Lecture Note-7

Integration using Cauchy's Residue Theorem (CRT)

Two main reasons account for the importance of integration in the complex plane. The practical reason is that complex integration can evaluate certain real integrals appearing in applications that are not accessible by real integral calculus. The theoretical reason is that some basic properties of analytic functions are difficult to prove by other methods. Complex integration also plays an important role in connections with special function, such as the gamma function, the error function, various polynomials and others, and the application of these functions in physics.

Cauchy's Integral Formula:

If a function $f(z)$ is analytic within and on a simple closed contour C and if z_0 is any point interior to C then,

$$\oint_C \frac{f(z)}{z-z_0}dz = 2\pi i\, f(z_0) \text{ - - - - - - - - - } \mathbf{(1)}$$

Special case : If z_0 is not an interior point of the contour C then $\oint_C \frac{f(z)}{z-z_0}dz = 0$.

Differentiating n-1 times w.r.to z_0

$$\oint_C \frac{f(z)}{(z-z_0)^n}dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0).$$

Definition of singular point (of an analytic function):

A point at which an analytic function $f(z)$ is not defined, i.e., at which $f'(z)$ fails to exist, called a singular point or pole or singularity of the function.

Example 7.1: If $f(z) = \frac{1}{(z+1)(z-3)}$, then $z=-1$, 3 are the singular points of $f(z)$.

Residue Finding Method:

If $f(z)$ is analytic inside and on a simple closed curve C except at pole or has singularity at $z = a$ of order 1, then

$$\text{Res}(a) = \lim_{z\to a}(z-a)\, f(z).$$

If $f(z)$ is analytic inside and on a simple closed curve C except at pole or has singularity at $z = a$ of order ***m***, then

$$\text{Res}(a) = \lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left\{(z-a)^m f(z)\right\}.$$

Cauchy Residue Theorem:

If $f(z)$ is analytic inside and on a simple closed curve C except at a finite number of n singular points $a_1, a_2, a_{3,} ..., a_n$ inside *C*, then

$$\oint_C f(z)\,dz = 2\pi i\left[\text{Res}\left(a_1\right)+\text{Res}\left(a_2\right)+...+\text{Res}\left(a_n\right)\right].$$

Example 7.2: Evaluate by CRT $\oint_C \frac{\sin \pi z}{(z-2)^2}\,dz$, where C is the circle $|z| = 3.$

Solution: For singular point, $(z-2)^2 = 0$

$$\Rightarrow z = 2$$

Singular point $z = 2$ is a pole of order 2. The point $z = 2$ lies inside the circle $|z| = 3$.

Residue at the point $z = 2$ is,

$$\text{Res}(z=2) = \lim_{z\to 2} \frac{1}{(2-1)!}\frac{d}{dz}\frac{\sin \pi z}{(z-2)^2}(z-2)^2$$

$$= \lim_{z\to 2} \frac{d}{dz}\sin \pi z$$

$$= \lim_{z\to 2} \pi\cos \pi z$$

$$= \pi.$$

So by CRT we know,

$$\oint_C \frac{\sin \pi z}{(z-2)^2}\,dz = 2\pi i\left(\text{Res(z=2)}\right) = 2\pi i(\pi) = 2\pi^2 i.$$

Example 7.3: Evaluate the contour integral $\oint_C \frac{dz}{z^3}$ by CRT, where C is the circle $|z+1| = 3.$

Solution: The poles or singularities of $\frac{1}{z^3}$ are as follows:

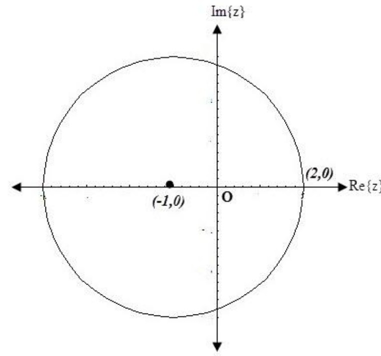
A pole of order 3 at $z = 0$. This pole lies inside the contour C.
Residue at the point $z = 0$ of order 3 is given by

$$\text{Re}\,s(z=0) = \lim_{z\to 0}\frac{1}{2!}\frac{d^2}{dz^2}\left\{z^3.\frac{1}{z^3}\right\}$$

$$= \lim_{z\to 0} \frac{1}{2!} \frac{d^2}{dz^2}\{1\}$$

$$= 0$$

So by CRT we know, $\oint_C \frac{dz}{z^3} = 2\pi i\left(\text{Res(z=0)}\right) = 2\pi i(0)\text{=0.}$



Sample Exercise Set on Cauchy residue theorem: 7.1

1. (i) Find all the singular points of the following functions , f(z) and show the points in the argand diagram, where f(z) = $\frac{1}{2z}$, $\frac{1}{z^2-4}$, $\frac{\sin z}{z}$, $\cot z$, $\frac{1}{z^6+1}$.

   (ii) Find all the singular points of the following functions , *f(z)* and show them in the argand diagram,then find corresponding residues : f(z) = $\frac{z^2+1}{z^2+z}$, $\frac{1}{z^3+i}$, $\frac{z^2+2}{z-4}$, $\frac{1}{z^6+1}$.

2. State Cauchy's integral formula and Cauchy's residue theorem (CRT). For each of the followings sketch the indicated path C and hence evaluate applying Cauchy's residue theorem (CRT), (if possible):

   (a) $\oint_C \frac{dz}{z-3i}$, C is the circle $|z| = 4$.

   (b) $\oint_C \frac{e^{-z}}{(z-1)^2}$, C consists of $|z| = 4$.

   (c) $\oint_C \frac{dz}{(z-6)^{10}}$, where C is the circle $|z| = 4$.

3. Evaluate the followings applying Cauchy's residue theorem (CRT) (if possible):

   (a) Evaluate the integrals along the contour as given in the figures:

   (i) $\oint_C \frac{2z}{(2z-i)^3}\, dz$ (Fig. 1), (ii) $\oint_C \frac{dz}{z^2-1}$ (Fig. 2), (iii) $\oint_C \frac{2z-1}{z^2-z} dz$ (Fig. 3).

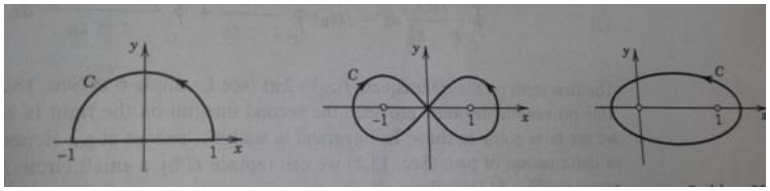


Fig.1 Fig.2 Fig.3

4. For the followings sketch the indicated path C and hence evaluate applying Cauchy's residue theorem (CRT) if possible:

(a) $\oint_C \dfrac{dz}{z^2+4}$, C is the contour as (i) $|z+2\,i| = 1$, (ii)$|z-2\,i| = 1$.

(b) $\oint_C \dfrac{\cos\left(\pi z^3\right)}{(z-1)(z-2)}\,dz$, where C is the circle $|z-3| = 4$,

(c) $\oint_C \dfrac{\sin 3z}{(z-\pi)^2}\,dz$; where C is the circle $|z| = 4$.

Application of Residue Theorem

Evaluation of Real Definite Integrals by Contour Integrals:

A large number of real definite integrals, whose evaluation by usual methods become sometimes very tedious, can be easily evaluated by using **Cauchy's Residue theorem**. For finding the integrals we take a suitable complex function $f(z)$ and closed curve C, then find the poles or singularity of the function $f(z)$ and calculate residues at those poles only which lie within the curve C. Then using Cauchy's residue theorem we have

$$\oint_C f(z)\,dz = 2\pi i\left[\text{sum of the residues of } f(z) \text{ at the poles within } C\right]$$

We call the curve, a contour and the process of integration along a contour is called contour integration.

(Improper Integral) Infinite real integrals of the form $\int_{-\infty}^{+\infty}\frac{f_1(x)}{f_2(x)}dx$ **or,** $\int_{0}^{+\infty}\frac{f_1(x)}{f_2(x)}dx$ where $f_1(x)$ and $f_2(x)$ are polynomials in x. Such integrals can be reduced to contour integrals, if

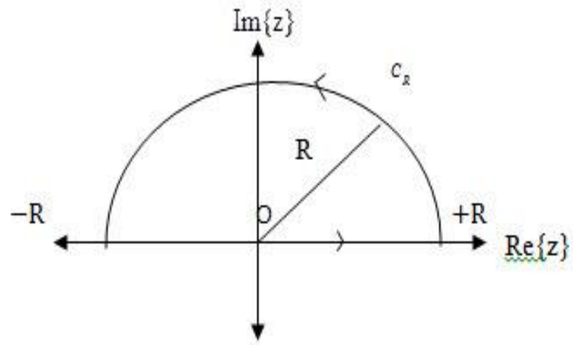
(i) $f_2(x)$ has no real roots.

(ii) The degree of $f_2(x)$ is greater than that of $f_1(x)$ by at least two.

Procedure to solve:

To evaluate such integrals we consider the contour integrals

$\oint_C \frac{f_1(z)}{f_2(z)}dz$ where C is the closed contour, consisting the real axis from $-R$ to R and the upper half C_R of the circle $|z| = R$ i.e.,



$$\oint_C \frac{f_1(z)}{f_2(z)}dz = \int_{-R}^{R}\frac{f_1(x)}{f_2(x)}dx + \int_{C_R}\frac{f_1(z)}{f_2(z)}dz \text{ - - - - } (1)$$

Now using CRT we get,

$$\oint_C \frac{f_1(z)}{f_2(z)}dz = 2\pi i \times (\text{sum of the residue at the poles within } C)$$

Then (1) becomes,

$$\int_{-R}^{R}\frac{f_1(x)}{f_2(x)}dx + \int_{C_R}\frac{f_1(z)}{f_2(z)}dz = 2\pi i \times (\text{sum of the residue at the poles within } C)$$

$$\Rightarrow \int_{-R}^{R}\frac{f_1(x)}{f_2(x)}dx = -\int_{C_R}\frac{f_1(z)}{f_2(z)}dz + 2\pi i \times (\text{sum of the residue at the poles within } C) \text{ - -- -- --- --- } (2)$$

$$\therefore \lim_{R\to\infty}\int_{-R}^{R}\frac{f_1(x)}{f_2(x)}dx = -\lim_{R\to\infty}\int_{C_R}\frac{f_1(z)}{f_2(z)}dz + 2\pi i \times (\text{sum of the residue at the poles within } C)$$

Now, on the semi circular path C_R, $|z| = R \Rightarrow z = Re^{i\theta}, (0 \le \theta \le \pi) \therefore dz = iRe^{i\theta}\, d\theta$. Then applying Jordan's Lemma,

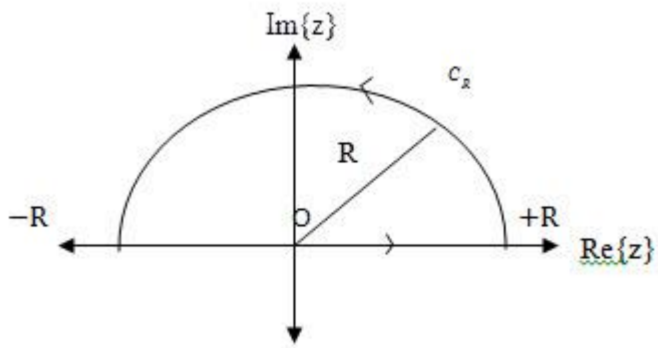
$$\lim_{R\to\infty}\int_{C_R}\frac{f_1(z)}{f_2(z)}dz=\lim_{R\to\infty}\int_0^{\pi}\frac{f_1(\mathrm{Re}^{i\theta})}{f_2(\mathrm{Re}^{i\theta})}Rie^{i\theta}d\theta=0$$

Then (2) reduces to

$$\therefore\int_{-\infty}^{\infty}\frac{f_1(x)}{f_2(x)}dx=2\pi i\times(\text{sum of the residues at the poles within } C)$$

Example 7.5: Evaluate $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{\left(x^2+4\right)^2}$ by using contour integration.

Solution:



We consider $\displaystyle\oint_C\frac{dz}{\left(z^2+4\right)^2}$ where C is the closed contour consisting of the semi circle C_R of radius R together with the part of the real axis –R to +R. i.e.,

$$\oint_C\frac{dz}{\left(z^2+4\right)^2}=\int_{-R}^{R}\frac{dx}{\left(x^2+4\right)^2}+\int_{C_R}\frac{dz}{\left(z^2+4\right)^2}\ \ldots\ldots(1)$$

Now the first integral has singularities or pole at $\left(z^2+4\right)^2=0$ i.e. $z=\pm 2i$ of order 2. But the only pole $z=+2i$ is inside the contour C. So,

$$\mathrm{Re}\,s(at\ z=+2i)=\lim_{z\to 2i}\frac{1}{1!}\frac{d}{dz}\left\{(z-2i)^2\cdot\frac{1}{(z-2i)^2(z+2i)^2}\right\}$$

$$=\lim_{z\to 2i}\frac{d}{dz}\left\{\frac{1}{(z+2i)^2}\right\}=\lim_{z\to 2i}\left\{\frac{-2}{(z+2i)^3}\right\}=\frac{-2}{(4i)^3}=\frac{1}{32i}$$

So by CRT,

$$\oint_C \frac{dz}{\left(z^2+4\right)^2} = 2\pi i \times \frac{1}{32i} = \frac{\pi}{16}$$

So equation (1) becomes

$$\int_{-R}^{R} \frac{dx}{\left(x^2+4\right)^2} + \int_{C_R} \frac{dz}{\left(z^2+4\right)^2} = \frac{\pi}{16}$$

By Jordan Lemma letting $R \to \infty$ and noting that the second integral in left hand side would become zero.

$$\lim_{z\to \mathrm{R}} z\, f(z) = \lim_{z\to \mathrm{R}} z \frac{1}{\left(z^2+4\right)^2} = \lim_{z\to \mathrm{R}} z \frac{1}{z^4\left(1+\frac{4}{z^2}\right)^2} = \lim_{z\to \mathrm{R}} \frac{1}{z^3\left(1+\frac{4}{z^2}\right)^2} = 0$$

$$\therefore \lim_{\mathrm{R}\to\infty} \int_{C_R} \frac{dz}{\left(z^2+4\right)^2} = 0.$$

Hence,

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{\left(x^2+4\right)^2} + \lim_{R\to\infty} \int_{C_R} \frac{dz}{\left(z^2+4\right)^2} = \frac{\pi}{16}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{\left(x^2+4\right)^2} + 0 = \frac{\pi}{16}$$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{\left(x^2+4\right)^2} = \frac{\pi}{16}.$$

Example 7.6: Evaluate $\int_0^{\infty} \frac{dx}{\left(x^4+16\right)}$ by using contour integration.

Solution: We consider $\oint_C \frac{dz}{\left(z^4+16\right)}$ where C is the closed contour consisting of the semi circle C_R of radius R together with the part of the real axis $-R$ to $+R$. i.e.,

$$\oint_C \frac{dz}{\left(z^4+16\right)} = \int_{-R}^{R} \frac{dx}{\left(x^4+16\right)} + \int_{C_R} \frac{dz}{\left(z^4+16\right)} \ldots\ldots(1)$$

The figure in the previous example should be considered here.

Now the first integral has singularities or poles at

$$z^4+16=0$$

$$\Rightarrow z^4 = -16 = 16e^{i\pi}$$

$$\therefore z_k = (16)^{\frac{1}{4}} e^{i\left(\frac{\pi+2k\pi}{4}\right)}, k = 0,1,2,3$$

When $k = 0,\ z_0 = 2e^{i\frac{\pi}{4}}$

$$k = 1,\ z_1 = 2e^{i\frac{3\pi}{4}}$$

$$k = 2,\ z_2 = 2e^{i\frac{5\pi}{4}}$$

$$k = 3,\ z_3 = 2e^{i\frac{7\pi}{4}}$$

i.e. there are four poles, but only two poles at z_0 and z_1 lie within the contour C. So,

$$\mathrm{B}_k = \mathrm{Res}\ (at\ z = z_k) = \lim_{z\to z_k}\left\{(z - z_k).\frac{1}{(z^4+16)}\right\} \quad , \ k = 0,1$$

$$= \lim_{z\to z_k}\left\{\frac{1}{4z_k^3}\right\}$$

$$\mathrm{B}_0 = \frac{1}{4z_0^3} = \frac{1}{4}z_0^{-3} = \frac{1}{4}e^{-i\frac{3\pi}{4}}$$

$$\mathrm{B}_1 = \frac{1}{4z_1^3} = \frac{1}{4}z_1^{-3} = \frac{1}{4}e^{-i\frac{9\pi}{4}}$$

So by CRT,

$$\oint_C \frac{dz}{(z^4+16)} = 2\pi i \times [\mathrm{B}_0 + \mathrm{B}_1] = \frac{\sqrt{2}\pi}{16}$$

So equation (1) becomes

$$\int_{-R}^{R} \frac{dx}{(x^4+16)} + \int_{C_R} \frac{dz}{(z^4+16)} = \frac{\sqrt{2}\pi}{16}.$$

By Jordan Lemma letting $R \to \infty$ and noting that the second integral in left hand side would become zero. Hence,

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{\left(x^4+16\right)}+\lim_{R\to\infty}\int_{c_R}\frac{dz}{\left(z^4+16\right)}=\frac{\sqrt{2}\pi}{16}$$

$$\Rightarrow\int_{-\infty}^{\infty}\frac{dx}{\left(x^4+16\right)}+0=\frac{\sqrt{2}\pi}{16}$$

$$\Rightarrow\int_{-\infty}^{\infty}\frac{dx}{\left(x^4+16\right)}=\frac{\sqrt{2}\pi}{16}$$

$$\int_{0}^{\infty}\frac{dx}{\left(x^4+16\right)}=\frac{\sqrt{2}\pi}{32}.$$

Matlab command for improper integral:

1. Evaluate $\int_0^\infty \frac{dx}{x^2+1}$,
`>> fun=@(x) 1./(x.^2+1);`
`>> q=integral(fun,0,inf)`
`q = 1.5708`
	2. Evaluate $\int_{-\infty}^\infty \frac{dx}{(x^2-2x+2)^2}$,
	`>> f=@(x) 1./(x.^2-2.*x+2).^2;`
	`>> q=integral(f,-inf,inf)`
	`q = 1.5708`

Sample Exercise Set on Improper Integral: 7.2

Integration of the form $\int_{-\infty}^{\infty} \frac{f_1(x)}{f_2(x)} dx$ (**improper integral**)

1. Evaluate the following improper integral using Cauchy's residue theorem (CRT):

(i) $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$, (ii) $\int_{0}^{\infty} \frac{dx}{x^2+1}$,

(iii) $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} dx$, (iv) $\int_{-\infty}^{\infty} \frac{dx}{\left(x^2-2x+2\right)^2}$,

(v) $\int_{0}^{\infty} \frac{x^2}{x^6+1} dx$.

Laurent series generalize Taylor series. Indeed, whereas a Taylor series has positive integer powers (and a constant term) and converges in a disc, a Laurent series is a series of positive and negative integer powers of $(z-z_0)$ and converges in an annulus (a circular ring) with center z_0 . Hence by a Laurent series we can represent a given function $f(z)$ that is analytic in an annulus and may have singularities outside the ring as well as in the "hole" of the annulus.

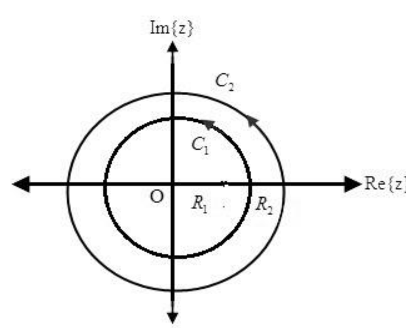
Laurent's Theorem:

Let $f(z)$ be analytic in a domain containing two concentric circles c_1 and c_2 with center z_0 , radii R_1 and R_2 , $(R_1 < R_2)$ and the annulus between them. Then $f(z)$ can be represented by the Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

$$= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

$$+ \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \frac{b_3}{(z-z_0)^3} + \cdots$$



The coefficients of Laurent series are given by the integrals

$$a_n = \frac{1}{2\pi i} \oint_c \frac{f(z^*)}{(z^*-z_0)^{n+1}}\, dz^* , \qquad b_n = \frac{1}{2\pi i} \oint_c (z^*-z_0)^{n-1}\, f(z^*)\, dz^*$$

The variable of integration is denoted by z^* , since z is used in Laurent series.

The existing negative power of $(z - z_0)$ is known as **principal part**. If there is finite number of terms in the principal part of $f(z)$ in the Laurent series expansion then the coefficient of $\left(\frac{1}{z-z_0}\right)$ is called the residue of $f(z)$ at pole $z = z_0$.

Laurent series expansion

Example: 7.7 Obtain Laurent series expansion of $f(z) = \dfrac{1}{(1+z^2)(z+2)}$ when (i) $1 < |z| < 2$, (ii) $|z| > 2$. .

Solution: (i) Since $1 < |z| < 2$

$$\Rightarrow \frac{1}{z} < 1 \text{ and } \frac{|z|}{2} < 1$$

$$\therefore \frac{1}{|z^2|} < 1 \text{ and } \frac{|z|}{2} < 1$$

Let $\frac{1}{(1+z^2)(z+2)} \equiv \frac{Az+B}{1+z^2} + \frac{C}{z+2}$

$\therefore 1 \equiv (\mathrm{A}z + \mathrm{B})(z + 2) + \mathrm{C}(1 + z^2)$

At, $z = -2,\ 5C = 1 \ \therefore \mathrm{C} = \frac{1}{5}$

Equating coefficients of z^2 ; $A + C = 0 \ \therefore A = -C = -\frac{1}{5}$

Equating coefficients of z ; $2A + B = 0 \ \therefore \mathrm{B} = -2\mathrm{A} = \frac{2}{5}$

$$\therefore \frac{1}{(1+z^2)(z+2)} = \frac{-\frac{1}{5}z+\frac{2}{5}}{1+z^2} + \frac{\frac{1}{5}}{z+2}$$

$$= \frac{2}{5}\frac{1}{1+z^2} - \frac{1}{5}\frac{z}{1+z^2} + \frac{1}{5}\frac{1}{z+2}$$

$$= \frac{2}{5}\frac{1}{z^2(1+\frac{1}{z^2})} - \frac{1}{5}\frac{z}{z^2(1+\frac{1}{z^2})} + \frac{1}{5}\frac{1}{2(1+\frac{z}{2})}$$

$$= \frac{2}{5z^2}(1 + \frac{1}{z^2})^{-1} - \frac{1}{5\mathrm{z}}(1 + \frac{1}{z^2})^{-1} + \frac{1}{10}(1 + \frac{z}{2})^{-1}$$

$$= \frac{2}{5z^2}\left(1-\frac{1}{z^2}+\frac{1}{z^4}-\frac{1}{z^6}+\cdots\right) - \frac{1}{5z}\left(1-\frac{1}{z^2}+\frac{1}{z^4}-\frac{1}{z^6}+\cdots\right)$$

$$+\frac{1}{10}(1-\frac{z}{2}+\frac{z^2}{4}-\cdots)$$

which is the required Laurent series.

(ii) For $|z| > 2$ we have $\frac{|z|}{2} > 1 \Rightarrow \frac{2}{|z|} < 1$

Also $\frac{1}{|z^2|} < 1$

$$\therefore \frac{1}{(1+z^2)(z+2)} = \frac{2}{5}\frac{1}{1+z^2} - \frac{1}{5}\frac{z}{1+z^2} + \frac{1}{5}\frac{1}{z+2}$$

$$= \frac{2}{5}\frac{1}{z^2(1+\frac{1}{z^2})} - \frac{1}{5}\frac{z}{z^2(1+\frac{1}{z^2})} + \frac{1}{5}\frac{1}{z(1+\frac{2}{z})}$$

$$= \frac{2}{5z^2}(1+\frac{1}{z^2})^{-1} - \frac{1}{5z}(1+\frac{1}{z^2})^{-1} + \frac{1}{5z}(1+\frac{2}{z})^{-1}$$

$$= \frac{2}{5z^2}\left(1-\frac{1}{z^2}+\frac{1}{z^4}-\frac{1}{z^6}+\cdots\right) - \frac{1}{5z}\left(1-\frac{1}{z^2}+\frac{1}{z^4}-\frac{1}{z^6}+\ \dots\dots\right)$$

$$+\frac{1}{5z}\left(1-\frac{2}{z}+\frac{4}{z^2}-\cdots\right).$$

which is the required Laurent series.

Sample Exercise Set on Laurent Series: 7.3

1. State Laurent series. Expand $f(z) = \dfrac{3z}{(z-1)(2-z)}$ in a Laurent series valid for

   (a) $|z| < 1$, (b) $1 < |z| < 2$, (c) $|z| > 2$, (d) $|z-1| > 2$ and (e) $0 < |z-1| < 1$.

2. Expand $f(z) = \dfrac{1}{z(z-2)}$ in a Laurent series valid for

   (a) $0 < |z| < 2$ and (b) $|z| > 2$.

3. Expand $f(z) = \dfrac{5z}{(z^2+1)(z+2)}$ in a Laurent series valid for

   (a) $1 < |z| < 2$ and (b) $|z| > 2$

4. Find the function, $f(z)$ and the region of convergence for the following series:

   a. $1 + z + z^2 + z^3 + \cdots$.

   b. $1 - z + z^2 - z^3 + \cdots$.

c. $1+2z+3z^2+4z^3+\cdots.$

d. $1-2z+3z^2-4z^3+\cdots.$

5. Given functions (i) $f(z)=\dfrac{z}{(z-1)(3-z)}$ [Figure: (a) and (b)] and (ii) $f(z)=\dfrac{z}{(z-1)(2-z)}$.

[Figure: (a) and (c)] Determine the region of convergence and the series for the following figures:

(a)	(b)	(c)