India International Mathematical Olympiad Training Camp 2016

–	Practice Tests
–	Practice Test 1
1	An acute-angled ABC ($AB < AC$) is inscribed into a circle ω. Let M be the centroid of ABC, and let AH be an altitude of this triangle. A ray MH meets ω at A'. Prove that the circumcircle of the triangle $A'HB$ is tangent to AB. *(A.I. Golovanov , A.Yakubov)*
2	Given that n is a natural number such that the leftmost digits in the decimal representations of 2^n and 3^n are the same, find all possible values of the leftmost digit.
3	Let a,b,c,d be real numbers satisfying $\lvert a\rvert, \lvert b\rvert, \lvert c\rvert, \lvert d\rvert > 1$ and $abc + abd + acd + bcd + a + b + c + d = 0$. Prove that $\frac{1}{a-1} + \frac{1}{b-1} + \frac{1}{c-1} + \frac{1}{d-1} > 0$
–	Practice Test 2
1	We say a natural number n is perfect if the sum of all the positive divisors of n is equal to $2n$. For example, 6 is perfect since its positive divisors $1, 2, 3, 6$ add up to $12 = 2 \times 6$. Show that an odd perfect number has at least 3 distinct prime divisors. *Note: It is still not known whether odd perfect numbers exist. So assume such a number is there and prove the result.*
2	Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $$f\left(x^2 + xf(y)\right) = xf(x + y)$$ for all reals x, y.
3	An equilateral triangle with side length 3 is divided into 9 congruent triangular cells as shown in the figure below. Initially all the cells contain 0. A *move* consists of selecting two adjacent cells (i.e., cells sharing a common boundary) and either increasing or decreasing the numbers in both the cells by 1 simultaneously. Determine all positive integers n such that after performing several such moves one can obtain 9 consecutive numbers $n, (n+1), \cdots, (n+8)$ in some order.

– Team Selection Tests

– Team Selection Test 1

1 Let ABC be an acute triangle with orthocenter H. Let G be the point such that the quadrilateral $ABGH$ is a parallelogram. Let I be the point on the line GH such that AC bisects HI. Suppose that the line AC intersects the circumcircle of the triangle GCI at C and J. Prove that $IJ = AH$.

2 Suppose that a sequence $a_1, a_2, \ldots$ of positive real numbers satisfies

$$a_{k+1} \geq \frac{ka_k}{a_k^2 + (k-1)}$$

for every positive integer k. Prove that $a_1 + a_2 + \ldots + a_n \geq n$ for every $n \geq 2$.

3 Let n be a natural number. A sequence $x_1, x_2, \cdots, x_{n^2}$ of n^2 numbers is called $n-$*good* if each x_i is an element of the set $\{1, 2, \cdots, n\}$ and the ordered pairs (x_i, x_{i+1}) are all different for $i = 1, 2, 3, \cdots, n^2$ (here we consider the subscripts modulo n^2). Two $n-$good sequences $x_1, x_2, \cdots, x_{n^2}$ and $y_1, y_2, \cdots, y_{n^2}$ are called *similar* if there exists an integer k such that $y_i = x_{i+k}$ for all $i = 1, 2, \cdots, n^2$ (again taking subscripts modulo n^2). Suppose that there exists a non-trivial permutation (i.e., a permutation which is different from the identity permutation) σ of $\{1, 2, \cdots, n\}$ and an $n-$ good sequence $x_1, x_2, \cdots, x_{n^2}$ which is similar to $\sigma(x_1), \sigma(x_2), \cdots, \sigma(x_{n^2})$. Show that $n \equiv 2 \pmod 4$.

– Team Selection Test 2

1 Suppose α, β are two positive rational numbers. Assume for some positive integers m, n, it is known that $\alpha^{\frac{1}{n}} + \beta^{\frac{1}{m}}$ is a rational number. Prove that each of $\alpha^{\frac{1}{n}}$ and $\beta^{\frac{1}{m}}$ is a rational number.

2 Let m and n be positive integers such that $m > n$. Define $x_k = \frac{m+k}{n+k}$ for $k = 1, 2, \ldots, n+1$. Prove that if all the numbers $x_1, x_2, \ldots, x_{n+1}$ are integers, then $x_1 x_2 \ldots x_{n+1} - 1$ is divisible by an odd prime.

3 For a finite set A of positive integers, a partition of A into two disjoint nonempty subsets A_1 and A_2 is *good* if the least common multiple of the elements in A_1 is equal to the greatest common divisor of the elements in A_2. Determine the minimum value of n such that there exists a set of n positive integers with exactly 2015 good partitions.

– Team Selection Test 3

1 Let n be a natural number. We define sequences $\langle a_i \rangle$ and $\langle b_i \rangle$ of integers as follows. We let $a_0 = 1$ and $b_0 = n$. For $i > 0$, we let

$$(a_i, b_i) = \begin{cases} (2a_{i-1}+1, b_{i-1}-a_{i-1}-1) & \text{if } a_{i-1} < b_{i-1}, \\ (a_{i-1}-b_{i-1}-1, 2b_{i-1}+1) & \text{if } a_{i-1} > b_{i-1}, \\ (a_{i-1}, b_{i-1}) & \text{if } a_{i-1} = b_{i-1}. \end{cases}$$

Given that $a_k = b_k$ for some natural number k, prove that $n+3$ is a power of two.

2 Let ABC be an acute triangle and let M be the midpoint of AC. A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC. Determine all possible values of $\frac{BT}{BM}$.

3 Let n be an odd natural number. We consider an $n \times n$ grid which is made up of n^2 unit squares and $2n(n+1)$ edges. We colour each of these edges either *red* or *blue*. If there are at most n^2 *red* edges, then show that there exists a unit square at least three of whose edges are *blue*.

– Team Selection Test 4

1 Let ABC be an acute triangle with circumcircle Γ. Let A_1, B_1 and C_1 be respectively the midpoints of the arcs BAC, CBA and ACB of Γ. Show that the inradius of triangle $A_1B_1C_1$ is not less than the inradius of triangle ABC.

2 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(x^3 + f(y)\right) = x^2 f(x) + y,$$

for all $x, y \in \mathbb{R}$. (Here $\mathbb{R}$ denotes the set of all real numbers.)

3 Let $\mathbb{N}$ denote the set of all natural numbers. Show that there exists two nonempty subsets A and B of $\mathbb{N}$ such that

- $A \cap B = \{1\}$;
- every number in $\mathbb{N}$ can be expressed as the product of a number in A and a number in B;
- each prime number is a divisor of some number in A and also some number in B;
- one of the sets A and B has the following property: if the numbers in this set are written as $x_1 < x_2 < x_3 < \cdots$, then for any given positive integer M there exists $k \in \mathbb{N}$ such that $x_{k+1} - x_k \geq M$.
- Each set has infinitely many composite numbers.