IMO Shortlist 2006

– Algebra

**1** A sequence of real numbers $a_0,\ a_1,\ a_2,\dots$ is defined by the formula

$$a_{i+1} = \lfloor a_i \rfloor \cdot \langle a_i \rangle \qquad \text{for} \quad i \geq 0;$$

here $a_0$ is an arbitrary real number, $\lfloor a_i \rfloor$ denotes the greatest integer not exceeding $a_i$, and $\langle a_i \rangle = a_i - \lfloor a_i \rfloor$. Prove that $a_i = a_{i+2}$ for $i$ sufficiently large.

*Proposed by Harmel Nestra, Estionia*

**2** The sequence of real numbers $a_0, a_1, a_2, \dots$ is defined recursively by

$$a_0 = -1, \qquad \sum_{k=0}^{n} \frac{a_{n-k}}{k+1} = 0 \quad \text{for} \quad n \geq 1.$$

Show that $a_n > 0$ for all $n \geq 1$.

*Proposed by Mariusz Skalba, Poland*

**3** The sequence $c_0, c_1, ..., c_n, ...$ is defined by $c_0 = 1, c_1 = 0$, and $c_{n+2} = c_{n+1} + c_n$ for $n \geq 0$. Consider the set $S$ of ordered pairs $(x, y)$ for which there is a finite set $J$ of positive integers such that $x = \sum_{j \in J} c_j$, $y = \sum_{j \in J} c_{j-1}$. Prove that there exist real numbers $\alpha$, $\beta$, and $M$ with the following property: An ordered pair of nonnegative integers $(x, y)$ satisfies the inequality

$$m < \alpha x + \beta y < M$$

if and only if $(x, y) \in S$.

*Remark:* A sum over the elements of the empty set is assumed to be 0.

**4** Prove the inequality:

$$\sum_{i<j} \frac{a_i a_j}{a_i + a_j} \leq \frac{n}{2(a_1 + a_2 + \cdots + a_n)} \cdot \sum_{i<j} a_i a_j$$

for positive reals $a_1, a_2, \dots, a_n$.

*Proposed by Dusan Dukic, Serbia*

**5** If $a, b, c$ are the sides of a triangle, prove that

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}}+\frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}}+\frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}}\leq 3$$

*Proposed by Hojoo Lee, Korea*

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**6** Determine the least real number $M$ such that the inequality

$$|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)|\leq M(a^2+b^2+c^2)^2$$

holds for all real numbers $a$, $b$ and $c$.

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– Combinatorics

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**1** We have $n \geq 2$ lamps $L_1, ..., L_n$ in a row, each of them being either on or off. Every second we simultaneously modify the state of each lamp as follows: if the lamp $L_i$ and its neighbours (only one neighbour for $i = 1$ or $i = n$, two neighbours for other $i$) are in the same state, then $L_i$ is switched off; otherwise, $L_i$ is switched on.
Initially all the lamps are off except the leftmost one which is on.
$(a)$ Prove that there are infinitely many integers $n$ for which all the lamps will eventually be off. $(b)$ Prove that there are infinitely many integers $n$ for which the lamps will never be all off.

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**2** Let $P$ be a regular 2006-gon. A diagonal is called *good* if its endpoints divide the boundary of $P$ into two parts, each composed of an odd number of sides of $P$. The sides of $P$ are also called *good*.
Suppose $P$ has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of $P$. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.

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**3** Let $S$ be a finite set of points in the plane such that no three of them are on a line. For each convex polygon $P$ whose vertices are in $S$, let $a(P)$ be the number of vertices of $P$, and let $b(P)$ be the number of points of $S$ which are outside $P$. A line segment, a point, and the empty set are considered as convex polygons of 2, 1, and 0 vertices respectively. Prove that for every real number $x$

$$\sum_{P} x^{a(P)}(1-x)^{b(P)}=1,$$

where the sum is taken over all convex polygons with vertices in $S$.

*Alternative formulation*:

Let $M$ be a finite point set in the plane and no three points are collinear. A subset $A$ of $M$ will be called round if its elements is the set of vertices of a convex $A-$gon $V(A)$. For each round subset let $r(A)$ be the number of points from $M$ which are exterior from the convex $A-$gon $V(A)$. Subsets with $0,1$ and 2 elements are always round, its corresponding polygons are the empty set, a point or a segment, respectively (for which all other points that are not vertices of the polygon are exterior). For each round subset $A$ of $M$ construct the polynomial

$$P_A(x) = x^{|A|}(1-x)^{r(A)}.$$

Show that the sum of polynomials for all round subsets is exactly the polynomial $P(x) = 1$.

*Proposed by Federico Ardila, Colombia*

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**4** A cake has the form of an $n$ x $n$ square composed of $n^2$ unit squares. Strawberries lie on some of the unit squares so that each row or column contains exactly one strawberry; call this arrangement $\mathcal{A}$.

Let $\mathcal{B}$ be another such arrangement. Suppose that every grid rectangle with one vertex at the top left corner of the cake contains no fewer strawberries of arrangement $\mathcal{B}$ than of arrangement $\mathcal{A}$. Prove that arrangement $\mathcal{B}$ can be obtained from $\mathcal{A}$ by performing a number of switches, defined as follows:

A switch consists in selecting a grid rectangle with only two strawberries, situated at its top right corner and bottom left corner, and moving these two strawberries to the other two corners of that rectangle.

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**5** An $(n,k)-$ tournament is a contest with $n$ players held in $k$ rounds such that:

$(i)$ Each player plays in each round, and every two players meet at most once.
$(ii)$ If player $A$ meets player $B$ in round $i$, player $C$ meets player $D$ in round $i$, and player $A$ meets player $C$ in round $j$, then player $B$ meets player $D$ in round $j$.

Determine all pairs $(n,k)$ for which there exists an $(n,k)-$ tournament.

*Proposed by Carlos di Fiore, Argentina*

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**6** A holey triangle is an upward equilateral triangle of side length $n$ with $n$ upward unit triangular holes cut out. A diamond is a $60^\circ - 120^\circ$ unit rhombus.
Prove that a holey triangle $T$ can be tiled with diamonds if and only if the

following condition holds: Every upward equilateral triangle of side length $k$ in $T$ contains at most $k$ holes, for $1 \leq k \leq n$.

*Proposed by Federico Ardila, Colombia*

**7** Consider a convex polyhedron without parallel edges and without an edge parallel to any face other than the two faces adjacent to it. Call a pair of points of the polyhedron *antipodal* if there exist two parallel planes passing through these points and such that the polyhedron is contained between these planes. Let $A$ be the number of antipodal pairs of vertices, and let $B$ be the number of antipodal pairs of midpoint edges. Determine the difference $A - B$ in terms of the numbers of vertices, edges, and faces.

*Proposed by Kei Irei, Japan*

– Geometry

**1** Let $ABC$ be triangle with incenter $I$. A point $P$ in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

**2** Let $ABC$ be a trapezoid with parallel sides $AB > CD$. Points $K$ and $L$ lie on the line segments $AB$ and $CD$, respectively, so that $AK/KB = DL/LC$. Suppose that there are points $P$ and $Q$ on the line segment $KL$ satisfying

$$\angle APB = \angle BCD \qquad \text{and} \qquad \angle CQD = \angle ABC.$$

Prove that the points $P$, $Q$, $B$ and $C$ are concyclic.

*Proposed by Vyacheslev Yasinskiy, Ukraine*

**3** Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \qquad \text{and} \qquad \angle ABC = \angle ACD = \angle ADE.$$

The diagonals $BD$ and $CE$ meet at $P$. Prove that the line $AP$ bisects the side $CD$.

*Proposed by Zuming Feng, USA*

**4** A point $D$ is chosen on the side $AC$ of a triangle $ABC$ with $\angle C < \angle A < 90^\circ$ in such a way that $BD = BA$. The incircle of $ABC$ is tangent to $AB$ and $AC$ at points $K$ and $L$, respectively. Let $J$ be the incenter of triangle $BCD$. Prove that the line $KL$ intersects the line segment $AJ$ at its midpoint.

**5** In triangle $ABC$, let $J$ be the center of the excircle tangent to side $BC$ at $A_1$ and to the extensions of the sides $AC$ and $AB$ at $B_1$ and $C_1$ respectively. Suppose that the lines $A_1B_1$ and $AB$ are perpendicular and intersect at $D$. Let $E$ be the foot of the perpendicular from $C_1$ to line $DJ$. Determine the angles $\angle BEA_1$ and $\angle AEB_1$.

*Proposed by Dimitris Kontogiannis, Greece*

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**6** Circles $w_1$ and $w_2$ with centres $O_1$ and $O_2$ are externally tangent at point $D$ and internally tangent to a circle $w$ at points $E$ and $F$ respectively. Line $t$ is the common tangent of $w_1$ and $w_2$ at $D$. Let $AB$ be the diameter of $w$ perpendicular to $t$, so that $A, E, O_1$ are on the same side of $t$. Prove that lines $AO_1$, $BO_2$, $EF$ and $t$ are concurrent.

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**7** In a triangle $ABC$, let $M_a$, $M_b$, $M_c$ be the midpoints of the sides $BC$, $CA$, $AB$, respectively, and $T_a$, $T_b$, $T_c$ be the midpoints of the arcs $BC$, $CA$, $AB$ of the circumcircle of $ABC$, not containing the vertices $A$, $B$, $C$, respectively. For $i \in \{a, b, c\}$, let $w_i$ be the circle with $M_iT_i$ as diameter. Let $p_i$ be the common external common tangent to the circles $w_j$ and $w_k$ (for all $\{i, j, k\} = \{a, b, c\}$) such that $w_i$ lies on the opposite side of $p_i$ than $w_j$ and $w_k$ do.
Prove that the lines $p_a$, $p_b$, $p_c$ form a triangle similar to $ABC$ and find the ratio of similitude.

*Proposed by Tomas Jurik, Slovakia*

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**8** Let $ABCD$ be a convex quadrilateral. A circle passing through the points $A$ and $D$ and a circle passing through the points $B$ and $C$ are externally tangent at a point $P$ inside the quadrilateral. Suppose that

$$\angle PAB + \angle PDC \leq 90^\circ \qquad \text{and} \qquad \angle PBA + \angle PCD \leq 90^\circ.$$

Prove that $AB + CD \geq BC + AD$.

*Proposed by Waldemar Pompe, Poland*

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**9** Points $A_1$, $B_1$, $C_1$ are chosen on the sides $BC$, $CA$, $AB$ of a triangle $ABC$ respectively. The circumcircles of triangles $AB_1C_1$, $BC_1A_1$, $CA_1B_1$ intersect the circumcircle of triangle $ABC$ again at points $A_2$, $B_2$, $C_2$ respectively ($A_2 \neq A, B_2 \neq B, C_2 \neq C$). Points $A_3$, $B_3$, $C_3$ are symmetric to $A_1$, $B_1$, $C_1$ with respect to the midpoints of the sides $BC$, $CA$, $AB$ respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

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**10** Assign to each side $b$ of a convex polygon $P$ the maximum area of a triangle that has $b$ as a side and is contained in $P$. Show that the sum of the areas assigned to the sides of $P$ is at least twice the area of $P$.

– Number Theory

**1** Determine all pairs $(x, y)$ of integers such that

$$1 + 2^x + 2^{2x+1} = y^2.$$

**2** For $x \in (0, 1)$ let $y \in (0, 1)$ be the number whose $n$-th digit after the decimal point is the $2^n$-th digit after the decimal point of $x$. Show that if $x$ is rational then so is $y$.

*Proposed by J.P. Grossman, Canada*

**3** We define a sequence $(a_1, a_2, a_3, \ldots)$ by

$$a_n = \frac{1}{n}\left(\left\lfloor\frac{n}{1}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + \cdots + \left\lfloor\frac{n}{n}\right\rfloor\right),$$

where $\lfloor x \rfloor$ denotes the integer part of $x$.

**a)** Prove that $a_{n+1} > a_n$ infinitely often.
**b)** Prove that $a_{n+1} < a_n$ infinitely often.

*Proposed by Johan Meyer, South Africa*

**4** Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let $k$ be a positive integer. Consider the polynomial $Q(x) = P(P(\ldots P(P(x)) \ldots))$, where $P$ occurs $k$ times. Prove that there are at most $n$ integers $t$ such that $Q(t) = t$.

**5** Find all integer solutions of the equation

$$\frac{x^7 - 1}{x - 1} = y^5 - 1.$$

**6** Let $a > b > 1$ be relatively prime positive integers. Define the weight of an integer $c$, denoted by $w(c)$ to be the minimal possible value of $|x| + |y|$ taken over all pairs of integers $x$ and $y$ such that

$$ax + by = c.$$

An integer $c$ is called a *local champion* if $w(c) \geq w(c \pm a)$ and $w(c) \geq w(c \pm b)$.

Find all local champions and determine their number.

*Proposed by Zoran Sunic, USA*

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**7** For all positive integers $n$, show that there exists a positive integer $m$ such that $n$ divides $2^m + m$.

*Proposed by Juhan Aru, Estonia*