A Few Elementary Properties of Polynomials

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Contents

1 Introduction

First we will define a few important terms. A **polynomial** $P(x)$ of degree n is generally of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \tag{1.1}$$

so that the **coefficient** of x^i is a_i, $0 \leq i \leq n$.

We can also write $P(x)$ in terms of its **roots**, r_i. The roots, or zeros, of a function $f(x)$ are the values of x for which $f(x) = 0$. As we will see, these are of great significance. Since the **degree** of $P(x)$, or the greatest power of x in the polynomial, is n (we write $\deg P = n$), we can prove that there are exactly n roots (not necessarily distinct). The **Fundamental Theorem of Algebra** (FTA) comes in handy, claiming that every polynomial has at least one zero. (For a proof of the FTA, see [1].) As a direct consequence of this theorem, we can write

$$P(x) = (x - r_1)P_1(x),$$

where $P_1(x)$ is a polynomial with degree $n-1$. If we apply FTA again, this time to $P_1(x)$, we get

$$P(x) = (x - r_1)(x - r_2)P_2(x),$$

where $P_2(x)$ is a polynomial with degree $n-2$. We can keep applying FTA until we get

$$P(x) = a_n(x - r_1)(x - r_2)\cdots(x - r_n), \tag{1.2}$$

which is our factored form of $P(x)$. It shows us that the roots are $r_1, r_2, \ldots, r_n$, since these are where $P(x)$ becomes zero.

A Few Introductory Problems

1. Show, by writing $P(x)$ in the two forms given in (1.1) and (1.2), that $a_0 = (-1)^n \cdot a_n(r_1 r_2 \cdots r_n)$.

2. (Canada 1988) For some integer a, the equations $1988x^2 + ax + 8891 = 0$ and $8891x^2 + ax + 1988 = 0$ share a common root. Find a.

3. (ARML[1] 1989) If $P(x)$ is a polynomial in x, and $x^{23} + 23x^{17} - 18x^{16} - 24x^{15} + 108x^{14} = (x^4 - 3x^2 - 2x + 9) \cdot P(x)$ for all values of x, compute the sum of the coefficients of $P(x)$.

[1] American Regions Math League

2 Vieta's Formulas

We define the *kth symmetric sum*, σ_k, of a set to be the sum of the elements multiplied k at a time. For example, the symmetric sums of w, x, y, and z are

$$\begin{aligned}\sigma_1 &= w + x + y + z\\ \sigma_2 &= wx + wy + wz + xy + xz + yz\\ \sigma_3 &= wxy + wxz + wyz + xyz\\ \sigma_4 &= wxyz.\end{aligned}$$

Vieta's Formulas state that

$$\sigma_k = (-1)^k \cdot \frac{a_{n-k}}{a_n}, \tag{2.1}$$

for $1 \le k \le n$. We can prove this inductively. The base case is $n = 1$, so let $f(x) = a_1 x + a_0$. Since the only one root is $-a_0/a_1$, the only possibility is $k = 1$. Plugging in $n = k = 1$ in Vieta's formulas, we get $(-1)^1 \cdot \frac{a_0}{a_1} = -a_0/a_1$, as desired.

Now the inductive step is to assume that it is true for $n = t$. If $n = t + 1$, we get

$$f(x) = a_{t+1}x^{t+1} + a_t x^t + \cdots + a_0.$$

Let $g(x)$ be a monic polynomial with degree t such that

$$f(x) = (a_{t+1})(x - r_{t+1})g(x),$$

where r_{t+1} is one of the $t + 1$ roots of $f(x) = 0$. Notice the roots of $g(x) = 0$ must be $r_1, r_2, \ldots, r_t$. Then we can write

$$g(x) = x^t - (r_1 + r_2 + \cdots + r_t)x^{t-1} + \cdots + (-1)^t(r_1 r_2 \cdots r_t).$$

In other words,

$$f(x) = (a_{t+1})(x - r_{t+1})(x^t - \sigma_1 x^{t-1} + \sigma_2 x^{t-2} + \cdots + (-1)^t \sigma_t).$$

Multiplying it out, we find that

$$f(x) = (a_{t+1})(x^{t+1} - x^t(r_1 + \cdots + r_t + r_{t+1}) + \cdots + (-1)^{t+1}(r_1 r_2 \cdots r_t r_{t+1}),$$

or

$$f(x) = a_{t+1}(x^{t+1} - \sigma_1 x^t + \cdots + (-1)^{t+1}\sigma_{t+1}),$$

which is what we wanted. □

Example 2.1. Determine the product of the roots of

$$50x^{50} + 49x^{49} + \cdots + x + 1 = 0.$$

Solution. From Vieta's Formulas, we have

$$\sigma_{50} = (-1)^{50} \cdot \frac{a_0}{a_{50}} = \boxed{\frac{1}{50}}.$$

□

Example 2.2. Find all ordered pairs (x, y, z) that satisfy

$$\begin{aligned} x + y + z &= 17, \\ xy + yz + xz &= 94, \\ xyz &= 168. \end{aligned}$$

Solution. We are given that $\sigma_1 = 17$, $\sigma_2 = 94$, and $\sigma_3 = 168$. Hence, x, y, and z are solutions to the equation $f(a) = a^3 - 17a^2 + 94a - 168 = 0$. Noticing that 7 is a factor of 168, we try $a = 7$ as a solution, and indeed $f(a)/(a-7) = a^2 - 10a + 24$, which can be factored as $(a-4)(a-6)$, so $f(a) = (a-4)(a-6)(a-7)$. Hence, all possible ordered pairs (x, y, z) are the six permutations of $(4, 6, 7)$. $\square$

Problems

4. Find all ordered triples (x, y, z) that satisfy

$$\begin{aligned} x + y - z &= 0, \\ zx - xy + yz &= 27, \\ xyz &= 54. \end{aligned}$$

5. (AIME[2] 2001) Find the sum of all roots, real and nonreal, of the equation $x^{2001} + \left(\frac{1}{2} - x\right)^{2001} = 0$, given that there are no multiple roots.

6. (USA 1984) The product of two of the four zeros of the quartic equation

$$x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$$

is -32. Find k.

7. (Canada 2001) There exists a quadratic polynomial $P(x)$ with integer coefficients such that: (a) both of its roots are positive integers, (b) the sum of its coefficients is prime, and (c) for some integer k, $P(k) = -55$. Show that one of its roots is 2, and find the other root.

[2] American Invitational Mathematics Examination

3 Tools for Finding the Roots of a Polynomial

It is common knowledge that the solutions to the quadratic polynomial $ax^2 + bx + c = 0$ are found by the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Unfortunately, the cubic and quartic formulas are so complicated that they are hardly worth memorizing, and it has even been proven that there exist no formulas for the roots of polynomials with degree greater than four. Naturally, we resort to other tactics for these polynomials.

The most simple tool we have for find the roots of a polynomial is the **Remainder Theorem**: The remainder upon dividing the polynomial $P(x)$ by $(x - a)$ is $P(a)$. The **Factor Theorem** follows directly from this: If r is a root of the polynomial $P(x)$, then the remainder upon dividing $P(x)$ by $(x - r)$ is 0. The proof of the Remainder Theorem is very simple. Let $Q(x)$ and r be the quotient and remainder, respectively, when $P(x)$ is divided by $(x - a)$. From the division algorithm, we obtain the identity

$$P(x) = Q(x)(x - a) + r.$$

Since an identity is true for all values of x, we can substitute $x = a$ above to find that $P(a) = r$, as desired.

Example 3.1. Find the remainder when $x^{2006} + x^{2005} + \cdots + x + 1$ is divided by $x + 1$.
Solution. By the Remainder Theorem, the remainder when divided by $x + 1$ is just $(-1)^{2006} + \cdots + (-1) + 1$. Since $(-1)^k = -1$ if k is odd and $(-1)^k = 1$ if k is even, the remainder is $\boxed{1}$. $\square$

Example 3.2. (The USSR Olympiad Problem Book) An unknown polynomial yields a remainder of 2 upon division by $x - 1$, and a remainder of 1 upon division by $x - 2$. What remainder is obtained if this polynomial is divided by $(x - 1)(x - 2)$?
Solution. Let this polynomial be $f(x)$. By the Remainder Theorem, $f(1) = 2$ and $f(2) = 1$. We need to find $r(x)$, where

$$f(x) = j(x)(x - 1)(x - 2) + r(x).$$

We know that $\deg r < \deg(x - 1)(x - 2) = 2$, so let $r(x) = ax + b$. Substituting $x = 1$ above, we find $f(1) = r(1) = a + b$ and $f(2) = r(2) = 2a + b$. Solving the equations

$$a + b = 2 \quad \text{and} \quad 2a + b = 1$$

simultaneously gives $a = -1$ and $b = 3$, so $r(x) = -x + 3$. $\square$

Coupled with the **Rational Root Theorem** (RRT), we can often use the Remainder Theorem to find the roots of a polynomial. RRT states that for a polynomial $P(x)$ with integer coefficients, any rational roots must be in the form p/q, where $|p|$ and $|q|$ are relatively prime, p divides a_0, and q divides a_n, where $\deg f = n$.

The proof of RRT is not difficult either. If p/q is a root of $P(x) = 0$, we must have

$$a_n \left(\frac{p}{q}\right)^n + a_{n-1} \left(\frac{p}{q}\right)^{n-1} + \cdots + a_0 = 0.$$

If we multiply through by q^n, we get

$$a_n p^n + a_{n-1} p^{n-1} q + \cdots + a_0 q^n = 0.$$

Modulo p, the left hand side becomes a_0q^n, so $a_0q^n \equiv 0 \pmod{p}$. Since we stated that p and q are relatively prime, we cannot have $q^n \equiv 0 \pmod{p}$. This means that p must divide a_0. When we consider the equation modulo q, we get $a_np^n \equiv 0 \pmod{q}$, and by the same argument, q must divide a_n, as desired.

Example 3.3. Find all roots of the equation $x^3 - 2x^2 - 5x + 6 = 0$.
Solution. Let's first try to look for rational roots. By RRT, all rational roots must be elements of the set $\{\pm 6, \pm 3, \pm 2, \pm 1\}$. How do we know which ones are roots? By the factor theorem, x^3-2x^2-5x+6 must be divisible by $x-r$, where r is a root. If we try $x = 1$, we find that $(x^3-2x^2-5x+6)/(x-1) = x^2-x-6$, so 1 is indeed a root. We can also factor $x^2 - x - 6$ to get $(x - 3)(x + 2)$, so the roots are -2, 1, and 3. □

Another tool we can use is **Descartes' Rule of Signs**, which tells us the possible numbers of positive and negative roots. The number of changes in sign (positive to negative or negative to positive) in $f(x)$ is the maximum number of positive roots and the number of sign changes in $f(-x)$ is the maximum number of negative roots. Also, the actual number of roots can only differ by a multiple of 2 from the maximum.

Example 3.4. If we have $f(x) = x^4 + 2x^3 - 25x^2 - 26x + 120$, there are 2 sign changes. Therefore, there are either 2 or 0 positive roots. Also, $f(-x) = x^4 - 2x^3 - 25x^2 + 26x + 120$ has 2 sign changes, so there are either 2 or 0 negative roots as well.

Yet another way to limit our search for roots is to use **upper and lower bounds**. If we have, for example, $f(x) = x^3 - 6x^2 + 11x - 6$, and we divide by $x - 7$, we get

$$\begin{array}{r|rrrr} 7 & 1 & -6 & 11 & -6 \\ & & 7 & 7 & 126 \\ \hline & 1 & 1 & 18 & 120 \end{array}$$

Since all the numbers in the bottom row (the coefficients of the quotient, and the remainder) are all positive, 7 is an upper bound on the roots; none of the roots can be greater than 7. Also, if we divide by $x + 1$, we get

$$\begin{array}{r|rrrr} -1 & 1 & -6 & 11 & -6 \\ & & -1 & 7 & -18 \\ \hline & 1 & -7 & 18 & -24 \end{array}$$

Since the numbers in the bottom row alternate in sign, -1 is a lower bound on the roots; none of the roots can be smaller than -1.

We can also use the **Single Bound Theorem**. Consider the monic polynomial $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, where a_i is real for $0 \le i \le n-1$. Let $M_1 = 1 + \max\{|a_0|, |a_1|, \ldots, |a_{n-1}|\}$ and $M_2 = \max\{1, |a_0| + |a_1| + \cdots + |a_{n-1}|\}$. Also, let $M = \min\{M_1, M_2\}$. The Single Bound Theorem states that all roots of $P(x)$ are between $-M$ and M. A proof is given in [3].

Another important fact is that all roots in the form $a + b\sqrt{c}$ come in "pairs". If $a + b\sqrt{c}$ is a root of a polynomial with rational coefficients, then $a - b\sqrt{c}$ must also be a root. This is also true if $c = -1$, in which case we have $a + bi$ and $a - bi$.

Example 3.5. A quartic polynomial with rational coefficients has roots $1 + \sqrt{5}$ and $\frac{1}{2} - i\frac{\sqrt{3}}{2}$. Find all other roots.
Solution. The other two roots must be $1 - \sqrt{5}$ and $\frac{1}{2} + i\frac{\sqrt{3}}{2}$. □

Problems

8. Find all roots of the equation $x^3 - 10x^2 + 23x - 14 = 0$.

9. Find all roots of the equation $x^3 - 9x^2 + 23x - 15 = 0$.

10. Show that all roots of the equation $x^4 - 14x^3 + 64x^2 - 114x + 63 = 0$ lie in the interval $(0, 14)$.

4 Transforming Polynomials

Example 4.1. If we have the polynomial $f(x)$ that has roots $r_1, r_2, \ldots, r_n$, how can we find the polynomial $g(x)$ that has roots $1/r_1, 1/r_2, \ldots, 1/r_n$? We know that

$$f\left(\frac{1}{1/r_1}\right) = 0,$$

so we find $g(x) = f(1/x) = a_n(1/x)^n + a_{n-1}(1/x)^{n-1} + \cdots + a_1(1/x) + a_0$. Multiplying by x^n to get a polynomial, we find

$$g(x) = a_n + a_{n-1}x + \cdots + a_1x^{n-1} + a_0x^n,$$

or

$$g(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n.$$

Notice that the coefficients of $g(x)$ are the same as the coefficients of $f(x)$, but reversed. □

Example 4.2. Consider again the polynomial $f(x)$. We can also find the polynomial $g(x)$ with roots m times the roots of $f(x)$ by substituting x/m. We have

$$f(x/m) = a_n(x/m)^n + a_{n-1}(x/m)^{n-1} + \cdots + a_1(x/m) + a_0.$$

Multiplying by m^n we find

$$g(x) = a_nx^n + a_{n-1}mx^{n-1} + \cdots + a_1m^{n-1}x + a_0m^n.$$

□

Example 4.3. Find the polynomial with roots that exceed the roots of $f(x) = 3x^3 - 14x^2 + x + 62 = 0$ by one.
Solution. Let $g(x) = f(x-1)$. If the roots of $f(x) = 0$ are r_i, $1 \le i \le 3$, then $g(r_i + 1) = f(r_i) = 0$, so $g(x)$ is the polynomial we need to find. A quick method to determine $g(x)$ is to keep dividing (using synthetic division) $f(x)$ by $x+1$ until we are left with a single number as the quotient. The remainder of the ith division will be the coefficient of x^{i-1}, $1 \le i \le 4$. This method is easily generalized to any polynomial of any degree. □

Problems

11. (Mu Alpha Theta 1991) If the roots of $f(x)$ as defined in Example 4.3 are a, b, c, determine

$$\frac{1}{a+3} + \frac{1}{b+3} + \frac{1}{c+3}.$$

12. Alice and Bob each have different cubic polynomials with leading coefficients equal to 1 (let them be $a(x)$ and $b(x)$, respectively). They find all roots, real and nonreal, of their respective polynomials and Alice remarks, "My polynomial has roots that are half the roots of your polynomial." Given that $a(x) = x^3 + 3x^2 + 3x + 7$, find $b(x)$.

5 Newton's Identities

Example 5.1. Let the roots of $x^3+2x^2+3x+4=0$ be a, b, c. Find $a^2+b^2+c^2$.
Solution. Recall that

$$(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca) \iff a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+bc+ca).$$

By Vieta's formulas, $a+b+c=-2$ and $ab+bc+ca=3$ so that $a^2+b^2+c^2 = \boxed{-2}$. $\square$

Newton's identities are nice generalizations to this principle. Let $s_n = r_1^n + r_2^n + \cdots + r_n^n$, where r_i are the roots of a polynomial $P(x)$, as defined in (1.2), and recall that a_i are the coefficients of x^i, as defined in (1.1). Then **Newton's identities**, or Newton's sums, are

$$\begin{aligned} a_n s_1 + a_{n-1} &= 0 \\ a_n s_2 + a_{n-1}s_1 + 2a_{n-2} &= 0 \\ a_n s_3 + a_{n-1}s_2 + a_{n-2}s_1 + 3a_{n-3} &= 0, \end{aligned}$$

and so on. More generally,

$$a_n s_d + a_{n-1}s_{d-1} + a_{n-2}s_{d-2} + \cdots + a_{n-d+1}s_1 + da_{n-d} = 0. \tag{5.1}$$

There are many proofs of Newton's identities. An inductive proof is given in [2].

We can use these in problems in which we have polynomials and we need the sum of the kth powers, but we can also apply Newton's sums in problems that have systems of equations, using the solutions to the system as the roots of a polynomial. This is illustrated well in problem 16.

Example 5.2. Find the sum of the fourth powers of the roots of the equation $7x^3-21x^2+9x+2=0$.
Solution. We need s_4. By Newton's identities,

$$\begin{aligned} 7s_1 - 21 &= 0, \\ s_1 &= 3; \\ 7s_2 - 21\,(3) + 2(9) &= 0, \\ s_2 &= \frac{45}{7}; \\ 7s_3 - 21\left(\frac{45}{7}\right) + 9\,(3) + 3(2) &= 0, \\ s_3 &= \frac{102}{7}; \\ 7s_4 - 21\left(\frac{102}{7}\right) + 9\left(\frac{45}{7}\right) + 2\,(3) + 4(0) &= 0, \end{aligned}$$

so $s_4 = \boxed{\dfrac{1695}{49}}$. $\square$

Problems

13. (AIME 2003) The roots of $x^4 - x^3 - x^2 - 1 = 0$ are a, b, c, and d. Find $p(a)+p(b)+p(c)+p(d)$, where $p(x) = x^6 - x^5 - x^3 - x^2 - x$.

14. (APMO[3] 2003) If the roots of the polynomial $P(x) = x^8 - 4x^7 + 7x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ are all positive real numbers, find all possible values of a_0.

15 (⋆). Find all solutions, real or complex, to the system of equations

$$\begin{aligned} x + y + z &= 3 \\ x^2 + y^2 + z^2 &= 3 \\ x^3 + y^3 + z^3 &= 3 \end{aligned}$$

[3] Asian Pacific Mathematical Olympiad

6 More Problems!

Here is a collection of difficult problems related to polynomials from national math olympiads around the world.

16. (AIME 1995) Find b if the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ has rational coefficients and 4 complex roots, two with sum $3 + 4i$ and the other two with product $13 + i$.

17. (Putnam 1939) If α, β, γ are the roots of $x^2 + ax^2 + bx + c = 0$, find the polynomial with roots $\alpha^3, \beta^3, \gamma^3$.

18. (Canada 1971) Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where the coefficients a_i are integers. If $p(0)$ and $p(1)$ are both odd, show that $p(x)$ has no integral roots.

19. (Canada 1996) If α, β, γ are the roots of $x^3 - x - 1 = 0$, find

$$\frac{1-\alpha}{1+\alpha} + \frac{1-\beta}{1+\beta} + \frac{1-\gamma}{1+\gamma}.$$

20. (IMO Shortlist 1988) Find the number of odd coefficients in the expansion of $(x^2 + x + 1)^n$.

21. (India 1995) Prove that there are infinitely many ordered pairs (a, b) of relatively prime nonzero integers such that the roots of the equations $x^2 + ax + b = 0$ and $x^2 + 2ax + b = 0$ are integers.

22. (USSR Olympiad Problem Book) Prove that if α and β are the roots of the equation $x^2 + px + 1 = 0$ and γ and δ are the roots of the equation $x^2 + qx + 1 = 0$, then

$$(\alpha - \gamma)(\beta - \gamma)(\alpha + \delta)(\beta + \delta) = q^2 - p^2.$$

23. (USA 1975) If $P(x)$ is a polynomial of degree n such that $P(n) = \dfrac{n}{n+1}$ for $n \in \{0, 1, 2, \ldots, n\}$, find $P(n + 1)$.

24. (USA 1977) Show that the product of the two real roots of the equation $x^4 + x^3 - 1 = 0$ is a root of the equation $x^6 + x^4 + x^3 - x^2 - 1 = 0$.

25. (IMO 1993) Let $f(x) = x^n + 5x^{n-1} + 3$, where n is an integer greater than 1. Prove that $f(x)$ cannot be expressed as the product of two polynomials with integer coefficients and degree greater than zero.

26. (Canada 1976) Let $P(x, y)$ be a symmetric polynomial that is divisible by $(x - y)$. Prove that $(x - y)^2$ divides $P(x, y)$.

27. (IMO Shortlist 1982) Let $f(x)$ be a monic polynomial with integer coefficients and degree 3. If the product of two of its roots is the third root, show that $p(1) + p(-1) - 2(1 + p(0))$ divides $2p(-1)$.

28. (Canada 1975) Let k be a positive integer. Find all polynomials with real coefficients which satisfy the equation

$$P(P(x)) = [P(x)]^k.$$

29. (IMO Shortlist 1982) A real number a exists such that the four distinct roots of $16x^4 - ax^3 + (2a+17)x^2 - ax + 16 = 0$ form a geometric progression. Find a.

30. (APMO 2001) Find all polynomials $f(x)$ with real coefficients such that r is rational if and only if $f(r)$ is rational for all real numbers r.

31. (IMO 1976) Let $P_1(x) = x^2 - 2$, and $P_{i+1} = P_1(P_i(x))$ for $i \in \{1, 2, 3, \ldots\}$. Prove that $P_n(x) = x$ has only distinct real roots for all n.

32. (IMO Shortlist 1981) If $P(x)$ is a polynomial with degree n such that $P(i) = \dfrac{(n+1-i)!i!}{(n+1)!}$, where $x \in \{0, 1, \ldots, n\}$, find $P(n+1)$.

33. (Canada 1970) Let $p(x)$ be a polynomial with integer coefficients such that there exist four distinct integers j such that $p(j) = 5$. Prove that there exists no integer k such that $p(k) = 8$.

34. (India 2003) Prove that the polynomial $8x^4 - 16x^3 + 16x^2 - 8x + r = 0$ has at least one real root for all real numbers r, and find the sum of the nonreal roots.

35. (IMO 1973) Find the minimum possible value of $a^2 + b^2$ if a and b are real numbers such that $x^4 + ax^3 + bx^2 + ax + 1 = 0$ has at least one real root.

7 Solutions to Problems

1. The result is clear if we substitute $x = 0$ into (1.1) and (1.2). $\square$

2. If we substract the equations from each other to eliminate the ax terms, we have $(8891-1988)x^2 - (8891 - 1988) = 0$ so that $x = \pm 1$. Substituting $x = 1$ into either of the equations we get $(8891 + 1988) + a = 0$, so $a = \boxed{-10879}$ causes both equations to have a common root. $\square$

3. It is important to realize that the sum of the coefficients of any polynomial is the polynomial evaluated at $x = 1$. In this case, we need $P(1)$. Plugging in $x = 1$ to the equation, we find that $P(1) = 90/5 = 18$. $\square$

4. Notice that if z wasn't negative, we would have symmetric expressions. Hence, let $w = -z$, so that

$$\begin{aligned} w + x + y &= 0, \\ wx + wy + xy &= -27, \\ wxy &= -54. \end{aligned}$$

Thus $\sigma_1 = 0$, $\sigma_2 = -27$, and $\sigma_3 = -54$, so w, x, and y are solutions to the equation $f(a) = a^3 - 27a + 54 = 0$. We can factor this to get $f(a) = (a + 6)(a - 3)^2$, so our solutions for (w, x, y) are $(-6, 3, 3)$, $(3, -6, 3)$, and $(3, 3, -6)$. Finally, the ordered triples (x, y, z) are $(3, 3, 6)$, $(-6, 3, -3)$, and $(3, -6, -3)$. $\square$

5. We can use the **Binomial Theorem** to expand $x^{2001} + \left(\frac{1}{2} - x\right)^{2001}$:

$$x^{2001} + \left[\left(\frac{1}{2}\right)^{2001} + 2001\left(\frac{1}{2}\right)^{2000}(-x) + \cdots + 2001\left(\frac{1}{2}\right)(-x)^{2000} + (-x)^{2001}\right].$$

Notice that x^{2001} disappears so that this polynomial has degree 2000. Thus we need $-\dfrac{a_{1999}}{a_{2000}}$ where a_i is the coefficient of x^i.

$$-\frac{a_{1999}}{a_{2000}} = -\frac{-\binom{2001}{2}\left(\frac{1}{2}\right)^2}{\binom{2001}{1}\left(\frac{1}{2}\right)} = \frac{\frac{2001\cdot 2000}{2}\cdot\frac{1}{2}}{2001},$$

and our answer is $\boxed{500}$. $\square$

6. Let the roots be r, s, t, and u. Assume without loss of generality that $rs = -32$. From Vieta's formulas we know that

$$\begin{aligned} r + s + t + u &= 18 && (\clubsuit) \\ rs + rt + ru + st + su + tu &= k && (\spadesuit) \\ rst + rsu + rtu + stu &= -200 && (\heartsuit) \\ rstu &= -1984 && (\diamondsuit) \end{aligned}$$

From ($\diamondsuit$) we find $tu = 62$. Factoring and substituting this into ($\heartsuit$), we get

$$rs(t+u) + tu(r+s) = -200,$$

or

$$-32(t+u) + 62(r+s) = -200.$$

Let $t+u = a$ and $r+s = b$. Now we have the system

$$\begin{aligned} -32a + 62b &= -200 \\ a + b &= 18 \end{aligned}$$

Solving, we get $a = 14$, $b = 4$. Now we substitute into ($\spadesuit$) to get

$$k = rs + (r+s)(t+u) + tu = -32 + (14)(4) + 62,$$

so $k = \boxed{86}$. $\square$

7. Let $P(x) = ax^2+bx+c = a(x-r_1)(x-r_2)$, where r_1, r_2 are the roots, and let $p = a+b+c$. Assume without loss of generality that $r_1 \le r_2$. Since $P(1) = a+b+c = p$, and $P(1) = a(1-r_1)(1-r_2)$, it clear that $a \in \{1, -1, p, -p\}$. However if $a = p$ then $(1-r_1)(1-r_2) = 1$ so $r_1 = r_2 = 0$ or $r_1 = r_2 = 2$, which is clearly impossible (because of condition (c)). Also if $a = -p$ then $(1-r_1)(1-r_2) = -1$ so $r_1 = 0$ and $r_2 = 2$, which is also a contradiction.

Now because $P(k) = a(k-r_1)(k-r_2) = -5 \cdot 11$, we must have one of the following:

$$\begin{cases} a = 1 \\ k - r_1 = 55 \\ k - r_2 = 1 \end{cases} \quad \text{or} \quad \begin{cases} a = 1 \\ k - r_1 = 11 \\ k - r_2 = 5 \end{cases}.$$

In the first case we get $r_2 = r_1 + 54$, $b = -2r_1 - 54$, and $c = r_1(r_1+54)$. Thus $r_1^2 + 52r_1 - (53+p) = 0$ so

$$r_1 = \frac{-52 \pm \sqrt{52^2 + 4(53+p)}}{2} = -26 \pm \sqrt{26^2 + 53 + p} = -26 \pm \sqrt{27^2 + p}.$$

Let $h^2 = 27^2 + p \iff p = (h+27)(h-27)$. Since p is prime $h - 27 = 1 \Rightarrow h = 28$, but then $p = 55$ which is not a prime. Therefore the first case doesn't work.

In the second case we find that $r_2 = r_1 + 6$ so $b = -2r_1 - 6$ and $c = r_1(r_1+6)$. Thus $p = 10(2r_1+6) + r_1^2 + 6r_1$ or

$$r_1^2 + 4r_1 - (5+p) = 0 \iff r = -2 \pm \sqrt{3^2 + p}.$$

Let $i^2 = 3^2 + p \iff p = (i+3)(i-3)$. Since p is prime we must have $i = 4$ and then $p = 7$. This works, and $r_1 = 2$, $r_2 = 8$.

In summary, the only polynomial $P(x)$ that satisfies all three conditions is $(x-2)(x-8)$ with roots 2 and 8. $\square$

8. By RRT, all roots must be factors of ± 14. We can check that $(x-7)$ is a factor, so we have $(x-7)(x^2-3x+2) = (x-7)(x-1)(x-2) = 0$, so the roots are $x = 1, 2, 7$. $\square$

9. It is not difficult to see that the factors are $(x-1),(x-3),(x-5)$, so the roots are $x = 1, 3, 5$. $\square$

10. Division by x yields

$$x^3 - 14x^2 + 64x - 114 + \frac{63}{x},$$

and the coefficients alternate in sign, so all roots are greater than 0. Division by $(x-14)$ yields

$$x^3 + 64x + 782 + \frac{11011}{x-14},$$

and the coefficients are positive so all roots are less than 14. $\square$

11 (Method 1). The polynomial with roots that exceed the roots of $f(x)$ by 3 is $g(x) := f(x-3)$. Using the method of Example 5.3 (divide $f(x)$ by $x+3$) we find

$$g(x) = 3x^3 - 41x^2 + 166x - 148.$$

Let $h(x)$ be the polynomial with roots that are reciprocals of those of $g(x)$. By the method of Example 5.1 we find

$$h(x) = -148x^3 + 166x^2 - 41x + 3.$$

The sum of its roots is

$$(-1)^1 \cdot \frac{166}{-148} = \boxed{\frac{83}{74}}.$$

$\square$

11 (Method 2). Let's write

$$\frac{1}{a+3} + \frac{1}{b+3} + \frac{1}{c+3}$$

with a common denominator. We get

$$\frac{(b+3)(c+3)+(a+3)(c+3)+(a+3)(b+3)}{(a+3)(b+3)(c+3)},$$

or

$$\frac{(ab+bc+ac)+6(a+b+c)+27}{abc+3(ab+ac+bc)+9(a+b+c)+27}.$$

We recognize this as

$$\frac{\sigma_2 + 6\sigma_1 + 27}{\sigma_3 + 3\sigma_2 + 9\sigma_1 + 27},$$

which we compute to be $\boxed{\frac{83}{74}}$, as before. $\square$

12. By Example 4.2, $b(x) = x^3 + 6x^2 + 12x + 56 = 0$. $\square$

13. Let $z = p(a) + p(b) + p(c) + p(d)$. Then

$$\begin{aligned} z &= (a^6 - a^5 - a^3 - a^2 - a) + (b^6 - b^5 - b^3 - b^2 - b) + (c^6 - c^5 - c^3 - c^2 - c) \\ &\quad + (d^6 - d^5 - d^3 - d^2 - d) \\ &= (a^6 + b^6 + c^6 + d^6) - (a^5 + b^5 + c^5 + d^5) + (a^3 + b^3 + c^3 + d^3) \\ &\quad - (a^2 + b^2 + c^2 + d^2) - (a + b + c + d) \\ &= s_6 - s_5 - s_3 - s_2 - s_1 \end{aligned}$$

Using Newton's sums with $d = 6$ on the roots of $x^4 - x^3 - x^2 - 1$, we get

$$\begin{aligned} a_4 s_6 + a_3 s_5 + a_2 s_4 + a_1 s_3 + a_0 s_2 + a_{-1} s_1 + 6a_{-2} &= 0 \\ s_6 - s_5 - s_4 + 0 s_3 - s_2 + 0 s_1 + 6 \cdot 0 &= 0 \\ s_6 - s_5 - s_4 - s_2 &= 0 \end{aligned}$$

Adding $s_4 - s_3 - s_1$ to both sides, we get

$$s_6 - s_5 - s_3 - s_2 - s_1 = s_4 - s_3 - s_1,$$

so $z = s_4 - s_3 - s_1$. We can use Newton's sums again to find that $s_1 = 1$, $s_3 = 4$, and $s_4 = 11$, so our answer is $11 - 4 - 1 = \boxed{6}$. □

14. Let the roots be $r_1, r_2, \ldots, r_8$. By Vieta's formulas, we have $\sigma_1 = r_1 + r_2 + \cdots + r_8 = 4$ and $\sigma_2 = 7$. The value of σ_2 does not seem particularly helpful in the form in which it is given, so we recall that Newton's identities give

$$a_8 s_2 + a_7 s_1 + 2a_6 = 0, \quad s_2 + (-4)(4) + 2(7) = 0,$$

so $s_2 = r_1^2 + r_2^2 + \cdots + r_8^2 = 2$. It seems that we have run into another dead end here, unless we recall the well-known Cauchy's Inequality, which states that, for real numbers $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$, we have

$$(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)^2 \le (x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n^2),$$

with equality if and only if $x_1/y_1 = x_2/y_2 = \cdots = x_8/y_8$. In this case, we put $x_1 = x_2 = \cdots = x_8 = 1$ and $y_i = r_i$ to get

$$(r_1 + r_2 + \cdots + r_n)^2 \le 8(r_1^2 + r_2^2 + \cdots + r_8^2).$$

Notice that we have equality, so we must have $r_1 = r_2 = \cdots = r_8 = 1/2$, and thus

$$a_0 = r_1 r_2 \cdots r_8 = \boxed{\frac{1}{256}}.$$

□

15. Assume without loss of generality that x, y, and z are the roots of a cubic polynomial with leading coefficient 1. Since $\sigma_1 = 3$, we find $a_2 = -3$. Hence, we know that the polynomial has the form

$$f(r) = r^3 - 3r^2 + a_1 r + a_0.$$

By Newton's sums, we have

$$\begin{aligned} a_3 s_2 + a_2 s_1 + 2a_1 &= 0, \\ (1)(3) + (-3)(3) + 2a_1 &= 0, \\ a_1 &= 3; \\ a_3 s_3 + a_2 s_2 + a_1 s_1 + 3a_0 &= 0, \\ (1)(3) + (-3)(3) + (3)(3) + 3a_0 &= 0, \\ a_0 &= -1. \end{aligned}$$

Thus

$$f(r) = r^3 - 3r^2 + 3r - 1 = (r-1)^3,$$

so the only solution is $\boxed{x = y = z = 1}$. $\square$

References

[1] Heinrich Dörrie. *100 Great Problems of Elementary Mathematics*. Dover, 1965.

[2] Z. Reichstein. An inductive proof of Newton's identities.

[3] John Kennedy. Some Polynomial Theorems.

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