Source: 2015 ISL G4

va2010
1145 posts

Jul 8, 2016, 2:59 am • 1 PM #1

Let ABC be an acute triangle and let M be the midpoint of AC. A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC. Determine all possible values of $\frac{BT}{BM}$.

This post has been edited 1 time. Last edited by v_Enhance, Jul 8, 2016, 3:23 am
Reason: improve title

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v_Enhance
4230 pos...

Jul 8, 2016, 3:00 am PM #2

whooo let's go bary

Denote by X the second intersection of $(BPMQ)$ with $\overline{AC}$. Let $T = (u, v, w)$ thus $a^2vw + b^2wu + c^2uv = 0$. Then $\overline{PT} \parallel \overline{BC} \implies P = (u : w + v : 0)$. Analogously, $Q = (0 : u + v, w)$. So,

$$AX = \frac{AB \cdot AP}{AM} = \frac{2c^2}{b}(v+w) \quad \text{and} \quad CX = \frac{CB \cdot CQ}{CM} = \frac{2a^2}{b}(v+u).$$

Adding these implies that $\frac{1}{2}b^2 = (v + w)c^2 + (v + u)a^2$.

However, by barycentric distance formula,

$$BT^2 = -a^2(v-1)w - b^2wv - c^2u(v-1) = a^2w + c^2u + \underbrace{-a^2vw - b^2wu - c^2uv}_{=0}.$$

Thus, adding gives $\frac{1}{2}b^2 + BT^2 = a^2 + c^2$, so $BT^2 = a^2 + c^2 - \frac{1}{2}b^2 = 2BM^2$, thus $BT/BM = \sqrt{2}$ is the only possible value.

This post has been edited 1 time. Last edited by v_Enhance, Jul 8, 2016, 3:01 am
Reason: smiley

TelvCohl
1986 pos...

Jul 8, 2016, 3:12 am • 1 PM #3

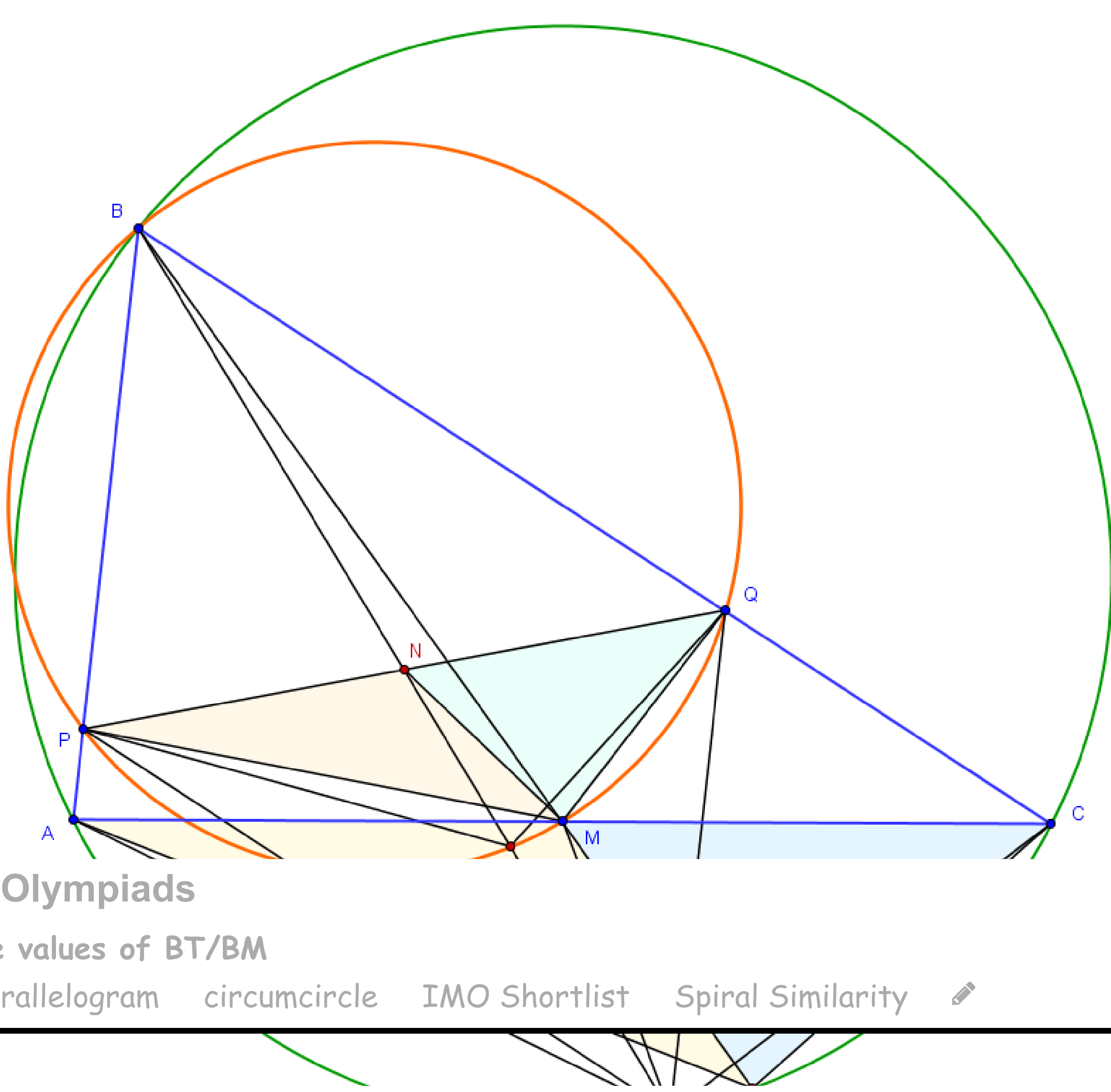
Let N be the midpoint of PQ and let $E \equiv BM \cap \odot(ABC)$, $S \equiv BN \cap \odot(BPQ)$. Since

$$\left\{ \begin{array}{l} \measuredangle MPQ = \measuredangle MBC = \measuredangle EAC, \measuredangle SPQ = \measuredangle SBC = \measuredangle TAC \\ \measuredangle MQP = \measuredangle MBA = \measuredangle ECA, \measuredangle SQP = \measuredangle SBA = \measuredangle TCA \end{array} \right\| \implies ACETM \overset{+}{\sim} PQMSN,$$

so $\measuredangle BMN = \measuredangle(EM, MN) = \measuredangle(TM, SN) = \measuredangle MTB \Longrightarrow BM$ is tangent to $\odot(MNT)$ at M, hence we conclude that

$$\frac{1}{2} \cdot BT^2 = BN \cdot BT = BM^2 \Longrightarrow \frac{BT}{BM} = \sqrt{2}.$$

Attachments:

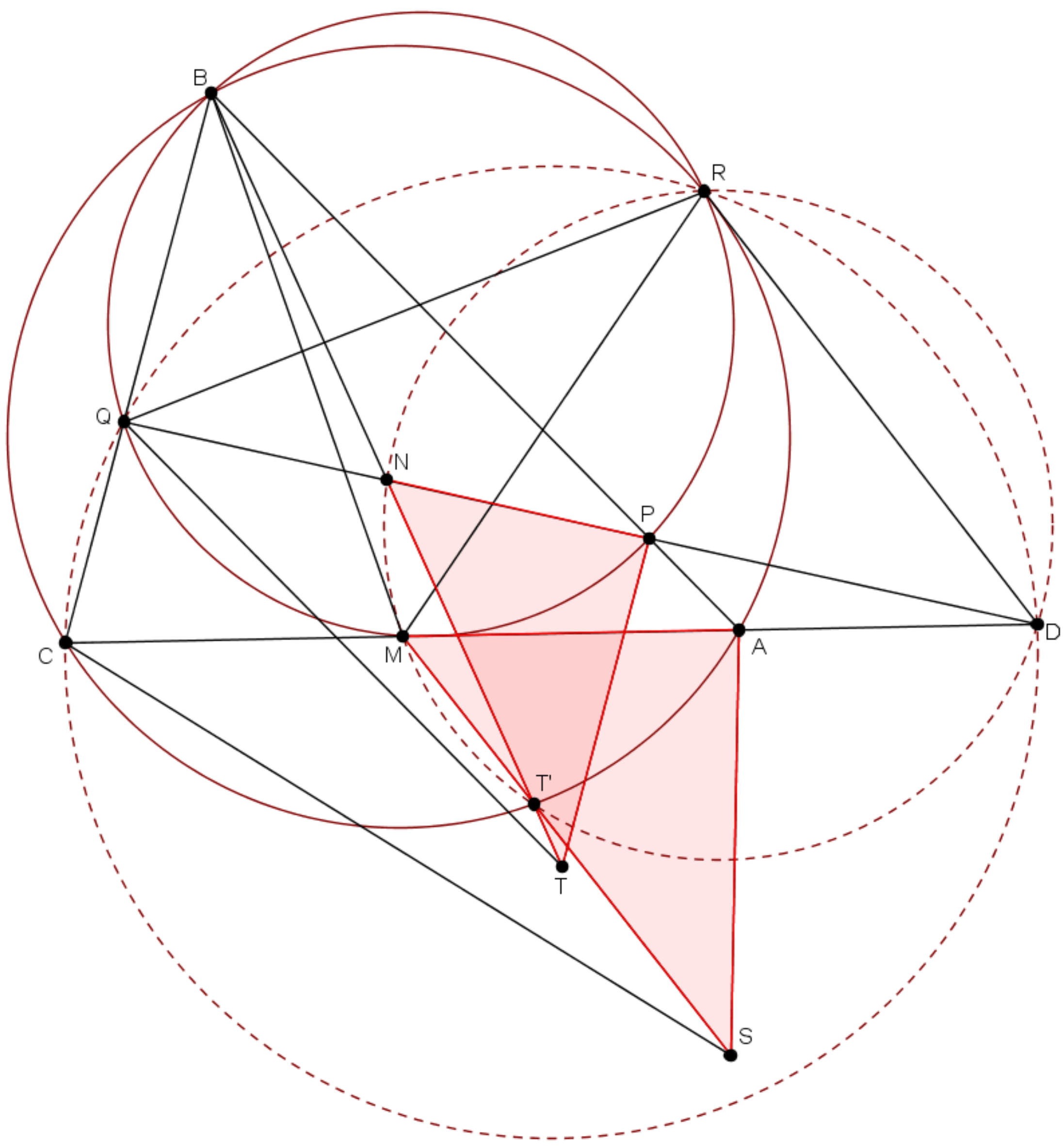


mjuk
192 posts

Jul 8, 2016, 3:16 am PM #4

Let $D = AC \cap PQ$. Let N be midpoint of PQ and let S be a point such that $\triangle ACS \overset{+}{\sim} \triangle PQT$.
Let R be Miquel point of $CAPQ \implies R \in \odot ABC$, $R \in \odot CDQ$. R is center of spiral similarity sending $CQ \to AP$, so it's also center of spiral similarity sending $MN \to AP \implies R \in \odot DMN$.
Let $T' = MS \cap BT$. Obviously $\triangle NPT \sim \triangle MAS \implies \angle DNT' = \angle DMT' \implies T' \in \odot DMN$.
$\angle BT'R = \angle NT'R = \angle NDR = \angle QDR = \angle QCR = \angle BCR \implies T' \in \odot ABC \implies T \equiv T'$
$\angle MDR = \angle CDR = \angle BQR = \angle BMR \implies BM$ touches $\odot TMN$
$\implies BM^2 = BN \cdot BT = \dfrac{BT^2}{2} \implies \dfrac{BT}{BM} = \sqrt{2}\blacksquare$

Attachments:



This post has been edited 4 times. Last edited by mjuk, Jul 8, 2016, 3:36 am

navi_0922...
143 posts

Jul 8, 2016, 9:37 am PM #5

Answer. $\sqrt{2}$.

Solution. Let us prove a well known spiral similarity lemma.

Lemma. Let AB and CD be two segments, and let lines AC and BD meet at X. Let circumcircle of ABX and CDX meet again at O. Then O is the center of the spiral similarity that carries AB to CD.

Proof of Lemma. Since $ABOX$ and $CDXO$ are cyclic. we have $\angle OBD = \angle OAC$ and $\angle OCA = \angle ODB$. It follows that $\triangle AOC$ and $\triangle BOD$ are similar, thus the result.

Back to the main proof. Suppose that T lies on (ABC). Let $PQ \cap BT = N$, $(ABC) \cap (APQ) = X \neq A$, $PQ \cap AC = Y$. Clearly since $BPTQ$ is a parallelogram, we have $BT = 2BN$. We will prove that points X, Y lies on (TMN).

By the lemma, X is the center of spiral similarity that maps segment PQ to AC. Note that this spiral similarity also maps the midpoint of PQ to midpoint of AC, which is N to M. So it also maps segment PN to AM, which implies $\triangle XMA \sim \triangle XNP$. Similarly, X is also the center of spiral similarity that maps AP to BQ. Notice that this spiral similarity also maps A to M and P to N because M and N are midpoints of AP and BQ respectively, so we also conclude that $\triangle XPA \sim \triangle XNM$. (Also known as the Averaging Principle.)

So using directed angles, $\angle(XM, MN) = \angle(XA, AP) = \angle(XT, TB) = \angle(XT, TN)$, so X lies on (TMN). Likewise, $\angle(YN, NX) = \angle(PN, NX) = \angle(AM, MX) = \angle(YM, MX)$, so Y lies on (TMN). Now, note that $\angle(BM, MX) = \angle(BA, AX) = \angle(PA, AX) = \angle(NM, MX)$, so BM is tangent to circle (TMN).

This means $2BM^2 = 2BN \cdot BT = BT^2$, which gives $\dfrac{BT}{BM} = \sqrt{2}$, as desired. Q.E.D

Jul 8, 2016, 12:19 pm PM #6

so BM is tangent to $\odot(XRM)$. Since $T \in \odot(XRM)$, $BM^2 = BR \cdot BT = \frac{BT^2}{2}$, so the answer is $\sqrt{2}$.

WizardMa...
326 posts

Jul 8, 2016, 3:16 pm PM #7

Bary was the first thing that came to my mind when I saw this one in the TST. Since v_Enhance already posted the bash I won't do it again but just remark that this is an excellent bary tutorial problem.

anantmud...
817 posts

Jul 8, 2016, 4:49 pm • 1 PM #8

Here is my approach.

Firstly observe that since B, P, M, Q are concyclic, the midpoint K of PQ varies on a line as P, Q vary on BA, BC respectively. Indeed, this follows since triangle MPQ has a fixed shape and spiral similarity moves each linear combination uniformly.

Let (BAM) meet BC again at X and (BCM) meet BA again at Y. Let U, V be the midpoints of AX, CY. Then, the locus of K is the line UV. Let L, N be the midpoints of BA, BC. Consider the circle $\gamma = (BLN)$ and $\omega = (B, \frac{BM}{\sqrt{2}})$. We claim that the line UV is the radical axis of ω and γ. This shall establish the result; $\frac{BT}{BM} = \frac{1}{\sqrt{2}}$.

Indeed, we define the function f from the Euclidean plane to the set of real numbers as follows: $f(Z) = p(Z,\gamma) - p(Z,\omega)$. It is clear that f is a linear function in Z. We want to establish that $f(U) = f(V) = 0$. Proving $f(U) = 0$ suffices, since $f(V) = 0$ shall follow analogously.

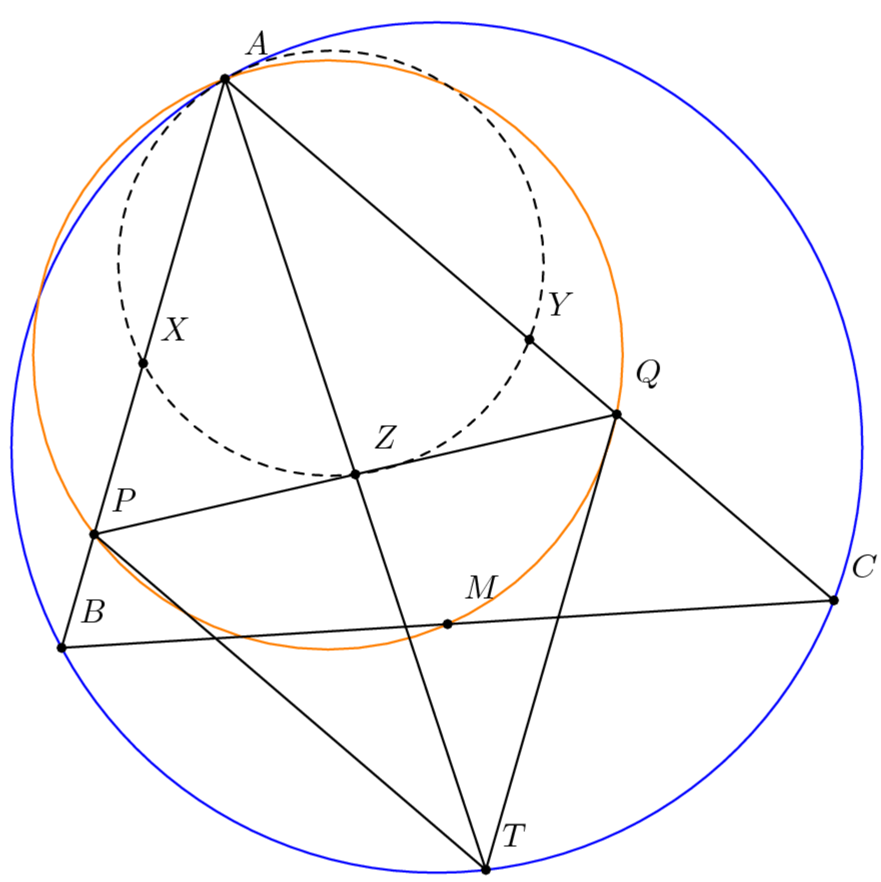
Notice that $f(U) = \frac{f(A)+f(X)}{2}$. We only need to ascertain ourselves that $f(A) = -f(X)$.

It is evident that $f(A) = -\frac{c^2}{2} + \frac{2a^2+2c^2-b^2}{8}$ and $f(X) = XB.XN - XB^2 + \frac{2a^2+2c^2-b^2}{8}$. It is equivalent to showing that $XB.XN - XB.XB = -\frac{2a^2-b^2}{4}$. Notice that $XB - XN = -(BX+XN) = -BN = -\frac{a}{2}$ and $XB = a - \frac{b^2}{2a}$ and hence the claim holds.

Comment. This problem came in India TST 2016 and Romania TST 2016 as per my knowledge.

kapilpavase
432 posts

Jul 8, 2016, 7:35 pm PM #9



angle chasing yields $\Delta BTC \sim \Delta AQT$ so that

$$\frac{BT}{AQ} = \frac{TC}{QT} = \frac{BC}{AT}$$

Now take X, Y, Z to be the midpts of AB, AC, PQ resp. Similar angle chasing gives $\Delta PMQ \sim \Delta AYM$ so that

$$\frac{AC}{PM} = \frac{AB}{MQ} = \frac{2AM}{PQ}$$

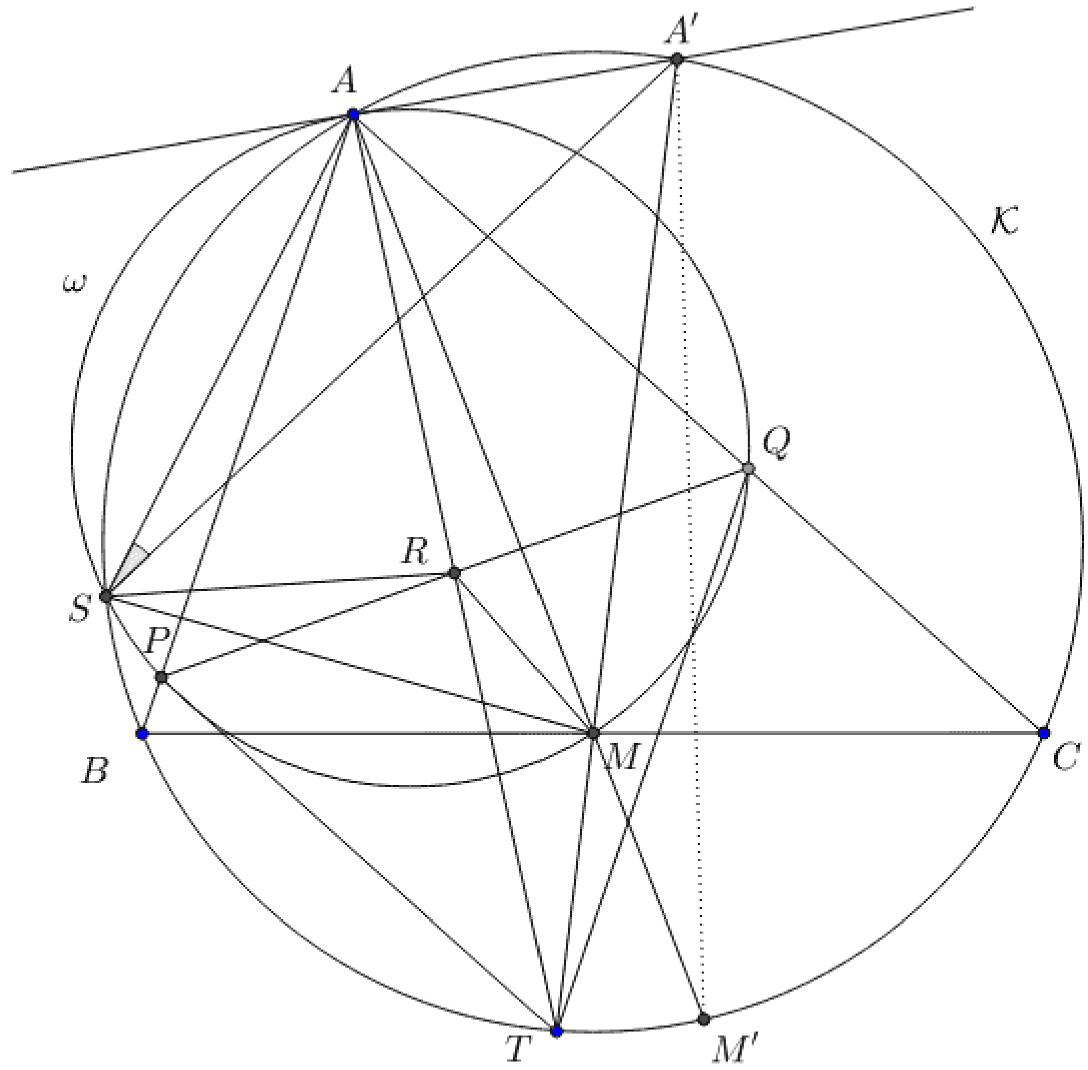
$$\frac{2AM.BC}{AT.PQ} = \frac{BT.AC}{AQ.PM} = \frac{TC.AB}{QT.MQ} = \frac{BT.AC+TC.AB}{AQ.PM+QT.MQ} = \frac{AT.BC}{AM.PQ}$$

so that $2AM^2 = AT^2$ as desired 🙂

Number1
303 posts

Jul 10, 2016, 3:10 am PM #12

Rename A and B.



Let tangent at A on ω cuts cicumcircle $\mathcal{K}$ of ABC at A' and let AM cut $\mathcal{K}$ at M'.
Let R halves PQ and let $\mathcal{K}$ meets ω second time at S. Then S is center of spiral similarity $\mathcal{P}$, which maps $\omega \longmapsto \mathcal{K}$ and

$$\begin{array}{ccc} P & \longmapsto & B \\ Q & \longmapsto & C \\ A & \longmapsto & A' \\ R & \longmapsto & M \\ M & \longmapsto & M' \end{array}$$

Say AR meets $A'M$ at T'. Since $\mathcal{P}$ send AR to $A'M$ and SA to SA' we have $\angle AT'A' = \angle(AR, AM') = \angle ASA'$, thus $T' \in \mathcal{K}$, and so $T' = T$.

Finally since $\mathcal{P}$ preserves angles we have

$$\angle RMA \stackrel{\mathcal{P}}{=} \angle MM'A' = \angle AM'A' = \angle ATA' = \angle RTM$$

and this means AT is tangent on circumcircle $\triangle MTR$, so $AM^2 = AT \cdot AR = \dfrac{AT^2}{2} \Longrightarrow \dfrac{AT}{AM} = \sqrt{2}$.

Ankoganit
624 posts

Jul 22, 2016, 3:37 pm PM #13

This is also India TST 2016 Day 3 Problem 2.

Quick Reply