USA Team Selection Test 2011

Day 1 June 9th

1 In an acute scalene triangle ABC, points D, E, F lie on sides BC, CA, AB, respectively, such that $AD \perp BC, BE \perp CA, CF \perp AB$. Altitudes AD, BE, CF meet at orthocenter H. Points P and Q lie on segment EF such that $AP \perp EF$ and $HQ \perp EF$. Lines DP and QH intersect at point R. Compute HQ/HR.

Proposed by Zuming Feng

2 In the nation of Onewaynia, certain pairs of cities are connected by roads. Every road connects exactly two cities (roads are allowed to cross each other, e.g., via bridges). Some roads have a traffic capacity of 1 unit and other roads have a traffic capacity of 2 units. However, on every road, traffic is only allowed to travel in one direction. It is known that for every city, the sum of the capacities of the roads connected to it is always odd. The transportation minister needs to assign a direction to every road. Prove that he can do it in such a way that for every city, the difference between the sum of the capacities of roads entering the city and the sum of the capacities of roads leaving the city is always exactly one.

Proposed by Zuming Feng and Yufei Zhao

3 Let p be a prime. We say that a sequence of integers $\{z_n\}_{n=0}^{\infty}$ is a [i]p-pod[/i] if for each $e \geq 0$, there is an $N \geq 0$ such that whenever $m \geq N$, p^e divides the sum

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} z_k.$$

Prove that if both sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ are p-pods, then the sequence $\{x_n y_n\}_{n=0}^{\infty}$ is a p-pod.

Day 2 June 10th

4 Find a real number t such that for any set of 120 points $P_1, \ldots P_{120}$ on the boundary of a unit square, there exists a point Q on this boundary with $|P_1Q| + |P_2Q| + \cdots + |P_{120}Q| = t$.

5 Let c_n be a sequence which is defined recursively as follows: $c_0 = 1$, $c_{2n+1} = c_n$ for $n \geq 0$, and $c_{2n} = c_n + c_{n-2^e}$ for $n > 0$ where e is the maximal nonnegative integer such that 2^e divides n. Prove that

$$\sum_{i=0}^{2^n-1} c_i = \frac{1}{n+2}\binom{2n+2}{n+1}.$$

6 A polynomial $P(x)$ is called *nice* if $P(0) = 1$ and the nonzero coefficients of $P(x)$ alternate between 1 and -1 when written in order. Suppose that $P(x)$ is nice, and let m and n be two relatively prime positive integers. Show that

$$Q(x) = P(x^n) \cdot \frac{(x^{mn}-1)(x-1)}{(x^m-1)(x^n-1)}$$

is nice as well.

Day 3 June 11th

7 Let ABC be an acute scalene triangle inscribed in circle Ω. Circle ω, centered at O, passes through B and C and intersects sides AB and AC at E and D, respectively. Point P lies on major arc BAC of Ω. Prove that lines BD, CE, OP are concurrent if and only if triangles PBD and PCE have the same incenter.

8 Let $n \geq 1$ be an integer, and let S be a set of integer pairs (a, b) with $1 \leq a < b \leq 2^n$. Assume $|S| > n \cdot 2^{n+1}$. Prove that there exists four integers $a < b < c < d$ such that S contains all three pairs (a, c), (b, d) and (a, d).

9 Determine whether or not there exist two different sets A, B, each consisting of at most 2011^2 positive integers, such that every x with $0 < x < 1$ satisfies the following inequality:

$$\left| \sum_{a \in A} x^a - \sum_{b \in B} x^b \right| < (1-x)^{2011}.$$