houssam9990

#10 Apr 7, 2016, 2:44 am

(1): $XT\cap YZ=P$.

proof:

let $TZ\cap XY=R, XZ\cap TY=W$.

apply pascal respectively to the hexagons:

$(TTXYYZ);(XXTZZY);(YYZXXT);(TTXZZY)$ to see that $XT\cap YZ\in AC\cap BD=P$.
next let G the intersection of the tangent at P to (ABP) and AB.
(2): $S\in\Gamma(G,r=GP)$.

proof:

on the one hand:

$XPG=XPB-GPB=180-AXT-XBD-BAC=ABD-BAC+180-AXT$

on the other hand $2AXT=180-BAD-ADC$, equating the latters yellds $XPG=GXP$,

hence $GX^2=GP^2=GB.GA\rightarrow G$ is on the radical axis of $(XYZ);(\Omega)$ i.e lays on the tangent of (XYZ) at S.

(3): $TSP=90$

proof:

$TSP=PSG+GST=90-1/2PGS+TXS=90-PXS+PXS=90$ which ends the proof.

This post has been edited 2 times. Last edited by houssam9990, Apr 7, 2016, 2:50 am

EulerMacaroni

#12 May 14, 2016, 4:33 am

First by Pascal on $XXTYYZ$ and $TTXZZY$, we get that A, C, and $XT\cap ZY$ are collinear. Applying Pascal again on $XXTZZY$ and $YYXTTZ$ gives that B, D, and $XT\cap ZY$ are collinear, so we conclude that $P\equiv XT\cap ZY$. Then

$$\angle ZYX=\frac{\angle ZOX}{2}=\frac{180^\circ-\angle ZBX}{2}=\frac{\angle ABC}{2}=\frac{180^\circ-\angle ADC}{2}=\frac{\angle TDY}{2}=\frac{180^\circ-\angle TOY}{2}=90^\circ-\angle TXY$$

so that $\angle YPZ=90^\circ$. If $F\equiv YZ\cap AB$, then

$$(A,B;F,X)\overset{Y}{=}(D,B;P,XY\cap BD)\overset{X}{=}(DX\cap\odot(XYZ),X;T,Y)=-1$$

which, combined with $\angle FPX=90^\circ$, gives that PF and PX are the internal and external angle bisectors of $\angle APB$. Then $\odot(FPX)$ is the P-Apollonius circle in $\triangle APB$, so its center M lies on the common internal tangent of $\odot(ASB)$ and $\odot(XYZ)$ by the radical axis theorem, hence S lies on $\odot(FPX)$ as well. Finally,

$$\angle SPT=\angle SPX=\angle SFX$$

and

$$\angle STP=180^\circ-\angle STX=\angle SYX=\angle SXF$$

whence $\angle PST=\angle FSX=90^\circ$, as desired.

Quick Reply

High School Olympiads

Two problems with nine point circle

Reply

Source: Own

buratinogigle

#1 Mar 21, 2016, 8:10 am

Problem 1. Let ABC be a triangle and d_1,d_2 are two perpendicular lines passing through circumcenter O of triangle ABC. A_1,A_2 are projections of A on d_1,d_2. Similarly, we have B_1,B_2,C_1,C_2. Prove that A_1A_2,B_1B_2,C_1C_2 bound a triangle whose incenter lies on nine point circle of triangle ABC.

Problem 2. Let ABC be a triangle and d_1,d_2 are two perpendicular lines passing through incenter I of triangle ABC. A_1,A_2 are projections of A on d_1,d_2. Similarly, we have B_1,B_2,C_1,C_2. Prove that A_1A_2,B_1B_2,C_1C_2 bound a triangle whose circumcenter lies on nine point circle of triangle ABC.

Luis González

#2 Mar 21, 2016, 9:12 am • 1

> *buratinogigle wrote:*
> **Problem 1.** Let ABC be a triangle and d_1,d_2 are two perpendicular lines passing through circumcenter O of triangle ABC. A_1,A_2 are projections of A on d_1,d_2. Similarly, we have B_1,B_2,C_1,C_2. Prove that A_1A_2,B_1B_2,C_1C_2 bound a triangle whose incenter lies on nine point circle of triangle ABC.

The perpendicularity $d_1 \perp d_2$ is unnecessary. Let D,E,F be the midpoints of $BC,CA,AB.$ Since $O \in d_1,$ then the reflection T_1 of A_1 across EF is the orthopole of d_1 WRT $\triangle ABC$ (well-known) and similarly the reflection T_2 of A_2 across EF is the orthopole of d_2 WRT $\triangle ABC \Longrightarrow A_1A_2$ is the reflection of T_1T_2 across $EF.$ Analogously B_1B_2 and C_1C_2 are the reflections of T_1T_2 across $FD,DE.$ Now according to the problem The incenter lies on circumcircle [Iran Second Round 95], the incenter of the triangle bounded by A_1A_2,B_1B_2,C_1C_2 lies on $\odot(DEF).$

Luis González

#3 Mar 21, 2016, 10:39 am • 1

> *buratinogigle wrote:*
> **Problem 2.** Let ABC be a triangle and d_1,d_2 are two perpendicular lines passing through incenter I of triangle ABC. A_1,A_2 are projections of A on d_1,d_2. Similarly, we have B_1,B_2,C_1,C_2. Prove that A_1A_2,B_1B_2,C_1C_2 bound a triangle whose circumcenter lies on nine point circle of triangle ABC.

Let $X \equiv B_1B_2 \cap C_1C_2,Y \equiv C_1C_2 \cap A_1A_2,Z \equiv A_1A_2 \cap B_1B_2.$ Let D,E,F,U,V,W be the midpoints of BC,CA,AB,IA,IB,IC and let A',B',C' be the tangency points of the incircle (I) with $BC,CA,AB.$ Since $\triangle VIC_1$ and $\triangle WIB_2$ are isosceles at V and $W,$ we get $\angle IC_1V=\angle VIC_1$ and $\angle IB_2W=\angle WIB_2 \Longrightarrow$ $\angle B_2IC_1=\angle WIB_2+\angle VIC_1+\angle VXW \Longrightarrow$ $\angle YXZ \equiv \angle VXW=90^{\circ}-(90^{\circ}+\tfrac{1}{2}\angle BAC-90^{\circ})=90^{\circ}-\tfrac{1}{2}\angle BAC=\angle B'A'C' \Longrightarrow X \in \odot(DVW)$ and similarly $\angle ZYX=\angle C'B'A' \Longrightarrow \triangle XYZ \sim \triangle A'B'C'.$ Moreover from cyclic $XVDW,$ we get $\angle DXV=\angle DWV=\angle ICA'=\angle IA'B' \Longrightarrow DX$ is the circumdiameter of $\triangle XYZ$ issuing from X and likewise EY and FZ are the circumdiameters of $\triangle XYZ$ issuing from $Y,Z \Longrightarrow J \equiv DX \cap EY \cap FZ$ is the circumcenter of $\triangle XYZ.$ Now from $\triangle XYZ \cup J \sim \triangle A'B'C' \cup I,$ we get $\angle FJE \equiv \angle ZJY=\angle B'IC'=\angle FDE$ (mod 180) $\Longrightarrow$ $J \in \odot(DEF).$

Quick Reply

High School Olympiads

The incenter lies on circumcircle [Iran Second Round 95]

Reply

Amir Hossein

#1 Nov 26, 2010, 1:13 am • 2

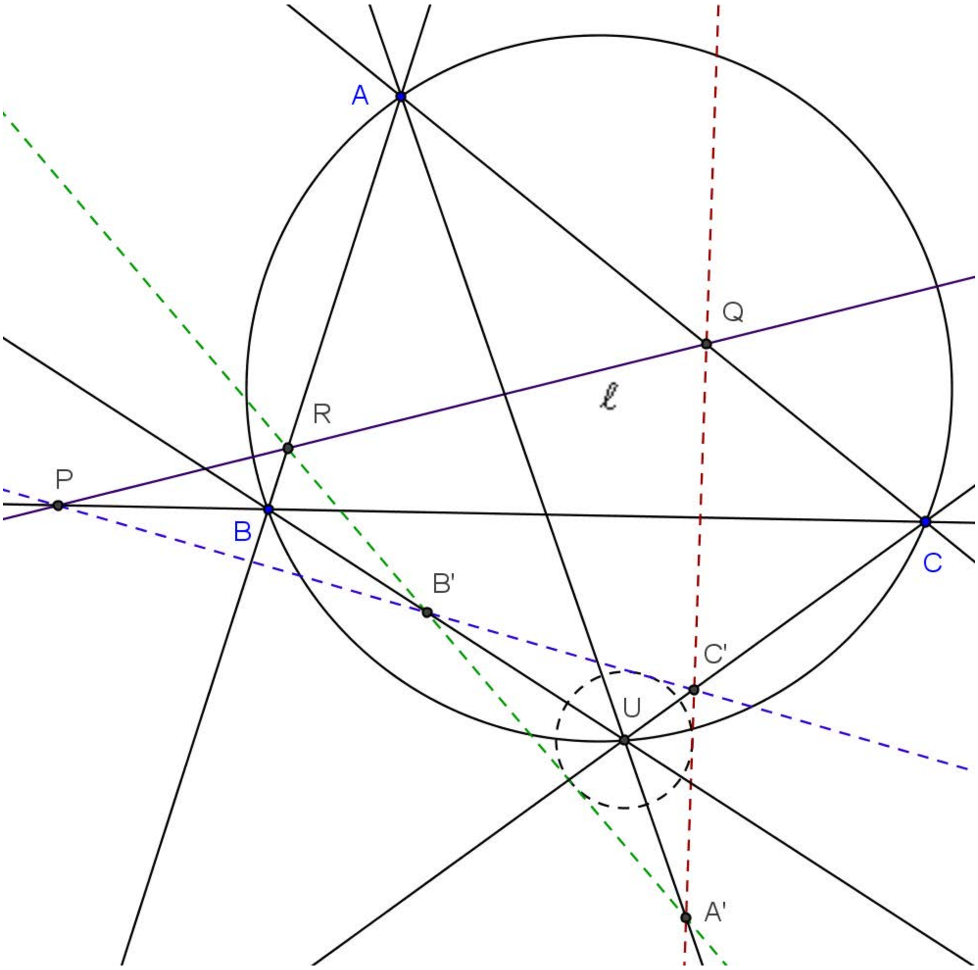
Let ABC be an acute triangle and let ℓ be a line in the plane of triangle $ABC.$ We've drawn the reflection of the line ℓ over the sides AB, BC and AC and they intersect in the points A', B' and C'. Prove that the incenter of the triangle $A'B'C'$ lies on the circumcircle of the triangle $ABC.$

Luis González

#2 Nov 26, 2010, 4:20 am • 3

Let us assume that ℓ cuts $\overline{AC}, \overline{AB}$ at Q, R and the extension of CB at $P.$ PB, RB become internal bisectors of $\angle RPB'$ and $\angle PRB' \Longrightarrow B$ is the incenter of $\triangle PRB' \Longrightarrow BB'$ is the internal bisector of $\angle A'B'C'$. QC and PC become external bisector and internal bisector of $\angle PQC'$ and $\angle QPC'$, respectively $\Longrightarrow C$ is the P-excenter of $\triangle PQC' \Longrightarrow CC'$ is the internal bisector of $\angle AC'B'$. Therefore, $U \equiv BB' \cap CC'$ is the incenter of $\triangle A'B'C'$. From the same fact that B and C are the incenter and P-excenter of $\triangle PRB'$ and $\triangle PQC'$, it follows that $\angle CBU = 90^\circ - \angle PRB$ and $\angle BCU = 90^\circ - \angle AQR.$ Then $\angle CBU + \angle BCU = 180^\circ - (\angle PRB + \angle AQR) = \angle BAC$ and the result follows.

Attachments:



jayme

#3 Jun 9, 2014, 8:12 pm

Dear Mathlinkers

Dear Mathlinkers,
in order to avoid some angle chasing, we can use the Beltrami's theorem.
Sincerely
Jean-Louis

v_Enhance
#4 Jun 15, 2014, 11:18 am

In the notation of a certain IMO 6, let ℓ_a, ℓ_b, ℓ_c denote the reflections of ℓ across $\overline{BC}$, $\overline{CA}$, $\overline{AB}$. Let $A' = \ell_b \cap \ell_c$ and define B, C similarly. Let I be the incenter of $\triangle A'B'C'$.
Indeed, let X and Y denote the reflections of A' across AB and AC. Then $AX = AY$, whence $\angle B'A'A = \angle YXA = \angle XYA = \angle C'A'A$, so $\overline{A'A}$ bisects $\angle A'B'C'$. Thus we discover that I is the concurrency point of $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$.

Using directed angles, compute

$$\measuredangle(\ell_B, \ell_C) = -\measuredangle(\ell_B, \ell) - \measuredangle(\ell, \ell_C) = -2\measuredangle(\overline{AC}, \ell) - 2\measuredangle(\ell, \overline{AB}) = 2\measuredangle(\overline{AC}, \overline{AB}).$$

It follows that $\angle B'A'C' = 180 - 2A$. Now $\angle B'IC' = 90^\circ + \angle BIC = 180^\circ - A$. It follows that A, B, I, C are concyclic. $\blacksquare$

Unfortunately, discovering this did not help me solve said #6.

jayme
#5 Jun 15, 2014, 12:11 pm

Dear Mathlinkers,
This is the Collings-Lalesco theorem...
http://perso.orange.fr/jl.ayme vol. 17 Une droite et un triangle

Sincerely
Jean-Louis

TelvCohl
#6 Oct 20, 2014, 10:46 pm • 1

The incenter of $\triangle A'B'C'$ is actually the pole of the Simson line which is parallel to ℓ .

Quick Reply

High School Olympiads

Lies On Radical Axis

Reply

Source: Own?

doxuanlong15052000

#1 Mar 15, 2016, 10:07 pm

Let ABC be a triangle with incircle (I) tangent to BC at D. The circle w passes through B and C and tangent to (I) at L. Then A lies on radical axis of w and $\odot(IDL)$.

This post has been edited 2 times. Last edited by Luis González, Mar 16, 2016, 8:38 am
Reason: Fixing notation

Luis González

#2 Mar 15, 2016, 10:32 pm • 4

Let λ be the circle tangent to AB, AC and internally tangent to w through its arc BC (not containing L). As A, L, X are the exsimilicenters of $(I), w$ and $\lambda,$ then by Monge & d'Alembert theorem they are collinear. Thus it is enough to prove that $LIDX$ is cyclic.

It's well-known that LD bisects $\angle BLC$ and XI bisects $\angle BXC,$ thus if we assume WLOG that $\angle LBC > \angle LCB,$ we get $\angle LXI = \frac{1}{2}\angle BXC - \angle BXL = \frac{1}{2}(180^\circ - \angle BLC) - \angle LCB = \frac{1}{2}(\angle LBC - \angle LCB) = \angle LDI \Longrightarrow LIDX$ is cyclic, as desired.

jjlim7

#3 Mar 20, 2016, 6:52 am

why does XI bisect BXC?
sorry i'm not my best nowadays

Luis González

#4 Mar 20, 2016, 7:21 am

Dear jjlim7, that's not a silly question. Please see Fairly difficult and incenter of triangle for some proofs. I've seen this fact referred as **Protassov theorem**.

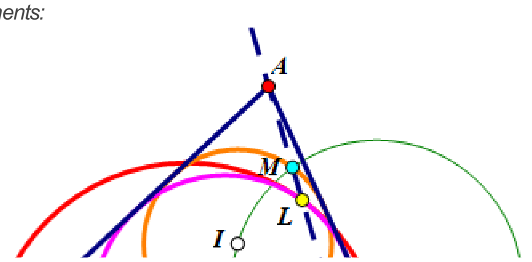
Lin_yangyuan

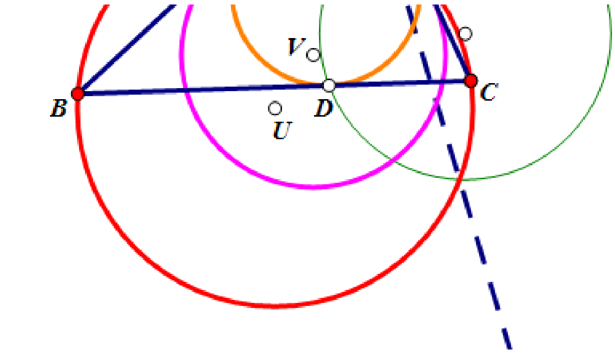
#5 Apr 25, 2016, 9:59 pm

> *doxuanlong15052000 wrote:*
> Let ABC be a triangle with incircle (I) tangent to BC at D. The circle w passes through B and C and tangent to (I) at L. Then A lies on radical axis of w and $\odot(IDL)$.

It's not a new conclusion and we can generalize it in this way: :)

Attachments:





Quick Reply

High School Olympiads

Fairly difficult

Reply

Source: Iran 1999

Omid Hatami
#1 Jun 10, 2004, 2:21 pm • 2

Given a triangle ABC.
Suppose that a circle ω passes through A and C, and intersects AB and BC in D and E.
A circle S is tangent to the segments DB and EB and externally tangent to the circle ω and lies inside of triangle ABC.
Suppose that the circle S is tangent to ω at M.
Prove that the angle bisector of the angle $\angle AMC$ passes through the incenter of triangle ABC.

This post has been edited 1 time. Last edited by Omid Hatami, Jun 10, 2004, 2:28 pm

grobber
#2 Jun 10, 2004, 2:27 pm

I seem to remember it from somewhere: it was weither posted before or I've struggled to solve it. Anyway, what I wanted to say was this: isn't S tangent to ω, BE and BD? (instead of BD you wrote ED).

sam-n
#3 Jun 10, 2004, 2:52 pm

yes I posted it before and noone manage to give a sol.official sol is really floundering but I know really elementary one.

Omid Hatami
#4 Jun 11, 2004, 10:19 am • 1

Last year,I heared from Gharakhani (One of the people in iranian team for IMO2004) that it has an elementary solution.But I don't know that solution.
So send that solution.

sprmnt21
#5 Jul 25, 2004, 8:49 pm

I proved a lemma that can solve quite strightforward the problem. I need some time to translate from italian to english the statment and the solution. In the meantime, if you can understand some italian you can have a look at

http://olimpiadi.sns.it/modules.php?op=modload&name=Forums&file=viewtopic&topic=2635&forum=5

Charlydif
#6 Jul 26, 2004, 8:14 am

I solved this problem some time ago using harmonic conjugates and some properties of projective geomtry........if i remember i will post the solution which was rteally nice.....
(I remember that g7 of isl 2002 can be solved similary)

sprmnt21
#7 Jul 31, 2004, 12:54 pm • 1

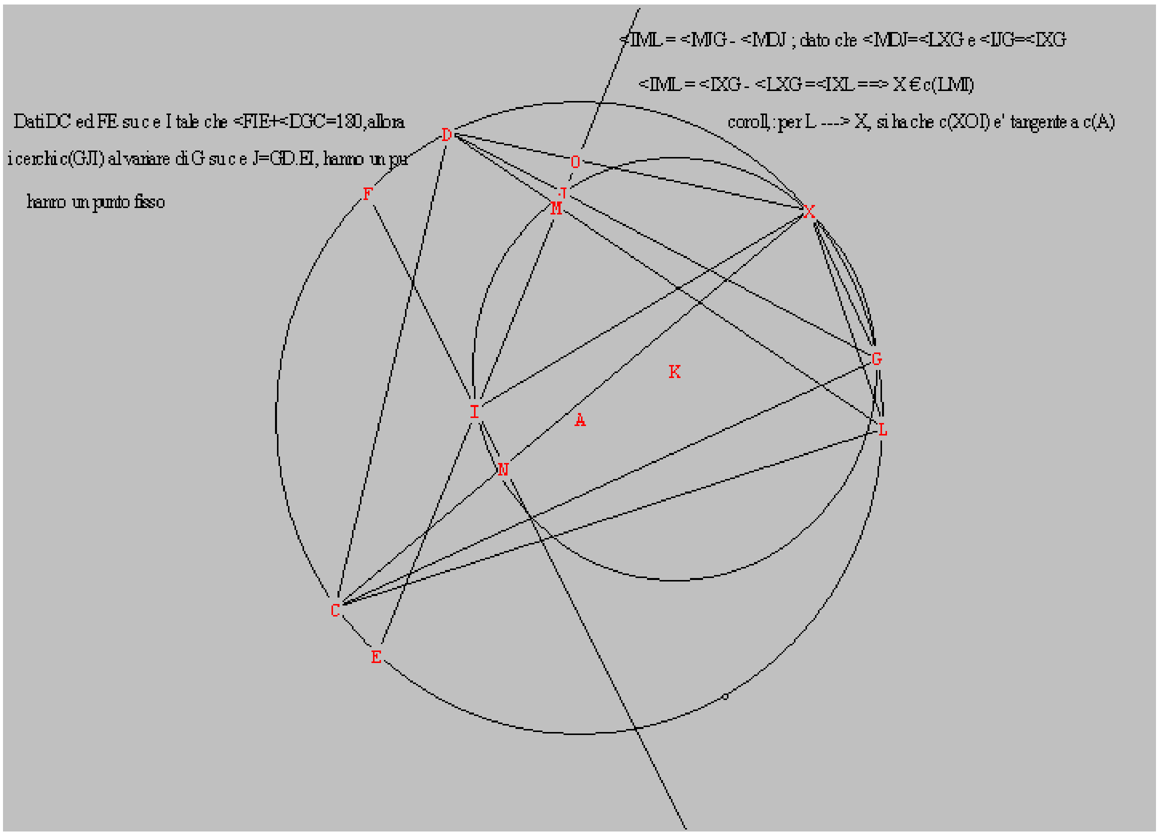
Iran99

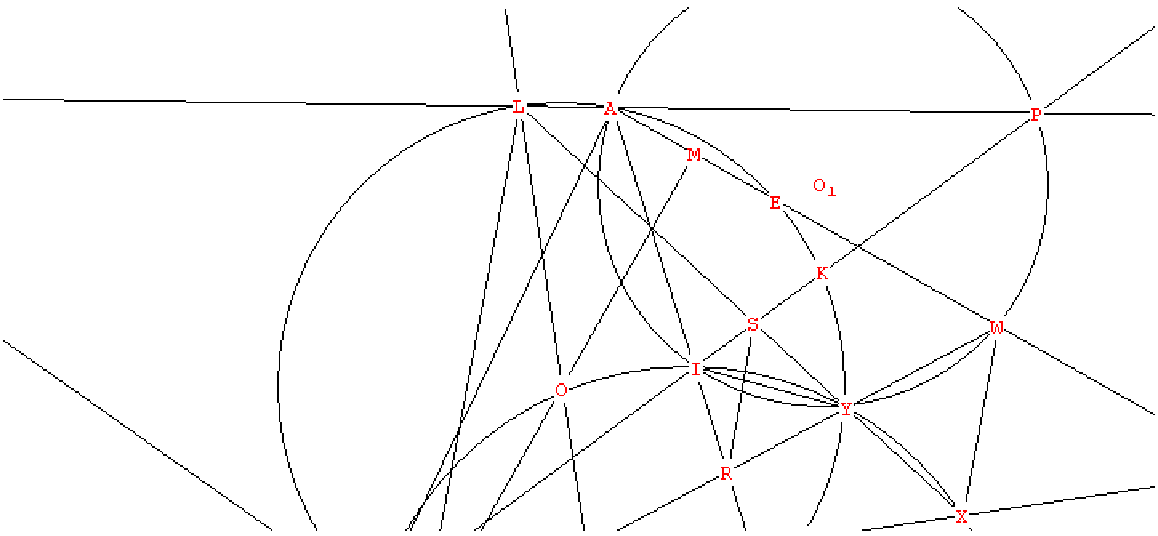
Let E and F be the intersections of c(O) with AC and BC resp.; M and N be the midpoints of AE and BF; G and L be the inters. Of circle c(O) with lines MO and NO resp.
If I is the incenter of ABC lets call P the inters. Between LA and BI and Q the inters. Between GB and AI. From the lemma RLxxxx we have that c(API) and c(BQI) intersect c(O) in the same point Y. Lets call W and X the intersection between c(API) with AC and c(BQI) with BC.
Let H and K be the intersection of c(O) with AI and BI. It is not too difficult to prove that LG//HK then LH=GK from which < IAP = < IBQ. Then ABPQ is cyclic and it easy to prove that P,W,X,Q are collinear and WX//LG.
If R and S are the inters. Of YG with AI and YL with BI from lemma RL250704 we have that YSIR is cyclic and tangent to c(O). Then SR//LG.
From < IYR = <ISR = <IKH = < BAH = < IAW it follows that W, Y and G are collinear. Similarly one can prove that X, Y and L are collinear too. So c(XYW) is tangent to c(O).
From < AWY = < RIY = < RSY = <YXW it follows that c(XYW) is tangent to AC and similarly one can prove that is tangent to BC too.
To conclude the proof observe tha < API = < AYI and < BQI = < BYI, but we know that < API = < BQI from which we have the thesis.

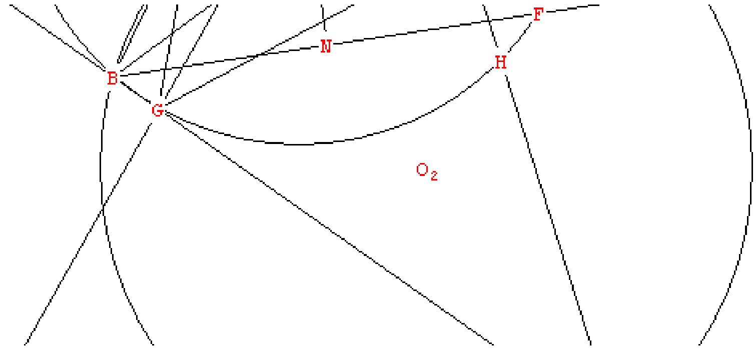
lemma RL250704

Dato un cerchio c e un triangolo ABC inscritto e siano R ed S due punti su c(ABC) e I un punto interno tale che < RIS sia supplementare ad < ABC. Se T = AC.SI provare che, al variare di C (fissati A e B) c(CTI) ha un punto fisso.

Attachments:







sigma

#8 Jul 31, 2004, 10:23 pm

sprmnt21 I don't understand the Lemma RLXXXX that you mean can you give me a proof

sprmnt21

#9 Aug 2, 2004, 12:31 am

Given D, C and F,E on a circe c(A) and I inside so that <FIE + <DGC = 180, where G is a generic point on the circle c(A). Then all the circles c(GJI), where J=GD.EI, as G varies on the circle, pass through a fixed point.

Proof.

If X is the (second) common point to c(A) and c(GJI) for a generic G and let L be a generic different position of G on c(A) and M=LD.EI, we have that <IML = <MJG-<MDJ. But given that <MDJ=<LXG and <IJG = <IXG then <IML=<IXG-<LXG=<IXL ==> X c(LMI) too.
As a corollary of this lemma we have that, when L=X, c(XOI) is tangent to c(A)

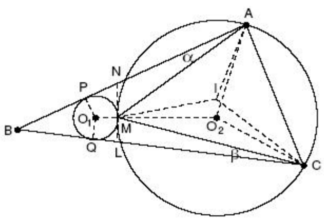
sigma

#10 Aug 8, 2004, 9:25 am

I post my solution here but i write in chinese, anybody help to translate?

Attachments:

证明：如图所示，I 为 $\triangle ABC$ 的内心，只须证 MI 平分 $\angle AMC$，设 $\angle MAB = \alpha, \angle MCB = \beta$。只须证 $\sin\angle AMI = \sin\angle CMI$。$\because$ $\sin\angle AMI = \frac{AI}{MI}\cdot\sin(\frac{A}{2}-\alpha), \sin\angle CMI = \frac{CI}{MI}\cdot\sin(\frac{C}{2}-\theta) \Longrightarrow \frac{\sin\angle AMI}{\sin\angle CMI} = \frac{AI}{CI}\cdot\frac{\sin(\frac{A}{2}-\alpha)}{\sin(\frac{C}{2}-\beta)} = \frac{\sin\frac{C}{2}\sin(\frac{A}{2}-\alpha)}{\sin\frac{A}{2}\sin(\frac{C}{2}-\beta)}$. 令 $x_1 = \frac{A}{2}, x_2 = \frac{A}{2}-\alpha, y_1 = \frac{C}{2}, y_2 = \frac{C}{2}-\beta$，则 $x_1, x_2, y_1, y_2 \in (0, \frac{\pi}{2}) \Longrightarrow \frac{\sin\angle AMI}{\sin\angle CMI} = \frac{\sin y_1 \sin x_2}{\sin x_1 \sin y_2}$ 故只须证：$\sin x_2 \sin y_1 = \sin x_1 \sin y_1 \quad (*)$ 在四边形 AO_1O_2P 中，设 $O_1M = O_1A = R, O_2M = OP = r$



$\angle NMA = \angle MCA = C-\beta \Longrightarrow \angle MNA = \pi - \angle NMA - \angle MAN = \pi+\beta-\alpha-C \quad \angle MO_2P = \angle MNA = \pi+\beta-\alpha-C \Longrightarrow MN = r\tan\frac{1}{2}\angle MO_2P = r\tan\frac{\pi+\beta-\alpha-C}{2} = r\cot\frac{C+\alpha-\beta}{2}, \quad MA = 2R\sin\angle MCA = 2R\sin(C-\beta) \quad \therefore \frac{MA}{MN} = \frac{2\cos^2\frac{C+\alpha-\beta}{2}}{\sin\alpha\sin(C-\beta)} = \frac{1+\cos(C+\alpha-\beta)}{\sin\alpha\sin(C-\beta)} = \frac{1+cos\alpha\cos(C-\beta)}{\sin\alpha\sin(C-\beta)} - 1$。在四边形 CO_1O_2Q 中，同理可得 $\frac{2R}{r} = \frac{1+\cos\beta\cos(A-\alpha)}{\sin\beta\sin(A-\alpha)} = 1$

$$\therefore \frac{1+\cos\alpha\cos(C-\beta)}{\sin\alpha\sin(C-\beta)} = \frac{1+\cos\beta\cos(A-\alpha)}{\sin\beta\sin(A-\alpha)}$$

$$\Longrightarrow \sin\beta\cos(C-\beta)\sin(A-\alpha)\cos\alpha - \cos\beta\sin(C-\beta)\cos(A-\alpha)\sin\alpha = \sin\alpha\sin(C-\beta) - \sin\beta\sin(A-\alpha)$$

$$\Longrightarrow \frac{1}{4}\{[\sin C - \sin(C-2\beta)][\sin A + \sin(A-2\alpha)] - [\sin C + \sin(C-2\beta)][\sin A - \sin(A-2\alpha)]\} =$$

$$\sin\alpha\sin(C-\beta) - \sin\beta\sin(A-\alpha)$$

$$\Longrightarrow \frac{1}{2}[\sin C\sin(A-2\alpha) - \sin A\sin(C-2\beta)] = \sin\alpha\sin(C-\beta) - \sin\beta\sin(A-\alpha)$$

由上面假设可知 $A = 2x_1, \alpha = x_1 - x_2, C = 2y_1, \beta = y_1 - y_2 \Longrightarrow A - 2\alpha = 2x_2, C - 2\beta = 2y_2, A - \alpha = x_1 + x_2, C - \beta = y_1 + y_2$ 则由上式 $\Longrightarrow \sin 2y_1 \sin 2x_2 - \sin 2x_1 \sin 2y_2 = 2\sin(x_1 - x_2)\sin(y_1 + y_2) - 2\sin(y_1 - y_2)\sin(x_1 + x_2) \Longrightarrow 2\sin x_2 \sin y_1 \cos x_2 \cos y_1 - 2\sin x_1 \sin y_2 \cos x_1 \cos y_2 = (\sin x_1 \cos x_2 - \cos x_1 \sin x_2)(\sin y_1 \cos y_2 + \cos y_1 \sin y_2) - (\sin y_1 \cos y_2 - \cos y_1 \sin y_2)(\sin x_1 \cos x_2 + \cos x_1 \sin x_2) = 2\sin x_1 \sin y_2 \cos x_2 \cos y_1 - 2\sin x_2 \sin y_1 \cos x_1 \cos y_2$
$\Longrightarrow \sin x_2 \sin y_1(\cos x_2 \cos y_1 + \cos x_1 \cos y_2) = \sin x_1 \sin y_2(\cos x_2 \cos y_1 + \cos x_1 \cos y_2), \quad \because \cos x_2 \cos y_1 + \cos x_1 \cos y_2 > 0 \quad \therefore$ 只能是 $\sin x_2 \sin y_1 = \sin x_1 \sin y_2$ 故 $(*)$ 式得证。从而有 MI 平分 $\angle AMC$ 即 $\triangle ABC$ 的内心 I 在 $\angle AMC$ 的平

分线上，原命题得证。

jayme

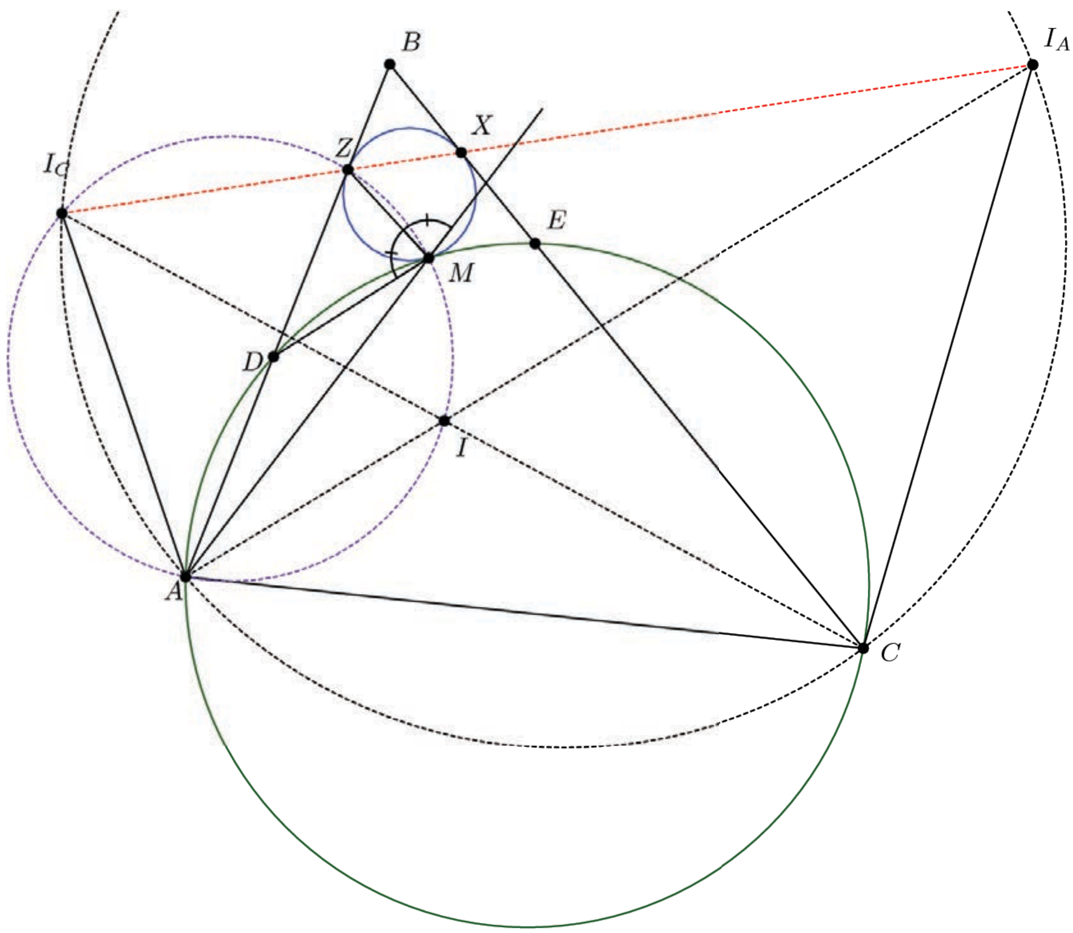
#11 Aug 2, 2008, 8:13 pm • 1

Dear Mathlinkers,
you can see my proof of this problem on my website http://perso.orange.fr/jl.ayme vol. 2 Un remarquable résultat de Vladimir Protassov.
Sincerely
Jean-Louis

drmzjoseph

#14 Mar 25, 2015, 12:10 pm • 3

S touch to AB and BC at Z and X respectively.
Let I_C be the C-excenter of $\triangle AEC$ and I_A the A-excenter of $\triangle ADC$
By Sawayama's Lemma (special case) we obtain I_C, Z, X and I_A are collinear.
$I \equiv AI_A \cap CI_C$ (incenter of $\triangle ABC$)
$\frac{1}{2}\angle AEC = \angle AI_AC = \angle CI_CA \Rightarrow ACI_AI_C$ is cyclic
ZM is exterior angle bisector of $\angle AMD \Rightarrow \angle ACD = 180^\circ - 2\angle DMZ$
$\Rightarrow 180^\circ - \angle AMZ = 180^\circ - \angle ACI_A = \angle AI_CZ \Rightarrow I_C, Z, M, I$ and A are concyclic.
(Because $\angle IAC = \angle ZAI = \angle II_CZ$)
$\Rightarrow \frac{1}{2}\angle AEC = \angle CI_CA = \angle AMI$ and also $\angle AMC = \angle AEC$
Q.E.D



This post has been edited 1 time. Last edited by drmzjoseph, Apr 19, 2015, 1:17 pm
Reason: Asymptote code

Quick Reply

High School Olympiads

incenter of triangle

Reply

Zeus93

#1 May 19, 2011, 7:56 pm • 1

$A, B \in$ circle (S), C is inside (S). Circle (S') is tangent to AC, BC and (S) at P, Q, R respectively. Prove that the circumcircle of triangle APR is go through the incenter of triangle ABC

Luis González

#2 May 21, 2011, 12:16 pm • 6

CB, CA cut (S) again at D, E and I, J are the incenters of $\triangle ABC, \triangle ABD$. By Sawayama's lemma, we have that PQ passes through J. Since B, I, J are collinear on the angle bisector of $\angle CBA$, we obtain

$\angle CPJ = 90^\circ - \frac{1}{2}\angle ACB = \angle AIJ \Longrightarrow P, A, I, J$ are concyclic $(\star)$

Let RA, RE cut (S') again at A', E'. Since R is the exsimilicenter of $(S) \sim (S')$, then $AE \parallel A'E' \Longrightarrow$ Arcs PA', PE' of (S') are congruent $\Longrightarrow RP$ bisects $\angle ARE$. On the other hand, angle chase gives

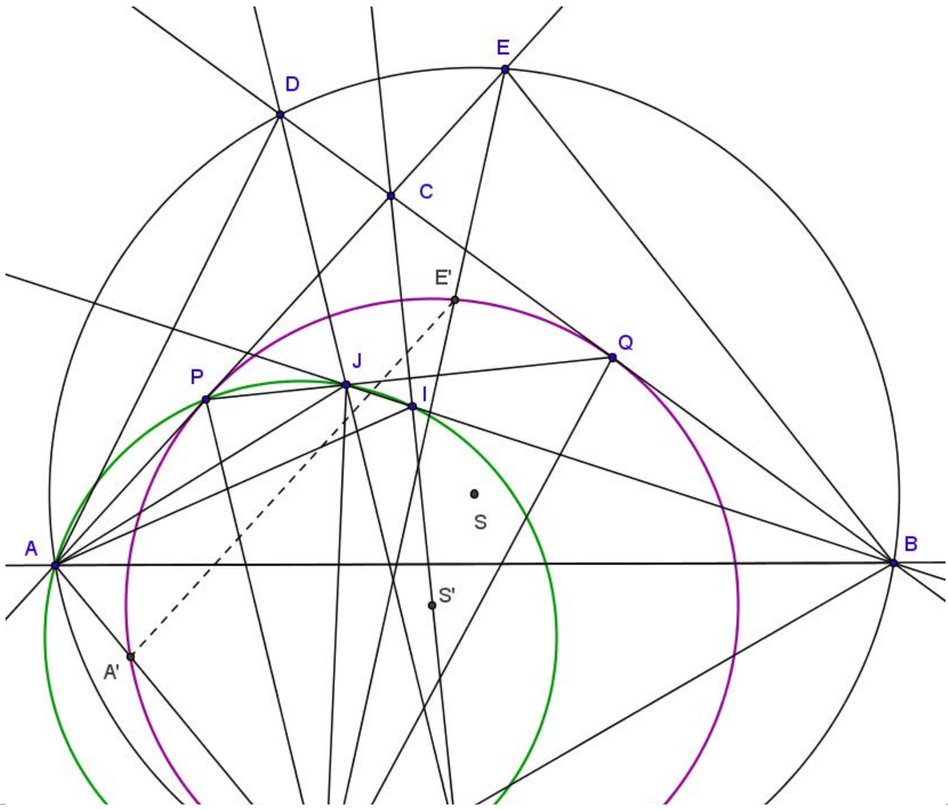
$\angle PJA = \angle CPQ - \angle CAJ = 90^\circ - \frac{1}{2}\angle ACB - \angle CAJ \Longrightarrow$

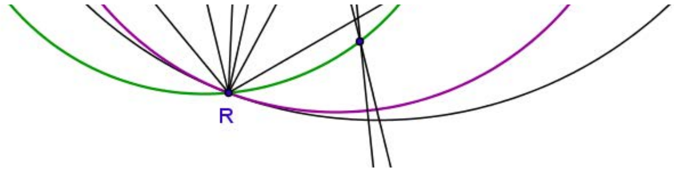
$2\angle PJA = 180^\circ - \angle ACB - 2(\angle EAB - \frac{1}{2}\angle DAB)$

$2\angle PJA = \angle EAB + \angle DBA - 2\angle EAB + \angle DAB = \angle DBA + \angle DAE = \angle ARE$

Since RP bisects $\angle ARE$, then $2\angle PJA = 2\angle PRA \Longrightarrow P, J, R, A$ are concyclic. Together with $(\star)$, we conclude that P, A, R, I are concyclic, as desired. In addition, this result yields that RI bisects $\angle ARB$, which is another celebrated property.

Attachments:





jayme

#3 May 21, 2011, 12:44 pm

Dear Mathlinkers,
this nice problem is used as a lemma to prove the Protassov's result.
For a synthetic proof see
http://perso.orange.fr/jl.ayme vol. 2 Un remarquable résultat de Vladimir Protassov p. 2
Sincerely
Jean-Louis

proglote

#4 Mar 10, 2014, 8:55 am • 1

By Sawayama's lemma, the incenter I_D of $\triangle ABD$ lies on (ARP) and on $PQ \implies \angle ARI_D = \angle CPQ = \angle PRQ$.
Therefore,

$$2\cdot\angle ARI_D \pm \angle PRI_D = \angle ARQ = \angle ARD + \angle DRQ = \angle ABD + \tfrac{1}{2}\angle BAD.$$

But $\pm\angle PRI_D = \pm\angle PAI_D = \frac{1}{2}\angle BAD - \angle CAB$, i.e. $\angle ARI_D = \frac{1}{2}\angle ABD + \frac{1}{2}\angle CAB \implies \angle ARI_D = \angle AII_D$ and $I \in (APRI_D)$.

Another interesting result is that, if $T = AD \cap BE$, then the incenter of ABT lies on IX, the radical axis of the two circles, for AI_DI_EB is cyclic.

mathuz

#5 Mar 10, 2014, 12:35 pm

curvilinear circles!
(S') is curvilinear circle of the triangles ABD, ABE.

jayme

#6 Jun 13, 2015, 12:12 pm

Dear Mathlinkers,

You can see : http://www.artofproblemsolving.com/community/c6h407366 p. 80

Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Nice geometry

Reply

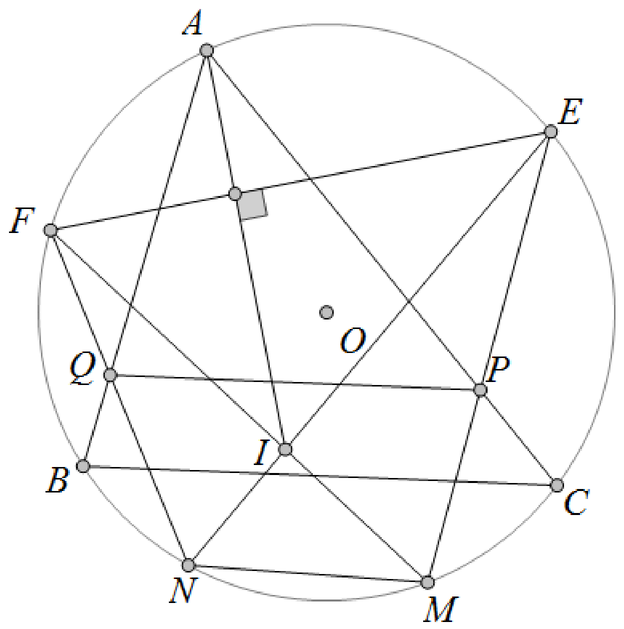
Source: Own

baopbc

#1 Mar 19, 2016, 8:40 pm • 2

Let ABC with O is the circumcentre. The bisectors of $\angle B, \angle C$ intersect (O) at E, F. P, Q lies on AC, AB such that: PQ parallel to BC. EP, FQ intersects (O) at M, N. FM intersects EN at I. Prove that AI is the bisector of $\angle BAC$.

Attachments:



quangMavis1999

#2 Mar 19, 2016, 9:19 pm • 5

By Pascal theorem for $B, F, E, C, M, N \Rightarrow S = BE \cap CF, I, K = BM \cap CN$ are collinear. So, we need to prove AK is bisector WRT $\angle BAC$.

We have

$\frac{AN}{BN} = \frac{AQ}{BQ} = \frac{AP}{PC} = \frac{AM}{CM} \Rightarrow \frac{BK}{CK} = \frac{BN}{CM} = \frac{AN}{AM} \Rightarrow \frac{\mathbf{d}_{K,AB}}{\mathbf{d}_{K,AC}} = \frac{[KAB]}{[KAC]}.\frac{AC}{AB} = \frac{BA.BK.\sin\angle ABK}{CA.CK.\sin\angle ACK}.\frac{AC}{AB} = \frac{BK.\sin\angle ABK}{CK.\sin\angle ACK} = \frac{AN}{AM}.\frac{\sin\angle ABK}{\sin\angle ACK} = 1$

We are done.

This post has been edited 1 time. Last edited by quangMavis1999, Mar 19, 2016, 9:23 pm

baopbc

#3 Mar 19, 2016, 9:49 pm

Thank quangMavis199 for your nice solution! But can you solve it without using Sin Theorem!

TelvCohl

#5 Mar 19, 2016, 10:14 pm • 5

Let $J \equiv BE \cap CF$ be the incenter of $\triangle ABC$. Since $E(A, B; F, N) = F(A, B; F, N) = (A, B; \infty, Q) = (A, C; \infty, P) = E(A, C; E, M) = F(A, C; E, M)$ so AJ passes through $I = EN \cap FM$ i.e. AI is the bisector of $\angle BAC$.

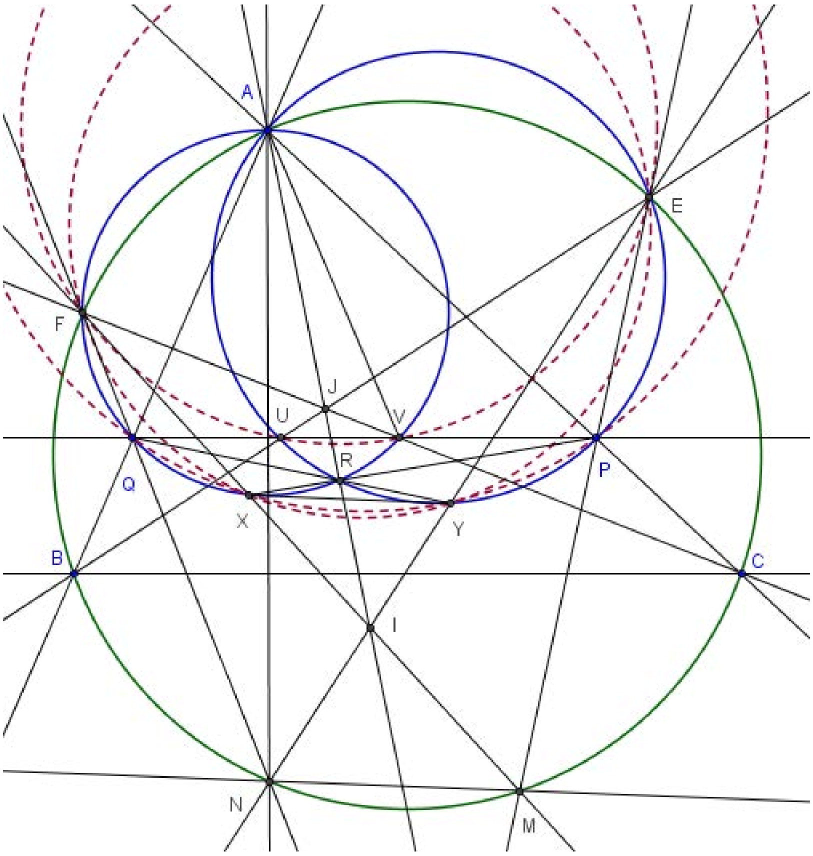
$E(A,C;E,M) = F(A,C;E,M)$, so AS passes through $P \equiv EN \cap FM$. i.e. AP is the bisector of $\angle BAC$

Luis González

#6 Mar 20, 2016, 2:31 am • 4

Let J be the incenter of $\triangle ABC$ and $U \equiv BE \cap PQ, V \equiv CF \cap PQ.$ Since $PQ \parallel BC,$ then by Reim's theorem, it follows that $AEPU, AFQV$ and $EFUV$ are cyclic, thus J is on the radical axis of $\odot(APE)$ and $\odot(AQF),$ i.e these circles meet again at $R \in AJ.$ If RP, RQ cut MF, NE at $X, Y,$ we get $\angle QXR = \angle QAR = \angle RAP = \angle RYP$ (mod 180º) $\Longrightarrow PQXY$ is cyclic $\Longrightarrow \angle IXY = \angle FAR - \angle RXY = \angle FAR - \angle RAV.$ But from $\angle BAV = \angle QFV = \angle NAC \Longrightarrow \angle NAR = \angle RAV \Longrightarrow$ $\angle FEN = \angle FAN = \angle FAR - \angle NAR = \angle FAR - \angle RAV = \angle IXY \Longrightarrow EFXY$ is cyclic $\Longrightarrow I \equiv FX \cap EY$ is on the radical axis AJ of $\odot(APE), \odot(AQF).$

Attachments:



buratinogigle

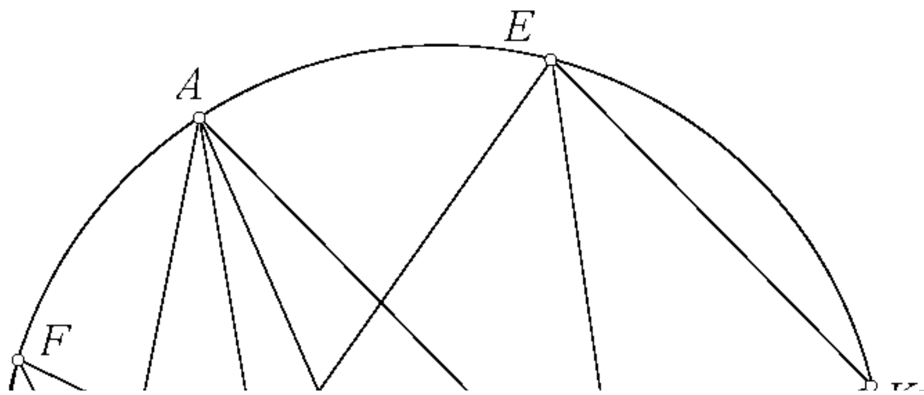
#7 Mar 20, 2016, 7:41 am • 3

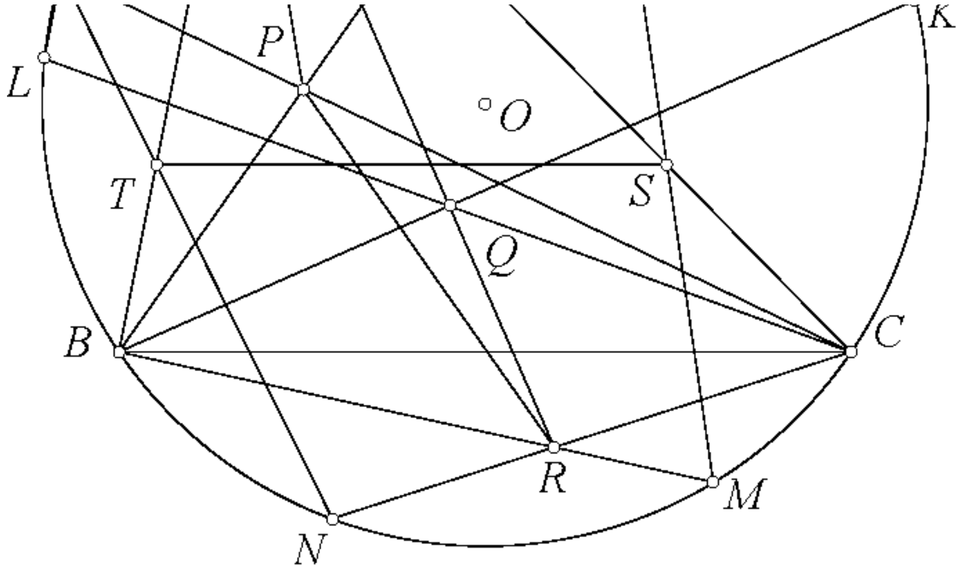
General problem. Let ABC be a triangle inscribed in circle (O) and P is any point. PB, PC cut (O) again at E, F. S, T lie on CA, AB such that $ST \parallel BC$. ES, FT cut (O) again at M, N. BM cuts CN at R. Provve that $\angle PAB = \angle RAC$.

Solution is base in idea of **Telv Cohl**.

Solution. Let Q be isogonal conjugate of P in triangle ABC, we will prove that A, Q, R are collinear, indeed. Let BQ, CQ cut (O) again at K, L then $EK \parallel CA$, $FL \parallel AB$. We have ratio chasing $B(AC, MK) = E(AC, MK) = (AC, S) = (AB, T) = E(AB, NL) = C(AB, NL)$ so A, Q, R are collinear.

Attachments:





Quick Reply

High School Olympiads

Geometry

Reply

Re1gnover
#1 Mar 18, 2016, 7:43 pm

Let $ABCD$ be a quadrilateral inscribed in circle (O). AC intersects BD at H. The line pass through H which perpendicular to OH intersects AD, BC at E, F. M, N be midpoints of BD, AC. MN intersects AD, BC at K, L. Prove that K, L, E, F are concyclic.

Luis González
#2 Mar 18, 2016, 10:51 pm

Let $J \equiv AD \cap BC$ and let S be the second intersection of $\odot(JAB)$ and $\odot(JCD)$; Miquel point of $ABCD.$ Since $ABCD$ is cyclic, then $S \in OH$ and $JS \perp OS$ (well-known) $\Longrightarrow JS \parallel EF.$ For any $ABCD,$ it's known that its Miquel point S is the isogonal conjugate (WRT $ABCD$) of the point at infinity of its Newton line $MN \Longrightarrow MN$ is parallel to the isogonal of JS WRT $\angle AJB.$ Hence $\angle KEF = \angle AJS = \angle KLF \Longrightarrow K, L, E, F$ are concyclic.

TelvCohl
#3 Mar 19, 2016, 7:07 am

Obviously, H, M, N, O lie on the circle with diameter HO, so EF and KL are antiparallel WRT $\angle MHN \Longrightarrow \angle KEF = \angle(AD, EF) = \angle(KL, BC) = \angle KLF$, hence we conclude that E, F, K, L are concyclic.

Vietnamisalwaysinmyheart
#4 Mar 19, 2016, 1:18 pm

> *Luis González wrote:*
> Let $J \equiv AD \cap BC$ and let S be the second intersection of $\odot(JAB)$ and $\odot(JCD)$; Miquel point of $ABCD.$ Since $ABCD$ is cyclic, then $S \in OH$ and $JS \perp OS$ (well-known) $\Longrightarrow JS \parallel EF.$ For any $ABCD,$ it's known that its Miquel point S is the isogonal conjugate (WRT $ABCD$) of the point at infinity of its Newton line $MN \Longrightarrow MN$ is parallel to the isogonal of JS WRT $\angle AJB.$ Hence $\angle KEF = \angle AJS = \angle KLF \Longrightarrow K, L, E, F$ are concyclic.

Can you explain the red part, please? Thanks!

sunken rock
#5 Mar 19, 2016, 1:42 pm

EF is tangent to (OMH), hence $\angle EHD = \angle NML = \angle CML \ (\ 1\ )$; with $\angle ACB = \angle ADB$ we infer $\angle CLM = \angle HED$, done.

Best regards,
sunken rock

Quick Reply

High School Olympiads

Equal segments

Reply

Source: Own

buratinogigle
#1 Mar 18, 2016, 9:18 pm • 1

Let ABC be a triangle inscribed in circle (O). Tangent at A of (O) intersects BC at T. Draw line ℓ passes through midpoint of AT and is parallel to A-symmedian of triangle ABC. Let P is a point on ℓ. PB, PC cut (O) again at Q, R. TQ, TR cut (O) again at S, T. Prove that $ST = BC$.

Luis González
#2 Mar 18, 2016, 10:25 pm • 2

The point T is named twice, so relabel U the intersection of the tangent of (O) through A with BC.

As the A-symmedian of $\triangle ABC$ is the polar of U WRT (O), then $UO \perp \ell \Longrightarrow \ell$ is the radical axis of (O) and $U \Longrightarrow PU^2 = PB \cdot PQ \Longrightarrow PU$ is tangent to $\odot(UBQ) \Longrightarrow \angle BQS = \angle BUP$ and similarly PU is tangent to $\odot(UCR) \Longrightarrow \angle CRT = \angle BUP \Longrightarrow \angle CRT = \angle BQS \Longrightarrow BTCS$ is an isosceles trapezoid with $ST = BC$.

buratinogigle
#3 Mar 18, 2016, 10:53 pm

Thank you dear Luis, I remembered in my memory there is a problem with midpoint of tangent as following

Let ABC be a triangle inscribed in circle (O). Tangent at A of (O) intersects BC at T. M is midpoint of AT. MB cuts (O) again at N. TN cuts (O) again at P. Prove that $PC \parallel AT$.

My way is the same as Luis, and I try to generalize this, I get above problem !

Quick Reply

High School Olympiads

6 points concyclic

Locked

Source: Singapore TST 2011

leeky

#1 Mar 18, 2016, 6:08 pm

Let P be a point in the interior of triangle ABC and let the lines AP, BP, CP meet BC, CA, AB at D, E, F respectively. Let the circles with diameters BC and AD intersect at points A', A''; circles with diameters CA and BE intersect at points B', B''; and circles with diameters AB and CF intersect at points C', C''. Prove that $A', A'', B', B'', C', C''$ lie on a common circle.

Luis González

#2 Mar 18, 2016, 9:56 pm

Discussed before at http://www.artofproblemsolving.com/community/c6h31399.

High School Olympiads

Prove 6 points lie on a circle

Reply

Source: Hard

Stun

#1 Mar 27, 2005, 1:44 pm

Given a triangle ABC and AA_0, BB_0, CC_0 are converge at M.
The circle with diameter BC intersect the circle with diameter AA_0 at A_1 and A_2. Similar we get B_1, B_2 and C_1, C_2. Prove that six points $A_1, A_2, B_1, B_2, C_1, C_2$ are lie on a circle

darij grinberg

#2 Mar 27, 2005, 2:40 pm

This was posted before at http://www.mathlinks.ro/Forum/viewtopic.php?t=3535 , but there is no correct solution yet.

darij

yetti

#3 Mar 27, 2005, 6:24 pm

Let A', B', C' be the midpoints of the sides BC, CA, AB and D, E, F the feet of altitudes AD, BE, CF from the vertices A, B, C. The circles $(A'), (B'), (C')$ with diameters BC, CA, AB obviously pairwise meet at the vertices $A = (B') \cap (C')$, $B = (C') \cap (A')$, $C = (A') \cap (B')$ and also at the feet of the altitudes $D = (B') \cap (C')$, $E = (C') \cap (A')$, $F = (A') \cap (B')$. Hence, the altitudes AD, BE, CF are the pairwise radical axes of the circle pairs $(B') - (C')$, $(C') - (A')$, $(A') - (B')$ and the orthocenter H is their radical center. Let P, Q, R be the midpoints of the cevians AA_0, BB_0, CC_0. Similar claim can be made for the circles $(P), (Q), (R)$ with diameters AA_0, BB_0, CC_0 - they pass through the feet of the altitudes and their radical center is the triangle orthocenter H. Actually, the 3 cevians AA_0, BB_0, CC_0 do not even have to be concurrent at all. For example, the circle (P) intersects the side BC at the point A_0 and at one other point D'. Since AA_0 is the circle diameter, the angle $\measuredangle AD'A_0 = 90^o$ is right and consequently, AD' is the altitude from the vertex A and the points $D' \equiv D$ are identical. Similarly, the circles $(Q), (R)$ with diameters BB_0, CC_0 pass through the feet E, F of the altitudes BE, CF. Consequently the power of the ortocenter to the circles $(P), (Q), (R)$ is $HA \cdot HD = HB \cdot HE = HC \cdot HF$, respectively, is the same and the orthocenter H therefore must be the radical center of these 3 circles. Of course, the power of the orthocenter to the circles $(A'), (B'), (C')$ is exactly the same as for the circles $(P), (Q), (R)$ and the orthocenter is thus the common radical center of all 6 circles $(A'), (B'), (C'), (P), (Q), (R)$, regardless of whether the cevians concur at a point M or not. But then the radical axis of any pair from these 6 circles passes through the orthocenter and the power of the orthocenter to all 6 circles is exactly the same, in particular,

$$HA_1 \cdot HA_2 = HB_1 \cdot HB_2 = HC_1 \cdot HC_2$$

As a result, the 6 points $A_1, A_2, B_1, B_2, C_1, C_2$ lie on a single circle, regardles of whether the cevians AA_0, BB_0, CC_0 are concurrent or not. In the original thread Geo4 [the points A1, A2, B1, B2, C1, C2 lie on a circle], grobber arrived to the same conclusion. Gotcha, Darij.

This post has been edited 1 time. Last edited by yetti, Mar 28, 2005, 3:26 am

darij grinberg

#4 Mar 27, 2005, 6:46 pm

> *yetti wrote:*
> in particular,
>
> $$HA_1 \cdot HA_2 = HB_1 \cdot HB_2 = HC_1 \cdot HC_2$$
>
> As a result, the 6 points $A_1, A_2, B_1, B_2, C_1, C_2$ lie on a single circle, regardles of whether the cevians

AA_0, BB_0, CC_0 are concurrent or not.

Sorry, Yetti - same error as Grobber did...

darij

yetti
#5 Mar 27, 2005, 7:16 pm

> *darij grinberg wrote:*
> Sorry, Yetti - same error as Grobber did...
>
> darij

All right, I see that I am wrong. Darn.

Yetti

Stun
#6 Mar 27, 2005, 8:26 pm

Phew!!! Thanks everyone , I think I finally got it.
Let X, Y, Z and U, V, T be the midpoint of BC, CA, AB and AA_0, BB_0, CC_0 respectively.
So we easily find that XU, YV, ZT are concurent at point K.
Hence K is also the center of $(A_1A_2B_1B_2)$ and $(B_1B_2C_1C_2)$ and $(C_1C_2A_1A_2)$ and we get the result

yetti
#7 Mar 28, 2005, 4:16 am

> *Stun wrote:*
> Phew!!! Thanks everyone , I think I finally got it. ...

After a good sleep, I got it as well. The centers of the circles $(P), (Q), (R)$ with the diameters AA_0, BB_0, CC_0, are the midpoints of these cevians, hence, they lie on the sides $B'C', C'A', A'B'$ of the medial triangle $\triangle A'B'C'$. From similarity of the triangle $\triangle ABC$ and its medial triangle,

$$\frac{C'P}{BA_0} = \frac{B'P}{CA_0}, \frac{A'Q}{CB_0} = \frac{C'Q}{AB_0}, \frac{B'R}{AC_0} = \frac{A'R}{BC_0}$$

Consequently,

$$\frac{C'P}{B'P} \cdot \frac{A'Q}{C'Q} \cdot \frac{B'R}{A'R} = \frac{BA_0}{CA_0} \cdot \frac{CB_0}{AB_0} \cdot \frac{AC_0}{BC_0}$$

i.e., the cevians $A'P, B'Q, C'R$ of the medial triangle $\triangle A'B'C'$ are concurrent iff the cevians AA_0, BB_0, CC_0 of the original triangle $\triangle ABC$ are concurrent. Due to

$$HA_1 \cdot HA_2 = HB_1 \cdot HB_2 = HC_1 \cdot HC_2$$

the point quadruples (B_1, B_2, C_1, C_2), (C_1, C_2, A_1, A_2), (A_1, A_2, B_1, B_2) are concyclic, and the centers of these 3 circles are $A_3 \equiv B'Q \cap C'R$, $B_3 \equiv C'R \cap A'P$, $C_3 \equiv A'P \cap B'Q$, respectively. Iff the cevians $A'P, B'Q, C'R$ concur, the centers A_3, B_3, C_3 are identical and because the 3 circles pairwise intersect in at least 2 points, they cannot be concentric, hence, they are identical.

Quick Reply

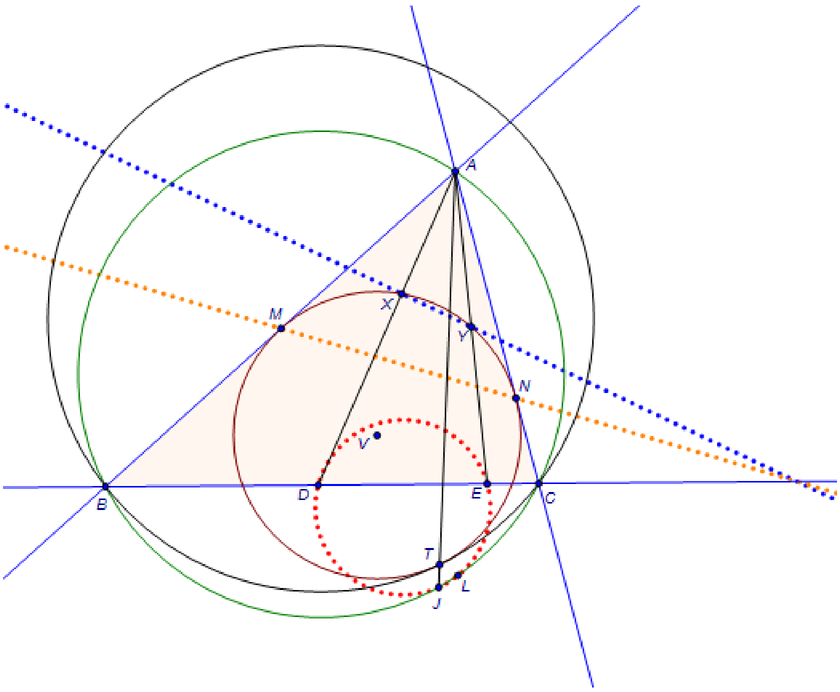
High School Olympiads

Prove fixed point(own)

Lin_yangyuan
#1 Mar 17, 2016, 6:46 pm

Given a fixed $\triangle ABC$ and a circle $\odot(U)$ passing through B,C. Let $\odot(V)$ be the circle tangent to AB, AC at M,N and tangent to $\odot(U)$ (internally) at T. X,Y are on $\odot(V)$ and XY, MN, BC are concurrent, $D \equiv AX \cap BC$□$E \equiv AY \cap BC$□$J \equiv AT \cap \odot(ABC)$.
Prove that:$\odot(JDE)$ cross a fixed point.

Attachments:



TelvCohl
#2 Mar 17, 2016, 9:31 pm • 1

Let P be the tangency point of the A-excircle of $\triangle ABC$ with BC and let $R \equiv AT \cap \odot(U)$. If Ω is the circle tangent to AB, AC and tangent to $\odot(U)$ internally, then from Monge & d'Alembert theorem we get the tangency point of Ω and $\odot(U)$ lies on AT, so R is the tangency point of Ω and $\odot(U) \Longrightarrow RP, RT$ are isogonal conjugate WRT $\angle BRC$ (well-known), hence if K is the intersection of BC and the tangent of $\odot(U)$ through T, then $\measuredangle RTK = \measuredangle RCT = \measuredangle RPK \Longrightarrow K \in \odot(PRT)$.

Since $G \equiv AT \cap BC$ is the radical center of $\odot(U), \odot(ABC)$ and $\odot(KPRT)$, so $AG \cdot JG = KG \cdot PG \Longrightarrow A, J, K, P$ are concyclic, hence if Z is the tangency point of the A-mixtilinear incircle of $\triangle ABC$ with $\odot(ABC)$, then we get $\measuredangle AJK = \measuredangle APK = \measuredangle AJZ \Longrightarrow J, K, Z$ are collinear.

From Geometry (a little hard, OWN) (**Lemma** at post #10) $\Longrightarrow TD, TE$ are isogonal conjugate WRT $\angle BTC$, so $KD \cdot KE = KT^2 = KB \cdot KC = KJ \cdot KZ \Longrightarrow Z \in \odot(DEJ)$. i.e. $\odot(DEJ)$ passes through a fixed point Z

Luis González
#3 Mar 17, 2016, 10:31 pm

Let $\odot(V)$ cut BC at L,K. Since $XY, MN, LK \equiv BC$ concur, then $X \mapsto Y, M \mapsto N, L \mapsto K$ is an involution on $\odot(V)$. Now projecting these points from A clearly forms an involutive pencil, in other words $D \mapsto E, B \mapsto C, L \mapsto K$ is an involution on $BC \Longrightarrow \odot(JDE), \odot(JBC), \odot(JLK)$ are coaxal for any J (not necessarily the J described in the problem)
$\Longrightarrow \odot(JDE)$ always goes through the second intersection of $\odot(ABC)$ and $\odot(JLK)$

$\Longrightarrow \odot(JDE)$ always goes through the second intersection of $\odot(ABC)$ and $\odot(JLK)$.

Lin_yangyuan

#4 Mar 17, 2016, 10:50 pm

Thanks for TelvCohl and Luis's replies，and your proofs are completely different from mine ，next time I'll post my proof.

Lin_yangyuan

#5 Mar 17, 2016, 11:15 pm

> *Luis González wrote:*
> Let $\odot(V)$ cut BC at $L, K.$ Since $XY, MN, LK \equiv BC$ concur, then $X \mapsto Y, M \mapsto N, L \mapsto K$ is an involution on $\odot(V).$ Now projecting these points from A clearly forms an involutive pencil, in other words $D \mapsto E, B \mapsto C, L \mapsto K$ is an involution on $BC \Longrightarrow \odot(JDE), \odot(JBC), \odot(JLK)$ are coaxal for any J (not necessarily the J described in the problem) $\Longrightarrow \odot(JDE)$ always goes through the second intersection of $\odot(ABC)$ and $\odot(JLK).$

Sorry Luis,maybe you misunderstand my expression ，only ABC is fixed and other point （like V） can move，then prove the conclusion.

Quick Reply

High School Olympiads

on the fixed line

Reply

mathmaths

#1 Mar 17, 2016, 6:48 pm

In acute triangle ABC,AD is the angle bisector of A,M is the midpoint of BC.P,Q are two moving points on segment AD such that □ABP=□CBQ respectively.Prove that the circumcenter of $\triangle PQM$ lies on a fixed line.

Luis González

#2 Mar 17, 2016, 9:36 pm

Let S and T be the midpoints of the arcs BC and BAC of $\odot(ABC)$. Cleary P and Q are inverse points under the inversion with center S and radius $SB=SC$ (for this just notice that SB is common tangent of $\odot(BAD)$ and $\odot(BPQ)$). Therefore $SP \cdot SQ=SM \cdot ST \Longrightarrow T \in \odot(PQM) \Longrightarrow$ the circumcenter of $\triangle PQM$ moves on the perpendicular bisector of $\overline{MT}$, certainly fixed.

Quick Reply

High School Olympiads

Concurrent lines, collinear points and line bisect segment

Reply

Source: Own

buratinogigle

#1 Mar 16, 2016, 11:26 am

Let ABC be a triangle with centroid G and P is an any point. Q is symmetric of midpoint PG through G. D, E, F are symmetric points of P through midpoints of BC, CA, AB, reps. A line passes through Q cuts BC, CA, AB at X, Y, Z, reps.

a) Prove that DX, EY, FZ are concurrent at point R.

b) Let a line ℓ passes through Q cuts RA, RB, RC at U, V, W. H, K, L lie on BC, CA, AB such that $UH \parallel PA, VK \parallel PB, WL \parallel PC$. Prove that H, K, L are collinear on line d.

c) Prove that d bisects segment PR.

Luis González

#2 Mar 16, 2016, 11:10 pm • 1

Consider an affine homology transforming P into the orthocenter of $\triangle ABC$. All the affine properties described in the problem are invariant. Clearly Q becomes circumcenter of $\triangle ABC$ and D, E, F are then the antipodes of A, B, C on (O). By Pascal theorem we can prove that $R \equiv DX \cap EY \cap FZ$ lies on (O) (see The point of concurrency lies on the circumcircle for a general configuration).

For b) and c) see A generalization of the Simson line theorem.

buratinogigle

#3 Mar 17, 2016, 11:00 am

Thank you dear Luis, actually I tried to find an affine proof for this problem but I do not gain achievements so I post on AoPS.

Quick Reply

High School Olympiads

The point of concurrency lies on the circumcircle

Reply

Source: own

Petry

#1 Jan 27, 2010, 3:33 am • 1

Hello!

Let P be a point inside of a triangle ABC, Γ is the circumcircle of ΔABC and d is a line through the point P. If $\{A, A'\} = AP \cap \Gamma$, $\{B, B'\} = BP \cap \Gamma$, $\{C, C'\} = CP \cap \Gamma$, $\{D\} = d \cap BC$, $\{E\} = d \cap CA$ and $\{F\} = d \cap AB$
then prove that the lines $A'D$, $B'E$ and $C'F$ are concurrent and the point of concurrency lies on the circle Γ.

Best regards, Petrisor Neagoe :)

Luis González

#2 Jan 27, 2010, 9:46 am

Let U be the second intersection of $A'D$ with Γ. By Pascal theorem for the non-convex hexagon $UA'ABCC'$ we get that $D \equiv A'U \cap BC, P \equiv AA' \cap CC'$ and $F' \equiv AB \cap C'U$ are collinear $\Longrightarrow F \equiv F' \Longrightarrow U \equiv A'D \cap C'F$. Again by Pascal theorem for the non-convex hexagon $UA'ACBB'$, the intersections $D \equiv A'U \cap BC, P \equiv AA' \cap BB'$ and $E' \equiv AC \cap B'U$ are collinear $\Longrightarrow E' \equiv E \Longrightarrow U \equiv A'D \cap B'E$.

Quick Reply

High School Olympiads

A generalization of the Simson line theorem

Reply

daothanhoai

#1 Apr 8, 2015, 7:26 am • 3

Problem 1:
Let ABC be a triangle, let a line L through circumecenter, let a point P lie on circumcircle. Let AP, BP, CP meets L at A_P, B_P, C_P. Denote A_0,B_0,C_0 are projection (mean perpendicular foot) of A_P, B_P, C_P to BC, CA, AB respectively. Then A_0, B_0, C_0 are collinear.
Problem 2:
The new line $\overline{A_0B_0C_0}$ bisect the orthocenter and P

- When L pass through P, this line is Simson line.

Reference:

[1] https://www.geogebra.org/student/m527653

[2] T. O. Dao, Advanced Plane Geometry, message 1781, September 26, 2014.

This post has been edited 1 time. Last edited by daothanhoai, Apr 8, 2015, 7:27 am

TelvCohl

#2 Apr 8, 2015, 10:56 pm • 5

My solution:

Let $X = \mathcal{L} \cap AC$ and M be the midpoint of AC .
Let Y, Z be the projection of P, B on AC, respectively .
Let H_A, H_B, H_C, H_P be the projection of A, B, C, P on $\mathcal{L}$, respectively .
Let A', B', C', P' be the orthopole of $\mathcal{L}$ WRT $\triangle BCP, \triangle CAP, \triangle ABP, \triangle ABC$, respectively .
Let R be the Poncelet point of $\{A, B, C, P\}$ (It's well-known that R is the midpoint of P and the orthocenter of $\triangle ABC$) .

From Euler-Poncelet points in cyclic quadrilateral (post #2 lemma) we get A', B', C', P' lie on a line τ .

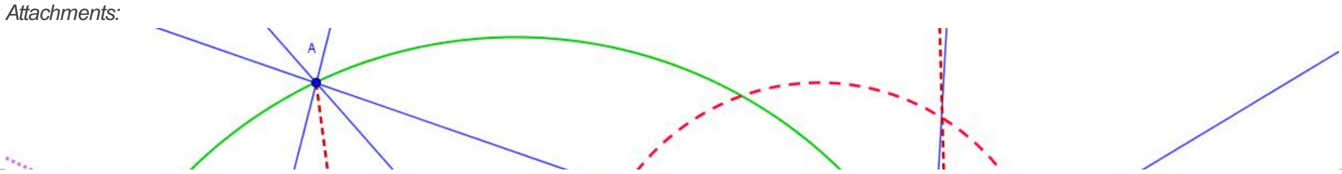
Since $\odot(A_PA_0C)$ is the pedal circle of A_P WRT $\triangle PAC$,
so from Fontene theorem we get $B' \in \odot(A_PA_0C) \Longrightarrow A_0, A_P, C, H_C, B'$ are concyclic .
Since $H_CA' \perp PB, H_CB' \perp PA, H_PA' \perp BC, H_PB' \perp AC$,
so $\angle B'H_CA' = \angle APB = \angle ACB = \angle B'H_PA' \Longrightarrow A', B', H_C, H_P$ are concyclic .
From Reim theorem and $H_PA' \parallel A_PA_0$ we get $A_0 \in A'B' \equiv \tau$ (similar discussion for B_0, C_0) .

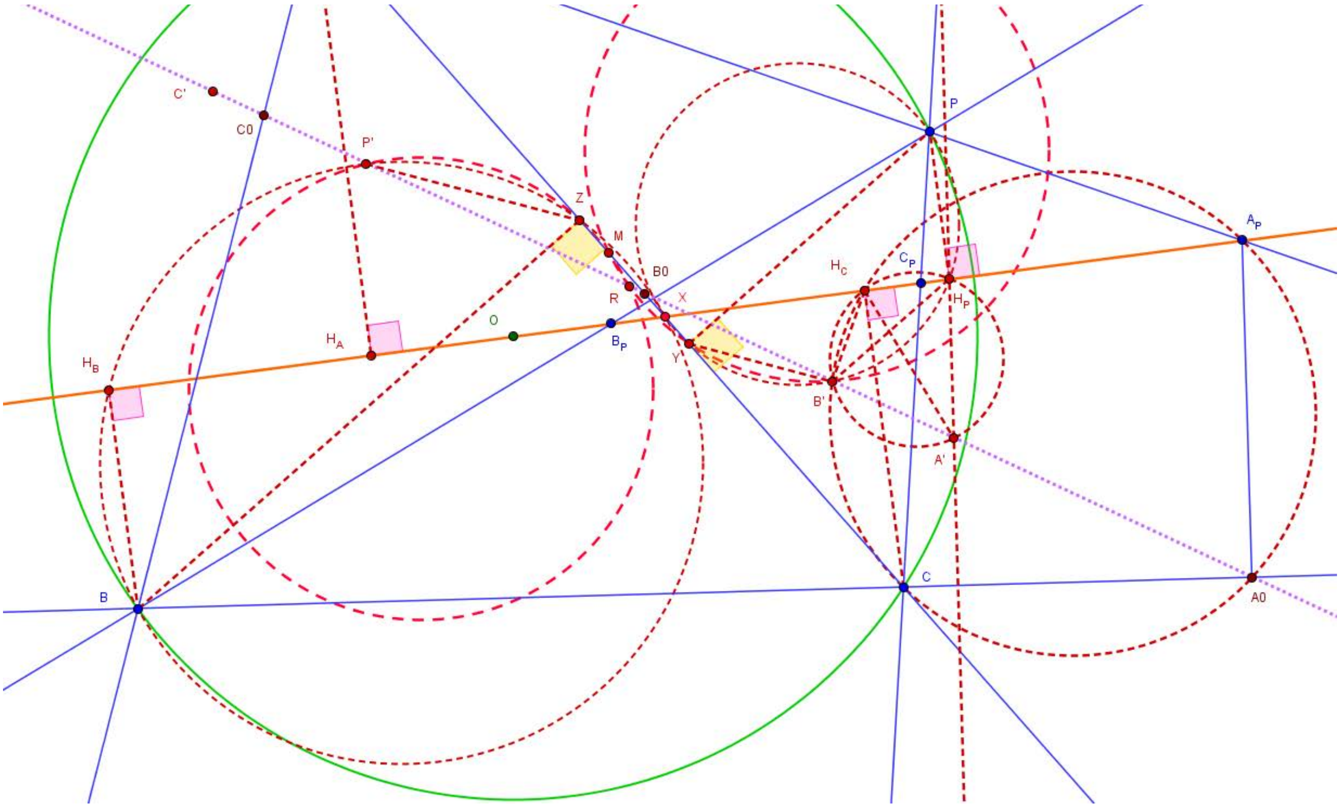
It's well-known that R lie on the 9-point circle of $\triangle ABC$,
so P', M, Z, R are concyclic at the 9-point circle of $\triangle ABC$.
Similarly B', M, Y, R are concyclic at the 9-point circle of $\triangle ABC$.
Since B, H_B, P', X, Z are concyclic at the pedal circle of X WRT $\triangle ABC$,
so $\angle AZP' = \angle XH_BP' = 90^\circ - \angle(AC, \tau)$ (notice that $H_BP' \perp AC$) .

Similarly we can prove $\angle CYB' = 90^\circ - \angle(AC, \tau) \Longrightarrow ZP' \parallel YB'$,
so from Reim theorem we get P', R, B' are collinear . i.e. $R \in \tau \equiv \overline{A_0B_0C_0}$

Q.E.D

Attachments:





Luis González

#3 Apr 19, 2015, 2:58 am • 2

Fix the line ℓ and animate P. The pencils PA, PB, PC are projective inducing a proyectivity on ℓ, i.e. the series A_p, B_p, C_p are projective $\Longrightarrow$ series A_0, B_0, C_0 are projective.

Let D, F be the antipodes of A, C on the circumcircle (O) and consider the case when $A_p \in BF$. If BC_p cuts (O) again at D', then by Pascal theorem for $APCFBD'$, it follows that $A_p, C_p, CF \cap AD'$ are collinear $\Longrightarrow D \equiv D' \Longrightarrow \angle C_pBA = 90^\circ \Longrightarrow B \equiv A_0 \equiv C_0 \Longrightarrow A_0 \mapsto C_0$ is a perspectivity $\Longrightarrow A_0C_0$ goes through a fixed point. When P coincides with $\{X, Y\} \equiv \ell \cap (O)$, then A_0C_0 becomes Simson lines of X, Y meeting at the orthopole T of $\ell \Longrightarrow T \in A_0C_0$ and similarly $T \in B_0C_0 \Longrightarrow A_0, B_0, C_0$ are collinear on a line τ passing through T.

Let H be the orthocenter of $\triangle ABC$ and let X be the midpoint of HP lying on 9-point circle (N). It's known that $T \in (N)$ when $O \in \ell$. Now since $X \barwedge P \barwedge A_0$ with fixed points at $(N) \cap BC$, then it follows that $X \mapsto A_0$ is a stereographic projection of (N) onto $BC \Longrightarrow X \in TA_0 \equiv \tau$.

tranquanghuy7198

#4 Aug 2, 2015, 5:22 pm • 3

> *daothanhoai wrote:*
>
> **Problem 1:**
> Let ABC be a triangle, let a line L through circumecenter, let a point P lie on circumcircle. Let AP, BP, CP meets L at A_P, B_P, C_P. Denote A_0,B_0,C_0 are projection (mean perpendicular foot) of A_P, B_P, C_P to BC, CA, AB respectively. Then A_0, B_0, C_0 are collinear.
> **Problem 2:**
> The new line $\overline{A_0B_0C_0}$ bisect the orthocenter and P
>
> - When L pass through P, this line is Simson line.

My solution:

If there exists a point W on the circumcircle of $\triangle ABC$, denote by d_W the Simson line of W WRT $\triangle ABC$

Now, let Q be the reflection of P WRT the line L and let M be the midpoint of PH (H is the orthocenter of $\triangle ABC$)
We will prove that: $MA_0 \parallel d_Q$. Indeed:

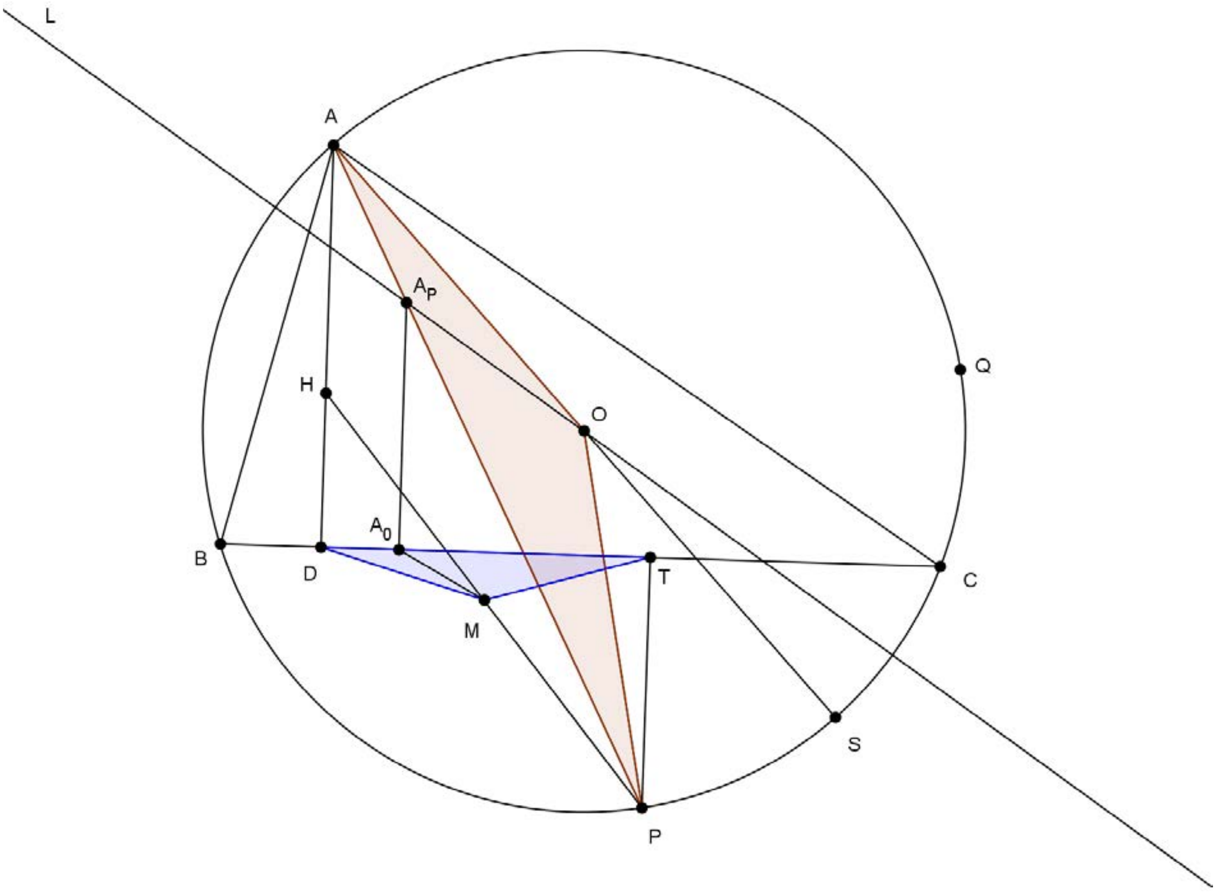
T, D are the projections of P, H on BC

$MP = MH \Rightarrow MD = MT$. On the other hand:

$$(TM, TD) = (d_P, d_S) = \frac{1}{2}.\widehat{\text{SP}} = (AS, AP) = (AO, AP) = -(PO, PA)(\text{mod}\pi)$$

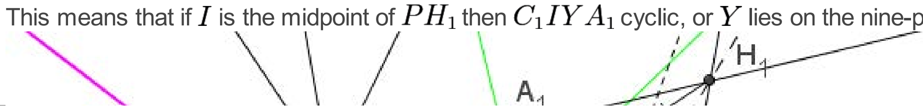
$\Rightarrow \triangle TMD \sim \triangle POA(1)$
We have: $\dfrac{A_0T}{A_0D} = \dfrac{A_PP}{A_PA}(\because AD \parallel A_PA_0 \parallel PT)(2)$
(1), (2) $\Rightarrow \triangle TMA_0 \sim \triangle POA_P$
$\Rightarrow (MT, MA_0) = (OA_P, OP) = \dfrac{1}{2}.\widehat{\text{QP}} = (d_P, d_Q)$ (mod π)
$\Rightarrow \boxed{MA_0 \parallel d_Q}$

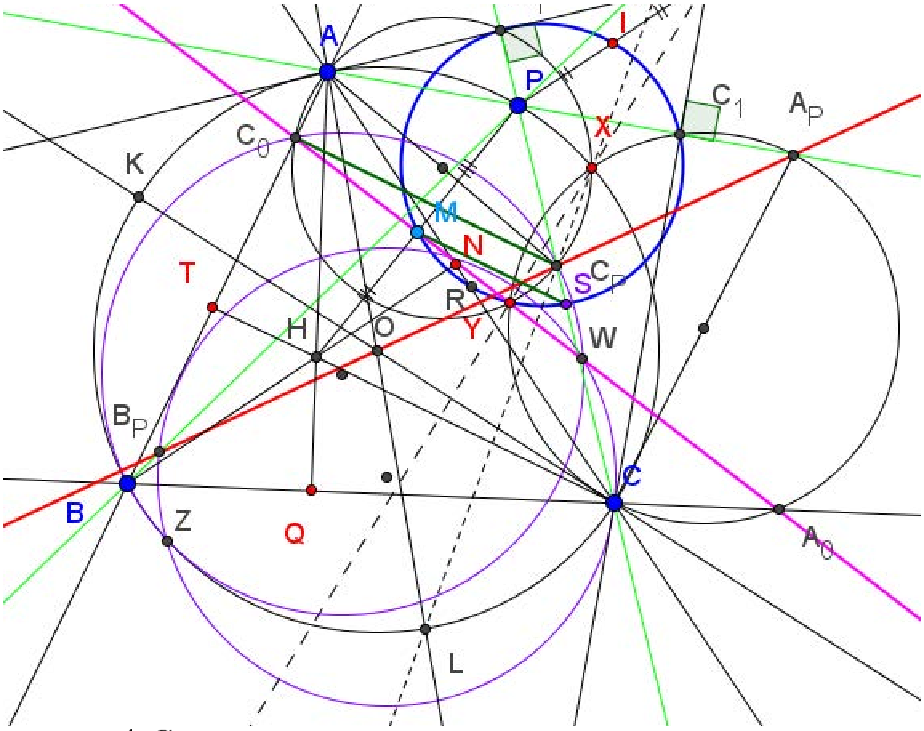
Similarly: $MB_0, MC_0 \parallel d_Q$
$\Rightarrow \overline{M, A_0, B_0, C_0}$ or $\overline{A_0, B_0, C_0}$ bisects HP
Q.E.D
Attachments:



daothanhoai
#5 Feb 11, 2016, 9:45 am • 1

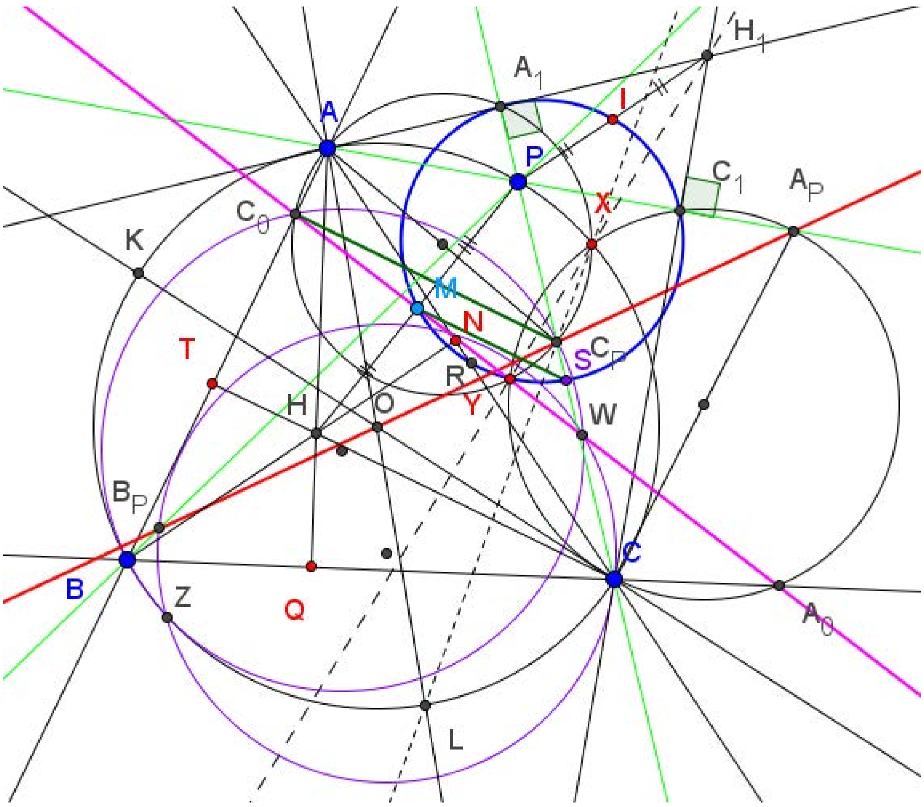
Tran Lam's proof:
Lemma 1 (well-known): The two circles with diameter AC_P, CA_P intersects at two points X, Y , one of them (say X) lies on (O), the second, (say Y), lies on the nine-point circle of $\triangle PAC$.
Proof of Lemma 1:
Let CO cut (O) again at K; KA_P cut (O) at X; XC_P cut (O) again at L, then by Pascal in hexagon $LAPCKX$ we get A, O, L collinear. Hence X lies on the circle with diameter CA_P as well as the circle with diameter AC_P. Let the circle with diameter AC_P cut PC at A_1, the circle with diameter CA_P cut PA at C_1, then AA_1, CC_1 are two altitudes of $\triangle PAC$. If they intersect at H_1 then H_1 obviously lies on the radical axis of the two spoken circles, thus H_1 lies on XY. Notice that $\angle C_1YA_1 = \angle C_1YX + \angle XYA_1 = \angle C_1CX + \angle XAH_1 = 360^0 - \angle AXC - \angle AH_1C = (180^0 - \angle AXC) + (180^0 - \angle AH_1C) = \angle APC + \angle APC = 2\angle APC$
This means that if I is the midpoint of PH_1 then C_1IYA_1 cyclic, or Y lies on the nine-point circle of $\triangle PAC$.





Lemma 2: A_0C_0 passes through Y.
Proof of Lemma 2:
Notice that $\angle XYC_0 = \angle XAC_0 = \angle XCA_0 = \angle XYA_0$. This implies that A_0, C_0, Y are collinear. Let AQ, CT the altitudes of $\triangle ABX$ with H the orthocenter. Easy to get A, H_1, C, H cyclic and the circle they lie on is the mirror of the circle O over AC. If R is the midpoint of AC and M midpoint of PH then there is no problem to see that R, M are the two common points of the nine-point circle of $\triangle PAC$ and the nine-point circle of $\triangle BAC$. By another words, M is the midpoint of PH. So if S is the midpoint of PC4 then $MS \parallel C_0C_P$. Now we are going to show that M lies on the line YC_0A_0. Notice that
$\angle XYM = \angle XYA_1 - \angle MYA_1 = \angle XAA_1 - (180 - \angle MSA_1) = \angle XAA_1 - (180^0 - \angle C_0C_PA_1) = \angle XAA_1 - \angle C_0AA_1 = \angle XYC_0$
This means that Y, M, C_0 are collinear or M lies on the line YC_0A_0 as desired.
Next, let the circle with diameter CBP cuts the circle with diameter BC_P intersect at W, Y . By lemma 1 Z lies on (O) and W lies on the nine-point circle of PBC. There is no problem to see that the nine-point circle of $\triangle PAC$ goes through S, M. By lemma 2 we have C_0, B_0, W collinear. As $\angle WMS = \angle WCS = \angle WC_0C_P$, but as $MS \parallel C_0C_P$, then W, M, C_0 are collinear. Hence all the points A_0, C_0, M, Y, B_0, W lie on a line d, this line bisects the line segment joining P and the orthocenter of $\triangle BAC$ (that is PH). Of course then the line l goes through P , then $X, C_P, A_P \equiv P$; $P_1 \equiv Y$, d is the Simson line of $\triangle ABC$.

Attachments:



This post has been edited 5 times. Last edited by daothanhoai, Feb 11, 2016, 10:15 am

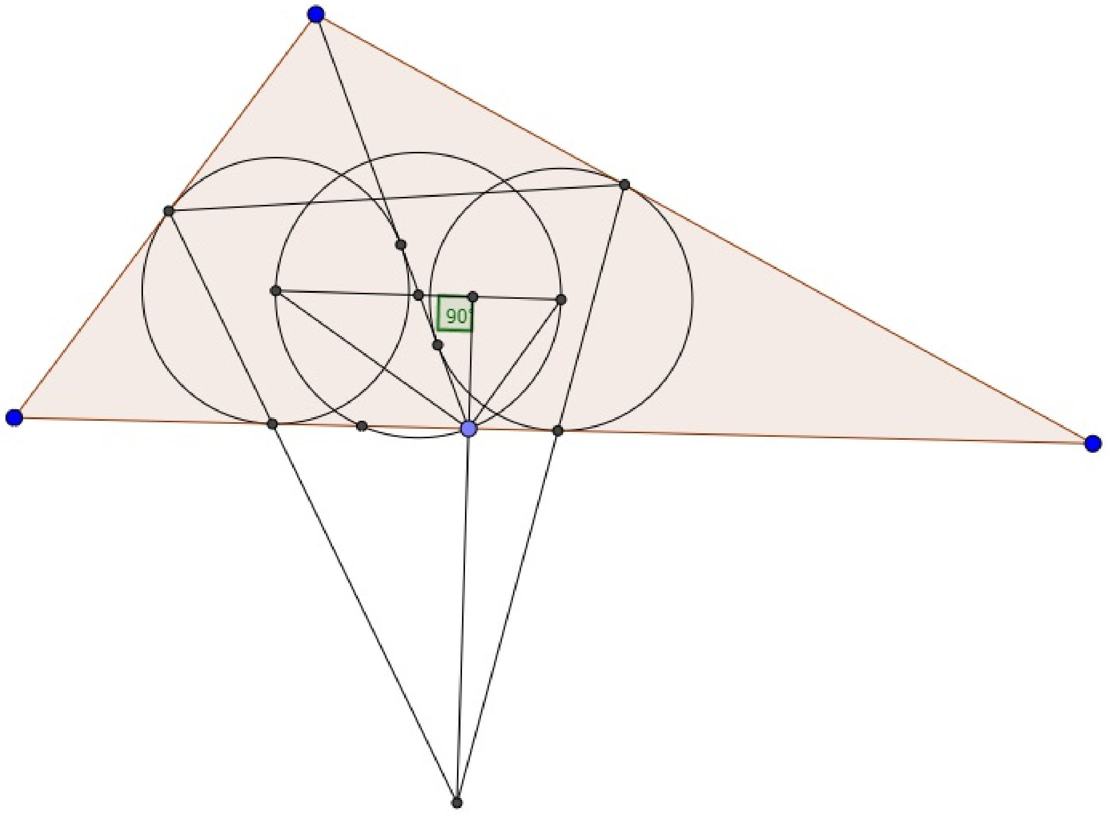
High School Olympiads

phantranhuongth

#1 Mar 16, 2016, 9:25 pm

Let ABC. D mobiles on BC. I is point center (ABD), J is point center ACD.
(a): Prove $(IJD$ through a fixed point.
(b) (I) cut AB, BC at P, M, (J) cuts AC, BC at N, Q. PM cut NQ at X.
Prove $(XD, IJ) = 90^0$.

Attachments:



Luis González

#2 Mar 16, 2016, 10:26 pm • 1

For convenience relabel I, J, K the incenters of $\triangle ABC, \triangle ABD, \triangle ACD$. If T is the tangency point of the incircle (I) of $\triangle ABC$ with BC, then $T \in \odot(DJK)$ (T is the fixed point) and moreover $MT = DQ$ (for a proof see two problems about cyclic quadrilateral problem 1).

From $MT = DQ$, we get $DM^2 - DQ^2 = TQ^2 - TM^2 = IQ^2 - IM^2$ and since $DM^2 = DJ^2 - MJ^2$ and $DQ^2 = DK^2 - QK^2 \Longrightarrow DJ^2 - DK^2 + QK^2 - QI^2 + MI^2 - MJ^2 = 0$. Thus by Carnot's theorem the perpendiculars from D, Q, M to JK, KI, IJ concur $\Longrightarrow XD \perp JK$.

doxuanlong15052000

#3 Mar 16, 2016, 10:58 pm • 1

My solution:

Let AI, BI cut XM, XN at U, V. Well-know that U lies on (ID) and V lies on (DJ). We have $\angle VQM = \frac{\angle C}{2} + 90^o$ and $\angle VUM = \frac{\angle ABD}{2} + \angle VDA = \frac{\angle C}{2} + 90^o$ then M,U,Q,V are concyclic $\Longrightarrow$ X lies on radical axis of (DI) and (DJ), so $(XD, IJ) = 90^0$.

Quick Reply

High School Olympiads

Mixtilinear Excircles Concurrence

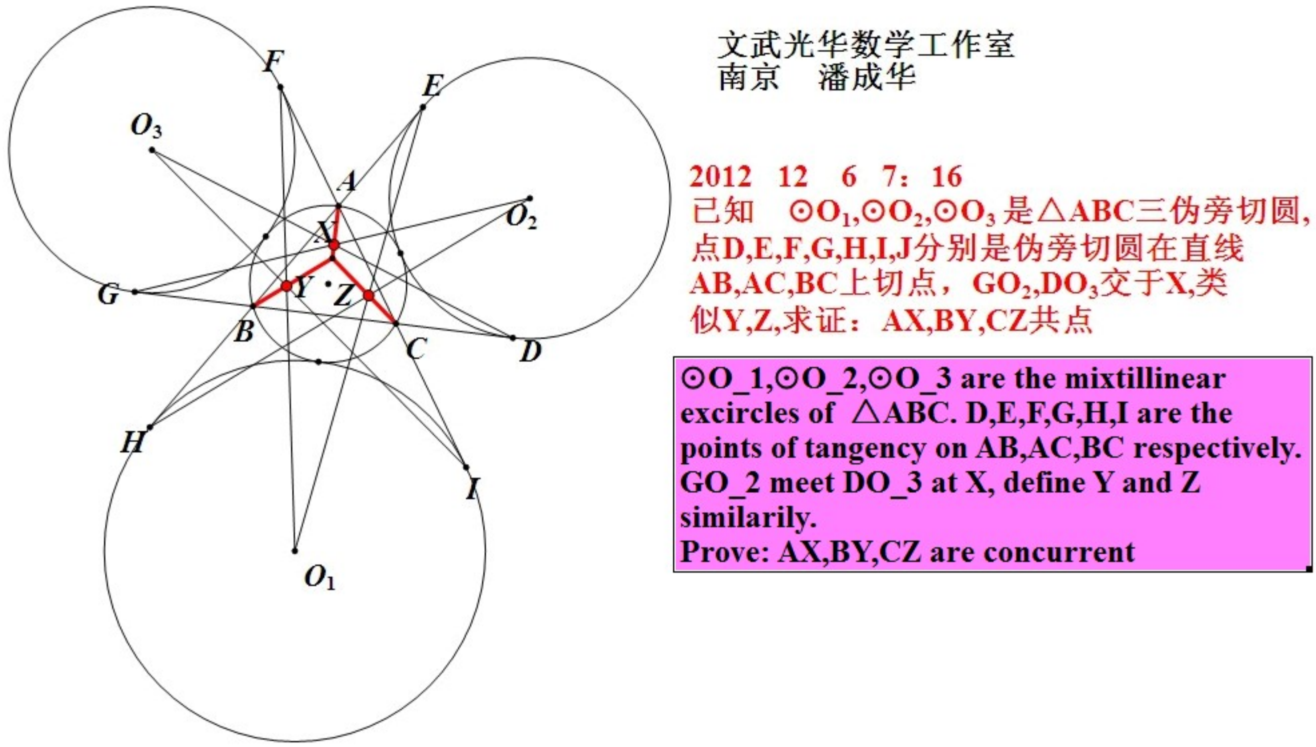
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Dec 6, 2012, 6:07 am • 1

See Attachment.
This is a problem proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



Luis González

#2 Mar 15, 2016, 7:29 am • 3

Let I and I_a, I_b, I_c be the incenter and 3 excenters of $\triangle ABC$ againts $A, B, C.$ We'll prove that AX, BY, CZ are in fact concurrent at X_{40} of $\triangle ABC;$ isogonal conjugate of its Bevan point Be (circumcenter of $\triangle I_aI_bI_c$).

It's well known that $DI_b \perp BI$ and $GI_c \perp CI.$ Thus if $J \equiv DI_b \cap GI_c,$ then $JI \perp BC$ as I_bI_c is clearly antiparallel to BC WRT $DI_b, GI_c.$ Now $\triangle DI_bO_2$ and $\triangle GI_cO_3$ are perspective with perspectrix $JI \Longrightarrow L \equiv O_2O_3 \cap I_bI_c \cap BC$ is the exsimilicenter of $(O_2) \sim (O_3).$ Thus if $U \equiv AI \cap O_2O_3,$ we have $(U, O_2, O_3, L) = I(A, I_b, I_c, L) = -1,$ i.e. U is the insimilicenter of $(O_2) \sim (O_3).$

Let V be the projection of U on $BC.$ From the trapezoid $GDO_2O_3,$ it's clear that X is midpoint of $\overline{UV}$ and as $DO_2 \parallel GO_3 \parallel UV,$ we get $(V, D, G, L) = (U, O_2, O_3, L) = -1 = (A, I_b, I_c, L) \Longrightarrow J, A, V$ are collinear. Now since $(\overline{UV} \parallel \overline{IJ}) \perp BC,$ their midpoints X, A' are collinear with $A.$ As A' is the center of $\odot(II_bI_c),$ it is the reflection of Be across $I_bI_c \Longrightarrow AX$ is the isogonal of ABe WRT $\triangle ABC$ and similarly BY and CZ are the isogonals of BBe and CBe WRT $\triangle ABC \Longrightarrow AX, BY, CZ$ concur at the isogonal conjugate X_{40} of $Be.$

High School Olympiads

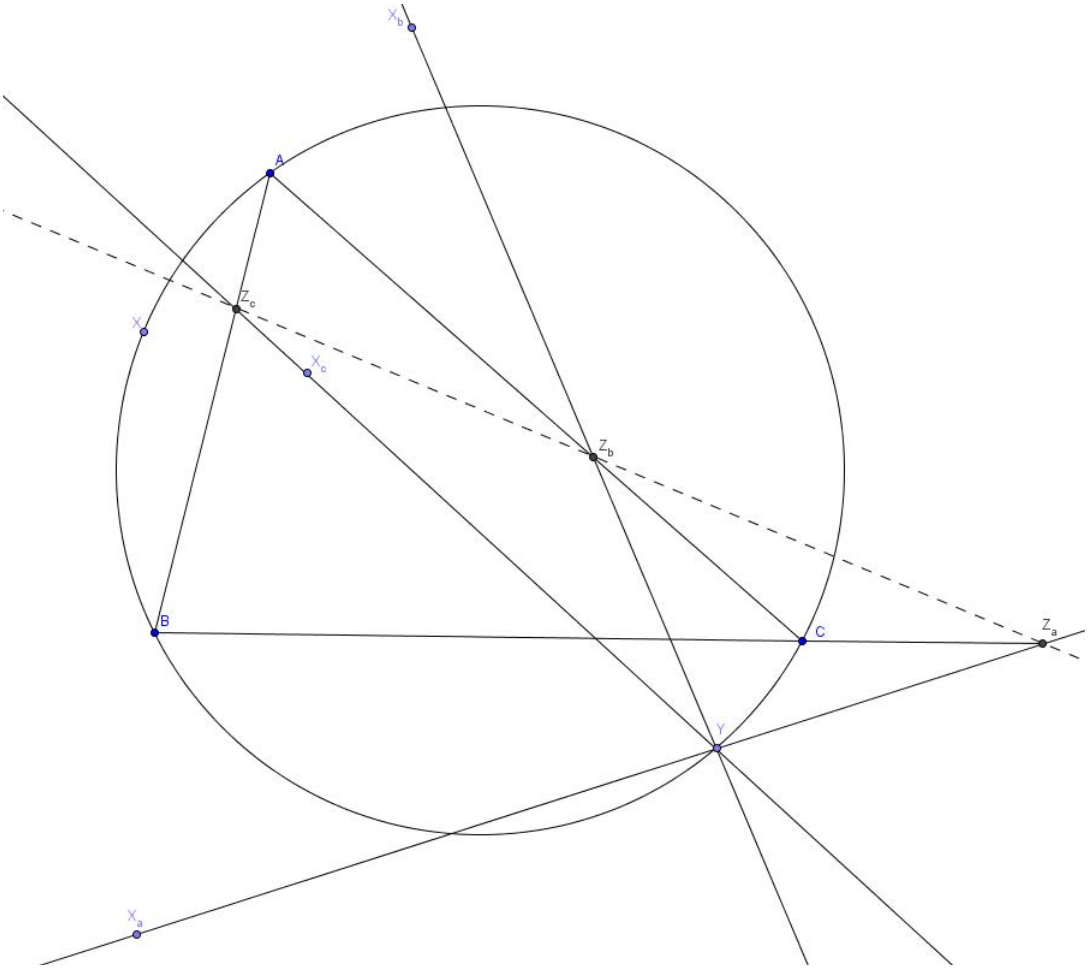
Collinear

Reply

Lockelox

#1 Mar 15, 2016, 1:45 am

Let X and Y are points on circumcircle of triangle ABC. Points X_a, X_b, X_c are symmetric to X with respect to the lines BC, AC, AB. Lines YX_a, YX_b, YX_c intersect BC, AC, AB in points Z_a, Z_b, Z_c. Prove that points Z_a, Z_b, Z_c are collinear.



XmL

#2 Mar 15, 2016, 5:55 am • 1

Interesting Configuration!

Lemma 1: $BX_c \cap CX_b \in (ABC)$.

Proof: Let $BX_c \cap CX_b = K$, then $\angle ACK = \angle ACX = \angle XBA = \angle ABK$, which means $K \in (ABC)$.

Lemma 2: Let YX_a intersect (ABC) again at W_a, similarly define W_b, W_c. Prove that AW_a, BW_b, CW_c concur on $\overline{X_aX_bX_c}$.

Proof: It suffices to show that $CW_c \cap BW_b \in X_bX_c$. In fact, this is an immediate consequence of Pascal's theorem on the hexagon $BYCKW_bW_c$, where $X_b = CK \cap YW_b$ by Lemma 1.

Main Proof: Let P denote the point of concurrence defined in Lemma 2. Applying Pascal's theorem on hexagon

Main Proof: Let P denote the point of concurrence defined in Lemma 2. Applying Pascal's theorem on hexagon AW_bCYBW_c, we have $P \in Z_bZ_c$. Symmetrically, we can obtain $P \in Z_bZ_a$. It follows that $Z_c, Z_a \in PZ_b \implies Z_a, Z_b, Z_c$ are collinear. $\square$

Does anyone know any results regarding the point P defined here?

This post has been edited 2 times. Last edited by XmL, Mar 15, 2016, 5:58 am
Reason: added question

Luis González

#3 Mar 15, 2016, 6:51 am • 1

I submitted this problem before at 3 collinear points in the style of Simson. See post #2 and the subsequent replies for a projective generalization

Quick Reply

High School Olympiads

3 collinear points in the style of Simson

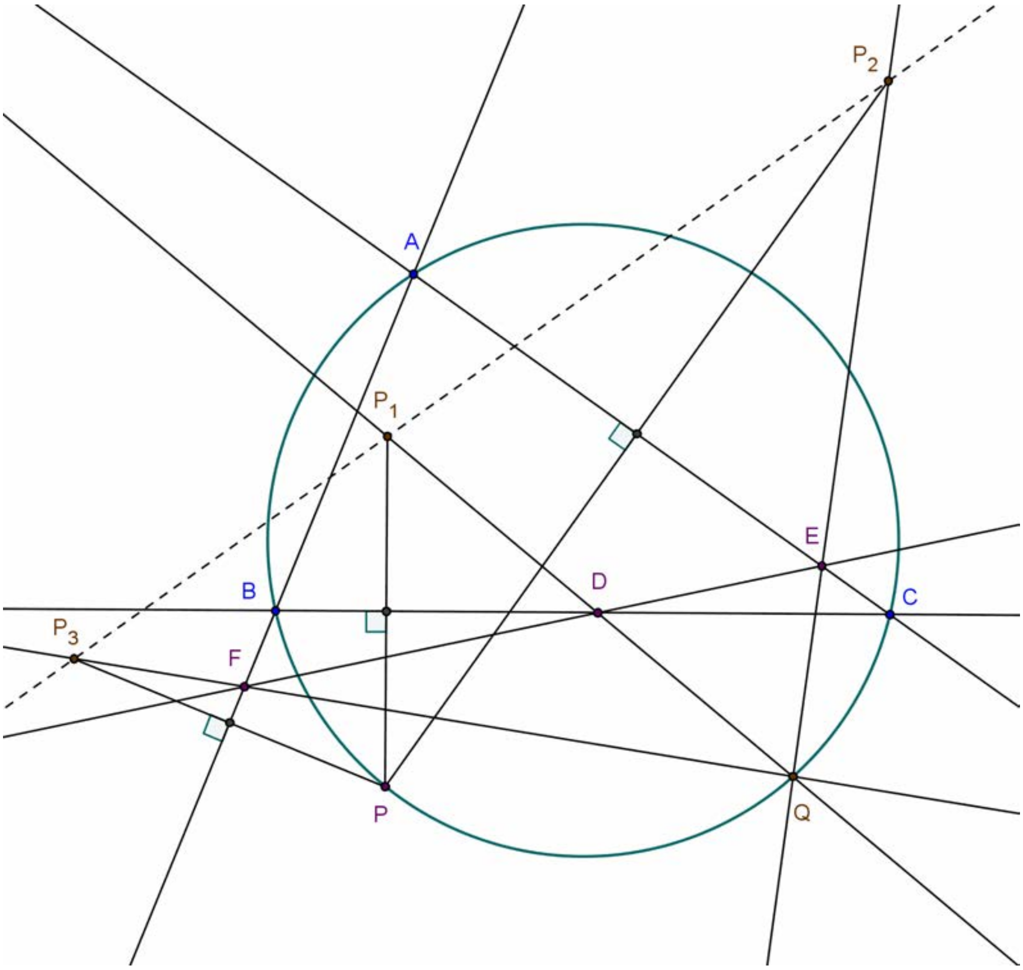
Reply

Luis González

#1 Jan 3, 2014, 9:51 am • 2

Let P, Q be 2 arbitrary points on the circumcircle of $\triangle ABC$. Let P_1, P_2, P_3 be the reflections of P across BC, CA, AB, respectively. QP_1, QP_2, QP_3 meet BC, CA, AB at D, E, F, respectively. Show that D, E, F are collinear.

Attachments:



proglote

#2 Jan 3, 2014, 12:20 pm

We have the following projective extension:

Let t be a line and T is a variable point on t. P, Q are fixed points and p, q are fixed lines. PT, QT cut p, q at P_0, Q_0. Then the points $PQ_0 \cap QP_0$ lie on a fixed line ℓ, and ℓ also contains the intersection of t and the axis of the projectivity sending $P_0 \mapsto Q_0$.

Taking t to be the line at infinity and p, q the Steiner lines of P, Q gives the original problem.

I still don't have a nice proof without cross-ratios.

Tarazmathboy

#3 Jan 3, 2014, 12:52 pm

can you solve without projective extention

jayme

#4 Jan 3, 2014, 12:58 pm

Dear Luis and Mathlinkers,
nice result. This line is known under "The Seimiya's line".
see for example,
http://perso.orange.fr/jl.ayme vol. 3 La droite de Seimiya
Sincerely
Jean-Louis

Luis González

#5 Jan 3, 2014, 11:35 pm • 2

Thanks dear Jean Louis for the reference, the source was unknown to me. Now, I would like present a proof of the generalized version mentioned by proglote.

Since $PT \equiv PP_0 \mapsto QT \equiv QQ_0$ is perspectivity, the series P_0, Q_0 with base lines p, q are projective $\Longrightarrow$ pencils PQ_0 and QP_0 are projective with double ray PQ when T goes to $PQ \cap t$, hence they are perspective $\Longrightarrow$ intersection $PQ_0 \cap QP_0$ runs on a line ℓ.

proglote

#6 Jan 3, 2014, 11:39 pm

Dear jayme, I think your article refers to the case when P, Q are inverse points, not arbitrary points on the circumcircle..

I also found another proof of the projective version:

As $P_0 \mapsto Q_0$ is a projectivity, the lines P_0Q_0 envelope a conic $\mathcal{E}$. Taking $T \in PQ$ we have that PQ is also tangent to $\mathcal{E}$. Let P^*, Q^* be intersections of the second tangents from P, Q to the conic. By Brianchon's theorem, all points of the form $PQ_0 \cap QP_0$ lie on P^*Q^*, as desired.

Luis González

#7 Jan 3, 2014, 11:58 pm

If $U \equiv t \cap p, V \equiv t \cap q$ then ℓ goes through intersections $P_1 \equiv VP \cap p$ and $Q_1 \equiv UQ \cap q$. Let $O \equiv p \cap q$ and OP, OQ cut t at M, N, respectively.

When T coincides with M or N, then P_0, Q_0 become O and its image in the corresponding serie, namely $X \equiv PN \cap p$ and $Y \equiv QM \cap q \Longrightarrow XY$ is proyective axis of $P_0 \mapsto Q_0$. Note that $P(M, N, U, V) = P(O, X, U, P_1)$ and $Q(M, N, U, V) = Q(Y, O, Q_1, V) \Longrightarrow (O, X, U, P_1) = (Y, O, Q_1, V) = (O, Y, V, Q_1) \Longrightarrow XY, t \equiv UV$ and $\ell \equiv P_1Q_1$ concur, as desired.

jayme

#8 Jan 4, 2014, 11:59 am

Dear Proglote and Mathlinkers,
in my article (http://perso.orange.fr/jl.ayme vol. 3 La droite de Seimiya p. 3) P and Q are not inverse. I think you have made a confusion with another article in the same volume intilted The Turner's line.
Sincerely
Jean-Louis

buratinogigle

#9 Mar 16, 2016, 12:51 am

Is this extension contained in the extension of Luis ?

Let ABC be a triangle inscribed in circle (O) and P, Q are two points on (O). D, E, F are projection of P on BC, CA, AB. X, Y, Z divide PD, PE, PF in the same ratios. Prove that QX, QY, QZ intersect BC, CA, AB, reps, follow three collinear points.

buratinogigle

#10 Mar 17, 2016, 1:38 pm

A consequence, when $PQ \parallel BC$ then EF bisects segment BC. I have a synthetic solution as following

Problem. Let ABC be a triangle with P, Q lie on (ABC) such that $PQ \parallel BC$. K, L are refletions of P through CA, AB. QK, QL cut CA, AB at E, F. Prove that EF bisects segment BC.

Proof. Follow property of isogonal lines K, L are reflection through AQ. Let AQ cuts BC at R. We have angle charsing, $\angle FPB = \angle FLB = \angle FLA - \angle BLA = \angle AFQ - \angle LAF - \angle BLA = \angle AFQ - \angle FBL = 180^\circ - \angle FAQ - \angle FQA - \angle FBP = \angle ABR + \angle ARB - \angle EQA - \angle AQP = \angle AQC - \angle AQE = \angle EQC$

Thus, we will prove that $\frac{EA}{EC} = \frac{FA}{FB}$ then by Menelaus then EF bisects BC. We have

$\frac{EA}{EC}.\frac{FB}{FC} = \frac{[QEA]}{[QEC]}.\frac{[PFB]}{[PFA]} = \frac{[QAE]}{[PFA]}.\frac{[PFB]}{[QEC]}$

$= \frac{d(A, QK).EQ}{d(A, FP).FP}.\frac{PB.PF}{QE.QC}$ (Because of $\angle FPB = \angle EQC$)

$= \frac{d(A, QK)}{d(A, FL)}$ (Becasue FL, FP reflect through AF and $BP = QC$)

$= 1$ (Because QK, QL reflect through AQ).

We are done.

baptiste

#11 Mar 17, 2016, 9:31 pm

we can prove a more general result which is:
Let D a homothetie og center P to the steiner's line and P'1, P'2 and P'3 construct also by the same homothetie. Then E' F' and D' are collinear.
this true because we can project PP1 on BC etc.. and if this is true for 3 different homothetie this true for all the other.
This is clairly true if we take the simson's line, P and the infinite line.

Quick Reply

High School Olympiads

Perspective

Reply

CEH

#1 Mar 13, 2016, 10:38 pm

Let ABC is triangle. We will draw three different rectangles ACFH,ABDE,BCKL which is out area of triangle ABC.
Define point X,Y,Z that (X is intersection of line EK and HL), (Y is intersection of line LH and DF), (Z is intersection of line DF and EK). Prove that triangle ABC and XYZ is perspective.

CEH

#2 Mar 14, 2016, 5:22 pm

Anyone can help me?

Luis González

#3 Mar 14, 2016, 10:12 pm

See http://www.artofproblemsolving.com/community/c6h266998 for the case when the rectangles are squares. The method for this problem is exactly the same.

CEH

#4 Mar 14, 2016, 10:25 pm

Thanks,Gonzalez.

Quick Reply

High School Olympiads

The windmill

Reply

Source: own

jayme

#1 Mar 26, 2009, 7:59 pm

Dear Mathlinkers,
let ABC be a triangle,
CB'B"A, AC'C"B, BA'A"C three squares erected externally on the sides of ABC,
and L, M, N respectively the points of intersection of A'B" and A"C', B'C" and B"A', C'A" and C"B'.
Prove that LMN is perspective with ABC.
Sincerely
Jean-Louis

Luis González

#2 Mar 27, 2009, 7:30 am

Let $P_a \equiv B'C'' \cap BC$ and define P_b and P_c cyclically. Lines $C''B$ and $B'C$ meet at the antipode M of A WRT the circumcircle of $\triangle ABC$. Hence, $MB = 2R\cos C$ and $MC = 2R\cos B$. By Menelaus' theorem for $\triangle MB'C''$, we get

$$\frac{P_aC}{P_aB} \cdot \frac{BC''}{MC''} \cdot \frac{MB'}{CB'} = 1 \Longrightarrow \frac{P_aC}{P_aB} = \frac{CA}{AB} \cdot \frac{AB + 2R\cos C}{CA + 2R\cos B} \quad (1)$$

By similar reasoning, we get the expressions

$$\frac{P_cA}{P_cB} = \frac{CA}{BC} \cdot \frac{BC + 2R\cos A}{CA + 2R\cos B} \quad (2)$$

$$\frac{P_bC}{P_bA} = \frac{BC}{AB} \cdot \frac{AB + 2R\cos C}{BC + 2R\cos A} \quad (3)$$

Now, multiplying the expressions (1), (2) and (3) together yields

$\frac{P_aC}{P_aB} \cdot \frac{P_cB}{P_cA} \cdot \frac{P_bA}{P_bC} = 1 \Longrightarrow P_a, P_b, P_c$ are collinear and the conclusion follows.

lym

#3 Mar 27, 2009, 8:06 am

in fact for $CB'B"A, AC'C"B, BA'A"C$ are Arbitrary rectangular□this Conclusion is Establishment .

Just $DesarguesTheorem$

This post has been edited 1 time. Last edited by lym, Mar 27, 2009, 8:09 am

Quick Reply

High School Olympiads

(ADE) tangent to BC

Reply

Source: Unknown

Lord.of.AMC

#1 Mar 14, 2016, 3:45 am

Points D and E are on sides AB and AC of triangle ABC such that $DE \parallel BC$. Line segments BE and CD meet the circumcircle of ADE at points M and N, respectively. Let P and Q be the intersection points of rays AM and AN with line segment BC, respectively. If $PQ = BC/2$, show that the circumcircle of ADE passes through the intersection of BC and the A-angle bisector.

Luis González

#2 Mar 14, 2016, 6:02 am

Since $\angle EBP = \angle MED = \angle MAD \Longrightarrow \odot(ABM)$ is tangent to BC and likewise $\odot(ACN)$ is tangent to BC. Since $PQ = \frac{1}{2}BC$, then clearly there is a point $X \in \overline{PQ}$ such that $PB = PX$ and $QC = QX \Longrightarrow$ $PX^2 = PB^2 = PM \cdot PA \Longrightarrow \odot(AMX)$ is tangent to BC and similarly $\odot(ANX)$ is tangent to $BC \Longrightarrow$ $\angle MXB = \angle MAX$ and $\angle NXC = \angle NAX \Longrightarrow \angle MAN = \angle NXC + \angle MXB = 180^\circ - \angle MXN \Longrightarrow$ $X \in \odot(ADMNE)$ and this circle is tangent to BC at X. Since $DE \parallel BC$, then X is the midpoint of its arc DE, i.e. AX bisects $\angle DAE \equiv \angle BAC$.

Quick Reply

High School Olympiads

Concurrent lines and collinear points

Reply

Source: Own

buratinogigle

#1 Mar 9, 2016, 1:31 pm • 2

Let ABC be a triangle with incircle (I) touches BC, CA, AB at D, E, F. The line passes through I and is perpendicular to IA cut CA, AB at A_1, A_2. A_c, A_b are symmetric of C, B through A_1, A_2. Circle (K_a) passes through A_b, A_c and is tangent to (I) at A_3. Similarly, we have B_3, C_3.

a) Prove that AA_3, BB_3, CC_3 are concurrent at point P.

b) Let AA_3 cut (I) again at A_4. Similarly, we have B_4, C_4. Prove that DA_4, EB_4, FC_4 are concurrent at Q.

c) Let P^*, Q^* are isogonal conjugate of P, Q, resp. Prove that line P^*Q^* passes through insimilicenter of (I) and (O).

Luis González

#2 Mar 13, 2016, 11:44 am • 3

Solution to problem a):

We use standard triangle notation $BC = a, CA = b, AB = c$ and s, r, R denote the semiperimeter, inradius and circumradius, resp. Since $AA_1 = AA_2 = \frac{bc}{s}$ (well-know), we get

$$BA_2 = c - \frac{bc}{s} = \frac{c(s-b)}{s} \Longrightarrow BA_b = \frac{2c(s-b)}{s} \Longrightarrow AA_b = c - \frac{2c(s-b)}{s} = \frac{c(2b-s)}{s}.$$

Likewise we obtain $AA_c = \dfrac{b(2c-s)}{s}$. Thus if (I) touches A_bA_c at X (see Concurrent and parallel lines problem a), we obtain

$$XA_b = FA_b = s - a - \frac{c(2b-s)}{s} = \frac{s(s-a+c)-2bc}{s} \ , \ XA_c = \frac{s(s-a+b)-2bc}{s}.$$

Let AA_3 cut BC and A_bA_c at D', X', resp and we define $E' \in CA, F' \in AB$ cyclically. Using the result of the problem Three concurrent lines for $\triangle A_3A_bA_c$, we get

$$\frac{X'A_b}{X'A_c} = \frac{XA_b{}^3}{XA_c{}^3} \Longrightarrow \frac{D'B}{D'C} = \frac{c}{b} \cdot \frac{AA_c}{AA_b} \cdot \frac{XA_b{}^3}{XA_c{}^3} = \frac{2c-s}{2b-s} \cdot \left[\frac{s(s-a+c)-2bc}{s(s-a+b)-2bc}\right]^3.$$

Hence, multiplying all the cyclic expressions together we get

$$\frac{D'B}{D'C} \cdot \frac{E'C}{E'A} \cdot \frac{F'A}{F'B} = \left[\frac{s(s-a+c)-2bc}{s(s-a+b)-2bc} \cdot \frac{s(s-b+a)-2ca}{s(s-b+c)-2ca} \cdot \frac{s(s-c+b)-2ab}{s(s-c+a)-2ab}\right]^3 = \frac{[s^2r^2(s^2+4r^2-16rR)]^3}{[s^2r^2(s^2+4r^2-16rR)]^3} = 1.$$

Thus by Ceva's theorem $AD' \equiv AA_3, BE' \equiv BB_3, CF' \equiv CC_3$ concur at P.

Luis González

#3 Mar 13, 2016, 11:45 am • 2

Solution to problems b) and c):

Assuming A_3, B_3, C_3 lie on the arcs EDF, FED, DFE of (I), then A_4, B_4, C_4 lie on its arcs EF, FD, DE, thus by Steinbart theorem, DA_4, EB_4, FC_4 concur at Q. Now consider a homology taking (I) into another circle whose center is the image Q' of Q. P' is then the Nagel point of $\triangle A'B'C' \Longrightarrow A', B', C', P', Q'$ and the Gergonne point Ge' of $\triangle A'B'C'$ lie on same conic (Feuerbach hyperbola of $\triangle A'B'C'$). Thus back in the primite figure A, B, C, P, Q, Ge lie on same conic $\mathcal{H}$ $\Longrightarrow P^*, Q^*$ and the insimilicenter of $(I) \sim (O)$ lie on the isogonal of $\mathcal{H}$ WRT $\triangle ABC$.

TelvCohl

#4 Mar 13, 2016, 9:29 pm • 2

Another solution to part (c) :

Let G_e be the Gergonne point of $\triangle ABC$ and let $B_5 \equiv CA \cap FQ, C_5 \equiv AB \cap EQ$. Since

$$B(A, G_e; P, Q) = (C_5, E; B_4, Q) \stackrel{F}{=} (F, E; B_4, C_4) \stackrel{E}{=} (F, B_5; Q, C_4) = C(G_e, A; Q, P) = C(A, G_e; P, Q),$$

so P, Q, G_e lie on a circumconic $\mathcal{C}$ of $\triangle ABC$, hence we conclude that P^*Q^* passes through the isogonal conjugate of G_e WRT $\triangle ABC$ which is the insimilicenter of $\odot(I) \sim \odot(O)$.

Quick Reply

High School Olympiads

Concurrent and parallel lines

Reply

Source: Own

buratinogigle
#1 Nov 14, 2015, 10:39 pm • 1

Let ABC be a triangle with excircles $(I_a),(I_b),(I_c)$. Line passes through I_a and is perpendicular to AI_a intersect CA,AB at A_1,A_2. A_c,A_b are symmetric of C,B through A_1,A_2, resp.

a) Prove that A_bA_c is tangent to (I_a) at A_a.

b) Similarly, we have B_b,C_c. $(I_a),(I_b),(I_c)$ touch BC,CA,AB at D,E,F, reps. Prove that DA_a,EB_b,FC_c are concurrent at K.

c) Let I,Ge,Na be incenter, Gergonne point, Nagel point of triangle ABC. Prove that $KN_a\parallel IGe$.

Luis González
#2 Nov 15, 2015, 1:29 am • 2

a) Let M,N,L be the midpoint of BC,CA,AB and redefine A_a as the second intersection of (I_a) with the parallel to IM through D. If the tangent of (I_a) at A_a cuts AC,AB at $A_c,A_b,$ then we need to show that A_1 and A_2 are the midpoints of CA_c and BA_b. Indeed if $J \equiv MI_a \cap A_bAc$ and $U \equiv BC \cap A_bA_c$, then $(MJ \parallel DA_a) \perp UI_a \Longrightarrow I_a$ is midpoint of MJ $\Longrightarrow MA_1JA_2$ is a parallelogram, even the Varignon parallelogram of BCA_cA_b since A_1,A_2 are unique on $AC,AB \Longrightarrow A_1$ and A_2 are the midpoints of CA_c and BA_b.

b) Let the incircle (I) of $\triangle ABC$ touch BC,CA,AB at $X,Y,Z,$ resp. It's well-known that $I_aM \parallel AX,$ so $DA_a \parallel AX$ and likewise $EB_b \parallel BY$ and $FC_c \parallel CZ.$ If P is the complement of Ge WRT $\triangle ABC,$ then it follows that DA_a,EB_b,FC_c concur at the reflection K of Ge on $P,$ in other words the anticomplement of Ge WRT $\triangle ABC.$

c) Since Na and K are the anticomplements of I and Ge WRT $\triangle ABC,$ then it follows that $KNa \parallel IGe.$

TelvCohl
#3 Mar 7, 2016, 6:56 pm • 2

Another proof of (a) :

Let I be the incenter of $\triangle ABC$ and let $\triangle XYZ$ be the intouch triangle of $\triangle ABC$. Let K be the midpoint of YZ and let the tangent of $\odot(I)$ passing through $T \equiv AX \cap \odot(I)$ cuts CA, AB at U, V, respectively. From $\measuredangle I_aA_2B = \measuredangle YXZ, \measuredangle A_2BI_a = \measuredangle ZYX \Longrightarrow \triangle A_2BI_a \sim \triangle XYZ$, so $\measuredangle I_aA_bA = \measuredangle YXK = \measuredangle TXZ$, hence if J is the incenter of $\triangle AUV$ then we get $\measuredangle JVA = \tfrac{1}{2}\measuredangle UVA = \measuredangle TXZ = \measuredangle I_aA_bA \Longrightarrow JV \parallel I_aA_b$. Analogously, we can prove JU is parallel to I_aA_c, so $\triangle AUV \cup J$ and $\triangle AA_cA_b \cup I_a$ are homothetic $\Longrightarrow A_bA_c$ is tangent to $\odot(I_a)$.

Quick Reply

High School Olympiads

Hard Geomeotry

Reply

Vietnamisalwaysinmyheart

#1 Mar 13, 2016, 8:34 am • 1

Given ΔABC. H is its orthocenter. Let EF intersects BC at G. Let K be the projection of H ontoAG.
Let L be midpoint of EF and N be the intersection of AD and EF. Let the bisector perpendicular of ND intersects HG at P
Prove that: (PND) and (GKL) are tangent to each other

Scorpion.k48

#2 Mar 13, 2016, 9:36 am

Where is E, F?

NHN

#3 Mar 13, 2016, 9:39 am

i think E,F,D foot of perpendicular, B,C,A

Luis González

#4 Mar 13, 2016, 9:43 am • 1

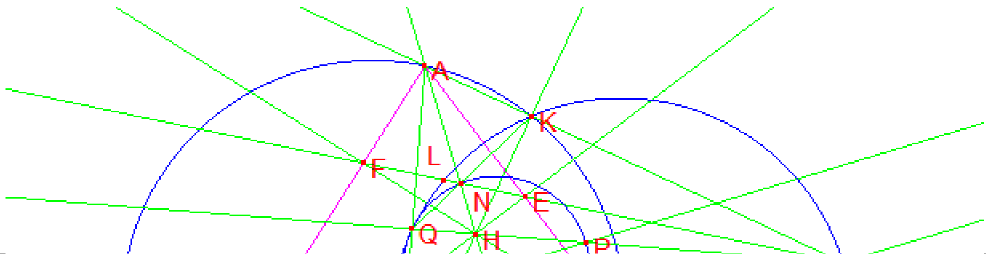
Let M be the midpoint of BC. By Brokart's theorem M is the orthocenter of $\triangle AHG$, i.e. GH is perpendicular to AM at X and MH is perpendicular to AG at K. Since AD is clearly the radical axis of $\odot(BCEF)$ and $\odot(GMXLK)$ with diameters $\overline{BC}, \overline{MG}$, then their intersections U, V are on AD. Moreover, GU, GV are tangents of $\odot(BCEF)$ as AD is the polar of G WRT $\odot(BCEF) \Longrightarrow GU = GV \Longrightarrow XH$ bisects $\angle UXV$. But since $(H, N, D, A) = -1$ and $\angle AXH = 90^\circ$, then XH also bisects $\angle DXN \Longrightarrow P$ is the midpoint of the arc ND of $\odot(XND)$. Now XN, XD are isogonals WRT $\angle UXV$ $\Longrightarrow \odot(XND)$ and $\odot(XUV)$ are tangent, in other words $\odot(PND)$ and $\odot(GKL)$ are tangent at X.

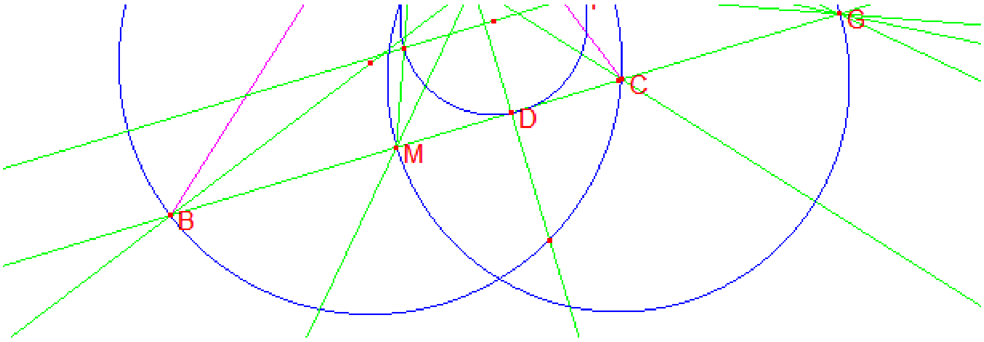
PROF65

#5 Mar 13, 2016, 7:55 pm

It s known that H is the symmetric of the antipode of A in the circle (ABC) wrt M,the midpoint of BC then M, H, K are collinear thus GH is orthogonal to AM let Q their intersection .we know that $(A, H; N, D) = -1$ but $AQ \perp QH$ hence QH is the bisector of $\widehat{NQD}$ besides P is on the bisector of ND so$P \in (QDP)$,let R the second intersection of the bisector of ND with the circle $(DNPQ)$ since $QR \perp QP \equiv QG$ then $R \in AM$ finaly, obviously $PR \parallel GM$ therefore the result follows.
R HAS
remark
I consider that it 's known that :
* K is the Miquel point of ABC with the collinear points E, F, G
** $LMKG$ cyclic indeed $GD \cdot GM = GC \cdot GB = GE \cdot GF = GN \cdot GL$ then $NLMD$ cyclic thus $\widehat{NLM} = \frac{\pi}{2}$
hence $LMKG$ cyclic

Attachments:





This post has been edited 1 time. Last edited by PROF65, Mar 13, 2016, 8:48 pm
Reason: remark

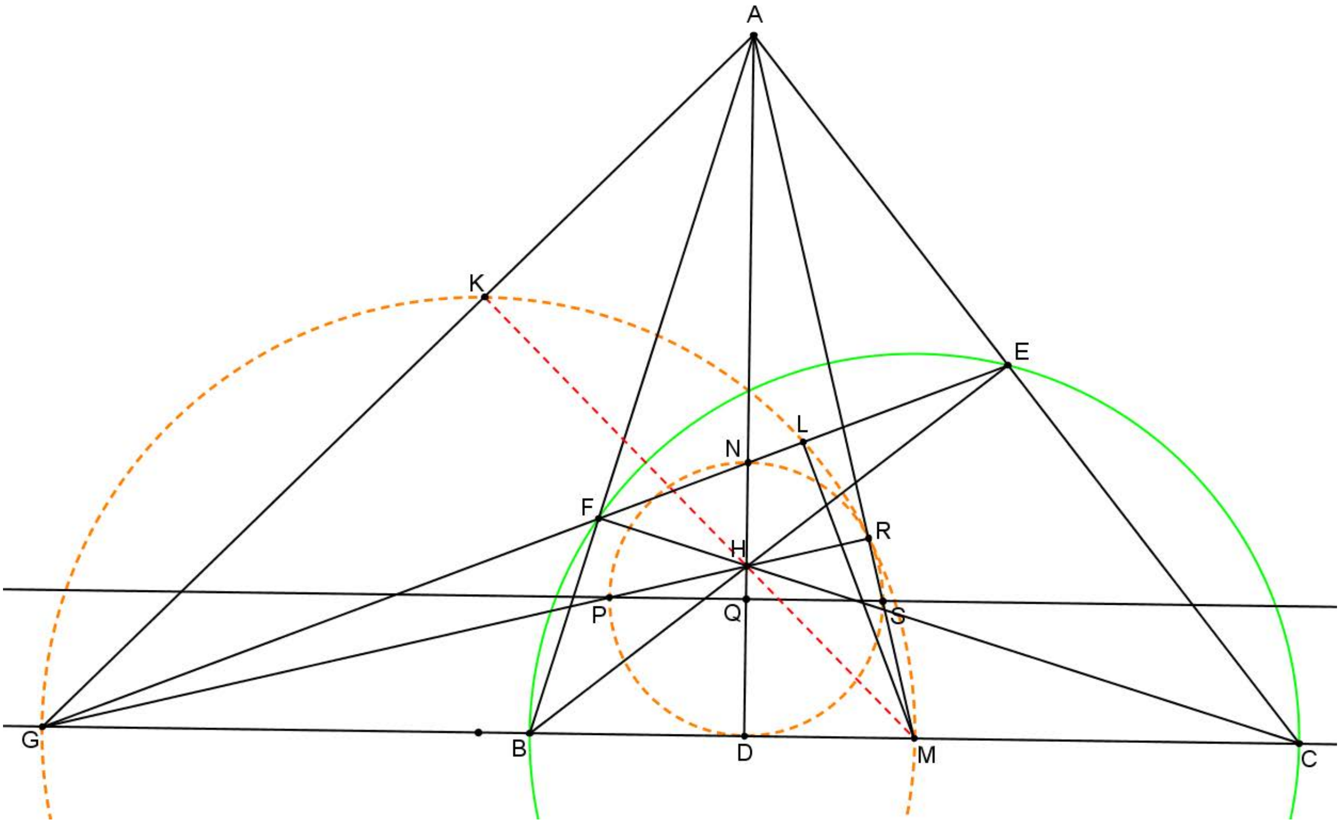
mjuk

#6 Mar 13, 2016, 10:17 pm

Let M, Q be midpoins of BC, ND, let $R = GH \cap AM$, $S = PQ \cap AM$.
By Brokard theorem in $BCEF$, $\angle GRM = \angle GKM = 90^\circ$, but $GLM = 90^\circ \to G, K, L, R, M$ are concyclic and K, H, M are collinear.
$(AHND) \overset{F}{=} (BCGD) = -1$ and $\angle ARH = 90^\circ \to \angle PRN = \angle PRD \to P$ is midpoint of arc ND in $\odot NRD \to R \in \odot PND$.
S lies on diameter PQ of $\odot PND$ and $\angle PRS = 90^\circ \to S \in \odot PND$.
Homothety taking $PS \to GM$ is centered at R and also takes $\odot PND \to \odot GKL$, but as $R \in \odot PND$ and $R \in \odot GKL$, we conclude that they are tangent to each other.

Didn't notice that above solution is same as mine.

Attachments:



This post has been edited 3 times. Last edited by mjuk, Mar 13, 2016, 10:23 pm
Reason: aa

Quick Reply

High School Olympiads

A progression and symmedians

Source: Indian Postal Coaching 2004

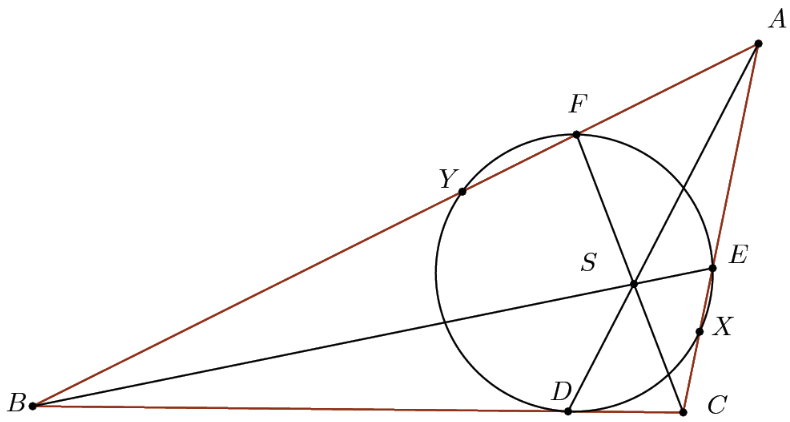
Rushil

#1 Sep 23, 2005, 4:15 pm

Suppose a circle passes through the feet of the symmedians of a non-isosceles triangle ABC , and is tangent to one of the sides. Show that $a^2+b^2, b^2+c^2, c^2+a^2$ are in geometric progression when taken in some order

Ankoganit

#2 Mar 11, 2016, 12:12 pm



Suppose the feet of symmedians from A, B, C are D, E, F in that order and S is the symmedian point (X_6). Also assume WLOG that (DEF) touches BC at D and intersects AC, AB a second time at X, Y respectively. Then using Power of point B wrt (DEF), we have

$$BY\cdot BF=BD^2 \implies BY=\frac{BD^2}{BF} \implies AY=AB-BY=AB-\frac{BD^2}{BF} \qquad (1)$$

Now suppose $BC=a, CA=b, AB=c$. It is well known that D divides BC in the ratio $c^2:b^2$, so we obtain $BD=\frac{c^2a}{b^2+c^2}$. Similarly one can obtain expression for BF, AF etc. Using these expressions and (1) we have, after some simplification,

$$AF\cdot AY=b^2\,c\,\frac{c^5-a^2\,c^3+b^2\,c^3+b^4\,c}{(a^2+b^2)\,(b^4+c^4+2\,b^2\,c^2)}$$

And similarly

$$AE\cdot AX=b\,c^2\cdot\frac{b^5+b\,c^4-a^2\,b^3+b^3\,c^2}{(a^2+c^2)\,(b^4+c^4+2\,b^2\,c^2)}$$

By power of point, we can equate these two, to obtain

$$-\,b\,c^2\cdot\frac{b^5+b\,c^4-a^2\,b^3+b^3\,c^2}{(a^2+c^2)\,(b^4+c^4+2\,b^2\,c^2)}+b^2\,c\,\frac{c^5-a^2\,c^3+b^2\,c^3+b^4\,c}{(a^2+b^2)\,(b^4+c^4+2\,b^2\,c^2)}=0$$
$$\iff -b^2\,c^2\,(c-b)\,(c+b)\,\frac{(a^4+a^2\,c^2+a^2\,b^2-c^4-c^2\,b^2-b^4)}{(c^2+b^2)^2\,(a^2+b^2)\,(a^2+c^2)}=0$$
$$\iff a^4-b^4-c^4+a^2\,b^2+a^2\,c^2-b^2\,c^2=0 \quad [\text{because the triangle is not isoceles}]$$

$$\iff \left(a^2+b^2\right)\left(a^2+c^2\right)=\left(b^2+c^2\right)^2$$
$$\iff a^2+b^2, b^2+c^2, c^2+a^2 \text{ are in G.P., as desired } \blacksquare$$

This post has been edited 1 time. Last edited by Ankoganit, Mar 11, 2016, 12:15 pm

Luis González

#3 Mar 13, 2016, 4:40 am

For other solutions see nice result related to the symmedial circle and Geometry Marathon-Olympiad level (problem #461)

Quick Reply

High School Olympiads

nice result related to the symmedial circle

Source: V.Thebault, AMM, E1035, 1952.

pohoatza
#1 Aug 29, 2007, 1:19 am

Show that if the circle passing through the feet of the symmedians of a non-isosceles triangle of sides a, b, c is tangent to one side, then the quantities b^2+c^2, c^2+a^2, a^2+b^2, arranged in some order, are consecutive terms of a geometric progression.

darij grinberg
#2 Aug 29, 2007, 1:24 am

Nice to know the real source of http://www.mathlinks.ro/Forum/viewtopic.php?t=53033 .
But since in the link above there are no replies and for not bumping it, I propose continuing the discussions here.

darij

Luis González
#3 Jul 27, 2011, 4:13 am • 2

K is the symmedian point of $\triangle ABC$ and K_A, K_B, K_C are the cevian traces of K on BC, CA, AB. Recalling that the general barycentric equation of the circle $\mathcal{C}$ is

$$\mathcal{C}(x,y,z)=a^2yz+b^2zx+c^2xy-(x+y+z)(px+qy+rz)=0$$

Substituting $K_A(0:b^2:c^2)$, $K_B(a^2:0:c^2)$ and $K_C(a^2:b^2:0)$ into the equation of $\mathcal{C}(x,y,z)$ and solving the three equations for p,q,r yields

$$p=\frac{b^2c^2}{2}\left(\frac{1}{c^2+a^2}+\frac{1}{a^2+b^2}-\frac{1}{b^2+c^2}\right)$$

$$q=\frac{c^2a^2}{2}\left(\frac{1}{b^2+c^2}+\frac{1}{a^2+b^2}-\frac{1}{c^2+a^2}\right)$$

$$r=\frac{a^2b^2}{2}\left(\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}-\frac{1}{a^2+b^2}\right)$$

If $\mathcal{C}$ is tangent to BC, then $C(0,y,z)=qy^2+rz^2-(q+r-a^2)yz=0$ has double roots. The product of the roots $y:z$ is $\frac{r}{q}$ and one of them is obviously $y:z=b^2:c^2 \Longrightarrow$ the other root is $\frac{rc^2}{qb^2}$. Thus, equating the roots gives $b^4q-c^4r=0$. Now, substituting the values of q,r found in the first part, we obtain

$$b^2\left(\frac{1}{b^2+c^2}+\frac{1}{a^2+b^2}-\frac{1}{c^2+a^2}\right)-c^2\left(\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}-\frac{1}{a^2+b^2}\right)=0$$

$$\frac{(b-c)(b+c)(b^4+c^4-a^4+b^2c^2-a^2b^2-a^2c^2)}{(b^2+c^2)(c^2+a^2)(a^2+b^2)}=0$$

$$(b-c)(b+c)[(b^2+c^2)^2-(c^2+a^2)(a^2+b^2)]=0$$

Therefore, the cevian circle $\mathcal{C}$ of K is tangent to $BC \Longleftrightarrow$ Either $b=c$, i.e. $\triangle ABC$ is isosceles with apex A, or $(b^2+c^2)^2=(c^2+a^2)(a^2+b^2)$ and the conclusion follows.

High School Olympiads

Geometry Marathon-Olympiad level

Locked

Source: 0

vaibhav2903
#1 Feb 14, 2010, 9:57 pm • 17

Hi mathlinkers!
I want to organise a Geometry marathon. This is to improve our skills in geometry. Whoever answers a pending problem can post a new problem. Let us start with easy ones and then go to complicated ones.

Different approaches are most welcome :).

Here is an easy one

1. Inradius of a triangle is equal to 1.find the sides of the triangle and P.T one of its angle is 90

[moderator edit: topic capitalized. :)]

Stephen
#2 Feb 14, 2010, 10:47 pm • 2

Well, I think you mean that the sides of the triangle are intergers.

Let the sides of the triangle are x, y, z.

Then by heron, $\frac{s(s-x)(s-y)(s-z)}{s^2} = 1^2 = 1$ where $s = \frac{x+y+z}{2}$.

So if we let $s - x = a, s - y = b, s - z = c$, then $a + b + c = abc$.

This is a famous problem. Since a, b, c are positive, $a, b, c = 1, 2, 3$.

So $x, y, z = 3, 4, 5$.

And it is obvious that one of that triangle's angle is 90 :)

Problem 2

In a convex hexagon $ABCDEF$, triangles ABC, CDE, EFA are similar.

Find conditions on these triangles under which triangle ACE is equilateral if and only if so is BDF.

vaibhav2903
#3 Feb 15, 2010, 8:42 pm • 4

can you just tell the problem clearly

Stephen
#4 Feb 16, 2010, 10:14 am • 1

Sorry. :(

I'll change the problem.

Problem 2

Let O be the circumcenter of an acute triangle ABC and let k be the circle with center S that is tangent to O at A and tangent to side BC at D.

Circle k meets AB and AC again at E and F respectively. The lines OS and ES meet k again at I and G.

Lines BO and IG intersect at H.

Prove that $GH = \frac{DF^2}{AF}$.

vaibhav2903

#5 Feb 16, 2010, 8:20 pm • 1

is the fig correct?[geogebra]29cd5e332189655049720ed1a0c527f8d822c43d[/geogebra]

Stephen

#6 Feb 16, 2010, 8:26 pm • 1

> vaibhav2903 wrote:
> is the fig correct?[geogebra]29cd5e332189655049720ed1a0c527f8d822c43d[/geogebra]

Sorry, I cannot see the picture. Can't you post it again by a file or something?

P.S. Since you sent me to post a solution, I'll give you some hints(if you can't understand it, sorry for my poor English).

In circle O, $< BAO =< ABO$. And in circle k, $< EAS =< AES$.

Since $< BAO =< EAS$, $< ABO =< AES$. So ES and BO are parallel.

And $< EAS =< AES =< AEG =< AIG$. So AE and GI are parallel too.

So we can know that $EBHG$ is a parallelogram. We can know that $GH = BE$.

Hint: $GH = BE$

vaibhav2903

#7 Feb 18, 2010, 7:17 pm • 1

since no 1 is able to post a solution fr this problem
lets do the next

problem 3:$ABCD$ is parellelogram and a st. line cuts AB at $\frac{AB}{3}$ and AD at $\frac{AD}{4}$ and AC at xAC.find x.

SOURBH

#8 Feb 18, 2010, 7:47 pm • 2

Solution to Problem number 3

Let the line l intersect AB at K,AD at L and AC at M.

As given $AK = \frac{AB}{3}$ and $AL = \frac{AD}{4}$

Extend line l to meet DC at Z (Obviously, line l is not parallel to DC)

$\triangle ZDL \sim \triangle KAL$

Hence $\frac{AL}{\ } = \frac{AK}{\ }$

Hence, $LD \qquad ZD$

$$\frac{1}{3} = \frac{\frac{x}{3}}{ZD}$$

$$ZD = x$$

By Menelaus Theorem , we get

$$\frac{DZ}{ZC} \cdot \frac{CM}{MA} \cdot \frac{AL}{LD} = 1$$

(Note that here I am not taking into consideration directed line segments)

$$\frac{1}{2} \cdot \frac{CM}{MA} \cdot \frac{1}{3} = 1$$

$$\frac{AM}{MC} = \frac{1}{6}$$

Problem 4

In $\triangle ABC$, $\angle BAC = 120$, Let AD be the angle bisector of $\angle BAC$

Express AD in terms of AB and BC

vaibhav2903

#9 Feb 18, 2010, 8:58 pm • 3

we know that

Click to reveal hidden text

problem 5

In a triangle ABC,AD is the feet of perpendicular to BC.the inradii of ADC,ADB and ABC are x, y, z .find the relation between x, y, z?

This post has been edited 3 times. Last edited by vaibhav2903, Feb 19, 2010, 8:37 am

SOURBH

#10 Feb 18, 2010, 9:28 pm • 2

Click to reveal hidden text

Agr_94_Math

#11 Feb 19, 2010, 5:02 pm • 2

$(x + y)c \sin B + xAC + yAB + xb \cos C + yc \cos B = 2zs.$

gokussj3

#12 Feb 19, 2010, 5:36 pm • 2

Is there anything simpler?

SOURBH

#13 Feb 19, 2010, 5:45 pm • 2

Click to reveal hidden text

Problem 6

Prove that the third pedal triangle is similar to the original triangle.[/hide]

vaibhav2903

#14 Feb 19, 2010, 6:19 pm • 2

hey,it is wrong the relation is $x^2 + y^2 = z^2$

---

vaibhav2903

#15 Feb 20, 2010, 2:12 pm • 2

still no one can answer it?

High School Olympiads

2A=3B ==> (a^2-b^2)(a^2+ac-b^2)=a^2c^2

Reply

Source: Titu Andreescu, University of Texas at Dallas, USA

sqing

#1 Mar 12, 2016, 3:45 pm

In triangle ABC, $2\angle A = 3\angle B$. Prove that

$$(a^2-b^2)(a^2+ac-b^2)=a^2c^2$$

here

Luis González

#2 Mar 13, 2016, 2:37 am

> sqing wrote:
> In triangle ABC, $2\angle A = 3\angle B$. Prove that
> $$(a^2-b^2)(a^2+ac-b^2)=b^2c^2$$

Take P on $\overline{BC}$ such that $\angle PAB=\tfrac{1}{3}\angle BAC \Longrightarrow \angle PAB=\tfrac{1}{2}\angle PBA \Longrightarrow PA^2=PB\cdot(PB+c)$ $(\star)$ (well-known). Moreover $\angle APC=\angle PAB+\angle PBA=\angle BAC \Longrightarrow \triangle ABC \sim \triangle PAC \Longrightarrow$

$$\frac{PA}{c}=\frac{b}{a} \Longrightarrow PA=\frac{bc}{a} \ \wedge \ \frac{PC}{b}=\frac{b}{a} \Longrightarrow PC=\frac{b^2}{a} \Longrightarrow PB=a-PC=\frac{a^2-b^2}{a}.$$

Substituting the latter expressions for PA, PB into $(\star)$, we obtain

$$\frac{b^2c^2}{a^2}=\frac{a^2-b^2}{a}\cdot\left(\frac{a^2-b^2}{a}+c\right) \Longrightarrow (a^2-b^2)\cdot(a^2+ac-b^2)=b^2c^2.$$

Quick Reply

High School Olympiads

Collinear

Reply

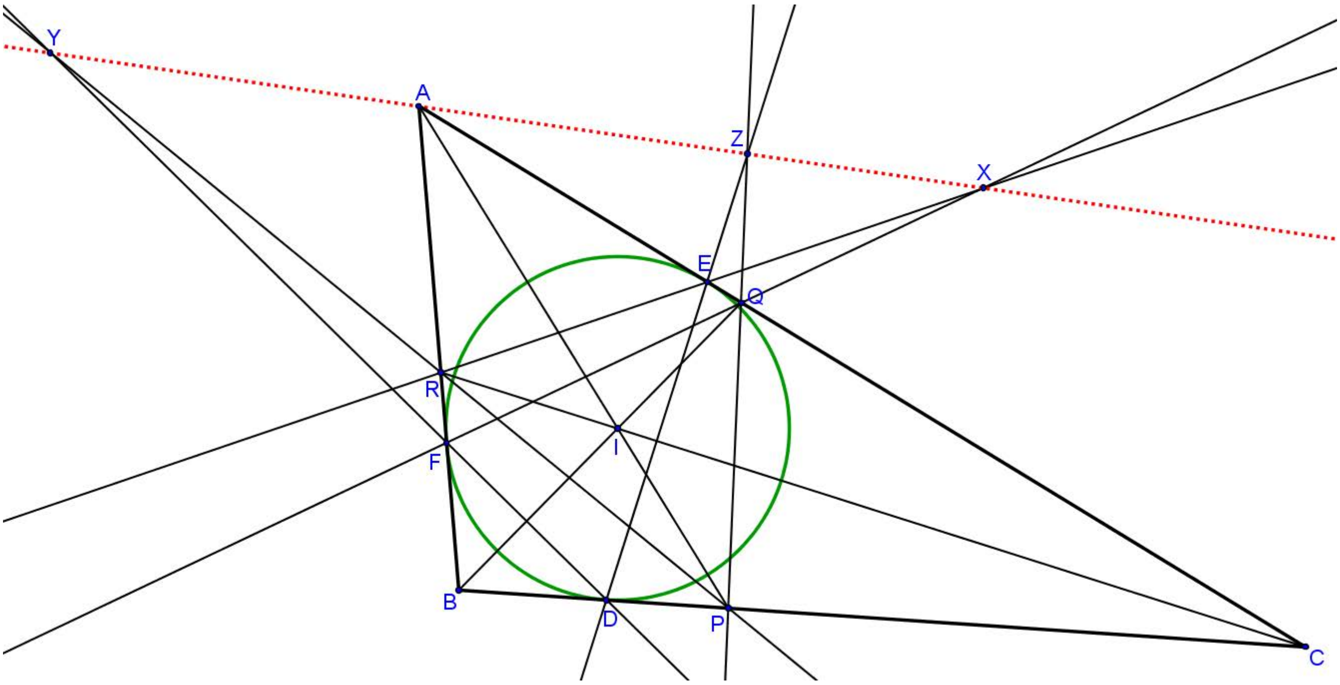
mjuk

#1 Mar 12, 2016, 9:53 pm

Let $\triangle ABC$ be a triangle with incenter I and intouch triangle $\triangle DEF$. Let $P = AI \cap BC$, $Q = BI \cap AC$, $R = CI \cap AB$. Let $X = ER \cap FQ$, $Y = DF \cap PR$, $Z = DE \cap PQ$.
Prove that A, X, Y, Z are collinear.

Remark

Attachments:



This post has been edited 1 time. Last edited by mjuk, Mar 12, 2016, 10:21 pm
Reason: remark

Luis González

#2 Mar 12, 2016, 10:43 pm • 1

Discussed before at http://www.artofproblemsolving.com/community/c6h1147406.

Quick Reply

High School Olympiads

collinear 4

Reply

ricarlos

#1 Oct 2, 2015, 12:50 am

Let ABC be a triangle. Let (D,D'), (E,E'), (F,F') two points on each side BC, AC and AB, respectively, such that (AD,BE,CF) and (AD',BE',CF') are concurrent. Prove that $(FE\cap F'E')$, $(FD\cap F'D')$, $(ED'\cap E'D)$ and C are collinear.

Luis González

#2 Oct 2, 2015, 1:33 am

Let $P \equiv AD \cap BE \cap CF$ and $Q \equiv AD' \cap BE' \cap CF'$. D,E,F,D',E',F' lie on a same conic (bicevian conic of P,Q). Thus by Pascal theorem for $EFDE'F'D'$, it follows that $X \equiv EF \cap E'F'$, $Y \equiv FD \cap F'D'$ and $Z \equiv ED' \cap E'D$ are collinear. Now, let $\mathcal{C}$ be the conic through A,B,C,P,Q, then EF and $E'F'$ are the polars of D,D' WRT $\mathcal{C} \Longrightarrow X$ is the pole of $DD' \equiv BC$ WRT $\mathcal{C} \Longrightarrow CX$ is tangent of $\mathcal{C}$ and likewise CY is tangent of $\mathcal{C} \Longrightarrow X,Y,Z,C$ are collinear on a tangent of $\mathcal{C}$.

PROF65

#3 Mar 13, 2016, 5:00 am

The first part is identical to **Luis** !
We know that D,D',E,E',F and F' lie on a conic then by applying Pascal to $DFED'F'E'$ yields $EF\cap E'F'$, $FD\cap F'D'$ and $ED'\cap E'D$ are collinear.
in the other hand $F(D,E,C,F')=-1, F'(D',E',C,F)=-1$ then $FD\cap F'D', FE\cap F'E'$ and C are collinear which end the proof.
R HAS

Quick Reply

High School Olympiads

Isogonal conjugate

Locked

cgsa4
#1 Mar 9, 2016, 10:12 pm

Let triangle ABC with incircle (I) and circumcircle (O). (I) tangent to BC, CA, AB at D, E, F respectively. EF intersect (O) at K, L. Let $AH \perp IK, AT \perp IL$, $H \in IK, T \in IL$. Prove that $\angle FDH = \angle EDT$

Luis González
#2 Mar 9, 2016, 10:53 pm

Posted before at http://www.artofproblemsolving.com/community/c6h1175175.

High School Olympiads

Incircle Isogonality

Reply

Source: Own

kormidscoler

#1 Dec 17, 2015, 11:13 pm

Suppose that there exists a triangle ABC with incenter I. Let the tangency points of BC, CA, AB wrt incircle I be D,E,F. Let's say that line EF meets with circumcircle O at X,Y (X,F,E,Y if you put them in order). Let G,H be perpendicular foot from A wrt IX, IY. Prove that angles GDF and HDE are equal.

Luis González

#2 Dec 18, 2015, 8:49 am

Let A',B',C' be the midpoints of $EF,FD,DE,$ resp. Inversion WRT (I) fixes D,E,F and swaps A,B,C and $A',B',C' \Longrightarrow \odot(A'B'C')$ goes to (O) and $\odot(EIFA)$ goes to $EF.$ Since $IG \cdot IX = IA' \cdot IA = IY \cdot IH,$ it follows that G,H are the inverses of X,Y in this inversion, therefore $\{G,H\} \equiv \odot(A'B'C') \cap \odot(IEF).$ Now it's known that in any $\triangle DEF$ with circumcenter $I,$ the inversion with center D and power $\tfrac{1}{2}DE \cdot DF$ followed by reflection on the angle bisector of $\angle EDF$ swaps $\odot(IEF)$ and the 9-point circle $\odot(A'B'C').$ As a result DG,DH are isogonals WRT $\angle EDF,$ i.e. $\angle GDF = \angle HDE.$

kormidscoler

#3 Dec 19, 2015, 12:30 pm

Um I solved it like that too, can u solve it without inversion?

Quick Reply

High School Olympiads

AR is parallel to IM

Reply

Source: OWN

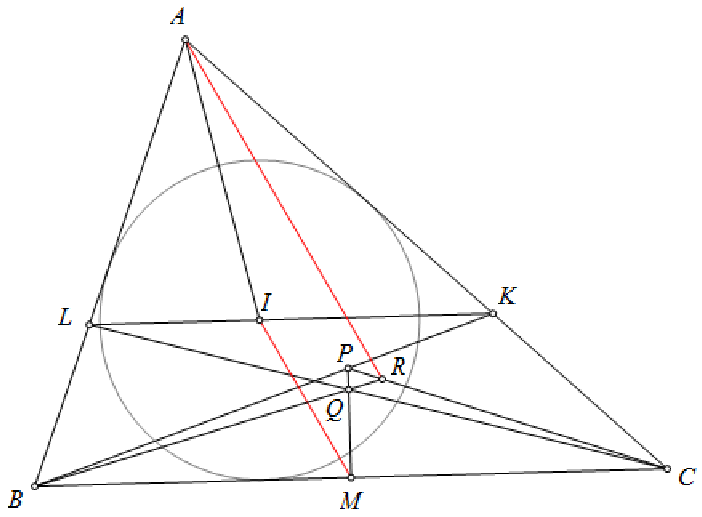
LeVietAn

#1 Mar 8, 2016, 10:07 am

Dear Mathlinkers,

Let ABC be triangle with incenter I. Let M be the midpoint of BC. The line through I parallel to BC intersects CA, AB at K, L respectively. The perpendicular bisector of BC intersects BK, CL at P, Q, respectively. The lines CP and BQ intersects at R. Prove that $AR \parallel IM$.

Attachments:



Luis González

#2 Mar 8, 2016, 11:05 am

Let the incircle (I) and the A-excircle touch BC at D, E, resp. D' is the antipode of D on (I) and the perpendicular bisector of BC cuts KL at X. Since M is midpoint of DE, it follows that X is the midpoint of ED', i.e. AXE is the A-Nagel cevian of $\triangle ABC$.

Since $KL \parallel BC$, then $S \equiv BK \cap CL$ is on the A-median AM. Moreover if $Y \equiv AM \cap KL$, from the complete $AKSL$, we get $X(Y, A, S, M) = -1 = X(I, D', D, M) \Longrightarrow S \in DX$. By symmetry R is clearly the reflection of S across XM $\Longrightarrow R \in XE$, i.e. AR is the A-Nagel cevian of $\triangle ABC$ and the conclusion follows.

XmL

#3 Mar 8, 2016, 12:07 pm

Let l denote the perpendicular bisector of BC. Reflect A over l, LK to obtain A', A''. Let D denote the feet of projection from I to BC, and E is the reflection of D over l. Since the reflection of D over LK lies on AE(Well-known), and $A'A''$ and AE are reflexive over LK, it follows that A', D, A'' are collinear.

Let $LC \cap KB = R'$, which is clearly the reflection of R over l. One can show that LKA'', CBA' are homothetic about R', so it follows that $R' \in A'A''$. Note that the axes of reflections l, LK are perpendicular, so reflecting $\overline{A'A''}$ over l gives us $\overline{AE}$. Thus it follows that $R \in AE$ and $AR||IM$ because $AE||IM$ is well-known.

Quick Reply

High School Olympiads

Nice one from geometry

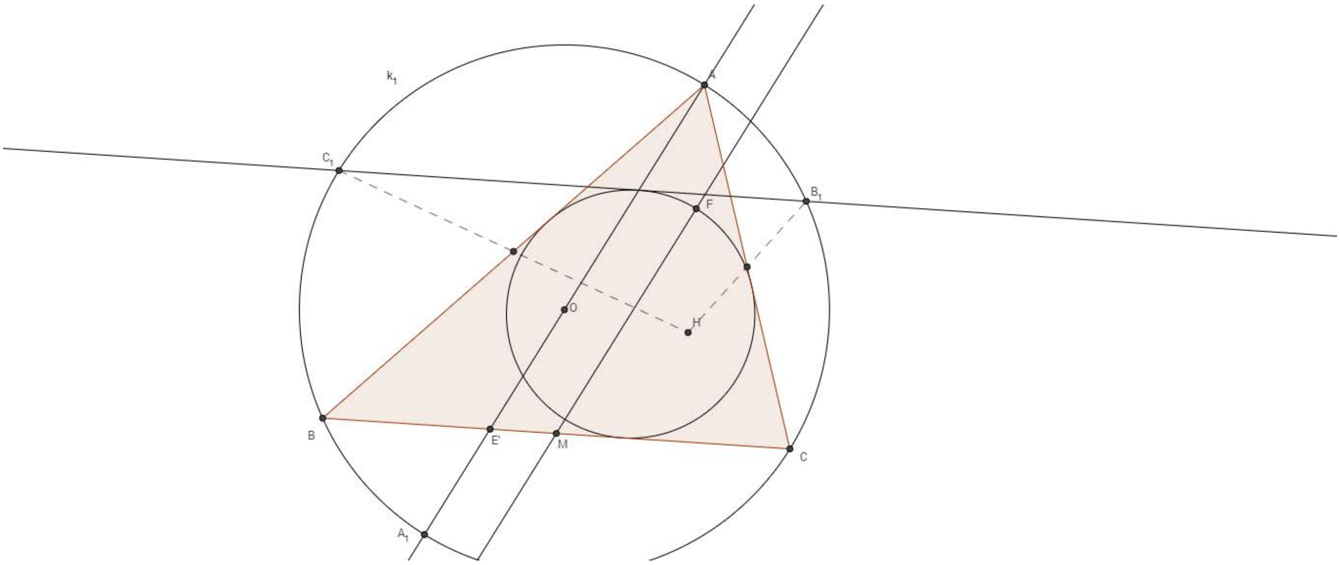
Source: My Own

mihajlon

#1 Mar 8, 2016, 2:52 am

Let $\triangle ABC$ be given, it's incircle k, circumcircle k_1 with center O, orthocenter H. Denote reflections of H WRT midpoints of AB and AC with C_1 and B_1. Let A-excircle touch BC at E' and $AE' \cap k_1 = \{A_1\}$. Prove that if F and M are $Feuerbach$ point and midpoint of BC, respectively, and C_1B_1 is tangent to k that $FM||A_1O$.

Attachments:



Luis González

#2 Mar 8, 2016, 3:52 am

Relabel $(I), (O)$ the incircle and circumcircle of $\triangle ABC$ with radii r, R. (I) touches BC at D and D' is the antipode of D on (I). Clearly B_1 and C_1 are the reflections of B and C on (O), therefore if B_1C_1 touches (I), then $D' \in B_1C_1 \Longrightarrow O \in \overline{AD'E'}$.

Redefine F as the intersection of AH with MI. $\overline{MIF}$ is the D-midline of $\triangle DD'E'$, i.e. $IF \parallel AD' \Longrightarrow IFAD'$ is a parallelogram, even a rhombus as AI bisects $\angle FAD' \equiv \angle HAO \Longrightarrow ID' = IF = r \Longrightarrow F \in (I)$. Since $MI \parallel AO$, then MI passes through the 9-point center N of $\triangle ABC$, lying on the segment $\overline{MI}$ (due to $MI \geq MN \Longleftrightarrow R - r \geq \frac{1}{2}R \Longleftrightarrow R \geq 2r) \Longrightarrow F \equiv (I) \cap \overrightarrow{NI} \Longrightarrow F$ coincides with the Feuerbach point of $\triangle ABC$ $\Longrightarrow FM \parallel OA$.

mihajlon

#3 Mar 8, 2016, 4:15 am

Thanks, very nice $Luis$.

High School Olympiads

Concurrent lines and collinear points

Source: Own

buratinogigle
#1 Mar 7, 2016, 12:05 pm

Let ABC be a triangle with incircle (I) touches BC, CA, AB at D, E, F. A_b, A_c lie on AB, AC such that $IA_b \perp IB, IA_c \perp IC$. Circle (K_a) passes through A_b, A_c and is tangent to (I) at A_1. Similarly, we have B_1, C_1.

a) Prove that AA_1, BB_1, CC_1 are concurrent at P.

b) Let AA_1, BB_1, CC_1 cut (I) again at A_2, B_2, C_2. Prove that DA_2, EB_2, FC_2 are concurrent at Q.

c) Prove that PQ passes through Nagel point of triangle ABC.

TelvCohl
#2 Mar 7, 2016, 3:41 pm • 1

Clearly, A_b, A_c lie on the reflection of BC in I, so A_bA_c is tangent to $\odot(I)$ at the reflection D_1 of D in I. Let X be the pole of A_1D_1 WRT $\odot(I)$ and let $D_2 \equiv EF \cap A_bA_c$. Since A_1X is tangent to $\odot(A_bA_1A_c)$ at A_1, so from $XA_1 = XD_1 \Longrightarrow A_1D_1$ is the bisector of $\angle A_cA_1A_b$, hence combine $(A_b, A_c; D_1, D_2) = -1$ we get $A_1D_1 \perp A_1D_2 \Longrightarrow D_2 \in DA_1$.

Since D_2 is the intersection of the polar of A, D_1 WRT $\odot(I)$, so D_2 lies on the polar τ of the Nagel point N_a of $\triangle ABC$ WRT $\odot(I)$, hence from $D(E, F; A_1, A_2) = -1$ we get DA_2 passes through the trilinear pole Q of τ WRT $\triangle DEF$. Similarly, we can prove Q lies on EB_2 and FC_2, so DA_2, EB_2, FC_2 are concurrent at Q.

Let $\triangle Q_DQ_EQ_F$ be the cevian triangle of Q WRT $\triangle DEF$. From Steinbart theorem $\Longrightarrow AA_1, BB_1, CC_1$ are concurrent at P. Since $D_1(E, F; N_a, D_2) = -1 = (E, F; Q_D, D_2)$, so Q_D lies on AN_a. Analogously, we can prove $Q_E \in BN_a$ and $Q_F \in CN_a$, so from Concurrent on OI line (**Lemma 1** at post #3) we conclude that P, Q, N_a are collinear.

Luis González
#3 Mar 8, 2016, 3:06 am • 1

Obviously A_bA_c is the tangent of (I) at X parallel to BC. Let $Y \equiv EF \cap A_bA_c$ and let M be the midpoint of $\overline{XY}$. Since $(A_b, A_c, X, Y) = -1$, then $MX^2 = MA_b \cdot MA_c \Longrightarrow M$ is on the radical axis of $(I), \odot(A_1A_bA_c)$, i.e. MA_1 is their common tangent. Thus since $\angle XA_1Y = 90^\circ$, it follows that D, A_1, Y are collinear.

Let $A_3 \equiv DA_2 \cap EF$ and define B_3 and C_3 cyclically. As EF is the polar of A WRT (I), then $Y(A_1, A_2, A_3, A) \equiv Y(D, A_2, A_3, A) = -1$, which means that AY is the polar of A_3 WRT $(I) \Longrightarrow A_3$ is on the polar AA_2 of Y WRT (I), i.e. AA_3 is the A-Nagel cevian of $\triangle ABC$ and likewise BB_3 and CC_3 are the Nagel cevians of $\triangle ABC$ concurring at the Nagel point $Na \Longrightarrow DA_2, EB_2, FC_2$ concur at the Ceva point Q of Ge, Na WRT $\triangle DEF$, where Ge is the Gergonne point of $\triangle ABC$. Thus according to problem collinear we conclude that AA_1, BB_1, CC_1 concur at a point P on NaQ.

Remark: Using the result of the problem Three concurrent lines for $\triangle A_bA_cA_1$, we obtain that P is the point $Na^3 \equiv ((s-a)^3 : (s-b)^3 : (s-c)^3)$, i.e. the barycentric cube of the Nagel point Na.

Quick Reply

High School Olympiads

collinear

Reply

Source: own

vankhea
#1 Jan 6, 2014, 9:46 am

Incircle (I) of ΔABC touch the sides BC, CA, AB at A', B', C' respectively. Let K be any point in plan. The rays KA', KB', KC' cuts $B'C', C'A', A'B'$ at A_1, B_1, C_1 and cuts incircle (I) at A_2, B_2, C_2 respectively.
Let P be concurrent point of AA_1, BB_1, CC_1 and let Q be concurrent point of AA_2, BB_2, CC_2.
Prove that K, P, Q collinear.

Luis González
#2 Jan 6, 2014, 11:03 am • 1

Let $B'K, C'K, B'C_2, C'B_2$ cut BC at Y, Z, U, V, respectively. By Desargues involution theorem for the quadrangle $B'C'B_2C_2$ and its circumcircle (I), we deduce that A' is a double point of the involution $Y \mapsto Z, U \mapsto V \Longrightarrow$ $B'(Z, Y, U, A') = C'(Y, Z, V, A') \Longrightarrow (Y, K, B_2, B_1) = (Z, K, C_2, C_1) \Longrightarrow$ pencils $B(C, B_1, B_2, K)$ and $C(B, C_1, C_2, K)$ are projective with double ray BC, hence $K, P \equiv BB_1 \cap CC_1$ and $Q \equiv BB_2 \cap CC_2$ are collinear.

P.S. The problem was also posted before at Hard geometry problem.

vankhea
#3 Jan 6, 2014, 11:20 am

Thank you sir Luis Gonzalez. I don't know this problem already posted.
Thanks you again

jayme
#4 Jan 6, 2014, 12:50 pm

Dear Mathlinkers,
just for information
1. Q is the Steinbart point...(http://perso.orange.fr/jl.ayme vol. 3 Les points de Steinbart-Rabinowitz, p. 2)
2. P can be proved by the cevian nest theorem (vol. 3 the cevian nest theorem)
I thik that a synthetic proof is possible ?
Sincerely
Jean-Louis

jayme
#5 Jan 6, 2014, 1:00 pm

Dear Mathlinkers,
I come back....
yes, by Pascal theorem we can prove that ABC, A1B1C1 and A2B2C2 share the same perspectric... then K, P and Q are collinear.
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Three concurrent lines

Source: Paul Yiu. Hyacinthos message #19848

Luis González

#1 May 10, 2012, 10:58 am • 1

Triangle $\triangle ABC$ is scalene with circumcircle (O). Let ω_A be the circle internally tangent to (O) at A and tangent to BC. Tangents from B, C to ω_A, different from BC, meet at A^*. Points B^* and C^* are defined cyclically. Show that the lines AA^*, BB^* and CC^* concur at the isogonal conjugate of the isotomic conjugate of the incenter I of $\triangle ABC$, i.e. the Kimberling center X_{33}.

r1234

#2 May 14, 2012, 6:04 pm

I have not solved the problem though, but got an equivalent problem.

ABC is a triangle. D is the mid-point of the arc BC not containing A. Draw the tangent at D to $\odot ABC$ which is parallel to BC. Draw two circles Γ_1, Γ_2 passing through A, B and A, C and touching the tangent drawn at D to $\odot ABC$. Suppose they intersect at A'. Similarly define B', C'. Show that AA', BB', CC' concur.

mlm95

#3 May 15, 2012, 11:42 am

how do you reach this equivalent problem?

the rest is not difficult you can boil down the problem into this one:

In triangle ABC, D is the foot of angle bisector of A on BC. Let A' be a point on BC such that $BD = A'C$. Prove that AA' and BB' and CC' are concurrent which is obvious.

r1234

#4 May 15, 2012, 5:38 pm

Dear mlm95,
I think your conclusion is not right. Can you check it,plaese?

paul1703

#5 May 15, 2012, 6:40 pm

What mlm said is true and in fact very easy (to see the fact that it is <=> to your problem), now how your problem is <=> to the stated one?

r1234

#6 May 16, 2012, 6:17 pm • 5

> r1234 wrote:
> ABC is a triangle. D is the mid-point of the arc BC not containing A. Draw the tangent at D to $\odot ABC$ which is parallel to BC. Draw two circles Γ_1, Γ_2 passing through A, B and A, C and touching the tangent drawn at D to $\odot ABC$. Suppose they intersect at A'. Similarly define B', C'. Show that AA', BB', CC' concur.

If we just invert the the figure wrt A then we can come back to the main statement of the problem. Now for proving my statement let ℓ be the tangent drawn at the midpoint of of the arc BC not containing A to $\odot ABC$. Now let $AB \cap \ell = X$ and $AC \cap \ell = Y$. Let D be the midpoint of the arc BC and the circles passing through A, B and A, C touches the line ℓ at P, Q, respectively. Note that $PX = XD$ and $QY = YD$. Let K be the midpoint of PQ. Note that AA' passes through K. Now it's easy to check that D, K are isotomic wrt the segment XY. Since $BC \parallel XY$ we conclude AA' is the isotomic line of AD wrt $\triangle ABC$. Hence AA', BB', CC' concur at the isotomic conjugate of incenter I of $\triangle ABC$.

Now, as said earlier, inverting this figure wrt A with some power gives the main statement of the problem. Hence AA^*, BB^*, CC^* concur at the isogonal conjugate of the isotomic conjugate of the incenter of $\triangle ABC$.

THANKS TO **mlm95**, and **paul1703** FOR THEIR OBSERVATIONS.

Luis González

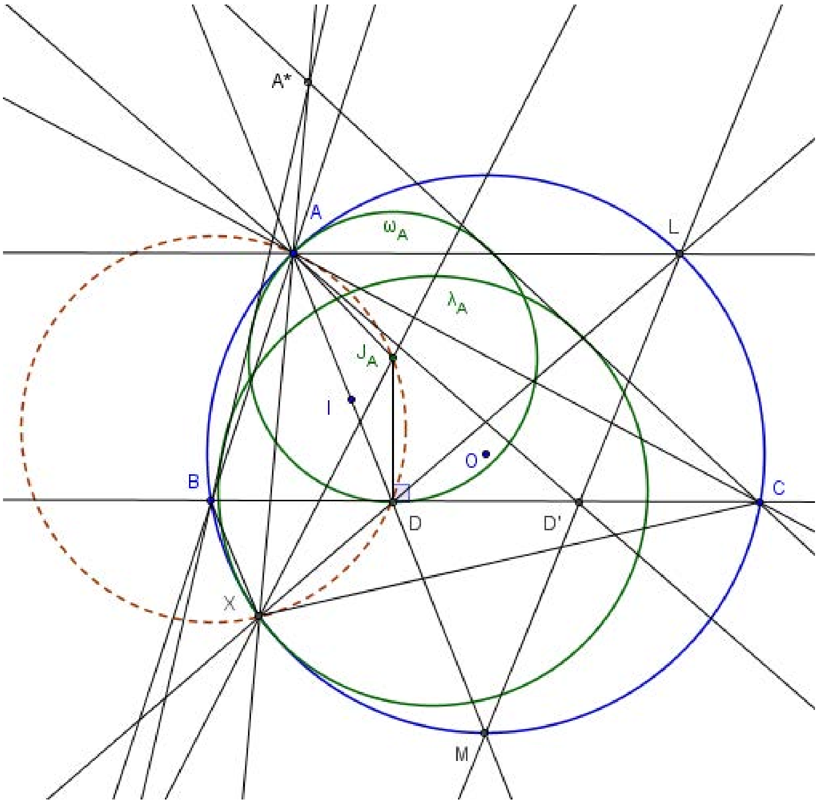
#7 Mar 2, 2016, 10:58 am

The inversive solution given by r1234 is very nice!. Here's another solution (no inversion):

Let D be the tangency point of $\omega_A \equiv (J_A)$ with BC. It's well known that AD is the internal bisector of $\angle BAC$, cutting (O) again at the midpoint M of its arc BC. Circle λ_A is tangent to BA^*, CA^* and internally tangent to the arc BMC (O) at X. Since A, A^* and X are the exsimilicenters of $(O) \sim \omega_A, \omega_A \sim \lambda_A$ and $\lambda_A \sim (O)$, then by Monge & d'Alembert theorem they are collinear.

On the other hand, it's well-known that XJ_A bisects $\angle BXC$ (see incenter of triangle and elsewhere). Hence assuming WLOG that $\angle ABC > \angle ACB$, we get $\angle AXJ_A = \frac{1}{2}\angle BXC - \angle BXA = \frac{1}{2}(180^\circ - \angle BAC) - \angle ACB = \frac{1}{2}(\angle ABC - \angle ACB) = \angle ADJ_A \Longrightarrow$ $AXDJ_A$ is cyclic $\Longrightarrow XJ_A$ bisects $\angle AXD$ due to $J_AA = J_AD \Longrightarrow XA, XD$ are isogonals WRT $\angle BXC$. Thus if XD cuts (O) again at L and ML cuts BC at D', then by symmetry $ALD'D$ is an isosceles trapezoid $\Longrightarrow D$ and D' are symmetric WRT the midpoint of BC and moreover $\angle XAM = \angle DLD' = \angle DAD' \Longrightarrow AA^* \equiv AX$ is the isogonal of the isotomic of $AD \equiv AI$ WRT $\triangle ABC$. Similarly for BB^* and CC^* and the conclusion follows.

Attachments:



Quick Reply

High School Olympiads

Cyclocevian Conjugate

Reply

ABCDE

#1 Mar 7, 2016, 9:34 pm

Let DEF be the pedal triangle of point P with respect to triangle ABC. Suppose that AD, BE, and CF concur at Q, and let R be the cyclocevian conjugate of Q with respect to ABC. Prove that P, Q, and R are collinear and that the line passing through them passes through a fixed point over all such points P.

Luis González

#2 Mar 7, 2016, 9:58 pm

Let P^* be the isogonal conjugate of $P.$ $\odot(DEF)$ cuts BC, CA, AB again at D^*, E^*, F^*; vertices of pedal triangle of $P^*.$ Thus $R \equiv AD^* \cap BE^* \cap CF^*.$ Now since $\triangle ABC$ and $\triangle D^*E^*F^*$ are orthologic with orthology centers P, P^* and perspective with perspector $R,$ by Sondat's theorem P, P^*, R are collinear. Similarly P, P^*, Q are collinear $\Longrightarrow P, P^*, Q, R$ are collinear. Now the fixed point is the De Longchamps point, i.e. orthocenter of antimedial triangle of $\triangle ABC.$ For a proof see Two difficult parallels (Lemma at post #3)

Quick Reply

High School Olympiads

Two difficult parallels

Reply

Source: seem to be difficult

jayme
#1 Jul 21, 2009, 5:01 pm

Dear Mathlinkers,
let ABC be a triangle, P a point so that the pedal triangle A'B'C' of P is also a cevian triangle, A"B"C" the antimedian triangle of ABC, L the orthocenter of A"B"C", A* the intersection of the A"-altitude of A"B"C" with AA' and cyclically B*, C*.
Prouve : A*B* // A'B'.
Sincerely
Jean-Louis

jayme
#2 Jul 26, 2009, 8:29 pm

Dear Mathlinkers,
any ideas concerning L (de Longchmps's point)?
Sincerely.
Jean-Louis

Luis González
#3 Jul 20, 2010, 11:22 pm

We shall show that P, its isogonal conjugate and the De Longchamps point of ABC are collinear iff the pedal triangle of P WRT ABC is perspective with ABC. Since I don't have a synthetic proof of this fact, I'll introduce this lemma apart

Lemma: Let P be a point of the plane of $\triangle ABC$ and P^{-1} be its isogonal conjugate. $L \equiv X_{20}$ is the De Longchamps point of $\triangle ABC$ (orthocenter of its antimedial triangle). Then, L, P, P^{-1} are collinear iff the pedal triangle of P WRT $\triangle ABC$ is perspective with $\triangle ABC$, i.e. if P moves on the Darboux cubic of $\triangle ABC$.

Using barycentric coordinates WRT $\triangle ABC$, the equation of the line ℓ passing through $P(u:v:w)$ and its isogonal conjugate $P^{-1}\left(\frac{a^2}{u}:\frac{b^2}{v}:\frac{c^2}{w}\right)$ is defined by

$$\ell \equiv u(c^2v^2-b^2w^2)x+v(a^2w^2-c^2u^2)y+w(b^2u^2-a^2v^2)z=0$$

If $L \equiv X_{20}$ lies on $\ell \Longrightarrow \sum_{\text{cyclic}}(S_AS_B+S_AS_C-S_BS_C)(c^2v^2-b^2w^2)x=0$

Then comparing this latter expression with the equation of the Darboux cubic:

$$\sum_{\text{cyclic}}(S_AS_B+S_AS_C-S_BS_C)(c^2y^2-b^2z^2)x=0$$

We conclude that L, P, P^{-1} are collinear and the proof of the lemma is completed.

This post has been edited 2 times. Last edited by Luis González, Jul 21, 2010, 1:43 am

Luis González
#4 Jul 21, 2010, 12:17 am

Let us restate the proposed problem as follows:

Problem: P is a point on the plane of $\triangle ABC$ such that its pedal triangle $\triangle A'B'C'$ is perspective with $\triangle ABC$. Let

$\triangle A_0B_0C_0$ be the antimedial triangle of $\triangle ABC$ and L be its orthocenter, i.e X_{20} of $\triangle ABC$. Lines AA', BB', CC' cut LA_0, LB_0, LC_0 at X, Y, Z, respectively. Then $\triangle XYZ$ and $\triangle A'B'C'$ are homothetic.

$\triangle ABC$ and $\triangle A'B'C'$ are perspective through perspector $Q \equiv AA' \cap BB' \cap CC'$ and orthologic through the orthology centers P and its isogonal conjugate P^{-1}. Hence, by Sondat theorem Q, P, P^{-1} are collinear. From the above lemma, we conclude that Q, L, P, P^{-1} are collinear. From $\triangle QA'P \sim \triangle QXL, \triangle QB'P \sim \triangle QYL, \triangle QC'P \sim \triangle QZL$ we get

$$\frac{\overline{QX}}{\overline{QA'}} = \frac{\overline{QY}}{\overline{QB'}} = \frac{\overline{QZ}}{\overline{QC'}} = \frac{\overline{QL}}{\overline{QP}}$$

Thus, $\triangle XYZ$ and $\triangle A'B'C'$ are centrally similar through center Q and similarity coefficient $\frac{\overline{QL}}{\overline{QP}} \Longrightarrow B'C' \parallel YZ$, $C'A' \parallel ZX$ and $A'B' \parallel XY. \square$

jayme

#5 Aug 2, 2010, 7:44 pm

Dear Luis and Mathlinkers,
I come back with the lemma...for which I haven't also a synthetic proof.
We know that the de Longchamps point L is a PC-point (L being the symmetric of the PC-point H (orthocenter) wrt O (center of the circumcircle)).
But how can we prove synthetically that L is the Ceva point of a PC-point?
Do some one know a construction of this PC-point?
Sincerely
Jean-Louis

jayme

#6 Sep 11, 2010, 5:02 pm

Dear Mathlinkers,
an article (last of three) concerning the first synthetic proof of the PC-lines and the de Longchamps point has been put on my website
http://perso.orange.fr/jl.ayme vol. 6 Pedal-cevian lines go through the de Longchamps's point, p.33
Sincerely
Jean-Louis

Quick Reply

High School Math

Geometry inequality

Reply

Isi

#1 Mar 6, 2016, 2:56 pm

A triangle ABC is given. A line is drawn such that it is parallel with BC and is tangent to the incircle. It meets AC and AB in D and E respectively. Prove that $8DE \leq AB + BC + AC$.

Luis González

#2 Mar 6, 2016, 10:58 pm • 1

It's from Italy TST 1999. See http://www.artofproblemsolving.com/community/c6h389382.

Quick Reply

High School Olympiads

Geometry inequality involving tangent to incircle

Reply

Source: Italy TST 1999

WakeUp

#1 Feb 2, 2011, 12:24 am

Let D and E be points on sides AB and AC respectively of a triangle ABC such that DE is parallel to BC and tangent to the incircle of ABC. Prove that

$$DE \leq \frac{1}{8}(AB+BC+CA)$$

Luis González

#2 Feb 2, 2011, 3:06 am • 1

$(I,r),(I_a,r_a)$ are the incircle and A-excircle of ABC. From $\triangle ABC \sim \triangle ADE$ with A-excircles $(I_a,r_a),(I,r)$, we get

$\frac{DE}{BC}=\frac{r}{r_a} \Longrightarrow DE=\frac{a\cdot r}{r_a}$. Hence, we must have $\frac{a+b+c}{8} \geq \frac{a\cdot r}{r_a}$

Using the identity $\frac{r}{r_a}=\frac{b+c-a}{a+b+c}$, the desired inequality becomes:

$\frac{(a+b+c)^2}{8}-a(b+c-a) \geq 0 \Longleftrightarrow \frac{(b+c-3a)^2}{8} \geq 0,$ which is true

Equality holds for $\triangle ABC$ with $b+c=3a,$ i.e. when $DE \cap (I)$ is the Nagel point

TheBernuli

#3 Jan 14, 2014, 9:20 pm

I think I have easier solution.

$BCED$ is tangent quadrilateral $\Longrightarrow DE=DB+EC-DC \leftrightarrow DE=\frac{ac+ab-a^2}{a+b+c}$

So, $\frac{ac+ab-a^2}{a+b+c} \leq \frac{1}{8}(a+b+c) \leftrightarrow (b+c-3a)^2 \geq 0$, and we are finished.

sunken rock

#4 Feb 28, 2014, 10:58 am

> *TheBernuli wrote:*
>
> I think I have easier solution.
> $BCED$ is tangent quadrilateral
>
> $\Longrightarrow DE=DB+EC-[color=\#FF0000][b]BC[/b][/color] \leftrightarrow DE=\frac{ac+ab-a^2}{a+b+c}$
>
> So, $\frac{ac+ab-a^2}{a+b+c} \leq \frac{1}{8}(a+b+c) \leftrightarrow (b+c-3a)^2 \geq 0$, and we are finished.

I did not get how does that come 'easier'! Do you have any secret formula to calculate BD, CE? Didn't you use the same $\frac{AD}{AB}=\frac{AE}{AC}=\frac{r}{r_a}$, thus rendering the calculations heavier??

Best regards,
sunken rock

Quick Reply

High School Olympiads

30 characterizations of the symmedian line

Reply

Luis González
#1 Sep 16, 2015, 11:15 am • 21

The following article compiles 30 different characterizations of the symmedian line of a triangle. Some results are already known and others are lifted from the AoPS forum and some personal notes. Proofs are left to the readers.

Keep in mind the purpose of this thread is not to collect the solutions. Enjoy!

Attachments:
Characterizations of the symmedian line.pdf (134kb)

jayme
#2 Sep 16, 2015, 1:25 pm

Dear Luis,
very interesting article...

Sincerely
Jean-Louis

Luis González
#3 Mar 6, 2016, 10:45 pm • 5

I was requested via PM to post the links of all problems lifted from AoPS. Here they are

(8) http://www.artofproblemsolving.com/community/c6h489545
(10) http://www.artofproblemsolving.com/community/c6h582385
(11) http://www.artofproblemsolving.com/community/c6h1138039
(14) http://www.artofproblemsolving.com/community/c6h1081584
(15) http://www.artofproblemsolving.com/community/c6h621835
(16) http://www.artofproblemsolving.com/community/c6h1100577
(17) http://www.artofproblemsolving.com/community/c6h264538
(18) http://www.artofproblemsolving.com/community/c6h396352
(20) http://www.artofproblemsolving.com/community/c6h435433
(21) http://www.artofproblemsolving.com/community/c6h364698
(22) http://www.artofproblemsolving.com/community/c6h501817
(23) http://www.artofproblemsolving.com/community/c6h242605
(24) http://www.artofproblemsolving.com/community/c6h446800
(25) http://www.artofproblemsolving.com/community/c6h1184665
(26) http://www.artofproblemsolving.com/community/c6h562068 (post #10)
(30) http://www.artofproblemsolving.com/community/c6h623408 (post # 9)

anditp
#4 Mar 7, 2016, 12:15 am

Anybody has the proof for 7??

GGPiku
#5 Mar 7, 2016, 1:16 am

> *anditp wrote:*

> Anybody has the proof for 7??

Hint: use that the circle with the diameter UV is the Apollonium's circle :)

anditp

#6 Mar 7, 2016, 1:25 am

Could you write the entire solution, please??

nikolapavlovic

#7 Mar 7, 2016, 1:41 am • 2

From the properties of the Apollonius circle we have
:$\frac{BS}{SC}=\frac{AB}{AC} \implies ABSC$ is harmonic $\implies$ the intersection of the tangents from B,C lies on AS and we are done.

This post has been edited 4 times. Last edited by Luis González, Mar 7, 2016, 2:43 am
Reason: Unhiding solution

buratinogigle

#8 Mar 10, 2016, 9:22 am

Here is a characterization of the symmedian line

http://artofproblemsolving.com/community/q1h1192618p5823900

Quick Reply

High School Olympiads

a new construction of the symmedian

Source: related to a problem from Mathley contest

CTK9CQT
#1 Jul 18, 2012, 8:39 pm

Let ABC be an acute triangle with its altitudes BE, CF. M is midpoint of BC, N is the intersection point of AM and EF. X is the projection of N on BC. Prove that AX is the symmedian of triangle ABC.

johnkwon0328
#2 Jul 18, 2012, 9:49 pm

Let X' as a point that is on BC and that makes AX' symmedian.
We have to prove that $X = X'$, so it is enough to show that $BN^2 - CN^2 = BX'^2 - CX'^2$.
It is well known that $BX' : CX' = c^2 : b^2$, and we can calculate BN^2 and CN^2 from Stewart's theorem in BEF and CEF as we know that $EF = a \cdot cosA$ and $EN : FN = c^2 : b^2$.

Click to reveal hidden text

yetti
#3 Jul 18, 2012, 11:21 pm • 1

WLOG, $AB > AC$. $(O), (P)$ is circumcircles of $\triangle ABC, \triangle AEF$, resp. Since $\triangle AEF \sim \triangle ABC$ are oppositely similar, $\angle AEP = \angle EBM = \angle MEB$. Then $BE \perp AE \Longrightarrow ME \perp PE$. Similrly, $MF \perp PF$, which means M is intersection of tangents of (P) at $E, F \Longrightarrow$ foot K of perpendicular from M to EF is midpoint of $|EF|$. AM, AK are medians of $\triangle ABC, \triangle AEF \Longrightarrow$ they are symmedians of oppositely similar $\triangle AEF, \triangle ABC$, resp. Since $\angle MXN = \frac{\pi}{2} = \angle MKN \Longrightarrow MNKX$ is cyclic $\Longrightarrow$
$\angle AKX = \angle AKN + \angle NKX = \angle AKN + (\pi - \angle XMN) = \pi \Longrightarrow X \in AK$.

High School Olympiads

acute non-isosceles triangle

Reply

aktyw19

#1 Mar 25, 2014, 10:05 pm

Let O- circumcenter of acute non-isosceles triangle ABC. Let P, Q such points on altitude AD, that OP and OQ perpendicular to AB and AC respectively. M- midpoint of BC, S- circumcenter of triangle OPQ. Prove, that $\angle BAS = \angle CAM$

ThirdTimeLucky

#2 Mar 26, 2014, 12:05 am

Lemma: In $\triangle ABC$, with circumcenter O, the projections of B, C onto AO are B', C' respectively. Let AD be the altitude, with $D \in BC$ and M the midpoint of BC. Then M is the circumcenter of $\triangle DB'C'$.

Proof: $BAB'D$ is cyclic with center K, the midpoint of AB. Therefore, perpendicular bisector of DB' is the perpendicular from K to $B'D$. We also have $\angle CAD = \angle OAB$ (since AO, AD are isogonal lines) and $\angle ABB' = \angle ADB'$ and so, $B'D \perp AC \implies$ perpendicular bisector of $B'D$ is parallel to AC and passes through $K \implies$ it also passes through M. Similarly, perpendicular bisector of $C'D$ also passes through M.

Back to problem: Let K, L be midpoints of AB, AC. Clearly P, Q are intersections of perpendicular bisectors of AB, AC respectively with AD, so we have $\angle PQO = \hat{C}, \angle OPQ = \hat{B}$ so $\angle AOP = \hat{C}$, and therefore $SO \perp OA$, i.e SO is antiparallel to BC. Let SO meet AB, AC at R, T respectively. Then $\angle ART = \hat{C}$ so $AK.AR = AO^2 = AP.AQ \implies RQ \perp AD$. Similarly, $TP \perp AD$. But AD is a circumcevian of $\triangle ART$, so by the lemma, S is midpoint of RT and so AS is the symmedian.

Luis González

#3 Mar 26, 2014, 12:26 am

$\angle POQ = \angle BAC, \angle OQP = \angle ACB = \angle AOP \Longrightarrow \triangle OPQ \sim \triangle ABC$ and AO is tangent of $\odot(OPQ)$. Hence, if the tangent of (O) at A cuts BC at E, we have $\angle DAS = \angle MEO$. But, from cyclic $AEMO, \angle MEO = \angle MAO \Longrightarrow$ $\angle DAS = \angle MAO \Longrightarrow AS, AM$ are isogonals WRT $\angle DAO \Longrightarrow AS, AM$ are isogonals WRT $\angle BAC$.

Quick Reply

High School Olympiads

Hard Geometry

Reply

GoJensenOrGoHome

#1 Sep 4, 2015, 3:34 am

Given a triangle ABC , let l be a line parallel to BC that cut AB and BC respectively in M and N. Now let P be the point of intersection of BN and MC. Let the circles passing through BMP and CNP intersects in a second point Q. Prove that $< BAQ =< PAC$.

Luis González

#2 Sep 4, 2015, 4:10 am • 1

As Q is the center of the spiral similarity that swaps $\overline{BM}$ and $\overline{NC}$, then $\triangle QBM \sim \triangle QNC \Longrightarrow$ $\text{dist}(Q, AB) : \text{dist}(Q, AC) = BM : NC = AB : AC \Longrightarrow AQ$ is the A-symmedian of $\triangle ABC$. Since AP is clearly the A-median of $\triangle ABC$, then AP and AQ are isogonals WRT $\angle BAC$ i.e. $\angle BAQ = \angle PAC$.

silouan

#3 Sep 6, 2015, 8:38 pm

It's from Balkan 2009 (but first time appeared in ML in 2007)
http://artofproblemsolving.com/community/c6h274320p1484879

Quick Reply

High School Olympiads

circumcircle and altitudes! CMO 2015 P4

Source: canadian mathematical olympiad

aditya21
#1 Apr 25, 2015, 2:29 am

Let ABC be an acute-angled triangle with circumcenter O. Let I be a circle with centre on the altitude from A in ABC, passing through vertex A and points P and Q on sides AB and AC. Assume that

$$BP.CQ = AP.AQ$$

Prove that I is tangent to the circumcircle of triangle BOC

This post has been edited 3 times. Last edited by aditya21, Apr 25, 2015, 2:30 am
Reason: e

ABCDE
#2 Apr 25, 2015, 2:49 am

Click to reveal hidden text

Luis González
#3 Apr 25, 2015, 7:32 am

Since P defines Q unambiguously, then it's enough to prove the converse, i.e. that the relation holds if $\odot(APQ)$ is tangent to $\odot(BOC)$.

Let D,E,F be the midpoints of BC,CA,AB. $(N) \equiv \odot(DEF)$ and $(K) \equiv \odot(OBC)$. It's known that AK,AN are isogonals WRT $\angle BAC$, thus the inversion $(A,\sqrt{AB \cdot AE})$ followed by symmetry WRT the angle bisector of $\angle BAC$ swaps (N) and $\odot(OBC)$. Thus $\odot(APQ)$ goes to the tangent $P'Q'$ of (N) antiparallel to BC that leaves A,N on a same side $\Longrightarrow P'Q'$ is the tangent of (N) at D. So denoting B_∞ and C_∞ the points at infinity of AC,AB, by involution properties, we get then $(P,B,A,C_\infty)=(P',E,B_\infty,A)$ and $(Q,C,A,B_\infty)=(Q',F,C_\infty,A)$. But $(P',E,B_\infty,A)=(Q',C_\infty,F,A)=(Q',F,C_\infty,A)^{-1} \Longrightarrow (P,B,A,C_\infty) \cdot (Q,C,A,B_\infty)=1 \Longrightarrow$ $BP \cdot CQ = AP \cdot AQ$.

Remark: From this proof we also get that the tangency point between $\odot(APQ)$ and $\odot(OBC)$ lies on the A-symmedian of $\triangle ABC$.

jayme
#4 Apr 25, 2015, 2:01 pm

Dear Mathlinkers,
this is just an observation...
Sometimes when a relation is given between two points in a triangle, we forget to present a geometrical construction of these two points... If yes, we will observed that we can have more than a solution...
Sincerely
Jean-Louis

TelvCohl
#5 Apr 25, 2015, 3:41 pm • 1

> jayme wrote:
> Dear Mathlinkers,
> this is just an observation...

Sometimes when a relation is given between two points in a triangle, we forget to present a geometrical construction of these two points... If yes, we will observed that we can have more than a solution...
Sincerely
Jean-Louis

It's not hard to find the construction of P and Q. From the condition (the center of $\odot(APQ)$ lie on A-altitude) we know PQ is anti-parallel to BC WRT $\angle A$. From $BP \cdot CQ = AP \cdot AQ$ we get there exist a point R on BC such that $RP \parallel AC$ and $RQ \parallel AB$. Notice that AR pass through the midpoint of PQ, so AR is A-symmedian of $\triangle ABC$, hence we get the construction of P and Q as following :

1. Let R be the intersection of A-symmedian of $\triangle ABC$ with BC.
2. Let $P \in AB, Q \in AC$ such that $RP \parallel AC, RQ \parallel AB$.

Actually, the tangent point of $\odot(OBC)$ and $\odot(APQ)$ is the projection of O on A-symmedian of $\triangle ABC$

jayme
#6 Apr 25, 2015, 5:37 pm • 1

Dear,
Thank for your precise answer correponding to the problem...
I was just speaking about only on the relation BP.CQ = AP. AQ which conduct more than one solution par P and Q...
Now with the restriction of the problem, it is OK.
Sincerely
Jean-Louis

drmzjoseph
#7 Apr 25, 2015, 10:55 pm

Let X be point such that $APXQ$ is an parallelogram
$\Rightarrow \frac{PB}{PX} = \frac{QX}{QC} \Rightarrow \triangle BPX \sim \triangle XQC \Rightarrow \angle BXP = \angle XCQ \Rightarrow B, X$ and C are collinear.
Easy see that $\angle APQ = \angle BCA \Rightarrow BPQC$ is cyclical, If Z is Miquel point of $\triangle ABC$ (Points P, Q, X) then $2\angle BAC = \angle BZC = \angle BOC$ and $\angle BXP = \angle BZP = \angle BCA = \angle QPA = \angle PQX = \angle PQZ + \angle ZCB$ Then Z is tangency point of $\odot(BOC)$ and $\odot(PAQ)$

This post has been edited 1 time. Last edited by drmzjoseph, Apr 25, 2015, 10:56 pm

drmzjoseph
#9 Apr 25, 2015, 11:16 pm

Similar problem
Source: ONEM - Peru 2014, problem 4 - category 3

Let ABC be an acute-angle triangle with circumcenter O, if D, E, F lies on the sides BC, CA and AB respectively, such that $BDEF$ is a parallelogram. Suppose that $DF^2 = AE.EC < \frac{AC^2}{4}$.
Prove that $\odot(AOC)$ and $\odot(DBF)$ are tangents

Remark

navredras
#10 Apr 26, 2015, 4:12 am

> *drmzjoseph wrote:*
> Let X be point such that $APXQ$ is an parallelogram
> $\Rightarrow \frac{PB}{PX} = \frac{QX}{QC} \Rightarrow \triangle BPX \sim \triangle XQC \Rightarrow \angle BXP = \angle XCQ \Rightarrow B, X$ and C are collinear.
> Easy see that $\angle APQ = \angle BCA \Rightarrow BPQC$ is cyclical, If Z is Miquel point of $\triangle ABC$ (Points P, Q, X) then $2\angle BAC = \angle BZC = \angle BOC$ and $\angle BXP = \angle BZP = \angle BCA = \angle QPA = \angle PQX = \angle PQZ + \angle ZCB$
> Then Z is tangency point of $\odot(BOC)$ and $\odot(PAQ)$

Why is $\angle APQ = \angle BCA$ and how the last equality ($\angle BXP = \angle PQZ + \angle ZCB$ helps us to determinate that Z is the point of tangency?

drmzjoseph

#11 Apr 26, 2015, 6:08 am

> navredras wrote:
>
> > drmzjoseph wrote:
> >
> > Let X be point such that $APXQ$ is an parallelogram
> > $\Rightarrow \frac{PB}{PX} = \frac{QX}{QC} \Rightarrow \triangle BPX \sim \triangle XQC \Rightarrow \angle BXP = \angle XCQ \Rightarrow B, X$ and C are collinear.
> > Easy see that $\angle APQ = \angle BCA \Rightarrow BPQC$ is cyclical, If Z is Miquel point of $\triangle ABC$ (Points P, Q, X) then $2\angle BAC = \angle BZC = \angle BOC$ and
> > $\angle BXP = \angle BZP = \angle BCA = \angle QPA = \angle PQX = \angle PQZ + \angle ZCB$ Then Z is tangency point of $\odot(BOC)$ and $\odot(PAQ)$
>
> Why is $\angle APQ = \angle BCA$ and how the last equality ($\angle BXP = \angle PQZ + \angle ZCB$ helps us to determinate that Z is the point of tangency?

Hello, thanks for read my solution.
1. If O_1 is the circumcenter of $\triangle PAQ \Rightarrow \angle APQ = 90^\circ - \angle O_1AQ = \angle BCA$
2, $\angle PZB = \angle PQZ + \angle ZCB$ i.e. $\odot(PZQ)$ and $\odot(BZC)$ are tangents (If we drawing a line tangent to $\odot(PZQ)$ is easy with angles see that is tangent to $\odot(BZC)$ too)

This post has been edited 1 time. Last edited by drmzjoseph, Apr 26, 2015, 6:12 am

Quick Reply

High School Olympiads

SRMC 2014

Reply

sebepkaly

#1 Jan 21, 2015, 12:52 pm

Let w be the circumcircle of non-isosceles acute triangle ABC. Tangent lines to w in A and B intersect at point S. Let M be the midpoint of AB, and H be the orthocenter of triangle ABC. The line HA intersects lines CM and CS at points M_a and S_a, respectively. The points M_b and S_b are defined analogously. Prove that M_aS_b and M_bS_a are the altitudes of triangle M_aM_bH.

This post has been edited 1 time. Last edited by sebepkaly, Jan 21, 2015, 4:11 pm

TelvCohl

#2 Jan 21, 2015, 2:43 pm • 1

My solution:

Let $D = AH \cap BC, E = BH \cap CA$.

It's well-known that CS is the isogonal conjugate of CM WRT $\angle ACB$,
so combine with $\triangle CAD \sim \triangle CBE \Longrightarrow \triangle CAD \cup S_a \cup M_a \sim \triangle CBE \cup M_b \cup S_b \; (\star)$,
hence we get $\angle DM_aC = \angle ES_bC \Longrightarrow M_a, M_b, S_a, S_b$ are concyclic .

From Steiner theorem and $(\star) \Longrightarrow \frac{AC^2}{AS_a \cdot AM_a} = \frac{DC^2}{DS_a \cdot DM_a} = \frac{EC^2}{ES_b \cdot EM_b} = \frac{BC^2}{BS_b \cdot BM_b}$,

so $\odot(ABDE)$ is coaxial with $\odot(S_aS_bM_aM_b)$ and the degenerate circle C ,
hence the center of $\odot(S_aS_bM_aM_b)$ lie on the line connecting C and the center M of $\odot(ABDE)$,
so we get M_aM_b is the diameter of $\odot(S_aS_bM_aM_b)$ and $M_aS_b \perp HM_b, M_bS_a \perp HM_a$.

Q.E.D

izat

#3 May 3, 2015, 8:52 pm

> *TelvCohl wrote:*
> My solution:
>
> Let $D = AH \cap BC, E = BH \cap CA$.
>
> It's well-known that CS is the isogonal conjugate of CM WRT $\angle ACB$,
> so combine with $\triangle CAD \sim \triangle CBE \Longrightarrow \triangle CAD \cup S_a \cup M_a \sim \triangle CBE \cup M_b \cup S_b \; (\star)$,
> hence we get $\angle DM_aC = \angle ES_bC \Longrightarrow M_a, M_b, S_a, S_b$ are concyclic .
>
> From Steiner theorem and $(\star) \Longrightarrow \frac{AC^2}{AS_a \cdot AM_a} = \frac{DC^2}{DS_a \cdot DM_a} = \frac{EC^2}{ES_b \cdot EM_b} = \frac{BC^2}{BS_b \cdot BM_b}$,
>
> so $\odot(ABDE)$ is coaxial with $\odot(S_aS_bM_aM_b)$ and the degenerate circle C ,
> hence the center of $\odot(S_aS_bM_aM_b)$ lie on the line connecting C and the center M of $\odot(ABDE)$,
> so we get M_aM_b is the diameter of $\odot(S_aS_bM_aM_b)$ and $M_aS_b \perp HM_b, M_bS_a \perp HM_a$.
>
> Q.E.D

I don't understand ,why circles are coaxial?

TelvCohl

#4 May 4, 2015, 10:28 am

> izat wrote:
> I don't understand ,why circles are coaxial?

$$\frac{AC^2}{AS_a \cdot AM_a} = \frac{DC^2}{DS_a \cdot DM_a} = \frac{EC^2}{ES_b \cdot EM_b} = \frac{BC^2}{BS_b \cdot BM_b}$$

means the ratio of the power of A, B, D, E WRT $\odot C$ and $\odot (S_a S_b M_a M_b)$ are the same ,

so A, B, D, E lie on a circle coaxial with the degenerate circle $\odot C$ and $\odot (S_a S_b M_a M_b)$.

Luis González

#5 May 4, 2015, 12:04 pm • 2

Let D, E be the feet of the altitudes on CB, CA. Z is the antipode of C on w and CM cuts AZ at L. Since $CDHE \cup S_a \sim CAZB \sim L \Longrightarrow \frac{HS_a}{S_a D} = \frac{ZL}{LA}$. But from parallelogram $HBZA$ with center M, we obtain $\frac{ZL}{LA} = \frac{HM_b}{M_b B} \Longrightarrow \frac{HS_a}{S_a D} = \frac{HM_b}{M_b B} \Longrightarrow M_b S_a \parallel BC \Longrightarrow M_b S_a \perp HM_a$ and similarly $M_a S_b \perp HM_b$.

Quick Reply

High School Olympiads

Geometry [2]

Reply

kaito_shinichi

#1 Jun 12, 2015, 2:00 am

Given triangle ABC. $AD \perp BC$ ($D \in BC$). M, N are midpoints of AB, AC. $(BMD) \cap (CND)$ at D, K. Prove that $\frac{KM}{KN} = \frac{AB}{AC}$

Luis González

#2 Jun 12, 2015, 7:06 am

By Miquel theorem, $AMKN$ is cyclic. Since $NC = ND$, then $MN \parallel BC$ is tangent of $\odot(CND)$ at N and similarly MN is tangent of $\odot(BMD) \Longrightarrow MN$ is common external tangent of $\odot(CND), \odot(BMD) \Longrightarrow$ their radical axis DK cuts MN at its midpoint P. Now since $\angle PAN = \angle PDN = \angle MNK = \angle MAK \Longrightarrow AK$ is A-symmedian of $\triangle AMN$ $\Longrightarrow KM : KN = AM : AN = AB : AC.$

professordad

#3 Jun 12, 2015, 7:27 am

By Miquel, $AMKN$ is cyclic so $\angle BMK = \angle ANK$. Also, $\angle MBK = \angle MDK = \angle NMK = \angle NAK$ so $\angle MBK = \angle NAK$. Thus $\triangle MBK \sim \triangle NAK$ and the result follows.

Quick Reply

High School Olympiads

symmedian property

Reply

hollandman

#1 Mar 15, 2009, 6:23 am

Let ABC be an acute triangle. Let M be the midpoint of the larger arc BC of its circumcircle. Let the internal bisector of $\angle BAC$ hit BC at D, and let MD meet the circumcircle again at P. Prove that AP is a symmedian.

Luis González

#2 Mar 15, 2009, 8:01 am

From $MB = MC,$ we obtain $\triangle MBD \sim \triangle PCD$ and $\triangle MCD \sim \triangle PBD$

$$\frac{PC}{MB} = \frac{CD}{MD}\ ,\ \frac{PB}{MC} = \frac{BD}{MD} \Longrightarrow \frac{PC}{PB} = \frac{CD}{BD}$$

On the other hand, $\frac{CD}{BD} = \frac{AC}{AB} \Longrightarrow \frac{PC}{PB} = \frac{AC}{AB}$

$\Longrightarrow P$ lies on the the A-Apollonius circle $\Longrightarrow AP$ is the A-symmedian.

sunken rock

#3 Nov 4, 2009, 6:03 pm

Let Q be the middle of BC, AQ intersects the circle at R, while MQ at N. AM being the external bisector of angle $\angle BAC$, so AM and AD are perpendicular, and A, D, Q and M are concyclic, hence $\angle QAD = \angle QMD$, so N is the middle of the arc PR and, consequently, AP symmedian.

Best regards,
sunken rock

jayme

#4 Nov 24, 2011, 5:03 pm

Dear Mathlinkers,
an article concerning the "Ayme's theorem" and this question have been put on my website.
http://perso.orange.fr/jl.ayme vol. 20 p. 28
You can use Google translator
Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Interesting geometry

Reply

truongtansang89

#1 Mar 13, 2011, 5:12 am

Let an acute triangle ABC with median CM. The bisectors of segments BC and

CA intersect CM at Q, P respectively. BQ intersects AP at I.

Prove that $\widehat{ACI}=\widehat{BCM}$

Luis González

#2 Mar 14, 2011, 11:58 am • 4

Let D be the reflection of C about $M.$ Lines AP,BQ cut DB,DA at $U,V,$ respectively. Then $BCVD$ and $ACUD$ are isosceles trapezoids with legs $CV=BD$ and $CU=AD,$ i.e. $\triangle ACV \sim \triangle BCU$ are isosceles with common apex C $\Longrightarrow \angle ACV=\angle BCU.$ If the tangents to the circumcircle of $\triangle ABC$ through A,B intersect at $E,$ then we have $\angle CAV=\angle BAE=\angle C$ and $\angle CBU=\angle ABE=\angle C.$ Thus, by Jacobi's theorem, it follows that lines CE,AU,BV concur at $I \Longrightarrow CI \equiv CE$ is the C-symmedian of $\triangle ABC \Longrightarrow \angle ACI=\angle BCM.$

Rijul saini

#3 Mar 16, 2011, 11:33 am

> *truongtansang89 wrote:*
> Let an acute triangle ABC with median CM. The bisectors of segments BC and
>
> CA intersect CM at Q, P respectively. BQ intersects AP at I.
>
> Prove that $\widehat{ACI}=\widehat{BCM}$

See this.

Quick Reply

High School Olympiads

Simedian

Reply

undefeatedturk
#1 Oct 2, 2011, 8:04 pm

M is midpoint of side BC of ABC triangle.($\angle BPC \neq 90$). O is circum center. P on $[MA$ such that $\angle BPC = 180 - \angle BAC$ BP and AC intersect E; CP and AB intersect F. N is midpoint of EF. D is the perpendicular from N to BC. Prove that AD is simedian of ABC and O, P, orthocenter of EDF are collinear.

Luis González
#2 Oct 2, 2011, 10:52 pm • 1

If AM cuts the circumcircle (O) again at Q, then $PCQB$ is a parallelogram $\Longrightarrow$ $AEPF$ and $ACQB$ are homothetic with homothetic center $A \Longrightarrow AEPF$ is cyclic with circumcircle (U) tangent to (O) at A. BC is then the polar of $N \equiv EF \cap AP$ WRT $(U) \Longrightarrow EF$ is the polar of D WRT $(U) \Longrightarrow DE, DF$ are tangents of $(U) \Longrightarrow AD$ is the A-symmedian of $\triangle AEF$, i.e. AD is the A-symmedian of $\triangle ABC$.

$UEDF$ is clearly a cyclic kite $\Longrightarrow$ Orthocenter T of the D-isosceles $\triangle DEF$ is the reflection of U about EF. Since $O(A, P, N, M) = -1, O(U, T, N, M) = -1$ and A, U, O are collinear, then O, P, T are collinear.

skytin
#3 Oct 3, 2011, 5:53 pm

Solution :
Let Q is intersection point of tangents to (O) thru B , C
Line thru Q and || BC intersect AB , AC at points B' , C'
N is Miquel point of triangle AB'C' and points Q , B , C on it's sides
Easy to see that N is on (O) and B'C intersect C'B at N
C'B' || BC , so M is on NA
Easy to see that A is center of homotety of BQCN and FPED
D is on QA
H is orthocenter of FDE , angle EFH = HEF = OBC = BCO
P is homotety center of BOC , EHF , so P is on OH . done

Quick Reply

High School Olympiads

Prove that AP is the A-symmedian of ABC.

Reply

vittasko

#1 Sep 1, 2010, 3:56 pm • 2

A triangle $\triangle ABC$ is given with altitude AD and let A' be, the reflexion of A with respect to the midpoint M of BC. Let E be, the orthogonal projection of A on the line segment BA' and let F be, the reflexion of E with respect to B. Prove that AP, where $P \equiv A'D \cap CF$, is the A-symmedian of $\triangle ABC$.

Kostas Vittas.

Luis González

#2 Sep 1, 2010, 11:06 pm • 2

Parallel line from A' to BC cuts AB, AC, AD at the reflections B', C', D' of A about B, C, D, i.e. $\triangle A'BC$ becomes the medial triangle of $\triangle AB'C'$. Thus, reflection F of E about B lies on the B'-altitude of $\triangle AB'C'$, being its midpoint. Consequently, lines $A'D$ and CF become the A- and B'- Schawtt line of $\triangle AB'C' \Longrightarrow P \equiv CF \cap A'D$ is the symmedian point of $\triangle AB'C'$. Since $\triangle ABC$ and $\triangle AB'C'$ are homothetic, then it follows that AP is the A-symmedian of $\triangle ABC$.

Quick Reply

High School Olympiads

symmedian

drEdrE

#1 Oct 10, 2012, 10:31 pm

Let ABC be a triangle, M,N be the midpoints of AB and AC, and P,Q be the intersections of the A-symmedian with BN and CM. Prove that the angles $\angle ABQ$ and $\angle ACP$ are equal.

Luis González

#2 Oct 10, 2012, 11:37 pm • 1

(X) is the circle through B,C tangent to AB, (Y) is the circle through C,A tangent to AB and (Z) is the circle through A,B tangent to AC. (X) and (Z) meet again at the 2nd Brocard point Ω_2 of $\triangle ABC \Longrightarrow \angle AB\Omega_2 = \omega$ is Brocard angle of $\triangle ABC$.

Let the tangents of the circumcircle of $\triangle ABC$ at B,C meet at D. DB and DC cut (Z) and (Y) again at U and V. $\angle AUB = \angle BAC = \angle DBC$ and $\angle AVC = \angle BAC = \angle DBC \Longrightarrow UAV \parallel BC$. Since $\triangle DBC$ is D-isosceles, then $BCVU$ is isosceles trapezoid $\Longrightarrow DB \cdot DU = DC \cdot DV \Longrightarrow$ A-symmedian AD is radical axis of $(Y),(Z)$. Since CM is radical axis of $(X),(Y)$ and $B\Omega_2$ is radical axis of $(X),(Z)$, then AD, CM an $B\Omega_2$ concur at the radical center Q of $(X),(Y),(Z) \Longrightarrow \angle ABQ \equiv \angle AB\Omega_2 = \omega$. By similar reasoning, CN is the C-cevian of the 1st Brocard point $\Omega_1 \Longrightarrow \angle ABQ = \angle ACP = \omega$.

Alternate formulation of the problem

vslmat

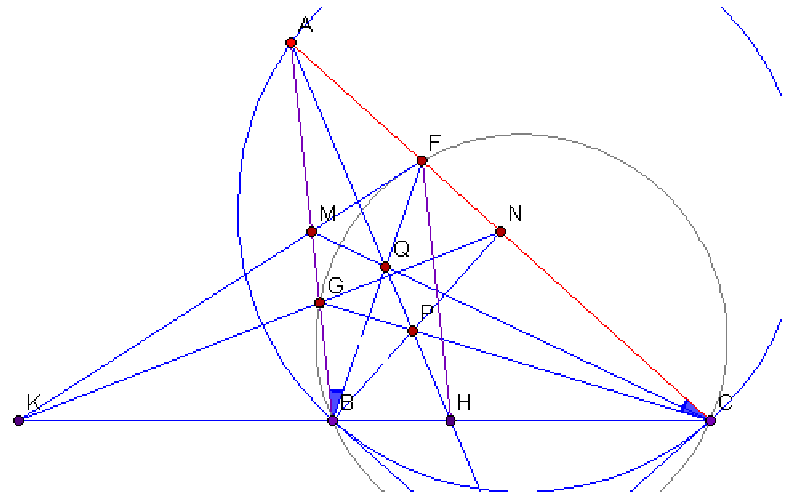
#3 Oct 21, 2012, 7:47 pm • 1

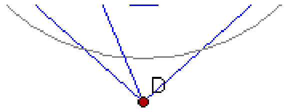
Another solution:
Let AH be the A-symmedian in ΔABC, H is on BC, CP cuts AB at G, BQ cuts AC at F.
Let GN cut BC at K, then $(K,H;B,C)$ is harmonic. If MF cuts BC at K', then $(K',H;B,C)$ is also harmonic, then $K' \equiv K$.

By Menelaus it is easy to get $\frac{AF}{FC} = \frac{GB}{GA} = \frac{KB}{KC} = \frac{HB}{HC}$, hence $HF \parallel AB$ and $GH \parallel AC$

Now notice that $\frac{AF}{AC} = \frac{BH}{BC}$, thus $AF.AC = \frac{AC^2.BH}{BC}$. Similarly $AG.AB = \frac{AB^2.CH}{BC}$, but as $\frac{BH}{HC} = \frac{AB^2}{AC^2}$, $AF.AC = AG.AB$, so $GFCB$ is cyclic and $\angle ABQ = \angle ACP$

Attachments:





Virgil Nicula

#4 Oct 22, 2012, 7:54 pm

See **PP3** from **here**.

Quick Reply

High School Olympiads

Mixtilinear incircles and symmedian

Reply

Source: own ?

jayme

#1 Nov 30, 2008, 5:13 pm

Dear Mathlinkers,
let ABC be a triangle, 0 the circumcircle of ABC, 2, 3 the B, C-mixtilinear incircles of ABC, B*, C* the point of contact of 2, 3 with 0,
B' the second point of intersection of BB* with 2,
Pb the tangent to 2 at B' which is parallel to the tangent to 0 at B,
C' the second point of intersection of CC* with 3,
Pc the tangent to 3 at C' which is parallel to the tangent to 0 at C,
and A" the meet point of 2 and 3.
Prove that AA" is the A-symmedian of ABC.
Sincerely
Jean-Louis

jayme

#2 Dec 1, 2008, 12:28 am

Dear Mathlinkers and Fang,
yes there is a typos in problem :

"A" is the meet point of Pb and Pc"

Sorry...
Sincerely
Jean-Louis

yetti

#3 Dec 1, 2008, 7:35 am

Tangents p_b, p_c of the mixtilinear incircles $(2), (3)$ at B', C' cut AB, AC at P, Q. Lines $BB'B^*, CC'C^*$ intersect at the external similarity center X_{56} of the circumcircle (O) and incircle (I). This is isogonal conjugate of the Nagel point Na. Thus, reflections t_b, t_c of the tangents $p_b = B'P, p_c = C'Q$ of $(2), (3)$ in the internal angle bisectors BI, CI are tangents of $(2), (3)$ parallel to AC, AB, which allows to calculate BP, CQ. Let t_b cut BA, BC at A_2, C_2 and let t_c cut CA, CB at A_3, B_3. Circles $(2), (3)$ are the B-, C-excircles of similar $\triangle A_2BC_2 \sim \triangle A_3B_3C$. Let (2) touch BA, BC at F_2, D_2 and let (3) touch CB, CA at D_3, E_3. $D_2E_3F_2D_3$ is a parallelogram, its diagonals F_2D_2, D_3E_3 cutting each other in half at the incenter I, $F_2E_3 \parallel D_3D_2 \equiv BC$.

$$\frac{BP}{CQ} = \frac{BC_2}{CB_3} = \frac{BD_2}{CD_3} = \frac{BF_2}{CE_3} = \frac{BA}{CA}$$

This means that $PQ \parallel BC$. Let tangents of the circumcircle (O) at B, C intersect at A_0. $\triangle A_0BC \sim \triangle A''PQ$ are centrally similar with similarity center A, having parallel sides, hence A, A'', A_0 are collinear and the line $AA''A_0$ is the A-symmedian.

Luis González

#4 Feb 24, 2016, 9:48 am

Let $p_b \equiv A''B', p_c \equiv A''C'$ cut AB, AC at U, V, resp. B-mixtilinear incircle (O_b) touches BC, BA at B_a, B_c and C-mixtilinear incircle (O_c) touches CB, CA at C_a, C_b. As B_aB_c, C_aC_b cut each other in half at the incenter I, then $B_aC_aB_cC_b$ is a parallelogram $\Longrightarrow B_cC_b \parallel BC$. If the B-excircle (I_b) touches BC at X, then we have $\triangle ABC \cup (I_b) \sim \triangle(BC, BA, p_b) \cup (O_b) \Longrightarrow \frac{BU}{BB_c} = \frac{BC}{BX} = \frac{a}{s}$ and similarly we have $\frac{CV}{CC_b} = \frac{a}{s} \Longrightarrow$ $UV \parallel B_cC_b \parallel BC$. Thus if the tangents of (O) at B, C meet at S, then $\triangle UVA''$ and $\triangle BCS$ are homothetic with center A $\Longrightarrow A'' \in AS$, i.e. AA'' is the A-symmedian.

Quick Reply

High School Olympiads

Pencil of circles from parallelograms

Reply

Source: me but probably old

yetti

#1 Nov 20, 2011, 12:02 am

D is arbitrary point on sideline BC of a $\triangle ABC$. Parallels to AB, AC through D cut AC, AB at E, F respectively.
(a) Show that $\odot(AEF)$ form a pencil of coaxal circles, when D moves.
(b) Identify, with proof, intersection of this pencil different from A WRT $\triangle ABC$.

mahanmath

#2 Nov 20, 2011, 1:00 am

Nice observation **yetti** !
Sketch of proof of both parts

panos_lo

#3 Nov 20, 2011, 7:58 pm

Slightly different: Inverting through A (by X' we call the the invert image of X) we can easily get that the point R satisfying that $AB'RC'$ is a parallelogram lies on $E'F'$ (follows immediately by Thales theorem). Thus, the invert image of R lies on the circle (A, F, E).
Note that this solution answers both questions.

panos_lo

#4 Nov 20, 2011, 8:02 pm

Mahanmath, could you please write the details of your solution?

mahanmath

#5 Nov 20, 2011, 8:40 pm

> panos_lo wrote:
> Mahanmath, could you please write the details of your solution?

The idea comes from barycentric coordinate

It's not hard to see $b.AE + c.AF = bc$, so if you put A at the origin of Cartesian coordinate system you'll find the perpendicular bisector of AE, AF should meet at fixed line . (it's just an easy computation for equation of those lines)

P.S

panos_lo

#6 Nov 20, 2011, 8:44 pm

Thanks a lot. I hope we see your article as soon as possible.

yetti

#7 Nov 20, 2011, 10:08 pm • 1

A-symmedian of $\triangle ABC$ cuts its circumcircle again at K. When $D \equiv B$, circle $\odot(AEF) \equiv \odot(AAB) \equiv (S)$ is tangent to AC at A. This circle cuts AK again at R and $\angle CBK = \angle CAK = \angle CAR = \angle ABR$.
$ABKC$ is harmonic $\Longrightarrow BC$ is B-symmedian of $\triangle ABK \Longrightarrow BR$ is its B-median. Similarly, when $D \equiv C$, circle $\odot(AEF) \equiv \odot(AAC) \equiv (T)$, tangent to AB at A, cuts AK also at its midpoint R.
Powers of E, F to $(S), (T)$ are in the same ratio $\frac{EA^2}{\overline{EA} \cdot \overline{EC}} = \frac{\overline{EA}}{\overline{EC}} = \frac{\overline{DB}}{\overline{DC}} = \frac{\overline{FB}}{\overline{FA}} = \frac{\overline{FA} \cdot \overline{FB}}{FA^2} \Longrightarrow \odot(AEF)$ is coaxal with $(S), (T) \Longrightarrow R \in \odot(AEF)$.

Luis González
#8 Dec 28, 2011, 8:52 am

Let M, N be the midpoints of AC, AB. $L \equiv AD \cap EF \cap MN$ is the center of the parallelogram $DEAF$. By Menelaus' theorem for $\triangle AMN$ cut by $\overline{ELF}$, we get

$$\frac{FN}{EM} = \frac{FA}{AE} \cdot \frac{LN}{ML} = \frac{DE}{DF} \cdot \frac{DB}{DC} = \frac{AB}{AC} \cdot \frac{DC}{DB} \cdot \frac{DB}{DC} = \frac{AB}{AC}$$

Circles $\odot(AEF)$ and $\odot(AMN)$ meet at A and the center P of the spiral similarity that takes $\overline{ME}$ into $\overline{NF}$. Thus $\frac{PN}{PM} = \frac{FN}{EM} = \frac{AB}{AC} = \frac{AN}{AM} \Longrightarrow AP$ is the A-symmedian of $\triangle AMN \Longrightarrow P$ is midpoint of the A-symmedian chord of the circumcircle $\odot(ABC)$. Hence, all circles $\odot(AEF)$ pass through the fixed points A, P. $\square$

Remark: Loci of the circumcenters U, V of $\triangle AEF$ and $\triangle DEF$ are two fixes lines. U obviously moves on the perpendicular bisector of $\overline{AP}$, while V moves on a line τ passing through the 9-point center of $\triangle ABC$. Alas, I don't have a synthetic proof of the 2nd assertion yet.

yetti
#9 Dec 28, 2011, 1:01 pm

> *Luis González wrote:*
> **Remark:** Loci of the circumcenters U, V of $\triangle AEF$ and $\triangle DEF$ are two fixes lines. U obviously moves on the perpendicular bisector of $\overline{AP}$, while V moves on a line τ passing through the 9-point center of $\triangle ABC$....

Let M, N be midpoints of AC, AB and $L \equiv AD \cap EF \cap MN$ diagonal intersection of parallelogram $DEAF$. Let K be centroid of $\triangle DEF \Longrightarrow \frac{\overline{DK}}{\overline{DL}} = \frac{2}{3} \Longrightarrow K$ is on fixed line $k \parallel MN$.
Let P be midpoint of the A-symmedian chord of $\triangle ABC$. Let AX be diameter of circumcircle (U) of $\triangle AEF$. U is on perpendicular bisector u of $AP \Longrightarrow X$ is on fixed line $x \perp AP$ through P.
But $EX \perp (AC \parallel DF)$ and $FX \perp (AB \parallel DE) \Longrightarrow X \in x$ is orthocenter of $\triangle DEF$.
$L \in MN$ moves with constant velocity, when $D \in BC$ moves with constant velocity $\Longrightarrow K \in k$ also moves with constant velocity.
$E \in AC, F \in AB$ move with constant velocity, when $D \in BC$ moves with constant velocity. Isosceles $\triangle UEF$ with $\angle FUE = 2\angle A$ remains similar $\Longrightarrow U \in u$ and $X \in x$ move with constant velocities as well.
Let (V) be circumcircle of $\triangle DEF \Longrightarrow \frac{\overline{XK}}{\overline{XV}} = \frac{2}{3} \Longrightarrow V$ is on fixed line v. Let (Q) be 9-point circle of $\triangle ABC$. When D is midpoint of BC, $V \equiv Q \Longrightarrow Q \in v$.

kingmathvn
#10 Mar 24, 2015, 3:18 pm

Luis: You confused FN/EM = AB/ AC????

Quick Reply

High School Olympiads

OK is perpendicular to BC

Reply

shinichiman
#1 Nov 11, 2013, 5:54 am

Let (O) be the circumscribed circle of ABC. CF, BE are the height of triangle ABC, respectively. M, S, N are the midpoint of BF, FE, CE respectively. The line from M which is perpendicular to BS intersect the line from M which is perpendicular to CS at K. Prove that $OK \perp BC$

shinichiman
#2 Nov 12, 2013, 3:28 am

Anyone ? ..

jayme
#3 Nov 12, 2013, 12:22 pm

Dear Mathlinkers,
a typo with two times M?
Sincerely
Jean-Louis

BlackSelena
#4 Nov 12, 2013, 12:28 pm

Let (O) be the circumscribed circle of ABC. CF, BE are the height of triangle ABC, respectively. M, S, N are the midpoint of BF, FE, CE respectively. The line from M which is perpendicular to BS intersect the line from N which is perpendicular to CS at K. Prove that $OK \perp BC$

Underlined part is the correct one.
Also this is a well-known problem. Just use Carnot theorem and you'll have the result

shinichiman
#5 Nov 12, 2013, 5:46 pm

> BlackSelena wrote:
> Let (O) be the circumscribed circle of ABC. CF, BE are the height of triangle ABC, respectively. M, S, N are the midpoint of BF, FE, CE respectively. The line from M which is perpendicular to BS intersect the line from N which is perpendicular to CS at K. Prove that $OK \perp BC$
>
> Underlined part is the correct one.
> Also this is a well-known problem. Just use Carnot theorem and you'll have the result

Hello BlackSelena, could you please prove this by Junior way!! Thank you.

jayme
#6 Nov 12, 2013, 6:55 pm

Dear Mathlinkers,
a possible way, is to resolve this problem with two orthologic triangles... I haven't explore this way...
Sincerely

Jean-Louis

Arab

#7 Nov 12, 2013, 9:55 pm • 2

Since $MK \perp BS, NK \perp CS$,we obtain
$BK^2 - SK^2 = BM^2 - SM^2$,
$CK^2 - SK^2 = CN^2 - SN^2$.
Consequently,

$$\begin{aligned} BK^2 - CK^2 &= (BM^2 - SM^2) - (CN^2 - SN^2) \\ &= \frac{1}{4}[(BF^2 - BE^2) - (CE^2 - CF^2)] \\ &= \frac{1}{4}[(BF^2 + CF^2) - (BE^2 + CE^2)] \\ &= \frac{1}{4}(BC^2 - BC^2) \\ &= 0 \end{aligned}$$

which implies that $BK = CK$,and hence K lies on the perpendicular bisector of BC,therefore,$OK \perp BC$,as desired.

Q.E.D.

shinichiman

#8 Nov 13, 2013, 11:09 am

Thank you very much.

TelvCohl

#9 Dec 25, 2014, 5:42 pm • 1

My solution:

Let T be the midpoint of BC .
Let B', C' be the projection of B, C on EF, respectively .

Easy to see B, C, E, F lie on a circle with center T .

Since $TS \perp EF$,
so S is the midpoint of $B'C'$,
hence we get $SE \cdot SC' = SF \cdot SB'$ . ... $(\star)$
Since $AF \cdot AB = AE \cdot AC$,
so combine with $(\star)$ we get AS is the radical axis of $\{\odot(BF), \odot(CE)\}$,
hence $AS \perp MN$.

Since the lines through B, C, S perpendicular to TM, TN, MN are concurrent at A ,
so we get $\triangle SBC$ and $\triangle TNM$ are orthologic and $TK \perp BC$.
ie. $OK \perp BC$

Q.E.D

Luis González

#10 Dec 26, 2014, 2:06 am • 1

More general: Keeping the same notations, let P be an arbitrary point on the A-symmedian of $\triangle ABC$ (not necessarily the midpoint S of EF). Perpendiculars from M, N to PB, PC meet at K. Then $OK \perp BC$.

Since $(MS \parallel BE) \perp AC$ and $(NS \parallel CF) \perp AB$, then S is orthocenter of $\triangle AMN \Longrightarrow ASP \perp MN \Longrightarrow \triangle ABC$ and $\triangle KNM$ are orthologic with orthology center P. Hence, perpendiculars from N, M, K to AC, AB, BC concur at the 2nd orthology center; the midpoint of $\overline{BC} \Longrightarrow K$ is on perpendicular bisector of $\overline{BC}$, i.e. $OK \perp BC$.

Arab

#11 Dec 26, 2014, 2:49 am

Another proof for **Luis' generalization**
Using the same idea in **#7** and noting that $AS \perp MN$, we obtain that

$$\begin{aligned} BK^2 - CK^2 &= (PK^2 + BM^2 - PM^2) - (PK^2 + CN^2 - PN^2) \\ &= (BM^2 - PM^2) - (CN^2 - PN^2) \\ &= (BM^2 - CN^2) - (PM^2 - PN^2) \\ &= (BM^2 - CN^2) - (SM^2 - SN^2) \\ &= (BM^2 + SN^2) - (CN^2 + SM^2) \\ &= \frac{1}{4}[(BF^2 + CF^2) - (BE^2 + CE^2)] \\ &= \frac{1}{4}(BC^2 - BC^2) \\ &= 0 \end{aligned}$$

meaning that $BK = CK$, so $OK \perp BC$, and we are done. $\blacksquare$

jayme

#12 Apr 18, 2015, 8:41 pm

Dear Mathlinkers,
it is easy to see that S is the orthocenter of the triangle AMN.
After this, the consideration of two orthologic triangle, gives the result...

Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Rectangular circumhyperbola and circle

Reply

daothanhoai

#1 Feb 1, 2015, 11:45 am

1. Let a rectangular hyperbola with center O, let P be a point on the plain such that the polar of P to the hyperbola meets the hyperbola at A, B. Then AB is common tangent of two circle (POA) and (POB).

Application we can show that: *In any triangle: The Circumcenter, the Nine point center, the Symmedian point and the Kiepert center lie on a circle*

2. Let a rectangular hyperbola with center O, let P be a point on the plain such that the polar of P to the hyperbola meets the hyperbola at A, B. Let A' be the reflection of A in O. Let M be the midpoint of AB and D be the reflection of M in B. Let E be the midpoint of PB. Then MD is common tangent of two circle $(A'ED)$ and $(A'EM)$.

Application we can show that: *In any triangle: The orthocenter, the Nine point center and Tarry point and midpoint of Brocard diameter lie on a circle.*

3. Let a rectangular hyperbola, F lie on the rectangular hyperbola, F' be reflection of F in center of the hyperbola. AB be two point lie on one brach of the rectangular hyperbola such that FF' through midpoint of AB. Then AB is common tangent of two circle $(FF'A)$ and $(FF'B)$.

Application we can show that: *Lester circle*

4. Let a rectangular hyperbola, let P be a point on the line isogonal Conjugate of the hyperbola. The polar of P relative to the hyperbola meets the hyperbola at HG. if K be a point on the rectangular hyperbola(or K lie on isogonal Conjugate of the hyperbola), K' be the inverse of K in circle whose diameter is HG. **Then four points: center of the hyperbola, and K, K'' and P lie on a circle.** The circle which diameter HG is a generalization of orthocentroidal circle. The circle $X(1)X(4)$ has all properties similarly with the orthocentroidal circle.

http://tube.geogebra.org/material/show/id/607917

Attachments:

Two circles through four triangle centers.pdf (179kb)

This post has been edited 4 times. Last edited by daothanhoai, Feb 2, 2015, 1:03 pm

---

TelvCohl

#2 Feb 1, 2015, 6:21 pm

Actually, I found 1 and 3 when I tried This problem (1 and 3 are almost the same theorem)

Let me rewrite the theorem as following

Theorem:

Let $\mathcal{H}$ be a rectangle hyperbola with center O .
Let F, F^* be the antipode of $\mathcal{H}$ and P be a point on FF^* .
Let the polar of P WRT $\mathcal{H}$ intersect $\mathcal{H}$ at A and B .

Then $\{P, O\}, \{F, F^*\}$ are the image of each other under the inversion WRT $\odot(AB)$.

Proof:

Let M be the midpoint of AB and H be the orthocenter of $\triangle FAB$.
Let X, Y, Z be the projection of A, B, F on BF, FA, AB, respectively .

Since FF^* pass through O and $P \in FF^*$

Since FF^* pass through O and $F' \in FF^*$,
so FF^* pass through the midpoint M of AB .
From $(P, M; F, F^*) = -1 \Longrightarrow MF \cdot MF^* = MP \cdot MO$ . ... (1)

Since $\mathcal{H}$ is the circum-rectangle hyperbola of $\triangle FAB$,
so the center O of $\mathcal{H}$ lie on the 9-point circle of $\triangle FAB$.
i.e. $O \in \odot(XYZ)$
Since the orthocenter H of $\triangle FAB$ lie on $\mathcal{H}$,
so ZY, ZX is the polar of X, Y WRT $\mathcal{H}$, respectively ,
hence from Seydewitz-Staudt theorem we get P, X, Y are collinear .
i.e. $P \equiv FM \cap XY$

Since $\angle MOX = \angle MYX = \angle MXP$,
so $\triangle MOX \sim \triangle MXP \Longrightarrow MP \cdot MO = MX^2 = MA^2 = MB^2$ . ... (2)

From $(1), (2) \Longrightarrow MF \cdot MF^* = MP \cdot MO = MA^2 = MB^2$.

Seydewitz-Staudt theorem :

Let $\mathcal{C}$ be a circumconic of $\triangle ABC$.
Let P be the pole of BC WRT $\mathcal{C}$ and ℓ be a line through P .

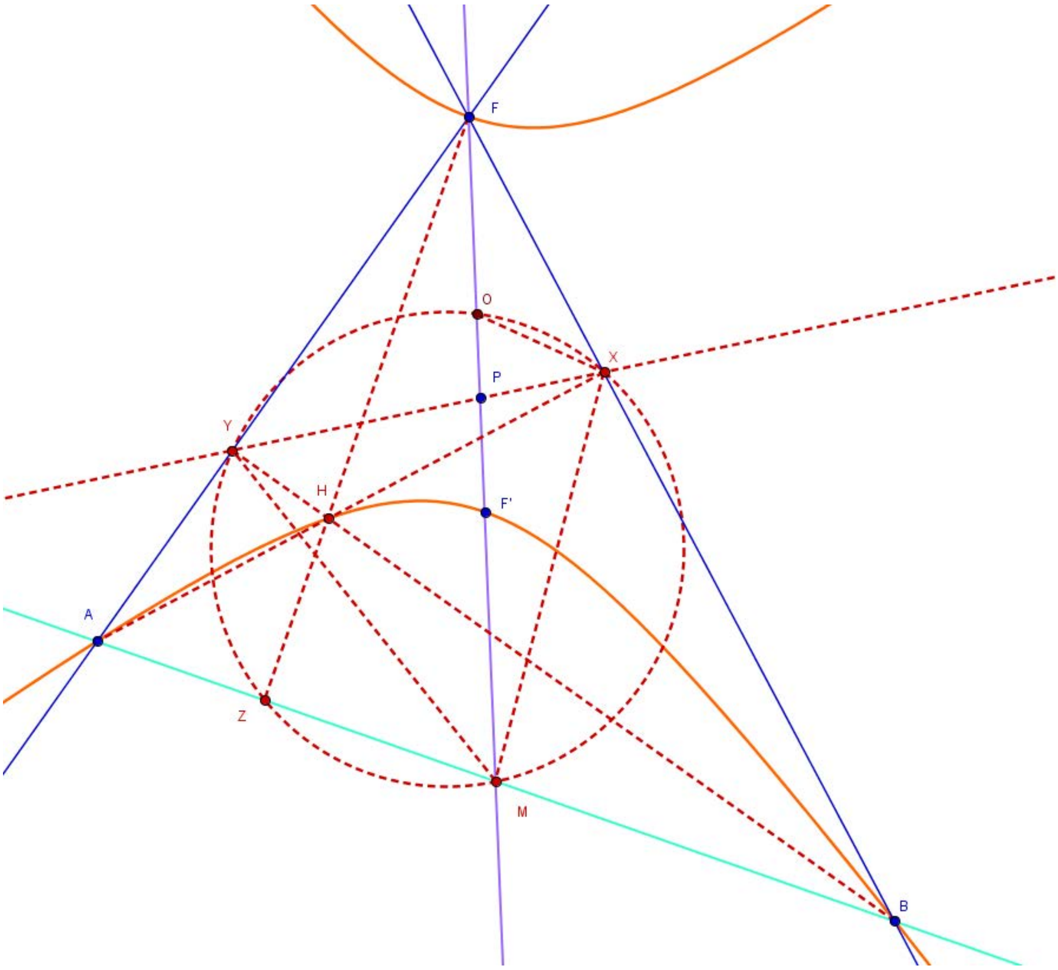
Then $X = \ell \cap AC, Y = \ell \cap AB$ are conjugate points WRT $\mathcal{C}$.

Conversely, if $X \in AC, Y \in AB$ are conjugate points WRT $\mathcal{C}$,
then XY pass through the pole P of BC WRT $\mathcal{C}$.

You can find the proof at here (thanks for **Luis González** gave me this link) :) .

EDIT: We can prove $P \in XY$ without using Seydewitz-Staudt theorem, just notice that the polar AB of P WRT $\mathcal{H}$ passes through Z, so the polar XY of Z WRT $\mathcal{H}$ passes through P. i.e. $P \in XY$

Attachments:



This post has been edited 1 time. Last edited by TelvCohl, Aug 10, 2015, 6:30 pm

daothanhoai

#3 Feb 1, 2015, 7:36 pm

See my proof:

Attachments:

Two circles through four triangle centers.pdf (179kb)

Luis González

#4 Feb 2, 2015, 12:13 am • 1

Another proof to the theorem mentioned by Telv:

Let $H \in \mathcal{H}$ be the orthocenter of $\triangle FAB$ and M the midpoint of $AB.$ AH, BH, FH cut BF, FA, AB at $X, Y, Z.$ Since AB and FF^* have conjugate directions WRT $\mathcal{H},$ then $M \in OP$ and since XY is the polar of Z WRT $\mathcal{H},$ then $P \in XY.$ Since F, F^* are antigonal conjugates WRT $\triangle HAB,$ then $\odot(F^*AB)$ is reflection of $\odot(FAB)$ on $AB \Longrightarrow F^*$ coincides with the projection of H on $FM,$ i.e. FXF^*HY is cyclic. Now, since $\odot(ABXY)$ and $\odot(FXY)$ are orthogonal, then FXF^*Y is harmonic $\Longrightarrow XP, XO$ are isogonals WRT $\angle FXF^* \Longrightarrow \odot(XPO)$ and $\odot(FXF^*)$ are tangent, i.e. $\odot(XPO)$ is orthogonal to the circle with diameter $\overline{AB}$ and the conclusion follows.

daothanhoai

#5 Feb 2, 2015, 8:18 am

Please let me known, WRT mean?

And maybe I mention Telv's theorem in http://forumgeom.fau.edu/FG2014volume14/FG201410.pdf ?

shinichiman

#6 Feb 2, 2015, 11:33 am • 1

> *daothanhoai wrote:*
>
> Please let me known, WRT mean?
>
> And maybe I mention Telv's theorem in http://forumgeom.fau.edu/FG2014volume14/FG201410.pdf ?

It means "with respect to".

TelvCohl

#7 Feb 2, 2015, 9:36 pm

Dear **daothanhoai**

After seeing the picture in PDF, I realized that you have a typo at **2.** .

> *daothanhoai wrote:*
>
> 2. Let a rectangular hyperbola with center O, let P be a point on the plain such that the polar of P to the hyperbola meets the hyperbola at A, B. Let A' be the reflection of A in O. Let M be the midpoint of AB and D be the reflection of M in B. Let E be the midpoint of PB. Then MD is common tangent of two circle $(A'ED)$ and $(A'EM)$.
>
> Application we can show that: *In any triangle: The orthocenter, the Nine point center and Tarry point and midpoint of Brocard diameter lie on a circle.*

E should be the midpoint of PD (not PB)

This follows by the theorem I mentioned above .

Since AB is the polar of P WRT the rectangle hyperbola ,
so OP pass through $M \Longrightarrow A'B \parallel OM \parallel EB$. i.e. A', B, E are collinear
From the theorem I mentioned above we get $BD^2 = BM^2 = MP \cdot MO = BE \cdot BA'$.
i.e. MD is the common tangent of $\odot(A'ED)$ and $\odot(A'EM)$

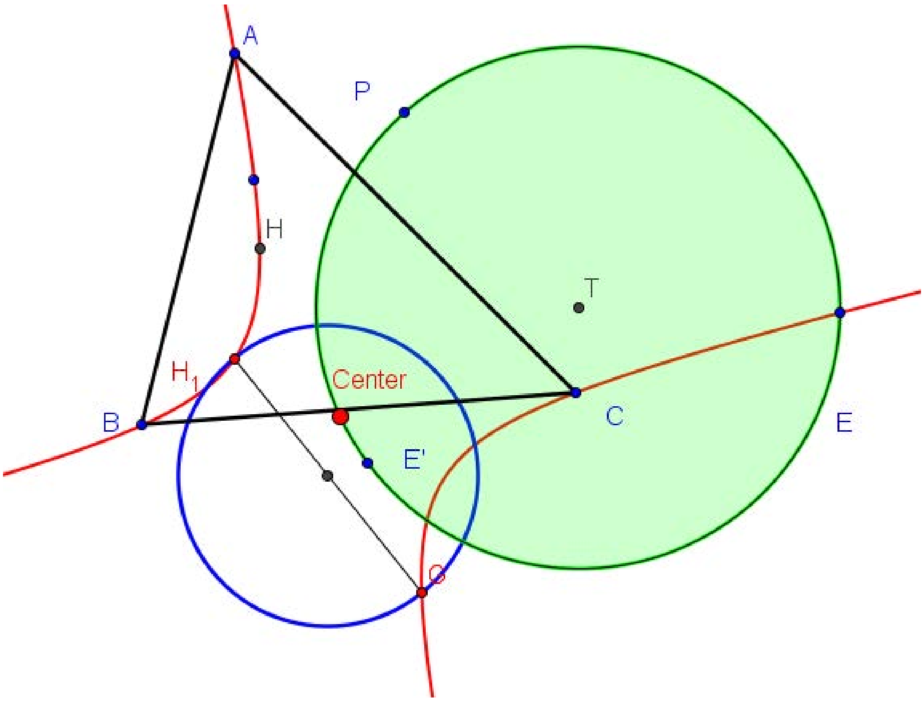
daothanhoai

#8 Feb 3, 2015, 11:37 pm

A generalization 4.

Let ABC be a rectangular circumhyperbola, let P be a point on the plain. The polar of P relative to the hyperbola meets the hyperbola at HG. if E be a point on the rectangular hyperbola(or E lie on isogonal Conjugate of the hyperbola), And E' be the inverse of E in circle whose diameter is HG. Then four points: The center of the hyperbola, and E,E' and P lie on a circle. The circle which diameter HG is a generalization of orthocentroidal circle. The circle X(1)X(4) has all properties similarly with the orthocentroidal circle.

Attachments:



Luis González

#9 Feb 4, 2015, 3:22 am • 1

Dao, your result 4 is again nothing but a consequence of the theorem mentioned in the posts #2 and #4. In fact, {E,E'} can be any pair of inverse points WRT the circle with diameter HG; 2 pairs of inverse points are concyclic.

Here is another related and useful result in these configurations. I found it some time ago: $\mathcal{H}$ is a rectangular hyperbola and $F, F^* \in \mathcal{H}$ are diametrically opposite. P is an arbitrary point on $\mathcal{H}$ other than F, F^*. Then the tangent of $\mathcal{H}$ at P is the P-symmedian of $\triangle PFF^*$.

IDMasterz

#10 Feb 8, 2015, 12:02 pm

> *Luis González wrote:*
>
> Here is another related and useful result in these configurations. I found it some time ago: $\mathcal{H}$ is a rectangular hyperbola and $F, F^* \in \mathcal{H}$ are diametrically opposite. P is an arbitrary point on $\mathcal{H}$ other than F, F^*. Then the tangent of $\mathcal{H}$ at P is the P-symmedian of $\triangle PFF^*$.

Is this the proof you had in mind?

Take F_1, F_1^* to be also two antipodal points. Let P^* be the antipode of P. Note that $\angle FPF_1 = \angle F_1P^*F = \angle F_1^*PF^*$. Hence, the angle bisector ℓ of $\angle FPF^*$ remains invariant as we move F. Let F coincide with P; we conclude PP, PO are isogonal w.r.t ℓ. So, the result follows.

TelvCohl

#11 Feb 8, 2015, 12:42 pm

We can get the result mentioned by **Luis** from the theorem mentioned in the posts #2 and #4 :

(I use the figure and notation in my post above)
From $MA^2 = MF \cdot MF^* \Longrightarrow MA$ is the tangent of $\odot(AFF^*)$,
so combine with $(P, M; F, F^*) = -1 \Longrightarrow AP$ is $A-$symmedian of $\triangle AFF^*$.

Done

IDMasterz
#12 Feb 8, 2015, 5:26 pm

> *TelvCohl wrote:*
> We can get the result mentioned by **Luis** from the theorem mentioned in the posts #2 and #4 :
>
> (I use the figure and notation in my post above)
> From $MA^2 = MF \cdot MF^* \Longrightarrow MA$ is the tangent of $\odot(AFF^*)$,
> so combine with $(P, M; F, F^*) = -1 \Longrightarrow AP$ is $A-$ symmedian of $\triangle AFF^*$.
>
> Done

So we could provide a backward proof of your properties using this one?

TelvCohl
#13 Apr 22, 2015, 10:48 am

> *Luis González wrote:*
> Here is another related and useful result in these configurations. I found it some time ago: $\mathcal{H}$ is a rectangular hyperbola and $F, F^* \in \mathcal{H}$ are diametrically opposite. P is an arbitrary point on $\mathcal{H}$ other than F, F^*. Then the tangent of $\mathcal{H}$ at P is the P-symmedian of $\triangle PFF^*$.

This simplified follows from the fact that $\mathcal{H}$ is the isogonal conjugate of the perpendicular bisector of FF^* WRT $\triangle PFF^*$.

Luis González
#14 Apr 22, 2015, 11:16 am

Yes Telv, that was exactly how I found the result. Quite simple
I resorted it in the solution of the problem Angle related to Fermat points and Isodynamic points and BTW we can also use it to prove r960.

Quick Reply

High School Olympiads

Prove that I,J, K are collinear.

Reply

hoangquan
#1 Jul 23, 2012, 10:19 am

Let M be a point in the plane of $\triangle ABC$, and d a line passing through M. Let I, J, K be the points where the reflections of lines MA, MB, MC with respect to d intersect lines BC, AC, AB respectively. Prove that I, J, K are collinear.

applepi2000
#2 Jul 23, 2012, 10:43 am

See here.

jayme
#3 Jul 23, 2012, 4:13 pm

Dear Mathlinkers,

this problem comes from
Goormaghtigh R., Sur une généralisation du théorème de Noyer, Droz-Farny et Neuberg

Sincerely
Jean-Louis

jayme
#4 May 1, 2015, 1:59 pm

Dear Mathlinkers,
you can see a complete study and more on

http://jl.ayme.pagesperso-orange.fr/Docs/Goormaghtigh.pdf

Sincerely
Jean-Louis

Luis González
#5 May 4, 2015, 7:46 am

Let X, Y, Z be the projections of M on BC, CA, AB and let P be the projection of M on JK.

From cyclic $MYPJ$ and $MZKP$, we get $\widehat{MPY} = \widehat{MJY}$ and $\widehat{MPZ} = \widehat{MKZ} \Longrightarrow$ $\widehat{YPZ} = \widehat{MJY} + \widehat{MKZ} = \widehat{BAC} + \widehat{KMJ}$. But since MJ, MK are the reflections of MB, MC on d, we deduce that $\widehat{KMJ} = \pi - \widehat{BMC} \Longrightarrow \widehat{YPZ} = \pi + \widehat{BAC} - \widehat{BMC} = \pi - \widehat{YXZ} \Longrightarrow P \in \odot(XYZ) \Longrightarrow JK$ touches the inconic $\mathcal{C}$ with focus M and pedal circle $\odot(XYZ)$. Similarly IJ touches $\mathcal{C} \Longrightarrow I, J, K$ are collinear on a tangent of $\mathcal{C}$.

MillenniumFalcon
#6 May 4, 2015, 9:07 am

I think it is supposed to be YPZ = KMJ - BAC, KMJ = BMC, Hence YPZ = BMC-BAC= ABM+ACM=YXZ, hence concyclic.

And i'm not very good at inconic, but why does IJ touch C as well?

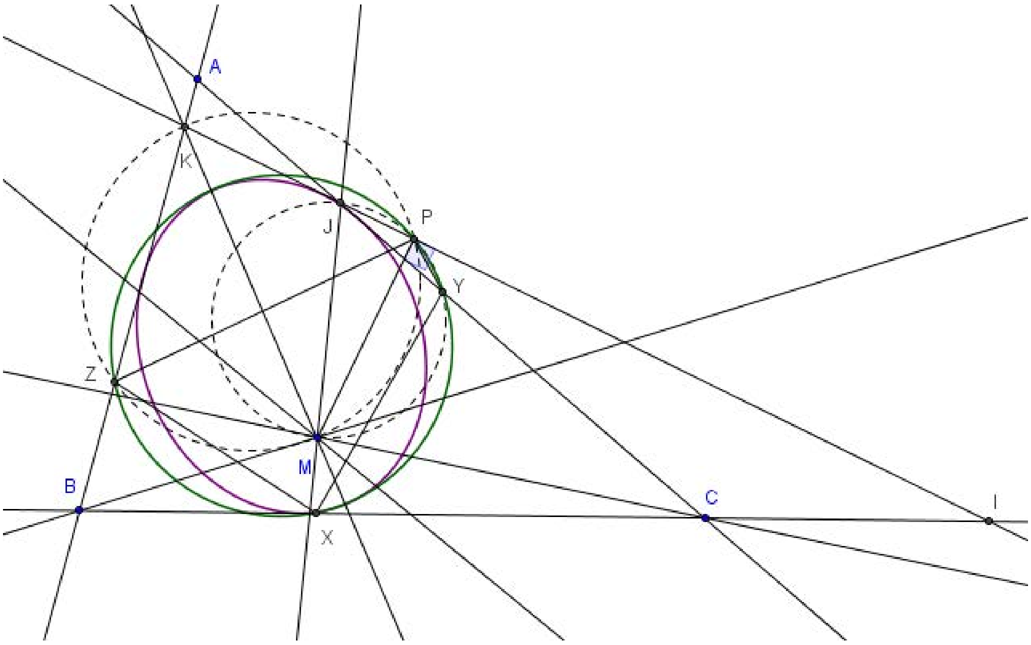
Luis González

#7 May 4, 2015, 9:51 am

MillenniumFalcon, since I'm not using oriented angles modulo 180º, the angle chase is figure-dependent. Referer to the diagram below.

Once you prove that JK touches $\mathcal{C},$ there's no need to prove it for IJ and $IK,$ as it simply follows by analogy due to the symmetry of the conditions. In general, given a central conic $\mathcal{C}$ with a focus $F,$ the projection of F on a tangent of $\mathcal{C}$ lies on its pedal circle, i.e. the circle with diameter its major axis.

Attachments:



MillenniumFalcon

#8 May 4, 2015, 4:02 pm

Ah ok. oops. Thanks.

TelvCohl

#10 May 7, 2015, 6:28 pm

Another solution :

Let JK cut BC at I^* .

From Dual Desargue Involution theorem $\Longrightarrow M(I^*, J; K, C) = M(A, B; C, K)$.

Since MI, MJ, MK is the reflection of MA, MB, MC in d, respectively ,
so $M(I, J; K, C) = M(A, B; C, K) = M(I^*, J; K, C) \Longrightarrow MI \equiv MI^* \Longrightarrow I \equiv I^*$.

Q.E.D

Quick Reply

High School Olympiads

Intresting property of isogonal conjugate points

Reply

paul1703

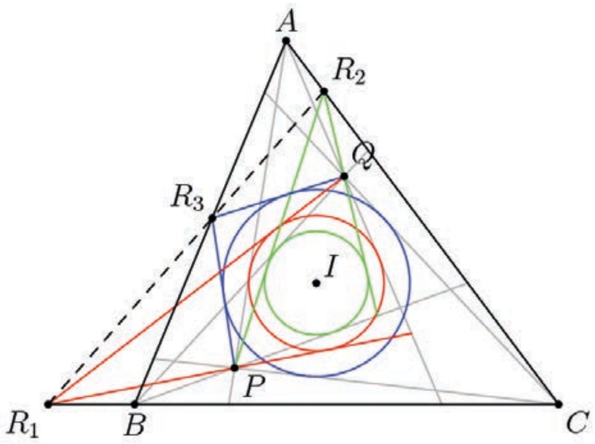
#1 Nov 10, 2011, 1:48 pm

Let P and Q be two isogonale conjugate points in triangle a ABC and let I be the incenter of the triangle. Consider the tangents from the points P and Q to the circle centered at I and tangent to lines AP, AQ prove that these tangents meet on BC.

nsato

#2 Nov 11, 2011, 1:50 am • 1

Nice problem! Let R_1 be the point in the problem. Do the same with vertices B and C, to get points R_2 and R_3. It looks like R_1, R_2, and R_3 are always collinear.



paul1703

#3 Nov 18, 2011, 4:39 pm

Maybe some brain-storming will help does anybody have any ideas??

skytin

#4 Nov 18, 2011, 8:40 pm

Hint :
Use Newton theorem for isogonal points

paul1703

#5 Nov 18, 2011, 11:20 pm

Please can you explain better?

skytin

#6 Nov 19, 2011, 12:43 am

If given four points A , B , C , D and P , Q are two isogonal points in ABCD, then midpoint of PQ lies on Gauss line of lines AB , BC , CD , AD.

Luis González

#7 Nov 19, 2011, 2:51 am • 1

> *nsato wrote:*
> Let R_1 be the point in the problem. Do the same with vertices B and C, to get points R_2 and R_3. It looks like $R_1, R_2,$ and R_3 are always collinear

This has been referred as "Goormaghtigh theorem" in Hyacinthos group. Furthermore, the line through these points is tangent to the inconic with foci P,Q. Check the following messages

http://tech.groups.yahoo.com/group/Hyacinthos/message/8745
http://tech.groups.yahoo.com/group/Hyacinthos/message/9301
http://tech.groups.yahoo.com/group/Hyacinthos/message/14761

rodinos

#8 Feb 15, 2015, 11:48 am

Luis,

Your links to Hyacinthos are in old format.

The new format is
https://groups.yahoo.com/neo/groups/Hyacinthos/conversations/messages/Number_of_message

So your links are:

https://groups.yahoo.com/neo/groups/Hyacinthos/conversations/messages/8745
https://groups.yahoo.com/neo/groups/Hyacinthos/conversations/messages/9301
https://groups.yahoo.com/neo/groups/Hyacinthos/conversations/messages/14761

Regards

Antreas

TelvCohl

#9 Mar 2, 2016, 3:40 pm • 3

Let M be the center of the spiral similarity of $PI \mapsto IQ$. Let X and Y be the points such that $\triangle MXA, \triangle MYB, \triangle MPI,$ $\triangle MIQ$ are directly similar. From $\triangle MXA \overset{+}{\sim} \triangle MPI \overset{+}{\sim} \triangle MIQ \Longrightarrow \triangle XPI \overset{+}{\sim} \triangle AIQ$, so $\measuredangle PAI = \measuredangle IAQ = \measuredangle PXI \Longrightarrow A, I, P, X$ are concyclic. Similarly, we can prove B, I, P, Y are concyclic. Let $Z \equiv AX \cap BY$. Since

$$\measuredangle BZA = \measuredangle ZBI + \measuredangle BIA + \measuredangle IAZ = \measuredangle YPI + \measuredangle BIA + \measuredangle IPX,$$

so combine $\measuredangle IPX = \measuredangle QIA$ and $\measuredangle YPI = \measuredangle BIQ$ we get $\measuredangle BZA = \measuredangle BCA \Longrightarrow Z$ lies on $\odot(ABC)$. On the other hand, from $\measuredangle MAZ = \measuredangle MBZ$ we get A, B, M, Z are concyclic, so M also lies on $\odot(ABC)$.

Let the tangent of $\odot(AIM)$ passing through I cuts BC at D and $N \equiv AI \cap \odot(ABC), U \equiv AI \cap BC, V \equiv MN \cap BC$. Let Φ be the inversion with center M and radius MI followed by reflection in MI. From $NA \cdot NU = NM \cdot NV = NI^2$ we get A, M, U, V are concyclic and $\triangle NMI \overset{-}{\sim} \triangle NIV$, so $\measuredangle MVD = \measuredangle MAI = \measuredangle MID \Longrightarrow D$ lies on $\odot(IMV)$, hence $\measuredangle MDI = \measuredangle MVI = \measuredangle MIA \Longrightarrow \triangle MAI \overset{+}{\sim} \triangle MID$. i.e. D is the image of A under Φ

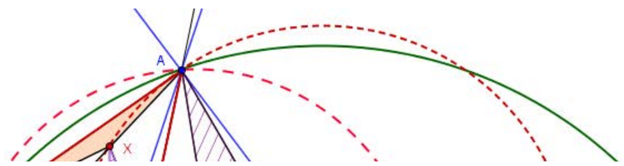
From $\triangle MPI \overset{+}{\sim} \triangle MIQ \Longrightarrow P, Q$ are the image of each other under Φ, so $\triangle MAP \overset{+}{\sim} \triangle MQD \Longrightarrow$

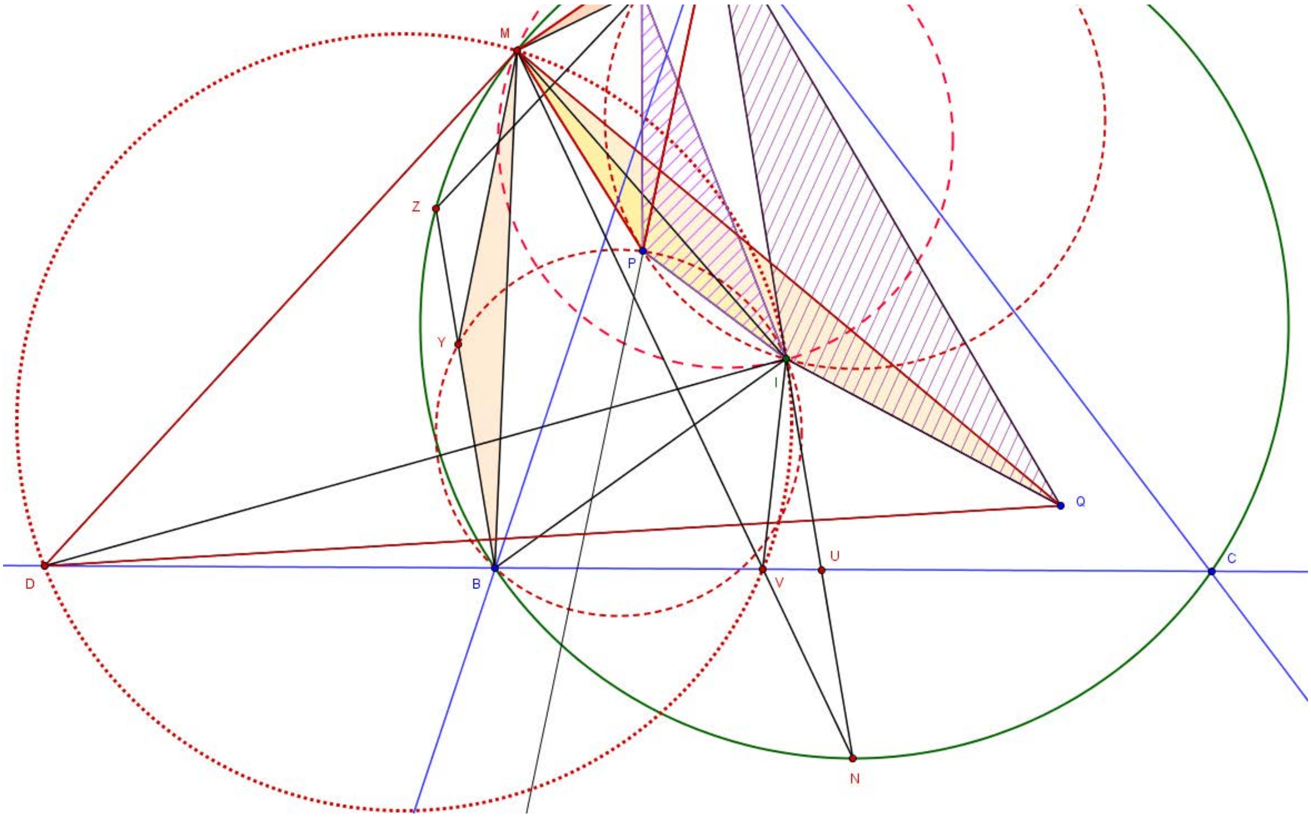
$$\measuredangle AQI = \measuredangle XIP = \measuredangle XAP = \measuredangle MAP - \measuredangle MAX = \measuredangle MQD - \measuredangle MQI = \measuredangle IQD,$$

hence QD and QA are symmetry WRT QI. Similarly, we can prove the reflection of AP in IP passes through D.

Remark : If we construct E, F similarly, then D, E, F are collinear (image of $\odot(ABC)$ under Φ).

Attachments:





Luis González

#10 Mar 2, 2016, 7:47 pm • 3

Let the second tangents from P, Q to the circle ω_I, centered at I tangent to AP, AQ, intersect at D. Let ℓ_B, τ_B be the tangents from B to ω_I. By dual of Desargues involution theorem for $APDQ$, it follows that $BA \mapsto BD, BP \mapsto BQ$, $\ell_B \mapsto \tau_B$ is an involution, which coincides then with the reflection across BI, because ℓ_B, τ_B and BP, BQ are symmetric WRT $BI \Longrightarrow BC \equiv BD$, i.e. $D \in BC$.

TelvCohl

#11 Mar 6, 2016, 12:48 am

Another approach :

Let the reflection of AP, BP, CP in IP cuts BC, CA, AB at D, E, F, respectively. From $\measuredangle BPC = \measuredangle FPE$ we get the isogonal conjugate of P WRT quadrilateral $BCEF$ is exist, so Q is the isogonal conjugate of P WRT quadrilateral $BCEF$, hence there is a conic $\mathcal{C}$ with focus P, Q and tangent to the sides of quadrilateral $BCEF$.

On the other hand, from hard geo (post #5) $\Longrightarrow EF$ is tangent to the incircle $\odot(I)$ of $\triangle ABC$, so EF is the forth common tangent of $\odot(I)$ and $\mathcal{C}$. Analogously, we can prove FD is tangent to $\odot(I)$ and $\mathcal{C}$, so we conclude that D, E, F are collinear and D, E, F lies on the reflection of AQ, BQ, CQ in IQ, respectively.

Luis González

#12 Mar 6, 2016, 8:21 am

> *Luis González wrote:*
>
> > *nsato wrote:*
> >
> > Let R_1 be the point in the problem. Do the same with vertices B and C, to get points R_2 and R_3. It looks like R_1, R_2, and R_3 are always collinear
>
> This has been referred as "Goormaghtigh theorem" in Hyacinthos group. Furthermore, the line through these points is tangent to the inconic with foci P,Q. Check the following messages
>
> http://tech.groups.yahoo.com/group/Hyacinthos/message/8745
> http://tech.groups.yahoo.com/group/Hyacinthos/message/9301
> http://tech.groups.yahoo.com/group/Hyacinthos/message/14761

See also http://www.artofproblemsolving.com/community/c6h490232 (post #5).

Quick Reply

High School Olympiads

Midpoints as center of the involutions

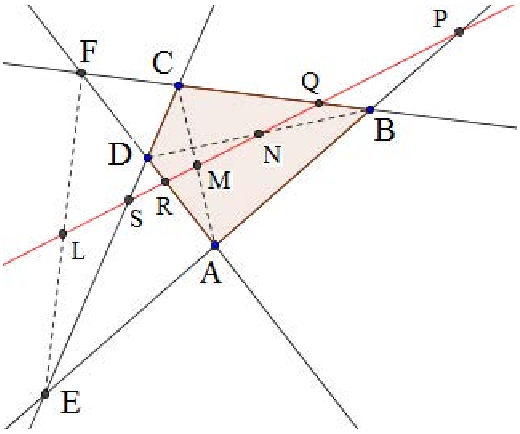
Reply

ricarlos

#1 Mar 5, 2016, 8:32 am

Let $ABCDEF$ be a complete quadrilateral. Let L, M, N be the midpoints of FE, AC and BD, respectively.
$P = LN \cap AB$,
$Q = LN \cap BC$,
$R = LN \cap AD$,
$S = LN \cap CD$. Prove that,
$LS \cdot LP = LR \cdot LQ = LM \cdot LN$,
$MQ \cdot MS = MR \cdot MP = MN \cdot ML$ and
$NS \cdot NR = NP \cdot NQ = NM \cdot NL$.

Attachments:



TelvCohl

#2 Mar 6, 2016, 12:44 am

Let $\mathcal{P}$ be the parabola which is tangent to AB, CD, AC, BD. Since M, N is the midpoint of AC, BD, respectively, so from Pure geometry (($\bigstar$) at post #2) we get MN is tangent to $\mathcal{P}$, hence $(E, P; A, B) = (E, S; C, D) \Longrightarrow$

$$(L, S; Q, R) \stackrel{F}{=} (E, S; C, D) = (E, P; A, B) \stackrel{F}{=} (L, P; R, Q)$$

$\Longrightarrow L$ is the center of the involution that swaps (P, S) and (Q, R) ... $(\star)$.

On the other hand, from Desargues' involution theorem (quadrilateral $ABCD$ cut by MN) we get $P \mapsto S, Q \mapsto R, M \mapsto N$ is an involution, so combine $(\star)$ we conclude that $LM \cdot LN = LP \cdot LS = LQ \cdot LR$.

Luis González

#3 Mar 6, 2016, 8:06 am

Let X, Y, Z be the midpoints of AB, BF, FA, resp. Clearly $N \in XY, L \in YZ$ and $M \in ZX$. Thus if $Y_\infty, Z_\infty, U_\infty$ denote the points at infinity of ZX, XY, MN, then by Desargues involution theorem for $YZY_\infty Z_\infty$ cut by the line MNL, it follows that $M \mapsto N, R \mapsto Q, L \mapsto U_\infty$ is an involution, i.e. L is the center of the involution that swaps (M, N) and (Q, R). But by Desargues involution theorem for $ABCD$ cut by MN, it follows that this involution swaps (P, S) as well and the conclusion follows.

Quick Reply

High School Olympiads

About orthocenter

Reply

Scorpion.k48

#1 Mar 3, 2016, 9:01 pm

Let $\triangle ABC$ with orthocenter H. $\overline{A_1, B_1, C_1}$ is tripolar line of H WRT $\triangle ABC$. Tangent at A of (O) cuts BC at A_2. Let M is Miquel point of $\triangle ABC$ WRT $\overline{A_1, B_1, C_1}$ and L is symmedian point of $\triangle ABC$. Prove that M is anti-Steiner point of HL WRT $\triangle ABC$

This post has been edited 1 time. Last edited by Scorpion.k48, Mar 4, 2016, 1:56 pm

TelvCohl

#5 Mar 3, 2016, 10:28 pm • 1

Solution to the first part :

Let T be the isotomic conjugate of H WRT $\triangle ABC$. Since the cevian triangle of T WRT $\triangle ABC$ is the pedal triangle of the anticomplement V of H WRT $\triangle ABC$ WRT $\triangle ABC$, so from Sondat's theorem we get the tripolar τ of T WRT $\triangle ABC$ is perpendicular to VT, hence notice T is the anticomplement of L WRT $\triangle ABC \Longrightarrow \tau$ is perpendicular to HL ... $(\star)$.

Since the Newton line of the complete quadrilateral $\mathbf{Q} \equiv \{\triangle ABC, \overline{A_1B_1C_1}\}$ is the complement of τ WRT $\triangle ABC$, so from $(\star)$ we get HL is the Steiner line of $\mathbf{Q}$, hence we conclude that M is the anti-steiner point of HL WRT $\triangle ABC$.

Remark : We can use the result of this problem to prove the following property of Kiepert hyperbola **:**

Given a $\triangle ABC$ with orthocenter H. Let P be a point on the Kiepert hyperbola of $\triangle ABC$ and Q be the isotomic conjugate of P WRT $\triangle ABC$. Then HP is perpendicular to the tripolar of Q WRT $\triangle ABC$.

Proof : First, notice that HP is perpendicular to the tripolar of Q WRT $\triangle ABC \Longleftrightarrow P$ lies on a fixed conic $\mathcal{C}$ passing through H and the Centroid of $\triangle ABC$ (the proof of this result is similar to Two surprising perpendiculars (post #11)), so we only need to prove $\mathcal{C}$ coincide with the Kiepert hyperbola of $\triangle ABC$.

Since the tripolar ς_F of the Fermat point F_e of $\triangle ABC$ coincide with its orthotransversal WRT $\triangle ABC$, so if ς_F cuts BC, CA, AB at D, E, F, respectively, then HF_e is the radical axis of $\odot(AD), \odot(BE), \odot(CF)$, hence HF_e is the Steiner line of the complete quadrilateral $\{\triangle ABC, \varsigma_F\} \Longrightarrow HF_e$ is perpendicular to the tripolar (WRT $\triangle ABC$) of the isotomic conjugate of F_e WRT $\triangle ABC \Longrightarrow F_e$ lies on $\mathcal{C}$ ... $(\bigstar)$.

From this problem $\Longrightarrow$ the tripolar of the isotomic conjugate of H WRT $\triangle ABC$ WRT $\triangle ABC$ is perpendicular to the tangent HL of the Kiepert hyperbola $\mathcal{K}$ passing through H, so combine $(\bigstar)$ we conclude that $\mathcal{C}$ coincide with $\mathcal{K}$.

Luis González

#6 Mar 4, 2016, 3:38 am • 1

Another solution to the first claim:

$\triangle A_0B_0C_0$ is the orthic triangle of $\triangle ABC$ and A', B', C' are the projections of A, B, C on B_0C_0, C_0A_0, A_0B_0. $\triangle L_aL_bL_c$ is the pedal triangle of L WRT $\triangle ABC$. Using the result of the problem Circumcircles of 3 triangles have two common points (post #4) for $\triangle A_0B_0C_0$, it follows that $\odot(AA_0A_1A'), \odot(BB_0B_1B'), \odot(CC_0B_1C')$ are coaxal with common radical axis passing through L. Denote by $-\varrho^2$ the power of L WRT these circles.

If AL cuts $\odot(AA_0A_1)$ again at U (projection of A_1 on AL and A_1U cuts $\overline{LL_b, LL_c}$ at L_2, L_3, we get then $LL_b \cdot LL_2 = LL_c \cdot LL_3 = LA \cdot LU = -\varrho^2$. Thus we conclude that $\overline{A_1B_1C_1}$ is the perspectrix of $\triangle ABC$ and the triangle $\triangle L_1L_2L_3$ formed by the images of L_a, L_b, L_c under inversion $(L, -\varrho^2)$. Thus according to "Kariya" parabola (post #4), we conclude that M is the anti-Steiner point of HL WRT $\triangle ABC$.

Quick Reply

High School Olympiads

Circumcircles of 3 triangles have two common points

Reply

Math-lover123
#1 Jan 1, 2014, 10:15 pm

Let ABC be a scalene triangle.
Denote by I_a the center of the excircle at side BC and
by A_1 its tangency point with the corresponding side.
Points I_b, I_c, B_1, C_1 are defined
analogously.
Show that the circumcircles of triangles AI_aA_1, BI_bB_1 and CI_cC_1 have two
common points.

Luis González
#2 Jan 2, 2014, 1:20 am

Let I be the incenter of $\triangle ABC$ and ℓ its trilinear polar that passes through $X \equiv BC \cap I_bI_c, Y \equiv CA \cap I_cI_a$ and $Z \equiv AB \cap I_aI_b$. Since $\angle XAI_a = \angle XA_1I_a = 90^\circ$, then the center O_a of $\odot(AA_1I_a)$ is the midpoint of $\overline{XI_a}$. Similarly, O_b, O_c are midpoints of $\overline{YI_b}, \overline{ZI_c}$. Since X, Y, Z are collinear, then O_a, O_b, O_c are also collinear on a line τ, which is the complement of the isotomic line of ℓ WRT $\triangle I_aI_bI_c$. Since $\overline{IA} \cdot \overline{II_a} = \overline{IB} \cdot \overline{II_b} = \overline{IC} \cdot \overline{II_c}$, it follows that $(O_a), (O_b), (O_c)$ are coaxal with common radical axis passing through I.

jayme
#3 Jan 2, 2014, 12:04 pm

Dear Mathlinkers, the common chord goes through the Mittenpunkt of ABC.
For this reason, D. Grinberg calls these two points "Mitten-Schröder punckts".
Sincerely
Jean-Louis

Luis González
#4 Jan 3, 2014, 1:52 am

In response to jayme's observation:

Denote M the Mittenpunkt of $\triangle ABC$, i.e. the symmedian point of $\triangle I_aI_bI_c$ and D, E, F the midpoints of I_bI_c, I_cI_a, I_aI_b, respectively. $\tau \equiv O_aO_bO_c$ is the complement of the isotomic line of I WRT $\triangle I_aI_bI_c$, thus in the $\triangle DEF$, it is the trilinear polar of the retrocenter M of $\triangle DEF$. The cevian triangle of M WRT $\triangle DEF$ is also pedal triangle of I, hence by Sondat's theorem IM is perpendicular to τ, which implies that M is on the radical axis of the aforementioned circles.

Quick Reply

High School Olympiads

"Kariya" parabola

Reply

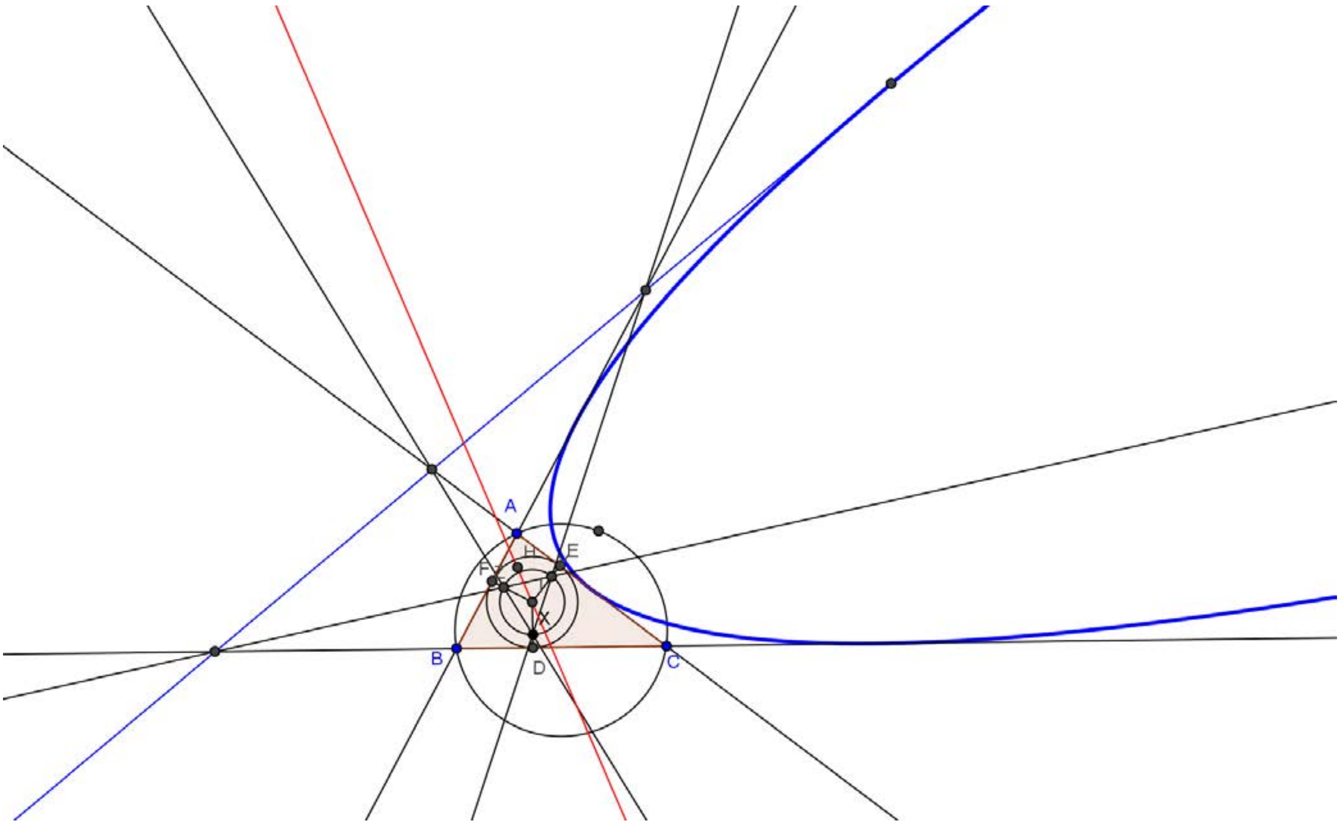
Source: Own

A-B-C

#1 Jul 10, 2015, 7:28 pm • 1

$\triangle ABC$, I is incenter. The incircle tangents to BC, CA, AB at D, E, F
Let X, Y, Z be three points on ID, IE, IF such that $\frac{IX}{ID} = \frac{IY}{IE} = \frac{IZ}{IF}$
Then $\triangle ABC$ and $\triangle XYZ$ are perspective according to Kariya's theorem.
Prove that the perspectrix of $\triangle ABC$ and $\triangle XYZ$ tangents to the parabola has directrix IH and focus T where H is orthocenter of $\triangle ABC$ and T is antisteiner point of IH WRT $\triangle ABC$

Attachments:



TelvCohl

#3 Jul 10, 2015, 8:21 pm • 3

Let $\overline{PQR}$ be the perspectrix of $\triangle ABC, \triangle XYZ$ and $\overline{D^*E^*F^*}$ be the perspectrix of $\triangle ABC$, $\triangle DEF$. Since $Q \mapsto R$ is a homography as X , Y , Z varies on ID , IE , IF, respectively , so $\overline{PQR}$ is tangent to a fixed conic $\mathcal{C}$. Notice $\overline{PQR}$ become the line ℓ_∞ at infinity when X , Y , Z coincide with the infinity point on ID , IE , IF, respetively, so ℓ_∞ is the tangent of $\mathcal{C} \Longrightarrow \mathcal{C}$ is a parabola . Since IH is the common radical axis of $\odot(AD^*)$, $\odot(BE^*)$, $\odot(CF^*)$, so the Newton line of the complete quadrilateral $\{$ BC , CA , AB , $\overline{D^*E^*F^*}$ $\}$ is perpendicular to $IH \Longrightarrow$ the focus of $\mathcal{C}$ is the anti-steiner point T of IH WRT $\triangle ABC$ ($\because$ the infinity point on the Newton line is the isogonal conjugate of the Miquel point of the complete quadrilateral) .

Luis González

#4 Jul 11, 2015, 5:59 am • 3

Generalization: P is arbitrary point on the plane of $\triangle ABC$ and $\triangle DEF$ is the pedal triangle of P WRT $\triangle ABC$. X, Y, Z lie on PD, PE, PF, such that $\overline{PD} \cdot \overline{PX} = \overline{PE} \cdot \overline{PY} = \overline{PF} \cdot \overline{PZ} = \varrho^2$. Then $\triangle ABC$ and $\triangle XYZ$ are perspective and their perspectrix touches the inscribed parabola with directrix PH and focus the anti-Steiner point of PH; where H is orthocenter of $\triangle ABC$.

Since $PD \cdot PX = \varrho^2$, then X is the pole of BC WRT $\odot(P, \varrho)$ and likewise Y and Z are the poles of CA, AB WRT $\odot(P, \varrho) \Longrightarrow \triangle ABC$ and $\triangle XYZ$ are perspective through the perspector of $\odot(P, \varrho)$ WRT $\triangle ABC \Longrightarrow U \equiv YZ \cap BC$, $V \equiv ZX \cap CA, W \equiv XY \cap AB$ are collinear on their perspectrix τ.

Since YZ and EF are antiparallel WRT PE, PF, then it follows that $PA \perp YZ$ and similarly $PB \perp ZX$ and $PC \perp XY$. Thus when X, Y, Z vary, YZ, ZX, XY have fixed directions $\Longrightarrow$ the series U, V, W are similar $\Longrightarrow UV \equiv \tau$ envelopes a fixed parabola $\mathcal{P}$ tangent to CA, AB. When Z coincides with the orthocenter of $\triangle PAB$, we clearly have $V \equiv A, U \equiv B$ $\Longrightarrow AB$ touches $\mathcal{P}$. When $X \equiv V \equiv D' \equiv PD \cap AC$, then $Y \equiv U \equiv E' \equiv PE \cap BC$ because $\overline{PE} \cdot \overline{PE'} = \overline{PD} \cdot \overline{PD'} \Longrightarrow D'E'$ touches $\mathcal{P}$. As a result, the directrix of $\mathcal{P}$ is the line through the orthocenters H and P of $\triangle ABC$ and $\triangle CD'E'$. Its focus is consequently the anti-Steiner point of PH WRT $\triangle ABC$.

Scorpion.k48

#5 Mar 23, 2016, 9:25 pm

Dear TelvCohl, why the infinity point on the Newton line is the isogonal conjugate of the Miquel point of the complete quadrilateral? Could you help me?

Quick Reply

High School Olympiads

Show that $XP=XL$

Reply

hnt12092000

#1 Mar 3, 2016, 9:25 am • 1

Can you solve my problem ?
Let ABC is a triangle. Circumcircle ABC is (O). (I) is incircle of the triangle ABC. $CI \cap AB = E$ and $BI \cap AC = F$, $EF \cap (O) = M, N$ such that E between M and F; $MI \cap BC = P$; $NI \cap BC = L$. X is Excircles of the triangle ABC at A. Show that $XP = XL$

Luis González

#2 Mar 3, 2016, 10:09 am • 2

Let Y and Z be the excenters of $\triangle ABC$ againts B and C. Thus I and (O) become orthocenter and 9-point circle of $\triangle XYZ$. From cyclic $IBZA$, we get $EB \cdot EA = EI \cdot EZ \Longrightarrow E$ has equal power WRT (O) and $\odot(IYZ) \Longrightarrow E$ is on their radical axis and so is F similarly $\Longrightarrow EF$ is radical axis of (O) and $\odot(IYZ) \Longrightarrow \{M, N\} \equiv (O) \cap \odot(IYZ)$.

Inversion with center I and power $IB \cdot IY = IC \cdot IZ$ swaps $(O), (J) \equiv \odot(XYZ)$ and $\odot(IYZ), BC \Longrightarrow \{P, L\} \equiv BC \cap \odot(XYZ)$. Now since $BC \equiv PL$ is antiparallel to YZ WRT XY, XZ, then $XJ \perp PL$, i.e. XJ is perpendicular bisector of $\overline{PL}$, or $XP = XL$.

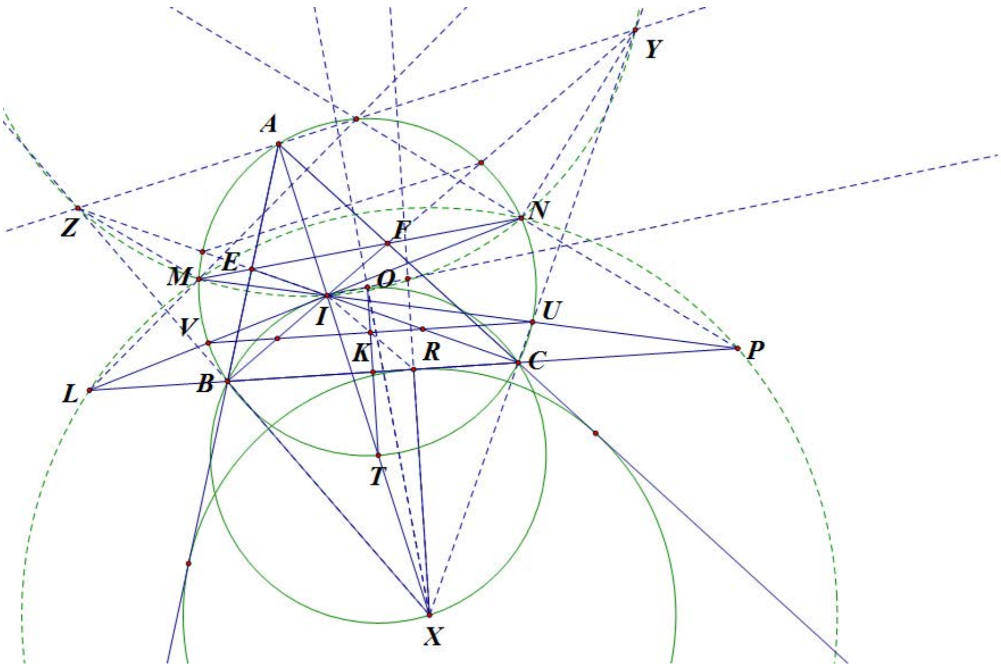
doxuanlong15052000

#4 Mar 3, 2016, 10:20 am • 2

My solution for this problem:
Let PM, NL cut (O) at U, V respectively, AI cuts (O) at T and (X) tangents to BC at R. From https://www.artofproblemsolving.com/community/c6t48f6h1184231_another_lemma[/urlPost #1, we have UV is average line of $\triangle IPL$. Since T is the midpoint of IX, then OT cuts IR at the midpoint of PR (Denote that K is the midpoint of IR then K lies on UV). On the other hand, K is the midpoint of UV, then R is the midpoint of PL, so $XL = XP$.

One large result: We have X is the circuncenter of $(LMNP)$.
Because UV is parallel to LP, then L, M, N, P are concyclic
Let Y and Z be the excenter of $\triangle ABC$ oppsite B and C respctively. Then I and O are the othocenter and Euler-circle center of $\triangle XYZ$. Let (O_1) be the circumcenter of $\triangle XYZ$ and (O_1') be the projection of (O_1) around YZ. From $EA.EB = EI.EZ$ and $FY.FI = FA.FC$, we have E and F lie on the radical exis of (O) and (O_1'), well-know that X, O, O_1' are collinear, then XO is perpendicular to EF. And since $XL = XP$, then X is the circumcenter of $(LMNP)$.

Attachments:



This post has been edited 2 times. Last edited by doxuanlong15052000, Mar 3, 2016, 10:29 am

Quick Reply

High School Olympiads

Locus of circumcenter

Reply

ThE-dArK-lOrD

#1 Mar 1, 2016, 11:58 pm

Given $\triangle ABC$, let D, E on it's circumcenter (O) such that DE is a diameter
Reflect DE with AB, BC, CA get 3 lines forming a triangle, let it's circumcenter be T
Proof that locus of T as D varies on (O) is a circle tangent to (O)

Luis González

#2 Mar 2, 2016, 12:29 am

For a more general problem see the topic Locus of a point on OI line.

Quick Reply

High School Olympiads

Locus of a point on OI line

Reply

Source: Own

TelvCohl
#1 Feb 28, 2015, 8:35 pm

Let ℓ be a line passing through the circumcenter O of $\triangle ABC$.
Let ℓ_A, ℓ_B, ℓ_C be the reflection of ℓ in BC, CA, AB, respectively .
Let $A^* = \ell_B \cap \ell_C, B^* = \ell_C \cap \ell_A, C^* = \ell_A \cap \ell_B$.
Let O^*, I^* be the circumcenter, incenter of $\triangle A^*B^*C^*$, respectively .
Let $k \in \mathbb{R}$ be a constant and P be a point such that $\overrightarrow{O^*P} : \overrightarrow{I^*P} = k$.

Prove that the locus of P is a circle tangent to $\odot(O)$ when ℓ rotates around O

Luis González
#2 Mar 1, 2015, 12:13 am • 1

From the problems The incenter lies on circumcircle [Iran Second Round 95] and On the reflections of a line wrt the sidelines of a triangle, $\triangle A^*B^*C^*$ is similar to the orthic triangle of $\triangle ABC, I^* \equiv AA^* \cap BB^*, CC^* \cap (O)$ and I^*O^* cuts (O) again at the Euler's reflection point X of $\triangle ABC.$

Let O_B, O_C be the reflections of O on CA, AB lying on $\ell_B, \ell_C.$ Since $\angle O_BA^*O_C$ is constant $\Longrightarrow A^*$ moves on a fixed circle that goes through O_B, O_C, A (because of $\angle O_BAO_C = 2 \cdot \angle BAC = \pi - \angle B^*A^*C^*$) and X (when ℓ coincides with the Euler line). Thus $\measuredangle I^*A^*X \equiv \measuredangle AA^*X$ is constant $\Longrightarrow \triangle A^*B^*C^* \cup \{X, P, I^*, O^*\}$ are all similar $\Longrightarrow$ $\frac{XI^*}{XP} = \varrho = \text{constant} \Longrightarrow P$ moves on image of (O) under homothety with center X and coefficient ϱ, i.e. a circle tangent to (O) at $X.$

TelvCohl
#3 Mar 1, 2015, 12:48 am

Dear **Luis**, thanks for your interest and nice solution
My solution is similar to yours but I prove " $\triangle A^*B^*C^* \cup \{X, P, I^*, O^*\}$ are all similar " in different way :

From Reflection of a line WRT the sides of a triangle $\Longrightarrow \odot(A^*B^*C^*) \perp \odot(ABC)$,
so X is the image of I^* under Inversion WRT $\odot(A^*B^*C^*) \Longrightarrow O^*I^* : O^*X$ is constant ,
hence we get $\triangle A^*B^*C^* \cup \{X, P, I^*, O^*\}$ are all similar .

P.S. When $P \equiv O^*$, the locus of P is the 9-point circle of the tangential triangle of $\triangle ABC$.

This post has been edited 2 times. Last edited by TelvCohl, May 30, 2015, 12:36 am

Quick Reply

High School Olympiads

Coaxality problem.

Reply

Source: Possibly own.

dothef1

#1 Feb 28, 2016, 8:56 pm

Let ABC be a triangle with orthocenter H, the $B-$ altitude and the C altitude meet the $A-$ median AM at X, Y respectively. And let D be the orthogonal projection of A on BC. Let H' be the antipode of H in $\odot(HBC)$. Prove that $\odot(BHY), \odot(CHX), \odot(H'DH)$ are coaxal.

This post has been edited 3 times. Last edited by Luis González, Feb 29, 2016, 7:34 am
Reason: Notation fixed

Luis González

#2 Feb 29, 2016, 6:15 am • 1

Let E, F be the feet of the altitudes on CA, AB. Tangent of $\odot(ABC) \equiv \odot(O, R)$ at A cuts BC at L and $U \equiv LO \cap AB$. Tangent of $\odot(HBC) \equiv \odot(J, R)$ at H' cuts BC at P and $V \equiv PJ \cap HB'$. Clearly $ABH'C$ is a parallelogram, thus by the symmetry WRT M, we get $\frac{VH'}{VB} = \frac{UA}{UC} \Longrightarrow$

$$\frac{VH'}{VB} = \frac{UA}{UC} = \frac{[OAL]}{[OCL]} = \frac{R \cdot AL}{OM \cdot LC} = \frac{R}{OM} \cdot \frac{AB}{AC} = \frac{AB}{AC \cdot \cos A} = \frac{AB}{AF} = \frac{YC}{YF},$$

which means that $VY \parallel AB \parallel H'C$, i.e. $VY \perp HC \Longrightarrow BHYV$ is cyclic on account of the right angles at $B, Y \Longrightarrow$ $\odot(BHY)$ goes through the projection S of H on PJ. Likewise $\odot(CHX)$ goes through S and S is obviously on $\odot(H'DHP) \Longrightarrow \odot(BHY), \odot(CHX)$ and $\odot(H'DH)$ are coaxal with common radical axis HS.

dothef1

#3 Feb 29, 2016, 6:58 pm

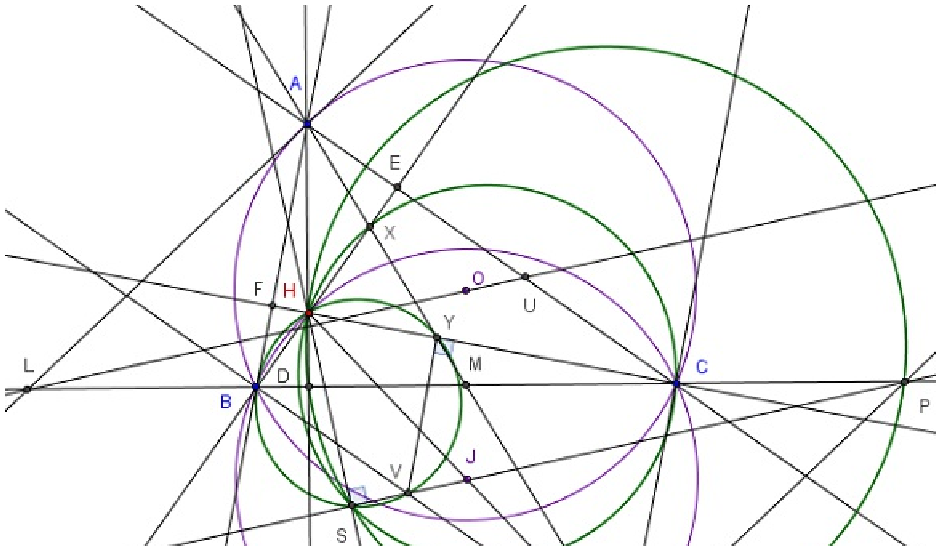
Sorry but what's HB' (there's no B') ? I guess you meant $H'C$?

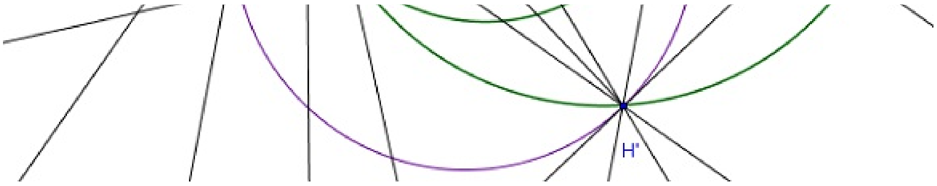
Luis González

#4 Mar 1, 2016, 8:59 am

Sorry it's a typo; I mean $V \equiv PJ \cap H'B$. Refer to the diagram below.

Attachments:



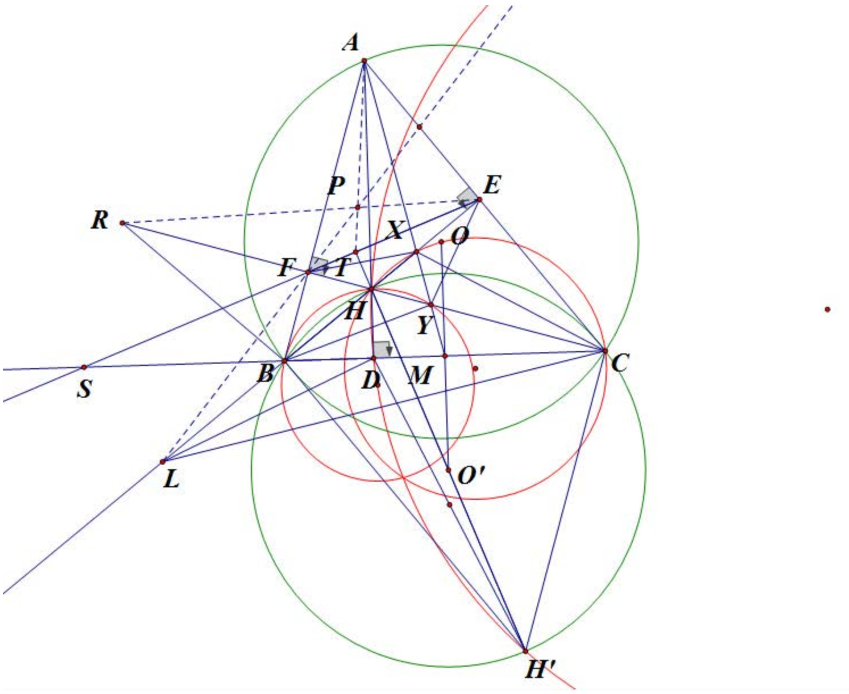


doxuanlong15052000

#5 Mar 1, 2016, 12:04 pm

My solution:
Let BH, CH cut CA, AB at point E and F respectively. Now, observe that the inversion with center H, power $HA.HD$:
$H' \mapsto T$, $D \mapsto A$, $(HDH') \mapsto AT$
$C \mapsto F$, $X \mapsto L$, $(HXC) \mapsto LF$
$Y \mapsto R$, $E \mapsto B$, $(HEY) \mapsto RE$
So, if $\odot(BHY), \odot(CHX), \odot(H'DH)$ are coaxal, then RE, LF, AT are concurrent and we can prove it.
Easy see that T lies on EF. We have $\angle HLF = \angle HCX$ and $\angle ABE = \angle ACF \Longrightarrow \angle BFL = \angle XCE$. Similar, we have $\angle YBM = \angle REF$ and $\angle AER = \angle ABY$. Use the Ceva' Theorem form Sin, we have done if $\dfrac{sinTAE}{sinTAF} = \dfrac{sinXCB}{sinXCA}.\dfrac{sinYBA}{sinYBM}$. But we have $\dfrac{sinXCB}{sinXCA}.\dfrac{sinYBA}{sinYBM} = \dfrac{CA}{BA}.\dfrac{MX}{XA}.\dfrac{YA}{YM} \Longrightarrow$ Done if $\dfrac{MX}{XA}.\dfrac{YA}{YM} = \dfrac{TE}{TF}$. But $\dfrac{TE}{TF} = \dfrac{HE}{HF}.\dfrac{BH'}{CH'} = \dfrac{HE}{HF}.\dfrac{AC}{AB} = \dfrac{CE}{BF}.\dfrac{AC}{AB}$. Note that $\dfrac{MX}{XA}.\dfrac{YA}{YM} = \dfrac{CE}{BF}.\dfrac{AC}{AB} \Longrightarrow$ We have done!

Attachments:



This post has been edited 2 times. Last edited by doxuanlong15052000, Mar 1, 2016, 12:06 pm

Quick Reply

High School Olympiads

Locus of isogonal conjugate

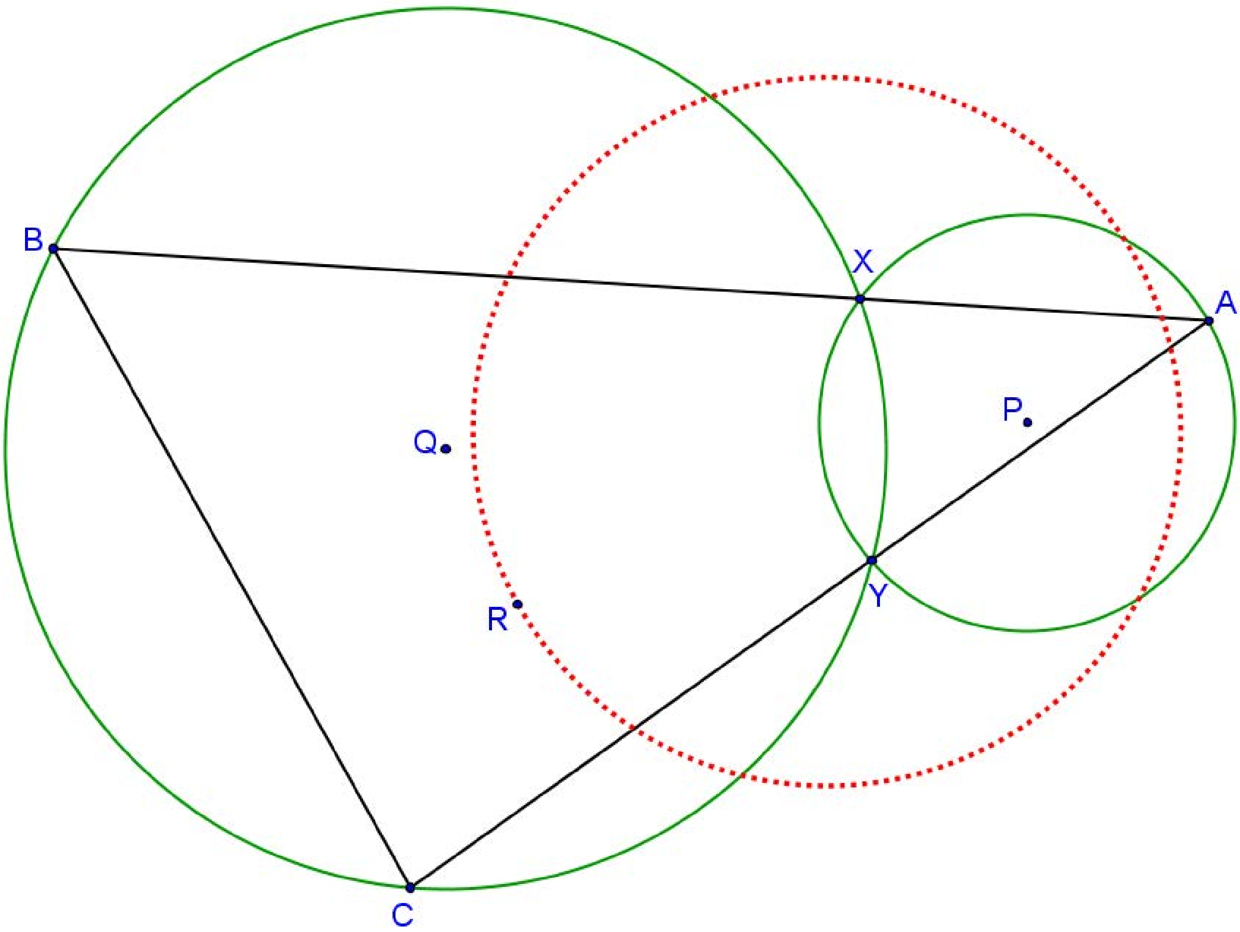
Reply

mjuk

#1 Feb 29, 2016, 11:15 pm

Let $(P),(Q)$ be two fixed circles intersecting at X,Y. Let A be a variable point on (P) and let $B=AX\cap(Q)$, $C=AY\cap(Q)$. Let R be isogonal conjugate of Q wrt. $\triangle ABC$. Prove that R lies on a fixed circle as A varies.

Attachments:



Luis González

#2 Mar 1, 2016, 12:49 am • 1

Let the tangent of (P) at Y cut (Q) again at $D. \angle DYC=\angle AXY=\angle BCY \Longrightarrow BDYC$ is an isosceles trapezoid with shoulders $BC=DY \Longrightarrow BC$ is constant $\Longrightarrow$ Q-isosceles $\triangle QBC$ are all congruent $\Longrightarrow$ $\angle ABR=\angle CBQ=\angle BCQ=\angle ACR=\theta=\text{const} \Longrightarrow BR,CR$ meet (Q) again at two fixed points $U,V.$ Now since $\angle BRC=\angle BAC+2\theta=\text{const},$ it follows that R moves on a circle that passes through $U,V.$

Quick Reply

High School Olympiads

Collinear

Reply

Source: Own

Scorpion.k48

#1 Feb 28, 2016, 10:58 pm

Let $\triangle ABC$ with orthocenter H .AH, BH, CH cut BC, CA, AB at A_0, B_0, C_0, res. Let A_1, B_1, C_1 is midpoint of BC, CA, AB. A_1B_1, A_1C_1 cut B_0C_0 at A_b, A_c, res. A_bC_1 cuts A_cB_1 at A_2. Similarity with B_2, C_2. Prove that A_2, B_2, C_2 are collinear.

TelvCohl

#3 Feb 29, 2016, 1:22 am • 1

Let O be the circumcenter of $\triangle ABC$ and let $X \equiv B_0C_0 \cap B_1C_1$. Since B_1C_1 and A_bA_c are antiparallel WRT $\angle B_1A_1C_1$, so B_1, C_1, A_b, A_c are concyclic, hence OA_2 is the Steiner line of the complete quadrilateral $\mathbf{Q} \equiv \{\triangle A_1B_1C_1, B_0C_0\}$.

Let S be the Miquel point of $\mathbf{Q}$ and let X_1 be the reflection of X in the midpoint of B_1C_1. From $XB_1 \cdot XC_1 = XA_b \cdot XA_c$ we get X lies on the radical axis A_1S of $\odot(A_1A_bA_c), \odot(A_1B_1C_1)$. On the other hand, since A_1X_1 is parallel to AX, so A_1X_1 is perpendicular to the Euler line OH of $\triangle A_1B_1C_1$, hence $S \equiv A_1X \cap \odot(A_1B_1C_1)$ is the Steiner point of $\triangle A_1B_1C_1$.

Since OA_2 is the Steiner line of S WRT $\triangle A_1B_1C_1$, so OA_2 is parallel to the Brocard axis $\tau_{\mathcal{B}}$ of $\triangle A_1B_1C_1$. Analogously, we can prove $OB_2 \parallel \tau_{\mathcal{B}}$ and $OC_2 \parallel \tau_{\mathcal{B}}$, so we conclude that $O, A_2, B_2\, C_2$ are collinear.

Luis González

#4 Feb 29, 2016, 12:44 pm • 1

More general: Consider $\triangle A_0B_0C_0$ and $\triangle A_1B_1C_1$ as cevian triangles of arbitrary points P_0, P_1. Defining A_2, B_2, C_2 in the same way, then A_2, B_2, C_2 are collinear and this line passes through the cross point of P_0, P_1 WRT $\triangle ABC$.

Proof: Let $\mathcal{C}$ be the conic through A, B, C, P_0, P_1. B_1 and A_c are the poles of A_1C_1 and A_0B_1 WRT $\mathcal{C} \Longrightarrow$ $U \equiv A_1C_1 \cap A_0B_1$ is the pole of B_1A_c WRT $\mathcal{C}$ and analogously $V \equiv A_1B_1 \cap A_0C_1$ is the pole of C_1A_b WRT $\mathcal{C} \Longrightarrow$ $A_2 \equiv B_1A_c \cap C_1A_b$ is the pole of $\tau_A \equiv UV$ WRT $\mathcal{C}$. τ_B, τ_C are defined cyclically.

Consider a homology taking P_1 into the centroid G of $\triangle ABC$, thus $\triangle A_1B_1C_1$ becomes medial triangle of $\triangle ABC$. Since $A_1A_0 \parallel B_1C_1$, then it follows that τ_A is the V-median of $\triangle VB_1C_1$, passing through the midpoint A' of B_1C_1 and the midpoint of $A_1A_0 \Longrightarrow \tau_A \parallel AP_0$ is the A_1-midline of $\triangle AA_0A_1$. Similarly τ_B, τ_C go through the midpoints B', C' of C_1A_1, A_1B_1 parallel to $BP_0, CP_0 \Longrightarrow \tau_A, \tau_B, \tau_C$ concur at the complement R of the complement of P_0 WRT $\triangle A_1B_1C_1$, clearly lying on P_0G. So back in the primitive figure, τ_A, τ_B, τ_C concur at a point R on $P_0P_1 \Longrightarrow A_2, B_2, C_2$ are collinear on the polar of R WRT $\mathcal{C}$. Since $R \in P_0P_1$, then $\overline{A_2B_2C_2}$ passes through the pole of P_0P_1 WRT $\mathcal{C}$, i.e. the cross point of P_0, P_1 WRT $\triangle ABC$.

Scorpion.k48

#5 Feb 29, 2016, 6:33 pm • 1

Prove that, we also have orthocenter of $\triangle A_0B_cC_b$ lie on $\overline{A_2, B_2, C_2}$.

This post has been edited 1 time. Last edited by Scorpion.k48, Feb 29, 2016, 6:33 pm

TelvCohl

#6 Feb 29, 2016, 7:12 pm

> *Scorpion.k48 wrote:*
> Prove that, we also have orthocenter of $\triangle A_0B_cC_b$ lie on $\overline{A_2, B_2, C_2}$.

From my proof above (post #3) we get A_2, B_2, C_2 lie on the Brocard axis $\eta_{\mathcal{B}}$ of $\triangle ABC$. Furthermore, there exist a parabola $\mathcal{P}$ with directrix $\eta_{\mathcal{B}}$ and tangent to the sides of $\triangle A_0B_0C_0, \triangle A_1B_1C_1$, so the orthocenter of the triangle formed by any 3 lines of $\{B_0C_0, C_0A_0, A_0B_0, B_1C_1, C_1A_1, A_1B_1\}$ lies on $\eta_{\mathcal{B}} \equiv \overline{A_2B_2C_2}$.

Quick Reply

High School Olympiads

Mixtilinear incircle, excircle

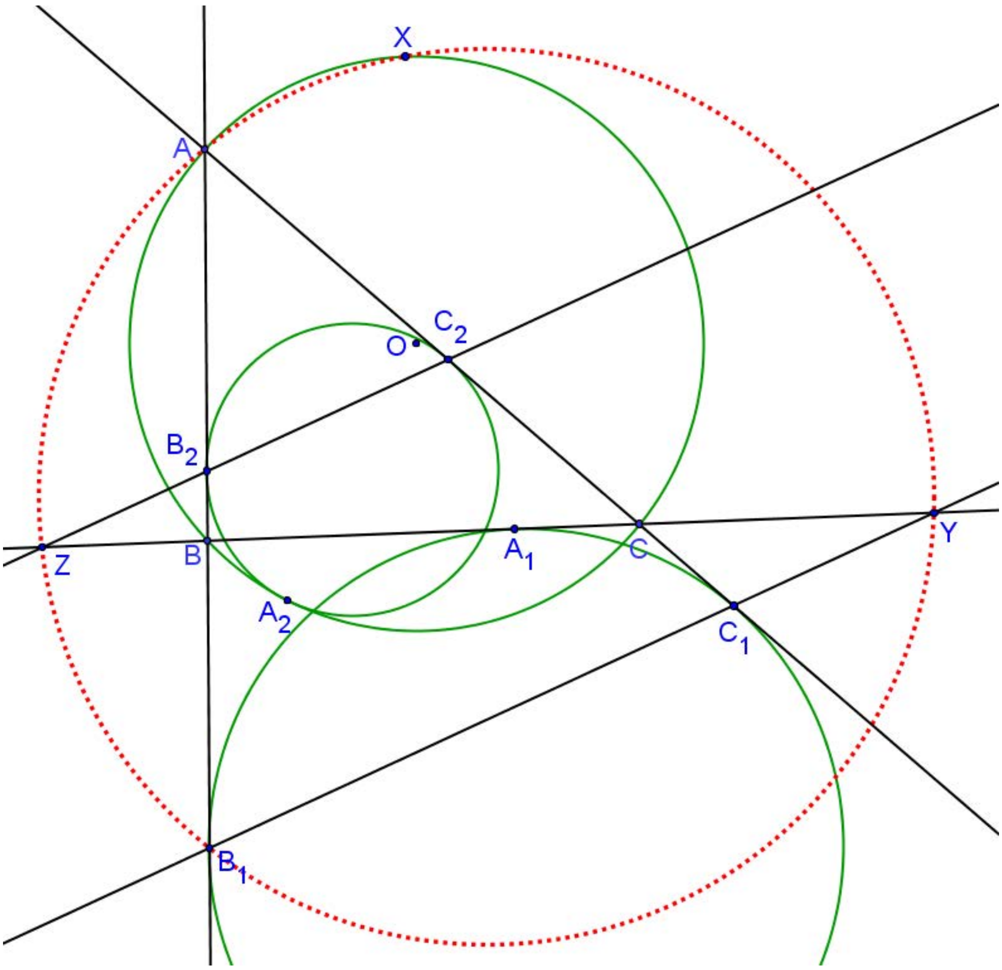
Reply

mjuk

#1 Feb 28, 2016, 10:56 pm

Let (O) be circumcircle of $\triangle ABC$. Let ω_a, Ω_a be A-mixtilinear incricle and A-excircle respectively. Let A_1, B_1, C_1 be touching points of Ω_a and BC, CA, AB respectively. Let A_2, B_2, C_2 be touching points of ω_a and $(O), AB, AC$ respectively. Let X be midpoint of arc BAC and let $Y = B_1C_1 \cap BC, Z = B_2C_2 \cap BC$.
Prove that A, X, Y, Z are concyclic.

Attachments:



Luis González

#2 Feb 29, 2016, 1:07 am

Let U be the foot of the external bisector of $\angle BAC$ and let V and W be the 2nd intersections of (O) with $\odot(AB_2C_2)$ and $\odot(AB_1C_1)$, respectively. These are precisely the images of X, Y, Z under the inversion with center A, power $AB \cdot AC$ followed by reflection on AX. Thus it's enough to show that U, V, W are collinear.

V and W are the centers of the spiral similarities that swaps BB_2, CC_2 and $BB_1, CC_1 \Longrightarrow$

$$\frac{WB}{WC} = \frac{BB_1}{CC_1} = \frac{s-c}{s-b}, \ \frac{VB}{VC} = \frac{BB_2}{CC_2} = \frac{\frac{c(s-b)}{s}}{\frac{b(s-c)}{s}} = \frac{c(s-b)}{b(s-c)} \Longrightarrow$$

$VB \quad WB \quad s-c \quad c(s-b) \quad c \quad UB$

$\frac{}{VC} \cdot \frac{}{WC} = \frac{}{s-b} \cdot \frac{}{b(s-c)} = \frac{}{b} = \frac{}{UC} \Longrightarrow U, V, W$ are collinear.

PVoltaire

#3 Feb 29, 2016, 1:34 am

Why does $\frac{VB}{VC} \cdot \frac{WB}{WC} = \frac{UB}{UC}$ imply that U, V, W are collinear?

Luis González

#4 Feb 29, 2016, 1:42 am

Another solution: Let I, I_a be the incenter and A-excenter of $\triangle ABC$ and let B', C' be the projections of B, C on AI. It's known that $I \in B_2C_2$ and B', C' are the two limiting points of $(I), (I_a)$, i.e. circle with diameter $B'C'$ is orthogonal to $(I), (I_a)$. Thus if $J_a \equiv B_1C_1 \cap AI$, the inversion with center A that swaps (I) and (I_a) takes $B' \mapsto C'$ and $I \mapsto J_a$, i.e. A is the center of the involution that swaps (B', C') and (I, J_a). Hence if $U \equiv AX \cap BC$, parallel projecting $\{A, I, B', C', J_a\}$ on BC from the point at infinity of AX gives that U is the center of the involution that swaps (Y, Z) and $(B, C) \Longrightarrow UY \cdot UZ = UB \cdot UC = UA \cdot UY \Longrightarrow AXYZ$ is cyclic.

Luis González

#5 Feb 29, 2016, 1:50 am • 1

> *PVoltaire wrote:*
>
> Why does $\frac{VB}{VC} \cdot \frac{WB}{WC} = \frac{UB}{UC}$ imply that U, V, W are collinear?

It follows from this well-known lemma (easy to prove): $ABCD$ is cyclic and $P \equiv AB \cap CD$. Then we have $\frac{PD}{PC} = \frac{AD}{AC} \cdot \frac{BD}{BC}$ or $\frac{PA}{PB} = \frac{DA}{DB} \cdot \frac{CA}{CB}$.

Quick Reply

High School Olympiads

Orthocenter and collinear

Reply

mamavuabo

#1 Feb 28, 2016, 4:22 am

Given triangle ABC inscribed in (O).The three attitudes are AD, BE, CF respectively.$AO \cap BC = N, FE \cap AD = M$.Let S be the midpoint of MN.Prove that AS bisects segment BC.

Luis González

#2 Feb 28, 2016, 9:42 am

Let $H \equiv BE \cap CF, L \equiv AO \cap EF, P \equiv EF \cap BC$ and let X denote the midpoint of $\overline{BC}$. $BCEF$ and $DNLM$ are cyclic with circumcircles (X) and (S). Since $(H, M, D, A) = -1$, it follows that PH is the polar of A WRT $(S) \Longrightarrow AS \perp PH$, but PH is also the polar of A WRT $(X) \Longrightarrow AX \perp PH \Longrightarrow A, S, X$ are collinear, i.e. AS bisects $\overline{BC}$.

Quick Reply

High School Olympiads

Prove □BAC=60

Reply

Source: Own

lym

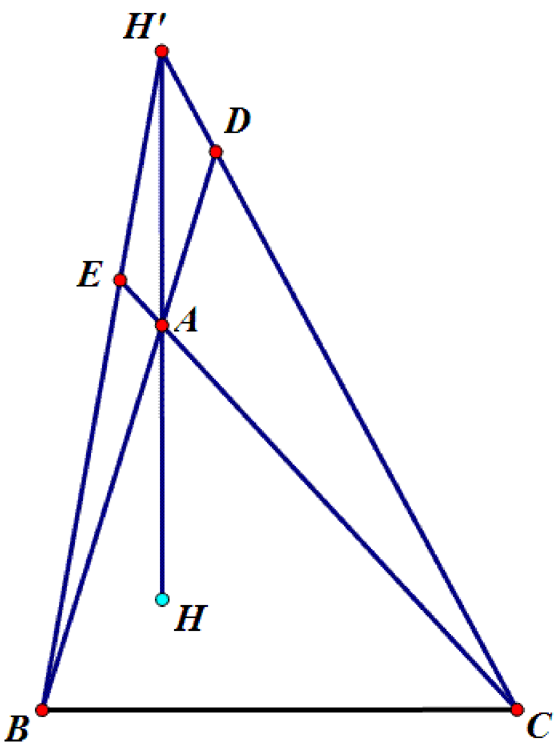
#1 Jan 12, 2010, 6:49 am

Figure, $AB < AC$, H is the orthocenter of □ABC, A is the midpoint of HH' . AB, AC intersect CH', BH' at D, E respectively .

Prove that□ □$BAC = 60 <=> BD = CE$.

Click to reveal hidden text

Attachments:



sunken rock

#2 Jan 15, 2010, 9:18 pm

The below proof was provided to me by my son, but he's too busy to type it by himself.

On CA produced we take $F|AF = AC$, on BA produced we take $G|AG = AB$. Obviously, $BCGF$ is an isosceles trapezoid, with $BG = CF$.

We shall get $EF = DG$ iff $\frac{EF}{CE} = \frac{DG}{BD}$ which, in turn is true iff $\frac{S_{BFH'}}{S_{BCH'}} = \frac{S_{CGH'}}{S_{BCH'}}$, the last equality being true iff $S_{BFH'} = S_{CGH'}$ ($*$).

Seing that BF and CG are parallel to the bisector of the $\angle BAC$, let's draw a parallel to this lie through H' and call P its intersection with FG.

The relation (*) is true iff $\frac{FP}{PG} = \frac{AC}{AB}$ (1), because $\triangle BAF \sim \triangle CAG$, but relation (1) shows that, H' is the symmetrical of the middle of the arc FG about FG, i.e. H' is the circumcenter of $\triangle AFG$, which happens iff $\angle BAC = 60°$.

The reverse of the problem, iff $\angle BAC = 60^\circ$, then H' is the circumcenter of the $\triangle AFG$, etc.

Note: Here S_{XYZ} stands for the area of $\triangle XYZ$.

Best regards,
sunken rock

lym

#3 Jan 18, 2010, 9:20 am

Thanks for your proof .
Is there anybody want to give different way? Nex I'll show how I design this problem .

Luis González

#4 Feb 27, 2016, 11:17 pm

Lemma (well-known): P is a point inside the parallelogram $ABCD$ and DP, BP cut AB, AD at $X, Y,$ respectively. Then $DX = BY \Longleftrightarrow PC$ bisects $\angle BPD. \blacksquare$

Back to the problem. Let M and L be the midpoints of $\overline{BC}$ and the arc BC of the circumcircle $(O) \equiv \odot(ABC).$ If T is the point forming the parallelogram $H'CTB,$ then from the previous lemma, we get $BD = CE \Longleftrightarrow AT$ bisects $\angle BAC \Longleftrightarrow A, T, L$ are collinear $\Longleftrightarrow \frac{AH'}{ML} = \frac{TH'}{TM} = 2 \Longleftrightarrow ML = \frac{1}{2}AH' = \frac{1}{2}AH = OM \Longleftrightarrow L$ is reflection of O on $BC \Longleftrightarrow 2\angle BAC = 180^\circ - \angle BAC \Longleftrightarrow \angle BAC = 60^\circ.$

Quick Reply

High School Olympiads

Symmedian (2)

Locked

mamavuabo

#1 Feb 26, 2016, 4:40 am

Given an acute triangle ABC inscribed in (O).Let the tangent lines to (O) through B, C intersect each other at $S.D$ is an arbitrary point lying on small arc $BC.AD \cap BS = E, AD \cap CS = F, BF \cap CE = R, RD \cap BC = U$.Prove that AU is $A-$ symmedian.

Luis González

#2 Feb 26, 2016, 5:12 am

Discussed before at http://www.artofproblemsolving.com/community/c6h1184665.

High School Olympiads

geometry

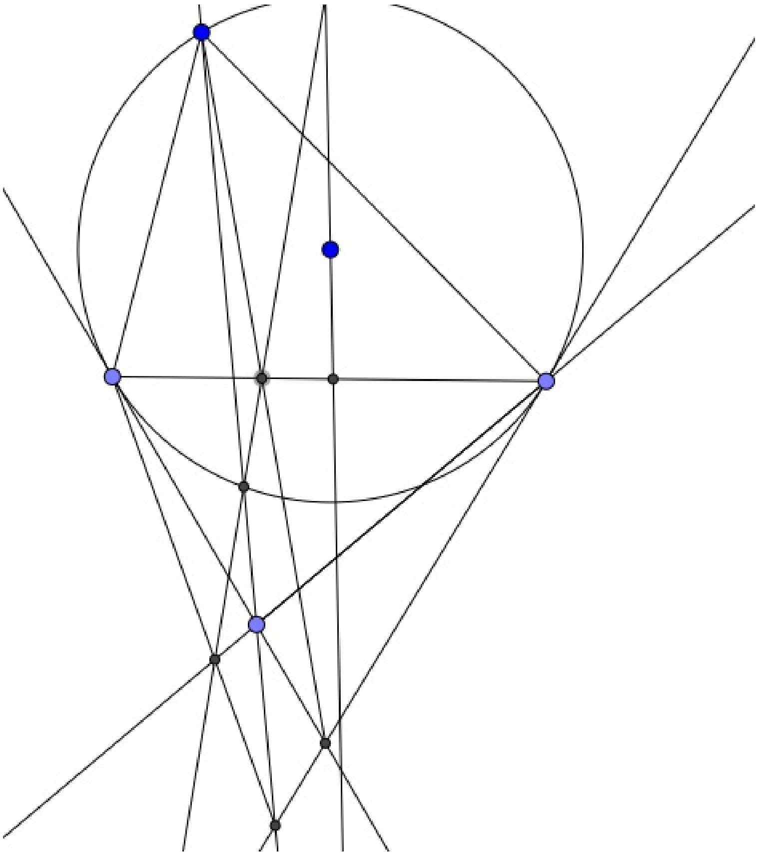
Reply

phantranhuongth

#1 Jan 13, 2016, 9:38 pm

Let ABC the inscribed circle (O). d pass A intersection tangent at B, C at M, N, intersection (O) at E. F is BN intersection CM. Prove EF passing one fixed point.

Attachments:



Luis González

#3 Jan 14, 2016, 4:22 am • 2

Let $P \equiv BM \cap CN, L \equiv MN \cap BC$ and $D \equiv AP \cap BC$. By Desargues involution theorem for the degenerate quadrilateral $BBCC$ cut by d, we get that L is fixed in the involution $M \mapsto N, A \mapsto E$, i.e. $(N, M, A, L) = (M, N, E, L)$. But $P(N, M, A, L) = P(C, B, D, L) \Longrightarrow (C, B, D, L) = (N, M, A, L) = (M, N, E, L)$, which means that $F \equiv BN \cap BM \cap DE \Longrightarrow EF$ always goes to the fixed point D; foot of the A-symmedian of $\triangle ABC$.

P.S. This configuration was also discussed at Cyclic quadrilateral + Tangent circles.

JackXD

#4 Jan 15, 2016, 11:00 pm

> *phantranhuongth wrote:*
>
> Let ABC the inscribed circle (O). d pass A intersection tangent at B, C at M, N, intersection (O) at E. F is BN intersection CM. Prove EF passing one fixed point.

Can you please improve your language a bit?

The problem cannot be made out from this.

Quick Reply

High School Olympiads

vietnam289
#1 Jan 2, 2016, 3:04 pm

Let ABC be a triangle with incenter I, circumcenter O. The line through I and perpendicular to BI intersects BC at B'. The line through I and perpendicular to CI intersects BC at C'. Let X be the radical center of three circles $(O); (B, BB'); (C, CC')$. Prove that AX is the A-symmedian.

A-B-C
#2 Jan 2, 2016, 6:58 pm

My solution
Lemma 1.$\triangle ABC$, ω_a is $A-$ mixtilinear incircle. ω_a is tangent to AB, AC at A_b, A_c.
$\odot(A, AA_b)$ intersect circumcircle (O) at M, N then MN is tangent to ω_a.
There is a proof in Euclidean geometry blog but it is in Vietnamese, here I translate it.
D is the tangency point of (O) and ω_a. AD intersects ω_a at T and intersects $\odot(A, AA_b)$ at K, L.
Since $\odot(A, AA_b)$ and ω_a are orthogonal, we obtain that $(KLDQ) = -1 \Rightarrow \overline{TA}.\overline{TD} = \overline{TK}.\overline{TL}$. Hence T lies on radical axes of (O) and $\odot(A, AA_b)$, this implies that T, M, N are collinear. D is exsimilcenter of ω_a and $(O) \Rightarrow$ tangent lines at T and A are parallel $\Rightarrow MN \equiv$ tangent line at T of ω_a.
Lemma 2. MN intersects AB, AC at P, Q.

$$\frac{AQ}{AC} = \frac{4c^2}{(a+b+c)^2}$$

Just using lemma 1 and notice that $AB.AP = AC.AQ = AT.AD = AA_b^2$ and $r_{\omega_a} = \frac{r}{\cos^2\frac{A}{2}}$.

Back to the main problem. B', C' can be considered as tangency points of $B, C-$ mixtilinear incircles and BC.
Radical line ℓ_b of (O) and $\odot(B, BB')$ intersects AB at S then $\frac{BS}{BA} = \frac{4a^2}{(a+b+c)^2}$ (according to lemma 2)
Radical line ℓ_c of (O) and $\odot(C, CC;)$ intersects AC at T then $\frac{CT}{CA} = \frac{4a^2}{(a+b+c)^2}$, likewise.
Therefore $\frac{BS}{BA} = \frac{CT}{CA}$. Now let W be intersection of tangent line at B, C of (O).
ℓ_b passes through S and parallel to WB, ℓ_c passes through T and parallel to WC.
$\Longrightarrow \ell_b \cap \ell_c$ lies on AT. This means AX is $A-$symmedian.

Luis González
#3 Feb 24, 2016, 10:17 am • 1

B-mixtilinear incircle (O_b) of $\triangle ABC$ touches BC at B' and touches (O) at Y. BY cuts (O_b) again at Y' and the tangent p_b of (O_b) at Y' cuts (O) at B_1, B_2. p_c is defined similarly. As p_b is parallel to the tangent of (O) at B, i.e. antiparallel to AC WRT BC, BA, it follows that $p_b \perp BO \Longrightarrow B$ is midpoint of the arc B_1B_2 of $(O) \Longrightarrow$
$BB_1{}^2 = BB_2{}^2 = BY \cdot BY' = BB'^2 \Longrightarrow \{B_1, B_2\} \in \odot(B, BB') \Longrightarrow p_b$ is the radical axis of $\odot(B, BB')$ and (O).
Likewise p_c is the radical axis of $\odot(C, CC')$ and $(O) \Longrightarrow X \equiv p_b \cap p_c$. Now according to the problem Mixtilinear incircles and symmedian, AX is the A-symmedian.

buratinogigle
#4 Feb 24, 2016, 5:39 pm

Here is general problem.

Let ABC be a triangle inscribed in circle (O). P, Q are two isogonal conjugate points on bisector of $\angle BAC$. AP cuts (PBC) again at J. M, N lies on BC such that $QM \parallel JB$ and $QN \parallel JC$. Prove that radical center of circles

(PBC) again at J. M, N lies on BC such that $QM \parallel JB$ and $QN \parallel JC$. Prove that radical center of circles $(B, BM), (C, CN)$ and (O) lies on A-symmedian of triangle ABC.

Luis González

#5 Feb 25, 2016, 4:57 am • 2

> *buratinogigle wrote:*
>
> Here is general problem.
>
> Let ABC be a triangle inscribed in circle (O). P, Q are two isogonal conjugate points on bisector of $\angle BAC$. AP cuts (PBC) again at J. M, N lies on BC such that $QM \parallel JB$ and $QN \parallel JC$. Prove that radical center of circles $(B, BM), (C, CN)$ and (O) lies on A-symmedian of triangle ABC.

Upon inverting with center A we get the following problem: P, Q are isogonal conjugates WRT $\triangle ABC$ lying on internal bisector of $\angle BAC$. $\odot(BPQ)$ and $\odot(CPQ)$ cut (O) again at N, M and the tangents of (O) at M, N cut AB, AC at U, V. $\odot(U, UM)$ and $\odot(V, VN)$ meet BC at $\{U_1, U_2\}$ and $\{V_1, V_2\}$, resp. Then the radical axis of $\odot(AU_1U_2)$ and $\odot(AV_1V_2)$ is the A-median of $\triangle ABC$.

Proof: Let J be the midpoint of the arc BC of (O). $\odot(BPQ)$ cuts BC, BA again at N', B' and $\odot(CPQ)$ cuts CB, CA again at M', C'. Since the inversion WRT $\odot(J, JB)$ swaps $(O), BC$ and P, Q (well-known), it follows that $N' \in JN$ and $M' \in JM \Longrightarrow \angle BB'N' = \angle BNJ = \angle BAJ \Longrightarrow B'N' \parallel AJ \equiv PQ$ and similarly $C'M' \parallel PQ$. Hence, by symmetry $B'C'M'N'$ is an isosceles trapezoid $\Longrightarrow B'C'$ is antiparallel to BC WRT AB, AC, i.e. $BCC'B'$ is cyclic. Hence if $R \equiv B'C' \cap BC$, we have $RB \cdot RC = RM' \cdot RN' \Longrightarrow R$ is on the radical axis MN of $\odot(MNN'M')$ and $(O) \Longrightarrow RM \cdot RN = RB' \cdot RC' \Longrightarrow B'C'MN$ is cyclic $\Longrightarrow B'N, C'M, PQ$ concur at the radical center K of $\odot(BPQ)$, $\odot(CPQ), \odot(B'C'MN)$.

Since N is the center of the spiral similarity that swaps $\overline{CN'}, \overline{AB'}$ and NP, NQ are isogonals WRT $\angle JNK$ (due to $PQ \parallel B'N'$), then we deduce that

$$\frac{VC}{VA} = \frac{NC^2}{NA^2} = \frac{NN'^2}{NB'^2} = \frac{NJ^2}{NK^2} = \frac{QJ}{QK} \cdot \frac{PJ}{PK}.$$

Similarly we have $\frac{UB}{UA} = \frac{QJ}{QK} \cdot \frac{PJ}{PK}$, which means that $UV \parallel BC$ and since (O) is orthogonal to $\odot(U, UM)$ and $\odot(V, VN)$, then it follows that the perpendicular bisector of BC is their radical axis $\Longrightarrow$ midpoint A_0 of BC has equal power WRT them, i.e. $A_0U_1 \cdot A_0U_2 = A_0V_1 \cdot A_0V_2 \Longrightarrow A_0$ is on the radical axis of $\odot(AU_1U_2)$ and $\odot(AV_1V_2)$.

Quick Reply

High School Olympiads

collinear with cyclic quadrilateral

Reply

Source: OWN

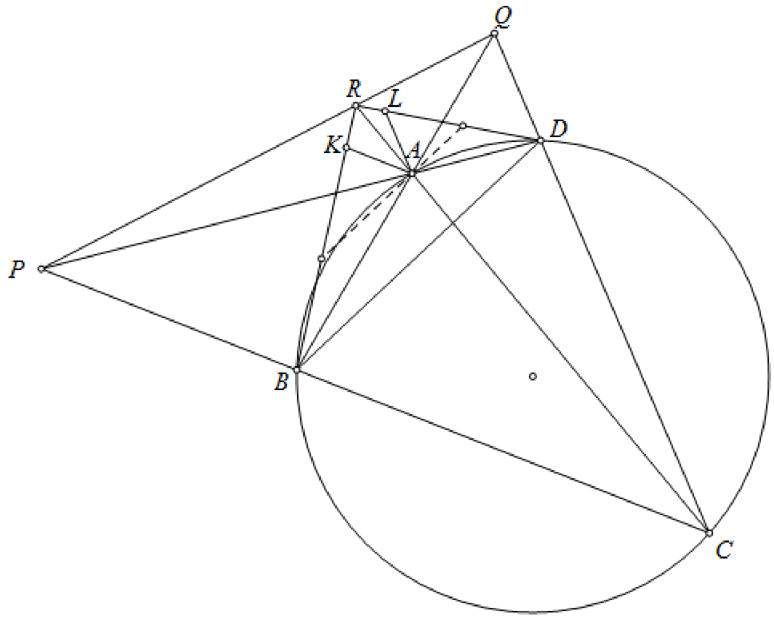
LeVietAn

#1 Feb 22, 2016, 8:33 pm

Dear Mathlinkers,

Given a cyclic quadrilateral $ABCD$. The lines AB and CD meet at P, the lines AD and BC meet at Q, and the diagonals AC and PQ meet at R. Choose the points K and L respectively on the lines BR and DR such that $AK \parallel BC$ and $AL \parallel CD$. Prove that A and the midpoint of segment BK, DL are collinear.

Attachments:



Luis González

#2 Feb 22, 2016, 9:39 pm • 1

This holds for any quadrilateral $ABCD$, not necessarily cyclic.

Let the parallel from A to BD cut BC and BR at T and M, respectively. Since $B(R, D, A, C) \equiv B(M, D, A, T) = -1$ $\Longrightarrow M$ is midpoint of $AT \Longrightarrow M$ is midpoint of BK. Similarly, if N is the midpoint of DL, we have $AN \parallel BD \Longrightarrow$ A, M, N are collinear on a parallel to BD.

Quick Reply

High School Olympiads

Mixtilinear circle

Locked

CEH

#1 Feb 22, 2016, 10:42 am

Let ABC is triangle and I is incentre and D is intersection point of line AI and BC. Let X is tangent point of A-mixtilinear incircle and a circumcircle of ABC. Define E,Y,F,Z similiarly. Prove that circumcircle of ADX,BEY,CFZ is coaxial circle.(Prove that three circumcenter is collinear)

Luis González

#2 Feb 22, 2016, 11:36 am

Discussed before at http://www.artofproblemsolving.com/community/c6h199419.

High School Olympiads

Mixtilinear incircles and a triad of coaxal circles

Reply

Source: me and yetti

pohoatza

#1 Apr 11, 2008, 3:01 am

Let ABC be a given triangle with the incenter I, and denote by X, Y, Z the intersections of the lines AI, BI, CI with the sides BC, CA, and AB, respectively. Consider $\mathcal{K}_a$ the circle tangent simultanously to the sidelines AB, AC, and internally to the circumcircle $\mathcal{C}(O)$ of ABC, and let A' be the tangency point of $\mathcal{K}_a$ with $\mathcal{C}$. Similarly, define B', and C'. Prove that the circumcircles of triangles AXA', BYB', and CZC' are coaxal (i.e. they two common points).

Vladimir Zajic (yetti on ML) and I have a neat synthetic proof by making use of inversion with respect to the incircle.

Luis González

#2 Jul 15, 2011, 11:05 am • 1

Let $\triangle I_aI_bI_c$ be the excentral triangle of $\triangle ABC.$ I and (O) become orthocenter and 9-point circle of $\triangle I_aI_bI_c.$ Thus (O,R) cuts I_bI_c at its midpoint $M,$ which is also midpoint of the arc BAC of $(O).$ According to problem 2.20 of 2005 mosp 2.20(g) 4.44(g), the points M,I,A' are collinear. Inversion with center I and power $\overline{IA}\cdot\overline{II_a}$ takes (O) into the circumcircle $(U,2R)$ of $\triangle I_aI_bI_c$ and the line BC into the circle $\odot(II_bI_c).$ Hence, ray $A'M$ cuts (U) at the inverse P of A' and ray IA cuts $\odot(II_bI_c)$ at the inverse D of $X.$ Clearly, P is the antipode of I_a WRT (U) and D is the reflection of I_a across $I_bI_c.$ Hence, the center U_a of the inverse $\odot(I_aDP)$ of $\odot(AXA')$ is the intersection of I_bI_c with the perpendicular to UI_a though $U.$ Likewise, the centers U_b,U_c of the inverses of $\odot(BYB'),\odot(CZC')$ are the intersections of $I_cI_a,\ I_aI_b$ with the perpendiculars to UI_b,UI_c through $U,$ respectively $\Longrightarrow U_a,U_b,U_c$ are collinear on the orthotransversal of U WRT $\triangle I_aI_bI_c.$ Since U has power $-4R^2$ WRT $(U_a),(U_b),(U_c),$ then we deduce that $(U_a),(U_b),(U_c)$ are coaxal with common radical axis the perpendicular from U to $U_aU_bU_c.$ Since $(U_a),(U_b),(U_c)$ are coaxal, then their inverses $\odot(AXA'),\odot(BYB'),\odot(CZC')$ under the referred inversion are also coaxal.

jayme

#3 Jul 15, 2011, 11:22 am

Dear Mathlinkers,
for a metric proof, you can see
http://perso.orange.fr/jl.ayme vol. 4 A new mixtilinear incircle adventure p. 36-43
Sincerely
Jean-Louis

Quick Reply

High School Math

Prove collinear.

Reply

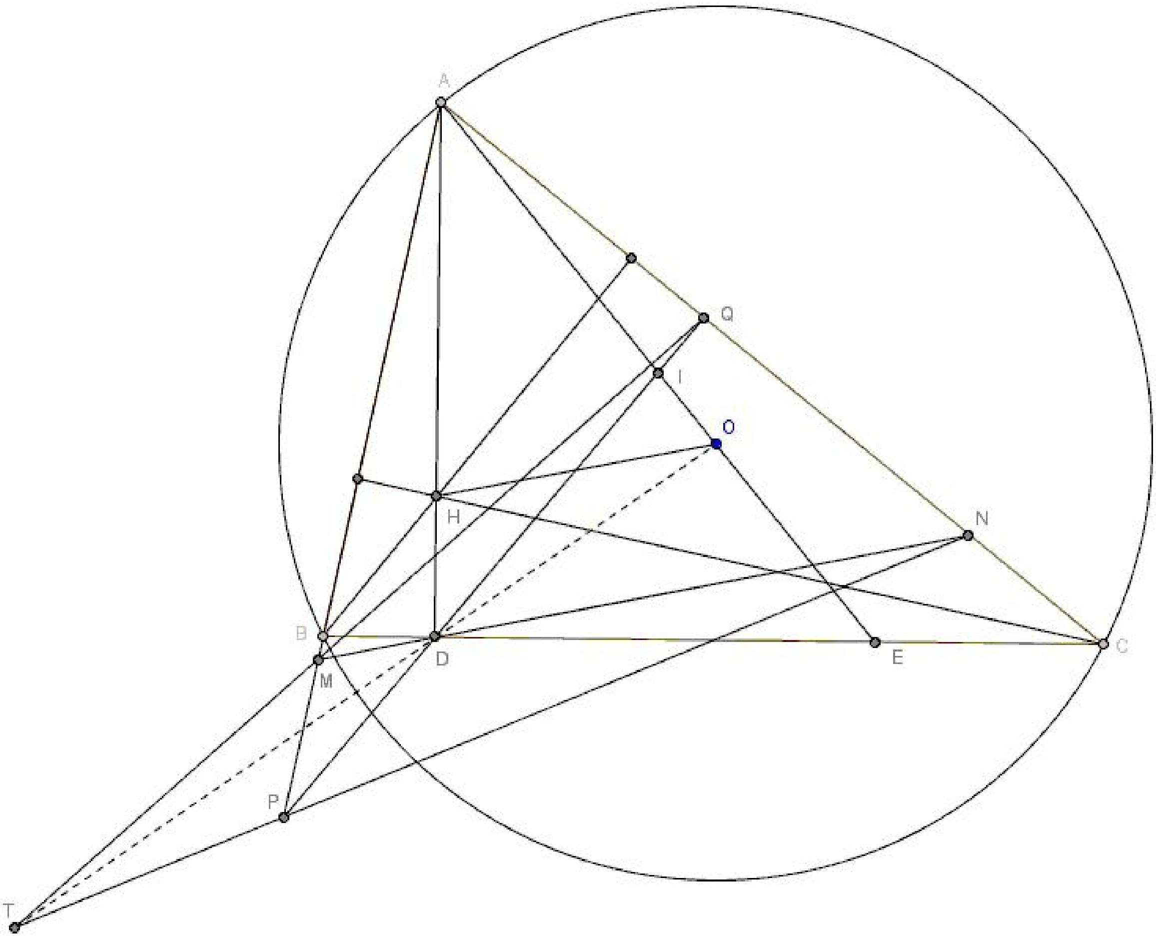
ducadongnoi

#1 Feb 21, 2016, 10:47 pm

Given a triangle ABC inscribed circle (O), orthocentre H, altitude AD. AO intersects BC at E, I is the midpoint of AE. Line through D and parallels to OH intersects AB, AC at M and N, line ID intersects AB, AC at P and Q. MQ intersects NP at T. Prove that T, D, O are collinear.

Sorry for my bad English!

Attachments:



Luis González

#2 Feb 22, 2016, 7:08 am • 1

Let U be the midpoint of BC and let AD, AO cut the circumcircle $\odot(ABC)$ again at X, Y, respectively. Since $\angle AXY = 90^\circ$, then $XY \parallel DE \Longrightarrow \frac{AD}{AX} = \frac{AE}{AY} = \frac{AI}{AO} \Longrightarrow OX \parallel DI$. Hence since $OU \parallel DA, OH \parallel DN$ and $OX \parallel DQ$, it follows that $O(D,H,X,U) = O(L,N,Q,A)$, where $L \equiv OD \cap AC$. As D is midpoint of HX (well-known), then $O(D,H,X,U) = -1 \Longrightarrow (L,N,Q,A) = -1$. But from the complete quadrilateral $MQNP$, we deduce that $T(D,N,Q,A) = -1 \Longrightarrow L \in TD$, i.e. T, D, O are collinear.

EinsteinXXI

#3 Mar 4, 2016, 8:08 pm

Here is my other solution :
Let G,F is the midpoints of OH, AH respectively.
Suppose OD meets AB, AC at K,L respectively.
Because G is the midpoint of OH, G is the center of Euler Circle of the triangle ABC.
So GD = GF. Hence angle GDF = angle GFD.
But angle GFD = angle OAH (because GF // OA).
So angle GDF = OAD (1).
On the other hand, we have ID = IA = IE (because angle ADE = 90* and I is the midpoint of the side AE).
So angle IDA = angle IAD = angle OAD (2).
According (1) and (2) we have angle GDF = angle IDA so I,D,G collinear.
We have D(HOGM) = -1 (because GH = GO and DM//OH).
So (AKPM) = -1 hence (AKMP) = -1.
Similarly, (ALNQ) = -1.
Hence MN, KL, PQ concurrence.
So T,K,L collinear (because T is the intersection of MN and PQ).
Hence O,D,T collinear (because O,D,L,K collinear).

This post has been edited 2 times. Last edited by EinsteinXXI, Mar 4, 2016, 8:11 pm
Reason: Sorry

Quick Reply

High School Olympiads

Mixtilinear incircle

Reply

CEH
#1 Feb 21, 2016, 2:27 pm

Let ABC is triangle. O and I is circumcentre and incentre of ABC. Let circle D,E,F is mixtilinear incircle of A,B,C. Let T is radical center of circle D,E,F. Prove that I,O,T is collinear.(What is ratio of TI and TO?)

Luis González
#3 Feb 21, 2016, 11:24 pm

See the threads mixtilinear incircles and radical centers. T is the midpoint of $IX,$ where X is the homothety center of the intouch triangle and excentral triangle of $\triangle ABC.$

Denote (O,R) and (I,r) the circumcircle and incircle of $\triangle ABC.$ If Be is the Bevan point of $\triangle ABC$ (reflection of I on O), then T is the exsimilicenter of (I,r) and $(Be,2R)$ (circumcircle of excentral triangle) $\Longrightarrow \frac{TO}{TI}=\frac{XBe}{XI}=\frac{2R}{r}.$

Quick Reply

High School Olympiads

mixtilinear incircles

Reply

wild_sloth

#1 Nov 26, 2010, 1:56 pm

prove: radical center of the three mixtilinear incircles of triangle ABC is on line IO

jayme

#2 Nov 26, 2010, 5:40 pm

Dear Mathlinkers,
for a synthetic proof, see
http://perso.orange.fr/jl.ayme vol. 4 A new mixtilinear incircle adventure II p. 11-13.
Sincerely
Jean-Louis

Luis González

#3 Nov 27, 2010, 9:49 pm • 1

Let $\triangle I_aI_bI_c$ be the excentral triangle of $\triangle ABC$. Incircle (I) and A-excircle (I_a) touch BC at X, Y, B- and C- mixtilinear incircles ω_b and ω_c touch BC at D, E and touch circumcircle (O) of $\triangle ABC$ at B_0, C_0. Thus, B_0D and C_0E bisect $\angle BB_0C$ and $\angle CC_0B \Longrightarrow B_0D$ and C_0E pass through the midpoint L of the arc BC of (O). Since BC is the image of (O) under the inversion with center L and radius $LB = LC$, it follows that $LB^2 = LD \cdot LB_0 = LE \cdot LC_0 \Longrightarrow L$ has equal power to ω_b and ω_c, thus radical axis τ_a of ω_b, ω_c goes through L and the midpoint U of DE. On the other hand, it's well-known that $DI \parallel BI_a$ and $EI \parallel CI_a \Longrightarrow \triangle DEI$ and $\triangle BCI_a$ are homothetic. Therefore, if M denotes the midpoint of BC, we deduce that $I_aM \parallel IU \ (\star)$.

Let $P \equiv I_aM \cap IX$. Since $\overline{MX} = -\overline{MY} \Longrightarrow XPYI_a$ is a parallelogram $\Longrightarrow M$ is the midpoint of PI_a. Together with $(\star)$, it follows that $IU \cap I_aX$ is the reflection of I about $U \Longrightarrow UL \equiv \tau_a$ is the I-midline of $\triangle IXI_a$. Now, since I_aX cuts IO at the exsimilicenter X_{57} of (I) and $\odot(I_aI_bI_c)$, then τ_a passes through the midpoint X_{999} of $\overline{IX_{57}}$. Analogously, radical axes ℓ_b, ℓ_c of mixtilinear incircles ω_c, ω_a and ω_a, ω_b pass through $X_{999} \Longrightarrow X_{999}$ is radical center of the three mixtilinear incircles $\omega_a, \omega_b, \omega_c$.

Quick Reply

High School Olympiads

radical centers

Reply

andria

#1 Jun 19, 2015, 12:51 am • 2

In $\triangle ABC$ with circumcenter O let X be radical center of three mixtilinear incircles and Y is radical center of three mixtilinear excircles prove that O is midpoint of XY.

TelvCohl

#4 Jun 19, 2015, 2:08 pm • 2

My solution :

Let I be the incenter of $\triangle ABC$ and $\triangle DEF$ be the intouch triangle of $\triangle ABC$.
Let I_a, I_b, I_c be the A-excenter, B-excenter, C-excenter of $\triangle ABC$, respectively .
Let $\omega_A, \omega_B, \omega_C$ be the A-mixtilinear incircle, B-mixtilinear incircle, C-mixtilinear incircle of $\triangle ABC$, respectively .
Let $\Omega_A, \Omega_B, \Omega_C$ be the A-mixtilinear excircle, B-mixtilinear excircle, C-mixtilinear excircle of $\triangle ABC$, respectively .
Let $\triangle A_1B_1C_1$ be the anti-complementary triangle of $\triangle I_aI_bI_c$ and $\triangle A_2B_2C_2$ be the mic-arc triangle of $\triangle ABC$.
Let A_2', B_2', C_2' be the antipode of A_2, B_2, C_2 in $\odot(O)$, resp and A_3, B_3, C_3 be the midpoint of A_1D, B_1E, C_1F, resp .
Let $A_b \equiv \omega_A \cap AB, A_c \equiv \omega_A \cap AC, A_b' \equiv \Omega_A \cap AB, A_c' \equiv \Omega_A \cap AC$ (define $B_a, B_c, C_a, C_b, B_a', B_c', C_a', C_b'$ similarly)
$A_4 \equiv \omega_A \cap \odot(O), B_4 \equiv \omega_B \cap \odot(O), C_4 \equiv \omega_C \cap \odot(O), A_5 \equiv \Omega_A \cap \odot(O), B_5 \equiv \Omega_B \cap \odot(O), C_5 \equiv \Omega_C \cap \odot(O)$.

Since the midline of isosceles trapezoid FDB_cB_a, DEC_aC_b is the radical axis of $\{\odot(I), \omega_B\}, \{\odot(I), \omega_C\}$, respectively ,
so the radical center of $\{\odot(I), \omega_B, \omega_C\}$ is the midpoint of $ID \Longrightarrow$ the midpoint D' of ID lie on the radical axis of $\{\omega_B, \omega_C\}$,
hence combine $A_2 \equiv B_4B_c \cap C_4C_b$ and $A_2B_c \cdot A_2B_4 = A_2C_b \cdot A_2C_4 \Longrightarrow A_2D'$ is the radical axis of $\{\omega_B, \omega_C\}$,
so the radical axis of $\{\omega_B, \omega_C\}$ passes through the midpoint of IT where T is the homothetic center of $\triangle DEF$ and $\triangle I_aI_bI_c$.

Similarly, the radical axis of $\{\omega_C, \omega_A\}$ passes through the midpoint of $IT \Longrightarrow X$ is the midpoint of IT (X_{999} in ETC) . ... $(\star)$

Since the midline of $FDB_c'B_a', DEC_a'C_b'$ is the radical axis of $\{\odot(I), \Omega_B\}, \{\odot(I), \Omega_C\}$, resp ,
so A_3 is the radical center of $\{\odot(I), \Omega_B, \Omega_C\} \Longrightarrow A_3$ lie on the radical axis of Ω_B and Ω_C ,
hence combine $A_2' \equiv B_5B_c' \cap C_5C_b'$ and $A_2'B_c' \cdot A_2'B_5 = A_2'C_b' \cdot A_2'C_5 \Longrightarrow A_2'A_3$ is the radical axis of $\{\Omega_B, \Omega_C\}$,
so the radical axis of $\{\Omega_B, \Omega_C\}$ passes through the homothetic center of $\triangle A_2'B_2'C_2'$ and $\triangle A_3B_3C_3$.

Similarly, $B_2'B_3$ is the radical axis of $\{\Omega_C, \Omega_A\} \Longrightarrow Y$ is the homothetic center of $\triangle A_2'B_2'C_2'$ and $\triangle A_3B_3C_3$.

Since I lie on the perpendicular bisector of B_1C_1, C_1A_1, A_1B_1, respectively ,
so I is the circumcenter of $\triangle A_1B_1C_1 \Longrightarrow I$ is the circumcenter of $\triangle A_3B_3C_3$.
($\because$ I is the circumcenter of $\triangle DEF$ and A_3, B_3, C_3 is the midpoint of A_1D, B_1E, C_1F, resp)
Notice that radius of $\odot(A_3B_3C_3)$ is $\frac{4R-r}{2}$ ($\because$ the radius of $\odot(A_1B_1C_1)$ is $4R$) $\Longrightarrow \frac{YI}{YO} = \frac{4R-r}{2R}$,
so combine $(\star) \Longrightarrow \frac{TI}{TI'} = \frac{r}{2R}$ (I' is the reflection of I in O) $\Longrightarrow \frac{XI}{XO} = \frac{r}{2R} \Longrightarrow X, Y$ are symmetry WRT O .

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, Jun 19, 2015, 6:40 pm

Luis González

#5 Jul 3, 2015, 4:44 am

Discussed before at http://www.artofproblemsolving.com/community/c6h392138.

Quick Reply

High School Olympiads

trigonometry

Reply

MR.129

#1 Mar 12, 2009, 9:53 pm

Point M is located inside triangle ABC. A_1,B_1,$ C_$ are foots of perpendiculars from M to BC,AC,AB respectively.Prove that:
$\cot \widehat{\mathrm{AA_1}B} + \cot \widehat{BB_1C} + \cot \widehat{CC_1A} = 0$

Luis González

#2 Jun 2, 2010, 8:46 am

M can be any point on the plane ABC if we consider oriented angles. Let $(u:v:w)$ be the barycentric coordinates of M with respect to $\triangle ABC$. Then the coordinates of A_1, B_1, C_1 (in Conway's notation) are: $A_1(0:a^2v+uS_C:a^2w+uS_B)$, $B_1(b^2u+vS_C:0:b^2w+vS_A)$ and $C_1(c^2u+wS_B:c^2v+wS_A:0)$.

Let α, β, γ denote the oriented angles between the cevians AA_1, BB_1, CC_1 and BC, CA, AB, respectively

$$\cot \alpha = \frac{-(uS_C+a^2v)S_B+(uS_B+a^2w)S_C}{-a^2S(u+v+w)} = \frac{vS_B-wS_C}{S(u+v+w)}$$

By cyclic permutation of elements we get $\cot \beta$ and $\cot \gamma$ as

$$\cot \beta = \frac{wS_C-uS_A}{S(u+v+w)} \ , \ \cot \gamma = \frac{uS_A-vS_B}{S(u+v+w)}$$

$$\Longrightarrow \cot \alpha + \cot \beta + \cot \gamma = \frac{vS_B-wS_C+wS_C-uS_A+uS_A-vS_B}{S(u+v+w)} = 0.$$

Quick Reply

High School Olympiads

geometry

Reply

phantranhuongth

#1 Feb 21, 2016, 9:53 am

Let ABC and a point M in ABC. Let $A', B', C' : MA' \perp BC, MB' \perp AC, ...$.
Prove that: $cot\angle AA'B + cot\angle BB'C + cot\angle CC'A = const$

Luis González

#2 Feb 21, 2016, 10:46 am

Let H be the orthocenter of $\triangle ABC$ and let D, E, F be the projections of A, B, C on $BC, CA, AB.$ Considering the segments $\overline{DA'}, \overline{EB'}, \overline{FC'}$ signed with regard to $D, E, F,$ we get

$$\cot \widehat{AA'B} = \frac{\overline{DA'}}{AD} = \frac{\overline{DA'}}{\frac{2[ABC]}{BC}} = \frac{\overline{DA'} \cdot BC}{2[ABC]} \Longrightarrow$$

$$\cot \widehat{AA'B} + \cot \widehat{BB'C} + \cot \widehat{CC'A} = \frac{\overline{DA'} \cdot BC + \overline{EB'} \cdot CA + \cdot \overline{FC'} \cdot AB}{2[ABC]}.$$

Using the result of the problem A generalization of Pythagorean theorem for M, H and their pedal triangles $\triangle A'B'C'$, $\triangle DEF$ WRT $\triangle ABC,$ it follows that the RHS of the latter expression equals 0 $\Longrightarrow$
$\cot \widehat{AA'B} + \cot \widehat{BB'C} + \cot \widehat{CC'A} = 0 = \text{const}.$

Luis González

#3 Feb 21, 2016, 10:30 pm

For another proof see http://www.artofproblemsolving.com/community/c6h264041.

Quick Reply

High School Olympiads

A generalization of Pythagorean theorem

Reply

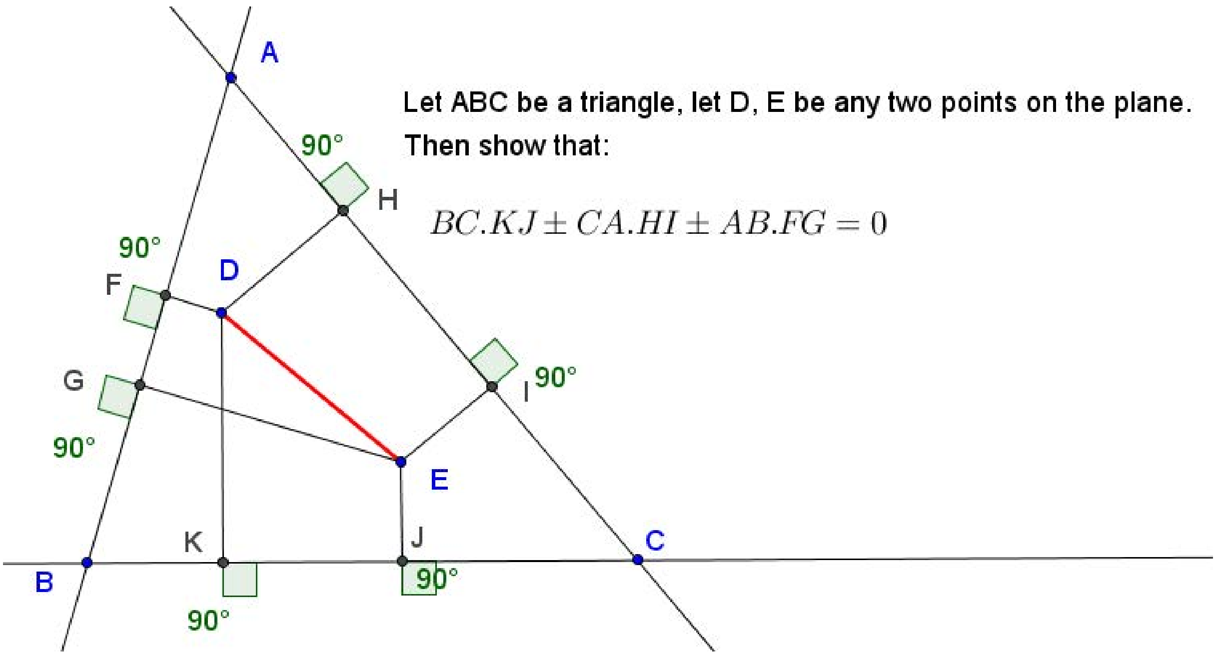
daothanhoai

#1 Sep 10, 2015, 12:54 am

I propose a generalization of Pythagorean theorem as follows:

When ABC is right at A, and EF is BC we have Pythagorean theorem

Attachments:



This post has been edited 1 time. Last edited by daothanhoai, Sep 10, 2015, 12:55 am

LeVietAn

#2 Sep 10, 2015, 1:43 am • 1

My Solution:

Lemma: Let Points A, B, X, Y, X', Y' such that $XX' \perp AB, YY' \perp AB, X' \in AB, Y' \in AB$ then $\overrightarrow{XY}.\overrightarrow{AB} = \overline{XY}.\overline{AB}$.

Proof: Well-know.

Back to main problem:

We have:

$\overline{BC}.\overline{KJ} + \overline{CA}.\overline{IH} + \overline{AB}.\overline{FG} = \overrightarrow{BC}.\overrightarrow{DE} + \overrightarrow{CA}.\overrightarrow{DE} + \overrightarrow{AB}.\overrightarrow{DE} = (\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB}).\overrightarrow{DE} = \overrightarrow{0}.\overrightarrow{DE} = 0.$

DONE

Luis González

#3 Sep 10, 2015, 1:46 am • 1

Let M_a, M_b, M_c denote the midpoints of $BC, CA, AB,$ respectively. Then we have $JB^2 - JC^2 = (JB + JC) \cdot (JB - JC) = BC \cdot (JB - JC) = \pm 2 \cdot BC \cdot JM_a$ and similarly $KB^2 - KC^2 = \pm 2 \cdot BC \cdot KM_a \Longrightarrow (JB^2 - JC^2) + (KB^2 - KC^2) = \pm 2 \cdot BC \cdot KJ.$ Now adding the cyclic expressions together, bearing in mind Carnot's theorem, we get $BC \cdot KJ \pm CA \cdot HI \pm AB \cdot FG = 0,$ as desired.

Quick Reply

High School Olympiads

Concurrent lines

Reply

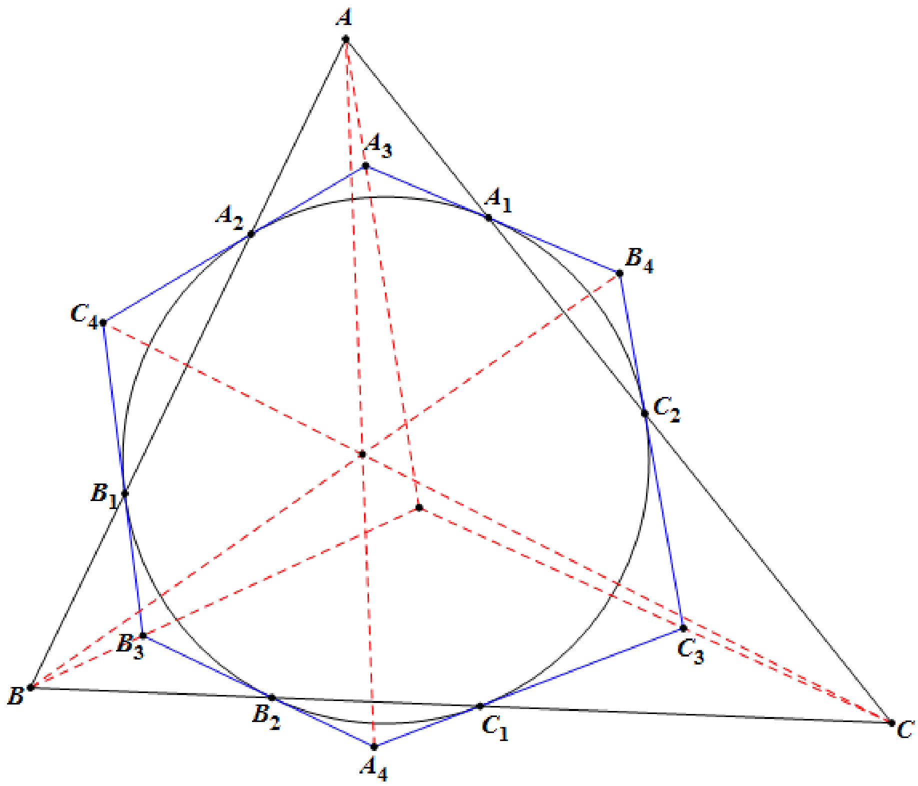
mjuk

#1 Feb 21, 2016, 12:51 am

Let ABC be a triangle and Ω a circle intersecting it's sides.
$B_2, C_1 = \Omega \cap BC$ such that B, B_2, C_1, C lie on line in that order.
$C_2, A_1 = \Omega \cap CA$ such that C, C_2, A_1, A lie on line in that order.
$A_2, B_1 = \Omega \cap AB$ such that A, A_2, B_1, B lie on line in that order.
Let $A_3 = A_1A_1 \cap A_2A_2, B_3 = B_1B_1 \cap B_2B_2, C_3 = C_1C_1 \cap C_2C_2$
Let $A_4 = B_2B_2 \cap C_1C_1, B_4 = C_2C_2 \cap A_1A_1, C_4 = A_2A_2 \cap B_1B_1$

(1) Prove that AA_3, BB_3, CC_3 are concurrent.
(2) Prove that AA_4, BB_4, CC_4 are concurrent.

Attachments:



This post has been edited 1 time. Last edited by mjuk, Feb 21, 2016, 1:02 am
Reason: picture edit

Luis González

#2 Feb 21, 2016, 1:16 am • 1

The concurrency of AA_4, BB_4, CC_4 is well-known; it's the perspector of Ω WRT $\triangle ABC$. For various proof you can see the thread concurrency of AA_3,BB_3,CC_3.

From the complete quadriltateral $A_1A_2B_1C_2$, we deduce that A, A_3 and the intersection $A_1B_1 \cap A_2C_2$ are collinear on the polar of $A_1A_2 \cap B_1C_2$ WRT Ω. Similarly BB_3 and CC_3 pass through $A_2B_2 \cap B_1C_1$ and $B_2C_2 \cap A_1C_1$. Now see the problem An ellipse and a point of concurrency.

Quick Reply

High School Olympiads

concurrency of AA_3,BB_3,CC_3

Reply

Source: it could have been posted before

shoki
#1 Nov 6, 2009, 10:11 pm

Let ΔABC be a triangle and (O) be a circle which intersects the sides BC, CA, AB at $\{A_1, A_2\}, \{B_1, B_2\}, \{C_1, C_2\}$. Also the intersection of the tangents at A_1, A_2 to (O) is A_3. The two other points (B_3, C_3) are defined similarly. Prove the concurrency of AA_3, BB_3, CC_3. I know a solution which is really ugly and I wanted to know if there exists other solutions. (Sorry if it has been posted before, I don't know the source).

Luis González
#2 Dec 15, 2009, 2:43 am • 1

Theorem: (O) is an arbitrary circle in the plane of $\triangle ABC$. P, Q, R are the poles of the sidelines BC, CA, AB with respect to (O). Then AP, BQ, CR concur.

Proof: $\delta(P,a)$ stands for the distance from P to the sideline BC and $\delta(P,b), \delta(P,c)$ are defined similarly. By 1st Salmon's theorem we have the following relations:

$$\frac{\overline{RO}}{\overline{PO}} = \frac{\delta(R,a)}{\delta(P,c)} \ , \ \frac{\overline{QO}}{\overline{PO}} = \frac{\delta(Q,a)}{\delta(P,b)} \Longrightarrow \frac{\delta(P,c)}{\delta(P,b)} = \frac{\delta(R,a)}{\delta(Q,a)} \cdot \frac{\overline{QO}}{\overline{RO}} \quad (1)$$

By cyclic exchange we have the relations:

$$\frac{\delta(Q,a)}{\delta(Q,c)} = \frac{\delta(P,b)}{\delta(R,b)} \cdot \frac{\overline{RO}}{\overline{PO}} \ . \ (2) \ , \ \frac{\delta(R,b)}{\delta(R,a)} = \frac{\delta(Q,c)}{\delta(P,c)} \cdot \frac{\overline{PO}}{\overline{QO}} \quad (3)$$

Multiplying (1), (2) and (3) together yields:

$$\frac{\delta(P,c)}{\delta(P,b)} \cdot \frac{\delta(Q,a)}{\delta(Q,c)} \cdot \frac{\delta(R,b)}{\delta(R,a)} = \frac{\delta(R,a)}{\delta(Q,a)} \cdot \frac{\delta(P,b)}{\delta(R,b)} \cdot \frac{\delta(Q,c)}{\delta(P,c)} \cdot \frac{\overline{QO}}{\overline{RO}} \cdot \frac{\overline{RO}}{\overline{PO}} \cdot \frac{\overline{PO}}{\overline{QO}}$$

$$\frac{\delta(P,c)}{\delta(P,b)} \cdot \frac{\delta(Q,a)}{\delta(Q,c)} \cdot \frac{\delta(R,b)}{\delta(R,a)} = 1 \Longrightarrow AP, BQ, CR$$ concur by Ceva's theorem.

shoki
#3 Dec 15, 2009, 2:56 am

I didn't check the computations but your idea was really NICE
ALL OF MY CONGRATS
That was a really nice idea to use this theorem (and what a useful theorem it is!)

Luis González
#4 Dec 15, 2009, 3:02 am

Thanks for your kind words dear shoki, indeed it is a very useful theorem but surprinsingly unknown as well. The problem is really nice. I was working on a solution since several days ago when a friend of mine sent me the generalization via private message and then I realized that you submitted it earlier.

Luis González

#5 Jan 9, 2010, 10:39 pm

We have an interesting concurrency when the subject circle concides with the 9-point circle.

Given a triangle $\triangle ABC$ and its 9-nine point circle (N). P, Q, R are the poles of the sidelines BC, CA, AB with respect to (N). Then AP, BQ, CR concur at the isotomic conjugate of X_{1078}.

Luis González

#6 Jan 10, 2010, 3:14 am

> Quote:
>
> Given a triangle $\triangle ABC$ and its 9-nine point circle (N). P, Q, R are the poles of the sidelines BC, CA, AB with respect to (N). Then AP, BQ, CR concur at the isotomic conjugate of X_{1078}.

The pole P of BC WRT (N) is the intersection of the tangent of (N) at the midpoint of BC and the perpendicular dropped from N to BC. Its anticomplement P' is then the intersection of the tangent τ_a of (O) at A and the perpendicular bisector ℓ_a of BC. Using barycentric coordinates with respect to $\triangle ABC$, we have:

$\ell_a \equiv (b^2-c^2)x + a^2y - a^2z = 0\ ,\ \tau_a \equiv c^2y + b^2z = 0$

$P' \equiv \ell_a \cap \tau_a \equiv (a^2(b^2+c^2) : b^2(c^2-b^2) : c^2(b^2-c^2))$

Therefore, $P(2b^2c^2 - b^4 - c^4 : a^2b^2 + a^2c^2 + b^2c^2 - c^4 : a^2b^2 + a^2c^2 + b^2c^2 - b^4)$

Coordinates of Q, R are found by cyclic permutation. Thus, AP, BQ, CR concur at

$$\left(\frac{1}{b^2c^2+c^2a^2+a^2b^2-a^4} : \frac{1}{b^2c^2+c^2a^2+a^2b^2-b^4} : \frac{1}{b^2c^2+c^2a^2+a^2b^2-c^4}\right)$$

Which is the isotomic conjugate of $X_{1078} \equiv (a^2b^2 + a^2c^2 + b^2c^2 - a^4)$

RSM

#7 Oct 21, 2011, 3:00 pm • 2

> Luis González wrote:
>
> **Theorem:** (O) is an arbitrary circle in the plane of $\triangle ABC$. P, Q, R are the poles of the sidelines BC, CA, AB with respect to (O). Then AP, BQ, CR concur.

Here is a synthetic solution for this theorem:-
Its enough to prove that intersection of the polars of A, B, C wrt (O) with BC, CA, AB are collinear. Suppose, A', B', C' are the inverse points of A, B, C wrt (O). A_1, B_1, C_1 are located on BC, CA, AB such that $A_1A' \perp OA$ and similar for the others. We have to prove that A_1, B_1, C_1 are collinear. Suppose, AA_2, BB_2, CC_2 are the altitudes of ABC. Consider the circles $\odot AA'A_2, \odot BB'B_2, \odot CC'C_2$. Clearly, the orthocenter H of ABC wrt these circles has same power and so has O. So these circles are co-axial. Their centers are midpoints of AA_1, BB_1, CC_1 and so they are collinear and this implies A_1, B_1, C_1 are collinear. So done.

swaqar

#8 Nov 4, 2011, 5:15 pm • 3

A friend of mine from New Zealand(Malcolm Granville, a former participant of IMO) remarked to me that he discovered this theorem while working on a special case of Nine Point Circle which appeared in some Romanian TST, as also remarked by Luis in his second post! The generalized statement, as he puts, is the following:

Given a triangle ABC and a conic ω. Let the poles of the lines of the sides be A', B' and C' corresponding to BC , AB and AC. Then, the lines AA' , BB' and CC' are concurrent.

Proof: Let the points $AB \cap A'B' = Z$, $AA' \cap BB' = P$, $AB \cap CC' = U$ and $A'B' \cap CC' = U'$ be in the projective plane.

We have $(A, B; Z, U) = (CA, CB; CZ, CU)$ and taking the poles of the lines in the last bundle, we have $(CA, CB; CZ, CU) = (B', A'; U', Z) = (A', B', Z, U') = (PA', PB', PZ, PU')$ and so the perspectivity from P sends A, B , Z and U to A' , B' , Z and U' respectively or P, U and U' are collinear which means $P \in CC'$.

The existence of Gergonne and Symmedian Point can be proved by this theorem, as observed by him.

VHCR

#9 Dec 12, 2011, 10:49 am

> *swaqar wrote:*
> We have $(A,B;Z,U)=(CA,CB;CZ,CU)$ and taking the poles of the lines in the last bundle, we have $(CA,CB;CZ,CU)=(B',A';U',Z)$.

Could you explain this part?

Dragonboy

#10 Jun 17, 2012, 10:27 am

> *RSM wrote:*
> Clearly, the orthocenter H of ABC wrt these circles has same power and so has O. So these circles are co-axial.

I'm not understanding this part. H can be the radical center of those three circles but how can you claim that those three circles are co axial?

RSM

#11 Jun 17, 2012, 11:32 am

I wrote that, both H and O have same power wrt the 3 circles. So clearly, they are co-axial.

Dragonboy

#12 Jun 17, 2012, 1:13 pm

> *RSM wrote:*
> I wrote that, both H and O have same power wrt the 3 circles. So clearly, they are co-axial.

Opps. Sorry i didn't notice that before

Blacklord

#13 Sep 26, 2015, 9:07 pm

Is there any suloution without polar??

Quick Reply

High School Olympiads

An ellipse and a point of concurrency

Petry

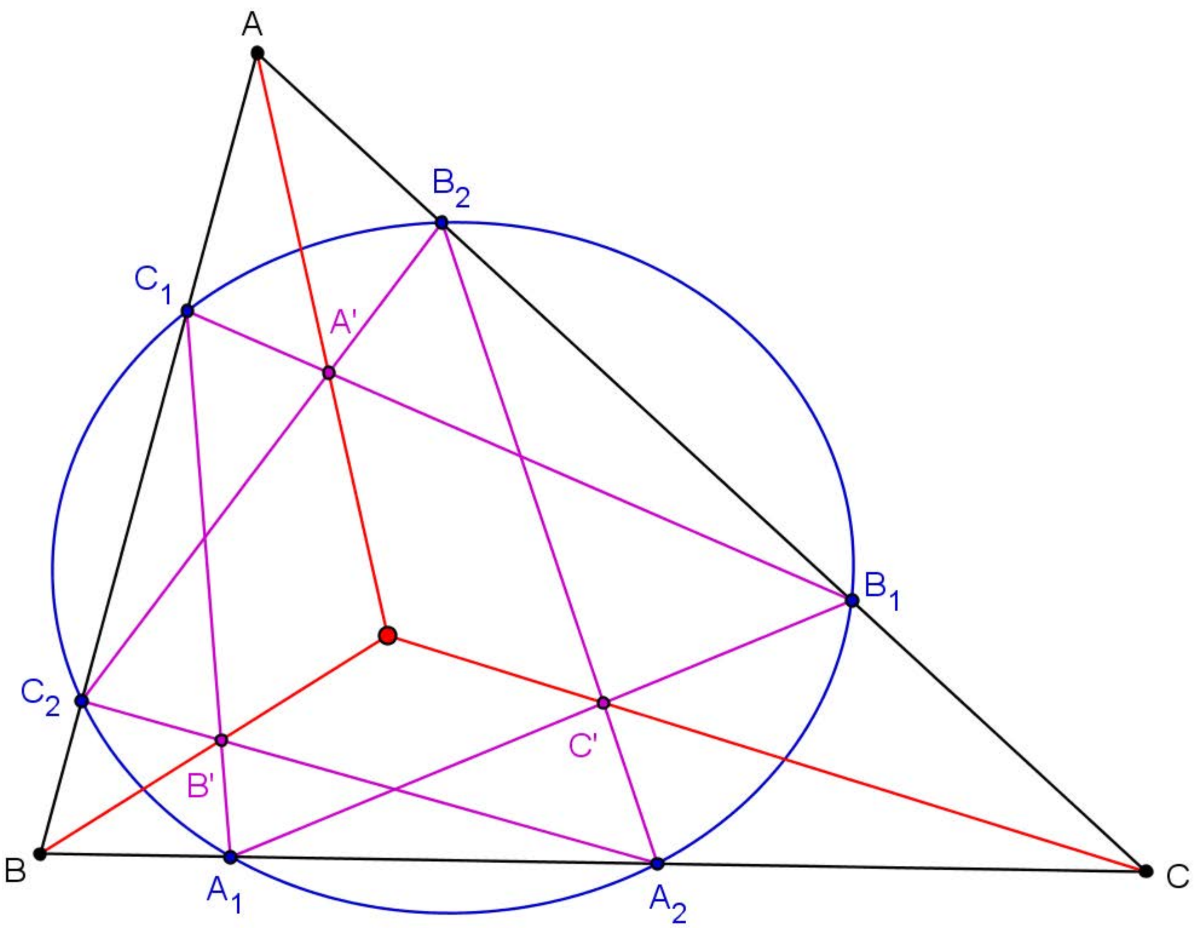
#1 Jul 21, 2010, 2:13 pm

Hello!

An ellipse intersects the sides of a triangle ABC at the points A_1, A_2, B_1, B_2, C_1, C_2
($A_1, A_2 \in BC$; $B_1, B_2 \in CA$; $C_1, C_2 \in AB$). Let's consider
$\{A'\} = B_1C_1 \cap B_2C_2$, $\{B'\} = C_1A_1 \cap C_2A_2$ and $\{C'\} = A_1B_1 \cap A_2B_2$.
Prove that the lines AA', BB' and CC' are concurrent.

Best regards,
Petrisor Neagoe :)

Attachments:



Luis González

#2 Jul 21, 2010, 8:57 pm

Substitute the ellipse for a general conic $\mathcal{H}.$ By Pascal theorem for the non-convex hexagon $C_1B_2A_1C_2B_1A_2$, the intersections $A_0 \equiv C_1B_2 \cap C_2B_1, C_0 \equiv B_2A_1 \cap B_1A_2$ and $B_0 \equiv A_1C_2 \cap A_2C_1$ are collinear. On the other hand, the line passing through $A \equiv B_1B_2 \cap C_1C_2$ and $A' \equiv C_2B_2 \cap C_1B_1$ is the polar of A_0 WRT $\mathcal{H}.$ Likewise, the lines BB' and CC' are the polars of B_0, C_0 WRT $\mathcal{H}.$ Since A_0, B_0, C_0 are collinear, then AA', BB', CC' concur at the pole of $A_0B_0C_0$ WRT $\mathcal{H}.$

Petry
#3 Jul 21, 2010, 11:10 pm

Dear Luis, thanks for your nice solution!

My solution:

First, it's known that
If a conic meet the sides BC, CA, AB of a triangle in three pairs of points A_1, A_2; B_1, B_2; C_1, C_2 then $\frac{A_1C}{A_1B}\cdot\frac{A_2C}{A_2B}\cdot\frac{B_1A}{B_1C}\cdot\frac{B_2A}{B_2C}\cdot\frac{C_1B}{C_1A}\cdot\frac{C_2B}{C_2A}=1$.

$\{A"\}=AA'\cap BC$, $\{B"\}=BB'\cap CA$ and $\{C"\}=CC'\cap AB$.
See the proposed problem posted here
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=255004.

So,
$\frac{A"B}{A"C}=\frac{\frac{C_1B}{C_1A}-\frac{C_2B}{C_2A}}{\frac{B_2C}{B_2A}-\frac{B_1C}{B_1A}}=\frac{\frac{C_1B\cdot C_2A-C_2B\cdot C_1A}{C_1A\cdot C_2A}}{\frac{B_2C\cdot B_1A-B_1C\cdot B_2A}{B_1A\cdot B_2A}}$ (1)

$\frac{B"C}{B"A}=\frac{\frac{A_1C}{A_1B}-\frac{A_2C}{A_2B}}{\frac{C_2A}{C_2B}-\frac{C_1A}{C_1B}}=\frac{\frac{A_1C\cdot A_2B-A_2C\cdot A_1B}{A_1B\cdot A_2B}}{\frac{C_2A\cdot C_1B-C_1A\cdot C_2B}{C_1B\cdot C_2B}}$ (2)

$\frac{C"A}{C"B}=\frac{\frac{B_1A}{B_1C}-\frac{B_2A}{B_2C}}{\frac{A_2B}{A_2C}-\frac{A_1B}{A_1C}}=\frac{\frac{B_1A\cdot B_2C-B_2A\cdot B_1C}{B_1C\cdot B_2C}}{\frac{A_2B\cdot A_1C-A_1B\cdot A_2C}{A_1C\cdot A_2C}}$ (3)

(1),(2),(3)$\Rightarrow \frac{A"B}{A"C}\cdot\frac{B"C}{B"A}\cdot\frac{C"A}{C"B}=\frac{B_1A\cdot B_2A}{C_1A\cdot C_2A}\cdot\frac{C_1B\cdot C_2B}{A_1B\cdot A_2B}\cdot\frac{A_1C\cdot A_2C}{B_1C\cdot B_2C}=$

$=\frac{A_1C}{A_1B}\cdot\frac{A_2C}{A_2B}\cdot\frac{B_1A}{B_1C}\cdot\frac{B_2A}{B_2C}\cdot\frac{C_1B}{C_1A}\cdot\frac{C_2B}{C_2A}=1\Rightarrow\frac{A"B}{A"C}\cdot\frac{B"C}{B"A}\cdot\frac{C"A}{C"B}=1\Rightarrow$

$\Rightarrow$ the lines AA', BB' and CC' are concurrent.

Best regards,
Petrisor Neagoe

P.S.
Let ABC be a triangle, $A_1B_1C_1$ is the P-cevian triangle of ABC and $A_2B_2C_2$ is the Q-cevian triangle of ABC. If $\{A'\}=B_1C_1\cap B_2C_2$, $\{B'\}=C_1A_1\cap C_2A_2$ and $\{C'\}=A_1B_1\cap A_2B_2$ then prove that the lines AA', BB' and CC' are concurrent.

For this problem, we can write a solution without a conic as tool.

Quick Reply

High School Olympiads

Radical Center is the Symmedian Point

Reply

ABCDE

#1 Feb 19, 2016, 11:18 pm • 1

Let ABC be a triangle and let D be the foot of the altitude from A to BC. Let X and Y be points on AB and AC different from A with $DX = DY = DA$. Let the circumcircle of $BCXY$ be ω_A, and define ω_B and ω_C similarly. Prove that the radical center of ω_A, ω_B, and ω_C is the symmedian point of ABC.

TelvCohl

#2 Feb 19, 2016, 11:56 pm • 2

Let $\triangle DEF$ be the orthic triangle of $\triangle ABC$. Let ω_B cuts BA, BC at B_a, B_c, respectively (define C_a and C_b similarly). From $\measuredangle FB_aE = \measuredangle EBF = \measuredangle ECF = \measuredangle FC_aE$ we get E, F, B_a, C_a are concyclic, so EF and B_aC_a are anti-parallel WRT $\angle A \Longrightarrow B_aC_a \parallel BC$. On the other hand, if the tangent τ_B of $\odot(ABC)$ passing through B cuts ω_C again at T_c, then $\measuredangle BT_cC_a = \measuredangle BAC \Longrightarrow T_cC_a \parallel BC$, so T_c lies on B_aC_a. Analogously, the intersection T_b of the tangent τ_C of $\odot(ABC)$ passing through C and B_aC_a lies on ω_B. Let $T \equiv \tau_B \cap \tau_C$. From $TB \cdot TT_c = TC \cdot TT_b$ we get T lies on the radical axis of ω_B and ω_C, so we conclude that the radical axis of ω_B, ω_C is the A-symmedian AT of $\triangle ABC$.

Luis González

#3 Feb 19, 2016, 11:57 pm • 2

Let E and F be the feet of the altitudes on CA, AB. $\odot(E, EB)$ cuts BC, BA again at R, S and $\odot(F, FC)$ cuts CA, CB at T, U. Since $\angle ESF = \angle EBA = \angle FCA = \angle FTE \Longrightarrow EFTS$ is cyclic $\Longrightarrow \angle TSF = \angle TEF = \angle ABC \Longrightarrow BC \parallel TS$. Thus if TS cuts ω_B, ω_C, again at S', T', we get $\angle RCS' = \angle RSS' = \angle SRC = \angle BAC$ and similarly $\angle UBT' = \angle BAC$. Hence if $X \equiv BT' \cap CS'$, then XB, XC are tangents of $\odot(ABC) \Longrightarrow BCS'T'$ is an isosceles trapezoid $\Longrightarrow XB \cdot XT' = XC \cdot XS' \Longrightarrow X$ has equal power WRT $\omega_B, \omega_C \Longrightarrow$ A-symmedian AX is the radical axis of ω_B, ω_C. Similarly the B-symmedian and C-symmedian are the radical axes ω_C, ω_A, and ω_A, ω_B and the conclusion follows.

mjuk

#4 Feb 20, 2016, 12:12 am • 1

Let ω_B intersect AB at Z, let radical axis of ω_A and ω_B intersect AB at S.

Let S' be intersection of C-symmedian and AB, then $\frac{S'A}{S'B} = \frac{b^2}{c^2}$, or $S'B = \frac{cb^2}{a^2+b^2}$ and $S'A = \frac{ca^2}{a^2+b^2}$

From sine law in $\triangle BDX$:

$\frac{BX}{\sin(2B-90)} = \frac{DX}{\sin(B)} = \frac{AD}{\sin(B)} = \frac{c}{\sin(90^\circ)} = c \rightarrow BX = -c\cos(2B)$, similarily $AZ = -c\cos(2A)$

From power of point: $SA(SA+AZ) = SB(SB+BX)$

Suffices to prove: $S'A(S'A+AZ) = S'B(S'B+BX)$, because this will imply $S = S'$.

$\leftrightarrow \frac{b^4}{(a^2+b^2)^2} - b^2c^2a^2 + b^2\cos(2A) = \frac{a^4c^2}{(a^2+b^2)^2} - \frac{a^2c^2}{a^2+b^2}$

$\leftrightarrow b^4 - b^4\cos(2A) - a^2b^2\cos(2A) = a^4 - a^4\cos(2B) - a^2b^2\cos(2A)$

$\leftrightarrow b^4\sin^2(A) + a^2b^2\sin^2(A) = a^4\sin^2(B) + a^2b^2\sin^2(B)$

But sine law implies that $b^4\sin^2(A) = a^2b^2\sin^2(B)$ and $a^4\sin^2(B) = a^2b^2\sin^2(A)$ hence $S = S'$.

This post has been edited 2 times. Last edited by Luis González, Feb 21, 2016, 12:17 am
Reason: Unhiding solution

buratinogigle

#5 Feb 20, 2016, 11:28 am

General problem. Let ABC be a triangle with altitudes AD, BE, CF. X, Y, Z divide AD, BE, CF in the same ratio. Projections of X on CA, AB and B, C lie on circle (ω_a). Similarly, we have circles (ω_b), (ω_c). Prove that Lemoine point of

Projections of X on CA, AB and B, C lie on circle (ω_a). Similarly, we have circles $(\omega_b), (\omega_c)$. Prove that Lemoine point of ABC is radical center of $(\omega_a), (\omega_b), (\omega_c)$.

Luis González

#6 Feb 21, 2016, 12:31 am • 1

> *buratinogigle wrote:*
>
> **General problem.** Let ABC be a triangle with altitudes $AD, BE, CF. X, Y, Z$ divide AD, BE, CF in the same ratio. Projections of X on CA, AB and B, C lie on circle (ω_a). Similarly, we have circles $(\omega_b), (\omega_c)$. Prove that Lemoine point of ABC is radical center of $(\omega_a), (\omega_b), (\omega_c)$.

More general: Let $\{U, R\} \in BC, \{Q, T\} \in CA, \{S, P\} \in AB,$ such that $PQRSTU$ is a Tucker hexagon in that order with ST, UP, QR parallel to $BC, CA, AB,$ respectively. Then the symmedian point of $\triangle ABC$ is the radical center of $\odot(BCQP), \odot(CASR)$ and $\odot(ABUT)$.

The proofs given in the posts #2 and #3 still works for this general configuration.

Dukejukem

#7 Feb 22, 2016, 10:36 pm • 1

> *ABCDE wrote:*
>
> Let ABC be a triangle and let D be the foot of the altitude from A to BC. Let X and Y be points on AB and AC different from A with $DX = DY = DA$. Let the circumcircle of $BCXY$ be ω_A, and define ω_B and ω_C similarly. Prove that the radical center of ω_A, ω_B, and ω_C is the symmedian point of ABC.

Let H be the projection of D onto AB. Note that X is the reflection of A in H; hence, $\text{pow}(A, \omega_A) = 2 \cdot AH \cdot AB = 2 \cdot AD^2$ where the last step follows from $\triangle ADH \sim \triangle ABD$.

Set $\lambda \equiv a^2 + b^2 + c^2$ and $\Delta \equiv [ABC]$. Let $\Omega \equiv \odot(ABC)$ and define $\mathcal{F} : \mathbb{R}^2 \to \mathbb{R}$ by $\mathcal{F}(X) \equiv \text{pow}(X, \omega_A) - \text{pow}(X, \Omega)$. It's well known that $\mathcal{F}$ is linear. Then since $K(a^2 : b^2 : c^2)$ are the barycentric coordinates of the symmedian point, we have

$$\mathcal{F}(K) = \frac{a^2}{\lambda}\mathcal{F}(A) + \frac{b^2}{\lambda}\mathcal{F}(B) + \frac{c^2}{\lambda}\mathcal{F}(C)$$

$$= \frac{a^2}{\lambda}\mathcal{F}(A)$$

$$= \frac{a^2 \text{pow}(A, \omega_A)}{\lambda}$$

$$= \frac{a^2\left(2 \cdot AD^2\right)}{\lambda}$$

$$= \frac{8\Delta^2}{\lambda}.$$

Hence, $\text{pow}(K, \omega_A) = \dfrac{8\Delta^2}{\lambda} + \text{pow}(K, \Omega)$, which is symmetric in a, b, c. Thus, K has equal power WRT $\omega_A, \omega_B, \omega_C$, as desired.

This post has been edited 2 times. Last edited by Dukejukem, Feb 22, 2016, 10:38 pm

Quick Reply

High School Olympiads

Monge Lines

Reply

nsato

#1 Feb 19, 2016, 10:31 pm

Given triangle ABC, let ϵ_A, ϵ_B, ϵ_C be the excircles, let μ_A, μ_B, μ_C be the mixtililinear incircles, and let ν_A, ν_B, ν_C be the mixtilinear excircles. Let P_1 be the external centre of similitude of ϵ_B and ϵ_C, etc., and let Q_1 be the external centre of similitude of μ_B and μ_C, etc. Then by Monge-d'Alembert, P_1, P_2, P_3 are collinear, and Q_1, Q_2, Q_3 are collinear.

(a) Show that P_1 is also the external centre of similitude of ν_B and ν_C.

(b) Show that lines $P_1P_2P_3$ and $Q_1Q_2Q_3$ are parallel.

This post has been edited 1 time. Last edited by nsato, Feb 22, 2016, 4:04 am
Reason: Correcting post.

TelvCohl

#2 Feb 19, 2016, 11:11 pm • 1

(a)

Let I be the incenter of $\triangle ABC$ and let I_b, I_c, J_b, J_c be the center of $\epsilon_B, \epsilon_C, \nu_B, \nu_C$, respectively. From Mannheim's theorem we get the tangency point Y of ν_B with BC lies on the perpendicular from I_b to BI. Similarly, we can prove $I_cZ \perp CI$ where $Z \equiv \nu_C \cap BC$. Since $T \equiv YI_b \cap ZI_c$ is the antipode of I in $\odot(I_bII_c)$, so notice BC and I_bI_c are anti-parallel WRT $\angle BIC$ we get $IT \perp BC$, hence from Desargue theorem ($\triangle YI_bJ_b$ and $\triangle ZI_cJ_c$) we conclude that P_1 lies on J_bJ_c. i.e. P_1 is also the exsimilicenter of ν_B and ν_C

Luis González

#4 Feb 20, 2016, 12:03 am • 1

(a) Clearly P_1, P_2, P_3 are the feet of the external bisectors of $\angle BAC, \angle ABC, \angle BCA$, i.e. $\overline{P_1P_2P_3}$ is the trilinear polar of I WRT $\triangle ABC$, which coincides with the polar of the isogonal conjugate L of the Gergonne point Ge WRT $\triangle ABC$ (well-known). If ν_A, ν_B, ν_C touch the circumcircle (O) at L_1, L_2, L_2, then $\triangle L_1L_2L_3$ is the circumcevian triangle of $L \Longrightarrow$ the perspectrix of $\triangle ABC$ and $\triangle L_1L_2L_3$ is the polar $\overline{P_1P_2P_3}$ of L WRT $(O) \Longrightarrow P_1 \in L_2L_3$, etc $\Longrightarrow P_1$ is also exsimilicenter of $\nu_B \sim \nu_C$.

(b) I think you mean that $\overline{P_1P_2P_3} \parallel \overline{Q_1Q_2Q_3}$. If μ_A, μ_B, μ_C touch (O) at T_1, T_2, T_3, then $\triangle T_1T_2T_3$ is the circumcevian triangle of the isogonal conjugate T of the Nagel point Na WRT $\triangle ABC \Longrightarrow \overline{Q_1Q_2Q_3}$ is the polar of T WRT $(O) \Longrightarrow$ $\overline{OIT} \perp \overline{Q_1Q_2Q_3}$ but $\overline{P_1P_2P_3} \perp \overline{OIL} \Longrightarrow (\overline{P_1P_2P_3} \parallel \overline{Q_1Q_2Q_3}) \perp OI$.

nsato

#5 Feb 22, 2016, 4:09 am

Of course, I did mean parallel in part (b). Thank you, TelvCohl and Luis. I was a bit afraid that part (a) was trivial, but it seems that it is not so obvious.

Quick Reply

High School Olympiads

Nice geometry

langkhach11112

#1 Feb 18, 2016, 11:09 am

D and E lie on side BC of triangle ABC such that $\angle BAD = \angle CAE$. M, N, H, K are perpendicular projections of B, C, D, E on AD, AE, AB, AC. Prove that A and the midpoint of segment HK, MN are collinear.

Luis González

#2 Feb 19, 2016, 4:34 am • 1

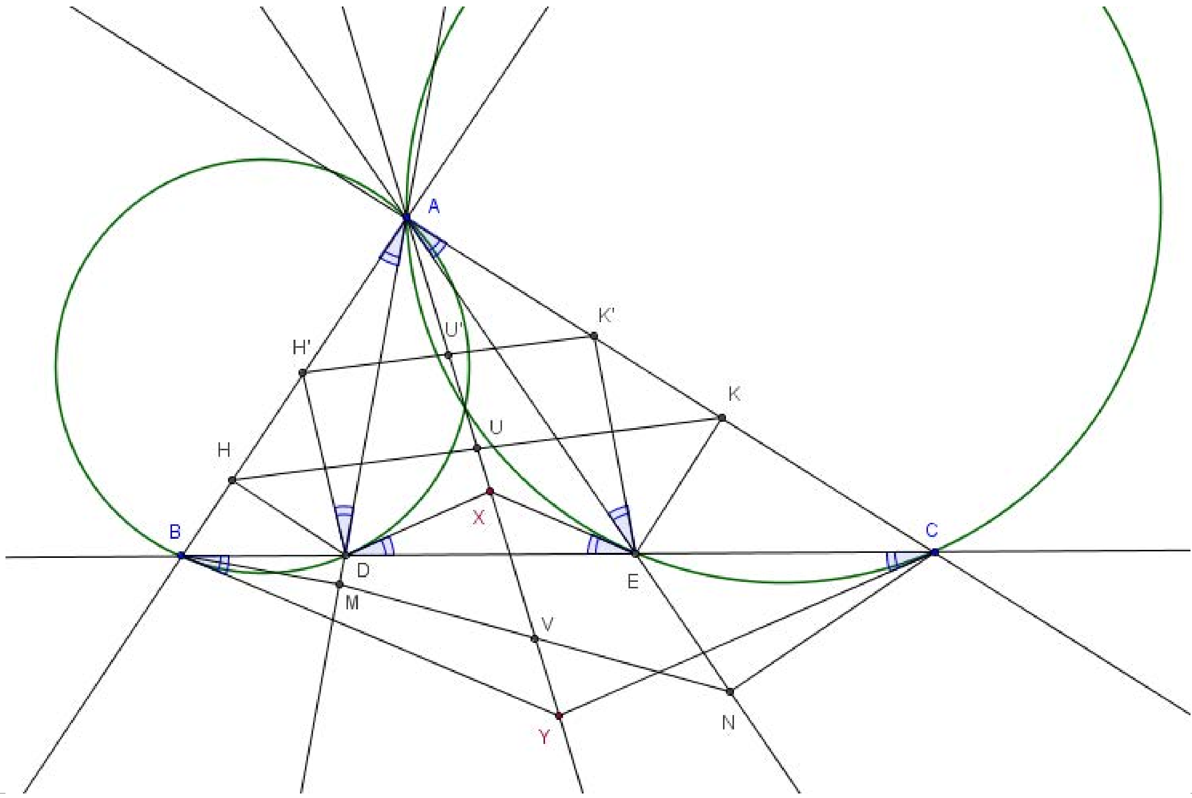
Lemma: In the plane of $\triangle ABC$, similar isosceles triangles $\triangle MBC, \triangle NCA, \triangle LAB$, with apices M, N, L, are constructed such that A, M are on the same side of BC and N, L are outside of $\triangle ABC$. Then AM passes through the midpoint of $\overline{NL}$. Likewise if N, L are constructed inwardly and M outwardly, then AM passes through the midpoint of $\overline{NL}$.

Proof: Since $\triangle MBC \sim \triangle LBA$ are spirally similar, then it follows that $\triangle ABC \sim \triangle LBM$ are also spirally similar $\Longrightarrow$ $\frac{LM}{AC} = \frac{BM}{BC}$. But from $\triangle NAC \sim \triangle MBC$, we have $\frac{AC}{BC} = \frac{AN}{BM} \Longrightarrow \frac{AN}{BM} = \frac{LM}{BM} \Longrightarrow AN = LM$. Analogously $AL = NM$, which means that $ANML$ is a parallelogam $\Longrightarrow$ AM passes through the midpoint of $\overline{NL}$.

Back to the main problem, denote $\angle BAD = \angle CAE = \theta$ and construct X and Y, such that $\angle XDE = \angle XED = \theta$ and $\angle YBC = \angle YCB = \theta$, such that X, Y are on different sides of BC and A, X are on the same side of BC. Clearly XD, XE are tangents of $\odot(ABD), \odot(ACE)$ and similarly YB, YC are tangents of $\odot(ABD), \odot(ACE)$, thus since $XD = XE$ and $YB = YC$, we deduce that A, X, Y are collinear on the radical axis of $\odot(ABD)$ and $\odot(ACE)$.

Let the perpendicular bisectors of AD, AE cut AH, AK at H', K'. Since $\triangle ADH \cup H' \sim \triangle AEK \cup K'$, we get $AH : AH' = AK : AK' \Longrightarrow HK \parallel H'K' \Longrightarrow AU$ passes through the midpoint U' of $\overline{H'K'}$. Using the lemma for $\triangle ADE$ together with the similar isosceles $\triangle XDE \sim \triangle K'EA \sim \triangle H'AD$, we get that $U' \in AX$, i.e. $U \in AX$. Similarly using the lemma for $\triangle ABC$, we get $V \in AY \Longrightarrow A, U, V$ are collinear, as desired.

Attachments:

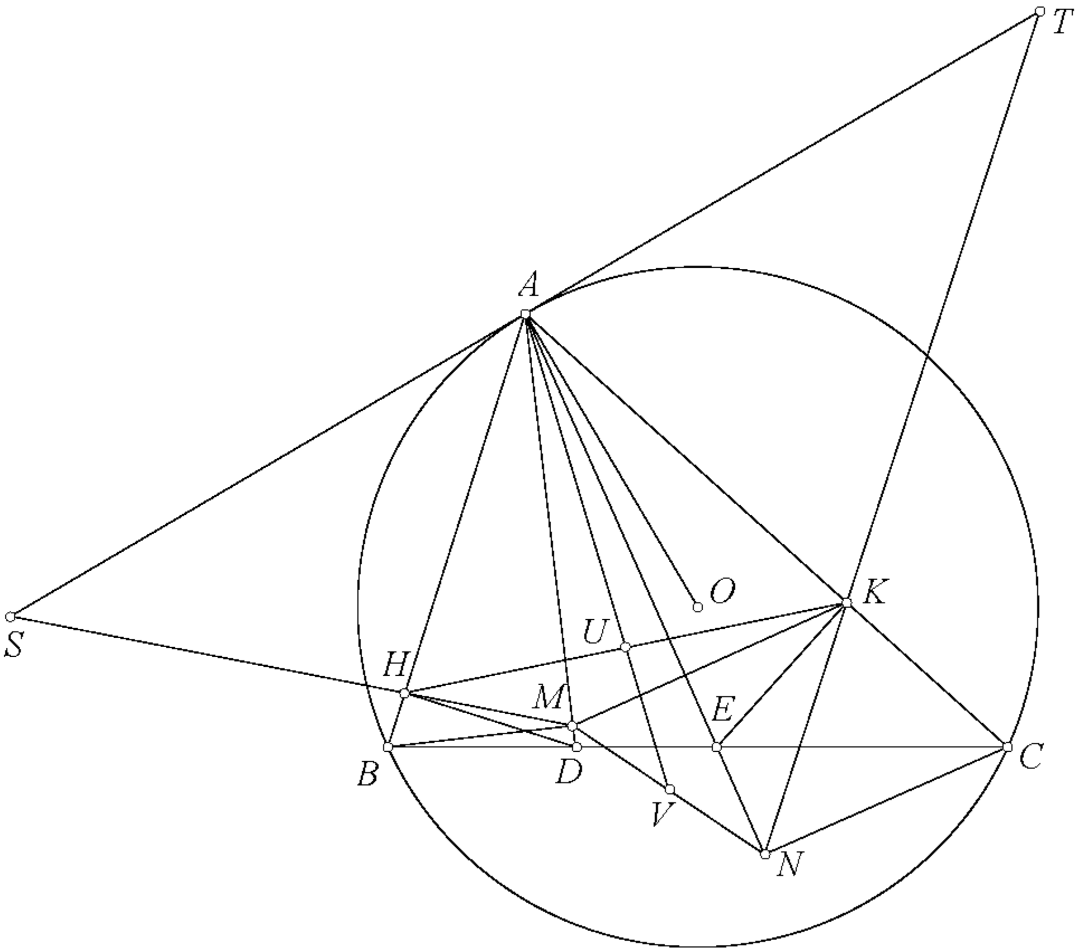


buratinogigle

#3 Feb 19, 2016, 12:43 pm • 1

Let (O) be circumcircle of triangle ABC. Tangent at A of (O) cut MH, NK at S, T. We easily seen $BHMD, CNEK$ are cyclic. Hence, $\angle SAH = \angle ACB = \angle ANK$ and $\angle S = \angle AHM - \angle HAS = \angle ADB - \angle ACB = \angle DAC = \angle EAB = \angle AEC - \angle ABC = \angle AKN - \angle TAC = \angle T$ Thus, triangles HAS and ANT are similar. Similarly, triangles KAT and AMS are similar. We get $SH.NT = AS.AT = SM.KT$ or $\frac{SH}{SM} = \frac{TK}{TN}$. From, $\angle S = \angle DAC = \angle EAB = \angle T$. We have triangles AMS and KMA are similar. We get $\frac{AS}{AK} = \frac{MS}{MA} = \frac{AT}{AK}$ so A is midpoint of ST. Combine $\frac{SH}{SM} = \frac{TK}{TN}$, follow ERIQ lemma then A and midpoints of HK, MN are collinear.

Attachments:



buratinogigle

#4 Feb 20, 2016, 11:39 am

General problem. Let ABC be a triangle with E, F lie on BC such that $\angle EAB = \angle FAC$. M, N lies on CA, AB such that $\angle FMA = \angle ENA$. AE, AF cut circles $(ENB), (FNC)$ again at P, Q. Prove that midpoints of MN, PQ and A are collinear.

My solution is still available for this case.

Quick Reply

High School Olympiads

Euler's Circle

Reply

AB2016

#1 Feb 18, 2016, 11:39 pm

Let ABC be an aqute-angle triangle. H is its ortochenter and E Euler's circle's centre. The bisectors of ACH and ABH meet in A_1. Analogously are defined B_1 and C_1.
Prove that

$$EA_1 + EB_1 + EC_1 = p - \frac{3R}{2},$$

where R, p are circumscribed circle's radius and the semiperimeter of ABC triangle.

This post has been edited 1 time. Last edited by AB2016, Feb 19, 2016, 1:37 am
Reason: Mistake

Luis González

#2 Feb 19, 2016, 12:39 am

That expression does not hold for all acute $\triangle ABC$. For instance for $a, b, c > R$ (E on the intersection of the 3 discs with diameters BC, CA, AB) the expression is $EA_1 + EB_1 + EC_1 = p - \frac{3}{2}R$.

Let $\triangle XYZ$ be the orthic triangle of $\triangle ABC$ and let U, V, W be the midpoints of BC, YZ, AH, resp. Cleary U, V, W, E are collinear on the perpendicular bisector of $\overline{YZ}$. In the cyclic $AYHZ$, it's known that the bisectors of $\angle ACH$ and $\angle ABH$ meet orthogonally on its Newton line $UVW \Longrightarrow A_1$ is the intersection of $\overline{UV}$ with the circle with diameter $\overline{BC} \Longrightarrow$
$EA_1 = UA_1 - UE = \frac{1}{2}(a - R)$. Similarly we have $EB_1 = \frac{1}{2}(b - R)$ and $EC_1 = \frac{1}{2}(c - R) \Longrightarrow$
$EA_1 + EB_1 + EC_1 = p - \frac{3}{2}R$.

AB2016

#3 Feb 19, 2016, 1:37 am

Yea, sorry, the problem was wrong.

Quick Reply

High School Olympiads

Orthocenter lies on PQ

Reply

Source: OWN

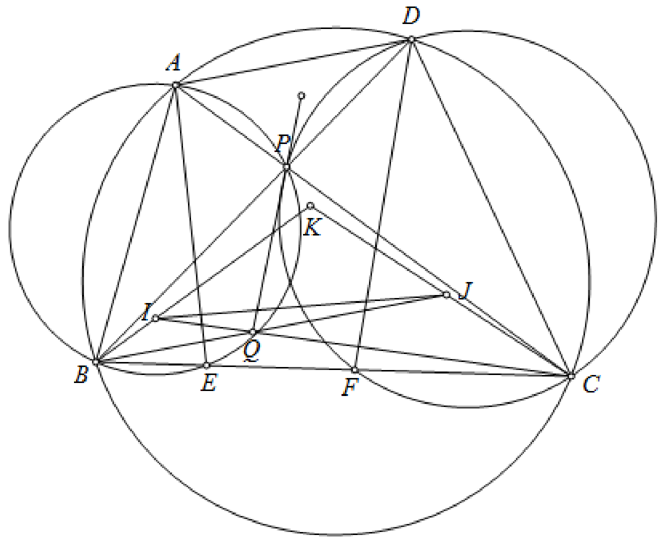
LeVietAn

#1 Feb 17, 2016, 11:19 am

Dear Mathlinkers,

Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at P. Suppose that the segment BC intersects the circumcircles of triangles PAB and PCD again at E and F, respectively. Let I and J respectively be the incenters of the incircles of the triangles EAB and FCD. Assume that BJ intersects CI at Q, and BI intersects CJ at K. Prove that the orthocenter of triangle KIJ lies on PQ.

Attachments:



Luis González

#2 Feb 18, 2016, 6:51 am • 1

Let I_1, I_2 be the incenters of $\triangle ABC, \triangle ABP$ and let J_1, J_2 be the incenters of $\triangle DCB, \triangle DCP. B, C, I_1, J_1$ lie on a same circle centered at the midpoint M of the arc BC of $\odot(ABCD)$ and likewise A, B, I, I_2 and B, C, J, J_2 lie respectively on circles centered at the midpoints of the arcs AB and CD of $\odot(PAB)$ and $\odot(PCD)$. Clearly $\overline{I_2PJ_2} \parallel I_1J_1$, and since $\triangle MI_1J_1$ is M-isosceles, then $I_1J_1J_2I_2$ is an isosceles trapezoid. Furthermore if $S \equiv CJ \cap II_2$ and $T \equiv BI \cap JJ_2$, we get $\angle I_1I_2S = \angle ABI = \angle CBI = \angle I_1J_1S \Longrightarrow S \in \odot(I_1J_1J_2I_2)$ and similarly $T \in \odot(I_1J_1J_2I_2) \Longrightarrow$ $\angle J_1SI_2 = \angle I_1TJ_2 \Longrightarrow IJTS$ is cyclic $\Longrightarrow \angle TI_1J_1 = \angle TSJ = \angle TIJ \Longrightarrow I_1J_1 \parallel IJ \Longrightarrow BIJC$ is cyclic.

Let H be the orthocenter of $\triangle KIJ$ and let HI, HJ cut BD, AC at X, Y, respectively. From $IH \perp CJ$, we get $\angle BXI = \angle 90^\circ - \angle BCJ - \angle DBC = \angle BAI \Longrightarrow ABIX$ is cyclic and likewise $DCJY$ is cyclic. Since $\angle AXD = \angle AIK = 90^\circ - \frac{1}{2}\angle APB = \angle DJK = \angle AYD$, then $AXYD$ is cyclic $\Longrightarrow$ $\angle DXY = \angle DAC = \angle DBC \Longrightarrow XY \parallel BC$. If HI, HJ cut BJ, CI at U, V, then $\triangle BIQ \cup U \sim \triangle CJQ \cup V$ (because $\angle BIU = \angle BAX = \angle CDY = \angle CJV) \Longrightarrow QU : QB = QV : QC \Longrightarrow UV \parallel BC \Longrightarrow$ $UV \parallel BC \parallel XY \Longrightarrow \triangle BUX$ and $\triangle CVY$ are perspective, thus by Desargues theorem, $P \equiv BX \cap CY$, $Q \equiv BU \cap CV$ and $H \equiv UX \cap VY$ are collinear.

Quick Reply

High School Olympiads

midpoint and cyclic

Locked

Paloe

#1 Feb 16, 2016, 7:23 am

Circle T_1 and T_2 intersect at A and B. Given point P in T_1 such that a tangent line from P intersect T_2 at Q and R. X, Y, Z is a projection from P onto AB, AQ, BR respectively. M is midpoint of AB. Prove that M, X, Y, Z is a cyclic.

Luis González

#2 Feb 16, 2016, 7:43 am • 1

Posted before at http://www.artofproblemsolving.com/community/c6h319596.

High School Olympiads

Circles and midpoint

Reply

Source: M,H-3

stergiu

#1 Dec 20, 2009, 12:15 am

Two circles meet at A and B.The tangent at a point P of one of the circles meets the other circle at Q and

R.The points X, Y, Z are the feet of the perpendiculars from P to AB, AQ, BR respectively.

Prove that the circle XYZ passes through the pidpoint M of AB.

Babis

kaka__2004

#2 Dec 29, 2009, 3:49 pm

how do you proove it?

Luis González

#3 Jan 21, 2010, 7:54 am

Let $C \equiv AQ \cap BR$ and M be the midpoint of AB. By simple angle chasing we have

$\angle ABP = \angle APQ$, $\angle CBA = \angle AQP$

$\Longrightarrow \angle PBR = 180^\circ - \angle CBA - \angle ABP = \angle QAP.$

This implies that $\angle APY = \angle BPZ,$ but the quadrilaterals $AXPY$ and $BXPZ$ are cyclic $\Longrightarrow \angle AXY = \angle BPZ \Longrightarrow$ lines XP and AB bisect $\angle YXZ$ internally and externally $(\star)$. Let U and V be the midpoints of PA and PB. Then $\triangle MUV$ is the medial triangle of $\triangle PAB \Longrightarrow MV = UA = UY, MU = VB = VZ$ and $\angle MVZ = \angle MUY = \angle APB + 2\angle BPZ$. Thus, $\triangle MUY$ and $\triangle ZVM$ are congruent by SAS. Which yields $MY = MZ,$ i.e. M lies on the perpendicular bisector of YZ. Together with $(\star)$, we conclude that $M \in \odot(XYZ)$.

Quick Reply

High School Olympiads

Pedal Triangle

Reply

Source: own

ABCDE

#1 Feb 16, 2016, 4:30 am • 1

Let P be a point in the plane of triangle ABC with pedal triangle DEF with respect to ABC. Suppose that AD intersects BP and CP at X and Y, respectively, and let EF intersect BC at Z. If ZX intersects AC at S and ZY intersects AB at T, prove that P lies on ST.

Luis González

#3 Feb 16, 2016, 5:30 am

This property is merely projective (i.e. it has nothing to do with pedal triangles); D, E, F can be arbitrary points on $BC, CA, AB.$ Letting $U \equiv PB \cap CA$ and $V \equiv PC \cap AB,$ we have $(S, C, U, A) = X(S, C, U, A) = (Z, C, B, D)$ and similarly we have $(T, V, B, A) = Y(T, V, B, A) = (Z, C, B, D) \Longrightarrow (S, C, U, A) = (T, V, B, A) \Longrightarrow P \in ST.$

Quick Reply

High School Olympiads

Fe again(1) --- Hard Feuerbach point Problem

Reply

Source: Own By 09.4.14ï¼Œwho want to tryï¼

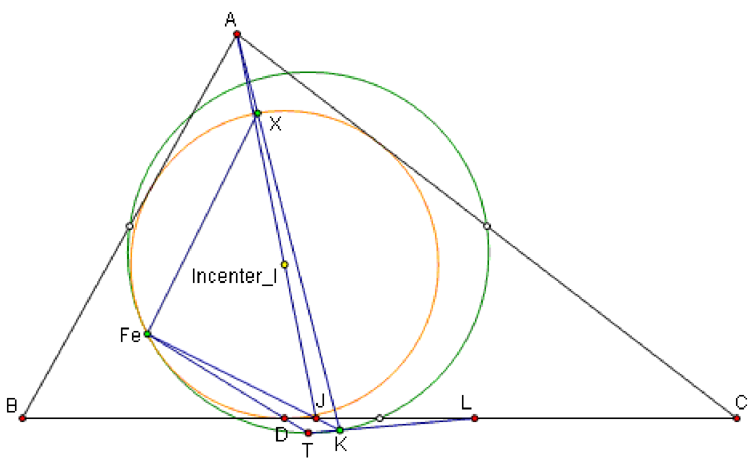
lym

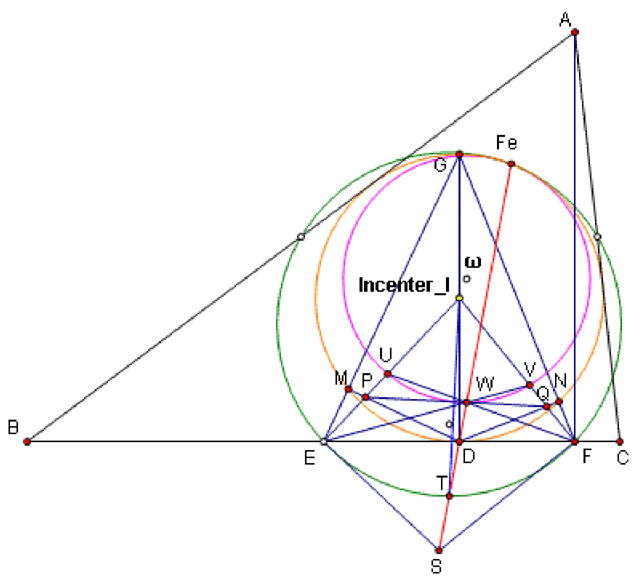
#1 Apr 14, 2009, 10:54 pm

The Incircle □I of ΔABC slice BC at D□the nine points circle---□ω of ΔABC meet BC at E, F□DI insecet □I at G . Fe is the Feuerbach point□E is the midpoint□L is on BC□and $EL = DE$□GE, GF separately insecet □I at M, N . And DM, DN separately insecet IE, IF at P, Q□IS is the diameter of □IEF□Prove that□

(1) : Fe, D, S is collinear .
(2) : Let DFe meet □ω at T□then IT□PQ .
(3) : Let DFe meet PQ at W□EW, FW separately insecet IF, IE at V, U□Then □ω□□I□□UVW are tangent in Fe .
(4) : AI insecet BC at J□let JFe, TL insecet at K□AK insecet □I at X□then XFe□KFe .

Attachments:





Luis González

#2 Feb 13, 2016, 11:30 am

Propositions (1),(2) and (3) follow from this general configuration:

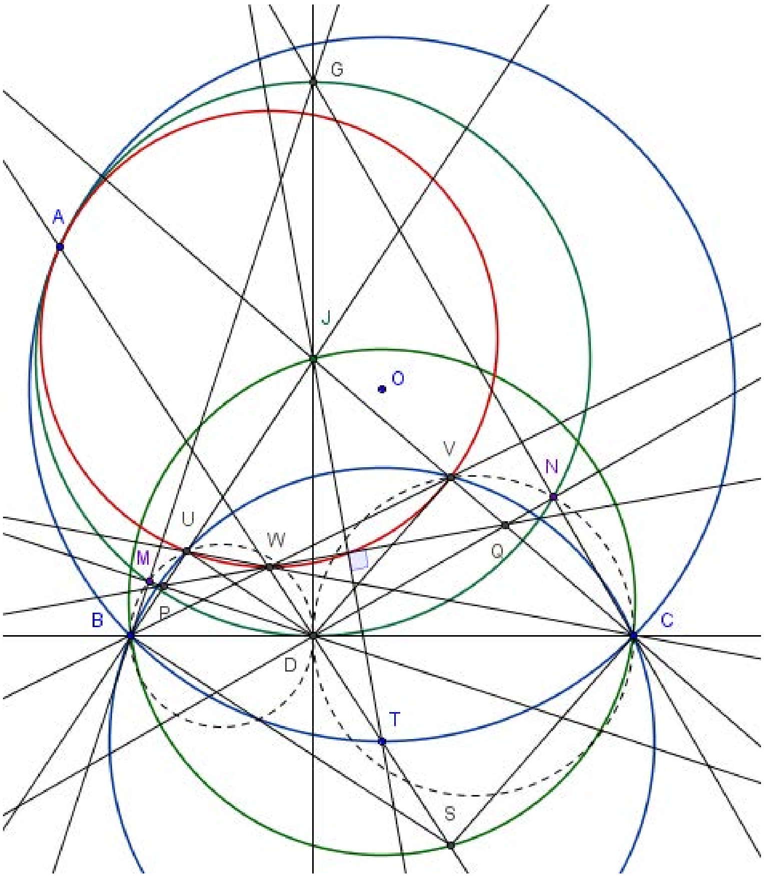
In a triangle $\triangle ABC,$ circle (J) is internally tangent to its circumcircle (O) at A and tangent to BC at D (it's well-known that AD is internal bisector of $\angle BAC$). G is the antipode of D on (J) and GB, GC cut (J) again at $M, N.$ DM, DN cut JB, JC at $P, Q.$ Then we have:
a) A, D and the antipode S of J on $\odot(JBC)$ are collinear.
b) If T is the midpoint of the arc BC of $(O),$ then $JT \perp PQ.$
c) If $W \equiv AD \cap PQ, U \equiv JB \cap CW$ and $V \equiv JC \cap BW,$ then $\odot(UVW)$ is tangent to $(O), (J)$ at $A.$

a) Redefine U and V as the projections of D on $JB, JC.$ From the problem Small problem about Incenter (post #6), we get that B, C, U, V lie on a same circle with center the midpoint T of the arc BC of $(O).$ Thus the perdendicular bisector of CV is the midparallel of $DV \parallel SC$ and the perpendicular bisector of BU is the midparallel of $DU \parallel SB \Longrightarrow T$ is the midpoint of $DS \Longrightarrow A, D, T, S$ are collinear.

b) Since $BMUD$ is cyclic on account of the right angles at $M, U,$ then $PB \cdot PU = PD \cdot PM \Longrightarrow P$ has equal power WRT (J) and $\odot(BUVC)$ and so does Q similarly $\Longrightarrow PQ$ is radical axis of (J) and $\odot(BUVC)$ perpendicular to their center line $JT,$ i.e. $JT \perp PQ.$

c) Redefine $W \equiv BV \cap CU.$ Since $WB \cdot WV = WC \cdot WU,$ then W has equal power WRT (J) and $\odot(BUVC),$ hence it lies on their radical axis $PQ.$ Now from Small problem about Incenter, we get that $W \in AD$ and $\odot(UVW)$ is tangent to $(O), (J)$ at $A.$

Attachments:



Luis González

#3 Feb 13, 2016, 11:31 am

Solution to proposition (4):

Let (N) and (I_a) be the 9-point circle and A-excircle of $\triangle ABC$ externally tangent at the A-Feuerbach point $Fa.$ Obviously (I_a) is tangent to BC at $L.$ Since $(NT \parallel I_aL) \perp BC \Longrightarrow Fa \in TL$ and since Fe is the exsimilicenter of $(I) \sim (N)$ and J is the insimilicenter of $(I_a) \sim (I),$ then by Monge & d'Alembert theorem, JFe passes through the insimilicenter Fa of $(I_a) \sim (N).$ Consequently K is identical with $Fa.$

Let S be the antipode of K on (N) and label τ_K, τ_S, τ_X the tangents of (N) at $K, S, X,$ respectively. Since A is the exsimilicenter of $(I) \sim (I_a)$ and K is the insimilicenter of $(N) \sim (I_a),$ it follows that $\overline{AXK}$ passes through the insimilicenter of $(I) \sim (N) \Longrightarrow \tau_X \parallel \tau_K.$ But obviously $\tau_S \parallel \tau_K \Longrightarrow \tau_X \parallel \tau_S \Longrightarrow X$ and S are homologous points under the direct homothety that takes (I) into $(N) \Longrightarrow Fe, X, S$ are collinear $\Longrightarrow \angle KFeX \equiv \angle KFeS = 90^\circ,$ i.e. $XFe \perp KFe.$

Quick Reply

High School Olympiads

Small problem about Incenter

Source: drmzjoseph asked me to post it

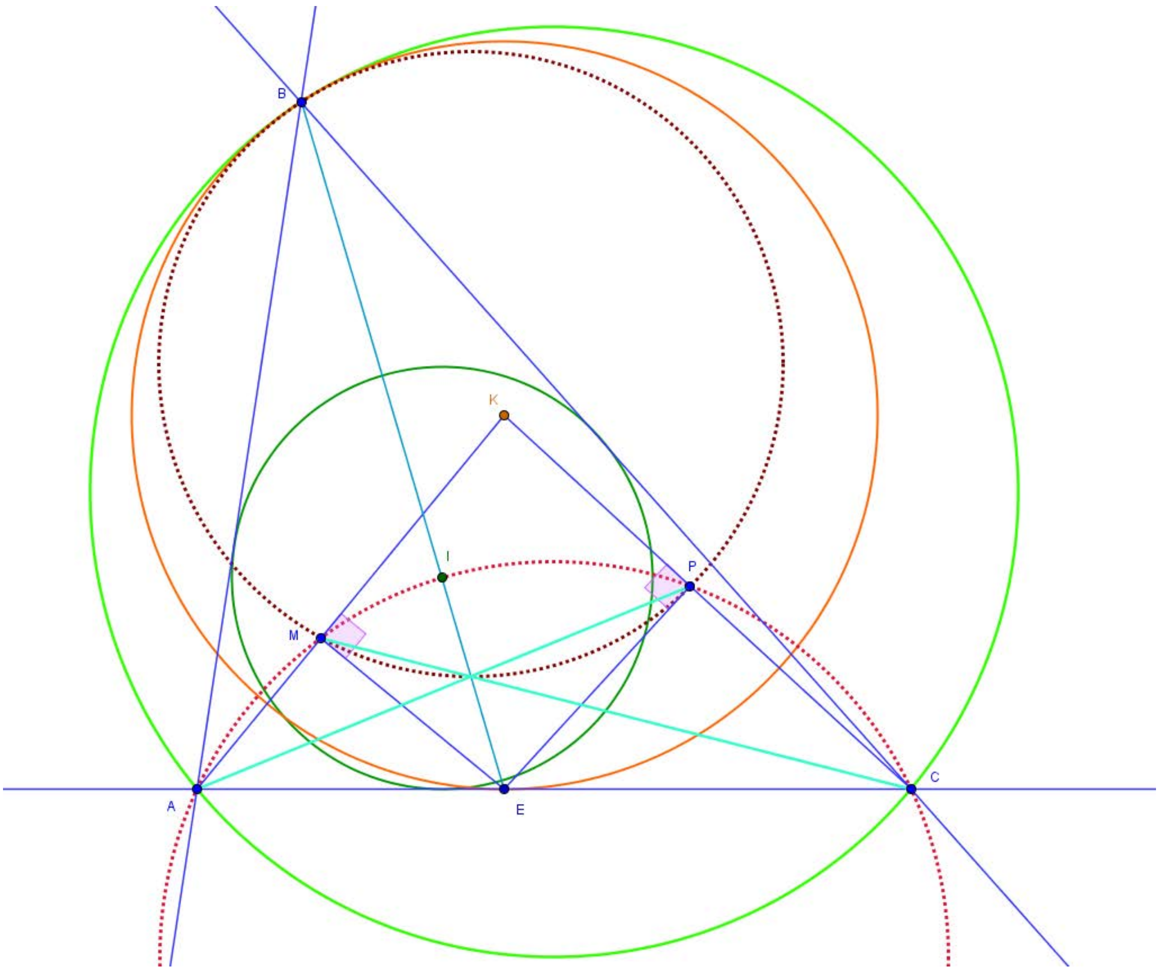
TelvCohl
#1 Sep 3, 2015, 11:54 am • 3

Let I be the Incenter of $\triangle ABC$ and $E \equiv BI \cap CA$.
Let $\odot(K)$ be the circle through B and tangent to CA at E .
Let M, P be the projection of E on $KA, KC,$ respectively .

Prove that

a) A, C, M, P, I are concyclic
b) $\odot(BMP)$ is tangent to $\odot(ABC)$ at B
c) $AP, BE, CM, \odot(BMP)$ are concurrent

Attachments:



THVSH
#4 Sep 3, 2015, 2:07 pm • 1

a) Let O be the circumcenter of $\triangle ABC$ and BD is an altitude of $\triangle ABC$, ($D \in AC$). We have $\angle KBE = \angle KEB = \angle DBE = \angle OBE \Longrightarrow K \in AO$. Hence, $\odot(K)$ is tangent to $\odot(ABC)$ at B. Let F be the symmetric of E wrt the midpoint of AC. The line through F and perpendicular to AC intersects BE at G. Let H be the midpoint of EG. From IMO Shortlist 2012, Geometry 4, A, C, K, G are concyclic and $\angle KAG = \angle KCG = 90^\circ$. So $HA = HM; HC = HP$. On the other hand, $\angle CAM = \angle KEM = \angle KPM \Longrightarrow A, C, P, M$ are concyclic. Thus, H is the center of $\odot(A, C, P, M)$. Since $HA = HC$ and $H \in BE$, we get H is the midpoint of arc AC of $\odot(O)$ not containing B. Therefore, $\odot(A, C, P, M) \equiv \odot(H, HA = HC)$ passes through incenter I.

b,c) Let $AP \cap CM = Q$. From (XYZ) and (AD) are tangent., we get $\odot(QMP)$ are tangent to $\odot(K)$ at a point B' and $B'Q$ passes through the point E (the projection of K on AC) and the point G (antipode of K in $\odot(KAC)$). So $EG \cap \odot(K) = \{E, B'\} \Longrightarrow B' \equiv B$. It means: BE passes through $Q \in \odot(B'MP) \equiv \odot(BMP)$ and $\odot(B, M, P, Q)$ is tangent to $\odot(K)$ at B, so it is also tangent to $\odot(ABC)$ at B.

jayme

#5 Sep 3, 2015, 3:57 pm

Dear Mathlinkers,
a way that I have quickly explored...
For (a)
1. let (KEMP) the circle xith diameter KE
2. let (BIC) a well known circle
3. by considering two time the three chords theorem (Monge or d'Alembert) and raisoning ad absurdum, it seems that work...

Sincerely
Jean-Louis

Luis González

#6 Sep 4, 2015, 8:52 am • 2

a) Let J be the midpoint of the arc AC of $\odot(ABC)$; center of $\odot(IAC)$ (well-known). KA, KC cut $(J) \equiv \odot(IAC)$ again at M', P'. Since $JA^2 = JC^2 = JE \cdot JB \Longrightarrow (J)$ is orthogonal to $(K) \Longrightarrow KE^2 = KA \cdot KM' = KC \cdot KP' \Longrightarrow M \equiv M', P \equiv P' \Longrightarrow A, C, M, P, I$ are concyclic.

b) Let $X \equiv MP \cap AC$. Since $\odot(KPEM)$ is clearly tangent to AC, we get $XE^2 = XM \cdot XP = XA \cdot XC \Longrightarrow X$ has equal power WRT $\odot(ABC)$ and $(K) \Longrightarrow XB$ is their common tangent, so $XB^2 = XE^2 = XM \cdot XP \Longrightarrow XB$ touches $\odot(BMP) \Longrightarrow \odot(BMP)$ is tangent to $\odot(ABC)$.

c) Let the tangents of (J) at A, C meet at T and let the circle (X) with center $X$and radius $XB = XE$ cut (J) at U, V. As (K) and $\odot(T, TA)$ are orthogonal to (X) and (J), then K, T lie on their radical axis UV, but T, K and $S \equiv AP \cap CM$ are collinear on the polar of X WRT (J). Thus since $(U, V, K, S) = -1 \Longrightarrow S$ is on the polar BE of K WRT (X). Now $SB \cdot SE = SU \cdot SV = SP \cdot SA = SM \cdot SC \Longrightarrow AC$ is the inverse of $\odot(BMP)$ under inversion with center S and power $SB \cdot SE \Longrightarrow S \in \odot(BMP)$.

linqaszayi

#7 Sep 6, 2015, 9:52 pm • 1

another way from a) to b),c):

since $ACMPI$ concyclic we get $\angle AMC = 90^o + \dfrac{1}{2}\angle B$ so $\angle EMC = \dfrac{1}{2}\angle B = \angle EBC$,thus $EMBC$ are concyclic.
similarly $BPEA$ are concyclic. assume the center of $\odot(BMP)$ is R ,then
$\angle RBC = \angle RBP + \angle PBC = 90^o - \angle BMP + \angle B - \angle PEC = 90^o - (180^o - \angle BCA - \angle EMP) + \angle B - \angle PEC = 90^o - \angle A$
so R is on OB and $\odot(BMP)$ is tangent to $\odot(ABC)$.
from the radical center theorem to $\odot(EMBC), \odot(BPEA), \odot(ACMPI)$ we get AP, BE, CM concur at Q;
$\angle MQP = 180^o - \angle PAC - \angle MCA = 180^o - \angle PBE - \angle MBE = 180^o - \angle PBM$,so Q is on $\odot(BMP)$.

This post has been edited 1 time. Last edited by linqaszayi, Sep 6, 2015, 9:53 pm

Quick Reply

High School Olympiads

Tangent circles

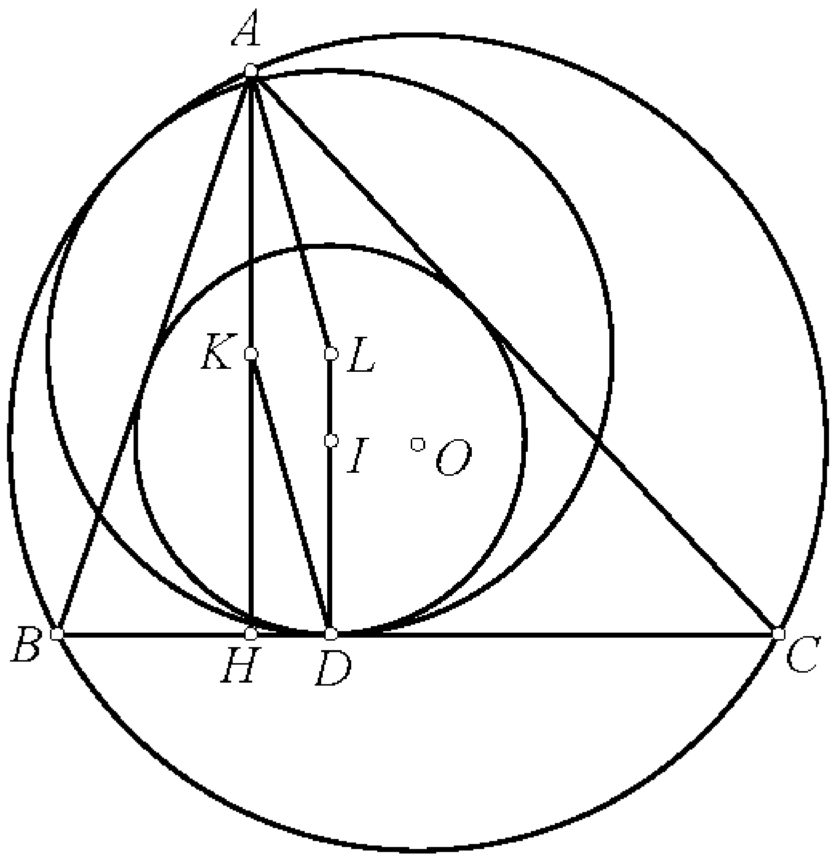
Source: Problem weekly, third week, December 2015

buratinogigle

#1 Feb 12, 2016, 12:08 pm • 2

Let ABC be a triangle inscribed in circle (O) with altitude AH. Incircle (I) touches BC at D. K is midpoint of AH. L is projection of K on ID. Prove that circle (L, LD) is tangent to (O).

Attachments:



IsoLyS

#2 Feb 12, 2016, 12:59 pm • 1

Let AA' be diameter of (O), and $A'I \cap (O) = T$. Denote midpoint of arcs $\widehat{\text{BC}}, \widehat{\text{BAC}}$ by M, N.
Also denote the foot of perpecdicular from I to AH by E, reflection of H wrt AI by H', and $DI \cap TN$ by F.
Since $AI \cdot IM = 2Rr$ and $\angle DIM = \angle IAA'$, $\triangle IDM \sim \triangle AIA'$, hence $\angle TA'A = \angle AMD$, which means $D \in TM$.
Since $\angle ITE = \angle IAH = \angle MAA' = \angle MTI$, we get T, E, D, M are collinear.
We know that $H' \in AA'$ and $IEHD$ is rectangular, hence $\angle IH'A = \angle IHA = \angle IDT$, which means $\triangle IDT \sim \triangle IH'A$.
Therefore since $\triangle ITM \sim \triangle IAA'$ and $MN \| DF$, $FD : MN = TD : TM = AH' : AA' = AH : MN$, hence L is midpoint of DF.
Since O is midpoint of MN, T, L, O are collinear, which means (L, LD) and (O, OM) are homothetic wrt T.
Therefore (L, LD) is tangent to (O) at T.

Luis González

#3 Feb 13, 2016, 12:01 am • 2

Let E, F be the tangency points of the incircle (I) with AC, AB and let M, N be the midpoints of the arcs BC and BAC of (O). If $\odot(AEIF)$ cuts (O) again at U, then it's known that M, D, U are collinear (see incenter I and touches BC side with D). Thus if DI cuts $\odot(AEIF)$ again at X, we have $\angle AUX = \angle AIX = \angle AMN \Longrightarrow X \in UN$. Since $AXDH$ is clearly a rectangle, then it follows that L is midpoint of DX, so from $DX \parallel MN \Longrightarrow U, L, O$ are collinear $\Longrightarrow$ $\odot(L, LD) \equiv \odot(UDX)$ is tangent to (O) at U.

mjuk

#4 Feb 13, 2016, 4:06 am • 1

Let X be midpoint of arc BAC and XY be diameter of (O). Let M be midpoint of BC. Line parallel to BC through A intersects XY at N and touches (L) at G. Let $S = GX \cap DY$. $\triangle GNX \sim \triangle YMD$ Proof
Now $\angle XSY = \angle GXN + \angle DYM = 90^\circ \rightarrow S \in (L)$ and $S \in (O)$.
Homothety centered at S, taking $GD \rightarrow XY$, also takes $(L) \rightarrow (O)$, hence they are tangent at S.

This post has been edited 1 time. Last edited by mjuk, Feb 13, 2016, 4:09 am
Reason: correction

FabrizioFelen

#5 Feb 13, 2016, 5:38 am • 2

My solution:

Let F be a point such that $F \in (O)$ and $AF \parallel BC$ and let Ω the circumference tangent to (O) in X such that $X \in \overset{\frown}{AB}$, also tangent to AF and BC in E and D' respectively $\Longrightarrow$ by Sawayama's lemma in $\triangle ABC$ we get $D'E$ passes through the incenter of $\triangle ABC \Longrightarrow E, D', I$ are collinear also that $D' = D$ and $AHDE$ is a rectangle $\Longrightarrow L$ is the midpoint of ED $\Longrightarrow \Omega = (L, LD) \Longrightarrow (L, LD)$ is tangent to (O)...

This post has been edited 3 times. Last edited by FabrizioFelen, Feb 13, 2016, 8:42 am

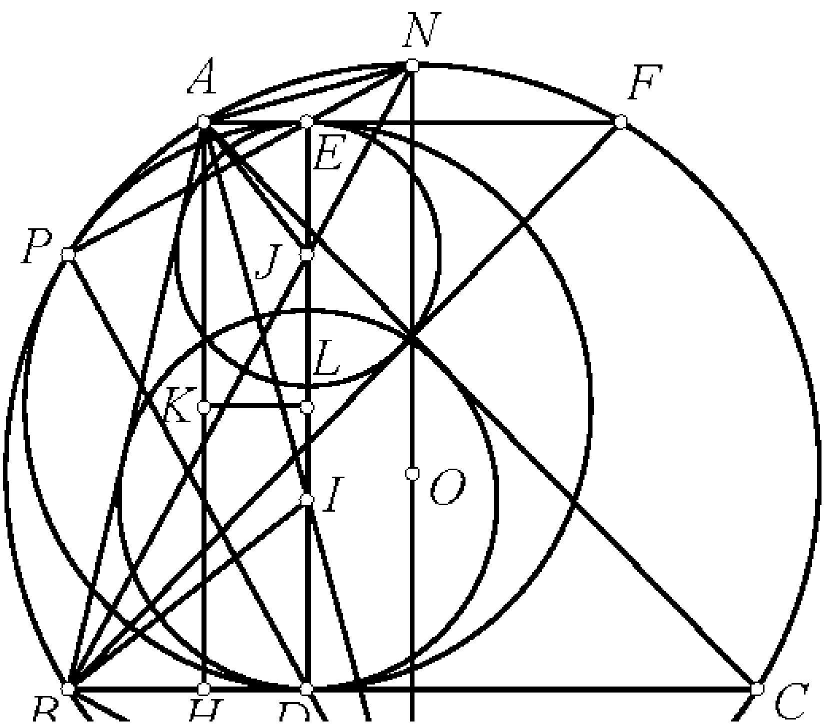
buratinogigle

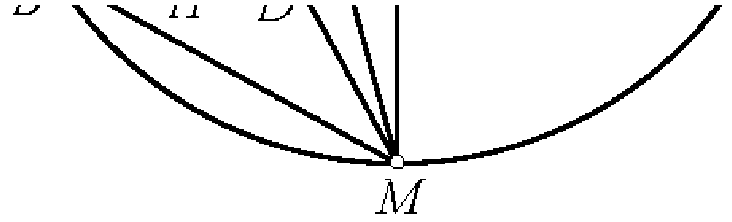
#6 Feb 13, 2016, 2:56 pm

Thank you for your interest, here is my solution.

Let DE be dimaeter of (L) easily seen $AE \parallel BC$. Let AE cut (O) again at F. Let (J) be incircle of FAB. We see $\angle AIB = 90^\circ + \frac{\angle C}{2} = 90^\circ + \frac{\angle AFB}{2} = \angle AJB$, this deduces $AJIB$ is cyclic. Hence, note that $AF \parallel BC$ so $\angle BID = 90^\circ - \angle IBD = \angle BAJ$, we get J, I, D are colinear. From this, we see (J) is tangent AF at E. Let AI, BJ cut (O) again at M, N. Easily seen MN is dimameter of (O). In other wise, $\angle ANJ = \angle BMI$ so the isocesles triangles $\triangle ANJ \sim \triangle BMI$. But two right triangles $\triangle AEJ \sim \triangle IDB$. Therefore, we get $\triangle ANE \sim \triangle IMD$. Thus, $\angle ANE = \angle IMD$ or NE and MD intersect at P on (O). Because MN is diameter of (O) so $\angle MPN = 90^\circ$, we deduce P lies on (L). From, $DE \parallel MN$ we get (L) is tangent to (O).

Attachments:





Quick Reply

High School Olympiads

incenter I and touches BC side with D

hEatLove

#1 Mar 28, 2011, 1:08 pm

I be the incenter of $\triangle ABC$ and incircle touches BC side with D . AI line intersects circumcircle of $\triangle ABC$ at E ; DE line intersects circumcircle of $\triangle ABC$ at F . Prove $\angle AFI = 90$

jgnr

#2 Mar 28, 2011, 5:20 pm

Let AI meets BC at M, AK is the diameter of (ABC), L is the foot of perpendicular from A to BC, and P is the midpoint of IL. It is easy to show that $\frac{AI}{IM} = \frac{b+c}{a} = \frac{LD}{DM}$. Since $\triangle AEC \sim \triangle ABM$ and $\triangle ABE \sim \triangle BME$, then we also have $\frac{AI}{IM} = \frac{AB}{BM} = \frac{AE}{EC} = \frac{AE}{EB} = \frac{BE}{EM} = \frac{IE}{EM}$. Therefore $\frac{IE}{EM} = \frac{LD}{DM}$, which gives $\frac{IE}{EM} \cdot \frac{MD}{DL} \cdot \frac{LP}{PI} = 1$. So P, D, E are collinear, and hence F, P, D, E are collinear. From IMO 2010 Q2, we know that KI and DP intersect at F. Thus $\angle AFI = \angle AFK = 90^\circ$, as desired.

Luis González

#3 Mar 28, 2011, 11:21 pm

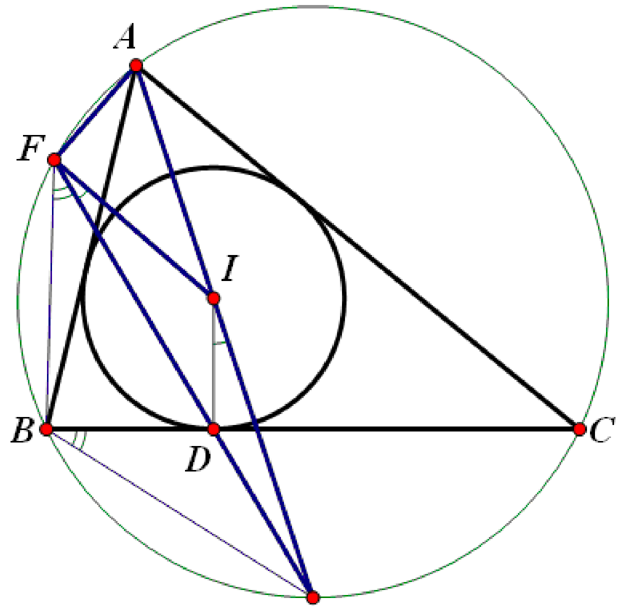
Let P be the antipode of E in the circumcircle (O) of $\triangle ABC$, i.e. midpoint of the arc BAC. Inversion with center E and radius $EB = EC = EI$ swaps (O) and the sideline $BC \Longrightarrow D, F$ are inverses and I is double. Hence $EI^2 = ED \cdot EF \Longrightarrow \odot(FDI)$ is tangent to EI through I, i.e. $\angle EID = \angle IFD$. But, $\angle EID = \angle AEP = \angle AFP \Longrightarrow \angle IFD = \angle AFP \Longrightarrow \angle AFI = 90^\circ$.

yunxiu

#4 Mar 29, 2011, 9:56 pm • 2

$\angle EBD = \angle BFE \Rightarrow \Delta EBD \sim \Delta EFB$, so $EI^2 = EB^2 = ED \cdot EF$. Hence
$\Delta EID \sim \Delta EFI \Rightarrow \angle EID = \angle EFI$, and we have
$\angle AFI = \angle AFB - \angle BFI = 180^\circ - \angle BEI - \angle EBD - \angle EID = 180^\circ - \angle IDB = 90^\circ$.

Attachments:



E

MeineMeinung

#5 Dec 10, 2015, 8:13 pm

Let X and Y be the point where the incircle touches AB and AC, respectively. Quite obvious that FD bisects $\angle BFC$. So, $\frac{FB}{FC} = \frac{DB}{DC} = \frac{BX}{CY}$.Then, since $\angle FBX = \angle FCY$, we get $\triangle FBX\ \triangle FCY$. Hence, F is the center of spiral similarity taking BX to CY and from a well-known property, we get that F must lies on circle $(AXIY)$ and hence $\angle AFI = 90^{\circ}$.

Quick Reply

High School Olympiads

A point on bisector ,hard for me!

Locked

Source: own

MRF2017

#1 Feb 11, 2016, 11:59 pm

Given a convex quadrilateral $ABCD$. I and J are incenters of the triangles ABC and ADC, respectively, and I_a and J_a are excenters of triangles ABC and ADC, respectively (inscribed in the angles BAC and DAC,respectively). Prove that the point of intersection of the lines $IJ_a$and JI_a lies on the bisector of the angle BCD.

Luis González

#2 Feb 12, 2016, 1:52 am

Posted before at http://www.artofproblemsolving.com/community/c6h1090064.

High School Olympiads

Hard geometry

Source: Komal,Jan 2015

andreiromania

#1 May 16, 2015, 6:12 pm

Let $ABCD$ be a convex quadrilateral. In the triangle ABC let I and J be the incenter and the excenter opposite to vertex A, respectively. In the triangle ACD let K and L be the incenter and the excenter opposite to vertex A, respectively. Show that the lines IL and JK, and the bisector of the angle BCD are concurrent.

This post has been edited 1 time. Last edited by andreiromania, May 16, 2015, 7:22 pm

andreiromania

#2 May 16, 2015, 7:14 pm

First,some quick notations.Let $Q = IL \cap JK$.Let $AB = a, BC = b, CD = c, DA = d, AC = e, \angle ACB = B, \angle ACD = D$.Also denote by r_{ABC} the inradius of ABC,by r_{eACD} the exradius corresponding to A of ACD,by $d(W, XY)$ the distance from W to XY,and by S_x the area of figure x.

We apply Menelaus' on $J - Q - K$ and triangle AIL to get that $\frac{AJ}{IJ}\frac{IQ}{LQ}\frac{LK}{AK} = 1$.By Sine Law we get $AI = \frac{1}{\sin(BAC/2)}\frac{a+e-b}{2}$,$AJ = \frac{1}{\sin(BAC/2)}\frac{a+e+b}{2}$,$AK = \frac{1}{\sin(DAC/2)}\frac{d+e-c}{2}$, $AI = \frac{1}{\sin(DAC/2)}\frac{d+e+c}{2}$ and also $IJ = \frac{1}{\sin(BAC/2)}b, KL = \frac{1}{\sin(DAC/2)}c$.

Quick calculations now give us $\frac{IQ}{LQ} = \frac{b(d+e-c)}{c(a+b+e)}$ and its "cousins" $\frac{IQ}{IL} = \frac{b(d+e-c)}{K}$ and $\frac{LQ}{IL} = \frac{c(a+b+e)}{K}$,where we denoted $K = ac + ec + bd + be$.

What we need to prove is that Q lies on BCD's bisector;this is done if we prove $d(Q, BC) = d(Q, DC)$.

Let's first calculate $d(Q, DC)$.We notice that I and L are on different sides of DC.Let's assume WLOG X is on the same side as I.Thus we can find that $d(Q, DC) = \frac{LQ}{LI}d(I, DC) - \frac{IQ}{IL}d(L, DC)(*)$.Now by Sine Law again we obtain $d(L, DC) = r_{eADC}$ and $d(I, DC) = r_{ABC} * \frac{sin(\frac{B}{2}+D)}{sin(\frac{B}{2})}$.Now we couple these with the well-known formulas $S_{ABC} = \frac{a+b+e}{2}r_{ABC}$,$S_{ACD} = \frac{d+e-c}{2}r_{eACD}$ and mash our formula $(*)$ by easy computations into: $d(Q, DC) = \frac{1}{K}(2cS_{ABC}\frac{sin(\frac{B}{2}+D)}{sin(\frac{B}{2})} - 2bS_{ADC})$.

Now on to $d(Q, BC)$;this time,we do not know for sure whether I and L lie on the same side of BC;but we can still assume $WLOG$ that they do;now obviously Q lies on the same side as both of them,so the formula for the distance is: $d(Q, BC) = \frac{LQ}{LI}d(I, BC) + \frac{IQ}{IL}d(L, BC)$.

By once again mashing and bashing by known formulae we arrive at $d(Q, BC) = \frac{1}{K}(2cS_{ABC} + 2bS_{ADC}\frac{cos(\frac{D}{2}+B)}{cos(\frac{D}{2})})$

Now that we have the formulae for both distances,proving that they are equal is reduced to a neatly-looking $2cS_{ABC}(1 - \frac{sin(\frac{B}{2}+D)}{sin(\frac{B}{2})}) = -2bS_{ACD}(1 + \frac{cos(\frac{D}{2}+B)}{cos(\frac{D}{2})})$.

By the area formula,$S_{ABC} = be\sin B, S_{ADC} = ce\sin D$ so after some simple simplification we are left to prove just that

$cos\frac{\sim}{2}(sin(\frac{\sim}{2}) - sin(\frac{\sim}{2}+D)) = -sin\frac{\sim}{2}(cos(\frac{\sim}{2}) - cos(\frac{\sim}{2}+B))$ which is standard trig manipulation, QED.

This post has been edited 4 times. Last edited by andreiromania, May 16, 2015, 7:24 pm

TelvCohl

#4 May 16, 2015, 7:22 pm

My solution :

Let $X = IL \cap JK$ and ℓ, τ be the bisector of $\angle DCB, \angle(\ell, CA)$, respectively .

Since $\angle(CI,\tau) = \angle(CK,\tau)$ and $\angle(CJ,\tau) = \angle(CL,\tau)$,
so we get $(CI, CJ; \ell, CA) = (CK, CL; CA, \ell)$ . ... $(\star)$

From the Dual of Desargue Involution theorem $\Longrightarrow (CI, CJ; CX, CA) = (CK, CL; CA, CX)$,
so combine $(\star)$ we get $CX \equiv \ell \Longrightarrow X \in \ell$. i.e. IL, JK and the bisector of $\angle DCB$ are concurrent

Q.E.D

andreiromania

#5 May 16, 2015, 7:25 pm

Note: the property is actually valid for any 4 points in the plane,not just for a convex quadrilateral.

tranquanghuy7198

#6 May 16, 2015, 10:22 pm

My solution.

Lemma. Given 4 angles $\alpha, \alpha', \beta, \beta'$ such that $\alpha + \beta = \alpha' + \beta' < 180$ and $\frac{sin\alpha}{sin\beta}$ = $\frac{sin\alpha'}{sin\beta'}$. We'll have $\alpha = \alpha', \beta = \beta'$

Proof.
Construct $\triangle CAB, \triangle C'A'B'$ such that $\angle A = \alpha, \angle B = \beta, \angle A' = \alpha', \angle B' = \beta' \Rightarrow \angle C = \angle C'$ (because $\alpha + \beta = \alpha' + \beta'$) and $\frac{CB}{CA} = \frac{CB'}{CA'}$
$\Rightarrow \triangle ABC \sim \triangle A'B'C'$ and the conclusion follows

tranquanghuy7198

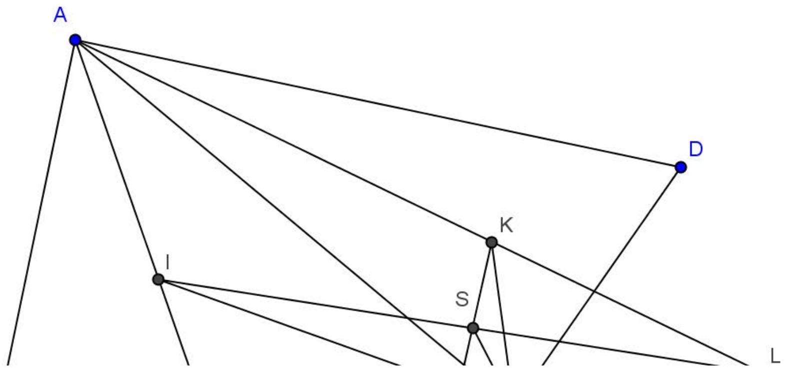
#7 May 16, 2015, 10:27 pm

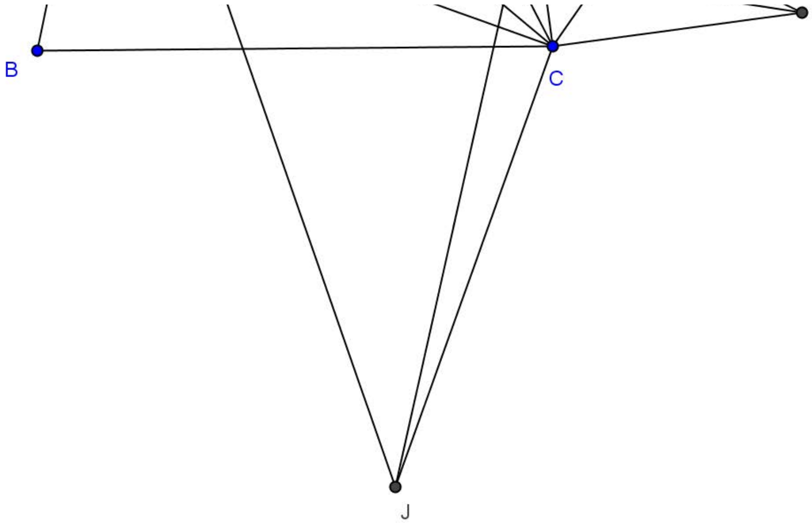
Back to our main problem.
$IL \cap JK = S$
Apply the Ceva theorem, we have:

$$\frac{sin\angle SCK}{sin\angle SCI} = \frac{sin\angle SKC}{sin\angle SKI}.\frac{sin\angle SIK}{sin\angle SIC} = \frac{sin\angle JKC}{sin\angle JKI}.\frac{sin\angle LIK}{sin\angle LIC} = (\frac{sin\angle JCK}{sin\angle JCI}.\frac{sin\angle JIC}{sin\angle JIK}).(\frac{sin\angle LKI}{sin\angle LKC}.\frac{sin\angle LCK}{sin\angle LCI}) = \frac{sin\angle JIC}{sin\angle JIK}.\frac{sin\angle LKI}{sin\angle LKC} = \frac{sin\angle AIC}{sin\angle AIK}.\frac{sin\angle AKI}{sin\angle AKC} = \frac{sin\angle ACI}{sin\angle ACK}$$

Now apply the lemma with the notice that $\angle SCK + \angle SCI = \angle ACI + \angle ACK$, we receive:
$\angle SCK = \angle ACI, \angle SCI = \angle ACK$ (1)
On the other hand: CI, CK are the angle bisectors of $\angle ACB, \angle ACD$ (2)
(1), (2) $\Rightarrow CS$ is the angle bisector of $\angle BCD$
Q.E.D

Attachments:





Alcumus526
#8 Aug 31, 2015, 2:59 pm

> tranquanghuy7198 wrote:
> $$(\frac{sin\angle JCK}{sin\angle JCI}.\frac{sin\angle JIC}{sin\angle JIK}).(\frac{sin\angle LKI}{sin\angle LKC}.\frac{sin\angle LCK}{sin\angle LCI})=\frac{sin\angle JIC}{sin\angle JIK}.\frac{sin\angle LKI}{sin\angle LKC}$$

How can you say that?? Where are other two fractions??

This post has been edited 1 time. Last edited by Alcumus526, Aug 31, 2015, 3:00 pm

kapilpavase
#9 Nov 10, 2015, 9:29 am

Actually this is very easy using harmonic divisions
Let $IJ\cap BC=E, KL\cap CD=F$.since cross ratios $[AIEJ],[ALFK]$ are same (harmonic) we have IL,EF,JK are concurrent say at X.
Next we have that $IF,EKAX$ are concurrent.Indeed this follows as pencil$[IA,IK,IF,IX]$ is harmonic
Now apply ceva in AEF and get $EX/XF=EC/CF$ so we are done.

This post has been edited 1 time. Last edited by kapilpavase, Nov 10, 2015, 9:30 am

Quick Reply

High School Olympiads

Tangent circles

Reply

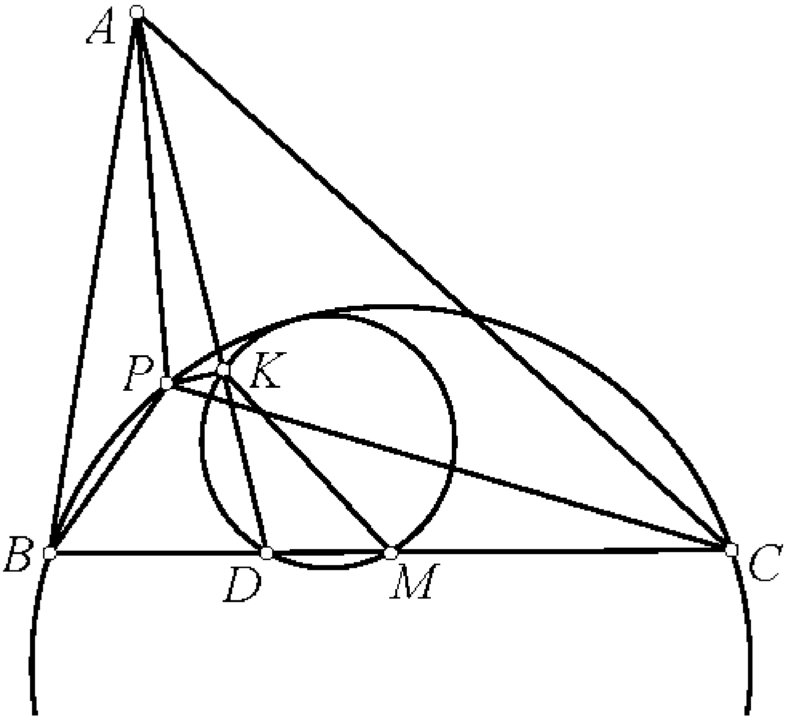
Source: Problem weekly, second week, December 2015

buratinogigle

#1 Feb 11, 2016, 4:55 pm

Let ABC be a triangle with symmedian AD and median AM. P is a point inside triangle ABC such that $\angle PBA = \angle PCA$. K is projection of P on AD. Prove that (KDM) and (PBC) are tangent.

Attachments:



TelvCohl

#2 Feb 11, 2016, 8:57 pm • 1

Let $E \equiv BP \cap CA, F \equiv CP \cap AB$. From $\angle PBA = \angle PCA$ we get B, C, E, F are concyclic, so the second intersection T of $\odot(BPC)$ and $\odot(AP) \equiv \odot(AKP)$ is the Miquel point of the complete quadrilateral $\{BC, EF, BE, CF\} \Longrightarrow \frac{TB}{TC} = \frac{BE}{CF} = \frac{AB}{AC} = \frac{BD^2}{CD^2}$, hence TD is the T-symmedian of $\triangle BTC \Longrightarrow \odot(DMT)$ is tangent to $\odot(BPC)$ at T.

On the other hand, since BC, EF, PT are concurrent at the radical center R of $\odot(BPC), \odot(EPF), \odot(BCEF)$, so from $P(R, A; B, C) = -1$ we get $G \equiv AP \cap DT$ lies on $\odot(BPC)$, hence $\measuredangle TKD = \measuredangle TPG = \measuredangle TMD \Longrightarrow D, K, M, T$ lie on a circle $\Longrightarrow \odot(DKM)$ and $\odot(BPC)$ are tangent to each other at T.

Luis González

#3 Feb 11, 2016, 10:31 pm • 1

Letting PB, PC cut AC, AB at Y, Z, then the problem is equivalent to Tangent circles (HSGS TST 2014) for the cyclic quadrilateral $BCYZ$.

buratinogigle

#4 Feb 12, 2016, 11:19 am

Thank you for your interest, I really created this problem from HSGS TST 2014 as Luis shown. This is also the general problem in the link http://www.artofproblemsolving.com/community/q4h1130448p5619094

Quick Reply

High School Olympiads

Tangent circles

Reply

Source: HSGS TST 2014

buratinogigle

#1 Dec 27, 2014, 1:45 pm • 1

Let $ABCD$ be cyclic quarilateral. M,N are midpoints of CD,AB. P lies on side CD such that $\frac{PD}{PC}=\frac{BD^2}{AC^2}$. AC cuts BD at E. H is projection of E on PN. Prove that circles (HMP) and (EDC) are tangent.

TelvCohl

#2 Dec 27, 2014, 6:56 pm • 3

My solution:

Let O be the center of $\odot(ABCD)$.
Let $X=AD \cap BC, Y=AB \cap CD, Z=\odot(ABE) \cap (CDE)$.

Easy to see E, Y, Z are collinear .

Since $\angle DZA=\angle DCA+\angle DBA=\angle DOA$,
so we get A, D, Z, O are concyclic ,
hence $\angle OZY=\angle OZD+\angle DZY=\angle OAD+\angle DCA=90^{\circ}$,
so we get O, M, N, Y, Z are lie on a circle with diameter OY .
From Brokard theorem we get O is the orthocenter of $\triangle EXY$,
so combine with $\angle OZY=90^{\circ}$ we get O, X, Z are collinear

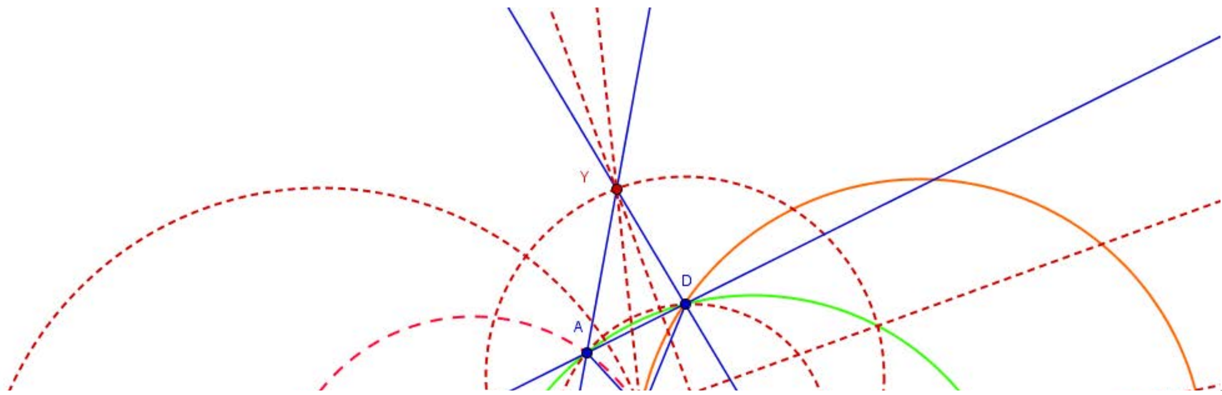
Since $PC:PD=AC^2:BD^2=XC^2:XD^2$,
so we get XP is $X-$ symmedian of $\triangle XCD$,
hence combine with $\triangle XAB \sim \triangle XCD$ we get $P \in XN$.
Since E, H, X, Z lie on a circle with diameter XE ,
so we get $\angle XHZ=\angle XEZ=\angle YOZ=\angle YMZ$.
ie. $Z \in \odot(HMP)$

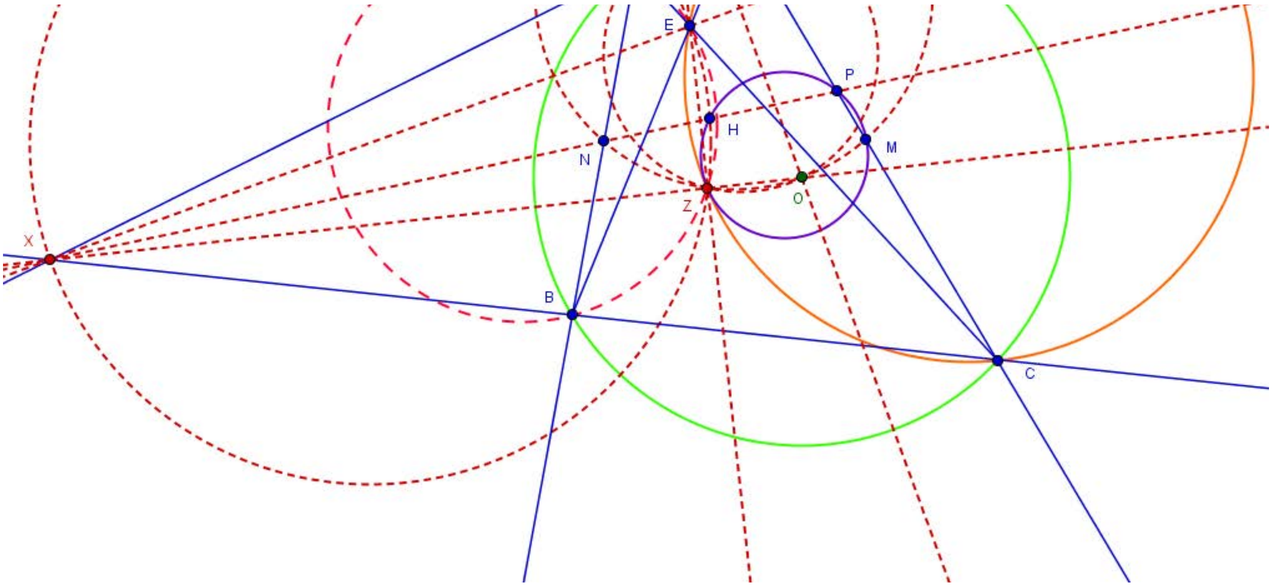
Since Z is the center of spiral similar of $AC \mapsto BD$,
so we get $PC:PD=AC^2:BD^2=ZC^2:ZD^2$,
hence ZP is $Z-$ symmedian of $\triangle ZCD$,
so we get $\angle CZM=\angle PZD$ and $\odot(HMPZ)$ is tangent to $\odot(EDCZ)$ at Z .

Q.E.D

By the way, what is HSGS ?

Attachments:





buratinogigle

#3 Dec 28, 2014, 12:32 am

Here is my solution

AD cuts BC at F. We have $\dfrac{PD}{PC} = \dfrac{BD^2}{AC^2} = \dfrac{FD^2}{FC^2}$ so FP is symmedian of FCD therefore FP passes through N. AB cuts CD at S. By Brokard theorem S, E, G are collinear and SE is perpendicular to OF at G. We have $\angle GMS = \angle GOS = \angle GEF = \angle FHG$ so $HGMP$ is cyclic. Tangent at G of (GCD) cuts CD at T. Notice $\triangle GAC \sim \triangle GBD$ we have $\dfrac{TC}{TD} = \dfrac{GC^2}{GD^2} = \dfrac{AC^2}{BD^2} = \dfrac{PC}{PD}$ so $(PG, CD) = -1$ deduce $TG^2 = TC.TD = TP.TM$ so TG tangent to (GPM). Thus, circles (HMP) and (EDC) are tangent at G.

Dear Telv Cohl HSGS is my school "**h**igh **s**chool for **g**ifted **s**tudent".

Best regards.

Attachments:

Figure2587.pdf (11kb)

Luis González

#4 Dec 28, 2014, 2:18 am • 2

Let $S \equiv AB \cap CD, T \equiv AD \cap BC$ and SE cuts $(O) \equiv \odot(ABCD)$ at U, V. Since SE, TE are the polars of T, S WRT (O), then SE is perpendicular to OT at K and $(U, V, K, S) = -1 \Longrightarrow SE \cdot SK = SU \cdot SV = SC \cdot SD \Longrightarrow K \in \odot(ECD)$ and likewise $K \in \odot(EAB) \Longrightarrow K$ is center of spiral similarity that swaps AC and $BD \Longrightarrow KC^2 : KD^2 = AC^2 : BD^2 = PC : PD \Longrightarrow KP$ is K-symmedian of $\triangle KDC \Longrightarrow KM, KP$ are isogonals WRT $\angle CKD \Longrightarrow \odot(KMP)$ touches $\odot(KDC)$. Now, it's enough to show that $H \in \odot(KMP)$.

If $L \equiv TE \cap CD$, then $(D, C, L, S) = -1 \Longrightarrow SL \cdot SM = SD \cdot SC = SE \cdot SK \Longrightarrow EKML$ is cyclic. But $TKHE$ is cyclic due to right angles at $H, K \Longrightarrow \angle THK = \angle TEK = \angle KMP \Longrightarrow H \in \odot(KMP)$.

Quick Reply

High School Olympiads

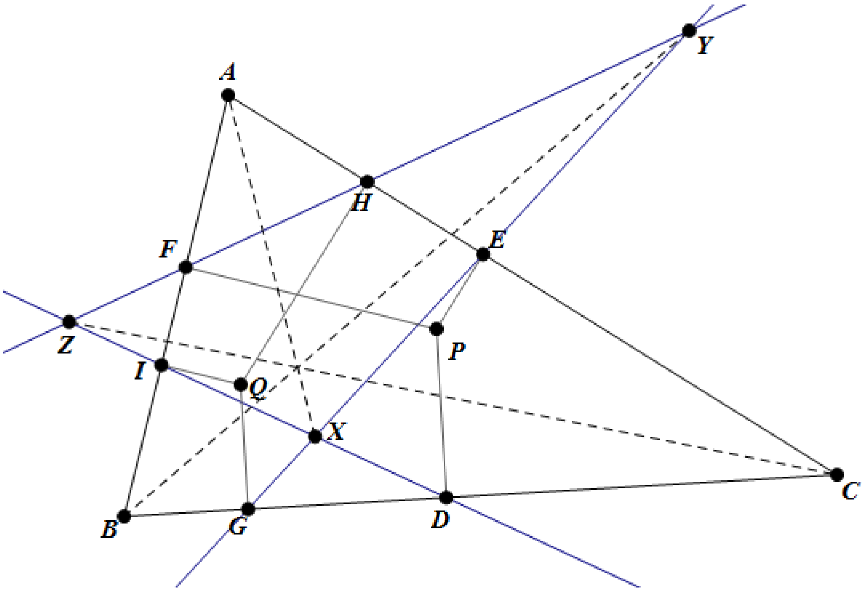
Concurrent lines

mjuk

#1 Feb 11, 2016, 1:51 am

Let P and Q be isogonal conjugates wrt. $\triangle ABC$, let $\triangle DEF$ and $\triangle GHI$ be pedal triangles of P and Q.
Let $X=ID\cap GE, Y=GE\cap HF, Z=HF\cap ID$.
Prove that AX, BY, CZ are concurrent.

Attachments:



Luis González

#2 Feb 11, 2016, 3:13 am • 2

It's well-known that D,E,F,G,H,I lie on a same circle (pedal circle of P,Q WRT $\triangle ABC$). Thus by Pascal theorem for the hexagon $FHEGDI$, the intersections $U\equiv FH\cap GD, V\equiv HE\cap DI$ and $W\equiv EG\cap IF$ are collinear, i.e. $\triangle ABC$ and the triangle $\triangle XYZ$ bounded by FH,EG,DI are perspective through $\overline{UVW}$. Thus by Desargues theorem AX,BY,CZ concur.

High School Olympiads

Concurrent on circumcircle

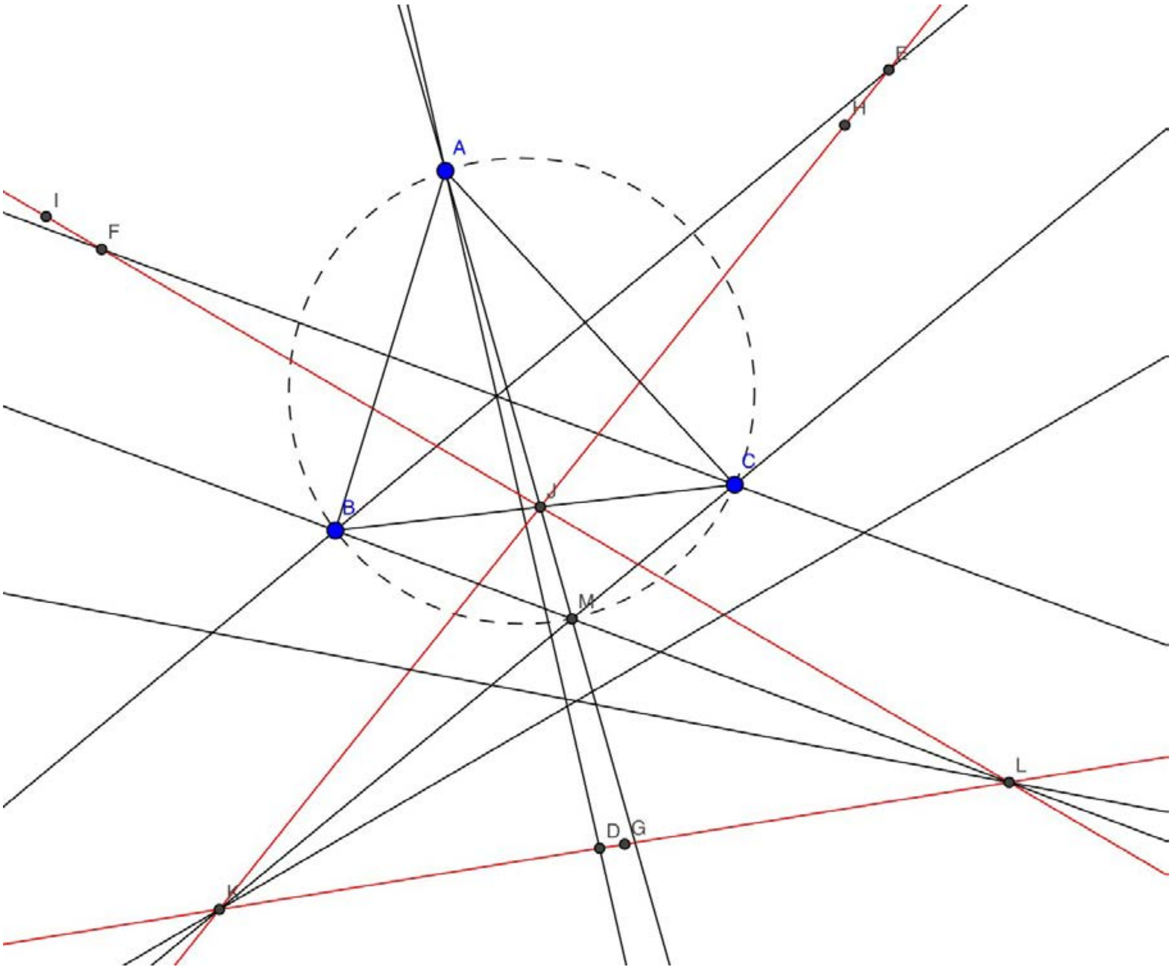
Reply

ThE-dArK-lOrD

#1 Feb 10, 2016, 10:40 pm

Given $\triangle ABC$
Let D, E, F are $A-, B-, C-$ excenter of $\triangle ABC$
Let G, H, I are points such that $\triangle GHI$ is anticomplementary triangle of $\triangle ABC$
Let $EH \cap FI = J, DG \cap EH = K, FI \cap DG = L$
Proof that
$1) J \in BC, K \in AB, L \in CA$
$2) AJ, BL, CK$ concurrent at point M on circumcircle of $\triangle ABC$

Attachments:



Luis González

#2 Feb 11, 2016, 2:33 am

According to my diagram the proposition 2) is false. In general, those lines are indeed concurrent but the concurrency point is not on the circumcircle of ABC. For 1) we have the following projective generalization:

P, Q are arbitrary points on the plane of $\triangle ABC$ and $\triangle P_AP_BP_C, \triangle Q_AQ_BQ_C$ are the anticevian triangles of P, Q WRT $\triangle ABC$. Then P_BQ_B, P_CQ_C meet at point A_0 on BC. Moreover if we define B_0 and C_0 cyclically, then AA_0, BB_0, CC_0 concur at the trilinear pole of PQ WRT $\triangle ABC$.

Proof: Let $\triangle DEF$ and $\triangle XYZ$ be the cevian triangles of P and Q WRT $\triangle ABC$, respectively. Consider a homology sending PQ to the line at infinity. Since $(P_A, A, D, P) = -1$, then P_A becomes midpoint of AD and similarly P_B, Q_B, P_C, Q_C become midpoints of BE, BY, CF, CZ, respectively. Thus P_BQ_B is the B-midline of $\triangle BEY$, intersecting BC at its midpoint A_0 and similarly $A_0 \in P_CQ_C$. So back in the primitive figure, it follows that $A_0 \equiv BC \cap P_BQ_B \cap P_CQ_C$ and A_0 is the harmonic conjugate of $PQ \cap BC$ WRT $B, C \Longrightarrow AA_0$ passes through the trilinear pole R of PQ WRT $\triangle ABC$ and similarly BB_0, CC_0 pass through R.

Quick Reply

High School Olympiads

Try this geometry

Source: Mock INMO by Viswanath-Anant-Kapil

Ankoganit

#1 Feb 10, 2016, 9:40 am

In ABC with circumcentre O the internal angle bisector of $\angle BAC$ meets the circumcircle of ΔBOC at a point P such that P, O are on same side of BC .Let O_1, O_2 be the circumcentres of $\Delta APB, \Delta APC$, let $AB \cap (APC) = U$ and $AC \cap (APB) = V$. Prove that the lines O_1V, O_2U and the external bisector of $\angle BAC$ are concurrent. (Here, (XYZ) means the circumcircle of ΔXYZ)

Luis González

#2 Feb 10, 2016, 11:48 am • 1

Let OP cut $(O_2), (O_1)$ again at $Y, Z,$ resp and let O_1V and O_2U meet at M. $\angle YAC = \angle OPC = \angle OBC = 90^\circ - \angle BAC \Longrightarrow \angle BAY = 90^\circ$ and likewise $\angle CAZ = 90^\circ \Longrightarrow AP$ also bisects $\angle YAZ$ and UY, VZ are diameters of $(O_2), (O_1)$. Since $\angle PUY = \angle PAY = \angle PAZ = \angle PVZ,$ it follows that $\angle O_2YP = \angle O_1ZP \Longrightarrow \triangle MYZ$ is isosceles with $MY = MZ \Longrightarrow M$ is midpoint of the arc YAZ of $\odot(YAZ)$, i.e. A, Y, Z, M are concyclic $\Longrightarrow \angle MAY = \angle O_1ZP = 90^\circ - \angle PAZ \Longrightarrow AM$ is external bisector of $\angle YAZ, \angle BAC$ $\Longrightarrow M$ is on the external bisector of $\angle BAC$.

houssam9990

#3 Feb 11, 2016, 2:44 am

claim : U, V, P are collinear.

proof:

$AUP + APV = ACP + ABV = 180 - APC - A/2 + 180 - A - AVB = 360 - (APC + APB + A + A/2) = BPC - A - A/2 = BOC - A - A/2 = A/2 = BAP = AUP + APU$

hence $APU = APV$ as desired.

now let $Q = O_1V \cap l$ where l is the A-external angle bisector.

we have:

$AQV = 180 - QAV - AVQ = AVO_1 - QAV = 90 - ABV - (90 - A/2) = A/2 - ABV = A/2 - (ABP - VBP) = A - ABP = A - UVA = AUV$

therefore U, A, V, Q are concyclic.

similarly, we define $R = O_2U \cap l$ and angle chase that easily $VUR = VAR$.

thus, $R, Q \in (AUV) \cap l$. hence we are done. (one may want to assume U, V are different from A)

This post has been edited 2 times. Last edited by houssam9990, Feb 11, 2016, 2:48 am

PROF65

#4 Feb 11, 2016, 3:11 am

Let E be the intersection of O_1V and O_2U.
$\widehat{O_1PO_2} = \widehat{O_1PA} + \widehat{APO_2} = \frac{\pi}{2} - \widehat{ABP} + \frac{\pi}{2} - \widehat{ACP} = \pi - \hat{A}$ and $\widehat{UO_2P} = \hat{A}, \widehat{VO_1P} = \hat{A}$ thus O_1PO_2E is parallelogram implies $UAVE$ is cyclic

we have also $\widehat{EO_1P} = \widehat{VO_1P} = \hat{A}, \widehat{VPO_1} = \frac{\pi}{2} - \frac{\hat{A}}{2}$ besides $\widehat{EO_2P} = \widehat{UO_2P} = \hat{A}, \widehat{UPO_2} = \frac{\pi}{2} - \frac{\hat{A}}{2}$ then U, V and P are colinear

$\widehat{EAV} = \widehat{EUV} = \widehat{O_2UP} = \frac{\pi}{2} - \frac{\hat{A}}{2}$ which means that E lies on the A-exterior bisector.

WCP

suli

#5 Feb 11, 2016, 5:48 am

Lemma. P, U, V collinear.

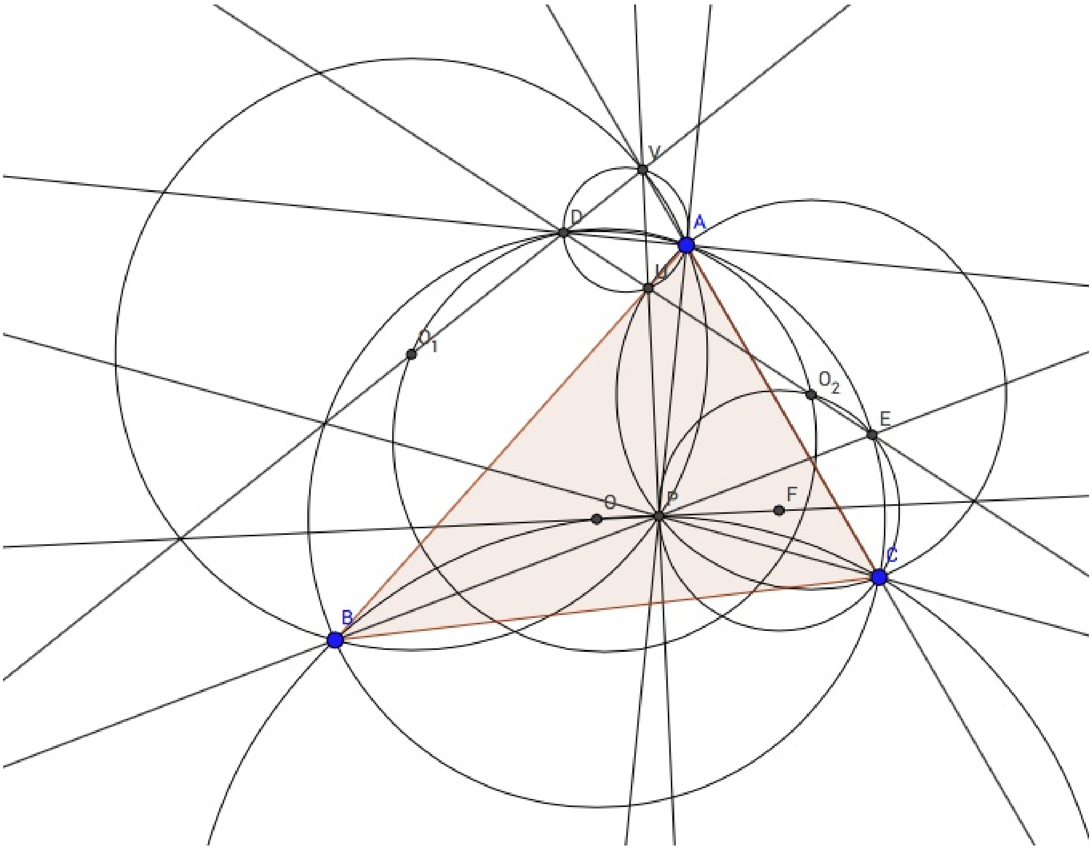
Proof. Clearly $180^\circ - \angle A$ and $\angle UPB = 180^\circ - \angle A$. But $\angle BPC = 2\angle A$, so $\angle UPV = 0^\circ$, so U, P, V collinear.

As a result, $90^\circ - \angle O_1VA = \angle VPA = \angle UPA = 90^\circ - \angle AUO_2$, so $\angle O_1VA = \angle AUO_2$. Thus if O_1V and O_2U intersect at D, then $AUVD$ is cyclic. Thus
$\angle DAU = \angle DVU = 90^\circ - (180^\circ - \angle VAP) = \angle VAP - 90^\circ = 90^\circ - \angle A/2$, so AD is external angle bisector of $\angle A$.

Now try proving the stronger result that A, B, C, D are concyclic

Hint

Attachments:



This post has been edited 4 times. Last edited by suli, Feb 11, 2016, 6:39 am

Quick Reply

High School Olympiads

The locus of P

Source: Mock INMO 2

Ankoganit

#1 Feb 10, 2016, 10:16 am

Let D be any point on the side BC of triangle ABC and let I_1, I_2 be the incentres of triangle ABD and triangle ACD respectively. Let the circumcircle of triangle I_1I_2D meet the segment AD again at P. Prove that as D varies, point P moves on a fixed circle.

(Suggested by Kapil)

Luis González

#2 Feb 10, 2016, 10:33 am • 2

P is clearly the intersection of AD with the common external tangent of $(I_1), (I_2)$, other than BC. Now if the incircle of $\triangle ABC$ touches AC at Y, then P lies on the fixed circle $\odot(A, AY)$ (see the problem Locus of intersection).

High School Olympiads

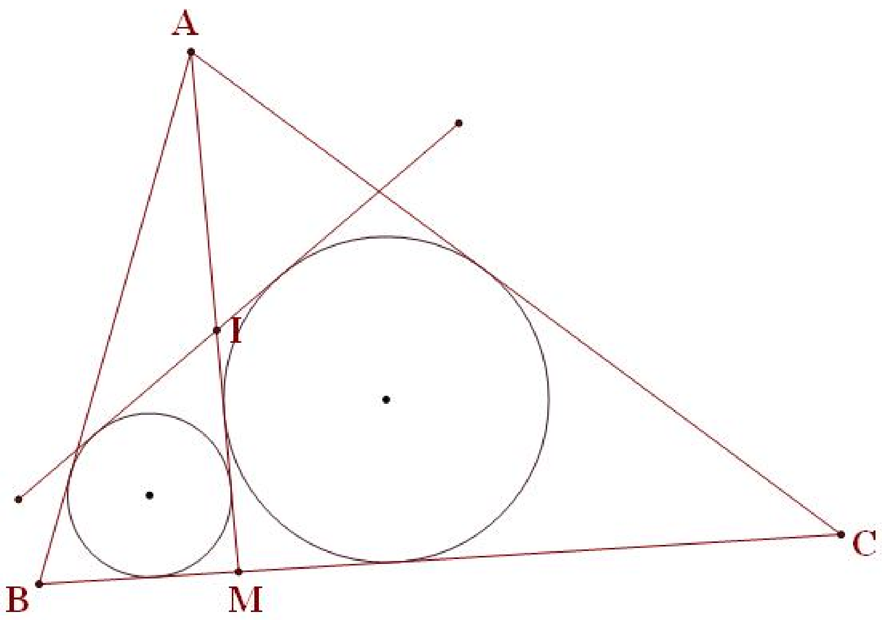
Locus of intersection

Reply

mathlink
#1 Jan 10, 2011, 10:58 am

Given a triangle ABC. Point M moves on the side BC. Let S and S' be the incircles of triangle AMB and AMC, respectively. The external tangent of S, S' ($\neq BC$) meet AM at I. Find the locus of I when M is changed.

Attachments:



Luis González
#2 Jan 10, 2011, 12:37 pm • 1

Assume that the common external tangent of $(S), (S')$, different from BC, cut $\overrightarrow{CB}$ at D. (S) and (S') touch AM at P, Q. Since $(S), (S')$ become incircle and D-excircle of $\triangle DIM$, it follows that $PM = IQ$. Therefore

$$AI = AQ - IQ = AQ - PM = \frac{1}{2}(AC + AM - MC) - \frac{1}{2}(AM + MB - AB)$$

$$\Longrightarrow AI = \frac{1}{2}(AB + AC - BC) = \text{const}.$$

Thus, locus of I is the circle centered at A that passes through the tangency points of the incircle with AB, AC.

Quick Reply

High School Olympiads

Perspective Triangles

Reply

ABCDE
#1 Feb 8, 2016, 12:50 pm

Let ABC be a triangle with incenter I. Let the orthocenters of IBC, ICA, and IAB be H_a, H_b, and H_c respectively. Prove that ABC is perspective with the anticomplimentary triangle of $H_aH_bH_c$.

TelvCohl
#2 Feb 8, 2016, 5:50 pm • 1

Let $\triangle XYZ$ be the anticomplementary triangle of $\triangle H_aH_bH_c$ and let $\triangle DEF, \triangle M_aM_bM_c$ be the intouch triangle, medial triangle of $\triangle ABC$, respectively. Since M_a, M_b, M_c is the pole of H_bH_c, H_cH_a, H_aH_b WRT $\odot(I)$, respectively, so notice $IM_a \perp H_bH_c \Longrightarrow IM_a \perp YZ$ we get the pole X_1 of YZ WRT $\odot(I)$ is the intersection of IM_a and M_bM_c.

Since I is complement of the Nagel point of $\triangle ABC$ WRT $\triangle ABC$, so $\frac{M_bX_1}{M_cX_1} = \frac{CD}{BD} \Longrightarrow A, D, X_1$ are collinear, hence their polar EF, BC, YZ WRT $\odot(I)$ are concurrent $\Longrightarrow YZ \cap BC$ lies on the perspectrix τ of $\triangle ABC, \triangle DEF$. Similarly, we can prove $ZX \cap CA, XY \cap AB$ lies on τ, so from Desargue theorem we conclude that AX, BY, CZ are concurrent.

Remark : Since the orthology center of $\{\triangle ABC, \triangle XYZ\}$ is the Nagel point of $\triangle ABC$, so the perspector of $\triangle ABC$ and $\triangle XYZ$ lies on the Feuerbach hyperbola (circum-rectangular hyperbola passing through Nagel point) of $\triangle ABC$.

ltf0501
#3 Feb 8, 2016, 11:01 pm

Why M_a, M_b, M_c is the pole of H_bH_c, H_cH_a, H_aH_b WRT $\odot(I)$???

Luis González
#4 Feb 9, 2016, 12:03 am • 3

There is more to say: $\triangle ABC$ and the anticomplementary triangle $\triangle XYZ$ of $\triangle H_aH_bH_c$ are perspective and the perspector $AX \cap BY \cap CZ$ is the isogonal conjugate of the centroid of the excentral triangle, i.e. X_{3062} of $\triangle ABC$.

Proof: Let $\triangle I_aI_bI_c$ be the excentral triangle of $\triangle ABC$ with centroid U. Let D be the midpoint of BC and let L be the common midpoint of XH_a, H_bH_c. Clearly AI_cBH_c and AI_bCH_b are parallelograms $\Longrightarrow DL$ is midparallel of $CH_b \parallel BH_c$ $\Longrightarrow I_aX \parallel DL \parallel CH_b \parallel BH_c \parallel I_bI_c$. Moreover, if the parallel I_aX from I_a to I_bI_c cuts $\odot(I_aI_bI_c)$ again at S, we get $I_aX = 2 \cdot DL = |CH_b - BH_c| = |AI_b - AI_c| = I_aS \Longrightarrow X$ is the reflection of S across $AI_a \Longrightarrow AX$ and AS are isogonals WRT $\angle BAC$. But notice that A and S are clearly homologous points under the homothety with center U that takes the circumcircle of $\triangle I_aI_bI_c$ into its 9-point circle $\odot(ABC)$, thus $U \in AS \Longrightarrow AX$ and AU are isogonals WRT $\triangle ABC$ and likewise BY and CZ are the isogonals of BU and CU WRT $\triangle ABC$. Hence we conclude that AX, BY, CZ concur at the isogonal conjugate of U WRT $\triangle ABC$, as desired.

Quick Reply

High School Olympiads

very nice and very hard!

Locked

Source: 10.4 Final Round of Sharygin geometry Olympiad 2015

MRF2017

#1 Feb 8, 2016, 9:52 pm

Let AA_1, BB_1, CC_1 be the altitudes of an acute-angled, nonisosceles triangle ABC, and A_2, B_2, C_2 be the touching points of sides BC, CA, AB with the correspondent excircles. It is known that line B_1C_1 touches the incircle of ABC. Prove that A_1 lies on the circumcircle of $A_2B_2C_2$.

Luis González

#2 Feb 8, 2016, 10:04 pm • 1

Already discussed at http://www.artofproblemsolving.com/community/c6h1187477.

High School Olympiads

Line tangent to incircle

Reply

Source: Inspired from Sharygin Geometry Olympiad 2015

livetolove212

#1 Jan 21, 2016, 12:22 am • 1

Let AA_1, BB_1, CC_1 be the altitudes of an acute-angled, nonisosceles triangle ABC, and A_2, B_2, C_2 be the touching points of sides BC, CA, AB with the correspondent excircles. Prove that B_1C_1 touches the incircle of ABC if and only if A_1 lies on the circumcircle of $A_2B_2C_2$.

Source.

anantmudgal09

#2 Jan 21, 2016, 12:53 am

Isn't this more like the actual Sharygin problem?

Edit:- Oh I see, sorry about that. Nice generalisation by the way.

This post has been edited 1 time. Last edited by anantmudgal09, Jan 21, 2016, 1:19 am

livetolove212

#3 Jan 21, 2016, 1:03 am

We have to prove "if and only if", which is not trivial

TelvCohl

#4 Jan 21, 2016, 2:18 am • 2

I'll just prove $A_1 \in \odot(A_2B_2C_2) \Longrightarrow B_1C_1$ touches the incircle of $\triangle ABC$.

Suppose the Bevan point B_e of $\triangle ABC$ doesn't coincide with A_2, then from $A_1 \in \odot(A_2B_2C_2)$ we get the isogonal conjugate of B_e WRT $\triangle ABC$ lies on the A-altitude of $\triangle ABC \Longrightarrow AB_e$ passes through the circumcenter O of $\triangle ABC$, hence notice B_e is the reflection of the incenter I of $\triangle ABC$ in O we get AO is the bisector of $\angle BAC \Longrightarrow AI$ coincide with the A-altitude of $\triangle ABC \Longrightarrow AB = AC$ (contradict the given condition).

From the discussion above we know $B_e \equiv A_2$. Let $\triangle A_3B_3C_3$ be the intouch triangle of $\triangle ABC$. Since $IA_3 = 2\text{dist}(O, BC) = AH$, so AHA_3I is a parallelogram $\Longrightarrow A_3H \perp B_3C_3$. From Passes through intersection of two tangent (**Lemma** at post #2) $\Longrightarrow$ the line connecting H and the reflection of I in B_3C_3 passes through the projection W of A_3 on B_3C_3, so H coincide with $W \Longrightarrow H \in B_3C_3$, hence from Newton theorem we conclude that B_1C_1 touches the incircle of $\triangle ABC$.

livetolove212

#5 Jan 21, 2016, 11:27 pm

Another proof for the converse.
Assume that A_2 is not Bevan point of triangle ABC. Let A_2' be Bevan point of $\triangle ABC$. Let K be the projection of A_2' onto AA_1. We get $A_1KA_2'A_2$ is a rectangle and K, B_2, C_2 lie on (AA_2'). Since the radical axes of $(AA_2'), (A_2B_2C_2A_1), (A_2A_1KA_2')$ concur at the radical center of 3 circles, we get KA_2', A_1A_2, B_2C_2 are concurrent. But $A_1A_2 \parallel KA_2'$ and B_2C_2 is not parallel to BC hence this is contradict.
Therefore A_2 is Bevan point of triangle ABC- the reflection of I wrt O. Let M be the midpoint of BC we get $OM = \frac{r}{2}$ hence $AH = r$ or H, I, M are collinear.
Let the bisector of $\angle B_1HC$ intersect AC at E, AB at F then $AE = AF$ and using ratio of power we get $(AEF), (AH), (O)$ are coaxal. The reflection of A through center of (AEF) is the intersection of HM and A-bisector