## Isotomic point of the height foot

geometry circumcircle



**Source**: All Russian MO 2015, grade 10, problem 7

**silouan**
*3804 posts*

*Aug 7, 2015, 6:43 pm* PM #1

In an acute-angled and not isosceles triangle $ABC$, we draw the median $AM$ and the height $AH$.
Points $Q$ and $P$ are marked on the lines $AB$ and $AC$, respectively, so that the $QM \perp AC$ and $PM \perp AB$.
The circumcircle of $PMQ$ intersects the line $BC$ for second time at point $X$. Prove that $BH = CX$.
M. Didin

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**TelvCohl**
*1991 posts*

*Aug 7, 2015, 8:09 pm* PM #2

My solution **:**

Let the tangent of $\odot(ABC)$ passing through $B, C$ meets each other at $T$ .
Let $S$ be the isogonal conjugate of $T$ WRT $\triangle ABC$ ($SB \parallel CA, SC \parallel AB$) .

From $PM \perp AB, TM \perp BC \Longrightarrow \angle TMP = \angle CBA = \angle TCP$ ,
so $C, P, T, M$ lie on a circle with diameter $CT \Longrightarrow P$ is the projection of $T$ on $CA$ .
Similarly, $Q$ is the projection of $T$ on $AB \Longrightarrow \odot(PMQ)$ is the pedal circle of $T$ WRT $\triangle ABC$ ,
so $X$ is the projection of $S$ on $BC$ due to $T, S$ share the same pedal circle (WRT $\triangle ABC$) $\Longrightarrow BH = CX$ .

Q.E.D

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**Luis González**
*3883 posts*

*Aug 8, 2015, 3:58 am* PM #3

Let $U \equiv QM \cap AC, V \equiv PM \cap AB. T$ is the midpoint of $PQ$ and $AM$ cuts $\odot(APQ)$ again at $N$ (reflection of orthocenter $M$ of $\triangle APQ$ on $PQ$). By symmetry, reflections $Y$ and $Z$ of $X$ on $PQ$ and $T$ lie on $\odot(APQ)$ and $PQ \parallel YZ \Longrightarrow \angle(XZ, XM) = \angle(ZN, ZX) = \angle ANZ = \angle AYZ \Longrightarrow AHY \perp BC$. But by Butterfly theorem for the cyclic $PQVU$, it follows that $TM \perp BC \Longrightarrow TM$ is X-midline of $\triangle XHY \Longrightarrow MX = MH$ or $BH = CX$.

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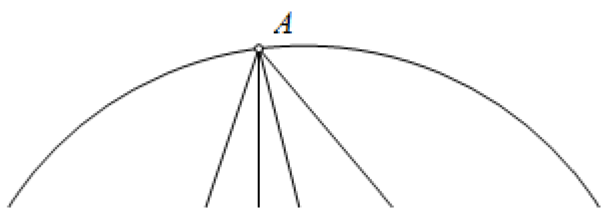
**livetolove212**
*793 posts*

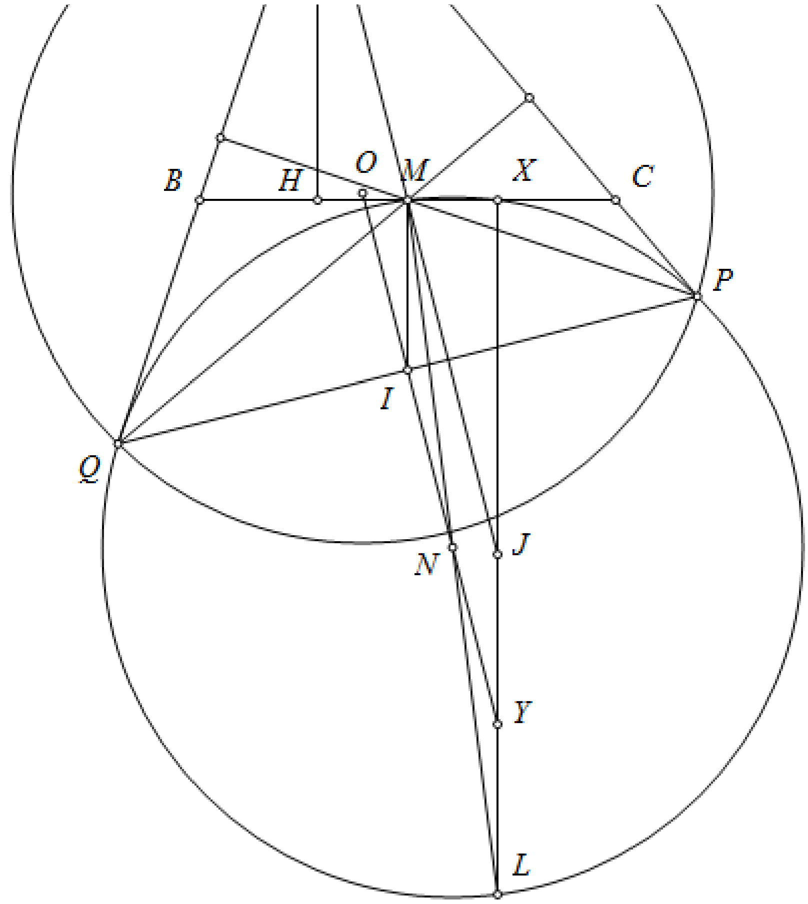
*Nov 17, 2015, 11:03 am* PM #4

Let $I$ be the midpoint of $PQ$. Since $M$ is the orthocenter of triangle $APQ$ and $MB = MC$ then applying butterfly theorem for the circle with diameter $PQ$, $IM \perp BC$. Let $J$ be the reflection of $A$ wrt $M, O$ be the circumcenter of triangle $APQ, N$ is the center of $(MPQ)$, $L$ be the antipode of $M$ wrt $(MPQ)$, $Y$ be the midpoint of $LJ$.

We have $NY \parallel = \frac{1}{2}MJ \parallel = \frac{1}{2}AM \parallel = IN$ hence $N$ is the midpoint of $IY$. This means $IY \parallel = MJ$ or $JY \parallel IM$. Therefore $LJ \perp BC$ at $X$. But $A$ and $J$ are symmetric wrt $M$ hence $MH = MX$ or $BH = CX$.

*Attachments:*






*This post has been edited 2 times. Last edited by livetolove212, Nov 17, 2015, 11:06 am*

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**NguyenHu...**
*8 posts*

*May 31, 2016, 6:21 pm* PM #5

This question is a little bit silly but how your guys can thinking like that ? ANY SECRET 😁

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**kapilpavase**
*432 posts*

*Jun 10, 2016, 6:43 pm • 1* PM #6

Let $PM \cap AB = D, QM \cap AP = E$ performing inversion wrt $M$ and ratio $\sqrt{-MD.MP}$ it amounts to showing that $H' = DE \cap BC$ & $X' = PQ \cap BC$ are equidistant from $M$.

But by converse of butterfly thm on $DQPE$ with $DQ$ and $EP$ as 'wings', it follows that $OM \perp BC$($O$ is centre of $\odot DQPE$. Now we see that the above thing that we wanted to prove is nothing but butterfly on $DQPE$ with $DE$ and $QP$ as wings. 🙂

*This post has been edited 2 times. Last edited by kapilpavase, Jun 10, 2016, 6:45 pm*

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**anantmudg...**
*839 posts*

*Jun 10, 2016, 10:31 pm • 1* PM #7

Here is my solution.

Reflecting $P, Q$ in $M$ the problem is equivalent to the following:

In triangle $ABC$ the feet from $A$ onto $BC$ is $H$ and $M$ is the midpoint of $BC$. Points $K, N$ are on $AB, AC$ such that $\angle AKM = \angle ANM = 90^\circ$. Let rays $MK, MN$ meet the lines through $B, C$ parallel to $AC, AB$ respectively at $P, Q$. Prove that $P, Q, H, M$ are concyclic.

For this newer statement, the proof is as follows: It suffices to prove that $H$ is the centre of a spiral similarity sending $KP$ to $NQ$ since we already know that $A, H, M, K, N$ are concyclic. Now, for this it suffices that $\dfrac{HK}{HN} = \dfrac{KP}{NQ}$.

Consider the following equalities:

$$\frac{HK}{HN} = \frac{\sin HNK}{\sin HKN} = \frac{\sin HAK}{\sin HAN} = \frac{\cos B}{\cos C}$$

And observe that $BM \cdot \cos B = BK$ and so $KP = BK \cdot \tan A$. Similarly, we observe that $CM \cdot \cos C = CN$ and $NQ = CN \cdot \tan A$.

Therefore, we conclude that

$$\frac{HK}{HN} = \frac{\cos B}{\cos C} = \frac{BM \cdot \tan A}{CM \cdot \tan A} \cdot \frac{\cos B}{\cos C} = \frac{KP}{NQ}$$

Thus, our claim holds. Now, it follows that points $P, Q, H, M$ are concyclic.

*This post has been edited 1 time. Last edited by anantmudgal09, Jun 10, 2016, 10:32 pm*
*Reason: Latex error*

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**v_Enhance**
*4253 posts*

*Jul 7, 2016, 8:33 pm* PM #8

Consider triangle $APQ$, which has orthocenter $M$. Extend line $BC$ to intersect the circle with diameter $PQ$ at $Z$, $Y$. Then $YZ$ also has midpoint $M$ by Butterfly theorem. By inversion at $M$, Butterfly theorem then implies $MH = MX$ too.

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*Quick Reply*