High School Olympiads

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numbers on a board, non-degenerate triangle

combinatorics combinatorics proposed triangle inequality

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Source: MEMO 2016 I2 combinatorics combinatorics proposed triangle inequality

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danepale
77 posts

Aug 24, 2016, 7:29 pm • 1
PM #1

There are $n \geq 3$ positive integers written on a board. A *move* consists of choosing three numbers a, b, c written from the board such that there exists a non-degenerate non-equilateral triangle with sides a, b, c and replacing those numbers with $a+b-c, b+c-a$ and $c+a-b$.

Prove that a sequence of moves cannot be infinite.

Diamondhead
78 posts

Aug 24, 2016, 7:32 pm
PM #2

Hint

Monovariant: product of all integers on a board.

Randomized99
27 posts

Aug 24, 2016, 7:34 pm • 1
PM #3

Use Schur for $r = 1$ to prove that product of all integers strictly decreases.

This post has been edited 1 time. Last edited by Randomized99, Aug 24, 2016, 7:35 pm

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CantonMath...
2336 posts

Aug 24, 2016, 8:13 pm • 1
PM #4

The above solution is too clever. Here is another solution

Solution

We claim that $a+b-c, b+c-a, c+a-b$ majorizes a, b, c. To prove this, without loss of generality assume that $a \leq b \leq c$; then, $a+b-c \leq a+c-b \leq b+c-a$, and

$$\begin{aligned} b+c-a &\geq c, \\ (b+c-a)+(a+c-b) = 2c &\geq c+b, \\ (b+c-a)+(a+c-b)+(a+b-c) &= c+b+a, \end{aligned}$$

so $b+c-a, a+c-b, a+b-c$ majorizes a, b, c, and it is also easy to see that if a, b, c are not all equal, then $b+c-a, a+c-b, a+b-c$ strictly majorizes a, b, c.

Thus, after each move, the n-tuple of numbers increases in lexicographical order (when sorted from increasing to decreasing). It is easy to see that the numbers are always positive integers, and thus there are only finitely many n-tuples possible. We are done.

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test20
459 posts

Aug 25, 2016, 5:48 pm • 1
PM #5

A very simple monovariant is the sum of the squares of all integers on the blackboard.

1. Since the sum S of all integers on the blackboard never changes, the sum of all squares is at most S^2.

2. $(a+b-c)^2+(b+c-a)^2+(c+a-b)^2$ is at least $a^2+b^2+c^2$, with equality occurring only for $a=b=c$. The claimed inequality is equivalent to $(a-b)^2+(a-c)^2+(b-c)^2 \geq 0$.

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rkm0959
1381 posts

Aug 26, 2016, 6:57 am PM #6

One can either

1. Get an Upper Bound of the sum of squares and prove that it strictly increases every time.

2. Get an Lower Bound of the product and prove that it strictly decreases every time.

1 is easy inequality, 2 is Schur with $r=1$. Not too difficult.

WizardMath
353 posts

Aug 26, 2016, 5:46 pm • 1 PM #7

We can just observe that we get a kind of majorized sequence of numbers in the end, id est, $a, b, c \cdots$ such that $c > a+b$ and so on.

Popescu
62 posts

Aug 27, 2016, 5:04 pm • 1 PM #8

What if the numbers are reals instead of integers?

Kezer
685 posts

Aug 30, 2016, 2:17 pm • 1 PM #9

I was stuck on this problem for a long time. Then, I gave up and wanted to check the solutions. Just at that moment, I realised the numbers on the board are positive integers and not real numbers and at that moment I understood why this problem was considered the easiest one of the MEMO...

Solution 1 *(Monovariant: Product of Numbers)*:
Consider the product of all numbers on the board. Take any three numbers a, b, c on the blackboard to make a move. We can write
$a=x+y, b=y+z, c=z+x$ due to Ravi. Then

$$(a+b-c)(b+c-a)(c+a-b)=8xyz \leq (x+y)(y+z)(z+x)=abc$$

by AM-GM. Equality never arises, as then we'd have to have $x=y=z$ and thus $a=b=c$, which we cannot use. Therefore, the product is strictly decreasing and as all numbers on the blackboard are integers, the product also remains an integer. Thus, it decreases by at least 1 each move and will therefore somewhen reach non-positivity, assuming there was an infinite sequence of moves. That is the desired contradiction.

Solution 2: *(Monovariant: Largest Number, Invariant: Sum of Numbers)*:
We'll proceed by induction.
Base case: For $n=3$: Take the numbers a, b, c. Let $a \geq b \geq c$. Then

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