Solutions of APMO 2014

Problem 1. For a positive integer m denote by $S(m)$ and $P(m)$ the sum and product, respectively, of the digits of m. Show that for each positive integer n, there exist positive integers $a_1, a_2, \ldots, a_n$ satisfying the following conditions:

$$S(a_1) < S(a_2) < \cdots < S(a_n) \text{ and } S(a_i) = P(a_{i+1}) \quad (i = 1, 2, \ldots, n).$$

(We let $a_{n+1} = a_1$.) (*Problem Committee of the Japan Mathematical Olympiad Foundation*)

Solution. Let k be a sufficiently large positive integer. Choose for each $i = 2, 3, \ldots, n$, a_i to be a positive integer among whose digits the number 2 appears exactly $k+i-2$ times and the number 1 appears exactly $2^{k+i-1} - 2(k+i-2)$ times, and nothing else. Then, we have $S(a_i) = 2^{k+i-1}$ and $P(a_i) = 2^{k+i-2}$ for each i, $2 \le i \le n$. Then, we let a_1 be a positive integer among whose digits the number 2 appears exactly $k+n-1$ times and the number 1 appears exactly $2^k - 2(k+n-1)$ times, and nothing else. Then, we see that a_1 satisfies $S(a_1) = 2^k$ and $P(a_1) = 2^{k+n-1}$. Such a choice of a_1 is possible if we take k to be large enough to satisfy $2^k > 2(k+n-1)$ and we see that the numbers $a_1, \ldots, a_n$ chosen this way satisfy the given requirements.

Problem 2. Let $S = \{1, 2, \ldots, 2014\}$. For each non-empty subset $T \subseteq S$, one of its members is chosen as its *representative*. Find the number of ways to assign representatives to all non-empty subsets of S so that if a subset $D \subseteq S$ is a disjoint union of non-empty subsets $A, B, C \subseteq S$, then the representative of D is also the representative of at least one of A, B, C. (*Warut Suksompong, Thailand*)

Solution. *Answer:* $108 \cdot 2014!$.

For any subset X let $r(X)$ denotes the representative of X. Suppose that $x_1 = r(S)$. First, we prove the following fact:

$$\text{If } x_1 \in X \text{ and } X \subseteq S, \text{ then } x_1 = r(X).$$

If $|X| \le 2012$, then we can write S as a disjoint union of X and two other subsets of S, which gives that $x_1 = r(X)$. If $|X| = 2013$, then let $y \in X$ and $y \ne x_1$. We can write X as a disjoint union of $\{x_1, y\}$ and two other subsets. We already proved that $r(\{x_1, y\}) = x_1$ (since $|\{x_1, y\}| = 2 < 2012$) and it follows that $y \ne r(X)$ for every $y \in X$ except x_1. We have proved the fact.

Note that this fact is true and can be proved similarly, if the ground set S would contain at least 5 elements.

There are 2014 ways to choose $x_1 = r(S)$ and for $x_1 \in X \subseteq S$ we have $r(X) = x_1$. Let $S_1 = S \setminus \{x_1\}$. Analogously, we can state that there are 2013 ways to choose $x_2 = r(S_1)$ and for $x_2 \in X \subseteq S_1$ we have $r(X) = x_2$. Proceeding similarly (or by induction), there are $2014 \cdot 2013 \cdots 5$ ways to choose $x_1, x_2, \ldots, x_{2010} \in S$ so that for all $i = 1, 2 \ldots, 2010$, $x_i = r(X)$ for each $X \subseteq S \setminus \{x_1, \ldots, x_{i-1}\}$ and $x_i \in X$.

We are now left with four elements $Y = \{y_1, y_2, y_3, y_4\}$. There are 4 ways to choose $r(Y)$. Suppose that $y_1 = r(Y)$. Then we clearly have $y_1 = r(\{y_1, y_2\}) = r(\{y_1, y_3\}) = r(\{y_1, y_4\})$. The only subsets whose representative has not been assigned yet are $\{y_1, y_2, y_3\}$, $\{y_1, y_2, y_4\}$, $\{y_1, y_3, y_4\}$, $\{y_2, y_3, y_4\}$, $\{y_2, y_3\}$, $\{y_2, y_4\}$, $\{y_3, y_4\}$. These subsets can be assigned in any way, hence giving $3^4 \cdot 2^3$ more choices.

In conclusion, the total number of assignments is $2014 \cdot 2013 \cdots 4 \cdot 3^4 \cdot 2^3 = 108 \cdot 2014!$.

Problem 3. Find all positive integers n such that for any integer k there exists an integer a for which $a^3 + a - k$ is divisible by n. (*Warut Suksompong, Thailand*)

Solution. *Answer:* All integers $n = 3^b$, where b is a nonnegative integer.

We are looking for integers n such that the set $A = \{a^3 + a \mid a \in \mathbf{Z}\}$ is a complete residue system by modulo n. Let us call this property by (*). It is not hard to see that $n = 1$ satisfies (*) and $n = 2$ does not.

If $a \equiv b \pmod{n}$, then $a^3 + a \equiv b^3 + b \pmod{n}$. So n satisfies (*) iff there are no $a, b \in \{0, \ldots, n-1\}$ with $a \neq b$ and $a^3 + a \equiv b^3 + b \pmod{n}$.

First, let us prove that 3^j satisfies (*) for all $j \geq 1$. Suppose that $a^3+a \equiv b^3+b \pmod{3^j}$ for $a \neq b$. Then $(a-b)(a^2+ab+b^2+1) \equiv 0 \pmod{3^j}$. We can easily check mod 3 that $a^2 + ab + b^2 + 1$ is not divisible by 3.

Next note that if A is not a complete residue system modulo integer r, then it is also not a complete residue system modulo any multiple of r. Hence it remains to prove that any prime $p > 3$ does not satisfy (*).

If $p \equiv 1 \pmod{4}$, there exists b such that $b^2 \equiv -1 \pmod{p}$. We then take $a = 0$ to obtain the congruence $a^3 + a \equiv b^3 + b \pmod{p}$.

Suppose now that $p \equiv 3 \pmod{4}$. We will prove that there are integers $a, b \not\equiv 0 \pmod{p}$ such that $a^2 + ab + b^2 \equiv -1 \pmod{p}$. Note that we may suppose that $a \not\equiv b \pmod{p}$, since otherwise if $a \equiv b \pmod{p}$ satisfies $a^2 + ab + b^2 + 1 \equiv 0 \pmod{p}$, then $(2a)^2 + (2a)(-a) + a^2 + 1 \equiv 0 \pmod{p}$ and $2a \not\equiv -a \pmod{p}$. Letting c be the inverse of b modulo p (i.e. $bc \equiv 1 \pmod{p}$), the relation is equivalent to $(ac)^2 + ac + 1 \equiv -c^2 \pmod{p}$. Note that $-c^2$ can take on the values of all non-quadratic residues modulo p. If we can find an integer x such that $x^2 + x + 1$ is a non-quadratic residue modulo p, the values of a and c will follow immediately. Hence we focus on this latter task.

Note that if $x, y \in \{0, \ldots, p-1\} = B$, then $x^2 + x + 1 \equiv y^2 + y + 1 \pmod{p}$ iff p divides $x + y + 1$. We can deduce that $x^2 + x + 1$ takes on $(p+1)/2$ values as x varies in B. Since there are $(p-1)/2$ non-quadratic residues modulo p, the $(p+1)/2$ values that $x^2 + x + 1$ take on must be 0 and all the quadratic residues.

Let C be the set of quadratic residues modulo p and 0, and let $y \in C$. Suppose that $y \equiv z^2 \pmod{p}$ and let $z \equiv 2w + 1 \pmod{p}$ (we can always choose such w). Then $y + 3 \equiv 4(w^2 + w + 1) \pmod{p}$. From the previous paragraph, we know that $4(w^2 + w + 1) \in C$. This means that $y \in C \implies y + 3 \in C$. Unless $p = 3$, the relation implies that all elements of B are in C, a contradiction. This concludes the proof.

Problem 4. Let n and b be positive integers. We say n is b-*discerning* if there exists a set consisting of n different positive integers less than b that has no two different subsets U and V such that the sum of all elements in U equals the sum of all elements in V.

(a) Prove that 8 is a 100-discerning.

(b) Prove that 9 is not 100–discerning.

(*Senior Problems Committee of the Australian Mathematical Olympiad Committee*)

Solution.

(a) Take $S = \{3, 6, 12, 24, 48, 95, 96, 97\}$, i.e.

$$S = \{3 \cdot 2^k : 0 \leq k \leq 5\} \cup \{3 \cdot 2^5 - 1, 3 \cdot 2^5 + 1\}.$$

As k ranges between 0 to 5, the sums obtained from the numbers $3 \cdot 2^k$ are $3t$, where $1 \le t \le 63$. These are 63 numbers that are divisible by 3 and are at most $3 \cdot 63 = 189$.

Sums of elements of S are also the numbers $95 + 97 = 192$ and all the numbers that are sums of 192 and sums obtained from the numbers $3 \cdot 2^k$ with $0 \le k \le 5$. These are 64 numbers that are all divisible by 3 and at least equal to 192. In addition, sums of elements of S are the numbers 95 and all the numbers that are sums of 95 and sums obtained from the numbers $3 \cdot 2^k$ with $0 \le k \le 5$. These are 64 numbers that are all congruent to -1 mod 3.

Finally, sums of elements of S are the numbers 97 and all the numbers that are sums of 97 and sums obtained from the numbers $3 \cdot 2^k$ with $0 \le k \le 5$. These are 64 numbers that are all congruent to 1 mod 3.

Hence there are at least $63 + 64 + 64 + 64 = 255$ different sums from elements of S. On the other hand, S has $2^8 - 1 = 255$ non-empty subsets. Therefore S has no two different subsets with equal sums of elements. Therefore, 8 is 100-discerning.

(b) Suppose that 9 is 100-discerning. Then there is a set $S = \{s_1, \dots, s_9\}$, $s_i < 100$ that has no two different subsets with equal sums of elements. Assume that $0 < s_1 < \dots < s_9 < 100$.

Let X be the set of all subsets of S having at least 3 and at most 6 elements and let Y be the set of all subsets of S having exactly 2 or 3 or 4 elements greater than s_3.

The set X consists of

$$\binom{9}{3} + \binom{9}{4} + \binom{9}{5} + \binom{9}{6} = 84 + 126 + 126 + 84 = 420$$

subsets of S. The set in X with the largest sums of elements is $\{s_4, \dots, s_9\}$ and the smallest sums is in $\{s_1, s_2, s_3\}$. Thus the sum of the elements of each of the 420 sets in X is at least $s_1 + s_2 + s_3$ and at most $s_4 + \dots + s_9$, which is one of $(s_4 + \dots + s_9) - (s_1 + s_2 + s_3) + 1$ integers. From the pigeonhole principle it follows that $(s_4 + \dots + s_9) - (s_1 + s_2 + s_3) + 1 \ge 420$, i.e.,

$$(s_4 + \dots + s_9) - (s_1 + s_2 + s_3) \ge 419. \tag{1}$$

Now let us calculate the number of subsets in Y. Observe that $\{s_4, \dots, s_9\}$ has $\binom{6}{2}$ 2-element subsets, $\binom{6}{3}$ 3-element subsets and $\binom{6}{4}$ 4-element subsets, while $\{s_1, s_2, s_3\}$ has exactly 8 subsets. Hence the number of subsets of S in Y equals

$$8\left(\binom{6}{2} + \binom{6}{3} + \binom{6}{4}\right) = 8(15 + 20 + 15) = 400.$$

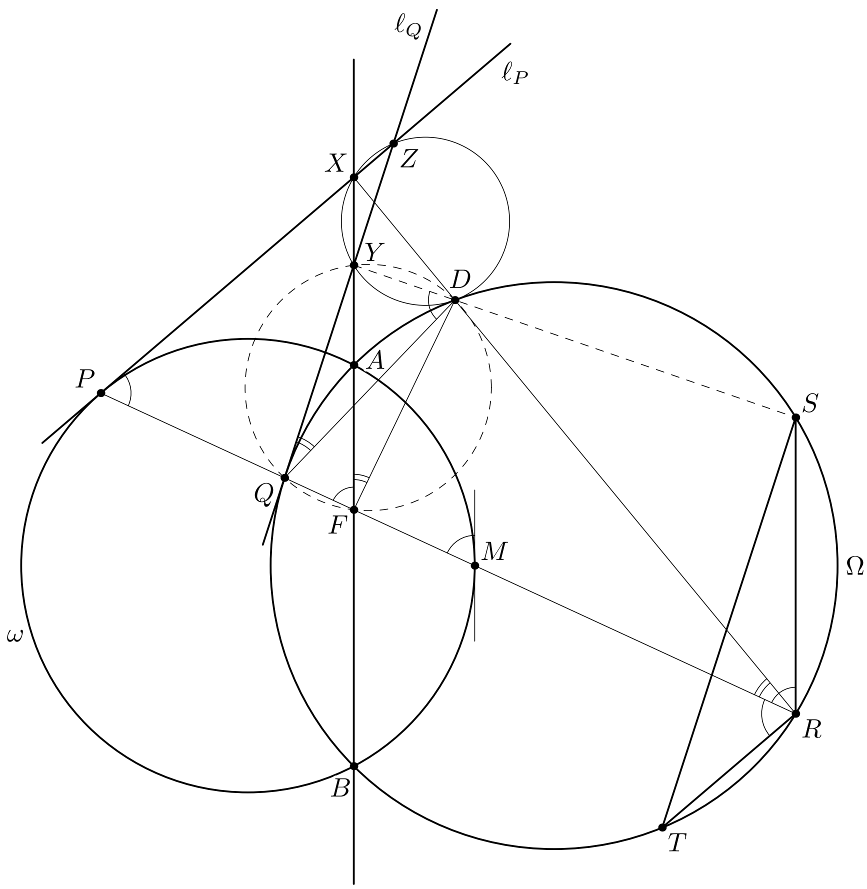
The set in Y with the largest sum of elements is $\{s_1, s_2, s_3, s_6, s_7, s_8, s_9\}$ and the smallest sum is in $\{s_4, s_5\}$. Again, by the pigeonhole principle it follows that $(s_1 + s_2 + s_3 + s_6 + s_7 + s_8 + s_9) - (s_4 + s_5) + 1 \ge 400$, i.e.,

$$(s_1 + s_2 + s_3 + s_6 + s_7 + s_8 + s_9) - (s_4 + s_5) \ge 399. \tag{2}$$

Adding (1) and (2) yields $2(s_6 + s_7 + s_8 + s_9) \ge 818$, so that $s_9 + 98 + 97 + 96 \ge s_9 + s_8 + s_7 + s_6 \ge 409$, i.e. $s_9 \ge 118$, a contradiction with $s_9 < 100$. Therefore, 9 is not 100-discerning.

Problem 5. Circles ω and Ω meet at points A and B. Let M be the midpoint of the arc AB of circle ω (M lies inside Ω). A chord MP of circle ω intersects Ω at Q (Q lies inside ω). Let ℓ_P be the tangent line to ω at P, and let ℓ_Q be the tangent line to Ω at Q. Prove that the circumcircle of the triangle formed by the lines ℓ_P, ℓ_Q, and AB is tangent to Ω. (*Ilya Bogdanov, Russia and Medeubek Kungozhin, Kazakhstan*)

Solution. Denote $X = AB \cap \ell_P$, $Y = AB \cap \ell_Q$, and $Z = \ell_P \cap \ell_Q$. Without loss of generality we have $AX < BX$. Let $F = MP \cap AB$.



Denote by R the second point of intersection of PQ and Ω; by S the point of Ω such that $SR \parallel AB$; and by T the point of Ω such that $RT \parallel \ell_P$. Since M is the midpoint of arc AB, the tangent ℓ_M at M to ω is parallel to AB, so $\angle(AB, PM) = \angle(PM, \ell_P)$. Therefore we have $\angle PRT = \angle MPX = \angle PFX = \angle PRS$. Thus the point Q is the midpoint of the arc TQS of Ω, hence $ST \parallel \ell_Q$. So the corresponding sides of the triangles RST and XYZ are parallel, and there exist a homothety h mapping RST to XYZ.

Let D be the second point of intersection of XR and Ω. We claim that D is the center of the homothety h; since $D \in \Omega$, this implies that the circumcircles of triangles RST and XYZ are tangent, as required. So, it remains to prove this claim. In order to do this, it suffices to show that $D \in SY$.

By $\angle PFX = \angle XPF$ we have $XF^2 = XP^2 = XA \cdot XB = XD \cdot XR$. Therefore, $\frac{XF}{XD} = \frac{XR}{XF}$, so the triangles XDF and XFR are similar, hence $\angle DFX = \angle XRF = \angle DRQ = \angle DQY$; thus the points D, Y, Q, and F are concyclic. It follows that $\angle YDQ = \angle YFQ = \angle SRQ = 180^\circ - \angle SDQ$ which means exactly that the points Y, D, and S are collinear, with D between S and Y.