China Girls Math Olympiad 2014

Day 1

1 In the figure of http://www.artofproblemsolving.com/Forum/download/file.php?id=50643&mode=view
$\odot O_1$ and $\odot O_2$ intersect at two points A, B.
The extension of O_1A meets $\odot O_2$ at C, and the extension of O_2A meets $\odot O_1$ at D,
and through B draw $BE \parallel O_2A$ intersecting $\odot O_1$ again at E.
If $DE \parallel O_1A$, prove that $DC \perp CO_2$.

2 Let $x_1, x_2, \ldots, x_n$ be real numbers, where $n \geq 2$ is a given integer, and let $\lfloor x_1 \rfloor, \lfloor x_2 \rfloor, \ldots, \lfloor x_n \rfloor$ be a permutation of $1, 2, \ldots, n$.
Find the maximum and minimum of $\sum\limits_{i=1}^{n-1} \lfloor x_{i+1} - x_i \rfloor$ (here $\lfloor x \rfloor$ is the largest integer not greater than x).

3 There are n students; each student knows exactly d girl students and d boy students ("knowing" is a symmetric relation). Find all pairs (n, d) of integers .

4 For an integer $m \geq 4$, let T_m denote the number of sequences $a_1, \ldots, a_m$ such that the following conditions hold:
(1) For all $i = 1, 2, \ldots, m$ we have $a_i \in \{1, 2, 3, 4\}$
(2) $a_1 = a_m = 1$ and $a_2 \neq 1$
(3) For all $i = 3, 4 \cdots, m, a_i \neq a_{i-1}, a_i \neq a_{i-2}$.

Prove that there exists a geometric sequence of positive integers $\{g_n\}$ such that for $n \geq 4$ we have that

$$g_n - 2\sqrt{g_n} < T_n < g_n + 2\sqrt{g_n}.$$

Day 2

5 Let a be a positive integer, but not a perfect square; r is a real root of the equation $x^3 - 2ax + 1 = 0$. Prove that $r + \sqrt{a}$ is an irrational number.

6 In acute triangle ABC, $AB > AC$. D and E are the midpoints of AB, AC respectively.
The circumcircle of ADE intersects the circumcircle of BCE again at P.
The circumcircle of ADE intersects the circumcircle BCD again at Q.
Prove that $AP = AQ$.

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7 Given a finite nonempty set X with real values, let $f(X) = \frac{1}{|X|} \sum\limits_{a\in X} a$, where $|X|$ denotes the cardinality of X. For ordered pairs of sets (A, B) such that $A \cup B = \{1, 2, \dots, 100\}$ and $A \cap B = \emptyset$ where $1 \leq |A| \leq 98$, select some $p \in B$, and let $A_p = A \cup \{p\}$ and $B_p = B - \{p\}$. Over all such (A, B) and $p \in B$ determine the maximum possible value of $(f(A_p) - f(A))(f(B_p) - f(B))$.

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8 Let n be a positive integer, and set S be the set of all integers in $\{1, 2, \dots, n\}$ which are relatively prime to n.
Set $S_1 = S \cap \left(0, \frac{n}{3}\right]$, $S_2 = S \cap \left(\frac{n}{3}, \frac{2n}{3}\right]$, $S_3 = S \cap \left(\frac{2n}{3}, n\right]$.
If the cardinality of S is a multiple of 3, prove that S_1, S_2, S_3 have the same cardinality.