# IMO 1974

Erfurt and East Berlin, German Democratic Republic

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### Day 1

$\boxed{1}$ Three players $A, B$ and $C$ play a game with three cards and on each of these 3 cards it is written a positive integer, all 3 numbers are different. A game consists of shuffling the cards, giving each player a card and each player is attributed a number of points equal to the number written on the card and then they give the cards back. After a number ($\geq 2$) of games we find out that A has 20 points, $B$ has 10 points and $C$ has 9 points. We also know that in the last game B had the card with the biggest number. Who had in the first game the card with the second value (this means the middle card concerning its value).

$\boxed{2}$ Let $ABC$ be a triangle. Prove that there exists a point $D$ on the side $AB$ of the triangle $ABC$, such that $CD$ is the geometric mean of $AD$ and $DB$, iff the triangle $ABC$ satisfies the inequality $\sin A \sin B \leq \sin^2 \frac{C}{2}$.

[hide="Comment"]*Alternative formulation, from IMO ShortList 1974, Finland 2:* We consider a triangle $ABC$. Prove that: $\sin(A)\sin(B) \leq \sin^2\left(\frac{C}{2}\right)$ is a necessary and sufficient condition for the existence of a point $D$ on the segment $AB$ so that $CD$ is the geometrical mean of $AD$ and $BD$.

$\boxed{3}$ Prove that for any n natural, the number

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} 2^{3k}$$

cannot be divided by 5.

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### Day 2

$\boxed{4}$ Consider decompositions of an $8 \times 8$ chessboard into $p$ non-overlapping rectangles subject to the following conditions: (i) Each rectangle has as many white squares as black squares. (ii) If $a_i$ is the number of white squares in the $i$-th rectangle, then $a_1 < a_2 < \ldots < a_p$. Find the maximum value of $p$ for which such a decomposition is possible. For this value of $p$, determine all possible sequences $a_1, a_2, \ldots, a_p$.

$\boxed{5}$ The variables $a, b, c, d$, traverse, independently from each other, the set of positive real values. What are the values which the expression

$$S = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$$

takes?

$\boxed{6}$ Let $P(x)$ be a polynomial with integer coefficients. We denote $\deg(P)$ its degree which is $\geq 1$. Let $n(P)$ be the number of all the integers $k$ for which we have $(P(k))^2 = 1$. Prove that $n(P) - \deg(P) \leq 2$.