ISI Entrance Examination 2015

1 Let $m_1 < m_2 < \dots m_{k-1} < m_k$ be k distinct positive integers such that their reciprocals are in arithmetic progression.

1.Show that $k < m_1 + 2$.
2. Give an example of such a sequence of length k for any positive integer k.

2 Let $y = x^2 + ax + b$ be a parabola that cuts the coordinate axes at three distinct points. Show that the circle passing through these three points also passes through $(0, 1)$.

3 Consider the set $S = 1, 2, 3, \dots, j$. Let $m(A)$ denote the maximum element of A. Prove that

$$\sum_{A \subseteq S} m(A) = (j-1)2^j + 1$$

4 Let $p(x) = x^7 + x^6 + b_5 x^5 + \cdots + b_0$ and $q(x) = x^5 + c_4 x^4 + \cdots + c_0$. If $p(i) = q(i)$ for $i = 1, 2, 3, \cdots, 6$. Show that there exists a negative integer r such that $p(r) = q(r)$.

5 If $0 < a_1 < \cdots < a_n$, show that the following equation has exactly n roots.

$$\frac{a_1}{a_1 - x} + \frac{a_2}{a_2 - x} + \frac{a_3}{a_3 - x} + \cdots + \frac{a_n}{a_n - x} = 2015$$

6 Find all $n \in \mathbb{N}$ so that 7 divides $5^n + 1$

7 Let $\gamma_1, \gamma_2, \gamma_3$ be three circles of unit radius which touch each other externally. The common tangent to each pair of circles are drawn (and extended so that they intersect) and let the triangle formed by the common tangents be $\triangle XYZ$. Find the length of each side of $\triangle XYZ$

8 Find all the functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$|f(x) - f(y)| = 2|x - y|$$