IMO 2010

Day 1

$\boxed{1}$ Find all function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ the following equality holds

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$

where $\lfloor a \rfloor$ is greatest integer not greater than a.

Proposed by Pierre Bornsztein, France

$\boxed{2}$ Given a triangle ABC, with I as its incenter and Γ as its circumcircle, AI intersects Γ again at D. Let E be a point on the arc BDC, and F a point on the segment BC, such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. If G is the midpoint of IF, prove that the meeting point of the lines EI and DG lies on Γ.

Proposed by Tai Wai Ming and Wang Chongli, Hong Kong

$\boxed{3}$ Find all functions $g : \mathbb{N} \to \mathbb{N}$ such that

$$(g(m) + n)(g(n) + m)$$

is perfect square for all $m, n \in \mathbb{N}$.

Proposed by Gabriel Carroll, USA

Day 2

4. Let P be a point interior to triangle ABC (with $CA \neq CB$). The lines AP, BP and CP meet again its circumcircle Γ at K, L, respectively M. The tangent line at C to Γ meets the line AB at S. Show that from $SC = SP$ follows $MK = ML$.

Proposed by Marcin E. Kuczma, Poland

5. Each of the six boxes B_1, B_2, B_3, B_4, B_5, B_6 initially contains one coin. The following operations are allowed

   Type 1) Choose a non-empty box B_j, $1 \leq j \leq 5$, remove one coin from B_j and add two coins to B_{j+1};

   Type 2) Choose a non-empty box B_k, $1 \leq k \leq 4$, remove one coin from B_k and swap the contents (maybe empty) of the boxes B_{k+1} and B_{k+2}.

   Determine if there exists a finite sequence of operations of the allowed types, such that the five boxes B_1, B_2, B_3, B_4, B_5 become empty, while box B_6 contains exactly $2010^{2010^{2010}}$ coins.

Proposed by Hans Zantema, Netherlands

6. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers, and s be a positive integer, such that

$$a_n = \max\{a_k + a_{n-k} \mid 1 \leq k \leq n-1\} \text{ for all } n > s.$$

   Prove there exist positive integers $\ell \leq s$ and N, such that

$$a_n = a_\ell + a_{n-\ell} \text{ for all } n \geq N.$$

Proposed by Morteza Saghafiyan, Iran