High School Olympiads

Hong Kong TST2 2016 P2

YanYau

#1 Oct 25, 2015, 3:44 pm

Let Γ be a circle and AB be a diameter. Let l be a line outside the circle, and is perpendicular to AB. Let X, Y be two points on l. If X', Y' are two points on l such that AX, BX' intersect on Γ and such that AY, BY' intersect on Γ. Prove that the circumcircles of triangles AXY and $AX'Y'$ intersect at a point on Γ other than A, or the three circles are tangent at A.

Luis González

#2 Oct 25, 2015, 11:25 pm • 3

Let $U \equiv AX \cap BX', V \equiv AY \cap BY'$ and let AX', AY' cut Γ again at $N, M,$ respectively. Clearly, A is center of inversion taking Γ into l and this carries $\odot(AXY)$ and $\odot(AX'Y')$ into UV and MN, respectively. Thus, it's enough to show that l, MN, UV concur.

From the inversion with center A, it follows that M, N, X', Y' are concyclic and since B is also center of inversion taking Γ into l, then U, V, X', Y' are concyclic. Thus UV, MN, l are pairwise radical axes of $\Gamma, \odot(MNX'Y'), \odot(UVX'Y')$ concurring at their radical center, as desired.

YanYau

#3 Oct 27, 2015, 2:19 pm

The solution I wrote during the test was along the lines of this, but in more detail:

Let $AX \cap BX' = X_0$, $AY \cap BY' = Y_0$, and $AB \cap l = I$

Notice that YY_0BI and XX_0BI are cyclic, then by PoP/Radical Lemma, XX_0YY_0 are cyclic.

Similarly, we can prove that $X'Y'X_0Y_0$ is cyclic.

Consider the radical center of Γ, XYX_0Y_0, and AXY. The radical center will be the intersection of XY and X_0Y_0, Let this point be C. Then the radical axis of Γ and AXY is AC.

Similarly, consider the radical center of Γ, $X'Y'X_0Y_0$, and $AX'Y'$, the radical center is the intersection of $X'Y'$ and X_0Y_0, which is again C. So the radical axis of Γ and $AX'Y'$ is also AC

Γ, AXY and $AX'Y'$ all share the same radical axis. Therefore they must all intersect on another point on Γ other than A.

The 3 circles are tangent when X_0Y_0 and l don't intersect. Which is achieved when X, Y are reflections across AB.

High School Olympiads

concyclic points

Locked

henderson

#1 Dec 1, 2015, 10:48 pm

Let ABC be a triangle and let X,Y,Z be the reflections of A,B,C in the opposite sides. Let X_b, X_c be the orthogonal projections of X on AC, AB, Y_c, Y_a be the orthogonal projections of X on BA, BC and Z_a, Z_b be the orthogonal projections of X on CB, CA, respectively. Prove that $X_b, X_c, Y_c, Y_a, Z_a, Z_b$ are concyclic.

Luis González

#2 Dec 1, 2015, 10:52 pm

Already discussed at http://www.artofproblemsolving.com/community/c6h1080693.

High School Olympiads

Six concyclic points

jayme

#1 Apr 22, 2015, 7:59 pm

Dear Mathlinkers,

1. ABC an acute triangle
2. XYZ the reflection triangle of ABC
3. Xb, Xc the orthogonal projections of X on AC, AB
4. Yc, Ya the orthogonal projections of Y on AB, BC
5. Za, Zb the orthogonal projections of Z on BC, AC.

Prove : Xb, Xc, Yc, Ya, Za, Zb are concyclic.

Sincerely
Jean-Louis

A-B-C

#2 Apr 22, 2015, 10:31 pm

Let D,E,F be midpoints of AX,BY,CZ
M,N are orthogonal projections of E,F on AB,AC
=> MN//BC
M,N are also midpoint of BYc and CZb
=> ZbYc//BC
XbXc//EF
E,F,B,C are concyclic
=> Zb,Xb,Yc,Xc are concyclic
Let the circle passes through these point be (Oa)
Similarly we have (Ob) and (Oc)
If these 6 points are not concyclic:
BC is radical axis of (Ob) and (Oc)
CA is radical axis of (Oc) and (Oa)
AB is radical axis of (Oa) and (Ob)
=> BC,CA,AB are concurrent=> So these points must be concyclic

Luis González

#3 Apr 22, 2015, 11:37 pm

Let D,E,F be the orthogonal projections of A,B,C on $BC,CA,AB.U,V$ are the orthogonal projections of E,F on AB,AC (midpoints of BY_c and CZ_b) $\Longrightarrow Y_cZ_b \parallel UV \parallel BC$ and likewise $Z_aX_c \parallel CA$ and $X_bY_a \parallel AB.$ Since $X_cX_b,$ Y_aY_c,Z_bZ_a are antiparallels to $BC,CA,AB,$ then it follows that $X_cX_bY_aY_cZ_bZ_a$ is a Tucker hexagon $\Longrightarrow X_b,X_c,Y_c,Y_a,$ Z_a,Z_b lie on a same circle Ω (a Tucker circle).

Luis González

#4 Apr 23, 2015, 12:29 am

In addition, the center of this circle is $X_{52};$ the orthocenter of the orthic triangle of $\triangle ABC.$

Let M be the midpoint of BC (midpoint of the 9-point circle arc EDF) and let J be the center of $\odot(EFZ_bY_c).$ Then $\angle EJF=2 \cdot \angle EY_cF=2 \cdot \angle EBF=\angle EMF \Longrightarrow EJFM$ is a rhombus $\Longrightarrow J$ is reflection of M on $EF.$ Thus, the perpendicular bisector of Y_cZ_b is the parallel τ_A from J to $DA \Longrightarrow \tau_A$ coincides with the Steiner line of M WRT $\triangle DEF,$ passing through the orthocenter X_{52} of $\triangle DEF.$ Similarly the perpendicular bisectors of Z_aX_c and X_bY_a pass through X_{52} $\Longrightarrow X_{52}$ is the center of Ω.

$\Longrightarrow X_{52}$ is the center of Ω.

TelvCohl
#5 Apr 23, 2015, 4:19 am

> *Luis González wrote:*
> In addition, the center of this circle is X_{52}; the orthocenter of the orthic triangle of $\triangle ABC$.

Let T be the center of $\odot(X_bX_cY_cY_aZ_aZ_b)$.
Let $\triangle H_aH_bH_c$ be the orthic triangle of $\triangle ABC$.

Since H_a lie on the perpendicular bisector of AX_b and AX_c ,
so H_a is the circumcenter of $\triangle AX_bX_c \Longrightarrow TH_a$ is the perpendicular bisector of X_bX_c ,
hence combine with $X_bX_c \parallel H_bH_c$ (anti-parallel to BC WRT $\angle A$) $\Longrightarrow TH_a \perp H_bH_c$.
Similarly we can prove $TH_b \perp H_cH_a$ and $TH_c \perp H_aH_b \Longrightarrow T$ is the orthocenter of $\triangle H_aH_bH_c$. i.e. T is X_{52} of $\triangle ABC$

Done

jayme
#6 Apr 23, 2015, 12:39 pm

Dear Mathlinkers,

Nice idea to think to a Tucker circle…

An outline of my proof

1. XbYa // AB
2. there is a circle passing through Xb, Ya and the two points of intersection of the circle with diameter AB
with the circles centered at D passing through A, centered at E passing through B.
This can be proved by a generalization of the Reim theorem.
3. We conclude withthe five circle theorem….

Sincerely
Jean-Louis

A-B-C
#9 Apr 23, 2015, 8:06 pm

I have a simple generaliztion of this problem:
D, E, F lie on AX, BY, CZ, respectively, such that:
$\frac{\overline{AD}}{\overline{AX}} = \frac{\overline{BE}}{\overline{BY}} = \frac{\overline{CF}}{\overline{CZ}} = k$
D_b, D_c are orthogonal projections of D on AC, AB
Similarly we have E_c, E_a, F_a, F_b
Then $D_b, D_c, E_c, E_a, F_a, F_b$ are concyclic
When k changes, locus of the center is the Euler line of orthic triangle.

Luis González
#11 Apr 23, 2015, 8:26 pm

> *A-B-C wrote:*
> I have a simple generaliztion of this problem:
> D, E, F lie on AX, BY, CZ, respectively, such that:
> $\frac{\overline{AD}}{\overline{AX}} = \frac{\overline{BE}}{\overline{BY}} = \frac{\overline{CF}}{\overline{CZ}} = k$
> D_b, D_c are orthogonal projections of D on AC, AB
> Similarly we have E_c, E_a, F_a, F_b
> Then $D_b, D_c, E_c, E_a, F_a, F_b$ are concyclic
> When k changes, locus of the center is the Euler line of orthic triangle.

As before, $D_cD_bE_aE_cF_bF_a$ is a Tucker hexagon, thus these points lie on a Tucker circle and it's known that the center of any Tucker circle of $\triangle ABC$ lies on its Brocard axis, which is not, in general the Euler line of its orthic triangle.

High School Olympiads

Nice, but easy

Reply

sunken rock

#1 Dec 1, 2015, 5:35 am • 2

The incenter of $\triangle ABC$ touches its sides at D, E, F respectively, and BE, CF intersect the incircle second time at M, N respectively. Extend $(MD$ to P so that $DP = 2MD$.
Prove that DE is tangent to the circle $\odot(DPN)$.

Best regards,
sunken rock

Luis González

#2 Dec 1, 2015, 8:06 am • 2

Let MN cut BC, CA at X, Y, resp. We have $\frac{YM}{YN} = \frac{EM^2}{EN^2}$ and $\frac{XN}{XM} = \frac{DN^2}{DM^2} \Longrightarrow \left[\frac{EM \cdot DN}{EN \cdot DM}\right]^2 = \frac{YM}{YN} \cdot \frac{XN}{XM} = (Y, M, N, X)$. This cross ratio equals that of an equilateral $\triangle ABC$ (as there is a homology sending $BE \cap CF$ to the center of the circle image of $\odot(DEF)$), thus clearly $(Y, M, N, X) = 4 \Longrightarrow \frac{EM}{EN} \cdot \frac{DN}{DM} = 2 \Longrightarrow \frac{EM}{EN} = \frac{DP}{DN}$. Since $\angle MEN = \angle NDP$, then $\triangle MEN \sim \triangle PDN \Longrightarrow \angle DPN = \angle EMN = \angle EDN \Longrightarrow DE$ touches $\odot(DPN)$.

TelvCohl

#3 Dec 1, 2015, 8:56 am • 1

Let U be the image of N under the homothety $\mathbf{H}(D, -\frac{1}{2})$ and let V be the midpoint of FD. Since $DU : DV = DN : FD = EN : EF$ ($\because FDNE$ is a harmonic quadrilateral), so combine $\measuredangle VDU = \measuredangle FEN$ we get $\triangle DUV \sim \triangle ENF$ (S.A.S) $\Longrightarrow \measuredangle DUV = \measuredangle ENF = \measuredangle EMF = \measuredangle DMV$ ($\because DEFM$ is a harmonic quadrilateral), hence $U \in \odot(DMV) \Longrightarrow \odot(DPN)$ is the image of $\odot(DMV)$ under the homothety $\mathbf{H}(D, -2)$ and $\odot(DPN)$ is tangent to $\odot(DMV)$ at D. On the other hand, it's well-known that DE is tangent to $\odot(DMV)$ at D, so we conclude that DE is tangent to $\odot(DPN)$.

Seventh

#4 Dec 1, 2015, 10:42 am • 2

Let r be the inradius of $\triangle ABC$, and ω be its incircle. Let $\pi : X \to \pi(X)$ be the inversion with center D and radius r.

Notice that $\pi(AFB)$ and $\pi(EMB)$ are both circumferences which pass trough D and $\pi(B)$, while $\pi(\omega)$ is a line trough $\pi(F)$ which is parallel to $\pi(B)\pi(C)$. Thus $D\pi(B) \parallel \pi(M)\pi(E)$, $\pi(M)\pi(E)$ is tangent to $\pi(AFB)$ at $\pi(F) \Rightarrow \pi(M)\pi(F) = \pi(F)\pi(E)$. In the same way, $\pi(N)\pi(E) = \pi(F)\pi(E)$, therefore $\pi(M)\pi(E) = 2\pi(E)\pi(N)$, while $\pi(M)D = 2D\pi(P)$, it means $D\pi(E) \parallel \pi(P)\pi(N) \Rightarrow DE$ is tangent to the circumcircle of $\triangle DNP$, as desired.

Quick Reply

High School Olympiads

equal ratios

Reply

Source: Vittas Kostas

silouan

#1 Nov 30, 2015, 4:19 am • 1

Let a triangle $\triangle\ ABC$ with circumcircle (O) and diameter AT with T on $\overset{\frown}{BC}$. An arbitrary line (ε) which is perpendicular to AT intersects $AB,\ AC$ at the points $K,\ L$ respectively and it is intersected by the tangents of (O) at $B,\ C$, at the points $P,\ Q$, respectively. Let
$S \equiv BL \cap CK$ and $X \equiv KL \cap TS$. Prove that $\frac{XK}{XL} = \frac{XP}{XQ}$

Luis González

#2 Nov 30, 2015, 7:33 am

Since KL is antiparallel to BC WRT AC, AB, then $BCLK$ is cyclic with circumcircle (J). Let $R \equiv KL \cap BC$ and let $M \equiv AR \cap (O)$ be the Miquel point of $BCLK$.

It's known that M is the inverse of S WRT $(J) \Longrightarrow JB^2 = JL^2 = JS \cdot JM \Longrightarrow BMLJ$ is cyclic $\Longrightarrow$ $\angle(ML, MB) = \angle(JL, JB) = 2\angle(CA, CB) = \angle(PL, PB) \Longrightarrow M \in \odot(PBJL)$ and similarly $M \in \odot(QCJK)$. Since $SB \cdot SL = SC \cdot SK$, it follows that $\overline{MSJ}$ is then radical axis of $\odot(PBL)$ and $\odot(QCK)$. But clearly $T \in \overline{MSJ}$ as $JM \perp \overline{AMR}$, thus $\overline{TSX}$ is radical axis of $\odot(PBL)$ and $\odot(QCK) \Longrightarrow XL \cdot XP = XK \cdot XQ$.

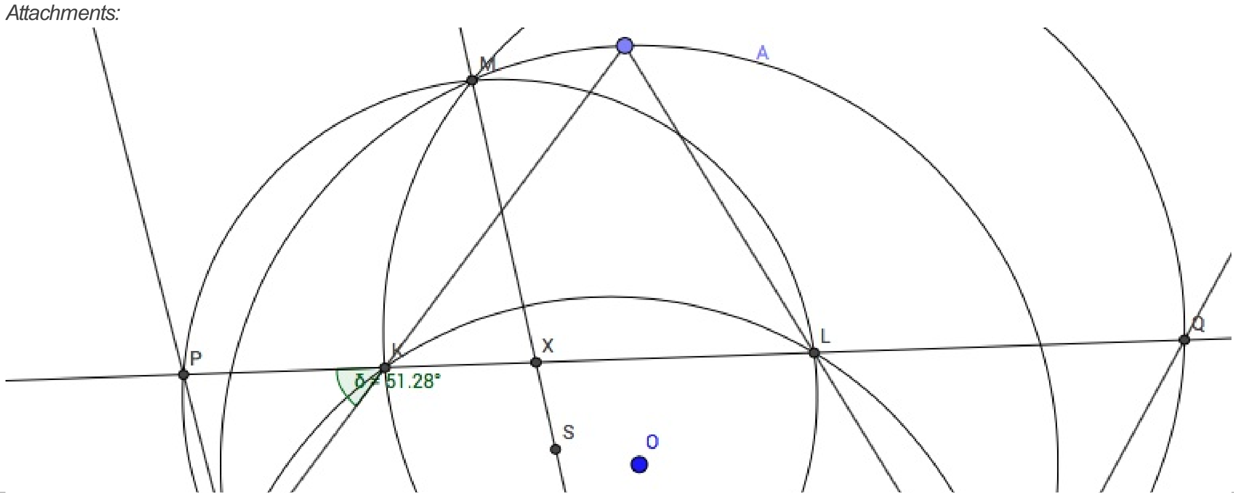
suli

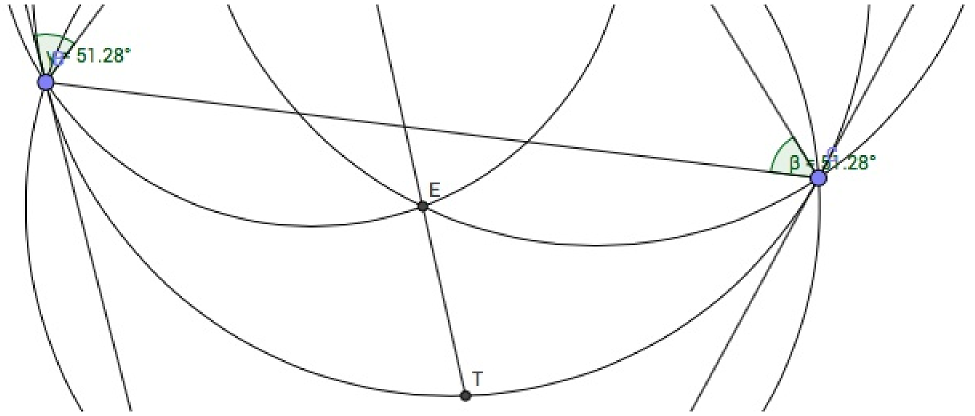
#3 Nov 30, 2015, 8:06 am

This has been one of the best geometry problems I have ever seen (and solved). Thank you for making my day.

Notice that by simple angle chasing $KLCB$ is cyclic.
Consider the circumcircles of PBL and QCK (the *good* circumcircles). Because $SK \cdot SC = SB \cdot SL$ by Power of a Point, S lies on the radical axis of the two circles.
Consider the center E of $KLCB$. By angle chasing we can prove it lies on both good circumcircles.
Consider the center of spiral similarity taking KL to BC. Because $KLCB$ is cyclic, $MLEB$ and $MKEC$ are also cyclic, so M also lies on both circumcircles. Hence EM is the radical axis of the good circumcircles.
But a well-known IMO problem asserts that $\angle EMA = 90^\circ$. (This is also proven by considering that the midpoint of KB gets mapped to LC under spiral similarity centered at M.)
As $\angle TMA = 90^\circ$ as well, T also lies on the radical axis. Hence X also lies on the radical axis, so $XK \cdot XQ = XL \cdot XP$, which is equivalent to what we want to prove.

Attachments:





This post has been edited 1 time. Last edited by suli, Nov 30, 2015, 8:06 am

Quick Reply

High School Olympiads

Geometry

Reply

Source: Unknown

prof.

#1 Nov 27, 2015, 1:09 am

Given a circle and the tangent t at point M. From every point A of straight line g that doesn't cut the circle and parallel with t, we draw tangents on given circle which cut the straight line t at points X and Y. Prove that the product $|XM| \cdot |YM|$ doesn't depend on position of point A.

Dukejukem

#2 Nov 27, 2015, 3:26 am

Let (O) be the circle in question, and let S, T be the tangency points of AX, AY with (O). Let H be the projection of M onto ST, and let U be the second intersection of MH with (O). Let K be the pole of g WRT (O).

Since $\measuredangle XOM = \measuredangle STM = \measuredangle HTM$, it follows that $\triangle XOM \sim \triangle MTH$ are similar right triangles. Similarly, $\triangle YOM \sim \triangle MSH$, and therefore $\frac{MX \cdot MY}{MO^2} = \frac{HM^2}{HS \cdot HT} = \frac{HM}{HU}$, where the last step follows from Power of a Point. Therefore, it is sufficient to show that $\frac{HM}{HU}$ is fixed.

Since the polars of K, S, T are concurrent at A, it follows that $K \in ST$. Moreover, $OK \perp g \implies K \in OM$. Then since MK, MU are isogonal WRT $\angle SMT$, it is well-known that $\triangle SMK \sim \triangle MUT$, and in particular, $MU = \frac{MS \cdot MT}{MK}$. Combined with the formula $HM = \frac{MS \cdot MT}{MO}$, it follows that $\frac{HM}{MU} = \frac{MK}{MO}$ is fixed. Thus, $\frac{HM}{HU}$ is fixed as well, and we are done. $\square$

suli

#3 Nov 27, 2015, 4:12 am

Let $AX = c, AY = b, XY = a, s = \dfrac{a+b+c}{2}, A = [AXY], r =$ the inradius of AXY, $h =$ distance from A to XY. Then

$$XM \cdot YM = (s-b)(s-c) = \frac{A^2}{s(s-a)} = \frac{A}{s} \cdot \frac{A}{s-a} = r \cdot (\frac{s}{A} - \frac{a}{A})^{-1} = r \cdot (r^{-1} - 2h^{-1})^{-1}$$

depends only on constants r and h.

This post has been edited 1 time. Last edited by suli, Nov 27, 2015, 4:13 am

Luis González

#4 Nov 27, 2015, 6:33 am

Label ω the object circle. B is a fixed point on g and the tangents from B to ω cut t at U, V. By dual of Desargues involution theorem for the degenerate quadrilateral $BUMV$ circumscribed to ω, we deduce that $AX \mapsto AY, AU \mapsto AV$, $AM \mapsto AB$ is an involution. Thus cutting this pencil with t, we get that $X \mapsto Y, U \mapsto V, M \mapsto \infty$ is an involution on $t \Longrightarrow$ M is the center of this involution $\Longrightarrow \overline{MX} \cdot \overline{MY}$ is constant.

P.S. Needless to say the property still holds for any conic ω. The proof is exactly the same.

Quick Reply

High School Olympiads

Symmetry

Locked

mamavuabo

#1 Nov 26, 2015, 5:09 am

Given triangle ABC circumscribed in (O), orthocenter $H.AH$ cuts BC at $D.E$ is symmetric with C through $D.HE$ cuts AB at F.Prove that $\widehat{FDO}=90$.

Luis González

#2 Nov 26, 2015, 6:47 am

Same as ISL 1996 (G3). See http://www.artofproblemsolving.com/community/c6h1133.

High School Olympiads

I need some pure geometry :))

Reply

Source: IMO Shortlist 1996 problem G3

grobber
#1 Oct 4, 2003, 9:44 am

Let O be the circumcenter and H the orthocenter of an acute-angled triangle ABC such that $BC > CA$. Let F be the foot of the altitude CH of triangle ABC. The perpendicular to the line OF at the point F intersects the line AC at P. Prove that $\measuredangle FHP = \measuredangle BAC$.

ya
#2 Oct 24, 2003, 7:11 am • 1

Let E be the midpoint of AC, G of OP.
angle OFP = angle OEP = 90
OFPG- inscribed in circle with center G
Let K be midpiont of OH.
It is obvious that K is the center ot the Euler's (also known as nine-point) circle for the triangle ABC.
Than K, G lie on the perpendicular bisector of the common chord FE.
angle FHP = 90- angle EFH
and angle EFH = angle EFC = angle ECF = 90- angle A
angle FHP = angle A
There you go!
Irina
P.S. I think I've seen this somewhere before... Was it an IMO problem? I'm too lazy to check...

grobber
#3 Oct 25, 2003, 1:10 am • 2

It's from a shortlist (can't remember which one); I don't know if it was actually given at an IMO, but I doubt it. Nice soln! The problem I was referring to (the one I said you could use in order to prove this) is the Butterfly problem.

Here's my soln:

Let T be the intersection between the altitude CH and the circumcircle of ABC. Let the chord FP (a chord in the circumcircle of ABC) cut the chord BT at Q. OF perpendicular to chord PF and O is the center of the circumcircle => F is the midpt of the chord PF and, because of the butterfly property, F must be the midpt of PQ (*). It's well-known that F is the midpt of HT (**). From (*) and (**) we get triangles FHP and FTQ equal, so HP || TQ=TB, so angle FHP=angle FTB=angle BAC Q.E.D.

sam-n
#4 Jun 12, 2004, 2:54 pm

u find it in our olympiad (14-th Iranian Mathematical Olympiad 1996/1997 (1375)september).
it's beatifuly solved by batterfly theorem.

darij grinberg
#5 Sep 1, 2004, 4:43 pm • 2

If somebody is still interested, I have another solution:

I use the **orthologic triangles theorem**, which states that if ABC and A'B'C' are two non-degenerate triangles, then the lines $A\ \overline{B'C'}$, $B\ \overline{C'A'}$, $C\ \overline{A'B'}$ concur if and only if the lines $A'\ \overline{BC}$, $B'\ \overline{CA}$, $C'\ \overline{AB}$ concur. Hereby, for any point P and any line g, the notion $P\ \overline{g}$ means the perpendicular from the point P to the line g.

For your problem, I will work with directed angles modulo 180°, and I will prove that < FHP = < CAB.

Let C' be the reflection of the point C in the line AB, or, equivalently, the reflection of the point C in the point F. Let also Z be the reflection of the point C in the point O. Then, the segment CZ is a diameter of the circumcircle of triangle ABC; hence, < CAZ = 90°, and thus $ZA\perp AC$. Similarly, $ZB\perp BC$.

Since the points O and F are the midpoints of the segments CZ and CC', we have OF || C'Z.

Now, since the point C' is the reflection of the point C in the line AB, we have < CAB = < BAC'. Thus, instead of proving < FHP = < CAB, it will be enough to show < FHP = < BAC'. But < FHP = < (FH; HP) = < (FH; AB) + < (AB; HP) = 90° + < (AB; HP), and < BAC' = < (AB; AC'). So we have to prove that 90° + < (AB; HP) = < (AB; AC'). This is equivalent to 90° = < (AB; AC') - < (AB; HP), what is obviously equivalent to 90° = < (HP; AC'). Thus, we must show that 90° = < (HP; AC'), i. e. we must show that $HP\perp AC'$. In other words, we must show that the point P lies on the line $H\overline{AC'}$.

Now, the point P is defined as the point of intersection of the lines $F\overline{OF}$ and AC. Since OF || C'Z, we can rewrite $F\overline{OF}$ as $F\overline{C'Z}$, and since $ZA\perp AC$, we can rewrite AC as $A\overline{ZA}$. Thus, we must prove that the point P, defined as the point of intersection of the lines $F\overline{C'Z}$ and $A\overline{ZA}$, lies on the line $H\overline{AC'}$. Or, simply, we have to prove that the lines $F\overline{C'Z}$, $A\overline{ZA}$, $H\overline{AC'}$ concur. By the orthologic triangles theorem, applied to the triangles FAH and AC'Z, this is equivalent to proving that the lines $A\overline{AH}$, $C'\overline{HF}$, $Z\overline{FA}$ concur. In order to prove this, we denote by S the point of intersection of the lines $A\overline{AH}$ and $C'\overline{HF}$, and try to show that this point S lies on the line $Z\overline{FA}$, i. e. that we have $ZS\perp FA$.

Well, since the point S lies on the line $A\overline{AH}$, we have $AS\perp AH$, and together with $AH\perp BC$, this gives AS || BC. Since the point S lies on the line $C'\overline{HF}$, we have $C'S\perp HF$, and since $HF\perp AB$, this yields C'S || AB. If the lines CS and AB meet at K, then from C'S || AB, we have CK : KS = CF : FC', and since CF : FC' = 1 (the point C' is the reflection of the point C in the point F), we have CK : KS = 1, too, so that the point K is the midpoint of the segment CS. On the other hand, AS || BC yields BK : KA = CK : KS, what now shows us that BK : KA = 1, and the point K is the midpoint of the segment AB. Thus, the segments AB and CS have the point K as their common midpoint, i. e. these segments bisect each other, and it follows that the quadrilateral ACBS is a parallelogram. Hence, not only AS || BC, but also BS || AC. Now, BS || AC together with $ZA\perp AC$ yields $ZA\perp BS$, while AS || BC together with $ZB\perp BC$ yields $ZB\perp AS$. Hence, the point Z lies on two of the three altitudes of the triangle ABS; this means that the point Z is the orthocenter of this triangle, and hence also lies on the third altitude. And this yields $ZS\perp AB$, or, in other words, $ZS\perp FA$. Proof complete.

Well, this is a really monstrous solution, but it doesn't use the butterfly theorem, does it?

Darij

This post has been edited 1 time. Last edited by darij grinberg, Mar 5, 2006, 4:26 pm

orl

#6 Sep 1, 2004, 5:54 pm

Have a look at page 27/52.

Virgil Nicula

#7 Sep 24, 2005, 2:23 am

See the problem $P3$ from http://www.mathlinks.ro/Forum/viewtopic.php?t=46146

yetti

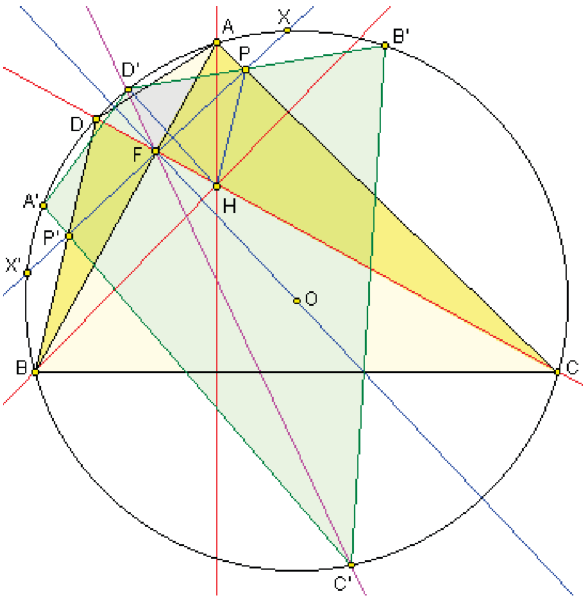
#8 Apr 5, 2006, 3:26 pm • 2

Let the altitude CF meet the circumcircle (O) of the triangle $\triangle ABC$ again at a point D and consider the cyclic quadrilateral ADBC with the diagonal intersection F. Let the perpendicular to OF at F meet AC at P, DB at P', the circumcircle arc DC opposite to the vertex B at X, and the circumcircle arc DC opposite to the vertex A at X'. Since $XX'\perp OF$, FX = FX'. By the butterfly theorem, FP = FP' as well, i.e., P' is a reflection of P in the line OF. Reflect the cyclic quadrilateral ADBC in the line OF into a cyclic quadrilateral A'D'B'C' with the same circumcircle (O) and the same diagonal intersection F. Then D'B' meets AC at P and A'C' meets DB at P'. (This is true for any cyclic quadrilateral ADBC, not necessarily with perpendicular diagonals $AB\perp CD$.)

D is a reflection of the orthocenter H of the triangle $\triangle ABC$ in the line AB, FH = FD. By symmetry, FD' = FD, hence FH = FD = FD' and the triangle $\triangle DD'H$ has right angle $\angle DD'H = 90^\circ$. But D' is a reflection of D in OF, hence $DD'\perp OF$, so that $OF\perp FP$ are midlines of this right angle triangle, i.e., $HD'\perp FP$. Consequently, the quadrilateral FHPD' is a kite, which means that the triangles $\triangle FHP\cong\triangle FD'P$ are (oppositely) congruent and $\angle FHP=\angle FD'P$. But obviously, $\angle FD'P\equiv\angle C'D'B'=\angle CDB=\angle CAB$, which is what we were supposed to prove.

Butterfly theorem can be proved in various ways, synthetically or by trigonometry. For example, see http://www.cut-theknot.org/pythagoras/Butterfly.shtml.

Attachments:



xeroxia

#9 Mar 25, 2011, 4:59 am

Unfortunately, there is a trigonometric solution. And I will write it in $\LaTeX$, tenaciously.
Let FP intersects the circumcircle of $\triangle AFC$ at J. We should show H, P, J, C are concyclic because $\angle FAC = \angle FJC$ and we are asked to show $\angle FHP = \angle FAP$.
This yields $FP \cdot FJ = FH \cdot FC = AF \cdot FB$.
Let $\angle FCA = \alpha$, $\angle FCB = \beta$, and $\angle AFP = \theta$.

$$\frac{AF}{FP} = \frac{\sin(90^\circ + \alpha - \theta)}{\sin(90^\circ - \alpha)}$$

$$\frac{FC}{FJ} = \frac{\sin(90^\circ - \alpha)}{\sin(\alpha + \theta)}$$

$$AF \cdot FC = FP \cdot FJ \cdot \frac{\cos(\alpha - \theta)}{\sin(\alpha + \theta)}$$

We will show $\dfrac{FC}{BF} = \dfrac{\cos(\alpha - \theta)}{\sin(\alpha + \theta)} = \dfrac{\cos\beta}{\sin\beta}$.

Let $R = 1$. Thus $AC = 2\cos\beta, BC = 2\cos\alpha, BF = 2\cos\alpha\sin\beta, AF = 2\cos\beta\sin\alpha, AB = 2\sin(\alpha + \beta)$.

So $MF = \sin(\beta - \alpha)$ and $OM = \cos(\alpha + \beta)$. And $\angle FOM = \angle AFP = \theta$. Then $\tan\theta = \dfrac{\sin(\beta - \alpha)}{\cos(\alpha + \beta)}$.

$$\frac{\cos(\alpha - \theta)}{\sin(\alpha + \theta)} = \frac{\cos\alpha\cos\theta + \sin\alpha\sin\theta}{\sin\alpha\cos\theta + \cos\alpha\sin\theta} = \frac{\cos\alpha + \sin\alpha\tan\theta}{\sin\alpha + \cos\alpha\tan\theta}.$$

$$\Rightarrow \frac{\cos\alpha + \sin\alpha\frac{\sin(\beta-\alpha)}{\cos(\alpha+\beta)}}{\sin\alpha + \cos\alpha\frac{\sin(\beta-\alpha)}{\cos(\alpha+\beta)}} = \frac{\cos\alpha\cos(\alpha+\beta) + \sin\alpha\sin(\beta-\alpha)}{\sin\alpha\cos(\alpha+\beta) + \cos\alpha\sin(\beta-\alpha)} \Rightarrow \frac{\cos\beta(\cos^2\alpha - \sin^2\alpha)}{\sin\beta(\cos^2\alpha - \sin^2\alpha)} = \frac{\cos\beta}{\sin\beta}\, Q.E.D$$

Luis González

#10 Mar 25, 2011, 10:53 pm • 2

Since $\angle HFA = \angle OFP = 90^\circ$ and $\angle HAF = \angle OAP$, it follows that O, H are isogonal conjugates with respect to $\triangle APF$. Consequently, if M, N denote the midpoints of AB, AC, then $\triangle FNM$ is the pedal triangle of O with respect to $\triangle APF \Longrightarrow HP \perp FN \Longrightarrow \angle FHP = \angle NFA$. Since $\triangle ANF$ is N-isosceles, then $\angle FHP = \angle BAC$.

mathreyes

#11 Feb 18, 2013, 12:44 am

> Luis González wrote:
> ...$\triangle FNM$ is the pedal triangle of O with respect to $\triangle APF \Longrightarrow HP \perp FN$...

why? I think this is not a useful reason to ensure that perpendicularity.

The real reason (for me, at least) is:

$\measuredangle ONP = \measuredangle OFP = 90 \Longrightarrow NPFO$ is cyclic, so $\measuredangle FNP = \measuredangle FOP$ but $\measuredangle NPH = \measuredangle OPF$.
Finally $\measuredangle FNP + \measuredangle NPH = \measuredangle FOP + \measuredangle OPF = 90$, so $HP \perp FN$.

(note that in my argument, there was no need to construct either point M nor pedal triangle of O.)

Mosquitall

#12 Jun 6, 2013, 4:44 pm • 1

Generalization:
Triangle ABC, and point F, such that $\angle BFC = \angle CFA = \gamma$, $\angle FAC = \beta$, point H is on CF and $\angle FHB = \beta$, point P is on AC and $\angle PHF = \beta$, point O with $\angle CBO = \angle OCB = \alpha$, $\alpha + \beta + \gamma = 180$. Then prove that $\angle PFA = \angle OFC$.

duanby

#13 Jun 7, 2013, 11:42 am

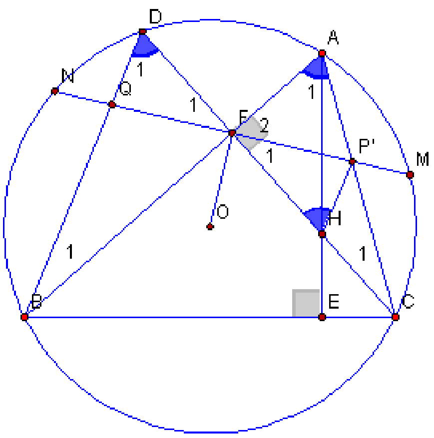
MY SOLUTION:
Let P' be the reflection of P wrt CF then P' is the isogonal conjugate point of O wrt ACF

vslmat

#14 Jul 8, 2013, 6:19 pm • 1

To avoid using the Butterfly theorem as well as advanced geometry, we can do this way:
Let CF cuts the circumcircle at D. On AC let's choose point P' so that $\angle FAP' = \angle BAC$, $P'F$ cuts BD at Q and cuts the circumcirle at M and N. Easy to see that $HP' \parallel BD$. As $DF = FH$ is a well known property, $QF = FP'$.
If we can prove that $P'M = NQ$ then F is the midpoint of MN and $OF \perp MN$, thus $P' \equiv P$ and we are done.
Now using sinus theorem we have
$\dfrac{P'C}{sinF_1} = \dfrac{FP'}{sinC_1}$ and $\dfrac{QD}{sinF_1} = \dfrac{QF}{sinD_1}$, thus $\dfrac{P'C}{QD} = \dfrac{sinD_1}{sinC_1}$. Similarly, we get $\dfrac{BQ}{AP'} = \dfrac{sinA_1}{sinB_1}$
Therefore, $\dfrac{BQ}{AP'} = \dfrac{P'C}{QD}$, or $BQ.QD = AP'.P'C$
But notice that $QD.BQ = NQ.QM$ and $AP'.P'C = P'M.P'N$, it follows that $NQ = P'M$
F is indeed the midpoint of MN and we are done.

Attachments:



XmL

#15 Jul 9, 2013, 4:47 am

Let the line through O perpendicular to AC meet AC, CF, AB at M, L, K respectively, thus

$\angle OKA = 90 - \angle A = \angle ACF$ and M is the midpoint of AC. From the first result we deduce that $\triangle ACF \sim \triangle LKF$. Since $\angle PFO = \angle CFA = 90$, thus $\angle AFP = \angle LFO$, which means that O, P are corresponding points concerning similar triangles LKF and ACF. Now let H' be the point that corresponds to H, thus H' is on FK and $\triangle FPH \sim \triangle FOH' \Rightarrow$ we now just need to prove $\angle OH'F = \angle A = \angle MFA \iff OH' \parallel MF \iff \frac{MO}{OK} = \frac{FH'}{KH'} = \frac{FH}{CH} (*)$.
Since $\triangle AMK \sim \triangle AFC$ and $\angle HAF = \angle MAO$, thus O, H are two corresponding points concerning those two triangles, which means that (*) is true. Q.E.D

This post has been edited 1 time. Last edited by XmL, Jul 12, 2013, 8:26 am

IDMasterz

#16 Jul 10, 2013, 9:30 pm

Let DEF be the orthic triangle of $\triangle ABC$.Since $\angle(OF, AF) = \angle FPH$ and we already have that O, H as isogonal wrt $\angle FAP$, we get H, O are isogonal conjugates wrt $\triangle AFP$. If we let M be the midpoint of AC, then note that $AH \perp FM$ (since they are the feet of the pedals from O). Now, M is the centre of $DFAC$ so $\angle MFC = 90 - A$ so $\angle FHP = \angle BAC$ as desired.

IDMasterz

#17 Jul 10, 2013, 9:37 pm

@mathreyes

It is well-known that for two isogonal conjugates X, Y, we have AX, BX, CX is perpendicular to the sides of the pedal triangles.

I realise now that my solution is basically the same as Luis's, sorry I posted on impulse hehe

Quick Reply

High School Olympiads

3 points on a line

Source: Cono Sur 2009 #3

fprosk

#1 Nov 17, 2015, 9:02 am

Let A, B, and C be three points such that B is the midpoint of segment AC and let P be a point such that $< PBC = 60$. Equilateral triangle PCQ is constructed such that B and Q are on different half=planes with respect to PC, and the equilateral triangle APR is constructed in such a way that B and R are in the same half-plane with respect to AP. Let X be the point of intersection of the lines BQ and PC, and let Y be the point of intersection of the lines BR and AP. Prove that XY and AC are parallel.

Luis González

#2 Nov 26, 2015, 2:40 am

Let O be the center of equilateral $\triangle APR$ and let M be the midpoint of PA. Since $\angle ABP = \angle AOP = 120^\circ \Longrightarrow B \in \odot(POA)$ and obviously RA, RP are tangents of $\odot(POA) \Longrightarrow BY$ is the B-symmedian of $\triangle PBA$, isogonal of its B-median $BM \Longrightarrow \angle PBY = \angle ABM = \angle PCA$. But as $\triangle PAC \cong \triangle PRQ$ are directly congruent, then $\angle PQR = \angle PCA \Longrightarrow \angle PQR = \angle PBY \Longrightarrow PQRB$ is cyclic $\Longrightarrow \angle RBQ = \angle RPQ = \angle APC \Longrightarrow PXBY$ is cyclic $\Longrightarrow \angle PYX = \angle PBX = \angle PRQ = \angle PAC \Longrightarrow XY \parallel AC$.

sunken rock

#3 Nov 26, 2015, 5:30 am

Very nice, as usual, Luis! A small remark for your proof: also BQ is the $B-$symmedian of $\triangle BCP$; with $BC = AB$, done!

Another idea:
A 90° rotation about P sends $\triangle PAC$ to $\triangle PRQ$ and B to T, midpoint of QR; since $\angle PBC = 60^\circ$ and that is the rotation angle value, T lies onto AC and $QR \parallel PB$. As $\triangle PTQ \equiv \triangle BTR$ (s.a.s. criterion), $PBRQ$ is an isosceles trapezoid, i.e. $\triangle BRQ \cong \triangle PQR \cong \triangle PCA$, hence $\angle RBQ = \angle APC \ (1 \ )$ and $\angle CBQ = \angle APB \ (2 \ )$. From (1) we infer $PYBX$ cyclic, i.e. $\angle YXB = \angle APB \ (3 \ )$; with (2), done.

Best regards,
sunken rock

This post has been edited 1 time. Last edited by sunken rock, Nov 26, 2015, 5:31 am

Quick Reply

High School Olympiads

Triangle inscribed in another triangle

Source: Cono Sur 2010 #5

fprosk

#1 Nov 17, 2015, 8:37 am

The incircle of triangle ABC touches sides BC, AC, and AB at D, E, and F respectively. Let ω_a, ω_b and ω_c be the circumcircles of triangles EAF, DBF, and DCE, respectively. The lines DE and DF cut ω_a at $E_a \neq E$ and $F_a \neq F$, respectively. Let r_A be the line E_aF_a. Let r_B and r_C be defined analogously. Show that the lines r_A, r_B, and r_C determine a triangle with its vertices on the sides of triangle ABC.

Luis González

#2 Nov 25, 2015, 9:18 am

Let X, Y, Z be the midpoints of BC, CA, AB, respectively. If BI cuts DE at U, we have $\angle BUD = \angle EDC - \angle IBC = 90^\circ - \frac{1}{2}\angle ACB - \frac{1}{2}\angle ABC = \frac{1}{2}\angle BAC = \angle IAE \Longrightarrow U \in \omega_a \Longrightarrow U \equiv E_a \Longrightarrow$ $\angle AE_aB \equiv \angle AE_aI = 90^\circ \Longrightarrow \angle AZE_a = 2\angle ABE_a = \angle ABC \Longrightarrow ZE_a \parallel BC \Longrightarrow E_a \in YZ$ and likewise $F_a \in YZ \Longrightarrow r_A \equiv YZ$ and similarly $r_B \equiv ZX$ and $r_C \equiv XY \Longrightarrow \triangle(r_A, r_B, r_C) \equiv \triangle XYZ.$

High School Olympiads

Isogonal conjugate points

Reply

Source: Own

buratinogigle

#1 Nov 24, 2015, 3:52 pm • 1

Let ABC be a triangle P,Q are two isogonal conjugate points. D,E,F are projection of P on BC,CA,AB. The line passes through D and are perpendicular to AD cuts QB,QC at A_b,A_c, reps. The line passes through E and are perpendicular to BE cuts QC,QA at B_c,B_a, reps. The line passes through F and are perpendicular to CF cuts QA,QB at C_a,C_b, reps. Prove Q and circumcenters of triangle $A_bB_cC_a,A_cB_aC_b$ are collinear.

Luis González

#2 Nov 25, 2015, 3:53 am • 2

Let X,Y,Z be the projections of Q on $BC,CA,AB.$ X,Y,Z,D,E,F lie on pedal circle of P,Q with radius $\varrho.$ Since $B_aE \perp BE$ and $AE \perp PE,$ then EB_a and EP are isogonals WRT $\angle BEA \Longrightarrow P,B_a$ are isogonal conjugates WRT $\triangle BEA \Longrightarrow \angle EBB_a=\angle ABP=\angle CBQ \Longrightarrow \angle QBB_a=\angle CBE.$ Since $\angle EXB=\angle AQB,$ then it follows that $\triangle BQB_a \sim \triangle BXE \Longrightarrow \frac{QB_a}{EX}=\frac{BQ}{BX}$ and by similar reasoning we have $\frac{QC_a}{FX}=\frac{CQ}{CX}.$ Thus

$$QB_a \cdot QC_a=\frac{BQ}{BX} \cdot \frac{CQ}{CX} \cdot EX \cdot FX=\frac{EX \cdot FX}{\cos \widehat{PDF} \cdot \cos \widehat{PDE}}=\frac{EX \cdot FX}{\sin \widehat{FEX} \cdot \sin \widehat{EFX}}=4\varrho^2,$$

which means that $\{B_a,C_a\}$ are inverse points WRT $\odot(Q,2\varrho).$ Similarly $\{C_b,A_b\}$ and $\{A_c,B_c\}$ are pairs of inverse points WRT $\odot(Q,2\varrho) \Longrightarrow \odot(A_bB_cC_a)$ and $\odot(A_cB_aC_b)$ are inverses WRT $\odot(Q,\varrho) \Longrightarrow$ their centers are collinear wih the inversion center $Q.$

Quick Reply

High School Olympiads

Concurrent with excircles

Reply

Source: Own

buratinogigle

#1 Nov 24, 2015, 4:11 pm

Let ABC be a triangle and excircles $(I_a),(I_b),(I_c)$ at vertex A,B,C touch BC,CA,AB at A_1,B_1,C_1. Lines A_1B_1,A_1C_1 cut (I_a) again at A_2,A_3. Similarly, we have B_2,B_3,C_2,C_3. Prove that lines A_2A_3,B_2B_3,C_2C_3 bound a triangle which is perspective to ABC.

Luis González

#2 Nov 25, 2015, 12:01 am • 1

Let B_1C_1,C_1A_1,A_1B_1 cut BC,CA,AB at $X,Y,Z.$ Obviously $\overline{XYZ}$ is the trilinear polar of the Nagel point of $\triangle ABC.$ Moreover if (I_a) touches CA,AB at $U,V,$ then UV hits BC at the harmonic conjugate of A_1 WRT $B,C,$ i.e. $X \in UV.$ Now projecting the line AX (not cutting (I_a)) to infinity and (I_a) into another circle, we get by obvious symmetry that $A_2A_3 \parallel UV \parallel B_1C_1,$ thus in the primitive figure, this means $X \in A_2A_3.$ In the same way, we have $Y \in B_2B_3$ and $Z \in C_2C_3 \Longrightarrow \triangle ABC$ and the triangle bounded by A_2A_3,B_2B_2,C_2C_3 are perspective through $\overline{XYZ}.$

TelvCohl

#3 Nov 25, 2015, 5:32 am • 1

Let $A_4 \equiv AA_1 \cap \odot(I_a), BB_4 \equiv BB_1 \cap \odot(I_b), C_4 \equiv CC_1 \cap \odot(I_c)$. Since AA_1,BB_1,CC_1 are concurrent (Nagel point N_a of $\triangle ABC$), so $(A_1B_1,A_1C_1;A_1A_4,BC)=-1 \Longrightarrow A_1A_2A_4A_3$ is a harmonic quadrilateral, hence $A^* \equiv A_2A_3 \cap BC$ is the pole of AA_1 WRT $\odot(I_a)$ and $(B,C;A_1,A^*)=-1 \Longrightarrow A^*$ lies on trilinear polar τ of N_a WRT $\triangle ABC$. Similarly, we can prove $B_2B_3 \cap CA, C_2C_3 \cap AB$ lie on τ, so from Desargue theorem we conclude that $\triangle ABC$ and the triangle with the sides A_2A_3,B_2B_3,C_2C_3 are perspective.

Quick Reply

High School Olympiads

An interesting perpendicularity

Reply

Source: Own ?

jayme
#1 Nov 20, 2015, 7:56 pm

Dear Mathlinkers,

1. ABCD a square
2. P a point on the segment AB
3. I, Ic, Id the incenters wrt the triangles PCD, PBC, PAD
4. (1) the circle through P, Ic, Id and I (well known)
5. Q the second point of intersection of (1) with AB.

Prove : IQ is perpendicular to IcId.

Sincerely
Jean-Louis

Shivaji_the_boss
#2 Nov 23, 2015, 10:40 pm

someone please prove.....

Luis González
#3 Nov 23, 2015, 11:54 pm

This is valid for any circumscribed quadrilateral $ABCD$, not necessarily square. Letting P be a point on $\overline{AB}$, the incenters I, I_c, I_d of $\triangle PCD, \triangle PBC, \triangle PAD$ and P lie on a same circle ω and if ω cuts AB again at Q, then $IQ \perp I_cI_d$.

Proof: Using a degenerate case of the configuration discussed at Several tangential quadrilaterals (see post #3), we get that $(I), (I_c), (I_d)$ have a common tangent τ. Thus if τ cuts PC, PD at U, V, then I becomes the P-excenter of $\triangle PUV \Longrightarrow$ $\angle I_cII_d \equiv \angle UIV = 90^\circ - \tfrac{1}{2}\angle CPD$. Since $\angle I_cPI_d = \tfrac{1}{2}(\angle BPC + \angle APD) + \angle CPD \Longrightarrow$ $\angle I_cPI_d = 90^\circ + \tfrac{1}{2}\angle CPD \Longrightarrow \angle I_cPI_d + \angle I_cII_d = 180^\circ \Longrightarrow P, I, I_c, I_d$ are concyclic.

Since $\angle QII_d = \angle QPI_d = \tfrac{1}{2}\angle DPA$ and $\angle II_dI_c = \angle IPI_c = \tfrac{1}{2}(\angle CPB + \angle CPD) \Longrightarrow$ $\angle QII_d + \angle II_dI_c = \tfrac{1}{2}(\angle DPA + \angle CPB + \angle CPD) = 90^\circ \Longrightarrow IQ \perp I_cI_d.$

Quick Reply

High School Olympiads

Several tangential quadrilaterals

Reply

Source: ELMO Shortlist 2011, G1; also ELMO #1

math154
#1 Jul 3, 2012, 9:50 am • 1

Let $ABCD$ be a convex quadrilateral. Let E, F, G, H be points on segments AB, BC, CD, DA, respectively, and let P be the intersection of EG and FH. Given that quadrilaterals $HAEP$, $EBFP$, $FCGP$, $GDHP$ all have inscribed circles, prove that $ABCD$ also has an inscribed circle.

Evan O'Dorney.

ACCCGS8
#2 Jul 3, 2012, 5:59 pm

We have:
(1): $AE + HP = AH + EP$
(2): $HP + DG = HD + PG$
(3): $EB + PF = EP + BF$
(4): $PF + GC = PG + FC$

Adding these above equations gives
$(AB + CD) - (BC + AD) = 2(HF - EG)$.

So it suffices to show that $HF = EG$, which I haven't done yet.

Luis González
#3 Jul 3, 2012, 11:46 pm • 1

The problem has a generalization; See the attachment below, $\ell_1, \ell_2, \ell_3, \ell_4$ are common external tangents of the referred circles (different from AB,BC,CD,DA), meeting each other at X, Y, Z, W. We prove that $ABCD$ has an incircle $\Longleftrightarrow XYZW$ has an incircle. Using the notations given by the diagram, we get

$$XY = P_1Q_1 - XM - YN = AB - AP_1 - BQ_1 - XM - YN \ \ (1)$$

$$WZ = P_3Q_3 - ZL - WO = DC - CP_3 - DQ_3 - ZL - WO \ \ (2)$$

$$YZ = P_2Q_2 - YN - ZL = BC - BQ_1 - CP_3 - YN - ZL \ \ (3)$$

$$XW = P_4Q_4 - XM - WO = AD - AP_1 - DQ_3 - XM - WO \ \ (4)$$

Combining $(1) + (2)$ and $(3) + (4)$ yields

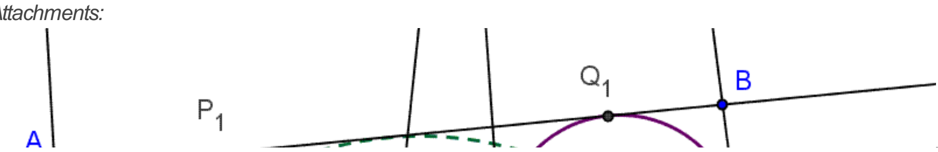
$$XY + WZ - (YZ + XW) = (AB + DC) - (BC + AD)$$

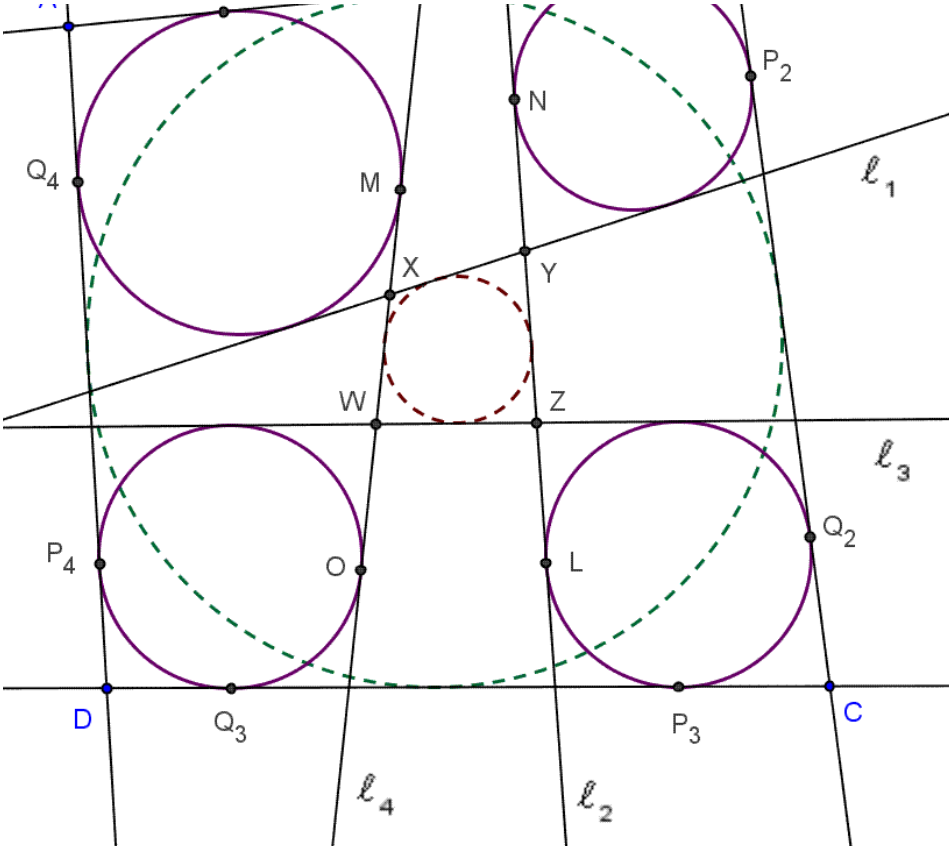
$$XY + WZ = YZ + XW \Longleftrightarrow AB + DC = BC + AD$$

$XYZW$ has an incircle $\Longleftrightarrow ABCD$ has an incircle.

P.S. See also the highlighted theorem in the topic Tangential quadrilateral.

Attachments:





ACCCGS8

#4 Jul 4, 2012, 5:02 pm

I managed to complete my proof.
Let the incentres of $BFPE$, $AEPH$, $HPGD$ and $PHCG$ be K, L, M and N, respectively. Observe that L, P, N are collinear, K, P, M are collinear, and that these lines are perpendicular. Note that $\angle KEL = \angle LHM = \angle MGN = \angle NFK = 90$ and that $\angle EPL = \angle HPL = \angle FPN = \angle GPN$. Let these angles equal x.

Now set up a Cartesian Coordinates system where LPN is the y-axis, KPM is the x-axis and P is the origin. Let $L = (0, a), M = (b, 0), N = (0, -c)$ and $K = (-d, 0)$.
Let $H = (p, ptan(90 - x))$. Using LH is perpendicular to MH, we obtain

$$x = \frac{a + btan(90 - x)}{(sec(90 - x))^2}$$

so the length of PH is $\frac{a + btan(90 - x)}{sec(90 - x)}$.
Similarly, we can determine the lengths of PG, PE and PF, and then it immediately follows that $HF = EG$.

The proof above is much nicer than this, but this is my ugly coordinates solution for anybody interested. Nice problem, though!

Quick Reply

High School Olympiads

About Isodynamic points and Isogonic points

Reply

USJL

#1 Nov 21, 2015, 11:18 pm

Given a triangle ΔABC. Let F_1 be the first-isogonic points, and S_1, S_2 be the first- and second-isodynamic points. Let $S_1^{'}$ be the reflection of S_1 wtr BC. Proof that $\angle AS_2S_1 = \angle F_1S_2S_1^{'}$

This post has been edited 1 time. Last edited by USJL, Nov 22, 2015, 12:46 pm
Reason: typo

TelvCohl

#3 Nov 22, 2015, 1:32 am • 2

Let B_1, C_1 be the second intersection of AB, AC with A-apollonius $\mathcal{A} \equiv \odot(AS_1S_1'S_2)$ of $\triangle ABC$, resp. Let T be the point s.t. $\triangle BTC$ is an equilateral triangle ($T \in AF_1$). Since B, C is the image of each other under the Inversion $\mathbf{I}(\mathcal{A})$, so B_1 and C_1 are symmetry WRT $BC \Longrightarrow B_1C_1 \parallel S_1S_1'$, hence AS_1' is the isogonal conjugate of AS_1 WRT $\angle BAC \Longrightarrow F_1 \in AS_1'$.

From $BS_1' : CS_1' = BS_2 : CS_2 \Longrightarrow C$ lies on the B-apollonius circle $\odot(V)$ of $\triangle BS_1'S_2$, so combine $\measuredangle CVB = \measuredangle CS_1'B + \measuredangle CS_2B = 120^\circ$ we get $\odot(V)$ is the circumcircle of $\triangle BTC$, hence S_1' and S_2 is the image of each other under the Inversion $\mathbf{I}(\odot(V)) \Longrightarrow F_1, S_2, T, V$ are concyclic ($\because F_1, S_1', T$ are collinear) $\Longrightarrow \measuredangle AS_2S_1 = \measuredangle AS_1'S_1 = \measuredangle F_1TV = \measuredangle F_1S_2S_1'$.

Luis González

#4 Nov 23, 2015, 10:46 am • 3

Generalization: O is circumcenter of $\triangle ABC$ and P is a point on its A-Apollonius circle Ω_A. Q is isogonal conjugate of P WRT $\triangle ABC$ and OP cuts Ω_A again at R. If M is the reflection of P on BC, then $\angle ARQ = \angle PRM$.

Proof: Let N, L be the reflections of P on AC, AB. Since P is on the A-Apollonius circle of $\triangle ABC$, then $\triangle MNL$ (homothetic to the pedal triangle of P) is M-isosceles $\Longrightarrow MNAL$ is a kite $\Longrightarrow AM \perp NL \Longrightarrow AM$ is the isogonal of AP $\Longrightarrow Q \in AM$. If AP cuts (O) again at J, then for any pair P, Q of isogonal conjugates, we have $\triangle AQO \sim \triangle PMJ$ (well-known). Therefore

$$\frac{OA}{AQ} = \frac{R}{AQ} = \frac{PJ}{PM} \Longrightarrow \frac{AP}{AQ} \cdot R = \frac{AP \cdot PJ}{PM}.$$

But since $(O) \perp \Omega_A$, then $AP \cdot PJ = OP \cdot PR$ and from $R^2 = OP \cdot OR$, we get $\triangle OAR \sim \triangle OPA$ yielding $\frac{AR}{AP} = \frac{R}{OP}$. Thus

$$\frac{AP}{AQ} \cdot \frac{R}{OP} = \frac{PR}{PM} \Longrightarrow \frac{PR}{PM} = \frac{AP}{AQ} \cdot \frac{AR}{AP} = \frac{AR}{AQ}.$$

Since $\angle RAQ = \angle RPM$, due to the cyclic $PARM$, then the latter expression reveals that $\triangle RAQ \sim \triangle RPM$ by SAS $\Longrightarrow \angle ARQ = \angle PRM$.

TelvCohl

#5 Nov 24, 2015, 4:40 am • 1

> *Luis González wrote:*
> **Generalization:** O is circumcenter of $\triangle ABC$ and P is a point on its A-Apollonius circle Ω_A. Q is isogonal conjugate of P WRT $\triangle ABC$ and OP cuts Ω_A again at R. If M is the reflection of P on BC, then $\angle ARQ = \angle PRM$.

Another approach :

From $P \in \Omega_A \Longrightarrow$ the pedal triangle $\triangle P_aP_bP_c$ of P WRT $\triangle ABC$ is an P_a -isosceles triangle, so $\measuredangle CQA = \measuredangle AQB$, hence if AQ cuts $\odot(BQC)$ at T then we get $\measuredangle TBC = \measuredangle BCT \Longrightarrow TO$ is the perpendicular bisector of BC. Since B and C is the image of each other under the Inversion $\mathbf{I}(\Omega_A)$, so if AB, AC cuts Ω_A again at B_1, C_1, respectively then we get $B_1C_1 \parallel PM \Longrightarrow AM$ is the isogonal conjugate of AP WRT $\angle BAC \Longrightarrow M$ lies on AQ.

From $CM : CR = BM : BR$ we know C lies on the B-apollonius circle $\odot(V)$ of $\triangle BMR$, so combine $\measuredangle CVB = \measuredangle CMB + \measuredangle CRB = \measuredangle BPC + \measuredangle CRB = \measuredangle BPC + \measuredangle CAB + \measuredangle CQB = 2\measuredangle CQB$ we get $\odot(V)$ is the circumcircle of $\triangle BQC$, hence we conclude that Q, R, T, V are concyclic $\Longrightarrow \measuredangle ARP = \measuredangle AMP = \measuredangle QTV = \measuredangle QRM$.

Quick Reply

High School Olympiads

Isogonal conjugate problem

Reply

Source: VMEO IV, problem 2

kaito_shinichi

#1 Nov 22, 2015, 2:36 am

Given triangle ABC and P, Q are two isogonal conjugate points in $\triangle ABC$. AP, AQ intersects (QBC) and (PBC) at M, N, respectively (M, N be inside triangle ABC)
1. Prove that M, N, P, Q locate on a circle - named (I)
2. $MN \cap PQ$ at J. Prove that IJ passed through a fixed line when P, Q changed

Luis González

#2 Nov 22, 2015, 4:21 am

AP cuts $\odot(PBC)$ and $\odot(QBC)$ again at Q', N', respectively and AQ cuts $\odot(PBC)$ and $\odot(QBC)$ again at M', P', respectively. It's known that the inversion with center A and power $AB \cdot AC$ followed by the symmetry on the angle bisector of $\angle BAC$ swaps $\odot(PBC)$ and $\odot(QBC)$, thus this swaps $\{P, P'\}, \{Q, Q'\}$ $\{M, M'\}, \{N, N'\} \Longrightarrow$ $AP \cdot AP' = AN \cdot AN' \Longrightarrow PN \parallel P'N'$, thus by Reim's theorem, it follows that P', Q', M', N' are concyclic $\Longrightarrow$ P, Q, M, N are concyclic.

PN, MQ, BC are pairwise radical axes of $\odot(ABC), \odot(PBC)$ and $\odot(QBC)$, concurring at their radical center R. Thus if AR cuts $\odot(ABC)$ again at X, we have $RX \cdot RA = RB \cdot RC = RP \cdot RN \Longrightarrow X \in \odot(APN) \Longrightarrow X$ is the Miquel point of the complete cyclic $PMQN \Longrightarrow I, J, X$ are collinear and $\overline{IJX} \perp AR$ (well-known), i.e. $\angle AXI = 90^\circ \Longrightarrow IJ$ passes through the antipode of A on $\odot(ABC)$, obviously fixed.

Quick Reply

High School Olympiads

Unexpected collinearity

Reply

sunken rock
#1 Nov 18, 2015, 11:44 pm

Let ABC be a $C-$right-angled triangle, CD its altitude, I_a, I_b the incenters of triangles CAD, CDB and O' the circumcenter of $\triangle CI_aI_b$.
Then IO' passes through the midpoint of AB, I being the incenter of $\triangle ABC$.

Comment: I came up with this while trying to solve other problem. Is it known?

Best regards,
sunken rock

Luis González
#2 Nov 19, 2015, 10:06 am

Let M be the midpoint of $AB.$ The incircle (I) and C-excircle touch AB at $X, Y,$ resp. It's known that D, X, I_a, I_b lie on a same circle ω (for the general configuration see two problems about cyclic quadrilateral (P1), incenters and cyclic, etc). Moreover, $O' \in \omega$ since $\angle I_aO'I_b = 2\angle I_aCI_b = 90^\circ$ and clearly X is the midpoint of the arc I_aDI_b of $\omega \Longrightarrow XO'$ is perpendicular bisector of $I_aI_b \Longrightarrow XO' \perp I_aI_b.$

On the other hand, easy angle chase reveals that $BI \perp CI_a$ and similarly $AI \perp CI_b \Longrightarrow I$ is orthocenter of $\triangle CI_aI_b \Longrightarrow CI \perp I_aI_b.$ Therefore $CI \parallel XO' \Longrightarrow CIXO'$ is a parallelogram. So if X' is the antipode of X on $(I),$ then it follows that $CX'IO'$ is also a parallelogram $\Longrightarrow IO' \parallel \overline{CX'Y} \Longrightarrow IO'$ is X-midline of $\triangle XYX'$ cutting XY at its midpoint $M,$ i.e. $M \in IO'.$

sunken rock
#3 Nov 19, 2015, 1:35 pm

As mentioned, I is the orthocenter of $\triangle CI_aI_b$ and $m(\widehat{I_bCI_a}) = 45^\circ$, and easy angle chase shows CI tangent to circles $\odot(CAI_a), \odot(BCI_b)$, consequently $CI^2 = II_a \cdot IA = II_b \cdot IB$, meaning that inversion of pole I and power IC^2 sends the circle $\odot(CI_aI_b)$ to $\odot(ABC)$, done.

Best regards,
sunken rock

suli
#4 Nov 20, 2015, 9:38 am

It is easy to prove that DI_AI_B and DCB are spirally similar, hence AI_AI_BB is cyclic and so angle chasing, some similar triangles, and the symmedian lemma prove that the midpoint of AB and O' lie on the symmedian of triangle II_AI_B through I.

jayme
#5 Nov 20, 2015, 3:53 pm

Dear S. and Mathlinkers,

just have a look at

http://jl.ayme.pagesperso-orange.fr/Docs/Le%20theoreme%20de%20Feuerbach-Ayme.pdf p. 20-23

Sincerely

Jean-Louis

Dukejukem

#6 Nov 21, 2015, 3:46 am

Denote $X_a \equiv BI \cap CI_a, X_b \equiv AI \cap CI_b$ and let ω be the circumcircle of $\triangle ACD.$

Straightforward angle chasing reveals that $\triangle CI_aX_b$ and $\triangle CI_bX_a$ are 45-45-90 triangles, and I is the orthocenter of $\triangle CI_aI_b.$ Then note that $\angle AX_bC = \angle ADC = 90^{\circ} \implies X_b \in \omega.$ But as AX_b bisects $\angle CAD$, X_b is the midpoint of arc $\widehat{CD}$ on $\omega \implies X_bC = X_bD.$ Similarly, $X_aC = X_aD$, and it follows that X_aX_b is the perpendicular bisector of $\overline{CD}.$ Hence, $X_aX_b \parallel AB$, so it is sufficient to show that IO' bisects $\overline{X_bX_c}.$ We will in fact show that $IX_aO'X_b$ is a parallelogram. Indeed, note that $X_aC = X_aI_b \implies X_aO'$ is the perpendicular bisector of $\overline{CI_b}.$ But since $IX_b \perp CI_b$, it follows that $X_aO' \parallel IX_b.$ Similarly, $X_bO' \parallel IX_a$, and we are done.

This post has been edited 1 time. Last edited by Dukejukem, Nov 21, 2015, 3:48 am

Quick Reply

High School Olympiads

Common tangents to the ex-circle and the circumcircle.

Reply

Source: Unknown

anantmudgal09

#1 Nov 19, 2015, 3:31 am

In $\triangle ABC$, let ω be the A-ex-circle and γ be the circumcircle. Let lines l_1, l_2 be tangent to both the circles externally. Let the bisectors of $\angle B, \angle C$ meet the opposite sides at M, N. Let MN meet γ again at X, Y with X closer to l_1 and Y closer to l_2.Prove that l_1, l_2 pass through X, Y respectively and furthermore, if l_1 meets BC at P and l_2 meets BC at Q then $\angle PAB = \angle QAC$.

(l_1 is closer to B than C and l_2 closer to C than B.)

Luis González

#2 Nov 19, 2015, 8:08 am

Discussed before at http://www.artofproblemsolving.com/community/c6h385175.

Quick Reply

High School Olympiads

Isosceles triangle with inscribed semicircle

Reply

Source: Cono Sur 2008 #5

fprosk

#1 Nov 18, 2015, 2:11 am

Let ABC be an isosceles triangle with base AB. A semicircle Γ is constructed with its center on the segment AB and which is tangent to the two legs, AC and BC. Consider a line tangent to Γ which cuts the segments AC and BC at D and E, respectively. The line perpendicular to AC at D and the line perpendicular to BC at E intersect each other at P. Let Q be the foot of the perpendicular from P to AB. Show that
$\frac{PQ}{CP}=\frac{1}{2}\frac{AB}{AC}$.

Luis González

#2 Nov 18, 2015, 4:05 am

Let M be the midpoint of AB and let J be the second intersection of $\odot(CDE)$ with CM (projection of P on CM). If r_C and R_C denote the radii of Γ and $\odot(CDE)$, then the power of the C-excenter M of $\triangle CDE$ WRT its circumcircle, using Euler's theorem, is $MJ \cdot MC=2R_C \cdot r_C=CP \cdot r_C \Longrightarrow \frac{PQ}{CP}=\frac{MJ}{CP}=\frac{r_C}{MC}=\frac{AM}{AC}=\frac{1}{2}\frac{AB}{AC}$.

Quick Reply

High School Olympiads

Thebault circle

Reply

Lin_yangyuan

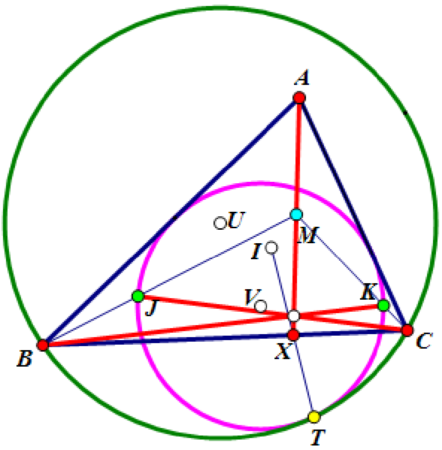
#1 Nov 15, 2015, 11:58 pm

it's not difficult actually

Attachments:

已知⊙U经过B,C，⊙V内切⊙U于T，且与AB,AC相切，I为内心，IT交BC于X，M为AX上一点，MB交⊙V于J，MC交⊙V于K，求证AX,BK,CJ共点.

Given △ABC and ⊙U passing through B, C . Let ⊙V be the circle tangent to AB, AC and tangent to ⊙(O) (internally) at T, I is the incenter of △ABC.X≡IT∩BC, M is onAX, J≡MB∩⊙V, K≡MC∩⊙V.
Prove that AX,BK,CJ are concurrent.



Luis González

#2 Nov 16, 2015, 12:23 am

Let (V) touch AC, AB at E, F and let P be the midpoint of the arc BC of (U). It's known that TI bisects $\angle BTC$ and that BC, PT, EF concur at a point Y (see Internally tangent circles and lines and concurrency for a proof). Thus $(B, C, X, Y) = -1 \Longrightarrow A(E, F, X, Y) = -1 \Longrightarrow AX$ is the polar of Y WRT (V). Now from here the problem is merely projective, i.e. projecting the line AY, not cutting (V), to infinity and (V) into antother circle, then the concurrency follows by obvious symmetry

Quick Reply

High School Olympiads

Triangle with circles touching incircle

Reply

Source: India Postals 2015

utkarshgupta
#1 Nov 15, 2015, 3:57 pm • 1

Let ABC be at triangle with incircle Γ. Let Γ_1, Γ_2, Γ_3 be three circles inside $\triangle ABC$ each of which is tangent to Γ and two sides of the triangle and their radii are $1, 4, 9$. Find the radius of Γ.

Luis González
#4 Nov 15, 2015, 10:42 pm • 2

Label r, r_1, r_2, r_3 the radii of $\Gamma, \Gamma_1, \Gamma_2, \Gamma_3$, respectively. $\Gamma \equiv (I)$ touches AB at F and Y is the projection of I_1 on IF. Thus

$$\sin \frac{A}{2} = \frac{IY}{II_1} = \frac{r - r_1}{r + r_1} \Longrightarrow \frac{r_1}{r} = \frac{1 - \sin \frac{A}{2}}{1 + \sin \frac{A}{2}} = \tan^2 \left(45^\circ - \frac{A}{4}\right) \Longrightarrow$$

$$\sqrt{\frac{r_1}{r}} = \tan \left(45^\circ - \frac{A}{4}\right) = \tan \left(\frac{90^\circ - \frac{A}{2}}{2}\right) = \tan \frac{\lambda_A}{2}.$$

Since $\lambda_A + \lambda_B + \lambda_C = 180^\circ$, then we have the well-known trigonometric identiy
$\tan \frac{\lambda_B}{2} \cdot \tan \frac{\lambda_C}{2} + \tan \frac{\lambda_C}{2} \cdot \tan \frac{\lambda_A}{2} + \tan \frac{\lambda_A}{2} \cdot \tan \frac{\lambda_B}{2} = 1 \Longrightarrow$

$$\sqrt{\frac{r_1}{r}} \cdot \sqrt{\frac{r_2}{r}} + \sqrt{\frac{r_2}{r}} \cdot \sqrt{\frac{r_3}{r}} + \sqrt{\frac{r_3}{r}} \cdot \sqrt{\frac{r_1}{r}} = 1 \Longrightarrow r = \sqrt{r_1 \cdot r_2} + \sqrt{r_2 \cdot r_3} + \sqrt{r_3 \cdot r_1}$$

$$\Longrightarrow r = \sqrt{1 \cdot 4} + \sqrt{4 \cdot 9} + \sqrt{9 \cdot 1} = 11.$$

Quick Reply

High School Olympiads

Not easy

izat

#1 Nov 14, 2015, 7:52 pm

Given four different circles T_1, T_2, T_3, T_4 .
T_1, T_3 are externally touch each other at X, and T_2, T_4 are also touch each other externally at X. Let T_1 and T_2, T_2 and T_3,T_3 and T_4, T_4 and T_1 intersect at A, B, C, D,respectively. Show that $AB * BC * XD^2 = CD * DA * XB^2$

Luis González

#2 Nov 14, 2015, 9:57 pm

Invert with center X and arbitrary power k^2, denoting inverse images with primes. Since T_1 and T_3 are tangent at X, then $B'C' \parallel D'A'$ and similarly $A'B' \parallel C'D' \Longrightarrow A'B'C'D'$ is a parallelogram. By inversion property we have $\frac{A'B'}{AB} = \frac{k^2}{XA \cdot XB}$ and the cyclic expressions, thus

$$\frac{AB \cdot BC}{A'B' \cdot B'C'} = \frac{XB^2 \cdot XA \cdot XC}{k^4} \quad , \quad \frac{CD \cdot DA}{C'D' \cdot D'A'} = \frac{XD^2 \cdot XA \cdot XC}{k^4}$$

Since $A'B' = C'D'$ and $B'C' = D'A' \Longrightarrow \dfrac{AB \cdot BC}{CD \cdot DA} = \dfrac{XB^2}{XD^2}$.

High School Olympiads

Plane geometry

kormidscoler

#1 Nov 14, 2015, 7:02 pm

There is a triangle ABC that has orthocenter H and circumcircle @ with a center O. An intersection between line AO and @ is D. Let X and Y be the foot of perpendicular each from H to lines BD and CD. Let M be a midpoint of BC. Let circumcenter of triangle MXY be J. Prove that HJ, and lines that are tangent to @ at B,C are concurrent.

Luis González

#2 Nov 14, 2015, 8:40 pm

Let X',Y' be the reflections of H across DB,DC and let AH,BH,CH cut $\odot(ABC)$ again at $P,Q,R,$ resp. Tangents of $\odot(ABC)$ at B,C meet at $T.$

Since BA,BC are the perpendicular bisectors of $HR,HP,$ then it follows that H,R,P,X' lie on a circle with center B and since $PR \parallel TB,$ then $(TB \parallel PR) \perp PX' \Longrightarrow TP=TX'.$ But by obvious symmetry $P \in \odot(T,TD) \Longrightarrow$ $X' \in \odot(T,TD)$ and likewise $Y' \in \odot(T,TD) \Longrightarrow T$ is circumcenter of $\triangle DX'Y'.$ Thus by the homothety $(H,\tfrac{1}{2}),$ we deduce that the circumcenter J of $\triangle MXY$ is the midpoint of HT and the conclusion follows.

kormidscoler

#3 Nov 14, 2015, 8:53 pm

Good solution, luis gonzalez. Can you find a solution using homothenter as H?

High School Olympiads

Coaxal circles

Reply

jayme
#1 Oct 13, 2014, 6:24 pm

Dear Mathlinkers,

1. ABC a triangle
2. P a point
3. A'B'C' the P-cevian triangle
4. Q the pivot point wrt ABC and A', B', C'
5. (1), (2), (3) the circumcircles wrt to AA'Q, BB'Q, CC'Q.

Prove : (1), (2) and (3) are coaxal.

Sincerely
Jean-Louis

aleksam
#2 Oct 13, 2014, 6:40 pm

Can you explain please, what is exactly Q?

jayme
#3 Oct 13, 2014, 6:44 pm

Dear Mathlinkers,
Q is the point of concurs of the circles (AB'C'), (BC'A') and (CA'B')...

Sincerely
Jean-Louis

Luis González
#4 Oct 28, 2014, 10:49 am • 2

Let τ_a, τ_b, τ_c denote the radical axes of the circumcircle (O) of $\triangle ABC$ with $\odot(AB'C'), \odot(BC'A'), \odot(CA'B')$, resp. τ_b, τ_c and QA' are pairwise radical axes of $(O), \odot(BC'A')$ and $\odot(CA'B')$ concurring at their radical center X and similarly we have $Y \equiv \tau_c \cap \tau_a \cap QB'$ and $Z \equiv \tau_a \cap \tau_b \cap QC'$. Hence if AX, BY, CZ cut (O) again at $U, V, W,$ we have $XU \cdot XA = XQ \cdot XA' \Longrightarrow U \in \odot(AQA')$ and similarly $V \in \odot(BQB')$ and $W \in \odot(CQC')$.

Since AA', BB', CC' concur at P and XA', YB', ZC' concur at Q, then by Cevian Nest Theorem, $AX \equiv AU$, $BY \equiv BV$ and $CZ \equiv CW$ concur at a point R. Thus, $\odot(AQA'), \odot(BQB')$ and $\odot(CQC')$ are coaxal with common radical axis QR.

TelvCohl
#5 Oct 21, 2015, 2:49 am • 1

We prove the stronger result as following :

Given a $\triangle ABC$ and a point P. Let $\triangle DEF$ be the cevian triangle of P WRT $\triangle ABC$ and let Q be the Miquel point of D, E, F WRT $\triangle ABC$. Then $\odot(AQD), \odot(BQE), \odot(CQF)$ are coaxial and the second intersection of these three circles is the isogonal conjugate (WRT $\triangle ABC$) of the antigonal conjugate of P WRT $\triangle ABC$.

Proof : Let X be the Miquel point of the complete quadrilateral $\{CA, AB, BP, CP\}$ (define Y, Z similarly). Since X is the

Proof : Let X be the Miquel point of the complete quadrilateral $\{CA, AB, BP, CP\}$ (define Y, Z similarly). Since X is the center of the spiral similarity of $BF \mapsto EC$ and $FA \mapsto PE$, so we get $XB \cdot XC = XE \cdot XF = XA \cdot XP$ and $\angle CXB$, $\angle EXF, \angle PXA$ share the same angle bisector ℓ, hence if Ψ_X is the Inversion with center X and factor $XB \cdot XC$ followed by the reflection in ℓ then $(B, C), (E, F), (A, P)$ is the image of each other under Ψ_X. From $Y \in \odot(APF)$ and $\odot(BCF)$ we get $\Psi_X(Y) \in \odot(PAE)$ and $\odot(CBE)$, so $\Psi_X(Y)$ coincide with Z. i.e. Ψ_X also maps Y, Z to each other

Let $W \equiv \odot(CAY) \cap \odot(ABZ)$. Since $\odot(CAY), \odot(ABZ)$ is the image of $\odot(BPZ), \odot(PCY)$, respectively under Ψ_X, so W is the image of D under $\Psi_X \Longrightarrow \triangle XDC \overset{+}{\sim} \triangle XBW$, hence we get $\measuredangle XCB = \measuredangle XWB \Longrightarrow W$ lies on $\odot(BCX)$. Furthermore, from $\triangle XDP \overset{+}{\sim} \triangle XAW$ we get $\measuredangle XPA = \measuredangle XWA$, so $W \in \odot(APX)$. Analogously, we can prove W lies on $\odot(BPY)$ and $\odot(CPZ)$, so we conclude that $\odot(BCX), \odot(CAY), \odot(ABZ), \odot(APX), \odot(BPY)$, $\odot(CPZ)$ have a common point W. ...... $(\bigstar)$

Let Φ_W be the Inversion with center W. From $(\bigstar)$ we get $\triangle\Phi_W(X)\Phi_W(Y)\Phi_W(Z)$ is the cevian triangle of $\Phi_W(P)$ WRT $\triangle\Phi_W(A)\Phi_W(B)\Phi_W(C)$. Since D lies on $\odot(CAZ)$ and $\odot(ABY)$, so $\Phi_W(D)$ lies on $\odot(\Phi_W(C)\Phi_W(A)\Phi_W(Z))$ and $\odot(\Phi_W(A)\Phi_W(B)\Phi_W(Y))$, hence $\Phi_W(D)$ is the Miquel point of the complete quadrilateral with the sides $\Phi_W(C)\Phi_W(A)$, $\Phi_W(A)\Phi_W(B), \Phi_W(B)\Phi_W(P), \Phi_W(C)\Phi_W(P)$ (similar discussion for $\Phi_W(E), \Phi_W(F)$).

From Three concurrent radical axes $\Longrightarrow \Phi_W(B)\Phi_W(F), \Phi_W(C)\Phi_W(E), \Phi_W(P)\Phi_W(D)$ are concurrent (at the isogonal conjugate of the complement (WRT $\triangle\Phi_W(B)\Phi_W(P)\Phi_W(C)$) of $\Phi_W(A)$ WRT $\triangle\Phi_W(B)\Phi_W(P)\Phi_W(C)$), so $\odot(WBF), \odot(WCE), \odot(WPD)$ are coaxial, hence notice $\odot(WBF), \odot(WCE)$ is the image of $\odot(DCE), \odot(DBF)$, respectively under Ψ_X we get $W \in \odot(AQD)$. Analogously, we can prove W lies on $\odot(BQE)$ and $\odot(CQF)$, so $\odot(AQD), \odot(BQE), \odot(CQF)$ are coaxial with common points Q, W.

Finally, from $\measuredangle BWC + \measuredangle CPB = \measuredangle BXC + \measuredangle CPB = \measuredangle BXP + \measuredangle PXC + \measuredangle CPB = \measuredangle(AB, CP) + \measuredangle(BP, CA)$ $+ \measuredangle(CP, BP) = \measuredangle BAC$ and similarly we get $\measuredangle CWA + \measuredangle APC = \measuredangle CBA, \measuredangle AWB + \measuredangle BPA = \measuredangle ACB$ we conclude that W is the isogonal conjugate (WRT $\triangle ABC$) of the antigonal conjugate of P WRT $\triangle ABC$.

Luis González

#6 Nov 4, 2015, 8:44 am • 1

> *TelvCohl wrote:*
>
> We prove the stronger result as following :
>
> Given a $\triangle ABC$ and a point P. Let $\triangle DEF$ be the cevian triangle of P WRT $\triangle ABC$ and let Q be the Miquel point of D, E, F WRT $\triangle ABC$. Then $\odot(AQD), \odot(BQE), \odot(CQF)$ are coaxial and the second intersection of these three circles is the isogonal conjugate (WRT $\triangle ABC$) of the antigonal conjugate of P WRT $\triangle ABC$.

Here's another approach. We use directed angles mod 180º throughout the proof.

From previous post #4 we known that $\odot(AQD), \odot(BQE), \odot(CQF)$ meet at a 2nd point R. Let $\odot(BQE)$ and $\odot(CQF)$ cut CA, AB again at Y, Z, resp. Thus we have $\angle(RQ, RZ) = \angle(FQ, FA) = \angle(EQ, EY) = \angle(RQ, RY) \Longrightarrow$ $R \in YZ$. Hence since $\angle(RB, RZ) = \angle(EB, EC)$ and $\angle(RY, RC) = \angle(FB, FC) \Longrightarrow$ $\angle(RB, RC) = \angle(EB, EC) + \angle(FB, FC) = \angle(AB, AC) + \angle(PB, PC)$. But if K is the antigonal conjugate of P and K^* the isogonal conjugate of K, we have $\angle(K^*B, K^*C) = \angle(AB, AC) + \angle(KC, KB) = \angle(AB + AC) + \angle(PB, PC) \Longrightarrow K^* \in \odot(RBC)$ and similarly K^* also lies on $\odot(RCA)$ and $\odot(RAB) \Longrightarrow R \equiv K^*$.

Quick Reply

High School Olympiads

complement (isot (K)) and isot (complement (K))

Reply

Source: Own

TelvCohl
#1 Oct 31, 2015, 8:47 pm • 1

Given a $\triangle ABC$ with symmedian point K. Let P be the complement (WRT $\triangle ABC$) of the isotomic conjugate of K WRT $\triangle ABC$ and let Q be the isotomic conjugate (WRT $\triangle ABC$) of the complement of K WRT $\triangle ABC$. Prove that P, Q are isogonal conjugate WRT $\triangle ABC$.

XmL
#2 Nov 1, 2015, 2:35 am • 1

Cross Ratio Bash:

We will show that AP, AQ are isogonals wrt $\triangle ABC$.

A' the reflection of A over M, the midpoint of BC. Let AK, AK' intersect BC at X, X', where K' is the isotomic conjugate of K wrt $\triangle ABC$. Let $KA' \cap BC = Y$, then one has $Y \in AQ$. Since AX, AM are isogonals wrt $\angle A$, it suffices to show $A(M, C; X, P) = A(X, B; M, Y)$

It is not hard to see that $AP||A'K'$, hence:

$$A(X, B; M, Y) = K(A, B; M, A') = B(A, K; C, A') = B(C, K'; A, C') = K'(M, B; A, A') = A'(M, B; X', K') = A(M, C; X, P)$$

This post has been edited 1 time. Last edited by XmL, Nov 1, 2015, 2:35 am
Reason: Format

Luis González
#3 Nov 1, 2015, 3:08 am • 3

More general: P, Q are isogonal conjugates WRT $\triangle ABC$. K is symmedian point of $\triangle ABC$, U is the isotomcomplement of Q WRT $\triangle ABC$ and V is the Ceva point of $\{P, K\}$ WRT $\triangle ABC$. Then U, V are isogonal conjugates WRT $\triangle ABC$.

Let $\triangle A'B'C'$ be the tangential triangle of $\triangle ABC$ and let $\triangle Q_AQ_BQ_C$ be the cevian triangle of Q WRT $\triangle ABC$. If the tangent of $\odot(AQ_BQ_C)$ at A cuts BC at X, then X is on the polar of P WRT $\triangle ABC$ (see the topic X,Y,Z lie on polar of Q), thus $A'P$ is the polar of X WRT $\odot(ABC)$ cutting BC at the harmonic conjugate X' of X WRT $\{B, C\} \Longrightarrow$ pencil $A(B, C, X, X') \equiv A(B, C, X, V)$ is harmonic $\Longrightarrow AV$ is the A-symmedian of $\triangle AQ_BQ_C$ isogonal of its A-median AU. Similarly BU, BV and CU, VC are isogonals WRT $\triangle ABC \Longrightarrow U, V$ are isogonal conjugates WRT $\triangle ABC$.

Quick Reply

High School Olympiads

X,Y,Z lie on polar of Q

Reply

euclideangeometry

#1 Sep 8, 2010, 10:48 pm

Dear everyone,

Let ABC be a triangle and P be an arbitrary point. Let triangle DEF be cevian triangle of P wrt ABC.
X is intersection of BC and tangent line at A on circumcircle of triangle AEF and define Y,Z similarly.
Let Q be the isogonal conjugate point of P wrt triangle ABC. Prove that X,Y,Z lie on polar of Q wrt circumcircle of ABC.

interesting but unsolved problem.

Luis González

#2 Sep 19, 2010, 3:11 am

We use barycentric coordinates WRT $\triangle ABC$. Let $(u:v:w)$ be the coordinates of P.

$$D(0:v:w)\ ,\ E(u:0:w)\ ,\ F(u:v:0)\ ,\ Q\left(\frac{a^2}{u}:\frac{b^2}{v}:\frac{c^2}{w}\right)$$

Equation of circle Γ_a passing through points A, F, E is given by

$$\Gamma_a \equiv a^2yz+b^2xz+c^2xy-(x+y+z)\left(\frac{uc^2}{u+v}y+\frac{ub^2}{u+w}z\right)=0$$

Thus, tangent ℓ_a to Γ_a through $A(1:0:0)$ is $\ell_a \equiv \left(\frac{vc^2}{u+v}\right)y+\left(\frac{wb^2}{u+w}\right)z=0$

Line ℓ_a intersects BC at point X with coordinates $X\left(0:\frac{wb^2}{u+w}:-\frac{vc^2}{u+v}\right)$

By cyclic exchange of elements, we obtain the coordinates of the intersections Y, Z as:

$$Y\left(\frac{wa^2}{v+w}:0:-\frac{uc^2}{v+u}\right)\ ,\ Z\left(\frac{va^2}{w+v}:-\frac{ub^2}{w+u}:0\right)$$

$$\Longrightarrow XYZ \equiv \frac{u(v+w)}{a^2}x+\frac{v(w+u)}{b^2}y+\frac{w(u+v)}{c^2}z=0$$

It remains to show that line XYZ coincides with the polar τ of Q WRT circumcircle (O) of $\triangle ABC$. We know that polar of Q WRT (O) is the perspectrix of $\triangle ABC$ and the circumcevian triangle $\triangle Q_aQ_bQ_c$ of Q, namely

$$Q_a\left(-\frac{a^2}{w+v}:\frac{b^2}{v}:\frac{c^2}{w}\right)\ ,\ Q_b\left(\frac{a^2}{u}:-\frac{b^2}{u+w}:\frac{c^2}{w}\right)\ ,\ Q_c\left(\frac{a^2}{u}:\frac{b^2}{v}:-\frac{c^2}{u+v}\right)$$

$$\Longrightarrow \tau \equiv \frac{u(v+w)}{a^2}x+\frac{v(w+u)}{b^2}y+\frac{w(u+v)}{c^2}z=0 \Longrightarrow XYZ \equiv \tau.\ \square$$

Luis González

#3 Oct 26, 2015, 9:16 am • 2

Here is a proof without barycentric coordinates:

Let M be the Miquel point of DEF WRT $\triangle ABC$. Since $\angle XAF = \angle AEF$, then $\angle XAM = \angle AEF + \angle FAM = \angle AMF + \angle FAM = \angle MFB = \angle MDC$ (mod 180º) $\Longrightarrow X \in \odot(AMD) \Longrightarrow$ A and X lie on a circle coaxal with $\odot(BDF)$ and $\odot(CDF) \Longrightarrow$ their powers WRT $\odot(BDF)$ and $\odot(CDE)$ are in the same ratio $\Longrightarrow$

$$\frac{XB \cdot XD}{XC \cdot XD} = \frac{AF \cdot AB}{AE \cdot AC} \Longrightarrow \frac{XB}{XC} = \frac{AB}{AC} \cdot \frac{AF}{AE} \quad (1).$$

Let $\triangle UVW$ be the circumcevian triangle of Q WRT $\triangle ABC$. VW, WU, UV cut BC, CA, AB at $X', Y', Z' \Longrightarrow$ $\overline{X'Y'Z'}$ is the polar of Q WRT $\odot(ABC)$. Since $\triangle CBW \sim \triangle CFA$, then $\frac{WB}{WC} = \frac{AF}{AC}$ and similarly $\frac{VB}{VC} = \frac{AB}{AE}$. Therefore

$$\frac{X'B}{X'C} = \frac{WB}{WC} \cdot \frac{VB}{VC} = \frac{AF}{AC} \cdot \frac{AB}{AE} = \frac{AB}{AC} \cdot \frac{AF}{AE} \quad (2).$$

From (1) and (2), we get $X \equiv X'$ and by similar reasoning $Y \equiv Y', Z \equiv Z' \Longrightarrow \overline{XYZ}$ is the polar of Q WRT $\odot(ABC)$. $\blacksquare$

Remark: Combining this result with the property found in the problem Perspective triangles from tangents, we get the following:

The polar of a point P WRT the circumcircle of a reference $\triangle ABC$ coincides with the trilinear polar of the isogonal conjugate of the isotomcomplement of the isogonal conjugate of P.

TelvCohl

#4 Oct 27, 2015, 6:37 am • 3

Let BP, CP cuts $\odot(AEF)$ again at P_B, P_C, respectively. Let $\triangle Q_aQ_bQ_c$ be the circumcevian triangle of Q WRT $\triangle ABC$. From Pascal theorem (for EP_BP_CFAA) we get BC, P_BP_C and the tangent of $\odot(AEF)$ passing through A are concurrent, so X lies on P_BP_C. Since $\measuredangle BAQ_b = \measuredangle BEC$ ($\because \triangle BEC \sim \triangle BAQ_b$) $= \measuredangle P_BAX$, so combine $\measuredangle ABQ_b = \measuredangle P_BBX$ we get Q_b is the isogonal conjugate of P_B WRT $\triangle AXB$, hence XQ_b is the isogonal conjugate of P_BP_C WRT $\angle DXA$.

Similarly, we can prove XQ_c is the isogonal conjugate of P_BP_C WRT $\angle DXA$, so X, Q_b, Q_c are collinear $\Longrightarrow X$ lies on the polar τ of Q WRT $\odot(ABC)$. Analogously, we can prove Y, Z lie on τ, so $\overline{XYZ}$ is the polar of Q WRT $\odot(ABC)$.

XmL

#5 Oct 27, 2015, 6:51 am

This is equivalent to this little lemma immediate through cross ratios, no creativity here:

Lemma: Let D denote the intersection of the tangents to (ABC) at B, C. E is on segment BC. $P \in DE$. Q is the isogonal conjugate of P wrt ABC. Let $A'B'C'$ denote the cevian triangle of Q wrt ABC. M is the midpoint of $C'B'$. Prove that AM, AE are isogonals wrt $\angle A$.

Proof: Apply a dilation of 2 from A on M and the midpoint of BC, which are mapped to M', F; Q, M', F are collinear by a well known property of complete quad. Since AF, AD are isogonals wrt $\angle A$. Thus it suffices to show that $A(P, E; D, B) = A(Q, M'; F, C)$.

Reflection over the bisector of $\angle B$:

$$A(P, E; D, B) = B(P, E; D, A) = B(B', A; F, C)$$

Since $B'M'||AB||CF$, therefore $B(B', A; F, C) = B'(M', Q; C, F) = A(M', Q; C, F) = A(Q, M'; F, C)$ and we are done.

This post has been edited 1 time. Last edited by XmL, Oct 27, 2015, 7:00 am
Reason: mistyped cevian

Quick Reply

High School Olympiads

The triangle with minimal area

Reply

quantummath
#1 Oct 30, 2015, 2:20 pm

Given two intersecting straight lines AB,AC and a point P between them , prove that, of all the straight lines, which pass through P and are terminated by AB,AC the one which is bisected at P, cuts off the triangle with minimum area.

Luis González
#2 Oct 31, 2015, 9:12 am • 1

Arbitrary line through P cuts AB,AC at M,N and let ℓ be the unique line through $P,$ cutting AB,AC at $X,Y,$ such that P is the midpoint of $\overline{XY}.$ Since $[AMN]=\tfrac{1}{2}AM \cdot AN \cdot \sin \widehat{A}$ and $[AXY]=\tfrac{1}{2}AX \cdot AY \cdot \sin \widehat{A},$ then it suffices to show that $AM \cdot AN \ge AX \cdot AY.$

WLOG assume that X is between A and $M,$ thus Y is on the extension of AN beyond $N.$ By Menelaus' theorem for $\triangle AXY$ cut by $\overline{MPN},$ we get

$$\frac{MX}{MA}=\frac{NY}{AN} \Longrightarrow \frac{AM-AX}{AM}=\frac{AY-AN}{AN} \Longrightarrow \frac{AX}{AM}+\frac{AY}{AN}=2 \Longrightarrow$$

$$\sqrt{\frac{AX}{AM} \cdot \frac{AY}{AN}} \le \frac{\frac{AX}{AM}+\frac{AY}{AN}}{2}=1 \Longrightarrow AM \cdot AN \ge AX \cdot AY.$$

Quick Reply

High School Olympiads

very very hard geometry

Reply

termas

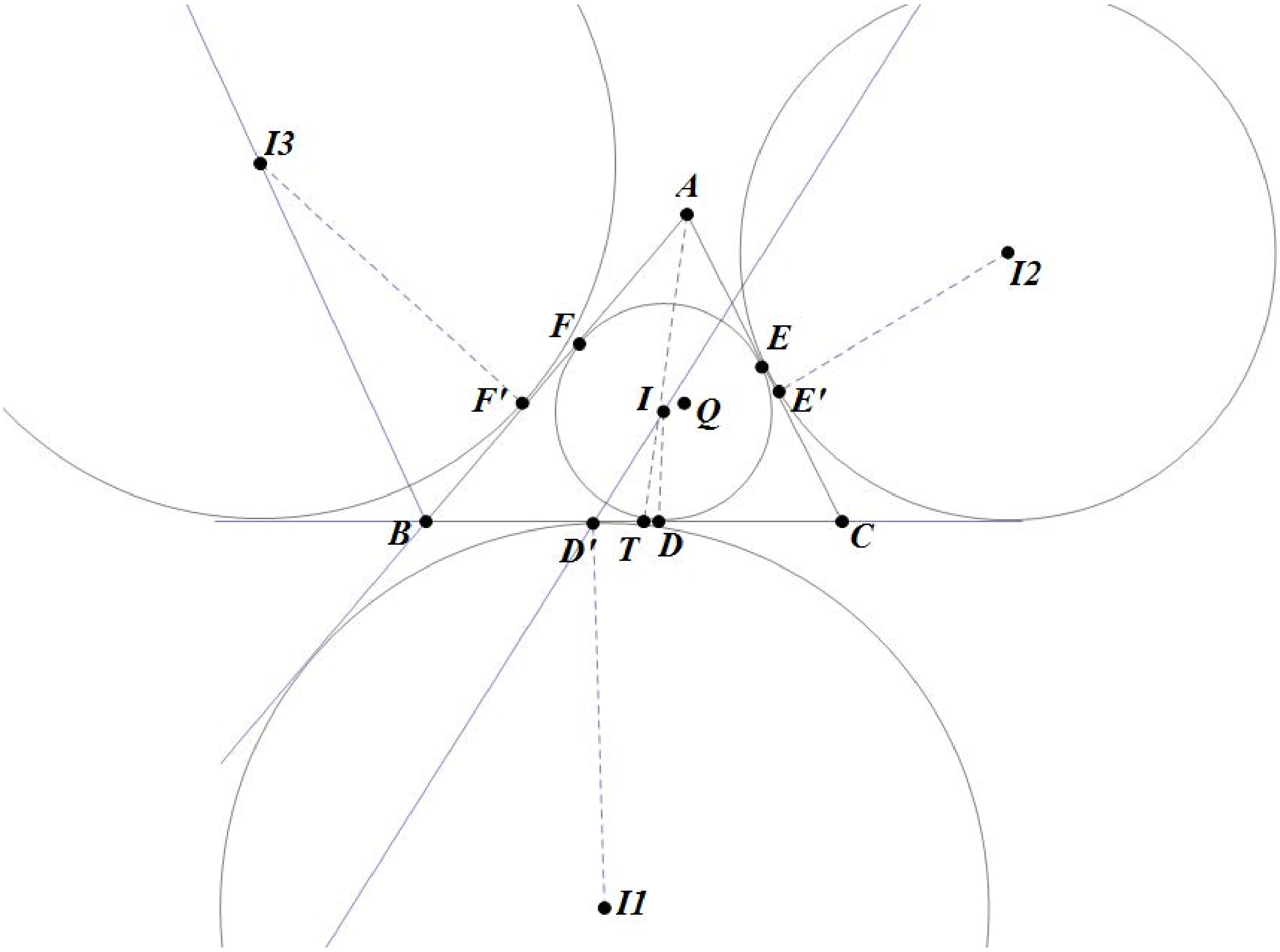
#1 Oct 30, 2015, 7:35 pm

The incircle of triangle ABC has center I and touchs the sides BC, CA, AB at the points D, E, F respectively,and Let the centers of the excircles tangent to BC, CA, AB be I_1, I_2, I_3 respectively, and the excircles be tangent to BC, CA, AB at D', E', F' respectively;let $Q = I_2E \bigcap I_3F$,and let AT is internal angle bisectors issued from A.we take $TN \perp BC$ meet ID' at point N, and M is midpoint QI

show that

$$\angle BAN = \angle CAM$$

Attachments:



Luis González

#2 Oct 31, 2015, 2:35 am

Let O, Na and Be be the circumcenter, Nagel point and Bevan point of $\triangle ABC$ (reflection of I on O). Isogonal conjugate of Na WRT $\triangle ABC$ is the exsimilicenter of incircle $\odot(I, r)$ and circumcircle $\odot(O, R)$ (well-known) and $Q \equiv I_2E \cap I_3F$ is the homothetic center of $\triangle DEF$ and $\triangle I_1I_2I_3$. i.e. exsimilicenter of their circumcircles $\odot(I, r)$ and $\odot(Be, 2R)$. Therefore we have

$$\frac{\overline{JI}}{\overline{JO}} = \frac{r}{R} \Longrightarrow \frac{\overline{OJ}}{\overline{OI}} = \frac{R}{R-r}, \frac{\overline{IJ}}{\overline{OI}} = \frac{r}{R-r}.$$

$$\frac{\overline{QBe}}{\overline{QI}} = \frac{2R}{r} \Longrightarrow \frac{\overline{QI} + 2\cdot\overline{OI}}{\overline{QI}} = \frac{2R}{r} \Longrightarrow \frac{\overline{QI}}{\overline{OI}} = \frac{2r}{2R-r} \Longrightarrow \frac{\overline{IM}}{\overline{IO}} = -\frac{r}{2R-r} \Longrightarrow$$

$$\frac{\overline{MJ}}{\overline{OI}} = \frac{Rr}{(R-r)(2R-r)} \Longrightarrow \frac{\overline{JM}}{\overline{JO}} = \frac{r}{2R-r} \Longrightarrow (I,M,O,J) = -1.$$

On the other hand, let X be the foot of the A-altitude of $\triangle ABC$ and ID' cuts AX at U. Then $(I,N,U,D') = (D,T,X,D') = (I,T,A,I_1) = -1 = (I,M,O,J) \Longrightarrow A(I,N,U,D') = A(I,M,O,J)$. Thus since $\{AU, AO\}$, and $\{AD', AJ\}$ are symmetric about AI (isogonals WRT $\angle BAC$), then it follows that AN, AM are symmetric about $AI \Longrightarrow \angle BAN = \angle CAM.$

XmL

#3 Oct 31, 2015, 6:42 am

Today must be opposite day.

Let AD' intersect ID, TN at X, Z; it is known that X lies on the incircle (I), so N is the midpoint of ZT, and thus AN bisects XI.

Dilate at I with ratio 2 and $A \mapsto I'$. Now it suffices to show lines $I'X, I'Q$ are symmetric over AI. Reflect X over AI to obtain Y, then $YEF, I'I_2I_3$ are homothetic thus Q, Y, I' are collinear and we are done.

Quick Reply

High School Olympiads

Sum of inradius

Locked

trunglqd91
#1 Oct 30, 2015, 10:22 pm

Let the circle (I,r) be in the circle (O,R) and they are tangent to each other at T. $\mathcal{D}$ is the tangency of (O) and (I) at T. A is varying in (O). From A, contruct 2 tangencies of (I) and cut $\mathcal{D}$ at B and C. Prove that the sum of $r(ABT)$ and $r(ACT)$ is constant which $r(ABT)$ is the inradius of $\triangle ABT$.

Luis González
#2 Oct 30, 2015, 11:56 pm

Already posted at http://www.artofproblemsolving.com/community/c6h575807.

High School Olympiads

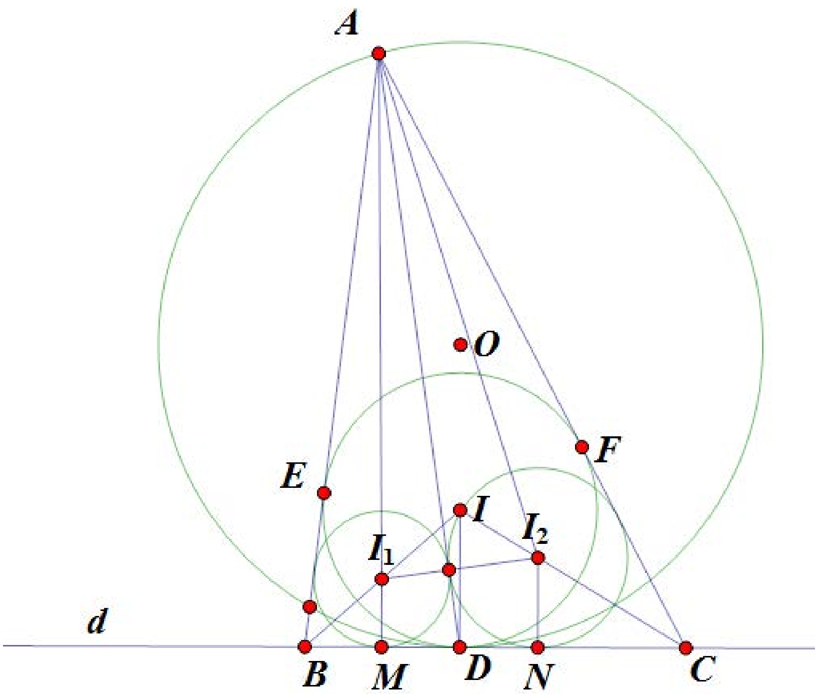
Constant

Reply

xiaotabuta

#1 Feb 12, 2014, 11:04 pm

$d, D, (O), (I)$ are fixed.
A moving in (O).
Tangent AB, AC with (I) (B, C are in d)
Prove that $r_{ABD} + r_{ACD}$ is constant with r is the radius of the incircle



Luis González

#2 Feb 13, 2014, 3:29 am • 2

Denote R and r the radii of (O) and (I), respectively. Let (I_1, r_1) and (I_2, r_2) touch AD at X and X'. Then

$$DX = \tfrac{1}{2}(AD - AB + DB) = \tfrac{1}{2}[AD - AB + \tfrac{1}{2}BC + \tfrac{1}{2}AB - \tfrac{1}{2}AC] =$$

$$= \tfrac{1}{2}\left[AD - \tfrac{1}{2}(AC + AB - BC)\right] = \tfrac{1}{2}(AD - AE).$$

Similarly, we have $DX' = \frac{1}{2}(AD - AE) \Longrightarrow X \equiv X'$, i.e. (I_1) and (I_2) are externally tangent at X.

Let DI_2, DI_1 cut (O) again at $U, V. \angle I_1DI_2 \equiv \angle VDU = 90^\circ \Longrightarrow UV$ is diameter of (O). Further, $\angle DUV = \angle(DI_1, d) = \angle ADI_1 = \angle DI_2I_1 \Longrightarrow UV \parallel I_1I_2 \perp DXA \Longrightarrow UV$ is perpendicular bisector of $\overline{AD}$ cutting it at its midpoint Y. Hence

$$\frac{I_1I_2}{UV} = \frac{r_1 + r_2}{2R} = \frac{DX}{DY} = \frac{\frac{1}{2}(AD - AE)}{\frac{1}{2}AD} \Longrightarrow \frac{r_1 + r_2}{2R} = 1 - \frac{AE}{AD} \quad (1).$$

By Casey's chord theorem (see theorem 1 at Casey's theorem and its applications) for (I) and (A) both internally tangent to (O, R), we get

$$AE = \frac{AD}{R}\sqrt{(R-0)\cdot(R-r)} \Longrightarrow \frac{AE}{AD} = \frac{\sqrt{R(R-r)}}{R} \ \ (2).$$

From (1) and (2), we get $r_1 + r_2 = 2R - 2\sqrt{R(R-r)} = \text{const}.$

Luis González

#3 Feb 13, 2014, 4:32 am • 1

See also the following threads:

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=62239
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=383713

Quick Reply

High School Olympiads

another concurrency problem (2.5.3/KK)

Reply

sergei93
#1 Apr 7, 2011, 11:11 am

Let $\triangle ABC$ be a triangle, and construct squares $ABB_1A_2, BCC_1B_2, CAA_1C_2$ externally on its sides. Prove that the perpendicular bisectors of the segments A_1A_2, B_1B_2, C_1C_2 are concurrent.

Luis González
#2 Apr 7, 2011, 1:09 pm

M is the midpoint of BC and D the reflection of B about $A.$ Then $\triangle ACD \cong \triangle AA_1A_2$ are congruent by SAS $\Longrightarrow$ $\angle AA_1A_2 = \angle ACD = \angle MAC,$ which implies that $AM \perp A_1A_2.$ Let $X \equiv A_2B_1 \cap C_2A_1, Y \equiv A_2B_1 \cap B_2C_1$ and $Z \equiv B_2C_1 \cap C_2A_1.$ AM and AX become A-altitude and A-circumdiameter of $\triangle AA_1A_2,$ i.e. AM, AK are isogonals WRT $\angle A_1AA_2.$ Consequently, AM, AK are isogonals WRT $\angle BAC \Longrightarrow AX$ is the A-symmedian of $\triangle ABC.$ Thus, $\triangle ABC$ and $\triangle XYZ$ are homothetic with homothetic center the symmedian point K of $\triangle ABC.$ Let G be the centroid of $\triangle ABC$ and let the perpendicular bisector ℓ_a of A_1A_2 cut GK at $U.$ Since ℓ_a passes through the circumcenter L of $\triangle AA_1A_2$ (midpoint of AX), it follows that $AG \parallel UL \Longrightarrow$

$$\frac{GU}{GK} = \frac{AL}{AK} = \frac{1}{2}\frac{AX}{AK} = \frac{1}{2}\left(\frac{KX - KA}{KA}\right) = \frac{k-1}{2}$$

Since $k = \frac{KX}{KA} > 1$ is the homothety coefficient of $\triangle ABC \sim \triangle XYZ,$ we conclude that the perpendicular bisectors of B_1B_2 and C_1C_2 cut GK at the same point $U.$ Hence, perpendicular bisectors of A_1A_2, B_1B_2, C_1C_2 concur at a point U lying on the line connecting the centroid and symmedian point of $\triangle ABC.$

P.S. The result is still true for the configuration of similar rectangles and its proof is analogous to the one given above.

nsato
#3 Apr 16, 2011, 3:10 am

See also
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=22321
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=22917

Quick Reply

High School Olympiads

Rectangles Perpendiculars Concurrence Again~~

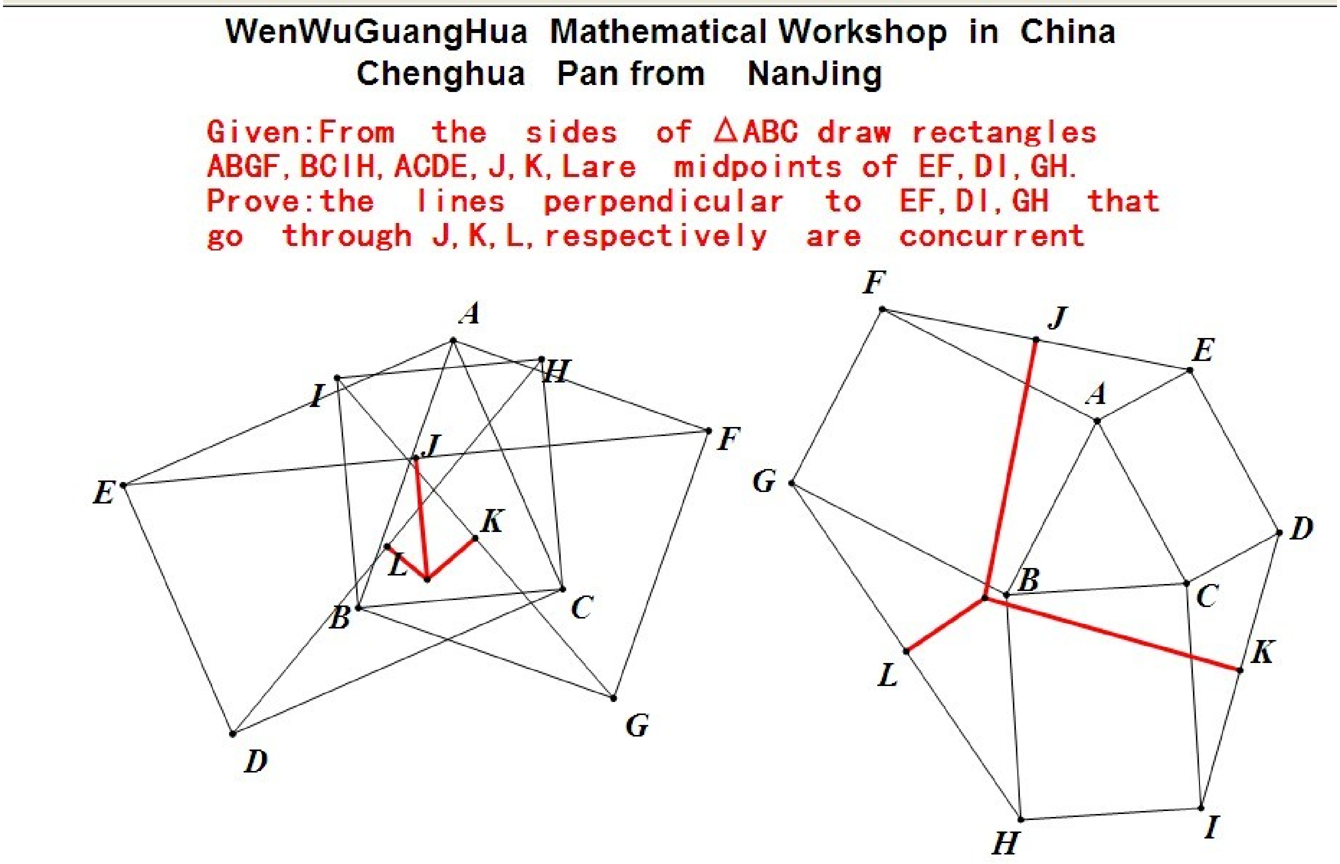
Source: (China) WenWuGuangHua Mathematics Workshop □□□□□□□□□

XmL

#1 Dec 22, 2012, 7:13 am

See Attachment.
These two problems are proposed by PCHP from WenWuGuangHua Mathematics Workshop in China

Attachments:



Luis González

#2 Dec 28, 2012, 10:23 am • 3

We assume that all rectangles are constructed outside $\triangle ABC$, the remaining case is treated analogously. Let $A' \equiv DE \cap FG, B' \equiv FG \cap HI, C' \equiv HI \cap DE.$ $\triangle ABC$ and $\triangle A'B'C'$ are homothetic with center $S \equiv AA' \cap BB' \cap CC'.$

AA' is the A'-circumdiameter of $\triangle A'EF \Longrightarrow$ perpendicular bisector of $\overline{EF}$ meets $\overline{AA'}$ at its midpoint O_a and the perpendicular τ_A from A' to EF is the isogonal of AA' WRT $\angle B'A'C'.$ Similarly, perpendiculars τ_B, τ_C from B', C' to GH, ID are the isogonals of BB' and CC' WRT $\angle C'B'A'$ and $\angle A'C'B' \Longrightarrow \tau_A, \tau_B, \tau_C$ concur at the isogonal conjugate S' of S WRT $\triangle A'B'C'.$

Perpendicular bisector of $\overline{EF}$ (parallel to A'S' through J) cuts SS' at $P.$ If $k<1$ denotes the homothetic coefficient of $\triangle ABC \sim \triangle A'B'C'$, we have

$$\frac{SP}{SS'} = \frac{SO_a}{SA'} = \frac{\frac{1}{2}(SA+SA')}{SA'} = \frac{k}{2} + 1.$$

Similarly, we conclude that the perpendicular bisectors of $\overline{GH}$ and $\overline{ID}$ meet the line SS' at the same point P.

drmzjoseph

#3 Feb 10, 2015, 2:41 pm

X a point arbitrary,
$AX^2 - BX^2 = FX^2 - GX^2$ (Theorem)
$BX^2 - CX^2 = HX^2 - IX^2$
$CX^2 - AX^2 = DX^2 - EX^2$
$\Rightarrow FX^2 + HX^2 + DX^2 = GX^2 + IX^2 + EX^2 \quad (1)$
(this is enough)

More problems related
* Prove that thes lines HE, GD, FI form a triangle in same perspective to ΔABC
* The perpendicular bisectors of HE, GD, FI are concurrent (of (1) is obvious)
* The lines perpendicular to DI, FE, GH from the midpoints of HE, GD, FI respectively are concurrent.
And http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=624770 (variation of this problem)

Quick Reply

High School Olympiads

Perpendicular bisectors concurrents

Source: Own Invention

drmzjoseph

#1 Feb 10, 2015, 2:00 pm

From the sides of ΔABC draw externally trapezoids isosceles and similar $ABGF, BCIH, ACDE$. (similarly in this order)
(Such that $AB \parallel GF;\quad BC \parallel HI;\quad AC \parallel DE$)
Prove that perpendicular bisectors of EF, DI, GH are concurrents.

TelvCohl

#2 Feb 10, 2015, 2:28 pm • 1

My solution:

Let $A^* = BH \cap CI, B^* = CD \cap AE, C^* = AF \cap BG$.
Let X, Y, Z be the midpoint of B^*C^*, C^*A^*, A^*B^*, respectively .
Let X^*, Y^*, Z^* be the midpoint of EF, GH, ID, respectively .
Let O, O^* be the circumcenter of $\triangle ABC, \triangle A^*B^*C^*$, respectively .

Since $A^*O \perp BC, B^* \perp CA, C^* \perp AB$,
so $\triangle ABC$ and $\triangle A^*B^*C^*$ are orthologic ,
hence the perpendicular from A, B, C to B^*C^*, C^*A^*, A^*B^* are concurrent at V .

Since $\frac{AX^*}{AX} = \frac{BY^*}{BY} = \frac{CZ^*}{CZ} = t$,

so the perpendicular bisector of EF, GH, ID are concurrent at $K \in VO^*$ such that $\frac{VK}{VO^*} = t$.

Q.E.D

High School Olympiads

Concurrent perp. bisectors

Locked

Source: Notes on Euclidean Geometry, by K Kedlaya

Ankoganit

#1 Oct 30, 2015, 7:53 pm

Let ABC be a triangle, and construct squares $ABB_1A_2, BCC_1B_2, CAA_1C_2,$ externally on its sides. Prove that the perpendicular bisectors of the segments $A_1A_2, B_1B_2, C_1C_2,$ are concurrent.

Ankoganit

#2 Oct 30, 2015, 9:29 pm

Anyone pls help im stuck

Geftus

#3 Oct 30, 2015, 11:16 pm

These assumptions are even too strong: instead of each square you can construct any rectangle! For proof you can use a magnificent theorem called British Flag Theorem (link)

Best regards, Geftus

Luis González

#4 Oct 30, 2015, 11:48 pm

Posted before at http://www.artofproblemsolving.com/community/c6h400834.

For generalizations see
http://www.artofproblemsolving.com/community/c6h513325
http://www.artofproblemsolving.com/community/c6h624770

High School Olympiads

Fixed sum

Reply

Source: own

andria

#1 Oct 29, 2015, 5:54 pm

Let F be the first ferma point of triangle ABC. Let $A'B'C'$ be the tangential triangle of ABC. τ is orthotransversal of F WRT ABC. Let X be an arbitrary point on τ. Prove that:
$\overline{\text{dis}(X,B'C')}+\overline{\text{dis}(X,A'C')}+\overline{\text{dis}(X,A'B')}$ is fixed. (the distances are signed)

Luis González

#2 Oct 29, 2015, 9:32 pm

Since F is on the Kiepert hyperbola of $\triangle ABC$, then its trilinear polar, which coincides with its orthotransversal τ, is perpendicular to the Euler line of $\triangle ABC$, which is the circumcenter-incenter line of the tangential $\triangle A'B'C'$. Now using the lemma posted in the problem Distance sum (see post #4 or #5 for a synthetic proof), we get the conslusion.

Quick Reply

High School Olympiads

Distance sum

Reply

Source: in Poncelet's porism, me

yetti

#1 Jun 13, 2008, 10:32 pm

Let $\mathbf{P} = \{\triangle ABC \mid (I), (O) = \text{const}\}$ be a set of triangles with a common incircle (I) and circumcircle (O). Let X be arbitrary fixed point inside or on the incircle and let $\triangle DEF$ be the pedal triangle of X with respect to a $\triangle ABC \in \mathbf{P}$, with $D \in BC, E \in CA, F \in AB$. Show that the sum $XD + XE + XF$ does not depend on the $\triangle ABC \in \mathbf{P}$.

yetti

#2 Jun 21, 2008, 12:25 am

1. When X is no longer inside of the incircle, $\overline{XD} + \overline{XE} + \overline{XF} = \text{const}$ holds for all $\triangle ABC \in \mathbf{P}$ only if the distances for each triangle in $\mathbf{P}$ are properly directed - one or two are negative outside of the triangle.

2. Locus of X such that $\overline{XD} + \overline{XE} + \overline{XF} = 0$?

Altheman

#3 Jun 21, 2008, 12:54 am

Euler's theorem for pedal triangle states that

[DEF]/[ABC]=|R^2-OP^2|/4R^2

Note OP^2-R^2 is the power of P with respect to the circumcircle.

I'm not sure if this helps or not.

The main idea of the proof is extending AP, BP, CP to hit the circumcircle again at X,Y,Z, then XYZ ~ DEF and a bunch of other similarities...

Luis González

#4 Mar 24, 2011, 3:32 am • 1

Lemma. The locus of points P, whose sum of oriented distances to the sidelines of $\triangle ABC$ is constant, is a perpendicular to the line connecting its incenter I and circumcenter O.

Proof. Let $(\alpha : \beta : \gamma)$ be the trilinear coordinates of P with respect to $\triangle ABC$. Assume that the sum of the oriented distances from P to BC, CA, AB is a constant k. Thus, the trilinear equation of its locus ℓ is given by

$$\ell \equiv \frac{\alpha \cdot S}{a\alpha + b\beta + c\gamma} + \frac{\beta \cdot S}{a\alpha + b\beta + c\gamma} + \frac{\gamma \cdot S}{a\alpha + b\beta + c\gamma} = k \ , \ \ S = 2|\triangle ABC|$$

$$\ell \equiv (S - ak)\alpha + (S - bk)\beta + (S - ck)\gamma = 0$$

Infinite point of ℓ for any k is indeed $X_{513} \equiv (b - c : c - a : a - b) \Longrightarrow \ell \perp IO.$

Now, back to the problem, let U be the fixed orthogonal projection of X on IO. Let D', E, F' denote the orthogonal projections of U on BC, CA, AB. M, N, L are the midpoints of BC, CA, AB and (I) touches BC at U. From the right trapezoid $IOMK$ with bases $OM \parallel IK$, we get

$$\overline{UD'} = \frac{\overline{OM} \cdot \overline{UI} + \overline{IK} \cdot \overline{UO}}{\ldots} = \frac{\overline{OM} \cdot \overline{UI} + r \cdot \overline{UO}}{\ldots}$$

$$\overline{UD'} = \frac{\quad}{IO} = \frac{\quad}{IO}$$

Similarly, we'll have the expressions

$$\overline{UE'} = \frac{\overline{ON} \cdot \overline{UI} + r \cdot \overline{UO}}{IO} \quad , \quad \overline{UF'} = \frac{\overline{OL} \cdot \overline{UI} + r \cdot \overline{UO}}{IO}$$

$$\Longrightarrow \overline{UD'} + \overline{UE'} + \overline{UF'} = \frac{(\overline{OM} + \overline{ON} + \overline{OL}) \cdot \overline{UI} + 3r \cdot \overline{UO}}{IO}$$

But, by Carnot's theorem we have $\overline{OM} + \overline{ON} + \overline{OL} = R + r$

$$\Longrightarrow \overline{UD'} + \overline{UE'} + \overline{UF'} = \frac{(R + r) \cdot \overline{UI} + 3r \cdot \overline{UO}}{IO} = \text{const}$$

Now, according to the previous lemma, we deduce that

$$\overline{XD} + \overline{XE} + \overline{XF} = \overline{UD'} + \overline{UE'} + \overline{UF'} = \text{const}.$$

Luis González

#5 Jan 3, 2015, 12:54 am • 1

Here is a synthetic proof to the lemma I referred in the previous post.

If we take points Y, Z on rays $\overrightarrow{CA}, \overrightarrow{BA}$, such that $CY = BZ = BC$, then $OI \perp YZ$ (this has been discussed many times before). If $\delta_A, \delta_B, \delta_C$ denote the oriented distances from P to BC, CA, AB, then using oriented areas, we get

$$[BZYC] = [PCB] + [PBZ] + [PZY] + [PYC] = \frac{BC \cdot (\delta_A + \delta_B + \delta_C)}{2} + [PZY].$$

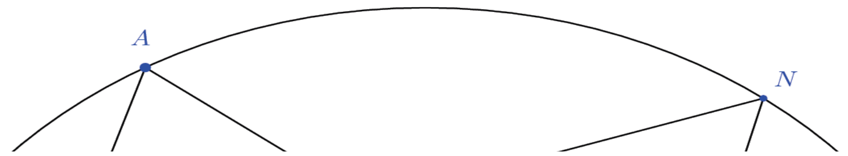
Hence, if $\delta_A + \delta_C + \delta_C$ is constant, then $[PZY]$ is constant $\Longrightarrow$ oriented distance from P to YZ is constant $\Longrightarrow$ P moves on a line parallel to YZ, i.e. perpendicular to OI.

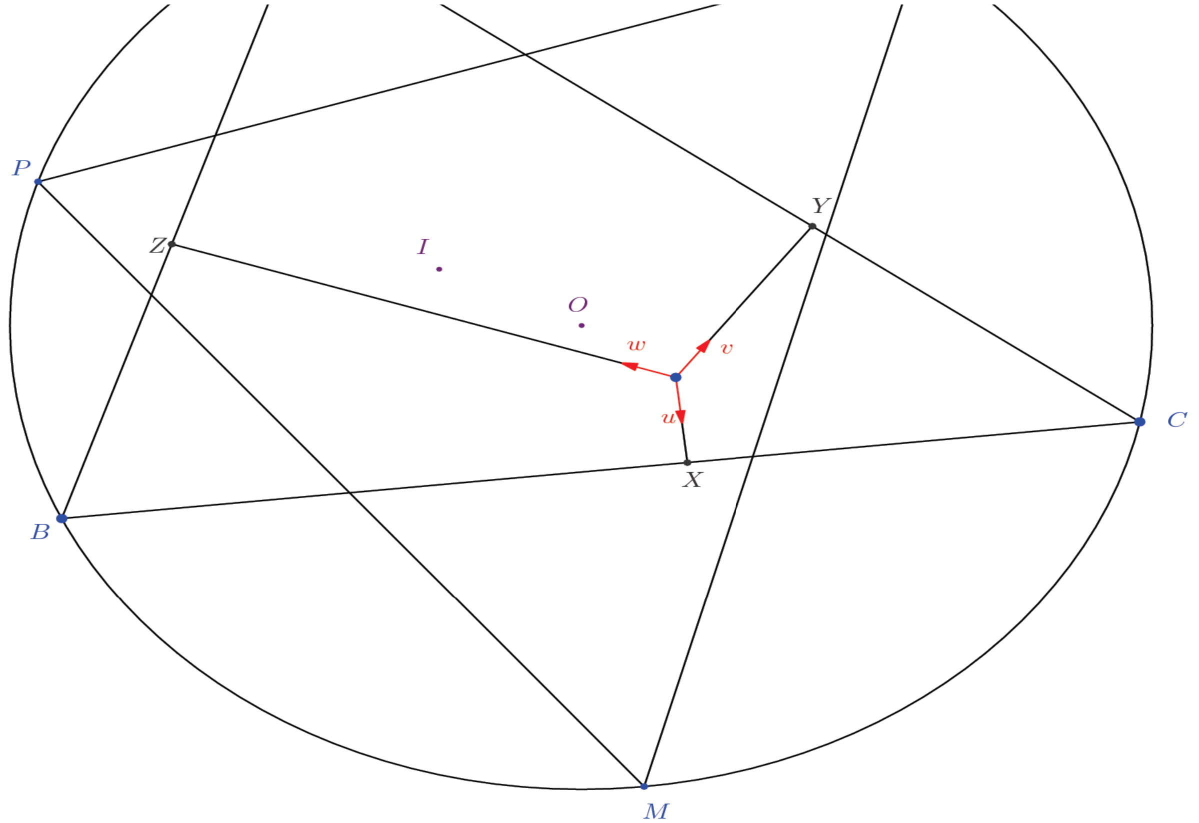
andria

#7 Oct 30, 2015, 7:14 pm • 1

> *Luis González wrote:*
> **Lemma.** The locus of points P, whose sum of oriented distances to the sidelines of $\triangle ABC$ is constant, is a perpendicular to the line connecting its incenter I and circumcenter O.

My proof:
Let $f(P)$ be the sum of distances of P from AB, AC, BC.
Let $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}$ be the vectors such that $|u| = |v| = |w| = 1$ and $\overrightarrow{u}.\overrightarrow{BC} = 0, \overrightarrow{v}.\overrightarrow{AC} = 0, \overrightarrow{w}.\overrightarrow{AB} = 0$ (see figure).
Consider two arbitrary points P, Q such that $f(P) = f(Q)$. Let X, Y, Z be the projections of P on BC, CA, AB respectively and X', Y', Z' be the projections of Q on BC, CA, AB respectively. Note that $\sum \overline{PX} = \sum \overline{QX'}$. Observe that:
$PX = \overrightarrow{PB}.\overrightarrow{u}$
$PY = \overrightarrow{PC}.\overrightarrow{v}$
$PZ = \overrightarrow{PA}.\overrightarrow{w}$
Hence $f(P) = \sum \overrightarrow{PB}.\overrightarrow{u}$ similarly $f(Q) = \sum \overrightarrow{QB}.\overrightarrow{u}$ so
$0 = f(P) - f(Q) = \overrightarrow{PQ}.(\overrightarrow{u} + \overrightarrow{v} + \overrightarrow{w}) \Longrightarrow PQ \perp \overrightarrow{u} + \overrightarrow{v} + \overrightarrow{w}$.
Let M, N, P be the midpoints of arcs BC, AC, AB respectively and G be the centroid of $\triangle MNP$ clearly I is orthocenter of $\triangle MNP$. Hence $\overrightarrow{u} + \overrightarrow{v} + \overrightarrow{w} = \frac{1}{R}(\overrightarrow{OM} + \overrightarrow{ON} + \overrightarrow{OP}) = 3\overrightarrow{OG} = \overrightarrow{OI} \Longrightarrow PQ \perp OI$.
Q.E.D





andria

#8 Oct 30, 2015, 7:25 pm

Generalization of above problem:

Let $n \geq 3$ be a positive integer and $A_1A_2\dots A_n$ be an arbitrary $n-gon$. then the locus of point P such that $\sum \text{dis}(P, A_iA_{i+1})$ is constat is a line perpendicular to $\overrightarrow{u_1}+\overrightarrow{u_2}+\dots+\overrightarrow{u_n}$ where $|u_1|=|u_2|=\dots=|u_n|=1$ and $\overrightarrow{u_i}.\overrightarrow{A_iA_{i+1}}=0$.

The solution in post #7# also works for this problem.

Quick Reply

High School Olympiads

Concurrent lines, through incenter and middle of arc

Reply

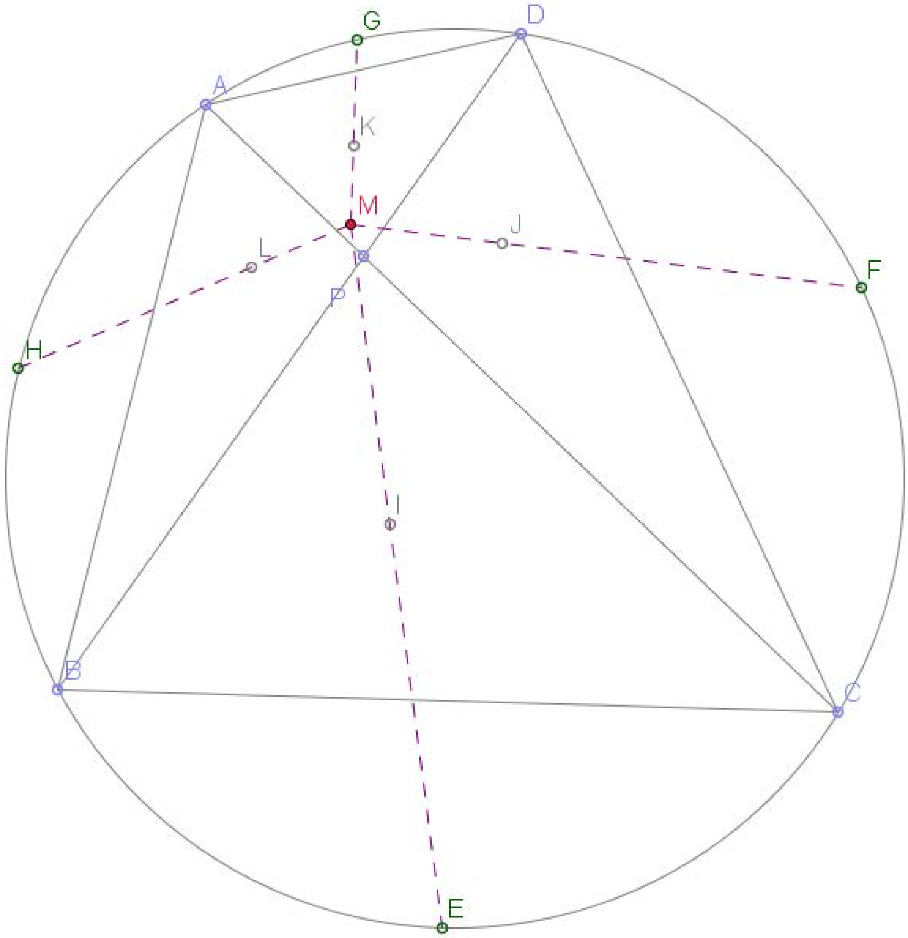
Source: A problem from Geometry in Figures.

Cezar

#1 Oct 29, 2015, 4:36 am

Let $ABCD$ be a cyclic quadrilateral. $AC \cap BD = P$. Let l be a special line that passes through the incenter of $\triangle BPC$ and midpoint of arc BC. Similarly define the special lines for $\triangle CPD, \triangle DPA, \triangle APB$. Prove that these four special lines are concurrent.

Attachments:



Luis González

#2 Oct 29, 2015, 10:59 am

Lemma: $\triangle ABC$ is a triangle with circumcircle (O) and U, V are the midpoints of its arc AC, AB. D is an arbitrary point on $\overline{BC}$ and I_1, I_2 are the incenters of $\triangle ABD, \triangle ACD$. Then I_1V, I_2U, OD concur.

Proof: Let AD cut (O) again at E and let $(J_1), (J_2)$ be the Thebault circles of the cevian ED of $\triangle EBC$. (J_1) touches the arc EB of (O) at X and (J_2) touches the arc EC of (O) at Y. It's known that U, I_2, X and V, I_1, Y are respectively collinear and that I_1I_2, J_1J_2, XY concur at a point T (see Ordinary and Thebault incircles or Five concurrent lines). Thus $\triangle I_1J_2Y$ and $\triangle I_2J_1X$ are perspective through $T \Longrightarrow$ by Desargues theorem $D \equiv I_1J_2 \cap I_2J_1, I_1Y \cap I_2X$ and $O \equiv XJ_1 \cap YJ_2$ are collinear, i.e. I_1V, I_2U, OD concur, as desired. ■

Using this lemma in the proposed problem, we immediately conclude that GK, FJ, EI, HL concur at a point M on OP, where O is the center of $\odot(ABCD)$.

andria

#3 Oct 29, 2015, 4:43 pm

this problem is from Russia see seven concurrent lines (post #3#)

jayme

#4 Oct 29, 2015, 5:00 pm

Dear Mathlinkers,

see a proof with the Ryokan Maruyama result and archive at

http://jl.ayme.pagesperso-orange.fr/Docs/Forme.pdf p. 5-11

Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Ordinary and Thebault incircles.

Reply

Source: Nice but difficult. Own.

yetti

#1 Dec 8, 2008, 3:04 am • 4

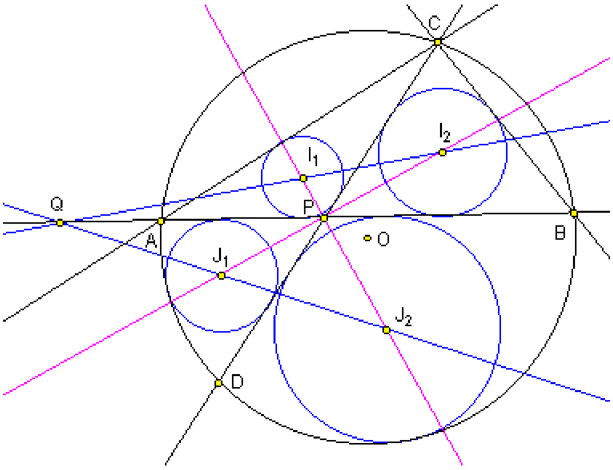
P is arbitrary point on the side AB of a $\triangle ABC$ with circumcircle (O) and CP cuts (O) again at D.

$(I_1), (I_2)$ are ordinary incircles of $\triangle APC, \triangle BPC$ with incenters I_1, I_2.

$(J_1), (J_2)$ are Thebault circles with centers J_1, J_2, tangent to the rays PD, PA resp PD, PB and internally tangent to the circumcircle (O).

Prove that the center lines I_1I_2, J_1J_2 intersect on the line AB.

Attachments:



encyclopedia

#2 Jan 8, 2009, 10:28 am

This problem is consequence of I_1J_1, I_2J_2, CP are concurrent, but I haven't proved it, yet

Flakky

#3 Jan 9, 2009, 2:42 am • 3

Denote $X = (J_1) \cap AB, Y = (J_2) \cap AB, X' = (I_1) \cap AB, Y' = (I_2) \cap AB, J_1J_2 \cap AB = T, I_1I_2 \cap AB = T'$ and $\angle BPC = \phi$

From Casey's theorem we have: $AX.BC + BX.AC = (CP + PX).AB \Leftrightarrow (AP - PX).BC + (BP + PX).AC$ $= (CP + PX).AB \Leftrightarrow PX = \dfrac{AP.BC + BP.AC - CP.AB}{AB + BC - AC}$

Analogically $PY = \dfrac{AP.BC + BP.AC - CP.AB}{AB + AC - BC}$

We find easily $PX' = \dfrac{PA + PC - AC}{2}, PY' = \dfrac{PC + PB - BC}{2}$

$TX = \dfrac{XY.J_1X}{J_2Y - J_1X}(\Delta TXJ_1 \sim \Delta TYJ_2)T'X' = \dfrac{X'Y'.I_1X'}{I_2Y' - I_1X'}(\Delta T'X'I_1 \sim \Delta T'Y'I_2)$

We have $XJ_1 = PX.tan\dfrac{\phi}{2}, YJ_2 = \dfrac{PY}{tan\frac{\phi}{2}}, X'I_1 = \dfrac{PX'}{tan\frac{\phi}{2}}, Y'I_2 = PY'.tan\dfrac{\phi}{2}$

So $TX + XP = \frac{\ldots}{\frac{PY}{tan\frac{\phi}{2}} - PX.tan\frac{\phi}{2}} + XP = \frac{\ldots}{PY - PX.tan^2\frac{\phi}{2}}$

Analogically $TX + XP = \frac{PX'.PY'(tan^2\frac{\phi}{2}+1)}{PY'.tan^2\frac{\phi}{2} - PX'}$. It's sufficient to prove that

$TX + XP = TX' + X'P \Leftrightarrow \frac{PX.PY(tan^2\frac{\phi}{2}+1)}{PY - PX.tan^2\frac{\phi}{2}}$

$= \frac{PX'.PY'(tan^2\frac{\phi}{2}+1)}{PY'.tan^2\frac{\phi}{2} - PX'} \Leftrightarrow tan^2\frac{\phi}{2} = \frac{PX'.PY(PX+PY')}{PY'.PX(PY+PX'}$

$\Leftrightarrow tan^2\frac{\phi}{2} = \frac{(PA+PC-AC)[2(PA.BC+BP.AC-PC.AB)+(AB+BC-AC)(PC+PB-BC)]}{(PC+PB-BC)[2(PA.BC+BP.AC-PC.AB)+(AB+AC-BC)(PA+PC-PB)]}$

$tan\frac{\phi}{2} = \frac{S_{PBC}}{\frac{BP+PC-BC}{2}.\frac{PB+PC+BC}{2}}, tan\frac{\phi}{2} = \frac{\frac{AP+PC-AC}{2}.\frac{PA+PC+AC}{2}}{S_{PAC}}$

$\Rightarrow tan^2\frac{\phi}{2} = \frac{PB(PA+PC+AC)(AP+PC-AC)}{PA(PB+PC+BC)(BP+PC+BC)}$

So it's sufficient to prove that $\frac{PB(PA+PC+AC)(AP+PC-AC)}{PA(PB+PC+BC)(BP+PC-BC)} =$

$\frac{(PA+PC-AC)[2(PA.BC+BP.AC-PC.AB)+(AB+BC-AC)(PC+PB-BC)]}{(PC+PB-BC)[2(PA.BC+BP.AC-PC.AB)+(AB+AC-BC)(PA+PC-PB)]}$

$\Leftrightarrow \frac{PB(PA+PC+AC)}{PA(BP+PC+BC)} = \frac{(2(PA.BC+BP.AC-PC.AB)+(AB+BC-AC)(PC+PB-BC)}{(2(PA.BC+BP.AC-PC.AB)+(AB+AC-BC)(PA+PC-PB)}(1)$

Denote $BC = a, AC = b, PA = c, PB = d, PC = e$

$(1) \Leftrightarrow \frac{d(c+e+b)}{c(d+a+e)} = \frac{ac+bd-ce-de+ab+cd+ae-be+d^2-a^2}{ac+bd-ce-de+ab+cd+be-ae+c^2-b^2}$

After some calculations(believe me they are not hard) we get:

$cd^3 - cda^2 - c^2e^2 + ace^2 - bce^2 + a^2c^2 + a^2bc + c^2ad + d^2ca - a^3c$

$= bde^2 - d^2e^2 - ade^2 + c^3d - b^2dc + b^2d^2 + b^2ad + d^2bc + dbc^2 - db^3(2)$

From Stewart's theorem we have: $a^2c + b^2d - cd(c+d) = e^2(c+d)$

We get that:

$(*)a^3c + ab^2d - acd(c+d) = e^2(ac+ad) \Leftrightarrow a^3c + ab^2d - ac^2d - acd^2 = e^2ac + e^2ad$

$(**)a^2bc + b^3d - cdb(c+d) = e^2(bc+bd) \Leftrightarrow a^2bc + b^3d - c^2db - cd^2b = e^2bc + e^2bd$

$(***)(a^2c + b^2d - cd(c+d))(c-d) = e^2(c+d)(c-d) \Leftrightarrow a^2c^2 + b^2dc + cd^3 + e^2d^2 =$

$c^2e^2 + c^3d + a^2cd + b^2d^2$

From $(*), (**), (***)$ we see that (2) is true, so (1) is true too. DONE!

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encyclopedia

#4 Jan 9, 2009, 6:39 pm • 1

Hi nice proof, I have an idea from your proof, If you can prove

$$\tan^2\frac{\phi}{2} = \frac{PX'.PY(PX+PY')}{PY'.PX(PY+PX')} \Rightarrow \frac{1}{\tan\frac{\phi}{2}}(\frac{1}{PX'}+\frac{1}{PY}) = \frac{1}{\cot\frac{\phi}{2}}(\frac{1}{PX}+\frac{1}{PY'})$$

This mean

$$\frac{1}{r_{I_1}} + \frac{1}{r_{J_2}} = \frac{1}{r_{I_2}} + \frac{1}{r_{J_1}}$$

This really nice indenity, more we can see

$$\frac{1}{r_{J_1}} - \frac{1}{r_{I_1}} = \frac{1}{r_{J_2}} - \frac{1}{r_{I_2}} \quad (1)$$

This is sufficient to show I_1J_1, I_2, J_2 and CP are concurrent, indeed Let $I_1J_1 \cap CP = S$ and $(I_1), (J_1)$ touch PC at M, N, resp, we have

$$\frac{SM}{r_{I_1}} = \frac{SN}{r_{J_1}} = \frac{MN}{r_{J_1} - r_{I_1}} = \frac{PM+PN}{r_{J_1} - r_{I_1}} \quad (2)$$

So

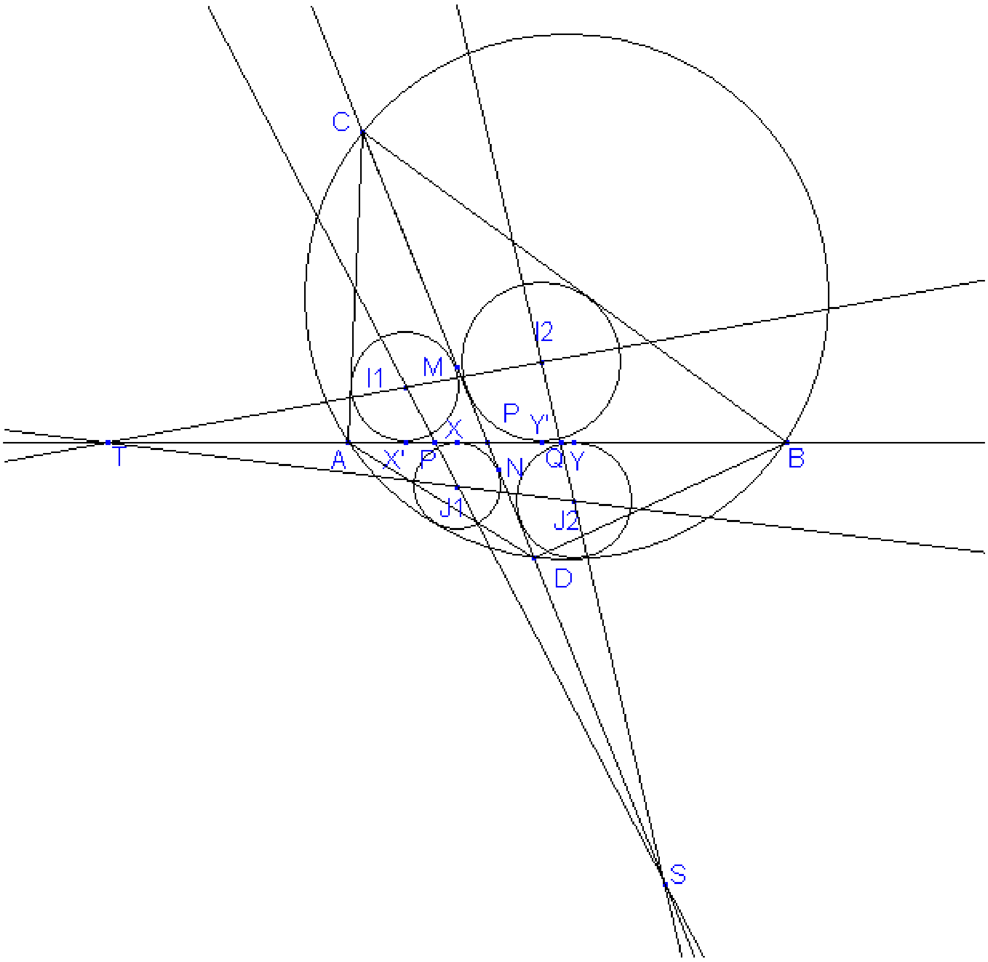
$$SP = SN + NP = \frac{\cdots}{r_{J_1} - r_{I_1}} + NP = \frac{\cdots}{r_{J_1} - r_{I_1}} = \frac{\cdots}{\frac{1}{r_{J_1}} - \frac{1}{r_{I_1}}} = \frac{\cdots}{\frac{1}{r_{J_1}} - \frac{1}{r_{I_1}}}$$

Similarly, let $S' = I_2J_2 \cap CP$ then

$$S'P = \frac{\tan\frac{\phi}{2} + \cot\frac{\phi}{2}}{\frac{1}{r_{J_2}} - \frac{1}{r_{I_2}}} \quad (3)$$

From $(1), (2), (3)$ derive $S \equiv S'$ more, if I_1J_1, I_2, J_2 and CP are concurrent at S let I_1J_1, I_2J_2 intersect BC at P, Q, easily seen $(I_1J_1PS) = (I_2J_2QS) = -1$ it is sufficient to show I_1I_1, J_1J_2, PQ are concurrent.

Attachments:



ma 29
#5 Jan 16, 2009, 7:41 am

> *yetti wrote:*
> P is arbitrary point on the side AB of a $\triangle ABC$ with circumcircle (O) and CP cuts (O) again at D.
>
> $(I_1), (I_2)$ are ordinary incircles of $\triangle APC, \triangle BPC$ with incenters I_1, I_2.
>
> $(J_1), (J_2)$ are Thebault circles with centers J_1, J_2, tangent to the rays PD, PA resp PD, PB and internally tangent to the circumcircle (O).
>
> Prove that the center lines I_1I_2, J_1J_2 intersect on the line AB.

Q at infinity when P is tangency point of C-excircle and AB.

yetti
#6 Jan 16, 2009, 1:41 pm • 1

For encyclopedia

Incircle (I) of $\triangle ABC$ touches AB at F. I_1I_2 is tangent at $T \in PC$ of a hyperbola $\mathcal{H}$ with foci C, F passing through A, B - see http://www.mathlinks.ro/viewtopic.php?p=1353036#1353036 (1st solution).
Parallel to AB through C cuts (O) again at E. J_1J_2 is tangent at $S \in PD$ of an ellipse $\mathcal{E}$ with foci C, E passing through A, B - see http://www.mathlinks.ro/viewtopic.php?p=1371790#1371790 (2nd page).

AB is the common chord of the ellipse $\mathcal{E}$ and hyperbola $\mathcal{H}$ with one common focus C, such that A, B are on the same side of the directrices e, h of $\mathcal{E}, \mathcal{H}$ with respect to C. Otherwise, these 2 conics are quite special: Their common chord AB is parallel to the ellipse major axis line CE and the other hyperbola focus F is on their common chord AB. Disregarding all this except for the one common focus C, the general theorem is:

Let $\mathcal{G}, \mathcal{H}$ be 2 arbitrary intersecting conics with one common focus C. Let g, h be the directrices of $\mathcal{G}, \mathcal{H}$ with respect to C. Let AB (and possibly A'B') be their common chord(s), such that A, B are either both on the same sides of g, h, or both on opposite sides of g, h. (The same condition for A', B'.) Let arbitrary line through the common focus C cut $\mathcal{G}$ at S, S' and $\mathcal{H}$ at T, T'. Then the tangents of $\mathcal{G}$ at S, S' cut the tangents of $\mathcal{H}$ at T, T' on AB or A'B'.

Better identification of the common chords

Leonhard Euler
#7 Jan 20, 2009, 2:35 pm • 2

Well known lemma: Let ABC be a triangle. (P) is circle that pass B, C and cut segment AB, AC at D, E, respectively. (Q) is circle that touch BD, CE and externally tangent to circle (P) at M. Bisector of $\angle BMC$ pass through incenter of triangle ABC.

Minor variation: Let ABC be a triangle and (P) is circle that pass B, C and cut ray AB beyoond A and ray AC beyond A at D, E, respectively. (Q) is a circle that touch AD, AE and internally tangent to circle (P) at M. Bisector of $\angle BMC$ pass through incenter of triangle ABC.

Problem: Let $(J_1), (J_2)$ touch (O) at M_1, M_2,resectively and let K, L be midpoint of arc BC, CA, respectively. KM_1 is bisector of $\angle CM_1B$. By lemma, KM_1 pass through I_2 and similarly, LM_2 pass through I_1. By applying Pascal theorem for A, K, M_1, B, L, M_2, we get $AK \cap BL, KM_1 \cap LM_2, AM_2 \cap BM_1$ are collinear, or equivalently, triangle I_1AM_2 and I_2BM_1 are perspective. Hence, By Desargues theorem, I_1I_2, AB, M_1M_2 are concurrent. M_1 and M_2 are external similitude center of $(J_1), (O)$ and $(J_2), (O)$, respectively. By Monge-D'Alembert theorem applying for $(J_1), (J_2), (O)$, M_1M_2 pass through X, where $X = J_1J_2 \cap AB$ is external similitude center of $(J_1), (J_2)$. Hence AB, J_1J_2, M_1M_2 are concurrent. Together with fact that I_1I_2, AB, M_1M_2 is concurrent, we get the result.

jayme
#8 May 31, 2012, 5:56 pm

Dear Mathlinkers,
for this difficult and nice problem, you can see

http://perso.orange.fr/jl.ayme vol. 20 "Shape and Mouvement" p. 17

Sincerely
Jean-Louis

simplependulum
#9 Jun 1, 2012, 4:26 pm • 1

The solution to this nice problem can be quite short ... if we are allowed to quote some famous results :

Let T_1 , T_2 be the tangency points of (J_1) , (J_2) with (ABC) ,
M_1 , M_2 the midpoints of minor arcs AC , BC respectively .

1. T_1T_2 , AB , J_1J_2 are concurrent .

Proof: Let $X = J_1J_2 \cap AB$, then X is also the intersection point of the two external common tangents . Consider the inversion at X that preserves (ABC) , (J_1) and (J_2) are sent to each other , we see that X sends T_1 to T_2 so these three lines are concurrent at X .

2. M_1 , I_1 , T_2 as well as M_2 , I_2 , T_1 are collinear .

Proof: omitted

Therefore , we just need to prove that I_1I_2 , AB , T_1T_2 are concurrent , but it can be done immediately because of Pascal , by proving I_1 , I_2 , $AB \cap T_1T_2$ are collinear .

This post has been edited 1 time. Last edited by simplependulum Jun 3, 2012, 3:36 pm

yetti

#10 Jun 1, 2012, 8:34 pm • 1

Another proof, using polarity transformation with center at the common focus of 2 conics and mapping them into 2 circles, is at http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=369005&p=2033012#p2033012.

Quick Reply

High School Olympiads

Five concurrent lines

Reply

Source: Thebault

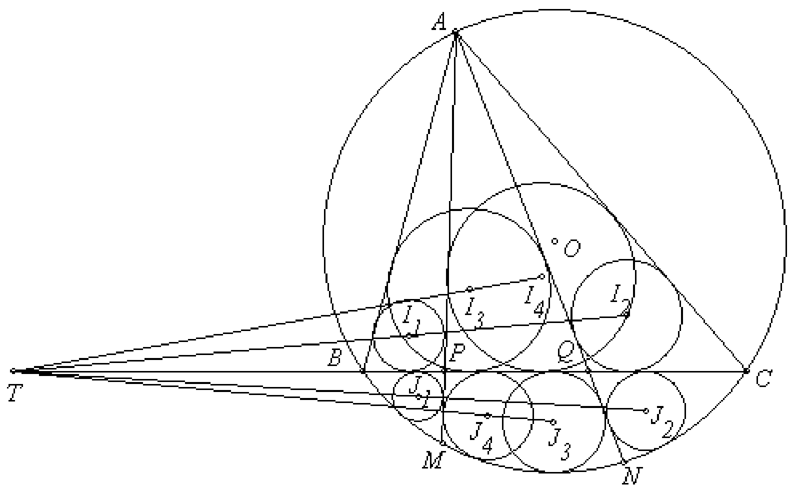
buratinogigle

#1 Jun 23, 2012, 4:07 pm • 3

Let ABC be a triangle with circumcircle (O). P, Q are two point on BC. AP, AQ cut (O) again at M, N, respectively. $(I_1), (I_2), (I_3), (I_4)$ are incircles of triangle PAB, QAC, QAB, PAC, respectively. $(J_1), (J_2), (J_3), (J_4)$ are Thebault circles tangent to the rays $PB, PM; QC, QN; QB, QN; PC, PM$, respectively and internally tangent to the circumcircle (O). Prove that $I_1I_2, I_3I_4, J_1J_2, J_3J_4$ and BC are concurrent.

See more Ordinary and Thebault incircles.

Attachments:



Luis González

#2 Jun 29, 2012, 12:36 pm • 5

Let $I \equiv BI_1I_3 \cap CI_2I_4$ be the incenter of $\triangle ABC$. Trivial angle chase reveals that $\angle BAI_1 = \angle IAI_4, \angle IAI_1 = \angle CAI_4$, $\angle IAI_3 = \angle CAI_2$ and $\angle BAI_3 = \angle IAI_2$, thus $(I_1, B, I, I_3) = (I_2, C, I, I_4) \Longrightarrow I_1I_2, I_3I_4$ and BC concur at a point T. For the same reason, T lies on the line connecting the A-excenters of $\triangle PAB$ and $\triangle QAC$ and the line connecting the A-excenters of $\triangle QAB$ and $\triangle PAC$.

On the other hand, let $(J_1), (J_2), (J_3), (J_4)$ touch (O) at K_1, K_2, K_3, K_4, respectively. Since K_1 and K_2 are the exsimilicenters of $(J_1) \sim (O)$ and $(J_2) \sim (O)$, then BC, J_1J_2 and K_1K_2 concur at the exsimilicenter T^* of $(J_1) \sim (J_2)$. If D, E denote the midpoints of the arcs CA, AB of (O), then according to the topic Sawayama, we deduce that $K_1 \in DI_4$, $K_2 \in EI_3, K_4 \in EI_1$ and $K_3 \in DI_2$. By Pascal theorem for the cyclic hexagon BDK_1K_2EC, the intersections $I_3 \equiv BD \cap EK_2, I_4 \equiv DK_1 \cap CE$ and $T^* \equiv K_1K_2 \cap BC$ are collinear $\Longrightarrow T \equiv T^*$. Analogously, by Pascal theorem for the cyclic hexagon BDK_3K_4EC, we deduce that $T \equiv I_1I_2 \cap BC \cap J_3J_4$. Thus, five lines $I_1I_2, I_3I_4, J_1J_2, J_3J_4$ and BC concur.

Quick Reply

High School Olympiads

Tangents form a hexagon circumscribed in a conic.

Locked

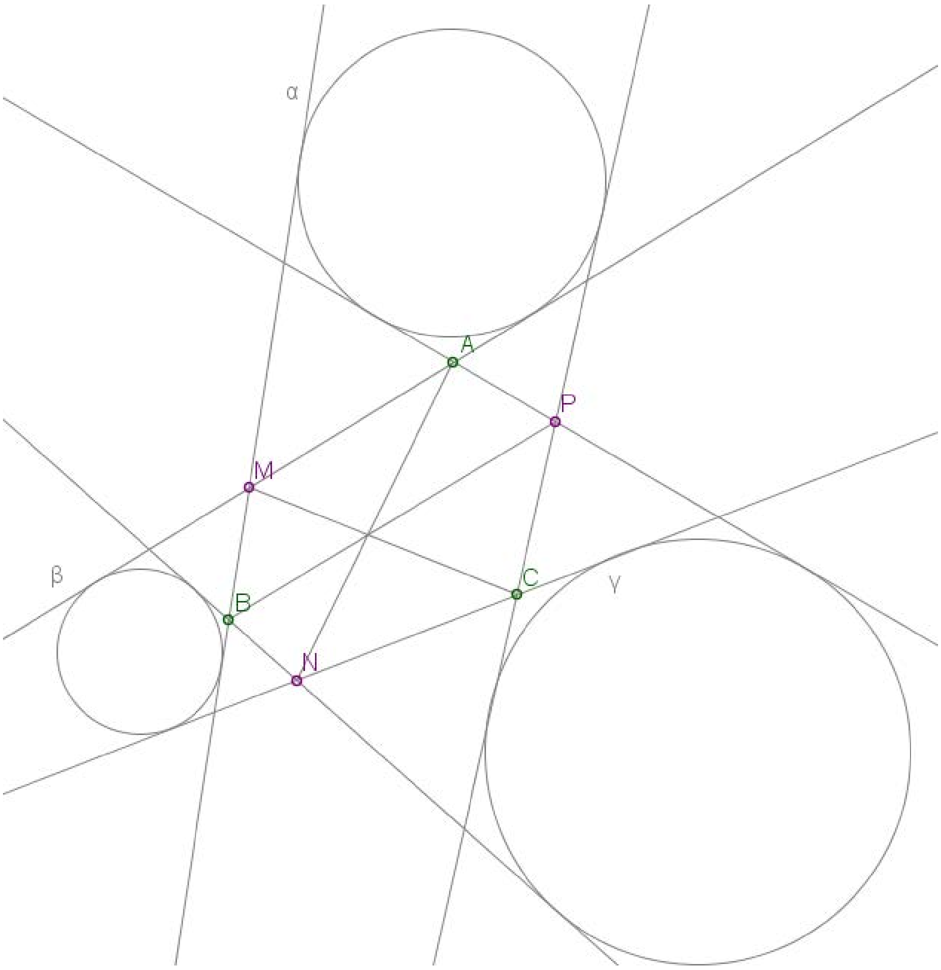
Source: A problem from Geometry in Figures

Cezar

#1 Oct 29, 2015, 5:13 am

Let α, β, γ be 3 circles not intersecting each other. Let A be the intersection of the internal tangent to α and β and the internal tangent to α and γ, similarly define B, C. And let M, N, P be the insimilicenters of (α, β), (β, γ), (γ, α). Prove that lines AN, CM, BP are concurrent.

Attachments:



Luis González

#2 Oct 29, 2015, 5:23 am

It's an old problem. See http://www.artofproblemsolving.com/community/c6h571511.

High School Olympiads

concurrent lines

Reply

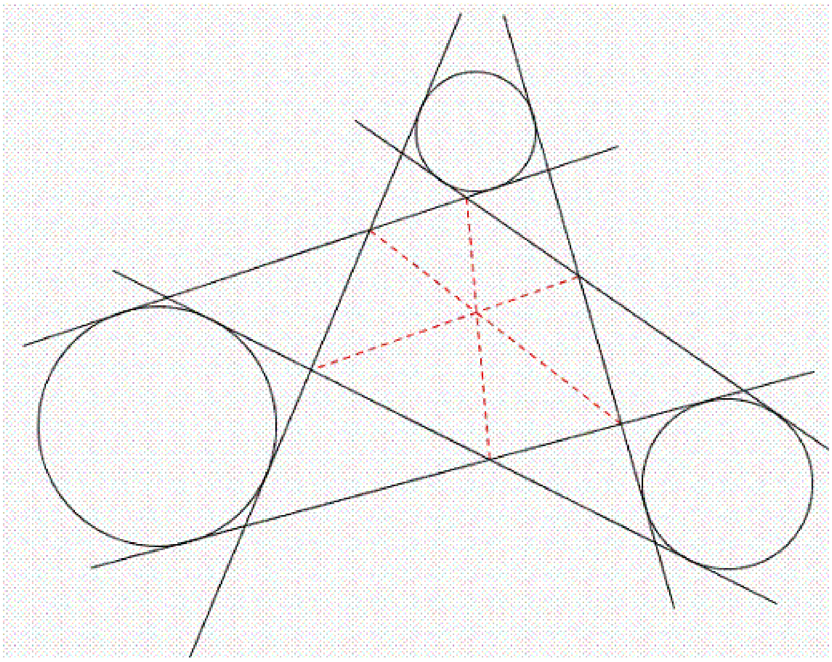
Source: iran 200?

nima-amini

#1 Jan 16, 2014, 11:51 pm

Prove red lines are concurrent.

Attachments:



Luis González

#2 Jan 17, 2014, 5:04 am • 1

Posted many times before; it's Iran PPCE 2007.

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=140096
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=278135
http://www.artofproblemsolving.com/Forum/viewtopic.php?t=362203

In fact, the problem is a particular case of the dual of the 4 conics theorem: Two of the common tangents to each pair of 3 conics are tangent to a same conic if and only if the intersections of the other pairs of tangents are collinear.

Quick Reply

High School Olympiads

Miquel point

Reply

ferma2000

#1 Oct 28, 2015, 11:09 pm

Dear mathlinkers;
M is miquel point of quadrilateral $ABCD$.
Claim:
M lies on nine point circle of a triangle formed by the intersections of diagonals of $ABCD$.
Warning: you mustn't use conics or projective geometry (inversion,...) in this topic
Best regards;

Luis González

#2 Oct 28, 2015, 11:35 pm • 1

Discussed before at Complete quadrilateral and Circumcenter on radical axis (equivalent problem). Also another proof can be seen in the book Geometry of conics by A.V. Akopyan & A. A. Zaslavsky (Theorem 4.12), page 110.

Quick Reply

High School Olympiads

Complete quadrilateral

Reply

404error

#1 May 28, 2015, 12:32 pm

Difficult. I have a proof using complex numbers but a Euclidean one would be nice.

Show that the Miquel point of a complete quadrilateral lies on the nine-point circle of the triangle determined by its 3 diagonals

TelvCohl

#2 May 28, 2015, 5:09 pm • 2

Problem :

Let $P_{14}P_{13}P_{12}P_{23}P_{34}P_{24}$ be the complete quadrilateral Γ formed by four lines $\ell_1, \ell_2, \ell_3, \ell_4$.
Let $T_1 \equiv P_{13}P_{24} \cap P_{14}P_{23}, T_2 \equiv P_{14}P_{23} \cap P_{12}P_{34}, T_3 \equiv P_{12}P_{34} \cap P_{13}P_{24}$ and F be the Miquel point of Γ .

Prove that F lie on the circumcircle of the medial triangle $\triangle M_1M_2M_3$ of $\triangle T_1T_2T_3$

Proof :

Let $X_1 \in \ell_1, X_2 \in \ell_2, X_3 \in \ell_3, X_4 \in \ell_4$ be the points such that $T_1X_1 \parallel \ell_2, T_1X_2 \parallel \ell_1, T_1X_3 \parallel \ell_4, T_1X_4 \parallel \ell_3$.

Easy to see X_1, X_2, X_3, X_4 are collinear at the T_1- midline of $\triangle T_1T_2T_3$.

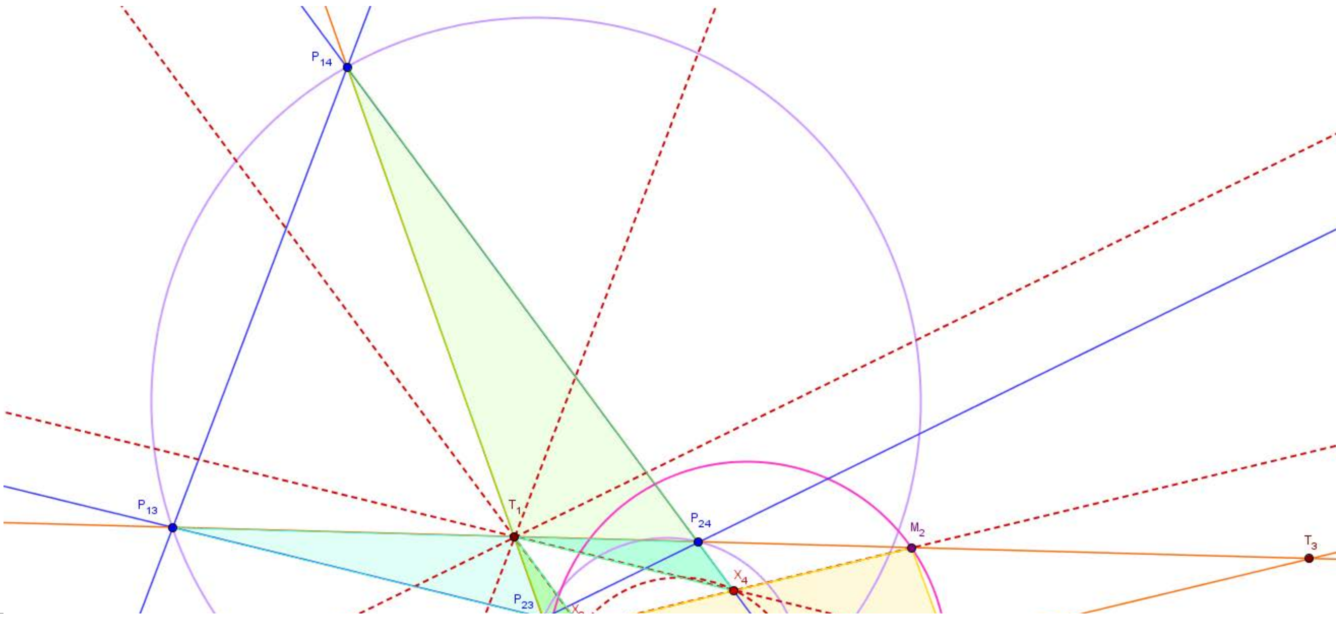
From $\triangle P_{23}X_3T_1 \sim \triangle T_1X_4P_{14}, \triangle P_{24}T_1X_4 \sim \triangle T_1P_{13}X_3 \Longrightarrow X_3P_{23} \cdot X_4P_{14} = X_3P_{13} \cdot X_4P_{24}$,
so $X_3P_{23} : P_{23}P_{13} = X_4P_{24} : P_{24}P_{14} \Longrightarrow \odot(P_{34}X_3X_4), \odot(P_{34}P_{23}P_{24}), \odot(P_{34}P_{13}P_{14})$ are coaxial ,
hence we get $F \in \odot(P_{34}X_3X_4) \Longrightarrow$ the reflection of F in $X_3X_4 \equiv M_2M_3$ lie on the Steiner line τ of Γ .

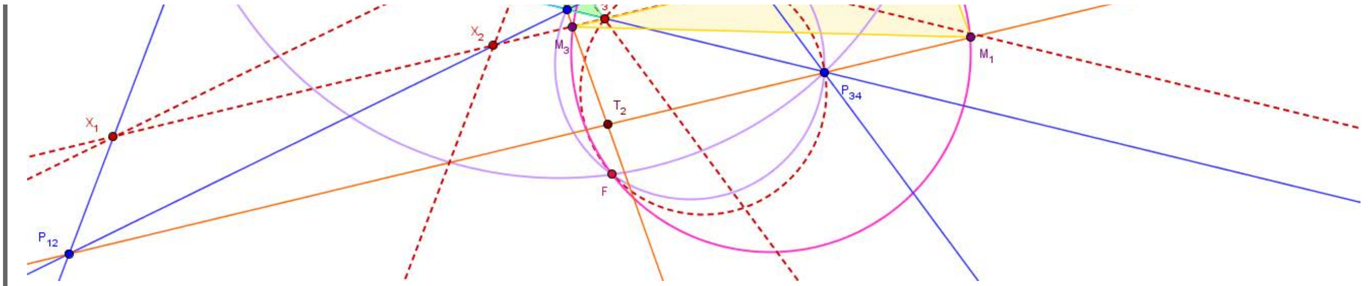
Similarly, we can prove the reflection of F in M_3M_1, M_1M_2 lie on τ ,
so τ is the Steiner line of F WRT $\triangle M_1M_2M_3 \Longrightarrow F \in \odot(M_1M_2M_3)$.

Q.E.D

Another approach

Attachments:





Quick Reply

High School Olympiads

Circumcenter on radical axis

Locked

Source: e-mail

yetti
#1 Apr 18, 2005, 9:50 pm

Let D, E, F be collinear points on the side lines BC, CA, AB of a triangle $\triangle ABC$. Let K, L, M be the midpoints of the segments AD, BE, CF and $(K), (L), (M)$ circles on the diameters AD, BE, CF. Show that:

(1) The circles $(K), (L), (M)$ belong to the same pencil (this is simple, but necessary for (2)).
(2) The circumcenter of the triangle formed by the lines AD, BE, CF is on the radical axis of this pencil (this is beautiful).

darij grinberg
#2 Apr 18, 2005, 10:58 pm

This is a problem from the 1st Round of the 5th MathLinks Contest. It's actually an old result of a rather famous mathematician [EDIT: this mathematician is Möbius], but I won't give any hints for now. Please don't post anything about the problem until April 24, 2005.

Darij

This post has been edited 1 time. Last edited by darij grinberg, Apr 26, 2005, 9:50 pm

pestich
#3 Apr 18, 2005, 11:18 pm

:beta:

This post has been edited 1 time. Last edited by pestich, May 11, 2005, 11:35 pm

Myth
#4 Apr 18, 2005, 11:25 pm

Ha-ha Hoda net - hodi s bubei.

Here is another one: if you don't know the answer, say "Pushkin", it works!

mecrazywong
#5 Apr 19, 2005, 6:33 pm

This problem is not hard at all, but the solution is quite long, yet straightforward...

pestich
#6 Apr 19, 2005, 8:52 pm

:theta:

This post has been edited 1 time. Last edited by pestich, May 11, 2005, 11:37 pm

yetti
#7 Apr 19, 2005, 10:03 pm

> pestich wrote:

> Yetti ,
>
> Are you trying to prove something here by taking the motto under my signature (the first one) too seriously?
>
> Maj. Pestich
>
> ---
> A wise man will investigate what a fool takes for granted.

Why would I do such a thing ? It is your motto, not mine. I do not even think that it is true. But I will not disturb you with the details (because the details would disturb you).

Yetti.

This post has been edited 1 time. Last edited by yetti, Nov 8, 2005, 7:31 am

Valentin Vornicu

#8 Apr 26, 2005, 7:20 pm

Continuation of this problem is here: http://www.mathlinks.ro/Forum/viewtopic.php?p=216159#p216159
Topic closed.

Myth

#9 Apr 26, 2005, 9:39 pm

There is another topic where two solutions were published.

High School Olympiads

Geometry

Locked

Blacklord

#1 Oct 28, 2015, 9:14 pm

ABC is a isosceles triangle (AB=AC).D is a point on BC such that the incircle of ABD and the A-excircle of ADC have the same radius x.
Prove that 4x=Hb
(Hb is equal to the length of the altitude from B of the triangle ABC.)

Luis González

#2 Oct 28, 2015, 9:56 pm

Posted before at http://www.artofproblemsolving.com/community/c6h291963 and surely elsewhere.

High School Olympiads

segment equality

Reply

77ant
#1 Jul 29, 2009, 1:02 am

For an isosceles triangle ABC with AB = BC. AH is an altitude and D is a point on AC. The incircle of triangle ABD and the excircle of triangle BDC, relative to CD, are equal circles with radius r, Prove that AH = 4r

Luis González
#2 Aug 2, 2009, 12:55 am • 1

> Quote:
> Let $\triangle ABC$ be an isosceles triangle with $AB = AC = L$. D is a point on BC, such that the radii of the incircle of $\triangle ABD$ and the A-exincircle of $\triangle ADC$ are equal to r. Show that the altitude h on the leg L is four times r.

Drop perpendiculars DP and DQ from D to AB and AC, respectively. Then, using the well-known formulae of the inradii and exinradii in terms of altitudes, we get

$$DP = \frac{r(L + AD + BD)}{L} \ , \ DQ = \frac{r(AD + L - DC)}{L}$$

$$DP + DQ = \frac{r(2L + 2AD + BD - DC)}{L}$$

On the other hand, $DP + DQ = h \Longrightarrow h = \dfrac{r(2L + 2AD + BD - DC)}{L}$ $(\star)$

Since these two circles are congruent, the tangent segments from D to both are equal.

$$L + DC - AD = AD + BD - L \ \Longrightarrow \ 2L = 2AD + BD - DC$$

Combining with $(\star)$ yields $h = \dfrac{r(2L + 2L)}{L} = 4r.$

Quick Reply

High School Olympiads

Equal angles

Locked

socrates

#1 Oct 28, 2015, 6:53 am

Let $ABCD$ be a parallelogram such that $\angle ABC > 90^\circ$, and $\mathcal{L}$ the line perpendicular to BC that passes through B. Suppose that the segment CD does not intersect $\mathcal{L}$. Of all the circumferences that pass through C and D, there is one that is tangent to $\mathcal{L}$ at P, and there is another one that is tangent to $\mathcal{L}$ at Q (where $P \neq Q$). If M is the midpoint of AB, prove that $\angle PMD = \angle QMD$.

Luis González

#2 Oct 28, 2015, 7:37 am

Already posted at http://www.artofproblemsolving.com/community/c6h526958.

High School Olympiads

An interesting construction with parallelogram

Source: Sankt Peterburg MO

Lyub4o
#1 Mar 28, 2013, 3:22 pm

Given parallelogram $ABCD$.A line l through B is perpendicular to BC.Two circles passing through D and C touch l at points P and Q.If M is the middle of AB, prove that $\angle PMD=\angle QMD$.

leader
#2 Mar 28, 2013, 5:39 pm

$DM\cap BC=K$ $CD\cap PQ=R$ First of $RQ^2=RD*RC=RP^2\Rightarrow RP=RQ$ next $BK/DA=BM/MA=1\Rightarrow BK=DA=BC$ and PQ is perpendicular bisector of CK note that D,C must be on the same side of PQ
$\angle PDQ=180-\angle DQP-\angle DPQ=180-\angle QCR-\angle PCR=180-\angle PCQ=180-\angle PKQ$
so $PKQD$ is cyclic. now $\angle PKD=\angle PQD=\angle RCQ$ and $\angle CQR=\angle KQP=\angle KDP$ so $\triangle DPK\sim\triangle CRQ$ therefore $DK/DP=CQ/QR$ since $KM/MD=MB/MA=1\Rightarrow KM=MD\Rightarrow DK=2*KM$ and since $PQ=2*QR$ the previous ratio becomes $KM/DP=QC/PQ=QK/PQ$ but since $\angle QKM=\angle QPD$ we have $\triangle QMK\sim\triangle QDP$ similarly $\triangle PMK\sim\triangle QPD$ so
$\angle PMD=180-\angle PMK=180-\angle PDQ=180-\angle QMK=\angle QMD$

Luis González
#3 Mar 29, 2013, 11:04 am

DC is radical axis of $\odot(DCP)$ and $\odot(DCQ)$, meeting their common tangent $\overline{PQ}$ at its midpoint N. If DN cuts the circumcircle (K) of $\triangle DPQ$ again at E, then $\angle EPQ=\angle EDQ=\angle CQP\Longrightarrow CQ\parallel EP$. Similarly, $CP\parallel EQ$, thus $EPCQ$ is a parallelogram $\Longrightarrow C$ is reflection of E on $N\Longrightarrow C$ is on the reflection of (K) about $PQ\Longrightarrow$ ray CB cuts (K) at the reflection F of C on B. Hence M is midpoint of FD and $EF\parallel PQ$, i.e. DE,DF are isogonals WRT $\angle PDQ\Longrightarrow$ DF is D-symmedian of $\triangle DPQ\Longrightarrow PQ,KM$ and the tangent of (K) at D meet at the pole U of DF WRT (K). Thus since $M(P,Q,U,D)$ is harmonic and $KMU\perp DM$, then MD bisects $\angle PMQ$, or $\angle PMD=\angle QMD$.

Quick Reply

High School Olympiads

Collinearity!

Reply

socrates

#1 Oct 28, 2015, 5:40 am

A point P lies on side AB of a convex quadrilateral $ABCD$. Let ω be the inscribed circumference of triangle CPD and I the centre of ω. It is known that ω is tangent to the inscribed circumferences of triangles APD and BPC at points K and L respectively. Let E be the point where the lines AC and BD intersect, and F the point where the lines AK and BL intersect. Prove that the points E, I, F are collinear.

Luis González

#2 Oct 28, 2015, 7:25 am

Let (U) and (V) be the incircles of $\triangle PAD$ and $\triangle PBC$. Since $(I) \equiv \omega$ and (U) are tangent to PD at the same point K $\Longrightarrow DA + DP - AP = DP + DC - PC \Longrightarrow DA + PC = DC + AP \Longrightarrow APDC$ is tangential with an incircle ω_1 and similarly $BPDC$ is tangential with an incircle ω_2.

Since A is the exsimilicenter of $(U) \sim \omega_1$ and C is the exsimilicenter of $(I) \sim \omega_1$, then by Monge & d'Alembert theorem, it follows that $X \equiv AC \cap IU$ is the exsimilicenter of $(I) \sim (U)$ and similarly $Y \equiv BD \cap IV$ is the exsimilicenter of $(I) \sim (V) \Longrightarrow XY$ is the positive homothety axis of $(I), (U), (V)$, cutting UV at the eximilicenter Z of $(U) \sim (V)$. Moreover, since K and L are the insimilicenters of $(I) \sim (U)$ and $(I) \sim (V)$, then $Z \in KL$. Now $\triangle AXK$ and $\triangle BYL$ are perspective through $Z \Longrightarrow$ by Desargues theorem, it follows that $I \equiv XK \cap YL, E \equiv XA \cap YB$ and $F \equiv AK \cap BL$ are collinear.

Quick Reply

High School Olympiads

A cyclic quadrilateral

Reply

socrates

#1 Oct 28, 2015, 5:47 am

Let A be a point outside of a circumference ω. Through A, two lines are drawn that intersect ω, the first one cuts ω at B and C, while the other one cuts ω at D and E (D is between A and E). The line that passes through D and is parallel to BC intersects ω at point $F \neq D$, and the line AF intersects ω at $T \neq F$. Let M be the intersection point of lines BC and ET, N the point symmetrical to A with respect to M, and K be the midpoint of BC. Prove that the quadrilateral $DEKN$ is cyclic.

Luis González

#2 Oct 28, 2015, 6:25 am

Since $\angle MAT = \angle TFD = \angle TED \Longrightarrow MA$ is tangent to $\odot(TAE) \Longrightarrow$
$MA^2 = MN^2 = MT \cdot ME = MB \cdot MC \Longrightarrow (N, B, C, A) = -1 \Longrightarrow AN \cdot AK = AB \cdot AC = AD \cdot AE \Longrightarrow$
$DEKN$ is cyclic.

Quick Reply

High School Olympiads

A concurrency

Reply

socrates

#1 Oct 28, 2015, 4:19 am

Let ABC be an acute triangle, and AA_1, BB_1, and CC_1 its altitudes. Let A_2 be a point on segment AA_1 such that $\angle BA_2C = 90^\circ$. The points B_2 and C_2 are defined similarly. Let A_3 be the intersection point of segments B_2C and BC_2. The points B_3 and C_3 are defined similarly. Prove that the segments A_2A_3, B_2B_3, and C_2C_3 are concurrent.

Luis González

#2 Oct 28, 2015, 5:37 am

This is a particular case of the projective configuration discussed in the thread The Equilateral triangle (post #4). See also Three points are collinear for another solution.

Quick Reply

High School Olympiads

Three points are collinear

Source: own

daothanhoai

#1 Sep 25, 2015, 1:07 am

Let ABC and $A_1B_1C_1$ be two triangles and they are perpective, the perspector is P_1. Let BC_1 meets CB_1 at A_2, define B_2, C_2 cyclically. Then show that ABC and $A_2B_2C_2$ are perpective and perspector is P_2. Then show that $A_1B_1C_1$ and $A_2B_2C_2$ are perpective and perspecto is P_3. Then show that P_1, P_2, P_3 are collinear.

Luis González

#2 Sep 25, 2015, 1:36 am • 1

This configuration was discussed in the thread The Equilateral triangle (see precisely post #3 and #4).

Stefan4024

#3 Sep 25, 2015, 2:01 am • 1

Let $AB \cap A_1B_1 \equiv D, CB \cap C_1B_1 \equiv E, AC \cap A_1C_1 \equiv F$. From the Desargus Theorem since ABC and $A_1B_1C_1$ are perspective centrally, they are also perspective axially, hence D, E, F are collinear points.

Now take $\triangle AA_1B_2$ and $\triangle BB_1A_2$ Since $BA_2 \cap AB_2 \equiv C_1$, $BB_1 \cap AA_1 \equiv P_1$ and $B_1A_2 \cap A_1B_2 \equiv C$ we have that the two triangles are perspective axially, since $P - C - C_1$ is a line. Again using Desargus Theorem they are perspective centrally and their perspector is D, hence A_2B_2 passes through D. Simularly we prove that B_2, C_2, E and C_2, A_2, F are collinear. Hence $\triangle ABC$ and $\triangle A_2B_2C_2$ are perspective axially, and also perspective centrally. Simularly $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ are prespective centrally.

Now on to the third part. Consider $\triangle AA_1A_2$ and $\triangle BB_1B_2$. They are perspective centrally, since A_1B_1, AB and A_2B_2 concur at D, hence they are also perspective axially. Now since $A_1A_2 \cap B_1B_2 \equiv P_3$, $AA_2 \cap BB_2 \equiv P_2$ and $AA_1 \cap BB_1 \equiv P_1$, we get that the axis of perspectivity is the line $P_1 - P_2 - P_3$.

Q.E.D.

This post has been edited 1 time. Last edited by Stefan4024, Sep 25, 2015, 2:04 am

High School Olympiads

A locus!

Reply

socrates

#1 Oct 28, 2015, 4:34 am

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two non-concentric circumferences such that $\mathcal{S}_1$ is inside $\mathcal{S}_2$. Let K be a variable point on $\mathcal{S}_1$. The line tangent to $\mathcal{S}_1$ at point K intersects $\mathcal{S}_2$ at points A and B. Let M be the midpoint of arc AB that is in the semiplane determined by AB that does not contain $\mathcal{S}_1$. Determine the locus of the point symmetric to M with respect to K.

Luis González

#2 Oct 28, 2015, 5:23 am

The tangent of $\mathcal{S}_2$ at M is obviously parallel to $AB,$ thus K and M are homologous points under the direct homothety that takes $\mathcal{S}_1$ into $\mathcal{S}_2 \Longrightarrow MK$ passes through the exsimilicenter H of $\mathcal{S}_1 \sim \mathcal{S}_2$ and $\tfrac{HK}{HM}$ is constant (homothety ratio). Consequently if P denotes the reflection of M on $K,$ then $\tfrac{HP}{HM}$ is a constant, say $k \Longrightarrow P$ describes a circle image of $\mathcal{S}_2$ under homothety with center H and ratio k.

Quick Reply

High School Olympiads

Equal segments

socrates
#1 Oct 28, 2015, 4:14 am

Let ABC be a triangle where $AB > BC$, and D and E be points on sides AB and AC respectively, such that DE and AC are parallel. Consider the circumscribed circumference of triangle ABC. A circumference that passes through points D and E is tangent to the arc AC that does not contain B at point P. Let Q be the reflection of point P with respect to the perpendicular bisector of AC. The segments BQ and DE intersect at X. Prove that $AX = XC$.

Luis González
#2 Oct 28, 2015, 4:47 am

Tangents of $\odot(ABC) \equiv (O)$ at B, P and DE concur at the radical center R of $\odot(PDE), \odot(BDE)$ and (O). If X' is the projection of O on DE, then $OPRBX'$ is cyclic on account of the right angles at B, P, X'. Since $RB = RP$, then it follows that $X'R, X'O$ bisect $\angle BX'P$, thus by symmetry BX' hits (O) again at the reflection Q of P across the perpendicular bisector of $AC \Longrightarrow X \equiv X' \Longrightarrow AX = XC$.

High School Math

circle and ellipse

Reply

AndrewTom

#1 Oct 12, 2015, 1:59 pm

Let Γ be a circle and AB a fixed diameter of Γ. Let CD be any chord of the circle perpendicular to AB. Let P and Q be the points on the line segment CD such that $CP = PQ = QD$.

Prove that the loci of P and Q is an ellipse (as CD varies) and determine the foci of the ellipse.

AndrewTom

#2 Oct 17, 2015, 1:49 am

Any ideas on this one?

vanstraelen

#3 Oct 18, 2015, 1:26 am

Given $A(-a,0)$ and $B(a,0)$.
The line $x = t$ cuts the circle in $C(t,\sqrt{a^2-t^2})$ and $D(t,-\sqrt{a^2-t^2})$.
Then $CD = 2\sqrt{a^2-t^2}$.
If $CP = PQ = QD$, then $P(t,\dfrac{\sqrt{a^2-t^2}}{3})$.
Eliminating t from this point: $x = t$ and $y = \dfrac{\sqrt{a^2-t^2}}{3}$,
gives us the equation of an ellipse: $\dfrac{x^2}{a^2}+\dfrac{y^2}{\frac{a^2}{9}} = 1$.
Focus $F(\dfrac{2\sqrt{2}a}{3},0)$.

Luis González

#4 Oct 28, 2015, 3:35 am

Here is another approach (projective):

Let O denote the center of Γ and let $\overline{XY}$ be the diameter of Γ perpendicular to AB. Let $M \equiv CD \cap AB$ be the midpoint of $\overline{CD}$. Since $CP = PQ = QD = \frac{1}{3}CD \Longrightarrow \frac{MP}{MC} = \frac{1}{3}$, thus if O_∞ is the point at infinity of XY, we have $(M,P,C,O_\infty) = \frac{1}{3} = \text{const} \Longrightarrow C \mapsto P$ is an affine homology fixing $AB \Longrightarrow P$ describes an ellipse $\mathcal{E}$ image of Γ under this affine homology. Since $\mathcal{E}$ is symmetric about AB, then Q is on $\mathcal{E}$ as well.

Clearly, the major axis of $\mathcal{E}$ is $\overline{AB}$ and its minor axis $\overline{UV}$ is along XY, i.e. $U,V \in \overline{XY}$, such that $XU = UV = VY \Longrightarrow OU = \frac{1}{3}OX$. So the distance d from its center O to each focus is given by $d^2 = \sqrt{OA^2 - OU^2} = \sqrt{OA^2 - \frac{1}{9}OA^2} \Longrightarrow d = \frac{2\sqrt{2}}{3} \cdot OA$.

Quick Reply

High School Olympiads

Easy Geometry

Reply

Source: My creation

FabrizioFelen

#1 Oct 27, 2015, 10:26 am • 2

Let $\triangle ABC$ be a triangle acute and AD, BE, CF the heights of $\triangle ABC$. Let Ω the circumference through B, F, E, C and Γ the circumference through A, F, E. Let P be a point in Ω such that PD is bisector of $\angle BPC$ and P, E lies on distinct half plane of BC. Let $Q = \Omega \cap PA$ and H be a orthocenter of $\triangle ABC$. Let G the intersection of the tangent through H to Γ and the tangent through Q to Ω. Prove that:E, F, G are collinear

Luis González

#2 Oct 27, 2015, 11:05 am • 1

Let PD and QH cut Ω again at M, N, resp. M is clearly midpoint of the arc BEC. Since $AH \cdot AD = AB \cdot AF = AP \cdot AQ \Longrightarrow PQHD$ is cyclic $\Longrightarrow \angle MNQ = \angle DPQ = \angle DHN \Longrightarrow MN \parallel HD \Longrightarrow$ N is midpoint of the arc BC of $\Omega \Longrightarrow \angle MQN = 90^\circ \Longrightarrow$ $\angle GQN = \angle QMN = 90^\circ - \angle MNQ = 90^\circ - \angle AHQ = \angle GHQ \Longrightarrow \triangle GQH$ is isosceles with legs $GH = GQ$ $\Longrightarrow G$ has equal power WRT Ω and Γ, thus it lies on their radical axis EF.

EulerMacaroni

#3 Oct 27, 2015, 11:36 am • 1

Hmm I have a slightly different solution...
As shown above, you can prove that QH hits the midpoint of arc BC not containing E. By a well-known configuration, the circle through Q and H tangent to Γ (call it ω) is also tangent to the tangent line through H. It suffices to show that this circle is tangent to Γ and the tangent through Q, whence the conclusion follows from the radical axis theorem on Γ, ω and Ω.

The center of ω clearly lies on line HI, which also contains the center of Γ, thus ω and Γ are tangent circles. The tangent through Q intersects Γ at points L and N; inversion about the point M (midpoint of arc LN not containing H) with radius ML swaps line LM and Γ. Moreover, reflecting Ω about the line through I perpendicular to HQ then applying a homothety at H implies that HQ passes through M, so that they are inverses, hence ω is tangent to line LN, and the conclusion follows . $\blacksquare$

Quick Reply

High School Olympiads

geometry

Reply

Source: OWN

LeVietAn

#1 Oct 26, 2015, 12:13 pm

Dear Mathlinkers,
Let ABC is a triangle and (O) is the circumcircle of it. The tangents lines at B, C of (O) intersect at T. Let $AT \cap BC = D$. The line perpendicular to BC at D intersects OA at E. Let H is orthocenter of triangle BCE. Prove that AH passing through the midpoint of BC.

drmzjoseph

#3 Oct 26, 2015, 1:09 pm • 2

Hello I'm back to AoPS

Here my solution:

Let M be the midpoint of BC, $AM \cap ED \equiv N$, the tangent at A to (O) cut at BC at X, so X is the pole of AT WRT $(O) \Rightarrow XO \perp AT \Rightarrow AX^2 - XD^2 = AO^2 - OD^2$
Also
$XE^2 = XA^2 + AE^2 = ED^2 + XD^2 \Rightarrow AX^2 - XD^2 = ED^2 - AE^2 = AO^2 - OD^2 = -\text{Power}(D) = BD.DC$

Since $ED \parallel OT$ and $AO^2 = OM.OT$ we get $EN.ED = AE^2$

$ED^2 - AE^2 = ED.ND = BD.DC$ i.e N is the orthocenter of $\triangle BEC$ i.e $N \equiv H$ is sufficient.

LeVietAn

#4 Oct 26, 2015, 10:03 pm • 2

Welcome back Welcome . Following interesting problem I would like to give you and my friends on AOPS .

Let ABC be an acute triangle with $AB > AC$. Let Γ be its cirumcircle, H its orthocenter. The internal bisector of the angle $\angle BHC$ meets the side BC at the point K. Let Q be the point on Γ such that $\angle HQA = 90°$. Prove that the circle of Apollonius of K in the triangle HKQ is tangent to the nine-point circle of triangle ABC.

(Note. The circle of Apollonius of K in the triangle HKQ is the locus of points X such that $\dfrac{XH}{XQ} = \dfrac{KH}{KQ}$.)

Source: http://jwilson.coe.uga.edu/emt725/Apollonius/Cir.html

Luis González

#5 Oct 27, 2015, 3:38 am

> LeVietAn wrote:
> Dear Mathlinkers,
> Let ABC is a triangle and (O) is the circumcircle of it. The tangents lines at B, C of (O) intersect at T. Let $AT \cap BC = D$. The line perpendicular to BC at D intersects OA at E. Let H is orthocenter of triangle BCE. Prove that AH passing through the midpoint of BC.

Let M be the midpoint of BC and redefine H as the intersection of AM and DE. Since $DHE \parallel TMO$ (both perpendicular to BC), we get

$$\frac{DH \cdot DE}{TM \cdot TO} = \frac{AD^2}{AT^2} \Longrightarrow \frac{DH \cdot DE}{TB^2} = \frac{AD^2}{AT^2} \quad (1).$$

$$\frac{AD}{AT} = \frac{DB}{TB} \cdot \frac{\sin \widehat{B}}{\sin \widehat{C}} = \frac{DB}{TB} \cdot \frac{b}{c} \Longrightarrow \frac{AD^2}{AT^2} = \frac{DB^2}{TB^2} \cdot \frac{b^2}{c^2} = \frac{DB^2}{TB^2} \cdot \frac{DC}{DB} = \frac{DB \cdot DC}{TB^2} \quad (2).$$

From (1) and (2) we get $DH \cdot DE = DB \cdot DC$, which means that H is orthocenter of $\triangle BCE$.

Dukejukem

#6 Oct 27, 2015, 4:35 am

Since $(ED \parallel OT) \perp BC$, it follows that $\frac{AE}{AO} = \frac{AD}{AT} \equiv k$. Then if S, T lie on AB, AC, respectively so that $\frac{AS}{AB} = \frac{AT}{AC} = k$, we have $\triangle AST \cup E \cup D \sim \triangle ABC \cup O \cup T$ by homothety. Therefore, the tangents to $\omega \equiv \odot(AST)$ at S and T meet at D. Meanwhile, if $X \equiv BT \cap CS$, the converse of Pascal's Theorem applied to ω for hexagon $ASSXTT$ yields $X \in \omega$. Hence, Brokard's Theorem applied to $ASXT$ implies that X is the orthocenter of $\triangle BEC$. Finally, note that X lies on the A-median, on account of $ST \parallel BC$. $\square$

This post has been edited 2 times. Last edited by Dukejukem, Oct 27, 2015, 4:40 am

Luis González

#7 Oct 27, 2015, 5:43 am • 1

> *LeVietAn wrote:*
>
> Let ABC be an acute triangle with $AB > AC$. Let Γ be its cirumcircle, H its orthocenter. The internal bisector of the angle $\angle BHC$ meets the side BC at the point K. Let Q be the point on Γ such that $\angle HQA = 90^\circ$. Prove that the circle of Apollonius of K in the triangle HKQ is tangent to the nine-point circle of triangle ABC.
>
> (Note. The circle of Apollonius of K in the triangle HKQ is the locus of points X such that $\dfrac{XH}{XQ} = \dfrac{KH}{KQ}$.)
>
> Source: http://jwilson.coe.uga.edu/emt725/Apollonius/Cir.html

Lemma: In a $\triangle ABC$, let D, E, F be the midpoints of BC, CA, AB. Internal bisector of $\angle BAC$ cuts BC at U and the perpendicular to BC through U cuts AD at J. Then the circle $\odot(J, JU)$ touches the 9-point circle $\odot(DEF)$.

Proof: Let H be the orthocenter of $\triangle ABC$ and let X the foot of the A-altitude. S and T are the midpoinst of the arcs DX and EF of $\odot(DEF)$. M is the midpoint of the arc BC of $\odot(ABC)$ and its reflection L on BC is clearly the midpoint of the arc BHC. Since A is the exsimilicenter of $\odot(DEF)$ and $\odot(BHC)$, it follows that A, T, L are collinear, thus since AD is the A-median of $\triangle ALM$, then JU cuts AL at the reflection V of U on J.

Let AT cut BC at W and $\odot(DEF)$ again at R. Since $(U, X, D, W) = A(U, X, J, V) = (U, \infty, J, V) = -1$ and RT bisects $\angle XRD$ externally, then RU bisects $\angle XRD$ internally $\Longrightarrow R, U, S$ are collinear. Together with $UV \parallel ST$, we deduce that $\odot(RUV) \equiv \odot(J, JU)$ is tangent to $\odot(DEF)$ at R.

Back to the problem. Let X, Y, Z be the feet of the altitudes on BC, CA, AB. As Q is the 2nd intersection of Γ and $\odot(AYZ)$, then Q is the center of the spiral similarity that swaps $\overline{BZ}$ and $\overline{CY} \Longrightarrow \frac{QB}{QC} = \frac{BZ}{CY} = \frac{HB}{HC} \Longrightarrow Q$ is on the H-Apollonius circle of $\triangle HBC \Longrightarrow \odot(QHX)$ is the H-Apollonius circle of $\triangle HBC$ centered on $BC \Longrightarrow$ tangent of $\odot(HKQ)$ at K is perpendicular to BC, cutting HQ at $J \Longrightarrow \odot(J, JK)$ is the K-Apollonius circle of $\triangle HKQ$.

It's well-known that HQ passes through the midpoint M of BC, i.e. HQ is the H-median of $\triangle HBC$, thus using the previous lemma in $\triangle HBC$, we conclude that $\odot(J, JK)$ touches its 9-point circle $\odot(XYZ)$.

XmL

#8 Oct 27, 2015, 7:42 am

Apply the dilation at A that sends E, D to O, T; let B, C be mapped to B', C'. Thus it suffices to prove M, the midpoint of BC is the orthocenter of $OB'C'$ which is equivalent to $TB' \cdot TC' = MT \cdot TO = TB^2$. This is obvious when we construct the antiparallel to BC wrt $\angle A$ through T, and it meets AC, AB at L, K. Thus $TB' \cdot TC' = TL \cdot TK = TB^2$.

Dukejukem

#9 Oct 29, 2015, 2:11 am

> LeVietAn wrote:
>
> Let ABC be an acute triangle with $AB > AC$. Let Γ be its cirumcircle, H its orthocenter. The internal bisector of the angle $\angle BHC$ meets the side BC at the point K. Let Q be the point on Γ such that $\angle HQA = 90^\circ$. Prove that the circle of Apollonius of K in the triangle HKQ is tangent to the nine-point circle of triangle ABC.

Lemma: Let $\triangle ABC$ be a triangle with orthocenter H and let M be the midpoint of $\overline{BC}$. Define $\Gamma \equiv \odot(ABC)$ and let ω be the circle of diameter $\overline{AH}$. A line τ passing through A cuts ω, Γ for a second time at X, Y, respectively. Then $MX = MY$.

Proof: Let A' be the antipode of A in Γ and let ℓ be the line passing through M and perpendicular to τ. Since $(HX \parallel A'Y \parallel \ell) \perp \tau$ and M is the midpoint of $\overline{HA'}$ (well-known), it follows that ℓ is the perpendicular bisector of $\overline{XY} \implies MX = MY$. ■

Corollary: Let H_b, H_c be the second intersections of AC, AB with ω, respectively, and let T be a midpoint of arc $\widehat{H_bH_c}$ on ω. Then $\odot(M, MT)$ is tangent to Γ.

Proof: Let AT cut Γ for a second time at U. Since T is a midpoint of arc $\widehat{H_bH_c}$ on ω, it follows that AT bisects $\angle H_bAH_c$. Therefore, AU bisects $\angle BAC$, implying that U is a midpoint of arc $\widehat{BC}$ on Γ. Then by the lemma, $\odot(M, MT) \equiv \odot(M, MU)$, which is tangent to Γ because M, U and the center of Γ are collinear. ■

Back to the problem at hand, let $\triangle H_aH_bH_c$ be the orthic triangle of $\triangle ABC$ and let M be the midpoint of $\overline{BC}$. Let γ be the nine-point circle of $\triangle ABC$ and let ρ be the K-Apollonius circle of $\triangle HKQ$.

It is well-known that H is the center of positive homothety that maps $\Gamma \mapsto \gamma$. Therefore, H is also the center of the negative inversion $\mathcal{I}$ that swaps Γ, γ. Let $K' \equiv \mathcal{I}(K), Q' \equiv \mathcal{I}(K)$, and note that $\mathcal{I}$ swaps $\triangle ABC$ and $\triangle H_aH_bH_c$. Therefore, $\angle HH_aQ' = \angle HAQ = 90^\circ \implies Q' \equiv M$. Hence, if X is any point with $X' \equiv \mathcal{I}(X)$, the inversive distance formula implies that

$$\frac{XH}{XQ} : \frac{KH}{KQ} = \frac{MH}{MX'} : \frac{MH}{MK'} = \frac{MK'}{MX'}.$$

Therefore, $X \in \rho \iff MK' = MX'$. It follows that $\mathcal{I}(\rho) \equiv \odot(M, MK')$. Finally, note that HK bisects $\angle BAC \implies HK'$ bisects $\angle H_bHH_c \implies K'$ is a midpoint of arc $\widehat{H_bH_c}$ on $\odot(AHH_bH_c)$. Thus, by the corollary to the lemma, $\mathcal{I}(\rho)$ and Γ are tangent, and we are done. □

This post has been edited 3 times. Last edited by Dukejukem, Oct 29, 2015, 2:20 am

livetolove212

#10 Dec 5, 2015, 4:06 pm

> LeVietAn wrote:
>
> Welcome back  . Following interesting problem I would like to give you and my friends on AOPS .
>
> Let ABC be an acute triangle with $AB > AC$. Let Γ be its cirumcircle, H its orthocenter. The internal bisector of the angle $\angle BHC$ meets the side BC at the point K. Let Q be the point on Γ such that $\angle HQA = 90^\circ$. Prove that the circle of Apollonius of K in the triangle HKQ is tangent to the nine-point circle of triangle ABC.
>
> (Note. The circle of Apollonius of K in the triangle HKQ is the locus of points X such that $\frac{XH}{XQ} = \frac{KH}{KQ}$.)
>
> Source: http://jwilson.coe.uga.edu/emt725/Apollonius/Cir.html

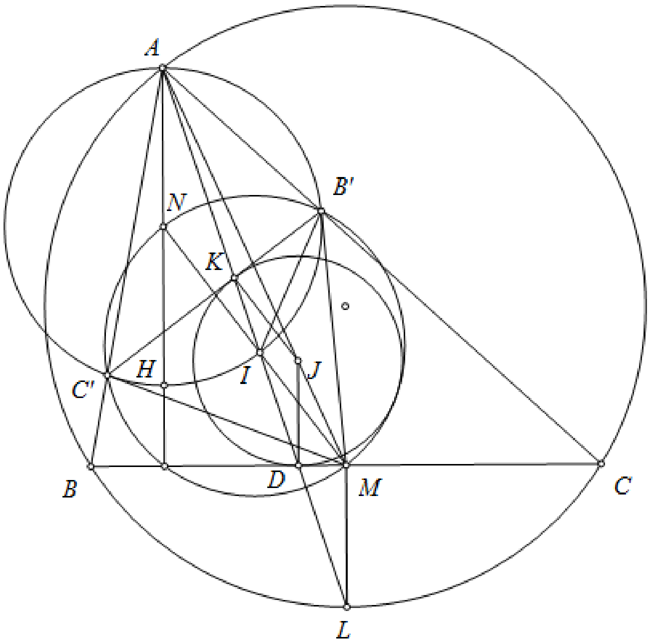
I use the same lemma as Luis's. Here is another proof for lemma.

Lemma. Given $\triangle ABC$, let AD be the internal bisector, M be the midpoint of BC. Line through D and perpendicular to BC intersects AM at J. Then (J, JD) is tangent to 9-point circle of triangle ABC.

Proof.
Let B', C' be the orthogonal projections of B, C onto AC, AB, respectively; K be the intersection of AD and $B'C'$, H be the orthocenter, N be the midpoint of AH, I be the incenter of triangle $MB'C'$, L be the midpoint of arc BC of (ABC).
We have $\angle B'IC' = 90^\circ + \frac{1}{2}\angle B'MC' = 180^\circ - \angle B'AC'$ then I lies on $(AB'C')$ or (N, NA). But $MB' = MC'$ then $IB' = IC'$. this means I lies on AD. Since N is the midpoint of arc $B'C'$ then M, I, N are collinear. But $NA = NI$ then using Thales theorem, $MI = ML$. On the other side, $\triangle AB'C' \cup I \sim \triangle ABC \cup L$ then
$\frac{JD}{MI} = \frac{AJ}{AM} = \frac{AD}{AL} = \frac{AK}{AI} = \frac{MI}{IK'}$, we get $JK = JD$ and $JK \perp B'C'$. Therefore (J, JD) is tangent to $B'C'$.

Since the incenter of triangle $MB'C'$ lies on KD then using Sawayama-Thebault theorem, (J, JD) is Thebault circle of triangle $MB'C'$ hence it is tangent to $(MB'C')$ or the 9-point circle of triangle ABC.

Attachments:



livetolove212

#11 Dec 5, 2015, 5:24 pm

Another proof.

Let H be the orthocenter of triangle ABC, B', C' be the orthogonal projections of B, C onto AC, AB, respectively; L be the midpoint of AH, E be the center of 9-point circle, ML cuts (AH) at T, N be the midpoint of arc BHC of (O_a)- the circumcircle of triangle BHC, P is the reflection of N wrt M.

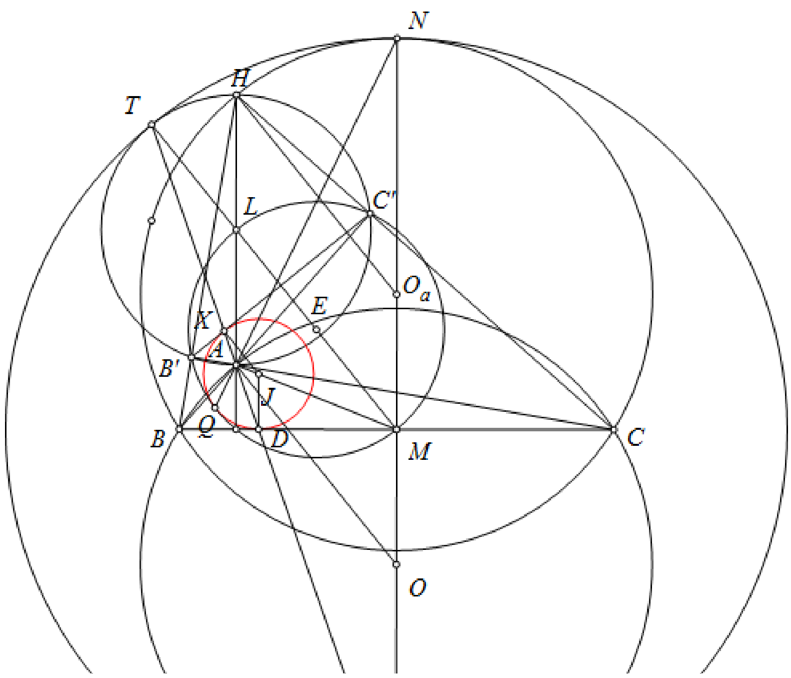
We have $MT = TL + LM = LH + HO_a = O_aN + O_aM = MN = MP$ then (M, MN) is tangent to $(AH), (O)$ and (O_a).

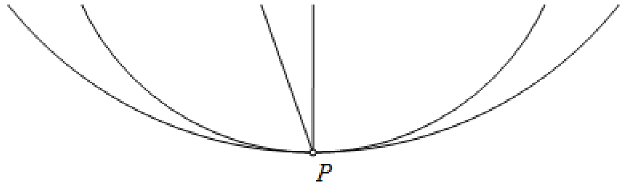
On the other side, $ALMO$ is a parallelogram and $LA = LT, OA = OP$ hence A, T, P are collinear. Let X, D be the intersections of TP and $B'C', BC$, respectively.

Consider the inversion $\mathcal{I}_A^{AC \cdot AB'}$: $(O_a) \leftrightarrow (E), BC \leftrightarrow (AH), B'C' \leftrightarrow (O)$ hence $P \leftrightarrow X, T \leftrightarrow D, N \leftrightarrow Q$. Since (M, MN) is tangent to $(AH), (O)$ and (O_a) we get (XQD) is tangent to $BC, B'C'$ and (E). Let J be the center of (XDQ) then $JD \perp BC, JX \perp B'C'$.

Since $PB = PC$ then AD is the bisector of $\angle BAC$. We have $\triangle BAC \cup D \cup P \sim \triangle B'AC' \cup X \cup T$ then $\dfrac{AD}{AP} = \dfrac{AX}{AT}$, but $JX \parallel TM, JD \parallel PM, JX = JD, MT = MP$ we obtain A, J, M are collinear. Therefore (J, JD) is tangent to 9-point circle of triangle ABC.

Attachments:





Quick Reply

High School Olympiads

Geometry, cyclic pentagon

Reply

Source: ?

Wolowizard

#1 Oct 25, 2015, 1:54 am

Pentagon $ABCDE$ is inscribed in circle with centre O. Label I_1, I_2, I_3, I_4 the incenters of triangles ABC ,AED, AEC,ABD respectively. Prove that if $OI_1=OI_2$ and $OI_3=OI_4$ than ABC and AED are congruent.

Luis González

#2 Oct 25, 2015, 2:52 am • 2

Label R the radius of (O) and r_1,r_2,r_3,r_4,r_5,r_6 the inradii of $\triangle ABC,\triangle AED,\triangle AEC,\triangle ABD,\triangle BCD,\triangle ECD,$ respectively. From Euler theorem $OI_i^2=R^2-2R \cdot r_i \ , \ r_i=1,2,3,4,$ we deduce that $r_1=r_2$ and $r_3=r_4$. But from Japanese theorem we obtain $r_1+r_3+r_6=r_2+r_4+r_5 \Longrightarrow r_5=r_6$. Since a scalene triangle is unambiguosly defined by the radii of its incircle, circumcircle and a side, then $\triangle BCD \cong \triangle EDC \Longrightarrow BC=ED$. Thus by same reasoning, it follows that $\triangle ABC \cong \triangle AED.$

Quick Reply

High School Olympiads

AG parallel to EH

leeky

#1 Oct 24, 2015, 1:37 pm • 1

O, H are the circumcenter, orthocenter of $\triangle ABC$ respectively. D is the foot of perpendicular from C to AB, M is the midpoint of AC, HM meets OD at G. Perpendicular to OD through D meets AC at E. Prove that AG is parallel to EH.

rkm0959

#3 Oct 24, 2015, 4:08 pm • 1

Wow, I bashed for 30 minutes, then made a mistake in writing down a coordinate, and bashed for 30 minutes again.

We toss the figure into the coordinate plane. $A(-a,0), B(b,0), C(0,c), D(0,0), M(-\frac{a}{2}, \frac{c}{2})$.

Now we do the perpendicular bisector bash to get $O(\frac{b-a}{2}, \frac{c^2-ab}{2c})$.

From $AH = 2OM$, we have $AH = \frac{c^2-ab}{c}$, so $H(0, \frac{ab}{c})$.

Let's calculate E. The line CA is $y = \frac{c}{a}x + c$, and the line perpendicular to OD at D is $y = \frac{ca-cb}{c^2-ab}x$.

Calculation gives $E(\frac{a(c^2-ab)}{a^2-c^2}, \frac{ac(a-b)}{a^2-c^2})$.

Let's calculate G. We have line HM is $y = \frac{2ab-c^2}{ac}x + \frac{ab}{c}$, and line OD is $y = \frac{c^2-ab}{cb-ca}x$.

Calculation gives $G(\frac{a^2(b-a)}{a^2-2ab+c^2}, \frac{a^2(c^2-ab)}{c(a^2-2ab+c^2)})$.

We have to prove that $AG \parallel EH$. Calculating the slope gives that both slopes are equal to $\frac{a}{c}$, so we are done.

Also, take note that the slope of MD is $-\frac{c}{a}$. Therefore, we have $AG \parallel EH \perp MD$.

This post has been edited 1 time. Last edited by rkm0959, Oct 24, 2015, 4:29 pm

TelvCohl

#4 Oct 24, 2015, 4:17 pm • 2

Lemma : Let P, Q be the isogonal conjugate WRT $\triangle ABC$. Let P_b be the projection of P on CA and let Q_c be the projection of Q on AB. Then AT is perpendicular to P_bQ_c where $T \equiv PQ_c \cap QP_b$.

Proof : Let Q_b be the projection of Q on CA and let P_c be the projection of P on AB. Let Q_b^*, Q_c^* be the antipode of Q_b, Q_c in the pedal circle of P, Q WRT $\triangle ABC$, respectively. From Pascal theorem (for $P_bQ_b^*Q_bP_cQ_c^*Q_c$) $\Longrightarrow PQ, P_bQ_c, P_cQ_b$ are concurrent, so from Desargue theorem ($\triangle PP_cQ_c$ and $\triangle QQ_bP_b$) we get $A, T, S \equiv PP_c \cap QQ_b$ are collinear. Since S is the antipode of A in $\odot(AP_cQ_b)$, so AT and the perpendicular from A to P_cQ_b are anti-parallel WRT $\angle BAC$, hence notice P_bQ_c is anti-parallel to P_cQ_b WRT $\angle BAC$ we conclude that $AT \perp P_bQ_c$.

Back to the main problem :

Let Z be the midpoint of BC. Let DE cuts BC at R and let R^* be the reflection of R in CH. From $\measuredangle ACO = \measuredangle HCB = \measuredangle R^*CD$ and $\measuredangle ADO = \measuredangle CDR = \measuredangle R^*DC$ we know O, R^* are isogonal conjugate WRT $\triangle ACD$, so AR^* is the isogonal conjugate of AO WRT $\angle BAC \Longrightarrow R^* \in AH$. Since $\measuredangle CRO = \measuredangle ZDO = \measuredangle ZDC + \measuredangle CDO = \measuredangle DAR^* + \measuredangle R^*DA = \measuredangle DR^*H = \measuredangle HRE$, so combine $\measuredangle ACO = \measuredangle HCR$ we get O, H are isogonal conjugate WRT $\triangle CER \Longrightarrow EH \perp DM$. On the other hand, from lemma we get $AG \perp DM$, so we conclude that $AG \parallel EH$.

Remark : This problem can be generalized as following :

Let P, Q be the isogonal conjugate WRT $\triangle ABC$. Let $\triangle P_aP_bP_c, \triangle Q_aQ_bQ_c$ be the pedal triangle of P, Q WRT $\triangle ABC$, respectively. Let the perpendicular from Q_c to PQ_c cuts CA at E and let $G \equiv PQ_c \cap QP_b$. Then $AG \parallel EQ$.

Proof : Let $R \equiv EQ_c \cap BC$ and let $\mathcal{C}$ be the inconic of $\triangle ABC$ with focus P, Q. Since EQ_c is tangent to $\mathcal{C}$ (well-known), so $\mathcal{C}$ is the inconic of $\triangle CER \Longrightarrow P, Q$ are isogonal conjugate WRT $\triangle CER \Longrightarrow EQ \perp P_bQ_c$. On the other hand, from lemma we get $AG \perp P_bQ_c$, so we conclude that $AG \parallel EQ$.

This post has been edited 1 time. Last edited by TelvCohl, Oct 24, 2015, 8:14 pm
Reason: Add remark

Luis González

#6 Oct 25, 2015, 1:17 am • 1

Since $\angle HDA = \angle ODE = 90^\circ$ and $\angle BAH = \angle CAO$, it follows that O, H are isogonal conjugates WRT $\triangle ADE$, thus it suffices to consider the following configuration:

In a $\triangle ABC$, let P be a point such that $PB \perp BC$ and Q is the isogonal conjugate of P WRT $\triangle ABC. Y$ is the projection of Q on AC and $M \equiv PY \cap BQ$. Then $AP \parallel CM$.

Proof: Let U and W be the antipodes of A and C on $\odot(ABC)$. When P varies on BW, the series P, Q, Y are obviously projective and P, Q, M are collinear when $P \equiv B$, thus the series P and M are projective $\Longrightarrow$ pencils AP and CM are projective. Hence it is enough to show that $AP \parallel CM$ holds for 3 positions of P.

When $P \equiv BW \cap AC$, then $M \in AC$, i.e. $AP \equiv CM$. When $P \equiv W$, then $M \equiv U \Longrightarrow (AP \parallel CM) \perp AC$. Finally when P is at infinity, then $Q \equiv U, Y \equiv C \Longrightarrow (AP \parallel CM) \perp BC$. Therefore $AP \parallel CM$ holds for any P, as desired.

P.S. This also proofs the generalization mentioned by Telv in the previous post.

Quick Reply

High School Olympiads

Perspective triangles from tangents

Reply

Source: Own

buratinogigle

#1 Nov 7, 2014, 10:06 am

Let ABC be a triangle and DEF is cevian triangle of a point P. Prove that the tangent at A, B, C of circumcircles of triangles AEF, BFD, CDE bound a triangle that is perpsective to ABC.

Reference ADGEOM.

TelvCohl

#2 Nov 7, 2014, 1:21 pm • 1

My solution :

Let $\triangle A'B'C'$ be a triangle formed by three tangents .
Let $X \equiv BC \cap B'C', Y \equiv CA \cap C'A', Z \equiv AB \cap A'B'$.

Since

$$\frac{\sin \angle BAX}{\sin \angle XAC} \cdot \frac{\sin \angle CBY}{\sin \angle YBA} \cdot \frac{\sin \angle ACZ}{\sin \angle ZCB} = \frac{\sin \angle AEF}{\sin \angle EFA} \cdot \frac{\sin \angle BFD}{\sin \angle FDB} \cdot \frac{\sin \angle CDE}{\sin \angle DEC} = \frac{AF}{AE} \cdot \frac{BD}{BF} \cdot \frac{CE}{CD} = 1,$$

so from Menelaus theorem we get X, Y, Z are collinear $\Longrightarrow \triangle A'B'C'$ and $\triangle ABC$ perspective (Desargue theorem).

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, Oct 31, 2015, 12:46 pm

TelvCohl

#3 Nov 8, 2014, 10:24 am • 1

Another solution:

Let $\triangle XYZ$ be the triangle formed by three tangents .
Let X', Y', Z' be the isogonal conjugate of X, Y, Z WRT $\triangle ABC$, respectively .

Easy to see $A \in Y'Z', B \in Z'X', C \in X'Y'$.

Since $\angle X'BD = \angle ABZ = \angle FDB$,
so we get $Z'X' \equiv BX' \parallel FD$.
Similarly, we can prove $X'Y' \parallel DE$ and $Y'Z' \parallel EF$,
so $\triangle X'Y'Z'$ and $\triangle DEF$ are homothetic ,
hence AX', BY', CZ' are concurrent at the isotomcomplement of P ,
so AX, BY, CZ are concurrent at the isogonal conjugate of the isotomcomplement of P .
i.e. AX, BY, CZ are concurrent at the isogonal conjugate of the crosspoint of $\{X(2), P\}$

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, Aug 21, 2015, 1:48 am

Luis González

#4 Dec 17, 2014, 10:44 am • 2

Let A', B', C' denote the midpoints of EF, FD, DE and let ℓ_A and s_A denote the tangent of $\odot(AEF)$ at A and the A-symmedian of $\triangle AEF$, resp. ℓ_B, s_B and ℓ_C, s_C are defined cyclically. Since AA', BB', CC' concur at the isotomcomplement of P, then s_A, s_B, s_C concur at the isogonal conjugate of the isotomcomplement K of P. Since $(s_A, \ell_A, AB, AC) = -1$, and ciclically, it follows that $\triangle ABC$ and $\triangle(\ell_A, \ell_B, \ell_C)$ are perspective with perspector K and perspectrix the trilinear polar of K.

Quick Reply

High School Olympiads

Two problems concerning nine-point circle

Reply

Radar

#1 Oct 24, 2015, 1:11 am

Let Ω be a circumcircle of triangle ABC and let ω be its nine-point circle. Let XY be a diameter of Ω.

a) Prove that if Simson lines of X and Y intersect at some K, then $K \in \omega$.

b) Let P, Q lying on XY be isogonal conjugates in $\triangle ABC$. Prove that their six-feet circle touches ω at K.

Luis González

#2 Oct 24, 2015, 3:06 am

Problem a) is well-known and has been posted many times before, e.g. Simson line of diametrically opposite points and elsewhere. For b) see two Yango's problems (problem 1).

FTW

#3 Oct 24, 2015, 3:14 am

What is a six-feet circle?

Quick Reply

High School Olympiads

Simson line of diametrically opposite points

Reply

vsalazar
#1 Dec 27, 2010, 4:52 am

Let Γ be the circumcircle of $\triangle ABC$, P, P' two diametrically opposite points on Γ and I the intersection of their Simson lines. Find the locus of I when P describes Γ.

Is very nice :) I made it up

Luis González
#2 Dec 27, 2010, 6:22 am • 1

Locus of I is the celebrated 9-point circle (N) of $\triangle ABC$. This follows from the fact that the oriented angle between the Simon lines p, p' of P, P' is half the measure of the oriented arc PP' of Γ and p, p' bisect (through U,V) the segments $\overline{HP}$ and $\overline{HP'}$ connecting the orthocenter H of $\triangle ABC$ with the poles P, P'. Thus UV is a diameter of the 9-point circle and $\angle UIV = 90^\circ \Longrightarrow I \in (N)$. Further, there's a strong generalization of this result:

• D is a fixed point in the plane of $\triangle ABC$ and a line ℓ through D cuts the circumcircle Γ at P, P'. Simson lines of P, P' WRT $\triangle ABC$ intersect at I. Then locus of I as ℓ spins around D is an ellipse $\mathcal{E}$ circumscribed in the pedal triangle of D WRT $\triangle ABC$. Its center is the midpoint of the segment connecting the orthocenter H with D and its axes have the same orientation of the asymptotes of equilateral circum-hyperbola $ABCD^{-1}$ where D^{-1} is the isogonal conjugate of D.

jayme
#3 Dec 27, 2010, 7:37 pm

Dear Mathlinkers,
only for history, the initial result comes from Steiner.
Sincerely
Jean-Louis

armpist
#4 Dec 27, 2010, 10:03 pm

> *jayme wrote:*
> Dear Mathlinkers,
> only for history, the initial result comes from Steiner.
> Sincerely
> Jean-Louis

Dear J-L

It is a well-known theorem of Goffart. If it were Steiner's own,
I would think that Goffart would have known this fact and would
not place it in N.A. as his own proposition.

I also want to propose to call this wonderful theorem as Goffart-Salazar.
Steiner just has enough theorems in his name.

M.T.

Attachments:

Attachments:
Goffart T..doc (280kb)

jayme

#5 Dec 28, 2010, 3:57 pm

Dear Armpist (M.) and Mathlinkers,
I can now precise my reference
Steiner , Crelle 53 (1857) p. 237.
I can read in German but the link is not very quick in order to have a copy of the passage in question perhaps some Mathlinkers...
Sincerely
Jean-Louis.

Quick Reply

High School Olympiads

two Yango's problems

Reply

littletush

#1 Jun 26, 2011, 11:45 am

1. In $\triangle ABC$ are a pair of isogonal conjugate points P and P'. The circuncenter O is on line PP'. Suppose that PJ, PQ, PK are perpendicular to AB, AC, BC respectively. Prove that the circumcircle of $\triangle JQK$ and $\triangle J'Q'K'$ are both tangent to the Euler's circle of $\triangle ABC$.

2. Two equal circle $\odot J$ and $\odot K$ intersect at A and B. $\odot P$ and $\odot Q$ are in $\odot J$, while they are tangent to $\odot J$ and $\odot K$, we know that $\odot P$ is tangent to $\odot Q$ at T. Prove that AT is perpendicular to BT.

Luis González

#2 Jun 27, 2011, 2:12 am

Problem 1 is known as 3rd Fontené theorem, which follows straightforwardly from the 1st Fontené theorem. Thus, let us prove the 1st Fontené theorem.

Theorem. P is an arbitrary point on the plane of $\triangle ABC$ with circumcircle (O). D, E, F are the midpoints of BC, CA, AB and P_1, P_2, P_3 are the orthogonal projections of P onto BC, CA, AB. Let $A_1 \equiv EF \cap P_2P_3$ and define B_1, C_1 cyclically. Then, P_1A_1, P_2B_1, P_3C_1 concur at a point $U \equiv \odot(P_1P_2P_3) \cap \odot(DEF)$, which, in general, is different from the Poncelet point of A, B, C, P. Furthermore, U is the anti-Steiner point of OP WRT $\triangle DEF$.

Proof. Orthogonal projection V of A onto OP is clearly the second intersection of the circumcircles of the cyclic quadrilaterals PP_2AP_3 and $OEAF$ with circumdiameters AP, AO. Since V is indeed the Miquel point of the complete quadrilateral bounded by AB, AC, EF, P_2P_3, it follows that $V \in \odot(FA_1P_3)$ $(\star)$. On the other hand, let PP_1 cut $\odot(AP_2P_3)$ again at T. Since AP is a diameter of $\odot(AP_2P_3)$, it follows that $\angle ATP$ is right, i.e. $AT \parallel EF$, i.e. EF is the perpendicular bisector of $TP_1 \Longrightarrow \angle TAF = \angle AFE$. But from $(\star)$ we deduce that $\angle A_1VP_3 = \angle AFE \Longrightarrow A_1 \in VT$. Further, since A_1 lies on the radical axis P_2P_3 of $\odot(AP_2P_3)$ and $\odot(P_1P_2P_3)$, then A_1 has equal power WRT $\odot(AP_2P_3)$ and $\odot(P_1P_2P_3)$. Consequently, if P_1A_1 cuts $\odot(P_1P_2P_3)$ again at U, it follows that $TUVP_1$ is an isosceles trapezoid with $UV \parallel TP_1 \Longrightarrow U$ is the reflection of V across EF. Since $\odot(AEF), \odot(DEF)$ are symmetric about EF, then $U \in \odot(DEF)$ is the anti-Steiner point of OP WRT DEF.

Now, let P^* be the isogonal conjugate of P WRT $\triangle ABC$. It's well known that P and P^* share the same pedal circle $\odot(P_1P_2P_3)$. If lines OP, OP^* are distinct, then the anti-Steiner point of OP^* WRT $\triangle DEF$ is the intersection U' of $\odot(DEF)$ and $\odot(P_1P_2P_3)$ different from U (Poncelet point of ABCP). As a result, $P^* \in OP \Longleftrightarrow U \equiv U' \Longleftrightarrow$ $\odot(P_1P_2P_2)$ and $\odot(DEF)$ are tangent through U.

P.S. For alternate approaches see Hard geometry and Pedal circle.

Luis González

#3 Jun 27, 2011, 4:22 am

Solution to problem 2.

Let M be the midpoint of $\overline{AB}$, which is the insimilicenter (center of symmetry) of $(J) \cong (K)$. Let (P) touch (K) and (J) at D, F. Thus, D is the exsimilicenter of $(P) \sim (K)$ and F is the insimilicenter of $(P) \sim (J)$. Hence, DF passes through the insimilicenter M of $(J) \cong (K)$. If DM cuts (J) again at D', we have $MF \cdot MD = MF \cdot MD' = MA^2$. Likewise, power of M WRT (Q) is $MA^2 \Longrightarrow$ circle with diameter $\overline{AB}$ is orthogonal to both $(P), (Q) \Longrightarrow T \in \omega$.

drmzjoseph

#4 Oct 31, 2015, 10:16 pm

> Luis González wrote:
> As a result, $P^* \in OP \Longleftrightarrow U \equiv U'$.

This is true, but maybe the reasonning isn't correct. Because the case If lines OP, OP^* are distinct, implies that $U \neq U'$ not means that $OP \equiv OP^* \Rightarrow U \equiv U'$ But if is true $U \equiv U' \Rightarrow OP \equiv OP^*$ according to reassoning.

Problem 1:

In $\triangle ABC$ are a pair of isogonal conjugate points P and P'. The circumcenter O lies on the line PP'. Let $\triangle XYZ$ be the pedal triangle of P. Prove that the $\odot(XYZ)$ is tangent to the Euler circle of $\triangle ABC$

My proof:

Denote A', B', C' the midpoints of BC, CA, AB respectively.
Denote γ the pedal circle of $\{P, P'\}$, denote $\mathcal{P}(X)$ the $\sqrt{\text{power}}$ of X on γ
Since O, P, P' is easy notice that $\dfrac{XA'}{\mathcal{P}(A')} = \dfrac{YB'}{\mathcal{P}(B')} = \dfrac{ZC'}{\mathcal{P}(C')}$

According to Purser's theorem(Is possible apply this, because A', B, C' are they all interiors or exteriors to γ): γ is tangent to $\odot(A'B'C')$ if only if $\mathcal{P}(A').B'C' \pm \mathcal{P}(B').C'A' \pm \mathcal{P}(C').A'B' = 0$
If only if $XA'.B'C' \pm YB'.C'A' \pm ZC'.A'B' = 0$ and this is true according to another geometry problem

This post has been edited 2 times. Last edited by drmzjoseph, Oct 31, 2015, 10:45 pm

Quick Reply

High School Olympiads

Concurrency on OI

Reply

Source: Experiments on GeoGebra

jlammy

#1 Oct 23, 2015, 7:02 pm • 1

Let DEF be the intouch triangle of $\triangle ABC$, which has incenter I and circumcenter O. Let (J_a) be the circle tangent to BC at D and tangent to (ABC) at T_a; define circles $(J_b), (J_c)$ and points T_b, T_c similarly. Prove that:

(a) AT_a, BT_b, CT_b concur on OI. [EDIT: OMDS THIS IS THE ISOGONAL MITTENPUNKT!]

(b) AJ_a, BJ_b, CJ_c concur.

Both these points seem to be triangle centers. Do the results still hold after extraversion?

This post has been edited 1 time. Last edited by jlammy, Oct 23, 2015, 7:08 pm
Reason: omds X_57

Luis González

#2 Oct 23, 2015, 8:39 pm

(a) The isogonal Mittenpunkt is the homothetic center of $\triangle ABC$ and the orthic triangle of $\triangle DEF$ (well-known), so if X is the projection of D on EF, it suffices to prove that A, X, T_a are collinear and similarly for the others. Since T_aD bisects $\angle BT_aC$, we have

$$\frac{\sin \widehat{BAT_a}}{\sin \widehat{CAT_a}} = \frac{T_aB}{T_aC} = \frac{DB}{DC} \ , \ \frac{\sin \widehat{BAX}}{\sin \widehat{CAX}} = \frac{XF}{XE} = \frac{BF}{CE} = \frac{DB}{DC} \Longrightarrow AX \equiv AT_a.$$

(b) Close, but no cigar. Geogebra says these lines are not concurrent.

Quick Reply

High School Olympiads

Geometry□a little hard□OWN□

Reply

Lin_yangyuan

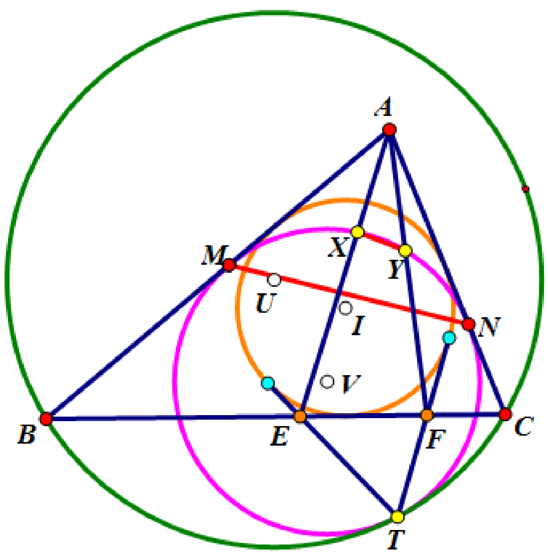
#1 Oct 18, 2015, 8:29 pm

This problem contains some conclusions I got before□a little hard□who have a try

Attachments:

Known △ ABC, circle U through B, C, circle V cut AB at M, cut AC at N, and cut circle U at T, through T as the inscribed circle tangent to BC at E, F, AE, AF cross circle V in X, Y.
Prove: XY, MN, BC total points.

已知△*ABC*，⊙*U*经过*B*,*C*，⊙*V*切*AB*于*M*，切*AC*于*N*，且与⊙*U*切于*T*，过*T*作内切圆作切线
交*BC*于*E*,*F*，*AE*,*AF*交⊙*V*于*X*,*Y*。
求证：*XY*,*MN*,*BC*共点.



GoJensenOrGoHome

#2 Oct 18, 2015, 10:06 pm

Sorry Lin_yangyuan, can you explain me what does "total points" mean?

Lin_yangyuan

#5 Oct 19, 2015, 6:28 pm

> *GoJensenOrGoHome wrote:*
> Sorry Lin_yangyuan, can you explain me what does "total points" mean?

sorry my english is poor ,it means three lines are concurrent

cgsa4
#6 Oct 21, 2015, 6:14 pm

I think the circle U must be the circumcircle of triangle ABC.

---

neelanjan
#7 Oct 21, 2015, 6:16 pm

what is total points .

---

amplreneo
#8 Oct 21, 2015, 6:46 pm

> *neelanjan wrote:*
> what is total points .

Lin_yangyuan clarified above.

> *Lin_yangyuan wrote:*
>> *GoJensenOrGoHome wrote:*
>> Sorry Lin_yangyuan, can you explain me what does "total points" mean?
>
> sorry my english is poor ,it means three lines are concurrent

---

neelanjan
#9 Oct 21, 2015, 7:19 pm

try it by homothety .

TelvCohl
#10 Oct 21, 2015, 9:49 pm • 1

Lemma : Given a $\triangle ABC$ and a circle $\odot(O)$ passing through B, C. Let $\odot(I)$ be the circle tangent to AB, AC and tangent to $\odot(O)$ (internally) at T. Let $E \equiv CA \cap \odot(I), F \equiv AB \cap \odot(I)$ and let $P \equiv EF \cap BC$. Let a line through P cuts $\odot(I)$ at Y, Z and $U \equiv AY \cap BC, V \equiv AZ \cap BC$. Then TU, TV are isogonal conjugate WRT $\angle BTC$.

Proof : Let the internal bisector of $\angle BTC$ cuts BC at Q. Let $Q_1 \equiv AQ \cap EF, Y_1 \equiv AY \cap EF, Z_1 \equiv AZ \cap EF$ and let the tangent of $\odot(I)$ passing through Y, Z cuts EF at Y_2, Z_2, respectively. From the problem a geometry lemma we get TP is the external bisector of $\angle BTC$, so $(E, F; P, Q_1) = (C, B; P, Q) = -1 \Longrightarrow AQ_1$ is the polar of P WRT $\odot(I)$. From Desargue involution theorem (for $YYZZ$) we get P is fixed under the involution that swaps $(E, F), (Y_2, Z_2)$, so

$$(Q, U; C, B) = (Q_1, Y_1; E, F) = (P, Y_2; E, F) = (P, Z_2; F, E) = (Q_1, Z_1; F, E) = (Q, V; B, C)$$

$\Longrightarrow U, V$ is the image of each other under the involution with fixed point Q and swaps (B, C), hence TU, TV are symmetry WRT TQ. i.e. TU, TV are isogonal conjugate WRT $\angle BTC$

Back to the main problem :

Since the internal bisector of $\angle BTC$ passes through the incenter I of $\triangle ABC$ (see incenter of triangle), so TE, TF are isogonal conjugate WRT $\angle BTC$, hence from the lemma we conclude that XY, MN, BC are concurrent.

Lin_yangyuan
#11 Oct 22, 2015, 7:07 pm

> *TelvCohl wrote:*
> **Lemma :** Given a $\triangle ABC$ and a circle $\odot(O)$ passing through B, C. Let $\odot(I)$ be the circle tangent to AB, AC and tangent to $\odot(O)$ (internally) at T. Let $E \equiv CA \cap \odot(I), F \equiv AB \cap \odot(I)$ and let $P \equiv EF \cap BC$. Let a line through P

cuts $\odot(I)$ at Y, Z and $U \equiv AY \cap BC, V \equiv AZ \cap BC$. Then TU, TV are isogonal conjugate WRT $\angle BTC$.

Proof : Let the internal bisector of $\angle BTC$ cuts BC at Q. Let $Q_1 \equiv AQ \cap EF, Y_1 \equiv AY \cap EF, Z_1 \equiv AZ \cap EF$ and let the tangent of $\odot(I)$ passing through Y, Z cuts EF at Y_2, Z_2, respectively. From the problem a geometry lemma we get TP is the external bisector of $\angle BTC$, so $(E, F; P, Q_1) = (C, B; P, Q) = -1 \Longrightarrow AQ_1$ is the polar of P WRT $\odot(I)$. From Desargue involution theorem (for $YYZZ$) we get P is fixed under the involution that swaps (E, F), (Y_2, Z_2), so

$$(Q, U; C, B) = (Q_1, Y_1; E, F) = (P, Y_2; E, F) = (P, Z_2; F, E) = (Q_1, Z_1; F, E) = (Q, V; B, C)$$

$\Longrightarrow U, V$ is the image of each other under the involution with fixed point Q and swaps (B, C), hence TU, TV are symmetry WRT TQ. i.e. TU, TV are isogonal conjugate WRT $\angle BTC$

Back to the main problem :

Since the internal bisector of $\angle BTC$ passes through the incenter I of $\triangle ABC$ (see incenter of triangle), so TE, TF are isogonal conjugate WRT $\angle BTC$, hence from the lemma we conclude that XY, MN, BC are concurrent.

The same solution 😄

Luis González

#12 Oct 23, 2015, 3:49 am

Let TE, TF cut (U) again at E', F'. From the problem parallel lines, we have $E'F' \parallel EF \Longrightarrow \odot(TEF)$ is tangent to (U) through $T \Longrightarrow (U), (V), \odot(TEF)$ forms a pencil. Thus if (V) cuts BC at B', C, then $(B, C), (B', C'), (E, F)$ are pairs of points in involution $\Longrightarrow (AB, AC), (AB', AC'), (AE, AF)$ are pairs of rays in involution, inducing an involution on (V) $\Longrightarrow X \mapsto Y, M \mapsto N, B' \mapsto C'$ is an involution on $(V) \Longrightarrow XY, MN$ and $B'C' \equiv BC$ concur at the pole of the involution.

Quick Reply

High School Olympiads

parallel lines

Reply

Source: Own

andria

#1 Jun 25, 2015, 2:15 pm • 3

In $\triangle ABC$ points D, E lie on AB, AC such that $BDEC$ is cyclic quadrilateral. ω is a circle which is tangent to AB, AC and also it is tangent to $\odot(BDEC)$ at M.point P varies on the arc BC(doesn't contain D); let the tangents from P to the incircle of $\triangle ABC$ intersect $\odot(BDEC)$ at S, T prove that $ST||BC$ if and if only $P \equiv M$.

Luis González

#2 Jul 5, 2015, 10:03 am • 3

Let the incircle (I) of $\triangle ABC$ touch BC at X and let PA, PX cut $\Omega \equiv \odot(BDEC)$ again at $Q, Y,$ respectively. By dual of Desagues involution theorem for $ABXC$ circumscribed to (I), it follows that $PB \mapsto PC, PS \mapsto PT, PX \mapsto PA$ is an involution $\Longrightarrow B \mapsto C, S \mapsto T, Y \mapsto Q$ is an involution on $\Omega \Longrightarrow BC, ST, QY$ concur at the pole of the involution. Therefore, if $ST \parallel BC$, then $QY \parallel ST \parallel BC \Longrightarrow PQ, PY$ are isogonals WRT $\angle SPT$. Since PX is the P-Nagel cevian of $\triangle PST$, then we deduce that Q is the tangency point of the P-mixtilinear incircle γ of $\triangle PST$ with Ω.

Since P and Q are the exsimilicenters of $(I) \sim \gamma$ and $\Omega \sim \gamma$, then by Monge & d'Alembert theorem for $(I), \Omega, \gamma,$ it follows that PQ passes through the exsimilicenter of K of $(I) \sim \Omega$. But by Monge & d'Alembert theorem for $(I), \Omega, \omega,$ it follows that A, K, M are collinear $\Longrightarrow A, Q, M$ are collinear $\Longrightarrow P \equiv M$. The converse is immediate.

Quick Reply

High School Olympiads

Focus of Steiner inellipse, Polar, Tripolar

Reply

Source: Own

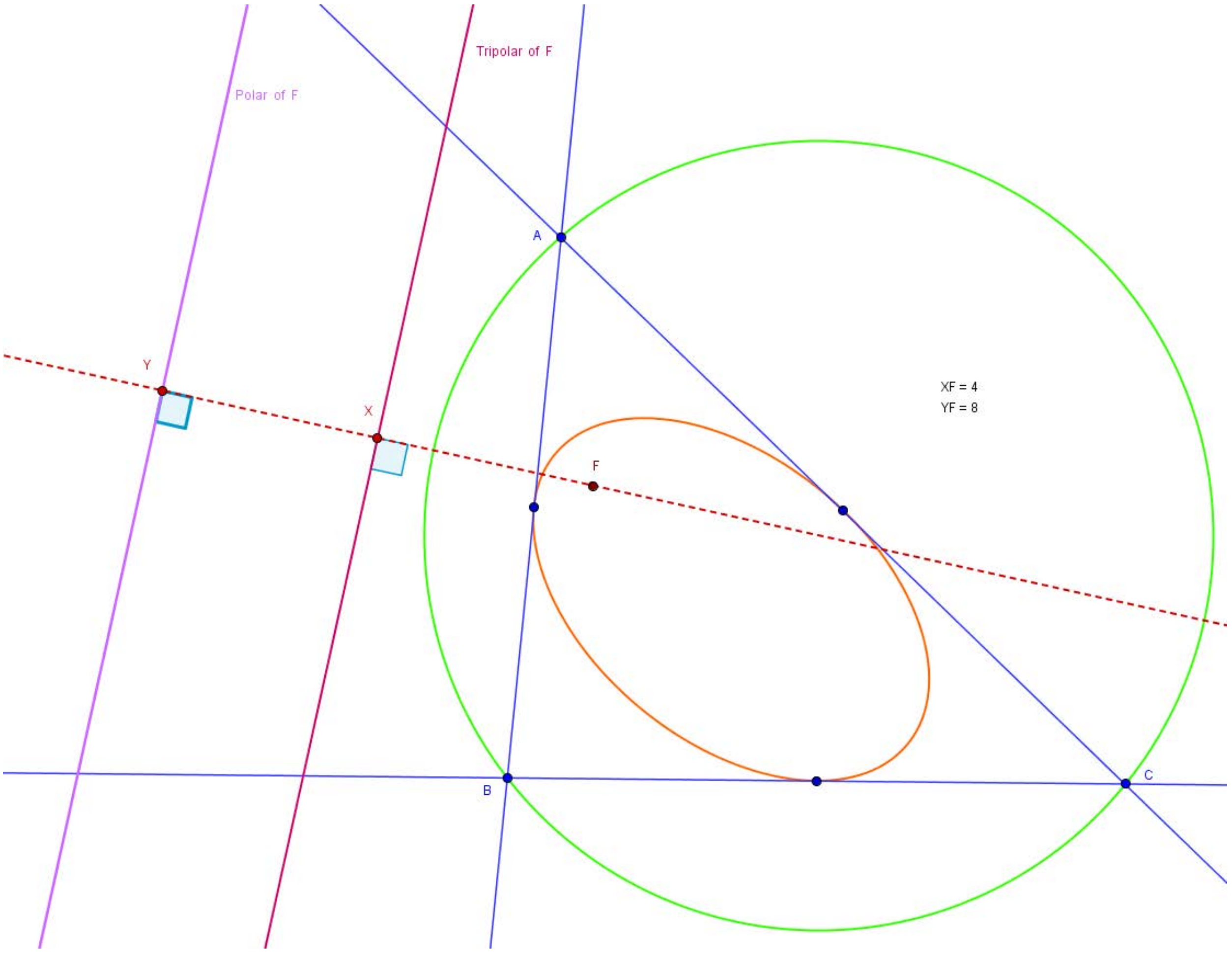
TelvCohl

#1 Aug 7, 2015, 6:47 am • 2

Let F be the focus of the Steiner inellipse of $\triangle ABC$.
Let τ be the polar of F WRT $\odot(ABC)$ and ϱ be the tripolar of F WRT $\triangle ABC$.

Prove that $\tau \parallel \varrho$ and $\text{dist}(F, \tau) = 2\text{dist}(F, \varrho)$

Attachments:



Luis González

#2 Oct 14, 2015, 10:58 am • 2

Lemma: P is arbitrary point on the plane of $\triangle ABC$ and $\triangle XYZ$ is the circumcevian triangle of P WRT $\triangle ABC.$ τ, ϱ and λ are the polar of P WRT $\odot(ABC)$, the trilinear polar of P WRT $\triangle ABC$ and the trilinear polar of P WRT $\triangle XYZ$, respectively. Then τ, ϱ, λ concur at a point Q and moreover the pencil formed by $\tau, \varrho, \lambda, QP$ is harmonic.

Proof: Denote U, V, W the intersections of PC with τ, ϱ and λ, respectively. Consider the homology sending P to the center of the conic image of $\odot(ABC)$. Thus $\triangle ABC$ and $\triangle XYZ$ become symmetric about $P \Longrightarrow \varrho, \lambda$ become symmetric about P and τ becomes the line at infinity $\Longrightarrow \tau \parallel \varrho \parallel \lambda$ and $\overline{PV} = -\overline{PW}$. So in the primivite figure, it follows that τ, ϱ, λ concur and $(U, V, W, P) = -1$.

Back to the problem, we use the previous lemma for $P \equiv F$ and for the sake of ease we keep all the previous notations. From the solution of the problem Construct a triangle given three special points, we get that F is the centroid of $\triangle XYZ$. As the trilinear polar of F WRT $\triangle XYZ$ is the line at infinity, then from the lemma we deduce that $\tau \parallel \varrho$ and $(U, V, \infty, F) = -1$ $\Longrightarrow V$ is midpoint of $\overline{FU} \Longrightarrow \text{dist}(F, \tau) = 2 \cdot \text{dist}(F, \varrho)$.

Quick Reply

High School Olympiads

Construct a triangle given three special points

Reply

Goutham

#1 Dec 31, 2011, 6:24 pm

Construct a triangle, by straight edge and compass, if the three points where the extensions of the medians intersect the circumcircle of the triangle are given.

Luis González

#2 Dec 31, 2011, 8:58 pm • 1

Label the unknown triangle $\triangle ABC$ with centroid G and symmedian point K. AG, BG, CG cut the circumcircle of $\triangle ABC$ again at the known points D, E, F. U is the centroid of $\triangle DEF$ and G_1, G_2, G_3 are the orthogonal projections of G on EF, FD, DE. From the cyclic quadrilaterals GG_1EG_3 and GG_1FG_2, we obtain $\angle GG_1G_2 = \angle CFD = \angle CAG = \angle KAB$. Analogously, we have $\angle GG_2G_1 = \angle KBA \Longrightarrow \triangle GG_1G_2 \sim \triangle KAB$. Likewise, $\triangle GG_1G_3 \sim \triangle KAC \Longrightarrow \triangle ABC \cup K \sim \triangle G_1G_2G_3 \cup G$, i.e. G is the symmedian point of $\triangle G_1G_2G_3$.

According to the topic symmedian point, the circumcenter of any triangle is the centroid of the antipedal triangle of its symmedian point. So we deduce that U is the center of the pedal circle of G WRT $\triangle DEF \Longrightarrow \odot(G_1G_2G_3)$ is the pedal circle of the conic $\mathcal{U}$ with center U tangent to EF, FD, DE, i.e. the Steiner inellipse $\mathcal{U}$ of $\triangle DEF$ tangent to its sides through their midpoints. Thus, the possible points G are the foci of $\mathcal{U}$.

The foci of $\mathcal{U}$ are indeed constructible using ruler and compass by defining an appropriate homology that takes $\mathcal{U}$ into a circle, e.g. see the exercise Construction of conic. Once the foci of $\mathcal{U}$ are constructed, then their circumcevian triangles WRT $\triangle DEF$ determine the two possible $\triangle ABC$.

Quick Reply

High School Olympiads

Concurrent related to a point on Incircle

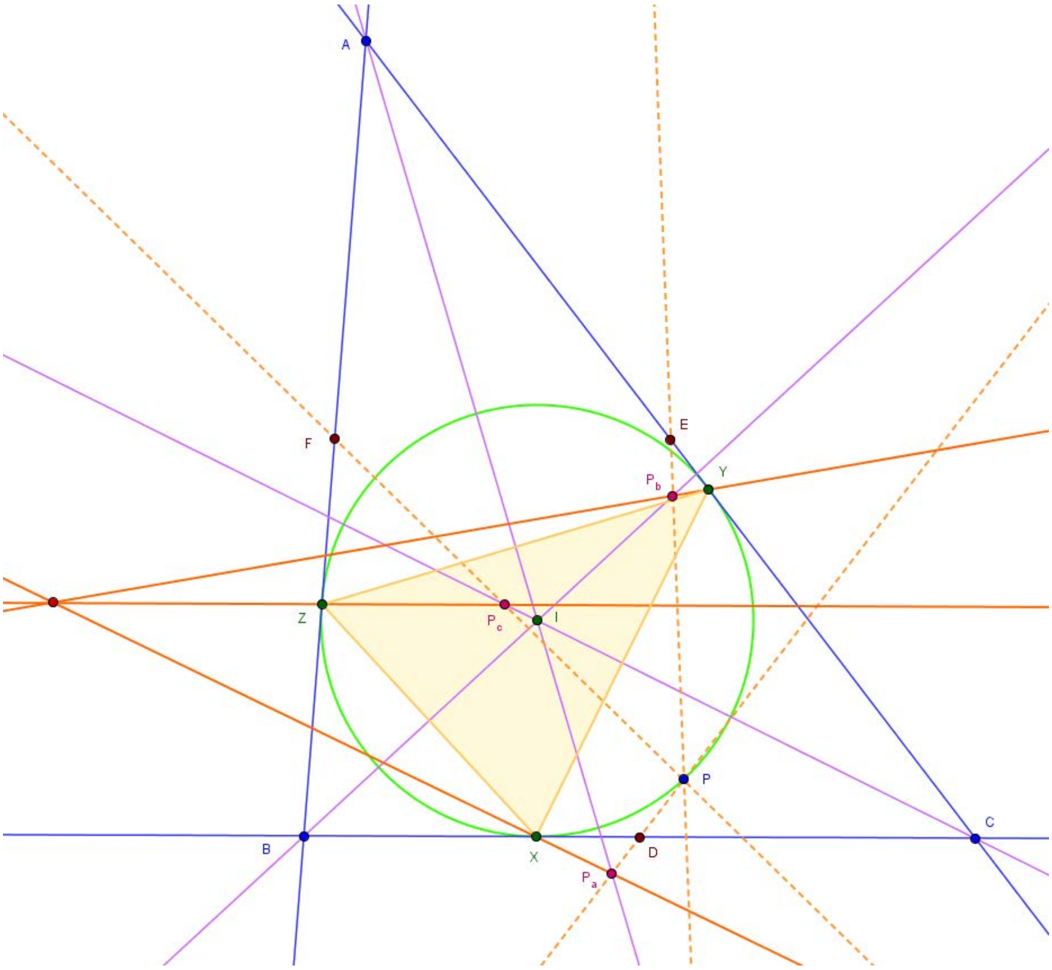
Reply

Source: Own

TelvCohl
#1 Oct 14, 2015, 7:53 am

Given a $\triangle ABC$ with incenter I. Let $\triangle DEF, \triangle XYZ$ be the medial triangle, intouch triangle of $\triangle ABC$, respectively. Let P be an arbitrary point on the incircle $\odot(I)$ of $\triangle ABC$ and $P_a \equiv PD \cap AI, P_b \equiv PE \cap BI, P_c \equiv PF \cap CI$. Prove that XP_a, YP_b, ZP_c are concurrent.

Attachments:



Luis González
#2 Oct 14, 2015, 8:59 am • 1

Dear Telv, this is a particular case of the configuration discussed in the problem On triangles with vertices on the internal angle bisectors.

Quick Reply

High School Olympiads

On triangles with vertices on the internal angle bisectors

Reply

Source: by Eric Daneels; prove synthetically!

pohoatza
#1 Aug 12, 2009, 2:50 am

Let ABC be a triangle and let I be its incenter. Let DEF and MNP be its intouch and medial triangles, respectively (i.e. D, E, F are the tangency points of the incircle with the sidelines; M, N, P are the midpoints of BC, CA, AB, respectively), and let X, Y, Z be points on the lines AI, BI, CI, respectively. Prove that triangles XYZ and DEF are perspective if and only if XYZ and MNP are perspective.

Luis González
#2 May 6, 2011, 11:02 am • 2

This is a beautiful generalization of the second mid-arc point, among other perspectors. Label $EF \equiv \tau_A, FD \equiv \tau_B$, $DE \equiv \tau_C, NP \equiv \ell_A, PM \equiv \ell_B, MN \equiv \ell_C$. Notation $\delta(U,\ell)$ stands for the distance from the point U to the line ℓ. It's well known that in any scalene triangle, the polar of one vertex with respect to the incircle, the midline referent to a second vertex and the inner angle bisector issuing from the third, concur. Thus, ED, PM, AI concur and FD, NM, AI concur. Consequently, any point X on the line AI will satisfy

$$\frac{\delta(X,\tau_C)}{\delta(X,\ell_B)} = \frac{\delta(I,\tau_C)}{\delta(I,\ell_B)} \ , \ \frac{\delta(X,\tau_B)}{\delta(X,\ell_C)} = \frac{\delta(I,\tau_B)}{\delta(I,\ell_C)} \Longrightarrow$$

$$\frac{\delta(X,\tau_C)}{\delta(X,\tau_B)} \cdot \frac{\delta(X,\ell_C)}{\delta(X,\ell_B)} = \frac{\delta(I,\tau_C)}{\delta(I,\tau_B)} \cdot \frac{\delta(I,\ell_C)}{\delta(I,\ell_B)} \quad (1)$$

Analogously, for points Y, Z on BI, CI we have the relations

$$\frac{\delta(Y,\tau_A)}{\delta(Y,\tau_C)} \cdot \frac{\delta(Y,\ell_A)}{\delta(Y,\ell_C)} = \frac{\delta(I,\tau_A)}{\delta(I,\tau_C)} \cdot \frac{\delta(I,\ell_A)}{\delta(I,\ell_C)} \quad (2)$$

$$\frac{\delta(Z,\tau_B)}{\delta(Z,\tau_A)} \cdot \frac{\delta(Z,\ell_B)}{\delta(Z,\ell_A)} = \frac{\delta(I,\tau_B)}{\delta(I,\tau_A)} \cdot \frac{\delta(I,\ell_B)}{\delta(I,\ell_A)} \quad (3)$$

Multiplying the equations (1), (2) and (3) yields

$$\left[\frac{\delta(X,\tau_C)}{\delta(X,\tau_B)} \cdot \frac{\delta(Y,\tau_A)}{\delta(Y,\tau_C)} \cdot \frac{\delta(Z,\tau_B)}{\delta(Z,\tau_A)}\right] \cdot \left[\frac{\delta(X,\ell_C)}{\delta(X,\ell_B)} \cdot \frac{\delta(Y,\ell_A)}{\delta(Y,\ell_C)} \cdot \frac{\delta(Z,\ell_B)}{\delta(Z,\ell_A)}\right] = 1$$

By the converse of Ceva's theorem in the intouch triangle $\triangle(\tau_A, \tau_B, \tau_C)$ and medial triangle $\triangle(\ell_A, \ell_B, \ell_C)$, we conclude that DX, EY, FZ concur $\Longleftrightarrow MX, NY, PZ$ concur.

Quick Reply

High School Olympiads

Proofs of Useful Identities and Inequalities in Geometry

Reply

BigSams
#1 Jun 30, 2011, 7:52 am • 16

The plan from the beginning was to:
(1) Compile the most Useful Identities and Inequalities in Geometry in a thread.
(2) Create an organized PDF of the collection, the first version of which is attached below.
(3) Open this thread to collect proofs of all of the results in the PDF.
(4) Create an organized PDF of the proofs in this thread.
(5) Post both PDFs in the Olympiad Articles forum.
(6) Continue a cycle of updates for both PDFs and threads.
There are a couple of other stages but I'll keep those a secret for now.

Rules:
(1) If you have comments or corrections for a specific post, PM that user.
(2) If you have comments or corrections for the PDF, PM me.
(3) Let's keep this thread clean and formal.
(4) Fresh proofs are welcome; all good (distinct) ones will be compiled. Obviously check previous posts to ensure no repeats.
(5) It is preferable that you attempt to post proofs of all results of a "set", as shown below.
(6) Synchronize with the notation in the PDF.

Note that for the Trigonometric Identities, post proofs that are as least cased-based as possible, for the sake of elegance.
I have written proofs in my notebooks for most of the results in the PDF and will post them later if they have not already been posted.
Here a few proofs to show how the thread should look.

4.1

4.2

4.3

4.4

Attachments:
Identities and Inequalities in Geometry - Version 1.0.pdf (237kb)

applepi2000
#2 Jun 30, 2011, 9:04 am • 2

4.7
4.8

GlassBead
#3 Jul 1, 2011, 12:01 am • 2

4.9

4.13

Should I provide an additional proof that $\tan\dfrac{A}{2}=\dfrac{r_a}{s}$?

Edit:
Additional proof

gaussintraining
#4 Jul 1, 2011, 5:25 am • 1

4.10

gaussintraining
#5 Jul 1, 2011, 8:21 am • 1

4.5

gaussintraining
#6 Jul 2, 2011, 5:48 am • 1

4.11
4.14

gaussintraining
#7 Jul 5, 2011, 9:06 am • 1

4.6
4.12

gaussintraining
#8 Jul 6, 2011, 7:05 am • 1

4.15
4.16

applepi2000
#9 Jul 10, 2011, 11:10 pm • 1

4.19

4.20

4.21

Almost done with 4, only 4.17, 4.18, and 4.22 left!

applepi2000
#10 Jul 11, 2011, 1:45 am • 1

6.3

6.4

GlassBead
#11 Jul 11, 2011, 2:19 am • 1

4.17

gaussintraining
#12 Jul 13, 2011, 7:40 pm • 1

4.18

applepi2000
#13 Jul 16, 2011, 6:50 am • 3

We have completed section 4:
4.22

5.1

BigSams
#14 Aug 14, 2011, 8:54 am

^Thanks everyone. I'll check them in detail later.
Other than the results with names and a few others, the remaining proofs are not difficult.. just tedious

I have realized that the Trigonometric Identities and their proofs can be found easily in thousands of textbooks and websites. For this reason (and because documenting the proofs would be a pain) the Trigonometric Identities section will be moved to the end as an Appendix without proofs. Most of the rest of the results are unusual and not tracked down easily altogether so their proofs will certainly be added. The best source for learning and understanding the Trigonometric Identities is AoPS Precalculus in my opinion. Otherwise, a quick Google search suffices.

Since the proofs posts seem to be going slowly, I am posting a set and will post more later. The distances between special points on the plane of a triangle can all be derived by some complex number geometry. Can someone type them up? Otherwise I'll do it sometime, but I'm pretty busy right now.

3.2

3.2.i and **3.2.j** are very closely related to some results in section 4, so they'll be removed and I didn't show the 1-line proofs.

ahaanomegas
#15 Aug 17, 2011, 3:53 am

@**BigSams**

Thank you very much for creating this thread! One question though: Did you mean 3.2 in your above post?
Moderator Edit: ^yes, thanks. Edited. See Rule 1 though.

3.3

applepi2000
#16 Aug 26, 2011, 2:44 am • 1

6.6

vivekrai
#17 Dec 19, 2011, 3:31 pm

I wish to have a **proof** for these :
For any point X in the plane :
$a\cdot XA^2+b\cdot XB^2+c\cdot XC^2=(a+b+c)\cdot XI^2+abc$
$3(XA^2+XB^2+XC^2)=9\cdot XG^2+(a^2+b^2+c^2)$

Learner94
#18 Dec 19, 2011, 5:21 pm

Consider $\triangle ABC$ and a point M in its complex plane. Let $A=z_1, B=z_2, C=z_3$ and $M=z$. M_A denotes the length of the median drawn from A.

Use $\frac{1}{\ }\sum^{3}|z-z_i|^2=3\left|z-\frac{z_1+z_2+z_3}{\ }\right|^2+\sum^{3}\left|z_i-\frac{z_1+z_2+z_3}{\ }\right|^2$

So we have $MA^2+MB^2+MC^2=3MG^2+GA^2+GB^2+GC^2$. Since $GA=\frac{2m_A}{3}$, We have $\sum_{cyc}GA^2=\frac{4}{9}\sum_{cyc}m_A^2=\frac{4}{9}\sum_{cyc}\frac{1}{4}\left(2(b^2+c^2)-a^2\right)=\frac{a^2+b^2+c^2}{3}$. (we have used the median formula $m_A^2=\frac{2(b^2+c^2)-a^2}{4}$.)

This post has been edited 3 times. Last edited by Learner94, Dec 19, 2011, 5:55 pm

vivekrai
#19 Dec 19, 2011, 5:43 pm

> *Learner94 wrote:*
> Consider $\triangle ABC$ and a point M in its complex plane. Let $A=z_1, B=z_2, C=z_3$ and $M=z$. M_A denotes the length of the median drawn from A.
>
> Use $\frac{1}{3}\sum_{j=1}^{3}|z-z_j|=3\left|z-\frac{z_1+z_2+z_3}{3}\right|+\sum_{j=1}^{3}\left|z_j-\frac{z_1+z_2+z_3}{3}\right|$

If I get this part then I would have surely done this. Could you highlight this part?

Learner94
#20 Dec 19, 2011, 5:53 pm

Sorry. I had some typo(edited).

For the proof just expand using $|z|^2=z\overline{z}$. Also this is true for all natural number n. (not just 3)

I would use complex number again to prove the other identity you posted. $I=\frac{az_1+bz_2+cz_3}{a+b+c}$. But may be a better proof is possible...

lehungvietbao
#21 Sep 27, 2012, 7:12 am

Dear mathlinkers, please post new results
I like this topic, it's very interesting .

ahaanomegas
#22 May 25, 2013, 6:36 am • 2

3.4
3.5

As a sidenote, I am surprised **Menelaus' Theorem** is not in the document. I'll do more later.

Section 3 progress

pi37
#23 May 28, 2013, 12:17 am • 1

3.6
3.7
3.14
3.15
3.16
3.17

fmasroor

#24 Jul 4, 2013, 6:32 am

> gaussintraining wrote:
> 4.5

Can you please prove 4.5a a bit better? I can't understand the proof you gave. Sorry

cause_im_batman

#25 Oct 27, 2013, 10:27 am

@fmasoor

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

and

$$\cos C = \cos\frac{2C}{2} = 1 - \sin^2\frac{C}{2}$$

Also ,

$$A + B = \pi - C \Rightarrow \frac{A+B}{2} = \frac{\pi}{2} - \frac{C}{2}$$

realcastle

#26 Feb 2, 2014, 7:47 pm

Click to reveal hidden text

Quick Reply

High School Math

distance from circumcengre to orthocentre

Reply

AndrewTom
#1 Oct 12, 2015, 3:27 am

Prove that $OH^2=9R^2+2r^2+8Rr-2s^2$.

AndrewTom
#2 Oct 13, 2015, 6:19 pm

Any ideas on this one?

Luis González
#3 Oct 14, 2015, 1:21 am • 1

Let O,H and G denote the circumcenter, orthocenter and centroid of $\triangle ABC,$ respectively. By Leibniz theorem for the circumcenter $O,$ we obtain

$$OG^2=\tfrac{1}{3}(OA^2+OB^2+OC^2)-\tfrac{1}{9}(a^2+b^2+c^2)=R^2-\tfrac{1}{9}(a^2+b^2+c^2).$$

Since $OG=\tfrac{1}{3}OH \Longrightarrow OH^2=9R^2-(a^2+b^2+c^2).$

Now using the well-known identity $a^2+b^2+c^2=2s^2-2r^2-8Rr,$ we get $OH^2=9R^2+2r^2+8Rr-2s^2,$ as desired.

AndrewTom
#4 Oct 14, 2015, 1:39 am

Thanks Luis.

I have not come across the identity $a^2+b^2+c^2=2s^2-2r^2-8Rr.$

How is it proved?

Luis González
#5 Oct 14, 2015, 2:55 am • 1

Check out the thread Proofs of Useful Identities and Inequalities in Geometry for a proof of that identity and many others.

Quick Reply

High School Math

distance from incentre to orthocentre

Reply

AndrewTom
#1 Oct 12, 2015, 3:28 am

Prove that $IH^2 = 4R^2 + 3r^2 + 4Rr - s^2$.

AndrewTom
#2 Oct 13, 2015, 6:19 pm

Any ideas on this one?

Luis González
#3 Oct 14, 2015, 2:16 am • 1

Let O, H, G, N and I denote the circumcenter, orthocenter, centroid, 9-point center and incenter of $\triangle ABC$, respectively. Noting that IN is the I-median of $\triangle IOH$ and using $IN = \frac{1}{2}R - r$ (Feuerbach theorem) we get the distance IH straightforwardly, however we are not using Feuerbach theorem in order to avoid circular reasoning.

By Leibniz theorem for I, we obtain $IG^2 = \frac{1}{3}(IA^2 + IB^2 + IC^2) - \frac{1}{9}(a^2 + b^2 + c^2)$.

Using the well-known formulae

$$IA = \sqrt{\frac{bc(s-a)}{s}}\ ,\ IB = \sqrt{\frac{ca(s-b)}{s}}\ ,\ IC = \sqrt{\frac{ab(s-c)}{s}},$$

expanding, simplifying terms and using $a^2 + b^2 + c^2 = 2s^2 - 2r^2 - 8Rr$, the expression becomes $IG^2 = \frac{1}{9}(s^2 + 5r^2 - 16Rr)$. Now we use Stewart theorem for the cevian IG of $\triangle IOH$ dividing $\overline{OH}$ in the ratio $GO : GH = -1 : 2$, bearing in mind the relations $OI^2 = R^2 - 2Rr$ (Euler theorem) and $OH^2 = 9R^2 + 2r^2 + 8Rr - 2s^2$ previously found in another topic.

$$IG^2 = \tfrac{1}{3}IH^2 + \tfrac{2}{3}IO^2 - \tfrac{2}{9}OH^2 \Longrightarrow IH^2 = 3 \cdot IG^2 - 2 \cdot IO^2 + \tfrac{2}{3}OH^2 \Longrightarrow$$

$$IH^2 = \tfrac{1}{3}(s^2 + 5r^2 - 16Rr) - 2(R^2 - 2Rr) + \tfrac{2}{3}(9R^2 + 2r^2 + 8Rr - 2s^2) = 4R^2 + 3r^2 + 4Rr - s^2.$$

Quick Reply

High School Math

nine point centre and incentres

Reply

AndrewTom

#1 Sep 22, 2015, 1:11 am

For $\triangle ABC$, N is the nine-point centre, I is the incentre, I_1, I_2, I_3 are the excentres opposite A, B, C respectively and R is the circumradius. Prove that

$$NI + NI_1 + NI_2 + NI_3 = 6R.$$

Luis González

#2 Sep 22, 2015, 1:14 am • 1

Discussed before at http://www.artofproblemsolving.com/community/c6h366606.

AndrewTom

#3 Sep 22, 2015, 1:32 pm

Thanks Luis.

Is it possible to do this without appealing to Feuerbach's theorem?

AndrewTom

#4 Sep 30, 2015, 5:08 am

Any ideas on doing this without using Feuerbach's theorem?

Luis González

#5 Oct 12, 2015, 2:31 am • 1

Indeed it can be solved without resorting to Feuerbach theorem. Essentially we have to prove that $NI = \frac{R}{2} - r$, $NI_1 = \frac{R}{2} + r_a$ and the cyclic expressions. We will only prove the relation $NI = \frac{R}{2} - r$ since the others can be proved analogously.

Let I, O, H be the incenter, circumcenter and orthocenter of $\triangle ABC$. Midpoint N of $\overline{OH}$ is 9-point center of $\triangle ABC \Longrightarrow IN$ is I-median of $\triangle IOH \Longrightarrow IN^2 = \frac{1}{2}(IH^2 + IO^2) - \frac{1}{4}OH^2$. Now using the well-known identities $OH^2 = 9R^2 + 2r^2 + 8Rr - 2s^2, IH^2 = 4R^2 + 3r^2 + 4Rr - s^2$ and $IO^2 = R^2 - 2Rr$, we get

$$IN^2 = \tfrac{1}{2}[(4R^2 + 3r^2 + 4Rr - s^2) + (R^2 - 2Rr)] - \tfrac{1}{4}[9R^2 + 2r^2 + 8Rr - 2s^2] =$$

$$= \tfrac{5}{2}R^2 + \tfrac{3}{2}r^2 + Rr - \tfrac{1}{2}s^2 - \tfrac{9}{4}R^2 - \tfrac{1}{2}r^2 - 2Rr + \tfrac{1}{2}s^2 = \tfrac{1}{4}R^2 + r^2 - Rr = \left(\tfrac{R}{2} - r\right)^2.$$

Quick Reply

High School Olympiads

Geometry

Locked

Blacklord

#1 Oct 13, 2015, 7:49 pm

Let F be a point in the triangle ABC such that AFB=AFC=120
Let AF intersect BC at D.La is the reflection of AD over BC
Lb and Lc defines similarly
Prove that La,Lb and Lc concur.

Luis González

#2 Oct 13, 2015, 9:32 pm

Discussed before at http://www.artofproblemsolving.com/community/c6h304719.

High School Olympiads

Torixeli

Reply

Elementaryyy
#1 Oct 7, 2009, 12:22 pm

Let ABC. T is Torixeli point. AT, BT, CT cut BC, CA, AB at A_1, B_1, C_1
T_1, T_2, T_3 is the symmetric poit of T by BC, CA, AB. Prove that A_1T_1, B_1T_2, C_1T_3 are concurrent

livetolove212
#2 Oct 7, 2009, 3:44 pm • 1

Denote T' the isogonal conjugate point of Torixeli point T, T'_a the reflection of T' across BC. We will show that $T'_a \in AT$.
For every point X which lies on AT. We have $\frac{S_{AXB}}{S_{AXC}} = \frac{BA.AT.\sin\angle BAT}{CA.AT.\sin\angle CAT} = \frac{BT}{CT}$
So to prove $T'_a \in AT$ we will prove $\frac{S_{ABT'_a}}{S_{ACT'_a}} = \frac{BT}{CT}$
Or $\frac{AB.BT'_a.\sin\angle ABT'_a}{AC.CT'.\sin\angle ACT'_a} = \frac{BT}{CT}(1)$
Put $\angle ABT = y, \angle ACT = z$ then $\frac{\sin z}{\sin y} = \frac{AC}{AB} = \frac{\sin C}{\sin B}(2)$
$(1) \Leftrightarrow \frac{\sin C}{\sin B}.\frac{\sin z}{\sin y}.\frac{\sin(B+y)}{\sin(C+z)} = \frac{\sin(C-z)}{\sin(B-y)}$
$\Leftrightarrow \sin^2 C.\sin(B+y).\sin(B-y) = \sin^2 B.\sin(C+z).\sin(C-z)$
$\Leftrightarrow \sin^2 C.(\cos 2B - \cos 2y) = \sin^2 B(\cos 2C - \cos 2z)$
$\Leftrightarrow \sin^2 C.(1 - 2\sin^2 B - 1 + 2\sin^2 y) = \sin^2 B(1 - 2\sin^2 C - 1 + 2\sin^2 z)$
$\Leftrightarrow \sin^2 C.\sin^2 y = \sin^2 B.\sin^2 z$ (It's true from (2))
Therefore T' lies on T_1T. Similarly we are done.

Attachments:
picture36.pdf (5kb)

28121941
#3 Oct 8, 2009, 2:48 am

I think it should be written Torricelli

Luis González
#4 Oct 10, 2009, 7:29 pm • 2

> *Quote:*
> $\triangle ABC$ is scalene and F, F' denote its 1st Fermat point and 1st isodynamic point. F_a, F_b, F_c are the reflections of F across BC, CA, AB. Prove that the cevian triangle $\triangle A''B''C''$ of F WRT $\triangle ABC$ is perspective with $\triangle F_aF_bF_c$ through F'.

Since pedal triangle of F' with respect to $\triangle ABC$ is equilateral, the reflections A', B', C' of F' about BC, CA, AB form another equilateral triangle centrally similar to its pedal triangle $\Longrightarrow$ Perpendiculars from A, B, C to the sidelines of the pedal triangle of F' concur then at the isogonal conjugate of F' WRT $\triangle ABC$. Therefore, $AX \perp B'C', BY \perp A'C'$ and $CZ \perp A'B'$ concur at F, where $X \in B'C', Y \in A'C'$ and $Z \in A'B'$.

Note that BC, CA, AB become perpendicular bisectors of $F'A', F'B', F'C'$, thus it follows that A, B, C are circumcenters of $\triangle F'B'C'$, $\triangle F'A'C'$ and $\triangle F'A'B' \Longrightarrow F$ is the circumcenter of $\triangle A'B'C' \Longrightarrow A' \in AF, B' \in BF$ and $C' \in CF$. Therefore, the reflections F_aF', F_bF', F_cF' of the lines AF, BF, CF across BC, CA, AB pass through $A'', B'', C'' \Longrightarrow$

$\triangle F_aF_bF_c$ and $\triangle A''B''C''$ are perspective through F'.

r1234

#5 May 6, 2012, 4:11 pm

> *Luis González wrote:*
> $\triangle ABC$ is scalene and F, F' denote its 1st Fermat point and 1st isodynamic point. F_a, F_b, F_c are the reflections of F across $BC, CA, AB.$ Prove that the cevian triangle $\triangle A''B''C''$ of F WRT $\triangle ABC$ is perspective with $\triangle F_aF_bF_c$ through $F'.$

Clearly F_a is the isogonal conjugate of A wrt $\triangle F'BC$. Now AA_1, AF' are isogonals.($A_1 = AF' \cap BC$). We know that F' lies on A-Apollonius circle. Hence $F'A'', F'A_1$ are isogonal, i.e $F'A'', F'A$ are isogonal $\Longrightarrow F', A'', F_a$ are collinear. Similarly F', B'', F_b are collinear and same thing holds for C. So done.

TelvCohl

#6 Oct 15, 2014, 10:33 am • 1

We can prove the stronger result:

Let P be a point and Q be the reflection of P with respect to BC ,
then AP, AQ are isogonal line of $\angle BAC$ iff P lie on A-Apollonius circle .

Proof:

Lemma :

Let P, P' be the isogonal conjugate of triangle ABC .
Let ℓ be the isogonal conjugate of AP WRT $\angle BPC$.
Let ℓ' be the isogonal conjugate of AP' WRT $\angle BP'C$.

Then ℓ, ℓ' are symmetry with respect to BC.

Proof of the lemma:

Let X be the intersection of BP' and CP .
Let Y lie on BC such that XA, XY are isogonal conjugate WRT $\angle BXC$.

Since CA, CY are isogonal conjugate WRT $\angle P'CP$,
so A, Y are isogonal conjugate of triangle CXP' .
i.e. $P'Y, P'A$ are isogonal conjugate WRT $\angle BP'C$.
Similarly, we can prove PY, PA are isogonal conjugate WRT $\angle BPC$.

Since $\angle PYB = \angle CYP'$,
so we get $PY, P'Y$ are symmetry with respect to BC .

Back to the main problem:

Let P' be the isogonal conjugate of P WRT to $\triangle ABC$.

If P lie on A-Apollonius circle

Since $PB : PC = \sin\angle PCB : \sin\angle PBC = \sin\angle P'CA : \sin\angle P'BA$,
so we get $\angle AP'B = \angle AP'C \Longrightarrow$ from lemma we get AP' pass through Q .

Similarly, we can prove the converse .

By the way,**Luis**'s method also work .

Quick Reply

High School Math

hyperbola and mid-point

Reply

AndrewTom

#1 Oct 12, 2015, 2:01 pm

A straight line intersects a hyperbola at the points K and L and the hyperbola's asymptotes at R and T. Prove that the mid-point of KL is also the mid-point of RT.

Luis González

#2 Oct 13, 2015, 11:13 am • 2

Let O denote the center of the hyperbola $\mathcal{H}$. B,C are the projections of L on OT, OR and D,E are the projections of K on OT,OR. It's known that all points on $\mathcal{H}$ verify that the product of their distances to the asymptotes is a constant $k^2 \Longrightarrow$ $LB \cdot LC = KD \cdot KE = k^2 \Longrightarrow [LBC] = [KDE] = \tfrac{1}{2}k^2 \cdot \sin \widehat{ROT}$. Thus by Euler theorem, denoting ϱ the radius of $\odot(ORT)$, we obtain

$$\frac{[LBC]}{[ORT]} = \frac{LT \cdot LR}{4\varrho^2}, \ \frac{[KDE]}{[ORT]} = \frac{KT \cdot KR}{4\varrho^2} \Longrightarrow LT \cdot LR = KT \cdot KR,$$

which clearly means that L and K are equidistant from the midpoint of RT, i.e. KL and RT have the same midpoint.

Projective solution

Quick Reply

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High School Olympiads

Coaxal circles

Reply

Source: me

yetti

#1 Aug 28, 2006, 2:43 pm

Let $\triangle ABC, \triangle A'B'C'$ be 2 triangles with the same circumcircle (O) and incircle (I). Let the lines AA', BB', CC' meet at $X \equiv BB' \cap CC',\ Y \equiv CC' \cap AA',\ Z \equiv AA' \cap BB'$. Let $(O'),\ (I')$ be the circumcircle and incircle of $\triangle XYZ$. Show that the circles $(O),\ (I),\ (O'),\ (I')$ are coaxal.

treegoner

#2 Aug 29, 2006, 8:32 am

What does coaxal here mean? You mean the centers are collinear?

Amir.S

#3 Aug 29, 2006, 8:45 am

it means that they have the same radicax axis.and with also collinear center

darij grinberg

#4 Aug 29, 2006, 2:53 pm

> yetti wrote:
> Let $\triangle ABC, \triangle A'B'C'$ be 2 triangles with the same circumcircle (O) and incircle (I). Let the lines AA', BB', CC' meet at $X \equiv BB' \cap CC',\ Y \equiv CC' \cap AA',\ Z \equiv AA' \cap BB'$. Let $(O'),\ (I')$ be the circumcircle and incircle of $\triangle XYZ$. Show that the circles $(O),\ (I),\ (O'),\ (I')$ are coaxal.

Peter Scholze doesn't think this is correct (and neither do I):

If we take A = A', B = B', C = C', then the lines AA', BB', CC' should be the tangents to the circumcircle of triangle ABC at the points A, B, C, so the triangle XYZ is the tangential triangle of triangle ABC. Then, you claim that the incircle and the circumcircle of triangle ABC and the circumcircle of the tangential triangle are coaxal, what is incorrect.

Darij

Amir.S

#5 Aug 29, 2006, 9:02 pm

yes,problem is wrong.

but,I proved 3 circles (I') , (I) , (O) are coaxal.

yetti

#6 Aug 29, 2006, 11:28 pm

I agree that the problem is not correct, only the circles (I), (O), (I') are coaxal, just as Amir.S says. Sorry.

Luis González

#7 Oct 13, 2015, 6:46 am

Poncelet theorem: Suppose circles ω_i belong to the same pencil and A_0 is a point on ω_0. The tangent to ω_1 from A_0 intersects ω_0 again at A_1, the tangent to ω_2 from A_1 intersects ω_0 again at A_2, etc, the tangent to ω_{i+1} from A_i intersects ω_0 again at A_{i+1}. Suppose that for some n, the points A_n and A_0 coincide. Then for any point B_0 on ω_0, the similarly constructed point B_n coincides with B_0.

This is a verbatim copy of Theorem 2.13 from the book Geometry of conics by A.V. Akopyan and A. A. Zaslavsky. The proposition and its proof can be found in pages 61 and 62.

In the proposed problem, by Poncelet theorem for $AA'B'B$, it follows that AA', BB' touch a circle ω of the pencil $(O),(I)$, thus ω is unique. Similarly by Poncelet theorem for $AA'C'C$, it follows that AA', CC' touch the same circle ω of the pencil $(O),(I) \Longrightarrow (I') \equiv \omega \Longrightarrow (O),(I),(I')$ are coaxal.

Quick Reply

High School Olympiads

cyclic quads

Reply

Merlinaeus
#1 Oct 13, 2015, 1:25 am

In cyclic quadrilateral $ABCD$ rays AB and DC intersect at point E, while segments AC and BD intersect at F. Point P is on ray EF such that angles BPE and CPE are congruent. Prove that $APCE$ and $BPDE$ are both cyclic quads

Grigory
#2 Oct 13, 2015, 1:50 am

$APCE$ and $BPDE$ are both cyclic quads ?? But E is on AC and BD

Merlinaeus
#3 Oct 13, 2015, 2:04 am

> Grigory wrote:
> $APCE$ and $BPDE$ are both cyclic quads ?? But E is on AC and BD

no, F is on AC and BD

Luis González
#4 Oct 13, 2015, 3:24 am

Let $G \equiv BC \cap AD$ and $M \equiv EF \cap BC.$ Since $P(B, C, M, G) = -1$ and PE bisects $\angle BPC,$ then it follows that $PG \perp PE \Longrightarrow GP$ is the polar of E WRT $\odot(ABCD) \Longrightarrow P$ and F are inverse points under the inversion with center E and power $EA \cdot EB = EC \cdot ED.$ As this inversion swaps A, B and $C, D,$ then $\odot(APCE)$ is the inverse of $BD.$ Similarly $BPDE$ is cyclic.

Merlinaeus
#5 Oct 13, 2015, 3:29 am

> Luis González wrote:
> Let $G \equiv BC \cap AD$ and $M \equiv EF \cap BC.$ Since $P(B, C, M, G) = -1$ and PE bisects $\angle BPC,$ then it follows that $PG \perp PE \Longrightarrow GP$ is the polar of E WRT $\odot(ABCD) \Longrightarrow P$ and F are inverse points under the inversion with center E and power $EA \cdot EB = EC \cdot ED.$ As this inversion swaps A, B and $C, D,$ then $\odot(APCE)$ is the inverse of $BD.$ Similarly $BPDE$ is cyclic.

thank you!
And I am hoping to find an elementary way without using inversion and harmonic division.

Stefan4024
#6 Oct 15, 2015, 1:20 am

> Luis González wrote:
> Let $G \equiv BC \cap AD$ and $M \equiv EF \cap BC.$ Since $P(B, C, M, G) = -1$ and PE bisects $\angle BPC,$ then it follows

that $PG \perp PE \implies GP$ is the polar of E WRT $\odot(ABCD) \implies P$ and F are inverse points under the inversion with center E and power $EA \cdot EB = EC \cdot ED$. As this inversion swaps A, B and C, D, then $\odot(APCE)$ is the inverse of BD. Similarly $BPDE$ is cyclic.

Are you sure that GP is the polar of E wrt $\odot(ABCD)$? Because it's well-known fact tht GF is the polar of E wrt $\odot(ABCD)$. Also how did you find out the power of the inversion?

It's enough to prove that A, B, P, F are cyclic and then a simple angle chase solves the problem. Actually using what you already found ("P and F are inverse points under the inversion with center E and power $EA \cdot EB = EC \cdot ED$.") is enough, but I can't get where this comes from. So could you elaborate a little bit?

This post has been edited 1 time. Last edited by Stefan4024, Oct 15, 2015, 1:21 am

hayoola

#7 Oct 17, 2015, 1:11 am

Let the rays AD, BC intersect at poin M and let j is the intersection between EF and BC we know that $(MjBC) = -1$ and we know that angelsCPj, BPj are equal so we find that MP is perpendicular to Pj by the known lemma if MP is perpendicular to EF the line MP passes from the circumcenter if cyclic $ABCD$ and the intersection between MO, EF is onthe intersection beween the circumcircle of triangelsAFB, DFC so $DFPC$ is cyclic so angels JPC, FDC are equal

MouN

#8 Oct 17, 2015, 4:10 am

Let the circle through C, D and F intersect the line EF at $P' \neq F$. Then by the power of E we have $EF \cdot EP' = ED \cdot EC = EA \cdot EB$ so $ABP'F$ is cyclic. Hence $\angle CP'F = 180^\circ - \angle BDC = 180^\circ - \angle BAC = \angle BP'F$ so $P' = P$. Now from the previous equality we have $\angle CPE = 180^\circ - \angle BAC = \angle CAE$ so $CPAE$ is cyclic. Similarly $BPDE$ is also cyclic.

Stefan4024

#9 Oct 17, 2015, 3:07 pm

Here's another solution:

Let $G \equiv BC \cap AD$. It's well-known that EF is the polar of G wrt $\odot(ABCD)$ and the other users proved that $PE \perp PG$. Now it's easy to prove that GP passes through O. If T_1 and T_2 are the two tangency points from G to $\odot(ABCD)$ then PG bisects $\angle T_1GT_2$, since it's a isoscelec triangle. But also OG bisects it since OT_1GT_2 is a deltoid. From this we have that P and G are the inverse points under the inversion wrt $\odot(ABCD)$.

Now from the power of point G wrt $\odot(ABCD)$ we have: $GC \cdot GB = OG^2 - R^2 = OG^2 - OT_1^2 = GT_1^2 = GO \cdot GP$. From this we have that $POCB$ is cyclic. Now a simple angle chase solves the problem. $\angle EPC = \frac{1}{2}\angle BPC = \frac{1}{2}\angle BOC = \angle BAC = \angle EAC$. Hence $APCE$ is cyclic. Similarly we prove that $BPDE$ is cyclic too.

Quick Reply

High School Olympiads

Problem about circumcenters.

Reply

phuocdinh_vn99

#1 Oct 12, 2015, 8:38 pm

Given triangle ABC, altitude AH. Let P be an arbitrary point on nine-point circle of triangle ABC. Let M, N be circumcenters of triangles PBH, PCH and Q be center of nine-point circle of triangle ABC. Prove that Q is midpoint of MN

Luis González

#2 Oct 12, 2015, 9:44 pm

The 9-point circle (Q) of $\triangle ABC$ is coaxal with $(M) \equiv \odot(PBH), (N) \equiv \odot(PCH)$ (as it passes through P, H) and goes through the midpoint of $BC \Longrightarrow (Q)$ is the midcircle of $(M), (N) \Longrightarrow Q$ is the midpoint of $\overline{MN}$.

jayme

#3 Oct 13, 2015, 7:52 pm

Dear Mathlinkers,
very elegant proof Dear Luis...

For more

http://jl.ayme.pagesperso-orange.fr/Docs/midcircle%20theorem.pdf

Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Locus, Isogonal conjugate, Four fixed angles

Reply

Source: Own

TelvCohl

#1 Oct 10, 2015, 1:08 am • 1

Given a fixed point A and a fixed line ℓ. Let $\theta, \alpha, \beta, \gamma$ be four fixed angles s.t. $\alpha + \beta + \gamma = 360^{\circ}$. Let B, C be the points on ℓ s.t. $\measuredangle BAC = \theta$. Let P be a point s.t. $\measuredangle BPC = \alpha, \measuredangle CPA = \beta, \measuredangle APB = \gamma$ and let Q be the isogonal conjugate of P WRT $\triangle ABC$. Find the locus of P, Q, respectively when B and C varies on ℓ.

Luis González

#2 Oct 10, 2015, 4:51 am • 2

Invert the figure with center A and arbitrary power, denoting inverse points with primes. The inverse of ℓ is then the fixed circle $(L) \equiv \odot(AB'C')$. Since $\angle AB'P' = \angle APB = \gamma$ and $\angle AC'P' = \angle CPA = \beta$ are constant, it follows that $P'B', P'C'$ cut (L) again at fixed points X, Y. Thus $\angle XP'Y \equiv \angle B'P'C' = 360^{\circ} - \beta - \gamma - \theta = \text{const} \Longrightarrow P'$ moves on a fixed circle Ω through $X, Y \Longrightarrow P$ moves on the inverse of Ω, i.e. a fixed circle.

By conformity, the angle between $B'C'$ and $B'Q'$ equals the angle between their inverses $(O) \equiv \odot(ABC)$ and $(O_c) \equiv \odot(ABQ) \Longrightarrow$
$\angle Q'B'C' = \angle OAO_c = (90^{\circ} - \angle ACB) + (\angle AQB - 90^{\circ}) = \angle AQB - \angle ACB = 180^{\circ} - \gamma$ and similarly we have $\angle Q'C'B' = 180^{\circ} - \beta$. Since $\angle B'AC' = \theta$ is constant, then $B'C'$ is constant, therefore all $\triangle Q'B'C'$ are directly congruent $\Longrightarrow$ all $\triangle Q'B'C' \cup L$ are directly congruent $\Longrightarrow LQ' = \text{const} \Longrightarrow Q'$ moves on a fixed circle ω with center L $\Longrightarrow Q$ moves on the inverse of ω, i.e. a fixed circle.

Quick Reply

High School Olympiads

Bicentric Q

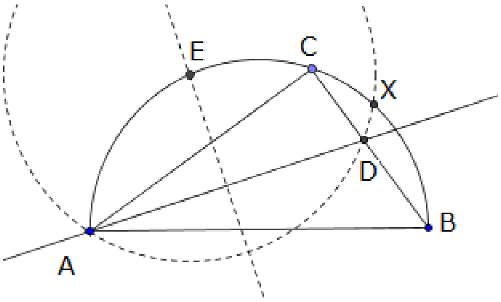
Reply

ricarlos

#1 Oct 9, 2015, 12:03 am

Let Ω be a semicircle with diameter AB. Let C and D be points on Ω and BC such that AD is bisector of $\angle BAC$. The perpendicular bisector of AD intersect Ω at E. A circle with center E and radius EA intersect Ω at $X \neq A$. Prove that $ABXC$ is a bicentric quadrilateral.

Attachments:



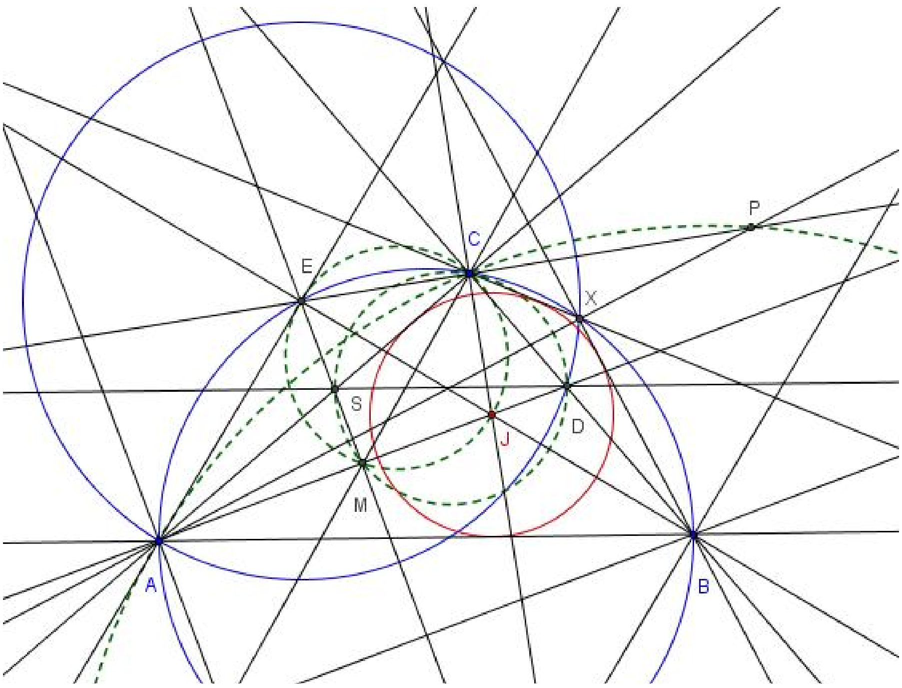
Luis González

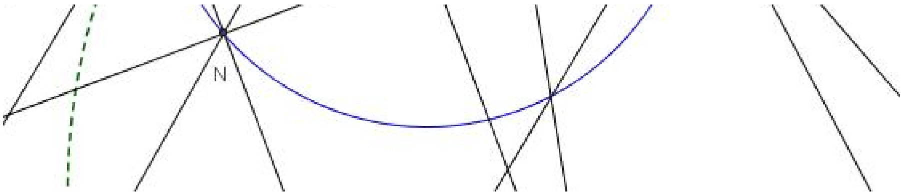
#2 Oct 9, 2015, 3:16 am

Let M be the midpoint of AD and let N be the midpoint of the arc BAC of Ω. EM cuts CA at S, EB cuts AD at J and AX cuts EC at P.

Since $CDMS$ is cyclic on account of the right angles at C, M, we have $\angle MDS = \angle MCA = \angle MAC = \angle MAB \Longrightarrow DS \parallel AB$. Together with $(MS \parallel NA) \perp AD$ and $MD \parallel NB$ (because of $\angle NBA = \angle MCA = \angle MAB$), it follows that $\triangle MSD$ and $\triangle NAB$ are homothetic with center $A \Longrightarrow C, M, N$ are collinear $\Longrightarrow \angle CMJ = \angle CNB = \angle CEJ$ $\Longrightarrow CEMJ$ is cyclic $\Longrightarrow \angle ECJ = \angle EMJ = 90^\circ$, i.e. $JC \perp PE$. Since $\angle ECA = \angle EAX$ (E is midpoint of the arc ACX), then AE is tangent of $\odot(PAC) \Longrightarrow EA^2 = EC \cdot EP \Longrightarrow JC$ is the polar of P WRT $\odot(E, EA) \Longrightarrow$ $C(J, A, X, P) = -1$. Together with $CJ \perp PE$, we deduce that CJ bisects $\angle ACX$. As AJ and BJ bisect $\angle BAC$ and $\angle ABX$, then J is incenter of $ABXC \Longrightarrow ABXC$ is bicentric.

Attachments:





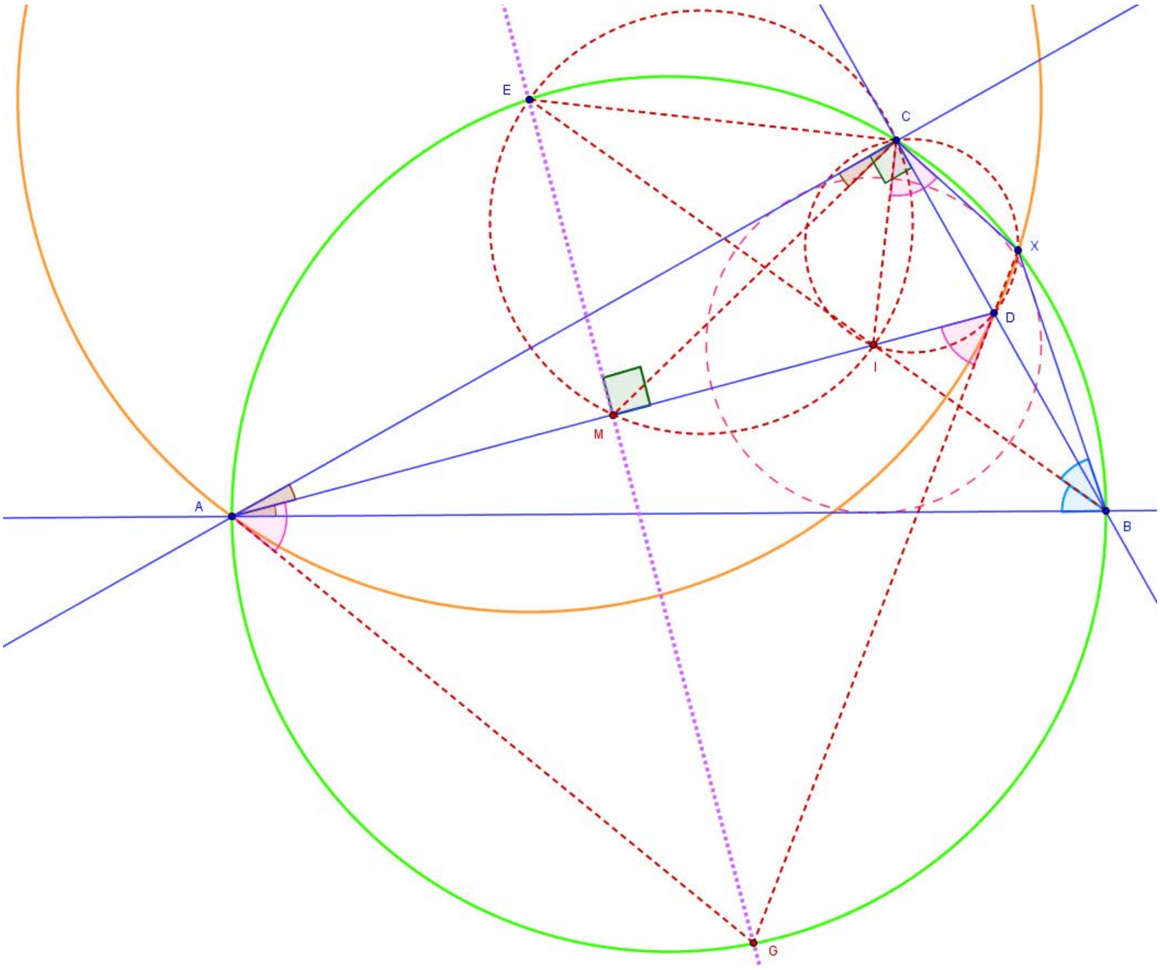
TelvCohl

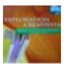
#3 Oct 9, 2015, 5:44 pm

Let M be the midpoint of AD and $G \equiv EM \cap \odot(ABC)$. Let I be the intersection of AD and the bisector of $\angle XBA$. From $\measuredangle AXD = \frac{1}{2}\measuredangle AED = \measuredangle AEG = \measuredangle AXG$ we get $D \in XG$, so $\measuredangle XBI = \frac{1}{2}\measuredangle XBA = \frac{1}{2}\measuredangle DGA = \measuredangle XBE \Longrightarrow B, I, E$ are collinear, hence notice $MA = MC = MD$ ($\because \angle ACB = 90^\circ$) we get $\measuredangle IEC = \measuredangle BAC = 2\measuredangle DAC = \measuredangle IMC \Longrightarrow C, E, I, M$ are concyclic.

On the other hand, from $\measuredangle CID = \measuredangle CEM = \measuredangle CXD$ we get C, D, I, X are concyclic, so $\measuredangle ICX = \measuredangle ADG = \measuredangle GAD = \measuredangle GAB + \measuredangle BAD = \measuredangle MEI + \measuredangle MAC = \measuredangle MCI + \measuredangle ACM = \measuredangle ACI \Longrightarrow I$ lies on the bisector of $\angle ACX \Longrightarrow I$ is the incenter of $ABXC$, hence we conclude that $ABXC$ is a bicentric quadrilateral.

Attachments:





livetolove212

#4 Dec 5, 2015, 8:44 pm

Generalization: Given triangle ABC inscribed in (O). Line through A and perpendicular to AB intersects the B-bisector at D. The perpendicular bisector of BD intersects arc AB of (O) at E. (E, EB) meets (O) again at X. Prove that $ABCX$ is a bicentric quadrilateral.

Quick Reply

High School Olympiads

A relation between Feuerbach triangle and Medial triangle

Locked

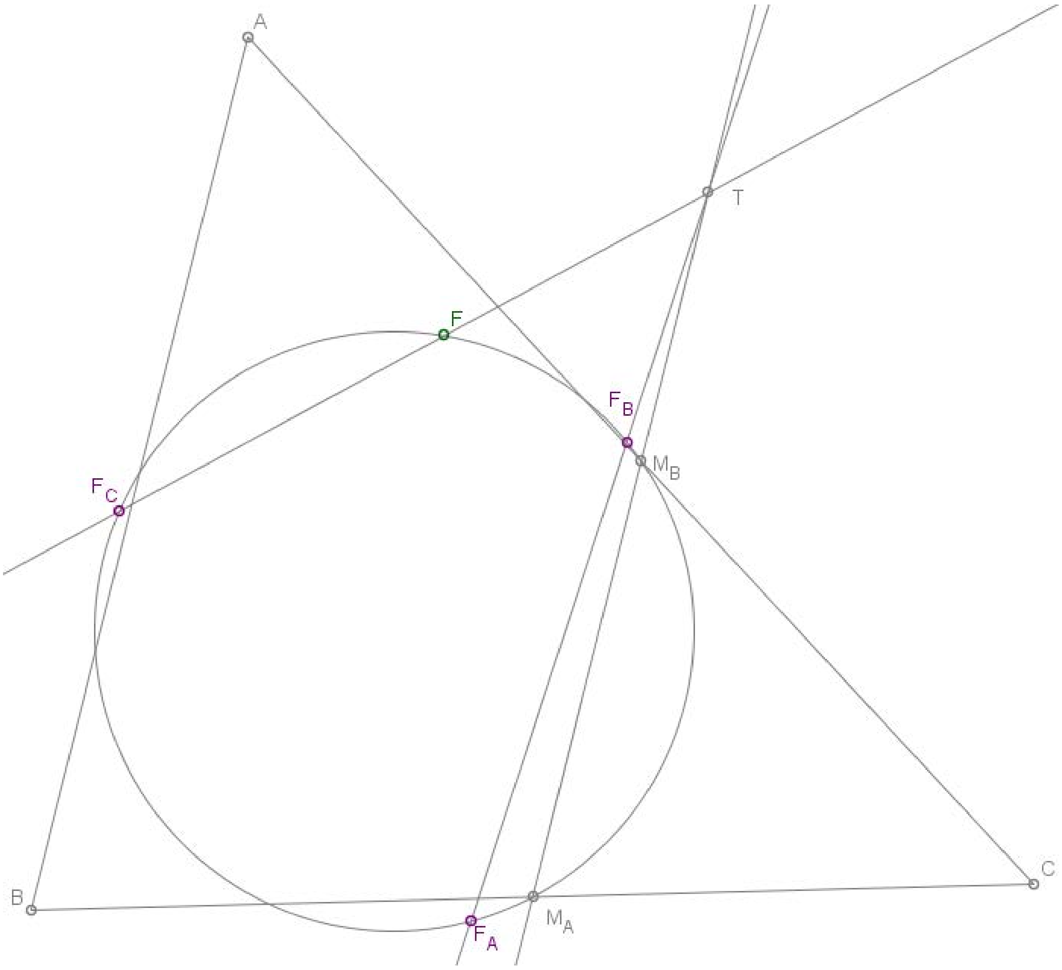
Source: From geometry in figures by Akopyan.

Cezar

#1 Oct 8, 2015, 1:33 am

Let F_A, F_B, F_C be the Feuerbach triangle of $\triangle ABC$, and F it's Feuerbach point. Let M_A, M_B be the midpoints of BC, AC.Prove that FF_C, F_BF_A, M_AM_B are concurrent.

Attachments:



Luis González

#2 Oct 8, 2015, 5:01 am

See http://www.artofproblemsolving.com/community/c6h399047 (posts #5, #6 and #11).

High School Olympiads

Nine point circle is tangent to incircle and three excircles

Reply

Source: China TST 2011 Day 1

yunxiu

#1 Mar 28, 2011, 1:35 pm

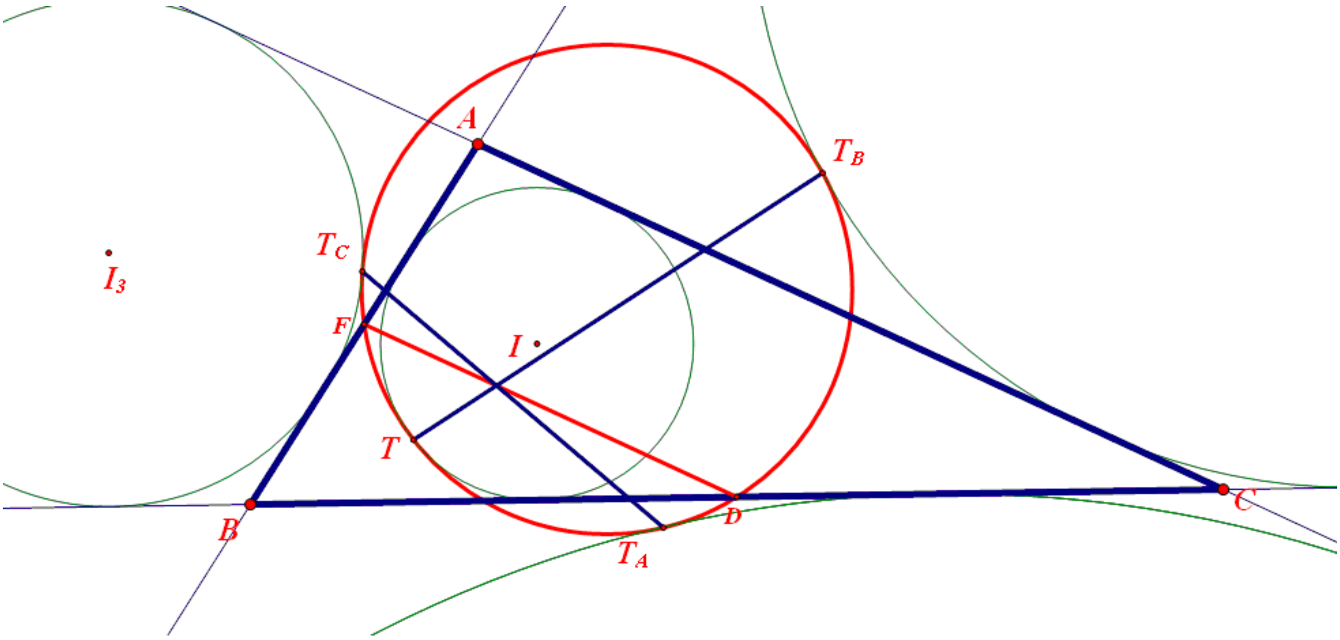
In $\triangle ABC$ we have $BC > CA > AB$. The nine point circle is tangent to the incircle, A-excircle, B-excircle and C-excircle at the points T, T_A, T_B, T_C respectively. Prove that the segments TT_B and lines T_AT_C intersect each other.

yunxiu

#2 Mar 29, 2011, 6:13 am

□□(Ye):If D is the midpoint of BC, F is the midpoint of AB, than $TT_B \cap T_AT_C$ is on the DF.

Attachments:



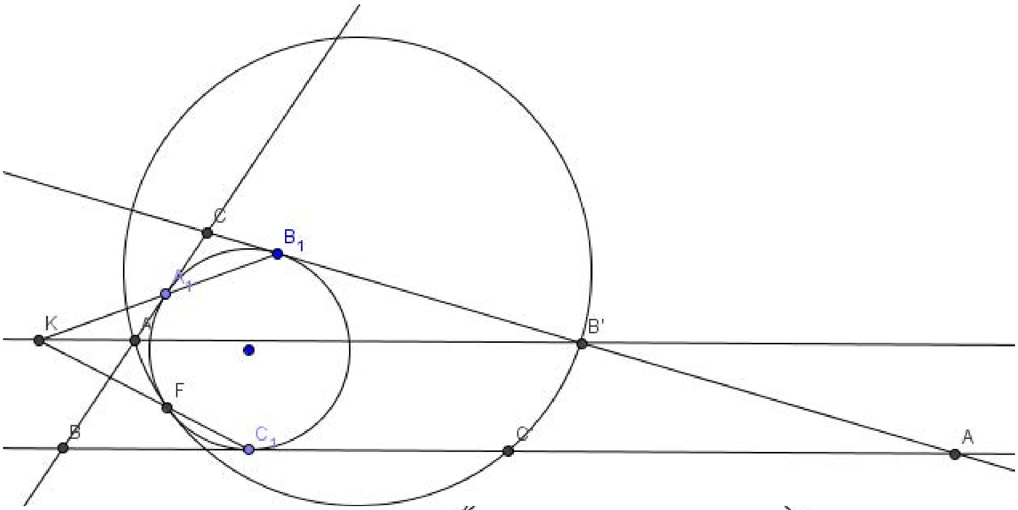
skytin

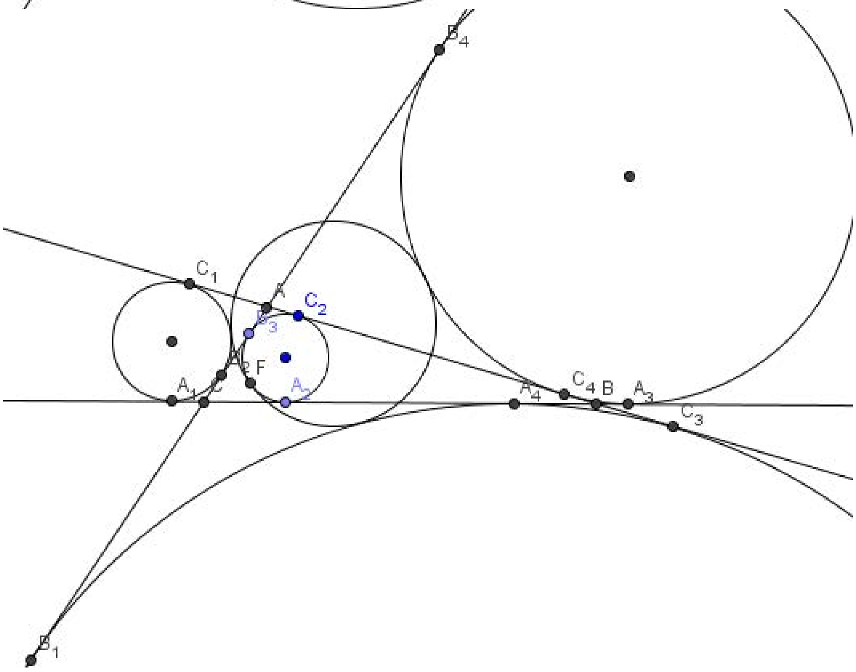
#3 Mar 31, 2011, 8:39 pm

Not very good solutuion :
Lemma 1 :
given quadrangle ABCD and line l , let l intersect AB , BC , CA , BD , AD , CD at points X_1 , X_2 , X_3 , X_4 , X_5 , X_6 , then for every point P circles (PX_1X_6) , (PX_2X_5) , (PX_3X_4) intersects at another point P'
and for points P_1 , P_2 , P_3 circles (P_1X_1X_6) , (P_2X_2X_5) , (P_3X_3X_4) have Radical center R and R is intersection of l and PP'
Lemma 2 :
let circle w tangent to sides AB , BC and CA at points C_1 , A_1 , B_1 , C' A' B' are midpoints of sides . w is tangent to (C'A'B') at point F , then lenes C_1F , A'B' , A_1B_1 intersects at one point
Descovered by F.Ivlev
Solution :
Let E is midpoint of CA
see picture
let A_1C_1 and A_2C_2 intersect DF at points X and X'
let C_3A_4 and A_3C_4 intersect DF at points Y and Y'
Use Lemma 2

So B_2T_B goes thru X , TB_3 goes thru X' , T_CB_4 goes thru Y' and B_1D goes thru Y
Use Reim's Theorem , so TX'XT_B and Y'YT_AT_C are cyclic
Let A_4C_3 intersect A_1C_1 and A_2C_2 at points P and Q
Let C_4A_3 intersect A_1C_1 and A_2C_2 at points P' and Q'
Well known that AQ'QQCPP' are cyclic and CA is diameter and Q'QPP' is rectangle , so Q'P and QP' are intersected at point E
And easy to see that D is on QP' and F is on Q'P
Use Lemma 1 for
quadrangle Q'QPP' and line DF , so Radical center of (TX'XT_B) , (Y'YT_AT_C) and Nine point circle is on line DF , let it is point R
So TT_b and T_AT_C intersects at point R on DF . Done

Attachments:





paul1703

#4 Mar 31, 2011, 9:56 pm

ca you prove the lemmas please? is the official solution on the internet?

Luis González

#5 Apr 2, 2011, 12:15 am • 1

Theorem: F is the Feuerbach point of $\triangle ABC$ and F_A, F_B, F_C are the three additional Feuerbach points against A, B, C. Then FF_A and F_BF_C intersect at the A-Schroeter point U_A of $\triangle ABC$ (intersection of the A-sidelines of the medial and orthic triangle).

Since F, F_A and the foot V_A of the A-internal bisector are collinear due to Monge & d'Alembert theorem, then it suffices to show that $F \in U_AV_A$. Using barycentric coordinates with respect to $\triangle ABC$, the equations of the A-sidelines of the medial and orthic triangle of ABC are then

$y+z-x=0$, $S_By+S_Cz-S_Ax=0 \Longrightarrow U_A \equiv (c^2-b^2 : c^2-a^2 : a^2-b^2)$

We verify that $U_A, V_A \equiv (0:b:c)$ and $F \equiv X_{11} \equiv [(b+c-a)(b-c)^2]$ are collinear

$$\begin{bmatrix} 0 & b & c \\ c^2-b^2 & c^2-a^2 & a^2-b^2 \\ (b+c-a)(b-c)^2 & (c+a-b)(c-a)^2 & (a+b-c)(a-b)^2 \end{bmatrix} = 0$$

The reasoning for F_B and F_C is analogous. Hence, we conclude that $U_A \equiv FF_A \cap F_BF_C$.

skytin

#6 Apr 2, 2011, 9:57 pm • 1

My solution to Luis' Theorem:
Lemma 2 :
let circle w tangent to sides AB , BC and CA at points C_1 , A_1 , B_1 , C' A' B' are midpoints of sides . w is tangent to (C'A'B') at point F , then lenes C_1F , A'B' , A_1B_1 intersects at one point
Descovered by F.Ivlev
Solution :
Let E is midpoint of CA
see picture
let A_1C_1 and A_2C_2 intersect DF at points X and X'
let C_3A_4 and A_3C_4 intersect DF at points Y and Y'
Use Lemma 2
So B_2T_B goes thru X , TB_3 goes thru X' , T_CB_4 goes thru Y' and B_1D goes thru Y
Use Reim's Theorem , so TX'XT_B and Y'YT_AT_C are cyclic
Let A_4C_3 intersect A_1C_1 and A_2C_2 at points P and Q
Let C_4A_3 intersect A_1C_1 and A_2C_2 at points P' and Q'

Let H_C and H_A are foots of C and A altitudes of triangle ABC
Let Line H_CH_A intersects segments C_4A_3 and A_4C_3 at points U and W
Easy to see that YY'UW is Isosceles trapezoid ,so YY'UW is cyclic
Well known that B_1B_4T_CT_A is cyclic , so after using Reim's Theorem (B_4B_1 || Y'Y) , so YY'T_CT_A is cyclic

Let C_5A_5 is second external tangent to (C_4B_4A_3) and (B_1A_4C_3)
Let prove that U is on T_CC_5 and W is on T_AA_5
Let CB and AB intersects C_5A_5 at points A* and C*
Let H_C*C* is Altitude of triangle BA*C*
Let lines CH_C and H_C*C* intersects at point F
Let S is foot of perpendicular from F on A*C*
Easy to see that EH_C = AE = EC , so angle H_CCE = EH_CC = SC*F = SH_CF (H_CFSC* is cyclic) , so H_C is on ES
Let M is midpoint of FI_A
Easy to see that M is on Perpendicular bissectors of segments A_4HC* and H_CC_3 and SA_5 , so M is center of (A_4H_CH_C*C_3) and centers of (H_CSA_5) and (A_4H_CH_C*C_3) are on Perpendicular bissector of segment SA_5
Let H_C' is second intersection point of (A_4H_CH_C*C_3) and (H_CSA_5) H_CH_C' is perpendicular to center line of this circles , so H_CH_C' is || to A*C*
Not hard to prove that H_CH_A is || to A*C* , so H_A is on H_CH_C'
EH_C = EH_A , so tangent to 9 point circle is || to H_CH_A || A*C* , so center of Homothety of circles (EH_CH_AT_A) and (B_1A_4A_5) is on EA_5 , so T_A is on EA_5
Angle AH_CE = EAH_C = BA*C* , so BH_CSA* is cyclic
Angle N_CT_AE + N_CDE = DH_CB = C*BA* = H_CSA* , so T_AH_CSA_5 is cyclic
So lines H_CH_AH_C' , T_AA_5 , A_4C_3 are Concurrent as Radical lines of circles (B_1A_4A_5) , (A_4H_CH_C*C_3) and (T_AH_CSA_5)
Easy to see that they concur at point W
Not hard to prove that T_AT_CC_5A_5 is cyclic , use Reim's Theorem , so T_AT_CUW is cyclic , so lines Y'Y , T_CT_A and UW will be Concurrent at Radical center of (YY'UW) , (YY'T_CT_A) , (T_AT_CUW)
Problem done

math154

#7 Apr 10, 2011, 10:21 am • 1

Here's a less geometric solution...

Let D, E, F be the intersections of the internal angle bisectors of $\triangle ABC$ with the sides. Then by d'Alembert's theorem, TDT_A, TET_B, TFT_C are lines (with points trivially in that order). Thus it suffices to show that segment TE intersects segment DF, i.e. $T \in \triangle BFD$ or $[TBF]+[TBD] < [BFD]$. It's well-known that

$$
\begin{aligned}
[IBC]:[ICA]:[IAB] &= a(1-\cos(B-C)) : b(1-\cos(C-A)) : c(1-\cos(A-B)) \\
&= a\sin^2(\beta-\gamma) : b\sin^2(\gamma-\alpha) : c\sin^2(\alpha-\beta) \\
&= (b+c-a)(b-c)^2 : (c+a-b)(c-a)^2 : (a+b-c)(a-b)^2,
\end{aligned}
$$

so the inequality is equivalent to

$$
\begin{aligned}
\frac{a}{a+b}\cdot\frac{c}{b+c} &> \frac{1}{\sum(b+c-a)(b-c)^2}\cdot\left(\frac{a}{a+b}(a+b-c)(a-b)^2+\frac{c}{b+c}(b+c-a)(b-c)^2\right) \\
ac\sum(b+c-a)(b-c)^2 &> a(b+c)(a+b-c)(a-b)^2+c(a+b)(b+c-a)(b-c)^2 \\
ac(c+a-b)(c-a)^2 &> ab(a+b-c)(a-b)^2+cb(b+c-a)(b-c)^2 \\
ac(c+a-b)((a-b)+(b-c))^2 &> ab(a+b-c)(a-b)^2+cb(b+c-a)(b-c)^2 \\
2ac(c+a-b)(a-b)(b-c) &> [ab(a+b-c)-ac(c+a-b)](a-b)^2+[cb(b+c-a)-ac(c+a-b)](b-c)^2 \\
2ac(c+a-b)(a-b)(b-c) &> a(a+b+c)(b-c)(a-b)^2+c(a+b+c)(b-a)(b-c)^2 \\
2ac(c+a-b) &> a(a+b+c)(a-b)+c(a+b+c)(c-b) \\
2ac(c+a-b) &> (a+b+c)(a^2-ab+c^2-cb+2ac)-2ac(a+b+c) \\
4ca(c+a) &> (a+b+c)(c+a)(c+a-b) \\
4ca &> (c+a)^2-b^2 \\
(a+b-c)(b+c-a) &> 0,
\end{aligned}
$$

where we have used the fact that $a>b>c$.

$\frac{d}{dt}\mathrm{Fail} > t$

dnkywin

#8 Apr 11, 2011, 11:42 pm • 1

My Solution:

Lemma 1: The nine-point center N lies in the region of the plane determined by rays IA and IC

Proof: It suffices to prove that N is in the half-plane determined by line IA containing C and that N is in the half-plane determined by line IC containing A.

To prove the first part, we let A be the origin and B be on the positive x-axis. We need to prove that $\overrightarrow{AI}\times\overrightarrow{AN}>0$. Note that ray AI can be expressed in the form $(k(1+\cos A), k\sin A)$, where $k>0$, and we get that the coordinates of N are $(R(\sin C+2\sin B\cos A)/2, R(\cos C+2\cos B\cos A)/2)$, which by product-to-sum is equal to $(R\sin C+R\sin(B-A)/2, R\cos(B-A)/2)$, so

$$
\begin{aligned}
\overrightarrow{AI}\times\overrightarrow{AN} &= kR/2[(1+\cos A)(\cos(B-A))-(\sin A)(2\sin C+\sin(B-A))] \\
&= kR/2[\cos(B-A)+\cos A\cos(B-A)-2\sin A\sin C-\sin A\sin(B-A)] \\
&= kR/2[\cos(B-A)+\cos B-(\cos(A-C)-\cos(A+C))] \\
&= kR/2(\cos(A-B)-\cos(A-C))>0
\end{aligned}
$$

Thus N lies in the half-plane determined by IA containing C. Similarly, N lying in the half-plane determined by IC containing A is equivalent to $\cos(C-B)-\cos(C-A)>0$, so we are done.
(end lemma)

Lemma 2: T_B is not in the portion of the plane determined by rays NT_A and NT_B.
Proof: If M is the midpoint of AH and N is the midpoint of BC, it is not hard to see that T_B and T_AT_C are on different sides of MN, which is a diameter of the nine-point circle.

Thus if we can prove that T is in the region of the plane subtended by rays NT_A and NT_C, then we are done, as then T and T_B lie on opposite sides of T_AT_C, so since they are concyclic, TT_B must intersect T_AT_C.

Since N is in the region of the plane determined by IA and IC, I must be in the region of the plane determined by rays NI_A and NI_C, where I_A and I_C are the A- and C- excenters, respectively. However, rays NI_A and NT_A point in the same direction and NI_C and NT_C point in the same direction. Thus I lies in the region of the plane determined by rays NT_A and NT_C. Since the incircle is smaller than the nine-point circle, T must be on ray NI, so T must be in the region of the plane determined by NT_A and NT_C.

yay

blackbelt14253

#9 Apr 14, 2011, 7:38 am • 2

Let N be the nine-point center, and let I, I_A, I_B, I_C be the incenter and the three excenters. Let D, E, F be the midpoints of the sides BC, CA, AB, respectively.

Since T, T_A, T_B, and T_C are concyclic, it suffices to show that T and T_B are on opposite sides of line T_CT_A.

Note that T is the intersection of ray $\overrightarrow{NI}$ with the nine point circle and that T_A is the intersection of ray $\overrightarrow{NI_A}$ with the nine-point circle (and similarly for the other excenters). It thus suffices to show that I lies inside non-reflex angle I_CNI_A, or that N lies inside non-reflex angle AIC.

To show that N and B lie on opposite sides of CI, let P be the point of intersection of the angle bisector of $\angle ACB$ and the perpendicular bisector of EF, the latter of which N must lie on. Drop perpendiculars from P to AC and BC, and let the feet of these perpendiculars be P_B and P_A, respectively. It is easy to see that P_A lies on the perpendicular bisector of EF. Thus P and C must lie on opposite sides of D on BC, since $AB < AC$. Since $CP_A = CP_B$ and $CD < CE$, P_B and C must lie on opposite sides of E on AC. Now note that $CP_A = CP_B$, and that $CD > CE$. Thus $P_AF < P_BE$. Noting also that $PP_A = PP_B$, we see that $PF < PE$, so this point P is too close to FD to be the nine-point center. Also note that this result implies that $\angle PDE > \angle PE$, and any other point on the perpendicular bisector of EF on the "wrong side" of CI will yield a greater $\angle PDE$ and a smaller $\angle PED$. Thus it is also impossible for any other point on the "wrong side" of CI on the perpendicular bisector of EF to be N. Thus N must lie on the claimed side of CI.

To show that N is on the claimed side of AI, we can repeat this process with the angle bisector of $\angle BAC$ and the perpendicular bisector of DE.

skytin

#10 Apr 23, 2011, 5:30 pm

Solution for Lemma 2
Today i found this Post :
http://nguyenvanlinh.files.wordpress.com/2010/11/fontene-theorem-and-some-corollaries.pdf
By Linh Nguyen Van
First problem is Generalisation of this Lemma
My solution to Fontene theorem 1 :
Let lines C_1B_1 and C_2B_2 intersects at point Q
Let (A_2B_2C_2) intersect A_2Q at points A_2 and X
Let A_2P intersect C_1B_1 at point A_3
Let (A_3C_2B_2) intersect C_1B_1 at points A_3A_4
Easy to see that QX*A_2Q = QA_3*QA_4 = QC_2*QB_2 , so A_2XA_3A_4 is cyclic , so angle A_2XA_4 = 90
Let line QP" intersects (PC_2A) at points Y and P"
Use Reim's Theorem , so C_2C_1QY are cyclic
Use Miquel Theorem , so Y is on (B_1C_1A)
Let make Simmetry of point Y wrt ine C_1B_1
Angle QYA_4 = 90 (QA_3*QA_4 = QB_2*QC_2 = QY*QP"A_4A_3YP" , so is cyclic) , so X -> Y and (B_1C_1A) -> (A_1C_1B_1) , so X is on 9point circle too . done

TelvCohl

#11 Oct 15, 2014, 9:26 pm • 2

> Luis González wrote:
>
> **Theorem:** F is the Feuerbach point of $\triangle ABC$ and F_A, F_B, F_C are the three additional Feuerbach points against A, B, C. Then FF_A and F_BF_C intersect at the A-Schroeter point U_A of $\triangle ABC$ (intersection of the A-sidelines of the medial and orthic triangle).

My solution to Luis's Theorem :

Lemma :

Let (O) be the circle tangent to $(O_1), (O_2)$ at A, B .
Let two common tangent (both external or internal) cut (O) at C, D and E, F .
Then AB, CE, DF are concurrent .

Proof of the lemma :

From 3 circles with common tangency point

we get there exist a circle (T) tangent to CE, DF and tangent to (O) at A .
Similarly, there exist a circle (S) tangent to CE, DF and tangent to (O) at B
By D'Alembert theorem we get AB, CE, DF are concurrent at the similarity center of $(T), (S)$.

Back to the main proof :

Let I, I_A, I_B, I_C be the incenter and the three excenters and N be the nine-point center .

Apply the lemma to $(N), (I), (I_A)$ we get FF_A pass through A-Schroeter point .
Apply the lemma to $(N), (I_B), (I_C)$ we get F_BF_C pass through A-Schroeter point .

Q.E.D

This post has been edited 2 times. Last edited by TelvCohl, Oct 8, 2015, 2:13 am

toto1234567890

#12 Jan 11, 2016, 7:54 pm

I think we just have to show that N is in $\triangle IAC$ which is very easy.

Quick Reply

High School Olympiads

Mixtilinear incircles concurrency

Locked

Source: 1st On-line Geometry Competition Day 2 Q2

leeky

#1 Oct 6, 2015, 3:02 pm

Let I,O be the incenter,circumcenter of $\triangle ABC$ respectively. X,Y,Z are the tangency points of the $A-,B-,C-$ mixtilinear incircles with the circumcircle of $\triangle ABC$ respectively. Let P be a point on IO, the second intersections of XP,YP,ZP with the $A-,B-,C-$ mixtilinear incircles are D,E,F respectively. Prove that AD,BE,CF are concurrent.

Luis González

#2 Oct 6, 2015, 9:45 pm

What is this On-line Geometry Competition?. As I see it, the majority of its problems are been lifted from AoPS. See http://www.artofproblemsolving.com/community/c6h612592.

High School Olympiads

About Mixtilinear Circle.

Reply

Jul

#1 Nov 4, 2014, 5:36 pm

Given triangle ABC and its circumcircle be (O). Let (w_a) be mixtilinear circle in angle A. Let (O) and (w_a) be internally tangent at X. Let XX_1 be diameter of (w_a). Similary, we have Y, Z, Y_1, Z_1. Prove that : AX_1, BY_1, CZ_1 are concurrent.

TelvCohl

#2 Nov 4, 2014, 6:45 pm • 3

Generalization :

Let I, O be the incenter, circumcenter of $\triangle ABC$, respectively.
Let (w_a) be A-Mixtilinear circle of $\triangle ABC$ and (w_a) tangent to (O) at X .
Let (w_b) be B-Mixtilinear circle of $\triangle ABC$ and (w_b) tangent to (O) at Y .
Let (w_c) be C-Mixtilinear circle of $\triangle ABC$ and (w_c) tangent to (O) at Z .
Let P be a point on OI and $X_1 = XP \cap (w_a), Y_1 = YP \cap (w_b), Z_1 = ZP \cap (w_c)$.

Then AX_1, BY_1, CZ_1 are concurrent

Proof :

Let $X_2 \equiv XP \cap (O), Y_2 \equiv YP \cap (O), Z_2 \equiv ZP \cap (O)$.
Let $X_3 \equiv AX_1 \cap (I), Y_3 \equiv BY_1 \cap (I), Z_3 \equiv CZ_1 \cap (I)$.
Let D, E, F be the tangency point of (I) with BC, CA, AB, respectively .
Let D_1, E_1, F_1 be the midpoint of arc BC, arc CA, arc AB, respectively .

It's well known $T = AX \cap BY \cap CZ \in OI$ (exsimilicenter of $(I) \sim (O)$) .

From Pascal theorem (for YF_1CZE_1B) we get $YF_1 \cap ZE_1$ lie on OI .
From Pascal theorem (for $YF_1Z_2ZE_1Y_2$) we get $E_1Y_2 \cap F_1Z_2 \in OI$.
Similarly, we can prove $D_1X_2 \cap E_1Y_2 \in OI \Longrightarrow D_1X_2, E_1Y_2, F_1Z_2$ are concurrent on OI . $(*)$

Since A, X is the exsimilicenter of $(I) \sim (w_a)$, $(w_a) \sim (O)$, respectively .
so we get D_1X_2, DX_3 is the corresponding line of (O), (I), respectively .
Similarly, we can prove $(I) \cup EY_3 \sim (O) \cup E_1Y_2$ and $(I) \cup FZ_3 \sim (O) \cup F_1Z_2$,
so from $(*)$ we get DX_3, EY_3, FZ_3 are concurrent (on OI) ,
hence from Steinbart theorem we get $AX_3 \equiv AX_1, BY_3 \equiv BY_1, CZ_3 \equiv CZ_1$ are concurrent .

Q.E.D

This post has been edited 1 time. Last edited by TelvCohl, Oct 6, 2015, 2:46 am

Quick Reply

High School Olympiads

Simson line perpendicular to Euler line

Reply

Source: 1st On-line Geometry Competition Day 1 Q2

leeky
#1 Oct 5, 2015, 4:49 pm • 3

Let P be a point on the Euler line of $\triangle ABC$ and Q be the isogonal conjugate of P with respect to $\triangle ABC$. BQ, CQ intersects the circumcircle of $\triangle ABC$ at E, F respectively. R is a point satisfying $\angle RFQ = \angle QFB$ and $\angle REQ = \angle QEC$. S is the second intersection of AR and the circumcircle of $\triangle ABC$. Prove that the Simson line of S with respect to $\triangle ABC$ is perpendicular to the Euler line of $\triangle ABC$.

Luis González
#2 Oct 6, 2015, 3:58 am

Let O and H be the circumcenter and orthocenter of $\triangle ABC.$ Since $\angle REB = \angle BEC = \angle BAC,$ then RE cuts the circumcircle (O) again at a fixed point Y and likewise RF cuts (O) again at a fixed point $Z.$ When P runs on $OH,$ clearly the pencils BE, CF are projective $\Longrightarrow$ pencils YE, ZF are projective. Thus it suffices to show that R is on the line AS for at least 3 positions of $P,$ where $S \in (O)$ is redefined as the pole of the Simson line perpendicular to $OH.$

When $P \equiv OH \cap BC,$ trivially $Q \equiv R \equiv A.$ When $Q \equiv O,$ then R clearly becomes circumcenter of $\triangle OEF \Longrightarrow AR$ meets (O) again at the anti-Steiner point of OH WRT the circumcevian triangle $\triangle DEF$ of $O,$ i.e. the pole S of the Simson line perpendicular to $OH.$

Finally, we assume the case $Q \equiv H.$ Let B', C' be the feet of the altitudes on $AC, AB. HERF$ is clearly a parallelogram, thus the homothety $\mathbf{H}(H, \tfrac{1}{2})$ carries AR into the Euler line of $\triangle AB'C'$ antiparallel to OH WRT $AB, AC \Longrightarrow AR$ passes through the isogonal conjugate S of the point at infinity of OH WRT $\triangle ABC \Longrightarrow R \in AS.$ Hence, we conclude that $R \in AS$ for any P on $OH.$

TelvCohl
#3 Oct 6, 2015, 10:07 am

See also here : Simson Line Perpendicular to Euler Line (lym) (Lemma 2 at post #2)

Quick Reply

High School Olympiads

Incenter lies on OI

Reply

Source: 1st On-line Geometry Competition Day 1 Q3

leeky
#1 Oct 5, 2015, 9:53 pm

I, O are the incenter and circumcenter of $\triangle ABC$ respectively. Denote (O_a) as the circle passing through B, C and tangent to the incircle, similarly define $(O_b), (O_c)$. Let A_b, A_c be the intersection of (O_a) with AB, AC respectively, similarly define B_c, B_a, C_a, C_b. Prove that the incenter of $\triangle XYZ$, which is the triangle formed by the three lines A_bA_c, B_cB_a, C_aC_b, lies on the line OI.

Luis González
#2 Oct 6, 2015, 1:32 am

There's more to say. The incenter J of $\triangle XYZ$ is the center X_{5045} of $\triangle ABC$, which lies on OI verifying $\frac{JI}{JO} = \frac{r}{4R+r}$, where R, r denote the radii of $(O), (I)$.

Let (I) touch BC at U and let M, S be the midpoints of BC and B_cC_b, resp. Clearly $\triangle XB_cC_b$ is X-isosceles, thus XS is internal bisector of $\angle YXZ$ cutting OI at J. From the problem 3 congruent circles (see post #3), we deduce that $BB_c = \frac{(s-b)(4R+r)r}{ab}$ and $CC_b = \frac{(s-c)(4R+r)r}{ac}$. Consequently, asumming WLOG that $b > c$, we obtain:

$$SM = \frac{1}{2}\left(\frac{(s-b)(4R+r)r}{ab} - \frac{(s-c)(4R+r)r}{ac}\right) = \frac{rs(4R+r)(b-c)}{2abc} \Longrightarrow$$

$$\frac{SM}{UM} = \frac{rs(4R+r)(b-c)}{2abc \cdot \frac{1}{2}(b-c)} = \frac{rs(4R+r)}{abc} = \frac{4R+r}{4R} \Longrightarrow$$

$$\frac{JO}{OI} = \frac{SM}{UM} = \frac{4R+r}{4R} \Longrightarrow \frac{JI}{JO} = \frac{r}{4R+r} \Longrightarrow J \equiv X_{5045}.$$

By similar reasoning, the other internal bisectors of $\triangle XYZ$ pass through X_{5045}, which is then its incenter.

TelvCohl
#3 Oct 6, 2015, 10:00 am

Let $\triangle T_aT_bT_c$ be the tangential triangle of $\triangle ABC$. From my proof to the problem Concurrent on OI line we get $\triangle XYZ$ and $\triangle T_aT_bT_c$ are homothetic and their homothety center J lies on OI, so notice O is the Incenter of $\triangle T_aT_bT_c$ we conclude that the Incenter of $\triangle XYZ$ lies on OI.

Quick Reply

High School Olympiads

3 congruent circles

Reply

Source: Own

Luis González

#1 Jul 30, 2015, 11:31 am • 2

$\triangle ABC$ is a triangle with incircle (I,r) and circumcircle (O,R). The circle passing through B,C and tangent to (I,r) cuts AC, AB again at A_C, A_B. Define similarly B_A, B_C and C_B, C_A. If $\varrho_A, \varrho_B, \varrho_C$ denote the radii of the excircles of $\triangle AA_CA_B, \triangle BB_AB_C, \triangle CC_BC_A$ againts $A,B,C,$ respectively, show that

$$\varrho_A = \varrho_B = \varrho_C = \frac{(4R+r)r}{4R}$$

TelvCohl

#2 Jul 30, 2015, 3:29 pm • 3

My solution :

Let $\triangle DEF$ be the intouch triangle of $\triangle ABC$ and M be the midpoint of EF .
Let J_a be the A-excenter of $\triangle AA_CA_B$ and H be the orthocenter of $\triangle DEF$.
Let I_a, I_b, I_c be the A-excenter, B-excenter, C-excenter of $\triangle ABC$, respectively .

Since $\triangle DEF \cup H$ and $\triangle I_aI_bI_c \cup I$ are homothetic ,

so $\frac{MI}{II_a} = \frac{1}{2} \cdot \frac{HD}{II_a} = \frac{1}{2} \cdot \frac{r}{2R} = \frac{r}{4R} \Longrightarrow \frac{MI_a}{II_a} = \frac{MI+II_a}{II_a} = \frac{4R+r}{4R}$ . ... $(\star)$

From Really Hard: To prove a rectangle (post #2 lemma) $\Longrightarrow J_a$ is the image of I_a under the inversion $\mathbf{I}(\odot(I))$,

so combine $(\star)$ we get $\frac{\varrho_A}{r} = \frac{AJ_a}{AI} = (A,\infty;J_a,I) = (M,I;I_a,\infty) = \frac{MI_a}{II_a} = \frac{4R+r}{4R} \Longrightarrow \varrho_A = \frac{(4R+r)r}{4R}$.

Q.E.D

Luis González

#3 Jul 30, 2015, 8:41 pm • 1

Thanks for your nice solution Telv. Here is mine

Let (I) touch BC, BA at X,Y. Inversion WRT $\odot(B,BX)$ takes circle through B,C tangent to (I) into a tangent of it, cutting BC, BA at the inverses P,Q of C, A_B. Thus we have

$$BP = \frac{(s-b)^2}{a} \Longrightarrow PX = (s-b) - \frac{(s-b)^2}{a} = \frac{(s-b)(s-c)}{a} \Longrightarrow \frac{PX}{PB} = \frac{s-c}{s-b}.$$

On the other hand, we have $(Q,B,Y,A) = (P,X,B,C) \Longrightarrow$

$$\frac{QB}{QY} \cdot \frac{AY}{AB} = \frac{PX}{PB} \cdot \frac{CB}{CX} \Longrightarrow \frac{QB}{QY} = \frac{ac}{(s-a)(s-b)}$$

But $(Q,B,Y,\infty) = (A_B,\infty,Y,B) \Longrightarrow \frac{QB}{QY} = \frac{BY}{A_BY} \Longrightarrow A_BY = \frac{(s-a)(s-b)^2}{ac}$

$$AA_B = AY - A_BY = (s-a) - \frac{(s-a)(s-b)^2}{ac} = (s-a) \cdot \frac{r(4R+r)}{ac}$$

Now since $\triangle AA_CA_B \sim \triangle ABC$, we obtain

$$\frac{\varrho_A}{r_A} = \frac{AA_B}{b} \Longrightarrow \varrho_A = \frac{rs}{s-a} \cdot (s-a) \cdot \frac{r(4R+r)}{abc} = \frac{r(4R+r)}{4R}.$$

Quick Reply

High School Olympiads

geometry

Locked

Tung-CHL

#1 Oct 5, 2015, 10:58 am

For triangle ABC with orthocenter H, inscribed center I. D,E,F is contactors of (I) on BC,CA,AB. A line d through D and perpendicular with EF. K is intersection of d and EF. Prove that KD is bisector of corner HKI.

Luis González

#2 Oct 5, 2015, 11:29 am

Discussed at http://www.artofproblemsolving.com/community/c6h614584 (see posts #4 and #7).

High School Olympiads

Mongoidal sphere

Locked

Source: Iberoamerican Olympiad for University Students 2011 - Problem 5

nunoarala

#1 Oct 3, 2015, 10:36 pm

Given are circles $\omega_1,\omega_2,\omega_3$ on the unit sphere S of $\mathbb{R}^3$. Suppose that for each pair (i,j) with $1\le i<j\le 3$ there are two maximal circles C_{ij} and C_{ji} such that both are tangent to ω_i and ω_j but none of them separates ω_i and ω_j. The maximal circles C_{ij} and C_{ji} meet at P_{ij} and P_{ji}.

Show that points P_{12}, P_{23}, P_{31}, P_{13}, P_{32} and P_{21} lie on the same maximal circle of S.

Luis González

#2 Oct 4, 2015, 9:31 pm

Posted earlier at http://www.artofproblemsolving.com/community/c6h1085894.

High School Olympiads

All points belong to the same circle

Source: Iberoamerican Competition (oimu 2011)

joao1

#1 May 7, 2015, 1:02 am

There are three circles w_1, w_2, w_3 on the unit sphere S of R^3. Suppose that for each pair (i,j), $1 \le i < j \le 3$ there are two great circles C_{ij} and C_{ji} of S such that both of them are tangent to w_i and w_j and none of them separate w_i and w_j. The intersection of great circles C_{ij} and C_{ji} are the points P_{ij} and P_{ji}.

Show that the points $P_{12}, P_{23}, P_{31}, P_{13}, P_{32}$ and P_{21} belong to the same great circle in S.

Luis González

#2 May 7, 2015, 5:02 am • 1

We consider a stereographic projection of the sphere S onto a plane. Taking P_{12} as the pole, then it suffices to show that the images $Q_{23}, Q_{31}, Q_{13}, Q_{32}, Q_{21}$ of $P_{23}, P_{31}, P_{13}, P_{32}, P_{21}$ are collinear.

$\omega_1, \omega_2, \omega_3$ go to circles $\Gamma_1, \Gamma_2, \Gamma_3$. Great circles $\mathcal{C}_{13}, \mathcal{C}_{32}$ go to circles $\mathcal{K}_{13}$ and $\mathcal{K}_{31}$ externally and internally tangent to both Γ_1, Γ_3 and bisecting the equatorial circle $\odot(Q_{21}, \varrho)$, i.e. $Q_{21} \in Q_{13}Q_{31}$. Same holds for the other pair of circles $\mathcal{K}_{23}, \mathcal{K}_{32}$.

Let X be the intersection of the common external tangents of Γ_1, Γ_3 and let Y be the intersection of the common external tangents of Γ_2, Γ_3. Tangency points $\{U, V\}$ and $\{S, T\}$ of $\mathcal{K}_{13}$ and $\mathcal{K}_{31}$ with Γ_1, Γ_3, respectively, are clearly inverses under the direct inversion with center X that swaps Γ_1 and $\Gamma_3 \Longrightarrow XU \cdot XV = XS \cdot XT \Longrightarrow X$ has equal power WRT $\mathcal{K}_{13}, \mathcal{K}_{31} \Longrightarrow X$ is on their radical axis $Q_{13}Q_{31} \Longrightarrow X, Q_{13}, Q_{31}, Q_{21}$ are collinear and similarly $Y, Q_{23}, Q_{32}, Q_{21}$ are collinear. But by Monge's theorem for $\Gamma_1, \Gamma_2, \Gamma_3$, the points X, Y, Q_{21} are collinear.Therefore $Q_{21}, Q_{13}, Q_{31}, Q_{23}, Q_{32}$ are collinear, as desired.

High School Olympiads

Tangent circles, equal segments

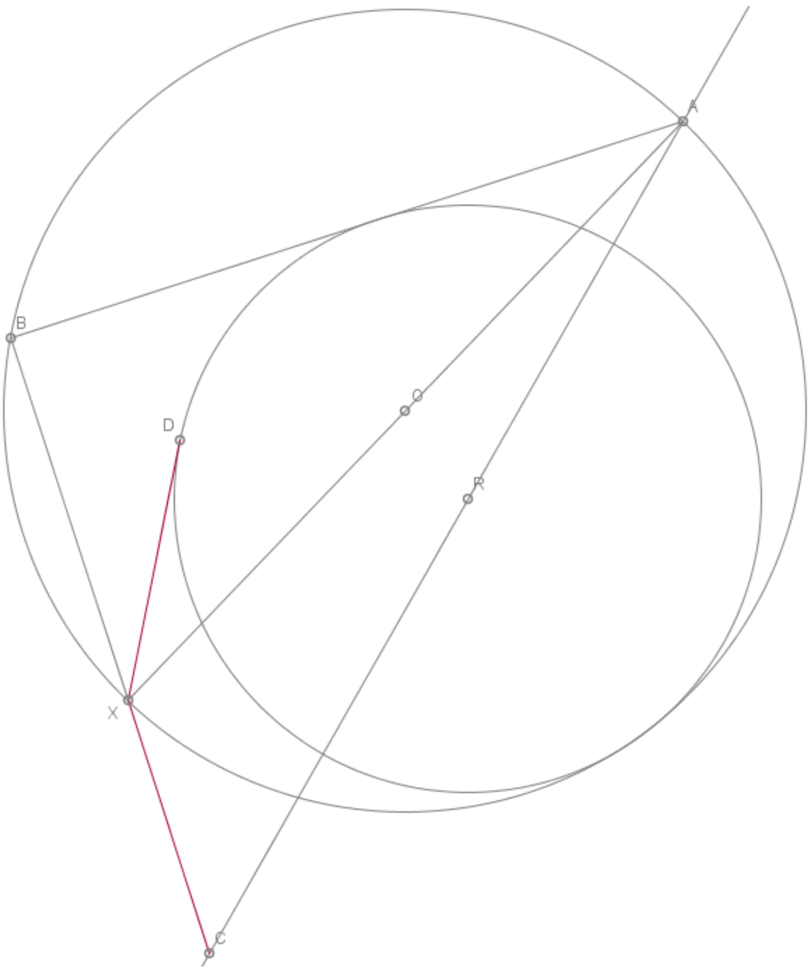
Source: A problem from Geometry in figures by Akopyan.

Cezar

#1 Oct 4, 2015, 3:04 am

Let α and β be two tangent circles, with centers O, R. Let X and A be two antipodal points on α.
Let $B \in \alpha$ such that AB tangent to β. Let $C = AR \cap BX$. Let $D \in \beta$ such that DX tangent to β. Prove $XD = XC$.

Attachments:



TelvCohl

#2 Oct 4, 2015, 2:25 pm

Let the reflection of AB in AR cuts $\odot(O)$ at E and let I be the incenter of $\triangle ABE$. Let $M \equiv AR \cap \odot(O)$ be the midpoint of arc BE and $T \equiv \odot(O) \cap \odot(R), K \equiv \odot(R) \cap AB, V \equiv AB \cap MX$. From Mannheim theorem $\Longrightarrow I$ is the projection of K on AR, so B, C, I, K lie on a circle with diameter CK. On the other hand, from homothety with center T that maps $\odot(R)$ $\mapsto \odot(O)$ we get $\measuredangle BTK = \measuredangle KTA$, so $\measuredangle BTK = \frac{1}{2}\measuredangle BEA = \measuredangle BIK$, hence B, C, I, K, T are concyclic. Since $\measuredangle CKV = \measuredangle MIB = 90^\circ - \frac{1}{2}\measuredangle BEA$, so combine $\measuredangle AVC = \measuredangle BMA = \measuredangle BEA$ (notice X is the orthocenter of $\triangle ACV$) we get $VC = VK \Longrightarrow V$ lies on the radical axis τ of $\odot(R)$ and the degenerate circle C, hence notice $VX \perp CR$ we conclude that $X \in \tau \Longrightarrow XC = XD$.

jayme

#3 Oct 4, 2015, 4:52 pm

#3 Oct 4, 2015, 4:52 pm

Dear Mathlinkers,
can some one refresh me by telling about the Mannheim's theorem...
Sincerely
Jean-Louis

liberator

#4 Oct 4, 2015, 6:37 pm

I think that this is the well known theorem that the segment connecting the contact points of the A-mixtilinear incircle with the triangle sides is perpendicular to the bisector of $\angle A$. This should not be confused with another theorem of Mannheim.

In this figure, β is the A-mixtilinear incircle of $\triangle ABE$, and K is one of the tangency points.

Luis González

#5 Oct 4, 2015, 9:27 pm

Let P and T be the tangency points of (R) with AB and (O), resp. It's known that TP is the internal bisector of $\angle ATB$ and TC is the external bisector of $\angle ATB$ (see for instance Concurrency with circumcenter). Thus if $M \equiv CP \cap XT$ we get, using the cyclic $PTCB, \angle MCT = \angle PBT = \angle AXT = \angle OTM$ and $\angle MCB = \angle PTB = \angle MTC \Longrightarrow$ $\odot(MCT)$ is tangent to BC, OT. Moreover $\angle PMT = \angle PBT + \angle PTB = \angle APT \Longrightarrow M \in (R) \Longrightarrow$ $XC^2 = XM \cdot XT = XD^2 \Longrightarrow XC = XD.$

Quick Reply

High School Olympiads

Concurrency with circumcenter

Reply

Mentalist

#1 Sep 7, 2015, 9:20 pm

In a triangle ABC with circumcenter O lets draw the exterior angle bisector of $\angle BAC$. Lets take points P and Q on this exterior angle bisector such that $PB \perp BC$ and $QC \perp BC$. Prove that lines AO, PC, QB are concurrent.

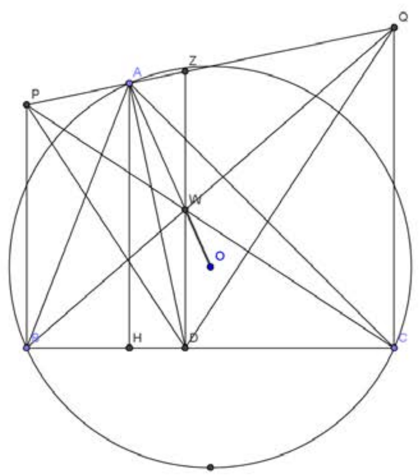
trunglqd91

#2 Sep 7, 2015, 11:42 pm

My solution:

Let $W = BQ \cap CP$.
AD is internal bisector of $\angle BAC (D \in BC)$.
$Z = DW \cap PQ$.
AH is the altitude of $\triangle ABC$.
Easy to see that $PBDA$ and $QADC$ is cyclic.
Thus $\angle BPD = \angle DQC$.
So $\triangle BPD \sim \triangle CQD$.
$\Longrightarrow \dfrac{BD}{PD} = \dfrac{CD}{QD} \Longrightarrow \dfrac{PD}{QD} = \dfrac{BD}{CD} = \dfrac{ZP}{ZQ}$.
Hence DZ is internal bisector of $\angle PDQ$.
But $\angle BDP = \angle CDQ$ Then $DW \perp BC$. $\Longrightarrow DW \parallel BP \parallel CQ$.
So we have $\dfrac{WZ}{QC} = \dfrac{PZ}{PQ} = \dfrac{BD}{BC} = \dfrac{DW}{QC}$
$\Longrightarrow W$ is the midpoint of DZ.
In the other hand $\angle ZAD = 90^o$.
$\Longrightarrow \triangle WAD$ is isosceles at W.
Hence $\angle WAQ = \angle ADB = 180^o - \angle BPA = \angle PAH \Longrightarrow \angle HAD = \angle WAD$
$\Longrightarrow AH, AW$ is isogonal conjugate WRT $\triangle ABC$.
So A, W, O. Done.

Attachments:



Dukejukem

#3 Sep 8, 2015, 3:37 am

Let A' be the antipode of A w.r.t. $\odot(ABC)$ and let l be the external bisector of $\angle BAC.$ Denote $P^* \equiv A'B \cap l$ and $Q^* \equiv A'C \cap l.$

Because $\angle ABP^* = \angle CBP = 90^{\circ}$, it follows that BP and BP^* are isogonal w.r.t. $\angle ABC.$ Meanwhile, it is clear that AP and AP^* are isogonal w.r.t. $\angle BAC.$ Therefore, P^* is the isogonal conjugate of P, and similarly Q^* is the isogonal conjugate of $Q.$

Now, observe that $\triangle P^*AB \sim \triangle Q^*AC$ because $\measuredangle PAB = \measuredangle CAQ$, and $\angle ABP^* = \angle ACQ^* = 90^{\circ}.$ Now, from the Law of Sines applied to $\triangle P^*AC$ and $\triangle P^*BC$ we have

$$\frac{P^*C}{\sin\angle P^*BC}\cdot\frac{P^*A}{\sin\angle P^*CA} = \frac{P^*B}{\sin\angle P^*CB}\cdot\frac{P^*C}{\sin\angle P^*AC} \implies \frac{\sin\angle P^*CA}{\sin\angle P^*CB} = \frac{P^*A}{P^*B}\cdot\frac{\sin\angle P^*AC}{\sin\angle P^*BC}.$$

Similarly, we obtain

$$\frac{\sin\angle Q^*BA}{\sin\angle Q^*BC} = \frac{Q^*A}{Q^*B}\cdot\frac{\sin\angle Q^*AB}{\sin\angle Q^*CB}.$$

Since $\frac{P^*A}{P^*B} = \frac{Q^*A}{Q^*B}$, it follows that

$$\frac{\sin\angle P^*CA}{\sin\angle P^*CB}\cdot\frac{\sin\angle Q^*BC}{\sin\angle Q^*BA} = \frac{\sin\angle P^*AC}{\sin\angle P^*BC}\cdot\frac{\sin\angle Q^*CB}{\sin\angle Q^*AB} = \frac{\sin\angle Q^*CB}{\sin\angle P^*BC},$$

where the last step follows because $\measuredangle P^*AC = \measuredangle BAQ^*.$ Now, if H is the projection of A onto BC, it is well-known that AH and AA' are isogonal w.r.t. $\angle BAC.$ Then because

$$\frac{\sin\angle Q^*CB}{\sin\angle P^*BC} = \frac{\sin\angle A'CB}{\sin\angle A'BC} = \frac{\sin\angle A'AB}{\sin\angle A'AC} = \frac{\sin\angle HAC}{\sin\angle HAB},$$

it follows from Trig Ceva that AH, BQ^*, CP^* are concurrent at some point $R.$ Therefore, AO, BQ, CP are concurrent at the isogonal conjugate of $R.$ $\square$

Luis González

#4 Sep 8, 2015, 5:48 am

Let AO cut $\odot(ABC)$ again at $X.$ Since $XB \perp AB$ and $XC \perp AC,$ it follows that $\angle XBC = \angle ABP$ and $\angle XCB = \angle ACQ.$ Since $\angle BAP = \angle CAQ,$ then by Jacobi's theorem, it follows that $AX \equiv AO, BQ, CP$ concur.

Mentalist

#5 Sep 11, 2015, 2:08 pm

Thanks for solutions 🙂

Quick Reply

High School Olympiads

tangent point of A-mixtilinear

Locked

huynguyen

#1 Oct 3, 2015, 4:50 am

Given an acute triangle ABC circumscribed in (O), incircle (I).P is an arbitrary point lying on the small arc BC.Let M,N be two points on (I) satisfying that MN is the polar of P wrt (I).PM,PN cut BC at D,E respectively.Let A-mixtilinear tangents (O) at R.Prove that (PDE) passes through R.

Luis González

#2 Oct 3, 2015, 5:18 am

Posted before at Mixtilinear incircles and somehow Poncelet's porism and for a generalization see Cevian and mixtilinear incircle.

High School Olympiads

Mixtilinear incircles and somehow Poncelet's porism

Source: me

pohoatza

#1 Feb 1, 2008, 5:19 am • 5

Let M be an arbitrary point on the circumcircle of ABC and let the tangents from this point to the incircle of the triangle meet the side BC at X_1, and X_2. Prove that the second intersection of the circumcircle of triangle MX_1X_2 with the circumcircle of ABC (different from M) is fixed and it coincides with the tangency point of the mixtilinear incircle in angle A with the circumcircle.

Note: The **mixtilinear incircle** in angle A is the circle tangent to the sides AB, AC and to the circumcircle of ABC internally.

This post has been edited 3 times. Last edited by pohoatza, Jul 2, 2008, 5:46 pm

yetti

#2 Feb 2, 2008, 6:30 pm • 4

Cosmin, I could not resist.

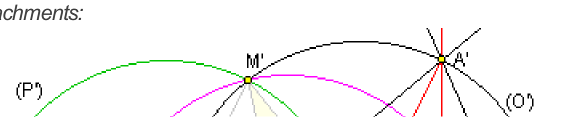
Let (I) be the common incircle of the $\triangle ABC, \triangle MX_1X_2$ with inradius r. Let D, E, F, Y_1, Y_2 be the tangency points of (I) with $BC \equiv X_1X_2, CA, AB, MX_1, MX_2$. Inversion with center I and power r^2 takes the vertices A, B, C, M, X_1, X_2 to the midpoints $A', B', C', M', X_1', X_2'$ of the sides $EF, FD, DE, Y_2Y_1, Y_1D, DY_2$ of the contact $\triangle DEF, \triangle DY_2Y_1$. The circumcircles $(O), (P)$ of the $\triangle ABC, \triangle MX_1X_2$ go into the circumcircles $(O'), (P')$, of the $\triangle A'B'C', \triangle M'X_1'X_2'$, identical with the 9-point circles of the two contact triangles with the common circumcircle (I), both with radius $\varrho = \frac{r}{2}$, congruent. The incircle tangents $BC \equiv X_1X_2, CA, AB, MX_1, MX_2$ go to the circles $\Gamma_a, \Gamma_b, \Gamma_c, \Omega_1, \Omega_2$ with diameters $ID = IE = IF = IX_1 = IX_2 = r$, all congruent to $(O'), (P')$. Since $(A'I \equiv AI) \perp (EF \parallel B'C'), B'I \perp C'A'$, $C'I \perp A'B'$, the inversion center I is the orthocenter of the $\triangle A'B'C'$ and similarly, the orthocenter of the $\triangle M'X_1'X_2'$. The congruent circles $(O'), \Gamma_b, \Gamma_c$ with radii $\varrho = \frac{r}{2}$ meet at A'. A circle (A', r) with center A' and radius r is tangent to all three at points diametrically opposite to A'.

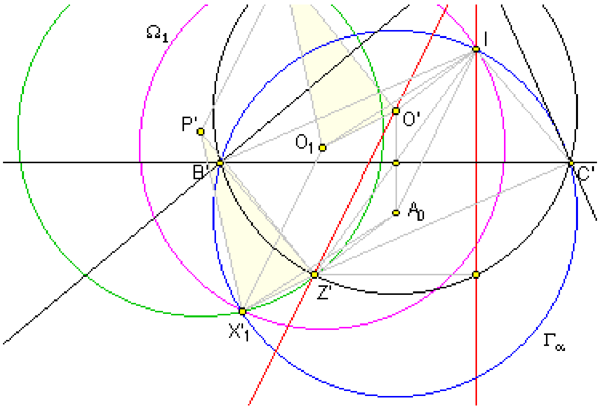
The internal bisector AI of the $\angle A$ passing through the inversion center I is carried into itself. The mixtilinear incircle (K) and the mixtilinear excircle (L) of the $\triangle ABC$ in the $\angle A$ are the only two circles centered on AI and tangent to $CA, AB, (O)$. Since the inversion center I is the similarity center of a circle and its inversion image, only the images $(K'), (L')$ of $(K), (L)$ are centered on AI and tangent to $\Gamma_b, \Gamma_c, (O')$. The mixtilinear excircle (L), lying outside of the circumcircle (O) and outside of the inversion circle (I), has both intersections with AI on the ray $(IL$. Since the inversion in (I) has positive power r^2, its image (L') also has both intersections with AI on the ray $(IL$ and it is centered on the ray $(IL$. It cannot be identical with the circle (A', r) centered on the opposite ray $(IA$. Therefore, the circle (A', r) is the inversion image of the mixtilinear incircle (K) in the $\angle A$. The inversion image of the tangency point Z of $(K), (O)$ is the tangency point Z' of $(A', r), (O')$, the diametrically opposite point of (O') with respect to A'.

Let A_0, B_0, C_0, O_1, O_2 be centers of the congruent circles $\Gamma_a, \Gamma_b, \Gamma_c, \Omega_1, \Omega_2$. Since Γ_a is a reflection of (O') is $B'C'$ and $A'O', A'I$ are isogonals of $A'B', A'C'$, the quadrilateral $B'Z'C'I$ is a parallelogram, therefore, its diagonals $B'C', IZ'$ cut each other at half at the midpoint of $B'C'$. Then the quadrilateral $IA_0Z'O'$ is also a parallelogram, because its diagonals $IZ', O'A_0$ cut each other in half at the midpoint of $B'C'$, and $IA_0 \parallel O'Z', IA_0 = O'Z' = \varrho$. Since $A_0I = A_0X_1' = O_1I = O_1X_1' = \varrho$, the quadrilateral $A_0IO_1X_1'$ is a rhombus and $O_1X_1' \parallel IA_0 \parallel O'Z'$. In addition, $O_1X_1' = O'Z' = \varrho$, hence, the quadrilateral $O_1X_1'Z'O'$ is a parallelogram and $O'O_1 \parallel Z'X_1', O'O_1 = Z'X_1'$. Obviously, $P'M' = P'X_1' = O_1M' = O_1X_1' = \varrho$, the quadrilateral $P'X_1'O_1M'$ is a rhombus and $P'X_1' \parallel M'O_1$.

As a result, the $\triangle P'X_1'Z' \cong \triangle M'O_1O'$ with $O'O_1 \parallel Z'X_1', O'O_1 = Z'X_1'$ and $P'X_1' \parallel M'O_1, P'X_1' = M'O_1 = \varrho$ are congruent by SAS and $P'Z' = M'O' = \varrho$. But this means that $Z' \in (P')$ and consequently, $Z \in (P)$, which is what we were supposed to prove.

Attachments:





pohoatza
#3 Feb 4, 2008, 4:43 pm • 1

Thank you for the nice solution, Vladimir! I was pretty sure you were going to *invert* things around here.

Erken
#4 Feb 4, 2008, 6:17 pm

What is your solution,pohoatza?

pohoatza
#5 Feb 4, 2008, 6:42 pm • 1

It appears that I had the same solution as Vladimir. It would be interesting to see a proof without inversion.

jayme
#6 Apr 5, 2009, 3:39 pm

Dear Cosmin and Mathlinkers,
a syntetical proof without inversion of this result can be found on on my website :
http://perso.orange.fr/jl.ayme vol. 4 A new mixtilinear incircle adventure III, p. 21
Sincerely
Jean-Louis

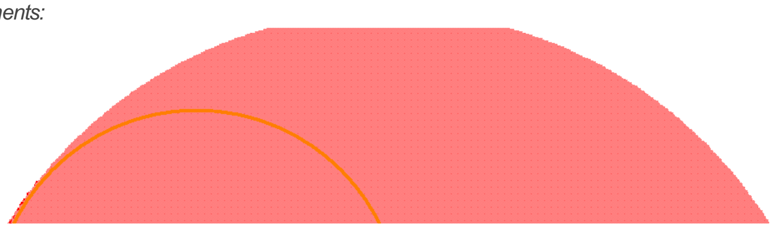
Fang-jh
#7 Apr 30, 2009, 4:19 pm

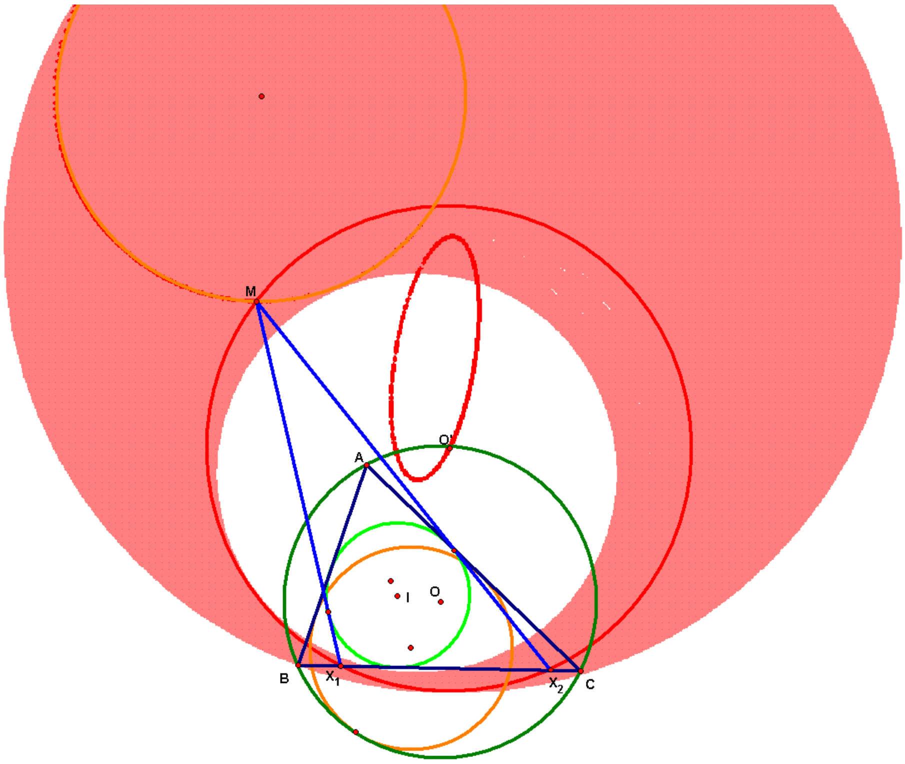
Mr Zhonghao Ye found an interesting result about the problem:
The circumcircle of triangle MX_1X_2 is always tangent to a fixed circle internally.
The circle is constructed as follows:
Let the mixtilinear incircle in angle A touch the circumcircle at A'. Let the tangent of the incircle which is parallel to BC cut the circumcircle at B', C'. then the fixed circle is the mixtilinear incircle in angle A' of triangle $A'B'C'$.

Fang-jh
#8 May 2, 2009, 8:56 pm

Another similar problem (Zhonghao Ye's discovery):
When moving point M lies on the another circle (call it ω) which is different from the circumcircle of triangle ABC. We have
(I): There exist two fixed circles which are tangent to the circumcircle of triangle MX_1X_2. (The two fixed circles are also tangent to ω)
(II):The locus of the circumcenter of triangle MX_1X_2 is an ellipse.

Attachments:





yetti

#9 May 10, 2009, 5:51 am • 1

This is not correct. The $\triangle ABC$ and its circumcircle $\odot(ABC)$ are irrelevant here, just a fixed tangent l of (I) is required. For $M \in \omega$, the circumcenter of $\triangle MX_1X_2$ lies on a conic; this can be either ellipse or hyperbola, even if (I) is outside of ω. It can be only hyperbola, when (I) is inside of ω. That's because $\odot(MX_1X_2)$ is tangent to 2 fixed circles $(F_1), (F_2)$, their centers being foci of the conic.

The point is to show that the foci F_1, F_2 are collinear with the center I of the circle (I). Moreover, the circles $(I), (F_1), (F_2)$ have a common similarity center, and when the arbitrary circle ω degenerates to an arbitrary line, this similarity center falls on the tangent l of (I).

The general problem has 2 important special cases:

(1) (I) and ω are incircle and circumcircle of some triangle $\triangle ABC$ with $l \equiv BC$. Then one of the circles $(F_1), (F_2)$, degenerates to a point on ω, the tangency point of the A-mixtilinear incircle of $\triangle ABC$.

(2) The arbitrary circle ω degenerates to a line. This leads to Thebault theorem and more:
Let ω be a line through the vertex A of a $\triangle ABC, (O), (I)$ its circumcircle and incircle, $(F_1), (F_2)$ circles tangent to ω, BC and internally tangent to (O). Then I, F_1, F_2 are collinar (Thebault theorem). In addition: If $A' \in \omega$ is arbitrary, tangents to (I) from A' cut BC at B', C', then circumcircle (O') of $\triangle A'B'C'$ is also tangent to $(F_1), (F_2)$.

See http://www.mathlinks.ro/viewtopic.php?t=275799 for an approach. The only bit not proved there follows from a simple lemma: Let (I, r) be an arbitrary circle and let $\mathcal{S}_1, \mathcal{S}_2$ be 2 concentric circles centered at an arbitrary point S, such that the difference of their radii is equal to r, but otherwise arbitrary. Let $(F_1), (F_2)$ be inversion images of $\mathcal{S}_1, \mathcal{S}_2$ in (I). Then the circles $(I), (F_1), (F_2)$ have a common similarity center.

livetolove212

#10 Nov 8, 2009, 7:29 pm • 5

Another synthetical proof:
First I will change the names of points (see in the figure)
Let $ME \cap (O) = \{P\}, MF \cap (O) = \{N\}$. It's easy to see that PN is tangent to (I), therefore (I) is the incircle of triangle MNP.
Denote $K = (MEF) \cap (O), L$ the midpoint of arc BC which contains A.
So we will show that K, I, L are collinear which follows that K is the tangency of A-Mixtilinear incircle and (O).
$\Leftrightarrow KI$ is the bisector of $\angle BKC (1)$
We have $\angle KBE = \angle KNC, \angle BEK = \angle KMF = \angle KCN$ therefore $\angle BKE = \angle CKN$
Thus $(1) \Leftrightarrow KI$ is the bisector of $\angle EKN. (2)$
Denote $MI \cap (MEF) = \{Q\}, MI \cap (O) = \{R\}$
We got $\angle EKQ = \angle EMQ = \angle RKN$

We get $\angle EKQ = \angle EMQ = \angle IKN$
Thus $(2) \Leftrightarrow KI$ is the bisector of $\angle RKQ$.
$\Leftrightarrow \dfrac{KR}{KQ} = \dfrac{IR}{IQ}(3)$
Since $\Delta PKN \sim \Delta EKF$ we get $\dfrac{KR}{KQ} = \dfrac{RP}{QE} = \dfrac{RI}{QI}$
Therefore (3) is true. We are done.

Attachments:
picture61.pdf (14kb)

TelvCohl

#11 Oct 26, 2014, 2:56 pm

My solution:

Let I_a be the A-excenter of $\triangle ABC$ and $T = AI \cap \odot(ABC)$.
Let I' be the incenter of $\triangle MX_1X_2$ and $K = \odot(BCII_a) \cap \odot(X_1X_2II')$.
Let $Y = AI \cap BC, Y' = MI' \cap BC, P = \odot(MX_1X_2) \cap (ABC)$.

Since $\angle IKI' = \angle IKI_a = 90$,
so K, I', I_a are collinear .
Since $(I, I'; Y', M) = (I, I_a; Y, A) = -1$,
so AM, BC, I_aK are concurrent at Z,
hence $AZ \cdot ZM = BZ \cdot ZC = I_aZ \cdot ZK$. i.e. A, K, M, I_a are concyclic
Consider $\{\odot(ABC), \odot(MX_1X_2), \odot(X_1X_2II')\}$ and $\{\odot(ABC), \odot(MX_1X_2), \odot(BCII_a)\}$
we get $S = BC \cap PM \cap IK$ is the radical center of $\{\odot(ABC), \odot(MX_1X_2), \odot(X_1X_2II'), \odot(BCII_a)\}$.

Since $SI \cdot SK = SP \cdot SM$,
so I, K, P, M are concyclic ,
hence $\angle TPI = \angle TPM + \angle MPI = \angle I_aKM + 180 - \angle IKM = 90$.
i.e. P lie on $\odot(IT)$ or P is the tangent point of $A-$ Mixtilinear circle with $\odot(ABC)$

Q.E.D

Quick Reply

High School Olympiads

Cevian and mixtilinear incircle

Source: me

yetti

#1 Jun 15, 2008, 1:43 pm • 3

Arbitrary line through the vertex A of a $\triangle ABC$ cuts its circumcircle (O) again at $K.$ Let X be an arbitrary point on the line $AK,$ outside of the triangle incircle $(I).$ Tangents to (I) from X cut BC at $Y, Z.$ Show that the circumcircle of the $\triangle KYZ$ cuts (O) again at the tangency point A^* of the mixtilinear incircle of the $\triangle ABC$ in the angle $\angle A.$

This post has been edited 1 time. Last edited by yetti, Jun 16, 2008, 12:37 pm

28121941

#2 Jun 16, 2008, 2:44 am

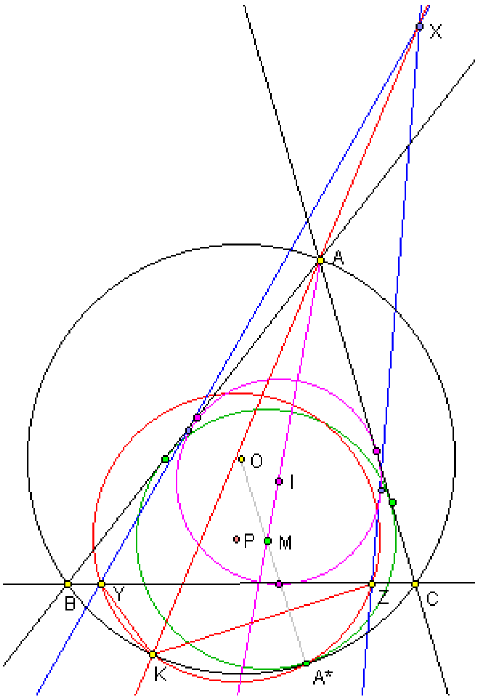
I am supposing the point A* is the point of tangency with (O) of a circle tangent to the sides of angle A and internally tangent to (O), right?

yetti

#3 Jun 16, 2008, 4:15 am • 1

That is correct, see the attached figure.

Attachments:



kaka__2004

#4 Jun 17, 2008, 8:15 pm

Nice! Could somebody solve it?

pohoatza

#5 Jul 2, 2008, 6:02 pm

#5 Jul 2, 2008, 6:02 pm

This is beautiful, Vladimir! It generalizes our previous problem http://www.mathlinks.ro/viewtopic.php?p=1022749#1022749.

My "idea" (hard to call it an *idea*) is to consider Y_K, Z_K the intersection points of the tangents from K to (I) with the sideline BC and to prove that the circumcircles of triangles KY_KZ_K, KYZ and ABC are concurrent. In this case, your problem follows from the one mentioned above. However, I couldn't find any synthetic proof for my assertion.

What is your proof?

Regards,
Cosmin

yetti

#6 Jul 3, 2008, 9:16 am

I forgot the proof, honestly, and I had to figure it out again. I do not have time today, give me a little more time (tomorrow :) and I promise you will laugh out loud.

yetti

#7 Jul 3, 2008, 12:41 pm • 1

Yes, it is generalization of your problem. I got it from this triangle construction: http://www.mathlinks.ro/viewtopic.php?t=207250, even though I did not need it there, your problem was sufficient.

Alright, let (I) be an arbitrary fixed circle with 2 fixed parallel tangent lines a, a' and let k be an arbitrary fixed line, intersecting the circle or not. Let $X \in k$ be arbitrary point of this line and let the tangents to (I) from X cut a at $Y, Z,$ so that (I) is incircle of the $\triangle XYZ.$ (This is not completely general, $X \in k$ has to be on the opposite side of a' than $(I),$ but proofs for the other cases - X between a, a' outside of (I), or X on the opposite side of a than (I) - are the same, except some incircle / excircles get switched.) Let S, S' be the intersections of k with a, a' and let a tangent to (I) from S' other than a' cut a at $T.$ The tangent $S'T$ is fixed. Let (I_x) be the X-excircle of the $\triangle XYZ.$ Since $S'T$ is a fixed tangent to $(I),$ parallel s to $S'T$ through S is tangent to $(I_x).$ Thus the X-excircle is tangent to 2 fixed lines $a, s,$ which means that it belongs to a file of circles with the common similarity center S and 2 common tangents $a, s.$ (A file of circles is concept dual to a pencil of circles.) Anyway, it is centered on the bisector i of the angle $\widehat{as}$ formed by the lines $a, s.$ Let M be the midpoint of $II_x.$ This is the circumcenter of the cyclic quadrilateral IYI_xZ and its circumcircle (M) is centered on a parallel $m \parallel i$ at half the distance from $I,$ than the angle bisector $i.$ But one point of the circle (M) is fixed, the incenter $I,$ therefore, the reflection $J \in (M)$ of I in the line m is also fixed. Thus the circle (M) belongs to a pencil of circles defined by the 2 fixed points $I, J.$ Let IJ cut $a \equiv YZ$ at $T'.$ Then $\overline{T'Y} \cdot \overline{T'Z} = \overline{T'I} \cdot \overline{T'J} = \text{const}.$ When X coincides with S', Y coincides with T and Z is at infinity as $Z^* \equiv a' \cap a.$ Since $\overline{T'T} \cdot \overline{T'Z^*} = \text{const}$ and $\overline{T'Z^*} = \infty,$ if follows that $\overline{T'T} = 0$ and $T' \equiv T$ are identical. Let $K \in k$ be an arbitrary fixed point (as opposed to $X \in k$ arbitrary movable) and let the fixed circumcircle of the $\triangle KIJ$ cut the fixed line KT again at $L.$ Then $\overline{TY} \cdot \overline{TZ} = \overline{TI} \cdot \overline{TJ} = \overline{TK} \cdot \overline{TL}$ and it follows that the quadrilateral $KYLZ$ is cyclic and moreover, its circumcircle passes through 2 fixed points $K, L,$ i.e., belongs to a pencil defined by these 2 points. Almost done.

Let (I) be the incircle and (O) the circumcircle of a $\triangle ABC.$ Let arbitrary fixed line k through A cut the circumcircle again at K (fixed point). Let $X \in k$ be arbitrary and let tangents to (I) from X cut the sideline $a \equiv BC$ at $Y, Z.$ Circumcircles of all triangles $\triangle KYZ$ form a pencil with 2 fixed points $K, L.$ Let A^* be the tangency point with (O) of the A-mixtilinear incircle of the $\triangle ABC$ in the angle $\angle A.$ When X coincides with A, then Y, Z coincide with $B, C,$ the circumcircle $\odot(KYZ) \equiv \odot(KBC)$ coincides with the circumcircle (O) and obviously, $A^* \in (O).$ When X coincides with $K,$ then by the problem http://www.mathlinks.ro/viewtopic.php?t=186117, circumcircle of the $\triangle KYZ$ passes through $A^*.$ Therefore, $K, L \equiv A^*$ are the 2 fixed points of the pencil of all circumcircles $\odot(KYZ),$ where $Y, Z \in BC$ are feet of the incircle tangents from $X \in k.$

This post has been edited 3 times. Last edited by yetti, Jul 4, 2008, 1:53 pm

28121941

#8 Jul 3, 2008, 2:13 pm • 1

Beautiful problem and, as usual, impressive solution of Yetti. Although after all the posted it is hard say some really interesting things, to follow the idea of Cosmin I was thinking in try to prove that the power of A* with respect to the circumcircles of KY_KZ_K and KYZ is 0 (the third one is obvious), but I have not yet achieved this.
Congratulations for a such nice problem!

jayme

#9 Apr 5, 2009, 3:45 pm • 1

Dear Yetti and Mathlinkers,

a new way for a proof of this result can be found on on my website :
http://perso.orange.fr/jl.ayme vol. 4 A new mixtilinear incircle adventure III, p. 24
Sincerely
Jean-Louis

lym

#10 Feb 17, 2011, 9:39 pm • 1

Let BC□XY□XZ touch $\odot I$ at D□E□F resp□XA intersec BC at J . Inversion with $\odot I$□
Delimit $XK \to \odot(IW), K \to K', Y \to Y', Z \to Z', J \to J'$ .Then EF pass thought W□W is a fixed point□
Y□Z are the midpoint of DE□DF resp□K' is on $\odot(IW)$□J' is the intersection of $\odot(ID)$ with $\odot(IW)$. Now□consider $\odot(K'Y'Z'), \odot(ID), \odot(DJ'K')$□their 3 radical axis $Y'Z', DJ', A'K'$(A' is on$\odot(K'Y'Z')$ and $\odot(DJ'K')$) are concurrent at T .

Obviously□T is the midpoint of DW□so T is a fixed point□and $TK' \cdot TA' = TY' \cdot TZ' = \frac{WE \cdot WF}{4} = \text{const}$.

It means that $\odot(K'Y'Z')$ pass thought a fixed point A' where is on $\odot(DJ'K')$□i.e. $\odot(KYZ)$ pass thought a fixed point A^* where is on $\odot(DJK)$. and A^* also is on $\odot(ABCK)$. Let A^*D intersect $\odot(ABC)$ at A''□then use **Reim's Theorem**□we get $AA'' \parallel BC$□acording to this character□we conclude A^* is the A-mixtilinear point . Done .

Wow□very short proof□Nice □

Little Gauss

#11 Jul 22, 2013, 1:08 pm

I really appreciated this beautiful problem. I gave this problem to this year's Korean IMO team members, and they found following amazing solutions. Hope that everyone can enjoy this!

The first solution is by Dong Ryul Kim (KOR2), who got 577777 in IMO 2012.

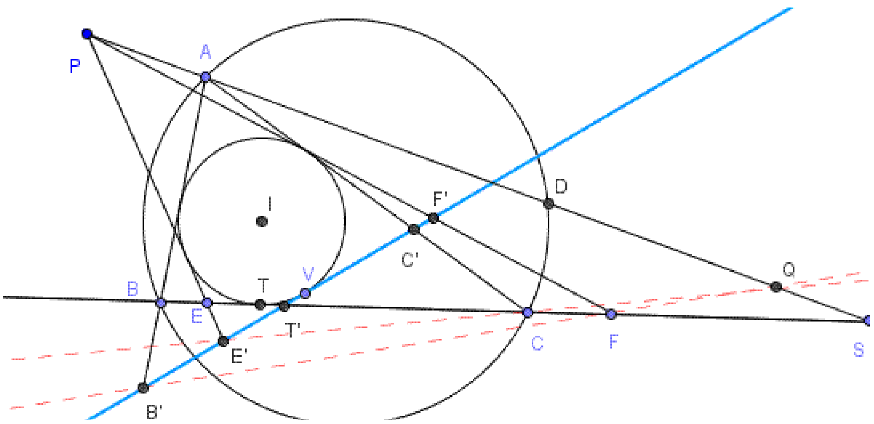
Let M be a mixtilinear point of triangle ABC, T be a tangent point of the incircle and BC, and A' be a point of reflection of A by the perpendicular bisector of BC. Let $S = PA \cap BC$.
It is well-known that M, T, A' is collinear. Since $AA' \parallel BC$, we can show that $TMDS$ is concyclic, so we have $TU \cdot US = MU \cdot UD$, where U is the intersection of MD and BC. Furthermore, $MU \cdot UD = BU \cdot UC$. Now the problem reduces to show $EU \cdot UF = TU \cdot US (= BU \cdot UC)$...(*).

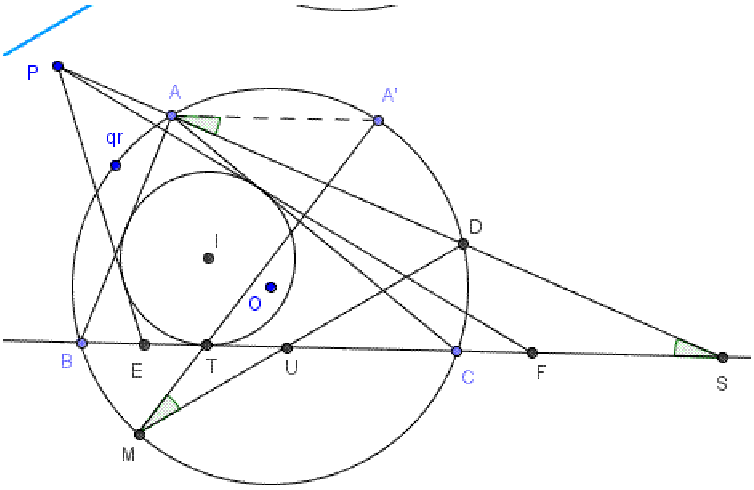
Here comes a fresh idea. Rotate the line BC slightly so that it still tangents to the incircle (I). Let this line be l, and its intersection with AB, AC, PE, PF be B', C', E', F'. Let T' be the intersection of BC and l. Then apply *Brianchon's theorem* on a hexagon $AB'E'PFC$. It follows that $AP, B'F, CE'$ are concurrent. Let this point be S' (denoted by Q in the above picture). By Menelaus' theorem on triangle $\triangle B'T'F$, we have $\frac{B'E'}{E'T'} \cdot \frac{T'C}{CF} \cdot \frac{FS'}{S'B'} = -1$.
Now take the limit as l approaches to BC. Maybe this part needs further verifications, but it seems almost certain that each point of the form x' converges to x and the length of each $x'y'$ converges to that of xy. As S' lies on PA and $B'F, CE'$ converges to the line BC, S' converges to S. Therefore we have the identity $\frac{BE}{ET} \cdot \frac{TC}{CF} \cdot \frac{FS}{SB} = -1$.

This is enough to show (*). Regard the line BC as the real line and take U as its origin. Denote the coordinate of the point X by x. Then we have $st = bc$ and need to show $ef = bc = st$. The above identity yields
$\frac{b-c}{e-t} \cdot \frac{t-c}{c-f} \cdot \frac{f-s}{s-b} = -1 \iff (b+c)(ef-st) + (s+t)(bc-ef) + (e+f)(st-bc) = 0$. From this we can easily conclude the proof.

I will post another solution later because of lack of time.

Attachments:





Luis González

#12 Jul 22, 2013, 11:40 pm • 1

Let D be the tangency point of the incircle (I) with BC. AX cuts BC at P. $\odot(DPK)$ passes through the tangency point M of the A-mixtilinear incircle with (O), see Tangent Circles+Centers Concyclic (the proof for the mixtilinear incircle is analogous).

By dual of Desargues theorem for the degenerate quadrilateral $ABDC$ circumscribed in the conic (I), the pencil $XB \mapsto XC, XY \mapsto XZ, XD \mapsto XA$ is involutive regardless of $X \Longrightarrow B \mapsto C, Y \mapsto Z, D \mapsto P$ is an involution on the line BC. The center R of this involution is then the intersection of BC with the radical axis MK of $\odot(KBC)$ and $\odot(KMDP)$. Thus $\overline{RY} \cdot \overline{RZ} = \overline{RD} \cdot \overline{RP} = \overline{RM} \cdot \overline{RK} \Longrightarrow M \in \odot(KYZ)$.

Quick Reply

High School Olympiads

Euler line and a circumcenter

Reply

Source: Own

SalaF
#1 Apr 25, 2015, 11:04 pm

Let ABC be a triangle with orthocenter H and circumcenter O. Let ℓ be a line perpendicular to OH which intersects the sides BC, CA, AB at X, Y, Z. Let $A_1B_1C_1$ be the triangle with sides AX, BY, CZ and let U be its circumcenter.
Prove that the lines AA_1, BB_1, CC_1 and HU are concurrent.

jayme
#2 May 2, 2015, 6:27 pm

Dear Mathlinkers,
any ideas?
Sincerely
Jean-Louis

BBeast
#3 May 2, 2015, 8:38 pm

The first part, about $AA1, BB1, CC1$ is obvious, because of Desargue's theorem for $\triangle A1B1C1$ and $\triangle ABC$.
Now we are left with proving HU passes through the center of perspectivity of $\triangle A1B1C1$ and $\triangle ABC$.

jayme
#4 May 6, 2015, 6:24 pm

Dear Mathlinkers,
yes the first part is a direct application of the Desargues theorem...
But the second part seems not to be so easy...
Any ideas?
Sincerely
Jean-Louis

VUThanhTung
#5 May 6, 2015, 6:54 pm

I tried the problem on computer and observed that the point of concurrence lies on the Kiepert hyperbola of $\triangle ABC$.

VUThanhTung
#6 May 6, 2015, 9:21 pm

Consider a more general scenario.

Given a triangle ABC and a fixed line d (we may assume d passes through the centroid G). Let l be a line perpendicular to d which intersects the sides BC, CA, AB at X, Y, Z. Let $A_1B_1C_1$ be the triangle with sides AX, BY, CZ.
Then by Desargue's theorem the lines AA_1, BB_1, CC_1 are concurrent at a point T.

I observe that the locus of T when l varies is always a circumhyperbola of $\triangle ABC$. This circumhyperbola always passes through the centroid G. If d is the Euler line, the hyperbola becomes the Kiepert hyperbola. Is the result is well-known?

Luis González
#7 May 6, 2015, 9:23 pm

> *VUThanhTung wrote:*
> I tried the problem on computer and observed that the point of concurrence lies on the Kiepert hyperbola of $\triangle ABC$.

This is easy to verify; the perspector of $\triangle ABC$ and $\triangle A_1B_1C_1$ is nothing but the tripole P of ℓ WRT $\triangle ABC$. Now it's known that $\ell \perp OH \Longleftrightarrow P$ is on the Kiepert hyperbola. The problem reduces then to prove the following: If P is a point on the Kiepert hyperbola then the circumcenter of its anticevian triangle, the orthocenter of $\triangle ABC$ and P are collinear.

Luis González
#8 May 6, 2015, 9:34 pm

> *VUThanhTung wrote:*
> Consider a more general scenario.
>
> Given a triangle ABC and a fixed line d (we may assume d passes through the centroid G). Let l be a line perpendicular to d which intersects the sides BC, CA, AB at X, Y, Z. Let $A_1B_1C_1$ be the triangle with sides AX, BY, CZ.
> Then by Desargue's theorem the lines AA_1, BB_1, CC_1 are concurrent at a point T.
>
> I observe that the locus of T when l varies is always a circumhyperbola of $\triangle ABC$. This circumhyperbola always passes through the centroid G. If d is the Euler line, the hyperbola becomes the Kiepert hyperbola. Is the result is well-known?

More general, if a line ℓ spins around a fixed point Q, then its tripole P WRT $\triangle ABC$ moves on conic through A, B, C. This is a known result, at least for me.

Let E, F be the intersections of PB, PC with AC, AB. As ℓ varies the series Y, Z are perspective and since $(A, C, E, Y) = -1$, and $(A, B, F, Z) = -1$, then the series E, F are also projective $\Longrightarrow$ pencils BE and CF are projective $\Longrightarrow P$ moves on a conic through B, C, which clearly also passes through A.

TelvCohl
#9 May 6, 2015, 9:42 pm

> *Luis González wrote:*
> More general, if a line ℓ spins around a fixed point Q, then its tripole P WRT $\triangle ABC$ moves on conic through A, B, C.

We can say more about this circumconic **:**

The locus of P is the isogonal conjugate (WRT $\triangle ABC$) of the tripolar of the isogonal conjugate of Q WRT $\triangle ABC$.

This post has been edited 1 time. Last edited by TelvCohl, Oct 1, 2015, 9:23 pm

TelvCohl
#10 Oct 1, 2015, 9:36 pm • 2

> *Luis González wrote:*
> The problem reduces then to prove the following: If P is a point on the Kiepert hyperbola then the circumcenter of its anticevian triangle, the orthocenter of $\triangle ABC$ and P are collinear.

Lemma 1 : Given a $\triangle ABC$ and a point P. Let $\triangle A^*B^*C^*$ be the circumcevian triangle of P WRT $\triangle ABC$. Let K, K^* be the symmedian point of $\triangle ABC, \triangle A^*B^*C^*$, respectively. Then K, P, K^* are collinear.

Proof : Let $T \equiv AK \cap \odot(ABC)$ and $T^* \equiv A^*K^* \cap \odot(A^*B^*C^*)$. Let the Lemoine axis of $\triangle ABC$ cuts BC, CA, AB at D, E, F, respectively and let the Lemoine axis of $\triangle A^*B^*C^*$ cuts B^*C^*, C^*A^*, A^*B^* at D^*, E^*, F^*, respectively. Let $A_1 \equiv BC \cap B^*C^*, B_1 \equiv CA \cap C^*A^*, C_1 \equiv AB \cap A^*B^*$ and let O be the circumcenter of $\triangle ABC$ ($\triangle A^*B^*C^*$).

Clearly, A_1, B_1, C_1 lie on the polar τ of P WRT $\odot(O)$. Since $ABTC$ and $A^*B^*T^*C^*$ are harmonic quadrilateral, so T, P, T^* are collinear $\Longrightarrow AK \cap A^*K^* \in \tau$. Similarly, we can prove $BK \cap B^*K^* \in \tau$ and $CK \cap C^*K^* \in \tau$. Since the tangent of $\odot(O)$ through B, C and AK are concurrent, so D lies on the polar of $AK \cap A^*K^*$ WRT $\odot(O)$. Similarly, we can prove D^* lies on the polar of $AK \cap A^*K^*$ WRT $\odot(O) \Longrightarrow D, P, D^*$ are collinear. Analogously, we can prove $P \in EE^*$ and $P \in FF^*$, so from Desargue theorem ($\triangle B_1EE^*$ and $\triangle C_1FF^*$) we get $\tau_: EF, E^*F^*$ are concurrent, hence their pole P, K, K^* WRT $\odot(O)$ are collinear.

Lemma 2 : Let P be a point on the Kiepert hyperbola of $\triangle ABC$. Let $\triangle P_aP_bP_c$ be the pedal triangle of P WRT $\triangle ABC$ and let $\triangle XYZ$ be the circumcevian triangle of P WRT $\triangle P_aP_bP_c$. Then P lies on the Kiepert hyperbola of $\triangle XYZ$.

Proof : Let Q be the isogonal conjugate of P WRT $\triangle ABC$. Let $\triangle Q_aQ_bQ_c, \triangle Q_AQ_BQ_C$ be the pedal triangle, circumcevian triangle of Q WRT $\triangle ABC$. Let R be the isogonal conjugate of Q WRT $\triangle Q_AQ_BQ_C$. Since $\triangle Q_AQ_BQ_C \cup R \sim \triangle Q_aQ_bQ_c \cup Q \cong \triangle XYZ \cup P$, so it suffices to prove Q lies on the Brocard axis of $\triangle Q_AQ_BQ_C$. Let O be the circumcenter of $\triangle ABC$. Let K, K_Q be the symmedian point of $\triangle ABC, \triangle Q_AQ_BQ_C$, respectively. From **Lemma 1** we get K, Q, K_Q are collinear, so notice Q lies on the Brocard axis OK of $\triangle ABC$ we conclude that $Q \in OK_Q$ (Brocard axis of $\triangle Q_AQ_BQ_C$).

Remark : There is a stronger result of **Lemma 2 :** If P is the Kiepert perspector of $\triangle ABC$ with angle θ, then P is the Kiepert perspector of $\triangle XYZ$ with angle $-\theta$ (but we don't need this stronger result in the proof).

Now we recall two well-known properties about conic as following **:**

Property 1 : Given a $\triangle ABC$ and two points P, Q. Let $\triangle P_aP_bP_c$ be the anticevian triangle of P WRT $\triangle ABC$ and let $\triangle Q_aQ_bQ_c$ be the anticevian triangle of Q WRT $\triangle ABC$. Then $P, Q, P_a, P_b, P_c, Q_a, Q_b, Q_c$ lie on a conic.

Property 2 : Given a $\triangle ABC$ and a point P. Let I, I_a, I_b, I_c be the incenter, A-excenter, B-excenter, C-excenter of $\triangle ABC$, respectively. Let $\mathcal{H}$ be a conic passing through I, I_a, I_b, I_c. Then the polar of P WRT $\mathcal{H}$ passes through P^* where P^* is the isogonal conjugate of P WRT $\triangle ABC$.

From **Property 1** and **Property 2** we get the following lemma **:**

Lemma 3 : Let $\mathcal{H}$ be a circum-rectangular hyperbola of $\triangle ABC$ and let P, Q be the points on $\mathcal{H}$. Let $\triangle DEF$ be the cevian triangle of Q WRT $\triangle ABC$ and let P^* be the isogonal conjugate of P WRT $\triangle DEF$. Then PP^* is tangent to $\mathcal{H}$.

Proof : Let I, I_a, I_b, I_c be the incenter, A-excenter, B-excenter, C-excenter of $\triangle DEF$, respectively. From **Property 1** we get $A, B, C, Q, I, I_a, I_b, I_c$ lie on a conic $\mathcal{C}$, but notice I is the orthocenter of $\triangle I_aI_bI_c$ we get $\mathcal{C}$ is a rectangular hyperbola $\Longrightarrow \mathcal{C} \equiv \mathcal{H}$, so from **Property 2** we conclude that P^* lies on the polar of P WRT $\mathcal{H}$. i.e. PP^* is tangent to $\mathcal{H}$

Back to the main problem :

Let H be the orthocenter of $\triangle ABC$. Let $\triangle XYZ$ be the anticevian triangle of P WRT $\triangle ABC$ and J be the circumcenter of $\triangle XYZ$. Let $\triangle A_1B_1C_1$ be the pedal triangle of P WRT $\triangle ABC$. Perform the Inversion with center P and denote V^* as the image of V (V is an arbitrary point). Obviously, $\triangle A^*B^*C^*$ is the pedal triangle of P WRT $\triangle A_1^*B_1^*C_1^*$ and $\triangle X^*Y^*Z^*$ is the pedal triangle of P WRT the medial triangle $\triangle A_2^*B_2^*C_2^*$ of $\triangle A_1^*B_1^*C_1^*$, so $PJ \equiv PJ^*$ passes through the isogonal conjugate Q^* of P WRT $\triangle A_2^*B_2^*C_2^*$.

Let $\triangle DEF$ be the anticomplementary triangle of $\triangle ABC$. Let $\triangle A_3^*B_3^*C_3^*$ be the cevian triangle of P WRT $\triangle A_1^*B_1^*C_1^*$. From $EF \parallel BC$ and $A \in EF \Longrightarrow$ the image of the line EF under the Inversion is the circle with diameter PA_3^*. Similarly, we can prove $\odot(PB_3^*), \odot(PC_3^*)$ is the image of the line FD, DE under the Inversion, respectively, so $\triangle D^*E^*F^*$ is the pedal triangle of P WRT $\triangle A_3^*B_3^*C_3^*$. Since H is the circumcenter of $\triangle DEF$, so we get $PH \equiv PH^*$ passes through the isogonal conjugate R^* of P WRT $\triangle A_3^*B_3^*C_3^*$.

Let $G^* \equiv A_1^*A_2^* \cap B_1^*B_2^* \cap C_1^*C_2^*$ be the Centroid of $\triangle A_1^*B_1^*C_1^*$. From **Lemma 2** we know P lies on the Kiepert hyperbola of $\triangle A_1^*B_1^*C_1^*$, so $A_1^*, B_1^*, C_1^*, G^*_:P$ lie on a rectangular hyperbola $\mathcal{K}$, hence from **Lemma 3** we conclude that $P, Q^*_: R^*$ lie on the tangent of $\mathcal{K}$ through $P \Longrightarrow P, J^*_: H^*$ are collinear $\Longrightarrow P, J, H$ are collinear.

Luis González

#11 Oct 1, 2015, 11:01 pm

> *Luis González wrote:*
>
> The problem reduces then to prove the following: If P is a point on the Kiepert hyperbola then the circumcenter of its anticevian triangle, the orthocenter of $\triangle ABC$ and P are collinear.

Javier García Capitán (pacoga in AoPS) has confirmed a generalization of this result. Unfortunately, we don't have a synthetic proof yet.

Generalization: P is a point on the plane of $\triangle ABC$ and $\triangle A'B'C'$ is the anticevian triángle of P, whose circumcenter is J.

Then PJ passes through the orthocenter of $\triangle ABC \iff$ either P is on Kiepert hyperbola or polar circle of $\triangle ABC$.

Quick Reply

User Profile

pacoga

Joined: September 8, 2005

Activity

Last Visited: Jul 27, 2015, 3:47 am

Total Posts: 44 (Click to view posts)
36 High School Olympia..
5 High School Math
2 College Math
1 LaTeX and Asymptote

Blog: pacoga's blog

Given: 1

Received: 5

High School Olympiads

Nice concurrency

Locked

Blacklord

#1 Sep 30, 2015, 11:13 pm

Let w be the circumcircle of the triangle ABC. T is the intersection of the lines which are tangent to w at B and C.AT cuts BC at M.P,Q on AB,AC are such that MP is parallel to AC and MQ is parallel to AB.
a. Prove that BCQP is cyclic.
b. Let A' be the circumcenter of BCQP.B' and C' defines similarly.prove that AA' ,BB' and CC' concur.

Luis González

#2 Oct 1, 2015, 3:06 am

This is a problem from China TST 2005. See Cyclic points and concurrency [1st Lemoine circle] or Circumcicle, tangents and parallel lines for various solutions. Topic locked.

High School Olympiads

Cyclic points and concurrency [1st Lemoine circle]

Reply

Source: China TST 2005

shobber

#1 Jun 27, 2006, 1:46 pm

Let ω be the circumcircle of acute triangle ABC. Two tangents of ω from B and C intersect at P, AP and BC intersect at D. Point E, F are on AC and AB such that $DE \parallel BA$ and $DF \parallel CA$.
(1) Prove that F, B, C, E are concyclic.

(2) Denote A_1 the centre of the circle passing through F, B, C, E. B_1, C_1 are difined similarly. Prove that AA_1, BB_1, CC_1 are concurrent.

yetti

#2 Aug 5, 2006, 11:33 pm

AP is the A-symmedian of the triangle $\triangle ABC$. Let O be the triangle circumcenter and K the symmedian point.

(1) AEDF is a parallelogram, hence its diagonals AD, EF cut each other in half. Since the midpoint of EF lies on the A-symmedian AD, EF is antiparallel to BC with respect to the angle $\angle A$, wich means that the points B, C, E, F are concyclic.

(2) Let parallels to the B-, C-symmedians BK, CK through the foot $D \in BC$ of the A-symmedian $AK \equiv AD \equiv AP$ meet the rays (AB, (AC at B', C'. The triangles $\triangle AB'C' \sim \triangle ABC$ are centrally similar with the similarity center A and D is the symmedian point of the triangle $\triangle AB'C'$. It immediately follows that the circumcircle (A_1) of the quadrilateral BCEF is the 1st Lemoine circle of the triangle $\triangle AB'C'$ centered at the midpoint X' of the segment DO', where O' is the circumcenter of this triangle. Therefore, AA_1 intersects the segment KO of the original triangle $\triangle ABC$ also at its midpoint X, the center of the 1st Lemoine circle of the original triangle. Simiarly, BB_1, CC_1 cut KO at its midpoint X, hence all three are concurrent at X.

alpha-beta

#3 Feb 5, 2009, 3:14 pm

can someone define 1st Lemoine circle or give some links?

mihai miculita

#4 Feb 5, 2009, 4:05 pm

The three parallels to the sides of a triangle ABC through the Lemoine point of the triangle ABC,
determine on the sides of triangle ABC, 6 concyclic points.
The circle of the 6 points is the 1-st Lemoine circle of triangle ABC.

Sardor

#5 May 27, 2014, 9:52 am

What's Lamoine point?
Please help me .

Quick Reply

High School Olympiads

Circumcicle, tangents and parallel lines.

Reply

Source: Italian Prelmo 2012

ctumeo

#1 Jun 3, 2012, 4:06 am

Let Z the circumcicle to acutangle triangle ABC.
The tangents to Z through B and C meet at P.
AP and BC meet at D.
Point E on AC and point F on AB are so that $DE \parallel BA$ and $DF \parallel CA$.

1) Prove that F, B, C, E lies on the same circle.

2) Let A_1 the center of circle through F, B, C, E and B_1, C_1 similarly definited.
Prove that AA_1, BB_1, CC_1 are concurrent.

Thanks

Luis González

#2 Jun 3, 2012, 6:28 am • 1

Posted before at Cyclic points and concurrency [1st Lemoine circle].

1) Since $\angle DEC = \angle DFB = \angle A$ and AD is the A-symmedian of $\triangle ABC$, we have that $\frac{DE}{DF} = \frac{\text{dist}(D,AC)}{\text{dist}(D,AB)} = \frac{AC}{AB} \Longrightarrow \triangle ABC \sim \triangle DFE$ are inversely similar by SAS $\Longrightarrow \angle DEF = \angle C \Longrightarrow$ $\angle FEC = \angle C + \angle A \Longrightarrow B, C, E, F$ are concyclic.

2) Let DF cut PC at M. Then $\angle FMC = \angle B \Longrightarrow M$ lies on (A_1). Similarly, $N \equiv DE \cap PB$ lies on (A_1). If MN cuts AC, AB at C^*, B^*, then C^*D is the C^* symmedian of $\triangle AB^*C^*$, since $\angle DCA = \angle DMB^*$ and $\triangle DMC \sim \triangle C^*B^*A$ are inversely similar. Likewise, B^*D is the B^* symmedian of $\triangle AB^*C^*$, thus D is symmedian point of $\triangle AB^*C^* \Longrightarrow (A_1)$ becomes the 1st Lemoine circle of $\triangle AB^*C^*$. Hence, A is exsimilicenter of (A_1) and the 1st Lemoine circle (X_{182}) of $\triangle ABC$, i.e. AA_1 passes through X_{182}. Analogously, BB_1 and CC_1 pass through X_{182}.

r1234

#3 Jun 3, 2012, 1:41 pm • 1

1) For the 1st part $AEDF$ is a parallelogram.So AD is the A-median of $\triangle AEF$.But AD is the A-symmedian of $\triangle ABC$.So $EFCB$ is cyclic.

2) First let me prove that the centre of Lemoine circle is the midpoint of the segment OS where O, S are circumcenter, Symmedian point of $\triangle ABC$ respectively.For this the lines through S parallel to BC meets AB, AC at P_c, P_b respectively.Similarly define Q_c, Q_a, R_a, R_b.Easy to check that $P_b, P_c, Q_a, Q_c, R_a, R_b$ lie on a circle and this is the so-called Lemoine circle. Now let $\triangle A_1B_1C_1$ be the tangential triangle of $\triangle ABC$.Note that $B_1C_1 \parallel P_bP_c$.Let $P_b' = B_1C_1 \cap SP_b, P_c' = SP_c \cap B_1C_1$.Since SA bisects P_bP_c so A is the midpoint of $P_b'P_c'$.Similarly define Q_a', Q_c', R_a', R_b'. Now note that O is circumcenter of the circle $P_b'P_c'Q_a'$....Hence S is the center of homothety of $(P_bP_cQ_a)$ and $(P_b'P_c'Q_a')$ with the ratio $2:1$. Hence center of the lemoin circle is the midpoint of OS.

Now back to the main problem

Note that circles $P_bP_cQ_cR_b$ and $EFBC$ are homothetic with the center A.So AA_1 passes through the center of the Lemoin circle.Hence AA_1, BB_1, CC_1 concur at the center of the Lemoin circle.

Quick Reply

High School Olympiads

Geometry involving Casey's theorem

AmirAlison

#1 Sep 30, 2015, 7:45 pm

Let O be the circumcentre of ABC.Suppose there are 2 circles u and v, inscribed in angle BAC with the following conditioins: u externally touches arc BOC of the circumcircle of triangle BOC and w internally touches the circumcircle of ABC. Prove that the radius of w is twice bigger than the radius of u(I would like to see the solution with Casey's theorem)

Luis González

#2 Sep 30, 2015, 10:58 pm

Let I be the incenter of $\triangle ABC.$ If the A-mixtilinear incircle w touches AC,AB at $Y,Z,$ then I is midpoint of $\overline{YZ}$ (well-known). So if u touches AC,AB at $U,V,$ it suffices to prove that UV is the A-midline of $\triangle AYZ,$ in other words that UV is perpendicular bisector of $\overline{AI}.$ See the problems middle and Prove that KL bisects BI.

High School Olympiads

middle

Reply

unt

#1 Jan 18, 2013, 7:24 pm

Let O circumcenter of $\triangle ABC$. Circle ω tangents side AB, BC at point K and L respectively and also tangent circumcircle of $\triangle AOC$.
Prove that line KL bisect segment BI, where I is incenter of $\triangle ABC$.

Luis González

#2 Jan 25, 2013, 11:14 am • 2

Redefine LK as the perpendicular bisector of $\overline{BI}$, thus we prove that the circle tangent to AB, BC through K, L touches the circle $\odot(AOC)$.

Let $\odot(AOC)$ cut BC again at P. $\angle APC = \angle AOC = 2\angle ABC \Longrightarrow \triangle PAB$ is P-isosceles $\Longrightarrow PO$ is perpendicular bisector of $\overline{BA}$, cutting the arc BA of (O) at its midpoint D. Since $DA = DB = DI$, then $D \in KL \Longrightarrow$ C-excenter D of $\triangle APC$ is on KL. By Sawayama's lemma, we conclude that the circle tangent to AB, BC through K, L is a Thebault circle of the cevian AB of $\triangle APC$ externally tangent to its circumcircle $\odot(AOC)$.

Quick Reply

High School Olympiads

Prove that KL bisects BI

Reply

trunglqd91
#1 Mar 29, 2015, 10:34 am

Let ABC is a triangle with circumcircle (O). Give a circle (W) tangent AB, BC at K, L, resp, and tangent the circumcircle (AOC). Prove that KL bisects BI with I is the centre of incircle (ABC)

TelvCohl
#2 Mar 29, 2015, 11:30 am

My solution:

Let $M \equiv BI \cap KL$ be the midpoint of KL .
Let A^*, C^* be the midpoint of BC, BA, respectively .
Let I_b be the B-excenter of $\triangle ABC$ and O^* be the projection of B on AC .
Let L^*, K^* be the projection of I_b on BA, BC, respectively .

Let Ψ be the composition of Inversion $\mathbf{I}(\sqrt{\frac{1}{2}BA \cdot BC})$ and Reflection $\mathbf{R}(BI)$.

Since $A \longleftrightarrow A^*, C \longleftrightarrow C^*, O \longleftrightarrow O^*$ under Ψ ,
so the image of $\odot(AOC)$ is the 9-point circle $\odot(A^*O^*C^*)$ of $\triangle ABC$ under Ψ ,
hence the image of $\odot(W)$ is B-excircle $\odot(I_b)$ under $\Psi \Longrightarrow L \longleftrightarrow L^*, K \longleftrightarrow K^*$ under Ψ ,
so we get $BL \cdot BL^* = BK \cdot BK^* = \frac{1}{2}BA \cdot BC$.

Since $Rt\triangle BML \sim Rt\triangle BL^*I_b$,
so $BM \cdot BI_b = BL \cdot BL^* = \frac{1}{2}BA \cdot BC$,
hence combine with $BI \cdot BI_b = BA \cdot BC$ we get M is the midpoint of BI ,
so KL is the perpendicular bisector of BI .

Q.E.D

andria
#3 Mar 29, 2015, 2:34 pm

My solution: let the parallel line from I to KL meet BA, BC at S, T we want to prove that KL is midline of $\triangle BTS$ note that S, T are tangency points of mixtilinear circle in front of A. apply an inversion with center B and power $\sqrt{BC \cdot BA}$ we know that $C \longleftrightarrow A$ under the inversion and inverse of mixtilinear circle is excircle of $\triangle ABC$ in front of $\angle B$ that touches BA, BC at S', T' and since $ac = 2Rh_B$ inverse of O is a reflection of A In the line BC call it O' let a line parallel to BC from O' intersect BA, BC at R, P so we get that $\odot ACO'$ is a nine point circle of $\triangle BRP$ also the inverse of W is a circle that it is tangent to BA, BC at K', L' and tangent to $\odot\triangle AO'C$; call it W' but note that from a well known fact the nine point circle of a triangle is tangent to its excircle from this fact we get that W' is excircle of $\triangle BRP$ now because $A'C'$ is midline of $\triangle BRP$ we get that T', S' are midpoints of BL', BK' so K, L are midpoints of BS, BT before the inversion and KL bisects BI.

Luis González
#4 Apr 10, 2015, 3:42 am

See http://www.artofproblemsolving.com/community/c6h517026 for a solution without inversion.

jayme
#5 May 10, 2015, 5:56 pm

Dear Mathlinkers

Dear Mathlinkers,
a direct proof... based on the key circle (CMI)...

1. O' the midpoint of the arc BC which doesn't contain O of (AOC)
M the second point of intersection de IO' with (O) ; it the contact point of (AOX) and (W)
(1) the circle (CMI)
X the point of intersection of KL and AI.
2. according to
http://jl.ayme.pagesperso-orange.fr/Docs/Un%20remarquable%20resultat%20de%20Vladimir%20Protassov.pdf p. 2-4
(1) goes through L and by angle chasing also through X
3. according to the Reim's theorem, JL // AB
4. in the same way, JK // BC
5. the parallelogram BLIK is a rhombus and we are done.

Sincerely
Jean-Louis

Quick Reply

High School Olympiads

Locus of Miquel point

Reply

Source: Own (Inspired from rodinos)

TelvCohl

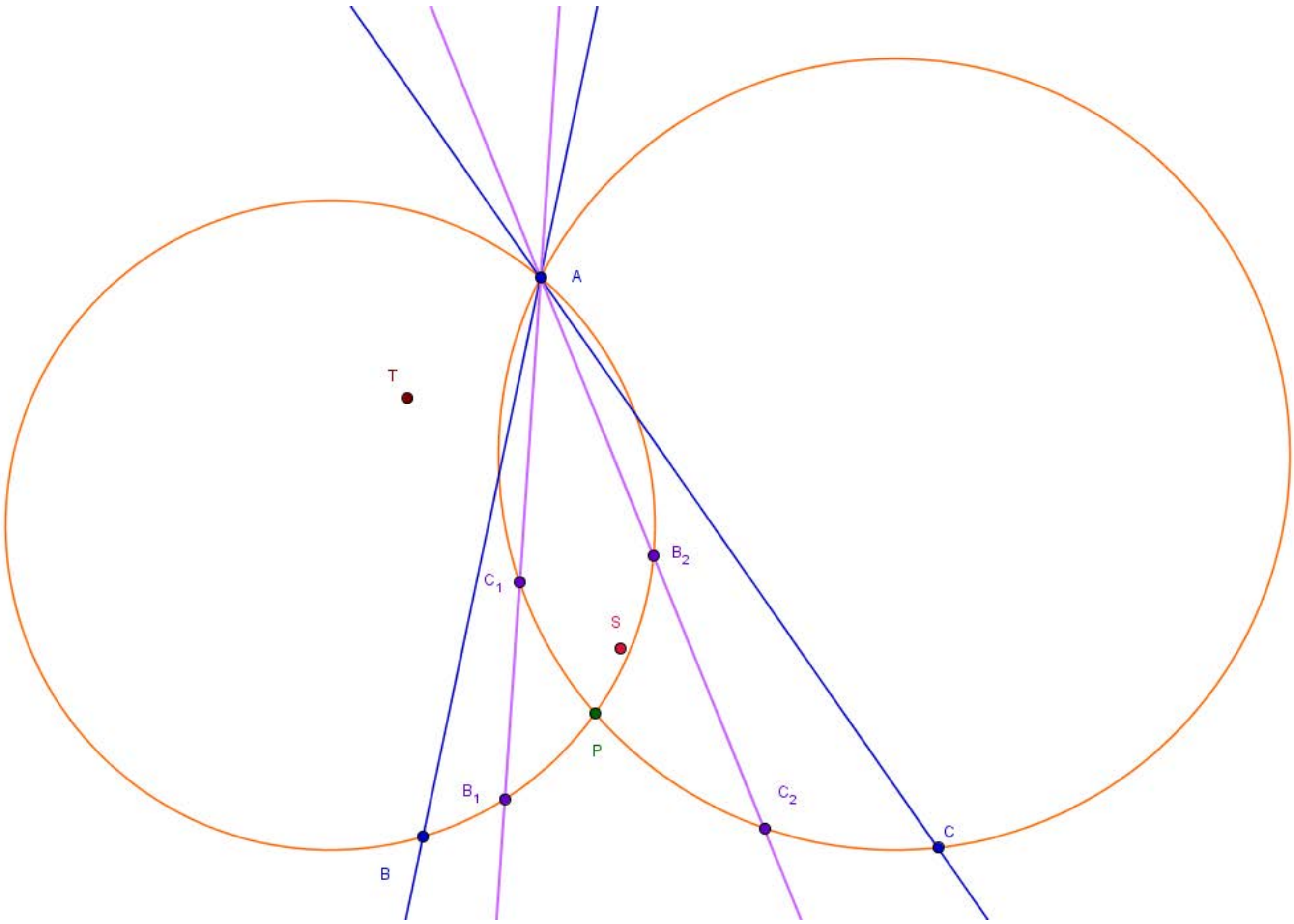
#1 Dec 11, 2014, 3:14 pm • 1

Let ℓ_1, ℓ_2 be two lines through A .
Let P be a point satisfy $\angle CPA = \angle APB$
Let $B_1 = \ell_1 \cap (PAB), C_1 = \ell_1 \cap (PAC), B_2 = \ell_2 \cap (PAB), C_2 = \ell_2 \cap (PAC)$.
Let T be the Miquel point of the complete quadrilateral $\{B_1C_1, B_2C_2, B_1C_2, B_2C_1\}$.
Let S be the Miquel point of the complete quadrilateral $\{B_1B_2, C_1C_2, B_1C_2, B_2C_1\}$.

(1) Find the locus of T when ℓ_2 varies
(2) Find the locus of S when ℓ_2 varies

Reference : Miquel point on the NPC

Attachments:



This post has been edited 1 time. Last edited by TelvCohl, Sep 30, 2015, 4:06 pm

Luis González

#2 Sep 30, 2015, 11:34 am • 2

The condition $\angle CPA = \angle APB$ is not necessary. We can consider the following general configuration: Two fixed circles Γ_1, Γ_2 meet at A, P and two lines ℓ_1, ℓ_2 pass through A. ℓ_1, ℓ_2 cut Γ_1 again at B_1, B_2 and ℓ_1, ℓ_2 cut Γ_2 again at C_1, C_2. Then the loci of the Miquel points T, S of $\{B_1C_1, B_2C_2, B_1C_2, B_2C_1\}$ and $\{B_1B_2, C_1C_2, B_1C_2, B_2C_1\}$, when ℓ_2 varies, are two distinct circles.

First off, we'll prove the following lemma for quadrilaterals:

Lemma: $ABCD$ is arbitrary quadrilateral and $P \equiv AC \cap BD$. $\odot(PAB), \odot(PCD)$ meet again at Q and $\odot(PAD)$, $\odot(PBC)$ meet again at R. PR cuts $\odot(PCD), \odot(PAB)$ again at U, V. Then $QAVB, QCUD$ are harmonic and R is the midpoint of $\overline{UV}$.

Invert with center P denoting inverse points with primes. $\odot(PAB), \odot(PBC), \odot(PCD), \odot(PDA)$ go to $A'B', B'C', C'D', D'A'$, resp $\Longrightarrow P' \equiv A'B' \cap C'D', R' \equiv B'C' \cap D'A'$ and $P'R'$ cuts $C'D', A'B'$ at U', V'. From the complete quadrangle $A'B'C'D'$, we get $P(B', C', Q', R') = P(B, A, Q, V) = -1 \Longrightarrow QAVB$ is harmonic and similarly $QCUD$ is also harmonic. Moreover $(R', U', V', P) = (R, U, V, \infty) = -1 \Longrightarrow R$ is midpoint of $\overline{UV}$.

Back to the problem. The locus of T can be figured out easily inverting with center A. Denoting inverse points with primes, Γ_1, Γ_2 go to lines $B_1{}'B_2{}', C_1{}'C_2{}'$ meeting at P' and $\odot(AC_1B_2), \odot(AB_1C_2)$ go to lines $C_1{}'B_2{}', B_1{}'C_2{}'$ meeting at T'. Now from the complete quadrilateral $B_1{}'C_1{}'B_2{}'C_2{}'$, it follows that $T'P'$ cuts $\ell_1 \equiv B_1{}'C_1{}'$ are the harmonic conjugate of A WRT $B_1{}', C_1{}' \Longrightarrow T'P'$ is fixed, consequently the locus of T is a circle ω_T passing through A, P.

Let $X \equiv B_1B_2 \cap C_1C_2$. Since P is the Miquel point of AC_1XB_2, then $P \equiv \odot(XB_1C_1) \cap \odot(XB_2C_2)$. If XS cuts $\odot(XB_1C_1)$ and $\odot(XB_2C_2)$ again at Y and Z, then from the previous lemma for $B_1C_1B_2C_2$, it follows that PB_1YC_1 is harmonic and S is midpoint of $\overline{YZ}$. Hence Y is fixed and since $\angle YZP = \angle XC_2P \equiv \angle C_1C_2P = \text{const}$, then Z moves on a fixed circle through $P, Y \Longrightarrow$ locus of S is the circle ω_S image of $\odot(ZPY)$ under homothety $\mathbf{H}(Y, \frac{1}{2})$.

Quick Reply

High School Olympiads

Prove that RP=RQ

Reply

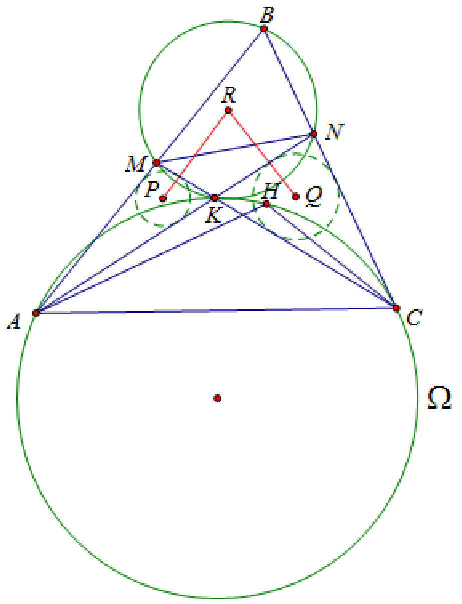
Source: OWN

LeVietAn

#1 Sep 28, 2015, 6:26 pm

Dear Mathlinkers,
Given an acute triangle ABC with $AB > BC$ and H is orthocenter. Let Ω is circumcircle of triangle ACH. Let K is midpoint of the arc AHC of Ω. Let $AB \cap CK = M, BC \cap AK = N$. Let P, Q are incenter of the triangles AMK, CNK, resp. If R is circumcenter of triangle BMN then prove that $RP = RQ$.

Attachments:



Luis González

#2 Sep 29, 2015, 11:43 am

Let E and Y be the midpoints of $\overline{AC}$ and the arc AC of $\odot(ABC)$. Clearly K is the reflection of Y on E and $BMKN$ is cyclic due to $\angle(KA, KC) = \angle(HA, HC) = \angle(BC, BA)$. If PKQ (parallel to AC) cuts BY at V, we get $\angle NKQ = \angle KAC = \frac{1}{2}\angle ABC = \angle NBV \Longrightarrow V \in \odot(BMKN)$.

Let S be the midpoint of KB. Thus internal bisectors AP, CQ of $\angle MAK, \angle NCK$ intersect at a point J on Newton line ES of the cyclic $BMKN$ (well-known). Hence since ES is the K-midline of $\triangle KBY \Longrightarrow JE$ passes through the midpoint U of VK. But as $PQ \parallel AC$, then JE is also J-median of $\triangle JPQ$, i.e. U is also midpoint of $PQ \Longrightarrow P, Q$ are isotomic points WRT $V, K \Longrightarrow RU$ is also perpendicular bisector of $\overline{PQ} \Longrightarrow RP = RQ$.

Quick Reply

High School Olympiads

ABCD square

Locked

Source: "Mathematical Excalibur" Magazine

leonardg

#1 Sep 29, 2015, 3:03 am

A splendor...

Attachments:

Problem 5. Let $ABCD$ be a square. Find the locus of points P in the plane, different from A, B, C, D such that

$$\angle APB + \angle CPD = 180^\circ.$$

Luis González

#2 Sep 29, 2015, 4:59 am

Posted many times before, e.g. Locus of point P, locus of P: APB+CPD=180 (ItaMO 2012), The locus of P with supplementary angles condition (Baltic Way 2001) and elsewhere.

High School Olympiads

Locus of point P

Locked

mitrov

#1 Apr 25, 2015, 11:16 pm

Let $ABCD$ be a square. Determine locus of point P distinct from A, B, C and D such that $\angle APB+\angle CPD=180$

Luis González

#2 Apr 25, 2015, 11:36 pm • 1

Discussed before at http://www.artofproblemsolving.com/community/c6h378178.

In general, for any quadrilateral $ABCD$, we have $\angle APB=\angle CPD \pmod \pi) \Longleftrightarrow$ pedal quadrilateral of P WRT $ABCD$ is cyclic $\Longleftrightarrow$ P is focus of a conic inscribed in $ABCD$ $\Longleftrightarrow$ P is on isoptic cubic of $ABCD$.

High School Olympiads

locus of P: APB+CPD=180

Reply

Source: ItaMO 2012, P5

Sayan
#1 May 21, 2012, 1:56 pm

$ABCD$ is a square. Describe the locus of points P, different from A, B, C, D, on that plane for which

$$\widehat{APB} + \widehat{CPD} = 180^\circ$$

Mathlover20
#2 May 21, 2012, 2:21 pm

> Sayan wrote:
> $ABCD$ is a square. Describe the locus of points P, different from A, B, C, D, on that plane for which
>
> $$\widehat{APB} + \widehat{CPD} = 180^\circ$$

Is it not the diagonal AC?

mavropnevma
#3 May 21, 2012, 3:10 pm

And why not also the diagonal BD? Indeed, by simple analytical geometry, the locus is made by the (open) diagonals (AC) and (BD).

EDIT. I stand corrected. The small arcs AB and CD also satisfy. In my analytical solution, this was the case when the two circles (for the arcs subtended from P) did coincide - and I wrongly dismissed it as a degenerate case, but in fact it contributes to the locus.

SCP
#4 May 22, 2012, 2:38 am

To have all points: points on circumcircle of $ABCD$ satisfy too on arcs AB, CD and the diagonals satisfy.
Now it is easy that moving the points parallel to AB gives a contradiction.

Hence we have proven tese are all.

Simo_the_Wolf
#5 May 22, 2012, 5:42 am

> SCP wrote:
> Now it is easy that moving the points parallel to AB gives a contradiction.

How? maybe in the exterior of the square is easier, but how you do it in the interior?

leader

#6 Aug 2, 2012, 5:00 pm

note that the circumcircles of CPD and APB are congruent note that if such circles intersect inside the square the intersections are on the diagonals and each diagonal point satisfies the given property
and if they intersect outside they have to intersect inside circle with diameter AB or CD so one of the given angles is obtuse but this cannot happen if the circles are different but congruent because they intersect outside the part of the 'small' part of the plane between the parallel lines BC and AD(by small i mean the one with the square in it) but the circles with diameter AB and CD are entirely inside that part. so the only left possibility is that the circles coincide which gives us the small arches AB and CD of circle $ABCD$.

Bob28

#7 Apr 26, 2013, 4:13 am

Can anyone show an analytical solution?

epsilonist

#8 Apr 26, 2013, 8:29 am

Co-ordinate-bash solution is not that bad.
First, we observe that if Γ_1 and Γ_2 are closed disks with diameters AB and CD, respectively, then $P \in \Gamma_1 \cup \Gamma_2$. Otherwise, the sum $\angle APB + \angle CPD$ would be less than π.

If L be the locus for the case $P \in \Gamma_1$, then the reflection of L across the line $y = \dfrac{1}{2}$ will be the locus for the other case.
Thus, we might assume first that $P \in \Gamma_1$
Now we let A, B, C, D have the co-ordinates $(0,1)$, $(1,1)$, $(1,0)$, $(0,0)$, respectively. Also, let $P = (x, y)$.
Case 1: $y \geq 1$
Reflect P over the line $y = \dfrac{1}{2}$ to a new point P'. By symmetry, we conclude that $\angle APB = \angle CP'D$.
Thus, $CPDP'$ is cyclic. (Note that it is not hard to see that the quadrilateral is not self-intersecting!)
So now, $y(y-1) = x(1-x)$ which means $\left(x - \dfrac{1}{2}\right)^2 + \left(y - \dfrac{1}{2}\right)^2 = \left(\dfrac{1}{\sqrt{2}}\right)^2$.
Case 2: $y < 1$
Define $P' := (x, y-1)$. Again, we conclude that $\angle APB = \angle CP'D$.
Thus, $CPDP'$ is cyclic. (Note that it is not hard to see that the quadrilateral is not self-intersecting!)
So now, $y(1-y) = x(1-x)$ which means $x = y$ or $x + y = 1$.

Hence, the nice locus follows!
The locus is the union of the arcs AB and CD of the circumcircle and the two diagonals of the square, except for those four points.

A nice problem! Is this Italian? The kids there must have been delighted solving this problem. :)

Best,
Epsie.

Bob28

#9 May 1, 2013, 8:37 pm

> *epsilonist wrote:*
> So now, $y(y-1) = x(1-x)$

Why?

Quick Reply

High School Olympiads

The locus of P with supplementary angles condition

Reply

Source: Baltic Way 2001

WakeUp

#1 Nov 18, 2010, 1:19 am

Given a rhombus $ABCD$, find the locus of the points P lying inside the rhombus and satisfying $\angle APD + \angle BPC = 180^\circ$.

yetti

#2 Nov 18, 2010, 2:29 am

Let $ABCD$ be just a parallelogram and $\angle APD + \angle CPB = 180^\circ \Longrightarrow$ circumcircles $(O_1), (O_3)$ of $\triangle APD, \triangle CPB$ are congruent. Translating $\triangle APD$ by $\overrightarrow{AB}$ into a $\triangle CP'B$ creates a cyclic quadrilateral $PBP'C$ and parallelogram $PABP'$ $\Longrightarrow$ circumcircles $(O_2), (O_3)$ of $\triangle BPA, \triangle CPB$ are also congruent. It follows that $\angle PCD = \angle DAP$, which means that isogonal conjugate P^* of P WRT $\triangle ACD$ is on perpendicular bisector of AC. As a result, the locus of P is a rectangular circum-hyperbola of the parallelogram $ABCD$, centered at its diagonal intersection E. If $ABCD$ is a rhombus, this hyperbola degenerates to its diagonals AC, BD.

Quick Reply

High School Olympiads

prove that the lines are concurrent

Reply

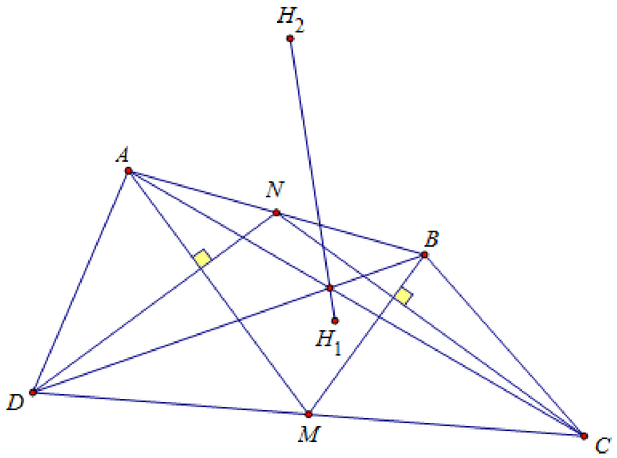
Source: OWN

LeVietAn

#1 Sep 28, 2015, 6:29 pm

Dear Mathlinkers,
Let convex quadrilateral $ABCD$ and M, N respectively midpoint of AB, CD such that $AM \perp DN, BM \perp CN$. Let H_1, H_2 be the orthocenter of triangle ABM, CDN, resp.
Prove that AC, BD and H_1H_2 are concurrent.

Attachments:



Luis González

#2 Sep 29, 2015, 4:48 am

General lemma: D is arbitrary point on the plane of $\triangle ABC$ and P is a point on the parallel from D to the A-median AM of $\triangle ABC$. Parallels from M to DB, DC cut PB, PC at Y, Z, resp. Then AM bisects $\overline{YZ}$.

When P varies, the pencils BP, CP are projective inducing a proyectivity $Y \mapsto Z$ between the lines MY, MZ. Since M is double when $P \equiv M$ and the point at infinity of MY goes to the point at infinity of MZ when $P \equiv D$, it follows that the series Y, Z are then similar $\Longrightarrow YZ$ has fixed direction, thus it suffices to prove that MA bisects YZ for a particular P.

Assume the case when $D \in YZ$ and let $E \equiv YZ \cap BC$. By Pappus theorem for $DBZMYC$ we deduce that $BZ \parallel CY$ and since $M(Y, Z, D, E) = -1 \Longrightarrow BZ \parallel CY \parallel DP \parallel AM \Longrightarrow AM$ is median of the trapezoid $BCYZ$ bisecting YZ, as desired.

When $A \in YZ$, we get the following corollary: $ABCD$ is quadrilateral and M, N are the midpoints of CD, AB. H is a point such that $HD \parallel MB$ and $HC \parallel MA$. If $P \equiv AC \cap DB$, then $HP \parallel MN$.$\blacksquare$

Back to the problem. From the previous corollary it immediately follows that H_1, H_2 and $AC \cap BD$ are collinear and moreover $\overline{PH_1H_2} \parallel MN$.

Quick Reply

High School Olympiads

X2, X5, X13, X14, X110

Source: Own

TelvCohl

#1 Mar 10, 2015, 1:09 pm • 1

Let G be the Centroid of $\triangle ABC$.
Let K be the Kiepert focus of $\triangle ABC$.
Let N be the 9-point center of $\triangle ABC$.
Let F_1 be the 1st Fermat point of $\triangle ABC$.
Let F_2 be the 2nd Fermat point of $\triangle ABC$.

Prove that N, K are isogonal conjugate of $\triangle GF_1F_2$

SalaF

#2 Apr 5, 2015, 9:06 pm • 4

Before posting my solution I want to specify - for purists' sake - that all the facts involved in the proof allow an elementary euclidean approach (which avoids the use of concis). However the considerable lenght of the aforesaid proof constrains me to concentrate solely on those results that I feel are not yet belonging to the substrate of "well-known facts". In fact, the demonstration might be viewed as a collection of known results that gather to provide our thesis; nevertheless I think that some of these could be considered quite original, hence my resolution to share them.

I'll rename the point K as E (referring to its definition as the Euler reflection point). The points O, H, K are respectively the circumcenter, orthocenter, Lemoine point of ABC. Let J_+, J_- be the isodynamics points (the isogonal conjugates respectively of F_1, F_2 wrt ABC .)
There are some well-known results about this configuration. First of all the "Lester circle" ones: the line F_1F_2 intercepts the Euler line at the reflection of O in G, whilst $\odot(GF_1F_2)$ touches OH at G (see, for an elementary proof, the paper here). Moreover the line F_1F_2 passes through K (well known; see Property of Kariya point); besides the line J_+J_- cincides with OK and $(OK, J_+J_-) = -1$, while the lines F_1J-, F_2J+ intersect at G (see Properties of Brokard and Isodynamic points).

There is a well-known results that states: E and G are inverse with respect the circle with diameter OK (Brocard circle). Despite being quite famous, I was not able to find any elementary proof on the web, so I'll have to write mine.
The first thing to be done is an inversion wrt $\odot(ABC)$; the image of the Brocard circle is a line perpendicular to OK passing through the image of K, which is easily verifiable to be the perpendicular bisector of J_+J_-. As the isodynamic points are the intersections of the Apollonius circles of ABC, this line passes through their centers, which are the intersections of BC, CA, AB with the tangent lines to $\odot(ABC)$ through A, B, C respectively. Now we can take the tangential triangle of ABC to be our new reference triangle and get a new problem:

Result # 1: Let ABC be a triangle whose incircle $I(\gamma)$ touches BC, CA, AB at D, E, F respectively. Let $D_1 = EF \cap BC$ and introduce similarly E_1, F_1. Let G be the first Shroeder point of ABC (that is the second common point of $\odot(AID), \odot(BIE), \odot(CIF)$, and also the inverse of the baricenter of DEF through γ). Finally let Fe be the Feuerbach point of ABC or equivalently the Euler reflection point of DEF.
Then the points G, Fe are simmetric with respect to $\overline{D_1E_1F_1} = t$ (the last one being the inverse of the Brocard circle of DEF wrt γ).

Proof: Let the circle $\odot(CIF)$ intersect again EF, DF at F_E, F_D respectively and CA, CB at F_A, F_B. We similarly define the points $E_D, E_F, D_E, D_F, E_A, E_C, D_B, D_C$. Then
$\angle(D_CD, D_CA) = \angle(ID, IA) = \angle(ID, BC) + \angle(BC, IA) = \angle(IA, EF) + \angle(BC, IA) = \angle(BC, EF)$ so that D_1DED_C is cyclic. As $CE = CD$, it's clear that $D_1D_C||DE$. Similarly, $DE||E_1E_C$. Besides
$\angle(GF_E, GI) + \angle(GI, GD_C) = \angle(FE, FI) + \angle(AI, AC) = 0$ so that F_E, G, D_C are collinear, as well as F_D, G, E_C; but then
$\angle(GE_C, D_C) = \angle(GF_D, GF_E) = \angle(FD, FE) = \angle(DC, DE) = \angle(D_1E_C, D_1D_C) \implies \quad G \in \odot(D_1DED_C)$.
On the other hand $\angle(GD_1, GE_1) = \angle(E_CD_1, E_CE_1) = \angle(FE_1, FD_1)$ so $\odot(GE_1D_1), \odot(FE_1D_1)$ are symmetric with respect to t, hence the symmetrical point of G wrt t is the concurrency point of the three circumferences $\odot(DE_1F_1), \odot(EF_1D_1), \odot(FE_1D_1)$. However this point is known to be exactly Fe (for instance see Feuerbach point: G, D,

X, Y concyclic).

Now we come back to the original problem and introduce some other points. The first is the point S, or the Steiner point of ABC (the point such that its Steiner line is parallel to OK). For the second one, we refer to the following lemma, whose proof can be found in the paper On rotation of a isogonal point:

Result #2: Let ABC be a triangle with X, X_1 and Y, Y_1 two pairs of points isogonal conjugate wrt ABC. Then the Miquel point of the complete quadrilateral formed by the lines $\{XY, XY_1, X_1Y, X_1Y_1\}$ lies on $\odot(ABC)$.

Taking $X = F_1$ and $Y = F_2$ (and recalling the very first observations of this treatment) we immediately see that the circles $\odot(GF_1F_2), \odot(GJ_+J_-), \odot(ABC)$ are concurrent at a point P. Moreover as G, E and J_+, J_- are inverse through the Brocard circle we have that $E \in \odot(GJ_+J_-)$. As J_+, J_- are also inverse through $\odot(ABC)$ this circle must be orthogonal to the latter and $(OK, J_+J_-) = -1$ implies that $K \in PE$.

Now we make a little excursus about trilinear polars. There is a well-known

Result #3: Let ABC be a triangle with X, X_1 and Y, Y_1 two pairs of points isogonal conjugate wrt ABC. Then X_1 belongs to the trilinear polar of Y if and only if Y_1 belongs to the trilinear polar of X (everithing wrt ABC).

For a reference see Trilinear polar (the english part). This has many applications: for instance the trilinear polar of every point on $\odot(ABC)$ passes through the symmedian point. Moreover the trilinear polar of H is perpendicular to OH (equivalently, contains the isogonal conjugate of E), so the trilinear polar of E contains the isogonal conjugate of $H \implies$ it's the line OK. Analogously, the trilinear polar of K (the center axis of the Apollonius circles) is perpendicular to OK (equivalently, contains the isogonal conjugate of S) so the trilinear polar of S is the line GK.

There is another interesting

Result #4: Let us be given a triangle ABC inscribed in a circle Γ and let P an arbitrary point on Γ. Let us call ℓ the trilinear polar of P wrt ABC ang G the baricenter of the triangle. Then $\angle(PA, PG) = \angle(\ell, BC)$}

Proof: Let P_A, P_B, P_C the cevian traces of P on BC, CA, AB and $A_1, B_1, C_1 = \ell \cap BC, CA, AB$. Let $K_AK_BK_C$ be the circumcevian triangle of K. Easy considerations (such as $S_A \in P_BP_C$, $(A_1A, A_1P, \ell, A_1S_A) = -1$ and $(A, K_A, K, S_A) = -1$ where S_A is the $A-$ vertex of the tangential triangle) lead to the collinearity of A_1, P, K_A.
Let us call D the midpoint of BC and let AD meet again Γ at G_A. From $(B, C, P_A, A_1) = -1$ we have $P_AD \cdot P_AA_1 = P_AB \cdot P_AC = P_AA \cdot P_AP$ so that $ADPA_1$ is cyclical. It follows that $\measuredangle DAP = \measuredangle DA_1P$; if A_1' is the intersection of PA_1 with the line parallel to BC conducted from K we also have $\measuredangle DAP = \measuredangle KA_1'P$. As $\measuredangle AG_AP = \measuredangle AK_AP$, it follows that $\triangle AG_AP \simeq \triangle A_1'K_AK$. Now we will show that in fact $\triangle K_AA_1K \simeq \triangle G_AGP \quad (\star)$; by the former observations, this is equivalent to showing that G is the image of A_1 in the similitude sending $\triangle AG_AP$ to $\triangle A_1'K_AK$. Equivalently, we want $\dfrac{AG}{GG_A} = \dfrac{A_1'A_1}{A_1K_A} = \dfrac{KK_A'}{K_A'K_A}$. This is independent of P, so it is sufficient to show $(\star)$ for a particular $P \in \Gamma$. If we take $P \equiv C$ this is obvious; indeed $(\star)$ in this case is equivalent to $\measuredangle KCK_A = \measuredangle G_AGC$, or $180° - \measuredangle CGA = \measuredangle KCB + \measuredangle KAB$, which trivially follows from G, K being isogonal conjugates.
Having $(\star)$ for every $P \in \Gamma$, we have $\angle(PA, PG) = \angle(KA_1, KA_1')$ (by the above similitude) $= \angle(\ell, BC)$, as we wanted.

This result easily give us that if $X, Y \in \odot(ABC)$ are such that $G \in XY$ then the trilinear polar of X is perpendicular to the Steiner line of Y. By the corollaries of **Result #3** we have that, for instance, the line EG intersect again $\odot(ABC)$ at a point L whose Steiner line wrt ABC is perpendicular to the trilinear polar of E, or OK. As E, G are inverse through the Brocard circle we have $\angle(EP, EL) = \angle(EK, EG) = \angle(KO, KG)$: but the angle subtended by two points is equal to the angle between their Steiner lines, so that the Steiner line of P is perpendicular to GK. By **Result #4** we have that PG intersects $\odot(ABC)$ at a point whose trilinear polar is parallel to GK and passes through K: but we have shown before that this point is S.
Let L_1, S_1 be the complements of L, S; clearly they belong to the nine point circle and S_1 is the midpoint of F_1F_2 (as OK contains the isogonal conjugates of F_1, F_2 and the two Fermat points are antigonal conjugate.) As their orthopoles are perpendicular, L_1 and S_1 are diametrically opposite on the nine point circle. If $M = F_1F_2 \cap OG$ it is easy to see that N is the common midpoint of L_1S_1, GM and GL_1, MS_1 (or equivalently EG, F_1F_2) are parallel to each other.

Now let A', P', S', E' be the symmetric points of A, P, S, E with respect to the Euler line. It is quite easy to see that the lines EA', BC are parallel and $\angle(EL, BC) = \angle(EL, EA') = \angle(SL, SA') = 90° + \angle(SL, LA') = 90° + \angle(SA, SS')$; this implies that the Steiner line of S' is parallel to EG. We also have $\angle(PP', PS) = \angle(SS', SP) = 90° + \angle(GE, GK)$ (angle between the Steiner lines of S', P) $= \angle(EK, OK)$ (as E, G are inverse through the Brocard circle) implying that the Steiner line of P' is perpendicular to EK.
Then $\angle(E'P, E'G) = \angle(EG, EP') = \angle(EL, EP') = \angle(EK, OK)$ (perpendicular to the Steiner lines of P', L).
However we have defined P as the center of the direct similarity which sends F_2G to KJ_- so that $\triangle PF_2G \simeq PKJ_-$ and $\angle(F_2P, F_2G) = \angle(KP, KJ_-) = \angle(EK, OK) = \angle(E'P, E'G)$. We conclude that E' belongs to the circle GF_1F_2.

Recalling that $EG//F_1F_2$ we only need a final result to conclude:

Lemma: Let ABC be a triangle and let the tangent line to $\odot(ABC)$ through A meet BC at D; D_1 is the midpoint of the segment AD. Let E be that point on $\odot(ABC)$ such that AE is parallel to BC and let E_1 be the symmetric of E in A. Then the points E_1, D_1 are isogonal conjugate with respect to ABC.

Proof : for every point P let P' the image of P after an inversion with center A and radius $\sqrt{AB \cdot AC}$ followed by a symmetry wrt the bisector of $\angle BAC$. Let P'' be the second intersection of AP' with $\odot(BCP')$; then P, P'' are isogonal conjugate wrt ABC (well-known, you can see here). Then if $P = D_1$ the respective P' is E_2, the symmetric of A in E. Moreover B, C, E_1, E_2 are concyclic (it's an isopsceles trapezoid), from which we have the thesis.

As N is the midpoint of M we have that in fact E, N are isogonal conjugate with respect to GF_1F_2 (applying the Lemma to the latter triangle).

Luis González

#4 Apr 25, 2015, 1:31 am • 3

Let O, H, L denote the circumcenter, orthocenter and symmedian point (Lemoine point) of $\triangle ABC. H, G, F_1, F_2, A, B, C$ lie on Kiepert hyperbola $\mathcal{K}$ of $\triangle ABC$; isogonal conjugate of Brocard axis OL.

Since F_1, F_2 are Kiepert perspectors $K(60^\circ)$ and $K(-60^\circ)$, then F_1F_2 passes through L and since L is the crosspoint of H, G, then L is the pole of HG WRT $\mathcal{K}$ (for proofs see the generalization discussed in the problem Schwatt's lines). Hence F_1F_2 hits HG at its midpoint D and according to the discussions at Rectangular circumhyperbola and circle, we have that GL is the G-symmedian of $\triangle GF_1F_2$ and GN touches its circumcircle $\odot(GF_1F_2)$.

If J is the midpoint of OL (center of Brocard circle Ω), then according to inverse wrt Brocard circle, K is the inverse of G WRT $\Omega \Longrightarrow \overline{JGK}$ is O-midline of $\triangle ODL \Longrightarrow GK \parallel F_1F_2 \Longrightarrow GK, GN$ are then isogonals WRT $\angle F_1GF_2$ $(\star)$.

Let GJ cut $\odot(GF_1F_2)$ again at X and DX cuts $\odot(GF_1F_2)$ again at U. Since GL is the polar of D WRT $\odot(GF_1F_2) \Longrightarrow G(L, D, X, U) = -1$ and since $GX \parallel LD$, then GU is G-median of $\triangle GDL \Longrightarrow GU$ cuts LD at its midpoint M and LU cuts XG at the reflection K^* of X on G. From $(L, D, F_1, F_2) = -1$, we get $ML^2 = MF_1 \cdot MF_2 = MG \cdot MU$ $\Longrightarrow \angle MLU = \angle UGL$, but as GM is D-midline of $\triangle ODL$ ($GM \parallel OL$), then $\angle UGL = \angle GLO \Longrightarrow$ $\angle GLO = \angle MLU \Longrightarrow \angle JLU = \angle GLM = \angle LGX \Longrightarrow JL$ touches $\odot(LGK^*) \Longrightarrow JL^2 = JG \cdot JK^* \Longrightarrow$ $K \equiv K^*$. Now since G is midpoint of XK and $F_2D \parallel XK$, then $F_2(G, D, K, X) = -1$, but $\angle GF_2X = \angle GDF_2 = |\angle GF_1F_2 - \angle GF_2F_1|$, i.e. F_2X touches $\odot(GDF_2) \Longrightarrow F_2K$ is the F_2-symmedian of $\triangle F_2GD$ isogonal conjugate of its median F_2N. Together with $(\star)$, we conclude that N, K are isogonal conjugates WRT $\triangle GF_1F_2$.

Luis González

#5 Sep 27, 2015, 11:05 am • 1

> *SalaF wrote:*
>
> **Result #4:** Let us be given a triangle ABC inscribed in a circle Γ and let P an arbitrary point on Γ. Let us call ℓ the trilinear polar of P wrt ABC ang G the baricenter of the triangle. Then $\angle(PA, PG) = \angle(\ell, BC)$

Here is another proof to this nice result:

Let PA cut BC at X and let PG cut Γ again at Q. It's known that the trilinear polar of a point on Γ passes through the symmedian point K of $\triangle ABC$, hence ℓ passes through K and the harmonic conjugate X' of X WRT B, C.

Let us animate P on Γ. $X \mapsto X'$ is an involution on BC and $P \mapsto Q$ is an involution on Γ with pole $G \Longrightarrow$ $X' \barwedge X \barwedge P \barwedge Q$. When $P \equiv M$, then X' goes to the point at infinity A_∞ of $BC \Longrightarrow$ pencils KA_∞, KX' and MA, MQ are proyective, thus it is enough to show that $\angle(PA, PG) = \angle(MA, MQ) = \angle(KX', KA_\infty) = \angle(\ell, BC)$ holds for at least 3 positions of P. When $P \equiv M$, clearly $\angle(PA, PG) = \angle(\ell, BC) = \pi$ and when $P \equiv B$, then $X \equiv X' \equiv B \Longrightarrow \ell$ becomes the B-symmedian isogonal conjugate of $BG \Longrightarrow \angle(PA, PG) = \angle(\ell, BC)$ and the same happens when $P \equiv C$. Therefore $\angle(PA, PG) = \angle(\ell, BC)$ holds for any P on Γ.

Quick Reply

High School Olympiads

Collinearity

Reply

Scorpion.k48
#1 Sep 27, 2015, 8:26 am

Let $\triangle ABC$. M is midpoint of BC. J lies on the internal angle bisector of $\widehat{BAC}$. $\triangle XYZ$ is pedal triangle of J WRT $\triangle ABC$. XJ cuts YZ at K. Prove that A, K, M are collinear.

Luis González
#2 Sep 27, 2015, 8:42 am • 1

Let AJ cut the circumcircle $\odot(ABC)$ again at D (midpoint of its arc BC). If U, V are the projections of D on AC, AB, then U, V, M are collinear on the Simson line of D WRT $\triangle ABC$. Now $\triangle DUV$ and $\triangle JYZ$ are homothetic with center A and they have corresponding cevians DM and JK due to $(DM \parallel JK) \perp BC \Longrightarrow A, K, M$ are collinear.

Dukejukem
#3 Sep 27, 2015, 8:45 am

Let ω be the circle with center J passing through Y, Z. Let τ be the line passing through A parallel to BC and denote $W \equiv YZ \cap \tau, K' \equiv YZ \cap AM.$

Since $AW \parallel BC$, we have $-1 = A(B, C; M, W) = (Z, Y; K', W)$, implying that W lies on the polar of K' WRT ω. But since AY, AZ are tangents to ω, it follows that K lies on the polar of A as well. Thus by La Hire's Theorem, A lies on the polar of K' as well. Hence, τ is the polar of K', implying that $JK' \perp \tau$, i.e. J, K', X are collinear. $\square$

This post has been edited 1 time. Last edited by Dukejukem, Sep 27, 2015, 8:47 am

tastymath75025
#4 Sep 27, 2015, 9:16 am • 2

another short one:

Let the parallel through A to BC be l. Let XJ and YZ meet l at P, Q. Let XJ meet YZ at R.

Note that $\angle APJ = 90^\circ$ so P is on circle $AZJY$. Since $AZJY$ is a kite, it is harmonic, so

(PA, PZ, PJ, PY) is harmonic, implying (Q, Z, R, Y) is harmonic by intersecting the pencil with line YZ. .But this implies that if AR meets BC at M', we have (B, M', C, ∞_{BC}) is harmonic, so $M = M'$ and R is on the median as desired.

This post has been edited 1 time. Last edited by tastymath75025, Sep 27, 2015, 9:17 am

Quick Reply

High School Olympiads

Perpendicular

Reply

Scorpion.k48

#1 Sep 26, 2015, 8:26 pm

Let $\{A, B\} = \odot(O_1) \cap \odot(O_2)$ and P, Q lies on $\odot(O_1), \odot(O_2)$, res such that $O_1P//O_2Q$. AB cuts O_1O_2, PQ at K, L, res. $\odot(QKL)$ cuts $\odot(O_2)$ again at R. Prove that $RP \perp RQ$.

TelvCohl

#2 Sep 26, 2015, 11:05 pm • 1

Let $S \equiv QO_2 \cap \odot(O_2), R^* \equiv PS \cap \odot(O_2)$. Let U, V be the exsimilicenter, insimilicenter of $\odot(O_1) \sim \odot(O_2)$, respectively. I'll only prove the case when $U \in PQ$ and $V \in PS$. Since U, V lie on the bisector of $\angle O_1AO_2$ ($\angle O_1BO_2$), so A, B, U, V lie on a circle with diameter UV. Since V is the insimilicenter of $\odot(O_1) \sim \odot(O_2)$, so from $AU \perp AV$ we get $VR^* \cdot VP = VA^2 = VK \cdot VU \Longrightarrow K, P, R^*, U$ are concyclic, hence $\measuredangle KR^*Q = \measuredangle KR^*V + 90^\circ = \measuredangle KUP + 90^\circ = \measuredangle KLQ \Longrightarrow K, L, Q, R^*$ are concyclic $\Longrightarrow R^* \equiv R \Longrightarrow RP \perp RQ$.

Luis González

#3 Sep 27, 2015, 1:45 am • 1

Let J be the insimilicenter of $(O_1) \sim (O_2)$ and $\overline{JQ}, \overline{JP}$ cut $(O_1), (O_2)$ at Y, Z, resp. As JQ, JP go through the antipodes of P, Q on $(O_1), (O_2)$, then PY, QZ are altitudes of $\triangle JPQ$ meeting at its orthocenter $M \Longrightarrow JM \perp PQ$ at S. Furthermore since PY, QZ, AB are pairwise radical axes of $(O_1), (O_2), \odot(PQZY) \Longrightarrow M \in AB$. Hence $\angle SQM = \angle MJZ = \angle MKZ \equiv \angle LKZ \Longrightarrow Z \in \odot(QLK) \Longrightarrow R \equiv Z \Longrightarrow \angle PRQ = 90^\circ$.

Quick Reply

High School Olympiads

Geometry Problem Involving the Use of One Angle

Locked

amplreneo
#1 Sep 26, 2015, 9:02 am

Let ABC be a triangle in which $\angle A = 60^\circ$. Let BE and CF be the bisectors of the angles $\angle B$ and $\angle C$ with E on AC and F on AB. Let M be the reflection of A in the line EF. Prove that M lies on BC.

Luis González
#2 Sep 26, 2015, 9:52 am

Discussed before at http://www.artofproblemsolving.com/community/c6h380706.

High School Olympiads

show that

Locked

jemima

#1 Dec 3, 2010, 6:03 pm • 1

let ABC be a triangle in which angle A = 60 . Let BE and CF be the angular bisectors of angles of B,C wiith E on AC and F on AB. Let m be the reflection of A in the ine EF. Prove that M lies on BC.

jgnr

#2 Dec 3, 2010, 7:10 pm

Let D be the point on BC such that $\angle BFD=\angle C$. So we have $\triangle BFD\sim\triangle BCA$, and hence $DF=\frac{AC}{BC}\cdot BF=AF$. Also note that $BD=\frac{AB}{BC}\cdot BF=\frac{c^2}{a+b}$. Similarly, let D' be the point on BC such that $\angle CED=\angle B$, then we have $D'E=AE$ and $CD'=\frac{b^2}{a+c}$. By cosine's law we get $a^2=b^2+c^2-bc$, therefore

$$BD+CD'=\frac{c^2}{a+b}+\frac{b^2}{a+c}=\frac{a(b^2+c^2)+b^3+c^3}{a^2+ab+bc+ca}=\frac{a(a^2+bc)+(b+c)(b^2+c^2-bc)}{a^2+ab+bc+ca}=\frac{a(a^2+bc)+(b+c)a^2}{a^2+ab+bc+ca}=a$$

Hence $D=D'$. Now we have $AF=FD$ and $AE=ED$, so $AEDF$ is a kite, which implies that $M=D$. Therefore M lies on BC.

Luis González

#3 Dec 3, 2010, 9:49 pm

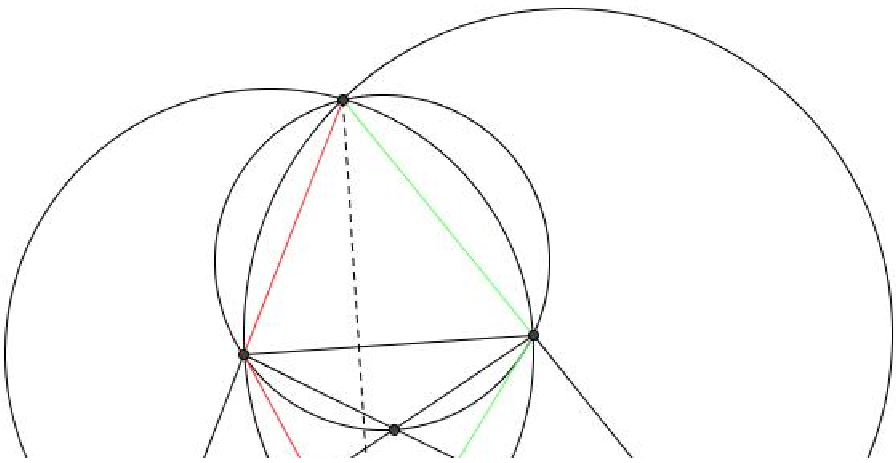
$I\equiv BE\cap CF$ is the incenter of $\triangle ABC$ and AI cuts BC and EF at $D,P,$ respectively. Because of $\angle FIE=120^\circ$, it follows that A,F,I,E lie on a circle with center $U\Longrightarrow BC$ is the polar of P WRT $(U)\Longrightarrow UP$ is perpendicular to BC through a point M'. Since cross ratio (I,A,P,D) is harmonic and $M'P\perp M'D,$ we deduce that $M'PU$ and BC bisect $\angle IM'A$ internally and externally. Then $UA=UI$ implies that $AUIM'$ is cyclic, but since P lies on the diagonal FE of the rhombus $FUEI$ formed by equilateral $\triangle UEI$ and $\triangle UFI,$ it follows that $PU=PI\Longrightarrow AUIM'$ is an isosceles trapezoid with symmetry axis $EF\Longrightarrow M'$ is the reflection of A about $EF.$

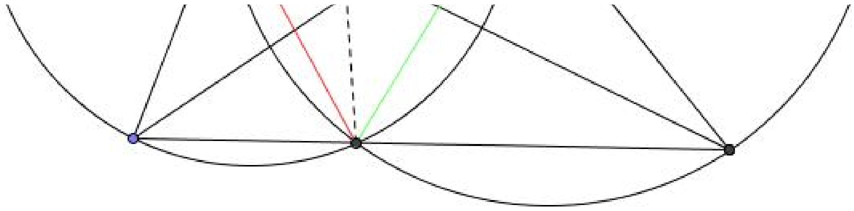
skytin

#4 Dec 3, 2010, 10:14 pm

It's very easy .

Attachments:





jemima
#5 Dec 3, 2010, 10:49 pm
thank you, any other proofs please

Layman conjecture
#6 Dec 5, 2010, 8:18 pm
jemima, these are the problems of the RMO 2010. If we find out who you are and which state you are from, you will be automatically disqualified.

SOURBH
#7 Dec 5, 2010, 11:31 pm
Angle-chasing FTW !!!

Drop perpendicular from A to EF and let it meet EF at O and BC at N.
I is the incentre
Intersection of AN and BE is Z and Intersection of AN and CF is X.

AFIE is cyclic.
IFE=IEF=30
OXF = 60= XIZ
ZNC= 60+B/2
FEA=90-B/2
AFE=30+B/2
OAE=B/2
AFC=ANC ..Thus AFNC is cyclic
NFC=B/2
NFO=OFA
Thus AO=ON thus N=M
DONE !!!

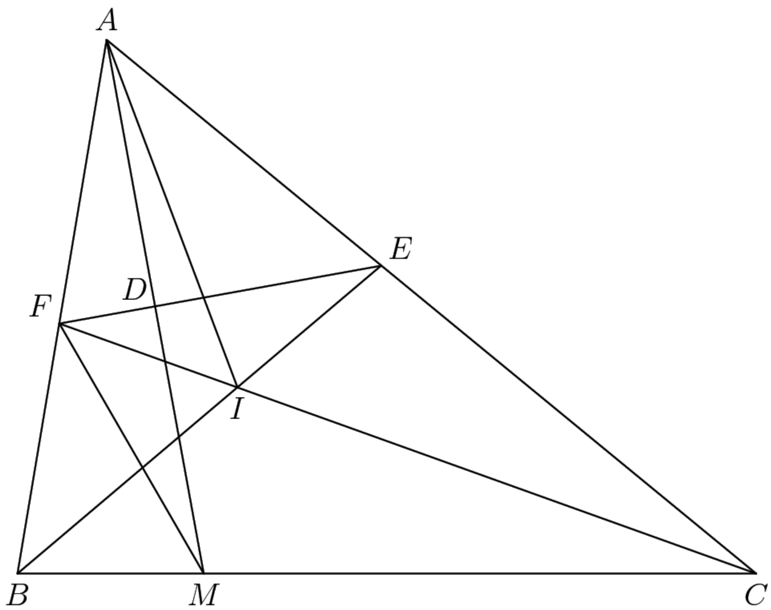
SOURBH
#8 Dec 6, 2010, 12:40 am
I request the moderators to completely ban Jemima from AOPS such that he is not even allowed to join the website again by creating a new ID....He has committed a grave offence and must be punished rigorosly ....

tulsidas
#9 Dec 6, 2010, 11:33 am
I completely agree with you all. This unscrupulous jemima should be thrown out of AOPS. He do not deserve to study mathematics. Shame on him.

Potla
#10 Dec 6, 2010, 10:38 pm
Here's my solution to the problem that I gave during the exam.

Let AD be the foot of perpendicular from A onto EF, and let it meet BC in M. Join FM.
Now, note that
$\angle FAE = 60^\circ \implies \angle BIC = 120^\circ$, ie $\angle FAE + \angle FIE = 180^\circ$,
Hence $AFIE$ is a cyclic quadrilateral. Now note that $\angle FAI = \angle FEI = \angle EAI = \angle IFE$ leads to $IF = IE$, ie

$\angle IFE = \angle IEM = 30°$.
Now, on the other hand we have,
$$\angle FAD = 90° - \angle AFD = 90° - \angle AIE = 90° - (\angle ABI + \angle IAB)$$
$$= \frac{1}{2}\angle ACB = \angle FCM;$$
Therefore $AFMC$ is also cyclic. So we get,
$\angle AFM = 180° - \angle ACB = 2\left(90° - \frac{1}{2}\angle ACB\right) = 2\angle AFD$, ie $\angle DFM = \angle DFA$.
Therefore considering $\triangle DFA$ and $\triangle DFM$, they are congruent from the SAS rule of congruency, leading to $AD = DM$.
Hence M is the reflection of A onto EF which lies on BC.



Hope it brings a bagful.

This post has been edited 1 time. Last edited by Potla, Dec 7, 2010, 12:59 am

siddharthanand
#11 Dec 6, 2010, 11:54 pm • 1

It's an embarrassing situation all the solutions to the given set of **RMO problems 2010 had been posted in December 3** where, the exams were conducted for us in December 5 . Its complete swindle
The Indian national mathematics , must come to know of it
Really despicable ! offence .
jemima must be punished so that in future such faults won't happen.

skytin
#12 Dec 6, 2010, 11:58 pm

> *siddharthanand wrote:*
> It's an embarrassing situation all the solutions to the given set of **RMO problems 2010 had been posted in December 3** where, the exams were conducted for us in December 5 . Its complete swindle
> The Indian national mathematics , must come to know of it
> Really despicable ! offence .
> jemima must be punished so that in future such faults won't happen.

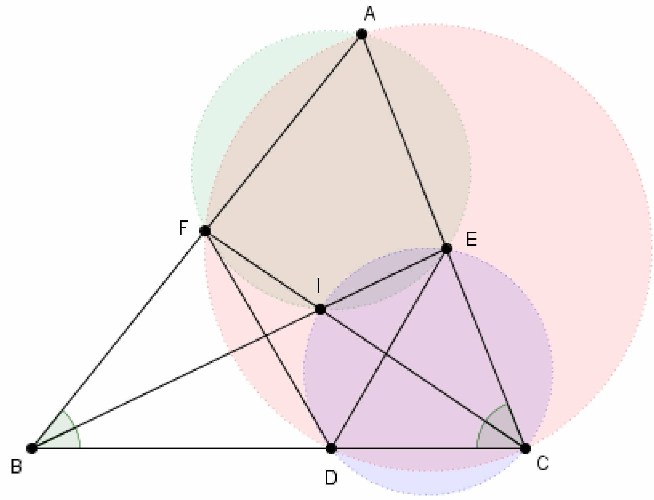
Yes , it is 9th grade of RMO 2010 , and i was on this olympiad

77ant
#13 Dec 7, 2010, 2:16 am

AFIE is cyclic. Circle AFC cut BC at D. (BF.BA)=(BI.BE)=(BD.BC) . IDCE is cyclic. angle EIC=angle EDC=60.
ABDE is cyclic. AF=FD, AE=ED (angle ACF= angle FCD, angle ABE= angle EBD). AFDE is kite.
Thus, D is the mirror of A in EF

Thus, D is the mirror of A in EF .

Attachments:



![avatar]

sankha012

#14 Jan 1, 2011, 12:58 pm • 2

Reflect B and C on FE to get B' and C'.It's sufficient to prove that B',A and C' are collinear.
After showing that $AEIF$ is cyclic,we get that $\angle IEF = \angle IFE = \frac{\pi}{6}$.Thus a rotation of $-\frac{\pi}{3}$ with the center E takes B to B' and a rotation of $\frac{\pi}{3}$ with the center F takes C to C'.Thus $\triangle BB'E$ and $\triangle CC'F$ are both equilateral.It follows that $AEBB'$ and $AFCC'$ are both cyclic.This implies $\angle BAB' = \angle CAC' = \frac{\pi}{3}$.Thus
$\angle B'AC' = \angle B'AB + \angle BAC + \angle CAC' = \frac{\pi}{3} + \frac{\pi}{3} + \frac{\pi}{3} = \pi$
QED

I failed to do this in the exam :(

This post has been edited 2 times. Last edited by sankha012, Jan 1, 2011, 2:08 pm

mahanmath

#15 Jan 1, 2011, 1:58 pm

> *jemima wrote:*
> let ABC be a triangle in which angle A = 60 . Let BE and CF be the angular bisectors of angles of B,C wiith E on AC and F on AB. Let m be the reflection of A in the ine EF. Prove that M lies on BC.

Also see this link :

Iran Second Round 2010

r1234

#16 Jan 8, 2011, 1:09 pm

there is another beautiful solution. extend EF and BC to meet in X. then prove that AX ix the external bisector of BAC. for this proof menelaus' theorem is required. the us e the fact that A=60 . and trivially get the result.

its not my own solution...

salgarkarap

#17 Jan 24, 2011, 9:30 am

whats wrong with you?
answer should be something like:

simson: feet of altitudes to sides of triangle from circumcircle point are collinear..
my version or rather extension: mirror image of a point in the corresponding sides are collinear,,
proof: apply homothety centered at that point of ratio 2.
Since A lies on circumcircle of EFI,its images in sides will be collinear..
But mirror image of a leg of angle in the bisector lies on the other leg…Hence image in EI and FI lie on BC..so will be that in EF..
I've selected for inmo and new here…can u help me???

High School Olympiads

Three collinear Orthocenters related to Orthotransversal

Reply

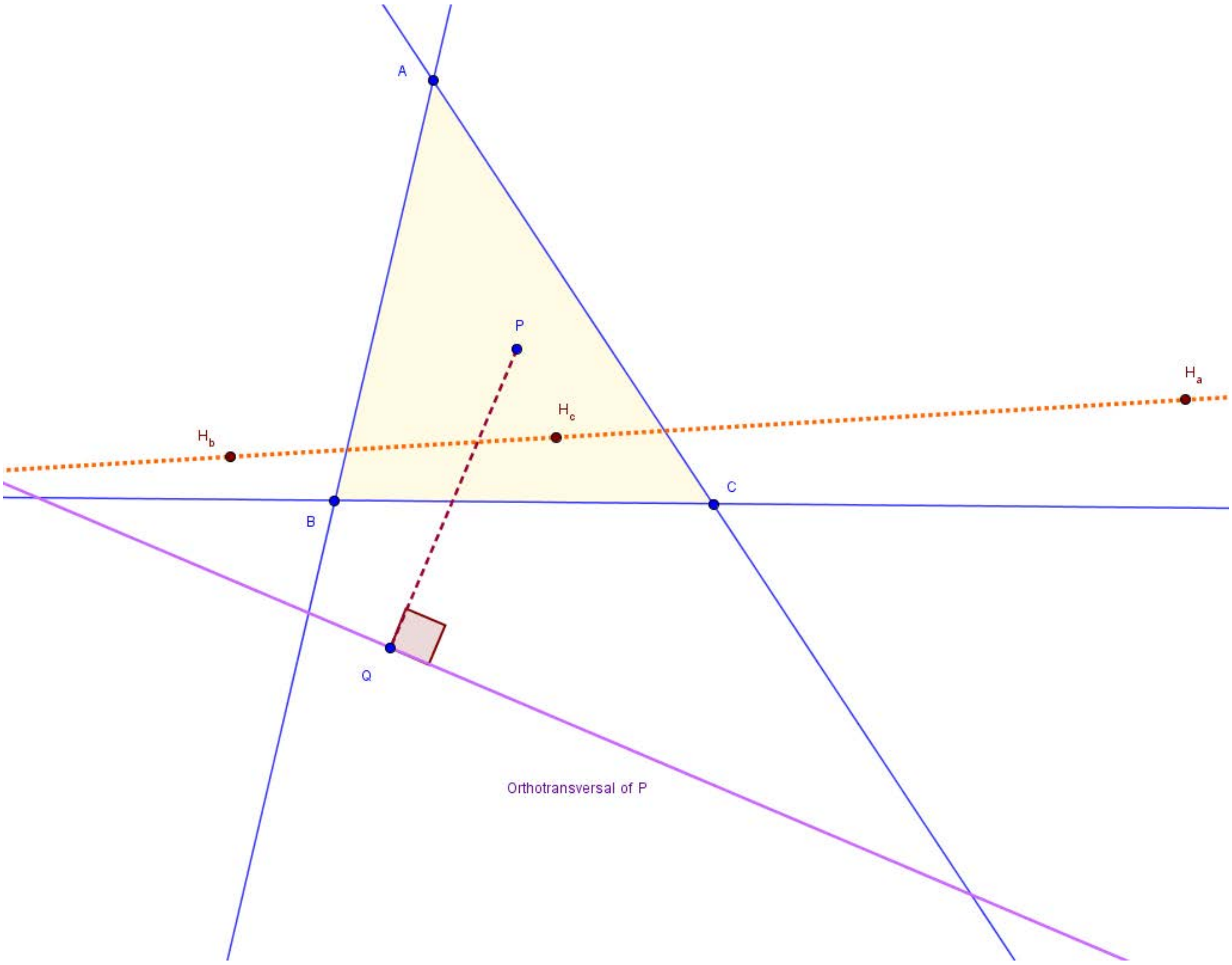
Source: Own

TelvCohl

#1 Sep 24, 2015, 12:06 am • 4

Given a $\triangle ABC$ and an arbitrary point P. Let Q be the projection of P on the Orthotransversal of P WRT $\triangle ABC$. Let H_a, H_b, H_c be the orthocenter of $\triangle APQ, \triangle BPQ, \triangle CPQ$, respectively. Prove that H_a, H_b, H_c are collinear.

Attachments:



Luis González

#2 Sep 24, 2015, 6:18 am • 7

Perpendiculars to PA, PB, PC at P cut BC, CA, AB at $X, Y, Z \Longrightarrow \overline{XYZ}$ is orthotransversal of P and let T_∞ denote the point at infinity of $\overline{XYZ}$. If $U \equiv CQ \cap AB$, we have $Q(A, B, C, T_\infty) = Q(A, B, U, Z) = Q(B, A, Z, U) = P(X, Y, Z, Q)$. But since PX, PY, PZ, PT_∞ are perpendiculars to PA, PB, PC, PQ, then $P(X, Y, Z, Q) = P(A, B, C, T_\infty) \Longrightarrow$ $P(A, B, C, T_\infty) = Q(A, B, C, T_\infty)$ $(*)$. Thus since PH_a, PH_b, PH_c, PQ are perpendicular to QA, QB, QC, T_∞ and QH_a, QH_b, QH_c, QP are perpendicular to PA, PB, PC, T_∞, it follows that $P(H_a, H_b, H_c, Q) = Q(A, B, C, T_\infty) = P(A, B, C, T_\infty) = Q(H_a, H_b, H_c, P) \Longrightarrow H_a, H_b, H_c$ are collinear.

Remark: The result $(*)$ means that the circum-conic through P, Q is a hyperbola with an asymptote parallel to the orthotransversal of P.

andria
#4 Sep 24, 2015, 1:40 pm

I think the following generalization works:
Let P be an arbitrary point on the plain of $\triangle ABC$. Let $\mathcal{H}$ be the equilateral hyperbola passing throw A, B, C, P. Let Q be an arbitrary point on $\mathcal{H}$. a perpendicular line from Q to AP cuts BC at A'. we define B', C' similarly. It's well known that A', B', C' are collinear. Let $PQ \cap \overline{A'B'C'} = R$. Let H_a, H_b, H_c be orthocenters of triangles ARP, BRP, CRP respectively. Prove that H_a, H_b, H_c are collinear.
P.S

TelvCohl
#5 Sep 24, 2015, 6:10 pm

> andria wrote:
> I think the following generalization works:
> Let P be an arbitrary point on the plain of $\triangle ABC$. Let $\mathcal{H}$ be the equilateral hyperbola passing throw A, B, C, P. Let Q be an arbitrary point on $\mathcal{H}$. a perpendicular line from Q to AP cuts BC at A'. we define B', C' similarly. It's well known that A', B', C' are collinear. Let $PQ \cap \overline{A'B'C'} = R$. Let H_a, H_b, H_c be orthocenters of triangles ARP, BRP, CRP respectively. Prove that H_a, H_b, H_c are collinear.

Let T_∞ be the infinity point on $\overline{A'B'C'}$. Since $PH_a \perp AR, PH_b \perp BR, PH_c \perp CR, RH_a \parallel QA', RH_b \parallel QB', RH_c \parallel QC'$ and $PQ \perp \overline{A'B'C'}$ (see Collinearity with Symmedian Point (post #16, post #17)), so $P(RH_a; H_bH_c) = R(T_\infty A; BC) = A(A'R; C'B') = Q(A'R; C'B') = R(H_aP; H_cH_b) = R(PH_a; H_bH_c) \Longrightarrow H_a, H_b, H_c$ are collinear.

andria
#6 Sep 24, 2015, 6:59 pm

Another property related to this problem:
Let P be an arbitrary point and ABC be a triangle. Let τ be the orthotransversal of P WRT $\triangle ABC$. Let Q be the projection of P onto τ. Let Q_a be the projection of Q on AP and A' be the projection of A on PQ. denote by Ω_a be the circumcircle of $\triangle PA'Q_a$. we define Ω_b, Ω_c similarly. H_a, H_b, H_c are orthocenters of APQ, BPQ, CPQ respectively.
1) prove that $\Omega_a, \Omega_b, \Omega_c$ are coaxal.
2) Let $\Omega_a \cap \Omega_b \cap \Omega_c = \{P, S\}$. Prove that H_a, H_b, H_c, S are collinear.

Luis González
#7 Sep 24, 2015, 9:19 pm • 1

> andria wrote:
> **Another property related to this problem:**
> Let P be an arbitrary point and ABC be a triangle. Let τ be the orthotransversal of P WRT $\triangle ABC$. Let Q be the projection of P onto τ. Let Q_a be the projection of Q on AP and A' be the projection of A on PQ. denote by Ω_a be the circumcircle of $\triangle PA'Q_a$. we define Ω_b, Ω_c similarly. H_a, H_b, H_c are orthocenters of APQ, BPQ, CPQ respectively.
> 1) prove that $\Omega_a, \Omega_b, \Omega_c$ are coaxal.
> 2) Let $\Omega_a \cap \Omega_b \cap \Omega_c = \{P, S\}$. Prove that H_a, H_b, H_c, S are collinear.

This simply follows from the collinearity of H_a, H_b, H_c. Note that PH_a, PH_b, PH_c are diameters of $\Omega_a, \Omega_b, \Omega_c$, thus their centers (midpoints of PH_a, PH_b, PH_c) are collinear $\Longrightarrow \Omega_a, \Omega_b, \Omega_c$ are coaxal and trivially $S \in \overline{H_aH_bH_c}$.

buratinogigle
#8 Sep 25, 2015, 10:36 pm • 1

Another generalization

Let ABC be a triangle and (P) is a circle. D, E, F are inversion images of A, B, C through (P). The lines passing through D, E, F and are perpendicular to PA, PB, PC which intersect BC, CA, AB at X, Y, Z, resp. X, Y, Z lie on line ℓ. S is polar of ℓ with respect to (P) and T is projection of S on ℓ. Prove that orthocenters of triangles AST, BST, CST are collinear.

Luis González

#9 Sep 25, 2015, 11:27 pm • 1

> buratinogigle wrote:
>
> Another generalization
>
> Let ABC be a triangle and (P) is a circle. D, E, F are inversion images of A, B, C through (P). The lines passing through D, E, F and are perpendicular to PA, PB, PC which intersect BC, CA, AB at X, Y, Z, resp. X, Y, Z lie on line ℓ. S is polar of ℓ with respect to (P) and T is projection of S on ℓ. Prove that orthocenters of triangles AST, BST, CST are collinear.

$A'X, B'Y, C'Z$ are the polars of A, B, C WRT $(P) \Longrightarrow \triangle UVW$ bounded by $A'X, B'Y, C'Z$ is the polar triangle of (P) WRT $\triangle ABC \Longrightarrow X, Y, Z$ are the poles of ASU, BSV, CSW WRT $(P) \Longrightarrow PX, PY, PZ$ are perpendicular to SA, SB, SC. Thus if T_∞ denotes the point at infinity of $\overline{XYZ}$, we have $S(A, B, C, T_\infty) = P(X, Y, Z, T) = T(A, B, C, T_\infty)$ and from here the proof runs exactly the same as the previous ones.

buratinogigle

#10 Oct 13, 2015, 3:15 pm

Thank Telv for nice problem. I have seen two corollaries as following

Corollary 1. Let ABC be a triangle and P is orthopole of line ℓ passing through its circumcenter. Q is projection of P on ℓ then orthocenters of triangles APQ, BPQ, CPQ are collinear.

I used inversion with pole P, I get quite strange problem

Corollary 2. Let ABC be a triangle with any point P. D, E, F lie on circles $(PBC), (PCA), (PAB)$ such that $PD \perp PA, PE \perp PB, PF \perp PC$. PQ is diameter of circle (DEF). PQ cuts perpendicular bisector of PA, PB, PC at X, Y, Z, reps. PA, PB, PC cut perpendicular bisector of PQ at U, V, W, reps. Prove that UX, VY, WZ are concurrent.

buratinogigle

#11 Oct 13, 2015, 3:54 pm

Corollary 3. If P lies on (ABC) and H is orthocenter of ABC then $PH \perp \overline{H_a, H_b, H_c}$.

TelvCohl

#12 Oct 13, 2015, 10:38 pm • 1

> buratinogigle wrote:
>
> **Corollary 3.** If P lies on (ABC) and H is orthocenter of ABC then $PH \perp \overline{H_a, H_b, H_c}$.

Proof : Let $R \equiv PH \cap \odot(ABC)$ and let R_a, R_b, R_c be the projection of R on BC, CA, AB, respectively. Let X, Y, Z be the projection of A, B, C on $\overline{R_aR_bR_c}$, respectively and let T be the orthopole of $\overline{R_aR_bR_c}$ WRT $\triangle ABC$. Since the figure $TXYZ$ and the figure $RR_aR_bR_c$ are homothetic, so $H_aH_b / H_bH_c = XY / YZ = R_aR_b / R_bR_c$.

On the other hand, we have $\measuredangle H_aQH_b = \measuredangle APB = \measuredangle ACB = \measuredangle R_bRR_a$ and similarly we get $\measuredangle H_aQH_c = \measuredangle R_cRR_a$, so the figure $QH_aH_bH_c$ is inversely similar to the figure $RR_aR_bR_c$, hence from $\measuredangle QH_bH_c = \measuredangle R_cR_bR = \measuredangle BAR = \measuredangle BPH$ we conclude that PH is perpendicular to $\overline{H_aH_bH_c}$.

Quick Reply

High School Olympiads

HK passes through the circumcenter

Reply

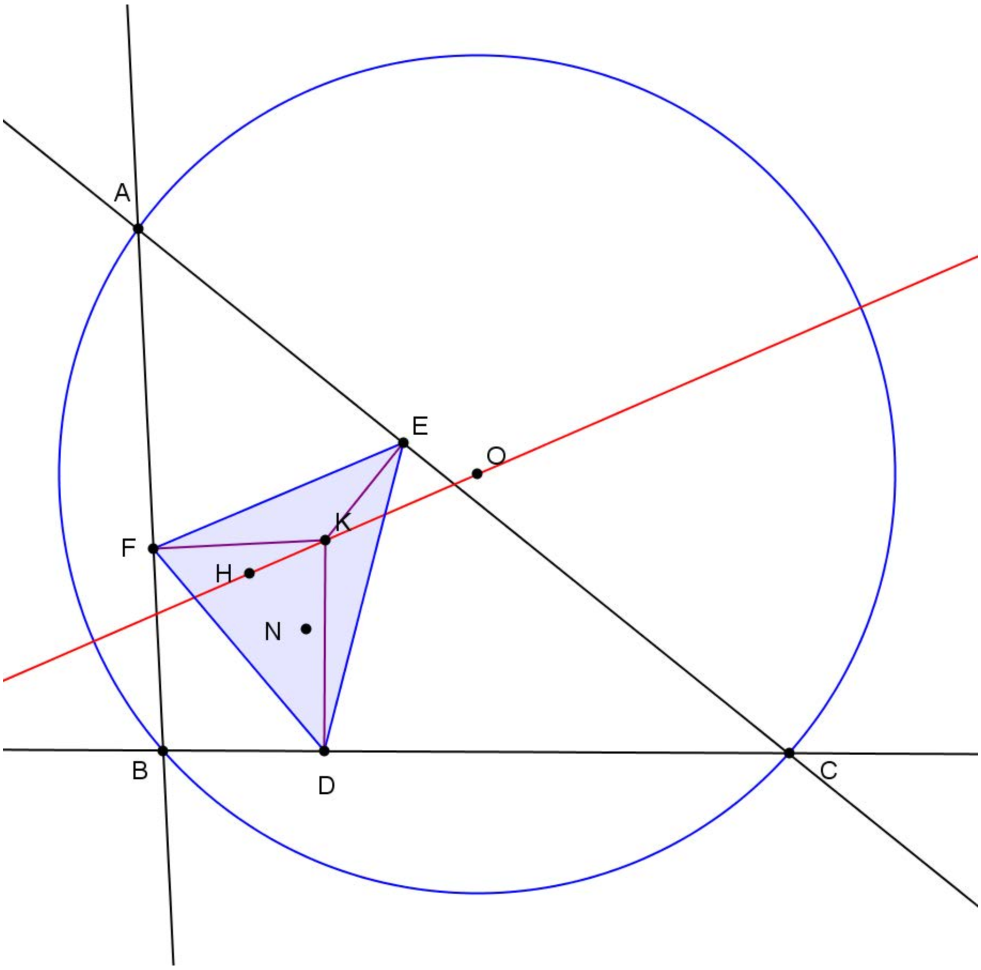
Source: Own?

THVSH

#1 Aug 5, 2015, 10:55 pm • 1

Let ABC be a triangle. K is the **Kosnita** point of $\triangle ABC$. $\triangle DEF$ is the pedal triangle of K wrt $\triangle ABC$. H is the orthocenter of $\triangle DEF$.
Prove that HK passes through the circumcenter O of $\triangle ABC$.

Attachments:



TelvCohl

#2 Aug 6, 2015, 1:38 am • 4

My solution **:**

Lemma :

Given an arbitrary point P and a $\triangle ABC$ with orthocenter H .
Let A^*, B^*, C^* be the antipode of A, B, C in $\odot(ABC)$, respectively .
Let ℓ_A be the perpendicular through A^* to AP (define ℓ_B and ℓ_C similarly) .
Let D be the point on ℓ_A such that $PD \parallel BC$ (define E and F similarly) .

Then D, E, F are collinear and $\overline{DEF} \perp PH$

Proof :

Let $\mathbf{P}(P, \odot)$ be the power of a point P WRT a circle $\odot$.
Let $\triangle XYZ$ be the circumcevian triangle of P WRT $\triangle ABC$.
Let $D^* = DP \cap AH, E^* = EP \cap BH, F^* = FP \cap CH$.

Easy to see $X \in \ell_A, Y \in \ell_B, Z \in \ell_C$.

Since A, D^*, X, D lie on a circle with diameter AD ,
so P is the radical center of $\odot(ABC), \odot(HD), \odot(AD) \Longrightarrow \mathbf{P}(P, \odot(HD)) = \mathbf{P}(P, \odot(ABC))$.
Similarly, we can prove $\mathbf{P}(P, \odot(HE)) = \mathbf{P}(P, \odot(ABC))$ and $\mathbf{P}(P, \odot(HF)) = \mathbf{P}(P, \odot(ABC))$,
so $\odot(HD), \odot(HE), \odot(HF)$ are coaxial (radical axis HP) $\Longrightarrow D, E, F$ are collinear and $\overline{DEF} \perp PH$.

From the lemma we get the following corollary $(\bigstar)$:

Let P, P^* be the isogonal conjugate of $\triangle ABC$.
Let $\mathcal{T}$ be the orthotransversal of P^* WRT $\triangle ABC$.
Let $\triangle P_AP_BP_C$ be the pedal triangle of P WRT $\triangle ABC$ and H be its orthocenter .

Then $\mathcal{T} \perp HP$

Proof :

Let $\mathcal{T}$ cut BC, CA, AB at D, E, F, respectively .
Let P_A^*, P_B^*, P_C^* be the antipode of P_A, P_B, P_C in $\odot(P_AP_BP_C)$, respectively .
Let $\boldsymbol{\ell_A}$ be the line through P_A^* and parallel to BC (define $\boldsymbol{\ell_B}$ and $\boldsymbol{\ell_C}$ similarly) .
Let D^* be a point on $\boldsymbol{\ell_A}$ such that $PD^* \parallel P_BP_C$ (define E^* and F^* similarly) .

From lemma we get D^*, E^*, F^* are collinear and $\overline{D^*E^*F^*} \perp HP$.

Since $P^*D \parallel P_BP_C \parallel PD^*, P^*E \parallel P_CP_A \parallel PE^*, P^*F \parallel P_AP_B \parallel PF^*$,
so $(D, D^*), (E, E^*), (F, F^*)$ are symmetry WRT the center of $\odot(P_AP_BP_C) \Longrightarrow \mathcal{T} \perp HP$.

Back to the main problem :

Let O_A, O_B, O_C be the reflection of O in BC, CA, AB, respectively .
Let O_a, O_b, O_c be the center of $\odot(BOC), \odot(COA), \odot(AOB)$, respectively .
Let N be the 9-point center of $\triangle ABC$ and $A^* \equiv O_bO_c \cap BC, B^* \equiv O_cO_a \cap CA, C^* \equiv O_aO_b \cap AB$.

Since K, O is the perspector, orthology center of $\triangle O_aO_bO_c, \triangle ABC$, respectively ,
so from Sondat's theorem $\Longrightarrow OK$ is perpendicular to the line τ through A^*, B^*, C^* .

Since A^* lie on the perpendicular bisector O_bO_c, BC of AO, OO_A, respectively ,
so A^* is the center of $\odot(AOO_A) \Longrightarrow AN \perp A^*N$ ($\because N$ is the midpoint of AO_A) .
Similarly, we can prove $BN \perp B^*N$ and $CN \perp C^*N \Longrightarrow \tau$ is the orthotransversal of N WRT $\triangle ABC$.
From the corollary $(\bigstar) \Longrightarrow HK \perp \tau$, so combine $OK \perp \tau$ we get O, H, K lie on the line perpendicular to τ .

Q.E.D

P.S. The fact $OK \perp \tau$ also can be proved by inversion (WRT $\odot(O)$).

Luis González

#3 Aug 6, 2015, 3:41 am • 2

Let the perpendiculars to NA, NB, NC at N cut BC, CA, AB at U, V, W, resp $\Longrightarrow \overline{UVW}$ is orthopolar of N WRT $\triangle ABC$. $\triangle NVW$ and $\triangle KFE$ are orthologic, being A the orthology center of $\triangle NVW$ WRT $\triangle KFE \Longrightarrow$ perpendiculars from F, E, K to $(NW \parallel DE), (NV \parallel DF), VW$ concur at the other orthology center $H \Longrightarrow KH \perp \overline{UVW}$ $(*)$.

On the other hand, let A', B', C' be the midpoints of OA, OB, OC. T denotes the orthocenter of $\triangle ABC$ and $X \equiv AT \cap BC$. Since $NA' \parallel AT$ and $\angle NXA = \angle OAX$, then $AA'NX$ is isosceles trapezoid, but $ANXU$ is cyclic on account of the right angles at $N, X \Longrightarrow AA'NXU$ is cyclic $\Longrightarrow UA' \perp OA$ is perpendicular bisector of OA and similarly VB' and WC' are perpendicular bisectors of OB and OC. Now, according to Geometry Problem (23), we have $OK \perp \overline{UVW}$. Together with $(*)$, it follows that O, K, H are collinear.

TelvCohl

TelvCohl

#4 Aug 6, 2015, 5:44 pm • 4

Remark : We can prove $\overline{HO} = 3\overline{HK}$ as following :

Let T be the orthocenter of $\triangle ABC$. Let K_a, H_a be the circumcenter, orthocenter of $\triangle AEF$, resp (define K_b, K_c, H_b, H_c similarly). Let $J \equiv AN \cap \odot(ABC)$ and K_a^* be the reflection of K_a in EF. From $\measuredangle JBC = \measuredangle NAC = \measuredangle BAK = \measuredangle FEK$ and $\measuredangle JCB = \measuredangle NAB = \measuredangle CAK = \measuredangle EFK$ we get $\triangle JBC \stackrel{-}{\sim} \triangle KEF$, so combine $\measuredangle CO_AB = \measuredangle BOC = 2\measuredangle BAC = \measuredangle FK_aE = \measuredangle EK_a^*F$ (and notice $O_AB = O_AC$ and $K_a^*E = K_a^*F$) $\Longrightarrow \triangle JBC \cup O_A \stackrel{-}{\sim} \triangle KEF \cup K_a^*$, hence $\measuredangle EKK_a^* = \measuredangle O_AJB = \measuredangle ACB = \measuredangle EKD \Longrightarrow K_a^* \in DK$. Since H_a is the reflection of K in the midpoint of EF, so $KK_a^*H_aK_a$ is a parallelogram, hence from $KK_a^* \perp BC$ we get $K_aH_a \perp BC \Longrightarrow K_aH_a$ passes through the midpoint M of KT ($\because K_a$ is the midpoint of KA). Similarly, we can prove $K_bH_b \perp CA, K_cH_c \perp AB$ and $M \in K_bH_b, M \in K_cH_c$.

Since H_a, H_b, H_c is the reflection of K in the midpoint of EF, FD, DE, resp, so $\triangle DEF$ and $\triangle H_aH_bH_c$ are congruent (and homothetic), hence combine with $H_aM \parallel DK, H_bM \parallel EK, H_cM \parallel FK \Longrightarrow \triangle DEF \cup K$, and $\triangle H_aH_bH_c \cup M$ are congruent (and homothetic).

Let P, Q be the 9-point center, circumcenter of $\triangle DEF$, resp. Let Q_a be the reflection of Q in EF and D^* be the reflection of D in K. From $KD^* \stackrel{\parallel}{=} DK \stackrel{\parallel}{=} MH_a \Longrightarrow H_aMKD^*$ is a parallelogram, so D^* is the reflection of M in the midpoint of EF, hence notice P is the midpoint of $DQ_a \Longrightarrow 2PK \stackrel{\parallel}{=} Q_aD^* \stackrel{\parallel}{=} QM$. Notice M, Q is the midpoint of KT, KN, resp we get $2MQ \stackrel{\parallel}{=} TN \Longrightarrow 4PK \stackrel{\parallel}{=} 2MQ \stackrel{\parallel}{=} TN$, so if $L \equiv HP \cap OT$ then we get $PK \stackrel{\parallel}{=} NL \Longrightarrow 3PK \stackrel{\parallel}{=} LO$ ($\because NO = NT$) $\Longrightarrow \overline{HO} : \overline{HK} = \overline{LO} : \overline{PK} = 3 : 1$.

buratinogigle

#5 Aug 10, 2015, 8:43 am • 1

This problem can be generalized for Euler line

Let ABC be a triangle with orthocenter H and P is a point on its Euler line ℓ. Q is isogonal conjugate of P. DEF is pedal triangle of Q and K is orthocenter of triangle DEF. QK cuts ℓ at L. R is a point on ℓ such that $(PL, RH) = -1$. Prove that $PK \parallel QR$.

Luis González

#6 Aug 10, 2015, 11:58 am • 2

> *buratinogigle wrote:*
> This problem can be generalized for Euler line
>
> Let ABC be a triangle with orthocenter H and P is a point on its Euler line ℓ. Q is isogonal conjugate of P. DEF is pedal triangle of Q and K is orthocenter of triangle DEF. QK cuts ℓ at L. R is a point on ℓ such that $(PL, RH) = -1$. Prove that $PK \parallel QR$.

Let A', B', C' denote the midpoints of EF, FD, DE, resp. $X \in \odot(DEF)$ is the projection of P on BC and PX cuts $\odot(DEF)$ again at the antipode P_A of $D \Longrightarrow P_A$ is reflection of K on A'. Thus if S is the midpoint of PK, then $A'S \parallel PP_A \parallel DQ$ and similarly we have $B'S \parallel EQ$ and $C'S \parallel FQ \Longrightarrow S$ is the complement of Q WRT $\triangle DEF \Longrightarrow A'S = \frac{1}{2}DQ$. Hence if $J \equiv DA' \cap QS$, then $JS : JQ = JA' : JD = A'S : DQ = -1 : 2. \Longrightarrow J$ is common centroid of $\triangle DEF$ and $\triangle PQK$.

Let Q_A, Q_B, Q_C be the reflections of Q on BC, CA, AB. Clearly $\triangle Q_AQ_BQ_C$ is homothetic to $\triangle DEF$ with centroid J^* on QJ. If Q_AJ^*, Q_BJ^*, Q_CJ^* cut $\odot(Q_AQ_BQ_C)$ again at J_A, J_B, J_C, then $\odot(Q_AJ_AQ), \odot(Q_BJ_BQ), \odot(Q_CJ_CQ)$ are coaxal with radical axis QJJ^*. Let U_A, U_B, U_C be the correponding centers of these circles. Since P is circumcenter of $\triangle Q_AQ_BQ_C$, then U_A is nothing but the intersection of BC with the perpendicular from P to Q_AJ_A. Since the pencil formed by Q_AJ_A, Q_AQ_C, Q_AQ_B and the parallel to Q_BQ_C from Q_A is harmonic, then the perpendiculars PU_A, PC, PB, PA to $Q_AJ_A, Q_AQ_C, Q_AQ_B, Q_BQ_C$ also formed a harmonic pencil $\Longrightarrow U_A$ is on the trilineal polar τ_P of P WRT $\triangle ABC$. Likewise $\{U_B, U_C\} \in \tau_P \Longrightarrow \tau_P \equiv U_AU_BU_C \perp QJ$ $(*)$.

On the other hand, according to Two surprising perpendiculars (see post #2), we have $QH \perp \tau_P$. Together with $(*)$, we deduce that Q, H, J are collinear, i.e. H is on Q-median QS of $\triangle PQK$. Thus if the parallel from Q to PK cuts ℓ at R^*, we get $Q(P, K, S, \infty) = Q(P, L, H, R^*) = -1 \Longrightarrow R \equiv R^* \Longrightarrow PK \parallel QR$.

Luis González

#7 Sep 3, 2015, 4:29 am • 1

From the previous generalization we get a straightforward proof of $HO = 3 \cdot HK$.

When P coincides with the 9-point center N of $\triangle ABC,$ we get that the line through K and the orthocenter T of $\triangle ABC$ goes through the midpoint M of $HN.$ Thus by Menelaus' theorem for $\triangle OHN$ cut by $\overline{TMK},$ we obtain $\frac{OK}{KH} = \frac{TO}{TN} \cdot \frac{MN}{HM} = 2 \cdot 1 = 2$ or $HO = 3 \cdot HK.$

Quick Reply

High School Math

circumcentre, incentres and excentres

Reply

AndrewTom
#1 Sep 22, 2015, 1:31 am

Prove that $OI^2 + OI_1^2 + OI_2^2 + OI_3^2 = 12R^2$

AndrewTom
#2 Sep 24, 2015, 2:24 pm

Any ideas on this one?

Luis González
#3 Sep 24, 2015, 11:37 pm • 1

Let r and r_1, r_2, r_3 denote the inradius and three exradii of $\triangle ABC$. By Euler's theorem we have $OI^2 = R^2 - 2R \cdot r$ and $OI_i{}^2 = R^2 + 2R \cdot r_i \ , i = 1, 2, 3.$ Thus adding these expressions together yields:

$$OI^2 + OI_1{}^2 + OI_2{}^2 + OI_3{}^2 = 4R^2 + 2R \cdot (r_1 + r_2 + r_3 - r) = 4R^2 + 8R^2 = 12R^2.$$

The last step follows from Steiner relation $r_1 + r_2 + r_3 = 4R + r.$

AndrewTom
#4 Sep 25, 2015, 4:37 am

Thanks Luis.

I am neither familiar with Euler's theorem for excentres nor Steiner's realtion and have posted them as separate problems.

Quick Reply

High School Math

nine-point centre and 90 degrees

Reply

AndrewTom
#1 Sep 21, 2015, 4:17 am

N is the nine-point centre of triangle ABC. If N lies on BC, prove that $|\ \angle B - \angle C\ | = 90^\circ$.

AndrewTom
#2 Sep 24, 2015, 3:05 am

Any ideas on this one?

Luis González
#3 Sep 24, 2015, 3:47 am • 1

Let D, E, F be the midpoints of BC, CA, AB and let X be the projection of A on $BC \Longrightarrow (N) \equiv \odot(DEFX)$ is the 9-point circle. Since $EF \parallel XD$, then $DXFE$ is an isosceles trapezoid $\Longrightarrow \widehat{EFX} = \widehat{FED} \Longrightarrow$ $\widehat{DFX} = |\widehat{EFX} - \widehat{EFD}| = |\widehat{FED} - \widehat{EFD}| = |\widehat{B} - \widehat{C}|$. Hence, it follows that $N \in BC \equiv XD \Longleftrightarrow \widehat{DFX} = 90^\circ$ $\Longleftrightarrow |\widehat{B} - \widehat{C}| = 90^\circ$.

Quick Reply