APMO 2000

1 Compute the sum: $\sum_{i=0}^{101} \frac{x_i^3}{1-3x_i+3x_i^2}$ for $x_i = \frac{i}{101}$.

2 Find all permutations $a_1, a_2, \dots, a_9$ of $1, 2, \dots, 9$ such that

$$a_1 + a_2 + a_3 + a_4 = a_4 + a_5 + a_6 + a_7 = a_7 + a_8 + a_9 + a_1$$

and

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 = a_4^2 + a_5^2 + a_6^2 + a_7^2 = a_7^2 + a_8^2 + a_9^2 + a_1^2$$

3 Let ABC be a triangle. Let M and N be the points in which the median and the angle bisector, respectively, at A meet the side BC. Let Q and P be the points in which the perpendicular at N to NA meets MA and BA, respectively. And O the point in which the perpendicular at P to BA meets AN produced.

Prove that QO is perpendicular to BC.

4 Let n, k be given positive integers with $n > k$. Prove that:

$$\frac{1}{n+1} \cdot \frac{n^n}{k^k(n-k)^{n-k}} < \frac{n!}{k!(n-k)!} < \frac{n^n}{k^k(n-k)^{n-k}}$$

5 Given a permutation $(a_0, a_1, \dots, a_n)$ of the sequence $0, 1, \dots, n$. A transportation of a_i with a_j is called legal if $a_i = 0$ for $i > 0$, and $a_{i-1} + 1 = a_j$. The permutation $(a_0, a_1, \dots, a_n)$ is called regular if after a number of legal transportations it becomes $(1, 2, \dots, n, 0)$.
For which numbers n is the permutation $(1, n, n-1, \dots, 3, 2, 0)$ regular?