Baltic Way 2009

**1** A polynomial $p(x)$ of degree $n \geq 2$ has exactly $n$ real roots, counted with multiplicity. We know that the coefficient of $x^n$ is 1, all the roots are less than or equal to 1, and $p(2) = 3^n$. What values can $p(1)$ take?

**2** Let $a_1, a_2, \ldots, a_{100}$ be nonnegative integers satisfying the inequality

$$a_1 \cdot (a_1 - 1) \cdot \ldots \cdot (a_1 - 20) + a_2 \cdot (a_2 - 1) \cdot \ldots \cdot (a_2 - 20) + \ldots + a_{100} \cdot (a_{100} - 1) \cdot \ldots \cdot (a_{100} - 20) \leq 100 \cdot 99 \cdot 98 \cdot \ldots$$

Prove that $a_1 + a_2 + \ldots + a_{100} \leq 9900$.

**3** Let $n$ be a given positive integer. Show that we can choose numbers $c_k \in \{-1, 1\}$ $(i \leq k \leq n)$ such that

$$0 \leq \sum_{k=1}^{n} c_k \cdot k^2 \leq 4.$$

**4** Determine all integers $n > 1$ for which the inequality

$$x_1^2 + x_2^2 + \ldots + x_n^2 \geq (x_1 + x_2 + \ldots + x_{n-1})x_n$$

holds for all real $x_1, x_2, \ldots, x_n$.

**5** Let $f_0 = f_1 = 1$ and $f_{i+2} = f_{i+1} + f_i$ for all $n \geq 0$. Find all real solutions to the equation
$$x^{2010} = f_{2009} \cdot x + f_{2008}$$

**6** Let $a$ and $b$ be integers such that the equation $x^3 - ax^2 - b = 0$ has three integer roots. Prove that $b = dk^2$, where $d$ and $k$ are integers and $d$ divides $a$.

**7** Suppose that for a prime number $p$ and integers $a, b, c$ the following holds:

$$6 \mid p + 1, \quad p \mid a + b + c, \quad p \mid a^4 + b^4 + c^4.$$

Prove that $p \mid a, b, c$.

**8**    Determine all positive integers $n$ for which there exists a partition of the set

$$\{n, n+1, n+2, \ldots, n+8\}$$

into two subsets such that the product of all elements of the first subset is equal to the product of all elements of the second subset.

**9**    Determine all positive integers $n$ for which $2^{n+1} - n^2$ is a prime number.

**10**    Let $d(k)$ denote the number of positive divisors of a positive integer $k$. Prove that there exist in finitely many positive integers $M$ that cannot be written as

$$M = \left(\frac{2\sqrt{n}}{d(n)}\right)^2$$

for any positive integer $n$.

**11**    Let $M$ be the midpoint of the side $AC$ of a triangle $ABC$, and let $K$ be a point on the ray $BA$ beyond $A$. The line $KM$ intersects the side $BC$ at the point $L$. $P$ is the point on the segment $BM$ such that $PM$ is the bisector of the angle $LPK$. The line $\ell$ passes through $A$ and is parallel to $BM$. Prove that the projection of the point $M$ onto the line $\ell$ belongs to the line $PK$.

**12**    In a quadrilateral $ABCD$ we have $AB \| CD$ and $AB = 2CD$. A line $\ell$ is perpendicular to $CD$ and contains the point $C$. The circle with centre $D$ and radius $DA$ intersects the line $\ell$ at points $P$ and $Q$. Prove that $AP \perp BQ$.

**13**    The point $H$ is the orthocentre of a triangle $ABC$, and the segments $AD, BE, CF$ are its altitudes. The points $I_1, I_2, I_3$ are the incentres of the triangles $EHF, FHD, DHE$ respectively. Prove that the lines $AI_1, BI_2, CI_3$ intersect at a single point.

**14**    For which $n \geq 2$ is it possible to find $n$ pairwise non-similar triangles $A_1, A_2, \ldots, A_n$ such that each of them can be divided into $n$ pairwise non-similar triangles, each of them similar to one of $A_1, A_2, \ldots, A_n$?

**15**    A unit square is cut into $m$ quadrilaterals $Q_1, \ldots, Q_m$. For each $i = 1, \ldots, m$ let $S_i$ be the sum of the squares of the four sides of $Q_i$. Prove that

$$S_1 + \ldots + S_m \geq 4$$

**16**   A [i]$n$-trnder walk[/i] is a walk starting at $(0,0)$, ending at $(2n,0)$ with no self intersection and not leaving the first quadrant, where every step is one of the vectors $(1,1)$, $(1,-1)$ or $(-1,1)$. Find the number of $n$-trnder walks.

**17**   Find the largest integer $n$ for which there exist $n$ different integers such that none of them are divisible by either of $7, 11$ or $13$, but the sum of any two of them is divisible by at least one of $7, 11$ and $13$.

**18**   Let $n > 2$ be an integer. In a country there are $n$ cities and every two of them are connected by a direct road. Each road is assigned an integer from the set $\{1, 2, \ldots, m\}$ (different roads may be assigned the same number). The *priority* of a city is the sum of the numbers assigned to roads which lead to it. Find the smallest $m$ for which it is possible that all cities have a different priority.

**19**   In a party of eight people, each pair of people either know each other or do not know each other. Each person knows exactly three of the others. Determine whether the following two conditions can be satisfied simultaneously:

 for any three people, at least two do not know each other;
for any four people there are at least two who know each other.

**20**   In the future city Baltic Way there are sixteen hospitals. Every night exactly four of them must be on duty for emergencies. Is it possible to arrange the schedule in such a way that after twenty nights every pair of hospitals have been on common duty exactly once?