China Girls Math Olympiad 2006

Day 1

1 Let $a > 0$, the function $f : (0, +\infty) \to R$ satisfies $f(a) = 1$, if for any positive reals x and y, there is

$$f(x)f(y) + f\left(\frac{a}{x}\right) f\left(\frac{a}{y}\right) = 2f(xy)$$

then prove that $f(x)$ is a constant.

2 Let O be the intersection of the diagonals of convex quadrilateral $ABCD$. The circumcircles of $\triangle OAD$ and $\triangle OBC$ meet at O and M. Line OM meets the circumcircles of $\triangle OAB$ and $\triangle OCD$ at T and S respectively.

Prove that M is the midpoint of ST.

3 Show that for any $i = 1, 2, 3$, there exist infinity many positive integer n, such that among n, $n+2$ and $n+28$, there are exactly i terms that can be expressed as the sum of the cubes of three positive integers.

4 8 people participate in a party.

(1) Among any 5 people there are 3 who pairwise know each other. Prove that there are 4 people who paiwise know each other.

(2) If Among any 6 people there are 3 who pairwise know each other, then can we find 4 people who pairwise know each other?

Day 2

5 The set $S = \{(a, b) \mid 1 \leq a, b \leq 5, a, b \in \mathbb{Z}\}$ be a set of points in the plane with integeral coordinates. T is another set of points with integeral coordinates in the plane. If for any point $P \in S$, there is always another point $Q \in T$, $P \neq Q$, such that there is no other integeral points on segment PQ. Find the least value of the number of elements of T.

6 Let $M = \{1, 2, \cdots, 19\}$ and $A = \{a_1, a_2, \cdots, a_k\} \subseteq M$. Find the least k so that for any $b \in M$, there exist $a_i, a_j \in A$, satisfying $b = a_i$ or $b = a_i \pm a_i$ (a_i and a_j do not have to be different) .

7 Given that $x_i > 0$, $i = 1, 2, \cdots, n$, $k \geq 1$. Show that:

$$\sum_{i=1}^{n} \frac{1}{1+x_i} \cdot \sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} \frac{x_i^{k+1}}{1+x_i} \cdot \sum_{i=1}^{n} \frac{1}{x_i^k}$$

8 Let p be a prime number that is greater than 3. Show that there exist some integers $a_1, a_2, \cdots a_k$ that satisfy:

$$-\frac{p}{2} < a_1 < a_2 < \cdots < a_k < \frac{p}{2}$$

making the product:

$$\frac{p-a_1}{|a_1|} \cdot \frac{p-a_2}{|a_2|} \cdots \frac{p-a_k}{|a_k|}$$

equals to 3^m where m is a positive integer.