IMO 1980

Austrian-Poland

Day 1

$\boxed{1}$ Given three infinite arithmetic progressions of natural numbers such that each of the numbers 1,2,3,4,5,6,7 and 8 belongs to at least one of them, prove that the number 1980 also belongs to at least one of them.

$\boxed{2}$ Let $\{x_n\}$ be a sequence of natural numbers such that

$$(a) 1 = x_1 < x_2 < x_3 < \dots; \quad (b) x_{2n+1} \le 2n \quad \forall n.$$

Prove that, for every natural number k, there exist terms x_r and x_s such that $x_r - x_s = k$.

$\boxed{3}$ Prove that the sum of the six angles subtended at an interior point of a tetrahedron by its six edges is greater than 540.

$\boxed{4}$ Prove that $\sum \frac{1}{i_1 i_2 \dots i_k} = n$ is taken over all non-empty subsets $\{i_1, i_2, \dots, i_k\}$ of $\{1, 2, \dots, n\}$. (The k is not fixed, so we are summing over all the $2^n - 1$ possible nonempty subsets.)

$\boxed{5}$ Let $A_1A_2A_3$ be a triangle and, for $1 \le i \le 3$, let B_i be an interior point of edge opposite A_i. Prove that the perpendicular bisectors of A_iB_i for $1 \le i \le 3$ are not concurrent.

Day 2

$\boxed{1}$ Given a sequence $\{a_n\}$ of real numbers such that $|a_{k+m} - a_k - a_m| \le 1$ for all positive integers k and m, prove that, for all positive integers p and q,

$$\left|\frac{a_p}{p} - \frac{a_q}{q}\right| < \frac{1}{p} + \frac{1}{q}.$$

$\boxed{2}$ Find the greatest natural number n such there exist natural numbers $x_1, x_2, \dots, x_n$ and natural $a_1 < a_2 < \dots < a_{n-1}$ satisfying the following equations for $i = 1, 2, \dots, n-1$:

$$x_1 x_2 \dots x_n = 1980 \quad \text{and} \quad x_i + \frac{1980}{x_i} = a_i.$$

$\boxed{3}$ Let S be a set of 1980 points in the plane such that the distance between every pair of them is at least 1. Prove that S has a subset of 220 points such that the distance between every pair of them is at least $\sqrt{3}$.

$\boxed{4}$ Let AB be a diameter of a circle; let t_1 and t_2 be the tangents at A and B, respectively; let C be any point other than A on t_1; and let $D_1D_2.E_1E_2$ be arcs on the circle determined by two lines through C. Prove that the lines AD_1 and AD_2 determine a segment on t_2 equal in length to that of the segment on t_2 determined by AE_1 and AE_2.

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Mariehamn (Finland)

Day 1

$\boxed{1}$ Let α, β and γ denote the angles of the triangle ABC. The perpendicular bisector of AB intersects BC at the point X, the perpendicular bisector of AC intersects it at Y. Prove that $\tan(\beta) \cdot \tan(\gamma) = 3$ implies $BC = XY$ (or in other words: Prove that a sufficient condition for $BC = XY$ is $\tan(\beta) \cdot \tan(\gamma) = 3$). Show that this condition is not necessary, and give a necessary and sufficient condition for $BC = XY$.

$\boxed{2}$ Define the numbers $a_0, a_1, \ldots, a_n$ in the following way:

$$a_0 = \frac{1}{2}, \quad a_{k+1} = a_k + \frac{a_k^2}{n} \quad (n > 1, k = 0, 1, \ldots, n-1).$$

Prove that

$$1 - \frac{1}{n} < a_n < 1.$$

$\boxed{3}$ Prove that the equation

$$x^n + 1 = y^{n+1},$$

where n is a positive integer not smaller then 2, has no positive integer solutions in x and y for which x and $n+1$ are relatively prime.

Day 2

$\boxed{1}$ Determine all positive integers n such that the following statement holds: If a convex polygon with with $2n$ sides $A_1A_2\ldots A_{2n}$ is inscribed in a circle and $n-1$ of its n pairs of opposite sides are parallel, which means if the pairs of opposite sides

$$(A_1A_2, A_{n+1}A_{n+2}), (A_2A_3, A_{n+2}A_{n+3}), \ldots, (A_{n-1}A_n, A_{2n-1}A_{2n})$$

are parallel, then the sides

$$A_nA_{n+1}, A_{2n}A_1$$

are parallel as well.

$\boxed{2}$ In a rectangular coordinate system we call a horizontal line parallel to the x -axis triangular if it intersects the curve with equation

$$y = x^4 + px^3 + qx^2 + rx + s$$

in the points A, B, C and D (from left to right) such that the segments AB, AC and AD are the sides of a triangle. Prove that the lines parallel to the x - axis intersecting the curve in four distinct points are all triangular or none of them is triangular.

$\boxed{3}$ Find the digits left and right of the decimal point in the decimal form of the number

$$(\sqrt{2} + \sqrt{3})^{1980}.$$

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Mersch (Luxembourg)

$\boxed{1}$ Let $p(x)$ be a polynomial with integer coefficients such that $p(0) = p(1) = 1$. We define the sequence $a_0, a_1, a_2, \ldots, a_n, \ldots$ that starts with an arbitrary nonzero integer a_0 and satisfies $a_{n+1} = p(a_n)$ for all $n \in \mathbb{N} \cup \{0\}$. Prove that $\gcd(a_i, a_j) = 1$ for all $i, j \in \mathbb{N} \cup \{0\}$.

$\boxed{2}$ Let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial with degree n and complex coefficients which satisfies

$$x \in \mathbb{R} \iff p(x) \in \mathbb{R}.$$

Show that $n = 1$

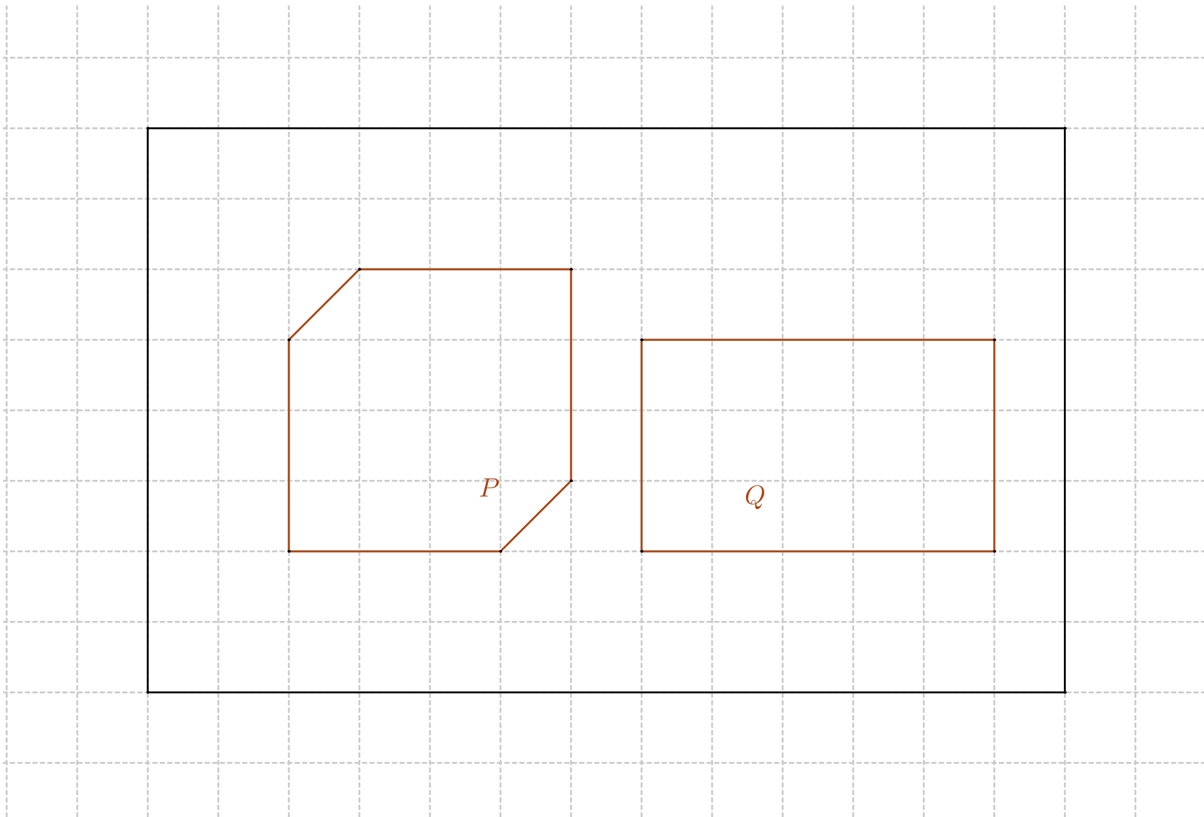
$\boxed{3}$ Two circles C_1 and C_2 are tangent at a point P. The straight line at D is tangent at A to one of the circles and cuts the other circle at the points B and C. Prove that the straight line PA is a bisector (interior or exterior) of the angle BPC.

$\boxed{4}$ Given a real number $x > 1$, prove that there exists a real number $y > 0$ such that

$$\lim_{n\to\infty} \underbrace{\sqrt{y + \sqrt{y + \cdots + \sqrt{y}}}}_{n \text{ roots}} = x.$$

$\boxed{5}$ In the Euclidean three-dimensional space, we call *folding* of a sphere S every partition of $S \setminus \{x, y\}$ into disjoint circles, where x and y are two points of S. A folding of S is called **linear** if the circles of the *folding* are obtained by the intersection of S with a family of parallel planes or with a family of planes containing a straight line D exterior to S. Is every *folding* of a sphere S **linear**?

$\boxed{6}$ Given the polygons P and Q as shown in the grid below, cut P into two polygons P_1 and P_2 such that, when pasted together differently, they form Q.



$\boxed{7}$ Prove that $4x^3 - 3x + 1 = 2y^2$ has at least 31 solutions in positive integers x, y with $x \leq 1980$.

Variant: Prove the equation $4x^3 - 3x + 1 = 2y^2$ has infinitely many solutions in positive integers x,y.

$\boxed{8}$ Prove that if (a, b, c, d) are positive integers such that $(a + 2^{\frac{1}{3}} b + 2^{\frac{2}{3}} c)^2 = d$ then d is a perfect square (i.e is the square of a positive integer).

$\boxed{9}$ Prove that is x, y are non negative integers then $5x \geq 7y$ if and only if there exist non-negative integers (a, b, c, d) such that

$$\begin{cases} x = a + 2b + 3c + 7d \\ y = b + 2c + 5d \end{cases}$$

$\boxed{10}$ The function f is defined on the set $\mathbb{Q}$ of all rational numbers and has values in $\mathbb{Q}$. It satisfies the conditions $f(1) = 2$ and $f(xy) = f(x)f(y) - f(x + y) + 1$ for all $x, y \in \mathbb{Q}$. Determine f (with proof)

11 A triangle (ABC) and a point D in its plane satisfy the relations

$$\frac{BC}{AD} = \frac{CA}{BD} = \frac{AB}{CD} = \sqrt{3}.$$

Prove that (ABC) is equilateral and D is its center.

12 There is a triangle ABC. Its circumcircle and its circumcentre are given. Show how the orthocentre of ABC may be constructed using only a straightedge (unmarked ruler). [The straightedge and paper may be assumed large enough for the construction to be completed]

13 Prove that the integer $145^n + 3114 \cdot 138^n$ is divisible by 1981 if $n = 1981$, and that it is not divisible by 1981 if $n = 1980$.

14 Let A be a fixed point in the interior of a circle ω with center O and radius r, where $0 < OA < r$. Draw two perpendicular chords BC, DE such that they pass through A. For which position of these cords does $BC + DE$ maximize?

15 Three points A, B, C are such that $B \in AC$. On one side of AC, draw the three semicircles with diameters AB, BC, CA. The common interior tangent at B to the first two semicircles meets the third circle E. Let U, V be the points of contact of the common exterior tangent to the first two semicircles. Evaluate the ratio $R = \frac{[EUV]}{[EAC]}$ as a function of $r_1 = \frac{AB}{2}$ and $r_2 = \frac{BC}{2}$, where $[X]$ denotes the area of polygon X.

16 In a pentagon Π in the plane, $M_1, ...M_5$ are the midpoints of the consecutive sides. Z_i is the centroid of the triangle $M_iM_{i+1}M_{i+3}$, where $i = 1, 2...5$ and it is understood that $M_{j\cdot 5} = M_j$ Given pentagon $Z_1Z_2Z_3Z_4Z_5$, determine the original pentagon Π.

17 Ten gamblers start playing with the same amount of money. In turn they cast five dice. If the dice show a total of n, the player must pay each other player $\frac{1}{n}$ times the sum which that player owns at the moment. They throw and pay one after the other. At the 10^{th} round (i.e. after each player has cast the five die once), the dice shows a total of 12 and after the payment it turns out that every player has exactly the same sum as he had in the beginning. Is it possible to determine the totals shown by the dice at the nine former rounds?

18 Do there exist $\{x, y\} \in \mathbb{Z}$ satisfying $(2x + 1)^3 + 1 = y^4$?

19 Find all pairs of solutions (x, y):

$$x^3 + x^2y + xy^2 + y^3 = 8(x^2 + xy + y^2 + 1).$$

$\boxed{20}$ The radii of the circumscribed circle and the inscribed circle of a regular n-gon, $n \geq 3$ are denoted by R_n and r_n, respectively. Prove that

$$\frac{r_n}{R_n} \geq \left(\frac{r_{n+1}}{R_{n+1}}\right)^2.$$

$\boxed{21}$ Let $ABCDEFGH$ be the rectangular parallelepiped where $ABCD$ and $EFGH$ are squares and the edges AE, BF, CG, DH are all perpendicular to the squares. Prove that if the 12 edges of the parallelepiped have integer lengths, the internal diagonal AG and the face diagonal AF cannot both have integer length.

$\boxed{22}$ Let p be a prime number. Prove that there is no number divisible by p in the $n-th$ row of Pascal's triangle if and only if n can be represented in the form $n = p^s q - 1$, where s and q are integers with $s \geq 0, 0 < q < p$.

$\boxed{23}$ Let a, b be positive real numbers, and let x, y be complex numbers such that $|x| = a$ and $|y| = b$. Find the minimal and maximal value of

$$\left|\frac{x+y}{1+x\overline{y}}\right|$$

$\boxed{24}$ Let k be the incircle and let l be the circumcircle of the triangle ABC. Prove that for each point A' of the circle l, there exists a triangle $(A'B'C')$, inscribed in the circle l and circumscribed about the circle k.