Lecture-2

Inverse Laplace transforms:

If the Laplace transform of a function $f(t)$ is $F(s)$ i.e., if $\mathcal{L}\{f(t)\} = F(s)$ then $f(t)$ is called the inverse Laplace transforms of $F(s)$ and we write

$$\mathcal{L}^{-1}\{F(s)\} = f(t).$$

Important formulae of Inverse Laplace transformation:

1	$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$	2	$\mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}, n = 0,1,2,\ldots$
3	$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$	4	$\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$
5	$\mathcal{L}^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at$	6	$\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$
7	$\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$		

Some workout examples on Inverse Laplace transformation:

Example: 1	$\mathcal{L}^{-1}\left\{\frac{s^2+1}{s^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}+\frac{1}{s^3}\right\} = 1 + \frac{t^2}{2!} = 1 + \frac{t^2}{2}.$
Example: 2	$\mathcal{L}^{-1}\left\{\frac{1}{2s-5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2(s-\frac{5}{2})}\right\} = \frac{1}{2}e^{\frac{5}{2}t}$
Example: 3	$\mathcal{L}^{-1}\left\{\frac{1}{s^2-16}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{4}\frac{4}{s^2-4^2}\right\} = \frac{1}{4}\sinh 4t$
Example: 4	$\mathcal{L}^{-1}\left\{\frac{2s}{s^2-9}\right\} = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2-3^2}\right\} = 2\cosh 3t$
Example: 5	$\mathcal{L}^{-1}\left\{\frac{4}{s-2}-\frac{s}{s^2-16}+\frac{4}{s^2-4}\right\}$ $= 4\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2-4^2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{2}{s^2-2^2}\right\}$ $= 4e^{2t} - \cosh 4t + 2\sinh 2t.$
Example: 6	$\mathcal{L}^{-1}\left\{\frac{5}{s}-\frac{3s}{s^2+16}+\frac{2}{s^2+4}\right\}$ $= 5\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$ $= 5 - 3\cos 4t + \sin 2t.$

First translation property:

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ **then** $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$.

Example: 01

$$\mathcal{L}^{-1}\left\{\frac{10}{(s+3)^4}\right\}$$
$$= 10\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^4}\right\}$$
$$= 10e^{-3t}\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}$$
$$= 10e^{-3t}\frac{t^3}{3!} = \frac{10}{6}e^{-3t}t^3.$$

Example: 02

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+1}\right\}$$
$$= e^{2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$
$$= e^{2t}\sin t.$$

Example: 03

$$\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\}$$
$$= e^{t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = e^{t}\cos 2t.$$

Example: 04

$$\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2-25}\right\}$$
$$= e^{-2t}\mathcal{L}^{-1}\left\{\frac{s}{s^2-25}\right\}$$
$$= e^{-2t}\cosh 5t.$$

Example: 05

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2-4}\right\}$$
$$= e^{-3t}\mathcal{L}^{-1}\left\{\frac{1}{s^2-2^2}\right\}$$
$$= e^{-3t}\mathcal{L}^{-1}\left\{\frac{1}{2}\frac{2}{s^2-2^2}\right\}$$
$$= \frac{1}{2}e^{-3t}\sinh 2t.$$

Example: 06

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4s+13}\right\} = \mathcal{L}^{-1}\left\{\frac{s+2-2}{(s+2)^2+3^2}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+3^2}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2+3^2}\right\}$$
$$= e^{-2t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} - 2e^{-2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+3^2}\right\}$$
$$= e^{-2t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} - \frac{2}{3}e^{-2t}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$
$$= e^{-2t}\cos 3t - \frac{2}{3}e^{-2t}\sin 3t.$$

Example: 07

$$\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+4s+13}\right\} = \mathcal{L}^{-1}\left\{\frac{2(s+2)-3}{(s+2)^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{2(s+2)-3}{(s+2)^2+9}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{2(s+2)}{(s+2)^2+3^2} - \frac{3}{(s+2)^2+3^2}\right\} = 2e^{-2t}\cos 3t - 3e^{-2t}\sin 3t.$$

Example: 08

$$\mathcal{L}^{-1}\left\{\frac{s}{(s+3)^5} - \frac{2s+7}{s^2+4s+29}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{s+3-3}{(s+3)^5} - \frac{2(s+2)+3}{(s+2)^2+5^2}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^5} - \frac{3}{(s+3)^5} - \frac{2(s+2)}{(s+2)^2+5^2} - \frac{3}{(s+2)^2+5^2}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^4} - \frac{3}{(s+3)^5} - 2\frac{(s+2)}{(s+2)^2+5^2} - \frac{3}{5}\frac{5}{(s+2)^2+5^2}\right\}$$
$$= e^{-3t}\frac{t^3}{3!} - 3e^{-3t}\frac{t^4}{4!} - 2e^{-2t}\cos 5t - \frac{3}{5}e^{-2t}\sin 5t.$$

Inverse Laplace transformation using partial fraction:

Example: 01 $\mathcal{L}^{-1}\left\{\frac{1}{s^2-5s+6}\right\}$ $=\mathcal{L}^{-1}\left\{\frac{1}{(s-3)(s-2)}\right\}$ $=\mathcal{L}^{-1}\left\{\frac{1}{(s-3)}-\frac{1}{(s-2)}\right\}$ $=e^{3t}-e^{2t}$.	Let, $\frac{1}{(s-3)(s-2)} \equiv \frac{A}{s-3}+\frac{B}{s-2}$ $\Rightarrow 1 = A(s-2)+B(s-3)$ $If\ s=2, B=-1$ and $if\ s=3, A=1$
Example: 02 $\mathcal{L}^{-1}\left\{\frac{3s+1}{(s+1)(s^2+1)}\right\}$ $=\mathcal{L}^{-1}\left\{\frac{3s+1}{(s+1)(s^2+1)}\right\}$ $=\mathcal{L}^{-1}\left\{\frac{-1}{s+1}+\frac{s+2}{s^2+1}\right\}$ $=\mathcal{L}^{-1}\left\{\frac{-1}{s+1}+\frac{s}{s^2+1}+\frac{2}{s^2+1}\right\}$ $=-e^{-t}+\cos t+2\sin t$.	Let, $\frac{3s+1}{(s+1)(s^2+1)} \equiv \frac{A}{s+1}+\frac{Bs+C}{s^2+1}$ $\Rightarrow 3s+1 = A(s^2+1)+(Bs+C)(s+1)$ Comparing both sides, we get $A+B=0, B+C=3\ and\ A+C=1$ By solving, we get $A=-1, B=1$ and $C=2$
Example: 03 $\mathcal{L}^{-1}\left\{\frac{4s+5}{(s-1)^2(s+2)}\right\}$ $=\mathcal{L}^{-1}\left\{\frac{1/3}{s-1}+\frac{3}{(s-1)^2}-\frac{1/3}{s+2}\right\}$ $=\frac{1}{3}e^t+3t\,e^t-\frac{1}{3}e^{-2t}$.	Let, $\frac{4s+5}{(s-1)^2(s+2)} \equiv \frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{C}{s+2}$ $\Rightarrow 4s+5 = A(s-1)(s+2)+B(s+2)$ $+C(s-1)^2$ Comparing both sides, we get $A+C=0, A+B-2C=4$ $and -2A+2B+C=5$ By solving, we get $A=\frac{1}{3}, B=3$ and $C=-\frac{1}{3}$.
Example: 04 $\mathcal{L}^{-1}\left\{\frac{3s^2+13s+26}{s(s^2+4s+13)}\right\}$ $=\mathcal{L}^{-1}\left\{\frac{2}{s}+\frac{s+5}{(s+2)^2+3^2}\right\}$ $=\mathcal{L}^{-1}\left\{\frac{2}{s}+\frac{s+2}{(s+2)^2+3^2}+\frac{3}{(s+2)^2+3^2}\right\}$ $=2+e^{-2t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\}$ $+e^{-2t}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$ $=2+e^{-2t}\cos 3t+e^{-2t}\sin 3t$	Let, $\frac{3s^2+13s+26}{s(s^2+4s+13)} \equiv \frac{A}{s}+\frac{Bs+C}{s^2+4s+13}$ $\Rightarrow 3s^2+13s+26$ $=A(s^2+4s+13)+(Bs+C)s$ Comparing both sides, we get $A+B=3, 4A+C=13\ and\ 13A=26$ By solving, we get $A=2, B=1$ and $C=5$.

Problem set: 2.1

Find the inverse Laplace transform of the following functions and also sketch $f(t)$:

(1-19) [if free hand sketching is getting complex then use MATLAB]

Using direct formula

1. $F(s) = \frac{1}{s-5}$, **Ans:** $f(t) = e^{5t}$.

2. $F(s) = \frac{1}{s^5}$, **Ans:** $f(t) = \frac{t^4}{24}$.

3. $F(s) = \frac{s^3-5s^2+6}{s^4}$, **Ans:** $f(t) = t^3 - 5t + 1$.

4. $F(s) = \frac{2+4s}{s^2+25}$, **Ans:** $f(t) = 4\cos 5t + \frac{2}{5}\sin 5t$.

5. $F(s) = \frac{3}{s^2+4}$, **Ans:** $f(t) = \frac{3}{2}\sin 2t$.

6. $F(s) = \frac{3}{s^2-4}$, **Ans:** $f(t) = \frac{3}{4}e^{2t} - \frac{3}{4}e^{-2t}$. (Using $\sinh x = \frac{e^x - e^{-x}}{2}$.)

First translation property

7. $F(s) = \frac{1}{(s-3)^4}$, **Ans:** $f(t) = e^{3t}\frac{t^3}{6}$.

8. $F(s) = \frac{3}{(s+2)^2+9}$, **Ans:** $f(t) = e^{-2t}\sin 3t$.

9. $F(s) = \frac{s-2}{(s-2)^2-16}$, **Ans:** $f(t) = \frac{e^{-2t}}{2} + \frac{e^{6t}}{2}$. (Using $\cosh x = \frac{e^x + e^{-x}}{2}$.)

10. $F(s) = \frac{s}{s^2+4s-9}$, **Ans:** $f(t) = e^{-2t}\left(\cosh(\sqrt{13}\, t) - \frac{2\sqrt{13}\sinh(\sqrt{13}\, t)}{13}\right)$.

11. $F(s) = \frac{5s-7}{s^2-6s+25}$, **Ans:** $f(t) = 5e^{3t}\left(\cos 4t + \frac{2}{5}\sin 4t\right)$.

12. $F(s) = \frac{s}{s^2-6s+10}$, **Ans:** $f(t) = e^{3t}(\cos t + 3\sin t)$.

Using partial fraction

Type unrepeated factors –

13. $F(s) = \frac{s+1}{s(s-2)(s+3)}$, **Ans:** $f(t) = \frac{3e^{2t}}{10} - \frac{2e^{-3t}}{15} - \frac{1}{6}$.

14. $F(s) = \frac{6}{(s+2)(s-4)}$, **Ans:** $f(t) = e^{4t} - e^{-2t}$.

15. $F(s) = \frac{6s-17}{s^2-5s+6}$, **Ans:** $f(t) = 5e^{2t} + e^{3t}$.

Type repeated factors –

16. $F(s) = \frac{s}{(s+1)^2}$, **Ans:** $f(t) = e^{-t} - t\,e^{-t}$.

17. $F(s) = \frac{7s^2+14s-9}{(s-1)^2(s-2)}$, **Ans:** $f(t) = -40e^t - 12te^t + 47e^{2t}$.

Type complex or irrational factors --

18. $F(s) = \frac{20}{(s^2+4s+1)(s+1)}$, **Ans:** $f(t) = 10e^{-2t}\left(\cosh(\sqrt{3}t) + \frac{\sqrt{3}\sinh(\sqrt{3}t)}{3}\right) - 10e^{-t}$.

19. $F(s) = \frac{s}{(s^2+4)(s-1)}$, **Ans:** $f(t) = \frac{2}{5}\sin 2t - \frac{1}{5}\cos 2t + \frac{1}{5}e^t$.

Inverse Laplace transformation associated with unit step function:

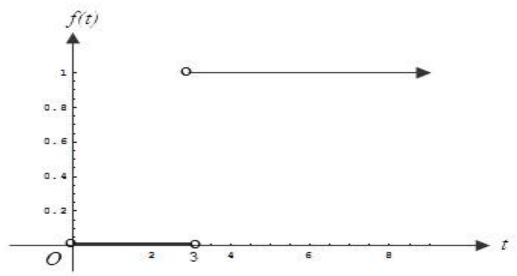
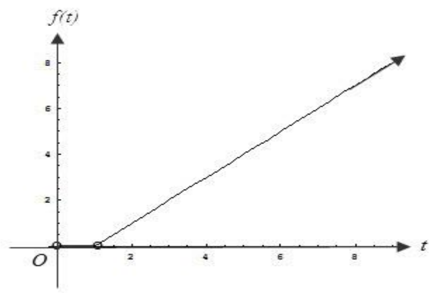
Laplace transform of **unit step function**is $\mathcal{L}\{u(t-a)\} = \mathcal{L}\{u_a(t)\} = \frac{e^{-as}}{s}$

$$\mathcal{L}\{f(t).u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}$$

So,$\mathcal{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = u(t-a) = u_a(t)$.

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$, then $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u_a(t) = f(t-a)u(t-a)$.

Some workout examples are given bellow:

Example 1: Find and sketch $f(t)$, where $f(t) = \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\}$. **Solution:** we know that $\mathcal{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = u(t-a) = u_a(t)$ **So,** $f(t) = \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\}$ $= u(t-3) = u_3(t) = \begin{cases} 0, & t < 3 \\ 1, & t > 3 \end{cases}$.	
Example 2: Find and sketch $f(t)$, where $f(t) = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2}\right\}$. **Solution:** Let,$F(s) = \frac{1}{s^2}$and $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t = f(t)$. We know that, $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u_a(t)$ So, $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2}\right\} = f(t-1)u_1(t) = (t-1)u_1(t)$ $= \begin{cases} 0, & t < 1 \\ t-1, & t > 1 \end{cases}$.	
Example 3: Find and sketch $f(t)$, where $f(t) = \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}$. **Solution:** Let,$F(s) = \frac{1}{s^2+1}$and $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t = f(t)$.	

We know that,

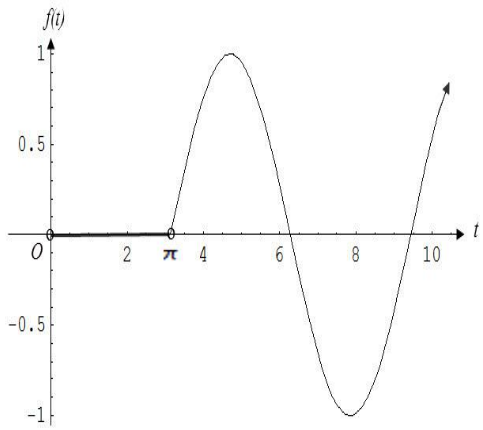
$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u_a(t)$

So, $\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\} = f(t-\pi)u_\pi(t)$

$= \sin(t-\pi)\, u_\pi(t)$

$= \begin{cases} 0, & t<\pi \\ -\sin(\pi - t), & t>\pi \end{cases}$

$= \begin{cases} 0, & t<\pi \\ -\sin\, t, & t>\pi \end{cases}.$



Example 4:

Find and sketch $f(t)$, where $f(t) = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2+\pi^2} + \frac{e^{-2s}}{s^2+\pi^2} + \frac{e^{-4s}}{s^2}\right\}$.

Solution:

$\mathcal{L}^{-1}\left\{\frac{1}{s^2+\pi^2}\right\} = \frac{1}{\pi}\sin(\pi t)$ and $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t.$

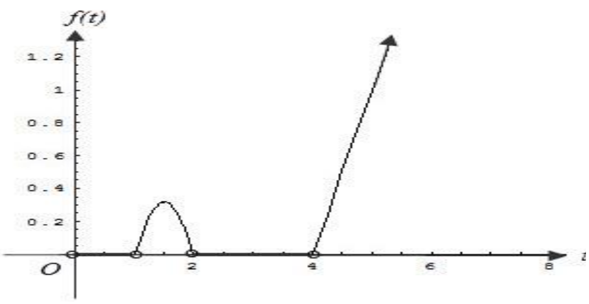
So, $f(t) = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2+\pi^2} + \frac{e^{-2s}}{s^2+\pi^2} + \frac{e^{-4s}}{s^2}\right\}$

$$= \frac{1}{\pi}\sin\big(\pi(t-1)\big)u_1(t) + \frac{1}{\pi}\sin\big(\pi(t-2)\big)u_2(t) + (t-4)u_4(t).$$

Since, $\sin\big(\pi(t-1)\big) = -\sin(\pi t)$ and $\sin\big(\pi(t-2)\big) = \sin(\pi t)$, so the first two terms cancel each other when $t>2$.

Hence, we obtain $f(t) =$

$$\begin{cases} 0, & 0<t<1 \\ -\frac{1}{\pi}\sin(\pi t), & 1<t<2 \\ 0, & 2<t<4 \\ t-4, & t>4 \end{cases}.$$



Problem set 2.2

Find inverse Laplace of the following functions and also sketch $f(t)$: (24-31)

Associated with unit step function

24. $F(s) = 3\left(\frac{e^{-5s}}{s}\right)$ **Ans:** $f(t) = 3\,u_5(t) = \begin{cases} 0; & 0 < t < 5 \\ 3; & t > 5 \end{cases}$

25. $F(s) = 4\left(\frac{e^{-3s}}{s^2}\right)$ **Ans:** $f(t) = 4(t-3)\,u_3(t) = \begin{cases} 0 & ; 0 < t < 3 \\ 4\,(t-3); & t > 3 \end{cases}$

26. $F(s) = \frac{se^{-\pi s}}{s^2+25}$ **Ans:** $f(t) = -\cos(5t)\;u_\pi(t) = \begin{cases} 0; & 0 < t < \pi \\ -\cos 5t\,; & t > \pi \end{cases}$

27. $F(s) = \frac{2(e^{-3s}-3e^{-4s})}{s}$ **Ans:** $f(t) = 2u_3(t) - 6u_4(t) = \begin{cases} 0, & 0 < t < 3 \\ 2, & 3 < t < 4 \\ -4, & t > 4 \end{cases}.$

28. $F(s) = \frac{5(e^{-\pi s}+e^{-2\pi s})}{s^2+25}$ **Ans:** $f(t) = (-\sin 5t)\;u_\pi(t) + (\sin 5t)\;u_{2\pi}(t)$

$$= \begin{cases} 0, & 0 < t < \pi \\ -\sin 5t\,, & \pi < t < 2\pi \\ 0, & t > 2\pi \end{cases}$$

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Associated with Dirac's delta function

29. $F(s) = 1$ **Ans:** $f(t) = \delta(t)$.

30. $F(s) = e^{-3s}$ **Ans:** $f(t) = \delta(t-3)$.

31. $F(s) = 25\,e^{-2s}$ **Ans:** $f(t) = 25\,\delta(t-2)$.