ECEN 227 - Introduction to Finite Automata and Discrete Mathematics

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Talk Overview

1. Introduction to binary relations
2. Properties of binary relations
3. Directed graphs

Outline

Relation

Relation

A binary relation between two sets A and B is a subset R of A x B.

Ex.

- S is the set of students at a university and C is the set of classes offered by the university.
- The relation E between S and C indicates whether a student is enrolled in a given class.
- Usually we can denote this relation as sEc.

Relations and Function

Recall that functions have more restrictions on the connection between the domain and the target as follows.

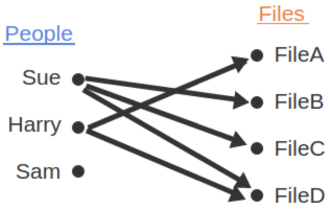
- Each element in the domain should point to one and only one element in the target.
- This is not the case in the relations

Arrow diagram for a relation

People = { Sue, Harry, Sam }

Files = { FileA, FileB, FileC, FileD }

Relation A: pAf if person p has access to file f



A = { (Sue, FileB) , (Sue, FileC) , (Sue, FileD) , (Harry, FileA) , (Harry, FileD) }

Matrix representation for a relation

People = { Sue, Harry, Sam }

Files = { File A, File B, File C, File D }

Relation A: pAf if person p has access to file f

$$
\begin{array}{cc}
 & \begin{array}{cccc} \text{File A} & \text{File B} & \text{File C} & \text{File D} \end{array} \\
\begin{array}{c} \text{Sue} \\ \text{Harry} \\ \text{Sam} \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{array}
$$

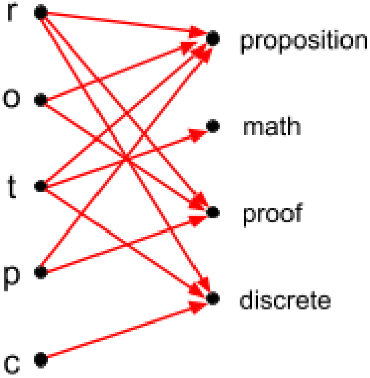
A = { (Sue, File B) (Sue, File C) (Sue, File D)
(Harry, File A) (Harry, File D) }

Excercise

Draw the arrow diagram and the matrix representation for the following relation. Define the set A = {r, o, t, p, c} and B = {discrete, math, proof, proposition}. Define the relation R ⊆ A x B such that (letter, word) is in the relation if that letter occurs somewhere in the word.

Excercise

Draw the arrow diagram and the matrix representation for the following relation. Define the set A = {r, o, t, p, c} and B = {discrete, math, proof, proposition}. Define the relation R ⊆ A x B such that (letter, word) is in the relation if that letter occurs somewhere in the word.

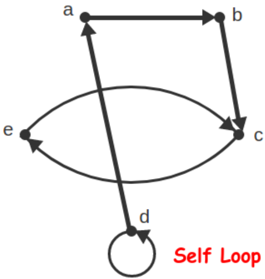


$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Binary Relation on a Set

- We can have a binary relation between a set A and itself.
- In this case we call it a binary relation on the set A.
- The result is a subset of A x A.
- The set A is called the domain of the binary relation.

Ex.



$A = \{ a, b, c, d, e \}$

$R \subseteq A \times A$

$R = \{ (a, b) (b, c) (e, c) (c, e) (d, a) (d, d) \}$

Excercise

Draw the arrow diagram for the following relation. The domain of relation D is {2, 3, 12, 16, 27, 48}. For x, y in the domain, xDy if y is an integer multiple of x.

Excercise

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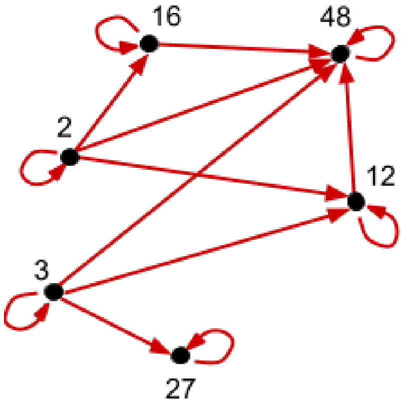
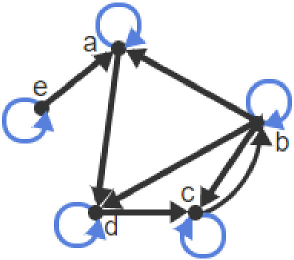


Outline

Binary relation R can be characterized by six properties. The properties are defined and illustrated using arrow diagrams.

- The relation R can be either reflexive or anti-reflexive or neither.
- The relation R can be either symmetric or anti-symmetric or neither.
- The relation R can be either transitive or not transitive.

Reflexive Relation

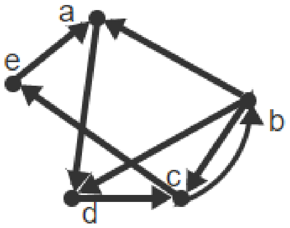


A = { a, b, c, d, e }

Relation R is reflexive if for all $x \in A$ $x R x$

a R a, b R b, c R c, d R d, and e R e

Anti-Reflexive Relation



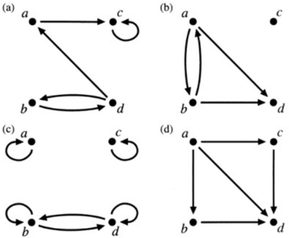
A = { a, b, c, d, e }

Relation R is
anti-reflexive
if for all $x \in A$
it is not true that
x R x

Excercise

Given the below relations indicate whether each relation is:

- reflexive, anti-reflexive, or neither



Excercise

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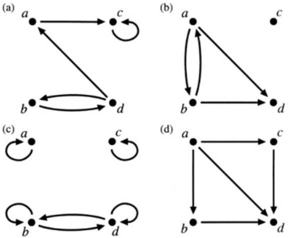
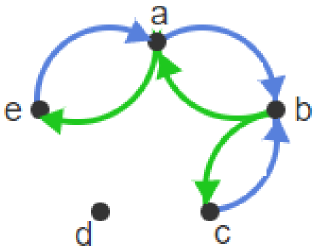


- (a) neither
- (b) anti-reflexive
- (c) reflexive
- (d) anti-reflexive

Symmetric Relation

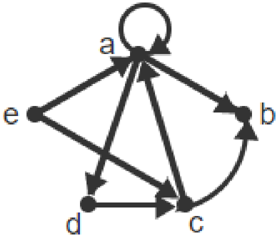


A = { a, b, c, d, e }

Relation R on A is
symmetric
if for all x, y ∈ A
x R y ↔ y R x

x R y ↔ y R x

Anti-Symmetric Relation



A = { a, b, c, d, e }

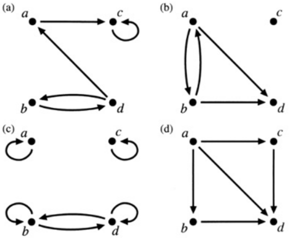
Relation R is
anti-symmetric
if for all x, y ∈ A
x R y and y R x → x = y

Note: there is no 

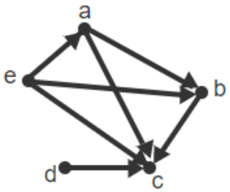
Excercise

Given the below relations indicate whether each relation is:

- symmetric, anti-symmetric, or neither



Transitive Relation



A = { a, b, c, d, e }

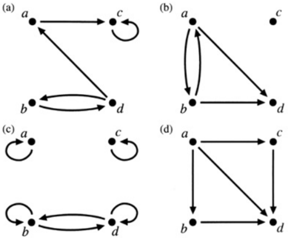
Relation R on A is
transitive if
for all x, y, z $\in$ A
if x R y and y R z,
then x R z

e R a and a R b $\longrightarrow$ e R b
e R b and b R c $\longrightarrow$ e R c
e R a and a R c $\longrightarrow$ e R c
a R b and b R c $\longrightarrow$ a R c

Excercise

Given the below relations indicate whether each relation is:

- transitive or not transitive



Excercise

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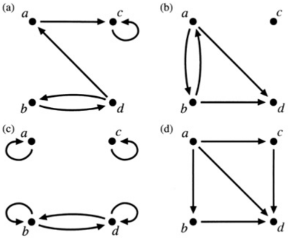


- (a) not transitive
- (b) transitive
- (c) transitive
- (d) transitive

Excercise 3

Given the below relation indicate whether the relation is:

- reflexive, anti-reflexive, or neither
- symmetric, anti-symmetric, or neither
- transitive or not transitive

The domain of the relation L is the set of all real numbers. For $x, y \in R$, xLy if $x < y$.

Excercise 3

Given the below relation indicate whether the relation is:

- reflexive, anti-reflexive, or neither
- symmetric, anti-symmetric, or neither
- transitive or not transitive

The domain of the relation L is the set of all real numbers. For $x, y \in R$, xLy if $x < y$.

Answer.

- anti-reflexive: For any real number x, it is always false that $x < x$.
- anti-symmetric: For any two real numbers x and y, it can not be true that $x < y$ and $y < x$.
- transitive: If $x < y$ and $y < z$, then $x < z$.

Excercise 3

Given the below relation indicate whether the relation is:

- reflexive, anti-reflexive, or neither
- symmetric, anti-symmetric, or neither
- transitive or not transitive

The domain of the relation L is the set of all real numbers. For $x, y \in R$, xLy if $x \leq y$.

Excercise 3

Given the below relation indicate whether the relation is:

- reflexive, anti-reflexive, or neither
- symmetric, anti-symmetric, or neither
- transitive or not transitive

The domain of the relation L is the set of all real numbers. For $x, y \in R$, xLy if $x \leq y$.

Answer.

- reflexive: For any real number x, it is always true that $x \leq x$.
- anti-symmetric: For any two real numbers x and y, if $x \leq y$ and $y \leq x$, then $x = y$.
- transitive: If $x \leq y$ and $y \leq z$, then $x \leq z$.

Outline

Graph

Graph is simply a relation over set. It is used widely in computer science topics.

Ex.

- Internet pages.
- Friends on facebook.
- Train/Bus stations.
- Communication network.
- etc.

Directed Graph (digraph)

Digraph

A directed graph (or digraph, for short) consists of a pair (V, E). V is a set of vertices, and E, a set of directed edges.

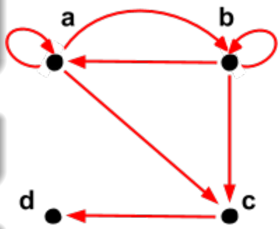
Vertex

An individual element of V is called a vertex.

Edge

A connection between two vertices.

Example

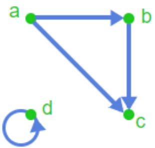


Graph Example

Graph G = (V, E)

V = { a, b, c, d }

E = { (a, b), (b, c), (a, c), (d, d) }



a is the tail of edge (a, b)
b is the head of edge (a, b)

The in-degree of c is 2
The out-degree of b is 1
The in-degree of d is 1

In-degree

The in-degree of a vertex is the number of edges pointing into it.

Out-degree

The out-degree of a vertex is the number of edges pointing out of it.

Walks and directed Graph

A walk in a directed graph G is a sequence of alternating vertices and edges that starts and ends with a vertex.

$$< v_0, v_1, v_2, \ldots, v_n >$$

Walk length

The length of a walk is l, the number of edges in the walk.

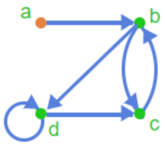
Open walk

An open walk is a walk in which the first and last vertices are different.

Closed walk

A closed walk is a walk in which the first and last vertices are the same.

Example



Walk:

⟨ a, b, c, b, d ⟩ walk length = 4

The walk is open because
the first and last vertices are not the same

Walk:

⟨ b, d, c, b, c, b ⟩ walk length = 5

Edge (c, b) occurs twice
so (c, b) is counted twice

This walk closed because the
first and last vertices are the same.

Closed walk of length 1: ⟨ d, d ⟩

Closed walk of length 0: ⟨ a ⟩

Definations

Trail

A trail is an open walk in which no edge occurs more than once.

Path

A path is a trail in which no vertex occurs more than once.

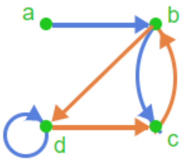
Circuit

A circuit is a closed walk in which no edge occurs more than once.

Cycle

A cycle is a circuit in which no vertex occurs more than once, except the first and last vertices which are the same.

Example



Walk:

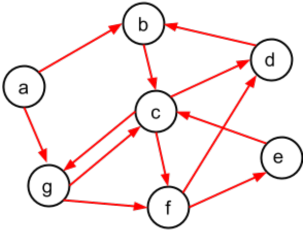
⟨ a, b, c, b, d ⟩ No edge occurs more than once. So this open walk is a trail.

b is reached twice
So this trail is not a path

⟨ b, d, c, b ⟩ No edge occurs more than once. So this closed walk is a circuit.

The circuit is a cycle because only the first and last vertices are repeated.

Excercise



- What is the in-degree of vertex d?
- What is the out-degree of vertex c?
- What is the head of edge (b, c)?
- What is the tail of edge (g, f)?
- List all the self-loops in the graph.
- Is <a, g, f, c, d> a walk in the graph? Is it a trail? Is it a path?
- Is <a, g, f, d, b> a walk in the graph? Is it a trail? Is it a path?
- Is <c, g, f, e> a circuit in the graph? Is it a cycle?

Questions