IMO Shortlist 2008

– Algebra

1 Find all functions $f:(0,\infty)\mapsto(0,\infty)$ (so f is a function from the positive real numbers) such that

$$\frac{\left(f(w)\right)^2+\left(f(x)\right)^2}{f(y^2)+f(z^2)}=\frac{w^2+x^2}{y^2+z^2}$$

for all positive real numbers w,x,y,z, satisfying $wx=yz$.

Author: Hojoo Lee, South Korea

2 **(a)** Prove that

$$\frac{x^2}{(x-1)^2}+\frac{y^2}{(y-1)^2}+\frac{z^2}{(z-1)^2}\geq 1$$

for all real numbers x, y, z, each different from 1, and satisfying $xyz=1$.

(b) Prove that equality holds above for infinitely many triples of rational numbers x, y, z, each different from 1, and satisfying $xyz=1$.

Author: Walther Janous, Austria

3 Let $S\subseteq\mathbb{R}$ be a set of real numbers. We say that a pair (f,g) of functions from S into S is a *Spanish Couple* on S, if they satisfy the following conditions:
(i) Both functions are strictly increasing, i.e. $f(x)<f(y)$ and $g(x)<g(y)$ for all x, $y\in S$ with $x<y$;

(ii) The inequality $f\left(g\left(g\left(x\right)\right)\right)<g\left(f\left(x\right)\right)$ holds for all $x\in S$.

Decide whether there exists a Spanish Couple - on the set $S=\mathbb{N}$ of positive integers; - on the set $S=\{a-\frac{1}{b}:a,b\in\mathbb{N}\}$

Proposed by Hans Zantema, Netherlands

4 For an integer m, denote by $t(m)$ the unique number in $\{1,2,3\}$ such that $m+t(m)$ is a multiple of 3. A function $f:\mathbb{Z}\to\mathbb{Z}$ satisfies $f(-1)=0$, $f(0)=1$, $f(1)=-1$ and $f\left(2^n+m\right)=f\left(2^n-t(m)\right)-f(m)$ for all integers m, $n\geq 0$ with $2^n>m$. Prove that $f(3p)\geq 0$ holds for all integers $p\geq 0$.

Proposed by Gerhard Woeginger, Austria

5 Let a, b, c, d be positive real numbers such that $abcd = 1$ and $a+b+c+d > \frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}$. Prove that

$$a+b+c+d < \frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}$$

Proposed by Pavel Novotn, Slovakia

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6 Let $f : \mathbb{R} \to \mathbb{N}$ be a function which satisfies $f\left(x+\frac{1}{f(y)}\right) = f\left(y+\frac{1}{f(x)}\right)$ for all x, $y \in \mathbb{R}$. Prove that there is a positive integer which is not a value of f.

Proposed by ymantas Darbnas (Zymantas Darbenas), Lithuania

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7 Prove that for any four positive real numbers a, b, c, d the inequality

$$\frac{(a-b)(a-c)}{a+b+c}+\frac{(b-c)(b-d)}{b+c+d}+\frac{(c-d)(c-a)}{c+d+a}+\frac{(d-a)(d-b)}{d+a+b} \geq 0$$

holds. Determine all cases of equality.

Author: Darij Grinberg (Problem Proposal), Christian Reiher (Solution), Germany

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– Combinatorics

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1 In the plane we consider rectangles whose sides are parallel to the coordinate axes and have positive length. Such a rectangle will be called a *box*. Two boxes *intersect* if they have a common point in their interior or on their boundary. Find the largest n for which there exist n boxes B_1, $\dots$, B_n such that B_i and B_j intersect if and only if $i \not\equiv j \pm 1 \pmod{n}$.

Proposed by Gerhard Woeginger, Netherlands

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2 Let $n \in \mathbb{N}$ and A_n set of all permutations $(a_1, \dots, a_n)$ of the set $\{1, 2, \dots, n\}$ for which

$$k|2(a_1+\cdots+a_k), \text{ for all } 1 \leq k \leq n.$$

Find the number of elements of the set A_n.

Proposed by Vidan Govedarica, Serbia

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3 In the coordinate plane consider the set S of all points with integer coordinates. For a positive integer k, two distinct points a, $B \in S$ will be called k-*friends* if there is a point $C \in S$ such that the area of the triangle ABC is equal to k. A set $T \subset S$ will be called k-*clique* if every two points in T are k-friends. Find the least positive integer k for which there exits a k-clique with more than 200 elements.

Proposed by Jorge Tipe, Peru

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4 Let n and k be positive integers with $k \geq n$ and $k - n$ an even number. Let $2n$ lamps labelled 1, 2, ..., $2n$ be given, each of which can be either *on* or *off*. Initially all the lamps are off. We consider sequences of steps: at each step one of the lamps is switched (from on to off or from off to on).

Let N be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off.

Let M be number of such sequences consisting of k steps, resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off, but where none of the lamps $n+1$ through $2n$ is ever switched on.

Determine $\frac{N}{M}$.

Author: Bruno Le Floch and Ilia Smilga, France

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5 Let $S = \{x_1, x_2, \ldots, x_{k+l}\}$ be a $(k+l)$-element set of real numbers contained in the interval $[0, 1]$; k and l are positive integers. A k-element subset $A \subset S$ is called *nice* if

$$\left| \frac{1}{k} \sum_{x_i \in A} x_i - \frac{1}{l} \sum_{x_j \in S \setminus A} x_j \right| \leq \frac{k+l}{2kl}$$

Prove that the number of nice subsets is at least $\dfrac{2}{k+l}\dbinom{k+l}{k}$.

Proposed by Andrey Badzyan, Russia

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6 For $n \geq 2$, let S_1, S_2, $\ldots$, S_{2^n} be 2^n subsets of $A = \{1, 2, 3, \ldots, 2^{n+1}\}$ that satisfy the following property: There do not exist indices a and b with $a < b$ and elements x, y, $z \in A$ with $x < y < z$ and y, $z \in S_a$, and x, $z \in S_b$. Prove that at least one of the sets S_1, S_2, $\ldots$, S_{2^n} contains no more than $4n$ elements.

Proposed by Gerhard Woeginger, Netherlands

– Geometry

1 Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of BC and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1 and C_2.

Prove that the six points A_1, A_2, B_1, B_2, C_1 and C_2 are concyclic.

Author: Andrey Gavrilyuk, Russia

2 Given trapezoid $ABCD$ with parallel sides AB and CD, assume that there exist points E on line BC outside segment BC, and F inside segment AD such that $\angle DAE = \angle CBF$. Denote by I the point of intersection of CD and EF, and by J the point of intersection of AB and EF. Let K be the midpoint of segment EF, assume it does not lie on line AB. Prove that I belongs to the circumcircle of ABK if and only if K belongs to the circumcircle of CDJ.

Proposed by Charles Leytem, Luxembourg

3 Let $ABCD$ be a convex quadrilateral and let P and Q be points in $ABCD$ such that $PQDA$ and $QPBC$ are cyclic quadrilaterals. Suppose that there exists a point E on the line segment PQ such that $\angle PAE = \angle QDE$ and $\angle PBE = \angle QCE$. Show that the quadrilateral $ABCD$ is cyclic.

Proposed by John Cuya, Peru

4 In an acute triangle ABC segments BE and CF are altitudes. Two circles passing through the point A anf F and tangent to the line BC at the points P and Q so that B lies between C and Q. Prove that lines PE and QF intersect on the circumcircle of triangle AEF.

Proposed by Davood Vakili, Iran

5 Let k and n be integers with $0 \leq k \leq n-2$. Consider a set L of n lines in the plane such that no two of them are parallel and no three have a common point. Denote by I the set of intersections of lines in L. Let O be a point in the plane not lying on any line of L. A point $X \in I$ is colored red if the open line segment OX intersects at most k lines in L. Prove that I contains at least $\dfrac{1}{2}(k+1)(k+2)$ red points.

Proposed by Gerhard Woeginger, Netherlands

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6 There is given a convex quadrilateral $ABCD$. Prove that there exists a point P inside the quadrilateral such that

$\angle PAB + \angle PDC = \angle PBC + \angle PAD = \angle PCD + \angle PBA = \angle PDA + \angle PCB = 90^\circ$

if and only if the diagonals AC and BD are perpendicular.

Proposed by Dusan Djukic, Serbia

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7 Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents to ω_1 and ω_2 intersect on ω.

Author: Vladimir Shmarov, Russia

– Number Theory

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1 Let n be a positive integer and let p be a prime number. Prove that if a, b, c are integers (not necessarily positive) satisfying the equations

$$a^n + pb = b^n + pc = c^n + pa$$

then $a = b = c$.

Proposed by Angelo Di Pasquale, Australia

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2 Let $a_1, a_2, \ldots, a_n$ be distinct positive integers, $n \geq 3$. Prove that there exist distinct indices i and j such that $a_i + a_j$ does not divide any of the numbers $3a_1, 3a_2, \ldots, 3a_n$.

Proposed by Mohsen Jamaali, Iran

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3 Let $a_0, a_1, a_2, \ldots$ be a sequence of positive integers such that the greatest common divisor of any two consecutive terms is greater than the preceding term; in symbols, $\gcd(a_i, a_{i+1}) > a_{i-1}$. Prove that $a_n \geq 2^n$ for all $n \geq 0$.

Proposed by Morteza Saghafian, Iran

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4 Let n be a positive integer. Show that the numbers

$$\binom{2^n-1}{0},\ \binom{2^n-1}{1},\ \binom{2^n-1}{2},\ \ldots,\ \binom{2^n-1}{2^{n-1}-1}$$

are congruent modulo 2^n to 1, 3, 5, $\ldots$, 2^n-1 in some order.

Proposed by Duskan Dukic, Serbia

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5 For every $n \in \mathbb{N}$ let $d(n)$ denote the number of (positive) divisors of n. Find all functions $f : \mathbb{N} \to \mathbb{N}$ with the following properties: - $d(f(x)) = x$ for all $x \in \mathbb{N}$. - $f(xy)$ divides $(x-1)y^{xy-1}f(x)$ for all x, $y \in \mathbb{N}$.

Proposed by Bruno Le Floch, France

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6 Prove that there are infinitely many positive integers n such that n^2+1 has a prime divisor greater than $2n+\sqrt{2n}$.

Author: Kestutis Cesnavicius, Lithuania
