USA Team Selection Test 2007

Day 1

1	Circles ω_1 and ω_2 meet at P and Q. Segments AC and BD are chords of ω_1 and ω_2 respectively, such that segment AB and ray CD meet at P. Ray BD and segment AC meet at X. Point Y lies on ω_1 such that $PY \parallel BD$. Point Z lies on ω_2 such that $PZ \parallel AC$. Prove that points Q, X, Y, Z are collinear.
2	Let n be a positive integer and let $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$ be two nondecreasing sequences of real numbers such that $$a_1 + \cdots + a_i \le b_1 + \cdots + b_i \text{ for every } i = 1, \ldots, n$$ and $$a_1 + \cdots + a_n = b_1 + \cdots + b_n.$$ Suppose that for every real number m, the number of pairs (i, j) with $a_i - a_j = m$ equals the numbers of pairs (k, ℓ) with $b_k - b_\ell = m$. Prove that $a_i = b_i$ for $i = 1, \ldots, n$.
3	Let θ be an angle in the interval $(0, \pi/2)$. Given that $\cos\theta$ is irrational, and that $\cos k\theta$ and $\cos[(k+1)\theta]$ are both rational for some positive integer k, show that $\theta = \pi/6$.

Day 2

4	Determine whether or not there exist positive integers a and b such that a does not divide $b^n - n$ for all positive integers n.
5	Triangle ABC is inscribed in circle ω. The tangent lines to ω at B and C meet at T. Point S lies on ray BC such that $AS \perp AT$. Points B_1 and C_1 lie on ray ST (with C_1 in between B_1 and S) such that $B_1T = BT = C_1T$. Prove that triangles ABC and AB_1C_1 are similar to each other.
6	For a polynomial $P(x)$ with integer coefficients, $r(2i-1)$ (for $i = 1, 2, 3, \ldots, 512$) is the remainder obtained when $P(2i-1)$ is divided by 1024. The sequence $$(r(1), r(3), \ldots, r(1023))$$

is called the *remainder sequence* of $P(x)$. A remainder sequence is called *complete* if it is a permutation of $(1, 3, 5, \ldots, 1023)$. Prove that there are no more than 2^{35} different complete remainder sequences.

–