = 1, \ldots, 10).$$

Find all 10-tuples $(v_1, v_2, \ldots, v_{10})$ that are solutions of this system.

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3 Find all polynomials $P(x)$ with integer coefficients such that $P(P(n) + n)$ is a prime number for infinitely many integers n.

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4 Let A, B, and F be positive integers, and assume $A < B < 2A$. A flea is at the number 0 on the number line. The flea can move by jumping to the right by A or by B. Before the flea starts jumping, Lavaman chooses finitely many intervals $\{m + 1, m + 2, \ldots, m + A\}$ consisting of A consecutive positive integers, and places lava at all of the integers in the intervals. The intervals must be chosen so that:

(i) any two distinct intervals are disjoint and not adjacent;
(ii) there are at least F positive integers with no lava between any two intervals; and
(iii) no lava is placed at any integer less than F.

Prove that the smallest F for which the flea can jump over all the intervals and avoid all the lava, regardless of what Lavaman does, is $F = (n-1)A + B$, where n is the positive integer such that $\frac{A}{n+1} \leq B - A < \frac{A}{n}$.

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5 Let $\triangle ABC$ be an acute-angled triangle with altitudes AD and BE meeting at H. Let M be the midpoint of segment AB, and suppose that the circumcircles of $\triangle DEM$ and $\triangle ABH$ meet at points P and Q with P on the same side of

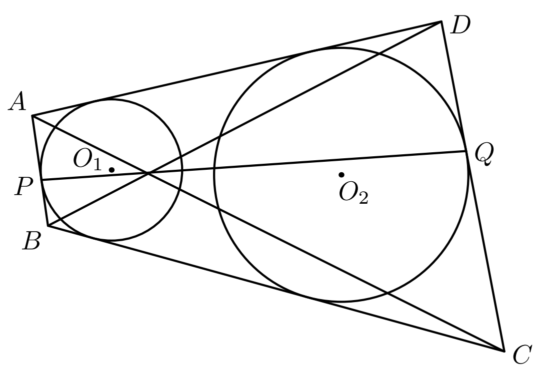
CH as A. Prove that the lines ED, PH, and MQ all pass through a single point on the circumcircle of $\triangle ABC$.

China Girls Math Olympiad 2013

Day 1

1 Let A be the closed region bounded by the following three lines in the xy plane: $x = 1, y = 0$ and $y = t(2x - t)$, where $0 < t < 1$. Prove that the area of any triangle inside the region A, with two vertices $P(t, t^2)$ and $Q(1, 0)$, does not exceed $\frac{1}{4}$.

2 As shown in the figure below, $ABCD$ is a trapezoid, $AB \parallel CD$. The sides DA, AB, BC are tangent to $\odot O_1$ and AB touches $\odot O_1$ at P. The sides BC, CD, DA are tangent to $\odot O_2$, and CD touches $\odot O_2$ at Q. Prove that the lines AC, BD, PQ meet at the same point.



3 In a group of m girls and n boys, any two persons either know each other or do not know each other. For any two boys and any two girls, there are at least one boy and one girl among them,who do not know each other. Prove that the number of unordered pairs of (boy, girl) who know each other does not exceed $m + \frac{n(n-1)}{2}$.

4 Find the number of polynomials $f(x) = ax^3 + bx$ satisfying both following conditions:
(i) $a, b \in \{1, 2, \ldots, 2013\}$;
(ii) the difference between any two of $f(1), f(2), \ldots, f(2013)$ is not a multiple of 2013.

Day 2

5 For any given positive numbers $a_1, a_2, \ldots, a_n$, prove that there exist positive numbers $x_1, x_2, \ldots, x_n$ satisfying $\sum_{i=1}^n x_i = 1$, such that for any positive numbers $y_1, y_2, \ldots, y_n$ with $\sum_{i=1}^n y_i = 1$, the inequality $\sum_{i=1}^n \frac{a_i x_i}{x_i + y_i} \geq \frac{1}{2} \sum_{i=1}^n a_i$ holds.

6 Let S be a subset of $\{0, 1, 2, \ldots, 98\}$ with exactly $m \geq 3$ (distinct) elements, such that for any $x, y \in S$ there exists $z \in S$ satisfying $x + y \equiv 2z \pmod{99}$. Determine all possible values of m.

7 As shown in the figure, $\odot O_1$ and $\odot O_2$ touches each other externally at a point T, quadrilateral $ABCD$ is inscribed in $\odot O_1$, and the lines DA, CB are tangent to $\odot O_2$ at points E and F respectively. Line BN bisects $\angle ABF$ and meets segment EF at N. Line FT meets the arc $\widehat{AT}$ (not passing through the point B) at another point M different from A. Prove that M is the circumcenter of $\triangle BCN$.

8 Let n (≥ 4) be an even integer. We label n pairwise distinct real numbers arbitrarily on the n vertices of a regular n-gon, and label the n sides clockwise as $e_1, e_2, \ldots, e_n$. A side is called *positive* if the numbers on both endpoints are increasing in clockwise direction. An unordered pair of distinct sides $\{e_i, e_j\}$ is called *alternating* if it satisfies both conditions:

(i) $2 \mid (i + j)$; and

(ii) if one rearranges the four numbers on the vertices of these two sides e_i and e_j in increasing order $a < b < c < d$, then a and c are the numbers on the two endpoints of one of sides e_i or e_j.

Prove that the number of alternating pairs of sides and the number of positive sides are of different parity.

China Girls Math Olympiad 2014

Day 1

1 In the figure of http://www.artofproblemsolving.com/Forum/download/file.php?id=50643&mode=view
$\odot O_1$ and $\odot O_2$ intersect at two points A, B.
The extension of O_1A meets $\odot O_2$ at C, and the extension of O_2A meets $\odot O_1$ at D,
and through B draw $BE \parallel O_2A$ intersecting $\odot O_1$ again at E.
If $DE \parallel O_1A$, prove that $DC \perp CO_2$.

2 Let $x_1, x_2, \dots, x_n$ be real numbers, where $n \ge 2$ is a given integer, and let $\lfloor x_1 \rfloor, \lfloor x_2 \rfloor, \dots, \lfloor x_n \rfloor$ be a permutation of $1, 2, \dots, n$.
Find the maximum and minimum of $\sum\limits_{i=1}^{n-1} \lfloor x_{i+1} - x_i \rfloor$ (here $\lfloor x \rfloor$ is the largest integer not greater than x).

3 There are n students; each student knows exactly d girl students and d boy students (”knowing” is a symmetric relation). Find all pairs (n, d) of integers .

4 For an integer $m \ge 4$, let T_m denote the number of sequences $a_1, \dots, a_m$ such that the following conditions hold:
(1) For all $i = 1, 2, \dots, m$ we have $a_i \in \{1, 2, 3, 4\}$
(2) $a_1 = a_m = 1$ and $a_2 \neq 1$
(3) For all $i = 3, 4 \cdots, m, a_i \neq a_{i-1}, a_i \neq a_{i-2}$.

Prove that there exists a geometric sequence of positive integers $\{g_n\}$ such that for $n \ge 4$ we have that

$$g_n - 2\sqrt{g_n} < T_n < g_n + 2\sqrt{g_n}.$$

Day 2

5 Let a be a positive integer, but not a perfect square; r is a real root of the equation $x^3 - 2ax + 1 = 0$. Prove that $r + \sqrt{a}$ is an irrational number.

6 In acute triangle ABC, $AB > AC$. D and E are the midpoints of AB, AC respectively.
The circumcircle of ADE intersects the circumcircle of BCE again at P.
The circumcircle of ADE intersects the circumcircle BCD again at Q.
Prove that $AP = AQ$.

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7 Given a finite nonempty set X with real values, let $f(X) = \frac{1}{|X|} \sum\limits_{a \in X} a$, where $|X|$ denotes the cardinality of X. For ordered pairs of sets (A, B) such that $A \cup B = \{1, 2, \dots, 100\}$ and $A \cap B = \emptyset$ where $1 \leq |A| \leq 98$, select some $p \in B$, and let $A_p = A \cup \{p\}$ and $B_p = B - \{p\}$. Over all such (A, B) and $p \in B$ determine the maximum possible value of $(f(A_p) - f(A))(f(B_p) - f(B))$.

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8 Let n be a positive integer, and set S be the set of all integers in $\{1, 2, \dots, n\}$ which are relatively prime to n.
Set $S_1 = S \cap \left(0, \frac{n}{3}\right]$, $S_2 = S \cap \left(\frac{n}{3}, \frac{2n}{3}\right]$, $S_3 = S \cap \left(\frac{2n}{3}, n\right]$.
If the cardinality of S is a multiple of 3, prove that S_1, S_2, S_3 have the same cardinality.

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China Girls Math Olympiad 2015

Day 1

1 Let $\triangle ABC$ be an acute-angled triangle with $AB > AC$, O be its circumcenter and D the midpoint of side BC. The circle with diameter AD meets sides AB, AC again at points E, F respectively. The line passing through D parallel to AO meets EF at M. Show that $EM = MF$.

2 Let $a \in (0,1)$,$f(x) = ax^3 + (1-4a)x^2 + (5a-1)x - 5a + 3$, $g(x) = (1-a)x^3 - x^2 + (2-a)x - 3a - 1$.
Prove that:For any real number x ,at least one of $|f(x)|$ and $|g(x)|$ not less than $a+1$.

3 In a 12×12 grid, colour each unit square with either black or white, such that there is at least one black unit square in any 3×4 and 4×3 rectangle bounded by the grid lines. Determine, with proof, the minimum number of black unit squares.

4 Let $g(n)$ be the greatest common divisor of n and 2015. Find the number of triples (a, b, c) which satisfies the following two conditions: 1) $a, b, c \in \{1, 2, ..., 2015\}$; 2) $g(a), g(b), g(c), g(a+b), g(b+c), g(c+a), g(a+b+c)$ are pairwise distinct.

Day 2

5 Determine the number of distinct right-angled triangles such that its three sides are of integral lengths, and its area is 999 times of its perimeter.
(Congruent triangles are considered identical.)

6 Let Γ_1 and Γ_2 be two non-overlapping circles. A, C are on Γ_1 and B, D are on Γ_2 such that AB is an external common tangent to the two circles, and CD is an internal common tangent to the two circles. AC and BD meet at E. F is a point on Γ_1, the tangent line to Γ_1 at F meets the perpendicular bisector of EF at M. MG is a line tangent to Γ_2 at G. Prove that $MF = MG$.

7 Let $x_1, x_2, \cdots, x_n \in (0,1)$, $n \geq 2$. Prove that

$$\frac{\sqrt{1-x_1}}{x_1} + \frac{\sqrt{1-x_2}}{x_2} + \cdots + \frac{\sqrt{1-x_n}}{x_n} < \frac{\sqrt{n-1}}{x_1 x_2 \cdots x_n}.$$

8 Let $n \geq 2$ be a given integer. Initially, we write n sets on the blackboard and do a sequence of moves as follows: choose two sets A and B on the blackboard such that none of them is a subset of the other, and replace A and B by $A \cap B$ and $A \cup B$. This is called a *move*.
Find the maximum number of moves in a sequence for all possible initial sets.

China Girls Math Olympiad 2016

Day 1

1 Let $n \geq 3$ be an integer. Put n^2 cards, each labelled $1, 2, \ldots, n^2$ respectively, in any order into n empty boxes such that there are exactly n cards in each box. One can perform the following operation: one first selects 2 boxes, takes out any 2 cards from each of the selected boxes, and then return the cards to the other selected box. Prove that, for any initial order of the n^2 cards in the boxes, one can perform the operation finitely many times such that the labelled numbers in each box are consecutive integers.

2 In $\triangle ABC, BC = a, CA = b, AB = c$, and Γ is its circumcircle. (1) Determine a necessary and sufficient condition on a, b and c if there exists a unique point $P(P \neq B, P \neq C)$ on the arc BC of Γ not passing through point A such that $PA = PB + PC$. (2) Let P be the unique point stated in (1). If AP bisects BC, prove that $\angle BAC < 60^\circ$.

3 Let m and n are relatively prime integers and $m > 1, n > 1$. Show that:There are positive integers a, b, c such that $m^a = 1 + n^b c$, and n and c are relatively prime.

4 Let n is a positive integers ,$a_1, a_2, \cdots, a_n \in \{0, 1, \cdots, n\}$. For the integer j $(1 \leq j \leq n)$,define b_j is the number of elements in the set $\{i|i \in \{1, \cdots, n\}, a_i \geq j\}$ .For example When $n = 3$,if $a_1 = 1, a_2 = 2, a_3 = 1$,then $b_1 = 3, b_2 = 1, b_3 = 0$. (1) Prove that

$$\sum_{i=1}^{n} (i + a_i)^2 \geq \sum_{i=1}^{n} (i + b_i)^2.$$

(2) Prove that

$$\sum_{i=1}^{n} (i + a_i)^k \geq \sum_{i=1}^{n} (i + b_i)^k,$$

for the integer $k \geq 3$.

Day 2

5 Define a sequence $\{a_n\}$ by

$$S_1 = 1,\ S_{n+1} = \frac{(2 + S_n)^2}{4 + S_n} (n = 1,\ 2,\ 3,\ \cdots).$$

Where S_n the sum of first n terms of sequence $\{a_n\}$.
For any positive integer n ,prove that

$$a_n \geq \frac{4}{\sqrt{9n+7}}.$$

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6 Find the greatest positive integer m, such that one of the 4 letters C, G, M, O can be placed in each cell of a table with m rows and 8 columns, and has the following property: For any two distinct rows in the table, there exists at most one column, such that the entries of these two rows in such a column are the same letter.

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7 In acute triangle $ABC, AB < AC$, I is its incenter, D is the foot of perpendicular from I to BC, altitude AH meets BI, CI at P, Q respectively. Let O be the circumcenter of $\triangle IPQ$, extend AO to meet BC at L. Circumcircle of $\triangle AIL$ meets BC again at N. Prove that $\frac{BD}{CD} = \frac{BN}{CN}$.

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8 Let $\mathbb{Q}$ be the set of rational numbers, $\mathbb{Z}$ be the set of integers. On the coordinate plane, given positive integer m, define

$$A_m = \left\{ (x,y) \mid x, y \in \mathbb{Q}, xy \neq 0, \frac{xy}{m} \in \mathbb{Z} \right\}.$$

For segment MN, define $f_m(MN)$ as the number of points on segment MN belonging to set A_m.

Find the smallest real number λ, such that for any line l on the coordinate plane, there exists a constant $\beta(l)$ related to l, satisfying: for any two points M, N on l,

$$f_{2016}(MN) \leq \lambda f_{2015}(MN) + \beta(l)$$

ELMO Shortlist 2013

– Algebra

1 Find all triples (f, g, h) of injective functions from the set of real numbers to itself satisfying

$$\begin{aligned} f(x+f(y)) &= g(x)+h(y) \\ g(x+g(y)) &= h(x)+f(y) \\ h(x+h(y)) &= f(x)+g(y) \end{aligned}$$

for all real numbers x and y. (We say a function F is *injective* if $F(a) \neq F(b)$ for any distinct real numbers a and b.)

Proposed by Evan Chen

2 Prove that for all positive reals a, b, c,

$$\frac{1}{a+\frac{1}{b}+1}+\frac{1}{b+\frac{1}{c}+1}+\frac{1}{c+\frac{1}{a}+1} \geq \frac{3}{\sqrt[3]{abc}+\frac{1}{\sqrt[3]{abc}}+1}.$$

Proposed by David Stoner

3 Find all $f: \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$, $f(x)+f(y) = f(x+y)$ and $f(x^{2013}) = f(x)^{2013}$.

Proposed by Calvin Deng

4 Positive reals a, b, and c obey $\frac{a^2+b^2+c^2}{ab+bc+ca} = \frac{ab+bc+ca+1}{2}$. Prove that

$$\sqrt{a^2+b^2+c^2} \leq 1+\frac{|a-b|+|b-c|+|c-a|}{2}.$$

Proposed by Evan Chen

5 Let a, b, c be positive reals satisfying $a+b+c = \sqrt[7]{a}+\sqrt[7]{b}+\sqrt[7]{c}$. Prove that $a^a b^b c^c \geq 1$.

Proposed by Evan Chen

6 Let a, b, c be positive reals such that $a+b+c=3$. Prove that

$$18\sum_{\text{cyc}}\frac{1}{(3-c)(4-c)}+2(ab+bc+ca)\geq 15.$$

Proposed by David Stoner

7 Consider a function $f:\mathbb{Z}\to\mathbb{Z}$ such that for every integer $n\geq 0$, there are at most $0.001n^2$ pairs of integers (x,y) for which $f(x+y)\neq f(x)+f(y)$ and $\max\{|x|,|y|\}\leq n$. Is it possible that for some integer $n\geq 0$, there are more than n integers a such that $f(a)\neq a\cdot f(1)$ and $|a|\leq n$?

Proposed by David Yang

8 Let a, b, c be positive reals with $a^{2014}+b^{2014}+c^{2014}+abc=4$. Prove that

$$\frac{a^{2013}+b^{2013}-c}{c^{2013}}+\frac{b^{2013}+c^{2013}-a}{a^{2013}}+\frac{c^{2013}+a^{2013}-b}{b^{2013}}\geq a^{2012}+b^{2012}+c^{2012}.$$

Proposed by David Stoner

9 Let a, b, c be positive reals, and let $\sqrt[2013]{\frac{3}{a^{2013}+b^{2013}+c^{2013}}}=P$. Prove that

$$\prod_{\text{cyc}}\left(\frac{(2P+\frac{1}{2a+b})(2P+\frac{1}{a+2b})}{(2P+\frac{1}{a+b+c})^2}\right)\geq\prod_{\text{cyc}}\left(\frac{(P+\frac{1}{4a+b+c})(P+\frac{1}{3b+3c})}{(P+\frac{1}{3a+2b+c})(P+\frac{1}{3a+b+2c})}\right).$$

Proposed by David Stoner

– Combinatorics

1 Let $n\geq 2$ be a positive integer. The numbers $1, 2, ..., n^2$ are consecutively placed into squares of an $n\times n$, so the first row contains $1, 2, ..., n$ from left to right, the second row contains $n+1, n+2, ..., 2n$ from left to right, and so on. The *magic square value* of a grid is defined to be the number of rows, columns, and main diagonals whose elements have an average value of $\frac{n^2+1}{2}$. Show that the magic-square value of the grid stays constant under the following two operations: (1) a permutation of the rows; and (2) a permutation of the columns. (The operations can be used multiple times, and in any order.)

Proposed by Ray Li

2 Let n be a fixed positive integer. Initially, n 1's are written on a blackboard. Every minute, David picks two numbers x and y written on the blackboard, erases them, and writes the number $(x+y)^4$ on the blackboard. Show that after $n-1$ minutes, the number written on the blackboard is at least $2^{\frac{4n^2-4}{3}}$.

Proposed by Calvin Deng

3 Let $a_1, a_2, ..., a_9$ be nine real numbers, not necessarily distinct, with average m. Let A denote the number of triples $1 \le i < j < k \le 9$ for which $a_i + a_j + a_k \ge 3m$. What is the minimum possible value of A?

Proposed by Ray Li

4 Let n be a positive integer. The numbers $\{1, 2, ..., n^2\}$ are placed in an $n \times n$ grid, each exactly once. The grid is said to be *Muirhead-able* if the sum of the entries in each column is the same, but for every $1 \le i, k \le n-1$, the sum of the first k entries in column i is at least the sum of the first k entries in column $i+1$. For which n can one construct a Muirhead-able array such that the entries in each column are decreasing?

Proposed by Evan Chen

5 There is a 2012×2012 grid with rows numbered $1, 2, \dots 2012$ and columns numbered $1, 2, \dots, 2012$, and we place some rectangular napkins on it such that the sides of the napkins all lie on grid lines. Each napkin has a positive integer thickness. (in micrometers!)

(a) Show that there exist 2012^2 unique integers $a_{i,j}$ where $i, j \in [1, 2012]$ such that for all $x, y \in [1, 2012]$, the sum

$$\sum_{i=1}^{x} \sum_{j=1}^{y} a_{i,j}$$

is equal to the sum of the thicknesses of all the napkins that cover the grid square in row x and column y.
(b) Show that if we use at most $500,000$ napkins, at least half of the $a_{i,j}$ will be 0.

Proposed by Ray Li

6 A 4×4 grid has its 16 cells colored arbitrarily in three colors. A *swap* is an exchange between the colors of two cells. Prove or disprove that it always takes at most three swaps to produce a line of symmetry, regardless of the grid's initial coloring.

Proposed by Matthew Babbitt

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7 A $2^{2014}+1$ by $2^{2014}+1$ grid has some black squares filled. The filled black squares form one or more snakes on the plane, each of whose heads splits at some points but never comes back together. In other words, for every positive integer n greater than 2, there do not exist pairwise distinct black squares s_1, s_2, \dots, s_n such that s_i and s_{i+1} share an edge for $i = 1, 2, \dots, n$ (here $s_{n+1} = s_1$).

What is the maximum possible number of filled black squares?

Proposed by David Yang

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8 There are 20 people at a party. Each person holds some number of coins. Every minute, each person who has at least 19 coins simultaneously gives one coin to every other person at the party. (So, it is possible that A gives B a coin and B gives A a coin at the same time.) Suppose that this process continues indefinitely. That is, for any positive integer n, there exists a person who will give away coins during the nth minute. What is the smallest number of coins that could be at the party?

Proposed by Ray Li

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9 Let f_0 be the function from $\mathbb{Z}^2$ to $\{0,1\}$ such that $f_0(0,0) = 1$ and $f_0(x,y) = 0$ otherwise. For each positive integer m, let $f_m(x,y)$ be the remainder when

$$f_{m-1}(x,y) + \sum_{j=-1}^{1}\sum_{k=-1}^{1} f_{m-1}(x+j, y+k)$$

is divided by 2.
Finally, for each nonnegative integer n, let a_n denote the number of pairs (x,y) such that $f_n(x,y) = 1$.
Find a closed form for a_n.

Proposed by Bobby Shen

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10 Let $N \geq 2$ be a fixed positive integer. There are $2N$ people, numbered $1, 2, ..., 2N$, participating in a tennis tournament. For any two positive integers i, j with $1 \leq i < j \leq 2N$, player i has a higher skill level than player j. Prior to the first round, the players are paired arbitrarily and each pair is assigned a unique court among N courts, numbered $1, 2, ..., N$.

During a round, each player plays against the other person assigned to his court (so that exactly one match takes place per court), and the player with higher skill wins the match (in other words, there are no upsets). Afterwards, for

$i = 2, 3, ..., N$, the winner of court i moves to court $i - 1$ and the loser of court i stays on court i; however, the winner of court 1 stays on court 1 and the loser of court 1 moves to court N.

Find all positive integers M such that, regardless of the initial pairing, the players $2, 3, \ldots, N + 1$ all change courts immediately after the Mth round.

Proposed by Ray Li

– Geometry

1 Let ABC be a triangle with incenter I. Let U, V and W be the intersections of the angle bisectors of angles A, B, and C with the incircle, so that V lies between B and I, and similarly with U and W. Let X, Y, and Z be the points of tangency of the incircle of triangle ABC with BC, AC, and AB, respectively. Let triangle UVW be the *David Yang triangle* of ABC and let XYZ be the *Scott Wu triangle* of ABC. Prove that the David Yang and Scott Wu triangles of a triangle are congruent if and only if ABC is equilateral.

Proposed by Owen Goff

2 Let ABC be a scalene triangle with circumcircle Γ, and let D,E,F be the points where its incircle meets BC, AC, AB respectively. Let the circumcircles of $\triangle AEF$, $\triangle BFD$, and $\triangle CDE$ meet Γ a second time at X, Y, Z respectively. Prove that the perpendiculars from A, B, C to AX, BY, CZ respectively are concurrent.

Proposed by Michael Kural

3 In $\triangle ABC$, a point D lies on line BC. The circumcircle of ABD meets AC at F (other than A), and the circumcircle of ADC meets AB at E (other than A). Prove that as D varies, the circumcircle of AEF always passes through a fixed point other than A, and that this point lies on the median from A to BC.

Proposed by Allen Liu

4 Triangle ABC is inscribed in circle ω. A circle with chord BC intersects segments AB and AC again at S and R, respectively. Segments BR and CS meet at L, and rays LR and LS intersect ω at D and E, respectively. The internal angle bisector of $\angle BDE$ meets line ER at K. Prove that if $BE = BR$, then $\angle ELK = \frac{1}{2}\angle BCD$.

Proposed by Evan Chen

5 Let ω_1 and ω_2 be two orthogonal circles, and let the center of ω_1 be O. Diameter AB of ω_1 is selected so that B lies strictly inside ω_2. The two circles tangent to ω_2, passing through O and A, touch ω_2 at F and G. Prove that $FGOB$ is cyclic.

Proposed by Eric Chen

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6 Let $ABCDEF$ be a non-degenerate cyclic hexagon with no two opposite sides parallel, and define $X = AB \cap DE$, $Y = BC \cap EF$, and $Z = CD \cap FA$. Prove that

$$\frac{XY}{XZ} = \frac{BE}{AD}\frac{\sin|\angle B - \angle E|}{\sin|\angle A - \angle D|}.$$

Proposed by Victor Wang

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7 Let ABC be a triangle inscribed in circle ω, and let the medians from B and C intersect ω at D and E respectively. Let O_1 be the center of the circle through D tangent to AC at C, and let O_2 be the center of the circle through E tangent to AB at B. Prove that O_1, O_2, and the nine-point center of ABC are collinear.

Proposed by Michael Kural

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8 Let ABC be a triangle, and let D, A, B, E be points on line AB, in that order, such that $AC = AD$ and $BE = BC$. Let ω_1, ω_2 be the circumcircles of $\triangle ABC$ and $\triangle CDE$, respectively, which meet at a point $F \neq C$. If the tangent to ω_2 at F cuts ω_1 again at G, and the foot of the altitude from G to FC is H, prove that $\angle AGH = \angle BGH$.

Proposed by David Stoner

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9 Let $ABCD$ be a cyclic quadrilateral inscribed in circle ω whose diagonals meet at F. Lines AB and CD meet at E. Segment EF intersects ω at X. Lines BX and CD meet at M, and lines CX and AB meet at N. Prove that MN and BC concur with the tangent to ω at X.

Proposed by Allen Liu

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10 Let $AB = AC$ in $\triangle ABC$, and let D be a point on segment AB. The tangent at D to the circumcircle ω of BCD hits AC at E. The other tangent from E to ω touches it at F, and $G = BF \cap CD$, $H = AG \cap BC$. Prove that $BH = 2HC$.

Proposed by David Stoner

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11 Let $\triangle ABC$ be a nondegenerate isosceles triangle with $AB = AC$, and let D, E, F be the midpoints of BC, CA, AB respectively. BE intersects the circumcircle of $\triangle ABC$ again at G, and H is the midpoint of minor arc BC.

$CF \cap DG = I, BI \cap AC = J$. Prove that $\angle BJH = \angle ADG$ if and only if $\angle BID = \angle GBC$.

Proposed by David Stoner

12 Let ABC be a nondegenerate acute triangle with circumcircle ω and let its incircle γ touch AB, AC, BC at X, Y, Z respectively. Let XY hit arcs AB, AC of ω at M, N respectively, and let $P \neq X, Q \neq Y$ be the points on γ such that $MP = MX, NQ = NY$. If I is the center of γ, prove that P, I, Q are collinear if and only if $\angle BAC = 90^\circ$.

Proposed by David Stoner

13 In $\triangle ABC$, $AB < AC$. D and P are the feet of the internal and external angle bisectors of $\angle BAC$, respectively. M is the midpoint of segment BC, and ω is the circumcircle of $\triangle APD$. Suppose Q is on the minor arc AD of ω such that MQ is tangent to ω. QB meets ω again at R, and the line through R perpendicular to BC meets PQ at S. Prove SD is tangent to the circumcircle of $\triangle QDM$.

Proposed by Ray Li

14 Let O be a point (in the plane) and T be an infinite set of points such that $|P_1P_2| \leq 2012$ for every two distinct points $P_1, P_2 \in T$. Let $S(T)$ be the set of points Q in the plane satisfying $|QP| \leq 2013$ for at least one point $P \in T$.

Now let L be the set of lines containing exactly one point of $S(T)$. Call a line ℓ_0 passing through O *bad* if there does not exist a line $\ell \in L$ parallel to (or coinciding with) ℓ_0.

\begin{enumerate}[(a)]
\item Prove that L is nonempty.
\item Prove that one can assign a line $\ell(i)$ to each positive integer i so that for every bad line ℓ_0 passing through O, there exists a positive integer n with $\ell(n) = \ell_0$.
\end{enumerate}

Proposed by David Yang

– Number Theory

1 Find all ordered triples of non-negative integers (a, b, c) such that $a^2 + 2b + c$, $b^2 + 2c + a$, and $c^2 + 2a + b$ are all perfect squares.

Proposed by Matthew Babbitt

2 For what polynomials $P(n)$ with integer coefficients can a positive integer be assigned to every lattice point in $\mathbb{R}^3$ so that for every integer $n \ge 1$, the sum of the n^3 integers assigned to any $n \times n \times n$ grid of lattice points is divisible by $P(n)$?

Proposed by Andre Arslan

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3 Define a *beautiful number* to be an integer of the form a^n, where $a \in \{3,4,5,6\}$ and n is a positive integer.
Prove that each integer greater than 2 can be expressed as the sum of pairwise distinct beautiful numbers.

Proposed by Matthew Babbitt

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4 Find all triples (a,b,c) of positive integers such that if n is not divisible by any prime less than 2014, then $n+c$ divides a^n+b^n+n.

Proposed by Evan Chen

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5 Let $m_1, m_2, ..., m_{2013} > 1$ be 2013 pairwise relatively prime positive integers and $A_1, A_2, ..., A_{2013}$ be 2013 (possibly empty) sets with $A_i \subseteq \{1, 2, ..., m_i - 1\}$ for $i = 1, 2, ..., 2013$. Prove that there is a positive integer N such that

$$N \le \left(2\left|A_1\right|+1\right)\left(2\left|A_2\right|+1\right)\cdots\left(2\left|A_{2013}\right|+1\right)$$

and for each $i = 1, 2, ..., 2013$, there does *not* exist $a \in A_i$ such that m_i divides $N - a$.

Proposed by Victor Wang

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6 Let $\mathbb{N}$ denote the set of positive integers, and for a function f, let $f^k(n)$ denote the function f applied k times. Call a function $f : \mathbb{N} \to \mathbb{N}$ *saturated* if

$$f^{f^{f(n)}(n)}(n) = n$$

for every positive integer n. Find all positive integers m for which the following holds: every saturated function f satisfies $f^{2014}(m) = m$.

Proposed by Evan Chen

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7 Let p be a prime satisfying $p^2 \mid 2^{p-1} - 1$, and let n be a positive integer. Define

$$f(x) = \frac{(x-1)^{p^n} - (x^{p^n} - 1)}{p(x-1)}.$$

Find the largest positive integer N such that there exist polynomials $g(x)$, $h(x)$ with integer coefficients and an integer r satisfying $f(x) = (x-r)^N g(x) + p \cdot h(x)$.

Proposed by Victor Wang

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8 We define the *Fibonacci sequence* $\{F_n\}_{n\ge 0}$ by $F_0 = 0$, $F_1 = 1$, and for $n \ge 2$, $F_n = F_{n-1} + F_{n-2}$; we define the *Stirling number of the second kind* $S(n,k)$ as the number of ways to partition a set of $n \ge 1$ distinguishable elements into $k \ge 1$ indistinguishable nonempty subsets.

For every positive integer n, let $t_n = \sum_{k=1}^{n} S(n,k) F_k$. Let $p \ge 7$ be a prime. Prove that

$$t_{n+p^{2p}-1} \equiv t_n \pmod p$$

for all $n \ge 1$.

Proposed by Victor Wang

ELMO Shortlist 2014

– Algebra

1 In a non-obtuse triangle ABC, prove that

$$\frac{\sin A \sin B}{\sin C} + \frac{\sin B \sin C}{\sin A} + \frac{\sin C \sin A}{\sin B} \geq \frac{5}{2}.$$

Proposed by Ryan Alweiss

2 Given positive reals a, b, c, p, q satisfying $abc = 1$ and $p \geq q$, prove that

$$p\left(a^2 + b^2 + c^2\right) + q\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq (p+q)(a+b+c).$$

Proposed by AJ Dennis

3 Let a, b, c, d, e, f be positive real numbers. Given that $def + de + ef + fd = 4$, show that

$$((a+b)de + (b+c)ef + (c+a)fd)^2 \geq \ 12(abde + bcef + cafd).$$

Proposed by Allen Liu

4 Find all triples (f, g, h) of injective functions from the set of real numbers to itself satisfying

$$\begin{aligned} f(x + f(y)) &= g(x) + h(y) \\ g(x + g(y)) &= h(x) + f(y) \\ h(x + h(y)) &= f(x) + g(y) \end{aligned}$$

for all real numbers x and y. (We say a function F is *injective* if $F(a) \neq F(b)$ for any distinct real numbers a and b.)

Proposed by Evan Chen

5 Let $\mathbb{R}^*$ denote the set of nonzero reals. Find all functions $f : \mathbb{R}^* \to \mathbb{R}^*$ satisfying

$$f(x^2 + y) + 1 = f(x^2 + 1) + \frac{f(xy)}{f(x)}$$

for all $x, y \in \mathbb{R}^*$ with $x^2 + y \neq 0$.

Proposed by Ryan Alweiss

6 Let a, b, c be positive reals such that $a+b+c=ab+bc+ca$. Prove that

$$(a+b)^{ab-bc}(b+c)^{bc-ca}(c+a)^{ca-ab} \ge a^{ca}b^{ab}c^{bc}.$$

Proposed by Sammy Luo

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7 Find all positive integers n with $n \ge 2$ such that the polynomial

$$P(a_1, a_2, ..., a_n) = a_1^n + a_2^n + ... + a_n^n - na_1a_2...a_n$$

in the n variables a_1, a_2, $\dots$, a_n is irreducible over the real numbers, i.e. it cannot be factored as the product of two nonconstant polynomials with real coefficients.

Proposed by Yang Liu

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8 Let a, b, c be positive reals with $a^{2014}+b^{2014}+c^{2014}+abc=4$. Prove that

$$\frac{a^{2013}+b^{2013}-c}{c^{2013}} + \frac{b^{2013}+c^{2013}-a}{a^{2013}} + \frac{c^{2013}+a^{2013}-b}{b^{2013}} \ge a^{2012}+b^{2012}+c^{2012}.$$

Proposed by David Stoner

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9 Let a, b, c be positive reals. Prove that

$$\sqrt{\frac{a^2(bc+a^2)}{b^2+c^2}} + \sqrt{\frac{b^2(ca+b^2)}{c^2+a^2}} + \sqrt{\frac{c^2(ab+c^2)}{a^2+b^2}} \ge a+b+c.$$

Proposed by Robin Park

– Combinatorics

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1 You have some cyan, magenta, and yellow beads on a non-reorientable circle, and you can perform only the following operations:
1. Move a cyan bead right (clockwise) past a yellow bead, and turn the yellow bead magenta.
2. Move a magenta bead left of a cyan bead, and insert a yellow bead left of where the magenta bead ends up.
3. Do either of the above, switching the roles of the words “magenta” and “left” with those of “yellow” and “right”, respectively.
4. Pick any two disjoint consecutive pairs of beads, each either yellow-magenta or magenta-yellow, appearing somewhere in the circle, and swap the orders of each pair.
5. Remove four consecutive beads of one color.

Starting with the circle: "yellow, yellow, magenta, magenta, cyan, cyan, cyan", determine whether or not you can reach
a) "yellow, magenta, yellow, magenta, cyan, cyan, cyan",
b) "cyan, yellow, cyan, magenta, cyan",
c) "magenta, magenta, cyan, cyan, cyan",
d) "yellow, cyan, cyan, cyan".

Proposed by Sammy Luo

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2 A $2^{2014}+1$ by $2^{2014}+1$ grid has some black squares filled. The filled black squares form one or more snakes on the plane, each of whose heads splits at some points but never comes back together. In other words, for every positive integer n greater than 2, there do not exist pairwise distinct black squares s_1, s_2, \dots, s_n such that s_i and s_{i+1} share an edge for $i=1,2,\dots,n$ (here $s_{n+1}=s_1$).

What is the maximum possible number of filled black squares?

Proposed by David Yang

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3 We say a finite set S of points in the plane is *very* if for every point X in S, there exists an inversion with center X mapping every point in S other than X to another point in S (possibly the same point).

(a) Fix an integer n. Prove that if $n \ge 2$, then any line segment $\overline{AB}$ contains a unique very set S of size n such that $A, B \in S$.

(b) Find the largest possible size of a very set not contained in any line.

(Here, an *inversion* with center O and radius r sends every point P other than O to the point P' along ray OP such that $OP \cdot OP' = r^2$.)

Proposed by Sammy Luo

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4 Let r and b be positive integers. The game of *Monis*, a variant of Tetris, consists of a single column of red and blue blocks. If two blocks of the same color ever touch each other, they both vanish immediately. A red block falls onto the top of the column exactly once every r years, while a blue block falls exactly once every b years.

(a) Suppose that r and b are odd, and moreover the cycles are offset in such a way that no two blocks ever fall at exactly the same time. Consider a period of rb years in which the column is initially empty. Determine, in terms of r and b, the number of blocks in the column at the end.

(b) Now suppose r and b are relatively prime and $r+b$ is odd. At time $t=0$, the column is initially empty. Suppose a red block falls at times $t=r, 2r, \dots, (b-1)r$

years, while a blue block falls at times $t = b, 2b, \ldots, (r-1)b$ years. Prove that at time $t = rb$, the number of blocks in the column is $|1 + 2(r-1)(b+r) - 8S|$, where

$$S = \left\lfloor \frac{2r}{r+b} \right\rfloor + \left\lfloor \frac{4r}{r+b} \right\rfloor + \ldots + \left\lfloor \frac{(r+b-1)r}{r+b} \right\rfloor.$$

Proposed by Sammy Luo

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5 Let n be a positive integer. For any k, denote by a_k the number of permutations of $\{1, 2, \ldots, n\}$ with exactly k disjoint cycles. (For example, if $n = 3$ then $a_2 = 3$ since (1)(23), (2)(31), (3)(12) are the only such permutations.) Evaluate

$$a_n n^n + a_{n-1} n^{n-1} + \cdots + a_1 n.$$

Proposed by Sammy Luo

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6 Let f_0 be the function from $\mathbb{Z}^2$ to $\{0, 1\}$ such that $f_0(0,0) = 1$ and $f_0(x, y) = 0$ otherwise. For each positive integer m, let $f_m(x, y)$ be the remainder when

$$f_{m-1}(x, y) + \sum_{j=-1}^{1} \sum_{k=-1}^{1} f_{m-1}(x+j, y+k)$$

is divided by 2.
Finally, for each nonnegative integer n, let a_n denote the number of pairs (x, y) such that $f_n(x, y) = 1$.
Find a closed form for a_n.

Proposed by Bobby Shen

– Geometry

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1 Let ABC be a triangle with symmedian point K. Select a point A_1 on line BC such that the lines AB, AC, A_1K and BC are the sides of a cyclic quadrilateral. Define B_1 and C_1 similarly. Prove that A_1, B_1, and C_1 are collinear.

Proposed by Sammy Luo

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2 $ABCD$ is a cyclic quadrilateral inscribed in the circle ω. Let $AB \cap CD = E$, $AD \cap BC = F$. Let ω_1, ω_2 be the circumcircles of AEF, CEF, respectively. Let $\omega \cap \omega_1 = G$, $\omega \cap \omega_2 = H$. Show that AC, BD, GH are concurrent.

Proposed by Yang Liu

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3 Let $A_1A_2A_3\cdots A_{2013}$ be a cyclic 2013-gon. Prove that for every point P not the circumcenter of the 2013-gon, there exists a point $Q \neq P$ such that $\frac{A_iP}{A_iQ}$ is constant for $i \in \{1, 2, 3, \cdots, 2013\}$.

Proposed by Robin Park

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4 Let $ABCD$ be a quadrilateral inscribed in circle ω. Define $E = AA \cap CD$, $F = AA \cap BC$, $G = BE \cap \omega$, $H = BE \cap AD$, $I = DF \cap \omega$, and $J = DF \cap AB$. Prove that GI, HJ, and the B-symmedian are concurrent.

Proposed by Robin Park

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5 Let P be a point in the interior of an acute triangle ABC, and let Q be its isogonal conjugate. Denote by ω_P and ω_Q the circumcircles of triangles BPC and BQC, respectively. Suppose the circle with diameter $\overline{AP}$ intersects ω_P again at M, and line AM intersects ω_P again at X. Similarly, suppose the circle with diameter $\overline{AQ}$ intersects ω_Q again at N, and line AN intersects ω_Q again at Y.

Prove that lines MN and XY are parallel.

(Here, the points P and Q are *isogonal conjugates* with respect to $\triangle ABC$ if the internal angle bisectors of $\angle BAC$, $\angle CBA$, and $\angle ACB$ also bisect the angles $\angle PAQ$, $\angle PBQ$, and $\angle PCQ$, respectively. For example, the orthocenter is the isogonal conjugate of the circumcenter.)

Proposed by Sammy Luo

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6 Let $ABCD$ be a cyclic quadrilateral with center O.
Suppose the circumcircles of triangles AOB and COD meet again at G, while the circumcircles of triangles AOD and BOC meet again at H.
Let ω_1 denote the circle passing through G as well as the feet of the perpendiculars from G to AB and CD.
Define ω_2 analogously as the circle passing through H and the feet of the perpendiculars from H to BC and DA.
Show that the midpoint of GH lies on the radical axis of ω_1 and ω_2.

Proposed by Yang Liu

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7 Let ABC be a triangle with circumcenter O. Let P be a point inside ABC, so let the points D, E, F be on BC, AC, AB respectively so that the Miquel point of DEF with respect to ABC is P. Let the reflections of D, E, F over the midpoints of the sides that they lie on be R, S, T. Let the Miquel point of RST with respect to the triangle ABC be Q. Show that $OP = OQ$.

Proposed by Yang Liu

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8 In triangle ABC with incenter I and circumcenter O, let A', B', C' be the points of tangency of its circumcircle with its A, B, C-mixtilinear circles, respectively. Let ω_A be the circle through A' that is tangent to AI at I, and define ω_B, ω_C similarly. Prove that $\omega_A, \omega_B, \omega_C$ have a common point X other than I, and that $\angle AXO = \angle OXA'$.

Proposed by Sammy Luo

9 Let P be a point inside a triangle ABC such that $\angle PAC = \angle PCB$. Let the projections of P onto BC, CA, and AB be X, Y, Z respectively. Let O be the circumcenter of $\triangle XYZ$, H be the foot of the altitude from B to AC, N be the midpoint of AC, and T be the point such that $TYPO$ is a parallelogram. Show that $\triangle THN$ is similar to $\triangle PBC$.

Proposed by Sammy Luo

10 We are given triangles ABC and DEF such that $D \in BC, E \in CA, F \in AB$, $AD \perp EF, BE \perp FD, CF \perp DE$. Let the circumcenter of DEF be O, and let the circumcircle of DEF intersect BC, CA, AB again at R, S, T respectively. Prove that the perpendiculars to BC, CA, AB through D, E, F respectively intersect at a point X, and the lines AR, BS, CT intersect at a point Y, such that O, X, Y are collinear.

Proposed by Sammy Luo

12 Let $AB = AC$ in $\triangle ABC$, and let D be a point on segment AB. The tangent at D to the circumcircle ω of BCD hits AC at E. The other tangent from E to ω touches it at F, and $G = BF \cap CD$, $H = AG \cap BC$. Prove that $BH = 2HC$.

Proposed by David Stoner

13 Let ABC be a nondegenerate acute triangle with circumcircle ω and let its incircle γ touch AB, AC, BC at X, Y, Z respectively. Let XY hit arcs AB, AC of ω at M, N respectively, and let $P \neq X, Q \neq Y$ be the points on γ such that $MP = MX, NQ = NY$. If I is the center of γ, prove that P, I, Q are collinear if and only if $\angle BAC = 90^\circ$.

Proposed by David Stoner

– Number Theory

1 Does there exist a strictly increasing infinite sequence of perfect squares $a_1, a_2, a_3, ...$ such that for all $k \in \mathbb{Z}^+$ we have that $13^k | a_k + 1$?

Proposed by Jesse Zhang

2 Define the Fibanocci sequence recursively by $F_1 = 1$, $F_2 = 1$ and $F_{i+2} = F_i + F_{i+1}$ for all i. Prove that for all integers $b, c > 1$, there exists an integer n such that the sum of the digits of F_n when written in base b is greater than c.

Proposed by Ryan Alweiss

3 Let t and n be fixed integers each at least 2. Find the largest positive integer m for which there exists a polynomial P, of degree n and with rational coefficients, such that the following property holds: exactly one of

$$\frac{P(k)}{t^k} \text{ and } \frac{P(k)}{t^{k+1}}$$

is an integer for each $k = 0, 1, ..., m$.

Proposed by Michael Kural

4 Let $\mathbb{N}$ denote the set of positive integers, and for a function f, let $f^k(n)$ denote the function f applied k times. Call a function $f : \mathbb{N} \to \mathbb{N}$ *saturated* if

$$f^{f^{f(n)}(n)}(n) = n$$

for every positive integer n. Find all positive integers m for which the following holds: every saturated function f satisfies $f^{2014}(m) = m$.

Proposed by Evan Chen

5 Define a *beautiful number* to be an integer of the form a^n, where $a \in \{3, 4, 5, 6\}$ and n is a positive integer.
Prove that each integer greater than 2 can be expressed as the sum of pairwise distinct beautiful numbers.

Proposed by Matthew Babbitt

6 Show that the numerator of

$$\frac{2^{p-1}}{p+1} - \left(\sum_{k=0}^{p-1} \frac{\binom{p-1}{k}}{(1-kp)^2}\right)$$

is a multiple of p^3 for any odd prime p.

Proposed by Yang Liu

7 Find all triples (a, b, c) of positive integers such that if n is not divisible by any prime less than 2014, then $n + c$ divides $a^n + b^n + n$.

Proposed by Evan Chen

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8 Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that:
(i) The greatest common divisor of the sequence $f(1), f(2), \dots$ is 1.
(ii) For all sufficiently large integers n, we have $f(n) \neq 1$ and

$$f(a)^n \mid f(a+b)^{a^{n-1}} - f(b)^{a^{n-1}}$$

for all positive integers a and b.

Proposed by Yang Liu

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9 Let d be a positive integer and let ε be any positive real. Prove that for all sufficiently large primes p with $\gcd(p-1, d) \neq 1$, there exists an positive integer less than p^r which is not a dth power modulo p, where r is defined by

$$\log r = \varepsilon - \frac{1}{\gcd(d, p-1)}.$$

Proposed by Shashwat Kishore

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10 Find all positive integer bases $b \geq 9$ so that the number

$$\frac{\overbrace{11\cdots1}^{n-1\ 1's}0\overbrace{77\cdots7}^{n-1\ 7's}8\overbrace{11\cdots1_b}^{n\ 1's}}{3}$$

is a perfect cube in base 10 for all sufficiently large positive integers n.

Proposed by Yang Liu

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11 Let p be a prime satisfying $p^2 \mid 2^{p-1} - 1$, and let n be a positive integer. Define

$$f(x) = \frac{(x-1)^{p^n} - (x^{p^n} - 1)}{p(x-1)}.$$

Find the largest positive integer N such that there exist polynomials $g(x)$, $h(x)$ with integer coefficients and an integer r satisfying $f(x) = (x-r)^N g(x) + p \cdot h(x)$.

Proposed by Victor Wang

National Math Olympiad (3rd Round) 2012

– Special Lesson's Exam (First Part)

1 Prove that the number of incidences of n distinct points on n distinct lines in plane is $\mathcal{O}(n^{\frac{4}{3}})$. Find a configuration for which $\Omega(n^{\frac{4}{3}})$ incidences happens.

2 Consider a set of n points in plane. Prove that the number of isosceles triangles having their vertices among these n points is $\mathcal{O}(n^{\frac{7}{3}})$. Find a configuration of n points in plane such that the number of equilateral triangles with vertices among these n points is $\Omega(n^2)$.

3 Prove that if n is large enough, among any n points of plane we can find 1000 points such that these 1000 points have pairwise distinct distances. Can you prove the assertion for n^α where α is a positive real number instead of 1000?

4 Prove that from an $n \times n$ grid, one can find $\Omega(n^{\frac{5}{3}})$ points such that no four of them are vertices of a square with sides parallel to lines of the grid. Imagine yourself as Erdos (!) and guess what is the best exponent instead of $\frac{5}{3}$!

– Special Lesson's Exam (Second Part)

1 Prove that for each coloring of the points inside or on the boundary of a square with 1391 colors, there exists a monochromatic regular hexagon.

2 Suppose $W(k,2)$ is the smallest number such that if $n \geq W(k,2)$, for each coloring of the set $\{1,2,...,n\}$ with two colors there exists a monochromatic arithmetic progression of length k. Prove that

$W(k,2) = \Omega(2^{\frac{k}{2}})$.

3 Prove that if n is large enough, then for each coloring of the subsets of the set $\{1,2,...,n\}$ with 1391 colors, two non-empty disjoint subsets A and B exist such that A, B and $A \cup B$ are of the same color.

4 Prove that if n is large enough, in every $n \times n$ square that a natural number is written on each one of its cells, one can find a subsquare from the main square such that the sum of the numbers is this subsquare is divisible by 1391.

– Number Theory Exam

1 $P(x)$ is a nonzero polynomial with integer coefficients. Prove that there exists infinitely many prime numbers q such that for some natural number n, $q|2^n+P(n)$.

Proposed by Mohammad Gharakhani

2 Prove that there exists infinitely many pairs of rational numbers $(\frac{p_1}{q},\frac{p_2}{q})$ with $p_1,p_2,q\in\mathbb{N}$ with the following condition:

$$|\sqrt{3}-\frac{p_1}{q}|<q^{-\frac{3}{2}},|\sqrt{2}-\frac{p_2}{q}|<q^{-\frac{3}{2}}.$$

Proposed by Mohammad Gharakhani

3 p is an odd prime number. Prove that there exists a natural number x such that x and $4x$ are both primitive roots modulo p.

Proposed by Mohammad Gharakhani

4 $P(x)$ and $Q(x)$ are two polynomials with integer coefficients such that $P(x)|Q(x)^2+1$.

a) Prove that there exists polynomials $A(x)$ and $B(x)$ with rational coefficients and a rational number c such that $P(x)=c(A(x)^2+B(x)^2)$.

b) If $P(x)$ is a monic polynomial with integer coefficients, Prove that there exists two polynomials $A(x)$ and $B(x)$ with integer coefficients such that $P(x)$ can be written in the form of $A(x)^2+B(x)^2$.

Proposed by Mohammad Gharakhani

5 Let p be a prime number. We know that each natural number can be written in the form

$$\sum_{i=0}^{t}a_ip^i(t,a_i\in\mathbb{N}\cup\{0\},0\le a_i\le p-1)$$

Uniquely.

Now let T be the set of all the sums of the form

$$\sum_{i=0}^{\infty}a_ip^i(0\le a_i\le p-1).$$

(This means to allow numbers with an infinite base p representation). So numbers that for some $N \in \mathbb{N}$ all the coefficients $a_i, i \geq N$ are zero are natural numbers. (In fact we can consider members of T as sequences $(a_0, a_1, a_2, ...)$ for which $\forall_{i\in\mathbb{N}} : 0 \leq a_i \leq p-1$.) Now we generalize addition and multiplication of natural numbers to this set so that it becomes a ring (it's not necessary to prove this fact). For example:

$1 + (\sum_{i=0}^{\infty}(p-1)p^i) = 1 + (p-1) + (p-1)p + (p-1)p^2 + ... = p + (p-1)p + (p-1)p^2 + ... = p^2 + (p-1)p^2 + (p-1)p^3 + ... = p^3 + (p-1)p^3 + ... = ...$

So in this sum, coefficients of all the numbers $p^k, k \in \mathbb{N}$ are zero, so this sum is zero and thus we can conclude that $\sum_{i=0}^{\infty}(p-1)p^i$ is playing the role of -1 (the additive inverse of 1) in this ring. As an example of multiplication consider

$$(1+p)(1+p+p^2+p^3+...) = 1+2p+2p^2+\cdots$$

Suppose p is 1 modulo 4. Prove that there exists $x \in T$ such that $x^2+1=0$.

Proposed by Masoud Shafaei

– Geometry Exam

1 Fixed points B and C are on a fixed circle ω and point A varies on this circle. We call the midpoint of arc BC (not containing A) D and the orthocenter of the triangle ABC, H. Line DH intersects circle ω again in K. Tangent in A to circumcircle of triangle AKH intersects line DH and circle ω again in L and M respectively. Prove that the value of $\frac{AL}{AM}$ is constant.

Proposed by Mehdi E'tesami Fard

2 Let the Nagel point of triangle ABC be N. We draw lines from B and C to N so that these lines intersect sides AC and AB in D and E respectively. M and T are midpoints of segments BE and CD respectively. P is the second intersection point of circumcircles of triangles BEN and CDN. l_1 and l_2 are perpendicular lines to PM and PT in points M and T respectively. Prove that lines l_1 and l_2 intersect on the circumcircle of triangle ABC.

Proposed by Nima Hamidi

3 Cosider ellipse ϵ with two foci A and B such that the lengths of it's major axis and minor axis are $2a$ and $2b$ respectively. From a point T outside of the ellipse, we draw two tangent lines TP and TQ to the ellipse ϵ. Prove that

$$\frac{TP}{TQ} \geq \frac{b}{a}.$$

Proposed by Morteza Saghafian

4 The incircle of triangle ABC for which $AB \neq AC$, is tangent to sides BC, CA and AB in points D, E and F respectively. Perpendicular from D to EF intersects side AB at X, and the second intersection point of circumcircles of triangles AEF and ABC is T. Prove that $TX \perp TF$.

Proposed By Pedram Safaei

5 Two fixed lines l_1 and l_2 are perpendicular to each other at a point Y. Points X and O are on l_2 and both are on one side of line l_1. We draw the circle ω with center O and radius OY. A variable point Z is on line l_1. Line OZ cuts circle ω in P. Parallel to XP from O intersects XZ in S. Find the locus of the point S.

Proposed by Nima Hamidi

– Combinatorics Exam

1 We've colored edges of K_n with $n-1$ colors. We call a vertex rainbow if it's connected to all of the colors. At most how many rainbows can exist?

Proposed by Morteza Saghafian

2 Suppose $s, k, t \in \mathbb{N}$. We've colored each natural number with one of the k colors, such that each color is used infinitely many times. We want to choose a subset $\mathcal{A}$ of $\mathbb{N}$ such that it has t disjoint monochromatic s-element subsets. What is the minimum number of elements of A?

Proposed by Navid Adham

3 In a tree with n vertices, for each vertex x_i, denote the longest paths passing through it by $l_i^1, l_i^2, ..., l_i^{k_i}$. x_i cuts those longest paths into two parts with $(a_i^1, b_i^1), (a_i^2, b_i^2), ..., (a_i^{k_i}, b_i^{k_i})$ vertices respectively. If $\max_{j=1,...,k_i}\{a_i^j \times b_i^j\} = p_i$, find the maximum and minimum values of $\sum_{i=1}^n p_i$.

Proposed by Sina Rezaei

4 **a)** Prove that for all $m, n \in \mathbb{N}$ there exists a natural number a such that if we color every 3-element subset of the set $\mathcal{A} = \{1, 2, 3, ..., a\}$ using 2 colors red and green, there exists an m-element subset of $\mathcal{A}$ such that all 3-element subsets of it are red or there exists an n-element subset of $\mathcal{A}$ such that all 3-element subsets of it are green.

b) Prove that for all $m, n \in \mathbb{N}$ there exists a natural number a such that if we color every k-element subset $(k > 3)$ of the set $\mathcal{A} = \{1, 2, 3, ..., a\}$ using 2 colors red and green, there exists an m-element subset of $\mathcal{A}$ such that all k-element subsets of it are red or there exists an n-element subset of $\mathcal{A}$ such that all k-element subsets of it are green.

– Algebra Exam

1 Suppose $0 < m_1 < ... < m_n$ and $m_i \equiv i(\mod 2)$. Prove that the following polynomial has at most n real roots. $(\forall 1 \leq i \leq n : a_i \in \mathbb{R})$.

$$a_0 + a_1x^{m_1} + a_2x^{m_2} + ... + a_nx^{m_n}.$$

2 Suppose $N \in \mathbb{N}$ is not a perfect square, hence we know that the continued fraction of $\sqrt{N}$ is of the form $\sqrt{N} = [a_0, \overline{a_1, a_2, ..., a_n}]$. If $a_1 \neq 1$ prove that $a_i \leq 2a_0$.

3 Suppose p is a prime number and $a, b, c \in \mathbb{Q}^+$ are rational numbers;

a) Prove that $\mathbb{Q}(\sqrt[p]{a} + \sqrt[p]{b}) = \mathbb{Q}(\sqrt[p]{a}, \sqrt[p]{b})$.

b) If $\sqrt[p]{b} \in \mathbb{Q}(\sqrt[p]{a})$, prove that for a nonnegative integer k we have $\sqrt[p]{\frac{b}{a^k}} \in \mathbb{Q}$.

c) If $\sqrt[p]{a} + \sqrt[p]{b} + \sqrt[p]{c} \in \mathbb{Q}$, then prove that numbers $\sqrt[p]{a}$, $\sqrt[p]{b}$ and $\sqrt[p]{c}$ are rational.

4 Suppose $f(z) = z^n + a_1z^{n-1} + ... + a_n$ for which $a_1, a_2, ..., a_n \in \mathbb{C}$. Prove that the following polynomial has only one positive real root like α

$$x^n + \Re(a_1)x^{n-1} - |a_2|x^{n-2} - ... - |a_n|$$

and the following polynomial has only one positive real root like β

$$x^n - \Re(a_1)x^{n-1} - |a_2|x^{n-2} - ... - |a_n|.$$

And roots of the polynomial $f(z)$ satisfy $-\beta \leq \Re(z) \leq \alpha$.

5 Let p be an odd prime number and let $a_1, a_2, ..., a_n \in \mathbb{Q}^+$ be rational numbers. Prove that

$$\mathbb{Q}(\sqrt[p]{a_1} + \sqrt[p]{a_2} + ... + \sqrt[p]{a_n}) = \mathbb{Q}(\sqrt[p]{a_1}, \sqrt[p]{a_2}, ..., \sqrt[p]{a_n}).$$

– Final Exam

1 Let G be a simple undirected graph with vertices $v_1, v_2, ..., v_n$. We denote the number of acyclic orientations of G with $f(G)$.

a) Prove that $f(G) \leq f(G - v_1) + f(G - v_2) + ... + f(G - v_n)$.

b) Let e be an edge of the graph G. Denote by G' the graph obtained by omiting e and making it's two endpoints as one vertex. Prove that $f(G) = f(G - e) + f(G')$.

c) Prove that for each $\alpha > 1$, there exists a graph G and an edge e of it such that
$\frac{f(G)}{f(G-e)} < \alpha$.
Proposed by Morteza Saghafian

2 Suppose S is a convex figure in plane with area 10. Consider a chord of length 3 in S and let A and B be two points on this chord which divide it into three equal parts. For a variable point X in $S - \{A, B\}$, let A' and B' be the intersection points of rays AX and BX with the boundary of S. Let S' be those points X for which $AA' > \frac{1}{3}BB'$. Prove that the area of S' is at least 6.

Proposed by Ali Khezeli

3 Prove that for each $n \in \mathbb{N}$ there exist natural numbers $a_1 < a_2 < ... < a_n$ such that $\phi(a_1) > \phi(a_2) > ... > \phi(a_n)$.

Proposed by Amirhossein Gorzi

4 We have n bags each having 100 coins. All of the bags have 10 gram coins except one of them which has 9 gram coins. We have a balance which can show weights of things that have weight of at most 1 kilogram. At least how many times shall we use the balance in order to find the different bag?

Proposed By Hamidreza Ziarati

5 We call the three variable polynomial P cyclic if $P(x, y, z) = P(y, z, x)$. Prove that cyclic three variable polynomials P_1, P_2, P_3 and P_4 exist such that for each cyclic three variable polynomial P, there exists a four variable polynomial Q such that $P(x, y, z) = Q(P_1(x, y, z), P_2(x, y, z), P_3(x, y, z), P_4(x, y, z))$.

Solution by Mostafa Eynollahzade and Erfan Salavati

6 **a)** Prove that $a > 0$ exists such that for each natural number n, there exists a convex n-gon P in plane with lattice points as vertices such that the area of P is less than an^3.

b) Prove that there exists $b > 0$ such that for each natural number n and each n-gon P in plane with lattice points as vertices, the area of P is not less than bn^2.

c) Prove that there exist $\alpha, c > 0$ such that for each natural number n and each n-gon P in plane with lattice points as vertices, the area of P is not less than $cn^{2+\alpha}$.

Proposed by Mostafa Eynollahzade

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7 The city of Bridge Village has some highways. Highways are closed curves that have intersections with each other or themselves in 4-way crossroads. Mr.Bridge Lover, mayor of the city, wants to build a bridge on each crossroad in order to decrease the number of accidents. He wants to build the bridges in such a way that in each highway, cars pass above a bridge and under a bridge alternately. By knowing the number of highways determine that this action is possible or not.

Proposed by Erfan Salavati

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8 **a)** Does there exist an infinite subset S of the natural numbers, such that $S \neq \mathbb{N}$, and such that for each natural number $n \notin S$, exactly n members of S are coprime with n?

b) Does there exist an infinite subset S of the natural numbers, such that for each natural number $n \in S$, exactly n members of S are coprime with n?

Proposed by Morteza Saghafian

National Math Olympiad (3rd Round) 2013

– Algebra

1 Let $a_0, a_1, \ldots, a_n \in \mathbb{N}$. Prove that there exist positive integers $b_0, b_1, \ldots, b_n$ such that for $0 \le i \le n : a_i \le b_i \le 2a_i$ and polynomial

$$P(x) = b_0 + b_1 x + \cdots + b_n x^n$$

is irreducible over $\mathbb{Q}[x]$.
(10 points)

2 Real numbers $a_1, a_2, \ldots, a_n$ add up to zero. Find the maximum of $a_1x_1 + a_2x_2 + \cdots + a_nx_n$ in term of a_i's, when x_i's vary in real numbers such that $(x_1 - x_2)^2 + (x_2 - x_3)^2 + \cdots + (x_{n-1} - x_n)^2 \le 1$.
(15 points)

3 For every positive integer $n \ge 2$, Prove that there is no $n-$tuple of distinct complex numbers $(x_1, x_2, \ldots, x_n)$ such that for each $1 \le k \le n$ following equality holds. $\prod_{\substack{1 \le i \le n \\ i \ne k}} (x_k - x_i) = \prod_{\substack{1 \le i \le n \\ i \ne k}} (x_k + x_i)$
(20 points)

4 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) \in \mathbb{Q}$ and

$$f(x + f(y)^2) = f(x + y)^2.$$

(25 points)

5 Prove that there is no polynomial $P \in \mathbb{C}[x]$ such that set $\{P(z) \mid |z| = 1\}$ in complex plane forms a polygon. In other words, a complex polynomial can't map the unit circle to a polygon.
(30 points)

– Number Theory

1 Let p a prime number and d a divisor of $p-1$. Find the product of elements in $\mathbb{Z}_p$ with order d. ($\mod p$).
(10 points)

2 Suppose that a, b are two odd positive integers such that $2ab+1 \mid a^2+b^2+1$. Prove that $a=b$.
(15 points)

3 Let $p>3$ a prime number. Prove that there exist $x, y \in \mathbb{Z}$ such that $p = 2x^2+3y^2$ if and only if $p \equiv 5, 11 \pmod{24}$
(20 points)

4 Prime $p = n^2+1$ is given. Find the sets of solutions to the below equation:

$$x^2-(n^2+1)y^2=n^2.$$

(25 points)

5 $p=3k+1$ is a prime number. For each $m \in \mathbb{Z}_p$, define function L as follow: $L(m) = \sum_{x\in\mathbb{Z}_p} \left(\frac{x(x^3+m)}{p}\right)$

a) For every $m \in \mathbb{Z}_p$ and $t \in {\mathbb{Z}_p}^*$ prove that $L(m) = L(mt^3)$. (5 points)

b) Prove that there is a partition of ${\mathbb{Z}_p}^* = A \cup B \cup C$ such that $|A| = |B| = |C| = \frac{p-1}{3}$ and L on each set is constant.

Equivalently there are a, b, c for which $L(x) = \begin{cases} a & x \in A \\ b & x \in B \\ c & x \in C \end{cases}$. (7 points)

c) Prove that $a+b+c=-3$. (4 points)

d) Prove that $a^2+b^2+c^2 = 6p+3$. (12 points)

e) Let $X = \frac{2a+b+3}{3}, Y = \frac{b-a}{3}$, show that $X, Y \in \mathbb{Z}$ and also show that :$p = X^2+XY+Y^2$. (2 points)

$({\mathbb{Z}_p}^* = \mathbb{Z}_p \setminus \{0\})$

– Geometry

1 Let $ABCDE$ be a pentagon inscribe in a circle (O). Let $BE \cap AD = T$. Suppose the parallel line with CD which passes through T which cut AB, CE at X, Y. If ω be the circumcircle of triangle AXY then prove that ω is tangent to (O).

2 Let ABC be a triangle with circumcircle (O). Let M, N be the midpoint of arc AB, AC which does not contain C, B and let M', N' be the point of tangency of incircle of $\triangle ABC$ with AB, AC. Suppose that X, Y are foot of perpendicular of A to MM', NN'. If I is the incenter of $\triangle ABC$ then prove that quadrilateral $AXIY$ is cyclic if and only if $b + c = 2a$.

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3 Suppose line ℓ and four points A, B, C, D lies on ℓ. Suppose that circles ω_1, ω_2 passes through A, B and circles ω_1', ω_2' passes through C, D. If $\omega_1 \perp \omega_1'$ and $\omega_2 \perp \omega_2'$ then prove that lines O_1O_2', O_2O_1', ℓ are concurrent where O_1, O_2, O_1', O_2' are center of $\omega_1, \omega_2, \omega_1', \omega_2'$.

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4 In a triangle ABC with circumcircle (O) suppose that A-altitude cut (O) at D. Let altitude of B, C cut AC, AB at E, F. H is orthocenter and T is midpoint of AH. Parallel line with EF passes through T cut AB, AC at X, Y. Prove that $\angle XDF = \angle YDE$.

---

5 Let ABC be triangle with circumcircle (O). Let AO cut (O) again at A'. Perpendicular bisector of OA' cut BC at P_A. P_B, P_C define similarly. Prove that :

I) Point P_A, P_B, P_C are collinear.

II) Prove that the distance of O from this line is equal to $\frac{R}{2}$ where R is the radius of the circumcircle.

– Combinatorics

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1 Assume that the following generating function equation is correct, prove the following statement: $\Pi_{i=1}^{\infty}(1 + x^{3i})\Pi_{j=1}^{\infty}(1 - x^{6j+3}) = 1$
Statement: The number of partitions of n to numbers not of the form $6k + 1$ or $6k - 1$ is equal to the number of partitions of n in which each summand appears at least twice.
(10 points)
Proposed by Morteza Saghafian

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2 How many rooks can be placed in an $n \times n$ chessboard such that each rook is threatened by at most $2k$ rooks?
(15 points)
Proposed by Mostafa Einollah zadeh

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3 n cars are racing. At first they have a particular order. At each moment a car may overtake another car. No two overtaking actions occur at the same time, and except moments a car is passing another, the cars always have an order.

A set of overtaking actions is called "small" if any car overtakes at most once. A set of overtaking actions is called "complete" if any car overtakes exactly once.
If F is the set of all possible orders of the cars after a small set of overtaking actions and G is the set of all possible orders of the cars after a complete set of overtaking actions, prove that

$$\mid F \mid = 2 \mid G \mid$$

(20 points)
Proposed by Morteza Saghafian

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4 We have constructed a rhombus by attaching two equal equilateral triangles. By putting $n-1$ points on all 3 sides of each triangle we have divided the sides to n equal segments. By drawing line segements between correspounding points on each side of the triangles we have divided the rhombus into $2n^2$ equal triangles.
We write the numbers $1, 2, \dots, 2n^2$ on these triangles in a way no number appears twice. On the common segment of each two triangles we write the positive difference of the numbers written on those triangles. Find the maximum sum of all numbers written on the segments.

(25 points)
Proposed by Amirali Moinfar

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5 Consider a graph with n vertices and $\frac{7n}{4}$ edges.
(a) Prove that there are two cycles of equal length.
(25 points)
(b) Can you give a smaller function than $\frac{7n}{4}$ that still fits in part (a)? Prove your claim.
We say function $a(n)$ is smaller than $b(n)$ if there exists an N such that for each $n > N$,$a(n) < b(n)$
(At most 5 points)
Proposed by Afrooz Jabal'ameli

– Final Exam

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1 An n-stick is a connected figure consisting of n matches of length 1 which are placed horizontally or vertically and no two touch each other at points other than their ends. Two shapes that can be transformed into each other by moving, rotating or flipping are considered the same.

An n-mino is a shape which is built by connecting n squares of side length 1 on their sides such that there's a path on the squares between each two squares of the n-mino.
Let S_n be the number of n-sticks and M_n the number of n-minos, e.g. $S_3 = 5$ And $M_3 = 2$.
(a) Prove that for any natural n, $S_n \geq M_{n+1}$.
(b) Prove that for large enough n we have $(2.4)^n \leq S_n \leq (16)^n$.
A **grid segment** is a segment on the plane of length 1 which it's both ends are integer points. A polystick is called **wise** if using it and it's rotations or flips we can cover all grid segments without overlapping, otherwise it's called **unwise**.
(c) Prove that there are at least 2^{n-6} different unwise n-sticks.
(d) Prove that any polystick which is in form of a path only going up and right is wise.
(e) Extra points: Prove that for large enough n we have $3^n \leq S_n \leq 12^n$

Time allowed for this exam was 2 hours.

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2 We define the distance between two circles $\omega, \omega'$by the length of the common external tangent of the circles and show it by $d(\omega, \omega')$. If two circles doesn't have a common external tangent then the distance between them is undefined. A point is also a circle with radius 0 and the distance between two cirlces can be zero.
(a) **Centroid.** n circles $\omega_1, \dots, \omega_n$ are fixed on the plane. Prove that there exists a unique circle $\overline{\omega}$ such that for each circle ω on the plane the square of distance between ω and $\overline{\omega}$ minus the sum of squares of distances of ω from each of the ω_is $1 \leq i \leq n$ is constant, in other words:

$$d(\omega, \overline{\omega})^2 - \frac{1}{n}\sum_{i=1}^{n} d(\omega_i, \omega)^2 = constant$$

(b) **Perpendicular Bisector.** Suppose that the circle ω has the same distance from ω_1, ω_2. Consider ω_3 a circle tangent to both of the common external tangents of ω_1, ω_2. Prove that the distance of ω from centroid of ω_1, ω_2 is not more than the distance of ω and ω_3. (If the distances are all defined)
(c) **Circumcentre.** Let C be the set of all circles that each of them has the same distance from fixed circles $\omega_1, \omega_2, \omega_3$. Prove that there exists a point on the plane which is the external homothety center of each two elements of C.
(d) **Regular Tetrahedron.** Does there exist 4 circles on the plane which the distance between each two of them equals to 1?

Time allowed for this problem was 150 minutes.

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3 Real function f **generates** real function g if there exists a natural k such that $f^k = g$ and we show this by $f \to g$. In this question we are trying to find some properties for relation $\to$, for example it's trivial that if $f \to g$ and $g \to h$ then $f \to h$.(transitivity)

(a) Give an example of two real functions f, g such that $f \neq g$,$f \to g$ and $g \to f$.
(b) Prove that for each real function f there exists a finite number of real functions g such that $f \to g$ and $g \to f$.
(c) Does there exist a real function g such that no function generates it, except for g itself?
(d) Does there exist a real function which generates both x^3 and x^5?
(e) Prove that if a function generates two polynomials of degree 1 P, Q then there exists a polynomial R of degree 1 which generates P and Q.

Time allowed for this problem was 75 minutes.

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4 A polygon A that doesn't intersect itself and has perimeter p is called **Rotund** if for each two points x, y on the sides of this polygon which their distance on the plane is less than 1 their distance on the polygon is at most $\frac{p}{4}$. (Distance on the polygon is the length of smaller path between two points on the polygon)
Now we shall prove that we can fit a circle with radius $\frac{1}{4}$ in any rotund polygon.
The mathematicians of two planets earth and Tarator have two different approaches to prove the statement. In both approaches by "inner chord" we mean a segment with both endpoints on the polygon, and "diagonal" is an inner chord with vertices of the polygon as the endpoints.
Earth approach: Maximal Chord
We know the fact that for every polygon, there exists an inner chord xy with a length of at most 1 such that for any inner chord $x'y'$ with length of at most 1 the distance on the polygon of x, y is more than the distance on the polygon of x', y'. This chord is called the **maximal chord**.
On the rotund polygon A_0 there's two different situations for maximal chord:
(a) Prove that if the length of the maximal chord is exactly 1, then a semicircle with diameter maximal chord fits completely inside A_0, so we can fit a circle with radius $\frac{1}{4}$ in A_0.
(b) Prove that if the length of the maximal chord is less than one we still can fit a circle with radius $\frac{1}{4}$ in A_0.
Tarator approach: Triangulation
Statement 1: For any polygon that the length of all sides is less than one and no circle with radius $\frac{1}{4}$ fits completely inside it, there exists a triangulation of it using diagonals such that no diagonal with length more than 1 appears in the triangulation.

Statement 2: For any polygon that no circle with radius $\frac{1}{4}$ fits completely inside it, can be divided into traingles that their sides are inner chords with length of at most 1.
The mathematicians of planet Tarator proved that if the second statement is true, for each rotund polygon there exists a circle with radius $\frac{1}{4}$ that fits completely inside it.
(c) Prove that if the second statement is true, then for each rotund polygon there exists a circle with radius $\frac{1}{4}$ that fits completely inside it.
They found out that if the first statement is true then the second statement is also true, so they put a bounty of a doogh on proving the first statement. A young earth mathematician named J.N., found a counterexample for statement 1, thus recieving the bounty.
(d) Find a 1392-gon that is counterexample for statement 1.
But the Tarators are not disappointed and they are still trying to prove the second statement.
(e) (Extra points) Prove or disprove the second statement.

Time allowed for this problem was 150 minutes.

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5 A subsum of n real numbers $a_1, \ldots, a_n$ is a sum of elements of a subset of the set $\{a_1, \ldots, a_n\}$. In other words a subsum is $\epsilon_1 a_1 + \cdots + \epsilon_n a_n$ in which for each $1 \leq i \leq n$,ϵ_i is either 0 or 1.
Years ago, there was a valuable list containing n real not necessarily distinct numbers and their $2^n - 1$ subsums. Some mysterious creatures from planet Tarator has stolen the list, but we still have the subsums.
(a) Prove that we can recover the numbers uniquely if all of the subsums are positive.
(b) Prove that we can recover the numbers uniquely if all of the subsums are non-zero.
(c) Prove that there's an example of the subsums for $n = 1392$ such that we can not recover the numbers uniquely.

Note: If a subsum is sum of element of two different subsets, it appears twice.
Time allowed for this question was 75 minutes.

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6 Planet Tarator is a planet in the Yoghurty way galaxy. This planet has a shape of convex 1392-hedron. On earth we don't have any other information about sides of planet tarator.
We have discovered that each side of the planet is a country, and has it's own currency. Each two neighbour countries have their own constant exchange rate, regardless of other exchange rates. Anybody who travels on land and crosses the border must change all his money to the currency of the destination country,

and there's no other way to change the money. Incredibly, a person's money may change after crossing some borders and getting back to the point he started, but it's guaranteed that crossing a border and then coming back doesn't change the money.
On a research project a group of tourists were chosen and given same amount of money to travel around the Tarator planet and come back to the point they started. They always travel on land and their path is a nonplanar polygon which doesn't intersect itself. What is the maximum number of tourists that may have a pairwise different final amount of money?

Note 1: Tourists spend no money during travel!
Note 2: The only constant of the problem is 1392, the number of the sides. The exchange rates and the way the sides are arranged are unknown. Answer must be a constant number, regardless of the variables.
Note 3: The maximum must be among all possible polyhedras.

Time allowed for this problem was 90 minutes.

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7 An equation $P(x) = Q(y)$ is called **Interesting** if P and Q are polynomials with degree at least one and integer coefficients and the equations has an infinite number of answers in $\mathbb{N}$.
An interesting equation $P(x) = Q(y)$ **yields in** interesting equation $F(x) = G(y)$ if there exists polynomial $R(x) \in \mathbb{Q}[x]$ such that $F(x) \equiv R(P(x))$ and $G(x) \equiv R(Q(x))$.
(a) Suppose that S is an infinite subset of $\mathbb{N} \times \mathbb{N}$.$S$ *is an answer* of interesting equation $P(x) = Q(y)$ if each element of S is an answer of this equation. Prove that for each S there's an interesting equation $P_0(x) = Q_0(y)$ such that if there exists any interesting equation that S is an answer of it, $P_0(x) = Q_0(y)$ yields in that equation.
(b) Define the degree of an interesting equation $P(x) = Q(y)$ by $max\{deg(P), deg(Q)\}$.
An interesting equation is called **primary** if there's no other interesting equation with lower degree that yields in it.
Prove that if $P(x) = Q(y)$ is a primary interesting equation and P and Q are monic then $(deg(P), deg(Q)) = 1$.

Time allowed for this question was 2 hours.

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8 Let $A_1A_2A_3A_4A_5$ be a convex 5-gon in which the coordinates of all of it's vertices are rational. For each $1 \le i \le 5$ define B_i the intersection of lines $A_{i+1}A_{i+2}$ and $A_{i+3}A_{i+4}$.
($A_i = A_{i+5}$) Prove that at most 3 lines from the lines A_iB_i ($1 \le i \le 5$) are concurrent.

Time allowed for this problem was 75 minutes.

National Math Olympiad (3rd Round) 2014

– Algebra

1 We have an equilateral triangle with circumradius 1. We extend its sides. Determine the point P inside the triangle such that the total lengths of the sides (extended), which lies inside the circle with center P and radius 1, is maximum.
(The total distance of the point P from the sides of an equilateral triangle is fixed)

Proposed by Erfan Salavati

2 Find all continuous function $f:\mathbb{R}^{\geq 0}\rightarrow \mathbb{R}^{\geq 0}$ such that :

$$f(xf(y))+f(f(y)) = f(x)f(y)+2 \;\forall x,y\in \mathbb{R}^{\geq 0}$$

Proposed by Mohammad Ahmadi

3 Let $p,q\in \mathbb{R}[x]$ such that $p(z)q(\overline{z})$ is always a real number for every complex number z. Prove that $p(x)=kq(x)$ for some constant $k\in\mathbb{R}$ or $q(x)=0$.

Proposed by Mohammad Ahmadi

4 For any $a,b,c>0$ satisfying $a+b+c+ab+ac+bc=3$, prove that

$$2\leq a+b+c+abc\leq 3$$

Proposed by Mohammad Ahmadi

5 We say $p(x,y)\in \mathbb{R}\left[x,y\right]$ is *good* if for any $y \neq 0$ we have $p(x,y) = p\left(xy,\frac{1}{y}\right)$. Prove that there are good polynomials $r(x,y),s(x,y)\in \mathbb{R}\left[x,y\right]$ such that for any good polynomial p there is a $f(x,y)\in \mathbb{R}\left[x,y\right]$ such that

$$f(r(x,y),s(x,y))= p(x,y)$$

Proposed by Mohammad Ahmadi

– Number Theory

1 Show that for every natural number n there are n natural numbers $x_1 < x_2 < ... < x_n$ such that

$$\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n} - \frac{1}{x_1 x_2 ... x_n} \in \mathbb{N} \cup 0$$

(15 points)

2 We say two sequence of natural numbers A=$(a_1, ..., a_n)$, B=$(b_1, ..., b_n)$are the exchange and we write $A \sim B$.
if $503 | a_i - b_i$ for all $1 \leq i \leq n$.
also for natural number r : $A^r = (a_1^r, a_2^r, ..., a_n^r)$.
Prove that there are natural number k, m such that : $i)250 \leq k$

$ii)$There are different permutations $\pi_1, ..., \pi_k$ from $\{1, 2, 3, ..., 502\}$ such that for $1 \leq i \leq k-1$ we have $\pi_i^m \sim \pi_{i+1}$

(15 points)

3 Let n be a positive integer. Prove that there exists a natural number m with exactly n prime factors, such that for every positive integer d the numbers in $\{1, 2, 3, \ldots, m\}$ of order d modulo m are multiples of $\phi(d)$.

(15 points)

4 $2 \leq d$ is a natural number. $B_{a,b}$=$\{a, a+b, a+2b, ..., a+db\}$ $A_{c,q}$=$\{cq^n | n \in \mathbb{N}\}$ Prove that for constant a, b, c, q, set of prime numbers p satisfying the following conditions is finite.

i) $p \nmid abcq$

ii) $A_{c,q} \equiv B_{a,b} (mod p)$

(15 points)

5 Can an infinite set of natural numbers be found, such that for all triplets (a, b, c) of it we have $abc + 1$ perfect square?

(20 points)

6 Prove that there are 100 natural number $a_1 < a_2 < ... < a_{99} < a_{100}$ ($a_i < 10^6$) such that A , A+A , 2A , A+2A , 2A + 2A are five sets apart ?

A = $\{ a_1, a_2, ..., a_{99}, a_{100} \}$

2A = $\{2a_i | 1 \leq i \leq 100\}$

A+A =$\{a_i + a_j | 1 \leq i < j \leq 100\}$

A + 2A = { $a_i + 2a_j | 1 \leq i, j \leq 100$}
2A + 2A = { $2a_i + 2a_j | 1 \leq i < j \leq 100$

(20 ponits)

– Combinatorics

1 Denote by g_n the number of connected graphs of degree n whose vertices are labeled with numbers $1, 2, ..., n$. Prove that $g_n \geq (\frac{1}{2}).2^{\frac{n(n-1)}{2}}$.
Note:If you prove that for $c < \frac{1}{2}$, $g_n \geq c.2^{\frac{n(n-1)}{2}}$, you will earn some point!

proposed by Seyed Reza Hosseini and Mohammad Amin Ghiasi

2 In a tennis tournament there are participants from n different countries. Each team consists of a coach and a player whom should settle in a hotel. The rooms considered for the settlement of coaches are different from players' ones. Each player wants to be in a room whose roommates are **all** from countries which have a defense agreement with the player's country. Conversely, each coach wants to be in a room whose roommates are **all** from countries which don't have a defense agreement with the coach's country. Find the minimum number of the rooms such that we can **always** grant everyone's desire.

proposed by Seyed Reza Hosseini and Mohammad Amin Ghiasi

3 We have a 10×10 table. T is a set of rectangles with vertices from the table and sides parallel to the sides of the table such that no rectangle from the set is a subrectangle of another rectangle from the set. t is the maximum number of elements of T.
(a) Prove that $t > 300$.
(b) Prove that $t < 600$.

Proposed by Mir Omid Haji Mirsadeghi and Kasra Alishahi

4 A **word** is formed by a number of letters of the alphabet. We show words with capital letters. A **sentence** is formed by a number of words. For example if $A = aa$ and $B = ab$ then the sentence AB is equivalent to $aaab$. In this language, A^n indicates $\underbrace{AA \cdots A}_{n}$. We have an equation when two sentences are equal. For example $XYX = YZ^2$ and it means that if we write the alphabetic letters forming the words of each sentence, we get two equivalent sequences of alphabetic letters. An equation is **simplified**, if the words of the left and the right side of the sentences of the both sides of the equation are different. Note that every word contains one alphabetic letter at least.

a)We have a simplified equation in terms of X and Y. Prove that both X and Y can be written in form of a power of a word like Z.(Z can contain only one alphabetic letter).

b) Words $W_1, W_2, \cdots, W_n$ are the answers of a simplified equation. Prove that we can produce these n words with fewer words.

c) n words $W_1, W_2, \cdots, W_n$ are the answers of a simplified system of equations. Define graph G with vertices $1, 2 \cdots, n$ such that i and j are connected if in one of the equations, W_i and W_j be the two words appearing in the right side of each side of the equation.($\cdots W_i = \cdots W_j$). If we denote by c the number of connected components of G, prove that these n words can be produced with at most c words.

Proposed by Mostafa Einollah Zadeh Samadi

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5 An n-mino is a connected figure made by connecting n 1×1 squares. Two polyminos are the same if moving the first we can reach the second. For a polymino P ,let $|P|$ be the number of 1×1 squares in it and ∂P be number of squares out of P such that each of the squares have at least on edge in common with a square from P.
(a) Prove that for every $x \in (0, 1)$:

$$\sum_P x^{|P|}(1-x)^{\partial P} = 1$$

The sum is on all different polyminos.
(b) Prove that for every polymino P, $\partial P \leq 2|P| + 2$
(c) Prove that the number of n-minos is less than 6.75^n.

Proposed by Kasra Alishahi

– Geometry

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1 In the circumcircle of triange $\triangle ABC$, AA' is a diameter.
We draw lines l' and l from A' parallel with Internal and external bisector of the vertex A. l' Cut out AB, BC at B_1 and B_2. l Cut out AC, BC at C_1 and C_2.
Prove that the circumcircles of $\triangle ABC$ $\triangle CC_1C_2$ and $\triangle BB_1B_2$ have a common point.

(20 points)

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2 $\triangle ABC$ is isosceles($AB = AC$). Points P and Q exist inside the triangle such that Q lies inside $\widehat{PAC}$ and $\widehat{PAQ} = \frac{\widehat{BAC}}{2}$. We also have $BP = PQ = CQ$.Let

X and Y be the intersection points of (AP, BQ) and (AQ, CP) respectively. Prove that quadrilateral $PQYX$ is cyclic. *(20 Points)*

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3 Distinct points B, B', C, C' lie on an arbitrary line ℓ. A is a point not lying on ℓ. A line passing through B and parallel to AB' intersects with AC in E and a line passing through C and parallel to AC' intersects with AB in F. Let X be the intersection point of the circumcircles of $\triangle ABC$ and $\triangle AB'C'(A \neq X)$. Prove that $EF \parallel AX$.

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4 D is an arbitrary point lying on side BC of $\triangle ABC$. Circle ω_1 is tangent to segments AD , BD and the circumcircle of $\triangle ABC$ and circle ω_2 is tangent to segments AD , CD and the circumcircle of $\triangle ABC$. Let X and Y be the intersection points of ω_1 and ω_2 with BC respectively and take M as the midpoint of XY. Let T be the midpoint of arc BC which does not contain A. If I is the incenter of $\triangle ABC$, prove that TM goes through the midpoint of ID.

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5 X and Y are two points lying on or on the extensions of side BC of $\triangle ABC$ such that $\widehat{XAY} = 90$. Let H be the orthocenter of $\triangle ABC$. Take X' and Y' as the intersection points of (BH, AX) and (CH, AY) respectively. Prove that circumcircle of $\triangle CYY'$,circumcircle of $\triangle BXX'$ and $X'Y'$ are concurrent.

– Final Exam

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1 In each of (a) to (d) you have to find a strictly increasing surjective function from A to B or prove that there doesn't exist any.
(a) $A = \{x | x \in \mathbb{Q}, x \leq \sqrt{2}\}$ and $B = \{x | x \in \mathbb{Q}, x \leq \sqrt{3}\}$
(b) $A = \mathbb{Q}$ and $B = \mathbb{Q} \cup \{\pi\}$
In (c) and (d) we say $(x, y) > (z, t)$ where $x, y, z, t \in \mathbb{R}$, whenever $x > z$ or $x = z$ and $y > t$.
(c) $A = \mathbb{R}$ and $B = \mathbb{R}^2$
(d) $X = \{2^{-x} | x \in \mathbb{N}\}$, then $A = X \times (X \cup \{0\})$ and $B = (X \cup \{0\}) \times X$

(e) If $A, B \subset \mathbb{R}$, such that there exists a surjective non-decreasing function from A to B and a surjective non-decreasing function from B to A , does there exist a surjective strictly increasing function from B to A?

Time allowed for this problem was 2 hours.

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2 Consider a flat field on which there exist a valley in the form of an infinite strip with arbitrary width ω. There exist a polyhedron of diameter d(Diameter in a polyhedron is the maximum distance from the points on the polyhedron) is in one side and a pit of diameter d on the other side of the valley. We want to roll the polyhedron and put it into the pit such that the polyhedron and the field always meet each other in one point at least while rolling (If the polyhedron and the field meet each other in one point at least then the polyhedron would not fall into the valley). For crossing over the bridge, we have built a rectangular bridge with a width of $\frac{d}{10}$ over the bridge. Prove that we can always put the polyhedron into the pit considering the mentioned conditions.

(You will earn a good score if you prove the decision for $\omega = 0$).

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3 (a) n is a natural number. $d_1, \dots, d_n, r_1, \dots, r_n$ are natural numbers such that for each i, j that $1 \le i < j \le n$ we have $(d_i, d_j) = 1$ and $d_i \ge 2$.
Prove that there exist an x such that
(i) $1 \le x \le 3^n$
(ii)For each $1 \le i \le n$

$$x \overset{d_i}{\not\equiv} r_i$$

(b) For each $\epsilon > 0$ prove that there exists natural N such that for each $n > N$ and each $d_1, \dots, d_n, r_1, \dots, r_n$ satisfying the conditions above there exists an x satisfying (ii) such that $1 \le x \le (2+\epsilon)^n$.

Time allowed for this exam was 75 minutes.

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4 Let P be a regular $2n$-sided polygon. A **rhombus-ulation** of P is dividing P into rhombuses such that no two intersect and no vertex of any rhombus is on the edge of other rhombuses or P.
(a) Prove that number of rhombuses is a function of n. Find the value of this function. Also find the number of vertices and edges of the rhombuses as a function of n.
(b) Prove or disprove that there always exists an edge e of P such that by erasing all the segments parallel to e the remaining rhombuses are connected.
(c) Is it true that each two rhombus-ulations can turn into each other using the following algorithm multiple times?
Algorithm: Take a hexagon -not necessarily regular- consisting of 3 rhombuses and re-rhombus-ulate the hexagon.
(d) Let $f(n)$ be the number of ways to rhombus-ulate P. Prove that:

$$\Pi_{k=1}^{n-1}(\binom{k}{2}+1) \le f(n) \le \Pi_{k=1}^{n-1} k^{n-k}$$

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5 A not necessary nonplanar polygon in $\mathbb{R}^3$ is called **Grid Polygon** if each of it's edges are parallel to one of the axes.

(a) There's a right angle between each two neighbour sides of the grid polygon, the plane containing this angle could be parallel to either xy plane, yz plane, or xz plane. Prove that parity of the number of the angles that the plane containing each of them is parallel to xy plane is equal to parity of the number of the angles that the plane containing each of them is parallel to yz plane and parity of the number of the angles that the plane containing each of them is parallel to zx plane.
(b) A grid polygon is called **Inscribed** if there's a point in the space that has an equal distance from all of the vertices of the polygon. Prove that any nonplanar grid hexagon is inscribed.
(c) Does there exist a grid 2014-gon without repeated vertices such that there exists a plane that intersects all of it's edges?
(d) If $a, b, c \in \mathbb{N} - \{1\}$, prove that a, b, c are sidelengths of a triangle iff there exists a grid polygon in which the number of it's edges that are parallel to x axis is a, the number of it's edges that are parallel to y axis is b and the number of it's edges that are parallel to z axis is c.

Time allowed for this exam was 1 hour.

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6 P is a monic polynomial of odd degree greater than one such that there exists a function $f : \mathbb{R} \to \mathbb{N}$ such that for each $x \in \mathbb{R}$,

$$f(P(x)) = P(f(x))$$

(a) Prove that there are a finite number of natural numbers in range of f.
(b) Prove that if f is not constant then the equation $P(x) - x = 0$ has at least two real solutions.
(c) For each natural $n > 1$ prove that there exists a function $f : \mathbb{R} \to \mathbb{N}$ and a monic polynomial of odd degree greater than one P such that for each $x \in \mathbb{R}$,

$$f(P(x)) = P(f(x))$$

and range of f contains exactly n different numbers.

Time allowed for this problem was 105 minutes.

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7 We have a machine that has an input and an output. The input is a letter from the finite set I and the output is a lamp that at each moment has one of the colors of the set $C = \{c_1, \ldots, c_p\}$.
At each moment the machine has an inner state that is one of the n members of finite set S. The function $o : S \to C$ is a surjective function defining that at each state, what color must the lamp be, and the function $t : S \times I \to S$ is a function defining how does giving each input at each state changes the state.

We only shall see the lamp and we have no direct information from the state of the car at current moment.
In other words a machine is $M = (S, I, C, o, t)$ such that S, I, C are finite, $t : S \times I \to S$, and $o : S \to C$ is surjective. It is guaranteed that for each two different inner states, there's a sequence of inputs such that the color of the lamp after giving the sequence to the machine at the first state is different from the color of the lamp after giving the sequence to the machine at the second state.
(a) The machine M has n different inner states. Prove that for each two different inner states, there's a sequence of inputs of length no more than $n-p$ such that the color of the lamp after giving the sequence to the machine at the first state is different from the color of the lamp after giving the sequence to the machine at the second state.
(b) Prove that for a machine M with n different inner states, there exists an algorithm with no more than n^2 inputs that starting at any unknown inner state, at the end of the algorithm the state of the machine at that moment is known.
Can you prove the above claim for $\frac{n^2}{2}$?

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8 The polynomials $k_n(x_1, \dots, x_n)$, where n is a non-negative integer, satisfy the following conditions

$$k_0 = 1$$

$$k_1(x_1) = x_1$$

$$k_n(x_1, \dots, x_n) = x_n k_{n-1}(x_1, \dots, x_{n-1}) + (x_n^2 + x_{n-1}^2) k_{n-2}(x_1, \dots, x_{n-2})$$

Prove that for each non-negative n we have $k_n(x_1, \dots, x_n) = k_n(x_n, \dots, x_1)$.

National Math Olympiad (3rd round) 2015

–	Algebra
1	x, y, z are three real numbers inequal to zero satisfying $x + y + z = xyz$. Prove that $$\sum\left(\frac{x^2-1}{x}\right)^2 \geq 4$$ *Proposed by Amin Fathpour*
2	Prove that there are no functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $\forall x, y \in \mathbb{R} : f(x^2 + g(y)) - f(x^2) + g(y) - g(x) \leq 2y$ and $f(x) \geq x^2$. *Proposed by Mohammad Ahmadi*
3	Does there exist an irreducible two variable polynomial $f(x, y) \in \mathbb{Q}[x, y]$ such that it has only four roots $(0, 1), (1, 0), (0, -1), (-1, 0)$ on the unit circle.
4	$p(x) \in \mathbb{C}[x]$ is a polynomial such that: $\forall z \in \mathbb{C}, \lvert z \rvert = 1 \Longrightarrow p(z) \in \mathbb{R}$ Prove that $p(x)$ is constant.
5	Find all polynomials $p(x) \in \mathbb{R}[x]$ such that for all $x \in \mathbb{R}$: $p(5x)^2 - 3 = p(5x^2 + 1)$ such that: $a)p(0) \neq 0$ $b)p(0) = 0$
6	$a_1, a_2, \ldots, a_n > 0$ are positive real numbers such that $\sum_{i=1}^{n} \frac{1}{a_i} = n$ prove that: $\sum_{i<j} \left(\frac{a_i - a_j}{a_i + a_j}\right)^2 \leq \frac{n^2}{2}\left(1 - \frac{n}{\sum_{i=1}^{n} a_i}\right)$
–	Number Theory
1	Prove that there are infinitely natural numbers n such that n can't be written as a sum of two positive integers with prime factors less than 1394.
2	$M_0 \subset \mathbb{N}$ is a non-empty set with a finite number of elements. Ali produces sets $M_1, M_2, ..., M_n$ in the following order: In step n, Ali chooses an element of M_{n-1} like b_n and defines M_n as $$M_n = \{b_n m + 1 \mid m \in M_{n-1}\}$$

Prove that at some step Ali reaches a set which no element of it divides another element of it.

3 Let $p > 5$ be a prime number and $A = \{b_1, b_2, \dots, b_{\frac{p-1}{2}}\}$ be the set of all quadratic residues modulo p, excluding zero. Prove that there doesn't exist any natural a, c satisfying $(ac, p) = 1$ such that set $B = \{ab_1 + c, ab_2 + c, \dots, ab_{\frac{p-1}{2}} + c\}$ and set A are disjoint modulo p.

This problem was proposed by Amir Hossein Pooya.

4 a, b, c, d, k, l are positive integers such that for every natural number n the set of prime factors of $n^k + a^n + c, n^l + b^n + d$ are same. prove that $k = l, a = b, c = d$.

5 $p > 30$ is a prime number. Prove that one of the following numbers is in form of $x^2 + y^2$.

$$p + 1, 2p + 1, 3p + 1, ..., (p - 3)p + 1$$

– Geometry

1 Let $ABCD$ be the trapezoid such that $AB \parallel CD$. Let E be an arbitrary point on AC. point F lies on BD such that $BE \parallel CF$. Prove that circumcircles of $\triangle ABF, \triangle BED$ and the line AC are concurrent.

2 Let ABC be a triangle with orthocenter H and circumcenter O. Let K be the midpoint of AH. point P lies on AC such that $\angle BKP = 90°$. Prove that $OP \parallel BC$.

3 Let ABC be a triangle. consider an arbitrary point P on the plain of $\triangle ABC$. Let R, Q be the reflections of P wrt AB, AC respectively. Let $RQ \cap BC = T$. Prove that $\angle APB = \angle APC$ if and if only $\angle APT = 90°$.

4 Let ABC be a triangle with incenter I. Let K be the midpoint of AI and $BI \cap \odot(\triangle ABC) = M, CI \cap \odot(\triangle ABC) = N$. points P, Q lie on AM, AN respectively such that $\angle ABK = \angle PBC, \angle ACK = \angle QCB$. Prove that P, Q, I are collinear.

5 Let ABC be a triangle with orthocenter H and circumcenter O. Let R be the radius of circumcircle of $\triangle ABC$. Let A', B', C' be the points on $\overrightarrow{AH}, \overrightarrow{BH}, \overrightarrow{CH}$ respectively such that $AH.AA' = R^2, BH.BB' = R^2, CH.CC' = R^2$. Prove that O is incenter of $\triangle A'B'C'$.

Iran Team Selection Test 2014

– TST 1

Day 1

1 suppose that O is the circumcenter of acute triangle ABC.
we have circle with center O that is tangent too BC that named w
suppose that X and Y are the points of intersection of the tangent from A to w with line BC(X and B are in the same side of AO) T is the intersection of the line tangent to circumcirle of ABC in B and the line from X parallel to AC. S is the intersection of the line tangent to circumcirle of ABC in C and the line from Y parallel to AB.
prove that ST is tangent ABC.

2 find all polynomials with integer coefficients that $P(\mathbb{Z})$ ={$p(a) : a \in \mathbb{Z}$} has a Geometric progression.

3 we named a $n*n$ table $selfish$ if we number the row and column with $0, 1, 2, 3, ..., n-1$.(from left to right an from up to down)
for every $\{i, j \in 0, 1, 2, ..., n-1\}$ the number of cell (i, j) is equal to the number of number i in the row j.
for example we have such table for $n = 5$
1 0 3 3 4
1 3 2 1 1
0 1 0 1 0
2 1 0 0 0
1 0 0 0 0
prove that for $n > 5$ there is no $selfish$ table

Day 2

4 Find the maximum number of Permutation of set $\{1, 2, 3, ..., 2014\}$ such that for every 2 different number a and b in this set at last in one of the permutation b comes exactly after a

5 n is a natural number. for every positive real numbers $x_1, x_2, ..., x_{n+1}$ such that $x_1x_2...x_{n+1} = 1$ prove that: $\sqrt[x_1]{n} + ... + \sqrt[x_{n+1}]{n} \geq n^{\sqrt[n]{x_1}} + ... + n^{\sqrt[n]{x_{n+1}}}$

6 I is the incenter of triangle ABC. perpendicular from I to AI meet AB and AC at B' and C' respectively .
Suppose that B'' and C'' are points on half-line BC and CB such that $BB'' = BA$ and $CC'' = CA$.
Suppose that the second intersection of circumcircles of $AB'B''$ and $AC'C''$ is T.
Prove that the circumcenter of AIT is on the BC.

– TST 2

Day 1

1 Consider a tree with n vertices, labeled with $1, \ldots, n$ in a way that no label is used twice. We change the labeling in the following way - each time we pick an edge that hasn't been picked before and swap the labels of its endpoints. After performing this action $n-1$ times, we get another tree with its labeling a permutation of the first graph's labeling.
Prove that this permutation contains exactly one cycle.

2 Point D is an arbitary point on side BC of triangle ABC. I,I_1 andI_2 are the incenters of triangles ABC,ABD and ACD respectively. $M \neq A$ and $N \neq A$ are the intersections of circumcircle of triangle ABC and circumcircles of triangles IAI_1 and IAI_2 respectively. Prove that regardless of point D, line MN goes through a fixed point.

3 prove for all $k > 1$ equation $(x+1)(x+2)...(x+k) = y^2$ has finite solutions.

Day 2

4 n is a natural number. We shall call a permutation $a_1, \ldots, a_n$ of $1, \ldots, n$ a quadratic(cubic) permutation if $\forall 1 \leq i \leq n-1$ we have $a_i a_{i+1} + 1$ is a perfect square(cube). (a) Prove that for infinitely many natural numbers n there exists a quadratic permutation. (b) Prove that for no natural number n exists a cubic permutation.

5 if $x, y, z > 0$ are postive real numbers such that $x^2+y^2+z^2 = x^2y^2+y^2z^2+z^2x^2$ prove that

$$((x-y)(y-z)(z-x))^2 \leq 2((x^2-y^2)^2+(y^2-z^2)^2+(z^2-x^2)^2)$$

6 Consider n segments in the plane which no two intersect and between their $2n$ endpoints no three are collinear. Is the following statement true?
Statement: There exists a simple $2n$-gon such that it's vertices are the $2n$ endpoints of the segments and each segment is either completely inside the polygon or an edge of the polygon.

– TST 3

Day 1

1 The incircle of a non-isosceles triangle ABC with the center I touches the sides BC, AC, AB at A_1, B_1, C_1 .
let AI, BI, CI meets BC, AC, AB at A_2, B_2, C_2.
let A' is a point on AI such that $A_1A' \perp B_2C_2$ .B', C' respectively.
prove that two triangle $A'B'C', A_1B_1C_1$ are equal.

2 is there a function $f : \mathbb{N} \to \mathbb{N}$ such that $i)\exists n \in \mathbb{N} : f(n) \neq n$ $ii)$ the number of divisors of m is $f(n)$ if and only if the number of divisors of $f(m)$ is n

3 let $m, n \in \mathbb{N}$ and $p(x), q(x), h(x)$ are polynomials with real Coefficients such that $p(x)$ is Descending.
and for all $x \in \mathbb{R}$ $p(q(nx + m) + h(x)) = n(q(p(x)) + h(x)) + m$.
prove that dont exist function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$ $f(q(p(x)) + h(x)) = f(x)^2 + 1$

Day 2

4 Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $x, y \in \mathbb{R}^+$,

$$f\left(\frac{y}{f(x+1)}\right) + f\left(\frac{x+1}{xf(y)}\right) = f(y)$$

5 Given a set $X = \{x_1, \ldots, x_n\}$ of natural numbers in which for all $1 < i \leq n$ we have $1 \leq x_i - x_{i-1} \leq 2$, call a real number a **good** if there exists $1 \leq j \leq n$ such that $2|x_j - a| \leq 1$. Also a subset of X is called **compact** if the average of its elements is a good number.
Prove that at least 2^{n-3} subsets of X are compact.

6 The incircle of a non-isosceles triangle ABC with the center I touches the sides BC at D.

let X is a point on arc BC from circumcircle of triangle ABC such that if E,F are feet of perpendicular from X on BI,CI and M is midpoint of EF we have $MB=MC$.
prove that $\widehat{BAD}=\widehat{CAX}$

Iran Team Selection Test 2015

– TST 1

Day 1

1 Find all polynomials $P, Q \in \mathbb{Q}[x]$ such that

$$P(x)^3 + Q(x)^3 = x^{12} + 1.$$

2 I_b is the B-excenter of the triangle ABC and ω is the circumcircle of this triangle. M is the middle of arc BC of ω which doesn't contain A. MI_b meets ω at $T \neq M$. Prove that

$$TB \cdot TC = TI_b^2.$$

3 Let $b_1 < b_2 < b_3 < \dots$ be the sequence of all natural numbers which are sum of squares of two natural numbers.
Prove that there exists infinite natural numbers like m which $b_{m+1} - b_m = 2015$.

Day 2

4 n is a fixed natural number. Find the least k such that for every set A of k natural numbers, there exists a subset of A with an even number of elements which the sum of it's members is divisible by n.

5 Let A be a subset of the edges of an $n \times n$ table. Let $V(A)$ be the set of vertices from the table which are connected to at least on edge from A and $j(A)$ be the number of the connected components of graph G which it's vertices are the set $V(A)$ and it's edges are the set A. Prove that for every natural number l:

$$\frac{l}{2} \leq min_{|A| \geq l}(|V(A)| - j(A)) \leq \frac{l}{2} + \sqrt{\frac{l}{2}} + 1$$

6 $ABCD$ is a circumscribed and inscribed quadrilateral. O is the circumcenter of the quadrilateral. E, F and S are the intersections of AB, CD , AD, BC and AC, BD respectively. E' and F' are points on AD and AB such that $A\hat{E}E' = E'\hat{E}D$ and $A\hat{F}F' = F'\hat{F}B$. X and Y are points on OE' and OF' such that $\frac{XA}{XD} = \frac{EA}{ED}$ and $\frac{YA}{YB} = \frac{FA}{FB}$. M is the midpoint of arc BD of (O) which contains A.
Prove that the circumcircles of triangles OXY and OAM are coaxal with the circle with diameter OS.

– TST 2

Day 1

1 a, b, c, d are positive numbers such that $\sum_{cyc} \frac{1}{ab} = 1$. Prove that : $abcd + 16 \geq 8\sqrt{(a+c)(\frac{1}{a} + \frac{1}{c})} + 8\sqrt{(b+d)(\frac{1}{b} + \frac{1}{d})}$

2 In triangle ABC(with incenter I) let the line parallel to BC from A intersect circumcircle of $\triangle ABC$ at A_1 let $AI \cap BC = D$ and E is tangency point of incircle with BC let $EA_1 \cap \odot(\triangle ADE) = T$ prove that $AI = TI$.

Day 2

4 Let $\triangle ABC$ be an acute triangle. Point Z is on A altitude and points X and Y are on the B and C altitudes out of the triangle respectively, such that: $\angle AYB = \angle BZC = \angle CXA = 90$
Prove that X,Y and Z are collinear, if and only if the length of the tangent drawn from A to the nine point circle of $\triangle ABC$ is equal with the sum of the lengths of the tangents drawn from B and C to the nine point circle of $\triangle ABC$.

5 We call a permutation $(a_1, a_2, \cdots, a_n)$ of the set $\{1, 2, \cdots, n\}$ "good" if for any three natural numbers $i < j < k$, $n \nmid a_i + a_k - 2a_j$ find all natural numbers $n \geq 3$ such that there exist a "good" permutation of a set $\{1, 2, \cdots, n\}$.

6 If a, b, c are positive real numbers such that $a + b + c = abc$ prove that

$$\frac{abc}{3\sqrt{2}} \left(\sum_{cyc} \frac{\sqrt{a^3 + b^3}}{ab + 1} \right) \geq \sum_{cyc} \frac{a}{a^2 + 1}$$

–	TST 3
Day 1	
1	Point A is outside of a given circle ω. Let the tangents from A to ω meet ω at S, T points X, Y are midpoints of AT, AS let the tangent from X to ω meet ω at $R \neq T$. points P, Q are midpoints of XT, XR let $XY \cap PQ = K, SX \cap TK = L$ prove that quadrilateral $KRLQ$ is cyclic.
2	Assume that a_1, a_2, a_3 are three given positive integers consider the following sequence: $a_{n+1} = \text{lcm}[a_n, a_{n-1}] - \text{lcm}[a_{n-1}, a_{n-2}]$ for $n \geq 3$ Prove that there exist a positive integer k such that $k \leq a_3 + 4$ and $a_k \leq 0$. ($[a, b]$ means the least positive integer such that$a \mid [a, b], b \mid [a, b]$ also because $\text{lcm}[a, b]$ takes only nonzero integers this sequence is defined until we find a zero number in the sequence)
3	$a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ are $2n$ positive real numbers such that $a_1, a_2, \cdots, a_n$ aren't all equal. And assume that we can divide $a_1, a_2, \cdots, a_n$ into two subsets with equal sums.similarly $b_1, b_2, \cdots, b_n$ have these two conditions. Prove that there exist a simple $2n$-gon with sides $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ and parallel to coordinate axises Such that the lengths of horizontal sides are among $a_1, a_2, \cdots, a_n$ and the lengths of vertical sides are among $b_1, b_2, \cdots, b_n$.(simple polygon is a polygon such that it doesn't intersect itself)
Day 2	
5	Prove that for each natural number d, There is a monic and unique polynomial of degree d like P such that $P(1)0$ and for each sequence like a_1,a_2, ... of real numbers that the recurrence relation below is true for them, there is a natural number k such that $0 = a_k = a_{k+1} = \ldots$: $P(n)a_1 + P(n-1)a_2 + \ldots + P(1)a_n = 0$ $n > 1$
6	AH is the altitude of triangle ABC and H' is the reflection of H trough the midpoint of BC. If the tangent lines to the circumcircle of ABC at B and C, intersect each other at X and the perpendicular line to XH' at H', intersects AB and AC at Y and Z respectively, prove that $\angle ZXC = \angle YXB$.

Iran TST 2016

Test 1 Day 1

1 Let m and n be positive integers such that $m > n$. Define $x_k = \frac{m+k}{n+k}$ for $k = 1, 2, \ldots, n+1$. Prove that if all the numbers $x_1, x_2, \ldots, x_{n+1}$ are integers, then $x_1 x_2 \ldots x_{n+1} - 1$ is divisible by an odd prime.

2 For a finite set A of positive integers, a partition of A into two disjoint nonempty subsets A_1 and A_2 is *good* if the least common multiple of the elements in A_1 is equal to the greatest common divisor of the elements in A_2. Determine the minimum value of n such that there exists a set of n positive integers with exactly 2015 good partitions.

3 Let $ABCD$ be a convex quadrilateral, and let P, Q, R, and S be points on the sides AB, BC, CD, and DA, respectively. Let the line segment PR and QS meet at O. Suppose that each of the quadrilaterals $APOS$, $BQOP$, $CROQ$, and $DSOR$ has an incircle. Prove that the lines AC, PQ, and RS are either concurrent or parallel to each other.

Test 1 Day 2

4 Let n be a fixed positive integer. Find the maximum possible value of

$$\sum_{1 \le r < s \le 2n} (s - r - n) x_r x_s,$$

where $-1 \le x_i \le 1$ for all $i = 1, \cdots, 2n$.

5 Let ABC be a triangle with $\angle C = 90^\circ$, and let H be the foot of the altitude from C. A point D is chosen inside the triangle CBH so that CH bisects AD. Let P be the intersection point of the lines BD and CH. Let ω be the semicircle with diameter BD that meets the segment CB at an interior point. A line through P is tangent to ω at Q. Prove that the lines CQ and AD meet on ω.

6 In a company of people some pairs are enemies. A group of people is called *unsociable* if the number of members in the group is odd and at least 3, and it is possible to arrange all its members around a round table so that every two neighbors are enemies. Given that there are at most 2015 unsociable groups, prove that it is possible to partition the company into 11 parts so that no two enemies are in the same part.

Proposed by Russia

Test 2 Day 1

1 Let ABC be an acute triangle and let M be the midpoint of AC. A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC. Determine all possible values of $\frac{BT}{BM}$.

2 Let a, b, c, d be positive real numbers such that $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} + \frac{1}{d+1} = 2$. Prove that

$$\sum_{cyc} \sqrt{\frac{a^2+1}{2}} \geq (3.\sum_{cyc} \sqrt{a}) - 8$$

3 Let n be a positive integer. Two players A and B play a game in which they take turns choosing positive integers $k \leq n$. The rules of the game are:

(i) A player cannot choose a number that has been chosen by either player on any previous turn.
(ii) A player cannot choose a number consecutive to any of those the player has already chosen on any previous turn.
(iii) The game is a draw if all numbers have been chosen; otherwise the player who cannot choose a number anymore loses the game.

The player A takes the first turn. Determine the outcome of the game, assuming that both players use optimal strategies.

Proposed by Finland

Test 2 Day 2

4 Let ABC be a triangle with $CA \neq CB$. Let D, F, and G be the midpoints of the sides AB, AC, and BC respectively. A circle Γ passing through C and tangent to AB at D meets the segments AF and BG at H and I, respectively. The points H' and I' are symmetric to H and I about F and G, respectively. The line $H'I'$ meets CD and FG at Q and M, respectively. The line CM meets Γ again at P. Prove that $CQ = QP$.

Proposed by El Salvador

6 Let $\mathbb{Z}_{>0}$ denote the set of positive integers. For any positive integer k, a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ is called [i]k-good[/i] if $\gcd(f(m)+n, f(n)+m) \le k$ for all $m \neq n$. Find all k such that there exists a k-good function.

Proposed by James Rickards, Canada

Test 3 Day 1

2 Let ABC be an arbitrary triangle and O is the circumcenter of $\triangle ABC$.Points X,Y lie on AB,AC,respectively such that the reflection of BC WRT XY is tangent to circumcircle of $\triangle AXY$.Prove that the circumcircle of triangle AXY is tangent to circumcircle of triangle BOC.

3 Let $p \neq 13$ be a prime number of the form $8k+5$ such that 39 is a quadratic non-residue modulo p. Prove that the equation

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 \equiv 0 \pmod{p}$$

has a solution in integers such that $p \nmid x_1x_2x_3x_4$.

Test 3 Day 2

4 Suppose that a sequence $a_1, a_2, \ldots$ of positive real numbers satisfies

$$a_{k+1} \ge \frac{ka_k}{a_k^2 + (k-1)}$$

for every positive integer k. Prove that $a_1 + a_2 + \ldots + a_n \ge n$ for every $n \ge 2$.

5 Let AD, BF, CE be altitudes of triangle ABC.Q is a point on EF such that $QF = DE$ and F is between E, Q.P is a point on EF such that $EP = DF$ and E is between P, F.Perpendicular bisector of DQ intersect with AB at X and perpendicular bisector of DP intersect with AC at Y.Prove that midpoint of BC lies on XY.

Iran Team Selection Test 2017

Test 1 Day 1

1 Let a, b, c, d be positive real numbers with $a+b+c+d=2$. Prove the following inequality:

$$\frac{(a+c)^2}{ad+bc}+\frac{(b+d)^2}{ac+bd}+4\geq 4\left(\frac{a+b+1}{c+d+1}+\frac{c+d+1}{a+b+1}\right).$$

Proposed by Mohammad Jafari

2 In the country of *Sugarland*, there are 13 students in the IMO team selection camp. 6 team selection tests were taken and the results have came out. Assume that no students have the same score on the same test.To select the IMO team, the national committee of math Olympiad have decided to choose a permutation of these 6 tests and starting from the first test, the person with the highest score between the remaining students will become a member of the team.The committee is having a session to choose the permutation.
Is it possible that all 13 students have a chance of being a team member?

Proposed by Morteza Saghafian

3 In triangle ABC let I_a be the A-excenter. Let ω be an arbitrary circle that passes through A, I_a and intersects the extensions of sides AB, AC (extended from B, C) at X, Y respectively. Let S, T be points on segments I_aB, I_aC respectively such that $\angle AXI_a=\angle BTI_a$ and $\angle AYI_a=\angle CSI_a$.Lines BT, CS intersect at K. Lines KI_a, TS intersect at Z.
Prove that X, Y, Z are collinear.

Proposed by Hooman Fattahi

Test 1 Day 2

4 We arranged all the prime numbers in the ascending order: $p_1=2<p_2<p_3<\cdots$.
Also assume that $n_1<n_2<\cdots$ is a sequence of positive integers that for all $i=1,2,3,\cdots$ the equation $x^{n_i}\equiv 2 \pmod{p_i}$ has a solution for x.
Is there always a number x that satisfies all the equations?

Proposed by Mahyar Sefidgaran , Yahya Motevasel

5 In triangle ABC, arbitrary points P, Q lie on side BC such that $BP = CQ$ and P lies between B, Q.The circumcircle of triangle APQ intersects sides AB, AC at E, F respectively.The point T is the intersection of EP, FQ.Two lines passing through the midpoint of BC and parallel to AB and AC, intersect EP and FQ at points X, Y respectively.
Prove that the circumcircle of triangle TXY and triangle APQ are tangent to each other.

Proposed by Iman Maghsoudi

6 In the unit squares of a transparent 1×100 tape, numbers $1, 2, \cdots, 100$ are written in the ascending order.We fold this tape on it's lines with arbitrary order and arbitrary directions until we reach a 1×1 tape with 100 layers.A permutation of the numbers $1, 2, \cdots, 100$ can be seen on the tape, from the top to the bottom.
Prove that the number of possible permutations is between 2^{100} and 4^{100}.
(*e.g.* We can produce all permutations of numbers $1, 2, 3$ with a 1×3 tape)

Proposed by Morteza Saghafian

Test 2 Day 1

1 $ABCD$ is a trapezoid with $AB \parallel CD$. The diagonals intersect at P. Let ω_1 be a circle passing through B and tangent to AC at A. Let ω_2 be a circle passing through C and tangent to BD at D. ω_3 is the circumcircle of triangle BPC.
Prove that the common chord of circles ω_1, ω_3 and the common chord of circles ω_2, ω_3 intersect each other on AD.

Proposed by Kasra Ahmadi

2 Find the largest number n that for which there exists n positive integers such that non of them divides another one, but between every three of them, one divides the sum of the other two.

Proposed by Morteza Saghafian

3 There are 27 cards, each has some amount of (1 or 2 or 3) shapes (a circle, a square or a triangle) with some color (white, grey or black) on them. We call a triple of cards a *match* such that all of them have the same amount of shapes or distinct amount of shapes, have the same shape or distinct shapes and have the same color or distinct colors. For instance, three cards shown in the figure are a *match* be cause they have distinct amount of shapes, distinct shapes but the same color of shapes.

What is the maximum number of cards that we can choose such that non of the triples make a *match*?

Proposed by Amin Bahjati

Test 2 Day 2

4 A $n+1$-tuple $(h_1, h_2, \cdots, h_{n+1})$ where $h_i(x_1, x_2, \cdots, x_n)$ are n variable polynomials with real coefficients is called *good* if the following condition holds:
For any n functions $f_1, f_2, \cdots, f_n : \mathbb{R} \to \mathbb{R}$ if for all $1 \le i \le n+1$, $P_i(x) = h_i(f_1(x), f_2(x), \cdots, f_n(x))$ is a polynomial with variable x, then $f_1(x), f_2(x), \cdots, f_n(x)$ are polynomials.

$a)$ Prove that for all positive integers n, there exists a *good* $n+1$-tuple $(h_1, h_2, \cdots, h_{n+1})$ such that the degree of all h_i is more than 1.

$b)$ Prove that there doesn't exist any integer $n > 1$ that for which there is a *good* $n+1$-tuple $(h_1, h_2, \cdots, h_{n+1})$ such that all h_i are symmetric polynomials.

Proposed by Alireza Shavali

5 k, n are two arbitrary positive integers. Prove that there exists at least $(k-1)(n-k+1)$ positive integers that can be produced by n number of k's and using only $+, -, \times, \div$ operations and adding parentheses between them, but cannot be produced using $n-1$ number of k's.

Proposed by Aryan Tajmir

6 Let $k > 1$ be an integer. The sequence $a_1, a_2, \cdots$ is defined as: $a_1 = 1, a_2 = k$ and for all $n > 1$ we have: $a_{n+1} - (k+1)a_n + a_{n-1} = 0$
Find all positive integers n such that a_n is a power of k.

Proposed by Amirhossein Pooya

Test 3 Day 1

1 Let $n > 1$ be an integer. Prove that there exists an integer $n-1 \ge m \ge \left\lfloor \frac{n}{2} \right\rfloor$ such that the following equation has integer solutions with $a_m > 0$:

$$\frac{a_m}{m+1} + \frac{a_{m+1}}{m+2} + \cdots + \frac{a_{n-1}}{n} = \frac{1}{\operatorname{lcm}(1, 2, \cdots, n)}$$

Proposed by Navid Safaei

2 Let P be a point in the interior of quadrilateral $ABCD$ such that:

$$\angle BPC = 2\angle BAC \quad , \quad \angle PCA = \angle PAD \quad , \quad \angle PDA = \angle PAC$$

Prove that:

$$\angle PBD = |\angle BCA - \angle PCA|$$

Proposed by Ali Zamani

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3 Find all functions $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ that satisfy the following conditions for all positive real numbers x, y, z :

$$f\left(f(x,y), z\right) = x^2y^2f(x,z)$$

$$f\left(x, 1 + f(x,y)\right) \geq x^2 + xyf(x,x)$$

Proposed by Mojtaba Zare, Ali Daei Nabi

Test 3 Day 2

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4 There are 6 points on the plane such that no three of them are collinear. It's known that between every 4 points of them, there exists a point that it's power with respect to the circle passing through the other three points is a constant value k.(Power of a point in the interior of a circle has a negative value.)
Prove that $k = 0$ and all 6 points lie on a circle.

Proposed by Morteza Saghafian

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5 Let $\{c_i\}_{i=0}^{\infty}$ be a sequence of non-negative real numbers with $c_{2017} > 0$. A sequence of polynomials if defined as:

$$P_{-1}(x) = 0 \ , \ P_0(x) = 1 \ , \ P_{n+1}(x) = xP_n(x) + c_nP_{n-1}(x)$$

Prove that there doesn't exist any integer $n > 2017$ and some real number c such that:

$$P_{2n}(x) = P_n(x^2 + c)$$

Proposed by Navid Safaei

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6 In triangle ABC let O and H be the circumcenter and the orthocenter. The point P is the reflection of A with respect to OH. Assume that P is not on the same side of BC as A. Points E,F lie on AB,AC respectively such that $BE=PC$,$CF=PB$. Let K be the intersection point of AP,OH. Prove that $\angle EKF=90^\circ$

Proposed by Iman Maghsoudi

Iran Team Selection Test 2018

Test 1 Day 1

1 Let $A_1, A_2, ..., A_k$ be the subsets of $\{1, 2, 3, ..., n\}$ such that for all $1 \leq i, j \leq k$:$A_i \cap A_j \neq \varnothing$. Prove that there are n distinct positive integers $x_1, x_2, ..., x_n$ such that for each $1 \leq j \leq k$:

$$lcm_{i \in A_j} \{x_i\} > lcm_{i \notin A_j} \{x_i\}$$

Proposed by Morteza Saghafian, Mahyar Sefidgaran

2 Determine the least real number k such that the inequality

$$\left(\frac{2a}{a-b}\right)^2 + \left(\frac{2b}{b-c}\right)^2 + \left(\frac{2c}{c-a}\right)^2 + k \geq 4\left(\frac{2a}{a-b} + \frac{2b}{b-c} + \frac{2c}{c-a}\right)$$

holds for all real numbers a, b, c.

Proposed by Mohammad Jafari

3 In triangle ABC let M be the midpoint of BC. Let ω be a circle inside of ABC and is tangent to AB, AC at E, F, respectively. The tangents from M to ω meet ω at P, Q such that P and B lie on the same side of AM. Let $X \equiv PM \cap BF$ and $Y \equiv QM \cap CE$. If $2PM = BC$ prove that XY is tangent to ω.

Proposed by Iman Maghsoudi

Test 1 Day 2

4 Let ABC be a triangle ($\angle A \neq 90^\circ$). BE, CF are the altitudes of the triangle. The bisector of $\angle A$ intersects EF, BC at M, N. Let P be a point such that $MP \perp EF$ and $NP \perp BC$. Prove that AP passes through the midpoint of BC.

Proposed by Iman Maghsoudi, Hooman Fattahi

5 Prove that for each positive integer m, one can find m consecutive positive integers like n such that the following phrase doesn't be a perfect power:

$$\left(1^3 + 2018^3\right)\left(2^3 + 2018^3\right) \cdots \left(n^3 + 2018^3\right)$$

Proposed by Navid Safaei

6 A simple graph is called "divisibility", if it's possible to put distinct numbers on its vertices such that there is an edge between two vertices if and only if number of one of its vertices is divisible by another one.

A simple graph is called "permutationary", if it's possible to put numbers $1, 2, ..., n$ on its vertices and there is a permutation π such that there is an edge between vertices i, j if and only if $i > j$ and $\pi(i) < \pi(j)$ (it's not directed!)

Prove that a simple graph is permutationary if and only if its complement and itself are divisibility.

Proposed by Morteza Saghafian

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Test 2 Day 1

1 Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the following conditions:
a. $x + f(y + f(x)) = y + f(x + f(y)) \quad \forall x, y \in \mathbb{R}$
b. The set $I = \left\{ \frac{f(x)-f(y)}{x-y} \mid x, y \in \mathbb{R}, x \neq y \right\}$ is an interval.

Proposed by Navid Safaei

2 Mojtaba and Hooman are playing a game. Initially Mojtaba draws 2018 vectors with zero sum. Then in each turn, starting with Mojtaba, the player takes a vector and puts it on the plane. After the first move, the players must put their vector next to the previous vector (the beginning of the vector must lie on the end of the previous vector).
At last, there will be a closed polygon. If this polygon is not self-intersecting, Mojtaba wins. Otherwise Hooman. Who has the winning strategy?

Proposed by Mahyar Sefidgaran, Jafar Namdar

3 Let $a_1, a_2, a_3, \cdots$ be an infinite sequence of distinct integers. Prove that there are infinitely many primes p that distinct positive integers i, j, k can be found such that $p \mid a_i a_j a_k - 1$.

Proposed by Mohsen Jamali

Test 2 Day 2

4 Call a positive integer "useful but not optimized " (!), if it can be written as a sum of distinct powers of 3 and powers of 5.
Prove that there exist infinitely many positive integers which they are not "useful but not optimized".
(e.g. $37 = (3^0 + 3^1 + 3^3) + (5^0 + 5^1)$ is a " useful but not optimized" number)

Proposed by Mohsen Jamali

5 Let ω be the circumcircle of isosceles triangle ABC ($AB = AC$). Points P and Q lie on ω and BC respectively such that $AP = AQ$ .AP and BC intersect at R. Prove that the tangents from B and C to the incircle of $\triangle AQR$ (different fromBC) are concurrent on ω.

Proposed by Ali Zamani, Hooman Fattahi

6 $a_1, a_2, \ldots, a_n$ is a sequence of positive integers that has at least $\frac{2n}{3} + 1$ distinct numbers and each positive integer has occurred at most three times in it. Prove that there exists a permutation $b_1, b_2, \ldots, b_n$ of a_i's such that all the n sums $b_i + b_{i+1}$ are distinct ($1 \le i \le n$, $b_{n+1} \equiv b_1$)

Proposed by Mohsen Jamali

Test 3 Day 1

1 Two circles $\omega_1(O)$ and ω_2 intersect each other at A, B ,and O lies on ω_2. Let S be a point on AB such that $OS \perp AB$. Line OS intersects ω_2 at P (other than O). The bisector of $A\hat{S}P$ intersects ω_1 at L (A and L are on the same side of the line OP). Let$K$ be a point on ω_2 such that $PS = PK$ (A and K are on the same side of the line OP). Prove that $SL = KL$.

Proposed by Ali Zamani

2 Find the maximum possible value of k for which there exist distinct reals $x_1, x_2, \ldots, x_k$ greater than 1 such that for all $1 \le i, j \le k$,

$$x_i^{\lfloor x_j \rfloor} = x_j^{\lfloor x_i \rfloor}.$$

Proposed by Morteza Saghafian

3 $n > 1$ and distinct positive integers $a_1, a_2, \ldots, a_{n+1}$ are given. Does there exist a polynomial $p(x) \in \mathbb{Z}[x]$ of degree $\le n$ that satisfies the following conditions?
a. $\forall_{1 \le i < j \le n+1} : \gcd(p(a_i), p(a_j)) > 1$
b. $\forall_{1 \le i < j < k \le n+1} : \gcd(p(a_i), p(a_j), p(a_k)) = 1$

Proposed by Mojtaba Zare

Test 3 Day 2

4 We say distinct positive integers $a_1, a_2, \dots, a_n$ are "good" if their sum is equal to the sum of all pairwise gcd's among them. Prove that there are infinitely many n s such that n good numbers exist.

Proposed by Morteza Saghafian

5 $2n-1$ distinct positive real numbers with sum S are given. Prove that there are at least $\binom{2n-2}{n-1}$ different ways to choose n numbers among them such that their sum is at least $\frac{S}{2}$.

Proposed by Amirhossein Gorzi

6 Consider quadrilateral $ABCD$ inscribed in circle ω. $P \equiv AC \cap BD$. E, F lie on sides AB, CD respectively such that $A\hat{P}E = D\hat{P}F$. Circles ω_1, ω_2 are tangent to ω at X, Y respectively and also both tangent to the circumcircle of $\triangle PEF$ at P. Prove that:

$$\frac{EX}{EY} = \frac{FX}{FY}$$

Proposed by Ali Zamani

Romania National Olympiad 2010

– Grade level 7

1 Let S be a subset with 673 elements of the set $\{1, 2, \ldots, 2010\}$. Prove that one can find two distinct elements of S, say a and b, such that 6 divides $a + b$.

2 Let $ABCD$ be a rectangle of centre O, such that $\angle DAC = 60^\circ$. The angle bisector of $\angle DAC$ meets DC at S. Lines OS and AD meet at L, and lines BL and AC meet at M. Prove that lines SM and CL are parallel.

3 Each of the small squares of a 50×50 table is coloured in red or blue. Initially all squares are red. A *step* means changing the colour of all squares on a row or on a column.
a) Prove that there exists no sequence of steps, such that at the end there are exactly 2011 blue squares.
b) Describe a sequence of steps, such that at the end exactly 2010 squares are blue.

Adriana & Lucian Dragomir

4 In the isosceles triangle ABC, with $AB = AC$, the angle bisector of $\angle B$ meets the side AC at B'. Suppose that $BB' + B'A = BC$. Find the angles of the triangle ABC.

Dan Nedeianu

– Grade level 8

1 Let a, b, c be integers larger than 1. Prove that

$$a(a-1) + b(b-1) + c(c-1) \leq (a+b+c-4)(a+b+c-5) + 4.$$

2 How many four digit numbers $\overline{abcd}$ simultaneously satisfy the equalities $a + b = c + d$ and $a^2 + b^2 = c^2 + d^2$?

3 Let $VABCD$ be a regular pyramid, having the square base $ABCD$. Suppose that on the line AC lies a point M such that $VM = MB$ and $(VMB) \perp (VAB)$. Prove that $4AM = 3AC$.

Mircea Fianu

4 Let a, b, c, d be positive integers, and let $p = a + b + c + d$. Prove that if p is a prime, then p is not a divisor of $ab - cd$.

Marian Andronache

– Grade level 9

1 In a triangle ABC denote by D, E, F the points where the angle bisectors of $\angle CAB, \angle ABC, \angle BCA$ respectively meet it's circumcircle.
a) Prove that the orthocenter of triangle DEF coincides with the incentre of triangle ABC.
b) Prove that if $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = 0$, then the triangle ABC is equilateral.

Marin Ionescu

2 Prove that there is a similarity between a triangle ABC and the triangle having as sides the medians of the triangle ABC if and only if the squares of the lengths of the sides of triangle ABC form an arithmetic sequence.

Marian Teler & Marin Ionescu

3 For any integer $n \geq 2$ denote by A_n the set of solutions of the equation

$$x = \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x}{3} \right\rfloor + \cdots + \left\lfloor \frac{x}{n} \right\rfloor.$$

a) Determine the set $A_2 \cup A_3$.
b) Prove that the set $A = \bigcup_{n \geq 2} A_n$ is finite and find $\max A$.

Dan Nedeianu & Mihai Baluna

4 Consider the set $\mathcal{F}$ of functions $f : \mathbb{N} \to \mathbb{N}$ (where $\mathbb{N}$ is the set of non-negative integers) having the property that

$$f(a^2 - b^2) = f(a)^2 - f(b)^2, \text{ for all } a, b \in \mathbb{N},\ a \geq b.$$

a) Determine the set $\{f(1) \mid f \in \mathcal{F}\}$.
b) Prove that $\mathcal{F}$ has exactly two elements.

Nelu Chichirim

– Grade level 10

1 Let $(a_n)_{n\geq 0}$ be a sequence of positive real numbers such that

$$\sum_{k=0}^{n} C_n^k a_k a_{n-k} = a_n^2, \text{ for any } n \geq 0.$$

Prove that $(a_n)_{n\geq 0}$ is a geometric sequence.

Lucian Dragomir

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2 Consider v, w two distinct non-zero complex numbers. Prove that

$$|zw + \bar{w}| \leq |zv + \bar{v}|,$$

for any $z \in \mathbb{C}, |z| = 1$, if and only if there exists $k \in [-1, 1]$ such that $w = kv$.

Dan Marinescu

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3 In the plane are given 100 points, such that no three of them are on the same line. The points are arranged in 10 groups, any group containing at least 3 points. Any two points in the same group are joined by a segment.
a) Determine which of the possible arrangements in 10 such groups is the one giving the minimal numbers of triangles.
b) Prove that there exists an arrangement in such groups where each segment can be coloured with one of three given colours and no triangle has all edges of the same colour.

Vasile Pop

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4 On the exterior of a non-equilateral triangle ABC consider the similar triangles ABM, BCN and CAP, such that the triangle MNP is equilateral. Find the angles of the triangles ABM, BCN and CAP.

Nicolae Bourbacut

– Grade level 11

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1 Let $a, b \in \mathbb{R}$ such that $b > a^2$. Find all the matrices $A \in \mathcal{M}_2(\mathbb{R})$ such that $\det(A^2 - 2aA + bI_2) = 0$.

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2 Let $A, B, C \in \mathcal{M}_n(\mathbb{R})$ such that $ABC = O_n$ and rank $B = 1$. Prove that $AB = O_n$ or $BC = O_n$.

3	Let $f:\mathbb{R}\to[0,\infty)$. Prove that $f(x+y)\ge(y+1)f(x),\ (\forall)x\in\mathbb{R}$ if and only if the function $g:\mathbb{R}\to[0,\infty),\ g(x)=e^{-x}f(x),\ (\forall)x\in\mathbb{R}$ is increasing.
4	Let $a\in\mathbb{R}_+$ and define the sequence of real numbers $(x_n)_n$ by $x_1=a$ and $x_{n+1}=\left\|x_n-\frac{1}{n}\right\|,\ n\ge 1$. Prove that the sequence is convergent and find it's limit.
–	Grade level 12

1 Let $f:\mathbb{R}\to\mathbb{R}$ be a monotonic function and $F:\mathbb{R}\to\mathbb{R}$ given by

$$F(x)=\int_0^x f(t)\ \mathrm{d}t.$$

Prove that if F has a finite derivative, then f is continuous.

Dorin Andrica & Mihai Piticari

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2 We say that a ring A has property (P) if any non-zero element can be written uniquely as the sum of an invertible element and a non-invertible element.
a) If in A, $1+1=0$, prove that A has property (P) if and only if A is a field.
b) Give an example of a ring that is not a field, containing at least two elements, and having property (P).

Dan Schwarz

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3 Let G be a finite group of order n. Define the set

$$H=\{x : x\in G \text{ and } x^2=e\},$$

where e is the neutral element of G. Let $p=|H|$ be the cardinality of H. Prove that
a) $|H\cap xH|\ge 2p-n$, for any $x\in G$, where $xH=\{xh : h\in H\}$.
b) If $p>\frac{3n}{4}$, then G is commutative.
c) If $\frac{n}{2}<p\le\frac{3n}{4}$, then G is non-commutative.

Marian Andronache

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4 Let $f:[-1,1]\to\mathbb{R}$ be a continuous function having finite derivative at 0, and

$$I(h)=\int_{-h}^{h} f(x)\ \mathrm{d}x,\ \ h\in[0,1].$$

Prove that

a) there exists $M > 0$ such that $|I(h) - 2f(0)h| \leq Mh^2$, for any $h \in [0,1]$.
b) the sequence $(a_n)_{n\geq 1}$, defined by $a_n = \sum_{k=1}^{n} \sqrt{k}|I(1/k)|$, is convergent if and only if $f(0) = 0$.

Calin Popescu

Romania National Olympiad 2011

–	Grade level 7
–	Grade level 8
–	Grade level 9
–	Grade level 10
–	Grade level 11
–	April 18th
1	A row of a matrix belonging to $\mathcal{M}_n(\mathbb{C})$ is said to be *permutable* if no matter how we would permute the entries of that row, the value of the determinant doesn't change. Prove that if a matrix has two *permutable* rows, then its determinant is equal to 0 .
2	Let $u : [a, b] \to \mathbb{R}$ be a continuous function that has finite left-side derivative $u_l'(x)$ in any point $x \in (a, b]$. Prove that the function u is monotonously increasing if and only if $u_l'(x) \geq 0$, for any $x \in (a, b]$.
3	Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous and strictly decreasing function with $g(\mathbb{R}) = (-\infty, 0)$. Prove that there are no continuous functions $f : \mathbb{R} \to \mathbb{R}$ with the property that there exists a natural number $k \geq 2$ so that : $\underbrace{f \circ f \circ \ldots \circ f}_{k \text{ times}} = g$.
4	Let A , $B \in \mathcal{M}_2(\mathbb{C})$ so that : $A^2 + B^2 = 2AB$. **a)** Prove that : $AB = BA$. **b)** Prove that : $\mathrm{tr}\,(A) = \mathrm{tr}\,(B)$.
–	Grade level 12

Romania National Olympiad 2012

–	IX
1	The altitude $[BH]$ dropped onto the hypotenuse of a triangle ABC intersects the bisectors $[AD]$ and $[CE]$ at Q and P respectively. Prove that the line passing through the midpoints of the segments $[QD]$ and $[PE]$ is parallel to the line AC .
2	Find all functions $f : \mathbb{R} \to \mathbb{R}$ with the following property: for any open bounded interval I, the set $f(I)$ is an open interval having the same length with I .
3	Prove that if $n \geq 2$ is a natural number and $x_1, x_2, \ldots, x_n$ are positive real numbers, then: $$4\left(\frac{x_1^3 - x_2^3}{x_1 + x_2} + \frac{x_2^3 - x_3^3}{x_2 + x_3} + \ldots + \frac{x_{n-1}^3 - x_n^3}{x_{n-1} + x_n} + \frac{x_n^3 - x_1^3}{x_n + x_1}\right) \leq \leq (x_1 - x_2)^2 + (x_2 - x_3)^2 + \ldots + (x_{n-1} - x$$
4	On a table there are $k \geq 2$ piles having $n_1, n_2, \ldots, n_k$ pencils respectively. A *move* consists in choosing two piles having a and b pencils respectively, $a \geq b$ and transferring b pencils from the first pile to the second one. Find the necessary and sufficient condition for $n_1, n_2, \ldots, n_k$, such that there exists a succession of moves through which all pencils are transferred to the same pile.
–	X
1	Let $M = \{x \in \mathbb{C} \,\|\, \|z\| = 1,\ \operatorname{Re} z \in \mathbb{Q}\}$. Prove that there exist infinitely many equilateral triangles in the complex plane having all affixes of their vertices in the set M .
2	Let a , b and c be three complex numbers such that $a + b + c = 0$ and $\|a\| = \|b\| = \|c\| = 1$. Prove that: $$3 \leq \|z - a\| + \|z - b\| + \|z - c\| \leq 4,$$ for any $z \in \mathbb{C}$, $\|z\| \leq 1$.

3 Let $a, b \in \mathbb{R}$ with $0 < a < b$. Prove that:

a) $2\sqrt{ab} \leq \frac{x+y+z}{3} + \frac{ab}{\sqrt[3]{xyz}} \leq a + b$, for $x, y, z \in [a, b]$.

b) $\left\{ \frac{x+y+z}{3} + \frac{ab}{\sqrt[3]{xyz}} \,|\, x, y, z \in [a, b] \right\} = [2\sqrt{ab}, a + b]$.

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4 Let n and m be two natural numbers, $m \geq n \geq 2$. Find the number of injective functions

$$f\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$$

such that there exists a unique number $i \in \{1, 2, \ldots, n-1\}$ for which $f(i) > f(i+1)$.

– XI

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1 Let $f, g\colon [0, 1] \to [0, 1]$ be two functions such that g is monotonic, surjective and $|f(x) - f(y)| \leq |g(x) - g(y)|$, for any $x, y \in [0, 1]$.

a) Prove that f is continuous and that there exists some $x_0 \in [0, 1]$ with $f(x_0) = g(x_0)$.
b) Prove that the set $\{x \in [0, 1] \,|\, f(x) = g(x)\}$ is a closed interval.

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2 Let n and k be two natural numbers such that $n \geq 2$ and $1 \leq k \leq n-1$. Prove that if the matrix $A \in \mathcal{M}_n(\mathbb{C})$ has exactly k minors of order $n-1$ equal to 0 , then $\det(A) \neq 0$.

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3 Let $A, B \in \mathcal{M}_4(\mathbb{R})$ such that $AB = BA$ and $\det(A^2 + AB + B^2) = 0$. Prove that:

$$\det(A + B) + 3\det(A - B) = 6\det(A) + 6\det(B) \ .$$

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4 Find all differentiable functions $f\colon [0, \infty) \to [0, \infty)$ for which $f(0) = 0$ and $f'(x^2) = f(x)$ for any $x \in [0, \infty)$.

– XII

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1 Let $f\colon [0,\infty) \to \mathbb{R}$ be a continuous function such that $\int_0^n f(x)f(n-x)\ \mathrm{d}x = \int_0^n f^2(x)\ \mathrm{d}x$, for any natural number $n \geq 1$. Prove that f is a periodic function.

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2 Let $(R,+,\cdot)$ be a ring and let f be a surjective endomorphism of R such that $[x,f(x)]=0$ for any $x \in R$, where $[a,b]=ab-ba$, $a,b \in R$. Prove that:

a) $[x,f(y)]=[f(x),y]$ and $x[x,y]=f(x)[x,y]$, for any $x,y \in R$;
b) If R is a division ring and f is different from the identity function, then R is commutative.

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3 Let $\mathcal{C}$ be the set of integrable functions $f\colon [0,1] \to \mathbb{R}$ such that $0 \leq f(x) \leq x$ for any $x \in [0,1]$. Define the function $V\colon \mathcal{C} \to \mathbb{R}$ by

$$V(f) = \int_0^1 f^2(x)\ \mathrm{d}x - \left(\int_0^1 f(x)\ \mathrm{d}x \right)^2 \ , \ f \in \mathcal{C} \ .$$

Determine the following two sets:

a) $\{V(f_a)\,|\,0 \leq a \leq 1\}$, where $f_a(x)=0$, if $0 \leq x \leq a$ and $f_a(x)=x$, if $a < x \leq 1$;
b) $\{V(f)\,|\,f \in \mathcal{C}\}$.

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4 Let m and n be two nonzero natural numbers. Determine the minimum number of distinct complex roots of the polynomial $\prod_{k=1}^m (f+k)$, when f covers the set of n^{th} - degree polynomials with complex coefficients.

Romania National Olympiad 2013

–	IX
1	A series of numbers is called complete if it has non-zero natural terms and any nonzero integer has at least one among multiple series. Show that the arithmetic progression is a complete sequence if and only if it divides the first term relationship.
2	Given $f:\mathbb{R}\to\mathbb{R}$ an arbitrary function and $g:\mathbb{R}\to\mathbb{R}$ a function of the second degree, with the property: for any real numbers m and n equation $f\left(x \right)=mx+n$ has solutions if and only if the equation $g\left(x \right)=mx+n$ has solutions Show that the functions f and g are equal.
3	Given P a point m inside a triangle acute-angled ABC and DEF intersections of lines with that AP, BP, CP with$\left[BC \right],\left[CA \right]$,respective $\left[AB \right]$ a) Show that the area of the triangle DEF is at most a quarter of the area of the triangle ABC b) Show that the radius of the circle inscribed in the triangle DEF is at most a quarter of the radius of the circle circumscribed on triangle $4ABC$.
4	Consider a nonzero integer number n and the function $f:\mathbb{N}\to \mathbb{N}$ by $$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{x-1}{2} + 2^{n-1} & \text{if } x \text{ is odd} \end{cases}.$$ Determine the set: $$A = \{ x\in \mathbb{N} \mid \underbrace{\left(f\circ f\circ ....\circ f \right)}_{n\ f\text{'s}}\left(x \right)=x \}.$$
–	X
1	Solve the following equation ${{2}^{{{\sin }^{4}}x-{{\cos }^{2}}x}}-{{2}^{{{\cos }^{4}}x-{{\sin }^{2}}x}}=\cos 2x$
2	To be considered the following complex and distinct a,b,c,d. Prove that the following affirmations are equivalent: i)For every $z\in \mathbb{C}$ the inequality takes place :$\left\| z-a \right\|+\left\| z-b \right\|\ge \left\| z-c \right\|+\left\| z-d \right\|$; ii)There is $t\in \left(0,1 \right)$ so that $c=ta+\left(1-t \right)b$ si $d=\left(1-t \right)a+tb$

3 Find all injective functions$f:\mathbb{Z}\to\mathbb{Z}$ that satisfy: $\left| f\left(x \right)-f\left(y \right) \right|\le \left| x-y \right|$,for any $x,y\in \mathbb{Z}$.

4 a)Prove that $\frac{1}{2}+\frac{1}{3}+...+\frac{1}{{{2}^{m}}}< m$, for any $m\in {{\mathbb{N}}^{*}}$.
b)Let ${{p}_{1}},{{p}_{2}},...,{{p}_{n}}$ be the prime numbers less than ${{2}^{100}}$. Prove that $\frac{1}{{{p}_{1}}}+\frac{1}{{{p}_{2}}}+...+\frac{1}{{{p}_{n}}}<10$

– XI

1 Given A, non-inverted matrices of order n with real elements, $n\ge 2$ and given ${{A}^{*}}$adjoin matrix A. Prove that $tr({{A}^{*}})\ne -1$ if and only if the matrix ${{I}_{n}}+{{A}^{*}}$ is invertible.

2 Whether m and n natural numbers, $m,n\ge 2$. Consider matrices, ${{A}_{1}},{{A}_{2}},...,{{A}_{m}}\in {{M}_{n}}(R)$ not all nilpotent. Demonstrate that there is an integer number $k>0$ such that ${{A}^{k}}_{1}+{{A}^{k}}_{2}+.....+{{A}^{k}}_{m}\ne {{O}_{n}}$

3 A function

$$\text{f:}\left(0\text{,}\infty \right)\to \left(0\text{,}\infty \right)$$

is called contract if, for every numbers $x,y\in \text{(0,}\infty \text{)}$ we have, $\underset{n\to \infty }{\mathop{\lim }}\,\left({{f}^{n}}\left(x \right)-{{f}^{n}}\left(y \right) \right)=0$ where ${{f}^{n}}=\underbrace{f\circ f\circ ...\circ f}_{n\ f\text{'s}}$
a) Consider

$$f:\left(0\text{,}\infty \right)\to \left(0\text{,}\infty \right)$$

a function contract, continue with the property that has a fixed point, that existing ${{x}_{0}}\in \left(0\text{,}\infty \right)$ there so that $f\left({{x}_{0}} \right)={{x}_{0}}$. Show that $f\left(x \right)>x$, for every $x\in \left(0\text{,}{{x}_{0}} \right)$ and $f\left(x \right)<x$, for every $x\in \left({{x}_{0}}\text{,}\infty \right)$.
b) Show that the given function

$$f\text{:}\left(0\text{,}\infty \right)\to \left(0\text{,}\infty \right)$$

given by $f\left(x \right)=x+\frac{1}{x}$ is contracted but has no fix number.

4 a) Consider

$$f\text{: }\!\![\!\!\text{ 0,}\infty)\to [0,\infty)$$

a differentiable and convex function .Show that $f\left(x \right)\le x$, for every $x\ge 0$, than ${f}'\left(x \right)\le 1$,for every $x\ge 0$
b) Determine

$$f\text{: }\!\![\!\!\text{ 0,}\infty)\to [0,\infty)$$

differentiable and convex functions which have the property that $f\left(0\right)=0$, and $f'\left(x\right)f\left(f\left(x\right)\right)=x$, for every $x\geq 0$

– XII

1 Determine continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that $\left(a^2+ab+b^2\right)\int\limits_a^b f\left(x\right)dx=3\int\limits_a^b x^2 f\left(x\right)dx$, for every $a,b\in\mathbb{R}$.

2 Given a ring $(A,+,\cdot)$ that meets both of the following conditions:
(1) A is not a field, and
(2) For every non-invertible element x of A, there is an integer $m>1$ (depending on x) such that $x=x^2+x^3+\ldots+x^{2^m}$.
Show that
(a) $x+x=0$ for every $x\in A$, and
(b) $x^2=x$ for every non-invertible $x\in A$.

3 Given $a\in(0,1)$ and C the set of increasing functions $f:[0,1]\to[0,\infty)$ such that $\int\limits_0^1 f(x)dx=1$. Determine: $(a)\max\limits_{f\in C}\int\limits_0^a f(x)dx$ $(b)\max\limits_{f\in C}\int\limits_0^a f^2(x)dx$

4 Given $n\geq 2$ a natural number, $(K,+,\cdot)$ a body with commutative property that $\underbrace{1+...+1}_{m}\neq 0, m=2,...,n$, $f\in K[X]$ a polynomial of degree n and G a subgroup of the additive group $(K,+,\cdot)$, $G\neq K$.Show that there is $a\in K$ so$f(a)\notin G$.

Romania Team Selection Test 2012

Day 1

1 Let $n_1, \ldots, n_k$ be positive integers, and define $d_1 = 1$ and $d_i = \frac{(n_1,\ldots,n_{i-1})}{(n_1,\ldots,n_i)}$, for $i \in \{2, \ldots, k\}$, where $(m_1, \ldots, m_\ell)$ denotes the greatest common divisor of the integers $m_1, \ldots, m_\ell$. Prove that the sums

$$\sum_{i=1}^{k} a_i n_i$$

with $a_i \in \{1, \ldots, d_i\}$ for $i \in \{1, \ldots, k\}$ are mutually distinct $\mod n_1$.

2 Let $ABCD$ be a cyclic quadrilateral such that the triangles BCD and CDA are not equilateral. Prove that if the Simson line of A with respect to $\triangle BCD$ is perpendicular to the Euler line of BCD, then the Simson line of B with respect to $\triangle ACD$ is perpendicular to the Euler line of $\triangle ACD$.

3 Let A and B be finite sets of real numbers and let x be an element of $A + B$. Prove that

$$|A \cap (x - B)| \leq \frac{|A - B|^2}{|A + B|}$$

where $A + B = \{a + b : a \in A, b \in B\}$, $x - B = \{x - b : b \in B\}$ and $A - B = \{a - b : a \in A, b \in B\}$.

4 Prove that a finite simple planar graph has an orientation so that every vertex has out-degree at most 3.

5 Let p and q be two given positive integers. A set of $p+q$ real numbers $a_1 < a_2 < \cdots < a_{p+q}$ is said to be balanced iff $a_1, \ldots, a_p$ were an arithmetic progression with common difference q and $a_p, \ldots, a_{p+q}$ where an arithmetic progression with common difference p. Find the maximum possible number of balanced sets, so that any two of them have nonempty intersection.

Comment: The intended problem also had "p and q are coprime" in the hypothesis. A typo when the problems where written made it appear like that in the exam (as if it were the only typo in the olympiad). Fortunately, the problem can be solved even if we didn't suppose that and it can be further generalized: we may suppose that a balanced set has $m+n$ reals $a_1 < \cdots < a_{m+n-1}$ so that $a_1, \ldots, a_m$ is an arithmetic progression with common difference p and $a_m, \ldots, a_{m+n-1}$ is an arithmetic progression with common difference q.

Day 2

1 Prove that for any positive integer $n \geq 2$ we have that

$$\sum_{k=2}^{n} \lfloor \sqrt[k]{n} \rfloor = \sum_{k=2}^{n} \lfloor \log_k n \rfloor.$$

2 Let $ABCD$ be a convex circumscribed quadrilateral such that $\angle ABC + \angle ADC < 180^\circ$ and $\angle ABD + \angle ACB = \angle ACD + \angle ADB$. Prove that one of the diagonals of quadrilateral $ABCD$ passes through the other diagonals midpoint.

3 Find the maximum possible number of kings on a 12×12 chess table so that each king attacks exactly one of the other kings (a king attacks only the squares that have a common point with the square he sits on).

4 Let k be a positive integer. Find the maximum value of

$$a^{3k-1}b + b^{3k-1}c + c^{3k-1}a + k^2 a^k b^k c^k,$$

where a, b, c are non-negative reals such that $a + b + c = 3k$.

Day 3

1 Let m and n be two positive integers greater than 1. Prove that there are m positive integers N_1 , ... , N_m (some of them may be equal) such that

$$\sqrt{m} = \sum_{i=1}^{m} (\sqrt{N_i} - \sqrt{N_i - 1})^{\frac{1}{n}}.$$

2 Let γ be a circle and l a line in its plane. Let K be a point on l, located outside of γ. Let KA and KB be the tangents from K to γ, where A and B are distinct points on γ. Let P and Q be two points on γ. Lines PA and PB intersect line l in two points R and respectively S. Lines QR and QS intersect the second time circle γ in points C and D. Prove that the tangents from C and D to γ are concurrent on line l.

3 Let a_1 , ... , a_n be positive integers and a a positive integer that is greater than 1 and is divisible by the product $a_1 a_2 \ldots a_n$. Prove that $a^{n+1} + a - 1$ is not divisible by the product $(a + a_1 - 1)(a + a_2 - 1) \ldots (a + a_n - 1)$.

4	Let S be a set of positive integers, each of them having exactly 100 digits in base 10 representation. An element of S is called *atom* if it is not divisible by the sum of any two (not necessarily distinct) elements of S. If S contains at most 10 atoms, at most how many elements can S have?

Day 4

1 Let ΔABC be a triangle. The internal bisectors of angles $\angle CAB$ and $\angle ABC$ intersect segments BC, respectively AC in D, respectively E. Prove that

$$DE \leq (3-2\sqrt{2})(AB+BC+CA).$$

2 Let $f, g : \mathbb{Z} \to [0, \infty)$ be two functions such that $f(n) = g(n) = 0$ with the exception of finitely many integers n. Define $h : \mathbb{Z} \to [0, \infty)$ by

$$h(n) = \max\{f(n-k)g(k) : k \in \mathbb{Z}\}.$$

Let p and q be two positive reals such that $1/p + 1/q = 1$. Prove that

$$\sum_{n\in\mathbb{Z}} h(n) \geq \left(\sum_{n\in\mathbb{Z}} f(n)^p\right)^{1/p} \left(\sum_{n\in\mathbb{Z}} g(n)^q\right)^{1/q}.$$

3 Determine all finite sets S of points in the plane with the following property: if $x, y, x', y' \in S$ and the closed segments xy and $x'y'$ intersect in only one point, namely z, then $z \in S$.

Day 5

1 Find all triples (a, b, c) of positive integers with the following property: for every prime p, if n is a quadratic residue $\mod p$, then $an^2 + bn + c$ is a quadratic residue $\mod p$.

2 Let n be a positive integer. Find the value of the following sum

$$\sum_{(n)} \sum_{k=1}^{n} e_k 2^{e_1+\cdots+e_k-2k-n},$$

where $e_k \in \{0, 1\}$ for $1 \leq k \leq n$, and the sum $\sum_{(n)}$ is taken over all 2^n possible choices of $e_1, \ldots, e_n$.

3 Let m and n be two positive integers for which $m < n$. n distinct points $X_1, \ldots, X_n$ are in the interior of the unit disc and at least one of them is on its border. Prove that we can find m distinct points $X_{i_1}, \ldots, X_{i_m}$ so that the distance between their center of gravity and the center of the circle is at least $\frac{1}{1+2m(1-1/n)}$.

Romania Team Selection Test 2013

Day 1

1 Given an integer $n \geq 2$, let a_n, b_n, c_n be integer numbers such that

$$\left(\sqrt[3]{2}-1\right)^n = a_n + b_n\sqrt[3]{2} + c_n\sqrt[3]{4}.$$

Prove that $c_n \equiv 1 \pmod{3}$ if and only if $n \equiv 2 \pmod{3}$.

2 Circles Ω and ω are tangent at a point P (ω lies inside Ω). A chord AB of Ω is tangent to ω at C; the line PC meets Ω again at Q. Chords QR and QS of Ω are tangent to ω. Let I, X, and Y be the incenters of the triangles APB, ARB, and ASB, respectively. Prove that $\angle PXI + \angle PYI = 90^\circ$.

3 Determine all injective functions defined on the set of positive integers into itself satisfying the following condition: If S is a finite set of positive integers such that $\sum\limits_{s\in S} \frac{1}{s}$ is an integer, then $\sum\limits_{s\in S} \frac{1}{f(s)}$ is also an integer.

4 Let n be an integer greater than 1. The set S of all diagonals of a $(4n-1)$-gon is partitioned into k sets, $S_1, S_2, \ldots, S_k$, so that, for every pair of distinct indices i and j, some diagonal in S_i crosses some diagonal in S_j; that is, the two diagonals share an interior point. Determine the largest possible value of k in terms of n.

Day 2

1 Suppose that a and b are two distinct positive real numbers such that $\lfloor na \rfloor$ divides $\lfloor nb \rfloor$ for any positive integer n. Prove that a and b are positive integers.

2 The vertices of two acute-angled triangles lie on the same circle. The Euler circle (nine-point circle) of one of the triangles passes through the midpoints of two sides of the other triangle. Prove that the triangles have the same Euler circle.

EDIT by pohoatza (in concordance with Luis' PS): Let ABC be a triangle with circumcenter Γ and nine-point center γ. Let X be a point on Γ and let Y, Z be on Γ so that the midpoints of segments XY and XZ are on γ. Prove that the midpoint of YZ is on γ.

3 Let S be the set of all rational numbers expressible in the form

$$\frac{(a_1^2+a_1-1)(a_2^2+a_2-1)\dots(a_n^2+a_n-1)}{(b_1^2+b_1-1)(b_2^2+b_2-1)\dots(b_n^2+b_n-1)}$$

for some positive integers $n, a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$. Prove that there is an infinite number of primes in S.

4 Let k be a positive integer larger than 1. Build an infinite set $\mathcal{A}$ of subsets of $\mathbb{N}$ having the following properties:

(a) any k distinct sets of $\mathcal{A}$ have exactly one common element;
(b) any $k+1$ distinct sets of $\mathcal{A}$ have void intersection.

Day 3

1 Let a and b be two square-free, distinct natural numbers. Show that there exist $c > 0$ such that

$$\left|\{n\sqrt{a}\} - \{n\sqrt{b}\}\right| > \frac{c}{n^3}$$

for every positive integer n.

2 Let γ a circle and P a point who lies outside the circle. Two arbitrary lines l and l' which pass through P intersect the circle at the points X, Y , respectively X', Y' , such that X lies between P and Y and X' lies between P and Y'. Prove that the line determined by the circumcentres of the triangles PXY' and $PX'Y$ passes through a fixed point.

3 Determine the largest natural number r with the property that among any five subsets with 500 elements of the set $\{1, 2, \dots, 1000\}$ there exist two of them which share at least r elements.

4 Let f and g be two nonzero polynomials with integer coefficients and $\deg f > \deg g$. Suppose that for infinitely many primes p the polynomial $pf + g$ has a rational root. Prove that f has a rational root.

Day 4

1 Fix a point O in the plane and an integer $n \geq 3$. Consider a finite family $\mathcal{D}$ of closed unit discs in the plane such that:
(a) No disc in $\mathcal{D}$ contains the point O; and
(b) For each positive integer $k < n$, the closed disc of radius $k+1$ centred at O

contains the centres of at least k discs in $\mathcal{D}$.
Show that some line through O stabs at least $\frac{2}{\pi}\log\frac{n+1}{2}$ discs in $\mathcal{D}$.

2 Let n be an integer larger than 1 and let S be the set of n-element subsets of the set $\{1, 2, \ldots, 2n\}$. Determine

$$\max_{A\in S}\left(\min_{x,y\in A, x\neq y}[x,y]\right)$$

where $[x, y]$ is the least common multiple of the integers x, y.

3 Given an integer $n \geq 2$, determine all non-constant polynomials f with complex coefficients satisfying the condition

$$1 + f(X^n + 1) = f(X)^n.$$

Day 5

1 Let n be a positive integer and let $x_1, \ldots, x_n$ be positive real numbers. Show that:

$$\min\left(x_1, \frac{1}{x_1}+x_2, \cdots, \frac{1}{x_{n-1}}+x_n, \frac{1}{x_n}\right) \leq 2\cos\frac{\pi}{n+2} \leq \max\left(x_1, \frac{1}{x_1}+x_2, \cdots, \frac{1}{x_{n-1}}+x_n, \frac{1}{x_n}\right)$$

2 Let K be a convex quadrangle and let l be a line through the point of intersection of the diagonals of K. Show that the length of the segment of intersection $l \cap K$ does not exceed the length of (at least) one of the diagonals of K.

3 Given a positive integer n, consider a triangular array with entries a_{ij} where i ranges from 1 to n and j ranges from 1 to $n-i+1$. The entries of the array are all either 0 or 1, and, for all $i > 1$ and any associated j , a_{ij} is 0 if $a_{i-1,j} = a_{i-1,j+1}$, and a_{ij} is 1 otherwise. Let S denote the set of binary sequences of length n, and define a map $f\colon S \to S$ via $f\colon (a_{11}, a_{12}, \cdots, a_{1n}) \to (a_{n1}, a_{n-1,2}, \cdots, a_{1n})$. Determine the number of fixed points of f.

Romania Team Selection Test 2014

Day 1

1 Let ABC be a triangle, let A', B', C' be the orthogonal projections of the vertices A ,B ,C on the lines BC, CA and AB, respectively, and let X be a point on the line AA'.Let γ_B be the circle through B and X, centred on the line BC, and let γ_C be the circle through C and X, centred on the line BC.The circle γ_B meets the lines AB and BB' again at M and M', respectively, and the circle γ_C meets the lines AC and CC' again at N and N', respectively.Show that the points M, M', N and N' are collinear.

2 Let $n \geq 2$ be an integer. Show that there exist $n+1$ numbers $x_1, x_2, \ldots, x_{n+1} \in \mathbb{Q} \setminus \mathbb{Z}$, so that $\{x_1^3\} + \{x_2^3\} + \cdots + \{x_n^3\} = \{x_{n+1}^3\}$, where $\{x\}$ is the fractionary part of x.

3 Let $A_0A_1A_2$ be a scalene triangle. Find the locus of the centres of the equilateral triangles $X_0X_1X_2$, such that A_k lies on the line $X_{k+1}X_{k+2}$ for each $k = 0, 1, 2$ (with indices taken modulo 3).

4 Let k be a nonzero natural number and m an odd natural number . Prove that there exist a natural number n such that the number $m^n + n^m$ has at least k distinct prime factors.

5 Let n be an integer greater than 1 and let S be a finite set containing more than $n+1$ elements.Consider the collection of all sets A of subsets of S satisfying the following two conditions :
(a) Each member of A contains at least n elements of S.
(b) Each element of S is contained in at least n members of A.

Determine $\max_A \min_B |B|$, as B runs through all subsets of A whose members cover S , and A runs through the above collection.

Day 2

1 Let ABC be a triangle and let X,Y,Z be interior points on the sides BC, CA, AB, respectively. Show that the magnified image of the triangle XYZ under a homothety of factor 4 from its centroid covers at least one of the vertices A, B, C.

2 Let a be a real number in the open interval $(0,1)$. Let $n \geq 2$ be a positive integer and let $f_n \colon \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = x + \frac{x^2}{n}$. Show that

$$\frac{a(1-a)n^2 + 2a^2n + a^3}{(1-a)^2n^2 + a(2-a)n + a^2} < (f_n \circ \cdots \circ f_n)(a) < \frac{an + a^2}{(1-a)n + a}$$

where there are n functions in the composition.

---

3 Determine all positive integers n such that all positive integers less than n and coprime to n are powers of primes.

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4 Let f be the function of the set of positive integers into itself, defi ned by $f(1) = 1$, $f(2n) = f(n)$ and $f(2n+1) = f(n) + f(n+1)$. Show that, for any positive integer n, the
number of positive odd integers m such that $f(m) = n$ is equal to the number of positive
integers **less or equal to** n and coprime to n.

[mod: the initial statement said less than n, which is wrong.]

Day 3

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1 Let ABC be an isosceles triangle, $AB = AC$, and let M and N be points on the sides BC and CA, respectively, such that $\angle BAM = \angle CNM$. The lines AB and MN meet at P. Show that the internal angle bisectors of the angles BAM and BPM meet at a point on the line BC.

---

2 For every positive integer n, let $\sigma(n)$ denote the sum of all positive divisors of n (1 and n, inclusive). Show that a positive integer n, which has at most two distinct prime factors, satisfies the condition $\sigma(n) = 2n - 2$ if and only if $n = 2^k(2^{k+1} + 1)$, where k is a non-negative integer and $2^{k+1} + 1$ is prime.

---

3 Determine the smallest real constant c such that

$$\sum_{k=1}^{n} \left(\frac{1}{k} \sum_{j=1}^{k} x_j \right)^2 \leq c \sum_{k=1}^{n} x_k^2$$

for all positive integers n and all positive real numbers $x_1, \cdots, x_n$.

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4 Let n be a positive integer and let A_n respectively B_n be the set of nonnegative integers $k < n$ such that the number of distinct prime factors of $\gcd(n,k)$ is even (respectively odd). Show that $|A_n| = |B_n|$ if n is even and $|A_n| > |B_n|$ if n is odd.

Example: $A_{10} = \{0, 1, 3, 7, 9\}$, $B_{10} = \{2, 4, 5, 6, 8\}$.

Day 4

1 Let $\triangle ABC$ be an acute triangle of circumcentre O. Let the tangents to the circumcircle of $\triangle ABC$ in points B and C meet at point P. The circle of centre P and radius $PB = PC$ meets the internal angle bisector of $\angle BAC$ inside $\triangle ABC$ at point S, and $OS \cap BC = D$. The projections of S on AC and AB respectively are E and F. Prove that AD, BE and CF are concurrent.

Author: Cosmin Pohoata

2 Let p be an odd prime number. Determine all pairs of polynomials f and g from $\mathbb{Z}[X]$ such that

$$f(g(X)) = \sum_{k=0}^{p-1} X^k = \Phi_p(X).$$

3 Let $n \in \mathbb{N}$ and S_n the set of all permutations of $\{1, 2, 3, ..., n\}$. For every permutation $\sigma \in S_n$ denote $I(\sigma) := \{i : \sigma(i) \leq i\}$.
Compute the sum $\sum_{\sigma \in S_n} \frac{1}{|I(\sigma)|} \sum_{i \in I(\sigma)} (i + \sigma(i))$.

Day 5

1 Let ABC a triangle and O his circumcentre.The lines OA and BC intersect each other at M ; the points N and P are defined in an analogous way.The tangent line in A at the circumcircle of triangle ABC intersect NP in the point X ; the points Y and Z are defined in an analogous way.Prove that the points X , Y and Z are collinear.

2 Let m be a positive integer and let A, respectively B, be two alphabets with m, respectively $2m$ letters. Let also n be an even integer which is at least $2m$. Let a_n be the number of words of length n, formed with letters from A, in which appear all the letters from A, each an even number of times. Let b_n be the number of words of length n, formed with letters from B, in which appear all the letters from B, each an odd number of times. Compute $\frac{b_n}{a_n}$.

3 Let n a positive integer and let $f\colon [0,1] \to \mathbb{R}$ an increasing function. Find the value of :

$$\max_{0 \le x_1 \le \cdots \le x_n \le 1} \sum_{k=1}^{n} f\left(\left| x_k - \frac{2k-1}{2n} \right| \right)$$

–	**Day 1**
1	Let ABC be a triangle, let O be its circumcenter, let A' be the orthogonal projection of A on the line BC, and let X be a point on the open ray AA' emanating from A. The internal bisectrix of the angle BAC meets the circumcircle of ABC again at D. Let M be the midpoint of the segment DX. The line through O and parallel to the line AD meets the line DX at N. Prove that the angles BAM and CAN are equal.
2	Let ABC be a triangle, and let r denote its inradius. Let R_A denote the radius of the circle internally tangent at A to the circle ABC and tangent to the line BC; the radii R_B and R_C are defined similarly. Show that $\frac{1}{R_A}+\frac{1}{R_B}+\frac{1}{R_C}\leq\frac{2}{r}$.
3	A Pythagorean triple is a solution of the equation $x^2+y^2=z^2$ in positive integers such that $x<y$. Given any non-negative integer n , show that some positive integer appears in precisely n distinct Pythagorean triples.
4	Let k be a positive integer congruent to 1 modulo 4 which is not a perfect square and let $a=\frac{1+\sqrt{k}}{2}$. Show that $\{\left\lfloor a^2n\right\rfloor-\left\lfloor a\left\lfloor an\right\rfloor\right\rfloor : n\in\mathbb{N}_{>0}\}=\{1,2,\ldots,\lfloor a\rfloor\}$.
5	Given an integer $N\geq 4$, determine the largest value the sum $$\sum_{i=1}^{\left\lfloor\frac{k}{2}\right\rfloor+1}\left(\left\lfloor\frac{n_i}{2}\right\rfloor+1\right)$$ may achieve, where $k,n_1,\ldots,n_k$ run through the integers subject to $k\geq 3$, $n_1\geq\ldots\geq n_k\geq 1$ and $n_1+\ldots+n_k=N$.
–	**Day 2**
1	Let a be an integer and n a positive integer . Show that the sum : $$\sum_{k=1}^{n}a^{(k,n)}$$ is divisible by n , where (x,y) is the greatest common divisor of the numbers x and y .

2 Let ABC be a triangle . Let A' be the center of the circle through the midpoint of the side BC and the orthogonal projections of B and C on the lines of support of the internal bisectrices of the angles ACB and ABC , respectively ; the points B' and C' are defined similarly . Prove that the nine-point circle of the triangle ABC and the circumcircle of $A'B'C'$ are concentric.

3 Given a positive real number t , determine the sets A of real numbers containing t , for which there exists a set B of real numbers depending on A , $|B| \geq 4$, such that the elements of the set $AB = \{ab \mid a \in A, b \in B\}$ form a finite arithmetic progression .

4 Consider the integral lattice $\mathbb{Z}^n$, $n \geq 2$, in the Euclidean n-space. Define a *line* in $\mathbb{Z}^n$ to be a set of the form $a_1 \times \cdots \times a_{k-1} \times \mathbb{Z} \times a_{k+1} \times \cdots \times a_n$ where k is an integer in the range $1, 2, \ldots, n$, and the a_i are arbitrary integers. A subset A of $\mathbb{Z}^n$ is called *admissible* if it is non-empty, finite, and every *line* in $\mathbb{Z}^n$ which intersects A contains at least two points from A. A subset N of $\mathbb{Z}^n$ is called *null* if it is non-empty, and every *line* in $\mathbb{Z}^n$ intersects N in an even number of points (possibly zero).
(a) Prove that every *admissible* set in $\mathbb{Z}^2$ contains a *null* set.
(b) Exhibit an *admissible* set in $\mathbb{Z}^3$ no subset of which is a *null* set .

– **Day 3**

1 Two circles γ and γ' cross one another at points A and B . The tangent to γ' at A meets γ again at C , the tangent to γ at A meets γ' again at C' , and the line CC' separates the points A and B . Let Γ be the circle externally tangent to γ , externally tangent to γ' , tangent to the line CC', and lying on the same side of CC' as B . Show that the circles γ and γ' intercept equal segments on one of the tangents to Γ through A .

2 Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be sequences of real numbers such that $a_0 > \frac{1}{2}$, $a_{n+1} \geq a_n$ and $b_{n+1} = a_n(b_n + b_{n+2})$ for all non-negative integers n . Show that the sequence $(b_n)_{n\geq 0}$ is bounded .

3 If k and n are positive integers , and $k \leq n$, let $M(n,k)$ denote the least common multiple of the numbers $n, n-1, \ldots, n-k+1$.Let $f(n)$ be the largest positive integer $k \leq n$ such that $M(n,1) < M(n,2) < \ldots < M(n,k)$. Prove that :
(a) $f(n) < 3\sqrt{n}$ for all positive integers n .
(b) If N is a positive integer , then $f(n) > N$ for all but finitely many positive integers n.

4 Given two integers $h \geq 1$ and $p \geq 2$, determine the minimum number of pairs of opponents an hp-member parliament may have, if in every partition of the parliament into h houses of p member each, some house contains at least one pair of opponents.

– **Day 4**

1 Let ABC and ABD be coplanar triangles with equal perimeters. The lines of support of the internal bisectrices of the angles CAD and CBD meet at P. Show that the angles APC and BPD are congruent.

2 Given an integer $k \geq 2$, determine the largest number of divisors the binomial coefficient $\binom{n}{k}$ may have in the range $n-k+1, \ldots, n$, as n runs through the integers greater than or equal to k.

3 Let n be a positive integer . If σ is a permutation of the first n positive integers , let $S(\sigma)$ be the set of all distinct sums of the form $\sum_{i=k}^{l} \sigma(i)$ where $1 \leq k \leq l \leq n$.
(a) Exhibit a permutation σ of the first n positive integers such that $|S(\sigma)| \geq \left\lfloor \frac{(n+1)^2}{4} \right\rfloor$.
(b) Show that $|S(\sigma)| > \frac{n\sqrt{n}}{4\sqrt{2}}$ for all permutations σ of the first n positive integers .

– **Day 5**

1 Let ABC be a triangle. Let P_1 and P_2 be points on the side AB such that P_2 lies on the segment BP_1 and $AP_1 = BP_2$; similarly, let Q_1 and Q_2 be points on the side BC such that Q_2 lies on the segment BQ_1 and $BQ_1 = CQ_2$. The segments P_1Q_2 and P_2Q_1 meet at R, and the circles P_1P_2R and Q_1Q_2R meet again at S, situated inside triangle P_1Q_1R. Finally, let M be the midpoint of the side AC. Prove that the angles P_1RS and Q_1RM are equal.

2 Let n be an integer greater than 1, and let p be a prime divisor of n. A confederation consists of p states, each of which has exactly n airports. There are p air companies operating interstate flights only such that every two airports in different states are joined by a direct (two-way) flight operated by one of these companies. Determine the maximal integer N satisfying the following condition: In every such confederation it is possible to choose one of the p air companies and N of the np airports such that one may travel (not necessarily

directly) from any one of the N chosen airports to any other such only by flights operated by the chosen air company.

3 Define a sequence of integers by $a_0 = 1$, and $a_n = \sum_{k=0}^{n-1} \binom{n}{k} a_k$, $n \geq 1$. Let m be a positive integer , let p be a prime , and let q and r be non-negative integers . Prove that :

$$a_{p^m q+r} \equiv a_{p^{m-1} q+r} \pmod{p^m}$$