Z - Transform

* $\boxed{x[n] = 3 \cdot 2^n \cdot u[n]}$

$$Z\{x[n]\} = 3 \cdot \frac{1}{(1 - 2z^{-1})}$$

* $\boxed{x[n] = -3^n u[n]}$

$$= -\frac{1}{1 - 3z^{-1}}$$

* $\boxed{x[n] = (-5)^n u[n]}$

$$= \frac{1}{1 - (-5)^n}$$

* $\boxed{x[n] = (-2)^n u[-n-1]}$

$$x[z] = \frac{1}{1 - (-2)^{z-1}}$$

* $\boxed{x[z] = \frac{1}{1 + 1/3 z^{-1}} \quad |z| > 1/3}$

$$x[n] = \left(-\frac{1}{3}\right)^n u[n]$$

* $\boxed{x[-z] = z^5}$

$$x[n] = \delta(n+5)$$

* $\boxed{x[z] = z^{-1}}$

$$\Rightarrow x[n] = \delta(n-1)$$

$$\boxed{x[n] = e^{-naT}u[n]}$$

$$x(z) = \sum_{n=-\infty}^{\infty} e^{-naT} u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} e^{-naT} z^{-n}$$

$$= \sum_{n=0}^{\infty} (e^{aT} z)^{-n}$$

$$= \frac{1}{1-e^{-aT}z^{-1}}$$

$$= \frac{z}{z-e^{-aT}}$$

$$\boxed{x[n] = \cos(\omega_0 n)\, u[n]}$$

$$x[n] = \frac{1}{2}(e^{i\omega_0})^n u[n] + \frac{1}{2}(e^{-i\omega_0})^n u[n]$$

$$= \frac{1}{2}(e^{i\omega_0})^n u[n] \overset{z}{\longleftrightarrow} \frac{1/2}{1-e^{i\omega_0}z^{-1}} \quad |z|>1$$

$$= \frac{1}{2}(e^{-i\omega_0})^n u[n] \overset{z}{\longleftrightarrow} \frac{1/2}{1-e^{-i\omega_0}z^{-1}} \quad |z|>1$$

$$\boxed{x[n] = \sin(\omega_0 n)\, u[n]}$$

$$\Rightarrow \delta[n] = \begin{cases} 1, & n=0 \\ 0, & n \neq 0 \end{cases}$$

$\cancel{\text{xDne}}$ $z\{\delta[n]\}$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = (1)\, z^0 = 1.$$

Exercise 8.1

1.

(a) $\boxed{\delta[n] \rightarrow 1}$

Now,

$$z\{\delta[n]\} = \sum_{n=-\infty}^{\infty} \delta[n] \cdot z^{-n}$$

$$= \delta[0] \cdot z^{0}$$

$$= 1 \quad \text{(Ans)}$$

(b) $\boxed{\delta[n-m] \rightarrow z^{-m}}$

$$z\{\delta[n-m]\} = \sum_{n=-\infty}^{\infty} \delta[n-m] \cdot z^{-n}$$

$$= \delta[m-m] \cdot z^{-m}$$

$$= \delta[0] \cdot z^{-m}$$

$$= 1 \cdot z^{-m}$$

$$= z^{-m} \quad \text{(Ans)}$$

(c) $\boxed{na^{n}u[n] \longleftrightarrow \frac{az}{(z-a)^{2}}}$

$$\cancel{z\{na^{n}u[n]\} \longleftrightarrow \frac{az(z+a)}{(z-a)^{3}}}$$

(c) $\boxed{n a^n u[n] \longleftrightarrow \frac{az}{(z-a)^2}}$

$$Z\{n a^n u[n]\} = -z \cdot \frac{d}{dz}\left(\frac{z}{z-a}\right)$$

$$= -z\left[\frac{\cancel{z}-a-\cancel{z}}{(z-a)^2}\right]$$

$$= \frac{az}{(z-a)^2} \quad (\text{A})$$

(d) $\boxed{n^2 a^n u[n] \longleftrightarrow \frac{az(z+a)}{(z-a)^3}}$

$$Z\{n^2 a^n u[n]\} = -z \cdot \frac{d}{dz}\left\{-z \cdot \frac{d}{dz}\left(\frac{z}{z-a}\right)\right\}$$

$$= -z \cdot \frac{d}{dz}\left\{-z\left[\frac{\cancel{z}-a-\cancel{z}}{(z-a)^2}\right]\right\}$$

$$= z \cdot \frac{d}{dz}\left(\frac{az}{(z-a)^2}\right)$$

$$= \frac{-az^2+a^2z+2az^2}{(z-a)^3}$$

$$= \frac{az(-z+a+2z)}{(z-a)^3}$$

$$= \frac{az(z+a)}{(z-a)^3} \quad (\text{A})$$

(d) $\boxed{X(z) = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{9}z^{-2}} \; ; \; |z| > \frac{1}{3}}$

$$= \frac{z^2 + \frac{1}{3}z}{z^2 - \frac{1}{9}}$$

$$= \frac{z\left(z+\frac{1}{3}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{3}\right)}$$

$$= \frac{z}{\left(z-\frac{1}{3}\right)}$$

inverse z transform

$$\therefore x[n] = \left(\frac{1}{3}\right)^n u[n] \quad \text{(Ans)}$$

(e) $\boxed{X(z) = \frac{1-az^{-1}}{z^{-1}-a} \, , \; |z| > \frac{1}{a}}$

$$X(z) = \frac{z-a}{1-az}$$

$$\frac{X(z)}{z} = \frac{z-a}{-a\left(z-\frac{1}{a}\right)z}$$

$$= -\frac{1}{a}\left[\frac{A}{z-\frac{1}{a}} + \frac{B}{z}\right]$$

$$\therefore A = \left[\frac{(z-a)\left(z-\frac{1}{a}\right)}{z\left(z-\frac{1}{a}\right)}\right]_{z=\frac{1}{a}}$$

$$= \frac{\frac{1}{a} - a}{1/a}$$

$$= \frac{1-a^2}{\cancel{a}} \times \frac{\cancel{a}}{1}$$

$$= 1 - a^2$$

$$\therefore B = \left[\frac{(z-a)\cancel{z}}{\cancel{z}(z - 1/a)} \right]_{z=0}$$

$$= \frac{-a}{-1/a} = a \times a = a^2$$

$$\therefore \frac{X(z)}{z} = \frac{1}{-a}\left[\frac{1-a^2}{z - 1/a} + \frac{a^2}{z} \right]$$

$$\Rightarrow X(z) = \left(\frac{1-a^2}{-a}\right) \cdot \left(\frac{z}{z - 1/a}\right) + \frac{a^2}{-a}$$

$$\Rightarrow X(z) = \left(\frac{1-a^2}{-a}\right) \frac{z}{z - 1/a} - a$$

inverse Z transformation

$$x[n] = \left(\frac{1-a^2}{-a}\right)\left(\frac{1}{a}\right)^n u[n] - a\,\delta[n]$$

$$= -a\,\delta[n] - \left(\frac{1-a^2}{a}\right)\left(1/a\right)^n u[n]$$

Ans

Inverse Z transform,

$$Z^{-1}[X(z)] = A\left[3z^{-1}\left(\frac{1}{1-0.75z^{-1}}\right)\right] - 2z^{-1}\left(\frac{1}{1-0.5z^{-1}}\right)$$

$$x[n] = A[3(0.75)^n u[n] - 2(0.5)^n u[n]]$$

(C) $\boxed{X(z) = \frac{z^{-1}}{(1+0.75z^{-1})(1-0.75z^{-1})^2}}$

$$\frac{z^{-1}}{(1+0.75z^{-1})(1-0.75z^{-1})^2} = \frac{A}{(1+0.75z^{-1})} + \frac{B}{(1-0.75z^{-1})} + \frac{C}{(1-0.75z^{-1})^2}$$

$$\Rightarrow z^{-1} = A(1-0.75z^{-1})^2 + B(1+0.75z^{-1})(1-0.75z^{-1}) + C(1+0.75z^{-1}) \text{ — (i)}$$

$$\Rightarrow z^{-1} = A\{1 - 2 \cdot 0.75z^{-1} + (0.75z^{-1})^2\} + B\{1-(0.75z^{-1})^2\} +$$

$$C(1+0.75z^{-1}) \text{ — (ii)}$$

Put $z^{-1} = \frac{1}{0.75}$

$$\Rightarrow \frac{1}{0.75} = C(1+1)$$

$$\Rightarrow 2C = \frac{1}{0.75}$$

Put $z^{-1} = -\frac{1}{0.75}$

$$\Rightarrow -\frac{1}{0.75} = A(2)^2 + 0 + 0$$

$$A = -\frac{1}{0.75 \times 4} = -\frac{1}{3}$$

From equation (11)

$$A+B+C=0$$

$$\Rightarrow B = -(A+C)$$

$$= -\left(-\frac{1}{3}+\frac{2}{3}\right)$$

$$= -\left(\frac{-1+2}{3}\right) = \frac{1}{-3}$$

$$\therefore X(z) = -\frac{1}{3}\frac{1}{(1+0.75z^{-1})} - \frac{1}{3}\frac{1}{(1-0.75z^{-1})} + -$$

Inverse z transformation

$$x[n] = -\frac{1}{3}(-0.75)^n u[n] - \frac{1}{3}z\left[\frac{1}{1-0.75z^{-1}}\right] + \frac{2}{3}\cancel{B}_n$$

$$[0.75)^n u[n]$$

(Ans)

(e) $\boxed{(n+1)\, u[n] \longleftrightarrow \frac{z^2}{(z-1)^2}}$

$= n+1\, u[n]$

$= nu[n] + u[n]$

$Z\left[\{nu[n]\} + u[n]\right\} =$

$= -z \cdot \frac{d}{dz}\left(\frac{z}{z-1}\right) + \frac{z}{z-1}$

$= -z\left[\frac{(z-1)-z}{(z-1)^2}\right] + \frac{z}{z-1}$

$= -z\left[\frac{-1}{(z-1)^2}\right] + \frac{z}{z-1}$

$= \frac{z}{(z-1)^2} + \frac{z}{(z-1)}$

$= \frac{z + z(z-1)}{(z-1)^2}$

$= \frac{z + z^2 - z}{(z-1)^2}$

$= \frac{z^2}{(z-1)^2}$ (Ans)

(2) Determine the sequence $x[n]$ with z-transform and hence sketch $x[n]$

$$X(z) = (1+2z)(1+3z^{-1})(1-z^{-1})$$

$$= (1+2z+3z^{-1}+6)(1-z^{-1})$$

$$= (1+2z+3z^{-1}+6-z^{-1}-2-3z^{-2}-6z^{-1})$$

$$= (1+2z+3z^{-1}+6-z^{-1}$$

$$= (5-4z^{-1}-3z^{2}+2z)$$

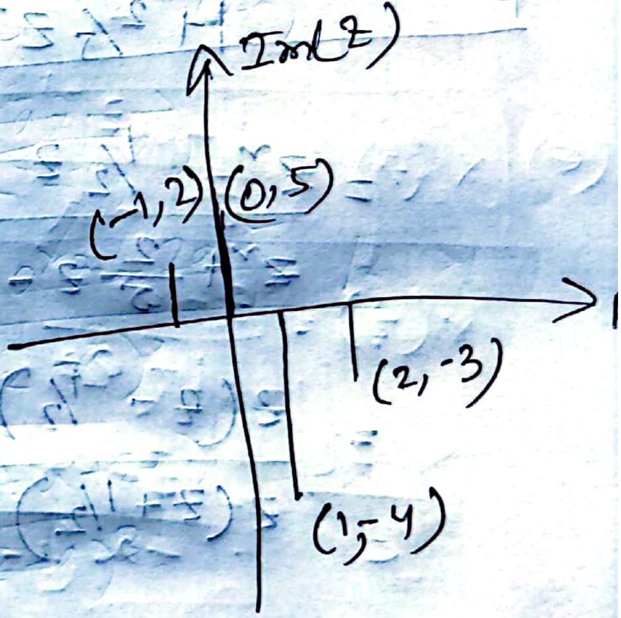
inverse of z transformation

$$z^{-1}\{X(z) = 5\delta[n]-4\delta[n-1]-3\delta[n-2]+2\delta[n+1]$$

$$x[n] = 5\delta[n]+2\delta[n+1]-4\delta[n-1]-3\delta[n-2] \quad (Ans)$$

Equivalently,

$$x[n] = \begin{cases} 2, & n=-1 \\ 5, & n=0 \\ -4, & n=1 \\ -3, & n=2 \end{cases}$$



3.

(a) $\boxed{X(z) = \dfrac{1}{1+\frac{1}{4}z^{-1}}, \; |z| > \frac{1}{4}}$

$\therefore x[n] = \left(\frac{1}{4}\right)^n u[n]$

(Ans)

(b) $\boxed{X(z) = \dfrac{1}{1+\frac{1}{4}z^{-1}}; \; |z| < \frac{1}{4}}$

$x[n] = -\left(-\frac{1}{4}\right)^n u[-n-1]$

(Ans)

(c) $\boxed{X(z) = \dfrac{1-\frac{1}{2}z^{-1}}{1+\frac{2}{5}z^{-1}-\frac{1}{20}z^{-2}}, \; |z| > \frac{1}{2}}$

$$= \frac{z^2 - \frac{1}{2}z}{z^2 + \frac{2}{5}z - \frac{1}{20}}$$

$$= \frac{z\left(z-\frac{1}{2}\right)}{z\left(z+\frac{1}{2}\right) - \frac{1}{10}\left(z+\frac{1}{20}\right)}$$

$$= \frac{z\left(z-\frac{1}{2}\right)}{\left(z+\frac{1}{2}\right)\left(z-\frac{1}{10}\right)}$$

$$\Rightarrow \frac{X(z)}{z} = \frac{A}{z+\frac{1}{2}} + \frac{B}{z-\frac{1}{10}}$$

$$A = \left[\frac{(z-1/2)\cancel{(z+1/2)}}{\cancel{(z+1/2)}(z-1/10)}\right]_{z=-\frac{1}{2}}$$

$$= \frac{-1/2 - \frac{1}{2}}{-1/2 - \frac{1}{10}} = \frac{5}{3}$$

$$B = \left[\frac{(z-1/2)(z-1/10)}{(z+1/2)(z-1/10)}\right]_{z=\frac{1}{10}}$$

$$= \frac{1/10 - 1/2}{1/10 + 1/2}$$

$$= \frac{-\frac{2}{5}}{3/5}$$

$$\boxed{= -\frac{2}{\cancel{5}} \times \frac{\cancel{5}}{3} = -\frac{2}{3}}$$

$$\therefore \frac{X(z)}{z} = \frac{\frac{5}{3}}{z+1/2} - \frac{\frac{2}{3}}{z-1/10}$$

$$\therefore X(z) = \frac{5/3\, z}{z+1/2} - \frac{\frac{2}{3} z}{z-1/10}$$

inverse

$$X[n] = 5/3\left(-1/2\right)^n u[n] - \frac{2}{3}\left(\frac{1}{10}\right)^n . u[n]$$

(Ans)

4.(a)

$$\boxed{X(z) = \frac{10z^{-2}}{(1+z^{-1})(1-z^{-1})^2} \; ; \; |z|>1}$$

$$X(z) = \frac{10z^{-2}}{(z+1)(z-1)^2}$$

$$\frac{X(z)}{10z^{-2}} = \frac{A}{(z+1)} + \frac{B}{(z-1)} + \frac{C}{(z-1)^2}$$

$$\therefore A = \left[\frac{\cancel{(z+1)}}{\cancel{(z+1)}(z-1)^2}\right]_{z=-1}$$

$$= \frac{1}{(-2)^2} = \frac{1}{4}$$

$$B = \left[\frac{(z-1)}{(z+1)(z-1)^2}\right]_{z=1}$$

$$= \left[\frac{1}{(z+1)(z-1)}\right]_{z=1}$$

$$= \left(\frac{1}{1-1}\right) = 0$$

$$C = \left[\frac{(z-1)^2}{(z+1)(z-1)^2}\right]_{z=1}$$

$$\frac{X(z)}{10z^{-2}} = \frac{1/4}{z+1} + \frac{1/2}{(z-1)^2}$$

$$x(z) = \frac{5}{2} z^{-2}(z+1) + 5z^{-2}(z-1)^2$$
$$= 5/2\left(z^{-1} + \frac{1}{z^2}\right) + 5\left(1-\frac{1}{z}\right)^2$$
$$x(z) = \frac{5}{2} z^{-1} + \frac{5}{2}\cdot\frac{1}{z^2} + 5\left(1-\frac{1}{z}\right)^2$$

inverse of z transform

$$x[n] = \frac{5}{2}(-1)^n u[n] + 5n u[n]$$

(Ans)

(b) $$\boxed{x(z) = \frac{1}{(1+0.5z^{-1})(1-0.5z^{-1})(1-z^{-1})}, \quad |z|>1}$$

$$x(z) = \frac{z^3}{(z+1/2)(z-1/2)(z-1)}$$

$$\frac{x(z)}{z} = \frac{z^2}{(z+1/2)(z-1/2)(z-1)}$$

$$= \frac{A}{z+1/2} + \frac{B}{z-1/2} + \frac{C}{z-1}$$

$$A = \left[\frac{z^2 \cancel{(z+1/2)}}{\cancel{(z+1/2)}(z-1/2)(z-1)}\right]_{z=-1/2} = \frac{1}{6}$$

$$B = \left[\frac{z^2}{(z+1/2)(z-1)}\right]_{z=1/2} \Rightarrow \frac{\frac{1}{4}}{1\times(-1/2)} = -\frac{1}{2}$$

$$C = \left[\frac{z^2}{(z+1/2)(z-1/2)}\right]_{z=1} = \frac{1}{3/4} = 4/3$$

$$\frac{X(z)}{z} = -\frac{\frac{1}{10}}{(z+1/2)} - \frac{\frac{1}{2}}{(z-1/2)} + \frac{\frac{4}{3}}{z-1}$$

$$\therefore X(z) = -\frac{1}{6} z\left(z+1/2\right)^{-1} - \frac{1}{2} z\left(z-1/2\right)^{-1} + 4/3\, z(z-1)^{-1}$$

Inverse Z transform

$$x[n] = \frac{1}{6}\left(-1/2\right)^n u[n] - 1/2\left(1/2\right)^n u[n] + 4/3\, u[n]$$

$$= \frac{1}{6}(-0.5)^n u[n] - \frac{1}{2}(0.5)^n u[n] + 4/3\, u[n]$$

(A)

(e) $\boxed{X(z) = \frac{1}{(1-z^{-1})(1-0.75z^{-1})}, \ |z|>1}$

$$\Rightarrow X(z) = \frac{z^2}{(z-1)(z-0.75)}$$

$$\Rightarrow \frac{X(z)}{z} = \frac{z}{(z-1)(z-0.75)}$$

$$= \frac{A}{z-1} + \frac{B}{z-0.75}$$

$$\therefore A = \left[\frac{z}{z-0.75}\right]_{z=1} = \frac{1}{1-0.75} = 4$$

$$B = \left[\frac{z}{z-1}\right]_{z=0.75} = \frac{0.75}{0.75-1}$$

so, $\frac{X(z)}{z} = \frac{4}{z-1} - \frac{3}{z-0.75}$

$X(z) = 4\left(\frac{z}{z-1}\right) - 3\left(\frac{z}{z-0.75}\right)$

inverse z transform

$x[n] = 4u[n] - 3(0.75)^n u[n]$ Ans.

(d) $\boxed{X(z) = \frac{A}{(1-0.75z^{-1})(1-0.5z^{-1})}}$

$\frac{X(z)}{A} = \frac{1}{(1-0.75z^{-1})(1-0.5z^{-1})}$

$\frac{1}{(1-0.75z^{-1})(1-0.5z^{-1})} = \frac{A}{(1-0.75z^{-1})} + \frac{B}{(1-0.5z^{-1})}$

$\Rightarrow 1 = A(1-0.5z^{-1}) + B(1-0.75z^{-1})$

$z^{-1} = \frac{1}{0.5}$

$\Rightarrow 1 = B\left(1 - \frac{0.75}{0.5}\right)$

$\Rightarrow B = \frac{1}{-0.5}$

$\therefore B = -2$

$z^{-1} = \frac{1}{0.75}$

$\Rightarrow 1 = \frac{A}{3}$

$\therefore A = 3$

$\therefore \frac{X(z)}{A} = \frac{3}{(1-0.75z^{-1})} + \frac{(-2)}{(1-0.5z^{-1})}$