China National Olympiad 2007

Day 1

1 Given complex numbers a, b, c, let $|a+b| = m, |a-b| = n$. If $mn \neq 0$, Show that
$$\max\{|ac+b|, |a+bc|\} \geq \frac{mn}{\sqrt{m^2+n^2}}$$

2 Show that:
1) If $2n-1$ is a prime number, then for any n pairwise distinct positive integers $a_1, a_2, \ldots, a_n$, there exists $i, j \in \{1, 2, \ldots, n\}$ such that
$$\frac{a_i+a_j}{(a_i, a_j)} \geq 2n-1$$

2) If $2n-1$ is a composite number, then there exists n pairwise distinct positive integers $a_1, a_2, \ldots, a_n$, such that for any $i, j \in \{1, 2, \ldots, n\}$ we have
$$\frac{a_i+a_j}{(a_i, a_j)} < 2n-1$$

Here (x, y) denotes the greatest common divisor of x, y.

3 Let $a_1, a_2, \ldots, a_{11}$ be 11 pairwise distinct positive integer with sum less than 2007. Let S be the sequence of $1, 2, \ldots, 2007$. Define an **operation** to be 22 consecutive applications of the following steps on the sequence S: on i-th step, choose a number from the sequense S at random, say x. If $1 \leq i \leq 11$, replace x with $x + a_i$; if $12 \leq i \leq 22$, replace x with $x - a_{i-11}$. If the result of **operation** on the sequence S is an odd permutation of $\{1, 2, \ldots, 2007\}$, it is an **odd operation**; if the result of **operation** on the sequence S is an even permutation of $\{1, 2, \ldots, 2007\}$, it is an **even operation**. Which is larger, the number of odd operation or the number of even permutation? And by how many?

Here $\{x_1, x_2, \ldots, x_{2007}\}$ is an even permutation of $\{1, 2, \ldots, 2007\}$ if the product $\prod_{i>j}(x_i - x_j)$ is positive, and an odd one otherwise.

Day 2

1 Let O, I be the circumcenter and incenter of triangle ABC. The incircle of $\triangle ABC$ touches BC, CA, AB at points D, E, F repsectively. FD meets CA at P, ED meets AB at Q. M and N are midpoints of PE and QF respectively. Show that $OI \perp MN$.

2 Let $\{a_n\}_{n\geq 1}$ be a bounded sequence satisfying

$$a_n < \sum_{k=a}^{2n+2006} \frac{a_k}{k+1} + \frac{1}{2n+2007} \quad \forall \quad n = 1, 2, 3, \ldots$$

Show that

$$a_n < \frac{1}{n} \quad \forall \quad n = 1, 2, 3, \ldots$$

3 Find a number $n \geq 9$ such that for any n numbers, not necessarily distinct, $a_1, a_2, \ldots, a_n$, there exists 9 numbers $a_{i_1}, a_{i_2}, \ldots, a_{i_9}, (1 \leq i_1 < i_2 < \ldots < i_9 \leq n)$ and $b_i \in 4, 7, i = 1, 2, \ldots, 9$ such that $b_1 a_{i_1} + b_2 a_{i_2} + \ldots + b_9 a_{i_9}$ is a multiple of 9.