Laurent Series (7.3)

Example: Obtain the Laurent Series expansion of $f(z) = \frac{1}{(1+z^2)(z+2)}$; $1 < |z| < 2$.

Solution: Given,

$$1 < |z| < 2$$

Now,

$$1 < |z|$$

$$\Rightarrow |z| > 1$$

$$\Rightarrow \frac{1}{|z|} < 1$$

$$\Rightarrow \left|\frac{1}{z}\right| < 1 \Rightarrow \left|\frac{1}{z}\right|^2 < 1 \Rightarrow \left|\frac{1}{z^2}\right| < 1$$

Property: $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, Property: $|a|^2 = |a^2|$

Again,

$$|z| < 2$$

$$\Rightarrow \frac{|z|}{2} < 1$$

$$\Rightarrow \left|\frac{z}{2}\right| < 1$$

Now, $f(z) = \frac{1}{(1+z^2)\,(z+2)} = \frac{Az+B}{1+z^2} + \frac{C}{z+2}$ ....(i)

Multiplying both side of equation (i) by $(1 + z^2)(z + 2)$ we get,

$$1 = (Az + B)(z + 2) + C(1 + z^2)$$

$$\Rightarrow 1 = Az^2 + 2Az + Bz + 2B + C + Cz^2$$

$$\Rightarrow 1 = (A + C)z^2 + (2A + B)z + (2B + C)$$

Equating coefficients from both sides,

$$A + \quad C = 0$$

$$2A + B \quad = 0$$

$$2B + C = 1$$

$$A = -\frac{1}{5}, B = \frac{2}{5}, C = \frac{1}{5}$$

So, from equation (i)

$$\frac{1}{(1 + z^2)\,(z + 2)} = \frac{-\frac{1}{5}z + \frac{2}{5}}{1 + z^2} + \frac{\frac{1}{5}}{z + 2}$$

$$\Rightarrow \frac{1}{(1+z^2)\,(z+2)} = \frac{-\frac{1}{5}z+\frac{2}{5}}{z^2\left(1+\frac{1}{z^2}\right)} + \frac{\frac{1}{5}}{2\left(1+\frac{z}{2}\right)}$$

$$= \frac{-\frac{1}{5}\ z}{z^2\left(1+\frac{1}{z^2}\right)} + \frac{\frac{2}{5}}{z^2\left(1+\frac{1}{z^2}\right)} + \frac{\frac{1}{5}}{2\left(1+\frac{z}{2}\right)}$$

$$= -\frac{1}{5z}\left(1+\frac{1}{z^2}\right)^{-1} + \frac{2}{5z^2}\left(1+\frac{1}{z^2}\right)^{-1} + \frac{1}{10}\left(1+\frac{z}{2}\right)^{-1}$$

Formula:

$$(1+z)^{-1} = 1 - z + z^2 - z^3 + z^4 - \cdots$$

$$(1-z)^{-1} = 1 + z + z^2 + z^3 + z^4 + \cdots$$

$$= -\frac{1}{5z}\left(1-\frac{1}{z^2}+\frac{1}{z^4}-\frac{1}{z^6}+\cdots\right) + \frac{2}{5z^2}\left(1-\frac{1}{z^2}+\frac{1}{z^4}-\frac{1}{z^6}+\cdots\right) + \frac{1}{10}\left(1-\frac{z}{2}+\frac{z^2}{4}-\cdots\right)$$

Ans.

Example: Obtain the Laurent Series expansion of $f(z) = \frac{1}{(1+z^2)(z+2)}$; $|z| > 2$

Solution: Given,

$$|z| > 2$$

$$\Rightarrow \frac{|z|}{2} > 1$$

$$\Rightarrow \frac{2}{|z|} < 1$$

$$\Rightarrow \left|\frac{2}{z}\right| < 1 \ldots (*)$$

$$\Rightarrow \left|\frac{1}{z}\right| < 1$$

$$\Rightarrow \left|\frac{1}{z}\right|^2 < 1$$

$$\Rightarrow \left|\frac{1}{z^2}\right| < 1 \ldots (**)$$

Logic: $2a < 1 \Rightarrow a < 1$

Now, $f(z) = \frac{1}{(1+z^2)(z+2)} = \frac{Az+B}{1+z^2} + \frac{C}{z+2}$ …………(i)

Multiplying both side of equation (i) by $(1+z^2)(z+2)$ we get,

$$1 = (Az + B)(z + 2) + C(1 + z^2)$$

$$\Longrightarrow 1 = Az^2 + 2Az + Bz + 2B + C + Cz^2$$

$$\Longrightarrow 1 = (A + C)z^2 + (2A + B)z + (2B + C)$$

Equating coefficients from both sides,

$$\begin{aligned} A + \quad\quad C &= 0 \\ 2A + B \quad\quad &= 0 \\ 2B + C &= 1 \end{aligned}$$

$$A = -\frac{1}{5}, B = \frac{2}{5}, C = \frac{1}{5}$$

So, from equation (i)

$$\frac{1}{(1 + z^2)\,(z + 2)} = \frac{-\frac{1}{5}z + \frac{2}{5}}{1 + z^2} + \frac{\frac{1}{5}}{z + 2}$$

$$= \frac{-\frac{1}{5}z + \frac{2}{5}}{z^2\left(1 + \frac{1}{z^2}\right)} + \frac{\frac{1}{5}}{z\left(1 + \frac{2}{z}\right)}$$

$$= \frac{-\frac{1}{5}\ z}{z^2\left(1+\frac{1}{z^2}\right)} + \frac{\frac{2}{5}}{z^2\left(1+\frac{1}{z^2}\right)} + \frac{\frac{1}{5}}{z\left(1+\frac{2}{z}\right)}$$

$$= -\frac{1}{5z}\left(1+\frac{1}{z^2}\right)^{-1} + \frac{2}{5z^2}\left(1+\frac{1}{z^2}\right)^{-1} + \frac{1}{5z}\left(1+\frac{2}{z}\right)^{-1}$$

$$-\frac{1}{5z}\left(1-\frac{1}{z^2}+\frac{1}{z^4}-\frac{1}{z^6}+\cdots\right) + \frac{2}{5z^2}\left(1-\frac{1}{z^2}+\frac{1}{z^4}-\frac{1}{z^6}+\cdots\right) + \frac{1}{5z}\left(1-\frac{2}{z}+\frac{4}{z^2}-\cdots\right)$$

Ans.