IMO 1968

Moscow, USSR

Day 1

1. Find all triangles whose side lengths are consecutive integers, and one of whose angles is twice another.

2. Find all natural numbers n the product of whose decimal digits is $n^2 - 10n - 22$.

3. Let a, b, c be real numbers with a non-zero. It is known that the real numbers $x_1, x_2, \ldots, x_n$ satisfy the n equations:
$$\begin{aligned} ax_1^2 + bx_1 + c &= x_2 \\ ax_2^2 + bx_2 + c &= x_3 \\ \cdots \quad \cdots \quad &\cdots \quad \cdots \\ ax_n^2 + bx_n + c &= x_1 \end{aligned}$$
Prove that the system has **zero**, <u>one</u> or *more than one* real solutions if $(b-1)^2 - 4ac$ is **negative**, equal to <u>zero</u> or *positive* respectively.

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Day 2

$\boxed{4}$ Prove that every tetrahedron has a vertex whose three edges have the right lengths to form a triangle.

$\boxed{5}$ Let f be a real-valued function defined for all real numbers, such that for some $a > 0$ we have

$$f(x+a) = \frac{1}{2} + \sqrt{f(x) - f(x)^2}$$

for all x. Prove that f is periodic, and give an example of such a non-constant f for $a = 1$.

$\boxed{6}$ Let n be a natural number. Prove that

$$\left\lfloor \frac{n+2^0}{2^1} \right\rfloor + \left\lfloor \frac{n+2^1}{2^2} \right\rfloor + \cdots + \left\lfloor \frac{n+2^{n-1}}{2^n} \right\rfloor = n.$$

[hide="Remark"]For any real number x, the number $\lfloor x \rfloor$ represents the largest integer smaller or equal with x.