All-Russian Olympiad 2012

– Grade level 9

Day 1

1 Let $a_1, \dots a_{11}$ be distinct positive integers, all at least 2 and with sum 407. Does there exist an integer n such that the sum of the 22 remainders after the division of n by $a_1, a_2, \dots, a_{11}, 4a_1, 4a_2, \dots, 4a_{11}$ is 2012?

2 A regular 2012-gon is inscribed in a circle. Find the maximal k such that we can choose k vertices from given 2012 and construct a convex k-gon without parallel sides.

3 Consider the parallelogram $ABCD$ with obtuse angle A. Let H be the feet of perpendicular from A to the side BC. The median from C in triangle ABC meets the circumcircle of triangle ABC at the point K. Prove that points K, H, C, D lie on the same circle.

4 The positive real numbers $a_1, \dots, a_n$ and k are such that $a_1 + \dots + a_n = 3k$, $a_1^2 + \dots + a_n^2 = 3k^2$ and $a_1^3 + \dots + a_n^3 > 3k^3 + k$. Prove that the difference between some two of $a_1, \dots, a_n$ is greater than 1.

Day 2

1 101 wise men stand in a circle. Each of them either thinks that the Earth orbits Jupiter or that Jupiter orbits the Earth. Once a minute, all the wise men express their opinion at the same time. Right after that, every wise man who stands between two people with a different opinion from him changes his opinion himself. The rest do not change. Prove that at one point they will all stop changing opinions.

2 The points A_1, B_1, C_1 lie on the sides sides BC, AC and AB of the triangle ABC respectively. Suppose that $AB_1 - AC_1 = CA_1 - CB_1 = BC_1 - BA_1$. Let I_A, I_B, I_C be the incentres of triangles AB_1C_1, A_1BC_1 and A_1B_1C respectively. Prove that the circumcentre of triangle $I_AI_BI_C$ is the incentre of triangle ABC.

3 Initially, ten consecutive natural numbers are written on the board. In one turn, you may pick any two numbers from the board (call them a and b) and replace them with the numbers $a^2 - 2011b^2$ and ab. After several turns, there were no initial numbers left on the board. Could there, at this point, be again, ten consecutive natural numbers?

4 In a city's bus route system, any two routes share exactly one stop, and every route includes at least four stops. Prove that the stops can be classified into two groups such that each route includes stops from each group.

– Grade level 10

Day 1

1 Let $a_1, \dots, a_{10}$ be distinct positive integers, all at least 3 and with sum 678. Does there exist a positive integer n such that the sum of the 20 remainders of n after division by $a_1, a_2, \dots, a_{10}, 2a_1, 2a_2, \dots, 2a_{10}$ is 2012?

2 The inscribed circle ω of the non-isosceles acute-angled triangle ABC touches the side BC at the point D. Suppose that I and O are the centres of inscribed circle and circumcircle of triangle ABC respectively. The circumcircle of triangle ADI intersects AO at the points A and E. Prove that AE is equal to the radius r of ω.

3 Any two of the real numbers a_1, a_2, a_3, a_4, a_5 differ by no less than 1. There exists some real number k satisfying

$$a_1 + a_2 + a_3 + a_4 + a_5 = 2k$$

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 = 2k^2$$

Prove that $k^2 \geq 25/3$.

4 Initially there are $n+1$ monomials on the blackboard: $1, x, x^2, \dots, x^n$. Every minute each of k boys simultaneously write on the blackboard the sum of some two polynomials that were written before. After m minutes among others there are the polynomials $S_1 = 1 + x, S_2 = 1 + x + x^2, S_3 = 1 + x + x^2 + x^3, \dots, S_n = 1 + x + x^2 + \dots + x^n$ on the blackboard. Prove that $m \geq \frac{2n}{k+1}$.

Day 2

1 101 wise men stand in a circle. Each of them either thinks that the Earth orbits Jupiter or that Jupiter orbits the Earth. Once a minute, all the wise men express their opinion at the same time. Right after that, every wise man who stands between two people with a different opinion from him changes his opinion himself. The rest do not change. Prove that at one point they will all stop changing opinions.

2 Does there exist natural numbers a, b, c all greater than 10^{10} such that their product is divisible by each of these numbers increased by 2012?

3 On a Cartesian plane, n parabolas are drawn, all of which are graphs of quadratic trinomials. No two of them are tangent. They divide the plane into many areas, one of which is above all the parabolas. Prove that the border of this area has no more than $2(n-1)$ corners (i.e. the intersections of a pair of parabolas).

4 The point E is the midpoint of the segment connecting the orthocentre of the scalene triangle ABC and the point A. The incircle of triangle ABC incircle is tangent to AB and AC at points C' and B' respectively. Prove that point F, the point symmetric to point E with respect to line $B'C'$, lies on the line that passes through both the circumcentre and the incentre of triangle ABC.

– Grade level 11

Day 1

1 Initially, there are 111 pieces of clay on the table of equal mass. In one turn, you can choose several groups of an equal number of pieces and push the pieces into one big piece for each group. What is the least number of turns after which you can end up with 11 pieces no two of which have the same mass?

2 Any two of the real numbers a_1, a_2, a_3, a_4, a_5 differ by no less than 1. There exists some real number k satisfying

$$a_1 + a_2 + a_3 + a_4 + a_5 = 2k$$

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 = 2k^2$$

Prove that $k^2 \geq 25/3$.

3 A plane is coloured into black and white squares in a chessboard pattern. Then, all the white squares are coloured red and blue such that any two initially white squares that share a corner are different colours. (One is red and the other is blue.) Let ℓ be a line not parallel to the sides of any squares. For every line segment I that is parallel to ℓ, we can count the difference between the length of its red and its blue areas. Prove that for every such line ℓ there exists a number C that exceeds all those differences that we can calculate.

4 Given is a pyramid $SA_1A_2A_3\dots A_n$ whose base is convex polygon $A_1A_2A_3\dots A_n$. For every $i = 1, 2, 3, \dots, n$ there is a triangle $X_iA_iA_{i+1}$ congruent to triangle SA_iA_{i+1} that lies on the same side from A_iA_{i+1} as the base of that pyramid. (You can assume a_1 is the same as a_{n+1}.) Prove that these triangles together cover the entire base.

Day 2

1 Given is the polynomial $P(x)$ and the numbers $a_1, a_2, a_3, b_1, b_2, b_3$ such that $a_1a_2a_3 \neq 0$. Suppose that for every x, we have

$$P(a_1x + b_1) + P(a_2x + b_2) = P(a_3x + b_3)$$

Prove that the polynomial $P(x)$ has at least one real root.

2 The points A_1, B_1, C_1 lie on the sides BC, CA and AB of the triangle ABC respectively. Suppose that $AB_1 - AC_1 = CA_1 - CB_1 = BC_1 - BA_1$. Let O_A, O_B and O_C be the circumcentres of triangles AB_1C_1, A_1BC_1 and A_1B_1C respectively. Prove that the incentre of triangle $O_AO_BO_C$ is the incentre of triangle ABC too.

3 On a circle there are $2n + 1$ points, dividing it into equal arcs ($n \geq 2$). Two players take turns to erase one point. If after one player's turn, it turned out that all the triangles formed by the remaining points on the circle were obtuse, then the player wins and the game ends.
Who has a winning strategy: the starting player or his opponent?

4 For a positive integer n define $S_n = 1! + 2! + \dots + n!$. Prove that there exists an integer n such that S_n has a prime divisor greater than 10^{2012}.