

## Lecture Note-6

### Complex Integration

#### Line integral in the complex plane

Complex definite integrals are called (complex) **line integrals**. They are written as

$$\int_C f(z) dz.$$

Here the **integrand**  $f(z)$  is integrated over a given curve  $C$ . This curve  $C$  in the complex plane is called the **path of integration**.

If  $C$  is a **closed path** (one whose terminal point coincides with its initial point),

then it is denoted by  $\oint_C f(z) dz$ .

**Partitioning of path  $C$ :** If  $C$  is a combination of  $C_1$  and  $C_2$  then,  $\int_C f(z) dz = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz$ .

We may represent  $C$  by a parametric representation  $z(t) = x(t) + i y(t)$   $a \leq t \leq b$ . That is,

$\int_C f(z) dz = \int_C f(z(t)) z'(t) dt$ . The sense of increasing  $t$  is called the **positive sense** on  $C$ .

**Note:** Parametric representation of any curve is not unique.

**Example 1:** Find and sketch the path whose orientation is given by  $z(t) = (1 + 3i)t$  ( $1 \leq t \leq 2$ ).

**Solution:**

$$z(t) = (1 + 3i)t \quad (1 \leq t \leq 2)$$

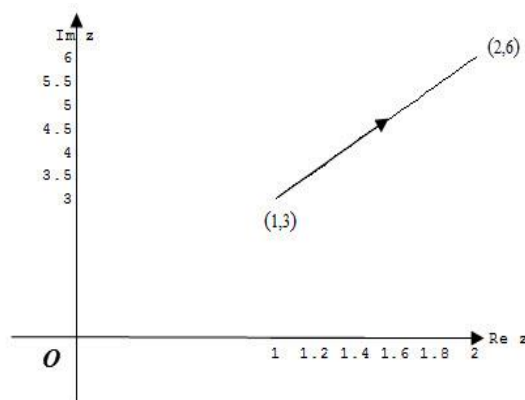
$$x(t) + i y(t) = t + i 3t$$

Comparing real and imaginary part, we get

$$x(t) = t, y(t) = 3t \quad (1 \leq t \leq 2).$$

$t$	$x$	$y$	$(x,y)$
1	1	3	(1,3)
2	2	6	(2,6)

So,  $z(t) = (1 + 3i)t$  ( $1 \leq t \leq 2$ ) represents the line segment from (1,3) to (2,6) in complex plane.



**Fig: 1**

**Example 2:** Find and sketch the path whose orientation is given by  $z(t) = 2e^{it}$  ( $0 \leq t < \pi$ ).

**Solution:**

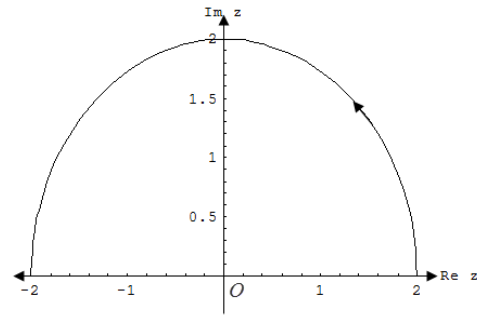
$$z(t) = 2e^{it} \quad (0 \leq t < \pi)$$

$$x(t) + i y(t) = 2 \cos(t) + i 2 \sin(t)$$

Comparing real and imaginary part,

we get  $x(t) = 2 \cos(t)$ ,  $y(t) = 2 \sin(t)$  ( $0 \leq t < \pi$ ).

So,  $z(t) = 2e^{it}$  ( $0 \leq t < \pi$ ) represents upper semicircle of radius 2 with center (0,0).



**Fig: 2**

**Example 3:** Sketch and represent the line segment from  $1 + i$  to  $4 - 2i$  parametrically.

**Solution:**

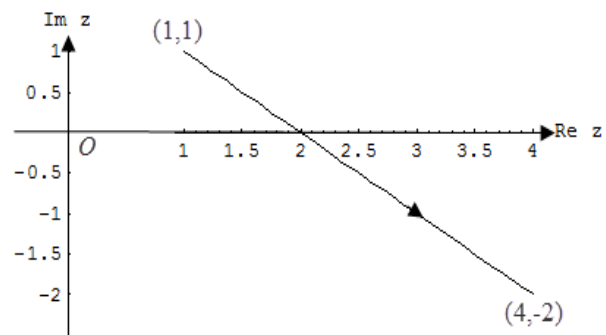
The equation of straight line passing through the points (1,1) to (4,-2) is,  $y - 1 = \left(\frac{-2-1}{4-1}\right)(x - 1)$

That is,  $y = -x + 2$

Let,  $x = t$  then  $y = -t + 2$  where  $t$  varies from  $t = 1$  to  $t = 4$ .

So, the parametric equation of line segment from  $1 + i$  to  $4 - 2i$  is,

$$x(t) = t, y(t) = -t + 2 \quad (1 \leq t \leq 4).$$



**Fig: 3**

**Example 4:** Sketch and represent unit circle (counterclockwise) parametrically.

**Solution:**

unit circle (counterclockwise)

That is,  $|z| = 1$  (counterclockwise)

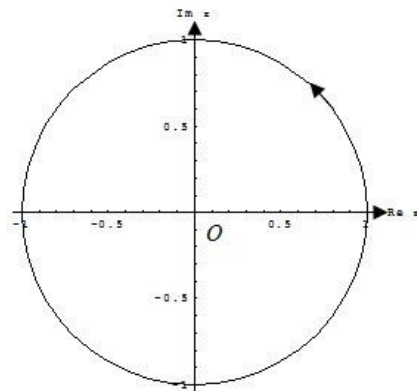
$$\text{Or, } |x + i y| = 1 \Rightarrow \sqrt{x^2 + y^2} = 1 \Rightarrow x^2 + y^2 = 1$$

.

Let,  $x = \cos t$  and  $y = \sin t$ ,

Then  $(\cos t)^2 + (\sin t)^2 = 1$  where

$t$  varies from  $t = 0$  to  $t = 2\pi$ .



**Fig: 4**

So, the parametric equation of unit circle

(counterclockwise) is,

$$x(t) = \cos t, y(t) = \sin t \quad (0 \leq t < 2\pi).$$

**Example 5:** Sketch the path  $C$  consisting of two line segments, one from  $z = 0$  to  $z = 2$  and other from  $z = 2$  to  $z = 3+i$ , hence evaluate  $\int_C f(z) dz$ , if  $f(z) = z^2$ .

**Solution:**

Given,  $C$  consists of two line segments, one from

$z = 0$  to  $z = 2$  and other from  $z = 2$  to  $z = 3+i$ .

**Along  $C_1$ :**

Equation of the line, which passes through  $(0,0)$  and  $(2,0)$ , is  $y = 0$

$$f(z) = z^2 = (x + iy)^2 = x^2 \quad [\text{using } y = 0]$$

We know,  $z = x + iy = x$ ,  $dz = dx$

and  $x$  varies from 0 to 2

$$\int_{C_1} f(z) dz = \int_0^2 x^2 dx = \frac{8}{3}.$$

**Along  $C_2$ :**

Equation of the line, which passes through  $(2,0)$  and  $(3,1)$  is,

$$y - 0 = \left( \frac{1-0}{3-2} \right) (x - 2) \Rightarrow y = x - 2.$$

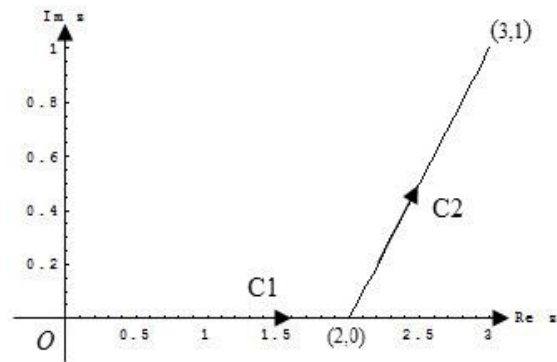
$$f(z) = z^2 = (x + iy)^2 = [(y + 2) + iy]^2 \quad [\text{using } x = y + 2]$$

We know,  $z = x + iy = y + 2 + iy$ ,  $dz = (1 + i)dy$  and  $y$  varies from 0 to 1.

$$\int_{C_2} f(z) dz = \int_0^1 [(y + 2) + iy]^2 (1 + i) dy = i \int_0^1 (4 + 4i - 2y^2 + 2iy^2 + 8iy) dy = \frac{10}{3} + \frac{26}{3}i$$

$$\text{Now, } \int_C f(z) dz = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz = 6 + \frac{26}{3}i.$$

**Example 6.** Sketch the path  $C$  from  $z = 0$  to  $z = 4 + 2i$  along the curve  $z = t^2 + it$  and hence evaluate  $\int_C f(z) dz$ , where  $f(z) = \bar{z}$ .



**Fig: 5**

**Solution:**

Given,  $z = 0$  to  $z = 4 + 2i$  and

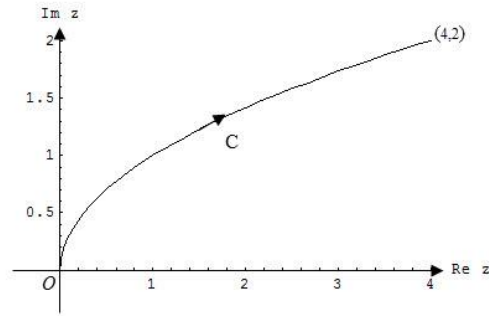
$$z = t^2 + it \Rightarrow x + iy = t^2 + it.$$

$$\therefore x = t^2 \text{ and } y = t$$

$$\text{Now, } f(z) = \bar{z} = x - iy = y^2 - iy$$

$$\text{and } z = x + iy = y^2 + iy$$

$$\Rightarrow dz = 2ydy + idy = (2y + i)dy$$

**Fig: 6**

Therefore,

$$\int_C f(z) dz = \int_0^2 (y^2 - iy)(2y + i) dy = \int_0^2 (2y^3 - iy^2 + y) dy = \left[ \frac{2y^4}{4} - i \frac{y^3}{3} + \frac{y^2}{2} \right]_0^2 = 10 - \frac{8}{3}i.$$

**Example 7:** Sketch the path  $C$  from  $z = -1 - i$  to  $z = 1 + i$  along the curve  $y = x^3$  and hence

evaluate  $\int_C f(z) dz$ , where  $f(z) = \begin{cases} y, & \text{when } y > 0 \\ 2, & \text{when } y < 0 \end{cases}$ .

**Solution:**

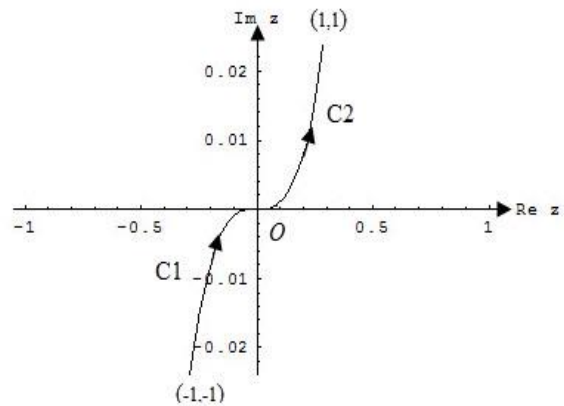
Given,  $C$  is the arc from  $z = -1 - i$  to  $z = 1 + i$

along the curve  $y = x^3$ .

$$f(z) = \begin{cases} y, & \text{when } y > 0 \\ 2, & \text{when } y < 0 \end{cases} = \begin{cases} x^3, & \text{when } x > 0 \\ 2, & \text{when } x < 0 \end{cases}$$

$$\text{and, } z = x + iy = x + ix^3, dz = (1 + 3x^2i)dx$$

$$\begin{aligned} \text{Now, } \int_C f(z) dz &= \int_{C1} f(z) dz + \int_{C2} f(z) dz \\ &= \int_{-1}^0 2 \cdot (1 + 3x^2i) dx + \int_0^1 x^3 (1 + 3x^2i) dx \\ &= \frac{9}{4} + \frac{5}{2}i. \end{aligned}$$

**Fig: 7**

**Example 8:** Sketch the path  $C$  from  $z = -1$  to  $z = 1$  along the upper half of the circle  $|z| = 1$  and

hence evaluate  $\int_C f(z) dz$ , where  $f(z) = \bar{z}$ .

**Solution:**

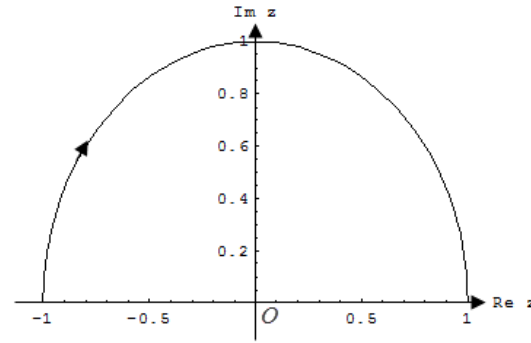
Given, C is the upper half of the circle  $|z|=1$

from  $z = -1$  to  $z = 1$ .

$|z|=1, z = 1 \cdot e^{i\theta}, dz = ie^{i\theta}d\theta$ , where

$\theta$  varies from  $\pi$  to  $0$ , and  $f(z) = \bar{z} = e^{-i\theta}$

Now,  $\int_C f(z) dz = \int_{\pi}^0 e^{-i\theta} ie^{i\theta} d\theta = -\pi i$ .



**Fig: 8**

**Matlab command to evaluate line integrals:**

1. Evaluate  $\int_C \operatorname{Re}(z) dz$ , where C is the shortest path from 0 to  $1+2i$  along  $z(t) = t + 2it, 0 \leq t \leq 1$ .

```
>> fun=@(z) real(z);
>> q=integral(fun,0,1+2i)
q = 0.5000 + 1.0000i
```

2. Evaluate  $\int_C \operatorname{Re}(z) dz$ , where C consists of the shortest path from  $z=0$  to  $z=1$  and then to  $z=1+2i$ .

```
>> fun=@(z) real(z);
>> q=integral(fun, 0,1+2i,'Waypoints',1)
q = 0.5000 + 2.0000i
```

3. Evaluate  $\int_C \bar{z} dz$ , where C is the line segment from  $z=2$  to  $z=2+3i$ .

```
>> fun=@(z) conj(z);
>> q=integral(fun,2,2+3i)
q = 4.5000 + 6.0000i
```

**Sample Exercise Set on Line Integral: 6****Sample Exercise**

1. Find and sketch the path and its orientation. Also classify whether the indicated points are interior, exterior or boundary of the following curves-(vi):

(i)  $z(t) = (1 + 3i)t \quad (1 \leq t \leq 4)$

(iii)  $z(t) = 3e^{it} \quad (0 \leq t \leq \pi)$

(ii)  $z(t) = (2 - i)t \quad (-2 \leq t \leq 2)$

(iv)  $z(t) = 3 \sin t + i 3 \cos t \quad (-\pi \leq t \leq \pi)$

(v)  $z(t) = 3 + i + 4e^{it} \quad (0 \leq t \leq 2\pi)$

(vi)  $z(t) = 6 \sin(t) + i 4 \cos(t) \quad (0 \leq t \leq 2\pi); (5,1)$

(vii)  $z(t) = 2 \sin(t) + i 3 \cos(t) + 3 + 2i, \quad (0 \leq t \leq 2\pi); (6,5)$

(viii)  $z(t) = 4 \cosh(t) + 3 i \sinh(t) .$

(ix)  $z(t) = 3 + 4i + (5 \cosh t + 2 i \sinh t).$

2. Sketch and represent them parametrically. Also classify whether the indicated points are interior, exterior or boundary of the following curves-(iii & iv):

(i) Line segment from  $-1 + 2i$  to  $4 - 2i$  ,

(ii) unit circle:  $|z| = 1$  (clockwise)

(iii)  $|z - 4i| = 3$  (counter clockwise) ; (1,6)

(iv)  $|z - 5 + i| = 4$  (counter clock wise) ; (1,2)

3. Sketch the path  $C$  from  $z = 0$  to  $z = 3i$  and hence evaluate  $\int_C z^2 dz$  .

4. Sketch the path  $C$  from  $z = 0$  to  $z = 3$  and hence evaluate  $\int_C \bar{z} dz$ .

5. Sketch the path  $C$  from  $z = 1$  to  $z = 4$  and hence evaluate  $\int_C (z + \bar{z}) \sin z dz$ .

6.  $\int_C \ln(z) dz$ ,  $C$  is the shortest path from  $i$  to  $2i$ .

7. Sketch the path  $C$ , which is the unit circle  $|z| = 2$  and hence evaluate  $\int_C (z + z^{-1}) dz$ .

- Sketch the corresponding paths and hence evaluate them (8-11):

8.  $\int_C (e^{2z} + \cos z) dz$ ,  $C$  is the shortest path from  $z = 2$  to  $z = 4$  .

9.  $\int_C (z \cdot \bar{z}) dz$ ,  $C$  is the path around the square with vertices  $0, 1, 1+i, i$  .

10.  $\int_C \left( \frac{1}{z-i} - \frac{2}{(z-i)^2} \right) dz$  ,  $C$  is the circle  $|z - i| = 3$  , clockwise.

11.  $\int_C (x + y - i x^2) dz$  ,  $C$  is the shortest path along the real axis from  $z = 0$  to  $z = 1$  and then along a line parallel to imaginary axis from  $z = 1$  to  $z = 1 + i$ .