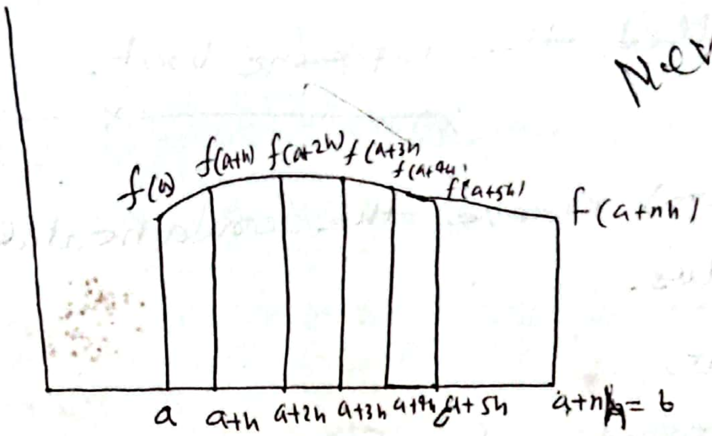
Define Integrals

(VI A)

Define integral $\int_a^b f(x)\,dx$



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Let $f(x)$ be a bounded single valued continuous function defined in the interval (a,b), ~~and~~ a and b being both finite quantities and $b>a$, and let the interval (a,b) be divided into n equal sub-intervals, each of length h, by the points

$a, a+h, a+2h, a+3h, a+4h, \ldots\ldots\ldots, a+nh$.

where $a+nh=b$

$$\Rightarrow h=\frac{b-a}{n}$$

then,

$$\lim_{h\to 0}\left[h\times f(a+h)+h\times f(a+2h)+h\times f(a+3h)+\cdots+h\times f(a+nh)\right]$$

$$=\lim_{h\to 0} h\left[f(a+h)+f(a+2h)+f(a+3h)+\cdots+f(a+nh)\right]$$

$$=\lim_{h\to 0} h\sum_{r=1}^{n} f(a+rh)$$

$$=\lim_{h\to 0}\frac{b-a}{n}\sum_{r=1}^{n} f\left\{a+r\cdot\frac{b-a}{n}\right\}$$

is defined as the definite integral of $f(x)$ with respect to x between the limit a and b and is denoted by the symbol

$$\int_a^b f(x)\,dx$$

where "a" is called the lower limit and [b]
is called the upper limit.

x———x———x

State and prove the fundamental theorem for Calculus.

<u>Answer:</u>

<u>Statement:</u> If $f(x)$ is integrable in (a,b) [...]
and if there exists a function $\phi(x)$ such that ϕ'[...]
$= f(x)$ in (a,b), then

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a)$$

<u>Proof:</u> Divide the interval (a,b) into n parts b[y]
intermediate points

$$a = x_0 < x_1 < x_2 < \cdots\cdots < x_n = b$$

Then we have, by the Mean Value theorem o[f]
Differential Calculus,

$$\phi(x_h) - \phi(x_{h-1}) = (x_h - x_{h-1})\,\phi'(p)$$

where $p \in (a,b)$ $[x_{h-1}$[...]

$$\Rightarrow (x_h - x_{h-1})\,\phi'(p) = \phi(x_h) - \phi(x_{h-1})$$

$$\Rightarrow \sum_{h=1}^{n} \phi'(p)\,\delta_h = \sum_{h=1}^{n} [\phi(x_h) - \phi(x_{h-1})]$$

where $\delta_h = x_h - x_{h-1}$

$$= \phi(x_1) - \phi(x_0) + \phi(x_2) - \phi(x_1) + \phi(x_3) - \phi(x_2) + \cdots + \phi(x_n) - \phi(x_{n-1})$$

$$= \phi(x_n) - \phi(x_0)$$

$$= \phi(b) - \phi(a)$$

$\therefore \lim_{\delta \to 0} \sum \phi'(P)\, \delta_h = \phi(b) - \phi(a)$, at

where δ is the greatest of sub-intervals

Sn: Since $f(x)$ and hence $\phi'(x)$ is integrable in (a, b), therefore

$$\lim_{\delta \to 0} \sum \phi'(P)\, \delta_h = \int_a^b \phi'(x)\, dx = \int_a^b f(x)\, dx.$$

$$\therefore \int_a^b f(x)\, dx = \phi(b) - \phi(a)$$

Proved

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Ex-1. $\int_a^b x^n dx$

Ans: $I = \int_a^b x^n dx$

$$= \left[\frac{x^{n+1}}{n+1}\right]_a^b = \frac{1}{n+1}\left[b^{n+1} - a^{n+1}\right] \text{ Ans}$$

Ex-2. $\int_0^{\pi/2} \cos^2 x\, dx = \frac{1}{2}\int_0^{\pi/2} (1+\cos 2x)\, dx = \left[\frac{x}{2} + \frac{\sin 2x}{4}\right]_0^{\pi/2} = \frac{\pi}{4} + \frac{1}{4}\sin\pi = \frac{\pi}{4}$

Ex-3. $\int_0^1 \frac{1-x}{1+x} dx = \int_0^1 \frac{2+x-1}{1+x} = \int_0^1 \frac{2}{1+x} - \int_0^1 dx = \left[2\log(1+x)\right]_0^1 - [x]_0^1 = 2\log 2 - 1$

Ex-4. $\int_0^a \frac{dx}{a^2+x^2} = \left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^a = \frac{1}{a}\tan^{-1} 1 - \frac{1}{a}\tan^{-1} 0 = \frac{1}{a} \cdot \frac{\pi}{4}$ Ans

Ex-1: $\int_0^1 \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx = \int_0^{\pi/2} z\, dz = \left[\frac{z^2}{2}\right]_0^{\pi/2} = \frac{\pi^2}{8}$

put $\sin^{-1} x = z$

$\frac{1}{\sqrt{1-x^2}} dx = dz$

x	1	0
z	$\pi/2$	0

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Ex-1. $I = \int_0^a \frac{dx}{a^2+x^2}$

$= \left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^a = \frac{1}{a}\tan^{-1}1 - \frac{1}{a}\tan^{-1}0 = \frac{1}{a}\frac{\pi}{4}$ Ans

Ex-2. $I = \int_0^a \sqrt{a^2-x^2}\,dx$ put $x = a\sin\theta$, $dx = a\cos\theta\,d\theta$

x	0	a
θ	0	$\pi/2$

$= \int_0^{\pi/2} \sqrt{a^2-a^2\sin^2\theta}\cdot a\cos\theta\,d\theta$

$= \frac{1}{2}a^2\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta = \frac{1}{2}a^2\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$

$= \frac{1}{2}a^2\left[\left(\frac{\pi}{2}+\frac{1}{2}\sin\pi\right) - \left(0+\frac{1}{2}\sin 0\right)\right]$

$= \frac{1}{4}\pi a^2$ Ans

v.v.g

Ex-3: Evaluate $\int_\alpha^\beta \sqrt{(x-\alpha)(\beta-x)}\,dx$

put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

$dx = 2(\beta-\alpha)\sin\theta\cos\theta\,d\theta$

x	α	β
θ	0	$\pi/2$

also

$x-\alpha = \alpha\cos^2\theta + \beta\sin^2\theta - \alpha$

$= \beta\sin^2\theta - \alpha(1-\cos^2\theta)$

$= \beta\sin^2\theta - \alpha\sin^2\theta = \sin^2\theta(\beta-\alpha)$

and

$\beta - x = \beta - \alpha\cos^2\theta - \beta\sin^2\theta$

$= (\beta-\alpha)\cos^2\theta$

$\therefore I = 2(\beta-\alpha)^2\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta$

$= \frac{1}{2}(\beta-\alpha)^2\int_0^{\pi/4}\sin^2 2\theta$

$= \frac{1}{4}(\beta-\alpha)^2\int(1-\cos 4\theta)\,d\theta$

$= \frac{1}{4}(\beta-\alpha)^2\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2}$

$= \frac{1}{4}(\beta-\alpha)^2\left[\frac{\pi}{2} - \frac{1}{4}\sin 2\pi\right]$

$= \frac{\pi}{8}(\beta-\alpha)^2$

$\cancel{x}$ $\cancel{x} = \alpha\cos^2\theta + \beta\sin^2\theta$

$\cancel{\alpha} = \alpha(1-\sin^2\theta) + \beta\sin^2\theta$

$0 = (\beta-\alpha)\sin^2\theta$

$\sin^2\theta = 0$

$\boxed{\theta = 0}$

$\beta = \alpha\cos^2\theta + \beta\sin^2\theta$

$\cos^2\theta = 0$

$\theta = \pi/2$

CHAPTER–VI (A)

Ex: 4 let $I = \int_{\alpha}^{\beta} \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$ $\beta > \alpha$

put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

$dx = 2(\beta-\alpha)\sin\theta\cos\theta$

also,

$x-\alpha = \alpha\cos^2\theta + \beta\sin^2\theta - \alpha$

$= (\beta-\alpha)\sin^2\theta$

$\beta - x = (\beta-\alpha)\cos^2\theta$

when $x = \alpha$, $(\beta-\alpha)\sin^2\theta = 0$

$\sin\theta = 0$

$\theta = 0$

Similarly

$x = \beta$, $(\beta-\alpha)\cos^2\theta = 0$

$\cos\theta = 0$

$\theta = \frac{\pi}{2}$

$$I = \int_0^{\pi/2} \frac{2(\beta-\alpha)\sin\theta\cos\theta\, d\theta}{(\beta-\alpha)\sin\theta\cos\theta}$$

$$= 2\int_0^{\pi/2} d\theta = 2[\theta]_0^{\pi/2} = 2\left(\frac{\pi}{2}-0\right) = \pi$$

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Ex: 5 show that $\int_0^{1/2} \frac{dx}{(1-2x^2)\sqrt{1-x^2}} = \frac{1}{2}\log(2+\sqrt{3})$

Solⁿ: $I = \int_0^{1/2} \frac{dx}{(1-2x^2)\sqrt{1-x^2}}$

x	0	$1/2$
θ	0	$\pi/6$

put $x = \sin\theta$

$dx = \cos\theta\, d\theta$

$$\therefore I = \int_0^{\pi/6} \frac{\cos\theta\, d\theta}{(1-2\sin^2\theta)\cos\theta} = \int_0^{\pi/6} \frac{d\theta}{\cos 2\theta} = \int_0^{\pi/6} \sec 2\theta\, d\theta$$

$$= \left[\frac{1}{2}\log\tan\left(\frac{\pi}{4}+\theta\right)\right]_0^{\pi/6} = \frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{\pi}{6}\right) - \frac{1}{2}\log\tan\left(\frac{\pi}{4}+0\right)$$

$$= \frac{1}{2}\left[\log\tan\frac{5}{12}\pi - \log\tan\frac{\pi}{4}\right]$$

$$= \frac{1}{2}\left[\log(2+\sqrt{3})\right]$$

Ex 16 show that $\int_0^{\pi/2} \sin^6\theta \cos^3\theta \, d\theta = \frac{2}{63}$

Solution: let $I = \int_0^{\pi/2} \sin^6\theta \cos^3\theta \, d\theta$

put $x = \sin\theta$

$dx = \cos\theta \, d\theta$

also, when $\theta = 0$ then $x = 0$ and $\theta = \pi/2$ t

$$I = \int_0^1 x^6 (1-x^2) \, dx$$

$$= \int_0^1 x^6 \, dx - \int_0^1 x^8 \, dx$$

$$= \frac{1}{7}\left[x^7\right]_0^1 - \frac{1}{9}\left[x^9\right]_0^1 = \frac{1}{7} - \frac{1}{9} = \frac{2}{63}$$

Examples. VI (A)

Evaluate the following integrals:

2 (ii) $\int_0^{2a} \sqrt{2ax - x^2} \, dx$

let $I = \int_0^{2a} \sqrt{2ax - x^2} \, dx$

$$= \int_0^{2a} \sqrt{a^2 - (x-a)^2} \, dx$$

$$= \left[\frac{x\sqrt{2ax - x^2}}{2} + \frac{a^2}{2} \sin^{-1}\frac{x-a}{a}\right]_0^{2a}$$

$$= \left[\frac{2a\sqrt{2a \cdot 2a - 4a^2}}{2} + \frac{a^2}{2} \sin^{-1}\frac{2a-a}{a}\right] - \left[0 + \frac{a^2}{2}\sin^{-1}\frac{a}{a}\right]$$

$$= \frac{a^2}{2} \sin^{-1} 1 + \frac{a^2}{2} \sin^{-1} 1$$

$$= a^2 \sin^{-1} 1$$

$$= a^2 \sin^{-1} \sin \pi/2$$

$$= \frac{a^2\pi}{2} \quad \text{Ans}$$

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2 (iii) $I = \int_1^{e^2} \frac{dx}{x(1+\log x)^2}$

put $1+\log x = \theta \longrightarrow$ $\log e^2 = 2\log e$

$\frac{1}{x}dx = d\theta$ $\qquad \log 1 = 0$

x	1	e^2
θ	1	3

$= \int_1^3 \frac{d\theta}{\theta^2}$

$= -\left[\frac{1}{\theta}\right]_1^3 = -\left[\frac{1}{3}-1\right] = -\left(\frac{1-3}{3}\right) = 2/3$ Ans

3 Let

$I = \int_0^1 x e^x dx$

Now, $I = \int x e^x dx$

$= x\int e^x dx - \int\left\{\frac{d}{dx}(x)\int e^x dx\right\}dx$

$= xe^x - \int e^x dx$

$= xe^x - e^x$

$\therefore \int_0^1 xe^x dx = \left[xe^x - e^x\right]_0^1 = \left(1.e^1 - e^1\right) - \left(0.e^0 - e^0\right)$

$= e - e + 1 = 1$ Ans

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4 (ii) $\int_0^1 \tan^{-1}x\, dx$

Let $I = \int \tan^{-1}x\, dx$

$= \tan^{-1}x\int dx - \int\left\{\frac{d}{dx}\tan^{-1}x\int dx\right\}dx$

$= x\tan^{-1}x - \int \frac{x}{1+x^2}dx$ $\qquad$ $1+x^2 = z$, $2x\,dx = dz$

$= x\tan^{-1}x - \frac{1}{2}\int \frac{1}{z}$

$= x\tan^{-1}x - \frac{1}{2}\log(1+x^2)$

$\therefore \int_0^1 \tan^{-1}x\, dx = \left[x\tan^{-1}x - \frac{1}{2}\log(1+x^2)\right]_0^1$

$= \left[1\tan^{-1}1 - \frac{1}{2}\log 2 - 0 + \frac{1}{2}\log 1\right]$

$= \frac{\pi}{4} - \frac{1}{2}\log 2$

5(i) $\int_0^{\pi} \sin mx \sin nx\, dx$

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$2\sin C \sin D = \cos(C-D) - \cos(C+D)$

let $I = \int \sin mx \sin nx\, dx$

$= \frac{1}{2}\int \{\cos(m-n)x - \cos(m+n)x\} dx$

$= \frac{1}{2} \sin(m-n)x \cdot \frac{1}{m-n} - \frac{1}{2}\sin(m+n)x \cdot \frac{1}{m+n}$

$\therefore \int_0^{\pi} \sin mx \sin nx\, dx = \frac{1}{2}\left[\frac{\sin(m-n)\pi}{m-n} - \frac{\sin(m+n)\pi}{m+n}\right]$

5(iii) $\int_0^{\pi/2} \sin x \sin 2x\, dx$

let $I = \int \sin x \sin 2x$

$= \frac{1}{2}\int (\cos x - \cos 3x)\, dx$

$= \frac{1}{2}\sin x - \frac{1}{6}\sin 3x$

$\therefore \int_0^{\pi/2} \sin x \sin 2x\, dx = \left[\frac{1}{2}\sin x - \frac{1}{6}\sin 3x\right]_0^{\pi/2}$

$= \frac{1}{2}\sin \pi/2 - 1/6 \sin 3\pi/2$

$= \frac{1}{2} + \frac{1}{6} = 2/3$

2. (ii) $\int_0^a \frac{dx}{(a^2+x^2)^{3/2}}$

Let $I = \int_0^a \frac{dx}{(a^2+x^2)^{3/2}}$

put $x = a\tan\theta$

$dx = a\sec^2\theta\, d\theta$

x	a	0
θ	$\pi/4$	0

$\therefore \int_0^{\pi/4} \frac{a\sec^2\theta\, d\theta}{a^3\sec^3\theta}$

$= \frac{1}{a^2}\int_0^{\pi/4} d\theta$

$= \frac{1}{a^2}\left[\theta\right]_0^{\pi/4}$

$= \frac{1}{a^2}\left(\pi/4 - 0\right)$

$= \frac{1}{a^2}\int_0^{\pi/4} \cos\theta\, d\theta$

$= \frac{1}{a^2}\left[\sin\theta\right]_0^{\pi/4}$

$= \frac{1}{a^2}\left[\sin\pi/4 - 0\right]$

$= \frac{1}{\sqrt{2}a^2}$ Ans.

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(ii) $\int_0^{\pi/2} \sin^2 x\cos^2 x\, dx = \frac{1}{4}\int_0^{\pi/2} \sin^2 2x\, dx$

$= \frac{1}{8}\int_0^{\pi/2} (1-\cos 4x)\, dx$

$= \frac{1}{8}\left[x - \frac{1}{4}\sin 4x\right]_0^{\pi/2}$

$= \frac{1}{8}\left[\pi/2 - \frac{1}{4}\sin 2\pi\right]$

$= \frac{\pi}{16} + \frac{1}{32}$

$= \frac{2\pi+1}{32}$ Ans.

11. (i) $\int_1^{\sqrt{e}} x \log x \, dx$

let $I = \int x \log x \, dx$

$= \log x \int x \, dx - \int \left(\frac{d}{dx}(\log x) \int x \, dx \right) dx$

$= \frac{1}{2}x^2 \log x - \int \frac{1}{x} \cdot \frac{1}{2}x^2 \, dx$

$= \frac{1}{2}x^2 \log x - \frac{1}{2} \int x \, dx$

$= \frac{1}{2}x^2 \log x - \frac{1}{4}x^2$

$\therefore \int_1^{\sqrt{e}} x \log x \, dx = \left[\frac{1}{2}x^2 \log x - \frac{1}{4}x^2 \right]_1^{\sqrt{e}}$

$= \frac{1}{2}(\sqrt{e})^2 \log \sqrt{e} - \frac{1}{4}(\sqrt{e})^2 - \frac{1}{2} \cdot 1^2 \log 1 + \frac{1}{4} \cdot 1^2$

$= \frac{1}{4}e - \frac{1}{4}e - 0 + \frac{1}{4}$

$= \frac{1}{4}$ Ans.

(ii) $\int_0^{\pi/2} x^2 \sin x \, dx$

let $I = \int x^2 \sin x \, dx$

$= x^2 \int \sin x \, dx - \int \left(\frac{d}{dx}(x^2) \int \sin x \, dx \right) dx$

$= -x^2 \cos x + \int 2x \cos x \, dx$

$= -x^2 \cos x + 2 \left[x \int \cos x \, dx - \int \left(\frac{d}{dx}(x) \int \cos x \, dx \right) dx \right.$

$$= -x^2\cos x + 2x\sin x - 2\int \sin x \, dx$$

$$= -x^2\cos x + 2x\sin x + 2\cos x$$

$$\therefore \int_0^{\pi/2} x^2 \sin x \, dx = \left[-x^2\cos x + 2x\sin x + 2\cos x\right]_0^{\pi/2}$$

$$= -\frac{\pi^2}{4}\cdot 0 + 2\cdot\frac{\pi}{2}\cdot 1 + 0 + 0 + 0 - 2\cdot 1$$

$$= \pi - 2 \quad \text{Ans.}$$

$$\int_0^{\pi/2} \sin\varphi \cos\varphi \sqrt{(a^2\sin^2\varphi + b^2\cos^2\varphi)}\, d\varphi$$

Let $\theta = a^2\sin^2\varphi + b^2\cos^2\varphi$

φ	0	$\pi/2$
θ	b^2	a^2

$$\therefore d\theta = 2(a^2 - b^2)\sin\varphi\cos\varphi \, d\varphi$$

$$\therefore \int_{b^2}^{a^2} \frac{1}{2(a^2-b^2)} \sqrt{\theta}\, d\theta = \frac{1}{2(a^2-b^2)}\cdot\frac{2}{3}\left[\theta^{3/2}\right]_{b^2}^{a^2}$$

$$= \frac{1}{3(a^2-b^2)}(a^3 - b^3)$$

$$= \frac{a^2 + ab + b^2}{3(a+b)} \quad \text{Ans.}$$

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$$= \int_1^2 \frac{dx}{x(1+2x)^2}$$

put $1+2x = z$

$\therefore 2\,dx = dz$

x	1	2
z	3	5

$$= \int_3^5 \frac{\frac{1}{2}\,dz}{\frac{z-1}{2} \cdot z^2}$$

$$= \int_3^5 \frac{1}{z^2(z-1)}$$

Now, $\frac{1}{(z-1)z^2} = \frac{A}{z-1} + \frac{Bz+C}{z^2}$

$\Rightarrow 1 = Az^2 + (Bz+C)(z-1)$

Equating the co-efficient like powers, we obtain

$C = -1$, $A+B = 0$, $-B+C = 0$

$\Rightarrow A = 1$ $\Rightarrow B = C$

$\therefore B = -1$

$$\therefore \int \left(\frac{1}{z-1} + \frac{-z-1}{z^2}\right) dz = \int \frac{dz}{z-1} - \int \frac{1}{z}\,dz - \int \frac{d}{z}$$

$$= \log(z-1) - \log z + \frac{1}{z}$$

$$\therefore \int_3^5 \frac{1}{z^2(z-1)} = \left[\log(z-1) - \log z + \frac{1}{z}\right]_3^5$$

$= \log 4 - \log 5 + \frac{1}{5} - \log 2 + \log 3 -$

$= 2\log 2 - \log 5 - \log 2 + \log 3 - \frac{2}{15}$

$= \log 2 - \log 5 + \log 3 - 2/15$

$= \log 6 - \log 5 - 2/15 = \log \frac{6}{5} - \frac{2}{15}$

(iii) $\int_0^{\pi/2} \frac{dx}{a+b\cos x}$ $[a>b>0]$

Solution
let, $I = \int \frac{dx}{a+b\cos x}$ $\boxed{a>b}$

$$= \int \frac{\sec^2 x/2 \, dx}{a + a\tan^2 x/2 + b - b\tan^2 x/2}$$

$$= \int \frac{\sec^2 x/2 \, dx}{(a+b) + (a-b)\tan^2 x/2}$$

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let $\sqrt{a-b}\tan x/2 = z$

$\therefore \sqrt{a-b}\sec^2 x/2 \, dx = 2\,dz$

$$\therefore \frac{1}{\sqrt{a-b}}\int \frac{2\,dz}{(a+b)+z^2} = 2\frac{1}{\sqrt{a-b}} \cdot \frac{1}{\sqrt{a+b}} \tan^{-1}\frac{z}{\sqrt{a+b}}$$

$$= \frac{2}{\sqrt{a^2-b^2}} \tan^{-1}\sqrt{\frac{a-b}{a+b}}\tan x/2$$

$$\therefore \int_0^{\pi/2} \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}}\left[\tan^{-1}\sqrt{\frac{a-b}{a+b}}\tan x/2\right]_0^{\pi/2}$$

$$= \frac{2}{\sqrt{a^2-b^2}} \tan^{-1}\sqrt{\frac{a-b}{a+b}} \cdot 1$$

$$= \frac{1}{\sqrt{a^2-b^2}} \cos^{-1} \frac{1-\frac{a-b}{a+b}}{1+\frac{a-b}{a+b}}$$

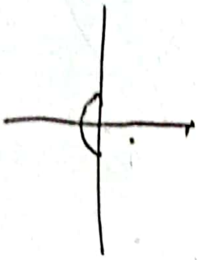
$$= \frac{1}{\sqrt{a^2-b^2}} \cos^{-1}\left(\frac{b}{a}\right)$$ Ans

(iv) $\displaystyle\int_0^{\pi} \frac{dx}{1-2a\cos x+a^2} \quad (0<a<1)$

Solution: let $\displaystyle I = \int_0^{\pi} \frac{dx}{1-2a\cos x+a^2}$

$$= \int_0^{\pi} \frac{\sec^2 \frac{x}{2}\, dx}{(1+a^2)(\tan^2\frac{x}{2}+1) - 2a(1-\tan^2\frac{x}{2})}$$

$$= \int_0^{\pi} \frac{\sec^2 \frac{x}{2}\, dx}{(1+a^2-2a)+(1+2a+a^2)\tan^2\frac{x}{2}}$$



put $(1+a)\tan\frac{x}{2} = z$

$\therefore (1+a)\sec^2\frac{x}{2}\, dx = 2dz$

x	0	π
z	0	∞

$$\int_0^{\infty} \frac{2\,dz/(1+a)}{(1-a)^2+z^2} = 2\frac{1}{(1+a)}\cdot\frac{1}{(1-a)}\tan^{-1}\frac{z}{1-a}\Big|_0^{\infty}$$

$$= \frac{2}{(1+a)(1-a)}\left[\tan^{-1}\infty - \tan^{-1}0\right]$$

$$= \frac{2\pi}{1-a^2}$$

	$0°$	$30°$	$45°$	$60°$	$90°$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt2}$	$\frac{\sqrt3}{2}$	1
cos	1	$\frac{\sqrt3}{2}$	$\frac{1}{\sqrt2}$	$\frac{1}{\sqrt3}$	0
tan	0	2	1		

Integral

Show that

13. $$\int_0^{\log 2} \frac{e^x}{1+e^x}\,dx = \log 3/2$$

solution: $$L.H.S = \int_0^{\log 2} \frac{e^x}{1+e^x}\,dx$$

put $1+e^x = z$

$\therefore e^x dx = dz$

x	0	$\log 2$
z	2	3

$$\therefore \int_2^3 \frac{dz}{z} = \log z \Big]_2^3$$

$$= \log 3/2 = R.H.S$$

Hence proved.

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14. $$\int_a^b \frac{\log x}{x}\,dx = \frac{1}{2}\log\left(\frac{b}{a}\right)\log(ab)$$

solution: $$L.H.S = \int_a^b \frac{\log x}{x}\,dx$$

put $\log x = z$

$\therefore \frac{1}{x}dx = dz$

x	a	b
z	$\log a$	$\log b$

$$\therefore \int_{\log a}^{\log b} z\,dz = \frac{1}{2}\left[z^2\right]_{\log a}^{\log b}$$

$$= \frac{1}{2}\left[(\log b)^2 - (\log a)^2\right]$$

$$= \frac{1}{2}(\log b - \log a)(\log b + \log a)$$

$$= \frac{1}{2}\log\left(\frac{b}{a}\right)\log(ab) = R.H.S$$

Hence proved.

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15. $\int_0^a \sin^{-1}\frac{2t}{1+t^2}\, dt = 2a\tan^{-1}a - \log(1+a^2)$

Solution: Let $I = \int \sin^{-1}\frac{2t}{1+t^2}\, dt$

put $t = \tan\theta$

$\therefore dt = \sec^2\theta\, d\theta$

$$\therefore I = \int \sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta} \cdot \sec^2\theta\, d\theta$$

$$= \int \sin^{-1}\sin 2\theta\, \sec^2\theta\, d\theta$$

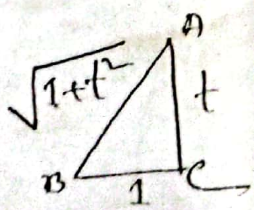
$$= 2\int \theta \sec^2\theta\, d\theta$$

$$= 2\left[\theta\int \sec^2\theta\, d\theta - \int\left(\frac{d}{d\theta}(\theta)\int \sec^2\theta\, d\theta\right) d\theta\right]$$

$$= 2\left[\theta\tan\theta - \int \tan\theta\, d\theta\right]$$

$$= 2\left[\theta\tan\theta - \log\sec\theta\right]$$

$$= 2\left[\tan^{-1}t \cdot t - \log\sec\tan^{-1}t\right]$$



$$= 2t\tan^{-1}t - 2\log(1+t^2)^{1/2}$$

$$\therefore \int_0^a \sin^{-1}\frac{2t}{1+t^2}\, dt = \left[2t\tan^{-1}t - \log(1+t^2)\right]_0^a$$

$$= 2a\tan^{-1}a - \log(1+a^2)$$

Hence proved.

16. (i) $\int_1^2 \sqrt{(x-1)(2-x)}\,dx = \pi/8$

Solution: let $I = \int_1^2 \sqrt{(x-1)(2-x)}\,dx$

put $x-1 = z^2$

$\therefore dx = 2z\,dz$

x	1	2
z	0	1

$\therefore \int_0^1 z.\sqrt{2-z^2-1}\,.\,2z\,dz$

$= 2\int_0^1 z^2\sqrt{1-z^2}\,dz$

Again, put $z = \sin\theta$

$dz = \cos\theta\,d\theta$

z	0	1
θ	0	$\pi/2$

$\therefore \int_0^{\pi/2} 2\sin^2\theta\cos\theta.\cos\theta\,d\theta = \frac{1}{2}\int_0^{\pi/2}\sin^2 2\theta\,d\theta$

$= \frac{1}{4}\int_0^{\pi/2}(1-\cos 4\theta)\,d\theta$

$= \frac{1}{4}\left[\theta - \frac{1}{16}\sin 4\theta\right]_0^{\pi/2}$

$= \frac{1}{4}\,\pi/2 - \frac{1}{16}\sin 2\pi$

$= \pi/8$

Hence proved.

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(11) $\int_8^{15} \frac{dx}{(x-3)\sqrt{x+1}} = \frac{1}{2}\log 5/3$

Solution: L.H.S $= \int_8^{15} \frac{dx}{(x-3)\sqrt{x+1}}$

Put $1+x = z^2$

$\therefore dx = 2z\,dz$

x	8	15
z	3	4

$\therefore \int_3^4 \frac{2z\,dz}{z\cdot(z^2-1-3)} = 2\int_3^4 \frac{dz}{z^2-4}$

$= 2\cdot\frac{1}{2\cdot 2}\log\frac{z-2}{z+2}\Big|_3^4$

$= \frac{1}{2}\log\frac{2}{6} - \frac{1}{2}\log\frac{1}{5}$

$= \frac{1}{2}\log\frac{2}{6}\times 5/1$

$= \frac{1}{2}\log\frac{5}{3}$

Hence proved.

12. $\int_0^a \frac{a^2-x^2}{(a^2+x^2)^2}\,dx = \frac{1}{2a}$.

Solution: L.H.S $= \int_0^a \frac{a^2-x^2}{(a^2+x^2)^2}\,dx = \int_0^a \left[\frac{1}{a^2+x^2} - \frac{2x^2}{(a^2+x^2)^2}\right]$

$= \int_0^a \frac{dx}{a^2+x^2} - 2\int_0^a \frac{x^2}{(a^2+x^2)^2}\,dx$

$= \frac{1}{a}\tan^{-1} x/a\Big|_0^a - 2\int_0^a \frac{x^2\,dx}{(a^2+x^2)^2}$

$= \frac{1}{a}\tan^{-1}1 - \frac{1}{a}\cdot 0 - 2\int_0^a \frac{x^2\,dx}{(a^2+x^2)^2}$

$$= \frac{\pi}{4a} - 2\int_0^a \frac{x^2}{(a^2+x^2)^2}\,dx$$

put $x = a\tan\theta$

$dx = a\sec^2\theta\, d\theta$

x	0	a
θ	0	$\pi/4$

$\therefore$ let $I = \int_0^{\pi/4} \frac{a^2\tan^2\theta}{a^4\sec^4\theta} \cdot a\sec^2\theta\, d\theta$

$$= \frac{1}{a}\int_0^{\pi/4} \sin^2\theta\, d\theta$$

$$= \frac{1}{2a}\int_0^{\pi/4} (1-\cos 2\theta)\, d\theta$$

$$= \frac{1}{2a}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/4}$$

$$= \frac{1}{2a}\left[\pi/4 - \frac{1}{2}\sin \pi/2\right]$$

$$= \frac{\pi}{8a} - \frac{1}{4a}$$

$$\therefore \int_0^a \frac{a^2-x^2}{(a^2+x^2)^2}\,dx = \frac{\pi}{4a} - \frac{\pi}{4a} + \frac{1}{2a}$$

$$= 1/2a$$

Hence proved.

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18. $\int_0^{3\pi/4} \frac{\sin x\, dx}{1+\cos^2 x} = \pi/4 + \tan^{-1} 1/\sqrt{2}$

Solⁿ: Let $I = \int_0^{3\pi/4} \frac{\sin x\, dx}{1+\cos^2 x}$ | Put $\cos x = z$

$\therefore dz = -\sin x\, dx$

x	$3\pi/4$	0
z	$-1/\sqrt{2}$	1

$\therefore \int_1^{-1/\sqrt{2}} \frac{-dz}{1+z^2} = -\tan^{-1} z \Big|_1^{-1/\sqrt{2}}$

$= -\tan^{-1}(-1/\sqrt{2}) + \tan^{-1} 1$

$= \pi/4 + \tan^{-1} 1/\sqrt{2} = \text{R.H.S}$

Hence proved.

17. $\int_0^{\pi/2} \cos^3 x \sqrt[4]{\sin x}\, dx = \frac{32}{65}$

Solⁿ: Let $I = \int_0^{\pi/2} \cos^3 x \sqrt[4]{\sin x}\, dx$

Put $\sin x = z$

$dz = \cos x\, dx$

x	0	$\pi/2$
z	0	1

$\therefore \int_0^1 (1-z^2)\, z^{1/4}\, dz = \int_0^1 (z^{1/4} - z^{9/4})\, dz$

$= \frac{4}{5}\left[z^{5/4}\right]_0^1 - \frac{4}{13}\left[z^{13/4}\right]_0^1$

$= \frac{4}{5} - \frac{4}{13}$

$= \frac{32}{65}$. Hence proved.

20. (i) $\displaystyle\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi}{2ab}$ $[a, b > 0]$

Solⁿ: let $\displaystyle I = \int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$

$$= \int_0^{\pi/2} \frac{\sec^2 x}{a^2 + b^2\tan^2 x}\, dx$$

put $z = b\tan x$

$\therefore dz = b\sec^2 x\, dx$

x	0	$\pi/2$
z	0	∞

$$\therefore \frac{1}{b}\int_0^{\infty} \frac{dz}{a^2+z^2} = \frac{1}{ab}\tan^{-1} z/a \Big|_0^{\infty}$$

$$= \frac{1}{ab}\left[\tan^{-1}\infty - \tan^{-1}0\right]$$

$$= \frac{\pi}{2ab}$$

Hence proved

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20. (ii) $\int_0^{\pi/4} \frac{\sin^2 x \cos^2 x}{(\sin^3 x + \cos^3 x)^2} dx = 1/6$

Solution: $L.H.S = \int_0^{\pi/4} \frac{\sin^2 x \cos^2 x \, dx}{(\sin^3 x + \cos^3 x)^2}$

$= \int_0^{\pi/4} \frac{\tan^2 x \sec^2 x \, dx}{(1+\tan^3 x)^2}$

Put $1+\tan^3 x = z$

$\therefore 3\tan^2 x \sec^2 x \, dx = dz$

x	0	$\pi/4$
z	1	2

$\therefore \frac{1}{3}\int_1^2 \frac{dz}{z^2} = -\left[\frac{1}{3z}\right]_1^2$

$= -\frac{1}{3}\left[\frac{1}{2} - \frac{1}{1}\right]$

$= -1/3 \cdot -1/2$

$= 1/6 = R.H.S$

Hence proved.

(I) $\int_0^{\pi/2} \frac{dx}{4+5\sin x} = \frac{1}{3}\log 2$

Sol^n: L.H.S $= \int_0^{\pi/2} \frac{\sec^2 x/2 \, dx}{4+4\tan^2 x/2 + 10\tan x/2}$

put $\tan x/2 = z$

$\therefore \sec^2 x/2 \, dx = 2dz$

x	0	$\pi/2$
z	0	$+1$

$\therefore \int_0^{-1} \frac{2dz}{4+4z^2+10z} = \frac{1}{2}\int_0^{+1} \frac{dz}{z^2+\frac{5}{2}z+1}$

$= \frac{1}{2}\int_0^{+1} \frac{dz}{(z+5/4)^2-(3/4)^2}$

$= \frac{1}{2}\cdot\frac{1}{2\cdot 3/4}\left[\log \frac{z+5/4-3/4}{z+5/4+3/4}\right]_0^{+1}$

$= \frac{1}{3}\left[\log \frac{z+1/2}{z+2}\right]_0^{+1}$

$= \frac{1}{3}\log \frac{+3/2}{1} - \frac{1}{3}\log(-1/4)$

$\frac{1}{3}\log \frac{2}{3} - \frac{1}{3}\log 4$

$= \frac{1}{3}\log 2 + \frac{1}{3}\log 3 - \frac{1}{3}\log 4$

$= \frac{1}{3}\log(+3/6) - \frac{1}{3}\log\frac{1}{4} = \frac{1}{3}\log \frac{3}{6} \times \frac{4}{1}$

$= \frac{1}{3}\log 2 + \frac{1}{3}\log 4 \qquad = \frac{1}{3}\log 2$

$= \frac{1}{3}\log 2$

Hence proved.

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(11) $\int_0^{\pi/2} \frac{dx}{5+4\sin x} = \frac{2}{3}\tan^{-1}\frac{1}{3}$

Solⁿ: L.H.S $= \int_0^{\pi/2} \frac{dx}{5+4\sin x}$

$= \int_0^{\pi/2} \frac{\sec^2 x/2 \, dx}{5+5\tan^2 x/2 + 8\tan x/2}$

Put $\tan x/2 = z$

$\therefore \sec^2 x/2 \, dx = 2dz$

x	0	$\pi/2$
z	0	$+1$

$\therefore \int_0^{+1} \frac{2\,dz}{5+5z^2+8z} = \frac{2}{5}\int_0^{+1} \frac{dz}{z^2+\frac{8}{5}z+1}$

$= \frac{2}{5}\int_0^{+1} \frac{dz}{(z+4/5)^2+(3/5)^2}$

$= \frac{2}{5}\cdot\frac{1}{3/5}\tan^{-1}\frac{z+4/5}{3/5}\Bigg|_0^{+1}$

$= \frac{2}{3}\tan^{-1}(+1/3)$

Hence proved.

22. (i) $\displaystyle\int_0^{\pi/2} \frac{dx}{5+3\cos x} = \frac{1}{2}\tan^{-1}\frac{1}{2}.$

Solution: Let $I = \displaystyle\int_0^{\pi/2} \frac{\sec^2 \frac{x}{2}\, dx}{5+5\tan^2\frac{x}{2}+3-3\tan^2\frac{x}{2}}$

$$= \int_0^{\pi/2} \frac{\sec^2\frac{x}{2}\, dx}{2\tan^2\frac{x}{2}+8}$$

Put $\tan\frac{x}{2} = z$

$\therefore \sec^2\frac{x}{2}\, dx = 2\,dz$

x	0	$\pi/2$
z	0	1

$$\therefore \int_0^1 \frac{2\,dz}{2z^2+8} = \int_0^1 \frac{dz}{z^2+4} =$$

$$= \frac{1}{2}\tan^{-1}\frac{z}{2}\Big|_0^1$$

$$= \frac{1}{2}\left[\tan^{-1}\frac{1}{2} - \tan^{-1}0\right]$$

$$= \frac{1}{2}\tan^{-1}\frac{1}{2}$$

Hence proved.

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22. (ii) $\int_0^{\pi/2} \frac{dx}{3+5\cos x} = \frac{1}{4}\log 3$ পূর্বের নিয়ম follow কর।

Solⁿ: L.H.S $= \int_0^{\pi/2} \frac{dx}{3+5\cos x}$

$$= \int_0^{\pi/2} \frac{\sec^2 x/2 \, dx}{3+3\tan^2 x/2 + 5 - 5\tan^2 x/2}$$

$$= \int_0^{\pi/2} \frac{\sec^2 x/2 \, dx}{8 - 2\tan^2 x/4}$$

Put $\tan x/2 = z$

$\therefore \sec^2 x/2 \, dx = 2dz$

x	0	$\pi/2$
z	0	1

$$\therefore \int_0^1 \frac{2\,dz}{8-2z^2} = \int_0^1 \frac{dz}{4-z^2}$$

$$= \frac{1}{2\cdot 2}\log \frac{2+z}{2-z}\Bigg|_0^1$$

$$= \frac{1}{4}\log\frac{3}{1} - \frac{1}{4}\log\frac{2}{2}$$

$$= \frac{1}{4}\log 3$$

Hence proved.

(iii) $\displaystyle\int_0^{\pi/2} \frac{dx}{1+4\cot^2 x} = \pi/6$

solution: $\displaystyle L.H.S = \int_0^{\pi/2} \frac{dx}{1+4\cot^2 x}$

$$= \int_0^{\pi/2} \frac{\operatorname{cosec}^2 x \, dx}{(1+\cot^2 x)(1+4\cot^2 x)}$$

put $\cot x = z$

$\therefore -\operatorname{cosec}^2 x dx = dz$

x	$\pi/2$	0
z	0	∞

$$\therefore \int_{\infty}^{0} \frac{-dz}{(1+z^2)(1+4z^2)} = \frac{1}{3}\int_0^{\infty}\left(\frac{4}{1+4z^2} - \frac{1}{1+z^2}\right) dz$$

$$= \frac{1}{3}\int_0^{\infty} \frac{1}{z^2+1/4} dz - \frac{1}{3}\int_0^{\infty}\frac{1}{1+z^2} dz = \frac{1}{3}\cdot 2\tan^{-1} 2z\Big|_0^{\infty} - \frac{1}{3}\tan^{-1} z\Big|_0^{\infty}$$

$$= \frac{1}{3}\left[2\cdot \pi/2 - 0\right] - \frac{1}{3}\left[\pi/2 - 0\right] = \frac{\pi}{3} - \frac{\pi}{6} = \frac{2\pi - \pi}{6} = \pi/6$$

$\therefore L.H.S = R.H.S$ (proved)

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$$\int_0^{\pi/2} \frac{dx}{1+\cos\theta\cos x} = \frac{\theta}{\sin\theta}$$

Soln: Let $I = \displaystyle\int_0^{\pi/2} \frac{\sec^2 x/2 \, dx}{1+\tan^2 x/2 + \cos\theta - \cos\theta \tan^2 x/2}$

$$= \int_0^{\pi/2} \frac{\sec^2 x/2 \, dx}{(1+\cos\theta) + (1-\cos\theta)\tan^2 x/2}$$

put $\sqrt{1-\cos\theta}\,\tan x/2 = z$

$\therefore \sqrt{(1-\cos\theta)}\,\sec^2 x/2 \, dx = 2\,dz$

$$= \int \frac{2/\sqrt{1-\cos\theta}\; dz}{(1+\cos\theta)+z^2}$$

$$= \frac{2}{\sqrt{1-\cos\theta}} \cdot \frac{1}{\sqrt{1+\cos\theta}} \tan^{-1} \frac{z}{\sqrt{1+\cos\theta}}$$

$$= \frac{2}{\sqrt{(1-\cos^2\theta)}} \tan^{-1} \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \tan x/2$$

$$= \frac{2}{\sin\theta} \tan^{-1} \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \tan x/2$$

$$= \frac{1}{\sin\theta} \cos^{-1}\cos\theta \tan x/2$$

$$= \frac{\theta}{\sin\theta} \tan x/2$$

$$\therefore \int_0^{\pi/2} \frac{dx}{1+\boxed{\cos\theta}\cos x} = \left[\frac{\theta}{\sin\theta}\tan^{x}/_{2}\right]_0^{\pi/2}$$

$$= \frac{\theta}{\sin\theta}\left[\tan\pi/_4 - \tan 0\right]$$

$$= \frac{\theta}{\sin\theta}$$

Hence proved.

$$\int_0^{\pi/2} \frac{\cos x\,dx}{(1+\sin x)(2+\sin x)} = \log 4/3$$

$$\frac{(1+\sin x+1)}{(1+\sin x)^2+(1+\sin x)}$$

Soln: let $I = \int_0^{\pi/2} \frac{\cos x\,dx}{(1+\sin x)^2+(1+\sin x)}$

put $1+\sin x = z$

$\therefore dz = \cos x\,dx$

x	0	$\pi/2$
z	1	2

$$\therefore \int_1^2 \frac{dz}{z^2+z}$$

$$= \int_1^2 \frac{(z+1)-z}{z(z+1)}\,dz = \int_1^2 \frac{1}{z}\,dz - \int \frac{1}{z+1}\,dz$$

$$= \log z\Big|_1^2 - \log(z+1)\Big|_1^2$$

$$= \log 2 - \log 1 - \log 3 + \log 2$$

$$= 2\log 2 - \log 3$$

$$= \log 4/3.$$

Hence proved.

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25. $\displaystyle\int_0^{\pi/4} \frac{\sin 2x}{\sin^4 x + \cos^4 x}\, dx = \pi/4$

Sol^n: $\displaystyle L.H.S = \int_0^{\pi/4} \frac{\sin 2x}{\sin^4 x + \cos^4 x}\, dx$

$$= 4 \int_0^{\pi/4} \frac{\sin 2x}{(1-\cos 2x)^2 + (1+\cos 2x)^2}\, dx$$

$$= 4 \int_0^{\pi/4} \frac{\sin 2x}{2 + 2\cos^2 2x}\, dx$$

$$= 2 \int_0^{\pi/4} \frac{\sin 2x}{1 + \cos^2 2x}\, dx$$

$$\therefore -2\int_1^0 \frac{\frac{1}{2}\, dz}{1+z^2} = \frac{1}{2}\int_0^1 \frac{dz}{1+z^2}$$

put $\cos 2x = z$

$\therefore -\sin 2x\, dx = \frac{1}{2}\, dz$

x	0	$\pi/4$
z	1	0

$$= 2 \cdot \frac{1}{2} \tan^{-1} z \Big|_0^1$$

$$= 2 \cdot \frac{1}{2} \left[\tan^{-1} 1 - \tan^{-1} 0\right]$$

$$= \pi/4 = R.H.S$$

Hence proved.

26(a) $\int_0^{\pi/2} \frac{dx}{(a^2\cos^2x+b^2\sin^2x)^2} = \frac{\pi}{4}\frac{a^2+b^2}{a^3b^3}$

$$\text{L.H.S} = \int_0^{\pi/2} \frac{dx}{(a^2\cos^2x+b^2\sin^2x)^2}$$

$$= \int_0^{\pi/2} \frac{\sec^4x\,dx}{(a^2+b^2\tan^2x)^2}$$

$$= \int_0^{\pi/2} \frac{(1+\tan^2x)\sec^2x\,dx}{(a^2+b^2\tan^2x)^2}$$

put $b\tan x = a\tan\theta$

$b\sec^2x\,dx = a\sec^2\theta\,d\theta$

$\sec^2x\,dx = \frac{a}{b}\sec^2\theta\,d\theta$

x	$\frac{\pi}{2}$	0
z	$\frac{\pi}{2}$	0

$$\therefore \int_0^{\pi/2} \frac{(1+\frac{a^2}{b^2}\tan^2\theta)\cdot\frac{a}{b}\sec^2\theta\,d\theta}{(a^2+a^2\tan^2\theta)^2}$$

$$= \frac{a}{b^3}\int_0^{\pi/2} \frac{(b^2+a^2\tan^2\theta)\sec^2\theta\,d\theta}{a^4(1+\tan^2\theta)^2}$$

$$= \frac{1}{b^3a^3}\int_0^{\pi/2} \frac{b^2+a^2\tan^2\theta}{\sec^2\theta}\,d\theta$$

$$= \frac{1}{a^3b^3}\int_0^{\pi/2}\frac{b^2}{\sec^2\theta}\,d\theta + \frac{1}{a^3b^3}\int_0^{\pi/2}\frac{a^2\tan^2\theta}{\sec^2\theta}\,d\theta$$

$$= \frac{1}{a^3b}\int_0^{\pi/2}\cos^2\theta\,d\theta + \frac{1}{ab^3}\int_0^{\pi/2}\sin^2\theta\,d\theta$$

$$= \frac{1}{2a^3b}\int_0^{\pi/2}(1+\cos2\theta)\,d\theta + \frac{1}{2b^3a}\int_0^{\pi/2}(1-\cos2\theta)\,d\theta$$

$$= \frac{1}{2a^3b}\left[\theta+\frac{\sin2\theta}{2}\right]_0^{\pi/2} + \frac{1}{2b^3a}\left[\theta-\frac{\sin2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{1}{2a^3b}\,\frac{\pi}{2} + \frac{1}{2b^3a}\,\frac{\pi}{2} = \frac{\pi}{4}\cdot\frac{a^2+b^2}{a^3b^3}$$

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27. $\int_2^e \left\{ \frac{1}{\log x} - \frac{1}{(\log x)^2} \right\} dx = e - \frac{2}{\log 2}$

Solution:

let $I = \int \left\{ \frac{1}{\log x} - \frac{1}{(\log x)^2} \right\} dx$

$= \frac{1}{\log x} \int dx - \int \left\{ \frac{d}{dx}\left(\frac{1}{\log x}\right) \int dx \right\} dx - \int$

$= \frac{1}{\log x} \cdot x + \cancel{\int \frac{dx}{(\log x)^2}} - \cancel{\int \frac{1}{(\log x)}}$

$= \frac{x}{\log x}$

$\therefore \int_2^e \left\{ \frac{1}{\log x} - \frac{1}{(\log x)^2} \right\} dx = \left[\frac{x}{\log x} \right]_2^e$

$= \frac{e}{\log e} - \frac{2}{\log 2}$

$= e - \frac{2}{\log 2}$

Hence proved

28(i) $\int_2^3 \frac{dx}{(x-1)\sqrt{x^2-2x}} = \pi/3$

Sol^n: let $I = \int_2^3 \frac{dx}{(x-1)\sqrt{x^2-2x}}$

let $x-1 = 1/z$

$dx = -\frac{1}{z^2} dz$

x	3	2
z	1/2	1

$= \int_1^{1/2} \frac{-\frac{1}{z^2} dz}{\frac{1}{z}\sqrt{(\frac{1}{z}+1)^2 - 2(\frac{1}{z}+1)}}$

$= -\int_1^{1/2} \frac{dz}{\sqrt{1-z^2}}$

$= \left[-\sin^{-1} z \right]_1^{1/2} = -\sin^{-1} 1/2 + \sin^{-1} 1$

$= -\pi/6 + \pi/2$

$= \frac{-\pi + 3\pi}{6} = \frac{2\pi}{6} = \pi/3$

28(ii) $\displaystyle\int_0^1 \frac{dx}{(1+x)\sqrt{1+2x-x^2}} = \frac{\pi}{4\sqrt{2}}$

Solution:

Let $\displaystyle I = \int_0^1 \frac{dx}{(1+x)\sqrt{1-x^2+2x}}$

put $1+x = \frac{1}{z}$

$dx = -\frac{1}{z^2}dz$

x	1	0
z	$\frac{1}{2}$	1

$\therefore \displaystyle\int_1^{1/2} \frac{-\frac{1}{z^2}dz}{\frac{1}{z}\sqrt{1+2(\frac{1}{z}-1)-(\frac{1}{z}-1)^2}}$

$\displaystyle = -\int_1^{1/2} \frac{dz}{\sqrt{\cancel{z^2}+2z-2z^2-1+2z-\cancel{z^2}}}$

$\displaystyle = -\int_1^{1/2} \frac{dz}{\sqrt{4z-1-2z^2}}$

$\displaystyle = -\frac{1}{\sqrt{2}}\int_1^{1/2} \frac{dz}{\sqrt{2z-\frac{1}{2}-z^2}}$

$\displaystyle = -\frac{1}{\sqrt{2}}\int_1^{1/2} \frac{dz}{\sqrt{(\frac{1}{\sqrt{2}})^2-(z-1)^2}}$

$\displaystyle = -\frac{1}{\sqrt{2}}\left[\sin^{-1}\sqrt{2}(z-1)\right]_1^{1/2}$

$= -\frac{1}{\sqrt{2}}\sin^{-1}\sqrt{2}\cdot\frac{1}{2} - \sin^{-1}\sqrt{2}(1-1)$

$= \frac{1}{\sqrt{2}}\sin^{-1}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\sin^{-1}\sin\frac{\pi}{4}$

$= \frac{\pi}{4\sqrt{2}}$

Hence proved

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Define Integrals

2 (i) $\lim_{n\to\infty}\left[\frac{1}{n+m}+\frac{1}{n+2m}+\cdots+\frac{1}{n+nm}\right]$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n+mr}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+m(r/n)}$$

$$= \int_0^1 \frac{dx}{1+mx}$$

$$= \frac{1}{m}\log(1+mx)\Big|_0^1$$

$$= \frac{1}{m}\log(1+m)$$

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2 (ii) $\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{n}{n^2+n^2}\right]$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{n}{n^2+r^2}$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{n}{n^2\{1+(\frac{r}{n})^2\}}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+(r/n)^2}$$

$$= \int_0^1\frac{dx}{1+x^2} = [\tan^{-1}x]_0^1 = \frac{\pi}{4}$$ Ans

2 (iii) $\lim_{n\to\infty}\left[\frac{1}{\sqrt{(n^2-1^2)}}+\frac{1}{\sqrt{(n^2-2^2)}}+\cdots+\frac{1}{\sqrt{\{n^2-(n-1)^2\}}}\right]$

$$= \lim_{n\to\infty}\sum_{r=1}^{n-1}\frac{1}{\sqrt{n^2-r^2}}$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n-1}\frac{1}{\sqrt{n^2(1-r^2/n^2)}}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n-1}\frac{1}{\sqrt{1-(r/n)^2}}$$

$$= \int_0^1\frac{dx}{\sqrt{1-x^2}}$$

$$= [\sin^{-1}x]_0^1 = \sin^{-1}1-\sin^{-1}0 = \frac{\pi}{2}$$ Ans

29 (iii)

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{(2n\cdot 1-1^2)}}+\frac{1}{\sqrt{(4n-2^2)}}+\cdots+\frac{1}{n}\right]$$

$$=\lim_{n\to\infty}\left[\frac{1}{\sqrt{2n\cdot 1-1^2}}+\frac{1}{\sqrt{2n\cdot 2-2^2}}+\cdots+\frac{1}{\sqrt{(2n\cdot n-n^2)}}\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{(2nr-r^2)}}$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{n^2\left(\frac{2r}{n}-\frac{r^2}{n^2}\right)}}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{\sqrt{2(r/n)-(r/n)^2}}$$

$$=\int_0^1\frac{dx}{\sqrt{2x-x^2}}$$

$$=\int_0^1\frac{dx}{\sqrt{1^2-(x^2-2x+1^2)}}$$

$$=\int_0^1\frac{dx}{\sqrt{1^2-(x-1)^2}}=\left[\sin^{-1}(x-1)\right]_0^1=\sin^{-1}\ldots$$

$$=\frac{\pi}{2}\ \text{Ans.}$$

29 (iv)

$$\lim_{n\to\infty}\left[\frac{1^2}{n^3+1^3}+\frac{2^2}{n^3+2^3}+\cdots+\frac{n^2}{2n^3}\right]$$

$$=\lim_{n\to\infty}\left[\frac{1^2}{n^3+1^3}+\frac{2^2}{n^3+2^3}+\cdots+\frac{n^2}{n^3+n^3}\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^2}{n^3+r^3}$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^2}{n^3(1+r^3/n^3)}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum\frac{(r/n)^2}{1+(r/n)^3}$$

$$=\int_0^1\frac{x^2\,dx}{1+x^3}=\frac{1}{3}\log(1+x^3)\Big|_0^1=\frac{1}{3}\log\ldots$$

(v)

$$\lim_{n\to\infty}\left[\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\cdots+\frac{1}{8n}\right]$$

$$=\lim_{n\to\infty}\left[\frac{n^2}{(n+0)^3}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\cdots+\frac{n^2}{(n+n)^3}\right]$$

$$=\lim_{n\to\infty}\sum_{r=0}^{n}\frac{n^2}{(n+r)^3}$$

$$=\lim_{n\to\infty}\sum_{r=0}^{n}\frac{n^2}{n^3(1+r/n)^3}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n}\frac{1}{(1+r/n)^3}$$

$$=\int_0^1\frac{dx}{(1+x)^3}=-\frac{1}{2}\left[\frac{1}{(1+x)^2}\right]_0^1=-\frac{1}{2}\left[\frac{1}{2^2}-\frac{1}{1}\right]=-\frac{1}{2}\left[\frac{1-4}{4}\right]$$

$$=-\frac{1}{2}\times-\frac{3}{4}=\frac{3}{8}\ \text{Ans}$$

(vi)

$$\lim_{n\to\infty}\left[\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\cdots+\frac{1}{8n}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}+\lim_{n\to\infty}\left[\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\cdots+\frac{n^2}{(n+n)^3}\right]$$

$$=0+\lim_{n\to\infty}\sum_{r=1}^{n}\frac{n^2}{(n+r)^3}=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{(1+r/n)^3}$$

$$=\int_0^1\frac{dx}{(1+x)^3}=3/8\ \text{Ans}$$

(vii)

$$\lim_{n\to\infty}\left[\frac{1}{n}+\frac{\sqrt{n^2-1^2}}{n^2}+\cdots+\frac{\sqrt{n^2-(n-1)^2}}{n^2}\right]$$

$$=\lim_{n\to\infty}\left[\frac{\sqrt{n^2-0^2}}{n^2}+\frac{\sqrt{n^2-1^2}}{n^2}+\cdots+\frac{\sqrt{n^2-(n-1)^2}}{n^2}\right]$$

$$=\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{\sqrt{n^2-r^2}}{n^2}$$

$$=\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{\sqrt{n^2(1-r^2/n^2)}}{n^2}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\sqrt{1-(r/n)^2}$$

$$=\int_0^1\sqrt{1-x^2}\,dx=\left[\frac{x\sqrt{1-x^2}}{2}+\frac{1}{2}\sin^{-1}x\right]_0^1$$

$$=\frac{\pi}{4}\ \text{Ans}$$

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29(vii)
$$\lim_{n\to\infty}\left[\frac{1}{n}+\frac{\sqrt{n^2-1^2}}{n^2}+\cdots+\frac{\sqrt{n^2-(n-1)^2}}{n^2}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}+\lim_{n\to\infty}\left[\frac{\sqrt{n^2-1^2}}{n^2}+\cdots+\frac{\sqrt{n^2-(n-1)^2}}{n^2}\right]$$

$$=0+\lim_{n\to\infty}\sum_{r=1}^{n-1}\sqrt{\frac{n^2-r^2}{n^2}}$$

Same as

29(viii)
$$\lim_{n\to\infty}\sum_{r=1}^{n-1}\frac{1}{n}\sqrt{\left(\frac{n+r}{n-r}\right)}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n-1}\sqrt{\frac{1+r/n}{1-r/n}}$$

$$=\int_0^1\sqrt{\frac{1+x}{1-x}}\,dx$$

$$=\int_0^1\frac{1+x}{\sqrt{1-x^2}}=\int_0^1\frac{1}{\sqrt{1-x^2}}dx+\int_0^1\frac{x}{\sqrt{1-x^2}}$$

$$\Rightarrow\left[\sin^{-1}x\right]_0^1-\frac{1}{2}\int_0^1\frac{-2x}{\sqrt{1-x^2}}dx$$

put $1-x^2=z$
$-2x\,dx=dz$
$x=0$
$x=1$

$$\Rightarrow\left(\frac{\pi}{2}-0\right)-\frac{1}{2}\int_1^0\frac{dz}{\sqrt{z}}$$

$$=\left(\frac{\pi}{2}-0\right)+\frac{1}{2}\int_0^1\frac{dz}{\sqrt{z}}$$

$$=\frac{\pi}{2}+\frac{1}{2}\,2\left[\sqrt{z}\right]_0^1=\frac{\pi}{2}+1$$ Ans

(x) $\lim_{n\to\infty} \left\{ \left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right) \cdots \cdots \left(1+\frac{n}{n}\right) \right\}^{1/n}$

Solution:

Let $y = \left\{ \left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right) \cdots \left(1+\frac{n}{n}\right) \right\}^{1/n}$

Taking Log on both sides, we get

$$\log y = \frac{1}{n}\left[\log\left(1+\frac{1}{n}\right) + \log\left(1+\frac{2}{n}\right) + \cdots + \log\left(1+\frac{n}{n}\right)\right]$$

$$\lim_{n\to\infty} \log y = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \log\left(1+\frac{r}{n}\right)$$

$$= \int_0^1 \log(1+x)\,dx$$

Now
$$\int_0 \log(1+x)\,dx = \log(1+x)\int dx - \int\left\{\frac{d}{dx}\log(1+x)\int dx\right\}dx$$
$$= x\log(1+x) - \int \frac{1}{1+x}\, x\,dx$$
$$= x\log(1+x) - \int \frac{x+1-1}{1+x}\,dx$$
$$= x\log(1+x) - \int dx + \int \frac{1}{1+x}\,dx$$
$$= x\log(1+x) - x + \log(1+x)$$

$$\therefore \int_0^1 \log(1+x)\,dx = \Big[x\log(1+x) - x + \log(1+x)\Big]_0^1$$
$$= \log 2 - \cancel{\log}\, 1 + \log 2$$
$$= \log 4 - \log e = \log \frac{4}{e}$$

$$\therefore \lim_{n\to\infty} \log y = \log \frac{4}{e}$$

$$\Rightarrow \lim_{n\to\infty} y = \frac{4}{e}$$

$$\Rightarrow \lim_{n\to\infty} \left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right\}^{1/n} = \frac{4}{e}$$

proved

33 (b) $\lim_{n\to\infty}\left\{\left(1+\frac{1^2}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\cdots\left(1+\frac{n^2}{n^2}\right)\right\}^{1/n}$

Solution: let

$$y = \left\{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\cdots\left(1+\frac{n^2}{n^2}\right)\right\}^{1/n}$$

Taking log on both sides, we get

$$\log y = \frac{1}{n}\left[\log\left(1+\frac{1}{n^2}\right)+\log\left(1+\frac{2^2}{n^2}\right)+\cdots\right]$$

$$\therefore \lim_{n\to\infty}\log y = \lim_{n\to\infty}\frac{1}{n}\sum\log\left(1+\frac{h^2}{n^2}\right)$$

$$= \int_0^1 \log(1+x^2)$$

Now $\int \log(1+x^2) = \log(1+x^2)\int dx - \int\left(\frac{d}{dx}\log(1+x^2)\cdots\right.$

$$= x\log(1+x^2) - \int \frac{1}{1+x^2}\times 2x \times x\,dx$$

$$= x\log(1+x^2) - \int \frac{2x^2}{1+x^2}\,dx$$

$$= x\log(1+x^2) - 2\int \frac{1+x^2-1}{1+x^2}\,dx$$

$$= x\log(1+x^2) - 2\left[\int dx - \int \frac{1}{1+x^2}\,dx\right]$$

$$= x\log(1+x^2) - 2x + 2\tan^{-1}x$$

$$\int_0^1 \log(1+x^2) = \left[x\log(1+x^2) - 2x + 2\tan^{-1}x\right]$$

$$= \log 2 - 2 + 2\cdot\pi/4$$

$$= \log 2 + \tfrac{1}{2}(\pi-4)$$

$$= \log 2 + \log e^{\frac{1}{2}(\pi-4)}$$

$$= \log 2e^{\frac{1}{2}(\pi-4)}$$

$$\therefore \lim_{n\to\infty}\log y = \log 2e^{\frac{1}{2}(\pi-4)}$$

$$\lim_{n\to\infty} y = 2e^{\frac{1}{2}(\pi-4)}$$

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(xi) $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{n+r}{n^2+r^2}$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1+r/n}{1+(r/n)^2}$

$= \int_0^1 \frac{1+x}{1+x^2}$

$= \int_0^1 \frac{1}{1+x^2}dx + \int_0^1 \frac{x}{1+x^2}dx$

$= [\tan^{-1}x]_0^1 + [\frac{1}{2}\log(1+x^2)]_0^1$

$= \frac{\pi}{4} + \frac{1}{2}\log 2$ Ans

(xii)

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$\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{(n+r)\sqrt{\{r(2n+r)\}}}$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{(1+r/n)\sqrt{\{r/n(2+r/n)\}}}$

$= \int_0^1 \frac{1}{(1+x)\sqrt{x(2+x)}}$

put $1+x = 1/z$

$dx = -1/z^2\,dz$

x	1	0
z	1/2	1

$= \int_1^{1/2} \frac{-1/z^2\,dz}{1/z\sqrt{(1/z-1)(2+1/z-1)}} = -\int_1^{1/2} \frac{dz}{\sqrt{(1+z)(1-z)}}$

$= -\int_1^{1/2} \frac{dz}{\sqrt{1-z^2}} = -[\sin^{-1}z]_1^{1/2} = -[\sin^{-1}1/2 - \sin^{-1}1]$

$= -\frac{\pi}{6} + \frac{\pi}{2} = \pi/3$ A

(xiii)

$\lim_{n\to\infty}\left[\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n}\right]$

$= \lim_{n\to\infty}\left[\frac{1}{n+0} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+2n}\right]$

$= \lim_{n\to\infty} \sum_{r=0}^{2n} \frac{1}{n+r} = \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{2n} \frac{1}{1+r/n} = \int_0^2 \frac{1}{1+x}$

$= [\log(1+x)]_0^2 = \log 3 - \log 1 = \log 3$

29 (xiii) $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{3n}\right]$

$= \lim_{n\to\infty}\frac{1}{n} + \lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{3n}\right]$

$= 0 + \lim_{n\to\infty}\sum_{r=1}^{2n}\frac{1}{n+r}$

$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}\frac{1}{1+r/n} = \int_0^2 \frac{1}{1+x}$

29 (xiv) $\lim_{n\to\infty}\frac{1}{\sqrt{n}}\left[1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}\right]$

$= \lim_{n\to\infty}\frac{1}{\sqrt{n}}\left[\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}\right]$

$= \lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{r=1}^{n}\frac{1}{\sqrt{r}}$

$= \lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{r=1}^{n}\frac{1}{\sqrt{n}\sqrt{r/n}}$

$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{\sqrt{r/n}}$

$= \int_0^1 \frac{dx}{\sqrt{x}}$

$= \left[2\sqrt{x}\right]_0^1 = 2$ Ans

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29 (xv) $\lim_{n\to\infty}\left[\frac{\sqrt{(n+1)}+\sqrt{(n+2)}+\cdots+\sqrt{2n}}{n\sqrt{n}}\right]$

$= \lim_{n\to\infty}\left[\frac{\sqrt{n+1}+\sqrt{n+2}+\cdots+\sqrt{n+n}}{n\sqrt{n}}\right]$

$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{n+r}}{n\sqrt{n}} = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\sqrt{1+r/n}$

$= \int_0^1 \sqrt{1+x}\,dx = \left[\frac{2}{3}(1+x)^{3/2}\right]_0^1$

$= 2/3\,(2^{3/2}-1) = 4/3\sqrt{2} - 2/3$ Ans

30 If $\int_0^a \frac{dx}{\sqrt{x+a}+\sqrt{x}} = \int_0^{\pi/4} \frac{\sin\theta\, d\theta}{\cos^2\theta}$, Find the value of a.

Solution: let

$$I_1 = \int_0^a \frac{dx}{\sqrt{x+a}+\sqrt{x}}$$

$$= \int_0^a \frac{\sqrt{x+a}-\sqrt{x}}{(\sqrt{x+a}+\sqrt{x})(\sqrt{x+a}-\sqrt{x})}\, dx$$

$$= \int_0^a \frac{\sqrt{x-a}-\sqrt{x}}{x+a-x}\, dx$$

$$= \int_0^a \frac{\sqrt{x+a}}{a} - \int_0^a \frac{\sqrt{x}}{a}\, dx$$

$$= \frac{1}{a}\left[\frac{2}{3}(x+a)^{3/2}\right]_0^a - \frac{1}{a}\left[\frac{2}{3}x^{3/2}\right]_0^a$$

$$= \frac{2}{3a}(2a)^{3/2} - \frac{2}{3a}a^{3/2} - \frac{2}{3a}a^{3/2}$$

$$= \frac{2}{3a}(2a)^{3/2} - 2\cdot\frac{3}{3a}a^{3/2}$$

$$= \frac{4(\sqrt{2}-1)}{3}\sqrt{a} \longrightarrow (1)$$

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Also,

$$I_2 = \int_0^{\pi/4} \frac{\sin\theta}{\cos^2\theta}\, d\theta$$

put
$\cos\theta = z$
$\sin\theta\, d\theta = -dz$

θ	$\pi/4$	0
z	$1/\sqrt{2}$	1

$$= \int_1^{1/\sqrt{2}} -\frac{dz}{z^2}$$

$$= \left[\frac{1}{z}\right]_1^{1/\sqrt{2}}$$

$$= \frac{1}{\frac{1}{\sqrt{2}}} - 1 = \sqrt{2}-1$$

According to question $I_1 = I_2$

$$\Rightarrow \sqrt{a}\,\frac{4(\cancel{\sqrt{2}-1})}{3} = \frac{(\cancel{\sqrt{2}-1})}{1}$$

$$a = 9/16$$

Ex-1 $\lim_{n\to\infty}\left\{\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right\}$

$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n+r}$

$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n(1+r/n)}$ put $\frac{1}{n}=h$

$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+\frac{r}{n}}$ $n\to\infty$ $h\to 0$

$= \lim_{h\to 0} h\sum_{r=1}^{n}\frac{1}{1+rh}$

$= \int_0^1 \frac{1}{1+x}\,dx$

$= [\log(1+x)]_0^1 = \log 2 - \log 1 = \log 2$

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Ex-2

$\lim_{n\to\infty}\left\{\left(1+\frac{1}{n^2}\right)^{2/n^2}\left(1+\frac{2^2}{n^2}\right)^{4/n^2}\left(1+\frac{3^2}{n^2}\right)^{6/n^2}\cdots\left(1+\frac{n^2}{n^2}\right.\right.$

Let $P=\left\{\left(1+\frac{1}{n^2}\right)^{2/n^2}\left(1+\frac{2^2}{n^2}\right)^{4/n^2}\left(1+3^2/n^2\right)^{6/n^2}\cdots\left(1+n^2/n^2\right.\right.$

Taking log on both sides, we get

$\log P = \frac{2}{n^2}\log\left(1+\frac{1}{n^2}\right)+\frac{4}{n^2}\log\left(1+\frac{2^2}{n^2}\right)+\frac{6}{n^2}\log$

$+\cdots+\frac{2n}{n^2}\log\left(1+n^2/n^2\right)$

$\Rightarrow \log P = \frac{2.1}{n^2}\log\left(1+\frac{1^2}{n^2}\right)+\frac{2.2}{n^2}\log\left(1+\frac{2^2}{n^2}\right)+\frac{2.3}{n^2}$

$+\cdots+\frac{2.n}{n^2}\log\left(1+\frac{n^2}{n^2}\right)$

$\Rightarrow \log P = \sum_{r=1}^{n}\frac{2r}{n^2}\log\left(1+\frac{r^2}{n^2}\right)$

$\Rightarrow \lim_{n\to\infty}\log P = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{2r}{n}\log\left(1+\frac{r^2}{n^2}\right)$

put $\frac{1}{n} = h \quad n \Rightarrow \infty \quad h \to 0$

$$= \lim_{h \to 0} h \sum_{n=1}^{n} 2nh \operatorname{Log}(1 + n^2h^2)$$

$$= \int_0^1 2x \operatorname{Log}(1+x^2)\, dx$$

$$= \int_1^2 \operatorname{Log} z \, dz$$

put $1 + x^2 = z$

$2x\,dx = dz$

$x = 0, \; z = 1$

$x = 1 \quad z = 2$

$$= [z \operatorname{Log} z]_1^2 - \int_1^2 \frac{z^2}{2z}\, dz$$

$$= \operatorname{Log} 4 - \operatorname{Log} 1 - \frac{1}{2}\int_1^2 z\, dz$$

$$= \operatorname{Log} 4 - \frac{1}{2}\left[\frac{z^2}{2}\right]_1^2$$

$$= \operatorname{Log} 4 - 1$$

$$= \operatorname{Log} 4 - \operatorname{Log} e$$

$$= \operatorname{Log} \frac{4}{e}$$

$$\lim_{n \to \infty} \operatorname{Log} P = \operatorname{Log} \frac{4}{e}$$

$$\lim_{n \to \infty} P = \frac{4}{e}$$

$$\lim_{n \to \infty} \left\{ \left(1 + \frac{1}{n^2}\right)^{2/n^2} \cdots \left(1 + \frac{n^2}{n^2}\right)^{2n/n^2} \right\} = \frac{4}{e}$$

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উদা-3 Prove that

$$\lim_{n\to\infty} \frac{1^m + 2^m + 3^m + \cdots + n^m}{n^{m+1}} = \frac{1}{m+1}$$

Solution:

$$\lim_{n\to\infty} \frac{1^m + 2^m + 3^m + \cdots + n^m}{n^{m+1}}$$

$$= \lim_{n\to\infty} \frac{1^m + 2^m + 3^m + \cdots + n^m}{n \cdot n^m}$$

$$= \lim_{n\to\infty} \frac{1}{n} \left[\frac{1^m + 2^m + 3^m + \cdots + n^m}{n^m} \right]$$

$$= \lim_{n\to\infty} \frac{1}{n} \left[\left(\frac{1}{n}\right)^m + \left(\frac{2}{n}\right)^m + \cdots + \left(\frac{n}{n}\right)^m \right]$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \left(\frac{r}{n}\right)^m$$

put $\frac{1}{n} = h$, $n \to \infty$, $h \to 0$

$$= \lim_{h\to 0} h \sum_{r=1}^{n} (rh)^m$$

$$= \int_0^1 x^m \, dx$$

$$= \left[\frac{x^{m+1}}{m+1} \right]_0^1$$

$$= \frac{1^{m+1}}{m+1}$$

$$= \frac{1}{m+1}$$

Proved