The Z-Transformation (Additional Problem)

Bubble sort: Bubble sort is a simple sorting algorithm used to sort lists. It is generally one of the first algorithms taught in computer science courses because it is a good algorithm to learn to build intuition about sorting. While sorting is a simple concept, it is a basic principle used in complex computer programs such as file search, data compression, and path finding.

The Bubble sort algorithm compares each pair of elements in an array and swaps them if they are out of order until the entire array is sorted. For each element in the list, the algorithm compares every pair of elements.

Following chart shows the number of elements in an array and corresponding maximum number of comparisons or swaps needed to sort an array:

No. of elements $n \geq 1$	Maximum no. of swaps needed $a[n]$
1	0
2	1
3	3
4	6
n	$\frac{n(n-1)}{2}$

Example: (i) For n=1,

Let A= [5], no swap is needed to sort.

(ii) For n=2,

Let A=[3,1], after one swap A=[1, 3] (sorted).

(iii) For n=3,

Let A=[c, b, a], then after 1st swap A=[b, c, a]

after 2^{nd} swap A=[b, a, c]

after 3^{rd} swap A=[a, b, c] (sorted).

If A=[b, c, a], then after 1^{st} swap A=[b, a, c]

After 2^{nd} swap A=[a, b, c] (sorted).

In the 1^{st} case, three swaps are needed and 2^{nd} case two swaps are needed. So the number found in recurrence relation is the maximum number of swaps to sort a list with n number of elements.

N.B. To sort an array or list, at least one element is needed in that array. So, for $n \leq 0$, $a[n] = 0$.

Example-1:The number of comparisons needed to sort an array of n elements by the method bubble sort (or by straight selection) can be expressed by the following recurrence relation $a[n+1] = a[n] + n, n \geq 1$ with initial condition $a[1] = 0$. Solve it and check your answer by direct substitution.

Solution: Given , $a[n+1] = a[n] + n$

Taking Z-transform we have

$$Z\{a[n+1]\} = Z\{a[n]\} + Z\{n\}$$

$$\rightarrow z\big[A(z) - a[0]\big] = A(z) + \frac{z}{(z-1)^2}$$

$$\rightarrow z\,A(z) - A(z) = \frac{z}{(z-1)^2} \qquad [\text{since, for } n \leq 0,\ a[n] = 0]$$

$$\rightarrow (z-1)\,A(z) = \frac{z}{(z-1)^2}$$

$$\rightarrow A(z) = \frac{z}{(z-1)^3}$$

Now, By Cauchy residue Theorem (CRT), we have-

The inverse Z-Transform of A(z) is $a[n] = \lim_{z \to 1}\left[\frac{1}{(3-1)!}\frac{d^2}{dz^2}\left((z-1)^3\frac{z^{n-1}.z}{(z-1)^3}\right)\right]$

$$= \lim_{z \to 1}\left[\frac{1}{(2)!}\frac{d^2}{dz^2}\left(z^n\right)\right]$$

$$= \frac{1}{2}\lim_{z \to 1}\left[\frac{d}{dz}\left(nz^{n-1}\right)\right]$$

$$= \frac{1}{2}\lim_{z \to 1}\left[\left(n(n-1)z^{n-2}\right)\right]$$

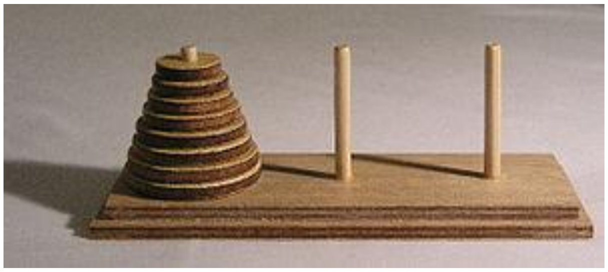
$$\therefore a[n] = \frac{n(n-1)}{2}$$

Now, for $n = 1$, we have $a[1] = 0$.

Tower of Hanoi puzzle: The Tower of Hanoi (also called the Tower of Brahma or Lucas' Tower and sometimes pluralized) is a mathematical game or puzzle. It consists of three rods and a number of disks of different sizes, which can slide onto any rod. The puzzle starts with the disks in a neat stack in ascending order of size on one rod, the smallest at the top, thus making a conical shape.

The objective of the puzzle is to move the entire stack to another rod, obeying the following simple rules:

1. Only one disk can be moved at a time.
2. Each move consists of taking the upper disk from one of the stacks and placing it on top of another stack or on an empty rod.
3. No larger disk may be placed on top of a smaller disk.



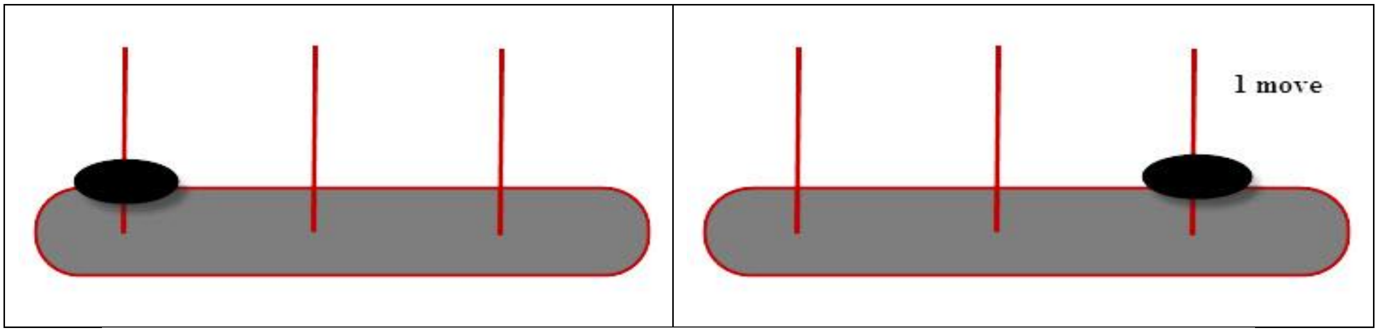
A model for tower of Hanoi puzzle

Following table shows the number of disk and corresponding number of moves to solve Hanoi puzzle:

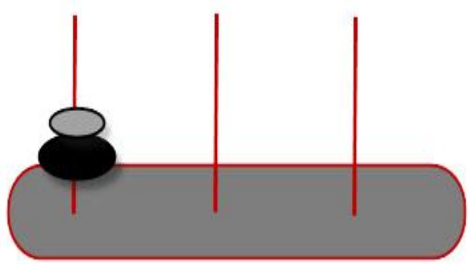
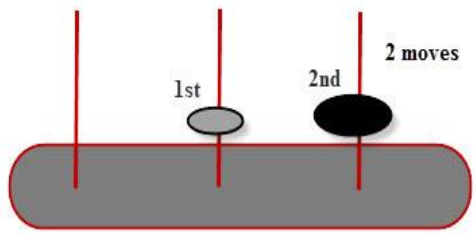
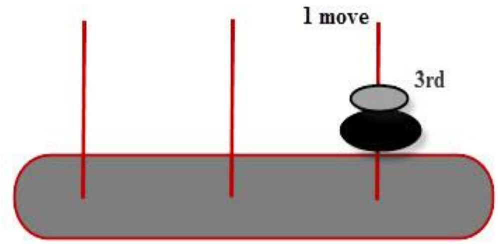
No. of disks $n \geq 1$	No. of moves needed $a[n]$
1	1
2	3
3	7
4	15
n	$2^n - 1$

N.B. To solve this puzzle, at least one disk is needed. So, for $n \leq 0$, $a[n] = 0$.

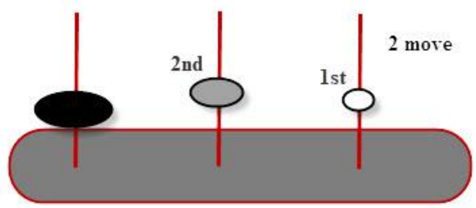
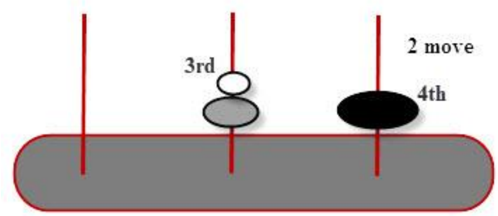
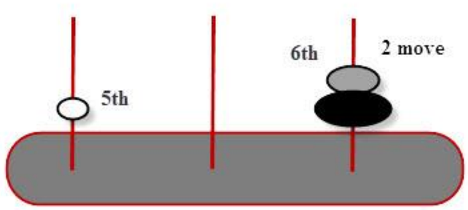
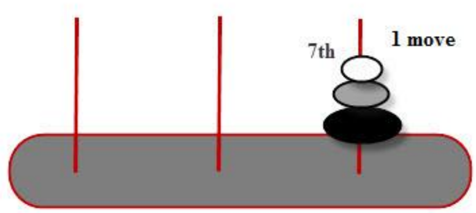
Example: $n = 1$



For n=2

For n=3

		
		Total number of moves= 2+2+2+1=7

Example-2 :The number of moves of disks, necessary to solve the Tower of Hanoi puzzle for n disks can be expressed by the following recurrence relation $a[n+1] = 2a[n] + 1,\ n \geq 1$ with initial condition $a[1] = 1$. Solve it and check your answer by direct substitution,

Solution: Given, $a[n+1] = 2a[n] + 1$

Taking Z-Transform we have-

$\rightarrow\ Z\{a[n+1]\} = Z\{2a[n]\} + Z\{1\}$

$\rightarrow z\big[A(z) - a[0]\big] = 2\,A(z) + \frac{z}{(z-1)}$

$\rightarrow (z-2)A(z) \quad = \frac{z}{(z-1)}$ $\qquad$ [since, for $n \leq 0,\ a[n] = 0$]

$\rightarrow \qquad A(z) \quad = \frac{z}{(z-1)(z-2)}$

Now by Cauchy Residue Theorem (CRT), we have-

The inverse Z-Transform of A(z) is-

$$a[n] = \lim_{z\to 1}\left[\left((z-1)\frac{z^{n-1}z}{(z-1)(z-2)}\right)\right] + \lim_{z\to 2}\left[\left((z-2)\frac{z^{n-1}z}{(z-1)(z-2)}\right)\right]$$

$$= \lim_{z\to 1}\left(\frac{z^n}{(z-2)}\right) + \lim_{z\to 2}\left(\frac{z^n}{(z-1)}\right)$$

$$= \frac{1^n}{(1-2)} + \frac{2^n}{(2-1)}$$

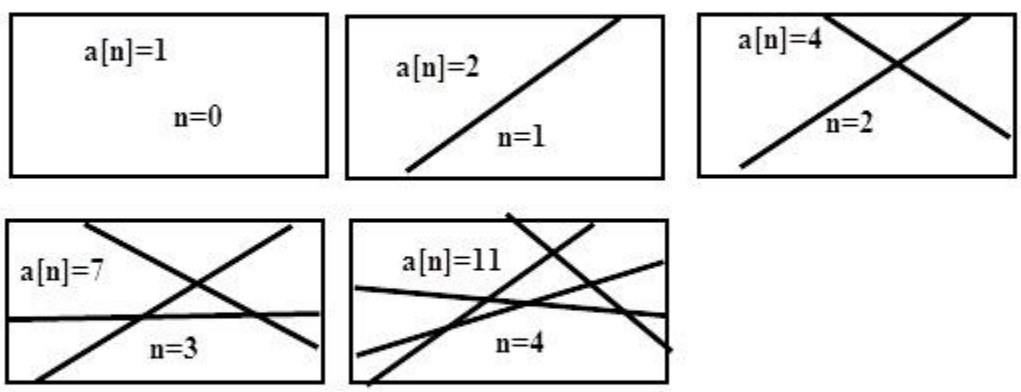
$$\therefore a[n] = 2^n - 1$$

Now, for n=1 , we have $a[1] = 1$.

Plane Divisions by Lines: The maximal number of regions into which n lines divide a plane are

$$a[n] = \frac{n(n+1)}{2} + 1$$

Following diagrams shows the number of lines (n) and corresponding number of divided regions $a[n]$, in a plane:



Example-3 :The maximum number of regions defined by n straight lines in the plane can be expressed by the following recurrence relation $a[n+1] = a[n] + n + 1,\ n \geq 0$ with initial condition $a[0] = 1$. Solve it and check your answer by direct substitution.

Solution: Given, $a[n+1] = a[n] + n + 1$

Taking Z-Transform we have,

$\rightarrow\ Z\{a[n+1]\} = Z\{a[n]\} + Z\{n\} + Z\{1\}$

$$\rightarrow z[A(z) - a[0]] = A(z) + \frac{z}{(Z-1)^2} + \frac{z}{z-1}$$

$$\rightarrow (z\text{-}1)\ A(z) = \frac{z}{(z-1)^2} + \frac{z}{(z-1)} + z$$

$$\rightarrow (z\text{-}1)\ A(z) = \frac{z^3-z^2+z}{(z-1)^2}$$

$$\rightarrow \quad A(z) = \frac{z^3-z^2+z}{(z-1)^3}$$

Now Following by **Example-1**, we have, The inverse Z-Transform of A(z) is, $a[n] = \frac{n(n+1)}{2} + 1$.

Now, for $n = 0$, $a[0] = 1$.

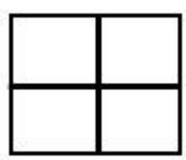
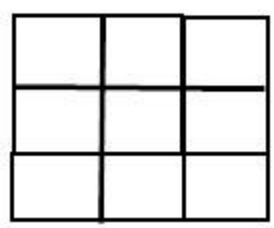
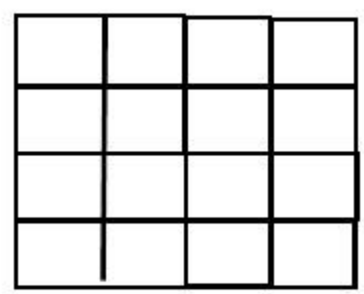
Number of squares in a square grid: If we determine the number of squares on smaller boards starting with one square we will readily discover a pattern that leads to a simple formula for a board of any number of squares.

A 1×1 square board obviously has only one square.

A 2x2 square board has 5 squares, the 4 basic ones and the one large 2x2 one.

A 3x3 square board has 14 squares, the smaller 9 plus, four 2x2's plus, one large 3x3 one.

A 4x4 square board has 30 squares, the smaller 16 plus, nine 3x3's, plus four 2x2's plus, one large 4x4 one.

 1×1	 2×2	 3×3	 4×4

N.B. If n is the number of grid and $a[n]$ is the number of square then, for $n \le 0$, $a[n] = 0$ and for $n \ge 1$, $a[n] = \frac{n(n+1)(2n+1)}{6}$.

Example-4 :The number of all squares in a square grid of dimension n can be expressed by the following recurrence relation $a[n+1] = a[n] + (n+1)^2, n \ge 1$ with initial condition $a[1] = 1$. Solve it and check your answer by direct substation.

Solution: Given , $a[n+1] = a[n] + (n+1)^2$

Taking Z-Transform we have

$\rightarrow\ Z\{a[n+1]\} = Z\{a[n]\}\ + Z\{(n+1)^2\}$

$\rightarrow$ z$\left[A(z) - a[0]\right] =$ A(z) $+ \frac{z+z^2}{(Z-1)^3} + \frac{2z}{(z-1)^2} + \frac{z}{z-1}$

$\rightarrow (z-1)\ A(z) = \frac{z+z^2}{(Z-1)^3} + \frac{2z}{(z-1)^2} + \frac{z}{z-1}$

$\rightarrow \quad A(z) \qquad = \frac{z^3+z^2}{(Z-1)^4}$

Now, By Cauchy-Residue Theorem (CRT) we have

The inverse Z-Transfor of A(z) is -

$$a[n] = \frac{n(n+1)(2n+1)}{6}$$

Now, for n= 1 ,we have , $a[1] = 1.$

[Reference: Title: The Recurrence Relations in Teaching Students of Informatics

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Exercise:

1. Find a closed formula for the generating function of the sequence:
   i. a[n+1] = 3 a[n]+2 a[n+1] ; a[0] = 1; a[1] = 2.
   ii. b[n+1] = 2 b[n] -3b[n+1] ; b[0] = 2; b[1]= 1.