ECEN 227 - Introduction to Finite Automata and Discrete Mathematics

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Talk Overview

1. Sum and product rules
2. The generalized product rule
3. Counting permutations
4. Counting subsets

Outline

1. Sum and product rules
2. The generalized product rule
3. Counting permutations
4. Counting subsets

Introduction

- Counting is an important mathematical tool to analyze many problems that arise in computer science.
  - Counting techniques are used to determine the number of valid passwords for a security system.
  - Counting techniques are used to calculate the nummber of addresses in a network.
  - Counting is also at the heart of discrete probability.
- The two most basic rules of counting are
  - Sum rule.
  - Product rule.
- These two rules applied in different combinations can be used to handle a wide range of counting problems.

Product Rule

Consider a restaurant that has a breakfast special that includes a drink, a main course, and a side. The set of choices for each category are:

$$\begin{gathered} \text{D} = \{\text{coffee, orange juice}\} \\ \text{M} = \{\text{pancakes, eggs}\} \\ \text{S} = \{\text{bacon, sausage, hash browns}\} \end{gathered}$$

Any particular breakfast selection can be described by a triplet indicating the choice of drink, main course, and side.

How many different selections you can make?

Product Rule

Breakfast Special:

Drink choices: Coffee, OJ

Main course choices: pancakes, eggs

Side choices: bacon, sausage, hash browns

Breakfast selections: ($\boxed{\text{Drink choice}}$, $\boxed{\text{Main course choice}}$, $\boxed{\text{Side choice}}$)

Select a drink:

Select a main course:

Select a side choice:

(	coffee	pancakes	bacon)
(	coffee	pancakes	sausage)
(	coffee	pancakes	hash browns)
(	coffee	eggs	bacon)
(	coffee	eggs	sausage)
(	coffee	eggs	hash browns)
(	OJ	pancakes	bacon)
(	OJ	pancakes	sausage)
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Number of breakfast selections $= 2 \cdot 2 \cdot 3 = 12$

Product Rule

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Main course choices: pancakes, eggs

Side choices: bacon, sausage, hash browns

Breakfast selections:	(Drink choice , Main course choice , Side choice)
Select a drink:	(coffee , pancakes , bacon)
Select a main course:	(coffee , pancakes , sausage)
Select a side choice:	(coffee , pancakes , hash browns)
	(coffee , eggs , bacon)
	(coffee , eggs , sausage)
	(coffee , eggs , hash browns)
	(OJ , pancakes , bacon)
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Number of breakfast selections = 2 · 2 · 3 = 12

$$|D \times M \times S| = |D| * |M| * |S| = 2 * 2 * 3 = 12$$

Product Rule

Theorem

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then,

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| * |A_2| * \cdots * |A_n|$$

Counting Strings with Product Rule

- If Σ is a set of characters (called an alphabet) then Σ^n is the set of all strings of length n whose characters come from the set Σ

Ex

- if $\Sigma = \{0, 1\}$, what is Σ^4? and what is $|\Sigma^4|$

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  - Σ^4 is 4 bit binary string. And $|\Sigma^4|$=$|\Sigma|$*$|\Sigma|$*$|\Sigma|$*$|\Sigma|$=16

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- if $\Sigma = \{a, b, c\}$, what is Σ^4? and what is $|\Sigma^4|$
  - Σ^4 is 4 character string over {a,b,c}. And $|\Sigma^4|$=$|\Sigma|$*$|\Sigma|$*$|\Sigma|$*$|\Sigma|$=81

Counting Strings with Product Rule

Define S to be the set of strings of length 5 that start and end with {a,b}. And the middle characters are from {a,b,c}

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Answer

$|S| = |\{a, b\} \times \{a, b, c\} \times \{a, b, c\} \times \{a, b, c\} \times \{a, b\}|$

$|S| = |\{a, b\}| * |\{a, b, c\}| * |\{a, b, c\}| * |\{a, b, c\}| * |\{a, b\}| = 2 * 3 * 3 * 3 * 2 = 108$

Sum Rule

Suppose a customer just orders a drink. The customer selects a hot drink or a cold drink.

The hot drink selections are H = {coffee, hot cocoa, tea}.
The cold drink selections are C = {milk, orange juice}.

How many different selections you can make?

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The total number of selections is $|H| + |C| = 3+2 = 5$.

Sum Rule

Theorem

Consider n sets, $A_1, A_2, \ldots, A_n$.
If the sets are mutually disjoint $(A_i \cap A_j = \phi$ *for* $i \neq j)$,

$$\text{Then, } |A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$$

Product and sum rule in combination: counting passwords.

Consider a system in which a password must be a string of length between 6 and 8. The characters can be any lower case letter or digit.

What is the total number of possiblities?

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Answer

- Let L be the set of all lower case letters and D be the set of digits.
- $|L| = 26$ and $|D| = 10$. The set of all allowed characters is C = L ∪ D.
- Since D ∩ L = ϕ, the sum rule can be applied to find the cardinality of C: $|C| = 26 + 10 = 36$
- The user must select a password of length 6 or 7 or 8. Denoted as A_6 or A_7 or A_8. The total number can be calculated as:

$$|A_6 \cup A_7 \cup A_8| = |A_6| + |A_7| + |A_8| = 36^6 + 36^7 + 36^8$$

Excercise

Consider the following definitions for sets of characters:

- Digits = { 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 }
- Letters = { a, b, c, . . . , z }
- Special characters = { *, &, \$, # }

Compute the number of passwords that satisfy the given constraints.

- Strings of length 6. Characters can be special characters, digits, or letters.

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  - $14.40^6 + 14.40^7 + 14.40^8$
- Strings of length 11. Where the first letter and the final letter is special character.

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- Strings of length 11. Where the first letter and the final letter is special character.
  - $4.40^9.4$

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Generalized Product Rule

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- Once the first place runner is determined, there are 19 possibilities left for the second place trophy
- Once the top two runners are determined, there are 18 possibilities for the third place trophy.
- The number of possibilities for the outcome of the race is $20 \cdot 19 \cdot 18 = 6840$.

Generalized Product Rule

Definition 8.3.1: Generalized product rule.

Consider a set S of sequences of k items. Suppose there are:

- n_1 choices for the first item.
- For every possible choice for the first item, there are n_2 choices for the second item.
- For every possible choice for the first and second items, there are n_3 choices for the third item.

$$\vdots$$

- For every possible choice for the first k-1 items, there are n_k choices for the k^{th} item

Then $|S| = n_1 \cdot n_2 \cdots n_k$.

Example

A family of four (2 parents and 2 kids) goes on a hiking trip. They have to pass a narrow trail and one by one. How many ways can they walk with a parent in the front and a parent in the rear?

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Desired sequence: (Parent, Child, Child, Parent)

Count sequences without repetitions

Parents = { Mom, Dad }

Children = { Sister, Brother }

(Parent, Child, Child, Parent)

(Mom, C, C, P)

(Mom, Sis, C, P) — (Mom, Sis, Bro, Dad)

(Mom, Bro, C, P) — (Mom, Bro, Sis, Dad)

(Dad, C, C, P)

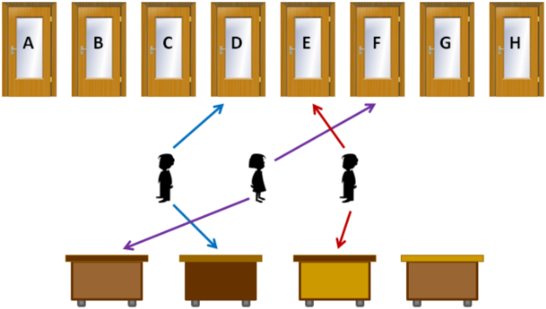
(Dad, Sis, C, P) — (Dad, Sis, Bro, Mom)

(Dad, Bro, C, P) — (Dad, Bro, Sis, Mom)

2 * 2 * 1 * 1 = 4 choices

Example 2

Three employees in a start-up. They rent an office space with 8 offices, anticipating growth. The office space comes with four desks. Each person can select an office and a desk. How many ways are there for the selection to be done



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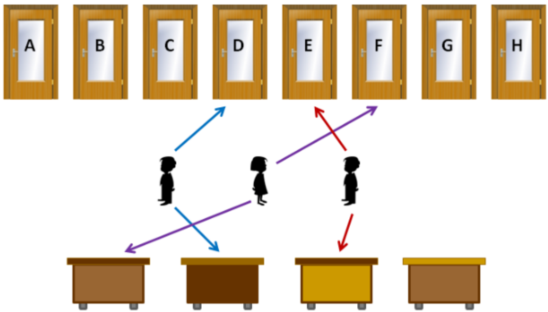

Overall the number of possible selections is: $(8 \cdot 4) \cdot (7 \cdot 3) \cdot (6 \cdot 2) = 8064$

Excercise 1

Consider the following definitions for sets of characters:

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- Letters $= \{$ a, b, c, $\ldots$,z $\}$
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Compute the number of passwords that satisfy the given constraints.

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  - $40 \cdot 39 \cdot 38 \cdot 37 \cdot 36 \cdot 35$
- Strings of length 6. Characters can be special characters, digits, or letters, with no repeated characters. The first character can not be a special character.

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  - $36 \cdot 39 \cdot 38 \cdot 37 \cdot 36 \cdot 35$

Excercise 2

How many strings are there over the set $\{a, b, c\}$ that have length 10 in which no two consecutive characters are the same? For example, the string "abcbcbabcb" would count and the strings "abbbcbabcb" and "aacbcbabcb" would not count.

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Answer.

3.2^9

Excercise 3

License plate numbers in a certain state consists of seven characters. The first character is a digit (0 through 9). The next four characters are capital letters (A through Z) and the last two characters are digits. Therefore, a license plate number in this state can be any string of the form:
Digit-Letter-Letter-Letter-Letter-Digit-Digit

- How many different license plate numbers are possible?

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- How many different license plate numbers are possible?
  - $10^3 * 26^4$
- How many license plate numbers are possible if no digit appears more than once?

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- How many different license plate numbers are possible?
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- How many license plate numbers are possible if no digit appears more than once?
    - $10 * 9 * 8 * 26^4$
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  - $10 * 9 * 8 * 26^4$
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  - 10*9*8*26*25*24*23

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- A common applications of the generalized product rule is in counting permutations
- An r-permutation is a sequence of r items with no repetitions, all taken from the same set.

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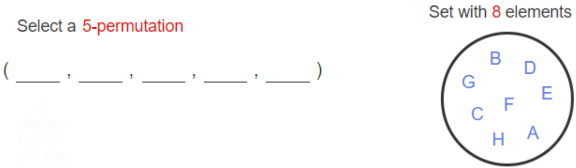

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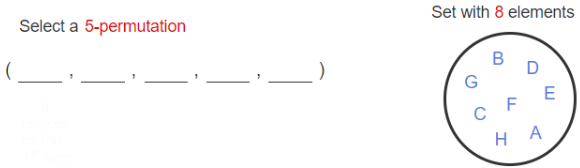


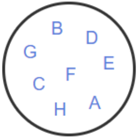
How many possibilities we have? $8*7*6*5*4 = 6720$

Counting Permutations

Select a 5-permutation

(_____ , _____ , _____ , _____ , _____)

Set with 8 elements



Note That

- (A,B,C,D,E) and (E,A,C,D,B) are two different permutations (possibilities).
- In other words, we care about the order within each permutation.

Counting Permutations

Let r and n be positive integers with r ≤ n. The number of r-permutations from a set with n elements is denoted by P(n, r):

$$P(n,r) = \frac{n!}{(n-r)!} = \frac{n(n-1)\ldots(n-r+1)\,\cancel{(n-r)}\,\cancel{(n-r-1)}\,\ldots\,\cancel{1}}{\cancel{(n-r)}\,\cancel{(n-r-1)}\,\ldots\,\cancel{1}} = n(n-1)\ldots(n-r+1)$$

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- Why n-r+1 ? Because, just before the last (rth) item is chosen, r - 1 items have already been chosen and there are n - (r - 1) = n - r + 1.

Example

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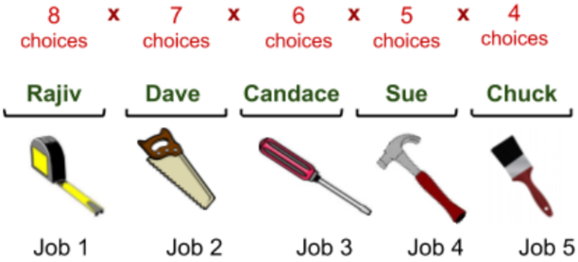

<u>8 Employees:</u>

- Sue
- Dave
- Chuck
- Rajiv
- Candace
- Jeremy
- Nelson
- Maureen

P(n,n)

- A permutation (without the parameter r) is a sequence that contains each element of a finite set exactly once. For example, the set {a, b, c} has six permutations:

(a, b, c)	(b, a, c)	(c, a, b)
(a, c, b)	(b, c, a)	(c, b, a)

The number of permutations of a finite set with n elements is

$$P(n, n) = n \times (n\text{-}1) \times \ldots \times 2 \times 1 = n!$$

Excercise 1

At a certain university in the U.S., all phone numbers are 7-digits long and start with either 824 or 825. Use P(n,r) Notation

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  - 2.P(10, 4)

Excercise 2

Consider the set $\{$John, Paul, George, Ringo$\}$. These four would like to sit on a bench together.

- How many possible seatings are there?

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Consider the set $\{\text{John, Paul, George, Ringo}\}$. These four would like to sit on a bench together.

- How many possible seatings are there?
  - $4!$
- Paul and John would like to sit next to each other. How many possible seatings are there?

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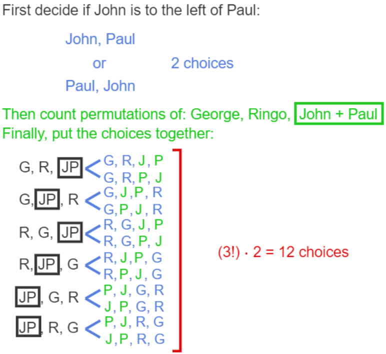
  - 3! * 2



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  - $2 \cdot 2 \cdot 8!$

Excercise 4

Count the number of different functions with the given domain, target and additional properties. Use P(n,r) Notation

- f: $\{0,1\}^7 \to \{0,1\}^7$.

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- f: $\{0,1\}^7 \rightarrow \{0,1\}^7$.
  - $(2^7)^{2^7}$
- f: $\{0,1\}^7 \rightarrow \{0,1\}^7$. The function f is one-to-one.

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  - $2^7!$
- f: $\{0,1\}^5 \to \{0,1\}^7$.

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Count the number of different functions with the given domain, target and additional properties. Use P(n,r) Notation

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Outline

1. Sum and product rules

2. The generalized product rule

3. Counting permutations

4. Counting subsets

R-Subset (R-Combinations)

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R-Subset (R-Combination)

R-subset

A subset of size r is called an r-subset.

Ex.

Let $S = \{a, b, c\}$.

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  - Only 3 $\Rightarrow \{a,b\}$, $\{a,c\}$, $\{b,c\}$

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  - (All of the above 6 pairs map to same subset {orange, blue, pink})

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The idea is to cancel the repeated permutations out of our counting using 6-to-1 rule.

K-to-1 Rule

How many permutations map to the selection {orange, blue, pink} ?

Chosen	Not Chosen
orange	red
blue	green
pink	

Number of ways to permute 3 colors

orange blue pink
orange pink blue
blue orange pink
blue pink orange
pink orange blue
pink blue orange

3! ways

{orange, blue, pink}

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3!-permutations map to one 3-subset

$$\text{Number of 3-subsets of colors} = \frac{P(5,3)}{3!} = \frac{5!}{3!2!} = 10$$

Counting Subsets

Counting subsets: 'n choose r' notation.

The number of ways of selecting an r-subset from a set of size n is:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$\binom{n}{r}$ is read "n choose r". The notation C(n, r) is sometimes used for $\binom{n}{r}$.

Excercise 1

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A teacher must select four members of the math club to participate in an upcoming competition. How many ways are there for her to make her selection if the club has 12 members?

- Since there is **no preference or order** among the 4 members
- In other word given one selection, swapping any pair within that selection give the same selection.
- The answer is $\binom{12}{4}$

Excercise 2

A file will be replicated on 3 different computers in a distributed network of 15 computers. How many ways are there to select the locations for the file?

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A file will be replicated on 3 different computers in a distributed network of 15 computers. How many ways are there to select the locations for the file?

- Suppose the three computers are C_4, C_7, C_{11}. There is is **no preference or order** among these three selection.
- In other word given one selection, swapping any pair within that selection give the same selection.
- The answer is $\binom{15}{3}$

Counting binary strings with a fixed number of 1's

How to count the number of 5-bit strings that have exactly two 1's?

2-subsets of { 1, 2, 3, 4, 5 }.

	1	2	3	4	5
{ 1, 2 }	1	1	0	0	0
{ 1, 3 }	1	0	1	0	0
{ 1, 4 }	1	0	0	1	0
{ 1, 5 }	1	0	0	0	1
{ 2, 3 }	0	1	1	0	0
{ 2, 4 }	0	1	0	1	0
{ 2, 5 }	0	1	0	0	1
{ 3, 4 }	0	0	1	1	0
{ 3, 5 }	0	0	1	0	1
{ 4, 5 }	0	0	0	1	1

# of 5-bit strings with exactly 2 1's

=

# of 2-subsets of { 1, 2, 3, 4, 5 }.

= $\binom{5}{2}$

Excercise 3

14 students have volunteered for a committee. Eight of them are seniors and six of them are juniors.

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- How many ways are there to select a committee with 3 seniors and 2 juniors?

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  - $\binom{8}{3}\binom{6}{2}$
- Suppose the committee must have five students (either juniors or seniors) and that one of the five must be selected as chair. How many ways are there to make the selection?

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  - $5 * \binom{14}{5}$

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  - generalized product rule (r-permutations) = P(n,r)
- Selection of r-items from a set of n-items **without repetition** and order within every selection does not matter
  - r-subset or r-combinations = C(n,r)

Questions