Integration using Cauchy's Residue theorem (CRT)

Theorem:

If $f(z)$ is $\boxed{\text{analytic}}$ inside and on a simple closed curve C except at a finite number of n singular points $a_1, a_2, a_3, \ldots, a_n$ inside C, then

$$\oint_C f(z)\,dz = 2\pi i\left[\text{Res}(a_1) + \text{Res}(a_2) + \cdots + \text{Res}(a_n)\right]$$

Residue finding method:

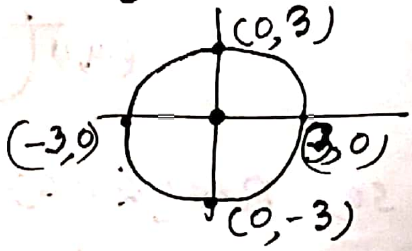
If $f(z)$ is analytic inside and on a simple closed curve C except at a pole or has singularity at $z=a$ of order m, then

$$\text{Res}(z=a) = \lim_{z\to a} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left\{(z-a)^m f(z)\right\}$$

Example 7.2:

Evaluate by CRT $\oint_C \frac{\sin \pi z}{(z-2)^2}\,dz$, $C: |z| = 3$

Soln: singular point $z=2$ of order 2. lies inside the circle $|z|=3$



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$$\operatorname{Res}(z=2) = \lim_{z \to 2} \frac{1}{(2-1)!} \frac{d}{dz} (z-2)^2 \frac{\sin \pi z}{(z-2)^2}$$

$$= \lim_{z \to 2} \frac{d}{dz} \sin(\pi z)$$

$$= \lim_{z \to 2} \pi \cos \pi z$$

$$= \pi \cos 2\pi$$

$$= \pi$$

So, $$\oint_c \frac{\sin \pi z}{(z-2)^2} dz = 2\pi i \left[\operatorname{Res}(z=2) \right]$$

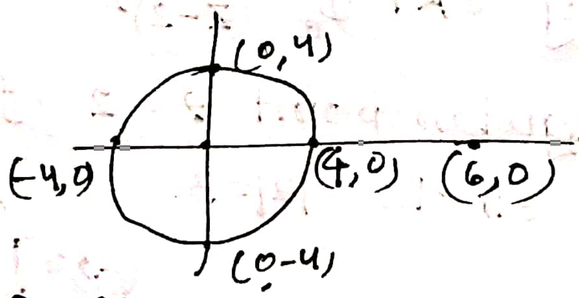
$$= 2\pi i \times \pi$$

$$= 2\pi^2 i \quad \text{(Ans)}$$

Exercise 7.1

2.(c) $\oint_c \frac{dz}{(z-6)^{10}}$, $c: |z|=4$

singular point $z=6$ of order 10. which lies outside the circle $|z|=4$



So, $\operatorname{Res}(z=6)=0$

$$\therefore \oint_c \frac{dz}{(z-6)^{10}} = 2\pi i \left[\operatorname{Res}(z=6) \right]$$

$$= 0 \quad \text{(Ans)}$$

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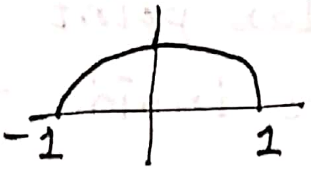
3.(a) $\oint_c \frac{2z}{(2z-i)^3}\,dz$;



singular point $z=\frac{i}{2}$ on $(0,1/2)$ of order 3 which lies inside the circle.

$$\text{Res}\left(z=\frac{i}{2}\right) = \lim_{z\to i/2} \frac{1}{2!}\frac{d^2}{dz^2}\frac{2z}{(2z-i)^3}\left(z-\frac{i}{2}\right)^3$$

$$= \lim_{z\to i/2} \frac{1}{2}\frac{d^2}{dz^2}\frac{2z}{(2z-i)^3}\cdot\frac{(2z-i)^3}{2^3}$$

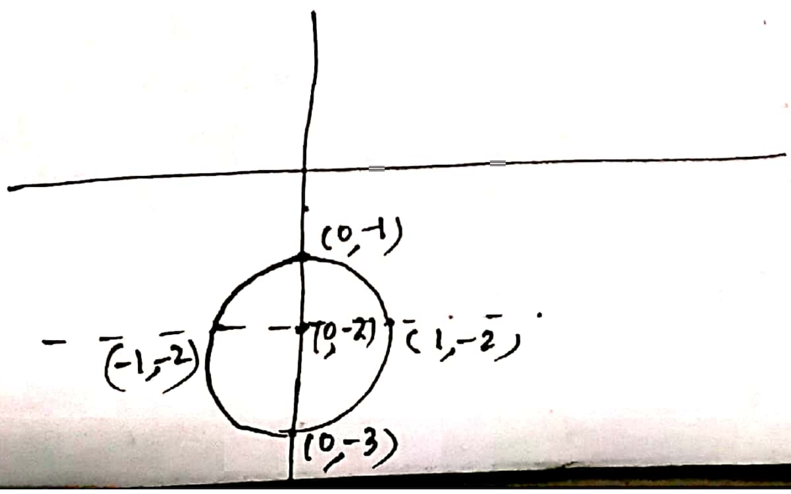
$$= \frac{1}{16}\lim_{z\to i/2}\frac{d^2}{dz^2}2z$$

$$= 0.$$

$$\oint_c \frac{2z}{(2z-i)^3}\,dz = 2\pi i\left[\text{Res}\left(z=i/2\right)\right]$$
$$= 0 \quad \text{(Ans)}$$

4.(a) $\oint_c \frac{dz}{z^2+4}$; c: $|z+2i| = 1$

C is the circle of center $(0,-2)$ and radius 1.



singular point $z = -2i$ or $(0, -2)$ of order 1 lies inside the circle.

$$Res(z = -2i) = \lim_{z \to -2i} \frac{1}{0!} (z+2i) \frac{1}{z^2+4}$$

$$= \lim_{z \to -2i} (z+2i) \frac{1}{(z+2i)(z-2i)}$$

$$= \lim_{z \to -2i} \frac{1}{z-2i}$$

$$= -\frac{1}{4i}$$

$$\oint_c \frac{dz}{z^2+4} = 2\pi i \left[Res(z = -2i) \right]$$

$$= -2\pi i \times \frac{1}{4i}$$

$$= -\frac{\pi}{2} \quad \text{(Ans)}$$

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Improper Integral:

exercise 7.2

(iv) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2-2x+2)^2}$

solution:

$$\oint_c \frac{dz}{(z^2-2z+2)^2} = \int_{-R}^{R} \frac{dx}{(x^2-2x+2)^2} + \int_{C_R} \frac{dz}{(z^2-2z+2)^2}$$

——— (i)

now, we have to evaluate

$$\oint_c \frac{dz}{(z^2-2z+2)^2}$$

$z^2-2z+2=0$

$\Rightarrow z = \frac{2\pm\sqrt{4-8}}{2}$

$z = \frac{2\pm 2i}{2}$

$= 1\pm i$

singular point $z = 1+i$ of order 2 lies in the upper half circle.

$$\text{Res}(z=1+i) = \lim_{z\to 1+i} \frac{1}{1!} \frac{d}{dz} \{z-(1+i)\}^2 \frac{1}{(z^2-2z+2)^2}$$

$$= \lim_{z\to 1+i} \frac{d}{dz} \frac{1}{\{z-(1+i)\}^2\{z-(1-i)\}^2} \{z-(1+i)\}^2$$

$$= \lim_{z\to 1+i} \frac{d}{dz} \frac{1}{\{z-(1-i)\}^2}$$

$$= \lim_{z\to 1+i} -2 \frac{1}{\{z-(1-i)\}^3}$$

$$= \frac{-2}{(1+i-1+i)^3} = \frac{-2}{(2i)^3} = \frac{-2}{8i^3} = \frac{-2}{-8i} = \frac{1}{4i}$$

$$\oint_c \frac{dz}{(z^2-2z+2)^2} = 2\pi i \times \frac{1}{4i} = \frac{\pi}{2}$$

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from (i) $\Rightarrow$

$$\int_{-R}^{R} \frac{dx}{(x^2-2x+2)^2} + \int_{C_R} \frac{dz}{(z^2-2z+2)^2} = \frac{\pi}{2}$$

$$\Rightarrow \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{(x^2-2x+2)^2} + \lim_{R\to\infty} \int_{C_R} \frac{dz}{(z^2-2z+2)^2} = \lim_{R\to\infty} \frac{\pi}{2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{(x^2-2x+2)^2} = \frac{\pi}{2}$$ [Using Jordan's lemma]

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