Variable: A Variable is a Quantity which can take any value assigned to each.

Variable $\longrightarrow x, y, z$

Variables are two kinds

(i) Dependent Variable (ii) Independent

Constant: A constant is a quantity which has a fixed value is called a constant.

$y = e^x \quad e \approx 2.828$

$\pi = \frac{22}{7}$

$\pi, e \rightarrow$ exponential number

Function: A Relation between Independent and dependent variable is called function.

dependent $\leftarrow y = x^2 \rightarrow$ Independent

" $\leftarrow y = f(x) \rightarrow$ "

" $\leftarrow P = f(x) \rightarrow$ "

$\frac{dy}{dx}$ $\rightarrow$ Independent

$y = x^2+5$ $\rightarrow$ dependent

$z = f(x, y)$ dependent, Independent

$P = 2q^2$

$P = f(q)$ P is a function of q

dependent $S = t^2 - 2$

$S = f(t) \rightarrow$ S is a function of t.

$\frac{dy}{dx}$ = differential Co-efficient

= derivative of y

$\rightarrow$ Independent

Differential equation:

An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a D. E.

or

An equation is involving a derivative is called a D. E.

Example

(i) $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + \frac{dx}{dy} = 0$

(ii) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

(iii) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

(iv) $\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0$

(v) $\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = \sin t$

(vi) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

D. E

Ordinary D. E | partial D. E

O.D.E

A D.E. involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an O.D.E.

OR

An equation is involving an ordinary derivative is called a O.D.E.

Ex: (i) $\frac{d^3y}{dx^3} + 5\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 10y = 0$

(ii) $\frac{dy}{dx} = \frac{\sqrt{1-x^2}}{\sqrt{2-y^2}}$

(iii) $\frac{d^3y}{dx^3} + 7\frac{d^2y}{dx^2} + 8\frac{dy}{dx} - 9y = \log x$

P.D.E [partial differential equation]

A differential equation involving partial derivatives of one or more dependent variables with respect to more than one independent variable is called a P.D.E.

OR

An equation is involving a partial derivative is called a P.D.E.

(i) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 2z$

(ii) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = kz$

(iii) $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$

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Order of the differential equation

The order of the highest ordered derivative involved in a differential equation is called the order of the differential eqn.

Ex :

(i) $\frac{d^3y}{dx^3} + \left(\frac{d^2y}{dx^2}\right)^4 + \frac{dy}{dx} + cy = 0$

This is an ordinary differential equation of the third order. Since the highest derivative involves in a third derivative.

$\frac{d^3y}{dx^3}$ — 3rd order.

$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3} \ldots \frac{d^ny}{dx^n}$

(ii) $\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right) = 0$

$\frac{d^2y}{dx^2}$ — Second order.

Degree of the D.E ;

The power of the highest ordered derivative in a differential equation is called the degree of the D.E.

(1) $\frac{dy}{dx} = \sqrt{\frac{d^2y}{dx^2} + 1}$

$\left(\frac{dy}{dx}\right)^2 = \frac{d^2y}{dx^2} + 1$

$\frac{d^2y}{dx^2}$ — 2nd order 1st degree

Linear Ordinary D.E

An ordinary differential equation involving no product of the dependent variable and or it's derivative or transcendental function of the dependent variable is called a linear O.D.E

— o —

(i) no product of the dependent variable

i.e $y^2, y^3, y^4, - - - -$ etc

(ii) no product of the dependent variable and it's derivative

i.e $y\frac{dy}{dx}, y\frac{d^2y}{dx^2}, y\frac{d^3y}{dx^3} - - -$ etc

(iii) no product derivative of the dependent variable

$\left(\frac{dy}{dx}\right)^2 \left(\frac{d^2y}{dx^2}\right)^2 - - -$ etc

(iv) Transcendental function of the dependent variable

$e^x = 1 + x + \frac{x^2}{2!} + - -$

e^y, $\sin y$, $\cos y$, $\tan y$ $\log y$

— — — etc

(i) $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + 5\frac{dy}{dx} + y = \cos x$

└── 3rd order 1st degree L. O. D. E.

(ii) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + y = 0$

└── 2nd Order 1st degree non-linear

(iii) $\frac{dy}{dx} = \frac{y}{\sqrt{y^2 - x^2}}$

└──→ 1st order 1st degree non-linear

Non-linear O. D. E

Same

Function:

(i) $y = x^2$ ⟶ polynomial function

(ii) $y = \frac{x-2}{x+3}$ → Rational function

(iii) $y = \tan x / \sin x$ → Triconomitical f^n

(iv) $y = e^x$, $y = 2^{-x}$ } exponential function

(v) $y = \log x$
$= \log_e x$
$x = e^y$ } Logarithm function

(vi) Transcendential function:

A function which can be expresses in a series is called transcendental function

$$\log(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\log(x) = (x-1) - \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} - \cdots$$

$$e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots$$

Existence and uniqueness theorem:

Statement: If in a differential equation

$$\frac{dy}{dx} = f(x,y),\ y(x_0) = y_0 \longrightarrow (1)$$

The function $f(x,y)$ and $\frac{\partial f}{\partial y}$ are defined at (x_0, y_0), then the Initial value problem (1) may have a unique solution.

Undefines:

(i) Division by zero is undefined

i.e $\frac{2}{0}$ is undefines

$\frac{2}{0}$ = undefines but $\lim\limits_{h \to 0} \frac{2}{x} = \infty$

(ii) Square root of a negative number is undefines

$\sqrt{-4} = 2i,\ i = \sqrt{-1}\quad i^2 = -1$

(iii) Logarithm of a non-positive number is undefines

$\text{Log}(-10) = x$ $\text{Log}(0) =$

$-10 = e^{x}$ $\text{Log}(1) = 0$

$\boxed{0 + \text{ve} \text{ বা } - \text{ve}}$

$-\text{ve} < \ > +\text{ve}$

0 ± ve ~~কোনোটাই~~ না।   0

সংজ্ঞায়িত না

Solution

Solution:

A quantity which satisfy a equation is called solution.

Solution

- General solⁿ (G.S)
- particular Solution (P.S)
- singular solution (S.S)

General solⁿ: If the solution of the nth order D.E involves n arbitrary Constants. Then it is called a General solution (G.S)

Ex: $y''+y=0 \Rightarrow y = c_1\cos x + c_2\sin x$
(G.S) G.S

particular solⁿ:

If the arbitrary Constant of the General solution of a differential equation are obtained from given Conditions, then the solution is Called a P.S.

$$\underbrace{y = 5e^{x} - 2e^{3x}}_{P.S}$$

Singular solⁿ:

The solution of the D.E which Cannot be obtain from its General solution is called a singular solution.

Problem

General problem (G.P)	Initial value problem (I.V.P)	Boundary value problem (B.V.P)

□ **Initial Condition:** (x এর একই মান হলে)

The condition involving the value of the dependent variable for the same value of the independent variable

□ **Boundary Condition:**

The condition involving the value of the dependent variable for the difference value of the Independent variable

□ **Initial value problem** (x এর এক এর অধিক মান হলে)

A problem involving one or more differential equation with one or more Initial conditions is called an I.V.P

$y''+y=0 \qquad y(0)=5 \quad y'(0)=3 \quad y''(0)=5$

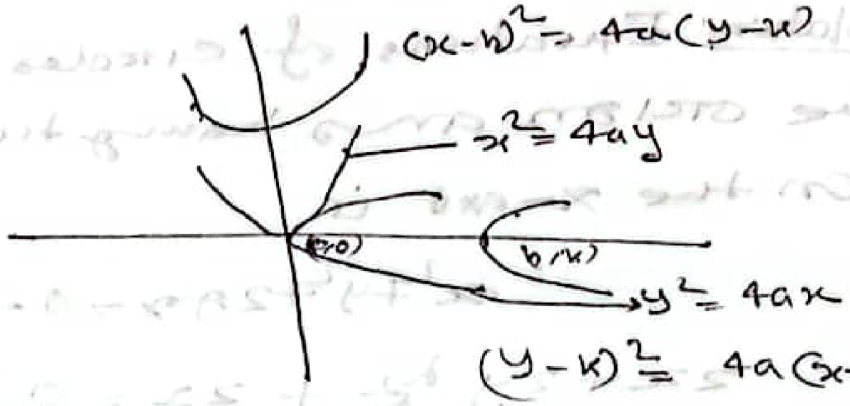
□ **Boundary value problem:**

A problem involving one or more differential equations with one or more Boundary conditions is called an B.V.P.

$$y''+y'+y=0 \qquad \left.\begin{aligned} y(0)&=5\\ y'(1)&=3\\ y''(2)&=4 \end{aligned}\right\}$$

② B.D-4

Ex-9 Find the D.E of all parabola's whose axes are parallel to the axis of y



Solution: The equation of parabola are

$(x-h)^2 = 4a(y-k) \longrightarrow (1)$

Diff. w.r.to x, we get

$2(x-h) = 4a\left(\frac{dy}{dx} - 0\right)$

$2 = 4a\frac{d^2y}{dx^2}$

$\frac{d^3y}{dx^3} = 0$ which is the D.E

Ex Find the D.E of all st. lines whose distance from the origin is unity.

Solution: The eqn of st. lines are $y = mx + c \rightarrow 1$

$\Rightarrow mx - y + c = 0$

Distance from the origin is

$\pm\frac{0-0+c}{\sqrt{1+m^2}} = 1$

$\Rightarrow c = \sqrt{1+m^2} \longrightarrow (11)$

From (1) and (11), we get

$y = mx + \sqrt{1+m^2}$

$\frac{dy}{dx} = m$

Putting the value of m in (11), we get

$y = x\frac{dy}{dx} \pm \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}$ which is the required D.E.

Ex.5 Find the D.E of all circles passing through the origin and having their centres on the x-axis.

Solution: Equations of circles passing through the origin and having their centres on the x-axis is

$$x^2+y^2+2gx=0 \longrightarrow (1)$$

$$2x+2y\frac{dy}{dx}+2g=0$$

$$g=-\left(x+y\frac{dy}{dx}\right)$$

putting g in (1), we get

$$x^2+y^2-2x\left(x+y\frac{dy}{dx}\right)=0$$

which is the required D.E

×

ΤΙΜΑ-1

Separable equations

An equation of the form

$$f_1(x)\, g_1(y)\, dx + f_2(x)\, g_2(y)\, dy = 0 \rightarrow (1)$$

is called a separable equation.

From (1), we get

$$f_1(x)\, g_1(y)\, dx + f_2(x)\, g_2(y)\, dy = 0$$

$$\Rightarrow \frac{f_1(x)}{f_2(x)} dx + \frac{g_2(y)}{g_1(y)} dy = 0$$

Integrating, we get

$$\int \frac{f_1(x)}{f_2(x)} dx + \int \frac{g_2(y)}{g_1(y)} dy = 0$$

$$F(x) + G(y) = c$$

where c is an arbitrary constant.

This is the General solution.

1. Solve the equation

$$(x-4)\, y^4\, dx - x^3 (y^2-3)\, dy = 0$$

Solution: Given that

$$(x-4)\, y^4\, dx - x^3 (y^2-3)\, dy = 0$$

This is a separable equation, separating the variable, we get

$$\frac{x-4}{x^3} dx - \frac{(y^2-3)}{y^4} dy = 0$$

$$\Rightarrow (x^{-2} - 4x^{-3})\, dx - (y^{-2} - 3y^{-4})\, dy = 0$$

Integrating, we get

$$\int (x^{-2} - 4x^{-3})\, dx - \int (y^{-2} - 3y^{-4})\, dy = 0$$

$$= -\frac{1}{x} + \frac{2}{x^2} + \frac{1}{y} - \frac{1}{y^3} = c$$

where c is an arbitrary cons

This is the General Solution

2 Solve the I.V.P that consists of the D.E

$x \sin y \, dx + (x^2+1) \cos y \, dy = 0 \quad y(1) = \pi/2$

Solution: Given that

$x \sin y \, dx + (x^2+1) \cos y \, dy = 0$

This is a separable equation, separating variables, we get

$$\frac{x}{x^2+1} dx + \frac{\cos y}{\sin y} dy = 0$$

Integrating, we get

$$\int \frac{x}{x^2+1} dx + \int \frac{\cos y}{\sin y} dy = 0$$

$$\Rightarrow \frac{1}{2} \log (x^2+1) + \log (\sin y) = \frac{1}{2} \log c$$

$$\Rightarrow \log (x^2+1) + \log (\sin y)^2 = \log c$$

$$\Rightarrow \log (x^2+1)(\sin^2 y) = \log c$$

$$(x^2+1)(\sin^2 y) = c \quad \text{—————} \quad (1)$$

where c is an arbitrary Constant.

This is the General Solution

putting $x = 1, y = \pi/2$ in eqn (1), we get

$(1+1) \cdot 1 = c \Rightarrow 2$

putting $c = 2$ in (1)

$(x^2+1) \sin^2 y = 2$

This is the particular soln

B.D

Ex.3 solve $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

Soln: given that

$$\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0 \longrightarrow (1)$$

This is the separable variables, separating variables, we get

$$\frac{\sec^2 x}{\tan x} dx + \frac{\sec^2 y}{\tan y} dy = 0$$

Integrating, we get

$$\int \frac{\sec^2 x}{\tan x} dx + \int \frac{\sec^2 y}{\tan y} dy = 0$$

$$= \log(\tan x) + \log \tan y = \log c$$

$$\tan x \tan y = c$$

③

Homogeneous equation

Homogeneous function:

A function $f(x,y)$ is said to be homogeneous of degree n in the variable x and if it can be expressed in the form

$$x^n \phi(y/x) \text{ or in the form } y^n \phi(x/y)$$

or

~~A function f(x~~

Ex: $M(x,y) = x^2 + xy + y^2$

$$= x^2 \{1 + \tfrac{y}{x} + (y/x)^2\}$$

$$= x^2 \phi_1(y/x)$$

$N(x,y) = x^2 - xy + y^2$

$$= x^2\{1 - (y/x) + (y/x)^2\}$$

$$= x^2 \phi_2(y/x)$$

$$\frac{M(x,y)}{N(x,y)} = \frac{x^{2-2}\ \phi_1(y/x)}{\cancel{x}\ \phi_2(y/x)} = x^0 \phi(y/x)$$

Homogeneous equation:

An equation of the form $M(x,y)dx + N(x,y)dy = 0$ is called homogeneous equation if $M(x,y)$ and $N(x,y)$ are both homogeneous function and they have the same degree

Ex: $(x^3+y^3)dx + 2x^2y\,dy = 0$

Q) Transform the homogeneous equation into a separable equation.

Solution: The homogeneous equation is

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$$

$$\Rightarrow \frac{dy}{dx} = \phi(y/x) \longrightarrow (1)$$

let $y = vx$

$$\left.\begin{aligned} y &= vx \\ \frac{dy}{dx} &= v + x\frac{dv}{dx} \end{aligned}\right\} \longrightarrow (2)$$

From (1) and (2), we get,

$$v + x\frac{dv}{dx} = \phi(v)$$

$$x\frac{dv}{dx} = \phi(v) - v$$

$$\frac{dv}{\phi(v)-v} = \frac{dx}{x}$$

This is the Separable equation.

Q) $(x\tan y/x + y)\,dx - x\,dy = 0$

$$\frac{dy}{dx} = \frac{x\tan y/x + y}{x} \longrightarrow (1)$$

Let, $y = vx$

$$\left.\begin{aligned} y &= vx \\ \frac{dy}{dx} &= v + x\frac{dv}{dx} \end{aligned}\right\} \longrightarrow (2)$$

From (1) and (2), we get

$$v + x\frac{dv}{dx} = \frac{x\tan v + vx}{x} = \tan v + v$$

$$\frac{dv}{\tan v} = \frac{dx}{x}$$

Integrating, we get

$$\int \frac{\cos v}{\sin v}\, dv = \int \frac{dx}{x}$$

$$\Rightarrow \log(\sin v) = \log x + \log c$$

$$\sin v = cx$$

$$\sin y/x = cx$$

This is the G. S

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4) $(y+\sqrt{x^2+y^2})\,dx - x\,dy = 0$

$$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x} \longrightarrow (1)$$

Let, $y = vx$

$$\left.\begin{aligned} y &= vx \\ \frac{dy}{dx} &= v + x\frac{dv}{dx} \end{aligned}\right\} \longrightarrow (2)$$

From (1) and (2), we get

$$v + x\frac{dv}{dx} = \frac{vx+\sqrt{x^2+v^2x^2}}{x}$$

$$x\frac{dv}{dx} = \sqrt{1+v^2}$$

$$\frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

Integrating, we get

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x}$$

$$\Rightarrow \log\{v+\sqrt{1+v^2}\} = \log x + \log c$$

$$v+\sqrt{1+v^2} = cx$$

$$\frac{y}{x} + \sqrt{\frac{y^2}{x^2}+1} = cx$$

$$y+\sqrt{y^2+x^2} = cx^2$$ This is the G.S

$$(x^3 + y^2\sqrt{x^2+y^2})\,dx - xy\sqrt{x^2+y^2}\,dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^3 + y^2\sqrt{x^2+y^2}}{xy\sqrt{x^2+y^2}} \rightarrow (1)$$

Let $y = vx$

$$\left.\begin{array}{l} y = vx \\ \frac{dy}{dx} = v + x\frac{dv}{dx} \end{array}\right\} \longrightarrow (2)$$

From (1) and (2), we get

$$v + x\frac{dv}{dx} = \frac{x^3 + v^2x^2\sqrt{x^2+v^2x^2}}{x\cdot vx\sqrt{x^2+v^2x^2}}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1+v^2\sqrt{1+v^2}}{v\sqrt{1+v^2}}$$

$$x\frac{dv}{dx} = \frac{1+v^2\sqrt{1+v^2}}{v\sqrt{1+v^2}} - v$$

$$x\frac{dv}{dx} = \frac{1+\cancel{v^2\sqrt{1+v^2}} - \cancel{v^2\sqrt{1+v^2}}}{v\sqrt{1+v^2}}$$

$$x\frac{dv}{dx} = \frac{1}{v\sqrt{1+v^2}}$$

$$v\sqrt{1+v^2}\,dv = \frac{dx}{x}$$

Integrating, we get

$$\int v\sqrt{1+v^2}\,dv = \int\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\frac{(1+v^2)^{3/2}}{3/2} = \log x + c$$

$$\Rightarrow \frac{1}{3}\left(1+\frac{y^2}{x^2}\right)^{3/2} = \log x + c$$

This is the G. S

Linear but not homogeneous

An equation of the form

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$ is called linear but not homogeneous.

Case (1)

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

If $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, then

$$\boxed{\begin{aligned} a_1x + b_1y &= v \\ a_2x + b_2y &= u \end{aligned}}$$

Problem (1)

$$\frac{dy}{dx} = \frac{y - x - 1}{y - x - 5} \longrightarrow (1)$$

Let $y - x = v$

$$\frac{dy}{dx} - 1 = \frac{dv}{dx}$$

$$\frac{dy}{dx} = 1 + \frac{dv}{dx} \longrightarrow (2)$$

From (1) and (2), we get

$$1 + \frac{dv}{dx} = \frac{v - 1}{v - 5}$$

$$\Rightarrow \frac{dv}{dx} = \frac{v-1}{v-5} - 1$$

$$\frac{dv}{dx} = \frac{v - 1 - v + 5}{v - 5} = \frac{4}{v-5}$$

$$(v - 5)\,dv = 4\,dx$$

Integrating, we get

$$\int (v - 5)\,dv = \int 4\,dx$$

$$\frac{v^2}{2} - 5v = 4x + c$$

$$\Rightarrow \frac{(y-x)^2}{2} - 5(y - x) = 4x + c$$

This is the G.S

Case (II) $\quad \dfrac{dy}{dx} = \dfrac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2} \longrightarrow (1)$

If $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$, Then

$$\left.\begin{aligned} x &= x'+h \\ y &= y'+k \\ \frac{dy}{dx} &= \frac{dy'}{dx'} \end{aligned}\right\} \longrightarrow (2)$$

From (1) and (2), we get,

$$\frac{dy'}{dx'} = \frac{a_1(x'+h)+b_1(y'+k)+c_1}{a_2(x'+h)+b_2(y'+k)+c_2}$$

$$\Rightarrow \frac{dy'}{dx'} = \frac{a_1x'+b_1y'+(a_1h+b_1k+c_1)}{a_2x'+b_2y'+(a_2h+b_2k+c_2)} \longrightarrow (3)$$

If we put,

$$a_1h+b_1k+c_1 = 0$$
$$a_2h+b_2k+c_2 = 0$$

~~By CT~~

$$\frac{h}{b_1c_2-b_2c_1} = \frac{k}{a_2c_1-a_1c_2} = \frac{1}{a_1b_2-a_2b_1}$$

Then equation (3) becomes.

$$\frac{dy'}{dx'} = \frac{a_1x'+b_1y'}{a_2x'+b_2y'} \longrightarrow (4)$$

Let

$$\left.\begin{aligned} y' &= vx' \\ \frac{dy'}{dx'} &= v+x'\frac{dv}{dx} \end{aligned}\right\} \longrightarrow (5)$$

B.D-13

Ex.1 solve $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$ ⟶ (1)

Here $a_1 = 1$, $b_1 = 2$

$a_2 = 2$ $b_2 = 1$

$\frac{a_1}{a_2} = \frac{1}{2}$ $\frac{b_1}{b_2} = \frac{2}{1}$

$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

let

$$\left.\begin{aligned} x &= x'+h \\ y &= y'+k \\ \frac{dy}{dx} &= \frac{dy'}{dx'} \end{aligned}\right\} \longrightarrow (2)$$

From (1) and (2), we get

$$\frac{dy'}{dx'} = \frac{x'+h+2(y'+k)-3}{2(x'+h)+y'+k-3}$$

$$\frac{dy'}{dx'} = \frac{x'+2y'+(h+2k-3)}{2x'+y'+(2h+k-3)} \longrightarrow (3)$$

If we put

$h+2k-3=0$

$2h+k-3=0$

$$\frac{h}{\quad} = \frac{k}{\quad} = \frac{1}{\quad}$$

i.e $h=1$, $k=1$, then

equation (3) becomes

$$\frac{dy'}{dx'} = \frac{x'+2y'}{2x'+y'}$$

$$\int \frac{v+2}{v^2-1}\,dv + \int \frac{dx'}{x'} = 0$$

$$\int \frac{v}{v^2-1}\,dv + 2\int \frac{1}{v^2-1}\,dv + \int \frac{dx'}{x'} = 0$$

$$= \frac{1}{2}\operatorname{Log}(v^2-1) + 2\cdot\frac{1}{2\cdot 1}\operatorname{Log}\frac{v-1}{v+1} + \operatorname{Log} x' = \operatorname{Log} c$$

$$\Rightarrow \log\left\{\sqrt{v^2-1}\left(\frac{v-1}{v+1}\right)x'\right\} = \log c$$

$$\Rightarrow \left(\sqrt{\frac{y'^2}{x'^2}-1}\right)\left(\frac{\frac{y'}{x'}-1}{\frac{y'}{x'}+1}\right)x' = c$$

$$\Rightarrow \sqrt{y'^2-x'^2}\left(\frac{y'-x'}{y'+x'}\right) = c$$

$$\Rightarrow \sqrt{(y-1)^2-(x-1)^2}\left(\frac{y-1-x+1}{y-1+x-1}\right) = c$$

where c is an arbitrary Constant

This is the G.S

P-13

(1) – (6). 7 (i) (ii) (iii) (iv)

$\boxed{7}$ $(5x+2y+1)dx+(2x+y+1)dy=0$

$$\Rightarrow \frac{dy}{dx} = -\frac{5x+2y+1}{2x+y+1} \longrightarrow (1)$$

let
$$\left.\begin{aligned} x &= x'+h \\ y &= y'+k \\ \frac{dy}{dx} &= \frac{dy'}{dx'} \end{aligned}\right\} \longrightarrow (2)$$

From (1) and (2), we get

$$\frac{dy'}{dx'} = -\frac{5(x'+h)+2(y'+k)+1}{2(x'+h)+(y'+k)+1}$$

$$\Rightarrow \frac{dy'}{dx'} = -\frac{5x'+2y'+(5h+2k+1)}{2x'+y'+(2h+k+1)} \longrightarrow (3)$$

If we put,

$$5h+2k+1=0$$
$$2h+k+1=0$$

$$\frac{h}{2-1} = \frac{k}{2-5} = \frac{1}{5-4} \Rightarrow h=1 \text{ and } k=-3$$

then the equation (3) becomes

$$\frac{dy'}{dx'} = -\frac{5x'+2y'}{2x'+y'} \longrightarrow (4)$$

let $y' = vx'$

$$\left.\frac{dy'}{dx'} = v + x'\frac{dv}{dx}\right\} \longrightarrow (5)$$

From (4) and (5), we get

$$v + x'\frac{dv}{dx'} = -\frac{5x'+2vx'}{2x'+vx'} = -\frac{5+2v}{2+v}$$

$$\Rightarrow x'\frac{dv}{dx'} = -\frac{5+2v}{2+v} - v = \frac{-5-2v-2v-v^2}{2+v}$$

$$\Rightarrow x'\frac{dv}{dx'} = -\frac{(v^2+4v+5)}{v+2}$$

$$\Rightarrow \frac{v+2}{v^2+4v+5}\,dv = -\frac{dx'}{x'}$$

Integrating, we get

$$\int \frac{v+2}{v^2+4v+5}\,dv + \int \frac{dx'}{x'} = 0$$

$$\Rightarrow \int \frac{\frac{1}{2}(2v+4)}{v^2+4v+5}\,dv + \int \frac{dx'}{x'} = 0$$

$$\Rightarrow \frac{1}{2}\log(v^2+4v+5) + \log x' = \log c$$

$$\Rightarrow x'\sqrt{v^2+4v+5} = c$$

$$\Rightarrow x'\sqrt{\frac{y'^2}{x'^2} + 4\frac{y'}{x'} + 5} = c$$

$$\Rightarrow \sqrt{y'^2 + 4x'y' + 5x'^2} = c$$

$$\Rightarrow \sqrt{(y+3)^2 + 4(x-1)(y+3) + 5(x-1)^2} = c$$

where c is an constant

B

uc-4

Exact and non-exact D.E

Exact differential equation

Method of grouping | Standard Grouping

$$d(uv) = u\,dv + v\,du$$

$$d(xy) = y\,dx + x\,dy$$

$$d(x^2y) = 2xy\,dx + x^2dy$$

$$2xy\,dx + x^2dy = d(x^2y)$$

$$\downarrow$$

$$M(x,y)\,dx + N(x,y)\,dy = du(x,y)$$

The expression $M(x,y)\,dx + N(x,y)\,dy$ is exact differential if there exists a function $u(x,y)$ such that

$$M(x,y)\,dx + N(x,y)\,dy = du(x,y)$$

If the expression

$M(x,y)\,dx + N(x,y)\,dy$ is exact differential then the equation

$M(x,y)\,dx + N(x,y)\,dy = 0$ is called

exact differential equation

$$\therefore\ du(x,y) = 0$$

Integrating, we get

$$u(x,y) = C$$

This is the general solution of the exact Differential equation

Theorem: state and prove the necessary and sufficient condition for exactness of a differential equation.

or

Find the condition $M(x,y)\,dx + N(x,y)\,dy = 0$ may be exact differential equation.

Answer: statement

The necessary and the sufficient condition for exactance of the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \text{ is}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

proof: Part-1 we assume that the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \text{ is exact}$$

Then ~~$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$~~ there exists a function $u(x,y)$ such that

$$M(x,y)\,dx + N(x,y)\,dy = du(x,y)$$

$$\Rightarrow M(x,y)\,dx + N(x,y)\,dy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Comparing on both sides, we get

$$M(x,y) = \frac{\partial u}{\partial x} \quad \text{——— (I)}$$

$$N(x,y) = \frac{\partial u}{\partial y} \quad \text{——— (II)}$$

From (1) $M(x,y) = \frac{\partial u}{\partial x}$ $\quad \therefore$ ~~$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$~~

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial x \partial y}$$

From (2) $N(x,y) = \frac{\partial u}{\partial y} \Rightarrow \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$

$$\therefore \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

which is the necessary condition.

(part-II) the sufficient condition:

Given $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, we have to show that

$M(x,y)\,dx + N(x,y)\,dy = 0$ is an exact differential equation

Let

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \longrightarrow (1)$$

we assume that there exists a function $u(x,y)$ such that

$$\frac{\partial u}{\partial x} = M(x,y) \longrightarrow (2)$$

Integrating (2) by partially w.r.to x, we get

$$\int \frac{\partial u}{\partial x} = \int M(x,y)$$

$$u = \int M\,\partial x + \phi(y)$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\int M\,\partial x + \frac{d\phi}{dy}$$

$$= \int \frac{\partial M}{\partial y}\,\partial x + \frac{d\phi}{dy}$$

$$= \int \frac{\partial N}{\partial x}\,\partial x + \frac{d\phi}{dy}$$

$$= N + \frac{d\phi}{dy}$$

$$N = \frac{\partial u}{\partial y} - \frac{d\phi}{dy}$$

$$M = \frac{\partial u}{\partial x}$$

Now,

$$M(x,y)\,dx + N(x,y)\,dy$$

$$= \frac{\partial u}{\partial x}dx + \left\{\frac{\partial u}{\partial y} - \frac{d\phi}{dy}\right\}dy$$

$$= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy - \frac{d\phi}{dy}dy$$

$$= du - d\phi$$

$$= d(u-\phi)$$

$$= dG \quad \text{where } G = u-\phi$$

$$\therefore M(x,y)\,dx + N(x,y)\,dy = dG,$$

So that $M(x,y)\,dx + N(x,y)\,dy = 0$ is an exact D.E.

Trajectories:

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Trajectories:

A curve which cuts every member of a given family of curves at a constant angle α is called an α-trajectory.

Orthogonal trajectories:

A curve that intersect a given family of curves at an angle $\theta = 90$, are called Orthogonal trajectories.

Find the orthogonal trajectories of the family of parabola $y = cx^2$

Solution: Given that, a family of Parabola

$$y = cx^2 \longrightarrow (1)$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = 2cx \longrightarrow (2)$$

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$(2) \div (1) \Rightarrow$

$$\frac{\frac{dy}{dx}}{y} = \frac{2cx}{cx^2} = \frac{2}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2y}{x}$$

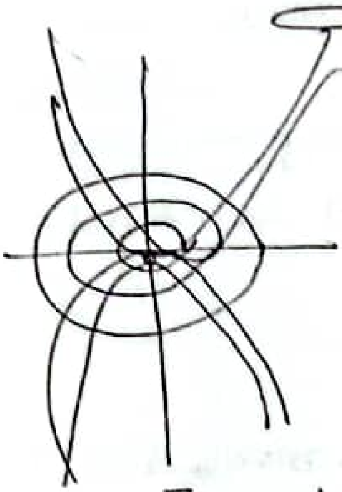
This is Differentiation equation of the family of curves (parabola)

Thus the D.E of orthogonal trajectories is

$$\frac{dy}{dx} = -\frac{x}{2y} \longrightarrow (3)$$

Separating the variable, we have

$2y\,dy = -x\,dx$



Integrating, we have

$$\frac{2y^2}{2} = -\frac{x^2}{2} + k^2$$

$$\Rightarrow x^2 + 2y^2 = k^2 \longrightarrow (4)$$

$$\Rightarrow \frac{x^2}{k^2} + \frac{y^2}{k^2/2} = 1$$ where k is an arbitrary constant

Equation (4) is a family of ellipse as it is the required family of orthogonal trajectories.

4. Find the orthogonal trajectories $x^2+y^2=c^2$

Solution: Given that, a family of circles

$$x^2+y^2 = c^2 \longrightarrow (1)$$

Diff. w.r.t. to x, we get

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

This is the D.E of the given family of circles (1)

Thus the D.E of orthogonal trajectories is

$$\frac{dy}{dx} = y/x \longrightarrow (2)$$

Separating and Integrating (2), we

$$\int \frac{dy}{y} = \int \frac{dx}{x} \Rightarrow \operatorname{Log} y = \operatorname{Log} x + \operatorname{Log} k$$

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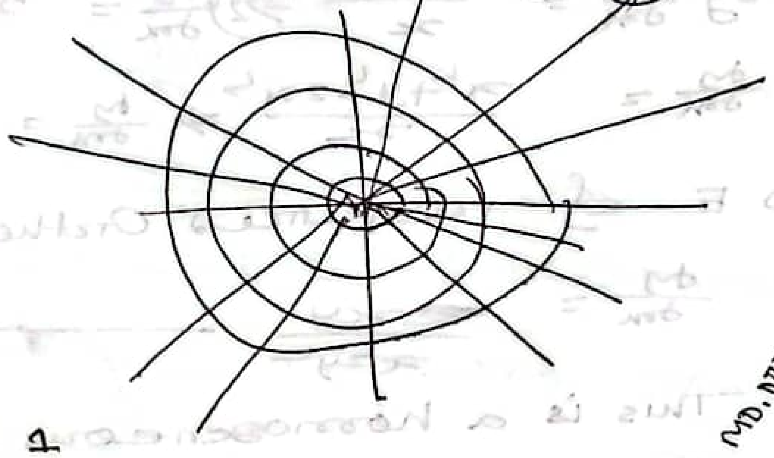
$\Rightarrow \log y = \log(xk) \Rightarrow y = kx$ ——— (3)

Equation (3) represent the orthogonal trajectories of (1) and it is a family of straight line



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Q. $cx^2 + y^2 = 1$

Given $cx^2+y^2 = 1$ —(1) i.e $c = \frac{1-y^2}{x^2}$

$$2cx + 2y\frac{dy}{dx} = 0 \quad \text{——(2)}$$

$$2\left\{\frac{1-y^2}{x^2}\right\}x + 2y\frac{dy}{dx} \Rightarrow \frac{1-y^2}{x} + y\frac{dy}{dx} = 0$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{1-y^2}{xy}}$$

the D.E of orthogonal trajectories is $\frac{dy}{dx} = \frac{xy}{1-y^2}$

$$\Rightarrow xy\,dx - (1-y^2)dy = 0$$

$$\Rightarrow xy\,dx - dy + y^2dy = 0$$

$$\Rightarrow x\,dx - \frac{dy}{y} + y\,dy = 0 \quad [\text{Dividing both side by } y]$$

$$\frac{x^2}{2} - \log y + \frac{y^2}{2} = k$$

$$x^2+y^2 - \log y^2 = 2k_1 = k \quad [2k_1 = k]$$

This represent a family of circle and it is the required trajectories of (1)

$x^2+y^2 = cx$ — (i) $\quad c = \frac{x^2+y^2}{x}$

$2x + 2y\frac{dy}{dx} = c$ — (ii)

$2x + 2y\frac{dy}{dx} = \frac{x^2+y^2}{x} \Rightarrow 2y\frac{dy}{dx} = \frac{x^2+y^2}{x} - 2x$

$\Rightarrow 2y\frac{dy}{dx} = \frac{x^2+y^2-2x^2}{x} \Rightarrow \frac{dy}{dx} = \frac{y^2-x^2}{2xy}$

The D.E of required Orthogonal trajectory

$$\frac{dy}{dx} = \frac{2xy}{x^2-y^2} \quad \text{——— (iii)}$$

This is a homogeneous equation of 2nd degree

let $y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = \frac{2xvx}{x^2-x^2v^2} = \frac{2v}{1-v^2}$

$x\frac{dv}{dx} = \frac{2v}{1-v^2} - v = \frac{2v-v+v^3}{1-v^2} = \frac{v^3+v}{1-v^2}$

$\frac{dx}{x} = \frac{1-v^2}{v^3+v}dv$

$\Rightarrow \frac{dx}{x} = \frac{(3v^2+1)\,dv}{v^3+v} - \frac{4v^2}{v^3+v})\,dv$

$= \text{Log}(3v^2+1) - 4$

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Find the orthogonal trajectories of the family of curves $y = cx^3$

Solution: $y = cx^3$

$$\frac{dy}{dx} = 3cx^2$$

$$\frac{dy}{dx} = \frac{3y}{x} = f(x,y) \text{ (say)}$$

This is the D.E of the family of curves

The D.E of required orthogonal trajectories

$$\frac{dy}{dx} = -\frac{1}{f(x,y)} = -\frac{x}{3y}$$

$$x\,dx + 3y\,dy = 0$$

$$\frac{x^2}{2} + \frac{3y^2}{2} = k^2$$

$$x^2 + 3y^2 = k^2$$

This is a equation of ellipse and it is required family of orthogonal trajectories.

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Find the Orthogonal trajectories to the curve

$$x^{\frac{4}{3}} + y^{\frac{4}{3}} = a^{\frac{4}{3}}$$

Oblique trajectories

Defⁿ: A curve that intersects a given family of curves at an angle $\theta \neq 90$ is called oblique trajectory.

Exam: $y = f(x) \Rightarrow \frac{dy}{dx} = g(x,y)$. The D.E of oblique t.

$$\frac{dy}{dx} = \frac{g(x,y) + \tan\alpha}{1 - g(x,y)\tan\alpha}$$

45- trajectory: A curve that intersects a given family of curves at an angle $\theta = 45°$ is called 45° trajectory.

Q. Find a family of oblique trajectories that intersects the family of st. lines $y = cx$ at angle 45°.

Solution: Given that,

$y = cx$ ——— (1) at angle 45°

$\frac{dy}{dx} = c$ ——— (2)

(2) ÷ 1 ⇒

$$\frac{dy}{dx} = \frac{y}{x} = f(x,y) \text{ ——— (3)}$$

This is the D.E of the given family of st. lines.

Thus the oblique trajectories is $\left[f(x,y) = y/x, \ \alpha = 45 \right.$

$$\frac{dy}{dx} = \frac{f(x,y) + \tan\alpha}{1 - f(x,y)\tan\alpha} = \frac{y/x + \tan 45°}{1 - y/x \tan 45°}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y}{x-y} \text{ ——— (4)}$$

This is the D.E of the oblique trajectories.

Eqⁿ (4) is a homogeneous of degree one

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let $y = vx$ ——— (5)

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \text{ ——— } (6)$$

Putting (5) (6) in (4)

$$v + x\frac{dv}{dx} = \frac{x+vx}{x-vx} = \frac{v+1}{1-v}$$

$$x\frac{dv}{dx} = \frac{v+1-v+v^2}{1-v} = \frac{v^2+1}{1-v}$$

$$x\frac{dv}{dx} = \frac{v^2+1}{1-v}$$

$$\frac{dx}{x} = \frac{1}{1+v^2}dv - \frac{v}{1+v^2}dv$$

$$\log x + \log k = \tan^{-1}v - \frac{1}{2}\log(v^2+1)$$

$$\Rightarrow 2\log(kx) = 2\tan^{-1}(y/x) - \log\left(\frac{y^2}{x^2}+1\right)$$

$$\Rightarrow \log(k^2x^2) = 2\tan^{-1}(y/x) - \log\left(\frac{x^2+y^2}{x^2}\right)$$

$$\Rightarrow \log(k^2x^2) + \log\left(\frac{x^2+y^2}{x^2}\right) = 2\tan^{-1}(y/x)$$

$$\Rightarrow \log k^2(x^2+y^2) = 2\tan^{-1}(y/x)$$

Where k is an arbitrary Constant which is the required oblique trajectory of the given curves. (1)

Page-77
Ross
16

Find a family of oblique trajectories that intersect the family of parabolas $y^2 = cx$ at angle $60°$

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Solution: Given that $y^2 = cx$ —— ① $\quad c = \frac{y^2}{x}$ —— ~~②~~

$$2y\frac{dy}{dx} = c$$

$$\Rightarrow 2y\frac{dy}{dx} = \frac{y^2}{x} \Rightarrow \frac{dy}{dx} = \frac{y}{2x} = f(x,y)$$

The D.E of the required oblique trajectories is

$$\frac{dy}{dx} = \frac{f(x,y) + \tan\alpha}{1 - f(x,y)\tan\alpha} = \frac{\frac{y}{2x} + \tan 60}{1 - \frac{y}{2x}\tan 60}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{y}{2x} + \sqrt{3}}{1 - \frac{y}{2x}\sqrt{3}} = \frac{y + 2x\sqrt{3}}{2x - \sqrt{3}y} \quad \text{—— ②}$$

This is the homogeneous equation of degree 1.

Let
$$y = vx$$
$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

putting this value in (2), we get

$$v + x\frac{dv}{dx} = \frac{vx + 2x\sqrt{3}}{2x - \sqrt{3}vx} = \frac{v + 2\sqrt{3}}{2 - \sqrt{3}v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v + 2\sqrt{3} - 2v + \sqrt{3}v^2}{2 - \sqrt{3}v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2\sqrt{3} - v + \sqrt{3}v^2}{2 - \sqrt{3}v}$$

$$\frac{dx}{x} = \frac{2 - \sqrt{3}v}{2\sqrt{3} - v + \sqrt{3}v^2}dv$$

$$\Rightarrow -\frac{1}{2}\frac{(2\sqrt{3}v - 4)}{\sqrt{3}v^2 - v + 2\sqrt{3}}dv - \frac{dx}{x} = 0$$

$$\Rightarrow -\frac{1}{2}\frac{(2\sqrt{3}v - 1)}{(\sqrt{3}v^2 - v + 2\sqrt{3})}dv + \frac{3}{2}\frac{dv}{\sqrt{3}\left(v^2 - \frac{v}{\sqrt{3}} + 2\right)} - \frac{dx}{x} = 0$$

$$\Rightarrow -\frac{1}{2}\frac{(2\sqrt{3}v - 1)dv}{\sqrt{3}v^2 - v + 2\sqrt{3}} - \frac{\sqrt{3}}{2}\frac{dv}{\left\{\left(v^2 - 2\cdot\frac{v}{2\sqrt{3}} + \left(\frac{1}{2\sqrt{3}}\right)^2\right\} + \frac{23}{12}} - \frac{dx}{x} = 0$$

$$\Rightarrow -\frac{1}{2}\frac{(2\sqrt{3}v - 1)dv}{\sqrt{3}v^2 - v + 2\sqrt{3}} + \frac{\sqrt{3}}{2}\frac{dv}{\left(v - \frac{1}{2\sqrt{3}}\right)^2 + \left(\frac{\sqrt{23}}{2\sqrt{3}}\right)^2} - \frac{dx}{x} = 0$$

Integrating, we get

$$\Rightarrow \frac{1}{2}\frac{(2\sqrt{3}v-1)\,dv}{\sqrt{3}v^2-v+2\sqrt{3}} - \frac{\sqrt{3}}{2}\frac{dv}{(v-\frac{1}{2\sqrt{3}})^2+(\frac{\sqrt{23}}{2\sqrt{3}})^2} + \frac{dx}{x} = 0$$

$$\Rightarrow \frac{1}{2}\log(\sqrt{3}v^2-v+2\sqrt{3}) - \frac{\sqrt{3}}{2}\cdot\frac{1}{\frac{\sqrt{23}}{2\sqrt{3}}}\tan^{-1}\frac{v-\frac{1}{2}\sqrt{3}}{\frac{\sqrt{23}}{2\sqrt{3}}} + \log x = \log c$$

$$\Rightarrow \frac{1}{2}\log\left(\sqrt{3}\frac{y^2}{x^2}-\frac{y}{x}+2\sqrt{3}\right) - \frac{\sqrt{3}}{2}\cdot\frac{2\sqrt{3}}{\sqrt{23}}\tan^{-1}\frac{y/x-\frac{\sqrt{3}}{2}}{\sqrt{23}} + \log x = \log c$$

Which is the required trajectory eqn (1)

[15] Find the 45° trajectories of the family of circles $x^2+y^2=c^2$

Solution. Given that $x^2+y^2=c^2$ —— (1) at an angle 45°

Diff ——————

$$2x+2y\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y} = f(x,y) \text{ (say)}$$

The D.E of required oblique trajectories

$$\frac{dy}{dx} = \frac{f(x,y)+\tan\alpha}{1-f(x,y)\tan\alpha}$$

$$= \frac{-x/y+1}{1-(-x/y)\cdot 1} \quad [\tan 45^\circ = 1]$$

$$\frac{dy}{dx} = \frac{y-x}{x+y} \text{ ——— } (2)$$

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let $y = vx$ ——— (3)

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ ——— (4)

$$v + x\frac{dv}{dx} = \frac{vx - x}{x + vx} = \frac{v-1}{v+1}$$

$$x\frac{dv}{dx} = \frac{v - 1 - v^2 - v}{v+1}$$

$$x\frac{dv}{dx} = -\frac{(v^2+1)}{v+1}$$

$$\frac{dx}{x} + \frac{v+1}{v^2+1}dv = 0$$

$$\Rightarrow \frac{dx}{x} + \frac{v}{v^2+1}dv + \frac{dv}{v^2+1} = 0$$

$$\log x + \frac{1}{2}\log(v^2+1) + \tan^{-1}v = c_1$$

$$\Rightarrow 2\log x + \log(v^2+1) + 2\tan^{-1}v = 2c_1 = c$$

$$\Rightarrow \log(x^2) + \log\left(\frac{y^2+x^2}{x^2}\right) + 2\tan^{-1}y/x = c$$

$$\Rightarrow \log(x^2+y^2) + 2\tan^{-1}y/x = c$$

which is the required oblique traje of the curves

OR

This is the 45° trajectories of the family of concentric circles $x^2+y^2 = c^2$

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12 Find the Value of k Such that the parabolas $y = 9x^2+k$ are the orthogonal trajectories of the family of ellipses $x^2+2y^2-y = c_2$.

Solution: Given that

$$x^2+2y^2-y = c_2 \quad \text{--- (i)}$$

$$2x + 4y\frac{dy}{dx} - \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2x}{4y-1}$$

This is the D.E of the family of Curves

Thus the D.E of orthogonal trajectories is

$$\frac{dy}{dx} = \frac{4y-1}{2x}$$

$$\frac{dy}{4y-1} = \frac{dx}{2x}$$

$$\frac{1}{4}\log(4y-1) = \frac{1}{2}\log x + \log k$$

$$\log(4y-1) = 2\log x + 4\log k$$

$$\log(4y-1) = \log x^2 + \log k^4 = \log x^2k^4$$

$$4y-1 = x^2k^4$$

$$y = \frac{1}{4} + \frac{1}{4}x^2k^4$$

where $k_1 = k^4$

$$y = \frac{1}{4} + \frac{1}{4}x^2k_1 \quad \text{--- (ii)}$$

Given $y = 9x^2+k$ --- (iii)

Comparing (ii) and (iii) we

$$k = \frac{1}{4}$$

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Find the orthogonal trajectories of the curves $y^2(a-x) = x^3$, prove that they reduce to the system $r^2 = b^2(3+\cos 2\theta)$

Solution: we have

$$y^2(a-x) = x^3 \quad \text{——— (1)}$$

$$\Rightarrow 2y(a-x)\frac{dy}{dx} - y^2 = 3x^2$$

$$\Rightarrow 2yp(a-x) - y^2 = 3x^2$$

$$(a-x) = \frac{3x^2+y^2}{2yp} \quad \text{——— (11)}$$

From (1) and (11),

$$\frac{x^3}{\cdot} = y^2\left(\frac{3x^2+y^2}{2yp}\right)$$

$$\frac{x^3}{y} = \frac{3x^2+y^2}{2p}$$

$$\frac{3x^2y+y^3}{2p} = x^3$$

For the orthogonal trajectories replace p by $-\frac{1}{p}$

$$\frac{3x^2y+y^3}{2(-\frac{1}{p})} = x^3$$

$$\Rightarrow p(3x^2y+y^3) = -2x^3$$

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(2004-2005)

$$\Rightarrow \frac{dy}{dx} = -\frac{2x^3}{3x^2y+y^3} \quad \text{——— (3)}$$

let $y = vx \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$ ——— (5)

(4)

putting (4) (5) in (3)

$$v + x\frac{dv}{dx} = -\frac{2x^3}{3x^2vx + v^3x^3} = -\frac{2}{3v+v^3}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-2-3v^2-v^4}{3v+v^3}$$

$$v^4+3v^2+2$$
$$= v^4+2v^2+v^2+2$$
$$= v^2(v^2+2)+(v^2+2)$$
$$(v^2+2)(v^2+1)$$

$$\Rightarrow \frac{3v+v^3}{(v^4+3v^2+2)}\,dv = -\frac{dx}{x}$$

$$\Rightarrow \left(\frac{2v}{1+v^2} - \frac{v}{v^2+2}\right) dv = 1\frac{dx}{x}$$

$$\text{Log}\,(1+v^2) - \frac{1}{2}\text{Log}\,(v^2+2) = -\text{Log}\,x + \text{Log}\,c$$

$$\Rightarrow x(1+v^2) = c\sqrt{v^2+2}$$

$$\Rightarrow x\left(1+y^2/x^2\right) = c\sqrt{y^2/x^2+2}$$

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$$x^2+y^2 = c\sqrt{2x^2+y^2}$$

Now, put $x = r\cos\theta$ and $y = r\sin\theta$

$$r^2 = x^2+y^2$$

$$r^2 = c\sqrt{2r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= cr\sqrt{2\cos^2\theta+\sin^2\theta}$$

$$r^2 = c^2\left\{\frac{1}{2}(2+2\cos 2\theta) + \frac{1}{2}(1-\cos 2\theta)\right\}$$

$$= c^2\left\{\frac{1}{2}(2+2\cos 2\theta + 1 - \cos 2\theta\right\}$$

$$= c^2\left\{\frac{1}{2}(3+\cos 2\theta)\right.$$

$$= b^2(3+\cos 2\theta$$

$$\left[b^2 = c^2/2\right.$$

$$r^2 = b^2(3+\cos 2\theta)$$ proved

Self - orthogonal trajectories

If the differential equation of a given family of curves and their orthogonal trajectories are identical, then the curves of the given family are called self-orthogonal trajectories.

Ex: Show that the system of curves

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$$ are self orthogonal.

Solution: Given that $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$ ——— (1)

$$\frac{2x}{a^2+\lambda}+\frac{2y}{b^2+\lambda}\frac{dy}{dx}=0 \qquad \left(\text{Since } p=\frac{dy}{dx}\right)$$

MD. NUR. ALAM
MATH
2004-2005

$$\frac{2x}{a^2+\lambda}+\frac{2yp}{b^2+\lambda}=0.$$

$$\Rightarrow \frac{2x(b^2+\lambda)+2yp(a^2+\lambda)}{(a^2+\lambda)(b^2+\lambda)}=0$$

$$\Rightarrow 2x(b^2+\lambda)+2yp(a^2+\lambda)=0.$$

$$\lambda=-\frac{b^2x+a^2yp}{x+yp}$$

$$\therefore a^2+\lambda=\frac{(a^2-b^2)x}{x+yp}, \quad b^2+\lambda=-\frac{(a^2-b^2)yp}{x+yp}$$

putting these value in eqn (1), we get

$$\Rightarrow \frac{x^2(x+yp)}{(a^2-b^2)x}-\frac{y^2(x+yp)}{(a^2-b^2)yp}=1$$

$$\Rightarrow \frac{x^2(x+yp)}{x}-\frac{y^2(x+yp)}{yp}=(a^2-b^2)$$

$$\Rightarrow x(x+yp)-y/p\,(x+yp)=a^2-b^2,$$

$$\Rightarrow (x+yp)(x-y/p)=a^2-b^2$$

replacing p by $-\frac{1}{p}$, O.T. of the D.E

$$\{x+y(-\tfrac{1}{p})\}\{x-\tfrac{y}{(-1/p)}\}=a^2-b^2$$

$\Rightarrow (x - y/p)(x + yp) = (a^2 - b^2)$ ——— (iii)

E_q^n (ii) and (iii) are identical

Hence the given family of curves is self - orthogonal

Q. Show that $\frac{x^2}{c} + \frac{y^2}{c-\lambda} = 1$ is self orthogonal trajectory

Solution: $\frac{x^2}{c} + \frac{y^2}{c-\lambda} = 1$ — (i)

$\frac{2x}{c} + \frac{2y}{c-\lambda} p = 0$ $\left[\frac{dy}{dx} = p \right.$

$c = \frac{\lambda x}{x + py}$ ——— (iii)

$c - \lambda = -\frac{\lambda y p}{x + py}$ ——— (iv)

~~Eqn~~ (iii) and (iv) in (ii)

$\Rightarrow \frac{x^2 + xyp}{\lambda x} - \frac{xy + y^2 p}{\lambda y p} = 0$

$\Rightarrow \frac{x + yp}{\lambda} - \frac{x + yp}{\lambda p} = 0$ ~~$\Rightarrow (px - y)(x + py) - \lambda p = 0$~~

$\Rightarrow (x + yp)\left(\frac{1}{\lambda} - \frac{1}{\lambda p}\right) = 0$ ~~(v) and (vi) are identic~~

$\Rightarrow (x + yp)(p - 1) = 0$

putting (iii) and (iv) in (i) ———

$\frac{x^2(x + py)}{\lambda x} + \frac{y^2(x + py)}{-\lambda p y} = 1$

$(x + py)(px - y) - \lambda p = 0$ ——— (v)

Orthogonal to ———

$\Rightarrow$

$(x - y/p)\left(-\frac{x}{p} - y\right) \cancel{+} \lambda \left(-\frac{1}{p}\right) = 0$