ECEN 227 - Introduction to Finite Automata and Discrete Mathematics

Dr. Mahmoud Nabil
mnmahmoud@ncat.edu

North Carolina A & T State University

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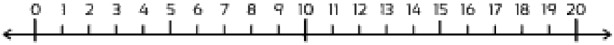
Talk Overview

1. The Division Algorithm
2. Modular Arithmetic
3. Prime factorizations
4. Primality Test

Outline

Number Theory Introduction

- Why do we use numbers basically? For Counting
- Addition and Multiplication operations are invented to support fast counting
- Subtraction and Division are then introduced as inverse operations for Addition and Multiplication.
- Operations are done on the number line.



Ex.

$5+3 = 8$ $\quad$ 8-3 $= 5$

$5\times3 = 15$ $\quad$ $15\div3 = 5$

Division

- We will focus our study on division when investigating the properties of integers.
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Number theory

Number theory is a branch of mathematics concerned with the study of integers. It forms the mathematical basis for modern cryptography.

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  - It means a can be as a product of two intgers one of them is b.
  - Or, $a = k \times b$
  - Then, $\frac{a}{b} = \frac{k \times b}{b} = k$
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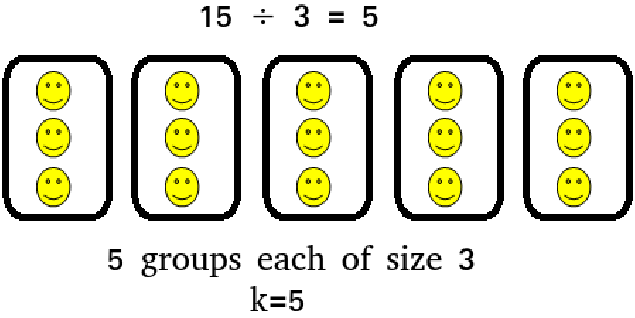
Divisibality

a is divisible by b (or b divides a) denoted by $b \mid a$ if there is an integer k such that $a = k \times b$

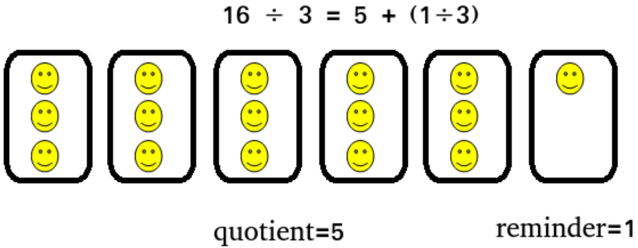
Divisibality

- $b \mid a$ read as b divides a.
- a can be divided into k groups each of size b if the division is possible.



Divisibality

- What if *b* can not divided *a*?



The division algorithm

Theorem

Let n be an integer and let d be a positive integer. Then, there are unique integers q and r, with $0 \le r \le d - 1$, *such that* $n = qd + r$.

Ex.

- $\frac{16}{3} \Rightarrow 16 = 5(3) + 1$
- quotient $= 5$ and reminder $= 1$

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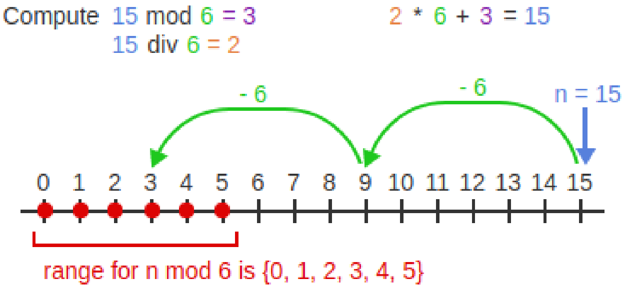
We say

- $16 \text{ div } 3 = 5$ (quotient)
- $16 \bmod 3 = 1$ (reminder)

Note that

Reminder is always positive

Computing div and mod.



Note that

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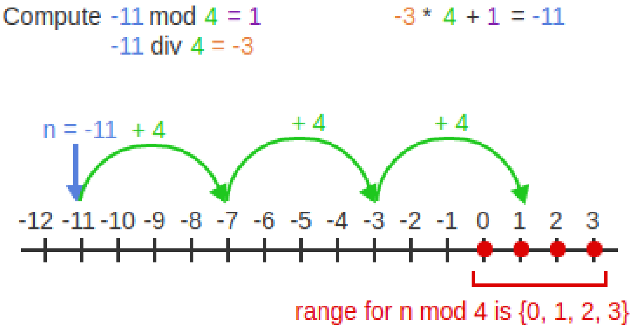
Computing div and mod for positive number.



Note that

Reminder is always positive

Computing div and mod for negative number.

Input: Integers n and d > 0.
Output: q = n div d, and r = n mod d.

Case 1: n ≥ 0.	Case 2: n < 0.

q := 0
r := n
While (r ≥ d)
    q := q + 1
    r := r - d
End-While
	q := 0
	r := n
	While (r < 0)
	    q := q - 1
	    r := r + d
	End-While

Divisibility and linear combinations

- A linear combination of two numbers is the sum of multiples of those numbers. For example, 3x - 7y and -2x + 4y are both linear combinations of x and y.

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Ex.

if 2 divides 10 and 2 divdes 20
Then 2 divdes any number in the form 10a+20b for any a and b.

Excercise

1. 344 mod 5

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   - $344 = 68 \times 5 + 4$, so $344 \text{ div } 5 = 68$.
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   - $(-344) = (-69) \times 5 + 1$, so $(-344) \bmod 5 = 1$.
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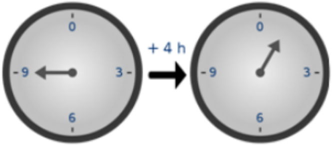
Outline

Modular Arithmetic

- In modular arithmetic, numbers "wrap around" upon reaching a given fixed quantity (this given quantity is known as the modulus) to leave a remainder.
- Imagine we are doing the arithmetic on circle instead of the number line.
- In modulo N, the result of any arithmetic operation takes values from 0 to N-1.



The 12-hour clock : **modulo 12**
If the time is 9:00 now, then 4 hours later it will be 1:00

9+4 =13
13 % 12= 1

Modular Arithmetic

- 1:00 and 13:00 hours are the same
- 1:00 and 25:00 hours are the same
- $1 \equiv 13 \bmod 12$
- $13 \equiv 25 \bmod 12$

$$a \equiv b \bmod n$$

- n is the modulus
- a is congruent to b modulo n
- a-b is an integer multiple of n (i.e., $n \mid (a-b)$)
- $a \bmod n = b \bmod n$

Example

- $38 \equiv 14 \mod 12$

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The same rule apply for negative numbers.

- $\text{-}8 \equiv 7 \bmod 5$

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Congruence Class Example

Integers modulo 5 can take values from $\{0, 1, 2, 3, 4\}$

$0 \equiv 5 \equiv 10 \equiv 15 \ldots \bmod 5$
$1 \equiv 6 \equiv 11 \equiv 16 \ldots \bmod 5$
$2 \equiv 7 \equiv 12 \equiv 17 \ldots \bmod 5$
$3 \equiv 8 \equiv 13 \equiv 18 \ldots \bmod 5$
$4 \equiv 9 \equiv 14 \equiv 19 \ldots \bmod 5$

We call the previous property as congruence class relation modulo 5.

Ring

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Ex.

- The set $Z_{13} = \{0, 1, 2, \ldots, 12\}$ is an arithmetic system modulo 13.
- The set $Z_{17} = \{0, 1, 2, \ldots, 16\}$ is an arithmetic system modulo 17.

Modular Arithmetic Operations

Addition

$[x + y] \bmod m = [(x \bmod m) + (y \bmod m)] \bmod m$

Multiplication

$[x * y] \bmod m = [(x \bmod m) * (y \bmod m)] \bmod m$

Exponentiation

$x^n \bmod m = [(x \bmod m)^n] \bmod m$

Excercise 1

Calculate the following:

- $(72 \times (65) + 211) \bmod 7$
- $38^7 \bmod 3$
- $44^{12} \bmod 6$

Excercise 2

Compute $3^{1000} \bmod 7$

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3^1 mod 7 $=3$

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$3^2 \bmod 7 = 2$

$3^3 \bmod 7 = 6$

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$3^3 \bmod 7 = 6$

$3^4 \bmod 7 = 4$

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$3^3 \bmod 7 = 6$

$3^4 \bmod 7 = 4$

$3^5 \bmod 7 = 5$

Excercise 2

Compute 3^{1000} mod 7

3^{1000} is hard to compute by hand but can we learn anything from trying small modular exponents of 3? (You can use calculator)

3^1 mod 7 =3

3^2 mod 7 =2

3^3 mod 7 =6

3^4 mod 7 =4

3^5 mod 7 =5

3^6 mod 7 =1

$$
\begin{aligned}
3^{1000}\ mod\ 7 &= 3^{6*166+4}\ mod\ 7 \\
&= [3^{6*166}\ mod\ 7 \times 3^4\ mod\ 7]\ mod\ 7 \\
&= [[3^6\ mod\ 7]^{166}\ mod\ 7] \times [3^4\ mod\ 7]\ mod\ 7 \\
&= [[1\ mod\ 7]^{166}\ mod\ 7] \times [3^4\ mod\ 7]\ mod\ 7 \\
&= 1 \times [3^4\ mod\ 7]\ mod\ 7 \\
&= 4
\end{aligned}
$$

Outline

Prime VS Composite Numbers

Prime Number

A prime number p is an integer that can be divided, without a remainder, only by itself and by 1.

Ex.

2,3,5,7,11,13

Composite Number

A positive integer is composite if it has a factor/divisor other than 1 or itself.

Ex.

$14 = 2 \times 7$
$10 = 2 \times 5$
$35 = 5 \times 7$

The Fundamental Theorem of Arithmetic

Theorem

Every positive integer other than 1 can be expressed uniquely as a product of prime numbers where the prime factors are written in increasing order.

Ex.

$1078 = 2 \times 7^2 \times 11$

The factors of 1078 are 2, 7, 11

- The multiplicity of 2 is 1
- The multiplicity of 7 is 2
- The multiplicity of 11 is 1

Greatest common divisor

GCD

The greatest common divisor (gcd) of non-zero integers x and y is the largest positive integer that is a factor of both x and y.

Ex.

GCD of 12 and 30

- Divisors of 12 are: 1, 2, 3, 4, **6** and 12
- Divisors of 30 are: 1, 2, 3, 5, **6**, 10, 15 and 30

The **Greatest Common Divisor** of 12 and 30 is **6**.

Least Common Multiple

LCM

The least common multiple (lcm) of non-zero integers x and y is the smallest positive integer that is an integer multiple of both x and y.

Ex.

LCM of 3 and 5:

- The multiples of 3 are: 3, 6, 9, 12, **15**, 18, ... etc
- The multiples of 5 are: 5, 10, **15**, 20, 25, ... etc

The **Least Common Multiple** of 3 and 5 is **15**

Calculating GCD and LCM Using Prime Factors

Let x and y be two positive integers with prime factorizations expressed using a common set of primes as:

$$x = p_1^{a_1} \times p_2^{a_2} \times \dots p_n^{a_n}$$
$$y = p_1^{b_1} \times p_2^{b_2} \times \dots p_n^{b_n}$$

$$\text{GCD}(x\ ,\ y) = p_1^{min(a_1,b_1)} \times p_2^{min(a_2,b_2)} \times \dots p_n^{min(a_n,b_n)}$$

$$\text{LCM}(x\ ,\ y) = p_1^{max(a_1,b_1)} \times p_2^{max(a_2,b_2)} \times \dots p_n^{max(a_n,b_n)}$$

Excercise

Some numbers and their prime factorizations are given below.

- $532 = 2^2 \times 7 \times 19$
- $648 = 2^3 \times 3^4$
- $1083 = 3 \times 19^2$
- $15435 = 3^2 \times 5 \times 7^3$

Use these prime factorizations to compute the following quantities.

1. gcd(532, 15435)
2. gcd(648, 1083)
3. lcm(532, 1083)
4. lcm(1083, 15435)

Outline

Checking a number is prime

- Primality test is an algorithm used to determine if a number is prime.

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Algorithm 2 Primality Test

Input:
Number N
Output: Prime or Not Prime

```
for i = 2 to N − 1 do
    if N is divisible by i (reminder is zero) then
        return "N is Not Prime"
    end if
end for
return "N is Prime"
```

Checking a number is prime

Theorem

If N is a composite number, then N has a factor greater than 1 and at most $\sqrt{N}$

Checking a number is prime

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Algorithm 4 Primality Test

Input:
Number N
Output: Prime or Not Prime

```
for i = 2 to √N do
    if N is divisible by i (reminder is zero) then
        return "N is Not Prime"
    end if
end for
return "N is Prime"
```

Questions