The Z-Transform

INTRODUCTION

A sequence is a list of numbers, sequences can be finite, like (1, 2, 3, 4) or infinite, like (1, 2, 3, 4, 5 . . .). We are interested in infinite sequences. These all have the general form $x_0, x_1, x_2, \ldots$ with the x_k's standing for the numbers in the sequence. We use the short hand:

$$(x_k)_{k=0}^{\infty} = (x_0, x_1, x_2, \ldots).$$

The Z-transform is a transform for sequences. Just like the Laplace transform takes a function of *t* and replaces it with another function of an auxiliary variable *s*, well, the Z-transform takes a sequence and replaces it with a function of an auxiliary variable, *z*. The reason for doing this is that it makes difference equations easier to solve, again, this is very like what happens with the Laplace transform, where taking the Laplace transform makes it easier to solve differential equations.

The subject of solving recurrence relations (difference equations) arise in many areas such as **combinatory, probability theory, discrete time control theory, economics** etc. There are several powerful methods available to solve these equations such as, summing factors, generating functions, **Z transformations**, Operator methods etc. It has only been in the last few decades that interest in the **Z transform** has evolved, mostly due to the rapid development of integrated circuit technology and microprocessor architecture. **Z-transform** techniques have now become a major tool in electrical, computer, and communication engineering.

In mathematics and signal processing, the **Z-transform** converts a discrete-time signal, which is a sequence of real or complex numbers, into a complex frequency domain representation.

Z-TRANSFORM

The z-transform of a sequence $x[n]$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}. \qquad (1)$$

This equation is, in general, an infinite sum or infinite power series, with z being a complex variable. Sometimes it is useful to consider Eq.(1) as an operator that transforms a sequence into a function, and we will refer to the *z-transform operator* $Z\{\,\}$, defined as

$$Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = X(z). \qquad (2)$$

With this interpretation, the z-transform operator is seen to transform the sequence $x[n]$ into the function $X(z)$, where z is a continuous complex variable. The correspondence between a sequence and its z-transform is indicated by the notation

$$x[n] \overset{z}{\leftrightarrow} X(z). \qquad (3)$$

The z-transform, as we have defined it in Eq.(1), is often referred to as the *two-sided or bilateral z-transform,* in contrast to the *one-sided or unilateral z-transform,* which is defined as

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}. \qquad (4)$$

Clearly, the bilateral and unilateral transform are equivalent only if $x[n] = 0$ for $n < 0$.

For any given sequence, the set of values of z for which the z-transform converges is called the ***region of convergence*** (**ROC**). The z-transform is most useful when the infinite sum can be expressed in closed form, i.e., when it can be summed and expressed as a simple mathematical formula. Among the most important and useful z-transforms are those for which $X(z)$ is a rational function inside the ROC, i.e.,

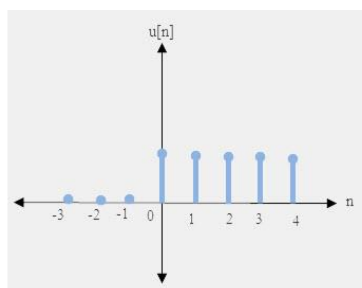
$$X(z) = \frac{P(z)}{Q(z)}, \qquad (5)$$

Where $P(z)$ and $Q(z)$ are polynomials in z. The values of z for which $X(z) = 0$ are called the **zeros** of $X(z)$, and the values of z for which $X(z)$ is infinite are referred to as the **poles** of $X(z)$. The poles of $X(z)$ for finite values of z are the roots of the denominator polynomial.

Discrete-Time Unit Step function

The definition and the wave form of the Discrete-Time Unit Step function $u[n]$ are shown as:

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$



where n is an integer.

<u>Properties and Theorems of z-Transform</u>

In the following discussion, $X(z)$ denotes the z-transform of $x[n]$, and the *ROC* of $X(z)$ is denoted by R_x; i.e.,

$$x[n] \overset{z}{\leftrightarrow} X(z), \quad \text{ROC} = R_x,$$

Linearity property

$$ax_1[n] + bx_2[n] \overset{z}{\leftrightarrow} aX_1(z) + bX_2(z), \qquad \text{ROC} = R_{x_1} \cap R_{x_2},$$

where a, b are arbitrary real or complex constants.

Shift of $x[n]$ in the Discrete-Time domain (Two-sided)

$$x[n-m] \overset{z}{\leftrightarrow} z^{-m}X(z),$$

$$\text{ROC} = R_x(\text{except for the possible addition or deletion of } z = 0 \text{ or } z = \infty)$$

The quantity m is an integer. If m is positive, the original sequence $x[n]$ is shifted right, and if m is negative, $x[n]$ is shifted left.

The derivation of this property follows directly from the z-transform expression in Eq.(1). If $y[n] = x[n-m]$, the corresponding z-transform is

$$Y(z) = \sum_{n=-\infty}^{\infty} x[n-m]\, z^{-n} = \sum_{k=-\infty}^{\infty} x[k] z^{-(m+k)} \qquad (k = n - m)$$

$$Y(z) = z^{-m} \sum_{k=-\infty}^{\infty} x[k] z^{-k} = z^{-m}\, X(z)$$

Left Shift of $x[n]$ in the Discrete-Time domain (One-sided),

$$x[n+m] \overset{z}{\leftrightarrow} z^{m}\left[X(z) - \sum_{k=0}^{m-1} x[k] z^{-k}\right],$$

Proof:

$$z\{x[n+m]\} = \sum_{n=0}^{\infty} x[n+m]\, z^{-n} = \sum_{k=m}^{\infty} x[k]\, z^{-(k-m)} \text{ , assuming } n + m = k$$

$$= z^{m}\left[\sum_{k=0}^{\infty} x[k]\, z^{-k} - \sum_{k=0}^{m-1} x[k]\, z^{-k} \right].$$

So, $Z\{x[n+1]\} = z\big[X(z) - x[0]\big]$, $Z\{x[n+2]\} = z^2\left[X(z) - x[0] - \frac{x[1]}{z}\right]$ and so on.

Right Shift of $x[n]$ in the Discrete-Time domain(one-sided)

This property is a generalization of the previous property, and allows use of non-zero values for $n < 0$.

$$x[n-m] \overset{z}{\leftrightarrow} z^{-m}X(z) + \sum_{k=1}^{m} x[-k]z^k,$$

Proof:

$$z\{x[n-m]\} = \sum_{n=0}^{\infty} x[n-m]\, z^{-n} = \sum_{k=-m}^{\infty} x[k]\, z^{-(m+k)} \text{ ,assuming } n-m=k$$

$$= z^{-m}\left[\sum_{k=-m}^{-1} x[k]\, z^{-k)} + \sum_{k=0}^{\infty} x[k]\, z^{-k}\right] = z^{-m}\left[X(z) + \sum_{k=1}^{m} x[-k]\, z^k\right].$$

So, $Z\{x[n-1]\} = z^{-1}[X(z) + x[-1]z]$, $Z\{x[n-2]\} = z^{-2}[X(z) + x[-1]z + x[-2]z^2]$ and so on.

Multiplication by a^n in the Discrete-Time domain

$$a^n x[n] \overset{z}{\leftrightarrow} X\left(\frac{z}{a}\right), \quad \text{ROC} = |a|R_x,$$

$$Z\{a^n x[n]\} = \sum_{n=-\infty}^{\infty} a^n x[n]\, z^{-n} = \sum_{n=-\infty}^{\infty} x[n]\left(\frac{z}{a}\right)^{-n} = X\left(\frac{z}{a}\right).$$

Multiplication by e^{-naT} in the Discrete-Time domain

$$e^{-naT}x[n] \overset{z}{\leftrightarrow} X(e^{aT}z), \quad \text{ROC} = |e^{-aT}|R_x.$$

Multiplication by n and n^2 in the Discrete-Time domain

$$nx[n] \overset{z}{\leftrightarrow} -z\frac{dX(z)}{dz}, \quad \text{ROC} = R_x.$$

$$n^2x[n] \overset{z}{\leftrightarrow} z\frac{d}{dz}X(z) + z^2\frac{d^2}{dz^2}X(z), \quad \text{ROC} = R_x.$$

By definition,

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}.$$

$$\frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} (-n)x[n]z^{-n-1} = -z^{-1}\sum_{n=-\infty}^{\infty} nx[n]z^{-n} = -z^{-1}Z\{nx[n]\}$$

$$Z\{nx[n]\} = \sum_{n=-\infty}^{\infty} nx[n]z^{-n} = -z\frac{d}{dz}X(z).$$

Differentiating one more time, we obtain the second pair.

Convolution in the Discrete-Time domain.

According to the convolution property,

$$x_1[n] * x_2[n] \overset{z}{\leftrightarrow} X_1(z)X_2(z), \quad \text{ROC contains } R_{x_1} \cap R_{x_2}$$

<u>The z-Transform of simple Discrete-Time sequence</u>

Example-1 The transform of the geometric sequence (**Right-sided exponential sequence).**

Consider the signal $x[n] = a^n u[n]$. The geometric sequence is defined as

$$x[n] = \begin{cases} 0 & n = -1, -2, -3 \dots. \\ a^n & n = 0, 1,2, \dots \dots \dots. \end{cases}$$

Because it is nonzero only for $n \geq 0$, this is an example of right-sided sequence. From Eq.(1),

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n]z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n \qquad (6)$$

The ROC is the range of values of z for which $|az^{-1}| < 1$, or equivalently, $|z| > |a|$. Inside the ROC, the infinite series converges to

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \qquad |z| > |a| \qquad (7)$$

For $a = 1, x[n]$ is the unit step sequence with z-transform

$$X(z) = \frac{1}{1 - z^{-1}}, \qquad |z| > 1. \qquad (8)$$

The pole-zero plot and the ROC for Example-1 are shown in Fig.1, where a "o" denotes the zero and an "x" the pole.

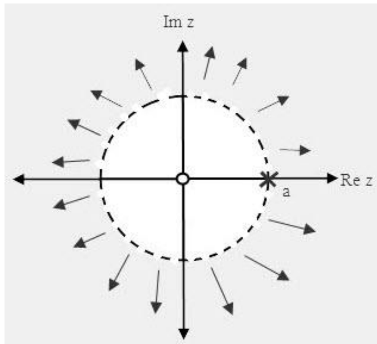


Fig. 1

Example-2 **Left-Sided Exponential sequence**

Consider the signal $x[n] = -a^n u[-n-1]$. Because it is nonzero only for $n \le -1$, this is a *left-sided* sequence. Then

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n \qquad (9)$$

If $|a^{-1}z| < 1$, or equivalently, $|z| < |a|$, the sum in Eq.(9) converges, and

$$X(z) = 1 - \frac{1}{1 - a^{-1}z} = \frac{z}{z-a}, \qquad |z| < |a| \qquad (10)$$

Comparing Eq.(7) and Eq.(10), we see that the infinite sums are different, but the algebraic expressions for $X(z)$ are identical. The z-transforms differ only in the ROC.

The pole-zero plot and the ROC for Example-2 are shown in Fig2.

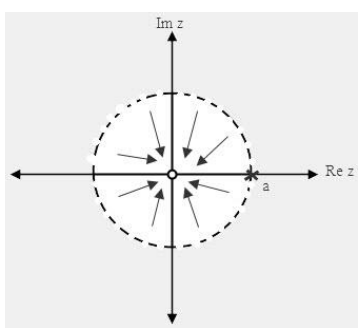


Fig. 2

Example-3 Sum of Two Exponential sequences

Consider a signal that is the sum of two real exponentials:

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n] + \left(-\tfrac{1}{3}\right)^n u[n]. \qquad (11)$$

The pole-zero plot and the ROC for Example-3 are shown in Fig.3.

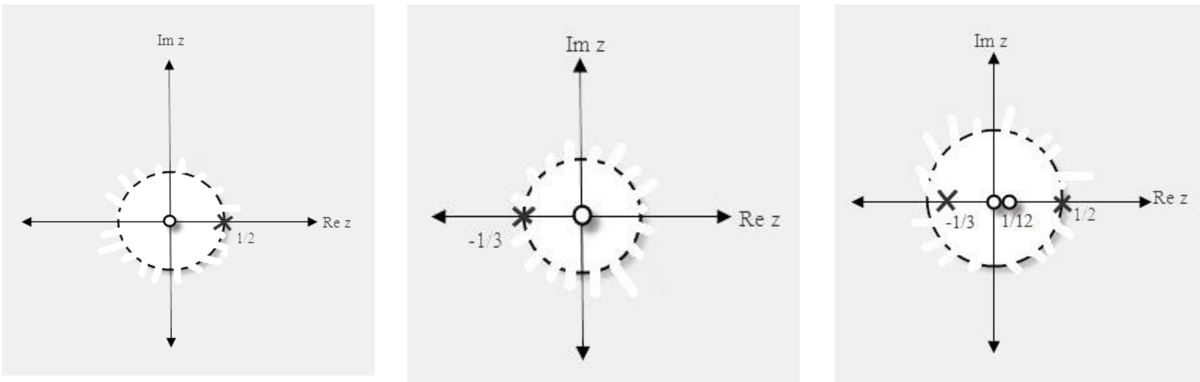


Fig.3

The z-transform is then

$$X(z) = \sum_{n=-\infty}^{\infty} \left\{\left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]\right\} z^{-n} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n]z^{-n} + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{3}\right)^n u[n]z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{1}{3}z^{-1}\right)^n = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}} = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)} \qquad (12)$$

For convergence of $X(z)$, both sums in Eq. (12) must converge, which requires both $\left|\frac{1}{2}z^{-1}\right| < 1$ and $\left|\left(-\frac{1}{3}\right)z^{-1}\right| < 1$ or, equivalently $|z| > \frac{1}{2}$ and $|z| > \frac{1}{3}$. Thus, the region of convergence is the region of overlap, $|z| > \frac{1}{2}$.

Example-4 Two-Sided Exponential sequence

Consider the sequence

$$x[n] = \left(-\tfrac{1}{3}\right)^n u[n] - \left(\tfrac{1}{2}\right)^n u[-n-1]. \qquad (13)$$

Note that this sequence grows exponentially as $n \to -\infty$. Using the general result, we obtain

$$\left(-\frac{1}{3}\right)^n u[n] \overset{z}{\leftrightarrow} \frac{1}{1+\frac{1}{3}z^{-1}}, \qquad |\mathrm{z}| > \frac{1}{3}$$

And

$$-\left(\frac{1}{2}\right)^n u[-n-1] \overset{z}{\leftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}, \qquad |\mathrm{z}| < \frac{1}{2}.$$

Thus by the linearity of the z-transform,

$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} \qquad \frac{1}{3} < |z|, \quad |z| < \frac{1}{2},$$

$$= \frac{2\left(1-\frac{1}{12}z^{-1}\right)}{\left(1+\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}. \qquad (14)$$

In this case, the ROC is the annular region $\frac{1}{3} < |z| < \frac{1}{2}$. Note that the rational function in this example is identical to the rational function in Example-3 and 4, but the ROC is different in two cases.

The pole-zero plot and the ROC for Example-4 are shown in Fig.4.

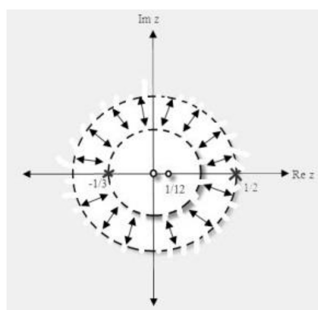


Fig.4

Example-5: Using the property of z-transform find $Z\{nx[n]\}$, where $x[n] = \left(-\frac{1}{3}\right)^n u[n]$.

Also find $Z\{n^2 x[n]\}$.

Solution: From the previous example we have

$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} = \frac{z}{z+\frac{1}{3}}, \quad \mathrm{ROC} = |\mathrm{z}| > \frac{1}{3}.$$

Therefore $Z\{nx[n]\} = -z\frac{d}{dz}X(z) = -\frac{1}{3}\frac{z}{\left(z+\frac{1}{3}\right)^2}$, ROC = $|z| > \frac{1}{3}$.

$$Z\{n^2x[n]\} = z\frac{d}{dz}X(z) + z^2\ \frac{d^2}{dz^2}X(z) = \frac{1}{3}\frac{z}{\left(z+\frac{1}{3}\right)^2} - \frac{2}{3}\frac{z^2}{\left(z+\frac{1}{3}\right)^3} = -\frac{1}{3}\frac{z\left(z-\frac{1}{3}\right)}{\left(z+\frac{1}{3}\right)^3}.$$

Example-6 **Finite-Length Sequence**

Consider the signal

$$x[n] = \begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n = \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}}\frac{z^N - a^N}{z-a} \qquad (15)$$

The ROC is determined by the set of values of z for which

$$\sum_{n=0}^{N-1} |az^{-1}|^n < \infty.$$

Since there are only finite number of nonzero terms, the sum will be finite as long as az^{-1} is finite, which in turn requires only that $|a|$ is finite, the ROC includes the entire z-plane, with the exception of the origin ($z = 0$).

Example-7 The Transform of the Discrete-Time Exponential Sequence
The discrete-time exponential sequence is defined as

$$x[n] = e^{-naT}u[n].$$

Then

$X(z) = \sum_{n=-\infty}^{\infty} e^{-naT}u[n]z^{-n} = \sum_{n=0}^{\infty} e^{-naT}z^{-n} = \sum_{n=0}^{\infty}(e^{aT}z)^{-n} = \frac{1}{1-e^{-aT}z^{-1}} = \frac{z}{z-e^{-aT}}$ (16).

ROC: $|e^{-aT}z^{-1}| < 1$ or, $|z| > |e^{-aT}|$

Example-8 The Transform of the Discrete-Time Cosine and Sine functions
Let

$$x[n] = \cos(\omega_0 n)u[n].$$

First, $x[n]$ is expressed as

$$x[n] = \frac{1}{2}\left(e^{i\omega_0}\right)^n u[n] + \frac{1}{2}\left(e^{-i\omega_0}\right)^n u[n].$$

Then using Eq.(16) and the exponential multiplication property, we see that

$$\frac{1}{2}\left(e^{i\omega_0}\right)^n u[n] \overset{z}{\leftrightarrow} \frac{\frac{1}{2}}{1-e^{i\omega_0}z^{-1}}, \qquad |z|>1$$

$$\frac{1}{2}\left(e^{-i\omega_0}\right)^n u[n] \overset{z}{\leftrightarrow} \frac{\frac{1}{2}}{1-e^{-i\omega_0}z^{-1}}, \qquad |z|>1$$

From the linearity property, it follows that

$$X(z) = \frac{\frac{1}{2}}{1-e^{i\omega_0}z^{-1}} + \frac{\frac{1}{2}}{1-e^{-i\omega_0}z^{-1}} \qquad |z|>1$$

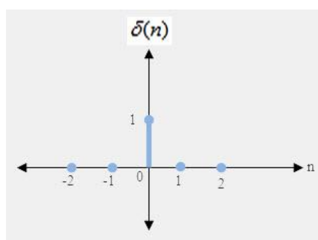
$$= \frac{1-\cos\omega_0\, z^{-1}}{1-2\cos\omega_0 z^{-1}+z^{-2}} \qquad |z|>1$$

$$X(z) = \frac{z^2 - z\cos\omega_0}{z^2-2z\cos\omega_0+1} \qquad (17)$$

Example-9 Similarly we can find z-transform for the function $x[n] = \sin(\omega_0 n)u[n]$.

Kronecker delta function or ***δ*** **function:** The Kronecker delta function, or δ function, is a generalized function, or distribution, on the real number line that is zero everywhere except at zero.

That is, $\delta[n] = \begin{cases} 1, & n=0 \\ 0, & n\neq 0 \end{cases}$



Example-10(a) **Delta function**

Find $Z\{\delta[n]\}$.

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n]z^{-n} = (1)z^0 = 1$$

Example-10(b) **Delta function**

Find $Z\{\delta[n+k]\}$.

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n+k] z^{-n} = (1) z^{k} = z^{k} \qquad (18)$$

Table-1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert$>1
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert$<1
4. $\delta[n-m]$	z^{-m}	All z except 0 (if m>0) or ∞ (if m<0)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $[\cos\omega_0 n]\, u[n]$	$\frac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]\, u[n]$	$\frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n[\cos\omega_0 n]\, u[n]$	$\frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n[\sin\omega_0 n]\, u[n]$	$\frac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$ $\quad \dfrac{1-a^N z^{-N}}{1-az^{-1}}$ $\quad |z| > 0$

Exercise- Prove all the z-transform of the sequences given in the table using definition.

PROPERTIES OF THE REGION OF CONVERGENCE FOR THE z-TRANSFORM

Property 1: The ROC is a ring or disk in the z-plane centered at the origin; i.e., $0 \le r_R < |z| < r_L \le \infty$.

Property 2: The Fourier transform of $x[n]$ converges absolutely if and only if the ROC of the z-transform of $x[n]$ includes the unit circle.

Property 3: The ROC cannot contain any poles.

Property 4: If $x[n]$ is a finite-duration sequence, i.e., a sequence that is zero except in a finite interval $-\infty < N_1 \le n \le N_2 < \infty$, then the ROC is the entire z-plane, except possibly $z = 0$ or $z = \infty$.

Property 5: If $x[n]$ is a right-sided sequence, i.e., a sequence that is zero for $n < N_1 < \infty$, the ROC extends outward from the outermost (i.e. largest magnitude) finite pole in $X(z)$ to $z = \infty$.

Property 6: If $x[n]$ is a left-sided sequence, i.e., a sequence that is zero for $n > N_2 > -\infty$ the ROC extends inward from the innermost (smallest magnitude) nonzero pole in $X(z)$ to $z = 0$.

Property 7: A two-sided sequence is an infinite-duration sequence that is neither right sided nor left sided. If $x[n]$ is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and, consistent with property 3, not containing any poles.

Stability, Causality, and the ROC

Consider a system with impulse response $h[n]$ for which the z-transform $H(z)$ has the pole-zero plot shown in Fig.5. There are three possible ROC's consistent with properties 1-7 that can be associated with this pole-zero plot. However, if we state in addition that the system is stable (or equivalently, that $h[n]$ is absolutely summable and therefore has a Fourier transform), then the ROC must include the unit circle. Thus, stability of the system and properties 1-7 imply that the ROC is the region $\frac{1}{2} < |z| < 2$. Note that as a consequence, $h[n]$ is two sided, and therefore, the system is not causal.

If we state instead that the system is causal, and therefore that $h[n]$ is right-sided, then property 5 would require that the ROC be the region $|z| > 2$. Under this condition, the system would not be stable; i.e., for this specific pole-zero plot(Fig.5), there is no ROC that imply that the system is both stable and causal.

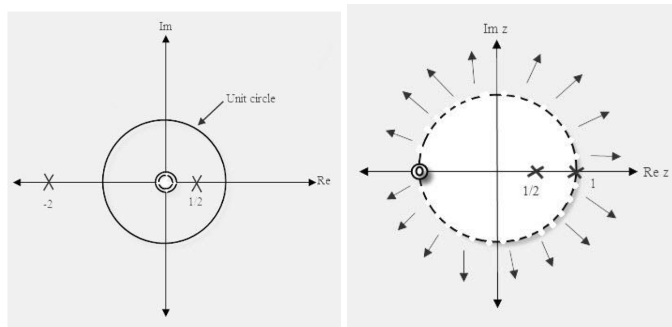


Fig. 5

THE INVERSE z-TRANSFORM

One of the important roles of the z-transform is in the analysis of discrete-time linear systems. Often, this analysis involves finding the z-transform of sequences and, after some manipulation of the algebraic expressions, finding the inverse z-transform. There are a number of formal and informal ways of determining the inverse z-transform from a given algebraic expression and associated ROC.

The inverse z-transform enables us to extract $x[n]$ from $X(z)$. It can be found by any of the following methods:

I. Partial Fraction Expansion
II. The Inversion Integral
III. Long Division of polynomials

Inspection method (using Table)

Let us consider the sequence of the form $x[n] = a^n u[n]$, which is used quite frequently. We know

$$a^n u[n] \overset{z}{\leftrightarrow} \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

The inverse z-transform of

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad \text{ROC} = |z| > \frac{1}{2} \quad \text{is} \quad x[n] = \left(\frac{1}{2}\right)^n u[n]. \qquad (19)$$

If the ROC associated with $X(z)$ in Eq. (19) had been $|z| < \frac{1}{2}$, we can write

$x[n] = -\left(\frac{1}{2}\right)^n u[-n-1].$

I. Partial fraction expansion

To see how to obtain a partial fraction expansion, let us assume that $X(z)$ is expressed as a ratio of polynomials in z^{-1}, i.e.,

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \tag{20}$$

Such z-transforms arise frequently in the study of linear time-invariant systems. To obtain the partial fraction expansion of $X(z)$, it is most convenient to note that $X(z)$ could be expressed in the form

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

Where the c_k's are the nonzero zeros of $X(z)$ and the d_k's are the nonzero poles of $X(z)$. If $M < N$ and the poles are all first order, then $X(z)$ can be expressed as

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}} \quad \text{where} \quad A_k = (1 - d_k z^{-1})X(z) \quad |z = d_k \tag{21}$$

Example-11

Consider the sequence $x[n]$ with z-transform

$$X(z) = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}, \qquad |z| > \frac{1}{2}$$

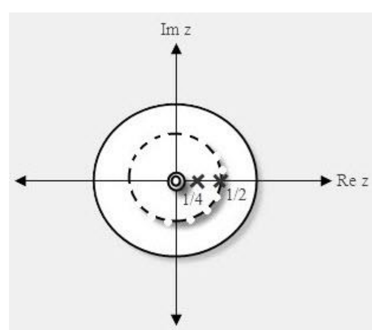


Fig.6

The pole-zero plot and the ROC for Example-11 are shown in Fig.6.

From the ROC, we see that $x[n]$ is a right-sided sequence. Since the poles are both first order, $X(z)$ can be expressed as

$$X(z) = \frac{A_1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{A_2}{\left(1 - \frac{1}{2}z^{-1}\right)}$$

$A_1 = \left(1 - \frac{1}{4}z^{-1}\right)X(z)\Big|_{z=\frac{1}{4}} = -1$ and $A_2 = \left(1 - \frac{1}{2}z^{-1}\right)X(z)\Big|_{z=\frac{1}{2}} = 2.$

Therefore,

$$X(z) = \frac{-1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{2}{\left(1 - \frac{1}{2}z^{-1}\right)}$$

Since $x[n]$ is right sided, the ROC for each term extends outward from the outermost pole. From Table and the linearity of the z-transform, it then follows that

$$x[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n].$$

If $M \geq N$ in Eq.(20), the complete partial fraction expansion would have the form

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}} \qquad (22)$$

Example-2 Consider the sequence $x[n]$ with z-transform

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$$

$$= \frac{(1 + z^{-1})^2}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})}, \qquad |z| > 1.$$

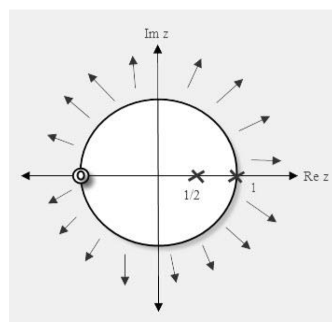


Fig.7

The pole-zero plot and the ROC for Example-8 are shown in Fig7.

From the ROC, it is clear that $x[n]$ is a right-sided sequence. Since M=N=2 and the poles are all first order, $X(z)$ can be represented as

$$X(z) = B_0 + \frac{A_1}{\left(1 - \frac{1}{2}z^{-1}\right)} + \frac{A_2}{(1 - z^{-1})} \qquad (23)$$

The constant B_0 can be found by long division:

$$\frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1\,\overline{\left)\begin{array}{l} 2 \\ z^{-2}+2z^{-1}+1 \\ \underline{z^{-2}-3z^{-1}+2} \\ \quad 5z^{-1}-1 \end{array}\right.}$$

Thus $X(z)$ can be expressed as

$$X(z)=2+\frac{-1+5z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)(1-z^{-1})} \qquad (24)$$

Now using Eq. (24), we obtain

$$A_1=\left[\frac{-1+5z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)(1-z^{-1})}\ \left(1-\frac{1}{2}z^{-1}\right)\right]\left(z=\frac{1}{2}\right)=-9 \quad \text{and}$$

$$A_2=\left[\frac{-1+5z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)(1-z^{-1})}\ (1-z^{-1})\right](z=1)=8.$$

Therefore,

$$X(z)=2-\frac{9}{\left(1-\frac{1}{2}z^{-1}\right)}+\frac{8}{(1-z^{-1})} \qquad (25)$$

From Table, we see that since the ROC is $|z|>1$,

$$2 \overset{z}{\leftrightarrow} 2\delta[n],$$

$$\frac{1}{1-\frac{1}{2}z^{-1}} \overset{z}{\leftrightarrow} \left(\frac{1}{2}\right)^n u[n],$$

$$\frac{1}{1-z^{-1}} \overset{z}{\leftrightarrow} u[n].$$

Thus, from the linearity of the z-transform,

$$x[n]=2\delta[n]-9\left(\frac{1}{2}\right)^n u[n]+8u[n].$$

Example-3 **Finite-Length Sequence**

Suppose $X(z)$ is given in the form

$$X(z) = z^2\left(1 - \frac{1}{2}z^{-1}\right)(1 + z^{-1})(1 - z^{-1}) \qquad (26)$$

Partial fraction is not appropriate. However, by multiplying the factors, we can express $X(z)$ as

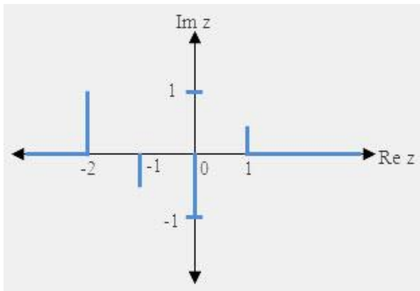
$$X(z) = z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$

Therefore, by inspection,

$$x[n] = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1].$$

Equivalently,

$$x[n] = \begin{cases} 1, & n = -2 \\ -\frac{1}{2}, & n = -1 \\ -1, & n = 0 \\ \frac{1}{2}, & n = 1 \\ 0, & \text{otherwise.} \end{cases}$$



II. The Inversion Integral

The inversion integral states that

$$x[n] = \frac{1}{2\pi i}\oint_C X(z)z^{n-1}\,dz,$$

Where C is the closed curve that encloses all poles of the integrant, and by Cauchy's residue theorem, this integral can be expressed as

$$x[n] = \sum_k \mathrm{Res}(\mathrm{z} = \mathrm{p}_k)\,[X(z)z^{n-1}] \quad ,$$

Where p_k represents a pole of $[X(z)z^{n-1}]$ and $\mathrm{Res}[X(z)z^{n-1}]$ represents a residue at $z = p_k$.

Example- Use the inversion integral method to find the Inverse z-transform of

$$X(z) = \frac{1 + 2z^{-1} + z^{-3}}{(1 - z^{-1})(1 - 0.75z^{-1})}.$$

Solution:

Multiplication of the numerator and denominator by z^3 yields

$$X(z) = \frac{z^3 + 2z^2 + 1}{z(z-1)(z-0.75)}.$$

Now, $x[n] = \sum_k \text{Res}\left[\frac{z^3+2z^2+1}{z(z-1)(z-0.75)} z^{n-1}\right]_{z=p_k}$

We are interested in the values of $x[0], x[1], x[2], ...$, that is , values of $n = 0, 1, 2, ...$

For $n = 0$, we get

$$x[0] = \sum_k \text{Res}(\text{z} = \text{p}_\text{k}) \left[\frac{z^3 + 2z^2 + 1}{z^2(z-1)(z-0.75)}\right]$$

$$= \text{Res}(\text{z} = 0) \left[\frac{z^3 + 2z^2 + 1}{z^2(z-1)(z-0.75)}\right] \quad +$$

$$+\text{Res}(\text{z} = 1) \left[\frac{z^3 + 2z^2 + 1}{z^2(z-1)(z-0.75)}\right] \quad + \text{Res}(\text{z} = 0.75) \left[\frac{z^3 + 2z^2 + 1}{z^2(z-1)(z-0.75)}\right]$$

$$x[0] = \left[\frac{d}{dz}\left\{\frac{z^3 + 2z^2 + 1}{(z-1)(z-0.75)}\right\}\right]_{z=0} + \left[\frac{z^3 + 2z^2 + 1}{z^2(z-0.75)}\right]_{z=1} + \left[\frac{z^3 + 2z^2 + 1}{z^2(z-1)}\right]_{z=0.75}$$

$x[0] = \frac{28}{9} + 16 - \frac{163}{9} = 1.$

For $n = 1$, we get

$$x[1] = \sum_k \text{Res}(\text{z} = \text{p}_\text{k}) \left[\frac{z^3 + 2z^2 + 1}{z(z-1)(z-0.75)}\right]$$

$$= \text{Res}(\text{z} = 0) \left[\frac{z^3 + 2z^2 + 1}{z(z-1)(z-0.75)}\right] \quad + \text{ Res}(\text{z} = 1) \left[\frac{z^3 + 2z^2 + 1}{z(z-1)(z-0.75)}\right]$$

$$+ \text{Res}(\text{z} = 0.75) \left[\frac{z^3 + 2z^2 + 1}{z(z-1)(z-0.75)}\right]$$

$$x[1] = \left[\frac{z^3 + 2z^2 + 1}{(z-1)(z-0.75)}\right]_{z=0} + \left[\frac{z^3 + 2z^2 + 1}{z(z-0.75)}\right]_{z=1} + \left[\frac{z^3 + 2z^2 + 1}{z(z-1)}\right]_{z=0.75}$$

$= \frac{4}{3} + 16 - \frac{163}{4} = \frac{15}{4}.$

For $n \geq 2$, there are no poles at $z = 0$, that is, the only poles are at $z = 1$ and $z = 0.75$. Therefore

$$x[n] = \sum_k \mathrm{Res}(\mathrm{z} = \mathrm{p_k}) \left[\frac{(z^3 + 2z^2 + 1)z^{n-2}}{(z-1)(z-0.75)}\right]$$

$$= \mathrm{Res}(\mathrm{z} = 1) \left[\frac{(z^3 + 2z^2 + 1)z^{n-2}}{(z-1)(z-0.75)}\right] + \mathrm{Res}(\mathrm{z} = 0.75) \left[\frac{(z^3 + 2z^2 + 1)z^{n-2}}{(z-1)(z-0.75)}\right]$$

$$x[n] = 16 - \frac{163}{9}(0.75)^n \text{ for } n \geq 2.$$

Now we can express $x[n]$ for all $n \geq 0$ as

$$x[n] = \frac{28}{9}\delta[n] + \frac{4}{3}\delta[n-1] + 16 - \frac{163}{9}(0.75)^n \,.$$

Example- Consider the z-transform

$$X(z) = \frac{1}{z - \frac{1}{4}}, \qquad |z| > \frac{1}{4}.$$

From the ROC, we identify this as corresponding to a right-sided sequence. We can rewrite $X(z)$ in the form

$$X(z) = \frac{z^{-1}}{1 - \frac{1}{4}z^{-1}} \qquad |z| > \frac{1}{4}.$$

$$X(z) = -4 + \frac{4}{1 - \frac{1}{4}z^{-1}} \qquad |z| > \frac{1}{4}$$

Now $x[n]$ can be expressed as

$$x[n] = -4\delta[n] + 4\left(\frac{1}{4}\right)^n u[n]. \qquad (27)$$

Alternatively, this problem can be solved by using the inversion integral method.

Example- Determine the inverse z-transform of

$$X(z) = \frac{1}{1 - 1.5\, z^{-1} + 0.5\, z^{-2}}$$

If

(a) ROC: $|z| > 1$
(b) ROC: $|z| < 0.5$
(c) ROC: $0.5 < |z| < 1$.

Solution: Using partial fraction,

$$X(z) = \frac{2}{1 - z^{-1}} - \frac{1}{1 - 0.5z^{-1}}$$

(a) In the case when ROC is $|z| > 1$, the signal $x[n]$ is causal and both terms are causal terms. Taking inverse z-transform, we get
$$x[n] = 2(1)^n u[n] - (0.5)^n u[n] = (2 - 0.5^n)u[n].$$
(b) When the ROC is $|z| < 0.5$, the signal $x[n]$ is anticausal. Thus both the terms result in anticausal components. Taking inverse z-transform, we get

$$x[n] = [-2 + (0.5)^n]u[-n - 1].$$

(c) In the case when ROC $0.5 < |z| < 1$ is a ring, which implies that the signal $x[n]$ is two-sided. Thus one of the terms corresponds to a causal signal and the other to an anticausal signal. Obviously, the given ROC is the overlapping of the regions $|z| > 0.5$ and $|z| < 1$. Hence the pole $p_2 = 0.5$ provides the causal part and the pole $p_1 = 1$ the anticausal. Thus

$$x[n] = -2(1)^n u[-n - 1] - (0.5)^n u[n].$$

The **convolution** property plays a particularly important role in the analysis of **LTI** (Linear Time-Invariant) systems. Specifically, as a consequence of this property, the z-transform of the output (i.e. $Y(z)$) of an LTI system is the product of the z-transform of the input (i.e. $X(z)$) and the z-transform of the system impulse response (i.e. $H(z)$). The z-transform of the impulse response of an LTI system is typically referred to as the system function.

Thus we can write

$$Y(z) = H(z)X(z).$$

We can find the discrete impulse response $h[n]$ by taking inverse z-transform of the discrete transfer function / system function $H(z)$, that is,

$$h[n] = Z^{-1}\{H(z)\}.$$

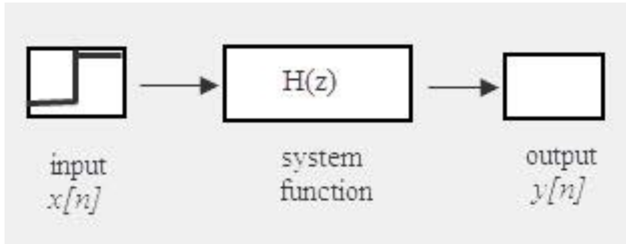


Table 2 SOME z-TRANSFORM PROPERTIES

Sequence	Transform	ROC
$x[n]$	$X(z)$	R_x
$x_1[n]$	$X_1(z)$	R_{x_1}
$x_2[n]$	$X_2(z)$	R_{x_2}
$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	$R_{x_1} \cap R_{x_2}$
$x[n - n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞.
$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\|R_x$
$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x, except for the possible addition or deletion of the origin or ∞.
$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	$R_{x_1} \cap R_{x_2}$

<u>Sample Exercises on Z-transform-8.1</u>

1. Prove the following z-transform pairs (using definition):

   (a) $\delta[n] \leftrightarrow 1$ (b) $\delta[n - m] \leftrightarrow z^{-m}$ (c) $na^n u[n] \leftrightarrow \frac{az}{(z-a)^2}$

   (d) $n^2 a^n u[n] \leftrightarrow \frac{az(z+a)}{(z-a)^3}$ (e) $(n + 1)u[n] \leftrightarrow \frac{z^2}{(z-1)^2}$.

2. Determine the sequence $x[n]$ with z-transform and hence sketch $x[n]$

$$X(z) = (1 + 2z)(1 + 3z^{-1})(1 - z^{-1}).$$

3. Determine the inverse z-transform for the following.

(a) $X(z) = \frac{1}{1+\frac{1}{4}z^{-1}}$, $|z| > \frac{1}{4}$.

(b) $X(z) = \frac{1-\frac{1}{2}z^{-1}}{1+\frac{2}{5}z^{-1}-\frac{3}{5}z^{-2}}$, $|z| > \frac{1}{2}$. (c) $X(z) = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{9}z^{-2}}$, $|z| > \frac{1}{3}$.

(d) $X(z) = \frac{1-az^{-1}}{z^{-1}-a}$, $|z| > |1/a|$.

4. Use partial fraction expansion and the inversion integral method to find $x[n]$

(a) $X(z) = \frac{10z^{-2}}{(1+z^{-1})(1-z^{-1})^2}$, $|z| > 1$. (b) $X(z) = \frac{1}{(1+0.5z^{-1})(1-0.50z^{-1})(1-z^{-1})}$, $|z| > 1$.

(c) $X(z) = \frac{z^{-1}}{(1+0.75z^{-1})(1-0.75z^{-1})^2}$, $|z| > 1$. (d) $X(z) = \frac{A}{(1-0.75z^{-1})(1-0.5z^{-1})}$, $|z| > 1$.

(e) $X(z) = \frac{1}{(1+z^{-1})(1-0.75z^{-1})}$, $|z| > 1$.

5. If $H(z) = \frac{1}{1-\frac{1}{4}z^{-2}}$ and $h[n] = A_1\alpha_1^n u[n] + A_2\alpha_2^n u[n]$, determine the values of A_1, A_2, α_1 and α_2.

Answer: 2. $x[n] = 5\delta[n] + 2\delta[n+1] - 4\delta[n-1] - 3\delta[n-2]$, 3(a). $\left(-\frac{1}{4}\right)^n u[n]$, 3(b). $\frac{15}{16}(-1)^n u[n] + \frac{1}{16}\left(\frac{3}{5}\right)^n u[n]$, 3(c)$(\frac{1}{3})^n u[n]$, 3(d). $-a\delta[n] - \frac{1-a^2}{a}\left(\frac{1}{a}\right)^n u[n]$, 4(a). $\frac{5}{2}(-1)^n u[n] - \frac{5}{2}u[n] + 5n\,u[n]$, 4(b). $\frac{1}{6}(-0.5)^n u[n] - \frac{1}{2}(0.5)^n u[n] + \frac{4}{3}u[n]$, 4(c). $\frac{1}{3}\left(\frac{3}{4}\right)^n u[n] + \frac{1}{2}n\left(\frac{3}{4}\right)^n u[n] - \frac{1}{3}\left(-\frac{3}{4}\right)^n u[n]$, , 4(d). $A[3(0.75)^n u[n] - 2(0.5)^n u[n]]$, 4(e). $\frac{4}{7}(-1)^n u[n] + \frac{3}{7}(0.75)^n u[n]$, 5. $A_1 = \frac{1}{2}, A_2 = \frac{1}{2}, \alpha_1 = \frac{1}{2}, \alpha_2 = -\frac{1}{2}$.

DIFFERENCE EQUATION

Difference equations are the discrete equivalent of differential equations. The terminology is similar and the methods of solution have much in common with each other. Difference equations arise whenever an independent variable can have only discrete values. They are growing importance in engineering in view of their association with discrete time systems based on the microprocessor.

Example - The difference equation describing the input-output relationship of a discrete-time system with zero initial conditions, is

$$2y[n] - 3y[n-1] - 2y[n-2] = x[n] + x[n-1] \qquad (1)$$

Compute:

a. The transfer function H(z)
b. The discrete-time impulse response $h[n]$.
c. The response when the input is the discrete unit step function $u[n]$.

Solution

a. Taking z-transform of both sides of Eq.(1), we obtain

$$2Y(z) - 3z^{-1}Y(z) - 2z^{-2}Y(z) = X(z) + z^{-1}X(z)$$

And thus

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-1}}{2 - 3z^{-1} - 2z^{-2}} = \frac{z^2 + z}{2z^2 - 3z - 2} \qquad (2)$$

b. To obtain the discrete-time impulse response $h[n]$, we need to compute the inverse z-transform of Eq.(2).

$$h[n] = \sum_k \mathrm{Res}(\mathrm{z} = \mathrm{p_k}) \left[\frac{(z^2 + z)z^{n-1}}{2\left(z + \frac{1}{2}\right)(z - 2)} \right]$$

$$= \left[\frac{(z+1)z^n}{2\left(z + \frac{1}{2}\right)} \right]_{z=2} + \left[\frac{(z+1)z^n}{2(z-2)} \right]_{z=-\frac{1}{2}}$$

$h[n] = \frac{3}{5}(2)^n - \frac{1}{10}\left(-\frac{1}{2}\right)^n.$

c. From $Y(z) = H(z)X(z)$, the transform $u[n] \overset{z}{\leftrightarrow} \frac{z}{z-1}$, and using the result of part(a) we obtain:

$$Y(z) = \frac{z^2 + z}{2z^2 - 3z - 2} \frac{z}{z - 1} = \frac{(z+1)z^2}{2\left(z + \frac{1}{2}\right)(z - 2)(z - 1)}$$

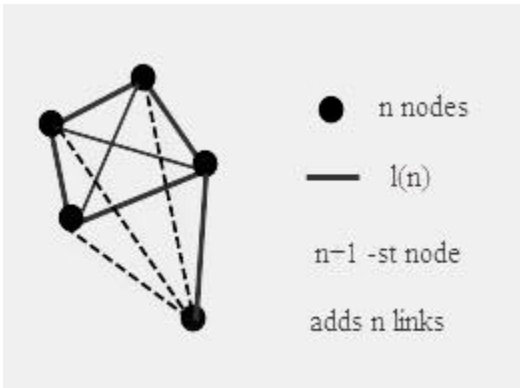
$$y[n] = \sum_k \mathrm{Res}(\mathrm{z} = \mathrm{p_k}) \left[\frac{(z+1)z^2 z^{n-1}}{2\left(z + \frac{1}{2}\right)(z - 2)(z - 1)} \right]$$

$$= \left[\frac{(z+1)z^{n+1}}{2(z-2)(z-1)}\right]_{z=-\frac{1}{2}} + \left[\frac{(z+1)z^{n+1}}{2\left(z+\frac{1}{2}\right)(z-1)}\right]_{z=2} + \left[\frac{(z+1)z^{n+1}}{2\left(z+\frac{1}{2}\right)(z-2)}\right]_{z=1}$$

$$y[n] = -\tfrac{1}{30}\left(-\tfrac{1}{2}\right)^{n} + \tfrac{6}{5}2^{n} - \tfrac{2}{3}.$$

Application of the z-Transform to <u>Connectivity</u>

Example: If we have *n* nodes connected to each other, how many total links do we have?
Solution:



If a new node is added to the previous *n* nodes, exactly *n* new links are created. If we call the number of initial links *l*(*n*) then the number of links between *n*+1 nodes will be

$$l(n+1) = l(n) + n\ .$$

This is a difference equation for *l* for which a closed form solution can be obtained with the z-transform. Let us define $Z\{l[n]\} = L(z)$. Taking z-transform of the above equation, we get

$$zL(z) - zl(0) = L(z) + \frac{z}{(z-1)^2}$$

There are zero links for zero nodes, thus $l(0) = 0$. Therefore we have

$$L(z) = \frac{z}{(z-1)^3}$$

which consists of a pole at $z = 1$ of order 3. Taking inverse z-transform,

$$l(n) = \frac{1}{2!}\left[\frac{d^2}{dz^2}\{L(z)(z-1)^3 z^{n-1}\}\right]_{z=1} = \frac{1}{2!}\left[\frac{d^2}{dz^2}z^n\right]_{z=1} = \frac{n(n-1)}{2}.$$

Application of the z-Transform to <u>Mathematics</u>

Example : The sequence 0, 1, 1, 2, 3, 5, 8, 13,… is said to form the Fibonacci numbers. Find the difference equation satisfied by them. Find an explicit formula for the nth Fibonacci number.

Solution:

It can be observed that sum of two consecutive numbers is the third number. Hence it satisfies the recurrence relation

$$x[n] = x[n+2] - x[n+1], \quad n \geq 0$$

with the initial conditions $x[0] = 0, x[1] = 1$.

So,

$$x[n+2] - x[n+1] - x[n] = 0$$

Now taking Z-transforms, we have

$$Z\{x[n+2] - x[n+1] - x[n]\} = 0$$

or, $z^2\left[X(z) - x[0] - \frac{x[1]}{z}\right] - z\left[X(z) - x[0]\right] - X(z) = 0$

or, $(z^2 - z - 1)X(z) = z$

Now, $X(z) = \frac{z}{z^2 - z - 1}$

$$x[n] = \sum_k \mathrm{Res}(\mathrm{z} = \mathrm{p_k})\left[\frac{z.z^{n-1}}{(z-z_1)(z-z_2)}\right] = \sum_k \mathrm{Res}(\mathrm{z} = \mathrm{p_k})\left[\frac{z^n}{(z-z_1)(z-z_2)}\right]$$

Where z_1 and z_2 are the poles of $X(z)$ and $z_1 = \frac{1+\sqrt{5}}{2}, z_2 = \frac{1-\sqrt{5}}{2}$.

$$x[n] = \frac{z_1^n}{z_1 - z_2} + \frac{z_2^n}{z_2 - z_1} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

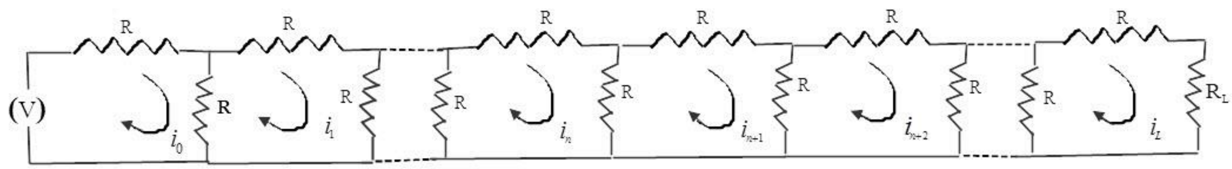
Example: Find a recurrence relation and initial conditions for the sequence $\{1, 5, 17, 53, 161, 485, ...\}$.

Solution:

Finding the recurrence relation would be easier if we had some context for the problem (like the Tower of Hanoi, for example). But we have only the sequence. Remember, the recurrence relation tells, how to get from previous terms to future terms. What is going on here? We could look at the differences between terms: 4, 12, 36, 108, …. Notice that these are growing by a factor of 3. Is the original sequence as well? 1·3=3, 5·3=15, 17·3=51 and so on. It appears that we always end up with 2 less than the next term.

So $a[n] = 3a[n-1] + 2$ is our recurrence relation and the initial condition is $a[0] = 1$.

Ladder Network



It is required to find the current in the n-th loop i_n for the ladder network shown in the Fig. Assume all the resistances except the load R_L have the same value R.

The equation for the loop with current i_{n+1} is

$$-Ri_n + 3Ri_{n+1} - Ri_{n+2} = 0$$

This equation is true for any n except -1 and $k-1$ (the beginning and the end loops). The equation with end conditions is sufficient to describe the network.

Applying z-transformation and dividing by R, we obtain

$$I(z) - 3zI(z) + 3zi_0 + z^2I(z) - z^2i_0 - zi_1 = 0, \qquad \text{or,} \, I(z) = \frac{z(zi_0 - 3i_0 + i_1)}{z^2 - 3z + 1}$$

Let us write the equation for the first loop

$2Ri_0 - Ri_1 = V$ or, $i_1 = 2i_0 - \frac{V}{R}$

$$i_n = Z^{-1}\{I(z)\} = \frac{1}{z_1 - z_2}[z_1^n(z_1 - A)i_0 - z_2^n(z_2 - A)],$$

$$\text{where } z_1 = \frac{3+\sqrt{5}}{2}, z_2 = \frac{3-\sqrt{5}}{2}, A = 1 + \frac{V}{Ri_0}$$

System of Difference Equations

Example: Solve, by z-transformation, the simultaneous difference equations,

$$u[n] = 3u[n-1] + 2v[n-1], \; v[n] = u[n-1] + 2v[n-1] \text{ with } u[-1] = 1, v[-1] = 2.$$

Solution: Taking z-transform of both the equations

$$(z-3)U(z) - 2V(z) = 7z \qquad (1)$$

$$U(z) + (2-z)V(z) = -5z \qquad (2)$$

Now using Cramer's rule, we get

$$U(z) = \frac{\begin{vmatrix} 7z & -2 \\ -5z & 2-z \end{vmatrix}}{\begin{vmatrix} z-3 & -2 \\ 1 & 2-z \end{vmatrix}} = \frac{z(7z-4)}{(z-4)(z-1)} \text{ and } V(z) = \frac{\begin{vmatrix} z-3 & 7z \\ 1 & -5z \end{vmatrix}}{\begin{vmatrix} z-3 & -2 \\ 1 & 2-z \end{vmatrix}} = \frac{z(5z-8)}{(z-4)(z-1)}$$

Hence, taking inverse z-trans

$$u[n] = 8(4)^n - 1 \text{ and } \mathrm{v}[n] = 4(4)^n + 1.$$

Sample exercises on application of Z-transform-8.2

1. Consider an LTI system with input $x[n]$ and output $y[n]$ that satisfies the difference equation with zero initial conditions
$$y[n] - 2y[n-1] + y[n-2] = x[n] - x[n-1].$$

Compute

(a) The transfer function $H(z)$.
(b) The discrete-time impulse response $h(n)$.
(c) The response when the input is the discrete unit step function $u[n]$.

Ans: (a)$H(z)$=$\frac{z}{z-1}$, (b)$h[n] = u[n]$ (c) $y[n] = (n+1)u[n]$,

2. Find the impulse response $h[n]$ of casual system having system function $H(z) = \frac{1+5z^{-1}}{1-2z^{-1}}$.
Ans: $h(n) = -\frac{5}{2}\delta[n] + \frac{7}{2}(2)^n u[n]$.
3. A discrete-time system is described by the difference equation
$$y[n] + y[n-1] = x[n]$$
Where $y[n] = 0$ for $n < 0$.
(a) Compute the transfer function $H(z)$.
(b) Compute the impulse response $h[n]$
(c) Compute the response when the input is $x[n] = 10$ for $n \geq 0$.
4. Given the discrete function
$$H(z) = \frac{z+2}{8z^2-2z-3}.$$
Write the difference equation that relates the output $y[n]$ to the input $x[n]$.
5. Find the recurrence relation and initial condition for the following sequences:
(a) $x[n] = \{3,5,11,21,43,85,...\}$ Ans: $a[n+2] = 2a[n] + a[n+1], a[0] = 3, a[1] = 5$.
(b) $x[n] = \{5,5,7,11,19,35,...\}$ Ans: $a[n+1] = a[n] + 2^n, a[0] = 5$.

6. Use the one-sided z-transform to determine $y[n], n \geq 0$ in the following cases:

(c) $6y[n] - 5y[n-1] + y[n-2] = \frac{1}{4^n}, n \geq 0; \quad y[-1] = 0, y[-2] = 0.$
Ans: $\frac{1}{2}\left(\frac{1}{4}\right)^n + \left(\frac{1}{2}\right)^n - \frac{4}{3}\left(\frac{1}{3}\right)^n.$

(d) $y[n] = \frac{1}{4}y[n-1] + x[n], \quad x[n] = \left(\frac{1}{5}\right)^n u[n], \; y[-1] = 1.$
Ans: $\frac{21}{4}\left(\frac{1}{4}\right)^n - 4\left(\frac{1}{5}\right)^n.$

(e) $y[n] = \frac{1}{4}y[n-2] + x[n], \quad x[n] = u[n], \; y[-1] = 0, y[-2] = 2.$
Ans: $\frac{4}{3} - \frac{1}{4}\left(\frac{1}{2}\right)^n + \frac{5}{12}\left(-\frac{1}{2}\right)^n.$

(f) $y[n] - y[n-1] - 2y[n-2] = 0, \quad y[-1] = 2, \; y[-2] = 7.$
Ans: $12(2)^n + 4(-1)^n.$

(g) $y[n+2] - 4y[n+1] + 4y[n] = 0, \quad y[0] = 0, y[1] = 6.$
Ans: $3 * \mathrm{n} * 2^{\mathrm{n}}.$

(h) $y[n+2] - 5y[n+1] + 4y[n] = 2^n, \quad y[0] = 1, y[1] = 0.$
Ans: $\frac{5}{3} - \frac{(4)^n}{6} - \frac{(2)^n}{2}.$

6. Assume that the population of a country in 2010 is 140 million and is growing at the rate of 2.5% a year.
   (a) Find a recurrence relation and initial condition for the population of the country *n* years after 2010?
   (b) Find an explicit formula for the population of the country *n* years after 2010.
   (c) Find the population of the country at the end of the year 2020.

7. Suppose that the number of bacteria in a colony doubles every hour.
   (a) Find a recurrence for the number of bacteria after *n* hours have elapsed.
   (b) If 100 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?
8. A deposit of Tk. 1,00,000 is made in an investment fund at the beginning of a year. On the last day of each year two dividends are awarded. The first dividend is 20% of the amount in the account during that year. The second dividend is 28% of the amount in the account in the previous year.
   (a) Find a recurrence relation for $\{P_n\}$, where P_n is the amount in the account after *n* years if no money has been withdrawn?
   (b) How much is in the account after *n* years if no money has been withdrawn?

Compute z-transform and inverse z-transform using MATLAB

Ex-1 Compute the z-transform of the function $f = \sin(n)$ using MATLAB

```
>> syms n x
>> f=sin(n);
>> ztrans(f, n, x)
ans =
(x*sin(1))/(x^2 - 2*cos(1)*x + 1)
```

Ex-2 Compute the z-transform of the function $f = a^n$ using MATLAB

```
>> syms a n x
>> f=a^n;
>> ztrans(f, x)
ans =
-x/(a - x)
```

Ex-3 Compute the inverse z-transform of the function $f = \frac{2z}{(z-2)^2}$ using MATLAB

```
>> syms k x
>> F=2*x/(x-2)^2;
>> iztrans(F, x, k)
 ans =
 2^k + 2^k*(k - 1)
```

Ex-4 Compute the inverse z-transform of the function $f = e^{\frac{a}{z}}$ using MATLAB

```
>> syms z a n
>> F=exp(a/z);
>> iztrans(F)
ans =
a^n/factorial(n)
```

Discrete-time systems described by difference equations (FIR and IIR)

Difference equation:

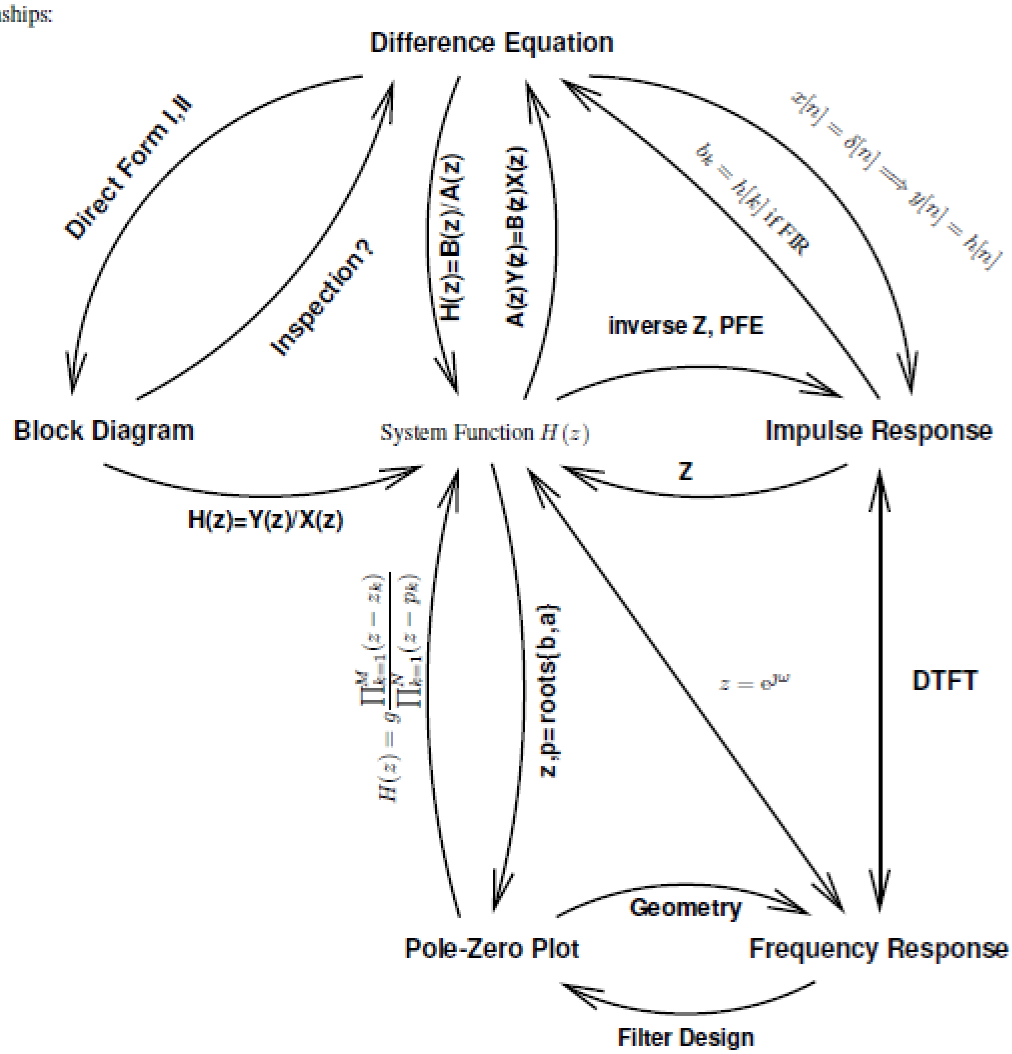
$$y[n] = -\sum_{k=1}^{N} a_k\, y[n-k] + \sum_{k=0}^{M} b_k\, x[n-k]$$

System function (in expanded polynomial and in factored polynomial forms):

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=1}^{N} a_k z^{-k}} = b_0 z^{N-M} \frac{\prod_{k=1}^{M}(z-z_k)}{\prod_{k=1}^{N}(z-p_k)}$$

Frequency magnitude response: $|\mathcal{H}(\omega)| = b_0 \frac{\prod_k |e^{j\omega} - z_k|}{\prod_k |e^{j\omega} - p_k|}$

Relationships:



Each representation corresponds to a type of input/output relationship, *e.g.*, convolution.