APMO 2001

1 For a positive integer n let $S(n)$ be the sum of digits in the decimal representation of n. Any positive integer obtained by removing several (at least one) digits from the right-hand end of the decimal representation of n is called a *stump* of n. Let $T(n)$ be the sum of all stumps of n. Prove that $n = S(n) + 9T(n)$.

2 Find the largest positive integer N so that the number of integers in the set $\{1, 2, \dots, N\}$ which are divisible by 3 is equal to the number of integers which are divisible by 5 or 7 (or both).

3 Two equal-sized regular n-gons intersect to form a $2n$-gon C. Prove that the sum of the sides of C which form part of one n-gon equals half the perimeter of C.

Alternative formulation:

Let two equal regular n-gons S and T be located in the plane such that their intersection $S \cap T$ is a $2n$-gon (with $n \geq 3$). The sides of the polygon S are coloured in red and the sides of T in blue.

Prove that the sum of the lengths of the blue sides of the polygon $S \cap T$ is equal to the sum of the lengths of its red sides.

4 A point in the plane with a cartesian coordinate system is called a *mixed point* if one of its coordinates is rational and the other one is irrational. Find all polynomials with real coefficients such that their graphs do not contain any mixed point.

5 Find the greatest integer n, such that there are $n + 4$ points A, B, C, D, $X_1, \dots, X_n$ in the plane with $AB \neq CD$ that satisfy the following condition: for each $i = 1, 2, \dots, n$ triangles ABX_i and CDX_i are equal.