USAMO 1999

Day 1 April 27th

1 Some checkers placed on an $n \times n$ checkerboard satisfy the following conditions:

(a) every square that does not contain a checker shares a side with one that does;

(b) given any pair of squares that contain checkers, there is a sequence of squares containing checkers, starting and ending with the given squares, such that every two consecutive squares of the sequence share a side.

Prove that at least $(n^2-2)/3$ checkers have been placed on the board.

2 Let $ABCD$ be a cyclic quadrilateral. Prove that

$$|AB-CD|+|AD-BC| \geq 2|AC-BD|.$$

3 Let $p>2$ be a prime and let a,b,c,d be integers not divisible by p, such that

$$\left\{\frac{ra}{p}\right\}+\left\{\frac{rb}{p}\right\}+\left\{\frac{rc}{p}\right\}+\left\{\frac{rd}{p}\right\}=2$$

for any integer r not divisible by p. Prove that at least two of the numbers $a+b$, $a+c$, $a+d$, $b+c$, $b+d$, $c+d$ are divisible by p.
(Note: $\{x\}=x-\lfloor x\rfloor$ denotes the fractional part of x.)

Day 2 April 27th

4 Let $a_1,a_2,\dots,a_n$ $(n>3)$ be real numbers such that

$$a_1+a_2+\cdots+a_n \geq n \qquad \text{and} \qquad a_1^2+a_2^2+\cdots+a_n^2 \geq n^2.$$

Prove that $\max(a_1,a_2,\dots,a_n) \geq 2$.

5 The Y2K Game is played on a 1×2000 grid as follows. Two players in turn write either an S or an O in an empty square. The first player who produces three consecutive boxes that spell SOS wins. If all boxes are filled without producing SOS then the game is a draw. Prove that the second player has a winning strategy.

6 Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle ω of triangle BCD meets CD at E. Let F be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle ACF meet line CD at C and G. Prove that the triangle AFG is isosceles.