ELMO Shortlist 2013

– Algebra

1 Find all triples (f, g, h) of injective functions from the set of real numbers to itself satisfying

$$\begin{aligned} f(x+f(y)) &= g(x)+h(y) \\ g(x+g(y)) &= h(x)+f(y) \\ h(x+h(y)) &= f(x)+g(y) \end{aligned}$$

for all real numbers x and y. (We say a function F is *injective* if $F(a) \neq F(b)$ for any distinct real numbers a and b.)

Proposed by Evan Chen

2 Prove that for all positive reals a, b, c,

$$\frac{1}{a+\frac{1}{b}+1}+\frac{1}{b+\frac{1}{c}+1}+\frac{1}{c+\frac{1}{a}+1} \geq \frac{3}{\sqrt[3]{abc}+\frac{1}{\sqrt[3]{abc}}+1}.$$

Proposed by David Stoner

3 Find all $f:\mathbb{R}\to\mathbb{R}$ such that for all $x, y \in \mathbb{R}$, $f(x)+f(y)=f(x+y)$ and $f(x^{2013})=f(x)^{2013}$.

Proposed by Calvin Deng

4 Positive reals a, b, and c obey $\frac{a^2+b^2+c^2}{ab+bc+ca}=\frac{ab+bc+ca+1}{2}$. Prove that

$$\sqrt{a^2+b^2+c^2} \leq 1+\frac{|a-b|+|b-c|+|c-a|}{2}.$$

Proposed by Evan Chen

5 Let a, b, c be positive reals satisfying $a+b+c=\sqrt[7]{a}+\sqrt[7]{b}+\sqrt[7]{c}$. Prove that $a^a b^b c^c \geq 1$.

Proposed by Evan Chen

6 Let a, b, c be positive reals such that $a+b+c=3$. Prove that

$$18\sum_{\text{cyc}}\frac{1}{(3-c)(4-c)}+2(ab+bc+ca)\ge 15.$$

Proposed by David Stoner

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7 Consider a function $f:\mathbb{Z}\to\mathbb{Z}$ such that for every integer $n\ge 0$, there are at most $0.001n^2$ pairs of integers (x,y) for which $f(x+y)\ne f(x)+f(y)$ and $\max\{|x|,|y|\}\le n$. Is it possible that for some integer $n\ge 0$, there are more than n integers a such that $f(a)\ne a\cdot f(1)$ and $|a|\le n$?

Proposed by David Yang

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8 Let a, b, c be positive reals with $a^{2014}+b^{2014}+c^{2014}+abc=4$. Prove that

$$\frac{a^{2013}+b^{2013}-c}{c^{2013}}+\frac{b^{2013}+c^{2013}-a}{a^{2013}}+\frac{c^{2013}+a^{2013}-b}{b^{2013}}\ge a^{2012}+b^{2012}+c^{2012}.$$

Proposed by David Stoner

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9 Let a, b, c be positive reals, and let $\sqrt[2013]{\frac{3}{a^{2013}+b^{2013}+c^{2013}}}=P$. Prove that

$$\prod_{\text{cyc}}\left(\frac{(2P+\frac{1}{2a+b})(2P+\frac{1}{a+2b})}{(2P+\frac{1}{a+b+c})^2}\right)\ge\prod_{\text{cyc}}\left(\frac{(P+\frac{1}{4a+b+c})(P+\frac{1}{3b+3c})}{(P+\frac{1}{3a+2b+c})(P+\frac{1}{3a+b+2c})}\right).$$

Proposed by David Stoner

– Combinatorics

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1 Let $n\ge 2$ be a positive integer. The numbers $1, 2, ..., n^2$ are consecutively placed into squares of an $n\times n$, so the first row contains $1, 2, ..., n$ from left to right, the second row contains $n+1, n+2, ..., 2n$ from left to right, and so on. The *magic square value* of a grid is defined to be the number of rows, columns, and main diagonals whose elements have an average value of $\frac{n^2+1}{2}$. Show that the magic-square value of the grid stays constant under the following two operations: (1) a permutation of the rows; and (2) a permutation of the columns. (The operations can be used multiple times, and in any order.)

Proposed by Ray Li

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2 Let n be a fixed positive integer. Initially, n 1's are written on a blackboard. Every minute, David picks two numbers x and y written on the blackboard, erases them, and writes the number $(x+y)^4$ on the blackboard. Show that after $n-1$ minutes, the number written on the blackboard is at least $2^{\frac{4n^2-4}{3}}$.

Proposed by Calvin Deng

3 Let $a_1, a_2, ..., a_9$ be nine real numbers, not necessarily distinct, with average m. Let A denote the number of triples $1 \le i < j < k \le 9$ for which $a_i + a_j + a_k \ge 3m$. What is the minimum possible value of A?

Proposed by Ray Li

4 Let n be a positive integer. The numbers $\{1, 2, ..., n^2\}$ are placed in an $n \times n$ grid, each exactly once. The grid is said to be *Muirhead-able* if the sum of the entries in each column is the same, but for every $1 \le i, k \le n-1$, the sum of the first k entries in column i is at least the sum of the first k entries in column $i+1$. For which n can one construct a Muirhead-able array such that the entries in each column are decreasing?

Proposed by Evan Chen

5 There is a 2012×2012 grid with rows numbered $1, 2, \dots 2012$ and columns numbered $1, 2, \dots, 2012$, and we place some rectangular napkins on it such that the sides of the napkins all lie on grid lines. Each napkin has a positive integer thickness. (in micrometers!)

(a) Show that there exist 2012^2 unique integers $a_{i,j}$ where $i, j \in [1, 2012]$ such that for all $x, y \in [1, 2012]$, the sum

$$\sum_{i=1}^{x} \sum_{j=1}^{y} a_{i,j}$$

is equal to the sum of the thicknesses of all the napkins that cover the grid square in row x and column y.
(b) Show that if we use at most $500,000$ napkins, at least half of the $a_{i,j}$ will be 0.

Proposed by Ray Li

6 A 4×4 grid has its 16 cells colored arbitrarily in three colors. A *swap* is an exchange between the colors of two cells. Prove or disprove that it always takes at most three swaps to produce a line of symmetry, regardless of the grid's initial coloring.

Proposed by Matthew Babbitt

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7 A $2^{2014}+1$ by $2^{2014}+1$ grid has some black squares filled. The filled black squares form one or more snakes on the plane, each of whose heads splits at some points but never comes back together. In other words, for every positive integer n greater than 2, there do not exist pairwise distinct black squares s_1, s_2, \dots, s_n such that s_i and s_{i+1} share an edge for $i=1,2,\dots,n$ (here $s_{n+1}=s_1$).

What is the maximum possible number of filled black squares?

Proposed by David Yang

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8 There are 20 people at a party. Each person holds some number of coins. Every minute, each person who has at least 19 coins simultaneously gives one coin to every other person at the party. (So, it is possible that A gives B a coin and B gives A a coin at the same time.) Suppose that this process continues indefinitely. That is, for any positive integer n, there exists a person who will give away coins during the nth minute. What is the smallest number of coins that could be at the party?

Proposed by Ray Li

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9 Let f_0 be the function from $\mathbb{Z}^2$ to $\{0,1\}$ such that $f_0(0,0)=1$ and $f_0(x,y)=0$ otherwise. For each positive integer m, let $f_m(x,y)$ be the remainder when

$$f_{m-1}(x,y)+\sum_{j=-1}^{1}\sum_{k=-1}^{1} f_{m-1}(x+j,y+k)$$

is divided by 2.
Finally, for each nonnegative integer n, let a_n denote the number of pairs (x,y) such that $f_n(x,y)=1$.
Find a closed form for a_n.

Proposed by Bobby Shen

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10 Let $N\geq 2$ be a fixed positive integer. There are $2N$ people, numbered $1,2,...,2N$, participating in a tennis tournament. For any two positive integers i,j with $1\leq i<j\leq 2N$, player i has a higher skill level than player j. Prior to the first round, the players are paired arbitrarily and each pair is assigned a unique court among N courts, numbered $1,2,...,N$.

During a round, each player plays against the other person assigned to his court (so that exactly one match takes place per court), and the player with higher skill wins the match (in other words, there are no upsets). Afterwards, for

$i = 2, 3, ..., N$, the winner of court i moves to court $i-1$ and the loser of court i stays on court i; however, the winner of court 1 stays on court 1 and the loser of court 1 moves to court N.

Find all positive integers M such that, regardless of the initial pairing, the players $2, 3, \ldots, N+1$ all change courts immediately after the Mth round.

Proposed by Ray Li

– Geometry

1 Let ABC be a triangle with incenter I. Let U, V and W be the intersections of the angle bisectors of angles A, B, and C with the incircle, so that V lies between B and I, and similarly with U and W. Let X, Y, and Z be the points of tangency of the incircle of triangle ABC with BC, AC, and AB, respectively. Let triangle UVW be the *David Yang triangle* of ABC and let XYZ be the *Scott Wu triangle* of ABC. Prove that the David Yang and Scott Wu triangles of a triangle are congruent if and only if ABC is equilateral.

Proposed by Owen Goff

2 Let ABC be a scalene triangle with circumcircle Γ, and let D,E,F be the points where its incircle meets BC, AC, AB respectively. Let the circumcircles of $\triangle AEF$, $\triangle BFD$, and $\triangle CDE$ meet Γ a second time at X, Y, Z respectively. Prove that the perpendiculars from A, B, C to AX, BY, CZ respectively are concurrent.

Proposed by Michael Kural

3 In $\triangle ABC$, a point D lies on line BC. The circumcircle of ABD meets AC at F (other than A), and the circumcircle of ADC meets AB at E (other than A). Prove that as D varies, the circumcircle of AEF always passes through a fixed point other than A, and that this point lies on the median from A to BC.

Proposed by Allen Liu

4 Triangle ABC is inscribed in circle ω. A circle with chord BC intersects segments AB and AC again at S and R, respectively. Segments BR and CS meet at L, and rays LR and LS intersect ω at D and E, respectively. The internal angle bisector of $\angle BDE$ meets line ER at K. Prove that if $BE = BR$, then $\angle ELK = \frac{1}{2}\angle BCD$.

Proposed by Evan Chen

5 Let ω_1 and ω_2 be two orthogonal circles, and let the center of ω_1 be O. Diameter AB of ω_1 is selected so that B lies strictly inside ω_2. The two circles tangent to ω_2, passing through O and A, touch ω_2 at F and G. Prove that $FGOB$ is cyclic.

Proposed by Eric Chen

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6 Let $ABCDEF$ be a non-degenerate cyclic hexagon with no two opposite sides parallel, and define $X = AB \cap DE$, $Y = BC \cap EF$, and $Z = CD \cap FA$. Prove that

$$\frac{XY}{XZ} = \frac{BE}{AD}\frac{\sin|\angle B - \angle E|}{\sin|\angle A - \angle D|}.$$

Proposed by Victor Wang

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7 Let ABC be a triangle inscribed in circle ω, and let the medians from B and C intersect ω at D and E respectively. Let O_1 be the center of the circle through D tangent to AC at C, and let O_2 be the center of the circle through E tangent to AB at B. Prove that O_1, O_2, and the nine-point center of ABC are collinear.

Proposed by Michael Kural

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8 Let ABC be a triangle, and let D, A, B, E be points on line AB, in that order, such that $AC = AD$ and $BE = BC$. Let ω_1, ω_2 be the circumcircles of $\triangle ABC$ and $\triangle CDE$, respectively, which meet at a point $F \neq C$. If the tangent to ω_2 at F cuts ω_1 again at G, and the foot of the altitude from G to FC is H, prove that $\angle AGH = \angle BGH$.

Proposed by David Stoner

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9 Let $ABCD$ be a cyclic quadrilateral inscribed in circle ω whose diagonals meet at F. Lines AB and CD meet at E. Segment EF intersects ω at X. Lines BX and CD meet at M, and lines CX and AB meet at N. Prove that MN and BC concur with the tangent to ω at X.

Proposed by Allen Liu

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10 Let $AB = AC$ in $\triangle ABC$, and let D be a point on segment AB. The tangent at D to the circumcircle ω of BCD hits AC at E. The other tangent from E to ω touches it at F, and $G = BF \cap CD$, $H = AG \cap BC$. Prove that $BH = 2HC$.

Proposed by David Stoner

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11 Let $\triangle ABC$ be a nondegenerate isosceles triangle with $AB = AC$, and let D, E, F be the midpoints of BC, CA, AB respectively. BE intersects the circumcircle of $\triangle ABC$ again at G, and H is the midpoint of minor arc BC.

$CF \cap DG = I, BI \cap AC = J$. Prove that $\angle BJH = \angle ADG$ if and only if $\angle BID = \angle GBC$.

Proposed by David Stoner

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12 Let ABC be a nondegenerate acute triangle with circumcircle ω and let its incircle γ touch AB, AC, BC at X, Y, Z respectively. Let XY hit arcs AB, AC of ω at M, N respectively, and let $P \neq X, Q \neq Y$ be the points on γ such that $MP = MX, NQ = NY$. If I is the center of γ, prove that P, I, Q are collinear if and only if $\angle BAC = 90^\circ$.

Proposed by David Stoner

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13 In $\triangle ABC$, $AB < AC$. D and P are the feet of the internal and external angle bisectors of $\angle BAC$, respectively. M is the midpoint of segment BC, and ω is the circumcircle of $\triangle APD$. Suppose Q is on the minor arc AD of ω such that MQ is tangent to ω. QB meets ω again at R, and the line through R perpendicular to BC meets PQ at S. Prove SD is tangent to the circumcircle of $\triangle QDM$.

Proposed by Ray Li

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14 Let O be a point (in the plane) and T be an infinite set of points such that $|P_1P_2| \le 2012$ for every two distinct points $P_1, P_2 \in T$. Let $S(T)$ be the set of points Q in the plane satisfying $|QP| \le 2013$ for at least one point $P \in T$.

Now let L be the set of lines containing exactly one point of $S(T)$. Call a line ℓ_0 passing through O *bad* if there does not exist a line $\ell \in L$ parallel to (or coinciding with) ℓ_0.

\begin{enumerate}[(a)]
\item Prove that L is nonempty.
\item Prove that one can assign a line $\ell(i)$ to each positive integer i so that for every bad line ℓ_0 passing through O, there exists a positive integer n with $\ell(n) = \ell_0$.
\end{enumerate}

Proposed by David Yang

– Number Theory

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1 Find all ordered triples of non-negative integers (a, b, c) such that $a^2 + 2b + c$, $b^2 + 2c + a$, and $c^2 + 2a + b$ are all perfect squares.

Proposed by Matthew Babbitt

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2 For what polynomials $P(n)$ with integer coefficients can a positive integer be assigned to every lattice point in $\mathbb{R}^3$ so that for every integer $n \geq 1$, the sum of the n^3 integers assigned to any $n \times n \times n$ grid of lattice points is divisible by $P(n)$?

Proposed by Andre Arslan

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3 Define a *beautiful number* to be an integer of the form a^n, where $a \in \{3, 4, 5, 6\}$ and n is a positive integer.
Prove that each integer greater than 2 can be expressed as the sum of pairwise distinct beautiful numbers.

Proposed by Matthew Babbitt

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4 Find all triples (a, b, c) of positive integers such that if n is not divisible by any prime less than 2014, then $n + c$ divides $a^n + b^n + n$.

Proposed by Evan Chen

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5 Let $m_1, m_2, ..., m_{2013} > 1$ be 2013 pairwise relatively prime positive integers and $A_1, A_2, ..., A_{2013}$ be 2013 (possibly empty) sets with $A_i \subseteq \{1, 2, ..., m_i - 1\}$ for $i = 1, 2, ..., 2013$. Prove that there is a positive integer N such that

$$N \leq (2|A_1| + 1)(2|A_2| + 1) \cdots (2|A_{2013}| + 1)$$

and for each $i = 1, 2, ..., 2013$, there does *not* exist $a \in A_i$ such that m_i divides $N - a$.

Proposed by Victor Wang

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6 Let $\mathbb{N}$ denote the set of positive integers, and for a function f, let $f^k(n)$ denote the function f applied k times. Call a function $f : \mathbb{N} \to \mathbb{N}$ *saturated* if

$$f^{f^{f(n)}(n)}(n) = n$$

for every positive integer n. Find all positive integers m for which the following holds: every saturated function f satisfies $f^{2014}(m) = m$.

Proposed by Evan Chen

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7 Let p be a prime satisfying $p^2 \mid 2^{p-1} - 1$, and let n be a positive integer. Define

$$f(x) = \frac{(x-1)^{p^n} - (x^{p^n} - 1)}{p(x-1)}.$$

Find the largest positive integer N such that there exist polynomials $g(x)$, $h(x)$ with integer coefficients and an integer r satisfying $f(x) = (x-r)^N g(x) + p \cdot h(x)$.

Proposed by Victor Wang

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8 We define the *Fibonacci sequence* $\{F_n\}_{n\ge 0}$ by $F_0 = 0$, $F_1 = 1$, and for $n \ge 2$, $F_n = F_{n-1} + F_{n-2}$; we define the *Stirling number of the second kind* $S(n,k)$ as the number of ways to partition a set of $n \ge 1$ distinguishable elements into $k \ge 1$ indistinguishable nonempty subsets.

For every positive integer n, let $t_n = \sum_{k=1}^{n} S(n,k)F_k$. Let $p \ge 7$ be a prime. Prove that

$$t_{n+p^{2p}-1} \equiv t_n \pmod p$$

for all $n \ge 1$.

Proposed by Victor Wang
