Canada National Olympiad 2002

1 Let S be a subset of $\{1, 2, \dots, 9\}$, such that the sums formed by adding each unordered pair of distinct numbers from S are all different. For example, the subset $\{1, 2, 3, 5\}$ has this property, but $\{1, 2, 3, 4, 5\}$ does not, since the pairs $\{1, 4\}$ and $\{2, 3\}$ have the same sum, namely 5.

What is the maximum number of elements that S can contain?

2 Call a positive integer n **practical** if every positive integer less than or equal to n can be written as the sum of distinct divisors of n.

For example, the divisors of 6 are 1, 2, 3, and 6. Since

$$1=\mathbf{1}, \quad 2=\mathbf{2}, \quad 3=\mathbf{3}, \quad 4=\mathbf{1}+\mathbf{3}, \quad 5=\mathbf{2}+\mathbf{3}, \quad 6=\mathbf{6},$$

we see that 6 is practical.

Prove that the product of two practical numbers is also practical.

3 Prove that for all positive real numbers a, b, and c,

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c$$

and determine when equality occurs.

4 Let Γ be a circle with radius r. Let A and B be distinct points on Γ such that $AB < \sqrt{3}r$. Let the circle with centre B and radius AB meet Γ again at C. Let P be the point inside Γ such that triangle ABP is equilateral. Finally, let the line CP meet Γ again at Q.

Prove that $PQ = r$.

5 Let $\mathbb{N} = \{0, 1, 2, \dots\}$. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$xf(y) + yf(x) = (x + y)f(x^2 + y^2)$$

for all x and y in $\mathbb{N}$.