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# Refinement on Learning

## Data Mining Theory (データマイニング工学)

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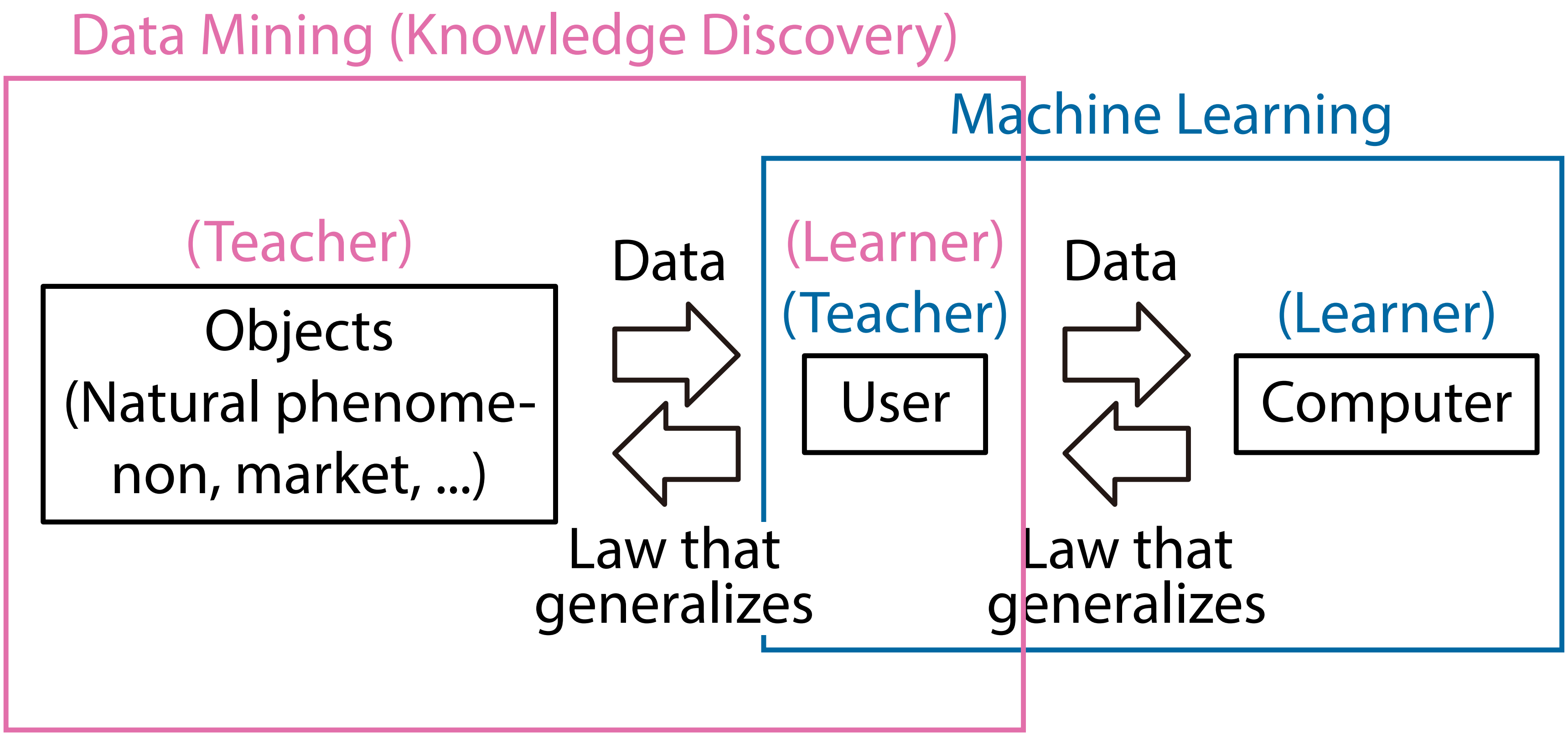
# Today's Outline

- Recap the main points of last week's lecture
- Consider the structure of a hypothesis space
  - Essential to efficiently search candidate hypotheses
- Understand the hypothesis space as a poset (半順序集合)
- Introduce the key concept of a refinement (精密化) operator to traverse the (structured) hypothesis space

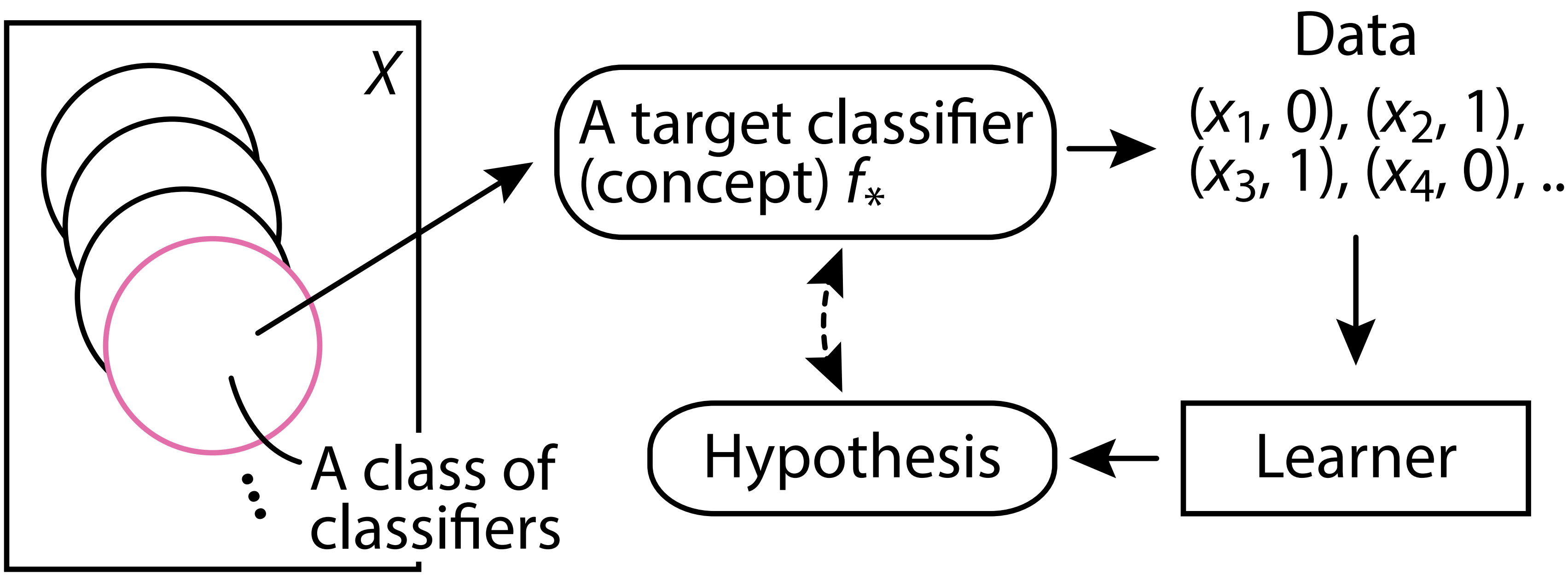
# Framework of Learning (ML vs DM)



# Formalization of Learning in Computational Manner

1. What are targets of learning? (学習対象)
   - Each target (concept) $C$ is a subset of the domain $X$ ($C \subseteq X$)
   - A concept space $\mathcal{C}$ is a collection of concepts ($\mathcal{C} \subseteq \mathcal{P}(X)$)
2. How to represent targets and hypotheses? （表現言語）
   - We use a hypothesis space $\mathcal{H}$
   - Each hypothesis $H \in \mathcal{H}$ represents a concept $\nu(H) \subseteq X$
3. How are data provided to a learner? （データ）
4. How does the learner work? (学習手順，アルゴリズム)
5. When can we say that the learner correctly learns the target? (学習の正当性)

# Learning Model



# Gold's Learning Model on Languages

- A concept space $\mathcal{C} \subseteq \{ A \mid A \subseteq \Sigma^* \}$ is chosen
- For a language $C \in \mathcal{C}$, an infinite sequence $\sigma = (x_1, y_1), (x_2, y_2), \dots$ is a complete presentation (完全提示) of $C$ if
  (i) $\{x_1, x_2, \dots\} = \Sigma^*$
  (ii) $y_i = 1 \iff x_i \in C$ for all $i$
- A learner is a procedure $M$ that receives $\sigma$ and generates an infinite sequence of hypotheses $\gamma = H_1, H_2, \dots$
- If $\gamma$ converges to some hypothesis $H$ and $\upsilon(H) = C$, we say that *$M$* identifies *$C$* in the limit (極限学習する)
  - If $M$ identifies any $C \in \mathcal{C}$ in the limit, *$M$* identifies *$\mathcal{C}$* in the limit

# Consistency of Hypotheses

- A language $C$ is inconsistent with $(x, y)$ (矛盾する) if $(y = 1 \text{ and } x \notin C)$ or $(y = 0 \text{ and } x \in C)$

- $C$ is consistent with $(x, y)$ if $C$ is not inconsistent with $(x, y)$

- For a set of examples $S = \{(x_1, y_1), \dots, (x_n, y_n)\}$,
  $C$ is consistent with $S$ ($C$ は $S$ に無矛盾)
  if $C$ is consistent with every $(x, y) \in S$

# Basic Strategy: Generate and Test

- Input: a complete presentation $\sigma$ of a language $C \in \mathcal{C}$
- Output: $\gamma = H_1, H_2, \ldots$

1. $i \leftarrow 1, S \leftarrow \emptyset$
2. **repeat**
3. $\quad S \leftarrow S \cup \{(x_i, y_i)\}$
4. $\quad$ **while** $\upsilon(H)$ is not consistent with $S$ **do**
5. $\quad\quad H \leftarrow$ the next hypothesis in the hypothesis space $\mathcal{H}$
6. $\quad$ **end while**
7. $\quad H_i \leftarrow H$ and output $H_i$
8. $\quad i \leftarrow i + 1$
9. **until forever**

# Power of Generate and Test Strategy and Its Problem

- For any class $\mathcal{C}$ of languages, Generate and Test strategy identifies $\mathcal{C}$ in the limit
  - That is, Generate and Test strategy identifies every language $C \in \mathcal{C}$ in the limit
- Unfortunately, this strategy is not realistic

# Power of Generate and Test Strategy and Its Problem

- For any class $\mathcal{C}$ of languages, Generate and Test strategy identifies $\mathcal{C}$ in the limit
  - That is, Generate and Test strategy identifies every language $C \in \mathcal{C}$ in the limit
- Unfortunately, this strategy is not realistic
- What is needed for more efficient learning?

→ An efficient search of candidate hypotheses is essential!

# Structurization of Hypothesis Space

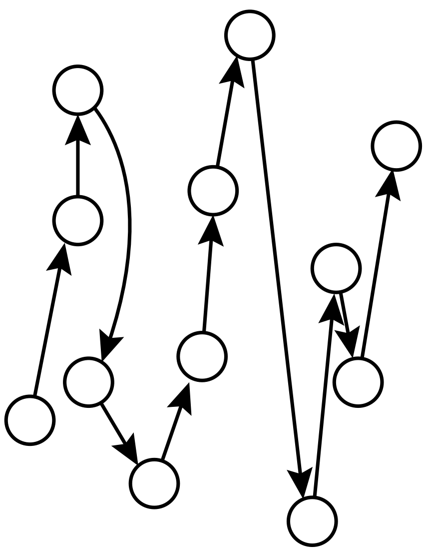
- To search hypotheses,
  (i) The structure of the hypothesis space $\mathcal{H}$
  (ii) An operator that enables to traverse the space

  are indispensable

1. The structured space is mathematically modeled as a poset (partially ordered set; 半順序集合)
2. As an operator, we use refinement (精密化)
   - For each hypothesis, a learner can “refine” it and derive a set of one level specific hypotheses

# Structurization of Hypothesis Space

No structure
in hypothesis space



# Structurization of Hypothesis Space

No structure
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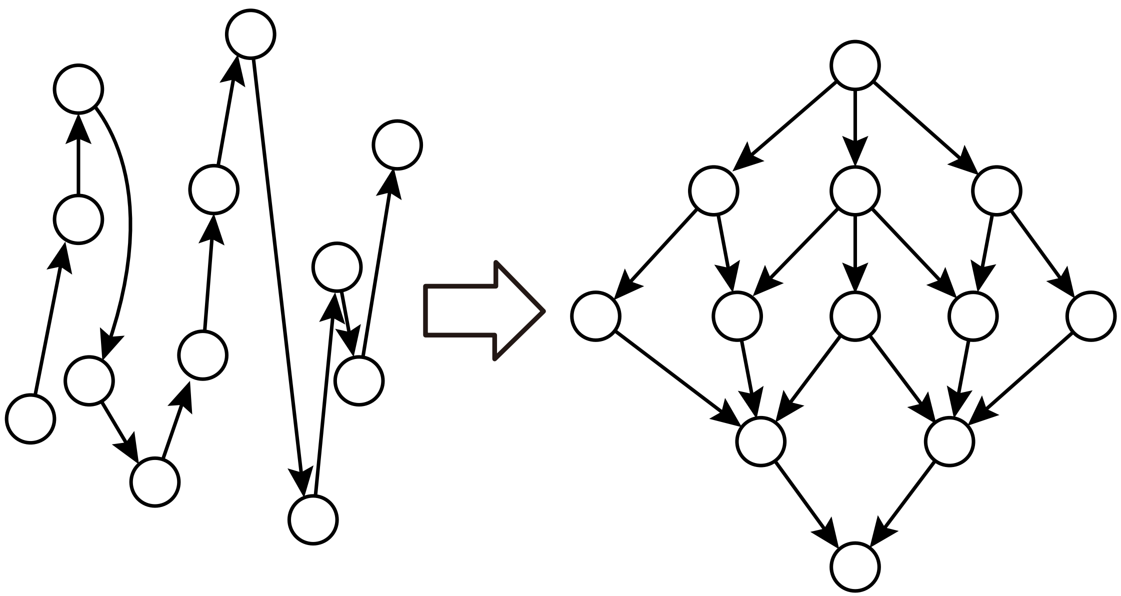
Poset with refinement

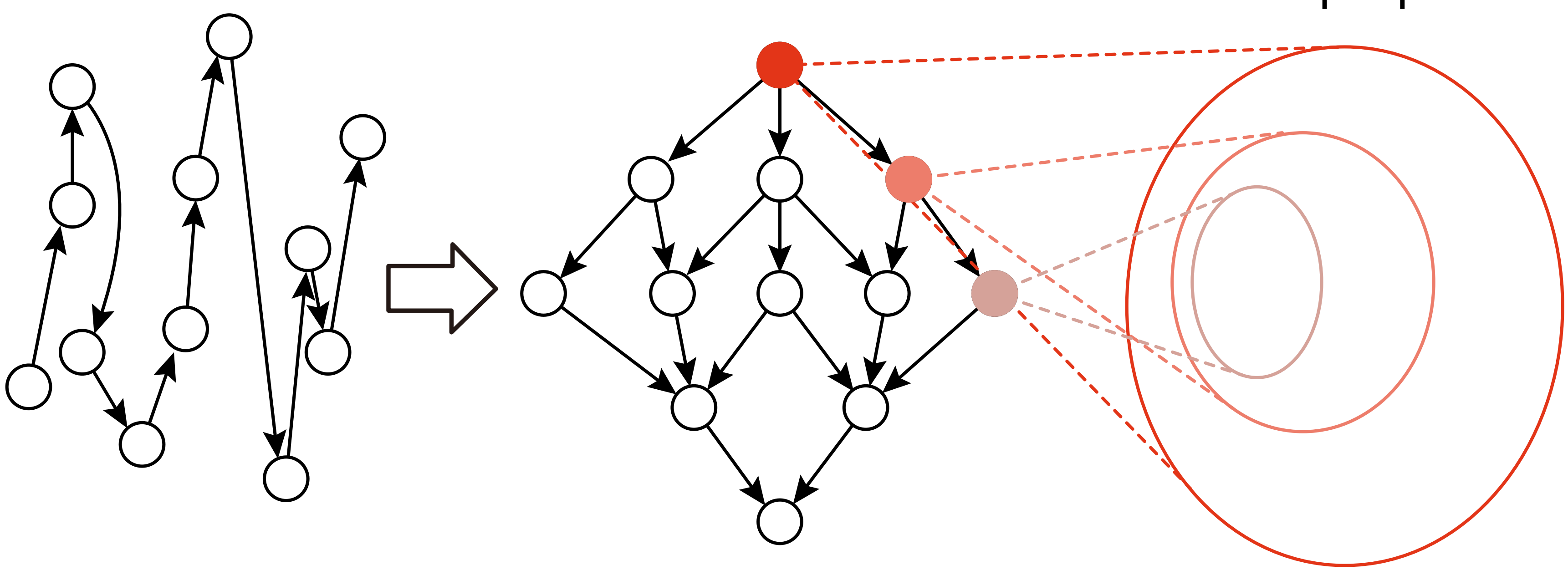


# Structurization of Hypothesis Space

No structure
in hypothesis space

Poset with refinement

Subset inclusion
relationships in
concept space



# Poset

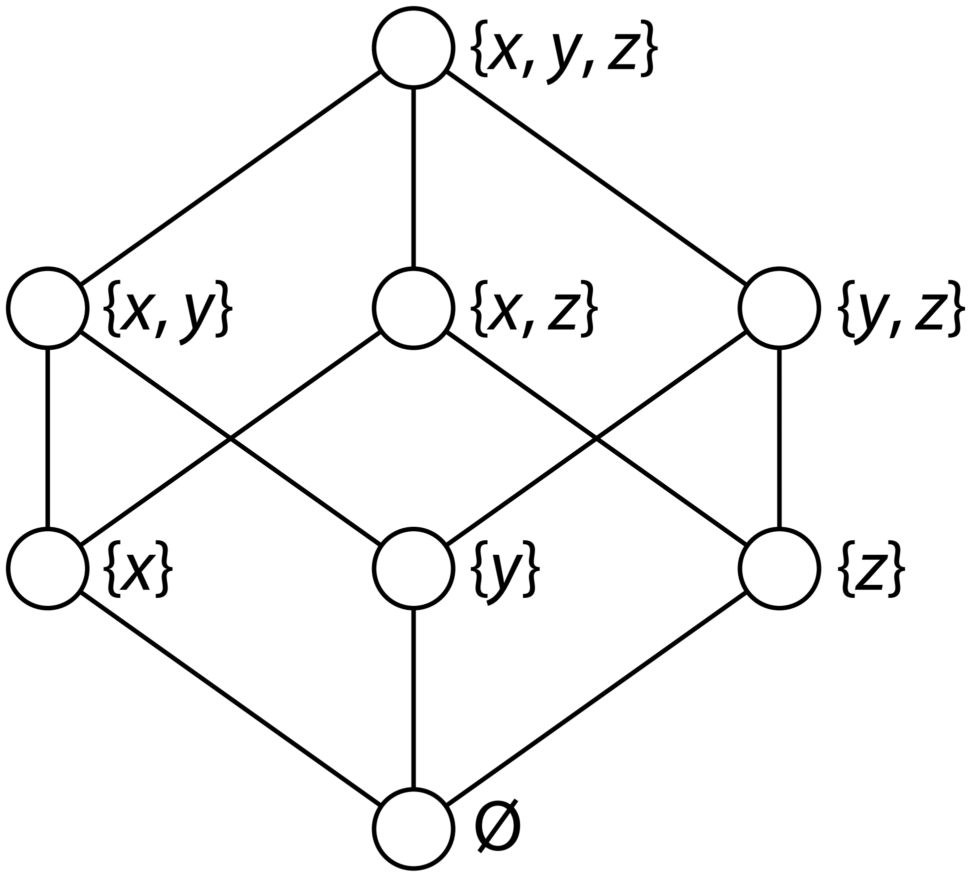
- A partial order (半順序) is a binary relation $\leq$ s.t.
  1. $x \leq x$ (reflexivity; 反射律)
  2. $(x \leq y \text{ and } y \leq x) \Rightarrow x = y$ (antisymmetry; 反対称律)
  3. $(x \leq y \text{ and } y \leq z) \Rightarrow x \leq z$ (transitivity; 推移律)

- A set $X$ with a partial order $\leq$, denoted as $(X, \leq)$, is called a partially ordered set (poset; 半順序集合)
  - The least upper bound (supremum; 最小上界) of $S \subseteq X$ is the least $x \in X$ s.t. $\forall s \in S, s \leq x$
  - The greatest lower bound (infimum; 最大下界) of $S \subseteq X$ is the greatest $x \in X$ s.t. $\forall s \in S, x \leq s$

# Lattice

- We use join "$\vee$" (結び) and meet "$\wedge$" (交わり)
  - $x \vee y = \sup\{x, y\}$ (join of $x$ and $y$)
    - For $S \subseteq X$, $\vee S = \sup S$
  - $x \wedge y = \inf\{x, y\}$ (meet of $x$ and $y$)
    - For $S \subseteq X$, $\wedge S = \inf S$
- A poset $(X, \leq)$ is a lattice (束) if $x \vee y$ and $x \wedge y$ exist for all $x, y \in X$
- Examples:
  - The power set $\mathcal{P}(X)$ of any set $X$ (we translate "$\subseteq$" as $\leq$)
  - The set of natural numbers $\mathbb{N}$ w.r.t "$\leq$"
  - The Cartesian product $\mathbb{N} \times \mathbb{N} = \{ (a, b) \mid a, b \in \mathbb{N} \}$, $(a, b) \leq (a', b')$ if $a \leq a'$ and $b \leq b'$

# The Power Set Is a Lattice



# Definition of Refinement

- Assume that our hypothesis space $(\mathcal{H}, \preceq)$ is a poset and

  $G \preceq H \Rightarrow \upsilon(G) \subseteq \upsilon(H)$

  $G \equiv H \Rightarrow \upsilon(G) = \upsilon(H)$

  - $\upsilon$ should be a homomorphism (準同型写像) that preserves structure between $\mathcal{C}$ and $\mathcal{H}$

- A refinement (精密化) is a mapping $\rho : \mathcal{H} \to 2^{\mathcal{H}}$ s.t.
  1. $\forall H \in \mathcal{H}, \rho(H)$ is finite
  2. $G \in \rho(H) \Rightarrow G \preceq H$
  3. $\forall H \in \mathcal{H},$ there is no infinite sequence $H_1, H_2, \ldots$ s.t. $H = H_1$ and $H_i \in \rho(H_{i+1})$

# Semantically Complete Refinement

- We write $X \xrightarrow{\rho} Y$ if $Y \in \rho(X)$
  - $\xrightarrow{*}$ is zero or more applications of $\xrightarrow{\rho}$
- A refinement $\rho$ is semantically complete (意味的に完全) if

$$\left\{ \upsilon(G) \;\middle|\; H \xrightarrow{*} G \right\} = \{ C \in \mathcal{C} \mid C \subset \upsilon(H) \}$$

  - Start from $H$, we can find any $C \subset \upsilon(H)$ by applying $\xrightarrow{\rho}$
  - If this condition is not satisfied, we will miss some concepts

# Pioneers of Refinement

- Refinement is (implicitly) used in various contexts
  - It can be viewed as an online construction of search space with tree-like structure
- It has been explicitly introduced in Model Inference System by Shapiro in 1981
  - E. Y. Shapiro, **An Algorithm That Infers Theories from Facts,** *IJCAI*, 1981
- Plotkin considered the opposite direction (from specific to general)
  - G. D. Plotkin, **A further note on inductive generalization**, *Machine Intelligence*, 1970

# Examples of Refinement

- Let us consider concrete examples of refinement and learning
- We use two simple examples:
  - Regular language (正則言語)
  - The set of pairs of natural numbers
    $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N} = \{ (a, b) \mid a, b \in \mathbb{N} \}$

# Regular Language (1/2)

- Given an alphabet $\Sigma$
  - For $a \in \Sigma$, $a^2 = aa, a^3 = aaa, \ldots$
  - $X^0 = \varnothing$, $X^n = \{ au \mid a \in X, u \in X^{n-1} \}$ $(n \geq 1)$
- For a regular expression (正則表現, RE) $H$, $\upsilon(H)$ is a regular language (正則言語)
  - $\varnothing$ is an RE; $\upsilon(\varnothing) = \varnothing$
  - $\forall a \in \Sigma$, $a$ is an RE; $\upsilon(a) = \{a\}$
  - If $X$ and $Y$ are REs,
    - $X + Y$ is an RE; $\upsilon(X + Y) = X \cup Y$ (union)
    - $XY$ is an RE; $\upsilon(XY) = \{ ab \mid a \in X, b \in Y \}$ (concatenation)
    - $X^*$ is an RE; $\upsilon(X^*) = \bigcup \{ X^n \mid n \geq 0 \}$ (Kleene closure; クリーネ閉包)

# Regular Language (2/2)

- Let $\Sigma = \{a_1, a_2, \ldots, a_n\}$
- We denote by $\top$ the language $(a_1 + a_2 + \cdots + a_n)^*$
  - $\upsilon(\top) = \Sigma^*$
  - The largest language over $\Sigma$
- Examples:
  - Assume that $\Sigma = \{\mathtt{a}, \mathtt{t}, \mathtt{g}, \mathtt{c}\}$
  - $\upsilon(\mathtt{at} + \mathtt{g}^*) = \{\varepsilon, \mathtt{at}, \mathtt{g}, \mathtt{gg}, \mathtt{ggg}, \ldots\}$
  - $\upsilon((\mathtt{a} + \mathtt{c})^*) = \{\varepsilon, \mathtt{a}, \mathtt{c}, \mathtt{aa}, \mathtt{ac}, \mathtt{ca}, \mathtt{cc}, \mathtt{aaa}, \ldots\}$
  - $\upsilon(\top) = \{\varepsilon, \mathtt{a}, \mathtt{t}, \mathtt{g}, \mathtt{c}, \mathtt{aa}, \mathtt{at}, \ldots\}$

# Refinement on Regular Languages

## (from P. D. Laird, Learning from Good and Bad Data, 1988)

1. $X \xrightarrow{\rho} X + X$
2. $X^* \xrightarrow{\rho} X^* X^*$
3. $X^* \xrightarrow{\rho} (X^*)^*$
4. $a \xrightarrow{\rho} \varnothing \quad (a \in \Sigma)$
5. $X^* \xrightarrow{\rho} X$
6. $X \xrightarrow{\rho} Y \Rightarrow X + Z \xrightarrow{\rho} Y + Z$
7. $X \xrightarrow{\rho} Y \Rightarrow Z + X \xrightarrow{\rho} Z + Y$
8. $X \xrightarrow{\rho} Y \Rightarrow X^* \xrightarrow{\rho} Y^*$
9. $X \xrightarrow{\rho} Y \Rightarrow XZ \xrightarrow{\rho} YZ$
10. $X \xrightarrow{\rho} Y \Rightarrow ZX \xrightarrow{\rho} ZY$

# Examples of Refinement on Regular Languages

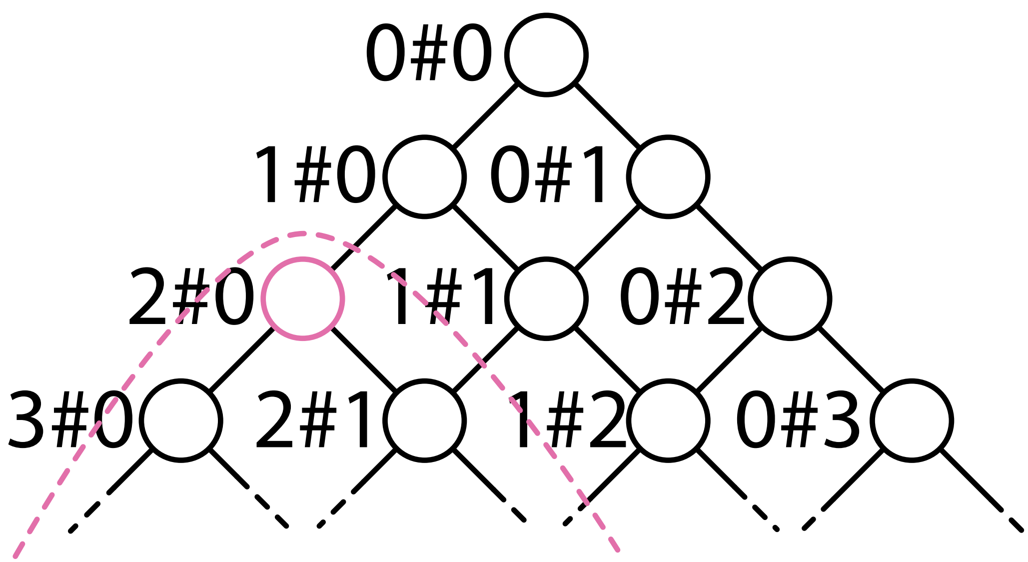
- Let $\Sigma = \{0, 1\}$
- $\top = (0 + 1)^* \xrightarrow{\rho} 0 + 1 \xrightarrow{\rho} \varnothing + 1 \xrightarrow{\rho} \varnothing + \varnothing$
- $\top = (0+1)^* \xrightarrow{\rho} (0+1)^*(0+1)^* \xrightarrow{\rho} (0+1)^*(0+1) \xrightarrow{\rho} (0+1)(0+1)$
  - $\upsilon((0 + 1)(0 + 1)) = \{00, 01, 10, 11\}$

# Efficient Learning with Refinement

$i \leftarrow 1, S \leftarrow \emptyset, H \leftarrow \top, Q \leftarrow \emptyset$ // $Q$ is a list of candidate hypotheses
**repeat**
    $S \leftarrow S \cup \{(x_i, y_i)\}$
    **while** $H$ is not consistent with $S$
        **if** $x \in \upsilon(H)$ for some $(x, 0) \in S$ **then**
            Append all $\rho(H)$ to the tail of $Q$
        **end if**
        $H \leftarrow$ the first hypothesis in $Q$, and remove it from $Q$
    **end while**
    $H_i \leftarrow H$ and output $H_i$
    $i \leftarrow i + 1$
**until forever**

# Hypothesis Space on $\mathbb{N}^2$

- $\mathcal{H} = \{\, a\#b \mid a, b \in \mathbb{N} \,\}$
- $a\#b \leq c\#d$ if $a \geq c$ and $b \geq d$
  - Note that we invert ≤ for mathematical convenience



# Refinement on $\mathbb{N}^2$

- We consider the following concept space $\mathcal{C}$:

  $\mathcal{C} = \{ \uparrow(a, b) \mid a, b \in \mathbb{N} \}$,

  where

  $\uparrow(a, b) = \{ (c, d) \in \mathbb{N}^2 \mid a \leq c, b \leq d \}$

  - A subset $O \subseteq \mathbb{N}^2$ s.t. $(a, b) \in O \Rightarrow \uparrow(a, b) \subseteq O$ is known to be open on the Alexandroff topology

- Define $\upsilon(a\#b) = \uparrow(a, b)$

- Refinement is given as follows:

  1. $a\#b \overset{\rho}{\to} (a + 1)\#b$
  2. $a\#b \overset{\rho}{\to} a\#(b + 1)$

# Refinement on Sets of $\mathbb{N}^2$

- We can further treat a (finite) set of $\uparrow(a, b)$ as a concept
- $\mathcal{H}_S = \{\, a_1\#b_1 + a_2\#b_2 + \cdots + a_n\#b_n \mid a_i, b_i, n \in \mathbb{N} \,\}$
- $\mathcal{C}_S = \{\, C \mid C \subseteq \mathcal{C} = \{\uparrow(a, b) \mid a, b \in \mathbb{N}\},\ C \text{ is finite} \,\}$
- $\upsilon(a_1\#b_1 + \cdots + a_n\#b_n) = \uparrow(a_1, b_1) \cup \cdots \cup \uparrow(a_n, b_n)$
- Refinement is given as follows:
  1. $a\#b \overset{\rho}{\to} (a+1)\#b$
  2. $a\#b \overset{\rho}{\to} a\#(b+1)$
  3. $X \overset{\rho}{\to} Y \Rightarrow X + Z \overset{\rho}{\to} Y + Z$ and $Z + X \overset{\rho}{\to} Z + Y$
  4. $X \overset{\rho}{\to} X + X$

# How about $\mathbb{R}$?

- Let us consider the set of real numbers $\mathbb{R}$
  - One of the most important objects in machine learning
- Each real number $x \in \mathbb{R}$ is represented as an infinite sequence
  - e.g., use infinite decimal expansions with $\Sigma = \{0, 1, \ldots, 9\}$
  - Let $\overline{x}$ be a representation of $x$
- Obviously, we cannot treat all elements in $\mathbb{R}$ as we cannot determine $x \in \mathbb{R}$ from $\overline{x}$ in finite time
- We can just treat prefixes of infinite sequences, and $\upsilon(w) = \{\, x \in \mathbb{R} \mid w \sqsubseteq \overline{x} \,\}$, which forms an open set on $\mathbb{R}$