APMO 1998

1 Let F be the set of all n-tuples $(A_1, \ldots, A_n)$ such that each A_i is a subset of $\{1, 2, \ldots, 1998\}$. Let $|A|$ denote the number of elements of the set A. Find

$$\sum_{(A_1,\ldots,A_n)\in F} |A_1 \cup A_2 \cup \cdots \cup A_n|$$

2 Show that for any positive integers a and b, $(36a + b)(a + 36b)$ cannot be a power of 2.

3 Let a, b, c be positive real numbers. Prove that

$$\left(1 + \frac{a}{b}\right)\left(1 + \frac{b}{c}\right)\left(1 + \frac{c}{a}\right) \geq 2\left(1 + \frac{a+b+c}{\sqrt[3]{abc}}\right).$$

4 Let ABC be a triangle and D the foot of the altitude from A. Let E and F lie on a line passing through D such that AE is perpendicular to BE, AF is perpendicular to CF, and E and F are different from D. Let M and N be the midpoints of the segments BC and EF, respectively. Prove that AN is perpendicular to NM.

5 Find the largest integer n such that n is divisible by all positive integers less than $\sqrt[3]{n}$.