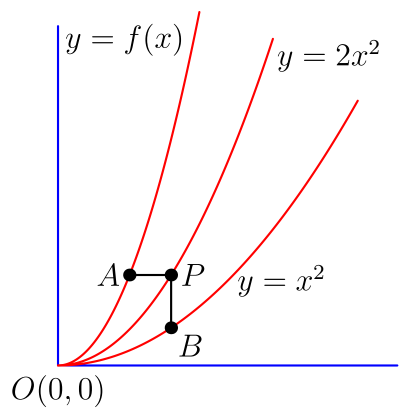
ISI B.Math Entrance Exam 2010

1 Prove that in each year , the 13^{th} day of some month occurs on a Friday .

2 In the accompanying figure , $y = f(x)$ is the graph of a one-to-one continuous function f . At each point P on the graph of $y = 2x^2$, assume that the areas OAP and OBP are equal . Here PA, PB are the horizontal and vertical segments . Determine the function f.



3 Show that , for any positive integer n , the sum of $8n + 4$ consecutive positive integers cannot be a perfect square .

4 If $a, b, c \in (0,1)$ satisfy $a + b + c = 2$, prove that
$\frac{abc}{(1-a)(1-b)(1-c)} \geq 8$

5 Let $a_1 > a_2 > ..... > a_r$ be positive real numbers .
Compute $\lim_{n\to\infty}(a_1^n + a_2^n + ... + a_r^n)^{\frac{1}{n}}$

6 Let each of the vertices of a regular 9-gon (polygon of 9 equal sides and equal angles) be coloured black or white . (a). Show that there are two adjacent verices of same colour. (b). Show there are three vertices of the same colour forming an isosceles triangle.

7 We are given $a, b, c \in \mathbb{R}$ and a polynomial $f(x) = x^3 + ax^2 + bx + c$ such that all roots (real or complex) of $f(x)$ have same absolute value. Show that $a = 0$ iff $b = 0$.

8 Let f be a real-valued differentiable function on the real line $\mathbb{R}$ such that $\lim_{x \to 0} \frac{f(x)}{x^2}$ exists, and is finite . Prove that $f'(0) = 0$.

9 Let $f(x)$ be a polynomial with integer co-efficients. Assume that 3 divides the value $f(n)$ for each integer n. Prove that when $f(x)$ is divided by $x^3 - x$, the remainder is of the form $3r(x)$ where $r(x)$ is a polynomial with integer coefficients.

10 Consider a regular heptagon (polygon of 7 equal sides and angles) $ABCDEFG$ as in the figure below:- (a). Prove $\frac{1}{\sin \frac{\pi}{7}} = \frac{1}{\sin \frac{2\pi}{7}} + \frac{1}{\sin \frac{3\pi}{7}}$ (b). Using (a) or otherwise, show that $\frac{1}{AG} = \frac{1}{AF} + \frac{1}{AE}$