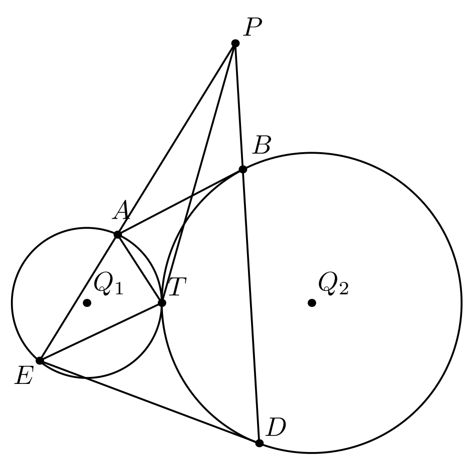
China Girls Math Olympiad 2012

Day 1 August 10th

1 Let $a_1, a_2, \ldots, a_n$ be non-negative real numbers. Prove that $\frac{1}{1+a_1}+\frac{a_1}{(1+a_1)(1+a_2)}+\frac{a_1a_2}{(1+a_1)(1+a_2)(1+a_3)}+\cdots+\frac{a_1a_2\cdots a_{n-1}}{(1+a_1)(1+a_2)\cdots(1+a_n)} \le 1$.

2 Circles Q_1 and Q_2 are tangent to each other externally at T. Points A and E are on Q_1, lines AB and DE are tangent to Q_2 at B and D, respectively, lines AE and BD meet at point P. Prove that
(1) $\frac{AB}{AT}=\frac{ED}{ET}$;
(2) $\angle ATP+\angle ETP=180^\circ$.

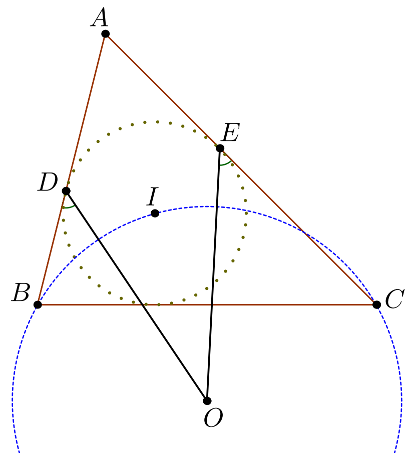


3 Find all pairs (a, b) of integers satisfying: there exists an integer $d \ge 2$ such that $a^n + b^n + 1$ is divisible by d for all positive integers n.

4 There is a stone at each vertex of a given regular 13-gon, and the color of each stone is black or white. Prove that we may exchange the position of two stones such that the coloring of these stones are symmetric with respect to some symmetric axis of the 13-gon.

Day 2 August 11th

5 As shown in the figure below, the in-circle of ABC is tangent to sides AB and AC at D and E respectively, and O is the circumcenter of BCI. Prove that $\angle ODB = \angle OEC$.



6 There are n cities, 2 airline companies in a country. Between any two cities, there is exactly one 2-way flight connecting them which is operated by one of the two companies. A female mathematician plans a travel route, so that it starts and ends at the same city, passes through at least two other cities, and each city in the route is visited once. She finds out that wherever she starts and whatever route she chooses, she must take flights of both companies. Find the maximum value of n.

7 Let $\{a_n\}$ be a sequence of nondecreasing positive integers such that $\frac{r}{a_r} = k+1$ for some positive integers k and r. Prove that there exists a positive integer s such that $\frac{s}{a_s} = k$.

8 Find the number of integers k in the set $\{0, 1, 2, \cdots, 2012\}$ such that $\binom{2012}{k}$ is a multiple of 2012.