Differentiation Rules and examples Equations and their graphs

The Basic Rules

The functions \(f(x)=c\) and \(g(x)=x^n\) where \(n\) is a positive integer are the building blocks from which all polynomials and rational functions are constructed. To find derivatives of polynomials and rational functions efficiently without resorting to the limit definition of the derivative, we must first develop formulas for differentiating these basic functions.

The Constant Rule

We first apply the limit definition of the derivative to find the derivative of the constant function, \(f(x)=c\). For this function, both \(f(x)=c\) and \(f(x+h)=c\), so we obtain the following result:f(x)=cf(x)=cf(x)=cf(x)=cf(x+h)=cf(x+h)=c

$$f'(x)=\lim_{h\to 0}f(x+h)-f(x)h=\lim_{h\to 0}c-ch=\lim_{h\to 0}0h=\lim_{h\to 0}0=0.f'(x)=\lim h\to 0f(x+h)-f(x)h=\lim h\to 0c-ch=\lim h\to 00h=\lim h\to 00=0.$$

The rule for differentiating constant functions is called the **constant rule**. It states that the derivative of a constant function is zero; that is, since a constant function is a horizontal line, the slope, or the rate of change, of a constant function is \(0\). We restate this rule in the following theorem.00. We restate this rule in the following theorem.

The Constant Rule

Let \(c\) be a constant. If \(f(x)=c\), then \(f′(c)=0.\)cc be a constant. If f(x)=cf(x)=c, then f'(c)=0.f′(c)=0.

Alternatively, we may express this rule as

$$ddx(c)=0.(3.3.1)(3.3.1)ddx(c)=0.$$

Example \(\PageIndex{1}\): Applying the Constant Rule3.3.13.3.1: Applying the Constant Rule

Find the derivative of \(f(x)=8.\)f(x)=8.f(x)=8.

Solution

This is just a one-step application of the rule: $f'(8)=0.$

Exercise $\PageIndex{1}$

Find the derivative of $g(x)=-3$.

Hint
Answer

The Power Rule

We have shown that

$$\dfrac{d}{dx}(x^2)=2x \text{ and } \dfrac{d}{dx}(x^{1/2})=\dfrac{1}{2}x^{-1/2}.$$

At this point, you might see a pattern beginning to develop for derivatives of the form $\dfrac{d}{dx}\left(x^n\right)$. We continue our examination of derivative formulas by differentiating power functions of the form $f(x)=x^n$ where n is a positive integer. We develop formulas for derivatives of this type of function in stages, beginning with positive integer powers. Before stating and proving the general rule for derivatives of functions of this form, we take a look at a specific case, $\dfrac{d}{dx}(x^3)$. As we go through this derivation, pay special attention to the portion of the expression in boldface, as the technique used in this case is essentially the same as the technique used to prove the general case.

Example $\PageIndex{2}$: Differentiating x^3

Find $\dfrac{d}{dx}\left(x^3\right)$.

Solution:

$\dfrac{d}{dx}(x^3)=\lim_{h\to 0}\dfrac{(x+h)^3-x^3}{h}$

$=\lim_{h\to 0}\dfrac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$	Notice that the first term in the other terms contain powers of
$=\lim_{h\to 0}\dfrac{3x^2h+3xh^2+h^3}{h}$	In this step the x^3 terms h
$=\lim_{h\to 0}\dfrac{h(3x^2+3xh+h^2)}{h}$	Factor out the common factor o
$=\lim_{h\to 0}(3x^2+3xh+h^2)$	After cancelling the common fa
$=3x^2$	Let h go to 0.

Exercise 3.3.2

Find $\dfrac{d}{dx}\left(x^4\right).$

Hint

Answer

As we shall see, the procedure for finding the derivative of the general form $f(x)=x^n$ is very similar. Although it is often unwise to draw general conclusions from specific examples, we note that when we differentiate $f(x)=x^3$, the power on x becomes the coefficient of x^2 in the derivative and the power on x in the derivative decreases by 1. The following theorem states that the **power rule** holds for all positive integer powers of x. We will eventually extend this result to negative integer powers. Later, we will see that this rule may also be extended first to rational powers of x and then to arbitrary powers of x. Be aware, however, that this rule does not apply to functions in which a constant is raised to a variable power, such as $f(x)=3^x$.

The Power Rule

Let n be a positive integer. If $f(x)=x^n$, then

$$f'(x)=nx^{n-1}. \tag{3.3.2}$$

Alternatively, we may express this rule as

$$\frac{d}{dx}\left(x^n\right)=nx^{n-1}. \tag{3.3.3}$$

Proof

For $f(x)=x^n$ where n is a positive integer, we have

$$f'(x)=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}.$$

Since

$$(x+h)^n=x^n+nx^{n-1}h+\binom{n}{2}x^{n-2}h^2+\binom{n}{3}x^{n-3}h^3+\ldots+nxh^{n-1}+h^n,$$

we see that

$$(x+h)^n-x^n=nx^{n-1}h+\binom{n}{2}x^{n-2}h^2+\binom{n}{3}x^{n-3}h^3+\ldots+nxh^{n-1}+h^n.$$

Next, divide both sides by h:

$$\frac{(x+h)^n-x^n}{h}=\frac{nx^{n-1}h+\binom{n}{2}x^{n-2}h^2+\binom{n}{3}x^{n-3}h^3+\ldots+nxh^{n-1}+h^n}{h}.$$

Thus,

$$\frac{(x+h)^n-x^n}{h}=nx^{n-1}+\binom{n}{2}x^{n-2}h+\binom{n}{3}x^{n-3}h^2+\ldots+nxh^{n-2}+h^{n-1}.$$

Finally,

$$f'(x)=\lim_{h\to 0}\left(nx^{n-1}+\binom{n}{2}x^{n-2}h+\binom{n}{3}x^{n-3}h^2+\ldots+nxh^{n-1}+h^n\right)$$

$$=nx^{n-1}.$$

□

Example \(\PageIndex{3}\): Applying the Power Rule 3.3.3

Find the derivative of the function $f(x)=x^{10}$ by applying the power rule.

Solution

Using the power rule with $n=10$, we obtain

$$f'(x)=10x^{10-1}=10x^9.$$

Exercise \(\PageIndex{3}\) 3.3.3

Find the derivative of $f(x)=x^7$.

Hint
Answer

The Sum, Difference, and Constant Multiple Rules

We find our next differentiation rules by looking at derivatives of sums, differences, and constant multiples of functions. Just as when we work with functions, there are rules that make it easier to find derivatives of functions that we add, subtract, or multiply by a constant. These rules are summarized in the following theorem.

Sum, Difference, and Constant Multiple Rules

Let $f(x)$ and $g(x)$ be differentiable functions and k be a constant. Then each of the following equations holds.

Sum Rule. The derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g.

$$\frac{d}{dx}(f(x)+g(x))=\frac{d}{dx}(f(x))+\frac{d}{dx}(g(x)); \tag{3.3.4}$$

that is,

$$\text{for } s(x)=f(x)+g(x),\ s'(x)=f'(x)+g'(x).$$

Difference Rule. The derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g :

$$\frac{d}{dx}(f(x)-g(x))=\frac{d}{dx}(f(x))-\frac{d}{dx}(g(x)); \tag{3.3.5}$$

that is,

$$\text{for } d(x)=f(x)-g(x),\ d'(x)=f'(x)-g'(x).$$

Constant Multiple Rule. The derivative of a constant c multiplied by a function f is the same as the constant multiplied by the derivative:

$$\frac{d}{dx}(kf(x))=k\frac{d}{dx}(f(x)); \tag{3.3.6}$$

that is,

$$\text{for } m(x)=kf(x),\ m'(x)=kf'(x). \tag{3.3.7}$$

Proof

We provide only the proof of the sum rule here. The rest follow in a similar manner.

For differentiable functions $f(x)$ and $g(x)$, we set $s(x)=f(x)+g(x)$. Using the limit definition of the derivative we have

$$s'(x)=\lim_{h\to 0}\frac{s(x+h)-s(x)}{h}.$$

By substituting $s(x+h)=f(x+h)+g(x+h)$ and $s(x)=f(x)+g(x),$ we obtain

$$s'(x)=\lim_{h\to 0}\frac{(f(x+h)+g(x+h))-(f(x)+g(x))}{h}.$$

Rearranging and regrouping the terms, we have

$$s'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right).$$

We now apply the sum law for limits and the definition of the derivative to obtain

$$s'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=f'(x)+g'(x).$$

□

Example 3.3.4: Applying the Constant Multiple Rule

Find the derivative of $g(x)=3x^2$ and compare it to the derivative of $f(x)=x^2.$

Solution

We use the power rule directly:

$$g'(x)=\frac{d}{dx}(3x^2)=3\frac{d}{dx}(x^2)=3(2x)=6x.$$

Since \(f(x)=x^2\) has derivative \(f′(x)=2x\), we see that the derivative of \(g(x)\) is 3 times the derivative of \(f(x)\). This relationship is illustrated in Figure.f(x)=x2f(x)=x2 has

derivative
f'(x)=2xf′(x)=2x, we see that the
derivative
of g(x)g(x) is 3 times the
derivative
of f(x)f(x)

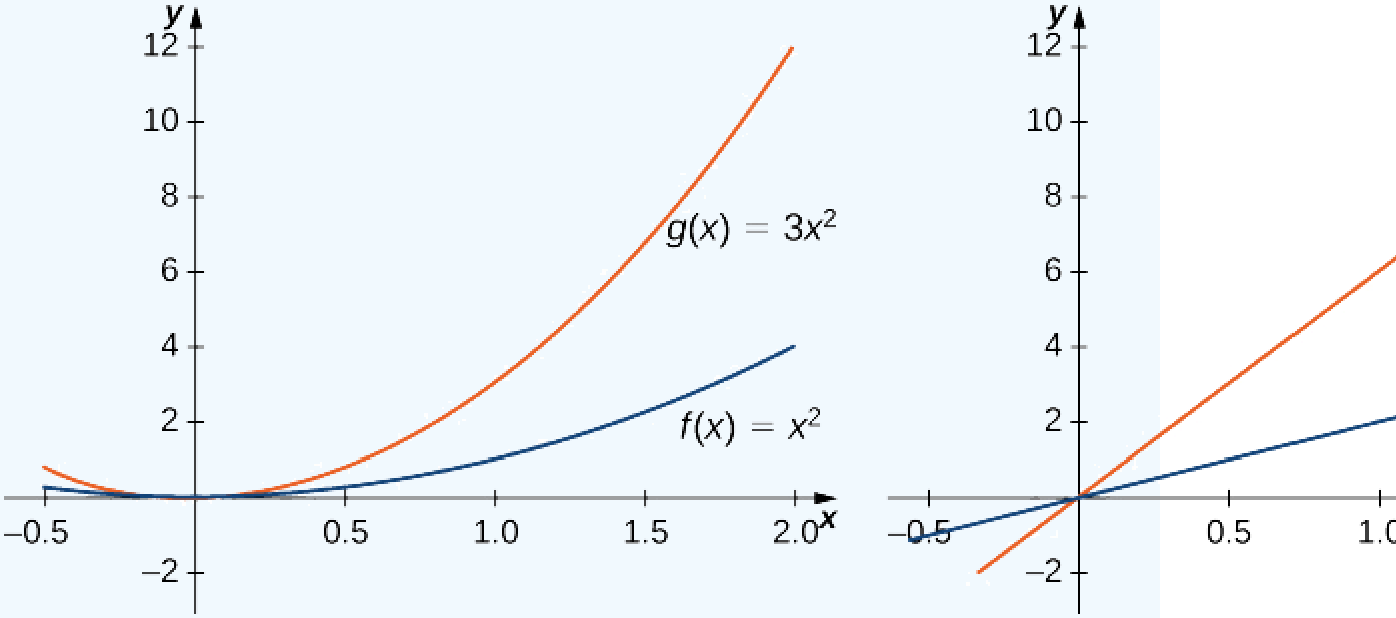

Figure \(\PageIndex{1}\): The derivative of \(g(x)\) is 3 times the derivative of \(f(x)\).3.3.13.3.1: The
derivative
of g(x)g(x) is 3 times the
derivative
of f(x)f(x)

Example \(\PageIndex{5}\): Applying Basic Derivative Rules3.3.53.3.5: Applying Basic
Derivative
Rules

Find the

derivative
of f(x)=2x5+7f(x)=2x5+7

Solution

We begin by applying the rule for differentiating the sum of two functions, followed by the rules for differentiating constant multiples of functions and the rule for differentiating powers. To better understand the

sequence
in which the
differentiation
rules are applied, we use Leibniz notation throughout the solution:

$f'(x)=\frac{d}{dx}(2x^5+7)=\frac{d}{dx}(2x^5)+\frac{d}{dx}(7)=2\frac{d}{dx}(x^5)+\frac{d}{dx}(7)=2(5x^4)+0=10x^4$ Apply the sum rule. Apply the constant multiple rule. Apply the power rule and the constant rule. Simplify. $f'(x)=ddx(2x5+7)=ddx(2x5)+ddx(7)$ Apply the sum rule. $=2ddx(x5)+ddx(7)$ Apply the constant multiple rule. $=2(5x4)+0$ Apply the power rule and the constant rule. $=10x4$ Simplify.

Exercise \(\PageIndex{4}\)3.3.43.3.4

Find the

derivative
of $f(x)=2x^3-6x^2+3.$ $f(x)=2x3-6x2+3.$

Hint

Answer

Example \(\PageIndex{6}\): Finding the Equation of a Tangent Line3.3.63.3.6: Finding the Equation of a
Tangent
Line

Find the equation of the line

tangent
to the graph of $f(x)=x^2-4x+6$ $f(x)=x2-4x+6$ $x=1$ $x=1$

Solution

To find the equation of the

tangent
line, we need a point and a
slope
. To find the point, compute

$$f(1)=1^2-4(1)+6=3.$$ $f(1)=12-4(1)+6=3.$

This gives us the point $(1,3)$. Since the slope of the tangent line at 1 is $f'(1)$, we must first find $f'(x)$. Using the definition of a derivative, we have $(1,3)$. Since the

slope of the tangent line at 1 is $f'(1)$ $f'(x)$. Using the definition of a derivative, we have

$$f'(x)=2x-4$$

so the

slope of the tangent line is $f'(1)=-2$. Using the point-slope formula, we see that the equation of the tangent line is

$$y-3=-2(x-1).$$

Putting the equation of the line in

slope-intercept form, we obtain

$$y=-2x+5.$$

Exercise \(\PageIndex{5}\) $3.3.5$

Find the equation of the line

tangent to the graph of $f(x)=3x^2-11$ $x=2$. Use the point-slope form.

Hint

Answer

The

Product Rule

Now that we have examined the basic rules, we can begin looking at some of the more advanced rules. The first one examines the

derivative of the product of two functions. Although it might be tempting to assume that the derivative of the product is the product of the derivatives, similar to the sum and difference rules, the **product rule** does not follow this pattern. To see why we cannot use this pattern, consider the function $f(x)=x^2$ f(x)=x2, whose derivative is $f'(x)=2x$ f′(x)=2x $\frac{d}{dx}(x)\cdot\frac{d}{dx}(x)=1\cdot 1=1.$ ddx(x)⋅ddx(x)=1⋅1=1.

Product Rule

Let \(f(x)\) and \(g(x)\) be differentiable functions. Then$f(x)$f(x)$g(x)$g(x) be

differentiable functions. Then

$$\frac{d}{dx}(f(x)g(x))=\frac{d}{dx}(f(x))\cdot g(x)+\frac{d}{dx}(g(x))\cdot f(x). \tag{3.3.8}$$

(3.3.8)ddx(f(x)g(x))=ddx(f(x))⋅g(x)+ddx(g(x))⋅f(x).

That is,

$$\text{if } p(x)=f(x)g(x), \text{then } p'(x)=f'(x)g(x)+g'(x)f(x).$$

if p(x)=f(x)g(x),then p′(x)=f′(x)g(x)+g′(x)f(x).

This means that the

derivative of a product of two functions is the derivative of the first function times the second function plus the derivative

of the second
function
times the first
function
.
Proof

We begin by assuming that \(f(x)\) and \(g(x)\) are differentiable functions. At a key point in this proof we need to use the fact that, since \(g(x)\) is differentiable, it is also continuous. In particular, we use the fact that since \(g(x)\) is continuous, \(\displaystyle
\lim_{h→0}g(x+h)=g(x).\)f(x)f(x)g(x)g(x) are

differentiable
functions. At a key point in this proof we need to use the fact that,
since g(x)g(x) is
differentiable
, it is also continuous. In particular, we use the fact that
since g(x)g(x)limh→0g(x+h)=g(x).limh→0g(x+h)=g(x).

By applying the

limit
definition of the
derivative
to p(x)=f(x)g(x),p(x)=f(x)g(x),
p'(x)=limh→0f(x+h)g(x+h)−f(x)g(x)h.p′(x)=limh→0f(x+h)g(x+h)−f(x)g(x)h
.

By adding and subtracting \(f(x)g(x+h)\) in the numerator, we
havef(x)g(x+h)f(x)g(x+h)

p'(x)=limh→0f(x+h)g(x+h)−f(x)g(x+h)+f(x)g(x+h)−f(x)g(x)h.p′(x)=limh→0f(x+h)g(x+h)−f(x)g(x+h)+f(x)g(x+h)−f(x)g(x)h.

After breaking apart this quotient and applying the

sum law for limits
, the
derivative
becomes
p'(x)=limh→0f(x+h)g(x+h)−f(x)g(x+h)h+limh→0f(x)g(x+h)−f(x)g(x)h.p′(x)=limh→0f(x+h)g(x+h)−f(x)g(x+h)h+limh→0f(x)g(x+h)−f(x)g(x)h.

Rearranging, we obtain

$$p'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\cdot g(x+h)\right)+\lim_{h\to 0}\left(\frac{g(x+h)-g(x)}{h}\cdot f(x)\right)=\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)\cdot\left(\lim_{h\to 0}g(x+h)\right)+\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right)\cdot f(x)$$
$$p'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\cdot g(x+h)\right)+\lim_{h\to 0}\left(\frac{g(x+h)-g(x)}{h}\cdot f(x)\right)=\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)\cdot\left(\lim_{h\to 0}g(x+h)\right)+\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right)\cdot f(x)$$

By using the continuity of \(g(x)\), the definition of the derivatives of \(f(x)\) and \(g(x)\), and applying the limit laws, we arrive at the product rule, $g(x)$ $g(x)$ $f(x)$ $f(x)$ $g(x)$ $g(x)$, and applying the

limit laws
, we arrive at the
product rule
,

$$p'(x)=f'(x)g(x)+g'(x)f(x).\quad p'(x)=f'(x)g(x)+g'(x)f(x).$$

□

Example \(\PageIndex{7}\): Applying the Product Rule to Constant Functions $3.3.7$ $3.3.7$: Applying the Product Rule to Constant Functions

For \(p(x)=f(x)g(x)\), use the product rule to find \(p′(2)\) if \(f(2)=3,\; f′(2)=−4,\; g(2)=1\), and \(g′(2)=6\). $p(x)=f(x)g(x)$ $p(x)=f(x)g(x)$, use the

product rule
to
find $p'(2)$ $p'(2)$ $f(2)=3, f'(2)=-4, g(2)=1$ $f(2)=3, f'(2)=-4, g(2)=1$ $g'(2)=6$ $g'(2)=6$

Solution

Since \(p(x)=f(x)g(x)\), \(p′(x)=f′(x)g(x)+g′(x)f(x),\) and hence $p(x)=f(x)g(x)$ $p(x)=f(x)g(x)$ $p'(x)=f'(x)g(x)+g'(x)f(x),$ $p'(x)=f'(x)g(x)+g'(x)f(x),$

$$p'(2)=f'(2)g(2)+g'(2)f(2)=(-4)(1)+(6)(3)=14.\quad p'(2)=f'(2)g(2)+g'(2)f(2)=(-4)(1)+(6)(3)=14.$$

Example \(\PageIndex{8}\): Applying the Product Rule to Binomials $3.3.8$ $3.3.8$: Applying the Product Rule

to Binomials

For \(p(x)=(x^2+2)(3x^3−5x),\) find \(p′(x)\) by applying the product rule. Check the result by first finding the product and then differentiating. $p(x)=(x^2+2)(3x^3-5x)$,p(x)=(x2+2)(3x3−5x), $p'(x)$p′(x) by applying the

product rule
. Check the result by first finding the product and then differentiating.

Solution

If we set \(f(x)=x^2+2\) and \(g(x)=3x^3−5x\), then \(f′(x)=2x\) and \(g′(x)=9x^2−5\).
Thus, $f(x)=x^2+2$f(x)=x2+2$g(x)=3x^3-5x$g(x)=3x3−5x$f'(x)=2x$f′(x)=2x$g'(x)=9x^2-5$g′(x)=9x2−5

$p'(x)=f'(x)g(x)+g'(x)f(x)=(2x)(3x^3-5x)+(9x^2-5)(x^2+2)$.p′(x)=f′(x)g(x)+g′(x)f(x)=(2x)(3x3−5x)+(9x2−5)(x2+2).

Simplifying, we have

$$p'(x)=15x^4+3x^2-10.$$ p′(x)=15x4+3x2−10.

To check, we see that \(p(x)=3x^5+x^3−10x\) and, consequently, \(p′(x)=15x^4+3x^2−10.\)$p(x)=3x^5+x^3-10x$p(x)=3x5+x3−10x$p'(x)=15x^4+3x^2-10$.p′(x)=15x4+3x2−10.

Exercise \(\PageIndex{6}\)3.3.63.3.6

Use the

product rule
to obtain the
derivative
of $p(x)=2x^5(4x^2+x)$.p(x)=2x5(4x2+x).

Hint

Answer

The

Quotient Rule

Having developed and practiced the product rule, we now consider differentiating quotients of functions. As we see in the following theorem, the derivative of the quotient is not the quotient of the derivatives; rather, it is the derivative of the function in the numerator times the function in the denominator minus the derivative of the function in the denominator times the function in the numerator, all divided by the square of the function in the denominator. In order to better grasp why we cannot simply take the quotient of the derivatives, keep in mind that

$$\frac{d}{dx}(x^2)=2x, \text{ not } \frac{\frac{d}{dx}(x^3)}{\frac{d}{dx}(x)}=\frac{3x^2}{1}=3x^2.$$

The Quotient Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then $f(x)$ $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{\frac{d}{dx}(f(x))\cdot g(x)-\frac{d}{dx}(g(x))\cdot f(x)}{(g(x))^2}. \tag{3.3.9}$$

That is, if

$$q(x)=\frac{f(x)}{g(x)}$$

then

$$q'(x)=\frac{f'(x)g(x)-g'(x)f(x)}{(g(x))^2}.$$

The proof of the

quotient rule is very similar to the proof of the product rule, so it is omitted here. Instead, we apply this new rule for finding derivatives in the next example.

Example \(\PageIndex{9}\): Applying the Quotient Rule 3.3.9: Applying the Quotient Rule

Use the quotient rule to find the derivative of $q(x)=\frac{5x^2}{4x+3}$.

Solution

Let \(f(x)=5x^2\) and \(g(x)=4x+3\). Thus, \(f′(x)=10x\) and \(g′(x)=4\). $f(x)=5x^2$ $g(x)=4x+3$ $f'(x)=10x$ $g'(x)=4$

Substituting into the quotient rule, we have

$$q'(x)=\frac{f'(x)g(x)-g'(x)f(x)}{(g(x))^2}=\frac{10x(4x+3)-4(5x^2)}{(4x+3)^2}.$$

Simplifying, we obtain

$$q'(x)=\frac{20x^2+30x}{(4x+3)^2}$$

Exercise \(\PageIndex{7}\) 3.3.7

Find the derivative of $h(x)=\frac{3x+1}{4x-3}$

Answer

Answer

It is now possible to use the

quotient rule
 to extend the
power rule
 to find derivatives of functions of the form x^kxkkk

Extended
Power Rule

If \(k\) is a negative integer, then$k$k

$$\frac{d}{dx}(x^k)=kx^{k-1}. \quad (3.3.10)$$(3.3.10)ddx(xk)=kxk−1.

Proof

If \(k\) is a negative integer, we may set \(n=−k\), so that n is a positive integer with \(k=−n\). Since for each positive integer \(n\),\(x^{−n}=\dfrac{1}{x^n}\), we may now apply the quotient rule by setting \(f(x)=1\) and \(g(x)=x^n\). In this case, \(f′(x)=0\) and \(g′(x)=nx^{n−1}\).
Thus,kk$n=-k$n=−k$k=-n$k=−nnn$x^{-n}=\frac{1}{x^n}$x−n=1xn, we may now apply the

quotient rule
 by
setting $f(x)=1$f(x)=1$g(x)=x^n$g(x)=xn$f'(x)=0$f′(x)=0$g'(x)=nx^{n-1}$g′(x)=nxn−1
$$\frac{d}{d}(x^{-n})=\frac{0(x^n)-1(nx^{n-1})}{(x^n)^2}.$$dd(x−n)=0(xn)−1(nxn−1)(xn)2.

Simplifying, we see that

$$\frac{d}{d}(x^{-n})=\frac{-nx^{n-1}}{x^{2n}}=-nx^{(n-1)-2n}=-nx^{-n-1}.$$dd(x−n)=−nxn−1x2n=−nx(n−1)−2n=−nx−n−1.

Finally, observe that since \(k=−n\), by substituting we have$k=-n$k=−n

$$\frac{d}{dx}(x^k)=kx^{k-1}.$$ddx(xk)=kxk−1.

□

Example \(\PageIndex{10}\): Using the Extended Power Rule$3.3.10$3.3.10:
Using the Extended
Power Rule

Find \(\dfrac{d}{dx}(x^{−4})\).$\frac{d}{dx}(x^{-4})$ddx(x−4)

Solution

By applying the extended

power rule
with $k=-4$

$$\frac{d}{dx}(x^{-4})=-4x^{-4-1}=-4x^{-5}.$$

Example \(\PageIndex{11}\): Using the Extended Power Rule and the Constant Multiple Rule 3.3.11: Using the Extended Power Rule and the Constant Multiple Rule

Use the extended

power rule
and the
constant multiple rule
to find $f(x)=\frac{6}{x^2}$

Solution

It may seem tempting to use the

quotient rule
to find this
derivative
, and it would certainly not be incorrect to do so. However, it is far easier to differentiate this
function
by first rewriting it as $f(x)=6x^{-2}$

$$\begin{aligned} f'(x)&=\frac{d}{dx}\left(\frac{6}{x^2}\right)=\frac{d}{dx}(6x^{-2}) && \text{Rewrite } \frac{6}{x^2} \text{ as } 6x^{-2}. \\ &=6\frac{d}{dx}(x^{-2}) && \text{Apply the constant multiple rule.} \\ &=6(-2x^{-3}) && \text{Use the extended power rule to differentiate } x^{-2}. \\ &=-12x^{-3} && \text{Simplify.} \end{aligned}$$

Exercise \(\PageIndex{8}\) 3.3.8

Find the

derivative
of $g(x)=\frac{1}{x^7}$ using the extended

power rule
.
Hint
Answer

Combining

Differentiation

Rules

As we have seen throughout the examples in this section, it seldom happens that we are called on to apply just one

differentiation
rule to find the
derivative
of a given
function
. At this point, by combining the
differentiation
rules, we may find the derivatives of any polynomial or
rational function
. Later on we will encounter more complex combinations of
differentiation
rules. A good rule of thumb to use when applying several rules is to apply the rules in reverse of the order in which we would evaluate the
function
.

Example \(\PageIndex{12}\): Combining Differentiation Rules3.3.123.3.12: Combining
Differentiation
Rules

For \(k(x)=3h(x)+x^2g(x)\), find
\(k′(x)\).$k(x)=3h(x)+x^2g(x)$k(x)=3h(x)+x2g(x)$k'(x)$k′(x)

Solution: Finding this

derivative
requires the
sum rule
, the

constant multiple rule
, and the
product rule
.

$k'(x)=\frac{d}{dx}(3h(x)+x^2g(x))=\frac{d}{dx}(3h(x))+\frac{d}{dx}(x^2g(x))$k′(x)=ddx(3h(x)+x2g(x))=ddx(3h(x))+ddx(x2g(x

$=3\frac{d}{dx}(h(x))+(\frac{d}{dx}(x^2)g(x)+\frac{d}{dx}(g(x))x^2)$=3ddx(h(x))+(ddx(x2)g(x)+ddx(g(x))x2)

$=3h'(x)+2xg(x)+g'(x)x^2$=3h′(x)+2xg(x)+g′(x)x2

Example \(\PageIndex{13}\): Extending the Product Rule3.3.133.3.13: Extending the Product Rule

For \(k(x)=f(x)g(x)h(x)\), express \(k′(x)\) in terms of \(f(x),g(x),h(x)\), and their derivatives.$k(x)=f(x)g(x)h(x)$k(x)=f(x)g(x)h(x)$k'(x)$k′(x)$f(x),g(x),h(x)$f(x),g(x),h(x)

Solution

We can think of the

function
$k(x)$k(x) as the product of the
function
$f(x)g(x)$f(x)g(x) and the
function
$h(x)$h(x)$k(x)=(f(x)g(x))\cdot h(x)$k(x)=(f(x)g(x))⋅h(x)

$k'(x)=\frac{d}{dx}(f(x)g(x))\cdot h(x)+\frac{d}{dx}(h(x))\cdot(f(x)g(x)).=(f'(x)g(x)+g'(x)f(x))h(x)+h'(x)f(x)g(x)=f'(x)g(x)h(x)+f(x)g'(x)h(x)+f(x)g(x)h'(x).$Apply the product rule to the product of $f(x)g(x)$ and $h(x)$.Apply the product rule to $f(x)g(x)$Simplify.k′(x)=ddx(f(x)g(x))⋅h(x)+ddx(h(x))⋅(f(x)g(x)).Apply the product rule to the product of f(x)g(x) and h(x).=(f′(x)g(x)+g′(x)f(x))h(x)+h′(x)f(x)g(x)Apply the product rule to f(x)g(x)=f′(x)g(x)h(x)+f(x)g′(x)h(x)+f(x)g(x)h′(x).Simplify.

Example \(\PageIndex{14}\): Combining the Quotient Rule and the Product Rule3.3.143.3.14: Combining the
Quotient Rule
and the
Product Rule

For \(h(x)=\dfrac{2x^3k(x)}{3x+2}\), find \(h′(x)\).h(x)=2x3k(x)3x+2h(x)=2x3k(x)3x+2h'(x)h′(x)

Solution

This procedure is typical for finding the

derivative
of a
rational function
.

h'(x)=ddx(2x3k(x))·(3x+2)−ddx(3x+2)·(2x3k(x))(3x+2)2=(6x2k(x)+k'(x)·2x3)(3x+2)−3(2x3k(x))(3x+2)2=−6x3k(x)+18x3k(x)+12x2k(x)+6x4k'(x)+4x3k'(x)(3x+2)2Apply the quotient rule.Apply the product rule to find ddx(2x3k(x)). Use ddx(3x+2)=3.Simplifyh′(x)=ddx(2x3k(x))·(3x+2)−ddx(3x+2)·(2x3k(x))(3x+2)2Apply the quotient rule.=(6x2k(x)+k′(x)·2x3)(3x+2)−3(2x3k(x))(3x+2)2Apply the product rule to find ddx(2x3k(x)). Use ddx(3x+2)=3.=−6x3k(x)+18x3k(x)+12x2k(x)+6x4k′(x)+4x3k′(x)(3x+2)2Simplify

Exercise \(\PageIndex{9}\)3.3.93.3.9

Find \(\dfrac{d}{dx}(3f(x)−2g(x)).\)ddx(3f(x)−2g(x)).ddx(3f(x)−2g(x)).

Hint
Answer
Example \(\PageIndex{15}\): Determining Where a Function Has a Horizontal Tangent3.3.153.3.15: Determining Where a
Function
Has a Horizontal
Tangent

Determine the values of \(x\) for which \(f(x)=x^3−7x^2+8x+1\) has a horizontal tangent line.xx for which $f(x)=x^3−7x^2+8x+1$f(x)=x3−7x2+8x+1 has a horizontal

tangent
line.

Solution

To find the values of \(x\) for which \(f(x)\) has a horizontal tangent line, we must solve \(f′(x)=0\).xx for which $f(x)$f(x) has a horizontal

tangent
line, we must solve $f'(x)=0$f′(x)=0.

Since $f'(x)=3x^2−14x+8=(3x−2)(x−4)$f′(x)=3x2−14x+8=(3x−2)(x−4),

we must solve $(3x−2)(x−4)=0$(3x−2)(x−4)=0. Thus we see that the

function
has horizontal
tangent
lines at $x=23$x=23 and $x=4$x=4 as shown in the following graph.

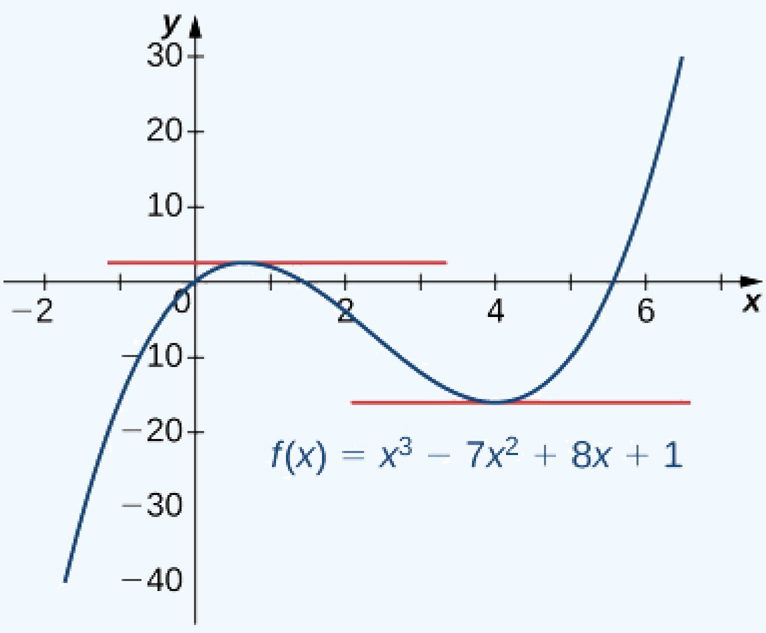

Figure 3.3.23.3.2: This

function
has horizontal
tangent
lines at $x=2/3$x=2/3 and $x=4$x=4.

Example $3.3.16$: Finding a Velocity

The position of an object on a coordinate axis at time t is given by $s(t)=\frac{t}{t^2+1}$. What is the

initial velocity
 of the object?

Solution

Since the

initial velocity
 is $v(0)=s'(0)$, begin by finding $s'(t)$ by applying the
quotient rule
:

$$s'(t)=\frac{1(t^2+1)-2t(t)}{(t^2+1)^2}=\frac{1-t^2}{(t^2+1)^2}.$$

After evaluating, we see that $v(0)=1$.

Exercise $3.3.10$

Find the values of x for which the line

tangent
 to the graph of $f(x)=4x^2-3x+2$ has a
tangent
 line parallel to the line $y=2x+3$.

Hint

Answer

Formula One Grandstands

Formula One car races can be very exciting to watch and attract a lot of spectators. Formula One track designers have to ensure sufficient grandstand space is available around the track to accommodate these viewers. However, car racing can be dangerous, and safety considerations are paramount. The grandstands must be placed where spectators will not be in danger should a driver lose control of a car (Figure $3.3.3$).

Figure $3.3.3$: The grandstand next to a straightaway of the Circuit de Barcelona-Catalunya race track, located where the spectators are not in danger.

Safety is especially a concern on turns. If a driver does not slow down enough before entering the turn, the car may slide off the racetrack. Normally, this just results in a wider turn, which slows the driver down. But if the driver loses control completely, the car may fly off the track entirely, on a path

tangent
 to the curve of the racetrack.

Suppose you are designing a new Formula One track. One section of the track can be modeled by the

function
 $f(x)=x^3+3x^2+x$ (Figure $3.3.4$). The current plan calls for grandstands to be built along the first straightaway and around a portion of the first curve. The plans call for the front corner of the grandstand to be located at the point $(-1.9,2.8)$. We want to determine whether this location puts the spectators in danger if a driver loses control of the car.

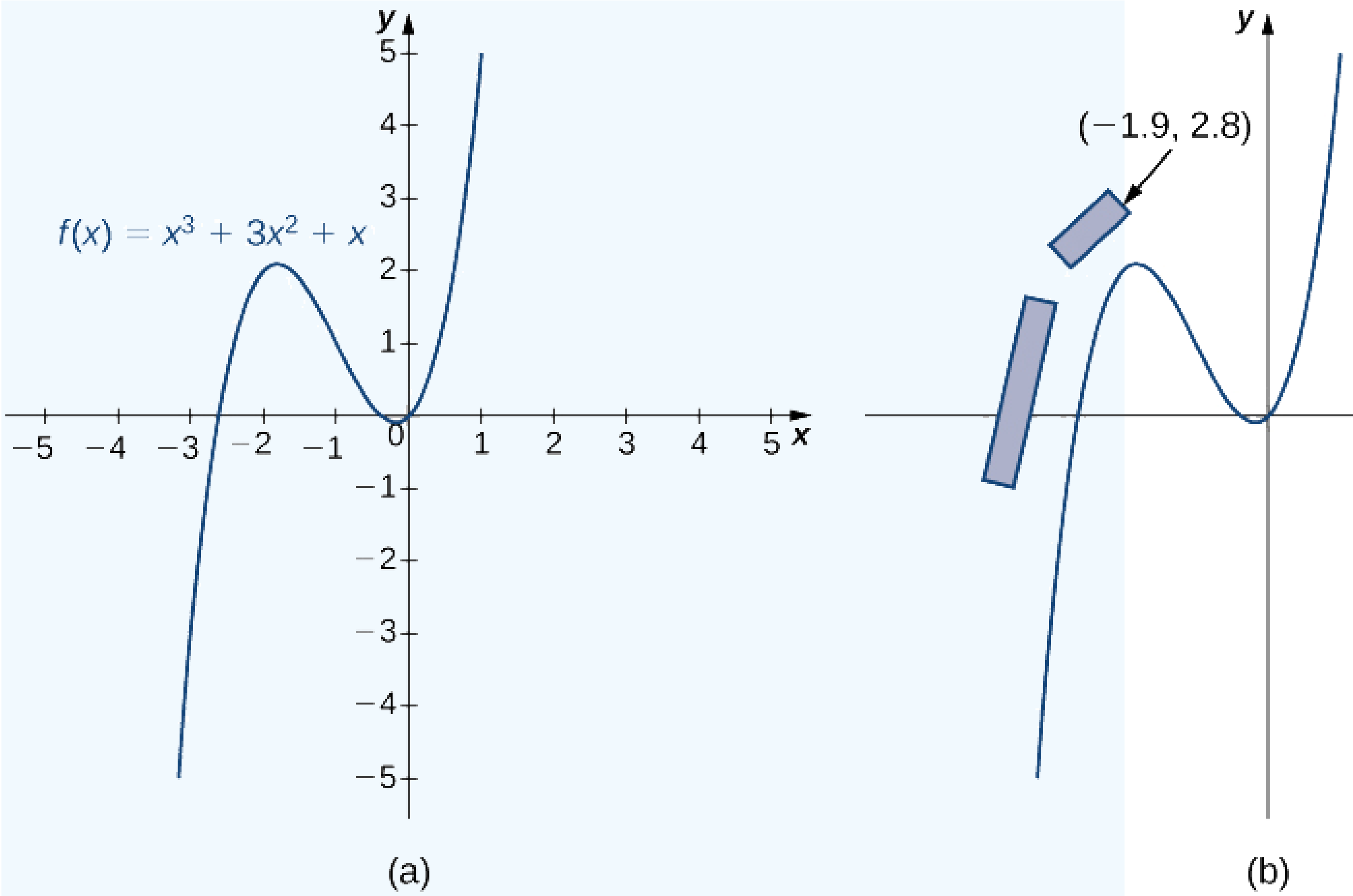

Figure 3.3.43.3.4: (a) One section of the racetrack can be modeled by the function
$f(x)=x^3+3x^2+x$f(x)=x3+3x2+x. (b) The front corner of the grandstand is located at $(−1.9,2.8$−1.9,2.8).

1. Physicists have determined that drivers are most likely to lose control of their cars as they are coming into a turn, at the point where the slope of the tangent line is 1. Find the (x,y)(x,y) coordinates of this point near the turn.
2. Find the equation of the tangent line to the curve at this point.
3. To determine whether the spectators are in danger in this scenario, find the xx-coordinate of the point where the tangent line crosses the line $y=2.8$y=2.8. Is this point safely to the right of the grandstand? Or are the spectators in danger?

4. What if a driver loses control earlier than the physicists project? Suppose a driver loses control at the point $(-2.5, 0.625)$. What is the slope of the tangent line at this point?
5. If a driver loses control as described in part 4, are the spectators safe?
6. Should you proceed with the current design for the grandstand, or should the grandstands be moved?