IMO 1967

Cetinje, Yugoslavia

Day 1

$\boxed{1}$ The parallelogram $ABCD$ has $AB = a, AD = 1$, $\angle BAD = A$, and the triangle ABD has all angles acute. Prove that circles radius 1 and center A, B, C, D cover the parallelogram if and only

$$a \leq \cos A + \sqrt{3} \sin A.$$

$\boxed{2}$ Prove that a tetrahedron with just one edge length greater than 1 has volume at most $\frac{1}{8}$.

$\boxed{3}$ Let k, m, n be natural numbers such that $m + k + 1$ is a prime greater than $n + 1$. Let $c_s = s(s + 1)$. Prove that

$$(c_{m+1} - c_k)(c_{m+2} - c_k) \ldots (c_{m+n} - c_k)$$

is divisible by the product $c_1 c_2 \ldots c_n$.

Day 2

$\boxed{4}$ $A_0B_0C_0$ and $A_1B_1C_1$ are acute-angled triangles. Describe, and prove, how to construct the triangle ABC with the largest possible area which is circumscribed about $A_0B_0C_0$ (so BC contains B_0, CA contains B_0, and AB contains C_0) and similar to $A_1B_1C_1$.

$\boxed{5}$ Let $a_1, \ldots, a_8$ be reals, not all equal to zero. Let

$$c_n = \sum_{k=1}^{8} a_k^n$$

for $n = 1, 2, 3, \ldots$. Given that among the numbers of the sequence (c_n), there are infinitely many equal to zero, determine all the values of n for which $c_n = 0$.

$\boxed{6}$ In a sports meeting a total of m medals were awarded over n days. On the first day one medal and $\frac{1}{7}$ of the remaining medals were awarded. On the second day two medals and $\frac{1}{7}$ of the remaining medals were awarded, and so on. On the last day, the remaining n medals were awarded. How many medals did the meeting last, and what was the total number of medals ?