Art of Problem Solving

2017 Iran Team Selection Test

Iran Team Selection Test 2017

Test 1 Day 1

1 Let a, b, c, d be positive real numbers with $a+b+c+d=2$. Prove the inequality:

$$\frac{(a+c)^2}{ad+bc}+\frac{(b+d)^2}{ac+bd}+4\geq 4\left(\frac{a+b+1}{c+d+1}+\frac{c+d+1}{a+b+1}\right)$$

Proposed by Mohammad Jafari

2 In the country of *Sugarland*, there are 13 students in the IMO team selection camp. 6 team selection tests were taken and the results have came out. Assume that no students have the same score on the same test.To select the IMO team, the national committee of math Olympiad have decided to choose a permutation of these 6 tests and starting from the first test, the person with the highest score between the remaining students will become a member of the team.The committee is having a session to choose the permutation.
Is it possible that all 13 students have a chance of being a team member?

Proposed by Morteza Saghafian

3 In triangle ABC let I_a be the A-excenter. Let ω be an arbitrary circle that passes through A, I_a and intersects the extensions of sides AB, AC (extended from B, C) at X, Y respectively. Let S, T be points on segments I_aB, I_aC respectively such that $\angle AXI_a=\angle BTI_a$ and $\angle AYI_a=\angle CSI_a$.Lines BT, CS intersect at K. Lines KI_a, TS intersect at Z.
Prove that X, Y, Z are collinear.

Proposed by Hooman Fattahi

Test 1 Day 2

4 We arranged all the prime numbers in the ascending order: $p_1=2<p_2<p_3<\cdots$.
Also assume that $n_1<n_2<\cdots$ is a sequence of positive integers that for all $i=1,2,3,\cdots$ the equation $x^{n_i}\equiv 2 \pmod{p_i}$ has a solution for x.
Is there always a number x that satisfies all the equations?

Proposed by Mahyar Sefidgaran , Yahya Motevasel

5 In triangle ABC, arbitrary points P,Q lie on side BC such that $BP=CQ$ and P lies between B,Q.The circumcircle of triangle APQ intersects sides AB,AC at E,F respectively.The point T is the intersection of EP,FQ.Two lines passing through the midpoint of BC and parallel to AB and AC, intersect EP and FQ at points X,Y respectively.
Prove that the circumcircle of triangle TXY and triangle APQ are tangent to each other.

Proposed by Iman Maghsoudi

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6 In the unit squares of a transparent 1×100 tape, numbers $1,2,\cdots,100$ are written in the ascending order.We fold this tape on it's lines with arbitrary order and arbitrary directions until we reach a 1×1 tape with 100 layers.A permutation of the numbers $1,2,\cdots,100$ can be seen on the tape, from the top to the bottom.
Prove that the number of possible permutations is between 2^{100} and 4^{100}.
(*e.g.* We can produce all permutations of numbers $1,2,3$ with a 1×3 tape)

Proposed by Morteza Saghafian

Test 2 Day 1

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1 $ABCD$ is a trapezoid with $AB \parallel CD$. The diagonals intersect at P. Let ω_1 be a circle passing through B and tangent to AC at A. Let ω_2 be a circle passing through C and tangent to BD at D. ω_3 is the circumcircle of triangle BPC.
Prove that the common chord of circles ω_1,ω_3 and the common chord of circles ω_2,ω_3 intersect each other on AD.

Proposed by Kasra Ahmadi

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2 Find the largest number n that for which there exists n positive integers such that non of them divides another one, but between every three of them, one divides the sum of the other two.

Proposed by Morteza Saghafian

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3 There are 27 cards, each has some amount of (1 or 2 or 3) shapes (a circle, a square or a triangle) with some color (white, grey or black) on them. We call a triple of cards a *match* such that all of them have the same amount of shapes or distinct amount of shapes, have the same shape or distinct shapes and have the same color or distinct colors. For instance, three cards shown in the figure are a *match* be cause they have distinct amount of shapes, distinct shapes but the same color of shapes.

What is the maximum number of cards that we can choose such that non of the triples make a *match*?

Proposed by Amin Bahjati

Test 2 Day 2

4 A $n+1$-tuple $(h_1, h_2, \cdots, h_{n+1})$ where $h_i(x_1, x_2, \cdots, x_n)$ are n variable polynomials with real coefficients is called *good* if the following condition holds:
For any n functions $f_1, f_2, \cdots, f_n : \mathbb{R} \to \mathbb{R}$ if for all $1 \le i \le n+1$, $P_i(x) = h_i(f_1(x), f_2(x), \cdots, f_n(x))$ is a polynomial with variable x, then $f_1(x), f_2(x), \cdots, f_n(x)$ are polynomials.

$a)$ Prove that for all positive integers n, there exists a *good* $n+1$-tuple $(h_1, h_2, \cdots, h_{n+1})$ such that the degree of all h_i is more than 1.

$b)$ Prove that there doesn't exist any integer $n > 1$ that for which there is a *good* $n+1$-tuple $(h_1, h_2, \cdots, h_{n+1})$ such that all h_i are symmetric polynomials.

Proposed by Alireza Shavali

5 k, n are two arbitrary positive integers. Prove that there exists at least $(k-1)(n-k+1)$ positive integers that can be produced by n number of k's and using only $+, -, \times, \div$ operations and adding parentheses between them, but cannot be produced using $n-1$ number of k's.

Proposed by Aryan Tajmir

6 Let $k > 1$ be an integer. The sequence $a_1, a_2, \cdots$ is defined as: $a_1 = 1, a_2 = k$ and for all $n > 1$ we have: $a_{n+1} - (k+1)a_n + a_{n-1} = 0$
Find all positive integers n such that a_n is a power of k.

Proposed by Amirhossein Pooya

Test 3 Day 1

1 Let $n > 1$ be an integer. Prove that there exists an integer $n-1 \ge m \ge \left\lfloor \frac{n}{2} \right\rfloor$ such that the following equation has integer solutions with $a_m > 0$:

$$\frac{a_m}{m+1} + \frac{a_{m+1}}{m+2} + \cdots + \frac{a_{n-1}}{n} = \frac{1}{\operatorname{lcm}(1, 2, \cdots, n)}$$

Proposed by Navid Safaei

2 Let P be a point in the interior of quadrilateral $ABCD$ such that:

$$\angle BPC = 2\angle BAC \quad , \quad \angle PCA = \angle PAD \quad , \quad \angle PDA = \angle PAC$$

Prove that:

$$\angle PBD = |\angle BCA - \angle PCA|$$

Proposed by Ali Zamani

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3 Find all functions $f:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ that satisfy the following conditions for all positive real numbers x,y,z :

$$f\left(f(x,y),z\right)=x^2y^2f(x,z)$$

$$f\left(x,1+f(x,y)\right)\ge x^2+xyf(x,x)$$

Proposed by Mojtaba Zare, Ali Daei Nabi

Test 3 Day 2

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4 There are 6 points on the plane such that no three of them are collinear. It's known that between every 4 points of them, there exists a point that it's power with respect to the circle passing through the other three points is a constant value k.(Power of a point in the interior of a circle has a negative value.)
Prove that $k=0$ and all 6 points lie on a circle.

Proposed by Morteza Saghafian

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5 Let $\{c_i\}_{i=0}^{\infty}$ be a sequence of non-negative real numbers with $c_{2017}>0$. A sequence of polynomials if defined as:

$$P_{-1}(x)=0\ ,\ P_0(x)=1\ ,\ P_{n+1}(x)=xP_n(x)+c_nP_{n-1}(x)$$

Prove that there doesn't exist any integer $n>2017$ and some real number c such that:

$$P_{2n}(x)=P_n(x^2+c)$$

Proposed by Navid Safaei

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6 In triangle ABC let O and H be the circumcenter and the orthocenter. The point P is the reflection of A with respect to OH. Assume that P is not on the same side of BC as A. Points E,F lie on AB,AC respectively such that $BE=PC$,$CF=PB$. Let K be the intersection point of AP,OH. Prove that $\angle EKF=90^\circ$

Proposed by Iman Maghsoudi

Iran TST 2016

Test 1 Day 1

1 Let m and n be positive integers such that $m > n$. Define $x_k = \frac{m+k}{n+k}$ for $k = 1, 2, \ldots, n+1$. Prove that if all the numbers $x_1, x_2, \ldots, x_{n+1}$ are integers, then $x_1 x_2 \ldots x_{n+1} - 1$ is divisible by an odd prime.

2 For a finite set A of positive integers, a partition of A into two disjoint nonempty subsets A_1 and A_2 is *good* if the least common multiple of the elements in A_1 is equal to the greatest common divisor of the elements in A_2. Determine the minimum value of n such that there exists a set of n positive integers with exactly 2015 good partitions.

3 Let $ABCD$ be a convex quadrilateral, and let P, Q, R, and S be points on the sides AB, BC, CD, and DA, respectively. Let the line segment PR and QS meet at O. Suppose that each of the quadrilaterals $APOS$, $BQOP$, $CROQ$, and $DSOR$ has an incircle. Prove that the lines AC, PQ, and RS are either concurrent or parallel to each other.

Test 1 Day 2

4 Let n be a fixed positive integer. Find the maximum possible value of

$$\sum_{1 \le r < s \le 2n} (s - r - n) x_r x_s,$$

where $-1 \le x_i \le 1$ for all $i = 1, \cdots, 2n$.

5 Let ABC be a triangle with $\angle C = 90^\circ$, and let H be the foot of the altitude from C. A point D is chosen inside the triangle CBH so that CH bisects AD. Let P be the intersection point of the lines BD and CH. Let ω be the semicircle with diameter BD that meets the segment CB at an interior point. A line through P is tangent to ω at Q. Prove that the lines CQ and AD meet on ω.

6 In a company of people some pairs are enemies. A group of people is called *unsociable* if the number of members in the group is odd and at least 3, and it is possible to arrange all its members around a round table so that every two neighbors are enemies. Given that there are at most 2015 unsociable groups, prove that it is possible to partition the company into 11 parts so that no two enemies are in the same part.

Proposed by Russia

Test 2 Day 1

1 Let ABC be an acute triangle and let M be the midpoint of AC. A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC. Determine all possible values of $\frac{BT}{BM}$.

2 Let a, b, c, d be positive real numbers such that $\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}+\frac{1}{d+1}=2$. Prove that

$$\sum_{cyc}\sqrt{\frac{a^2+1}{2}} \geq (3.\sum_{cyc}\sqrt{a}) - 8$$

3 Let n be a positive integer. Two players A and B play a game in which they take turns choosing positive integers $k \leq n$. The rules of the game are:

(i) A player cannot choose a number that has been chosen by either player on any previous turn.
(ii) A player cannot choose a number consecutive to any of those the player has already chosen on any previous turn.
(iii) The game is a draw if all numbers have been chosen; otherwise the player who cannot choose a number anymore loses the game.

The player A takes the first turn. Determine the outcome of the game, assuming that both players use optimal strategies.

Proposed by Finland

Test 2 Day 2

4 Let ABC be a triangle with $CA \neq CB$. Let D, F, and G be the midpoints of the sides AB, AC, and BC respectively. A circle Γ passing through C and tangent to AB at D meets the segments AF and BG at H and I, respectively. The points H' and I' are symmetric to H and I about F and G, respectively. The line $H'I'$ meets CD and FG at Q and M, respectively. The line CM meets Γ again at P. Prove that $CQ = QP$.

Proposed by El Salvador

6 Let $\mathbb{Z}_{>0}$ denote the set of positive integers. For any positive integer k, a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ is called [i]k-good[/i] if $\gcd(f(m) + n, f(n) + m) \le k$ for all $m \neq n$. Find all k such that there exists a k-good function.

Proposed by James Rickards, Canada

Test 3 Day 1

2 Let ABC be an arbitrary triangle and O is the circumcenter of $\triangle ABC$.Points X, Y lie on AB, AC,respectively such that the reflection of BC WRT XY is tangent to circumcircle of $\triangle AXY$.Prove that the circumcircle of triangle AXY is tangent to circumcircle of triangle BOC.

3 Let $p \neq 13$ be a prime number of the form $8k + 5$ such that 39 is a quadratic non-residue modulo p. Prove that the equation

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 \equiv 0 \pmod p$$

has a solution in integers such that $p \nmid x_1x_2x_3x_4$.

Test 3 Day 2

4 Suppose that a sequence $a_1, a_2, \ldots$ of positive real numbers satisfies

$$a_{k+1} \ge \frac{ka_k}{a_k^2 + (k-1)}$$

for every positive integer k. Prove that $a_1 + a_2 + \ldots + a_n \ge n$ for every $n \ge 2$.

5 Let AD, BF, CE be altitudes of triangle ABC.Q is a point on EF such that $QF = DE$ and F is between E, Q.P is a point on EF such that $EP = DF$ and E is between P, F.Perpendicular bisector of DQ intersect with AB at X and perpendicular bisector of DP intersect with AC at Y.Prove that midpoint of BC lies on XY.

Iran Team Selection Test 2015

– TST 1

Day 1

1 Find all polynomials $P, Q \in \mathbb{Q}[x]$ such that

$$P(x)^3 + Q(x)^3 = x^{12} + 1.$$

2 I_b is the B-excenter of the triangle ABC and ω is the circumcircle of this triangle. M is the middle of arc BC of ω which doesn't contain A. MI_b meets ω at $T \neq M$. Prove that

$$TB \cdot TC = TI_b^2.$$

3 Let $b_1 < b_2 < b_3 < \dots$ be the sequence of all natural numbers which are sum of squares of two natural numbers.
Prove that there exists infinite natural numbers like m which $b_{m+1} - b_m = 2015$.

Day 2

4 n is a fixed natural number. Find the least k such that for every set A of k natural numbers, there exists a subset of A with an even number of elements which the sum of it's members is divisible by n.

5 Let A be a subset of the edges of an $n \times n$ table. Let $V(A)$ be the set of vertices from the table which are connected to at least on edge from A and $j(A)$ be the number of the connected components of graph G which it's vertices are the set $V(A)$ and it's edges are the set A. Prove that for every natural number l:

$$\frac{l}{2} \leq min_{|A| \geq l}(|V(A)| - j(A)) \leq \frac{l}{2} + \sqrt{\frac{l}{2}} + 1$$

6 $ABCD$ is a circumscribed and inscribed quadrilateral. O is the circumcenter of the quadrilateral. E, F and S are the intersections of AB, CD , AD, BC and AC, BD respectively. E' and F' are points on AD and AB such that $A\hat{E}E' = E'\hat{E}D$ and $A\hat{F}F' = F'\hat{F}B$. X and Y are points on OE' and OF' such that $\frac{XA}{XD} = \frac{EA}{ED}$ and $\frac{YA}{YB} = \frac{FA}{FB}$. M is the midpoint of arc BD of (O) which contains A.
Prove that the circumcircles of triangles OXY and OAM are coaxal with the circle with diameter OS.

– TST 2

Day 1

1 a, b, c, d are positive numbers such that $\sum_{cyc} \frac{1}{ab} = 1$. Prove that : $abcd + 16 \geq 8\sqrt{(a+c)(\frac{1}{a} + \frac{1}{c})} + 8\sqrt{(b+d)(\frac{1}{b} + \frac{1}{d})}$

2 In triangle ABC(with incenter I) let the line parallel to BC from A intersect circumcircle of $\triangle ABC$ at A_1 let $AI \cap BC = D$ and E is tangency point of incircle with BC let $EA_1 \cap \odot(\triangle ADE) = T$ prove that $AI = TI$.

Day 2

4 Let $\triangle ABC$ be an acute triangle. Point Z is on A altitude and points X and Y are on the B and C altitudes out of the triangle respectively, such that: $\angle AYB = \angle BZC = \angle CXA = 90$
Prove that X,Y and Z are collinear, if and only if the length of the tangent drawn from A to the nine point circle of $\triangle ABC$ is equal with the sum of the lengths of the tangents drawn from B and C to the nine point circle of $\triangle ABC$.

5 We call a permutation $(a_1, a_2, \cdots, a_n)$ of the set $\{1, 2, \cdots, n\}$ "good" if for any three natural numbers $i < j < k$, $n \nmid a_i + a_k - 2a_j$ find all natural numbers $n \geq 3$ such that there exist a "good" permutation of a set $\{1, 2, \cdots, n\}$.

6 If a, b, c are positive real numbers such that $a + b + c = abc$ prove that

$$\frac{abc}{3\sqrt{2}}\left(\sum_{cyc}\frac{\sqrt{a^3+b^3}}{ab+1}\right) \geq \sum_{cyc}\frac{a}{a^2+1}$$

– TST 3

Day 1

1 Point A is outside of a given circle ω. Let the tangents from A to ω meet ω at S, T points X, Y are midpoints of AT, AS let the tangent from X to ω meet ω at $R \neq T$. points P, Q are midpoints of XT, XR let $XY \cap PQ = K, SX \cap TK = L$ prove that quadrilateral $KRLQ$ is cyclic.

2 Assume that a_1, a_2, a_3 are three given positive integers consider the following sequence: $a_{n+1} = \text{lcm}[a_n, a_{n-1}] - \text{lcm}[a_{n-1}, a_{n-2}]$ for $n \geq 3$
Prove that there exist a positive integer k such that $k \leq a_3 + 4$ and $a_k \leq 0$.
($[a, b]$ means the least positive integer such that$a \mid [a, b], b \mid [a, b]$ also because $\text{lcm}[a, b]$ takes only nonzero integers this sequence is defined until we find a zero number in the sequence)

3 $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ are $2n$ positive real numbers such that $a_1, a_2, \cdots, a_n$ aren't all equal. And assume that we can divide $a_1, a_2, \cdots, a_n$ into two subsets with equal sums.similarly $b_1, b_2, \cdots, b_n$ have these two conditions. Prove that there exist a simple $2n$-gon with sides $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ and parallel to coordinate axises Such that the lengths of horizontal sides are among $a_1, a_2, \cdots, a_n$ and the lengths of vertical sides are among $b_1, b_2, \cdots, b_n$.(simple polygon is a polygon such that it doesn't intersect itself)

Day 2

5 Prove that for each natural number d, There is a monic and unique polynomial of degree d like P such that $P(1)0$ and for each sequence like a_1,a_2, ... of real numbers that the recurrence relation below is true for them, there is a natural number k such that $0 = a_k = a_{k+1} = ...$: $P(n)a_1 + P(n-1)a_2 + ... + P(1)a_n = 0$ $n > 1$

6 AH is the altitude of triangle ABC and H' is the reflection of H trough the midpoint of BC. If the tangent lines to the circumcircle of ABC at B and C, intersect each other at X and the perpendicular line to XH' at H', intersects AB and AC at Y and Z respectively, prove that $\angle ZXC = \angle YXB$.

Iran Team Selection Test 2014

– TST 1

Day 1

1 suppose that O is the circumcenter of acute triangle ABC.
we have circle with center O that is tangent too BC that named w
suppose that X and Y are the points of intersection of the tangent from A to w with line BC(X and B are in the same side of AO) T is the intersection of the line tangent to circumcirle of ABC in B and the line from X parallel to AC. S is the intersection of the line tangent to circumcirle of ABC in C and the line from Y parallel to AB.
prove that ST is tangent ABC.

2 find all polynomials with integer coefficients that $P(\mathbb{Z})$ =$\{p(a) : a \in \mathbb{Z}\}$ has a Geometric progression.

3 we named a $n*n$ table $selfish$ if we number the row and column with $0, 1, 2, 3, ..., n-1$.(from left to right an from up to down)
for every $\{i, j \in 0, 1, 2, ..., n-1\}$ the number of cell (i, j) is equal to the number of number i in the row j.
for example we have such table for $n = 5$
1 0 3 3 4
1 3 2 1 1
0 1 0 1 0
2 1 0 0 0
1 0 0 0 0
prove that for $n > 5$ there is no $selfish$ table

Day 2

4 Find the maximum number of Permutation of set $\{1, 2, 3, ..., 2014\}$ such that for every 2 different number a and b in this set at last in one of the permutation b comes exactly after a

5 n is a natural number. for every positive real numbers $x_1, x_2, ..., x_{n+1}$ such that $x_1x_2...x_{n+1} = 1$ prove that: $\sqrt[x_1]{n} + ... + \sqrt[x_{n+1}]{n} \geq n^{\sqrt[n]{x_1}} + ... + n^{\sqrt[n]{x_{n+1}}}$

6 I is the incenter of triangle ABC. perpendicular from I to AI meet AB and AC at B' and C' respectively .
Suppose that B'' and C'' are points on half-line BC and CB such that $BB'' = BA$ and $CC'' = CA$.
Suppose that the second intersection of circumcircles of $AB'B''$ and $AC'C''$ is T.
Prove that the circumcenter of AIT is on the BC.

– TST 2

Day 1

1 Consider a tree with n vertices, labeled with $1, \ldots, n$ in a way that no label is used twice. We change the labeling in the following way - each time we pick an edge that hasn't been picked before and swap the labels of its endpoints. After performing this action $n-1$ times, we get another tree with its labeling a permutation of the first graph's labeling.
Prove that this permutation contains exactly one cycle.

2 Point D is an arbitary point on side BC of triangle ABC. I,I_1 andI_2 are the incenters of triangles ABC,ABD and ACD respectively. $M \neq A$ and $N \neq A$ are the intersections of circumcircle of triangle ABC and circumcircles of triangles IAI_1 and IAI_2 respectively. Prove that regardless of point D, line MN goes through a fixed point.

3 prove for all $k > 1$ equation $(x+1)(x+2)...(x+k) = y^2$ has finite solutions.

Day 2

4 n is a natural number. We shall call a permutation $a_1, \ldots, a_n$ of $1, \ldots, n$ a quadratic(cubic) permutation if $\forall 1 \leq i \leq n-1$ we have $a_i a_{i+1} + 1$ is a perfect square(cube). (a) Prove that for infinitely many natural numbers n there exists a quadratic permutation. (b) Prove that for no natural number n exists a cubic permutation.

5 if $x, y, z > 0$ are postive real numbers such that $x^2+y^2+z^2 = x^2y^2+y^2z^2+z^2x^2$ prove that

$$((x-y)(y-z)(z-x))^2 \leq 2((x^2-y^2)^2+(y^2-z^2)^2+(z^2-x^2)^2)$$

6 Consider n segments in the plane which no two intersect and between their $2n$ endpoints no three are collinear. Is the following statement true?
Statement: There exists a simple $2n$-gon such that it's vertices are the $2n$ endpoints of the segments and each segment is either completely inside the polygon or an edge of the polygon.

– TST 3

Day 1

1 The incircle of a non-isosceles triangle ABC with the center I touches the sides BC, AC, AB at A_1, B_1, C_1 .
let AI, BI, CI meets BC, AC, AB at A_2, B_2, C_2.
let A' is a point on AI such that $A_1A' \perp B_2C_2$ .B', C' respectively.
prove that two triangle $A'B'C', A_1B_1C_1$ are equal.

2 is there a function $f : \mathbb{N} \to \mathbb{N}$ such that $i) \exists n \in \mathbb{N} : f(n) \neq n$ $ii)$ the number of divisors of m is $f(n)$ if and only if the number of divisors of $f(m)$ is n

3 let $m, n \in \mathbb{N}$ and $p(x), q(x), h(x)$ are polynomials with real Coefficients such that $p(x)$ is Descending.
and for all $x \in \mathbb{R}$ $p(q(nx + m) + h(x)) = n(q(p(x)) + h(x)) + m$.
prove that dont exist function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$ $f(q(p(x)) + h(x)) = f(x)^2 + 1$

Day 2

4 Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $x, y \in \mathbb{R}^+$,

$$f\left(\frac{y}{f(x+1)}\right) + f\left(\frac{x+1}{xf(y)}\right) = f(y)$$

5 Given a set $X = \{x_1, \ldots, x_n\}$ of natural numbers in which for all $1 < i \leq n$ we have $1 \leq x_i - x_{i-1} \leq 2$, call a real number a **good** if there exists $1 \leq j \leq n$ such that $2|x_j - a| \leq 1$. Also a subset of X is called **compact** if the average of its elements is a good number.
Prove that at least 2^{n-3} subsets of X are compact.

6 The incircle of a non-isosceles triangle ABC with the center I touches the sides BC at D.

let X is a point on arc BC from circumcircle of triangle ABC such that if E,F are feet of perpendicular from X on BI,CI and M is midpoint of EF we have $MB=MC$.
prove that $\widehat{BAD}=\widehat{CAX}$

Iran Team Selection Test 2013

– TST 1

Day 1

1 In acute-angled triangle ABC, let H be the foot of perpendicular from A to BC and also suppose that J and I are excenters oposite to the side AH in triangles ABH and ACH. If P is the point that incircle touches BC, prove that I, J, P, H are concyclic.

2 Find the maximum number of subsets from $\{1, ..., n\}$ such that for any two of them like A, B if $A \subset B$ then $|B - A| \geq 3$. (Here $|X|$ is the number of elements of the set X.)

3 For nonnegative integers m and n, define the sequence $a(m, n)$ of real numbers as follows. Set $a(0, 0) = 2$ and for every natural number n, set $a(0, n) = 1$ and $a(n, 0) = 2$. Then for $m, n \geq 1$, define

$$a(m, n) = a(m - 1, n) + a(m, n - 1).$$

Prove that for every natural number k, all the roots of the polynomial $P_k(x) = \sum_{i=0}^{k} a(i, 2k + 1 - 2i)x^i$ are real.

Day 2

4 m and n are two nonnegative integers. In the Philosopher's Chess, The chessboard is an infinite grid of identical regular hexagons and a new piece named the Donkey moves on it as follows:

Starting from one of the hexagons, the Donkey moves m cells in one of the 6 directions, then it turns 60 degrees clockwise and after that moves n cells in this new direction until it reaches it's final cell.

At most how many cells are in the Philosopher's chessboard such that one cannot go from anyone of them to the other with a finite number of movements of the Donkey?

Proposed by Shayan Dashmiz

5 Do there exist natural numbers a, b and c such that $a^2 + b^2 + c^2$ is divisible by $2013(ab + bc + ca)$?

Proposed by Mahan Malihi

6 Points A, B, C and D lie on line l in this order. Two circular arcs C_1 and C_2, which both lie on one side of line l, pass through points A and B and two circular arcs C_3 and C_4 pass through points C and D such that C_1 is tangent to C_3 and C_2 is tangent to C_4. Prove that the common external tangent of C_2 and C_3 and the common external tangent of C_1 and C_4 meet each other on line l.

Proposed by Ali Khezeli

– TST 2

Day 1

7 Nonnegative real numbers $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$ are such that $p_1 + \cdots + p_n = q_1 + \cdots + q_n$
Among all the matrices with nonnegative entries having p_i as sum of the i-th row's entries and q_j as sum of the j-th column's entries, find the maximum sum of the entries on the main diagonal.

8 Find all Arithmetic progressions $a_1, a_2, ...$ of natural numbers for which there exists natural number $N > 1$ such that for every $k \in \mathbb{N}$:

$a_1 a_2 ... a_k \mid a_{N+1} a_{N+2} ... a_{N+k}$

9 find all functions $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ such that f is increasing and also:

$f(f(x) + 2g(x) + 3f(y)) = g(x) + 2f(x) + 3g(y)$

$g(f(x) + y + g(y)) = 2x - g(x) + f(y) + y$

Day 2

10 On each edge of a graph is written a real number,such that for every even tour of this graph,sum the edges with signs alternatively positive and negative is zero.prove that one can assign to each of the vertices of the graph a real number such that sum of the numbers on two adjacent vertices is the number on the edge between them.(tour is a closed path from the edges of the graph that may have repeated edges or vertices)

11 Let a, b, c be sides of a triangle such that $a \geq b \geq c$. prove that:

$\sqrt{a(a+b-\sqrt{ab})} + \sqrt{b(a+c-\sqrt{ac})} + \sqrt{c(b+c-\sqrt{bc})} \geq a+b+c$

12 Let $ABCD$ be a cyclic quadrilateral that inscribed in the circle ω.Let I_1, I_2 and r_1, r_2 be incenters and radii of incircles of triangles ACD and ABC,respectively.assume that $r_1 = r_2$. let ω' be a circle that touches AB, AD and touches ω at T. tangents from A, T to ω meet at the point K.prove that I_1, I_2, K lie on a line.

– TST 3

Day 1

13 P is an arbitrary point inside acute triangle ABC. Let A_1, B_1, C_1 be the reflections of point P with respect to sides BC, CA, AB. Prove that the centroid of triangle $A_1B_1C_1$ lies inside triangle ABC.

14 we are given n rectangles in the plane. Prove that between $4n$ right angles formed by these rectangles there are at least $[4\sqrt{n}]$ distinct right angles.

15 a) Does there exist a sequence $a_1 < a_2 < \dots$ of positive integers, such that there is a positive integer N that $\forall m > N$, a_m has exactly $d(m) - 1$ divisors among a_is?

b) Does there exist a sequence $a_1 < a_2 < \dots$ of positive integers, such that there is a positive integer N that $\forall m > N$, a_m has exactly $d(m) + 1$ divisors among a_is?

Day 2

16 The function $f : \mathbb{Z} \to \mathbb{Z}$ has the property that for all integers m and n

$$f(m) + f(n) + f(f(m^2 + n^2)) = 1.$$

We know that integers a and b exist such that $f(a) - f(b) = 3$. Prove that integers c and d can be found such that $f(c) - f(d) = 1$.

Proposed by Amirhossein Gorzi

17 In triangle ABC, AD and AH are the angle bisector and the altitude of vertex A, respectively. The perpendicular bisector of AD, intersects the semicircles with diameters AB and AC which are drawn outside triangle ABC in X and Y, respectively. Prove that the quadrilateral $XYDH$ is concyclic.

Proposed by Mahan Malihi

18 A special kind of parallelogram tile is made up by attaching the legs of two right isosceles triangles of side length 1. We want to put a number of these tiles on the floor of an $n \times n$ room such that the distance from each vertex of each tile to the sides of the room is an integer and also no two tiles overlap. Prove that at least an area n of the room will not be covered by the tiles.

Proposed by Ali Khezeli

Iran Team Selection Test 2012

– Exam 1

Day 1

1 Find all positive integers $n \geq 2$ such that for all integers i, j that $0 \leq i, j \leq n$, $i + j$ and $\binom{n}{i} + \binom{n}{j}$ have same parity.

Proposed by Mr.Etesami

2 Consider ω is circumcircle of an acute triangle ABC. D is midpoint of arc BAC and I is incenter of triangle ABC. Let DI intersect BC in E and ω for second time in F. Let P be a point on line AF such that PE is parallel to AI. Prove that PE is bisector of angle BPC.

Proposed by Mr.Etesami

3 Let n be a positive integer. Let S be a subset of points on the plane with these conditions:

$i)$ There does not exist n lines in the plane such that every element of S be on at least one of them.

$ii)$ for all $X \in S$ there exists n lines in the plane such that every element of $S - X$ be on at least one of them.

Find maximum of $\mid S \mid$.

Proposed by Erfan Salavati

Day 2

1 Consider $m + 1$ horizontal and $n + 1$ vertical lines $(m, n \geq 4)$ in the plane forming an $m \times n$ table. Cosider a closed path on the segments of this table such that it does not intersect itself and also it passes through all $(m-1)(n-1)$ interior vertices (each vertex is an intersection point of two lines) and it doesn't pass through any of outer vertices. Suppose A is the number of vertices such that the path passes through them straight forward, B number of the table squares that only their two opposite sides are used in the path, and C number of the table squares that none of their sides is used in the path. Prove that

$$A = B - C + m + n - 1.$$

Proposed by Ali Khezeli

2 The function $f:\mathbb{R}^{\geq 0} \longrightarrow \mathbb{R}^{\geq 0}$ satisfies the following properties for all $a,b \in \mathbb{R}^{\geq 0}$:

a) $f(a)=0 \Leftrightarrow a=0$

b) $f(ab)=f(a)f(b)$

c) $f(a+b) \leq 2\max\{f(a),f(b)\}$.

Prove that for all $a,b \in \mathbb{R}^{\geq 0}$ we have $f(a+b) \leq f(a)+f(b)$.

Proposed by Masoud Shafaei

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3 The pentagon $ABCDE$ is inscribed in a circle w. Suppose that w_a,w_b,w_c,w_d,w_e are reflections of w with respect to sides AB,BC,CD,DE,EA respectively. Let A' be the second intersection point of w_a,w_e and define B',C',D',E' similarly. Prove that

$$2 \leq \frac{S_{A'B'C'D'E'}}{S_{ABCDE}} \leq 3,$$

where S_X denotes the surface of figure X.

Proposed by Morteza Saghafian, Ali khezeli

– Exam 2

Day 1

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1 Is it possible to put $\binom{n}{2}$ consecutive natural numbers on the edges of a complete graph with n vertices in a way that for every path (or cycle) of length 3 where the numbers a,b and c are written on its edges (edge b is between edges c and a), b is divisible by the greatest common divisor of the numbers a and c?

Proposed by Morteza Saghafian

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2 Let $g(x)$ be a polynomial of degree at least 2 with all of its coefficients positive. Find all functions $f:\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that

$$f(f(x)+g(x)+2y)=f(x)+g(x)+2f(y) \quad \forall x,y \in \mathbb{R}^+.$$

Proposed by Mohammad Jafari

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3 Suppose $ABCD$ is a parallelogram. Consider circles w_1 and w_2 such that w_1 is tangent to segments AB and AD and w_2 is tangent to segments BC and CD. Suppose that there exists a circle which is tangent to lines AD and DC and externally tangent to w_1 and w_2. Prove that there exists a circle which is tangent to lines AB and BC and also externally tangent to circles w_1 and w_2.

Proposed by Ali Khezeli

Day 2

1 For positive reals a, b and c with $ab + bc + ca = 1$, show that

$$\sqrt{3}(\sqrt{a} + \sqrt{b} + \sqrt{c}) \le \frac{a\sqrt{a}}{bc} + \frac{b\sqrt{b}}{ca} + \frac{c\sqrt{c}}{ab}.$$

Proposed by Morteza Saghafian

2 Points A and B are on a circle ω with center O such that $\frac{\pi}{3} < \angle AOB < \frac{2\pi}{3}$. Let C be the circumcenter of the triangle AOB. Let l be a line passing through C such that the angle between l and the segment OC is $\frac{\pi}{3}$. l cuts tangents in A and B to ω in M and N respectively. Suppose circumcircles of triangles CAM and CBN, cut ω again in Q and R respectively and theirselves in P (other than C). Prove that $OP \perp QR$.

Proposed by Mehdi E'tesami Fard, Ali Khezeli

3 We call a subset B of natural numbers *loyal* if there exists natural numbers $i \le j$ such that $B = \{i, i+1, ..., j\}$. Let Q be the set of all *loyal* sets.

Now for every subset $A = \{a_1 < a_2 < ... < a_k\}$ of $\{1, 2, ..., n\}$ we set

$f(A) = \max_{1 \le i \le k-1} a_{i+1} - a_i$, and $g(A) = \max_{B \subseteq A, B \in Q} |B|$.

And we define

$F(n) = \sum_{A \subseteq \{1,2,...,n\}} f(A)$ and $G(n) = \sum_{A \subseteq \{1,2,...,n\}} g(A)$.

Prove that there exists $m \in \mathbb{N}$ such that for each natural number $n > m$, we have $F(n) > G(n)$.

(By $|A|$ we mean the number of elements of A and if $|A| \le 1$, we define $f(A)$ to be zero).

Proposed by Javad Abedi

– Exam 3

Day 1

1 Consider a regular 2^k-gon with center O and label its sides clockwise by $l_1, l_2, ..., l_{2^k}$. Reflect O with respect to l_1, then reflect the resulting point with respect to l_2

and do this process until the last side. Prove that the distance between the final point and O is less than the perimeter of the 2^k-gon.

Proposed by Hesam Rajabzade

2 Do there exist 2000 real numbers (not necessarily distinct) such that all of them are not zero and if we put any group containing 1000 of them as the roots of a monic polynomial of degree 1000, the coefficients of the resulting polynomial (except the coefficient of x^{1000}) be a permutation of the 1000 remaining numbers?

Proposed by Morteza Saghafian

3 Find all integer numbers x and y such that:

$$(y^3 + xy - 1)(x^2 + x - y) = (x^3 - xy + 1)(y^2 + x - y).$$

Proposed by Mahyar Sefidgaran

Day 2

1 Suppose p is an odd prime number. We call the polynomial $f(x) = \sum_{j=0}^{n} a_j x^j$ with integer coefficients i-remainder if $\sum_{p-1|j, j>0} a_j \equiv i \pmod{p}$. Prove that the set $\{f(0), f(1), ..., f(p-1)\}$ is a complete residue system modulo p if and only if polynomials $f(x), (f(x))^2, ..., (f(x))^{p-2}$ are 0-remainder and the polynomial $(f(x))^{p-1}$ is 1-remainder.

Proposed by Yahya Motevassel

2 Let n be a natural number. Suppose A and B are two sets, each containing n points in the plane, such that no three points of a set are collinear. Let $T(A)$ be the number of broken lines, each containing $n-1$ segments, and such that it doesn't intersect itself and its vertices are points of A. Define $T(B)$ similarly. If the points of B are vertices of a convex n-gon (are in *convex position*), but the points of A are not, prove that $T(B) < T(A)$.

Proposed by Ali Khezeli

3 Let O be the circumcenter of the acute triangle ABC. Suppose points A', B' and C' are on sides BC, CA and AB such that circumcircles of triangles $AB'C', BC'A'$ and $CA'B'$ pass through O. Let ℓ_a be the radical axis of the circle with center B' and radius $B'C$ and circle with center C' and radius $C'B$. Define ℓ_b and ℓ_c similarly. Prove that lines ℓ_a, ℓ_b and ℓ_c form a triangle such that it's orthocenter coincides with orthocenter of triangle ABC.

Proposed by Mehdi E'tesami Fard

Iran Team Selection Test 2011

Day 1

1 In acute triangle ABC angle B is greater than$C$. Let M is midpoint of BC. D and E are the feet of the altitude from C and B respectively. K and L are midpoint of ME and MD respectively. If KL intersect the line through A parallel to BC in T, prove that $TA = TM$.

2 Find all natural numbers n greater than 2 such that there exist n natural numbers $a_1, a_2, \ldots, a_n$ such that they are not all equal, and the sequence $a_1a_2, a_2a_3, \ldots, a_na_1$ forms an arithmetic progression with nonzero common difference.

3 There are n points on a circle $(n > 1)$. Define an "interval" as an arc of a circle such that it's start and finish are from those points. Consider a family of intervals F such that for every element of F like A there is almost one other element of F like B such that $A \subseteq B$ (in this case we call A is sub-interval of B). We call an interval maximal if it is not a sub-interval of any other interval. If m is the number of maximal elements of F and a is number of non-maximal elements of F, prove that $n \geq m + \frac{a}{2}$.

Day 2

4 Define a finite set A to be 'good' if it satisfies the following conditions:

-**(a)** For every three disjoint element of A, like a, b, c we have $\gcd(a, b, c) = 1$;

-**(b)** For every two distinct $b, c \in A$, there exists an $a \in A$, distinct from b, c such that bc is divisible by a.

Find all good sets.

5 Find all surjective functions $f : \mathbb{R} \to \mathbb{R}$ such that for every $x, y \in \mathbb{R}$, we have

$$f(x + f(x) + 2f(y)) = f(2x) + f(2y).$$

6 The circle ω with center O has given. From an arbitrary point T outside of ω draw tangents TB and TC to it. K and H are on TB and TC respectively.

a) B' and C' are the second intersection point of OB and OC with ω respectively. K' and H' are on angle bisectors of $\angle BCO$ and $\angle CBO$ respectively such that $KK' \perp BC$ and $HH' \perp BC$. Prove that K, H', B' are collinear if and only if H, K', C' are collinear.

b) Consider there exist two circle in TBC such that they are tangent two each other at J and both of them are tangent to ω.and one of them is tangent to TB at K and other one is tangent to TC at H. Prove that two quadrilateral $BKJI$ and $CHJI$ are cyclic (I is incenter of triangle OBC).

Day 3

7 Find the locus of points P in an equilateral triangle ABC for which the square root of the distance of P to one of the sides is equal to the sum of the square root of the distance of P to the two other sides.

8 Let p be a prime and k a positive integer such that $k \le p$. We know that $f(x)$ is a polynomial in $\mathbb{Z}[x]$ such that for all $x \in \mathbb{Z}$ we have $p^k | f(x)$.

(a) Prove that there exist polynomials $A_0(x), \ldots, A_k(x)$ all in $\mathbb{Z}[x]$ such that

$$f(x) = \sum_{i=0}^{k} (x^p - x)^i p^{k-i} A_i(x),$$

(b) Find a counter example for each prime p and each $k > p$.

9 We have n points in the plane such that they are not all collinear. We call a line ℓ a 'good' line if we can divide those n points in two sets A, B such that the sum of the distances of all points in A to ℓ is equal to the sum of the distances of all points in B to ℓ. Prove that there exist infinitely many points in the plane such that for each of them we have at least $n+1$ good lines passing through them.

Day 4

10 Find the least value of k such that for all $a, b, c, d \in \mathbb{R}$ the inequality

$$\sqrt{(a^2+1)(b^2+1)(c^2+1)} + \sqrt{(b^2+1)(c^2+1)(d^2+1)} + \sqrt{(c^2+1)(d^2+1)(a^2+1)} + \sqrt{(d^2+$$

$$\ge 2(ab + bc + cd + da + ac + bd) - k$$

holds.

11 Let ABC be a triangle and A', B', C' be the midpoints of BC, CA, AB respectively. Let P and P' be points in plane such that $PA = P'A', PB = P'B', PC = P'C'$. Prove that all PP' pass through a fixed point.

12 Suppose that $f : \mathbb{N} \to \mathbb{N}$ is a function for which the expression $af(a)+bf(b)+2ab$ for all $a, b \in \mathbb{N}$ is always a perfect square. Prove that $f(a) = a$ for all $a \in \mathbb{N}$.

Iran Team Selection Test 2010

Day 1

1 Let $f:\mathbb{N}\to\mathbb{N}$ be a non-decreasing function and let n be an arbitrary natural number. Suppose that there are prime numbers $p_1,p_2,\dots,p_n$ and natural numbers $s_1,s_2,\dots,s_n$ such that for each $1\le i\le n$ the set $\{f(p_ir+s_i)|r=1,2,\dots\}$ is an infinite arithmetic progression. Prove that there is a natural number a such that

$$f(a+1),f(a+2),\dots,f(a+n)$$

form an arithmetic progression.

2 Find all non-decreasing functions $f:\mathbb{R}^+\cup\{0\}\to\mathbb{R}^+\cup\{0\}$ such that for each $x,y\in\mathbb{R}^+\cup\{0\}$

$$f\left(\frac{x+f(x)}{2}+y\right)=2x-f(x)+f(f(y)).$$

3 Find all two-variable polynomials $p(x,y)$ such that for each $a,b,c\in\mathbb{R}$:

$$p(ab,c^2+1)+p(bc,a^2+1)+p(ca,b^2+1)=0$$

Day 2

4 S,T are two trees without vertices of degree 2. To each edge is associated a positive number which is called length of this edge. Distance between two arbitrary vertices v,w in this graph is defined by sum of length of all edges in the path between v and w. Let f be a bijective function from leaves of S to leaves of T, such that for each two leaves u,v of S, distance of u,v in S is equal to distance of $f(u),f(v)$ in T. Prove that there is a bijective function g from vertices of S to vertices of T such that for each two vertices u,v of S, distance of u,v in S is equal to distance of $g(u)$ and $g(v)$ in T.

5 Circles W_1,W_2 intersect at P,K. XY is common tangent of two circles which is nearer to P and X is on W_1 and Y is on W_2. XP intersects W_2 for the second time in C and YP intersects W_1 in B. Let A be intersection point of

BX and CY. Prove that if Q is the second intersection point of circumcircles of ABC and AXY

$$\angle QXA = \angle QKP$$

6 Let M be an arbitrary point on side BC of triangle ABC. W is a circle which is tangent to AB and BM at T and K and is tangent to circumcircle of AMC at P. Prove that if $TK||AM$, circumcircles of APT and KPC are tangent together.

Day 3

7 Without lifting pen from paper, we draw a polygon in such away that from every two adjacent sides one of them is vertical.
In addition, while drawing the polygon all vertical sides have been drawn from up to down. Prove that this polygon has cut itself.

8 Let ABC an isosceles triangle and $BC > AB = AC$. D, M are respectively midpoints of BC, AB. X is a point such that $BX \perp AC$ and $XD||AB$. BX and AD meet at H. If P is intersection point of DX and circumcircle of AHX (other than X), prove that tangent from A to circumcircle of triangle AMP is parallel to BC.

9 Sequence of real numbers $a_0, a_1, \dots, a_{1389}$ are called concave if for each $0 < i < 1389$, $a_i \geq \frac{a_{i-1}+a_{i+1}}{2}$. Find the largest c such that for every concave sequence of non-negative real numbers:

$$\sum_{i=0}^{1389} i a_i^2 \geq c \sum_{i=0}^{1389} a_i^2$$

Day 4

10 In every 1×1 square of an $m \times n$ table we have drawn one of two diagonals. Prove that there is a path including these diagonals either from left side to the right side, or from the upper side to the lower side.

11 Let O, H be circumcenter and orthogonal center of triangle ABC. M, N are midpoints of BH and CH. BB' is diagonal of circumcircle. If $HONM$ is a cyclic quadrilateral, prove that $B'N = \frac{1}{2}AC$.

12 Prove that for each natural number m, there is a natural number N such that for each b that $2 \leq b \leq 1389$ sum of digits of N in base b is larger than m.

2009 Iran Team Selection Tes.

Iran Team Selection Test 2009

Day 1

1	Let ABC be a triangle and A' , B' and C' lie on BC , CA and AB respectively such that the incenter of $A'B'C'$ and ABC are coincide and the inradius of $A'B'C'$ is half of inradius of ABC . Prove that ABC is equilateral .	khashi70 view topic
2	Let a be a fix natural number . Prove that the set of prime divisors of $2^{2^n} + a$ for $n = 1, 2, \cdots$ is infinite	khashi70 view topic
3	Suppose that a,b,c be three positive real numbers such that $a + b + c = 3$. Prove that : $$\frac{1}{2+a^2+b^2} + \frac{1}{2+b^2+c^2} + \frac{1}{2+c^2+a^2} \le \frac{3}{4}$$	khashi70 view topic

Day 2

4	Find all polynomials f with integer coefficient such that, for every prime p and natural numbers u and v with the condition: $$p \mid uv - 1$$ we always have $p \mid f(u)f(v) - 1$.	khashi70 view topic
5	ABC is a triangle and AA' , BB' and CC' are three altitudes of this triangle . Let P be the feet of perpendicular from C' to $A'B'$, and Q is a point on $A'B'$ such that $QA = QB$. Prove that : $\angle PBQ = \angle PAQ = \angle PC'C$	khashi70 view topic
6	We have a closed path on a vertices of a $n \times n$ square which pass from each vertice exactly once . prove that we have two adjacent vertices such that if we cut the path from these points then length of each pieces is not less than quarter of total path .	khashi70 view topic

Day 3

7	Suppose three direction on the plane . We draw 11 lines in each direction . Find maximum number of the points on the plane which are on three lines .	khashi70 view topic
8	Find All Polynomials $P(x,y)$ such that for all reals x, y we have $$P(x^2, y^2) = P\left(\frac{(x+y)^2}{2}, \frac{(x-y)^2}{2}\right).$$	khashi70 view topic
9	In triangle ABC, D, E and F are the points of tangency of incircle with the center of I to BC, CA and AB respectively. Let M be the foot of the perpendicular from D to EF. P is on DM such that $DP = MP$. If H is the orthocenter of BIC, prove that PH bisects EF.	khashi70 view topic

Day 4

10	Let ABC be a triangle and $AB \ne AC$. D is a point on BC such that $BA = BD$ and B is between C and D . Let I_c be center of the circle which touches AB and the extensions of AC and BC . CI_c intersect the circumcircle of ABC again at T . If $\angle TDI_c = \frac{\angle B + \angle C}{4}$ then find $\angle A$	khashi70 view topic
11	Let n be a positive integer. Prove that	

$$3^{\frac{5^{2^n}-1}{2^{n+2}}} \equiv (-5)^{\frac{3^{2^n}-1}{2^{n+2}}} \pmod{2^{n+4}}.$$

khashi70 view topic

12 T is a subset of $1, 2, ..., n$ which has this property : for all distinct $i, j \in T$, $2j$ is not divisible by i . Prove that :
$ |T| \leq \frac {4}{9}n + {\mathrm{log}}_2n_} + 2$

khashi70 view topic

Iran Team Selection Test 2008

Day 1

1 Find all functions $f:\mathbb{R}\longrightarrow\mathbb{R}$ such that for each $x,y\in\mathbb{R}$:

$$f(xf(y))+y+f(x)=f(x+f(y))+yf(x)$$

2 Suppose that I is incenter of triangle ABC and l' is a line tangent to the incircle. Let l be another line such that intersects AB,AC,BC respectively at C',B',A'. We draw a tangent from A' to the incircle other than BC, and this line intersects with l' at A_1. B_1,C_1 are similarly defined. Prove that AA_1,BB_1,CC_1 are concurrent.

3 Suppose that T is a tree with k edges. Prove that the k-dimensional cube can be partitioned to graphs isomorphic to T.

Day 2

4 Let P_1,P_2,P_3,P_4 be points on the unit sphere. Prove that $\sum_{i\neq j}\frac{1}{|P_i-P_j|}$ takes its minimum value if and only if these four points are vertices of a regular pyramid.

5 Let $a,b,c>0$ and $ab+bc+ca=1$. Prove that:

$$\sqrt{a^3+a}+\sqrt{b^3+b}+\sqrt{c^3+c}\geq 2\sqrt{a+b+c}.$$

6 Prove that in a tournament with 799 teams, there exist 14 teams, that can be partitioned into groups in a way that all of the teams in the first group have won all of the teams in the second group.

Day 3

7 Let S be a set with n elements, and F be a family of subsets of S with 2^{n-1} elements, such that for each $A,B,C\in F$, $A\cap B\cap C$ is not empty. Prove that the intersection of all of the elements of F is not empty.

8 Find all polynomials p of one variable with integer coefficients such that if a and b are natural numbers such that $a+b$ is a perfect square, then $p(a)+p(b)$ is also a perfect square.

9 I_a is the excenter of the triangle ABC with respect to A, and AI_a intersects the circumcircle of ABC at T. Let X be a point on TI_a such that $XI_a^2 = XA.XT$. Draw a perpendicular line from X to BC so that it intersects BC in A'. Define B' and C' in the same way. Prove that AA', BB' and CC' are concurrent.

Day 4

10 In the triangle ABC, $\angle B$ is greater than $\angle C$. T is the midpoint of the arc BAC from the circumcircle of ABC and I is the incenter of ABC. E is a point such that $\angle AEI = 90^\circ$ and $AE \parallel BC$. TE intersects the circumcircle of ABC for the second time in P. If $\angle B = \angle IPB$, find the angle $\angle A$.

11 k is a given natural number. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for each $m, n \in \mathbb{N}$ the following holds:

$$f(m) + f(n) \mid (m+n)^k$$

12 In the acute-angled triangle ABC, D is the intersection of the altitude passing through A with BC and I_a is the excenter of the triangle with respect to A. K is a point on the extension of AB from B, for which $\angle AKI_a = 90^\circ + \frac{3}{4}\angle C$. I_aK intersects the extension of AD at L. Prove that DI_a bisects the angle $\angle AI_aB$ iff $AL = 2R$. (R is the circumradius of ABC)

Iran Team Selection Test 2007

Day 1

1 In triangle ABC, M is midpoint of AC, and D is a point on BC such that $DB = DM$. We know that $2BC^2 - AC^2 = AB.AC$. Prove that

$$BD.DC = \frac{AC^2.AB}{2(AB + AC)}$$

2 Let A be the largest subset of $\{1, \ldots, n\}$ such that for each $x \in A$, x divides at most one other element in A. Prove that

$$\frac{2n}{3} \leq |A| \leq \left\lceil \frac{3n}{4} \right\rceil.$$

3 Find all solutions of the following functional equation:

$$f(x^2 + y + f(y)) = 2y + f(x)^2.$$

Day 2

1 In an isosceles right-angled triangle shaped billiards table , a ball starts moving from one of the vertices adjacent to hypotenuse. When it reaches to one side then it will reflect its path. Prove that if we reach to a vertex then it is not the vertex at initial position
By Sam Nariman

2 Find all monic polynomials $f(x)$ in $\mathbb{Z}[x]$ such that $f(\mathbb{Z})$ is closed under multiplication.
By Mohsen Jamali

3 Let ω be incircle of ABC. P and Q are on AB and AC, such that PQ is parallel to BC and is tangent to ω. AB, AC touch ω at F, E. Prove that if M is midpoint of PQ, and T is intersection point of EF and BC, then TM is tangent to ω.
By Ali Khezeli

Day 3

1 Does there exist a a sequence $a_0, a_1, a_2, \ldots$ in $\mathbb{N}$, such that for each $i \neq j, (a_i, a_j) = 1$, and for each n, the polynomial $\sum_{i=0}^{n} a_i x^i$ is irreducible in $\mathbb{Z}[x]$?
By Omid Hatami

2 Suppose n lines in plane are such that no two are parallel and no three are concurrent. For each two lines their angle is a real number in $[0, \frac{\pi}{2}]$. Find the largest value of the sum of the $\binom{n}{2}$ angles between line.
By Aliakbar Daemi

3 O is a point inside triangle ABC such that $OA = OB + OC$. Suppose B', C' be midpoints of arcs $\overset{\frown}{AOC}$ and AOB. Prove that circumcircles COC' and BOB' are tangent to each other.

Day 4

1 Find all polynomials of degree 3, such that for each $x, y \geq 0$:

$$p(x+y) \geq p(x) + p(y)$$

2 Triangle ABC is isosceles ($AB = AC$). From A, we draw a line ℓ parallel to BC. P, Q are on perpendicular bisectors of AB, AC such that $PQ \perp BC$. M, N are points on ℓ such that angles $\angle APM$ and $\angle AQN$ are $\frac{\pi}{2}$. Prove that

$$\frac{1}{AM} + \frac{1}{AN} \leq \frac{2}{AB}$$

3 Let P be a point in a square whose side are mirror. A ray of light comes from P and with slope α. We know that this ray of light never arrives to a vertex. We make an infinite sequence of $0, 1$. After each contact of light ray with a horizontal side, we put 0, and after each contact with a vertical side, we put 1. For each $n \geq 1$, let B_n be set of all blocks of length n, in this sequence.
a) Prove that B_n does not depend on location of P.
b) Prove that if $\frac{\alpha}{\pi}$ is irrational, then $|B_n| = n + 1$.