IMO 1970

Day 1

$\boxed{1}$ M is any point on the side AB of the triangle ABC. r, r_1, r_2 are the radii of the circles inscribed in ABC, AMC, BMC. q is the radius of the circle on the opposite side of AB to C, touching the three sides of AB and the extensions of CA and CB. Similarly, q_1 and q_2. Prove that $r_1 r_2 q = r q_1 q_2$.

$\boxed{2}$ We have $0 \leq x_i < b$ for $i = 0, 1, \ldots, n$ and $x_n > 0, x_{n-1} > 0$. If $a > b$, and $x_n x_{n-1} \ldots x_0$ represents the number A base a and B base b, whilst $x_{n-1} x_{n-2} \ldots x_0$ represents the number A' base a and B' base b, prove that $A'B < AB'$.

$\boxed{3}$ The real numbers $a_0, a_1, a_2, \ldots$ satisfy $1 = a_0 \leq a_1 \leq a_2 \leq \ldots . b_1, b_2, b_3, \ldots$ are defined by $b_n = \sum_{k=1}^{n} \frac{1 - \frac{a_{k-1}}{a_k}}{\sqrt{a_k}}$.

a.) Prove that $0 \leq b_n < 2$.

b.) Given c satisfying $0 \leq c < 2$, prove that we can find a_n so that $b_n > c$ for all sufficiently large n.

Day 2

1. Find all positive integers n such that the set $\{n, n+1, n+2, n+3, n+4, n+5\}$ can be partitioned into two subsets so that the product of the numbers in each subset is equal.

2. In the tetrahedron $ABCD, \angle BDC = 90^o$ and the foot of the perpendicular from D to ABC is the intersection of the altitudes of ABC. Prove that:

$$(AB + BC + CA)^2 \leq 6(AD^2 + BD^2 + CD^2).$$

When do we have equality?

3. Given 100 coplanar points, no three collinear, prove that at most 70% of the triangles formed by the points have all angles acute.