HW 06

1\. a) $X \cap Y$ is always countable.

Since this is the intersection of 2 sets, and $|Y| > |X|$, then $|X \cap Y| \leq |X| < |Y|$, thus based on the ordering of cardinality, it must be countable.

b) $X \cup Y$ is always uncountable.

The cardinality of the union of a set must be at least as great as the cardinality of the larger set. Thus, since Y is uncountable, $X \cup Y$ must also be uncountable.

c) $\bigcup_{i \in X} A_i$ is always uncountable.

Since $\bigcup$ is non-empty, the combined elements of one or more uncountable sets must also be uncountable since only the union of multiple countable sets can create another countable set.

d) $\bigcap_{i \in X} A_i$ is sometimes countable.

Countable Ex. $A_i = \mathbb{R}$ over $[i-1, i)$

↳ Thus, none of the A_i sets has any overlapping elements, but individually, they are still uncountable by diagonalization. Since the empty set is countable, this is valid.

Uncountable Ex. $A_i = \mathbb{R}$

↳ Then $\bigcap_{i \in X} A_i = \mathbb{R}$, and since $\mathbb{R}$ is uncountable this is an uncountable intersection

e) $\bigcup_{i \in Y} B_i$ is sometimes countable

Countable Ex. $B_i = \mathbb{N}$

↳ Thus, no matter the amount of B_i, the union is still all natural numbers, so the set is countable.

Uncountable Ex. $B_i =$ a distinct real number from $[0, 1)$

↳ Since there are as uncountable number of B_i's, we can represent the set $\mathbb{R}$ from $[0, 1)$. Thus, the union is all the real numbers from 0 to 1 exclusive, which is uncountable.

f) $\bigcap_{i \in Y} B_i$ is always countable.

Since the intersection of many countable sets cannot create a set with more elements than any individual set. Since B_i is countable, the intersection must also be countable.

2. a) Finite, all possible divisors of a finite number are less that number in magnitude.

b) Countably Infinite, all numbers are represented by $y = 8x$ where $x \in \mathbb{Z}$.
Since $y = 8x$ is invertible, it shares a bijection with $\mathbb{Z}$, which is countable.

c) Uncountably Infinite, we can use diagonalization to show that we can never enumerate every function from $\mathbb{N}$ to $\mathbb{N}$.

d) Countably Infinite, we can enumerate over all strings based on length since there are only a finite # of combinations at each length. Or we can think of them as a 26-base # system.

e) Uncountably Infinite, we are creating the power set of a countable set, which was proved in the notes to be uncountable.

f) Uncountably Infinite, we can use diagonalization to prove that we can't enumerate all possible infinite strings. If we assign the first 10 letters to a digit, this is just like the $\mathbb{R}$ proof.

3. a) Consider the program & subroutine as follows:

```
testHalt(P, x)
    return testFixed(P')

P'(y):
    if y==x
        run P(x)
        return y
    return some input not y
```

Thus testFixed(P') returns true if and only if P'(y) returns on input x, and false otherwise, which would solve the halting problem.

b) Consider the program as follows:

```
testFixed(P):
    if testFixedorNull(P) == Null:
        return False
    else:
        return True
```

Since testFixed(P) is uncomputable, we have shown that testFixedorNull is uncomputable by reduction.

c) testFixedorNull(P, x):

```
F(y):
  returns the bijective natural number for y
if testFixed&Natural(P', F(x)) == Null:
    return Null
else
    return x (since F is invertible)
```

```
P'(z):
return F(P(F^-1(x)))
```

P' essentially checks to see that P(x) returns x, thus we have proved testFixed&Natural is uncomputable by reduction.

4. a) Consider the following program & subroutine:

```
testHalt(F, x):
    return P(F', x, 0):
```

```
F'(i):
  F(x)
  return 0
```

returns true if F(x) halts, false if it doesn't, so it is a reduction from the Halting problem.

b)

```
testHalt(Q, x):
    return ¬P(F', G')
```

```
F'(y):
  loop forever
```

```
G'(y):
  if y==x:
    Q(x)
    return 0
  else:
    loop forever
```

We have defined behavior at all inputs except x. At x, if Q(x) doesn't halt, P should return true, if it does halt, it returns false by definition of P, thus we have a proof by reduction.

5. a) $3 \cdot 2^{18}$

b) $104!$

c) i. $7!/4!$

ii. $7!/4$

d) i. $5!$

ii. $15 \cdot 4!$

e) 25^8

f) $\frac{20!}{2^{10}}$ or $\binom{20}{2} * \binom{18}{2} * \cdots * \binom{2}{2}$