Romania National Olympiad 2006

– Grade level 7

– April 17th

1 Let ABC be a triangle and the points M and N on the sides AB respectively BC, such that $2 \cdot \frac{CN}{BC} = \frac{AM}{AB}$. Let P be a point on the line AC. Prove that the lines MN and NP are perpendicular if and only if PN is the interior angle bisector of $\angle MPC$.

2 A square of side n is formed from n^2 unit squares, each colored in red, yellow or green. Find minimal n, such that for each coloring, there exists a line and a column with at least 3 unit squares of the same color (on the same line or column).

3 In the acute-angle triangle ABC we have $\angle ACB = 45^\circ$. The points A_1 and B_1 are the feet of the altitudes from A and B, and H is the orthocenter of the triangle. We consider the points D and E on the segments AA_1 and BC such that $A_1D = A_1E = A_1B_1$. Prove that

a) $A_1B_1 = \sqrt{\frac{A_1B^2 + A_1C^2}{2}}$;

b) $CH = DE$.

4 Let A be a set of positive integers with at least 2 elements. It is given that for any numbers $a > b$, $a, b \in A$ we have $\frac{[a,b]}{a-b} \in A$, where by $[a,b]$ we have denoted the least common multiple of a and b. Prove that the set A has *exactly* two elements.

Marius Gherghu, Slatina

– Grade level 8

– April 17th

1 We consider a prism with 6 faces, 5 of which are circumscriptible quadrilaterals. Prove that all the faces of the prism are circumscriptible quadrilaterals.

2 Let n be a positive integer. Prove that there exists an integer k, $k \geq 2$, and numbers $a_i \in \{-1, 1\}$, such that

$$n = \sum_{1 \leq i < j \leq k} a_i a_j.$$

3 Let $ABCDA_1B_1C_1D_1$ be a cube and P a variable point on the side $[AB]$. The perpendicular plane on AB which passes through P intersects the line AC' in Q. Let M and N be the midpoints of the segments $A'P$ and BQ respectively.

a) Prove that the lines MN and BC' are perpendicular if and only if P is the midpoint of AB.

b) Find the minimal value of the angle between the lines MN and BC'.

4 Let $a, b, c \in \left[\frac{1}{2}, 1\right]$. Prove that

$$2 \leq \frac{a+b}{1+c} + \frac{b+c}{1+a} + \frac{c+a}{1+b} \leq 3.$$

selected by Mircea Lascu

– Grade level 9

– April 17th

1 Find the maximal value of

$$\left(x^3+1\right)\left(y^3+1\right),$$

where $x, y \in \mathbb{R}$, $x + y = 1$.

Dan Schwarz

2 Let ABC and DBC be isosceles triangle with the base BC. We know that $\measuredangle ABD = \frac{\pi}{2}$. Let M be the midpoint of BC. The points E, F, P are chosen such that $E \in (AB)$, $P \in (MC)$, $C \in (AF)$, and $\measuredangle BDE = \measuredangle ADP = \measuredangle CDF$. Prove that P is the midpoint of EF and $DP \perp EF$.

3 We have a quadrilateral $ABCD$ inscribed in a circle of radius r, for which there is a point P on CD such that $CB = BP = PA = AB$.

(a) Prove that there are points A, B, C, D, P which fulfill the above conditions.

(b) Prove that $PD = r$.

Virgil Nicula

4 $2n$ students ($n \geq 5$) participated at table tennis contest, which took 4 days. In every day, every student played a match. (It is possible that the same pair meets twice or more times, in different days) Prove that it is possible that the contest ends like this:

- there is only one winner;

- there are 3 students on the second place;

- no student lost all 4 matches.

How many students won only a single match and how many won exactly 2 matches? (In the above conditions)

– Grade level 10

– April 17th

1 Let M be a set composed of n elements and let $\mathcal{P}(M)$ be its power set. Find all functions $f : \mathcal{P}(M) \to \{0, 1, 2, \ldots, n\}$ that have the properties

(a) $f(A) \neq 0$, for $A \neq \phi$;

(b) $f(A \cup B) = f(A \cap B) + f(A \Delta B)$, for all $A, B \in \mathcal{P}(M)$, where $A \Delta B = (A \cup B) \setminus (A \cap B)$.

2 Prove that for all $a, b \in \left(0, \frac{\pi}{4}\right)$ and $n \in \mathbb{N}^*$ we have

$$\frac{\sin^n a + \sin^n b}{(\sin a + \sin b)^n} \geq \frac{\sin^n 2a + \sin^n 2b}{(\sin 2a + \sin 2b)^n}.$$

3 Prove that among the elements of the sequence $\left(\lfloor n\sqrt{2}\rfloor + \lfloor n\sqrt{3}\rfloor\right)_{n\geq 0}$ are an infinity of even numbers and an infinity of odd numbers.

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4 Let $n \in \mathbb{N}$, $n \geq 2$. Determine n sets A_i, $1 \leq i \leq n$, from the plane, pairwise disjoint, such that:

(a) for every circle $\mathcal{C}$ from the plane and for every $i \in \{1, 2, \ldots, n\}$ we have $A_i \cap \text{Int}\,(\mathcal{C}) \neq \phi$;

(b) for all lines d from the plane and every $i \in \{1, 2, \ldots, n\}$, the projection of A_i on d is not d.

– Grade level 11

– April 17th

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1 Let A be a $n \times n$ matrix with complex elements and let $A^\star$ be the classical adjoint of A. Prove that if there exists a positive integer m such that $(A^\star)^m = 0_n$ then $(A^\star)^2 = 0_n$.

Marian Ionescu, Pitesti

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2 We define a *pseudo-inverse* $B \in \mathcal{M}_n(\mathbb{C})$ of a matrix $A \in \mathcal{M}_n(\mathbb{C})$ a matrix which fulfills the relations

$$A = ABA \quad \text{and} \quad B = BAB.$$

a) Prove that any square matrix has at least a pseudo-inverse.

b) For which matrix A is the pseudo-inverse unique?

Marius Cavachi

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3 We have in the plane the system of points $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$, which have different centers of mass. Prove that there is a point P such that

$$PA_1 + PA_2 + \ldots + PA_n = PB_1 + PB_2 + \ldots + PB_n.$$

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4 Let $f : [0, \infty) \to \mathbb{R}$ be a function such that for any $x > 0$ the sequence $\{f(nx)\}_{n\geq 0}$ is increasing.

a) If the function is also continuous on $[0,1]$ is it true that f is increasing?

b) The same question if the function is continuous on $\mathbb{Q} \cap [0, \infty)$.

– Grade level 12

– April 17th

1 Let $\mathcal{K}$ be a finite field. Prove that the following statements are equivalent:

(a) $1 + 1 = 0$;

(b) for all $f \in \mathcal{K}[X]$ with $\deg f \geq 1$, $f\left(X^2\right)$ is reducible.

2 Prove that

$$\lim_{n\to\infty} n\left(\frac{\pi}{4} - n\int_0^1 \frac{x^n}{1+x^{2n}}\, dx\right) = \int_0^1 f(x)\, dx,$$

where $f(x) = \frac{\arctan x}{x}$ if $x \in (0,1]$ and $f(0) = 1$.

Dorin Andrica, Mihai Piticari

3 Let G be a finite group of n elements ($n \geq 2$) and p be the smallest prime factor of n. If G has only a subgroup H with p elements, then prove that H is in the center of G.

Note. The center of G is the set $Z(G) = \{a \in G \,|\, ax = xa,\, \forall x \in G\}$.

4 Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 f(x) dx = 0.$$

Prove that there is $c \in (0,1)$ such that

$$\int_0^c x f(x) dx = 0.$$

Cezar Lupu, Tudorel Lupu