IMO 1969

Bucharest, Romania

Day 1

1. Prove that there are infinitely many positive integers m, such that $n^4 + m$ is not prime for any positive integer n.

2. Let $f(x) = \cos(a_1 + x) + \frac{1}{2}\cos(a_2 + x) + \frac{1}{4}\cos(a_3 + x) + \ldots + \frac{1}{2^{n-1}}\cos(a_n + x)$, where a_i are real constants and x is a real variable. If $f(x_1) = f(x_2) = 0$, prove that $x_1 - x_2$ is a multiple of π.

3. For each of $k = 1, 2, 3, 4, 5$ find necessary and sufficient conditions on $a > 0$ such that there exists a tetrahedron with k edges length a and the remainder length 1.

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Day 2

$\boxed{4}$ C is a point on the semicircle diameter AB, between A and B. D is the foot of the perpendicular from C to AB. The circle K_1 is the incircle of ABC, the circle K_2 touches CD, DA and the semicircle, the circle K_3 touches CD, DB and the semicircle. Prove that K_1, K_2 and K_3 have another common tangent apart from AB.

$\boxed{5}$ Given $n > 4$ points in the plane, no three collinear. Prove that there are at least $\frac{(n-3)(n-4)}{2}$ convex quadrilaterals with vertices amongst the n points.

$\boxed{6}$ Given real numbers $x_1, x_2, y_1, y_2, z_1, z_2$ satisfying $x_1 > 0, x_2 > 0, x_1y_1 > z_1^2$, and $x_2y_2 > z_2^2$, prove that:

$$\frac{8}{(x_1 + x_2)(y_1 + y_2) - (z_1 + z_2)^2} \leq \frac{1}{x_1y_1 - z_1^2} + \frac{1}{x_2y_2 - z_2^2}.$$

Give necessary and sufficient conditions for equality.