Romania National Olympiad 2008

–	Grade level 7
–	April 30th
1	Let ABC be an acute angled triangle with $\angle B > \angle C$. Let D be the foot of the altitude from A on BC, and let E be the foot of the perpendicular from D on AC. Let F be a point on the segment (DE). Show that the lines AF and BF are perpendicular if and only if $EF \cdot DC = BD \cdot DE$.
2	A rectangle can be divided by parallel lines to its sides into 200 congruent squares, and also in 288 congruent squares. Prove that the rectangle can also be divided into 392 congruent squares.
3	Let p, q, r be 3 prime numbers such that $5 \leq p < q < r$. Knowing that $2p^2 - r^2 \geq 49$ and $2q^2 - r^2 \leq 193$, find p, q, r.
4	Let $ABCD$ be a rectangle with center O, $AB \neq BC$. The perpendicular from O to BD cuts the lines AB and BC in E and F respectively. Let M, N be the midpoints of the segments CD, AD respectively. Prove that $FM \perp EN$.
–	Grade level 8
1	A tetrahedron has the side lengths positive integers, such that the product of any two opposite sides equals 6. Prove that the tetrahedron is a regular triangular pyramid in which the lateral sides form an angle of at least 30 degrees with the base plane.
2	a) We call *admissible sequence* a sequence of 4 even digits in which no digits appears more than two times. Find the number of admissible sequences. b) For each integer $n \geq 2$ we denote d_n the number of possibilities of completing with even digits an array with n rows and 4 columns, such that (1) any row is an admissible sequence; (2) the sequence 2, 0, 0, 8 appears exactly ones in the array. Find the values of n for which the number $\frac{d_{n+1}}{d_n}$ is an integer.

3 Let $a, b \in [0,1]$. Prove that

$$\frac{1}{1+a+b} \leq 1 - \frac{a+b}{2} + \frac{ab}{3}.$$

4 Let $ABCDA'B'C'D'$ be a cube. On the sides $(A'D')$, $(A'B')$ and $(A'A)$ we consider the points M_1, N_1 and P_1 respectively. On the sides (CB), (CD) and (CC') we consider the points M_2, N_2 and P_2 respectively. Let d_1 be the distance between the lines M_1N_1 and M_2N_2, d_2 be the distance between the lines N_1P_1 and N_2P_2, and d_3 be the distance between the lines P_1M_1 and P_2M_2. Suppose that the distances d_1, d_2 and d_3 are pairwise distinct. Prove that the lines M_1M_2, N_1N_2 and P_1P_2 are concurrent.

– Grade level 9

1 Find functions $f : \mathbb{N} \to \mathbb{N}$, such that $f(x^2 + f(y)) = xf(x) + y$, for $x, y \in \mathbb{N}$.

2 a) Prove that

$$\frac{1}{2} + \frac{1}{3} + ... + \frac{1}{2^{2n}} > n,$$

for all positive integers n.

b) Prove that for every positive integer n we have $\min\left\{k \in \mathbb{Z}, k \geq 2 \mid \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} > n\right\} > 2^n$.

3 Let n be a positive integer and let a_i be real numbers, $i = 1, 2, \dots, n$ such that $|a_i| \leq 1$ and $\sum_{i=1}^n a_i = 0$.
Show that $\sum_{i=1}^n |x - a_i| \leq n$, for every $x \in \mathbb{R}$ with $|x| \leq 1$.

4 On the sides of triangle ABC we consider points $C_1, C_2 \in (AB), B_1, B_2 \in (AC), A_1, A_2 \in (BC)$ such that triangles A_1, B_1, C_1 and $A_2B_2C_2$ have a common centroid.
Prove that sets $[A_1, B_1] \cap [A_2B_2], [B_1C_1] \cap [B_2C_2], [C_1A_1] \cap [C_2A_2]$ are not empty.

– Grade level 10

1 Let ABC be a triangle and the points $D \in (BC)$, $E \in (CA)$, $F \in (AB)$ such that

$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB}.$$

Prove that if the circumcenters of the triangles DEF and ABC coincide then ABC is equilateral.

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2 Let a, b, c be 3 complex numbers such that

$$a|bc| + b|ca| + c|ab| = 0.$$

Prove that

$$|(a-b)(b-c)(c-a)| \geq 3\sqrt{3}|abc|.$$

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3 Let $A = \{1, 2, \ldots, 2008\}$. We will say that set X is an r-set if $\emptyset \neq X \subset A$, and $\sum_{x \in X} x \equiv r \pmod 3$. Let X_r, $r \in \{0, 1, 2\}$ be the set of r-sets.

Find which one of X_r has the most elements.

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4 We consider the proposition $p(n)$: $n^2 + 1$ divides $n!$, for positive integers n. Prove that there are infinite values of n for which $p(n)$ is true, and infinite values of n for which $p(n)$ is false.

– Grade level 11

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1 Let $f : (0, \infty) \to \mathbb{R}$ be a continous function such that the sequences $\{f(nx)\}_{n \geq 1}$ are nondecreasing for any real number x. Prove that f is nondecreasing.

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2 Let A be a $n \times n$ matrix with complex elements. Prove that $A^{-1} = \overline{A}$ if and only if there exists an invertible matrix B with complex elements such that $A = B^{-1} \cdot \overline{B}$.

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3 Let $f : \mathbb{R} \to \mathbb{R}$ be a function, two times derivable on $\mathbb{R}$ for which there exist $c \in \mathbb{R}$ such that

$$\frac{f(b) - f(a)}{b - a} \neq f'(c),$$

for all $a \neq b \in \mathbb{R}$.

Prove that $f''(c) = 0$.

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4 Let $A=(a_{ij})_{1\leq i,j\leq n}$ be a real $n\times n$ matrix, such that $a_{ij}+a_{ji}=0$, for all i,j. Prove that for all non-negative real numbers x,y we have

$$\det(A+xI_n)\cdot\det(A+yI_n)\geq\det(A+\sqrt{xy}I_n)^2.$$

– Grade level 12

1 Let $a>0$ and $f:[0,\infty)\to[0,a]$ be a continuous function on $(0,\infty)$ and having Darboux property on $[0,\infty)$. Prove that if $f(0)=0$ and for all nonnegative x we have

$$xf(x)\geq\int_0^x f(t)dt,$$

then f admits primitives on $[0,\infty)$.

2 Let $f:[0,1]\to\mathbb{R}$ be a derivable function, with a continuous derivative f' on $[0,1]$. Prove that if $f\left(\frac{1}{2}\right)=0$, then

$$\int_0^1\left(f'(x)\right)^2dx\geq 12\left(\int_0^1 f(x)dx\right)^2.$$

3 Let A be a unitary finite ring with n elements, such that the equation $x^n=1$ has a unique solution in A, $x=1$. Prove that

a) 0 is the only nilpotent element of A;

b) there exists an integer $k\geq 2$, such that the equation $x^k=x$ has n solutions in A.

4 Let $\mathcal{G}$ be the set of all finite groups with at least two elements.

a) Prove that if $G\in\mathcal{G}$, then the number of morphisms $f:G\to G$ is at most $\sqrt[p]{n^n}$, where p is the largest prime divisor of n, and n is the number of elements in G.

b) Find all the groups in $\mathcal{G}$ for which the inequality at point a) is an equality.