IMO 2007

Ha Noi, Vietnam

Day 1 - *25 July 2007*

$\boxed{1}$ Real numbers a_1, a_2, $\ldots$, a_n are given. For each i, $(1 \le i \le n)$, define

$$d_i = \max\{a_j \mid 1 \le j \le i\} - \min\{a_j \mid i \le j \le n\}$$

and let $d = \max\{d_i \mid 1 \le i \le n\}$.

(a) Prove that, for any real numbers $x_1 \le x_2 \le \cdots \le x_n$,

$$\max\{|x_i - a_i| \mid 1 \le i \le n\} \ge \frac{d}{2}. \qquad (*)$$

(b) Show that there are real numbers $x_1 \le x_2 \le \cdots \le x_n$ such that the equality holds in (*).

Author: Michael Albert, New Zealand

$\boxed{2}$ Consider five points A, B, C, D and E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line passing through A. Suppose that ℓ intersects the interior of the segment DC at F and intersects line BC at G. Suppose also that $EF = EG = EC$. Prove that ℓ is the bisector of angle DAB.

Author: Charles Leytem, Luxembourg

$\boxed{3}$ In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a *clique* if each two of them are friends. (In particular, any group of fewer than two competitiors is a clique.) The number of members of a clique is called its *size*.

Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged into two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

Author: Vasily Astakhov, Russia

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Day 2 - *26 July 2007*

$\boxed{4}$ In triangle ABC the bisector of angle BCA intersects the circumcircle again at R, the perpendicular bisector of BC at P, and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoint of AC is L. Prove that the triangles RPK and RQL have the same area.

Author: Marek Pechal, Czech Republic

$\boxed{5}$ Let a and b be positive integers. Show that if $4ab-1$ divides $(4a^2-1)^2$, then $a=b$.

[hide="How a lemma of an ISL problem was selected for IMO"]Strictly this IMO problem does not correspond to any ISL problem 2007. This is rather a lemma of ISL 2007, number theory problem N6. But as the IMO Problem Selection Committee appreciated this problem so much they chose to select this lemma as IMO problem, re-classifying [url=http://www.mathlinks.ro/viewtopic.php? Shortlist Number Theory Problem N6[/url] by just using its key lemma from hard to medium. [url=http://www.imo-register.org.uk/2007-report.html]Source: UK IMO Report.[/url]

Edited by Orlando Dhring

Author: Kevin Buzzard and Edward Crane, United Kingdom

$\boxed{6}$ Let n be a positive integer. Consider

$$S = \{(x,y,z) \mid x,y,z \in \{0,1,\ldots,n\}, x+y+z>0\}$$

as a set of $(n+1)^3-1$ points in the three-dimensional space. Determine the smallest possible number of planes, the union of which contains S but does not include $(0,0,0)$.

Author: Gerhard Wginger, Netherlands