IMO 1991

Day 1

$\boxed{1}$ Given a triangle ABC, let I be the center of its inscribed circle. The internal bisectors of the angles A, B, C meet the opposite sides in A', B', C' respectively. Prove that

$$\frac{1}{4} < \frac{AI \cdot BI \cdot CI}{AA' \cdot BB' \cdot CC'} \leq \frac{8}{27}.$$

$\boxed{2}$ Let $n > 6$ be an integer and $a_1, a_2, \cdots, a_k$ be all the natural numbers less than n and relatively prime to n. If

$$a_2 - a_1 = a_3 - a_2 = \cdots = a_k - a_{k-1} > 0,$$

prove that n must be either a prime number or a power of 2.

$\boxed{3}$ Let $S = \{1, 2, 3, \cdots, 280\}$. Find the smallest integer n such that each n-element subset of S contains five numbers which are pairwise relatively prime.

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Day 2

1. Suppose G is a connected graph with k edges. Prove that it is possible to label the edges $1, 2, \ldots, k$ in such a way that at each vertex which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is equal to 1.

   [hide="Graph-Definition"]A **graph** consists of a set of points, called vertices, together with a set of edges joining certain pairs of distinct vertices. Each pair of vertices u, v belongs to at most one edge. The graph G is connected if for each pair of distinct vertices x, y there is some sequence of vertices $x = v_0, v_1, v_2, \cdots, v_m = y$ such that each pair v_i, v_{i+1} $(0 \leq i < m)$ is joined by an edge of G.

2. Let ABC be a triangle and P an interior point of ABC. Show that at least one of the angles $\angle PAB$, $\angle PBC$, $\angle PCA$ is less than or equal to 30°.

3. An infinite sequence $x_0, x_1, x_2, \ldots$ of real numbers is said to be **bounded** if there is a constant C such that $|x_i| \leq C$ for every $i \geq 0$. Given any real number $a > 1$, construct a bounded infinite sequence $x_0, x_1, x_2, \ldots$ such that

   $$|x_i - x_j||i - j|^a \geq 1$$

   for every pair of distinct nonnegative integers i, j.