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Postal Coaching

2010

Set 1

$\boxed{1}$ Let γ, Γ be two concentric circles with radii r, R with $r < R$. Let $ABCD$ be a cyclic quadrilateral inscribed in γ. If $\overrightarrow{AB}$ denotes the Ray starting from A and extending indefinitely in $B's$ direction then Let $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}, \overrightarrow{DA}$ meet Γ at the points C_1, D_1, A_1, B_1 respectively. Prove that

$$\frac{[A_1B_1C_1D_1]}{[ABCD]} \geq \frac{R^2}{r^2}$$

where [.] denotes area.

$\boxed{2}$ Find all non-negative integers m, n, p, q such that

$$p^m q^n = (p+q)^2 + 1.$$

$\boxed{3}$ Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $\boxed{1}$ $f(1) = 1$ $\boxed{2}$ $f(m+n)(f(m) - f(n)) = f(m-n)(f(m) + f(n))$ $\forall\ m, n \in \mathbb{Z}$

$\boxed{4}$ Prove that the following statement is true for two natural nos. m, n if and only $v(m) = v(n)$ where $v(k)$ is the highest power of 2 dividing k. $\exists$ a set A of positive integers such that (i) $x, y \in \mathbb{N}, |x-y| = m \implies x \in A$ or $y \in A$ (ii) $x, y \in \mathbb{N}, |x-y| = n \implies x \notin A$ or $y \notin A$

$\boxed{5}$ A point P lies on the internal angle bisector of $\angle BAC$ of a triangle $\triangle ABC$. Point D is the midpoint of BC and PD meets the external angle bisector of $\angle BAC$ at point E. If F is the point such that $PAEF$ is a rectangle then prove that PF bisects $\angle BFC$ internally or externally.

$\boxed{6}$ Let a, b, c denote the sides of a triangle and $[ABC]$ the area of the triangle as usual.

(a) If $6[ABC] = 2a^2 + bc$, determine A, B, C. (b) For all triangles, prove that $3a^2 + 3b^2 - c^2 \geq 4\sqrt{3}[ABC]$.

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Set 2

1 A polynomial $P(x)$ with real coecients and of degree $n \geq 3$ has n real roots $x_1 < x_2 < \cdots < x_n$ such that

$$x_2 - x_1 < x_3 - x_2 < \cdots < x_n - x_{n-1}$$

Prove that the maximum value of $|P(x)|$ on the interval $[x_1, x_n]$ is attained in the interval $[x_{n-1}, x_n]$.

2 Call a triple (a, b, c) of positive integers a **nice** triple if a, b, c forms a non-decreasing arithmetic progression, $gcd(b, a) = gcd(b, c) = 1$ and the product abc is a perfect square. Prove that given a nice triple, there exists some other nice triple having at least one element common with the given triple.

3 In a quadrilateral $ABCD$, we have $\angle DAB = 110°, \angle ABC = 50°$ and $\angle BCD = 70°$. Let M, N be the mid-points of AB and CD respectively. Suppose P is a point on the segment MN such that $\frac{AM}{CN} = \frac{MP}{PN}$ and $AP = CP$. Find $\angle APC$.

4 How many ordered triples (a, b, c) of positive integers are there such that none of a, b, c exceeds 2010 and each of a, b, c divides $a + b + c$?

5 For any positive real numbers a, b, c, prove that

$$\sum_{cyclic} \frac{(b+c)(a^4 - b^2c^2)}{ab + 2bc + ca} \geq 0$$

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Set 3

$\boxed{1}$ In a family there are four children of dierent ages, each age being a positive integer not less than 2 and not greater than 16. A year ago the square of the age of the eldest child was equal to the sum of the squares of the ages of the remaining children. One year from now the sum of the squares of the youngest and the oldest will be equal to the sum of the squares of the other two. How old is each child?

$\boxed{2}$ In a circle with centre at O and diameter AB, two chords BD and AC intersect at E. F is a point on AB such that $EF \perp AB$. FC intersects BD in G. If $DE = 5$ and $EG = 3$, determine BG.

$\boxed{3}$ Find all natural numbers n such that the number $n(n+1)(n+2)(n+3)$ has exactly three different prime divisors.

$\boxed{4}$ Five distinct points A, B, C, D and E lie in this order on a circle of radius r and satisfy $AC = BD = CE = r$. Prove that the orthocentres of the triangles ACD, BCD and BCE are the vertices of a right-angled triangle.

$\boxed{5}$ Let a, b, c be integers such that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 3$$

Prove that abc is a cube of an integer.

$\boxed{6}$ Students have taken a test paper in each of $n \geq 3$ subjects. It is known that in any subject exactly three students got the best score, and for any two subjects exactly one student got the best scores in both subjects. Find the smallest n for which the above conditions imply that exactly one student got the best score in each of the n subjects.

$\boxed{7}$ Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that for every $n \geq 2$,

$$f(f(n-1)) = f(n+1) - f(n)?$$

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Set 4

$\boxed{1}$ Let A, B, C, D be four distinct points in the plane such that the length of the six line segments AB, AC, AD, BC, BD, CD form a 2-element set a, b. If $a > b$, determine all the possible values of $\frac{a}{b}$.

$\boxed{2}$ Let $a_1, a_2, \dots, a_n$ be real numbers lying in $[-1, 1]$ such that $a_1 + a_2 + \cdots + a_n = 0$. Prove that there is a $k \in \{1, 2, \dots, n\}$ such that $|a_1 + 2a_2 + 3a_3 + \cdots + ka_k| \leq \frac{2k+1}{4}$.

$\boxed{3}$ Prove that a prime p is expressible in the form $x^2 + 3y^2; x, y \in Z$ if and only if it is expressible in the form $m^2 + mn + n^2; m, n \in Z$.Can p be replaced by a natural number n?

$\boxed{4}$ Let C_1, C_2 be two circles in the plane intersecting at two distinct points. Let P be the midpoint of a variable chord AB of C_2 with the property that the circle on AB as diameter meets C_1 at a point T such that PT is tangent to C_1 . Find the locus of P .

$\boxed{5}$ Prove that there exist a set of 2010 natural numbers such that product of any 1006 numbers is divisible by product of remaining 1004 numbers.

$\boxed{6}$ Let $n > 1$ be an integer.

A set $S \subseteq \{0, 1, 2, \cdots, 4n-1\}$ is called sparse if for any $k \in \{0, 1, 2, \cdots, n-1\}$ the following two conditions are satised:

(a) The set $S \cap \{4k-2, 4k-1, 4k, 4k+1, 4k+2\}$ has at most two elements;

(b) The set $S \cap \{4k+1, 4k+2, 4k+3\}$ has at most one element. Prove that there are exactly $8 \cdot 7^{n-1}$ sparse subsets.

Set 5

1 Let $n \in \mathbb{N}$ and A_n set of all permutations $(a_1, \ldots, a_n)$ of the set $\{1, 2, \ldots, n\}$ for which

$$k | 2(a_1 + \cdots + a_k), \text{ for all } 1 \le k \le n.$$

Find the number of elements of the set A_n.

Proposed by Vidan Govedarica, Serbia

2 Let M be an interior point of a $\triangle ABC$ such that $\angle AMB = 150^\circ, \angle BMC = 120^\circ$. Let P, Q, R be the circumcentres of the $\triangle AMB, \triangle BMC, \triangle CMA$ respectively. Prove that $[PQR] \ge [ABC]$.

3 In a group of k people, some are acquainted with each other and some are not. Every evening, one person invites all his acquaintances to a party and introduces them to each other(if they have not already acquainted). Suppose that after each person has arranged at least one party, some two people do not know each other. Prove that they do not meet each other in the next party.

4 For each $n \in \mathbb{N}$, let $S(n)$ be the sum of all numbers in the set $\{1, 2, 3, \cdots, n\}$ which are relatively prime to n.

(a) Show that $2 \cdot S(n)$ is not a perfect square for any n.

(b) Given positive integers m, n, with odd n, show that the equation $2 \cdot S(x) = y^n$ has at least one solution (x, y) among positive integers such that $m|x$.

5 Let p be a prime and $Q(x)$ be a polynomial with integer coecients such that $Q(0) = 0$, $Q(1) = 1$ and the remainder of $Q(n)$ is either 0 or 1 when divided by p, for every $n \in \mathbb{N}$. Prove that $Q(x)$ is of degree at least $p - 1$.

6 Solve the equation for positive integers m, n:

$$\left\lfloor \frac{m^2}{n} \right\rfloor + \left\lfloor \frac{n^2}{m} \right\rfloor = \left\lfloor \frac{m}{n} + \frac{n}{m} \right\rfloor + mn$$

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Set 6

1. Does there exist an increasing sequence of positive integers $a_1, a_2, \cdots$ with the following two properties?

   (i) Every positive integer n can be uniquely expressed in the form $n = a_j - a_i$,

   (ii) $\frac{a_k}{k^3}$ is bounded.

2. Suppose $\triangle ABC$ has circumcircle Γ, circumcentre O and orthocentre H. Parallel lines α, β, γ are drawn through the vertices A, B, C, respectively. Let α', β', γ' be the reections of α, β, γ in the sides BC, CA, AB, respectively.

   (a) Show that α', β', γ' are concurrent if and only if α, β, γ are parallel to the Euler line OH.

   (b) Suppose that α', β', γ' are concurrent at the point P . Show that Γ bisects OP .

3. Determine the smallest odd integer $n \geq 3$, for which there exist n rational numbers $x_1, x_2, ..., x_n$ with the following properties:

   (a)
   $$\sum_{i=1}^{n} x_i = 0, \ \sum_{i=1}^{n} x_i^2 = 1.$$

   (b)
   $$x_i \cdot x_j \geq -\frac{1}{n} \ \forall \ 1 \leq i, j \leq n.$$

4. $\triangle ABC$ has semiperimeter s and area F . A square $PQRS$ with side length x is inscribed in ABC with P and Q on BC, R on AC, and S on AB. Similarly, y and z are the sides of squares two vertices of which lie on AC and AB, respectively. Prove that
   $$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq \frac{s(2 + \sqrt{3})}{2F}$$

5. Find the rst integer $n > 1$ such that the average of $1^2, 2^2, \cdots, n^2$ is itself a perfect square.

6. Find all polynomials P with integer coecients which satisfy the property that, for any relatively prime integers a and b, the sequence $\{P(an + b)\}_{n \geq 1}$ contains an innite number of terms, any two of which are relatively prime.