Chapter-5

Coordinate systems

Cartesian coordinates:

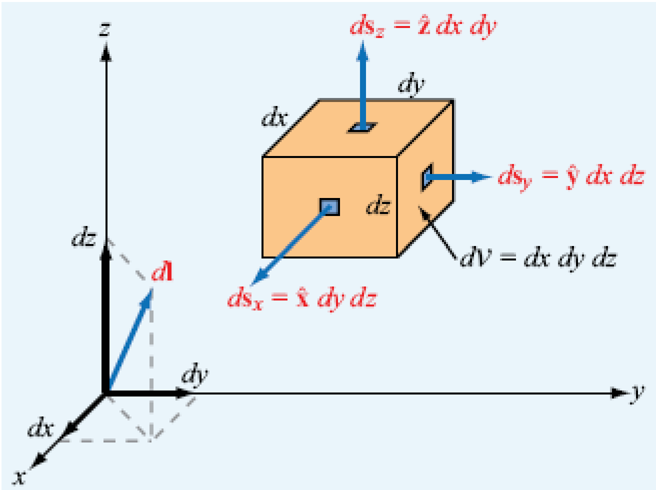

Figure: Differential length, area, and volume in Cartesian coordinates.

Cartesian variables x, y, z

Vector representation, $\vec{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$

Magnitude of $\vec{A}$ is , $|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

Position vector, $\overrightarrow{OP} = \hat{x}x_1 + \hat{y}y_1 + \hat{z}z_1$ for $P = (x_1, y_1, z_1)$

Base vector properties: $\hat{x}\cdot\hat{x} = \hat{y}\cdot\hat{y} = \hat{z}\cdot\hat{z} = 1$ and $\hat{x}\cdot\hat{y} = \hat{y}\cdot\hat{x} = \hat{z}\cdot\hat{x} = 0$

$\hat{x}\times\hat{y} = \hat{z}, \hat{y}\times\hat{z} = \hat{x}, \hat{z}\times\hat{x} = \hat{y}$

Dot product $\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$

Cross product $\vec{A}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{x}\left(A_yB_z - A_zB_y\right) - \hat{y}\left(A_xB_z - A_zB_x\right) + \hat{z}\left(A_xB_y - A_yB_x\right)$

Differential length, $d\vec{l} = \hat{x}\, dx + \hat{y}\, dy + \hat{z}\, dz$

Differential surface area, $d\vec{s}_x = \hat{x}\, dy\, dz,\ d\vec{s}_y = \hat{y}\, dz\, dx,\ d\vec{s}_z = \hat{z}\, dx\, dy$

Differential volume, $dV = dx\, dy\, dz$

Cylindrical coordinates:

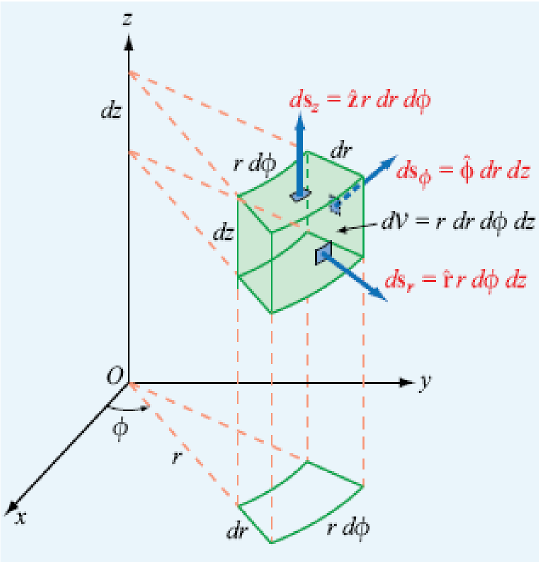

Figure: Differential length, area, and volume in cylindrical coordinates.

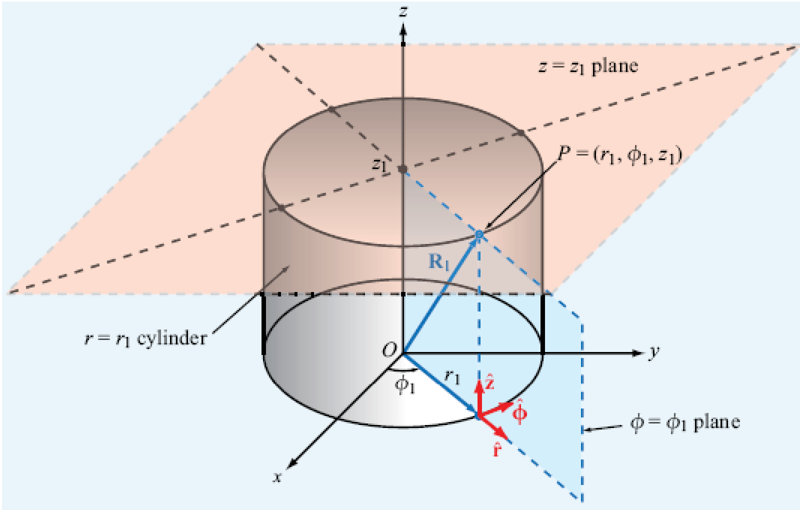

Figure: Point $P(r_1, \phi_1, z_1)$ in cylindrical coordinates; r_1 is the radial distance from the origin in the xy plane, ϕ_1 is the angle in xy plane measured from the x axis toward the y axis, and z_1 is the vertical distance from the xy plane.

Cylindrical variables r, ϕ, z

Vector representation, $\vec{A} = \hat{r}A_r + \hat{\phi}A_\phi + \hat{z}A_z$

Magnitude of $\vec{A}$ is , $|\vec{A}| = \sqrt{A_r^2 + A_\phi^2 + A_z^2}$

Position vector, $\overrightarrow{OP} = \hat{r}r_1 + \hat{z}z_1$ for $P = (r_1, \phi_1, z_1)$

Base vector properties: $\hat{r}\cdot\hat{r} = \hat{\phi}\cdot\hat{\phi} = \hat{z}\cdot\hat{z} = 1$ and $\hat{r}\cdot\hat{\phi} = \hat{\phi}\cdot\hat{z} = \hat{z}\cdot\hat{r} = 0$

$$\hat{r}\times\hat{\phi} = \hat{z}, \hat{\phi}\times\hat{z} = \hat{r}, \hat{z}\times\hat{r} = \hat{\phi}$$

Dot product $\vec{A}\cdot\vec{B} = A_r B_r + A_\phi B_\phi + A_z B_z$

Cross product

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{r} & \hat{\phi} & \hat{z} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix} = \hat{r}(A_\phi B_z - A_z B_\phi) - \hat{\phi}\,(A_r B_z - A_z B_r) + \hat{z}(A_r B_\phi - A_\phi B_r)$$

Differential length, $d\vec{l} = \hat{r}dr + \hat{\phi}\, r d\phi + \hat{z}\, dz$

Differential surface area, $d\vec{s}_r = \hat{r}\, r d\phi\, dz,\ d\vec{s}_\phi = \hat{\phi}\, dr dz,\ d\vec{s}_z = \hat{z}\, dr\, r d\phi$

Differential volume, $dV = dr\, rd\phi\, dz$

Spherical coordinates:

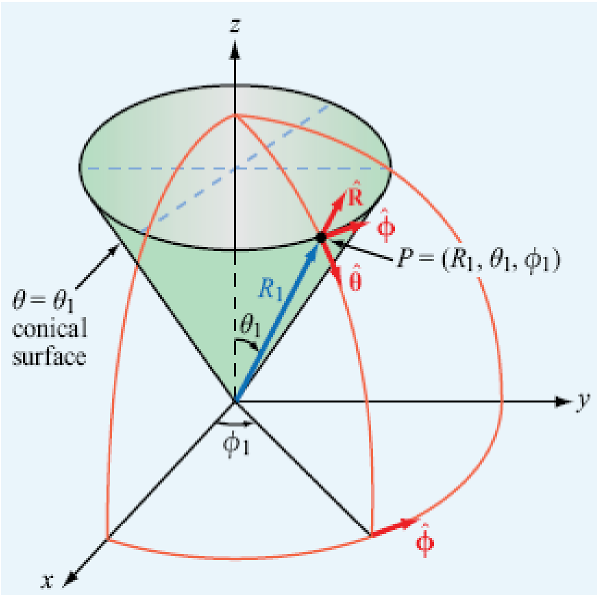

Figure: point $P(R_1, \theta_1, \phi_1)$ in spherical coordinates.

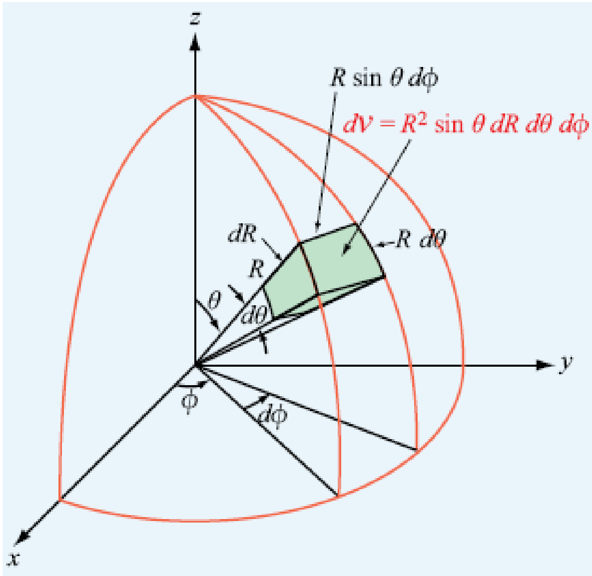

Figure: Differential volume in spherical coordinates.

Spherical variables R, θ, ϕ

Vector representation, $\vec{A} = \hat{R}A_r + \hat{\theta}A_\theta + \hat{\phi}A_\phi$

Magnitude of $\vec{A}$ is , $|\vec{A}| = \sqrt{A_R{}^2 + A_\theta{}^2 + A_\phi{}^2}$

Position vector, $\overrightarrow{OP} = \hat{R}R_1$ for $P = (R_1, \theta_1, \phi_1)$

Base vector properties: $\hat{R}\cdot\hat{R} = \hat{\theta}\cdot\hat{\theta} = \hat{\phi}\cdot\hat{\phi} = 1$ and $\hat{R}\cdot\hat{\theta} = \hat{\theta}\cdot\hat{\phi} = \hat{\phi}\cdot\hat{R} = 0$

$\hat{R}\times\hat{\theta} = \hat{\phi},\ \hat{\theta}\times\hat{\phi} = \hat{R}\ ,\hat{\phi}\times\hat{R} = \hat{\theta}$

Dot product $\vec{A}\cdot\vec{B} = A_R B_R + A_\theta B_\theta + A_\phi B_\phi$

Cross product

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{R} & \hat{\theta} & \hat{\phi} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix} = \hat{R}\left(A_\theta B_\phi - A_\phi B_\theta\right) - \hat{\theta}\left(A_R B_\phi - A_\phi B_R\right) + \hat{\phi}\left(A_R B_\theta - A_\theta B_R\right)$$

Differential length, $d\vec{l} = \hat{R}\, dR + \hat{\theta}\, R\, d\theta + \hat{\phi}\, R\sin\theta\ d\phi$

Differential surface area, $d\vec{s}_R = \hat{R}\ Rd\theta\ R\sin\theta\ \ d\phi,\ \ d\vec{s}_\theta = \hat{\theta}\ dR\ R\sin\theta\ \ d\phi,$

$$d\vec{s}_\phi = \hat{\phi}\ dR\ Rd\theta$$

Differential volume, $dV = dR\ R\ d\theta\ R\sin\theta\ d\phi$

Transformation (Cartesian coordinates to cylindrical and vice versa):

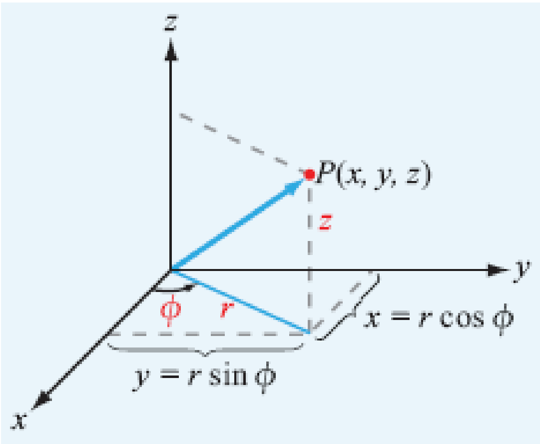

Figure: Interrelationships between Cartesian coordinates (x, y, z) and cylindrical coordinates(r, ϕ, z).

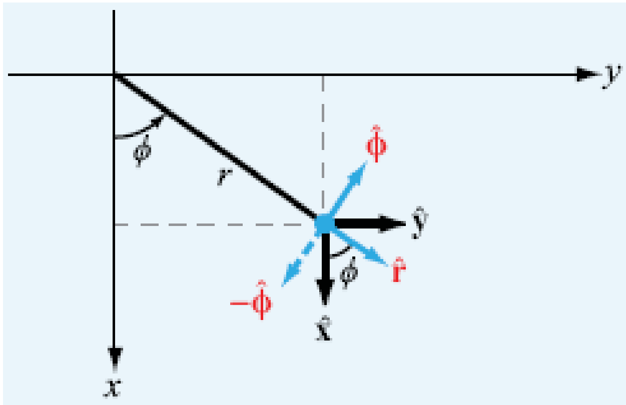

Figure: Interrelationships between base vectors $(\hat{x}, \hat{y})$ and $(\hat{r}, \hat{\phi})$.

We know, $x = r\cos\phi$, $y = r\sin\phi$ and z = z

$r = \sqrt{x^2 + y^2}$, $\phi = \arctan\left(\dfrac{y}{x}\right)$, $z = z$; $0 \le r < \infty$, $0 \le \phi < 2\pi$, $-\infty < z < \infty$

		A_r	A_ϕ	A_z
		$\hat{r}$	$\hat{\phi}$	$\hat{z}$
A_x	$\hat{x}$	$\cos\phi$	$-\sin\phi$	0
A_y	$\hat{y}$	$\sin\phi$	$\cos\phi$	0
A_z	$\hat{z}$	0	0	1

That is, $\hat{r} = \hat{x}\cos\phi + \hat{y}\sin\phi$, $\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$ and $\hat{z} = \hat{z}$.

$A_r = A_x\cos\phi + A_y\sin\phi$, $A_\phi = -A_x\sin\phi + A_y\cos\phi$ and $A_z = A_z$.

$\hat{x} = \hat{r}\cos\phi - \hat{\phi}\sin\phi$, $\hat{y} = \hat{r}\sin\phi + \hat{\phi}\cos\phi$, $\hat{z} = \hat{z}$

$A_x = A_r\cos\phi - A_\varphi\sin\phi$, $A_y = A_r\sin\phi + A_\phi\cos\phi$, $A_z = A_z$

Transformation (Spherical coordinates to cylindrical coordinates and vice versa):

We know, $r = R\sin\theta$, $z = R\cos\theta$ and $\phi = \phi$

$R = \sqrt{r^2 + z^2}$,$\theta = \arctan\left(\frac{r}{z}\right)$, $\phi = \phi$; $0 \le r < \infty$, $0 \le \phi < 2\pi$, $-\infty < z < \infty$

		A_R	A_θ	A_ϕ
		$\hat{R}$	$\hat{\theta}$	$\hat{\phi}$
A_r	$\hat{r}$	$\sin\theta$	$\cos\theta$	0
A_z	$\hat{z}$	$\cos\theta$	$-\sin\theta$	0
A_ϕ	$\hat{\phi}$	0	0	1

That is, $\hat{r} = \hat{R}\sin\theta + \hat{\theta}\cos\theta$, $\hat{z} = \hat{R}\cos\theta - \hat{\theta}\sin\theta$ and $\hat{\phi} = \hat{\phi}$.

$A_r = A_R\sin\theta + A_\theta\cos\theta$, $A_z = A_R\cos\theta - A_\theta\sin\theta$ and $A_\phi = A_\phi$.

$\hat{R} = \hat{r}\sin\theta - \hat{z}\cos\theta$, $\quad \hat{\theta} = \hat{r}\cos\theta - \hat{z}\sin\theta$, $\quad \hat{\phi} = \phi$

$A_R = A_r\sin\theta - A_z\cos\theta$, $\quad A_\theta = A_r\cos\theta - A_z\sin\theta$, $\quad A_\phi = A_\phi$

Transformation (spherical coordinates to Cartesian coordinates and vice versa):

We know, $x = R\sin\theta\cos\phi$, $y = R\sin\theta\,\sin\phi$ and $z = R\cos\theta$;

$R = \sqrt{x^2 + y^2 + z^2}$, $\theta = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right)$, $\phi = arc\ tan\left(\frac{y}{x}\right)$;

$0 \le r < \infty$, $0 \le \phi < 2\pi$, $-\infty < z < \infty$

		A_R	A_θ	A_ϕ
		$\hat{R}$	$\hat{\theta}$	$\hat{\phi}$
A_x	$\hat{x}$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
A_y	$\hat{y}$	$\sin\theta\sin\phi$	$\cos\theta\ \sin\phi$	$\cos\phi$
A_z	$\hat{z}$	$\cos\theta$	$-\sin\theta$	0

That is,

$\hat{x} = \hat{R}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi$, $\quad \hat{y} = \hat{R}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi$

and $\hat{z} = \hat{R}\cos\theta - \hat{\theta}\sin\theta$.

$A_x = A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$,

$A_y = A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$

and $A_z = A_R\cos\theta - A_\theta\sin\theta$..

$\hat{R} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$

$\hat{\theta} = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta$

$\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$

$A_R = A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z\cos\theta$

$A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$

$A_\phi = -A_x\sin\phi + A_y\cos\phi$

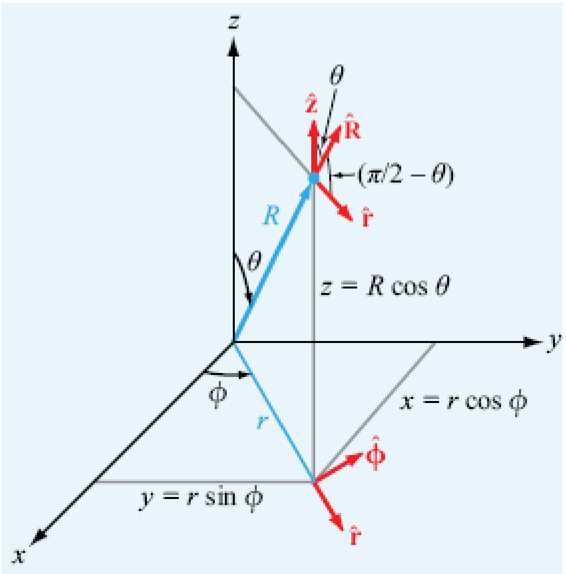

Figure: Interrelationships between (x, y, z) and (R, θ, ϕ)

Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r = \sqrt{x^2+y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_r = A_x\cos\phi + A_y\sin\phi$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = r\cos\phi$ $y = r\sin\phi$ $z = z$	$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_x = A_r\cos\phi - A_\phi\sin\phi$ $A_y = A_r\sin\phi + A_\phi\cos\phi$ $A_z = A_z$
Cartesian to spherical	$R = \sqrt{x^2+y^2+z^2}$ $\theta = \tan^{-1}[\sqrt{x^2+y^2}/z]$ $\phi = \tan^{-1}(y/x)$	$\hat{\mathbf{R}} = \hat{\mathbf{x}}\sin\theta\cos\phi$ $+ \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta\cos\phi$ $+ \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$	$A_R = A_x\sin\theta\cos\phi$ $+ A_y\sin\theta\sin\phi + A_z\cos\theta$ $A_\theta = A_x\cos\theta\cos\phi$ $+ A_y\cos\theta\sin\phi - A_z\sin\theta$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$
Spherical to Cartesian	$x = R\sin\theta\cos\phi$ $y = R\sin\theta\sin\phi$ $z = R\cos\theta$	$\hat{\mathbf{x}} = \hat{\mathbf{R}}\sin\theta\cos\phi$ $+ \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{R}}\sin\theta\sin\phi$ $+ \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_x = A_R\sin\theta\cos\phi$ $+ A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$ $A_y = A_R\sin\theta\sin\phi$ $+ A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$
Cylindrical to spherical	$R = \sqrt{r^2+z^2}$ $\theta = \tan^{-1}(r/z)$ $\phi = \phi$	$\hat{\mathbf{R}} = \hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$	$A_R = A_r\sin\theta + A_z\cos\theta$ $A_\theta = A_r\cos\theta - A_z\sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$r = R\sin\theta$ $\phi = \phi$ $z = R\cos\theta$	$\hat{\mathbf{r}} = \hat{\mathbf{R}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_r = A_R\sin\theta + A_\theta\cos\theta$ $A_\phi = A_\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$

Example 1: Transform $\left(\sqrt{2}, \frac{3\pi}{4}, 3\right)$ from cylindrical coordinates to Cartesian coordinates.

Solution: $x = r\cos\phi = \sqrt{2}\cos\frac{3\pi}{4} = -1,$

$y = r\sin\phi = \sqrt{2}\sin\frac{3\pi}{4} = 1,$

$z = z = 3$

So, Cartesian point $(x, y, z) = (-1,1,3)$

Example 2: Transform $(-1,1,3)$ from Cartesian coordinate to cylindrical coordinate.

Solution: $r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$$\phi = \tan^{-1}\left(\frac{1}{-1}\right) = \tan^{-1}(-1) = \frac{3\pi}{4}$$

$z = z = 3$

Example 3: Transform $\left(1,0,\sqrt{3}\right)$ from Cartesian coordinate to spherical coordinate.

Solution: $R = \sqrt{x^2 + y^2 + z^2} = \sqrt{1^2 + 0^2 + (\sqrt{3})^2} = \sqrt{4} = 2$

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right) = tan^{-1}\left(\frac{\sqrt{1^2 + 0^2}}{(\sqrt{3})^2}\right) = \frac{\pi}{6}$$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{0}{1} = 0$$

So, the spherical point $(R, \theta, \phi) = \left(2, \frac{\pi}{6}, 0\right)$

Example 4: Transform $\left(2, \frac{\pi}{6}, 0\right)$ from spherical coordinate to Cartesian coordinate.

Solution: $x = R\sin\theta\cos\phi = 2\ \sin\frac{\pi}{6}\ \cos 0 = 1$

$$y = R\sin\theta\sin\phi = 2\ \sin\frac{\pi}{6}\ \sin 0 = 0$$

$$z = R\cos\theta = 2\cos\frac{\pi}{6} = \sqrt{3}$$

So, the Cartesian point $(x, y, z) = \left(1,0,\sqrt{3}\right)$

Example 5: Transform $\left(2, \frac{\pi}{6}, 0\right)$ from spherical coordinate to cylindrical coordinate.

Solution: $r = R\sin\theta = 2\sin\frac{\pi}{6} = 1$

$$z = R\cos\theta = 2\cos\frac{\pi}{6} = \sqrt{3}$$

$$\phi = \phi = 0$$

So, the cylindrical pint $(r, z, \phi) = \left(1, \sqrt{3}, 0\right)$

Example 6: Transform $\left(1, \sqrt{3}, 0\right)$ from cylindrical coordinate to spherical coordinate.

Solution: $R = \sqrt{r^2 + z^2} = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$

$$\theta = \tan^{-1}\left(\frac{r}{z}\right) = tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$\phi = \phi = 0$$

So, the spherical point $(R, \theta, \phi) = \left(2, \frac{\pi}{6}, 0\right)$

Example 7: Express vector $\vec{A} = \hat{x}(x + y) + \hat{y}(y - x) + \hat{z}z$ in spherical coordinate.

Solution: We know in spherical coordinate

$$\vec{A} = \widehat{R}\,A_R + \hat{\theta}A_\theta + \hat{\phi}A_\phi$$

$$A_R = A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$$

$$= (x + y)\sin\theta\cos\phi + (y - x)\sin\theta\sin\phi + z\cos\theta$$

$$= (R\sin\theta\cos\phi + R\sin\theta\sin\phi)\sin\theta\cos\phi$$

$$+ (\,R\sin\theta\sin\phi - R\sin\theta\cos\phi)\sin\theta\sin\phi + \; R\cos\theta\cos\theta$$

$$= R\sin^2\theta\cos^2\phi + R\sin^2\theta\sin\phi\cos\phi + R\;\sin^2\theta\sin^2\phi - R\sin^2\theta\sin\phi\cos\phi$$

$$+ R\cos^2\theta$$

$$= R\sin^2\theta\,(\cos^2\phi + \sin^2\phi) + R\cos^2\theta$$

$$= R\sin^2\theta + R\cos^2\theta = R$$

$$A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$$

$$= (x + y)\cos\theta\cos\phi + (y - x)\cos\theta\sin\phi - z\sin\theta$$

$$= (R\sin\theta\cos\phi + R\sin\theta\sin\phi)\cos\theta\cos\phi$$

$$+ (\,R\sin\theta\sin\phi - R\sin\theta\cos\phi)\cos\theta\sin\phi \;\; - R\sin\theta\cos\theta$$

$$= R\sin\theta\cos\theta\cos^2\phi + R\sin\theta\cos\theta\sin\phi\cos\phi + R\sin\theta\cos\theta\sin^2\phi$$

$$- R\sin\theta\cos\theta\sin\phi\cos\phi - R\sin\theta\cos\theta$$

$$= \; R\sin\theta\cos\theta\,(\sin^2\phi + \cos^2\phi) - \; R\sin\theta\cos\theta$$

$$= 0$$

$A_\phi = -A_x \sin\phi + A_y \cos\phi$

$= -(x+y)\sin\phi + (y-x)\cos\phi$

$= -(R\sin\theta\cos\phi + R\sin\theta\sin\phi)\sin\phi + (R\sin\theta\sin\phi - R\sin\theta\cos\phi)\cos\phi$

$= -R\sin\theta\sin\phi\cos\phi - R\sin\theta\sin^2\phi + R\sin\theta\sin\phi\cos\phi - R\sin\theta\cos^2\phi$

$= -R\sin\theta(\sin^2\phi + \cos^2\phi) = -R\sin\theta$

$\therefore \vec{A} = \hat{R}\,R - \hat{\phi}\,R\sin\theta$

Example 8: Express vector $\vec{A} = \hat{R}\,R - \hat{\phi}\,R\sin\theta$ in Cartesian form.

Solution: We know in Cartesian coordinate $\vec{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$

$A_x = A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$

$= R\sin\theta\cos\phi + 0.\cos\theta\cos\phi + R\sin\theta\sin\phi$

$= x + y$

$A_y = A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$

$= R\sin\theta\sin\phi + 0.\cos\theta\sin\phi - R\sin\theta\cos\phi$

$= y - x$

$A_z = A_R\cos\theta - A_\theta\sin\theta$

$= R\cos\theta - 0.\sin\theta$

$= z$

$\therefore \vec{A} = \hat{x}(x+y) + \hat{y}(y-x) + \hat{z}z$

Example 9: Transform vector $\vec{A} = \hat{x}(x+y) + \hat{y}(y-x) + \hat{z}z$ to cylindrical coordinates.

Solution: We know, cylindrical coordinate $\vec{A} = \hat{r}A_r + \hat{\phi}A_\phi + \hat{z}A_z$

$A_r = A_x\cos\phi + A_y\sin\phi = (x+y)\cos\phi + (y-x)\sin\phi$

$= (r\cos\phi + r\sin\phi)\cos\phi + (r\sin\phi - r\cos\phi)\sin\phi$

$= r\cos^2\phi + r\sin\phi\cos\phi + r\sin^2\phi - r\sin\phi\cos\phi$

$= r(\cos^2\phi + \sin^2\phi) = r$

$A_\phi = -A_x\sin\phi + A_y\cos\phi$

$= -(x+y)\sin\phi + (y-x)\cos\phi$

$= -(r\cos\phi + r\sin\phi)\sin\phi + (r\sin\phi - r\cos\phi)\cos\phi$

$= -r\cos\phi\sin\phi - r\sin^2\phi + r\cos\phi\sin\phi - r\,cos^2\phi$

$= -r(\sin^2\phi + cos^2\phi)$

$= -r$

$A_z = z$

$\therefore \vec{A} = \hat{r}r - \hat{\phi}r + \hat{z}z$

Example 10: Express vector $\vec{A} = \hat{\mathrm{r}}\, r - \hat{\phi}\, \mathrm{r} + \hat{\mathrm{z}}\, z$ in Cartesian coordinate.

Solution: We know in Cartesian coordinate $\vec{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$

$A_x = A_r \cos\varphi - A_\varphi \sin\varphi$

$= r\cos\varphi + \mathrm{r}\sin\varphi$

$= x + y$

$A_y = A_r \sin\varphi + A_\varphi \cos\varphi$

$= r\sin\varphi - r\cos\varphi$

$= y - x$

$A_z = z$

Hence, $\therefore \vec{A} = \hat{\mathrm{x}}\,(x + y) + \hat{\mathrm{y}}\,(y - x\,) + \hat{\mathrm{z}}\,z$

Example 11: Transform the vector $\vec{A} = \hat{r}3\cos\phi - \hat{\phi}2r + \hat{z}z$ to spherical coordinate.

Solution: We know in spherical coordinate $\vec{A} = \hat{R}\,A_R + \hat{\theta}A_\theta + \hat{\phi}A_\phi$

$A_R = A_r \sin\theta + A_z \cos\theta$

$= 3\cos\phi\sin\theta + z\cos\theta = 3\cos\phi\sin\theta + R\cos\theta\cos\theta$

$A_\theta = A_r \cos\theta - A_z \sin\theta$

$= 3\cos\phi\ \cos\theta - z\sin\theta = 3\cos\phi\ \cos\theta - R\cos\theta\sin\theta$

$A_\phi = A_\phi = -2r = -2R\sin\theta$

$\therefore \vec{A} = \hat{R}\,(3\cos\phi\sin\theta + R\cos^2\theta) - \hat{\theta}2R\sin\theta + \hat{\phi}(3\cos\phi\ \cos\theta - R\cos\theta\sin\theta)$

Example 12: Transform the vector $\hat{A} = \hat{R}\cos\phi + \hat{\theta}\sin\phi + \hat{\phi}\sin^2\theta$ at the point $\left(3, \frac{\pi}{2}, \pi\right)$ to cylindrical coordinates.

Solution: Given that, $\hat{A} = \hat{R}\cos\phi + \hat{\theta}\sin\phi + \hat{\phi}\sin^2\theta$ at the point $\left(3, \frac{\pi}{2}, \pi\right)$.

Here, $R = 3$, $\theta = \frac{\pi}{2}$, $\phi = \pi$ then r =Rsin$\theta = 3$, $\phi = \pi$ and z =Rcos$\theta = 0$

We get, $\left(3, \frac{\pi}{2}, \pi\right) \to (3, \pi, 0)$

Now, from given equation $A_R = \cos\varphi = -1, A_\theta = \sin\varphi = 0,\ A_\varphi = \sin^2\theta = 1$

We know,

$\hat{A} = A_r\hat{r} + A_\phi\hat{\phi} + A_z\hat{z}$

$\therefore\ A_r = A_R\sin\theta + A_\theta\cos\theta = -1,\ A_\phi = A_\phi = 1$ and $A_z = A_R\cos\theta - A_\theta\sin\theta = 0$

Hence, $\hat{A} = -\hat{r} + \hat{\phi}$.

Example 13: Transform the vector $\vec{A} = \hat{r}\cos\phi - \hat{\phi}\sin\phi + \hat{z}\sin\phi\cos\phi$ at the point $\left(2, \frac{\pi}{4}, 2\right)$.

to spherical coordinates.

Solution: Given that , $\vec{A} = \hat{r}\cos\phi - \hat{\phi}\sin\phi + \hat{z}\sin\phi\cos\phi$ at the point $\left(2, \frac{\pi}{4}, 2\right)$.

Here, $A_r = \cos\phi,\ A_\phi = -\sin\phi,$ and $A_z = \cos\phi\sin\phi$ and r =2, $\phi = \frac{\pi}{4},\ z = 2$

We get, $R = \sqrt{r^2 + z^2} = \sqrt{4+4} = \sqrt{8},\quad \theta = \tan^{-1}\left(\frac{r}{z}\right) = \tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4},\quad \phi = \frac{\pi}{4}$

$\therefore (R, \theta, \phi) = \left(\sqrt{8}, \frac{\pi}{4}, \frac{\pi}{4}\right)$

We know, $\vec{A} = A_R\hat{R} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$

Here, $A_R = A_r\sin\theta + A_z\cos\theta$, $A_\theta = A_r\cos\theta - A_z\sin\theta$ and $A_\phi = A_\phi$.

$A_R = \cos\phi\sin\theta + \cos\phi\sin\phi\cos\theta$, $A_\theta = A_r\cos\phi\cos\theta - \cos\phi\sin\phi\sin\theta$ and $A_\phi = -\sin\phi$.

$\therefore\ \vec{A} = \left(\cos\phi\sin\theta + \cos\phi\sin\phi\cos\theta\right)\hat{R} + \left(A_r\cos\phi\cos\theta - \cos\phi\sin\phi\sin\theta\right)\hat{\theta} + \left(-\sin\phi\right)\hat{\phi}$

At the point $\left(\sqrt{8}, \frac{\pi}{4}, \frac{\pi}{4}\right)$ $\quad \vec{A} = .853\hat{R} + 0.1464\hat{\theta} - \frac{1}{\sqrt{2}}\hat{\phi}$.

Sample exercise-5.1

1. **Convert the coordinates of the following points from Cartesian to cylindrical and spherical coordinates:**

a. $p_1 = (-1, \sqrt{3}, -2\sqrt{3})$ **Ans:** $\left(2, \frac{2\pi}{3}, -2\sqrt{3}\right)$ and $\left(4, \frac{5\pi}{6}, \frac{2\pi}{3}\right)$.

b. $p_2 = (4, 0, -4)$ **Ans:** $(4, 0, -4)$ and $\left(4\sqrt{2}, \frac{3\pi}{4}, 0\right)$.

c. $p_3 = (\sqrt{8}, -\sqrt{8}, 4)$ **Ans:** $\left(4, \frac{7\pi}{4}, 4\right)$ and $\left(4\sqrt{2}, \frac{\pi}{4}, \frac{7\pi}{4}\right)$.

2. **Convert the coordinates of the following points from cylindrical to Cartesian and spherical coordinates:**

a. $p_1 = (2, \frac{2\pi}{3}, 2\sqrt{3})$ **Ans:** $(-1, \sqrt{3}, 2\sqrt{3})$ and $\left(4, \frac{\pi}{6}, \frac{2\pi}{3}\right)$.

b. $p_2 = (\sqrt{3}, 0, -1)$ **Ans:** $(\sqrt{3}, 0, -1)$ and $\left(2, \frac{2\pi}{3}, 0\right)$.

c. $p_3 = (4\sqrt{3}, \pi, -4)$ **Ans:** $(-4\sqrt{3}, 0, -4)$ and $\left(8, \frac{2\pi}{3}, \pi\right)$.

3. **Convert the coordinates of the following points from spherical to Cartesian and cylindrical coordinates:**

a. $p_1 = (3, \frac{3\pi}{4}, \frac{5\pi}{3})$ **Ans:** $\left(\frac{3\sqrt{2}}{4}, -\frac{3\sqrt{6}}{4}, \frac{-3}{\sqrt{2}}\right)$ and $\left(\frac{3}{\sqrt{2}}, \frac{5\pi}{3}, \frac{-3}{\sqrt{2}}\right)$.

b. $p_2 = (4, \frac{\pi}{2}, \frac{\pi}{4})$ **Ans:** $(2\sqrt{2}, 2\sqrt{2}, 0)$ and $\left(4, \frac{\pi}{4}, 0\right)$.

c. $p_3 = (1, \frac{\pi}{4}, \frac{\pi}{2})$ **Ans:** $\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{2}, \frac{1}{\sqrt{2}}\right)$.

4. Transform the following vectors into cylindrical coordinates at the indicated points:

a. $\boldsymbol{A} = \hat{\mathrm{x}}(x+y) + \hat{\mathrm{y}}(x+y) + \hat{\mathrm{z}}\,\mathrm{z}$ at $p = (4, 0, -4)$ **Ans:** $\boldsymbol{A} = 4\,\hat{\mathrm{r}} + 4\,\widehat{\phi} - 4\,\hat{\mathrm{z}}$.

b. $\boldsymbol{B} = \widehat{\mathrm{R}}\sin\theta + \hat{\theta}\cos\theta + \widehat{\phi}\cos^2\phi$ at $p = (2, \frac{\pi}{2}, \frac{\pi}{4})$ **Ans:** $\boldsymbol{B} = \hat{\mathrm{r}} + \frac{1}{2}\widehat{\phi}$.

5. Transform the following vectors into spherical coordinates at the indicated points:

a. $\boldsymbol{A} = \hat{\mathrm{x}}y - \hat{\mathrm{y}}x$ at $p = (1, -1, 0)$ **Ans:** $\boldsymbol{A} = -\sqrt{2}\,\widehat{\phi}$.

b. $\boldsymbol{B} = \hat{\mathrm{z}}\sin\phi$ at $p = (2, \frac{\pi}{4}, 2)$ **Ans:** $\boldsymbol{B} = \frac{1}{2}\widehat{\mathrm{R}} - \frac{1}{2}\hat{\theta}$.

6. Transform the following vectors into cartesian coordinates at the indicated points:

a. $\boldsymbol{A} = -\widehat{\phi}R\sin\theta$ at $p = (1, \frac{\pi}{2}, 0)$ **Ans:** $\boldsymbol{A} = -\hat{\mathrm{y}}$.

b. $\boldsymbol{B} = -\hat{\mathrm{r}}\cos\phi - \widehat{\phi}\sin\phi + \hat{\mathrm{z}}\,\mathrm{z}$ at $p = (2, \frac{2\pi}{3}, 2\sqrt{3})$ **Ans:** $\boldsymbol{B} = \frac{1}{2}\hat{\mathrm{x}} + \frac{\sqrt{3}}{2}\,\hat{\mathrm{y}} + 2\sqrt{3}\,\hat{\mathrm{z}}$.