ECEN 227 - Introduction to Finite Automata and Discrete Mathematics

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Talk Overview

Outline

Introduction

Set

A set is a collection of objects.

Elements

The objects in a set are called elements.

Ex.

$$A = \{1, 5, 3, 9\}$$

- We call the previous statement as roster notation.

Introduction

The set N | The set F | The set M

N = { 2, 4, 6, 10 }

This set has four real-number elements

F = { Watermelon, Strawberry, Banana }

This set has three fruit elements

M = { 2, Strawberry, Monkey }

Set elements may be of different varieties

Empty and Null Sets

Empty set

The set with no elements is called the empty set and is denoted by the symbol ϕ.

Null set

The empty set is sometimes referred to as the null set and can also be denoted by $\{\}$.

Ex.

- $A = \{\}$
- $B = \phi$

Finite and Infinite Sets

Finite set

A finite set has a finite number of elements.

Infinite set

An infinite set has an infinite number of elements.

Ex.

- $B = \{1, 3, 5, \ldots, 99\}$ finite set
- $C = \{3, 6, 9, 12, ...\}$ infinite set

Set Cardinality

Set Cardinality

The cardinality of a finite set A, denoted by $|A|$, is the number of elements in A.

Ex.

- $A = \{1, 3, 5, 9\}$ $\quad |A| = 4$
- $B = \{1, 3, 5, \ldots, 99\}$ $\quad |B| = 50$

Belonging

- The symbol $\in$ is used to indicate that an element is in a set.
- The symbol $\notin$ indicates that an element is not in a set.

Ex.

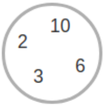
$$A = \{1, 4, 7\}$$

- $1 \in A$
- $2 \notin A$

Note that, capital letters will be used as variables denoting sets, and lower case letters will be used for elements in the set.

Example

The set A



$A = \{ 2, 3, 6, 10 \}$
$\quad = \{ 3, 2, 10, 6 \}$

Order does not matter in listing elements

$|A| = 4$

|A| is the cardinality of A, which is the number of elements in A

The cardinality is finite $\Rightarrow$ A is finite set

$2 \in A$
$5 \notin A$

$\in$ indicated that an element is in a set
$\notin$ indicates that an element is *not* in a set

The empty set

$\varnothing = \{ \}$

The empty set has no elements and is denoted $\varnothing$

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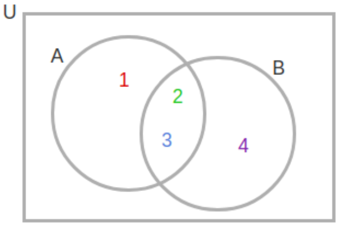
Mathematical Sets

Set	Symbol	Examples of elements
N is the set of ***natural numbers***, which includes all integers greater than or equal to 0.	**N**	0, 1, 2, ...
Z is the set of all integers.	**Z**	..., -2, -1, 0, 1, 2, ...
Q is the set of ***rational numbers***, which includes all real numbers that can be expressed as a/b, where a and b are integers and b ≠ 0.	**Q**	0, 1/2, 5.23, -5/3
R is the set of real numbers.	**R**	0, 1/2, 5.23, -5/3, π, $\sqrt{2}$

Venn Diagram

Venn Diagram

A Venn diagram is a drawing illustration of the relationships between and among sets.



$A = \{1, 2, 3\}$

$1 \in A$ $\quad$ $4 \notin A$

$2 \in A$

$3 \in A$

$B = \{2, 3, 4\}$

Note That

The universal set, usually denoted by the variable U, is a set that contains all elements in Venn Diagram.

Set Builder Notation

- A set is defined by specifying that the set includes all elements in a larger set that also satisfy certain conditions.

Ex.

$$C = \{x \in Z : \quad 0 < x < 100 \text{ and x is prime}\}$$

- The colon symbol ":" is read "such that".
- The description for C above would read:

  "C includes all x in integers such that $0 < x < 100$ and x is prime".

Subset and Proper Subset

Subset

If every element in B is also an element of A, then B is a subset of A, denoted as $B \subseteq A$.

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Proper Subset

If $B \subseteq A$ and there is an element of A that is not an element of B (i.e., $B \neq A$), then B is a proper subset of A, denoted as $B \subset A$.

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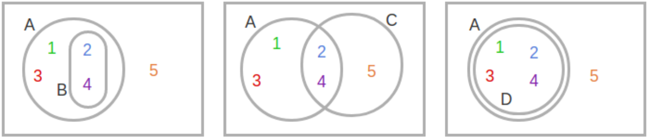

$A = \{1, 2, 3, 4\}$

$B = \{2, 4\}$

$B \subseteq A$

$3 \in A \quad 3 \notin B$

$B \subset A$

$A = \{1, 2, 3, 4\}$

$C = \{2, 4, 5\}$

$5 \in C \quad 5 \notin A$

$C \nsubseteq A$

$A = \{1, 2, 3, 4\}$

$D = \{1, 2, 3, 4\}$

$A \subseteq D, \; D \subseteq A \Rightarrow A = D$

Excercise

Which of the following statements are always true for any two sets A and B?

- If $A \subset B$, then $A \subseteq B$.

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  - False
- If $A \subset B$, then $A \neq B$.

Excercise

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    - True
- If $A = B$, then $A \subseteq B$.
    - True
- If $A = B$, then $A \subset B$.
    - False
- If $A \subset B$, then $A \neq B$.
    - True

Outline

Set of sets

- It is possible that the elements of a set are themselves sets.

Ex.

$$A = \{\{1,2\}, \phi, \{1,2,3\}, \{1\}\}$$

Mark as True or False

- $\{1,2\} \in A$

Set of sets

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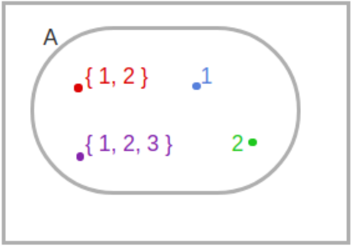
- $\{1,2\} \in A$
  - True
- $\{1\} \in A$
  - True
- $1 \in A$
  - False
- $\{1\} \subseteq A$
  - False

Set of Sets



$A = \{\{1, 2\}, 1, 2, \{1, 2, 3\}\}$

$\{1, 2\} \in A$

$1 \in A$ $\quad |A| = 4$

$2 \in A$

$\{1, 2, 3\} \in A$

The cardinality of set A = { {1, 2}, 1, 2, {1, 2, 3} } is 4. The elements are {1, 2}, 1, 2, and {1, 2, 3}.

Power Set

Power Set

The power set of a set A, denoted P(A), is the set of all subsets of A. For example, if $A = \{1, 2, 3\}$, then:

$$\mathrm{P(A)} = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

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Ex.

A = { ○, □, △ }
List all subsets:

size 0 { ∅ ,

size 1 {○} , {□} , {△} ,

size 2 {○, □} , {○, △} , {□, △} ,

size 3 {○, □, △} } = P(A) (power set of A)

P(A) = { ∅ , {○} , {□} , {△} , {○, □} , {○, △} , {□, △} , {○, □, △} }

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P(A) = { ∅, {○}, {□}, {△}, {○, □}, {○, △}, {□, △}, {○, □, △} }

Can you guess the cardanality of the power set?

Cardinality of Power Set

Theorem

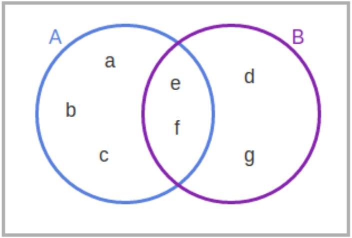
Let A be a finite set of cardinality n. Then the cardinality of the power set of A is 2^n, or $|P(A)| = 2^n$.

Ex. What is the cardinality of P($\{1, 2, 3, 4, 5, 6\}$)?

Outline

Intersetion Operation

- The intersection of A and B, denoted $A \cap B$ and read "A intersect B",
- It is the set of elements that are elements of both A and B.



A = { a, b, c, e, f }

B = { d, e, f, g }

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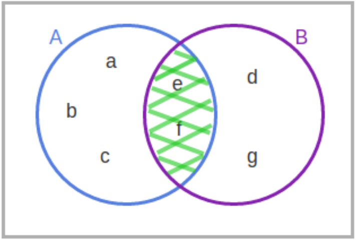

$A = \{a, b, c, e, f\}$

$B = \{d, e, f, g\}$

$A \cap B = \{e, f\}$

Union Operation

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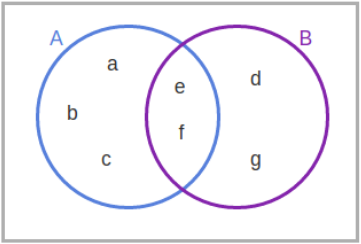

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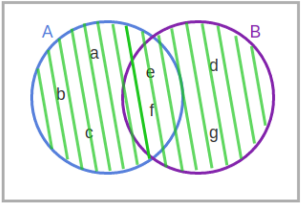

A = { a, b, c, e, f }
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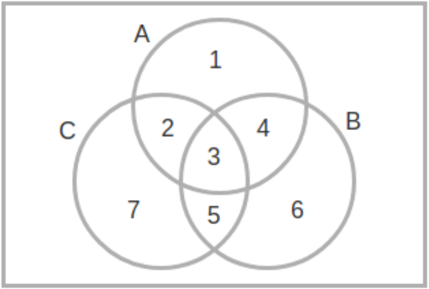
$A \cup B$ = { a, b, c, e, f, d, g }

Excercise 1

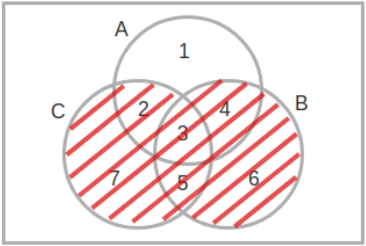


A = { 1, 2, 3, 4 }

B = { 3 , 4 , 5 , 6 }

C = { 2 , 3 , 5 , 7 }

Excercise 1



$A = \{1, 2, 3, 4\}$

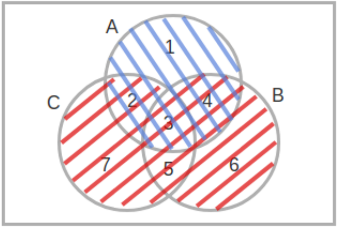
$B = \{3, 4, 5, 6\}$

$C = \{2, 3, 5, 7\}$

$A \cap \underbrace{(B \cup C)}$

$B \cup C = \{2, 3, 4, 5, 6, 7\}$

Excercise 1



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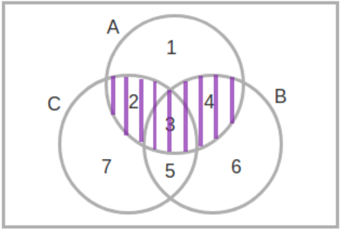

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$\underbrace{A}\cap\underbrace{(B \cup C)}$

$B \cup C = \{2, 3, 4, 5, 6, 7\}$

$A = \{1, 2, 3, 4\}$

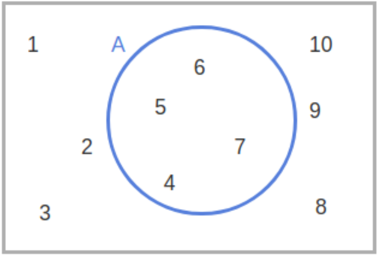
$A \cap (B \cup C) = \{2, 3, 4\}$

Outline

Set Complement

- The complement of a set A, denoted $\overline{A}$, is the set of all elements in U that are not elements of A.
- An alternative definition of $\overline{A}$ is U - A.

Ex.



U = { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 }

A = { 4, 5, 6, 7 }

The universal set U is {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. The set A is {4, 5, 6, 7}.

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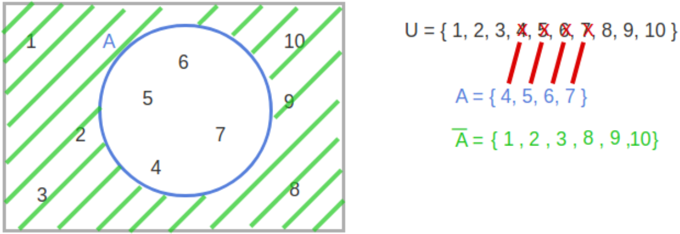


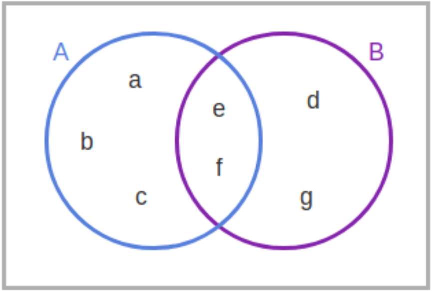
The complement of A is found by removing the elements of A from U. Therefore, the complement of A is {1, 2, 3, 8, 9, 10}.

Outline

Set Difference

- The difference between two sets A and B, denoted A - B, is the set of elements that are in A but not in B.

Ex.



A = { a, b, c, e, f }

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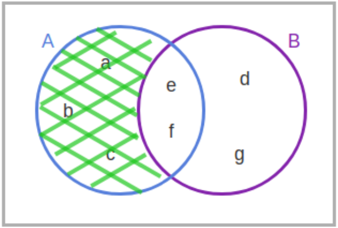


$A = \{ a, b, c, \cancel{e}, \cancel{f} \}$

$B = \{ d, e, f, g \}$

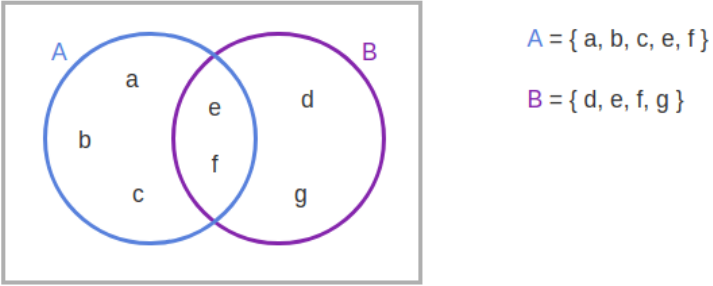
$A - B = \{ a, b, c \}$

To determine A - B, find the elements that are in both A and B (e and f) and remove those elements from A. A - B = {a, b, c}.

Symmetric Difference

- The symmetric difference between two sets A and B, denoted $A \oplus B$, is the set of elements that are a member of exactly one of A and B but not both.

Ex.



The set A is {a, b, c, e, f} and the set B is {d, e, f, g}.

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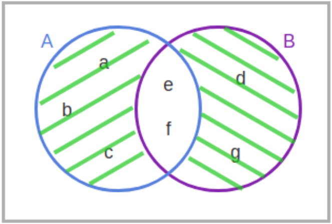


$A = \{a, b, c, \cancel{e}, \cancel{f}\}$

$B = \{d, \cancel{e}, \cancel{f}, g\}$

$A \oplus B = \{a, b, c, d, g\}$

To determine $A \oplus B$, remove the elements that are in both A and B (e and f) and take the remaining elements that are in A or B. $A \oplus B = \{a, b, c, d, g\}$

Notes on Set Difference

- The difference operation is not commutative. A - B $\neq$ B - A.
- The symmetric difference is commutative. A $\oplus$ B = B $\oplus$ A.
- An alternative definition of the set difference operation is:

$$A - B = A \cap \overline{B}$$

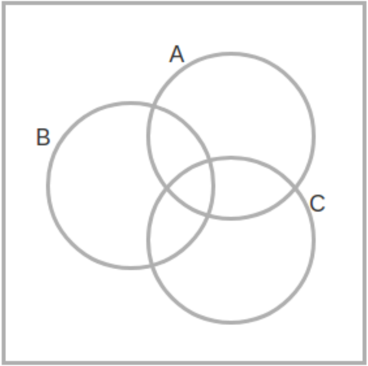
- An alternative definition of the symmetric difference operation is:

$$A \oplus B = (A - B) \cup (B - A)$$

Operations Summary

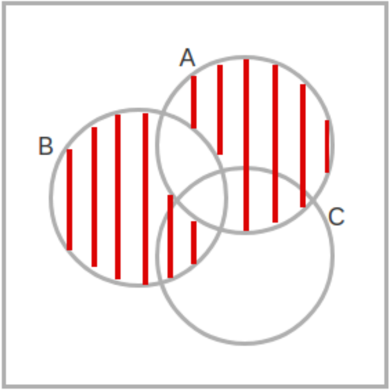
Operation	Notation	Description
Intersection	$A \cap B$	$\{ x : x \in A \text{ and } x \in B \}$
Union	$A \cup B$	$\{ x : x \in A \text{ or } x \in B \text{ or both} \}$
Difference	$A - B$	$\{ x : x \in A \text{ and } x \notin B \}$
Symmetric difference	$A \oplus B$	$\{ x : x \in A - B \text{ or } x \in B - A \}$
Complement	$\overline{A}$	$\{ x : x \notin A \}$

Excercise 1



$(\overline{A \oplus B}) \cap C$

Excercise 1



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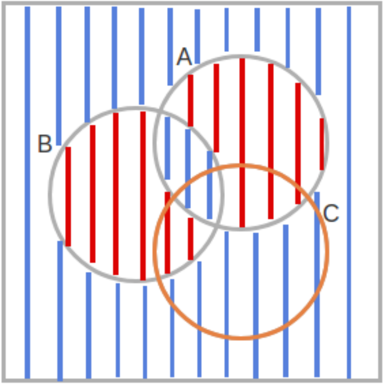
$A \oplus B$

Excercise 1



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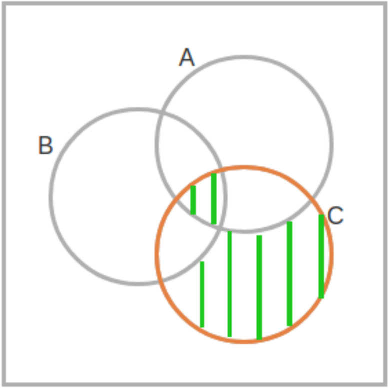

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$$(\overline{A \oplus B}) \cap C$$

Excercise 2

Sets E through H are defined as follows.

- E = $\{x \in Z\text{: } x \text{ is odd}\}$
- F = $\{x \in Z^+\text{: } x \leq 7\}$
- G = $\{x \in Z\text{: } x < 7\}$
- H = $\{x \in Z^+\text{: } x \leq 6\}$

Indicate whether each statement is true or false.

Ex.

- $G \subseteq H$

Excercise 2

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Ex.

- $G \subseteq H$
  - False
- $E \cup F \subseteq \mathbf{R}$

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Indicate whether each statement is true or false.

Ex.

- $G \subseteq H$
  - False
- E ∪ F ⊆ **R**
  - True
- $\{\{0\}\} \subseteq P(G)$

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  - True
- $\{\{0\}\} \subseteq P(G)$
  - True
- $\{0\} \subseteq P(G)$
  - False

Outline

Ordered Pair

Ordered Pair

An ordered pair of elements is written (x, y) where the order of elements matters.

Notes

- $(x, y) \neq (y, x)$ unless $x = y$.
- By contrast, $\{x, y\} = \{y, x\}$.
- An ordered list of n items is called an ordered n-tuple.

Ordered Pair

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An ordered pair of elements is written (x, y) where the order of elements matters.

Notes

- $(x, y) \neq (y, x)$ unless $x = y$.
- By contrast, $\{x, y\} = \{y, x\}$.
- An ordered list of n items is called an ordered n-tuple.

  Ex.

  - (w, x, y, z) is an ordered 4-tuple.
  - (u, w, x, y, z) is an ordered 5-tuple.

Cartesian product

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Cartesian product of A and B, denoted A x B, is the set of all ordered pairs in which the first entry is in A and the second entry is in B.

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$$A \times B = \{ (a, b) : a \in A \text{ and } b \in B \}$$

Notes

- A x B is the same as B x A, unless A = B.
- If A and B are finite sets, then $|A \times B| = |A| \, . \, |B|$

Finite Sets Cartesian Product

$B = \{a, b, c\}$

$A = \{1, 2\}$

	a	b	c
1	(1 , a)	(1 , b)	(1 , c)
2	(2 , a)	(2 , b)	(2 , c)

$A \times B =$
$\{(1, a), (1, b), (1, c)\}$
$(2, a), (2, b), (2, c))$

Finite Sets Cartesian Product

A = { 1, 2 }

B = { a, b, c }

	1	2
a	(a , 1)	(a , 2)
b	(b , 1)	(b , 2)
c	(c , 1)	(c , 2)

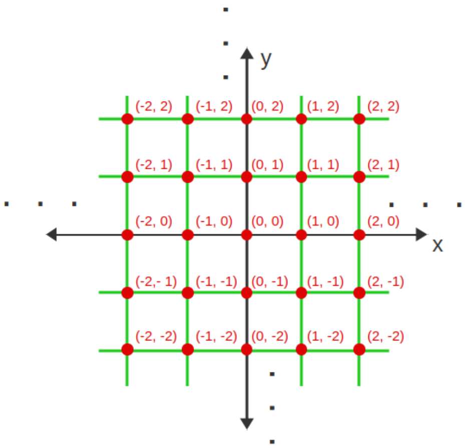
B x A

{(a , 1) , (a , 2) ,
(b , 1) , (b , 2) ,
(c , 1) , (c , 2) }

InFinite Sets Cartesian Product

Z = the set of all integers

Z x Z = { (x, y): x and y are integers }



The set $Z \times Z$ forms an infinite grid of points when plotted on the x-y plane.

Self Cartesian Product

- A × A ≡ A^2 or more generally:

$$A^k = \underbrace{A \times \cdots \times A}_{\text{k times}}$$

Ex.

- if A $= \{0, 1\}$
- $A^k = \{0, 1\}^3 = \{ (0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1) \}$

Strings

- If A is a set of symbols or characters, then A^n can be written without parentheses and commas (i.e., called string).

Ex.

Strings

- If A is a set of symbols or characters, then A^n can be written without parentheses and commas (i.e., called string).

Ex.

- $\{0,1\}^3$ is 3-bit binary string "000" to "111".
- $\{0,1\}^n$ is n-bit binary string.

Excercise

Given the following sets express the result as strings.

- A $=$ {a}
- B $=$ {b, c}
- C $=$ {a, b, d}

Questions

- $A \times (B \cup C)$

Excercise

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  - {aa, ab, ac, ad}
- $(A \times B) \cup (A \times C)$

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  - {aa, ab, ac, ad}
- $(A \times B) \cup (A \times C)$
  - {aa, ab, ac, ad}
- $P(A \times B)$
  - { ϕ, {ab}, {ac}, {ab,ac} }

Outline

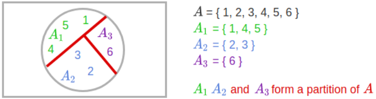
Partitions

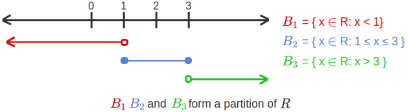
Disjoint Sets

Two sets, A and B, are said to be disjoint if their intersection is empty
$$(A \cap B = \phi).$$

- $A_1, A_2, \ldots, A_n$ is a partition for a non-empty set A if all of the following conditions hold:

- $A = A_1 \cup A_2 \cup \cdots \cup A_n$.
- For all i, $A_i \subseteq A$.
- For all i, $A_i \neq \phi$
- $A_1, A_2, \ldots, A_n$ are pairwise disjoint.





Excercise

Let sets A through F be defined as follows.

- $A = \{000\}$
- $B = \{111\}$
- $C = \{0x : x \in \{0,1\}^2\}$
- $D = \{01x : x \in \{0,1\}\}$
- $E = \{1x : x \in \{0,1\}^2\}$
- $F = \{00x : x \in \{0,1\}\}$

What are the partitions of the set $\{0,1\}^3$ using one or more of the sets defined above?

Excercise

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Sol:

- C, E
- E, D, F

Questions