Sharygin Geometry Olympiad 2012

1 In triangle ABC point M is the midpoint of side AB, and point D is the foot of altitude CD. Prove that $\angle A = 2\angle B$ if and only if $AC = 2MD$.

2 A cyclic n-gon is divided by non-intersecting (inside the n-gon) diagonals to $n-2$ triangles. Each of these triangles is similar to at least one of the remaining ones. For what n this is possible?

3 A circle with center I touches sides AB, BC, CA of triangle ABC in points C_1, A_1, B_1. Lines AI, CI, B_1I meet A_1C_1 in points X, Y, Z respectively. Prove that $\angle YB_1Z = \angle XB_1Z$.

4 Given triangle ABC. Point M is the midpoint of side BC, and point P is the projection of B to the perpendicular bisector of segment AC. Line PM meets AB in point Q. Prove that triangle QPB is isosceles.

5 On side AC of triangle ABC an arbitrary point is selected D. The tangent in D to the circumcircle of triangle BDC meets AB in point C_1; point A_1 is defined similarly. Prove that $A_1C_1 \parallel AC$.

6 Point C_1 of hypothenuse AC of a right-angled triangle ABC is such that $BC = CC_1$. Point C_2 on cathetus AB is such that $AC_2 = AC_1$; point A_2 is defined similarly. Find angle AMC, where M is the midpoint of A_2C_2.

7 In a non-isosceles triangle ABC the bisectors of angles A and B are inversely proportional to the respective sidelengths. Find angle C.

8 Let BM be the median of right-angled triangle $ABC(\angle B = 90^\circ)$. The incircle of triangle ABM touches sides AB, AM in points A_1, A_2; points C_1, C_2 are defined similarly. Prove that lines A_1A_2 and C_1C_2 meet on the bisector of angle ABC.

9 In triangle ABC, given lines l_b and l_c containing the bisectors of angles B and C, and the foot L_1 of the bisector of angle A. Restore triangle ABC.

10 In a convex quadrilateral all sidelengths and all angles are pairwise different.
a) Can the greatest angle be adjacent to the greatest side and at the same time the smallest angle be adjacent to the smallest side?

b) Can the greatest angle be non-adjacent to the smallest side and at the same time the smallest angle be non-adjacent to the greatest side?

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11 Given triangle ABC and point P. Points A', B', C' are the projections of P to BC, CA, AB. A line passing through P and parallel to AB meets the circumcircle of triangle $PA'B'$ for the second time in point C_1. Points A_1, B_1 are defined similarly. Prove that
a) lines AA_1, BB_1, CC_1 concur;
b) triangles ABC and $A_1B_1C_1$ are similar.

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12 Let O be the circumcenter of an acute-angled triangle ABC. A line passing through O and parallel to BC meets AB and AC in points P and Q respectively. The sum of distances from O to AB and AC is equal to OA. Prove that $PB + QC = PQ$.

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13 Points A, B are given. Find the locus of points C such that C, the midpoints of AC, BC and the centroid of triangle ABC are concyclic.

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14 In a convex quadrilateral $ABCD$ suppose $AC \cap BD = O$ and M is the midpoint of BC. Let $MO \cap AD = E$. Prove that $\frac{AE}{ED} = \frac{S_{\triangle ABO}}{S_{\triangle CDO}}$.

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15 Given triangle ABC. Consider lines l with the next property: the reflections of l in the sidelines of the triangle concur. Prove that all these lines have a common point.

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16 Given right-angled triangle ABC with hypothenuse AB. Let M be the midpoint of AB and O be the center of circumcircle ω of triangle CMB. Line AC meets ω for the second time in point K. Segment KO meets the circumcircle of triangle ABC in point L. Prove that segments AL and KM meet on the circumcircle of triangle ACM.

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17 A square $ABCD$ is inscribed into a circle. Point M lies on arc BC, AM meets BD in point P, DM meets AC in point Q. Prove that the area of quadrilateral $APQD$ is equal to the half of the area of the square.

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18 A triangle and two points inside it are marked. It is known that one of the triangles angles is equal to $58°$, one of two remaining angles is equal to $59°$, one of two given points is the incenter of the triangle and the second one is its circumcenter. Using only the ruler without partitions determine where is each of the angles and where is each of the centers.

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19 Two circles with radii 1 meet in points X, Y, and the distance between these points also is equal to 1. Point C lies on the first circle, and lines CA, CB are tangents to the second one. These tangents meet the first circle for the second time in points B', A'. Lines AA' and BB' meet in point Z. Find angle XZY.

20 Point D lies on side AB of triangle ABC. Let ω_1 and Ω_1, ω_2 and Ω_2 be the incircles and the excircles (touching segment AB) of triangles ACD and BCD. Prove that the common external tangents to ω_1 and ω_2, Ω_1 and Ω_2 meet on AB.

21 Two perpendicular lines pass through the orthocenter of an acute-angled triangle. The sidelines of the triangle cut on each of these lines two segments: one lying inside the triangle and another one lying outside it. Prove that the product of two internal segments is equal to the product of two external segments.

22 A circle ω with center I is inscribed into a segment of the disk, formed by an arc and a chord AB. Point M is the midpoint of this arc AB, and point N is the midpoint of the complementary arc. The tangents from N touch ω in points C and D. The opposite sidelines AC and BD of quadrilateral $ABCD$ meet in point X, and the diagonals of $ABCD$ meet in point Y. Prove that points X, Y, I and M are collinear.

23 An arbitrary point is selected on each of twelve diagonals of the faces of a cube.The centroid of these twelve points is determined. Find the locus of all these centroids.

24 Given are n $(n > 2)$ points on the plane such that no three of them are collinear. In how many ways this set of points can be divided into two non-empty subsets with non-intersecting convex envelops?