Nonlinear Equations in One Variable
Lecture-1

Objective:

To find the root of a nonlinear equation in on variable with the help of different methods .

Methodologies:

Five methods can be used to find roots of nonlinear equation in one variable. They are-

1. **Graphical Method (to find the nature of the roots)**
2. **Bisection Method**
3. **The Secant Method**
4. **Newton-Raphson Method**
5. **Fixed Point Iteration Method**

What is nonlinear equation?

An equation in which one or more terms have a variable of degree 2 or higher is called a nonlinear equation.

Example of nonlinear equations

1. $x^3 + 4x - 3 = 0$
2. $x^2 + x + 2 = 25$

Number of Real Roots by Graphical Method

- **A polynomial equation** $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$ **of degree *n* has exactly *n* roots.**

- **Some of them are real and others are complex.**

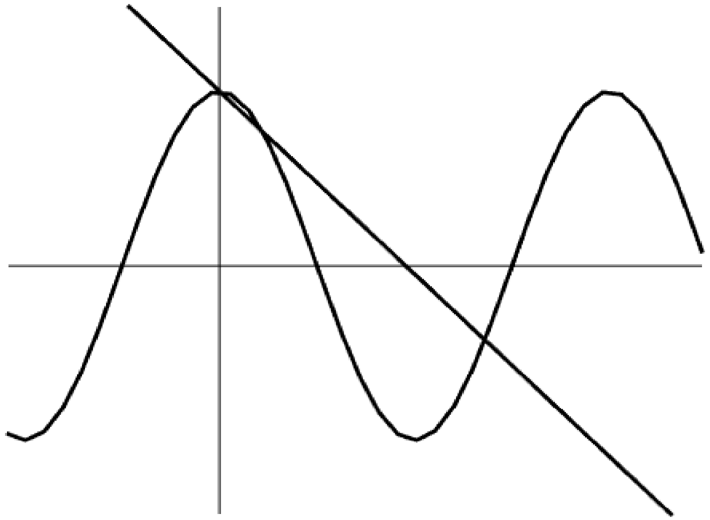
- **Geometrically, if the graph of** $y = f(x)$ **crosses the *x*-axis at x = a , then x = a is a real root of** $f(x) = 0$

- **Now we shall consider graphically to find the number of real roots and its location.**

- **Rewrite the equation** $f(x)=0$ **as** $f_1(x)=f_2(x)$

- **At the point of intersection** $x=x_1$ **(say) of the graphs** $y=f_1(x)$ **and** $y=f_2(x)$ **that is** $f_1(x_1)=f_2(x_1)$ **and hence** $x=x_1$ **is a root of the equation.**

- **Thus the number of intersections of the two graphs will be the number of real roots (See the following figure)**

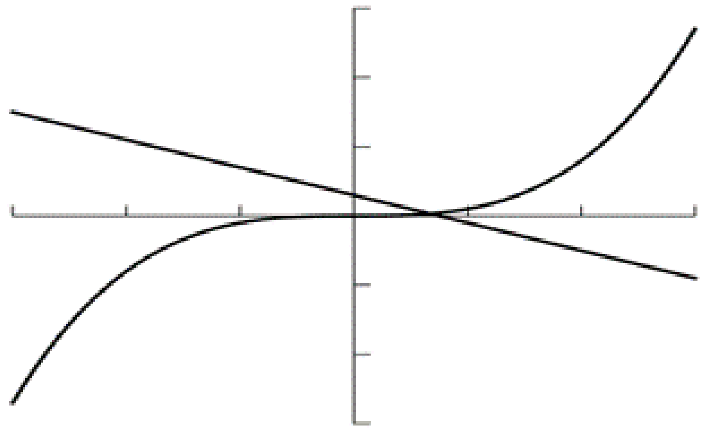


Problems and Solutions

Example: (a) Find the number of real roots of $x^3 + 4x - 3 = 0$ by graphical method

(b) Find the nmber of complex roots, if any.

(c) Use MATLAB function "roots" to find all the roots including complex roots.



$(a) \quad x^3 + 4x - 3 = 0$

$\Longrightarrow x^3 = 3 - 4x$

Let, $f_1(x) = x^3$ and $f_2(x) = 3 - 4x$

There is one point of intersection here in the plot. So the number of real root is 1

(b) It is a polynomial equation of degree three and hence the total number of roots is three.

So, number of complex roots = Total number of roots - Number of real roots

=3-1=2.

(c) >> p=[1 0 4 -3] % entry of cubic polynomial

p = 1 0 4 -3

>> Roots = roots(p)

Roots =

-0.3368 + 2.0833i

-0.3368 - 2.0833i

0.6736 + 0.0000i

Exercise

Given the following polynomial equations and an interval.

a. $x^3 - 5x + 1 = 0$; $[2, 3]$,

b. $x^3 + x^2 - 2x - 5 = 0$; $[1,2]$,

c. $x^4 - 2x - 5 = 0$; $[0, 2]$,

d. $x^4 + x^2 - 80 = 0$; $[2.90, 2.92]$.

- Find the number of real roots of the equation by graphical method. Find also the number of complex roots, if any.
- Write MATLAB commands "**roots**" to find all the roots including complex roots.

Location of Roots

- To locate the roots of $f(x)=0$ first study the graph of $y=f(x)$ as shown below (Fig-4.1) .

- If we can find two values of *x*, one for which $f(x)$ is positive, and one for which $f(x)$ is negative, then the curve must have crossed the *x*-axis and so must have passed through a root of the equation $f(x)=0$

- In general, if $f(x)=0$ is continuous in $[a,b]$ and $f(a)$ and $f(b)$ are opposite in signs i.e., $f(a)f(b)<0$, then there exists odd number of real roots (at least one root) of $f(x)=0$ in (a,b)

- But the only exception where it does not work is when curve touches the *x*-axis. For this case, the existence of a root can be determined by the sign of $f'(x)$ in the interval (a,b) containing the root and it will satisfy the condition $f'(a)f'(b)<0$

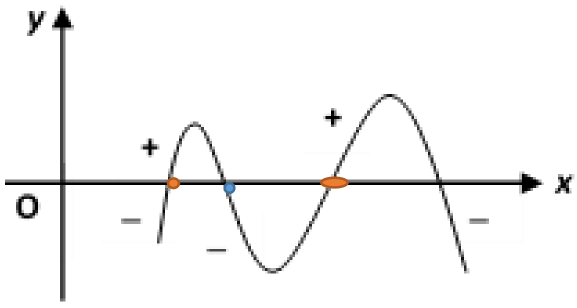


Fig 4.1

Example: The equation $(x-1)e^{x}=x$ has two real roots. For each root, find an interval where it lies.

Solution: Consider the values of function $f(x)=(x-1)e^{x}-x=y$ for different values of x:

x	$f(x)$
-2	1.59
-1	0.26
0	-1
1	-1
2	5.39

From the above table, we see that $f(-1)f(0)=-0.26<0$, a root lies in $(-1,\ 0)$,

and $f(1)f(2)=-5.39<0$, a root lies in $(1,2)$

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Techniques to find real roots

Method of Bisection

Algorithm:

Step-1: If $f(x)$ be continuous in $[a,b]$, Choose two approximations a and b such that $f(a)f(b)<0$.

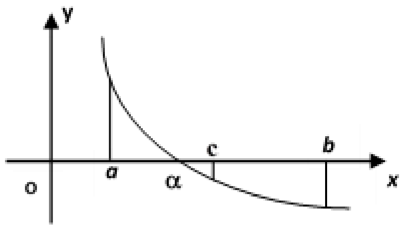

Fig 4.2Bisection Method

Step-2: Evaluate the mid-point c of $[a,b]$ such that $c=(a+b)/2$

Step-3: If $f(c)=0$ we conclude that c is a root of $f(x)=0$ .If $f(c)\neq 0$ and

(i) If $f(a)f(c)<0$ the root is in (a,c) or

(ii) If $f(c)f(b)<0$ the root is in (c,b)

Step-4: By designating the new interval of root as $[a_1, b_1]$ we can calculate the next iterate x_1 by the formula $x_{n+1} = \frac{a_n + b_n}{2}$, $n = 1, 2, 3, \ldots$

Step-5: Repeat the process until $|x_{n+1} - x_n| \leq \varepsilon$,where ε is the specified accuracy

Problems and Solutions

Ex 1(a): Find the root of $x^3 - 5x + 1 = 0$ using bisection method (3steps) in the interval [2, 3] and find the updated interval.

Solution: Given $f(x) = x^3 - 5x - 1$

a	b	f(a)	f(b)	$\boldsymbol{c = \frac{a+b}{2}}$	f(c)	update
2	3	-1	13	$c = \frac{2+3}{2} = 2.5$	4.125	b=c
2	2.5	-1	4.125	$c = \frac{2+2.5}{2} = 2.25$	1.14	b=c
2	2.25	-1	1.14	$c = \frac{2+2.25}{2} = 2.125$	-0.029	a=c
2.125	2.25					

So the updated interval is [2.125,2.25]

Class work

Ex 1(c): Find the root of $x^4 - 2x - 5 = 0$ using bisection method (2steps) in the interval [0, 2] and find the updated interval (new smaller interval).

Solution: Given $f(x) = x^4 - 2x - 5$

a	b	f(a)	f(b)	$c = \frac{a+b}{2}$	f(c)	update
				$c =$		
				$c =$		

So the updated interval is [,]

Problems and Solutions

Example: Find the root of $f(x) = x^2 - 3$ using bisection method with accuracy $\varepsilon = 0.01$ in the interval [1, 2].

Solution:

A	*b*	*f(a)*	*f(b)*	*c = (a + b)/2*	*f(c)*	*f(a)*f(c)*	*Update*	*b − a*
1.0	2.0	-2.0	1.0	1.5	-0.75	> 0	a = c	0.5
1.5	2.0	-0.75	1.0	1.75	0.062	< 0	b = c	0.25
1.5	1.75	-0.75	0.0625	1.625	-0.359	> 0	a = c	0.125
1.625	1.75	-0.3594	0.0625	1.6875	-0.1523	> 0	a = c	0.0625
1.6875	1.75	-0.1523	0.0625	1.7188	-0.0457	> 0	a = c	0.0313
1.7188	1.75	-0.0457	0.0625	1.7344	0.0081	< 0	b = c	0.0156
1.71988	1.7344	-0.0457	0.0081	1.7266	-0.0189	> 0	a = c	0.0078

Exercise

Given the following polynomial equations and an interval.

a. $x^3 - 5x + 1 = 0$; $[2, 3]$,

b. $x^3 + x^2 - 2x - 5 = 0$; $[1,2]$,

c. $x^4 - 2x - 5 = 0$; $[0, 2]$,

d. $x^4 + x^2 - 80 = 0$; $[2.90,2.92]$.

Apply bisection method two times in the given interval to find the new smaller interval of this root.

Advantages and Drawbacks: Bisection Method

Advantages:

- ❑ **Always Convergent.**

Drawbacks:

- ❑ **Slow convergence.**
- ❑ **If one of the initial guess is closer to the root , the convergence is slower.**

Outcome

By applying Bisection method, nonlinear equations in one variable can be solved to find roots (approximately), although it has few drawbacks.

Multiple Choice Questions

1. Find the degree of the polynomial $x^4 + x^2 - 80 = 0.$

a) 3 , b) 2 , c) 4 , d) 5

2. Find the number of real roots of the polynomial $x^3 + x^2 - 2x - 5 = 0.$

a) 3 , b) 1 , c) 4 , d) 5

3. Find the number of complex roots of the polynomial $x^4 + x^2 - 80 = 0$

a) 3 , b) 2 , c) 4 , d) 5

4. If $f(x) = x^3 - 5x + 1 = 0$ **is continuous on the interval** $[2, 3]$, What is the value of $\boldsymbol{f}(\mathbf{2})$?

a) -3 , b) -1 , c) 4 , d) 0

5. If $f(x)$ is continuous in $[a, b]$ then what is the condition that there exists odd number of real roots(at least one root) of $f(x)$=0 ?

a) $f(a)f(b) < 0$ **, b)** $f(a)f(b) > 0$ **, c)** $f(a) < 0$ **,d) Neither**