1
1999 National Contests: Problems and Solutions

1.1 Belarus

National Olympiad, Fourth Round

Problem 10.1 Determine all real numbers a such that the function $f(x) = \{ax + \sin x\}$ is periodic. Here $\{y\}$ is the fractional part of y.

Solution: The solutions are $a = \frac{r}{\pi}$, $r \in \mathbb{Q}$.

First, suppose $a = \frac{r}{\pi}$ for some $r \in \mathbb{Q}$; write $r = \frac{p}{q}$ with p, $q \in \mathbb{Z}$, $q > 0$. Then

$$\begin{aligned} f(x + 2q\pi) &= \left\{ \frac{p}{q\pi}(x + 2q\pi) + \sin(x + 2q\pi) \right\} \\ &= \left\{ \frac{p}{q\pi}x + 2p + \sin x \right\} \\ &= \left\{ \frac{p}{q\pi}x + \sin x \right\} = f(x) \end{aligned}$$

so f is periodic with period $2q\pi$.

Now, suppose f is periodic; then there exists $p > 0$ such that $f(x) = f(x+p)$ for all $x \in \mathbb{R}$. Then $\{ax+\sin x\} = \{ax+ap+\sin(x+p)\}$ for all $x \in \mathbb{R}$; in other words $g(x) = ap+\sin(x+p)-\sin x$ is an integer for all x. But g is continuous, so there exists $k \in \mathbb{Z}$ such that $g(x) = k$ for all $x \in \mathbb{R}$. Rewriting this gives

$$\sin(x + p) - \sin x = k - ap \qquad \text{for all } x \in \mathbb{R}.$$

Letting $x = y,\ y + p,\ y + 2p,\ \ldots,\ y + (n - 1)p$ and summing gives

$$\sin(y + np) - \sin y = n(k - ap) \qquad \text{for all } y \in \mathbb{R} \text{ and } n \in \mathbb{N}.$$

Since the left hand side of this equation is bounded by 2, we conclude that $k = ap$ and $\sin(x + p) = \sin x$ for all $x \in \mathbb{R}$. In particular, $\sin\left(\frac{\pi}{2} + p\right) = \sin\left(\frac{\pi}{2}\right) = 1$ and hence $p = 2m\pi$ for some $m \in \mathbb{N}$. Thus $a = \frac{k}{p} = \frac{k}{2m\pi} = \frac{r}{\pi}$ with $r = \frac{k}{2m} \in \mathbb{Q}$, as desired.

Problem 10.2 Prove that for any integer $n > 1$ the sum S of all divisors of n (including 1 and n) satisfies the inequalities

$$k\sqrt{n} < S < \sqrt{2k}n,$$

where k is the number of divisors of n.

Solution: Let the divisors of n be $1 = d_1 < d_2 < \cdots < d_k = n$;

then $d_i d_{k+1-i} = n$ for each i. Thus

$$S = \sum_{i=1}^{k} d_i = \sum_{i=1}^{k} \frac{d_i + d_{k+1-i}}{2} > \sum_{i=1}^{k} \sqrt{d_i d_{k+1-i}} = k\sqrt{n},$$

giving the left inequality. (The inequality is strict because equality does not hold for $\frac{d_1+d_k}{2} \geq \sqrt{d_1 d_k}$.) For the right inequality, let $S_2 = \sum_{i=1}^{k} d_i^2$ and use the Power Mean Inequality to get

$$\frac{S}{k} = \frac{\sum_{i=1}^{k} d_i}{k} \leq \sqrt{\frac{\sum_{i=1}^{k} d_i^2}{k}} = \sqrt{\frac{S_2}{k}} \quad \text{so} \quad S \leq \sqrt{kS_2}.$$

Now

$$\frac{S_2}{n^2} = \sum_{i=1}^{k} \frac{d_i^2}{n^2} = \sum_{i=1}^{k} \frac{1}{d_{k+1-i}^2} \leq \sum_{j=1}^{n} \frac{1}{j^2} < \frac{\pi^2}{6}$$

since $d_1, \ldots, d_k$ are distinct integers between 1 and n. Therefore

$$S \leq \sqrt{kS_2} < \sqrt{\frac{kn^2\pi^2}{6}} < \sqrt{2k}n.$$

Problem 10.3 There is a 7×7 square board divided into 49 unit cells, and tiles of three types: 3×1 rectangles, 3-unit-square corners, and unit squares. Jerry has infinitely many rectangles and one corner, while Tom has only one square.

(a) Prove that Tom can put his square somewhere on the board (covering exactly one unit cell) in such a way that Jerry can not tile the rest of the board with his tiles.

(b) Now Jerry is given another corner. Prove that no matter where Tom puts his square (covering exactly one unit cell), Jerry can tile the rest of the board with his tiles.

Solution:

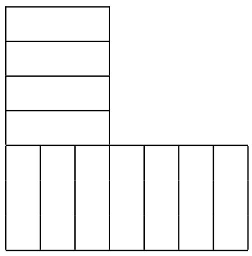
(a) Tom should place his square on the cell marked `X` in the boards below.

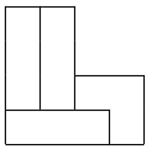
1	2	3	1	2	3	1
2	3	1	2	3	1	2
3	1	2	3	1	2	3
1	2	3	1	2	3	1
2	3	1	X	3	1	2
3	1	2	3	1	2	3
1	2	3	1	2	3	1

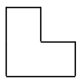
1	3	2	1	3	2	1
2	1	3	2	1	3	2
3	2	1	3	2	1	3
1	3	2	1	3	2	1
2	1	3	X	1	3	2
3	2	1	3	2	1	3
1	3	2	1	3	2	1

The grid on the left contains 17 1's, 15 2's and 16 3's; since every 3×1 rectangle contains a 1, a 2 and a 3, Jerry's corner must cover a 3 and two 1's; thus it must be oriented like a Γ. But every such corner covers a 1, a 2 and a 3 in the right grid, as does any 3×1 rectangle. Since the right grid also contains 17 1's, 15 2's and 16 3's, Jerry cannot cover the 48 remaining squares with his pieces.

(b) The following constructions suffice.







The first figure can be rotated and placed on the 7×7 board so that Tom's square falls into its blank, untiled region. Similarly, the second figure can be rotated and placed within the remaining untiled 4×4 region so that Tom's square is still uncovered; and finally, the single corner can be rotated and placed without overlapping Tom's square.

Problem 10.4 A circle is inscribed in the isosceles trapezoid $ABCD$. Let the circle meet diagonal AC at K and L (with K between

A and L). Find the value of

$$\frac{AL \cdot KC}{AK \cdot LC}.$$

First Solution:

Lemma. *Suppose we have a (not necessarily isosceles) trapezoid $ABCD$ circumscribed about a circle with radius r, where the circle touches sides AB, BC, CD, DA at points P, Q, R, S, respectively. Let line AC intersect the circle at K and L, with K between A and L. Also write $m = AP$ and $n = CR$. Then*

$$AK \cdot LC = mn + 2r^2 - \sqrt{(mn + 2r^2)^2 - (mn)^2}$$

and

$$AL \cdot KC = mn + 2r^2 + \sqrt{(mn + 2r^2)^2 - (mn)^2}.$$

Proof: Assume without loss of generality that $AB \parallel CD$, and orient the trapezoid so that lines AB and CD are horizontal. Let $t = AK$, $u = KL$, and $v = LC$; also let $\sigma = t + v$ and $\pi = tv$. By Power of a Point, we have $t(t + u) = m^2$ and $v(v + u) = n^2$; multiplying these gives $\pi(\pi + u\sigma + u^2) = m^2n^2$. Also, A and C are separated by $m + n$ horizontal distance and $2r$ vertical distance; thus $AC^2 = (m + n)^2 + (2r)^2$. Then

$$\begin{aligned}
(m + n)^2 + (2r)^2 &= AC^2 = (t + u + v)^2 \\
m^2 + 2mn + n^2 + 4r^2 &= t(t + u) + v(v + u) + 2\pi + u\sigma + u^2 \\
m^2 + 2mn + n^2 + 4r^2 &= m^2 + n^2 + 2\pi + u\sigma + u^2 \\
2mn + 4r^2 - \pi &= \pi + u\sigma + u^2.
\end{aligned}$$

Multiplying by π on both sides we have

$$\pi(2mn + 4r^2 - \pi) = \pi(\pi + u\sigma + u^2) = (mn)^2,$$

a quadratic in π with solutions

$$\pi = mn + 2r^2 \pm \sqrt{(mn + 2r^2)^2 - (mn)^2}.$$

But since $m^2n^2 = t(t + u)v(v + u) \geq t^2v^2$, we must have $mn \geq \pi$. Therefore $AK \cdot LC = \pi = mn + 2r^2 - \sqrt{(mn + 2r^2)^2 - (mn)^2}$. And since $(AK \cdot AL) \cdot (CK \cdot CL) = m^2 \cdot n^2$, we have $AL \cdot KC = \frac{m^2n^2}{\pi} = mn + 2r^2 + \sqrt{(mn + 2r^2)^2 - (mn)^2}$. ■

As in the lemma, assume that $AB \parallel CD$ and let the given circle be tangent to sides AB, BC, CD, DA at points P, Q, R, S, respectively. Also define $m = AP = PB = AS = BQ$ and $n = DR = RC = DS = CQ$.

Drop perpendicular $\overline{AX}$ to line CD. Then $AD = m + n$, $DX = |m-n|$, and $AX = 2r$. Then by the Pythagorean Theorem on triangle ADX, we have $(m+n)^2 = (m-n)^2 + (2r)^2$ which gives $mn = r^2$.

Using the lemma, we find that $AK \cdot LC = (3 - 2\sqrt{2})r^2$ and $AL \cdot KC = (3 + 2\sqrt{2})r^2$. Thus $\frac{AL \cdot KC}{AK \cdot LC} = 17 + 12\sqrt{2}$.

Second Solution: Suppose $A'B'C'D'$ is a square with side length s, and define K', L' analagously to K and L. Then $A'C' = s\sqrt{2}$ and $K'L' = s$, and $A'L' = K'C' = s\frac{\sqrt{2}+1}{2}$ and $A'K' = L'C' = s\frac{\sqrt{2}-1}{2}$. Thus

$$\frac{A'L' \cdot K'C'}{A'K' \cdot L'C'} = \frac{(\sqrt{2}+1)^2}{(\sqrt{2}-1)^2} = (\sqrt{2}+1)^4 = 17 + 12\sqrt{2}.$$

Consider an arbitrary isosceles trapezoid $ABCD$ with inscribed circle ω; assume $AB \parallel CD$. Since no three of A, B, C, D are collinear, there is a projective transformation τ taking $ABCD$ to a parallelogram $A'B'C'D'$. This map takes ω to a conic ω' tangent to the four sides of $A'B'C'D'$. Let $P = BC \cap AD$, and let ℓ be the line through P parallel to line AB; then τ maps ℓ to the line at ∞. Since ω does not intersect ℓ, ω' is an ellipse. Thus by composing τ with an affine transformation (which preserves parallelograms) we may assume that ω' is a circle. Let W, X, Y, Z be the tangency points of ω to sides AB, BC, CD, DA respectively, and W', X', Y', Z' their images under τ. By symmetry line WY passes through the intersection of lines BC and AD, and line XZ is parallel to lines AB and CD; thus $W'Y' \parallel B'C' \parallel A'D'$ and $X'Z' \parallel A'B' \parallel C'D'$. But ω' is tangent to the parallel lines $A'B'$ and $C'D'$ at W' and Y', so $\overline{W'Y'}$ is a diameter of ω' and $W'Y' \perp A'B'$; thus $B'C' \perp A'B'$ and $A'B'C'D'$ is a rectangle. Since $A'B'C'D'$ has an inscribed circle it must be a square. Thus we are in the case considered at the beginning of the problem; if K' and L' are the intersections of line $A'C'$ with ω', with K' between A' and L', then $\frac{A'L' \cdot K'C'}{A'K' \cdot L'C'} = 17 + 12\sqrt{2}$. Now τ maps $\{K, L\} = AC \cap \omega$ to $\{K', L'\} = A'C' \cap \omega'$ (but perhaps not in that order). If $\tau(K) = K'$ and $\tau(L) = L'$, then since projective transformations preserve cross-ratios, we would have

$$\frac{AL \cdot KC}{AK \cdot LC} = \frac{A'L' \cdot K'C'}{A'K' \cdot L'C'} = 17 + 12\sqrt{2}.$$

But if instead $\tau(K) = L'$ and $\tau(L) = K'$, then we would obtain $\frac{AL \cdot KC}{AK \cdot LC} = \frac{1}{17+12\sqrt{2}} < 1$, impossible since $AL > AK$ and $KC > LC$. It follows that $\frac{AL \cdot KC}{AK \cdot LC} = 17 + 12\sqrt{2}$, as desired.

Problem 10.5 Let P and Q be points on the side AB of the triangle ABC (with P between A and Q) such that $\angle ACP = \angle PCQ = \angle QCB$, and let $\overline{AD}$ be the angle bisector of $\angle BAC$. Line AD meets lines CP and CQ at M and N respectively. Given that $PN = CD$ and $3\angle BAC = 2\angle BCA$, prove that triangles CQD and QNB have the same area.

Solution: Since $3\angle BAC = 2\angle ACB$,

$$\angle PAN = \angle NAC = \angle ACP = \angle PCQ = \angle QCD.$$

Let θ equal this common angle measure. Thus $ACNP$ and $ACDQ$ are cyclic quadrilaterals, so

$$\theta = \angle ANP = \angle CQD = \angle CPN.$$

From angle-angle-side congruency we deduce that $\triangle NAP \cong \triangle CQD \cong \triangle PCN$. Hence $CP = CQ$, and by symmetry we have $AP = QB$. Thus, $[CQD] = [NAP] = [NQB]$.

Problem 10.6 Show that the equation

$$\{x^3\} + \{y^3\} = \{z^3\}$$

has infinitely many rational non-integer solutions. Here $\{a\}$ is the fractional part of a.

Solution: Let $x = \frac{3}{5}(125k+1)$, $y = \frac{4}{5}(125k+1)$, $z = \frac{6}{5}(125k+1)$ for any integer k. These are never integers because 5 does not divide $125k + 1$. Moreover

$$125x^3 = 3^3(125k+1)^3 \equiv 3^3 \pmod{125},$$

so 125 divides $125x^3 - 3^3$ and $x^3 - \left(\frac{3}{5}\right)^3$ is an integer; thus $\{x^3\} = \frac{27}{125}$. Similarly $\{y^3\} = \frac{64}{125}$ and $\{z^3\} = \frac{216}{125} - 1 = \frac{91}{125} = \frac{27}{125} + \frac{64}{125}$, and therefore $\{x^3\} + \{y^3\} = \{z^3\}$.

Problem 10.7 Find all integers n and real numbers m such that the squares of an $n \times n$ board can be labelled 1, 2, ..., n^2 with each number appearing exactly once in such a way that

$$(m-1)a_{ij} \le (i+j)^2 - (i+j) \le ma_{ij}$$

for all $1 \le i, j \le n$, where a_{ij} is the number placed in the intersection of the ith row and jth column.

Solution: Either $n = 1$ and $2 \le m \le 3$ or $n = 2$ and $m = 3$. It is easy to check that these work using the constructions below.

$$\boxed{1} \qquad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$$

Now suppose we are given a labelling of the squares $\{a_{ij}\}$ which satisfies the given conditions. By assumption $a_{11} \ge 1$ so

$$m - 1 \le (m-1)a_{11} \le (1+1)^2 - (1+1) = 2$$

and $m \le 3$. On the other hand $a_{nn} \le n^2$ so

$$4n^2 - 2n = (n+n)^2 - (n+n) \le ma_{nn} \le mn^2$$

and $m \ge \frac{4n^2-2n}{n^2} = 4 - \frac{2}{n}$. Thus $4 - \frac{2}{n} \le m \le 3$ which implies the result.

Problem 11.1 Evaluate the product

$$\prod_{k=0}^{2^{1999}} \left(4\sin^2 \frac{k\pi}{2^{2000}} - 3\right).$$

Solution: For simplicity, write $f(x) = \sin\left(\frac{x\pi}{2^{2000}}\right)$.

At $k = 0$, the expression inside the parentheses equals -3. Recognizing the triple-angle formula $\sin(3\theta) = 4\sin^3\theta - 3\sin\theta$ at play, and noting that $f(k) \ne 0$ when $1 \le k \le 2^{1999}$, we can rewrite the given product as

$$-3\prod_{k=1}^{2^{1999}} \frac{\sin\left(\frac{3k\pi}{2^{2000}}\right)}{\sin\left(\frac{k\pi}{2^{2000}}\right)} \quad \text{or} \quad -3\prod_{k=1}^{2^{1999}} \frac{f(3k)}{f(k)}. \tag{1}$$

Now

$$\prod_{k=1}^{2^{1999}} f(3k) = \prod_{k=1}^{\frac{2^{1999}-2}{3}} f(3k) \cdot \prod_{k=\frac{2^{1999}+1}{3}}^{\frac{2^{2000}-1}{3}} f(3k) \cdot \prod_{\frac{2^{2000}+2}{3}}^{2^{1999}} f(3k).$$

Since $\sin\theta = \sin(\pi-\theta) = -\sin(\pi+\theta)$, we have $f(x) = f(2^{2000}-x) = -f(x-2^{2000})$. Hence, letting $S_i = \{k \mid 1 \le k \le 2^{1999}, k \equiv i \pmod 3\}$ for $i = 0, 1, 2$, the last expression equals

$$\begin{aligned}
&\prod_{k=1}^{\frac{2^{1999}-2}{3}} f(3k) \cdot \prod_{k=\frac{2^{1999}+1}{3}}^{\frac{2^{2000}-1}{3}} f(2^{2000}-3k) \cdot \prod_{\frac{2^{2000}+2}{3}}^{2^{1999}} \left(-f(3k-2^{2000})\right) \\
&\quad = \prod_{k\in S_0} f(k) \cdot \prod_{k\in S_1} f(k) \cdot \prod_{k\in S_2} (-f(k)) \\
&\quad = (-1)^{\frac{2^{1999}+1}{3}} \prod_{k=1}^{2^{1999}} f(k) = -\prod_{k=1}^{2^{1999}} f(k).
\end{aligned}$$

Combined with the expression in (1), this implies that the desired product is $(-3)(-1) = 3$.

Problem 11.2 Let m and n be positive integers. Starting with the list 1, 2, 3, ..., we can form a new list of positive integers in two different ways.

(i) We first erase every mth number in the list (always starting with the first); then, in the list obtained, we erase every nth number. We call this *the first derived list.*

(ii) We first erase every nth number in the list; then, in the list obtained, we erase every mth number. We call this *the second derived list.*

Now, we call a pair (m, n) *good* if and only if the following statement is true: if some positive integer k appears in both derived lists, then it appears in the same position in each.

(a) Prove that $(2, n)$ is good for any positive integer n.

(b) Determine if there exists any good pair (m, n) such that $2 < m < n$.

Solution: Consider whether some positive integer j is in the first derived list. If it is congruent to $1 \pmod m$, then $j + mn$ is as well so they are both erased. If not, then suppose it is the t-th number remaining after we've erased all the multiples of m. There are n multiples of m erased between j and $j+mn$, so $j+mn$ is the $(t+mn-n)$-th number remaining after we've erased all the multiples of m. But either t and $t+mn-n$ are both congruent to $1 \pmod n$ or both *not* congruent to $1 \pmod n$. Hence j is erased after our second pass if and only if $j+mn$ is as well.

A similar argument applies to the second derived list. Thus in either derived list, the locations of the erased numbers repeat with period mn; and also, among each mn consecutive numbers exactly $mn-(m+n-1)$ remain. (In the first list, $n+\left(\left\lfloor\frac{mn-n-1}{n}\right\rfloor+1\right) = n+\left(m-1+\left\lfloor\frac{-1}{n}\right\rfloor+1\right) = m+n-1$ of the first mn numbers are erased; similarly, $m+n-1$ of the first mn numbers are erased in the second list.)

These facts imply that the pair (m,n) is good if and only if when any $k \le mn$ is in both lists, it appears at the same position.

(a) Given a pair $(2,n)$, the first derived list (up to $k=2n$) is $4,6,8,\ldots,2n$. If n is even, the second derived list is $3,5,\ldots,n-1,n+2,n+4,\ldots,2n$. And if n is odd, the second derived list is $3,5,\ldots,n-2,n,n+3,n+5,\ldots,2n$. In either case the first and second lists' common elements are the even numbers between $n+2$ and $2n$ inclusive. Each such $2n-i$ (with $i<\frac{n-1}{2}$) is the $(n-1-i)$-th number on both lists, showing that $(2,n)$ is good.

(b) Such a pair exists—in fact, the simplest possible pair $(m,n)=(3,4)$ suffices. The first derived list (up to $k=12$) is $3,5,6,9,11,12$ and the second derived list is $3,4,7,8,11,12$. The common elements are $3,11,12$, and these are all in the same positions.

Problem 11.3 Let a_1, a_2, $\ldots$, a_{100} be an ordered set of numbers. At each move it is allowed to choose any two numbers a_n, a_m and change them to the numbers

$$\frac{a_n^2}{a_m}-\frac{n}{m}\left(\frac{a_m^2}{a_n}-a_m\right) \quad \text{and} \quad \frac{a_m^2}{a_n}-\frac{m}{n}\left(\frac{a_n^2}{a_m}-a_n\right)$$

respectively. Determine if it is possible, starting with the set with $a_i=\frac{1}{5}$ for $i=20$, 40, 60, 80, 100 and $a_i=1$ otherwise, to obtain a set consisting of integers only.

Solution: After transforming a_n to $a_n'=\frac{a_n^2}{a_m}-\frac{n}{m}\left(\frac{a_m^2}{a_n}-a_m\right)$ and

a_m to $a'_m = \frac{a_m^2}{a_n} - \frac{m}{n}\left(\frac{a_n^2}{a_m} - a_n\right)$, we have

$$\begin{aligned}
&\frac{a'_n}{n} + \frac{a'_m}{m} \\
&= \left[\left(\frac{1}{n}\cdot\frac{a_n^2}{a_m} - \frac{1}{m}\cdot\frac{a_m^2}{a_n}\right) + \frac{a_m}{m}\right] \\
&\quad + \left[\left(\frac{1}{m}\cdot\frac{a_m^2}{a_n} - \frac{1}{n}\cdot\frac{a_n^2}{a_m}\right) + \frac{a_n}{n}\right] \\
&= \frac{a_n}{n} + \frac{a_m}{m}.
\end{aligned}$$

Thus the quantity $\sum_{i=1}^{100} \frac{a_i}{i}$ is invariant under the given operation. At the beginning, this sum equals

$$I_1 = \sum_{i=1}^{99} \frac{a_i}{i} + \frac{1}{500}.$$

When each of the numbers $\frac{a_1}{1}, \frac{a_2}{2}, \ldots, \frac{a_{99}}{99}$ is written as a fraction in lowest terms, none of their denominators are divisible by 125; while 125 *does* divide the denominator of $\frac{1}{500}$. Thus when written as a fraction in lowest terms, I_1 must have a denominator divisible by 125.

Now suppose by way of contradiction that we could make all the numbers equal to integers $b_1, b_2, \ldots, b_{100}$ in that order. Then in $I_2 = \sum_{i=1}^{100} \frac{b_i}{i}$, the denominator of each of the fractions $\frac{b_i}{i}$ is not divisible by 125. Thus when I_2 is written as a fraction in lowest terms, its denominator is not divisible by 125 either. But then I_2 cannot possibly equal I_1, a contradiction. Therefore we can never obtain a set consisting of integers only.

Problem 11.4 A circle is inscribed in the trapezoid $ABCD$. Let K, L, M, N be the points of intersections of the circle with diagonals AC and BD respectively (K is between A and L and M is between B and N). Given that $AK \cdot LC = 16$ and $BM \cdot ND = \frac{9}{4}$, find the radius of the circle.

Solution: Let the circle touch sides AB, BC, CD, DA at P, Q, R, S, respectively, and let r be the radius of the circle. Let $w = AS = AP$; $x = BP = BQ$; $y = CQ = CR$; and $z = DR = DS$. As in problem 11.4, we have $wz = xy = r^2$ and thus $wxyz = r^4$. Also observe that from the lemma in problem 11.4, $AK \cdot LC$ depends only on r and $AP \cdot CR$; and $BM \cdot ND$ depends only on r and $BP \cdot DR$.

Now draw a parallelogram $A'B'C'D'$ circumscribed about the same circle, with points P', Q', R', S' defined analagously to P, Q, R, S, such that $A'P' = C'R' = \sqrt{wy}$. Draw points K', L', M', N' analagously to K, L, M, N. Then since $A'P' \cdot C'R' = wy$, by the observation in the first paragraph we must have $A'K' \cdot L'C' = AK \cdot LC = 16$; therefore $A'K' = L'C' = 4$. And as with quadrilateral $ABCD$, we have $A'P' \cdot B'P' \cdot C'R' \cdot D'R' = r^4 = wxyz$. Thus $B'P' \cdot D'R' = xz$ and again by the observation we must have $B'M' \cdot N'D' = BM \cdot ND = \frac{9}{4}$. Therefore $B'M' = N'D' = \frac{3}{2}$.

Then if O is the center of the circle, we have $A'O = 4 + r$ and $S'O = r$. By the Pythagorean Theorem $A'S' = \sqrt{8r + 16}$; similarly, $S'D' = \sqrt{3r + \frac{9}{4}}$. Since $A'S' \cdot S'D' = r^2$, we have

$$(8r + 16)\left(3r + \frac{9}{4}\right) = r^4,$$

which has positive solution $r = 6$ and, by Descartes' rule of signs, no other positive solutions.

Problem 11.5 Find the greatest real number k such that for any triple of positive real numbers a, b, c such that

$$kabc > a^3 + b^3 + c^3,$$

there exists a triangle with side legths a, b, c.

Solution: Equivalently, we want the greatest real number k such that for any a, b, $c > 0$ with $a + b \le c$, we have

$$kabc \le a^3 + b^3 + c^3.$$

First pick $b = a$ and $c = 2a$. Then we must have

$$2ka^3 \le 10a^3 \Longrightarrow k \le 5.$$

On the other hand, suppose $k = 5$. Then writing $c = a + b + x$, expanding $a^3 + b^3 + c^3 - 5abc$ gives

$$2a^3 + 2b^3 - 2a^2b - 2ab^2 + abx + 3(a^2 + b^2)x + 3(a + b)x^2 + x^3.$$

But $2a^3 + 2b^3 - 2a^2b - 2ab^2 \ge 0$ (either by rearrangement, by AM-GM, or from the inequality $(a + b)(a - b)^2 \ge 0$); and the other terms are nonnegative. Thus $a^3 + b^3 + c^3 - 5abc \ge 0$, as desired.

Problem 11.6 Find all integers x and y such that

$$x^6 + x^3y = y^3 + 2y^2.$$

Solution: The only solutions are (x, y) equals $(0, 0)$, $(0, -2)$, and $(2, 4)$.

If $x = 0$ then $y = 0$ or -2; if $y = 0$ then $x = 0$. Now assume that both x and y are nonzero, and rewrite the given equation as $x^3(x^3 + y) = y^2(y + 2)$.

We first show that $(x, y) = (ab, 2b^3)$, (ab, b^3), or $(ab, \frac{b^3}{2})$ for some integers a, b. Suppose some prime p divides y exactly $m > 0$ times (that is, y is divisible by p^m but not p^{m+1}). Then since $x^6 = y^3 + 2y^2 - x^3y$, p must divide x as well — say, $n > 0$ times.

First suppose $p > 2$; then it divides the right hand side $y^2(y + 2)$ exactly $2m$ times. If $3n < m$ then p divides the left hand side $x^3(x^3 + y)$ exactly $6n$ times so that $6n = 2m$, a contradiction. And if $3n > m$ then p divides the left hand side exactly $3n + m$ times so that $3n + m = 2m$ and $3n = m$, a contradiction. Therefore $3n = m$.

Now suppose $p = 2$. If $m > 1$, then 2 divides the right hand side exactly $2m + 1$ times. If $3n < m$ then 2 divides the left hand side $6n$ times so that $6n = 2m + 1 > 2m$, a contradiction. If $3n > m$, then 2 divides the left hand side $3n + m$ times so that $3n + m = 2m + 1$ and $3n = m + 1$. Or finally, we could have $3n = m$.

We wish to show that $(x, y) = (ab, 2b^3)$, (ab, b^3), or $(ab, \frac{b^3}{2})$. If 2 divides y only once, then from before (since $3n = m$ when $p > 2, m > 0$) we have $y = 2b^3$ and $x = ab$ for some a, b. And if 2 divides y more than once, then (since $3n = m$ when $p > 2, m > 0$ and since $3n = m$ or $m + 1$ when $p = 2, m > 1$) we either have $(x, y) = (ab, b^3)$ or $(x, y) = (ab, \frac{b^3}{2})$.

Now simply plug these possibilities into the equation. We then either have $a^6 + a^3 = b^3 + 2$, $a^6 + 2a^3 = 8b^3 + 8$, or $8a^6 + 4a^3 = b^3 + 4$.

In the first case, if $a > 1$ then $b^3 = a^6 + a^3 - 2$ and some algebra verifies that $(a^2 + 1)^3 > b^3 > (a^2)^3$, a contradiction; if $a < 0$ then we have $(a^2)^3 > b^3 > (a^2 - 1)^3$. Thus either $a = 0$ and $x = 0$ or $a = 1$ and $b = 0$. But we've assumed $x, y \neq 0$, so this case yields no solutions.

In the second case, if $a > 0$ then $(a^2 + 1)^3 > (2b)^3 > (a^2)^3$. If $a < -2$ then $(a^2)^3 > (2b)^3 > (a^2 - 1)^3$. Thus either $a = -2, -1$, or 0; and these yield no solutions either.

Finally, in the third case when $a > 1$ then $(2a^2+1)^3 > b^3 > (2a^2)^3$. When $a < -1$ then $(2a^2)^3 > b^3 > (2a^2-1)^3$. Thus either $a = -1, 0$, or 1; this yields both $(a,b) = (-1,0)$ and $(a,b) = (1,2)$. Only the latter gives a solution where $x, y \neq 0$ — namely, $(x,y) = (2,4)$. This completes the proof.

Problem 11.7 Let O be the center of circle ω. Two equal chords AB and CD of ω intersect at L such that $AL > LB$ and $DL > LC$. Let M and N be points on AL and DL respectively such that $\angle ALC = 2\angle MON$. Prove that the chord of ω passing through M and N is equal to AB and CD.

Solution: We work backward. Suppose that P is on minor arc $\widehat{AC}$ and Q is on minor arc $\widehat{BD}$ such that $PQ = AB = CD$, where line PQ hits $\overline{AL}$ at M' and $\overline{DL}$ at N'. We prove that $\angle ALC = 2\angle M'ON'$.

Say that the midpoints of $\overline{AB}$, $\overline{PQ}$, $\overline{CD}$ are T_1, T_2, and T_3. $\overline{CD}$ is the image of $\overline{AB}$ under the rotation about O through angle $\angle T_1OT_3$; this angle also equals the measure of $\widehat{AC}$, which equals $\angle ALC$. Also, by symmetry we have $\angle T_1OM' = \angle M'OT_2$ and $\angle T_2ON' = \angle N'OT_3$. Therefore

$$\begin{aligned}\angle ALC &= \angle T_1OT_3 = \angle T_1OT_2 + \angle T_2OT_3\\ &= 2(\angle M'OT_2 + \angle T_2ON') = 2\angle M'ON',\end{aligned}$$

as claimed.

Now back to the original problem. Since $\angle T_1OT_3 = \angle ALC$, $\angle T_1OL = \frac{1}{2}T_1OT_3 = \frac{1}{2}\angle ALC$. Then since $\angle MON = \frac{1}{2}\angle ALC = \angle T_1OL$, M must lie on $\overline{T_1L}$. Then look at the rotation about O that sends T_1 to M; it sends A to some P on $\widehat{AC}$, and B to some point Q on $\widehat{BD}$. Then $\overline{PQ}$ is a chord with length AB, passing through M on $\overline{AL}$ and N' on $\overline{DL}$. From the previous work, we know that $\angle ALC = 2\angle MON'$; and since $\angle ALC = 2\angle MON$, we must have $N = N'$. Thus the length of the chord passing through M and N indeed equals AB and CD, as desired.

IMO Selection Tests

Problem 1 Find all functions $h : \mathbb{Z} \to \mathbb{Z}$ such that

$$h(x+y) + h(xy) = h(x)h(y) + 1$$

for all x, $y \in \mathbb{Z}$.

Solution: There are three possible functions:

$$h(n) = 1;$$
$$h(2n) = 1, \quad h(2n+1) = 0;$$
$$h(n) = n + 1.$$

Plugging $(x, y) = (0, 0)$ into the functional equation, we find that

$$h(0)^2 - 2h(0) + 1 = 0$$

and hence $h(0) = 1$. Plugging in $(x, y) = (1, -1)$ then yields

$$h(0) + h(-1) = h(1)h(-1) + 1$$

and

$$h(-1) = h(1)h(-1),$$

and thus either $h(-1) = 0$ or $h(1) = 1$.

First suppose that $h(1) \neq 1$; then $h(-1) = 0$. Then plugging in $(x, y) = (2, -1)$ and $(x, y) = (-2, 1)$ yields $h(1) + h(-2) = 1$ and $h(-2) = h(-2)h(1) + 1$. Substituting $h(-2) = 1 - h(1)$ into the second equation, we find that

$$1 - h(1) = (1 - h(1))h(1) + 1,$$
$$h(1)^2 - 2h(1) = 0, \quad \text{and} \quad h(1)(h(1) - 2) = 0,$$

implying that $h(1) = 0$ or $h(1) = 2$.

Thus, $h(1) = 0$, 1, or 2. Plugging $y = 1$ into the equation for each of these cases shows that h must be one of the three functions presented.

Problem 2 Let a, b, $c \in \mathbb{Q}$, $ac \neq 0$. Given that the equation $ax^2 + bxy + cy^2 = 0$ has a non-zero solution of the form

$$(x, y) = (a_0 + a_1 \sqrt[3]{2} + a_2 \sqrt[3]{4}, b_0 + b_1 \sqrt[3]{2} + b_2 \sqrt[3]{4})$$

with a_i, $b_i \in \mathbb{Q}$, $i = 0$, 1, 2, prove that it has also has a non-zero rational solution.

Solution: Let $(\alpha, \beta) = (a_0 + a_1 \sqrt[3]{2} + a_2 \sqrt[3]{4}, b_0 + b_1 \sqrt[3]{2} + b_2 \sqrt[3]{4})$ be the given solution, and suppose without loss of generality that β is non-zero. Then $\frac{\alpha}{\beta}$ is a root to the polynomial

$$at^2 + bt + c = 0.$$

Also, $\frac{\alpha}{\beta}$ is of the form $c_0 + c_1\sqrt[3]{2} + c_2\sqrt[3]{4}$ for some rationals c_0, c_1, c_2. But because it is a root to a quadratic with rational coefficients, it must also be of the form $d + e\sqrt{f}$ for rationals d, e, f.

Thus we have $(c_0 - d) + c_1\sqrt[3]{2} + c_2\sqrt[3]{4} = e\sqrt{f}$, so the quantity $\left(c_0' + c_1\sqrt[3]{2} + c_2\sqrt[3]{4}\right)^2$ must be an integer (where we write $c_0' = c_0 - d$). After expanding this square, the coefficients of $\sqrt[3]{2}$ and $\sqrt[3]{4}$ are $2(c_2^2 + c_0'c_1)$ and $2c_0'c_2 + c_1^2$, respectively; these quantities must equal zero. From $2c_0'c_2 + c_1^2 = 0$ we have $(c_0'c_1)^2 = -2c_0'^3c_2$; and from $c_2^2 + c_0'c_1 = 0$ we have $(c_0'c_1)^2 = c_2^4$. Thus $-2c_0'^3c_2 = c_2^4$. This implies that either $c_2 = 0$ or $c_2 = -\sqrt[3]{2}c_0'$; in the latter case, since c_2 is rational we must still have $c_2 = c_0' = 0$.

Then $c_1 = 0$ as well, and $\frac{\alpha}{\beta} = c_0$ is *rational.* Thus $(x, y) = (\frac{\alpha}{\beta}, 1)$ is a non-zero rational solution to the given equation.

Problem 3 Suppose a and b are positive integers such that the product of all divisors of a (including 1 and a) is equal to the product of all divisors of b (including 1 and b). Does it follow that $a = b$?

Solution: Yes, it follows that $a = b$. Let $d(n)$ denote the number of divisors of n; then the product of all divisors of n is

$$\prod_{k|n} k = \sqrt{\prod_{k|n} k \cdot \prod_{k|n} \frac{n}{k}} = \sqrt{\prod_{k|n} n} = n^{\frac{d(n)}{2}}.$$

Thus the given condition implies that $a^{d(a)}$ and $b^{d(b)}$ equal the same number N. Since N is both a perfect $d(a)$-th power and a perfect $d(b)$-th power, it follows that it is also a perfect ℓ-th power of some number t, where $\ell = \operatorname{lcm}(d(a), d(b))$. Then $a = t^{\frac{\ell}{d(a)}}$ and $b = t^{\frac{\ell}{d(b)}}$ are both powers of the same number t as well.

Now if a is a bigger power of t than b, then it must have more divisors than b; but then $t^{\frac{\ell}{d(a)}} < t^{\frac{\ell}{d(b)}}$, a contradiction. Similarly a cannot be a smaller power of t than b. Therefore $a = b$, as claimed.

Problem 4 Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{1}{1+ab} + \frac{1}{1+bc} + \frac{1}{1+ca} \geq \frac{3}{2}.$$

Solution: Using the AM-HM inequality or the Cauchy-Schwarz inequality, we have

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq \frac{9}{x+y+z}$$

for $x,\ y,\ z \geq 0$. Also, notice that $a^2+b^2+c^2 \geq ab+bc+ca$ since this inequality is equivalent to $\frac{1}{2}(a-b)^2+\frac{1}{2}(b-c)^2+\frac{1}{2}(c-a)^2 \geq 0$. Thus,

$$\begin{aligned}\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ca} &\geq \frac{9}{3+ab+bc+ca}\\ &\geq \frac{9}{3+a^2+b^2+c^2} \geq \frac{3}{2},\end{aligned}$$

as desired.

Problem 5 Suppose triangle T_1 is similar to triangle T_2, and the lengths of two sides and the angle between them of T_1 are proportional to the lengths of two sides and the angle between them of T_2 (but not necessarily the corresponding ones). Must T_1 be congruent to T_2?

Solution: The triangles are not necessarily congruent. Say the vertices of T_1 are A, B, C with $AB = 4$, $BC = 6$, and $CA = 9$, and say that $\angle BCA = k\angle ABC$.

Then let the vertices of T_2 be D, E, F where $DE = \frac{8k}{3}$, $EF = 4k$, and $FD = 6k$. Triangles ABC and DEF are similar in that order, so $\angle EFD = \angle BCA = k\angle ABC$; also, $EF = k \cdot AB$ and $FD = k \cdot BC$. Therefore these triangles satisfy the given conditions.

Now since $AB < AC$ we have $\angle BCA < \angle ABC$ and $k < 1$; so $DE = \frac{8k}{3} < \frac{8}{3} < AB$. Thus triangles ABC and DEF are not congruent, as desired.

Problem 6 Two real sequences $x_1,\ x_2,\ \ldots,$ and $y_1,\ y_2,\ \ldots,$ are defined in the following way:

$$x_1 = y_1 = \sqrt{3}, \quad x_{n+1} = x_n + \sqrt{1+x_n^2}, \quad y_{n+1} = \frac{y_n}{1+\sqrt{1+y_n^2}}$$

for all $n \geq 1$. Prove that $2 < x_n y_n < 3$ for all $n > 1$.

First Solution: Let $z_n = \frac{1}{y_n}$ and notice that the recursion for y_n is equivalent to

$$z_{n+1} = z_n + \sqrt{1+z_n^2}.$$

Also note that $z_2 = \sqrt{3} = x_1$; since the x_i and z_i satisfy the same recursion, this means that $z_n = x_{n-1}$ for all $n > 1$. Thus,

$$x_n y_n = \frac{x_n}{z_n} = \frac{x_n}{x_{n-1}}.$$

Because the x_i are increasing, for $n > 1$ we have $x_{n-1}^2 \geq x_1^2 = 3 > \frac{1}{3} \Rightarrow 2x_{n-1} > \sqrt{1 + x_{n-1}^2} \Rightarrow 3x_{n-1} > x_n$. Also, $\sqrt{1 + x_{n-1}^2} > x_{n-1} \Rightarrow x_n > 2x_{n-1}$. Therefore,

$$2 < x_n y_n = \frac{x_n}{x_{n-1}} < 3,$$

as desired.

Second Solution: Writing $x_n = \tan a_n$ for $0^\circ < a_n < 90^\circ$, we have

$$\begin{aligned} x_{n+1} &= \tan a_n + \sqrt{1 + \tan^2 a_n} = \tan a_n + \sec a_n \\ &= \frac{1 + \sin a_n}{\cos a_n} = \tan\left(\frac{90^\circ + a_n}{2}\right). \end{aligned}$$

Since $a_1 = 60^\circ$, we have $a_2 = 75^\circ$, $a_3 = 82.5^\circ$, and in general $a_n = 90^\circ - \frac{30^\circ}{2^{n-1}}$. Thus

$$x_n = \tan\left(90^\circ - \frac{30^\circ}{2^{n-1}}\right) = \cot\left(\frac{30^\circ}{2^{n-1}}\right) = \cot\theta_n,$$

where $\theta_n = \frac{30^\circ}{2^{n-1}}$.

Similar calculation shows that

$$y_n = \tan 2\theta_n = \frac{2\tan\theta_n}{1 - \tan^2\theta_n},$$

implying that

$$x_n y_n = \frac{2}{1 - \tan^2\theta_n}.$$

Since $0^\circ < \theta_n < 45^\circ$, we have $0 < \tan^2\theta_n < 1$ and $x_n y_n > 2$. And since for $n > 1$ we have $\theta_n < 30^\circ$, we also have $\tan^2\theta_n < \frac{1}{3}$ so that $x_n y_n < 3$.

Note: From the closed forms for x_n and y_n in the second solution, we can see the relationship $y_n = \frac{1}{x_{n-1}}$ used in the first solution.

Problem 7 Let O be the center of the excircle of triangle ABC opposite A. Let M be the midpoint of $\overline{AC}$, and let P be the intersection of lines MO and BC. Prove that if $\angle BAC = 2\angle ACB$, then $AB = BP$.

First Solution: Since O is the excenter opposite A, we know that O is equidistant from lines AB, BC, and CA. We also know that line AO bisects angle BAC. Thus $\angle BAO = \angle OAC = \angle ACB$. Letting D be the intersection of $\overline{AO}$ and $\overline{BC}$, we then have $\angle DAC = \angle ACD$ and hence $DC = AD$.

Consider triangles OAC and ODC. From above their altitudes from O are equal, and their altitudes from C are also clearly equal. Thus, $OA/OD = [OAC]/[ODC] = AC/DC$.

Next, because M is the midpoint of $\overline{AC}$ we have $[OAM] = [OMC]$ and $[PAM] = [PMC]$, and hence $[OAP] = [OPC]$ as well. Then

$$\frac{OA}{OD} = \frac{[OAP]}{[ODP]} = \frac{[OPC]}{[ODP]} = \frac{PC}{DP}.$$

Thus, $\frac{AC}{DC} = \frac{OA}{OD} = \frac{PC}{DP}$, and $\frac{AC}{CP} = \frac{DC}{DP} = \frac{AD}{DP}$. By the Angle Bisector Theorem, $\overline{AP}$ bisects $\angle CAD$.

It follows that $\angle BAP = \angle BAD + \angle DAP = \angle ACP + \angle PAC = \angle APB$, and therefore $BA = BP$, as desired.

Second Solution: Let R be the midpoint of the arc BC (not containing A) of the circumcircle of triangle ABC; and let I be the incenter of triangle ABC. We have $\angle RBI = \frac{1}{2}(\angle CAB + \angle ABC) = \frac{1}{2}(180^\circ - \angle BRI)$. Thus $RB = RI$ and similarly $RC = RI$, and hence R is the circumcenter of triangle BIC. But since $\angle IBO = 90^\circ = \angle ICO$, quadrilateral $IBOC$ is cyclic and R is also the circumcenter of triangle BCO.

Let lines AO and BC intersect at Q. Since M, O, and P are collinear we may apply Menelaus' Theorem to triangle AQC to get

$$\frac{AM}{CM}\frac{CP}{QP}\frac{QO}{AO} = 1.$$

But $\frac{AM}{CM} = 1$, and therefore $\frac{CP}{PQ} = \frac{AO}{QO}$.

And since R lies on $\overline{AO}$ and $\overline{QO}$, we have

$$\frac{AO}{QO} = \frac{AR+RO}{QR+RO} = \frac{AR+RC}{CR+RQ},$$

which in turn equals $\frac{AC}{CQ}$ since triangles ARC and CRQ are similar; and $\frac{AC}{CQ} = \frac{AC}{AQ}$ since we are given that $\angle BAC = 2\angle ACB$; i.e.,

$\angle QAC = \angle QCA$ and $CQ = AQ$. Thus we have shown that $\frac{CP}{PQ} = \frac{AC}{AQ}$. By the Angle-Bisector Theorem, this implies that line AP bisects $\angle QAC$, from which it follows that $\angle BAP = \frac{3}{2}\angle ACB = \angle BPA$ and $AB = BP$.

Problem 8 Let O, O_1 be the centers of the incircle and the excircle opposite A of triangle ABC. The perpendicular bisector of $\overline{OO_1}$ meets lines AB and AC at L and N respectively. Given that the circumcircle of triangle ABC touches line LN, prove that triangle ABC is isosceles.

Solution: Let M be the midpoint of arc $\widehat{BC}$ not containing A. Angle-chasing gives $\angle OBM = \frac{1}{2}(\angle A + \angle B) = \angle BOM$ and hence $MB = MO$.

Since $\angle OBC = \frac{\angle B}{2}$ and $\angle CBO_1 = \frac{1}{2}(\pi - \angle B)$, we have $\angle OBO_1$ is a right angle. And since we know both that M lies on line AOO_1 (the angle bisector of $\angle A$) and that $MB = MO$, it follows that $\overline{BM}$ is a median to the hypotenuse of right triangle OBO_1 and thus M is the midpoint of $\overline{OO_1}$.

Therefore, the tangent to the circumcircle of ABC at M must be perpendicular to line AM. But this tangent is also parallel to line BC, implying that AM, the angle bisector of $\angle A$, is perpendicular to line BC. This can only happen if $AB = AC$, as desired.

Problem 9 Does there exist a bijection f of

(a) a plane with itself

(b) three-dimensional space with itself

such that for any distinct points A, B line AB and line $f(A)f(B)$ are perpendicular?

Solution:

(a) Yes: simply rotate the plane 90° about some axis perpendicular to it. For example, in the xy-plane we could map each point (x, y) to the point $(y, -x)$.

(b) Suppose such a bijection existed. Label the three-dimensional space with x-, y-, and z-axes; given any point $P = (x_0, y_0, z_0)$, we also view it as the vector p from $(0, 0, 0)$ to (x_0, y_0, z_0). Then the given condition says that

$$(a-b)\cdot(f(a)-f(b))=0$$

for any vectors a, b.

Assume without loss of generality that f maps the origin to itself; otherwise, $g(p)=f(p)-f(0)$ is still a bijection and still satisfies the above equation. Plugging $b=(0,0,0)$ into the equation above we have $a\cdot f(a)=0$ for all a. Then the above equation reduces to

$$a\cdot f(b)+b\cdot f(a)=0.$$

Given any vectors a, b, c and any reals m, n we then have

$$m\,(a\cdot f(b)+b\cdot f(a))=0$$
$$n\,(a\cdot f(c)+c\cdot f(a))=0$$
$$a\cdot f(mb+nc)+(mb+nc)\cdot f(a)=0.$$

Adding the first two equations and subtracting the third gives

$$a\cdot(mf(b)+nf(c)-f(mb+nc))=0.$$

Since this must be true for any vector a, we must have $f(mb+nc)=mf(B)+nf(C)$. Therefore f is linear, and it is determined by how it transforms the unit vectors $\mathbf{i}=(1,0,0)$, $\mathbf{j}=(0,1,0)$, and $\mathbf{k}=(0,0,1)$. If $f(\mathbf{i})=(a_1,a_2,a_3)$, $f(\mathbf{j})=(b_1,b_2,b_3)$, and $f(\mathbf{k})=(c_1,c_2,c_3)$, then for a vector x we have

$$f(x)=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_3 \\ a_3 & b_3 & c_3 \end{bmatrix} x.$$

Applying $f(a)\cdot a=0$ with $a=\mathbf{i},\mathbf{j},\mathbf{k}$, we have $a_1=b_2=c_3=0$. Then applying $a\cdot f(b)+b\cdot f(a)$ with $(a,b)=(\mathbf{i},\mathbf{j}), (\mathbf{j},\mathbf{k}), (\mathbf{j},\mathbf{k})$ we have $b_1=-a_2$, $c_1=-a_3$, $c_2=-b_3$. But then the determinant of the array in the equation is

$$a_2b_3c_1+a_3b_1c_2=-a_2b_3a_3+a_3a_2b_3=0,$$

so there exist constants k_1, k_2, k_3 not all zero such that $k_1f(\mathbf{i})+k_2f(\mathbf{j})+k_3f(\mathbf{k})=0$. But then $f(k_1,k_2,k_3)=0=f(0,0,0)$, contradicting the assumption that f was a bijection!

Therefore our original assumption was false, and no such bijection exists.

Problem 10 A word is a finite sequence of two symbols a and b. The number of the symbols in the word is said to be the length of the word. A word is called 6-*aperiodic* if it does not contain a subword of the form *cccccc* for any word c. Prove that $f(n) > \left(\frac{3}{2}\right)^n$, where $f(n)$ is the total number of 6-aperiodic words of length n.

Solution: Rather than attempting to count all such words, we add some restrictions and count only some of the 6-aperiodic words. Also, instead of working with a's and b's we'll work with 0's and 1's.

The Thue-Morse sequence is defined by $t_0 = 0$, $t_1 = 1$, $t_{2n+1} = 1 - t_{2n}$, and $t_{2n} = t_n$. These properties can be used to show that the only subwords of the form $cc \dots c$ are 00 and 11.

We restrict the 6-aperiodic words in a similar spirit. Call a word $x_1 x_2 \dots x_n$ of length n *6-countable* if it satisfies the following conditions:

(i) $x_{5i} = x_i$ for $1 \le i$.

(ii) $x_{5i-1} = 1 - x_{5i}$ for $1 \le i \le \frac{n}{5}$.

(iii) If $(x_{5i+2}, x_{5i+3}, x_{5i+4}) = (1, 0, 1)$ [or $(0, 1, 0)$], then $(x_{5i+7}, x_{5i+8}, x_{5i+9}) \ne (0, 1, 0)$ [or $(1, 0, 1)$].

Lemma 1. *Every 6-countable word is 6-aperiodic.*

Proof: Suppose by way of contradiction that some 6-countable word contains a subword of the form *cccccc*, where the strings c appear in the positions x_j through $x_{j+\ell-1}$; $x_{j+\ell}$ through $x_{j+2\ell-1}$; and so on up to $x_{j+5\ell}$ through $x_{j+6\ell-1}$. Pick a word with the smallest such ℓ.

If $5 \mid \ell$, then look at the indices i between j and $j + \ell - 1$ such that $5 \mid i$; say they are $5i_1, 5i_2, \dots, 5i_{\ell/5}$. Then $x_{5i_1} x_{5i_2} \dots x_{5i_{\ell/5}}$, $x_{5i_1+\ell} x_{5i_2+\ell} \dots x_{5i_{\ell/5}+\ell}$, $\dots$, $x_{5i_1+5\ell} x_{5i_2+5\ell} \dots x_{5i_{\ell/5}+5\ell}$ all equal the same string c'; then (using the first condition of countability) the subword starting at x_{i_1} and ending on $x_{i_{\ell/5}+\ell}$ is of the form $c'c'c'c'c'c'$. But this contradicts the minimal choice of ℓ; therefore, we *can't* have $5 \mid \ell$.

Now, suppose that in the first appearance of c some two adjacent characters a_j, a_{j+1} were equal. Then since $5 \nmid \ell$, one of $j, j+\ell, j+2\ell, \dots, j+4\ell$ is $4 \pmod 5$ — say, $j + k\ell$. Then $a_{j+k\ell}$, $a_{j+k\ell+1}$ must be the same since $a_j a_{j+1} = a_{j+k\ell} a_{j+k\ell+1}$; but they must also be different from the second condition of 6-countability! Because this is impossible, it follows that the characters in c alternate between 0 and 1.

A similar argument, though, shows that $a_{j+\ell-1}$ and $a_{j+\ell}$ must be different; hence c is of the form $1010\ldots10$ or $0101\ldots01$. But this would imply that our word violated the third condition of 6-countability—a contradiction. Therefore our original assumption was false, and any 6-countable word is 6-aperiodic. ■

Lemma 2. *Given a positive integer m, there are more than $\left(\frac{3}{2}\right)^{5m}$ 6-countable words of length $5m$.*

Proof: Let α_m be the number of length-$5m$ 6-countable words. To create a length-$5m$ 6-countable word $x_1x_2\ldots x_{5m}$, we can choose each of the "three-strings" $x_1x_2x_3$, $x_6x_7x_8$, …, $x_{5m-4}x_{5m-3}x_{5m-2}$ to be any of the eight strings 000, 001, 010, 011, 100, 101, 110, or 111—taking care that no two adjacent strings are 010 and 101. Some quick counting then shows that $\alpha_1 = 8 > \left(\frac{3}{2}\right)^5$ and $\alpha_2 = 64 - 2 = 62 > \left(\frac{3}{2}\right)^{10}$.

Let β_m be the number of length-$5m$ 6-countable words whose last three-string is 101; by symmetry, this also equals the number of length-$5m$ 6-countable words whose last three-string is 010. Also let γ_m be the number of length-$5m$ 6-countable words whose last three-string is *not* 101; again by symmetry, this also equals the number of length-$5m$ 6-countable words whose last three-string isn't 010. Note that $\alpha_m = \gamma_m + \beta_m$.

For $m \geq 1$, observe that $\gamma_m = \beta_{m+1}$ because to any length-$5m$ word whose last three-string isn't 010, we can append the three-string 101 (as well as two other pre-determined numbers); and given a length-$5(m+1)$ word whose last three- string is 101, we can reverse this construction. Similar arguing shows that $\gamma_{m+1} = 6(\gamma_m + \beta_m) + \gamma_m$; the $6(\gamma_m + \beta_m)$ term counts the words whose last three-string is neither 010 nor 101, and the γ_m term counts the words whose last three-string is 010. Combined, these recursions give

$$\gamma_{m+2} = 7\gamma_{m+1} + 6\gamma_m$$

$$\beta_{m+2} = 7\beta_{m+1} + 6\beta_m$$

$$\alpha_{m+2} = 7\alpha_{m+1} + 6\alpha_m.$$

Now if $\alpha_{m+1} > \left(\frac{3}{2}\right)^{5m+5}$ and $\alpha_m > \left(\frac{3}{2}\right)^{5m}$, then

$$\begin{aligned}\alpha_{m+2} &= 7\alpha_{m+1} + 6\alpha_m \\ &> \left(\frac{3}{2}\right)^{5m}\left(7\cdot\left(\frac{3}{2}\right)^{5} + 6\right) \\ &> \left(\frac{3}{2}\right)^{5m}\left(\frac{3}{2}\right)^{10} = \left(\frac{3}{2}\right)^{5(m+2)}.\end{aligned}$$

Then since $\alpha_m > \left(\frac{3}{2}\right)^{5m}$ is true for $m = 1, 2$, by induction it is true for all positive integers m. ■

The lemma proves the claim for $n = 5m$. Now suppose we are looking at length-$(5m + i)$ words, where $m \geq 0$ and $i = 1, 2, 3,$ or 4. Then given any length-$5m$ 6-countable word, we can form a length-$(5m+i)$ word by choosing $x_{5m+1}, x_{5m+2}, x_{5m+3}$ to be anything. (For convenience, we say there is exactly $\alpha_0 = 1 \geq \left(\frac{3}{2}\right)^0$ length-0 6-countable word: the "empty word.") Thus there are at least $2\alpha_m > \left(\frac{3}{2}\right)^{5m+1}$, $4\alpha_m > \left(\frac{3}{2}\right)^{5m+2}$, $8\alpha_m > \left(\frac{3}{2}\right)^{5m+3}$, and $8\alpha_m > \left(\frac{3}{2}\right)^{5m+4}$ 6-countable length-$(5m+1)$, -$(5m+2)$, -$(5m+3)$, and -$(5m+4)$ words, respectively. This completes the proof.

Problem 11 Determine all positive integers n, $n \geq 2$, such that $\binom{n-k}{k}$ is even for $k = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$.

Solution: Lucas's Theorem states that for integers

$$n = n_r p^r + n_{r-1}p^{r-1} + \cdots + n_0$$

and

$$m = m_r p^r + m_{r-1}p^{r-1} + \cdots + m_0$$

written in base p for a prime p, we have

$$\binom{n}{m} \equiv \binom{n_r}{m_r}\binom{n_{r-1}}{m_{r-1}}\cdots\binom{n_0}{b_0} \pmod{p}.$$

With $p = 2$, the binary representation of $n = 2^s - 1$ we have $n_r = n_{r-1} = \cdots = n_0 = 1$. Then for any $0 \leq m \leq 2^s - 1$ each $\binom{n_i}{m_i} = 1$, and thus $\binom{n}{m} \equiv 1 \cdot 1 \cdot \cdots \cdot 1 \equiv 1 \pmod{2}$.

This implies that n must be one less than a power of 2, or else one of $n - k$ will equal such a number $2^s - 1$ and then $\binom{n-k}{k}$ will be odd.

In fact, all such $n = 2^s - 1$ *do* work: for $k = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$, there is at least one 0 in the binary representation of $n - k$ (not counting leading zeros, of course). And whenever there is a 0 in the binary representation of $n - k$, there is a 1 in the corresponding digit of k. Then the corresponding $\binom{(n-k)_i}{k_i}$ equals 0, and by Lucas's Theorem $\binom{n-k}{k}$ is even.

Therefore, $n = 2^s - 1$ for integers $s \geq 2$.

Problem 12 A number of n players took part in a chess tournament. After the tournament was over, it turned out that among any four players there was one who scored differently against the other three (i.e., he got a victory, a draw, and a loss). Prove that the largest possible n satisfies the inequality $6 \leq n \leq 9$.

Solution:

Let $A_1 \Rightarrow A_2 \Rightarrow \cdots \Rightarrow A_n$ denote "A_1 beats A_2, A_2 beats A_3, …, A_{n-1} beats A_n," and let $X \mid Y$ denote "X draws with Y."

First we show it is possible to have the desired results with $n = 6$: call the players A, B, C, D, E, F. Then let

$$A \Rightarrow B \Rightarrow C \Rightarrow D \Rightarrow E \Rightarrow A,$$

$$F \Rightarrow A,\ F \Rightarrow B,\ F \Rightarrow C,\ F \Rightarrow D,\ F \Rightarrow E,$$

and have all other games end in draws. Visually, we can view this arrangement as a regular pentagon $ABCDE$ with F at the center. There are three different types of groups of 4, represented by $ABCD$, $ABCF$, and $ABDF$; in these three respective cases, B (or C), A, and A are the players who score differently from the other three.

Alternatively, let

$$A \Rightarrow B \Rightarrow C \Rightarrow D \Rightarrow E \Rightarrow F \Rightarrow A,$$

$$B \Rightarrow D \Rightarrow F \Rightarrow B, \quad C \Rightarrow A \Rightarrow E \Rightarrow C,$$

$$A \mid D,\ B \mid E,\ C \mid F.$$

In this arrangement there are three different types of groups of four, represented by $\{A, B, C, D\}$, $\{A, B, D, E\}$, and $\{A,\ B,\ D,\ F\}$. (If the players are arranged in a regular hexagon, these correspond to a trapezoid-shaped group, a rectangle-shaped group, and a diamond-shaped group.) In these three cases, A, B (or D), and A (or D) are the players who score differently against the other three.

Now we show it is impossible to have the desired results with $n = 10$ and thus all $n \geq 10$; suppose by way of contradiction it *was* possible. First we prove that all players draw exactly 4 times.

To do this, draw a graph with n vertices representing the players, and draw an edge between two vertices if they drew in their game. If V has degree 3 or less, then look at the remaining 6 or more vertices it is not adjacent to. By Ramsey's Theorem, either three of them (call them X, Y, Z) are all adjacent or all not adjacent. But then in the group $\{V, X, Y, Z\}$, none of the players draws exactly once with the other players, a contradiction.

Thus each vertex has degree at least 4; we now prove that every vertex has degree *exactly* 4. Suppose by way of contradiction that some vertex A was adjacent to at least 5 vertices B, C, D, E, F. None of these vertices can be adjacent to two others; for example, if B was adjacent to C and D then in $\{A, B, C, D\}$ each vertex draws at least twice—but some player must draw exactly once in this group. Now in the group $\{B, C, D, E\}$ some pair must draw: without loss of generality, say B and C. In the group $\{C, D, E, F\}$ some pair must draw as well; since C can't draw with D, E, or F from our previous observation, assume without loss of generality that E and F draw.

Now in $\{A, B, C, D\}$ vertex D must beat one of B, C and lose to the other; without loss of generality, say D loses to B and beats C. Looking at $\{A, D, E, F\}$, we can similarly assume that D beats E and loses to F. Next, in $\{A, C, D, E\}$ players C and E can't draw; without loss of generality, say C beats E. And then in $\{A, C, E, F\}$, player C must lose to F. But then in $\{C, D, E, F\}$ no player scores differently against the other three players—a contradiction.

Now suppose A were adjacent to B, C, D, E, and without loss of generality assume $B \mid C$; then ABC is a triangle. For each J besides A, B, C, look at the group $\{A, B, C, J\}$: J must draw with one of A, B, C. By the Pigeonhole Principle, one of A, B, C draws with at least three of the J and thus has degree at least 5. But this is impossible from above.

It follows that it is *impossible* for n to be at least 10. But since n can be 6, the maximum n is between 6 and 9, as desired.

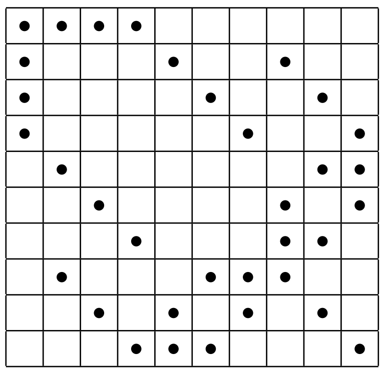
1.2 Brazil

Problem 1 Let $ABCDE$ be a regular pentagon such that the star region $ACEBD$ has area 1. Let AC and BE meet at P, and let BD and CE meet at Q. Determine $[APQD]$.

Solution: Let $R = AD \cap BE$, $S = AC \cap BD$, $T = CE \cap AD$. Now $\triangle PQR \sim \triangle CAD$ because they are corresponding triangles in regular pentagons $QTRPS$ and $ABCDE$, and since $\triangle CAD \sim \triangle PAR$ as well we have $\triangle PQR \cong \triangle PAR$. Thus, $[APQD] = \frac{[APQD]}{[ACEBD]} = \frac{2[APR]+[PQR]+[RQT]}{5[APR]+[PQR]+2[RQT]} = \frac{3[APR]+[RQT]}{6[APR]+2[RQT]} = \frac{1}{2}$.

Problem 2 Given a 10×10 board, we want to remove n of the 100 squares so that no 4 of the remaining squares form the corners of a rectangle with sides parallel to the sides of the board. Determine the minimum value of n.

Solution: The answer is 66. Consider the diagram below, in which a colored circle represents a square that has *not* been removed. The diagram demonstrates that n can be 66:



Now we proceed to show that n is at least 66. Suppose, for contradiction, that it is possible with $n = 65$. Denote by a_i the number of squares left in row i ($i = 1, 2, \ldots, 10$); in row i, there are $\binom{a_i}{2}$ pairs of remaining squares. If no four remaining squares form the corners of a rectangle, then the total number $N = \sum_{i=1}^{10} \binom{a_i}{2}$ must not exceed $\binom{10}{2} = 45$. But note that, with a fixed $\sum_{i=1}^{10} a_i = 35$, the minimum of $\sum_{i=1}^{10} \binom{a_i}{2}$ is attained when and only when no two a_i's differ by more than 1. Thus, $45 = \sum_{i=1}^{10} \binom{a_i}{2} \geq 5 \cdot \binom{4}{2} + 5 \cdot \binom{3}{2} = 45$, i.e., this minimum *is* attained here, implying that five of the a_i's equal 4 and the rest equal 3. Then it is easy to see that aside from permutations of the row and columns, the first five rows of the board must be as follows:

•	•	•	•						
•				•	•	•			
	•			•			•	•	
		•			•		•		•
			•			•		•	•

We inspect this figure and notice that it is now impossible for another row to contain at least 3 remaining squares without forming the vertices of a rectangle with sides parallel to the sides of the board. This is a contradiction, since each of the remaining 5 rows is supposed to have 3 remaining squares. Thus, it is impossible for n to be less than 66, and we are done.

Problem 3 The planet Zork is spherical and has several cities. Given any city A on Zork, there exists an antipodal city A' (i.e., symmetric with respect to the center of the sphere). In Zork, there are roads joining pairs of cities. If there is a road joining cities P and Q, then there is a road joining P' and Q'. Roads don't cross each other, and any given pair of cities is connected by some sequence of roads. Each city is assigned a value, and the difference between the values of every pair of connected cities is at most 100. Prove that there exist two antipodal cities with values differing by at most 100.

Solution: Let $[A]$ denote the value assigned to city A. Name the pairs of cities

$$(Z_1, Z_1'), (Z_2, Z_2'), (Z_3, Z_3'), \ldots, (Z_n, Z_n')$$

with

$$0 \le [Z_i] - [Z_i'] \text{ for all } i.$$

Since any given pair of cities is connected by some sequence of roads, there must exist a, b such that Z_a and Z_b' are connected by a single road. Then Z_a' and Z_b are also connected by a single road. Thus, $[Z_a] - [Z_b'] \le 100$ and $[Z_b] - [Z_a'] \le 100$. Adding, we have

$$[Z_a] - [Z_a'] + [Z_b] - [Z_b'] \le 200.$$

Hence, either $0 \le [Z_a] - [Z'_a] \le 100$ or $0 \le [Z_b] - [Z'_b] \le 100$; in either case, we are done.

Problem 4 In Tumbolia there are n soccer teams. We want to organize a championship such that each team plays exactly once with each other team. All games take place on Sundays, and a team can't play more than one game in the same day. Determine the smallest positive integer m for which it is possible to realize such a championship in m Sundays.

Solution: Let a_n be the smallest positive integer for which it is possible to realize a championship between n soccer teams in a_n Sundays. For $n > 1$, it is necessary that $a_n \ge 2\lceil \frac{n}{2} \rceil - 1$; otherwise the total number of games played would not exceed $(2\lceil \frac{n}{2} \rceil - 2) \cdot \lfloor \frac{n}{2} \rfloor \le \frac{(n-1)^2}{2} < \binom{n}{2}$, a contradiction.

On the other hand, $2\lceil \frac{n}{2} \rceil - 1$ days suffice. Suppose that $n = 2t + 1$ or $2t + 2$; number the teams from 1 to n and the Sundays from 1 to $2t + 1$. On the i-th Sunday, let team i either sit out (if n is odd) or play team $2t + 2$ (if n is even); and have any other team j play with the team $k \ne 2t + 2$ such that $j + k \equiv 2i \pmod{2t + 1}$. Then each team indeed plays every other team, as desired.

Problem 5 Given a triangle ABC, show how to construct, with straightedge and compass, a triangle $A'B'C'$ with minimal area such that A', B', C' lie on AB, BC, CA, respectively, $\angle B'A'C' = \angle BAC$, and $\angle A'C'B' = \angle ACB$.

Solution:

All angles are directed modulo $180°$.

For convenience, call any triangle $A'B'C'$ "*zart*" if A', B', C' lie on lines AB, BC, CA, respectively, and $\triangle ABC \sim \triangle A'B'C'$. The problem is, then, to construct the *zart* triangle with minimal area.

Suppose we have any *zart* triangle, and let P be the point (different from A') where the circumcircles of triangles $AA'C'$ and $BB'A'$ meet. Then

$$\angle B'PC' = 360° - \angle A'PB' - \angle C'PA'$$

$$= 360° - (180° - \angle CBA) - (180° - \angle BAC) = 180° - \angle ACB,$$

so P also lies on the circumcircle of triangle $CC'B'$.

Next,

$$\angle PAB = \angle PC'A' = \angle B'C'A' - \angle B'C'P$$
$$= \angle B'CC' - \angle B'CP' = \angle PCA,$$

and with similar reasoning we have

$$\angle PAB = \angle PC'A' = \angle PCA = \angle PB'C' = \angle PBC.$$

There is a unique point P (one of the Brocard points) satisfying $\angle PAB = \angle PBC = \angle PCA$, and thus P is fixed—*independent* of the choice of triangle $A'B'C'$. And since it is the corresponding point in similar triangles ABC and $A'B'C'$, we have

$$[A'B'C'] = [ABC]\left(\frac{PA'}{PA}\right)^2.$$

Thus $[A'B'C']$ is minimal when PA' is minimal, which occurs when $PA' \perp AB$ (and analogously, when $PB' \perp PC$ and $PC' \perp PA$). Thus, the *zart* triangle with minimal area is the pedal triangle $A'B'C'$ of P to triangle ABC. This triangle is indeed similar to triangle ABC; letting $\theta = \angle PAB$ be the Brocard angle, it is the image of triangle ABC under a rotation through $\theta - 90°$, followed by a homothety of ratio —sin θ|.

To construct this triangle, first draw the circles $\{X : \angle BXA = \angle BCA + \angle CAB\}$ and $\{Y : \angle CYB = \angle CAB + \angle ABC\}$ and let P' be their point of intersection (different from B); then we also have $\angle AP'C = \angle ABC + \angle BCA$. Then

$$\angle P'AB = 180° - \angle ABP' - \angle BP'A =$$
$$180° - (\angle ABC - \angle P'BC) - (\angle BCA + \angle CAB) = \angle P'BC,$$

and similarly $\angle P'BC = \angle P'CA$. Therefore $P = P'$. Finally, drop the perpendiculars from P' to the sides of triangle ABC to form A', B', C'. This completes the construction.

1.3 Bulgaria

National Olympiad, Third Round

Problem 1 Find all triples (x, y, z) of natural numbers such that y is a prime number, y and 3 do not divide z, and $x^3 - y^3 = z^2$.

Solution: Rewrite the equation in the form

$$(x-y)(x^2+xy+y^2) = z^2.$$

Any common divisor of $x-y$ and x^2+xy+y^2 also divides both z^2 and $(x^2+xy+y^2)-(x+2y)(x-y) = 3y^2$. But z^2 and $3y^2$ are relatively prime by assumption, hence $(x-y)$ and (x^2+xy+y^2) must be relatively prime as well. Therefore, both $(x-y)$ and (x^2+xy+y^2) are perfect squares.

Now writing $a = \sqrt{x-y}$, we have

$$x^2+xy+y^2 = (a^2+y)^2 + (a^2+y)y + y^2 = a^4 + 3a^2y + 3y^2$$

and

$$4(x^2+xy+y^2) = (2a^2+3y)^2 + 3y^2.$$

Writing $m = 2\sqrt{x^2+xy+y^2}$ and $n = 2a^2+3y$, we have

$$m^2 = n^2 + 3y^2$$

or

$$(m-n)(m+n) = 3y^2,$$

so $(m-n, m+n) = (1, 3y^2)$, $(3, y^2)$, or $(y, 3y)$.

In the first case, $2n = 3y^2 - 1$ and $4a^2 = 2n - 6y = 3y^2 - 6y - 1$ is a square, which is impossible modulo 3.

In the third case, $n = y < 2a^2 + 3y = n$, a contradiction.

In the second case, we have $4a^2 = 2n - 6y = y^2 - 6y - 3 < (y-3)^2$. And when $y \geq 10$ we have $y^2 - 6y - 3 > (y-4)^2$, hence $y = 2, 3, 5$, or 7. In this case we have $a = \frac{\sqrt{y^2-6y-3}}{2}$, which is real only when $y = 7$, $a = 1$, $x = y + a^2 = 8$, and $z = 13$. This yields the unique solution $(x, y, z) = (8, 7, 13)$.

Problem 2 A convex quadrilateral $ABCD$ is inscribed in a circle whose center O is inside the quadrilateral. Let $MNPQ$ be the quadrilateral whose vertices are the projections of the intersection point of the diagonals AC and BD onto the sides of $ABCD$. Prove that $2[MNPQ] \le [ABCD]$.

Solution: The result actually holds even when $ABCD$ is not cyclic. We begin by proving the following result:

Lemma. *If $\overline{XW}$ is an altitude of triangle XYZ, then* $\frac{XW}{YZ} \le \frac{1}{2}\tan\left(\frac{\angle Y + \angle Z}{2}\right)$.

Proof: X lies on an arc of a circle determined by $\angle YXZ = 180^\circ - \angle Y - \angle Z$. Its distance from $\overline{YZ}$ is maximized when it is at the center of this arc, which occurs when $\angle Y = \angle Z$; and at this point, $\frac{XW}{YZ} = \frac{1}{2}\tan\left(\frac{\angle Y + \angle Z}{2}\right)$. ■

Suppose M, N, P, Q are on sides AB, BC, CD, DA, respectively. Also let T be the intersection of $\overline{AC}$ and $\overline{BD}$.

Let $\alpha = \angle ADB$, $\beta = \angle BAC$, $\gamma = \angle CAD$, $\delta = \angle DBA$. From the lemma, $MT \le \frac{1}{2}AB \cdot \tan\left(\frac{\beta+\delta}{2}\right)$ and $QT \le \frac{1}{2}AD \cdot \tan\left(\frac{\alpha+\gamma}{2}\right)$; also, $\angle MTQ = 180^\circ - \angle QAM = 180^\circ - \angle DAB$. Thus $2[MTQ] = MT \cdot QT \sin\angle MTQ \le \frac{1}{4}\tan\left(\frac{\alpha+\gamma}{2}\right)\tan\left(\frac{\beta+\delta}{2}\right) AB \cdot AD \sin\angle DAB$. But since $\frac{\alpha+\gamma}{2} + \frac{\beta+\delta}{2} = 90^\circ$, this last expression exactly equals $\frac{1}{4}AB \cdot AD \sin\angle DAB = \frac{1}{2}[ABD]$. Thus, $2[MTQ] \le \frac{1}{2}[ABD]$.

Likewise, $2[NTM] \le \frac{1}{2}[BCA]$, $[PTN] \le \frac{1}{2}[CDB]$, and $[QTP] \le \frac{1}{2}[DAC]$. Adding these four inequalities shows that $2[MNPQ]$ is at most

$$\frac{1}{2}([ABD] + [CDB]) + \frac{1}{2}([BCA] + [DAC]) = [ABCD],$$

as desired.

Problem 3 In a competition 8 judges marked the contestants by *pass* or *fail*. It is known that for any two contestants, two judges marked both with *pass*; two judges marked the first contestant with *pass* and the second contestant with *fail*; two judges marked the first contestant with *fail* and the second contestant with *pass*; and finally, two judges marked both with *fail*. What is the largest possible number of contestants?

Solution: For a rating r (either *pass* or *fail*), let $\overline{r}$ denote the opposite rating. Also, whenever a pair of judges agree on the rating for some contestant, call this an "agreement." We first prove that any two judges share at most three agreements; suppose by way of contradiction this were false.

Then assume without loss of generality that the judges (labeled with numbers) mark the first four contestants (labeled with letters) as follows in the left table:

	A	B	C	D
1	a	b	c	d
2	a	b	c	d
3	a	$\overline{b}$		
4	a	$\overline{b}$		
5	$\overline{a}$	$\overline{b}$		
6	$\overline{a}$	$\overline{b}$		
7	$\overline{a}$	b		
8	$\overline{a}$	b		

	A	B	C	D
1	a	b	c	d
2	a	b	c	d
3	a	$\overline{b}$	$\overline{c}$	$\overline{d}$
4	a	$\overline{b}$	$\overline{c}$	$\overline{d}$
5	$\overline{a}$	$\overline{b}$		
6	$\overline{a}$	$\overline{b}$		
7	$\overline{a}$	b		
8	$\overline{a}$	b		

	A	B	C	D
1	a	b	c	d
2	a	b	c	d
3	a	$\overline{b}$	$\overline{c}$	$\overline{d}$
4	a	$\overline{b}$	$\overline{c}$	$\overline{d}$
5	$\overline{a}$	$\overline{b}$	c	d
6	$\overline{a}$	$\overline{b}$	c	d
7	$\overline{a}$	b		
8	$\overline{a}$	b		

Applying the given condition to contestants A and C, judges 3 and 4 must both give C the rating $\overline{c}$; similarly, they must both give D the rating $\overline{d}$. Next, applying the condition to contestants B and C, judges 5 and 6 must both give C the rating c; similarly, they must both give D the rating d. But now the condition fails for contestants C and D, a contradiction.

Thus each pair of judges agrees on at most three ratings, as claimed; thus there are at most $3 \cdot \binom{8}{2} = 84$ agreements between all the judges. On the other hand, for each contestant exactly four judges mark him with *pass* and exactly four judges mark him with *fail*, hence there are $\binom{4}{2} + \binom{4}{2} = 12$ agreements per contestant. It follows that there are at most $\frac{84}{12} = 7$ contestants; and as the following table shows (with 1 representing *pass* and 0 representing *fail*), it is indeed possible to have exactly 7 contestants:

	A	B	C	D	E	F	G
1	1	1	1	1	1	1	1
2	1	1	1	0	0	0	0
3	1	0	0	1	1	0	0
4	1	0	0	0	0	1	1
5	0	1	0	1	0	0	1
6	0	1	0	0	1	1	0
7	0	0	1	1	0	1	0
8	0	0	1	0	1	0	1

Problem 4 Find all pairs (x, y) of integers such that

$$x^3 = y^3 + 2y^2 + 1.$$

Solution: When $y^2 + 3y > 0$, $(y+1)^3 > x^3 > y^3$. Thus we must have $y^2 + 3y \leq 0$, and $y = -3, -2, -1$, or 0 — yielding the solutions $(x, y) = (1, 0)$, $(1, -2)$, and $(-2, -3)$.

Problem 5 Let B_1 and C_1 be points on the sides AC and AB of triangle ABC. Lines BB_1 and CC_1 intersect at point D. Prove that a circle can be inscribed inside quadrilateral AB_1DC_1 if and only if the incircles of the triangles ABD and ACD are tangent to each other.

Solution: Say the incircle of triangle ABD is tangent to $\overline{AD}$ at T_1 and that the incircle of triangle ACD is tangent to $\overline{AD}$ at T_2; then $DT_1 = \frac{1}{2}(DA + DB - AB)$ and $DT_2 = \frac{1}{2}(DA + DC - AC)$.

First suppose a circle can be inscribed inside AB_1DC_1. Let it be tangent to sides AB_1, B_1D, DC_1, C_1A at points E, F, G, H, respectively. Using equal tangents, we have

$$\begin{aligned} AB - BD &= (AH + HB) - (BF - DF) \\ &= (AH + BF) - (BF - DF) = AH + DF \end{aligned}$$

and similarly $AC - CD = AE + DG$. But $AH + DF = AE + DG$ by equal tangents, implying that $AB - BD = AC - CD$ and thus $DA + DB - AB = DA + DC - AC$. Therefore $DT_1 = DT_2$, $T_1 = T_2$, and the two given incircles are tangent to each other.

Next suppose the two incircles are tangent to each other. Then $DA + DB - AB = DA + DC - AC$. Let ω be the incircle of ABB_1, and let D' be the point on $\overline{BB_1}$ (different from B_1) such that line CD' is tangent to ω. Suppose by way of contradiction that $D \neq D'$. From the result in the last paragraph, we know that the incircles of triangles ABD' and ACD' are tangent and hence $D'A + D'B - AB = D'A + D'C - AC$. Then since $DB - AB = DC - AC$ and $D'B - AB = D'C - AC$, we must have $DB - D'B = DC - D'C$ by subtraction. Thus $DD' = |DB - D'B| = |DC - D'C|$. But then the triangle inequality fails in triangle $DD'C$, a contradiction. This completes the proof.

Problem 6 Each interior point of an equilateral triangle of side 1 lies in one of six congruent circles of radius r. Prove that

$$r \geq \frac{\sqrt{3}}{10}.$$

Solution: From the condition, we also know that every point inside or *on* the triangle lies inside or *on* one of the six circles.

Define $R = \frac{1}{1+\sqrt{3}}$. Orient the triangle so that A is at the top, B is at the bottom-left, and C is at the bottom-right (so that $\overline{BC}$ is horizontal). Draw point W on $\overline{AB}$ such that $WA = R$; then draw point X directly below W such that $WX = R$. Then in triangle WXB, $WB = 1 - R = \sqrt{3}R$ and $\angle BWX = 30°$, implying that $XB = R$ as well. Similarly draw Y on $\overline{AC}$ such that $YA = R$, and Z directly below Y such that $YZ = ZC = R$.

In triangle AWY, $\angle A = 60°$ and $AW = AY = R$, implying that $WY = R$. This in turn implies that $XZ = R$ and that $WX = YZ = R\sqrt{2}$.

Now suppose by way of contradiction that we could cover the triangle with six congruent circles of radius $r < \frac{\sqrt{3}}{10}$. The points A, B, C, W, X, Y, Z each lie on or inside one of the circles. But any two of these points are at least $R > 2r$ apart, so they must lie on or inside *different* circles. Thus there are at least seven circles, a contradiction.

National Olympiad, Fourth Round

Problem 1 A rectangular parallelepiped has integer dimensions. All of its faces of are painted green. The parallelepiped is partitioned into unit cubes by planes parallel to its faces. Find all possible measurements of the parallelepiped if the number of cubes without a green face is one third of the total number of cubes.

Solution: Let the parallelepiped's dimensions be a, b, c; they must all be at least 3 or else every cube has a green face. Then the condition is equivalent to

$$3(a-2)(b-2)(c-2) = abc,$$

or

$$3 = \frac{a}{a-2} \cdot \frac{b}{b-2} \cdot \frac{c}{c-2}.$$

If all the dimensions are at least 7, then $\frac{a}{a-2} \cdot \frac{b}{b-2} \cdot \frac{c}{c-2} \leq \left(\frac{7}{5}\right)^3 = \frac{343}{125} < 3$, a contradiction. Thus one of the dimensions — say, a — equals 3, 4, 5, or 6. Assume without loss of generality that $b \leq c$.

When $a = 3$ we have $bc = (b-2)(c-2)$, which is impossible.

When $a = 4$, rearranging the equation yields $(b-6)(c-6) = 24$. Thus $(b, c) = (7, 30)$, $(8, 18)$, $(9, 14)$, or $(10, 12)$.

When $a = 5$, rearranging the equation yields $(2b-9)(2c-9) = 45$. Thus $(b, c) = (5, 27)$, $(6, 12)$, or $(7, 9)$.

And when $a = 6$, rearranging the equation yields $(b-4)(c-4) = 8$. Thus $(b, c) = (5, 12)$ or $(6, 8)$.

Therefore the parallelepiped may measure $4 \times 7 \times 30$, $4 \times 8 \times 18$, $4 \times 9 \times 14$, $4 \times 10 \times 12$, $5 \times 5 \times 27$, $5 \times 6 \times 12$, $5 \times 7 \times 9$, or $6 \times 6 \times 8$.

Problem 2 Let $\{a_n\}$ be a sequence of integers such that for $n \geq 1$

$$(n-1)a_{n+1} = (n+1)a_n - 2(n-1).$$

If 2000 divides a_{1999}, find the smallest $n \geq 2$ such that 2000 divides a_n.

Solution: First, we note that the sequence $a_n = 2n - 2$ works. Then writing $b_n = a_n - (2n-2)$ gives the recursion

$$(n-1)b_{n+1} = (n+1)b_n.$$

Some calculations show that $b_3 = 3b_2$, $b_4 = 6b_2$, $b_5 = 10b_2$ — and in general, that $b_n = \frac{n(n-1)}{2}b_2$ for $n \geq 2$. Thus when $n \geq 2$, the solution to the original equation is of the form

$$a_n = 2(n-1) + \frac{n(n-1)}{2}c$$

for some constant c; plugging in $n = 2$ shows that $c = a_2 - 2$ is an integer.

Now, since $2000 \mid a_{1999}$ we have $2(1999-1) + \frac{1999 \cdot 1998}{2} \cdot c \equiv 0 \Longrightarrow -4 + 1001c \equiv 0 \Longrightarrow c \equiv 4 \pmod{2000}$. Then $2000 \mid a_n$ exactly when

$$\begin{aligned} 2(n-1) + 2n(n-1) &\equiv 0 \pmod{2000} \\ \Longleftrightarrow \ (n-1)(n+1) &\equiv 0 \pmod{1000}. \end{aligned}$$

$(n-1)(n+1)$ is divisible by 8 exactly when n is odd; and it is divisible by 125 exactly when either $n-1$ or $n+1$ is divisible by 125. The smallest $n \geq 2$ satisfying these requirements is $n = 249$.

Problem 3 The vertices of a triangle have integer coordinates and one of its sides is of length $\sqrt{n}$, where n is a square-free natural number. Prove that the ratio of the circumradius to the inradius of the triangle is an irrational number.

Solution: Label the triangle ABC; let r, R, K be the inradius, circumradius, and area of the triangle; let $a = BC$, $b = CA$, $c = AB$ and write $a = p_1\sqrt{q_1}$, $b = p_2\sqrt{q_2}$, $c = p_3\sqrt{q_3}$ for positive integers p_i, q_i with q_i square-free. By Pick's Theorem ($K = I + \frac{1}{2}B - 1$), K is rational. Also, $R = \frac{abc}{4K}$ and $r = \frac{2K}{a+b+c}$. Thus $\frac{R}{r} = \frac{abc(a+b+c)}{8K^2}$ is rational if and only if $abc(a+b+c) = a^2bc + ab^2c + abc^2$ is rational. Let this quantity equal m, and assume by way of contradiction that m *is* rational.

We have $a^2bc = m_1\sqrt{q_2q_3}$, $ab^2c = m_2\sqrt{q_3q_1}$, and $abc^2 = m_3\sqrt{q_1q_2}$ for positive integers m_1, m_2, m_3. Then $m_1\sqrt{q_2q_3} + m_2\sqrt{q_3q_1} = m - m_3\sqrt{q_1q_2}$. Squaring both sides, we find that

$$m_1^2q_2q_3 + m_2^2q_3q_1 + 2m_1m_2q_3\sqrt{q_1q_2} = m^2 + m_3^2 - 2mm_3\sqrt{q_1q_2}.$$

If $\sqrt{q_1q_2}$ is not rational, then the coefficients of $\sqrt{q_1q_2}$ must be the same on both sides; but this is impossible since $2m_1m_2q_3$ is positive while $-2mm_3$ is not.

Hence $\sqrt{q_1q_2}$ is rational. Since q_1 and q_2 are square-free, this can only be true if $q_1 = q_2$. Similarly, $q_2 = q_3$.

Assume without loss of generality that $BC = \sqrt{n}$ so that $q_1 = q_2 = q_3 = n$ and $p_1 = 1$. Also assume that A is at $(0,0)$, B is at (w,x), and C is at (y,z). By the triangle inequality, we must have $p_2 = p_3$ and hence

$$w^2 + x^2 = y^2 + z^2 = p_2^2n$$
$$(w-y)^2 + (x-z)^2 = n.$$

Notice that

$$n = (w-y)^2 + (x-z)^2 \equiv w^2 + x^2 + y^2 + z^2 = 2p_2^2n \equiv 0 \pmod{2},$$

so n is even. Thus w and x have the same parity; and y and z have the same parity. Then w, x, y, z must *all* have the same parity since $w^2 + x^2 \equiv y^2 + z^2 \pmod{4}$. But then $n = (w-y)^2 + (x-z)^2 \equiv 0 \pmod{4}$, contradicting the assumption that n is square-free.

Therefore our original assumption was false; and the ratio of the circumradius to the inradius is indeed always irrational.

Note: Without the condition that n is square-free, the ratio *can* be rational. For example, the points $(i, 2j - i)$ form a grid of points $\sqrt{2}$ apart. In this grid, we can find a $3\sqrt{2}$-$4\sqrt{2}$-$5\sqrt{2}$ right triangle by choosing, say, the points $(0, 0)$, $(3, 3)$, and $(7, -1)$. Then $q_1 = q_2 = q_3$, and the ratio is indeed rational.

Problem 4 Find the number of all natural numbers n, $4 \leq n \leq 1023$, whose binary representations do not contain three consecutive equal digits.

Solution: A *binary string* is a finite string of digits, all either 0 or 1. Call such a string (perhaps starting with zeroes) *valid* if it does not contain three consecutive equal digits. Let a_n represent the number of valid n-digit strings; let s_n be the number of valid strings starting with two equal digits; and let d_n be the number of valid strings starting with two different digits. Observe that $a_n = s_n + d_n$ for all n.

An $(n+2)$-digit string starting with 00 is valid if and only if its last n digits form a valid string starting with 1; similarly, an $(n+2)$-digit string starting with 11 is valid if and only if its last n digits form a valid string starting with 0. Thus, $s_{n+2} = a_n = s_n + d_n$.

An $(n+2)$-digit string starting with 01 is valid if and only if its last n digits form a valid string starting with 00, 01, or 10; similarly, an $(n+2)$-digit string starting with 10 is valid if and only if its last n digits form a valid string starting with 11, 01, or 10. Thus, $d_{n+2} = s_n + 2d_n$.

Solving these recursions gives

$$s_{n+4} = 3s_{n+2} - s_n \quad \text{and} \quad d_{n+4} = 3d_{n+2} - d_n,$$

which when added together yield

$$a_{n+4} = 3a_{n+2} - a_n.$$

Thus we can calculate initial values of a_n and then use the recursion to find other values:

n	1	2	3	4	5	6	7	8	9	10
a_n	2	4	6	10	16	26	42	68	110	178

Now of the a_n valid n-digit strings, only half start with 1; thus only half are binary representations of positive numbers. Therefore exactly

$$\frac{1}{2}(a_1 + a_2 + \cdots + a_{10}) = 231$$

numbers between 1 and 1023 have the desired property; and ignoring 1, 2, and 3, we find that the answer is 228.

Problem 5 The vertices A, B and C of an acute-angled triangle ABC lie on the sides B_1C_1, C_1A_1 and A_1B_1 of triangle $A_1B_1C_1$ such that $\angle ABC = \angle A_1B_1C_1$, $\angle BCA = \angle B_1C_1A_1$, and $\angle CAB = \angle C_1A_1B_1$. Prove that the orthocenters of the triangle ABC and triangle $A_1B_1C_1$ are equidistant from the circumcenter of triangle ABC.

Solution: Let H and H_1 be the orthocenters of triangles ABC and $A_1B_1C_1$, respectively; and let O, O_A, O_B, O_C be the circumcenters of triangles ABC, A_1BC, AB_1C, and ABC_1, respectively.

First note that $\angle BA_1C = \angle C_1A_1B_1 = \angle CAB = 180^\circ - \angle CHB$, showing that BA_1CH is cyclic; moreover, $O_AA_1 = \frac{BC}{2\sin\angle BA_1C} = \frac{CB}{2\sin\angle CAB} = OA$ so circles ABC and BA_1CH have the same radius. Similarly, CB_1AH and AC_1BH are cyclic with circumradius OA. Then $\angle HBC_1 = 180^\circ - \angle C_1AH = \angle HAB_1 = 180^\circ - \angle B_1CH = \angle HCA_1$; thus angles $\angle HO_CC_1$, $\angle HO_AA_1$, $\angle HO_BB_1$ are equal as well.

Let $\angle(\vec{r}_1, \vec{r}_2)$ denote the angle between rays $\vec{r}_1$ and $\vec{r}_2$. Since $O_AC = O_AB = HB = HC$, quadrilateral BO_ACH is a rhombus and hence a parallelogram. Then

$$\begin{aligned}
\angle(\overrightarrow{OA}, \overrightarrow{HO_A}) &= \angle(\overrightarrow{OA}, \overrightarrow{OB}) + \angle(\overrightarrow{OB}, \overrightarrow{HO_A}) \\
&= 2\angle ACB + \angle(\overrightarrow{CO_A}, \overrightarrow{HO_A}) \\
&= 2\angle ACB + \angle CO_AH \\
&= 2\angle ACB + 2\angle CBH \\
&= 2\angle ACB + 2(90^\circ - \angle ACB) \\
&= 180^\circ.
\end{aligned}$$

Similarly, $\angle(\overrightarrow{OB}, \overrightarrow{HO_B}) = \angle(\overrightarrow{OC}, \overrightarrow{HO_C}) = 180^\circ$. Combining this result with $\angle HO_AA_1 = \angle HO_BB_1 = \angle HO_CC_1$ from above, we find that

$$\angle(\overrightarrow{OA}, \overrightarrow{O_A A_1}) = \angle(\overrightarrow{OB}, \overrightarrow{O_B B_1}) = \angle(\overrightarrow{OC}, \overrightarrow{O_C C_1}).$$

Let this common angle be θ.

We now use complex numbers with the origin at O, letting p denote the complex number representing point P. Since HBO_AC is a parallelogram we have $o_A = b + c$ and we can write $a_1 = b + c + xa$ where $x = \operatorname{cis}\theta$. We also have $b_1 = c + a + xb$ and $c_1 = a + b + xc$ for the same x. We can rewrite these relations as

$$a_1 = a + b + c + (x-1)a,$$

$$b_1 = a + b + c + (x-1)b,$$

$$c_1 = a + b + c + (x-1)c.$$

Thus the map sending z to $a + b + c + (x-1)z = h + (x-1)z$ is a spiral similarity taking triangle ABC into triangle $A'B'C'$. It follows that this map also takes H to H_1, so

$$h_1 = h + (x-1)h = xh$$

and $OH_1 = |h_1| = |x||h| = |h| = OH$, as desired.

Problem 6 Prove that the equation

$$x^3 + y^3 + z^3 + t^3 = 1999$$

has infinitely many integral solutions.

Solution: Observe that $(m-n)^3 + (m+n)^3 = 2m^3 + 6mn^2$. Now suppose we want a general solution of the form

$$(x, y, z, t) = (a - b, a + b, \frac{c}{2} - \frac{d}{2}, \frac{c}{2} + \frac{d}{2})$$

for integers a, b and odd integers c, d. One simple solution to the given equation is $(x, y, z, t) = (10, 10, -1, 0)$, so try setting $a = 10$ and $c = -1$. Then

$$(x, y, z, t) = (10 - b, 10 + b, -\frac{1}{2} - \frac{d}{2}, -\frac{1}{2} + \frac{d}{2})$$

is a solution exactly when

$$(2000 + 60b^2) - \frac{1 + 3d^2}{4} = 1999 \iff d^2 - 80b^2 = 1.$$

The second equation is a Pell's equation with solution $(d_1, b_1) = (9, 1)$; and we can generate infinitely many more solutions by setting $(d_{n+1}, b_{n+1}) = (9d_n + 80b_n, 9b_n + d_n)$ for $n = 1, 2, 3, \ldots$; this follows from a general recursion $(p_{n+1}, q_{n+1}) = (p_1 p_n + q_1 q_n D, p_1 q_n + q_1 p_n)$ for generating solutions to $p^2 - Dq^2 = 1$ given a nontrivial solution (p_1, q_1).

A quick check also shows that each d_n is odd. Thus since there are infinitely many solutions (b_n, d_n) to the Pell's equation (and with each d_n odd), there are infinitely many integral solutions

$$(x_n, y_n, z_n, t_n) = (10 - b_n, 10 + b_n, -\frac{1}{2} - \frac{d_n}{2}, -\frac{1}{2} + \frac{d_n}{2})$$

to the original equation.

1.4 Canada

Problem 1 Find all real solutions to the equation $4x^2-40\lfloor x\rfloor+51=0$, where $[x]$ denotes the greatest integer less than or equal to x.

Solution: Note that $(2x-3)(2x-17) = 4x^2-40x+51 \leq 4x^2-40\lfloor x\rfloor+51=0$, which gives $1.5 \leq x \leq 8.5$ and $1 \leq \lfloor x\rfloor \leq 8$. Then

$$x=\frac{\sqrt{40\lfloor x\rfloor-51}}{2},$$

so it is necessary that

$$\lfloor x\rfloor=\left\lfloor\frac{\sqrt{40\lfloor x\rfloor-51}}{2}\right\rfloor.$$

Testing $\lfloor x\rfloor \in \{1,2,3,\ldots,8\}$ into this equation, we find that only $\lfloor x\rfloor = 2,6,7,$ and 8 work. Thus the only solutions for x are $\frac{\sqrt{29}}{2}, \frac{\sqrt{189}}{2}, \frac{\sqrt{229}}{2}, \frac{\sqrt{269}}{2}$.

Problem 2 Let ABC be an equilateral triangle of altitude 1. A circle, with radius 1 and center on the same side of AB as C, rolls along the segment AB; as it rolls, it always intersects both $\overline{AC}$ and $\overline{BC}$. Prove that the length of the arc of the circle that is inside the triangle remains constant.

Solution: Let ω be "the circle." Let O be the center of ω. Let ω intersect segments $\overline{AC}$ and $\overline{BC}$ at M and N, respectively. Let the circle through O, C, and M intersect $\overline{BC}$ again at P. Now $\angle PMO = 180^\circ - \angle OCP = 60^\circ = \angle MCO = \angle MPO$, so $OP = OM = 1$, and P coincides with N. Thus, $\angle MON = \angle MOP = \angle MCP = 60^\circ$. Therefore, the angle of the arc of ω that is inside the triangle ABC is constant, and hence the length of the arc must be constant as well.

Problem 3 Determine all positive integers n such that $n = d(n)^2$, where $d(n)$ denotes the number of positive divisors of n (including 1 and n).

Solution: Label the prime numbers $p_1 = 2, p_2 = 3, \ldots$. Since n is a perfect square, we have

$$n=\prod_{i=1}^{\infty} p_i^{2a_i}, \quad d(n)=\prod_{i=1}^{\infty}(2a_i+1).$$

Then $d(n)$ is odd and so is n, whence $a_1 = 0$. Since $\frac{d(n)}{\sqrt{n}} = 1$, we have

$$\prod_{i=1}^{\infty} \frac{2a_i + 1}{p_i^{a_i}} = 1.$$

By Bernoulli's inequality, we have $p_i^{a_i} \geq (p_i - 1)a_i + 1 > 2a_i + 1$ for all primes $p_i \geq 5$ that divide n. Also, $3^{a_2} \geq 2a_2 + 1$ with equality only when $a_2 \in \{0, 1\}$. Thus, for equality to hold above, we must have $a_1 = a_3 = a_4 = a_5 = \cdots = 0$ and $a_2 \in \{0, 1\}$; therefore, $n \in \{1, 9\}$ are the only solutions.

Problem 4 Suppose $a_1, a_2, \ldots, a_8$ are eight distinct integers from the set $\mathcal{S} = \{1, 2, \ldots, 17\}$. Show that there exists an integer $k > 0$ such that the equation $a_i - a_j = k$ has at least three different solutions. Also, find a specific set of 7 distinct integers $\{b_1, b_2, \cdots, b_7\}$ from $\mathcal{S}$ such that the equation

$$b_i - b_j = k$$

does not have three distinct solutions for any $k > 0$.

Solution: For the first part of this problem, assume without loss of generality that $a_1 < a_2 < \cdots < a_8$; also assume, for the purpose of contradiction, that there does *not* exist an integer $k > 0$ such that the equation $a_i - a_j = k$ has at least three different solutions. Let $\delta_i = a_{i+1} - a_i$ for $i = 1, 2, \ldots, 7$. Then

$$16 \geq a_8 - a_1 = \delta_1 + \ldots + \delta_7 \geq 1 + 1 + 2 + 2 + 3 + 3 + 4 = 16$$

(for otherwise three of the δ_i's would be equal, a contradiction). Since equality must hold, $\Pi = (\delta_1, \delta_2, \ldots, \delta_7)$ must be a permutation of $(1, 1, 2, 2, 3, 3, 4)$.

Say we have a "m-n pair" if some $(\delta_i, \delta_{i+1}) = (m, n)$ or (n, m). Note that we cannot have any 1-1 or 1-2 pairs (δ_i, δ_{i+1}); otherwise we'd have $a_{i+2} - a_i = 2$ or 3, giving at least three solutions to $a_i - a_j = 2$ or 3. Nor can we have two 1-3 pairs because then, along with $\delta_i = 4$, we'd have three solutions to $a_i - a_j = 4$. Then considering what entries each 1 is next to, we see that we must have

$$\Pi = (1, 4, \ldots, 3, 1) \quad \text{or} \quad (1, 4, 1, 3, \ldots)$$

(or these lists backwards).

But now we can't have any 2-2 pairs; otherwise, along with the 1-3 pair and the $\delta_i = 4$, we'd have three solutions to $a_i - a_j = 4$. Thus we have either

$$\Pi = (1, 4, 2, 3, 2, 3, 1) \quad \text{or} \quad (1, 4, 1, 3, 2, 3, 2)$$

(or these lists backwards). In either case there are at least four solutions to $a_i - a_j = 5$, a contradiction.

Thus, regardless of the $\{a_1, a_2, \ldots, a_8\}$ that we choose, for some integer $k \in \{2, 3, 4, 5\}$ the equation $a_i - a_j = k$ has at least three different solutions.

For the second part of the problem, let $(b_1, b_2, \ldots, b_7) = (1,\ 2,\ 4,\ 9,\ 14,\ 16,\ 17)$. Each of 1, 2, 3, 5, 7, 8, 12, 13, and 15 is the difference of exactly two pairs of the b_i, and each of 10, 14, and 16 is the difference of exactly one pair of the b_i. But no number is the difference of more than two such pairs, and hence the set $\{b_1, b_2, \ldots, b_7\}$ suffices.

Problem 5 Let x, y, z be non-negative real numbers such that

$$x + y + z = 1.$$

Prove that

$$x^2y + y^2z + z^2x \le \frac{4}{27}.$$

and determine when equality occurs.

Solution: Assume without loss of generality that $x = \max\{x, y, z\}$.

- If $x \ge y \ge z$, then

$$\begin{aligned} x^2y + y^2z + z^2x &\le x^2y + y^2z + z^2x + z(xy + (x - y)(y - z)) \\ &= (x + z)^2y = 4\left(\frac{1}{2} - \frac{1}{2}y\right)\left(\frac{1}{2} - \frac{1}{2}y\right)y \le \frac{4}{27}, \end{aligned}$$

  where the last inequality follows from AM-GM. Equality occurs if and only if $z = 0$ (from the first inequality) and $y = \frac{1}{3}$, in which case $(x, y, z) = \left(\frac{2}{3}, \frac{1}{3}, 0\right)$.
- If $x \ge z \ge y$, then

$$\begin{aligned} x^2y + y^2z + z^2x &= x^2z + z^2y + y^2x - (x - z)(z - y)(x - y) \\ &\le x^2z + z^2y + y^2x \le \frac{4}{27}, \end{aligned}$$

  where the second inequality is true from the result we proved for $x \ge y \ge z$ (except with y and z reversed). Equality holds in the first inequality only when two of x, y, z are equal; and in the second only when $(x, z, y) = \left(\frac{2}{3}, \frac{1}{3}, 0\right)$. Since these conditions can't both be true, the inequality is actually strict in this case.

Therefore the inequality is indeed true, and equality holds when (x, y, z) equals $\left(\frac{2}{3}, \frac{1}{3}, 0\right)$, $\left(\frac{1}{3}, 0, \frac{2}{3}\right)$, or $\left(0, \frac{2}{3}, \frac{1}{3}\right)$.

1.5 China

Problem 1 Let ABC be an acute triangle with $\angle C > \angle B$. Let D be a point on side BC such that $\angle ADB$ is obtuse, and let H be the orthocenter of triangle ABD. Suppose that F is a point inside triangle ABC and is on the circumcircle of triangle ABD. Prove that F is the orthocenter of triangle ABC if and only if both of the following are true: $HD \parallel CF$, and H is on the circumcircle of triangle ABC.

Solution: All angles are directed modulo 180°. First observe that if P is the orthocenter of triangle UVW, then

$$\begin{aligned}\angle VPW &= (90^\circ - \angle PWV) + (90^\circ - \angle WVP)\\ &= \angle WVU + \angle UWV = 180^\circ - \angle VUW.\end{aligned}$$

First suppose that F is the orthocenter of triangle ABC. Then

$$\angle ACB = 180^\circ - \angle AFB = 180^\circ - \angle ADB = \angle AHB,$$

so $ACHB$ is cyclic. And lines CF and HD are both perpendicular to side AB, so they are parallel.

Conversely, suppose that $HD \parallel CF$ and that H is on the circumcircle of triangle ABC. Since $AFDB$ and $AHCB$ are cyclic,

$$\angle AFB = \angle ADB = 180^\circ - \angle AHB = 180^\circ - \angle ACB.$$

Thus F is an intersection point of the circle defined by $\angle AFB = 180^\circ - \angle ACB$ and the line defined by $CF \perp AB$. But there are only two such points: the orthocenter of triangle ABC and the reflection of C across line AB. The latter point lies outside of triangle ABC, and hence F must indeed be the orthocenter of triangle ABC.

Problem 2 Let a be a real number. Let $\{f_n(x)\}$ be a sequence of polynomials such that $f_0(x) = 1$ and $f_{n+1}(x) = xf_n(x) + f_n(ax)$ for $n = 0, 1, 2, \ldots$.

(a) Prove that

$$f_n(x) = x^n f_n\left(\frac{1}{x}\right)$$

for $n = 0, 1, 2, \ldots$.

(b) Find an explicit expression for $f_n(x)$.

Solution: When $a = 1$, we have $f_n(x) = (x+1)^n$ for all n, and part (a) is easily checked. Now assume that $a \neq 1$.

Observe that f_n has degree n and always has constant term 1. Write $f_n(x) = c_0 + c_1 x + \cdots + c_n x^n$; we prove by induction on n that

$$(a^i - 1)c_i = (a^{n+1-i} - 1)c_{i-1}$$

for $0 \le i \le n$ (where we let $c_{-1} = 0$).

The base case $n = 0$ is clear. Now suppose that $f_{n-1}(x) = b_0 + b_1 x + \cdots + b_{n-1}x^{n-1}$ satisfies the claim: specifically, we know $(a^i - 1)b_i = (a^{n-i} - 1)b_{i-1}$ and $(a^{n+1-i} - 1)b_{i-2} = (a^{i-1} - 1)b_{i-1}$ for $i \ge 1$.

For $i = 0$, the claim states $0 = 0$. For $i \ge 1$, the given recursion gives $c_i = b_{i-1} + a^i b_i$ and $c_{i-1} = b_{i-2} + a^{i-1}b_{i-1}$. Then the claim is equivalent to

$$\begin{aligned}
&(a^i - 1)c_i = (a^{n+1-i} - 1)c_{i-1}\\
&\iff (a^i - 1)(b_{i-1} + a^i b_i) = (a^{n+1-i} - 1)(b_{i-2} + a^{i-1}b_{i-1})\\
&\iff (a^i - 1)b_{i-1} + a^i(a^i - 1)b_i\\
&\qquad\qquad = (a^{n+1-i} - 1)b_{i-2} + (a^n - a^{i-1})b_{i-1}\\
&\iff (a^i - 1)b_{i-1} + a^i(a^{n-i} - 1)b_{i-1}\\
&\qquad\qquad = (a^{i-1} - 1)b_{i-1} + (a^n - a^{i-1})b_{i-1}\\
&\iff (a^n - 1)b_{i-1} = (a^n - 1)b_{i-1},
\end{aligned}$$

so it is true.

Now by telescoping products, we have

$$\begin{aligned}
c_i = \frac{c_i}{c_0} &= \prod_{k=1}^{i} \frac{c_k}{c_{k-1}}\\
&= \prod_{k=1}^{i} \frac{a^{n+1-k} - 1}{a^k - 1} = \frac{\prod_{k=n+1-i}^{n}(a^k - 1)}{\prod_{k=1}^{i}(a^k - 1)}\\
&= \frac{\prod_{k=i+1}^{n}(a^k - 1)}{\prod_{k=1}^{n-i}(a^k - 1)} = \prod_{k=1}^{n-i} \frac{a^{n+1-k} - 1}{a^k - 1}\\
&= \prod_{k=1}^{n-i} \frac{c_k}{c_{k-1}} = \frac{c_{n-i}}{c_0} = c_{n-i},
\end{aligned}$$

giving our explicit form. Also, $f_n(x) = x^n f_n\left(\frac{1}{x}\right)$ if and only if $c_i = c_{n-i}$ for $i = 0, 1, \ldots, n$, and from above this is indeed the case. This completes the proof.

Problem 3 There are 99 space stations. Each pair of space stations is connected by a tunnel. There are 99 two-way main tunnels, and all the other tunnels are strictly one-way tunnels. A group of 4 space stations is called *connected* if one can reach each station in the group from every other station in the group without using any tunnels other than the 6 tunnels which connect them. Determine the maximum number of connected groups.

Solution: In this solution, let $f(x) = \frac{x(x-1)(x-2)}{6}$, an extension of the definition of $\binom{x}{3}$ to all real numbers x.

In a group of 4 space stations, call a station *troublesome* if three one-way tunnels lead toward it or three one-way tunnels lead out of it. In each group there is at most one troublesome station of each type for a count of at most two troublesome stations. Also, if a station is troublesome in a group, that group is not connected.

Label the stations $1, 2, \ldots, 99$. For $i = 1, 2, \ldots, 99$, let a_i one-way tunnels point into station i and b_i one-way tunnels point out. Station i is troublesome in $\binom{a_i}{3} + \binom{b_i}{3}$ groups of four. Adding over all stations, we obtain a total count of $\sum_{i=1}^{198}\left(\binom{a_i}{3} + \binom{b_i}{3}\right)$. This equals $\sum_{i=1}^{198} f(x_i)$ for nonnegative integers $x_1, x_2, \ldots, x_{198}$ with $\sum_{i=1}^{198} x_i = 96\cdot 99$. Without loss of generality, say that $x_1, x_2, \ldots, x_k$ are at least 1 and $x_{k+1}, x_{k+2}, \ldots, x_{198}$ are zero. Since $f(x)$ is convex as a function of x for $x \geq 1$, this is at least $k\binom{96\cdot 99/k}{2}$. Also, $mf(x) \geq f(mx)$ when $m \leq 1$ and $mx \geq 2$. Letting $m = k/198$ and $mx = 96\cdot 99/198 = 48$, we find that our total count is at least $198\binom{48}{2}$. Since each unconnected group of 4 stations has at most two troublesome stations, there are at least $99\binom{48}{3}$ unconnected groups of four and at most $\binom{99}{4} - 99\binom{48}{3}$ connected groups.

All that is left to show is that this maximum can be attained. Arrange the stations around a circle, and put a two-way tunnel between any two adjacent stations; otherwise, place a one-way tunnel running from station A to station B if and only if A is 3, 5, $\ldots$, or 97 stations away clockwise from B. In this arrangement, every station is troublesome $2\binom{48}{3}$ times. It is easy to check that under this arrangement, no unconnected group of four stations contains two adjacent stations. And suppose that station A is troublesome in a group of four stations A, B, C, D with B closest and D furthest away clockwise from A. If one-way tunnels lead from A to the other tunnels, three one-way tunnels must lead to D from the other tunnels; and if one-way tunnels lead to A from the other tunnels, three one-way tunnels must lead from B to the other tunnels. Thus every unconnected group of four stations has exactly two troublesome stations. Hence equality holds in the previous paragraph, and there are indeed exactly $\binom{99}{4} - 99\binom{48}{3}$ connected groups.

Problem 4 Let m be a positive integer. Prove that there are integers a, b, k, such that both a and b are odd, $k \geq 0$, and

$$2m = a^{19} + b^{99} + k \cdot 2^{1999}.$$

Solution: The key observation is that if $\{t_1, \cdots, t_n\}$ equals $\{1, 3, 5, \ldots, 2^n - 1\}$ modulo 2^n, then $\{t_1^s, \cdots, t_n^s\}$ does as well for any odd positive integer s. To show this, note that for $i \neq j$,

$$t_i^s - t_j^s = (t_i - t_j)(t_i^{s-1} + t_i^{s-2}t_j + \cdots + t_j^{s-1}).$$

Since $t_i^{s-1} + t_i^{s-2}t_j + \cdots + t_j^{s-1}$ is an odd number, $t_i \equiv t_j \iff t_i^s \equiv t_j^s \pmod{2^n}$.

Therefore there exists an odd number a_0 such that $2m - 1 \equiv a_0^{19} \pmod{2^{1999}}$. Hence if we pick $a \equiv a_0 \pmod{2^{1999}}$ sufficiently negative so that $2m - 1 - a^{19} > 0$, then

$$(a, b, k) = \left(a, 1, \frac{2m - 1 - a^{19}}{2^{1999}}\right)$$

is a solution to the equation.

Problem 5 Determine the maximum value of λ such that if $f(x) = x^3 + ax^2 + bx + c$ is a cubic polynomial with all its roots nonnegative, then

$$f(x) \geq \lambda(x - a)^3$$

for all $x \geq 0$. Find the equality condition.

Solution: Let α, β, γ be the three roots. Without loss of generality, suppose that $0 \leq \alpha \leq \beta \leq \gamma$. We have

$$x - a = x + \alpha + \beta + \gamma \geq 0 \quad \text{and} \quad f(x) = (x - \alpha)(x - \beta)(x - \gamma).$$

If $0 \le x \le \alpha$, then (applying the arithmetic-mean geometric mean inequality) to obtain the first inequality below)

$$-f(x) = (\alpha - x)(\beta - x)(\gamma - x) \le \frac{1}{27}(\alpha + \beta + \gamma - 3x)^3$$
$$\le \frac{1}{27}(x + \alpha + \beta + \gamma)^3 = \frac{1}{27}(x - a)^3,$$

so that $f(x) \ge -\frac{1}{27}(x-a)^3$. Equality holds exactly when $\alpha - x = \beta - x = \gamma - x$ in the first inequality and $\alpha + \beta + \gamma - 3x = x + \alpha + \beta + \gamma$ in the second; that is, when $x = 0$ and $\alpha = \beta = \gamma$.

If $\beta \le x \le \gamma$, then (again applying AM-GM to obtain the first inequality below)

$$-f(x) = (x - \alpha)(x - \beta)(\gamma - x) \le \frac{1}{27}(x + \gamma - \alpha - \beta)^3$$
$$\le \frac{1}{27}(x + \alpha + \beta + \gamma)^3 = \frac{1}{27}(x - a)^3,$$

so that again $f(x) \ge -\frac{1}{27}(x-a)^3$. Equality holds exactly when $x - \alpha = x - \beta = \gamma - x$ in the first inequality and $x + \gamma - \alpha - \beta = x + \alpha + \beta + \gamma$; that is, when $\alpha = \beta = 0$ and $\gamma = 2x$.

Finally, when $\alpha < x < \beta$ or $x > \gamma$ then

$$f(x) > 0 \ge -\frac{1}{27}(x - a)^3.$$

Thus, $\lambda = -\frac{1}{27}$ works. From the above reasoning we can find that λ must be at most $-\frac{1}{27}$ or else the inequality fails for the polynomial $f(x) = x^2(x-1)$ at $x = \frac{1}{2}$. Equality occurs when either $\alpha = \beta = \gamma$ and $x = 0$; or $\alpha = \beta = 0$, γ any nonnegative real, and $x = \frac{\gamma}{2}$.

Problem 6 A $4 \times 4 \times 4$ cube is composed of 64 unit cubes. The faces of 16 unit cubes are to be colored red. A coloring is called *interesting* if there is exactly 1 red unit cube in every $1 \times 1 \times 4$ rectangular box composed of 4 unit cubes. Determine the number of interesting colorings. (Two colorings are different even if one can be transformed into another by a series of rotations.)

Solution: Pick one face of the cube as our bottom face. For each unit square A on the bottom face, we consider the vertical $1 \times 1 \times 4$

rectangular box with A at its bottom. Suppose the i-th unit cube up (counted from A) in the box is colored; then write the number i in A.

Each interesting coloring is mapped one-to-one to a 4×4 *Latin square* on the bottom face. (In an $n \times n$ Latin square, each row and column contains each of n symbols $a_1, \ldots, a_n$ exactly once.) Conversely, given a Latin square we can reverse this construction. Therefore, to solve the problem, we only need to count the number of distinct 4×4 Latin squares.

Note that switching rows of a Latin square will generate another Latin square. Thus if our four symbols are a, b, c, d, then each of the $4! \cdot 3!$ arrangements of the first row and column correspond to the same number of Latin squares. Therefore there are $4! \cdot 3! \cdot x$ four-by-four Latin squares, where x is the number of Latin squares whose first row and column both contain the symbols a, b, c, d in that order. The entry in the second row and second column equals either a, c, or d, yielding the Latin squares

$$\begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix}, \quad \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & b & a \\ d & c & a & b \end{bmatrix},$$

$$\begin{bmatrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{bmatrix}, \quad \begin{bmatrix} a & b & c & d \\ b & d & a & c \\ c & a & d & b \\ d & c & b & a \end{bmatrix}.$$

Thus $x = 4$, and there are $4! \cdot 3! \cdot 4 = 576$ four-by-four Latin squares, and 576 interesting colorings.

1.6 Czech and Slovak Republics

Problem 1 In the fraction

$$\frac{29 \div 28 \div 27 \div \cdots \div 16}{15 \div 14 \div 13 \div \cdots \div 2}$$

parentheses may be repeatedly placed anywhere in the numerator, granted they are also placed on the identical locations in the denominator.

(a) Find the least possible integral value of the resulting expression.

(b) Find all possible integral values of the resulting expression.

Solution:

(a) The resulting expression can always be written (if we refrain from canceling terms) as a ratio $\frac{A}{B}$ of two integers A and B satisfying

$$AB = (2)(3)\cdots(29) = 29! = 2^{25}\cdot 3^{13}\cdot 5^6\cdot 7^4\cdot 11^2\cdot 13^2\cdot 17\cdot 19\cdot 23\cdot 29.$$

(To find these exponents, we could either count primes directly factor by factor, or use the rule that

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots \tag{1}$$

is the exponent of p in $n!$.)

The primes that have an odd exponent in the factorization of 29! cannot "vanish" from the ratio $\frac{A}{B}$ even after making any cancellations. For this reason no integer value of the result can be less than

$$H = 2 \cdot 3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 = 1,292,646.$$

On the other hand,

$$\begin{aligned}
&\frac{29 \div (28 \div 27 \div \cdots \div 16)}{15 \div (14 \div 13 \div \cdots \div 2)} \\
&\quad = \frac{29 \cdot 14}{15 \cdot 28} \cdot \frac{(27)(26)\cdots(16)}{(13)(12)\cdots(2)} \\
&\quad = \frac{29 \cdot 14^2}{28} \cdot \frac{27!}{(15!)^2} \\
&\quad = 29 \cdot 7 \cdot \frac{2^{23} \cdot 3^{13} \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23}{(2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13)^2} = H.
\end{aligned}$$

(Again it helps to count exponents in factorials using (1).) The number H is thus the desired least value.

(b) Let's examine the products A and B more closely. In each of the fourteen pairs of numbers

$$\{29, 15\}, \{28, 14\}, \ldots, \{16, 2\},$$

one of the numbers is a factor in A and the other is a factor in B. The resulting value V can then be written as a product

$$\left(\frac{29}{15}\right)^{\epsilon_1}\left(\frac{28}{14}\right)^{\epsilon_2}\cdots\left(\frac{16}{2}\right)^{\epsilon_{14}},$$

where each ϵ_i equals ± 1, and where $\epsilon_1 = 1$ and $\epsilon_2 = -1$ no matter how the parentheses are placed. Since the fractions $\frac{27}{13}$, $\frac{26}{12}$, $\ldots$, $\frac{16}{2}$ are greater than 1, the resulting value V (whether an integer or not) has to satisfy the estimate

$$V \le \frac{29}{15} \cdot \frac{14}{28} \cdot \frac{27}{13} \cdot \frac{26}{12} \cdot \ldots \cdot \frac{16}{2} = H,$$

where H is number determined in part (a). It follows that H is the *only* possible integer value of V!

Problem 2 In a tetrahedron $ABCD$ we denote by E and F the midpoints of the medians from the vertices A and D, respectively. (The median from a vertex of a tetrahedron is the segment connecting the vertex and the centroid of the opposite face.) Determine the ratio of the volumes of tetrahedrons $BCEF$ and $ABCD$.

Solution: Let K and L be the midpoints of the edges BC and AD, and let A_0, D_0 be the centroids of triangles BCD and ABC, respectively. Both medians AA_0 and DD_0 lie in the plane AKD, and their intersection T (the centroid of the tetrahedron) divides them in $3:1$ ratios. T is also the midpoint of $\overline{KL}$, since $\vec{T} = \frac{1}{4}(\vec{A}+\vec{B}+\vec{C}+\vec{D}) = \frac{1}{2}(\frac{1}{2}(\vec{A}+\vec{D})+\frac{1}{2}(\vec{B}+\vec{C})) = \frac{1}{2}(\vec{K}+\vec{L})$. It follows that $\frac{ET}{AT} = \frac{FT}{DT} = \frac{1}{3}$, and hence $\triangle ATD \sim \triangle ETF$ and $EF = \frac{1}{3}AD$. Since the plane BCL bisects both segments AD and EF into halves, it also divides both tetrahedrons $ABCD$ and $BCEF$ into two parts of equal volume. Let G be the midpoint of $\overline{EF}$; the corresponding volumes than satisfy

$$\frac{[BCEF]}{[ABCD]} = \frac{[BCGF]}{[BCLD]} = \frac{GF}{LD} \cdot \frac{[BCG]}{[BCL]} = \frac{1}{3}\frac{KG}{KL} = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}.$$

Problem 3 Show that there exists a triangle ABC for which, with the usual labelling of sides and medians, it is true that $a \neq b$ and $a + m_a = b + m_b$. Show further that there exists a number k such that for each such triangle $a + m_a = b + m_b = k(a + b)$. Finally, find all possible ratios $a : b$ of the sides of these triangles.

Solution: We know that

$$m_a^2 = \frac{1}{4}(2b^2 + 2c^2 - a^2), \quad m_b^2 = \frac{1}{4}(2a^2 + 2c^2 - b^2),$$

so

$$m_a^2 - m_b^2 = \frac{3}{4}(b^2 - a^2).$$

As $m_a - m_b = b - a \neq 0$ by hypothesis, it follows that $m_a + m_b = \frac{3}{4}(b + a)$. From the system of equations

$$\begin{aligned} m_a - m_b &= b - a \\ m_a + m_b &= \frac{3}{4}(b + a) \end{aligned}$$

we find $m_a = \frac{1}{8}(7b - a)$, $m_b = \frac{1}{8}(7a - b)$, and

$$a + m_a = b + m_b = \frac{7}{8}(a + b).$$

Thus $k = \frac{7}{8}$.

Now we examine for what $a \neq b$ there exists a triangle ABC with sides a, b and medians $m_a = \frac{1}{8}(7b - a), m_b = \frac{1}{8}(7a - b)$. We can find all three side lengths in the triangle AB_1G, where G is the centroid of the triangle ABC and B_1 is the midpoint of the side AC:

$$AB_1 = \frac{b}{2}, \quad AG = \frac{2}{3}m_a = \frac{2}{3} \cdot \frac{1}{8}(7b - a) = \frac{1}{12}(7b - a),$$
$$B_1G = \frac{1}{3}m_b = \frac{1}{3} \cdot \frac{1}{8}(7a - b) = \frac{1}{24}(7a - b).$$

Examining the triangle inequalities for these three lengths, we get the condition

$$\frac{1}{3} < \frac{a}{b} < 3,$$

from which the value $\frac{a}{b} = 1$ has to be excluded by assumption. This condition is also sufficient: once the triangle AB_1G has been constructed, it can always be completed to a triangle ABC with $b = AC, m_a = AA_1, m_b = BB_1$. Then from the equality $m_a^2 - m_b^2 = \frac{3}{4}(b^2 - a^2)$ we would also have $a = BC$.

Problem 4 In a certain language there are only two letters, A and B. The words of this language satisfy the following axioms:

(i) There are no words of length 1, and the only words of length 2 are AB and BB.

(ii) A sequence of letters of length $n > 2$ is a word if and only if it can be created from some word of length less than n by the following construction: all letters A in the existing word are left unchanged, while each letter B is replaced by some word. (While performing this operation, the B's do not all have to be replaced by the same word.)

Show that for any n the number of words of length n equals

$$\frac{2^n + 2 \cdot (-1)^n}{3}.$$

Solution: Let us call any finite sequence of letters A, B a "string." From here on, we let $\cdots$ denote a (possibly empty) string, while $***$ will stand for a string consisting of identical letters. (For example, $\underbrace{B *** B}_{k}$ is a string of k B's.)

We show that an arbitrary string is a word if and only if it satisfies the following conditions: (a) the string terminates with the letter B; and (b) it either starts with the letter A, or else starts (or even wholly consists of) an even number of B's.

It is clear that these conditions are necessary: they are satisfied for both words AB and BB of length 2, and they are likewise satisfied by any new word created by the construction described in (ii) if they are satisfied by the words in which the B's are replaced.

We now show by induction on n that, conversely, any string of length n satisfying the conditions is a word. This is clearly true for $n = 1$ and $n = 2$. If $n > 2$, then a string of length n satisfying the conditions must have one of the forms

$$AA \cdots B,\ AB \cdots B,\ \underbrace{B *** B}_{2k} A \cdots B,\ \underbrace{B *** B}_{2k+2},$$

where $2 \leq 2k \leq n-2$. We have to show that these four types of strings arise from the construction in (ii) in which the B's are replaced by strings (of lengths less than n) satisfying the condition — that is, by *words* in view of the induction hypothesis.

The word $AA\cdots B$ arises as $A(A\cdots B)$ from the word AB. The word $AB\cdots B$ arises either as $A(B\cdots B)$ from the word AB, or as $(AB)(\cdots B)$ from the word BB, depending on whether its initial letter A is followed by an even or an odd number of B's. The word $\underbrace{B***B}_{2k}A\cdots B$ arises as $(B***B)(A\cdots B)$ from the word BB, and the word $\underbrace{B***B}_{2k+2}$ as $(\underbrace{B***B}_{2k})(BB)$ from the word BB. This completes the proof by induction.

Now we show that the number p_n of words of length n is indeed given by the formula

$$p_n = \frac{2^n + 2\cdot(-1)^n}{3}.$$

It is clearly true for $n = 1$ and 2 since $p_1 = 0$ and $p_2 = 2$; and the formula will then follow by induction if we can show that $p_{n+2} = 2^n + p_n$ for each n. But this recursion is obvious because each word of length $n+2$ is either of the form $A\cdots B$ where $\cdots$ is any of 2^n strings of length n; or of the form $BB\cdots$ where $\cdots$ is any of the p_n words of length n.

Problem 5 In the plane an acute angle APX is given. Show how to construct a square $ABCD$ such that P lies on side BC and P lies on the bisector of angle BAQ where Q is the intersection of ray PX with CD.

Solution: Consider the ration by 90° around the point A that maps B to D, and the points P, C, D into some points P', C', D', respectively. Since $\angle PAP' = 90°$, it follows from the nature of exterior angle bisectors that AP' bisects $\angle QAD'$. Consequently, the point P' has the same distance from $\overline{AD'}$ and $\overline{AQ}$, equal to the side length s of square $ABCD$. But this distance is also the length of the altitude AD in triangle AQP'; then since the altitudes from A and P' in this triangle are equal, we have $AQ = P'Q$. Since we can construct P', we can also construct Q as the intersection of line PX with the perpendicular bisector of the segment AP'. The rest of the construction is obvious, and it is likewise clear that the resulting square $ABCD$ has the required property.

Problem 6 Find all pairs of real numbers a and b such that the system of equations

$$\frac{x+y}{x^2+y^2}=a, \qquad \frac{x^3+y^3}{x^2+y^2}=b$$

has a solution in real numbers (x, y).

Solution: If the given system has a solution (x, y) for $a = A$, $b = B$, then it clearly also has a solution (kx, ky) for $a = \frac{1}{k}A$, $b = kB$, for any $k \neq 0$. It follows that the existence of a solution of the given system depends only on the value of the product ab.

We therefore begin by examining the values of the expression

$$P(u,v) = \frac{(u+v)(u^3+v^3)}{(u^2+v^2)^2}$$

where the numbers u and v are normalized by the condition $u^2+v^2 = 1$. This condition implies that

$$\begin{aligned} P(u,v) &= (u+v)(u^3+v^3) = (u+v)^2(u^2-uv+v^2) \\ &= (u^2+2uv+v^2)(1-uv) = (1+2uv)(1-uv). \end{aligned}$$

Under the condition $u^2 + v^2 = 1$ the product uv can attain all values in the interval $[-\frac{1}{2}, \frac{1}{2}]$ (if $u = \cos\alpha$ and $v = \sin\alpha$, then $uv = \frac{1}{2}\sin 2\alpha$). Hence it suffices to find the range of values of the function $f(t) = (1+2t)(1-t)$ on the interval $t \in [-\frac{1}{2}, \frac{1}{2}]$. From the formula

$$f(t) = -2t^2 + t + 1 = -2\left(t - \frac{1}{4}\right)^2 + \frac{9}{8}$$

it follows that this range of values is the closed interval with endpoints $f\left(-\frac{1}{2}\right) = 0$ and $f\left(\frac{1}{4}\right) = \frac{9}{8}$.

This means that if the given system has a solution, its parameters a and b must satisfy $0 \le ab \le \frac{9}{8}$, where the equality $ab = 0$ is possible only if $x + y = 0$ (then, however, $a = b = 0$).

Conversely, if a and b satisfy $0 < ab \le \frac{9}{8}$, by our proof there exist numbers u and v such that $u^2 + v^2 = 1$ and $(u+v)(u^3+v^3) = ab$. Denoting $a' = u + v$ and $b' = u^3 + v^3$, the equality $a'b' = ab \neq 0$ implies that both ratios $\frac{a'}{a}$ and $\frac{b}{b'}$ have the same value $k \neq 0$. But then $(x, y) = (ku, kv)$ is clearly a solution of the given system for the parameter values a and b.

1.7 France

Problem 1

(a) What is the maximum volume of a cylinder that is inside a given cone and has the same axis of revolution as the cone? Express your answer in terms of the radius R and height H of the cone.

(b) What is the maximum volume of a ball that is inside a given cone? Again, express your answer in terms of R and H.

(c) Given fixed values for R and H, which of the two maxima you found is bigger?

Solution: Let $\ell = \sqrt{R^2 + H^2}$ be the slant height of the given cone; also, orient the cone so that its base is horizontal and its tip is pointing upward.

(a) Intuitively, the cylinder with maximum volume rests against the base of the cone, and the center of the cylinder's base coincides with the center of the cone's base. The top face of the cylinder cuts off a smaller cone at the top of the original cone. If the cylinder has radius r, then the smaller cone has height $r \cdot \frac{H}{R}$ and the cylinder has height $h = H - r \cdot \frac{H}{R}$. Then the volume of the cylinder is

$$\pi r^2 h = \pi r^2 H \left(1 - \frac{r}{R}\right) = 4\pi R^2 H \left(\frac{r}{2R} \cdot \frac{r}{2R} \cdot \left(1 - \frac{r}{R}\right)\right).$$

And by AM-GM on $\frac{r}{2R}$, $\frac{r}{2R}$, and $1 - \frac{r}{R}$ this is at most

$$4\pi R^2 H \cdot \frac{1}{27}\left(\frac{r}{2R} + \frac{r}{2R} + \left(1 - \frac{r}{R}\right)\right)^3 = \frac{4}{27}\pi R^2 H,$$

with equality when $r/2R = 1 - r/R \iff r = \frac{2}{3}R$.

(b) Intuitively, the sphere with maximum volume is tangent to the base and lateral face of the cone; and its center lies on the cone's axis. Say the sphere has radius r.

Take a planar cross-section of the cone slicing through its axis; this cuts off a triangle from the cone and a circle from the sphere. The triangle's side lengths are ℓ, ℓ, and $2R$; and its height (from the side of length $2R$) is H. The circle has radius r and is the incircle of this triangle.

The area K of the triangle is $\frac{1}{2}(2R)(H) = RH$ and its semiperimeter is $s = R + \ell$. Then since $K = rs$ we have $r = \frac{RH}{R+\ell}$, and thus the volume of the sphere is

$$\frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{RH}{R+\ell}\right)^3.$$

(c) We claim that when $h/R = \sqrt{3}$ or $2\sqrt{6}$, the two volumes are equal; when $\sqrt{3} < h/R < 2$, the sphere has larger volume; and when $0 < h/R < \sqrt{3}$ or $2 < h/R$, the cylinder has larger volume.

We wish to compare $\frac{4}{27}\pi R^2 H$ and $\frac{4}{3}\pi\left(\frac{RH}{R+\ell}\right)^3$; equivalently, multiplying by $\frac{27}{4\pi R^2 H}(R+\ell)^3$, we wish to compare $(R+\ell)^3$ and $9RH^2 = 9R(\ell^2 - R^2)$. Writing $\phi = \ell/R$, this is equivalent to comparing $(1+\phi)^3$ and $9(\phi^2-1)$. Now,

$$(1+\phi)^3 - 9(\phi^2-1) = \phi^3 - 6\phi^2 + 3\phi + 10 = (\phi+1)(\phi-2)(\phi-5).$$

Thus when $\phi = 2$ or 5, the volumes are equal; when $2 < \phi < 5$, the sphere has larger volume; and when $1 < \phi < 2$ or $5 < \phi$, the cylinder has larger volume. Comparing R and H instead of R and ℓ yields the conditions stated before.

Problem 2 Find all integer solutions to $(n+3)^n = \sum_{k=3}^{n+2} k^n$.

Solution: $n = 2$ and $n = 3$ are solutions to the equations; we claim they are the only ones.

First observe that the function $f(n) = \left(\frac{n+3}{n+2}\right)^n = \left(1 + \frac{1}{n+2}\right)^n$ is an increasing function for $n > 0$. To see this, note that the derivative of $\ln f(n)$ with respect to n is $\ln\left(1+\frac{1}{n+2}\right) - \frac{n}{(n+2)(n+3)}$. By the Taylor expansion,

$$\begin{aligned}\ln\left(1+\frac{1}{n+2}\right) &= \sum_{j=1}^{\infty} \frac{1}{(n+2)^{2j}}\left[\frac{1}{2j-1}(n+2) - \frac{1}{2j}\right] \\ &> \frac{2(n+2)-1}{2(n+2)^2}\end{aligned}$$

and hence

$$\begin{aligned}\frac{d}{dn}\ln f(n) &= \ln\left(\frac{n+3}{n+2}\right) - \frac{n}{(n+2)(n+3)} \\ &> \frac{2(n+2)-1}{2(n+2)^2} - \frac{n}{(n+2)^2} = \frac{3}{2(n+2)^2} > 0.\end{aligned}$$

Thus $\ln f(n)$ and therefore $f(n)$ is indeed increasing.

Now, notice that if $f(n) > 2$ then we have

$$\left(\frac{2}{1}\right)^n > \left(\frac{3}{2}\right)^n > \cdots > \left(\frac{n+3}{n+2}\right)^n > 2$$

so that

$$(n+3)^n > 2(n+2)^n > \cdots > 2^j(n+3-j)^n > \cdots > 2^n \cdot (3)^n.$$

Then

$$3^n + 4^n + \cdots + (n+2)^n < \left(\frac{1}{2^n} + \frac{1}{2^{n-1}} + \cdots + \frac{1}{2}\right)(n+3)^n$$
$$= \left(1 - \frac{1}{2^n}\right)(n+3)^n < (n+3)^n,$$

so the equality does not hold.

Then since $2 < f(6) < f(7) < \cdots$, the equality must fail for all $n \geq 6$. Quick checks show it also fails for $n = 1, 4, 5$ (in each case, one side of the equation is odd while the other is even). Therefore the only solutions are $n = 2$ and $n = 3$.

Problem 3 For which acute-angled triangle is the ratio of the shortest side to the inradius maximal?

Solution: Let the sides of the triangle have lengths $a \leq b \leq c$; let the angles opposite them be A, B, C; let the semiperimeter be $s = \frac{1}{2}(a+b+c)$; and let the inradius be r. Without loss of generality say the triangle has circumradius $R = \frac{1}{2}$ and that $a = \sin A$, $b = \sin B$, $c = \sin C$.

The area of the triangle equals both $rs = \frac{1}{2}r(\sin A + \sin B + \sin C)$ and $abc/4R = \frac{1}{2}\sin A \sin B \sin C$. Thus

$$r = \frac{\sin A \sin B \sin C}{\sin A + \sin B + \sin C}$$

and

$$\frac{a}{r} = \frac{\sin A + \sin B + \sin C}{\sin B \sin C}.$$

Since $A = 180° - B - C$, $\sin A = \sin(B+C) = \sin B \cos C + \cos B \sin C$ and we also have

$$\frac{a}{r} = \cot B + \csc B + \cot C + \csc C.$$

Note that $f(x) = \cot x + \csc x$ is a decreasing function along the interval $0° < x < 90°$. Now there are two cases: $B \leq 60°$, or $B > 60°$.

If $B \leq 60^\circ$, then assume that $A = B$; otherwise the triangle with angles $A' = B' = \frac{1}{2}(A + B) \leq B$ and $C' = C$ has a larger ratio a'/r'. Then since $C < 90^\circ$ we have $45^\circ < A \leq 60^\circ$. Now,

$$\frac{a}{r} = \frac{\sin A + \sin B + \sin C}{\sin B \sin C} = \frac{2\sin A + \sin(2A)}{\sin A \sin(2A)} = 2\csc(2A) + \csc A.$$

Now $\csc x$ has second derivative $\csc x(\csc^2 x + \cot^2 x)$, which is strictly positive when $0^\circ < x < 180^\circ$; thus both $\csc x$ and $\csc(2x)$ are strictly convex along the interval $0^\circ < x < 90^\circ$. Therefore $g(A) = 2\csc(2A) + \csc A$, a convex function in A, is maximized in the interval $45^\circ \leq A \leq 60^\circ$ at one of the endpoints. Since $g(45^\circ) = 2 + \sqrt{2} < 2\sqrt{3} = g(60^\circ)$, it is maximized when $A = B = C = 60^\circ$.

As for the case when $B > 60^\circ$, since $C > B > 60^\circ$, the triangle with $A' = B' = C' = 60^\circ$ has a larger ratio a'/r'. Therefore the maximum ratio is $2\sqrt{3}$, attained with an equilateral triangle.

Problem 4 There are 1999 red candies and 6661 yellow candies on a table, made indistinguishable by their wrappers. A *gourmand* applies the following algorithm until the candies are gone:

(a) If there are candies left, he takes one at random, notes its color, eats it, and goes to (b).

(b) If there are candies left, he takes one at random, notes its color, and

- (i) if it matches the last one eaten, he eats it also and returns to (b).
- (ii) if it does not match the last one eaten, he wraps it up again, puts it back, and goes to (a).

Prove that all the candies will eventually be eaten. Find the probability that the last candy eaten is red.

Solution: If there are finitely many candies left at any point, then at the next instant the gourmand must perform either step (a), part (i) of step (b), or part (ii) of step (b). He eats a candy in the first two cases; in the third case, he returns to step (a) and eats a candy. Since there are only finitely many candies, the gourmand must eventually eat all the candies.

We now prove by induction on the total number of candies that if we start with $r > 0$ red candies and $y > 0$ yellow candies immediately before step (a), then the probability is $\frac{1}{2}$ that the last candy eaten is red.

Suppose that the claim is true for all smaller amounts of candy. After the gourmand first completes steps (a) and (b) exactly once, suppose there are r' red candies and y' yellow candies left; we must have $r' + y' < r + y$. The chances that $r' = 0$ is

$$\frac{r}{r+y} \cdot \frac{r-1}{r+y-1} \cdot \cdots \cdot \frac{1}{y+1} = \frac{1}{\binom{r+y}{r}}.$$

Similarly, the chances that $y' = 0$ is $\frac{1}{\binom{r+y}{y}} = \frac{1}{\binom{r+y}{r}}$. (In the case $r = y = 1$, this proves the claim.)

Otherwise, the probability is $1 - \frac{2}{\binom{r+y}{r}}$ that both r' and y' are still positive. By the induction hypothesis in this case the last candy is equally likely to be red as it is yellow. Thus the overall probability that the last candy eaten is red is

$$\underbrace{\frac{1}{\binom{r+y}{r}}}_{y'=0} + \underbrace{\frac{1}{2}\left(1 - \frac{2}{\binom{r+y}{r}}\right)}_{r',y'>0} = \frac{1}{2}.$$

This completes the inductive step, and the proof.

Problem 5 With a given triangle, form three new points by reflecting each vertex about the opposite side. Show that these three new points are collinear if and only if the the distance between the orthocenter and the circumcenter of the triangle is equal to the diameter of the circumcircle of the triangle.

Solution: Let the given triangle be ABC and let the reflections of A, B, C across the corresponding sides be D, E, F. Let A', B', C' be the midpoints of $\overline{BC}, \overline{CA}, \overline{AB}$, and as usual let G, H, O denote the triangle's centroid, orthocenter, and circumcenter. Let triangle $A''B''C''$ be the triangle for which A, B, C are the midpoints of $B''C'', C''A'', A''B''$, respectively. Then G is the centroid and H is the circumcenter of triangle $A''B''C''$. Let $D', E,' F'$ denote the projections of O on the lines $B''C'', C''A'', A''B''$, respectively.

Consider the homothety h with center G and ratio $-1/2$. It maps A, B, C, A'', B'', C'' into A', B', C', A, B, C, respectively. Note that $A'D' \perp BC$ since O is the orthocenter of triangle $A'B'C'$. This implies $AD : A'D' = 2 : 1 = GA : GA'$ and $\angle DAG = \angle D'A'G$. We conclude that $h(D) = D'$. Similarly, $h(E) = E'$ and $h(F) = F'$. Thus, D, E, F are collinear if and only if D', E', F' are collinear. Now D', E', F' are the projections of O on the sides $B''C'', C''A'', A''B''$, respectively. By Simson's theorem, they are collinear if and only if O lies on the circumcircle of triangle $A''B''C''$. Since the circumradius of triangle $A''B''C''$ is $2R$, O lies on its circumcircle if and only if $OH = 2R$.

1.8 Hong Kong (China)

Problem 1 Let $PQRS$ be a cyclic quadrilateral with $\angle PSR = 90°$, and let H and K be the respective feet of perpendiculars from Q to lines PR and PS. Prove that line HK bisects $\overline{QS}$.

First Solution: Since $\overline{QK}$ and $\overline{RS}$ are both perpendicular to $\overline{PS}$, $\overline{QK}$ is parallel to $\overline{RS}$ and thus $\angle KQS = \angle RSQ$. Since $PQRS$ is cyclic, $\angle RSQ = \angle RPQ$. Since $\angle PKQ = \angle PHQ = 90°$, $PKHQ$ is also cyclic and it follows that $\angle RPQ = \angle HPQ = \angle HKQ$. Thus, $\angle KQS = \angle HKQ$; since triangle KQS is right, it follows that line HK bisects $\overline{QS}$.

Second Solution: The Simson line from Q with respect to $\triangle PRS$ goes through H, K, and the foot F of the perpendicular from Q to $\overleftrightarrow{RS}$. Thus, line HK is line FK, a diagonal in rectangle $SFQK$, so it bisects the other diagonal, $\overline{QS}$.

Problem 2 The base of a pyramid is a convex nonagon. Each base diagonal and each lateral edge is colored either black or white. Both colors are used at least once. (Note that the sides of the base are not colored.) Prove that there are three segments colored the same color which form a triangle.

Solution: Let us assume the contrary. From the pigeonhole principle, 5 of the lateral edges must be of the same color; assume they are black, and say they are the segments from the vertex V to B_1, B_2, B_3, B_4, and B_5 where $B_1B_2B_3B_4B_5$ is a convex pentagon (and where the B_i's are not necessarily adjacent vertices of the nonagon). The $\overline{B_iB_{i+1}}$ (where $B_6 = B_1$) cannot all be sides of the nonagon, so without loss of generality suppose that B_1B_2 is colored. Then because triangle VB_iB_j cannot have three sides colored black, each segment B_1B_2, B_2B_4, B_4B_1 must be white; but then triangle $B_1B_2B_4$ has three sides colored white, a contradiction.

Problem 3 Let s and t be nonzero integers, and let (x, y) be any ordered pair of integers. A move changes (x, y) to $(x - t, y - s)$. The pair (x, y) is *good* if after some (possibly zero) number of moves it becomes a pair of integers that are not relatively prime.

(a) Determine if (s, t) is a good pair;

(b) Prove that for any s, t there exists a pair (x, y) which is not good.

Solution:

(a) Let us assume that (s, t) is not good. Then, after one move, we have $(s-t, t-s)$, so we may assume without loss of generality that $s - t = 1$ and $t - s = -1$ since these numbers must be relatively prime. Then $s + t$ cannot equal 0 because it is odd; also, $s + t = (s-t)+2t \neq (s-t)+0 = 1$, and $s+t = (t-s)+2s \neq (t-s)+0 = -1$. Hence some prime p divides $s+t$. After $p-1$ moves, (s, t) becomes $(s - (p - 1)t, t - (p - 1)s) \equiv (s + t, t + s) \equiv (0, 0) \pmod{p}$, a contradiction. Thus (s, t) is good.

(b) Let x and y be integers which satisfy $sx - ty = g$, where $g = \gcd(s, t)$. Dividing by g, we find $s'x - t'y = 1$, so $\gcd(x, y) = 1$. Now, suppose by way of contradiction that after k moves some prime p divides both $x - kt$ and $y - ks$. We then have

$$\begin{aligned} 0 &\equiv x - kt \equiv y - ks \\ &\Longrightarrow 0 \equiv s(x - kt) \equiv t(y - ks) \\ &\Longrightarrow 0 \equiv sx - ty = g \pmod{p}. \end{aligned}$$

Thus p divides g, which divides s and t, so the first equation above becomes $0 \equiv x \equiv y \pmod{p}$; but x and y are relatively prime, a contradiction. Thus (x, y) is not good.

Problem 4 Let f be a function defined on the positive reals with the following properties:

(i) $f(1) = 1$;

(ii) $f(x + 1) = xf(x)$;

(iii) $f(x) = 10^{g(x)}$,

where $g(x)$ is a function defined on the reals satisfying

$$g(ty + (1 - t)z) \leq tg(y) + (1 - t)g(z)$$

for all real y, z and any $0 \leq t \leq 1$.

(a) Prove that

$$t[g(n) - g(n - 1)] \leq g(n + t) - g(n) \leq t[g(n + 1) - g(n)]$$

where n is an integer and $0 \leq t \leq 1$.

(b) Prove that
$$\frac{4}{3} \le f\left(\frac{1}{2}\right) \le \frac{4\sqrt{2}}{3}.$$

Solution:

(a) Setting $t = \frac{1}{2}$ in the given inequality, we find that $g(\frac{1}{2}(y+z)) \le \frac{1}{2}(g(y)+g(z))$. Now fix t (perhaps not equal to $\frac{1}{2}$) constant; letting $y = n - t$ and $z = n + t$ in $g(\frac{1}{2}(y+z)) \le \frac{1}{2}(g(y)+g(z))$ gives $g(n) \le \frac{1}{2}(g(n-t)+g(n+t))$, or
$$g(n) - g(n-t) \le g(n+t) - g(n). \tag{1}$$

Plugging in $z = n$, $y = n - 1$ into the given inequality gives $g(t(n-1)+(1-t)n) \le tg(n-1)+(1-t)g(n)$, or
$$t[g(n) - g(n-1)] \le g(n) - g(n-t).$$

Combining this with (1) proves the inequality on the left side. And the inequality on the right side follows from the given inequality with $z = n$, $y = n + 1$.

(b) From (ii), $f(\frac{3}{2}) = \frac{1}{2}f(\frac{1}{2})$, and $f(\frac{5}{2}) = \frac{3}{2}f(\frac{3}{2}) = \frac{3}{4}f(\frac{1}{2})$. Also, $f(2) = 1 \cdot f(1) = 1$, and $f(3) = 2f(2) = 2$. Now, if we let $n = 2$ and $t = \frac{1}{2}$ in the inequality in part (a), we find $\frac{1}{2}[g(2) - g(1)] \le g\left(\frac{5}{2}\right) - g(2) \le \frac{1}{2}[g(3) - g(2)]$. Exponentiating with base 10 yields $\sqrt{\frac{f(2)}{f(1)}} \le \frac{f\left(\frac{5}{2}\right)}{f(2)} \le \sqrt{\frac{f(3)}{f(2)}}$, or $1 \le f\left(\frac{5}{2}\right) \le \sqrt{2}$. Plugging in $f\left(\frac{5}{2}\right) = \frac{3}{4}f(\frac{1}{2})$ yields the desired result.

1.9 Hungary

Problem 1 I have $n \geq 5$ real numbers with the following properties:

(i) They are nonzero, but at least one of them is 1999.

(ii) Any four of them can be rearranged to form a geometric progression.

What are my numbers?

Solution: First suppose that the numbers are all nonnegative. If $x \leq y \leq z \leq w \leq v$ are any five of the numbers, then x, y, z, w; x, y, z, v; x, y, w, v; x, z, w, v; and y, z, w, v must all be geometric progressions. Comparing each two successive progressions in this list we find that $x = y = z = w = v$. Thus all our numbers are equal.

If some numbers are negative in our original list, replace each number x by $|x|$. The geometric progression property is preserved, and thus from above all the values $|x|$ are equal. Hence, each original number was 1999 or -1999. And because $n \geq 5$, some three numbers are equal. But no geometric progression can be formed from three -1999s and a 1999, or from three 1999s and a -1999. Therefore all the numbers must be equal — to 1999.

Problem 2 Let ABC be a right triangle with $\angle C = 90°$. Two squares S_1 and S_2 are inscribed in triangle ABC such that S_1 and ABC share a common vertex C, and S_2 has one of its sides on AB. Suppose that $[S_1] = 441$ and $[S_2] = 440$. Calculate $AC + BC$.

Solution: Let $S_1 = CDEF$ and $S_2 = KLMN$ with D and K on $\overline{AC}$ and N on $\overline{BC}$. Let $s_1 = 21, s_2 = \sqrt{440}$ and $a = BC, b = CA, c = AB$. Using ratios between similar triangles AED, ABC, EBF we get $c = AB = AE + EB = c(s_1/a + s_1/b)$ or $s_1(1/a + 1/b) = 1$. Since triangles ABC, AKL, NBM are similar we have $c = AB = AL + LM + MB = s_2(b/a + 1 + a/b)$ and $s_2 = abc/(ab + c^2)$. Then

$$\frac{1}{s_2^2} - \frac{1}{s_1^2} = \left(\frac{1}{c} + \frac{c}{ab}\right)^2 - \left(\frac{1}{a} + \frac{1}{b}\right)^2$$
$$= \left(\frac{1}{c^2} + \frac{c^2}{a^2b^2} + \frac{2}{ab}\right) - \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{2}{ab}\right) = \frac{1}{c^2}.$$

Thus $c = 1/\sqrt{1/s_2^2 - 1/s_1^2} = 21\sqrt{440}$. Solving $s_2 = abc/(ab + c^2)$ for ab yields $ab = s_2c^2/(c - s_2) = 21^2 \cdot 22$. Finally, $AC + BC = a + b = ab/s_1 = 21 \cdot 22 = 462$.

Problem 3 Let O and K be the centers of the respective spheres tangent to the faces, and the edges, of a right pyramid whose base is a 2 by 2 square. Determine the volume of the pyramid if O and K are equidistant from the base.

Solution: Let r, R be the spheres' respective radii. Let the pyramid have base $ABCD$, vertex P, and height h. By symmetry, O and K lie on the altitude through P.

Take a cross-section of the pyramid with a plane perpendicular to the base, cutting the base at a line through its center parallel to $\overline{AB}$. It cuts off an isosceles triangle from the pyramid with base 2 and legs $\sqrt{h^2+1}$; the triangle's incircle is the cross-section of the sphere centered at O and hence has radius r. On the one hand, the area of this triangle is the product of its inradius and semiperimeter, or $\frac{1}{2}r(2 + 2\sqrt{h^2+1})$. On the other hand, it equals half of the product of its base and height, or $\frac{1}{2} \cdot 2 \cdot h$. Setting these quantities equal, we have $r = (\sqrt{h^2+1} - 1)/h$.

Next, by symmetry the second sphere is tangent to $\overline{AB}$ at its midpoint M. Then since K must be distance r from plane $ABCD$, we have $R^2 = KM^2 = r^2 + 1$. Furthermore, if the second sphere is tangent to $\overline{AP}$ at N, then by equal tangents we have $AN = AM = 1$.

Then $PN = PA - 1 = \sqrt{h^2+2} - 1$. Also, $PK = h + r$ if K is on the opposite side of plane $ABCD$ as O, and it equals $h - r$ otherwise. Thus

$$\begin{aligned} PK^2 &= PN^2 + NK^2 \\ (h \pm r)^2 &= (\sqrt{h^2+2} - 1)^2 + (r^2 + 1) \\ \pm\, 2rh &= 4 - 2\sqrt{h^2+2}. \end{aligned}$$

Recalling that $r = (\sqrt{h^2+1} - 1)/h$, this gives

$$\pm\,(\sqrt{h^2+1} - 1) = 2 - \sqrt{h^2+2}.$$

This equation has the unique solution $h = \sqrt{7}/3$. Thus the volume of the pyramid is $\frac{1}{3} \cdot 4 \cdot \frac{\sqrt{7}}{3} = 4\sqrt{7}/9$.

Problem 4 For any given positive integer n, determine (as a function of n) the number of ordered pairs (x, y) of positive integers such that

$$x^2 - y^2 = 10^2 \cdot 30^{2n}.$$

Further prove that the number of such pairs is never a perfect square.

Solution: Since $10^2 \cdot 30^{2n}$ is even, x and y must have the same parity. Then (x, y) is a valid solution if and only if $(u, v) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right)$ is a pair of positive integers that satisfies $u > v$ and $uv = 5^2 \cdot 30^{2n}$. Now $5^2 \cdot 30^{2n} = 2^{2n} \cdot 3^{2n} \cdot 5^{2n+2}$ has exactly $(2n+1)^2(2n+3)$ factors; thus without the condition $u > v$ there are exactly $(2n+1)^2(2n+3)$ such pairs (u, v). Exactly one pair has $u = v$, and by symmetry half of the remaining pairs have $u > v$; and it follows that there are $\frac{1}{2}\left((2n+1)^2(2n+3) - 1\right) = (n+1)(4n^2+6n+1)$ valid pairs.

Now suppose that $(n+1)(4n^2+6n+1)$ were a square. Since $n+1$ and $4n^2+6n+1 = (4n+2)(n+1) - 1$ are coprime, $4n^2+6n+1$ must be a square as well; but $(2n+1)^2 < 4n^2+6n+1 < (2n+2)^2$, a contradiction.

Problem 5 For $0 \leq x, y, z \leq 1$, find all solutions to the equation

$$\frac{x}{1+y+zx} + \frac{y}{1+z+xy} + \frac{z}{1+x+yz} = \frac{3}{x+y+z}.$$

Solution: Assume $x + y + z > 0$, since otherwise the equation is meaningless. $(1-z)(1-x) \geq 0 \Rightarrow 1 + zx \geq x + z$, and hence $x/(1+y+zx) \leq x/(x+y+z)$. Doing this for the other two fractions yields that the left hand side is at most $(x+y+z)/(x+y+z) \leq 3/(x+y+z)$. If equality holds, we must have in particular that $x + y + z = 3 \Rightarrow x = y = z = 1$. We then verify that this is indeed a solution.

Problem 6 The midpoints of the edges of a tetrahedron lie on a sphere. What is the maximum volume of the tetrahedron?

Solution: Let the sphere have center O. First let A, B, C be any points on its surface. Then $[OAB] = \frac{1}{2} OA \cdot OB \sin \angle AOB \leq \frac{1}{2} r^2$. Likewise, the height from C to plane OAB is at most $CO = r$, whence tetrahedron $OABC$ has maximum volume $r^3/6$. Now, if $\{A, A'\}, \{B, B'\}, \{C, C'\}$ are pairs of antipodal points on the sphere, the octahedron $ABCA'B'C'$ can be broken up into 8 such tetrahedra with vertex O and therefore has maximum volume $4r^3/3$. Equality holds for a regular octahedron.

In the situation of the problem, shrink the tetrahedron T (with volume V) by a factor of $1/2$ about each vertex to obtain four tetrahedra, each with volume $V/8$. Then the six midpoints form an octahedron with volume $V/2$. Moreover, the segment connecting two opposite vertices C and D of this octahedron has T's centroid P as its midpoint. If $O \neq P$ then line OP is a perpendicular bisector of each segment, and then all these segments must lie in the plane through P perpendicular to line OP; but then $V/2 = 0$. Otherwise, the midpoints form three pairs of antipodal points, whose volume is at most $4r^3/3$ from the last paragraph. Therefore $V \leq 8r^3/3$, with equality for a regular tetrahedron.

Problem 7 A positive integer is written in each square of an n^2 by n^2 chess board. The difference between the numbers in any two adjacent squares (sharing an edge) is less than or equal to n. Prove that at least $\lfloor n/2 \rfloor + 1$ squares contain the same number.

Solution: Consider the smallest and largest numbers a and b on the board. They are separated by at most $n^2 - 1$ squares horizontally and $n^2 - 1$ vertically, so there is a path from one to the other with length at most $2(n^2 - 1)$. Then since any two successive squares differ by at most n, we have $b - a \leq 2(n^2 - 1)n$. But since all numbers on the board are integers lying between a and b, only $2(n^2 - 1)n + 1$ distinct numbers can exist; and because $n^4 > (2(n^2 - 1)n + 1)(n/2)$, more than $n/2$ squares contain the same number, as needed.

Problem 8 One year in the 20th century, Alex noticed on his birthday that adding the four digits of the year of his birth gave his actual age. That same day, Bernath—who shared Alex's birthday but was not the same age as him—also noticed this about his own birth year and age. That day, both were under 99. By how many years do their ages differ?

Solution: Let c be the given year. Alex's year of birth was either $18\underline{uv}$ or $19\underline{uv}$ respectively (where u and v are digits), and thus either $c = 18\underline{uv} + (9 + u + v) = 1809 + 11u + 2v$ or $c = 19\underline{uv} + (10 + u + v) = 1910 + 11u + 2v$.

Similarly, let Bernath's year of birth end in the digits u', v'. Alex and Bernath could not have been born in the same century. Otherwise, we would have $11u + 2v = 11u' + 2v' \Rightarrow 2(v - v') = 11(u' - u)$; thus either $(u, v) = (u', v')$ or else $|v - v'| \geq 11$, which are both impossible. Then without loss of generality say Alex was born in the 1800s, and that $1809 + 11u + 2v = 1910 + 11u' + 2v' \Rightarrow 11(u - u') + 2(v - v') = 101 \Rightarrow u - u' = 9, v - v' = 1$. The difference between their ages then equals $19\underline{u'v'} - 18\underline{uv} = 100 + 10(u' - u) + (v' - v) = 9$.

Problem 9 Let ABC be a triangle and D a point on the side AB. The incircles of the triangles ACD and CDB touch each other on $\overline{CD}$. Prove that the incircle of ABC touches $\overline{AB}$ at D.

Solution: Suppose that the incircle of a triangle XYZ touches sides YZ, ZX, XY at U, V, W. Then (using equal tangents) $XY + YZ + ZX = (YW + YU) + (XW + ZU) + XZ = (2YU) + (XZ) + XZ$, and $YU = \frac{1}{2}(XY + YZ - ZX)$.

Thus if the incircles of triangles ACD and CDB touch each other at E, then $AD + DC - CA = 2DE = BD + DC - CB \Rightarrow AD - CA = (AB - AD) - BC \Rightarrow AD = \frac{1}{2}(CA + AB - BC)$. But if the incircle of ABC is tangent to $\overline{AB}$ at D', then $AD' = \frac{1}{2}(CA + AB - BC)$ as well—so $D = D'$, as desired.

Problem 10 Let R be the circumradius of a right pyramid with a square base. Let r be the radius of the sphere touching the four lateral faces and the circumsphere. Suppose that $2R = (1 + \sqrt{2})r$. Determine the angle between adjacent faces of the pyramid.

Solution: Let P be the pyramid's vertex, $ABCD$ the base, and M, N the midpoints of sides AB, CD. By symmetry, both spheres are centered along the altitude from P. Plane PMN intersects the pyramid in triangle PMN and meets the spheres in great circles. Let the smaller circle have center O; it is tangent to $\overline{PM}$, $\overline{PN}$, and the large circle at some points U, V, W. Again by symmetry W is lies on the altitude from P, implying that it is diametrically opposite P on the larger circle. Thus $OP = 2R - r = \sqrt{2}r$, triangle OUP is a 45°-45°-90° triangle, and $\angle OPU = \angle OPV = 45°$. Therefore triangle NPM is isosceles right, and the distance from P to plane $ABCD$ equals $BC/2$.

Hence one can construct a cube with P as its center and $ABCD$ as a face; this cube can be decomposed into six pyramids congruent to $PABCD$. In particular, three such pyramids have a vertex at A; so three times the dihedral angle between faces PAB, PAD forms one revolution, and this angle is $2\pi/3$. Stated differently, say the three pyramids are $PABD$, $PADE$, $PAEB$; let P' be the midpoint of $\overline{AP}$, and let B', D', E' be points on planes PAB, PAD, PAE such that lines $B'P'$, $D'P'$, $E'P'$ are all perpendicular to line AP. The desired angle is the angle between any two of these lines. But since these three lines all lie in one plane (perpendicular to line AP), this angle must be $2\pi/3$.

Problem 11 Is there a polynomial $P(x)$ with integer coefficients such that $P(10) = 400, P(14) = 440$, and $P(18) = 520$?

Solution: If P exists, then by taking the remainder modulo $(x-10)(x-14)(x-18)$ we may assume P is quadratic. Writing $P(x) = ax^2+bx+c$, direct computation reveals $P(x+4)+P(x-4)-2P(x) = 32a$ for all x. Plugging in $x = 14$ gives $40 = 32a$, which is impossible since a must be an integer. Therefore no such polynomial exists.

Problem 12 Let a, b, c be positive numbers and $n \geq 2$ be an integer such that $a^n + b^n = c^n$. For which k is it possible to construct an obtuse triangle with sides a^k, b^k, c^k?

Solution: First, $a, b < c$. Thus for $m > n$ we have $c^m = c^{m-n}(a^n + b^n) > a^{m-n}a^n + b^{m-n}b^n = a^m + b^m$, while for $m < n$ we have $c^m = c^{m-n}(a^n + b^n) < a^{m-n}a^n + b^{m-n}b^n = a^m + b^m$. Now, a triangle with sides a^k, b^k, c^k exists iff $a^k + b^k > c^k$, and it is then obtuse iff $(a^k)^2 + (b^k)^2 < (c^k)^2$, i.e. $a^{2k} + b^{2k} < c^{2k}$. From our first observation, these correspond to $k < n$ and $2k > n$, respectively; and hence $n/2 < k < n$.

Problem 13 Let $n > 1$ be an arbitrary real number and k be the number of positive primes less than or equal to n. Select $k+1$ positive integers such that none of them divides the product of all the others. Prove that there exists a number among the $k+1$ chosen numbers which is bigger than n.

Solution: Suppose otherwise; then our chosen numbers $a_1, \ldots, a_{k+1}$ have a total of at most k distinct prime factors (i.e. the primes less than or equal to n). Let $o_p(a)$ denote the highest value of d such that $p^d \mid a$. Also let $q = a_1 a_2 \cdots a_{k+1}$. Then for each prime p, $o_p(q) = \sum_{i=1}^{k+1} o_p(a_i)$, and it follows that there can be at most one "hostile" value of i for which $o_p(a_i) > o_p(q)/2$. Since there are at most k primes which divide q, there is some i which is not hostile for any such prime. Then $2o_p(a_i) \leq o_p(q) \Rightarrow o_p(a_i) \leq o_p(q/a_i)$ for each prime p dividing q, implying that $a_i \mid q/a_i$, a contradiction.

Problem 14 The polynomial $x^4 - 2x^2 + ax + b$ has four distinct real roots. Show that the absolute value of each root is smaller than $\sqrt{3}$.

Solution: Let the roots be p, q, r, s. We have $p+q+r+s = 0, pq + pr+ps+qr+qs+rs = -2$, and hence $p^2+q^2+r^2+s^2 = 0^2-2(-2) = 4$. But by Cauchy-Schwarz, $(1+1+1)(q^2+r^2+s^2) \geq (q+r+s)^2$ for any real q, r, s; furthermore, since q, r, s must be distinct, the inequality becomes strict. Thus $4 = p^2 + q^2 + r^2 + s^2 > p^2 + (-p)^2/3 = 4p^2/3$ or $|p| < \sqrt{3}$, and the same argument holds for q, r, s.

Problem 15 Each side of a convex polygon has integral length and the perimeter is odd. Prove that the area of the polygon is at least $\sqrt{3}/4$.

Solution:

Lemma 1. *If $0 \leq x, y \leq 1$, then*

$$\sqrt{1-x^2} + \sqrt{1-y^2} \geq \sqrt{1-(x+y-1)^2}.$$

Proof: Squaring and subtracting $2 - x^2 - y^2$ from both sides gives the equivalent inequality $2\sqrt{(1-x^2)(1-y^2)} \geq -2(1-x)(1-y)$, which is true since the left side is nonnegative and the right is at most 0. ■

Lemma 2. *If $x_1 + \cdots + x_n \leq n - 1/2$ and $0 \leq x_i \leq 1$ for each i, then $\sum_{i=1}^{n} \sqrt{1-x_i^2} \geq \sqrt{3}/2$.*

Proof: Use induction on n. In the case $n = 1$, the statement is clear. If $n > 1$, then either $\min(x_1, x_2) \leq 1/2$ or $x_1 + x_2 > 1$. In the first case we immediately have $\max(\sqrt{1-x_1^2}, \sqrt{1-x_2^2}) \geq \sqrt{3}/2$; in the second case we can replace x_1, x_2 by the single number $x_1 + x_2 - 1$ and use the induction hypothesis together with the previous lemma. ■

Now consider our polygon. Let P, Q be vertices such that $l = PQ$ is maximal. The polygon consists of two paths from P to Q, each of integer length $\geq l$; these lengths are distinct since the perimeter is odd. Then the greater of the two lengths is $m \geq l + 1$. Position the polygon in the coordinate plane with $P = (0,0), Q = (l,0)$ and the longer path in the upper half-plane. Since each side of the polygon has integer length, we can divide this path into line segments of length 1. Let the endpoints of these segments, in order, be $P_0 = P, P_1 = (x_1, y_1), P_2 = (x_2, y_2), \ldots, P_m = Q$. There exists some r such that y_r is maximal; then either $r \geq x_r + 1/2$ or $(m - r) \geq (\ell - x_r) + 1/2$. Assume the former (otherwise, just reverse the choices of P and Q). We already know that $y_1 \geq 0$, and by the maximal definition of l we must have $x_1 \geq 0$ as well; then since the polygon is convex we must have $y_1 \leq y_2 \leq \ldots \leq y_r$ and $x_1 \leq x_2 \leq \ldots \leq x_r$. But $y_{i+1} - y_i = \sqrt{1 - (x_{i+1} - x_i)^2}$, so

$$y_r = \sum_{i=0}^{r-1} (y_{i+1} - y_i) = \sum_{i=0}^{r-1} \sqrt{1 - (x_{i+1} - x_i)^2} \geq \sqrt{3}/2$$

by the second lemma. And we must have $l \geq 1$, implying that triangle PP_rQ has area at least $\sqrt{3}/4$. Since this triangle lies within the polygon (by convexity), we are done.

Problem 16 Determine if there exists an infinite sequence of positive integers such that

(i) no term divides any other term;

(ii) every pair of terms has a common divisor greater than 1, but no integer greater than 1 divides all of the terms.

Solution: The desired sequence exists. Let $p_0, p_1, \ldots$ be the primes greater than 5 in order, and let $q_{3i} = 6, q_{3i+1} = 10, q_{3i+2} = 15$ for each nonnegative integer i. Then let $s_i = p_i q_i$ for all $i \geq 0$. The sequence $s_0, s_1, s_2, \ldots$ clearly satisfies (i) since s_i is not even divisible by p_j for $i \neq j$. For the first part of (ii), any two terms have their indices both in $\{0,1\}$, both in $\{0,2\}$, or both in $\{1,2\}$ (mod 3), so they have a common divisor of 2, 3, or 5, respectively. For the second part, we just need to check that no prime divides all the s_i; this holds since $2 \nmid s_2, 3 \nmid s_1, 5 \nmid s_0$, and no prime greater than 5 divides more than one s_i.

Problem 17 Prove that, for every positive integer n, there exists a polynomial with integer coefficients whose values at $1, 2, \ldots, n$ are different powers of 2.

Solution: We may assume $n \geq 4$. For each $i = 1, 2, \ldots, n$, write $\prod_{j=1, j\neq i}^{n}(i-j) = 2^{q_i} m_i$ for positive integers q_i, m_i with m_i odd. Let L be the least common multiple of all the q_i, and let $r_i = L/q_i$. For each i, there are infinitely many powers of 2 which are congruent to 1 modulo $|m_i^{r_i}|$. (Specifically, by Euler's theorem, $2^{\phi(|m_i^{r_i}|)j} \equiv 1 \pmod{|m_i^{r_i}|}$ for all $j \geq 0$.) Thus there are infinitely many integers c_i such that $c_i m_i^{r_i} + 1$ is a power of 2; choose one. Then define

$$P(x) = \sum_{i=1}^{n} c_i \left(\prod_{\substack{j=1 \\ j\neq i}}^{n} (x-j) \right)^{r_i} + 2^L.$$

For each $k, 1 \leq k \leq n$, in the sum each term $\left(\prod_{j=1, j\neq i}^{n}(x-j)\right)^{r_i}$ vanishes for all $i \neq k$. Then

$$P(k) = c_k \left(\prod_{\substack{j=1 \\ j\neq k}}^{n} (k-j) \right)^{r_i} + 2^L = 2^L (c_i m_i^{r_i} + 1),$$

a power of 2. Moreover, by choosing the c_i appropriately, we can guarantee that these values are all distinct, as needed.

Problem 18 Find all integers $N \geq 3$ for which it is possible to choose N points in the plane (no three collinear) such that each triangle formed by three vertices on the convex hull of the points contains exactly one of the points in its interior.

Solution: First, if the convex hull is a k-gon, then it can be divided into $k-2$ triangles each containing exactly one chosen point; and since no three of the points are collinear, the sides and diagonals of the convex hull contain no chosen points on their interiors, giving $N = 2k - 2$.

Now we construct, by induction on $k \geq 3$, a convex k-gon with a set S of $k-2$ points inside such that each triangle formed by vertices of the k-gon contains exactly one point of S in its interior. The case $k = 3$ is easy. Now, assume we have a k-gon $P_1 P_2 \ldots P_k$ and a set S. Certainly we can choose Q such that $P_1 P_2 \ldots P_k Q$ is a convex $(k+1)$-gon. Let

R move along the line segment from P_k to Q. Initially (at $R = P_k$), for any indices $1 \le i < j < k$, the triangle P_iP_jR internally contains a point of S by assumption; if R is moved a sufficiently small distance d_{ij}, this point still lies inside triangle P_iP_jR. Now fix a position of R such that P_kR is less than the minimum d_{ij}; $P_1P_2 \dots P_kR$ is a convex $(k+1)$-gon. Let P be an interior point of the triangle bounded by lines P_1P_k, RP_{k-1}, P_kR. We claim the polygon $P_1P_2 \dots P_kR$ and the set $S \cup \{P\}$ satisfy our condition. If we choose three of the P_i, they form a triangle containing a point of S by hypothesis, and no others; any triangle P_iP_jR $(i, j < k)$ contains only the same internal point as triangle $P_iP_jP_k$; and each triangle P_iP_kR contains only P. This completes the induction step.

1.10 Iran

First Round

Problem 1 Suppose that $a_1 < a_2 < \cdots < a_n$ are real numbers. Prove that

$$a_1a_2^4 + a_2a_3^4 + \cdots + a_na_1^4 \geq a_2a_1^4 + a_3a_2^4 + \cdots + a_1a_n^4.$$

First Solution: We prove the claim by induction on n. For $n = 2$, the two sides are equal; now suppose the claim is true for $n-1$, i.e.,

$$a_1a_2^4 + a_2a_3^4 + \cdots + a_{n-1}a_1^4 \geq a_2a_1^4 + a_3a_2^4 + \cdots + a_1a_{n-1}^4.$$

Then the claim for n will follow from the inequality

$$a_{n-1}a_n^4 + a_na_1^4 - a_{n-1}a_1^4 \geq a_na_{n-1}^4 + a_1a_n^4 - a_1a_{n-1}^4$$

(Notice that this is precisely the case for $n = 3$.) Without loss of generality, suppose $a_n - a_1 = 1$; otherwise, we can divide each of a_1, a_{n-1}, a_n by $a_n - a_1 > 0$ without affecting the truth of the inequality. Then by Jensen's inequality for the convex function x^4, we have

$$a_1^4(a_n - a_{n-1}) + a_n^4(a_{n-1} - a_1) \geq (a_1(a_n - a_{n-1}) + a_n(a_{n-1} - a_1))^4$$
$$= (a_{n-1}(a_n - a_1))^4 = a_{n-1}^4(a_n - a_1),$$

which rearranges to yield our desired inequality.

Second Solution: We use an elementary method to prove the case $n = 3$. Define

$$p(x, y, z) = xy^4 + yz^4 + zx^4 - yx^4 - zy^4 - xz^4.$$

We wish to prove that $p(x, y, z) \geq 0$ when $x \leq y \leq z$. Since $p(x, x, z) = p(x, y, y) = p(z, y, z) = 0$, we know that $(y - x)(z - y)(z - x)$ divides $p(x, y, z)$. In fact,

$$\begin{aligned}
p(x, y, z) &= yz^4 - zy^4 + zx^4 - xz^4 + xy^4 - yx^4 \\
&= zy(z^3 - y^3) + xz(x^3 - z^3) + xy(y^3 - x^3) \\
&= zy(z^3 - y^3) + xz(y^3 - z^3) + xz(x^3 - y^3) + xy(y^3 - x^3) \\
&= z(y - x)(z^3 - y^3) + x(z - y)(x^3 - y^3)
\end{aligned}$$

$$= (y-x)(z-y)\left(z(z^2+zy+y^2) - x(x^2+xy+y^2)\right)$$

$$= (y-x)(z-y)\left((z^3-x^3) + y^2(z-x) + y(z^2-x^2)\right)$$

$$= (y-x)(z-y)(z-x)\left(z^2+zx+x^2+y^2+yz+yx\right)$$

$$= \frac{1}{2}(y-x)(z-y)(z-x)\left((x+y)^2+(y+z)^2+(z+x)^2\right) \geq 0,$$

as desired.

Problem 2 Suppose that n is a positive integer. The n-tuple $(a_1, \ldots, a_n)$ of positive integers is said to be *good* if $a_1 + \cdots + a_n = 2n$ if for every k between 1 and n, no k of the n integers add up to n. Find all n-tuples that are good.

First Solution: Call an n-tuple of positive integers *proper* if the integers add up to $2n$.

Without loss of generality, we suppose that $a_1 \leq a_2 \leq \cdots \leq a_n = M$. If $M \leq 2$ then $a_1 = \cdots = a_n = 2$ and this leads to the solution $(2, 2, \ldots, 2)$ for odd n. Now we suppose that $M \geq 3$. Since the average of $\{a_1, \ldots, a_n\}$ is 2, we must have $a_1 = 1$. Now say we have a proper n-tuple $S = (a_1, \ldots, a_n)$, where

$$1 = a_1 = \cdots = a_i < a_{i+1} \leq a_{i+2} \leq \cdots \leq a_n = M.$$

Lemma. *If $i \geq \max\{a_n - a_{i+1}, a_{i+1}\}$, then S is not good.*

Proof: Suppose that we have a balance and weights $a_1, a_2, \ldots, a_n$. We put a_n on the left hand side of the balance, then put a_{n-1} on the right hand side, and so on — adding the heaviest available weight to the lighter side (or if the sides are balanced, we add it to the left hand side). Before we put x_{i+1} on the balance, the difference between the two sides is between 0 and x_n. After we put x_{i+1} on the balance, the difference between the two sides is no greater than $\max\{x_n - x_{i+1}, x_{i+1}\}$. Now we have enough 1's to put on the lighter side to balance the two sides. Since the the total weight is $2n$ an even number, there will have even number of 1's left and we can split them to balance the sides, i.e., there is a subtuple A with its sum equal to n and thus S is not good. ■

Note that

$$2n = a_1 + \cdots + a_n \geq i + 2(n - i - 1) + M \iff i \geq M - 2. \quad (1)$$

Now we consider the following cases:

(i) $M = 3$ and $i = 1$. Then $S = (1, \underbrace{2, \ldots, 2}_{n-2}, 3)$. If $n = 2m$ and $m \geq 2$, then $A = (\underbrace{2, \ldots, 2}_{m})$ has sum $2m = n$ and thus S is not good; if $n = 2m + 1$ and $m \geq 1$, then $A = (\underbrace{2, \ldots, 2}_{m-1}, 3)$ has sum $2m + 1 = n$ and thus S is not good. Therefore $(1, 3)$ is the only good tuple for $M = 3, i = 1$.

(ii) $M = 3$ and $i \geq 2$. Then $a_{i+1} = 2$ or 3 and $i \geq \max\{a_n - a_{i+1}, a_{i+1}\}$ — implying that S is not good from our lemma — unless $i = 2$ and $a_{i+1} = 3$. But then $S = (1, 1, 3, 3)$, and $(1, 3)$ has sum $4 = n$.

(iii) $M \geq 4$ and $a_{i+1} = 2$. Then from (1), $i \geq M - 2 = \max\{a_n - a_{i+1}, a_{i+1}\}$. By the lemma S is not good.

(iv) $M \geq 4$, $a_{i+1} > 2$, and $i + 1 \neq n$. Since $a_{i+1} \neq 2$ equality does not hold in (1), and thus $i \geq M - 1 \geq \max\{a_n - a_{i+1}, a_{i+1}\}$ (and S is not good) unless $a_{i+1} = M$. In this case, $S = (1, \ldots, 1, M, \ldots, M)$. Note that

$$2n = i + (n - i)M \geq i + 4(n - i)$$

so that $i \geq \frac{2}{3}n$. Then the remaining $n - i \geq 2$ values M have sum at most $\frac{4}{3}n$, and hence $M \leq \frac{2}{3}n$. Thus

$$i \geq 2n/3 \geq M = \max\{a_n - a_{i+1}, a_{i+1}\},$$

and by the lemma S is not good.

(v) $M \geq 4$ and $i+1 = n$. Then we have the good n-tuple $(1, \ldots, 1, n+1)$.

Therefore the only possible good n-tuples are $(1, 1, \ldots, 1, n+1)$ and $(2, 2, \ldots, 2)$, and the second n-tuple is good if and only if n is odd.

Second Solution: Say a proper n-tuple has "subsum" m if some k of the integers $(0 \leq k \leq n)$ add up to m.

Lemma. *Every proper n-tuple besides* $(2,\ 2,\ \ldots,\ 2)$ *has subsums* 0, $1,\ \ldots,\ n-1$ *(and possibly others). Furthermore, if a proper n-tuple besides* $(2, 2, \ldots, 2)$ *contains a* 2, *it has subsum n as well.*

Proof: If $n = 1$ the claim is trivial. Now assume the claims are true for $n - 1$; we prove each is true for n as well. Suppose we have an n-tuple N besides $(2, 2, \ldots, 2)$.

If there is a 2 in the n-tuple, then the other $n - 1$ integers form a proper $(n-1)$-tuple besides $(2, 2, \ldots, 2)$. By the induction hypothesis, this $(n-1)$-tuple has subsums $0, \ldots, n-2$. Remembering the original 2, our complete n-tuple N has subsums $0, \ldots, n$.

Otherwise, suppose there is no 2 in N. We prove by induction on $k < n$ that there is a subtuple A_k of k numbers with subsums $0, \ldots, k$. Since the average value of the integers in N is 2, we must have at least one 1 in N, proving the case for $k = 1$. Now assume the claim is true for $k - 1$; the elements of A_{k-1} are each at least 1, so they add up to at least $k - 1$. Then the inequality

$$(k-1)(k-n) < 0$$

implies

$$2n < (k+1)(n-k+1) + (k-1),$$

so that the average value of $N \setminus A_{k-1}$ is less than $k + 1$. Therefore at least one of the other $n - k + 1$ integers x is at most k, implying that $A_{k-1} \cup (x)$ has subsums $0, \ldots, k$. This completes the inductive step and the proof of our lemma. ■

From our lemma, the only possible good n-tuple with a 2 is $(2, 2, \ldots, 2)$, and this is good if and only if n is odd. Every other n-tuple N has a subtuple A_{n-1} of $n - 1$ integers with subsums $0, \ldots, n-1$. If these integers are not all 1, then the remaining integer is at most n and N must have subsum n. Therefore, the only other possible good n-tuple is $(1, 1, \ldots, 1, n+1)$, which is indeed always a good n-tuple.

Third Solution: Suppose we have a good n-tuple $(a_1, \ldots, a_n)$, and consider the sums $a_1,\ a_1 + a_2,\ \ldots,\ a_1 + a_2 + \cdots + a_{n-1}$. All these sums are between 0 and $2n$ exclusive; thus if any of the sums is 0 (mod n), it equals n and we have a contradiction. Also, if any two are congruent modulo n, we can subtract these two sums to obtain another partial sum that equals n, a contradiction again. Therefore, the sums must all be nonzero and distinct modulo n.

Specifically, $a_2 \equiv a_1 + \cdots + a_k \pmod{n}$ for some $k \geq 1$. If $k > 1$ then we can subtract a_2 from both sides to find a partial sum that equals n. Therefore $k = 1$ and $a_1 \equiv a_2 \pmod{n}$. Similarly, all the a_i are congruent modulo n. From here, easy algebra shows that the presented solutions are the only ones possible.

Problem 3 Let I be the incenter of triangle ABC and let AI meet the circumcircle of ABC at D. Denote the feet of the perpendiculars from I to BD and CD by E and F, respectively. If $IE+IF = AD/2$, calculate $\angle BAC$.

Solution: A well-known fact we will use in this proof is that $DB = DI = DC$. In fact, $\angle BDI = \angle C$ gives $\angle DIB = (\angle A + \angle B)/2$ while $\angle IBD = (\angle A + \angle B)/2$. Thus $DB = DI$, and similarly $DC = DI$.

Let $\theta = \angle BAD$. Then

$$\begin{aligned}\frac{1}{4} ID \cdot AD &= \frac{1}{2} ID \cdot (IE + IF)\\ &= \frac{1}{2} BD \cdot IE + \frac{1}{2} CD \cdot IF = [BID] + [DIC]\\ &= \frac{ID}{AD}([BAD] + [DAC]) = \frac{1}{2} ID \cdot (AB + AC) \cdot \sin\theta,\end{aligned}$$

whence $\frac{AD}{AB+AC} = 2\sin\theta$.

Let X be the point on $\overrightarrow{AB}$ different from A such that $DX = DA$. Since $\angle XBD = \angle DCA$ and $\angle DXB = \angle XAD = \angle DAC$, we have $\triangle XBD \cong \triangle ACD$, and $BX = AC$. Then $2\sin\theta = \frac{AD}{AB+AC} = \frac{AD}{AB+BX} = \frac{AD}{AX} = \frac{1}{2\cos\theta}$, so that $2\sin\theta\cos\theta = \frac{1}{2}$, and $\angle BAC = 2\theta = 30^\circ$ or 150°.

Problem 4 Let ABC be a triangle with $BC > CA > AB$. Choose points D on $\overline{BC}$ and E on $\overrightarrow{BA}$ such that

$$BD = BE = AC.$$

The circumcircle of triangle BED intersects $\overline{AC}$ at P and the line BP intersects the circumcircle of triangle ABC again at Q. Prove that $AQ + QC = BP$.

First Solution: Except where indicated, all angles are directed modulo 180°.

Let Q' be the point on line BP such that $\angle BEQ' = \angle DEP$. Then

$$\angle Q'EP = \angle AED - \angle BEQ' + \angle DEP = \angle BED.$$

Since $BE = BD$, $\angle BED = \angle EDB$; since $BEPD$ is cyclic, $\angle EDB = \angle EPB$. Therefore $\angle Q'EP = \angle EPB = \angle EPQ'$ and $Q'P = Q'E$.

Since $BEPD$ and $BAQC$ are cyclic, we have

$$\angle BEQ' = \angle DEP = \angle DBP = \angle CAQ,$$

$$\angle Q'BE = \angle QBA = \angle QCA.$$

Combining this with $BE = AC$ yields that triangles EBQ' and ACQ are congruent. Thus $BQ' = QC$ and $EQ' = AQ$. Therefore

$$AQ + QC = EQ' + BQ' = PQ' + BQ',$$

which equals BP if Q' is between B and P.

Since E is on $\overrightarrow{BA}$ and P is on $\overrightarrow{CA}$, E and P are on the same side of $\overline{BC}$ and thus $\overline{BD}$. And since D is on $\overrightarrow{BC}$ and P is on $\overrightarrow{AC}$, D and P are on the same side of $\overline{BA}$ and thus $\overline{BE}$. Thus, $BEPD$ is cyclic in that order and (using undirected angles) $\angle BEQ' = \angle DEP < \angle BEP$. It follows that Q' lies on segment BP, as desired.

Second Solution: Since $BEPD$ and $BAQC$ are cyclic, we have

$$\angle PED = \angle PBD = \angle QBC = \angle QAC$$

and

$$\angle EPD = \pi - \angle DBE = \pi - \angle CBA = \angle AQC,$$

which together imply $\triangle PED \sim \triangle QAC$. Then

$$\frac{AC \cdot EP}{DE} = AQ$$

and

$$\frac{AC \cdot PD}{DE} = QC.$$

As in the first solution, $BEPD$ is cyclic in that order, so Ptolemy's Theorem implies that

$$BD \cdot EP + BE \cdot PD = BP \cdot DE.$$

Substituting $BD = BE = AC$ we have

$$\frac{AC \cdot EP}{DE} + \frac{AC \cdot PD}{DE} = BP,$$

or $AQ + QC = BP$, as desired.

Problem 5 Suppose that n is a positive integer and let

$$d_1 < d_2 < d_3 < d_4$$

be the four smallest positive integer divisors of n. Find all integers n such that

$$n = d_1^2 + d_2^2 + d_3^2 + d_4^2.$$

Solution: The answer is $n = 130$. Note that $x^2 \equiv 0 \pmod 4$ when x is even, and $1 \pmod 4$ when x is odd.

If n is odd, then all the d_i are odd and $n \equiv d_1^2 + d_2^2 + d_3^2 + d_4^2 \equiv 1 + 1 + 1 + 1 \equiv 0 \pmod 4$, a contradiction. Thus $2 \mid n$.

If $4 \mid n$ then $d_1 = 1$ and $d_2 = 2$, and $n \equiv 1 + 0 + d_3^2 + d_4^2 \not\equiv 0 \pmod 4$, a contradiction. Thus $4 \nmid n$.

Therefore $\{d_1, d_2, d_3, d_4\} = \{1, 2, p, q\}$ or $\{1, 2, p, 2p\}$ for some odd primes p, q. In the first case $n \equiv 3 \pmod 4$, a contradiction. Thus $n = 5(1 + p^2)$ and $5 \mid n$, so $p = d_3 = 5$ and $n = 130$.

Problem 6 Suppose that $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$ are two 0-1 sequences. The difference $d(A, B)$ between A and B is defined to be the number of i's for which $a_i \neq b_i$ $(1 \leq i \leq n)$. Suppose that A, B, C are three 0-1 sequences and that $d(A, B) = d(A, C) = d(B, C) = d$.

(a) Prove that d is even.

(b) Prove that there exists an 0-1 sequence D such that

$$d(D, A) = d(D, B) = d(D, C) = \frac{d}{2}.$$

Solution:

(a) Modulo 2, we have

$$\begin{aligned} d(A, B) &= (a_1 - b_1) + (a_2 - b_2) + \cdots + (a_n - b_n) \\ &\equiv (a_1 + a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_n). \end{aligned}$$

Thus,

$$3d \equiv d(A, B) + d(B, C) + d(C, A) = 2(\sum a_i + \sum b_i + \sum c_i),$$

so d must be divisible by 2.

(b) Define D as follows: for each i, if $a_i = b_i = c_i$, then let $d_i = a_i = b_i = c_i$. Otherwise, two of a_i, b_i, c_i are equal; let d_i equal that value. We claim this sequence D satisfies the requirements.

Let α be the number of i for which $a_i \neq b_i$ and $a_i \neq c_i$ (that is, for which a_i is "unique"). Define β and γ similarly, and note that $d(A, D) = \alpha$, $d(B, D) = \beta$, and $d(C, D) = \gamma$. We also have

$$d = d(A, B) = \alpha + \beta$$
$$d = d(B, C) = \beta + \gamma$$
$$d = d(C, A) = \gamma + \alpha.$$

Thus, $\alpha = \beta = \gamma = \frac{d}{2}$, as desired.

Second Round

Problem 1 Define the sequence $\{x_n\}_{n \geq 0}$ by $x_0 = 0$ and

$$x_n = \begin{cases} x_{n-1} + \dfrac{3^{r+1} - 1}{2}, & \text{if } n = 3^r(3k + 1), \\ \\ x_{n-1} - \dfrac{3^{r+1} + 1}{2}, & \text{if } n = 3^r(3k + 2), \end{cases}$$

where k and r are nonnegative integers. Prove that every integer appears exactly once in this sequence.

First Solution: We prove by induction on $t \geq 1$ that

(i) $\{x_0, x_1, \ldots, x_{3^t - 2}\} = \left\{-\frac{3^t - 3}{2}, -\frac{3^t - 1}{2}, \ldots, \frac{3^t - 1}{2}\right\}$.

(ii) $x_{3^t - 1} = -\frac{3^t - 1}{2}$.

These claims imply the desired result, and they are easily verified for $t = 1$. Now supposing they are true for t, we show they are true for $t + 1$.

For any positive integer m, write $m = 3^r(3k + s)$ for nonnegative integers r, k, s, with $s \in \{1, 2\}$; and define $r_m = r$ and $s_m = s$.

Then for $m < 3^t$, observe that

$$r_m = r_{m+3^t} = r_{m+2\cdot 3^t}$$
$$s_m = s_{m+3^t} = r_{m+2\cdot 3^t},$$

so that

$$x_m - x_{m-1} = x_{3^t+m} - x_{3^t+m-1} = x_{2\cdot 3^t+m} - x_{2\cdot 3^t+m-1}.$$

Adding these equations from $m = 1$ to $m = k < 3^t$, we have

$$x_k = x_{3^t+k} - x_{3^t}$$

$$x_k = x_{2\cdot 3^t+k} - x_{2\cdot 3^t}.$$

Now, setting $n = 3^t$ in the recursion and using (ii) from the induction hypothesis, we have $x_{3^t} = 3^t$ — and

$$\{x_{3^t}, \ldots, x_{2\cdot 3^t-2}\} = \left\{\tfrac{3^t+3}{2}, \ldots, \tfrac{3^{t+1}-1}{2}\right\}$$

$$x_{2\cdot 3^t-1} = \tfrac{3^t+1}{2}.$$

Then setting $n = 2 \cdot 3^t$ in the recursion we have $x_{2\cdot 3^t} = -3^t$ — giving

$$\{x_{2\cdot 3^t}, \ldots, x_{3^{t+1}-2}\} = \left\{-\tfrac{3^{t+1}-3}{2}, \ldots, -\tfrac{3^t+1}{2}\right\}$$

$$x_{2\cdot 3^{t+1}-1} = -\tfrac{3^{t+1}-1}{2}.$$

Combining this with (i) and (ii) from the induction hypothesis proves the claims for $t + 1$. This completes the proof.

Second Solution: For $n_i \in \{-1, 0, 1\}$, let the number

$$[n_m n_{m-1} \cdots n_0]$$

in "base $\bar{3}$" equal $\sum_{i=0}^{m} n_i \cdot 3^i$. It is simple to prove by induction on k that the base $\bar{3}$ numbers with at most k digits equal

$$\left\{-\frac{3^k-1}{2}, -\frac{3^k-3}{2}, \ldots, \frac{3^k-1}{2}\right\},$$

which implies every integer has a unique representation in base $\bar{3}$.

Now we prove by induction on n that if $n = a_m a_{m-1} \ldots a_0$ in base 3, then $x_n = [b_m b_{m-1} \ldots b_0]$ in base $\bar{3}$, where $b_i = -1$ if $a_i = 2$ and $b_i = a_i$ for all other cases.

For the base case, $x_0 = 0 = [0]$. Now assume the claim is true for $n - 1$. First suppose that $n = 3^r(3k + 1)$. Then

$$n = a_m a_{m-1} \ldots a_i \; 1 \; \underbrace{0\,0 \ldots\, 0}_{r}$$

$$\frac{3^{r+1}-1}{2} = \underbrace{1\,1 \ldots 1}_{r+1} = [\underbrace{1\,1 \ldots 1}_{r+1}]$$

$$n - 1 = a_m a_{m-1} \ldots a_i \; 0 \; \underbrace{2\,2 \ldots 2}_{r}$$

$$x_{n-1} = [b_m b_{m-1} \dots b_i \ 0 \ \underbrace{-1\,-1 \ \dots \ -1}_{r}],$$

so that

$$x_n = [b_m b_{m-1} \dots b_i \ 0 \ \underbrace{-1\,-1 \ \dots \ -1}_{r}] + [\underbrace{1\ 1 \dots 1}_{r+1}]$$
$$= [b_m b_{m-1} \dots b_i \ 1 \ \underbrace{0\ 0 \ \dots \ 0}_{r}].$$

Now suppose that $n = 3^r(3k+2)$. Then

$$n = a_m a_{m-1} \dots a_i \ 2 \ \underbrace{0\ 0 \dots 0}_{r}$$
$$n - 1 = a_m a_{m-1} \dots a_i \ 1 \ \underbrace{2\ 2 \dots 2}_{r}$$
$$x_{n-1} = [b_m b_{m-1} \dots b_i \ 1 \ \underbrace{-1\,-1 \ \dots \ -1}_{r}].$$

Also,

$$-\frac{3^{r+1}+1}{2} = -(\underbrace{1\ 1 \dots 1}_{r} \ 2)$$
$$= -3^r - 3^{r-1} - \dots - 3 - 2$$
$$= -3^{r+1} + 3^r + 3^{r-1} + \dots + 3 + 1$$
$$= [-1 \ \underbrace{1\ 1 \dots 1}_{r+1}].$$

Therefore

$$x_n = [b_m b_{m-1} \dots b_i \ 1 \ \underbrace{-1\,-1 \ \dots \ -1}_{r}] + [-1 \ \underbrace{1\ 1 \dots 1}_{r+1}]$$
$$= [b_m b_{m-1} \dots b_i \ -1 \ \underbrace{0\ 0 \ \dots \ 0}_{r}].$$

In either case, the claim is true for n, completing the induction.

And since all integers appear exactly once in base $\bar{3}$, they appear exactly once in $\{x_n\}_{n \geq 0}$, as desired.

Problem 2 Suppose that $n(r)$ denotes the number of points with integer coordinates on a circle of radius $r > 1$. Prove that

$$n(r) < 6\sqrt[3]{\pi r^2}.$$

Solution: Consider a circle of radius r containing n lattice points; we must prove that $n < 6\sqrt[3]{\pi r^2}$.

Since $r > 1$ and $6\sqrt[3]{\pi} > 8$, we may assume $n > 8$. Label the n lattice points on the circle $P_1, P_2, \ldots, P_n$ in counterclockwise order. Since the sum of the (counterclockwise) arcs P_1P_3, P_2P_4, P_nP_2 is 4π, one of the arcs P_iP_{i+2} has measure at most $\frac{4\pi}{n}$; assume without loss of generality it is arc P_1P_3.

Consider a triangle ABC inscribed in an arc of angle $\frac{4\pi}{n}$; clearly its area is maximized by moving A and C to the endpoints of the arc and then moving B to the midpoint (where the distance to line AC is greatest). Then $\angle CAB = \angle BCA = \frac{\pi}{n}$ and $\angle ABC = 180^\circ - \frac{2\pi}{n}$, so

$$\begin{aligned}
[ABC] &= \frac{abc}{4r} = \frac{(2r\sin\frac{\pi}{n})(2r\sin\frac{2\pi}{n})(2r\sin\frac{\pi}{n})}{4r} \\
&\le \frac{(2r\frac{\pi}{n})(2r\frac{2\pi}{n})(2r\frac{\pi}{n})}{4r} \\
&= \frac{4r^2\pi^3}{n^3}.
\end{aligned}$$

Since triangle $P_1P_2P_3$ is inscribed in an arc of measure $\frac{4\pi}{n}$, by the preceding argument, $[P_1P_2P_3] \le \frac{4r^2\pi^3}{n^3}$. But since P_1, P_2, and P_3 are lattice points, the area $[P_1P_2P_3]$ is at least $\frac{1}{2}$ (this can be proven by either Pick's Formula $K = I + \frac{1}{2}B - I$ or the "determinant formula" $K = \frac{1}{2}\left|x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3\right|$). Therefore,

$$\begin{aligned}
&\frac{1}{2} \le [P_1P_2P_3] \le \frac{4r^2\pi^3}{n^3} \\
&\Longrightarrow n^3 \le 8r^2\pi^3 \\
&\Longrightarrow n \le \sqrt[3]{8r^2\pi^3} = 2\pi\sqrt[3]{r^2} < 6\sqrt[3]{\pi r^2},
\end{aligned}$$

as desired.

Problem 3 Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y$$

for all $x, y \in \mathbb{R}$.

Solution: Let $(x, y) = (x, x^2)$. Then

$$f(f(x) + x^2) = f(0) + 4x^2 f(x). \qquad (1)$$

Let $(x, y) = (x, -f(x))$. Then

$$f(0) = f(x^2 + f(x)) - 4f(x)^2. \qquad (2)$$

Adding (1) and (2) gives $4f(x)(f(x) - x^2) = 0$. This implies that for each individual x, either $f(x) = 0$ or $f(x) = x^2$. (Alternatively, plugging $y = \frac{x^2 - f(x)}{2}$ into the original equation also yields this result.) Clearly $f(x) = 0$ and $f(x) = x^2$ satisfy the given equation; we now show that f cannot equal some combination of the two functions.

Suppose that there is an $a \neq 0$ such that $f(a) = 0$. Plugging in $x = a$ into the original equation, we have

$$f(y) = f(a^2 - y).$$

If $y \neq \frac{a^2}{2}$, then $y^2 \neq (a^2 - y)^2$ so $f(y) = f(a^2 - y) = 0$. Thus $f(y) = 0$ for all $y \neq \frac{a^2}{2}$. And by choosing $x = 2a$ or some other value in the original equation, we can similarly show that $f(\frac{a^2}{2}) = 0$.

Therefore $f(x) = 0$ for all x or $f(x) = x^2$ for all x, as claimed.

Problem 4 In triangle ABC, the angle bisector of $\angle BAC$ meets BC at D. Suppose that ω is the circle which is tangent to BC at D and passes through A. Let M be the second point of intersection of ω and AC. Let P be the second point of intersection of ω and BM. Prove that P lies on a median of triangle ABD.

Solution: Extend $\overline{AP}$ to meet $\overline{BD}$ at E. We claim that $BE = ED$ and thus $\overline{AP}$ is a median of triangle ABD, as desired. In fact,

$$BE = ED \iff BE^2 = ED^2 = EP \cdot EA$$

$$\iff \triangle BEP \sim \triangle AEB \iff \angle EBP = \angle BAE.$$

Let N be the second intersection of ω with AB. Using directed angles and arc measures, since $\overline{AD}$ bisects the angle between lines AN and AC, we have $\widehat{DM} = \widehat{ND}$ and

$$\angle BAE = \angle NAP = \frac{\widehat{ND} - \widehat{PD}}{2} = \frac{\widehat{DM} - \widehat{PD}}{2} = \angle DBM = \angle EBP,$$

as desired.

Problem 5 Let ABC be a triangle. If we paint the points of the plane in red and green, prove that either there exist two red points which are one unit apart or three green points forming a triangle congruent to ABC.

First Solution: We call a polygon or a segment green (red) if the vertices of the polygon or the segment are all green (red).

Suppose that there is no red unit segment. We prove that there is a green triangle congruent to triangle ABC. If the whole plane is green, the proof is trivial.

Now we further suppose that there is a red point R on the plane. We claim that there is a green equilateral triangle with unit side length. In fact, let ω be the circle with center R and radius $\sqrt{3}$. Then ω is not all red, since otherwise we could find a red unit segment. Let G be a green point on ω. Let ω_1 and ω_2 be two unit circles centered at R and G, respectively, and let ω_1 and ω_2 meet at P and Q. Then both P and Q must be green and triangle PQG is a green unit equilateral triangle.

Let $G_1G_2G_3$ be a green unit equilateral triangle. Construct a triangle $G_1X_1Y_1$ that is congruent to triangle ABC. If both X_1 and Y_1 are green, we are done. Without loss of generality, we assume that $X_1\ Y_1$ is red. Translate triangle $G_1G_2G_3$ by $\overrightarrow{G_1Y_1}$ to obtain triangle $Y_1Y_2Y_3$. Then both Y_2 and Y_3 are green. Similarly, translate triangle $G_1G_2G_3$ by $\overrightarrow{G_1X_1}$ to obtain triangle $X_1X_2X_3$. Then at least one of X_2 and X_3 is green (since X_2X_3 cannot be a red unit segment). Without loss of generality, say X_2 is green. Now triangle $G_2X_2Y_2$ is a green triangle and congruent to triangle $G_1X_1Y_1$ (translated by $\overrightarrow{G_1G_2}$) and thus congruent to triangle ABC, as desired.

Second Solution: Suppose by way of contradiction there were no such red or green points, and say the sides of triangle ABC are a, b, and c.

First we prove no red segment has length a. If XY were a red segment of length a, then the unit circles around X and Y must be completely green. Now draw Z so that $\triangle XYZ \cong \triangle ABC$; the unit circle around Z must be completely *red*, or else it would form an illegal triangle with the corresponding points around X and Y. But on this unit circle we can find a red unit segment, a contradiction.

Now, the whole plane cannot be green so there must be some red point R. The circle ω around R with radius a must be completely green. Then pick two points D, E on ω with $DE = a$, and construct F outside ω so that $\triangle DEF \cong \triangle ABC$ (we can do this since $a \le b, c$); F must be red. Thus if we rotate DE around R, F forms a completely red circle of radius greater than a — and on this circle we can find two red points distance a apart, a contradiction.

Third Round

Problem 1 Suppose that $S = \{1, 2, \ldots, n\}$ and that $A_1, A_2, \ldots, A_k$ are subsets of S such that for every $1 \le i_1, i_2, i_3, i_4 \le k$, we have

$$|A_{i_1} \cup A_{i_2} \cup A_{i_3} \cup A_{i_4}| \le n - 2.$$

Prove that $k \le 2^{n-2}$.

Solution: For a set T, let $|T|$ denote the numbers of elements in T. We call a set $T \subset S$ *2-coverable* if $T \subseteq A_i \cup A_j$ for some i and j (not necessarily distinct). Among the subsets of S that are not 2-coverable, let A be a subset with minimum $|A|$.

Consider the family of sets $S_1 = \{A \cap A_1, A \cap A_2, \ldots, A \cap A_k\}$. ($A \cap A_i$ might equal $A \cap A_j$, but we ignore any duplicate sets.) Since A is not 2-coverable, if $X \in S_1$, then $A - X \notin S_1$. Thus at most half the subsets of $|A|$ are in S_1, and $|S_1| \le 2^{|A|-1}$.

On the other hand, let $B = S - A$ and consider the family of sets $S_2 = \{B \cap A_1, B \cap A_2, \ldots, B \cap A_k\}$. We claim that if $X \in S_2$, then $B - X \notin S_2$. Suppose on the contrary that both $X, B - X \in S_2$ for some $X = B \cap A_\ell$ and $B - X = B \cap A_{\ell'}$. By the minimal definiton of A there are A_i and A_j such that $A_i \cup A_j = A \setminus \{m\}$ for some i, j, and m. Then

$$|A_\ell \cup A_{\ell'} \cup A_i \cup A_j| = n - 1,$$

a contradiction. Thus we assumption is false and $|S_2| \le 2^{|B|} - 1 = 2^{n-|A|-1}$.

Since every set A_i is uniquely determined by its intersection with sets A and $B = S - A$, it follows that $|A| \le |B| \cdot |C| \le 2^{n-2}$.

Problem 2 Let ABC be a triangle and let ω be a circle passing through A and C. Sides AB and BC meet ω again at D and E, respectively. Let γ be the incircle of the circular triangle EBD and let S be its center. Suppose that γ touches the arc DE at M. Prove that the angle bisector of $\angle AMC$ passes through the incenter of triangle ABC.

First Solution: We work backward. Let I be the incenter of triangle ABC. Let N be the midpoint of arc AC on ω that is opposite to B, and let $\overrightarrow{NI}$ meet with ω again at M'. Then $\overline{M'N}$ bisects $\angle AM'C$. We claim that γ is tangent to ω at M', and our desired results follows.

To prove our claim, we are going to do some heavy trigonometry calculations. Let ℓ be the line tangent to ω at M', and let ℓ meet AB and AC at P and Q, respectively. Let ω' be the incircle of triangle PBQ. We are reduced to proving that ω' is tangent to $\overline{PQ}$ at M'.

Let $\angle IAM' = a, \angle M'AB = x, \angle M'CI = b, \angle BCM' = y$. Then $\angle CAI = \angle IAB = x + a$ and $\angle ICA = \angle BCI = y + b$. Since PM' is tangent to ω, we have $\angle PM'A = \angle M'CA = 2b + y$ and thus $\angle BPQ = \angle PM'A + \angle M'AP = 2b + x + y$. Applying the law of sines to triangle PAM', we have

$$\frac{PM'}{\sin x} = \frac{AM'}{\sin(2b + x + y)} \iff PM' = \frac{AM' \sin x}{\sin(2b + x + y)}. \qquad (1)$$

Similarly,

$$M'Q = \frac{M'C \sin y}{\sin(2a + x + y)}. \qquad (2)$$

And, applying the law of sines to triangle $AM'C$ gives

$$\frac{AM'}{\sin(2b + y)} = \frac{M'C}{\sin(2a + x)}. \qquad (3)$$

Combining (1), (2), and (3) we have

$$\frac{PM'}{M'Q} = \frac{\sin x \sin(2b + y) \sin(2a + x + y)}{\sin y \sin(2a + x) \sin(2b + x + y)}. \qquad (4)$$

Now, observe that for a triangle XYZ with inradius r, and with incircle Γ touching XY at T, we have

$$\frac{XT}{TY} = \frac{r \cot \frac{\angle X}{2}}{r \cot \frac{\angle Y}{2}} = \frac{\cot \frac{\angle X}{2}}{\cot \frac{\angle Y}{2}}.$$

Thus it suffices to prove that $\frac{PM'}{M'Q} = \frac{\cot \angle BPQ}{\cot \angle BQP}$, or equivalently (from (4)) any of the following statements:

$$\frac{\sin x \sin(2b + y) \sin(2a + x + y)}{\sin y \sin(2a + x) \sin(2b + x + y)} = \frac{\cot \frac{2b+x+y}{2}}{\cot \frac{2a+x+y}{2}}$$

$$\frac{\sin x \sin(2b + y) \sin \frac{2a+x+y}{2} \cos \frac{2a+x+y}{2}}{\sin y \sin(2a + x) \sin \frac{2b+x+y}{2} \cos \frac{2b+x+y}{2}} = \frac{\cos \frac{2b+x+y}{2} \sin \frac{2a+x+y}{2}}{\cos \frac{2a+x+y}{2} \sin \frac{2b+x+y}{2}}$$

$$\sin x \sin(2b+y)\cos^2\frac{2a+x+y}{2} = \sin y \sin(2a+x)\cos^2\frac{2b+x+y}{2}$$

$$\begin{aligned} &\sin x \sin(2b+y)(\cos(2a+x+y)+1) \\ &= \sin y \sin(2a+x)(\cos(2b+x+y)+1), \end{aligned}$$

or equivalently that $L = R$ where

$$\begin{aligned} R &= \sin x \sin(2b+y)\cos(2a+x+y) \\ &\quad - \sin y \sin(2a+x)\cos(2b+x+y), \\ L &= \sin y \sin(2a+x) - \sin x \sin(2b+y). \end{aligned}$$

Note that

$$\begin{aligned} R &= 1/2[\ \sin x\,(\sin(2b+2y+x+2a) + \sin(2b-2a-x))\ ] \\ &\quad -1/2[\ \sin y\,(\sin(2b+2x+y+2a) + \sin(2a-2b-y))\ ] \\ &= -1/4[\ \cos(2a+2b+2x+2y) - \cos(2a+2b+2y) \\ &\qquad + \cos(2b-2a) - \cos(2b-2a-2x) \\ &\qquad - \cos(2a+2b+2x+2y) + \cos(2b+2a+2x) \\ &\qquad - \cos(2b-2a) + \cos(2a-2b-2y)\ ] \\ &= -1/4[\ \cos(2a+2b+2x) - \cos(2b-2a-2x) \\ &\qquad + \cos(2a-2b-2y) - \cos(2a+2b+2y)\ ] \\ &= 1/2[\ \sin(2b)\sin(2a+2x) - \sin(2a)\sin(2b+2y)\ ] \\ &= 2[\ \underbrace{\sin b \sin(a+x)}_{(i)} \cos b \cos(a+x) \\ &\qquad - \underbrace{\sin a \sin(b+y)}_{(ii)} \cos a \cos(b+y)\ ]. \end{aligned}$$

To simplify this expression, note that $\overline{MI}, \overline{AI}$, and $\overline{CI}$ meet at I and that $\angle IMA = \angle IMC$. Then the trigonometric form of Ceva's Theorem gives

$$\begin{aligned} &\frac{\sin\angle IMA \sin\angle IAC \sin\angle ICM}{\sin\angle IAM \sin\angle ICA \sin\angle IMC} = 1 \\ &\iff \sin a \sin(b+y) = \sin b \sin(a+x). \end{aligned}$$

Swapping quantities (i) and (ii) above thus yields

$$\begin{aligned}
R &= 2\left(\sin a \sin(b+y)\cos b\cos(a+x)\right.\\
&\qquad \left. - \sin b \sin(a+x)\cos a\cos(b+y)\right)\\
&= \cos b \sin(b+y)[\sin(2a+x) - \sin x]\\
&\qquad - \sin b\cos(b+y)[\sin(2a+x) + \sin x]\\
&= \sin(2a+x)[\ \sin(b+y)\cos b - \sin b\cos(b+y)\ ]\\
&\quad - \sin x[\ \sin(b+y)\cos b + \sin b\cos(b+y)\ ]\\
&= \sin(2a+x)\sin y - \sin x\sin(2b+y) = L,
\end{aligned}$$

as desired.

Second Solution: Let O and R be the center and radius of ω, r be the radius of γ, and I be the incenter of triangle ABC. Extend lines AI and CI to hit ω at M_A and M_C respectively; also let line AD be tangent to γ at F, and let line CE be tangent to γ at G. Finally, let d be the length of the exterior tangent from M_A to ω. Notice that since line AM_A bisects $\angle DAC$, we have $DM_A = M_AC$; similarly, $EM_C = M_CA$.

Applying Generalized Ptolemy's Theorem to the "circles" M_A, C, D, and γ externally tangent to ω gives

$$CG \cdot DM_A = M_AC \cdot DF + d \cdot CD$$
$$d^2 = M_AC^2 \left(\tfrac{CG-DF}{CD}\right)^2.$$

Note that d^2 equals the power of M_A with respect to γ, so $d^2 = M_AS^2 - r^2$.

By Stewart's Theorem on cevian M_AM in triangle SOM_A, we also have

$$M_AS^2 \cdot OM + M_AO^2 \cdot MS = M_AM^2 \cdot SO + SM \cdot MO \cdot SO$$
$$M_AS^2 \cdot R + R^2 \cdot r = M_AM^2 \cdot (R+r) + r \cdot R \cdot (R+r)$$
$$M_AM^2(R+r) = (M_AS^2 - r^2)R = d^2R,$$

Combining the two equations involving d^2, we find

$$M_AC^2\left(\frac{CG-DF}{CD}\right)^2 = \frac{M_AM^2(R+r)}{R}$$

$$\left(\frac{M_AM}{M_AC}\right)^2 = \left(\frac{R}{R+r}\right)\left(\frac{CG-DF}{CD}\right)^2.$$

Similarly,

$$\left(\frac{M_CM}{M_CA}\right)^2 = \left(\frac{R}{R+r}\right)\left(\frac{AF-EG}{AE}\right)^2.$$

But

$$CG - DF = (CG + GB) - (DF + FB) = CB - DB$$

and similarly

$$AF - EG = (AF + FB) - (EG + GB) = AB - BE.$$

Furthermore, because $ACDE$ is cyclic some angle-chasing gives $\angle BDC = \angle AEC$ and $\angle DCB = \angle BAE$, so $\triangle CBD \sim \triangle ABE$ and

$$\frac{CG-DF}{CD} = \frac{CB-DB}{CD} = \frac{AB-BE}{EA} = \frac{AF-EG}{AE}.$$

Therefore we have $\frac{M_AM}{M_AC} = \frac{M_CM}{M_CA} \Longrightarrow \frac{\sin\angle MAM_A}{\sin\angle M_AAC} = \frac{\sin\angle M_CCM}{\sin\angle ACM_C}$. But by the trigonometric form of Ceva's theorem in triangle AMC applied to lines AM_A, CM_C, and MI, we have

$$\frac{\sin\angle MAM_A}{\sin\angle M_AAC}\cdot\frac{\sin\angle ACM_C}{\sin\angle M_CCM}\cdot\frac{\sin\angle CMI}{\sin\angle IMA} = 1$$

so that

$$\sin\angle CMI = \sin\angle IMA \quad\Longrightarrow\quad \angle CMI = \angle IMA$$

since $\angle AMC < 180°$. Therefore, line MI bisects $\angle AMC$, so the angle bisector of $\angle AMC$ indeed passes through the incenter I of triangle ABC.

Problem 3 Suppose that $C_1, C_2, \ldots, C_n$ are circles of radius 1 in the plane such that no two of them are tangent and the subset of the plane formed by the union of these circles is connected (i.e., for any partition of $\{1, 2, \ldots, n\}$ into nonempty subsets A and B, $\bigcup_{a\in A} C_a$ and $\bigcup_{b\in B} C_b$ are not disjoint). Prove that $|S| \geq n$, where

$$S = \bigcup_{1\leq i<j\leq n} C_i \cap C_j,$$

the intersection points of the circles. (Each circle is viewed as the set of points on its circumference, not including its interior.)

Solution: Let $T = \{C_1, C_2, \ldots, C_n\}$. For every $s \in S$ and $C \in T$ define

$$f(s, C) = \begin{cases} 0, & \text{if } s \notin C, \\ \frac{1}{k}, & \text{if } s \in C, \end{cases}$$

where k is the number of circles passing through s (including C). Thus

$$\sum_{C \in T} f(s, C) = 1$$

for every $s \in S$.

On the other hand, for a fixed circle $C \in T$ let $s_0 \in S \cap C$ be a point such that

$$f(s_0, C) = \min\{f(s, C) \mid s \in S \cap C\}.$$

Suppose that $C, C_2, \ldots C_k$ be the circles which pass through s_0. Then C meets $C_2, \ldots, C_k$ again in distinct points $s_2, \ldots, s_k$. Therefore

$$\sum_{s \in C} f(s, C) \geq \frac{1}{k} + \frac{k-1}{k} = 1.$$

We have

$$|S| = \sum_{s \in S} \sum_{C \in T} f(s, C) = \sum_{C \in T} \sum_{s \in S} f(s, C) \geq n,$$

as desired.

Problem 4 Suppose that $-1 \leq x_1, x_2, \ldots, x_n \leq 1$ are real numbers such that $x_1 + x_2 + \ldots + x_n = 0$. Prove that there exists a permutation σ of $\{1, 2, \ldots, n\}$ such that, for every $1 \leq p \leq q \leq n$,

$$|x_{\sigma(p)} + \cdots + x_{\sigma(q)}| < 2 - \frac{1}{n}.$$

Also prove that the expression on the right hand side cannot be replaced by $2 - \frac{4}{n}$.

Solution: If $n = 1$ then $x_1 = 0$, and the permutation $\sigma(1) = 1$ suffices; if $n = 2$ then $|x_1|, |x_2| \leq 1$ and $|x_1 + x_2| = 0$, and the permutation $(\sigma(1), \sigma(2)) = (1, 2)$ suffices. Now assume $n \geq 3$.

View the x_i as vectors; the problem is equivalent to saying that if we start at a point on the number line, we can travel along the n vectors $x_1, x_2, \ldots, x_n$ in some order so that we stay within an interval $(m, m + 2 - \frac{1}{n}]$.

Call x_i "long" if $|x_i| \geq 1 - \frac{1}{n}$ and call it "short" otherwise. Also call x_i "positive" if $x_i \geq 0$ and "negative" if $x_i < 0$. Suppose without loss of generality that there are at least as many long positive vectors as long negative vectors — otherwise, we could replace each x_i by $-x_i$. We make our trip in two phases:

(i) First, travel alternating along long positive vectors and long negative vectors until no long negative vectors remain. Suppose at some time we are at a point P. Observe that during this leg of our trip, traveling along a pair of vectors changes our position by at most $\frac{1}{n}$ in either direction. Thus if we travel along $2t \leq n$ vectors after P, we stay within $\frac{t}{n} \leq \frac{1}{2}$ of P; and if we travel along $2t+1$ vectors after P, we stay within $\frac{t}{n} + 1 \leq \frac{3}{2} < 2 - \frac{1}{n}$ of P. Therefore, during this phase, we indeed stay within an interval $I = \left(m, m + 2 - \frac{1}{n}\right]$ of length $2 - \frac{1}{n}$.

(ii) After phase (i), we claim that as long as vectors remain unused and we are inside I, there is an unused vector we can travel along while remaining in I; this implies we can finish the trip while staying in I.

If there are no positive vectors, then we can travel along any negative vector, and vice versa. Thus assume there are positive *and* negative vectors remaining; since all the long negative vectors were used in phase (i), only short negative vectors remain.

Now if we are to the right of $m + 1 - \frac{1}{n}$, we can travel along a short negative vector without reaching or passing m. And if we are on or to the left of $m + 1 - \frac{1}{n}$, we can travel along a positive vector (short or long) without passing $m + 2 - \frac{1}{n}$.

Therefore it is possible to complete our journey, and it follows that the desired permutation σ indeed exists.

However, suppose $\frac{1}{n}$ is changed to $\frac{4}{n}$. This bound is never attainable for $n = 1$, and it is not always attainable when $n = 2$ (when $x_1 = 1$, $x_2 = -1$, for example).

And if $n = 2k+1 \geq 3$ or $2k+2 \geq 4$, suppose that $x_1 = x_2 = \cdots = x_k = 1$ and $x_{k+1} = x_{k+2} = \cdots = x_{2k+1} = -\frac{k}{k+1}$ — if n is even, we can let $x_n = 0$ and ignore this term in the permutation.

If two adjacent numbers in the permutation are equal then their sum is either $2 \geq 2 - \frac{4}{n}$ or $-2 \cdot \frac{k}{k+1} \leq -2 + \frac{4}{n}$. Therefore in the permutation, the vectors must alternate between $-\frac{k}{k+1}$ and 1, starting and ending with $-\frac{k}{k+1}$.

But then the outer two vectors add up to $-2 \cdot \frac{k}{k+1}$, so the middle $2k-1$ vectors add up to $2 \cdot \frac{k}{k+1} \geq 2 - \frac{4}{n}$, a contradiction. Therefore, $\frac{1}{n}$ cannot be replaced by $\frac{4}{n}$.

Problem 5 Suppose that $r_1, \ldots, r_n$ are real numbers. Prove that there exists $S \subseteq \{1, 2, \ldots, n\}$ such that

$$1 \leq |S \cap \{i, i+1, i+2\}| \leq 2,$$

for $1 \leq i \leq n-2$, and

$$\left| \sum_{i \in S} r_i \right| \geq \frac{1}{6} \sum_{i=1}^{n} |r_i|.$$

Solution: Let $S = \sum_{i=1}^{n} |r_i|$ and for $i = 0, 1, 2$, define

$$s_i = \sum_{r_j \geq 0,\, j \equiv i} r_j,$$

$$t_i = \sum_{r_j < 0,\, j \equiv i} r_j,$$

where congruences are taken modulo 3. Then we have $S = s_1 + s_2 + s_3 - t_1 - t_2 - t_3$, and $2S$ equals

$$(s_1 + s_2) + (s_2 + s_3) + (s_3 + s_1) - (t_1 + t_2) - (t_2 + t_3) - (t_3 + t_1).$$

Therefore there are $i_1 \neq i_2$ such that either $s_{i_1} + s_{i_2} \geq s/3$ or $t_{i_1} + t_{i_2} \leq -s/3$ or both. Without loss of generality, we assume that $s_{i_1} + s_{i_2} \geq s/3$ and $|s_{i_1} + s_{i_2}| \geq |t_{i_1} + t_{i_2}|$. Thus $s_{i_1} + s_{i_2} + t_{i_1} + t_{i_2} \geq 0$. We have

$$[s_{i_1} + s_{i_2} + t_{i_1}] + [s_{i_1} + s_{i_2} + t_{i_2}] \geq s_{i_1} + s_{i_2} \geq s/3.$$

Therefore at least one of $s_{i_1} + s_{i_2} + t_{i_1}$ and $s_{i_1} + s_{i_2} + t_{i_2}$ is bigger or equal to $s/6$ and we are done.

1.11 Ireland

Problem 1 Find all the real values of x which satisfy

$$\frac{x^2}{(x+1-\sqrt{x+1})^2} < \frac{x^2+3x+18}{(x+1)^2}.$$

Solution: We must have $x \in (-1,0) \cup (0,\infty)$ for the quantities above to be defined. Make the substitution $y = \sqrt{x+1}$, so that $y \in (0,1) \cup (1,\infty)$ and $x = y^2 - 1$. Then the inequality is equivalent to

$$\begin{aligned}
&\frac{(y^2-1)^2}{(y^2-y)^2} < \frac{(y^2-1)^2+3(y^2-1)+18}{y^4} \\
&\iff \frac{(y+1)^2}{y^2} < \frac{y^4+y^2+16}{y^4} \\
&\iff (y+1)^2y^2 < y^4+y^2+16 \\
&\iff 2y^3 < 16,
\end{aligned}$$

so the condition is satisfied exactly when $y < 2$; i.e., exactly when $y \in (0,1) \cup (1,2)$, which is equivalent to $x \in (-1,0) \cup (0,3)$.

Problem 2 Show that there is a positive number in the Fibonacci sequence which is divisible by 1000.

Solution: In fact, for any natural number n, there exist infinitely many positive Fibonacci numbers divisible by n.

The Fibonacci sequence is defined thus: $F_0 = 0$, $F_1 = 1$, and $F_{k+2} = F_{k+1} + F_k$ for all $k \geq 0$. Consider ordered pairs of consecutive Fibonacci numbers $(F_0, F_1), (F_1, F_2), \ldots$ taken modulo n. Since the Fibonacci sequence is infinite and there are only n^2 possible ordered pairs of integers modulo n, two such pairs (F_j, F_{j+1}) must be congruent: $F_i \equiv F_{i+m}$ and $F_{i+1} \equiv F_{i+m+1} \pmod{n}$ for some i and m.

If $i \geq 1$ then $F_{i-1} \equiv F_{i+1} - F_i \equiv F_{i+m+1} - F_{i+m} \equiv F_{i+m-1} \pmod{n}$; similarly $F_{i+2} \equiv F_{i+1} + F_i \equiv F_{i+m+1} + F_{i+m} \equiv F_{i+2+m} \pmod{n}$. Continuing similarly, we have $F_j \equiv F_{j+m} \pmod{n}$ for all $j \geq 0$. In particular, $0 = F_0 \equiv F_m \equiv F_{2m} \equiv \cdots \pmod{n}$, so the numbers F_m, $F_{2m}, \ldots$ are all positive Fibonacci numbers divisible by n. Applying this to $n = 1000$, we are done. (In fact, the smallest such m is 750.)

Problem 3 Let D, E, F be points on the sides BC, CA, AB, respectively, of triangle ABC such that $AD \perp BC$, $AF = FB$, and BE is the angle bisector of $\angle B$. Prove that AD, BE, CF are concurrent if and only if

$$a^2(a-c) = (b^2 - c^2)(a+c),$$

where $a = BC$, $b = CA$, $c = AB$.

Solution: By Ceva's Theorem, the cevians AD, BE, CF in $\triangle ABC$ are concurrent if and only if (using directed line segments)

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

In this problem, $\frac{AF}{FB} = 1$, and $\frac{CE}{EA} = \frac{a}{c}$. Thus AD, BE, CF are concurrent if and only if $\frac{BD}{DC} = \frac{c}{a}$.

This in turn is true if and only if $BD = \frac{ac}{a+c}$ and $DC = \frac{a^2}{a+c}$. Because $AB^2 - BD^2 = BD^2 = AC^2 - CD^2$, this last condition hold exactly when the following equations are true:

$$\begin{aligned}
AB^2 - \left(\frac{ac}{a+c}\right)^2 &= AC^2 - \left(\frac{a^2}{a+c}\right)^2 \\
(a+c)^2c^2 - a^2c^2 &= (a+c)^2b^2 - a^4 \\
a^4 - a^2c^2 &= (b^2 - c^2)(a+c)^2 \\
a^2(a-c) &= (b^2 - c^2)(a+c).
\end{aligned}$$

Therefore the three lines concur if and only if the given equation holds, as desired.

Alternatively, applying the law of cosines gives

$$\frac{BD}{DC} = \frac{c\cos B}{b\cos C} = \frac{c}{b} \cdot \frac{a^2+c^2-b^2}{2ac} \cdot \frac{2ab}{a^2+b^2-c^2} = \frac{a^2+c^2-b^2}{a^2+b^2-c^2}.$$

Again, this equals $\frac{c}{a}$ exactly when the given equation holds.

Problem 4 A 100 by 100 square floor is to be tiled. The only available tiles are rectangular 1 by 3 tiles, fitting exactly over three squares of the floor.

(a) If a 2 by 2 square is removed from the center of the floor, prove that the remaining part of the floor can be tiled with available tiles.

(b) If, instead, a 2 by 2 square is removed from the corner, prove that the remaining part of the floor cannot be tiled with the available tiles.

Solution: Choose a coordinate system so that the corners of the square floor lie along the lattice points $\{(x, y) \mid 0 \leq x, y \leq 100, x, y \in \mathbb{Z}\}$. Denote the rectanglar region $\{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$ by $[(a, c) - (b, d)]$.

(a) It is evident that any rectangle with at least one dimension divisible by 3 can be tiled. First tile the four rectangles

$$[(0,0) - (48,52)], \quad [(0,52) - (52,100)],$$
$$[(52,48) - (100,100)], \quad \text{and} \quad [(48,0) - (100,52)].$$

The only part of the board left untiled is $[(48, 48) - (52, 52)]$. But recall that the central region $[(49, 49) - (51, 51)]$ has been removed. It is obvious that the remaining portion can be tiled.

(b) Assume without loss of generality that $[(0, 0) - (2, 2)]$ is the 2 by 2 square which is removed. Label each remaining square $[(x, y) - (x + 1, y + 1)]$ with the number $L(x, y) \in \{0, 1, 2\}$ such that $L(x, y) \equiv x + y \pmod 3$. There are 3333 squares labeled 0, 3331 squares labeled 1, and 3332 squares labeled 2. However, each 1 by 3 tile covers an equal number of squares of each label. Therefore, the floor cannot be tiled.

Problem 5 Define a sequence u_n, $n = 0, 1, 2, \ldots$ as follows: $u_0 = 0$, $u_1 = 1$, and for each $n \geq 1$, u_{n+1} is the smallest positive integer such that $u_{n+1} > u_n$ and $\{u_0, u_1, \ldots, u_{n+1}\}$ contains no three elements which are in arithmetic progression. Find u_{100}.

Solution: Take any nonnegative integer n (e.g., 100) and express it in base-2 (e.g., $100 = 1100100_2$). Now interpret that sequence of 1's and 0's as an integer in base-3 (e.g., $1100100_3 = 981$). Call that integer t_n (e.g., $t_{100} = 981$).

We now prove that $t_n = u_n$ by strong induction on n. It is obvious that $t_0 = u_0$ and that $t_1 = u_1$. Now assume that $t_k = u_k$ for all $k < n$. We shall show that $t_n = u_n$.

First we show that $u_n \leq t_n$ by proving that, in the sequence $t_0, t_1, t_2, \ldots, t_n$, no three numbers form an arithmetic progression.

Pick any three numbers $0 \leq \alpha < \beta < \gamma \leq n$, and consider t_α, t_β, and t_γ in base-3. In the addition of t_α and t_γ, since both t_α and t_γ consist of only 1's and 0's, no carrying can occur. But $t_\alpha \neq t_\gamma$, so they must differ in at least one digit. In that digit in the sum $t_\alpha + t_\gamma$ must lie a "1." On the other hand, t_β consists of only 1's and 0's, so $2t_\beta$ consists of only 2's and 0's. Thus, the base-3 representations of $t_\alpha + t_\gamma$ and $2t_\beta$ are different: the former contains a "1" while the latter does not. Thus, $t_\alpha + t_\gamma \neq 2t_\beta$ for any choice of α, β, γ, so among $t_0, t_1, t_2, \ldots, t_n$, no three numbers are in arithmetic progression. Hence $u_n \leq t_n$.

Next we show that $u_n \geq t_n$ by showing that for all $k \in \{t_{n-1} + 1, t_{n-1} + 2, \ldots, t_n - 1\}$, there exist numbers a and b such that $t_a + k = 2t_b$. First note that k must contain a 2 in its base-3 representation, because the t_i are the only nonnegative integers consisting of only 1's and 0's in base-3. Therefore, we can find two numbers a and b with $0 \leq t_a < t_b < k$ such that:

- whenever k has a "0" or a "1" in its base-3 representation, t_a and t_b each also have the same digit in the corresponding positions in their base-3 representations;
- whenever k has a "2" in its base-3 representation, t_a has a "0" in the corresponding position in its base-3 representation, but t_b has a "1" in the corresponding position in its base-3 representation.

The t_a and t_b, thus constructed, satisfy $t_a < t_b < k$ while $t_a + k = 2t_b$, so t_a, t_b, k form an arithmetic progression. Thus, $u_n \geq t_n$. Putting this together with the previous result, we have forced $u_n = t_n$; hence $u_{100} = t_{100} = 981$.

Problem 6 Solve the system of equations

$$y^2 - (x+8)(x^2+2) = 0$$
$$y^2 - (8+4x)y + (16+16x-5x^2) = 0.$$

Solution: We first check that the solutions $(x, y) = (-2, -6)$ and $(-2, 6)$ both work and are the only solutions with $x = -2$.

We substitute $y^2 = (x+8)(x^2+2)$ into $y^2+16+16x-5x^2 = 4(x+2)y$ to get $4(x+2)y = x^3+3x^2+18x+32 = (x+2)(x^2+x+16)$. The case $x = -2$ has already been finished, so to deal with the case $x \neq -2$, we write

$$4y = x^2 + x + 16.$$

Squaring both sides, we have

$$16y^2 = x^4 + 2x^3 + 33x^2 + 32x + 256,$$

but from the first original equation we have

$$16y^2 = 16x^3 + 128x^2 + 32x + 256;$$

subtracting these two equations, we have $x^4 - 14x^3 - 95x^2 = 0$, or $x^2(x+5)(x-19) = 0$. Thus, $x \in \{0, -5, 19\}$. We use the equation $4y = x^2 + x + 16$ to find the corresponding y's.

In this way we find that the only solutions (x, y) are $(-2, -6)$, $(-2, 6)$, $(0, 4)$, $(-5, 9)$, and $(19, 99)$; it can be checked that each of these pairs works.

Problem 7 A function $f : \mathbb{N} \to \mathbb{N}$ satisfies

(i) $f(ab) = f(a)f(b)$ whenever the greatest common divisor of a and b is 1;

(ii) $f(p + q) = f(p) + f(q)$ for all prime numbers p and q.

Prove that $f(2) = 2$, $f(3) = 3$, and $f(1999) = 1999$.

Solution: Let us agree on the following notation: we shall write $(\mathrm{i})_{a,b}$ when we plug (a, b) (where a and b are relatively prime) into (i), and $(\mathrm{ii})_{p,q}$ when we plug (p, q) (where p and q are primes) into (ii).

First we find $f(1)$, $f(2)$, and $f(4)$. By $(\mathrm{i})_{1,b}$ we find $f(1) = 1$. By $(\mathrm{i})_{2,3}$ we find $f(6) = f(2)f(3)$; by $(\mathrm{ii})_{3,3}$ we get $f(6) = 2f(3)$; thus,

$$f(2) = 2.$$

Now by $(\mathrm{ii})_{2,2}$ we have

$$f(4) = 4.$$

Next we find some useful facts. From $(\mathrm{ii})_{3,2}$ and $(\mathrm{ii})_{5,2}$, respectively, we obtain

$$f(5) = f(3) + 2,\ f(7) = f(5) + 2 = f(3) + 4.$$

Now we can find $f(3)$. By $(\mathrm{ii})_{5,7}$ we have $f(12) = f(5) + f(7) = 2f(3) + 6$; from $(\mathrm{i})_{4,3}$ we have $f(12) = 4f(3)$; we solve for $f(3)$ to find

$$f(3) = 3.$$

Then using the facts from the previous paragraph, we find

$$f(5) = 5,\ f(7) = 7.$$

We proceed to find $f(13)$ and $f(11)$. By $(\mathrm{i})_{3,5}$, we have $f(15) = 15$. From $(\mathrm{ii})_{13,2}$ and $(\mathrm{ii})_{11,2}$, respectively, we find

$$f(13) = f(15) - f(2) = 13,\ f(11) = f(13) - f(2) = 11.$$

Finally, we can calculate $f(1999)$. By using (i) repeatedly with 2, 7, 11, and 13, we find $f(2002) = f(2 \cdot 7 \cdot 11 \cdot 13) = f(2)f(7)f(11)f(13) = 2002$. Noting that 1999 is a prime number; from $(\mathrm{ii})_{1999,3}$ we obtain

$$f(1999) = f(2002) - f(3) = 1999,$$

and we have finished.

Problem 8 Let a, b, c, d be positive real numbers whose sum is 1. Prove that

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+d} + \frac{d^2}{d+a} \geq \frac{1}{2}$$

with equality if and only if $a = b = c = d = 1/4$.

Solution: Apply the Cauchy-Schwarz inequality to find $[(a+b) + (b+c) + (c+d) + (d+a)]\left(\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+d} + \frac{d^2}{d+a}\right) \geq (a+b+c+d)^2$, which is equivalent to

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+d} + \frac{d^2}{d+a} \geq \frac{1}{2}(a+b+c+d) = \frac{1}{2},$$

with equality if and only if $\frac{a+b}{a} = \frac{b+c}{b} = \frac{c+d}{d} = \frac{d+a}{a}$, i.e., if and only if $a = b = c = d = \frac{1}{4}$.

Problem 9 Find all positive integers m such that the fourth power of the number of positive divisors of m equals m.

Solution: If the given condition holds for some integer m, then m must be a perfect fourth power and we may write its prime factorization as $m = 2^{4a_2}3^{4a_3}5^{4a_5}7^{4a_7}\cdots$ for nonnegative integers $a_2, a_3, a_5, a_7, \ldots$. Now the number of positive divisors of m equals $(4a_2+1)(4a_3+1)(4a_5+1)(4a_7+1)\cdots$; this is odd, so m is odd and $a_2 = 0$. Thus,

$$1 = \frac{4a_3+1}{3^{a_3}} \cdot \frac{4a_5+1}{5^{a_5}} \cdot \frac{4a_7+1}{7^{a_7}} \cdot \cdots = x_3x_5x_7\cdots,$$

where we write $x_p = \frac{4a_p+1}{p^{a_p}}$ for each p.

When $a_3 = 1$, $x_3 = \frac{5}{3}$; when $a_3 = 0$ or 2, $x_3 = 1$. And by Bernoulli's inequality, when $a_3 > 2$ we have

$$3_3^a = (8+1)^{a_3/2} > 8(a_3/2) + 1 = 4a_3 + 1$$

so that $x_3 < 1$.

When $a_5 = 0$ or 1, $x_5 = 1$; and by Bernoulli's inequality, when $a_5 \geq 2$ we have

$$5^{a_5} = (24+1)^{a_5/2} \geq 24a_5/2 + 1 = 12a_5 + 1$$

so that $x_5 \leq \frac{4a_5+1}{12a_5+1} \leq \frac{9}{25}$.

Finally, for any $p > 5$ when $a_p = 0$ we have $x_p = 1$; when $a_p = 1$ we have $p^{a_p} = p > 5 = 4a_p + 1$ so that $x_p < 1$; and when $a_p > 0$ then again by Bernoulli's inequality we have

$$p^{a_p} > 5^{a_p} > 12a_p + 1$$

so that as above $x_p < \frac{9}{25}$.

Now if $a_3 \neq 1$ then we have $x_p \leq 1$ for all p; but since $1 = x_2x_3x_5\cdots$ we must actually have $x_p = 1$ for all p. This means that $a_3 \in \{0, 2\}$, $a_5 \in \{0, 1\}$, and $a_7 = a_{11} = \cdots = 0$; so that $m = 1^4$, $(3^2)^4$, 5^4, or $(3^2 \cdot 5)^4$.

Otherwise, if $a_3 = 1$ then $3 \mid m = 5^4(4a_5+1)^4(4a_7+1)^4\cdots$. Then for some prime $p' \geq 5$, $3 \mid 4a_{p'} + 1$ so that $a_{p'} \geq 2$; from above we have $x_{p'} \leq \frac{9}{25}$. But then $x_3x_5x_7\cdots \leq \frac{5}{3}\frac{9}{27} < 1$, a contradiction.

Thus the only solutions are 1, 5^4, 3^8, and $3^8 \cdot 5^4$; and these can be easily verified by inspection.

Problem 10 Let $ABCDEF$ be a convex hexagon such that $AB = BC$, $CD = DE$, $EF = FA$, and

$$\angle ABC + \angle CDE + \angle EFA = 360^\circ.$$

Prove that the respective perpendiculars from A, C, E to FB, BD, DF are concurrent.

First Solution: The result actually holds even without the given angle condition. Let $\mathcal{C}_1$ be the circle with center B and radius $AB = BC$, $\mathcal{C}_2$ the circle with center D and radius $CD = DE$, and $\mathcal{C}_3$ the circle with center F and radius $EF = FA$. The line through A and perpendicular to line FB is the radical axis of circles $\mathcal{C}_3$ and $\mathcal{C}_1$, the line through C and perpendicular to line BD is the radical axis of circles $\mathcal{C}_1$ and $\mathcal{C}_2$, and the line through E and perpendicular to line DF is the radical axis of circles $\mathcal{C}_2$ and $\mathcal{C}_3$. The result follows because these three radical axes meet at the radical center of the three circles.

Second Solution: We first establish two lemmas:

Lemma 1. *Given points $W \neq Y$ and $X \neq Z$, lines WY and XZ are perpendicular if and only if*

$$XW^2 - WZ^2 = XY^2 - YZ^2. \tag{$*$}$$

Proof: Introduce Cartesian coordinates such that $W = (0.0)$, $X = (1, 0)$, $Y = (x_1, y_1)$, and $Z = (x_2, y_2)$. Then (*) becomes

$$x_1^2 + y_1^2 - x_2^2 - y_2^2 = (x_1 - 1)^2 + y_1^2 - (x_2 - 1)^2 - y_2^2,$$

which upon cancellation yields $x_1 = x_2$. This is true if and only if line YZ is perpendicular to the x-axis WX. ■

If P is the intersection of the perpendiculars from A and C to lines FB and BD, respectively, then the lemma implies that

$$PF^2 - PB^2 = AF^2 - AB^2$$

and

$$PB^2 - PD^2 = CB^2 - CD^2.$$

From the given isosceles triangles, we have $EF = FA$, $AB = BC$, and $CD = DE$. Subtracting the first equation from the second then gives

$$PD^2 - PF^2 = ED^2 - EF^2.$$

Hence line PE is also perpendicular to line DF, which completes the proof.

1.12 Italy

Problem 1 Given a rectangular sheet with sides a and b, with $a > b$, fold it along a diagonal. Determine the area of the triangle that passes over the edge of the paper.

Solution: Let $ABCD$ be a rectangle with $AD = a$ and $AB = b$. Let D' be the reflection of D across line AC, and let $E = AD' \cap BC$. We wish to find $[CD'E]$. Since $AB = CD'$, $\angle ABE = \angle CD'E = 90°$, and $\angle BEA = \angle D'EC$, triangles ABE and $CD'E$ are congruent. Thus $AE = EC$ and $CE^2 = AE^2 = AB^2 + BE^2 = b^2 + (a - CE)^2$. Hence $CE = \frac{a^2+b^2}{2a}$. It follows that

$$[CD'E] = [ACD'] - [ACE] = \frac{ab}{2} - \frac{b}{2} \cdot CE = \frac{b(a^2 - b^2)}{4a}.$$

Problem 2 A positive integer is said to be *balanced* if the number of its decimal digits equals the number of its distinct prime factors (for instance 15 is balanced, while 49 is not). Prove that there are only finitely many balanced numbers.

Solution: For $n > 15$, consider the product of the first n primes. The first sixteen primes have product

$$(2\cdot 53)(3\cdot 47)(5\cdot 43)(7\cdot 41)\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 37 > 100^4\cdot 10^8 = 10^{16},$$

while the other $n - 16$ primes are each at least 10. Thus the product of the first n primes is greater than 10^n.

Then if x has n digits and is balanced, then it it as at least the product of the first n primes. If $n \geq 16$ then from the previous paragraph x would be greater than 10^n and would have at least $n+1$ digits, a contradiction. Thus x can have at most 15 digits, implying that the number of balanced numbers is finite.

Problem 3 Let $\omega, \omega_1, \omega_2$ be three circles with radii r, r_1, r_2, respectively, with $0 < r_1 < r_2 < r$. The circles ω_1 and ω_2 are internally tangent to ω at two distinct points A and B and meet in two distinct points. Prove that $\overline{AB}$ contains an intersection point of ω_1 and ω_2 if and only if $r_1 + r_2 = r$.

Solution: Let O be the center of ω, and note that the centers C, D of ω_1, ω_2 lie on $\overline{OA}$ and $\overline{OB}$, respectively. Let E be a point on $\overline{AB}$

such that $CE \parallel OB$. Then $\triangle ACE \sim \triangle AOB$. Hence $AE = CE$ and E is on ω_1. We need to prove that $r = r_1 + r_2$ if and only if E is on ω_2.

Note that $r = r_1 + r_2$ is equivalent to

$$OD = OB - BD = r - r_2 = r_1 = AC,$$

that is $CEDO$ is a parallelogram or $DE \parallel AO$. Hence $r = r_1 + r_2$ if and only if $\triangle BDO \sim \triangle BOA$ or $BD = DE$, that is, E is on ω_2.

Problem 4 Albert and Barbara play the following game. On a table there are 1999 sticks: each player in turn must remove from the table some sticks, provided that the player removes at least one stick and at most half of the sticks remaining on the table. The player who leaves just one stick on the table loses the game. Barbara moves first. Determine for which of the players there exists a winning strategy.

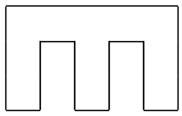
Solution: Call a number k *hopeless* if a player faced with k sticks has no winning strategy. If k is hopeless, then so is $2k+1$: a player faced with $2k+1$ sticks can only leave a pile of $k+1, k+2, \ldots,$ or $2k$ sticks, from which the other player can leave k sticks. Then since 2 is hopeless, so are $5, 11, \ldots, 3 \cdot 2^n - 1$ for all $n \geq 0$. Conversely, if $3 \cdot 2^n - 1 < k < 3 \cdot 2^{n+1} - 1$, then given k sticks a player can leave $3 \cdot 2^n - 1$ sticks and force a win. Since 1999 is not of the form $3 \cdot 2^n - 1$, it is not hopeless and hence Barbara has a winning strategy.

Problem 5 On a lake there is a village of pile-built dwellings, set on the nodes of an $m \times n$ rectangular array. Each dwelling is an endpoint of exactly p bridges which connect the dwelling with one or more of the adjacent dwellings (here adjacent means with respect to the array, hence diagonal connection is not allowed). Determine for which values of m, n, p it is possible to place the bridges so that from any dwelling one can reach any other dwelling. (Clearly, two adjacent dwellings can be connected by more than one bridge).

Solution: Suppose it is possible to place the bridges in this manner, and set the villages along the lattice points $\{(a, b) \mid 1 \leq a \leq m, 1 \leq b \leq n\}$. Color the dwellings cyan and magenta in a checkerboard fashion, so that every bridge connects a cyan dwelling with a magenta dwelling. Since each dwelling is at the end of the same number of bridges (exactly p of them), the number of cyan dwellings must equal the number of magenta dwellings; thus $2 \mid mn$.

Obviously $mn = 2$ works for all values of p. And for $p = 1$, we cannot have $mn > 2$ because otherwise if any two dwellings A and B are connected, then they cannot be connected to any other dwellings. Similarly, if $m = 1, n > 2$ (or $n = 1, m > 2$) then p bridges must connect $(1, 1)$ and $(1, 2)$ (or $(1, 1)$ and $(2, 1)$); but then neither of these dwellings is connected to any other dwellings, a contradiction.

Now assume that $2 \mid mn$ with $m, n > 1$ and $p > 1$; assume without loss of generality that $2 \mid m$. Build a sequence of bridges starting at $(1, 1)$, going up to $(1, n)$, right to (m, n), down to $(m, 1)$, and left to $(m - 1, 1)$; and then weaving back to $(1, 1)$ by repeatedly going from $(k, 1)$ up to $(k, n - 1)$ left to $(k - 1, n - 1)$ down to $(k - 1, 1)$ and left to $(k - 2, 1)$ for $k = m - 1, m - 3, \ldots, 3$. (The sideways E below shows this construction for $m = 6$, $n = 4$.)



So far we have built two bridges leading out of every dwelling, and any dwelling can be reached from any other dwelling. For the remaining $p - 2$ bridges needed for each dwelling, note that our sequence contains exactly mn bridges, an even number; so if we build *every other* bridge in our sequence, and do this $p - 2$ times, then exactly p bridges come out of every dwelling.

Thus either $mn = 2$ and p equals any value; or $2 \mid mn$ with $m, n, p > 1$.

Problem 6 Determine all triples (x, k, n) of positive integers such that

$$3^k - 1 = x^n.$$

Solution: $(3^k - 1, k, 1)$ for all positive integers k, and $(2, 2, 3)$.

The case of $n = 1$ is obvious. Now, n cannot be even because then 3 could not divide $3^k = \left(x^{\frac{n}{2}}\right)^2 + 1$ (since no square is congruent to 2 modulo 3); and also, we must have $x \neq 1$.

Assume that $n > 1$ is odd and $x \geq 2$. Then $3^k = (x+1) \sum_{i=0}^{n-1} (-x)^i$ implying that both $x + 1$ and $\sum_{i=0}^{n-1} (-x)^i$ are powers of 3. Then since

$x+1 \le x^2-x+1 \le \sum_{i=0}^{n-1}(-x)^i$, we must have $0 \equiv \sum_{i=0}^{n-1}(-x)^i \equiv n \pmod{x+1}$, so that $x+1 \mid n$. Specifically, this means that $3 \mid n$.

Writing $x' = x^{\frac{n}{3}}$, we have $3^k = x'^3+1 = (x'+1)(x'^2-x'+1)$. As before $x'+1$ must equal some power of 3, say 3^t. But then $3^k = (3^t-1)^3+1 = 3^{3t}-3^{2t+1}+3^{t+1}$, which is strictly between 3^{3t-1} and 3^{3t} for $t>1$. Therefore we must have $t=1$, $x'=2$, and $k=2$, giving the solution $(x,k,n)=(2,2,3)$.

Problem 7 Prove that for each prime p the equation

$$2^p+3^p=a^n$$

has no integer solutions (a,n) with $a,n>1$.

Solution: When $p=2$ we have $a^n=13$, which is impossible. When p is odd, then $5 \mid 2^p+3^p$; then since $n>1$, we must have $25 \mid 2^p+3^p$. Then

$$2^p+(5-2)^p \equiv 2^p+\left(\binom{p}{1}5\cdot(-2)^{p-1}+(-2)^p\right) \equiv 5p\cdot 2^{p-1} \pmod{25},$$

so $5 \mid p$. Thus we must have $p=5$, but then $a^n=2^5+3^5=5^2\cdot 11$ has no solutions.

Problem 8 Points D and E are given on the sides AB and AC of triangle ABC such that $DE \parallel BC$ and $\overline{DE}$ is tangent to the incircle of ABC. Prove that

$$DE \le \frac{AB+BC+CA}{8}.$$

Solution: Let $BC=a$, $CA=b$, $AB=c$. Also let $h=\frac{2[ABC]}{a}$ be the distance from A to line BC and let $r=\frac{2[ABC]}{a+b+c}$ the inradius of triangle ABC; note that $\frac{h-2r}{h}=\frac{b+c-a}{a+b+c}$.

Let $x=b+c-a$, $y=c+a-b$, $z=a+b-c$. Then

$$(x+y+z)^2 \ge (2\sqrt{x(y+z)})^2 = 4x(y+z)$$

by AM-GM, which implies that $(a+b+c)^2 \ge 8(b+c-a)a$, or

$$\frac{b+c-a}{a+b+c}\cdot a \le \frac{a+b+c}{8} \Longrightarrow \frac{h-2r}{h}\cdot BC \le \frac{AB+BC+CA}{8}.$$

But since $DE \parallel BC$, we have $\frac{DE}{BC}=\frac{h-2r}{h}$; substituting this into the above inequality gives the desired result.

Problem 9

(a) Find all the strictly monotonic functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + f(y)) = f(x) + y, \qquad \text{for all } x, y \in \mathbb{R}.$$

(b) Prove that for every integer $n > 1$ there do not exist strictly monotonic functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + f(y)) = f(x) + y^n, \qquad \text{for all } x, y \in \mathbb{R}.$$

Solution:

(a) The only such functions are $f(x) = x$ and $f(x) = -x$. Setting $x = y = 0$ gives $f(f(0)) = f(0)$, while setting $x = -f(0), y = 0$ gives $f(-f(0)) = f(0)$. Since f is strictly monotonic it is injective, so $f(0) = -f(0)$ and thus $f(0) = 0$. Next, setting $x = 0$ gives $f(f(y)) = y$ for all y.

Suppose f is increasing. If $f(x) > x$ then $x = f(f(x)) > f(x)$, a contradiction; if $f(x) < x$ then $x = f(f(x)) < f(x)$, a contradiction. Thus $f(x) = x$ for all x.

Next suppose that f is decreasing. Plugging in $x = -f(t)$, $y = t$, and then $x = 0$, $y = -t$ shows that $f(-f(t)) = f(f(-t)) = -t$, so $f(t) = -f(-t)$ for all t. Now given x, if $f(x) < -x$ then $x = f(f(x)) > f(-x) = -f(x)$, a contradiction. And if $f(x) > -x$ then $x = f(f(x)) < f(-x) = -f(x)$, a contradiction. Hence we must have $f(x) = -x$ for all x.

Therefore either $f(x) = x$ for all x or $f(x) = -x$ for all x; and it is easy to check that these two functions work.

(b) Since f is strictly monotonic, it is injective. Then for $y \neq 0$ we have $f(y) \neq f(-y)$ so that $f(x+f(y)) \neq f(x+f(-y))$ and hence $f(x) + y^n \neq f(x) + (-y)^n$; thus, n can't be even.

Now suppose there is such an f for odd n; then by arguments similar to those in part (a), we find that $f(0) = 0$ and $f(f(y)) = y^n$. Specifically, $f(f(1)) = 1$. If f is increasing then as in part (a) we have $f(1) = 1$; then $f(2) = f(1 + f(1)) = f(1) + 1^n = 2$ and $2^n = f(f(2)) = f(2) = 2$, a contradiction. If f is decreasing, then as in part (a) we have $f(1) = -1$; then $f(2) = f(1 + f(-1)) = f(1) + (-1)^n = -2$ and $2^n = f(f(2)) = f(-2) = -f(2) = 2$, a contradiction.

Problem 10 Let X be a set with $|X| = n$, and let $A_1, A_2, \ldots, A_m$ be subsets of X such that

(a) $|A_i| = 3$ for $i = 1, 2, \ldots, m$.

(b) $|A_i \cap A_j| \leq 1$ for all $i \neq j$.

Prove that there exists a subset of X with at least $\lfloor \sqrt{2n} \rfloor$ elements, which does not contain A_i for $i = 1, 2, \ldots, m$.

Solution: Let A be a subset of X containing no A_i, and having the maximum number of elements subject to this condition. Let k be the size of A. By assumption, for each $x \in X - A$, there exists $i(x) \in \{1, \ldots, m\}$ such that $A_{i(x)} \subseteq A \cup \{x\}$. Let $L_x = A \cap A_{i(x)}$, which by the previous observation must have 2 elements. Since $|A_i \cap A_j| \leq 1$ for $i \neq j$, the L_x must all be distinct. Now there are $\binom{k}{2}$ 2-element subsets of A, so there can be at most $\binom{k}{2}$ sets L_x. Thus $n - k \leq \binom{k}{2}$ or $k^2 + k \geq 2n$. It follows that

$$k \geq \frac{1}{2}(-1 + \sqrt{1 + 8n}) > \sqrt{2n} - 1,$$

that is, $k \geq \lfloor \sqrt{2n} \rfloor$.

1.13 Japan

Problem 1 You can place a stone at each of 1999×1999 squares on a grid pattern. Find the minimum number of stones you must place such that, when an arbitrary blank square is selected, the total number of stones placed in the corresponding row and column is at least 1999.

Solution: Place stones in a checkerboard pattern on the grid, so that stones are placed on the four corner squares. This placement satisfies the condition and contains $1000 \times 1000 + 999 \times 999 = 1998001$ stones. We now prove this number is minimal.

Suppose the condition is satisfied. Assume without loss of generality that the j-th column contains k stones, and every other row or column also contains at least k stones. For each of the k stones in the j-th column, the row containing that stone must contain at least k stones by our minimal choice of k. And for each of the $1999 - k$ blank squares in the j-th column, to satisfy the given condition there must be at least $1999 - k$ stones in the row containing that square. Thus total number of stones is at least

$$k^2 + (1999 - k)^2 = 2\left(k - \frac{1999}{2}\right)^2 + \frac{1999^2}{2} \geq \frac{1999^2}{2} = 1998000.5,$$

and it follows that there indeed must be at least 1998001 stones.

Problem 2 Let $f(x) = x^3 + 17$. Prove that for each natural number n, $n \geq 2$, there is a natural number x for which $f(x)$ is divisible by 3^n but not by 3^{n+1}.

Solution: We prove the result by induction on n. If $n = 2$, then $x = 1$ suffices. Now suppose that the claim is true for $n \geq 2$ — that is, there is a natural number y such that $y^3 + 17$ is divisible by 3^n but not 3^{n+1}. We prove that the claim is true for $n + 1$.

Suppose we have integers a, m such that a is not divisible by 3 and $m \geq 2$. Then $a^2 \equiv 1 \pmod 3$ and thus $3^m a^2 \equiv 3^m \pmod{3^{m+1}}$. Also, since $m \geq 2$ we have $3m - 3 \geq 2m - 1 \geq m + 1$. Hence

$$(a + 3^{m-1})^3 \equiv a^3 + 3^m a^2 + 3^{2m-1} a + 3^{3m-3} \equiv a^3 + 3^m \pmod{3^{m+1}}.$$

Since $y^3 + 17$ is divisible by 3^n, it is congruent to either 0, 3^n, or $2 \cdot 3^n$ modulo 3^{n+1}. Since 3 does not divide 17, 3 cannot divide y either.

Hence applying our result from the previous paragraph twice — once with $(a, m) = (y, n)$ and once with $(a, m) = (y + 3^{n-1}, n)$ — we find that 3^{n+1} must divide either $(y + 3^{n-1})^3 + 17$ or $(y + 2 \cdot 3^{n-1})^3 + 17$.

Hence there exists a natural number x' not divisible by 3 such that $3^{n+1} \mid x'^3 + 17$. If 3^{n+2} does not divide $x'^3 + 17$, we are done. Otherwise, we claim the number $x = x' + 3^n$ suffices. Since $x = x' + 3^{n-1} + 3^{n-1} + 3^{n-1}$, the result from two paragraphs ago tells us that $x^3 \equiv x'^3 + 3^n + 3^n + 3^n \equiv x'^3 \pmod{3^{n+1}}$. Thus $3^{n+1} \mid x^3 + 17$ as well. On the other hand, since $x = x' + 3^n$, we have $x^3 \equiv x'^3 + 3^{n+1} \not\equiv x'^3 \pmod{3^{n+2}}$. It follows that 3^{n+2} does *not* divide $x^3 + 17$, as desired. This completes the inductive step.

Problem 3 From a set of $2n + 1$ weights (where n is a natural number), if any one weight is excluded, then the remaining $2n$ weights can be divided into two sets of n weights that balance each other. Prove that all the weights are equal.

Solution: Label the weights $a_1, a_2, \ldots, a_{2n+1}$. Then for each j, $1 \le j \le 2n$, we have

$$c_1^{(j)} a_1 + c_2^{(j)} a_2 + \cdots + c_{2n}^{(j)} a_{2n} = a_{2n+1}$$

where $c_j^{(j)} = 0$, n of the other $c_i^{(j)}$ equal 1, and the remaining $c_i^{(j)}$ equal -1.

Thus we have $2n$ equations in the variables $a_1, a_2, \ldots, a_{2n}$. Clearly $(a_1, a_2, \ldots, a_{2n}) = (a_{2n+1}, a_{2n+1}, \ldots, a_{2n+1})$ is a solution to this system of equations. By Kramer's Rule, this solution is unique if and only if the determinant of the matrix

$$\begin{bmatrix} c_1^{(1)} & c_2^{(1)} & \cdots & c_{2n}^{(1)} \\ c_1^{(2)} & c_2^{(2)} & \cdots & c_{2n}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ c_1^{(2n)} & c_2^{(2n)} & \cdots & c_{2n}^{(2n)} \end{bmatrix}$$

is nonzero. We show this is true by proving that this determinant is odd.

If we add an integer m to any single integer in the matrix, its determinant changes by m multiplied by the corresponding cofactor. Specifically, if m is even then the parity of the determinant does not change. Thus the parity of the presented determinant is the same as the parity of the determinant

$$\begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{vmatrix}.$$

This matrix has eigenvector $(1, 1, \ldots, 1)$ corresponding to the eigenvalue $2n - 1$; and eigenvectors $(1, -1, 0, \ldots, 0, 0)$, $(0, 1, -1, \ldots, 0, 0)$, $\ldots$, $(0, 0, 0, \ldots, 1, -1)$ corresponding to eigenvalue -1. These $2n$ eigenvectors are linearly independent, so the matrix's characteristic polynomial is $p(x) = (x - (2n-1))(x+1)^{2n-1}$. Hence its determinant $p(0) = -(2n - 1)$ is odd, as desired.

Problem 4 Prove that

$$f(x) = (x^2 + 1^2)(x^2 + 2^2)(x^2 + 3^2) \cdots (x^2 + n^2) + 1$$

cannot be expressed as a product of two integral-coefficient polynomials with degree greater than 0.

Solution: The claim is obvious when $n = 1$. Now assume $n \geq 2$ and suppose by way of contradiction that $f(x)$ *could* be expressed as such a product $g(x)h(x)$ with

$$g(x) = a_0 + a_1 x + \cdots + a_\ell x^\ell,$$
$$h(x) = b_0 + b_1 x + \cdots + b_{\ell'} x^{\ell'},$$

where $\ell, \ell' > 0$ and the coefficients a_i and b_i are integers.

For $m = \pm 1, \pm 2, \ldots, \pm n$, since $(mi)^2 + m^2 = 0$ we have $1 = f(mi) = g(mi)h(mi)$. But since g and h have integer coefficients, $g(mi)$ equals either $1, -1, i$, or $-i$. Moreover, since the imaginary part of

$$g(mi) = (a_0 - a_2 m^2 + a_4 m^4 - \cdots) + m(a_1 - a_3 m^2 + a_5 m^4 - \cdots)i$$

is a multiple of m, $g(mi)$ must equal ± 1 for $m \neq \pm 1$. Going further, since $1 = g(mi)h(mi)$ we have $g(mi) = h(mi) = \pm 1$ for $m \neq \pm 1$.

Then by the factor theorem,

$$g(x) - h(x) = (x^2 + 2^2)(x^2 + 3^2) \cdots (x^2 + n^2)k(x)$$

for some integer-coefficient polynomial $k(x)$ with degree at most 1. Since $(g(i), h(i))$ equals $(1, -1), (-1, 1), (i, -i)$, or $(-i, i)$, we have

$$2 \geq |g(i) - h(i)| = (-1 + 2^2)(-1 + 3^2) \cdots (-1 + n^2)|k(i)|,$$

and hence we must have $k(i) = 0$. Since $k(x)$ has degree at most 1, this implies that $k(x) = 0$ for *all* x and that $g(x) = h(x)$ for all x. But then $a_0^2 = g(0)h(0) = f(0) = (1^2)(2^2) \cdots (n^2) + 1$, which is impossible.

Problem 5 For a convex hexagon $ABCDEF$ whose side lengths are all 1, let M and m be the maximum and minimum values of the three diagonals AD, BE, and CF. Find all possible values of m and M.

Solution: We claim that the possible values are $\sqrt{3} \leq M \leq 3$ and $1 \leq m \leq 2$.

First we show all such values are attainable. Continuously transform $ABCDEF$ from an equilateral triangle ACE of side length 2, into a regular hexagon of side length 1, and finally into a segment of length 3 (say, by enlarging the diagonal AD of the regular hexagon while bringing B, C, E, F closer to line AD). Then M continuously varies from $\sqrt{3}$ to 2 to 3. Similarly, by continuously transforming $ABCDEF$ from a 1×2 rectangle into a regular hexagon, we can make m vary continuously from 1 to 2.

Now we prove no other values are attainble. First, we have $AD \leq AB + BC + CD = 3$, and similarly $BE, CF \leq 3$ so that $M \leq 3$.

Next, suppose by way of contradiction that $m < 1$ and say without loss of generality that $AD < 1$. Since $AD < AB = BC = CD = 1$,

$$\angle DCA < \angle DAC, \ \angle ABD < \angle ADB,$$
$$\angle CBD = \angle CDB, \ \angle BCA = \angle BAC.$$

Therefore,

$$\angle CDA + \angle BAD = \angle CDB + \angle BDA + \angle BAC + \angle CAD$$
$$> \angle CBD + \angle DBA + \angle BCA + \angle ACD = \angle CBA + \angle BCD.$$

Consequently $\angle CDA + \angle BAD > 180^\circ$ and likewise $\angle EDA + \angle FAD > 180^\circ$. But then

$$\angle CDE + \angle BAF = \angle CDA + \angle EDA + \angle BAD + \angle FAD > 360^\circ,$$

which is impossible since $ABCDEF$ is convex. Hence $m \geq 1$.

Next we demonstrate that $M \geq \sqrt{3}$ and $m \leq 2$. Since the sum of the six interior angles in $ABCDEF$ is $720°$, some pair of adjacent angles has sum greater than or equal to $240°$ and some pair has sum less than or equal to $240°$. Thus it suffices to prove that $CF \geq \sqrt{3}$ when $\angle A + \angle B \geq 240°$, and that $CF \leq 2$ when $\angle A + \angle B \leq 240°$.

Suppose by way of contradiction that $\angle A + \angle B \geq 240°$ and $CF < \sqrt{3}$. By the law of cosines, $CF^2 = BC^2 + BF^2 - 2BC \cdot BF \cos \angle FBC$. Thus if we fix A, B, F and decrease $\angle ABC$, we decrease $\angle FBC$ and CF; similarly, by fixing A, B, C and decreasing $\angle BAF$, we decrease CF. Therefore, it suffices to prove that $\sqrt{3} \geq CF$ when $\angle A + \angle B = 240°$. And likewise, it suffices to prove that $CF \leq 2$ when $\angle A + \angle B = 240°$.

Now suppose that $\angle A + \angle B$ *does* equal $240°$. Let lines AF and BC intersect at P, and set $x = PA$ and $y = PB$. Since $\angle A + \angle B = 240°$, $\angle P = 60°$. Then applying the law of cosines to triangles PAB and PCF yields

$$1 = AB^2 = x^2 + y^2 - xy$$

and

$$CF^2 = (x+1)^2 + (y+1)^2 - (x+1)(y+1) = 2 + x + y.$$

Therefore, we need only find the possible values of $x + y$ given that $x^2 + y^2 - xy = 1$ and $x, y \geq 0$. These conditions imply that $(x+y)^2 + 3(x-y)^2 = 4$, $x + y \geq 0$, and $|x - y| \leq x + y$. Hence

$$1 = \frac{1}{4}(x+y)^2 + \frac{3}{4}(x-y)^2 \leq (x+y)^2 \leq (x+y)^2 + 3(x-y)^2 = 4,$$

so $1 \leq x + y \leq 2$ and $\sqrt{3} \leq CF \leq 2$. This completes the proof.

1.14 Korea

Problem 1 Let R and r be the circumradius and inradius of triangle ABC respectively, and let R' and r' be the circumradius and inradius of triangle $A'B'C'$ respectively. Prove that if $\angle C = \angle C'$ and $Rr' = R'r$, then the triangles are similar.

Solution: Let ω be the circumcircle of triangle ABC. By scaling, rotating, and translating, we may assume that $A = A'$, $B = B'$, $R = R'$, $r = r'$ and that C, C' lie on the same arc $\widehat{AB}$ of ω. If the triangles were similar before these transformations, they still remain similar; so it suffices to prove they are now congruent.

Since $r = \frac{1}{2}(AC + BC - AB)\cot(\angle C)$ and $r' = \frac{1}{2}(A'C' + B'C' - A'B')\cot(\angle C') = \frac{1}{2}(A'C' + B'C' - AB)\cot(\angle C)$, we must have $AC + BC = A'C' + B'C'$ and hence $AB + BC + CA = A'B' + B'C' + C'A'$. Then the area of triangle ABC is $\frac{1}{2}r(AB + BC + CA)$, which thus equals the area of triangle $A'B'C'$, $\frac{1}{2}r'(A'B' + B'C' + C'A')$. Since these triangles share the same base $\overline{AB}$, we know that the altitudes from C and C' to $\overline{AB}$ are equal. This implies that $\triangle ABC$ is congruent to either $\triangle A'B'C'$ or $\triangle B'A'C'$, as desired.

Problem 2 Suppose $f : \mathbb{Q} \to \mathbb{R}$ is a function satisfying

$$|f(m+n) - f(m)| \le \frac{n}{m}$$

for all positive rational numbers n and m. Show that for all positive integers k,

$$\sum_{i=1}^{k} |f(2^k) - f(2^i)| \le \frac{k(k-1)}{2}.$$

Solution: It follows from the condition $|f(m+n) - f(m)| \le \frac{n}{m}$ that

$$\left|f(2^{i+1}) - f(2^i)\right| \le \frac{2^{i+1} - 2^i}{2^i} = 1.$$

Therefore, for $k > i$,

$$\left|f(2^k) - f(2^i)\right| \le \sum_{j=i}^{k-1} \left|f(2^{j+1}) - f(2^j)\right| \le k - i.$$

From the above inequality, we obtain

$$\sum_{i=1}^{k} \left|f(2^k) - f(2^i)\right| = \sum_{i=1}^{k-1} \left|f(2^k) - f(2^i)\right| \le \sum_{i=1}^{k-1} (k-i) = \frac{k(k-1)}{2}.$$

This completes the proof.

Problem 3 Find all positive integers n such that $2^n - 1$ is a multiple of 3 and $\frac{2^n-1}{3}$ is a divisor of $4m^2+1$ for some integer m.

Solution: The answer is all 2^k where $k = 1, 2, \ldots$.

First, it is easy to conclude (using Fermat's Little Theorem, or by simple observation) that if $3 \mid 2^n - 1$, then n must be even.

Suppose by way of contradiction that $\ell \ge 3$ is a positive odd divisor of n. Then $2^\ell - 1$ is not divisible by 3 but it *is* a divisor of $2^n - 1$, so it is a divisor of $4m^2 + 1$ as well. On the other hand, $2^\ell - 1$ has a prime divisor p of the form $4r + 3$. Then $(2m)^2 \equiv -1 \pmod{4r+3}$; but a standard number theory result states that a square cannot equal -1 modulo a prime of the form $4r + 3$.

Therefore n is indeed of the form 2^k for $k \ge 1$. For such n, we have

$$\frac{2^n - 1}{3} = (2^{2^1} + 1)(2^{2^2} + 1)(2^{2^3} + 1) \cdots (2^{2^{k-1}} + 1).$$

The factors on the right side are all relatively prime to 2 since they are all odd. They are also Fermat numbers, and another result from number theory states that they are relatively prime. (Suppose that some prime p divided both $2^{2^a} + 1$ and $2^{2^b} + 1$ for $a < b$. Then $2^{2^a} \equiv 2^{2^b} \equiv -1 \pmod{p}$. But then $-1 \equiv 2^{2^b} = \left(2^{2^a}\right)^{2^{b-a}} \equiv \left((-1)^2\right)^{2^{b-a-1}} \equiv 1 \pmod{p}$, implying that $p = 2$; again, this is impossible.) Therefore by the Chinese Remainder Theorem, there is a positive integer c simultaneously satisfying

$$c \equiv 2^{2^{i-1}} \pmod{2^{2^i} + 1} \qquad \text{for all } i = 1, 2, \ldots, k-1$$

and $c \equiv 0 \pmod{2}$. Putting $c = 2m$, $4m^2 + 1$ is a multiple of $\frac{2^n-1}{3}$, as desired.

Problem 4 Suppose that for any real x with $|x| \ne 1$, a function $f(x)$ satisfies

$$f\left(\frac{x-3}{x+1}\right) + f\left(\frac{3+x}{1-x}\right) = x.$$

Find all possible $f(x)$.

Solution: Set $t = \frac{x-3}{x+1}$ so that $x = \frac{3+t}{1-t}$. Then the given equation can be rewritten as

$$f(t) + f\left(\frac{t-3}{t+1}\right) = \frac{3+t}{1-t}.$$

Similarly, set $t = \frac{3+x}{1-x}$ so that $x = \frac{t-3}{t+1}$ and $\frac{x-3}{x+1} = \frac{3+t}{1-t}$. Again we can rewrite the given equation, this time as

$$f\left(\frac{3+t}{1-t}\right) + f(t) = \frac{t-3}{t+1}.$$

Adding these two equations we have

$$\frac{8t}{1-t^2} = 2f(t) + f\left(\frac{t-3}{t+1}\right) + f\left(\frac{3+t}{1-t}\right) = 2f(t) + t,$$

so that

$$f(t) = \frac{4t}{1-t^2} - \frac{t}{2},$$

and some algebra verifies that this solution works.

Problem 5 Consider a permutation $(a_1, a_2, \ldots, a_6)$ of $1, 2, \ldots, 6$ such that the minimum number of transpositions needed to transform $(a_1, a_2, a_3, a_4, a_5, a_6)$ to $(1, 2, 3, 4, 5, 6)$ is four. Find the number of such permutations.

Solution: Given distinct numbers $b_1, b_2, \ldots, b_k$ between 1 and n, in a k-cycle with these numbers b_1 is mapped to one of the other $k-1$ numbers; its image is mapped to one of the $k-2$ remaining numbers; and so on until the remaining number is mapped to b_1. Hence there are $(k-1)(k-2)\cdots(1) = (k-1)!$ cycles of length k involving these numbers.

Any permutation which can be achieved with four transpositions is even, so a permutation satisfying the given conditions must be either (i) the identity permutation, (ii) a composition of two transpositions, (iii) a 3-cycle, (iv) a composition of a 2-cycle and a 4-cycle, (v) a composition of two 3-cycles, or (vi) a 5-cycle. Permutations of type (i), (ii), and (iii) can be attained with fewer transpositions from our observations above. Conversely, any even permutation that can be achieved with zero or two transpositions is of these three types. Hence the permutations described in the problem statement are precisely those of types (iv), (v), and (vi). For type-(iv) permutations, there are $\binom{6}{2} = 15$ ways to assign which cycle each of $1, 2, \ldots, 6$ belongs; and there are $(2-1)!(4-1)! = 6$ ways to rearrange them within the cycles, for a total of $15 \cdot 6 = 90$ permutations. For type-(v) permutations, there are $\frac{1}{2}\binom{6}{3} = 10$ ways to assign which cycle each number belongs to (since $\binom{6}{3}$ counts each such permutation twice, once in the form $(a\,b\,c)(d\,e\,f)$ and again in the form $(d\,e\,f)(a\,b\,c)$). And there are $(3-1)!(3-1)! = 4$ ways to rearrange the numbers within these two cycles for a total of $10 \cdot 4 = 40$ type-(v) permutations. Finally, for type-(v) permutations there are $\binom{6}{5} = 6$ ways to choose which five numbers are cycled, and $(5-1)! = 24$ different cycles among any five numbers. This gives a total of $6 \cdot 24 = 144$ type-(v) permutations, and altogether

$$90 + 40 + 144 = 274$$

permutations which can be attained with four permutations, but no less.

Problem 6 Let $a_1, a_2, \cdots, a_{1999}$ be nonnegative real numbers satisfying the following two conditions:

(a) $a_1 + a_2 + \cdots + a_{1999} = 2$;

(b) $a_1a_2 + a_2a_3 + \cdots + a_{1998}a_{1999} + a_{1999}a_1 = 1$.

Let $S = a_1^2 + a_2^2 + \cdots + a_{1999}^2$. Find the maximum and minimum possible values of S.

Solution: Without loss of generality assume that a_{1999} is the minimum a_i. We may also assume that $a_1 > 0$. From the given equations we have

$$\begin{aligned}
4 &= (a_1 + a_2 + \cdots + a_{1999})^2 \\
&\geq (a_1 + a_2 + \cdots + a_{1999})^2 - (a_1 - a_2 + a_3 - \cdots - a_{1998} + a_{1999})^2 \\
&= 4(a_1 + a_3 + \cdots + a_{1999})(a_2 + a_4 + \cdots + a_{1998}) \\
&\geq 4(a_1a_2 + a_2a_3 + \cdots + a_{1998}a_{1999}) \\
&\qquad +4(a_1a_4 + a_2a_5 + \cdots + a_{1996}a_{1999}) \\
&\qquad +4a_1(a_6 + a_8 + \cdots + a_{1998}) \\
&= 4(1 - a_{1999}a_1) + 4(a_1a_4 + a_2a_5 \cdots + a_{1996}a_{1999}) \\
&\qquad +4a_1(a_6 + a_8 + \cdots + a_{1998})
\end{aligned}$$

$$= 4 + 4(a_1a_4 + a_2a_5 + \cdots + a_{1996}a_{1999})$$
$$+4a_1(a_6 + a_8 + \cdots + a_{1998} - a_{1999})$$
$$\geq 4.$$

Hence equality must hold in the first and third inequality. Thus we must have

(i) $a_1 + a_3 + \cdots + a_{1999} = a_2 + a_4 + \cdots + a_{1998} = 1$

(ii) $a_1a_4 = a_2a_5 = \cdots = a_{1996}a_{1999} = 0$

(iii) $a_6 + a_8 + \cdots + a_{1998} = a_{1999}$.

Condition (ii) implies $a_4 = 0$; from (iii) we get $a_6 = a_8 = \cdots = a_{1998} = 0$. Thus from (i), we have $a_2 = 1$, and from (b), we have $a_1 + a_3 = 1$. Applying these to the first given condition (a), we have

$$a_4 + a_5 + \cdots + a_{1999} = 0,$$

so that $a_4 = a_5 = \cdots = a_{1999} = 0$. Therefore

$$\begin{aligned} S &= a_1^2 + a_2^2 + a_3^2 \\ &= a_1^2 + 1 + (1 - a_1)^2 \qquad \text{since } a_2 = a_1 + a_3 = 1 \\ &= 2(a_1^2 - a_1 + 1) \\ &= 2\left(a_1 - \frac{1}{2}\right)^2 + \frac{3}{2}. \end{aligned}$$

Thus S has maximum value 2 attained when $a_1 = 1$, and minimum value $\frac{3}{2}$ when $a_1 = \frac{1}{2}$.

1.15 Poland

Problem 1 Let D be a point on side BC of triangle ABC such that $AD > BC$. Point E on side AC is defined by the equation

$$\frac{AE}{EC} = \frac{BD}{AD - BC}.$$

Show that $AD > BE$.

First Solution: Fix the points B, C, D and the distance AD, and let A vary; its locus is a circle with center D. From the equation, the ratio $\frac{AE}{EC}$ is fixed; therefore, $\lambda = \frac{EC}{AC}$ is also fixed. Since E is the image of A under a homothety about C with ratio λ, the locus of all points E is the image of the locus of A under this homothety — a circle centered on $\overline{BC}$. Then BE has its unique maximum when E is the intersection of the circle with line BC farther from B. If we show that $AD = BE$ in this case then we are done (the original inequality $AD > BE$ will be strict because equality can only hold in this degenerate case). Indeed, in this case the points B, D, C, E, A are collinear in that order; our equation gives

$$AE \cdot (AC - BD) = AE \cdot (AD - BC) = EC \cdot BD$$

$$\Rightarrow AE \cdot AC = (AE + EC) \cdot BD = AC \cdot BD$$

$$\Rightarrow AE = BD \Rightarrow AD = BE.$$

Second Solution: Let F be the point on $\overline{AD}$ such that $FA = BC$, and let line BF hit side AC at E'. By the law of sines we have $AE' = FA \cdot \frac{\sin \angle AFE'}{\sin \angle FE'A} = CB \cdot \frac{\sin \angle DFB}{\sin \angle CE'F}$ and $E'C = CB \cdot \frac{\sin \angle E'BC}{\sin \angle CE'B} = CB \cdot \frac{\sin \angle FBD}{\sin \angle CE'F}$. Hence $\frac{AE'}{E'C} = \frac{\sin \angle DFB}{\sin \angle FBD} = \frac{DB}{FD} = \frac{BD}{AD - BC} = \frac{AE}{EC}$, and $E' = E$.

Let ℓ be the line passing through A parallel to side BC. Draw G on ray BC such that $BG = AD$ and $CG = FD$; and let lines GE and ℓ intersect at H. Triangles ECG and EAH are similar, so $AH = CG \cdot \frac{AE}{EC} = FD \cdot \frac{AE}{EC}$.

By Menelaus' Theorem applied to triangle CAD and line EFB, we have

$$\frac{CE \cdot AF \cdot DB}{EA \cdot FD \cdot BC} = 1.$$

Thus $AH = FD \cdot \frac{AE}{EC} = FD \cdot \frac{AF \cdot DB}{FD \cdot BC} = DB \cdot \frac{AF}{BC} = DB$, implying that quadrilateral $BDAH$ is a parallelogram and that $BH = AD$. It follows that triangle BHG is isosceles with $BH = BG = AD$; and since $\overline{BE}$ in a cevian in this triangle, we must have $BE < AD$, as desired.

Problem 2 Given are nonnegative integers $a_1 < a_2 < \cdots < a_{101}$ smaller than 5050. Show that one can choose four distinct integers a_k, a_l, a_m, a_n such that

$$5050 \mid (a_k + a_l - a_m - a_n).$$

Solution: First observe that the a_i are all distinct modulo 5050 since they are all between 0 and 5050. Now consider all sums $a_i + a_j, i < j$; there are $\binom{101}{2} = 5050$ such sums. If any two such sums, $a_k + a_l$ and $a_m + a_n$, are congruent mod 5050, we are done. (In this case, all four indices would indeed be distinct: if, for example, $k = m$, then we would also have $l = n$ since all a_i are different mod 5050, but we chose the pairs $\{k, l\}$ and $\{m, n\}$ to be distinct.) The only other possibility is that these sums occupy every possible congruence class mod 5050. Then, adding all such sums gives $100(a_1 + a_2 + \cdots + a_{101}) \equiv 0 + 1 + \cdots + 5049 = 2525 \cdot 5049 \pmod{5050}$. Since the number on the left side is even but $2525 \cdot 5049$ is odd, we get a contradiction.

Problem 3 For a positive integer n, let $S(n)$ denote the sum of its digits. Prove that there exist distinct positive integers $\{n_i\}_{1 \le i \le 50}$ such that

$$n_1 + S(n_1) = n_2 + S(n_2) = \cdots = n_{50} + S(n_{50}).$$

Solution: We show by induction on k that there exist positive integers $n_1, \ldots, n_k$ with the desired property. For $k = 1$ the statement is obvious. For $k > 1$, we have (by induction) numbers $m_1 < \cdots < m_{k-1}$ with the desired property. Note that we can make all m_i arbitrarily large, e.g. by adding some large power of 10 to all of them (which preserves our property). Then, choose m with $1 \le m \le 9$ and $m \equiv m_1 + 1 \pmod 9$; recall that $S(x) \equiv x \pmod 9$. Then we have $m_1 - m + S(m_1) - S(m) + 11 = 9\ell$ for some integer ℓ; by choosing the m_i large enough we can ensure $10^\ell > m_{k-1}$. Now let $n_i = 10^{\ell+1} + m_i$ for $i < k$ and $n_k = m + 10^{\ell+1} - 10$. Now it is obvious that $n_i + S(n_i) = n_j + S(n_j)$ for $i, j < k$, and

$$\begin{aligned} n_1 + S(n_1) &= (10^{l+1} + m_1) + (1 + S(m_1)) \\ &= (m_1 + S(m_1) + 1) + 10^{l+1} \\ &= (9\ell + S(m) + m - 10) + 10^{\ell+1} \\ &= (m + 10^{l+1} - 10) + (9\ell + S(m)) \\ &= n_k + S(n_k), \end{aligned}$$

as needed.

Problem 4 Find all integers $n \geq 2$ for which the system of equations

$$\begin{aligned} x_1^2 + x_2^2 + 50 &= 16x_1 + 12x_2 \\ x_2^2 + x_3^2 + 50 &= 16x_2 + 12x_3 \\ &\cdots\cdots\cdots \\ x_{n-1}^2 + x_n^2 + 50 &= 16x_{n-1} + 12x_n \\ x_n^2 + x_1^2 + 50 &= 16x_n + 12x_1 \end{aligned}$$

has a solution in integers $(x_1, x_2, \ldots, x_n)$.

Solution: Answer: $3 \mid n$.

We rewrite the equation $x^2 + y^2 + 50 = 16x + 12y$ as $(x-8)^2 + (y-6)^2 = 50$, whose integer solutions are

$$(7,-1),\ (7,13),\ (9,-1),\ (9,13),\ (3,1),\ (3,11)$$
$$(13,1),\ (13,11),\ (1,5),\ (1,7),\ (15,5),\ (15,7).$$

Thus every pair (x_i, x_{i+1}) (where $x_{n+1} = x_1$) must be one of these. If $3 \mid n$ then just let $x_{3i} = 1, x_{3i+1} = 7, x_{3i+2} = 13$ for each i. Conversely, if a solution exists, consider the pairs (x_i, x_{i+1}) which occur; every pair's first coordinate is the second coordinate of another pair, and vice versa, which reduces the above possibilities to $(1,7), (7,13), (13,1)$. It follows that the x_i must form a repeating sequence $1, 7, 13, 1, 7, 13, \ldots$, which is only possible when $3 \mid n$.

Problem 5 Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be integers. Prove that

$$\sum_{1 \leq i < j \leq n} (|a_i - a_j| + |b_i - b_j|) \leq \sum_{1 \leq i,j \leq n} |a_i - b_j|.$$

Solution: Define $f_{\{a,b\}}(x) = 1$ if either $a \le x < b$ or $b \le x < a$, and 0 otherwise. Observe that when a, b are integers, $|a-b| = \sum_x f_{\{a,b\}}(x)$ where the sum is over all integers (the sum is valid since only finitely many terms are nonzero). Now suppose $a_\le$ is the number of values of i for which $a_i \le x$, and $a_>, b_\le, b_>$ are defined analogously. We have $(a_\le - b_\le) + (a_> - b_>) = (a_\le + a_>) - (b_\le + b_>) = n - n = 0 \Rightarrow (a_\le - b_\le)(a_> - b_>) \le 0$. Thus $a_\le a_> + b_\le b_> \le a_\le b_> + a_> b_\le$. But $a_\le a_> = \sum_{i<j} f_{\{a_i,a_j\}}(x)$ since both sides count the same set of pairs, and the other terms reduce similarly, giving

$$\sum_{1\le i<j\le n} f_{\{a_i,a_j\}}(x) + f_{\{b_i,b_j\}}(x) \le \sum_{1\le i,j\le n} f_{\{a_i,b_j\}}(x).$$

Now summing over all integers x and using our first observation, we get the desired inequality. Equality holds iff the above inequality is an equality for all x, which is true precisely when the a_i equal the b_i in some order.

Problem 6 In a convex hexagon $ABCDEF$, $\angle A + \angle C + \angle E = 360^\circ$ and

$$AB \cdot CD \cdot EF = BC \cdot DE \cdot FA.$$

Prove that $AB \cdot FD \cdot EC = BF \cdot DE \cdot CA$.

First Solution: Construct point G so that triangle GBC is similar to triangle FBA (and with the same orientation). Then $\angle DCG = 360^\circ - (\angle GCB + \angle BCD) = \angle DEF$ and $\frac{GC}{CD} = \frac{FA \cdot \frac{BC}{AB}}{CD} = \frac{FE}{ED}$, so triangles DCG, DEF are similar.

Now $\frac{AB}{BF} = \frac{CB}{BG}$ by similar triangles, and $\angle ABC = \angle ABF + \angle FBC = \angle CBG + \angle FBC = \angle FBG$; thus $\triangle ABC \sim \triangle FBG$, and likewise $\triangle EDC \sim \triangle FDG$. Then

$$\frac{AB}{CA} \cdot \frac{EC}{DE} \cdot \frac{FD}{BF} = \frac{FB}{GF} \cdot \frac{FG}{DF} \cdot \frac{FD}{BF} = 1$$

as needed.

Second Solution: Invert about F with some radius r. The original equality becomes

$$\frac{A'B' \cdot r^2}{A'F \cdot B'F} \cdot \frac{C'D' \cdot r^2}{C'F \cdot D'F} \cdot \frac{r^2}{E'F} = \frac{B'C' \cdot r^2}{B'F \cdot C'F} \cdot \frac{D'E' \cdot r^2}{D'F \cdot E'F} \cdot \frac{r^2}{A'F}$$

or $\frac{A'B'}{B'C'} = \frac{E'D'}{D'C'}$. The original angle condition is $\angle FAB + \angle BCF + \angle FCD + \angle DEF = 360°$; using directed angles, this turns into $\angle A'B'F + \angle FB'C' + \angle C'D'F + \angle FD'E' = 360°$, or $\angle A'B'C' = \angle E'D'C'$. Thus triangles $A'B'C', E'D'C'$ are similar, giving $\frac{A'B'}{A'C'} = \frac{E'D'}{E'C'}$ or, equivalently,

$$\frac{A'B' \cdot r^2}{A'F \cdot B'F} \cdot \frac{r^2}{D'F} \cdot \frac{E'C' \cdot r^2}{C'F \cdot E'F} = \frac{r^2}{B'F} \cdot \frac{D'E' \cdot r^2}{D'F \cdot E'F} \cdot \frac{C'A' \cdot r^2}{A'F \cdot C'F}.$$

Inverting back, we see that we are done.

Third Solution: Position the hexagon in the complex plane and let $a = B - A, b = C - B, \ldots, f = A - F$. The product identity implies that $|ace| = |bdf|$, and the angle equality implies $\frac{-b}{a} \cdot \frac{-d}{c} \cdot \frac{-f}{e}$ is positive real; hence $ace = -bdf$. Also $a + b + c + d + e + f = 0$; Multiplying this by ad and adding $ace + bdf = 0$ gives

$$a^2d + abd + acd + ad^2 + ade + adf + ace + bdf = 0$$

which factors to $a(d + e)(c + d) + d(a + b)(f + a) = 0$. Thus $|a(d + e)(c + d)| = |d(a + b)(f + a)|$, which is what we wanted.

1.16 Romania

National Olympiad

Problem 7.1 Determine the side lengths of a right triangle if they are integers and the product of the leg lengths is equal to three times the perimeter.

Solution: One of the leg lengths must be divisible by 3; let the legs have lengths $3a$ and b and let the hypotenuse have length c, where a, b, and c are positive integers. From the given condition we have $3ab = 3(3a+b+c)$, or $c = ab - 3a - b$. By the Pythagorean theorem, we have $(3a)^2 + b^2 = c^2 = (ab - 3a - b)^2$, which simplifies to

$$ab[(a-2)(b-6)-6] = 0.$$

Since $a, b > 0$, we have $(a, b) \in \{(3, 12), (4, 9), (5, 8), (8, 7)\}$, and therefore the side lengths of the triangle are either $(9, 12, 15)$, $(8, 15, 17)$, or $(7, 24, 25)$.

Problem 7.2 Let a, b, c be nonzero integers, $a \neq c$, such that

$$\frac{a}{c} = \frac{a^2 + b^2}{c^2 + b^2}.$$

Prove that $a^2 + b^2 + c^2$ cannot be a prime number.

Solution: Cross-multiplying and factoring, we have $(a-c)(b^2 - ac) = 0$. Since $a \neq c$, we have $ac = b^2$. Now, $a^2 + b^2 + c^2 = a^2 + (2ac - b^2) + c^2 = (a+c)^2 - b^2 = (a+b+c)(a-b+c)$. Also, $|a|, |c|$ cannot both be 1. Then $a^2+b^2+c^2 > |a|+|b|+|c| \geq |a+b+c|, |a-b+c|$, whence $a^2 + b^2 + c^2$ cannot be a prime number.

Problem 7.3 Let $ABCD$ be a convex quadrilateral with $\angle BAC = \angle CAD$ and $\angle ABC = \angle ACD$. Rays AD and BC meet at E and rays AB and DC meet at F. Prove that

(a) $AB \cdot DE = BC \cdot CE$;

(b) $AC^2 < \frac{1}{2}(AD \cdot AF + AB \cdot AE)$.

Solution:

(a) Because $\angle BAC + \angle CBA = \angle ECA$, we have $\angle ECD = \angle BAC$. Then $\triangle CDE \sim \triangle ACE$, and $\frac{CE}{DE} = \frac{AE}{CE}$. But since $\overline{AC}$ is the angle bisector of $\angle A$ in triangle ABE, we also have $\frac{AE}{CE} = \frac{AB}{BC}$. Thus $\frac{CE}{DE} = \frac{AB}{BC}$, whence $AB \cdot DE = BC \cdot CE$.

(b) Note that $\overline{AC}$ is an angle bisector of both triangle ADF and triangle AEB. Thus it is enough to prove that if $\overline{XL}$ is an angle bisector in an arbitrary triangle XYZ, then $XL^2 < XY \cdot XZ$. Let M be the intersection of $\overrightarrow{XL}$ and the circumcircle of triangle XYZ. Because $\triangle XYL \sim \triangle XMZ$, we have $XL^2 < XL \cdot XM = XY \cdot XZ$, as desired.

Problem 7.4 In triangle ABC, D and E lie on sides BC and AB, respectively, F lies on side AC such that $EF \parallel BC$, G lies on side BC such that $EG \parallel AD$. Let M and N be the midpoints of $\overline{AD}$ and $\overline{BC}$, respectively. Prove that

(a) $\dfrac{EF}{BC} + \dfrac{EG}{AD} = 1;$

(b) the midpoint of $\overline{FG}$ lies on line MN.

Solution:

(a) Since $EF \parallel BC$, $\triangle AEF \sim \triangle ABC$ and $\frac{EF}{BC} = \frac{AE}{AB}$. Similarly, since $EG \parallel AD$, $\triangle BEG \sim \triangle BAD$ and $\frac{EG}{AD} = \frac{EB}{AB}$. Hence $\frac{EF}{BC} + \frac{EG}{AD} = 1$.

(b) Let lines AN, EF intersect at point P, and let Q be the point on line BC such that $PQ \parallel AD$. Since $BC \parallel EF$ and N is the midpoint of $\overline{BC}$, P is the midpoint of $\overline{EF}$. Then vector EP equals both vectors PF and GQ, and $PFQG$ is a parallelogram. Thus the midpoint X of $\overline{FG}$ must also be the midpoint of $\overline{PQ}$. But then since M is the midpoint of $\overline{AD}$ and $AD \parallel PQ$, points M, X, N must be collinear.

Problem 8.1 Let $p(x) = 2x^3 - 3x^2 + 2$, and let

$$S = \{p(n) \mid n \in \mathbb{N}, n \leq 1999\},$$
$$T = \{n^2 + 1 \mid n \in \mathbb{N}\},$$
$$U = \{n^2 + 2 \mid n \in \mathbb{N}\}.$$

Prove that $S \cap T$ and $S \cap U$ have the same number of elements.

Solution: Note that $|S \cap T|$ is the number of squares of the form $2n^3 - 3n^2 + 1 = (n-1)^2(2n+1)$ where $n \in \mathbb{N}, n \leq 1999$. And for

$n \leq 1999$, $(n-1)^2(2n+1)$ is a square precisely when either $n = 1$ or when $n \in \{\frac{1}{2}(k^2-1) \mid k = 1, 3, 5, \ldots, 63\}$. Thus, $|S \cap T| = 33$.

Next, $|S \cap U|$ is the number of squares of the form $2n^3 - 3n^2 = n^2(2n-3)$ where $n \in \mathbb{N}, n \leq 1999$. And for $n \leq 1999$, $n^2(2n-3)$ is a square precisely when either $n = 0$ or when $n \in \{\frac{1}{2}(k^2+3) \mid k = 1, 3, 5, \ldots, 63\}$. Thus $|S \cap U| = 33$ as well, and we are done.

Problem 8.2

(a) Let $n \geq 2$ be a positive integer and

$$x_1, y_1, x_2, y_2, \ldots, x_n, y_n$$

be positive real numbers such that

$$x_1 + x_2 + \cdots + x_n \geq x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

Prove that

$$x_1 + x_2 + \cdots + x_n \leq \frac{x_1}{y_1} + \frac{x_2}{y_2} + \cdots + \frac{x_n}{y_n}.$$

(b) Let a, b, c be positive real numbers such that

$$ab + bc + ca \leq 3abc.$$

Prove that

$$a^3 + b^3 + c^3 \geq a + b + c.$$

Solution:

(a) Applying the Cauchy-Schwarz inequality and then the given inequality, we have

$$\left(\sum_{i=1}^{n} x_i\right)^2 \leq \sum_{i=1}^{n} x_iy_i \cdot \sum_{i=1}^{n} \frac{x_i}{y_i} \leq \sum_{i=1}^{n} x_i \cdot \sum_{i=1}^{n} \frac{x_i}{y_i}.$$

Dividing both sides by $\sum_{i=1}^{n} x_i$ yields the desired inequality.

(b) By the AM-HM inequality on a, b, c we have

$$a + b + c \geq \frac{9abc}{ab + bc + ca} \geq \frac{9abc}{3abc} = 3.$$

Then, since the given condition is equivalent to $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq 3$, we have $a + b + c \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Hence setting $x_1 = a, x_2 = b, x_3 = c$ and $y_1 = \frac{1}{a^2}, y_2 = \frac{1}{b^2}, y_3 = \frac{1}{c^2}$ in the result from part (a) gives $a + b + c \leq a^3 + b^3 + c^3$, as desired.

Problem 8.3 Let $ABCDA'B'C'D'$ be a rectangular box, let E and F be the feet of perpendiculars from A to lines $A'D$ and $A'C$ respectively, and let P and Q be the feet of perpendiculars from B' to lines $A'C'$ and $A'C$ respectively. Prove that

(a) planes AEF and $B'PQ$ are parallel;

(b) triangles AEF and $B'PQ$ are similar.

Solution:

(a) Let $(P_1P_2\ldots P_k)$ denote the plane containing points $P_1, P_2, \ldots, P_k$. First observe that quadrilateral $A'B'CD$ is a parallelogram and thus lies in a single plane.

We are given that $AE \perp A'D$. Also, line AE is contained in plane $(ADD'A)$, which is perpendicular to line CD. Hence $AE \perp CD$ as well, and therefore $AE \perp (A'B'CD)$ and $AE \perp A'C$. And since we know that $A'C \perp AF$, we have $A'C \perp (AEF)$ and $A'C \perp EF$.

Likewise, $B'Q \perp A'C$. And since lines $EF, B'Q$, and $A'C$ all lie in plane $(A'B'CD)$, it follows that $EF \parallel B'Q$. In a similar way we deduce that $AF \parallel PQ$. Hence the planes (AEF) and $(B'PQ)$ are parallel, as desired.

(b) Since $EF \parallel B'Q$ and $FA \parallel QP$, we have $\angle EFA = \angle PQB'$. Furthermore, from above $AE \perp EF$ and likewise $B'P \perp PQ$, implying that $\angle AEF = \angle B'PQ = 90^\circ$ as well. Therefore $\triangle AEF \sim \triangle B'PQ$, as desired.

Problem 8.4 Let $SABC$ be a right pyramid with equilateral base ABC, let O be the center of ABC, and let M be the midpoint of $\overline{BC}$. If $AM = 2SO$ and N is a point on edge SA such that $SA = 25SN$, prove that planes ABP and SBC are perpendicular, where P is the intersection of lines SO and MN.

Solution: Let $AB = BC = CA = s$. Then some quick calculations show that $AO = \frac{\sqrt{3}}{3}s$, $AM = \frac{\sqrt{3}}{2}s$, $AS = \frac{5}{\sqrt{48}}s$, and $AN = \frac{24}{5\sqrt{48}}s$. Then $AO \cdot AM = AN \cdot AS = \frac{1}{2}s^2$, whence $MONS$ is a cyclic quadrilateral. Thus, $\angle MNS = 90^\circ$, and P is the orthocenter of triangle AMS. Let Q be the intersection of lines AP and MS. Note that $\angle AMB = \angle AQM = \angle QMB = 90^\circ$. From repeated applications of the Pythagorean theorem, we have $AB^2 = AM^2 +$

$MB^2 = AQ^2 + QM^2 + MB^2 = AQ^2 + QB^2$, whence $\angle AQB = 90°$. Now $AQ \perp QB$ and $AQ \perp QM$, so line AQ must be perpendicular to plane SBC. Then since plane ABP contains line AQ, planes ABP and SBC must be perpendicular.

Problem 9.1 Let ABC be a triangle with angle bisector $\overline{AD}$. One considers the points M, N on rays AB and AC respectively, such that $\angle MDA = \angle ABC$ and $\angle NDA = \angle BCA$. Lines AD and MN meet at P. Prove that

$$AD^3 = AB \cdot AC \cdot AP.$$

Solution: Since $\triangle ADB \sim \triangle AMD$, $\frac{AD}{AB} = \frac{AM}{AD}$. Also, $\angle MAN + \angle NDM = \pi$, whence $AMDN$ is cyclic. Since $\angle DCA = \angle ADN = \angle AMN$, $\triangle ADC \sim \triangle APM$, and $\frac{AD}{AP} = \frac{AC}{AM}$. Therefore,

$$\frac{AD}{AB}\frac{AD}{AC}\frac{AD}{AP} = \frac{AM}{AD}\frac{AD}{AC}\frac{AC}{AM} = 1.$$

Problem 9.2 For $a, b > 0$, denote by $t(a, b)$ the positive root of the equation

$$(a+b)x^2 - 2(ab-1)x - (a+b) = 0.$$

Let $M = \{(a,b) \mid a \neq b,\ t(a,b) \leq \sqrt{ab}\}$. Determine, for $(a,b) \in M$, the minimum value of $t(a,b)$.

Solution: Consider the polynomial $P(x) = (a+b)x^2 - 2(ab-1)x - (a+b) = 0$. Since $a + b \neq 0$, the product of its roots is $-\frac{a+b}{a+b} = -1$. Hence P must have a unique positive root $t(a,b)$ and a unique negative root. Since the leading coefficient of $P(x)$ is positive, the graph of $P(x)$ is positive for $x > t(a,b)$ and negative for $0 \leq x < t(a,b)$ (since in the latter case, x is between the two roots). Thus, the condition $t(a,b) \leq \sqrt{ab}$ is equivalent to $P(\sqrt{ab}) \geq 0$, or

$$(ab-1)(a+b-2\sqrt{ab}) \geq 0.$$

But $a + b > 2\sqrt{ab}$ by AM-GM, where the inequality is sharp since $a \neq b$. Thus $t(a,b) \leq \sqrt{ab}$ exactly when $ab \geq 1$.

Now using the quadratic formula, we find that

$$t(a,b) = \frac{ab-1}{a+b} + \sqrt{\left(\frac{ab-1}{a+b}\right)^2 + 1}.$$

Thus given $ab \geq 1$, we have $t(a,b) \geq 1$ with equality when $ab = 1$.

Problem 9.3 In the convex quadrilateral $ABCD$ the bisectors of angles A and C meet at I. Prove that there exists a circle inscribed in $ABCD$ if and only if

$$[AIB] + [CID] = [AID] + [BIC].$$

Solution: It is well known that a circle can be inscribed in a convex quadrilateral $ABCD$ if and only if $AB + CD = AD + BC$. The bisector of angle A consists of those points lying inside $\angle BAD$ equidistant from lines AB and AD; similarly, the bisector of angle C consists of those points lying inside $\angle BCD$ equidistant from lines BC and BD.

Suppose $ABCD$ has an incircle. Then its center is equidistant from all four sides of the quadrilateral, so it lies on both bisectors and hence equals I. If we let r denote the radius of the incircle, then we have

$$[AIB] + [CID] = r(AB + CD) = r(AD + BC) = [AID] + [BIC].$$

Conversely, suppose that $[AIB] + [CID] = [AID] + [BIC]$. Let $d(I, \ell)$ denote the distance from I to any line ℓ, and write $x = d(I, AB) = d(I, AD)$ and $y = d(I, BC) = d(I, CD)$. Then

$$[AIB] + [CID] = [AID] + [BIC]$$

$$AB \cdot x + CD \cdot y = AD \cdot x + BC \cdot y$$

$$x\,(AB - AD) = y\,(BC - CD).$$

If $AB = AD$, then $BC = CD$ and it follows that $AB + CD = AD + BC$. Otherwise, suppose that $AB > AD$; then $BC > CD$ as well. Consider the points $A' \in \overline{AB}$ and $C' \in \overline{BC}$ such that $AD = AA'$ and $CD = CC'$. By SAS, we have $\triangle AIA' \cong \triangle AID$ and $\triangle DCI \cong \triangle C'IC$. Hence $IA' = ID = IC'$. Furthermore, subtracting $[AIA'] + [DCI] = [AID] + [C'IC]$ from both sides of our given condition, we have $[A'IB] = [C'IB]$ or $IA' \cdot IB \cdot \sin \angle A'IB = IC' \cdot IB \cdot \sin \angle CIB$. Thus $\angle A'IB = \angle C'IB$, and hence $\triangle A'IB \cong C'IB$ by SAS.

Thus $\angle IBA' = \angle IBC'$, implying that I lies on the angle bisector of $\angle ABC$. Therefore $x = d(I, AB) = d(I, BC) = y$, and the circle centered at I with radius $x = y$ is tangent to all four sides of the quadrilateral.

Problem 9.4

(a) Let $a, b \in \mathbb{R}$, $a < b$. Prove that $a < x < b$ if and only if there exists $0 < \lambda < 1$ such that $x = \lambda a + (1 - \lambda)b$.

(b) The function $f : \mathbb{R} \to \mathbb{R}$ has the property:

$$f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in \mathbb{R}$, $x \neq y$, and all $0 < \lambda < 1$. Prove that one cannot find four points on the function's graph that are the vertices of a parallelogram.

Solution:

(a) No matter what x is, there is a unique value $\lambda = \frac{b-x}{b-a}$ such that $x = \lambda a + (1 - \lambda)b$; and $0 < \frac{b-x}{b-a} < 1 \iff a < x < b$, which proves the claim.

(b) The condition is Jensen's inequality and shows that the function f is strictly convex. Stated geometrically, whenever $x < t < y$ the point $(t, f(t))$ lies strictly below the line joining $(x, f(x))$ and $(y, f(y))$. Suppose there were a parallelogram on the graph of f whose vertices, from left to right, have x-coordinates a, b, d, c. Then either $(b, f(d))$ or $(d, f(d))$ must lie on or above the line joining $(a, f(a))$ and $(c, f(c))$, a contradiction.

Problem 10.1 Find all real numbers x and y satisfying

$$\begin{aligned}\frac{1}{4^x} + \frac{1}{27^y} &= \frac{5}{6}\\ \log_{27} y - \log_4 x &\geq \frac{1}{6}\\ 27^y - 4^x &\leq 1.\end{aligned}$$

Solution: First, for the second equation to make sense we must have $x, y > 0$ and thus $27^y > 1$. Now from the third equation we have

$$\frac{1}{27^y} \geq \frac{1}{4^x + 1},$$

which combined with the first equation gives

$$\frac{1}{4^x} + \frac{1}{4^x + 1} \leq \frac{5}{6},$$

whence $x \geq \frac{1}{2}$. Similarly, the first and third equations also give

$$\frac{5}{6} \leq \frac{1}{27^y - 1} + \frac{1}{27^y},$$

whence $y \leq \frac{1}{3}$. If either $x > \frac{1}{2}$ or $y < \frac{1}{3}$, we would have $\log_{27} y - \log_4 x < \frac{1}{6}$, contradicting the second given equation. Thus, the only solution is $(x, y) = \left(\frac{1}{2}, \frac{1}{3}\right)$, which indeed satisfies all three equations.

Problem 10.2 A plane intersects edges AB, BC, CD, DA of the regular tetrahedron $ABCD$ at points M, N, P, Q, respectively. Prove that

$$MN \cdot NP \cdot PQ \cdot QM \geq AM \cdot BN \cdot CP \cdot DQ.$$

Solution: By the law of cosines in triangle MBN, we have

$$MN^2 = MB^2 + BN^2 - MB \cdot BN \geq MB \cdot BN.$$

Similarly, $NP^2 \geq CN \cdot CP$, $PN^2 \geq DP \cdot DQ$, and $MQ^2 \geq AQ \cdot AM$. Multiplying these inequalities yields

$$(MN \cdot NP \cdot PQ \cdot MQ)^2 \geq$$
$$(BM \cdot CN \cdot DP \cdot AQ) \cdot (AM \cdot BN \cdot CP \cdot DQ).$$

Now the given plane is different from plane (ABC) and (ADC). Thus if it intersects line AC at some point T, then points M, N, T must be collinear—because otherwise, the only plane containing M, N, T would be plane (ABC). Therefore it intersects line AC at most one point T, and by Menelaus' Theorem applied to triangle ABC and line MNT we have

$$\frac{AM \cdot BN \cdot CT}{MB \cdot NC \cdot TA} = 1.$$

Similarly, P, Q, T are collinear and

$$\frac{AQ \cdot DP \cdot CT}{QD \cdot PC \cdot TA} = 1.$$

Equating these two fractions and cross-multiplying, we find that

$$AM \cdot BN \cdot CP \cdot DQ = BM \cdot CN \cdot DP \cdot AQ.$$

This is true even if the plane does not actually intersect line AC: in this case, we must have $MN \parallel AC$ and $PQ \parallel AC$, in which case ratios of similar triangles show that $AM \cdot BN = BM \cdot CN$ and $CP \cdot DQ = DP \cdot AQ$.

Combining this last equality with the inequality from the first paragraph, we find that

$$(MN \cdot NP \cdot PQ \cdot QM)^2 \geq (AM \cdot BN \cdot CP \cdot DQ)^2,$$

which implies the desired result.

Problem 10.3 Let a, b, c $(a \neq 0)$ be complex numbers. Let z_1 and z_2 be the roots of the equation $az^2 + bz + c = 0$, and let w_1 and w_2 be the roots of the equation

$$(a + \overline{c})z^2 + (b + \overline{b})z + (\overline{a} + c) = 0.$$

Prove that if $|z_1|, |z_2| < 1$, then $|w_1| = |w_2| = 1$.

Solution: We begin by proving that $\operatorname{Re}(b)^2 \leq |a + \overline{c}|^2$. If $z_1 = z_2 = 0$, then $b = 0$ and the claim is obvious. Otherwise, write $a = m + ni$ and $c = r + si$; and write $z_1 = x + yi$ where $t = |z_1| = \sqrt{x^2 + y^2} < 1$. Also note that

$$r^2 + s^2 = |c|^2 = |az_1z_2|^2 < |a|^2|z_1|^2 = (m^2 + n^2)t^2. \qquad (1)$$

Assume WLOG that $z_1 \neq 0$. Then $|\operatorname{Re}(b)| = |\operatorname{Re}(-b)| = |\operatorname{Re}(az_1 + c/z_1)| = |\operatorname{Re}(az_1) + \operatorname{Re}(c/z_1)|$; that is,

$$\begin{aligned}
|\operatorname{Re}(b)| &= |(mx - ny) + (rx + sy)/t^2| \\
&= |x(m + r/t^2) + y(s/t^2 - n)| \\
&\leq \sqrt{x^2 + y^2}\sqrt{(m + r/t^2)^2 + (s/t^2 - n)^2} \\
&= t\sqrt{(m + r/t^2)^2 + (s/t^2 - n)^2},
\end{aligned}$$

where the inequality follows from Cauchy-Schwarz. Proving our claim then reduces to showing that

$$\begin{aligned}
&t^2\left((m + r/t^2)^2 + (s/t^2 - n)^2\right) \leq (m + r)^2 + (n - s)^2 \\
&\iff (mt^2 + r)^2 + (st^2 - n)^2 \leq t^2\left((m + r)^2 + (n - s)^2\right) \\
&\iff (r^2 + s^2)(1 - t^2) < (m^2 + n^2)(t^4 - t^2) \\
&\iff (1 - t^2)\left((m^2 + n^2)t^2 - (r^2 + s^2)\right).
\end{aligned}$$

But $1-t^2>0$ by assumption, and $(m^2+n^2)t^2-(r^2+s^2)>0$ from (1); therefore our claim is true.

Now since $|c/a|=|z_1z_2|<1$, we have $|c|<|a|$ and $a+\overline{c}\neq 0$. Then by the quadratic equation, the roots to $(a+\overline{c})z^2+(b+\overline{b})z+(\overline{a}+c)=0$ are given by

$$\frac{-(b+\overline{b})\pm\sqrt{(b+\overline{b})^2-4(a+\overline{c})(\overline{a}+c)}}{2(a+\overline{c})},$$

or (dividing the numerator and denominator by 2)

$$\frac{-\operatorname{Re}(b)\pm\sqrt{\operatorname{Re}(b)^2-|a+\overline{c}|^2}}{a+\overline{c}}=\frac{-\operatorname{Re}(b)\pm i\sqrt{|a+\overline{c}|^2-\operatorname{Re}(b)^2}}{a+\overline{c}}.$$

When evaluating either root, the absolute value of the numerator is $\sqrt{\operatorname{Re}(b)^2+(|a+\overline{c}|^2-\operatorname{Re}(b)^2)}=|a+\overline{c}|$; and the absolute value of the denominator is clearly $|a+\overline{c}|$ as well. Therefore indeed $|w_1|=|w_2|=1$, as desired.

Problem 10.4

(a) Let $x_1,y_1,x_2,y_2,\ldots,x_n,y_n$ be positive real numbers such that

(i) $x_1y_1<x_2y_2<\cdots<x_ny_n$;

(ii) $x_1+x_2+\cdots+x_k\geq y_1+y_2+\cdots+y_k$ for all $k=1,2,\ldots,n$.

Prove that

$$\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\leq\frac{1}{y_1}+\frac{1}{y_2}+\cdots+\frac{1}{y_n}.$$

(b) Let $A=\{a_1,a_2,\ldots,a_n\}\subset\mathbb{N}$ be a set such that for all distinct subsets $B,C\subseteq A$, $\sum_{x\in B}x\neq\sum_{x\in C}x$. Prove that

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}<2.$$

Solution:

(a) Let $\pi_i=\frac{1}{x_iy_i}$, $\delta_i=x_i-y_i$ for all $1\leq i\leq n$. We are given that $\pi_1>\pi_2>\cdots>\pi_n>0$ and that $\sum_{i=1}^k\delta_i\geq 0$ for all $1\leq k\leq n$. Note that

$$\sum_{k=1}^n\left(\frac{1}{y_k}-\frac{1}{x_k}\right)=\sum_{k=1}^n\pi_k\delta_k$$

$$= \pi_n \sum_{i=1}^{n} \delta_i + \sum_{k=1}^{n-1} (\pi_k - \pi_{k+1}) (\delta_1 + \delta_2 + \cdots + \delta_k) \geq 0,$$

as desired.

(b) Assume without loss of generality that $a_1 < a_2 < \cdots < a_n$, and let $y_i = 2^{i-1}$ for all i. Clearly,

$$a_1 y_1 < a_2 y_2 < \cdots < a_n y_n.$$

For any k, the $2^k - 1$ sums made by choosing at least one of the numbers $a_1, a_2, \ldots, a_k$ are all distinct. Hence the largest of them, $\sum_{i=1}^{k} a_i$, must be at least $2^k - 1$. Thus for all $k = 1, 2, \ldots, n$ we have

$$a_1 + a_2 + \cdots + a_k \geq 2^k - 1 = y_1 + y_2 + \cdots + y_k.$$

Then by part (a), we must have

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} < \frac{1}{y_1} + \frac{1}{y_2} + \cdots + \frac{1}{y_n} = 2 - \frac{1}{2^{n-1}} < 2,$$

as desired.

IMO Selection Tests

Problem 1

(a) Show that out of any 39 consecutive positive integers, it is possible to choose one number with the sum of its digits divisible by 11.

(b) Find the first 38 consecutive positive integers, none with the sum of its digits divisible by 11.

Solution: Call an integer "deadly" if its sum of digits is divisible by 11, and let $d(n)$ equal the sum of the digits of a positive integer n.

If n ends in a 0, then the numbers $n, n+1, \ldots, n+9$ differ only in their units digits, which range from 0 to 9; hence $d(n), d(n+1), \ldots, d(n+9)$ is an arithmetic progression with common difference 1. Thus if $d(n) \not\equiv 1 \pmod{11}$, then one of these numbers is deadly.

Next suppose that if n ends in $k \geq 0$ nines. Then $d(n+1) = d(n) + 1 - 9k$: the last k digits of $n+1$ are 0's instead of 9's, and the next digit to the left is 1 greater than the corresponding digit in n.

Finally, suppose that n ends in a 0 and that $d(n) \equiv d(n+10) \equiv 1 \pmod{11}$. Since $d(n) \equiv 1 \pmod{11}$, we must have $d(n+9) \equiv$

$10 \pmod{11}$. If $n+9$ ends in k 9's, then we have $2 \equiv d(n+10) - d(n+9) \equiv 1 - 9k \Longrightarrow k \equiv 6 \pmod{11}$.

(a) Suppose we had 39 consecutive integers, none of them deadly. One of the first ten must end in a 0: call it n. Since none of $n, n+1, \ldots, n+9$ are deadly, we must have $d(n) \equiv 1 \pmod{11}$. Similarly, $d(n+10) \equiv 1 \pmod{11}$ and $d(n+20) \equiv 1 \pmod{11}$. From our third observation above, this implies that both $n+9$ and $n+19$ must end in at least six 9's. But this is impossible, because $n+10$ and $n+20$ can't both be multiples of one million!

(b) Suppose we have 38 consecutive numbers $N, N+1, \ldots, N+37$, none of which is deadly. By an analysis similar to that in part (a), none of the first nine can end in a 0; hence, $N+9$ must end in a 0, as must $N+19$ and $N+29$. Then we must have $d(N+9) \equiv d(N+19) \equiv 1 \pmod{11}$. Therefore $d(N+18) \equiv 10 \pmod{11}$; and furthermore, if $N+18$ ends in k 9's we must have $k \equiv 6 \pmod{11}$. The smallest possible such number is 999999, yielding the 38 consecutive numbers $999981, 999982, \ldots, 1000018$. And indeed, none of these numbers is deadly: their sums of digits are congruent to $1, 2, \ldots, 10, 1, 2, \ldots, 10, 1, 2, \ldots, 10, 2, 3, \ldots, 9$, and $10 \pmod{11}$, respectively.

Problem 2 Let ABC be an acute triangle with angle bisectors $\overline{BL}$ and $\overline{CM}$. Prove that $\angle A = 60^\circ$ if and only if there exists a point K on $\overline{BC}$ ($K \neq B, C$) such that triangle KLM is equilateral.

Solution: Let I be the intersection of lines BL and CM. Then $\angle BIC = 180^\circ - \angle ICB - \angle CBI = 180^\circ - \frac{1}{2}(\angle C + \angle B) = 180^\circ - \frac{1}{2}(180^\circ - \angle A) = 90^\circ + \angle A$, and thus $\angle BIC = 120^\circ$ if and only if $\angle A = 60^\circ$.

For the "only if" direction, suppose that $\angle A = 60^\circ$. Then let K be the intersection of $\overline{BC}$ and the internal angle bisector of $\angle BIC$; we claim that triangle KLM is equilateral. Since $\angle BIC = 120^\circ$, we know that $\angle MIB = \angle KIB = 60^\circ$. And since $\angle IBM = \angle IBK$ and $IB = IB$, by ASA congruency we have $\triangle IBM \cong \triangle IBK$; in particular, $IM = IK$. Similarly, $IL = IK$; and since $\angle KIL = \angle LIM = \angle MIK = 120^\circ$, we know that triangle KLM is equilateral.

For the "if" direction, suppose that K is on $\overline{BC}$ and triangle KLM is equilateral. Consider triangles BLK and BLM: $BL = BL$, $LM = LK$, and $\angle MBL = \angle KBL$. There is no SSA congruency, but we do then know that either $\angle LKB + \angle BML = 180^\circ$ or $\angle LKB = \angle BML$. But since $\angle KBM < 90^\circ$ and $\angle MLK = 60^\circ$, we know that $\angle LKB + \angle BML > 210^\circ$. Thus $\angle LKB = \angle BML$, whence $\triangle BLK \cong \triangle BLM$, and $BK = BM$. It follows that $IK = IM$. Similarly, $IL = IK$, and I is the circumcenter of triangle KLM. Thus $\angle LIM = 2\angle LKM = 120^\circ$, giving $\angle BIC = \angle LIM = 120^\circ$ and $\angle A = 60^\circ$.

Problem 3 Show that for any positive integer n, the number

$$S_n = \binom{2n+1}{0} \cdot 2^{2n} + \binom{2n+1}{2} \cdot 2^{2n-2} \cdot 3 + \cdots + \binom{2n+1}{2n} \cdot 3^n$$

is the sum of two consecutive perfect squares.

Solution: Let $\alpha = 1+\sqrt{3}$, $\beta = 1-\sqrt{3}$, and $T_n = \frac{1}{2}\left(\alpha^{2n+1} + \beta^{2n+1}\right)$. Note that $\alpha\beta = -2$, $\frac{\alpha^2}{2} = 2 + \sqrt{3}$, and $\frac{\beta^2}{2} = 2 - \sqrt{3}$. Also, applying the binomial expansion to $(1+\sqrt{3})^n$ and $(1-\sqrt{3})^n$, we find that $T_n = \sum_{k=0}^{n} \binom{2n+1}{2k} 3^k$—which is an integer for all n.

Applying the binomial expansion to $(2+\sqrt{3})^{2n+1}$ and $(2-\sqrt{3})^{2n+1}$ instead, we find that

$$\begin{aligned} S_n &= \frac{\left(\frac{\alpha^2}{2}\right)^{2n+1} + \left(\frac{\beta^2}{2}\right)^{2n+1}}{4} \\ &= \frac{\alpha^{4n+2} + \beta^{4n+2}}{2^{2n+3}} \\ &= \frac{\alpha^{4n+2} + 2(\alpha\beta)^{2n+1} + \beta^{4n+2}}{2^{2n+3}} + \frac{1}{2} \\ &= \frac{\left(\alpha^{2n+1} + \beta^{2n+1}\right)^2}{2^{2n+3}} + \frac{1}{2} \\ &= \frac{T_n^2}{2^{2n+1}} + \frac{1}{2}. \end{aligned}$$

Thus $2^{2n+1} S_n = T_n^2 + 2^{2n}$. Then $2^{2n} \mid T_n^2$ but $2^{2n+1} \nmid T_n^2$, and hence $T_n \equiv 2^n \pmod{2^{n+1}}$. Therefore

$$S_n = \frac{T_n^2}{2^{2n+1}} + \frac{1}{2} = \left(\frac{T_n - 2^n}{2^{n+1}}\right)^2 + \left(\frac{T_n + 2^n}{2^{n+1}}\right)^2$$

is indeed the sum of two consecutive perfect squares.

Problem 4 Show that for all positive real numbers $x_1, x_2, \cdots, x_n$ such that

$$x_1 x_2 \cdots x_n = 1,$$

the following inequality holds:

$$\frac{1}{n-1+x_1} + \frac{1}{n-1+x_2} + \cdots + \frac{1}{n-1+x_n} \leq 1.$$

First Solution: Let $a_1 = \sqrt[n]{x_1}$, $a_2 = \sqrt[n]{x_2}$, $\ldots$, $a_n = \sqrt[n]{x_n}$. Then $a_1 a_2 \cdots a_n = 1$ and

$$\frac{1}{n-1+x_k} = \frac{1}{n-1+a_k^n} = \frac{1}{n-1+\frac{a_k^{n-1}}{a_1 \cdots a_{k-1} a_{k+1} \cdots a_n}}$$

$$\leq \frac{1}{n-1+\frac{(n-1)a_k^{n-1}}{a_1^{n-1}+\cdots+a_{k-1}^{n-1}+a_{k+1}^{n-1}+\cdots+a_n^{n-1}}}$$

by the AM-GM Inequality. It follows that

$$\frac{1}{n-1+x_k} \leq \frac{a_1^{n-1}+\cdots+a_{k-1}^{n-1}+a_{k+1}^{n-1}+\cdots+a_n^{n-1}}{(n-1)(a_1^{n-1}+a_2^{n-1}+\cdots+a_n^{n-1})}.$$

Summing up yields $\sum_{k=1}^{n} \frac{1}{n-1+x_k} \leq 1$, as desired.

Second Solution: Let $f(x) = \frac{1}{n-1-x}$; we wish to prove that $\sum_{i=1}^{n} f(x_i) \leq 1$. Note that

$$f(y) + f(z) = \frac{2(n-1)+y+z}{(n-1)^2+yz+(y+z)(n-1)}.$$

Suppose that any of our x_i does not equal 1; then we have $x_j < 1 < x_k$ for some j, k. If $f(x_j) + f(x_k) \leq \frac{1}{n-1}$, then all the other $f(x_i)$ are less than $\frac{1}{n-1}$. But then $\sum_{i=1}^{n} f(x_i) < 1$ and we are done.

Otherwise, $f(x_j) + f(x_k) > \frac{1}{n-1}$. Now set $x_j' = 1$ and $x_k' = x_j x_k$; then $x_j' x_k' = x_j x_k$, while $x_j < 1 < x_k \Rightarrow (1-x_j)(x_k-1) > 0 \Rightarrow x_j + x_k > x_j' + x_k'$. Let $a = 2(n-1)$, $b = (n-1)^2 + x_j x_k = (n-1)^2 + x_j' x_k'$, and $c = \frac{1}{n-1}$; also let $m = x_j + x_k$ and $m' = x_j' + x_k'$. Then we have

$$f(x_j) + f(x_k) = \frac{a+cm}{b+m} \quad \text{and} \quad f(x_j') + f(x_k') = \frac{a+cm'}{b+m'}.$$

Now $\frac{a+cm}{b+m} > c \Rightarrow a + cm > (b+m)c \Rightarrow \frac{a}{b} > c$; and from here,

$$(a-bc)(m-m') > 0 \Rightarrow \frac{a+cm'}{b+m'} > \frac{a+cm}{b+m}$$

$$\Rightarrow f(x_j') + f(x_k') = f(x_j) + f(x_k).$$

Hence as long as no pair $f(x_j) + f(x_k) \le \frac{1}{n-1}$ and the x_i do not all equal 1, we can continually replace pairs x_j and x_k (neither equal to 1) by 1 and $x_j x_k$. This keeps the product $x_1 x_2 \cdots x_n$ equal to 1 while increasing $\sum_{i=1}^n f(x_i)$. Then eventually our new $\sum_{i=1}^n f(x_i) \le 1$, which implies that our original $\sum_{i=1}^n f(x_i)$ was also at most 1. This completes the proof.

Third Solution: Suppose, for the sake of contradiction, that $\dfrac{1}{n-1+x_1} + \dfrac{1}{n-1+x_2} + \cdots + \dfrac{1}{n-1+x_n} > 1$. Letting $y_i = x_i/(n-1)$ for $i = 1, 2, \ldots, n$, we have

$$\frac{1}{1+y_1} + \frac{1}{1+y_2} + \cdots + \frac{1}{1+y_n} > n-1$$

and hence

$$\begin{aligned}
\frac{1}{1+y_1} &> \left(1 - \frac{1}{1+y_2}\right) + \left(1 - \frac{1}{1+y_3}\right) + \cdots + \left(1 - \frac{1}{1+y_n}\right) \\
&= \frac{y_2}{1+y_2} + \frac{y_3}{1+y_3} + \cdots + \frac{y_n}{1+y_n} \\
&> (n-1) \sqrt[n-1]{\frac{y_2 y_3 \cdots y_n}{(1+y_2)(1+y_3)\cdots(1+y_n)}}.
\end{aligned}$$

We have analagous inequalities with $\frac{1}{1+y_2}, \frac{1}{1+y_n}, \ldots, \frac{1}{1+y_n}$ on the left hand side; multiplying these n inequalities together gives

$$\begin{aligned}
&\prod_{k=1}^n \frac{1}{1+y_k} > (n-1)^n \frac{y_1 y_2 \cdots y_n}{(1+y_1)(1+y_2)\cdots(1+y_n)} \\
&1 > ((n-1)y_1)((n-1)y_2)\cdots((n-1)y_n) = x_1 x_2 \cdots x_n,
\end{aligned}$$

a contradiction.

Problem 5 Let $x_1, x_2, \ldots, x_n$ be distinct positive integers. Prove that

$$x_1^2 + x_2^2 + \cdots + x_n^2 \ge \frac{(2n+1)(x_1 + x_2 + \cdots + x_n)}{3}.$$

Solution: Assume without loss of generality that $x_1 < x_2 < \cdots < x_n$. We will prove that $3x_k^2 \ge 2(x_1 + x_2 + \cdots + x_{k-1}) + (2k+1)x_k$; then, summing this inequality over $k = 1, 2, \ldots, n$, we will have the desired inequality.

First, $x_1 + x_2 + \cdots + x_{k-1} \leq (x_k - (k-1)) + (x_k - (k-2)) + \cdots + (x_k - 1) = (k-1)x_k - \frac{k(k-1)}{2}$. Thus,

$$2(x_1 + x_2 + \cdots + x_{k-1}) + (2k+1)x_k \leq (4k-1)x_k - k(k-1).$$

Now

$$3x_k^2 - [(4k-1)x_k - k(k-1)] = x_k(3x_k - 4k + 1) + k(k-1),$$

which is minimized at $x_k = \frac{2}{3}k$. Then since $x_k \geq k$,

$$x_k(3x_k - 4k + 1) + k(k-1) \geq k(3k - 4k + 1) + k(k-1) = 0$$

so

$$3x_k^2 \geq (4k-1)x_k - k(k-1) \geq 2(x_1 + x_2 + \cdots + x_{k-1}) + (2k+1)x_k,$$

and we have finished.

Problem 6 Prove that for any integer n, $n \geq 3$, there exist n positive integers $a_1, a_2, \ldots, a_n$ in arithmetic progression, and n positive integers $b_1, b_2, \ldots, b_n$ in geometric progression, such that

$$b_1 < a_1 < b_2 < a_2 < \cdots < b_n < a_n.$$

Give one example of such progressions $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ each having at least 5 terms.

Solution: Our strategy is to find progressions where $b_n = a_{n-1} + 1$ and $b_{n-1} = a_{n-2} + 1$. Write $d = a_{n-1} - a_{n-2}$. Then for all $2 \leq i, j \leq n-1$ we have $b_{i+1} - b_i \leq b_n - b_{n-1} = d$, so that $b_j = b_n + \sum_{i=j}^{n-1}(b_i - b_{i+1}) > a_{n-1} + (n-j)d = a_{j-1}$.

And if we ensure that $b_1 < a_1$, then $b_j = b_1 + \sum_{i=1}^{j-1}(b_{i+1} - b_i) \leq a_1 + (j-1)d = a_j$ for all j, so the chain of inequalities is satisfied.

Let $b_1, b_2, \ldots, b_n$ equal $k^{n-1}, k^{n-2}(k+1), \ldots, k^0(k+1)^{n-1}$, where k is a value to be determined later. Also set $a_{n-1} = b_n - 1$ and $a_{n-2} = b_{n-1} - 1$, and define the other a_i accordingly. Then $d = a_n - a_{n-1} = b_n - b_{n-1} = (k+1)^{n-2}$, and $a_1 = (k+1)^{n-2}(k+3-n) - 1$. Thus, we need only pick k such that

$$(k+1)^{n-2}(k+3-n) - 1 - k^{n-1} > 0.$$

Viewing the left hand side as a polynomial in k, the coefficient of k^{n-1} is zero but the coefficient of k^{n-2} is 1. Therefore, it is positive for sufficiently large k and we can indeed find satisfactory sequences $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$.

For $n = 5$, we seek k such that

$$(k+1)^3(k-2) - 1 - k^4 > 0.$$

Computation shows that $k = 5$ works, yielding

$$625 < 647 < 750 < 863 < 900 < 1079 < 1080 < 1295 < 1296 < 1511.$$

Problem 7 Let a be a positive real number and $\{x_n\}$ $(n \geq 1)$ be a sequence of real numbers such that $x_1 = a$ and

$$x_{n+1} \geq (n+2)x_n - \sum_{k=1}^{n-1} kx_k,$$

for all $n \geq 1$. Show that there exists a positive integer n such that $x_n > 1999!$.

Solution: We will prove by induction on $n \geq 1$ that

$$x_{n+1} > \sum_{k=1}^{n} kx_k > a \cdot n!.$$

For $n = 1$, we have $x_2 \geq 3x_1 > x_1 = a$.

Now suppose that the claim holds for all values up through n. Then

$$\begin{aligned} x_{n+2} &\geq (n+3)x_{n+1} - \sum_{k=1}^{n} kx_k \\ &= (n+1)x_{n+1} + 2x_{n+1} - \sum_{k=1}^{n} kx_k \\ &> (n+1)x_{n+1} + 2\sum_{k=1}^{n} kx_k - \sum_{k=1}^{n} kx_k = \sum_{k=1}^{n+1} kx_k, \end{aligned}$$

as desired. Furthermore, $x_1 > 0$ by definition and $x_2, x_3, \ldots, x_n$ are also positive by the induction hypothesis; thus $x_{n+2} > (n+1)x_{n+1} > (n+1)(a \cdot n!) = a \cdot (n+1)!$. This completes the inductive step.

Therefore for sufficiently large n, we have $x_{n+1} > n! \cdot a > 1999!$.

Problem 8 Let O, A, B, C be variable points in the plane such that $OA = 4, OB = 2\sqrt{3}$ and $OC = \sqrt{22}$. Find the maximum possible area of triangle ABC.

Solution: We first look for a tetrahedron $MNPQ$ with the following properties: (i) if H is the foot of the perpendicular from M to plane (NPQ), then $HN = 4$, $HP = 2\sqrt{3}$, and $HQ = \sqrt{22}$; and (ii) lines MN, MP, MQ are pairwise perpendicular.

If such a tetrahedron exists, then let $O = H$ and draw triangle ABC in plane (NPQ). We have $MA = \sqrt{MO^2 + OA^2} = \sqrt{MH^2 + HN^2} = MN$, and similarly $MB = MP$ and $MC = MQ$. Hence

$$\begin{aligned}[ABCM] &\leq \frac{1}{3}[ABM] \cdot MC \leq \frac{1}{3} \cdot \left(\frac{1}{2}MA \cdot MB\right) \cdot MC \\ &= \frac{1}{3} \cdot \left(\frac{1}{2}MN \cdot MP\right) \cdot MQ = [MNPQ],\end{aligned}$$

and therefore the maximum possible area of triangle $[ABC]$ is $[NPQ]$.

It remains to find tetrahedron $MNPQ$. Let $x = MH$; then $MN = \sqrt{x^2+16}$, $MP = \sqrt{x^2+12}$, and $MQ = \sqrt{x^2+22}$. By the Pythagorean Theorem on triangle MHN, we have $NH = 4$. Next let lines NH and PQ intersect at R; then in similar right triangles MHN and MRN, we have $MR = MH \cdot \frac{MN}{NH} = \frac{1}{4}(x^2+16)$.

Since $MN \perp (MPQ)$ we have $MN \perp PQ$; and since $MH \perp (NPQ)$ we have $MH \perp PQ$ as well. Hence $PQ \perp (MNHR)$, so that $\overline{MR}$ is an altitude in the right triangle MPQ. Therefore $MR \cdot PQ = 2[MPQ] = MP \cdot MQ$, or (after squaring both sides)

$$\sqrt{\frac{(x^2+16)^2}{16} - (x^2+16)}\sqrt{x^2+12+x^2+22} = \sqrt{x^2+12}\sqrt{x^2+22}.$$

Setting $4y = x^2 + 16$ and squaring both sides, we obtain

$$(y^2 - 4y)(8y+2) = (4y-4)(4y+6)(y-6)(4y^2+y-2) = 0.$$

Since $y = \frac{1}{4}(x^2+16) > 4$, the only solution is $y = 6 \Longrightarrow x = \sqrt{8}$. Then by taking $MN = \sqrt{24}, MP = \sqrt{20}$, $MQ = \sqrt{30}$, we get the required tetrahedron.

Then $[MNPQ]$ equals both $\frac{1}{3}MH \cdot [MPQ]$ and $\frac{1}{6}MN \cdot MP \cdot MQ$. Setting these two expressions equal, we find that the maximum area of $[ABC]$ is

$$[NPQ] = \frac{MN \cdot MP \cdot MQ}{2 \cdot MH} = 15\sqrt{2}.$$

Problem 9 Let a, n be integers and let p be a prime such that $p > |a| + 1$. Prove that the polynomial $f(x) = x^n + ax + p$ cannot be represented as a product of two nonconstant polynomials with integer coefficients.

Solution: Let z be a complex root of the polynomial. We shall prove that $|z| > 1$. Suppose $|z| \leq 1$. Then, $z^n + az = -p$, we deduce that

$$p = |z^n + az| = |z||z^{n-1} + a| \leq |z^{n-1}| + |a| \leq 1 + |a|,$$

which contradicts the hypothesis.

Now, suppose $f = gh$ is a decomposition of f into nonconstant polynomials with integer coefficients. Then $p = f(0) = g(0)h(0)$, and either $|g(0)| = 1$ or $|h(0)| = 1$. Assume without loss of generality that $|g(0)| = 1$. If $z_1, z_2, \ldots, z_k$ are the roots of g then they are also roots of f. Therefore

$$1 = |g(0)| = |z_1 z_2 \cdots z_k| = |z_1||z_2| \cdots |z_k| > 1,$$

a contradiction.

Problem 10 Two circles meet at A and B. Line ℓ passes through A and meets the circles again at C and D respectively. Let M and N be the midpoints of arcs $\widehat{BC}$ and $\widehat{BD}$, which do not contain A, and let K be the midpoint of $\overline{CD}$. Prove that $\angle MKN = 90^\circ$.

Solution: All angles are directed modulo 180°. Let M' be the reflection of M across K. Then triangles MKC and $M'KD$ are congruent in that order, and $M'D = MC$. Because M is the midpoint of $\widehat{BC}$, we have $M'D = MC = MB$; and similarly, because N is the midpoint of $\widehat{BD}$ we have $BN = DN$. Next, $\angle MBN = (180^\circ - \angle ABM) + (180^\circ - \angle NBA) = \angle MCA + \angle ADN = \angle M'DA + \angle ADN = \angle M'DN$. Hence $\triangle M'DN \cong \triangle MBN$, and $MN = M'N$. Therefore $\overline{NK}$ is the median to the base of isosceles triangle MNM', so it is also an altitude and $NK \perp MK$.

Problem 11 Let $n \geq 3$ and $A_1, A_2, \ldots, A_n$ be points on a circle. Find the greatest number of acute triangles having vertices in these points.

Solution: Without loss of generality assume the points $A_1, A_2, \ldots, A_n$ are ordered in that order counterclockwise; also, take indices modulo n os that $A_{n+1} = A_1, A_{n+2} = A_2$, and so on. Denote by A_iA_j

the arc of the circle starting from A_i and ending in A_j in the counterclockwise direction; let $m(A_iA_j)$ denote the angle measure of the arc; and call an arc A_iA_j obtuse if $m(A_iA_j) \geq 180°$. Obviously, $m(A_iA_j) + m(A_jA_i) = 360°$, and thus at least one of the arcs A_iA_j and A_jA_i is obtuse. Let x_s be the number of obtuse arcs each having exactly $s-1$ points along their interiors. If $s \neq \frac{n}{2}$, then for each i at least one of the arcs A_iA_{i+s} or $A_{i+s}A_i$ is obtuse; summing over all i, we deduce that

$$x_s + x_{n-s} \geq n \tag{1}$$

for every $s \neq \frac{n}{2}$; and similar reasoning shows that this inequality also holds even when $s = \frac{n}{2}$. For all s, equality holds if and only if there are no diametrically opposite points A_i, A_{i+s}.

The number of non-acute triangles $A_iA_jA_k$ equals the number of non-acute angles $\angle A_iA_jA_k$. And for each obtuse arc A_iA_k containing $s-1$ points in its interior, there are $n-s-1$ non-acute angles $A_iA_jA_k$: namely, with those A_j in the interior of arc A_kA_i. It follows that the number N of non-acute triangles is

$$N = x_1(n-2) + x_2(n-3) + \ldots + x_{n-3}\cdot 2 + x_{n-2}\cdot 1 + x_{n-1}\cdot 0.$$

By regrouping terms and using (1) we obtain

$$\begin{aligned} N &\geq \sum_{s=1}^{\frac{n-1}{2}} (s-1)\cdot(x_{n-s}+x_s) \\ &\geq n\left(1+2+\cdots+\frac{n-3}{2}\right) = \frac{n(n-1)(n-3)}{8} \end{aligned}$$

if n is odd, and

$$\begin{aligned} N &\geq \sum_{s=1}^{\frac{n-2}{2}} (s-1)\cdot(x_{n-s}+x_s) + \frac{n-2}{2}x_{n/2} \\ &\geq n\left(1+2+\cdots+\frac{n-4}{2}\right) + \frac{n-2}{2}\cdot\frac{n}{2} = \frac{n(n-2)^2}{8} \end{aligned}$$

if n is even.

Equality is obtained when there are no diametrically opposite points, and when $x_k = 0$ for $k < \frac{n}{2}$. When n is odd, for instance, this happens when the points form a regular n-gon; and when n is even, equality occurs when $m(A_1A_2) = m(A_2A_3) = \cdots = m(A_{n-1}A_n) = \frac{360°}{n} + \epsilon$ where $0 < \epsilon < \frac{360°}{n^2}$.

Finally, note that the total number of triangles having vertices in the n points is $\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$. Subtracting the minimum values of N found above, we find that the maximum number of acute angles is $\frac{(n-1)n(n+1)}{24}$ if n is odd, and $\frac{(n-2)n(n+2)}{24}$ if n is even.

Problem 12 The scientists at an international conference are either *native* or *foreign*. Each native scientist sends exactly one message to a foreign scientist and each foreign scientist sends exactly one message to a native scientist, although at least one native scientist does not receive a message. Prove that there exists a set S of native scientists and a set T of foreign scientists such that the following conditions hold: (i) the scientists in S sent messages to exactly those foreign scientists who were not in T (that is, every foreign scientist not in T received at least one message from somebody in S, but none of the scientists in T received any messages from scientists in S); and (ii) the scientists in T sent messages to exactly those native scientists not in S.

Solution: Let A be the set of native scientists and B be the set of foreign scientists. Let $f\colon A \to B$ and $g\colon B \to A$ be the functions defined as follows: $f(a)$ is the foreign scientist receiving a message from a, and $g(b)$ is the native scientist receiving a message from b. If such subsets S, T exist we must have $T = B - f(S)$; hence we have to prove that there exists a subset $S \subseteq A$ such that $A - S = g(B - f(S))$.

For each subset $X \subseteq A$, let $h(X) = A - g(B - f(X))$. If $X \subseteq Y$, then $f(X) \subseteq f(Y) \Longrightarrow B - f(Y) \subseteq B - f(X) \Longrightarrow g(B - f(Y)) \subseteq g(B - f(X)) \Longrightarrow A - g(B - f(X)) \subseteq A - g(B - f(Y)) \Longrightarrow h(X) \subseteq h(Y)$.

Let $M = \{X \subseteq A \mid h(X) \subseteq X\}$. The set M is nonempty, since $A \in M$. Furthermore, it is given that g is not surjective, so that some native scientist a_0 is never in $g(B - f(X))$ and thus always in $h(X)$ for all $X \subseteq A$. Thus every subset in M contains a_0, so that $S = \bigcap_{X \in M} X$ is nonempty.

From the definition of S we have $h(S) \subseteq S$. And from the monotony of h it follows that $h(h(S)) \subseteq h(S)$; thus, $h(S) \in M$ and $S \subset h(S)$. Combining these results, we have $S = h(S)$, as desired.

Problem 13 A polyhedron P is given in space. Determine whether there must exist three edges of P that can be the sides of a triangle.

Solution: The answer is “yes.” Assume, for the purpose of contradiction, that there exists a polyhedron P in which no three edges can form the sides of a triangle. Let the edges of P be $E_1, E_2, E_3, \ldots, E_n$, in non-increasing order of length; let e_i be the length of E_i. Consider the two faces that share E_1: for each of those faces, the sum of the lengths of all its edges except E_1 is greater than e_1. Therefore,

$$e_2 + e_3 + \cdots + e_n > 2e_1.$$

But, since we are assuming that no three edges of P can form the sides of a triangle, we have $e_{i+1} + e_{i+2} \leq e_i$ for $i = 1, 2, \ldots, n-2$. Hence,

$$\begin{aligned} &2(e_2 + e_3 + \cdots + e_n) \\ &\quad = e_2 + (e_2 + e_3) + (e_3 + e_4) + \cdots + (e_{n-1} + e_n) + e_n \\ &\quad \leq e_2 + (e_1) + (e_2) + \cdots + (e_{n-2}) + e_n, \end{aligned}$$

so

$$e_2 + e_3 + \cdots + e_n \leq e_1 + e_2 - e_{n-1} < e_1 + e_1 + 0 = 2e_1,$$

a contradiction. Thus, our assumption was incorrect and some three edges *can* be the sides of a triangle.

1.17 Russia

Fourth round

Problem 8.1 A father wishes to take his two sons to visit their grandmother, who lives 33 kilometers away. He owns a motorcycle whose maximum speed is 25 km/h. With one passenger, its maximum speed drops to 20 km/h. (He cannot carry two passengers.) Each brother walks at a speed of 5 km/h. Show that all three of them can reach the grandmother's house in 3 hours.

Solution: Have the father drive his first son 24 kilometers, which takes $\frac{6}{5}$ hours; then drive back to meet his second son 9 kilometers from home, which takes $\frac{3}{5}$ hours; and finally drive his second son $\frac{6}{5}$ more hours.

Each son spends $\frac{6}{5}$ hours riding 24 kilometers, and $\frac{9}{5}$ hours walking 9 kilometers. Thus they reach their grandmother's house in exactly 3 hours — as does the father, who arrives at the same time as his second son.

Problem 8.2 The natural number A has the following property: the sum of the integers from 1 to A, inclusive, has decimal expansion equal to that of A followed by three digits. Find A.

Solution: We know that

$$\begin{aligned} k &= (1 + 2 + \cdots + A) - 1000A \\ &= \frac{A(A+1)}{2} - 1000A = A\left(\frac{A+1}{2} - 1000\right) \end{aligned}$$

is between 0 and 999, inclusive. If $A < 1999$ then k is negative. If $A \geq 2000$ then $\frac{A+1}{2} - 1000 \geq \frac{1}{2}$ and $k \geq 1000$. Therefore $A = 1999$, and indeed $1 + 2 + \cdots + 1999 = 1999000$.

Problem 8.3 On sides BC, CA, AB of triangle ABC lie points A_1, B_1, C_1 such that the medians A_1A_2, B_1B_2, C_1C_2 of triangle $A_1B_1C_1$ are parallel to AB, BC, CA, respectively. Determine in what ratios the points A_1, B_1, C_1 divide the sides of ABC.

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[1] Problems are numbered as they appeared in the contests. Problems that appeared more than once in the contests are only printed once in this book.

First Solution: A_1, B_1, C_1 divide sides BC, CA, AB in $1:2$ ratios (so that $\frac{BA_1}{A_1C} = \frac{1}{2}$, and so on).

Lemma. *In any triangle XYZ, the medians can be translated to form a triangle. Furthermore, the medians of this new triangle are parallel to the sides of triangle XYZ.*

Proof: Let x, y, z denote the vectors $\overrightarrow{YZ}$, $\overrightarrow{ZX}$, $\overrightarrow{XY}$ respectively; then $x + y + z = \overrightarrow{0}$. Also, the vectors representing the medians of triangle XYZ are $m_x = z + \frac{x}{2}$, $m_y = x + \frac{y}{2}$, $m_z = y + \frac{z}{2}$. These vectors add up to $\frac{3}{2}(x+y+z) = \overrightarrow{0}$, so the medians indeed form a triangle.

Furthermore, the vectors representing the medians of the new triangle are $m_x + \frac{m_y}{2} = x + y + z - \frac{3}{4}y = -\frac{3}{4}y$, and similarly $-\frac{3}{4}z$ and $-\frac{3}{4}x$. Therefore, these medians are parallel to XZ, YX, and ZY. ■

Let D, E, F be the midpoints of sides BC, CA, AB, and let l_1, l_2, l_3 be the segments A_1A_2, B_1B_2, C_1C_2.

Since l_1, l_2, l_3 are parallel to AB, BC, CA, the medians of the triangle formed by l_1, l_2, l_3 are parallel to CF, AD, BE. But from the lemma, they are also parallel to B_1C_1, C_1A_1, A_1B_1.

Therefore, $BE \parallel A_1B_1$, and hence $\triangle BCE \sim \triangle A_1CB_1$. Then

$$\frac{B_1C}{AC} = \frac{1}{2} \cdot \frac{B_1C}{EC} = \frac{1}{2} \cdot \frac{A_1C}{BC} = \frac{1}{2}\left(1 - \frac{A_1B}{CB}\right).$$

Similarly

$$\frac{C_1A}{BA} = \frac{1}{2}\left(1 - \frac{B_1C}{AC}\right)$$
$$\frac{A_1B}{CB} = \frac{1}{2}\left(1 - \frac{C_1A}{BA}\right).$$

Solving these three equations gives

$$\frac{B_1C}{AC} = \frac{C_1A}{BA} = \frac{A_1B}{CB} = \frac{1}{3},$$

as claimed; and it is straightforward to verify with the above equations that these ratio indeed work.

Second Solution: As above, we know that $A_1B_1 \parallel BE$, $B_1C_1 \parallel AD$, $C_1A_1 \parallel CF$.

Let A', B', C' be the points dividing the sides BC, CA, AB in $1:2$ ratios — since $\frac{CA'}{CB} = \frac{CB'}{\frac{1}{2}CA}$, we know $A'B' \parallel BE \parallel A_1B_1$, and so on.

Suppose by way of contradiction that A_1 were closer to B than A'. Then since $A_1B_1 \parallel A'B'$, B_1 is farther from C than B'. Similarly, C_1 is closer to A than C', and A_1 is *farther* from B than A' — a contradiction.

Likewise, A_1 cannot be farther from B than A'. Thus $A_1 = A'$, $B_1 = B'$, and $C_1 = C'$.

Problem 8.4 We are given 40 balloons, the air pressure inside each of which is unknown and may differ from balloon to balloon. It is permitted to choose up to k of the balloons and equalize the pressure in them (to the arithmetic mean of their respective original pressures). What is the smallest k for which it is always possible to equalize the pressures in all of the balloons?

Solution: $k = 5$ is the smallest such value.

First suppose that $k = 5$. Note that we can equalize the pressure in any 8 balloons: first divide them into two groups of four $\{A, B, C, D\}$ and $\{E, F, G, H\}$ and equalize the pressure in each group; then equalize the pressure in $\{A, B, E, F\}$ and $\{C, D, G, H\}$.

Then divide the 40 balloons into eight "5-groups" of five and equalize the pressure in each group. Then form five new groups of eight — containing one balloon from each "5-group" — and equalize the pressure in each of these new groups.

Now suppose that $k \leq 4$. Let $b_1, b_2, \ldots, b_{40}$ denote the original air pressures inside the balloons. It is simple to verify that the pressure in each balloon can always be written as a linear combination $a_1b_1 + \cdots + a_{40}b_{40}$, where the a_i are rational with denominators not divisible by any primes except 2 and 3. Thus if we the b_j are linearly independent over the rationals (say, if $b_j = e^j$), we can never obtain

$$\frac{1}{40}b_1 + \frac{1}{40}b_2 + \cdots + \frac{1}{40}b_{40}$$

in a balloon. In this case, we can never equalize the pressures in all 40 balloons.

Problem 8.5 Show that the numbers from 1 to 15 cannot be divided into a group A of 2 numbers and a group B of 13 numbers in such a way that the sum of the numbers in B is equal to the product of the numbers in A.

Solution: Suppose by way of contradiction this were possible, and let a and b be the two numbers in A. Then we have

$$(1+2+\cdots+15)-a-b=ab$$
$$120=ab+a+b$$
$$121=(a+1)(b+1),$$

Since a and b are integers between 1 and 15, the only possible solution to this equation is $(a,b)=(10,10)$. But a and b must be distinct, a contradiction.

Problem 8.6 Given an acute triangle ABC, let A_1 be the reflection of A across the line BC, and let C_1 be the reflection of C across the line AB. Show that if A_1, B, C_1 lie on a line and $C_1B=2A_1B$, then $\angle CA_1B$ is a right angle.

Solution: By the given reflections, we have $\triangle ABC \cong \triangle ABC_1 \cong \triangle A_1BC$.

Since $\angle B$ is acute, C_1 and A lie on the same side of BC. Thus C_1 and A_1 lie on opposite sides of BC as well.

Then since C_1, B, A_1 lie on a line we have

$$\begin{aligned}180^\circ &= \angle C_1BA + \angle ABC + \angle CBA_1\\ &= \angle ABC + \angle ABC + \angle ABC,\end{aligned}$$

so that $\angle ABC = 60^\circ$. Also we know that

$$C_1B = 2A_1B \Longrightarrow CB = 2AB,$$

implying that triangle ABC is a 30°-60°-90° triangle and $\angle CA_1B = \angle BAC = 90^\circ$.

Problem 8.7 In a box lies a complete set of 1×2 dominoes. (That is, for each pair of integers i,j with $0\le i\le j\le n$, there is one domino with i on one square and j on the other.) Two players take turns selecting one domino from the box and adding it to one end of an open (straight) chain on the table, so that adjacent dominoes have the same numbers on their adjacent squares. (The first player's move may be any domino.) The first player unable to move loses. Which player wins with correct play?

Solution: The first player has a winning strategy. If $n=0$, this is clear. Otherwise, have the first player play the domino $(0,0)$ and suppose the second player plays $(0, a)$; then have the first player play (a, a).

At this point, the second player faces a chain whose ends are either 0 or a; also, the domino $(0, k)$ is on the table if and only if the domino (a, k) is on the table. In such a "good" situation, if the second player plays $(0, k)$ the first player can play (k, a) next to it; and if the second player plays (a, k) the first player can play $(k, 0)$. In both cases, the same conditions for a "good" situation occur.

Therefore the first player can always play a domino with this strategy, forcing the second player to lose.

Problem 8.8 An open chain of 54 squares of side length 1 is made so that each pair of consecutive squares is joined at a single vertex, and each square is joined to its two neighbors at opposite vertices. Is it possible to cover the surface of a $3 \times 3 \times 3$ cube with this chain?

Solution: It is not possible; suppose by way of contradiction it were.

Create axes so that the cube has corners at $(3i, 3j, 3k)$ for $i, j, k \in \{0, 1\}$, and place the chain onto the cube. Imagine that every two adjacent squares in the chain are connected by pivots, and also let the start and end vertices of the chain be "pivots."

Consider some pivot P at (x, y, z); then the next pivot Q in the chain is either at $(x, y \pm 1, z \pm 1)$, $(x \pm 1, y, z \pm 1)$, or $(x \pm 1, y \pm 1, z)$. In any case, the sum of the coordinates of P has the same parity as the sum of the coordinates of Q — and hence *all* the pivots' sums of coordinates have the same parity. Suppose without loss of generality the sums are even.

Form a graph whose vertices are the lattice points on the cube with even sums of coordinates; and join two vertices with an edge if the two lattice points are opposite corners of a unit square. Every square in our chain contains one of these edges — but since there are exactly 54 such edges (one across each unit square on the cube's surface), and 54 squares in our chain, every edge is used exactly once. Then as we travel from pivot to pivot along our chain, we create an Eulerian path visiting all the edges. But four vertices — at $(0, 0, 0)$, $(0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 0)$ — have odd degree 3, so this is impossible.

Problem 9.1 Around a circle are written all of the positive integers from 1 to N, $N \geq 2$, in such a way that any two adjacent integers have at least one common digit in their decimal expansions. Find the smallest N for which this is possible.

Solution: $N = 29$. Since 1 must be adjacent to two numbers, we must have $N \geq 11$. But then 9 must be adjacent to two numbers, and the next smallest numbers containing 9 as a digit are 19 and 29. Therefore $N \geq 29$, and indeed $N = 29$ suffices:

$$19, 9, 29, 28, 8, 18, 17, 7, 27, \ldots, 13, 3, 23, 2, 22, 21, 20, 12, 11, 10, 1.$$

Problem 9.2 In triangle ABC, points D and E are chosen on side CA such that $AB = AD$ and $BE = EC$ (E lying between A and D). Let F be the midpoint of the arc BC of the circumcircle of ABC. Show that B, E, D, F lie on a circle.

Solution: Let I be the incenter of triangle ABC, and notice that

$$\begin{aligned}\angle BIC &= 180^\circ - \angle ICB - \angle CBI \\ &= 180^\circ - \frac{\angle B}{2} - \frac{\angle C}{2} \\ &= 90^\circ + \frac{\angle A}{2}.\end{aligned}$$

Also, since $AD = AB$ we have $\angle ADB = 90^\circ - \frac{\angle A}{2}$ and $\angle BDC = 180^\circ - \angle ADB = 90^\circ + \frac{\angle A}{2}$. Therefore, $BIDC$ is cyclic.

Some angle-chasing shows that that B, I, and C lie on a circle with center F. Thus D lies on this circle, $FD = FC$, and $\angle FDC = \angle DCF$.

Also, since $BE = EC$, we have $\angle CBE = \angle C$. Combining these facts, we have

$$\begin{aligned}180^\circ - \angle EDF &= \angle FDC \\ &= \angle DCF \\ &= \angle ACF \\ &= \angle C + \frac{\angle A}{2} \\ &= \angle CBE + \angle FBC \\ &= \angle FBE.\end{aligned}$$

Therefore $BEDF$ is cyclic, as desired.

Problem 9.3 The product of the positive real numbers x, y, z is 1. Show that if

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq x+y+z,$$

then

$$\frac{1}{x^k}+\frac{1}{y^k}+\frac{1}{z^k} \geq x^k+y^k+z^k$$

for all positive integers k.

First Solution: Write $x=\frac{a}{b}$, $y=\frac{b}{c}$, $z=\frac{c}{a}$ for some positive numbers a, b, c. (For example, we could take $a=1, b=\frac{1}{x}, c=\frac{1}{xy}$.) The given equation becomes

$$\begin{aligned} &\frac{b}{a}+\frac{c}{b}+\frac{a}{c} \geq \frac{a}{b}+\frac{b}{c}+\frac{c}{a} \\ &\iff a^2b+b^2c+c^2a \geq ab^2+bc^2+ca^2 \\ &\iff 0 \geq (a-b)(b-c)(c-a). \end{aligned}$$

For any positive integer k, write $A=a^k$, $B=b^k$, $C=c^k$. Then $a>b \iff A>B$ and $a<b \iff A<B$, and so on. Thus we also know that $0 \geq (A-B)(B-C)(C-A)$, and

$$\begin{aligned} &0 \geq (A-B)(B-C)(C-A) \\ &\iff \frac{B}{A}+\frac{C}{B}+\frac{A}{C} \geq \frac{A}{B}+\frac{B}{C}+\frac{C}{A} \\ &\iff \frac{1}{x^k}+\frac{1}{y^k}+\frac{1}{z^k} \geq x^k+y^k+z^k, \end{aligned}$$

as desired.

Second Solution: The inequality

$$0 \geq (a-b)(b-c)(c-a)$$

might spark this realization: dividing through by abc we have

$$0 \geq (x-1)(y-1)(z-1).$$

Indeed,

$$\begin{aligned} &(x-1)(y-1)(z-1) = xyz+x+y+z-xy-yz-zx-1 \\ &= x+y+z-\frac{1}{z}-\frac{1}{x}-\frac{1}{y} \leq 0. \end{aligned}$$

Therefore, we have

$$\begin{aligned} (x-1)(y-1)(z-1) &\le 0 \\ \Rightarrow (x^k-1)(y^k-1)(z^k-1) &\le 0 \\ \Rightarrow x^k+y^k+z^k &\ge \frac{1}{x^k}+\frac{1}{y^k}+\frac{1}{z^k}, \end{aligned}$$

as desired.

Problem 9.4 A maze consists of an 8×8 grid, in each 1×1 cell of which is drawn an arrow pointing up, down, left or right. The top edge of the top right square is the exit from the maze. A token is placed on the bottom left square, and then is moved in a sequence of turns. On each turn, the token is moved one square in the direction of the arrow. Then the arrow in the square the token moved from is rotated 90° clockwise. If the arrow points off of the board (and not through the exit), the token stays put and the arrow is rotated 90° clockwise. Prove that sooner or later the token will leave the maze.

Solution: Suppose by way of contradiction the token did not leave the maze. Let *position* denote the set-up of the board, including both the token's location and the directions of all the arrows. Since the token moves infinitely many times inside the maze, and there are only finitely many positions, some position must repeat.

During the "cycle time" between two occurrences of this position, suppose the token visits some square S. Then the arrow on S must make at least four 90° rotations: thus at some point during the cycle time, the token must visit all the squares adjacent to S. It follows that the token visits *all* the squares on the board during the cycle time.

Specifically, the token visits the upper-right square during the cycle time; but at some point, this square's arrow will point out of the maze. Then when the token lands on this square it will exit — a contradiction.

Problem 9.5 Each square of an infinite grid is colored in one of 5 colors, in such a way that every 5-square (Greek) cross contains one square of each color. Show that every 1×5 rectangle also contains one square of each color.

Solution: Label the centers of the grid squares with coordinates, and suppose that square $(0, 0)$ is colored maroon. The Greek cross centered at $(1, 1)$ must contain a maroon-colored square. However, the squares $(0, 1)$, $(1, 0)$, and $(1, 1)$ cannot be maroon because each of these squares is in a Greek cross with $(0, 0)$. Thus either $(1, 2)$ or $(2, 1)$ is maroon — without loss of generality, say $(1, 2)$.

Then by a similar analysis on square $(1, 2)$ and the Greek cross centered at $(2, 1)$, one of the squares $(2, 0)$ and $(3, 1)$ must be maroon. $(2, 0)$ is in a Greek cross with $(0, 0)$ though, so $(3, 1)$ is maroon.

Repeating the analysis on square $(2, 0)$ shows that $(2, -1)$ is maroon; and spreading outward, every square of the form $(i + 2j, 2i - j)$ is maroon. But since these squares are the centers of Greek crosses that tile the plane, no other squares can be maroon. And since no two of these squares are in the same 1×5 rectangle, no two maroon squares can be in the same 1×5 rectangle.

The same argument applies to all the other colors — lavender, tickle-me-pink, green, neon orange. Therefore the five squares in each 1×5 rectangle have distinct colors, as desired.

Problem 9.7 Show that each natural number can be written as the difference of two natural numbers having the same number of prime factors.

Solution: If n is even, then we can write it as $(2n) - (n)$.

Now suppose n is odd, and let d be the smallest odd prime that does not divide n. Then write $n = (dn) - ((d - 1)n)$. The number dn contains exactly one more prime factor than n. As for $(d - 1)n$, it is divisible by 2 since $d - 1$ is even; but its odd factors are less than d so they all divide n. Therefore $(d - 1)n$ also contains exactly one more prime factor than n, and dn and $(d - 1)n$ have the same number of prime factors.

Problem 9.8 In triangle ABC, with $AB > BC$, points K and M are the midpoints of sides AB and CA, and I is the incenter. Let P be the intersection of the lines KM and CI, and Q the point such that $QP \perp KM$ and $QM \parallel BI$. Prove that $QI \perp AC$.

Solution: Draw point S on ray CB such that $CS = CA$. Let P' be the midpoint of AS. Since triangle ACS is isosceles, P' lies on CI;

and since P' and M are midpoints of AS and AC, we have $P'M \parallel SC$. It follows that $P = P'$.

Let the incircle touch BC, CA, AB at D, E, F respectively. Writing $a = BC$, $b = CA$, $c = AB$, and $s = \frac{1}{2}(a+b+c)$, we have

$$SD = SC - DC = b - (s-c) = \frac{1}{2}(b+c-a) = FA,$$
$$BF = s - b = DB,$$
$$AP = PS.$$

Therefore
$$\frac{SD}{DB}\frac{BF}{FA}\frac{AP}{PS} = 1,$$

and by Menelaus' Theorem applied to triangle ABS, P lies on line DF.

Then triangle PDE is isosceles, and $\angle DEP = \angle PDE = \angle FEA = 90^\circ - \frac{\angle A}{2}$ while $\angle CED = 90^\circ - \frac{\angle C}{2}$. Therefore

$$\angle PEA = 180^\circ - \angle DEP - \angle CED = 90^\circ - \frac{\angle B}{2}.$$

Now let Q' be the point such that $Q'I \perp AC$, $Q'M \parallel BI$. Then $\angle Q'EP = 90^\circ - \angle PEA = \frac{\angle B}{2}$.

But we also know that $\angle Q'MP = \angle IBC$ (from parallel lines $BC \parallel MP$ and $IB \parallel Q'M$), and $\angle IBC = \frac{\angle B}{2}$ as well. Therefore $\angle Q'MP = \angle Q'EP$, quadrilateral $Q'EMP$ is cyclic, and $\angle Q'PM = \angle Q'EM = 90^\circ$. Therefore $Q = Q'$, and QI is indeed perpendicular to AC.

Problem 10.2 In the plane is given a circle ω, a point A inside ω, and a point B not equal to A. Consider all possible triangles BXY such that X and Y lie on ω and A lies on the chord XY. Show that the circumcenters of these triangles all lie on a line.

Solution: We use directed distances. Let O be the circumcenter and R be the circumradius of triangle BXY. Drop the perpendicular OO' to line AB.

The power of A with respect to circle BXY equals both $AX \cdot AY$ and $AO^2 - R^2$. Therefore

$$BO' - O'A = \frac{BO'^2 - O'A^2}{BO' + O'A}$$

$$= \frac{(BO^2 - O'O^2) - (OA^2 - OO'^2)}{AB}$$
$$= \frac{XA \cdot AY}{AB}$$

which is constant since $AX \cdot AY$ also equals the power of A with respect to ω.

Since $BO' - O'A$ and $BO' + O'A = AB$ are constant, BO' and $O'A$ are constant as well. Thus O' is fixed regardless of the choice of X and Y. Therefore O lies on the line through O' perpendicular to AB, as desired.

Problem 10.3 In space are given n points in general position (no three points are collinear and no four are coplanar). Through any three of them is drawn a plane. Show that for any $n-3$ points in space, there exists one of the drawn planes not passing through any of these points.

Solution: Call the given n points *given* and the $n-3$ points *random*, and call all these points "level-0." Since there are more given points than random points, one of the given points is not random: say, A. Draw a plane not passing through A, and for each of the other points P let (P) be the intersection of AP with this plane. Call these points (P) level-1.

Since no four given points were coplanar, no three of the level-1 given points map to collinear points on this plane; and since no three given points were collinear, no two of the level-1 given points map to the same point on this plane. Thus we have $n-1$ level-1 given points and at most $n-3$ level-1 random points.

Now perform a similar operation — since there are more level-1 given points than random points, one of them is not random: say, (B). Draw a line not passing through (B), and for each of the other points (P) let $((P))$ be the intersection of $B(P)$ with this plane. Call these points $((P))$ level-2.

Since no three level-1 given points were collinear, all of the level-2 given points are distinct. Thus we have $n-2$ level-2 given points but at most $n-3$ level-2 random points. Therefore one of these given points $((C))$ is not random.

Consider the drawn plane ABC. If it contained some level-0 random point — say, Q — then (Q) would be collinear with (B) and (C), and thus $((Q)) = ((C))$, a contradiction. Therefore plane ABC does not pass through any of the level-0 random points, as desired.

Problem 10.5 Do there exist 10 distinct integers, the sum of any 9 of which is a perfect square?

Solution: Yes, there do exist 10 such integers. Write $S = a_1 + \cdots + a_{10}$, and consider the linear system of equations

$$\begin{aligned} S - a_1 &= 9 \cdot 1^2 \\ S - a_2 &= 9 \cdot 2^2 \\ &\vdots \\ S - a_{10} &= 9 \cdot 10^2. \end{aligned}$$

Adding all these gives

$$9S = 9 \cdot (1^2 + 2^2 + \cdots + 10^2)$$

so that

$$a_i = S - 9i^2 = 1^2 + 2^2 + \cdots + 10^2 - 9i^2.$$

Then all the a_i's are distinct integers, and any nine of them add up to perfect square.

Problem 10.6 The incircle of triangle ABC touches sides BC, CA, AB at A_1, B_1, C_1, respectively. Let K be the point on the circle diametrically opposite C_1, and D the intersection of the lines B_1C_1 and A_1K. Prove that $CD = CB_1$.

Solution: Draw D' on B_1C_1 such that $CD' \parallel AB$. Then $\angle D'CB_1 = \angle C_1AB_1$ and $\angle CD'B_1 = \angle AC_1B_1$, implying that $\triangle AB_1C_1 \sim \triangle CB_1D'$.

Thus triangle CB_1D' is isosceles and $CD' = CB_1$. But $CB_1 = CA_1$, so that triangle CA_1D' is isosceles also. And since $\angle D'CA_1 = 180° - \angle B$, we have $\angle CA_1D' = \frac{\angle B}{2}$.

But note that

$$\begin{aligned} \angle CA_1K &= \angle A_1C_1K \\ &= 90° - \angle C_1KA_1 \end{aligned}$$

$$= 90^\circ - \angle C_1A_1B$$
$$= \frac{\angle B}{2}$$

also. Therefore D' lies on A_1K and by definition it lies on B_1C_1. Hence $D' = D$.

But from before $CD' = CB_1$; thus $CD = CB_1$, as desired.

Problem 10.7 Each voter in an election marks on a ballot the names of n candidates. Each ballot is placed into one of $n+1$ boxes. After the election, it is observed that each box contains at least one ballot, and that for any $n+1$ ballots, one in each box, there exists a name which is marked on all of these ballots. Show that for at least one box, there exists a name which is marked on all ballots in the box.

Solution: Suppose by way of contradiction that in every box, no name is marked on all the ballots. Label the boxes $1, 2, \ldots, n$, and look at an arbitrary ballot from the first box.

Suppose it has n "chosen" names Al, Bob, ..., Zed. By assumption, some ballot in the second box does not have the name Al on it; some ballot in the third box does not have the name Bob on it; and so on, so some ballot in the $(i+1)$-th box does not have the i-th chosen name on it. But then on these $n+1$ ballots, one from each box, there is no name marked on all the ballots — a contradiction.

Problem 10.8 A set of natural numbers is chosen so that among any 1999 consecutive natural numbers, there is a chosen number. Show that there exist two chosen numbers, one of which divides the other.

Solution: Draw a large table with 1999 columns and 2000 rows. In the first row write $1, 2, \ldots, 1999$.

Define the entries in future rows recursively as follows: suppose the entries in row i are $k+1$, $k+2$, $\ldots$, $k+1999$, and that their product is M. Then fill row $i+1$ with $M+k+1$, $M+k+2$, $\ldots$, $M+k+1999$. All the entries in row $i+1$ are bigger than the entries in row i; furthermore, every entry divides the entry immediately below it (and therefore *all* the entries directly below it).

In each row there are 1999 consecutive numbers, and hence each row contains a chosen number. Then since we have 2000 rows, there are two chosen numbers in the same column — and one of them divides another, as desired.

Problem 11.1 The function $f(x)$ is defined on all real numbers. It is known that for all $a > 1$, the function $f(x) + f(ax)$ is continuous. Show that $f(x)$ is continuous.

First Solution: We know that for $a > 1$, the functions

$$P(x) = f(x) + f(ax),$$
$$Q(x) = f(x) + f(a^2x),$$
$$P(ax) = f(ax) + f(a^2x)$$

are all continuous. Thus the function

$$\frac{1}{2}\left(P(x) + Q(x) - P(ax)\right) = f(x)$$

is continuous as well.

Problem 11.3 In a class, each boy is friends with at least one girl. Show that there exists a group of at least half of the students, such that each boy in the group is friends with an odd number of the girls in the group.

Solution: We perform strong induction on the total number of students. The base case of zero students is obvious.

Now suppose that we know the claim is true for any number of students less than n (where $n > 0$), and we wish to prove it for n. Since there must be at least one girl, pick any girl from the n students. We now partition the class into three subsets: $A =$ this girl, $B =$ this girl's male friends, and $C =$ everybody else.

Because we are using strong induction, the induction hypothesis states that there must be a subset C' of C, with at least $\frac{|C|}{2}$ students, such that any boy in C' is friends with an odd number of girls in C'.

Let B_O be the set of boys in B who are friends with an odd number of girls in C', and let B_E be the set of boys in B who are friends with an even number of girls in C'. Then there are two possible cases:

(i) $|B_O| \geq \frac{|A \cup B|}{2}$.

The set $S = B_O \cup C'$ will realize the claim, i.e., S will have at least $\frac{n}{2}$ elements, and each boy in S will be friends with an odd number of girls in S.

(ii) $|A \cup B_E| \geq \frac{|A \cup B|}{2}$.

The set $T = A \cup B_E \cup C'$ will realize the claim. T will have at least $\frac{n}{2}$ elements; each boy in C' will be friends with an odd number of girls in C' but not the girl in A; and each boy in B_E will be friends with an even number of girls in C' and the girl in A — making a total of an odd number of girls.

Thus the induction is complete.

Note: With a similar proof, it is possible to prove a slightly stronger result: suppose each boy in a class is friends with at least one girl, and that every boy has a parity, either "even" or "odd." Then there is a group of at least half the students, such that each boy in the group is friends with the same parity of girls as his own parity. (By letting all the boys' parity be "odd," we have the original result.)

Problem 11.4 A polyhedron is circumscribed about a sphere. We call a face big if the projection of the sphere onto the plane of the face lies entirely within the face. Show that there are at most 6 big faces.

Solution:

Lemma. *Given a sphere of radius R, let a "slice" of the sphere be a portion cut off by two parallel planes. The surface area of the sphere contained in this slice is $2\pi RW$, where W is the distance between the planes.*

Proof: Orient the sphere so that the slice is horizontal. Take an infinitesimal horizontal piece of this slice, shaped like a frustrum (a small sliver from the bottom of a radially symmetric cone). Say it has width w, radius r, and slant height ℓ; then its lateral surface area (for infinitesimal w) is $2\pi r\ell$. But if the side of the cone makes an angle θ with the horizontal, then we have $\ell \sin\theta = w$ and $R \sin\theta = r$ so that the surface area also equals $2\pi Rw$. Adding over all infinitesimal pieces, the complete slice has lateral surface area $2\pi RW$, as desired. ■

Say that the inscribed sphere has radius R and center O. For each big face F in the polyhedron, project the sphere onto F to form a circle k. Then connect k with O to form a cone. Because these cones don't share any volume, they hit the sphere's surface in several non-overlapping circular regions.

Each circular region is a slice of the sphere with width $R(1-\frac{1}{2}\sqrt{2})$, and it contains $2\pi R^2(1-\frac{1}{2}\sqrt{2}) > \frac{1}{7}(4\pi R^2)$ of the sphere's surface area. Thus each circular region takes up more than $\frac{1}{7}$ of the surface area of the sphere, implying there must be less than 7 such regions and therefore at most six big faces.

Problem 11.5 Do there exist real numbers a, b, c such that for all real numbers x, y,

$$|x+a|+|x+y+b|+|y+c|>|x|+|x+y|+|y|?$$

Solution: No such numbers exist; suppose they did. Let $y=-b-x$. Then for all real x we have

$$|x+a|+|-b-x+c|>|x|+|-b|+|-b-x|.$$

If we pick x sufficiently negative, this gives

$$\begin{aligned}&(-x-a)+(-b-x+c)>(-x)+|b|+(-b-x)\\&\Rightarrow -a+c>|b|\geq 0,\end{aligned}$$

so $c>a$. On the other hand, if we pick x sufficiently positive, this gives

$$\begin{aligned}&(x+a)+(b+x-c)>(x)+|b|+(b+x)\\&\Rightarrow a-c>|b|\geq 0,\end{aligned}$$

so $c<a$ as well — a contradiction.

Problem 11.6 Each cell of a 50×50 square is colored in one of four colors. Show that there exists a cell which has cells of the same color directly above, directly below, directly to the left, and directly to the right of it (though not necessarily adjacent to it).

Solution: By the pigeonhole principle, at least one-quarter of the squares (625) are the same color: say, red.

Of these red squares, at most 50 are the topmost red squares of their columns, and at most 50 are the bottommost red squares of their columns. Similarly, at most 50 are the leftmost red squares in their rows and at most 50 are the rightmost red squares in their rows. This gives at most 200 squares; the remaining 425 or more red squares then have red squares directly above, directly below, directly to the left, and directly to the right of them.

Problem 11.8 A polynomial with integer coefficients has the property that there exist infinitely many integers which are the value of the polynomial evaluated at more than one integer. Prove that there exists at most one integer which is the value of the polynomial at exactly one integer.

Solution: First observe that the polynomial cannot be constant. Now let $P(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0$ be the polynomial with $c_n \neq 0$. The problem conditions imply that n is even and at least 2, and we can assume without loss of generality that $c_n > 0$.

Since $P(x)$ has positive leading coefficient and it is not constant, there exists a value N such that $P(x)$ is decreasing for all $x < N$. Also consider the pairs of integers (s,t) with $s < t$ and $P(s) = P(t)$; since there are infinitely many pairs, there must be infinitely many with $s < N$.

Now for any integer k, look at the polynomial $P(x) - P(k - x)$. Some algebra shows that the coefficient of x^n is zero and that the coefficient of x^{n-1} is $f(k) = 2c_{n-1} + c_n(nk)$.

Let K be the largest integer such that $f(K) < 0$ (such an integer exists because from assumptions made above, $c_n \cdot n > 0$). Then for sufficiently large t we have

$$P(t) < P(K-t) < P(K-1-t) < \cdots$$

and

$$P(t) \geq P(K+1-t) > P(K+2-t) > P(K+3-t) > \cdots > P(N).$$

Therefore we must have $s = K+1-t$ and $P(t) - P(K+1-t) = 0$ for infinitely many values of t. But since P has finite degree, this implies that $P(x) - P(K+1-x)$ is *identically* zero.

Then if $P(a) = b$ for some integers a, b, we also have $P(K+1-a) = b$. Therefore there is at most one value b that could possibly be the value of $P(x)$ at exactly one integer x — specifically, $b = P(\frac{K+1}{2})$.

Fifth round

Problem 9.1 In the decimal expansion of A, the digits occur in increasing order from left to right. What is the sum of the digits of

$9A$?

Solution: Write $A = \underline{a_1a_2\ldots a_k}$. Then since $9A = 10A - A$, by performing the subtraction

$$\begin{array}{rcccccc} & a_1 & a_2 & a_3 & \cdots & a_k & 0 \\ - & & a_1 & a_2 & \cdots & a_{k-1} & a_k \\ \hline \end{array}$$

we find that the digits of $9A$ are

$$a_1,\ a_2 - a_1,\ a_3 - a_2,\ \ldots,\ a_{k-1} - a_{k-2},\ a_k - a_{k-1} - 1,\ 10 - a_k,$$

and that these digits add up to $10 - 1 = 9$.

Problem 9.3 Let S be the circumcircle of triangle ABC. Let A_0 be the midpoint of the arc BC of S not containing A, and C_0 the midpoint of the arc AB of S not containing C. Let S_1 be the circle with center A_0 tangent to BC, and let S_2 be the circle with center C_0 tangent to AB. Show that the incenter I of ABC lies on a common external tangent to S_1 and S_2.

Solution: We prove a more general result: I lies on a common external tangent to S_1 and S_2, parallel to AC.

Drop the perpendicular from A_0 to BC, hitting at P, and drop the perpendicular from A_0 to AC, hitting circle S_1 at Q (with Q closer to AC than A_0).

Note that A, I, and A_0 are collinear. Then

$$\angle CIA_0 = \angle CAI + \angle ICA = \frac{\angle A + \angle C}{2} = \angle A_0CI,$$

so that $IA_0 = CA_0$.

Next, from circle S_1 we know that $A_0Q = A_0P$.

Finally,

$$\angle IA_0Q = \angle AA_0Q = 90^\circ - \angle CAA_0 = \frac{1}{2}(180^\circ - \angle CAB) = \angle CA_0P.$$

Thus, triangles IA_0Q and CA_0P are congruent. Then

$$\angle IQA_0 = \angle CPA_0 = 90^\circ,$$

so that IQ is tangent to S_1 at Q. Furthermore, since A_0Q is perpendicular to both IQ and AC, we have $IQ \parallel AC$.

Therefore, the line through I parallel to AC is tangent to S_1; by a similar argument it is tangent to S_2; and thus it is a common external tangent to S_1 and S_2, as claimed.

Problem 9.4 The numbers from 1 to 1000000 can be colored black or white. A permissible move consists of selecting a number from 1 to 1000000 and changing the color of that number and each number not relatively prime to it. Initially all of the numbers are black. Is it possible to make a sequence of moves after which all of the numbers are colored white?

First Solution: It is possible. We begin by proving the following lemma:

Lemma. *Given a set S of positive integers, there is a subset $T \subseteq S$ such that every element of S divides an odd number of elements in T.*

Proof: We prove the claim by induction on $|S|$, the number of elements in S. If $|S| = 1$ then let $T = S$.

If $|S| > 1$, then say the smallest element of S is a. Look at the set $S' = S \setminus \{a\}$ — the set of the largest $|S| - 1$ elements in S. By induction there is a subset $T' \subseteq S'$ such that every element in S' divides an odd number of elements in T'.

If a also divides an odd number of elements in T', then the set $T = T'$ suffices. Otherwise, the set $T = T' \cup \{a\}$ suffices: a divides an odd number of elements in T; the other elements are bigger than a and can't divide it, and therefore still divide an odd number of elements in T. This completes the induction and the proof of the lemma. ■

Now, write each number $n > 1$ in its prime factorization

$$p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$$

for distinct primes p_i and positive integers a_i. Then notice that the color of n will always be the same as the color of $P(n) = p_1 p_2 \cdots p_k$.

Apply the lemma to the set $S = \bigcup_{i=2}^{1000000} P(i)$ to find a subset $T \subseteq S$ such that every element of S divides an odd number of elements in T. For each $q \in S$, let $t(q)$ equal the number of elements in T that q divides, and let $u(q)$ equal the number of primes dividing q.

Select all the numbers in T, and consider how the color of a number $n > 1$ changes. The number of elements in T not relatively prime to

n equals

$$\sum_{q|P(n),\, q>1} (-1)^{u(q)+1} t(q)$$

by the Inclusion-Exclusion Principle: if $q \mid P(n)$ is divisible by exactly $m > 0$ primes, then it is counted $\binom{m}{1} - \binom{m}{2} + \binom{m}{3} - \cdots = 1$ time in the sum. (For example, if $n = 6$ then the number of elements in T divisible by 2 or 3 equals $t(2) + t(3) - t(6)$.)

But by the definition of T, each of the values $t(q)$ is odd. Then since there are $2^k - 1$ divisors $q > 1$ of $P(n)$, the above quantity is the sum of $2^k - 1$ odd numbers and is odd itself. Therefore after selecting T, every number $n > 1$ will switch color an odd number of times and will turn white.

Finally, select 1 to turn 1 white, and we are done.

Note: In fact, a slight modification of the above proof shows that T is unique, which with some work implies that there is *exactly* one way to make all the numbers white by only selecting square-free numbers at most once each (other methods are different only trivially, either by selecting a number twice or by selecting numbers that aren't square-free).

Second Solution: Yes, it is possible. We prove a more general statement, where we replace 1000000 in the problem by some arbitrary positive integer m, and where we focus on the numbers divisible by just a few primes instead of all the primes.

Lemma. *For a finite set of distinct primes $S = \{p_1, p_2, \ldots, p_n\}$, let $Q_m(S)$ be the set of numbers between 2 and m divisible only by primes in S. The elements of $Q_m(S)$ can be colored black or white; a permissible move consists of selecting a number in $Q_m(S)$ and changing the color of that number and each number not relatively prime to it. Then it is possible to reverse the coloring of $Q_m(S)$ by selecting several numbers in a subset $R_m(S) \subseteq Q_m(S)$.*

Proof: We prove the lemma by induction on n. If $n = 1$, then selecting p_1 suffices. Now suppose $n > 1$, and assume without loss of generality that the numbers are all black to start with.

Let $T = \{p_1, p_2, \ldots, p_{n-1}\}$, and define t to be the largest integer such that $tp_n \le m$. We can assume $t \ge 1$ because otherwise we could ignore p_n and just use use the smaller set T, and we'd be done by our induction hypothesis.

Now select the numbers in $R_m(T)$, $R_t(T)$, and $p_n R_t(T) = \{p_n x \mid x \in R_t(T)\}$, and consider the effect of this action on a number y:

- *y is not a multiple of p_n.* Selecting the numbers in $R_m(T)$ makes y white. Then if selecting $x \in R_t(T)$ changes y's color, selecting xp_n will change it back so that y will become white.
- *y is a power of p_n.* Selecting the numbers in $R_m(T)$ and $R_t(T)$ has no effect on y, but each of the $|R_t(T)|$ numbers in $xR_t(T)$ changes y's color.
- *$p_n \mid y$ but y is not a power of p_n.* Selecting the numbers in $R_m(T)$ makes y white. Since $y \neq p_n^i$, it is divisible by some prime in T so selecting the numbers in $R_t(T)$ makes y black again. Finally, each of the $|R_t(T)|$ numbers in $xR_t(T)$ changes y's color.

Therefore, all the multiples of p_n are the same color (black if $|R_t(T)|$ is even, white if $|R_t(T)|$ is odd), while all the other numbers in $Q_m(S)$ are white. If the multiples of p_n are still black, we can select p_n to make them white, and we are done. ■

Now back to the original problem: set $m = 1000000$ and let S be the set of all primes under 1000000. Then from the lemma, we can select numbers between 2 and 1000000 so that all the numbers $2, 3, \ldots, 1000000$ are white. And finally, we finish off by selecting 1.

Problem 9.5 An equilateral triangle of side length n is drawn with sides along a triangular grid of side length 1. What is the maximum number of grid segments on or inside the triangle that can be marked so that no three marked segments form a triangle?

Solution: The grid is made up of $\frac{n(n+1)}{2}$ small equilateral triangles of side length 1. In each of these triangles, at most 2 segments can be marked so we can mark at most $\frac{2}{3} \cdot \frac{3n(n+1)}{2} = n(n+1)$ segments in all. Every segment points in one of three directions, so we can achieve the maximum $n(n+1)$ by marking all the segments pointing in two of the directions.

Problem 9.6 Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. Prove that for every natural number n,

$$\sum_{k=1}^{n^2} \{\sqrt{k}\} \leq \frac{n^2-1}{2}.$$

Solution: We prove the claim by induction on n. For $n = 1$, we have $0 \leq 0$. Now supposing that the claim is true for n, we prove it is true for $n+1$.

Each of the numbers $\sqrt{n^2+1}$, $\sqrt{n^2+2}$, $\ldots$, $\sqrt{n^2+2n}$ is between n and $n+1$, and thus

$$\begin{aligned} \{\sqrt{n^2+i}\} &= \sqrt{n^2+i} - n \\ &< \sqrt{n^2+i+\frac{i^2}{4n^2}} - n \\ &= \frac{i}{2n}. \end{aligned}$$

Therefore we have

$$\begin{aligned} \sum_{k=1}^{(n+1)^2} \{\sqrt{k}\} &= \sum_{k=1}^{n^2} \{\sqrt{k}\} + \sum_{k=n^2+1}^{(n+1)^2} \{\sqrt{k}\} \\ &< \frac{n^2-1}{2} + \frac{1}{2n}\sum_{i=1}^{2n} i + 0 \\ &= \frac{n^2-1}{2} + \frac{2n+1}{2} \\ &= \frac{(n+1)^2-1}{2}, \end{aligned}$$

completing the inductive step and the proof.

Problem 9.7 A circle passing through vertices A and B of triangle ABC intersects side BC again at D. A circle passing through vertices B and C intersects side AB again at E, and intersects the first circle again at F. Suppose that the points A, E, D, C lie on a circle centered at O. Show that $\angle BFO$ is a right angle.

Solution: Since $AEDC$ is cyclic with O as its center,

$$\begin{aligned} \angle COA = 2\angle CDA &= \angle CDA + \angle CEA \\ &= (180^\circ - \angle ADB) + (180^\circ - \angle BEC). \end{aligned}$$

Since $BDFA$ and $BEFC$ are cyclic, $\angle ADB = \angle AFB$ and $\angle BEC = \angle BFC$. Hence

$$\angle COA = 360^\circ - \angle AFB - \angle BFC = \angle CFA.$$

Hence $AFOC$ is cyclic. Therefore

$$\angle OFA = 180^\circ - \angle ACO = 180^\circ - \frac{180^\circ - \angle COA}{2} = 90^\circ + \angle CDA.$$

Since $ABDF$ is cyclic,

$$\angle OFA + \angle AFB = 90^\circ + \angle CDA + \angle ADB = 270^\circ.$$

Hence $\angle BFO = 90^\circ$, as desired.

Problem 9.8 A circuit board has 2000 contacts, any two of which are connected by a lead. The hooligans Vasya and Petya take turns cutting leads: Vasya (who goes first) always cuts one lead, while Petya cuts either one or three leads. The first person to cut the last lead from some contact loses. Who wins with correct play?

Solution: Petya wins with correct play; arrange the contacts in a circle and label them 1, 2, ..., 2000, and let (x, y) denote the lead between contacts x and y (where labels are taken modulo 2000).

If Vasya disconnects $(a, 1000 + a)$, Petya can disconnect $(500 + a, 1500 + a)$; otherwise, if Vasya disconnects (a, b), Petya can disconnect the three leads $(a + 500, b + 500)$, $(a + 1000, b + 1000)$, and $(a+1500, b+1500)$. Notice that in each case, Petya and Vasya tamper with different contacts.

Using this strategy, after each of Petya's turns the circuit board is symmetrical under 90°, 180°, and 270° rotations, ensuring that he can always make the above moves — for example, if $(a + 1500, b + 1500)$ were already disconnected during Petya's turn, then (a, b) must have been as well before Vasya's turn.

Also, Petya can never lose, because if he disconnected the last lead (x, y) from some contact x, then Vasya must have already disconnected the last lead $(x - 1500, y - 1500)$, $(x - 1000, y - 1000)$, or $(x - 500, y - 500)$ from some other contact, a contradiction.

Problem 10.1 Three empty bowls are placed on a table. Three players A, B, C, whose order of play is determined randomly, take turns putting one token into a bowl. A can place a token in the first or second bowl, B in the second or third bowl, and C in the third or first bowl. The first player to put the 1999th token into a bowl loses. Show that players A and B can work together to ensure that C will lose.

Solution: Suppose A plays only in the first bowl until it contains 1998 tokens, then always plays in the second bowl; and suppose B plays only in the third bowl until it contains 1998 tokens, then always plays in the second bowl as well.

Suppose by way of contradiction that C doesn't lose. Without loss of generality, say the first bowl fills up to 1998 tokens before the third bowl does — call this point in time the "critical point."

First suppose the third bowl never contains 1998 tokens. Then at most 999 round pass after the critical point since during each round, the third bowl gains 2 tokens (one from B, one from C). But then A plays at most 999 tokens into the second bowl and doesn't lose; thus nobody loses, a contradiction.

Thus the third bowl does contain 1998 tokens some $k \leq 999$ more rounds after the critical point. After this k-th round A has played at most k tokens into the second bowl, and B has possibly played at most one token into the second bowl during the k-th round; so the second bowl has at most 1000 tokens. However, the first and third bowls each have 1998 tokens, so during the next round C will lose.

Problem 10.2 Find all infinite bounded sequences $a_1, a_2, \ldots$ of positive integers such that for all $n > 2$,

$$a_n = \frac{a_{n-1} + a_{n-2}}{\gcd(a_{n-1}, a_{n-2})}.$$

Solution: The only such sequence is $2, 2, 2, \ldots$.

Let $g_n = \gcd(a_n, a_{n+1})$. Then g_{n+1} divides both a_{n+1} and a_{n+2}, so it divides $g_n a_{n+2} - a_{n+1} = a_n$ as well. Thus g_{n+1} divides both a_n and a_{n+1}, and it divides their greatest common divisor g_n.

Therefore, the g_i form a nonincreasing sequence of positive integers and eventually equal some positive constant g. At this point, the a_i satisfy the recursion

$$g a_n = a_{n-1} + a_{n-2}.$$

If $g = 1$, then $a_n = a_{n-1} + a_{n-2} > a_{n-1}$ so the sequence is increasing and unbounded.

If $g \geq 3$, then $a_n = \frac{a_{n-1}+a_{n-2}}{g} < \frac{a_{n-1}+a_{n-2}}{2} \leq \max\{a_{n-1}, a_{n-2}\}$. Similarly, $a_{n+1} < \max\{a_{n-1}, a_n\} \leq \max\{a_{n-2}, a_{n-1}\}$, so that $\max\{a_n, a_{n+1}\} < \max\{a_{n-2}, a_{n-1}\}$. Therefore the maximum values of successive pairs of terms form an infinite decreasing sequence of positive integers, a contradiction.

Thus $g = 2$ and eventually we have $2a_n = a_{n-1} + a_{n-2}$ or $a_n - a_{n-1} = -\frac{1}{2}(a_{n-1} - a_{n-2})$. This implies that $a_i - a_{i-1}$ converges to 0 and that the a_i are eventually constant as well. From $2a_n = a_{n-1} + a_{n-2}$, this constant must be 2.

Now if $a_n = a_{n+1} = 2$ for $n > 1$, then $\gcd(a_{n-1}, a_n) = \gcd(a_{n-1}, 2)$ either equals 1 or 2. Since

$$2 = a_{n+1} = \frac{a_{n-1} + a_n}{\gcd(a_{n-1}, 2)},$$

this either implies $a_{n-1} = 0$ — which is impossible — or $a_{n-1} = 2$. Therefore all the a_i equal 2, and this sequence indeed works.

Problem 10.3 The incircle of triangle ABC touches sides AB, BC, CA at K, L, M, respectively. For each two of the incircles of AMK, BKL, CLM is drawn the common external tangent not lying along a side of ABC. Show that these three tangents pass through a single point.

Solution: Let D, E, F be the midpoints of minor arcs MK, KL, LM of the incircle, respectively; and let S_1, S_2, S_3 be the incircles of triangles AMK, BKL, and CLM, respectively.

Since AK is tangent to the incircle, $\angle AKD = \angle KLD = \angle KMD = \angle DKM$; similarly, $\angle AMD = \angle DMK$. Thus, D is the incenter of AMK and the center of S_1.

Likewise, E is center of S_2 and F is the center of S_3. By the result proved in Problem 9.3, the incenter I of triangle KLM lies on a common external tangent to S_1 and S_2. But it does not lie on AB, so it must lie on the other external tangent. Similarly, the common external tangent to S_2 and S_3 (not lying on BC) passes through I, as does the common external tangent to S_3 and S_1 (not lying on CA); so the three tangents all pass through I, as desired.

Problem 10.4 An $n \times n$ square is drawn on an infinite checkerboard. Each of the n^2 cells contained in the square initially contains a token. A move consists of jumping a token over an adjacent token (horizontally or vertically) into an empty square; the token jumped over is removed. A sequence of moves is carried out in such a way that at the end, no further moves are possible. Show that at least $\frac{n^2}{3}$ moves have been made.

Solution: At the end of the game no two adjacent squares contain tokens: otherwise (since no more jumps are possible) they would have to be in an infinitely long line of tokens, which is not allowed. Then during the game, each time a token on square A jumps over another token on square B, imagine putting a 1×2 domino over squares A and B. At the end, every tokenless square on the checkerboard is covered by a tile; so no two uncovered squares are adjacent. We now prove there must be at least $\frac{n^2}{3}$ dominoes, implying that at least $\frac{n^2}{3}$ moves have been made:

Lemma. *If an $n \times n$ square board is covered with 1×2 rectangular dominoes (possibly overlapping, and possibly with one square off the board) in such a way that no two uncovered squares are adjacent, then at least $\frac{n^2}{3}$ tiles are on the board.*

Proof: Call a pair of adjacent squares on the checkerboard a "tile." If a tile contains two squares on the border of the checkerboard, call it an "outer tile"; otherwise, call it an "inner tile."

Now for each domino D, consider any tile it partly covers. If this tile is partly covered by exactly m dominoes, say D *destroys* $\frac{1}{m}$ of that tile. Adding over all the tiles that D lies on, we find the total quantity a of outer tiles that D destroys, and the total quantity b of inner tiles that D destroys. Then say that D scores $1.5a + b$ points.

Consider a vertical domino D wedged in the upper-left corner of the chessboard; it partly destroys two horizontal tiles. But one of the two squares immediately to D's right must be covered; so if D destroys all of one horizontal tile, it can only destroy at most half of the other.

Armed with this type of analysis, some quick checking shows that any domino scores at most 6 points; and that any domino scoring 6 points must lie completely on the board, not be wedged in a corner, not overlap any other dominoes, and not have either length-1 edge hit another domino.

Now in a valid arrangement of dominoes, every tile is destroyed completely; since there are $4(n-1)$ outer tiles and $2(n-1)(n-2)$ inner tiles, this means that a total of $1.5 \cdot 4(n-1)+2(n-1)(n-2) = 2(n^2-1)$ points are scored. Therefore, there must be at least $\lceil \frac{2(n^2-1)}{6} \rceil = \lceil \frac{n^2-1}{3} \rceil$ dominoes.

Suppose by way of contradiction that we have *exactly* $\frac{n^2-1}{3}$ dominoes. First, for this to be an integer 3 cannot divide n. Second, the restrictions described two paragraphs ago must hold for every domino.

Suppose we have any horizontal domino not at the bottom of the chessboard; one of the two squares directly below it must be covered. But to satisfy our restrictions, it must be covered by a horizontal domino (not a vertical one). Thus we can find a chain of horizontal dominoes stretching to the bottom of the board, and similarly we can follow this chain to the top of the board.

Similarly, if there is any vertical domino then some chain of vertical dominoes stretches across the board. But we can't have both a horizontal *and* a vertical chain, so all the dominoes must have the same orientation: say, horizontal.

Now to cover the tiles in any given row while satisfying the restrictions, we must alternate between blank squares and horizontal dominoes. In the top row, since no dominoes are wedged in a corner we must start and end with blank squares; thus we must have $n \equiv 1 \pmod{3}$. But then in the second row, we must start with a horizontal domino (to cover the top-left vertical tiles); then after alternating between dominoes and blank squares, the end of the row will contain two blank squares—a contradiction. Thus it is impossible to cover the chessboard with exactly $\frac{n^2-1}{3}$ dominoes, and indeed at least $\frac{n^2}{3}$ dominoes are needed. ■

Note: When n is even, there is a simpler proof of the main result: split the n^2 squares of the board into 2×2 mini-boards, each containing four (overlapping) 1×2 tiles. At the end of the game, none of these n^2 tiles can contain two checkers (since no two checkers can be adjacent at the end of the game). But any jump removes a checker from at most three full tiles; therefore, there must be at least $\frac{n^2}{3}$ moves.

Sadly, a similar approach for odd n yields a lower bound of only $\frac{n^2-n-1}{3}$ moves. For large enough n though, we can count the number of tokens that end up completely *outside* the $(n+2) \times (n+2)$ area around the checkerboard — each made a jump that freed at most two full tiles, and from here we can show that $\frac{n^2}{3}$ moves are necessary.

Problem 10.5 The sum of the decimal digits of the natural number n is 100, and that of $44n$ is 800. What is the sum of the digits of $3n$?

Solution: The sum of the digits of $3n$ is 300.

Let $S(x)$ denote the sum of the digits of x. Then $S(a+b)$ equals $S(a)+S(b)$, minus nine times the number of carries in the addition $a+b$. Therefore, $S(a+b) \le S(a)+S(b)$; applying this repeatedly, we have $S(a_1 + \cdots + a_k) \le S(a_1) + \cdots + S(a_k)$.

Also note that for a digit $d \le 2$ we have $S(44d) = 8d$; for $d = 3$ we have $S(8d) = 6 < 8d$; and for $d \ge 4$, $44d \le 44(9)$ has at most 3 digits so its sum is at most $27 < 8d$.

Now write $n = \sum n_i \cdot 10^i$, so that the n_i are the digits of n in base 10. Then

$$\sum 8n_i = S(44n) \le \sum S(44n_i \cdot 10^i) = \sum S(44n_i) \le \sum 8n_i,$$

so equality must occur in the second inequality — that is, each of the n_i must equal 0, 1, or 2. But then each digit of $3n$ is simply three times the corresponding digit of n, and $S(3n) = 3S(n) = 300$, as claimed.

Problem 10.7 The positive real numbers x and y satisfy

$$x^2 + y^3 \ge x^3 + y^4.$$

Show that $x^3 + y^3 \le 2$.

Solution: Equivalently we can prove that if $x^3 + y^3 > 2$, then

$$x^2 + y^3 < x^3 + y^4.$$

First notice that $\sqrt{\frac{x^2+y^2}{2}} \le \sqrt[3]{\frac{x^3+y^3}{2}}$ by the Power-Mean Inequality, implying that

$$\begin{aligned} x^2 + y^2 &\le (x^3 + y^3)^{2/3} \cdot 2^{1/3} \\ &< (x^3 + y^3)^{2/3}(x^3 + y^3)^{1/3} \\ &= x^3 + y^3, \end{aligned}$$

or $x^2 - x^3 < y^3 - y^2$. But $0 \le y^2(y-1)^2 \Rightarrow y^3 - y^2 \le y^4 - y^3$, so that

$$\begin{aligned} x^2 - x^3 &< y^4 - y^3 \\ \Rightarrow x^2 + y^3 &< x^3 + y^4, \end{aligned}$$

as desired.

Problem 10.8 In a group of 12 people, among every 9 people one can find 5 people, any two of whom know each other. Show that there exist 6 people in the group, any two of whom know each other.

Solution: Suppose by way of contradiction that no 6 people know each other. Draw a complete graph with twelve vertices corresponding to the people, labeling the people (and their corresponding vertices) $A, B, \ldots, L$. Color the edge between two people red if they know each other, and blue otherwise. Then among every nine vertices there is at least one red K_5; and among any six vertices there is at least one blue edge.

We prove that there are no blue cycles of odd length in this graph. Suppose, for sake of contradiction, that there is a blue cycle of length (i) 3 or 5, (ii) 7, (iii) 9, or (iv) 11.

(i) First suppose there is a blue 3-cycle (say, ABC) or a blue 5-cycle (without loss of generality, $ABCDE$). In the first case, there is a blue edge among $DEFGHI$ (sat, DE); then any red K_5 contains at most one vertex from $\{A, B, C\}$ and at most one vertex from $\{D, E\}$. In the second case, any K_5 still contains at most two vertices from $\{A, B, C, D, E\}$.

Now, $FGHIJK$ contains some other blue edge: without loss of generality, say FG is blue. Now for each edge V_1V_2 in $HIJKL$, there must be a red K_5 among $ABCDEFGV_1V_2$. From before, this K_5 can contain at most two vertices from $\{A, B, C, D, E\}$; and it contains at most one vertex from each of $\{F, G\}$, $\{V_1\}$, and $\{V_2\}$. Therefore V_1 and V_2 must be connected by a red edge, so $HIJKL$ is a red K_5. Now $FHIJKL$ cannot be a red K_6, so without loss of generality suppose FH is blue. Similarly, $GHIJKL$ cannot be a red K_6, so without loss of generality either GH or GI is blue. In either case, $ABCDEFGHI$ must contain some red K_5. If GH is blue then this K_5 contains at most four vertices, two from $\{A, B, C, D, E\}$ and one from each of $\{F, G, H\}$ and $\{I\}$; and if GI is blue then this K_5 again contains at most four vertices, two from $\{A, B, C, D, E\}$ and one from each of $\{F, H\}$ and $\{G, I\}$. Either possibility yields a contradiction.

(ii) If there is some blue 7-cycle, say without loss of generality it is $ABCDEFG$. As before, any K_5 contains at most three vertices from $\{A, B, \ldots, G\}$, so $HIJKL$ must be a red K_5. Now for each of the $\binom{5}{2} = 10$ choices of pairs $\{V_1, V_2\} \subset \{H, I, J, K, L\}$, there must be a red K_5 among $ABCDEFGV_1V_2$; so for each edge in $HIJKL$, some red triangle in $ABCDEFG$ forms a red K_5 with that edge. But $ABCDEFG$ contains at most 7 red triangles: ACE, BDF, $\ldots$, and GBD. Thus some triangle corresponds to two edges. Without loss of generality, either ACE corresponds to both HI and HJ; or ACE corresponds to both HI and JK. In either case, $ACEHIJ$ is a red K_6, a contradiction.

(iii) Next suppose that there is some blue 9-cycle; then among these nine vertices there can be no red K_5, a contradiction.

(iv) Finally, suppose that there is some blue 11-cycle; without loss of generality, say it is $ABCDEFGHIJK$. There is a red K_5 among $\{A, B, C, D, E, F, G, H, I\}$, which must be $ACEGI$. Likewise, $DFHJA$ must be a red K_5, so AC, AD, $\ldots$, AH are all red. Similarly, *every* edge in $ABCDEFGHI$ is red except for those in the blue 11-cycle.

Now among $\{A, B, C, D, E, F, G, H, L\}$ there is some red K_5, either $ACEGL$ or $BDFHL$. Without loss of generality, assume the former. Then since $ACEGLI$ and $ACEGLJ$ can't be red 6-cycles, AI and AJ must be blue. But then AIJ is a blue 3-cycle, a contradiction. ■

Thus there are indeed no blue cycles of odd length, so the blue edges form a bipartite graph: that is, the twelve vertices can be partitioned into two groups $\mathcal{G}_1$ and $\mathcal{G}_2$ containing no blue edges. One of these groups, say $\mathcal{G}_1$, has at least 6 vertices; but then $\mathcal{G}_1$ is a red K_6, a contradiction. Therefore our original assumption was false; there *is* some red K_6, so some six people do indeed know each other.

Problem 11.1 Do there exist 19 distinct natural numbers which add to 1999 and which have the same sum of digits?

Solution: No such integers exist; suppose by way of contradiction they did.

The average of the numbers is $\frac{1999}{19} < 106$, so one number is at most 105 and has digit sum at most 18.

Every number is congruent to its digit sum modulo 9, so all the numbers and their digit sums are congruent modulo 9 — say, congruent to k. Then $19k \equiv 1999 \Rightarrow k \equiv 1 \pmod 9$, so the common digit sum is either 1 or 10.

If it is 1 then all the numbers equal 1, 10, 100, or 1000 so that some two are equal — which is not allowed. Thus the common digit sum is 10. Note that the twenty smallest numbers with digit sum 10 are:

$$19, 28, 37, \ldots, 91, 109, 118, 127, \ldots, 190, 208.$$

The sum of the first nine numbers is $(10 + 20 + \cdots + 90) + (9 + 8 + \cdots + 1) = 450 + 45 = 495$, while the sum of the next nine numbers is $(900)+(10+20+\cdots+80)+(9+8+7+\cdots+1) = 900+360+45 = 1305$, so the first eighteen numbers add up to 1800.

Since $1800 + 190 \neq 1999$, the largest number among the nineteen must be at least 208. But then the smallest eighteen numbers add up to at least 1800, giving a total sum of at least $2028 > 1999$, a contradiction.

Problem 11.2 At each rational point on the real line is written an integer. Show that there exists a segment with rational endpoints, such that the sum of the numbers at the endpoints does not exceed twice the number at the midpoint.

First Solution: Let $f : \mathbb{Q} \to \mathbb{Z}$ be the function that maps each rational point to the integer written at that point. Suppose by way of contradiction that for all $q, r \in \mathbb{Q}$,

$$f(q) + f(r) > 2f\left(\frac{q+r}{2}\right).$$

For $i \geq 0$, let $a_i = \frac{1}{2^i}$ and $b_i = -\frac{1}{2^i}$. We shall prove that for some k, $f(a_k)$ and $f(b_k)$ are both less than $f(0)$. Suppose that for some i, $f(a_i) \geq f(0)$. Now we apply the condition:

$$f(a_{i+1}) < \frac{f(a_i) + f(0)}{2} \leq f(a_i).$$

Since the range of f is the integers, $f(a_{i+1}) \leq f(a_i) - 1$ as long as $f(a_i) \geq f(0)$. Therefore, there exists some m such that $f(a_m) < f(0)$. Then

$$f(a_{m+1}) < \frac{f(a_m) + f(0)}{2} < \frac{2f(0)}{2},$$

so $f(a_i) < f(0)$ for $i \geq m$.

Similarly, there exists n such that $f(b_i) < f(0)$ for $i \geq n$. Now if we just take $k = \max\{m, n\}$, we have a contradiction:

$$f(a_k) + f(b_k) < 2f(0).$$

Second Solution: Define f as in the first solution, and suppose by way of contradiction that there was no such segment; rewrite the inequality in the first solution as

$$f(p) - f\left(\frac{p+q}{2}\right) > f\left(\frac{p+q}{2}\right) - f(q).$$

For a continuous function, this would be equivalent to saying that f is strictly convex; however, f is not continuous. But we can still show a similar result for the set $\mathbb{F} = \left\{\frac{i}{2^j} \mid i, j \in \mathbb{Z}, j \geq 0\right\}$, fractions whose denominators are powers of 2. For convenience, write P_x to represent $(x, f(x))$ on the graph of f in the xy-plane. Then we have the following result:

Lemma. *For all $a, b, c \in \mathbb{F}$ with b between a and c, P_b is below the segment connecting P_a and P_c.*

Proof: Equivalently we can prove that the average rate of change of f in the interval $[a, b]$ is smaller than the average rate of change of f in the interval $[b, c]$ — that is,

$$\frac{f(b) - f(a)}{b - a} < \frac{f(c) - f(b)}{c - a}.$$

Partition $[a, b]$ and $[b, c]$ into sub-intervals of equal length δ. For example, if $a = \frac{\alpha}{2^j}$, $b = \frac{\beta}{2^j}$, and $c = \frac{\gamma}{2^j}$, we could use $\delta = \frac{1}{2^j}$.

Let $\Delta_x = \frac{f(x+\delta) - f(x)}{\delta}$, the average rate of change of f in the interval $[x, x+\delta]$. Then apply our inequality to find that

$$\Delta_a < \Delta_{a+\delta} < \cdots < \Delta_{c-\delta}.$$

Thus,

$$\begin{aligned}
\frac{f(b) - f(a)}{b - a} &\leq \max\{\Delta_a, \Delta_{a+\delta}, \ldots, \Delta_{b-\delta}\} \\
&= \Delta_{b-\delta} \\
&< \Delta_b \\
&= \min\{\Delta_b, \Delta_{b+\delta}, \ldots, \Delta_{c-\delta}\}
\end{aligned}$$

$$\leq \frac{f(c) - f(b)}{c - b},$$

as desired. ■

Now consider all numbers $x \in \mathbb{F}$ between 0 and 1. Since P_x lies below the segment connecting P_0 and P_1, we have $f(x) \leq \max\{f(0), f(1)\}$.

Pick some number $k \in \mathbb{F}$ between 0 and 1. For $k < x < 1$, P_x must lie above the line connecting P_0 and P_k; otherwise, P_k would be above the segment connecting P_0 and P_x, contradicting our lemma. Similarly, for $0 < x < k$, P_x must lie above the line connecting P_k and P_1.

Since there are infinitely many values $x \in \mathbb{F}$ in the interval $(0, 1)$ but $f(x)$ is bounded from above and below in this interval, some three points have the same y-coordinate – contradicting our lemma. Therefore our original assumption was false and the segment described in the problem *does* exist.

Problem 11.3 A circle inscribed in quadrilateral $ABCD$ touches sides DA, AB, BC, CD at K, L, M, N, respectively. Let S_1, S_2, S_3, S_4 be the incircles of triangles AKL, BLM, CMN, DNK, respectively. The common external tangents to S_1 and S_2, to S_2 and S_3, to S_3 and S_4, and to S_4 and S_1, not lying on the sides of $ABCD$, are drawn. Show that the quadrilateral formed by these tangents is a rhombus.

Solution: Let P be the intersection of the two common external tangents involving S_1, and let Q, R, S be the intersections of the pairs of tangents involving S_2, S_3, S_4, respectively.

As in problem 10.3, the centers of S_1, S_2, S_3, S_4 are the midpoints of arcs KL, LM, MN, NK, respectively. AB does not pass through the incenter I of triangle KLM, so by the result proved in problem 9.3 the other external tangent PQ must pass through I and be parallel to KM. Likewise, $RS \parallel KM$ so we have $PQ \parallel RS$.

Similarly, $QR \parallel LN \parallel SP$, so $PQRS$ is a parallelogram.

Let $\langle X \mid \omega \rangle$ denote the length of the tangent from point X to circle ω, and let $\langle \omega_1 \mid \omega_2 \rangle$ denote the length of the external tangent to circles ω_1 and ω_2. Then we also know

$$AB = \langle A \mid S_1 \rangle + \langle S_1 \mid S_2 \rangle + \langle S_2 \mid B \rangle$$

$$= \langle A \mid S_1 \rangle + \langle S_1 \mid P \rangle + PQ + \langle Q \mid S_2 \rangle + \langle S_2 \mid B \rangle$$

and three analogous equations. Substituting these into $AB + CD = BC + DA$, which is true since $ABCD$ is circumscribed about a circle, we find that $PQ + RS = QR + SP$.

But since $PQRS$ is a parallelogram, $PQ = RS$ and $QR = SP$, implying that $PQ = QR = RS = SP$ and that $PQRS$ is a rhombus.

Problem 11.5 Four natural numbers have the property that the square of the sum of any two of the numbers is divisible by the product of the other two. Show that at least three of the four numbers are equal.

Solution: Suppose by way of contradiction four such numbers did exist, and pick a counterexample a, b, c, d with minimum sum $a+b+c+d$. If some prime p divided both a and b, then from $a \mid (b+c)^2$ and $a \mid (b+d)^2$ we know that p divides c and d as well: but then $\frac{a}{p}$, $\frac{b}{p}$, $\frac{c}{p}$, $\frac{d}{p}$ are a counter-example with smaller sum. Therefore, the four numbers are pairwise relatively prime.

Suppose that some prime $p > 2$ divided a. Then since a divides each of $(b+c)^2, (c+d)^2, (d+b)^2$, we know that p divides $b+c$, $c+d$, $d+b$. Hence p divides $(b+c)+(c+d)+(d+b)$ and thus $b+c+d$. Therefore $p \mid (b+c+d)-(b+c) = d$, and similarly $p \mid c$ and $p \mid b$, a contradiction.

Thus each of a, b, c, d are powers of 2. But since they are pairwise relatively prime, three of them must equal 1 — a contradiction. Therefore our original assumption was false, and no such counterexample exists.

Problem 11.6 Show that three convex polygons in the plane cannot be intersected by a single line if and only if for each of the polygons, there exists a line intersecting none of the polygons, such that the given polygon lies on the opposite side of the line from the other two.

Solution: In this proof, "polygon" refers to both the border and interior of a polygon – the problem statement is not affected by this assumption, because a line hitting the interior of a polygon must hit its border as well.

Suppose that some line ℓ intersects all three polygons; orient the figure to make ℓ horizontal, and say it hits the polygons (from left to right) at A, B, and C. Any line m not hitting any of the polygons is either parallel to ℓ; hits ℓ to the left of B; or hits ℓ to the right of B. In all of these cases, m does not separate B from both A and C, so m cannot separate the polygon containing B from the other polygons. (In the first two cases B and C are not separated; and in the first and third cases A and B are not separated.)

To prove the other direction, we begin by proving an intuitively obvious but nontrivial lemma:

Lemma. *Given two non-intersecting polygons, there is a line that separates them.*

Let V be the convex hull of the two polygons. If all its vertices are in one polygon, then this polygon contains the other — a contradiction. Also, for any four vertices A, B, C, D in that order on V (not necessarily adjacent), since AC and BD intersect we cannot have A and C in one polygon and B and D in the other. Thus one run of adjacent vertices $V_1, \ldots, V_m$ is in one polygon P; and the remaining vertices $W_1, \ldots, W_n$ are in the other polygon Q.

Then V_1V_m is contained in polygon P, so line V_1V_m does not intersect Q; therefore we can simply choose a line extremely close to V_1V_m that doesn't hit P, and separates P and Q. ■

Now call the polygons T, U, V, and suppose no line intersects all three. Then every two polygons are disjoint — if M was in $T \cup U$ and $N \neq M$ was in V, then the line MN hits all three polygons.

Triangulate the convex hull H of T and U (that is, divide it into triangles whose vertices are vertices of H). If V intersects H at some point M, then M is on or inside of these triangles, XYZ. Without loss of generality say $X \in T$ and $Y, Z \in U$ (otherwise both triangle XYZ and M are inside either T or U, so this polygon intersects V). Then line XM intersects both T and V; and since it hits YZ, it intersects U as well – a contradiction.

Thus H is disjoint from V, and from the lemma we can draw a line separating the two — and thus separating T and U from V, as desired. We can repeat this construction for T and U, so we are done.

Problem 11.7 Through vertex A of tetrahedron $ABCD$ passes a plane tangent to the circumscribed sphere of the tetrahedron. Show that the lines of intersection of the plane with the planes ABC, ACD,

ABD form six equal angles if and only if

$$AB \cdot CD = AC \cdot BD = AD \cdot BC.$$

Solution: Perform an inversion about A with arbitrary radius r. Since the given plane P is tangent to the circumscribed sphere of $ABCD$, the sphere maps to a plane parallel to P containing B', C', D', the images of B, C, D under inversion. Planes P, ABC, ACD, and ABD stay fixed under the inversion since they all contain A.

Now, since $C'D'$ is in a plane parallel to P, plane $ACD = AC'D'$ intersects P in a line parallel to $C'D'$. More rigorously, complete parallelogram $C'D'AX$. Then X is both in plane $AC'D' = ACD$ and in plane P (since $PX \parallel C'D'$), so the intersection of ACD and P is the line PX, parallel to $C'D'$.

Similarly, plane ADB intersects P in a line parallel to $D'B'$, and plane ABC intersects P in a line parallel to $B'C'$. These lines form six equal angles if and only if $C'D'$, $D'B'$, $B'C'$ form equal angles: that is, if triangle $C'D'B'$ is equilateral and $C'D' = D'B' = B'C'$. Under the inversion distance formula, this is true if and only if

$$\frac{CD \cdot r^2}{AC \cdot AD} = \frac{DB \cdot r^2}{AD \cdot AB} = \frac{BC \cdot r^2}{AB \cdot AC},$$

which (multiplying by $\frac{AB \cdot AC \cdot AD}{r^2}$) is equivalent to

$$AB \cdot CD = AC \cdot BD = AD \cdot BC,$$

as desired.

1.18 Slovenia

Problem 1 The sequence of real numbers $a_1, a_2, a_3, \ldots$ satisfies the initial conditions $a_1 = 2, a_2 = 500, a_3 = 2000$ as well as the relation

$$\frac{a_{n+2} + a_{n+1}}{a_{n+1} + a_{n-1}} = \frac{a_{n+1}}{a_{n-1}}$$

for $n = 2, 3, 4, \ldots$. Prove that all the terms of this sequence are positive integers and that 2^{2000} divides the number a_{2000}.

Solution: From the recursive relation it follows that $a_{n+2}a_{n-1} = a_{n+1}^2$ for $n = 2, 3, \ldots$. No term of our sequence can equal 0, and hence it is possible to write

$$\frac{a_{n+2}}{a_{n+1}a_n} = \frac{a_{n+1}}{a_n a_{n-1}}$$

for $n = 2, 3, \ldots$. It follows by induction that the value of the expression $\frac{a_{n+1}}{a_n a_{n-1}}$ is constant, namely equal to $\frac{a_3}{a_2 a_1} = 2$. Thus $a_{n+2} = 2a_n a_{n+1}$ and all terms of the sequence are positive integers.

From this new relation, we also know that $\frac{a_{n+1}}{a_n}$ is an even integer for all positive integers n. Write $a_{2000} = \frac{a_{2000}}{a_{1999}} \frac{a_{1999}}{a_{1998}} \cdots \frac{a_2}{a_1} \cdot a_1$. In this product each of the 1999 fractions is divisible by 2, and $a_1 = 2$ is even as well. Thus a_{2000} is indeed divisible by 2^{2000}.

Problem 2 Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the condition

$$f(x - f(y)) = 1 - x - y$$

for all $x, y \in \mathbb{R}$.

Solution: For $x = 0, y = 1$ we get $f(-f(1)) = 0$. For $y = -f(1)$ it follows that $f(x) = 1 + f(1) - x$. Writing $a = 1 + f(1)$ and $f(x) = a - x$, we have

$$1 - x - y = f(x - f(y)) = a - x + f(y) = 2a - x - y$$

so that $a = \frac{1}{2}$. And indeed, the function $f(x) = \frac{1}{2} - x$ satisfies the functional equation.

Problem 3 Let E be the intersection of the diagonals in cyclic quadrilateral $ABCD$, and let F and G be the midpoints of sides AB and CD, respectively. Prove that the three lines through G, F, E perpendicular to $\overline{AC}, \overline{BD}, \overline{AD}$, respectively, intersect at one point.

Solution: All angles are directed modulo 180°. Drop perpendicular $\overline{GP}$ to diagonal AC and perpendicular $\overline{FQ}$ to diagonal BD. Let R be the intersection of lines PG and FQ, and let H be the foot of the perpendicular from E to side AD. We wish to prove that H, E, R are collinear.

Since F and G are midpoints of corresponding sides in similar triangles DEC and ABE (with opposite orientations), triangles DPE and AQE are similar with opposite orientations as well. Thus $\angle DPE = \angle EQA$ and therefore $AQPD$ is a cyclic quadrilateral. And because $\angle EQR = 90^\circ = \angle EPR$, the quadrilateral $EQRP$ is cyclic, too. So

$$\angle ADQ = \angle APQ = \angle EPQ = \angle ERQ.$$

It follows that $\angle DEH = 90^\circ - \angle ADQ = 90^\circ - \angle ERQ = \angle QER$; and since D, E, Q are collinear then H, E, R must be as well.

Problem 4 Three boxes with at least one marble in each are given. In a *step* we choose two of the boxes, doubling the number of marbles in one of the boxes by taking the required number of marbles from the other box. Is it always possible to empty one of the boxes after a finite number of steps?

Solution: Without loss of generality suppose that the number of marbles in the boxes are a, b, and c with $a \le b \le c$. Write $b = qa + r$ where $0 \le r < a$ and $q \ge 1$. Then express q in binary:

$$q = m_0 + 2m_1 + \cdots + 2^k m_k,$$

where each $m_i \in \{0, 1\}$ and $m_k = 1$. Now for each $i = 0, 1, \ldots, k$, add $2^i a$ marbles to the first box: if $m_i = 1$ take these marbles from the second box; otherwise take them from the third box. In this way we take at most $(2^k - 1)a < qa \le b \le c$ marbles from the third box and exactly qa marbles from the second box altogether.

In the second box there are now $r < a$ marbles left. Thus the box with the least number of marbles now contains less than a marbles. Then by repeating the described procedure, we will eventually empty one of the boxes.

1.19 Taiwan

Problem 1 Determine all solutions (x, y, z) of positive integers such that

$$(x+1)^{y+1} + 1 = (x+2)^{z+1}.$$

Solution: Let $a = x+1$, $b = y+1$, $c = z+1$. Then $a, b, c \geq 2$ and

$$a^b + 1 = (a+1)^c$$
$$((a+1) - 1)^b + 1 = (a+1)^c.$$

Taking either equation mod $(a+1)$ yields $(-1)^b + 1 \equiv 0$, so b is odd. Then taking the second equation mod $(a+1)^2$ after applying the binomial expansion yields

$$\binom{b}{1}(a+1)(-1)^{b-1} + (-1)^b + 1 \equiv 0 \pmod{(a+1)^2}$$

so $a + 1 \mid b$ and a is even.

On the other hand, taking the first equation mod a^2 after applying the binomial expansion yields

$$1 \equiv \binom{c}{1} a + 1 \pmod{a^2}$$

so c is divisible by a and is even as well. Write $a = 2a_1$ and $c = 2c_1$. Then

$$2^b a_1^b = a^b = (a+1)^c - 1 = ((a+1)^{c_1} - 1)((a+1)^{c_1} + 1).$$

It follows that $\gcd((a+1)^{c_1} - 1, (a+1)^{c_1} + 1) = 2$. Therefore, using the fact that $2a_1$ is a divisor of $(a+1)^{c_1} - 1$, we may conclude that

$$(a+1)^{c_1} - 1 = 2a_1^b$$
$$(a+1)^{c_1} + 1 = 2^{b-1}.$$

We must have $2^{b-1} > 2a_1^b \Rightarrow a_1 = 1$. Then these equations give $c_1 = 1$ and $b = 3$, and therefore the only solution is $(x, y, z) = (1, 2, 1)$.

Problem 2 There are 1999 people participating in an exhibition. Out of any 50 people, at least 2 do not know each other. Prove that we can find at least 41 people who each know at most 1958 other people.

Solution: Let Y be the set of people who know at least 1959 other people, and let $N(p)$ denote the set of people whom p knows. Assume by way of contradiction that less than 41 people each know at most 1958 people; then $|Y| \geq 1959$. We now show that some 50 people all know each other, a contradiction.

Pick a person $y_1 \in Y$ and write $B_1 = N(y_1)$ with $|B_1| \geq 1959$. Then $|B_1| + |Y| > 1999$, and there is a person $y_2 \in B_1 \cap Y$.

Now write $B_2 = N(y_1) \cap N(y_2)$ with $|B_2| = |B_1| + |N(y_2)| - |B_1 \cup N(y_2)| \geq 1959 + 1959 - 1999 = 1999 - 40 \cdot 2$. Then $|B_2| + |Y| > 1999$, and there is a person $y_3 \in B_2 \cap Y$.

Now continue similarly: suppose we have $j \leq 48$ different people $y_1, y_2, \ldots, y_j$ in Y who all know each other; and suppose that $B_j = N(y_1) \cap N(y_2) \cap \cdots \cap N(y_j)$ has at least $1999 - 40j \geq 79 > 40$ elements. Then $|B_j| + |Y| > 1999$, and there is a person $y_{j+1} \in B_j \cap Y$; and $B_{j+1} = B_j \cap N(y_{j+1})$ has at least $|B_j| + |N(y_{j+1})| - |B_j \cup N(y_{j+1})| \geq (1999 - 40j) + 1959 - 1999 = 1959 - 40(j+1) > 0$ elements, and we can continue onward.

Thus we can find 49 people $y_1, y_2, \ldots, y_{49}$ such that $B_{49} = N(y_1) \cap N(y_2) \cap \cdots \cap N(y_{49})$ is nonempty. Thus there is a person $y_{50} \in B_{49}$; but then any two people from $y_1, y_2, \ldots, y_{50}$ know each other, a contradiction.

Problem 3 Let P^* denote all the odd primes less than 10000, and suppose $p \in P^*$. For each subset $S = \{p_1, p_2, \cdots, p_k\}$ of P^*, with $k \geq 2$ and not including p, there exists a $q \in P^* \setminus S$ such that

$$q + 1 \mid (p_1 + 1)(p_2 + 1) \cdots (p_k + 1).$$

Find all such possible values of p.

Solution: A "Mersenne prime" is a prime of the form $2^n - 1$ for some positive integer n. Notice that if $2^n - 1$ is prime then $n > 1$ and n is prime because otherwise we could either write (if n were even) $n = 2m$ and $2^n - 1 = (2^m - 1)(2^m + 1)$, or (if n were odd) $n = ab$ for odd a, b and $2^n - 1 = (2^a - 1)(2^{(b-1)a} + 2^{(b-2)a} + \cdots + 2^a + 1)$. Then some calculations show that the set T of Mersenne primes less than 10000 is

$$\{M_2, M_3, M_5, M_7, M_{13}\} = \{3, 7, 31, 127, 8191\},$$

where $M_p = 2^p - 1$. ($2^{11} - 1$ is not prime: it equals $23 \cdot 89$.) We claim this is the set of all possible values of p.

If some prime p is *not* in T, then look at the set $S = T$. Then there must be some prime $q \notin S$ less than 10000 such that

$$q + 1 \mid (M_2 + 1)(M_3 + 1)(M_5 + 1)(M_7 + 1)(M_{13} + 1) = 2^{30}.$$

Thus, $q + 1$ is a power of 2 and q is a Mersenne prime less than 10000 — and therefore $q \in T = S$, a contradiction.

On the other hand, suppose p is in T. Suppose we have a set $S = \{p_1, p_2, \ldots, p_k\} \subset P^*$ not including p, with $k \geq 2$ and $p_1 < p_2 < \cdots < p_k$. Suppose by way of contradiction that for all $q \in P^*$ such that $q + 1 \mid (p_1 + 1) \cdots (p_k + 1)$, we have $q \in S$. Then

$$4 \mid (p_1 + 1)(p_2 + 1) \Longrightarrow M_2 \in S$$

$$8 \mid (M_2 + 1)(p_2 + 1) \Longrightarrow M_3 \in S$$

$$32 \mid (M_2 + 1)(M_3 + 1) \Longrightarrow M_5 \in S$$

$$128 \mid (M_2 + 1)(M_5 + 1) \Longrightarrow M_7 \in S$$

$$8192 \mid (M_3 + 1)(M_5 + 1)(M_7 + 1) \Longrightarrow M_{13} \in S.$$

Then p, a Mersenne prime under 10000, must be in S — a contradiction. Therefore there *is* some prime $q < 10000$ not in S with $q + 1 \mid (p_1 + 1) \cdots (p_k + 1)$, as desired. This completes the proof.

Problem 4 The altitudes through the vertices A, B, C of an acute-angled triangle ABC meet the opposite sides at D, E, F, respectively, and $AB > AC$. The line EF meets BC at P, and the line through D parallel to EF meets the lines AC and AB at Q and R, respectively. Let N be a point on the side BC such that $\angle NQP + \angle NRP < 180^\circ$. Prove that $BN > CN$.

Solution: Let M be the midpoint of BC. We claim that P, Q, M, R are concyclic. Given this, we would have

$$\angle MQP + \angle MRP = 180^\circ > \angle NQP + \angle NRP.$$

This can only be true if N is between M and C; then $BN > CN$, as desired.

Since $\angle BEC = \angle BFC = 90^\circ$, we observe that the points B, C, E, F are concyclic and thus $PB \cdot PC = PE \cdot PF$. Also, the points D, E, F, M lie on the nine-point circle of triangle ABC so that

$PE \cdot PF = PD \cdot PM$. (Alternatively, it's easy to show that $DEFM$ is cyclic with some angle-chasing). These two equations yield

$$PB \cdot PC = PD \cdot PM. \tag{1}$$

On the other hand, since $\triangle AEF \sim \triangle ABC$ and $QR \parallel EF$, we have $\angle RBC = \angle AEF = \angle CQR$. Thus $CQBR$ is cyclic and

$$DQ \cdot DR = DB \cdot DC. \tag{2}$$

Now let $MB = MC = a$, $MD = d$, $MP = p$. Then we have $PB = p + a$, $DB = a + d$, $PC = p - a$, $CD = a - d$, $DP = p - d$. Then equation (1) implies

$$\begin{aligned} (p+a)(p-a) &= (p-d)p \\ \implies a^2 &= dp \\ \implies (a+d)(a-d) &= (p-d)d, \end{aligned}$$

or equivalently

$$DB \cdot DC = DP \cdot DM. \tag{3}$$

Combining (2) and (3) yields $DQ \cdot DR = DP \cdot DM$, so that the points P, Q, M, R are concyclic, as claimed.

Problem 5 There are 8 different symbols designed on n different T-shirts, where $n \geq 2$. It is known that each shirt contains at least one symbol, and for any two shirts, the symbols on them are not all the same. Also, for any k symbols, $1 \leq k \leq 7$, the number of shirts containing at least one of the k symbols is even. Find the value of n.

Solution: Let X be the set of 8 different symbols, and call a subset S of X "stylish" if some shirt contains exactly those symbols in S. Look at a stylish set A with the minimal number of symbols $|A| \geq 1$; since $n \geq 2$, we must have $|A| \leq 7$. Then all the other $n-1$ stylish sets contain at least one of the $k = 8 - |A|$ symbols in $X \setminus A$, so $n - 1$ is even and n is odd.

Observe that any nonempty subset $S \subseteq X$ contains an odd number of stylish subsets: For $S = X$ this number is n; and for $|S| \leq 7$, an even number t of stylish sets contain some element of $X \setminus S$, so the remaining *odd* number $n - t$ of stylish sets are contained in S.

Then every nonempty subset of X is stylish. Otherwise, pick a minimal non-stylish subset $S \subseteq X$. Its only stylish subsets are its

$2^{|S|}-2$ proper subsets, which are all stylish by the minimal definition of S; but this is an *even* number, which is impossible. Thus there must be $2^8 - 1 = 255$ T-shirts; and indeed, given any k symbols $(1 \le k \le 7)$, an even number $2^8 - 2^{8-k}$ of T-shirts contain at least one of these k symbols.

1.20 Turkey

Problem 1 Let ABC be an isosceles triangle with $AB = AC$. Let D be a point on $\overline{BC}$ such that $BD = 2DC$, and let P be a point on $\overline{AD}$ such that $\angle BAC = \angle BPD$. Prove that

$$\angle BAC = 2\angle DPC.$$

Solution: Draw X on $\overline{BP}$ such that $BX = AP$. Then $\angle ABX = \angle ABP = \angle DPB - \angle PAB = \angle CAB - \angle PAB = \angle CAP$. And since $AB = CA$ and $BX = AP$, by SAS we have $\triangle ABX \cong \triangle CAP$. Hence $[ABX] = [CAP]$, and also $\angle DPC = 180^\circ - \angle CPA = 180^\circ - \angle AXB = \angle PXA$.

Next, since $BD = 2CD$, the distance from B to line AD is twice the distance from C to line AD. Therefore $[ABP] = 2[CAP] \Longrightarrow [ABX] + [AXP] = 2[ABX]$. Hence $[AXP] = [ABX]$ and $XP = BX = AP$. Hence $\angle PXA = \angle XAP$, and $\angle BAC = \angle BPD = \angle PXA + \angle XAP = 2\angle PXA = 2\angle DPC$, as desired.

Problem 2 Prove that

$$(a + 3b)(b + 4c)(c + 2a) \geq 60abc$$

for all real numbers $0 \leq a \leq b \leq c$.

Solution: By AM-GM we have $a + b + b \geq 3\sqrt[3]{ab^2}$; multiplying this and the analagous inequalities yields $(a+2b)(b+2c)(c+2a) \geq 27abc$. Then

$$\begin{aligned}
&(a + 3b)(b + 4c)(c + 2a) \\
&\geq \left(a + \frac{1}{3}a + \frac{8}{3}b\right)\left(b + \frac{2}{3}b + \frac{10}{3}c\right)(c + 2a) \\
&= \frac{20}{9}(a + 2b)(b + 2c)(c + 2a) \geq 60abc,
\end{aligned}$$

as desired.

Problem 3 The points on a circle are colored in three different colors. Prove that there exist infinitely many isosceles triangles with vertices on the circle and of the same color.

First Solution: Partition the points on the circle into infinitely many regular 13-gons. In each 13-gon, by the Pigeonhole Principle there are at least 5 vertices of the same color: say, red. Later we use some extensive case analysis to show that among these 5 vertices, some three form an isosceles triangle. Then for each 13-gon there is a monochrome isosceles triangle; so there are infinitely many monochrome isosceles triangles, as desired.

It suffices now to prove the following claim:

Claim *Suppose 5 vertices of a regular 13-gon are colored red. Then some three red vertices form an isosceles triangle.*

Proof: Suppose none of these 5 vertices did form an isosceles triangle. Label the vertices $P_0, \ldots, P_{12}$ (with indices taken modulo 13); first we prove that P_i and P_{i+2} cannot both be red. Assume they could be, and say without loss of generality that P_{12} and P_1 were red; then P_{10}, P_0, and P_3 cannot be red. Furthermore, at most one vertex from each pair (P_{11}, P_4), (P_4, P_7), and (P_7, P_8) is red since each of these pairs forms an isosceles triangle with P_1. Similarly, at most one vertex from each pair (P_2, P_9), (P_9, P_6), and (P_6, P_5) is red. Now three vertices from $\{P_{11}, P_4, P_7, P_8\} \cup \{P_2, P_9, P_6, P_5\}$ are red; assume without loss of generality that two vertices from $\{P_{11}, P_4, P_7, P_8\}$ are. Vertices P_4 and P_8 can't both be red because they form an isosceles triangle with P_{12}; so vertices P_{11} and P_7 must be red. But then any remaining vertex forms an isosceles triangle with some two of P_1, P_7, P_{11}, P_{12}, so we can't have five red vertices, a contradiction.

Next we prove that P_i and P_{i+1} can't be red. If so, suppose without loss of generality that P_6 and P_7 are red. Then P_4, P_5, P_8, and P_9 cannot be red from the result in the last paragraph. P_0 cannot be red either, because triangle $P_0P_6P_7$ is isosceles. Now each pair (P_3, P_{11}) and (P_{11}, P_1) contains at most one red vertex because triangles $P_3P_7P_{11}$ and $P_1P_6P_{11}$ are isosceles. Also, P_1 and P_3 can't both be red from the result in the last paragraph. Thus at most one of $\{P_1, P_3, P_{11}\}$ can be red; similarly, at most one of $\{P_{12}, P_{10}, P_2\}$ can be red. But then we have at most four red vertices, again a contradiction.

Thus if P_i is red then $P_{i-2}, P_{i-1}, P_{i+1}, P_{i+2}$ cannot be red; but then we can have at most four red vertices, a contradiction. ■

Second Solution: Suppose we have $k \geq 1$ colors and a number $n \geq 3$. Then Van der Warden's theorem states that we can find N such that for any coloring of the numbers $1, 2, \ldots, N$ in the k colors, there are n numbers in arithmetic progression which are colored the same. Apply this theorem with $k = n = 3$ to find such an N, and partition the points on the circle into infinitely many regular N-gons rather than 13-gons. For each N-gon $P_1P_2\ldots P_N$, there exist i, j, k (between 1 and N) in arithmetic progression such that P_i, P_j, P_k are all the same color. Hence triangle $P_iP_jP_k$ is a monochrome isosceles triangle. It follows that since we have infinitely many such N-gons, there are infinitely many monochrome isosceles triangles.

Problem 4 Let $\angle XOY$ be a given angle, and let M and N be two points on the rays OX and OY, respectively. Determine the locus of the midpoint of $\overline{MN}$ as M and N varies along the rays OX and OY such that $OM + ON$ is constant.

Solution: Let $\hat{x}$ and $\hat{y}$ be the unit vectors pointing along rays OX and OY. Suppose we want $OM + ON$ to equal the constant k; then when $OM = c$ we have $ON = k - c$, and thus the midpoint of $\overline{MN}$ is $\frac{1}{2}(c\hat{x} + (k - c)\hat{y})$. As c varies from 0 to k, this traces out the line segment connecting $\frac{1}{2}k\hat{x}$ with $\frac{1}{2}k\hat{y}$; that is, the segment $\overline{M'N'}$ where $OM' = ON' = \frac{1}{2}k$, $M' \in \overrightarrow{OX}$, and $N' \in \overrightarrow{OY}$.

Problem 5 Some of the vertices of the unit squares of an $n \times n$ chessboard are colored such that any $k \times k$ square formed by these unit squares has a colored point on at least one of its sides. If $l(n)$ denotes the minimum number of colored points required to ensure the above condition, prove that

$$\lim_{n\to\infty} \frac{l(n)}{n^2} = \frac{2}{7}.$$

Solution: For each colored point P, consider any 1×1 square of the board it lies on. If this square contains m colored points, say that P gains $\frac{1}{m}$ points from that square. Adding over all the 1×1 squares that P lies on, we find the total number of points that P accrues.

Any colored point on the edge of the chessboard gains at most 2 points. As for a colored point P on the chessboard's interior, the 2×2 square centered at P must have a colored point Q on its border. Then P and Q both lie on some unit square, which P gains at most half a point from; thus P accrues at most $\frac{7}{2}$ points.

Therefore any colored point collects at most $\frac{7}{2}$ points, and $l(n)$ colored points collectively accrue at most $\frac{7}{2}l(n)$ points. But for the given condition to hold, the total number of points accrued must be n^2. It follows that $\frac{7}{2}l(n) \geq n^2$ and thus $\frac{l(n)}{n^2} \geq \frac{2}{7}$.

Now, given some $n \times n$ board, embed it as the corner of an $n' \times n'$ board where $7 \mid n'+1$ and $n \leq n' \leq n+6$. To each 7×7 grid of vertices on the $n' \times n'$ board, color the vertices as below:

$$\begin{array}{ccccccc}
\bullet & \circ & \bullet & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \bullet & \circ & \bullet \\
\circ & \bullet & \circ & \bullet & \circ & \circ & \circ \\
\bullet & \circ & \circ & \circ & \circ & \bullet & \circ \\
\circ & \circ & \bullet & \circ & \bullet & \circ & \circ \\
\circ & \bullet & \circ & \circ & \circ & \circ & \bullet \\
\circ & \circ & \circ & \bullet & \circ & \bullet & \circ
\end{array}$$

Then any $k \times k$ square on the chessboard has a colored point on at least one of its sides. Since we color $\frac{2}{7}(n'+1)^2$ vertices in this coloring, we have

$$l(n) \leq \frac{2}{7}(n'+1)^2 \leq \frac{2}{7}(n+7)^2$$

so that

$$\frac{l(n)}{n^2} \leq \frac{2}{7}\left(\frac{n+7}{n}\right)^2.$$

As $n \to \infty$, the right hand side becomes arbitrarily close to $\frac{2}{7}$. Since from before $\frac{l(n)}{n^2} \geq \frac{2}{7}$ for all n, this implies that $\lim_{n\to\infty} \frac{l(n)}{n^2}$ exists and equals $\frac{2}{7}$.

Problem 6 Let $ABCD$ be a cyclic quadrilateral, and let L and N be the midpoints of diagonals AC and BD, respectively. Suppose that $\overline{BD}$ bisects $\angle ANC$. Prove that $\overline{AC}$ bisects $\angle BLD$.

Solution: Suppose we have *any* cyclic quadrilateral $ABCD$ where L and N are the midpoints of $\overline{AC}$ and $\overline{BD}$. Perform an inversion about B with arbitrary radius; A, D, C map to collinear points A', D', C', while N maps to the point N' such that D' is the midpoint of $\overline{BN'}$.

There are only two points X on line $A'D'$ such that $\angle BXN' = \angle BA'N'$: the point A' itself, and the reflection of A' across D'. Then $\angle ANB = \angle BNC \iff \angle BA'N' = \angle BC'N' \iff A'D' = D'C' \iff \frac{AD}{BA \cdot BD} = \frac{DC}{BD \cdot BC} \iff AD \cdot BC = BA \cdot DC$.

Similarly, $\angle BLA = \angle DLA \iff AD \cdot BC = BA \cdot DC$. Therefore $\angle ANB = \angle BNC \iff \angle BLA = \angle DLA$; that is, $\overline{BD}$ bisects $\angle ANC$ if and only if $\overline{AC}$ bisects $\angle BLD$, which implies the claim.

Problem 7 Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that the set

$$\left\{ \frac{f(x)}{x} \mid x \in \mathbb{R} \text{ and } x \neq 0 \right\}$$

is finite and

$$f(x - 1 - f(x)) = f(x) - x - 1$$

for all $x \in \mathbb{R}$.

Solution: First we show that the set $\{x - f(x) \mid x \in \mathbb{R}\}$ is finite. If not, there exist infinitely many $k \neq 1$ such that for some x_k, $x_k - f(x_k) = k$. But then

$$\frac{f(k-1)}{k-1} = \frac{f(x_k - 1 - f(x_k))}{k-1} = \frac{f(x_k) - x_k - 1}{k-1} = -1 - \frac{2}{k-1}.$$

Since k takes on infinitely many values, $\frac{f(k-1)}{k-1}$ does as well—a contradiction.

Now choose x_0 so that $|x - f(x)|$ is maximal for $x = x_0$. Then for $y = x_0 - 1 - f(x_0)$ we have

$$y - f(y) = y - (f(x_0) - x_0 - 1) = 2(x_0 - f(x_0)).$$

Then because of the maximal definition of x_0, we must have $y - f(y) = x_0 - f(x_0) = 0$. Therefore $f(x) = x$ for all x, and this function indeed satisfies the given conditions.

Problem 8 Let the area and the perimeter of a cyclic quadrilateral C be A_C and P_C, respectively. If the area and the perimeter of the quadrilateral which is tangent to the circumcircle of C at the vertices of C are A_T and P_T, respectively, prove that

$$\frac{A_C}{A_T} \geq \left(\frac{P_C}{P_T}\right)^2.$$

Solution: Let the outer quadrilateral be $EFGH$ with angles $\angle E = 2\alpha_1$, $\angle F = 2\alpha_2$, $\angle G = 2\alpha_3$, $\angle H = 2\alpha_4$; also let the circumcircle of C have radius r and center O. Say that sides EF, FG, GH, HE are tangent to C at I, J, K, L.

In right triangle EIO, we have $IO = r$ and $\angle OEI = \alpha_1$ so that $EI = r\cot\alpha_1$. After finding $IF, FJ, \ldots, LE$ similarly, we find that $P_T = 2r\sum_{i=1}^4 \cot\alpha_i$. Also, $[EFO] = \frac{1}{2}EF \cdot IO = \frac{1}{2}EF \cdot r$; finding $[FGO], [GHO], [HEO]$ similarly shows that $A_T = \frac{1}{2}P_T \cdot r$.

As for quadrilateral $IJKL$, note that $IJ = 2r\sin\angle IKJ = 2r\sin\angle FIJ = 2r\sin(90^\circ - \alpha_2) = 2r\cos\alpha_2$. After finding JK, KL, LI in a similar manner we have $P_C = 2r\sum_{i=1}^4 \cos\alpha_i$. Also note that $\angle IOJ = 180^\circ - \angle JFI = 180^\circ - 2\alpha_2$, and hence $[IOJ] = \frac{1}{2}OI \cdot OJ \sin\angle IOJ = \frac{1}{2}r^2\sin(2\alpha_2) = r^2\sin\alpha_2\cos\alpha_2$. Adding this to the analogous expressions for $[JOK], [KOL], [LOI]$, we find that $A_C = r^2\sum_{i=1}^4 \sin\alpha_i\cos\alpha_i$.

Therefore the inequality we wish to prove is

$$\begin{aligned}
&A_C \cdot P_T^2 \geq A_T \cdot P_C^2 \\
\iff\ & r^2\sum_{i=1}^4 \sin\alpha_i\cos\alpha_i \cdot P_T^2 \geq \left(\frac{1}{2}P_T \cdot r\right) \cdot 4r^2\left(\sum_{i=1}^4 \cos\alpha_i\right)^2 \\
\iff\ & P_T \cdot \sum_{i=1}^4 \sin\alpha_i\cos\alpha_i \geq 2r \cdot \left(\sum_{i=1}^4 \cos\alpha_i\right)^2 \\
\iff\ & \sum_{i=1}^4 \cot\alpha_i \cdot \sum_{i=1}^4 \sin\alpha_i\cos\alpha_i \geq \left(\sum_{i=1}^4 \cos\alpha_i\right)^2.
\end{aligned}$$

But this is true by the Cauchy-Schwarz inequality $\sum a_i^2 \sum b_i^2 \geq \left(\sum a_i b_i\right)^2$ applied with each $a_i = \sqrt{\cot\alpha_i}$ and $b_i = \sqrt{\sin\alpha_i\cos\alpha_i}$.

Problem 9 Prove that the plane is not a union of the inner regions of finitely many parabolas. (The outer region of a parabola is the union of the lines on the plane not intersecting the parabola. The inner region of a parabola is the set of points on the plane that do not belong to the outer region of the parabola.)

Solution: Suppose by way of contradiction we could cover the plane with the inner regions of finitely many parabolas — say, n of them. Choose some fixed positive acute angle $\theta < \left(\frac{360}{2n}\right)^\circ$.

Take any of the parabolas and (temporarily) choose a coordinate system so that it satisfies the equation $y = ax^2$ with $a \geq 0$ (and where our coordinates are chosen to scale, so that one unit along the y-axis has the same length as one unit along the x-axis). Draw the tangents to the parabola at $x = \pm\frac{\cot\theta}{2a}$; these lines have slopes $2ax = \pm\cot\theta$. These lines meet on the y-axis at an angle of 2θ, forming a V-shaped region in the plane that contains the inner region of the parabola.

Performing the above procedure with all the parabolas, we obtain n V-shaped regions covering the entire plane. Again choose an x-axis, and say the rays bordering these regions make angles ϕ_j and $\phi_j + 2\theta$ with the positive x-axis (with angles taken modulo 360°). Then since $2n\theta < 360°$, there is some angle ϕ' not in any of the intervals $[\phi_j, \phi_j + 2\theta]$. Then consider the line passing through the origin and making angle of ϕ' with the positive x-axis; far enough out, the points on this line cannot lie in any of the V-shaped regions, a contradiction. Thus our original assumption was false, and we *cannot* cover the plane with the inner regions of finitely many parabolas.

1.21 Ukraine

Problem 1 Let $P(x)$ be a polynomial with integer coefficients. The sequence $\{x_n\}_{n\geq 1}$ satisfies the conditions $x_1 = x_{2000} = 1999$, and $x_{n+1} = P(x_n)$ for $n \geq 1$. Calculate

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{1999}}{x_{2000}}.$$

Solution: Write $a_i = x_i - x_{i-1}$ for each i, where we take subscripts (of both the x_i and a_i) modulo 1999. Since $c-d$ divides $P(c)-P(d)$ for integers c and d, we have that $a_i = x_i - x_{i-1}$ divides $P(x_i)-P(x_{i-1}) = a_{i+1}$ for all i.

First suppose that all the $a_i \neq 0$. Then $|a_{i+1}| \geq |a_i|$ for all i but also $|a_1| = |a_{2000}|$; hence all the $|a_i|$ equal the same value $m > 0$. But if n of the $a_1, a_2, \ldots, a_{1999}$ equal $m \neq 0$ and the other $1999 - n$ equal $-m$, then their sum $0 = x_{1999} - x_0 = a_1 + a_2 + \cdots + a_{1999}$ equals $m(2n - 1999) \neq 0$, a contradiction.

Thus for some k we have $a_k = 0$; then since a_k divides a_{k+1}, we have $a_{k+1} = 0$ and similarly $a_{k+2} = 0$, and so on. Thus all the x_i are equal and the given expression equals 1999.

Problem 2 For real numbers $0 \leq x_1, x_2, \ldots, x_6 \leq 1$ prove the inequality

$$\frac{x_1^3}{x_2^5 + x_3^5 + x_4^5 + x_5^5 + x_6^5 + 5} + \frac{x_2^3}{x_1^5 + x_3^5 + x_4^5 + x_5^5 + x_6^5 + 5} + \cdots + \frac{x_6^3}{x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + 5} \leq \frac{3}{5}.$$

Solution: The condition $0 \leq x_1, x_2, \ldots, x_6 \leq 1$ implies that the left hand side of the inequality is at most

$$\sum_{i=1}^{6} \frac{x_i^3}{x_1^5 + x_2^5 + \cdots + x_6^5 + 4} = \frac{x_1^3 + x_2^3 + \cdots + x_6^3}{x_1^5 + x_2^5 + \cdots + x_6^5 + 4}.$$

For $t \geq 0$ we have $\frac{t^5+t^5+t^5+1+1}{5} \geq t^3$ by AM-GM. Adding up the six resulting inequalities for $t = x_1, x_2, \ldots, x_6$ and dividing by $(x_1^5 + x_2^5 + \cdots + x_6^5 + 4)$ shows that the above expression is at most $\frac{3}{5}$.

Problem 3 Let $\overline{AA_1}, \overline{BB_1}, \overline{CC_1}$ be the altitudes of an acute triangle ABC, and let O be an arbitrary point inside the triangle $A_1B_1C_1$.

Let M, N, P, Q, R, S be the orthogonal projections of O onto lines AA_1, BC, BB_1, CA, CC_1, AB, respectively. Prove that lines MN, PQ, RS are concurrent.

Solution: Observe that three lines passing through different vertices of a triangle are concurrent if and only if their reflections across the corresponding angle bisectors are also concurrent; this is easily proved using the trigonometric form of Ceva's Theorem.

Let A_0, B_0, C_0 be the centers of rectangles OMA_1N, OPB_1Q, OSC_1R, respectively. Under the homothety with center O and ratio $\frac{1}{2}$, triangle $A_1B_1C_1$ maps to triangle $A_0B_0C_0$. Then since lines AA_1, BB_1, CC_1 are the angle bisectors of triangle $A_1B_1C_1$ (easily proved with angle-chasing), the angle bisectors of triangle $A_0B_0C_0$ are parallel to lines AA_1, BB_1, CC_1.

Because OMA_1N is a rectangle, diagonals OA_1 and MN are reflections of each across the line through A_0 parallel to line AA_1. From above, this line is precisely the angle bisector of $\angle C_0A_0B_0$ in triangle $A_0B_0C_0$. Similarly, lines OB_1 and OC_1 are reflections of lines PQ and RS across the other angle bisectors. Then since lines OA_1, OB_1, OC_1 concur at O, from our initial observation lines MN, PQ, RS concur as well.

1.22 United Kingdom

Problem 1 I have four children. The age in years of each child is a positive integer between 2 and 16 inclusive and all four ages are distinct. A year ago the square of the age of the oldest child was equal to sum of the squares of the ages of the other three. In one year's time the sum of the squares of the ages of the oldest and the youngest children will be equal to the sum of the squares of the other two children. Decide whether this information is sufficient to determine their ages uniquely, and find all possibilities for their ages.

Solution: Let the children's present ages be $a+1$, $b+1$, $c+1$, and $d+1$. We are given that $1 \le a < b < c < d \le 15$; note that $b \le 13$ so that $b - a \le 12$. We are also given

$$d^2 = a^2 + b^2 + c^2 \tag{1}$$

and

$$(d+2)^2 + (a+2)^2 = (b+2)^2 + (c+2)^2. \tag{2}$$

Subtracting (1) from (2) gives $4(a+d) + a^2 = 4(b+c) - a^2$, or

$$a^2 = 2(b + c - a - d). \tag{3}$$

Then a must be even since its square is even. Furthermore, since $d > c$,

$$a^2 = 2(b - a + (c - d)) < 2(b - a) < 24,$$

and hence either $a = 2$ and $a = 4$.

If $a = 4$ then, since $a^2 < 2(b - a)$, we have $2b > a^2 + 2a = 24$ so that $b > 12$. This forces $b = 13$, $c = 14$, and $d = 15$, which contradicts the given conditions.

Thus $a = 2$. Equation (3) gives $b + c - d = 4$, so substituting $a = 2$ and $d = b + c - 4$ into (1) and simplifying yields

$$(b-4)(c-4) = 10 = 1 \cdot 10 = 2 \cdot 5.$$

Therefore we have $(b, c) = (5, 14)$ or $(6, 9)$, in which cases $d = 15$ and $d = 11$ respectively.

Hence the only possible solutions are $(a, b, c, d) = (2, 5, 14, 15)$ or $(2, 6, 9, 11)$, and these indeed satisfy (1) and (2). It follows that there is no unique solution, and it is not possible to determine the childrens' ages.

Problem 2 A circle has diameter $\overline{AB}$ and X is a fixed point on the segment AB. A point P, distinct from A and B, lies on the circle. Prove that, for all possible positions of P,

$$\frac{\tan \angle APX}{\tan \angle PAX}$$

is a constant.

Solution: Let Q be the projection of X onto $\overline{AP}$. Note that $\angle APB = 90°$, and thus $\tan \angle PAX = \frac{PB}{PA}$. Also, $XQ \parallel PB$ so $\triangle AQX \sim \triangle APB$. Therefore,

$$\tan \angle APX = \frac{QX}{QP} = \frac{\frac{AX \cdot BP}{AB}}{\frac{BX \cdot AP}{AB}} = \frac{AX \cdot BP}{BX \cdot AP},$$

and

$$\frac{\tan \angle APX}{\tan \angle PAX} = \frac{AX}{BX}$$

is fixed.

Problem 3 Determine a positive constant c such that the equation

$$xy^2 - y^2 - x + y = c$$

has exactly three solutions (x, y) in positive integers.

Solution: When $y = 1$ the left hand side is 0. Thus we can rewrite our equation as

$$x = \frac{y(y-1) + c}{(y+1)(y-1)}.$$

The numerator is congruent to $-1(-2)+c$ modulo $y+1$, and it is also congruent to c modulo $y-1$. Hence we must have $c \equiv -2 \pmod{y+1}$ and $c \equiv 0 \pmod{y-1}$. Since $c = y-1$ satisfies these congruences, we must have $c \equiv y - 1 \pmod{\operatorname{lcm}(y-1, y+1)}$. When y is even, $\operatorname{lcm}(y-1, y+1) = y^2 - 1$; when y is odd, $\operatorname{lcm}(y-1, y+1) = \frac{1}{2}(y^2 - 1)$.

Then for $y = 2, 3, 11$ we have $c \equiv 1 \pmod 3$, $c \equiv 2 \pmod 4$, $c \equiv 10 \pmod{60}$. Hence, we try setting $c = 10$. For x to be an integer we must have $y - 1 \mid 10 \Rightarrow y = 2, 3, 6,$ or 11; these values give $x = 4, 2, \frac{2}{7}$, and 1 respectively. Thus there *are* exactly three solutions in positive integers, namely $(x, y) = (4, 2), (2, 3)$, and $(1, 11)$.

Problem 4 Any positive integer m can be written uniquely in base 3 form as a string of 0's, 1's and 2's (not beginning with a zero). For example,

$$98 = 81 + 9 + 2 \times 3 + 2 \times 1 = (10122)_3.$$

Let $c(m)$ denote the sum of the cubes of the digits of the base 3 form of m; thus, for instance

$$c(98) = 1^3 + 0^3 + 1^3 + 2^3 + 2^3 = 18.$$

Let n be any fixed positive integer. Define the sequence $\{u_r\}$ as

$$u_1 = n, \text{ and } u_r = c(u_{r-1}) \text{ for } r \geq 2.$$

Show that there is a positive integer r such that $u_r = 1, 2$, or 17.

Solution: If m has $d \geq 5$ digits then we have $m \geq 3^{d-1} = (80+1)^{(d-1)/4} \geq 80 \cdot \frac{d-1}{4} + 1 > 8d$ by Bernoulli's inequality. Thus $m > c(m)$.

If $m > 32$ has 4 digits in base 3, then $c(m) \leq 2^3 + 2^3 + 2^3 + 2^3 = 32 < m$. And if $27 \leq m \leq 32$, then m starts with the digits 10 in base 3 and $c(m) < 1^3 + 0^3 + 2^3 + 2^3 = 17 < m$.

Therefore $0 < c(m) < m$ for all $m \geq 27$, and hence eventually we have some positive $u_s < 27$. Since u_s has at most three digits, u_{s+1} can only equal $8, 16, 24, 1, 9, 17, 2, 10$, or 3. If it equals 1, 2, or 17 we are already done; if it equals 3 or 9 then $u_{s+2} = 1$; and otherwise a simple check shows that u_r will eventually equal 2:

$$\left.\begin{array}{r} 8 = (22)_3 \\ 24 = (220)_3 \end{array}\right\} \to 16 = (121)_3 \to 10 = (101)_3 \to 2.$$

Problem 5 Consider all functions $f : \mathbb{N} \to \mathbb{N}$ such that

(i) for each positive integer m, there is a unique positive integer n such that $f(n) = m$;

(ii) for each positive integer n, $f(n+1)$ is either $4f(n) - 1$ or $f(n) - 1$.

Find the set of positive integers p such that $f(1999) = p$ for some function f with properties (i) and (ii).

Solution: Imagine hopping along a sidewalk whose blocks are marked from left to right with the positive integers, where at time n we stand on the block marked $f(n)$. Note that if $f(n) - f(n+1) > 0$ then $f(n) - f(n+1) = 1$; that is, whenever we move to the left we move exactly one block. And whenever we move to the right from $f(n)$ we must move to $4f(n) - 1$.

Then suppose that we are at block $f(a)$ and that $f(a) - 1$ is unvisited; if we move to the right (that is, if $f(a+1) > f(a)$) then at some point we must pass through block $f(a)$ to reach block $f(a) - 1$ again— which is not allowed. Thus we must have $f(a+1) = f(a) - 1$.

Therefore our path is completely determined by the value of $f(1)$: because whenever we are at $f(n)$, if $f(n) - 1 > 0$ is unvisited we must have $f(n + 1) = f(n) - 1$. And otherwise, we must have $f(n + 1) = 4f(n) - 1$.

If $f(1) = 1$ then consider the function f defined as follows: whenever $2^k \le n < 2^{k+1}$, set $f(n) = (3 \cdot 2^k - 1) - n$. It is bijective since for $n = 2^k, 2^k + 1, \dots, 2^{k+1} - 1$ we have $f(n) = 2^{k+1} - 1, 2^{k+1} - 2, \dots, 2^k$; and a quick check shows it satisfies condition (ii) as well. Thus from the previous paragraph this is the *only* function with $f(1) = 1$, and in this case sice $2^{10} \le 1999 < 2^{11}$ we have $f(1999) = (3 \cdot 2^{10} - 1) - 1999 = 1072$.

If $f(1) = 2$ then consider instead the function f defined as follows: whenever $4^k \le n < 3 \cdot 4^k$, set $f(n) = (4^{k+1} - 1) - n$; and if $3 \cdot 4^k \le n < 4^{k+1}$ set $f(n) = (7 \cdot 4^k - 1) - n$. Again, we can check that this function satisfies the conditions; and again, this must be the only function with $f(1) = 2$. In this case since $4^5 \le 1999 < 3 \cdot 4^5$, we have $f(1999) = (4^6 - 1) - 1999 = 2096$.

Finally, suppose that $f(1) \ge 3$; first we must visit $f(1) - 1, f(1) - 2, \dots, 1$. It follows that $f(n) = 3$ and $f(n + 2) = 1$ for some n. But then $f(n + 3) = 4 \cdot 1 - 1 = 3 = f(n)$, a contradiction.

Therefore the only possible values of $f(1999)$ are 1072 and 2096.

Problem 6 For each positive integer n, let $S_n = \{1, 2, \dots, n\}$.

(a) For which values of n is it possible to express S_n as the union of two non-empty disjoint subsets so that the elements in the two subsets have equal sum?

(b) For which values of n is it possible to express S_n as the union of three non-empty disjoint subsets so that the elements in the three subsets have equal sum?

Solution:

(a) Let $\sigma(T)$ denote the sum of the elements in a set T. For the condition to hold $\sigma(S_n) = \frac{n(n+1)}{2}$ must be even, and hence we must have $n = 4k - 1$ or $4k$ where $k \in \mathbb{N}$. For such n, let A consist of the second and third elements of each of the sets

$\{n, n-1, n-2, n-3\}$, $\{n-4, n-5, n-6, n-7\}$, ..., $\{4, 3, 2, 1\}$ (or if $n = 4k-1$, the last set in this grouping will be $\{3, 2, 1\}$); and let $B = S_n \setminus A$. Then $\sigma(A) = \sigma(B)$, as desired.

(b) For the condition to hold, $\sigma(S_n) = \frac{n(n+1)}{2}$ must be divisible by 3; furthermore, the construction is impossible for $n = 3$. Thus n must be of the form $3k+2$ or $3k+3$ where $k \in \mathbb{N}$. We prove all such n work by induction on n. We have $S_5 = \{5\} \cup \{1, 4\} \cup \{2, 3\}$, $S_6 = \{1, 6\} \cup \{2, 5\} \cup \{3, 4\}$, $S_8 = \{8, 4\} \cup \{7, 5\} \cup \{1, 2, 3, 6\}$, and $S_9 = \{9, 6\} \cup \{8, 7\} \cup \{1, 2, 3, 4, 5\}$. Now suppose that we can partition S_{n-6} into $A \cup B \cup C$ with $\sigma(A) = \sigma(B) = \sigma(C)$; then $\sigma(A \cup \{n-5, n\}) = \sigma(B \cup \{n-4, n-1\}) = \sigma(C \cup \{n-3, n-2\})$, completing the inductive step and the proof of our claim.

Problem 7 Let $ABCDEF$ be a hexagon which circumscribes a circle ω. The circle ω touches sides AB, CD, EF at their respective midpoints P, Q, R. Let ω touch sides BC, DE, FA at X, Y, Z respectively. Prove that lines PY, QZ, RX are concurrent.

Solution: Let O be the center of ω. Since P is the midpoint of $\overline{AB}$, $AP = PB$; then by equal tangents, $ZA = AP = PB = BX$. Thus $\angle ZOA = \angle AOP = \angle POB = \angle BOX$. It follows that $\angle ZOP = \angle POX$, and hence $\angle ZYP = \angle PYX$. Therefore line YP is the angle bisector of $\angle XYZ$. Similarly lines XR and ZQ are the angle bisectors of $\angle ZXY$ and $\angle YZX$, and therefore lines PY, QZ, RX meet at the incenter of triangle XYZ.

Problem 8 Some three non-negative real numbers p, q, r satisfy

$$p + q + r = 1.$$

Prove that

$$7(pq + qr + rp) \le 2 + 9pqr.$$

Solution: Given a function f of three variables, let $\sum_{\text{cyc}} f(p, q, r)$ denote the "cyclic sum" $f(p, q, r) + f(q, r, p) + f(r, p, q)$; for example, $\sum_{\text{cyc}}(pqr + p) = 3pqr + p + q + r$. Since $p + q + r = 1$ the inequality is equivalent to

$$7(pq + qr + rp)(p + q + r) \le 2(p + q + r)^3 + 9pqr$$
$$\iff 7 \sum_{\text{cyc}} \left(p^2 q + pq^2 + pqr\right)$$

$$\leq 9pqr + \sum_{\text{cyc}} \left(2p^3 + 6p^2q + 6pq^2 + 4pqr\right)$$

$$\Longleftrightarrow \sum_{\text{cyc}} p^2q + \sum_{\text{cyc}} pq^2 \leq \sum_{\text{cyc}} 2p^3 = \sum_{\text{cyc}} \frac{2p^3 + q^3}{3} + \sum_{\text{cyc}} \frac{p^3 + 2q^3}{3},$$

and this last inequality is true by weighted AM-GM.

Problem 9 Consider all numbers of the form $3n^2 + n + 1$, where n is a positive integer.

(a) How small can the sum of the digits (in base 10) of such a number be?

(b) Can such a number have the sum of its digits (in base 10) equal to 1999?

Solution:

(a) Let $f(n) = 3n^2 + n + 1$. When $n = 8$, the sum of the digits of $f(8) = 201$ is 3. Suppose that some $f(m)$ had a smaller sum of digits; then the last digit of $f(m)$ must be either 0, 1, or 2. However, for any n, $f(n) = n(n+3) + 1 \equiv 1 \pmod 2$; thus $f(m)$ must have units digit 1.

Because $f(n)$ can never equal 1, this means we must have $3m^2+m+1 = 10^k+1$ for some positive integer k, and $m(3m+1) = 10^k$. Since m and $3m+1$ are relatively prime, and $m < 3m+1$, we must either have $(m, 3m+1) = (1, 10^k)$—which is impossible—or $(m, 3m+1) = (2^k, 5^k)$. For $k = 1$, $5^k \neq 3 \cdot 2^k + 1$; and for $k > 1$, we have $5^k = 5^{k-2} \cdot 25 > 2^{k-2} \cdot (12+1) \geq 3 \cdot 2^k + 1$. Therefore, $f(m)$ *can't* equal $10^k + 1$, and 3 is indeed the minimum value for the sum of digits.

(b) Consider $n = 10^{222} - 1$. $f(n) = 3 \cdot 10^{444} - 6 \cdot 10^{222} + 3 + 10^{222}$. Thus, its decimal expansion is

$$2\,\underbrace{9\ldots9}_{221}\,5\,\underbrace{0\ldots0}_{221}\,3,$$

and the sum of the digits in $f(10^{222} - 1)$ is 1999.

1.23 United States of America

Problem 1 Some checkers placed on an $n \times n$ checkerboard satisfy the following conditions:

(i) every square that does not contain a checker shares a side with one that does;

(ii) given any pair of squares that contain checkers, there is a sequence of squares containing checkers, starting and ending with the given squares, such that every two consecutive squares of the sequence share a side.

Prove that at least $\frac{n^2-2}{3}$ checkers have been placed on the board.

Solution: It suffices to show that if m checkers are placed so as to satisfy condition (b), then the number of squares they either cover or are adjacent to is at most $3m+2$. But this is easily seen by induction: it is obvious for $m = 1$, and if m checkers are so placed, some checker can be removed so that the remaining checkers still satisfy (b); they cover at most $3m-1$ squares, and the new checker allows us to count at most 3 new squares (since the square it occupies was already counted, and one of its neighbors is occupied).

Note. The exact number of checkers required is known for $m \times n$ checkerboards with m small, but only partial results are known in the general case. Contact the authors for more information.

Problem 2 Let $ABCD$ be a convex cyclic quadrilateral. Prove that

$$|AB - CD| + |AD - BC| \geq 2|AC - BD|.$$

First Solution: Let E be the intersection of $\overline{AC}$ and $\overline{BD}$. Then the triangles ABE and DCE are similar, so if we let $x = AE, y = BE, z = AB$, then there exists k such that $kx = DE, ky = CE, kz = CD$. Now

$$|AB - CD| = |k-1|z$$

and

$$|AC - BD| = |(kx + y) - (ky + x)| = |k-1| \cdot |x - y|.$$

Since $|x-y| \le z$ by the triangle inequality, we conclude $|AB-CD| \ge |AC-BD|$, and similarly $|AD-BC| \ge |AC-BD|$. These two inequalities imply the desired result.

Second Solution: Let $2\alpha, 2\beta, 2\gamma, 2\delta$ be the measures of the arcs subtended by AB, BC, CD, DA, respectively, and take the radius of the circumcircle of $ABCD$ to be 1. Assume without loss of generality that $\beta \le \delta$. Then $\alpha+\beta+\gamma+\delta = \pi$, and (by the Extended Law of Sines)

$$|AB-CD| = 2\,|\sin\alpha - \sin\gamma| = 4\left|\sin\frac{\alpha-\gamma}{2}\right|\left|\cos\frac{\alpha+\gamma}{2}\right|$$

and

$$\begin{aligned}|AC-BD| &= 2\,|\sin(\alpha+\beta) - \sin(\beta+\gamma)| \\ &= 4\left|\sin\frac{\alpha-\gamma}{2}\right|\left|\cos\left(\frac{\alpha+\gamma}{2}+\beta\right)\right|.\end{aligned}$$

Since $0 \le \frac{1}{2}(\alpha+\gamma) \le \frac{1}{2}(\alpha+\gamma)+\beta \le \frac{\pi}{2}$ (by the assumption $\beta \le \delta$) and the cosine function is nonnegative and decreasing on $[0, \frac{\pi}{2}]$, we conclude that $|AB-CD| \ge |AC-BD|$, and similarly $|AD-BC| \ge |AC-BD|$.

Problem 3 Let $p > 2$ be a prime and let a, b, c, d be integers not divisible by p, such that

$$\left\{\frac{ra}{p}\right\} + \left\{\frac{rb}{p}\right\} + \left\{\frac{rc}{p}\right\} + \left\{\frac{rd}{p}\right\} = 2$$

for any integer r not divisible by p. Prove that at least two of the numbers $a+b$, $a+c$, $a+d$, $b+c$, $b+d$, $c+d$ are divisible by p. Here, for real numbers x, $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x.

Solution: For convenience, we write $[x]$ for the unique integer in $\{0, \ldots, p-1\}$ congruent to x modulo p. In this notation, the given condition can be written

$$[ra] + [rb] + [rc] + [rd] = 2p \quad \text{for all } r \text{ not divisible by } p. \qquad (1)$$

The conditions of the problem are preserved by replacing a, b, c, d with ma, mb, mc, md for any integer m relatively prime to p. If we choose m so that $ma \equiv 1 \pmod{p}$ and then replace a, b, c, d with $[ma], [mb], [mc], [md]$, respectively, we end up in the case $a = 1$ and

$b, c, d \in \{1, \ldots, p-1\}$. Applying (1) with $r = 1$, we see moreover that $a + b + c + d = 2p$.

Now observe that

$$[(r+1)x] - [rx] = \begin{cases} [x] & [rx] < p - [x] \\ -p + [x] & [rx] \geq p - [x]. \end{cases}$$

Comparing (1) applied to two consecutive values of r and using the observation, we see that for each $r = 1, \ldots, p-2$, two of the quantities

$$p - a - [ra], p - b - [rb], p - c - [rc], p - d - [rd]$$

are positive and two are negative. We say that a pair (r, x) is *positive* if $[rx] < p - [x]$ and *negative* otherwise; then for each $r < p - 1$, $(r, 1)$ is positive, so exactly one of $(r, b), (r, c), (r, d)$ is also positive.

Lemma. *If $r_1, r_2, x \in \{1, \ldots, p-1\}$ have the property that (r_1, x) and (r_2, x) are negative but (r, x) is positive for all $r_1 < r < r_2$, then*

$$r_2 - r_1 = \left\lfloor \frac{p}{x} \right\rfloor \quad \textit{or} \quad r_2 - r_1 = \left\lfloor \frac{p}{x} \right\rfloor + 1.$$

Proof: Note that (r', x) is negative if and only if $\{r'x + 1, r'x + 2, \ldots, (r'+1)x\}$ contains a multiple of p. In particular, exactly one multiple of p lies in $\{r_1 x, r_1 x + 1, \ldots, r_2 x\}$. Since $[r_1 x]$ and $[r_2 x]$ are distinct elements of $\{p - [x], \ldots, p-1\}$, we have

$$p - x + 1 < r_2 x - r_1 x < p + x - 1,$$

from which the lemma follows. ■

$[rx]$	9	10	**0**	1	2	3	4	5	6	7	8	9	10	**0**
is (r, x) + or $-$?	$-$			$+$			$+$			$+$			$-$	
r	**3**			4			5			6			**7**	

(The above diagram illustrates the meanings of *positive* and *negative* in the case $x = 3$ and $p = 11$. Note that the difference between 7 and 3 here is $\lfloor \frac{p}{x} \rfloor + 1$. The next r such that (r, x) is negative is $r = 10$; $10 - 7 = \lfloor \frac{p}{x} \rfloor$.)

Recall that exactly one of $(1, b)$, $(1, c)$, $(1, d)$ is positive; we may as well assume $(1, b)$ is positive, which is to say $b < \frac{p}{2}$ and $c, d > \frac{p}{2}$. Put $s_1 = \lfloor \frac{p}{b} \rfloor$, so that s_1 is the smallest positive integer such that (s_1, b) is negative. Then exactly one of (s_1, c) and (s_1, d) is positive, say the former. Since s_1 is also the smallest positive integer such that (s_1, c) is positive, or equivalently such that $(s_1, p - c)$ is negative, we have

$s_1 = \lfloor \frac{p}{p-c} \rfloor$. The lemma states that consecutive values of r for which (r, b) is negative differ by either s_1 or $s_1 + 1$. It also states (when applied with $x = p - c$) that consecutive values of r for which (r, c) is positive differ by either s_1 or $s_1 + 1$. From these observations we will show that (r, d) is always negative.

r	1		s_1	$s_1 + 1$		s'	$s' + 1$		s	$s + 1 \stackrel{?}{=} t$
(r, b)	$+$		$-$	$+$		$-$	$+$		$-$	$-?$
(r, c)	$-$	$\ldots$	$+$	$-$	$\ldots$	$+$	$-$	$\ldots$	$-$	$+?$
(r, d)	$-$		$-$	$-$		$-$	$-$		$+$	$-?$

Indeed, if this were not the case, there would exist a smallest positive integer $s > s_1$ such that (s, d) is positive; then (s, b) and (s, c) are both negative. If s' is the last integer before s such that (s', b) is negative (possibly equal to s_1), then (s', d) is negative as well (by the minimal definition of s). Also,

$$s - s' = s_1 \quad \text{or} \quad s - s' = s_1 + 1.$$

Likewise, if t were the next integer after s' such that (t, c) were positive, then

$$t - s' = s_1 \quad \text{or} \quad t - s' = s_1 + 1.$$

From these we deduce that $|t - s| \leq 1$. However, we can't have $t \neq s$ because then both (s, b) and (t, b) would be negative—and any two values of r for which (r, b) is negative differ by at least $s_1 \geq 2$, a contradiction. (The above diagram shows the hypothetical case when $t = s+1$.) But nor can we have $t = s$ because we already assumed that (s, c) is negative. Therefore we *can't* have $|t - s| \leq 1$, contradicting our findings and thus proving that (r, d) is indeed always negative.

Now if $d \neq p - 1$, then the unique $s \in \{1, \ldots, p - 1\}$ such that $[ds] = 1$ is not equal to $p - 1$; and (s, d) is positive, a contradiction. Thus $d = p - 1$ and $a + d$ and $b + c$ are divisible by p, as desired.

Problem 4 Let $a_1, a_2, \ldots, a_n$ $(n > 3)$ be real numbers such that

$$a_1 + a_2 + \cdots + a_n \geq n \qquad \text{and} \qquad a_1^2 + a_2^2 + \cdots + a_n^2 \geq n^2.$$

Prove that $\max(a_1, a_2, \ldots, a_n) \geq 2$.

Solution: Let $b_i = 2 - a_i$, and let $S = \sum b_i$ and $T = \sum b_i^2$. Then the given conditions are that

$$(2-b_1)+\cdots+(2-b_n)\geq n$$

and

$$(4-4b_1+b_1^2)+\cdots+(4-4b_n+b_n^2)\geq n^2,$$

which is to say $S \leq n$ and $T \geq n^2-4n+4S$.

From these inequalities, we obtain

$$T \geq n^2-4n+4S \geq (n-4)S+4S = nS.$$

On the other hand, if $b_i > 0$ for $i = 1,\ldots,n$, then certainly $b_i < \sum b_i = S \leq n$, and so

$$T = b_1^2+\cdots+b_n^2 < nb_1+\cdots+nb_n = nS.$$

Thus we cannot have $b_i > 0$ for $i = 1,\ldots,n$, so $b_i \leq 0$ for some i; then $a_i \geq 2$ for that i, proving the claim.

Note: The statement is false when $n \leq 3$. The example $a_1 = a_2 = \cdots = a_{n-1} = 2$, $a_n = 2-n$ shows that the bound cannot be improved. Also, an alternate approach is to show that if $a_i \leq 2$ and $\sum a_i \geq n$, then $\sum a_i^2 \leq n^2$ (with the equality case just mentioned), by noticing that replacing a pair a_i, a_j with $2, a_i+a_j-2$ increases the sum of squares.

Problem 5 The Y2K Game is played on a 1×2000 grid as follows. Two players in turn write either an S or an O in an empty square. The first player who produces three consecutive boxes that spell SOS wins. If all boxes are filled without producing SOS then the game is a draw. Prove that the second player has a winning strategy.

Solution: Call a partially filled board *stable* if there is no SOS and no single move can produce an SOS; otherwise call it *unstable*. For a stable board call an empty square *bad* if either an S or an O played in that square produces an unstable board. Thus a player will lose if the only empty squares available to him are bad, but otherwise he can at least be guaranteed another turn with a correct play.

Claim: A square is bad if and only if it is in a block of 4 consecutive squares of the form S – – S.

Proof: If a square is bad, then an O played there must give an unstable board. Thus the bad square must have an S on one side and an empty square on the other side. An S played there must also give an unstable board, so there must be another S on the other side of the empty square. ∎

From the claim it follows that there are always an even number of bad squares. Thus the second player has the following winning strategy:

(a) If the board is unstable at any time, play the winning move; otherwise continue as below.

(b) On the first move, play an S at least four squares away from either end and at least seven squares from the first player's first move. (The board is long enough that this is possible.)

(c) On the second move, play an S three squares away from the second player's first move, so that the squares in between are empty and so that the board remains stable. (Regardless of the first player's second move, this must be possible on at least one side.) This produces two bad squares; whoever plays in one of them first will lose. Thus the game will not be a draw.

(d) On any subsequent move, play in a square which is not bad—keeping the board stable, of course. Such a square will always exist because if the board is stable, there will be an odd number of empty squares and an even number of bad squares.

Since there exist bad squares after the second player's second move, the game cannot end in a draw; and since the second player can always leave the board stable, the first player cannot win. Therefore eventually the second player will win.

Note: Some other names for the S – – S block, from submitted solutions, included arrangement, combo, configuration, formation, pattern, sandwich, segment, situation, and trap. (Thanks to Alexander Soifer and Zvezdelina Stankova-Frenkel for passing these along.)

Problem 6 Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle ω of triangle BCD meets CD at E. Let F be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle ACF meet line CD at C and

G. Prove that the triangle AFG is isosceles.

Solution: We will show that $FA = FG$. Let H be the center of the escribed circle of triangle ACD opposite vertex A. Then H lies on the angle bisector AF. Let K be the point where this escribed circle touches CD. By a standard computation using equal tangents, we see that $CK = \frac{1}{2}(AD + CD - AC)$. By a similar computation in triangle BCD, we see that $CE = \frac{1}{2}(BC + CD - BD) = CK$. Therefore $E = K$ and $F = H$.

Since F is now known to be an excenter, we have that FC is the external angle bisector of $\angle DCA = \angle GCA$. Therefore

$$\angle GAF = \angle GCF = \frac{\pi}{2} - \frac{1}{2}\angle GCA = \frac{\pi}{2} - \frac{1}{2}\angle GFA.$$

We conclude that the triangle GAF is isosceles with $FA = FG$, as desired.

1.24 Vietnam

Problem 1 Solve the system of equations

$$\begin{aligned}(1+4^{2x-y})\cdot 5^{1-2x+y} &= 1+2^{2x-y+1}\\ y^3+4x+1+\ln(y^2+2x) &= 0.\end{aligned}$$

Solution: The only solution is $(x, y) = (0, -1)$.

First solve the first equation for $t = 2x-y$. Multiplying the equation by 5^{t-1} yields

$$(1-5^{t-1})+4(4^{t-1}-10^{t-1})=0.$$

This has the obvious solution $t = 1$. There are no other solutions: if $t > 1$ then both $1-5^{t-1}$ and $4^{t-1}-10^{t-1}$ are negative; and if $t < 1$ then both these terms are positive. Therefore, $2x - y = 1$.

Substitute $2x = y + 1$ into the second equation to get

$$y^3+2y+3+\ln(y^2+y+1)=0.$$

This has the not-so-obvious solution $y = -1$. To prove this is the only solution, it suffices to show that $f(y) = y^3 + 2y + 3 + \ln(y^2 + y + 1)$ is always increasing. Its derivative is

$$f'(y)=3y^2+2+\frac{2y+1}{y^2+y+1}.$$

But we know that

$$\begin{aligned}&2(y+1)^2+1>0\\ &\Rightarrow 2y+1>-2(y^2+y+1)\\ &\Rightarrow \frac{2y+1}{y^2+y+1}>-2,\end{aligned}$$

where we can safely divide by $y^2 + y + 1 = (y + \frac{1}{2})^2 + \frac{3}{4} > 0$. Thus $f'(y) > 3y^2 > 0$ for all y, as desired.

Problem 2 Let A', B', C' be the respective midpoints of the arcs BC, CA, AB, not containing points A, B, C, respectively, of the circumcircle of the triangle ABC. The sides BC, CA, AB meet the pairs of segments

$$\{C'A', A'B'\},\ \{A'B', B'C'\},\ \{B'C', C'A'\}$$

at the pairs of points

$$\{M,N\},\ \{P,Q\},\ \{R,S\},$$

respectively. Prove that $MN = PQ = RS$ if and only if the triangle ABC is equilateral.

Solution: If ABC is equilateral then $MN = PQ = RS$ by symmetry.

Now suppose that $MN = PQ = RS$. Observe that $\angle NMA' = \angle BMS = \frac{1}{2}(\widehat{BC'} + \widehat{CA'}) = \frac{1}{2}(\angle C + \angle A)$ and similarly $\angle C'SR = \angle MSB = \frac{1}{2}(\angle A + \angle C)$. Furthermore, $\angle A'B'C' = \angle A'B'B + \angle BB'C' = \frac{1}{2}(\angle A + \angle C)$ as well.

Thus $MB = SB$, and also $\triangle C'RS \sim \triangle C'A'B' \sim \triangle NA'M$. Next, by the law of sines in triangles $C'SB$ and $A'MB$ we have

$$C'S = SB \cdot \frac{\sin \angle C'BS}{\sin \angle SC'B} = SB \cdot \frac{\sin \frac{\angle C}{2}}{\sin \frac{\angle A}{2}}$$

and

$$MA' = MB \cdot \frac{\sin \angle A'BM}{\sin \angle MA'B} = MB \cdot \frac{\sin \frac{\angle A}{2}}{\sin \frac{\angle C}{2}},$$

giving $\frac{C'S}{MA'} = \left(\frac{\sin \frac{\angle C}{2}}{\sin \frac{\angle A}{2}}\right)^2$.

Next, because $\triangle C'RS \sim \triangle C'A'B'$ we have $RS = A'B' \cdot \frac{C'S}{C'B'}$; and because $\triangle NA'M \sim \triangle C'A'B'$ we have $MN = B'C' \cdot \frac{MA'}{B'A'}$. Therefore since $RS = MN$ we have

$$\begin{aligned}
&A'B' \cdot \frac{C'S}{C'B'} = B'C' \cdot \frac{MA'}{B'A'} \\
&\Longrightarrow \frac{C'S}{MA'} = \left(\frac{B'C'}{A'B'}\right)^2 = \left(\frac{\sin \frac{1}{2}(\angle B + \angle C)}{\sin \frac{1}{2}(\angle B + \angle A)}\right)^2 = \left(\frac{\cos \frac{\angle A}{2}}{\cos \frac{\angle C}{2}}\right)^2 \\
&\Longrightarrow \left(\frac{\sin \frac{\angle C}{2}}{\sin \frac{\angle A}{2}}\right)^2 = \left(\frac{\cos \frac{\angle A}{2}}{\cos \frac{\angle C}{2}}\right)^2 \\
&\Longrightarrow \left(\sin \frac{\angle C}{2} \cos \frac{\angle C}{2}\right)^2 = \left(\sin \frac{\angle A}{2} \cos \frac{\angle A}{2}\right)^2 \\
&\Longrightarrow \frac{1}{4}\sin^2 \angle C = \frac{1}{4}\sin^2 \angle A \\
&\Longrightarrow \sin \angle C = \sin \angle A.
\end{aligned}$$

Since $\angle A+\angle C < 180°$, we must have $\angle A = \angle C$. Similarly $\angle A = \angle B$, and therefore triangle ABC is equilateral.

Problem 3 For $n = 0, 1, 2, \ldots$, let $\{x_n\}$ and $\{y_n\}$ be two sequences defined recursively as follows:

$$x_0 = 1,\ x_1 = 4,\ x_{n+2} = 3x_{n+1} - x_n;$$

$$y_0 = 1,\ y_1 = 2,\ y_{n+2} = 3y_{n+1} - y_n.$$

(a) Prove that $x_n^2 - 5y_n^2 + 4 = 0$ for all non-negative integers n.

(b) Suppose that a, b are two positive integers such that $a^2 - 5b^2 + 4 = 0$. Prove that there exists a non-negative integer k such that $x_k = a$ and $y_k = b$.

Solution: We first prove by induction on k that for $k \geq 0$, we have $(x_{k+1}, y_{k+1}) = (\frac{3x_k+5y_k}{2}, \frac{x_k+3y_k}{2})$. For $k = 0$ we have $(4, 2) = (\frac{3+5}{2}, \frac{1+3}{2})$, and for $k = 1$ we have $(11, 5) = (\frac{12+10}{2}, \frac{4+6}{2})$. Now assuming it's true for k and $k + 1$, we know that

$$(x_{k+3}, y_{k+3}) = (3x_{k+2} - x_{k+1}, 3y_{k+2} - y_{k+1}).$$

Substituting the expressions for $x_{k+2}, x_{k+1}, y_{k+2}, y_{k+1}$ from the induction hypothesis, this equals

$$\left(\frac{3(3x_{k+1} - x_k) + 5(3y_{k+1} - y_k)}{2}, \frac{(3x_{k+1} - x_k) + 3(3y_{k+1} - y_k)}{2}\right)$$
$$= \left(\frac{3x_{k+2} + 5y_{k+2}}{2}, \frac{x_{k+2} + 3y_{k+2}}{2}\right),$$

completing the induction and the proof of our claim.

(a) We prove the claim by induction; for $n = 0$ we have $1-5+4 = 0$. Now assuming it is true for n, we prove (with the help of our above observation) that it is true for $n + 1$:

$$\begin{aligned} & x_{n+1}^2 - 5y_{n+1}^2 \\ & = \left(\frac{3x_n + 5y_n}{2}\right)^2 - 5\left(\frac{x_n + 3y_n}{2}\right)^2 \\ & = \frac{9x_n^2 + 30x_ny_n + 25y_n^2}{4} - 5 \cdot \frac{x_n^2 + 6x_ny_n + 9y_n^2}{4} \\ & = \frac{4x_n^2 - 20y_n^2}{4} = x_n^2 - 5y_n^2 = -4, \end{aligned}$$

as desired.

(b) Suppose by way of contradiction that $a^2 - 5b^2 + 4 = 0$ for integers $a, b > 0$, and that there did *not* exist k such that $(x_k, y_k) = (a, b)$. Choose a counterexample that minimizes $a + b$.

Note that $0 \equiv a^2 - 5b^2 + 4 \equiv a - b \pmod 2$. Next, $a^2 = 5b^2 - 4 < 9b^2 \Rightarrow a < 3b$. And there are no counterexamples with $a = 1$ or 2; thus $a^2 > 5$ and $0 = 5a^2 - 25b^2 + 20 < 5a^2 - 25b^2 + 4a^2 \Rightarrow 3a > 5b$.

Therefore $a' = \frac{3a-5b}{2}$ and $b' = \frac{3b-a}{2}$ are positive integers. Then since $a^2 - 5b^2 = -4$, some quick algebra shows that $a'^2 - 5b'^2 = -4$ as well; but $a' + b' = \frac{3a-5b}{2} + \frac{3b-a}{2} = a - b < a + b$. It follows from the minimal definition of (a, b) that there must exist some (a_k, b_k) equal to (a', b').

But then it is easy to verify that $(a, b) = (\frac{3a'+5b'}{2}, \frac{a'+3b'}{2}) = (a_{k+1}, b_{k+1})$, a contradiction! This completes the proof.

Problem 4 Let a, b, c be real numbers such that $abc + a + c = b$. Determine the greatest possible value of the expression

$$P = \frac{2}{a^2+1} - \frac{2}{b^2+1} + \frac{3}{c^2+1}.$$

Solution: The condition is equivalent to $b = \frac{a+c}{1-ac}$, which suggests making the substitutions $A = \operatorname{Tan}^{-1} a$ and $C = \operatorname{Tan}^{-1} c$; then we have $b = \tan(A + C)$ and

$$\begin{aligned}
P &= \frac{2}{\tan^2 A + 1} - \frac{2}{\tan^2(A+C) + 1} + \frac{3}{\tan^2 C + 1} \\
&= 2\cos^2 A - 2\cos^2(A+C) + 3\cos^2 C \\
&= (2\cos^2 A - 1) - (2\cos^2(A+C) - 1) + 3\cos^2 C \\
&= \cos(2A) - \cos(2A + 2C) + 3\cos^2 C \\
&= 2\sin(2A + C)\sin C + 3\cos^2 C.
\end{aligned}$$

Letting $u = |\sin C|$, this expression is at most

$$\begin{aligned}
2u + 3(1 - u^2) &= -3u^2 + 2u + 3 \\
&= -3\left(u - \frac{1}{3}\right)^2 + \frac{10}{3} \leq \frac{10}{3}.
\end{aligned}$$

Equality can be achieved when $\sin(2A+C) = 1$ and $\sin C = \frac{1}{3}$, which gives $(a, b, c) = (\frac{\sqrt{2}}{2}, \sqrt{2}, \frac{\sqrt{2}}{4})$. Thus the maximum value of P is $\frac{10}{3}$.

Problem 5 In the three-dimensional space let Ox, Oy, Oz, Ot be four nonplanar distinct rays such that the angles between any two of them have the same measure.

(a) Determine this common measure.

(b) Let Or be another ray different from the above four rays. let $\alpha, \beta, \gamma, \delta$ be the angles formed by Or with Ox, Oy, Oz, Ot, respectively. Put

$$p = \cos\alpha + \cos\beta + \cos\gamma + \cos\delta,$$
$$q = \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta.$$

Prove that p and q remain constant as Or rotates about the point O.

Solution: Put O at the origin, and say the four rays hit the unit sphere at A, B, C, D. Then $ABCD$ is a regular tetrahedron, and (letting X also represent the vector $\overrightarrow{OX}$) we have $A+B+C+D = 0$.

(a) Say the common angle is ϕ. Then

$$0 = A \cdot (A+B+C+D) = A \cdot A + A \cdot (B+C+D) = 1 + 3\cos\phi,$$

so $\phi = \cos^{-1}\left(-\frac{1}{3}\right)$.

(b) Without loss of generality say that Or hits the unit sphere at $U = (1, 0, 0)$; also write $A = (x_1, y_1, z_1)$, and so on. Then

$$\begin{aligned} p &= U \cdot A + U \cdot B + U \cdot C + U \cdot D \\ &= U \cdot (A+B+C+D) \\ &= U \cdot \vec{0} = 0, \end{aligned}$$

a constant. Also, $(x_1, x_2, x_3, x_4) = (\cos\alpha, \cos\beta, \cos\gamma, \cos\delta)$ and $q = \sum x_i^2$. The following lemma then implies that q will always equal $\frac{4}{3}$:

Lemma. *Suppose we are given a regular tetrahedron T inscribed in the unit sphere and with vertices (x_i, y_i, z_i) for $1 \le i \le 4$. Then we have $\sum x_i^2 = \sum y_i^2 = \sum z_i^2 = \frac{4}{3}$ and $\sum x_i y_i = \sum y_i z_i = \sum z_i x_i = 0$.*

Proof: This is easily verified when the vertices are at

$$A_0 = (0, 0, 1), B_0 = (\tfrac{2\sqrt{2}}{3}, 0, -\tfrac{1}{3}),$$
$$C_0 = (-\tfrac{\sqrt{2}}{3}, \tfrac{\sqrt{6}}{3}, -\tfrac{1}{3}), D_0 = (-\tfrac{\sqrt{2}}{3}, -\tfrac{\sqrt{6}}{3}, -\tfrac{1}{3}).$$

Now assume these equations are true for a tetrahedron $ABCD$, and rotate it about the z-axis through an angle θ. Then each (x_i, y_i, z_i) becomes $(x'_i, y'_i, z'_i) = (x_i \cos\theta - y_i \sin\theta, x_i \sin\theta + y_i \cos\theta, z_i)$, and

$$\sum x_i'^2 = \cos^2\theta \sum x_i^2 - 2\sin\theta\cos\theta \sum x_i y_i + \sin^2\theta \sum y_i^2 = \frac{4}{3}$$
$$\sum y_i'^2 = \sin^2\theta \sum x_i^2 + 2\sin\theta\cos\theta \sum x_i y_i + \cos^2\theta \sum y_i^2 = \frac{4}{3}$$
$$\sum z_i'^2 = \sum z_i^2 = \frac{4}{3}$$
$$\sum x'_i y'_i = \sin\theta\cos\theta \sum (x_i^2 - y_i^2) + (\cos^2\theta - \sin^2\theta) \sum x_i y_i = 0$$
$$\sum y'_i z'_i = \sin\theta \sum x_i z_i + \cos\theta \sum y_i z_i = 0$$
$$\sum z'_i x'_i = \cos\theta \sum z_i x_i - \sin\theta \sum z_i y_i = 0.$$

Similarly, the equations remain true after rotating $ABCD$ about the y- and z-axes.

Now, first rotate our given tetrahedron T about the z-axis until one vertex is in the yz-plane; next rotate it about the x-axis until this vertex is at $(0, 0, 1)$; and finally rotate it about the z-axis again until the tetrahedron corresponds with the initial tetrahedron $A_0B_0C_0D_0$ described above. Since we *know* the above equations are true for $A_0B_0C_0D_0$, if we reverse the rotations to return to T the equations will remain true, as claimed. ■

Problem 6 Let $\mathcal{S} = \{0, 1, 2, \ldots, 1999\}$ and $\mathcal{T} = \{0, 1, 2, \ldots\}$. Find all functions $f : \mathcal{T} \to \mathcal{S}$ such that

(i) $f(s) = s$ for all $s \in \mathcal{S}$.

(ii) $f(m + n) = f(f(m) + f(n))$ for all $m, n \in \mathcal{T}$.

Solution: Suppose that $f(2000) = 2000-t$, where $1 \le t \le 2000$. We prove by induction on n that for all $n \ge 2000$, we have $f(n) = f(n-t)$. By assumption it is true for $n = 2000$. Then assuming it is true for

n, we have

$$f(n+1) = f(f(n) + f(1)) = f(f(n-t) + f(1)) = f(n-t+1),$$

completing the inductive step. Therefore the function is completely determined by the value of $f(2000)$, and it follows that there are at most 2000 such functions.

Conversely, given any $c = 2000 - t \in S$, let f be the function such that $f(s) = s$ for all $s \in \mathcal{S}$ while $f(n) = f(n-t)$ for all $n \geq 2000$. We prove by induction on $m+n$ that condition (ii) holds. If $m+n \leq 2000$ then $m, n \in \mathcal{S}$ and the claim is obvious. Otherwise, $m + n > 2000$. Again, if $m, n \in \mathcal{S}$ the claim is obvious; otherwise assume without loss of generality that $n \geq 2000$. Then

$$f(m+n) = f(m+n-t) = f(f(m) + f(n-t)) = f(f(m) + f(n)),$$

where the first and third equalities come from our periodic definition of f, and the second equality comes from the induction hypothesis. Therefore there are exactly 2000 functions f.

Problem 7 For $n = 1, 2, \ldots$, let $\{u_n\}$ be a sequence defined by

$$u_1 = 1,\ u_2 = 2,\ u_{n+2} = 3u_{n+1} - u_n.$$

Prove that

$$u_{n+2} + u_n \geq 2 + \frac{u_{n+1}^2}{u_n}$$

for all n.

Solution: We first prove by induction that for $n \geq 1$, we have $u_n u_{n+2} = u_{n+1}^2 + 1$. Since $u_3 = 5$, for $n = 1$ we have $1 \cdot 5 = 2^2 + 1$, as desired.

Now assuming our claim is true for n, we have

$$\begin{aligned}
u_{n+2}^2 + 1 &= u_{n+2}(3u_{n+1} - u_n) + 1 \\
&= 3u_{n+1}u_{n+2} - (u_n u_{n+2} - 1) \\
&= 3u_{n+1}u_{n+2} - u_{n+1}^2 \\
&= u_{n+1}(3u_{n+2} - u_{n+1}) = u_{n+1}u_{n+3},
\end{aligned}$$

so it is true for $n+1$ as well and thus all $n \geq 1$.

Therefore, for all $n \geq 1$ we have

$$u_{n+2} + u_n = \frac{u_{n+1}^2 + 1}{u_n} + u_n = \frac{u_{n+1}^2}{u_n} + \left(u_n + \frac{1}{u_n}\right) \geq \frac{u_{n+1}^2}{u_n} + 2,$$

where $u_n + \frac{1}{u_n} \geq 2$ by AM-GM. This proves the inequality.

Problem 8 Let ABC be a triangle inscribed in circle ω. Construct all points P in the plane (ABC) and not lying on ω, with the property that the lines PA, PB, PC meet ω again at points A', B', C' such that $A'B' \perp A'C'$ and $A'B' = A'C'$.

Solution: All angles are directed modulo $180°$. We solve a more general problem: suppose we have some fixed triangle DEF and we want to find all points P such that when $A' = PA \cap \omega$, $B' = PB \cap \omega$, $C' = PC \cap \omega$ then triangles $A'B'C'$ and DEF are similar with the same orientations. (In other words, we want $\angle B'C'A' = \angle EFD$ and $\angle C'A'B' = \angle FDE$.)

Given X, Y on ω, let $\angle\widehat{XY}$ denote the angle $\angle XZY$ for any other point Z on ω. Now given a point P, we have $\angle BPA = \angle\widehat{BA} + \angle\widehat{B'A'} = \angle BCA + \angle B'C'A'$ and $\angle CPB = \angle\widehat{CB} + \angle\widehat{C'B'} = \angle CAB + \angle C'A'B'$. Thus $\angle B'C'A' = \angle EFD$ if and only if $\angle BPA = \angle BCA + \angle EFD$, while $\angle C'A'B' = \angle FDE$ if and only if $\angle CPB = \angle CAB + \angle FDE$. Therefore our desired point P is the intersection point, different than B, of the two circles $\{P' \mid \angle BP'A = \angle BCA + \angle EFD\}$ and $\{P' \mid \angle CP'B = \angle CAB + \angle FDE\}$.

Now back to our original problem: we wish to find P such that triangle $A'B'C'$ is a $45°$-$45°$-$90°$ triangle with $\angle C'A'B' = 90°$. Because our angles are directed, there are two possible orientations for such a triangle: either $\angle A'B'C' = 45°$ or $\angle A'B'C' = -45°$. Applying the above construction twice with triangle DEF defined appropriately yields the two desired possible locations of P.

Problem 9 Consider real numbers a, b such that $a \neq 0, a \neq b$ and all roots of the equation

$$ax^3 - x^2 + bx - 1 = 0$$

are real and positive. Determine the smallest possible value of the expression

$$P = \frac{5a^2 - 3ab + 2}{a^2(b-a)}.$$

Solution: When the roots of the equation are all $\sqrt{3}$, we have $a = \frac{1}{3\sqrt{3}}$, $b = \sqrt{3}$, and $P = 12\sqrt{3}$. We prove that $12\sqrt{3}$ is minimal.

Say the roots to $ax^3 - x^2 + bx - 1$ are $p = \tan A$, $q = \tan B$, and $r = \tan C$ with $0 < A, B, C < 90°$. Then

$$\begin{aligned} ax^3 - x^2 + bx - 1 &= a(x-p)(x-q)(x-r) \\ &= ax^3 - a(p+q+r)x^2 + a(pq+qr+rp)x - a(pqr). \end{aligned}$$

Thus $a = \frac{1}{p+q+r} = \frac{1}{pqr} > 0$ and $p + q + r = pqr$. Then

$$\begin{aligned} r &= \frac{p+q}{pq-1} \\ &= -\tan(A+B) \\ &= \tan(180° - A - B), \end{aligned}$$

so $A + B + C = 180°$. Then since $\tan x$ is convex for $0 < x < 90°$, we have

$$\frac{1}{a} = \tan A + \tan B + \tan C \geq 3\tan 60° = 3\sqrt{3},$$

so $a \leq \frac{1}{3\sqrt{3}}$.

Also notice that

$$\frac{b}{a} = pq + qr + rp \geq 3\sqrt[3]{p^2q^2r^2} = 3\sqrt[3]{\frac{1}{a^2}} \geq 9 > 1,$$

so $b > a$. Thus the denominator of P is always positive and is an increasing function of b, while the numerator of P is a decreasing function of b. Therefore, for a constant a, P is a decreasing function of b.

Furthermore,

$$\begin{aligned} &(p-q)^2 + (q-r)^2 + (r-p)^2 \geq 0 \\ &\implies (p+q+r)^2 \geq 3(pq+qr+rp) \\ &\implies \frac{1}{a^2} \geq \frac{3b}{a} \implies b \leq \frac{1}{3a}, \end{aligned}$$

and

$$P \geq \frac{5a^2 - 3a(\frac{1}{3a}) + 2}{a^2(\frac{1}{3a} - a)} = \frac{5a^2 + 1}{\frac{a}{3} - a^3}.$$

Thus for $0 < a \le \frac{1}{3\sqrt{3}}$, it suffices to show that

$$\begin{aligned} &5a^2 + 1 \ge 12\sqrt{3}\left(\frac{a}{3} - a^3\right) = 4\sqrt{3}a - 12\sqrt{3}a^3 \\ \iff\ &12\sqrt{3}a^3 + 5a^2 - 4\sqrt{3}a + 1 \ge 0 \\ \iff\ &3\left(a - \frac{1}{3\sqrt{3}}\right)(4\sqrt{3}a^2 + 3a - \sqrt{3}) \ge 0 \\ \iff\ &4\sqrt{3}a^2 + 3a - \sqrt{3} \le 0. \end{aligned}$$

But the last quadratic has one positive root

$$\frac{-3 + \sqrt{57}}{8\sqrt{3}} \ge \frac{-3 + 7}{8\sqrt{3}} = \frac{1}{2\sqrt{3}} > \frac{1}{3\sqrt{3}},$$

so it is indeed negative when $0 < a \le \frac{1}{3\sqrt{3}}$. This completes the proof.

Problem 10 Let $f(x)$ be a continuous function defined on $[0, 1]$ such that

(i) $f(0) = f(1) = 0$;

(ii) $2f(x) + f(y) = 3f\left(\dfrac{2x + y}{3}\right)$ for all $x, y \in [0, 1]$.

Prove that $f(x) = 0$ for all $x \in [0, 1]$.

Solution: We prove by induction on k that $f\left(\frac{m}{3^k}\right) = 0$ for all integers $k \ge 0$ and all integers $0 \le m \le 3^k$. The given conditions show this claim is true for $k = 0$; now assuming it is true for $k - 1$, we prove it is true for k.

If $m \equiv 0 \pmod 3$ then $f\left(\frac{m}{3^k}\right) = f\left(\frac{\frac{m}{3}}{3^{k-1}}\right) = 0$ by the induction hypothesis.

If $m \equiv 1 \pmod 3$, then $1 \le m \le 3^k - 2$ and

$$3f\left(\frac{m}{3^k}\right) = 2f\left(\frac{\frac{m-1}{3}}{3^{k-1}}\right) + f\left(\frac{\frac{m+2}{3}}{3^{k-1}}\right) = 0 + 0 = 0.$$

Thus $f\left(\frac{m}{3^k}\right) = 0$.

And if $m \equiv 2 \pmod 3$, then $2 \le m \le 3^k - 1$ and

$$3f\left(\frac{m}{3^k}\right) = 2f\left(\frac{\frac{m+1}{3}}{3^{k-1}}\right) + f\left(\frac{\frac{m-2}{3}}{3^{k-1}}\right) = 0 + 0 = 0.$$

Hence $f\left(\frac{m}{3^k}\right) = 0$, finishing our induction.

Now, for any $x \in [0,1]$ we can form a sequence of numbers of the form $\frac{m}{3^k}$ whose limit is x; then since f is continuous, it follows that $f(x) = 0$ for all $x \in [0,1]$, as desired.

Problem 11 The base side and the altitude of a right regular hexagonal prism $ABCDEF - A'B'C'D'E'F'$ are equal to a and h respectively.

(a) Prove that six planes

$$(AB'F'), (CD'B'), (EF'D'), (D'EC), (F'AE), (B'CA)$$

touch the same sphere.

(b) Determine the center and the radius of the sphere.

Solution:

(a) Let O be the center of the prism. $(AB'F')$ is tangent to a unique sphere centered at O. Now the other five planes are simply reflections and rotations of $(AB'F')$ with respect to O; and since the sphere remains fixed under these transformations, it follows that all six planes are tangent to this same sphere.

(b) From part (a), the center of the sphere is the center O of the prism. Let P be the midpoint of $\overline{AE}$ and let P' be the midpoint of $\overline{A'E'}$. Also let Q be the midpoint of $\overline{PF'}$, and let $\overline{OR}$ be the perpendicular from O to line PF'. Note that P, P', Q, R, O, F' all lie in one plane.

It is straightforward to calculate that $F'P' = \frac{a}{2}$ and $QO = \frac{3a}{4}$. Also, since $QO \parallel F'P'$ we have $\angle RQO = \angle PF'P'$; combined with $\angle ORQ = \angle PP'F' = 90^\circ$, this gives $\triangle ORQ \sim \triangle PP'F'$. Hence the radius of the sphere is

$$OR = PP' \cdot \frac{OQ}{PF'} = h \cdot \frac{\frac{3a}{4}}{\sqrt{\left(\frac{a}{2}\right)^2 + h^2}} = \frac{3ah}{2\sqrt{a^2 + 4h^2}}.$$

Problem 12 For $n = 1, 2, \ldots$, two sequences $\{x_n\}$ and $\{y_n\}$ are defined recursively by

$$x_1 = 1,\ y_1 = 2,\ x_{n+1} = 22y_n - 15x_n,\ y_{n+1} = 17y_n - 12x_n.$$

(a) Prove that x_n and y_n are not equal to zero for all $n = 1, 2, \ldots$.

(b) Prove that each sequence contains infinitely many positive terms and infinitely many negative terms.

(c) For $n = 1999^{1945}$, determine whether x_n and y_n are divisible by 7 or not.

Solution:

(a) The recursive equation for x_{n+1} gives $y_n = \frac{1}{22}(15x_n + x_{n+1})$ and thus also $y_{n+1} = \frac{1}{22}(15x_{n+1} + x_{n+2})$. Substituting these expressions into the other recursive equation gives

$$x_{n+2} = 2x_{n+1} - 9x_n$$

and similarly we can also find

$$y_{n+2} = 2y_{n+1} - 9y_n.$$

Quick computation gives $x_2 = 29$ and $y_2 = 22$. Then x_1, x_2 are odd, and if x_n, x_{n+1} are odd then x_{n+2} must be as well; thus all the x_n are odd and hence nozero. Similarly, all the y_n are congruent to $2 \pmod 4$ and thus nonzero as well.

(b) Note that $x_{n+3} = 2(2x_{n+1} - 9x_n) - 9x_{n+1} = -5x_{n+1} - 18x_n$. Thus if x_n, x_{n+1} are positive (or negative) then x_{n+3} is negative (or positive). Hence among every four consecutive terms x_n there is some positive number and some negative number. Therefore there are infinitely many positive terms and infinitely many negative terms in this sequence; and a similar proof holds for the y_n.

(c) All congruences are taken modulo 7 unless stated otherwise. Note that $x_1 \equiv x_2 \equiv 1$. Now if $x_n \equiv x_{n+1}$ and $x_n \not\equiv 0$, then we have $(x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5}) \equiv (0, 5x_n, 3x_n, 3x_n)$ and $5x_n \not\equiv 0$, $3x_n \not\equiv 0$. This implies that $x_3, x_7, x_{11}, \ldots$ are all divisible by 7 but no other x_n are. Since $1999^{1945} \equiv 3^{1945} \equiv 3 \cdot 9^{1944/2} \equiv 3 \pmod 4$, we indeed have $7 \mid x_n$ when $n = 1999^{1945}$.

Now suppose by way of contradiction that $7 \mid y_{n'}$ for some n', and choose the minimal such n'. From the recursion for y_n, we have $y_n \equiv y_{n+1} + 3y_{n+2}$. Now if $n' \geq 5$ then $y_{n'-2} \equiv y_{n'-1}$, $y_{n'-3} \equiv 4y_{n'-1}$, and $y_{n'-4} \equiv 0$ — contradicting the minimal choice of n'. Thus we have $n' \leq 4$, but we have $(y_1, y_2, y_3, y_4) \equiv (2, 1, 5, 1)$. Therefore no term is divisible by 7; and specifically, $7 \nmid y_n$ when $n = 1999^{1945}$.