India

International Mathematical Olympiad Training Camp

2002

$\boxed{1}$ Let A, B and C be three points on a line with B between A and C. Let $\Gamma_1, \Gamma_2, \Gamma_3$ be semicircles, all on the same side of AC and with AC, AB, BC as diameters, respectively. Let l be the line perpendicular to AC through B. Let Γ be the circle which is tangent to the line l, tangent to Γ_1 internally, and tangent to Γ_3 externally. Let D be the point of contact of Γ and Γ_3. The diameter of Γ through D meets l in E. Show that $AB = DE$.

$\boxed{2}$ Show that there is a set of 2002 consecutive positive integers containing exactly 150 primes. (You may use the fact that there are 168 primes less than 1000)

$\boxed{3}$ Let $X = \{2^m 3^n | 0 \leq m,\ n \leq 9\}$. How many quadratics are there of the form $ax^2 + 2bx + c$, with equal roots, and such that a, b, c are distinct elements of X?

$\boxed{4}$ Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Show that there exist points D, E, and F on sides BC, CA, and AB respectively such that

$$OD + DH = OE + EH = OF + FH$$

and the lines AD, BE, and CF are concurrent.

$\boxed{5}$ Let a, b, c be positive reals such that $a^2 + b^2 + c^2 = 3abc$. Prove that

$$\frac{a}{b^2c^2} + \frac{b}{c^2a^2} + \frac{c}{a^2b^2} \geq \frac{9}{a+b+c}$$

$\boxed{6}$ Determine the number of n-tuples of integers $(x_1, x_2, \cdots, x_n)$ such that $|x_i| \leq 10$ for each $1 \leq i \leq n$ and $|x_i - x_j| \leq 10$ for $1 \leq i, j \leq n$.

$\boxed{7}$ Given two distinct circles touching each other internally, show how to construct a triangle with the inner circle as its incircle and the outer circle as its nine point circle.

$\boxed{8}$ Let $\sigma(n) = \sum_{d|n} d$, the sum of positive divisors of an integer $n > 0$.

(a) Show that $\sigma(mn) = \sigma(m)\sigma(n)$ for positive integers m and n with $gcd(m, n) = 1$

(b) Find all positive integers n such that $\sigma(n)$ is a power of 2.

$\boxed{9}$ On each day of their tour of the West Indies, Sourav and Srinath have either an apple or an orange for breakfast. Sourav has oranges for the first m days, apples for the next m days, followed by oranges for the next m days, and so on. Srinath has oranges for the first n days, apples for the next n days, followed by oranges for the next n days, and so on. If $\gcd(m, n) = 1$, and if the tour lasted for mn days, on how many days did they eat the same kind of fruit?

10 Let T denote the set of all ordered triples (p, q, r) of nonnegative integers. Find all functions $f : T \to \mathbb{R}$ satisfying

$$f(p,q,r) = \begin{cases} 0 & \text{if } pqr = 0, \\ 1 + \frac{1}{6}\left(f(p+1,q-1,r) + f(p-1,q+1,r)\right. & \\ +f(p-1,q,r+1) + f(p+1,q,r-1) & \\ \left.+f(p,q+1,r-1) + f(p,q-1,r+1)\right) & \text{otherwise} \end{cases}$$

for all nonnegative integers p, q, r.

11 Let ABC be a triangle and P an exterior point in the plane of the triangle. Suppose the lines AP, BP, CP meet the sides BC, CA, AB (or extensions thereof) in D, E, F, respectively. Suppose further that the areas of triangles PBD, PCE, PAF are all equal. Prove that each of these areas is equal to the area of triangle ABC itself.

12 Let a, b be integers with $0 < a < b$. A set $\{x, y, z\}$ of non-negative integers is *olympic* if $x < y < z$ and if $\{z - y, y - x\} = \{a, b\}$. Show that the set of all non-negative integers is the union of pairwise disjoint olympic sets.

13 Let ABC and PQR be two triangles such that

(a) P is the mid-point of BC and A is the midpoint of QR.

(b) QR bisects $\angle BAC$ and BC bisects $\angle QPR$

Prove that $AB + AC = PQ + PR$.

14 Let p be an odd prime and let a be an integer not divisible by p. Show that there are $p^2 + 1$ triples of integers (x, y, z) with $0 \leq x, y, z < p$ and such that $(x + y + z)^2 \equiv axyz \pmod{p}$

15 Let $x_1, x_2, \ldots, x_n$ be arbitrary real numbers. Prove the inequality

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \cdots + \frac{x_n}{1+x_1^2+\cdots+x_n^2} < \sqrt{n}.$$

16 Is it possible to find 100 positive integers not exceeding 25,000, such that all pairwise sums of them are different?

17 Let n be a positive integer and let $(1 + iT)^n = f(T) + ig(T)$ where i is the square root of -1, and f and g are polynomials with real coefficients. Show that for any real number k the equation $f(T) + kg(T) = 0$ has only real roots.

18 Consider the square grid with $A = (0, 0)$ and $C = (n, n)$ at its diagonal ends. Paths from A to C are composed of moves one unit to the right or one unit up. Let C_n (n-th catalan

number) be the number of paths from A to C which stay on or below the diagonal AC. Show that the number of paths from A to C which cross AC from below at most twice is equal to $C_{n+2} - 2C_{n+1} + C_n$

19 Let ABC be an acute triangle. Let DAC, EAB, and FBC be isosceles triangles exterior to ABC, with $DA = DC, EA = EB$, and $FB = FC$, such that

$$\angle ADC = 2\angle BAC, \quad \angle BEA = 2\angle ABC, \quad \angle CFB = 2\angle ACB.$$

Let D' be the intersection of lines DB and EF, let E' be the intersection of EC and DF, and let F' be the intersection of FA and DE. Find, with proof, the value of the sum

$$\frac{DB}{DD'} + \frac{EC}{EE'} + \frac{FA}{FF'}.$$

20 Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{c+a}{c+b} + \frac{a+b}{a+c} + \frac{b+c}{b+a}$$

21 Given a prime p, show that there exists a positive integer n such that the decimal representation of p^n has a block of 2002 consecutive zeros.