India
National Olympiad
2005

$\boxed{1}$ Let M be the midpoint of side BC of a triangle ABC. Let the median AM intersect the incircle of ABC at K and L, K being nearer to A than L. If $AK = KL = LM$, prove that the sides of triangle ABC are in the ratio $5 : 10 : 13$ in some order.

$\boxed{2}$ Let α and β be positive integers such that $\dfrac{43}{197} < \dfrac{\alpha}{\beta} < \dfrac{17}{77}$. Find the minimum possible value of β.

$\boxed{3}$ Let p, q, r be positive real numbers, not all equal, such that some two of the equations

$$\begin{aligned} px^2 + 2qx + r &= 0qx^2 + 2rx + \not p \\ 0rx^2 + 2px + q &= 0. \end{aligned}$$

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have a common root, say α. Prove that

a) α is real and negative;

b) the remaining third quadratic equation has non-real roots.

All possible 6-digit numbers, in each of which the digits occur in nonincreasing order (from left to right, e.g. 877550) are written as a sequence in increasing order. Find the 2005-th number in this sequence.

Let x_1 be a given positive integer. A sequence $\{x_n\}_{n\geq 1}$ of positive integers is such that x_n, for $n \geq 2$, is obtained from x_{n-1} by adding some nonzero digit of x_{n-1}. Prove that

a) the sequence contains an even term;

b) the sequence contains infinitely many even terms.

Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(x^2 + yf(z)) = xf(x) + zf(y),$$

for all $x, y, z \in \mathbb{R}$.