ELMO Shortlist 2014

–	Algebra
1	In a non-obtuse triangle ABC, prove that $$\frac{\sin A\sin B}{\sin C}+\frac{\sin B\sin C}{\sin A}+\frac{\sin C\sin A}{\sin B}\ge\frac{5}{2}.$$ *Proposed by Ryan Alweiss*
2	Given positive reals a,b,c,p,q satisfying $abc=1$ and $p\ge q$, prove that $$p\left(a^2+b^2+c^2\right)+q\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge(p+q)(a+b+c).$$ *Proposed by AJ Dennis*
3	Let a,b,c,d,e,f be positive real numbers. Given that $def+de+ef+fd=4$, show that $$((a+b)de+(b+c)ef+(c+a)fd)^2\ge\ 12(abde+bcef+cafd).$$ *Proposed by Allen Liu*
4	Find all triples (f,g,h) of injective functions from the set of real numbers to itself satisfying $$\begin{aligned}f(x+f(y))&=g(x)+h(y)\\ g(x+g(y))&=h(x)+f(y)\\ h(x+h(y))&=f(x)+g(y)\end{aligned}$$ for all real numbers x and y. (We say a function F is *injective* if $F(a)\neq F(b)$ for any distinct real numbers a and b.) *Proposed by Evan Chen*
5	Let $\mathbb{R}^*$ denote the set of nonzero reals. Find all functions $f:\mathbb{R}^*\to\mathbb{R}^*$ satisfying $$f(x^2+y)+1=f(x^2+1)+\frac{f(xy)}{f(x)}$$ for all $x,y\in\mathbb{R}^*$ with $x^2+y\neq 0$. *Proposed by Ryan Alweiss*

6 Let a, b, c be positive reals such that $a+b+c=ab+bc+ca$. Prove that

$$(a+b)^{ab-bc}(b+c)^{bc-ca}(c+a)^{ca-ab} \ge a^{ca}b^{ab}c^{bc}.$$

Proposed by Sammy Luo

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7 Find all positive integers n with $n \ge 2$ such that the polynomial

$$P(a_1, a_2, ..., a_n) = a_1^n + a_2^n + ... + a_n^n - n a_1 a_2 ... a_n$$

in the n variables $a_1, a_2, \dots, a_n$ is irreducible over the real numbers, i.e. it cannot be factored as the product of two nonconstant polynomials with real coefficients.

Proposed by Yang Liu

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8 Let a, b, c be positive reals with $a^{2014}+b^{2014}+c^{2014}+abc=4$. Prove that

$$\frac{a^{2013}+b^{2013}-c}{c^{2013}} + \frac{b^{2013}+c^{2013}-a}{a^{2013}} + \frac{c^{2013}+a^{2013}-b}{b^{2013}} \ge a^{2012}+b^{2012}+c^{2012}.$$

Proposed by David Stoner

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9 Let a, b, c be positive reals. Prove that

$$\sqrt{\frac{a^2(bc+a^2)}{b^2+c^2}} + \sqrt{\frac{b^2(ca+b^2)}{c^2+a^2}} + \sqrt{\frac{c^2(ab+c^2)}{a^2+b^2}} \ge a+b+c.$$

Proposed by Robin Park

– Combinatorics

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1 You have some cyan, magenta, and yellow beads on a non-reorientable circle, and you can perform only the following operations:
1. Move a cyan bead right (clockwise) past a yellow bead, and turn the yellow bead magenta.
2. Move a magenta bead left of a cyan bead, and insert a yellow bead left of where the magenta bead ends up.
3. Do either of the above, switching the roles of the words "magenta" and "left" with those of "yellow" and "right", respectively.
4. Pick any two disjoint consecutive pairs of beads, each either yellow-magenta or magenta-yellow, appearing somewhere in the circle, and swap the orders of each pair.
5. Remove four consecutive beads of one color.

Starting with the circle: "yellow, yellow, magenta, magenta, cyan, cyan, cyan", determine whether or not you can reach
a) "yellow, magenta, yellow, magenta, cyan, cyan, cyan",
b) "cyan, yellow, cyan, magenta, cyan",
c) "magenta, magenta, cyan, cyan, cyan",
d) "yellow, cyan, cyan, cyan".

Proposed by Sammy Luo

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2 A $2^{2014}+1$ by $2^{2014}+1$ grid has some black squares filled. The filled black squares form one or more snakes on the plane, each of whose heads splits at some points but never comes back together. In other words, for every positive integer n greater than 2, there do not exist pairwise distinct black squares s_1, s_2, \dots, s_n such that s_i and s_{i+1} share an edge for $i=1,2,\dots,n$ (here $s_{n+1}=s_1$).

What is the maximum possible number of filled black squares?

Proposed by David Yang

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3 We say a finite set S of points in the plane is *very* if for every point X in S, there exists an inversion with center X mapping every point in S other than X to another point in S (possibly the same point).

(a) Fix an integer n. Prove that if $n \ge 2$, then any line segment $\overline{AB}$ contains a unique very set S of size n such that $A, B \in S$.

(b) Find the largest possible size of a very set not contained in any line.

(Here, an *inversion* with center O and radius r sends every point P other than O to the point P' along ray OP such that $OP \cdot OP' = r^2$.)

Proposed by Sammy Luo

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4 Let r and b be positive integers. The game of *Monis*, a variant of Tetris, consists of a single column of red and blue blocks. If two blocks of the same color ever touch each other, they both vanish immediately. A red block falls onto the top of the column exactly once every r years, while a blue block falls exactly once every b years.

(a) Suppose that r and b are odd, and moreover the cycles are offset in such a way that no two blocks ever fall at exactly the same time. Consider a period of rb years in which the column is initially empty. Determine, in terms of r and b, the number of blocks in the column at the end.

(b) Now suppose r and b are relatively prime and $r+b$ is odd. At time $t=0$, the column is initially empty. Suppose a red block falls at times $t=r, 2r, \dots, (b-1)r$

years, while a blue block falls at times $t = b, 2b, \dots, (r-1)b$ years. Prove that at time $t = rb$, the number of blocks in the column is $|1 + 2(r-1)(b+r) - 8S|$, where

$$S = \left\lfloor \frac{2r}{r+b} \right\rfloor + \left\lfloor \frac{4r}{r+b} \right\rfloor + \dots + \left\lfloor \frac{(r+b-1)r}{r+b} \right\rfloor.$$

Proposed by Sammy Luo

5 Let n be a positive integer. For any k, denote by a_k the number of permutations of $\{1, 2, \dots, n\}$ with exactly k disjoint cycles. (For example, if $n = 3$ then $a_2 = 3$ since (1)(23), (2)(31), (3)(12) are the only such permutations.) Evaluate

$$a_n n^n + a_{n-1} n^{n-1} + \dots + a_1 n.$$

Proposed by Sammy Luo

6 Let f_0 be the function from $\mathbb{Z}^2$ to $\{0, 1\}$ such that $f_0(0, 0) = 1$ and $f_0(x, y) = 0$ otherwise. For each positive integer m, let $f_m(x, y)$ be the remainder when

$$f_{m-1}(x, y) + \sum_{j=-1}^{1} \sum_{k=-1}^{1} f_{m-1}(x + j, y + k)$$

is divided by 2.
Finally, for each nonnegative integer n, let a_n denote the number of pairs (x, y) such that $f_n(x, y) = 1$.
Find a closed form for a_n.

Proposed by Bobby Shen

– Geometry

1 Let ABC be a triangle with symmedian point K. Select a point A_1 on line BC such that the lines AB, AC, A_1K and BC are the sides of a cyclic quadrilateral. Define B_1 and C_1 similarly. Prove that A_1, B_1, and C_1 are collinear.

Proposed by Sammy Luo

2 $ABCD$ is a cyclic quadrilateral inscribed in the circle ω. Let $AB \cap CD = E$, $AD \cap BC = F$. Let ω_1, ω_2 be the circumcircles of AEF, CEF, respectively. Let $\omega \cap \omega_1 = G$, $\omega \cap \omega_2 = H$. Show that AC, BD, GH are concurrent.

Proposed by Yang Liu

3 Let $A_1A_2A_3\cdots A_{2013}$ be a cyclic 2013-gon. Prove that for every point P not the circumcenter of the 2013-gon, there exists a point $Q \neq P$ such that $\frac{A_iP}{A_iQ}$ is constant for $i \in \{1, 2, 3, \cdots, 2013\}$.

Proposed by Robin Park

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4 Let $ABCD$ be a quadrilateral inscribed in circle ω. Define $E = AA \cap CD$, $F = AA \cap BC$, $G = BE \cap \omega$, $H = BE \cap AD$, $I = DF \cap \omega$, and $J = DF \cap AB$. Prove that GI, HJ, and the B-symmedian are concurrent.

Proposed by Robin Park

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5 Let P be a point in the interior of an acute triangle ABC, and let Q be its isogonal conjugate. Denote by ω_P and ω_Q the circumcircles of triangles BPC and BQC, respectively. Suppose the circle with diameter $\overline{AP}$ intersects ω_P again at M, and line AM intersects ω_P again at X. Similarly, suppose the circle with diameter $\overline{AQ}$ intersects ω_Q again at N, and line AN intersects ω_Q again at Y.

Prove that lines MN and XY are parallel.

(Here, the points P and Q are *isogonal conjugates* with respect to $\triangle ABC$ if the internal angle bisectors of $\angle BAC$, $\angle CBA$, and $\angle ACB$ also bisect the angles $\angle PAQ$, $\angle PBQ$, and $\angle PCQ$, respectively. For example, the orthocenter is the isogonal conjugate of the circumcenter.)

Proposed by Sammy Luo

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6 Let $ABCD$ be a cyclic quadrilateral with center O.
Suppose the circumcircles of triangles AOB and COD meet again at G, while the circumcircles of triangles AOD and BOC meet again at H.
Let ω_1 denote the circle passing through G as well as the feet of the perpendiculars from G to AB and CD.
Define ω_2 analogously as the circle passing through H and the feet of the perpendiculars from H to BC and DA.
Show that the midpoint of GH lies on the radical axis of ω_1 and ω_2.

Proposed by Yang Liu

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7 Let ABC be a triangle with circumcenter O. Let P be a point inside ABC, so let the points D, E, F be on BC, AC, AB respectively so that the Miquel point of DEF with respect to ABC is P. Let the reflections of D, E, F over the midpoints of the sides that they lie on be R, S, T. Let the Miquel point of RST with respect to the triangle ABC be Q. Show that $OP = OQ$.

Proposed by Yang Liu

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8 In triangle ABC with incenter I and circumcenter O, let A', B', C' be the points of tangency of its circumcircle with its A, B, C-mixtilinear circles, respectively. Let ω_A be the circle through A' that is tangent to AI at I, and define ω_B, ω_C similarly. Prove that $\omega_A, \omega_B, \omega_C$ have a common point X other than I, and that $\angle AXO = \angle OXA'$.

Proposed by Sammy Luo

9 Let P be a point inside a triangle ABC such that $\angle PAC = \angle PCB$. Let the projections of P onto BC, CA, and AB be X, Y, Z respectively. Let O be the circumcenter of $\triangle XYZ$, H be the foot of the altitude from B to AC, N be the midpoint of AC, and T be the point such that $TYPO$ is a parallelogram. Show that $\triangle THN$ is similar to $\triangle PBC$.

Proposed by Sammy Luo

10 We are given triangles ABC and DEF such that $D \in BC, E \in CA, F \in AB$, $AD \perp EF, BE \perp FD, CF \perp DE$. Let the circumcenter of DEF be O, and let the circumcircle of DEF intersect BC, CA, AB again at R, S, T respectively. Prove that the perpendiculars to BC, CA, AB through D, E, F respectively intersect at a point X, and the lines AR, BS, CT intersect at a point Y, such that O, X, Y are collinear.

Proposed by Sammy Luo

12 Let $AB = AC$ in $\triangle ABC$, and let D be a point on segment AB. The tangent at D to the circumcircle ω of BCD hits AC at E. The other tangent from E to ω touches it at F, and $G = BF \cap CD$, $H = AG \cap BC$. Prove that $BH = 2HC$.

Proposed by David Stoner

13 Let ABC be a nondegenerate acute triangle with circumcircle ω and let its incircle γ touch AB, AC, BC at X, Y, Z respectively. Let XY hit arcs AB, AC of ω at M, N respectively, and let $P \neq X, Q \neq Y$ be the points on γ such that $MP = MX, NQ = NY$. If I is the center of γ, prove that P, I, Q are collinear if and only if $\angle BAC = 90^\circ$.

Proposed by David Stoner

– Number Theory

1 Does there exist a strictly increasing infinite sequence of perfect squares $a_1, a_2, a_3, \ldots$ such that for all $k \in \mathbb{Z}^+$ we have that $13^k | a_k + 1$?

Proposed by Jesse Zhang

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2 Define the Fibanocci sequence recursively by $F_1 = 1$, $F_2 = 1$ and $F_{i+2} = F_i + F_{i+1}$ for all i. Prove that for all integers $b, c > 1$, there exists an integer n such that the sum of the digits of F_n when written in base b is greater than c.

Proposed by Ryan Alweiss

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3 Let t and n be fixed integers each at least 2. Find the largest positive integer m for which there exists a polynomial P, of degree n and with rational coefficients, such that the following property holds: exactly one of

$$\frac{P(k)}{t^k} \text{ and } \frac{P(k)}{t^{k+1}}$$

is an integer for each $k = 0, 1, ..., m$.

Proposed by Michael Kural

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4 Let $\mathbb{N}$ denote the set of positive integers, and for a function f, let $f^k(n)$ denote the function f applied k times. Call a function $f : \mathbb{N} \to \mathbb{N}$ *saturated* if

$$f^{f^{f(n)}(n)}(n) = n$$

for every positive integer n. Find all positive integers m for which the following holds: every saturated function f satisfies $f^{2014}(m) = m$.

Proposed by Evan Chen

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5 Define a *beautiful number* to be an integer of the form a^n, where $a \in \{3, 4, 5, 6\}$ and n is a positive integer.
Prove that each integer greater than 2 can be expressed as the sum of pairwise distinct beautiful numbers.

Proposed by Matthew Babbitt

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6 Show that the numerator of

$$\frac{2^{p-1}}{p+1} - \left(\sum_{k=0}^{p-1} \frac{\binom{p-1}{k}}{(1-kp)^2}\right)$$

is a multiple of p^3 for any odd prime p.

Proposed by Yang Liu

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7 Find all triples (a, b, c) of positive integers such that if n is not divisible by any prime less than 2014, then $n + c$ divides $a^n + b^n + n$.

Proposed by Evan Chen

8 Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that:
(i) The greatest common divisor of the sequence $f(1), f(2), \dots$ is 1.
(ii) For all sufficiently large integers n, we have $f(n) \neq 1$ and

$$f(a)^n \mid f(a+b)^{a^{n-1}} - f(b)^{a^{n-1}}$$

for all positive integers a and b.

Proposed by Yang Liu

9 Let d be a positive integer and let ε be any positive real. Prove that for all sufficiently large primes p with $\gcd(p-1, d) \neq 1$, there exists an positive integer less than p^r which is not a dth power modulo p, where r is defined by

$$\log r = \varepsilon - \frac{1}{\gcd(d, p-1)}.$$

Proposed by Shashwat Kishore

10 Find all positive integer bases $b \geq 9$ so that the number

$$\frac{\overbrace{11\cdots1}^{n-1\ 1's}0\overbrace{77\cdots7}^{n-1\ 7's}8\overbrace{11\cdots1}^{n\ 1's}{}_b}{3}$$

is a perfect cube in base 10 for all sufficiently large positive integers n.

Proposed by Yang Liu

11 Let p be a prime satisfying $p^2 \mid 2^{p-1} - 1$, and let n be a positive integer. Define

$$f(x) = \frac{(x-1)^{p^n} - (x^{p^n} - 1)}{p(x-1)}.$$

Find the largest positive integer N such that there exist polynomials $g(x)$, $h(x)$ with integer coefficients and an integer r satisfying $f(x) = (x-r)^N g(x) + p \cdot h(x)$.

Proposed by Victor Wang