National Math Olympiad (Second Round) 2003

Day 1

1 We call the positive integer n a $3-$*stratum* number if we can divide the set of its positive divisors into 3 subsets such that the sum of each subset is equal to the others. a) Find a $3-$stratum number. b) Prove that there are infinitely many $3-$stratum numbers.

2 In a village, there are n houses with $n > 2$ and all of them are not collinear. We want to generate a water resource in the village. For doing this, point A is *better* than point B if the sum of the distances from point A to the houses is less than the sum of the distances from point B to the houses. We call a point *ideal* if there doesnt exist any *better* point than it. Prove that there exist at most 1 *ideal* point to generate the resource.

3 n volleyball teams have competed to each other (each 2 teams have competed exactly 1 time.). For every 2 distinct teams like A, B, there exist exactly t teams which have lost their match with A, B. Prove that $n = 4t + 3$. (Notabene that in volleyball, there doesnt exist tie!)

Day 2

1 Let $x, y, z \in \mathbb{R}$ and $xyz = -1$. Prove that:

$$x^4 + y^4 + z^4 + 3(x + y + z) \geq \frac{x^2}{y} + \frac{x^2}{z} + \frac{y^2}{x} + \frac{y^2}{z} + \frac{z^2}{x} + \frac{z^2}{y}.$$

2 $\angle A$ is the least angle in ΔABC. Point D is on the arc BC from the circumcircle of ΔABC. The perpendicular bisectors of the segments AB, AC intersect the line AD at M, N, respectively. Point T is the meet point of BM, CN. Suppose that R is the radius of the circumcircle of ΔABC. Prove that:

$$BT + CT \leq 2R.$$

3 We have a chessboard and we call a 1×1 square a room. A robot is standing on one arbitrary vertex of the rooms. The robot starts to move and in every one movement, he moves one side of a room. This robot has 2 memories A, B.

At first, the values of A, B are 0. In each movement, if he goes up, 1 unit is added to A, and if he goes down, 1 unit is waned from A, and if he goes right, the value of A is added to B, and if he goes left, the value of A is waned from B. Suppose that the robot has traversed a traverse (!) which hasnt intersected itself and finally, he has come back to its initial vertex. If $v(B)$ is the value of B in the last of the traverse, prove that in this traverse, the interior surface of the shape that the robot has moved on its circumference is equal to $|v(B)|$.