

All-Russian Olympiad

— grade 9 day 1

1 In country some cities are connected by oneway flights(There are no more then one flight between two cities). City A called "available" for city B , if there is flight from B to A , maybe with some transfers. It is known, that for every 2 cities P and Q exist city R , such that P and Q are available from R . Prove, that exist city A , such that every city is available for A .

2 $ABCD$ is an isosceles trapezoid with $BC \parallel AD$. A circle ω passing through B and C intersects the side AB and the diagonal BD at points X and Y respectively. Tangent to ω at C intersects the line AD at Z . Prove that the points X , Y , and Z are collinear.

3 There are 100 dwarfs with weight $1, 2, \dots, 100$. They sit on the left riverside. They can not swim, but they have one boat with capacity 100. River has strong river flow, so every dwarf has power only for one passage from right side to left as oarsman. On every passage can be only one oarsman. Can all dwarfs get to right riverside?

4 Are there infinite increasing sequence of natural numbers, such that sum of every 2 different numbers are relatively prime with sum of every 3 different numbers?

— grade9 day 2

1 There are $n > 3$ different natural numbers, less than $(n - 1)!$ For every pair of numbers Ivan divides biggest on lowest and write integer quotient (for example, 100 divides $7 = 14$) and write result on the paper. Prove, that not all numbers on paper are different.

2 a, b, c - different natural numbers. Can we build quadratic polynomial $P(x) = kx^2 + lx + m$, with k, l, m are integer, $k > 0$ that for some integer points it get values a^3, b^3, c^3 ?

3 In the scalene triangle ABC , $\angle ACB = 60$ and Ω is its circumcircle. On the bisectors of the angles BAC and CBA points A', B' are chosen respectively such that $AB' \parallel BC$ and $BA' \parallel AC$. $A'B'$ intersects with Ω at D, E . Prove that triangle CDE is isosceles. (A. Kuznetsov)

4	Every cell of 100×100 table is colored black or white. Every cell on table border is black. It is known, that in every 2×2 square there are cells of two colors. Prove, that exist 2×2 square that is colored in chess order.
—	Grade 10
1	$f_1(x) = x^2 + p_1x + q_1$, $f_2(x) = x^2 + p_2x + q_2$ are two parabolas. l_1 and l_2 are two not parallel lines. It is known, that segments, that cutted on the l_1 by parabolas are equals, and segments, that cutted on the l_2 by parabolas are equals too. Prove, that parabolas are equals.
2	Let ABC be an acute angled isosceles triangle with $AB = AC$ and circumcentre O . Lines BO and CO intersect AC , AB respectively at B' , C' . A straight line l is drawn through C' parallel to AC . Prove that the line l is tangent to the circumcircle of $\triangle B'OC$.
3	There are 3 heaps with 100, 101, 102 stones. Ilya and Kostya play next game. Every step they take one stone from some heap, but not from same, that was on previous step. They make his steps in turn, Ilya make first step. Player loses if can not make step. Who has winning strategy?
5	n is composite. $1 < a_1 < a_2 < \dots < a_k < n$ - all divisors of n . It is known, that $a_1 + 1, \dots, a_k + 1$ are all divisors for some m (except 1, m). Find all such n .
8	In a non-isosceles triangle ABC , O and I are circumcenter and incenter, respectively. B' is reflection of B with respect to OI and lies inside the angle ABI . Prove that the tangents to circumcircle of $\triangle BB'I$ at B' , I intersect on AC . (A. Kuznetsov)
—	Grade 11
1	$S = \sin 64x + \sin 65x$ and $C = \cos 64x + \cos 65x$ are both rational for some x . Prove, that for one of these sums both summands are rational too.
2	Same as Grade 10 P2
3	There are n positive real numbers on the board a_1, \dots, a_n . Someone wants to write n real numbers b_1, \dots, b_n , such that: $b_i \geq a_i$ If $b_i \geq b_j$ then $\frac{b_i}{b_j}$ is integer. Prove that it is possible to write such numbers with the condition
$b_1 \cdots b_n \leq 2^{\frac{n-1}{2}} a_1 \cdots a_n.$	

- 4 Magicman and his helper want to do some magic trick. They have special card desk. Back of all cards is common color and face is one of 2017 colors.
Magic trick: magicman go away from scene. Then viewers should put on the table $n > 1$ cards in the row face up. Helper looks at these cards, then he turn all cards face down, except one, without changing order in row. Then magicman returns on the scene, looks at cards, then show on the one card, that lays face down and names it face color.
What is minimal n such that magicman and his helper can has strategy to make magic trick successfully?
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- 5 $P(x)$ is polynomial with degree $n \geq 2$ and nonnegative coefficients. a, b, c - sides for some triangle. Prove, that $\sqrt[n]{P(a)}$, $\sqrt[n]{P(b)}$, $\sqrt[n]{P(c)}$ are sides for some triangle too.
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- 6 In the 200×200 table in some cells lays red or blue chip. Every chip "see" other chip, if they lay in same row or column. Every chip "see" exactly 5 chips of other color. Find maximum number of chips in the table.
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- 7 There is number N on the board. Every minute Ivan makes next operation: takes any number a written on the board, erases it, then writes all divisors of a except a (Can be same numbers on the board). After some time on the board there are N^2 numbers.
For which N is it possible?
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- 8 Given a convex quadrilateral $ABCD$. We denote I_A, I_B, I_C and I_D centers of $\omega_A, \omega_B, \omega_C$ and ω_D , inscribed in the triangles DAB, ABC, BCD and CDA , respectively. It turned out that $\angle BI_A A + \angle I_C I_A I_D = 180^\circ$. Prove that $\angle BI_B A + \angle I_C I_B I_D = 180^\circ$. (A. Kuznetsov)