Iran's Geometry Problems

Problems and Solutions from Contests

2014-2015

This booklet is prepared by Hirad Aalipanah, Iman Maghsoudi.
With special thanks to Morteza Saghafian, Mahdi Etesami Fard,
Davood Vakili, Erfan Salavati.
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www.ysc.ac.ir - www.igo-official.ir

The first Iranian Geometry Olympiad was held simultaneously in Tehran and Isfahan on September 4th, 2014 with over 300 participants. This competition had two levels, junior and senior which each level had 5 problems. The contestants solved problems in 4 hours and 30 minutes.
In the end, the highest ranked participants in each level awarded with gold ruler, silver ruler or bronze ruler respectively.
This booklet have the problems of this competition plus other geometry problems used in other Iranian mathematical competition since summer of 2014 till spring of 2015.
This year the second Iranian Geometry Olympiad will be held in Tehran on September 3th, 2015. We tend to provide online presence for those who are interested from other countries. Those who wish to participate can contact Mr. Salavati for more information at erfan.salavati@gmail.com

Iranian Geometry Olympiads website: www.igo-official.ir

Problems

1.**(Geometry Olympiad(Junior and Senior level))** In a right triangle ABC we have $\angle A = 90^\circ$,$\angle C = 30^\circ$. Denot by C the circle passing through A which is tangent to BC at the midpoint.Assume that C intersects AC and the circumcircle of ABC at N and M respectively. Prove that $MN \perp BC$.

Proposed by Mahdi Etesami Fard

2.**(Geometry Olympiad(Junior Level))** The inscribed circle of $\triangle ABC$ touches BC, AC and AB at D, E and F respectively. Denote the perpendicular foots from F, E to BC by K, L respectively. Let the second intersection of these perpendiculars with the incircle be M, N respectively. Show that $\frac{S_{\triangle BMD}}{S_{\triangle CND}} = \frac{DK}{DL}$

Proposed by Mahdi Etesami Fard

3.(**Geometry Olympiad (Junior Level)**) Each of Mahdi and Morteza has drawn an inscribed 93-gon. Denote the first one by $A_1A_2...A_{93}$ and the second by $B_1B_2...B_{93}$. It is known that $A_iA_{i+1} \parallel B_iB_{i+1}$ for $1 \leqslant i \leqslant 93$ ($A_{93} = A_1, B_{93} = B_1$). Show that $\frac{A_iA_{i+1}}{B_iB_{i+1}}$ is a constant number independent of i.

Proposed by Morteza Saghafian

4.(**Geometry Olympiad (Junior Level)**) In a triangle ABC we have $\angle C = \angle A + 90^\circ$. The point D on the continuation of BC is given such that $AC = AD$. A point E in the side of BC in which A doesnt lie is chosen such that

$$\angle EBC = \angle A, \angle EDC = \frac{1}{2}\angle A$$

Prove that $\angle CED = \angle ABC$.

Proposed by Morteza Saghafian

5.(**Geometry Olympiad (Junior Level)**) Two points X, Y lie on the arc BC of the circumcircle of $\triangle ABC$ (this arc does not contain A) such that $\angle BAX = \angle CAY$. Let M denotes the midpoint of the chord AX . Show that $BM + CM > AY$

Proposed by Mahan Tajrobekar

6.**(Geometry Olympiad(Senior level))** In a quadrilateral $ABCD$ we have $\angle B = \angle D = 60^\circ$. Consider the line whice is drawn from M, the midpoint of AD, parallel to CD. Assume this line intersects BC at P. A point X lies on CD such that $BX = CX$. Prove that $AB = BP \Leftrightarrow \angle MXB = 60^\circ$

Proposed by Davood Vakili

7.**(Geometry Olympiad(Senior level))** An acute-angled triangle ABC is given. The circle with diameter BC intersects AB, AC at E, F respectively. Let M be the midpoint of BC and P the intersection point of AM and EF. X is a point on the arc EF and Y the second intersection point of XP with circle mentioned above. Show that $\angle XAY = \angle XYM$.

Proposed by Ali Zooelm

8.**(Geometry Olympiad(Senior level))** The tangent line to circumcircle of the acute-angled triangle ABC ($AC > AB$) at A intersects the continuation of BC at P. We denote by O the circumcenter of ABC. X is a point OP such that $\angle AXP = 90^\circ$. Two points E, F respectively on AB, AC at the same side of OP are chosen such that

$$\angle EXP = \angle ACX, \ \angle FXO = \angle ABX$$

If K, L denote the intersection points of EF with the circumcircle of $\triangle ABC$, show that OP is tangent to the circumcircle of $\triangle KLX$.

Proposed by Mahdi Etesami Fard

9.**(Geometry Olympiad(Senior level))** Two points P, Q lie on the side BC of triangle ABC and have the same distance to the midpoint. The pependiculars from$P$, Q tp BC intesects AC, AB at E, F respectively. LEt M be the intersection point of $PF$and EQ. If H_1 and H_2 denote the orthocenter of $\triangle BFP$ and $\triangle CEQ$ recpectively, show that $AM \perp H_1H_2$.

Proposed by Mahdi Etesami Fard

10.**(IGO Short list)**Suppose that I is incenter of $\triangle ABC$ and CI inresects AB at D.In circumcircle of $\triangle ABC$, T is midpoint of arc BAC and BI intersect this circle at M. If MD intersects AT at N, prove that: $BM \parallel CN$.

Proposed by Ali Zooelm

Solutions

1.**(Geometry Olympiad(Junior and Senior Level))** In a right triangle ABC we have $\angle A = 90^\circ$, $\angle C = 30^\circ$. Denot by C the circle passing through A which is tangent to BC at the midpoint.Assume that C intersects AC and the circumcircle of ABC at N and M respectively. Prove that $MN \perp BC$.

Proposed by Mahdi Etesami Fard

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solution.

Let K midpoint of side BC. Therefore:

$$AK = KC \Rightarrow \angle KAC = \angle NKC = 30^\circ$$

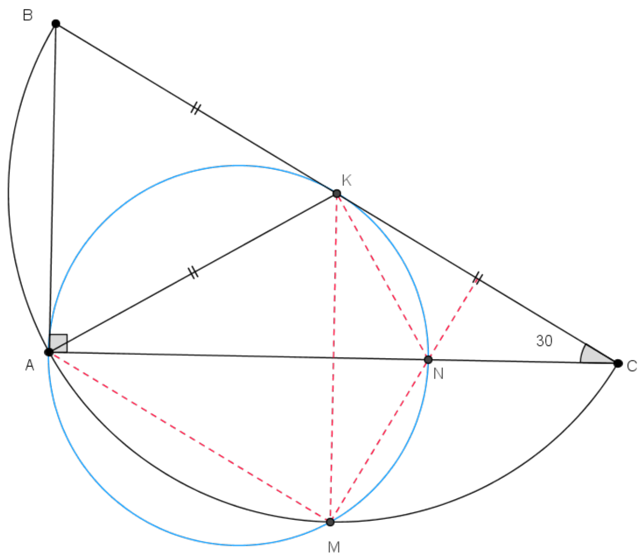
$$\angle ANK = \angle NKC + \angle ACB = 60^\circ$$

A, K, N, M lie on circle (C). Therefore:

$$\angle KAN = \angle KMN = 30^\circ, \angle AMK = 60^\circ$$

We know that K is the circumcenter of $\triangle ABC$. So we can say $KM = KC = AK$. Therefore $\triangle AKM$ is equilateral.(because of $\angle AMK = 60^\circ$). So $\angle AKM = 60^\circ$. We know that $\angle AKB = 60^\circ$, so we have $\angle MKC = 60^\circ$. On the other hand:

$$\angle KMN = 30^\circ \Rightarrow MN \perp BC$$



2.**(Geometry Olympiad(Junior Level))** The inscribed circle of $\triangle ABC$ touches BC, AC and AB at D, E and F respectively. Denote the perpendicular foots from F, E to BC by K, L respectively. Let the second intersection of these perpendiculars with the incircle be M, N respectively. Show that $\frac{S_{\triangle BMD}}{S_{\triangle CND}} = \frac{DK}{DL}$

Proposed by Mahdi Etesami Fard

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solution.

Let I be the incenter of $\triangle ABC$. We know that

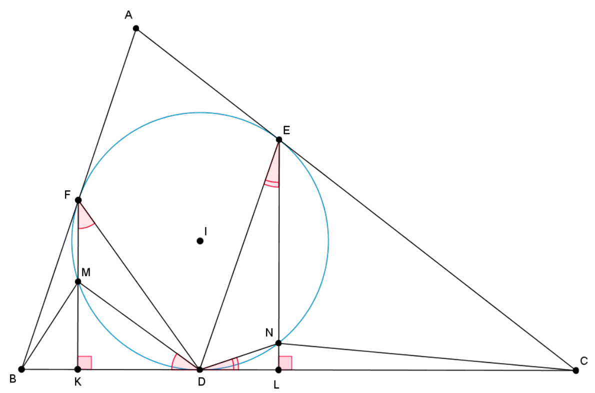
$$\left.\begin{array}{l}\angle BFK = 90^\circ - \angle B \\ \angle BFD = 90^\circ - \frac{1}{2}\angle B\end{array}\right\} \Rightarrow \angle DFM = \frac{1}{2}\angle B$$

But $\angle DFM = \angle MDK$. Therefore

$$\angle MDK = \frac{1}{2}\angle B$$

Hense $\triangle MDK$ and $\triangle BID$ are similar (same angles) and $\frac{MK}{DK} = \frac{r}{BD}$. In the same way we have $\frac{NL}{DL} = \frac{r}{CD}$. Therefore

$$r = \frac{MK \cdot BD}{DK} = \frac{NL \cdot CD}{DL} \Rightarrow \frac{area\ of\ \triangle BMD}{area\ of\ \triangle CND} = \frac{MK \cdot BD}{NL \cdot CD} = \frac{DK}{DL}$$



3.(**Geometry Olympiad (Junior Level)**) Each of Mahdi and Morteza has drawn an inscribed 93-gon. Denote the first one by $A_1A_2...A_{93}$ and the second by $B_1B_2...B_{93}$. It is known that $A_iA_{i+1} \parallel B_iB_{i+1}$ for $1 \leqslant i \leqslant 93$ ($A_{93} = A_1, B_{93} = B_1$). Show that $\frac{A_iA_{i+1}}{B_iB_{i+1}}$ is a constant number independent of i.

Proposed by Morteza Saghafian

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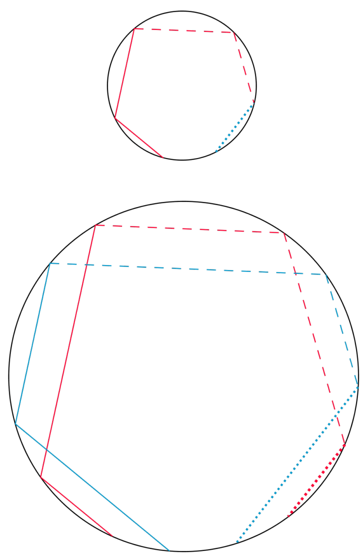
solution.

We draw a 93-gon similar with the second 93-gon in the circumcircle of the first 93-gon (so the sides of the second 93-gon would be multiplying by a constant number c). Now we have two 93-gons witch are inscribed in the same circle and apply the problem's conditions. We name this 93-gons $A_1A_2...A_{93}$ and $C_1C_2...C_{93}$.

We know that $A_1A_2 \parallel C_1C_2$. Therefore $\overset{\frown}{A_1C_1}=\overset{\frown}{A_2C_2}$ but they lie on the opposite side of each other. In fact, $\overset{\frown}{A_iC_i}=\overset{\frown}{A_{i+1}C_{i+1}}$ and they lie on the opposite side of each other for all $1 \leqslant i \leqslant 93$ ($\overset{\frown}{A_{94}C_{94}}=\overset{\frown}{A_1C_1}$). Therefore $\overset{\frown}{A_1C_1}$ and $\overset{\frown}{A_1C_1}$ lie on the opposite side of each other. So $\overset{\frown}{A_1C_1}= 0°$ or $180°$. This means that the 93-gons are coincident or reflections of each other across the center. So $A_iA_{i+1} = C_iC_{i+1}$ for $1 \leqslant i \leqslant 93$. Therefore, $\frac{A_iA_{i+1}}{B_iB_{i+1}} = c$.



4.(**Geometry Olympiad (Junior Level)**) In a triangle ABC we have $\angle C = \angle A + 90°$. The point D on the continuation of BC is given such that $AC = AD$. A point E in the side of BC in which A doesnt lie is chosen such that

$$\angle EBC = \angle A, \angle EDC = \frac{1}{2}\angle A$$

Prove that $\angle CED = \angle ABC$.

Proposed by Morteza Saghafian

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solution.

Suppose M is the midpoint of CD. Hense AM is the perpendicular bisector of CD. AM intersects DE and BE at P, Q respectively. Therefore, $PC = PD$. We have

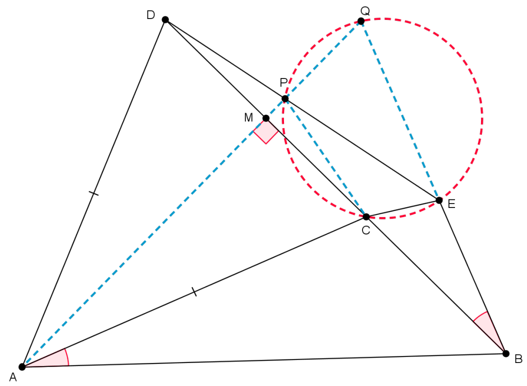
$$\angle EBA + \angle CAB = \angle A + \angle B + \angle A = 180° - \angle C + \angle A = 90°$$

Hense $AC \perp BE$. Thus in $\triangle ABQ$, BC, AC are altitudes. This means C is the orthocenter of this triangle and

$$\angle CQE = \angle CQB = \angle A = \frac{1}{2}\angle A + \frac{1}{2}\angle A = \angle PDC + \angle PCD = \angle CPE$$

Hense $CPQE$ is cyclic. Therefore

$$\angle CED = \angle CEP = \angle CQP = \angle CQA = \angle CBA = \angle B.$$



5.(**Geometry Olympiad (Junior Level)**) Two points X, Y lie on the arc BC of the circumcircle of $\triangle ABC$ (this arc does not contain A) such that $\angle BAX = \angle CAY$. Let M denotes the midpoint of the chord AX . Show that

$$BM + CM > AY$$

Proposed by Mahan Tajrobekar

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solution.

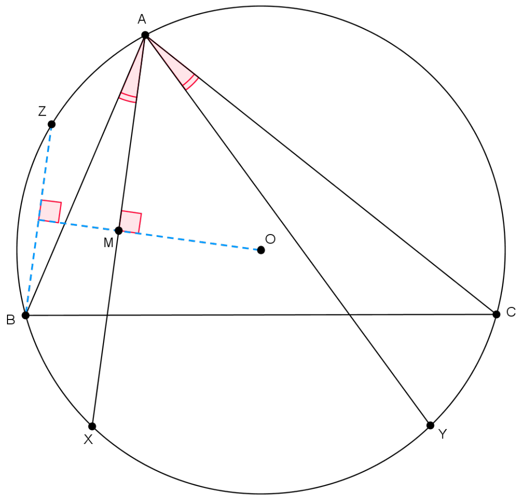
O is the circumcenter of $\triangle ABC$, so $OM \perp AX$. We draw a perpendicular line from B to OM. This line intersects with the circumcircle at Z. Since $OM \perp BZ$, OM is the perpendicular bisector of BZ. This means $MZ = MB$. By using triangle inequality we have

$$BM + MC = ZM + MC > CZ$$

But $BZ \parallel AX$, thus

$$\overset{\frown}{AZ} = \overset{\frown}{BX} = \overset{\frown}{CY} \Rightarrow \overset{\frown}{ZAC} = \overset{\frown}{YCA} \Rightarrow CZ = AY$$

Hense $BM + CM > AY$.



6.**(Geometry Olympiad(Senior level))** In a quadrilateral $ABCD$ we have $\angle B = \angle D = 60°$. Consider the line whice is drawn from M, the midpoint of AD, parallel to CD. Assume this line intersects BC at P. A point X lies on CD such that $BX = CX$. Prove that:

$$AB = BP \Leftrightarrow \angle MXB = 60°$$

Proposed by Davood Vakili

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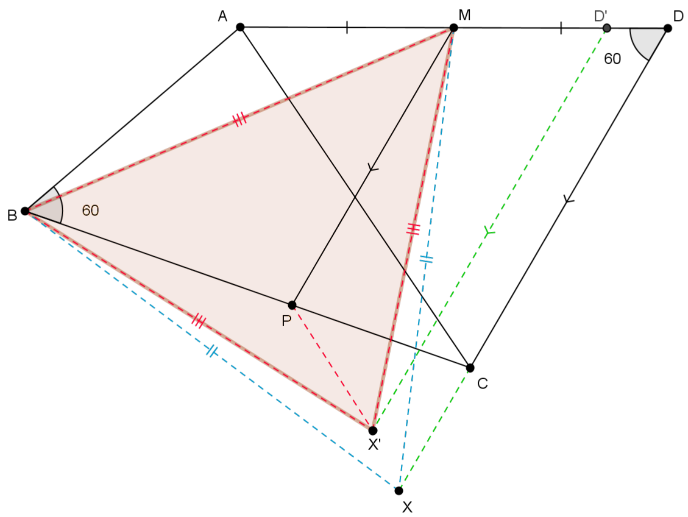
solution.

Suppose X' is a point such that $\triangle MBX'$ is equilateral.(X' and X lie on the same side of MB) It's enough to show that:

$$AB = BP \;\Leftrightarrow\; X' \equiv X$$

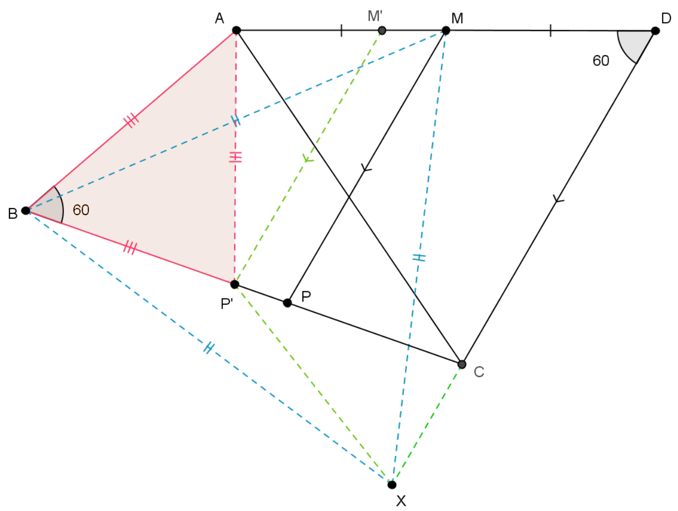


We want to prove that if $AB = BP$ then $\angle MXB = 60°$.
$AB = BP$ therefore $\triangle ABP$ is equilateral. We know that $\angle ABP = \angle MBX' = 60°$, Therefore $\angle ABM = \angle PBX'$. On the other hand $AB = BP, BM = BX'$ therefore $\triangle BAM$ and $\triangle BPX'$ are equal.

$$\angle X'PM = 360° - \angle MPB - \angle BPX' = 360° - \angle DCB - \angle BAM' = 120°$$

$MP \parallel DC$, so we can say $\angle PMD = 120°$. If we draw the line passing through X' such that be parallel with CD and this line intersects AD in D', then quadrilateral $MPX'D'$ is isosceles trapezoid. Therefore $PX' = MD'$. In the other hand $PX' = AM = MD$ (becauese $\triangle BAM$ and $\triangle BPX'$ are equal.) According to the statements we can say $MD' = MD$. In other words, $D' \equiv D$ and X' lie on CD. Therefore both of X and X' lie on intersection of DC and perpendicular bisector of MB, so $X' \equiv X$.

Now we prove if $\angle MXB = 60°$ then $AB = BP$.
Let P' such that $\triangle MP'X$ be equilateral.(P' and X be on the same side of AB) It's enough to show that $P' \equiv P$.



Draw the line passing through P' such that be parallel with CD. Suppose that this line intersects AD in M'.

$$\angle XP'M' = 360° - \angle M'P'B - \angle BP'X = 360° - \angle DCA - \angle BAM = 120°$$

Also $\angle P'M'D = 120°$. Therefore quadrilateral $XP'M'D$ is isosceles trapezoid and $DM' = P'X = AM = DM$. So we can say $M' \equiv M \Rightarrow P' \equiv P$.

7.**(Geometry Olympiad(Senior level))** An acute-angled triangle ABC is given. The circle with diameter BC intersects AB, AC at E, F respectively. Let M be the midpoint of BC and P the intersection point of AM and EF. X is a point on the arc EF and Y the second intersection point of XP with circle mentioned above. Show that $\angle XAY = \angle XYM$.

Proposed by Ali Zooelm

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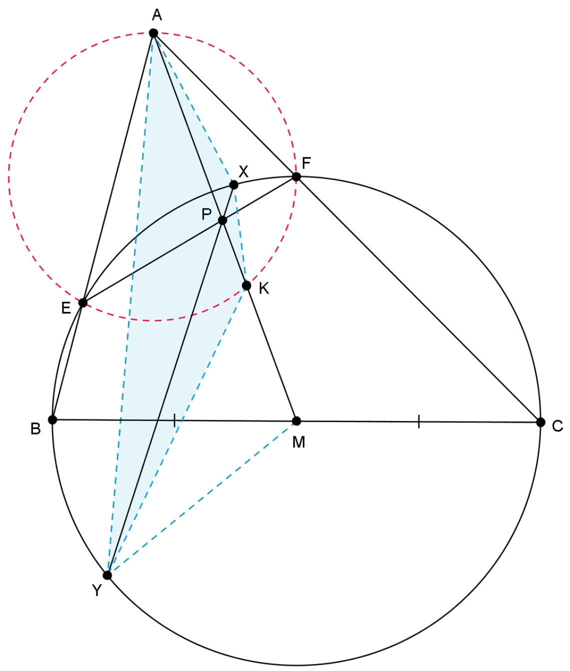
solution.
Suppose point K is intersection AM and circumcircle of $\triangle AEF$. MF tangent to circumcircle of $\triangle AEF$ at F.
(because of $\angle MFC = \angle MCF = \angle AEF$). Therefore $MF^2 = MK.MA$. In the other hand, $MY = MF$ so $MY^2 = MK.MA$. It means

$$\angle MYK = \angle YAM \tag{1}$$

Also $AP.PK = PE.PF = PX.PY$ therefore $AXKY$ is(...??) .Therefore

$$\angle XAY = \angle XYK \tag{2}$$

According to equation 1 and 2 we can say $\angle XAY = \angle XYM$.



8.**(Geometry Olympiad(Senior level))** The tangent line to circumcircle of the acute-angled triangle ABC ($AC > AB$) at A intersects the continuation of BC at P. We denote by O the circumcenter of ABC. X is a point OP such that $\angle AXP = 90°$. Two points E, F respectively on AB, AC at the same side of OP are chosen such that

$$\angle EXP = \angle ACX, \ \angle FXO = \angle ABX$$

If K, L denote the intersection points of EF with the circumcircle of $\triangle ABC$, show that OP is tangent to the circumcircle of $\triangle KLX$.

Proposed by Mahdi Etesami Fard

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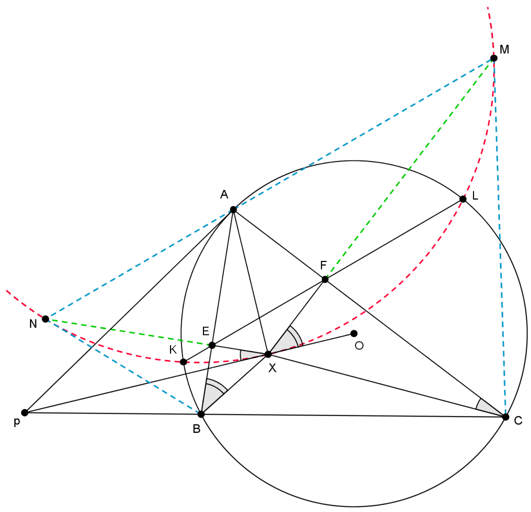
solution.

Let M and N on continuation of XF and XE such that M, L, X, N, K lie on same circle. We have to prove $\angle AMX = \angle ACX$. In other hand, $\angle ACX = \angle NXP$ so we have to prove $\angle ACX = \angle NMX$.



We know that $XF.FM = FL.FK = AF.FC$. Therefore $AMCX$ is cyclic and $\angle AMX = \angle ACX$. similarly we can say $ANBX$ is cyclic. Now it's enough to show that $\angle AMX = \angle NMX$. In other words, we have to show that A, N, M lie on same line. we know that $ANBX$ is cyclic therefore:

$$\angle NAM = \angle NAE + \angle A + \angle FAM = \angle EXB + \angle A + \angle CXF$$

$$= \angle A + 180° - \angle BXC + \angle ABX + \angle ACX$$

$$= \angle A + 180° - \angle BXC + \angle BXC - \angle A = 180°$$

9.**(Geometry Olympiad(Senior level))** Two points P, Q lie on the side BC of triangle ABC and have the same distance to the midpoint. The pependiculars from$P$, Q tp BC intesects AC, AB at E, F respectively. LEt M be the intersection point of $PF$and EQ. If H_1 and H_2 denote the orthocenter of $\triangle BFP$ and $\triangle CEQ$ recpectively, show that $AM \perp H_1H_2$.

Proposed by Mahdi Etesami Fard

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solution.

First we show that if we move P and Q, the line AM doesn't move. To show that we calculate $\frac{\sin\angle A_1}{\sin\angle A_2}$. By the law of sines in $\triangle AFM$ and $\triangle AEM$ we have

$$\frac{\sin\angle A_1}{\sin\angle A_2} = \frac{\sin\angle F_1}{\sin\angle E_1} \cdot \frac{FM}{EM} \tag{3}$$

also, for $\triangle FBP$ and $\triangle CEQ$ we have

$$\left.\begin{array}{l} \sin\angle F_1 = \frac{BP}{PF} \cdot \sin\angle B \\ \sin\angle E_1 = \frac{CQ}{EQ} \cdot \sin\angle C \end{array}\right\} \Rightarrow \frac{\sin\angle F_1}{\sin\angle E_1} = \frac{\sin\angle B}{\sin\angle C} \cdot \frac{EQ}{FP} \tag{4}$$

from (3) and (4) we have

$$\frac{\sin\angle A_1}{\sin\angle A_2} = \frac{\sin\angle B}{\sin\angle C} \cdot \frac{EQ}{FP} \cdot \frac{FM}{EM} \tag{5}$$

$\triangle FMQ$ and $\triangle EMP$ are similar, thus

$$\frac{FM}{FP} = \frac{FQ}{FQ+EP}, \ \frac{EQ}{EM} = \frac{FQ+EP}{EP}$$

with putting this into (5) we have

$$\frac{\sin\angle A_1}{\sin\angle A_2} = \frac{\sin\angle B}{\sin\angle C}\cdot\frac{FQ}{EP} \tag{6}$$

on the other hand

$$\left.\begin{array}{l}\tan\angle B = \frac{FQ}{BQ}\\ \tan\angle C = \frac{EP}{CP}\\ BQ = CP\end{array}\right\} \Rightarrow \frac{FQ}{EP} = \frac{\tan\angle B}{\tan\angle C}$$

if we put this in (6) we have

$$\frac{\sin\angle A_1}{\sin\angle A_2} = \frac{\sin\angle B}{\sin\angle C}\cdot\frac{\tan\angle B}{\tan\angle C}$$

wich is constant.

now we show that H_1H_2s are parallel. consider α the angle between H_1H_2 and BC. Hense we have

$$\tan\alpha = \frac{H_2P - H_1Q}{QP} \tag{7}$$

H_1 and H_2 are the orthometers of $\triangle BFP$ and $\triangle CQE$ respectively. Thus we have

$$QF\cdot H_1Q = BQ\cdot QP \Rightarrow H_1Q = \frac{BQ\cdot QP}{FQ}$$

$$EP\cdot H_2P = CP\cdot PQ \Rightarrow H_2P = \frac{CP\cdot PQ}{EP}$$

but $CP = BQ$. Thus

$$H_2P - H_1Q = \frac{PQ\cdot BQ\cdot(FQ-EP)}{EP\cdot FQ}$$

by putting this in (7) :

$$\tan\alpha = \frac{BQ\cdot(FQ-EP)}{EP\cdot FQ} = \frac{BQ}{EP} - \frac{BQ}{FQ} = \frac{CP}{EP} - \frac{BQ}{FQ}$$

$$\Rightarrow \tan\alpha = \cot\angle B - \cot\angle C \tag{8}$$

hense $\tan\alpha$ is constant, thus H_1H_2s are parallel.

Soppuse θ is the angle between AM and BC. we have to show

$$\tan\alpha \cdot \tan\theta = 1$$

let AM intersects with BC at X. We have

$$\frac{BX}{CX} = \frac{\sin\angle A_1}{\sin\angle A_2} \cdot \frac{\sin\angle C}{\sin\angle B} \Rightarrow \frac{BX}{CX} = \frac{\tan\angle B}{\tan\angle C}$$

let D be the foot of the altitude drawn from A. We have

$$\frac{BX}{CX} = \frac{\tan\angle B}{\tan\angle C} = \frac{\frac{AD}{BD}}{\frac{AD}{CD}} = \frac{CD}{BD} \Rightarrow BD = CX$$

$$\tan\theta = \frac{AD}{DX} = \frac{AD}{CD - CX} = \frac{AD}{CD - BD} = \frac{1}{\frac{CD}{AD} - \frac{BD}{AD}} = \frac{1}{\cot\angle B - \cot\angle C}$$

this equality and (8) implies that $AM \perp H_1H_2$.