Art of Problem Solving
2008 China National Olympiad

China National Olympiad 2008

Day 1

1 Suppose $\triangle ABC$ is scalene. O is the circumcenter and A' is a point on the extension of segment AO such that $\angle BA'A = \angle CA'A$. Let point A_1 and A_2 be foot of perpendicular from A' onto AB and AC. H_A is the foot of perpendicular from A onto BC. Denote R_A to be the radius of circumcircle of $\triangle H_A A_1 A_2$. Similiarly we can define R_B and R_C. Show that:

$$\frac{1}{R_A}+\frac{1}{R_B}+\frac{1}{R_C}=\frac{2}{R}$$

where R is the radius of circumcircle of $\triangle ABC$.

2 Given an integer $n \geq 3$, prove that the set $X = \{1, 2, 3, \ldots, n^2 - n\}$ can be divided into two non-intersecting subsets such that neither of them contains n elements $a_1, a_2, \ldots, a_n$ with $a_1 < a_2 < \ldots < a_n$ and $a_k \leq \frac{a_{k-1}+a_{k+1}}{2}$ for all $k = 2, \ldots, n-1$.

3 Given a positive integer n and $x_1 \leq x_2 \leq \ldots \leq x_n, y_1 \geq y_2 \geq \ldots \geq y_n$, satisfying

$$\sum_{i=1}^{n} i x_i = \sum_{i=1}^{n} i y_i$$

Show that for any real number α, we have

$$\sum_{i=1}^{n} x_i [i\alpha] \geq \sum_{i=1}^{n} y_i [i\alpha]$$

Here $[\beta]$ denotes the greastest integer not larger than β.

Day 2

1 Let A be an infinite subset of $\mathbb{N}$, and n a fixed integer. For any prime p not dividing n, There are infinitely many elements of A not divisible by p. Show that for any integer $m > 1, (m, n) = 1$, There exist finitely many elements of A, such that their sum is congruent to 1 modulo m and congruent to 0 modulo n.

2 Find the smallest integer n satisfying the following condition: regardless of how one colour the vertices of a regular n-gon with either red, yellow or blue, one can always find an isosceles trapezoid whose vertices are of the same colour.

3 Find all triples (p, q, n) that satisfy

$$q^{n+2} \equiv 3^{n+2} (\mod p^n), \quad p^{n+2} \equiv 3^{n+2} (\mod q^n)$$

where p, q are odd primes and n is an positive integer.