HW 11

1. $A^TA = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \in \mathbb{R}^{m\times n} = I_n$ $\qquad \lambda_i = \langle a_i, a_i \rangle = \|a_i\|^2$

$\sqrt{\lambda_i} = \sigma_i = \|a_i\|$

Thus, <u>C</u>

2. a) $\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix} \rangle = 5 - 1 + 4 = 0$

b) $\underbrace{\begin{bmatrix} -1/\sqrt{14} & 1/\sqrt{3} & 5/\sqrt{42} \\ 3/\sqrt{14} & 1/\sqrt{3} & -1/\sqrt{42} \\ 2/\sqrt{14} & -1/\sqrt{3} & 4/\sqrt{42} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \sqrt{14} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{42} \end{bmatrix}}_{D} = A$

c) $A = U\Sigma V^T$

$A^TA = D^TB^TBD = D^TD = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 42 \end{bmatrix}$ $\qquad \lambda_1 = 42 \quad \lambda_2 = 14 \quad \lambda_3 = 3$

$U_R = AV_R\Sigma_R^{-1}$ $\qquad V = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = V^T$ $\qquad \Sigma = \begin{bmatrix} \sqrt{42} & 0 & 0 \\ 0 & \sqrt{14} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/42 & 0 & 0 \\ 0 & 1/14 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 0 & 1/14 & 0 \\ 0 & 0 & 1/3 \\ 1/42 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} -1 & 1 & 5 \\ 3 & 1 & -1 \\ 2 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 5/\sqrt{42} & -1/\sqrt{14} & 1/\sqrt{3} \\ -1/\sqrt{42} & 3/\sqrt{14} & 1/\sqrt{3} \\ 4/\sqrt{42} & 2/\sqrt{14} & -1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 5/\sqrt{42} & -1/\sqrt{14} & 1/\sqrt{3} \\ -1/\sqrt{42} & 3/\sqrt{14} & 1/\sqrt{3} \\ 4/\sqrt{42} & 2/\sqrt{14} & -1/\sqrt{3} \end{bmatrix}$

d) $A = \begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3/\sqrt{50} & 0 \\ 0 & 1/\sqrt{50} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = (5)(\sqrt{2}) \begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix} \begin{bmatrix} 3/\sqrt{50} & 0 \\ 0 & 1/\sqrt{50} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$= \begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

e) $A^TA = \begin{bmatrix} -1 & 1 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 32 \end{bmatrix}$ $\qquad V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\qquad \Sigma = \begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$

$U = \begin{bmatrix} -1 & 4 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{32} & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{32} & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 4/\sqrt{32} & -1/\sqrt{2} \\ 4/\sqrt{32} & 1/\sqrt{2} \end{bmatrix}$

$A = \begin{bmatrix} 4/\sqrt{32} & -1/\sqrt{2} \\ 4/\sqrt{32} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

3. a) i.

b) iii.

c) $\alpha = 2$ ← check

d) iv.

4. a) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad A^TA = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{bmatrix}$

$\|A_F\| = \sqrt{\sum_{i=1}\sum_{j=1} |A_{ij}|^2}$ $\quad \sqrt{tr(A^TA)} = \sqrt{a^2+c^2+b^2+d^2}$

$= \sqrt{a^2+b^2+c^2+d^2} =$

b) $\|A_F\| = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |A_{ij}|^2}$ $\quad \|A_F^T\| = \sqrt{\sum_{j=1}^{m}\sum_{i=1}^{n} |A^T_{ji}|^2} = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |A^T_{ji}|^2}$ when $A^T_{ji} = A_{ij}$

From def. of transpose, $A_{ij} = A^T_{ji}$. so every entry present in A is also present in A^T. The sum of the squares will be the same if all the entries are the same.

c) $\|A\|_F = \sqrt{trace(A^TA)}$ $\quad \|UA\|_F = \|(UA)^T\|_F = \|A^TU^T\|_F = \|A^TV\|_F$

↳ $\|A\|_F$ from note, then $\|A^TV\|_F = \|A_F\| = \|A^T\|_F$

d) $\|U\Sigma V^T\|_F = \|\Sigma\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} |\sigma_{ij}|^2} = \sqrt{\sum_{i=1}^{n} \sigma_i^2} =$ proved