# India
## Regional Mathematical Olympiad
### 2004

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1. Consider in the plane a circle $\Gamma$ with centre O and a line l not intersecting the circle. Prove that there is a unique point Q on the perpendicular drawn from O to line l, such that for any point P on the line l, PQ represents the length of the tangent from P to the given circle.

2. Positive integers are written on all the faces of a cube, one on each. At each corner of the cube, the product of the numbers on the faces that meet at the vertex is written. The sum of the numbers written on the corners is 2004. If T denotes the sum of the numbers on all the faces, find the possible values of T.

3. Let $\alpha$ and $\beta$ be the roots of the equation $x^2 + mx - 1 = 0$ where $m$ is an odd integer. Let $\lambda_n = \alpha^n + \beta^n, n \geq 0$ Prove that (A) $\lambda_n$ is an integer (B) gcd ( $\lambda_n, \lambda_{n+1}) = 1$ .

4. Prove that the number of triples $(A, B, C)$ where $A, B, C$ are subsets of $\{1, 2, \ldots, n\}$ such that $A \cap B \cap C = \phi$ , $A \cap B \neq \phi$, $C \cap B \neq \phi$ is $7^n - 2.6^n + 5^n$.

5. Let ABCD be a quadrilateral; X and Y be the midpoints of AC and BD respectively and lines through X and Y respectively parallel to BD, AC meet in O. Let P,Q,R,S be the midpoints of AB, BC, CD, DA respectively. Prove that

   (A) APOS and APXS have the same area (B) APOS, BQOP, CROQ, DSOR have the same area.

6. Let $p_1, p_2, \ldots$ be a sequence of primes such that $p_1 = 2$ and for $n \geq 1, p_{n+1}$ is the largest prime factor of $p_1 p_2 \ldots p_n + 1$ . Prove that $p_n \neq 5$ for any $n$.

7. Let $x$ and $y$ be positive real numbers such that $y^3 + y \leq x - x^3$. Prove that

   (A) $y < x < 1$ (B) $x^2 + y^2 < 1$.