ECEN 227 - Introduction to Finite Automata and Discrete Mathematics

ECEN 227

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Talk Overview

1. Introduction to proofs
2. Prove by Exhaustion
3. Direct Proof
4. Proof by Contrapositive
5. Indirect Proof
6. Proof by Cases

Outline

Introduction

Theorem

A theorem is a statement that can be proven to be true.

Axiom

It is a statement which is accepted without question, and which has no proof.

Proof

A proof is of a series of steps, each of which follows logically from assumptions, axioms, or from previously proven statements, whose final step should result in the statement or the theorem being proven.

Introduction

- One of the hardest parts of writing proofs is knowing where to start.
- Proofs have common patterns, we will cover:
  - Proof by Exhaustion.
  - Direct proof.
  - Proof by contrapositive.
  - Proof by contradiction.
  - Proof by cases.
- Coming up with proofs requires trial and error, even for experienced mathematicians.

Example

Theorem

Every positive integer is less than or equal to its square.

Proof.

- Let x be an integer $x > 0$. Name a generic object in the domain and state given assumptions about the object
- Since x is an integer and $x > 0$, then $x \geq 1$. State reasoning in complete sentence
- Since $x > 0$, we can multiply both sides of the inequality by x to get:

$$x * 1 \leq x * x.$$

- Simplify the expression we get

$$x \leq x^2.$$

□

Outline

Prove by Exhaustion

- For universal statements, if the domain is small, it may be easiest to prove the statement by checking each element individually.

Theorem

if $n \in \{-1, 0, 1\}$ *then* $n^2 = |n|$

Proof.

- $n = -1: \quad (-1)^2 = 1 = |-1|.$
- $n = 0: \quad (0)^2 = 0 = |0|.$
- $n = 1: \quad (1)^2 = 1 = |1|.$

$\square$

Counter example

- A counterexample is an assignment of values to variables.
- A counterexample can be used to show a universal statement is false. (disproof)

Ex

" If n is an integer greater than 1, then $(1.1)^n < n^{10}$ ".

For n = 686, the statement is false because

$$(1.1)^{686} > 686^{10}$$

Outline

Direct Proof

Used to proof Conditional Statements such as $p \rightarrow c$ are correct.

Direct Proof

In a direct proof of a conditional statement, the hypothesis p is assumed to be true and the conclusion c is proven as a direct result of the assumption.

Direct Proof

Theorem

if x and y are positive real numbers then:

$$\frac{x}{y} + \frac{y}{x} \geq 2$$

Proof.

$\because$ x and y are real numbers

Direct Proof

Theorem

if x and y are positive real numbers then:

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Proof.

$\because$ x and y are real numbers

$\therefore$ $x - y$ is also a real number.

Direct Proof

Theorem

if x and y are positive real numbers then:

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Proof.

$\because$ x and y are real numbers

$\therefore x - y$ is also a real number.

$\therefore (x-y)^2 \geq 0$, the square of any real number is greater than or equal to 0.

Direct Proof

Theorem

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Proof.

$\because$ x and y are real numbers

$\therefore x - y$ is also a real number.

$\therefore (x - y)^2 \geq 0$, the square of any real number is greater than or equal to 0.

$\therefore x^2 - 2xy + y^2 \geq 0$

Direct Proof

Theorem

if x and y are positive real numbers then:

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Proof.

$\because$ x and y are real numbers

$\therefore x - y$ is also a real number.

$\therefore (x-y)^2 \geq 0$, the square of any real number is greater than or equal to 0.

$\therefore x^2 - 2xy + y^2 \geq 0$

$\therefore \frac{x}{y} - 2 + \frac{y}{x} \geq 0$ divide both sides of the inequality by xy

Direct Proof

Theorem

if x and y are positive real numbers then:

$$\frac{x}{y} + \frac{y}{x} \geq 2$$

Proof.

$\because$ x and y are real numbers

$\therefore$ $x - y$ is also a real number.

$\therefore$ $(x-y)^2 \geq 0$, the square of any real number is greater than or equal to 0.

$\therefore$ $x^2 - 2xy + y^2 \geq 0$

$\therefore$ $\frac{x}{y} - 2 + \frac{y}{x} \geq 0$ divide both sides of the inequality by xy

$\therefore$ $\frac{x}{y} + \frac{y}{x} \geq 2$ Adding 2 to both sides

$\square$

Outline

Proof by Contrapositive

- Used to proof Conditional Statements such as $p \rightarrow c$ are correct.
- Remember if $p \rightarrow c$ then $\neg c \rightarrow \neg p$ (i.e., contrapositive)

Proof by Contrapositive

In a proof by contrapositive of a conditional statement, the conclusion c is assumed to be false (i.e., $\neg c = true$) and the hypothesis p is proven as false (i.e., $\neg p = true$).

Proof by Contrapositive (Example 1)

Theorem

If $3n + 7$ is an odd integer, then n is an even integer

Proof.

Assume:

n is an odd integer negation of conclusion

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Then:

$n = 2k + 1$ for some integer k

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Proof.

Assume:

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Then:

$n = 2k + 1$ for some integer k

$$\begin{aligned} 3n + 7 &= 3(2k + 1) + 7 \\ &= 6k + 3 + 7 \\ &= 6k + 10 \\ &= 2(3k + 5) \end{aligned}$$

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Theorem

If $3n + 7$ is an odd integer, then n is an even integer

Proof.

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Since k is an integer, $3k + 5$ is also an integer

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Since k is an integer, $3k + 5$ is also an integer

2(x) is an even integer for any integer x

Therefore: $3n + 7$ is an even integer. if n is odd integer

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Therefore: $3n + 7$ is an even integer. if n is odd integer

□

Proof by Contrapositive (Example 2)

Theorem

For every integer x, if x^2 is even, then x is even.

Proof.

Assume:

x is an odd integer   negation of conclusion

Proof by Contrapositive (Example 2)

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For every integer x, if x^2 is even, then x is even.

Proof.

Assume:

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Then:

x = 2k+1

$x^2 = (2k + 1)^2$

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$x^2 = 4k^2 + 4k + 1$

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$x^2 = 2(2k^2 + 2k) + 1$

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x^2 is odd negation of hypothesis

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Proof by Contrapositive (Example 3)

Theorem

For every positive real number r, if r is irrational, then $\sqrt{r}$ is also irrational.

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Proof.

Assume:

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Proof by Contrapositive (Example 3)

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For every positive real number r, if r is irrational, then $\sqrt{r}$ is also irrational.

Proof.

Assume:

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Then:

$\sqrt{r} = \frac{x}{y}$

Proof by Contrapositive (Example 3)

Theorem

For every positive real number r, if r is irrational, then $\sqrt{r}$ is also irrational.

Proof.

Assume:

$\sqrt{r}$ is rational number   negation of conclusion

Then:

$\sqrt{r} = \frac{x}{y}$

$r = \frac{x^2}{y^2}$ Squaring both sides

Proof by Contrapositive (Example 3)

Theorem

For every positive real number r, if r is irrational, then $\sqrt{r}$ is also irrational.

Proof.

Assume:

$\sqrt{r}$ is rational number negation of conclusion

Then:

$\sqrt{r} = \frac{x}{y}$

$r = \frac{x^2}{y^2}$ Squaring both sides

Note : x and y are integers, also x^2 and y^2 are both integers.
Since $y \neq 0$, y^2 is also non-zero. The number r is equal to the ratio of two integers in which the denominator is non-zero.

Proof by Contrapositive (Example 3)

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For every positive real number r, if r is irrational, then $\sqrt{r}$ is also irrational.

Proof.

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r is rational negation of hypothesis

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Note : x and y are integers, also x^2 and y^2 are both integers.
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r is rational   negation of hypothesis

□

Outline

Proof by Contradiction (Indirect Proof)

proof by contradiction

A proof by contradiction starts by assuming that the theorem is false and then shows that some logical inconsistency arises as a result of this assumption.

- Unlike direct proofs a proof by contradiction can be used to prove theorems that are not conditional statements.

Ex. To prove the statement $p \to q$ then the beginning assumption is $p \wedge \neg q$ which is logically equivalent to $\neg(p \to q)$.

Proof by Contradiction (Example 1)

Theorem

If a and b are positive real numbers then $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$

Proof.

Assume:

1. $a > 0, b > 0$
2. $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$

Proof by Contradiction (Example 1)

Theorem

If a and b are positive real numbers then $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$

Proof.

Assume:

1. $a > 0, b > 0$
2. $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$

Then:

$\therefore (\sqrt{a} + \sqrt{b})^2 = (\sqrt{a+b})^2$ Squaring both sides of 2

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$\therefore (\sqrt{a}^2 + 2\sqrt{ab} + \sqrt{b}^2) = a + b$

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$\therefore (\sqrt{a}^2 + 2\sqrt{ab} + \sqrt{b}^2) = a + b$

$\therefore (\sqrt{a}^2 + 2\sqrt{ab} + \sqrt{b}^2) = a + b$

$\therefore a + 2\sqrt{ab} + b = a + b$ Subtract a+b

Proof by Contradiction (Example 1)

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Proof.

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$\therefore a + 2\sqrt{ab} + b = a + b$ Subtract a+b

$\therefore 2\sqrt{ab} = 0$

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If a and b are positive real numbers then $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$

Proof.

Assume:

1. $a > 0, b > 0$
2. $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$

Then:

$\therefore (\sqrt{a} + \sqrt{b})^2 = (\sqrt{a+b})^2$ Squaring both sides of 2

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$\therefore a + 2\sqrt{ab} + b = a + b$ Subtract a+b

$\therefore 2\sqrt{ab} = 0$

Either a = 0 or b = 0, Contradiction with 1

□

Proof by Contradiction (Example 2)

Theorem

Among any group of 25 people, there must be at least three who are all born in the same month.

Proof by Contradiction (Example 2)

Theorem

p: group of 25 people,
q: there must be at least three who are all born in the same month.
$p \to q$

Proof by Contradiction (Example 2)

Theorem

- x_1 : # *of people in Jan*
- x_2 : # *of people in Feb*
- ...
- x_{12} : # *of people in Dec*
- $x_1 + x_2 + \cdots + x_{12} = 25$
- $(x_1 + x_2 + \cdots + x_{12} = 25) \rightarrow ((x_1 \geq 3) \vee \ldots \vee (x_{12} \geq 3))$

Proof by Contradiction (Example 2)

Proof.

Assume:

1. $(x_1 + x_2 + \cdots + x_{12} = 25)$
2. $((x_1 \leq 2) \wedge \ldots \wedge (x_{12} \leq 2))$

Then.

$(x_1 + x_2 + \cdots + x_{12}) \leq (2 + x_2 + \cdots + x_{12})$

$(x_1 + x_2 + \cdots + x_{12}) \leq (2 + 2 + \cdots + x_{12})$

$(x_1 + x_2 + \cdots + x_{12}) \leq 24$

Contradiction with 1. □

Outline

Proof by cases

- A proof by cases of a universal statement such as $\forall x P(x)$ breaks the domain for the variable x into different cases and gives a different proof for each case.
- Every value in the domain must be included in at least one case.

Example 1

Theorem

For every integer x, $x^2 - x$ is an even integer.

Proof.

Case 1 x is even: $x = 2k$ for some integer k

$$\begin{aligned} x^2 - x &= (2k)^2 - 2k \\ &= 4k^2 - 2k \\ &= 2(2k^2 + k) \\ &= 2d \end{aligned}$$

$\therefore$ theorem is correct for Case 1 $\square$

Example 1

Theorem

For every integer x, $x^2 - x$ is an even integer.

Proof.

Case 2 x is odd: x = 2k +1 for some integer k

$$\begin{aligned} x^2 - x &= (2k+1)^2 - (2k+1) \\ &= 4k^2 + 4k + 1 - (2k+1) \\ &= 4k^2 + 2k \\ &= 2(2k^2 + k) \\ &= 2d \end{aligned}$$

∴ theorem is correct for Case 2 □

Example 2

Theorem

For any real number x, $|x + 5| - x > 1$

Proof.

Case 1. $(x + 5) \geq 0$: Therefore : $|x + 5| = (x - 5)$

$$\begin{aligned} |x + 5| - x &= (x + 5) - x \\ &= 5 > 1 \end{aligned}$$

$\therefore$ theorem is correct for Case 1

Example 2

Theorem

For any real number x, $|x+5| - x > 1$

Proof.

Case 2. $(x+5) \leq 0$: Therefore : $|x+5| = -(x-5)$

$$
\begin{aligned}
|x+5| - x &= -(x+5) - x \\
&= 2(-x) - 5
\end{aligned}
$$

$\because (x+5) \leq 0$
$\because x \leq -5$
$\because -x \geq 5$
$\because 2(-x) - 5 \geq 5 > 1$
$\therefore$ theorem is correct for Case 2 $\square$

Example 3

Theorem

Consider a group of six people. Each pair of people are either friends or enemies with each other. Then there are three people in the group who are all mutual friends or all mutual enemies.

Proof.

On board.

□

Questions