National Math Olympiad (3rd Round) 1993

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1 Prove that there exist infinitely many positive integers which can't be represented as sum of less than 10 odd positive integers' perfect squares.

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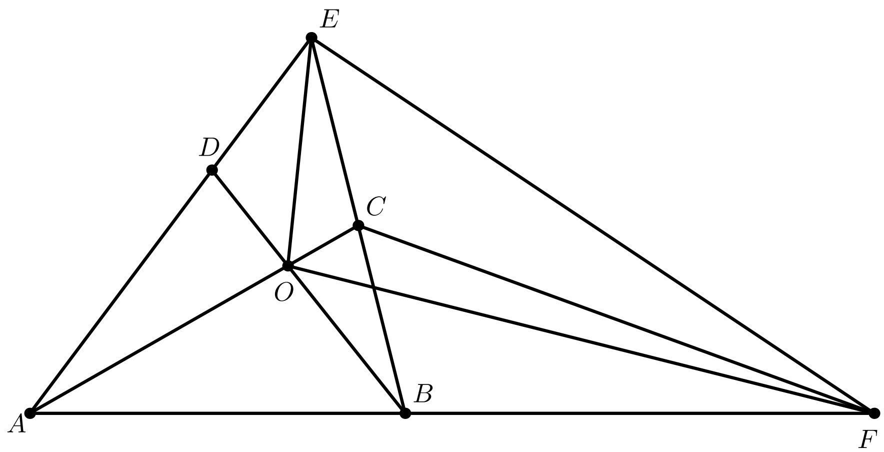
2 In the figure below, area of triangles AOD, DOC, and AOB is given. Find the area of triangle OEF in terms of area of these three triangles.



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4 Prove that there exists a subset S of positive integers such that we can represent each positive integer as difference of two elements of S in exactly one way.

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5 In a convex quadrilateral $ABCD$, diagonals AC and BD are equal. We construct four equilateral triangles with centers O_1, O_2, O_3, O_4 on the sides sides AB, BC, CD, DA outside of this quadrilateral, respectively. Show that $O_1O_3 \perp O_2O_4$.

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6 Let $x_1, x_2, \ldots, x_{12}$ be twelve real numbers such that for each $1 \leq i \leq 12$, we have $|x_i| \geq 1$. Let $I = [a, b]$ be an interval such that $b - a \leq 2$. Prove that number of the numbers of the form $t = \sum_{i=1}^{12} r_i x_i$, where $r_i = \pm 1$, which lie inside the interval I, is less than 1000.

National Math Olympiad (3rd Round) 2005

Day 1

1 Suppose $a, b, c \in \mathbb{R}^+$. Prove that :

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2 \geq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

2 Suppose $\{x_n\}$ is a decreasing sequence that $\lim_{n\to\infty} x_n = 0$. Prove that $\sum(-1)^n x_n$ is convergent

3 Find all $\alpha > 0$ and $\beta > 0$ that for each $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n) \in \mathbb{R^+}^n$ that:

$$(\sum x_i^\alpha)(\sum y_i^\beta) \geq \sum x_i y_i$$

4 Suppose $P, Q \in \mathbb{R}[x]$ that $deg\ P = deg\ Q$ and $PQ' - QP'$ has no real root. Prove that for each $\lambda \in \mathbb{R}$ number of real roots of P and $\lambda P + (1-\lambda)Q$ are equal.

5 Suppose $a, b, c \in \mathbb{R}^+$and

$$\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}=2$$

Prove that $ab+ac+bc \leq \frac{3}{2}$

6 Suppose $A \subseteq \mathbb{R}^m$ is closed and non-empty. Let $f : A \to A$ is a lipchitz function with constant less than 1. (ie there exist $c < 1$ that $|f(x)-f(y)| < c|x-y|,\ \forall x, y \in A$). Prove that there exists a unique point $x \in A$ such that $f(x) = x$.

Day 2

1 From each vertex of triangle ABC we draw 3 arbitary parrallell lines, and from each vertex we draw a perpendicular to these lines. There are 3 rectangles that one of their diagnals is triangle's side. We draw their other diagnals and call them ℓ_1, ℓ_2 and ℓ_3.

a) Prove that ℓ_1, ℓ_2 and ℓ_3 are concurrent at a point P.

b) Find the locus of P as we move the 3 arbitary lines.

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2 Suppose O is circumcenter of triangle ABC. Suppose $\frac{S(OAB)+S(OAC)}{2} = S(OBC)$. Prove that the distance of O (circumcenter) from the radical axis of the circumcircle and the 9-point circle is

$$\frac{a^2}{\sqrt{9R^2-(a^2+b^2+c^2)}}$$

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3 Prove that in acute-angled traingle ABC if r is inradius and R is radius of circumcircle then:

$$a^2+b^2+c^2 \geq 4(R+r)^2$$

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4 Suppose in triangle ABC incircle touches the side BC at P and $\angle APB = \alpha$. Prove that :

$$\frac{1}{p-b}+\frac{1}{p-c}=\frac{2}{rtg\alpha}$$

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5 Suppose H and O are orthocenter and circumcenter of triangle ABC. ω is circumcircle of ABC. AO intersects with ω at A_1. A_1H intersects with ω at A' and A'' is the intersection point of ω and AH. We define points B', B'', C' and C'' similiarly. Prove that $A'A''$, $B'B''$ and $C'C''$ are concurrent in a point on the Euler line of triangle ABC.

Day 3

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1 Find all $n, p, q \in \mathbb{N}$ that:

$$2^n+n^2=3^p7^q$$

---

2 Let $a \in \mathbb{N}$ and $m = a^2 + a + 1$. Find the number of $0 \le x \le m$ that:

$$x^3 \equiv 1 (\text{mod } m)$$

---

3 $p(x)$ is an irreducible polynomial in $\mathbb{Q}[x]$ that $\deg p$ is odd. $q(x), r(x)$ are polynomials with rational coefficients that $p(x) | q(x)^2 + q(x).r(x) + r(x)^2$. Prove that

$$p(x)^2 | q(x)^2 + q(x).r(x) + r(x)^2$$

---

4 k is an integer. We define the sequence $\{a_n\}_{n=0}^{\infty}$ like this:

$$a_0 = 0, \quad a_1 = 1, \quad a_n = 2ka_{n-1} - (k^2+1)a_{n-2} \ \ (n \ge 2)$$

p is a prime number that $p \equiv 3 (\text{mod } 4)$
a) Prove that $a_{n+p^2-1} \equiv a_n (\text{mod } p)$
b) Prove that $a_{n+p^3-p} \equiv a_n (\text{mod } p^2)$

---

5 Let $a, b, c \in \mathbb{N}$ be such that $a, b \ne c$. Prove that there are infinitely many prime numbers p for which there exists $n \in \mathbb{N}$ that $p | a^n + b^n - c^n$.

Day 4

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1 We call the set $A \in \mathbb{R}^n$ CN if and only if for every continuous $f : A \to A$ there exists some $x \in A$ such that $f(x) = x$.
a) Example: We know that $A = \{x \in \mathbb{R}^n | |x| \le 1\}$ is CN.
b) The circle is not CN.

Which one of these sets are CN?

1) $A = \{x \in \mathbb{R}^3 | |x| = 1\}$

2) The cross $\{(x, y) \in \mathbb{R}^2 | xy = 0, \ |x| + |y| \le 1\}$

3) Graph of the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \sin\frac{1}{x} \text{ if } x \ne 0, \ f(0) = 0$$

---

2 n vectors are on the plane. We can move each vector forward and backeard on the line that the vector is on it. If there are 2 vectors that their endpoints concide we can omit them and replace them with their sum (If their sum is nonzero). Suppose with these operations with 2 different method we reach to a vector. Prove that these vectors are on a common line

3 $f(n)$ is the least number that there exist a $f(n)-$mino that contains every $n-$mino.
Prove that $10000 \leq f(1384) \leq 960000$.
Find some bound for $f(n)$

4 a) Year 1872 Texas
3 gold miners found a peice of gold. They have a coin that with possibility of $\frac{1}{2}$ it will come each side, and they want to give the piece of gold to one of themselves depending on how the coin will come. Design a fair method (It means that each of the 3 miners will win the piece of gold with possibility of $\frac{1}{3}$) for the miners.

b) Year 2005, faculty of Mathematics, Sharif university of Technolgy
Suppose $0 < \alpha < 1$ and we want to find a way for people name A and B that the possibity of winning of A is α. Is it possible to find this way?

c) Year 2005 Ahvaz, Takhti Stadium
Two soccer teams have a contest. And we want to choose each player's side with the coin, But we don't know that our coin is fair or not. Find a way to find that coin is fair or not?

d) Year 2005,summer
In the National mathematical Oympiad in Iran. Each student has a coin and must find a way that the possibility of coin being TAIL is α or no. Find a way for the student.

Day 5

1 An airplane wants to go from a point on the equator, and at each moment it will go to the northeast with speed v. Suppose the radius of earth is R.
a) Will the airplane reach to the north pole? If yes how long it will take to reach the north pole?
b) Will the airplne rotate finitely many times around the north pole? If yes how many times?

2 We define a relation between subsets of $\mathbb{R}^n$. $A \sim B \iff$ we can partition A, B in sets $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$(i.e $A = \bigcup_{i=1}^{n} A_i,\ B = \bigcup_{i=1}^{n} B_i, A_i \cap A_j = \emptyset,\ B_i \cap B_j = \emptyset$) and $A_i \simeq B_i$.
Say the the following sets have the relation $\sim$ or not ?

a) Natural numbers and composite numbers.
b) Rational numbers and rational numbers with finite digits in base 10.
c) $\{x \in \mathbb{Q} | x < \sqrt{2}\}$ and $\{x \in \mathbb{Q} | x < \sqrt{3}\}$
d) $A = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$ and $A \setminus \{(0, 0)\}$

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3 For each $m \in \mathbb{N}$ we define $rad\ (m) = \prod p_i$, where $m = \prod p_i^{\alpha_i}$.

abc Conjecture
Suppose $\epsilon > 0$ is an arbitrary number, then there exist K depinding on ϵ that for each 3 numbers $a, b, c \in \mathbb{Z}$ that $gcd(a, b) = 1$ and $a + b = c$ then:

$$max\{|a|, |b|, |c|\} \leq K(rad\ (abc))^{1+\epsilon}$$

Now prove each of the following statements by using the abc conjecture :
a) Fermat's last theorem for $n > N$ where N is some natural number.
b) We call $n = \prod p_i^{\alpha_i}$ strong if and only $\alpha_i \geq 2$.
c) Prove that there are finitely many n such that $n,\ n + 1,\ n + 2$ are strong.
d) Prove that there are finitely many rational numbers $\frac{p}{q}$ such that:

$$\left|\sqrt[3]{2} - \frac{p}{q}\right| < \frac{2^{1384}}{q^3}$$

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4 Suppose we have some proteins that each protein is a sequence of 7 "AMINO-ACIDS" $A,\ B,\ C,\ H,\ F,\ N$. For example $AFHNNNHAFFC$ is a protein. There are some steps that in each step an amino-acid will change to another one. For example with the step $NA \to N$ the protein $BANANA$ will cahnge to $BANNA$("in Persian means workman"). We have a set of allowed steps that each protein can change with these steps. For example with the
set of steps:
1) $AA \longrightarrow A$
2) $AB \longrightarrow BA$
3) $A \longrightarrow$ null
Protein $ABBAABA$ will change like this:
$ABB\underline{AA}BA$
$\underline{AB}BABA$

$B\underline{AB}ABA$
$BB\underline{AA}BA$
$BB\underline{AB}A$
$BBB\underline{AA}$
$BBB\underline{A}$
BBB
You see after finite steps this protein will finish it steps.
Set of allowed steps that for them there exist a protein that may have infinitely many steps is dangerous. Which of the following allowed sets are dangerous?
a) $NO \longrightarrow OONN$
b) $\begin{cases} HHCC \longrightarrow HCCH \\ CC \longrightarrow CH \end{cases}$
c) Design a set of allowed steps that change $\underbrace{AA\dots A}_{n} \longrightarrow \underbrace{BB\dots B}_{2^n}$
d) Design a set of allowed steps that change $\underbrace{A\dots A}_{n}\underbrace{B\dots B}_{m} \longrightarrow \underbrace{CC\dots C}_{mn}$
You see from c and d that we acn calculate the functions $F(n) = 2^n$ and $G(M,N) = mn$ with these steps. Find some other calculatable functions with these steps. (It has some extra mark.)

National Math Olympiad (3rd Round) 2002

1 Let $a, b, c \in \mathbb{R}^n, a+b+c=0$ and $\lambda > 0$. Prove that

$$\prod_{cycle} \frac{|a|+|b|+(2\lambda+1)|c|}{|a|+|b|+|c|} \geq (2\lambda+3)^3$$

2 $f : \mathbb{R} \longrightarrow \mathbb{R}^+$ is a non-decreasing function. Prove that there is a point $a \in \mathbb{R}$ that

$$f(a+\frac{1}{f(a)}) < 2f(a)$$

3 a_n is a sequence that $a_1 = 1, a_2 = 2, a_3 = 3$, and

$$a_{n+1} = a_n - a_{n-1} + \frac{a_n^2}{a_{n-2}}$$

Prove that for each natural n, a_n is integer.

4 a_n (n is integer) is a sequence from positive reals that

$$a_n \geq \frac{a_{n+2}+a_{n+1}+a_{n-1}+a_{n-2}}{4}$$

Prove a_n is constant.

5 ω is circumcirlce of triangle ABC. We draw a line parallel to BC that intersects AB, AC at E, F and intersects ω at U, V. Assume that M is midpoint of BC. Let ω' be circumcircle of UMV. We know that $R(ABC) = R(UMV)$. ME and ω' intersect at T, and FT intersects ω' at S. Prove that EF is tangent to circumcircle of MCS.

6 M is midpoint of $BC.P$ is an arbitary point on BC. C_1 is tangent to big circle.Suppose radius of C_1 is r_1
Radius of C_4 is equal to radius of C_1 and C_4 is tangent to BC at P. C_2 and C_3 are tangent to big circle and line BC and circle C_4.
`http://aycu01.webshots.com/image/4120/2005120338156776027_rs.jpg`
Prove :

$$r_1 + r_2 + r_3 = R$$

(R radius of big circle)

7 In triangle ABC, AD is angle bisector (D is on BC) if $AB + AD = CD$ and $AC + AD = BC$, what are the angles of ABC?

8 Circles C_1 and C_2 are tangent to each other at K and are tangent to circle C at M and N. External tangent of C_1 and C_2 intersect C at A and B. AK and BK intersect with circle C at E and F respectively. If AB is diameter of C, prove that EF and MN and OK are concurrent. (O is center of circle C.)

9 Let M and N be points on the side BC of triangle ABC, with the point M lying on the segment BN, such that $BM = CN$. Let P and Q be points on the segments AN and AM, respectively, such that $\measuredangle PMC = \measuredangle MAB$ and $\measuredangle QNB = \measuredangle NAC$. Prove that $\measuredangle QBC = \measuredangle PCB$.

10 H, I, O, N are orthogonal center, incenter, circumcenter, and Nagelian point of triangle ABC. I_a, I_b, I_c are excenters of ABC corresponding vertices A, B, C. S is point that O is midpoint of HS. Prove that centroid of triangles $I_aI_bI_c$ and SIN concide.

11 In an $m \times n$ table there is a policeman in cell $(1, 1)$, and there is a thief in cell (i, j). A move is going from a cell to a neighbor (each cell has at most four neighbors). Thief makes the first move, then the policeman moves and ... For which (i, j) the policeman can catch the thief?

12 We have a bipartite graph G (with parts X and Y). We orient each edge arbitrarily. *Hessam* chooses a vertex at each turn and reverse the orientation of all edges that v is one of their endpoint. Prove that with these steps we can reach to a graph that for each vertex v in part X, $\deg^+(v) \geq \deg^-(v)$ and for each vertex in part Y, $\deg^+ v \leq \deg^- v$

13 f, g are two permutations of set $X = \{1, \dots, n\}$. We say f, g have common points iff there is a $k \in X$ that $f(k) = g(k)$.
a) If $m > \frac{n}{2}$, prove that there are m permutations $f_1, f_2, \dots, f_m$ from X that for each permutation $f \in X$, there is an index i that f, f_i have common points.
b) Prove that if $m \leq \frac{n}{2}$, we can not find permutations $f_1, f_2, \dots, f_m$ satisfying the above condition.

14 A subset S of $\mathbb{N}$ is *eventually linear* iff there are $k, N \in \mathbb{N}$ that for $n > N, n \in S \iff k|n$. Let S be a subset of $\mathbb{N}$ that is closed under addition. Prove that S is eventually linear.

15 Let A be be a point outside the circle C, and AB and AC be the two tangents from A to this circle C. Let L be an arbitrary tangent to C that cuts AB and AC in P and Q. A line through P parallel to AC cuts BC in R. Prove that while L varies, QR passes through a fixed point. :)

16 For positive a, b, c,

$$a^2 + b^2 + c^2 + abc = 4$$

Prove $a + b + c \leq 3$

17 Find the smallest natural number n that the following statement holds :
Let A be a finite subset of $\mathbb{R}^2$. For each n points in A there are two lines including these n points. All of the points lie on two lines.

18 Find all continious $f : \mathbb{R} \longrightarrow \mathbb{R}$ that for any x, y

$$f(x) + f(y) + f(xy) = f(x + y + xy)$$

19 I is incenter of triangle ABC. Incircle of ABC touches AB, AC at X, Y. XI intersects incircle at M. Let $CM \cap AB = X'$. L is a point on the segment $X'C$ that $X'L = CM$. Prove that A, L, I are collinear iff $AB = AC$.

20 $a_0 = 2, a_1 = 1$ and for $n \geq 1$ we know that : $a_{n+1} = a_n + a_{n-1}$ m is an even number and p is prime number such that p divides $a_m - 2$. Prove that p divides $a_{m+1} - 1$.

21 Excircle of triangle ABC corresponding vertex A, is tangent to BC at P. AP intersects circumcircle of ABC at D. Prove

$$r(PCD) = r(PBD)$$

whcih $r(PCD)$ and $r(PBD)$ are inradii of triangles PCD and PBD.

22 15000 years ago Tilif ministry in Persia decided to define a code for $n \geq 2$ cities. Each code is a sequence of $0, 1$ such that no code start with another code. We know that from 2^m calls from foreign countries to Persia 2^{m-a_i} of them where from the i-th city (So $\sum_{i=1}^{n} \frac{1}{2^{a_i}} = 1$). Let l_i be length of code assigned to i-th city. Prove that $\sum_{i=1}^{n} \frac{l_i}{2^i}$ is minimum iff $\forall i,\ l_i = a_i$

23 Find all polynomials p with real coefficients that if for a real a,$p(a)$ is integer then a is integer.

24 A, B, C are on circle $\mathcal{C}$. I is incenter of ABC , D is midpoint of arc BAC. W is a circle that is tangent to AB and AC and tangent to $\mathcal{C}$ at P. (W is in $\mathcal{C}$) Prove that P and I and D are on a line.

25 An ant walks on the interior surface of a cube, he moves on a straight line. If ant reaches to an edge the he moves on a straight line on cube's net. Also if he reaches to a vertex he will return his path.
a) Prove that for each beginning point ant can has infinitely many choices for his direction that its path becomes periodic.
b) Prove that if if the ant starts from point A and its path is periodic, then for each point B if ant starts with this direction, then his path becomes periodic.

National Math Olympiad (3rd Round) 2003

1 suppose this equation: x ¡sup¿2¡/sup¿ +y ¡sup¿2¡/sup¿ +z ¡sup¿2¡/sup¿ =w ¡sup¿2¡/sup¿ . show that the solution of this equation (if w,z have same parity) are in this form:
x=2d(XZ-YW), y=2d(XW+YZ),z=d(X ¡sup¿2¡/sup¿ +Y ¡sup¿2¡/sup¿ -Z ¡sup¿2¡/sup¿ -W ¡sup¿2¡/sup¿),w=d(X ¡sup¿2¡/sup¿ +Y ¡sup¿2¡/sup¿ +Z ¡sup¿2¡/sup¿ +W ¡sup¿2¡/sup¿)

2 assume ABCD a convex quadrilatral. P and Q are on BC and DC respectively such that angle BAP= angle DAQ .prove that [ADQ]=[ABP]
([ABC] means its area) iff the line which crosses through the orthocenters of these traingles , is perpendicular to AC.

3 assume that A is a finite subset of prime numbers, and a is an positive integer. prove that there are only finitely many positive integers m s.t: prime divisors of aˆm-1 are contained in A.

4 XOY is angle in the plane.A,B are variable point on OX,OY such that 1/OA+1/OB=1/K (k is constant).draw two circles with diameter OA and OB.prove that common external tangent to these circles is tangent to the constant circle(ditermine the radius and the locus of its center).

5 Let p be an odd prime number. Let S be the sum of all primitive roots modulo p. Show that if $p-1$ isn't squarefree (i. e., if there exist integers k and m with $k>1$ and $p-1=k^2m$), then $S \equiv 0 \mod p$.

If not, then what is S congruent to $\mod p$?

6 let the incircle of a triangle ABC touch BC,AC,AB at A1,B1,C1 respectively. M and N are the midpoints of AB1 and AC1 respectively. MN meets A1C1 at T . draw two tangents TP and TQ through T to incircle. PQ meets MN at L and B1C1 meets PQ at K . assume I is the center of the incircle .

prove IK is parallel to AL

7 $f_1, f_2, \ldots, f_n$ are polynomials with integer coefficients. Prove there exist a reducible $g(x)$ with integer coefficients that $f_1+g, f_2+g, \ldots, f_n+g$ are irreducible.

8 A positive integer n is said to be a *perfect power* if $n = a^b$ for some integers a, b with $b > 1$. (a) Find 2004 perfect powers in arithmetic progression. (b) Prove that perfect powers cannot form an infinite arithmetic progression.

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9 Does there exist an infinite set S such that for every $a, b \in S$ we have $a^2+b^2-ab \mid (ab)^2$.

---

10 let p be a prime and a and n be natural numbers such that (p^a -1)/ (p-1) = 2 ^n
find the number of natural divisors of na. :)

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11 assume that X is a set of n number.and $0 \leq k \leq n$.the maximum number of permutation which acting on X st every two of them have at least k component in common,is $a_{n,k}$.and the maximum nuber of permutation st every two of them have at most k component in common,is $b_{n,k}$.
a)proeve that :$a_{n,k} \cdot b_{n,k-1} \leq n!$
b)assume that p is prime number,determine the exact value of $a_{p,2}$.

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12 There is a lamp in space.(Consider lamp a point)
Do there exist finite number of equal sphers in space that the light of the lamp can not go to the infinite?(If a ray crash in a sphere it stops)

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13 here is the most difficult and the most beautiful problem occurs in 21th iranian (2003) olympiad
assume that P is n-gon ,lying on the plane ,we name its edge 1,2,..,n.
if S=s1,s2,s3,.... be a finite or infinite sequence such that for each i, si is in {1,2,...,n},
we move P on the plane according to the S in this form: at first we reflect P through the s1
(s1 means the edge which iys number is s1)then through s2 and so on like the figure below.
a)show that there exist the infinite sequence S sucth that if we move P according to S we cover all the plane
b)prove that the sequence in a) isn't periodic.
c)assume that P is regular pentagon ,which the radius of its circumcircle is 1,and D is circle ,with radius 1.00001 ,arbitrarily in the plane .does exist a sequence S such that we move P according to S then P reside in D completely?

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14 n \geq 6 is an integer. evaluate the minimum of f(n) s.t: any graph with n vertices and f(n) edge contains two cycle which are distinct(also they have no comon vertice)?

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15 Assume $m \times n$ matrix which is filled with just 0, 1 and any two row differ in at least $n/2$ members, show that $m \leq 2n$.
(for example the diffrence of this two row is only in one index
110
100)

Edited by Myth

16 Segment AB is fixed in plane. Find the largest n, such that there are n points $P_1, P_2, \ldots, P_n$ in plane that triangles ABP_i are similar for $1 \leq i \leq n$. Prove that all of P_i's lie on a circle.

17 A simple calculator is given to you. (It contains 8 digits and only does the operations +,-,*,/,$\sqrt{}$) How can you find $3^{\sqrt{2}}$ with accuracy of 6 digits.

18 In tetrahedron $ABCD$, radius four circumcircles of four faces are equal. Prove that $AB = CD$, $AC = BD$ and $AD = BC$.

19 An integer n is called a good number if and only if $|n|$ is not square of another intger. Find all integers m such that they can be written in infinitely many ways as sum of three different good numbers and product of these three numbers is square of an odd number.

20 Suppose that M is an arbitrary point on side BC of triangle ABC. B_1, C_1 are points on AB, AC such that $MB = MB_1$ and $MC = MC_1$. Suppose that H, I are orthocenter of triangle ABC and incenter of triangle MB_1C_1. Prove that A, B_1, H, I, C_1 lie on a circle.

21 Let ABC be a triangle. W_a is a circle with center on BC passing through A and perpendicular to circumcircle of ABC. W_b, W_c are defined similarly. Prove that center of W_a, W_b, W_c are collinear.

22 Let $a_1 = a_2 = 1$ and

$$a_{n+2} = \frac{n(n+1)a_{n+1} + n^2 a_n + 5}{n+2} - 2$$

for each $n \in \mathbb{N}$. Find all n such that $a_n \in \mathbb{N}$.

23 Find all homogeneous linear recursive sequences such that there is a T such that $a_n = a_{n+T}$ for each n.

24	A, B are fixed points. Variable line l passes through the fixed point C. There are two circles passing through A, B and tangent to l at M, N. Prove that circumcircle of AMN passes through a fixed point.
25	Let A, B, C, Q be fixed points on plane. M, N, P are intersection points of AQ, BQ, CQ with BC, CA, AB. D', E', F' are tangency points of incircle of ABC with BC, CA, AB. Tangents drawn from M, N, P (not triangle sides) to incircle of ABC make triangle DEF. Prove that DD', EE', FF' intersect at Q.
26	Circles C_1, C_2 intersect at P. A line Δ is drawn arbitrarily from P and intersects with C_1, C_2 at B, C. What is locus of A such that the median of AM of triangle ABC has fixed length k.
27	$S \subset \mathbb{N}$ is called a square set, iff for each $x, y \in S$, $xy+1$ is square of an integer. a) Is S finite? b) Find maximum number of elements of S.
28	There are n points in $\mathbb{R}^3$ such that every three form an acute angled triangle. Find maximum of n.
29	Let $c \in \mathbb{C}$ and $A_c = \{p \in \mathbb{C}[z] \mid p(z^2+c) = p(z)^2 + c\}$. a) Prove that for each $c \in C$, A_c is infinite. b) Prove that if $p \in A_1$, and $p(z_0) = 0$, then $\|z_0\| < 1.7$. c) Prove that each element of A_c is odd or even. Let $f_c = z^2 + c \in \mathbb{C}[z]$. We see easily that $B_c := \{z, f_c(z), f_c(f_c(z)), \dots\}$ is a subset of A_c. Prove that in the following cases $A_c = B_c$. d) $\|c\| > 2$. e) $c \in \mathbb{Q} \backslash \mathbb{Z}$. f) c is a non-algebraic number g) c is a real number and $c \notin [-2, \frac{1}{4}]$.

National Math Olympiad (3rd Round) 2004

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1 We say $m \circ n$ for natural m,n $\Longleftrightarrow$
nth number of binary representation of m is 1 or mth number of binary representation of n is 1.
and we say $m \bullet n$ if and only if m, n doesn't have the relation $\circ$
We say $A \subset \mathbb{N}$ is golden $\Longleftrightarrow \forall U, V \subset A$ that are finite and arenot empty and $U \cap V = \emptyset$,There exist $z \in A$ that $\forall x \in U, y \in V$ we have $z \circ x, z \bullet y$
Suppose $\mathbb{P}$ is set of prime numbers.Prove if $\mathbb{P} = P_1 \cup ... \cup P_k$ and $P_i \cap P_j = \emptyset$ then one of $P_1, ..., P_k$ is golden.

---

2 A is a compact convex set in plane. Prove that there exists a point $O \in A$, such that for every line XX' passing through O, where X and X' are boundary points of A, then
$$\frac{1}{2} \leq \frac{OX}{OX'} \leq 2.$$

---

3 Suppose $V = \mathbb{Z}_2^n$ and for a vector $x = (x_1, ..x_n)$ in V and permutation σ.We have $x_\sigma = (x_{\sigma(1)}, ..., x_{\sigma(n)})$
Suppose $n = 4k+2, 4k+3$ and $f : V \to V$ is injective and if x and y differ in more than $n/2$ places then $f(x)$ and $f(y)$ differ in more than $n/2$ places.
Prove there exist permutaion σ and vector v that $f(x) = x_\sigma + v$

---

4 We have finite white and finite black points that for each 4 oints there is a line that white points and black points are at different sides of this line.Prove there is a line that all white points and black points are at different side of this line.

---

5 assume that k,n are two positive integer $k \leq n$count the number of permutation $\{ 1, \ldots, n\}$ st for any $1 \leq i, j \leq k$and any positive integer m we have $f^m(i) \neq j$ (f^m meas iterarte function,)

---

6 assume that we have a n*n table we fill it with 1,...,n such that each number exists exactly n times prove that there exist a row or column such that at least $\sqrt{n}$ diffrent number are contained.

---

7 Suppose F is a polygon with lattice vertices and sides parralell to x-axis and y-axis.Suppose $S(F), P(F)$ are area and perimeter of F.
Find the smallest k that: $S(F) \leq k.P(F)^2$

---

8 $\mathbb{P}$ is a n-gon with sides $l_1, ..., l_n$ and vertices on a circle. Prove that no n-gon with this sides has area more than $\mathbb{P}$

---

9 Let ABC be a triangle, and O the center of its circumcircle.
Let a line through the point O intersect the lines AB and AC at the points M and N, respectively. Denote by S and R the midpoints of the segments BN and CM, respectively.
Prove that $\measuredangle ROS = \measuredangle BAC$.

---

10 $f : \mathbb{R}^2 \to \mathbb{R}^2$ is injective and surjective. Distance of X and Y is not less than distance of $f(X)$ and $f(Y)$. Prove for A in plane:

$$S(A) \geq S(f(A))$$

where $S(A)$ is area of A

---

11 assume that ABC is acute traingle and AA' is median we extend it until it meets circumcircle at A". let AP_a be a diameter of the circumcircle. the pependicular from A' to AP_a meets the tangent to circumcircle at A" in the point X_a; we define X_b, X_c similary . prove that X_a, X_b, X_c are one a line.

---

12 $\mathbb{N}_{10}$ is generalization of $\mathbb{N}$ that every hypernumber in $\mathbb{N}_{10}$ is something like: $\overline{...a_2a_1a_0}$ with $a_i \in 0, 1..9$
(Notice that $\overline{...000} \in \mathbb{N}_{10}$)
Also we easily have $+, *$ in $\mathbb{N}_{10}$.
first k number of $a * b$= first k nubmer of (first k number of a * first k number of b)
first k number of $a + b$= first k nubmer of (first k number of a + first k number of b)
Fore example $\overline{...999} + \overline{...0001} = \overline{...000}$
Prove that every monic polynomial in $\mathbb{N}_{10}[x]$ with degree d has at most d^2 roots.

---

13 Suppose f is a polynomial in $\mathbb{Z}[X]$ and m is integer .Consider the sequence a_i like this $a_1 = m$ and $a_{i+1} = f(a_i)$ find all polynomials f and alll integers m that for each i:

$$a_i | a_{i+1}$$

---

14 We define $f:\mathbb{N}\to\mathbb{N}$, $f(n)=\sum_{k=1}^{n}(k,n)$.

a) Show that if $\gcd(m,n)=1$ then we have $f(mn)=f(m)\cdot f(n)$;

b) Show that $\sum_{d|n} f(d)=nd(n)$.

15 This problem is easy but nobody solved it.
point A moves in a line with speed v and B moves also with speed v' that at every time the direction of move of B goes from A.We know $v\geq v'$.If we know the point of beginning of path of A, then B must be where at first that B can catch A.

16 Let ABC be a triangle . Let point X be in the triangle and AX intersects BC in Y . Draw the perpendiculars YP,YQ,YR,YS to lines CA,CX,BX,BA respectively. Find the necessary and sufficient condition for X such that $PQRS$ be cyclic .

17 Let $p=4k+1$ be a prime. Prove that p has at least $\frac{\phi(p-1)}{2}$ primitive roots.

18 Prove that for any n, there is a subset $\{a_1,\ldots,a_n\}$ of $\mathbb{N}$ such that for each subset S of $\{1,\ldots,n\}$, $\sum_{i\in S} a_i$ has the same set of prime divisors.

19 Find all integer solutions of $p^3=p^2+q^2+r^2$ where p,q,r are primes.

20 $p(x)$ is a polynomial in $\mathbb{Z}[x]$ such that for each $m,n\in\mathbb{N}$ there is an integer a such that $n\mid p(a^m)$. Prove that 0 or 1 is a root of $p(x)$.

21 $a_1,a_2,\ldots,a_n$ are integers, not all equal. Prove that there exist infinitely many prime numbers p such that for some k

$$p\mid a_1^k+\ldots+a_n^k.$$

22 Suppose that $\mathcal{F}$ is a family of subsets of X. A,B are two subsets of X s.t. each element of $\mathcal{F}$ has non-empty intersection with A,B. We know that no subset of X with $n-1$ elements has this property. Prove that there is a representation A,B in the form $A=\{a_1,\ldots,a_n\}$ and $B=\{b_1,\ldots,b_n\}$, such that for each $1\leq i\leq n$, there is an element of $\mathcal{F}$ containing both a_i,b_i.

23	$\mathcal{F}$ is a family of 3-subsets of set X. Every two distinct elements of X are exactly in k elements of $\mathcal{F}$. It is known that there is a partition of $\mathcal{F}$ to sets X_1, X_2 such that each element of $\mathcal{F}$ has non-empty intersection with both X_1, X_2. Prove that $\|X\| \leq 4$.
24	In triangle ABC, points M, N lie on line AC such that $MA = AB$ and $NB = NC$. Also K, L lie on line BC such that $KA = KB$ and $LA = LC$. It is know that $KL = \frac{1}{2}BC$ and $MN = AC$. Find angles of triangle ABC.
25	Finitely many convex subsets of $\mathbb{R}^3$ are given, such that every three have non-empty intersection. Prove that there exists a line in $\mathbb{R}^3$ that intersects all of these subsets.
26	Finitely many points are given on the surface of a sphere, such that every four of them lie on the surface of open hemisphere. Prove that all points lie on the surface of an open hemisphere.
27	$\Delta_1, \ldots, \Delta_n$ are n concurrent segments (their lines concur) in the real plane. Prove that if for every three of them there is a line intersecting these three segments, then there is a line that intersects all of the segments.
28	Find all prime numbers p such that $p = m^2 + n^2$ and $p \mid m^3 + n^3 - 4$.
29	Incircle of triangle ABC touches AB, AC at P, Q. BI, CI intersect with PQ at K, L. Prove that circumcircle of ILK is tangent to incircle of ABC if and only if $AB + AC = 3BC$.
30	Find all polynomials $p \in \mathbb{Z}[x]$ such that $(m, n) = 1 \Rightarrow (p(m), p(n)) = 1$

National Math Olympiad (3rd Round) 2006

– Number Theory

1 n is a natural number. d is the least natural number that for each a that $gcd(a, n) = 1$ we know $a^d \equiv 1 \pmod{n}$. Prove that there exist a natural number that $\text{ord}_n b = d$

2 n is a natural number that $\frac{x^n+1}{x+1}$ is irreducible over $\mathbb{Z}_2[x]$. Consider a vector in $\mathbb{Z}_2^n$ that it has odd number of 1's (as entries) and at least one of its entries are 0. Prove that these vector and its translations are a basis for $\mathbb{Z}_2^n$

3 L is a fullrank lattice in $\mathbb{R}^2$ and K is a sub-lattice of L, that $\frac{A(K)}{A(L)} = m$. If m is the least number that for each $x \in L$, mx is in K. Prove that there exists a basis $\{x_1, x_2\}$ for L that $\{x_1, mx_2\}$ is a basis for K.

4 a, b, c, t are antural numbers and $k = c^t$ and $n = a^k - b^k$.
a) Prove that if k has at least q different prime divisors, then n has at least qt different prime divisors.
b)Prove that $\varphi(n)$ id divisible by $2^{\frac{t}{2}}$

5 For each n, define $L(n)$ to be the number of natural numbers $1 \leq a \leq n$ such that $n \mid a^n - 1$. If $p_1, p_2, \ldots, p_k$ are the prime divisors of n, define $T(n)$ as $(p_1 - 1)(p_2 - 1) \cdots (p_k - 1)$.
a) Prove that for each $n \in \mathbb{N}$ we have $n \mid L(n)T(n)$.
b) Prove that if $\gcd(n, T(n)) = 1$ then $\varphi(n) = L(n)T(n)$.

6 a) $P(x), Q(x)$ are polynomials with rational coefficients and $P(x)$ is not the zero polynomial. Prove that there exist a non-zero polynomial $Q(x) \in \mathbb{Q}[x]$ that

$$P(x)|Q(R(x))$$

b) P, Q are polynomial with integer coefficients and P is monic. Prove that there exist a monic polynomial $Q(x) \in \mathbb{Z}[x]$ that

$$P(x)|Q(R(x))$$

– Algebra

1 For positive numbers $x_1, x_2, \dots, x_s$, we know that $\prod_{i=1}^{s} x_k = 1$. Prove that for each $m \geq n$

$$\sum_{k=1}^{s} x_k^m \geq \sum_{k=1}^{s} x_k^n$$

2 Find all real polynomials that

$$p(x + p(x)) = p(x) + p(p(x))$$

3 Find all real x, y, z that

$$\begin{cases} x + y + zx = \frac{1}{2} \\ y + z + xy = \frac{1}{2} \\ z + x + yz = \frac{1}{2} \end{cases}$$

4 $p(x)$ is a real polynomial that for each $x \geq 0$, $p(x) \geq 0$. Prove that there are real polynomials $A(x), B(x)$ that $p(x) = A(x)^2 + xB(x)^2$

5 Find the biggest real number k such that for each right-angled triangle with sides a, b, c, we have

$$a^3 + b^3 + c^3 \geq k\left(a + b + c\right)^3.$$

6 P, Q, R are non-zero polynomials that for each $z \in \mathbb{C}$, $P(z)Q(\bar{z}) = R(z)$.
a) If $P, Q, R \in \mathbb{R}[x]$, prove that Q is constant polynomial.
b) Is the above statement correct for $P, Q, R \in \mathbb{C}[x]$?

– Linear Algebra

1 Suppose that $A \in \mathcal{M}_n(\mathbb{R})$ with $\text{Rank}(A) = k$. Prove that A is sum of k matrices $X_1, \dots, X_k$ with $\text{Rank}(X_i) = 1$.

2 $f:\mathbb{R}^{n}\longrightarrow\mathbb{R}^{m}$ is a non-zero linear map. Prove that there is a base $\{v_{1},\dots,v_{n}m\}$ for $\mathbb{R}^{n}$ that the set $\{f(v_{1}),\dots,f(v_{n})\}$ is linearly independent, after ommitting Repetitive elements.

3 Suppose (u,v) is an inner product on $\mathbb{R}^{n}$ and $f:\mathbb{R}^{n}\longrightarrow\mathbb{R}^{n}$ is an isometry, that $f(0)=0$.
1) Prove that for each u,v we have $(u,v)=(f(u),f(v)$
2) Prove that f is linear.

4 $f:\mathbb{R}^{n}\longrightarrow\mathbb{R}^{n}$ is a bijective map, that Image of every $n-1$-dimensional affine space is a $n-1$-dimensional affine space.
1) Prove that Image of every line is a line.
2) Prove that f is an affine map. (i.e. $f=goh$ that g is a translation and h is a linear map.)

– Geometry

1 Prove that in triangle ABC, radical center of its excircles lies on line GI, which G is Centroid of triangle ABC, and I is the incenter.

2 ABC is a triangle and R,Q,P are midpoints of AB,AC,BC. Line AP intersects RQ in E and circumcircle of ABC in F. T,S are on RP,PQ such that $ES\perp PQ,ET\perp RP$. F' is on circumcircle of ABC that FF' is diameter. The point of intersection of AF' and BC is E'. S',T' are on AB,AC that $E'S'\perp AB,E'T'\perp AC$. Prove that TS and $T'S'$ are perpendicular.

3 In triangle ABC, if L,M,N are midpoints of AB,AC,BC. And H is orthogonal center of triangle ABC, then prove that

$$LH^{2}+MH^{2}+NH^{2}\leq\frac{1}{4}(AB^{2}+AC^{2}+BC^{2})$$

4 Circle $\Omega(O,R)$ and its chord AB is given. Suppose C is midpoint of arc AB. X is an arbitrary point on the cirlce. Perpendicular from B to CX intersects circle again in D. Perpendicular from C to DX intersects circle again in E. We draw three lines $\ell_{1},\ell_{2},\ell_{3}$ from A,B,E parralell to OX,OD,OC. Prove that these lines are concurrent and find locus of concurrncy point.

5 M is midpoint of side BC of triangle ABC, and I is incenter of triangle ABC, and T is midpoint of arc BC, that does not contain A. Prove that

$$\cos B+\cos C=1\Longleftrightarrow MI=MT$$

–	Combinatorics
1	Let A be a family of subsets of $\{1, 2, \ldots, n\}$ such that no member of A is contained in another. Sperners Theorem states that $\|A\| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$. Find all the families for which the equality holds.
2	Let B be a subset of $\mathbb{Z}_3^n$ with the property that for every two distinct members $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ of B there exist $1 \leq i \leq n$ such that $a_i \equiv b_i + 1 \pmod{3}$. Prove that $\|B\| \leq 2^n$.
3	Let C be a (probably infinite) family of subsets of $\mathbb{N}$ such that for every chain $C_1 \subset C_2 \subset \ldots$ of members of C, there is a member of C containing all of them. Show that there is a member of C such that no other member of C contains it!
4	Let D be a family of s-element subsets of $\{1 \ldots, n\}$ such that every k members of D have non-empty intersection. Denote by $D(n, s, k)$ the maximum cardinality of such a family. a) Find $D(n, s, 4)$. b) Find $D(n, s, 3)$.
5	Let E be a family of subsets of $\{1, 2, \ldots, n\}$ with the property that for each $A \subset \{1, 2, \ldots, n\}$ there exist $B \in F$ such that $\frac{n-d}{2} \leq \|A \triangle B\| \leq \frac{n+d}{2}$. (where $A \triangle B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference). Denote by $f(n, d)$ the minimum cardinality of such a family. a) Prove that if n is even then $f(n, 0) \leq n$. b) Prove that if $n - d$ is even then $f(n, d) \leq \lceil \frac{n}{d+1} \rceil$. c) Prove that if n is even then $f(n, 0) = n$
6	The National Foundation of Happiness (NFoH) wants to estimate the happiness of people of country. NFoH selected n random persons, and on every morning asked from each of them whether she is happy or not. On any two distinct days, exactly half of the persons gave the same answer. Show that after k days, there were at most $n - \frac{n}{k}$ persons whose yes answers equals their no answers.
–	Final Exam
1	A regular polyhedron is a polyhedron that is convex and all of its faces are regular polygons. We call a regular polhedron a "*Choombam*" iff none of its

faces are triangles.
a) prove that each choombam can be inscribed in a sphere.
b) Prove that faces of each choombam are polygons of at most 3 kinds. (i.e. there is a set $\{m, n, q\}$ that each face of a choombam is n-gon or m-gon or q-gon.)
c) Prove that there is only one choombam that its faces are pentagon and hexagon. (Soccer ball)
`http://aycu08.webshots.com/image/5367/2001362702285797426_rs.jpg`
d) For $n > 3$, a prism that its faces are 2 regular n-gons and n squares, is a choombam. Prove that except these choombams there are finitely many choombams.

---

2 A liquid is moving in an infinite pipe. For each molecule if it is at point with coordinate x then after t seconds it will be at a point of $p(t, x)$. Prove that if $p(t, x)$ is a polynomial of t, x then speed of all molecules are equal and constant.

---

3 For $A \subset \mathbb{Z}$ and $a, b \in \mathbb{Z}$. We define $aA + b := \{ax + b | x \in A\}$. If $a \neq 0$ then we calll $aA + b$ and A to similar sets. In this question the Cantor set C is the number of non-negative integers that in their base-3 representation there is no 1 digit. You see

$$C = (3C)\dot{\cup}(3C + 2) \qquad (1)$$

(i.e. C is partitioned to sets $3C$ and $3C + 2$). We give another example $C = (3C)\dot{\cup}(9C + 6)\dot{\cup}(3C + 2)$.

A representation of C is a partition of C to some similiar sets. i.e.

$$C = \bigcup_{i=1}^{n} C_i \qquad (2)$$

and $C_i = a_i C + b_i$ are similar to C.
We call a representation of C a primitive representation iff union of some of C_i is not a set similar and not equal to C.
Consider a primitive representation of Cantor set. Prove that
a) $a_i > 1$.
b) a_i are powers of 3.
c) $a_i > b_i$
d) (1) is the only primitive representation of C.

---

4 The image shown below is a cross with length 2. If length of a cross of length k it is called a k-cross. (Each k-cross ahs $6k + 1$ squares.)

http://aycu08.webshots.com/image/4127/2003057947601864020_th.jpg
a) Prove that space can be tiled with 1-crosses.
b) Prove that space can be tiled with 2-crosses.
c) Prove that for $k \geq 5$ space can not be tiled with k-crosses.

---

5 A calculating ruler is a ruler for doing algebric calculations. This ruler has three arms, two of them are sationary and one can move freely right and left. Each of arms is gradient. Gradation of each arm depends on the algebric operation ruler does. For eaxample the ruler below is designed for multiplying two numbers. Gradations are logarithmic.
http://aycu05.webshots.com/image/5604/2000468517162383885_rs.jpg
For working with ruler, (e.g for calculating $x.y$) we must move the middle arm that the arrow at the beginning of its gradation locate above the x in the lower arm. We find y in the middle arm, and we will read the number on the upper arm. The number written on the ruler is the answer.
1) Design a ruler for calculating x^y. Grade first arm (x) and (y) from 1 to 10.
2) Find all rulers that do the multiplication in the interval $[1, 10]$.
3) Prove that there is not a ruler for calculating $x^2 + xy + y^2$, that its first and second arm are grade from 0 to 10.

---

6 Assume that C is a convex subset of $\mathbb{R}^d$. Suppose that $C_1, C_2, \dots, C_n$ are translations of C that $C_i \cap C \neq \emptyset$ but $C_i \cap C_j = \emptyset$. Prove that

$$n \leq 3^d - 1$$

Prove that $3^d - 1$ is the best bound.
P.S. In the exam problem was given for $n = 3$.

---

7 We have finite number of distinct shapes in plane. A "*convex Kearting*" of these shapes is covering plane with convex sets, that each set consists exactly one of the shapes, and sets intersect at most in border.
http://aycu30.webshots.com/image/4109/2003791140004582959_th.jpg
In which case Convex kearting is possible?
1) Finite distinct points
2) Finite distinct segments
3) Finite distinct circles

---

8 We mean a traingle in $\mathbb{Q}^n$, 3 points that are not collinear in $\mathbb{Q}^n$

a) Suppose that ABC is triangle in $\mathbb{Q}^n$. Prove that there is a triangle $A'B'C'$ in $\mathbb{Q}^5$ that $\angle B'A'C' = \angle BAC$.

b) Find a natural m that for each traingle that can be embedded in $\mathbb{Q}^n$ it can be embedded in $\mathbb{Q}^m$.
c) Find a triangle that can be embedded in $\mathbb{Q}^n$ and no triangle similar to it can be embedded in $\mathbb{Q}^3$.
d) Find a natural m' that for each traingle that can be embedded in $\mathbb{Q}^n$ then there is a triangle similar to it, that can be embedded in $\mathbb{Q}^m$.
You must prove the problem for $m = 9$ and $m' = 6$ to get complete mark. (Better results leads to additional mark.)

National Math Olympiad (3rd Round) 2007

– Algebra&Analysis

– August 26th

1 Let a, b be two complex numbers. Prove that roots of $z^4 + az^2 + b$ form a rhombus with origin as center, if and only if $\frac{a^2}{b}$ is a non-positive real number.

2 a, b, c are three different positive real numbers. Prove that:

$$\left|\frac{a+b}{a-b} + \frac{b+c}{b-c} + \frac{c+a}{c-a}\right| > 1$$

3 Find the largest real T such that for each non-negative real numbers a, b, c, d, e such that $a + b = c + d + e$:

$$\sqrt{a^2 + b^2 + c^2 + d^2 + e^2} \geq T(\sqrt{a} + \sqrt{b} + \sqrt{c} + \sqrt{d} + \sqrt{e})^2$$

4 a) Let $n_1, n_2, \dots$ be a sequence of natural number such that $n_i \geq 2$ and $\epsilon_1, \epsilon_2, \dots$ be a sequence such that $\epsilon_i \in \{1, 2\}$. Prove that the sequence:

$$\sqrt[n_1]{\epsilon_1 + \sqrt[n_2]{\epsilon_2 + \dots + \sqrt[n_k]{\epsilon_k}}}$$

is convergent and its limit is in $(1, 2]$. Define $\sqrt[n_1]{\epsilon_1 + \sqrt[n_2]{\epsilon_2 + \dots}}$ to be this limit.
b) Prove that for each $x \in (1, 2]$ there exist sequences $n_1, n_2, \dots \in \mathbb{N}$ and $n_i \geq 2$ and $\epsilon_1, \epsilon_2, \dots$, such that $n_i \geq 2$ and $\epsilon_i \in \{1, 2\}$, and $x = \sqrt[n_1]{\epsilon_1 + \sqrt[n_2]{\epsilon_2 + \dots}}$

5 Prove that for two non-zero polynomials $f(x, y), g(x, y)$ with real coefficients the system:

$$\begin{cases} f(x, y) = 0 \\ g(x, y) = 0 \end{cases}$$

has finitely many solutions in $\mathbb{C}^2$ if and only if $f(x, y)$ and $g(x, y)$ are coprime.

–	Geometry
–	August 27th
1	Let ABC, l and P be arbitrary triangle, line and point. A', B', C' are reflections of A, B, C in point P. A'' is a point on $B'C'$ such that $AA'' \parallel l$. B'', C'' are defined similarly. Prove that A'', B'', C'' are collinear.
2	a) Let ABC be a triangle, and O be its circumcenter. BO and CO intersect with AC, AB at B', C'. $B'C'$ intersects the circumcircle at two points P, Q. Prove that $AP = AQ$ if and only if ABC is isosceles. b) Prove the same statement if O is replaced by I, the incenter.
3	Let I be incenter of triangle ABC, M be midpoint of side BC, and T be the intersection point of IM with incircle, in such a way that I is between M and T. Prove that $\angle BIM - \angle CIM = \frac{3}{2}(\angle B - \angle C)$, if and only if $AT \perp BC$.
4	Let ABC be a triangle, and D be a point where incircle touches side BC. M is midpoint of BC, and K is a point on BC such that $AK \perp BC$. Let D' be a point on BC such that $\frac{D'M}{D'K} = \frac{DM}{DK}$. Define ω_a to be circle with diameter DD'. We define ω_B, ω_C similarly. Prove that every two of these circles are tangent.
5	Let ABC be a triangle. Squares AB_cB_aC, CA_bA_cB and BC_aC_bA are outside the triangle. Square $B_cB'_cB'_aB_a$ with center P is outside square AB_cB_aC. Prove that BP, C_aB_a and A_cB_c are concurrent.
–	Number Theory
–	August 28th
1	Let n be a natural number, such that $(n, 2(2^{1386}-1)) = 1$. Let $\{a_1, a_2, \ldots, a_{\varphi(n)}\}$ be a reduced residue system for n. Prove that: $$n \mid a_1^{1386} + a_2^{1386} + \ldots + a_{\varphi(n)}^{1386}$$
2	Let m, n be two integers such that $\varphi(m) = \varphi(n) = c$. Prove that there exist natural numbers $b_1, b_2, \ldots, b_c$ such that $\{b_1, b_2, \ldots, b_c\}$ is a reduced residue system with both m and n.

3 Let n be a natural number, and $n=2^{2007}k+1$, such that k is an odd number. Prove that
$$n \not| 2^{n-1}+1$$

4 Find all integer solutions of
$$x^4+y^2=z^4$$

5 A hyper-primitive root is a k-tuple $(a_1,a_2,\dots,a_k)$ and $(m_1,m_2,\dots,m_k)$ with the following property:
For each $a\in\mathbb{N}$, that $(a,m)=1$, has a unique representation in the following form:
$$a\equiv a_1^{\alpha_1}a_2^{\alpha_2}\dots a_k^{\alpha_k}\pmod{m}\qquad 1\leq\alpha_i\leq m_i$$

Prove that for each m we have a hyper-primitive root.

6 Something related to this problem (`http://www.mathlinks.ro/Forum/viewtopic.php?p=845756#845756`):
Prove that for a set $S\subset\mathbb{N}$, there exists a sequence $\{a_i\}_{i=0}^{\infty}$ in S such that for each n, $\sum_{i=0}^{n}a_ix^i$ is irreducible in $\mathbb{Z}[x]$ if and only if $|S|\geq 2$.
By Omid Hatami

– Final Exam

– September 4th

1 Consider two polygons P and Q. We want to cut P into some smaller polygons and put them together in such a way to obtain Q. We can translate the pieces but we can not rotate them or reflect them. We call P,Q equivalent if and only if we can obtain Q from P(which is obviously an equivalence relation).
`http://i3.tinypic.com/4lrb43k.png`
a) Let P,Q be two rectangles with the same area(their sides are not necessarily parallel). Prove that P and Q are equivalent.
b) Prove that if two triangles are not translation of each other, they are not equivalent.
c) Find a necessary and sufficient condition for polygons P,Q to be equivalent.

2 We call the mapping $\Delta : \mathbb{Z}\backslash\{0\} \longrightarrow \mathbb{N}$, a degree mapping if and only if for each $a, b \in \mathbb{Z}$ such that $b \neq 0$ and $b \not| a$ there exist integers r, s such that $a = br + s$, and $\Delta(s) < \Delta(b)$.
a) Prove that the following mapping is a degree mapping:

$$\delta(n) = \text{Number of digits in the binary representation of } n$$

b) Prove that there exist a degree mapping Δ_0 such that for each degree mapping Δ and for each $n \neq 0$, $\Delta_0(n) \leq \Delta(n)$.
c) Prove that $\delta = \Delta_0$
`http://i16.tinypic.com/4qntmd0.png`

---

3 We call a set A a good set if it has the following properties:
1. A consists circles in plane.
2. No two element of A intersect.
Let A, B be two good sets. We say A, B are equivalent if we can reach from A to B by moving circles in A, making them bigger or smaller in such a way that during these operations each circle does not intersect with other circles.
Let a_n be the number of inequivalent good subsets with n elements. For example $a_1 = 1, a_2 = 2, a_3 = 4, a_4 = 9$.
`http://i5.tinypic.com/4r0x81v.png`
If there exist a, b such that $Aa^n \leq a_n \leq Bb^n$, we say growth ratio of a_n is larger than a and is smaller than b.
a) Prove that growth ratio of a_n is larger than 2 and is smaller than 4.
b) Find better bounds for upper and lower growth ratio of a_n.

---

4 In the following triangular lattice distance of two vertices is length of the shortest path between them. Let $A_1, A_2, \ldots, A_n$ be constant vertices of the lattice. We want to find a vertex in the lattice whose sum of distances from vertices is minimum. We start from an arbitrary vertex. At each step we check all six neighbors and if sum of distances from vertices of one of the neighbors is less than sum of distances from vertices at the moment we go to that neighbor. If we have more than one choice we choose arbitrarily, as seen in the attached picture.

Obviusly the algorithm finishes
a) Prove that when we can not make any move we have reached to the problem's answer.
b) Does this algorithm reach to answer for each connected graph?

---

5 Look at these fractions. At firs step we have $\frac{0}{1}$ and $\frac{1}{0}$, and at each step we write

$\frac{a+b}{c+d}$ between $\frac{a}{b}$ and $\frac{c}{d}$, and we do this forever

$$\begin{array}{ccccccccc}
\frac{0}{1} & & & & & & & & \frac{1}{0} \\
\frac{0}{1} & & & & \frac{1}{1} & & & & \frac{1}{0} \\
\frac{0}{1} & & \frac{1}{2} & & \frac{1}{1} & & \frac{2}{1} & & \frac{1}{0} \\
\frac{0}{1} & \frac{1}{3} & \frac{1}{2} & \frac{2}{3} & \frac{1}{1} & \frac{3}{2} & \frac{2}{1} & \frac{3}{1} & \frac{1}{0} \\
 & & & & \dots & & & &
\end{array}$$

a) Prove that each of these fractions is irreducible.
b) In the plane we have put infinitely many circles of diameter 1, over each integer on the real line, one circle. The inductively we put circles that each circle is tangent to two adjacent circles and real line, and we do this forever. Prove that points of tangency of these circles are exactly all the numbers in part a(except $\frac{1}{0}$).
`http://i2.tinypic.com/4m8tmbq.png`
c) Prove that in these two parts all of positive rational numbers appear.
If you don't understand the numbers, look at here (`http://upload.wikimedia.org/wikipedia/commons/2/21/Arabic_numerals-en.svg`).

---

6 Scientist have succeeded to find new numbers between real numbers with strong microscopes. Now real numbers are extended in a new larger system we have an order on it (which if induces normal order on $\mathbb{R}$), and also 4 operations addition, multiplication,... and these operation have all properties the same as $\mathbb{R}$.
`http://i14.tinypic.com/4tk6mnr.png`
a) Prove that in this larger system there is a number which is smaller than each positive integer and is larger than zero.
b) Prove that none of these numbers are root of a polynomial in $\mathbb{R}[x]$.

---

7 A ring is the area between two circles with the same center, and width of a ring is the difference between the radii of two circles.
`http://i18.tinypic.com/6cdmvi8.png`
a) Can we put uncountable disjoint rings of width 1(not necessarily same) in the space such that each two of them can not be separated.
`http://i19.tinypic.com/4qgx30j.png`
b) What's the answer if 1 is replaced with 0?

---

8 In this question you must make all numbers of a clock, each with using 2, exactly 3 times and Mathematical symbols. You are not allowed to use English

alphabets and words like sin or lim or a, b and no other digits.
http://i2.tinypic.com/5x73dza.png

National Math Olympiad (3rd Round) 2008

– Algebra

1 Suppose that $f(x) \in \mathbb{Z}[x]$ be an irreducible polynomial. It is known that f has a root of norm larger than $\frac{3}{2}$. Prove that if α is a root of f then $f(\alpha^3+1) \neq 0$.

2 Find the smallest real K such that for each $x, y, z \in \mathbb{R}^+$:

$$x\sqrt{y} + y\sqrt{z} + z\sqrt{x} \leq K\sqrt{(x+y)(y+z)(z+x)}$$

3 Let (b_0, b_1, b_2, b_3) be a permutation of the set $\{54, 72, 36, 108\}$. Prove that $x^5 + b_3x^3 + b_2x^2 + b_1x + b_0$ is irreducible in $\mathbb{Z}[x]$.

4 Let $x, y, z \in \mathbb{R}^+$ and $x + y + z = 3$. Prove that:

$$\frac{x^3}{y^3+8} + \frac{y^3}{z^3+8} + \frac{z^3}{x^3+8} \geq \frac{1}{9} + \frac{2}{27}(xy + xz + yz)$$

5 Prove that the following polynomial is irreducible in $\mathbb{Z}[x, y]$:

$$x^{200}y^5 + x^{51}y^{100} + x^{106} - 4x^{100}y^5 + x^{100} - 2y^{100} - 2x^6 + 4y^5 - 2$$

– Number Theory

1 Let $k > 1$ be an integer. Prove that there exists infinitely many natural numbers such as n such that:

$$n | 1^n + 2^n + \ldots + k^n$$

2 Prove that there exists infinitely many primes p such that:

$$13 | p^3 + 1$$

3 Let P be a regular polygon. A regular sub-polygon of P is a subset of vertices of P with at least two vertices such that divides the circumcircle to equal arcs. Prove that there is a subset of vertices of P such that its intersection with each regular sub-polygon has even number of vertices.

---

4 Let u be an odd number. Prove that $\frac{3^{3u}-1}{3^u-1}$ can be written as sum of two squares.

---

5 Find all polynomials $f \in \mathbb{Z}[x]$ such that for each $a, b, x \in \mathbb{N}$

$$a+b+c|f(a)+f(b)+f(c)$$

– Combinatorics

---

1 Prove that the number of pairs (α, S) of a permutation α of $\{1, 2, \dots, n\}$ and a subset S of $\{1, 2, \dots, n\}$ such that

$$\forall x \in S : \alpha(x) \notin S$$

is equal to $n!F_{n+1}$ in which F_n is the Fibonacci sequence such that $F_1 = F_2 = 1$

---

2 Prove that the number permutations α of $\{1, 2, \dots, n\}$ s.t. there does not exist $i < j < n$ s.t. $\alpha(i) < \alpha(j+1) < \alpha(j)$ is equal to the number of partitions of that set.

---

3 Prove that for each n:

$$\sum_{k=1}^{n} \binom{n+k-1}{2k-1} = F_{2n}$$

---

4 Let S be a sequence that:

$$\begin{cases} S_0 = 0 \\ S_1 = 1 \\ S_n = S_{n-1} + S_{n-2} + F_n \quad (n > 1) \end{cases}$$

such that F_n is Fibonacci sequence such that $F_1 = F_2 = 1$. Find S_n in terms of Fibonacci numbers.

---

5 n people decide to play a game. There are $n-1$ ropes and each of its two ends are in hand of one of the players, in such a way that ropes and players form a tree. (Each person can hold more than rope end.)

At each step a player gives one of the rope ends he is holding to another player. The goal is to make a path of length $n-1$ at the end.

But the game regulations change before game starts. Everybody has to give one of his rope ends only two one of his neighbors. Let a and b be minimum steps for reaching to goal in these two games. Prove that $a=b$ if and only if by removing all players with one rope end (leaves of the tree) the remaining people are on a path. (the remaining graph is a path.)
http://i37.tinypic.com/2l9h1tv.png

– Complex numbers

1 Prove that for $n>0$ and $a\neq 0$ the polynomial $p(z)=az^{2n+1}+bz^{2n}+\bar{b}z+\bar{a}$ has a root on unit circle

2 Let $g,f:\mathbb{C}\longrightarrow\mathbb{C}$ be two continuous functions such that for each $z\neq 0$, $g(z)=f(\frac{1}{z})$. Prove that there is a $z\in\mathbb{C}$ such that $f(\frac{1}{z})=f(-\bar{z})$

3 For each $c\in\mathbb{C}$, let $f_c(z,0)=z$, and $f_c(z,n)=f_c(z,n-1)^2+c$ for $n\geq 1$.
a) Prove that if $|c|\leq\frac{1}{4}$ then there is a neighborhood U of origin such that for each $z\in U$ the sequence $f_c(z,n),n\in\mathbb{N}$ is bounded.
b) Prove that if $c>\frac{1}{4}$ is a real number there is a neighborhood U of origin such that for each $z\in U$ the sequence $f_c(z,n),n\in\mathbb{N}$ is unbounded.

– Geometry

1 Let ABC be a triangle with $BC>AC>AB$. Let A',B',C' be feet of perpendiculars from A,B,C to BC,AC,AB, such that $AA'=BB'=CC'=x$. Prove that:
a) If $ABC\sim A'B'C'$ then $x=2r$
b) Prove that if A',B' and C' are collinear, then $x=R+d$ or $x=R-d$.

(In this problem R is the radius of circumcircle, r is radius of incircle and $d=OI$)

2 Let l_a, l_b, l_c be three parallel lines passing through A, B, C respectively. Let l'_a be reflection of l_a into BC. l'_b and l'_c are defined similarly. Prove that l'_a, l'_b, l'_c are concurrent if and only if l_a is parallel to Euler line of triangle ABC.

3 Let $ABCD$ be a quadrilateral, and E be intersection points of AB, CD and AD, BC respectively. External bisectors of DAB and DCB intersect at P, external bisectors of ABC and ADC intersect at Q and external bisectors of AED and AFB intersect at R. Prove that P, Q, R are collinear.

4 Let ABC be an isosceles triangle with $AB = AC$, and D be midpoint of BC, and E be foot of altitude from C. Let H be orthocenter of ABC and N be midpoint of CE. AN intersects with circumcircle of triangle ABC at K. The tangent from C to circumcircle of ABC intersects with AD at F.
Suppose that radical axis of circumcircles of CHA and CKF is BC. Find $\angle BAC$.

5 Let D, E, F be tangency point of incircle of triangle ABC with sides BC, AC, AB. DE and DF intersect the line from A parallel to BC at K and L. Prove that the Euler line of triangle DKL passes through Feuerbach point of triangle ABC.

– Final Exam

1 Police want to arrest on the famous criminals of the country whose name is Kaiser. Kaiser is in one of the streets of a square shaped city with n vertical streets and n horizontal streets. In the following cases how many police officers are needed to arrest Kaiser?
`http://i38.tinypic.com/2i1icec_th.png` `http://i34.tinypic.com/28rk4s3_th.png`
a) Each police officer has the same speed as Kaiser and every police officer knows the location of Kaiser anytime.
b) Kaiser has an infinite speed (finite but with no bound) and police officers can only know where he is only when one of them see Kaiser.
Everybody in this problem (including police officers and Kaiser) move continuously and can stop or change his path.

2 Consider six arbitrary points in space. Every two points are joined by a segment. Prove that there are two triangles that can not be separated.
`http://i38.tinypic.com/35n615y.png`

3 a) Prove that there are two polynomials in $\mathbb{Z}[x]$ with at least one coefficient larger than 1387 such that coefficients of their product is in the set $\{-1, 0, 1\}$.

b) Does there exist a multiple of x^2-3x+1 such that all of its coefficient are in the set $\{-1,0,1\}$

---

4 =A subset S of $\mathbb{R}^2$ is called an algebraic set if and only if there is a polynomial $p(x,y)\in\mathbb{R}[x,y]$ such that

$$S=\{(x,y)\in\mathbb{R}^2|p(x,y)=0\}$$

Are the following subsets of plane an algebraic sets?
1. A square
`http://i36.tinypic.com/28uiaep.png`
2. A closed half-circle
`http://i37.tinypic.com/155m155.png`

---

5 a) Suppose that $RBR'B'$ is a convex quadrilateral such that vertices R and R' have red color and vertices B and B' have blue color. We put k arbitrary points of colors blue and red in the quadrilateral such that no four of these $k+4$ point (except probably $RBR'B'$) lie one a circle. Prove that exactly one of the following cases occur?
1. There is a path from R to R' such that distance of every point on this path from one of red points is less than its distance from all blue points.
2. There is a path from B to B' such that distance of every point on this path from one of blue points is less than its distance from all red points.
We call these two paths the blue path and the red path respectively.

Let n be a natural number. Two people play the following game. At each step one player puts a point in quadrilateral satisfying the above conditions. First player only puts red point and second player only puts blue points. Game finishes when every player has put n points on the plane. First player's goal is to make a red path from R to R' and the second player's goal is to make a blue path from B to B'.
b) Prove that if $RBR'B'$ is rectangle then for each n the second player wins.
c) Try to specify the winner for other quadrilaterals.

---

6 There are five research labs on Mars. Is it always possible to divide Mars to five connected congruent regions such that each region contains exactly on research lab.
`http://i37.tinypic.com/f2iq8g.png`

---

7 A graph is called a *self-intersecting* graph if it is isomorphic to a graph whose every edge is a segment and every two edges intersect. Notice that no edge

contains a vertex except its two endings.
a) Find all n's for which the cycle of length n is self-intersecting.
b) Prove that in a self-intersecting graph $|E(G)| \leq |V(G)|$.
c) Find all self-intersecting graphs.
http://i35.tinypic.com/x43s5u.png

8 In an old script found in ruins of Perspolis is written:

```
This script has been finished in a year whose 13th power is
258145266804692077858261512663
You should know that if you are skilled in Arithmetics you will
    know the year this script is finished easily.
```

Find the year the script is finished. Give a reason for your answer.

National Math Olympiad (3rd Round) 2009

1 Suppose $n > 2$ and let $A_1, \dots, A_n$ be points on the plane such that no three are collinear.
(a) Suppose $M_1, \dots, M_n$ be points on segments $A_1A_2, A_2A_3, \dots, A_nA_1$ respectively. Prove that if $B_1, \dots, B_n$ are points in triangles $M_2A_2M_1, M_3A_3M_2, \dots, M_1A_1M_n$ respectively then

$$|B_1B_2| + |B_2B_3| + \dots + |B_nB_1| \leq |A_1A_2| + |A_2A_3| + \dots + |A_nA_1|$$

Where $|XY|$ means the length of line segment between X and Y.

(b) If X, Y and Z are three points on the plane then by H_{XYZ} we mean the half-plane that it's boundary is the exterior angle bisector of angle $X\hat{Y}Z$ and doesn't contain X and Z ,having Y crossed out.
Prove that if $C_1, \dots, C_n$ are points in $H_{A_nA_1A_2}, H_{A_1A_2A_3}, \dots, H_{A_{n-1}A_nA_1}$ then

$$|A_1A_2| + |A_2A_3| + \dots + |A_nA_1| \leq |C_1C_2| + |C_2C_3| + \dots + |C_nC_1|$$

Time allowed for this problem was 2 hours.

---

2 Permutation π of $\{1, \dots, n\}$ is called **stable** if the set $\{\pi(k) - k | k = 1, \dots, n\}$ is consisted of exactly two different elements.
Prove that the number of stable permutation of $\{1, \dots, n\}$ equals to $\sigma(n) - \tau(n)$ in which $\sigma(n)$ is the sum of positive divisors of n and $\tau(n)$ is the number of positive divisors of n.

Time allowed for this problem was 75 minutes.

---

3 An arbitary triangle is partitioned to some triangles homothetic with itself. The ratio of homothety of the triangles can be positive or negative.
Prove that sum of all homothety ratios equals to 1.

Time allowed for this problem was 45 minutes.

---

4 Does there exists two functions $f, g : \mathbb{R} \to \mathbb{R}$ such that: $\forall x \neq y : |f(x) - f(y)| + |g(x) - g(y)| > 1$

Time allowed for this problem was 75 minutes.

---

5 A ball is placed on a plane and a point on the ball is marked.
Our goal is to roll the ball on a polygon in the plane in a way that it comes

back to where it started and the marked point comes to the top of it. Note that We are not allowed to rotate without moving, but only rolling.
Prove that it is possible.

Time allowed for this problem was 90 minutes.

---

6 Let z be a complex non-zero number such that $Re(z), Im(z) \in \mathbb{Z}$.
Prove that z is uniquely representable as $a_0+a_1(1+i)+a_2(1+i)^2+\cdots+a_n(1+i)^n$ where $n \geq 0$ and $a_j \in \{0,1\}$ and $a_n = 1$.

Time allowed for this problem was 1 hour.

---

7 A sphere is inscribed in polyhedral P. The faces of P are coloured with black and white in a way that no two black faces share an edge.
Prove that the sum of surface of black faces is less than or equal to the sum of the surface of the white faces.

Time allowed for this problem was 1 hour.

---

8 Sone of vertices of the infinite grid $\mathbb{Z}^2$ are missing. Let's take the remainder as a graph. Connect two edges of the graph if they are the same in one component and their other components have a difference equal to one. Call every connected component of this graph a **branch**.
Suppose that for every natural n the number of missing vertices in the $(2n+1)\times(2n+1)$ square centered by the origin is less than $\frac{n}{2}$.
Prove that among the branches of the graph, exactly one has an infinite number of vertices.

Time allowed for this problem was 90 minutes.

National Math Olympiad (3rd Round) 2010

Day 1

1 suppose that polynomial $p(x)=x^{2010}\pm x^{2009}\pm ...\pm x\pm 1$ does not have a real root. what is the maximum number of coefficients to be -1?(14 points)

2 a,b,c are positive real numbers. prove the following inequality:
$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{(a+b+c)^2}\ge \frac{7}{25}(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{a+b+c})^2$
(20 points)

3 prove that for each natural number n there exist a polynomial with degree $2n+1$ with coefficients in $\mathbb{Q}[x]$ such that it has exactly 2 complex zeros and it's irreducible in $\mathbb{Q}[x]$.(20 points)

4 For each polynomial $p(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_1x+a_0$ we define it's derivative as this and we show it by $p'(x)$:

$$p'(x)=na_nx^{n-1}+(n-1)a_{n-1}x^{n-2}+...+2a_2x+a_1$$

a) For each two polynomials $p(x)$ and $q(x)$ prove that:(3 points)

$$(p(x)q(x))'=p'(x)q(x)+p(x)q'(x)$$

b) Suppose that $p(x)$ is a polynomial with degree n and $x_1,x_2,...,x_n$ are it's zeros. prove that:(3 points)

$$\frac{p'(x)}{p(x)}=\sum_{i=1}^{n}\frac{1}{x-x_i}$$

c) $p(x)$ is a monic polynomial with degree n and $z_1,z_2,...,z_n$ are it's zeros such that:

$$|z_1|=1,\quad \forall i\in\{2,..,n\}:|z_i|\le 1$$

Prove that $p'(x)$ has at least one zero in the disc with length one with the center z_1 in complex plane. (disc with length one with the center z_1 in complex plane: $D=\{z\in \mathbb{C}:|z-z_1|\le 1\}$)(20 points)

5 x,y,z are positive real numbers such that $xy+yz+zx=1$. prove that: $3-\sqrt{3}+\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\geq(x+y+z)^2$
(20 points)

the exam time was 6 hours.

Day 2

1 suppose that $a=3^{100}$ and $b=5454$. how many zs in $[1,3^{99})$ exist such that for every c that $gcd(c,3)=1$, two equations $x^z\equiv c$ and $x^b\equiv c \pmod{a}$ have the same number of answers?($\frac{100}{6}$ points)

2 R is a ring such that $xy=yx$ for every $x,y\in R$ and if $ab=0$ then $a=0$ or $b=0$. if for every Ideal $I\subset R$ there exist $x_1,x_2,..,x_n$ in R (n is not constant) such that $I=(x_1,x_2,...,x_n)$, prove that every element in R that is not 0 and it's not a unit, is the product of finite irreducible elements.($\frac{100}{6}$ points)

3 If p is a prime number, what is the product of elements like g such that $1\leq g\leq p^2$ and g is a primitive root modulo p but it's not a primitive root modulo p^2, modulo p^2?($\frac{100}{6}$ points)

4 sppose that $\sigma_k:\mathbb{N}\longrightarrow\mathbb{R}$ is a function such that $\sigma_k(n)=\sum_{d|n}d^k$. $\rho_k:\mathbb{N}\longrightarrow\mathbb{R}$ is a function such that $\rho_k*\sigma_k=\delta$. find a formula for ρ_k.($\frac{100}{6}$ points)

5 prove that if p is a prime number such that $p=12k+\{2,3,5,7,8,11\}(k\in\mathbb{N}\cup\{0\})$, there exist a field with p^2 elements.($\frac{100}{6}$ points)

6 g and n are natural numbers such that $gcd(g^2-g,n)=1$ and $A=\{g^i|i\in\mathbb{N}\}$ and $B=\{x\equiv(n)|x\in A\}$(by $x\equiv(n)$ we mean a number from the set $\{0,1,...,n-1\}$ which is congruent with x modulo n). if for $0\leq i\leq g-1$ $a_i=|[\frac{ni}{g},\frac{n(i+1)}{g})\cap B|$ prove that $g-1|\sum_{i=0}^{g-1}ia_i$.(the symbol $|$ $|$ means the number of elements of the set)($\frac{100}{6}$ points)

the exam time was 4 hours

Day 3

1 1. In a triangle ABC, O is the circumcenter and I is the incenter. X is the reflection of I to O. A_1 is foot of the perpendicular from X to BC. B_1 and C_1 are defined similarly. prove that AA_1,BB_1 and CC_1 are concurrent.(12 points)

2 in a quadrilateral $ABCD$, E and F are on BC and AD respectively such that the area of triangles AED and BCF is $\frac{4}{7}$ of the area of $ABCD$. R is the intersection point of digonals of $ABCD$. $\frac{AR}{RC}=\frac{3}{5}$ and $\frac{BR}{RD}=\frac{5}{6}$.
a) in what ratio does EF cut the digonals?(13 points)
b) find $\frac{AF}{FD}$.(5 points)

---

3 in a quadrilateral $ABCD$ digonals are perpendicular to each other. let S be the intersection of digonals. K,L,M and N are reflections of S to AB,BC,CD and DA. BN cuts the circumcircle of SKN in E and BM cuts the circumcircle of SLM in F. prove that $EFLK$ is concyclic.(20 points)

---

4 in a triangle ABC, I is the incenter. BI and CI cut the circumcircle of ABC at E and F respectively. M is the midpoint of EF. C is a circle with diameter EF. IM cuts C at two points L and K and the arc BC of circumcircle of ABC (not containing A) at D. prove that $\frac{DL}{IL}=\frac{DK}{IK}$.(25 points)

---

5 In a triangle ABC, I is the incenter. D is the reflection of A to I. the incircle is tangent to BC at point E. DE cuts IG at P (G is centroid). M is the midpoint of BC. prove that
a) $AP||DM$.(15 points)
b) $AP=2DM$. (10 points)

---

6 In a triangle ABC, $\angle C=45$. AD is the altitude of the triangle. X is on AD such that $\angle XBC=90-\angle B$ (X is in the triangle). AD and CX cut the circumcircle of ABC in M and N respectively. if tangent to circumcircle of ABC at M cuts AN at P, prove that P,B and O are collinear.(25 points)

the exam time was 4 hours and 30 minutes.

Day 4

---

1 suppose that $\mathcal{F}\subseteq X^{(k)}$ and $|X|=n$. we know that for every three distinct elements of $\mathcal{F}$ like A,B,C, at most one of $A\cap B$,$B\cap C$ and $C\cap A$ is ϕ. for $k\le \frac{n}{2}$ prove that:
a) $|\mathcal{F}|\le max(1,4-\frac{n}{k})\times \binom{n-1}{k-1}$.(15 points)
b) find all cases of equality in a) for $k\le \frac{n}{3}$.(5 points)

---

2 suppose that $\mathcal{F}\subseteq \bigcup_{j=k+1}^{n}X^{(j)}$ and $|X|=n$. we know that $\mathcal{F}$ is a sperner family and it's also H_k. prove that: $\sum_{B\in\mathcal{F}}\frac{1}{\binom{n-1}{|B|-1}}\le 1$

(15 points)

3 suppose that $\mathcal{F} \subseteq p(X)$ and $|X| = n$. we know that for every $A_i, A_j \in \mathcal{F}$ that $A_i \supseteq A_j$ we have $3 \leq |A_i| - |A_j|$. prove that: $|\mathcal{F}| \leq \lfloor \frac{2^n}{3} + \frac{1}{2}\binom{n}{\lfloor \frac{n}{2} \rfloor} \rfloor$

(20 points)

4 suppose that $\mathcal{F} \subseteq X^{(K)}$ and $|X| = n$. we know that for every three distinct elements of $\mathcal{F}$ like A, B and C we have $A \cap B \not\subset C$.

a)(10 points) Prove that :

$$|\mathcal{F}| \leq \binom{k}{\lfloor \frac{k}{2} \rfloor} + 1$$

b)(15 points) if elements of $\mathcal{F}$ do not necessarily have k elements, with the above conditions show that:

$$|\mathcal{F}| \leq \binom{n}{\lceil \frac{n-2}{3} \rceil} + 2$$

5 suppose that $\mathcal{F} \subseteq p(X)$ and $|X| = n$. prove that if $|\mathcal{F}| > \sum_{i=0}^{k-1} \binom{n}{i}$ then there exist $Y \subseteq X$ with $|Y| = k$ such that $p(Y) = \mathcal{F} \cap Y$ that $\mathcal{F} \cap Y = \{F \cap Y : F \in \mathcal{F}\}$(20 points)
you can see this problem also here:
COMBINATORIAL PROBLEMS AND EXERCISES-SECOND EDITION-by LASZLO LOVASZ-AMS CHELSEA PUBLISHING- chapter 13- problem 10(c)!!!

6 Suppose that X is a set with n elements and $\mathcal{F} \subseteq X^{(k)}$ and $X_1, X_2, ..., X_s$ is a partition of X. We know that for every $A, B \in \mathcal{F}$ and every $1 \leq j \leq s$, $E = B \cap (\bigcup_{i=1}^{j} X_i) \neq A \cap (\bigcup_{i=1}^{j} X_i) = F$ shows that none of E, F contains the other one. Prove that

$$|\mathcal{F}| \leq \max_{\sum_{i=1}^{S} w_i = k} \prod_{j=1}^{s} \binom{|X_j|}{w_j}$$

(15 points)

Exam time was 5 hours and 20 minutes.

Day 5

1 **two variable ploynomial**

$P(x,y)$ is a two variable polynomial with real coefficients. degree of a monomial means sum of the powers of x and y in it. we denote by $Q(x,y)$ sum of monomials with the most degree in $P(x,y)$.
(for example if $P(x,y)=3x^4y-2x^2y^3+5xy^2+x-5$ then $Q(x,y)=3x^4y-2x^2y^3$.)
suppose that there are real numbers x_1,y_1,x_2 and y_2 such that $Q(x_1,y_1)>0$, $Q(x_2,y_2)<0$
prove that the set $\{(x,y)|P(x,y)=0\}$ is not bounded.
(we call a set S of plane bounded if there exist positive number M such that the distance of elements of S from the origin is less than M.)

time allowed for this question was 1 hour.

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2 **rolling cube**

a,b and c are natural numbers. we have a $(2a+1)\times(2b+1)\times(2c+1)$ cube. this cube is on an infinite plane with unit squares. you call roll the cube to every side you want. faces of the cube are divided to unit squares and the square in the middle of each face is coloured (it means that if this square goes on a square of the plane, then that square will be coloured.)
prove that if any two of lengths of sides of the cube are relatively prime, then we can colour every square in plane.

time allowed for this question was 1 hour.

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3 **points in plane**

set A containing n points in plane is given. a *copy* of A is a set of points that is made by using transformation, rotation, homogeneity or their combination on elements of A. we want to put n *copies* of A in plane, such that every two copies have exactly one point in common and every three of them have no common elements.
a) prove that if no 4 points of A make a parallelogram, you can do this only using transformation. (A doesn't have a parallelogram with angle 0 and a parallelogram that it's two non-adjacent vertices are one!)
b) prove that you can always do this by using a combination of all these things.

time allowed for this question was 1 hour and 30 minutes

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4 **carpeting**

suppose that S is a figure in the plane such that it's border doesn't contain any lattice points. suppose that x,y are two lattice points with the distance 1 (we call a point lattice point if it's coordinates are integers). suppose that we can cover the plane with copies of S such that x, y always go on lattice points (you can rotate or reverse copies of S). prove that the area of S is equal to lattice points inside it.

time allowed for this question was 1 hour.

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5 **interesting sequence**

n is a natural number and $x_1, x_2, ...$ is a sequence of numbers 1 and -1 with these properties:

it is periodic and its least period number is $2^n - 1$. (it means that for every natural number j we have $x_{j+2^n-1} = x_j$ and $2^n - 1$ is the least number with this property.)

There exist distinct integers $0 \leq t_1 < t_2 < ... < t_k < n$ such that for every natural number j we have

$$x_{j+n} = x_{j+t_1} \times x_{j+t_2} \times ... \times x_{j+t_k}$$

Prove that for every natural number s that $s < 2^n - 1$ we have

$$\sum_{i=1}^{2^n-1} x_i x_{i+s} = -1$$

Time allowed for this question was 1 hours and 15 minutes.

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6 **polyhedral**

we call a 12-gon in plane good whenever:
first, it should be regular, second, it's inner plane must be filled!!, third, it's center must be the origin of the coordinates, forth, it's vertices must have points $(0, 1)$,$(1, 0)$,$(-1, 0)$ and $(0, -1)$.
find the faces of the massivest polyhedral that it's image on every three plane xy,yz and zx is a good 12-gon.
(it's obvios that centers of these three 12-gons are the origin of coordinates for three dimensions.)

time allowed for this question is 1 hour.

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7 **interesting function**

S is a set with n elements and $P(S)$ is the set of all subsets of S and $f : P(S) \to \mathbb{N}$
is a function with these properties:
for every subset A of S we have $f(A) = f(S - A)$.

for every two subsets of S like A and B we have $max(f(A), f(B)) \geq f(A \cup B)$ prove that number of natural numbers like x such that there exists $A \subseteq S$ and $f(A) = x$ is less than n.

time allowed for this question was 1 hours and 30 minutes.

8 [b]numbers $n^2 + 1$[/b]

Prove that there are infinitely many natural numbers of the form $n^2 + 1$ such that they don't have any divisor of the form $k^2 + 1$ except 1 and themselves.

time allowed for this question was 45 minutes.

Day 6

1 Prove that the group of orientation-preserving symmetries of the cube is isomorphic to S_4 (the group of permutations of $\{1, 2, 3, 4\}$).(20 points)

2 prove the third sylow theorem: suppose that G is a group and $|G| = p^e m$ which p is a prime number and $(p, m) = 1$. suppose that a is the number of p-sylow subgroups of G ($H < G$ that $|H| = p^e$). prove that $a|m$ and $p|a - 1$.(Hint: you can use this: every two p-sylow subgroups are conjugate.)(20 points)

3 suppose that $G < S_n$ is a subgroup of permutations of $\{1, ..., n\}$ with this property that for every $e \neq g \in G$ there exist exactly one $k \in \{1, ..., n\}$ such that $g.k = k$. prove that there exist one $k \in \{1, ..., n\}$ such that for every $g \in G$ we have $g.k = k$.(20 points)

4 a) prove that every discrete subgroup of $(\mathbb{R}^2, +)$ is in one of these forms:
i-$\{0\}$.
ii-$\{mv | m \in \mathbb{Z}\}$ for a vector v in $\mathbb{R}^2$.
iii-$\{mv + nw | m, n \in \mathbb{Z}\}$ for tho linearly independent vectors v and w in $\mathbb{R}^2$.(lattice L)
b) prove that every finite group of symmetries that fixes the origin and the lattice L is in one of these forms: $\mathcal{C}_i$ or $\mathcal{D}_i$ that $i = 1, 2, 3, 4, 6$ ($\mathcal{C}_i$ is the cyclic group of order i and $\mathcal{D}_i$ is the dyhedral group of order i).(20 points)

5 suppose that p is a prime number. find that smallest n such that there exists a non-abelian group G with $|G| = p^n$.

SL is an acronym for Special Lesson. this year our special lesson was Groups and Symmetries.

the exam time was 5 hours.

National Math Olympiad (3rd Round) 2011

– Topology

1 (a) We say that a hyperplane H that is given with this equation

$$H = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid a_1x_1 + \dots + a_nx_n = b\}$$

($a = (a_1, \dots, a_n) \in \mathbb{R}^n$ and $b \in \mathbb{R}$ constant) bisects the finite set $A \subseteq \mathbb{R}^n$ if each of the two halfspaces $H^+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid a_1x_1 + \dots + a_nx_n > b\}$ and $H^- = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid a_1x_1 + \dots + a_nx_n < b\}$ have at most $\lfloor \frac{|A|}{2} \rfloor$ points of A.

Suppose that $A_1, \dots, A_n$ are finite subsets of $\mathbb{R}^n$. Prove that there exists a hyperplane H in $\mathbb{R}^n$ that bisects all of them at the same time.

(b) Suppose that the points in $B = A_1 \cup \dots \cup A_n$ are in general position. Prove that there exists a hyperplane H such that $H^+ \cap A_i$ and $H^- \cap A_i$ contain exactly $\lfloor \frac{|A_i|}{2} \rfloor$ points of A_i.

(c) With the help of part (b), show that the following theorem is true: Two robbers want to divide an open necklace that has d different kinds of stones, where the number of stones of each kind is even, such that each of the robbers receive the same number of stones of each kind. Show that the two robbers can accomplish this by cutting the necklace in at most d places.

2 Prove that these three statements are equivalent:

(a) For every continuous function $f : S^n \to \mathbb{R}^n$, there exists an $x \in S^n$ such that $f(x) = f(-x)$.

(b) There is no antipodal mapping $f : S^n \to S^{n-1}$.

(c) For every covering of S^n with closed sets $A_0, \dots, A_n$, there exists an index i such that $A_i \cap -A_i \neq \emptyset$.

– Combinatorics

1 prove that if graph G is a tree, then there is a vertex that is common between all of the longest paths.

proposed by Sina Rezayi

2 prove that the number of permutations such that the order of each element is a multiple of d is $\frac{n!}{(\frac{n}{d})!d^{\frac{n}{d}}}\prod_{i=0}^{\frac{n}{d}-1}(id+1)$.

proposed by Mohammad Mansouri

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3 Suppose that $p(n)$ is the number of partitions of a natural number n. Prove that there exists $c>0$ such that $P(n)\ge n^{c\cdot\log n}$.

proposed by Mohammad Mansouri

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4 We say the point i in the permutation σ **ongoing** if for every $j<i$ we have $\sigma(j)<\sigma(i)$.

a) prove that the number of permutations of the set $\{1,....,n\}$ with exactly r ongoing points is $s(n,r)$.

b) prove that the number of n-letter words with letters $\{a_1,....,a_k\},a_1<.....<a_k$. with exactly r ongoing points is $\sum_m \binom{k}{m}S(n,m)s(m,r)$.

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5 Suppose that n is a natural number. we call the sequence $(x_1,y_1,z_1,t_1),(x_2,y_2,z_2,t_2),.....,(x_s,y$ of $\mathbb{Z}^4$ **good** if it satisfies these three conditions:

i) $x_1=y_1=z_1=t_1=0$.

ii) the sequences x_i,y_i,z_i,t_i be strictly increasing.

iii) $x_s+y_s+z_s+t_s=n$. (note that s may vary).

Find the number of good sequences.

proposed by Mohammad Ghiasi

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6 Every bacterium has a horizontal body with natural length and some nonnegative number of vertical feet, each with nonnegative (!) natural length, that lie below its body. In how many ways can these bacteria fill an $m\times n$ table such that no two of them overlap?

proposed by Mahyar Sefidgaran

– Number Theory

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1 Suppose that $S\subseteq\mathbb{Z}$ has the following property: if $a,b\in S$, then $a+b\in S$. Further, we know that S has at least one negative element and one positive element. Is the following statement true?
There exists an integer d such that for every $x\in\mathbb{Z}$, $x\in S$ if and only if $d|x$.
proposed by Mahyar Sefidgaran

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2 Let n and k be two natural numbers such that k is even and for each prime p if $p|n$ then $p-1|k$. let $\{a_1, ..., a_{\phi(n)}\}$ be all the numbers coprime to n. What's the remainder of the number $a_1^k + ... + a_{\phi(n)}^k$ when it's divided by n?
proposed by Yahya Motevassel

3 Let k be a natural number such that $k \geq 7$. How many (x, y) such that $0 \leq x, y < 2^k$ satisfy the equation $73^{73^x} \equiv 9^{9^y} \pmod{2^k}$?

Proposed by Mahyar Sefidgaran

4 Suppose that n is a natural number and n is not divisible by 3. Prove that
$(n^{2n} + n^n + n + 1)^{2n} + (n^{2n} + n^n + n + 1)^n + 1$
has at least $2d(n)$ distinct prime factors where $d(n)$ is the number of positive divisors of n.
proposed by Mahyar Sefidgaran

5 Suppose that k is a natural number. Prove that there exists a prime number in $\mathbb{Z}_{[i]}$ such that every other prime number in $\mathbb{Z}_{[i]}$ has a distance at least k with it.

6 a is an integer and p is a prime number and we have $p \geq 17$. Suppose that $S = \{1, 2, ..., p-1\}$ and $T = \{y | 1 \leq y \leq p-1, ord_p(y) < p-1\}$. Prove that there are at least $4(p-3)(p-1)^{p-4}$ functions $f : S \longrightarrow S$ satisfying $\sum_{x \in T} x^{f(x)} \equiv a$ $(mod\ p)$.
proposed by Mahyar Sefidgaran

– Geometry

1 We have 4 circles in plane such that any two of them are tangent to each other. we connect the tangency point of two circles to the tangency point of two other circles. Prove that these three lines are concurrent.

proposed by Masoud Nourbakhsh

2 In triangle ABC, ω is its circumcircle and O is the center of this circle. Points M and N lie on sides AB and AC respectively. ω and the circumcircle of triangle AMN intersect each other for the second time in Q. Let P be the intersection point of MN and BC. Prove that PQ is tangent to ω iff $OM = ON$.

proposed by Mr.Etesami

3 In triangle ABC, X and Y are the tangency points of incircle (with center I) with sides AB and AC respectively. A tangent line to the circumcircle of triangle ABC (with center O) at point A, intersects the extension of BC at D. If D, X and Y are collinear then prove that D, I and O are also collinear.

proposed by Amirhossein Zabeti

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4 A variant triangle has fixed incircle and circumcircle. Prove that the radical center of its three excircles lies on a fixed circle and the circle's center is the midpoint of the line joining circumcenter and incenter.

proposed by Masoud Nourbakhsh

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5 Given triangle ABC, D is the foot of the external angle bisector of A, I its incenter and I_a its A-excenter. Perpendicular from I to DI_a intersects the circumcircle of triangle in A'. Define B' and C' similarly. Prove that AA', BB' and CC' are concurrent.

proposed by Amirhossein Zabeti

– Algebra

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1 We define the recursive polynomial $T_n(x)$ as follows: $T_0(x) = 1$ $T_1(x) = x$ $T_{n+1}(x) = 2xT_n(x) + T_{n-1}(x)$ $\forall n \in \mathbb{N}$.

a) find $T_2(x), T_3(x), T_4(x)$ and $T_5(x)$.

b) find all the roots of the polynomial $T_n(x)$ $\forall n \in \mathbb{N}$.

Proposed by Morteza Saghafian

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2 For nonnegative real numbers x, y, z and t we know that $|x-y|+|y-z|+|z-t|+|t-x| = 4$.
Find the minimum of $x^2 + y^2 + z^2 + t^2$.

proposed by Mohammadmahdi Yazdi, Mohammad Ahmadi

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3 We define the polynomial $f(x)$ in $\mathbb{R}[x]$ as follows: $f(x) = x^n + a_{n-2}x^{n-2} + a_{n-3}x^{n-3} + ... + a_1x + a_0$
Prove that there exists an i in the set $\{1, ..., n\}$ such that we have $|f(i)| \geq \frac{n!}{\binom{n}{i}}$.

proposed by Mohammadmahdi Yazdi

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4 For positive real numbers a, b and c we have $a + b + c = 3$. Prove $\frac{a}{1+(b+c)^2} + \frac{b}{1+(a+c)^2} + \frac{c}{1+(a+b)^2} \leq \frac{3(a^2+b^2+c^2)}{a^2+b^2+c^2+12abc}$.

proposed by Mohammad Ahmadi

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5 $f(x)$ is a monic polynomial of degree 2 with integer coefficients such that $f(x)$ doesn't have any real roots and also $f(0)$ is a square-free integer (and is not 1 or -1). Prove that for every integer n the polynomial $f(x^n)$ is irreducible over $\mathbb{Z}[x]$.

proposed by Mohammadmahdi Yazdi

– Final

1 A regular dodecahedron is a convex polyhedra that its faces are regular pentagons. The regular dodecahedron has twenty vertices and there are three edges connected to each vertex. Suppose that we have marked ten vertices of the regular dodecahedron.

a) prove that we can rotate the dodecahedron in such a way that at most four marked vertices go to a place that there was a marked vertex before.

b) prove that the number four in previous part can't be replaced with three.

proposed by Kasra Alishahi

2 **a)** Prove that for every natural numbers n and k, we have monic polynomials of degree n, with integer coefficients like $A = \{P_1(x), ..., P_k(x)\}$ such that no two of them have a common factor and for every subset of A, the sum of elements of A has all its roots real.

b) Are there infinitely many monic polynomial of degree n with integer coefficients like $P_1(x), P_2(x), ...$ such that no two of them have a common factor and the sum of a finite number of them has all it's roots real?

proposed by Mohammad Mansouri

3 We have connected four metal pieces to each other such that they have formed a tetragon in space and also the angle between two connected metal pieces can vary.
In the case that the tetragon can't be put in the plane, we've marked a point on each of the pieces such that they are all on a plane. Prove that as the tetragon varies, that four points remain on a plane.

proposed by Erfan Salavati

4 The escalator of the station **champion butcher** has this property that if m persons are on it, then it's speed is $m^{-\alpha}$ where α is a fixed positive real number. Suppose that n persons want to go up by the escalator and the width of the stairs is such that all the persons can stand on a stair. If the length of the escalator is l, what's the least time that is needed for these persons to go up? Why?

proposed by Mohammad Ghiasi

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5 Suppose that α is a real number and $a_1 < a_2 < .....$ is a strictly increasing sequence of natural numbers such that for each natural number n we have $a_n \leq n^{\alpha}$. We call the prime number q golden if there exists a natural number m such that $q|a_m$. Suppose that $q_1 < q_2 < q_3 < .....$ are all the golden prime numbers of the sequence $\{a_n\}$.

a) Prove that if $\alpha = 1.5$, then $q_n \leq 1390^n$. Can you find a better bound for q_n?

b) Prove that if $\alpha = 2.4$, then $q_n \leq 1390^{2n}$. Can you find a better bound for q_n?

[i]part **a** proposed by mahyar sefidgaran by an idea of this question that the nth prime number is less than 2^{2n-2}
part **b** proposed by mostafa einollah zade[/i]

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6 We call two circles in the space fighting if they are intersected or they are clipsed. Find a good necessary and sufficient condition for four distinct points A, B, A', B' such that each circle passing through A, B and each circle passing through A', B' are fighting circles.

proposed by Ali Khezeli

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7 Suppose that $f : P(\mathbb{N}) \longrightarrow \mathbb{N}$ and A is a subset of $\mathbb{N}$. We call f A-predicting if the set $\{x \in \mathbb{N} | x \notin A, f(A \cup x) \neq x\}$ is finite. Prove that there exists a function that for every subset A of natural numbers, it's A-predicting.

proposed by Sepehr Ghazi-Nezami

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8 We call the sequence $d_1, ..., d_n$ of natural numbers, not necessarily distinct, **covering** if there exists arithmetic progressions like $c_1 + kd_1$,....,$c_n + kd_n$ such that every natural number has come in at least one of them. We call this sequence **short** if we can not delete any of the $d_1, ..., d_n$ such that the resulting sequence be still covering.
a) Suppose that $d_1, ..., d_n$ is a short covering sequence and suppose that we've covered all the natural numbers with arithmetic progressions $a_1 + kd_1, ..., a_n + kd_n$, and suppose that p is a prime number that p divides $d_1, ..., d_k$ but it does not divide $d_{k+1}, ..., d_n$. Prove that the remainders of $a_1, ..., a_k$ modulo p contains all the numbers $0, 1, ..., p - 1$.
b) Write anything you can about covering sequences and short covering sequences in the case that each of $d_1, ..., d_n$ has only one prime divisor.

National Math Olympiad (3rd Round) 2012

–	Special Lesson's Exam (First Part)
1	Prove that the number of incidences of n distinct points on n distinct lines in plane is $\mathcal{O}(n^{\frac{4}{3}})$. Find a configuration for which $\Omega(n^{\frac{4}{3}})$ incidences happens.
2	Consider a set of n points in plane. Prove that the number of isosceles triangles having their vertices among these n points is $\mathcal{O}(n^{\frac{7}{3}})$. Find a configuration of n points in plane such that the number of equilateral triangles with vertices among these n points is $\Omega(n^2)$.
3	Prove that if n is large enough, among any n points of plane we can find 1000 points such that these 1000 points have pairwise distinct distances. Can you prove the assertion for n^α where α is a positive real number instead of 1000?
4	Prove that from an $n\times n$ grid, one can find $\Omega(n^{\frac{5}{3}})$ points such that no four of them are vertices of a square with sides parallel to lines of the grid. Imagine yourself as Erdos (!) and guess what is the best exponent instead of $\frac{5}{3}$!
–	Special Lesson's Exam (Second Part)
1	Prove that for each coloring of the points inside or on the boundary of a square with 1391 colors, there exists a monochromatic regular hexagon.
2	Suppose $W(k,2)$ is the smallest number such that if $n\ge W(k,2)$, for each coloring of the set $\{1,2,...,n\}$ with two colors there exists a monochromatic arithmetic progression of length k. Prove that $W(k,2)=\Omega(2^{\frac{k}{2}})$.
3	Prove that if n is large enough, then for each coloring of the subsets of the set $\{1,2,...,n\}$ with 1391 colors, two non-empty disjoint subsets A and B exist such that A, B and $A\cup B$ are of the same color.
4	Prove that if n is large enough, in every $n\times n$ square that a natural number is written on each one of its cells, one can find a subsquare from the main square such that the sum of the numbers is this subsquare is divisible by 1391.

– Number Theory Exam

1 $P(x)$ is a nonzero polynomial with integer coefficients. Prove that there exists infinitely many prime numbers q such that for some natural number n, $q|2^n + P(n)$.

Proposed by Mohammad Gharakhani

2 Prove that there exists infinitely many pairs of rational numbers $(\frac{p_1}{q}, \frac{p_2}{q})$ with $p_1, p_2, q \in \mathbb{N}$ with the following condition:

$$|\sqrt{3} - \frac{p_1}{q}| < q^{-\frac{3}{2}}, |\sqrt{2} - \frac{p_2}{q}| < q^{-\frac{3}{2}}.$$

Proposed by Mohammad Gharakhani

3 p is an odd prime number. Prove that there exists a natural number x such that x and $4x$ are both primitive roots modulo p.

Proposed by Mohammad Gharakhani

4 $P(x)$ and $Q(x)$ are two polynomials with integer coefficients such that $P(x)|Q(x)^2+1$.

a) Prove that there exists polynomials $A(x)$ and $B(x)$ with rational coefficients and a rational number c such that $P(x) = c(A(x)^2 + B(x)^2)$.

b) If $P(x)$ is a monic polynomial with integer coefficients, Prove that there exists two polynomials $A(x)$ and $B(x)$ with integer coefficients such that $P(x)$ can be written in the form of $A(x)^2 + B(x)^2$.

Proposed by Mohammad Gharakhani

5 Let p be a prime number. We know that each natural number can be written in the form

$$\sum_{i=0}^{t} a_i p^i (t, a_i \in \mathbb{N} \cup \{0\}, 0 \le a_i \le p-1)$$

Uniquely.

Now let T be the set of all the sums of the form

$$\sum_{i=0}^{\infty} a_i p^i (0 \le a_i \le p-1).$$

(This means to allow numbers with an infinite base p representation). So numbers that for some $N \in \mathbb{N}$ all the coefficients $a_i, i \geq N$ are zero are natural numbers. (In fact we can consider members of T as sequences $(a_0, a_1, a_2, ...)$ for which $\forall_{i\in\mathbb{N}} : 0 \leq a_i \leq p-1$.) Now we generalize addition and multiplication of natural numbers to this set so that it becomes a ring (it's not necessary to prove this fact). For example:

$1 + (\sum_{i=0}^{\infty}(p-1)p^i) = 1 + (p-1) + (p-1)p + (p-1)p^2 + ... = p + (p-1)p + (p-1)p^2 + ... = p^2 + (p-1)p^2 + (p-1)p^3 + ... = p^3 + (p-1)p^3 + ... = ...$

So in this sum, coefficients of all the numbers $p^k, k \in \mathbb{N}$ are zero, so this sum is zero and thus we can conclude that $\sum_{i=0}^{\infty}(p-1)p^i$ is playing the role of -1 (the additive inverse of 1) in this ring. As an example of multiplication consider

$$(1+p)(1+p+p^2+p^3+...) = 1 + 2p + 2p^2 + \cdots$$

Suppose p is 1 modulo 4. Prove that there exists $x \in T$ such that $x^2 + 1 = 0$.

Proposed by Masoud Shafaei

– Geometry Exam

1 Fixed points B and C are on a fixed circle ω and point A varies on this circle. We call the midpoint of arc BC (not containing A) D and the orthocenter of the triangle ABC, H. Line DH intersects circle ω again in K. Tangent in A to circumcircle of triangle AKH intersects line DH and circle ω again in L and M respectively. Prove that the value of $\frac{AL}{AM}$ is constant.

Proposed by Mehdi E'tesami Fard

2 Let the Nagel point of triangle ABC be N. We draw lines from B and C to N so that these lines intersect sides AC and AB in D and E respectively. M and T are midpoints of segments BE and CD respectively. P is the second intersection point of circumcircles of triangles BEN and CDN. l_1 and l_2 are perpendicular lines to PM and PT in points M and T respectively. Prove that lines l_1 and l_2 intersect on the circumcircle of triangle ABC.

Proposed by Nima Hamidi

3 Cosider ellipse ϵ with two foci A and B such that the lengths of it's major axis and minor axis are $2a$ and $2b$ respectively. From a point T outside of the ellipse, we draw two tangent lines TP and TQ to the ellipse ϵ. Prove that

$$\frac{TP}{TQ} \geq \frac{b}{a}.$$

Proposed by Morteza Saghafian

4 The incircle of triangle ABC for which $AB \neq AC$, is tangent to sides BC, CA and AB in points D, E and F respectively. Perpendicular from D to EF intersects side AB at X, and the second intersection point of circumcircles of triangles AEF and ABC is T. Prove that $TX \perp TF$.

Proposed By Pedram Safaei

5 Two fixed lines l_1 and l_2 are perpendicular to each other at a point Y. Points X and O are on l_2 and both are on one side of line l_1. We draw the circle ω with center O and radius OY. A variable point Z is on line l_1. Line OZ cuts circle ω in P. Parallel to XP from O intersects XZ in S. Find the locus of the point S.

Proposed by Nima Hamidi

– Combinatorics Exam

1 We've colored edges of K_n with $n-1$ colors. We call a vertex rainbow if it's connected to all of the colors. At most how many rainbows can exist?

Proposed by Morteza Saghafian

2 Suppose $s, k, t \in \mathbb{N}$. We've colored each natural number with one of the k colors, such that each color is used infinitely many times. We want to choose a subset $\mathcal{A}$ of $\mathbb{N}$ such that it has t disjoint monochromatic s-element subsets. What is the minimum number of elements of A?

Proposed by Navid Adham

3 In a tree with n vertices, for each vertex x_i, denote the longest paths passing through it by $l_i^1, l_i^2, ..., l_i^{k_i}$. x_i cuts those longest paths into two parts with $(a_i^1, b_i^1), (a_i^2, b_i^2), ..., (a_i^{k_i}, b_i^{k_i})$ vertices respectively. If $\max_{j=1,...,k_i}\{a_i^j \times b_i^j\} = p_i$, find the maximum and minimum values of $\sum_{i=1}^{n} p_i$.

Proposed by Sina Rezaei

4 **a)** Prove that for all $m, n \in \mathbb{N}$ there exists a natural number a such that if we color every 3-element subset of the set $\mathcal{A} = \{1, 2, 3, ..., a\}$ using 2 colors red and green, there exists an m-element subset of $\mathcal{A}$ such that all 3-element subsets of it are red or there exists an n-element subset of $\mathcal{A}$ such that all 3-element subsets of it are green.

b) Prove that for all $m, n \in \mathbb{N}$ there exists a natural number a such that if we color every k-element subset $(k > 3)$ of the set $\mathcal{A} = \{1, 2, 3, ..., a\}$ using 2 colors red and green, there exists an m-element subset of $\mathcal{A}$ such that all k-element subsets of it are red or there exists an n-element subset of $\mathcal{A}$ such that all k-element subsets of it are green.

– Algebra Exam

1 Suppose $0 < m_1 < ... < m_n$ and $m_i \equiv i (\mod 2)$. Prove that the following polynomial has at most n real roots. $(\forall 1 \le i \le n : a_i \in \mathbb{R})$.

$$a_0 + a_1 x^{m_1} + a_2 x^{m_2} + ... + a_n x^{m_n}.$$

2 Suppose $N \in \mathbb{N}$ is not a perfect square, hence we know that the continued fraction of $\sqrt{N}$ is of the form $\sqrt{N} = [a_0, \overline{a_1, a_2, ..., a_n}]$. If $a_1 \neq 1$ prove that $a_i \le 2a_0$.

3 Suppose p is a prime number and $a, b, c \in \mathbb{Q}^+$ are rational numbers;

a) Prove that $\mathbb{Q}(\sqrt[p]{a} + \sqrt[p]{b}) = \mathbb{Q}(\sqrt[p]{a}, \sqrt[p]{b})$.

b) If $\sqrt[p]{b} \in \mathbb{Q}(\sqrt[p]{a})$, prove that for a nonnegative integer k we have $\sqrt[p]{\frac{b}{a^k}} \in \mathbb{Q}$.

c) If $\sqrt[p]{a} + \sqrt[p]{b} + \sqrt[p]{c} \in \mathbb{Q}$, then prove that numbers $\sqrt[p]{a}$, $\sqrt[p]{b}$ and $\sqrt[p]{c}$ are rational.

4 Suppose $f(z) = z^n + a_1 z^{n-1} + ... + a_n$ for which $a_1, a_2, ..., a_n \in \mathbb{C}$. Prove that the following polynomial has only one positive real root like α

$$x^n + \Re(a_1)x^{n-1} - |a_2|x^{n-2} - ... - |a_n|$$

and the following polynomial has only one positive real root like β

$$x^n - \Re(a_1)x^{n-1} - |a_2|x^{n-2} - ... - |a_n|.$$

And roots of the polynomial $f(z)$ satisfy $-\beta \le \Re(z) \le \alpha$.

5 Let p be an odd prime number and let $a_1, a_2, ..., a_n \in \mathbb{Q}^+$ be rational numbers. Prove that

$$\mathbb{Q}(\sqrt[p]{a_1} + \sqrt[p]{a_2} + ... + \sqrt[p]{a_n}) = \mathbb{Q}(\sqrt[p]{a_1}, \sqrt[p]{a_2}, ..., \sqrt[p]{a_n}).$$

– Final Exam

1 Let G be a simple undirected graph with vertices $v_1, v_2, ..., v_n$. We denote the number of acyclic orientations of G with $f(G)$.

a) Prove that $f(G) \leq f(G - v_1) + f(G - v_2) + ... + f(G - v_n)$.

b) Let e be an edge of the graph G. Denote by G' the graph obtained by omiting e and making it's two endpoints as one vertex. Prove that $f(G) = f(G - e) + f(G')$.

c) Prove that for each $\alpha > 1$, there exists a graph G and an edge e of it such that

$\frac{f(G)}{f(G-e)} < \alpha$.

Proposed by Morteza Saghafian

2 Suppose S is a convex figure in plane with area 10. Consider a chord of length 3 in S and let A and B be two points on this chord which divide it into three equal parts. For a variable point X in $S - \{A, B\}$, let A' and B' be the intersection points of rays AX and BX with the boundary of S. Let S' be those points X for which $AA' > \frac{1}{3}BB'$. Prove that the area of S' is at least 6.

Proposed by Ali Khezeli

3 Prove that for each $n \in \mathbb{N}$ there exist natural numbers $a_1 < a_2 < ... < a_n$ such that $\phi(a_1) > \phi(a_2) > ... > \phi(a_n)$.

Proposed by Amirhossein Gorzi

4 We have n bags each having 100 coins. All of the bags have 10 gram coins except one of them which has 9 gram coins. We have a balance which can show weights of things that have weight of at most 1 kilogram. At least how many times shall we use the balance in order to find the different bag?

Proposed By Hamidreza Ziarati

5 We call the three variable polynomial P cyclic if $P(x, y, z) = P(y, z, x)$. Prove that cyclic three variable polynomials P_1, P_2, P_3 and P_4 exist such that for each cyclic three variable polynomial P, there exists a four variable polynomial Q such that $P(x, y, z) = Q(P_1(x, y, z), P_2(x, y, z), P_3(x, y, z), P_4(x, y, z))$.

Solution by Mostafa Eynollahzade and Erfan Salavati

6 **a)** Prove that $a > 0$ exists such that for each natural number n, there exists a convex n-gon P in plane with lattice points as vertices such that the area of P is less than an^3.

b) Prove that there exists $b>0$ such that for each natural number n and each n-gon P in plane with lattice points as vertices, the area of P is not less than bn^2.

c) Prove that there exist $\alpha, c>0$ such that for each natural number n and each n-gon P in plane with lattice points as vertices, the area of P is not less than $cn^{2+\alpha}$.

Proposed by Mostafa Eynollahzade

7 The city of Bridge Village has some highways. Highways are closed curves that have intersections with each other or themselves in 4-way crossroads. Mr.Bridge Lover, mayor of the city, wants to build a bridge on each crossroad in order to decrease the number of accidents. He wants to build the bridges in such a way that in each highway, cars pass above a bridge and under a bridge alternately. By knowing the number of highways determine that this action is possible or not.

Proposed by Erfan Salavati

8 **a)** Does there exist an infinite subset S of the natural numbers, such that $S\neq \mathbb{N}$, and such that for each natural number $n\notin S$, exactly n members of S are coprime with n?

b) Does there exist an infinite subset S of the natural numbers, such that for each natural number $n\in S$, exactly n members of S are coprime with n?

Proposed by Morteza Saghafian

National Math Olympiad (3rd Round) 2013

– Algebra

1 Let $a_0, a_1, \dots, a_n \in \mathbb{N}$. Prove that there exist positive integers $b_0, b_1, \dots, b_n$ such that for $0 \le i \le n : a_i \le b_i \le 2a_i$ and polynomial

$$P(x) = b_0 + b_1 x + \dots + b_n x^n$$

is irreducible over $\mathbb{Q}[x]$.
(10 points)

2 Real numbers $a_1, a_2, \dots, a_n$ add up to zero. Find the maximum of $a_1 x_1 + a_2 x_2 + \dots + a_n x_n$ in term of a_i's, when x_i's vary in real numbers such that $(x_1 - x_2)^2 + (x_2 - x_3)^2 + \dots + (x_{n-1} - x_n)^2 \le 1$.
(15 points)

3 For every positive integer $n \ge 2$, Prove that there is no $n-$tuple of distinct complex numbers $(x_1, x_2, \dots, x_n)$ such that for each $1 \le k \le n$ following equality holds. $\prod_{\substack{1 \le i \le n \\ i \ne k}} (x_k - x_i) = \prod_{\substack{1 \le i \le n \\ i \ne k}} (x_k + x_i)$
(20 points)

4 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) \in \mathbb{Q}$ and

$$f(x + f(y)^2) = f(x + y)^2.$$

(25 points)

5 Prove that there is no polynomial $P \in \mathbb{C}[x]$ such that set $\{P(z) \mid |z| = 1\}$ in complex plane forms a polygon. In other words, a complex polynomial can't map the unit circle to a polygon.
(30 points)

– Number Theory

1 Let p a prime number and d a divisor of $p - 1$. Find the product of elements in $\mathbb{Z}_p$ with order d. ($\mod p$).
(10 points)

2 Suppose that a, b are two odd positive integers such that $2ab+1 \mid a^2+b^2+1$. Prove that $a=b$.
(15 points)

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3 Let $p>3$ a prime number. Prove that there exist $x, y \in \mathbb{Z}$ such that $p = 2x^2+3y^2$ if and only if $p \equiv 5, 11 \pmod{24}$
(20 points)

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4 Prime $p=n^2+1$ is given. Find the sets of solutions to the below equation:

$$x^2-(n^2+1)y^2=n^2.$$

(25 points)

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5 $p=3k+1$ is a prime number. For each $m \in \mathbb{Z}_p$, define function L as follow: $L(m)=\sum_{x\in\mathbb{Z}_p}\left(\frac{x(x^3+m)}{p}\right)$

a) For every $m \in \mathbb{Z}_p$ and $t \in {\mathbb{Z}_p}^*$ prove that $L(m)=L(mt^3)$. (5 points)

b) Prove that there is a partition of ${\mathbb{Z}_p}^* = A \cup B \cup C$ such that $|A|=|B|=|C|=\frac{p-1}{3}$ and L on each set is constant.

Equivalently there are a, b, c for which $L(x)=\begin{cases} a & x\in A \\ b & x \in B \\ c & x\in C\end{cases}$. (7 points)

c) Prove that $a+b+c=-3$. (4 points)

d) Prove that $a^2+b^2+c^2=6p+3$. (12 points)

e) Let $X=\frac{2a+b+3}{3}, Y=\frac{b-a}{3}$, show that $X, Y \in \mathbb{Z}$ and also show that $:p = X^2+XY+Y^2$. (2 points)

$({\mathbb{Z}_p}^* = \mathbb{Z}_p \setminus \{0\})$

– Geometry

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1 Let $ABCDE$ be a pentagon inscribe in a circle (O). Let $BE \cap AD = T$. Suppose the parallel line with CD which passes through T which cut AB, CE at X, Y. If ω be the circumcircle of triangle AXY then prove that ω is tangent to (O).

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2 Let ABC be a triangle with circumcircle (O). Let M, N be the midpoint of arc AB, AC which does not contain C, B and let M', N' be the point of tangency of incircle of $\triangle ABC$ with AB, AC. Suppose that X, Y are foot of perpendicular of A to MM', NN'. If I is the incenter of $\triangle ABC$ then prove that quadrilateral $AXIY$ is cyclic if and only if $b + c = 2a$.

3 Suppose line ℓ and four points A, B, C, D lies on ℓ. Suppose that circles ω_1, ω_2 passes through A, B and circles ω'_1, ω'_2 passes through C, D. If $\omega_1 \perp \omega'_1$ and $\omega_2 \perp \omega'_2$ then prove that lines $O_1O'_2, O_2O'_1, \ell$ are concurrent where O_1, O_2, O'_1, O'_2 are center of $\omega_1, \omega_2, \omega'_1, \omega'_2$.

4 In a triangle ABC with circumcircle (O) suppose that A-altitude cut (O) at D. Let altitude of B, C cut AC, AB at E, F. H is orthocenter and T is midpoint of AH. Parallel line with EF passes through T cut AB, AC at X, Y. Prove that $\angle XDF = \angle YDE$.

5 Let ABC be triangle with circumcircle (O). Let AO cut (O) again at A'. Perpendicular bisector of OA' cut BC at P_A. P_B, P_C define similarly. Prove that :

I) Point P_A, P_B, P_C are collinear.

II) Prove that the distance of O from this line is equal to $\frac{R}{2}$ where R is the radius of the circumcircle.

– Combinatorics

1 Assume that the following generating function equation is correct, prove the following statement: $\Pi_{i=1}^{\infty}(1 + x^{3i})\Pi_{j=1}^{\infty}(1 - x^{6j+3}) = 1$
Statement: The number of partitions of n to numbers not of the form $6k+1$ or $6k-1$ is equal to the number of partitions of n in which each summand appears at least twice.
(10 points)
Proposed by Morteza Saghafian

2 How many rooks can be placed in an $n \times n$ chessboard such that each rook is threatened by at most $2k$ rooks?
(15 points)
Proposed by Mostafa Einollah zadeh

3 n cars are racing. At first they have a particular order. At each moment a car may overtake another car. No two overtaking actions occur at the same time, and except moments a car is passing another, the cars always have an order.

A set of overtaking actions is called "small" if any car overtakes at most once. A set of overtaking actions is called "complete" if any car overtakes exactly once.
If F is the set of all possible orders of the cars after a small set of overtaking actions and G is the set of all possible orders of the cars after a complete set of overtaking actions, prove that

$$\mid F \mid= 2 \mid G \mid$$

(20 points)
Proposed by Morteza Saghafian

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4 We have constructed a rhombus by attaching two equal equilateral triangles. By putting $n-1$ points on all 3 sides of each triangle we have divided the sides to n equal segments. By drawing line segements between correspounding points on each side of the triangles we have divided the rhombus into $2n^2$ equal triangles.
We write the numbers $1, 2, \dots, 2n^2$ on these triangles in a way no number appears twice. On the common segment of each two triangles we write the positive difference of the numbers written on those triangles. Find the maximum sum of all numbers written on the segments.

(25 points)
Proposed by Amirali Moinfar

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5 Consider a graph with n vertices and $\frac{7n}{4}$ edges.
(a) Prove that there are two cycles of equal length.
(25 points)
(b) Can you give a smaller function than $\frac{7n}{4}$ that still fits in part (a)? Prove your claim.
We say function $a(n)$ is smaller than $b(n)$ if there exists an N such that for each $n > N$,$a(n) < b(n)$
(At most 5 points)
Proposed by Afrooz Jabal'ameli

– Final Exam

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1 An n-stick is a connected figure consisting of n matches of length 1 which are placed horizontally or vertically and no two touch each other at points other than their ends. Two shapes that can be transformed into each other by moving, rotating or flipping are considered the same.

An n-mino is a shape which is built by connecting n squares of side length 1 on their sides such that there's a path on the squares between each two squares of the n-mino.
Let S_n be the number of n-sticks and M_n the number of n-minos, e.g. $S_3 = 5$ And $M_3 = 2$.
(a) Prove that for any natural n, $S_n \geq M_{n+1}$.
(b) Prove that for large enough n we have $(2.4)^n \leq S_n \leq (16)^n$.
A **grid segment** is a segment on the plane of length 1 which it's both ends are integer points. A polystick is called **wise** if using it and it's rotations or flips we can cover all grid segments without overlapping, otherwise it's called **unwise**.
(c) Prove that there are at least 2^{n-6} different unwise n-sticks.
(d) Prove that any polystick which is in form of a path only going up and right is wise.
(e) Extra points: Prove that for large enough n we have $3^n \leq S_n \leq 12^n$

Time allowed for this exam was 2 hours.

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2 We define the distance between two circles $\omega, \omega'$by the length of the common external tangent of the circles and show it by $d(\omega, \omega')$. If two circles doesn't have a common external tangent then the distance between them is undefined. A point is also a circle with radius 0 and the distance between two cirlces can be zero.
(a) **Centroid.** n circles $\omega_1, \dots, \omega_n$ are fixed on the plane. Prove that there exists a unique circle $\overline{\omega}$ such that for each circle ω on the plane the square of distance between ω and $\overline{\omega}$ minus the sum of squares of distances of ω from each of the ω_is $1 \leq i \leq n$ is constant, in other words:

$$d(\omega, \overline{\omega})^2 - \frac{1}{n}\sum_{i=1}^{n} d(\omega_i, \omega)^2 = constant$$

(b) **Perpendicular Bisector.** Suppose that the circle ω has the same distance from ω_1, ω_2. Consider ω_3 a circle tangent to both of the common external tangents of ω_1, ω_2. Prove that the distance of ω from centroid of ω_1, ω_2 is not more than the distance of ω and ω_3. (If the distances are all defined)
(c) **Circumcentre.** Let C be the set of all circles that each of them has the same distance from fixed circles $\omega_1, \omega_2, \omega_3$. Prove that there exists a point on the plane which is the external homothety center of each two elements of C.
(d) **Regular Tetrahedron.** Does there exist 4 circles on the plane which the distance between each two of them equals to 1?

Time allowed for this problem was 150 minutes.

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3 Real function f **generates** real function g if there exists a natural k such that $f^k = g$ and we show this by $f \to g$. In this question we are trying to find some properties for relation $\to$, for example it's trivial that if $f \to g$ and $g \to h$ then $f \to h$.(transitivity)

(a) Give an example of two real functions f, g such that $f \neq g$,$f \to g$ and $g \to f$.
(b) Prove that for each real function f there exists a finite number of real functions g such that $f \to g$ and $g \to f$.
(c) Does there exist a real function g such that no function generates it, except for g itself?
(d) Does there exist a real function which generates both x^3 and x^5?
(e) Prove that if a function generates two polynomials of degree 1 P, Q then there exists a polynomial R of degree 1 which generates P and Q.

Time allowed for this problem was 75 minutes.

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4 A polygon A that doesn't intersect itself and has perimeter p is called **Rotund** if for each two points x, y on the sides of this polygon which their distance on the plane is less than 1 their distance on the polygon is at most $\frac{p}{4}$. (Distance on the polygon is the length of smaller path between two points on the polygon)
Now we shall prove that we can fit a circle with radius $\frac{1}{4}$ in any rotund polygon.
The mathematicians of two planets earth and Tarator have two different approaches to prove the statement. In both approaches by "inner chord" we mean a segment with both endpoints on the polygon, and "diagonal" is an inner chord with vertices of the polygon as the endpoints.
Earth approach: Maximal Chord
We know the fact that for every polygon, there exists an inner chord xy with a length of at most 1 such that for any inner chord $x'y'$ with length of at most 1 the distance on the polygon of x, y is more than the distance on the polygon of x', y'. This chord is called the **maximal chord**.
On the rotund polygon A_0 there's two different situations for maximal chord:
(a) Prove that if the length of the maximal chord is exactly 1, then a semicircle with diameter maximal chord fits completely inside A_0, so we can fit a circle with radius $\frac{1}{4}$ in A_0.
(b) Prove that if the length of the maximal chord is less than one we still can fit a circle with radius $\frac{1}{4}$ in A_0.
Tarator approach: Triangulation
Statement 1: For any polygon that the length of all sides is less than one and no circle with radius $\frac{1}{4}$ fits completely inside it, there exists a triangulation of it using diagonals such that no diagonal with length more than 1 appears in the triangulation.

Statement 2: For any polygon that no circle with radius $\frac{1}{4}$ fits completely inside it, can be divided into traingles that their sides are inner chords with length of at most 1.
The mathematicians of planet Tarator proved that if the second statement is true, for each rotund polygon there exists a circle with radius $\frac{1}{4}$ that fits completely inside it.
(c) Prove that if the second statement is true, then for each rotund polygon there exists a circle with radius $\frac{1}{4}$ that fits completely inside it.
They found out that if the first statement is true then the second statement is also true, so they put a bounty of a doogh on proving the first statement. A young earth mathematician named J.N., found a counterexample for statement 1, thus recieving the bounty.
(d) Find a 1392-gon that is counterexample for statement 1.
But the Tarators are not disappointed and they are still trying to prove the second statement.
(e) (Extra points) Prove or disprove the second statement.

Time allowed for this problem was 150 minutes.

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5 A subsum of n real numbers $a_1, \dots, a_n$ is a sum of elements of a subset of the set $\{a_1, \dots, a_n\}$. In other words a subsum is $\epsilon_1 a_1 + \dots + \epsilon_n a_n$ in which for each $1 \le i \le n$,ϵ_i is either 0 or 1.
Years ago, there was a valuable list containing n real not necessarily distinct numbers and their $2^n - 1$ subsums. Some mysterious creatures from planet Tarator has stolen the list, but we still have the subsums.
(a) Prove that we can recover the numbers uniquely if all of the subsums are positive.
(b) Prove that we can recover the numbers uniquely if all of the subsums are non-zero.
(c) Prove that there's an example of the subsums for $n = 1392$ such that we can not recover the numbers uniquely.

Note: If a subsum is sum of element of two different subsets, it appears twice.
Time allowed for this question was 75 minutes.

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6 Planet Tarator is a planet in the Yoghurty way galaxy. This planet has a shape of convex 1392-hedron. On earth we don't have any other information about sides of planet tarator.
We have discovered that each side of the planet is a country, and has it's own currency. Each two neighbour countries have their own constant exchange rate, regardless of other exchange rates. Anybody who travels on land and crosses the border must change all his money to the currency of the destination country,

and there's no other way to change the money. Incredibly, a person's money may change after crossing some borders and getting back to the point he started, but it's guaranteed that crossing a border and then coming back doesn't change the money.
On a research project a group of tourists were chosen and given same amount of money to travel around the Tarator planet and come back to the point they started. They always travel on land and their path is a nonplanar polygon which doesn't intersect itself. What is the maximum number of tourists that may have a pairwise different final amount of money?

Note 1: Tourists spend no money during travel!
Note 2: The only constant of the problem is 1392, the number of the sides. The exchange rates and the way the sides are arranged are unknown. Answer must be a constant number, regardless of the variables.
Note 3: The maximum must be among all possible polyhedras.

Time allowed for this problem was 90 minutes.

7 An equation $P(x) = Q(y)$ is called **Interesting** if P and Q are polynomials with degree at least one and integer coefficients and the equations has an infinite number of answers in $\mathbb{N}$.
An interesting equation $P(x) = Q(y)$ **yields in** interesting equation $F(x) = G(y)$ if there exists polynomial $R(x) \in \mathbb{Q}[x]$ such that $F(x) \equiv R(P(x))$ and $G(x) \equiv R(Q(x))$.
(a) Suppose that S is an infinite subset of $\mathbb{N} \times \mathbb{N}$.$S$ *is an answer* of interesting equation $P(x) = Q(y)$ if each element of S is an answer of this equation. Prove that for each S there's an interesting equation $P_0(x) = Q_0(y)$ such that if there exists any interesting equation that S is an answer of it, $P_0(x) = Q_0(y)$ yields in that equation.
(b) Define the degree of an interesting equation $P(x) = Q(y)$ by $max\{deg(P), deg(Q)\}$.
An interesting equation is called **primary** if there's no other interesting equation with lower degree that yields in it.
Prove that if $P(x) = Q(y)$ is a primary interesting equation and P and Q are monic then $(deg(P), deg(Q)) = 1$.

Time allowed for this question was 2 hours.

8 Let $A_1A_2A_3A_4A_5$ be a convex 5-gon in which the coordinates of all of it's vertices are rational. For each $1 \le i \le 5$ define B_i the intersection of lines $A_{i+1}A_{i+2}$ and $A_{i+3}A_{i+4}$.
$(A_i = A_{i+5})$ Prove that at most 3 lines from the lines A_iB_i $(1 \le i \le 5)$ are concurrent.

Time allowed for this problem was 75 minutes.

National Math Olympiad (3rd Round) 2014

– Algebra

1 We have an equilateral triangle with circumradius 1. We extend its sides. Determine the point P inside the triangle such that the total lengths of the sides (extended), which lies inside the circle with center P and radius 1, is maximum.
(The total distance of the point P from the sides of an equilateral triangle is fixed)

Proposed by Erfan Salavati

2 Find all continuous function $f:\mathbb{R}^{\geq 0}\rightarrow \mathbb{R}^{\geq 0}$ such that :

$$f(xf(y))+f(f(y)) = f(x)f(y)+2 \;\forall x,y\in \mathbb{R}^{\geq 0}$$

Proposed by Mohammad Ahmadi

3 Let $p,q\in \mathbb{R}[x]$ such that $p(z)q(\overline{z})$ is always a real number for every complex number z. Prove that $p(x)=kq(x)$ for some constant $k \in \mathbb{R}$ or $q(x)=0$.

Proposed by Mohammad Ahmadi

4 For any $a,b,c>0$ satisfying $a+b+c+ab+ac+bc=3$, prove that

$$2\leq a+b+c+abc \leq 3$$

Proposed by Mohammad Ahmadi

5 We say $p(x,y)\in \mathbb{R}\left[x,y\right]$ is *good* if for any $y \neq 0$ we have $p(x,y) = p\left(xy,\frac{1}{y}\right)$. Prove that there are good polynomials $r(x,y) ,s(x,y)\in \mathbb{R}\left[x,y\right]$ such that for any good polynomial p there is a $f(x,y)\in \mathbb{R}\left[x,y\right]$ such that

$$f(r(x,y),s(x,y))= p(x,y)$$

Proposed by Mohammad Ahmadi

– Number Theory

1 Show that for every natural number n there are n natural numbers $x_1 < x_2 < ... < x_n$ such that

$$\frac{1}{x_1}+\frac{1}{x_2}+...+\frac{1}{x_n}-\frac{1}{x_1x_2...x_n} \in \mathbb{N} \cup 0$$

(15 points)

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2 We say two sequence of natural numbers A=$(a_1,...,a_n)$, B=$(b_1,...,b_n)$are the exchange and we write $A \sim B$.
if $503|a_i-b_i$ for all $1 \le i \le n$.
also for natural number r : $A^r=(a_1^r,a_2^r,...,a_n^r)$.
Prove that there are natural number k,m such that : $i)250 \le k$

$ii)$There are different permutations $\pi_1,...,\pi_k$ from $\{1,2,3,...,502\}$ such that for $1 \le i \le k-1$ we have $\pi_i^m \sim \pi_{i+1}$

(15 points)

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3 Let n be a positive integer. Prove that there exists a natural number m with exactly n prime factors, such that for every positive integer d the numbers in $\{1,2,3,\ldots,m\}$ of order d modulo m are multiples of $\phi(d)$.

(15 points)

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4 $2 \le d$ is a natural number. $B_{a,b}$=$\{a,a+b,a+2b,...,a+db\}$ $A_{c,q}$=$\{cq^n|n \in \mathbb{N}\}$
Prove that for constant a,b,c,q, set of prime numbers p satisfying the following conditions is finite.

i) $p \nmid abcq$

ii) $A_{c,q} \equiv B_{a,b} (mod p)$

(15 points)

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5 Can an infinite set of natural numbers be found, such that for all triplets (a,b,c) of it we have $abc+1$ perfect square?

(20 points)

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6 Prove that there are 100 natural number $a_1 < a_2 < ... < a_{99} < a_{100}$ ($a_i < 10^6$) such that A , A+A , 2A , A+2A , 2A + 2A are five sets apart ?

A = \{ $a_1,a_2,...,a_{99},a_{100}$ \}

2A = $\{2a_i|1 \le i \le 100\}$

A+A =$\{a_i+a_j|1 \le i < j \le 100\}$

A + 2A = $\{\ a_i + 2a_j | 1 \leq i, j \leq 100\}$
2A + 2A = $\{\ 2a_i + 2a_j | 1 \leq i < j \leq 100$

(20 ponits)

– Combinatorics

1 Denote by g_n the number of connected graphs of degree n whose vertices are labeled with numbers $1, 2, ..., n$. Prove that $g_n \geq (\frac{1}{2}).2^{\frac{n(n-1)}{2}}$.
Note:If you prove that for $c < \frac{1}{2}$, $g_n \geq c.2^{\frac{n(n-1)}{2}}$, you will earn some point!
proposed by Seyed Reza Hosseini and Mohammad Amin Ghiasi

2 In a tennis tournament there are participants from n different countries. Each team consists of a coach and a player whom should settle in a hotel. The rooms considered for the settlement of coaches are different from players' ones. Each player wants to be in a room whose roommates are **all** from countries which have a defense agreement with the player's country. Conversely, each coach wants to be in a room whose roommates are **all** from countries which don't have a defense agreement with the coach's country. Find the minimum number of the rooms such that we can **always** grant everyone's desire.
proposed by Seyed Reza Hosseini and Mohammad Amin Ghiasi

3 We have a 10×10 table. T is a set of rectangles with vertices from the table and sides parallel to the sides of the table such that no rectangle from the set is a subrectangle of another rectangle from the set. t is the maximum number of elements of T.
(a) Prove that $t > 300$.
(b) Prove that $t < 600$.
Proposed by Mir Omid Haji Mirsadeghi and Kasra Alishahi

4 A **word** is formed by a number of letters of the alphabet. We show words with capital letters. A **sentence** is formed by a number of words. For example if $A = aa$ and $B = ab$ then the sentence AB is equivalent to $aaab$. In this language, A^n indicates $\underbrace{AA \cdots A}_{n}$. We have an equation when two sentences are equal. For example $XYX = YZ^2$ and it means that if we write the alphabetic letters forming the words of each sentence, we get two equivalent sequences of alphabetic letters. An equation is **simplified**, if the words of the left and the right side of the sentences of the both sides of the equation are different. Note that every word contains one alphabetic letter at least.

a)We have a simplified equation in terms of X and Y. Prove that both X and Y can be written in form of a power of a word like Z.(Z can contain only one alphabetic letter).

b) Words $W_1, W_2, \cdots, W_n$ are the answers of a simplified equation. Prove that we can produce these n words with fewer words.

c) n words $W_1, W_2, \cdots, W_n$ are the answers of a simplified system of equations. Define graph G with vertices $1, 2 \cdots, n$ such that i and j are connected if in one of the equations, W_i and W_j be the two words appearing in the right side of each side of the equation.($\cdots W_i = \cdots W_j$). If we denote by c the number of connected components of G, prove that these n words can be produced with at most c words.

Proposed by Mostafa Einollah Zadeh Samadi

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5 An n-mino is a connected figure made by connecting n 1×1 squares. Two polyminos are the same if moving the first we can reach the second. For a polymino P ,let $|P|$ be the number of 1×1 squares in it and ∂P be number of squares out of P such that each of the squares have at least on edge in common with a square from P.
(a) Prove that for every $x \in (0, 1)$:

$$\sum_P x^{|P|}(1-x)^{\partial P} = 1$$

The sum is on all different polyminos.
(b) Prove that for every polymino P, $\partial P \leq 2|P| + 2$
(c) Prove that the number of n-minos is less than 6.75^n.

Proposed by Kasra Alishahi

– Geometry

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1 In the circumcircle of triange $\triangle ABC$, AA' is a diameter.
We draw lines l' and l from A' parallel with Internal and external bisector of the vertex A. l' Cut out AB, BC at B_1 and B_2. l Cut out AC, BC at C_1 and C_2.
Prove that the circumcircles of $\triangle ABC$ $\triangle CC_1C_2$ and $\triangle BB_1B_2$ have a common point.

(20 points)

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2 $\triangle ABC$ is isosceles($AB = AC$). Points P and Q exist inside the triangle such that Q lies inside $\widehat{PAC}$ and $\widehat{PAQ} = \frac{\widehat{BAC}}{2}$. We also have $BP = PQ = CQ$.Let

X and Y be the intersection points of (AP, BQ) and (AQ, CP) respectively. Prove that quadrilateral $PQYX$ is cyclic. *(20 Points)*

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3 Distinct points B, B', C, C' lie on an arbitrary line ℓ. A is a point not lying on ℓ. A line passing through B and parallel to AB' intersects with AC in E and a line passing through C and parallel to AC' intersects with AB in F. Let X be the intersection point of the circumcircles of $\triangle ABC$ and $\triangle AB'C'(A \neq X)$. Prove that $EF \parallel AX$.

---

4 D is an arbitrary point lying on side BC of $\triangle ABC$. Circle ω_1 is tangent to segments AD , BD and the circumcircle of $\triangle ABC$ and circle ω_2 is tangent to segments AD , CD and the circumcircle of $\triangle ABC$. Let X and Y be the intersection points of ω_1 and ω_2 with BC respectively and take M as the midpoint of XY. Let T be the midpoint of arc BC which does not contain A. If I is the incenter of $\triangle ABC$, prove that TM goes through the midpoint of ID.

---

5 X and Y are two points lying on or on the extensions of side BC of $\triangle ABC$ such that $\widehat{XAY} = 90$. Let H be the orthocenter of $\triangle ABC$. Take X' and Y' as the intersection points of (BH, AX) and (CH, AY) respectively. Prove that circumcircle of $\triangle CYY'$,circumcircle of $\triangle BXX'$ and $X'Y'$ are concurrent.

– Final Exam

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1 In each of (a) to (d) you have to find a strictly increasing surjective function from A to B or prove that there doesn't exist any.
(a) $A = \{x|x \in \mathbb{Q}, x \leq \sqrt{2}\}$ and $B = \{x|x \in \mathbb{Q}, x \leq \sqrt{3}\}$
(b) $A = \mathbb{Q}$ and $B = \mathbb{Q} \cup \{\pi\}$
In (c) and (d) we say $(x, y) > (z, t)$ where $x, y, z, t \in \mathbb{R}$, whenever $x > z$ or $x = z$ and $y > t$.
(c) $A = \mathbb{R}$ and $B = \mathbb{R}^2$
(d) $X = \{2^{-x}|x \in \mathbb{N}\}$, then $A = X \times (X \cup \{0\})$ and $B = (X \cup \{0\}) \times X$

(e) If $A, B \subset \mathbb{R}$, such that there exists a surjective non-decreasing function from A to B and a surjective non-decreasing function from B to A , does there exist a surjective strictly increasing function from B to A?

Time allowed for this problem was 2 hours.

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2 Consider a flat field on which there exist a valley in the form of an infinite strip with arbitrary width ω. There exist a polyhedron of diameter d(Diameter in a polyhedron is the maximum distance from the points on the polyhedron) is in one side and a pit of diameter d on the other side of the valley. We want to roll the polyhedron and put it into the pit such that the polyhedron and the field always meet each other in one point at least while rolling (If the polyhedron and the field meet each other in one point at least then the polyhedron would not fall into the valley). For crossing over the bridge, we have built a rectangular bridge with a width of $\frac{d}{10}$ over the bridge. Prove that we can always put the polyhedron into the pit considering the mentioned conditions.

(You will earn a good score if you prove the decision for $\omega = 0$).

---

3 (a) n is a natural number. $d_1, \ldots, d_n, r_1, \ldots, r_n$ are natural numbers such that for each i, j that $1 \leq i < j \leq n$ we have $(d_i, d_j) = 1$ and $d_i \geq 2$.
Prove that there exist an x such that
(i) $1 \leq x \leq 3^n$
(ii)For each $1 \leq i \leq n$

$$x \overset{d_i}{\not\equiv} r_i$$

(b) For each $\epsilon > 0$ prove that there exists natural N such that for each $n > N$ and each $d_1, \ldots, d_n, r_1, \ldots, r_n$ satisfying the conditions above there exists an x satisfying (ii) such that $1 \leq x \leq (2 + \epsilon)^n$.

Time allowed for this exam was 75 minutes.

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4 Let P be a regular $2n$-sided polygon. A **rhombus-ulation** of P is dividing P into rhombuses such that no two intersect and no vertex of any rhombus is on the edge of other rhombuses or P.
(a) Prove that number of rhombuses is a function of n. Find the value of this function. Also find the number of vertices and edges of the rhombuses as a function of n.
(b) Prove or disprove that there always exists an edge e of P such that by erasing all the segments parallel to e the remaining rhombuses are connected.
(c) Is it true that each two rhombus-ulations can turn into each other using the following algorithm multiple times?
Algorithm: Take a hexagon -not necessarily regular- consisting of 3 rhombuses and re-rhombus-ulate the hexagon.
(d) Let $f(n)$ be the number of ways to rhombus-ulate P. Prove that:

$$\Pi_{k=1}^{n-1}(\binom{k}{2} + 1) \leq f(n) \leq \Pi_{k=1}^{n-1} k^{n-k}$$

---

5 A not necessary nonplanar polygon in $\mathbb{R}^3$ is called **Grid Polygon** if each of it's edges are parallel to one of the axes.

(a) There's a right angle between each two neighbour sides of the grid polygon, the plane containing this angle could be parallel to either xy plane, yz plane, or xz plane. Prove that parity of the number of the angles that the plane containing each of them is parallel to xy plane is equal to parity of the number of the angles that the plane containing each of them is parallel to yz plane and parity of the number of the angles that the plane containing each of them is parallel to zx plane.
(b) A grid polygon is called **Inscribed** if there's a point in the space that has an equal distance from all of the vertices of the polygon. Prove that any nonplanar grid hexagon is inscribed.
(c) Does there exist a grid 2014-gon without repeated vertices such that there exists a plane that intersects all of it's edges?
(d) If $a, b, c \in \mathbb{N} - \{1\}$, prove that a, b, c are sidelengths of a triangle iff there exists a grid polygon in which the number of it's edges that are parallel to x axis is a, the number of it's edges that are parallel to y axis is b and the number of it's edges that are parallel to z axis is c.

Time allowed for this exam was 1 hour.

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6 P is a monic polynomial of odd degree greater than one such that there exists a function $f : \mathbb{R} \to \mathbb{N}$ such that for each $x \in \mathbb{R}$,

$$f(P(x)) = P(f(x))$$

(a) Prove that there are a finite number of natural numbers in range of f.
(b) Prove that if f is not constant then the equation $P(x) - x = 0$ has at least two real solutions.
(c) For each natural $n > 1$ prove that there exists a function $f : \mathbb{R} \to \mathbb{N}$ and a monic polynomial of odd degree greater than one P such that for each $x \in \mathbb{R}$,

$$f(P(x)) = P(f(x))$$

and range of f contains exactly n different numbers.

Time allowed for this problem was 105 minutes.

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7 We have a machine that has an input and an output. The input is a letter from the finite set I and the output is a lamp that at each moment has one of the colors of the set $C = \{c_1, \ldots, c_p\}$.
At each moment the machine has an inner state that is one of the n members of finite set S. The function $o : S \to C$ is a surjective function defining that at each state, what color must the lamp be, and the function $t : S \times I \to S$ is a function defining how does giving each input at each state changes the state.

We only shall see the lamp and we have no direct information from the state of the car at current moment.
In other words a machine is $M = (S, I, C, o, t)$ such that S, I, C are finite, $t : S \times I \to S$, and $o : S \to C$ is surjective. It is guaranteed that for each two different inner states, there's a sequence of inputs such that the color of the lamp after giving the sequence to the machine at the first state is different from the color of the lamp after giving the sequence to the machine at the second state.
(a) The machine M has n different inner states. Prove that for each two different inner states, there's a sequence of inputs of length no more than $n-p$ such that the color of the lamp after giving the sequence to the machine at the first state is different from the color of the lamp after giving the sequence to the machine at the second state.
(b) Prove that for a machine M with n different inner states, there exists an algorithm with no more than n^2 inputs that starting at any unknown inner state, at the end of the algorithm the state of the machine at that moment is known.
Can you prove the above claim for $\frac{n^2}{2}$?

---

8 The polynomials $k_n(x_1, \ldots, x_n)$, where n is a non-negative integer, satisfy the following conditions

$$k_0 = 1$$

$$k_1(x_1) = x_1$$

$$k_n(x_1, \ldots, x_n) = x_n k_{n-1}(x_1, \ldots, x_{n-1}) + (x_n^2 + x_{n-1}^2) k_{n-2}(x_1, \ldots, x_{n-2})$$

Prove that for each non-negative n we have $k_n(x_1, \ldots, x_n) = k_n(x_n, \ldots, x_1)$.

National Math Olympiad (3rd round) 2015

–	Algebra
1	x, y, z are three real numbers inequal to zero satisfying $x + y + z = xyz$. Prove that $$\sum\left(\frac{x^2-1}{x}\right)^2 \geq 4$$ *Proposed by Amin Fathpour*
2	Prove that there are no functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $\forall x, y \in \mathbb{R} : f(x^2 + g(y)) - f(x^2) + g(y) - g(x) \leq 2y$ and $f(x) \geq x^2$. *Proposed by Mohammad Ahmadi*
3	Does there exist an irreducible two variable polynomial $f(x, y) \in \mathbb{Q}[x, y]$ such that it has only four roots $(0, 1), (1, 0), (0, -1), (-1, 0)$ on the unit circle.
4	$p(x) \in \mathbb{C}[x]$ is a polynomial such that: $\forall z \in \mathbb{C}, \lvert z \rvert = 1 \Longrightarrow p(z) \in \mathbb{R}$ Prove that $p(x)$ is constant.
5	Find all polynomials $p(x) \in \mathbb{R}[x]$ such that for all $x \in \mathbb{R}$: $p(5x)^2 - 3 = p(5x^2 + 1)$ such that: $a)p(0) \neq 0$ $b)p(0) = 0$
6	$a_1, a_2, \ldots, a_n > 0$ are positive real numbers such that $\sum_{i=1}^n \frac{1}{a_i} = n$ prove that: $\sum_{i<j} \left(\frac{a_i-a_j}{a_i+a_j}\right)^2 \leq \frac{n^2}{2}\left(1 - \frac{n}{\sum_{i=1}^n a_i}\right)$
–	Number Theory
1	Prove that there are infinitely natural numbers n such that n can't be written as a sum of two positive integers with prime factors less than 1394.
2	$M_0 \subset \mathbb{N}$ is a non-empty set with a finite number of elements. Ali produces sets $M_1, M_2, ..., M_n$ in the following order: In step n, Ali chooses an element of M_{n-1} like b_n and defines M_n as $$M_n = \{b_n m + 1 \mid m \in M_{n-1}\}$$

Prove that at some step Ali reaches a set which no element of it divides another element of it.

3 Let $p > 5$ be a prime number and $A = \{b_1, b_2, \dots, b_{\frac{p-1}{2}}\}$ be the set of all quadratic residues modulo p, excluding zero. Prove that there doesn't exist any natural a, c satisfying $(ac, p) = 1$ such that set $B = \{ab_1 + c, ab_2 + c, \dots, ab_{\frac{p-1}{2}} + c\}$ and set A are disjoint modulo p.

This problem was proposed by Amir Hossein Pooya.

4 a, b, c, d, k, l are positive integers such that for every natural number n the set of prime factors of $n^k + a^n + c, n^l + b^n + d$ are same. prove that $k = l, a = b, c = d$.

5 $p > 30$ is a prime number. Prove that one of the following numbers is in form of $x^2 + y^2$.

$$p + 1, 2p + 1, 3p + 1, ..., (p - 3)p + 1$$

– Geometry

1 Let $ABCD$ be the trapezoid such that $AB \parallel CD$. Let E be an arbitrary point on AC. point F lies on BD such that $BE \parallel CF$. Prove that circumcircles of $\triangle ABF, \triangle BED$ and the line AC are concurrent.

2 Let ABC be a triangle with orthocenter H and circumcenter O. Let K be the midpoint of AH. point P lies on AC such that $\angle BKP = 90°$. Prove that $OP \parallel BC$.

3 Let ABC be a triangle. consider an arbitrary point P on the plain of $\triangle ABC$. Let R, Q be the reflections of P wrt AB, AC respectively. Let $RQ \cap BC = T$. Prove that $\angle APB = \angle APC$ if and if only $\angle APT = 90°$.

4 Let ABC be a triangle with incenter I. Let K be the midpoint of AI and $BI \cap \odot(\triangle ABC) = M, CI \cap \odot(\triangle ABC) = N$. points P, Q lie on AM, AN respectively such that $\angle ABK = \angle PBC, \angle ACK = \angle QCB$. Prove that P, Q, I are collinear.

5 Let ABC be a triangle with orthocenter H and circumcenter O. Let R be the radius of circumcircle of $\triangle ABC$. Let A', B', C' be the points on $\overrightarrow{AH}, \overrightarrow{BH}, \overrightarrow{CH}$ respectively such that $AH.AA' = R^2, BH.BB' = R^2, CH.CC' = R^2$. Prove that O is incenter of $\triangle A'B'C'$.

National Math Olympiad (3rd Round) 2016

1 The sequence (a_n) is defined as:

$$a_1 = 1007$$

$$a_{i+1} \geq a_i + 1$$

Prove the inequality:

$$\frac{1}{2016} > \sum_{i=1}^{2016} \frac{1}{a_{i+1}^2 + a_{i+2}^2}$$

2 Find all function $f : \mathbb{N} \to \mathbb{N}$ such that for all $a, b \in \mathbb{N}$, $(f(a)+b)f(a+f(b)) = (a+f(b))^2$

3 Do there exists many infinitely points like $(x_1, y_1), (x_2, y_2), \ldots$ such that for any sequences like $b_1, b_2, \ldots$ of real numbers there exists a polynomial $P(x, y) \in R[x, y]$ such that we have for all $i : P(x_i, y_i) = b_i$

– Geometry

1 Let ABC be an arbitrary triangle,P is the intersection point of the altitude from C and the tangent line from A to the circumcircle. The bisector of angle A intersects BC at D . PD intersects AB at K, if H is the orthocenter then prove : $HK \perp AD$

2 Let ABC be an arbitrary triangle. Let E, E be two points on AB, AC respectively such that their distance to the midpoint of BC is equal. Let P be the second intersection of the triangles ABC, AEF circumcircles . The tangents from E, F to the circumcircle of AEF intersect each other at K. Prove that : $\angle KPA = 90$

3 Let ABC be a triangle and let AD, BE, CF be its altitudes . FA_1, DB_1, EC_1 are perpendicular segments to BC, AC, AB respectively.
Prove that : ABC $A_1B_1C_1$

– Number Theory

1 Let F be a subset of the set of positive integers with at least two elements and $P(x)$ be a polynomial with integer coefficients such that for any two distinct elements of F like a and b, the following two conditions hold

- $a+b \in F$, and
- $\gcd(P(a), P(b)) = 1$.

Prove that $P(x)$ is a constant polynomial.

---

2 Let P be a polynomial with integer coefficients. We say P is *good* if there exist infinitely many prime numbers q such that the set

$$X = \{P(n) \mod q : \quad n \in \mathbb{N}\}$$

has at least $\frac{q+1}{2}$ members.
Prove that the polynomial $x^3 + x$ is good.

---

3 Let m be a positive integer. The positive integer a is called a *golden residue* modulo m if $\gcd(a, m) = 1$ and $x^x \equiv a \pmod{m}$ has a solution for x. Given a positive integer n, suppose that a is a golden residue modulo n^n. Show that a is also a golden residue modulo n^{n^n}.

Proposed by Mahyar Sefidgaran

– Combinatorics

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1 Find the number of all permutations of $\{1, 2, \cdots, n\}$ like p such that there exists a unique $i \in \{1, 2, \cdots, n\}$ that :

$$p(p(i)) \geq i$$

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2 Is it possible to divide a 7×7 table into a few connected parts of cells with the same perimeter?
(A group of cells is called connected if any cell in the group, can reach other cells by passing through the sides of cells.)

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3 There are 24 robots on the plane. Each robot has a $70°$ field of view. What is the maximum number of observing relations?
(Observing is a one-sided relation)

– Algebra

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1 Let $P(x) \in \mathbb{Z}[X]$ be a polynomial of degree 2016 with no rational roots. Prove that there exists a polynomial $T(x) \in \mathbb{Z}[X]$ of degree 1395 such that for all distinct (not necessarily real) roots of $P(x)$ like (α, β) :

$$T(\alpha) - T(\beta) \notin \mathbb{Q}$$

Note: $\mathbb{Q}$ is the set of rational numbers.

2 Let $a, b, c \in \mathbb{R}^+$ and $abc = 1$ prove that:
$\frac{a+b}{(a+b+1)^2} + \frac{b+c}{(b+c+1)^2} + \frac{c+a}{(c+a+1)^2} \geq \frac{2}{a+b+c}$

3 Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all positive real numbers x, y :

$$f(y)f(x + f(y)) = f(x)f(xy)$$

– Geometry

1 In triangle ABC , w is a circle which passes through B, C and intersects AB, AC at E, F respectively. BF, CE intersect the circumcircle of ABC at B', C' respectively. Let A' be a point on BC such that $\angle C'A'B = \angle B'A'C$.
Prove that if we change w, then all the circumcircles of triangles $A'B'C'$ passes through a common point.

2 Given $\triangle ABC$ inscribed in (O) an let I and I_a be it's incenter and A-excenter ,respectively.
Tangent lines to (O) at C, B intersect the angle bisector of A at M, N ,respectively.
Second tangent lines through M, N intersect (O) at X, Y.

Prove that $XYII_a$ is cyclic.

3 Given triangle $\triangle ABC$ and let D, E, F be the foot of angle bisectors of A, B, C ,respectively. M, N lie on EF such that $AM = AN$. Let H be the foot of A-altitude on BC.
Points K, L lie on EF such that triangles $\triangle AKL, \triangle HMN$ are correspondingly similiar (with the given order of vertices) such that $AK \nparallel HM$ and $AK \nparallel HN$.

Show that: $DK = DL$

– Number Theory

1 Let p, q be prime numbers (q is odd). Prove that there exists an integer x such that:

$$q|(x+1)^p - x^p$$

If and only if

$$q \equiv 1 \pmod p$$

2 We call a function g *special* if $g(x) = a^{f(x)}$ (for all x) where a is a positive integer and f is polynomial with integer coefficients such that $f(n) > 0$ for all positive integers n.

A function is called an *exponential polynomial* if it is obtained from the product or sum of special functions. For instance, $2^x 3^{x^2+x-1} + 5^{2x}$ is an exponential polynomial.

Prove that there does not exist a non-zero exponential polynomial $f(x)$ and a non-constant polynomial $P(x)$ with integer coefficients such that

$$P(n)|f(n)$$

for all positive integers n.

3 A sequence $P = \{a_n\}$ is called a Permutation of natural numbers (positive integers) if for any natural number m, there exists a unique natural number n such that $a_n = m$.

We also define $S_k(P)$ as: $S_k(P) = a_1 + a_2 + \cdots + a_k$ (the sum of the first k elements of the sequence).

Prove that there exists infinitely many distinct Permutations of natural numbers like $P_1, P_2, \cdots$ such that:

$$\forall k, \forall i < j : S_k(P_i)|S_k(P_j)$$

– Combinatorics

1 In an election, there are 1395 candidates and some voters. Each voter, arranges all the candidates by the priority order.

We form a directed graph with 1395 vertices, an arrow is directed from U to V when the candidate U is at a higher level of priority than V in more than half of the votes. (otherwise, there's no edge between U, V)

Is it possible to generate all complete directed graphs with 1395 vertices?

2 A 100×100 table is given. At the beginning, every unit square has number "0" written in them. Two players playing a game and the game stops after 200 steps (each player plays 100 steps).

In every step, one can choose a row or a column and add 1 to the written number in all of it's squares $\pmod 3$.
First player is the winner if more than half of the squares (5000 squares) have the number "1" written in them,
Second player is the winner if more than half of the squares (5000 squares) have the number "0" written in them. Otherwise, the game is draw.

Assume that both players play at their best. What will be the result of the game ?

Proposed by Mahyar Sefidgaran

3 A 30×30 table is given. We want to color some of it's unit squares such that any colored square has at most k neighbors. (Two squares (i,j) and (x,y) are called neighbors if $i-x, j-y \equiv 0,-1,1 \pmod{30}$ and $(i,j) \neq (x,y)$. Therefore, each square has exactly 8 neighbors)
What is the maximum possible number of colored squares if:

$a) k = 6$

$b) k = 1$