

November 22, 2015



Refinement on Learning

Data Mining Theory (データマイニング工学)

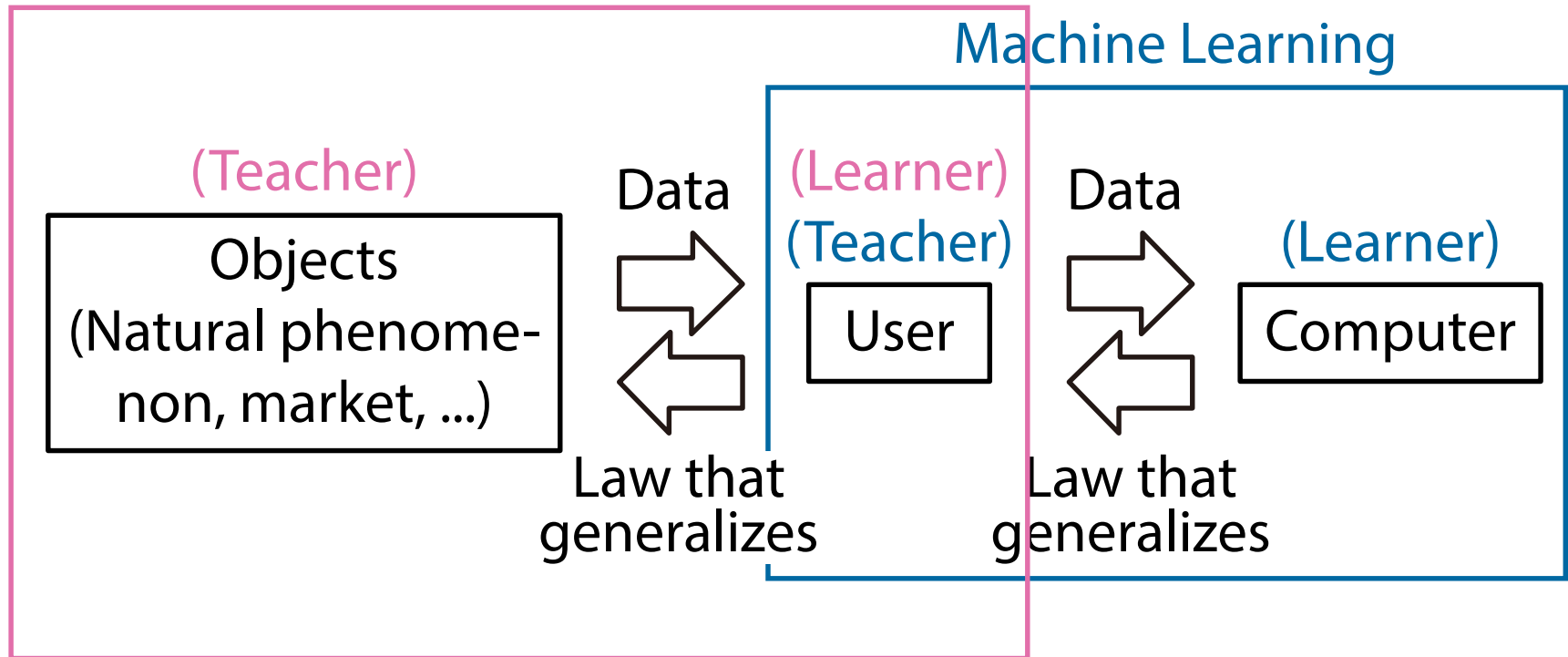
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Today's Outline

- Recap the main points of last week's lecture
- Consider the **structure** of a hypothesis space
 - Essential to efficiently **search candidate hypotheses**
- Understand the hypothesis space as a **poset** (半順序集合)
- Introduce the key concept of a **refinement** (精密化) operator to traverse the (structured) hypothesis space

Framework of Learning (ML vs DM)

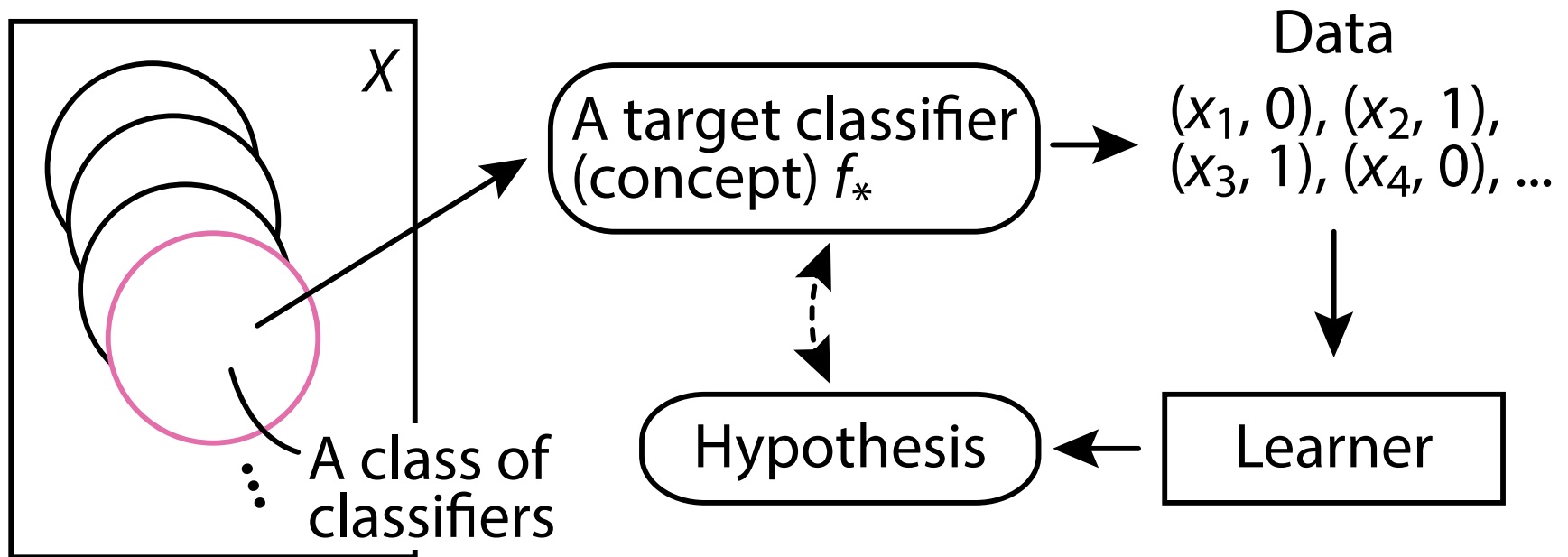
Data Mining (Knowledge Discovery)



Formalization of Learning in Computational Manner

1. What are **targets** of learning? (学習対象)
 - Each target (**concept**) C is a subset of the domain X ($C \subseteq X$)
 - A **concept space** \mathcal{C} is a collection of concepts ($\mathcal{C} \subseteq \mathcal{P}(X)$)
2. How to **represent** targets and hypotheses? (表現言語)
 - We use a **hypothesis space** \mathcal{H}
 - Each hypothesis $H \in \mathcal{H}$ represents a concept $\mathcal{U}(H) \subseteq X$
3. How are **data** provided to a learner? (データ)
4. How does the learner **work**? (学習手順, アルゴリズム)
5. When can we say that the learner **correctly** learns the target? (学習の正当性)

Learning Model



Gold's Learning Model on Languages

- A concept space $\mathcal{C} \subseteq \{ A \mid A \subseteq \Sigma^* \}$ is chosen
- For a language $C \in \mathcal{C}$, an infinite sequence $\sigma = (x_1, y_1), (x_2, y_2), \dots$ is a **complete presentation** (完全提示) of C if
 - (i) $\{x_1, x_2, \dots\} = \Sigma^*$
 - (ii) $y_i = 1 \iff x_i \in C$ for all i
- A **learner** is a procedure M that receives σ and generates an infinite sequence of hypotheses $\gamma = H_1, H_2, \dots$
- If γ converges to some hypothesis H and $v(H) = C$, we say that **M identifies C in the limit** (極限学習する)
 - If M identifies any $C \in \mathcal{C}$ in the limit,
 M identifies \mathcal{C} in the limit

Consistency of Hypotheses

- A language C is **inconsistent with** (x, y) (矛盾する) if $(y = 1 \text{ and } x \notin C) \text{ or } (y = 0 \text{ and } x \in C)$
- C is **consistent with** (x, y) if C is not inconsistent with (x, y)
- For a set of examples $S = \{(x_1, y_1), \dots, (x_n, y_n)\}$, C is **consistent with** S (C は S に無矛盾) if C is consistent with every $(x, y) \in S$

Basic Strategy: Generate and Test

- Input: a complete presentation σ of a language $C \in \mathcal{C}$
- Output: $\gamma = H_1, H_2, \dots$

1. $i \leftarrow 1, S \leftarrow \emptyset$
2. repeat
3. $S \leftarrow S \cup \{(x_i, y_i)\}$
4. while $v(H)$ is not consistent with S do
5. $H \leftarrow$ the next hypothesis in the hypothesis space \mathcal{H}
6. end while
7. $H_i \leftarrow H$ and output H_i
8. $i \leftarrow i + 1$
9. until forever

Power of Generate and Test Strategy and Its Problem

- For any class \mathcal{C} of languages, **Generate and Test strategy identifies \mathcal{C} in the limit**
 - That is, Generate and Test strategy identifies every language $C \in \mathcal{C}$ in the limit
- Unfortunately, this strategy is not realistic

Power of Generate and Test Strategy and Its Problem

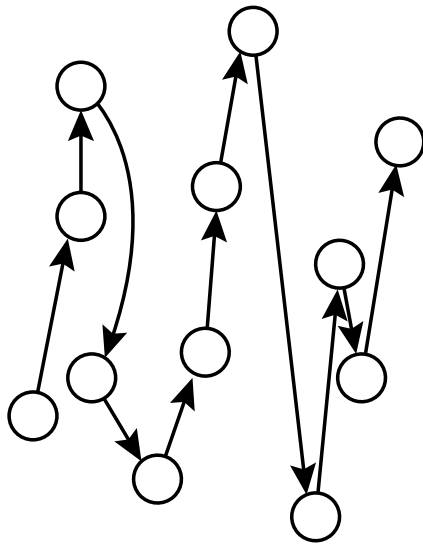
- For any class \mathcal{C} of languages, **Generate and Test strategy identifies \mathcal{C} in the limit**
 - That is, Generate and Test strategy identifies every language $C \in \mathcal{C}$ in the limit
 - Unfortunately, this strategy is not realistic
 - What is needed for more efficient learning?
- An **efficient search of candidate hypotheses** is essential!

Structurization of Hypothesis Space

- To search hypotheses,
 - (i) The **structure** of the hypothesis space \mathcal{H}
 - (ii) An **operator** that enables to traverse the spaceare indispensable
- The structured space is mathematically modeled as a **poset** (partially ordered set; 半順序集合)
- As an operator, we use **refinement** (精密化)
 - For each hypothesis, a learner can “refine” it and derive a set of one level specific hypotheses

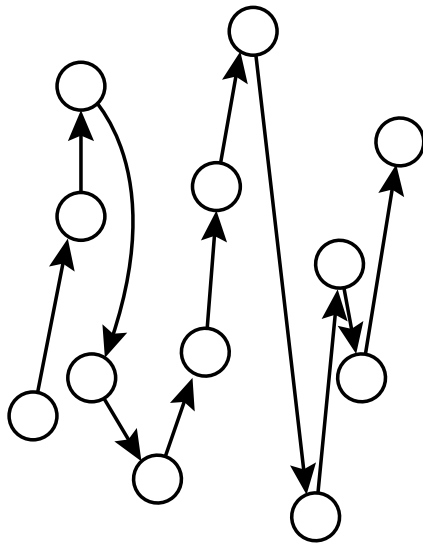
Structurization of Hypothesis Space

No structure
in hypothesis space

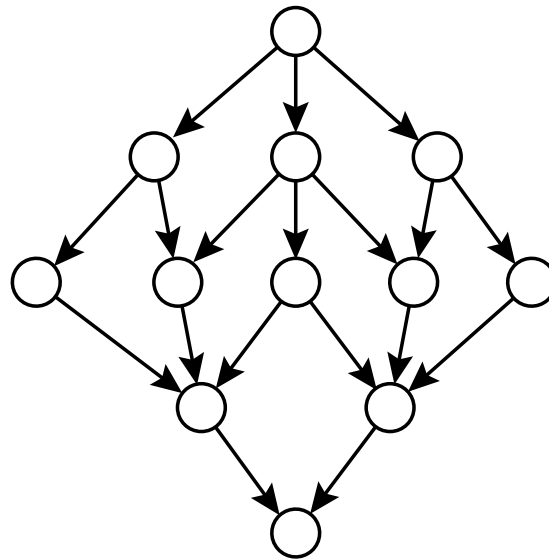
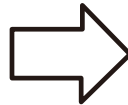


Structurization of Hypothesis Space

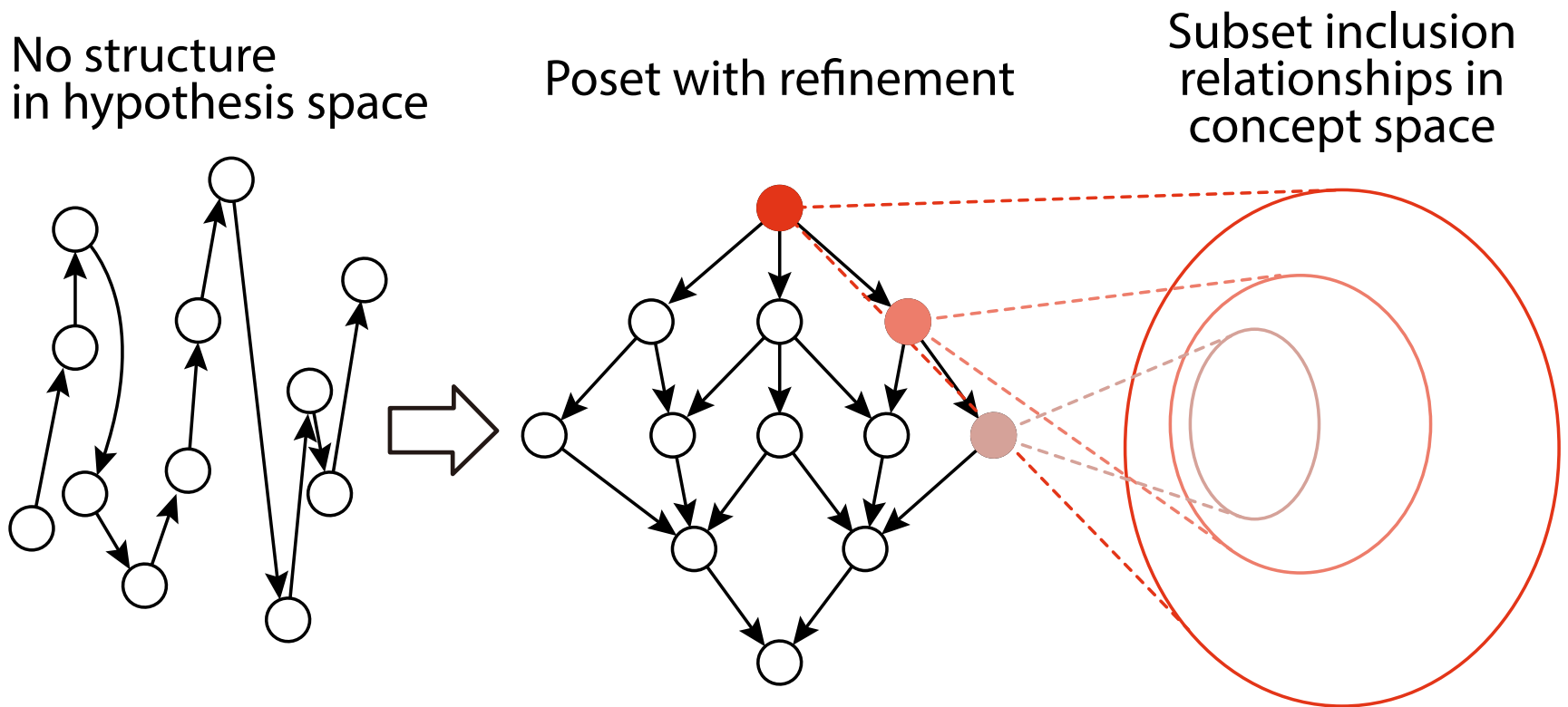
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Poset with refinement



Structurization of Hypothesis Space



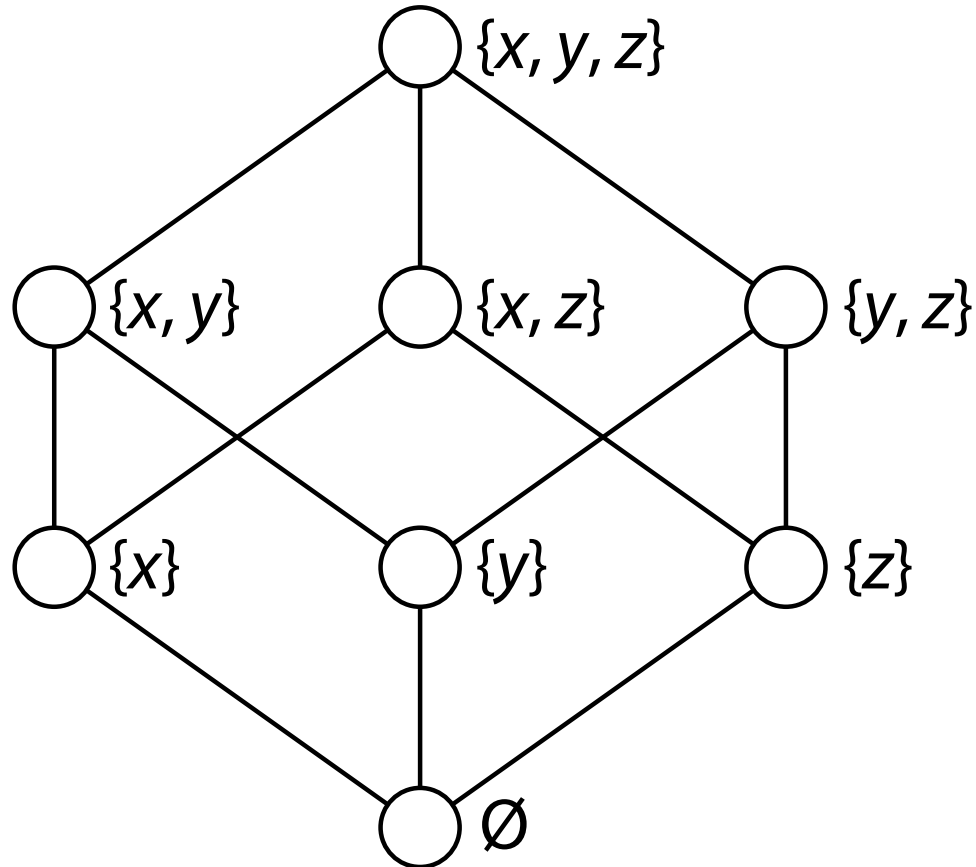
Poset

- A **partial order** (半順序) is a binary relation \leq s.t.
 1. $x \leq x$ (reflexivity; 反射律)
 2. $(x \leq y \text{ and } y \leq x) \Rightarrow x = y$ (antisymmetry; 反对称律)
 3. $(x \leq y \text{ and } y \leq z) \Rightarrow x \leq z$ (transitivity; 推移律)
- A set X with a partial order \leq , denoted as (X, \leq) , is called a **partially ordered set** (poset; 半順序集合)
 - The **least upper bound** (supremum; 最小上界) of $S \subseteq X$ is the least $x \in X$ s.t. $\forall s \in S, s \leq x$
 - The **greatest lower bound** (infimum; 最大下界) of $S \subseteq X$ is the greatest $x \in X$ s.t. $\forall s \in S, x \leq s$

Lattice

- We use **join** “ \vee ” (結び) and **meet** “ \wedge ” (交わり)
 - $x \vee y = \sup\{x, y\}$ (join of x and y)
 - For $S \subseteq X$, $\vee S = \sup S$
 - $x \wedge y = \inf\{x, y\}$ (meet of x and y)
 - For $S \subseteq X$, $\wedge S = \inf S$
- A poset (X, \leq) is a **lattice** (束) if $x \vee y$ and $x \wedge y$ exist for all $x, y \in X$
- Examples:
 - The power set $\mathcal{P}(X)$ of any set X (we translate “ \subseteq ” as \leq)
 - The set of natural numbers \mathbb{N} w.r.t “ \leq ”
 - The Cartesian product $\mathbb{N} \times \mathbb{N} = \{(a, b) \mid a, b \in \mathbb{N}\}$,
 $(a, b) \leq (a', b')$ if $a \leq a'$ and $b \leq b'$

The Power Set Is a Lattice



Definition of Refinement

- Assume that our hypothesis space (\mathcal{H}, \preceq) is a poset and
$$G \preceq H \Rightarrow \nu(G) \subseteq \nu(H)$$
$$G \equiv H \Rightarrow \nu(G) = \nu(H)$$
 - ν should be a **homomorphism** (準同型写像) that preserves structure between \mathcal{C} and \mathcal{H}
- A **refinement** (精密化) is a mapping $\rho : \mathcal{H} \rightarrow \mathcal{H}^2$ s.t.
 1. $\forall H \in \mathcal{H}, \rho(H)$ is finite
 2. $G \in \rho(H) \Rightarrow G \preceq H$
 3. $\forall H \in \mathcal{H}$, there is no infinite sequence H_1, H_2, \dots s.t.
$$H = H_1 \text{ and } H_i \in \rho(H_{i+1})$$

Semantically Complete Refinement

- We write $X \xrightarrow{\rho} Y$ if $Y \in \rho(X)$
 - $\xrightarrow{*}$ is zero or more applications of $\xrightarrow{\rho}$
- A refinement ρ is **semantically complete** (意味的に完全) if
$$\left\{ \nu(G) \mid H \xrightarrow{*} G \right\} = \left\{ C \in \mathcal{C} \mid C \subset \nu(H) \right\}$$
 - Start from H , we can find any $C \subset \nu(H)$ by applying $\xrightarrow{\rho}$
 - If this condition is not satisfied, we will miss some concepts

Pioneers of Refinement

- Refinement is (implicitly) used in various contexts
 - It can be viewed as an online construction of search space with tree-like structure
- It has been explicitly introduced in **Model Inference System** by Shapiro in 1981
 - E. Y. Shapiro, **An Algorithm That Infers Theories from Facts**, *IJCAI*, 1981
- Plotkin considered the opposite direction (from specific to general)
 - G. D. Plotkin, **A further note on inductive generalization**, *Machine Intelligence*, 1970

Examples of Refinement

- Let us consider concrete examples of refinement and learning
- We use two simple examples:
 - Regular language (正則言語)
 - The set of pairs of natural numbers
 $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N} = \{ (a, b) \mid a, b \in \mathbb{N} \}$

Regular Language (1/2)

- Given an alphabet Σ
 - For $a \in \Sigma$, $a^2 = aa$, $a^3 = aaa, \dots$
 - $X^0 = \emptyset$, $X^n = \{ au \mid a \in X, u \in X^{n-1} \} \ (n \geq 1)$
- For a **regular expression** (正則表現, RE) H , $v(H)$ is a **regular language** (正則言語)
 - \emptyset is an RE; $v(\emptyset) = \emptyset$
 - $\forall a \in \Sigma$, a is an RE; $v(a) = \{a\}$
 - If X and Y are REs,
 - $X + Y$ is an RE; $v(X + Y) = X \cup Y$ (union)
 - XY is an RE; $v(XY) = \{ ab \mid a \in X, b \in Y \}$ (concatenation)
 - X^* is an RE; $v(X^*) = \bigcup \{ X^n \mid n \geq 0 \}$
(**Kleene closure**; クリーネ閉包)

Regular Language (2/2)

- Let $\Sigma = \{a_1, a_2, \dots, a_n\}$
- We denote by \top the language $(a_1 + a_2 + \dots + a_n)^*$
 - $u(\top) = \Sigma^*$
 - The largest language over Σ
- Examples:
 - Assume that $\Sigma = \{a, t, g, c\}$
 - $u(at + g^*) = \{\epsilon, at, g, gg, ggg, \dots\}$
 - $u((a + c)^*) = \{\epsilon, a, c, aa, ac, ca, cc, aaa, \dots\}$
 - $u(\top) = \{\epsilon, a, t, g, c, aa, at, \dots\}$

Refinement on Regular Languages

(from P. D. Laird, Learning from Good and Bad Data, 1988)

1. $X \xrightarrow{\rho} X + X$
2. $X^* \xrightarrow{\rho} X^* X^*$
3. $X^* \xrightarrow{\rho} (X^*)^*$
4. $a \xrightarrow{\rho} \emptyset \quad (a \in \Sigma)$
5. $X^* \xrightarrow{\rho} X$
6. $X \xrightarrow{\rho} Y \Rightarrow X + Z \xrightarrow{\rho} Y + Z$
7. $X \xrightarrow{\rho} Y \Rightarrow Z + X \xrightarrow{\rho} Z + Y$
8. $X \xrightarrow{\rho} Y \Rightarrow X^* \xrightarrow{\rho} Y^*$
9. $X \xrightarrow{\rho} Y \Rightarrow XZ \xrightarrow{\rho} YZ$
10. $X \xrightarrow{\rho} Y \Rightarrow ZX \xrightarrow{\rho} ZY$

Examples of Refinement on Regular Languages

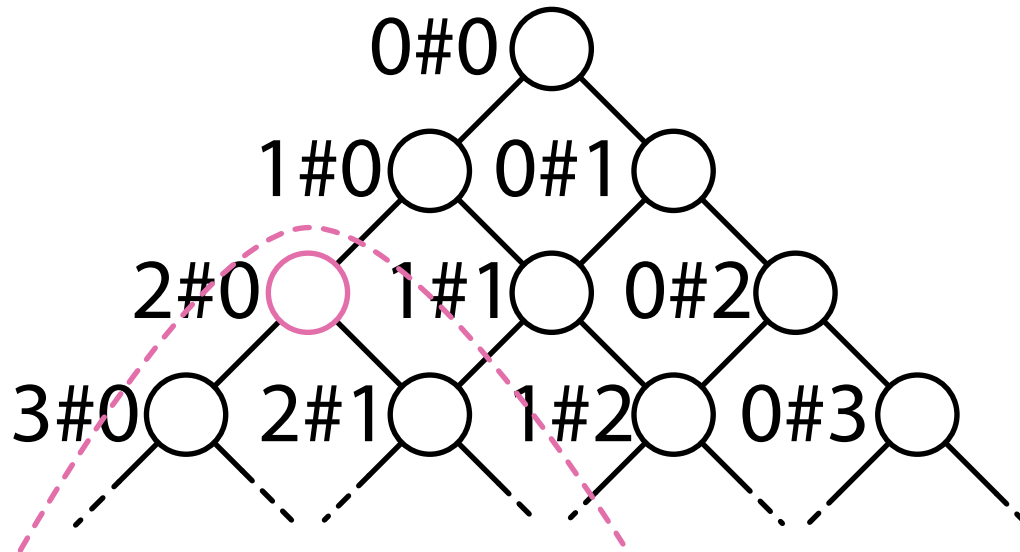
- Let $\Sigma = \{0, 1\}$
- $T = (0 + 1)^* \xrightarrow{\rho} 0 + 1 \xrightarrow{\rho} \emptyset + 1 \xrightarrow{\rho} \emptyset + \emptyset$
- $T = (0 + 1)^* \xrightarrow{\rho} (0 + 1)^*(0 + 1)^* \xrightarrow{\rho} (0 + 1)^*(0 + 1) \xrightarrow{\rho} (0 + 1)(0 + 1)$
 - $v((0 + 1)(0 + 1)) = \{00, 01, 10, 11\}$

Efficient Learning with Refinement

1. $i \leftarrow 1, S \leftarrow \emptyset, H \leftarrow T, Q \leftarrow \emptyset$ // Q is a list of candidate hypotheses
2. repeat
3. $S \leftarrow S \cup \{(x_i, y_i)\}$
4. while H is not consistent with S
5. if $x \in v(H)$ for some $(x, o) \in S$ then
6. Append all $\rho(H)$ to the tail of Q
7. end if
8. $H \leftarrow$ the first hypothesis in Q , and remove it from Q
9. end while
10. $H_i \leftarrow H$ and output H_i
11. $i \leftarrow i + 1$
12. until forever

Hypothesis Space on \mathbb{N}^2

- $\mathcal{H} = \{a\#b \mid a, b \in \mathbb{N}\}$
- $a\#b \leq c\#d$ if $a \geq c$ and $b \geq d$
 - Note that we invert \leq for mathematical convenience



Refinement on \mathbb{N}^2

- We consider the following concept space \mathcal{C} :

$$\mathcal{C} = \{ \uparrow(a, b) \mid a, b \in \mathbb{N} \},$$

where

$$\uparrow(a, b) = \{ (c, d) \in \mathbb{N}^2 \mid a \leq c, b \leq d \}$$

- A subset $O \subseteq \mathbb{N}^2$ s.t. $(a, b) \in O \Rightarrow \uparrow(a, b) \subseteq O$ is known to be **open** on the **Alexandroff topology**
- Define $\nu(a\#b) = \uparrow(a, b)$
- Refinement is given as follows:
 1. $a\#b \xrightarrow{\rho} (a+1)\#b$
 2. $a\#b \xrightarrow{\rho} a\#(b+1)$

Refinement on Sets of \mathbb{N}^2

- We can further treat a (finite) set of $\uparrow(a, b)$ as a concept
- $\mathcal{H}_S = \{ a_1 \# b_1 + a_2 \# b_2 + \cdots + a_n \# b_n \mid a_i, b_i, n \in \mathbb{N} \}$
- $\mathcal{C}_S = \{ C \mid C \subseteq \mathcal{C} = \{ \uparrow(a, b) \mid a, b \in \mathbb{N} \}, C \text{ is finite} \}$
- $u(a_1 \# b_1 + \cdots + a_n \# b_n) = \uparrow(a_1, b_1) \cup \cdots \cup \uparrow(a_n, b_n)$
- Refinement is given as follows:
 1. $a \# b \xrightarrow{\rho} (a + 1) \# b$
 2. $a \# b \xrightarrow{\rho} a \# (b + 1)$
 3. $X \xrightarrow{\rho} Y \Rightarrow X + Z \xrightarrow{\rho} Y + Z \text{ and } Z + X \xrightarrow{\rho} Z + Y$
 4. $X \xrightarrow{\rho} X + X$

How about \mathbb{R} ?

- Let us consider the set of **real numbers** \mathbb{R}
 - One of the most important objects in machine learning
- Each real number $x \in \mathbb{R}$ is represented as an **infinite sequence**
 - e.g., use infinite **decimal expansions** with $\Sigma = \{0, 1, \dots, 9\}$
 - Let \bar{x} be a representation of x
- Obviously, we cannot treat all elements in \mathbb{R} as we cannot determine $x \in \mathbb{R}$ from \bar{x} in finite time
- We can just treat **prefixes** of infinite sequences, and $u(w) = \{x \in \mathbb{R} \mid w \sqsubseteq \bar{x}\}$, which forms an **open set** on \mathbb{R}