Art of Problem Solving

2006 Iran MO (2nd round)

National Math Olympiad (Second Round) 2006

Day 1

1 Let C_1, C_2 be two circles such that the center of C_1 is on the circumference of C_2. Let C_1, C_2 intersect each other at points M, N. Let A, B be two points on the circumference of C_1 such that AB is the diameter of it. Let lines AM, BN meet C_2 for the second time at A', B', respectively. Prove that $A'B' = r_1$ where r_1 is the radius of C_1.

2 Determine all polynomials $P(x, y)$ with real coefficients such that

$$P(x+y, x-y) = 2P(x, y) \qquad \forall x, y \in \mathbb{R}.$$

3 In the night, stars in the sky are seen in different time intervals. Suppose for every k stars $(k > 1)$, at least 2 of them can be seen in one moment. Prove that we can photograph $k-1$ pictures from the sky such that each of the mentioned stars is seen in at least one of the pictures.
(The number of stars is finite. Define the moments that the n^{th} star is seen as $[a_n, b_n]$ that $a_n < b_n$.)

Day 2

1 **a.)** Let $m > 1$ be a positive integer. Prove there exist finite number of positive integers n such that $m + n | mn + 1$.

b.) For positive integers $m, n > 2$, prove that there exists a sequence $a_0, a_1, \cdots, a_k$ from positive integers greater than 2 that $a_0 = m$, $a_k = n$ and $a_i + a_{i+1} | a_i a_{i+1} + 1$ for $i = 0, 1, \cdots, k-1$.

2 Let $ABCD$ be a convex cyclic quadrilateral. Prove that: $a)$ the number of points on the circumcircle of $ABCD$, like M, such that $\frac{MA}{MB} = \frac{MD}{MC}$ is 4. $b)$ The diagonals of the quadrilateral which is made with these points are perpendicular to each other.

3 Some books are placed on each other. Someone first, reverses the upper book. Then he reverses the 2 upper books. Then he reverses the 3 upper books and continues like this. After he reversed all the books, he starts this operation from the first. Prove that after finite number of movements, the books become exactly like their initial configuration.