Balkan MO 2008

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1 Given a scalene acute triangle ABC with $AC > BC$ let F be the foot of the altitude from C. Let P be a point on AB, different from A so that $AF = PF$. Let H, O, M be the orthocenter, circumcenter and midpoint of $[AC]$. Let X be the intersection point of BC and HP. Let Y be the intersection point of OM and FX and let OF intersect AC at Z. Prove that F, M, Y, Z are concyclic.

2 Is there a sequence $a_1, a_2, \ldots$ of positive reals satisfying simoultaneously the following inequalities for all positive integers n:
a) $a_1 + a_2 + \ldots + a_n \leq n^2$
b) $\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} \leq 2008$?

3 Let n be a positive integer. Consider a rectangle $(90n+1) \times (90n+5)$ consisting of unit squares. Let S be the set of the vertices of these squares. Prove that the number of distinct lines passing through at least two points of S is divisible by 4.

4 Let c be a positive integer. The sequence $a_1, a_2, \ldots$ is defined as follows $a_1 = c$, $a_{n+1} = a_n^2 + a_n + c^3$ for all positive integers n. Find all c so that there are integers $k \geq 1$ and $m \geq 2$ so that $a_k^2 + c^3$ is the mth power of some integer.