Fifth Iranian Geometry Olympiad

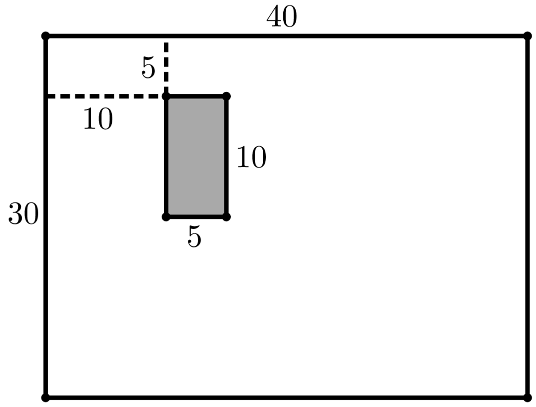
The problems along with their solutions

Contents

Elementary Level

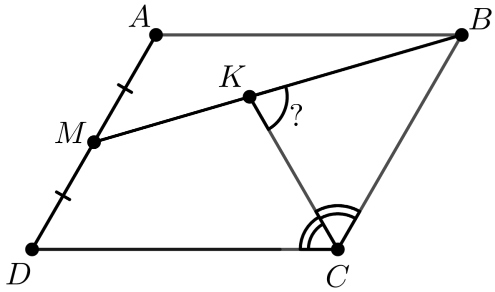
Problems

1. As shown below, there is a 40×30 paper with a filled 10×5 rectangle inside of it. We want to cut out the filled rectangle from the paper using four straight cuts. Each straight cut is a straight line that divides the paper into two pieces, and we keep the piece containing the filled rectangle. The goal is to minimize the total length of the straight cuts. How to achieve this goal, and what is that minimized length? Show the correct cuts and write the final answer. There is no need to prove the answer.



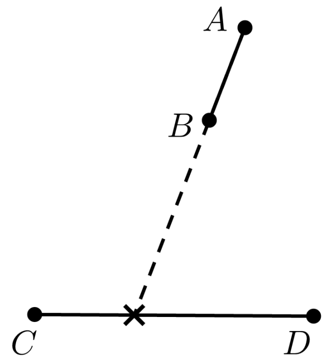
2. Convex hexagon $A_1A_2A_3A_4A_5A_6$ lies in the interior of convex hexagon $B_1B_2B_3B_4B_5B_6$ such that $A_1A_2 \parallel B_1B_2$, $A_2A_3 \parallel B_2B_3$,..., $A_6A_1 \parallel B_6B_1$. Prove that the areas of simple hexagons $A_1B_2A_3B_4A_5B_6$ and $B_1A_2B_3A_4B_5A_6$ are equal. (A simple hexagon is a hexagon which does not intersect itself.)

3. In the given figure, $ABCD$ is a parallelogram. We know that $\angle D = 60°$, $AD = 2$ and $AB = \sqrt{3}+1$. Point M is the midpoint of AD. Segment CK is the angle bisector of C. Find the angle CKB.

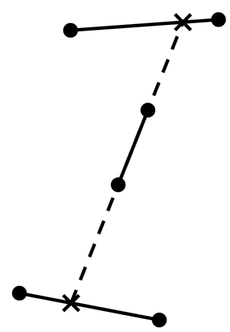


4. There are two circles with centers O_1, O_2 lie inside of circle ω and are tangent to it. Chord AB of ω is tangent to these two circles such that they lie on opposite sides of this chord. Prove that $\angle O_1AO_2 + \angle O_1BO_2 > 90°$.

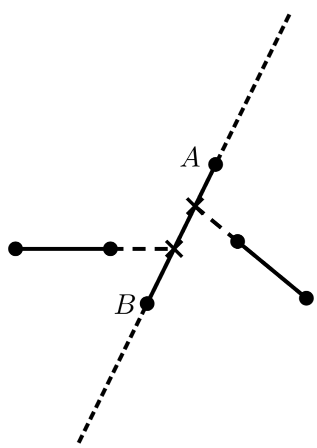
5. There are some segments on the plane such that no two of them intersect each other (even at the ending points). We say segment AB **breaks** segment CD if the extension of AB cuts CD at some point between C and D.



(a) Is it possible that each segment when extended from both ends, breaks exactly one other segment from each way?

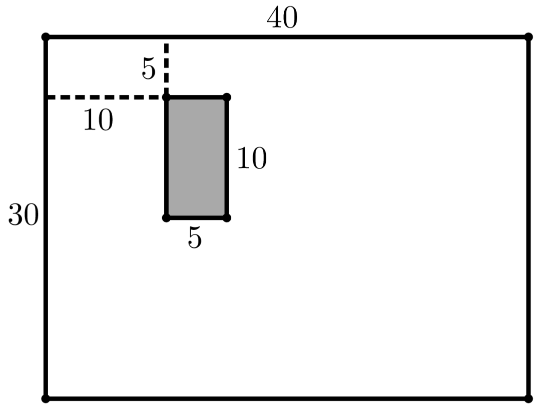


(b) A segment is called **surrounded** if from both sides of it, there is exactly one segment that breaks it. (*e.g.* segment AB in the figure.) Is it possible to have all segments to be surrounded?



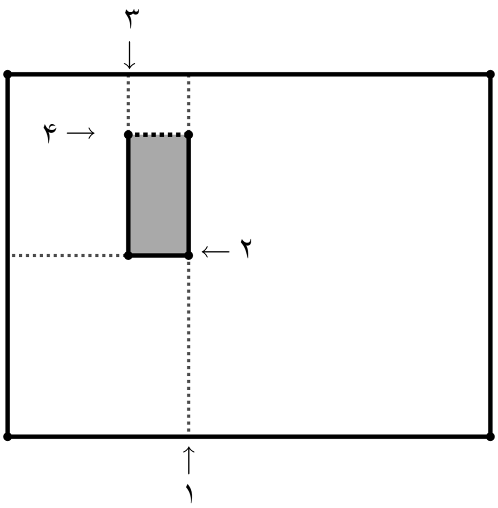
Solutions

1. As shown below, there is a 40×30 paper with a filled 10×5 rectangle inside of it. We want to cut out the filled rectangle from the paper using four straight cuts. Each straight cut is a straight line that divides the paper into two pieces, and we keep the piece containing the filled rectangle. The goal is to minimize the total length of the straight cuts. How to achieve this goal, and what is that minimized length? Show the correct cuts and write the final answer. There is no need to prove the answer.



Proposed by Morteza Saghafian

Solution. The answer is 65. Here is an example of the solution:



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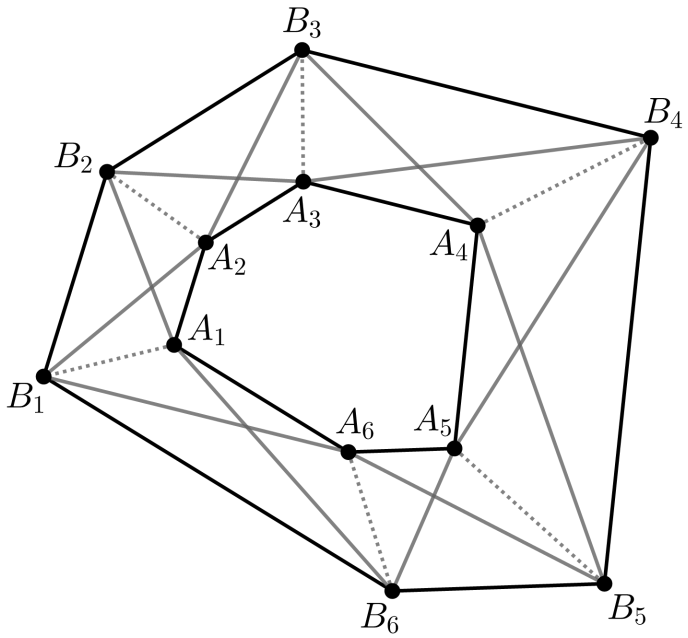
2. Convex hexagon $A_1A_2A_3A_4A_5A_6$ lies in the interior of convex hexagon $B_1B_2B_3B_4B_5B_6$ such that $A_1A_2 \parallel B_1B_2$, $A_2A_3 \parallel B_2B_3$,..., $A_6A_1 \parallel B_6B_1$. Prove that the areas of simple hexagons $A_1B_2A_3B_4A_5B_6$ and $B_1A_2B_3A_4B_5A_6$ are equal. (A simple hexagon is a hexagon which does not intersect itself.)

Proposed by Mahdi Etesamifard - Hirad Aalipanah

Solution. As you can see, we have divided the area between two polygons into 6 trapezoids. In each trapezoid is it easy to see that the triangles which have the same area (like $B_1A_1A_2$ and $B_2A_1A_2$) each belongs to one of the simple hexagons. Therefore, if we add up their areas and add the common area (the area of $A_1A_2A_3A_4A_5A_6$) to them, we can conclude that the areas of the two simple hexagons are equal.



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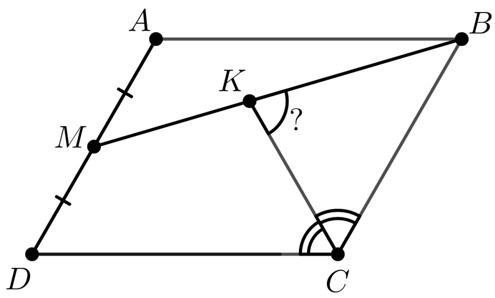
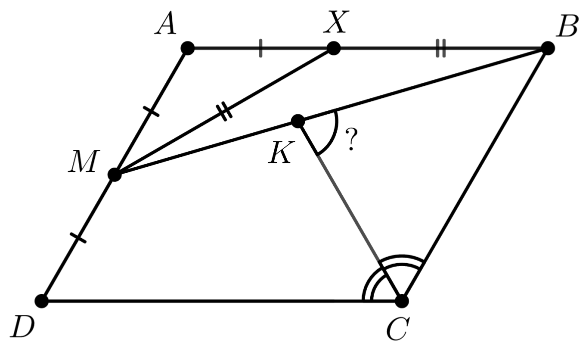
3. In the given figure, $ABCD$ is a parallelogram. We know that $\angle D = 60°$, $AD = 2$ and $AB = \sqrt{3} + 1$. Point M is the midpoint of AD. Segment CK is the angle bisector of C. Find the angle CKB.



Proposed by Mahdi Etesamifard

Solution 1. Let X be a point on AB such that $AX = 1$ and $XB = \sqrt{3}$. We know that $\angle MAX = 120°$. Therefore by Pythagoras theorem we know that $MX = \sqrt{3}$. So we have $\angle MBX = 15°$ and $\angle CBK = 45°$. Hence, $\angle CKB = 180° - 60° - 45° = 75°$.



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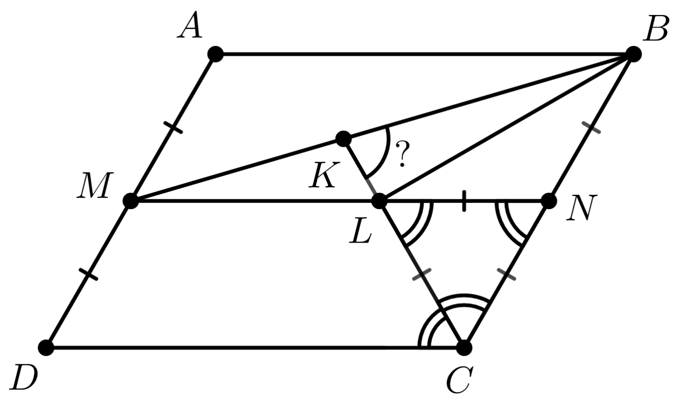
Solution 2. Let N be the midpoint of side BC. MN intersects CK at L. It's clear that the triangle CNL is equilateral. Therefore, we have $LN = CN = NB$. So, BCL is a right-angled triangle. Because of Pythagoras's theorem we have $BL = \sqrt{3}$. On the other hand, we have $ML = \sqrt{3}$ and $\angle BLN = 30°$. Because of that, we have $\angle LBM = 15°$ and so we have $\angle CBK = 30° + 15° = 45°$. Hence, $\angle CKB = 180° - 60° - 45° = 75°$.



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4. There are two circles with centers O_1, O_2 lie inside of circle ω and are tangent to it. Chord AB of ω is tangent to these two circles such that they lie on opposite sides of this chord. Prove that $\angle O_1AO_2 + \angle O_1BO_2 > 90°$.

Proposed by Iman Maghsoudi

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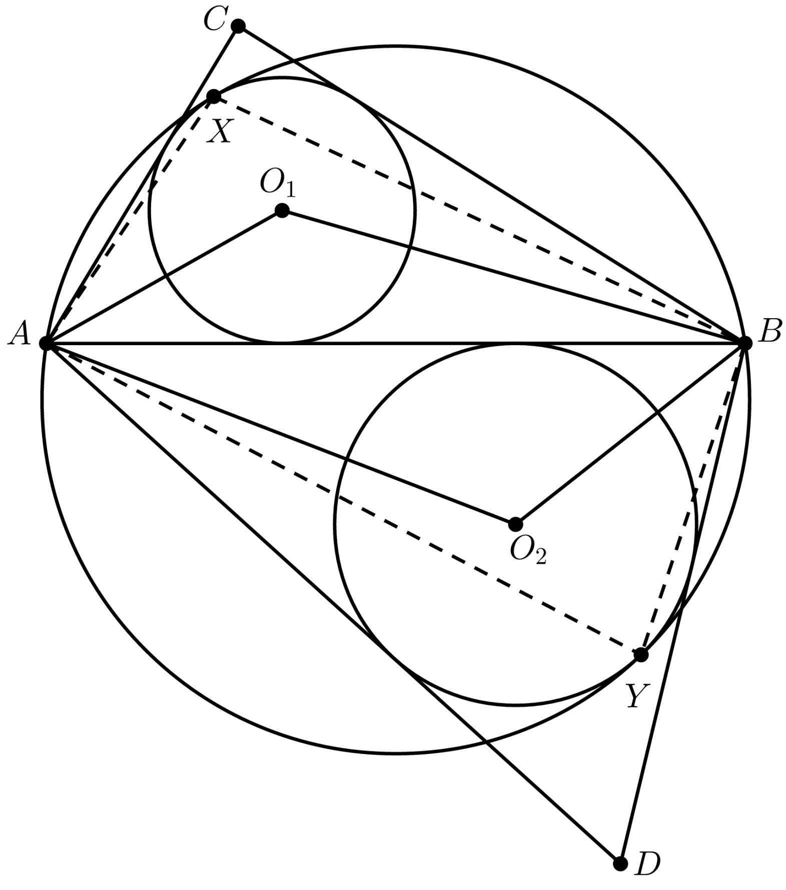
Solution. Let AC, BC be tangents from A, B to the circle with center O_1 and AD, BD be tangents from A, B to the circle with center O_2. It's enough to show that $\angle CAD + \angle CBD > 180°$. Or to show that $\angle ACB + \angle ADB < 180°$.

We know that C, D lie on the outside of circle ω. Therefore, we can always say that $\angle ACB < \angle AXB$ and $\angle ADB < \angle AYB$ because of the exterior angles. But we know that $\angle AXB + \angle AYB = 180°$. Hense, we can conclude that $\angle ACB + \angle ADB < 180°$ and the statement is proven.



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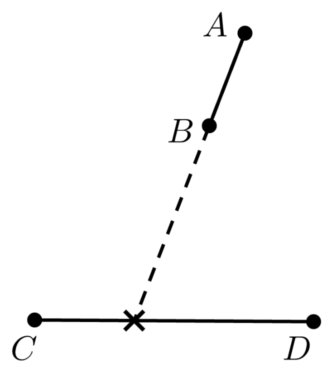
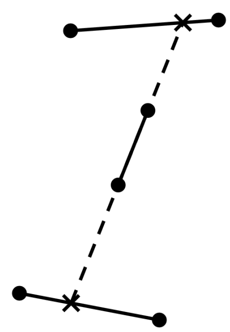
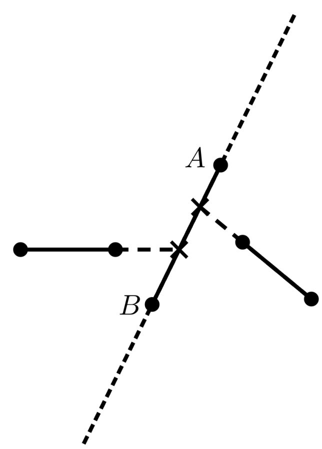
5. There are some segments on the plane such that no two of them intersect each other (even at the ending points). We say segment AB **breaks** segment CD if the extension of AB cuts CD at some point between C and D.





(a) Is it possible that each segment when extended from both ends, breaks exactly one other segment from each way?

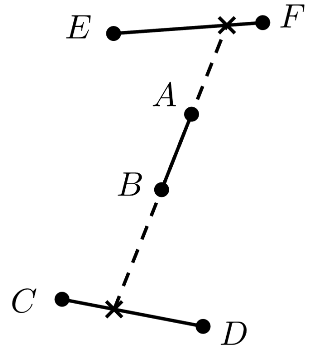


(b) A segment is called **surrounded** if from both sides of it, there is exactly one segment that breaks it. (*e.g.* segment AB in the figure.) Is it possible to have all segments to be surrounded?

Proposed by Morteza Saghafian

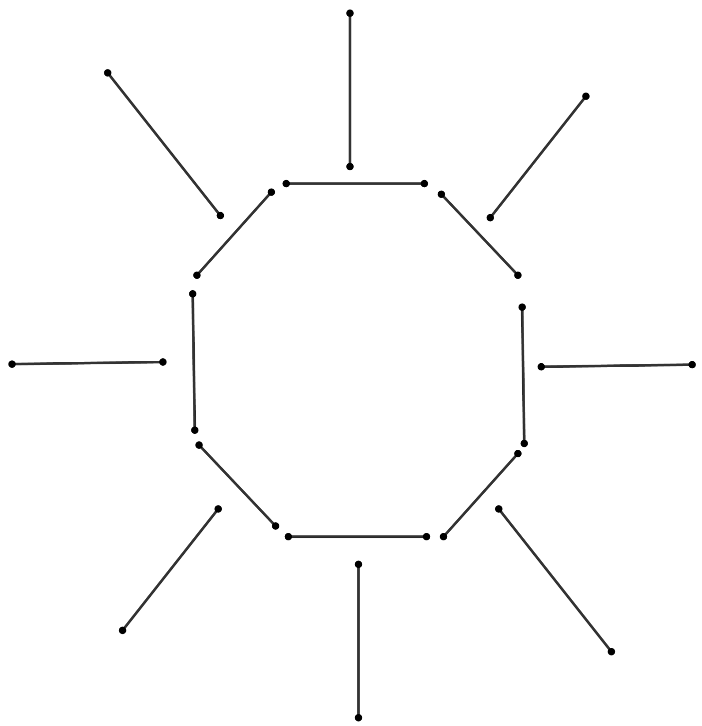
Solution.

(a) No. Consider the convex hull of the endpoints of these segments. Let A be a vertex of the convex hull, where AB is one of the segments.



We know that there exist segments CD, EF as in the figure. So A lies inside the convex hull of C, D, E, F and therefore it cannot be a vertex of the main convex hull. Contradiction! □

(b) Yes. The figure below shows that it is possible for all segments to be surrounded.



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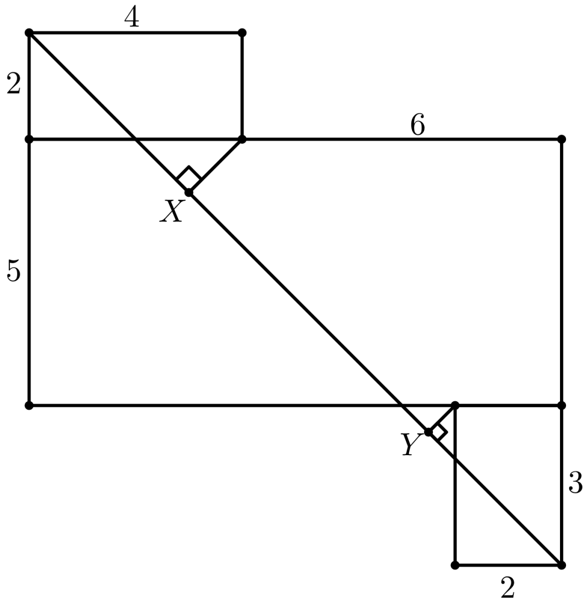
Intermediate Level

Problems

1. There are three rectangles in the following figure. The lengths of some segments are shown. Find the length of the segment XY.



2. In convex quadrilateral $ABCD$, the diagonals AC and BD meet at the point P. We know that $\angle DAC = 90^\circ$ and $2\angle ADB = \angle ACB$. If we have $\angle DBC + 2\angle ADC = 180^\circ$ prove that $2AP = BP$.

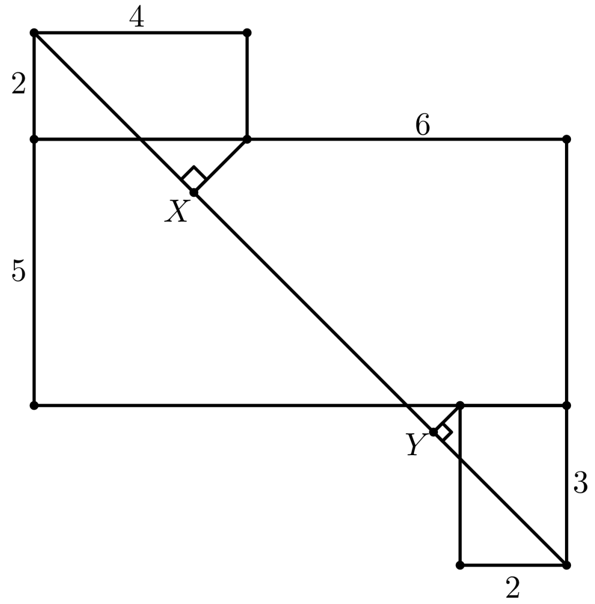
3. Let ω_1, ω_2 be two circles with centers O_1 and O_2, respectively. These two circles intersect each other at points A and B. Line O_1B intersects ω_2 for the second time at point C, and line O_2A intersects ω_1 for the second time at point D . Let X be the second intersection of AC and ω_1. Also Y is the second intersection point of BD and ω_2. Prove that $CX = DY$.

4. We have a polyhedron all faces of which are triangle. Let P be an arbitrary point on one of the edges of this polyhedron such that P is not the midpoint or endpoint of this edge. Assume that $P_0 = P$. In each step, connect P_i to the centroid of one of the faces containing it. This line meets the perimeter of this face again at point P_{i+1}. Continue this process with P_{i+1} and the other face containing P_{i+1}. Prove that by continuing this process, we cannot pass through all the faces. (The centroid of a triangle is the point of intersection of its medians.)

5. Suppose that $ABCD$ is a parallelogram such that $\angle DAC = 90^\circ$. Let H be the foot of perpendicular from A to DC, also let P be a point along the line AC such that the line PD is tangent to the circumcircle of the triangle ABD. Prove that $\angle PBA = \angle DBH$.

Solutions

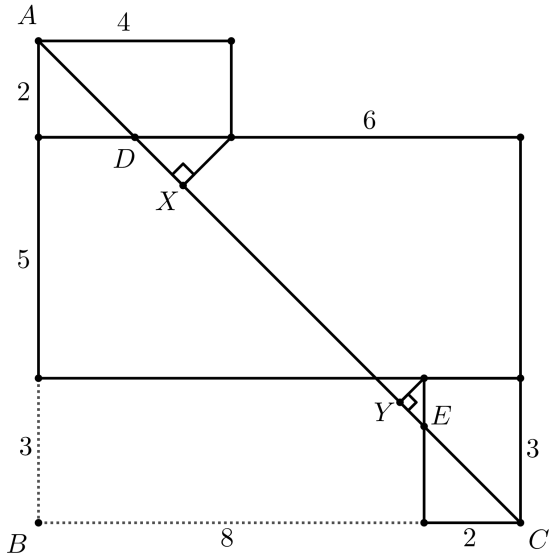
1. There are three rectangles in the following figure. The lengths of some segments are shown. Find the length of the segment XY.



Proposed by Hirad Aalipanah

Let us continue the rectangular sides to get the ABC triangle. Because $AB = BC$ we can say that $\angle BCA = \angle BAC = 45°$. Therefore, we can determine some of the segments using the Pythagoras's theorem such as $AD = 2\sqrt{2}$, $DX = \sqrt{2}$, $CE = 2\sqrt{2}$ and $EY = \frac{\sqrt{2}}{2}$. So, we have

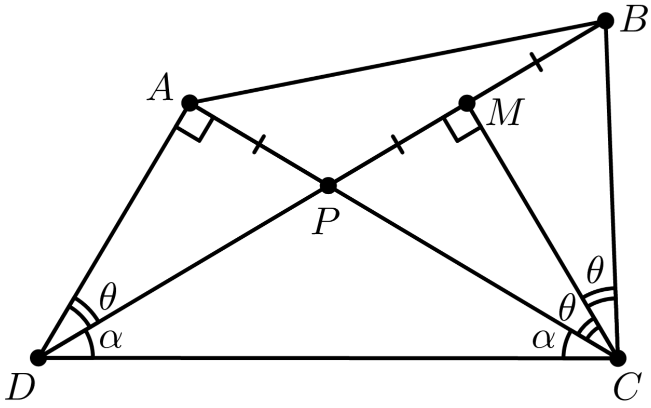
$$XY = AC - AD - DX - CE - EY = 10\sqrt{2} - 2\sqrt{2} - \sqrt{2} - 2\sqrt{2} - \frac{\sqrt{2}}{2} = \frac{9\sqrt{2}}{2}$$


A
4
2
6
D
X
5
Y
E
3
3
B
8
2
C

2. In convex quadrilateral $ABCD$, the diagonals AC and BD meet at the point P. We know that $\angle DAC = 90°$ and $2\angle ADB = \angle ACB$. If we have $\angle DBC + 2\angle ADC = 180°$ prove that $2AP = BP$.

Proposed by Iman Maghsoudi

Solution.



Let M be the intersection point of the angle bisector of $\angle PCB$ with segment PB. Since $\angle PCM = \angle PDA = \theta$ and $\angle APD = \angle MPC$, we get that $\triangle PMC \sim \triangle PAD$, which means $\angle PMC = 90°$.
Now in triangle CPB, the angle bisector of vertex C is the same as the altitude from C, this means CPB is an isosceles triangle and so $PM = MB, PC = CB$.
In triangle DBC, we have

$$\widehat{DBC} + 2\theta + \widehat{PCD} + \widehat{PDC} = 180°.$$

This along with the assumption that $\angle DBC + 2\angle ADC = 180°$, implies $\angle PCD = \angle PDC$. Therefore $PC = PD$ and so $\triangle PMC \cong \triangle PAD$, hence $AP = PM = \frac{PB}{2}$. ■

3. Let ω_1, ω_2 be two circles with centers O_1 and O_2, respectively. These two circles intersect each other at points A and B. Line O_1B intersects ω_2 for the second time at point C, and line O_2A intersects ω_1 for the second time at point D . Let X be the second intersection of AC and ω_1. Also Y is the second intersection point of BD and ω_2. Prove that $CX = DY$.

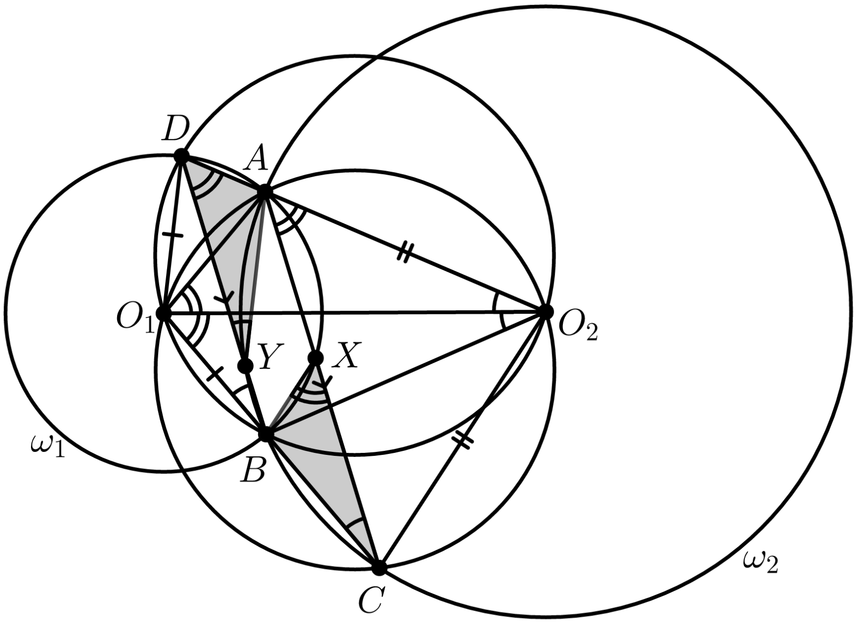
Proposed by Alireza Dadgarnia

Solution. First, we use a well-known lemma.

Lemma. *Let $PQRS$ be a convex quadrilateral with $RQ = RS$, $\angle RPQ = \angle RPS$ and $PQ \neq PS$. Then $PQRS$ is cyclic.*

Proof. Assume the contrary, and let $P' \neq P$ be the intersection point of the circle passing through R, S, Q with line PR.
Since $P'QRS$ is cyclic and $RQ = RS$, we get $\angle SP'R = \angle QP'R$. Now let's considerate on triangles $SP'P$ and $QP'P$. In these two triangles we have $\angle SP'P = \angle QP'P$ and also $\angle P'PQ = \angle P'PS$. This means these two triangles are congruent, hence $PQ = PS$, which is a contradiction. So the lemma is proved.

Back to the problem.



Triangles ADY and BXC are similar, because

$$\widehat{ADY} = \widehat{BXC} = 180^\circ - \widehat{BXA},$$

and

$$\widehat{DYA} = \widehat{BCX} = 180^\circ - \widehat{AYB}.$$

Note that O_2 lies on the angle bisector of $\angle AO_1B$, $O_2A = O_2C$ and also $O_1A \neq O_1C$. So we can use the lemma and conclude that O_1AO_2C is cyclic. Similarly, we get that O_2BO_1D is cyclic.

$$\widehat{AYD} = 180^\circ - \widehat{AYB} = \widehat{O_1CA} = \widehat{O_1O_2A} = \widehat{O_1BD}.$$

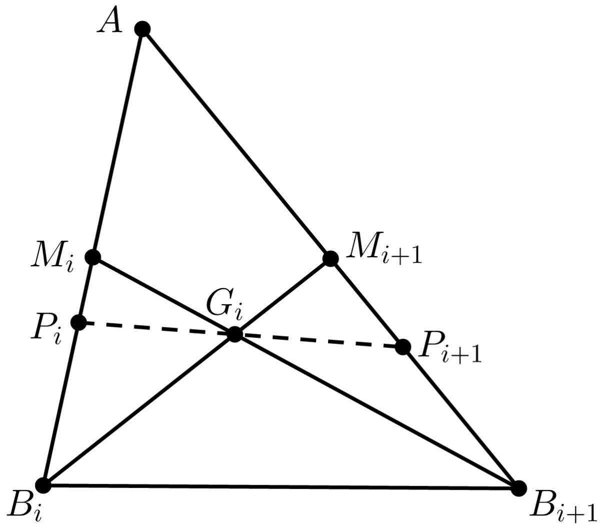
Which means $AC \parallel BD$ and so $AY = BC$. But since $\triangle ADY \sim \triangle BXC$, we get that these two triangles are congruent and so $CX = DY$. ■

4. We have a polyhedron all faces of which are triangle. Let P be an arbitrary point on one of the edges of this polyhedron such that P is not the midpoint or endpoint of this edge. Assume that $P_0 = P$. In each step, connect P_i to the centroid of one of the faces containing it. This line meets the perimeter of this face again at point P_{i+1}. Continue this process with P_{i+1} and the other face containing P_{i+1}. Prove that by continuing this process, we cannot pass through all the faces. (The centroid of a triangle is the point of intersection of its medians.)

Proposed by Mahdi Etesamifard - Morteza Saghafian

Solution. Suppose that AB is the edge that P lies on. Let M be the midpoint of AB and without loss of generality, assume that P lies between B and M. We will prove that it is impossible to pass through a face which doesn't contain A. (Such face exists in any polyhedron)

Let $B = B_0, B_1, B_2, \ldots$ be the vertices adjacent to A in this order. Let M_i be the midpoint of AB_i. By using induction, we prove that for each i, P_i lies on edge AB_i, between B_i and M_i. For $i = 0$ the claim is true. Now assume the claim for i and consider the triangle AB_iB_{i+1} with centroid G_i.



Since P_i lies between M_i and B_i, we get that P_iG_i lies between M_iG_i and B_iG_i, which are the medians of this triangle. So P_{i+1} lies on AB_{i+1}, between M_{i+1} and B_{i+1}. So the claim is proved.

We proved that P_i's lie on AB_i's, so the sequence of points P_i goes around A and therefore does not pass through a face which doesn't contain A. ■

5. Suppose that $ABCD$ is a parallelogram such that $\angle DAC = 90°$. Let H be the foot of perpendicular from A to DC, also let P be a point along the line AC such that the line PD is tangent to the circumcircle of the triangle ABD. Prove that $\angle PBA = \angle DBH$.

Proposed by Iman Maghsoudi

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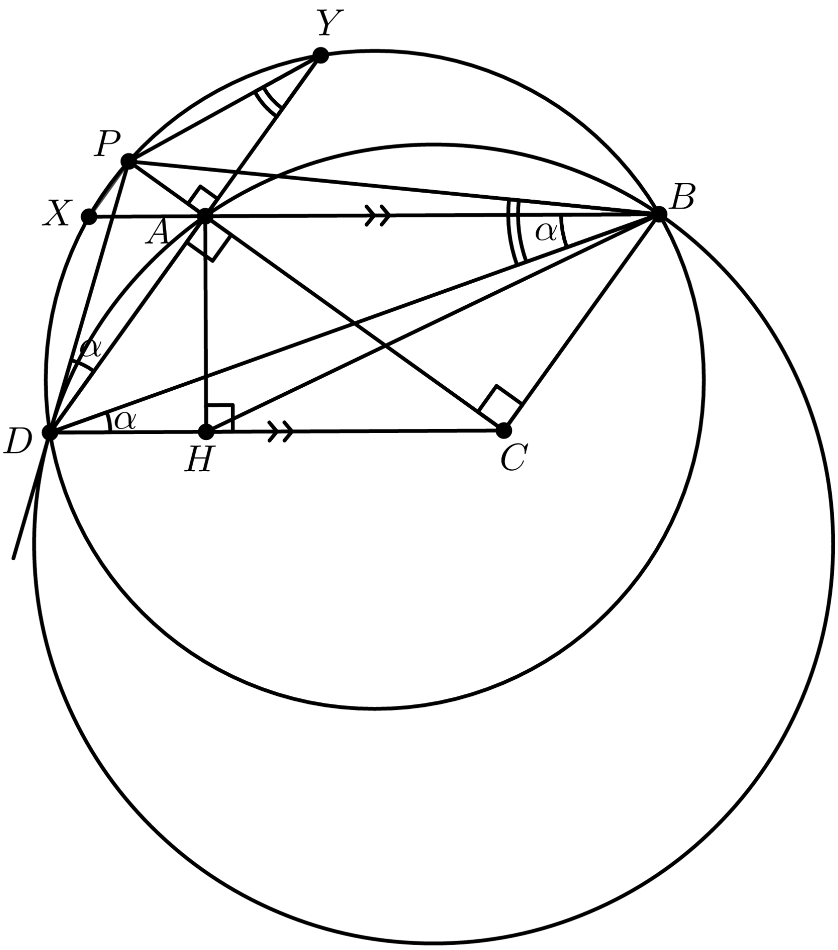

Suppose that AB, AD meet the circumcircle of triangle PDB for the second time at points X, Y respectively. Let $\angle CDB = \alpha$ and $\angle ADB = \theta$. Therefore, we have $\angle ABD = \alpha$, and so $\angle ADP = \alpha$.

Also $\angle PDB = \angle PXB = \alpha + \theta$, and $\angle PAX = \angle ACD = \angle DAH$. Which implies

$$\overset{\triangle}{APX} \sim \overset{\triangle}{ADH} \Longrightarrow \frac{AP}{AH} = \frac{AX}{AD},$$

$$\overset{\triangle}{XAD} \sim \overset{\triangle}{YAB} \Longrightarrow \frac{AY}{AB} = \frac{AX}{AD},$$

$$\Longrightarrow \frac{AP}{AH} = \frac{AY}{AB}.$$

Now since $\angle HAB = \angle PAY = 90°$, It can be written that $\overset{\triangle}{APY} \sim \overset{\triangle}{AHB}$.

$$\Longrightarrow \widehat{HBA} = \widehat{PYA} = \widehat{PBD} \implies \widehat{PBA} = \widehat{DBH}.$$

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Advanced Level

Problems

1. Two circles ω_1, ω_2 intersect each other at points A, B. Let PQ be a common tangent line of these two circles with $P \in \omega_1$ and $Q \in \omega_2$. An arbitrary point X lies on ω_1. Line AX intersects ω_2 for the second time at Y. Point $Y' \neq Y$ lies on ω_2 such that $QY = QY'$. Line $Y'B$ intersects ω_1 for the second time at X'. Prove that $PX = PX'$

2. In acute triangle ABC, $\angle A = 45°$. Points O, H are the circumcenter and the orthocenter of ABC, respectively. D is the foot of altitude from B. Point X is the midpoint of arc AH of the circumcircle of triangle ADH that contains D. Prove that $DX = DO$.

3. Find all possible values of integer $n > 3$ such that there is a convex n-gon in which, each diagonal is the perpendicular bisector of at least one other diagonal.

4. Quadrilateral $ABCD$ is circumscribed around a circle. Diagonals AC, BD are not perpendicular to each other. The angle bisectors of angles between these diagonals, intersect the segments AB, BC, CD and DA at points K, L, M and N. Given that $KLMN$ is cyclic, prove that so is $ABCD$.

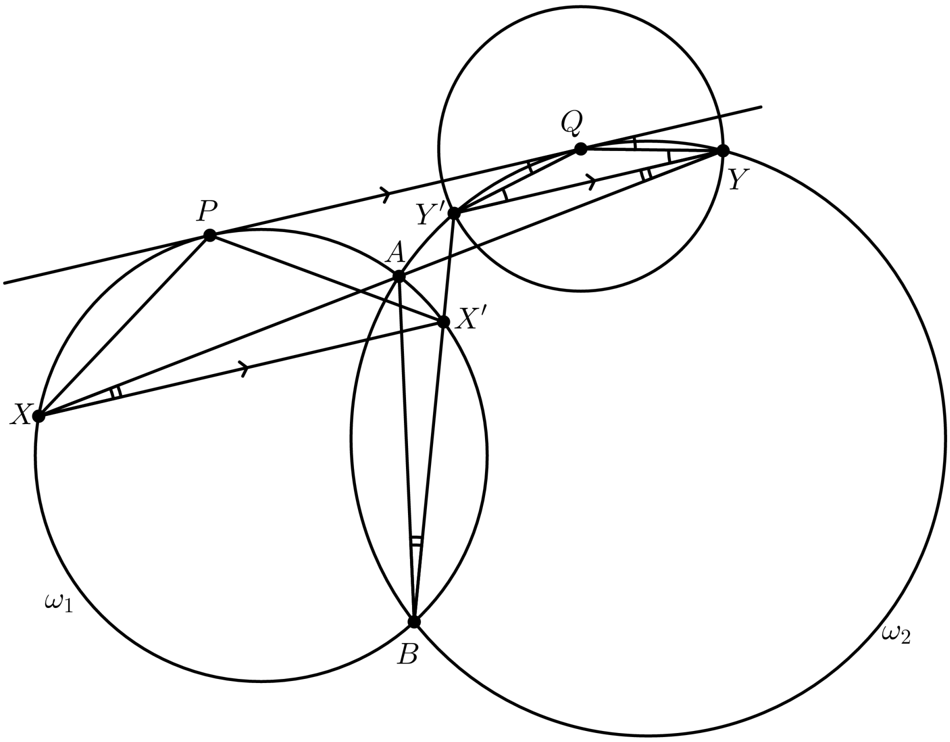
5. $ABCD$ is a cyclic quadrilateral. A circle passing through A, B is tangent to segment CD at point E. Another circle passing through C, D is tangent to AB at point F. Point G is the intersection point of AE, DF, and point H is the intersection point of BE, CF. Prove that the incenters of triangles AGF, BHF, CHE, DGE lie on a circle.

Solutions

1. Two circles ω_1, ω_2 intersect each other at points A, B. Let PQ be a common tangent line of these two circles with $P \in \omega_1$ and $Q \in \omega_2$. An arbitrary point X lies on ω_1. Line AX intersects ω_2 for the second time at Y. Point $Y' \neq Y$ lies on ω_2 such that $QY = QY'$. Line $Y'B$ intersects ω_1 for the second time at X'. Prove that $PX = PX'$

Proposed by Morteza Saghafian

Solution.

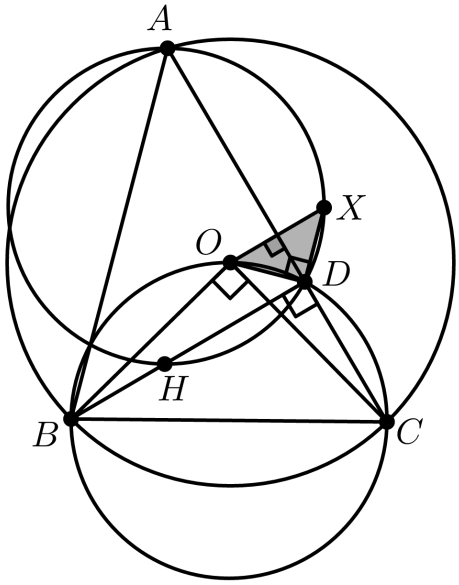


$QY = QY'$ implies $\angle QYY' = \angle QY'Y$. Considering circle ω_2, we have $\angle QYY' = \angle Y'QP$. This means $YY' \parallel PQ$.
We also have $\angle Y'YA = \angle Y'BA$ and $\angle ABX' = \angle AXX'$. This means $XX' \parallel YY' \parallel PQ$.
Therefore $\angle PXX' = \angle X'PQ = \angle PX'X$, so $PX = PX'$ ■

2. In acute triangle ABC, $\angle A = 45°$. Points O, H are the circumcenter and the orthocenter of ABC, respectively. D is the foot of altitude from B. Point X is the midpoint of arc AH of the circumcircle of triangle ADH that contains D. Prove that $DX = DO$.

Proposed by Fatemeh Sajadi

Solution.



Since $\angle AXH = 90°$ and $XA = XH$, we conclude that $\angle AHX = 45° = \angle ADX$.
Also $\angle BOC = 2\angle A = 90°$, therefore points O, D lie on a circle with diameter BC. This implies

$$\widehat{ODA} = \widehat{OBC} = 45° \implies \widehat{ODX} = 90°.$$

But note that

$$\widehat{ACH} = 90° - \widehat{A} = 45° = \frac{1}{2}\widehat{AXH}.$$

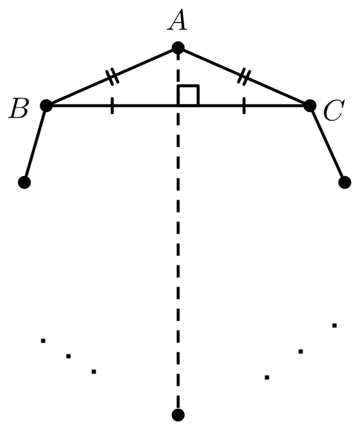
This alongside with $XA = XH$ means X is the circumcenter of triangle ACH and so $XA = XC$. Thus OX is the perpendicular bisector of AC and so $OX \perp AC$. Now in triangle ODX, the angle bisector of vertex D is the same as the altitude from D, hence it is an isosceles triangle with $DX = DO$. ■

3. Find all possible values of integer $n > 3$ such that there is a convex n-gon in which, each diagonal is the perpendicular bisector of at least one other diagonal.

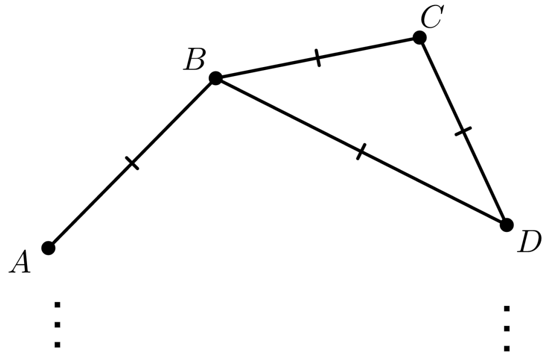
Proposed by Mahdi Etesamifard

Solution. Let m be the total number of the perpendicular bisectors of all diagonals in the given n-gon. The statement of the problem implies that m is not less than the number of diagonals. But it is clear that the total number of perpendicular bisectors of the diagonals does not exceed the number of diagonals! Hence, we conclude that each diagonal is the perpendicular bisector of exactly one other diagonal. Since the perpendicular bisector of a diagonal is a unique line, we get that for each diagonal d, there is exactly one diagonal d' such that d' is the perpendicular bisector of d.

Consider three adjacent vertices B, A, C of the n-gon, where A lies between B and C. BC is a diagonal of the n-gon, and only diagonals that contain A have an intersection point with BC. Specially, the diagonal which is the perpendicular bisector of BC passes through A. Hence $AB = AC$. Using this similar idea, it is deduced that all sides of this n-gon have the same length.



Similar to the previous part, consider four adjacent points of the n-gon, A, B, C, D with the given order. If $n > 4$, then AD is a diagonal of the n-gon, and the only diagonals that contain B or C, have an intersection point with AD. Therefore either B or C lie on the perpendicular bisector of AD. Without loss of generality, assume that $BA = BD$. According to the previous argument, $BA = BC = CD$. Thus triangle BCD is an equilateral and so $\angle BCD = 60°$. (In the other case we would have $\angle ABC = 60°$.)



This implies that between any two adjacent vertices, there is one that has a 60 degree angle. Hence there is at least $\frac{n}{2}$ angles of $60°$ in this n-gon.
It is known that the total number of 60 degree angles in an n-gon with $n > 3$ is at most 2. So we must have $\frac{n}{2} \leq 2$ which means $n \leq 4$, a contradiction.
Clearly, any rhombus satisfies the desired property. So the answer is $\boxed{n = 4}$. ■

4. Quadrilateral $ABCD$ is circumscribed around a circle. Diagonals AC, BD are not perpendicular to each other. The angle bisectors of angles between these diagonals, intersect the segments AB, BC, CD and DA at points K, L, M and N. Given that $KLMN$ is cyclic, prove that so is $ABCD$.

Proposed by Nikolai Beluhov (Bulgaria)

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Solution. Let P be the intersection point of AC, BD. First we claim that KL and MN are not parallel. Assume the contrary, that $KL \parallel MN$. Since $KLMN$ is cyclic, we have $KN = ML$, and $PK = PL,\ PM = PN$. We also have

$$\frac{KP}{PM} = \frac{PL}{PN}.$$

Let $AP = x, BP = y, CP = z$ and $DP = t$. Also let $\angle APB = 2\alpha$ and $\angle BPC = 2\theta$. We have

$$KP = \frac{xy}{x+y}\cos\alpha,\ PM = \frac{zt}{z+t}cos\alpha \implies \frac{KP}{PM} = \frac{\frac{1}{z}+\frac{1}{t}}{\frac{1}{x}+\frac{1}{y}}.$$

Similarly,

$$\frac{PL}{PN} = \frac{\frac{1}{x}+\frac{1}{t}}{\frac{1}{y}+\frac{1}{z}}.$$

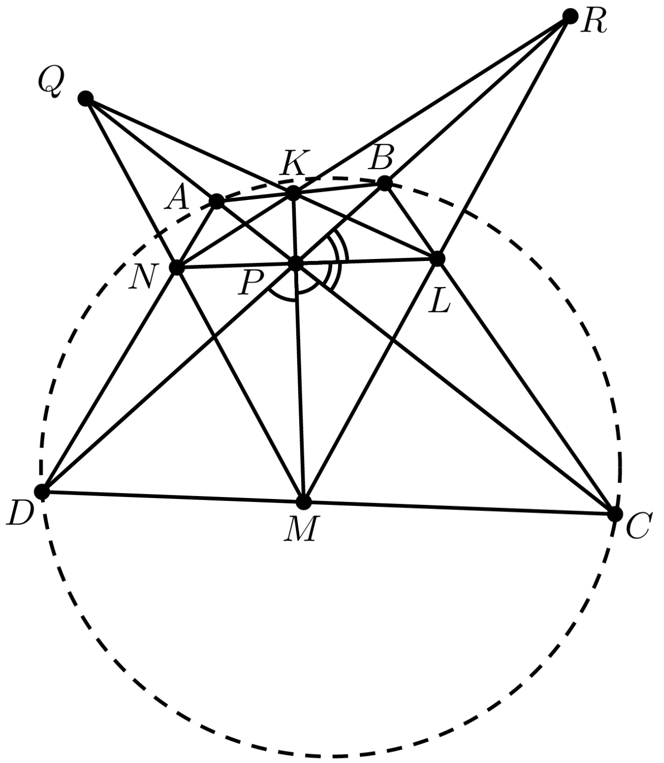
Since $\frac{KP}{PM} = \frac{PL}{PN}$, with a little of calculation, we shall have

$$\frac{1}{yz}+\frac{1}{z^2}+\frac{1}{zt} = \frac{1}{tx}+\frac{1}{x^2}+\frac{1}{xy} \implies \left(\frac{1}{x}-\frac{1}{y}\right)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right) = 0.$$

Which means $x = z$. But note that $PK = PL$ implies

$$\frac{xy}{x+y}\cos\alpha = \frac{yz}{y+z}\cos\theta.$$

But $\theta = 90^\circ - \alpha$, so we must have $\alpha = \theta = 45^\circ$, and so $AC \perp BD$, which is a contradiction. Therefore the claim is proved. With the similar idea, we can show that KN and LM are not parallel.



By Menelaus' theorem, KL and MN meet at a point Q on AC such that $\frac{AQ}{QC} = \frac{AP}{PC}$ and LM and NK meet at a point R on BD such that $\frac{BR}{RD} = \frac{BP}{PD}$.

Let the incircle ω of $ABCD$ touch its sides at K', L', M', and N'. By Brianchon's theorem, AL', CK', and BD are concurrent. By Ceva's and Menelaus' theorems, K', L', and Q are collinear. Analogously, M', N', and Q are collinear and $L'M'$ and $N'K'$ meet at R.

By Brianchon's theorem, $K'M'$ and $L'N'$ meet at P. It follows the diagonals and opposite sides of both $KLMN$ and $K'L'M'N'$ intersect at the vertices of $\triangle PQR$. Therefore, both the circumcircle of $KLMN$ and ω coincide with the polar circle of $\triangle PQR$.

Since K is a common point of AB and ω, $K \equiv K'$. Analogously, $L \equiv L'$, $M \equiv M'$, and $N \equiv N'$. Hence the angle bisector KM of AC and BD makes equal angles with AB and CD and $ABCD$ is cyclic, as needed. ■

5. $ABCD$ is a cyclic quadrilateral. A circle passing through A, B is tangent to segment CD at point E. Another circle passing through C, D is tangent to AB at point F. Point G is the intersection point of AE, DF, and point H is the intersection point of BE, CF. Prove that the incenters of triangles AGF, BHF, CHE, DGE lie on a circle.

Proposed by Le Viet An (Vietnam)

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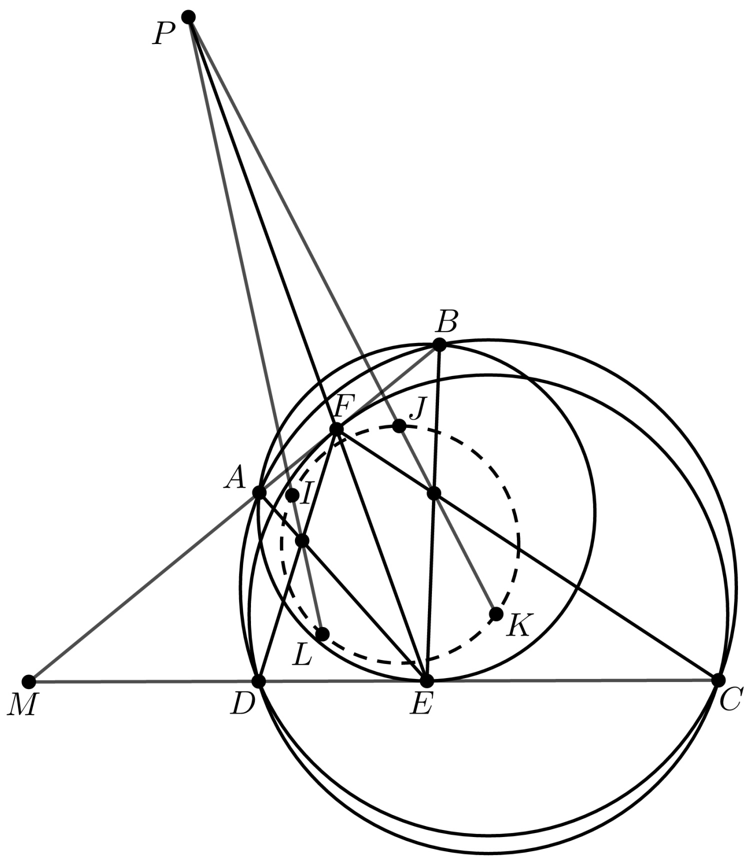
Solution.



Let I, J, K, L be the incenters of the triangles AGF, BHF, CHE, DGE respectively. Let ω be the circumcircle of $ABCD$. In case of $AB \parallel CD$, we would conclude that $ABCD$ is an isosceles trapezoid and it is easy to see that $IJKL$ is also an isosceles trapezoid.

So assume that $AB \not\parallel CD$ and let M be the intersection point of rays BA and CD. Since $ABCD$ is cyclic, it is obtained that

$$MA \cdot MB = MD \cdot MC = \mathcal{P}_M(\omega)$$

Since ME is tangent to $\odot ABE$, we get

$$\widehat{MEA} = \widehat{MBE}.$$

We also have $ME^2 = MA \cdot MB = \mathcal{P}_M(\odot ABE)$ and $MF^2 = MD \cdot MC = \mathcal{P}_M(\odot CDF)$, which implies $ME = MF$, and so $\widehat{MEF} = \widehat{MFE}$. Therefore,

$$\widehat{AEF} = \widehat{MEF} - \widehat{MEA} = \widehat{MFE} - \widehat{MBE} = \widehat{BEF}.$$

The latest equation means that EF is the interior angle bisector of $\angle AEB$. Similarly, FE is the interior angle bisector of $\angle CFD$.

Note that H, J, K are collinear and $\angle FJH = 90^\circ + \frac{\angle FBH}{2}$. Thus

$$\begin{aligned}
\widehat{FJK} &= 90^\circ + \frac{\widehat{MBE}}{2} = 90^\circ + \frac{\widehat{MEA}}{2} \\
&= 90^\circ + \frac{180^\circ - \widehat{AEC}}{2} = 180^\circ - \frac{\widehat{AEC}}{2} \\
&= 180^\circ - \frac{\widehat{AEB} + \widehat{BEC}}{2} = 180^\circ - \left(\widehat{FEB} + \widehat{BEK}\right) \\
&= 180^\circ - \widehat{FEK}
\end{aligned}$$

This results in that $EFJK$ is cyclic. With similar arguments, $EFIL$ is also cyclic.
Since EF is the interior angle bisector of $\angle GEH$ and $\angle GFH$, it is easy to see that triangles GEF and HEF are congruent. Therefore $EG = EH$ and $FG = FH$, and so $\frac{GE}{GF} = \frac{HE}{HF} = k$. Consider three lines, the exterior angle bisector of vertex G in $\triangle GEF$, the exterior angle bisector of vertex H in $\triangle HEF$ and the line EF. According to the latest equation, there is two cases:

- These three lines are pairwise parallel. This means $EFJK$ and $EFIL$ are isosceles trapezoids. Hence the segments EF, JK and IL have the same perpendicular bisector and so $IJKL$ is an isosceles trapezoid. $\square$
- These three points are concurrent at a point P where $\frac{PE}{PF} = k$. Now we simply have

$$PJ \cdot PK = \mathcal{P}_P(\odot EFJK) = PE \cdot PF = \mathcal{P}_P(\odot EFIL) = PI \cdot PL.$$

  Which means $IJKL$ is cyclic. $\square$

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