November 25, 2015

Discretization and Learning

Data Mining Theory (データマイニング工学)

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Today's Outline

- Recap the main points of last week's lecture
- Discretization on learning
- Real number computation (実数計算)
- Learning = computing = discretization?
- All slides are at:
  `http://mahito.info/materials.html`

Structurization of Hypothesis Space

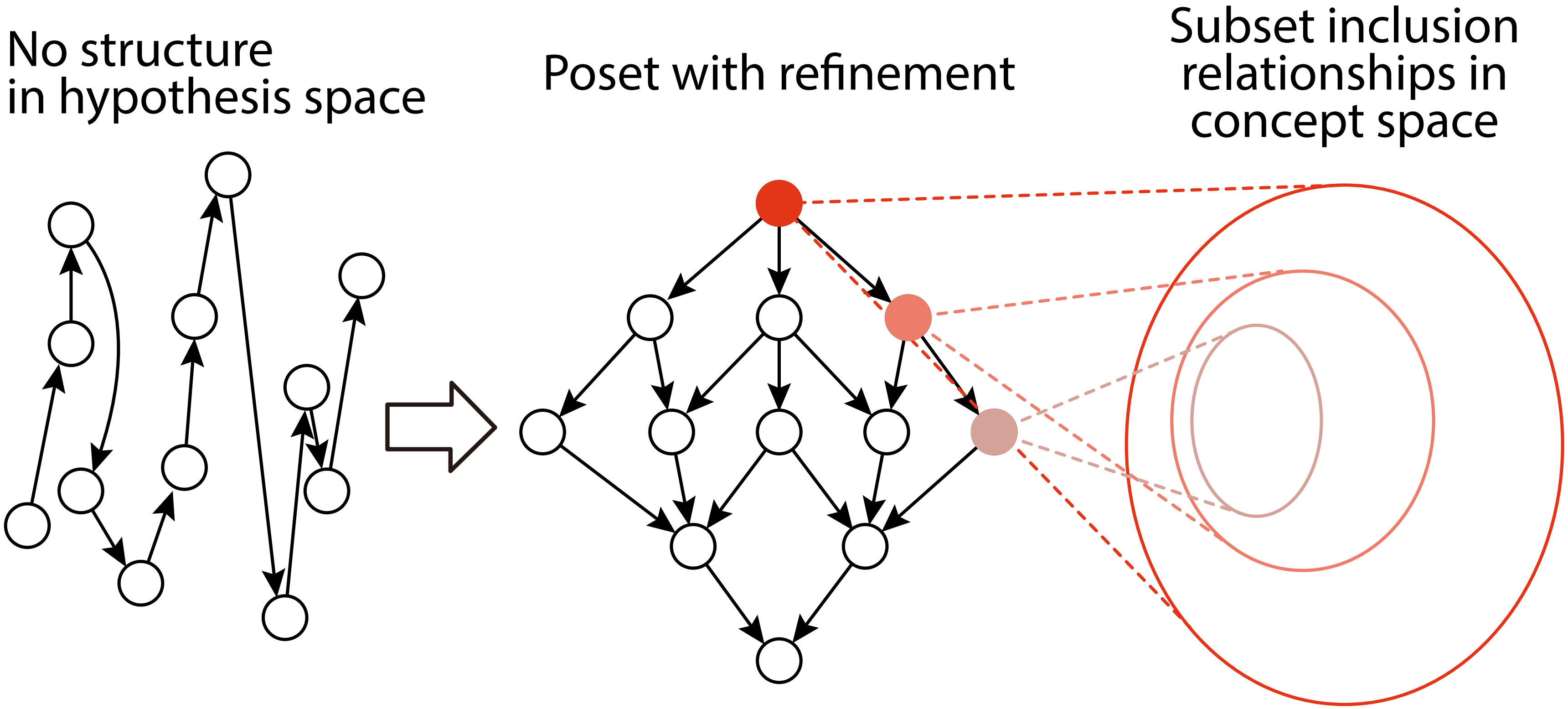
- To search hypotheses,
  (i) The structure of the hypothesis space $\mathcal{H}$
  (ii) An operator that enables to traverse the space

  are indispensable

- The structured space is mathematically modeled as a poset (partially ordered set; 半順序集合)

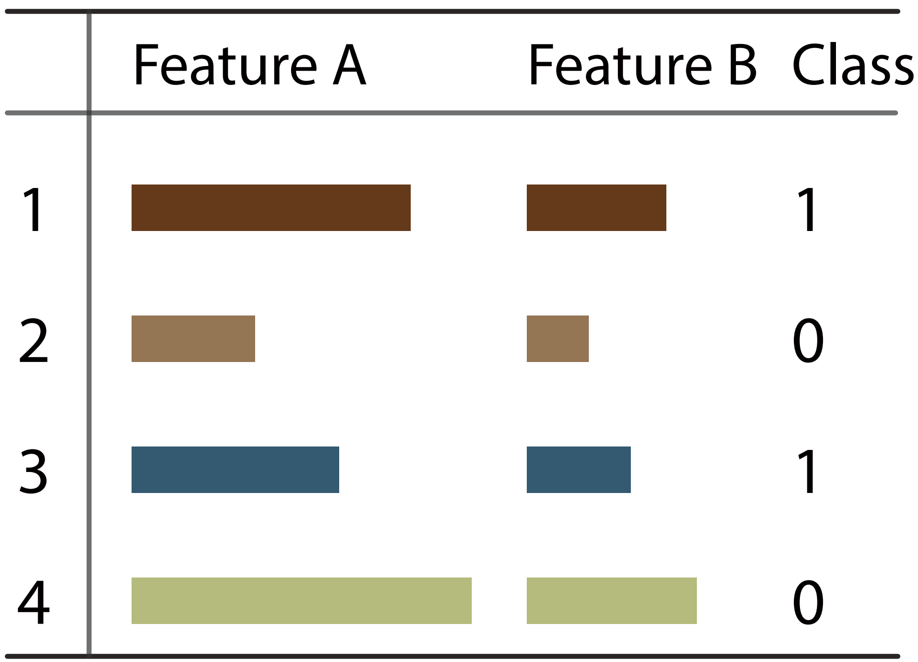
- As an operator, we use refinement (精密化)
  - For each hypothesis, a learner can “refine” it and derive a set of one level-specific hypotheses

Structurization of Hypothesis Space



Discretization and Learning

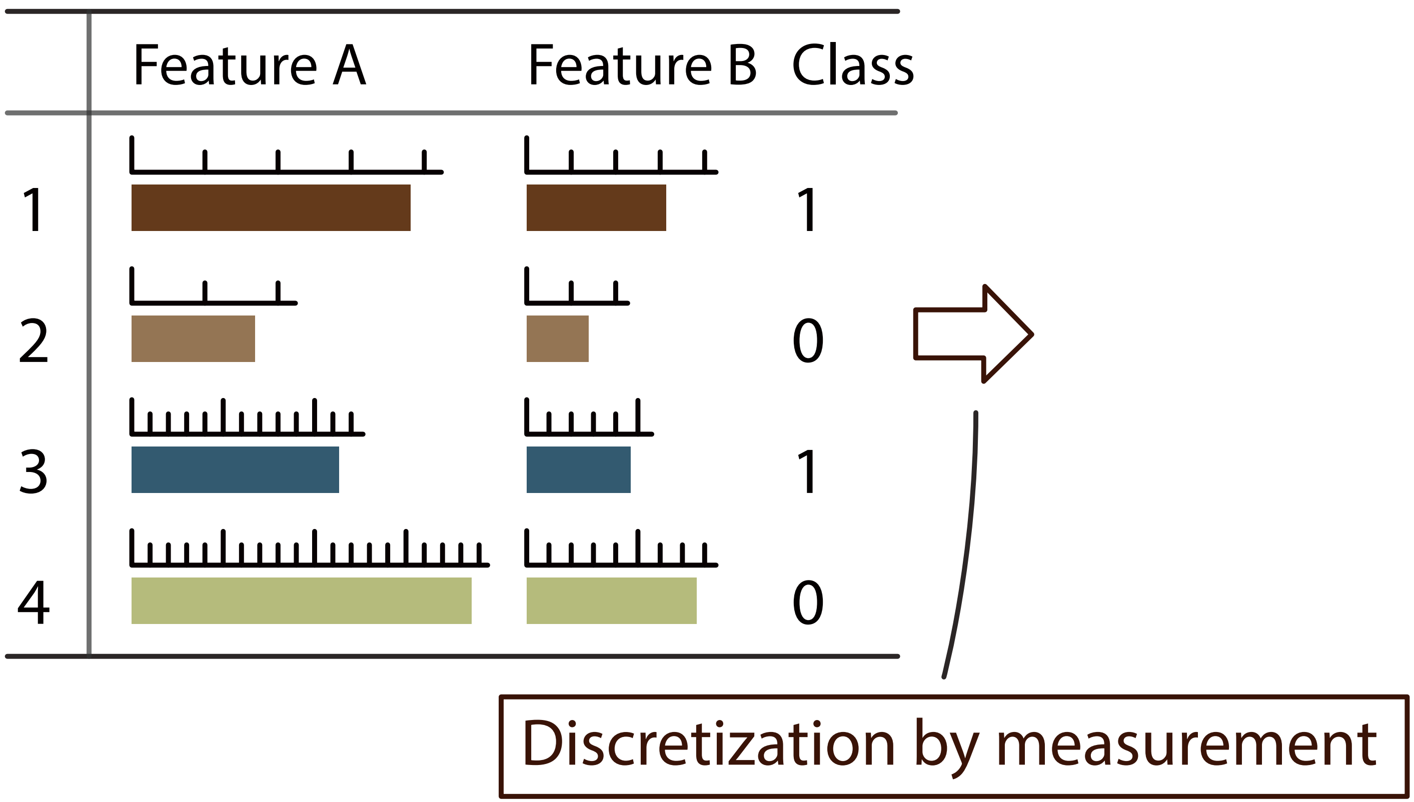
Analog data (reals)



Goal: Learning of a classifier

Discretization and Learning

Analog data (reals)



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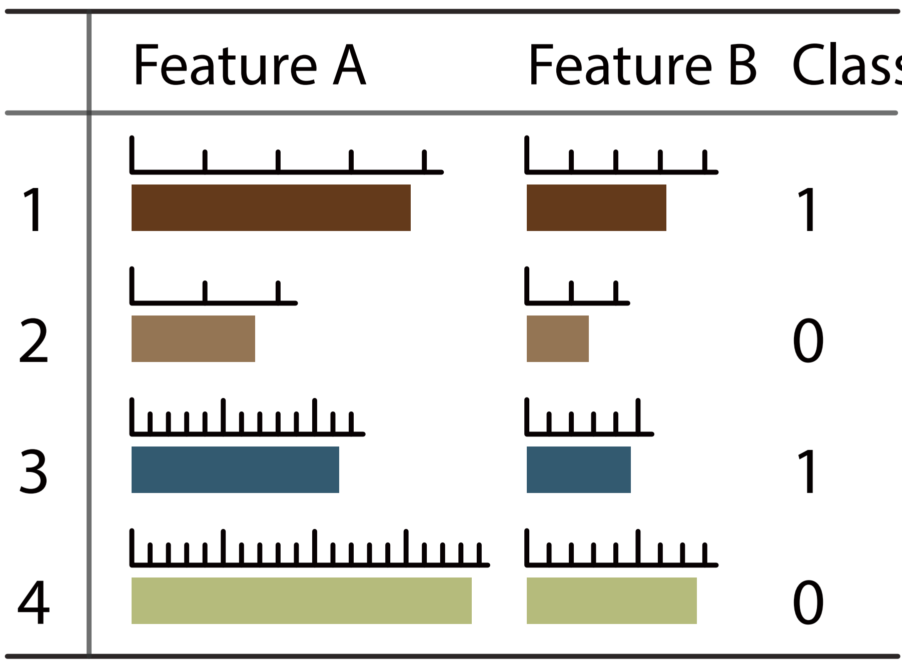

Digital data (rationals)

Feature A	Feature B	Class
1.5	0.6	1
0.6	0.2	0
1.1	0.4	1
1.8	0.7	0

Discretization by measurement

Goal: Learning of a classifier

Discretization and Learning

Analog data (reals)

	Feature A	Feature B	Class
1	1.539582…	0.6069…	1
2	0.676711…	0.2655…	0
3	1.111577…	0.4998…	1
4	1.871501…	0.7569…	0

Digital data (rationals)

Feature A	Feature B	Class
1.5	0.6	1
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Discretization by measurement

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Discretization and Learning

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Data in theory

Digital data (rationals)

Feature A	Feature B	Class
1.5	0.6	1
0		
1.1	0.4	1
1.8	0.7	0

Data on computer

Discretization by measurement

Goal: Learning of a classifier

Fatal Error Caused by Discretization

- Solve the system of equations [Schröder, 2003]

  $40157959.0\,x + 67108865.0\,y = 1$

  $67108864.5\,x + 112147127.0\,y = 0$

  - We can solve by the well-known formula:

    $$x = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11} a_{22} - a_{21} a_{12}}, \quad y = \frac{b_2 a_{11} - b_1 a_{21}}{a_{11} a_{22} - a_{21} a_{12}}$$

- Computation by floating point arithmetic with double precision variables (IEEE 754):

  $x = 112147127, \quad y = -67108864.5$

- Correct solution:

  $x = 224294254, \quad y = -134217729$

Treat Data as Intervals

Analog data (reals)

	Feature A	Feature B	Class
1	1.539582…	0.6069…	1
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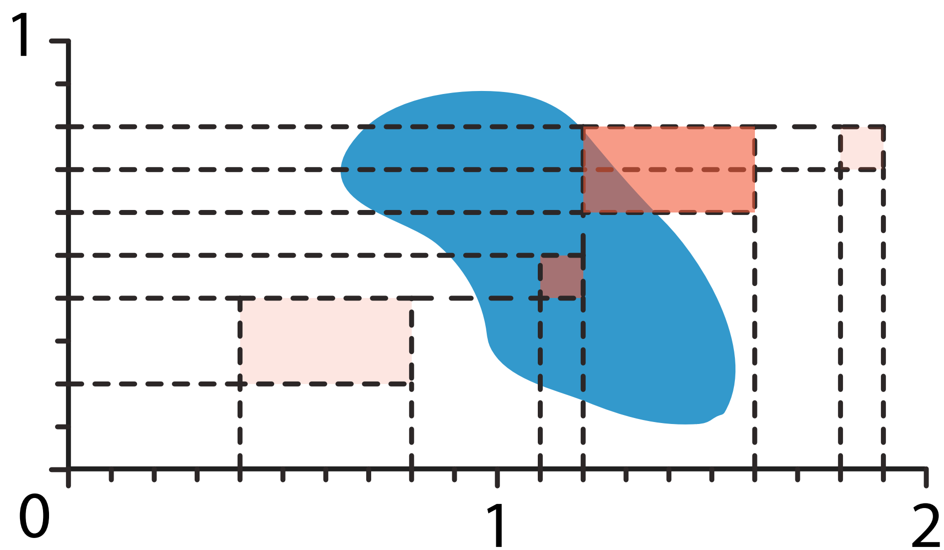
Digital data (rationals)

Feature A	Feature B	Class
1.2~1.6	0.6~0.8	1
0.4~0.8	0.2~0.4	0
1.1~1.2	0.4~0.5	1
1.8~1.9	0.7~0.8	0

Discretization by measurement

Goal: Learning of a classifier

Geometric Point of View



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- Discretized data are intervals in $\mathbb{R}^d$
  - The width of an interval corresponds to the error of a data point
- A learner finds a set intersecting intervals of class 1

How to Compute Real Numbers?

- Consider computation of $f(x) = 3 \cdot x$
- For example: $f(1/3) = 3 \cdot 1/3$
- Since $1/3 = 0.33333\ldots$, a computer should output $0.99999\ldots$ (or $1.00000\ldots$)
- However, it cannot output any digit since:
  - If an input is $0.333\ldots$ forever, the output is $0.999\ldots$
  - If an input is $0.333\ldots34$ at some point, the output is $1.000\ldots02$
- Thus the computer cannot determine even the first digit at any moment

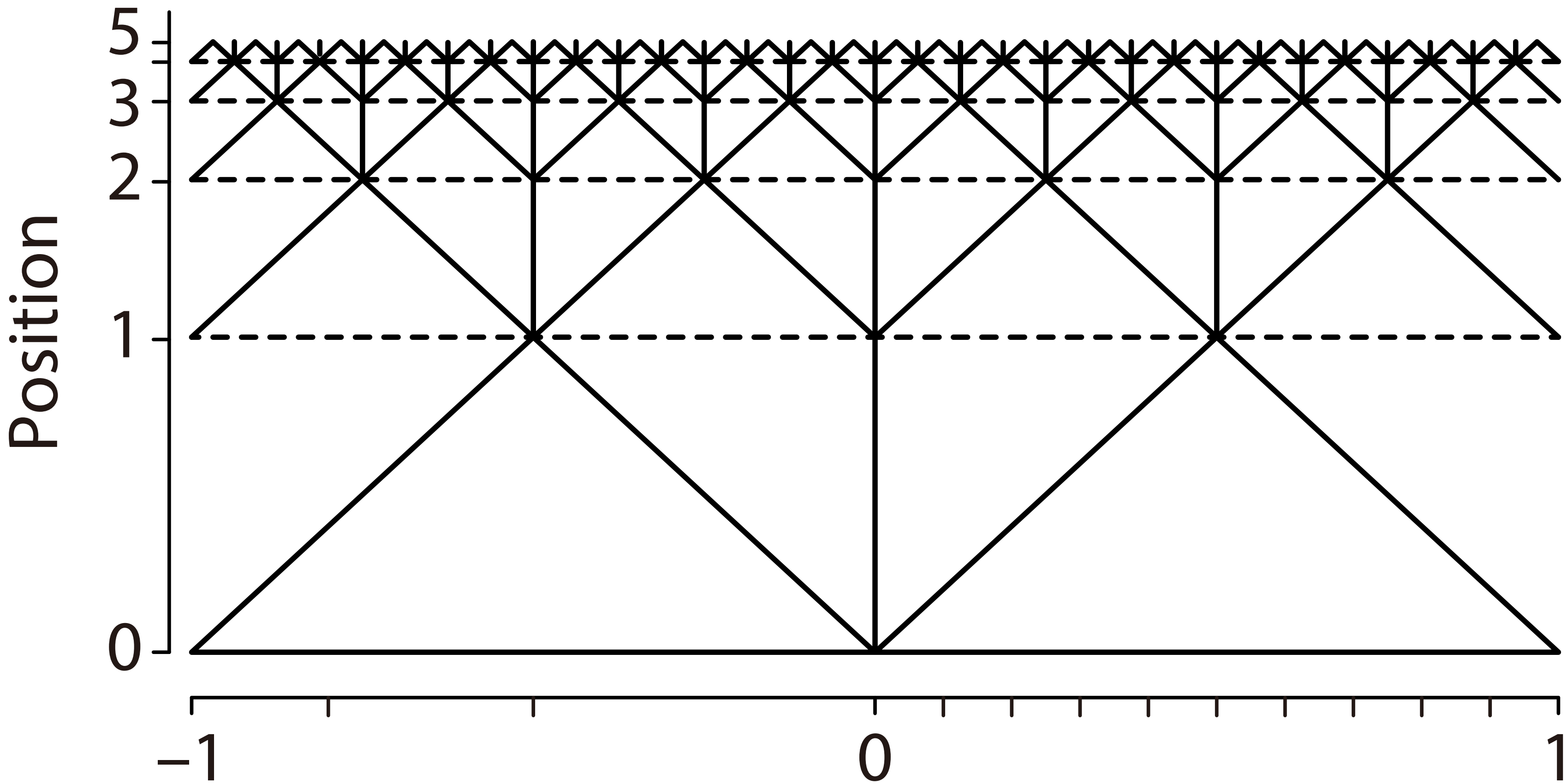
What is Problem in Real Number Computation?

- The problem is caused by the representation of real numbers (実数表現)
- Decimal representation (10進表現) lacks redundancy (冗長性)
  - We need more sequences that represent the same number
- **Solution:** signed digit representation (符号付き2進数)
  - Use three symbols: 1, 0, and $\bar{1}$ ($\bar{1}$ means -1) and defined as:

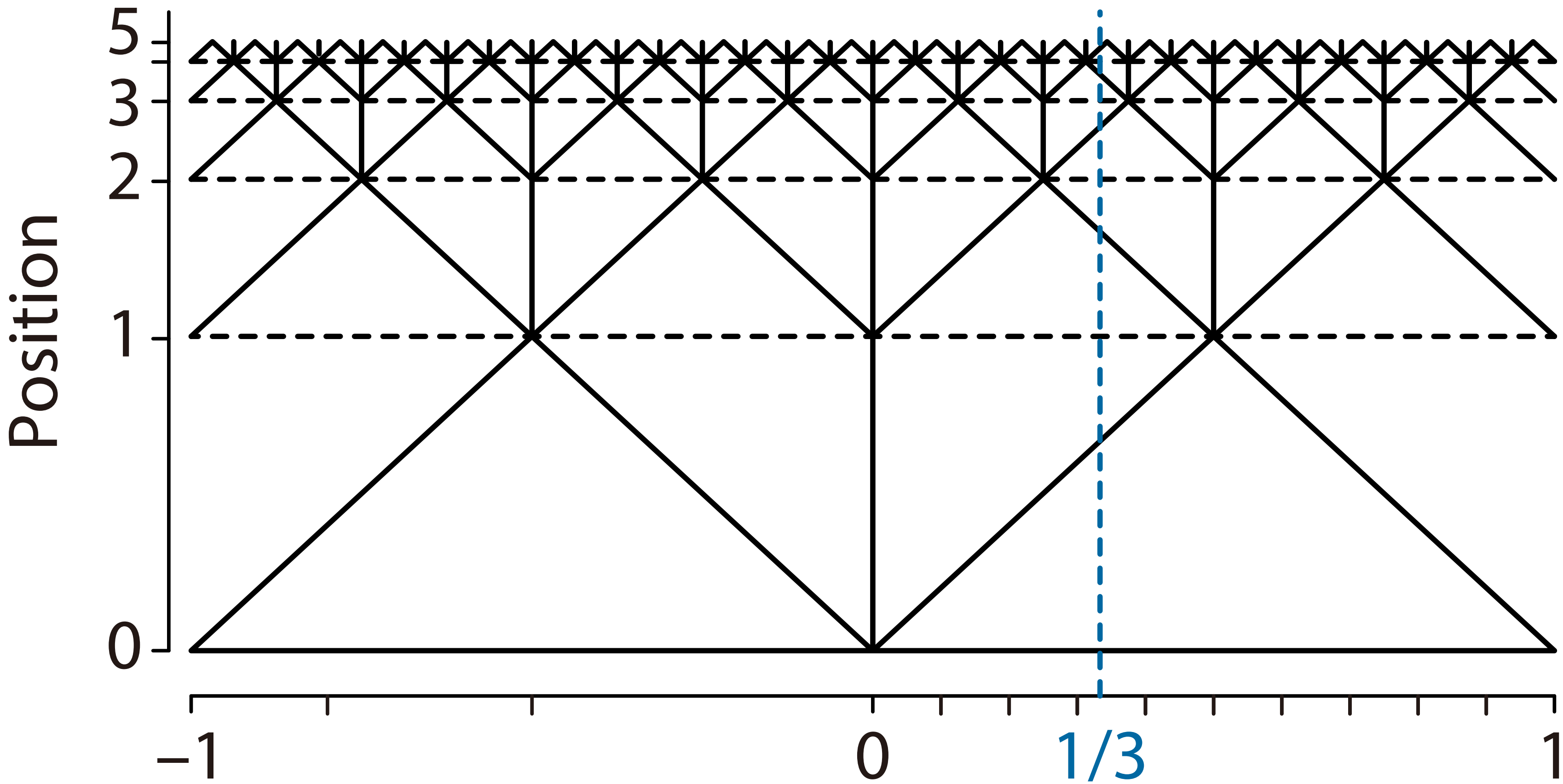
$$\rho(a_1 a_2 \dots) = \sum_{i=1}^{\infty} a_i \cdot 2^{-i}$$

    - Same as the binary representation if we use only 0 and 1

Signed Digit Representation



Signed Digit Representation



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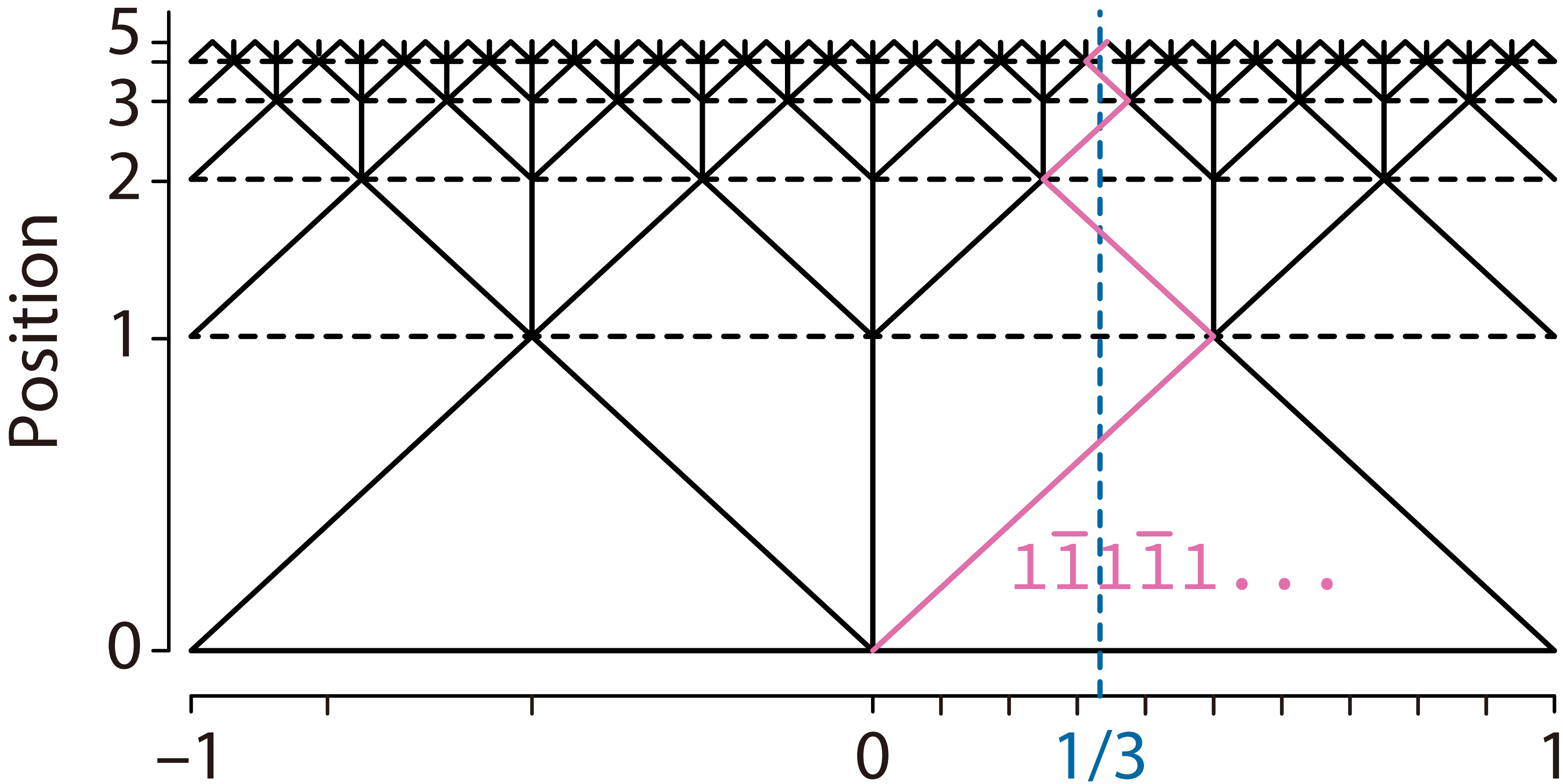

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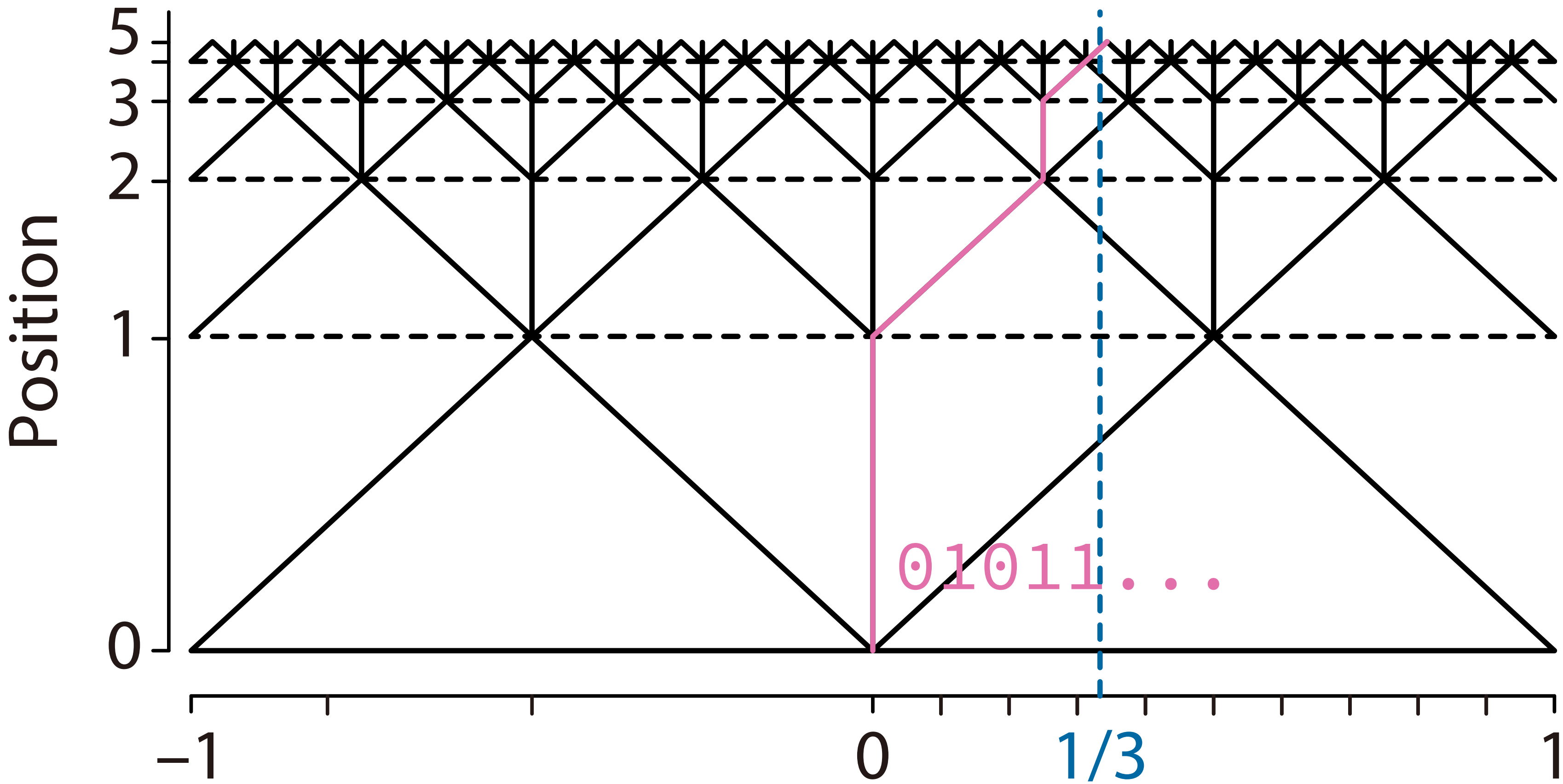


Gray Code

- Using signed digit representation, we can achieve computation over reals in a natural sense

- Another interesting representation is Gray code (グレイコード) by Frank Gray (1947) and Émile Baudot (1878)
  - Originally, another binary encoding of natural numbers
    - Important in applications of conversion between analog and digital information [Knuth, 2005]

- Gray codes for natural numbers:

	0	1	2	3	4	5	6	7
Binary	000	001	010	011	100	101	110	111
Gray	000	001	011	010	110	111	101	100

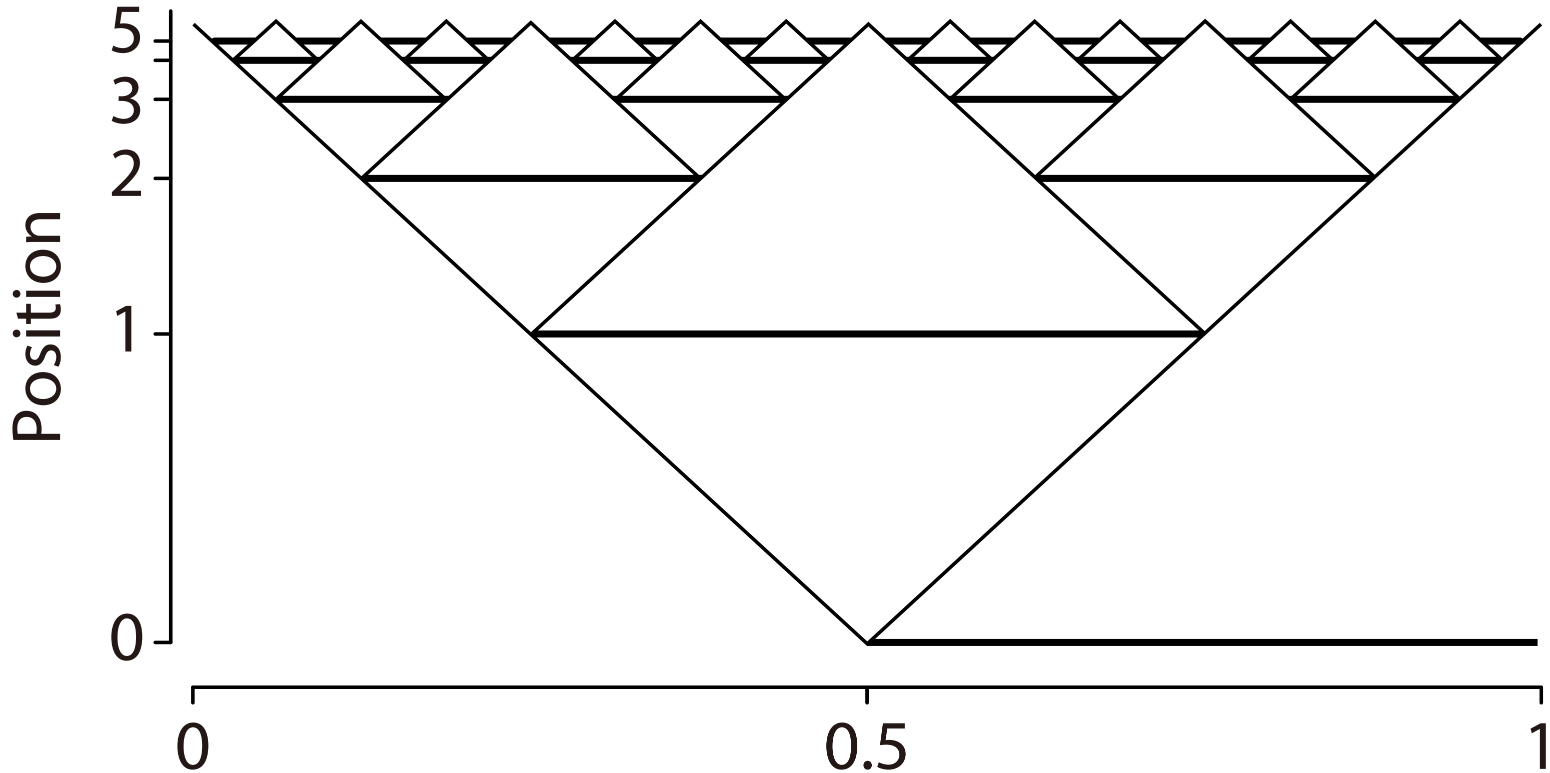
Gray Code Embedding

- Gray code can be used for real number representation
  - We use three symbols 0, 1, and $\perp$

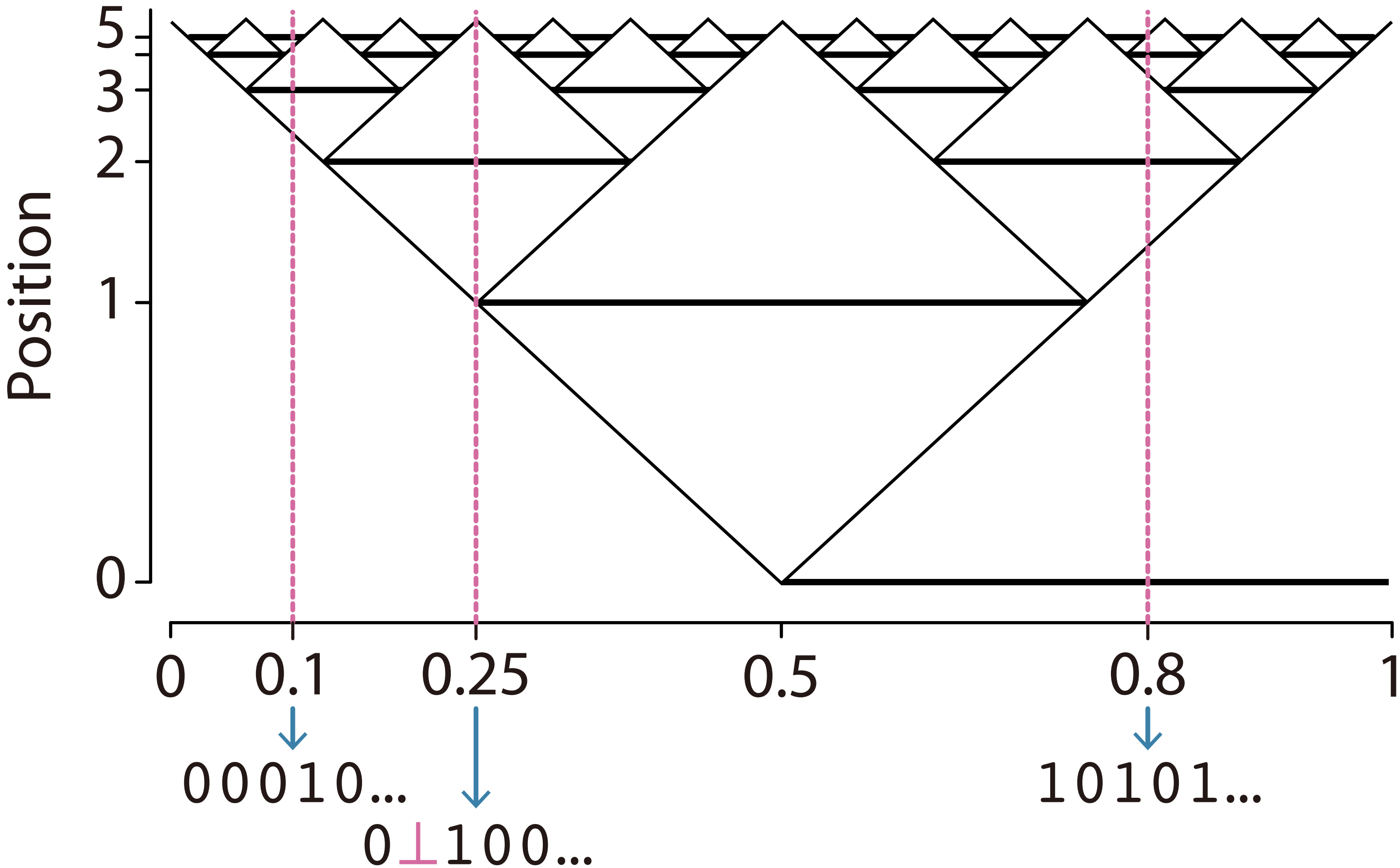
- The Gray code embedding (グレイコード埋め込み) is an injection γ_G that maps $x \in [0, 1]$ to an infinite sequence $p_0 p_1 p_2 \ldots$, where
  - $p_i := 1$ if $2^{-i}m - 2^{-(i+1)} < x < 2^{-i}m + 2^{-(i+1)}$ for an odd m,
  - $p_i := 0$ if the same holds for an even m,
  - $p_i := \perp$ if $x = 2^{-i}m - 2^{-(i+1)}$ for some integer m

- Power of representations for real number computation:
  Gray code = signed digit representation [Dusky, 2002]
  > binary representation

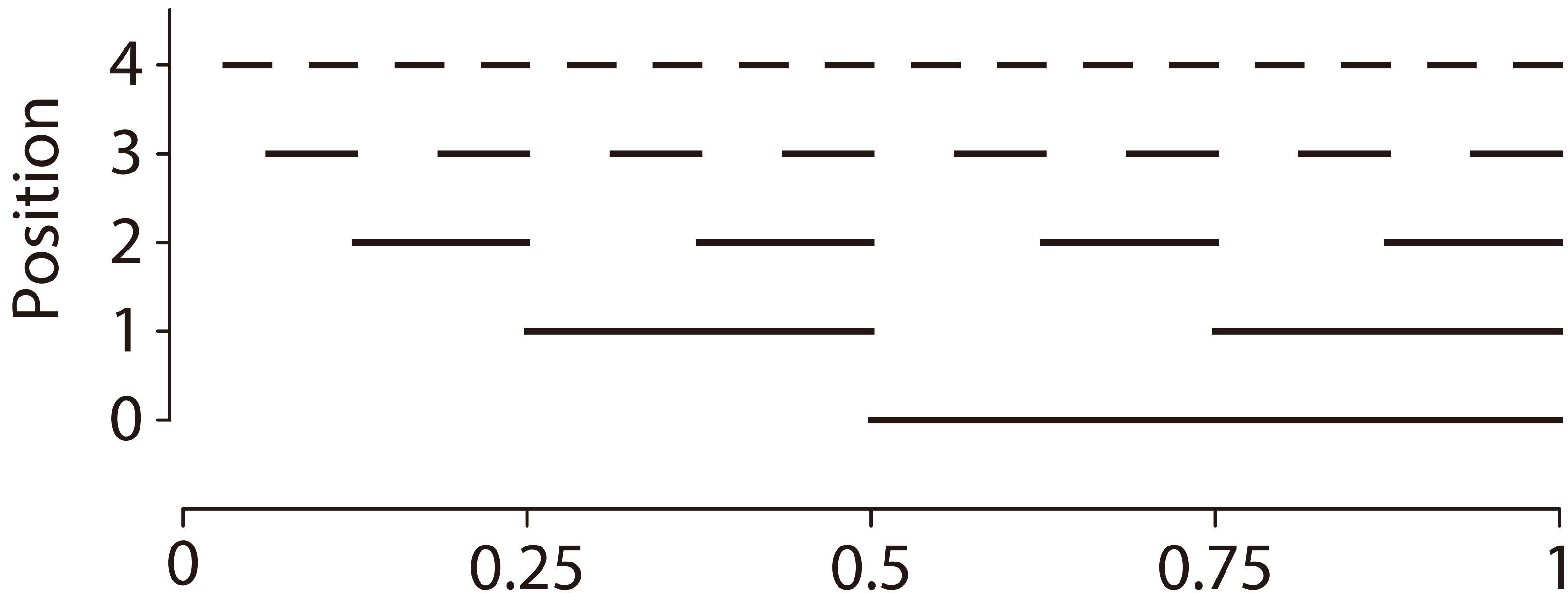
Gray Code on Reals



Gray Code on Reals



Binary Representation



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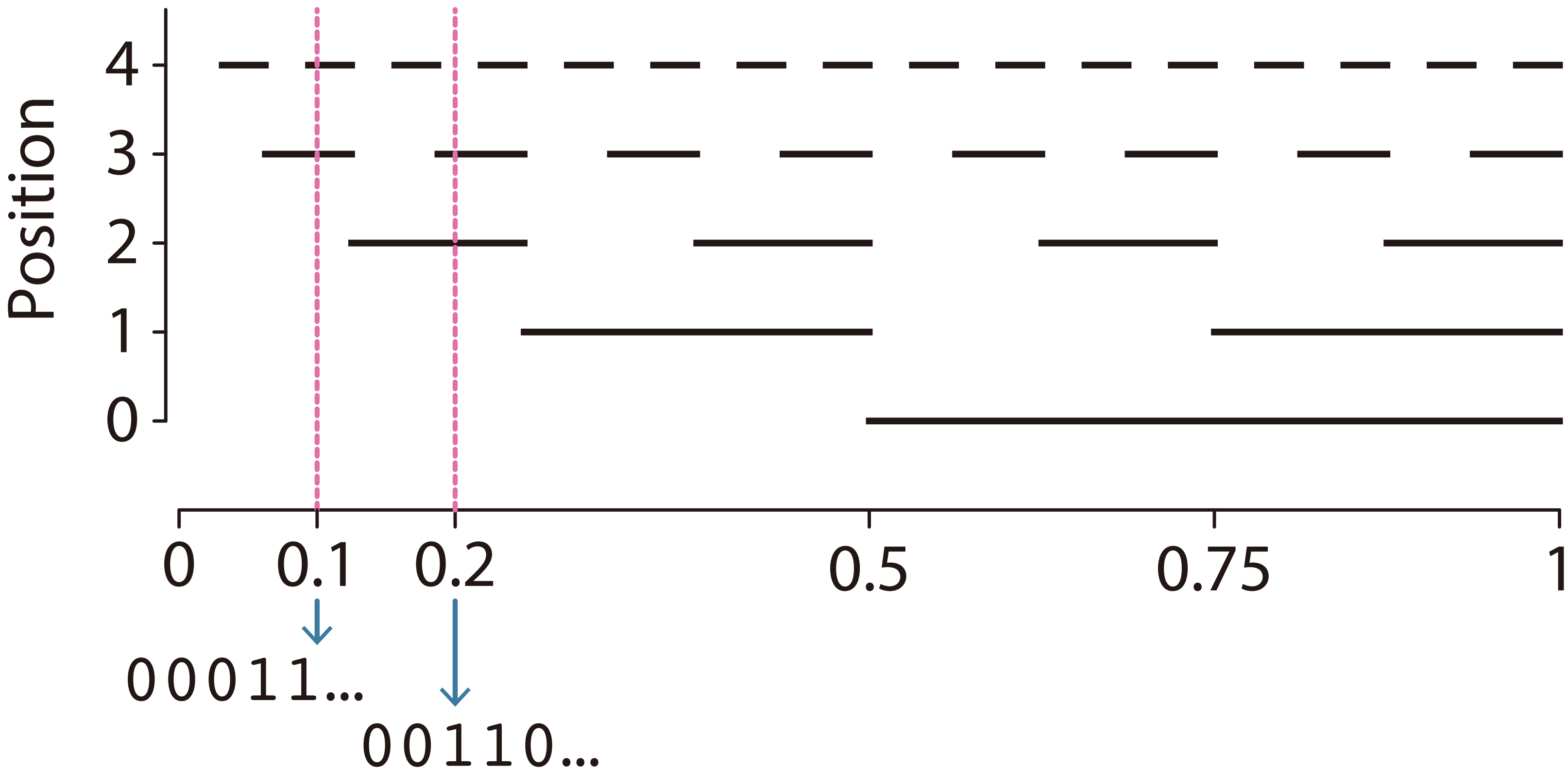
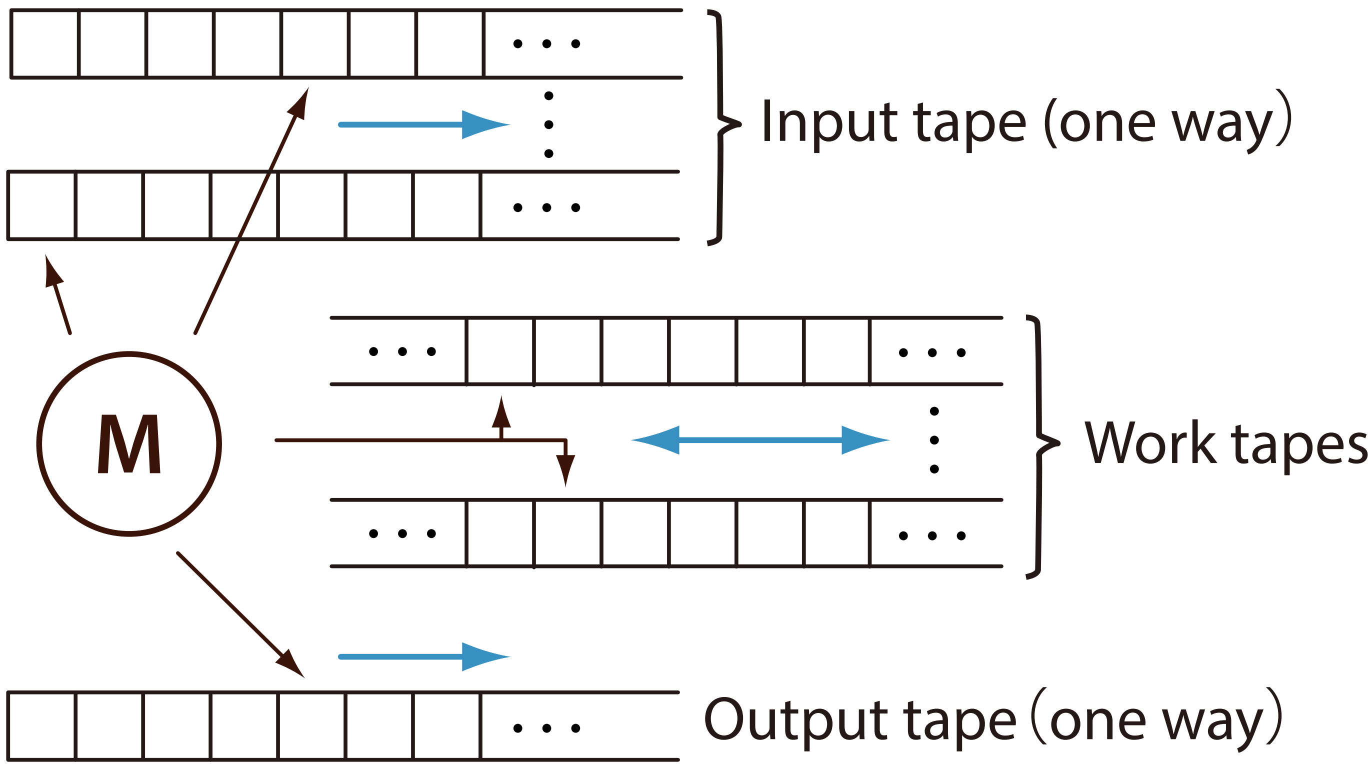


Computation via Type-2 Machine

- Computation of real numbers is realized as conversion between their representations (infinite sequences)
- Computation on infinite sequences in Σ^ω is formulated using Type-2 machine

$$\begin{tikzcd}
\Sigma^\omega \arrow[r, "g"] \arrow[d, "\xi"'] & \Sigma^\omega \arrow[d, "\zeta"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Type-2 Machine



Discretization and Learning

- In finite time, a computer (Type-2 machine) receives a finite prefix (接頭辞) of an infinite sequence that represents a real number
  - The input is thus discretized (離散化)

- A computer continues to output succeeding digits of output which is getting closer to the true value

Discretization and Learning

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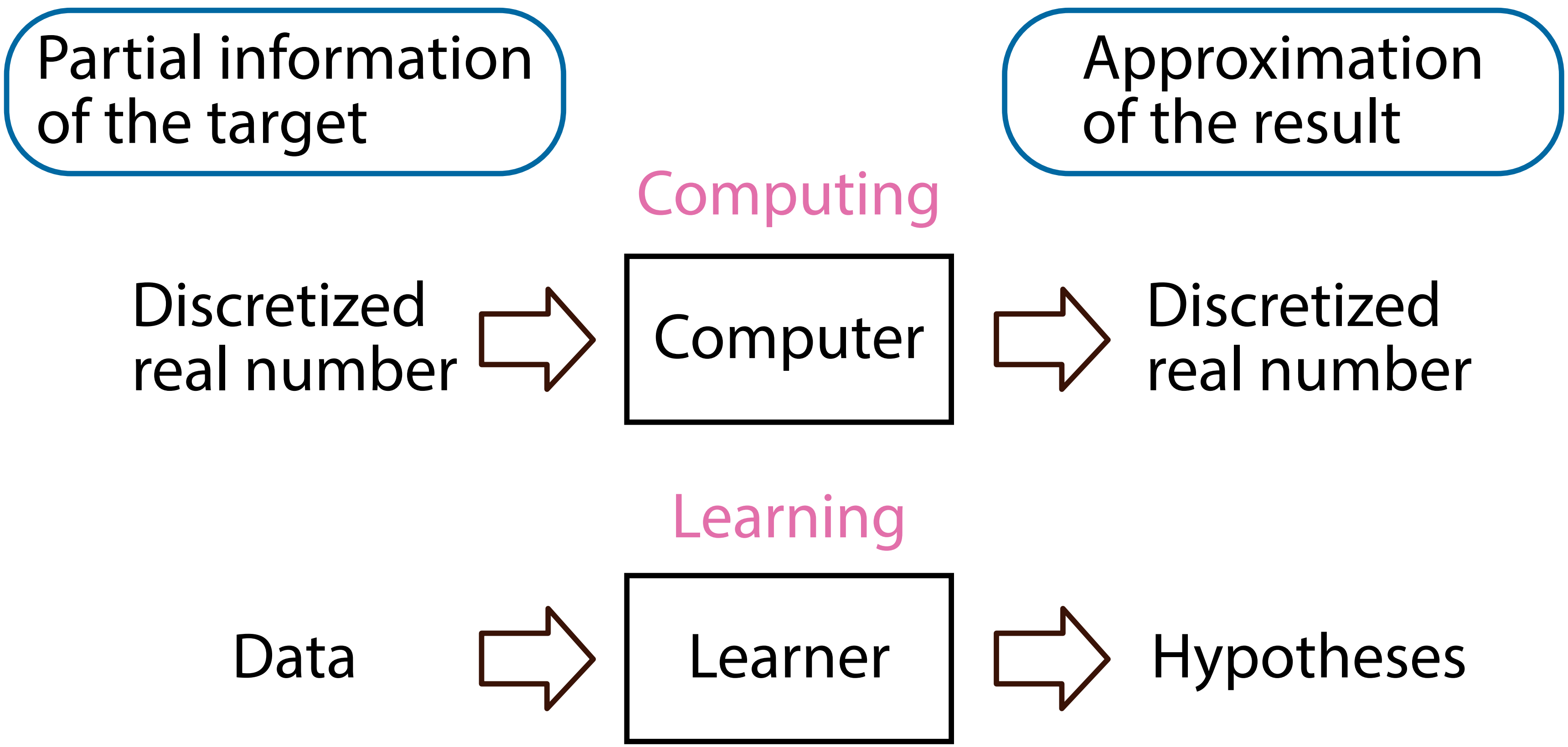
- *This is similar to the mechanism of learning*
  - Discretized approximation (in **computing**)
  - Partial information of concepts (in **learning**)

Discretization and Learning



Real Number Computation as Learning

- Concept (learning target): a real number $x \in \mathbb{R}$
- Hypothesis: a finite sequence $H = a_1 a_2 \ldots a_k$
  - A hypothesis H represents an interval $\upsilon(H)$
- Data: prefixes of $x = \rho(a_1 a_2 \ldots)$
- Correctness:
  - **Consistency**: H is always consistent with x, i.e., $x \in \upsilon(H)$
  - Instead of convergence in identification in the limit,
    we have **effectivity**:
    For a sequence of hypotheses $w_1, w_2, w_3, \ldots,$
    $\upsilon(w_i) \supseteq \upsilon(w_{i+1})$ always holds

Summary of Real Number Computation in Machine Learning Framework

Target	Real number
Representation	Gray code/signed digit representation
Data	Prefix (Discretized value, interval)
Algorithm	Depends on functions
Correctness	Consistency & Effectivity

Example: Binary Representation

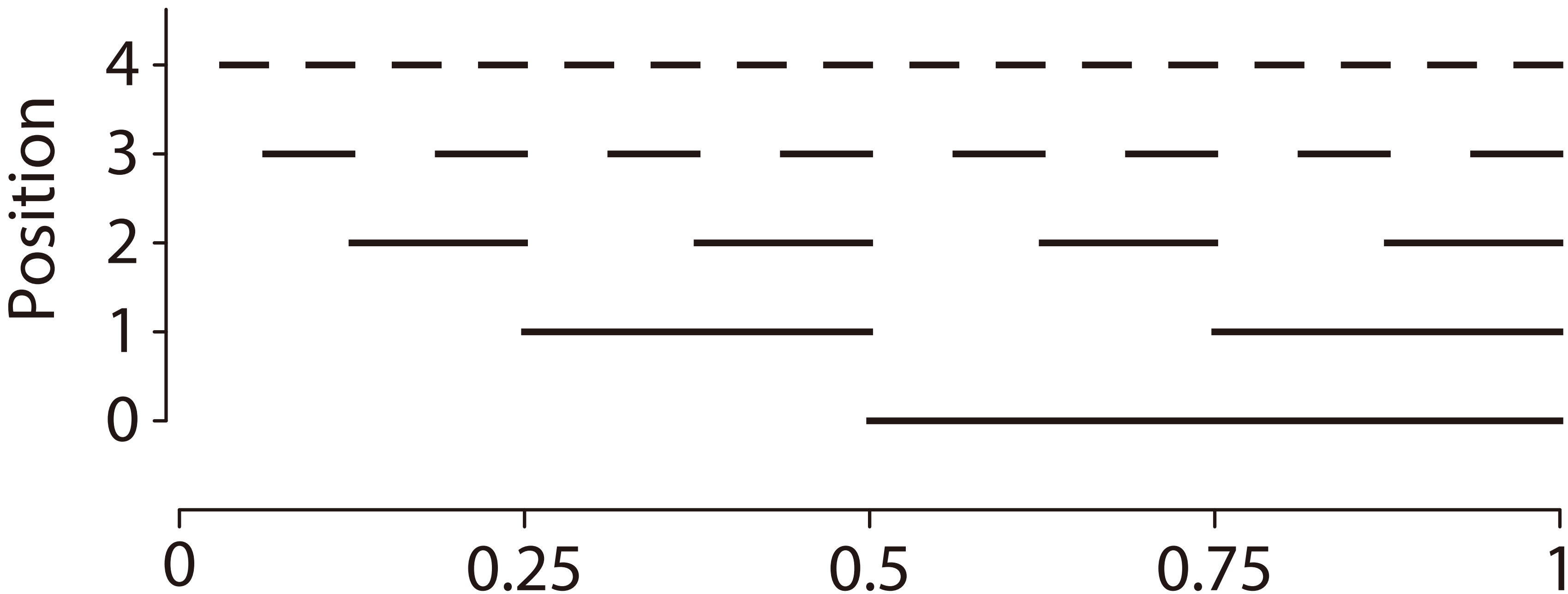
- $\Sigma = \{\mathtt{0}, \mathtt{1}\}$
- Binary representation $\rho : \Sigma^\omega \to [0, 1]$:

  $$\rho(a_1 a_2 \ldots) = \sum_{i=1}^{\infty} a_i \cdot 2^{-i}$$

- Binary representation for finite sequence
  $\upsilon : \Sigma^* \to \mathcal{P}([0, 1])$:

  $$\begin{aligned}\upsilon(a_1 a_2 \ldots a_k) &= [\, \rho(a_1 a_2 \ldots a_k \mathtt{000} \ldots),\ \rho(a_1 a_2 \ldots a_k \mathtt{111} \ldots)] \\ &= \left[\sum_{i=1}^{k} a_i \cdot 2^{-i}, \sum_{i=1}^{k} a_i \cdot 2^{-i} + 2^{-k} \right]\end{aligned}$$

Binary Representation



Example: Signed Digit Representation

- $\Sigma = \{\mathtt{0}, \mathtt{1}, \bar{\mathtt{1}}\}$
- Signed digit representation $\rho : \Sigma^\omega \to [0, 1]$:

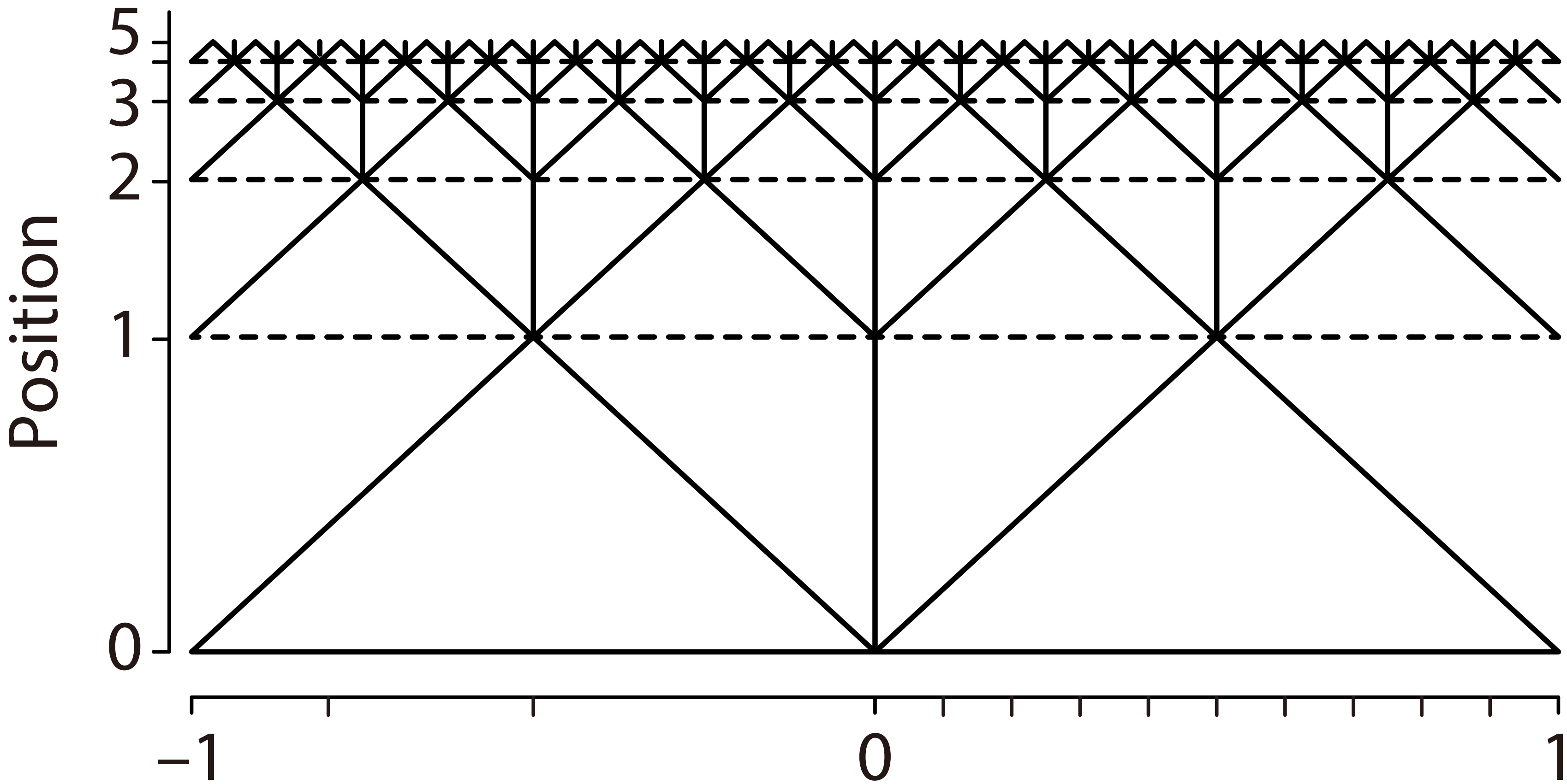
  $$\rho(a_1 a_2 \ldots) = \sum_{i=1}^{\infty} a_i \cdot 2^{-i}$$

- Signed digit representation for finite sequence $\upsilon : \Sigma^* \to \mathcal{P}([0, 1])$:

  $$\upsilon(a_1 a_2 \ldots a_k) = [\,\rho(a_1 a_2 \ldots a_k \bar{\mathtt{1}}\bar{\mathtt{1}}\bar{\mathtt{1}} \ldots),\ \rho(a_1 a_2 \ldots a_k \mathtt{1}\mathtt{1}\mathtt{1} \ldots)]$$

  $$= \left[\sum_{i=1}^{k} a_i \cdot 2^{-i} - 2^{-k}, \sum_{i=1}^{k} a_i \cdot 2^{-i} + 2^{-k}\right]$$

Signed Digit Representation



Refinement

- Refinement of signed digit representation is simple:
  1. $w \xrightarrow{\rho} w0$
  2. $w \xrightarrow{\rho} w1$
  3. $w \xrightarrow{\rho} w\bar{1}$
- "Learning with refinement"
  = "Real number computation"

Efficient Learning with Refinement

$i \leftarrow 1, S \leftarrow \varnothing, H \leftarrow \top, Q \leftarrow \varnothing$ // Q is a list of candidate hypotheses
repeat
    $S \leftarrow S \cup \{(x_i, y_i)\}$
    while H is not consistent with S
        if $x \in \upsilon(H)$ for some $(x, 0) \in S$ **then**
            Append all $\rho(H)$ to the tail of Q
        end if
        $H \leftarrow$ the first hypothesis in Q, and remove it from Q
    end while
    $H_i \leftarrow H$ and output H_i
    $i \leftarrow i + 1$
until forever

Conclusion

- Computing and learning have been studied in different fields
- However, if we consider computation over $\mathbb{R}$, there is a close connection between computing and learning
- This is still a developing field
  - No textbook!
  - Some interesting papers:
    - de Brecht, M., **Topological and Algebraic Aspects of Algorithmic Learning Theory**, *PhD thesis* (2010)
    - Sugiyama, M. and Hirowatari, E. and Tsuiki, H. and Yamamoto, A., **Learning Figures with the Hausdorff Metric by Fractals—Towards Computable Binary Classification**, *Machine Learning* (2012)

Take-Home Massages

1. Learning $\simeq$ Computing on $\mathbb{R}$ $\neq$ Computing on $\mathbb{N}$
2. Representation of objects is essential
3. Structure of hypothesis space is crucial for efficiency
4. We are learners in data mining