ECEN 227 - Introduction to Finite Automata and Discrete Mathematics

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Talk Overview

1. Introduction
2. Floor and Cieling
3. Function Properties
4. Function Inverse
5. Function Decomposition

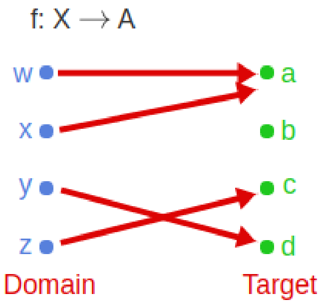
Outline

Introduction

Function

A function f that maps elements of a set X to elements of a set Y, is a subset of X $\times$ Y such that for every $x \in X$, there is exactly one y $\in$ Y for which (x, y) $\in$ f

Arrow Diagram of Function



X = { w, x, y, z }

A = { a, b, c, d }

f = { (w, a), (x, a), (y, d), (z, c) }

- f: X → Y means f is a function from X to Y.

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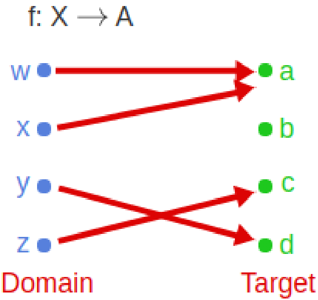

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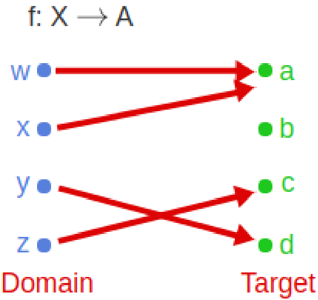

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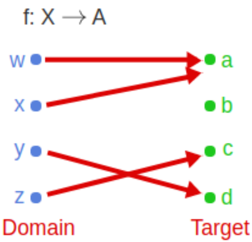
- The set X is called the domain of f.
- The set Y is the target of f.

Well defined function

Well defined function

If f maps an element of the domain to zero elements or more than one element of the target, then f is not well-defined. (i.e., Not a function)



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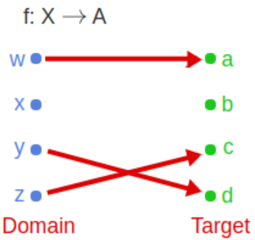

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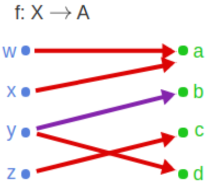

$X = \{ w, x, y, z \}$

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$f = \{ (w, a), (x, a), (y, d), (z, c), (y, b) \}$

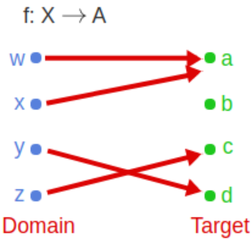
f is no longer a function because (y, b), (y, d) $\in$ f.

(Not a Function)

Range

Range

For function f: $X \to Y$, an element y is in the range of f if and only if there is an $x \in X$ such that $(x, y) \in f$.



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Range: $\{a, c, d\}$

Excercise on Function Range

Express the range of each function using roster notation.

- Let $A = \{2, 3, 4, 5\}$.
  f: $A \rightarrow Z$ such that $f(x) = 2x - 1$.

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  - $\{3, 5, 7, 9\}$
- Let $A = \{2, 3, 4, 5\}$.
  f: $A \times A \rightarrow Z$, where $f(x,y) = x+y$.

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  - $\{3, 5, 7, 9\}$
- Let A = $\{2, 3, 4, 5\}$.
  f: A x A $\rightarrow$ Z, where f(x,y) = x+y.
  - $\{4, 5, 6, 7, 8, 9, 10\}$

Function Equality

Two functions, f and g, are equal if

- f and g have the same domain.
- f and g have the same target.
- $f(x) = g(x)$ for every element x in the domain.

Excercise on Function Equality

Ex. Indicate if f and g are equal fuctions

- **f**: Z → Z, where f(x) = x^2
  g: Z → Z, where g(x) = $|x|^2$.

Excercise on Function Equality

Ex. Indicate if f and g are equal fuctions

- **f**: Z → Z, where f(x) $= x^2$
  g: Z → Z, where g(x) $= |x|^2$.
  - $f = g$
- **f**: R → Z, where f(x) $= x^2$
  g: Z → Z, where g(x) $= x^2$.

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  - $f \neq g$ different domains
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- **f**: Z → Z, where f(x) = x^3
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  - $f \neq g$ because, f(-2) = -8, and g(-2) = 8.
- **f**: Z x Z → Z, where f(x,y) = $|x + y|$
  g: Z x Z → Z, where g(x,y) = $|x| + |y|$.

Excercise on Function Equality

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  g: Z x Z → Z, where g(x,y) = $|x| + |y|$.
  - $f \neq g$ because, f(-2,2) = 0, and g(-2,2) = 4.

Outline

Excercise on Function Range

Express the range of each function using roster notation.

Floor function

The floor function maps a real number to the nearest integer in the downward direction.

$$\text{floor: } \mathrm{R} \rightarrow \mathrm{Z}$$
$$\text{floor}(\mathrm{x}) = \lfloor x \rfloor$$

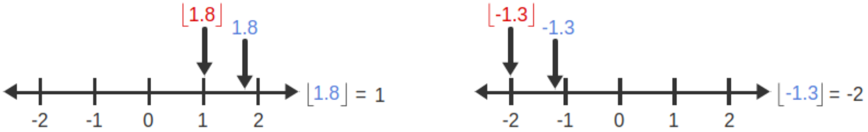
Cieling function

The floor function maps a real number to the nearest integer in the upward direction.

$$\text{ceil: } \mathrm{R} \rightarrow \mathrm{Z}$$
$$\text{ceil}(\mathrm{x}) = \lceil x \rceil$$

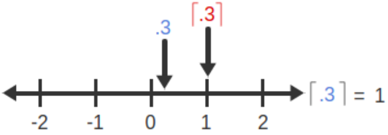
Examples

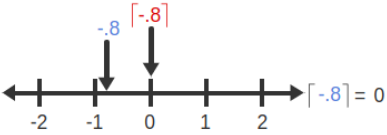
To compute the floor function slide *down* to nearest integer:



Examples

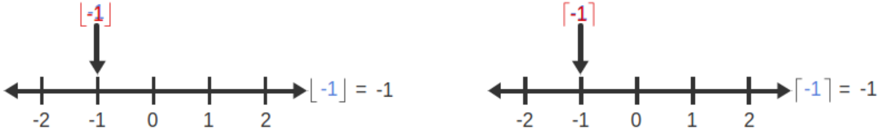
To compute the ceiling function slide *up* to nearest integer:





Examples

The ceiling and floor of an integer are the same:



Outline

Function Properties

One-to-one

A function f: $X \to Y$ is one-to-one or injective if $x1 \neq x2$ implies that $f(x1) \neq f(x2)$.

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A function f: $X \rightarrow Y$ is onto or surjective if the range of f is equal to the target Y.

Bijective

A function is bijective or (one-to-one correspondence) if it is both one-to-one and onto.

In formal definations

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Every element in the target is covered by one or less elements from the domain.

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Onto

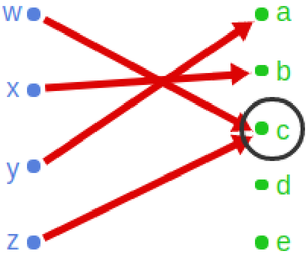
Every element in the target is covered by one or more elements from the domain.

Bijective

Every element in the target is covered by exactly one element from the domain.

Function Properties Examples

f: X $\rightarrow$ A



X = { w, x, y, z }

A = { a, b, c, d, e }

f is not one-to-one because
f(w) = f(z) = c

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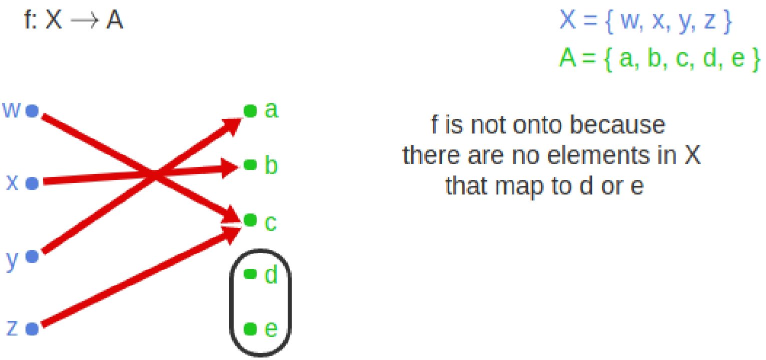
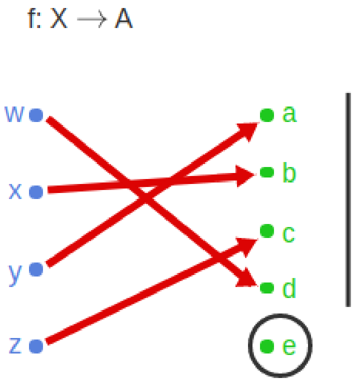
Function Properties Examples



f is not onto because no elements in X map to d or e.

Function Properties Examples



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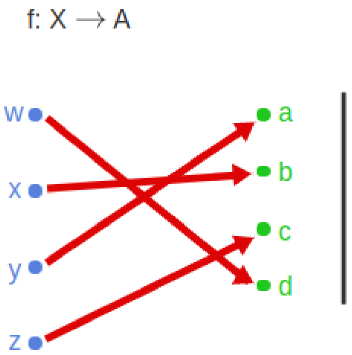

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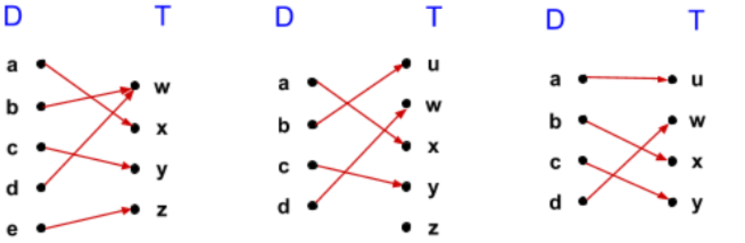
Now f is one-to-one and onto

Now f is one-to-one and onto. f is a bijection.

Relative sizes of the domain and target



onto
$|D| \geq |T|$

one-to-one
$|D| \leq |T|$

bijection
$|D| \leq |T|$ and $|D| \geq |T|$
$|D| = |T|$

Excercise 1

- Let f be a function whose domain is $\{0,1\}^3$ and whose target is $\{0,1\}^2$.

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  - No
- Is it possible that f is onto?

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Ex.

- Is it possible that f is one-to-one?
  - No
- Is it possible that f is onto?
  - Yes

Excercise 2

- For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

Ex.

- f: R → R. f(x) = x^2

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- f: Z → Z. f(x) $= 5x - 4$

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- f: Z → Z. f(x) $= 5x - 4$
  - One-to-one
  - Not onto.

Outline

Function Inverse

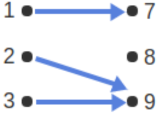
- If a function f: $X \to Y$ is a bijection, then the inverse of f is obtained by exchanging the first and second entries in each pair in f.
- The inverse of f is denoted by f^{-1}

$$f^{-1} = \{ \text{ (y, x) : (x, y) } \in \text{ f } \}.$$

Example 1

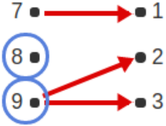
$X = \{1, 2, 3\}$

$Y = \{7, 8, 9\}$



f: $X \to Y$

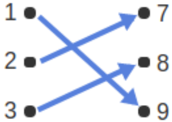
$f = \{(1, 7), (2, 9), (3, 9)\}$



$f^{-1} = \{(7, 1), (9, 2), (9, 3)\}$

f^{-1} is not a function. f does not have an inverse.

Example 2



$g: X \to Y$

$g = \{ (1, 9), (2, 7), (3, 8) \}$

$g^{-1} = Y \to X$

$g^{-1} = \{ (7, 2), (8, 3), (9, 1) \}$

g^{-1} is a function. g has an inverse defined by

$g^{-1}(7) = 2$ $\quad$ $g^{-1}(8) = 3$ $\quad$ $g^{-1}(9) = 1$

Excercise 1

- For each of the following functions, indicate whether the function has a well-defined inverse. If the inverse is well-defined, give the input/output relationship of f^{-1}.

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- f: R $\rightarrow$ R. f(x) $= x^2$

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  - $f^{-1}(x) = \sqrt[3]{x}$
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  - $f^{-1}(x) = x + 4$
- f: $Z \to Z$. $f(x) = 5x - 4$
  - One-to-one
  - Not onto.
  - f^{-1} is not well defined

Excercise 2

$f : \{0,1\}^3 \to \{0,1\}^3$. The output of is obtained by taking the input string and reversing the bits. For example, $f(011) = 110$

- Indicate whether f has a well-defined inverse and write f^{-1} if exists.

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$f : \{0,1\}^3 \to \{0,1\}^3$. The output of is obtained by taking the input string and reversing the bits. For example, $f(011) = 110$

- Indicate whether f has a well-defined inverse and write f^{-1} if exists.

Sol:

- f has a well-defined inverse.
- $f^{-1} = f$

Outline

1. Introduction
2. Floor and Cieling
3. Function Properties
4. Function Inverse
5. Function Decomposition

Function Decomposition

Function Decomposition

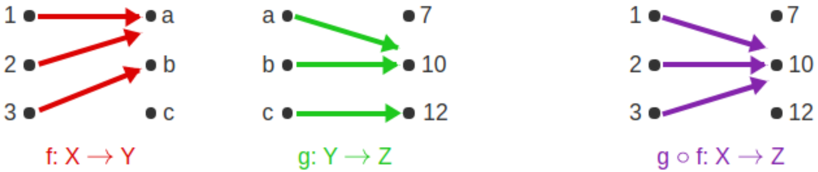
Let $f\colon X \rightarrow Y$ and $g\colon Y \rightarrow Z$.
The composition of g with f, denoted $g \circ f$, is the function $(g \circ f)\colon X \rightarrow Z$, such that for all $x \in X$, $(g \circ f)(x) = g(f(x))$.

Example

$X = \{1, 2, 3\}$

$Y = \{a, b, c\}$

$Z = \{7, 10, 12\}$



Notes I

- $f \circ g$ is not the same as $g \circ f$.

Ex.

$$\text{f: } R^+ \to R^+, \text{ f(x)} = x^3$$
$$\text{g: } R^+ \to R^+, \text{ g(x)} = x + 2$$

- $(f \circ g)(x) = f(g(x)) = (x+2)^3$
- $(g \circ f)(x) = g(f(x)) = x^3 + 2$

Notes II

- It is possible to compose more than two functions.
- Composition is associative.

$$f \circ g \circ h = (f \circ g) \circ h = f \circ (g \circ h) = f(g(h(x)))$$

Ex.

$$\text{f: } R^+ \rightarrow R^+,\ \text{f(x)} = x^3$$
$$\text{g: } R^+ \rightarrow R^+,\ \text{g(x)} = x + 2$$
$$\text{h: } R^+ \rightarrow R^+,\ \text{h(x)} = x - 1$$

- $(f \circ g)(x) = f(g(x)) = (x + 2)^3$
- $(f \circ g \circ h)(x) = f(g(h(x))) = (x + 1)^3$

Identity Function

Identity Function

The identity function always maps a set onto itself and maps every element onto itself.

- The identity function on A, denoted $I_A : A \to A$, is defined as $I_A(a) = a$, for all $a \in A$.

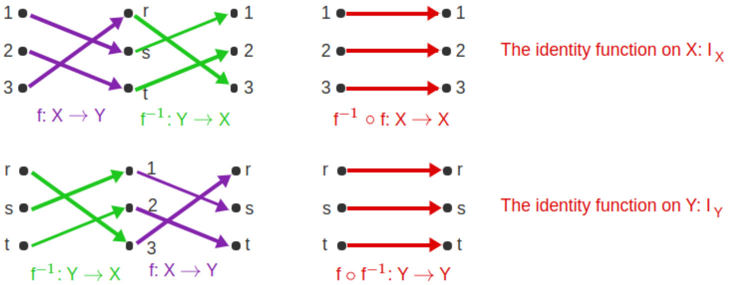
Note That

Let f: $A \to B$ be a bijection. Then $f^{-1} \circ f = I_A$ and $f \circ f^{-1} = I_B$.

Example

$X = \{1, 2, 3\}$

$Y = \{r, s, t\}$



The composition of f with the inverse of f has domain Y and target Y and maps each element to itself and is therefore the identity function on Y.

Questions