Lecture 7

Gradient and Directional derivative

Objective

- To discuss about gradient
- To discuss about directional derivative
- To calculate Laplacian of a scalar function

Gradient

The Gradient Vector: grad

The gradient of a scalar function $T(u_1, u_2, u_3)$ is given by

$$\text{gradT} = \boldsymbol{\nabla}\,\text{T} = \frac{\hat{u}_1}{h_1}\frac{\partial T}{\partial u_1} + \frac{\hat{u}_2}{h_2}\frac{\partial T}{\partial u_2} + \frac{\hat{\mathrm{u}}_3}{h_3}\frac{\partial T}{\partial u_3}\,.$$

where h_i are scale factors and $\hat{u}_i$ are the unit vectors along u_i, $(i = 1, 2, 3)$.

- **For Cartesian coordinates** $u_1 = x, u_2 = y, u_3 = z, h_1 = h_2 = h_3 = 1,$
- **For cylindrical coordinates** $u_1 = r, u_2 = \varphi, u_3 = z, h_1 = h_3 = 1$ and $h_2 = r$
- **For spherical coordinates** $u_1 = R, u_2 = \theta,\ u_3 = \varphi, h_1 = 1, h_2 = R\&\ h_3 = R\sin\theta.$

The Gradient Vector: grad

Example. Find the gradient of the following scalar functions:

$T(x, y, z) = \frac{xyz}{(x^2+y^2+z^2)}$ at the point $(1, 1,1)$.

For Cartesian coordinates, $\text{gradT} = \nabla T = \hat{\mathrm{x}}\frac{\partial T}{\partial x} + \hat{\mathrm{y}}\frac{\partial T}{\partial y} + \hat{\mathrm{z}}\frac{\partial T}{\partial z}\cdot$

$$= \hat{\mathrm{x}}\frac{\partial}{\partial x}\left(\frac{xyz}{(x^2 + y^2 + z^2)}\right) + \hat{\mathrm{y}}\frac{\partial}{\partial y}\left(\frac{xyz}{(x^2 + y^2 + z^2)}\right) + \hat{\mathrm{z}}\frac{\partial}{\partial z}\left(\frac{xyz}{(x^2 + y^2 + z^2)}\right)$$

$$= \hat{\mathrm{x}}\frac{(x^2 + y^2 + z^2)yz - 2x^2yz}{(x^2 + y^2 + z^2)^2} + \hat{\mathrm{y}}\frac{(x^2 + y^2 + z^2)xz - 2y^2xz}{(x^2 + y^2 + z^2)^2} + \hat{\mathrm{z}}\frac{(x^2 + y^2 + z^2)xy - 2z^2xy}{(x^2 + y^2 + z^2)^2}$$

At the point $(1, 1,1)$, $\nabla T = \frac{1}{9}(\hat{\mathrm{x}} + \hat{\mathrm{y}} + \hat{\mathrm{z}})\cdot$

The Gradient Vector: grad

Example. Find the gradient of the following scalar functions:

$T(r, \phi, z) = \frac{z\cos\phi}{(1+r^2)}$, at the point $(1, \pi, 2)$

For cylindrical coordinates, $\quad \text{gradT} = \boldsymbol{\nabla}\, T = \frac{\hat{\mathrm{r}}}{1}\frac{\partial T}{\partial r} + \frac{\hat{\phi}}{r}\frac{\partial T}{\partial \phi} + \frac{\hat{\mathrm{z}}}{1}\frac{\partial T}{\partial z}$

$$= \hat{\mathrm{r}}\frac{\partial}{\partial r}\left(\frac{z\cos\phi}{(1+r^2)}\right) + \hat{\phi}\frac{1}{\mathrm{r}}\frac{\partial}{\partial \phi}\left(\frac{z\cos\phi}{(1+r^2)}\right) + \hat{\mathrm{z}}\frac{\partial}{\partial z}\left(\frac{z\cos\phi}{(1+r^2)}\right)$$

$$= -\hat{\mathrm{r}}\frac{2zr\cos\phi}{(1+r^2)^2} - \hat{\phi}\frac{1}{\mathrm{r}}\frac{z\sin\phi}{(1+r^2)} + \hat{\mathrm{z}}\frac{\cos\phi}{(1+r^2)}\cdot$$

At the point $(1, \pi, 2), \boldsymbol{\nabla}\, T = \hat{\mathrm{r}} - \hat{\mathrm{z}}\frac{1}{2}\cdot$

The Gradient Vector: grad

Example. Find the gradient of the following scalar functions:

$T(R,\theta,\phi) = R\cos\theta\sin\phi$, at the point $\left(2,\frac{\pi}{2},\frac{\pi}{4}\right)\cdot$

- **For spherical coordinates,** $\mathrm{grad}T = \nabla T = \frac{\hat{\mathrm{R}}}{1}\frac{\partial T}{\partial R} + \frac{\hat{\theta}}{R}\frac{\partial T}{\partial \theta} + \frac{\hat{\phi}}{R\sin\theta}\frac{\partial T}{\partial \phi}$

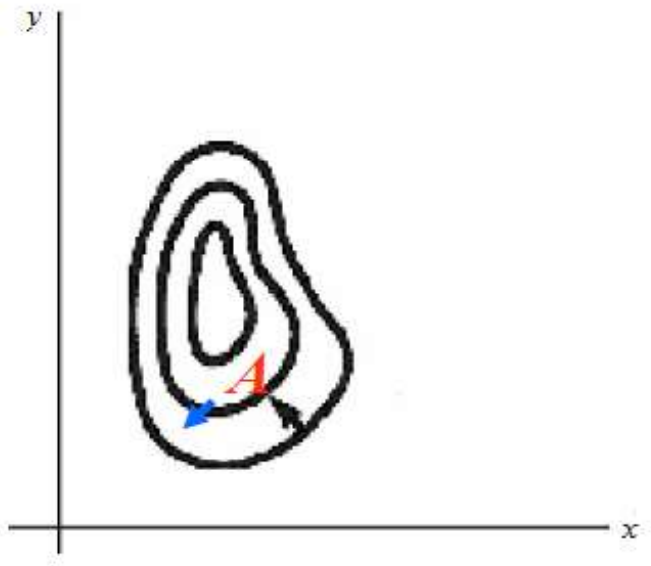
$$= \hat{\mathrm{R}}\frac{\partial}{\partial R}(R\cos\theta\sin\phi) + \frac{\hat{\theta}}{R}\frac{\partial}{\partial \theta}(R\cos\theta\sin\phi) + \frac{\hat{\phi}}{R\sin\theta}\frac{\partial}{\partial \phi}(R\cos\theta\sin\phi)$$

$$= \hat{\mathrm{R}}\cos\theta\sin\phi - \hat{\theta}\sin\theta\sin\phi + \frac{\hat{\phi}}{R\sin\theta}R\cos\theta\cos\phi.$$

At the point $\left(2,\frac{\pi}{2},\frac{\pi}{4}\right)$,$\nabla T = -\hat{\theta}\frac{1}{\sqrt{2}}$.

Directional Derivatives

For a given function $T = T(u_1, u_2)$, the **directional derivative** in the direction of a unit vector is the **gradient vector** at a point A



- magnitude = the largest directional derivative, and
- pointing in the direction in which this largest directional derivative occurs, is known as the **gradient vector**.

Hence, the component of ∇T in the direction of a vector $\mathbf{d}$ is equal to $\nabla T \cdot \mathbf{d}$ and it is called the directional derivative of T in the direction of $\mathbf{d}$.

Directional Derivatives

Example: Find the directional derivative of $T(x,y,z) = xy^2 - z^2$ at the point $(1,-1,4\,)$ in the direction $\mathbf{d} = \hat{\mathrm{x}} - \hat{\mathrm{y}} + 4\,\hat{\mathrm{z}}$.

Solution: $\boldsymbol{\nabla}\,\mathrm{T} = \hat{\mathrm{x}}\dfrac{\partial}{\partial x}(xy^2 - z^2) + \hat{\mathrm{y}}\dfrac{\partial}{\partial y}(xy^2 - z^2) + \hat{\mathrm{z}}\dfrac{\partial}{\partial z}(xy^2 - z^2)$

$$= \hat{\mathrm{x}}\;y^2 + \hat{\mathrm{y}}\;2xy + \hat{\mathrm{z}}\frac{\partial}{\partial z}(-2z)$$

At the point $(1,-1,4\,)$, $\boldsymbol{\nabla}\,T = \hat{\mathrm{x}} - 2\,\hat{\mathrm{y}} - 8\,\hat{\mathrm{z}}$

Now, the unit vector in the direction of $\hat{\mathrm{x}} - \hat{\mathrm{y}} + 4\,\hat{\mathrm{z}}$ is

$$\hat{\mathrm{a}} = \frac{\hat{\mathrm{x}} - \hat{\mathrm{y}} + 4\,\hat{\mathrm{z}}}{\sqrt{1+1+16}} = \frac{1}{\sqrt{18}}\hat{\mathrm{x}} - \frac{1}{\sqrt{18}}\hat{\mathrm{y}} + \frac{4}{\sqrt{18}}\hat{\mathrm{z}}$$

Then the required directional derivative is,

$$\boldsymbol{\nabla}\,T \cdot \hat{\mathrm{a}} = (\hat{\mathrm{x}} - 2\,\hat{\mathrm{y}} - 8\,\hat{\mathrm{z}}) \cdot \left(\frac{1}{\sqrt{18}}\hat{\mathrm{x}} - \frac{1}{\sqrt{18}}\hat{\mathrm{y}} + \frac{4}{\sqrt{18}}\hat{\mathrm{z}}\right) = \frac{1}{\sqrt{18}} + \frac{2}{\sqrt{18}} - \frac{32}{\sqrt{18}} = -\frac{29}{\sqrt{18}}\,.$$

Directional Derivatives

Example: Find the directional derivative of $T(r, \phi, z) = \frac{1}{2} e^{-\frac{r}{5}} \cos\phi$ at the point $\left(2, \frac{\pi}{4}, 3\right)$ in the direction $\hat{\mathrm{r}}$.

Solution: $\nabla T = \hat{\mathrm{r}} \frac{\partial}{\partial r}\left(\frac{1}{2} e^{-r/5} \cos\phi\right) + \widehat{\phi} \frac{1}{\mathrm{r}} \frac{\partial}{\partial \phi}\left(\frac{1}{2} e^{-r/5} \cos\phi\right) + \hat{\mathrm{z}} \frac{\partial}{\partial z}\left(\frac{1}{2} e^{-r/5} \cos\phi\right)$

$$= -\hat{\mathrm{r}} \frac{1}{10} e^{-r/5} \cos\phi - \widehat{\phi} \frac{1}{2\mathrm{r}} e^{-r/5} \sin\phi$$

At the point $\left(2, \frac{\pi}{4}, 3\right)$, $\nabla T = -\hat{\mathrm{r}} \frac{1}{10\sqrt{2}} e^{-2/5} - \widehat{\phi} \frac{1}{4\sqrt{2}} e^{-2/5}$

Then the required directional derivative is, $\nabla T \cdot \hat{\mathrm{r}} = -\frac{1}{10\sqrt{2}} e^{-2/5}$

Directional Derivatives

Example: Find the directional derivative of $T(R,\theta,\phi)=\frac{1}{R}\sin^2\theta$ at the point $\left(5,\frac{\pi}{4},\frac{\pi}{2}\right)$ in the direction $\widehat{\mathrm{R}}$.

Solution: $\nabla T=\widehat{\mathrm{R}}\frac{\partial}{\partial R}\left(\frac{1}{R}\sin^2\theta\right)+\frac{\widehat{\theta}}{R}\frac{\partial}{\partial\theta}\left(\frac{1}{R}\sin^2\theta\right)+\frac{\widehat{\phi}}{R\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{R}\sin^2\theta\right)$

$$=-\widehat{\mathrm{R}}\frac{1}{R^2}\sin^2\theta+\widehat{\theta}\frac{1}{R^2}2\sin\theta\cos\theta$$

At the point $\left(5,\frac{\pi}{4},\frac{\pi}{2}\right)$, $\nabla T=-\widehat{\mathrm{R}}\frac{1}{50}+\widehat{\theta}\frac{1}{25}$

Then the required directional derivative is, $\nabla T\cdot\widehat{\mathrm{R}}=-\frac{1}{50}$.

Laplacian Operator

Laplacian operator: $\boldsymbol{\nabla}^2 = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$

Laplace Equation: $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\mathrm{T}) = \boldsymbol{\nabla}^2 T(u_1, u_2, u_3) = 0$

The Laplacian of a scalar function T in different coordinate system are defined as follows:

In Cartesian coordinates $\boldsymbol{\nabla}^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$,

In Cylindrical coordinates $\boldsymbol{\nabla}^2 T = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}$ and

In Spherical coordinates $\boldsymbol{\nabla}^2 T = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial T}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{R^2 \sin^2\theta}\left(\frac{\partial^2 T}{\partial \phi^2}\right)$.

Laplacian Operator

Example: Find the Laplacian of the scalar function $\mathrm{T} = \frac{3}{\mathrm{x}^2+\mathrm{y}^2}$.

Solution: In Cartesian co-ordinates we know the Laplacian is

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(3(x^2+y^2)^{-1})\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}(3(x^2+y^2)^{-1})\right) + \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}(3(x^2+y^2)^{-1})\right)$$

$$= \frac{\partial}{\partial x}(-3(x^2+y^2)^{-2}.2x) + \frac{\partial}{\partial y}(-3(x^2+y^2)^{-2}.2y) + 0$$

$$= 24x^2(x^2+y^2)^{-3} - 6(x^2+y^2)^{-2} + 24y^2(x^2+y^2)^{-3} - 6(x^2+y^2)^{-2}$$

$$= \frac{12}{(x^2+y^2)^2}$$

Laplacian Operator

Example: Find the Laplacian of the scalar function $\mathrm{T} = 5\mathrm{e}^{-\mathrm{r}}\cos\phi$.

Solution: In Cylindrical coordinates we know the Laplacian is

$$\nabla^2 T = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(5\mathrm{e}^{-\mathrm{r}}\cos\phi)\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(\frac{\partial}{\partial \phi}(5\mathrm{e}^{-\mathrm{r}}\cos\phi)\right) + \frac{\partial}{\partial \mathrm{z}}\left(\frac{\partial}{\partial \mathrm{z}}(5\mathrm{e}^{-\mathrm{r}}\cos\phi)\right)$$

$$= \frac{5\cos\phi}{r}\frac{\partial}{\partial r}(-r\, e^{-r}) + \frac{5\mathrm{e}^{-\mathrm{r}}}{r^2}\frac{\partial}{\partial \phi}(-sin\phi) + 0$$

$$= -\frac{5\cos\phi}{r}[-e^{-r} - r\, e^{-r}] - \frac{5\mathrm{e}^{-\mathrm{r}}}{r^2}cos\phi$$

$$= \frac{-5e^{-r}\cos\phi}{r} + 5\mathrm{e}^{-\mathrm{r}}\cos\phi - \frac{5\mathrm{e}^{-\mathrm{r}}}{r^2}cos\phi.$$

Sample Exercise

1. Find the gradient of the following scalar functions at the indicated point:

(a) $T(x,y,z) = 2x^3 y\, z + \mathrm{y}^2\, x^2 - 5\, \frac{y}{z}$ at the point $(0, 2, -1)$.

(b) $T(r,\phi,z) = \frac{z+\sin\phi}{r}$, at the point $\left(2, \frac{3\pi}{2}, 1\right) \cdot$

(c) $T(R,\theta,\phi) = R^2 \cos\phi \sin\theta$, at the point $\left(2, \frac{\pi}{4}, \frac{2\pi}{3}\right) \cdot$

2. (a) Find the directional derivative of $T(x,y,z) = x^2 y - xz$ at the point $(1, 0, 2\,)$in the direction $\mathbf{d} = \hat{\mathrm{x}} - 2\hat{\mathrm{y}} - 6\,\hat{\mathrm{z}}$.

(b) Find the directional derivative of $T(r,\phi,z) = r^3 \cos\phi$ at the point $\left(2, \frac{\pi}{4}, 1\right)$ in the direction $\hat{\mathrm{r}}$.

(c) Find the directional derivative of $T(R,\theta,\phi) = \frac{1}{R}\cos^2\theta$ at the point $\left(1, \frac{\pi}{4}, \frac{\pi}{2}\right)$ in the direction $\widehat{\mathrm{R}} - \hat{\theta}$.

3. Find the Laplacian of the following scalar functions:

(a) $\mathrm{T} = 4y^2 z^2 \cdot$ **(b)** $\mathrm{T} = xy + zx \cdot$ **(c)** $\mathrm{T} = 10\mathrm{r}^3 \cos 2\phi$

Sample MCQ

- Given $\mathrm{T}(x, y, z) = 2x^3 y\, z + \mathrm{y}^2\, x^2 - 5\,\frac{y}{z}$ at the point $(0, 2, -1)$. grad T $= \mathbf{\nabla}\mathrm{T} = ?$

a) $\nabla\mathrm{T} = -5\,\hat{\mathrm{y}} + 10\hat{\mathrm{z}}$ b) $\nabla\mathrm{T} = 5\,\hat{\mathrm{y}} + 10\hat{\mathrm{z}}$ c) $\nabla\mathrm{T} = 5\,\hat{\mathrm{y}} - 10\hat{\mathrm{z}}$ d) $\nabla\mathrm{T} = -5\,\hat{\mathrm{y}} - 10\hat{\mathrm{z}}$

- Which one is the directional derivative (D.D.) of $T(x, y, z) = x^2 y - xz$ at the point $(1, 0, 2)$ in the direction $\mathbf{d} = \hat{\mathrm{x}} - 2\hat{\mathrm{y}} - 6\,\hat{\mathrm{z}}$?

  a) $\frac{6}{\sqrt{41}}$ b) $\frac{2}{\sqrt{41}}$ c) $\frac{-2}{\sqrt{41}}$ d) $\frac{1}{\sqrt{41}}$

- Which one of the following is the directional derivative of $T(x, y, z) = x^2 y\, z$ at the point $(1, 0, 2)$ in the direction $= \hat{\mathrm{y}}$?

  a) 1 b) 2 c) -1 d) 3

Outcome

After this lecture student will know

- About the idea of gradient and directional derivative
- How to find directional derivative in the direction of a given vector
- How to find Laplacian of a scalar function

Next class

- Divergence, curl