



# Art of Problem Solving

## 2017 USAMO

USAMO 2017

Day 1

April 19th

1 Prove that there are infinitely many distinct pairs  $(a, b)$  of relatively prime integers  $a > 1$  and  $b > 1$  such that  $a^b + b^a$  is divisible by  $a + b$ .

2 Let  $m_1, m_2, \dots, m_n$  be a collection of  $n$  positive integers, not necessarily distinct. For any sequence of integers  $A = (a_1, \dots, a_n)$  and any permutation  $w = w_1, \dots, w_n$  of  $m_1, \dots, m_n$ , define an  $[i]A$ -inversion $[/i]$  of  $w$  to be a pair of entries  $w_i, w_j$  with  $i < j$  for which one of the following conditions holds:

- $a_i \geq w_i > w_j$
- $w_j > a_i \geq w_i$ , or
- $w_i > w_j > a_i$ .

Show that, for any two sequences of integers  $A = (a_1, \dots, a_n)$  and  $B = (b_1, \dots, b_n)$ , and for any positive integer  $k$ , the number of permutations of  $m_1, \dots, m_n$  having exactly  $k$   $A$ -inversions is equal to the number of permutations of  $m_1, \dots, m_n$  having exactly  $k$   $B$ -inversions.

3 Let  $ABC$  be a scalene triangle with circumcircle  $\Omega$  and incenter  $I$ . Ray  $AI$  meets  $\overline{BC}$  at  $D$  and meets  $\Omega$  again at  $M$ ; the circle with diameter  $\overline{DM}$  cuts  $\Omega$  again at  $K$ . Lines  $MK$  and  $BC$  meet at  $S$ , and  $N$  is the midpoint of  $\overline{IS}$ . The circumcircles of  $\triangle KID$  and  $\triangle MAN$  intersect at points  $L_1$  and  $L_2$ . Prove that  $\Omega$  passes through the midpoint of either  $\overline{IL_1}$  or  $\overline{IL_2}$ .

*Proposed by Evan Chen*

Day 2

April 20th

4 Let  $P_1, P_2, \dots, P_{2n}$  be  $2n$  distinct points on the unit circle  $x^2 + y^2 = 1$ , other than  $(1, 0)$ . Each point is colored either red or blue, with exactly  $n$  red points and  $n$  blue points. Let  $R_1, R_2, \dots, R_n$  be any ordering of the red points. Let  $B_1$  be the nearest blue point to  $R_1$  traveling counterclockwise around the circle starting from  $R_1$ . Then let  $B_2$  be the nearest of the remaining blue points to  $R_2$  travelling counterclockwise around the circle from  $R_2$ , and so on, until we have labeled all of the blue points  $B_1, \dots, B_n$ . Show that the number of counterclockwise arcs of the form  $R_i \rightarrow B_i$  that contain the point  $(1, 0)$  is independent of the way we chose the ordering  $R_1, \dots, R_n$  of the red points.

# Art of Problem Solving

## 2017 USAMO

- 5 Let  $\mathbf{Z}$  denote the set of all integers. Find all real numbers  $c > 0$  such that there exists a labeling of the lattice points  $(x, y) \in \mathbf{Z}^2$  with positive integers for which:
- only finitely many distinct labels occur, and
  - for each label  $i$ , the distance between any two points labeled  $i$  is at least  $c^i$ .

- 6 Find the minimum possible value of

$$\frac{a}{b^3 + 4} + \frac{b}{c^3 + 4} + \frac{c}{d^3 + 4} + \frac{d}{a^3 + 4}$$

given that  $a, b, c, d$  are nonnegative real numbers such that  $a + b + c + d = 4$ .



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USAMO 2016

**Day 1**                      April 19th

- 1**                      Let  $X_1, X_2, \dots, X_{100}$  be a sequence of mutually distinct nonempty subsets of a set  $S$ . Any two sets  $X_i$  and  $X_{i+1}$  are disjoint and their union is not the whole set  $S$ , that is,  $X_i \cap X_{i+1} = \emptyset$  and  $X_i \cup X_{i+1} \neq S$ , for all  $i \in \{1, \dots, 99\}$ . Find the smallest possible number of elements in  $S$ .

- 2**                      Prove that for any positive integer  $k$ ,

$$(k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

is an integer.

- 3**                      Let  $\triangle ABC$  be an acute triangle, and let  $I_B, I_C$ , and  $O$  denote its  $B$ -excenter,  $C$ -excenter, and circumcenter, respectively. Points  $E$  and  $Y$  are selected on  $\overline{AC}$  such that  $\angle ABY = \angle CBY$  and  $\overline{BE} \perp \overline{AC}$ . Similarly, points  $F$  and  $Z$  are selected on  $\overline{AB}$  such that  $\angle ACZ = \angle BCZ$  and  $\overline{CF} \perp \overline{AB}$ . Lines  $\overleftrightarrow{I_B F}$  and  $\overleftrightarrow{I_C E}$  meet at  $P$ . Prove that  $\overline{PO}$  and  $\overline{YZ}$  are perpendicular.  
*Proposed by Evan Chen and Telv Cohl*

**Day 2**                      April 20th

- 4**                      Find all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that for all real numbers  $x$  and  $y$ ,

$$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2.$$

- 5**                      An equilateral pentagon  $AMNPQ$  is inscribed in triangle  $ABC$  such that  $M \in \overline{AB}$ ,  $Q \in \overline{AC}$ , and  $N, P \in \overline{BC}$ . Let  $S$  be the intersection of  $\overleftrightarrow{MN}$  and  $\overleftrightarrow{PQ}$ . Denote by  $\ell$  the angle bisector of  $\angle MSQ$ .  
Prove that  $\overline{OI}$  is parallel to  $\ell$ , where  $O$  is the circumcenter of triangle  $ABC$ , and  $I$  is the incenter of triangle  $ABC$ .

- 6**                      Integers  $n$  and  $k$  are given, with  $n \geq k \geq 2$ . You play the following game against an evil wizard.

# Art of Problem Solving

## 2016 USAMO

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The wizard has  $2n$  cards; for each  $i = 1, \dots, n$ , there are two cards labeled  $i$ . Initially, the wizard places all cards face down in a row, in unknown order.

You may repeatedly make moves of the following form: you point to any  $k$  of the cards. The wizard then turns those cards face up. If any two of the cards match, the game is over and you win. Otherwise, you must look away, while the wizard arbitrarily permutes the  $k$  chosen cards and then turns them back face-down. Then, it is your turn again.

We say this game is *winnable* if there exist some positive integer  $m$  and some strategy that is guaranteed to win in at most  $m$  moves, no matter how the wizard responds.

For which values of  $n$  and  $k$  is the game winnable?

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USAMO 2015

### Day 1

- 1 Solve in integers the equation

$$x^2 + xy + y^2 = \left( \frac{x+y}{3} + 1 \right)^3.$$

- 2 Quadrilateral  $APBQ$  is inscribed in circle  $\omega$  with  $\angle P = \angle Q = 90^\circ$  and  $AP = AQ < BP$ . Let  $X$  be a variable point on segment  $\overline{PQ}$ . Line  $AX$  meets  $\omega$  again at  $S$  (other than  $A$ ). Point  $T$  lies on arc  $AQB$  of  $\omega$  such that  $\overline{XT}$  is perpendicular to  $\overline{AX}$ . Let  $M$  denote the midpoint of chord  $\overline{ST}$ . As  $X$  varies on segment  $\overline{PQ}$ , show that  $M$  moves along a circle.

- 3 Let  $S = \{1, 2, \dots, n\}$ , where  $n \geq 1$ . Each of the  $2^n$  subsets of  $S$  is to be colored red or blue. (The subset itself is assigned a color and not its individual elements.) For any set  $T \subseteq S$ , we then write  $f(T)$  for the number of subsets of  $T$  that are blue.
- Determine the number of colorings that satisfy the following condition: for any subsets  $T_1$  and  $T_2$  of  $S$ ,

$$f(T_1)f(T_2) = f(T_1 \cup T_2)f(T_1 \cap T_2).$$

### Day 2 [b]

- 4 Steve is piling  $m \geq 1$  indistinguishable stones on the squares of an  $n \times n$  grid. Each square can have an arbitrarily high pile of stones. After he finished piling his stones in some manner, he can then perform *stone moves*, defined as follows. Consider any four grid squares, which are corners of a rectangle, i.e. in positions  $(i, k), (i, l), (j, k), (j, l)$  for some  $1 \leq i, j, k, l \leq n$ , such that  $i < j$  and  $k < l$ . A stone move consists of either removing one stone from each of  $(i, k)$  and  $(j, l)$  and moving them to  $(i, l)$  and  $(j, k)$  respectively, or removing one stone from each of  $(i, l)$  and  $(j, k)$  and moving them to  $(i, k)$  and  $(j, l)$  respectively.
- Two ways of piling the stones are equivalent if they can be obtained from one another by a sequence of stone moves.
- How many different non-equivalent ways can Steve pile the stones on the grid?



# Art of Problem Solving

## 2015 USAMO

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- 5 Let  $a, b, c, d, e$  be distinct positive integers such that  $a^4 + b^4 = c^4 + d^4 = e^5$ . Show that  $ac + bd$  is a composite number.
- 
- 6 Consider  $0 < \lambda < 1$ , and let  $A$  be a multiset of positive integers. Let  $A_n = \{a \in A : a \leq n\}$ . Assume that for every  $n \in \mathbb{N}$ , the set  $A_n$  contains at most  $n\lambda$  numbers. Show that there are infinitely many  $n \in \mathbb{N}$  for which the sum of the elements in  $A_n$  is at most  $\frac{n(n+1)}{2}\lambda$ . (A multiset is a set-like collection of elements in which order is ignored, but repetition of elements is allowed and multiplicity of elements is significant. For example, multisets  $\{1, 2, 3\}$  and  $\{2, 1, 3\}$  are equivalent, but  $\{1, 1, 2, 3\}$  and  $\{1, 2, 3\}$  differ.)
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USAMO 2014

Day 1

April 29th

1

Let  $a, b, c, d$  be real numbers such that  $b - d \geq 5$  and all zeros  $x_1, x_2, x_3$ , and  $x_4$  of the polynomial  $P(x) = x^4 + ax^3 + bx^2 + cx + d$  are real. Find the smallest value the product  $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$  can take.

2

Let  $\mathbb{Z}$  be the set of integers. Find all functions  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  such that

$$xf(2f(y) - x) + y^2f(2x - f(y)) = \frac{f(x)^2}{x} + f(yf(y))$$

for all  $x, y \in \mathbb{Z}$  with  $x \neq 0$ .

3

Prove that there exists an infinite set of points

$$\dots, P_{-3}, P_{-2}, P_{-1}, P_0, P_1, P_2, P_3, \dots$$

in the plane with the following property: For any three distinct integers  $a, b$ , and  $c$ , points  $P_a, P_b$ , and  $P_c$  are collinear if and only if  $a + b + c = 2014$ .

Day 2

April 30th

4

Let  $k$  be a positive integer. Two players  $A$  and  $B$  play a game on an infinite grid of regular hexagons. Initially all the grid cells are empty. Then the players alternately take turns with  $A$  moving first. In his move,  $A$  may choose two adjacent hexagons in the grid which are empty and place a counter in both of them. In his move,  $B$  may choose any counter on the board and remove it. If at any time there are  $k$  consecutive grid cells in a line all of which contain a counter,  $A$  wins. Find the minimum value of  $k$  for which  $A$  cannot win in a finite number of moves, or prove that no such minimum value exists.

5

Let  $ABC$  be a triangle with orthocenter  $H$  and let  $P$  be the second intersection of the circumcircle of triangle  $AHC$  with the internal bisector of the angle  $\angle BAC$ . Let  $X$  be the circumcenter of triangle  $APB$  and  $Y$  the orthocenter of triangle  $APC$ . Prove that the length of segment  $XY$  is equal to the circumradius of triangle  $ABC$ .

6

Prove that there is a constant  $c > 0$  with the following property: If  $a, b, n$  are positive integers such that  $\gcd(a + i, b + j) > 1$  for all  $i, j \in \{0, 1, \dots, n\}$ , then

$$\min\{a, b\} > c^n \cdot n^{\frac{n}{2}}.$$



# Art of Problem Solving

## 2013 USAMO

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USAMO 2013

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Day 1

April 30th

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- 1 In triangle  $ABC$ , points  $P, Q, R$  lie on sides  $BC, CA, AB$  respectively. Let  $\omega_A, \omega_B, \omega_C$  denote the circumcircles of triangles  $AQR, BRP, CPQ$ , respectively. Given the fact that segment  $AP$  intersects  $\omega_A, \omega_B, \omega_C$  again at  $X, Y, Z$ , respectively, prove that  $YX/XZ = BP/PC$ .
- 
- 2 For a positive integer  $n \geq 3$  plot  $n$  equally spaced points around a circle. Label one of them  $A$ , and place a marker at  $A$ . One may move the marker forward in a clockwise direction to either the next point or the point after that. Hence there are a total of  $2n$  distinct moves available; two from each point. Let  $a_n$  count the number of ways to advance around the circle exactly twice, beginning and ending at  $A$ , without repeating a move. Prove that  $a_{n-1} + a_n = 2^n$  for all  $n \geq 4$ .
- 
- 3 Let  $n$  be a positive integer. There are  $\frac{n(n+1)}{2}$  marks, each with a black side and a white side, arranged into an equilateral triangle, with the biggest row containing  $n$  marks. Initially, each mark has the black side up. An *operation* is to choose a line parallel to the sides of the triangle, and flipping all the marks on that line. A configuration is called *admissible* if it can be obtained from the initial configuration by performing a finite number of operations. For each admissible configuration  $C$ , let  $f(C)$  denote the smallest number of operations required to obtain  $C$  from the initial configuration. Find the maximum value of  $f(C)$ , where  $C$  varies over all admissible configurations.
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Day 2

May 1st

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- 4 Find all real numbers  $x, y, z \geq 1$  satisfying
- $$\min(\sqrt{x + xyz}, \sqrt{y + xyz}, \sqrt{z + xyz}) = \sqrt{x - 1} + \sqrt{y - 1} + \sqrt{z - 1}.$$
- 
- 5 Given positive integers  $m$  and  $n$ , prove that there is a positive integer  $c$  such that the numbers  $cm$  and  $cn$  have the same number of occurrences of each non-zero digit when written in base ten.
-



6

Let  $ABC$  be a triangle. Find all points  $P$  on segment  $BC$  satisfying the following property: If  $X$  and  $Y$  are the intersections of line  $PA$  with the common external tangent lines of the circumcircles of triangles  $PAB$  and  $PAC$ , then

$$\left(\frac{PA}{XY}\right)^2 + \frac{PB \cdot PC}{AB \cdot AC} = 1.$$



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# Art of Problem Solving

## 2012 USAMO

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USAMO 2012

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**Day 1**

April 24th

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**1** Find all integers  $n \geq 3$  such that among any  $n$  positive real numbers  $a_1, a_2, \dots, a_n$  with  $\max(a_1, a_2, \dots, a_n) \leq n \cdot \min(a_1, a_2, \dots, a_n)$ , there exist three that are the side lengths of an acute triangle.

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**2** A circle is divided into 432 congruent arcs by 432 points. The points are colored in four colors such that some 108 points are colored Red, some 108 points are colored Green, some 108 points are colored Blue, and the remaining 108 points are colored Yellow. Prove that one can choose three points of each color in such a way that the four triangles formed by the chosen points of the same color are congruent.

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**3** Determine which integers  $n > 1$  have the property that there exists an infinite sequence  $a_1, a_2, a_3, \dots$  of nonzero integers such that the equality

$$a_k + 2a_{2k} + \dots + na_{nk} = 0$$

holds for every positive integer  $k$ .

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**Day 2**

April 25th

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**4** Find all functions  $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$  (where  $\mathbb{Z}^+$  is the set of positive integers) such that  $f(n!) = f(n)!$  for all positive integers  $n$  and such that  $m - n$  divides  $f(m) - f(n)$  for all distinct positive integers  $m, n$ .

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**5** Let  $P$  be a point in the plane of  $\triangle ABC$ , and  $\gamma$  a line passing through  $P$ . Let  $A', B', C'$  be the points where the reflections of lines  $PA, PB, PC$  with respect to  $\gamma$  intersect lines  $BC, AC, AB$  respectively. Prove that  $A', B', C'$  are collinear.

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**6** For integer  $n \geq 2$ , let  $x_1, x_2, \dots, x_n$  be real numbers satisfying

$$x_1 + x_2 + \dots + x_n = 0, \quad \text{and} \quad x_1^2 + x_2^2 + \dots + x_n^2 = 1.$$

For each subset  $A \subseteq \{1, 2, \dots, n\}$ , define

$$S_A = \sum_{i \in A} x_i.$$



# Art of Problem Solving

## 2012 USAMO

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(If  $A$  is the empty set, then  $S_A = 0$ .)

Prove that for any positive number  $\lambda$ , the number of sets  $A$  satisfying  $S_A \geq \lambda$  is at most  $2^{n-3}/\lambda^2$ . For which choices of  $x_1, x_2, \dots, x_n, \lambda$  does equality hold?

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USAMO 2011

Day 1

April 27th

- 1 Let  $a, b, c$  be positive real numbers such that  $a^2 + b^2 + c^2 + (a + b + c)^2 \leq 4$ . Prove that

$$\frac{ab + 1}{(a + b)^2} + \frac{bc + 1}{(b + c)^2} + \frac{ca + 1}{(c + a)^2} \geq 3.$$

- 2 An integer is assigned to each vertex of a regular pentagon so that the sum of the five integers is 2011. A turn of a solitaire game consists of subtracting an integer  $m$  from each of the integers at two neighboring vertices and adding  $2m$  to the opposite vertex, which is not adjacent to either of the first two vertices. (The amount  $m$  and the vertices chosen can vary from turn to turn.) The game is won at a certain vertex if, after some number of turns, that vertex has the number 2011 and the other four vertices have the number 0. Prove that for any choice of the initial integers, there is exactly one vertex at which the game can be won.

- 3 In hexagon  $ABCDEF$ , which is nonconvex but not self-intersecting, no pair of opposite sides are parallel. The internal angles satisfy  $\angle A = 3\angle D$ ,  $\angle C = 3\angle F$ , and  $\angle E = 3\angle B$ . Furthermore  $AB = DE$ ,  $BC = EF$ , and  $CD = FA$ . Prove that diagonals  $\overline{AD}$ ,  $\overline{BE}$ , and  $\overline{CF}$  are concurrent.

Day 2

April 28th

- 4 Consider the assertion that for each positive integer  $n \geq 2$ , the remainder upon dividing  $2^{2^n}$  by  $2^n - 1$  is a power of 4. Either prove the assertion or find (with proof) a counterexample.

- 5 Let  $P$  be a given point inside quadrilateral  $ABCD$ . Points  $Q_1$  and  $Q_2$  are located within  $ABCD$  such that
- $$\angle Q_1BC = \angle ABP, \quad \angle Q_1CB = \angle DCP, \quad \angle Q_2AD = \angle BAP, \quad \angle Q_2DA = \angle CDP.$$
- Prove that  $\overline{Q_1Q_2} \parallel \overline{AB}$  if and only if  $\overline{Q_1Q_2} \parallel \overline{CD}$ .

- 6 Let  $A$  be a set with  $|A| = 225$ , meaning that  $A$  has 225 elements. Suppose further that there are eleven subsets  $A_1, \dots, A_{11}$  of  $A$  such that  $|A_i| = 45$  for  $1 \leq i \leq 11$  and  $|A_i \cap A_j| = 9$  for  $1 \leq i < j \leq 11$ . Prove that  $|A_1 \cup A_2 \cup \dots \cup A_{11}| \geq 165$ , and give an example for which equality holds.



# Art of Problem Solving

## 2010 USAMO

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USAMO 2010

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Day 1                      April 27th

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- 1                      Let  $AXYZB$  be a convex pentagon inscribed in a semicircle of diameter  $AB$ . Denote by  $P, Q, R, S$  the feet of the perpendiculars from  $Y$  onto lines  $AX, BX, AZ, BZ$ , respectively. Prove that the acute angle formed by lines  $PQ$  and  $RS$  is half the size of  $\angle XOZ$ , where  $O$  is the midpoint of segment  $AB$ .
- 
- 2                      There are  $n$  students standing in a circle, one behind the other. The students have heights  $h_1 < h_2 < \dots < h_n$ . If a student with height  $h_k$  is standing directly behind a student with height  $h_{k-2}$  or less, the two students are permitted to switch places. Prove that it is not possible to make more than  $\binom{n}{3}$  such switches before reaching a position in which no further switches are possible.
- 
- 3                      The 2010 positive numbers  $a_1, a_2, \dots, a_{2010}$  satisfy the inequality  $a_i a_j \leq i + j$  for all distinct indices  $i, j$ . Determine, with proof, the largest possible value of the product  $a_1 a_2 \dots a_{2010}$ .
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Day 2                      April 28th

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- 4                      Let  $ABC$  be a triangle with  $\angle A = 90^\circ$ . Points  $D$  and  $E$  lie on sides  $AC$  and  $AB$ , respectively, such that  $\angle ABD = \angle DBC$  and  $\angle ACE = \angle ECB$ . Segments  $BD$  and  $CE$  meet at  $I$ . Determine whether or not it is possible for segments  $AB, AC, BI, ID, CI, IE$  to all have integer lengths.
- 
- 5                      Let  $q = \frac{3p-5}{2}$  where  $p$  is an odd prime, and let
- $$S_q = \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{5 \cdot 6 \cdot 7} + \dots + \frac{1}{q(q+1)(q+2)}$$
- Prove that if  $\frac{1}{p} - 2S_q = \frac{m}{n}$  for integers  $m$  and  $n$ , then  $m - n$  is divisible by  $p$ .
- 
- 6                      A blackboard contains 68 pairs of nonzero integers. Suppose that for each positive integer  $k$  at most one of the pairs  $(k, k)$  and  $(-k, -k)$  is written on the blackboard. A student erases some of the 136 integers, subject to the condition that no two erased integers may add to 0. The student then scores one point for each of the 68 pairs in which at least one integer is erased. Determine, with proof, the largest number  $N$  of points that the student can guarantee to score regardless of which 68 pairs have been written on the board.
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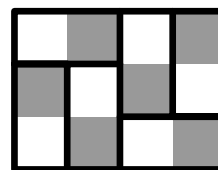
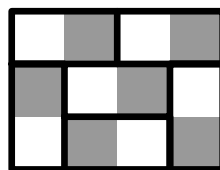
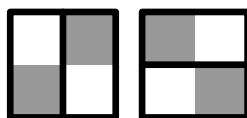
USAMO 2009

Day 1

April 28th

- 1** Given circles  $\omega_1$  and  $\omega_2$  intersecting at points  $X$  and  $Y$ , let  $\ell_1$  be a line through the center of  $\omega_1$  intersecting  $\omega_2$  at points  $P$  and  $Q$  and let  $\ell_2$  be a line through the center of  $\omega_2$  intersecting  $\omega_1$  at points  $R$  and  $S$ . Prove that if  $P, Q, R$  and  $S$  lie on a circle then the center of this circle lies on line  $XY$ .
- 2** Let  $n$  be a positive integer. Determine the size of the largest subset of  $\{-n, -n+1, \dots, n-1, n\}$  which does not contain three elements  $a, b, c$  (not necessarily distinct) satisfying  $a + b + c = 0$ .
- 3** We define a *chessboard polygon* to be a polygon whose sides are situated along lines of the form  $x = a$  or  $y = b$ , where  $a$  and  $b$  are integers. These lines divide the interior into unit squares, which are shaded alternately grey and white so that adjacent squares have different colors. To tile a chessboard polygon by dominoes is to exactly cover the polygon by non-overlapping  $1 \times 2$  rectangles. Finally, a *tasteful tiling* is one which avoids the two configurations of dominoes shown on the left below. Two tilings of a  $3 \times 4$  rectangle are shown; the first one is tasteful, while the second is not, due to the vertical dominoes in the upper right corner.

Distasteful tilings



- a) Prove that if a chessboard polygon can be tiled by dominoes, then it can be done so tastefully.
- b) Prove that such a tasteful tiling is unique.

Day 2

April 29th

- 4 For  $n \geq 2$  let  $a_1, a_2, \dots, a_n$  be positive real numbers such that

$$(a_1 + a_2 + \dots + a_n) \left( \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right) \leq \left( n + \frac{1}{2} \right)^2.$$

Prove that  $\max(a_1, a_2, \dots, a_n) \leq 4 \min(a_1, a_2, \dots, a_n)$ .

- 5 Trapezoid  $ABCD$ , with  $\overline{AB} \parallel \overline{CD}$ , is inscribed in circle  $\omega$  and point  $G$  lies inside triangle  $BCD$ . Rays  $AG$  and  $BG$  meet  $\omega$  again at points  $P$  and  $Q$ , respectively. Let the line through  $G$  parallel to  $\overline{AB}$  intersect  $\overline{BD}$  and  $\overline{BC}$  at points  $R$  and  $S$ , respectively. Prove that quadrilateral  $PQRS$  is cyclic if and only if  $\overline{BG}$  bisects  $\angle CBD$ .

- 6 Let  $s_1, s_2, s_3, \dots$  be an infinite, nonconstant sequence of rational numbers, meaning it is not the case that  $s_1 = s_2 = s_3 = \dots$ . Suppose that  $t_1, t_2, t_3, \dots$  is also an infinite, nonconstant sequence of rational numbers with the property that  $(s_i - s_j)(t_i - t_j)$  is an integer for all  $i$  and  $j$ . Prove that there exists a rational number  $r$  such that  $(s_i - s_j)r$  and  $(t_i - t_j)/r$  are integers for all  $i$  and  $j$ .



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# Art of Problem Solving

## 2008 USAMO

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USAMO 2008

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Day 1

April 29th

- 
- 1** Prove that for each positive integer  $n$ , there are pairwise relatively prime integers  $k_0, k_1, \dots, k_n$ , all strictly greater than 1, such that  $k_0 k_1 \dots k_n - 1$  is the product of two consecutive integers.
- 
- 2** Let  $ABC$  be an acute, scalene triangle, and let  $M$ ,  $N$ , and  $P$  be the midpoints of  $\overline{BC}$ ,  $\overline{CA}$ , and  $\overline{AB}$ , respectively. Let the perpendicular bisectors of  $\overline{AB}$  and  $\overline{AC}$  intersect ray  $AM$  in points  $D$  and  $E$  respectively, and let lines  $BD$  and  $CE$  intersect in point  $F$ , inside of triangle  $ABC$ . Prove that points  $A$ ,  $N$ ,  $F$ , and  $P$  all lie on one circle.
- 
- 3** Let  $n$  be a positive integer. Denote by  $S_n$  the set of points  $(x, y)$  with integer coordinates such that
- $$|x| + \left| y + \frac{1}{2} \right| < n.$$
- A path is a sequence of distinct points  $(x_1, y_1), (x_2, y_2), \dots, (x_\ell, y_\ell)$  in  $S_n$  such that, for  $i = 2, \dots, \ell$ , the distance between  $(x_i, y_i)$  and  $(x_{i-1}, y_{i-1})$  is 1 (in other words, the points  $(x_i, y_i)$  and  $(x_{i-1}, y_{i-1})$  are neighbors in the lattice of points with integer coordinates). Prove that the points in  $S_n$  cannot be partitioned into fewer than  $n$  paths (a partition of  $S_n$  into  $m$  paths is a set  $\mathcal{P}$  of  $m$  nonempty paths such that each point in  $S_n$  appears in exactly one of the  $m$  paths in  $\mathcal{P}$ ).
- 

Day 2

April 30th

- 
- 4** Let  $\mathcal{P}$  be a convex polygon with  $n$  sides,  $n \geq 3$ . Any set of  $n - 3$  diagonals of  $\mathcal{P}$  that do not intersect in the interior of the polygon determine a *triangulation* of  $\mathcal{P}$  into  $n - 2$  triangles. If  $\mathcal{P}$  is regular and there is a triangulation of  $\mathcal{P}$  consisting of only isosceles triangles, find all the possible values of  $n$ .
- 
- 5** Three nonnegative real numbers  $r_1, r_2, r_3$  are written on a blackboard. These numbers have the property that there exist integers  $a_1, a_2, a_3$ , not all zero, satisfying  $a_1 r_1 + a_2 r_2 + a_3 r_3 = 0$ . We are permitted to perform the following operation: find two numbers  $x, y$  on the blackboard with  $x \leq y$ , then erase  $y$  and write  $y - x$  in its place. Prove that after a finite number of such operations, we can end up with at least one 0 on the blackboard.
-



6

At a certain mathematical conference, every pair of mathematicians are either friends or strangers. At mealtime, every participant eats in one of two large dining rooms. Each mathematician insists upon eating in a room which contains an even number of his or her friends. Prove that the number of ways that the mathematicians may be split between the two rooms is a power of two (i.e., is of the form  $2^k$  for some positive integer  $k$ ).

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USAMO 2007

Day 1

April 24th

- 1 Let  $n$  be a positive integer. Define a sequence by setting  $a_1 = n$  and, for each  $k > 1$ , letting  $a_k$  be the unique integer in the range  $0 \leq a_k \leq k - 1$  for which  $a_1 + a_2 + \dots + a_k$  is divisible by  $k$ . For instance, when  $n = 9$  the obtained sequence is  $9, 1, 2, 0, 3, 3, 3, \dots$ . Prove that for any  $n$  the sequence  $a_1, a_2, \dots$  eventually becomes constant.
- 2 A square grid on the Euclidean plane consists of all points  $(m, n)$ , where  $m$  and  $n$  are integers. Is it possible to cover all grid points by an infinite family of discs with non-overlapping interiors if each disc in the family has radius at least 5?
- 3 Let  $S$  be a set containing  $n^2 + n - 1$  elements, for some positive integer  $n$ . Suppose that the  $n$ -element subsets of  $S$  are partitioned into two classes. Prove that there are at least  $n$  pairwise disjoint sets in the same class.

Day 2

April 25th

- 4 An *animal* with  $n$  cells is a connected figure consisting of  $n$  equal-sized cells[1]. A *dinosaur* is an animal with at least 2007 cells. It is said to be *primitive* if its cells cannot be partitioned into two or more dinosaurs. Find with proof the maximum number of cells in a primitive dinosaur.  
  
(1) Animals are also called *polyominoes*. They can be defined inductively. Two cells are *adjacent* if they share a complete edge. A single cell is an animal, and given an animal with  $n$  cells, one with  $n + 1$  cells is obtained by adjoining a new cell by making it adjacent to one or more existing cells.
- 5 Prove that for every nonnegative integer  $n$ , the number  $7^{7^n} + 1$  is the product of at least  $2n + 3$  (not necessarily distinct) primes.
- 6 Let  $ABC$  be an acute triangle with  $\omega$ ,  $S$ , and  $R$  being its incircle, circumcircle, and circumradius, respectively. Circle  $\omega_A$  is tangent internally to  $S$  at  $A$  and tangent externally to  $\omega$ . Circle  $S_A$  is tangent internally to  $S$  at  $A$  and tangent internally to  $\omega$ . Let  $P_A$  and  $Q_A$  denote the centers of  $\omega_A$  and  $S_A$ , respectively. Define points  $P_B, Q_B, P_C, Q_C$  analogously. Prove that

$$8P_AQ_A \cdot P_BQ_B \cdot P_CQ_C \leq R^3,$$



# Art of Problem Solving

## 2007 USAMO

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with equality if and only if triangle  $ABC$  is equilateral.

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USAMO 2006

### Day 1

- 1 Let  $p$  be a prime number and let  $s$  be an integer with  $0 < s < p$ . Prove that there exist integers  $m$  and  $n$  with  $0 < m < n < p$  and

$$\left\{ \frac{sm}{p} \right\} < \left\{ \frac{sn}{p} \right\} < \frac{s}{p}$$

if and only if  $s$  is not a divisor of  $p - 1$ .

Note: For  $x$  a real number, let  $[x]$  denote the greatest integer less than or equal to  $x$ , and let  $\{x\} = x - [x]$  denote the fractional part of  $x$ .

- 2 For a given positive integer  $k$  find, in terms of  $k$ , the minimum value of  $N$  for which there is a set of  $2k + 1$  distinct positive integers that has sum greater than  $N$  but every subset of size  $k$  has sum at most  $\frac{N}{2}$ .

- 3 For integral  $m$ , let  $p(m)$  be the greatest prime divisor of  $m$ . By convention, we set  $p(\pm 1) = 1$  and  $p(0) = \infty$ . Find all polynomials  $f$  with integer coefficients such that the sequence

$$\{p(f(n^2)) - 2n\}_{n \geq 0}$$

is bounded above. (In particular, this requires  $f(n^2) \neq 0$  for  $n \geq 0$ .)

### Day 2

- 4 Find all positive integers  $n$  such that there are  $k \geq 2$  positive rational numbers  $a_1, a_2, \dots, a_k$  satisfying  $a_1 + a_2 + \dots + a_k = a_1 \cdot a_2 \cdots a_k = n$ .

- 5 A mathematical frog jumps along the number line. The frog starts at 1, and jumps according to the following rule: if the frog is at integer  $n$ , then it can jump either to  $n + 1$  or to  $n + 2^{m_n+1}$  where  $2^{m_n}$  is the largest power of 2 that is a factor of  $n$ . Show that if  $k \geq 2$  is a positive integer and  $i$  is a nonnegative integer, then the minimum number of jumps needed to reach  $2^i k$  is greater than the minimum number of jumps needed to reach  $2^i$ .

- 6 Let  $ABCD$  be a quadrilateral, and let  $E$  and  $F$  be points on sides  $AD$  and  $BC$ , respectively, such that  $\frac{AE}{ED} = \frac{BF}{FC}$ . Ray  $FE$  meets rays  $BA$  and  $CD$  at  $S$  and  $T$ , respectively. Prove that the circumcircles of triangles  $SAE$ ,  $SBF$ ,  $TCF$ , and  $TDE$  pass through a common point.
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# Art of Problem Solving

## 2005 USAMO

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USAMO 2005

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**Day 1**                      April 19th

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**1**                      Determine all composite positive integers  $n$  for which it is possible to arrange all divisors of  $n$  that are greater than 1 in a circle so that no two adjacent divisors are relatively prime.

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**2**                      Prove that the system

$$\begin{aligned}x^6 + x^3 + x^3y + y &= 147^{157} \\ x^3 + x^3y + y^2 + y + z^9 &= 157^{147}\end{aligned}$$

has no solutions in integers  $x$ ,  $y$ , and  $z$ .

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**3**                      Let  $ABC$  be an acute-angled triangle, and let  $P$  and  $Q$  be two points on its side  $BC$ . Construct a point  $C_1$  in such a way that the convex quadrilateral  $APBC_1$  is cyclic,  $QC_1 \parallel CA$ , and  $C_1$  and  $Q$  lie on opposite sides of line  $AB$ . Construct a point  $B_1$  in such a way that the convex quadrilateral  $APCB_1$  is cyclic,  $QB_1 \parallel BA$ , and  $B_1$  and  $Q$  lie on opposite sides of line  $AC$ . Prove that the points  $B_1$ ,  $C_1$ ,  $P$ , and  $Q$  lie on a circle.

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**Day 2**                      April 20th

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**4**                      Legs  $L_1, L_2, L_3, L_4$  of a square table each have length  $n$ , where  $n$  is a positive integer. For how many ordered 4-tuples  $(k_1, k_2, k_3, k_4)$  of nonnegative integers can we cut a piece of length  $k_i$  from the end of leg  $L_i$  ( $i = 1, 2, 3, 4$ ) and still have a stable table?

(The table is *stable* if it can be placed so that all four of the leg ends touch the floor. Note that a cut leg of length 0 is permitted.)

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**5**                      Let  $n$  be an integer greater than 1. Suppose  $2n$  points are given in the plane, no three of which are collinear. Suppose  $n$  of the given  $2n$  points are colored blue and the other  $n$  colored red. A line in the plane is called a *balancing line* if it passes through one blue and one red point and, for each side of the line, the number of blue points on that side is equal to the number of red points on the same side. Prove that there exist at least two balancing lines.

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- 6 For  $m$  a positive integer, let  $s(m)$  be the sum of the digits of  $m$ . For  $n \geq 2$ , let  $f(n)$  be the minimal  $k$  for which there exists a set  $S$  of  $n$  positive integers such that  $s(\sum_{x \in X} x) = k$  for any nonempty subset  $X \subset S$ . Prove that there are constants  $0 < C_1 < C_2$  with

$$C_1 \log_{10} n \leq f(n) \leq C_2 \log_{10} n.$$



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# Art of Problem Solving

## 2004 USAMO

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USAMO 2004

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Day 1

April 27th

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- 1 Let  $ABCD$  be a quadrilateral circumscribed about a circle, whose interior and exterior angles are at least 60 degrees. Prove that

$$\frac{1}{3}|AB^3 - AD^3| \leq |BC^3 - CD^3| \leq 3|AB^3 - AD^3|.$$

When does equality hold?

- 
- 2 Suppose  $a_1, \dots, a_n$  are integers whose greatest common divisor is 1. Let  $S$  be a set of integers with the following properties:

- (a) For  $i = 1, \dots, n$ ,  $a_i \in S$ .
- (b) For  $i, j = 1, \dots, n$  (not necessarily distinct),  $a_i - a_j \in S$ .
- (c) For any integers  $x, y \in S$ , if  $x + y \in S$ , then  $x - y \in S$ .

Prove that  $S$  must be equal to the set of all integers.

- 
- 3 For what real values of  $k > 0$  is it possible to dissect a  $1 \times k$  rectangle into two similar, but noncongruent, polygons?

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Day 2

April 28th

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- 4 Alice and Bob play a game on a 6 by 6 grid. On his or her turn, a player chooses a rational number not yet appearing in the grid and writes it in an empty square of the grid. Alice goes first and then the players alternate. When all squares have numbers written in them, in each row, the square with the greatest number in that row is colored black. Alice wins if she can then draw a line from the top of the grid to the bottom of the grid that stays in black squares, and Bob wins if she can't. (If two squares share a vertex, Alice can draw a line from one to the other that stays in those two squares.) Find, with proof, a winning strategy for one of the players.

- 
- 5 Let  $a, b, c > 0$ . Prove that  $(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3$ .

- 
- 6 A circle  $\omega$  is inscribed in a quadrilateral  $ABCD$ . Let  $I$  be the center of  $\omega$ . Suppose that

$$(AI + DI)^2 + (BI + CI)^2 = (AB + CD)^2.$$





# Art of Problem Solving

## 2004 USAMO

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Prove that  $ABCD$  is an isosceles trapezoid.

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USAMO 2003

Day 1

April 29th

**1** Prove that for every positive integer  $n$  there exists an  $n$ -digit number divisible by  $5^n$  all of whose digits are odd.

**2** A convex polygon  $\mathcal{P}$  in the plane is dissected into smaller convex polygons by drawing all of its diagonals. The lengths of all sides and all diagonals of the polygon  $\mathcal{P}$  are rational numbers. Prove that the lengths of all sides of all polygons in the dissection are also rational numbers.

**3** Let  $n \neq 0$ . For every sequence of integers

$$A = a_0, a_1, a_2, \dots, a_n$$

satisfying  $0 \leq a_i \leq i$ , for  $i = 0, \dots, n$ , define another sequence

$$t(A) = t(a_0), t(a_1), t(a_2), \dots, t(a_n)$$

by setting  $t(a_i)$  to be the number of terms in the sequence  $A$  that precede the term  $a_i$  and are different from  $a_i$ . Show that, starting from any sequence  $A$  as above, fewer than  $n$  applications of the transformation  $t$  lead to a sequence  $B$  such that  $t(B) = B$ .

Day 2

April 30th

**4** Let  $ABC$  be a triangle. A circle passing through  $A$  and  $B$  intersects segments  $AC$  and  $BC$  at  $D$  and  $E$ , respectively. Lines  $AB$  and  $DE$  intersect at  $F$ , while lines  $BD$  and  $CF$  intersect at  $M$ . Prove that  $MF = MC$  if and only if  $MB \cdot MD = MC^2$ .

**5** Let  $a, b, c$  be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2} + \frac{(2b+c+a)^2}{2b^2+(c+a)^2} + \frac{(2c+a+b)^2}{2c^2+(a+b)^2} \leq 8.$$

**6** At the vertices of a regular hexagon are written six nonnegative integers whose sum is  $2003^{2003}$ . Bert is allowed to make moves of the following form: he may pick a vertex and replace the number written there by the absolute value of the



# Art of Problem Solving

## 2003 USAMO

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difference between the numbers written at the two neighboring vertices. Prove that Bert can make a sequence of moves, after which the number 0 appears at all six vertices.

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USAMO 2002

Day 1

May 3rd

**1** Let  $S$  be a set with 2002 elements, and let  $N$  be an integer with  $0 \leq N \leq 2^{2002}$ . Prove that it is possible to color every subset of  $S$  either black or white so that the following conditions hold:

- (a) the union of any two white subsets is white;
- (b) the union of any two black subsets is black;
- (c) there are exactly  $N$  white subsets.

**2** Let  $ABC$  be a triangle such that

$$\left(\cot \frac{A}{2}\right)^2 + \left(2 \cot \frac{B}{2}\right)^2 + \left(3 \cot \frac{C}{2}\right)^2 = \left(\frac{6s}{7r}\right)^2,$$

where  $s$  and  $r$  denote its semiperimeter and its inradius, respectively. Prove that triangle  $ABC$  is similar to a triangle  $T$  whose side lengths are all positive integers with no common divisors and determine these integers.

**3** Prove that any monic polynomial (a polynomial with leading coefficient 1) of degree  $n$  with real coefficients is the average of two monic polynomials of degree  $n$  with  $n$  real roots.

Day 2

May 4th

**4** Let  $\mathbb{R}$  be the set of real numbers. Determine all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all pairs of real numbers  $x$  and  $y$ .

**5** Let  $a, b$  be integers greater than 2. Prove that there exists a positive integer  $k$  and a finite sequence  $n_1, n_2, \dots, n_k$  of positive integers such that  $n_1 = a$ ,  $n_k = b$ , and  $n_i n_{i+1}$  is divisible by  $n_i + n_{i+1}$  for each  $i$  ( $1 \leq i < k$ ).

**6** I have an  $n \times n$  sheet of stamps, from which I've been asked to tear out blocks of three adjacent stamps in a single row or column. (I can only tear along the perforations separating adjacent stamps, and each block must come out of the



# Art of Problem Solving

## 2002 USAMO

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sheet in one piece.) Let  $b(n)$  be the smallest number of blocks I can tear out and make it impossible to tear out any more blocks. Prove that there are real constants  $c$  and  $d$  such that

$$\frac{1}{7}n^2 - cn \leq b(n) \leq \frac{1}{5}n^2 + dn$$

for all  $n > 0$ .

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USAMO 2001

**Day 1**                      May 1st

**1**                      Each of eight boxes contains six balls. Each ball has been colored with one of  $n$  colors, such that no two balls in the same box are the same color, and no two colors occur together in more than one box. Determine, with justification, the smallest integer  $n$  for which this is possible.

**2**                      Let  $ABC$  be a triangle and let  $\omega$  be its incircle. Denote by  $D_1$  and  $E_1$  the points where  $\omega$  is tangent to sides  $BC$  and  $AC$ , respectively. Denote by  $D_2$  and  $E_2$  the points on sides  $BC$  and  $AC$ , respectively, such that  $CD_2 = BD_1$  and  $CE_2 = AE_1$ , and denote by  $P$  the point of intersection of segments  $AD_2$  and  $BE_2$ . Circle  $\omega$  intersects segment  $AD_2$  at two points, the closer of which to the vertex  $A$  is denoted by  $Q$ . Prove that  $AQ = D_2P$ .

**3**                      Let  $a, b, c \geq 0$  and satisfy

$$a^2 + b^2 + c^2 + abc = 4.$$

Show that

$$0 \leq ab + bc + ca - abc \leq 2.$$

**Day 2**                      May 2nd

**4**                      Let  $P$  be a point in the plane of triangle  $ABC$  such that the segments  $PA$ ,  $PB$ , and  $PC$  are the sides of an obtuse triangle. Assume that in this triangle the obtuse angle opposes the side congruent to  $PA$ . Prove that  $\angle BAC$  is acute.

**5**                      Let  $S$  be a set of integers (not necessarily positive) such that

(a) there exist  $a, b \in S$  with  $\gcd(a, b) = \gcd(a - 2, b - 2) = 1$ ;

(b) if  $x$  and  $y$  are elements of  $S$  (possibly equal), then  $x^2 - y$  also belongs to  $S$ .

Prove that  $S$  is the set of all integers.

**6**                      Each point in the plane is assigned a real number such that, for any triangle, the number at the center of its inscribed circle is equal to the arithmetic mean of the three numbers at its vertices. Prove that all points in the plane are assigned the same number.

USAMO 2000

Day 1

May 2nd

1

Call a real-valued function  $f$  *very convex* if

$$\frac{f(x) + f(y)}{2} \geq f\left(\frac{x+y}{2}\right) + |x-y|$$

holds for all real numbers  $x$  and  $y$ . Prove that no very convex function exists.

2

Let  $S$  be the set of all triangles  $ABC$  for which

$$5\left(\frac{1}{AP} + \frac{1}{BQ} + \frac{1}{CR}\right) - \frac{3}{\min\{AP, BQ, CR\}} = \frac{6}{r},$$

where  $r$  is the inradius and  $P, Q, R$  are the points of tangency of the incircle with sides  $AB, BC, CA$ , respectively. Prove that all triangles in  $S$  are isosceles and similar to one another.

3

A game of solitaire is played with  $R$  red cards,  $W$  white cards, and  $B$  blue cards. A player plays all the cards one at a time. With each play he accumulates a penalty. If he plays a blue card, then he is charged a penalty which is the number of white cards still in his hand. If he plays a white card, then he is charged a penalty which is twice the number of red cards still in his hand. If he plays a red card, then he is charged a penalty which is three times the number of blue cards still in his hand. Find, as a function of  $R, W$ , and  $B$ , the minimal total penalty a player can amass and all the ways in which this minimum can be achieved.

Day 2

May 2nd

4

Find the smallest positive integer  $n$  such that if  $n$  squares of a  $1000 \times 1000$  chessboard are colored, then there will exist three colored squares whose centers form a right triangle with sides parallel to the edges of the board.

5

Let  $A_1A_2A_3$  be a triangle and let  $\omega_1$  be a circle in its plane passing through  $A_1$  and  $A_2$ . Suppose there exist circles  $\omega_2, \omega_3, \dots, \omega_7$  such that for  $k = 2, 3, \dots, 7$ ,  $\omega_k$  is externally tangent to  $\omega_{k-1}$  and passes through  $A_k$  and  $A_{k+1}$ , where  $A_{n+3} = A_n$  for all  $n \geq 1$ . Prove that  $\omega_7 = \omega_1$ .

6 Let  $a_1, b_1, a_2, b_2, \dots, a_n, b_n$  be nonnegative real numbers. Prove that

$$\sum_{i,j=1}^n \min\{a_i a_j, b_i b_j\} \leq \sum_{i,j=1}^n \min\{a_i b_j, a_j b_i\}.$$



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# Art of Problem Solving

## 1999 USAMO

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USAMO 1999

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**Day 1**                      April 27th

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- 1**                      Some checkers placed on an  $n \times n$  checkerboard satisfy the following conditions:
- (a) every square that does not contain a checker shares a side with one that does;
- (b) given any pair of squares that contain checkers, there is a sequence of squares containing checkers, starting and ending with the given squares, such that every two consecutive squares of the sequence share a side.

Prove that at least  $(n^2 - 2)/3$  checkers have been placed on the board.

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- 2**                      Let  $ABCD$  be a cyclic quadrilateral. Prove that

$$|AB - CD| + |AD - BC| \geq 2|AC - BD|.$$

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- 3**                      Let  $p > 2$  be a prime and let  $a, b, c, d$  be integers not divisible by  $p$ , such that

$$\left\{ \frac{ra}{p} \right\} + \left\{ \frac{rb}{p} \right\} + \left\{ \frac{rc}{p} \right\} + \left\{ \frac{rd}{p} \right\} = 2$$

for any integer  $r$  not divisible by  $p$ . Prove that at least two of the numbers  $a + b, a + c, a + d, b + c, b + d, c + d$  are divisible by  $p$ .  
(Note:  $\{x\} = x - \lfloor x \rfloor$  denotes the fractional part of  $x$ .)

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**Day 2**                      April 27th

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- 4**                      Let  $a_1, a_2, \dots, a_n$  ( $n > 3$ ) be real numbers such that

$$a_1 + a_2 + \dots + a_n \geq n \quad \text{and} \quad a_1^2 + a_2^2 + \dots + a_n^2 \geq n^2.$$

Prove that  $\max(a_1, a_2, \dots, a_n) \geq 2$ .

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- 5**                      The Y2K Game is played on a  $1 \times 2000$  grid as follows. Two players in turn write either an S or an O in an empty square. The first player who produces three consecutive boxes that spell SOS wins. If all boxes are filled without producing SOS then the game is a draw. Prove that the second player has a winning strategy.
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# Art of Problem Solving

## 1999 USAMO

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Let  $ABCD$  be an isosceles trapezoid with  $AB \parallel CD$ . The inscribed circle  $\omega$  of triangle  $BCD$  meets  $CD$  at  $E$ . Let  $F$  be a point on the (internal) angle bisector of  $\angle DAC$  such that  $EF \perp CD$ . Let the circumscribed circle of triangle  $ACF$  meet line  $CD$  at  $C$  and  $G$ . Prove that the triangle  $AFG$  is isosceles.



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USAMO 1998

Day 1

April 28th

- 1** Suppose that the set  $\{1, 2, \dots, 1998\}$  has been partitioned into disjoint pairs  $\{a_i, b_i\}$  ( $1 \leq i \leq 999$ ) so that for all  $i$ ,  $|a_i - b_i|$  equals 1 or 6. Prove that the sum

$$|a_1 - b_1| + |a_2 - b_2| + \dots + |a_{999} - b_{999}|$$

ends in the digit 9.

- 2** Let  $\mathcal{C}_1$  and  $\mathcal{C}_2$  be concentric circles, with  $\mathcal{C}_2$  in the interior of  $\mathcal{C}_1$ . From a point  $A$  on  $\mathcal{C}_1$  one draws the tangent  $AB$  to  $\mathcal{C}_2$  ( $B \in \mathcal{C}_2$ ). Let  $C$  be the second point of intersection of  $AB$  and  $\mathcal{C}_1$ , and let  $D$  be the midpoint of  $AB$ . A line passing through  $A$  intersects  $\mathcal{C}_2$  at  $E$  and  $F$  in such a way that the perpendicular bisectors of  $DE$  and  $CF$  intersect at a point  $M$  on  $AB$ . Find, with proof, the ratio  $AM/MC$ .

- 3** Let  $a_0, a_1, \dots, a_n$  be numbers from the interval  $(0, \pi/2)$  such that

$$\tan(a_0 - \frac{\pi}{4}) + \tan(a_1 - \frac{\pi}{4}) + \dots + \tan(a_n - \frac{\pi}{4}) \geq n - 1.$$

Prove that

$$\tan a_0 \tan a_1 \dots \tan a_n \geq n^{n+1}.$$

Day 2

April 28th

- 4** A computer screen shows a  $98 \times 98$  chessboard, colored in the usual way. One can select with a mouse any rectangle with sides on the lines of the chessboard and click the mouse button: as a result, the colors in the selected rectangle switch (black becomes white, white becomes black). Find, with proof, the minimum number of mouse clicks needed to make the chessboard all one color.

- 5** Prove that for each  $n \geq 2$ , there is a set  $S$  of  $n$  integers such that  $(a - b)^2$  divides  $ab$  for every distinct  $a, b \in S$ .

- 6** Let  $n \geq 5$  be an integer. Find the largest integer  $k$  (as a function of  $n$ ) such that there exists a convex  $n$ -gon  $A_1 A_2 \dots A_n$  for which exactly  $k$  of the quadrilaterals  $A_i A_{i+1} A_{i+2} A_{i+3}$  have an inscribed circle. (Here  $A_{n+j} = A_j$ .)

USAMO 1997

Day 1

May 1st

- 1 Let  $p_1, p_2, p_3, \dots$  be the prime numbers listed in increasing order, and let  $x_0$  be a real number between 0 and 1. For positive integer  $k$ , define

$$x_k = \begin{cases} 0 & \text{if } x_{k-1} = 0, \\ \left\{ \frac{p_k}{x_{k-1}} \right\} & \text{if } x_{k-1} \neq 0, \end{cases}$$

where  $\{x\}$  denotes the fractional part of  $x$ . (The fractional part of  $x$  is given by  $x - [x]$  where  $[x]$  is the greatest integer less than or equal to  $x$ .) Find, with proof, all  $x_0$  satisfying  $0 < x_0 < 1$  for which the sequence  $x_0, x_1, x_2, \dots$  eventually becomes 0.

- 2 Let  $ABC$  be a triangle. Take points  $D, E, F$  on the perpendicular bisectors of  $BC, CA, AB$  respectively. Show that the lines through  $A, B, C$  perpendicular to  $EF, FD, DE$  respectively are concurrent.

- 3 Prove that for any integer  $n$ , there exists a unique polynomial  $Q$  with coefficients in  $\{0, 1, \dots, 9\}$  such that  $Q(-2) = Q(-5) = n$ .

Day 2

May 2nd

- 4 To *clip* a convex  $n$ -gon means to choose a pair of consecutive sides  $AB, BC$  and to replace them by the three segments  $AM, MN$ , and  $NC$ , where  $M$  is the midpoint of  $AB$  and  $N$  is the midpoint of  $BC$ . In other words, one cuts off the triangle  $MBN$  to obtain a convex  $(n+1)$ -gon. A regular hexagon  $\mathcal{P}_6$  of area 1 is clipped to obtain a heptagon  $\mathcal{P}_7$ . Then  $\mathcal{P}_7$  is clipped (in one of the seven possible ways) to obtain an octagon  $\mathcal{P}_8$ , and so on. Prove that no matter how the clippings are done, the area of  $\mathcal{P}_n$  is greater than  $\frac{1}{3}$ , for all  $n \geq 6$ .

- 5 Prove that, for all positive real numbers  $a, b, c$ , the inequality

$$\frac{1}{a^3 + b^3 + abc} + \frac{1}{b^3 + c^3 + abc} + \frac{1}{c^3 + a^3 + abc} \leq \frac{1}{abc}$$

holds.

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Suppose the sequence of nonnegative integers  $a_1, a_2, \dots, a_{1997}$  satisfies

$$a_i + a_j \leq a_{i+j} \leq a_i + a_j + 1$$

for all  $i, j \geq 1$  with  $i + j \leq 1997$ . Show that there exists a real number  $x$  such that  $a_n = \lfloor nx \rfloor$  (the greatest integer  $\leq nx$ ) for all  $1 \leq n \leq 1997$ .

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USAMO 1996

Day 1

May 2nd

- 1 Prove that the average of the numbers  $n \sin n^\circ$  ( $n = 2, 4, 6, \dots, 180$ ) is  $\cot 1^\circ$ .
- 2 For any nonempty set  $S$  of real numbers, let  $\sigma(S)$  denote the sum of the elements of  $S$ . Given a set  $A$  of  $n$  positive integers, consider the collection of all distinct sums  $\sigma(S)$  as  $S$  ranges over the nonempty subsets of  $A$ . Prove that this collection of sums can be partitioned into  $n$  classes so that in each class, the ratio of the largest sum to the smallest sum does not exceed 2.
- 3 Let  $ABC$  be a triangle. Prove that there is a line  $\ell$  (in the plane of triangle  $ABC$ ) such that the intersection of the interior of triangle  $ABC$  and the interior of its reflection  $A'B'C'$  in  $\ell$  has area more than  $\frac{2}{3}$  the area of triangle  $ABC$ .

Day 2

May 2nd

- 4 An  $n$ -term sequence  $(x_1, x_2, \dots, x_n)$  in which each term is either 0 or 1 is called a *binary sequence of length  $n$* . Let  $a_n$  be the number of binary sequences of length  $n$  containing no three consecutive terms equal to 0, 1, 0 in that order. Let  $b_n$  be the number of binary sequences of length  $n$  that contain no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that  $b_{n+1} = 2a_n$  for all positive integers  $n$ .
- 5 Let  $ABC$  be a triangle, and  $M$  an interior point such that  $\angle MAB = 10^\circ$ ,  $\angle MBA = 20^\circ$ ,  $\angle MAC = 40^\circ$  and  $\angle MCA = 30^\circ$ . Prove that the triangle is isosceles.
- 6 Determine (with proof) whether there is a subset  $X$  of the integers with the following property: for any integer  $n$  there is exactly one solution of  $a + 2b = n$  with  $a, b \in X$ .



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USAMO 1995

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- April 27th
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- 1** Let  $p$  be an odd prime. The sequence  $(a_n)_{n \geq 0}$  is defined as follows:  $a_0 = 0$ ,  $a_1 = 1, \dots, a_{p-2} = p - 2$  and, for all  $n \geq p - 1$ ,  $a_n$  is the least positive integer that does not form an arithmetic sequence of length  $p$  with any of the preceding terms. Prove that, for all  $n$ ,  $a_n$  is the number obtained by writing  $n$  in base  $p - 1$  and reading the result in base  $p$ .
- 
- 2** A calculator is broken so that the only keys that still work are the  $\sin$ ,  $\cos$ , and  $\tan$  buttons, and their inverses (the  $\arcsin$ ,  $\arccos$ , and  $\arctan$  buttons). The display initially shows 0. Given any positive rational number  $q$ , show that pressing some finite sequence of buttons will yield the number  $q$  on the display. Assume that the calculator does real number calculations with infinite precision. All functions are in terms of radians.
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- 3** Given a nonisosceles, nonright triangle  $ABC$ , let  $O$  denote the center of its circumscribed circle, and let  $A_1$ ,  $B_1$ , and  $C_1$  be the midpoints of sides  $BC$ ,  $CA$ , and  $AB$ , respectively. Point  $A_2$  is located on the ray  $OA_1$  so that  $OAA_1$  is similar to  $OA_2A$ . Points  $B_2$  and  $C_2$  on rays  $OB_1$  and  $OC_1$ , respectively, are defined similarly. Prove that lines  $AA_2$ ,  $BB_2$ , and  $CC_2$  are concurrent, i.e. these three lines intersect at a point.
- 
- 4** Suppose  $q_0, q_1, q_2, \dots$  is an infinite sequence of integers satisfying the following two conditions:
- (i)  $m - n$  divides  $q_m - q_n$  for  $m > n \geq 0$ ,
  - (ii) there is a polynomial  $P$  such that  $|q_n| < P(n)$  for all  $n$
- Prove that there is a polynomial  $Q$  such that  $q_n = Q(n)$  for all  $n$ .
- 
- 5** Suppose that in a certain society, each pair of persons can be classified as either *amicable* or *hostile*. We shall say that each member of an amicable pair is a *friend* of the other, and each member of a hostile pair is a *foe* of the other. Suppose that the society has  $n$  persons and  $q$  amicable pairs, and that for every set of three persons, at least one pair is hostile. Prove that there is at least one member of the society whose foes include  $q(1 - 4q/n^2)$  or fewer amicable pairs.
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