Need to solve this problem...can someone help??

RMO combinatorics

Source: rmo

nuke4peace
7 posts

Sep 5, 2016, 2:55 pm #1

On a stormy night 10 guests came to dinner party and left their shoes outside the room.After the dinner there was a blackout,and the guest leaving one by one,put on at random,any pair of shoes big enough for their feet(Each pair of shoes stay together.)Any guest who could not find a pair big enough spent the night there.What is the largest number of guests who might have had to spend the night there?

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rafayaashary1
861 posts

Sep 5, 2016, 8:21 pm #2

Let us index the guests as $a_1, a_2, \cdots, a_{10}$, and let their respective shoes be $r_1, l_1 \geq r_2, l_2 \geq \cdots \geq r_{10}, l_{10}$

For someone with index i to be "trapped," all of either $r_1, r_2, \cdots, r_j$ or $l_1, l_2, \cdots, l_k$ (or both) must be taken, where j is the highest index such that $r_i = r_j$ and k is similarly defined.

If it is the former condition, we say a person is "right-trapped." If it is the latter, we say "left-trapped." Someone who satisfies both is said to be both.

Consider the "right-trapped" person with the highest j value and the "left-trapped" person with the highest k value. Denote these $j_{\max}$ and $k_{\max}$, respectively.

Let t be the number of "trapped" people overall.

We have, by shoe-count $j_{\max} + k_{\max} \leq 2(10 - t)$
But we also have trivially that $j_{\max}, k_{\max} \geq t$
So $2t \leq j_{\max} + k_{\max} \leq 2(10 - t) \implies t \leq 5$

But 5 is easy to find a construction for:
a_{10} takes l_1, r_{10}
a_9 takes l_2, r_9
a_8 takes l_3, r_8
a_7 takes l_4, r_7
a_6 takes l_5, r_6
But then $a_1, a_2, a_3, a_4, \text{ and}, a_5$ are all left-trapped.

So the answer is $\boxed{5}$

Note: When we compare r's and l's, we are comparing their size. So while the symbol stands for a shoe, we also use it to stand for a shoe's size. (Better than introducing even more variables, right?) :/

PS: Feel free to ask for clarification; I was in a bit of a time crunch, and wasn't able to explain as much as I would like to :|

This post has been edited 4 times. Last edited by rafayaashary1, Sep 5, 2016, 11:50 pm
Reason: final edit: format, grammar, and notation

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nuke4peace
7 posts

Sep 7, 2016, 2:08 pm PM #3

Thanks for the solution...but if u get spare time then can u explain in detail...(in the question it says that a pair of shoes will stay together,)

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rafayaashary1
861 posts

Sep 7, 2016, 9:00 pm PM #4

Ahh, I didn't see that condition :/
Luckily, the proof is actually a case of the above, and I will try to post it at the nearest opportunity :)

I believe the answer remains $\boxed{5}$

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rafayaashary1
861 posts

Sep 8, 2016, 4:21 am PM #5

Let us index the guests as $a_1, a_2, \cdots, a_{10}$, and let their respective shoes be $s_1 \ge s_2 \ge \cdots \ge s_{10}$
(we say $s_1 > s_2$ if s_1 is larger in size, and do similarly for the other relations)

For someone with index i to be "trapped," all of $s_1, s_2, \cdots s_j$ must be taken, where j is the highest index such that $s_i = s_j$

Consider the "trapped" person with the highest j value, which we will call j_{max}

Let t be the number of "trapped" people overall.

We have, by shoe-pair-count $j_{\text{max}} \le 10 - t$
(in order for all of person $i's$ options to be taken, at least as many people must have left)

But we also have that $j_{\text{max}} \ge t$
(the highest index of a shoe with size at least person $i's$ shoes size must have index at least i when arranged in decreasing order, since everyone's shoes fit on the way to the party, and their original shoes are WLOG at their index)

So $t \le 10 - t \implies t \le 5$

But 5 is easy to find a construction for:
a_{10} takes s_1
a_9 takes s_2
a_8 takes s_3
a_7 takes s_4
a_6 takes s_5
And then $a_1, a_2, a_3, a_4, \text{ and}, a_5$ are all "trapped." $\square$

So the answer is $\boxed{5}$