India
ISI B.Stat Entrance Exam
2007

$\boxed{1}$ Suppose a is a complex number such that

$$a^2 + a + \frac{1}{a} + \frac{1}{a^2} + 1 = 0$$

If m is a positive integer, find the value of

$$a^{2m} + a^m + \frac{1}{a^m} + \frac{1}{a^{2m}}$$

$\boxed{2}$ Use calculus to find the behaviour of the function

$$y = e^x \sin x \qquad -\infty < x < +\infty$$

and sketch the graph of the function for $-2\pi \leq x \leq 2\pi$. Show clearly the locations of the maxima, minima and points of inflection in your graph.

$\boxed{3}$ Let $f(u)$ be a continuous function and, for any real number u, let $[u]$ denote the greatest integer less than or equal to u. Show that for any $x > 1$,

$$\int_1^x [u]([u]+1)f(u)du = 2\sum_{i=1}^{[x]} i \int_i^x f(u)du$$

$\boxed{4}$ Show that it is not possible to have a triangle with sides a, b, and c whose medians have length $\frac{2}{3}a$, $\frac{2}{3}b$ and $\frac{4}{5}c$.

$\boxed{5}$ Show that

$$-2 \leq \cos\theta \left(\sin\theta + \sqrt{\sin^2\theta + 3}\right) \leq 2$$

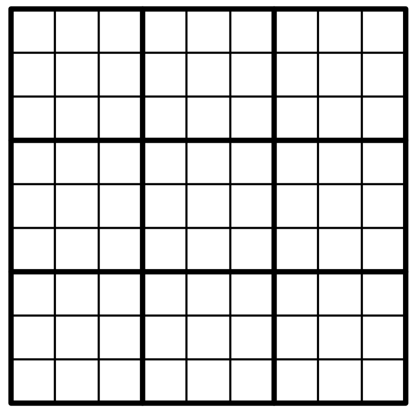
for all values of θ.

$\boxed{6}$ Let $S = \{1, 2, \cdots, n\}$ where n is an odd integer. Let f be a function defined on $\{(i,j) : i \in S, j \in S\}$ taking values in S such that (i) $f(s,r) = f(r,s)$ for all $r, s \in S$ (ii) $\{f(r,s) : s \in S\} = S$ for all $r \in S$

Show that $\{f(r,r) : r \in S\} = S$

$\boxed{7}$ Consider a prism with triangular base. The total area of the three faces containing a particular vertex A is K. Show that the maximum possible volume of the prism is $\sqrt{\frac{K^3}{54}}$ and find the height of this largest prism.

$\boxed{8}$ The following figure shows a $3^2 \times 3^2$ grid divided into 3^2 subgrids of size 3×3. This grid has 81 cells, 9 in each subgrid.



Now consider an $n^2 \times n^2$ grid divided into n^2 subgrids of size $n \times n$. Find the number of ways in which you can select n^2 cells from this grid such that there is exactly one cell coming from each subgrid, one from each row and one from each column.

$\boxed{9}$ Let $X \subset \mathbb{R}^2$ be a set satisfying the following properties: (i) if (x_1, y_1) and (x_2, y_2) are any two distinct elements in X, then

$$\text{either,} \quad x_1 > x_2 \text{ and } y_1 > y_2 \text{ or,} \quad x_1 < x_2 \text{ and } y_1 < y_2$$

(ii) there are two elements (a_1, b_1) and (a_2, b_2) in X such that for any $(x, y) \in X$,

$$a_1 \leq x \leq a_2 \text{ and } b_1 \leq y \leq b_2$$

(iii) if (x_1, y_1) and (x_2, y_2) are two elements of X, then for all $\lambda \in [0, 1]$,

$$(\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2) \in X$$

Show that if $(x, y) \in X$, then for some $\lambda \in [0, 1]$,

$$x = \lambda a_1 + (1 - \lambda)a_2, y = \lambda b_1 + (1 - \lambda)b_2$$

$\boxed{10}$ Let A be a set of positive integers satisfying the following properties: (i) if m and n belong to A, then $m + n$ belong to A; (ii) there is no prime number that divides all elements of A.

(a) Suppose n_1 and n_2 are two integers belonging to A such that $n_2 - n_1 > 1$. Show that you can find two integers m_1 and m_2 in A such that $0 < m_2 - m_1 < n_2 - n_1$ (b) Hence show that there are two consecutive integers belonging to A. (c) Let n_0 and $n_0 + 1$ be two consecutive integers belonging to A. Show that if $n \geq n_0^2$ then n belongs to A.