3rd Year Communications and Electronics Department

Antennas and Waveguides Engineering- ELC3050

Rectangular Cavity Resonator Assignment

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Objective:

We aim to identify the two first resonance Frequencies for a Rectangular Cavity Resonator, and calculate their respective bandwidth and Quality factor.

Resonance Frequencies

Generally the resonance frequency for TE_{mnl} or TM_{mnl} mode is:

$$f_{res,mnl} = \frac{c}{2\sqrt{\varepsilon_r}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{l}{d}\right)^2}$$

We know that the first resonance frequency is always TE_{101} mode which is the dominant mode, the resonant frequency for TE_{101} is:

$$f_{res,101} = \frac{c}{2\sqrt{\varepsilon_r}} \sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{d}\right)^2}$$

<u>But the next resonance frequency depends on the ratios between the 3 dimensions of the cavity.</u>

1-If $d \gg a, b$ and $a \gg b$ then the next resonance frequency corresponds to TE_{102} mode whose resonance frequency is

$$f_{res,102} = \frac{c}{2\sqrt{\varepsilon_r}} \sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{2}{d}\right)^2}$$

2-But if the three dimensions are close to each other, then the next mode is TE_{011} whose resonance frequency is:

$$f_{res,011} = \frac{c}{2\sqrt{\epsilon_r}} \sqrt{\left(\frac{1}{b}\right)^2 + \left(\frac{1}{d}\right)^2}$$

Quality Factor Calculations

The Quality Factor:

1-Quality Factor Due to Conductor walls losses, it depends on the mode

$$Q_{conductor} = \frac{\omega_{res} E_{stored}}{P_{loss}}$$

$$Q_{\text{conductor}} = \frac{2\omega(W_e)}{P_{\text{loss}}\ (\text{watt})} = \frac{\omega_o \varepsilon\left(\iiint \sum |E_{\text{crosssection}}|^2 dv\right)}{R_s \iint_{\text{walls}} \left|H_{\text{tangential}}\right|^2 ds}$$

2-Quality Factor Due to the Dielectric Filling: it doesn't depend on the mode

$$Q_{dielectric} = \frac{1}{\tan(\delta)}$$

So the Total Quality factor of the Cavity will be

$$Q_{total} = Q_{conuctor} || Q_{dielectric}$$

We will return to the quality factor calculations in more detail.

Bandwidth Calculations:

We know that the fractional Bandwidth $BW_{fractional} = \frac{1}{Q_{total}}$

$$\therefore BW_{fractional} = \frac{BW_{actual}}{f_{res}} = \frac{1}{Q_{total}}$$

$$\therefore BW_{actual} = \frac{f_{res}}{Q_{total}}$$

<u>Now we return to the Quality Factor Calculations:</u>

<u>Quality factor Calculations for TE_{10l} modes</u>

We get the field Expressions for the $\boldsymbol{TE_{10l}}$ mode:

$$H_z = H_z^+ e^{-j\beta z} - H_z^- e^{+j\beta z} = -2jH_{z0}\cos(k_x x)\sin(\beta z)$$
$$E_y = E_y^+ e^{-j\beta z} - E_y^- e^{+j\beta z} = -2jE_{y0}\sin(k_x x)\sin(\beta z) = -\frac{2\omega\mu}{k_x} H_{z0}\sin(k_x x)\sin(\beta z)$$
$$H_x = H_x^+ e^{-j\beta z} + H_x^- e^{+j\beta z} = 2H_{x0}\sin(k_x x)\cos(\beta z) = \frac{j2\beta}{k_x} H_{z0}\sin(k_x x)\cos(\beta z)$$

Then to get the stored Electric energy (W_e):

$$W_e = \frac{\epsilon}{4}\iiint_v |E_{crosssectional}|^2 dv$$

$$W_e = \frac{\epsilon}{4}\int_0^a\int_0^b\int_0^d \left|E_y\right|^2 dxdydz = \frac{\varepsilon abd}{16} E_0^2$$

Then to get the power losses due to conducting walls:

$$P_{loss} = \frac{R_s}{2}\iint_{walls} \left|H_{tangential}\right|^2 ds$$

$$P_{loss} = \frac{R_s}{2}\left[2\int_0^a\int_0^d (|H_z|^2 + |H_x|^2)dxdz + 2\int_0^b\int_0^d (|H_z|^2)dydz + 2\int_0^a\int_0^b (|H_x|^2)dxdy\right]$$

So we get

$$P_{loss} = \frac{R_s E_0^2 \lambda^2}{8\eta^2}\left(\frac{\ell^2 ab}{d^2} + \frac{bd}{a^2} + \frac{\ell^2 a}{2d} + \frac{d}{2a}\right)$$

So finally we get the quality factor due to conductive walls to be for TE_{10l} mode:

$$Q_{conductor} = \frac{2\omega(W_e)}{P_{loss}(\ watt)} = \frac{(kad)^3 b\eta}{2\pi^2 R_s}\frac{1}{(2\ell^2 a^3 b + 2bd^3 + \ell^2 a^3 d + ad^3)}$$

We will use this previous expression to calculate the quality factor for both TE_{101} and TE_{102} modes and hence get the BW.

Now we want to get the Q expression for the possible $\boldsymbol{TE_{011}}$ mode:

First we get the field expressions:

From the boundary conditions we get:

$$H_z^+ = -H_z^-$$

$$E_x^+ = -E_x^-$$

$$H_y^+ = H_y^-$$

So the field expressions will be:

$$H_z = H_z^+ e^{-j\beta z} - H_z^- e^{j\beta z} = 2H_{zo} cos(\frac{\pi y}{b}) sin(\frac{\pi z}{d})$$

$$E_x = E_x^+ e^{-j\beta z} - E_x^- e^{j\beta z} = \frac{2\omega\mu b}{\pi} H_{zo} sin(\frac{\pi y}{b}) sin(\frac{\pi z}{d})$$

$$H_y = H_y^+ e^{-j\beta z} + H_y^- e^{j\beta z} = \frac{2j\beta b}{\pi} H_{z0} sin(\frac{\pi y}{b}) cos(\frac{\pi z}{d})$$

Now we will go through the same steps we went through with TE_{10l} calculations:

To get the stored Electric energy (W_e):

$$W_e = \frac{\varepsilon}{4} \iiint_v |E_{crosssectional}|^2 dv$$

$$W_e = \frac{\varepsilon}{4} \int_0^a \int_0^b \int_0^d |E_x|^2 dxdydz = \frac{\omega^2 \mu^2 \varepsilon}{4\pi^2} a\, b^3 d\, H_{zo}^2$$

To get the power loss due to the conducting walls:

$$P_{loss} = \frac{R_s}{2} \iint_{walls} \left|H_{tangential}\right|^2 ds$$

$$P_{loss} = \frac{R_s}{2} \left[2\int_0^a \int_0^d |H_z|^2 dxdz + 2\int_0^b \int_0^d \left[|H_z|^2 + \left|H_y\right|^2\right] dydz + 2\int_0^a \int_0^b \left|H_y\right|^2 dxdy \right]$$

$$P_{loss} = R_s \left[2ad + bd + \frac{b^3}{d} + \frac{2b^3 a}{d^2}\right] H_{zo}^2$$

So finally we get the Quality factor due to conductive walls for TE_{011} mode to be:

$$Q_{conductor} = \frac{2\omega(W_e)}{P_{\text{loss}}\,(\text{ watt })} = \frac{\omega^3 \mu^2 \varepsilon}{2\pi^2 R_s} \frac{ab^3 d}{2ad + bd + \frac{b^3}{d} + \frac{2b^3 a}{d^2}}$$

Now we have the Quality factor due to conductive walls for all the possible modes TE_{101}, TE_{102}, TE_{011}

The other component of the quality factor is due to the dielectric filling which doesn’t depend on the mode and only depends on the dielectric filling and its loss tangent.

$$Q_{dielectric} = \frac{1}{\tan(\delta)}$$

And the total Quality factor for every mode will be:

$$Q_{total} = Q_{dielectric} || Q_{conductor}$$

We can then easily calculate the bandwidth as previously stated:

$$BW = \frac{f_{res}}{Q_{total}}$$

Example:

Problem (3)
A rectangular cavity having $a = 2$ cm, $b = 1$ cm and a length of 6 cm is filled with a dielectric of relative permittivity $\varepsilon_r = 2.5 - j0.0001$ at the resonant frequency of the TE_{101} mode. The cavity is made of copper ($\sigma = 5.8 \times 10^7 \text{mho/m}$). Find the quality factor for the TE_{101} mode.

From the previous givens we get:

$$\varepsilon_r = 2.5, \qquad \tan(\delta) = \frac{0.0001}{2.5} = 4 \times 10^{-5}, \sigma = 5.8 \times 10^7, a = 2\ cm, b = 1\ cm, d = 6\ cm$$

By inputting the previous parameters to our code we get:

```
tan(δ) : 4*10^-5

Enter The Dielectric Constant :2.5

σ : 5.8*10^7

a in Cm : 2

b in Cm : 1

d in Cm : 6

First Resonance Frequency corresponds to TE101 Mode

First Resonance Frequency = 5000000000 Hz

Its Quality Factor= 4.531296e+03

Its Fractional Bandwidth= 2.206874e-02 %

Its Actual Bandwidth : 1.103437e+06 Hz

Next Resonance frequency corresponds to TE102 Mode

Second Resonance Frequency = 5.700877e+09 Hz

Its Quality Factor = 5.073636e+03

Its Fractional Bandwidth : 1.970973e-02 %

Its Actual Bandwidth : 1.123628e+06 Hz
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Summary of formulae:

For TE_{101} mode:

$$f_{res,101} = \frac{c}{2\sqrt{\varepsilon_r}}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{d}\right)^2}$$

$$Q_{conductor} = \frac{(kad)^3 b\eta}{2\pi^2 R_s}\frac{1}{(2a^3b + 2bd^3 + a^3d + ad^3)}$$

For TE_{102} mode:

$$f_{res,102} = \frac{c}{2\sqrt{\varepsilon_r}}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{2}{d}\right)^2}$$

$$Q_{conductor} = \frac{(kad)^3 b\eta}{2\pi^2 R_s}\frac{1}{(8a^3b + 2bd^3 + 4a^3d + ad^3)}$$

For TE_{011} mode:

$$f_{res,011} = \frac{c}{2\sqrt{\varepsilon_r}}\sqrt{\left(\frac{1}{b}\right)^2 + \left(\frac{1}{d}\right)^2}$$

$$Q_{conductor} = \frac{\omega^3\mu^2\varepsilon}{2\pi^2 R_s}\frac{ab^3d}{2ad + bd + \frac{b^3}{d} + \frac{2b^3a}{d^2}}$$

For all modes:

$$Q_{dielectric} = \frac{1}{\tan(\delta)}$$

$$Q_{total} = Q_{dielectric}||Q_{conductor}$$

$$BW_{fractional} = \frac{1}{Q_{total}}$$

$$BW_{actual} = \frac{f_{res}}{Q_{total}}$$

The used code:

```
%Defining the Speed of light and Constants
c=3*(10^8);
Muo=4*pi*10^-7;
eo=(10^-9)/(36*pi);
%***************Taking Cavity Parameters as input************%
%Enter the loss tangent and Calculating Qdielectric%
tandel=input("tan(?) : ");
Qd=1/tandel;
%Enter the Dielectric Constant%
er=input("Enter The Dielectric Constant :");
cr=c/sqrt(er);
%Enter The Conductivity of the walls Metal%
cond=input("? : ");
%Taking Cavity Dimensions as input
a=input("a in Cm : ")*(10^-2);
b=input("b in Cm : ")*(10^-2);
d=input("d in Cm : ")*(10^-2);
%************************************************************%
%***************Calculating for the first resonance
Frequency*************%
%Calculating First Res Freq TE101 and its Qulaity Factor%
fc_101=calcResFreq(a,b,d,1,0,1,cr);
fprintf("First Resonance Frequency corresponds to TE101 Mode\n");
fprintf("First Resonance Frequency = %d Hz\n",fc_101);
k_101=2*pi*fc_101/cr;
Eta=sqrt(Muo/(eo*er));
Rs_101=sqrt(pi*fc_101*Muo/cond);
Q_101=((k_101*a*d)^3)*b*Eta/((2*Rs_101*(pi^2))*(2*(a^3)*b+2*b*(d^3)+(a^3)*d
+a*(d^3)));
Q=(Qd*Q_101)/(Qd+Q_101);
fprintf("Its Quality Factor= %d\n",Q);
fprintf("Its Fractional Bandwidth= %d %% \n",100/Q);
fprintf("Its Actual Bandwidth : %d Hz\n", fc_101/Q);
%Calculating for TE102%
fc_102=calcResFreq(a,b,d,1,0,2,cr);
k_102=2*pi*fc_102/cr;
Rs_102=sqrt(pi*fc_102*Muo/cond);
Q_102=((k_102*a*d)^3)*b*Eta/((2*Rs_102*(pi^2))*(8*(a^3)*b+2*b*(d^3)+4*(a^3)
*d+a*(d^3)));
%Calculating for TE011%
fc_011=calcResFreq(a,b,d,0,1,1,cr);
w_011=2*pi*fc_011;
Rs_011=sqrt(pi*fc_011*Muo/cond);
Q_011=(w_011^3)*(Muo^2)*eo*er*(b^3)*a*d/(2*(pi^2)*Rs_011*(2*a*d+b*d+(b^3)/d
+2*(b^3)*a/(d^2)));
if (fc_102<=fc_011)
    %The next resonance is TE102%
    fprintf("Next Resonance frequency corresponds to TE102 Mode\n");
    fprintf("Second Resonance Frequency = %d Hz\n",fc_102);
    Q=(Qd*Q_102)/(Qd+Q_102);
    fprintf("Its Quality Factor = %d\n",Q);
    fprintf("Its Fractional Bandwidth : %d %% \n", 100/Q);
    fprintf("Its Actual Bandwidth : %d Hz\n", fc_102/Q);
else
    %The next resonance is TE011%
    fprintf("Next Resonance frequency corresponds to TE011 Mode\n");
    fprintf("Second Resonance Frequency = %d Hz\n",fc_011);
    Q=(Qd*Q_011)/(Qd+Q_011);
    fprintf("Its Quality Factor = %d\n",Q);
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    fprintf("Its Fractional Bandwidth : %d %% \n", 100/Q);
    fprintf("Its Actual Bandwidth : %d Hz\n", fc_011/Q);
end
function fc = calcResFreq(a,b,d,m,n,l,cr)
kx=(m*pi/a);
ky=(n*pi/b);
B=(l*pi/d);
k=sqrt((kx^2)+(ky^2)+(B^2));
fc=cr*k/(2*pi);
end
```

References

DAVID M. POZAR, "Microwave Resonators," in *Microwave engineering*, S.l.: JOHN WILEY & SONS, 2021, pp. 284–288.