Art of Problem Solving
2004 Iran MO (2nd round)

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National Math Olympiad (Second Round) 2004

Day 1

1 ABC is a triangle and $\angle A = 90°$. Let D be the meet point of the interior bisector of $\angle A$ and BC. And let I_a be the $A-$excenter of $\triangle ABC$. Prove that:

$$\frac{AD}{DI_a} \le \sqrt{2} - 1.$$

2 Let $f : \mathbb{R}^{\geq 0} \to \mathbb{R}$ be a function such that $f(x) - 3x$ and $f(x) - x^3$ are ascendant functions. Prove that $f(x) - x^2 - x$ is an ascendant function, too.
(We call the function $g(x)$ ascendant, when for every $x \le y$ we have $g(x) \le g(y)$.)

3 The road ministry has assigned 80 informal companies to repair 2400 roads. These roads connect 100 cities to each other. Each road is between 2 cities and there is at most 1 road between every 2 cities. We know that each company repairs 30 roads that it has agencies in each 2 ends of them. Prove that there exists a city in which 8 companies have agencies.

Day 2

4 $\mathbb{N}$ is the set of positive integers. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that for every pair $(m, n) \in \mathbb{N}^2$ we have that:

$$f(m) + f(n) \mid m + n.$$

5 The interior bisector of $\angle A$ from $\triangle ABC$ intersects the side BC and the circumcircle of ΔABC at D, M, respectively. Let ω be a circle with center M and radius MB. A line passing through D, intersects ω at X, Y. Prove that AD bisects $\angle XAY$.

6 We have a $m \times n$ table and $m \ge 4$ and we call a 1×1 square a room. When we put an alligator coin in a room, it menaces all the rooms in his column and his adjacent rooms in his row. What's the minimum number of alligator coins required, such that each room is menaced at least by one alligator coin? (Notice that all alligator coins are vertical.)