Art of Problem Solving
2009 USAMO

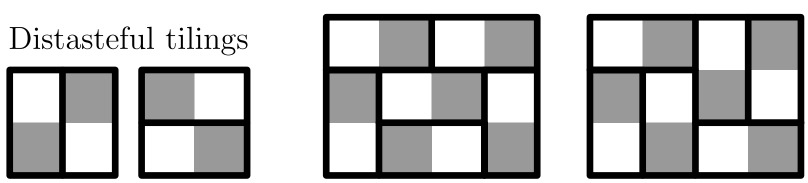
USAMO 2009

Day 1 April 28th

1 Given circles ω_1 and ω_2 intersecting at points X and Y, let ℓ_1 be a line through the center of ω_1 intersecting ω_2 at points P and Q and let ℓ_2 be a line through the center of ω_2 intersecting ω_1 at points R and S. Prove that if P, Q, R and S lie on a circle then the center of this circle lies on line XY.

2 Let n be a positive integer. Determine the size of the largest subset of $\{-n, -n+1, \ldots, n-1, n\}$ which does not contain three elements a, b, c (not necessarily distinct) satisfying $a+b+c=0$.

3 We define a *chessboard polygon* to be a polygon whose sides are situated along lines of the form $x=a$ or $y=b$, where a and b are integers. These lines divide the interior into unit squares, which are shaded alternately grey and white so that adjacent squares have different colors. To tile a chessboard polygon by dominoes is to exactly cover the polygon by non-overlapping 1×2 rectangles. Finally, a *tasteful tiling* is one which avoids the two configurations of dominoes shown on the left below. Two tilings of a 3×4 rectangle are shown; the first one is tasteful, while the second is not, due to the vertical dominoes in the upper right corner.



a) Prove that if a chessboard polygon can be tiled by dominoes, then it can be done so tastefully.

b) Prove that such a tasteful tiling is unique.

Day 2 April 29th

4 For $n \geq 2$ let $a_1, a_2, \ldots a_n$ be positive real numbers such that

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \leq \left(n + \frac{1}{2}\right)^2.$$

Prove that $\max(a_1, a_2, \ldots, a_n) \leq 4\min(a_1, a_2, \ldots, a_n)$.

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5 Trapezoid $ABCD$, with $\overline{AB} \parallel \overline{CD}$, is inscribed in circle ω and point G lies inside triangle BCD. Rays AG and BG meet ω again at points P and Q, respectively. Let the line through G parallel to $\overline{AB}$ intersects $\overline{BD}$ and $\overline{BC}$ at points R and S, respectively. Prove that quadrilateral $PQRS$ is cyclic if and only if $\overline{BG}$ bisects $\angle CBD$.

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6 Let $s_1, s_2, s_3, \ldots$ be an infinite, nonconstant sequence of rational numbers, meaning it is not the case that $s_1 = s_2 = s_3 = \ldots$. Suppose that $t_1, t_2, t_3, \ldots$ is also an infinite, nonconstant sequence of rational numbers with the property that $(s_i - s_j)(t_i - t_j)$ is an integer for all i and j. Prove that there exists a rational number r such that $(s_i - s_j)r$ and $(t_i - t_j)/r$ are integers for all i and j.

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