NZ Math Olympiad Training 2005

Assignment 3: Geometry

Problems

1. Given a triangle ABC with $\angle A = 70^\circ$. Let I be the incentre of ABC. Suppose that $CA + AI = BC$. Find $\angle B$.

2. A circle ω centred at O touches the sides of an angle with the vertex A at points B and C. A point M is chosen on the largest arc of ω with the endpoints B and C so that it is different from B and C and does not lie on AO. The lines BM and CM intersect AO at points P and Q, respectively. Let K be the foot of perpendicular drawn from P on AC, and L be the foot of perpendicular drawn from Q on AB. Prove that $OM \perp KL$.

3. The inscribed circle of triangle ABC touches AB, BC, and CA at points L, N, E, respectively. The line LE intersects the line BC at H, and LN intersects the line AC at J, and the points H, J, N, E lie on the same side of the line AB. Let O and P be the midpoints of the segments EJ and NH, respectively. Given that $\text{Area}(ABOP) = u^2$ and $\text{Area}(COP) = v^2$, find $\text{Area}(HJNE)$.

4. For which positive integers n the inequality
$$\sin n\alpha + \sin n\beta + \sin n\gamma < 0$$
holds for all α, β, γ, which are the angles of an acute-angled triangle?

5. Let $ABCD$ be a convex quadrilateral. Prove that
$$\text{Area}(ABCD) \leq \frac{(AB)^2 + (BC)^2 + (CD)^2 + (DA)^2}{4}.$$

6. Three circles $\omega_1, \omega_2, \omega_3$, each of radius r, pass through the point S and touch internally a circle ω of radius R at points T_1, T_2, T_3, respectively. Prove that the line T_1T_2 passes through the point of intersection of the circles ω_1 and ω_2, different from S.

7. (Shortlist, 2004) The circle Γ and the line ℓ do not intersect. Let AB be the diameter of Γ perpendicular to ℓ, with B closer to ℓ than A. An arbitrary point C, different from A and B is chosen on Γ. The line AC intersects ℓ at D. The line DE is tangent to Γ at E, with B and E on the same side of AC. Let BE intersect ℓ at F and let AF intersect Γ at $G \neq A$. Prove that the reflection of G in AB lies on the line CF.

8. (Shortlist, 2004) Let O be the circumcentre of an acute-angled triangle ABC with $\angle B < \angle C$. The line AO meets the side BC at D. The circumcentres of the triangles ABD and ACD are E and F, respectively. Extend the sides BA and CA beyond A, and choose on the respective extensions points G and H such that $AG = AC$ and $AH = AB$. Prove that the quadrilateral $EFGH$ is a rectangle if and only if $\angle ACB - \angle ABC = 60°$.

9. (Shortlist, 2004) Let $\mathcal{P}$ be a convex polygon. Prove that there is a convex hexagon which is contained in $\mathcal{P}$ and which occupies at least 75 percent of the area of $\mathcal{P}$.[1]

10. (Shortlist, 2004) A cyclic quadrilateral $ABCD$ is given. The lines AD and BC intersect at E, with C between B and E; the diagonals AC and BD intersect at F. Let M be the midpoint of the side CD, and let $N \neq M$ be a point on the circumcircle of the triangle ABM such that $AN/BN = AM/BM$. Prove that the points E, F and N are collinear.

[1] This problem was considered by the Jury as the most beautiful. However it was the first to be rejected as Jury was affraid that it might be known.