EGMO 2013

Day 1	April 10th

1 The side BC of the triangle ABC is extended beyond C to D so that $CD = BC$. The side CA is extended beyond A to E so that $AE = 2CA$. Prove that, if $AD = BE$, then the triangle ABC is right-angled.

2 Determine all integers m for which the $m \times m$ square can be dissected into five rectangles, the side lengths of which are the integers $1, 2, 3, \ldots, 10$ in some order.

3 Let n be a positive integer.

(a) Prove that there exists a set S of $6n$ pairwise different positive integers, such that the least common multiple of any two elements of S is no larger than $32n^2$.

(b) Prove that every set T of $6n$ pairwise different positive integers contains two elements the least common multiple of which is larger than $9n^2$.

Day 2	April 11th

4 Find all positive integers a and b for which there are three consecutive integers at which the polynomial
$$P(n) = \frac{n^5 + a}{b}$$
takes integer values.

5 Let Ω be the circumcircle of the triangle ABC. The circle ω is tangent to the sides AC and BC, and it is internally tangent to the circle Ω at the point P. A line parallel to AB intersecting the interior of triangle ABC is tangent to ω at Q.

Prove that $\angle ACP = \angle QCB$.

6 Snow White and the Seven Dwarves are living in their house in the forest. On each of 16 consecutive days, some of the dwarves worked in the diamond mine while the remaining dwarves collected berries in the forest. No dwarf performed both types of work on the same day. On any two different (not necessarily consecutive) days, at least three dwarves each performed both types of work. Further, on the first day, all seven dwarves worked in the diamond mine.

Prove that, on one of these 16 days, all seven dwarves were collecting berries.