National Math Olympiad (Second Round) 1999

Day 1

1 Does there exist a positive integer that is a power of 2 and we get another power of 2 by swapping its digits? Justify your answer.

2 ABC is a triangle with $\angle B > 45^\circ$, $\angle C > 45^\circ$. We draw the isosceles triangles CAM, BAN on the sides AC, AB and outside the triangle, respectively, such that $\angle CAM = \angle BAN = 90^\circ$. And we draw isosceles triangle BPC on the side BC and inside the triangle such that $\angle BPC = 90^\circ$. Prove that ΔMPN is an isosceles triangle, too, and $\angle MPN = 90^\circ$.

3 We have a 100×100 garden and weve plant 10000 trees in the 1×1 squares (exactly one in each.). Find the maximum number of trees that we can cut such that on the segment between each two cut trees, there exists at least one uncut tree.

Day 2

1 Find all positive integers m such that there exist positive integers $a_1, a_2, \ldots, a_{1378}$ such that:

$$m = \sum_{k=1}^{1378} \frac{k}{a_k}.$$

2 Let ABC be a triangle and points P, Q, R be on the sides AB, BC, AC, respectively. Now, let A', B', C' be on the segments PR, QP, RQ in a way that $AB||A'B'$, $BC||B'C'$ and $AC||A'C'$. Prove that:

$$\frac{AB}{A'B'} = \frac{S_{PQR}}{S_{A'B'C'}}.$$

Where S_{XYZ} is the surface of the triangle XYZ.

3 Let $A_1, A_2, \cdots, A_n$ be n distinct points on the plane ($n > 1$). We consider all the segments A_iA_j where $i < j \leq n$ and color the midpoints of them. What's the minimum number of colored points? (In fact, if k colored points coincide, we count them 1.)