China National Olympiad 2004

Day 1

1 Let $EFGH, ABCD$ and $E_1F_1G_1H_1$ be three convex quadrilaterals satisfying:

i) The points E, F, G and H lie on the sides AB, BC, CD and DA respectively, and $\frac{AE}{EB} \cdot \frac{BF}{FC} \cdot \frac{CG}{GD} \cdot \frac{DH}{HA} = 1$;
ii) The points A, B, C and D lie on sides H_1E_1, E_1F_1, F_1, G_1 and G_1H_1 respectively, and $E_1F_1||EF, F_1G_1||FG, G_1H_1||GH, H_1E_1||HE$.

Suppose that $\frac{E_1A}{AH_1} = \lambda$. Find an expression for $\frac{F_1C}{CG_1}$ in terms of λ.

Xiong Bin

2 Let c be a positive integer. Consider the sequence $x_1, x_2, \ldots$ which satisfies $x_1 = c$ and, for $n \geq 2$,

$$x_n = x_{n-1} + \left\lfloor \frac{2x_{n-1} - (n+2)}{n} \right\rfloor + 1$$

where $\lfloor x \rfloor$ denotes the largest integer not greater than x. Determine an expression for x_n in terms of n and c.

Huang Yumin

3 Let M be a set consisting of n points in the plane, satisfying:
i) there exist 7 points in M which constitute the vertices of a convex heptagon;
ii) if for any 5 points in M which constitute the vertices of a convex pentagon, then there is a point in M which lies in the interior of the pentagon.
Find the minimum value of n.

Leng Gangsong

Day 2

1 For a given real number a and a positive integer n, prove that:
i) there exists exactly one sequence of real numbers $x_0, x_1, \ldots, x_n, x_{n+1}$ such that

$$\begin{cases} x_0 = x_{n+1} = 0, \\ \frac{1}{2}(x_i + x_{i+1}) = x_i + x_i^3 - a^3, \ i = 1, 2, \ldots, n. \end{cases}$$

ii) the sequence $x_0, x_1, \ldots, x_n, x_{n+1}$ in i) satisfies $|x_i| \leq |a|$ where $i = 0, 1, \ldots, n+1$.

Liang Yengde

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2 For a given positive integer $n \geq 2$, suppose positive integers a_i where $1 \leq i \leq n$ satisfy $a_1 < a_2 < \ldots < a_n$ and $\sum_{i=1}^{n} \frac{1}{a_i} \leq 1$. Prove that, for any real number x, the following inequality holds

$$\left(\sum_{i=1}^{n} \frac{1}{a_i^2 + x^2} \right)^2 \leq \frac{1}{2} \cdot \frac{1}{a_1(a_1 - 1) + x^2}$$

Li Shenghong

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3 Prove that every positive integer n, except a finite number of them, can be represented as a sum of 2004 positive integers: $n = a_1 + a_2 + \cdots + a_{2004}$, where $1 \leq a_1 < a_2 < \cdots < a_{2004}$, and $a_i \mid a_{i+1}$ for all $1 \leq i \leq 2003$.

Chen Yonggao
