# Lecture-3

## Application of Laplace transformation

The Laplace transformation is useful in solving differential equations. There are four steps to follow, such as

Differential equation

↓

Apply Laplace Transformation

↓

Use the Initial Values

↓

Solve the algebraic equation for $Y(s)$

↓

Apply Inverse Laplace Transformation.

**Important formulae**

$$\mathcal{L}\{\dot{f}(t)\} = \mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0).$$

$\mathcal{L}\{\ddot{f}(t)\} = \mathcal{L}\left\{\frac{d^2 f(t)}{dt^2}\right\} = s^2F(s) - sf(0) - \dot{f}(0)$ where $f(0)$, and $\dot{f}(0)$are the initial values of $f$ and$\dot{f}$.

$$\mathcal{L}\{\dddot{f}(t)\} = \mathcal{L}\left\{\frac{d^3 f(t)}{dt^3}\right\} = s^3F(s) - s^2f(0) - s\dot{f}(0) - \ddot{f}(0).$$

The general case for the Laplace transform of an $n^{\text{th}}$ derivative is

$$\mathcal{L}\{f^n(t)\} = \mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$

**Solving Ordinary Differential equations with constant coefficients:**

The Laplace transform is useful in solving linear ordinary differential equations with constant coefficients. Having obtained expressions for the Laplace transforms of derivatives, we are now in a position to use Laplace transform and also inverse Laplace transform methods to solve ordinary differential equations with constant coefficients. To illustrate this, consider the general second-order differential equation

$$\frac{d^2y}{dt^2} + \alpha\frac{dy}{dt} + \beta y = f(t) \quad \text{or} \quad \ddot{y}(t) + \alpha\dot{y}(t) + \beta y(t) = f(t) \qquad (1)$$

Where, $\propto$ and $\beta$ are constants, subject to initial conditions

$$y(0) = A, \dot{y}(0) = B \qquad (2)$$

where A and B are given constants. On taking the Laplace transform of both sides and using condition (2), we obtain the algebraic equation for determination of $\mathcal{L}\{y(t)\} = Y(s)$. The required solution is then obtained by finding the inverse Laplace transform of $Y(s)$. The method is easily extended for the higher order differential equations.

**Example:**
Solve the differential equation or the initial value problem
$\ddot{y}(t) + y(t) = e^t, \qquad y(0) = 1, \dot{y}(0) = -2.$
**Solution:**
Given,

$\ddot{y}(t) + y(t) = e^t,$

$\Rightarrow \mathcal{L}\{\ddot{y}(t)\} + \mathcal{L}\{y(t)\} = \mathcal{L}\{e^t\}$ [applying Laplace transformation]

$\Rightarrow s^2Y(s) - sy(0) - \dot{y}(0) + Y(s) = \frac{1}{s-1}$ [let, $\mathcal{L}\{y(t)\} = Y(s)$]

$\Rightarrow s^2Y(s) - s(1) - (-2) + Y(s) = \frac{1}{s-1}$ [using the initial values]

$\Rightarrow Y(s) = \frac{s^2-3s+3}{(s^2+1)\,(s-1)}$ [solving the equation]

$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{A}{s-1} + \frac{Bs+C}{s^2+1}\right\}$ [applying inverse Laplace transformation and using partial fraction]

$$\frac{s^2 - 3s + 3}{(s^2 + 1)\,(s - 1)} \equiv \frac{A}{s - 1} + \frac{Bs + C}{s^2 + 1}$$
$$\Rightarrow s^2 - 3s + 3 \equiv A(s^2 + 1) + (Bs + C)(s - 1)$$
$$\Rightarrow s^2 - 3s + 3 \equiv (A + B)s^2 + (C - B)s + A - C$$

Equating coefficients

$$A + B = 1$$
$$C - B = -3$$
$$A - C = 3$$

Solving we get, $A = \frac{1}{2}$, $B = \frac{1}{2}$, $C = -\frac{5}{2}$

$$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{\frac{1}{2}}{s - 1} + \frac{\frac{s}{2}}{s^2 + 1} - \frac{\frac{5}{2}}{s^2 + 1}\right\}$$

$\Rightarrow y(t) = \frac{e^t}{2} + \frac{1}{2}\cos t - \frac{5}{2}\sin t.$

Therefore the solution of the differential equation is $y(t) = \frac{e^t}{2} + \frac{1}{2}\cos t - \frac{5}{2}\sin t.$

**Example:** A resistance $R$ in series with inductance $L$ is connected with e.m.f $E(t) = t$. The current $i(t)$ is given by

$$L\frac{di}{dt}+Ri=t; \qquad i(0)=0$$

Use Laplace transform to find the current $i(t)$.

**Solution:**

Given,

$$L\frac{di}{dt}+Ri=t$$

$\Rightarrow L\mathcal{L}\left\{\frac{di}{dt}\right\}+R\mathcal{L}\{i\}=\mathcal{L}\{t\}$ [applying Laplace transformation]

$\Rightarrow LsI(s)-L.i(0)+RI(s)=\frac{1}{s^2}$ [let, $\mathcal{L}\{i(t)\}=I(s)$]

$\Rightarrow LsI(s)-L(0)+RI(s)=\frac{1}{s^2}$ [using the initial values]

$\Rightarrow I(s)=\frac{1}{(Ls+\mathrm{R})s^2}$ [Solving the equation]

$\Rightarrow \mathcal{L}^{-1}\{I(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{(Ls+\mathrm{R})s^2}\right\}$ [applying inverse Laplace transformation]

$\Rightarrow i(t)=\mathcal{L}^{-1}\left\{\frac{A}{s}+\frac{B}{s^2}+\frac{C}{(Ls+\mathrm{R})}\right\}$[ using partial fraction]

$$\frac{1}{(Ls+\mathrm{R})s^2}\equiv\frac{A}{s}+\frac{B}{s^2}+\frac{C}{(Ls+\mathrm{R})}$$

$$\Rightarrow 1\equiv As(Ls+R)+B(Ls+R)+Cs^2$$

$$\Rightarrow 1\equiv(LA+C)s^2+(AR+BL)s+BR$$

Equating coefficients

$$LA+C=0$$

$$AR+BL=0$$

$$BR=1$$

Solving we get, $A=-\frac{L}{R^2}, B=\frac{1}{R}, C=\frac{L^2}{R^2}$

Hence,

$$i(t)=\mathcal{L}^{-1}\left\{-\frac{L}{R^2}\frac{1}{s}+\frac{1}{R}\frac{1}{s^2}+\frac{L^2}{R^2}\frac{1}{(Ls+\mathrm{R})}\right\}$$

$$\Rightarrow i(t)=\mathcal{L}^{-1}\left\{-\frac{L}{R^2}\frac{1}{s}+\frac{1}{R}\frac{1}{s^2}+\frac{L^2}{R^2L}\frac{1}{\left(s+\frac{R}{L}\right)}\right\}$$

$$\Rightarrow i(t)=-\frac{L}{R^2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+\frac{1}{R}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}+\frac{L}{R^2}\mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{R}{L}\right)}\right\}$$

$$\Rightarrow i(t)=-\frac{L}{R^2}+\frac{t}{R}+\frac{L}{R^2}e^{-\frac{R}{L}t}.$$

**Example:**

An inductor of 2 henrys, a resistor of 16 ohms and a capacitor of .02 farads are connected in series with an e.m.f. of E volts. At $t=0$ the charge on the capacitor and current in the circuit are zero. Find the charge and current at any time $t>0$ if E= $300$ (volts).

Let $q(t)$ and $i(t)$ be the instantaneous charge and current respectively at time t. By Kirchhoff's law's, we have

$$2\frac{di}{dt} + 16i + \frac{q}{0.02} = E$$

$$\Rightarrow 2\frac{d^2q}{dt^2} + 16\frac{dq}{dt} + 50q = E \quad \text{[since } i = \frac{dq}{dt}\text{]}$$

$$\Rightarrow 2\frac{d^2q}{dt^2} + 16\frac{dq}{dt} + 50q = E \ldots\ldots (1)$$

With the initial conditions

$$q(0) = 0, i(0) = \dot{q}(0) = 0.$$

**Solution:**

If $E = 300$, then equation (1) becomes

$$\frac{d^2q}{dt^2} + 8\frac{dq}{dt} + 25q = 150$$

$\Rightarrow \mathcal{L}\left\{\frac{d^2q}{dt^2}\right\} + 8\mathcal{L}\left\{\frac{dq}{dt}\right\} + 25\mathcal{L}\{q\} = \mathcal{L}\{150\}$ [applying Laplace transform]

$\Rightarrow \{s^2Q(s) - s\,q(0) - \dot{q}(0)\} + 8\{sQ(s) - q(0)\} + 25Q(s) = \frac{150}{s}$ [let, $\mathcal{L}\{q(t)\} = Q(s)$]

$\Rightarrow \{s^2Q(s) - s\,.0 - 0\} + 8\{sQ(s) - 0\} + 25Q(s) = \frac{150}{s}$ [using the initial values]

$$\Rightarrow s^2Q(s) + 8sQ(s) + 25Q(s) = \frac{150}{s}$$

$$\Rightarrow (s^2 + 8s + 25)Q = \frac{150}{s}$$

$\Rightarrow Q(s) = \frac{150}{s(s^2+8s+25)}$ [ solving the equation]

$$\Rightarrow \mathcal{L}^{-1}\{Q(s)\} = \mathcal{L}^{-1}\left\{\frac{150}{s(s^2 + 8s + 25)}\right\}$$

$\Rightarrow q(t) = \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{Bs+C}{(s^2+8s+25)}\right\}$ [ using partial fraction]

$$\frac{150}{s(s^2 + 8s + 25)} \equiv \frac{A}{s} + \frac{Bs + C}{(s^2 + 8s + 25)}$$

$$\Rightarrow 150 \equiv A(s^2 + 8s + 25) + (Bs + C)s$$

$$\Rightarrow 150 \equiv (A + B)s^2 + (8A + C)s + 25A$$

Equating coefficients

$$A + B = 0$$

$$8A + C = 0$$

$$25A = 150$$

Solving we get, $A = 6, B = -6, C = -48$

$$q(t) = \mathcal{L}^{-1}\left\{\frac{6}{s} - \frac{6s+48}{(s^2+8s+25)}\right\}$$

$$\Rightarrow q(t) = \mathcal{L}^{-1}\left\{\frac{6}{s} - \frac{6(s+4)+24}{(s+4)^2+9}\right\}$$

$$\Rightarrow q(t) = \mathcal{L}^{-1}\left\{\frac{6}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{6(s+4)}{(s+4)^2+9}\right\} - \mathcal{L}^{-1}\left\{\frac{24}{(s+4)^2+9}\right\}$$

$q(t) = 6 - 6e^{-4t}Cos\, t - 8\, e^{-4t}\sin 3t$ [applying inverse Laplace transformation]

$i(t) = \frac{dq}{dt} = 50\, e^{-4t}\sin 3t.$

**Example:**

The current $i(t)$ in an electrical circuit is given by the DE, $\frac{d^2i}{dt^2} + 2\frac{di}{dt} = \begin{cases} 0, & 0 < t < 10 \\ 1, & 10 < t < 20 \\ 0, & t > 20 \end{cases}$

$i(0) = \frac{di}{dt}(0) = 0.$ Determine current $i(t)$.

**Solution:** Using unit step function the DE becomes,

$$\frac{d^2i}{dt^2} + 2\frac{di}{dt} = u(t-10) - u(t-20) \ldots\ldots..(1).$$

$\Rightarrow \mathcal{L}\left\{\frac{d^2i}{dt^2}\right\} + 2\mathcal{L}\left\{\frac{di}{dt}\right\} = \mathcal{L}\{u(t-10)\} - \mathcal{L}\{u(t-20)\}$ [applying Laplace transform]

$$\Rightarrow \left\{s^2I(s) - si(0) - \frac{di}{dt}(0)\right\} + 2\{sI(s) - i(0)\} = \frac{e^{-10s}}{s} - \frac{e^{-20s}}{s}$$

$[\because \mathcal{L}\{f(t).u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}$

$$\Rightarrow I(s)\big(s(s+2)\big) = \frac{e^{-10s}}{s} - \frac{e^{-20s}}{s}$$

$$\Rightarrow I(s) = \frac{1}{s(s+2)}\left(\frac{e^{-10s}}{s} - \frac{e^{-20s}}{s}\right)$$

Applying inverse Laplace transform, we get

$$\mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s+2)}\left(\frac{e^{-10s}}{s} - \frac{e^{-20s}}{s}\right)\right\} \ldots\ldots\ldots\ldots(2).$$

We know $\mathcal{L}^{-1}\{e^{-as}G(s)\} = u(t-a).g(t-a)$ and $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+2)}\right\} = \frac{t}{2} + \frac{e^{-2t}}{4} - \frac{1}{4}.$

$$\therefore i(t) = \left[\frac{(t-10)}{2} + \frac{e^{-2(t-10)}}{4} - \frac{1}{4}\right]u(t-10) - \left[\frac{(t-20)}{2} + \frac{e^{-2(t-20)}}{4} - \frac{1}{4}\right]u(t-20).$$

$$= \begin{cases} 0, & 0 < t < 10 \\ \dfrac{(t-10)}{2} + \dfrac{e^{-2(t-10)}}{4} - \dfrac{1}{4}, & 10 < t < 20\,. \\ 5 + \dfrac{e^{-2(t-10)}}{4} - \dfrac{e^{-2(t-20)}}{4}, & t > 20 \end{cases}$$

**Problem set 3.1**

**Apply Laplace transform to solve the following ordinary differential equations and hence justify your answer, where $\dot{y} \equiv \frac{dy(t)}{dt}$ and $\ddot{y} \equiv \frac{d^2y(t)}{dt^2}$ : (1-12)**

1. $\dot{y}(t) = 3;\quad y(0) = 2.$ **Ans:** $y(t) = 3t + 2.$
2. $\dot{y}(t) = 4t;\quad y(0) = 1.$ **Ans:** $y(t) = 2\,t^2 + 1.$
3. $\dot{y}(t) = 2t - 1;\quad y(0) = 3.$ **Ans:** $y(t) = t\,(t-1) + 3.$
4. $\dot{y}(t) = t^2;\quad y(0) = 4.$ **Ans:** $y(t) = \frac{t^3}{3} + 4.$
5. $\dot{y}(t) = e^{2t};\quad y(0) = 2.$ **Ans:** $y(t) = \frac{e^{2t}}{2} + \frac{3}{2}.$
6. $\dot{y}(t) + y(t) = 2\,;\quad y(0) = 0.$ **Ans:** $y(t) = 2 - 2\,e^{-t}.$
7. $\ddot{y}(t) = 5\,;\quad y(0) = 1, \dot{y}(0) = 2.$ Ans: $y(t) = \frac{t\,(5t+4)}{2} + 1.$
8. $\ddot{y}(t) - 2\,\dot{y}(t) = \cos t\,;\quad y(0) = 0, \dot{y}(0) = 1.$

   **Ans:** $f(t) = \frac{7}{10}\,e^{2t} - \frac{2}{5}\left(\sin t + \frac{1}{2}\cos t\right) - \frac{1}{2}.$
9. $\ddot{y}(t) + 3\,\dot{y}(t) - y(t) = e^t;\quad y(0) = \dot{y}(0) = 0.$

   **Ans:**

   $f(t) = \frac{1}{3}\,e^t - \frac{1}{3}\,e^{\frac{-3t}{2}}\left[\cosh\left(\frac{\sqrt{13}}{2}\,t\right) + \frac{5}{\sqrt{13}}\sinh\left(\frac{\sqrt{13}}{2}\,t\right)\right].$
10. $\ddot{y}(t) - 7\dot{y}(t) + 12y(t) = 0, y(0) = 2, \dot{y}(0) = 1.$

    **Ans:** $y(t) = -5e^{4t} + 7e^{3t}.$
11. $\ddot{y}(t) + y(t) = \begin{cases} t, & 0 < t < 1 \\ 0, & t > 1 \end{cases}, y(0) = 0, \dot{y}(0) = 0.$

    **Ans:** $y(t) = t - \sin t$ if $0 < t < 1$ and $\cos(t-1) + \sin(t-1) - \sin t$ if $t > 1.$

**Shifted data problems:**

This is a short name for initial value problem with initial conditions referring to some later instant $t = t_0$ instead of $t = 0$. In this case, the conditions $y(0)$ and $y'(0)$ occurring in the Laplace transform approach cannot be used immediately.

12. $y'' + y = 2t, \quad y\left(\frac{\pi}{4}\right) = \frac{\pi}{2}, \quad y'\left(\frac{\pi}{4}\right) = 2 - \sqrt{2}$.

**Solution:** set $t = \bar{t} + \frac{\pi}{4}$ so that $t = \frac{\pi}{4}$ gives $\bar{t} = 0$ and then Laplace transform becomes applicable throughout.

Now, the shifted problem is

$\bar{y}'' + \bar{y} = 2\left(\bar{t} + \frac{\pi}{4}\right), \ \bar{y}(0) = \frac{\pi}{2}$ and $\bar{y}'(0) = 2 - \sqrt{2}$.

Using Laplace transform on both sides we obtain

$$s^2\,\bar{Y}(s) - s\bar{y}(0) - \bar{y}'(0) = \frac{2}{s^2} + \frac{\pi}{2}\frac{1}{s}$$

$$\Rightarrow \bar{Y}(s) = \frac{2}{s^2\,(s^2+1)} + \frac{\pi}{2}\,\frac{1}{s\,(s^2+1)} + \frac{\pi}{2}\,\frac{s}{s^2+1} + \left(2 - \sqrt{2}\right)\frac{1}{s^2+1}$$

Applying inverse Laplace on both sides,

$$\Rightarrow \bar{y}(\bar{t}) = 2\,(\bar{t} - \sin\bar{t}) + \frac{\pi}{2}\,(1 - \cos\bar{t}) + \frac{\pi}{2}\cos\bar{t} + \left(2 - \sqrt{2}\right)\sin\bar{t}$$

Substituting $\bar{t} = t - \frac{\pi}{4}$ we obtain the solution

$$y(t) = 2t - \sin t + \cos t.$$

**Solving Simultaneous Ordinary Differential Equations by Laplace Transform**

**Example:**

$$\begin{cases} \frac{dx(t)}{dt} = 2x(t) - 3y(t) \\ \frac{dy(t)}{dt} = y(t) - 2x(t) \end{cases} \text{ subject to } x(0) = 8, y(0) = 3.$$

**Solution:**

Taking the Laplace transforms of both equations

$$\Rightarrow \mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = 2\mathcal{L}\{x(t)\} - 3\mathcal{L}\{y(t)\}$$

$$\mathcal{L}\left\{\frac{dy(t)}{dt}\right\} = \mathcal{L}\{y(t)\} - 2\mathcal{L}\{x(t)\}$$

$$\Rightarrow sX(s) - x(0) = 2X(s) - 3Y(s)$$

$$sY(s) - y(0) = Y(s) - 2X(s)$$

$$\Rightarrow (s-2)X(s) + 3Y(s) = 8$$

$2X(s) + (s-1)Y(s) = 3$ [using initial condition and rearranging]

Now solving this two equations simultaneously using **Cramer's rule** and partial fraction we get,

$$X(s) = \frac{\begin{vmatrix} 8 & 3 \\ 3 & s-1 \end{vmatrix}}{\begin{vmatrix} s-2 & 3 \\ 2 & s-1 \end{vmatrix}} = \frac{8s-17}{s^2-3s-4} = \frac{8s-17}{(s+1)(s-4)} = \frac{5}{s+1} + \frac{3}{s-4}$$

$$Y(s) = \frac{\begin{vmatrix} s-2 & 8 \\ 2 & 3 \end{vmatrix}}{\begin{vmatrix} s-2 & 3 \\ 2 & s-1 \end{vmatrix}} = \frac{3s-22}{s^2-3s-4} = \frac{3s-22}{(s+1)(s-4)} = \frac{5}{s+1} - \frac{2}{s-4}$$

Now taking inverse Laplace transform we get,

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{5}{s+1} + \frac{3}{s-4}\right\}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{5}{s+1} - \frac{2}{s-4}\right\}$$

$$\Rightarrow y(t) = 5e^{-t} + 3e^{4t}$$

$$y(t) = 5e^{-t} - 2e^{4t}$$

## Problem set 3.2

**Solve the following system of differential equations where $x(t) \equiv x$, $y(t) \equiv y$, $\dot{y} \equiv \frac{dy(t)}{dt}$ and $\dot{x} \equiv \frac{dx(t)}{dt}$, using Laplace transformation. Also justify your answers. (13-16)**

13. $\dot{x} = y$

    $\dot{y} = 16x$; $x(0) = 0, y(0) = 4.$

    Answer: $x(t) = \sinh 4t$, $y(t) = 4\cosh 4t$

14. $\dot{x} = -4y$

    $\dot{y} = x$; $x(0) = 2, y(0) = 0.$

    Answer: $x(t) = 2\cos 2t$, $y(t) = \sin 2t$

15. $\dot{x} = 2x + y$

    $\dot{y} = 4x + 2y$; $x(0) = 1, y(0) = 6.$

    Answer: $x(t) = e^{2t}(\cosh 2t + 3\sinh 2t), y(t) = e^{2t}(6\cosh 2t + 2\sinh 2t).$

16. $\dot{x} = 3x + y$

    $\dot{y} = 4x + 3y$; $x(0) = 3, y(0) = 2.$

    Answer: $x(t) = e^{3t}(3\cosh 2t + \sinh 2t)$, $y(t) = e^{3t}(2\cosh 2t + 6\sinh 2t)$

**Problem set 3.3 (Application)**

**General talk:**

The Laplace transform is widely used in the following science and engineering field** .

1. Analysis of electronic circuits.

2. System modeling.

3. Digital signal processing.

4. Nuclear physics.

5. Process control.

The following examples highlights the importance of laplace transform in different engineering fields.

**Problem:**

The following example based on the concepts from nuclear physics. Consider the following first

order linear differential equation

$$\frac{dN}{dt} = -\lambda N \dots \dots \dots (1)$$

This equation is the fundamental relationship describing radioactive decay, where $N = N(t)$

represents the number of undecayed atoms remaining in a sample of a radioactive isotope

at time $t$ and $\lambda$ is the decay constant.

We can use laplace transform to solve this equation $(1)$.

Rearranging the above equation $(1)$ we get,

$$\frac{dN}{dt} + \lambda N = 0 \dots \dots \dots (2)$$

Taking laplace transform on both sides of $(2)$

$$s\, L(N) - N(0) + \lambda\, L(N) = 0$$
$$\Rightarrow s\, \overline{N} - N_0 + \lambda\, \overline{N} = 0$$

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** Applications of Laplace Transform in Engineering Fields.

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Where $L(N) = \bar{N}$ and $N(0) = N_0$

$$\Rightarrow \bar{N} = \frac{N_0}{s + \lambda}$$

Now taking inverse laplace transform on both sides we get,

$$N(t) = N_0\, e^{-\lambda t}.$$

Which is indeed the correct form for radioactive decay.

**EXAMPLE 3 Four-Terminal *RLC*-Network**

Find the output voltage response in Fig. 135 if $= 20\ \Omega$, $L = 1$ H, $C = 10^{-4}$ F, the input is $\delta(t)$ (a unit impulse at time $t = 0$), and current and charge are zero at time $t = 0$.

*Solution.* To understand what is going on, note that the network is an $LC$-circuit to which two wires at $A$ and $B$ are attached for recording the voltage $v(t)$ on the capacitor. Recalling from Sec. 2.9 that current $i(t)$ and charge $q(t)$ are related by $i = q' = dq/dt$, we obtain the model

$$Li' + i + \frac{q}{C} = Lq'' + q\ ' + \frac{q}{C} = q'' + 20q' + 10{,}000q = \delta(t).$$

From (1) and (2) in Sec. 6.2 and (5) in this section we obtain the subsidiary equation for $Q(s) = \mathcal{L}(q)$

$$(s^2 + 20s + 10{,}000)Q = 1. \qquad \text{Solution} \qquad Q = \frac{1}{(s+10)^2 + 9900}.$$

By the first shifting theorem in Sec. 6.1 we obtain from $Q$ damped oscillations for $q$ and $v$; rounding $9900 \approx 99.50^2$, we get (Fig. 135)

$$q = \mathcal{L}^{-1}(Q) = \frac{1}{99.50}e^{-10t}\sin 99.50t \qquad \text{and} \qquad v = \frac{q}{C} = 100.5e^{-10t}\sin 99.50t. \qquad \blacksquare$$

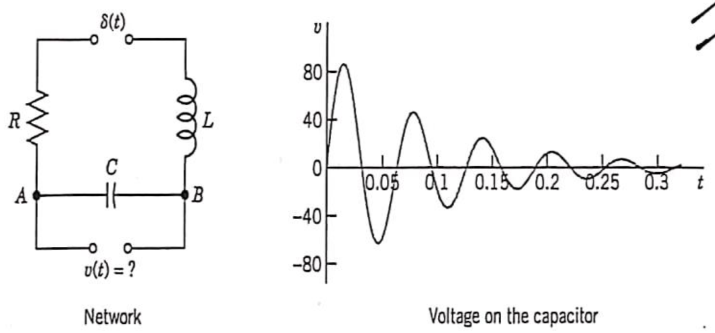


Fig. 135. Network and output voltage in Example 3

**EXAMPLE 4** **Unrepeated Complex Factors. Damped Forced Vibrations**

Solve the initial value problem for a damped mass–spring system acted upon by a sinusoidal force for some time interval (Fig. 136),

$$y'' + 2y' + 2y = r(t), \quad r(t) = 10 \sin 2t \text{ if } 0 < t < \pi \text{ and } 0 \text{ if } t > \pi; \qquad y(0) = 1, \quad y'(0) = -5.$$

*Solution.* From Table 6.1, (1), (2) in Sec. 6.2, and the second shifting theorem in Sec. 6.3, we obtain the subsidiary equation

$$(s^2 \quad - s + 5) + 2(s \quad - 1) + 2 \quad = 10\frac{2}{s^2+4}(1 - e^{-\pi s}).$$

We collect the -terms, $(s^2 + 2s + 2)$ , take $-s + 5 - 2 = -s + 3$ to the right, and solve,

$$= \frac{20}{(s^2+4)(s^2+2s+2)} - \frac{20e^{-\pi s}}{(s^2+4)(s^2+2s+2)} + \frac{s-3}{s^2+2s+2}. \tag{6}$$

For the last fraction we get from Table 6.1 and the first shifting theorem

$$\mathcal{L}^{-1}\left\{\frac{s+1-4}{(s+1)^2+1}\right\} = e^{-t}(\cos t - 4 \sin t). \tag{7}$$

In the first fraction in (6) we have unrepeated complex roots, hence a partial fraction representation

$$\frac{20}{(s^2+4)(s^2+2s+2)} = \frac{As+B}{s^2+4} + \frac{Ms+N}{s^2+2s+2}.$$

Multiplication by the common denominator gives

$$20 = (As + B)(s^2 + 2s + 2) + (Ms + N)(s^2 + 4).$$

We determine $A, B, M, N$. Equating the coefficients of each power of $s$ on both sides gives the four equations

$$\text{(a)} \quad [s^3]: \quad 0 = A + M \qquad\qquad \text{(b)} \quad [s^2]: \quad 0 = 2A + B + N$$

$$\text{(c)} \quad [s]: \quad 0 = 2A + 2B + 4M \qquad \text{(d)} \quad [s^0]: \quad 20 = 2B + 4N.$$

We can solve this, for instance, obtaining $M = -A$ from (a), then $A = B$ from (c), then $N = -3A$ from (b), and finally $A = -2$ from (d). Hence $A = -2$, $B = -2$, $M = 2$, $N = 6$, and the first fraction in (6) has the representation

$$\frac{-2s-2}{s^2+4} + \frac{2(s+1)+6-2}{(s+1)^2+1}. \quad \text{Inverse transform:} \quad -2\cos 2t - \sin 2t + e^{-t}(2\cos t + 4 \sin t). \tag{8}$$

The sum of this inverse and (7) is the solution of the problem for $0 < t < \pi$, namely (the sines cancel),

$$y(t) = 3e^{-t}\cos t - 2\cos 2t - \sin 2t \qquad \text{if } 0 < t < \pi. \tag{9}$$

In the second fraction in (6), taken with the minus sign, we have the factor $e^{-\pi s}$, so that from (8) and the second shifting theorem (Sec. 6.3) we get the inverse transform of this fraction for $t > 0$ in the form

$$+2\cos(2t - 2\pi) + \sin(2t - 2\pi) - e^{-(t-\pi)}[2\cos(t-\pi) + 4\sin(t-\pi)]$$
$$= 2\cos 2t + \sin 2t + e^{-(t-\pi)}(2\cos t + 4\sin t).$$

The sum of this and (9) is the solution for $t > \pi$,

$$y(t) = e^{-t}[(3 + 2e^{\pi})\cos t + 4e^{\pi}\sin t] \qquad \text{if } t > \pi. \tag{10}$$

Figure 136 shows (9) (for $0 < t < \pi$) and (10) (for $t > \pi$), a beginning vibration, which goes to zero rapidly because of the damping and the absence of a driving force after $t = \pi$. ■

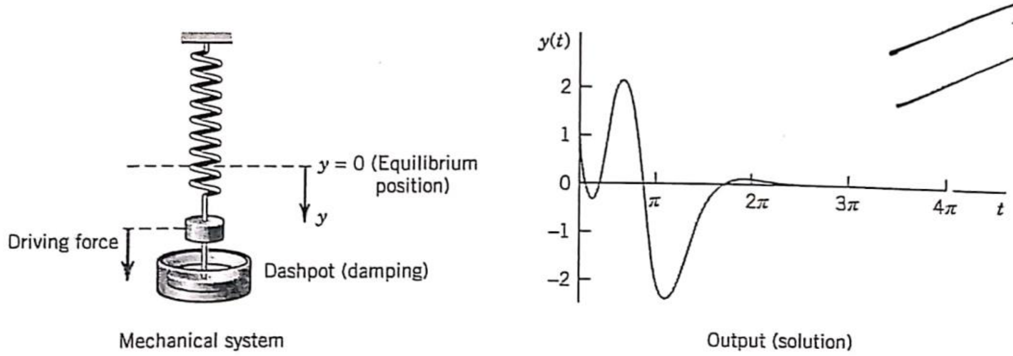


Fig. 136. Example 4

**EXAMPLE 5** **Response of a Damped Vibrating System to a Single Square Wave**

Using convolution, determine the response of the damped mass–spring system modeled by

$$y'' + 3y' + 2y = r(t), \qquad r(t) = 1 \text{ if } 1 < t < 2 \text{ and } 0 \text{ otherwise}, \qquad y(0) = y'(0) = 0.$$

This system with an **input** (a driving force) ***that acts for some time only*** (Fig. 143) has been solved by partial fraction reduction in Sec. 6.4 (Example 1).

***Solution y onvolution.*** The transfer function and its inverse are

$$Q(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}, \qquad \text{hence} \qquad q(t) = e^{-t} - e^{-2t}.$$

Hence the convolution integral (3) is (except for the limits of integration)

$$y(t) = \int q(t-\tau)\cdot 1\, d\tau = \int [e^{-(t-\tau)} - e^{-2(t-\tau)}]\, d\tau = e^{-(t-\tau)} - \frac{1}{2}e^{-2(t-\tau)}.$$

Now comes an important point in handling convolution. $r(\tau) = 1$ if $1 < \tau < 2$ only. Hence if $t < 1$, the integral is zero. If $1 < t < 2$, we have to integrate from $\tau = 1$ (not 0) to $t$. This gives (with the first two terms from the upper limit)

$$y(t) = e^{-0} - \tfrac{1}{2}e^{-0} - (e^{-(t-1)} - \tfrac{1}{2}e^{-2(t-1)}) = \tfrac{1}{2} - e^{-(t-1)} + \tfrac{1}{2}e^{-2(t-1)}.$$

If $t > 2$, we have to integrate from $\tau = 1$ to 2 (not to $t$). This gives

$$y(t) = e^{-(t-2)} - \tfrac{1}{2}e^{-2(t-2)} - \left(e^{-(t-1)} - \tfrac{1}{2}e^{-2(t-1)}\right).$$

Figure 143 shows the input (the square wave) and the interesting output, which is zero from 0 to 1, then increases, reaches a maximum (near 2.6) after the input has become zero (why ), and finally decreases to zero in a monotone fashion. ■

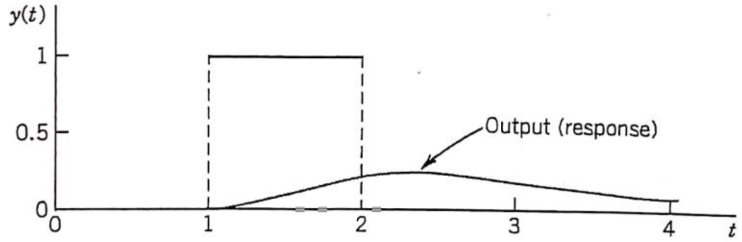


**Fig. 143.** Square wave and response in Example 5

**XAMPLE 6 A Volterra Integral Equation of the Second Kind**

Solve the Volterra integral equation of the second kind[3]

$$y(t) - \int_0^t y(\tau) \sin(t - \tau)\, d\tau = t.$$

***Solution.*** From (1) we see that the given equation can be written as a convolution, $y - y \quad \sin t = t$. Writing $Y = \mathcal{L}(y)$ and applying the convolution theorem, we obtain

$$Y(s) - Y(s)\frac{1}{s^2 + 1} = Y(s)\frac{s^2}{s^2 + 1} = \frac{1}{s^2}.$$

The solution is

$$Y(s) = \frac{s^2 + 1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4} \qquad \text{and gives the answer} \qquad y(t) = t + \frac{t^3}{6}.$$