Pan African 2008

Day 1

$\boxed{1}$ Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x + y) \le f(x) + f(y) \le x + y$ for all $x, y \in \mathbb{R}$.

$\boxed{2}$ Let C_1 be a circle with centre O, and let AB be a chord of the circle that is not a diameter. M is the midpoint of AB. Consider a point T on the circle C_2 with diameter OM. The tangent to C_2 at the point T intersects C_1 at two points. Let P be one of these points. Show that $PA^2 + PB^2 = 4PT^2$.

$\boxed{3}$ Let a, b, c be three positive integers such that $a < b < c$. Consider the the sets A, B, C and X, defined as follows: $A = \{1, 2, \dots, a\}$, $B = \{a + 1, a + 2, \dots, b\}$, $C = \{b + 1, b + 2, \dots, c\}$ and $X = A \cup B \cup C$. Determine, in terms of a, b and c, the number of ways of placing the elements of X in three boxes such that there are x, y and z elements in the first, second and third box respectively, knowing that: i) $x \le y \le z$; ii) elements of B cannot be put in the first box; iii) elements of C cannot be put in the third box.

Pan African 2008

Day 2

1. Let x and y be two positive reals. Prove that $xy \le \frac{x^{n+2}+y^{n+2}}{x^n+y^n}$ for all non-negative integers n.

2. A set of positive integers X is called *connected* if $|X| \ge 2$ and there exist two distinct elements m and n of X such that m is a divisor of n. Determine the number of connected subsets of the set $\{1, 2, \ldots, 10\}$.

3. Prove that for all positive integers n, there exists a positive integer m which is a multiple of n and the sum of the digits of m is equal to n.