ECEN 227 - Introduction to Finite Automata and Discrete Mathematics

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Talk Overview

1. Sequences
2. Recurrence relations
3. Summation
4. Mathematical induction

Outline

1. Sequences
2. Recurrence relations
3. Summation
4. Mathematical induction

Sequence

Sequence

A sequence is a special type of function in which the domain is a consecutive set of integers.

The general form of a sequence: $S_1, S_2, S_3, \ldots, S_n$

S_1 is the first term and the 1 in the subscript is the initial index.

S_2 is the second term and the 2 in the subscript is the second index.

S_n is the nth term and the n in the subscript is the nth index.

Ex.

2, 4, 6, 8, 10

Finite and Infinite Sequence

When the sequence goes on forever it is called an infinite sequence, otherwise it is a finite sequence.

Ex.

1, 2, 3, 4, ... (Infinite Sequence)

20, 25, 30, 35, ... (Infinite Sequence)

1, 3, 5, 7 (Finite Sequence)

7, 6, 5, 4, 3, 2, 1 (Finite Sequence)

Increasing and non-decreasing sequences.

$2, < 4, < 5, < 6$ Increasing *and* non-decreasing

$2, \leq 4, \leq 4, \leq 6$ Non-decreasing *but not* increasing

$=$

Decreasing and non-increasing sequences.

$6, > \ 5, > \ 4, > \ 2$ Decreasing *and* non-increasing

$6, \geq \ 4, \geq \ 4, \geq \ 2$ Non-increasing *but not* decreasing

=

Excercise I

Give the first ten terms of the following sequences. You can assume that the sequences start with an index of 1. Logs are to base 2.
Indicate whether the sequence is increasing, decreasing, non-increasing, or non-decreasing.

- The n^{th} term is $\lceil \sqrt{n} \rceil$.
- The first two terms in the sequence are 1. The rest of the terms are the sum of the two preceding terms.
- The n^{th} term is $1/\mathrm{n}$.
- The n^{th} term is 3.

Excercise II

Give the first ten terms of the following sequences. You can assume that the sequences start with an index of 1. Logs are to base 2.
Indicate whether the sequence is increasing, decreasing, non-increasing, or non-decreasing.

- The n^{th} term is n^2.
- The n^{th} term is $\lceil log(n) \rceil$
- The n^{th} term is $2^{\lceil log(n) \rceil}$
- The n^{th} term is $\lceil \frac{-n}{2} \rceil$

Arithemtic Sequence

Arithemtic Sequence

An arithmetic sequence is a sequence of real numbers where each term after the initial term (a_0) is found by taking the previous term and adding a fixed number called the common difference (d).

- An arithmetic sequence can be finite or infinite.
- n^{th} term = a_0 + d × (n) Assuming initial index is zero

Ex.

- Initial term (a_0) = 1
- Common Difference (d) = 3
- 1, 4, 7, 10, ...
- n^{th} term = 1 + 3 × (n)

Example

Suppose a person inherits a collection of 500 baseball cards and decides to continue growing the collection at a rate of 10 additional cards each week. **Describe the process as a sequence**

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Answer.

- a_n is the number of cards in the collection after n weeks of collecting. Since the collection starts with 500 cards, $a_0 = 500$.
- The sequence a_n is an arithmetic sequence with an initial value of 500 and a common difference of 10.
- After n weeks of collecting, $a_n = 500 + 10n$.

Geometric Sequence

Geometric Sequence

A geometric sequence is a sequence of real numbers where each term after the initial term (a_0) is found by taking the previous term and multiplying by a fixed number called the common ratio (r).

- A geometric sequence can be finite or infinite.
- n^{th} term $= a_0 \times \mathrm{r}^n$ Assuming initial index is zero

Ex.

- Initial term $(a_0) = 4$,
- Common Ratio $(\mathrm{r}) = 1/2$
- 4, 2, 1, 1/2, 1/4, ...
- n^{th} term $= 4 \times (1/2)^n$ Assuming initial index is zero

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Suppose \$1000 is stored in a bank account that earns 6% annual interest compounded monthly. **Describe the process as a sequence**

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Suppose \$1000 is stored in a bank account that earns 6% annual interest compounded monthly. **Describe the process as a sequence**

- Since the interest rate is annual and compounded monthly, (6/12)% of the current amount is added to the account each month.
- a_0=1000 is the initial balance in the account, and a_n is the balance in the account after n months of earning interest. Each month, the balance in the account is 1.005 times the amount that was in the account in the previous month.
- The sequence a_n is a geometric sequence with $a_n = 1000 \times (1.005)^n$

Outline

1 Sequences

2 Recurrence relations

3 Summation

4 Mathematical induction

Recurrence relation

Fibonacci sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

The next number is found by adding the two numbers before it together.

Rule is: $a_n = a_{n-1} + a_{n-2}$

Recurrence relation

A rule that defines a term $\mathbf{a}_n$ as a function of previous terms in the sequence is called a recurrence relation.

Recurrence Relation for Arithemtic Sequence

- a_0 = a (initial value)
- a_n = d + a_{n-1} for n 1 (recurrence relation)
- Initial value = a. Common difference = d.

Ex.

- Initial term = 1
- Common Difference = 4
- 1, 5, 9, 13, ...

Recurrence Relation for Geometric Sequence

- $a_0 = \mathrm{a}$ (initial value)
- $a_n = \mathrm{r} \times a_{n-1}$ for n 1 (recurrence relation)
- Initial value = a. Common ratio = r.

Ex.

- Initial term = 1
- Common ratio = 2
- 1, 2, 4, 8, 16, ...

Arithemtic vs Geometric Sequence

	Arithemtic Sequence	Geometric Sequence
n^{th} term iterative	$a_n = a_0 + d \times n$	$a_n = a_0 \times r^n$
n^{th} term recursive	$a_n = \text{d} + a_{n-1}$	$a_n = \text{r} \times a_{n-1}$

Example

An individual takes out a \$20,000 car loan. The interest rate for the loan is 3%, compounded monthly. He wishes to make a monthly payment of \$500. **Define an to be the amount of outstanding debt after n months recursively.**

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Answer.

$$a_0 = \$20000$$
$$a_n = (1.0025) \times a_{n-1} - 500$$

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Summation

- Summation notation is used to express the sum of terms in a numerical sequence.
- Consider a sequence:

$$a_s, a_{s+1}, \ldots, a_t$$

The notation to express the sum of the terms in that sequence is:

$$\sum_{i=s}^{i=t} a_i = a_s + a_{s+1} + \cdots + a_t$$

- The variable i is called the index of the summation.
- The variable s is the lower limit
- The variable t is the upper limit of the summation.

Example 1

Suppose we want to write a summation for the sequence

$$n^2 \text{ for n} = 1,\ 2, \ldots 5$$

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Suppose we want to write a summation for the sequence

$$n^2 \text{ for n} = 1,\ 2, \ldots 5$$

Then:

$$\sum_{j=1}^{j=5} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$$

Example 2

It is important to use parentheses if you have more than one term.

$$\sum_{j=1}^{j=4}(j^2+1) = (1^2+1)+(2^2+1)+(3^2+1)+(4^2+1)$$

$$\sum_{j=1}^{j=4} j^2+1 = (\sum_{j=1}^{j=4} j^2)+1 = (1^2)+(2^2)+(3^2)+(4^2)+1$$

Excercise 1

Evaluate the following summations.

- $\sum_{k=-1}^{k=4} k^2$

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Evaluate the following summations.

- $\sum_{k=-1}^{k=4} k^2$
  - $(-1)^2 + 0^2 + 1^2 + 2^2 + 3^2 + 4^2 = 31$
- $\sum_{k=0}^{k=4} 2^k$

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  - $(-1)^2 + 0^2 + 1^2 + 2^2 + 3^2 + 4^2 = 31$
- $\sum_{k=0}^{k=4} 2^k$
  - $2^0 + 2^1 + 2^3 + 2^4 = 31$
- $\sum_{k=-3}^{k=2} k^3$

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- $\sum_{k=0}^{k=4} 2^k$
  - $2^0 + 2^1 + 2^3 + 2^4 = 31$
- $\sum_{k=-3}^{k=2} k^3$
  - $(-3)^3 + (-2)^3 + (-1)^3 + 0^3 + 1^3 + 2^3 = -27$

Pulling out a final term from a summations

$$\sum_{j=m}^{j=n} a_j = \sum_{j=m}^{j=n-1} a_j + a_n$$

Ex.

Pulling out a final term from a summations

$$\sum_{j=m}^{j=n} a_j = \sum_{j=m}^{j=n-1} a_j + a_n$$

Ex.

$$\sum_{j=1}^{n} (j+1)^2 = \underbrace{(1+1)^2 + (2+1)^2 + \ldots + ((n-1)+1)^2}_{} + (n+1)^2$$

$$= \sum_{j=1}^{n-1} (j+1)^2 + (n+1)^2$$

Excercise

Pull out the final term from the following summations.

- $\displaystyle\sum_{j=0}^{j=n+2} 2^{j-1}$

Excercise

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- $\sum_{k=0}^{k=m+2} (k^2 - 4k + 1)$

Excercise

Pull out the final term from the following summations.

- $\sum_{j=0}^{j=n+2} 2^{j-1}$
  - $\sum_{j=0}^{j=n+1} 2^{j-1} + 2^{n+1}$
- $\sum_{k=0}^{k=m+2} (k^2 - 4k + 1)$
  - $\sum_{k=0}^{k=m+1} (k^2 - 4k + 1) + (m+2)^2 - 4(m+2) + 1$

Change of variables in summations

$$\sum_{j=1}^{j=n} (j+2)^3$$

We can substitute a term in the summation to get a reduced summation.

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Change of variables in summations

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We can substitute a term in the summation to get a reduced summation.

Let k = j+2

Three steps to be taken:

- Replace term in the summation.
- Determine the new upper limit.
- Determine the new lower limit.

$$\sum_{k=3}^{k=n+2} (k)^3$$

Excercise

Pull out the final term from the following summations.

- Substitute variable j for k, where j = k - 1, in the summation
  $\sum_{k=0}^{k=n-1} 2^{k-2}$

Excercise

Pull out the final term from the following summations.

- Substitute variable j for k, where j = k - 1, in the summation $\sum_{k=0}^{k=n-1} 2^{k-2}$
  - $\sum_{j=-1}^{j=n-2} 2^{j-1}$
- Substitute variable k for j, where k = j - 4, in the summation $\sum_{j=4}^{j=17} (2j+4)$

Excercise

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- Substitute variable j for k, where j = k - 1, in the summation
  $$\sum_{k=0}^{k=n-1} 2^{k-2}$$
  - $$\sum_{j=-1}^{j=n-2} 2^{j-1}$$
- Substitute variable k for j, where k = j - 4, in the summation
  $$\sum_{j=4}^{j=17} (2j+4)$$
  - $$\sum_{k=0}^{k=13} (2k+12)$$

Closed forms for sums

Closed form form a summuation

A closed form for a sum is a mathematical expression that expresses the value of the sum without summation notation.

Ex.

$$\sum_{k=1}^{k=n} k = \frac{n(n+1)}{2}$$

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Ex.

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- Arithemtic sequences have a closed form.
- Geometric sequences have a closed form.

Known Sequences Sum Closed Form

- $\sum_{k=1}^{k=n} k = \frac{n(n+1)}{2}$
- $\sum_{k=1}^{k=n} c = c \times n$
- $\sum_{k=1}^{k=n} ck = c \times \frac{n(n+1)}{2}$

Known Sequences Sum Closed Form

Arithemtic Sequence

For any integer n $\geq$ 1:

$$\sum_{k=0}^{k=n-1} a + kd = a \times n + \frac{d(n-1)n}{2}$$

Geometric Sequence

For any real number r $\neq$ 1 and any integer n $\geq$ 1:

$$\sum_{k=0}^{k=n-1} ar^k = \frac{a(r^n-1)}{r-1}$$

Excercise 1

- $\sum_{k=0}^{k=100} (3 + 5k)$

Excercise 1

- $\sum_{k=0}^{k=100}(3+5k)$
  - $\sum_{k=0}^{k=100} 3 + \sum_{k=0}^{k=100} 5k = 3 \times 101 + \frac{5\times100\times101}{2} = 303 + 25250 = 25553$
- $\sum_{k=0}^{k=100} 3 \times (1.1)^k$

Excercise 1

- $\sum_{k=0}^{k=100}(3+5k)$
  - $\sum_{k=0}^{k=100} 3 + \sum_{k=0}^{k=100} 5k = 3 \times 101 + \frac{5\times100\times101}{2} = 303 + 25250 = 25553$
- $\sum_{k=0}^{k=100} 3 \times (1.1)^k$
  - $\frac{3(1.1^{101}-1)}{0.1} \approx 454730.2072$

Excercise 2

A Silicon Valley company purchases 3 new cars at the end of every month. Let a_n denote the number of cars he has after n months. Let $a_0 = 23$.

- What is a_8?

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- What is a_8?
  - $23 + 3 \times 8 = 47$
- If it pays \$50 each month to have each car maintained, what is the total amount that it has paid for maintenance after 2 years? Note that the company purchases the new cars at the end of each month, so during the first month, he is only maintaining 23 cars.

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- If it pays \$50 each month to have each car maintained, what is the total amount that it has paid for maintenance after 2 years? Note that the company purchases the new cars at the end of each month, so during the first month, he is only maintaining 23 cars.
  - $50 \sum_{i=0}^{i=23} (23 + 3i) = 50[23 \times 24 + \frac{3 \times 23 \times 24}{2}]$

Excercise 2

A population of rabbits on a farm grows by 12% each year. Define a sequence r_n describing the rabbit population at the end of each year. Suppose that the sequence starts with $r_0 = 30$.

- Give a mathematical expression for r_{12}

Excercise 2

A population of rabbits on a farm grows by 12% each year. Define a sequence r_n describing the rabbit population at the end of each year. Suppose that the sequence starts with $r_0 = 30$.

- Give a mathematical expression for r_{12}
  - $30(1.12)^{12}$
- If each rabbit consumes 10 pounds of rabbit food each year, then how much rabbit food is consumed in 10 years? For simplicity, you can omit the food consumed by the baby rabbits born in a given year. For example, suppose the farm starts tabulating rabbit food on January 1, 2012 at which time the rabbit population is 30. You will count the food consumed by those 30 rabbits during 2012. You won't count the food consumed by the rabbits born in 2012 until after January 1, 2013.

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  - $10 \sum_{i=0}^{i=9} 30(1.12)^i = 300(\frac{1.12^{10}-1}{0.12})$

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Mathematical induction

- Suppose that one day a genie grants you three wishes that will expire by the end of the day.

Mathematical induction

- Suppose that one day a genie grants you three wishes that will expire by the end of the day.
- You make two wishes and then for your third wish, you wish for three more wishes the next day.

Given the fact that you can always use your third wish to renew your wishes for the next day, it is possible to prove that from that first day onward, you can have three wishes every day for the rest of your life.

Mathematically we want to prove that on day n you will have three wishes.

Induction Proof

- Induction is a proof technique that is especially useful for proving statements about elements in a sequence.
- An inductive proof establishes that some statement parameterized by n is true, for any positive integer n.

The two components of an inductive proof.

- **The base case** establishes that the theorem is true for the first value in the sequence.

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  The genie grants you three wishes on day 1
- **The inductive step** establishes that if the theorem is true for k, then the theorem also holds for $k + 1$.
  If you have three wishes on day k, then you can get three wishes for day $k+1$.

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  If you have three wishes on day k, then you can get three wishes for day k+1.

Note That.

Inductive step is doing direct proof for the statement $P(K) \rightarrow P(K+1)$

Principle of mathematical induction.

Let $S(n)$ be a statement parameterized by a positive integer n. Then $S(n)$ is true for all positive integers n, if:

1. $S(1)$ is true (the base case).
2. For all $k \in \mathbf{Z}^+$, $S(k)$ implies $S(k+1)$ (the inductive step).

Why it works?

- P(1) is true.
- For all k ≥ 1, P(k) implies P(k + 1).

  k = 1 P(1) implies P(2) $\Rightarrow$ P(2) is true

Conclusion: P(1) is true

$P(1)$ is true, and for all $k \geq 1$, $P(k)$ implies $P(k+1)$. Setting $k = 1$ means that $P(1)$ implies $P(2)$.

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- $P(1)$ is true.
- For all $k \geq 1$, $P(k)$ implies $P(k+1)$.

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Conclusion: $P(1) \wedge P(2)$ is true

Therefore, $P(2)$ is also true.

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  $k = 1$ $P(1)$ implies $P(2)$ $\Rightarrow$ $P(2)$ is true

  $k = 2$ $P(2)$ implies $P(3)$ $\Rightarrow$ $P(3)$ is true

  $k = 3$ $P(3)$ implies $P(4)$ $\Rightarrow$ $P(4)$ is true

Conclusion: $P(1) \wedge P(2) \wedge P(3) \wedge P(4)$ is true

For $k = 2$, $P(2)$ implies $P(3)$. Therefore, $P(3)$ is also true. For $k = 3$, $P(3)$ implies $P(4)$. Therefore, $P(4)$ is also true.

Why it works?

- $P(1)$ is true.
- For all $k \geq 1$, $P(k)$ implies $P(k+1)$.

$k = 1$ $P(1)$ implies $P(2)$ $\Rightarrow$ $P(2)$ is true
$k = 2$ $P(2)$ implies $P(3)$ $\Rightarrow$ $P(3)$ is true
$k = 3$ $P(3)$ implies $P(4)$ $\Rightarrow$ $P(4)$ is true
$\vdots$

Since $P(k)$ implies $P(k+1)$ for all $k \geq 1$, the process can be continued up to any $n \geq 1$

Conclusion: $P(1) \wedge P(2) \wedge P(3) \wedge P(4)$ is true

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- For all k ≥ 1, P(k) implies P(k + 1).

k = 1 P(1) implies P(2) $\Rightarrow$ P(2) is true

k = 2 P(2) implies P(3) $\Rightarrow$ P(3) is true

k = 3 P(3) implies P(4) $\Rightarrow$ P(4) is true

$\vdots$

Since P(k) implies P(k + 1) for all k ≥ 1,the process can be continued up to any n ≥ 1

Conclusion: P(1) $\wedge$ P(2) $\wedge$ P(3) $\wedge$ P(4) ... is true

$\boxed{\forall n \geq 1 \text{ P(n) is true.}}$

Therefore, for all $n \geq 1$, $P(n)$ is true.

Example 1 (Proving Identity)

Theorem

$$\sum_{j=1}^{j=n} j = \frac{n(n+1)}{2}$$

Proof.

Base case:

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Proof.

Base case:

When n $= 1$, the left side of the equation is $\sum_{j=1}^{j=1}$ j $= 1$

When n $= 1$, the right side of the equation is $1(1+1)/2 = 1$.

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Direct proof of P(k) → P(k+1) (Next Slide)

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$\square$

Inductive Step Proof

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we need to prove that $\sum_{j=1}^{j=k+1} j = \frac{(k+1)(k+2)}{2}$

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Proof

Starting with the left side of the equation to be proven:

$$\sum_{j=1}^{j=k+1} j = \sum_{j=1}^{j=k} j + (\mathrm{k}+1)$$ pulling out the last term

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Proof

Starting with the left side of the equation to be proven:

$$\sum_{j=1}^{j=k+1} j = \sum_{j=1}^{j=k} j + (\mathrm{k}+1)$$

pulling out the last term

$$\sum_{j=1}^{j=k+1} j = \frac{k(k+1)}{2} + (\mathrm{k}+1)$$

$$\sum_{j=1}^{j=k+1} j = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2} = RHS$$

Example 2 (Proving Identity)

Theorem

$$\sum_{j=1}^{j=n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof.

Base case:

Example 2 (Proving Identity)

Theorem

$$\sum_{j=1}^{j=n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof.

Base case:

When n $= 1$, the left side of the equation is $\sum_{j=1}^{j=1} j^2 = 1$

When n $= 1$, the right side of the equation is $\frac{1(1+1)(2.1+1)}{6} = 1$.

Example 2 (Proving Identity)

Theorem

$$\sum_{j=1}^{j=n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof.

Base case:

When n = 1, the left side of the equation is $\sum_{j=1}^{j=1} j^2 = 1$

When n = 1, the right side of the equation is $\frac{1(1+1)(2.1+1)}{6} = 1$.

Inductive step:

Direct proof of P(k) → P(k+1) (Next Slide)

□

Inductive Step Proof

Assume

Inductive Step Proof

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$$\sum_{j=1}^{j=k} j^2 = \frac{k(k+1)(2k+1)}{6}$$

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we need to prove that $\sum_{j=1}^{j=k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

Proof

Inductive Step Proof

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we need to prove that $\sum_{j=1}^{j=k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

Proof

Starting with the left side of the equation to be proven:

$$\sum_{j=1}^{j=k+1} j^2 = \sum_{j=1}^{j=k} j^2 + (k+1)^2$$

pulling out the last term

Inductive Step Proof

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$$\sum_{j=1}^{j=k+1} j^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$$

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$$\sum_{j=1}^{j=k} j^2 = \frac{k(k+1)(2k+1)}{6}$$

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$$\sum_{j=1}^{j=k+1} j^2 = \frac{(k+1)(k(2k+1)+6(k+1))}{6} = \frac{(k+1)(2k^2+7k+6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6} = \text{RHS}$$

Example 3 (Proving Inequality)

Theorem

for $n \geq 4$, $2^n \geq 3n$

Proof.

Base case:

Example 3 (Proving Inequality)

Theorem

$$\textit{for } n \geq 4,\ 2^n \geq 3n$$

Proof.

Base case:

When n $= 4$, the left side of the equation is $2^4 = 16$

When n $= 4$, the right side of the equation is 3x4 $= 12$.

$16 \geq 12$

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$16 \geq 12$

Inductive step:

Direct proof of P(k) → P(k+1) (Next Slide)

□

Inductive Step Proof

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$2^k \geq 3k$

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$2^k \geq 3k$

we need to prove that $2^{k+1} \geq 3(k+1)$

Proof

Inductive Step Proof

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$2^k \geq 3k$

we need to prove that $2^{k+1} \geq 3(k+1)$

Proof

Starting with the left side of the equation to be proven:

$2^{k+1} = 2^k \times 2$

Inductive Step Proof

Assume

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Inductive Step Proof

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Proof

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$2^{k+1} = 2^k \times 2$

$2^{k+1} \geq 3k \times 2$

$2^{k+1} \geq 3k + 3k$

Inductive Step Proof

Assume

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$2^{k+1} = 2^k \times 2$

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$2^{k+1} \geq 3k + 3$

Inductive Step Proof

Assume

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we need to prove that $2^{k+1} \geq 3(k+1)$

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Starting with the left side of the equation to be proven:

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$2^{k+1} \geq 3k \times 2$

$2^{k+1} \geq 3k + 3k$

$2^{k+1} \geq 3k + 3$

$2^{k+1} \geq 3(k+1) = RHS$

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Example 4 (Arithemtic Sequences)

Theorem

For any integer $n \geq 1$:

$$\sum_{j=0}^{j=n-1} a + jd = a \times n + \frac{d(n-1)n}{2}$$

Proof.

Base case:

Example 4 (Arithemtic Sequences)

Theorem

For any integer $n \geq 1$:

$$\sum_{j=0}^{j=n-1} a + jd = a \times n + \frac{d(n-1)n}{2}$$

Proof.

Base case:

When n $=$ 1, the left side of the equation is $\sum_{j=0}^{j=0} a + jd =$ a

When n $=$ 1, the right side of the equation is $a \times 1 + \frac{d(1-1).1}{2} =$ a.

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Inductive step:

Direct proof of P(k) → P(k+1) (Next Slide)

Inductive Step Proof

Assume

Inductive Step Proof

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$$\sum_{j=0}^{j=k-1} (a + jd) = ak + \frac{d(k-1)k}{2}$$

Inductive Step Proof

Assume

$$\sum_{j=0}^{j=k-1} (a + jd) = ak + \frac{d(k-1)k}{2}$$

we need to prove that $\sum_{j=0}^{j=k} a + jd = a(k+1) + \frac{d(k)(k+1)}{2}$

Proof

Inductive Step Proof

Assume

$$\sum_{j=0}^{j=k-1}(a+jd) = ak + \frac{d(k-1)k}{2}$$

we need to prove that $\sum_{j=0}^{j=k} a+jd = a(k+1) + \frac{d(k)(k+1)}{2}$

Proof

Starting with the left side of the equation to be proven:

$$\sum_{j=0}^{j=k}(a+jd) = \sum_{j=0}^{j=k-1}(a+jd) + a + kd$$ pulling out the last term

$$\sum_{j=0}^{j=k}(a+jd) = ak + \frac{d(k-1)k}{2} + a + kd$$ using our assumption

$$\sum_{j=0}^{j=k}(a+jd) = a(k+1) + \frac{d(k-1)k}{2} + \frac{2kd}{2}$$

$$\sum_{j=0}^{j=k}(a+jd) = a(k+1) + \frac{kd(k-1+2)}{2} = a(k+1) + \frac{kd(k+1)}{2} = \text{RHS}$$

Example 5 (Geometric Sequence)

Theorem

For any real number $r \neq 1$ and any integer $n \geq 1$:

$$\sum_{j=0}^{j=n-1} ar^j = \frac{a(r^n - 1)}{r - 1}$$

Proof.

Base case:

Example 5 (Geometric Sequence)

Theorem

For any real number $r \neq 1$ and any integer $n \geq 1$:

$$\sum_{j=0}^{j=n-1} ar^j = \frac{a(r^n - 1)}{r - 1}$$

Proof.

Base case:

When n $= 1$, the left side of the equation is $\sum_{j=0}^{j=0} ar^j = \text{a}$

When n $= 1$, the right side of the equation is $\frac{a(r^1 - 1)}{r - 1} = \text{a}$.

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For any real number $r \neq 1$ and any integer $n \geq 1$:

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Base case:

When n = 1, the left side of the equation is $\sum_{j=0}^{j=0} ar^j = a$

When n = 1, the right side of the equation is $\frac{a(r^1-1)}{r-1} = a$.

Inductive step:

Direct proof of P(k) → P(k+1) (Next Slide)

□

Inductive Step Proof

Assume

Inductive Step Proof

Assume

$$\sum_{j=0}^{j=k-1} ar^j = \frac{a(r^k - 1)}{r-1}$$

Inductive Step Proof

Assume

$$\sum_{j=0}^{j=k-1} ar^j = \frac{a(r^k-1)}{r-1}$$

we need to prove that $\sum_{j=0}^{j=k} ar^j = \frac{a(r^{k+1}-1)}{r-1}$

Proof

Inductive Step Proof

Assume

$$\sum_{j=0}^{j=k-1} ar^j = \frac{a(r^k-1)}{r-1}$$

we need to prove that $\sum_{j=0}^{j=k} ar^j = \frac{a(r^{k+1}-1)}{r-1}$

Proof

Starting with the left side of the equation to be proven:

$$\sum_{j=0}^{j=k} ar^j = \sum_{j=0}^{j=k-1} ar^j + ar^k$$ pulling out the last term

$$\sum_{j=0}^{j=k} ar^j = \frac{a(r^k-1)}{r-1} + ar^k$$ using our assumption

$$\sum_{j=0}^{j=k} ar^j = \frac{a(r^k-1)}{r-1} + \frac{ar^k(r-1)}{r-1}$$

$$\sum_{j=0}^{j=k} ar^j = \frac{ar^k-a}{r-1} + \frac{ar^{k+1}-ar^k}{r-1} = \frac{a(r^{k+1}-1)}{r-1} = \text{RHS}$$

Example 6 (Divisibility proof)

Theorem

For every positive integer n, 3 divides $2^{2n} - 1$

Proof.

Base case:

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Theorem

For every positive integer n, 3 divides $2^{2n} - 1$

Proof.

Base case:

When n $= 1$: $2^{2\times 1} - 1 = 3$.

Since 3 evenly divides 3, the theorem holds for the case n $= 1$.

Example 6 (Divisibility proof)

Theorem

For every positive integer n, 3 divides $2^{2n} - 1$

Proof.

Base case:

When n = 1: $2^{2\times 1} - 1 = 3$.

Since 3 evenly divides 3, the theorem holds for the case n = 1.

Inductive step:

Direct proof of $P(k) \rightarrow P(k+1)$ (Next Slide)

$\square$

Inductive Step Proof

Assume

Inductive Step Proof

Assume

3 divides $2^{2k} - 1$

Inductive Step Proof

Assume

3 divides $2^{2k} - 1$

we need to prove that 3 divides $2^{2k+2} - 1$

Proof

Inductive Step Proof

Assume

3 divides $2^{2k} - 1$

we need to prove that 3 divides $2^{2k+2} - 1$

Proof

$2^{2k+2} - 1 = 2^{2k} \times 2^2 - 1$

$2^{2k+2} - 1 = 4 \times (2^{2k} - 1 + 1) - 1$

$2^{2k+2} - 1 = 4 \times (2^{2k} - 1) + 4 - 1$

$4 \times (2^{2k} - 1) + 3$

$$\underbrace{4 \times (2^{2k} - 1)}_{\text{divsible by 3}} + \underbrace{3}_{\text{divsible by 3}}$$

Example 7 (Recurrence proof)

Theorem

- $g_0 = 1$
- $g_n = 3g_{n-1} + 2n$, *for any* $n \geq 1$.

Then for any $n \geq 0$,
$$g_n = \frac{5}{2}(3^n) - n - \frac{3}{2}$$

Proof.

Base case:

Example 7 (Recurrence proof)

Theorem

- $g_0 = 1$
- $g_n = 3g_{n-1} + 2n$, *for any* $n \geq 1$.

Then for any $n \geq 0$,

$$g_n = \frac{5}{2}(3^n) - n - \frac{3}{2}$$

Proof.

Base case:

$g_0 = \frac{5}{2}(3^0) - 0 - \frac{3}{2} = 1$

Theorem is correct at the base case

Example 7 (Recurrence proof)

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- $g_0 = 1$
- $g_n = 3g_{n-1} + 2n$, *for any* $n \geq 1$.

Then for any $n \geq 0$,

$$g_n = \frac{5}{2}(3^n) - n - \frac{3}{2}$$

Proof.

Base case:

$g_0 = \frac{5}{2}(3^0) - 0 - \frac{3}{2} = 1$

Theorem is correct at the base case **Inductive step:**

Direct proof of P(k) → P(k+1) (Next Slide)

□

Inductive Step Proof

Assume

Inductive Step Proof

Assume

$$g_k = \frac{5}{2}(3^k) - k - \frac{3}{2}$$

Inductive Step Proof

Assume

$$g_k = \frac{5}{2}(3^k) - k - \frac{3}{2}$$

we need to prove that $g_{k+1} = \frac{5}{2}(3^{k+1}) - (k+1) - \frac{3}{2}$

$$g_{k+1} = \frac{5}{2}(3^{k+1}) - k - \frac{5}{2}$$

Proof

Inductive Step Proof

Assume

$$g_k = \frac{5}{2}(3^k) - k - \frac{3}{2}$$

we need to prove that $g_{k+1} = \frac{5}{2}(3^{k+1}) - (k+1) - \frac{3}{2}$

$$g_{k+1} = \frac{5}{2}(3^{k+1}) - k - \frac{5}{2}$$

Proof

Starting with the left side of the equation to be proven:

$g_{k+1} = 3g_k + 2k$

$g_{k+1} = 3(\frac{5}{2}(3^k) - k - \frac{3}{2}) + 2(k+1)$

$g_{k+1} = (\frac{5}{2}(3^{k+1}) - 3k - \frac{9}{2}) + 2(k+1)$

$g_{k+1} = \frac{5}{2}(3^{k+1}) - k - \frac{5}{2} = RHS$

Questions