Chapter-4
Eigenvalues and Eigenvectors

Let $A = (a_{ij})_{n\times n}$ is a square matrix. A non-zero vector $\boldsymbol{V}$ in $\mathbb{R}^n$ is called an eigenvector of A if $A\boldsymbol{V}$ is a scalar multiple of $\boldsymbol{V}$; that is $A\boldsymbol{V} = \lambda\boldsymbol{V}$ for some scalar λ. The scalar λ is called an eigenvalue of A and $\boldsymbol{V}$ is called the eigenvector of A corresponding to λ.

Example: The vector $\boldsymbol{V} = \begin{bmatrix}1\\2\end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix}3 & 0\\8 & -1\end{bmatrix}$ corresponding to the eigenvalue $\lambda = 3$.

$$A\boldsymbol{V} = \begin{bmatrix}3 & 0\\8 & -1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix} = 3\begin{bmatrix}1\\2\end{bmatrix} = \lambda\boldsymbol{V}.$$

Characteristic matrix:

Provided that A is a square matrix of order $n \times n$. Then the matrix $A - \lambda I$ is called the characteristic matrix where λ is scalar and I is the unit matrix.

Example:

$$A = \begin{bmatrix}2 & 2 & 1\\1 & 3 & 1\\1 & 2 & 2\end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix}2 & 2 & 1\\1 & 3 & 1\\1 & 2 & 2\end{bmatrix} - \lambda\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}2-\lambda & 2 & 1\\1 & 3-\lambda & 1\\1 & 2 & 2-\lambda\end{bmatrix}$$

is the characteristic matrix.

Characteristic polynomial:

The determinant $|A - \lambda I|$ results a polynomial of λ, which is called characteristic polynomial of matrix A. Following is an example of characteristic polynomial of λ of degree 3, the order of the matrix A,

$$|A - \lambda I| = \begin{vmatrix}2-\lambda & 2 & 1\\1 & 3-\lambda & 1\\1 & 2 & 2-\lambda\end{vmatrix}$$

$$= (2-\lambda)(6 - 5\lambda + \lambda^2) - 2(1-\lambda) + 1(2 - 3 + \lambda)$$

$$= \lambda^3 - 7\lambda^2 + 11\lambda - 5.$$

Characteristic equation:

The equation $|A - \lambda I| = 0$ is called characteristic equation for matrix A.

For example, $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$ is characteristic polynomial for the above matrix A.

Characteristic roots or eigenvalues:

The roots of the characteristic equation $|A - \lambda I| = 0$ are called characteristic roots of matrix A.

$\therefore \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$
$\Longrightarrow (\lambda - 1)(\lambda - 1)(\lambda - 5) = 0$
$\Longrightarrow \lambda = 1,1,5$
So, the characteristic roots or eigenvalues are 1,1 and 5

Example:

Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & 5 & 2 \end{bmatrix}$

Solution:
The characteristic matrix of A is,

$$A - \lambda I = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & 5 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1-\lambda & 2 & -1 \\ 0 & -2-\lambda & 0 \\ 0 & 5 & 2-\lambda \end{bmatrix}$$

The characteristic polynomial of A is $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & -2-\lambda & 0 \\ 0 & 5 & 2-\lambda \end{vmatrix}$
The characteristic equation of A is $(\lambda - 1)(\lambda + 2)(\lambda - 2) = 0$
So, the characteristic roots or the Eigenvalues of A is $\lambda = 1, -2, 2$
Now by definition $\boldsymbol{V} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ is an Eigenvector of A corresponding to the Eigenvalue λ if and only if $\boldsymbol{V}$ is a non-trivial solution of $(A - \lambda I)\boldsymbol{V} = 0$

So, $\begin{bmatrix} 1-\lambda & 2 & -1 \\ 0 & -2-\lambda & 0 \\ 0 & 5 & 2-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$

When $\lambda = 1$, then

$\begin{bmatrix} 0 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$

Forming a linear system, we have
$0 \cdot v_1 + 2 \cdot v_2 - 1 \cdot v_3 = 0$
$0 \cdot v_1 - 3 \cdot v_2 + 0 \cdot v_3 = 0$
$0 \cdot v_1 + 5 \cdot v_2 + 1 \cdot v_3 = 0$
Solving we get $v_2 = v_3 = 0$

MatLab command for finding eigenvalues and eigenvectors:

```
>> A=[1 2 -1;0 -2 0;0 5 2];
>> [v,d]=eig(sym(A));
>> eigenvalues=eig(A)'
eigenvalues =
    1    2   -2
>> eigenvectors=double(v)
eigenvectors =
    1.0000   -1.0000    0.8667
         0         0   -0.8000
         0    1.0000    1.0000
```

Hence v_1 is a free variable. Let, $v_1 = a$, where a is any real number. Therefore, the eigenvector of A corresponding to the eigenvalue $\lambda = 1$ are the non-zero vectors of the form $\boldsymbol{V} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$. In particular, if $a = 1$, then $\boldsymbol{V} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an Eigenvector corresponding to the Eigenvalue of $\lambda = 1$

Again, when $\lambda = -2$, we find

$$\begin{bmatrix} 3 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 5 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

Forming a linear system, we have

$3 \cdot v_1 + 2 \cdot v_2 - 1 \cdot v_3 = 0$

$0 \cdot v_1 - 0 \cdot v_2 + 0 \cdot v_3 = 0$

$0 \cdot v_1 + 5 \cdot v_2 + 4 \cdot v_3 = 0$

This system has one free variable. Let $v_3 = b$.

$\therefore v_2 = -\frac{4b}{5}$ and $\therefore v_1 = -\frac{13b}{15}$

Therefore, $\boldsymbol{V} = \begin{bmatrix} -\frac{13b}{15} \\ -\frac{4b}{5} \\ b \end{bmatrix}$.

In particular, let $b = -15$

So, $\boldsymbol{V} = \begin{bmatrix} 13 \\ 12 \\ -15 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = -2$

When $\lambda = 2$,

$$\begin{bmatrix} -1 & 2 & -1 \\ 0 & -4 & 0 \\ 0 & 5 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

Forming a linear system, we have

$-1 \cdot v_1 + 2 \cdot v_2 - 1 \cdot v_3 = 0$

$0 \cdot v_1 - 4 \cdot v_2 + 0 \cdot v_3 = 0$

$0 \cdot v_1 + 5 \cdot v_2 + 0 \cdot v_3 = 0$

Hence, $v_2 = 0$ and v_3 is free variable. Let $v_3 = c$ then we have $v_1 = -c$

Therefore, $V = \begin{bmatrix} -c \\ 0 \\ c \end{bmatrix}$

In particular, if $c = 1$ we have $\boldsymbol{V} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Example: Solve the following system of differential equation using eigenvalue and eigenvector.

$$\begin{cases} \dot{x_1}(t) = -1.5x_1(t) + 0.5x_2(t) \\ \dot{x_2}(t) = x_1(t) - x_2(t) \end{cases} \text{ with } x_1(0) = 5, x_2(0) = 4.$$

where $\dot{x_1}(t) = \frac{dx_1}{dt}$ and $\dot{x_2}(t) = \frac{dx_2}{dt}$.

Solution:

Let, $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ and $\dot{X}(t) = \begin{pmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \end{pmatrix}$

So, $X(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$.

We write, $A = \begin{pmatrix} -1.5 & 0.5 \\ 1 & -1 \end{pmatrix}$.

Now the system of differential equation can be written as

$\dot{X}(t) = AX(t)$

Let λ and $\boldsymbol{V}$ be the eigenvalue and eigenvector of A respectively and C is an integral constant then we have the solution of the form,

$X(t) = CVe^{\lambda t}$.

The characteristic matrix of A is $A - \lambda I = \begin{pmatrix} -1.5 & 0.5 \\ 1 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} -1.5 - \lambda & 0.5 \\ 1 & -1 - \lambda \end{pmatrix}$$

The characteristic polynomial of A is $|A - \lambda I| = \begin{vmatrix} -1.5 - \lambda & 0.5 \\ 1 & -1 - \lambda \end{vmatrix}$

The characteristic equation of A is $(\lambda + 1.5)(\lambda + 1) - 0.5 = 0$

$$\Rightarrow (\lambda^2 + 2.5\lambda + 1) = 0$$

$$\Rightarrow (\lambda + 0.5)(\lambda + 2) = 0$$

So the characteristic roots or the eigenvalues of A is $\lambda = -0.5, -2$

Now by definition $\boldsymbol{V} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue λ if and only if $\boldsymbol{V}$ is a non-trivial solution of $(A - \lambda I)\boldsymbol{V} = 0$

So, $\begin{pmatrix} -1.5 - \lambda & 0.5 \\ 1 & -1 - \lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

When $\lambda = -0.5$, then $\begin{pmatrix} -1 & 0.5 \\ 1 & -0.5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

Forming a linear system, we have

$-v_1 + 0.5v_2 = 0$

$v_1 - 0.5v_2 = 0$

Solving the above system, we get $-v_1 + 0.5v_2 = 0.$

Here v_2 is a free variable. Let $v_2 = a$

$v_1 = a$ and $v_2 = 2a$.

Therefore, the eigenvector of A corresponding to the eigenvalue $\lambda = -0.5$ are the non-zero vectors of the form $\boldsymbol{V}_1 = \begin{pmatrix} a \\ 2a \end{pmatrix}$.

Again, when $\lambda = -2,$ then $\begin{pmatrix} 0.5 & 0.5 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

Forming a linear system, we have

$0.5v_1 + 0.5v_2 = 0$

$v_1 + v_2 = 0$

Solving the above system, we get $0.5v_1 + 0.5v_2 = 0.$

Here v_2 is a free variable. Let $v_2 = b$

Thus, we get $v_1 = -b$ and $v_2 = b$.

Therefore, the eigenvector of A corresponding to the eigenvalue $\lambda = -2$ are the non-zero vectors of the form $\boldsymbol{V}_2 = \begin{pmatrix} -b \\ b \end{pmatrix}$.

So, the solution of the system of differential equation can be written as,

$X(t) = C_1V_1e^{\lambda_1 t} + C_2V_2e^{\lambda_2 t}$

$\Rightarrow \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1 \begin{pmatrix} a \\ 2a \end{pmatrix} e^{-0.5t} + C_2 \begin{pmatrix} -b \\ b \end{pmatrix} e^{-2t}$

$\Rightarrow \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = C_1 \begin{pmatrix} a \\ 2a \end{pmatrix} + C_2 \begin{pmatrix} -b \\ b \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 5 \\ 4 \end{pmatrix} = C_1 \begin{pmatrix} a \\ 2a \end{pmatrix} + C_2 \begin{pmatrix} -b \\ b \end{pmatrix}$

We can write,

$aC_1 - bC_2 = 5$

$2aC_1 + bC_2 = 4$

Solving the system for C_1 and C_2, we have, $C_1 = 3/a$ and $C_2 = -2/b$. Therefore,

$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \frac{3}{a} \begin{pmatrix} a \\ 2a \end{pmatrix} e^{-0.5t} - \frac{2}{b} \begin{pmatrix} -b \\ b \end{pmatrix} e^{-2t}$

In particular, if $a = 1,\ b = 1$ then

$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-0.5t} - 2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t}$

$\Rightarrow \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 3e^{-0.5t} \\ 6e^{-0.5t} \end{pmatrix} - \begin{pmatrix} -2e^{-2t} \\ 2e^{-2t} \end{pmatrix}$

$x_1(t) = 3e^{-0.5t} + 2e^{-2t}$

$x_2(t) = 6e^{-0.5t} - 2e^{-2t}$

Sample Exercise-4.1

1. Find the eigenvalues and eigenvectors of the following matrices

a. $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Ans: $\lambda_1 = -1, \lambda_2 = 3, \boldsymbol{V}_1 = (-a, a)^T, \boldsymbol{V}_2 = (b, b)^T$
$\boldsymbol{V}_1|_{a=1} = (-1, 1)^T, \boldsymbol{V}_2|_{b=1} = (1, 1)^T$

b. $\begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$

Ans: $\lambda_1 = 2, \lambda_2 = -1, \boldsymbol{V}_1 = (-a, a)^T, \boldsymbol{V}_2 = (0, b)^T$
$\boldsymbol{V}_1|_{a=1} = (-1, 1)^T, \boldsymbol{V}_2|_{b=1} = (0, 1)^T$

c. $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

Ans: $\lambda_1 = -1, \lambda_2 = 5, \boldsymbol{V}_1 = (-2a, a)^T, \boldsymbol{V}_2 = (b, b)^T$
$\boldsymbol{V}_1|_{a=1} = (-2, 1)^T, \boldsymbol{V}_2|_{b=1} = (1, 1)^T$

d. $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & -5 & 2 \end{bmatrix}$

Ans: $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -2$
$\boldsymbol{V}_1 = (a, 0, 0)^T, \boldsymbol{V}_2 = (-b, 0, b)^T, \boldsymbol{V}_3 = (-0.2c, 0.8c, c)^T$
$\boldsymbol{V}_1|_{a=1} = (1,0,0)^T, \boldsymbol{V}_2|_{b=1} = (-1, 0,1)^T, \boldsymbol{V}_2|_{c=1} = (-0.2, 0.8,1)^T$

e. $\begin{bmatrix} 2 & 3 & 3 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$

Ans: $\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -1$
$\boldsymbol{V}_1 = (0, -a, a)^T, \boldsymbol{V}_2 = (-b, -b, b)^T, \boldsymbol{V}_3 = \left(-\frac{2c}{3}, -\frac{c}{3}, c\right)^T$
$\boldsymbol{V}_1|_{a=1} = (0, -1,1)^T, \boldsymbol{V}_2|_{b=1} = (-1, -1,1)^T, \boldsymbol{V}_2|_{c=1} = \left(-\frac{2}{3}, -\frac{1}{3}, 1\right)^T$

2. Solve the following system of differential equations using eigenvalue and eigenvector where $\dot{x_1}(t) = \frac{dx_1}{dt}$ and $\dot{x_2}(t) = \frac{dx_2}{dt}$.

a. $\begin{cases} \dot{x_1}(t) = x_1(t) + 2x_2(t) \\ \dot{x_2}(t) = 3x_1(t) + 2x_2(t) \end{cases}$
with $x_1(0) = 0, x_2(0) = -4.$

Ans.
$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = -\frac{8}{5a} e^{-t} \begin{pmatrix} -a \\ a \end{pmatrix} - \frac{4}{5b} e^{4t} \begin{pmatrix} 2b \\ 3b \end{pmatrix}$$
$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = -\frac{8}{5} e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \frac{4}{5} e^{4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

b. $\begin{cases} \dot{x_1}(t) = -5x_1(t) + x_2(t) \\ \dot{x_2}(t) = 4x_1(t) - 2x_2(t) \end{cases}$
with $x_1(0) = 1, x_2(0) = 2.$

Ans.
$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \frac{3}{5} e^{-t} \begin{pmatrix} 1 \\ 4 \end{pmatrix} - \frac{2}{5} e^{-6t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

c. $\begin{cases} \dot{x_1}(t) = x_2(t) \\ \dot{x_2}(t) = 1.5x_1(t) - 2.5x_2(t) \end{cases}$
with $x_1(0) = -4, x_2(0) = 9.$

Ans.
$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = -\frac{22}{7} e^{-3t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} - \frac{3}{7} e^{0.5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

<u>Cayley-Hamilton Theorem:</u>

Every square matrix is a zero of its characteristic polynomial.

Or,

Every square matrix satisfies its characteristic equation

$$i.e\ A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots \ldots\ldots + + a_1A + a_0I = 0$$

Example: Verify the Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Solution: The characteristic matrix of A is $A - \lambda I = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix}$$

Therefore, the characteristic equation of the matrix A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$
$$\Rightarrow \lambda^3 - \lambda^2 - 15\lambda - 15 = 0$$

Now in order to verify Cayley –Hamilton theorem we have to show that

$$A^3 - A^2 - 15A - 15I = 0$$

So, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 14 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 44 & 33 & 53 \\ 33 & 6 & 21 \\ 53 & 21 & 41 \end{bmatrix}$$

$$\therefore A^3 - A^2 - 15A - 15I$$

$$= \begin{bmatrix} 44 & 33 & 53 \\ 33 & 6 & 21 \\ 53 & 21 & 41 \end{bmatrix} - \begin{bmatrix} 14 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{bmatrix} - 15 \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} - 15 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore A^3 - A^2 - 15A - 15I = 0$

Hence Cayley-Hamilton theorem is verified.

Example: Using Cayley-Hamilton theorem find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Solution: The characteristic matrix of A is $A - \lambda I = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix}$$

Therefore, the characteristic equation of the matrix A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - \lambda^2 - 15\lambda - 15 = 0$$

Now according to the Cayley –Hamilton theorem, we have

$A^3 - A^2 - 15A - 15I = 0$

$\Rightarrow A^2 - A - 15I - 15A^{-1} = 0$

$\Rightarrow A^{-1} = \frac{1}{15}[A^2 - A - 15I]$

Here,

$$A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{15}\left[\begin{bmatrix} 14 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} - 15 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]$$

$$= \frac{1}{15}\begin{bmatrix} -2 & 1 & 5 \\ 1 & -8 & 5 \\ 5 & 5 & -5 \end{bmatrix}.$$

Example: Using Cayley-Hamilton theorem find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

The characteristic matrix of A is $A - \lambda I = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1-\lambda & 2 & 2 \\ 3 & 1-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{bmatrix}.$$

Therefore, the characteristic equation of the matrix A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 3 & 1-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 3\lambda^2 - 5\lambda + 1 = 0$$

Now according to the Cayley –Hamilton theorem, we have

$$A^3 - 3A^2 - 5A + I = 0$$

$\Rightarrow A^2 - 3A - 5I + A^{-1} = 0$

$A^{-1} = 3A + 5I - A^2$

$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 6 & 4 \\ 6 & 7 & 6 \\ 3 & 4 & 3 \end{bmatrix}$

$\therefore A^{-1} = 3\begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 9 & 6 & 4 \\ 6 & 7 & 6 \\ 3 & 4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & -6 \\ -2 & -1 & 5 \end{bmatrix}$

Sample Exercise-4.2

State the Cayley-Hamilton theorem. Hence find the inverse of the following matrices using Cayley – Hamilton theorem and verify your result.

a. $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ Ans. $A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ Ans. $A^{-1} = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & -6 \\ -2 & -1 & 5 \end{bmatrix}$

c. $A = \begin{bmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{bmatrix}$ Ans. $A^{-1} = \begin{bmatrix} -2 & 5 & -3 \\ -8 & 17 & -10 \\ 5 & -10 & 6 \end{bmatrix}$

d. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$ Ans. $A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$

e. $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$ Ans. $A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & -2 \\ 1 & 0 & -1 \end{bmatrix}$

Vector

Vector Spaces

Vectors in $\mathbb{R}^2$:

The set of all ordered pairs of real numbers is called two-dimensional vector space and is denoted by $\mathbb{R}^2$.

$\mathbb{R}^2 = \{(x, y): x, y \in \mathbb{R}\}$, where $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

Example: $(1,2); (2,-1); (3,4); (\sqrt{3}, 5); (0, \mathrm{e}) \in \mathbb{R}^2$

Vectors in $\mathbb{R}^3$:

The set of all ordered triplets of real numbers is called three-dimensional vector space and is denoted by $\mathbb{R}^3$

$\mathbb{R}^3 = \{(x, y, z): x, y, z \in \mathbb{R}\}$, where $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$

Vectors in $\mathbb{R}^n$:

If n is a positive integer then the set of all ordered $\boldsymbol{n}$ triplets of real numbers is called vector n-space and is denoted by $\mathbb{R}^n$ and if $\boldsymbol{u} \in \mathbb{R}^n$; $u = (\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots \ldots, \boldsymbol{u}_n)$, then $\boldsymbol{u}$ is called a n-dimensional vector in $\mathbb{R}^n$. A particular $\boldsymbol{n}$ triplets in $\mathbb{R}^{\mathbf{n}}$ is called co-ordinates of point.

Addition of two vectors in $\mathbb{R}^2$:

If **a**$= (a_1, a_2)$ and $\boldsymbol{b} = (b_1, b_2)$ be two vectors in $\mathbb{R}^2$ then $\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1, a_2 + b_2)$

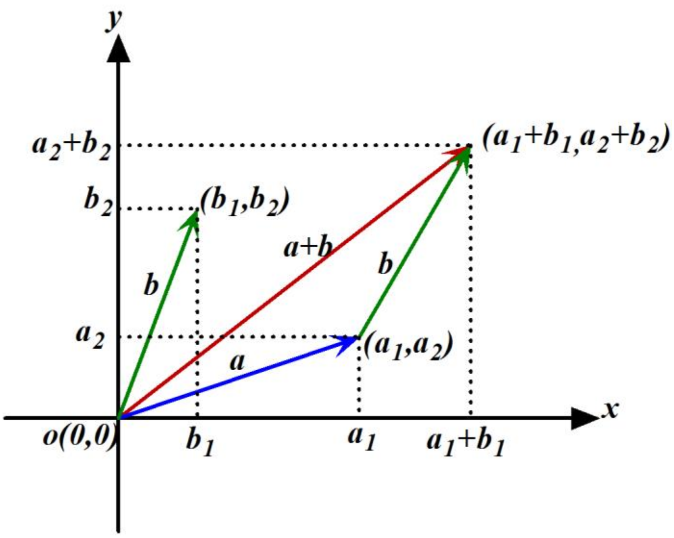

Fig: addition of two vectors.

Subtraction of two vectors in $\mathbb{R}^2$:

If **a**$= (a_1, a_2)$ and $\boldsymbol{b} = (b_1, b_2)$ be two vectors in $\mathbb{R}^2$ then $\boldsymbol{a} - \boldsymbol{b} = (a_1 - b_1, a_2 - b_2)$

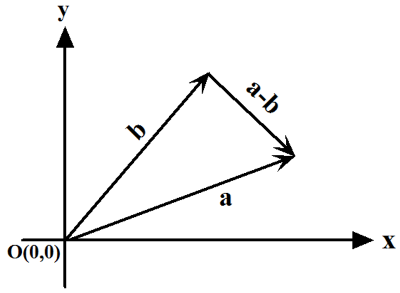

Fig: Subtraction of two vectors.

Scalar multiplication of vectors in $\mathbb{R}^2$:

Let $k \in \mathbb{R}$ be a scalar and $\boldsymbol{u} \in \mathbb{R}^2$ where $\boldsymbol{u} = (u_1, u_2)$ is a vector in $\mathbb{R}^2$ then

$$k\boldsymbol{u} = k(\boldsymbol{u}_1, \boldsymbol{u}_2) = (k\boldsymbol{u}_1, k\boldsymbol{u}_2)$$

Zero Vector:

The vector whose components are all zero is called the zero vector and is denoted by $\mathbf{0}$ and defined by $\mathbf{0} = (0,0, \dots\dots,0)$

Dot or Inner product of two non-zero vectors:

Let $\boldsymbol{u}(u_1, u_2)$ and $\boldsymbol{v}(v_1, v_2)$ are two non-zero vectors and θ be the angle between. The dot product of $\boldsymbol{u}$ and $\boldsymbol{v}$ is denoted by $\boldsymbol{u}, \boldsymbol{v}$ and defined by $\boldsymbol{u}.\boldsymbol{v} = \|\boldsymbol{u}\|\|\boldsymbol{v}\|\cos\theta = u_1v_1 + u_2v_2$

Parallel vectors in $\mathbb{R}^2$:

Let $\boldsymbol{u}, \boldsymbol{v} \epsilon \mathbb{R}^2$, then the vector u is called parallel vector of v if $\boldsymbol{u} = k\boldsymbol{v}$. If the non-zero scalar $k > 0$ then they are in the same directed and if $k < 0$ then they are in opposite directed.

Perpendicular vectors in $\mathbb{R}^2$:

Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^2$ then $\boldsymbol{u}$ and $\boldsymbol{v}$ are said to be perpendicular (or orthogonal) if $\boldsymbol{u}.\boldsymbol{v} = 0$

Example: Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^2$ where $\boldsymbol{u} = (3, -4), \boldsymbol{v} = (8,6)$; $\boldsymbol{u} \cdot \boldsymbol{v} = 3 \cdot 8 + (-4) \cdot 6 = 0$

Hence $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular vector.

Distance between two vectors in $\mathbb{R}^2$:

Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^2$ where $\boldsymbol{u}(u_1, u_2)$ and $\boldsymbol{v}(v_1, v_2)$, then the distance between $\boldsymbol{u}$ and $\boldsymbol{v}$, denoted by $d(\boldsymbol{u}, \boldsymbol{v})$ are defined by $d(\boldsymbol{u}, \boldsymbol{v}) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$

Example: If $\boldsymbol{u} = (5,4), \boldsymbol{v} = (1,1)$ then $d(\boldsymbol{u}, \boldsymbol{v}) = \sqrt{(5-1)^2 + (4-1)^2} = \sqrt{16+9} = 5$

Norm or Length in $\mathbb{R}^2$:

Let $\boldsymbol{u} \in \mathbb{R}^2$, where $\boldsymbol{u} = (u_1, u_2)$ then the norm or length or modulus of $\boldsymbol{u}$ denoted by $\|\boldsymbol{u}\|$ and is defined by $\|\boldsymbol{u}\| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}} = \sqrt{u_1^2 + u_2^2}$

Example: Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$; where $\boldsymbol{u} = (1,0,-1)$, $\boldsymbol{v} = (2,-3,1)$ then find

$(i) 2\boldsymbol{u} + 3\boldsymbol{v}$ $\quad (ii) \boldsymbol{u} \cdot \boldsymbol{v}$ $\quad$ (iii) $\|\boldsymbol{v}\|$ $\quad (iv)\ \|\boldsymbol{u} - \boldsymbol{v}\|$ $\quad (v) d(\boldsymbol{u}, \boldsymbol{v})$

Solution:

$(i)\ 2\boldsymbol{u} + 3\boldsymbol{v} = 2(1,0,-1) + 3(2,-3,1) = (2,0,-2) + (6,-9,3) = (8,-9,1)$

$(ii)\ \boldsymbol{u} \cdot \boldsymbol{v} = 1 \cdot 2 + 0 \cdot (-3) + (-1) \cdot 1 = 2 - 1 = 1$

$(iii)\ \|\boldsymbol{v}\| = \sqrt{4+9+1} = \sqrt{14}$

$(iv)\ \boldsymbol{u} - \boldsymbol{v} = (1-2, 0+3, -1-1) = (-1,3,-2)$

$\therefore \|\boldsymbol{u} - \boldsymbol{v}\| = \sqrt{(-1)^2 + 3^2 + (-2)^2} = \sqrt{14}$

$(v) d(\boldsymbol{u}, \boldsymbol{v}) = \sqrt{(1-2)^2 + (0+3)^2 + (-1-1)^2} = \sqrt{14}$.

Vector Spaces:

Let F be a field of scalars and $\boldsymbol{V}$ be a non-empty set of vectors. If $\boldsymbol{V}$ contains the following rules of vector addition and scalar multiplication, then $\boldsymbol{V}$ is called vector space over F.

In vector addition:

$A_1: \boldsymbol{u}, \boldsymbol{v} \in V \Rightarrow (\boldsymbol{u} + \boldsymbol{v}) \in \boldsymbol{V}$

$A_2: \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V} \Rightarrow (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$

$A_3: \boldsymbol{u}, \boldsymbol{0} \in \boldsymbol{V} \Rightarrow (\boldsymbol{u} + \boldsymbol{0}) = (\boldsymbol{0} + \boldsymbol{u}) = \boldsymbol{u},$ here $\boldsymbol{0}$ is the zero vector

$A_4: \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V} \Rightarrow \boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$

$A_5: \boldsymbol{u} \in \boldsymbol{V} \Rightarrow -\boldsymbol{u} \in \boldsymbol{V} \Rightarrow \boldsymbol{u} + (-\boldsymbol{u}) = (-\boldsymbol{u}) + \boldsymbol{u} = 0$

In scalar multiplication:

$M_1: a \in F$ and $\boldsymbol{u} \in \boldsymbol{V} \Rightarrow a\boldsymbol{u} \in \boldsymbol{V}$

$M_2: a \in F$ and $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V} \Rightarrow a(\boldsymbol{u} + \boldsymbol{v}) = a\boldsymbol{u} + a\boldsymbol{v}$

$M_3: a, b \in F$ and $\boldsymbol{u} \in \boldsymbol{V} \Rightarrow (a + b)\boldsymbol{u} = a\boldsymbol{u} + b\boldsymbol{u}$

$M_4: a, b \in F$ and $\boldsymbol{u} \in \boldsymbol{V} \Rightarrow (ab)\boldsymbol{u} = a(b\boldsymbol{u})$

$M_3: 1 \in F \ \Rightarrow 1.\boldsymbol{u} = \boldsymbol{u}.1 = \boldsymbol{u};$ $\quad \boldsymbol{u} \in F$

Subspace:

Let $\boldsymbol{W}$ be a non empty subset of a vector space $\boldsymbol{V}$ over the field F. We call $\boldsymbol{W}$ a subspace of $\boldsymbol{V}$ if and only if $\boldsymbol{W}$ is a vector space over the field F under the laws of vector addition and scalar multiplication defined on $\boldsymbol{V}$, or equivalently, $\boldsymbol{W}$ is a subspace of $\boldsymbol{V}$ wherever $w_1, w_2 \in \boldsymbol{W}$ and $\alpha, \beta \in F$ implies that $\alpha w_1 + \beta w_2 \in \boldsymbol{W}$.

Example: Show that $S = \{(a, b, c) \mid a, b, c \in \mathbb{R}$ and $a + b + c = 0\}$ is a subspace of the vector space $\mathbb{R}^{\mathbf{3}}$**.**

Solution: Here $\boldsymbol{0} \in \mathbb{R}^{\mathbf{3}}$, $\boldsymbol{0} = (0,0,0) \in S$ $\qquad \therefore 0 + 0 + 0 = 0$

Hence S is non empty set. Again, let $\boldsymbol{u} = (a_1, b_1, c_1) \in S$ and $\boldsymbol{v} = (a_2, b_2, c_2) \in S$

$a_1 + b_1 + c_1 = 0$ and $a_2 + b_2 + c_2 = 0$

For any scalars α, β we get $\alpha\boldsymbol{u} + \beta\boldsymbol{v} = \alpha(a_1, b_1, c_1) + \beta(a_2, b_2, c_2)$

$$= (\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2)$$

Now, $a + b + c = (\alpha a_1 + \beta a_2 + \alpha b_1 + \beta b_2 + \alpha c_1 + \beta c_2)$

$$= \alpha(a_1 + b_1 + c_1) + \beta(a_2 + b_2 + c_2) = \alpha.0 + \beta.0 = 0$$

$\therefore \alpha\boldsymbol{u} + \beta\boldsymbol{v} \in S$. Hence S is a subspace of $\mathbb{R}^{\mathbf{3}}$ **.**

Linear combination, Dependency and Independency of Vector

Linear combination:

Let $\boldsymbol{V}(F)$ be a vector space, where $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \dots \boldsymbol{v}_n \in \boldsymbol{V}$ and $\alpha_1, \alpha_2, \alpha_3 \dots \alpha_n \in F$. If $\alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2 + \alpha_3\boldsymbol{v}_3 + \cdots + \alpha_n\boldsymbol{v}_n = \boldsymbol{u} \in \boldsymbol{V}$, then $\boldsymbol{u}$ is called linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \dots \boldsymbol{v}_n$.

Linear dependency of vectors:

Let $\boldsymbol{V}(F)$ be a vector space, where $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \dots \boldsymbol{v}_n \in \boldsymbol{V}$ and $\alpha_1, \alpha_2, \alpha_3 \dots \alpha_n \in F$. If $\alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2 + \alpha_3\boldsymbol{v}_3 + \cdots + \alpha_n\boldsymbol{v}_n = \mathbf{0}$, and at least one of the element of the set $\{\alpha_1, \alpha_2, \alpha_3 \dots \alpha_n\}$ is not zero, then the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \dots \boldsymbol{v}_n$ are linearly dependent.

Linear independency of vectors:

Let $\boldsymbol{V}(F)$ be a vector space and $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \dots \boldsymbol{v}_n \in \boldsymbol{V}$ and $\alpha_1, \alpha_2, \alpha_3, \dots \alpha_n \in F$. If $\alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2 + \alpha_3\boldsymbol{v}_3 + \cdots + \alpha_n\boldsymbol{v}_n = \mathbf{0}$, and all of the elements of the set $\{\alpha_1, \alpha_2, \alpha_3 \dots \alpha_n\}$ are zero , then the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \dots \boldsymbol{v}_n$ are linearly independent.

Example: Write the vector $\boldsymbol{u} = (1, -2,5)$ as a linear combination of the vectors $\boldsymbol{u_1} = (1,1,1)$, $\boldsymbol{u_2} = (1,2,3)$ and $\boldsymbol{u_3} = (2, -1,1)$.

Solution: Let $\alpha_1\, \boldsymbol{u_1} + \alpha_2\, \boldsymbol{u_2} + \alpha_3\, \boldsymbol{u_3} = \boldsymbol{u}$ **;** where $\alpha_1, \alpha_2, \alpha_3$ are scalars.

$$\alpha_1(1,1,1) + \alpha_2(1,2,3) + \alpha_3(2, -1,1) = (1, -2,5)$$

$$(\alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 - \alpha_3, \alpha_1 + 3\alpha_2 + \alpha_3) = (1, -2,5)$$

Equating corresponding components

$$\alpha_1 + \alpha_2 + 2\alpha_3 = 1$$
$$\alpha_1 + 2\alpha_2 - \alpha_3 = -2$$
$$\alpha_1 + 3\alpha_2 + \alpha_3 = 5$$

Solving using elementary row operation, we get $\alpha_1 = -6, \alpha_2 = 3$ and $\alpha_3 = 2$.

$\therefore -6\, \boldsymbol{u_1} + 3\boldsymbol{u_2} + 2\boldsymbol{u_3} = \boldsymbol{u}$

Hence $\boldsymbol{u}$ is a linear combination of the vectors $\boldsymbol{u_1}, \boldsymbol{u_2}$ and $\boldsymbol{u_3}$.

Example: Test whether the vectors $\boldsymbol{u_1} = (1, 0, 1)$, $\boldsymbol{u_2} = (-3, 2, 6)$ & $\boldsymbol{u_3} = (4, 5, -2)$ are linearly dependent or independent.

Solution: Let $\alpha_1\, \boldsymbol{u_1} + \alpha_2\, \boldsymbol{u_2} + \alpha_3\, \boldsymbol{u_3} = \mathbf{0}$ **;** where $\alpha_1, \alpha_2, \alpha_3$ are scalars.

$$\alpha_1(1, 0, 1) + \alpha_2(-3, 2, 6) + \alpha_3(4, 5, -2) = (0, 0, 0)$$

$$(\alpha_1 - 3\alpha_2 + 4\alpha_3, 0 + 2\alpha_2 + 5\alpha_3, \alpha_1 + 6\alpha_2 - 2\alpha_3) = (0, 0, 0)$$

We can write from above equation

$$\alpha_1 - 3\alpha_2 + 4\alpha_3 = 0$$
$$0 + 2\alpha_2 + 5\alpha_3 = 0$$
$$\alpha_1 + 6\alpha_2 - 2\alpha_3 = 0$$

Solving the above linear system, we get $\alpha_1 = 0$, $\alpha_2 = 0$ and $\alpha_3 = 0$.
Hence the vectors $\boldsymbol{u_1}, \boldsymbol{u_2}$ and $\boldsymbol{u_3}$ are linearly independent

Sample Exercise-4.3

Write the vector $\boldsymbol{u}$ as a linear combination of the vectors $\boldsymbol{u_1}$, $\boldsymbol{u_2}$ and $\boldsymbol{u_3}$ **,** where

1. $\boldsymbol{u} = (5, 9, 5)$, $\boldsymbol{u_1} = (1, -1, 3)$, $\boldsymbol{u_2} = (2, 1, 4)$, $\boldsymbol{u_3} = (3, 2, 5)$**.**
2. $\boldsymbol{u} = (6, 20, 2)$, $\boldsymbol{u_1} = (1, 2, 3)$, $\boldsymbol{u_2} = (1, 3, -2)$, $\boldsymbol{u_3} = (1, 4, 1)$**.**
3. $\boldsymbol{u} = (4, 2, 1, 0)$, $\boldsymbol{u_1} = (3, 1, 0, 1)$, $\boldsymbol{u_2} = (1, 2, 3, 1)$, $\boldsymbol{u_3} = (0, 3, 6, 6)$**.**

Test whether the following vectors are linearly independent or dependent.

4. $\boldsymbol{u_1} = (2, -1, 4)$, $\boldsymbol{u_2} = (3, 6, 2)$, $\boldsymbol{u_3} = (2, 10, -4)$
5. $\boldsymbol{u_1} = (3, 0, 1, -1)$, $\boldsymbol{u_2} = (2, -1, 0, 1)$, $\boldsymbol{u_3} = (1, 1, 1, -2)$
6. $\boldsymbol{u_1} = (1, -1, 2)$, $\boldsymbol{u_2} = (3, -5, 1)$, $\boldsymbol{u_3} = (2, 7, 8)$, $\boldsymbol{u_4} = (-1, 1, 1)$
7. $\boldsymbol{u_1} = (2, 1, 2)$, $\boldsymbol{u_2} = (0, 1, -1)$, $\boldsymbol{u_3} = (4, 3, 3)$.
8. $\boldsymbol{u_1} = (2, 0, -1)$, $\boldsymbol{u_2} = (1, 1, 0)$, $\boldsymbol{u_3} = (0, -1, 1)$