Canada National Olympiad 2016

1 The integers $1, 2, 3, \ldots, 2016$ are written on a board. You can choose any two numbers on the board and replace them with their average. For example, you can replace 1 and 2 with 1.5, or you can replace 1 and 3 with a second copy of 2. After 2015 replacements of this kind, the board will have only one number left on it.

(a) Prove that there is a sequence of replacements that will make the final number equal to 2.

(b) Prove that there is a sequence of replacements that will make the final number equal to 1000.

2 Consider the following system of 10 equations in 10 real variables $v_1, \ldots, v_{10}$:

$$v_i = 1 + \frac{6v_i^2}{v_1^2 + v_2^2 + \cdots + v_{10}^2} \qquad (i = 1, \ldots, 10).$$

Find all 10-tuples $(v_1, v_2, \ldots, v_{10})$ that are solutions of this system.

3 Find all polynomials $P(x)$ with integer coefficients such that $P(P(n) + n)$ is a prime number for infinitely many integers n.

4 Let A, B, and F be positive integers, and assume $A < B < 2A$. A flea is at the number 0 on the number line. The flea can move by jumping to the right by A or by B. Before the flea starts jumping, Lavaman chooses finitely many intervals $\{m+1, m+2, \ldots, m+A\}$ consisting of A consecutive positive integers, and places lava at all of the integers in the intervals. The intervals must be chosen so that:

(i) any two distinct intervals are disjoint and not adjacent;
(ii) there are at least F positive integers with no lava between any two intervals; and
(iii) no lava is placed at any integer less than F.

Prove that the smallest F for which the flea can jump over all the intervals and avoid all the lava, regardless of what Lavaman does, is $F = (n-1)A + B$, where n is the positive integer such that $\frac{A}{n+1} \le B - A < \frac{A}{n}$.

5 Let $\triangle ABC$ be an acute-angled triangle with altitudes AD and BE meeting at H. Let M be the midpoint of segment AB, and suppose that the circumcircles of $\triangle DEM$ and $\triangle ABH$ meet at points P and Q with P on the same side of

CH as A. Prove that the lines ED, PH, and MQ all pass through a single point on the circumcircle of $\triangle ABC$.