Nonlinear Equations in One Variable

(The Secant Method ,Newton-Raphson Method)

Lecture-2

Objective:

To find the root of a nonlinear equation in on variable with the help of secant methods and Newton-Raphson methods.

The Secant Method

In **Secant** method two values of x near the root is used and the root is approximated by the x-intercept of the secant line (chord) joining the two points.

Algorithm:

Step-1: Calculate the next estimated of the root from two initial guess $[x_n, x_{n+1}]$ using the following formula

$$x_{n+2} = x_{n+1} - \frac{(x_{n+1} - x_n)}{f(x_{n+1}) - f(x_n)} f(x_{n+1}) \qquad n \geq 1$$

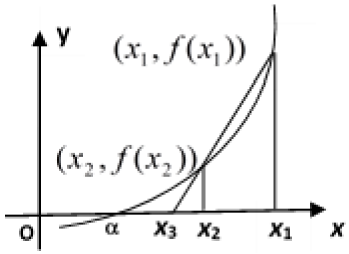

Fig.4.3 The Secant Method

Step-2: In selecting, x_1 and x_2 care should be taken so that , x_2 is closer to the root than x_1 to get rapid convergence. This can be achieved by selecting x_1 and x_2 such that $|f(x_2)| < |f(x_1)|$

Step-3: Repeat the process until $|x_{n+1} - x_n| \leq \varepsilon$, where ε is the specified accuracy.

Problems and Solutions

Example: Find the root of $f(x) = \cos x + 2\sin x + x^2$ using secant method initiating with $x_1 = 0$ and $x_2 = -0.1$ with tolerancy/accuracy $\varepsilon = 0.001$

Solution:

n	x_{n+2}	$f(x_{n+2})$	$\lvert x_{n+2} - x_{n+1} \rvert$
1	-0.5136	0.1522	0.4136
2	-0.6100	0.0457	0.0964
3	-0.6514	0.0065	0.0414
4	-0.6582	0.0013	0.0068
5	-0.6598	0.0006	0.0016
6	0.6595	0.0002	0.0003

Ex 1(a): $\boldsymbol{x^3 - 5x + 1 = 0}$; $[\mathbf{2, 3}]$ Apply secant method to estimate the root correct to 2 d.p. in the last interval acquired by using bisection method.

Solution: $f(x) = x^3 - 5x + 1$ **and the last interval [2.125, 2.25] (see the previous slide)**

$$\boldsymbol{x_0 = 2.125 \;\&\; x_1 = 2.25}$$

Now according to secant method $\boldsymbol{x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1)}$

$$\boldsymbol{x_2 = 2.25 - \frac{(2.25 - 2.125)}{f(2.25) - f(2.125)} f(2.25)}$$

$$\boldsymbol{x_2 = 2.25 - \frac{0.125}{1.14 - (-0.029)} 1.14 = 2.13}$$

Now similarly $\boldsymbol{x_3 = x_2 - \frac{(x_2 - x_1)}{f(x_2) - f(x_1)} f(x_2)}$**=2.13**$\boldsymbol{-\frac{(2.13 - 2.25)}{f(2.13) - f(2.25)} f(2.13)}$**=2.13**$\boldsymbol{-\frac{(-0.12)}{0.014 - 1.14} 0.014}$**=2.13**

Root 2.13

Exercise

Given the following polynomial equations and an interval.

a. $x^3 - 5x + 1 = 0$; $[2, 3]$,

b. $x^3 + x^2 - 2x - 5 = 0;\ [1,2], \mathrm{H.W.}$

c. $x^4 - 2x - 5 = 0;\ [0, 2]$,

d. $x^4 + x^2 - 80 = 0;\ [2.90, 2.92]$.

Apply secant method (2 steps x2 and x3) to estimate the root correct to 2 d.p. in the last interval acquired by using bisection method(2 steps).

Advantages and Drawbacks: The secant Method

Advantages:

- It converges faster.

Drawbacks:

- Division by zero.
- Root jumping.

Outcome

By applying Secant method, nonlinear equations in one variable can be solved to find roots (approximately) of the equation, And it converges faster than bisection method, although it has few drawbacks.

Newton-Raphson Method

In this method, the root of the equation $f(x) = 0$ is approximated by the x-intercept of the tangent line through a guess value x_0 .Newton-Raphson formula can be written as follows

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad , n = 0, 1, 2, 3, \cdot \cdot \cdot$$

Algorithm:

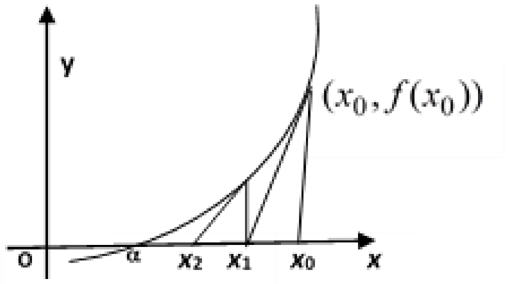

Fig.4.4 Newton-Raphson Method

Step-1: Calculate $f'(x)$ symbolically.

Step-2: Substitute $f(x)$ and $f'(x)$ in the formula.

Step-3: Choose a suitable starting value for x_0

Step-4: Repeat the process until $|x_{n+1} - x_n| \leq \varepsilon$, where ε is the specified accuracy.

Problems and Solutions

Example: Consider the function $f(x) = \sin x + 3x - 1$.

i. Perform one iteration using Newton-Raphson formula for finding its root near $x = 0$.

ii. Write MATLAB syntax for finding the root in $[0, 1]$ using MATLAB function “**fzero**”.

Solution:

i. $f(x) = \sin x + 3x - 1$; $x_0 = 0$

$$f'(x) = \cos x + 3$$

$$f(0) = -1, \quad f'(0) = 4$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{f(0)}{f'(0)} = -\frac{-1}{4} = 0.25$$

ii. MATLAB code:

```
>> F=@(x) sin(x)+3*x-1;
>> Sol = fzero(F, [0,1])

Sol =
   0.2507
```

Class work

Example: Consider the function $f(x) = x^3 + x^2 - 2x - 5;\ [1,2]$.

i. Perform one iteration using Newton-Raphson formula for finding its root.

ii. Write MATLAB syntax for finding the root in $[0, 1]$ using MATLAB function "**fzero**".

Solution:

i. $f(x) = x^3 + x^2 - 2x - 5;\ \ x_0 = 1$

$f'(x) =$

$$f(1) = -5, \quad f'(1) = 3$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{-5}{3} = 2.67$$

ii. MATLAB code:

```
>> F=@(x)
>> Sol = fzero(F, [])
```

Exercise

Given the following polynomial equations and an interval.

a. $x^3 - 5x + 1 = 0$; $[2, 3]$,

b. $x^3 + x^2 - 2x - 5 = 0$; $[1,2]$,

c. $x^4 - 2x - 5 = 0$; $[0, 2]$,

d. $x^4 + x^2 - 80 = 0$; $[2.90,2.92]$.

- ❑ Write down an iteration formula based on Newton-Raphson method.
- ❑ Perform one iteration starting using the above formula with a suitable value in the given interval to estimate the root to 2 d.p.
- ❑ Write down MATLAB codes to execute the iteration four times.

Advantages and Drawbacks: Newton-Raphson Method

Advantages:

- ❑ It converges faster.
- ❑ Requires only one guess.

Drawbacks:

- ❑ Division by zero.
- ❑ Root jumping.
- ❑ Inflection point issue.

Outcome

By applying Newton-Raphson method, nonlinear equations in one variable can be solved to find roots (approximately) of the equation, although it has few drawbacks.

1. What is the formula to find the root of the polynomial by Secant method?

a) $x_{n+1} = \dfrac{a_n + b_n}{2}$, **b)** $x_{n+2} = x_{n+1} - \dfrac{(x_{n+1} - x_n)}{f(x_{n+1}) - f(x_n)} f(x_{n+1})\Big)$, **c)** $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$ **,d) Neither**

2. The next iterative value of the root of $x^2 - 4 = 0$ using secant method, if the initial guesses are 3 and 4, is-

a) 2.2857 , b) 2.5000 , c) 5.5000 , d) 5.7143

3. What is the formula to find the root of the polynomial by Newton-Raphson method?

a) $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$, **b)** $x_{n+1} = \dfrac{a_n + b_n}{2}$ **, c)** $y - f(x_0) = f'(x_0)\,(x - x_0)$ **d) Neither**

4. What is the derivative of the function $f(x) = 2\cos 3x + 2 - x$?

a) $f'(x) = -6\sin 3x - 1$ **,b)** $f'(x) = -4\sin 2x - 1$ **, c)** $f'(x) = -\sin 2x - 1$ **d) Neither**

5. The next iterative value of the root of $x^2 - 4 = 0$ **using Newton Raphson method, if the initial guess is 3, is–**

a) 2.166 , b) 2.5000 , c) 5.5000 , d) 5.7143