Chapter - 6
Gradient, Divergence and Curl

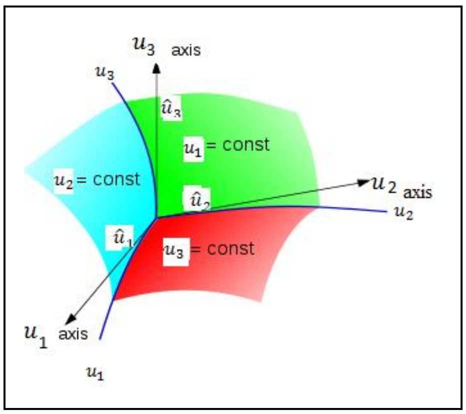
1. The Gradient Vector: grad

Consider a room in which the temperature is given by a scalar field, $T(u_1, u_2, u_3)$ at each point (u_1, u_2, u_3) (assume that the temperature does not change over time.). At each point in the room, the gradient of T at that point will show the direction in which the temperature rises most quickly. The magnitude of the gradient will determine how fast the temperature rises in that direction.

A vector field, called the gradient, written: gradT or $\mathbf{\nabla}$ T, where $\mathbf{\nabla}$ is called 'del', can be associated with a scalar field T.

At every point, the direction of the vector field ($\mathbf{\nabla}$ T) is

- orthogonal to the scalar field contour (C) and
- in the direction of the maximum rate of change of $T.\hat{u}_3$



The gradient of a scalar function $T(u_1, u_2, u_3)$ is given by

$$\text{gradT} = \mathbf{\nabla}\,\text{T} = \frac{\hat{u}_1}{h_1}\frac{\partial T}{\partial u_1} + \frac{\hat{u}_2}{h_2}\frac{\partial T}{\partial u_2} + \frac{\hat{u}_3}{h_3}\frac{\partial T}{\partial u_3}\cdot$$

where h_i are scale factors and $\hat{u}_i$ are the unit vectors along u_i, $(i = 1, 2, 3)$. For cartesian coordinates $u_1 = x, u_2 = y, u_3 = z, h_1 = h_2 = h_3 = 1$, for cylindrical coordinates $u_1 = r, u_2 = \varphi, u_3 = z, h_1 = h_3 = 1$ and $h_2 = r$ and for spherical coordinates $u_1 = R, u_2 = \theta,\ u_3 = \varphi, h_1 = 1, h_2 = R\&\ h_3 = R\sin\theta$.

Example 1.1. Find the gradient of the following scalar functions:

(a) $T(x, y, z) = \frac{xyz}{(x^2+y^2+z^2)}$ at the point $(1, 1,1)$.

For Cartesian coordinates, $\text{gradT} = \boldsymbol{\nabla}\, T = \hat{\mathrm{x}}\frac{\partial T}{\partial x} + \hat{\mathrm{y}}\frac{\partial T}{\partial y} + \hat{\mathrm{z}}\frac{\partial T}{\partial z}\cdot$

$$= \hat{\mathrm{x}}\frac{\partial}{\partial x}\left(\frac{xyz}{(x^2+y^2+z^2)}\right) + \hat{\mathrm{y}}\frac{\partial}{\partial y}\left(\frac{xyz}{(x^2+y^2+z^2)}\right) + \hat{\mathrm{z}}\frac{\partial}{\partial z}\left(\frac{xyz}{(x^2+y^2+z^2)}\right)$$

$$= \hat{\mathrm{x}}\frac{(x^2+y^2+z^2)yz - 2x^2yz}{(x^2+y^2+z^2)^2} + \hat{\mathrm{y}}\frac{(x^2+y^2+z^2)xz - 2y^2xz}{(x^2+y^2+z^2)^2} + \hat{\mathrm{z}}\frac{(x^2+y^2+z^2)xy - 2z^2xy}{(x^2+y^2+z^2)^2}$$

At the point $(1, 1,1), \boldsymbol{\nabla}\, T = \frac{1}{9}(\hat{\mathrm{x}} + \hat{\mathrm{y}} + \hat{\mathrm{z}})\cdot$

(b) $T(r, \phi, z) = \frac{z\cos\phi}{(1+r^2)}$, at the point $(1, \pi, 2)$

For cylindrical coordinates, $\text{gradT} = \boldsymbol{\nabla}\, T = \frac{\hat{\mathrm{r}}}{1}\frac{\partial T}{\partial r} + \frac{\hat{\phi}}{r}\frac{\partial T}{\partial \phi} + \frac{\hat{\mathrm{z}}}{1}\frac{\partial T}{\partial z}$

$$= \hat{\mathrm{r}}\frac{\partial}{\partial r}\left(\frac{z\cos\phi}{(1+r^2)}\right) + \hat{\phi}\frac{1}{\mathrm{r}}\frac{\partial}{\partial \phi}\left(\frac{z\cos\phi}{(1+r^2)}\right) + \hat{\mathrm{z}}\frac{\partial}{\partial z}\left(\frac{z\cos\phi}{(1+r^2)}\right)$$

$$= -\hat{\mathrm{r}}\frac{2zr\cos\phi}{(1+r^2)^2} - \hat{\phi}\frac{1}{\mathrm{r}}\frac{z\sin\phi}{(1+r^2)} + \hat{\mathrm{z}}\frac{\cos\phi}{(1+r^2)}\cdot$$

At the point $(1, \pi, 2), \boldsymbol{\nabla}\, T = \hat{\mathrm{r}} - \hat{\mathrm{z}}\frac{1}{2}\cdot$

(c) $T(R, \theta, \phi) = R\cos\theta\sin\phi$, at the point $\left(2, \frac{\pi}{2}, \frac{\pi}{4}\right)\cdot$

For cylindrical coordinates, $\text{gradT} = \boldsymbol{\nabla}\, T = \frac{\hat{\mathrm{R}}}{1}\frac{\partial T}{\partial R} + \frac{\hat{\theta}}{R}\frac{\partial T}{\partial \theta} + \frac{\hat{\phi}}{R\sin\theta}\frac{\partial T}{\partial \phi}$

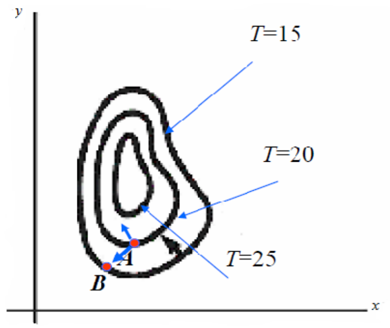
$$= \hat{\mathrm{R}}\frac{\partial}{\partial R}(R\cos\theta\sin\phi) + \frac{\hat{\theta}}{R}\frac{\partial}{\partial \theta}(R\cos\theta\sin\phi) + \frac{\hat{\phi}}{R\sin\theta}\frac{\partial}{\partial \phi}(R\cos\theta\sin\phi)$$

$$= \hat{\mathrm{R}}\cos\theta\sin\phi - \hat{\theta}\sin\theta\sin\phi + \frac{\hat{\phi}}{R\sin\theta}R\cos\theta\cos\phi.$$

At the point $\left(2, \frac{\pi}{2}, \frac{\pi}{4}\right)$, $\boldsymbol{\nabla}\, T = -\hat{\theta}\frac{1}{\sqrt{2}}$.

2. Directional Derivatives:

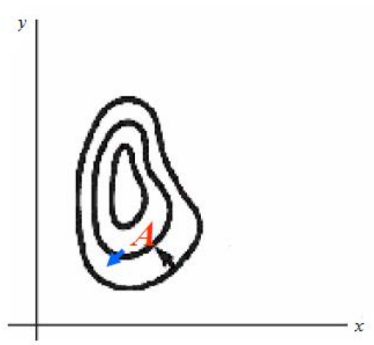
Consider the temperature T at various points of a heated metal plate. Some contours for T are shown in the following diagram.



We are interested in how T changes from one point to another. The rate of change of T in the direction specified by AB is given by $(20-15)/AB = 5/AB$, examples of **directional derivative.**

In general, for a given function $T = T(u_1, u_2)$, **the directional derivative** in the direction of a unit vector is the **gradient vector** at a point A

- magnitude = the largest directional derivative, and
- pointing in the direction in which this largest directional derivative occurs, is known as the **gradient vector**.



Hence, the component of ∇T in the direction of a vector **d** is equal to $\nabla T \cdot \mathbf{d}$ and it is called the directional derivative of T in the direction of **d**.

Example 2.1 Find the directional derivative of $T(x, y, z) = xy^2 - z^2$ at the point $(1, -1, 4)$ in the direction $\mathbf{d} = \hat{x} - \hat{y} + 4\,\hat{z}$.

Solution: $\nabla \mathrm{T} = \hat{x}\frac{\partial}{\partial x}(xy^2 - z^2) + \hat{y}\frac{\partial}{\partial y}(xy^2 - z^2) + \hat{z}\frac{\partial}{\partial z}(xy^2 - z^2)$

$$= \hat{x}\, y^2 + \hat{y}\, 2xy + \hat{z}(-2z)$$

At the point $(1,-1,4\,)$, $\boldsymbol{\nabla}\, T = \hat{\mathrm{x}} - 2\,\hat{\mathrm{y}} - 8\,\hat{\mathrm{z}}$

Now, the unit vector in the direction of $\hat{\mathrm{x}} - \hat{\mathrm{y}} + 4\,\hat{\mathrm{z}}$ is $\hat{\mathrm{a}} = \frac{\hat{\mathrm{x}}-\hat{\mathrm{y}}+4\,\hat{\mathrm{z}}}{\sqrt{1+1+16}} = \frac{1}{\sqrt{18}}\hat{\mathrm{x}} - \frac{1}{\sqrt{18}}\hat{\mathrm{y}} + \frac{4}{\sqrt{18}}\hat{\mathrm{z}}$

Then the required directional derivative is, $\boldsymbol{\nabla}\, T \cdot \hat{\mathrm{a}} = (\hat{\mathrm{x}} - 2\,\hat{\mathrm{y}} - 8\,\hat{\mathrm{z}}) \cdot \left(\frac{1}{\sqrt{18}}\hat{\mathrm{x}} - \frac{1}{\sqrt{18}}\hat{\mathrm{y}} + \frac{4}{\sqrt{18}}\hat{\mathrm{z}}\right)$

$$= \frac{1}{\sqrt{18}} + \frac{2}{\sqrt{18}} - \frac{32}{\sqrt{18}} = -\frac{29}{\sqrt{18}}.$$

Example 2.2 Find the directional derivative of $T(r,\phi,z) = \frac{1}{2}e^{-\frac{r}{5}}\cos\phi$ at the point $\left(2,\frac{\pi}{4},3\right)$ in the direction $\hat{\mathrm{r}}$.

Solution: $\boldsymbol{\nabla}\, T = \hat{\mathrm{r}}\frac{\partial}{\partial r}\left(\frac{1}{2}e^{-r/5}\cos\phi\right) + \hat{\phi}\frac{1}{\mathrm{r}}\frac{\partial}{\partial\phi}\left(\frac{1}{2}e^{-r/5}\cos\phi\right) + \hat{\mathrm{z}}\frac{\partial}{\partial z}\left(\frac{1}{2}e^{-r/5}\cos\phi\right)$

$$= -\hat{\mathrm{r}}\frac{1}{10}e^{-r/5}\cos\phi - \hat{\phi}\frac{1}{2\mathrm{r}}e^{-r/5}\sin\phi$$

At the point $\left(2,\frac{\pi}{4},3\,\right)$, $\boldsymbol{\nabla}\, T = -\hat{\mathrm{r}}\frac{1}{10\sqrt{2}}e^{-2/5} - \hat{\phi}\frac{1}{4\sqrt{2}}e^{-2/5}$

Then the required directional derivative is, $\boldsymbol{\nabla}\, T \cdot \hat{\mathrm{r}} = -\frac{1}{10\sqrt{2}}e^{-2/5}$.

Example 2.3 Find the directional derivative of $T(R,\theta,\phi) = \frac{1}{R}\sin^2\theta$ at the point $\left(5,\frac{\pi}{4},\frac{\pi}{2}\right)$ in the direction $\hat{\mathrm{R}}$.

Solution: $\boldsymbol{\nabla}\, T = \hat{\mathrm{R}}\frac{\partial}{\partial R}\left(\frac{1}{R}\sin^2\theta\right) + \frac{\hat{\theta}}{R}\frac{\partial}{\partial\theta}\left(\frac{1}{R}\sin^2\theta\right) + \frac{\hat{\phi}}{R\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{R}\sin^2\theta\right)$

$$= -\hat{\mathrm{R}}\frac{1}{R^2}\sin^2\theta + \hat{\theta}\frac{1}{R^2}2\sin\theta\cos\theta$$

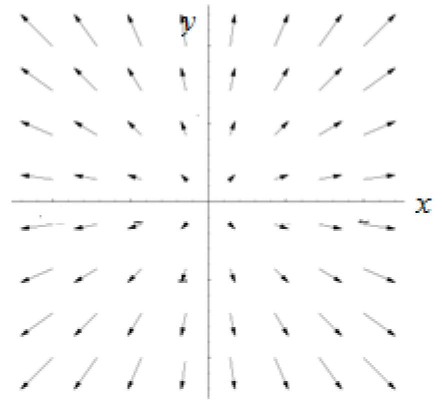
At the point $\left(5,\frac{\pi}{4},\frac{\pi}{2}\right)$, $\boldsymbol{\nabla}\, T = -\hat{\mathrm{R}}\frac{1}{50} + \hat{\theta}\frac{1}{25}$

Then the required directional derivative is, $\boldsymbol{\nabla}\, T \cdot \hat{\mathrm{R}} = -\frac{1}{50}$.

3. The divergence and curl of a vector function

Consider air as it is heated or cooled. The velocity of the air at each point defines a vector field. While air is heated in a region, it expands in all directions, and thus the velocity field points outward from that region. The divergence of the velocity field in that region would thus have a positive value. While the air is cooled and thus contracting, the divergence of the velocity has a negative value.

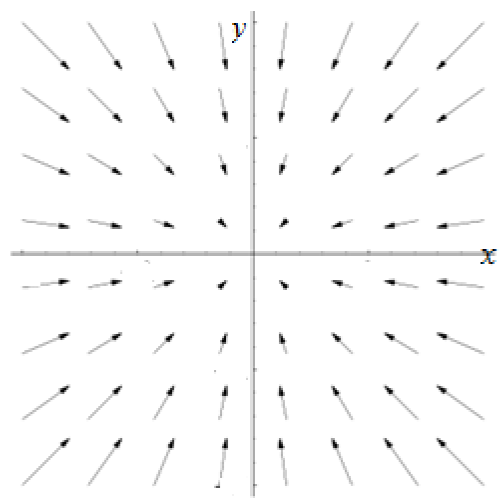
The divergence of a vector field is relatively easy to understand intuitively. Imagine that the vector field **A** pictured below gives the velocity of some fluid flow. It appears that the fluid is exploding outward from the origin.



$$\nabla \cdot \mathbf{A} > 0$$

This expansion of fluid flowing with velocity field $\boldsymbol{A}$ is captured by the divergence of $\boldsymbol{A}$, which we denote div $\boldsymbol{A}$ or mathematically $\nabla \cdot \mathbf{A}$. The divergence of the above vector field is positive since the flow is expanding.

In contrast, the below vector field represents fluid flowing so that it compresses as it moves toward the origin. Since this compression of fluid is the opposite of expansion, the divergence of this vector field is negative.



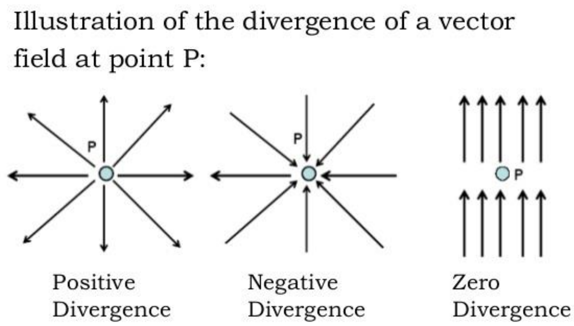
$$\nabla \cdot \mathbf{A} < 0$$

Lastly, a **solenoidal vector field** (also known as an **incompressible vector field**, a **divergence-free vector field**) is a vector field $\boldsymbol{A}$ with divergence zero at all points in the field. That is

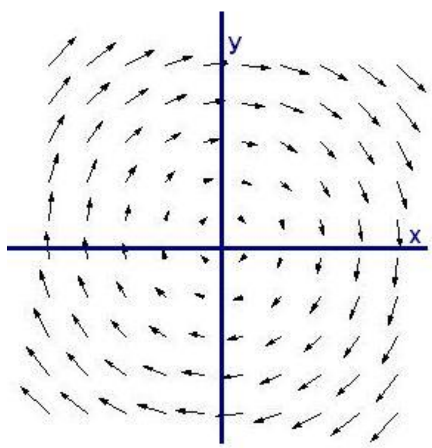
$$\nabla \cdot \mathbf{A} = \mathbf{0}$$

Hence the illustration of the divergence of a vector field at any point P is given below:

DIVERGENCE OF A VECTOR



The curl of a vector field captures the idea of how a flow may rotate. Imagine that the below vector field F represents fluid flow. The vector field indicates that the fluid is circulating around a central axis. This rotation of fluid flowing with velocity field $\boldsymbol{A}$ is captured by the curl of $\boldsymbol{A}$, which we denote curl$\mathbf{A}$ or mathematically $\boldsymbol{\nabla} \times \mathbf{A}$.



If curl$\mathbf{A} = 0$ then $\mathbf{A}$ is called **conservative or irrotational**.

Now, let $\mathbf{A} = \hat{\mathrm{u}}_1\, A_1(u_1, u_2, u_3) + \hat{\mathrm{u}}_2\, A_2(u_1, u_2, u_3) + \hat{\mathrm{u}}_3\, A_3(u_1, u_2, u_3)$ is a vector. The divergence and the curl is given by

$$\mathrm{div}\mathbf{A} = \boldsymbol{\nabla} \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\partial}{\partial u_2}(h_1 A_2 h_3) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right] \cdot$$

and $\mathrm{curl}\mathbf{A} = \nabla \times \mathbf{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} \hat{u}_1 h_1 & \hat{u}_2 h_2 & \hat{u}_3 h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$,

- The derivation can be found in advanced calculus books.

Example 3.1 Determine divergence and curl. Also check each of the following vector fields solenoidal, conservative or both.

(a) $\mathbf{A} = \hat{x}x^2 + \hat{y}\,2xy$

$$\mathrm{div}\mathbf{A} = \nabla \cdot \mathbf{A} = \left[\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(2xy)\right] = 2x + 2x = 4x.$$

$$\mathrm{curl}\mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 2xy & 0 \end{vmatrix} = \hat{z}\,2y.$$

∴ **A** is not solenoidal or conservative.

(b) $\mathbf{A} = \hat{r}\frac{\sin\phi}{r^2} + \hat{\phi}\frac{\cos\phi}{r^2}$

$$\mathrm{div}\mathbf{A} = \nabla \cdot \mathbf{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(A_1h_2h_3) + \frac{\partial}{\partial u_2}(h_1A_2h_3) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right]$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(\frac{\sin\phi}{r^2}\,r\right) + \frac{\partial}{\partial \phi}\left(\frac{\cos\phi}{r^2}\right) + \frac{\partial}{\partial z}(0 \cdot r)\right] = -\frac{\sin\phi}{r^3} - \frac{\sin\phi}{r^3} = -\frac{2\sin\phi}{r^3}.$$

$$\mathrm{curl}\mathbf{A} = \nabla \times \mathbf{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} \hat{u}_1 h_1 & \hat{u}_2 h_2 & \hat{u}_3 h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix} = \frac{1}{r}\begin{vmatrix} \hat{r} & \hat{\phi}\,r & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \frac{\sin\phi}{r^2} & \frac{\cos\phi}{r^2}r & 0 \end{vmatrix} = -\hat{z}\frac{2\cos\phi}{r^2}.$$

∴ **A** is not solenoidal or conservative.

(c) $\mathbf{A} = \hat{R}(Re^{-R})$

$$\mathrm{div}\mathbf{A} = \nabla \cdot \mathbf{A} = \frac{1}{R^2\sin\theta}\left[\frac{\partial}{\partial R}(Re^{-R} \cdot R^2\sin\theta) + \frac{\partial}{\partial \theta}(0 \cdot R\sin\theta) + \frac{\partial}{\partial \phi}(0 \cdot R)\right] = e^{-R}(3 - R).$$

$\nabla \times \mathbf{A} = \frac{1}{R^2\sin\theta}\begin{vmatrix} \hat{R} & \hat{\theta}\,R & \hat{\phi}\,R\sin\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ Re^{-R} & R \cdot 0 & R\sin\theta \cdot 0 \end{vmatrix} = 0.$ ∴ **A** is conservative but not solenoidal.

Example 3.2 Test whether $\mathbf{A} = \hat{\mathrm{x}}(y^2\cos x + z^3) + \hat{\mathrm{y}}(2y\sin x - y) + \hat{\mathrm{z}}(3xz^2 + 2)$is a conservative force field. If conservative, find the scalar potential Tsuch that $\mathbf{A} = \boldsymbol{\nabla} T$. Hence find the work done in moving an object in this field from $(0, 1, -1)$ to $\left(\frac{\pi}{2}, -1, 2\right)\cdot$

Solution: We know for conservative force field curl$\mathbf{A} = \boldsymbol{\nabla} \times \mathbf{A} = 0$

$$\boldsymbol{\nabla} \times \mathbf{A} = \begin{vmatrix} \hat{\mathrm{x}} & \hat{\mathrm{y}} & \hat{\mathrm{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2\cos x + z^3 & 2y\sin x - y & 3xz^2 + 2 \end{vmatrix} = 0$$

Hence **A** is conservative force field.

Let $T(x, y, z)$be a scalar potential of **A,** i.e. $\mathbf{A} = \boldsymbol{\nabla} T \because\therefore\; T(x, y, z) = \int \mathbf{A}\cdot d\mathbf{l}$

$$= \int [\hat{\mathrm{x}}(y^2\cos x + z^3) + \hat{\mathrm{y}}(2y\sin x - y) + \hat{\mathrm{z}}(3xz^2 + 2)].[\hat{\mathrm{x}}\,dx + \hat{\mathrm{y}}\,dy + \hat{\mathrm{z}}dz]$$

$$= \int [(y^2\cos x + z^3)dx + \;(2y\sin x - y)dy + (3xz^2 + 2)dz]$$

$= \int d\left(xz^3 + y^2\sin x + 2z - \frac{y^2}{2}\right) = xz^3 + y^2\sin x + 2z - \frac{y^2}{2} + c,\;\; c$ is a constant.

Now, work done, $= \int_{(0,1,-1)}^{\left(\frac{\pi}{2},-1,2\right)} \mathbf{A}\cdot d\mathbf{l} = \left[xz^3 + y^2\sin x + 2z - \frac{y^2}{2}\right]_{(0,1,-1)}^{\left(\frac{\pi}{2},-1,2\right)} = 4\pi + 7\cdot$

Example 3.3 Test whether $\mathbf{A} = \hat{\mathrm{r}}(\cos\phi) + \hat{\phi}(z\sin\phi)$is a conservative force field. If conservative, find the scalar potential Tsuch that $\mathbf{A} = \boldsymbol{\nabla} T$. Hence find the work done in moving an object in this field from $(0, \pi, -1)$ to $(1, 2\pi, 2)\cdot$

Solution: We know for conservative force field curl$\mathbf{A} = \boldsymbol{\nabla} \times \mathbf{A} = 0$

$$\boldsymbol{\nabla} \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathrm{r}} & \hat{\phi}\,\mathrm{r} & \hat{\mathrm{z}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ \cos\phi & -r\sin\phi & 0 \end{vmatrix} = 0$$

Hence **A** is conservative force field.

Let $T(r, \phi, z)$ be a scalar potential of **A,** i.e.$\mathbf{A} = \boldsymbol{\nabla} T \because\therefore\; T(r, \phi, z) = \int \mathbf{A}\cdot d\mathbf{l}$

$$= \int \left[\hat{\mathrm{r}}(\cos\phi) + \hat{\phi}(-\sin\phi)\right].\left[\hat{\mathrm{r}}\,dr + \hat{\phi}\,r\,d\phi + \hat{\mathrm{z}}dz\right]$$

$$= \int [\cos\phi\,dr - \;\mathrm{r}\sin\phi\,d\phi]$$

$= \int d(r\cos\phi) = r\cos\phi + c,\;\; c$ is a constant.

Now, work done,

$$\int_{(0,\pi,-1)}^{(1,2\pi,2)} \mathbf{A}\cdot d\mathbf{l} = [r\cos\phi]_{(0,\pi,-1)}^{(1,2\pi,2)} = 1\cdot$$

Example 3.4 Test whether $\mathbf{A} = \widehat{\mathrm{R}}(\sin\theta\cos\phi) + \hat{\theta}(\cos\theta\cos\phi) + \widehat{\phi}(-\sin\phi)$is a conservative force field. If conservative, find the scalar potential$T$such that $\mathbf{A} = \boldsymbol{\nabla}\, T$. Hence find the work done in moving an object in this field from $\left(0,\frac{\pi}{4},\frac{\pi}{3}\right)$ to $\left(1,\frac{\pi}{2},\pi\right)\cdot$

Solution: We know for conservative force field curl$\mathbf{A} = \boldsymbol{\nabla}\times\mathbf{A} = 0$

$$\boldsymbol{\nabla}\times\mathbf{A} = \frac{1}{R^2\sin\theta}\begin{vmatrix} \widehat{\mathrm{R}} & \hat{\theta}\,\mathrm{R} & \widehat{\phi}\,\mathrm{R}\sin\theta \\ \dfrac{\partial}{\partial R} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ \sin\theta\cos\phi & \mathrm{R}\cos\theta\cos\phi & -\mathrm{R}\sin\theta\sin\phi \end{vmatrix} = 0$$

Hence$\mathbf{A}$ is conservative force field.

Let$T(R,\theta,\phi)$be a scalar potential of $\mathbf{A}$**,** i.e.$\mathbf{A} = \boldsymbol{\nabla}\, T \because\therefore\ T(R,\theta,\phi) = \int \mathbf{A}\cdot d\mathbf{l}$

$$= \int \left[\widehat{\mathrm{R}}(\sin\theta\cos\phi) + \hat{\theta}(\cos\theta\cos\phi) + \widehat{\phi}(-\sin\phi)\right].\left[\widehat{\mathrm{R}}\,dR + \hat{\theta}\,R\,d\theta + \widehat{\phi}\,\mathrm{R}\sin\theta\,d\phi\right]$$

$$= \int [(\sin\theta\cos\phi)dR + (\cos\theta\cos\phi)R\,d\theta + (-\sin\phi)\mathrm{R}\sin\theta\,d\phi]$$

$= \int d(R\sin\theta\cos\phi) = R\sin\theta\cos\phi + c$, c is a constant.

Now, work done, $\int_{\left(0,\frac{\pi}{4},\frac{\pi}{3}\right)}^{\left(1,\frac{\pi}{2},\pi\right)} \mathbf{A}\cdot d\mathbf{l} = [R\sin\theta\cos\phi]_{\left(0,\frac{\pi}{4},\frac{\pi}{3}\right)}^{\left(1,\frac{\pi}{2},\pi\right)} = -1\cdot$

4. Laplacian operator

The Laplacian of a scalar function is defined as the divergence of the gradient of that function.

The differential operator $\boldsymbol{\nabla}^2 = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$ is called Laplacian operator, and

$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\mathrm{T}) = \boldsymbol{\nabla}^2 T(u_1,u_2,u_3) = 0$ is called Laplace Equation.

The Laplacian of a scalar function T in different coordinate system are defined as follows:

In Cartesian coordinates $\boldsymbol{\nabla}^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$,

in cylindrical coordinates $\boldsymbol{\nabla}^2 T = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{\partial^2 T}{\partial z^2}$ and

in Spherical coordinates $\boldsymbol{\nabla}^2 T = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial T}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\left(\frac{\partial^2 T}{\partial\phi^2}\right)\cdot$

Example 4.1 Find the Laplacian of the scalar function $\mathrm{T} = \frac{3}{\mathrm{x}^2+\mathrm{y}^2}$.

Solution: In Cartesian co-ordinates we know the Laplacian is

$$\boldsymbol{\nabla}^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

$$= \frac{\partial}{\partial x}(\frac{\partial}{\partial x}(3(x^2+y^2)^{-1})) + \frac{\partial}{\partial y}(\frac{\partial}{\partial y}(3(x^2+y^2)^{-1})) + \frac{\partial}{\partial z}(\frac{\partial}{\partial z}(3(x^2+y^2)^{-1}))$$

$$= \frac{\partial}{\partial x}(-3(x^2+y^2)^{-2}.2x) + \frac{\partial}{\partial y}(-3(x^2+y^2)^{-2}.2y) + 0$$

$$= 24x^2(x^2+y^2)^{-3} - 6(x^2+y^2)^{-2} + 24y^2(x^2+y^2)^{-3} - 6(x^2+y^2)^{-2}$$

$$= \frac{12}{(x^2+y^2)^2}$$

Example 4.2 Find the Laplacian of the scalar function $\mathrm{T} = 5\mathrm{e}^{-\mathrm{r}}\cos\phi$.

Solution: In Cylindrical coordinates we know the Laplacian is

$$\boldsymbol{\nabla}^2 T = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(5\mathrm{e}^{-\mathrm{r}}\cos\phi)\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(\frac{\partial}{\partial \phi}(5\mathrm{e}^{-\mathrm{r}}\cos\phi)\right) + \frac{\partial}{\partial \mathrm{z}}\left(\frac{\partial}{\partial \mathrm{z}}(5\mathrm{e}^{-\mathrm{r}}\cos\phi)\right)$$

$$= \frac{5\cos\phi}{r}\frac{\partial}{\partial r}(-r\,e^{-r}) + \frac{5\mathrm{e}^{-\mathrm{r}}}{r^2}\frac{\partial}{\partial \phi}(-sin\phi) + 0$$

$$= -\frac{5\cos\phi}{r}[-e^{-r} - r\,e^{-r}] - \frac{5\mathrm{e}^{-\mathrm{r}}}{r^2}cos\phi$$

$$= \frac{-5e^{-r}\cos\phi}{r} + 5\mathrm{e}^{-\mathrm{r}}\cos\phi - \frac{5\mathrm{e}^{-\mathrm{r}}}{r^2}cos\phi.$$

Example 4.3 Find the Laplacian of the scalar function $\mathrm{T} = 10\mathrm{e}^{-\mathrm{R}}\sin\theta$.

Solution: In Spherical coordinates we know the Laplacian is,

$$\boldsymbol{\nabla}^2 T = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial T}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\left(\frac{\partial^2 T}{\partial \phi^2}\right).$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}(10\mathrm{e}^{-\mathrm{R}}\sin\theta)\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}(10\mathrm{e}^{-\mathrm{R}}\sin\theta)\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial}{\partial\phi}(10\mathrm{e}^{-\mathrm{R}}\sin\theta)\right)$$

$$= \frac{10\sin\theta}{R^2}\frac{\partial}{\partial R}(-R^2e^{-R}) + \frac{10\mathrm{e}^{-\mathrm{R}}}{R^2\sin\theta}\frac{\partial}{\partial\theta}(sin\theta cos\theta) + 0$$

$$= -\frac{10\sin\theta}{R^2}\frac{\partial}{\partial R}(-R^2e^{-R}) + \frac{10\mathrm{e}^{-\mathrm{R}}}{R^2\sin\theta}\frac{\partial}{\partial\theta}(sin\theta cos\theta) + 0$$

$$= -\frac{10\sin\theta}{R^2}[2Re^{-R} - R^2e^{-R}] - \frac{10\mathrm{e}^{-\mathrm{R}}}{2R^2\sin\theta}\frac{\partial}{\partial\theta}(sin2\theta)$$

$$= -\frac{10\sin\theta}{R^2}[2Re^{-R} - R^2e^{-R}] - \frac{10\mathrm{e}^{-\mathrm{R}}}{R^2\sin\theta}cos2\theta.$$

Sample exercise - 6.1

1. Find the gradient of the following scalar functions at the indicated point:

(a) $T(x,y,z) = 2x^3y\,z + \mathrm{y}^2\,x^2 - 5\,\frac{y}{z}$ at the point $(0,2,-1)$. · Ans: $\boldsymbol{\nabla}\mathrm{T} = 5\,\hat{\mathrm{y}} + 10\hat{\mathrm{z}}$
(b) $T(r,\phi,z) = \frac{z+\sin\phi}{r}$, at the point $\left(2,\frac{3\pi}{2},1\right)$· Ans: $\boldsymbol{\nabla}\mathrm{T} = \frac{1}{2}\,\hat{\mathrm{z}}$
(c) $T(R,\theta,\phi) = R^2\cos\phi\sin\theta$, at the point $\left(2,\frac{\pi}{4},\frac{2\pi}{3}\right)$·
Ans: $\boldsymbol{\nabla}\mathrm{T} = -\sqrt{2}\,\widehat{\mathrm{R}} - \frac{1}{\sqrt{2}}\hat{\theta} - \sqrt{3}\,\hat{\phi}$

2. (a) Find the directional derivative (D. D.) of $T(x,y,z) = x^2y - xz$ at the point $(1,0,2)$in the direction $\mathbf{d} = \hat{\mathrm{x}} - 2\hat{\mathrm{y}} - 6\,\hat{\mathrm{z}}$. Ans: $\boldsymbol{\nabla}\mathrm{T} = \hat{\mathrm{x}}\,(2\mathrm{xy} - \mathrm{z}) + \hat{\mathrm{y}}\,\mathrm{x}^2 - \hat{\mathrm{z}}\,x$ and D. D. is $\frac{2}{\sqrt{41}}$·

(b) Find the directional derivative (D. D.) of $T(r,\phi,z) = r^3\cos\phi$ at the point $\left(2,\frac{\pi}{4},1\right)$ in the direction $\hat{\mathrm{r}}$. Ans: D.D. is $\frac{12}{\sqrt{2}}$·

(c) Find the directional derivative (D. D.) of $T(R,\theta,\phi) = \frac{1}{R}\cos^2\theta$ at the point $\left(1,\frac{\pi}{4},\frac{\pi}{2}\right)$ in the direction $\widehat{\mathrm{R}} - \hat{\theta}$. Ans: D. D. is $-\frac{1}{2\sqrt{2}} + \frac{1}{\sqrt{2}}$·

3. Determine divergence and curl. Also check each of the following vector fields solenoidal, conservative or both.

(a) $\mathbf{A} = \hat{x}\, zy^3 + \hat{y}\, 2y\sin(xy) + \hat{z}\, 3x^2 \ln z$

Ans: $\nabla . \mathbf{A} = \frac{3}{z}x^2 + 2\sin xy + 2xy\cos xy$ and $\nabla \times \mathbf{A} = \hat{y}\,(y^3 - 6x\,\mathrm{ln z}) + \hat{z}\,(2y^2\cos xy - 3y^2 z)\cdot$

(b) $\mathbf{A} = \hat{r}\frac{\sin\phi}{r} + \hat{\phi}\,\frac{\cos\phi}{r^2}$ Ans: $\nabla . \mathbf{A} = -\frac{\sin\phi}{r^3}$ and $\nabla \times \mathbf{A} = -\hat{z}\frac{\cos\phi}{\mathrm{r}}(\frac{1}{r^2} + \frac{1}{r})\cdot$

(c) $\mathbf{A} = \hat{R}\cos\theta + \hat{\theta}(R - \sin\theta)$ Ans: $\nabla . \mathbf{A} = \cot\theta$ and $\nabla \times \mathbf{A} = \hat{\phi}\, 2$.

4. (a) Test whether $\mathbf{A} = \hat{x}\,(\sin y + 1) + \hat{y}\,(2yz + x\cos y\,) + \hat{z}\,(y^2 - 3)$ is a conservative force field. If conservative, find the scalar potential Tsuch that $\mathbf{A} = \nabla$ T. Hence find the work done in moving an object in this field from $(1, -1, 5)$ to $\left(2, \frac{\pi}{2}, 1\right)\cdot$ Ans: $T = x\sin y + y^2 z + x - 3z + c$

(b) Test whether $\mathbf{A} = \hat{r}\,(2rz - \cos\phi) + \hat{\phi}\,(\sin\phi) + \hat{z}\,(r^2)$ is a conservative force field. If conservative, find the scalar potential Tsuch that $\mathbf{A} = \nabla$ T. Hence find the work done in moving an object in this field from $(1, 0, -1)$ to $(1, \pi, 0)\cdot$ Ans: $T = \mathrm{r}^2\mathrm{z} - \mathrm{r}\cos\phi + c$

(c) Test whether $\mathbf{A} = \hat{R}(\cos\theta) + \hat{\theta}(-\sin\theta)$ is a conservative force field. If conservative, find the scalar potential Tsuch that $\mathbf{A} = \nabla$ T. Hence find the work done in moving an object in this field from $\left(0, \frac{\pi}{4}, \frac{\pi}{3}\right)$ to $\left(1, \frac{\pi}{2}, 0\right)\cdot$ Ans: $T = R\cos\theta + c$

5. Find the Laplacian of the following scalar functions:

(a) $\mathrm{T} = 4y^2z^2\cdot$ Ans: $8(y^2 + z^2)$

(b) $\mathrm{T} = xy + zx\cdot$ Ans: $0\cdot$

(c) $\mathrm{T} = 10\mathrm{r}^3\cos 2\phi\cdot$ Ans: $50r\cos 2\phi\cdot$

(d) $\mathrm{T} = \frac{2}{R^3}\cos\theta\sin\phi\cdot$ Ans: $\frac{2}{R^5}\cos\theta\sin\phi\,(4 - \csc^2\theta)\cdot$

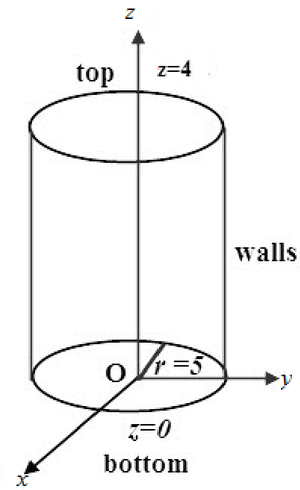
MATLAB for Gradient, divergence and curl

Find the gradient of the following scalar functions $T(x, y, z) = \frac{3}{(x^2+z^2)}\cdot$	>>syms x y z >>gradient(3/(x^2 + z^2), [x, y, z])
Determine divergence of $\mathbf{A} = \hat{x}\, x^2 + \hat{y}\, 2xy$	>>syms x y z >>divergence([x^2, 2*y^2, 0], [x, y, z])
Determine curl of $\mathbf{A} = \hat{x}\, x + \hat{y}\, 2y^2 + \hat{z}\, 3z^3$	>>syms x y z >>curl([x, 2*y^2, 3*z^3], [x, y, z])
Find the Laplacian of $\mathrm{T} = 4\mathrm{xy}^2\mathrm{z}^3$.	>> syms x y z >> laplacian(4*x*y^2*z^3)

Gauss's-Divergence Theorem and Stoke's Theorem

1. Some Prerequisite Example

Example 1.1. For a vector function $\mathbf{A} = \hat{\mathrm{r}}r^2 + \hat{\mathrm{z}}\, 2z$, find surface integral for the circular cylindrical region enclosed by $r = 5, z = 0, z = 4$.



Solution

$$\oint_s \mathbf{A}\cdot d\mathbf{s} = \int_{\substack{\text{top}\\\text{face}}} \mathbf{A}\cdot d\mathbf{s} + \int_{\substack{\text{bottom}\\\text{face}}} \mathbf{A}\cdot d\mathbf{s} + \int_{\substack{\text{walls}\\\text{face}}} \mathbf{A}\cdot d\mathbf{s}$$

i. Top face: $z = 4,\ \mathbf{A} = \hat{\mathrm{r}}r^2 + \hat{\mathrm{z}}\, 8$and $d\mathbf{s} = \hat{\mathrm{z}}\, rdrd\phi$

$$\therefore \int_{\substack{\text{top}\\\text{face}}} \mathbf{A}\cdot d\mathbf{s} = 8\int_0^{2\pi}\int_0^5 rdrd\phi = 200\pi.$$

ii. Bottom face: $z = 0,\ \mathbf{A} = \hat{\mathrm{r}}r^2$ and $d\boldsymbol{s} = -\hat{\mathrm{z}}\, rdrd\phi \therefore \int_{\substack{\text{bottom}\\\text{face}}} \mathbf{A}\cdot d\mathbf{s} = 0$

iii. Walls face: $r = 5,\ \mathbf{A} = \hat{\mathrm{r}}\, 25 + \hat{\mathrm{z}}2z$ and $d\boldsymbol{s} = \hat{\mathrm{r}}\, rdzd\phi$

$$\therefore \int_{\substack{\text{walls}\\\text{face}}} \mathbf{A}\cdot d\mathbf{s} = 25\int_0^{2\pi}\int_0^4 5dzd\phi = 1000\pi.$$

Total, $\oint_s \ \mathbf{A}\cdot d\mathbf{s} = 1200\pi$

Example 1.2. Evaluate $\iiint_E 16\, z\, dV$ where E is the upper half of the sphere

$x^2 + y^2 + z^2 = 1.$

Solution
Since we are taking the upper half of the sphere the limits for the variables are,

$$0 \le R \le 1$$
$$0 \le \theta \le \frac{\pi}{2}$$
$$0 \le \varphi \le 2\pi$$

The integral is then,

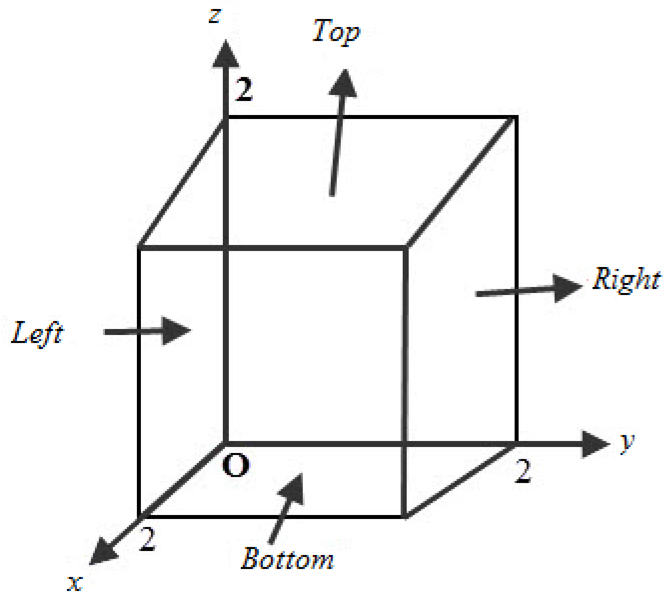
$$\begin{aligned}
\iiint_E 16z\, dV &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{1} R^2 \sin\theta (16R\cos\theta)\, dR d\varphi\, d\theta \\
&= \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{1} 8R^3 \sin 2\theta\, dR\, d\varphi d\theta \\
&= \int_0^{\frac{\pi}{2}}\int_0^{2\pi} 2\sin 2\theta d\varphi\, d\theta \\
&= \int_0^{\frac{\pi}{2}} 4\pi \sin 2\theta\, d\theta \\
&= -2\pi \cos 2\theta \Big|_0^{\frac{\pi}{2}} = 4\pi.
\end{aligned}$$

2. Gauss's divergence theorem

Statement: The surface integral of the normal component of a vector function **A** taken arround a closed surfce S is equal to the integral of the divergence of **A** taken over the volume V enclosed by the surface S.

Mathematically, $\int_V \nabla \cdot \mathbf{A}\, dv = \oint_S \mathbf{A} \cdot d\mathbf{s}$

Example 2.1. For the vector field $\mathbf{A} = \hat{x}\, xz - \hat{y}\, yz^2 - \hat{z}\, xy$, verify the divergence theorem by computing **(a)** the total outward flux flowing through the surface of a cube centered at the origin and with sides equal to 2 units each and parallel to the Cartesian axes, **(b)** the integral of $\nabla \cdot \mathbf{A}$ over the cube's volume.



Solution

We first evaluate the surface integral over the six faces.

i. Front face: $x = 2, d\mathbf{s} = \hat{x}\, dydz \therefore \int_{\substack{\text{front}\\ \text{face}}} \mathbf{A} \cdot d\mathbf{s} = \int_0^2 \int_0^2 xzdydz = 8.$

ii. Back face: $x = 0, d\mathbf{s} = -\hat{x}\, dydz \therefore \int_{\substack{\text{back}\\ \text{face}}} \mathbf{A} \cdot d\mathbf{s} = 0.$

iii. Right face: $y = 2, d\mathbf{s} = \hat{y}\, dxdz \therefore \int_{\substack{\text{right}\\ \text{face}}} \mathbf{A} \cdot d\mathbf{s} = \int_0^2 \int_0^2 -yz^2dxdz = -\frac{32}{3}.$

iv. Left face: $y = 0, d\mathbf{s} = -\hat{y}\, dxdz \therefore \int_{\substack{\text{left}\\ \text{face}}} \mathbf{A} \cdot d\mathbf{s} = 0.$

v. Top face: $z = 2, d\mathbf{s} = \hat{z}\, dxdy \therefore \int_{\substack{\text{top}\\ \text{face}}} \mathbf{A} \cdot d\mathbf{s} = \int_0^2 \int_0^2 -xydydx = -4.$

vi. Bottom face: $z = 0, d\mathbf{s} = -\hat{z}\, dxdy \therefore \int_{\substack{\text{bottom}\\ \text{face}}} \mathbf{A} \cdot d\mathbf{s} = \int_0^2 \int_0^2 xydydx = 4.$

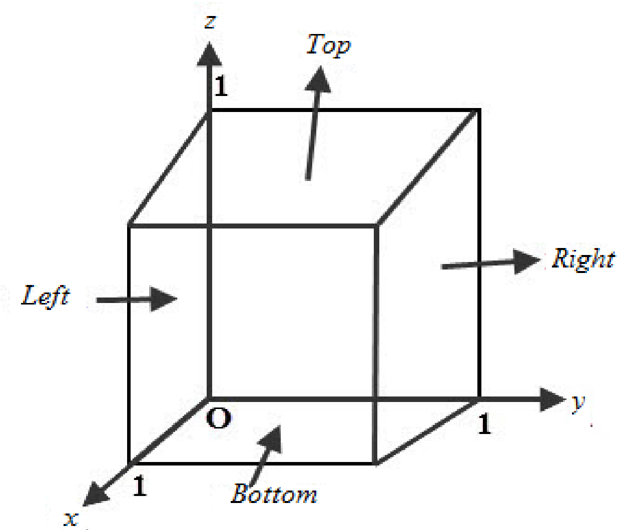
Adding the above six values, we have $\oint_S \mathbf{A} \cdot d\mathbf{s} = 8 + 0 - \frac{32}{3} + 0 - 4 + 4 = -\frac{8}{3}\cdot$

Now the divergence of **A** is $\frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(-yz^2) + \frac{\partial}{\partial z}(-xy) = z - z^2.$

Hence $\int_V \nabla \cdot \mathbf{A}\, dv = \int_0^2 \int_0^2 \int_0^2 (z - z^2)dx\, dy\, dz = -\frac{8}{3}\cdot$

which is the same as the result of the closed surface integral. The divergence theorem is therefore verified.

Example 2.2. Given $A = \hat{x}\, x^2 + \hat{y}\, xy + \hat{z}\, yz$,verify the divergence theorem over a cubeone unit on each side. The cube is situated in the first octant of the Cartesian coordinate system with one corner at the origin.



Solution

We first evaluate the surface integral over the six faces.

i. Front face: $x = 1, d\mathbf{s} = \hat{x}\, dydz$

$$\int_{\substack{\text{front}\\ \text{face}}} \mathbf{A} \cdot d\mathbf{s} = \int_0^1 \int_0^1 dydz = 1.$$

ii. Back face: $x = 0, d\mathbf{s} = -\hat{x}\, dydz$

$$\int_{\substack{\text{back}\\ \text{face}}} \mathbf{A} \cdot d\mathbf{s} = 0.$$

iii. Right face: $y = 1, d\mathbf{s} = \hat{y}\, dxdz$

$$\int_{\substack{\text{right}\\ \text{face}}} \mathbf{A} \cdot d\mathbf{s} = \int_0^1 \int_0^1 x\, dxdz = \frac{1}{2}.$$

iv. Left face: $y = 0, d\mathbf{s} = -\hat{y}\, dxdz$

$$\int_{\substack{\text{left}\\ \text{face}}} \mathbf{A} \cdot d\mathbf{s} = 0.$$

v. Top face: $z = 1, d\mathbf{s} = \hat{z} dxdy$

$$\int_{\substack{\text{top}\\ \text{face}}} \mathbf{A} \cdot d\mathbf{s} = \int_0^1 \int_0^1 y\, dydx = \frac{1}{2}.$$

vi. Bottom face: $z = 0, d\mathbf{s} = -\hat{z}\, dxdy$

$$\int_{\substack{\text{bottom}\\ \text{face}}} \mathbf{A} \cdot d\mathbf{s} = 0$$

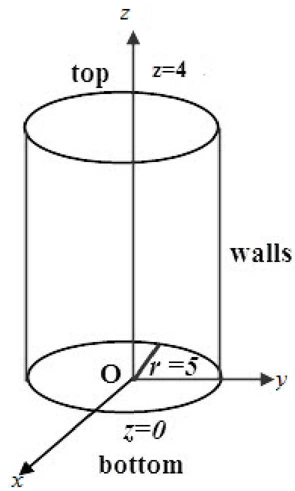
Adding the above six values, we have $\oint_S \mathbf{A} \cdot d\mathbf{s} = 1 + 0 + \frac{1}{2} + 0 + \frac{1}{2} + 0 = 2.$

Now the divergence of **A** is $\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(yz) = 3x + y.$

Hence $\int_V \nabla \cdot \mathbf{A}\, dv = \int_0^1 \int_0^1 \int_0^1 (3x + y) dx\, dy\, dz = 2.$

which is the same as the result of the closed surface integral. The divergence theorem is therefore verified.

Example 2.3. For a vector function $\mathbf{A} = \hat{r}r^2 + \hat{z}\,2z$, verify for the circular cylindrical region enclosed by $r = 5, z = 0, z = 4$.



Solution

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial}{\partial \phi}(A_\phi) + \frac{1}{r}\frac{\partial}{\partial z}(rA_z)$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(rr^2) + 0 + \frac{1}{r}\frac{\partial}{\partial z}(r2z) = \frac{1}{r}\frac{\partial}{\partial r}(r^3) + 2 = (3r+2)$$

$$\int_V \nabla \cdot \mathbf{A}\, dv = \int_{z=0}^{4}\int_{\varphi=0}^{2\pi}\int_{r=0}^{5}(3r+2)\, r dr d\phi dz = 1200\pi.$$

$$\oint_s \mathbf{A}\cdot d\mathbf{s} = \int_{\substack{\text{top}\\\text{face}}} \mathbf{A}\cdot d\mathbf{s} + \int_{\substack{\text{bottom}\\\text{face}}} \mathbf{A}\cdot d\mathbf{s} + \int_{\substack{\text{walls}\\\text{face}}} \mathbf{A}\cdot d\mathbf{s}$$

i. Top face: $z = 4$, $\mathbf{A} = \hat{r}r^2 + \hat{z}\,8$and $d\mathbf{s} = \hat{z}\, r dr d\phi$

$$\therefore \int_{\substack{\text{top}\\\text{face}}} \mathbf{A}\cdot d\mathbf{s} = 8\int_0^{2\pi}\int_0^{5} r dr d\phi = 200\pi.$$

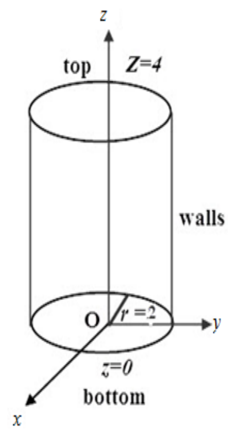
ii. Bottom face: $z = 0$, $\mathbf{A} = \hat{r}r^2$ and $d\boldsymbol{s} = -\hat{z}\, r dr d\phi \therefore \int_{\substack{\text{bottom}\\\text{face}}} \mathbf{A}\cdot d\mathbf{s} = 0.$

iii. Walls face: $r = 5$, $\mathbf{A} = \hat{r}\,25 + \hat{z}2z$ and $d\boldsymbol{s} = \hat{r}\, r dz d\phi$

$$\therefore \int_{\substack{\text{walls}\\\text{face}}} \mathbf{A} \cdot d\mathbf{s} = 25 \int_0^{2\pi} \int_0^4 5 dz d\phi = 1000\pi.$$

Total, $\oint_s \mathbf{A} \cdot d\mathbf{s} = 1200\pi = \int_V \nabla \cdot \mathbf{A}\, dv$ (verified).

Example 2.4 A vector field $\mathbf{A} = \hat{\mathrm{r}}\, 10e^{-r} - \hat{\mathrm{z}}\, 3z$, verify the divergence theorem for the cylindrical region enclosed by $r = 2, z = 0$ and $z = 4$.



Solution

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial}{\partial \phi}\left(A_\phi\right) + \frac{1}{r}\frac{\partial}{\partial z}(rA_z)$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(r10e^{-r}) + 0 - \frac{1}{r}\frac{\partial}{\partial z}(3zr) = \frac{1}{r}10e^{-r}(1-r) - 3.$$

$$\int_V \nabla \cdot \mathbf{A} dv = \int_0^4 \int_0^{2\pi} \int_0^2 \left(\frac{1}{r}10e^{-r}(1-r) - 3\right) r dr d\phi dz$$

$$= 10 \int_0^4 \int_0^{2\pi} \int_0^2 e^{-r}(1-r) dr d\phi dz - 3 \int_0^4 \int_0^{2\pi} \int_0^2 r dr d\phi dz = \frac{160\pi}{e^2} - 48\pi.$$

i. Top face: $z = 4,\ \mathbf{A} = \hat{\mathrm{r}}\, 10e^{-r} - \hat{\mathrm{z}}\, 12$ and $d\boldsymbol{s} = \hat{\mathrm{z}}\, r dr d\phi$

$$\therefore \int_{\substack{\text{top}\\\text{face}}} \mathbf{A} \cdot d\mathbf{s} = -12 \int_0^{2\pi} \int_0^2 r dr d\phi = -48\pi.$$

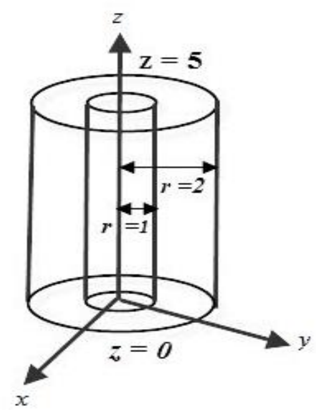
ii. Bottom face: $z = 0,\ \mathbf{A} = \hat{\mathrm{r}}\, 10e^{-r}$ and $d\boldsymbol{s} = \hat{\mathrm{z}}\, r dr d\phi \therefore \int_{\substack{\text{bottom}\\\text{face}}} \mathbf{A} \cdot d\mathbf{s} = 0.$

iii. Walls face: $r = 2,\ \mathbf{A} = \hat{\mathrm{r}}\, 10e^{-2} - \hat{\mathrm{z}}\, 3z$ and and $d\mathbf{s} = \hat{\mathrm{r}}\, r dz d\phi$

$$\therefore \int_{\substack{\text{walls}\\ \text{face}}} \mathbf{A}\cdot d\mathbf{s} = 20\int_0^{2\pi}\int_0^4 e^{-2}dzd\phi = \frac{160\pi}{e^2}.$$

Total $= \frac{160\pi}{e^2} + 0 - 48\pi = \frac{160\pi}{e^2} - 48\pi.$

Example 2.5 A vector field $\mathbf{A} = \hat{\mathrm{r}}r^3$ exists in the region between two concentric cylindrical surfaces defined by $r = 1$ and $r = 2$, with both cylinders extending between $z = 0$ and $z = 5$. Verify the divergence theorem by evaluating the following:(a) $\oint_s \mathbf{A}\cdot d\mathbf{s}$ and (b) $\int_v \nabla\cdot\mathbf{A}dv$.



Solution:

$$\nabla\cdot\mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial}{\partial\phi}(A_\phi) + \frac{1}{r}\frac{\partial}{\partial z}(rA_z)$$

$= \frac{1}{r}\frac{\partial}{\partial r}(rr^3) + 0 + \frac{1}{r}\frac{\partial}{\partial\phi}(0) = \frac{1}{r}\frac{\partial}{\partial r}(r^4) = 4r^2.$

$\int_V \nabla\cdot\mathbf{A}dv = \int_0^5\int_0^{2\pi}\int_1^2 4r^2rdrd\varphi dz = \int_0^5\int_0^{2\pi}\int_1^2 4r^3drd\phi dz = \int_0^5\int_0^{2\pi}[r^4]_1^2 d\phi dz = 150\pi.$

i. Top face: $z = 5$, $\mathbf{A} = \hat{\mathrm{r}}r^3$and $d\boldsymbol{s} = \hat{\mathrm{z}}\,rdrd\phi$ $\int_{\substack{\text{top}\\ \text{face}}} \mathbf{A}\cdot d\mathbf{s} = 0.$

ii. Bottom face: $z = 0$, $\mathbf{A} = \hat{\mathrm{r}}r^3$ and $d\boldsymbol{s} = -\hat{\mathrm{z}}rdrd\phi$ $\int_{\substack{\text{bottom}\\ \text{face}}} \mathbf{A}\cdot d\mathbf{s} = 0.$

iii. Outside surface: $r = 2, \mathbf{A} = \hat{\mathrm{r}}\ 8$ and $d\boldsymbol{s} = \hat{\mathrm{r}}rdzd\phi$

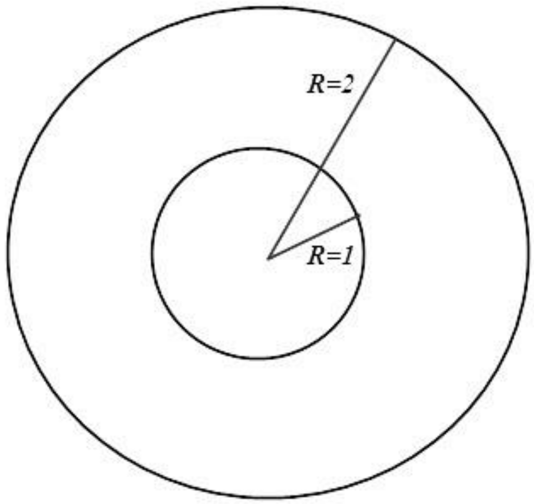
$$\therefore \int_{\text{outside}} \mathbf{A}\cdot d\mathbf{s} = 16\int_0^{2\pi}\int_0^5 dzd\phi = 160\pi.$$

iv. Inside surface: $r = 1, \mathbf{A} = \hat{\mathrm{r}}$ and $d\boldsymbol{s} = -\hat{\mathrm{r}}rdzd\phi$

$$\therefore \int_{\text{intside}} \mathbf{A}\cdot d\mathbf{s} = -\int_0^{2\pi}\int_0^5 dzd\phi = -10\pi.$$

Adding all surface, $160\pi - 10\pi = 150\pi$.

Example 2.6 For the vector field $A = \hat{\mathrm{R}}\, 3R^2,$ evalute both sides the divergence theorem for the region enclosed between spharical shells defined by $R = 1$ and $R = 2$.



Solution

At the outer surface $d\mathbf{s} = \hat{\mathrm{R}}(R_2)^2 \sin\theta \, d\theta d\phi$ and we get $\mathbf{A} \cdot d\mathbf{s} = 3(R_2)^4 \sin\theta \, d\theta d\phi$

$\therefore \oint_s \mathbf{A} \cdot d\mathbf{s} = \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} 3 \times 2^4 \times \sin\theta \, d\theta \, d\phi = 192\pi.$

At the inner surface $d\mathbf{s} = -\hat{\mathrm{R}}(R_2)^2 \sin\theta \, d\theta d\phi$ and we get $\mathbf{A} \cdot d\mathbf{s} = 3(R_2)^4 \sin\theta \, d\theta d\phi$

$\therefore \oint_s \mathbf{A} \cdot d\mathbf{s} = -\int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} 3 \times 1^4 \times \sin\theta \, d\theta \, d\phi = -12\pi.$

$$\text{Total} = 192\pi - 12\pi = 180\pi.$$

Again,

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2 \sin\theta}\left[\frac{\partial}{\partial R}(3R^2 \cdot R^2 \sin\theta) + \frac{\partial}{\partial\theta}(0 \cdot R\sin\theta) + \frac{\partial}{\partial\phi}(0 \cdot R)\right] = 12R$$

Now, Outer sphere: $\int_V \nabla \cdot \mathbf{A} \, dv = \int_V 12R \, dv =$

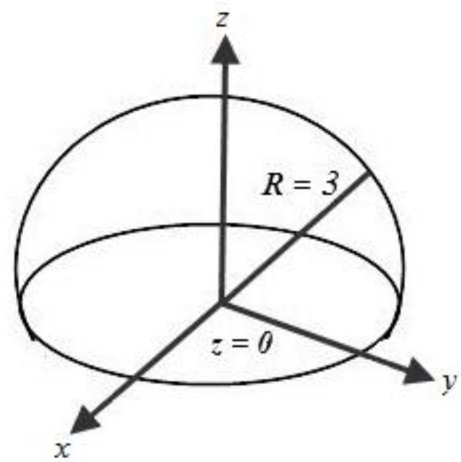
$$\int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} \int_{R=0}^{2} 12 \, R^3 \sin\theta \, dR \, d\phi \, d\theta = 192\pi$$

Inner sphere: $\int_V \nabla \cdot \mathbf{A} \, dv = \int_V 12R \, dv =$

$$\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\int_{R=0}^{1} 12\, R^3 \sin\theta\, dR\, d\phi\, d\theta = 12\pi$$

Total $= 192\pi - 12\pi = 180\pi$.

Example 2.7 Find $\oint_s \mathbf{A}\cdot d\mathbf{s}$ over the surface of a hemispherical region that is the top half of a sphere of radius 3 centered at $(0,0,0)$ with its flat base coinciding with the xy plane. Also verify divergence theorem where $\mathbf{A} = \hat{z}\, z$.



Solution

Given that $\boldsymbol{A} = \hat{z}\, R\cos\theta\ (0 \le \theta \le \pi)$.

In spherical coordinates $\mathbf{A} = \widehat{\mathrm{R}}\, R\cos^2\theta - \hat{\theta}\, \mathrm{R}\cos\theta\sin\theta$

Over the hemisphere surface $d\mathbf{s} = \widehat{\mathrm{R}}\, R^2\sin\theta\, d\theta d\phi$ and we get $\mathbf{A}\cdot d\mathbf{s} = R^3\cos^2\theta\sin\theta\, d\theta d\phi$

$\therefore \oint_s \mathbf{A}\cdot d\mathbf{s} = \int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}} R^3\cos^2\theta\sin\theta\, d\theta\, d\phi = \frac{2}{3}\pi R^3 = 18\pi.\ (R=3)$

Now, $\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{1}{R^2\sin\theta}\left[\frac{\partial}{\partial R}(R\cos^2\theta\cdot R^2\sin\theta) + \frac{\partial}{\partial\theta}(-\mathrm{R}\cos\theta\sin\theta\cdot R\sin\theta) + \frac{\partial}{\partial\phi}(0\cdot R)\right]$

$$\frac{1}{R^2\sin\theta}[3R^2\cos^2\theta\sin\theta - R^2(2\sin\theta\cos^2\theta - \sin^3\theta)] = 3\cos^2\theta - 2\cos^2\theta + \sin^2\theta = 1$$

and $dv = dl_R dl_\theta dl_\varphi = dR(R)d\theta(R\sin\theta)d\phi$

So, $\int_V \boldsymbol{\nabla}\cdot\mathbf{A}\, dv = \int_V dv = \int_{\theta=0}^{\frac{\pi}{2}}\int_{\varphi=0}^{2\pi}\int_{R=0}^{3} R^2\sin\theta\, dR\, d\phi\, d\theta = \frac{2}{3}\pi 3^3 = 18\pi = \oint_s \mathbf{A}\cdot d\mathbf{s}\cdot$

Sample exercise - 6.2

1. For the vector field $\mathbf{A} = \hat{x}\, xy + \hat{y}\, y^2 z + \hat{z}\, xz$, verify the divergence theorem by computing **(a)** the total outward flux flowing through the surface of a cube centered at the origin and with sides equal to 2 units each and parallel to the Cartesian axes, **(b)** the integral of $\nabla \cdot \mathbf{A}$ over the cube's volume.

2. For a vector function $\mathbf{A} = \hat{r} r^2 + \hat{z}\, 3z$, verify for the circular cylindrical region enclosed by $r = 1, z = 0, z = 4$. Ans: 20π

3. A vector field $\mathbf{A} = \hat{r}\, r^2$ exists in the region between two concentric cylindrical surfaces defined by $r = 2$ and $r = 3$, with both cylinders extending between $z = 0$ and $z = 3$. Verify the divergence theorem by evaluating the following: (a) $\oint_s \mathbf{A} \cdot d\mathbf{s}$ and (b) $\int_v \nabla \cdot \mathbf{A}\, dv$. Ans: 114π

4. Find $\oint_s \mathbf{A} \cdot d\mathbf{s}$ over the surface of a hemispherical region that is the top half of a sphere of radius 4 centered at $(0, 0, 0)$ with its flat base coinciding with the xy plane. Also verify divergence theorem. where $\mathbf{A} = \hat{R}\, R \cos\theta$. Ans: 64π

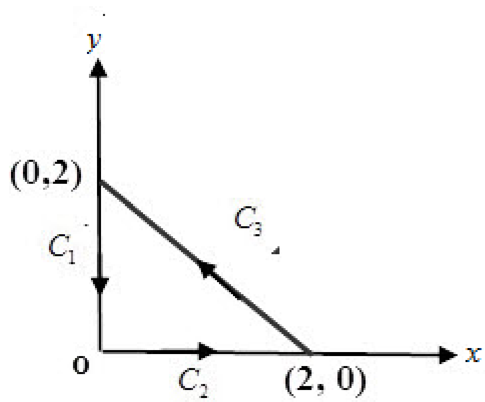
5. For the vector field $A = \hat{R}\, 2R^3$, evaluate both sides of the divergence theorem for the region enclosed between spherical shells defined by $R = 2$ and $R = 3$. Ans: 1688π

3. Stokes's Theorem:

Let S be the open surface (two-sided) and C be the closed boundary of S, the vector field $\mathbf{A}$ is continuous on S. Then

$$\int_{s} (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_{c} \mathbf{A} \cdot d\mathbf{l}\,.$$

Example 3.1. Assume that a vector field $\mathbf{A} = \hat{x}(2x^2 + y^2) + \hat{y}(xy - y^2)$, (a) find $\oint_c \mathbf{A} \cdot d\mathbf{l}$ around the triangular contour, (b) find $\int_s (\nabla \times \mathbf{A}) \cdot d\mathbf{s}$ over triangular arc, (c) verify Stokes's theorem and (d) can $\mathbf{A}$ be expressed as gradient of a scalar? Explain.



Solution

(a) $d\boldsymbol{l} = \hat{x}\, dx + \hat{y}\, dy \therefore \mathbf{A} \cdot d\mathbf{l} = (2x^2 + y^2)dx + (xy - y^2)dy$

Path c_1; $x = 0, dx = 0, \oint_{c_1} \mathbf{A} \cdot d\mathbf{l} = -\int_2^0 y^2 dy = \frac{8}{3}\cdot$

Path c_2; $y = 0, dy = 0, \oint_{c_2} \mathbf{A} \cdot d\mathbf{l} = \int_0^2 2x^2 dx = \frac{16}{3}\cdot$

Path c_3; $y = 2 - x, dy = -dx,$

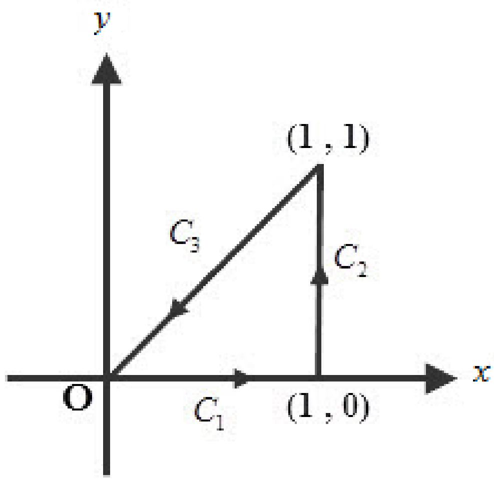
$$\oint_{c_3} \mathbf{A} \cdot d\mathbf{l} = \int_2^0 [\{2x^2 + (2-x)^2\} - \{x(2-x) - (2-x)^2\}]dx = -\frac{28}{3}\cdot$$

Total, $\oint_c \mathbf{A} \cdot d\mathbf{l} = -\frac{4}{3}\cdot$

Now, $\nabla \times \mathbf{A} = -\hat{z}\, y, d\boldsymbol{s} = \hat{z}\, dxdy \therefore \int_s (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = -\int_0^2 \int_0^{2-x} ydydx = -\frac{4}{3}\cdot$

(c) Stokes's theorem is verified. (d) No, $\because \nabla \times \mathbf{A} \neq 0$.

Example 3.2. Assume that a vector field $\mathbf{A} = \hat{x}\, xy - \hat{y}(x^2 + 2y^2)$, (a) find $\oint_c \mathbf{A} \cdot d\mathbf{l}$ around the triangular contour, (b) find $\int_s (\nabla \times \mathbf{A}) \cdot d\mathbf{s}$ over triangular arc.



Solution

(a) $d\mathbf{l} = \hat{x}\, dx + \hat{y}\, dy \therefore \mathbf{A} \cdot d\mathbf{l} = xy\, dx - (x^2 + 2y^2)\, dy$

Path c_1; $y = 0,\ dy = 0, \oint_{c_1} \mathbf{A} \cdot d\mathbf{l} = \int_0^1 xy\, dx - (x^2 + 2y^2)\, dy = 0 \cdot$

Path c_2; $x = 1,\ dx = 0, \oint_{c_2} \mathbf{A} \cdot d\mathbf{l} = \int_0^1 xy\, dx - (x^2 + 2y^2)\, dy$

$$= \int_0^1 -(1 + 2y^2) dy = -\frac{5}{3} \cdot$$

Path c_3; $y = x,\ dy = dx,$

$$\oint_{c_3} \mathbf{A} \cdot d\mathbf{l} = \int_1^0 xy\, dx - (x^2 + 2y^2)\, dy = \int_1^0 y^2\, dy - (y^2 + 2y^2)\, dy = \frac{2}{3} \cdot$$

$$\text{Total, } \oint_c \mathbf{A} \cdot d\mathbf{l} = -\frac{5}{3} + \frac{2}{3} = -1 \cdot$$

Now, $\nabla \times \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -(x^2 + 2y^2) & 0 \end{vmatrix} = -3x\, \hat{z}, d\boldsymbol{s} = \hat{z}\, dxdy$

$$\therefore \int_s (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = -\int_0^1 \int_0^x 3x\, dydx = -1 \cdot$$

Stokes's theorem is verified.

Example 3.3. Repeat **Ex3.2.** for the contour shown below.

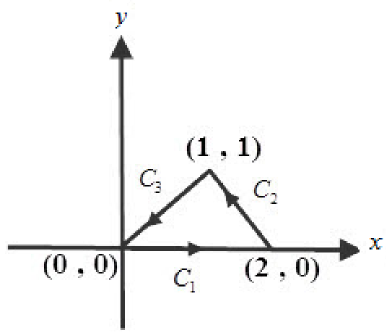
Solution

(a) $d\boldsymbol{l} = \hat{\mathrm{x}}\, dx + \hat{\mathrm{y}}\, dy \therefore \mathbf{A} \cdot d\mathbf{l} = xy\, dx - (x^2 + 2y^2)\, dy$

Path c_1; $y = 0,\ dy = 0, \oint_{c_1} \mathbf{A} \cdot d\mathbf{l} = \int_0^2 xy\, dx - (x^2 + 2y^2)\, dy = 0 \cdot$

Path c_2; $y = 2 - x,\ dy = -dx,$

$$\oint_{c_3} \mathbf{A} \cdot d\mathbf{l} = \int_0^1 xy\, dx - (x^2 + 2y^2)\, dy = \int_2^1 x(2-x)\, dx + (x^2 + 2(2-x)^2)\, dx = -\frac{11}{3} \cdot$$



Path c_3; $y = x,\ dy = dx,$

$$\oint_{c_3} \mathbf{A} \cdot d\mathbf{l} = \int_1^0 xy\, dx - (x^2 + 2y^2)\, dy = \int_1^0 y^2\, dy - (y^2 + 2y^2)\, dy = \frac{2}{3}$$

$$\text{Total, } \oint_c \mathbf{A} \cdot d\mathbf{l} = -\frac{11}{3} + \frac{2}{3} = -3 \cdot$$

$$\text{Now, } \nabla \times \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -(x^2 + 2y^2) & 0 \end{vmatrix} = -3x\, \hat{\mathrm{z}}, d\boldsymbol{s} = \hat{\mathrm{z}}\, dxdy$$

$$\therefore \int_s (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = -\int_0^1 \int_y^{2-y} 3x\, dxdy = -3 \cdot$$

Stokes's theorem is verified.

Example 3.4. For vector field $\mathbf{A} = \hat{\mathrm{z}} \frac{\cos\phi}{r}$, verify Stokes's theorem for a segment of a cylindrical surface defined by $r = 2, \pi/3 \le \phi \le \pi/2$, and $0 \le z \le 3$.

Solution

Stokes's theorem states that $\int_s (\nabla \times \mathbf{A}) \cdot d\boldsymbol{s} = \oint_c \mathbf{A} \cdot d\mathbf{l}$

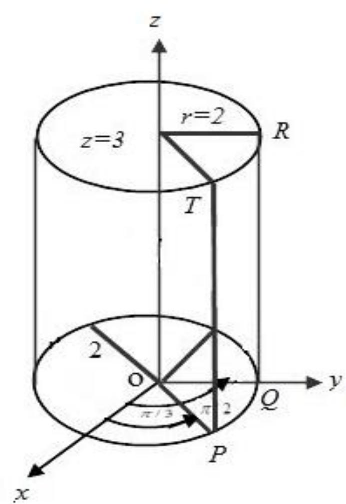
LHS: With $\mathbf{A}$ having only a component $A_z = \frac{\cos\phi}{r}$,use of the expression for $\nabla \times \mathbf{A}$ in cylindrical coordinates,

$$\nabla \times \mathbf{A} = \hat{\mathrm{r}}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{\mathrm{z}}\frac{1}{r}\left(\frac{\partial (rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right)$$

$$= \hat{\mathrm{r}}\frac{1}{r}\frac{\partial\left(\frac{\cos\phi}{r}\right)}{\partial\phi} - \hat{\phi}\frac{\partial\left(\frac{\cos\phi}{r}\right)}{\partial r} = -\hat{\mathrm{r}}\left(\frac{\sin\phi}{r^2}\right) + \hat{\phi}\left(\frac{\cos\phi}{r^2}\right)$$

The integral of $\nabla \times \mathbf{A}$ over the specified surface S is

$$\int_s (\nabla \times \mathbf{A}) \cdot d\boldsymbol{s} = \int_0^3 \int_{\pi/3}^{\pi/2} \left[-\hat{\mathrm{r}}\left(\frac{\sin\phi}{r^2}\right) + \hat{\phi}\left(\frac{\cos\phi}{r^2}\right)\right] \cdot \hat{\mathrm{r}}\, r d\phi dz = -\int_0^3 \int_{\pi/3}^{\pi/2} \frac{\sin\phi}{r} d\phi dz = -\frac{3}{4}\cdot$$



RHS: The surface S is bounded by contour $c = PQRT$ shown in figure above. The direction of c is chosen so that it is compatible with the surface normal $\hat{\mathrm{r}}$ by the right-hand rule. Hence,

$$\oint_c \mathbf{A} \cdot d\mathbf{l} = \int_P^Q \mathbf{A}_{PQ} \cdot d\mathbf{l} + \int_Q^R \mathbf{A}_{QR} \cdot d\mathbf{l} + \int_R^T \mathbf{A}_{RT} \cdot d\mathbf{l} + \int_T^P \mathbf{A}_{TP} \cdot d\mathbf{l}$$

where $\boldsymbol{A}_{PQ}$, $\boldsymbol{A}_{QR}$, $\boldsymbol{A}_{RT}$ and $\boldsymbol{A}_{TP}$ are the field $\boldsymbol{A}$ along segments PQ, QR, RT and $T\,P$respectively. Over segment PQ the dot product of $\boldsymbol{A}_{PQ} = \hat{\mathrm{z}}\frac{cos\phi}{r}$and $d\mathbf{l} = \hat{\phi}\, rd\phi$ is zero, and the same is true for segment RT. Over segment QR, $\phi = \pi/2$; hence, $\mathbf{A}_{QR} = \hat{\mathrm{z}}\frac{cos\pi/2}{2} = 0.$ For the last segment, $\mathbf{A}_{TP} = \hat{\mathrm{z}}\frac{cos\pi/3}{2} = \hat{\mathrm{z}}/4$ and $d\mathbf{l} = \hat{\mathrm{z}}dz$. Hence,

$$\oint_c \mathbf{A}\cdot d\mathbf{l} = \int_T^P \left(\hat{z}\cdot\frac{1}{4}\right)\hat{z}dz = \int_3^0 \frac{1}{4}dz = -\frac{3}{4}\cdot$$

which is the same as the result obtained by evaluating the left-hand side of Stokes's equation.

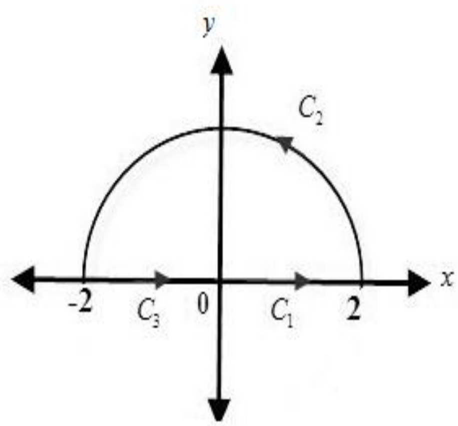
Example 3.5. Assume that a vector field $\mathbf{A} = \hat{r}\, r\cos\phi + \hat{\phi}\sin\phi$, (a) find $\oint_c \mathbf{A}\cdot d\mathbf{l}$ over the semicircular contour, and (b) find $\int_s (\nabla\times\mathbf{A})\cdot d\mathbf{s}$ over the surface of the semicircle.

Solution

(a) $d\mathbf{l} = \hat{r}\,dr + \hat{\phi}\,rd\phi \therefore \mathbf{A}\cdot d\mathbf{l} = r\cos\phi\,dr + \sin\phi\, r\,d\phi$

Path c_1; $\phi = 0,\ d\phi = 0,\ \oint_{c_1} \mathbf{A}\cdot d\mathbf{l} = \int_0^2 r\cos\phi\,dr = 2.$

Path c_2; $r = 2,\ dr = 0,\ \oint_{c_2} \mathbf{A}\cdot d\mathbf{l} = \int_0^\pi \sin\phi\, r\,d\phi = 4.$



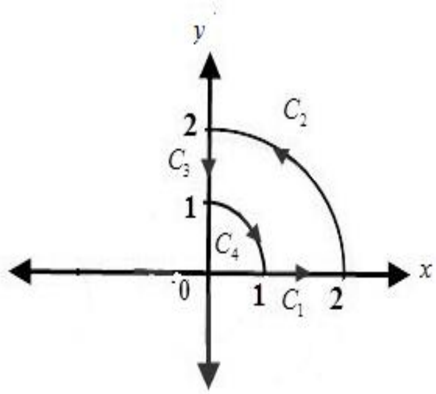
Path c_3; $\phi = \pi,\ d\phi = 0,$

$\oint_{c_3} \mathbf{A}\cdot d\mathbf{l} = \int_2^0 r\cos\varphi\,dr = -\int_2^0 r\,dr = 2\cdot$ Total, $\oint_c \mathbf{A}\cdot d\mathbf{l} = 2 + 4 + 2 = 8\cdot$

Now, $\nabla\times\mathbf{A} = \nabla\times\mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ r\cos\phi & r\sin\phi & 0 \end{vmatrix} = \hat{z}\frac{1}{r}(\sin\phi + r\sin\phi), d\boldsymbol{s} = \hat{z}\, r\,drd\phi$

$\therefore \int_s (\nabla\times\mathbf{A})\cdot d\mathbf{s} = \int_0^\pi\int_0^2 (\sin\phi + r\sin\phi)\,drd\phi = 8\cdot$ Stokes's theorem is verified.

Example 3.6. Repeat **Ex3.5.** for the contour shown below.



Solution

(a) $d\mathbf{l} = \hat{\mathrm{r}}\, dr + \hat{\phi}\, r d\phi \therefore \mathbf{A} \cdot d\mathbf{l} = \ r\cos\phi\, dr + \sin\phi\, r\, d\phi$

Path c_1; $\phi = 0,\ d\phi = 0,\ \oint_{c_1} \mathbf{A} \cdot d\mathbf{l} = \int_1^2 r\cos\phi\, dr = \frac{3}{2}.$

Path c_2; $r = 2,\ dr = 0,\ \oint_{c_2} \mathbf{A} \cdot d\mathbf{l} = \int_0^{\frac{\pi}{2}} \sin\phi\, r\, d\phi = 2.$

Path c_3; $\phi = \frac{\pi}{2},\ d\phi = 0, \oint_{c_3} \mathbf{A} \cdot d\mathbf{l} = \int_2^1 r\cos\phi\, dr = 0 \cdot$

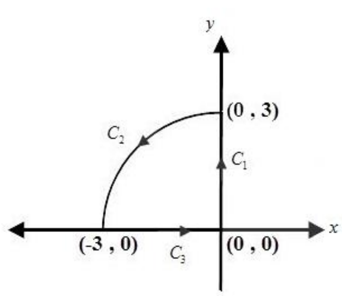
Path c_4; $r = 1,\ dr = 0,\ \oint_{c_2} \mathbf{A} \cdot d\mathbf{l} = \int_{\frac{\pi}{2}}^{0} \sin\phi\, r\, d\phi = -1.$

Total, $\oint_c \mathbf{A} \cdot d\mathbf{l} = \frac{3}{2} + 0 + 2 - 1 = \frac{5}{2} \cdot$

Now, $\boldsymbol{\nabla} \times \mathbf{A} = \boldsymbol{\nabla} \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathrm{r}} & r\hat{\phi} & \hat{\mathrm{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ r\cos\phi & r\sin\phi & 0 \end{vmatrix} = \hat{\mathrm{z}}\frac{1}{r}(\sin\phi + r\sin\phi), d\boldsymbol{s} = \hat{\mathrm{z}}\, r\, dr d\phi$

$\therefore \int_s (\boldsymbol{\nabla} \times \mathbf{A}) \cdot d\mathbf{s} = \int_0^{\pi/2}\int_1^2 (\sin\phi + r\sin\phi)\, dr d\phi = \frac{5}{2} \cdot$ Stokes's theorem is verified.

Example 3.7. Assume that a vector field $\mathbf{A} = \hat{\mathrm{r}}\cos\phi + \hat{\phi}\sin\phi$, (a) find $\oint_c \mathbf{A} \cdot d\mathbf{l}$ over the path comprising a quarter section of a circle, and (b) find $\int_s (\boldsymbol{\nabla} \times \mathbf{A}) \cdot d\mathbf{s}$ over the surface of the quarter section.



Solution

(a) $d\mathbf{l} = \hat{\mathrm{r}}\, dr + \hat{\phi}\, r d\phi \therefore \mathbf{A} \cdot d\mathbf{l} = \cos\phi\, dr + \sin\phi\, r d\phi$

Path c_1; $\phi = \frac{\pi}{2},\ d\phi = 0,\ \oint_{c_1} \mathbf{A} \cdot d\mathbf{l} = \int_0^3 \cos\phi\, dr = \int_0^3 \cos\frac{\pi}{2} dr = 0.$

Path c_2; $r = 3,\ dr = 0, \oint_{c_2} \mathbf{A} \cdot d\mathbf{l} = \int_{\pi/2}^{\pi} \sin\phi\, r d\phi = 3.$

Path c_3; $\phi = \pi,\ d\phi = 0, \therefore \oint_{c_3} \mathbf{A} \cdot d\mathbf{l} = \int_3^0 \cos\phi\, dr = -\int_3^0 dr = 3 \cdot$

Total, $\oint_c \mathbf{A} \cdot d\mathbf{l} = 0 + 3 + 3 = 6 \cdot$

Now, $\nabla \times \mathbf{A} = \nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathrm{r}} & r\hat{\phi} & \hat{\mathrm{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \cos\phi & r\sin\phi & 0 \end{vmatrix} = \hat{\mathrm{z}}\frac{1}{\mathrm{r}} 2\sin\phi\, , d\mathbf{s} = \hat{\mathrm{z}}\, r dr d\phi$

$\therefore \int_s (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \int_{\pi/2}^{\pi} \int_0^3 2\sin\phi\, dr d\phi = 6 \cdot \therefore$ Stokes's theorem is verified.

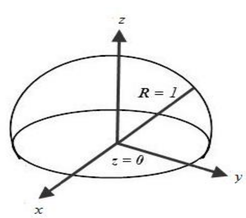
Example 3.8. Verify Stokes's theorem for the vector field, $\mathbf{A} = \hat{\mathrm{R}} \cos\theta + \hat{\phi} \sin\theta$ by evaluating it on the hemisphere of unit radius.

Solution

$$\nabla \times \mathbf{A} = \frac{\mathbf{1}}{R^2 \sin\theta}\begin{vmatrix} \hat{\mathrm{R}} & R\,\hat{\theta} & R\sin\theta\,\hat{\phi} \\ \dfrac{\partial}{\partial R} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ \cos\theta & R \cdot 0 & R\sin\theta\sin\theta \end{vmatrix}$$

$$= \frac{\hat{\mathrm{R}}\, 2\mathrm{R}\sin\theta\cos\theta - \hat{\theta}\, R\sin^2\theta + \hat{\phi}\, R\sin^2\theta}{R^2\sin\theta} \cdot$$

and $d\mathbf{s} = \hat{\mathrm{R}}(R d\theta)(R\sin\theta\, d\phi) \therefore\ (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = 2\mathrm{R}\sin\theta\cos\theta\, d\theta\, d\phi,\ (R = 1).$



$$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \int_{\theta=0}^{\pi/2} \int_{\varphi=0}^{2\pi} 2\sin\theta\cos\theta\, d\phi d\theta$$

$$= 4\pi \int_0^{\pi/2} \sin\theta\cos\theta\, d\theta = 4\pi \int_0^1 u\, du = 2\pi.$$

Again, $d\mathbf{l} = \hat{\phi}\, R\sin\theta\, d\phi \therefore \mathbf{A} \cdot d\mathbf{l} = R\sin^2\theta\, d\phi = d\phi$, ($R = 1$ & $\theta = \pi/2$).

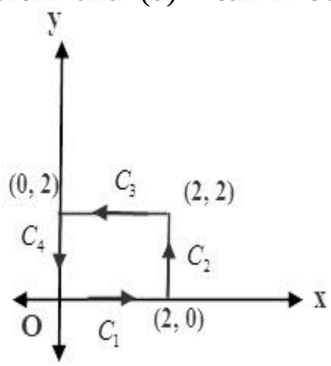
$\oint_c \mathbf{A} \cdot d\mathbf{l} = \int_0^{2\pi} d\phi = 2\pi. \therefore$ Stokes's theorem is verified.

Sample exercise – 6.3

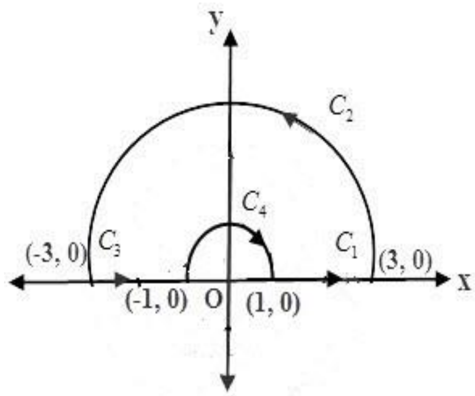
1. Verify Stokes's theorem for the vector field $\mathbf{A} = \hat{\phi}\, 3\sin\frac{\phi}{2}$ by evaluating it on the hemisphere of radius 2. Ans: 24

2. For vector field $\mathbf{A} = \hat{z}\cos\varphi$, verify Stokes's theorem for a segment of a cylindrical surface defined by $r = 5, \pi/4 \le \varphi \le \pi/2$, and $0 \le z \le 4$. Ans: $-2\sqrt{2}$

3. Assume that a vector field $\mathbf{A} = \hat{x}(x^2 - y^2) + \hat{y}(x^2 - xy)$, (a) find $\oint_c \mathbf{A} \cdot d\mathbf{l}$ around the rectangular contour, (b) find $\int_s (\nabla \times \mathbf{A}) \cdot d\mathbf{s}$ over rectangular arc, (c) verify Stokes's theorem and (d) can $\mathbf{A}$ be expressed as gradient of a scalar? Explain. Ans: 12



4. Assume that a vector field, $\mathbf{A} = \hat{r}\, r\sin\phi + \hat{\phi}\cos\phi$, (a) find $\oint_c \mathbf{A} \cdot d\mathbf{l}$ over the semicircular contour shown below, and (b) find $\int_s (\nabla \times \mathbf{A}) \cdot d\mathbf{s}$ over the surface of the semicircles. Ans: 0



Book: Fundamentals of Applied Electromagnetics (6th Edition)
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