Canada National Olympiad 2016

1 The integers $1, 2, 3, \ldots, 2016$ are written on a board. You can choose any two numbers on the board and replace them with their average. For example, you can replace 1 and 2 with 1.5, or you can replace 1 and 3 with a second copy of 2. After 2015 replacements of this kind, the board will have only one number left on it.

(a) Prove that there is a sequence of replacements that will make the final number equal to 2.

(b) Prove that there is a sequence of replacements that will make the final number equal to 1000.

2 Consider the following system of 10 equations in 10 real variables $v_1, \ldots, v_{10}$:

$$v_i = 1 + \frac{6v_i^2}{v_1^2 + v_2^2 + \cdots + v_{10}^2} \qquad (i = 1, \ldots, 10).$$

Find all 10-tuples $(v_1, v_2, \ldots, v_{10})$ that are solutions of this system.

3 Find all polynomials $P(x)$ with integer coefficients such that $P(P(n) + n)$ is a prime number for infinitely many integers n.

4 Let A, B, and F be positive integers, and assume $A < B < 2A$. A flea is at the number 0 on the number line. The flea can move by jumping to the right by A or by B. Before the flea starts jumping, Lavaman chooses finitely many intervals $\{m + 1, m + 2, \ldots, m + A\}$ consisting of A consecutive positive integers, and places lava at all of the integers in the intervals. The intervals must be chosen so that:

(i) any two distinct intervals are disjoint and not adjacent;
(ii) there are at least F positive integers with no lava between any two intervals; and
(iii) no lava is placed at any integer less than F.

Prove that the smallest F for which the flea can jump over all the intervals and avoid all the lava, regardless of what Lavaman does, is $F = (n-1)A + B$, where n is the positive integer such that $\frac{A}{n+1} \leq B - A < \frac{A}{n}$.

5 Let $\triangle ABC$ be an acute-angled triangle with altitudes AD and BE meeting at H. Let M be the midpoint of segment AB, and suppose that the circumcircles of $\triangle DEM$ and $\triangle ABH$ meet at points P and Q with P on the same side of

CH as A. Prove that the lines ED, PH, and MQ all pass through a single point on the circumcircle of $\triangle ABC$.

Canada National Olympiad 2015

1 Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of positive integers. Find all functions f, defined on $\mathbb{N}$ and taking values in $\mathbb{N}$, such that $(n-1)^2 < f(n)f(f(n)) < n^2 + n$ for every positive integer n.

2 Let ABC be an acute-angled triangle with altitudes AD, BE, and CF. Let H be the orthocentre, that is, the point where the altitudes meet. Prove that

$$\frac{AB \cdot AC + BC \cdot CA + CA \cdot CB}{AH \cdot AD + BH \cdot BE + CH \cdot CF} \leq 2.$$

3 On a $(4n+2) \times (4n+2)$ square grid, a turtle can move between squares sharing a side.The turtle begins in a corner square of the grid and enters each square exactly once, ending in the square where she started. In terms of n, what is the largest positive integer k such that there must be a row or column that the turtle has entered at least k distinct times?

4 Let ABC be an acute-angled triangle with circumcenter O. Let I be a circle with centre on the altitude from A in ABC, passing through vertex A and points P and Q on sides AB and AC. Assume that

$$BP.CQ = AP.AQ$$

Prove that I is tangent to the circumcircle of triangle BOC

5 Let p be a prime number for which $\frac{p-1}{2}$ is also prime, and let a, b, c be integers not divisible by p. Prove that there are at most $1 + \sqrt{2p}$ positive integers n such that $n < p$ and p divides $a^n + b^n + c^n$.

Canada National Olympiad 2014

– April 2nd

1 Let $a_1, a_2, \dots, a_n$ be positive real numbers whose product is 1. Show that the sum

$$\frac{a_1}{1+a_1} + \frac{a_2}{(1+a_1)(1+a_2)} + \frac{a_3}{(1+a_1)(1+a_2)(1+a_3)} + \cdots + \frac{a_n}{(1+a_1)(1+a_2)\cdots(1+a_n)}$$

is greater than or equal to $\frac{2^n-1}{2^n}$.

2 Let m and n be odd positive integers. Each square of an m by n board is coloured red or blue. A row is said to be red-dominated if there are more red squares than blue squares in the row. A column is said to be blue-dominated if there are more blue squares than red squares in the column. Determine the maximum possible value of the number of red-dominated rows plus the number of blue-dominated columns. Express your answer in terms of m and n.

3 Let p be a fixed odd prime. A p-tuple $(a_1, a_2, a_3, \dots, a_p)$ of integers is said to be *good* if

- **(i)** $0 \le a_i \le p-1$ for all i, and
- **(ii)** $a_1 + a_2 + a_3 + \cdots + a_p$ is not divisible by p, and
- **(iii)** $a_1a_2 + a_2a_3 + a_3a_4 + \cdots + a_pa_1$ is divisible by p.

Determine the number of good p-tuples.

4 The quadrilateral $ABCD$ is inscribed in a circle. The point P lies in the interior of $ABCD$, and $\angle PAB = \angle PBC = \angle PCD = \angle PDA$. The lines AD and BC meet at Q, and the lines AB and CD meet at R. Prove that the lines PQ and PR form the same angle as the diagonals of $ABCD$.

5 Fix positive integers n and $k \ge 2$. A list of n integers is written in a row on a blackboard. You can choose a contiguous block of integers, and I will either add 1 to all of them or subtract 1 from all of them. You can repeat this step as often as you like, possibly adapting your selections based on what I do. Prove that after a finite number of steps, you can reach a state where at least $n-k+2$ of the numbers on the blackboard are all simultaneously divisible by k.

Canada National Olympiad 2013

1 Determine all polynomials $P(x)$ with real coefficients such that

$$(x+1)P(x-1)-(x-1)P(x)$$

is a constant polynomial.

2 The sequence $a_1, a_2, \ldots, a_n$ consists of the numbers $1, 2, \ldots, n$ in some order. For which positive integers n is it possible that the $n+1$ numbers $0, a_1, a_1+a_2, a_1+a_2+a_3, \ldots, a_1+a_2+\cdots+a_n$ all have di fferent remainders when divided by $n+1$?

3 Let G be the centroid of a right-angled triangle ABC with $\angle BCA = 90^\circ$. Let P be the point on ray AG such that $\angle CPA = \angle CAB$, and let Q be the point on ray BG such that $\angle CQB = \angle ABC$. Prove that the circumcircles of triangles AQG and BPG meet at a point on side AB.

4 Let n be a positive integer. For any positive integer j and positive real number r, define $f_j(r)$ and $g_j(r)$ by

$$f_j(r) = \min(jr, n) + \min\left(\frac{j}{r}, n\right), \text{ and } g_j(r) = \min(\lceil jr \rceil, n) + \min\left(\left\lceil \frac{j}{r} \right\rceil, n\right),$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Prove that

$$\sum_{j=1}^{n} f_j(r) \leq n^2 + n \leq \sum_{j=1}^{n} g_j(r)$$

for all positive real numbers r.

5 Let O denote the circumcentre of an acute-angled triangle ABC. Let point P on side AB be such that $\angle BOP = \angle ABC$, and let point Q on side AC be such that $\angle COQ = \angle ACB$. Prove that the reflection of BC in the line PQ is tangent to the circumcircle of triangle APQ.

Canada National Olympiad 2012

1 Let x, y and z be positive real numbers. Show that $x^2 + xy^2 + xyz^2 \ge 4xyz - 4$.

2 For any positive integers n and k, let $L(n, k)$ be the least common multiple of the k consecutive integers $n, n+1, \ldots, n+k-1$. Show that for any integer b, there exist integers n and k such that $L(n, k) > bL(n+1, k)$.

3 Let $ABCD$ be a convex quadrilateral and let P be the point of intersection of AC and BD. Suppose that $AC + AD = BC + BD$. Prove that the internal angle bisectors of $\angle ACB$, $\angle ADB$ and $\angle APB$ meet at a common point.

4 A number of robots are placed on the squares of a finite, rectangular grid of squares. A square can hold any number of robots. Every edge of each square of the grid is classified as either passable or impassable. All edges on the boundary of the grid are impassable. You can give any of the commands up, down, left, or right.

All of the robots then simultaneously try to move in the specified direction. If the edge adjacent to a robot in that direction is passable, the robot moves across the edge and into the next square. Otherwise, the robot remains on its current square. You can then give another command of up, down, left, or right, then another, for as long as you want. Suppose that for any individual robot, and any square on the grid, there is a finite sequence of commands that will move that robot to that square. Prove that you can also give a finite sequence of commands such that all of the robots end up on the same square at the same time.

5 A bookshelf contains n volumes, labelled 1 to n, in some order. The librarian wishes to put them in the correct order as follows. The librarian selects a volume that is too far to the right, say the volume with label k, takes it out, and inserts it in the k-th position. For example, if the bookshelf contains the volumes $1, 3, 2, 4$ in that order, the librarian could take out volume 2 and place it in the second position. The books will then be in the correct order $1, 2, 3, 4$.

(a) Show that if this process is repeated, then, however the librarian makes the selections, all the volumes will eventually be in the correct order.
(b) What is the largest number of steps that this process can take?

Canada National Olympiad 2011

1 Consider 70-digit numbers with the property that each of the digits $1, 2, 3, ..., 7$ appear 10 times in the decimal expansion of n (and $8, 9, 0$ do not appear). Show that no number of this form can divide another number of this form.

2 Let $ABCD$ be a cyclic quadrilateral with opposite sides not parallel. Let X and Y be the intersections of AB, CD and AD, BC respectively. Let the angle bisector of $\angle AXD$ intersect AD, BC at E, F respectively, and let the angle bisectors of $\angle AYB$ intersect AB, CD at G, H respectively. Prove that $EFGH$ is a parallelogram.

3 Amy has divided a square into finitely many white and red rectangles, each with sides parallel to the sides of the square. Within each white rectangle, she writes down its width divided by its height. Within each red rectangle, she writes down its height divided by its width. Finally, she calculates x, the sum of these numbers. If the total area of white equals the total area of red, determine the minimum of x.

4 Show that there exists a positive integer N such that for all integers $a > N$, there exists a contiguous substring of the decimal expansion of a, which is divisible by 2011.
Note. A contiguous substring of an integer a is an integer with a decimal expansion equivalent to a sequence of consecutive digits taken from the decimal expansion of a.

5 Let d be a positive integer. Show that for every integer S, there exists an integer $n > 0$ and a sequence of n integers $\epsilon_1, \epsilon_2, ..., \epsilon_n$, where $\epsilon_i = \pm 1$ (not necessarily dependent on each other) for all integers $1 \le i \le n$, such that $S = \sum_{i=1}^{n} \epsilon_i (1 + id)^2$.

Canada National Olympiad 2010

1 For all natural n, an n-staircase is a figure consisting of unit squares, with one square in the first row, two squares in the second row, and so on, up to n squares in the n^{th} row, such that all the left-most squares in each row are aligned vertically.
Let $f(n)$ denote the minimum number of square tiles requires to tile the n-staircase, where the side lengths of the square tiles can be any natural number. e.g. $f(2) = 3$ and $f(4) = 7$.
(a) Find all n such that $f(n) = n$.
(b) Find all n such that $f(n) = n + 1$.

2 Let A, B, P be three points on a circle. Prove that if a, b are the distances from P to the tangents at A, B respectively, and c is the distance from P to the chord AB, then $c^2 = ab$.

3 Three speed skaters have a friendly "race" on a skating oval. They all start from the same point and skate in the same direction, but with different speeds that they maintain throughout the race. The slowest skater does 1 lap per minute, the fastest one does 3.14 laps per minute, and the middle one does L laps a minute for some $1 < L < 3.14$. The race ends at the moment when all three skaters again come together to the same point on the oval (which may differ from the starting point.) Determine the number of different choices for L such that exactly 117 passings occur before the end of the race.
Note: A passing is defined as when one skater passes another one. The beginning and the end of the race when all three skaters are together are not counted as passings.

4 Each vertex of a finite graph can be coloured either black or white. Initially all vertices are black. We are allowed to pick a vertex P and change the colour of P and all of its neighbours. Is it possible to change the colour of every vertex from black to white by a
sequence of operations of this type?
Note: A finite graph consists of a finite set of vertices and a finite set of edges between vertices. If there is an edge between vertex A and vertex B, then A and B are neighbours of each other.

5 Let $P(x)$ and $Q(x)$ be polynomials with integer coefficients. Let $a_n = n! + n$. Show that if $\frac{P(a_n)}{Q(a_n)}$ is an integer for every n, then $\frac{P(n)}{Q(n)}$ is an integer for every integer n such that $Q(n) \neq 0$.

Canada National Olympiad 2009

1 Given an $m \times n$ grid with unit squares coloured either black or white, a black square in the grid is *stranded* if there is some square to its left in the same row that is white and there is some square above it in the same column that is white.
Find a closed formula for the number of $2 \times n$ grids with no stranded black square.
Note that n is any natural number and the formula must be in terms of n with no other variables.

2 Two circles of different radii are cut out of cardboard. Each circle is subdivided into 200 equal sectors. On each circle 100 sectors are painted white and the other 100 are painted black. The smaller circle is then placed on top of the larger circle, so that their centers coincide. Show that one can rotate the small circle so that the sectors on the two circles line up and at least 100 sectors on the small circle lie over sectors of the same color on the big circle.

3 Define $f(x, y, z) = \frac{(xy+yz+zx)(x+y+z)}{(x+y)(y+z)(z+x)}$.
Determine the set of real numbers r for which there exists a triplet of positive real numbers satisfying $f(x, y, z) = r$.

4 Find all ordered pairs of integers (a, b) such that $3^a + 7^b$ is a perfect square.

5 A set of points is marked on the plane, with the property that any three marked points can be covered with a disk of radius 1. Prove that the set of all marked points can be covered with a disk of radius 1.

Canada National Olympiad 2008

– March 26th

1 $ABCD$ is a convex quadrilateral for which AB is the longest side. Points M and N are located on sides AB and BC respectively, so that each of the segments AN and CM divides the quadrilateral into two parts of equal area. Prove that the segment MN bisects the diagonal BD.

2 Determine all functions f defined on the set of rational numbers that take rational values for which

$$f(2f(x)+f(y)) = 2x+y,$$

for each x and y.

3 Let a, b, c be positive real numbers for which $a+b+c=1$. Prove that

$$\frac{a-bc}{a+bc}+\frac{b-ca}{b+ca}+\frac{c-ab}{c+ab}\le\frac{3}{2}.$$

4 Determine all functions f defined on the natural numbers that take values among the natural numbers for which

$$(f(n))^p \equiv n \mod f(p)$$

for all $n \in \mathbf{N}$ and all prime numbers p.

5 A *self-avoiding rook walk* on a chessboard (a rectangular grid of unit squares) is a path traced by a sequence of moves parallel to an edge of the board from one unit square to another, such that each begins where the previous move ended and such that no move ever crosses a square that has previously been crossed, i.e., the rook's path is non-self-intersecting.

Let $R(m,n)$ be the number of self-avoiding rook walks on an $m\times n$ (m rows, n columns) chessboard which begin at the lower-left corner and end at the upper-left corner. For example, $R(m,1)=1$ for all natural numbers m; $R(2,2)=2$; $R(3,2)=4$; $R(3,3)=11$. Find a formula for $R(3,n)$ for each natural number n.

Canada National Olympiad 2007

1 What is the maximum number of non-overlapping 2×1 dominoes that can be placed on a 8×9 checkerboard if six of them are placed as shown? Each domino must be placed horizontally or vertically so as to cover two adjacent squares of the board.

2 You are given a pair of triangles for which two sides of one triangle are equal in length to two sides of the second triangle, and the triangles are similar, but not necessarily congruent. Prove that the ratio of the sides that correspond under the similarity is a number between $\frac{1}{2}(\sqrt{5}-1)$ and $\frac{1}{2}(\sqrt{5}+1)$.

3 Suppose that f is a real-valued function for which

$$f(xy) + f(y-x) \geq f(y+x)$$

for all real numbers x and y.
a. Give a nonconstant polynomial that satisfies the condition. b. Prove that $f(x) \geq 0$ for all real x.

4 For two real numbers a, b, with $ab \neq 1$, define the $*$ operation by

$$a * b = \frac{a+b-2ab}{1-ab}.$$

Start with a list of $n \geq 2$ real numbers whose entries x all satisfy $0 < x < 1$. Select any two numbers a and b in the list; remove them and put the number $a * b$ at the end of the list, thereby reducing its length by one. Repeat this procedure until a single number remains.
a. Prove that this single number is the same regardless of the choice of pair at each stage. b. Suppose that the condition on the numbers x is weakened to $0 < x \leq 1$. What happens if the list contains exactly one 1?

5 Let the incircle of triangle ABC touch sides BC, CA and AB at D, E and F, respectively. Let ω, ω_1, ω_2 and ω_3 denote the circumcircles of triangle ABC, AEF, BDF and CDE respectively.

Let ω and ω_1 intersect at A and P, ω and ω_2 intersect at B and Q, ω and ω_3 intersect at C and R.
a. Prove that ω_1, ω_2 and ω_3 intersect in a common point.
b. Show that PD, QE and RF are concurrent.

Canada National Olympiad 2006

1 Let $f(n,k)$ be the number of ways of distributing k candies to n children so that each child receives at most 2 candies. For example $f(3,7)=0, f(3,6)=1, f(3,4)=6$. Determine the value of $f(2006,1)+f(2006,4)+\ldots+f(2006,1000)+f(2006,1003)+\ldots+f(2006,4012)$.

2 Let ABC be acute triangle. Inscribe a rectangle $DEFG$ in this triangle such that $D\in AB, E\in AC, F\in BC, G\in BC$. Describe the locus of (i.e., the curve occupied by) the intersections of the diagonals of all possible rectangles $DEFG$.

3 In a rectangular array of nonnegative reals with m rows and n columns, each row and each column contains at least one positive element. Moreover, if a row and a column intersect in a positive element, then the sums of their elements are the same. Prove that $m=n$.

4 Consider a round-robin tournament with $2n+1$ teams, where each team plays each other team exactly one. We say that three teams X, Y and Z, form a *cycle triplet* if X beats Y, Y beats Z and Z beats X. There are no ties.
a)Determine the minimum number of cycle triplets possible.
b)Determine the maximum number of cycle triplets possible.

5 The vertices of a right triangle ABC inscribed in a circle divide the circumference into three arcs. The right angle is at A, so that the opposite arc BC is a semicircle while arc BC and arc AC are supplementary. To each of three arcs, we draw a tangent such that its point of tangency is the mid point of that portion of the tangent intercepted by the extended lines AB, AC. More precisely, the point D on arc BC is the midpoint of the segment joining the points D' and D'' where tangent at D intersects the extended lines AB, AC. Similarly for E on arc AC and F on arc AB. Prove that triangle DEF is equilateral.

Canada National Olympiad 2005

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1 An equilateral triangle of side length n is divided into unit triangles. Let $f(n)$ be the number of paths from the triangle in the top row to the middle triangle in the bottom row, such that adjacent triangles in a path share a common edge and the path never travels up (from a lower row to a higher row) or revisits a triangle. An example is shown on the picture for $n = 5$. Determine the value of $f(2005)$.

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2 Let (a, b, c) be a Pythagorean triple, i.e. a triplet of positive integers with $a^2 + b^2 = c^2$.

a) Prove that $\left(\frac{c}{a} + \frac{c}{b}\right)^2 > 8$. b) Prove that there are no integer n and Pythagorean triple (a, b, c) satisfying $\left(\frac{c}{a} + \frac{c}{b}\right)^2 = n$.

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3 Let S be a set of $n \geq 3$ points in the interior of a circle. a) Show that there are three distinct points $a, b, c \in S$ and three distinct points A, B, C on the circle such that a is (strictly) closer to A than any other point in S, b is closer to B than any other point in S and c is closer to C than any other point in S. b) Show that for no value of n can four such points in S (and corresponding points on the circle) be guaranteed.

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4 Let ABC be a triangle with circumradius R, perimeter P and area K. Determine the maximum value of: $\frac{KP}{R^3}$.

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5 Let's say that an ordered triple of positive integers (a, b, c) is [i]n-powerful[/i] if $a \leq b \leq c, \gcd(a, b, c) = 1$ and $a^n + b^n + c^n$ is divisible by $a + b + c$. For example, $(1, 2, 2)$ is 5-powerful. a) Determine all ordered triples (if any) which are n-powerful for all $n \geq 1$. b) Determine all ordered triples (if any) which are 2004-powerful and 2005-powerful, but not 2007-powerful.

Art of Problem Solving
2004 Canada National Olympiad

Canada National Olympiad 2004

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1 Find all ordered triples (x, y, z) of real numbers which satisfy the following system of equations:

$$\begin{cases} xy &= z - x - y \\ xz &= y - x - z \\ yz &= x - y - z \end{cases}$$

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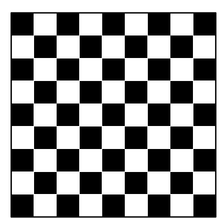
2 How many ways can 8 mutually non-attacking rooks be placed on the 9×9 chessboard (shown here) so that all 8 rooks are on squares of the same color?

(Two rooks are said to be attacking each other if they are placed in the same row or column of the board.)



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3 Let A, B, C, D be four points on a circle (occurring in clockwise order), with $AB < AD$ and $BC > CD$. The bisectors of angles BAD and BCD meet the circle at X and Y, respectively. Consider the hexagon formed by these six points on the circle. If four of the six sides of the hexagon have equal length, prove that BD must be a diameter of the circle.

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4 Let p be an odd prime. Prove that:

$$\sum_{k=1}^{p-1} k^{2p-1} \equiv \frac{p(p+1)}{2} \pmod{p^2}$$

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5 Let T be the set of all positive integer divisors of 2004^{100}. What is the largest possible number of elements of a subset S of T such that no element in S divides any other element in S?

Canada National Olympiad 2003

1 Consider a standard twelve-hour clock whose hour and minute hands move continuously. Let m be an integer, with $1 \leq m \leq 720$. At precisely m minutes after 12:00, the angle made by the hour hand and minute hand is exactly 1°. Determine all possible values of m.

2 Find the last three digits of the number $2003^{2002^{2001}}$.

3 Find all real positive solutions (if any) to

$$x^3 + y^3 + z^3 = x + y + z, \text{ and}$$
$$x^2 + y^2 + z^2 = xyz.$$

4 Prove that when three circles share the same chord AB, every line through A different from AB determines the same ratio $XY : YZ$, where X is an arbitrary point different from B on the rst circle while Y and Z are the points where AX intersects the other two circles (labeled so that Y is between X and Z).

5 Let S be a set of n points in the plane such that any two points of S are at least 1 unit apart.
Prove there is a subset T of S with at least $\frac{n}{7}$ points such that any two points of T are at least $\sqrt{3}$ units apart.

Canada National Olympiad 2002

1 Let S be a subset of $\{1, 2, \dots, 9\}$, such that the sums formed by adding each unordered pair of distinct numbers from S are all different. For example, the subset $\{1, 2, 3, 5\}$ has this property, but $\{1, 2, 3, 4, 5\}$ does not, since the pairs $\{1, 4\}$ and $\{2, 3\}$ have the same sum, namely 5.

What is the maximum number of elements that S can contain?

2 Call a positive integer n **practical** if every positive integer less than or equal to n can be written as the sum of distinct divisors of n.

For example, the divisors of 6 are 1, 2, 3, and 6. Since

$$1=\mathbf{1}, \quad 2=\mathbf{2}, \quad 3=\mathbf{3}, \quad 4=\mathbf{1}+\mathbf{3}, \quad 5=\mathbf{2}+\mathbf{3}, \quad 6=\mathbf{6},$$

we see that 6 is practical.

Prove that the product of two practical numbers is also practical.

3 Prove that for all positive real numbers a, b, and c,

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c$$

and determine when equality occurs.

4 Let Γ be a circle with radius r. Let A and B be distinct points on Γ such that $AB < \sqrt{3}r$. Let the circle with centre B and radius AB meet Γ again at C. Let P be the point inside Γ such that triangle ABP is equilateral. Finally, let the line CP meet Γ again at Q.

Prove that $PQ = r$.

5 Let $\mathbb{N} = \{0, 1, 2, \dots\}$. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$xf(y) + yf(x) = (x + y)f(x^2 + y^2)$$

for all x and y in $\mathbb{N}$.