ECEN 227 - Introduction to Finite Automata and Discrete Mathematics

Dr. Mahmoud Nabil
mnmahmoud@ncat.edu

North Carolina A & T State University

January 27, 2020

Talk Overview

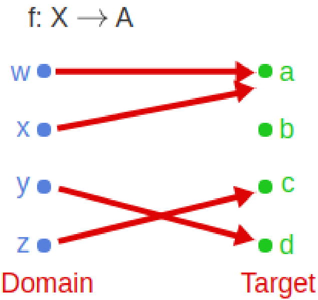
Outline

Introduction

Function

A function f that maps elements of a set X to elements of a set Y, is a subset of X × Y such that for every $x \in X$, there is exactly one y ∈ Y for which (x, y) ∈ f

Arrow Diagram of Function



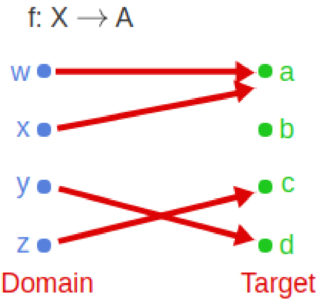
X = { w, x, y, z }

A = { a, b, c, d }

f = { (w, a), (x, a), (y, d), (z, c) }

- f: X → Y means f is a function from X to Y.

Arrow Diagram of Function



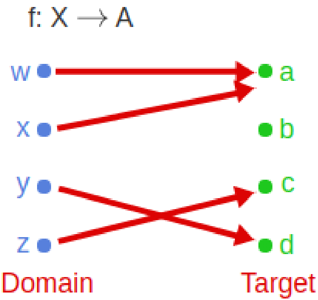
$X = \{ w, x, y, z \}$

$A = \{ a, b, c, d \}$

$f = \{ (w, a), (x, a), (y, d), (z, c) \}$

- f: X → Y means f is a function from X to Y.
- The set X is called the domain of f.

Arrow Diagram of Function



$X = \{ w, x, y, z \}$

$A = \{ a, b, c, d \}$

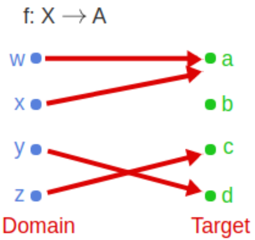
$f = \{ (w, a), (x, a), (y, d), (z, c) \}$

- f: X → Y means f is a function from X to Y.
- The set X is called the domain of f.
- The set Y is the target of f.

Well defined function

Well defined function

f should map every element in the domain to exactly one element in the target to be well defined function.



X = { w, x, y, z }

A = { a, b, c, d }

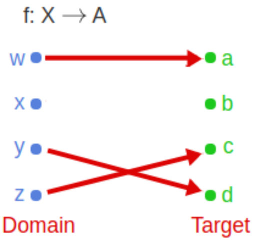
f = { (w, a), (x, a), (y, d), (z, c) }

(Well Defined Function)

Well defined function

Well defined function

f should map every element in the domain to exactly one element in the target to be well defined function.



X = { w, x, y, z }

A = { a, b, c, d }

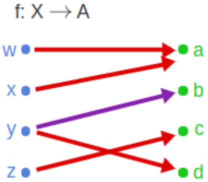
f = { (w, a) , (y, d), (z, c) }

(Not a Function)

Well defined function

Well defined function

f should map every element in the domain to exactly one element in the target to be well defined function.



X = { w, x, y, z }

A = { a, b, c, d }

f = { (w, a), (x, a), (y, d), (z, c), (y, b) }

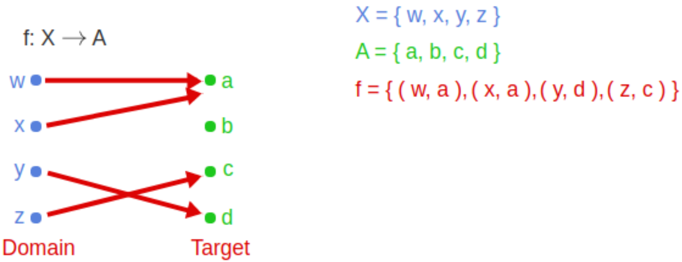
f is no longer a function because (y, b), (y, d) ∈ f.

(Not a Function)

Range

Range

For function f: $X \to Y$, an element y is in the range of f if and only if there is an $x \in X$ such that $(x, y) \in f$.



Range: $\{a, c, d\}$

Excercise on Function Range

Express the range of each function using roster notation.

- Let $A = \{2, 3, 4, 5\}$.
  f: $A \rightarrow Z$ such that $f(x) = 2x - 1$.

Excercise on Function Range

Express the range of each function using roster notation.

- Let $A = \{2, 3, 4, 5\}$.
  f: $A \rightarrow Z$ such that $f(x) = 2x - 1$.
  - $\{3, 5, 7, 9\}$
- Let $A = \{2, 3, 4, 5\}$.
  f: $A \times A \rightarrow Z$, where $f(x,y) = x+y$.

Excercise on Function Range

Express the range of each function using roster notation.

- Let A = {2, 3, 4, 5}.
  f: A → Z such that f(x) = 2x - 1.
  - {3, 5, 7, 9}
- Let A = {2, 3, 4, 5}.
  f: A x A → Z, where f(x,y) = x+y.
  - {4, 5, 6, 7, 8, 9, 10}

Function Equality

Two functions, f and g, are equal if

- f and g have the same domain.
- f and g have the same target.
- $f(x) = g(x)$ for every element x in the domain.

Excercise on Function Equality

Ex. Indicate if f and g are equal fuctions

- **f**: Z → Z, where f(x) = x^2
  g: Z → Z, where g(x) = $|x|^2$.

Excercise on Function Equality

Ex. Indicate if f and g are equal fuctions

- **f**: Z → Z, where f(x) = x^2
  g: Z → Z, where g(x) = $|x|^2$.
  - $f = g$
- **f**: R → Z, where f(x) = x^2
  g: Z → Z, where g(x) = x^2.

Excercise on Function Equality

Ex. Indicate if f and g are equal fuctions

- **f**: Z → Z, where f(x) $= x^2$
  g: Z → Z, where g(x) $= |x|^2$.
  - $f = g$
- **f**: R → Z, where f(x) $= x^2$
  g: Z → Z, where g(x) $= x^2$.
  - $f \neq g$ different domains
- **f**: Z → Z, where f(x) $= x^3$
  g: Z → Z, where g(x) $= |x|^3$.

Excercise on Function Equality

Ex. Indicate if f and g are equal fuctions

- **f**: Z → Z, where f(x) $= x^2$
  g: Z → Z, where g(x) $= |x|^2$.
  - $f = g$
- **f**: R → Z, where f(x) $= x^2$
  g: Z → Z, where g(x) $= x^2$.
  - $f \neq g$ different domains
- **f**: Z → Z, where f(x) $= x^3$
  g: Z → Z, where g(x) $= |x|^3$.
  - $f \neq g$ because, f(-2) = -8, and g(-2) = 8.
- **f**: Z x Z → Z, where f(x,y) $= |x + y|$
  g: Z x Z → Z, where g(x,y) $= |x| + |y|$.

Excercise on Function Equality

Ex. Indicate if f and g are equal fuctions

- **f**: Z → Z, where f(x) = x^2
  g: Z → Z, where g(x) = $|x|^2$.
  - $f = g$
- **f**: R → Z, where f(x) = x^2
  g: Z → Z, where g(x) = x^2.
  - $f \neq g$ different domains
- **f**: Z → Z, where f(x) = x^3
  g: Z → Z, where g(x) = $|x|^3$.
  - $f \neq g$ because, f(-2) = -8, and g(-2) = 8.
- **f**: Z x Z → Z, where f(x,y) = $|x + y|$
  g: Z x Z → Z, where g(x,y) = $|x| + |y|$.
  - $f \neq g$ because, f(-2,2) = 0, and g(-2,2) = 4.

Outline

Excercise on Function Range

Express the range of each function using roster notation.

Floor function

The floor function maps a real number to the nearest integer in the downward direction.

$$\text{floor: } R \rightarrow Z$$
$$\text{floor}(x) = \lfloor x \rfloor$$

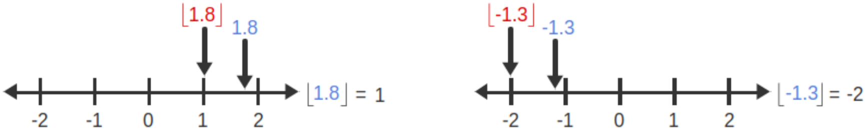
Cieling function

The floor function maps a real number to the nearest integer in the upward direction.

$$\text{ceil: } R \rightarrow Z$$
$$\text{ceil}(x) = \lceil x \rceil$$

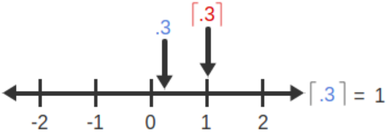
Examples

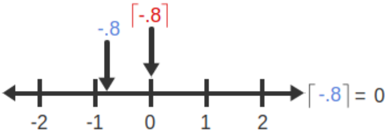
To compute the floor function slide *down* to nearest integer:



Examples

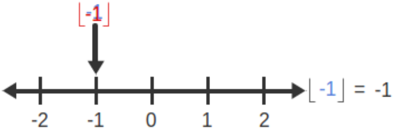
To compute the ceiling function slide *up* to nearest integer:

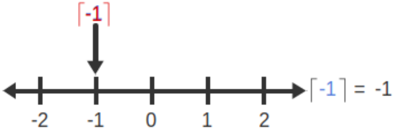




Examples

The ceiling and floor of an integer are the same:





Outline

Function Properties

One-to-one

A function f: $X \to Y$ is one-to-one or injective if $x_1 \neq x_2$ implies that $f(x_1) \neq f(x_2)$.

Function Properties

One-to-one

A function f: X $\rightarrow$ Y is one-to-one or injective if $x_1 \neq x_2$ implies that f(x_1) $\neq$ f(x_2).

Onto

A function f: X $\rightarrow$ Y is onto or surjective if the range of f is equal to the target Y.

Function Properties

One-to-one

A function f: $X \to Y$ is one-to-one or injective if $x_1 \neq x_2$ implies that $f(x_1) \neq f(x_2)$.

Onto

A function f: $X \to Y$ is onto or surjective if the range of f is equal to the target Y.

Bijective

A function is bijective or (one-to-one correspondence) if it is both one-to-one and onto.

In formal definations

One-to-one

Every element in the target is covered by one or less elements from the domain.

In formal definations

One-to-one

Every element in the target is covered by one or less elements from the domain.

Onto

Every element in the target is covered by one or more elements from the domain.

In formal definations

One-to-one

Every element in the target is covered by one or less elements from the domain.

Onto

Every element in the target is covered by one or more elements from the domain.

Bijective

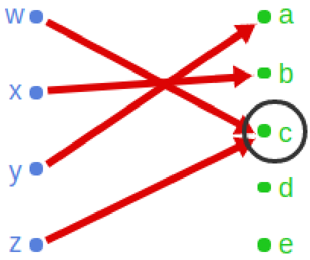
Every element in the target is covered by exactly one element from the domain.

Function Properties Examples

$f: X \to A$

$X = \{ w, x, y, z \}$

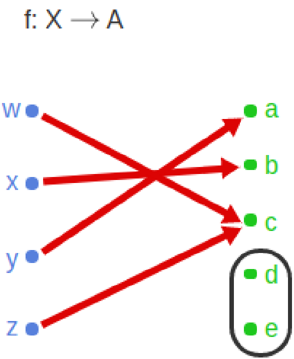
$A = \{ a, b, c, d, e \}$



f is not one-to-one because
$f(w) = f(z) = c$

f is not one-to-one because $f(w) = f(z) = c$.

Function Properties Examples

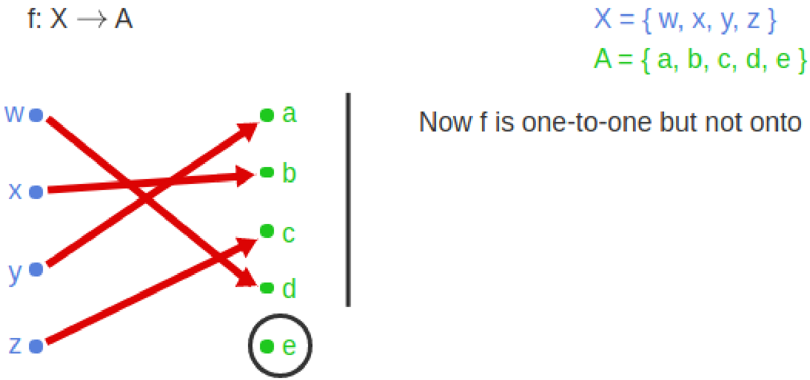


X = { w, x, y, z }
A = { a, b, c, d, e }

f is not onto because
there are no elements in X
that map to d or e

f is not onto because no elements in X map to d or e.

Function Properties Examples



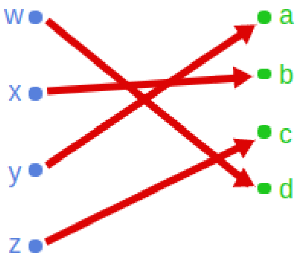
Now f is one-to-one but not onto.

Function Properties Examples

$f: X \to A$

$X = \{ w, x, y, z \}$

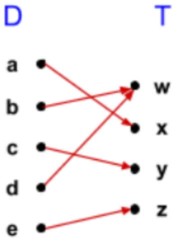
$A = \{ a, b, c, d \}$



Now f is one-to-one and onto

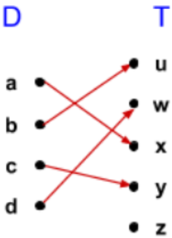
Now f is one-to-one and onto. f is a bijection.

Relative sizes of the domain and target



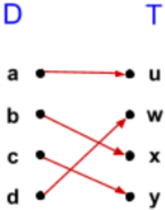
onto

$|D| \geq |T|$



one-to-one

$|D| \leq |T|$



bijection

$|D| \leq |T|$ and $|D| \geq |T|$

$|D| = |T|$

Excercise 1

- Let f be a function whose domain is $\{0,1\}^3$ and whose target is $\{0,1\}^2$.

Ex.

- Is it possible that f is one-to-one?

Excercise 1

- Let f be a function whose domain is $\{0,1\}^3$ and whose target is $\{0,1\}^2$.

Ex.

- Is it possible that f is one-to-one?
  - No
- Is it possible that f is onto?

Excercise 1

- Let f be a function whose domain is $\{0,1\}^3$ and whose target is $\{0,1\}^2$.

Ex.

- Is it possible that f is one-to-one?
  - No
- Is it possible that f is onto?
  - Yes

Excercise 2

For each of the functions below, indicate whether the function is onto, one-to-one, neither or both.

Ex.

- $f : \{0,1\}^3 \to \{0,1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, f(001) = 101 and f(110) = 110

Excercise 2

For each of the functions below, indicate whether the function is onto, one-to-one, neither or both.

Ex.

- $f : \{0, 1\}^3 \rightarrow \{0, 1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, f(001) = 101 and f(110) = 110
    - Neither one-to-one nor onto.

Excercise 2

For each of the functions below, indicate whether the function is onto, one-to-one, neither or both.

Ex.

- $f : \{0,1\}^3 \to \{0,1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, f(001) = 101 and f(110) = 110
  - Neither one-to-one nor onto.
- $f : \{0,1\}^3 \to \{0,1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example f(011) = 110.

Excercise 2

For each of the functions below, indicate whether the function is onto, one-to-one, neither or both.

Ex.

- $f : \{0,1\}^3 \to \{0,1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, f(001) = 101 and f(110) = 110
  - Neither one-to-one nor onto.
- $f : \{0,1\}^3 \to \{0,1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example f(011) = 110.
  - One-to-one and onto.

Excercise 2

- For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

Ex.

- f: R → R. f(x) $= x^2$

Excercise 2

- For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

Ex.

- f: R → R. f(x) $= x^2$
  - Not onto.
  - Not one to one.
- f: R → R. f(x) $= x^3$

Excercise 2

- For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

Ex.

- f: R → R. f(x) $= x^2$
  - Not onto.
  - Not one to one.
- f: R → R. f(x) $= x^3$
  - One to one
  - Onto.
- h: Z → Z. $h(x) = x^3$

Excercise 2

- For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

Ex.

- f: R → R. f(x) $= x^2$
  - Not onto.
  - Not one to one.
- f: R → R. f(x) $= x^3$
  - One to one
  - Onto.
- h: Z → Z. $h(x) = x^3$
  - Not onto.
  - One to one.

Excercise 2

- For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

Ex.

- f: R → R. f(x) $= x^2$
  - Not onto.
  - Not one to one.
- f: R → R. f(x) $= x^3$
  - One to one
  - Onto.
- h: Z → Z. $h(x) = x^3$
  - Not onto.
  - One to one.
- f: Z → Z. f(x) $= x - 4$

Excercise 2

- For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

Ex.

- f: R → R. f(x) $= x^2$
  - Not onto.
  - Not one to one.
- f: R → R. f(x) $= x^3$
  - One to one
  - Onto.
- h: Z → Z. $h(x) = x^3$
  - Not onto.
  - One to one.
- f: Z → Z. f(x) $= x - 4$
  - Onto.
  - One to one.
- f: Z → Z. f(x) $= 5x - 4$

Excercise 2

- For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

Ex.

- f: R → R. $f(x) = x^2$
  - Not onto.
  - Not one to one.
- f: R → R. $f(x) = x^3$
  - One to one
  - Onto.
- h: Z → Z. $h(x) = x^3$
  - Not onto.
  - One to one.
- f: Z → Z. $f(x) = x - 4$
  - Onto.
  - One to one.
- f: Z → Z. $f(x) = 5x - 4$
  - One-to-one
  - Not onto.

Outline

Function Inverse

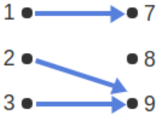
- If a function f: X → Y is a bijection, then the inverse of f is obtained by exchanging the first and second entries in each pair in f.
- The inverse of f is denoted by f^{-1}

$$f^{-1} = \{\ (y, x) : (x, y) \in f\ \}.$$

Example 1

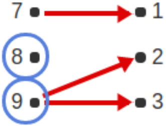
$X = \{1, 2, 3\}$

$Y = \{7, 8, 9\}$



$f: X \to Y$

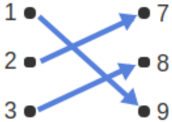
$f = \{(1, 7), (2, 9), (3, 9)\}$



$f^{-1} = \{(7, 1), (9, 2), (9, 3)\}$

f^{-1} is not a function. f does not have an inverse.

Example 2



$g: X \rightarrow Y$

$g = \{ (1, 9), (2, 7), (3, 8) \}$

$g^{-1} = Y \rightarrow X$

$g^{-1} = \{ (7, 2), (8, 3), (9, 1) \}$

g^{-1} is a function. g has an inverse defined by

$g^{-1}(7) = 2$ $g^{-1}(8) = 3$ $g^{-1}(9) = 1$

Excercise 1

- For each of the following functions, indicate whether the function has a well-defined inverse. If the inverse is well-defined, give the input/output relationship of f^{-1}.

Ex.

- f: R $\rightarrow$ R. $f(x) = x^2$

Excercise 1

- For each of the following functions, indicate whether the function has a well-defined inverse. If the inverse is well-defined, give the input/output relationship of f^{-1}.

Ex.

- f: R → R. $f(x) = x^2$
  - Not onto.
  - Not one to one.
  - f^{-1} is not well defined
- f: R → R. $f(x) = x^3$

Excercise 1

- For each of the following functions, indicate whether the function has a well-defined inverse. If the inverse is well-defined, give the input/output relationship of f^{-1}.

Ex.

- f: R → R. $f(x) = x^2$
  - Not onto.
  - Not one to one.
  - f^{-1} is not well defined
- f: R → R. $f(x) = x^3$
  - One to one
  - Onto.
  - $f^{-1}(x) = \sqrt[3]{x}$
- h: Z → Z. $h(x) = x^3$

Excercise 1

- For each of the following functions, indicate whether the function has a well-defined inverse. If the inverse is well-defined, give the input/output relationship of f^{-1}.

Ex.

- f: R → R. $f(x) = x^2$
  - Not onto.
  - Not one to one.
  - f^{-1} is not well defined
- f: R → R. $f(x) = x^3$
  - One to one
  - Onto.
  - $f^{-1}(x) = \sqrt[3]{x}$
- h: Z → Z. $h(x) = x^3$
  - Not onto.
  - One to one.
  - f^{-1} is not well defined

Excercise 1

- For each of the following functions, indicate whether the function has a well-defined inverse. If the inverse is well-defined, give the input/output relationship of f^{-1}.

Ex.

- f: R → R. $f(x) = x^2$
  - Not onto.
  - Not one to one.
  - f^{-1} is not well defined
- f: R → R. $f(x) = x^3$
  - One to one
  - Onto.
  - $f^{-1}(x) = \sqrt[3]{x}$
- h: Z → Z. $h(x) = x^3$
  - Not onto.
  - One to one.
  - f^{-1} is not well defined
- f: Z → Z. $f(x) = x - 4$

Excercise 1

- For each of the following functions, indicate whether the function has a well-defined inverse. If the inverse is well-defined, give the input/output relationship of f^{-1}.

Ex.

- f: R → R. $f(x) = x^2$
  - Not onto.
  - Not one to one.
  - f^{-1} is not well defined
- f: R → R. $f(x) = x^3$
  - One to one
  - Onto.
  - $f^{-1}(x) = \sqrt[3]{x}$
- h: Z → Z. $h(x) = x^3$
  - Not onto.
  - One to one.
  - f^{-1} is not well defined
- f: Z → Z. $f(x) = x - 4$
  - Onto.
  - One to one.
  - $f^{-1}(x) = x + 4$
- f: Z → Z. $f(x) = 5x - 4$

Excercise 1

- For each of the following functions, indicate whether the function has a well-defined inverse. If the inverse is well-defined, give the input/output relationship of f^{-1}.

Ex.

- f: R $\rightarrow$ R. $f(x) = x^2$
  - Not onto.
  - Not one to one.
  - f^{-1} is not well defined
- f: R $\rightarrow$ R. $f(x) = x^3$
  - One to one
  - Onto.
  - $f^{-1}(x) = \sqrt[3]{x}$
- h: Z $\rightarrow$ Z. $h(x) = x^3$
  - Not onto.
  - One to one.
  - f^{-1} is not well defined
- f: Z $\rightarrow$ Z. $f(x) = x - 4$
  - Onto.
  - One to one.
  - $f^{-1}(x) = x + 4$
- f: Z $\rightarrow$ Z. $f(x) = 5x - 4$
  - One-to-one
  - Not onto.
  - f^{-1} is not well defined

Excercise 2

$f : \{0,1\}^3 \to \{0,1\}^3$. The output of is obtained by taking the input string and reversing the bits. For example, $f(011) = 110$

- Indicate whether f has a well-defined inverse and write f^{-1} if exists.

Excercise 2

$f : \{0, 1\}^3 \to \{0, 1\}^3$. The output of is obtained by taking the input string and reversing the bits. For example, $f(011) = 110$

- Indicate whether f has a well-defined inverse and write f^{-1} if exists.

Sol:

- f has a well-defined inverse.
- $f^{-1} = f$

Outline

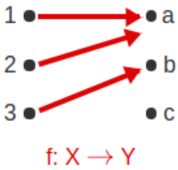
Composition of functions

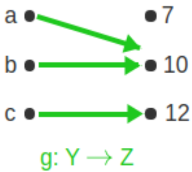
Composition of functions

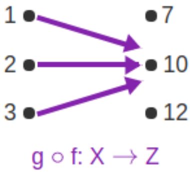
Let f: $X \to Y$ and g: $Y \to Z$.
The composition of g with f, denoted $g \circ f$, is the function $(g \circ f)\colon X \to Z$, such that for all $x \in X$, $(g \circ f)(x) = g(f(x))$.

Example

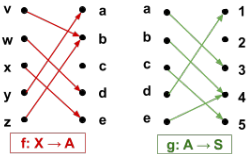
X = { 1, 2, 3 }
Y = { a, b, c }
Z = { 7, 10, 12 }





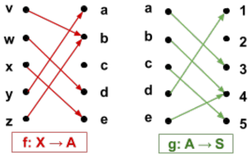


Excercise



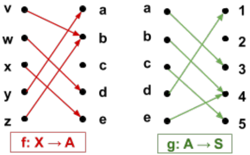
- What is the domain of g ∘ f?

Excercise



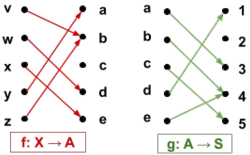
- What is the domain of $g \circ f$?
  - $X = \{v, w, x, y, z\}$

Excercise



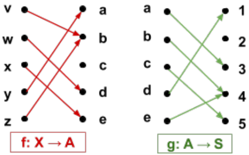
- What is the domain of $g \circ f$?
  - $X = \{v, w, x, y, z\}$
- What is the target of $g \circ f$?

Excercise



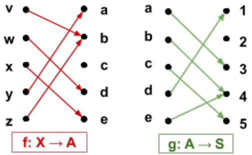
- What is the domain of g ∘ f?
  - X = {v, w, x, y, z}
- What is the target of g ∘ f?
  - S = {1, 2, 3, 4, 5}

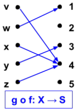
Excercise



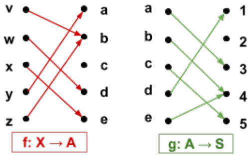
- What is the domain of $g \circ f$?
  - $X = \{v, w, x, y, z\}$
- What is the target of $g \circ f$?
  - $S = \{1, 2, 3, 4, 5\}$
- Give the arrow diagram for $g \circ f$.

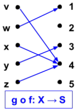
Excercise



- What is the domain of g ∘ f?
  - X = {v, w, x, y, z}
- What is the target of g ∘ f?
  - S = {1, 2, 3, 4, 5}
- Give the arrow diagram for g ∘ f.
  - 

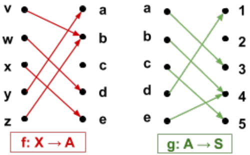
Excercise

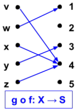


- What is the domain of g ∘ f?
  - X = {v, w, x, y, z}
- What is the target of g ∘ f?
  - S = {1, 2, 3, 4, 5}
- Give the arrow diagram for g ∘ f.
  - 

- What is the range of g ∘ f?

Excercise



- What is the domain of g ∘ f?
  - X = {v, w, x, y, z}
- What is the target of g ∘ f?
  - S = {1, 2, 3, 4, 5}
- Give the arrow diagram for g ∘ f.
  - 

- What is the range of g ∘ f?
  - {1, 3, 4}

Notes I

- $f \circ g$ is not the same as $g \circ f$.

Ex.

$$\text{f: } R^+ \to R^+, \text{ f(x)} = x^3$$
$$\text{g: } R^+ \to R^+, \text{ g(x)} = x + 2$$

- $(f \circ g)(x) = f(g(x)) = (x+2)^3$
- $(g \circ f)(x) = g(f(x)) = x^3 + 2$

Notes II

- It is possible to compose more than two functions.
- Composition is associative.

$$f \circ g \circ h = (f \circ g) \circ h = f \circ (g \circ h) = f(g(h(x)))$$

Ex.

$$\text{f: } R^+ \to R^+,\ \text{f(x)} = x^3$$
$$\text{g: } R^+ \to R^+,\ \text{g(x)} = x + 2$$
$$\text{h: } R^+ \to R^+,\ \text{h(x)} = x - 1$$

- $(f \circ g)(x) = f(g(x)) = (x + 2)^3$
- $(f \circ g \circ h)(x) = f(g(h(x))) = (x + 1)^3$

Identity Function

Identity Function

The identity function always maps a set onto itself and maps every element onto itself.

- The identity function on A, denoted $I_A : A \to A$, is defined as $I_A(a) = a$, for all $a \in A$.

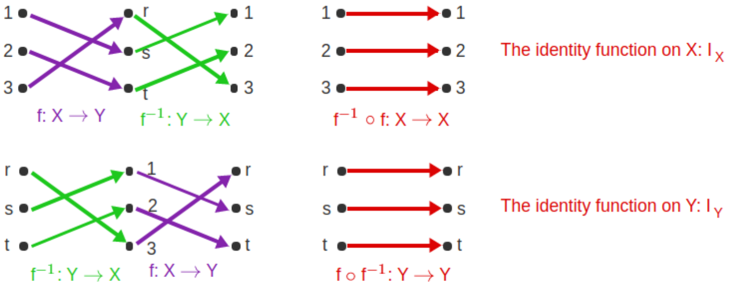
Note That

Let f: $A \to B$ be a bijection. Then $f^{-1} \circ f = I_A$ and $f \circ f^{-1} = I_B$.

Example

X = { 1, 2, 3 }

Y = { r, s, t }



The composition of f with the inverse of f has domain Y and target Y and maps each element to itself and is therefore the identity function on Y.

Questions