Geometry Problem Set

National Camp 2018

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1 Basic Stuffs

The problems that are listed below are your tools for solving tougher olympiad problems, be sure to know these by heart.

☐ **Problem 1.1.** *Prove that the diagonals of a rhombus are perpendicular.*

☐ **Problem 1.2.** *Let L, M be the midpoints of BC and CA of $\triangle ABC$ respectively. Prove that $AL = BM \iff AC = BC$.*

☐ **Problem 1.3.** *Let P, Q, R, S be four points on a plane. Prove that* [1] $PR \perp QS \iff PQ^2 - QR^2 = PS^2 - RS^2$.

☐ **Problem 1.4.** *Let the circles ω_1 and ω_2 meet at X, Y. Two lines l_1, l_2 through X intersect ω_1, ω_2 at P_1, P_2 and Q_1, Q_2 respectively. Prove that $\triangle YP_1Q_1$ and $\triangle YP_2Q_2$ are similar.*

Note: This little and easy problem might seem very trivial, but this can be very useful in dealing with harder problems. Yufei Zhao's 3 lemmas in geometry for further reading.

☐ **Problem 1.5.**

1. *Prove that for all $\triangle ABC$ the following relations are true:*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

(R is the circumradius)

2. *In $\triangle ABC$, P lies on BC. Prove that* [2]

$$\frac{BP}{CP} = \frac{AB \times sin\angle BAP}{AC \times sin\angle PAC}$$

☐ **Problem 1.6.** *Let P and Q be arbitrary points on sides BC and CA respectively. Let the internal bisectors of $\angle CAP$ and $\angle CBQ$ meet at R. Prove that $\angle AQB + \angle APB = 2\angle ARB$.*

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[1] This is often called **Perpendicularity Lemma** in olympiad folklore

[2] This is a very important lemma!

☐ **Problem 1.7.** *Let P, Q, R be points on sides BC, CA, AB of $\triangle ABC$. Prove that the perpendiculars to the sides at these points are concurrent if and only if $BP^2 + CQ^2 + AR^2 = PC^2 + QA^2 + RB^2$.*

☐ **Problem 1.8.** *Let D, E, F are the midpoints of BC, CA, AB resp. Prove that $\angle CAD = \angle ABE \iff \angle AFC = \angle ADB$.*

☐ **Problem 1.9.** *Let the angle bisector of $\angle BAC$ meets $\odot ABC$ at A and X resp. Prove that $XI = XB = XC = XI_a$ where I is the incenter and I_a is the excenter opposite to A of $\triangle ABC$.*

Note: This is important as well.

☐ **Problem 1.10.** *Let circles S_1 and S_2 meet at points A and B. An arbitrary line passing through A intersects S_1 and S_2 at P and Q resp. Prove $\dfrac{BP}{BQ}$ is constant.*

☐ **Problem 1.11.** *Let L, M, N are the midpoints of BC, CA, AB and AD, BE, CF are altitudes of $\triangle ABC$. Prove that*

- *O is the orthocenter of $\triangle LMN$.*
- *H is the incenter of $\triangle DEF$.*
- *D, E, F, L, M, N all lie on a circle.*
- *The center of this circle is the midpoint of OH.*
- *Let $BO \cap \odot ABC = Q$. Prove that $AQCH$ is a parallelogram*
- *Prove that $AH = a \cot A = 2R \cos A$ (R is the circumradius) and $HD = 2 \cos B \cos C$*
- *Prove that the reflection of H on BC lies on the circumcenter.*
- *Prove that the reflection of the **Euler Line**[3] on the sides of $\triangle ABC$ concur at the circumcirle.*

☐ **Problem 1.12.** *In $\triangle ABC$, $\angle BAC = 90^\circ$, AD is an altitude. The circle with center A and radius AD meets $\odot ABC$ at U and V. Prove that UV passes through the midpoint of AD.*

☐ **Problem 1.13.** *Let the incircle and excircle (opposite to A) of $\triangle ABC$ meet BC at D and E resp. Suppose F is the antipode of D wrt the incircle.*

1. *Prove that A, F, E are collinear.*
2. *M be the midpoint of DE. Prove that MI meets AD at it's midpoint.*

☐ **Problem 1.14.** *Let the incircle of $\triangle ABC$ meets AB and AC at X and Y resp. BI and CI meet XY at P and Q respectively. Prove that $BPQC$ is cyclic. (In fact $BP \perp CP$ and $BQ \perp CQ$)*

[3] It is the line joining the orthocenter and the circumcenter

☐ **Problem 1.15.** *If four points A, C, B, D lie on a line in this order satisfying the property that* $\frac{AC}{BC} = \frac{AD}{BD}$, *then A, B, C, D are in harmonic order. Prove that if A, B, C, D are in harmonic order and M is the midpoint of AB, then*

1. *$MA^2 = MC.MD$ and $DA.DB = DC.DM$.*
2. *If P is a point s.t $\angle APB = 90°$, then PA and PB are two bisectors of $\angle CPD$.*
3. *Suppose Q is point in the plane. Let a line l meets QA, QB, QC, QD at four points A_1, B_1, C_1, D_1 respectively. Then prove that A_1, B_1, C_1, D_1 are also in harmonic order.*

Note: This is the one of the most important lemma or theorem what you may call it, in bamming projective problems. For further reading go to Alexander Remorov's Projective Geometry handout.

☐ **Problem 1.16.** *AD is an altitude of $\triangle ABC$. E, F are on AC, AB so that AD, BE, CF are concurrent. Prove $\angle EDA = \angle FDA$.*

☐ **Problem 1.17.** *Let AD be an altitude of $\triangle ABC$ and $E \in \odot ABC$ so that $AE \parallel BC$. Prove that D, G, E are collinear where G is the centroid of $\triangle ABC$.*

☐ **Problem 1.18.** *Let O be the circumcenter of $\triangle ABC$ and A', B', C' are reflections of A on BC, CA, AB resp. Prove that AA', BB', CC' are concurrent.*

☐ **Problem 1.19.** *Let D, E are on sides AC, AB of $\triangle ABC$ resp. such that $BE = CD$. Let $\odot ABC \cap \odot ADE = P$. Prove that $PB = PC$.*

☐ **Problem 1.20.** *Let a line PQ touch circle S_1 and S_2 at P and Q resp. Prove that the radical axis of S_1 and S_2 passes through the midpoint of PQ.*

☐ **Problem 1.21.** *Let $\omega_1, \omega_2, omega_3$ are 3 circles. Prove that the 3 radical axis of ω_1 and ω_2,ω_2 and ω_3,ω_3 and ω_1 are either concurrent or parallel.*

☐ **Problem 1.22.** *Two equal-radius circles ω_1 and ω_2 are centered at points O_1 and O_2 . A point X is reflected through O_1 and O_2 to get points A_1 and A_2. The tangents from A_1 to ω_1 touch ω_1 at points P_1 and Q_1, and the tangents from A_2 to ω_2 touch ω_2 at points P_2 and Q_2. If P_1Q_1 and P_2Q_2 intersect at Y , prove that Y is equidistant from A_1 and A_2.*

☐ **Problem 1.23.** *Let BD, CE be the altitudes of $\triangle ABC$ and M be the midpoint of BC. If the ray MH meet $\odot ABC$ at point K, prove that AK, BC, DE are concurrent.*

☐ **Problem 1.24.** *Two circle ω and Γ touches one another internally at P with ω inside of Γ. Let AB be a chord of Γ which touches ω at D. Let $PD \cap \Gamma = Q$. Prove that $QA = QB$.*

□ **Problem 1.25.** *Let AD be a symmedian of $\triangle ABC$ with D on $\odot ABC$. Let M be the midpoint of AD. Prove that $\angle BMD = \angle CMD$ and A, M, O, D are cyclic where O is the circumcenter of $\triangle ABC$.*

□ **Problem 1.26.** *Let A, B be two fixed points and let P be varying point such that $\frac{PA}{PB}$ is constant. Prove that the locus of P is a circle.*

□ **Problem 1.27.** *Prove that $r_1 + r_2 + r_3 = 4R + r$ (R, r, r_1, r_2, r_3 are the circumradius, inradius and three exradiuses respectively of a triangle)*

□ **Problem 1.28.** *Let M be the midpoint of the altitude BE in $\triangle ABC$ and suppose that the excircle opposite to B touches AC at Y. Then MY goes through the incenter I.*

□ **Problem 1.29.** *Let ABC be a triangle, and draw isosceles triangles $\triangle DBC, \triangle AEC, \triangle ABF$ external to $\triangle ABC$ (with $BC; CA; AB$ as their respective bases). Prove that the lines through $A; B; C$ perpendicular to $EF; FD; DE$, respectively, are concurrent.*

□ **Problem 1.30.** *In a triangle ABC we have $AB = AC$. A circle which is internally tangent with the circumscribed circle of the triangle is also tangent to the sides $AB; AC$ in the points P, respectively Q. Prove that the midpoint of PQ is the center of the inscribed circle of the triangle ABC*

□ **Problem 1.31.** ***Nagel Point*** *N: If the Excircles of ABC touch $BC; CA; AB$ at $D; E; F$, then the intersection point of $AD; BE; CF$ is called the* ***Nagel Point*** *N. Prove that*

1. *$I; G; N$ are collinear. (G centroid, I incenter.)*
2. *$GN = 2 \cdot IG$.*
3. ***Speiker center*** *S: The incircle of the medial triangle is called the Speiker circle, and its center is* ***Speiker center*** *S. Prove that S is the midpoint of IN.*

2 Olympiad Problems

The problems below are not sorted by difficulty. These are really nice problems, so try all of them :)

□ **Problem 2.1.** *Let PB and PC are tangent to $\odot ABC$. Let D, E, F are projection of A on BC, PB, PC resp. Prove that $AD^2 = AE \times AF$.*

□ **Problem 2.2.** *Let D and E are on AB and AC s.t $DE \parallel BC$. P is an arbitrary point inside $\triangle ADE$. $PB, PC \cap DE = F, G$. Let $\odot PDG \cap \odot PFE = Q$. Prove that A, P, Q are collinear.*

□ **Problem 2.3.** *Let AB and CD be chords in a circle of center O with A, B, C, D distinct , and with the lines AB and CD meeting at a right angle at point E. Let also M and N be the midpoints of AC and BD respectively . If $MN \perp OE$, prove that $AD \parallel BC$*

□ **Problem 2.4.** *Circles $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at A and B. Let $M \in AB$. A line through M (different from AB) cuts circles $\mathcal{C}_1$ and $\mathcal{C}_2$ at Z, D, E, C respectively such that $D, E \in ZC$. Perpendiculars at B to the lines EB, ZB and AD respectively cut circle $\mathcal{C}_2$ in F, K and N. Prove that $KF = NC$.*

□ **Problem 2.5.** *Let D be a point on side AC of triangle ABC. Let E and F be points on the segments BD and BC respectively, such that $\angle BAE = \angle CAF$. Let P and Q be points on BC and BD respectively, such that EP and FQ are both parallel to CD. Prove that $\angle BAP = \angle CAQ$.*

□ **Problem 2.6.** *In the non-isosceles triangle ABC an altitude from A meets side BC in D . Let M be the midpoint of BC and let N be the reflection of M in D. The circumcirle of triangle AMN intersects the side AB in $P \neq A$ and the side AC in $Q \neq A$. Prove that AN, BQ and CP are concurrent.*

□ **Problem 2.7.** *In triangle ABC, the interior and exterior angle bisectors of $\angle BAC$ intersect the line BC in D and E, respectively. Let F be the second point of intersection of the line AD with the circumcircle of the triangle ABC. Let O be the circumcenter of the triangle $ABC$and let D' be the reflection of D in O. Prove that $\angle D'FE = 90$.*

□ **Problem 2.8.** *Let $ABCD$ be a convex quadrilateral such that the line BD bisects the angle ABC. The circumcircle of triangle ABC intersects the sides AD and CD in the points P and Q, respectively. The line through D and parallel to AC intersects the lines BC and BA at the points R and S, respectively. Prove that the points P, Q, R and S lie on a common circle.*

□ **Problem 2.9.** *The incircle of triangle ABC touches BC, CA, AB at points A_1, B_1, C_1, respectively. The perpendicular from the incenter I to the median from vertex C meets the line A_1B_1 in point K. Prove that CK is parallel to AB.*

□ **Problem 2.10.** *Let X be an arbitrary point inside the circumcircle of a triangle ABC. The lines BX and CX meet the circumcircle in points K and L respectively. The line LK intersects BA and AC at points E and F respectively. Find the locus of points X such that the circumcircles of triangles AFK and AEL touch.*

□ **Problem 2.11.** *Let BD be a bisector of triangle ABC. Points I_a, I_c are the incenters of triangles ABD, CBD respectively. The line I_aI_c meets AC in point Q. Prove that $\angle DBQ = 90°$.*

□ **Problem 2.12.** *Given right-angled triangle ABC with hypotenuse AB. Let M be the midpoint of AB and O be the center of circumcircle ω of triangle CMB. Line AC meets ω for the second time in point K. Segment KO meets the circumcircle of triangle ABC in point L. Prove that segments AL and KM meet on the circumcircle of triangle ACM.*

□ **Problem 2.13.** *Let BN be median of triangle ABC. M is a point on BC. S lies on BN such that $MS \parallel AB$. P is a point such that $SP \perp AC$ and $BP \parallel AC$. MP cuts AB at Q. Prove that $QB = QP$.*

□ **Problem 2.14.** *Let $ABCD$ be a convex quadrilateral with AB parallel to CD. Let P and Q be the midpoints of AC and BD, respectively. Prove that if $\angle ABP = \angle CBD$, then $\angle BCQ = \angle ACD$.*

□ **Problem 2.15.** *Point P lies inside a triangle ABC. Let D, E and F be reflections of the point P in the lines BC, CA and AB, respectively. Prove that if the triangle DEF is equilateral, then the lines AD, BE and CF intersect in a common point.*

□ **Problem 2.16.** *Let $\triangle ABC$ be an acute angled triangle. The circle with diameter AB intersects the sides AC and BC at points E and F respectively. The tangents drawn to the circle through E and F intersect at P. Show that P lies on the altitude through the vertex C.*

□ **Problem 2.17.** *Let γ be circle and let P be a point outside γ. Let PA and PB be the tangents from P to γ (where $A, B \in \gamma$). A line passing through P intersects γ at points Q and R. Let S be a point on γ such that $BS \parallel QR$. Prove that SA bisects QR*

□ **Problem 2.18.** *Given is a convex quadrilateral $ABCD$ with $AB = CD$. Draw the triangles ABE and CDF outside $ABCD$ so that $\angle ABE = \angle DCF$ and $\angle BAE = \angle FDC$. Prove that the midpoints of $\overline{AD}$, $\overline{BC}$ and $\overline{EF}$ are collinear*

□ **Problem 2.19.** *Let P be a point out of circle C. Let PA and PB be the tangents to the circle drawn from C. Choose a point K on AB . Suppose that the circumcircle of triangle PBK intersects C again at T. Let P' be the reflection of P with respect to A. Prove that*

$$\angle PBT = \angle P'KA$$

□ **Problem 2.20.** *Consider a circle C_1 and a point O on it. Circle C_2 with center O, intersects C_1 in two points P and Q. C_3 is a circle which is externally tangent to C_2 at R and internally tangent to C_1 at S and suppose that RS passes through Q. Suppose X and Y are second intersection points of PR and OR with C_1. Prove that QX is parallel with SY.*

□ **Problem 2.21.** *In triangle ABC we have $\angle A = \frac{\pi}{3}$. Construct E and F on continue of AB and AC respectively such that $BE = CF = BC$. Suppose that EF meets circumcircle of $\triangle ACE$ in K. ($K \not\equiv E$). Prove that K is on the bisector of $\angle A$*

□ **Problem 2.22.** *In triangle ABC, $\angle A = 90°$ and M is the midpoint of BC. Point D is chosen on segment AC such that $AM = AD$ and P is the second meet point of the circumcircles of triangles $\Delta AMC, \Delta BDC$. Prove that the line CP bisects $\angle ACB$*

□ **Problem 2.23.** *Let C_1, C_2 be two circles such that the center of C_1 is on the circumference of C_2. Let C_1, C_2 intersect each other at points M, N. Let A, B be two points on the circumference of C_1 such that AB is the diameter of it. Let lines AM, BN meet C_2 for the second time at A', B', respectively. Prove that $A'B' = r_1$ where r_1 is the radius of C_1.*

□ **Problem 2.24.** *Given a triangle ABC, let P lie on the circumcircle of the triangle and be the midpoint of the arc BC which does not contain A. Draw a straight line l through P so that l is parallel to AB. Denote by k the circle which passes through B, and is tangent to l at the point P. Let Q be the second point of intersection of k and the line AB (if there is no second point of intersection, choose $Q = B$). Prove that $AQ = AC$.*

□ **Problem 2.25.** *Let $ABCD$ be a cyclic quadrilateral in which internal angle bisectors $\angle ABC$ and $\angle ADC$ intersect on diagonal AC. Let M be the midpoint of AC. Line parallel to BC which passes through D cuts BM at E and circle $ABCD$ in F ($F \neq D$). Prove that $BCEF$ is parallelogram*

□ **Problem 2.26.** *The side BC of the triangle ABC is extended beyond C to D so that $CD = BC$. The side CA is extended beyond A to E so that $AE = 2CA$. Prove that, if $AD = BE$, then the triangle ABC is right-angled*

□ **Problem 2.27.** *$ABCD$ is a cyclic quadrilateral inscribed in the circle Γ with AB as diameter. Let E be the intersection of the diagonals AC and BD. The tangents to Γ at the points C, D meet at P. Prove that $PC = PE$*

□ **Problem 2.28.** *The quadrilateral $ABCD$ is inscribed in a circle. The point P lies in the interior of $ABCD$, and $\angle PAB = \angle PBC = \angle PCD = \angle PDA$. The lines AD and BC meet at Q, and the lines AB and CD meet at R. Prove that the lines PQ and PR form the same angle as the diagonals of $ABCD$*

□ **Problem 2.29.** *Let $ABCD$ be a cyclic quadrilateral with opposite sides not parallel. Let X and Y be the intersections of AB, CD and AD, BC respectively. Let the angle bisector of $\angle AXD$ intersect AD, BC at E, F respectively, and let the angle bisectors of $\angle AYB$ intersect AB, CD at G, H respectively. Prove that $EFGH$ is a parallelogram.*

□ **Problem 2.30.** *Triangle ABC is given with its centroid G and cicumcentre O is such that GO is perpendicular to AG. Let A' be the second intersection of AG with circumcircle of triangle ABC. Let D be the intersection of lines CA' and AB and E the intersection of lines BA' and AC. Prove that the circumcentre of triangle ADE is on the circumcircle of triangle ABC*

□ **Problem 2.31.** *Let M be the midpoint of the side AC of $\triangle ABC$. Let $P \in AM$ and $Q \in CM$ be such that $PQ = \frac{AC}{2}$. Let (ABQ) intersect with BC at $X \neq B$ and (BCP) intersect with BA at $Y \neq B$. Prove that the quadrilateral $BXMY$ is cyclic.*

□ **Problem 2.32.** *Let be given a triangle ABC and its internal angle bisector BD ($D \in BC$). The line BD intersects the circumcircle Ω of triangle ABC at B and E. Circle ω with diameter DE cuts Ω again at F. Prove that BF is the symmedian line of triangle ABC.*

□ **Problem 2.33.** *ΔABC is a triangle such that $AB \neq AC$. The incircle of ΔABC touches BC, CA, AB at D, E, F respectively. H is a point on the segment EF such that $DH \perp EF$. Suppose $AH \perp BC$, prove that H is the orthocenter of ΔABC.*

□ **Problem 2.34.** *Let ABC be a triangle and let P be a point on the angle bisector AD, with D on BC. Let E, F and G be the intersections of AP, BP and CP with the circumcircle of the triangle, respectively. Let H be the intersection of EF and AC, and let I be the intersection of EG and AB. Determine the geometric place of the intersection of BH and CI when P varies*

□ **Problem 2.35.** *Let $D; E; F$ be the points on the sides $BC; CA; AB$ respectively, of $\triangle ABC$. Let $P; Q; R$ be the second intersection of $AD; BE; CF$ respectively, with the cricumcircle of $\triangle ABC$. Show that*

$$\frac{AD}{PD} + \frac{BE}{QE} + \frac{CF}{RF} \geq 9$$

□ **Problem 2.36.** *Points D and E lie on sides AB and AC of triangle ABC such that $DE \parallel BC$. Let P be an arbitrary point inside ABC. The lines PB and PC intersect DE at F and G, respectively. If O_1 is the circumcenter of PDG and O_2 is the circumcenter of PFE, show that $AP \parallel O_1O_2$.*

□ **Problem 2.37.** *Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$*

□ **Problem 2.38.** *Let O and I be the circumcenter and incenter of triangle ABC, respectively. Let ωA be the excircle of triangle ABC opposite to A; let it be tangent to AB, AC, BC at K, M, N, respectively. Assume that the midpoint of segment KM lies on the circumcircle of triangle ABC. Prove that $O; N; I$ are collinear.*

□ **Problem 2.39.** *Let $ABCD$ be a cyclic quadrilateral. Let $AB \cap CD = P$ and $AD \cap BC = Q$. Let the tangents from Q meet the circumcircle of $ABCD$ at E and F. Prove that $P; E; F$ are collinear.*