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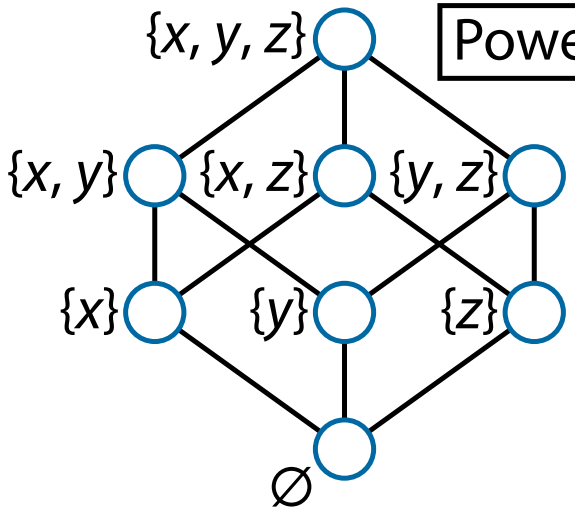
ISIT 2016

Information Decomposition on Structured Space

Mahito Sugiyama (Osaka Univ.)
Hiroyuki Nakahara (RIKEN), Koji Tsuda (UTokyo)

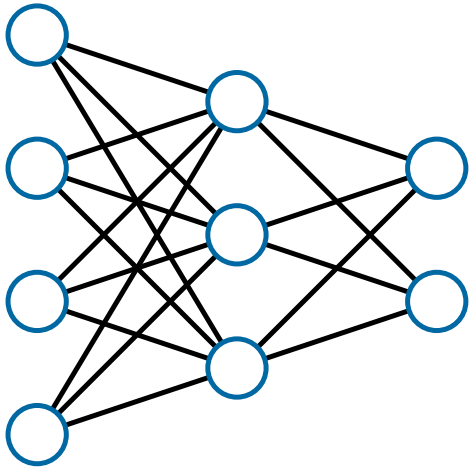
Contributions

- We build information geometry for posets (partially ordered sets)
  - Decomposition of KL divergence
- Key observations:
  - θ-coordinate → principal ideals (lower sets) → p-coordinate
  - p-coordinate → principal filters (upper sets) → η-coordinate
- Code: `https://git.io/decomp`

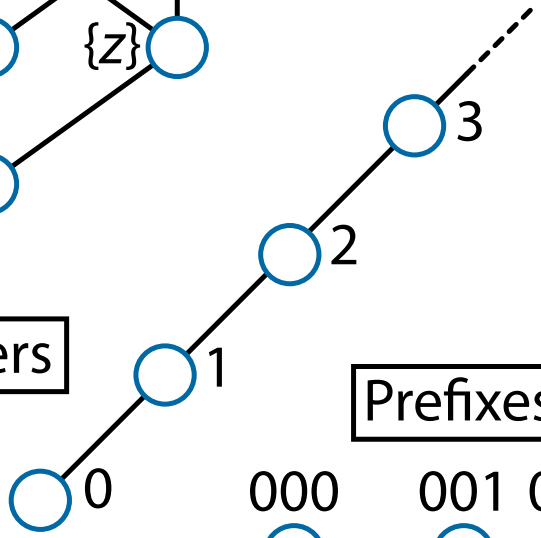


Power set

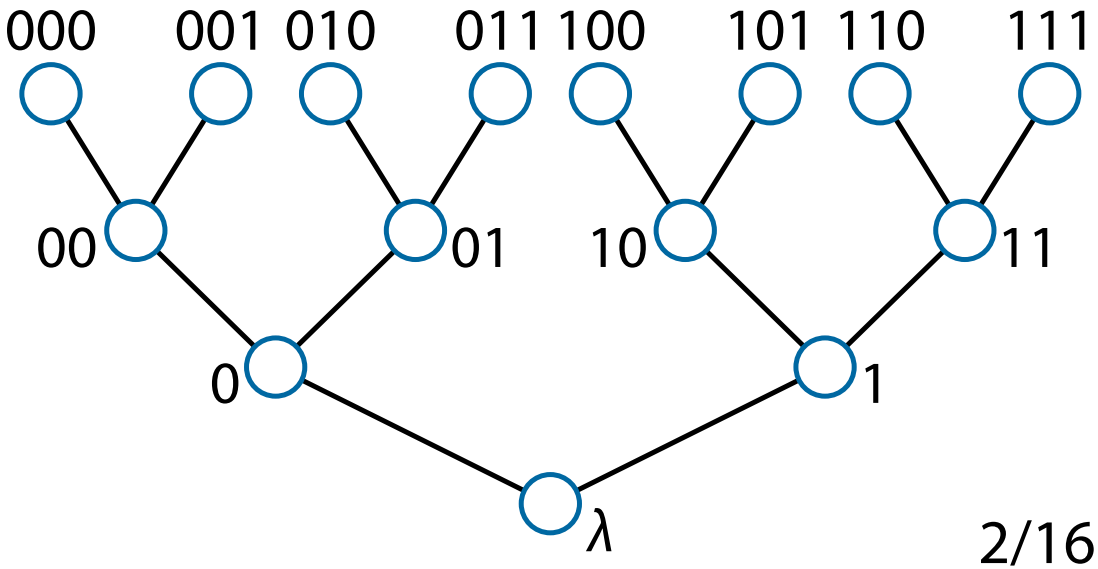
Neural networks



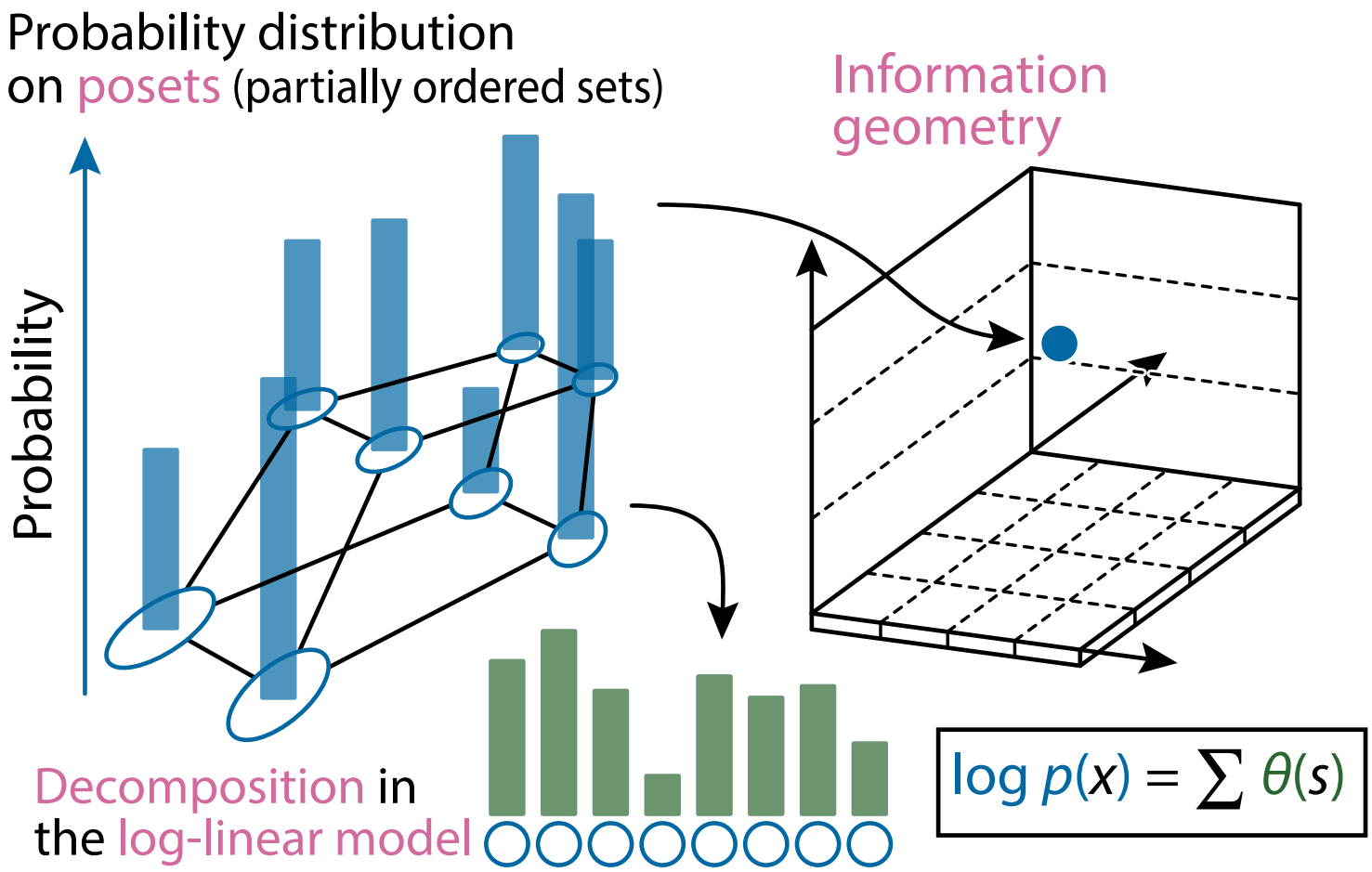
Positive integers



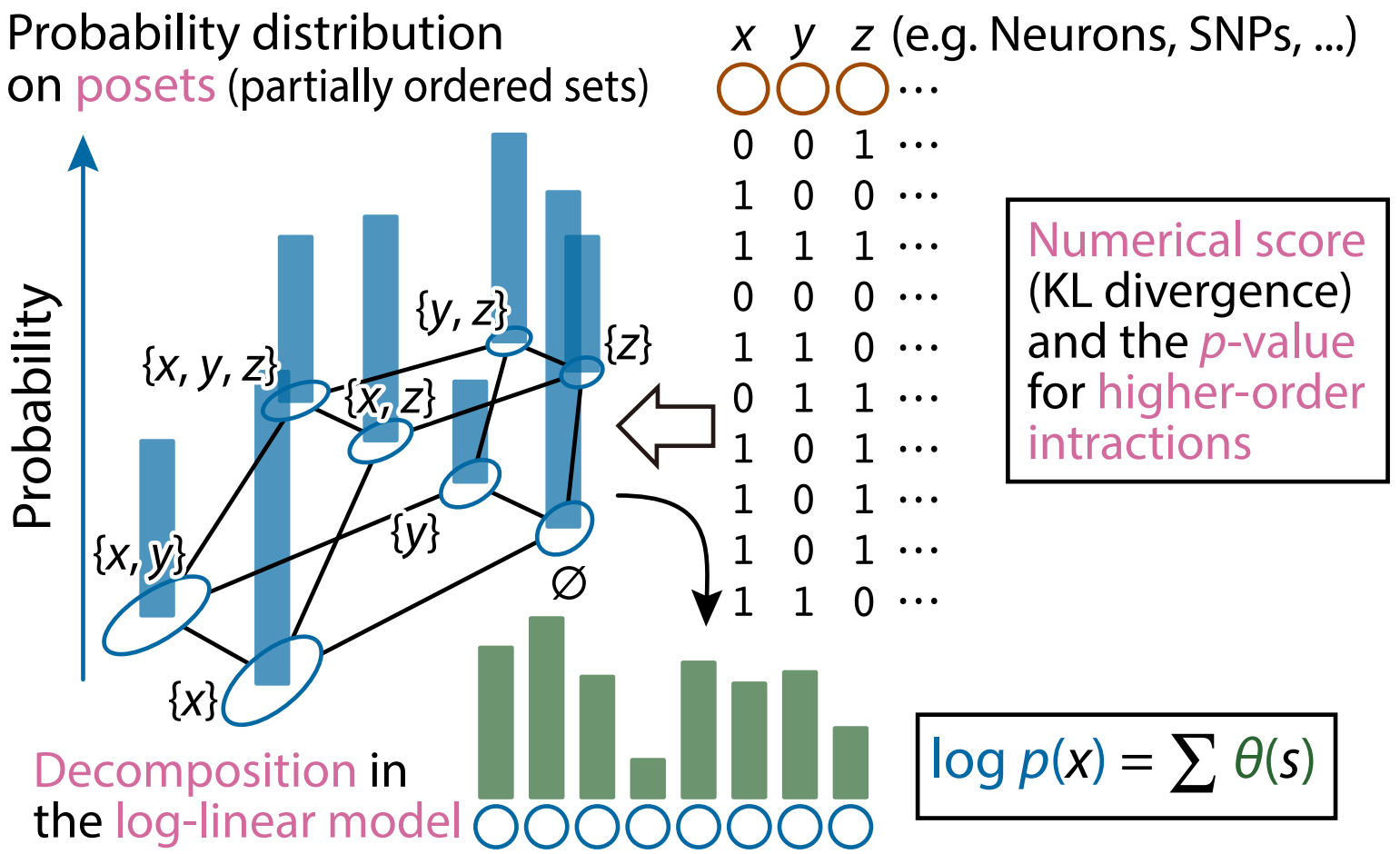
Prefixes



Summary


Probability distribution
on posets (partially ordered sets)
Probability
Information
geometry
Decomposition in
the log-linear model
$\log p(x) = \sum \theta(s)$

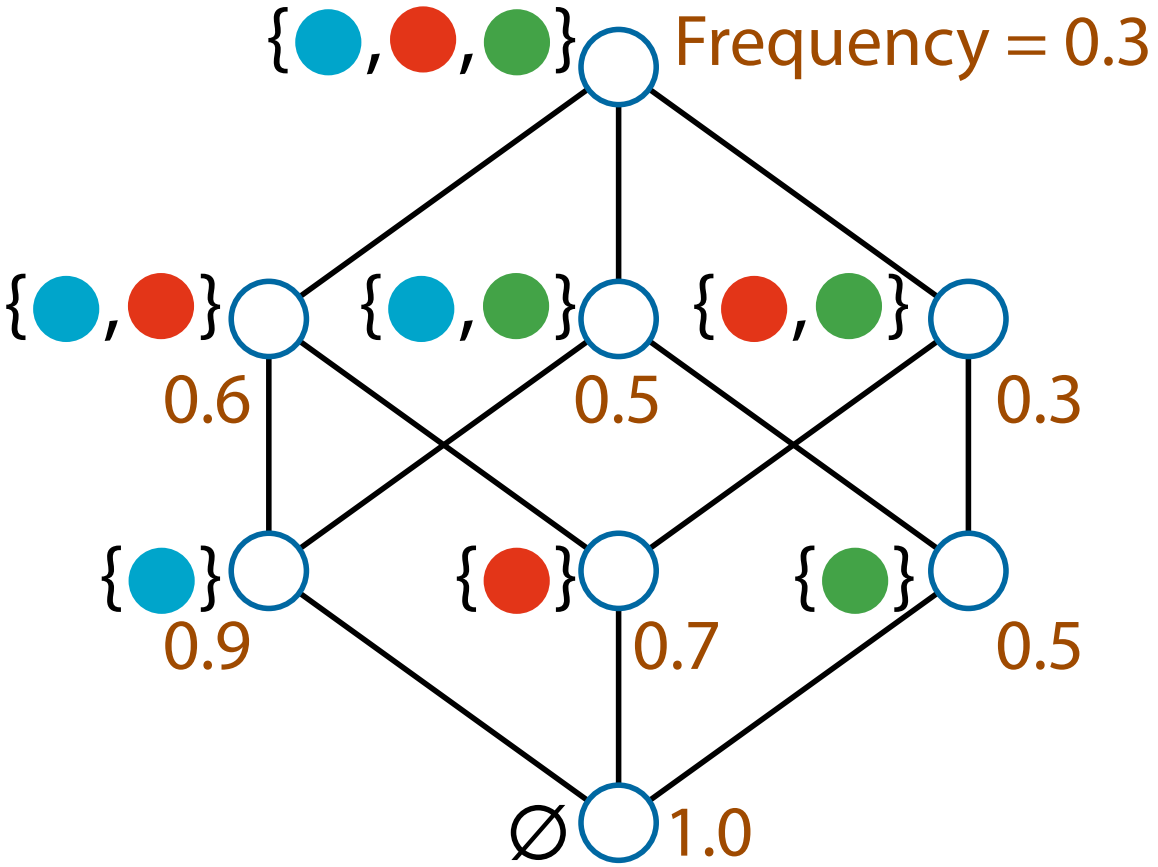
Summary


Probability distribution
on posets (partially ordered sets)
Probability
{x, y, z}
{y, z}
{z}
{x, z}
{x, y}
{y}
∅
{x}
x y z (e.g. Neurons, SNPs, ...)
0 0 1 ⋯
1 0 0 ⋯
1 1 1 ⋯
0 0 0 ⋯
1 1 0 ⋯
0 1 1 ⋯
1 0 1 ⋯
1 0 1 ⋯
1 0 1 ⋯
1 1 0 ⋯
Numerical score
(KL divergence)
and the p-value
for higher-order
intractions
Decomposition in
the log-linear model
log p(x) = ∑ θ(s)

Transaction database

ID 1:	1	1	0
ID 2:	1	1	1
ID 3:	1	1	0
ID 4:	1	1	1
ID 5:	1	1	0
ID 6:	1	0	1
ID 7:	1	0	1
ID 8:	1	1	1
ID 9:	1	0	0
ID10:	0	1	0

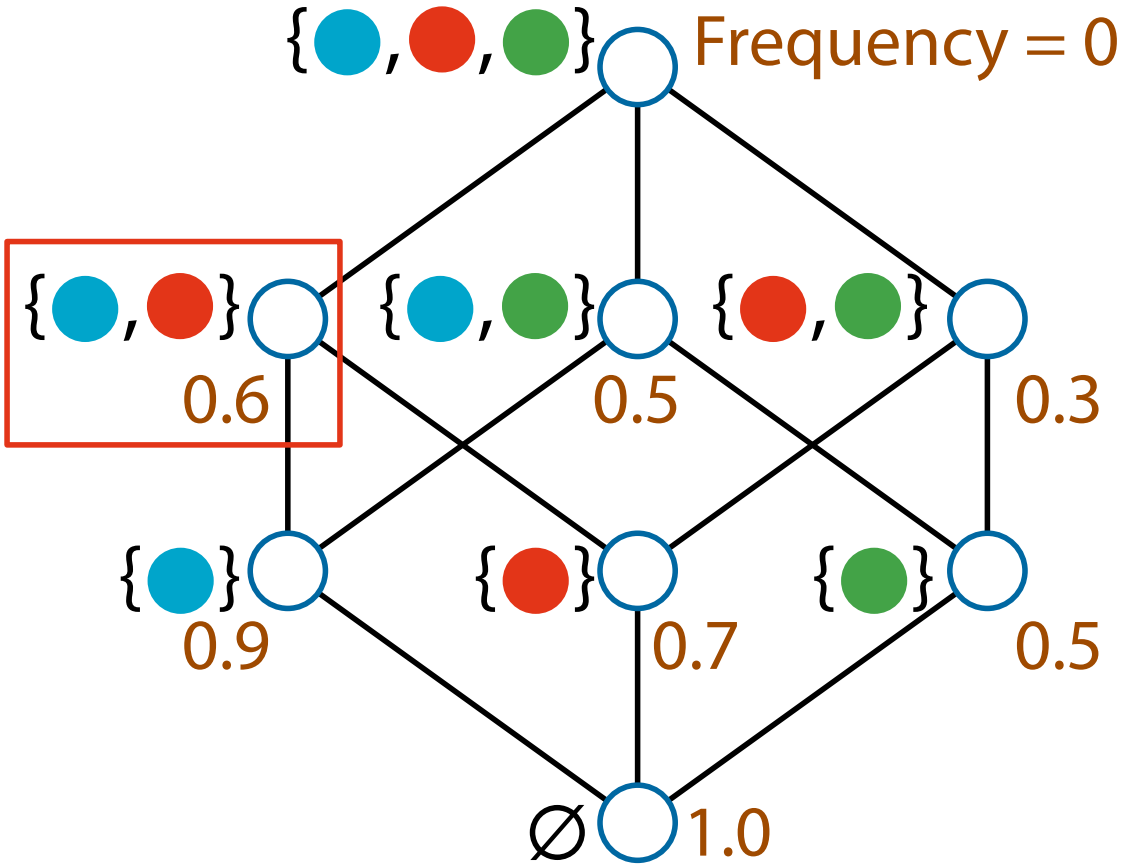
Itemset lattice



Transaction database

ID 1:	1	1	0
ID 2:	1	1	1
ID 3:	1	1	0
ID 4:	1	1	1
ID 5:	1	1	0
ID 6:	1	0	1
ID 7:	1	0	1
ID 8:	1	1	1
ID 9:	1	0	0
ID10:	0	1	0

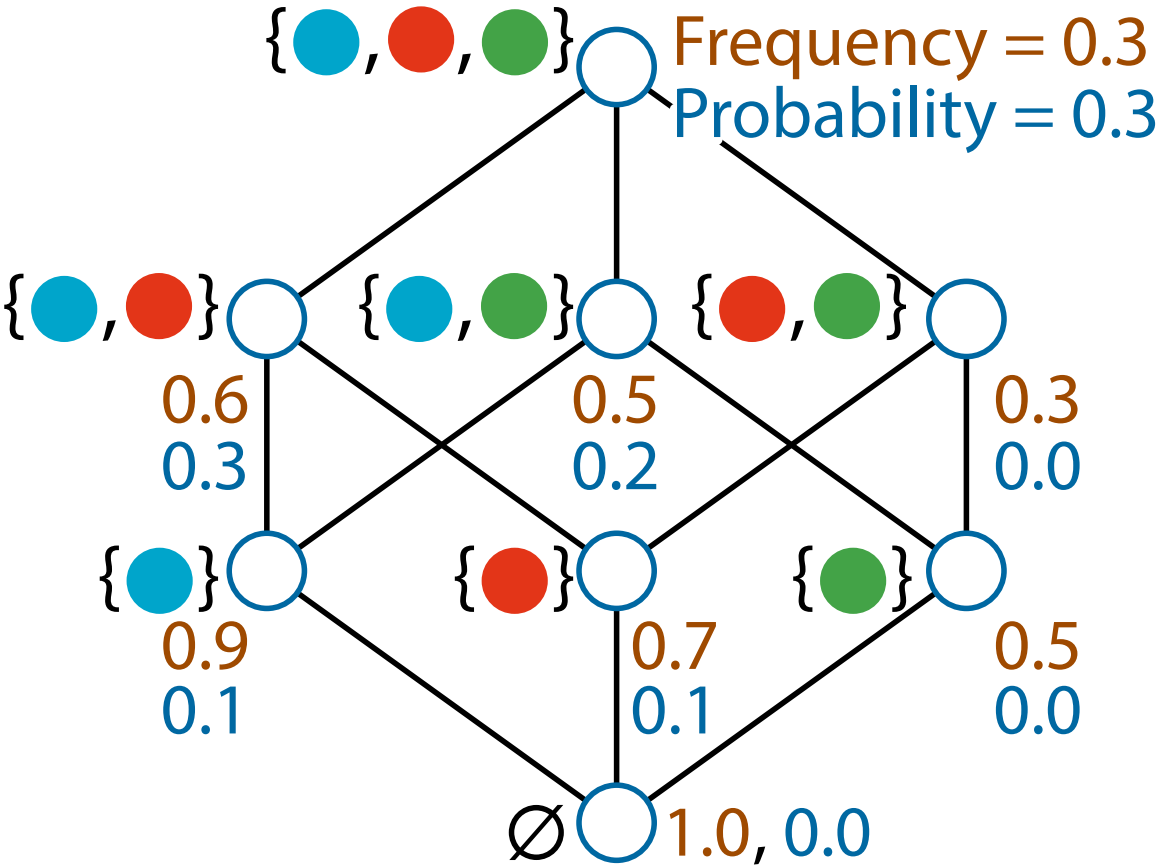
Itemset lattice



Transaction database

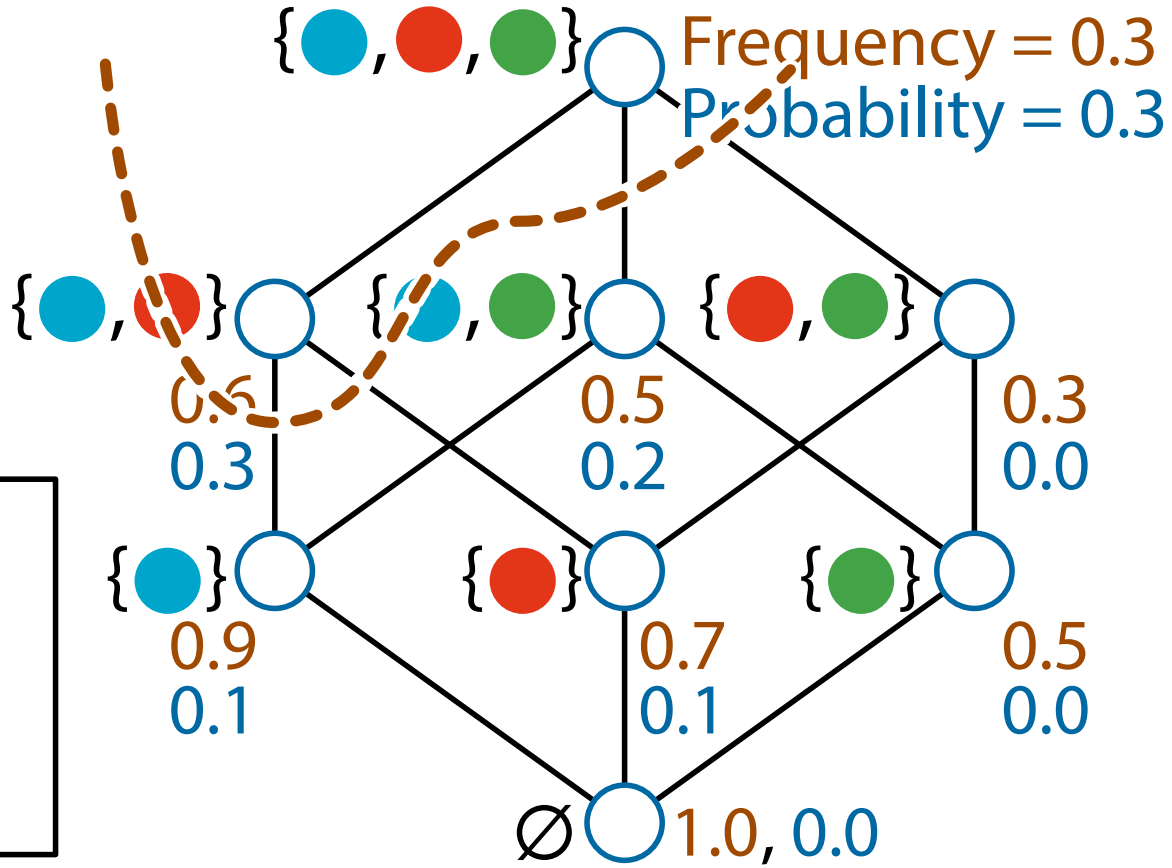
ID 1:	1	1	0
ID 2:	1	1	1
ID 3:	1	1	0
ID 4:	1	1	1
ID 5:	1	1	0
ID 6:	1	0	1
ID 7:	1	0	1
ID 8:	1	1	1
ID 9:	1	0	0
ID10:	0	1	0

Itemset lattice



Upward =
Pattern mining

Itemset lattice

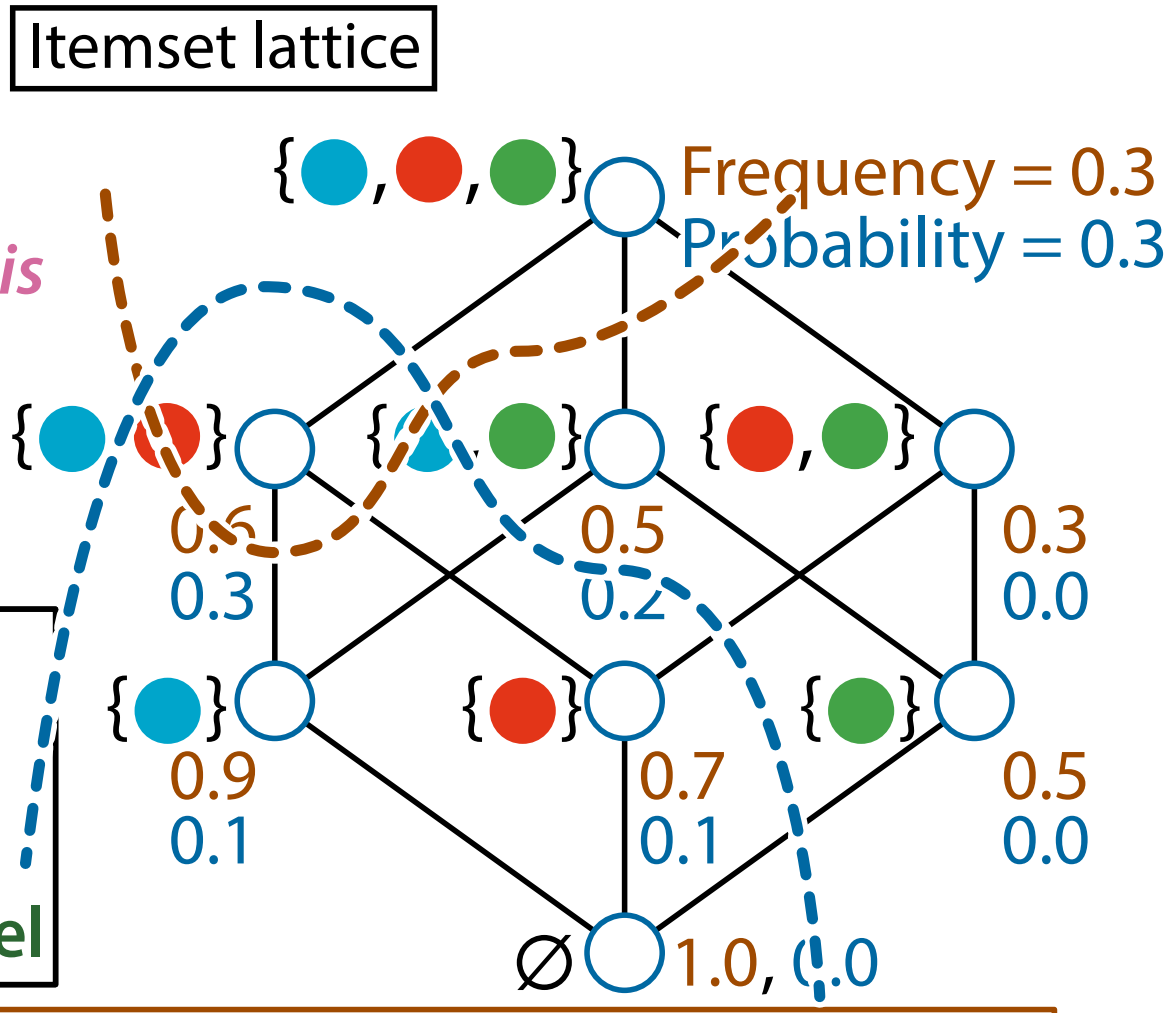


η: Frequency

p: Probability

$$\eta(\{●,●\}) = p(\{●,●\}) + p(\{●,●,●\})$$

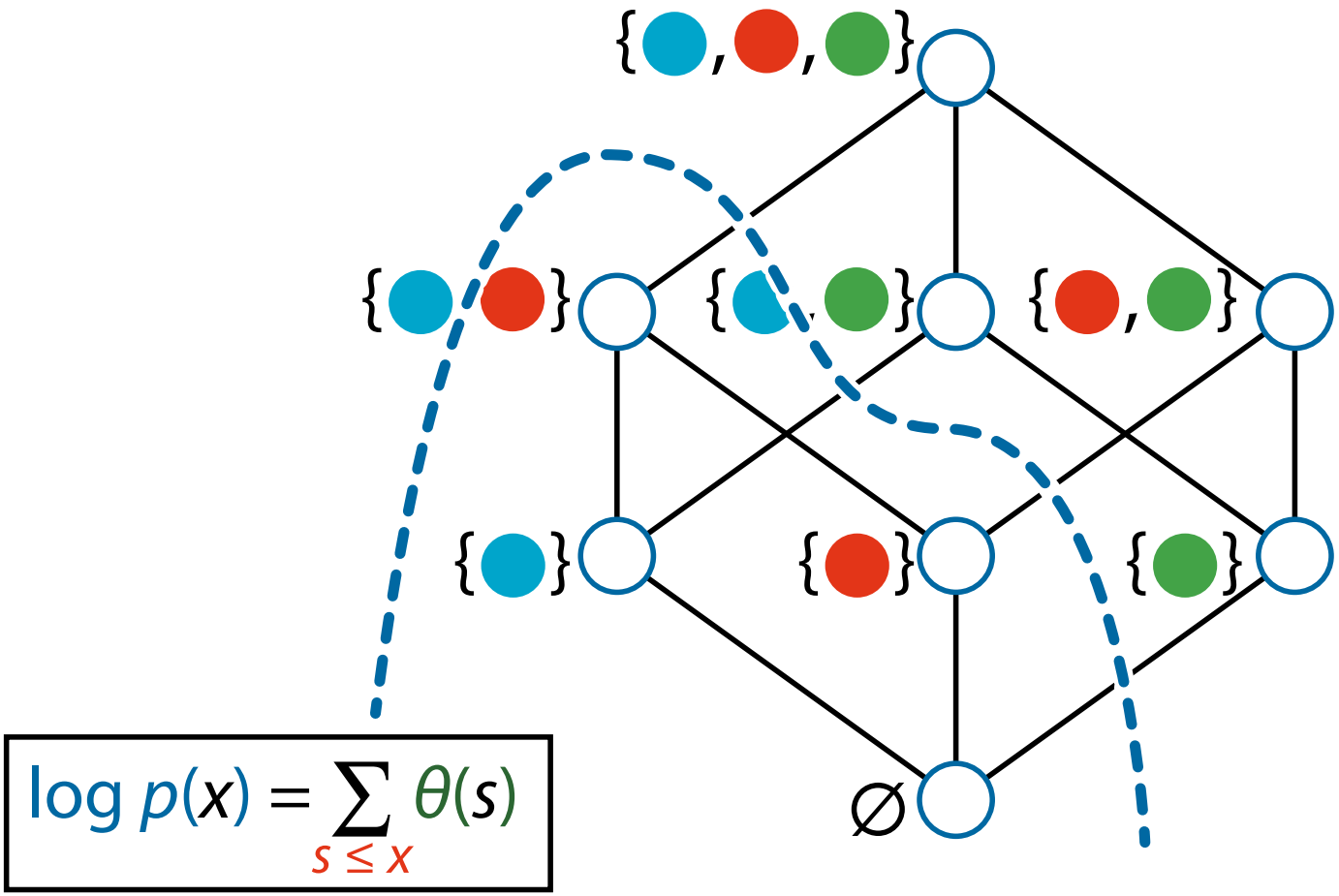
Upward =
Pattern mining
Downward =
Log-linear analysis

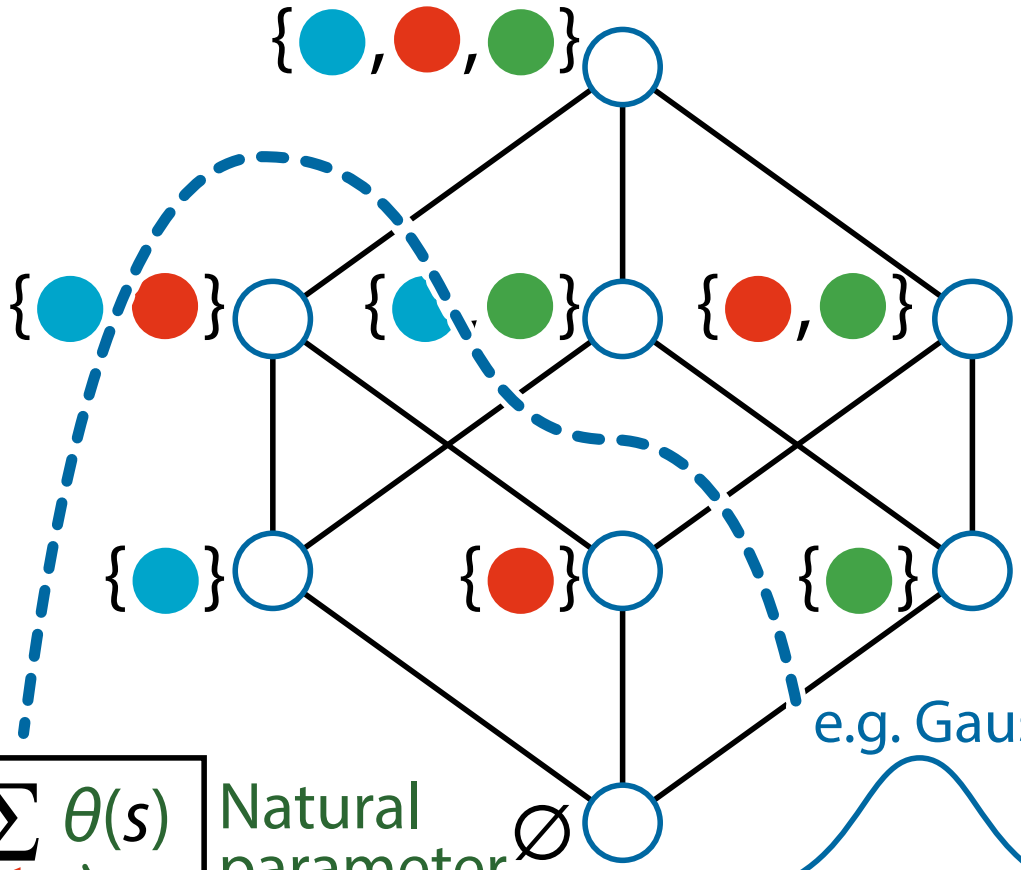


η: Frequency
p: Probability
θ: Coefficient of **log-linear model**

$$\eta(\{●,●\}) = p(\{●,●\}) + p(\{●,●,●\})$$

$$\log p(\{●,●\}) = \theta(\{●,●\}) + \theta(\{●\}) + \theta(\{●\}) + \theta(\emptyset)$$


{ , , }
{ , }
{ , }
{ , }
{ }
{ }
{ }
∅
log p(x) = ∑ θ(s)
s ≤ x



$$\log p(x) = \sum_{s \le x} \theta(s)$$

Natural parameter

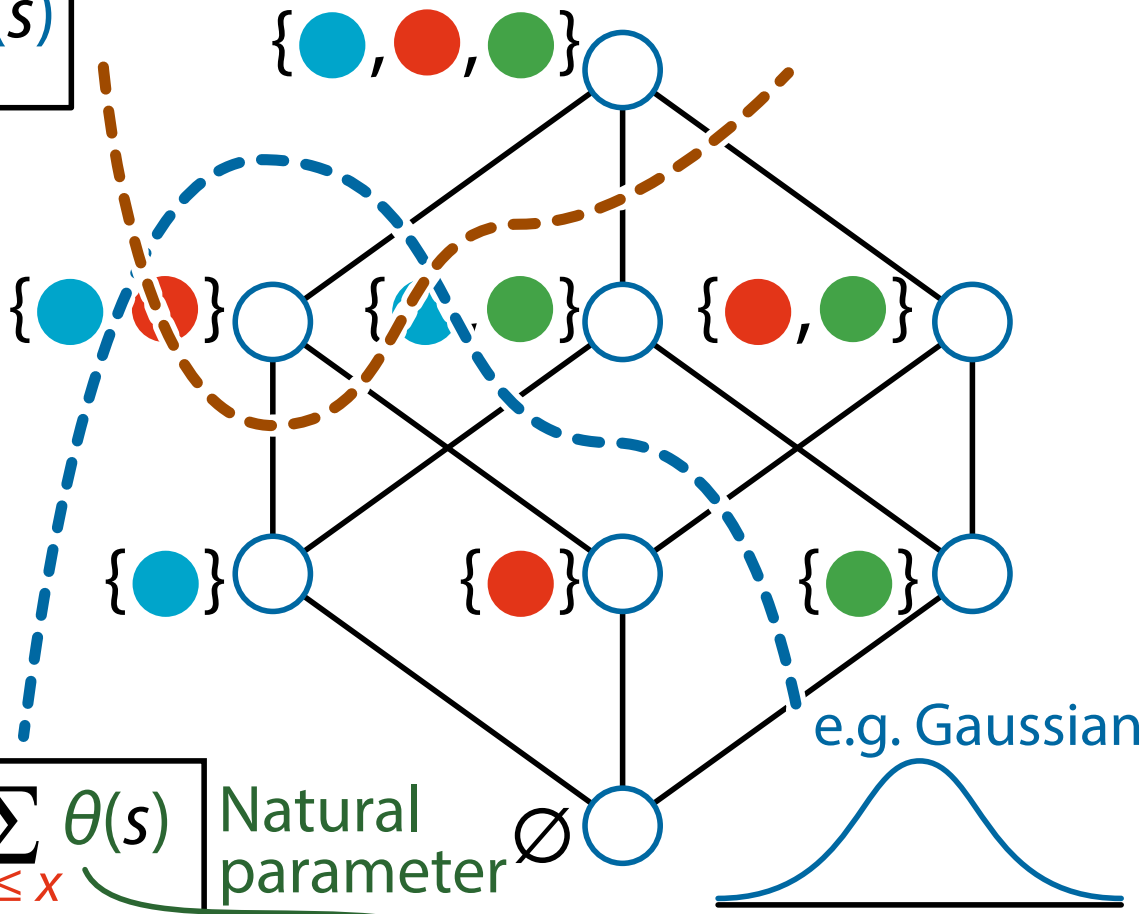
Exponential family:

$$p(x) = \exp\Big(\sum \theta(s) F_s(x) - \psi(\theta) \Big)$$

$$\eta(x) = \sum_{s \geq x} p(s)$$

$$\eta(x) = \mathbb{E}[\, F_X(s) \,]$$

Sufficient statistics of exponential family

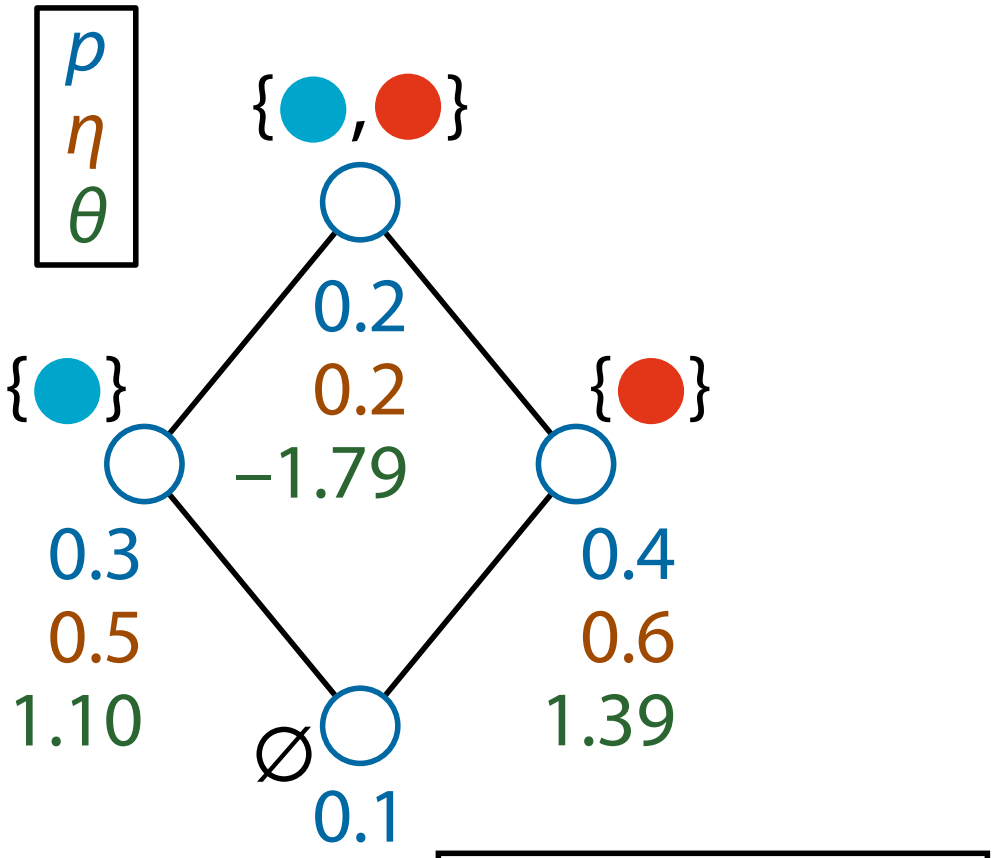


$$\log p(x) = \sum_{s \leq x} \theta(s)$$

Natural parameter

Exponential family: $p(x) = \exp\left(\sum \theta(s) F_s(x) - \psi(\theta) \right)$

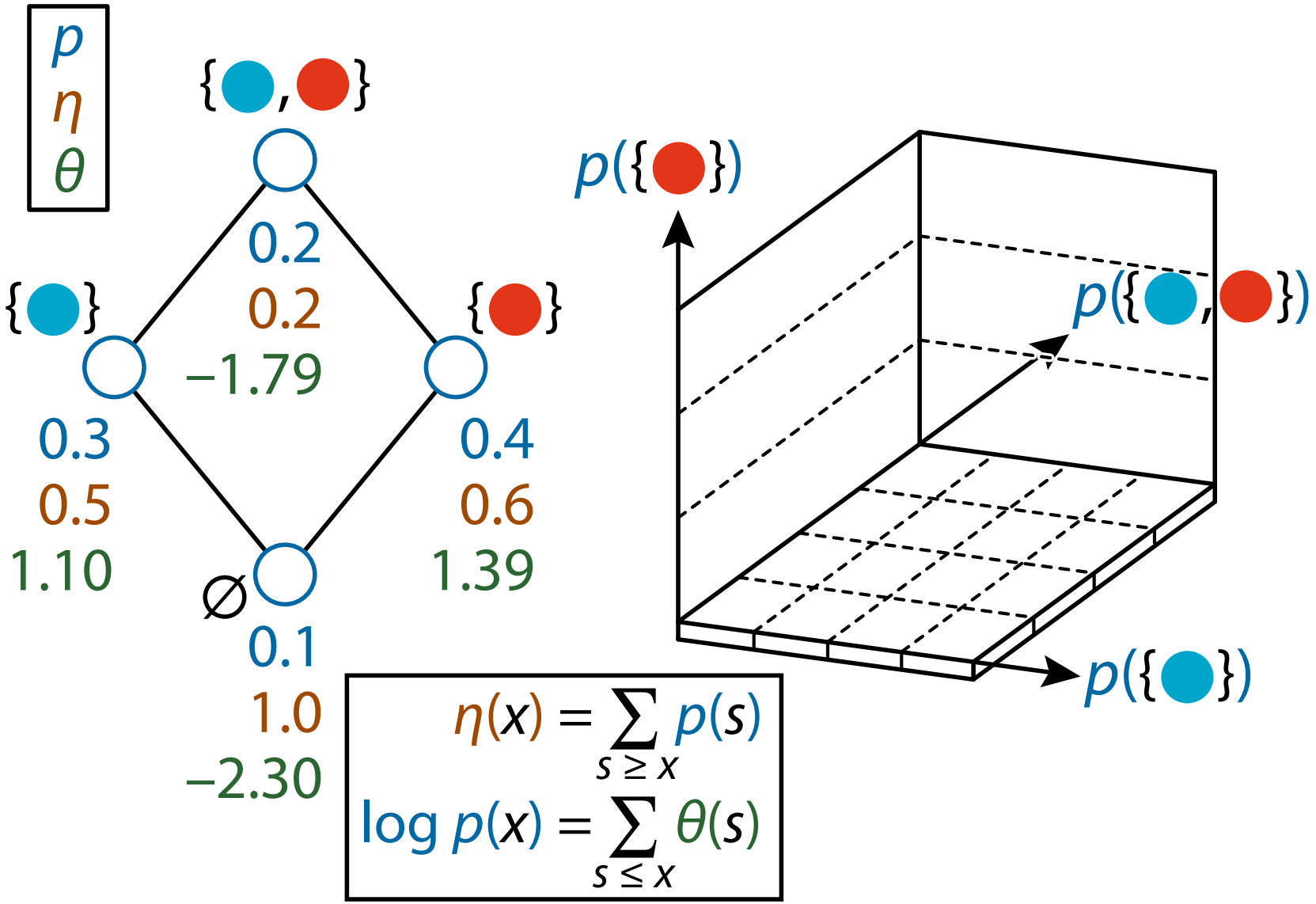
Triple for each node



$$\eta(x) = \sum_{s \geq x} p(s)$$

$$\log p(x) = \sum_{s \leq x} \theta(s)$$

Triple for each node

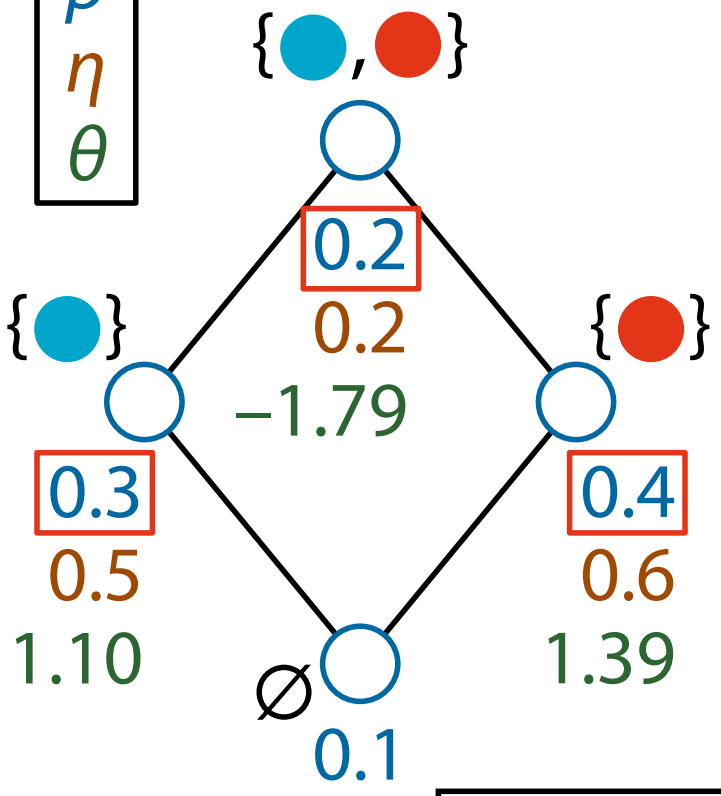


$$\eta(x) = \sum_{s \geq x} p(s)$$

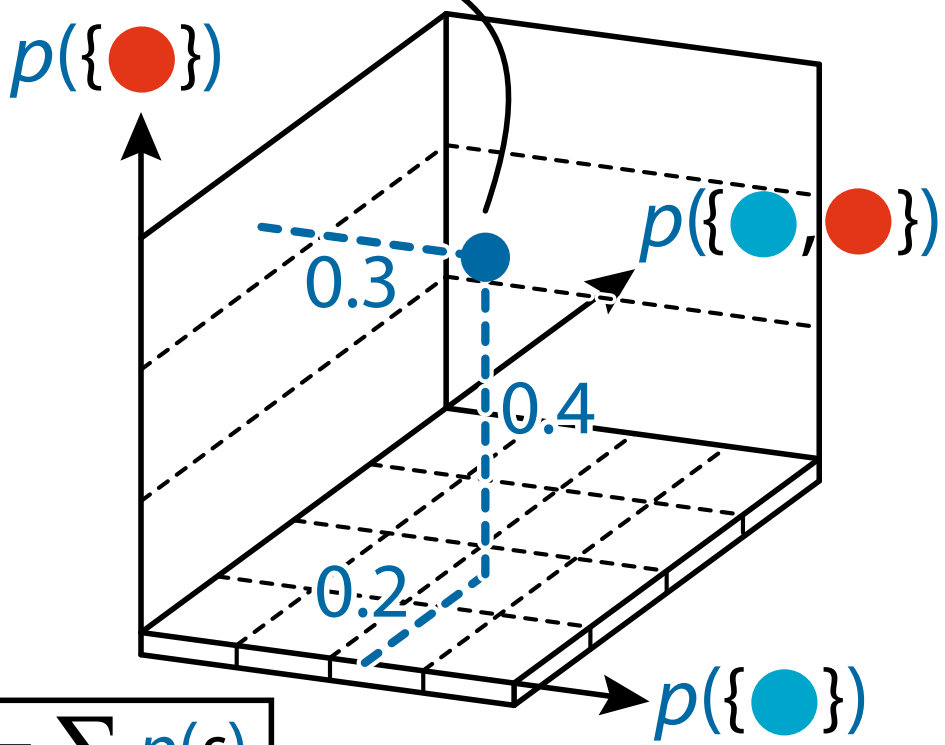
$$\log p(x) = \sum_{s \leq x} \theta(s)$$

Triple for each node

p
η
θ



Probability distribution is a "point" in 3D space



$$\eta(x) = \sum_{s \geq x} p(s)$$

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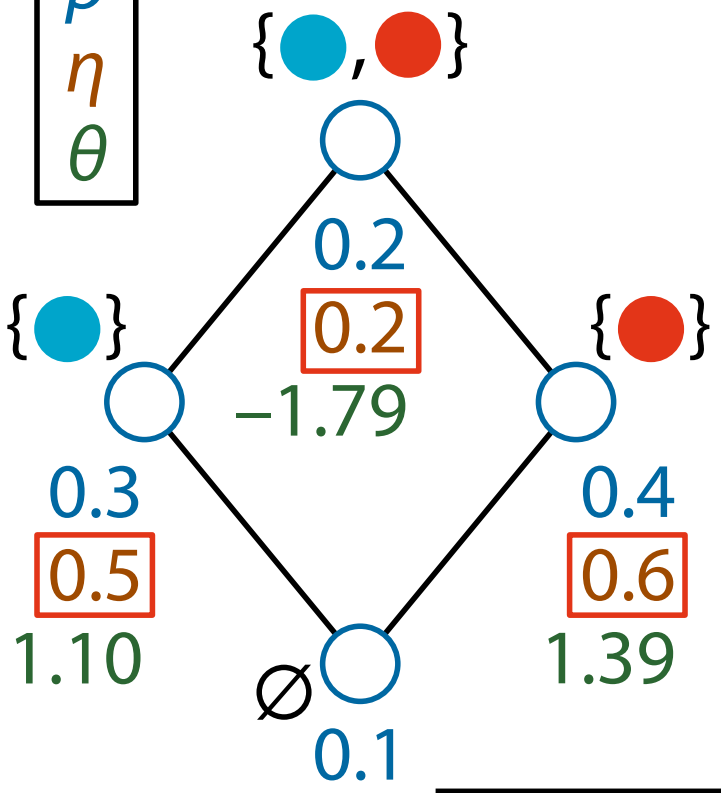
Triple for each node

p
η
θ



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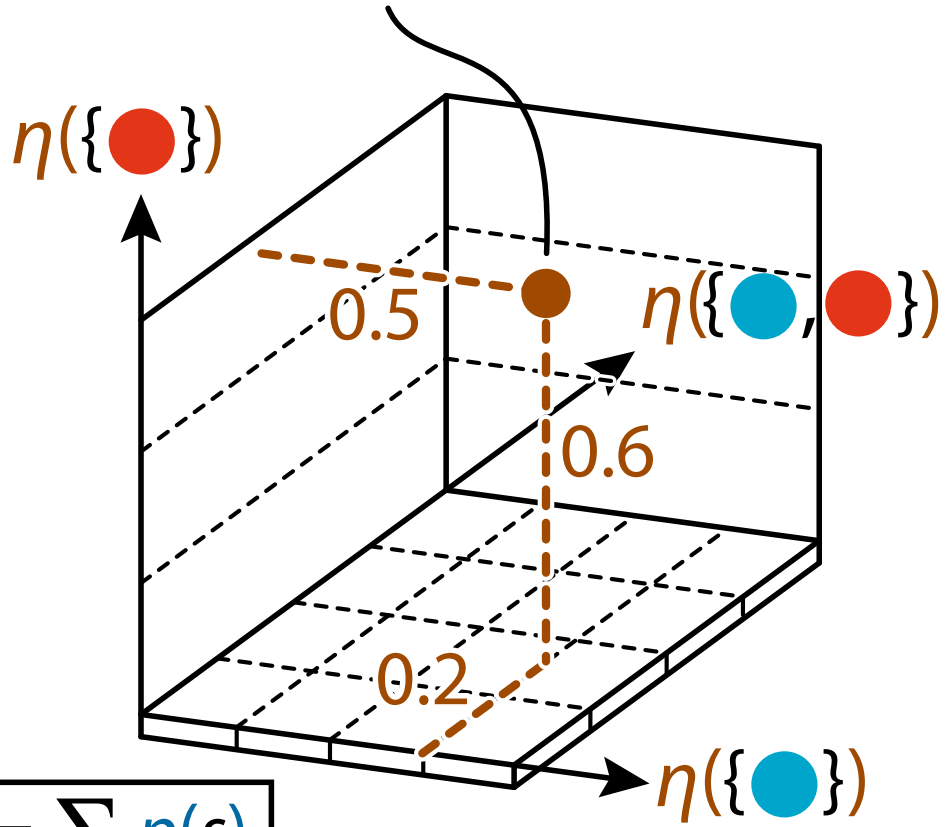


$$\eta(x) = \sum_{s \geq x} p(s)$$

$$\log p(x) = \sum_{s \leq x} \theta(s)$$

Triple for each node

p
η
θ

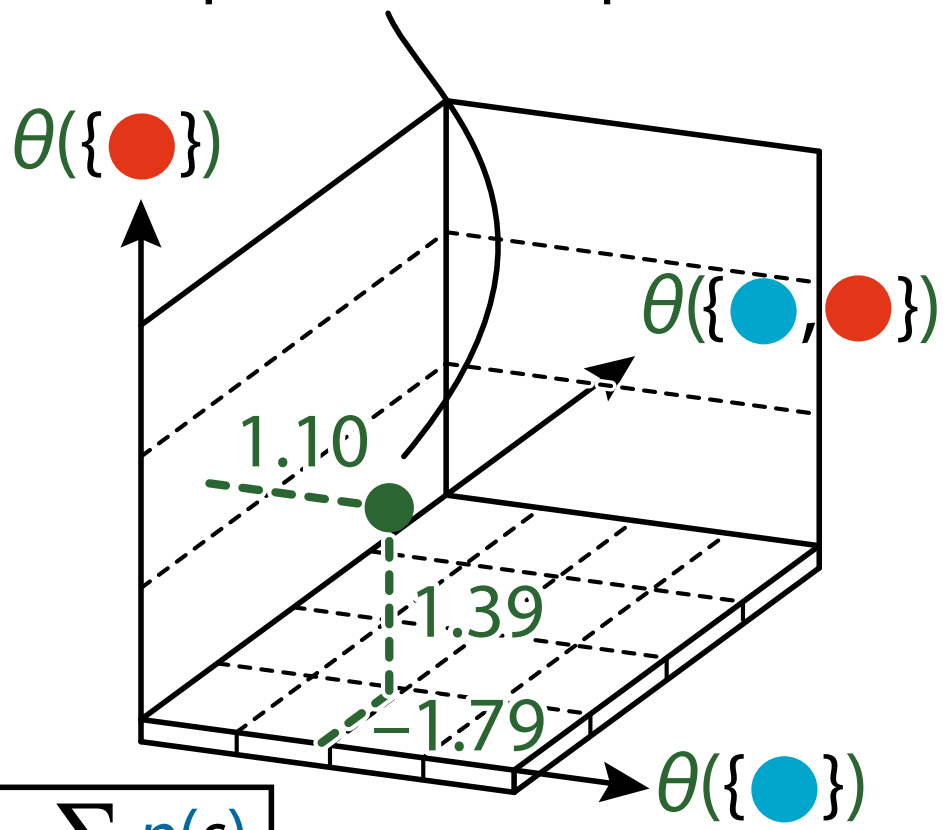
{●,●}
0.2
0.2
−1.79

{●}
0.3
0.5
1.10

{●}
0.4
0.6
1.39

∅
0.1
1.0
−2.30

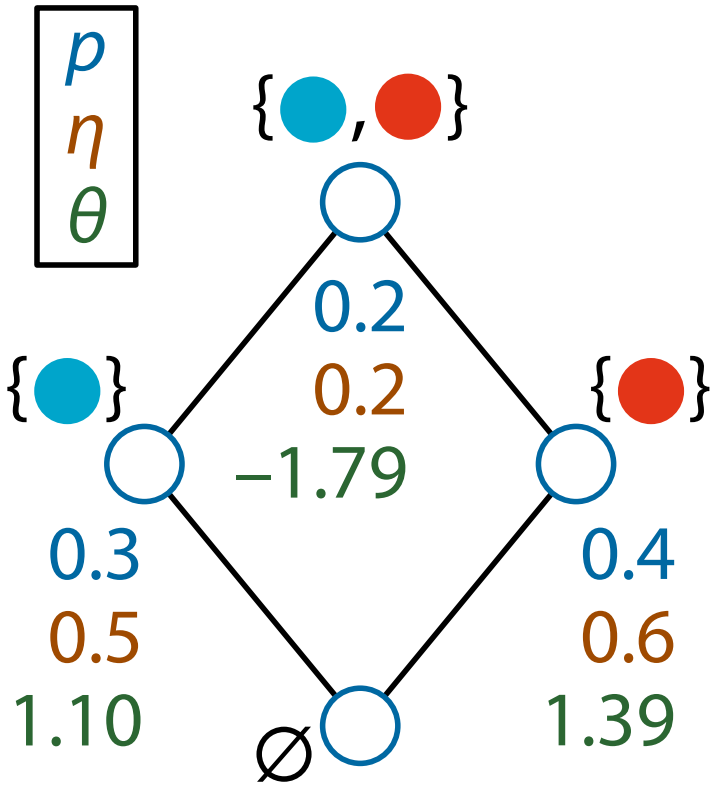
Probability distribution is a "point" in 3D space

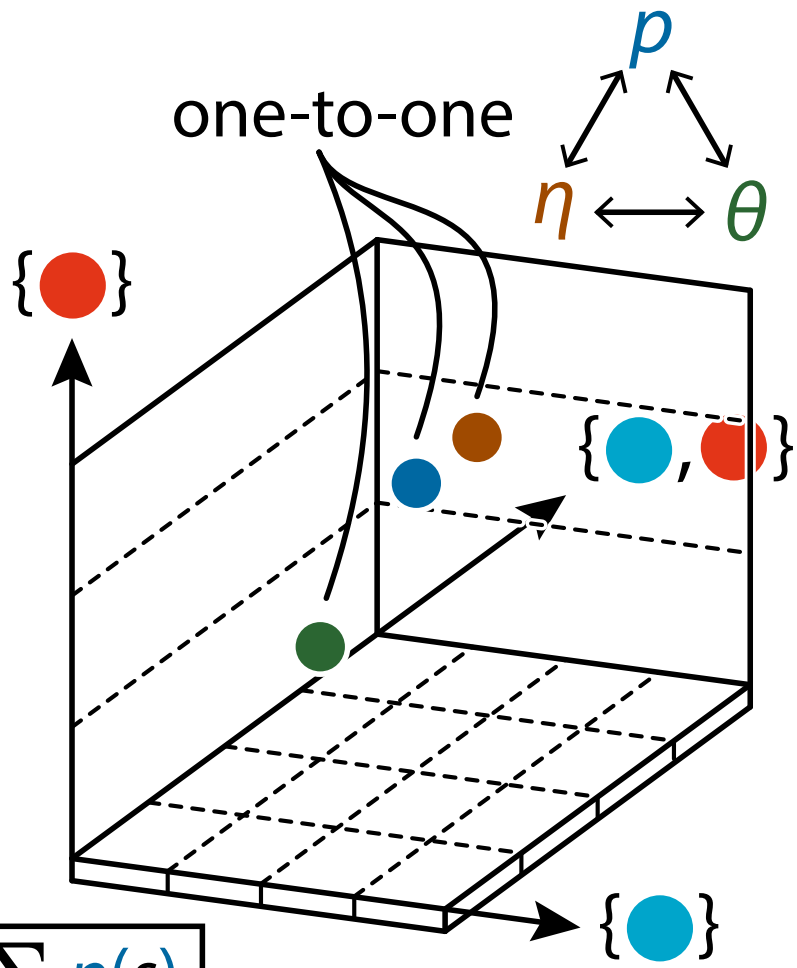


$$\eta(x) = \sum_{s \geq x} p(s)$$

$$\log p(x) = \sum_{s \leq x} \theta(s)$$

Triple for each node





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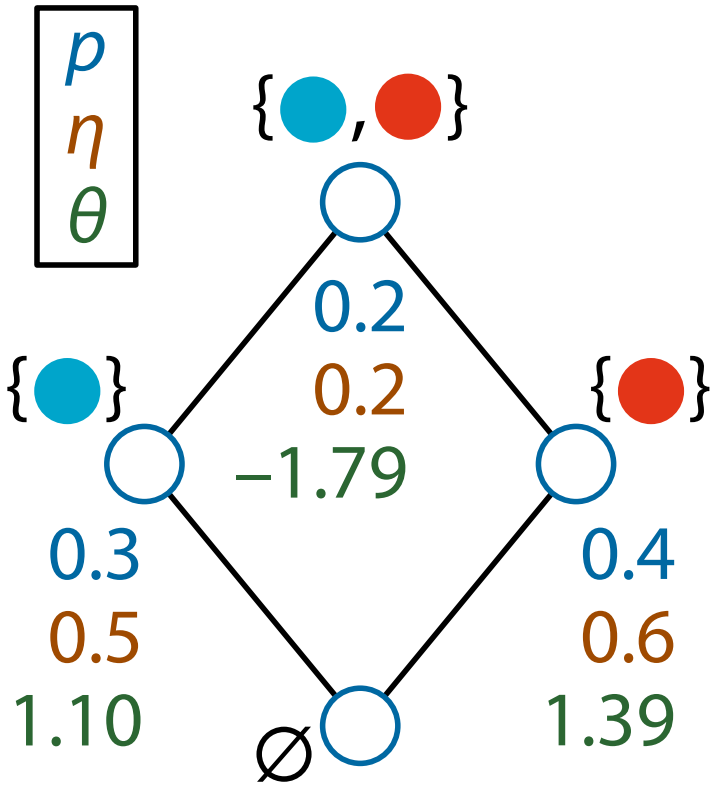
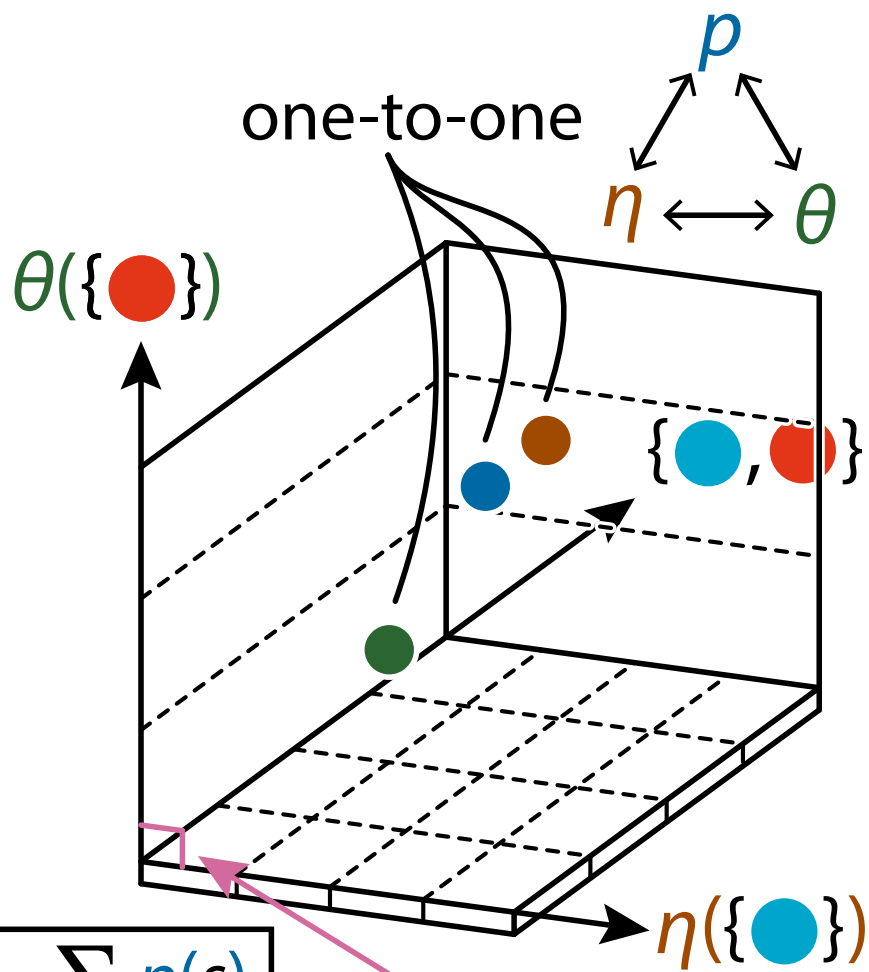
$$\log p(x) = \sum_{s \leq x} \theta(s)$$

Triple for each node

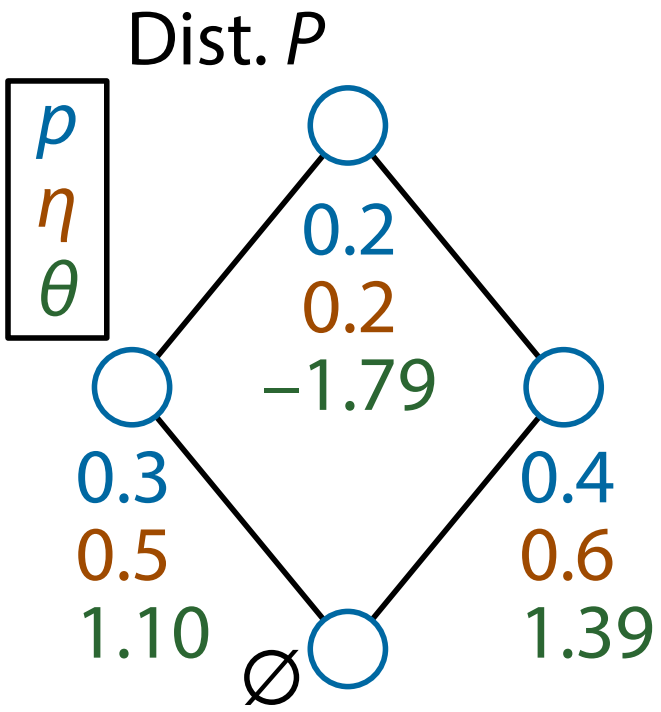


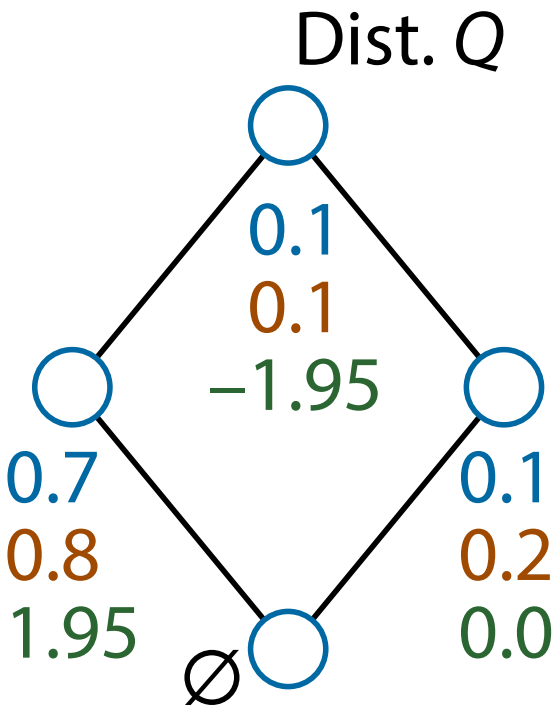


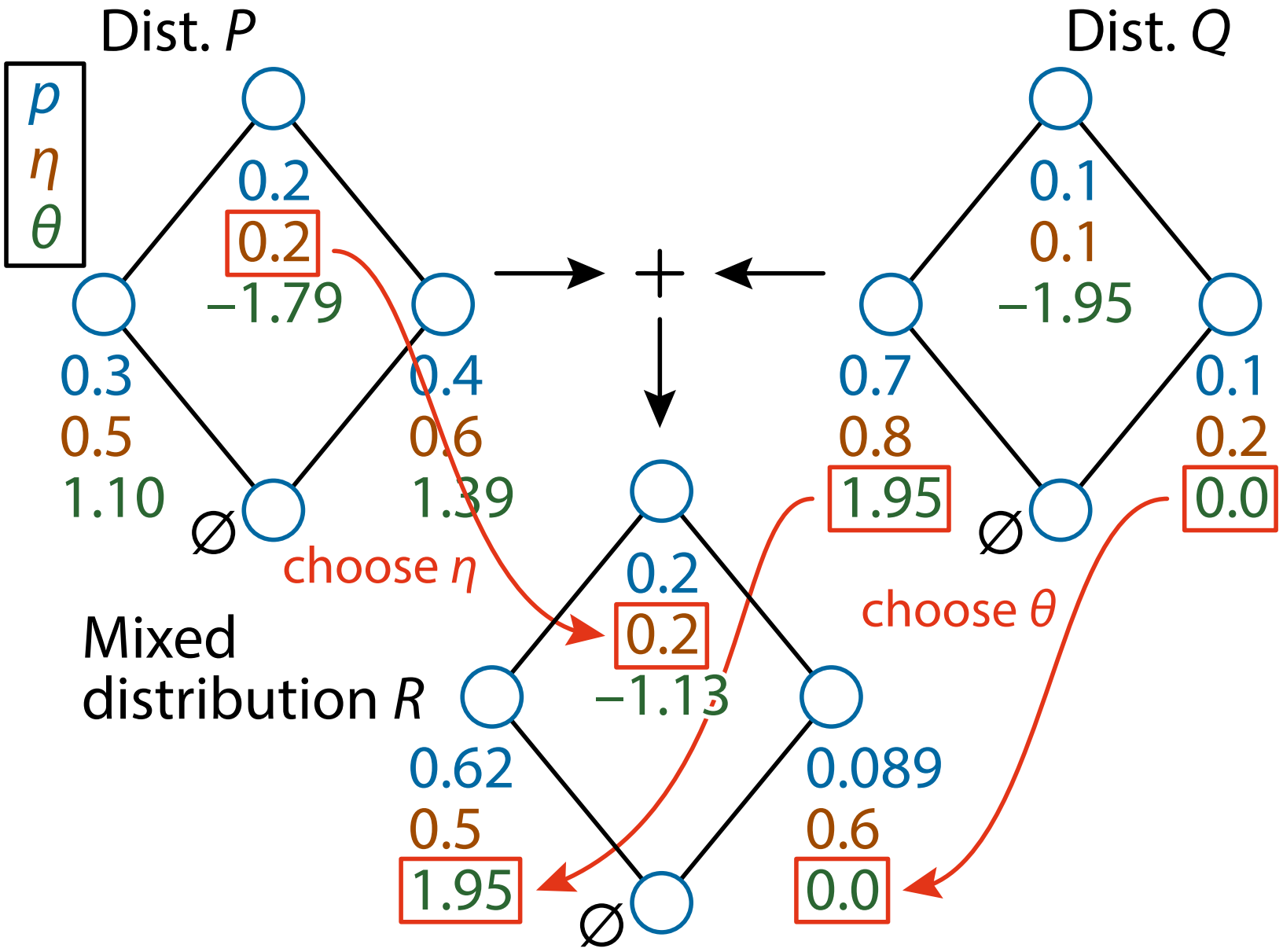
$$\eta(x) = \sum_{s \geq x} p(s)$$

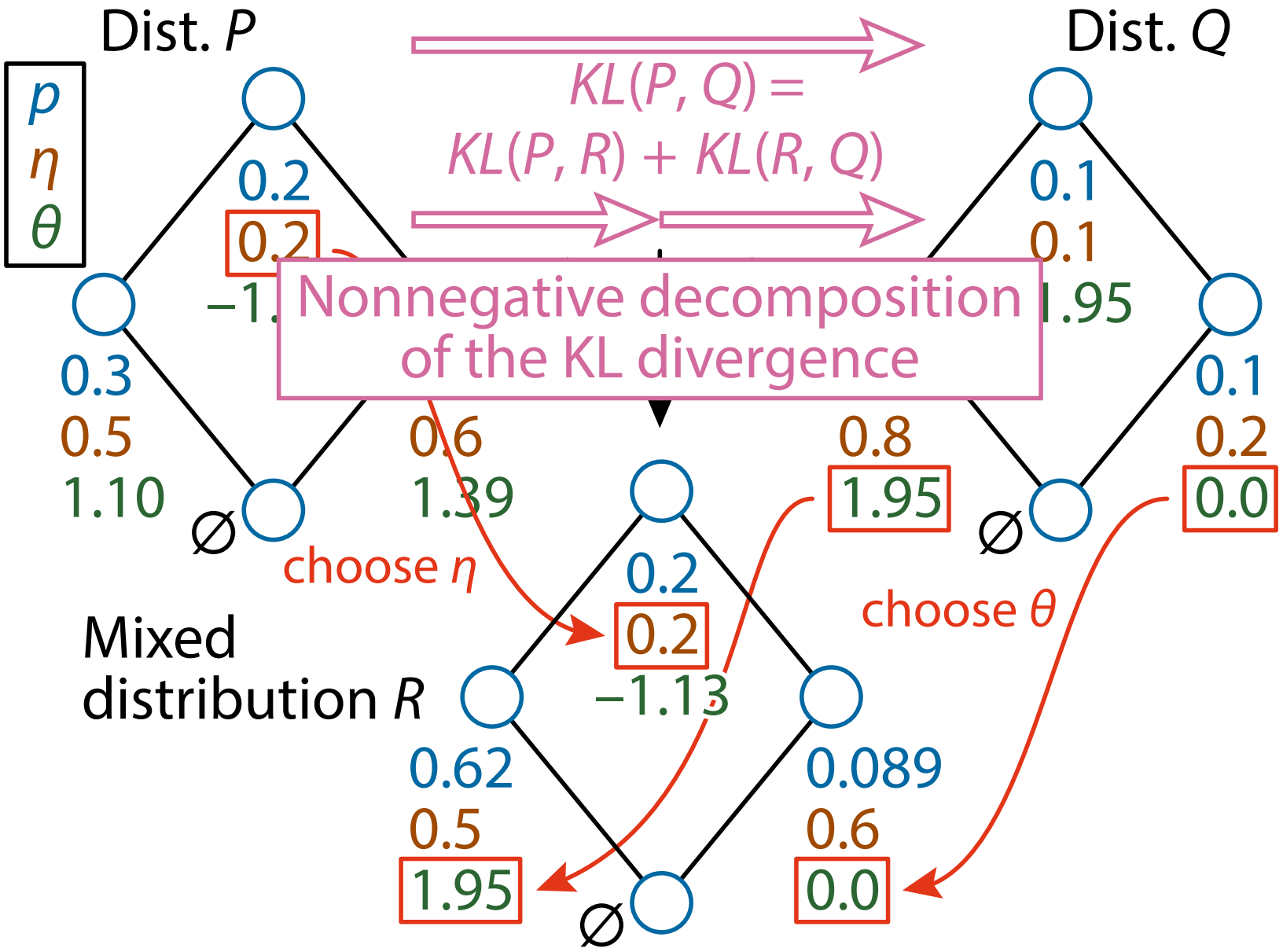
$$\log p(x) = \sum_{s \leq x} \theta(s)$$

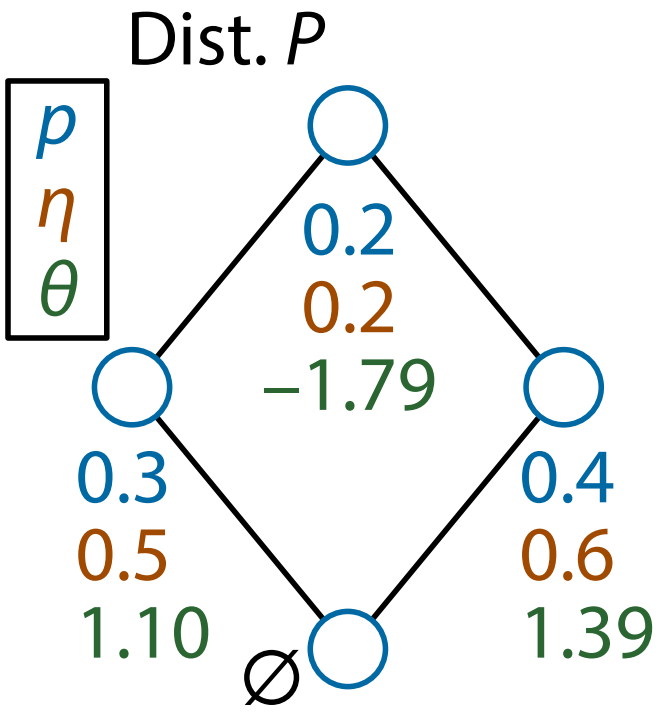
θ and η are dually orthogonal

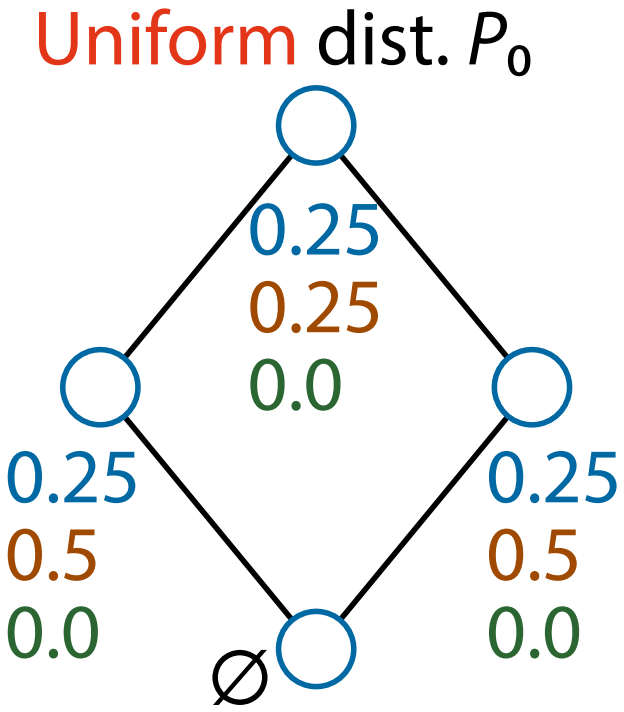

Dist. P
p
η
θ
0.2
0.2
−1.79
0.3
0.5
1.10
0.4
0.6
1.39
∅

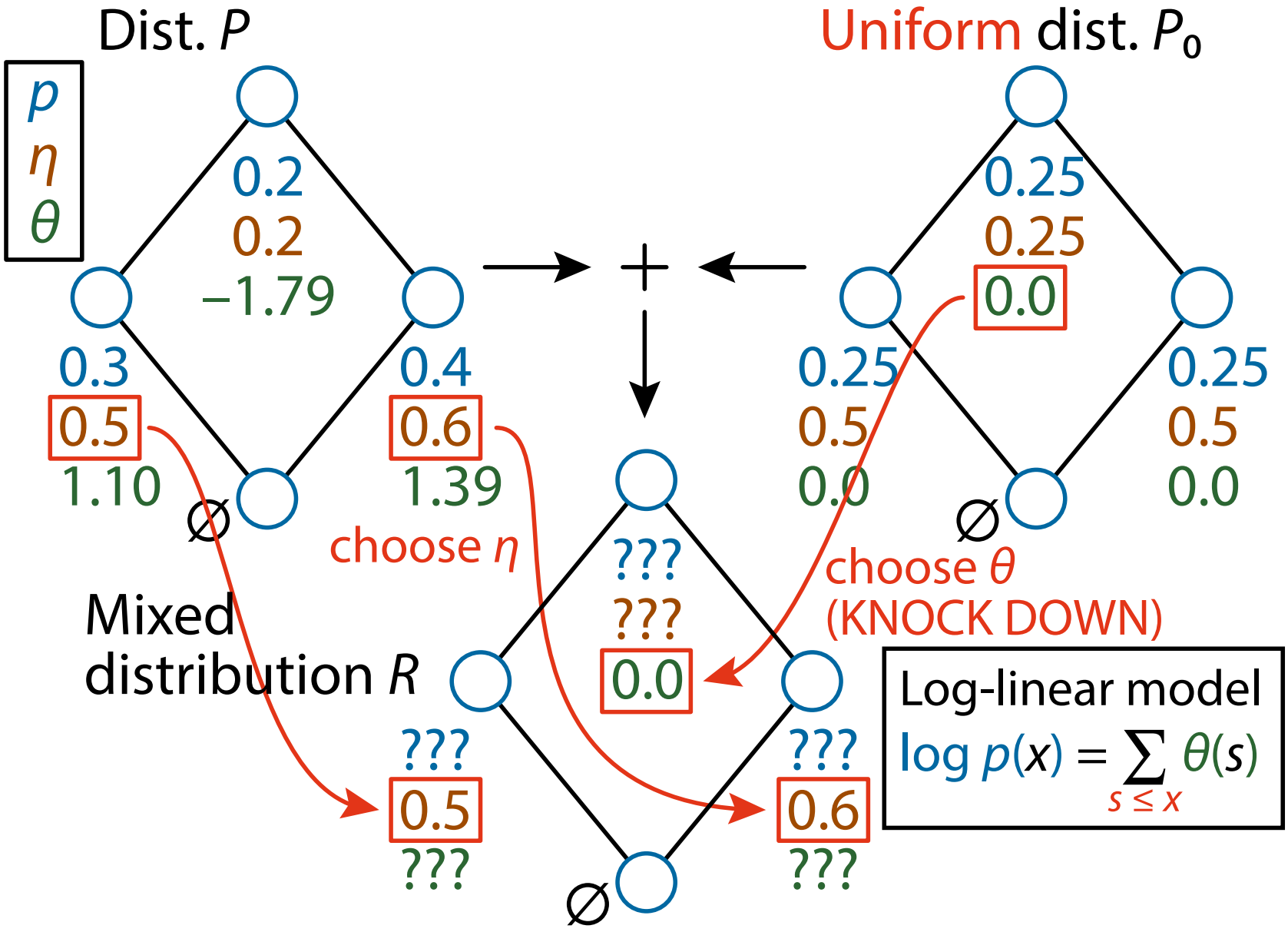

Dist. Q
0.1
0.1
−1.95
0.7
0.8
1.95
0.1
0.2
0.0
∅

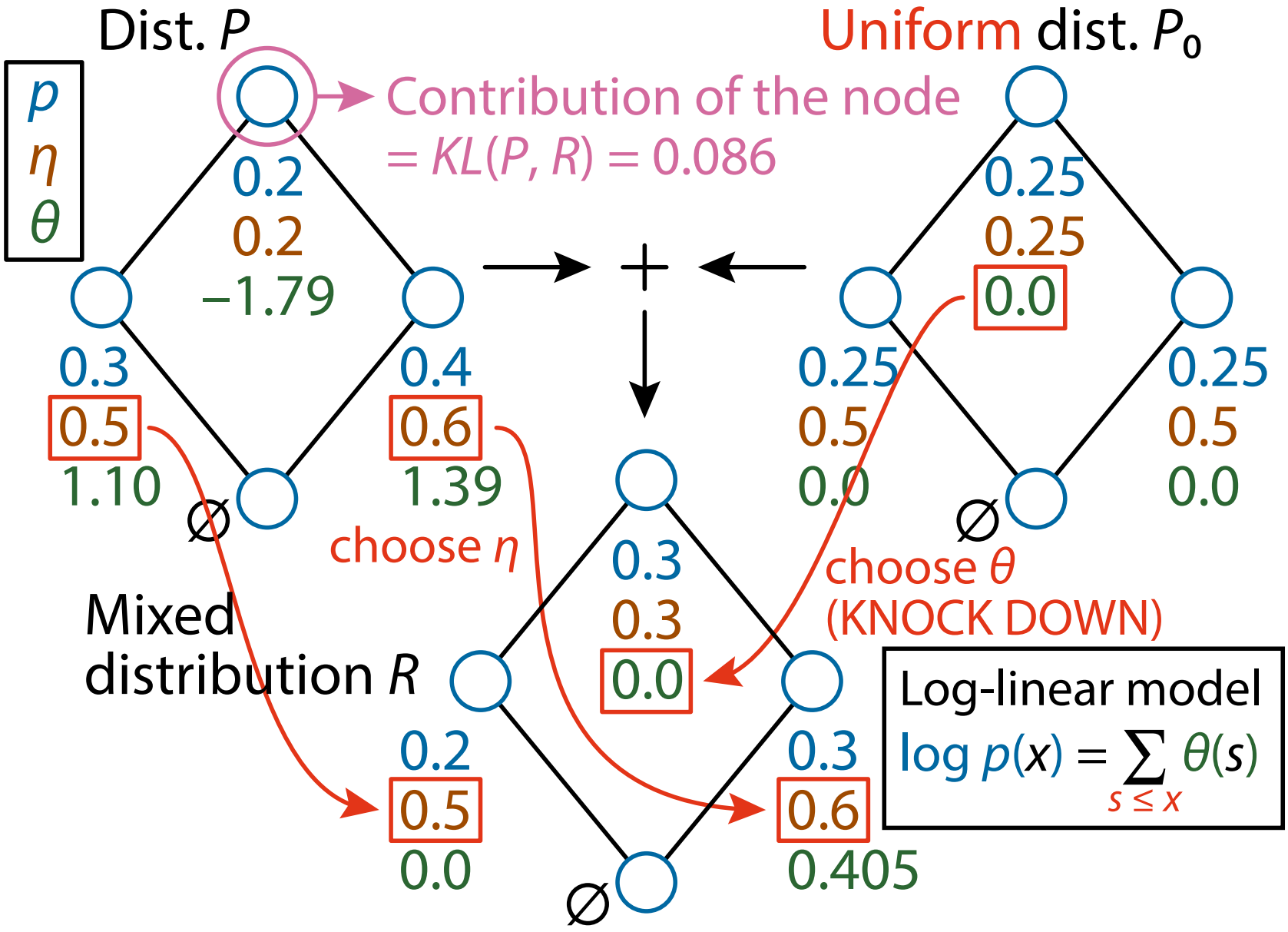

Dist. P
Dist. Q
p
η
θ
0.2
0.2
−1.79
0.3
0.5
1.10
0.4
0.6
1.39
∅
0.1
0.1
−1.95
0.7
0.8
1.95
0.1
0.2
0.0
∅
choose η
choose θ
Mixed
distribution R
0.2
0.2
−1.13
0.62
0.5
1.95
0.089
0.6
0.0
∅

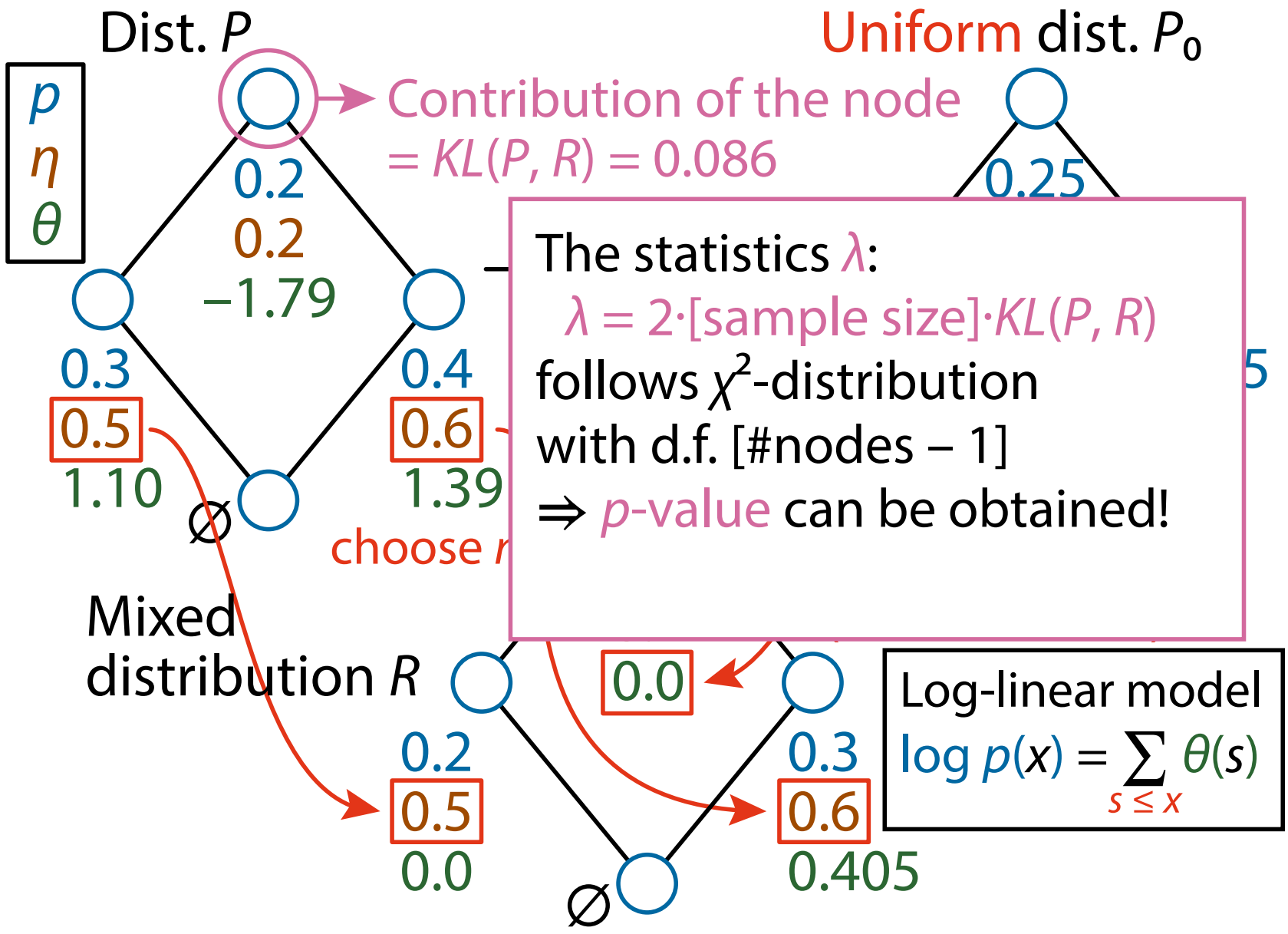

Dist. P
Dist. Q
p
η
θ
KL(P, Q) =
KL(P, R) + KL(R, Q)
Nonnegative decomposition
of the KL divergence
0.2
0.2
0.1
0.1
0.3
0.5
1.10
0.6
1.39
0.8
1.95
0.1
0.2
0.0
choose η
choose θ
Mixed
distribution R
0.2
0.2
−1.13
0.62
0.5
1.95
0.089
0.6
0.0


Dist. P
p
η
θ
0.2
0.2
−1.79
0.3
0.5
1.10
0.4
0.6
1.39
∅


Uniform dist. P_0
0.25
0.25
0.0
0.25
0.5
0.0
0.25
0.5
0.0
∅


Dist. P
Uniform dist. P_0
p
η
θ
0.2
0.2
−1.79
0.3
0.5
1.10
0.4
0.6
1.39
∅
0.25
0.25
0.0
0.25
0.5
0.0
0.25
0.5
0.0
∅
choose η
choose θ
(KNOCK DOWN)
Mixed
distribution R
???
???
0.0
???
0.5
???
???
0.6
???
∅
Log-linear model
$\log p(x) = \sum_{s \le x} \theta(s)$

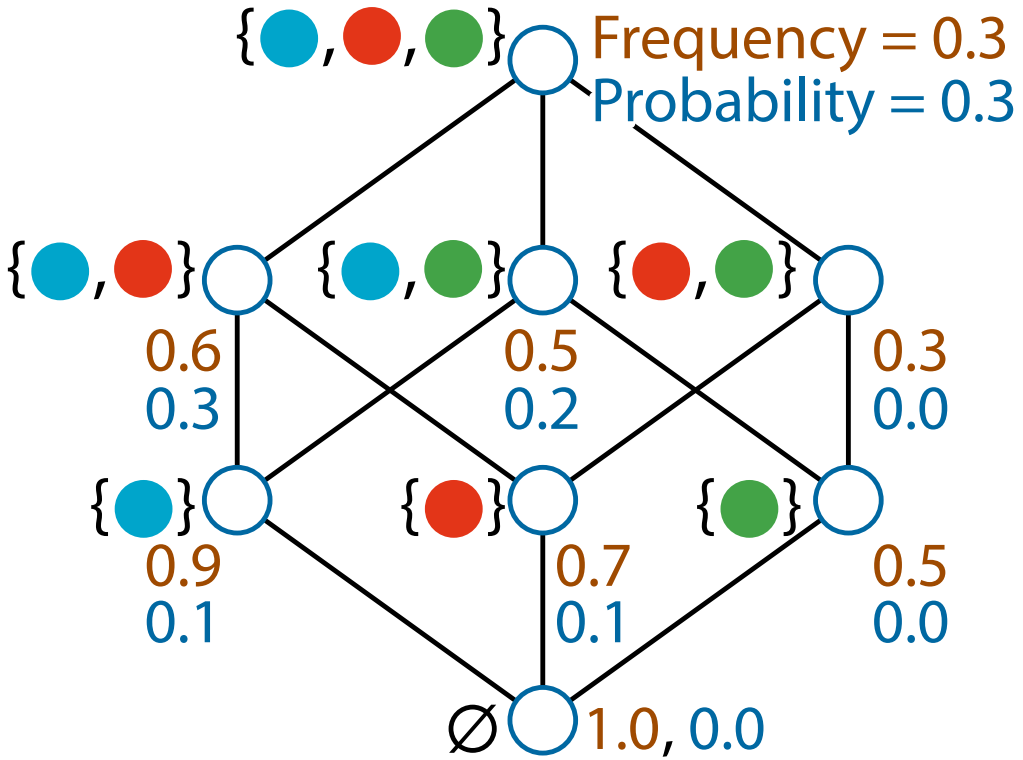

Dist. P
Uniform dist. P0
p
η
θ
Contribution of the node
= KL(P, R) = 0.086
0.2
0.2
−1.79
0.3
0.5
1.10
0.4
0.6
1.39
∅
0.25
0.25
0.0
0.25
0.5
0.0
0.25
0.5
0.0
∅
choose η
choose θ
(KNOCK DOWN)
Mixed
distribution R
0.3
0.3
0.0
0.2
0.5
0.0
0.3
0.6
0.405
∅
Log-linear model
log p(x) = Σ θ(s)
s ≤ x


Dist. P
Uniform dist. P0
p
η
θ
Contribution of the node
= KL(P, R) = 0.086
0.2
0.2
−1.79
0.25
0.3
0.5
1.10
0.4
0.6
1.39
choose r
The statistics λ:
λ = 2·[sample size]·KL(P, R)
follows χ2-distribution
with d.f. [#nodes – 1]
⇒ p-value can be obtained!
Mixed
distribution R
0.0
0.2
0.5
0.0
0.3
0.6
0.405
Log-linear model
log p(x) = Σ θ(s)
s ≤ x

Make a Poset from Data

Dataset

ID 1:	1	1	0
ID 2:	1	1	1
ID 3:	1	1	0
ID 4:	1	1	1
ID 5:	1	1	0
ID 6:	1	0	1
ID 7:	1	0	1
ID 8:	1	1	1
ID 9:	1	0	0
ID10:	0	1	0

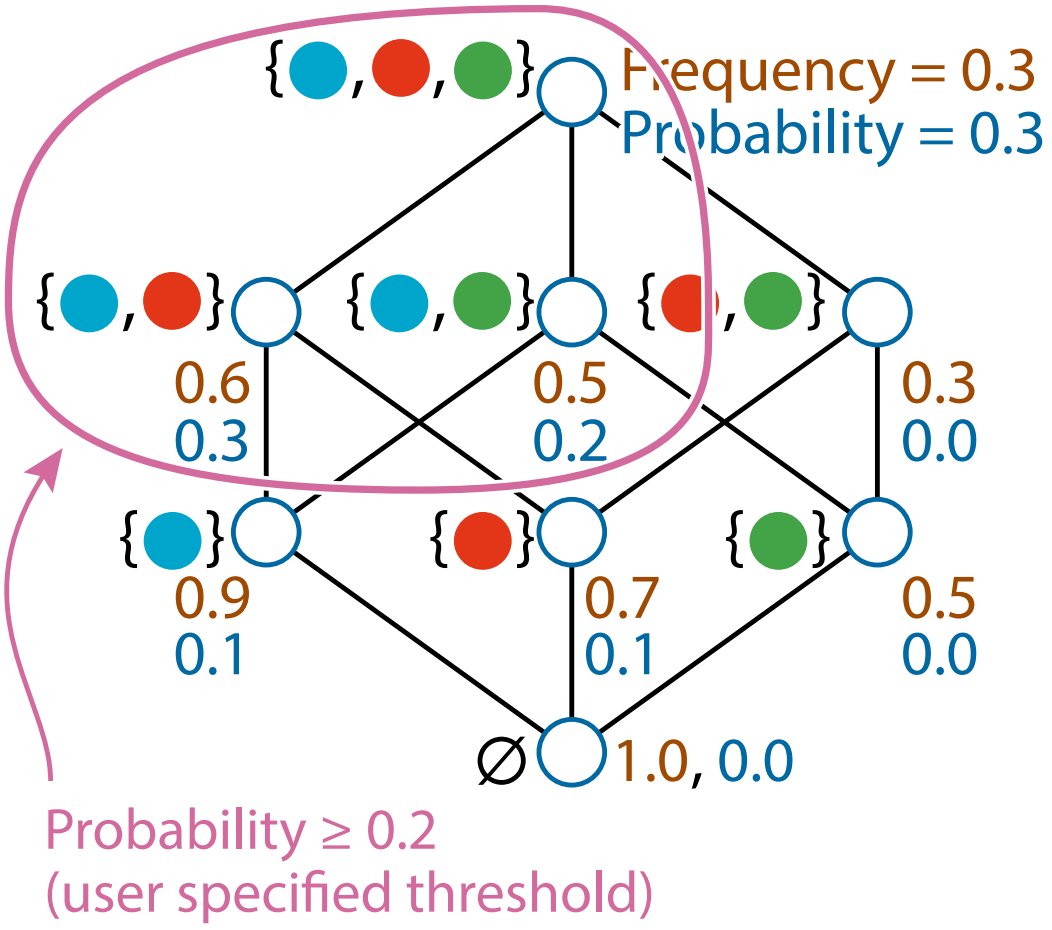


Number of nodes = $2^{\#\text{features}}$
⇒combinatorial explosion!

Make a Poset from Data

Dataset

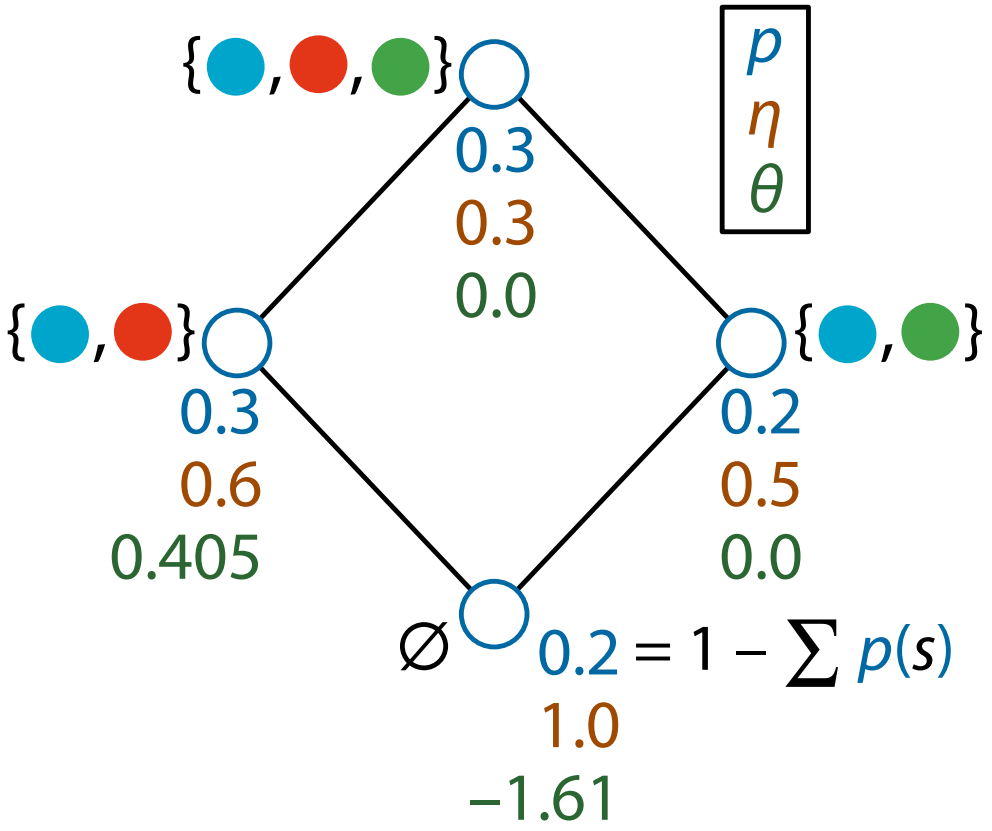
ID 1:	1	1	0
ID 2:	1	1	1
ID 3:	1	1	0
ID 4:	1	1	1
ID 5:	1	1	0
ID 6:	1	0	1
ID 7:	1	0	1
ID 8:	1	1	1
ID 9:	1	0	0
ID10:	0	1	0



Remove Nodes with Probability 0

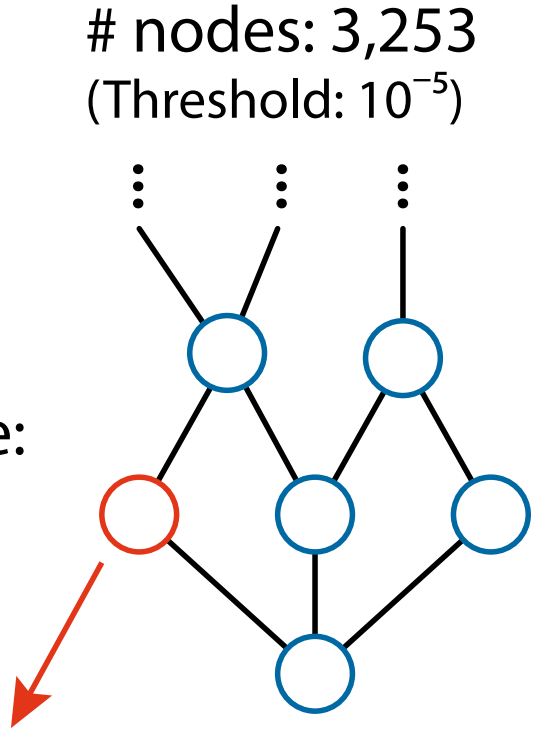
Dataset

ID 1:	1	1	0
ID 2:	1	1	1
ID 3:	1	1	0
ID 4:	1	1	1
ID 5:	1	1	0
ID 6:	1	0	1
ID 7:	1	0	1
ID 8:	1	1	1
ID 9:	1	0	0
ID10:	0	1	0



Example on Real Data (kosarak)

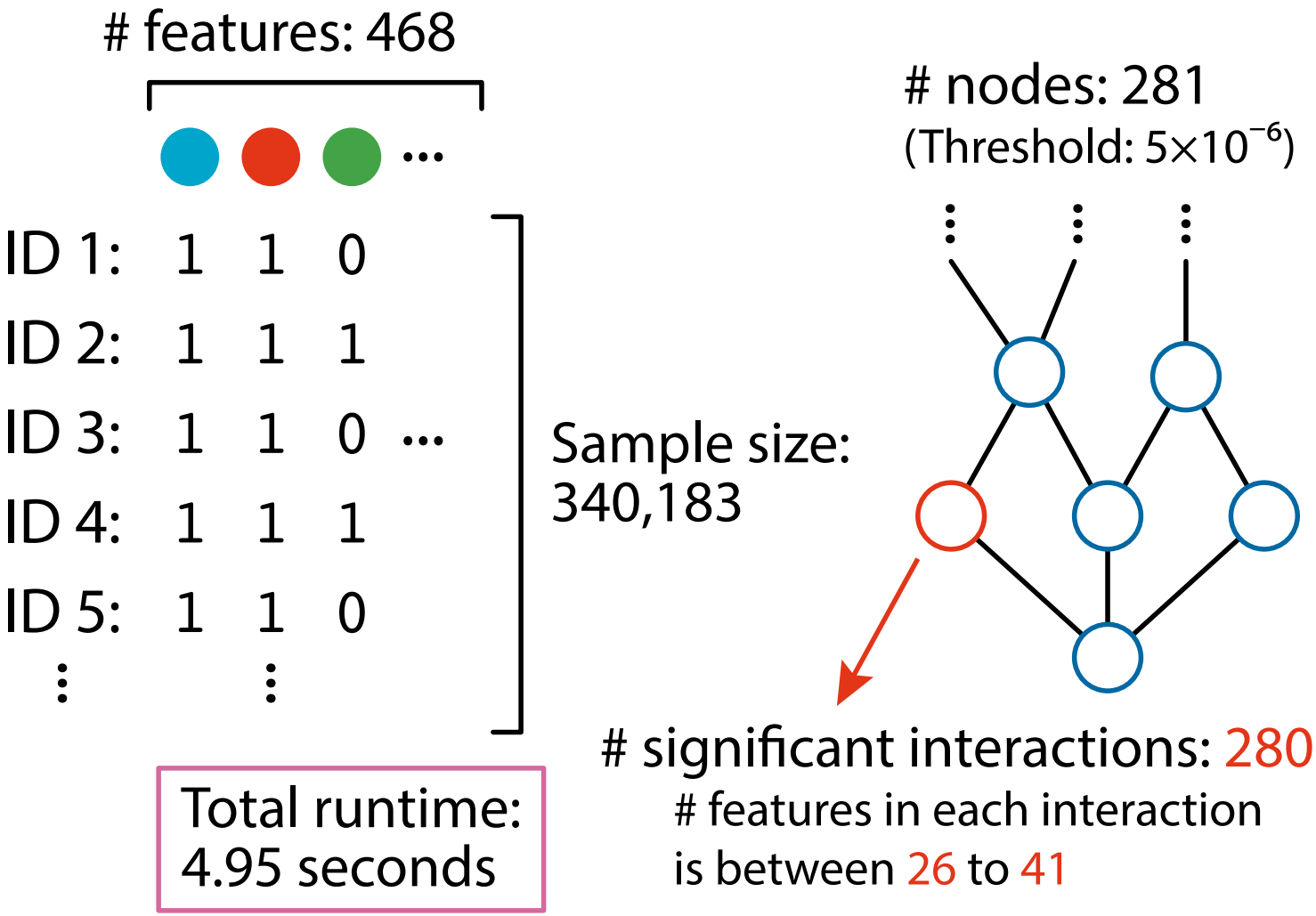
Total runtime:
4.95 seconds



significant interactions: 583

- Single feature: 537
- Pairwise interactions: 41
- Triple interactions: 5

Example on Real Data (accidents)



Conclusion

- We build information geometry for posets (partially ordered sets)
  - Natural connection between the information geometric dual coordinates and the partial order structure
  - Code: `https://git.io/decomp`
- We can decompose a probability distribution and asses the significance of any-order interactions
- Our results generalizes the following:
  - S. Amari, *Information geometry on hierarchy of probability distributions*, IEEE Trans. on Information Theory (2001)
  - H. Nakahara, S. Amari, *Information-geometric measure for neural spikes*, Neural Computation (2002)