APMO 2008

1 Let ABC be a triangle with $\angle A < 60^\circ$. Let X and Y be the points on the sides AB and AC, respectively, such that $CA + AX = CB + BX$ and $BA + AY = BC + CY$. Let P be the point in the plane such that the lines PX and PY are perpendicular to AB and AC, respectively. Prove that $\angle BPC < 120^\circ$.

2 Students in a class form groups each of which contains exactly three members such that any two distinct groups have at most one member in common. Prove that, when the class size is 46, there is a set of 10 students in which no group is properly contained.

3 Let Γ be the circumcircle of a triangle ABC. A circle passing through points A and C meets the sides BC and BA at D and E, respectively. The lines AD and CE meet Γ again at G and H, respectively. The tangent lines of Γ at A and C meet the line DE at L and M, respectively. Prove that the lines LH and MG meet at Γ.

4 Consider the function $f : \mathbb{N}_0 \to \mathbb{N}_0$, where $\mathbb{N}_0$ is the set of all non-negative integers, defined by the following conditions :

(i) $f(0) = 0$; (ii) $f(2n) = 2f(n)$ and (iii) $f(2n + 1) = n + 2f(n)$ for all $n \geq 0$.

(a) Determine the three sets $L = \{n|f(n) < f(n+1)\}$, $E = \{n|f(n) = f(n+1)\}$, and $G = \{n|f(n) > f(n + 1)\}$. (b) For each $k \geq 0$, find a formula for $a_k = \max\{f(n) : 0 \leq n \leq 2^k\}$ in terms of k.

5 Let a, b, c be integers satisfying $0 < a < c - 1$ and $1 < b < c$. For each k, $0 \leq k \leq a$, Let $r_k, 0 \leq r_k < c$
be the remainder of kb when divided by c. Prove that the two sets $\{r_0, r_1, r_2, \cdots , r_a\}$ and $\{0, 1, 2, \cdots , a\}$ are different.