Legendre Decomposition for Tensors

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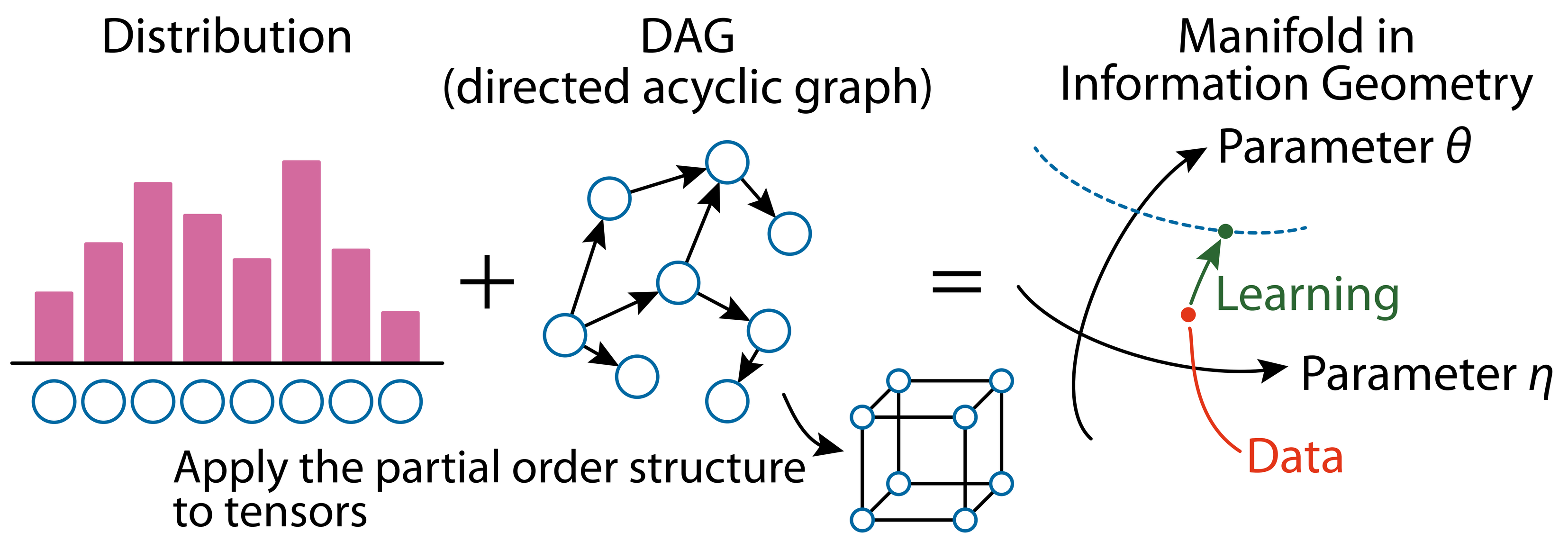
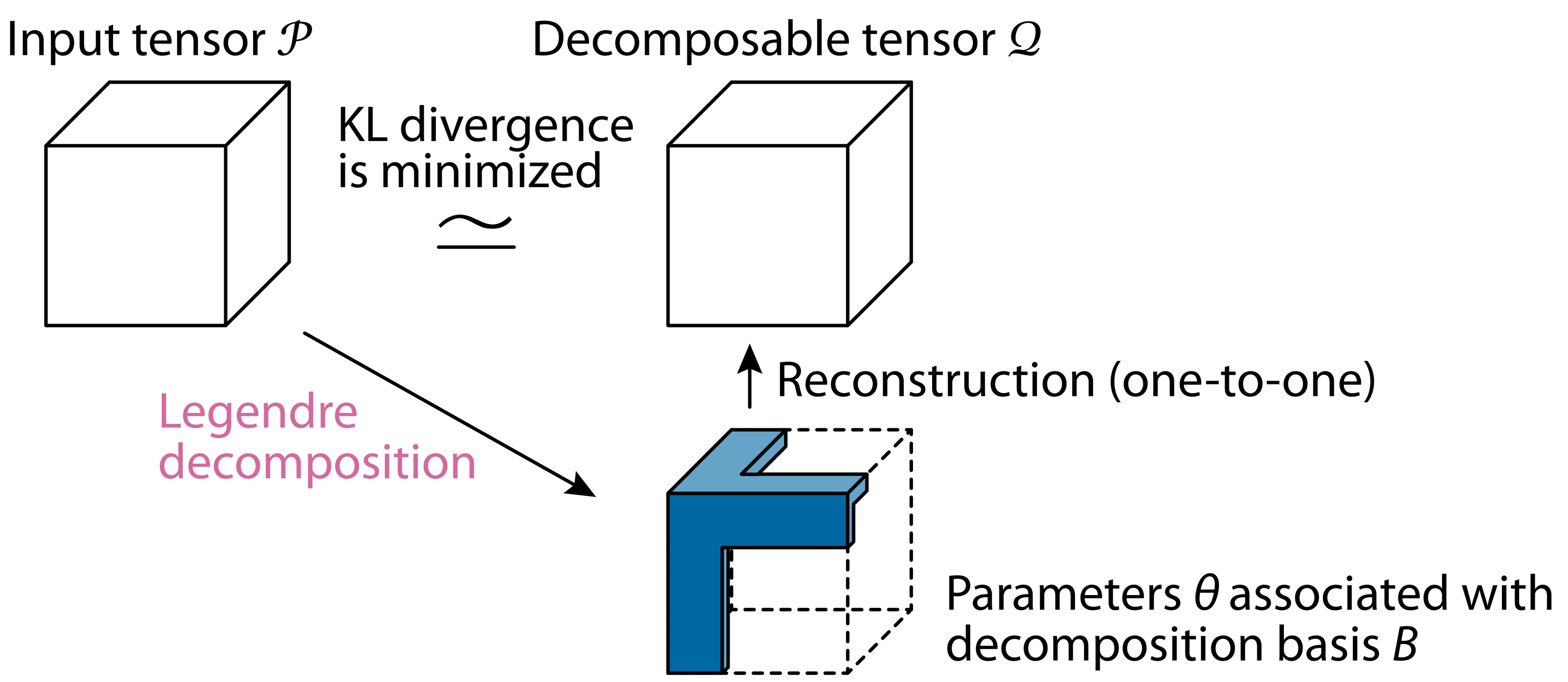
Our Approach



Summary

- We present Legendre decomposition for tensors
  - A new nonnegative decomposition method
  - A tensor is factorized into a multiplicative combination of parameters
- Our proposal is theoretically supported by information geometry
  - The reconstructed tensor is unique and always minimizes the KL divergence from an input tensor



Properties of Legendre Decomposition

- Given $\mathcal{P} \in \mathbb{R}_{\geq 0}^{I_1 \times I_2 \times \dots \times I_N}$, Legendre decomposition finds $\mathcal{Q}$, where
  - (i) $\mathcal{Q}$ always exists,
  - (ii) $\mathcal{Q}$ is unique, and
  - (iii) $\mathcal{Q}$ is the best approximation in the sense of the KL divergence:

$$\mathcal{Q} = \operatorname{argmin}_{\mathcal{R} \in \boldsymbol{\mathcal{S}}_B} D_{\mathrm{KL}}(\mathcal{P}, \mathcal{R}),$$

$$\boldsymbol{\mathcal{S}}_B = \left\{ \mathcal{R} \in \mathbb{R}_{\geq 0}^{I_1 \times I_2 \times \dots \times I_N} \;\middle|\; \mathcal{R} \text{ is fully decomposable with } B \right\}$$

Legendre Decomposition

Input matrix:

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix} \xrightarrow{\text{Legendre decomposition}} \begin{bmatrix} q_{11} & q_{12} & q_{13} & q_{14} \\ q_{21} & q_{22} & q_{23} & q_{24} \\ q_{31} & q_{32} & q_{33} & q_{34} \\ q_{41} & q_{42} & q_{43} & q_{44} \end{bmatrix}$$

Reconstructed matrix:

Decomposition basis (arbitrary subset of indices)

↕ θ: parameter

$$\begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \\ \theta_{21} & \theta_{22} & 0 & \\ \theta_{31} & \theta_{32} & \boxed{0} & \\ & & & \end{bmatrix} \qquad \begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} \\ \theta_{21} & \theta_{22} & 0 & 0 \\ \theta_{31} & \theta_{32} & 0 & 0 \\ \theta_{41} & \theta_{42} & 0 & 0 \end{bmatrix}$$

$$q_{ij} = \prod_{k \leq i,\, l \leq j} \exp(\theta_{kl})$$

↕ η: constraint

$$\begin{bmatrix} & & & \\ & \boxed{q_{22}} & q_{23} & q_{24} \\ & q_{32} & q_{33} & q_{34} \\ & q_{42} & q_{43} & q_{44} \end{bmatrix} \qquad \begin{bmatrix} \eta_{11} & \eta_{12} & \eta_{13} & \eta_{14} \\ \eta_{21} & \eta_{22} & & \\ \eta_{31} & \eta_{32} & & \\ \eta_{41} & \eta_{42} & & \end{bmatrix}$$

$$\eta_{ij} = \sum_{k \geq i,\, l \geq j} q_{kl}$$

Input matrix:

$$\begin{bmatrix} \hat{\eta}_{11} & \hat{\eta}_{12} & \hat{\eta}_{13} & \hat{\eta}_{14} \\ \hat{\eta}_{21} & \hat{\eta}_{22} & & \\ \hat{\eta}_{31} & \hat{\eta}_{32} & & \\ \hat{\eta}_{41} & \hat{\eta}_{42} & & \end{bmatrix} \xrightarrow{\text{Legendre decomposition}} \begin{bmatrix} \eta_{11} & \eta_{12} & \eta_{13} & \eta_{14} \\ \eta_{21} & \eta_{22} & & \\ \eta_{31} & \eta_{32} & & \\ \eta_{41} & \eta_{42} & & \end{bmatrix}$$

Reconstructed matrix:

Find $\boldsymbol{\theta}$ s.t. $\hat{\boldsymbol{\eta}} = \boldsymbol{\eta}$

Information Geometry

- Consider statistical manifold: $\boldsymbol{\mathcal{S}} = \{P \mid 0 < p_v < 1 \text{ and } \sum p_v = 1\}$
  - Each distribution P corresponds to a tensor $\mathcal{P}$; p_v is a probability
- $\boldsymbol{\mathcal{S}}$ is dually flat as (θ, η) is connected via Legendre transformation:
  $\theta = \nabla\varphi(\eta)$, $\quad \eta = \nabla\psi(\theta)$
  with two convex functions
  $\psi(\theta) = -\theta_\perp = -\log p_\perp$, $\quad \varphi(\eta) = \sum_{v \in \Omega} p_v \log p_v$
  - $\perp$ is the least element and corresponds to $(1, 1, \dots, 1)$
- Riemannian metric g, Fisher information (gradient), is given as
  $g(\theta) = \nabla\nabla\psi(\theta)$, $\quad g(\eta) = \nabla\nabla\varphi(\eta)$,

$$g_{uv}(\theta) = \frac{\partial \eta_u}{\partial \theta_v} = \mathbf{E}\left[\frac{\partial \log p_w}{\partial \theta_u}\frac{\partial \log p_w}{\partial \theta_v}\right] = \sum_{w \in \Omega} \zeta(u, w)\zeta(v, w)p_w - \eta_u\eta_v,$$

$$g_{uv}(\eta) = \frac{\partial \theta_u}{\partial \eta_v} = \mathbf{E}\left[\frac{\partial \log p_w}{\partial \eta_u}\frac{\partial \log p_u}{\partial \eta_v}\right] = \sum_{w \in \Omega} \mu(w, u)\mu(w, v)p_w^{-1}$$

  - We use zeta function ζ and Möbius function μ

Legendre Decomposition as Projection

- The e-flat and m-flat submanifolds:

$$\boldsymbol{\mathcal{S}}_B = \{\, \mathcal{Q} \in \boldsymbol{\mathcal{S}} \mid \theta_v = 0 \text{ for all } v \in \Omega \setminus B \,\}$$

$$\boldsymbol{\mathcal{S}}'_B = \{\, \mathcal{Q} \in \boldsymbol{\mathcal{S}} \mid \eta_v = \hat{\eta}_v \text{ for all } v \in B \,\}$$

- Legendre decomposition finds the intersection $\boldsymbol{\mathcal{S}}_B \cap \boldsymbol{\mathcal{S}}'_B$
  - From some $P \in \boldsymbol{\mathcal{S}}_B$ to $\boldsymbol{\mathcal{S}}_B \cap \boldsymbol{\mathcal{S}}'_B$ is called e-projection

Experiments on MNIST