Euclidean Geometry

1. Basic Geometric Entities and Properties

- **Points and Coordinates**: A point in 2D is represented as (x, y). The distance between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by the Euclidean distance formula:

$$\text{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- **Lines**: A line can be represented in several forms:
  - *Slope-intercept form*: $y = mx + c$, where m is the slope and c is the y-intercept.
  - *General form*: $ax + by + c = 0$.
  - Slope of a line through points (x_1, y_1) and (x_2, y_2): $m = \frac{y_2 - y_1}{x_2 - x_1}$.
  - Parallel lines have equal slopes; perpendicular lines have slopes $m_1 \cdot m_2 = -1$.

- **Distance from a Point to a Line**: For a point $P(x_0, y_0)$ and line $ax + by + c = 0$, the distance is:

$$\text{Distance} = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

2. Vectors and Dot/Cross Products

- **Vector Representation**: A vector from point $A(x_1, y_1)$ to $B(x_2, y_2)$ is $\vec{AB} = (x_2 - x_1, y_2 - y_1)$.

- **Dot Product**: For vectors $\vec{A} = (a_1, a_2)$ and $\vec{B} = (b_1, b_2)$, the dot product is:

$$\vec{A} \cdot \vec{B} = a_1 b_1 + a_2 b_2$$

  The angle θ between vectors satisfies:

$$\cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|}$$

- **Cross Product Magnitude (2D)**: For vectors $\vec{A}$ and $\vec{B}$, the cross product magnitude is:

$$|\vec{A} \times \vec{B}| = a_1 b_2 - a_2 b_1$$

  This gives the area of the parallelogram formed by the vectors. The sign indicates orientation (positive for counter-clockwise, negative for clockwise).

3. Triangles

- **Area of a Triangle**:
  - Using base and height: Area $= \frac{1}{2} \cdot \text{base} \cdot \text{height}$.
  - Using coordinates of vertices $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ (Shoelace formula):

$$\text{Area} = \frac{1}{2} \left| x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \right|$$

  - Using sides a, b, c and semi-perimeter $s = \frac{a+b+c}{2}$ (Heron's formula):

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

- **Pythagorean Theorem**: In a right triangle with legs a, b and hypotenuse c:

$$a^2 + b^2 = c^2$$

- **Law of Cosines**: For a triangle with sides a, b, c and angle C opposite side c:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

- **Law of Sines**: For sides a, b, c and opposite angles A, B, C:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

4. Circles

- **Equation of a Circle**: With center (h, k) and radius r:

$$(x-h)^2 + (y-k)^2 = r^2$$

- **Circle-Line Intersection**: For a line $ax + by + c = 0$ and circle $(x-h)^2 + (y-k)^2 = r^2$, solve by substituting the line equation into the circle equation, forming a quadratic equation in one variable.
- **Circle-Circle Intersection**: For two circles with centers (h_1, k_1), (h_2, k_2) and radii r_1, r_2, find intersection points by solving the system of their equations. The distance between centers d determines the number of intersections:
  - $d > r_1 + r_2$: No intersection.
  - $d = r_1 + r_2$: One intersection (tangent externally).
  - $|r_1 - r_2| < d < r_1 + r_2$: Two intersections.

5. Polygons

- **Area of a Polygon**: For a polygon with vertices $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, use the Shoelace formula:

$$\text{Area} = \frac{1}{2} \left| \sum_{i=1}^{n-1} (x_i y_{i+1} - y_i x_{i+1}) + (x_n y_1 - y_n x_1) \right|$$

- **Convex Polygon**: A polygon is convex if all interior angles are less than $180°$. Check convexity by ensuring the cross product of consecutive edges has consistent sign.
- **Point in Polygon**: Use the ray-casting algorithm to determine if a point lies inside a polygon by counting intersections of a ray from the point with polygon edges.

6. Angles and Trigonometry

- **Angle Between Two Lines**: For lines with slopes m_1 and m_2, the angle θ is:

$$\tan\theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

- **Rotation of a Point**: To rotate point (x, y) around the origin by angle θ:

$$x' = x\cos\theta - y\sin\theta, \quad y' = x\sin\theta + y\cos\theta$$

7. Computational Geometry Techniques

- **Line Segment Intersection**: Check if two line segments AB and CD intersect by computing the orientations of points using cross products. Segments intersect if A and B lie on opposite sides of line CD, and C and D lie on opposite sides of line AB.
- **Convex Hull**: Use algorithms like Graham's scan or Andrew's monotone chain to find the smallest convex polygon containing all points. Time complexity: $O(n \log n)$.
- **Closest Pair of Points**: Use a divide-and-conquer approach to find the closest pair of points in a set in $O(n \log n)$ time.

8. Numerical Precision

- **Floating-Point Issues**: Use a small epsilon (e.g., 10^{-9}) for comparing floating-point numbers to handle precision errors.
- **Integer Coordinates**: When possible, work with integer coordinates to avoid precision issues, especially for distances and areas.

9. Common Problem Types in CP

- **Intersection Problems**: Compute intersections of lines, segments, or circles.
- **Area and Perimeter**: Calculate areas of triangles, polygons, or regions defined by intersections.
- **Distance Problems**: Find minimum or maximum distances between points, lines, or geometric objects.
- **Angle Calculations**: Compute angles between lines or vectors for orientation or rotation tasks.
- **Convex Hull and Containment**: Solve problems involving point inclusion or minimum enclosing shapes.