Lecture 10

Divergence Theorem

Gauss's divergence theorem

Statement:

The surface integral of the normal component of a vector function **A** taken around a closed surface S is equal to the integral of the divergence of **A** taken over the volume V enclosed by the surface S.

Mathematically,

$$\int_V \nabla \cdot \mathbf{A}\, dv = \oint_S \mathbf{A} \cdot d\mathbf{s}$$

Gauss's divergence theorem

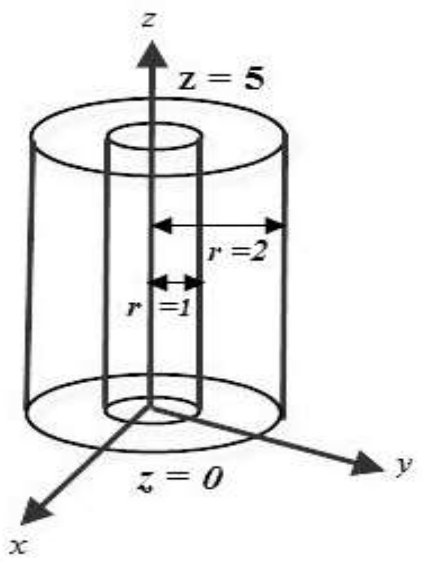
Example:

A vector field $\mathbf{A} = \hat{\mathbf{r}}r^3$ exists in the region between two concentric cylindrical surfaces defined by $r = 1$ and $r = 2$, with both cylinders extending between $z = 0$ and $z = 5$. Verify the divergence theorem by evaluating the following:

(a) $\oint_s \mathbf{A} \cdot d\mathbf{s}$ and (b) $\int_v \nabla \cdot \mathbf{A} dv$.

Gauss's divergence theorem

Solution:



Step 1: Evaluate the surface integral over the all faces.

i. Top face: $z = 5,\ \mathbf{A} = \hat{r}r^3$and $d\boldsymbol{s} = \hat{z}\,rdrd\phi$ $\int_{\substack{\text{top}\\\text{face}}} \mathbf{A}\cdot d\boldsymbol{s} = 0.$

Gauss's divergence theorem

Solution:

ii. Bottom face: $z = 0$, $\mathbf{A} = \hat{\mathrm{r}} r^3$ and $d\boldsymbol{s} = -\hat{\mathrm{z}} r dr d\phi$

$$\int_{\substack{\text{bottom}\\\text{face}}} \mathbf{A}\cdot d\boldsymbol{s} = 0.$$

iii. Outside surface: $r = 2, \mathbf{A} = \hat{\mathrm{r}}\, 8$ and $d\boldsymbol{s} = \hat{\mathrm{r}} r dz d\phi$

$$\therefore \int_{\text{outside}} \mathbf{A}\cdot d\boldsymbol{s} = 16 \int_0^{2\pi}\int_0^{5} dz d\phi = 160\pi.$$

iv. Inside surface: $r = 1, \mathbf{A} = \hat{\mathrm{r}}$ and $d\boldsymbol{s} = -\hat{\mathrm{r}} r dz d\phi$

$$\therefore \int_{\text{intside}} \mathbf{A}\cdot d\boldsymbol{s} = -\int_0^{2\pi}\int_0^{5} dz d\phi = -10\pi.$$

Gauss's divergence theorem

Solution:

Step 2: Adding the above four values.

Thus, we have $\oint_s \mathbf{A} \cdot d\mathbf{s} = 160\pi - 10\pi = 150\pi$

Step 3: Find the divergence of **A**.

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial}{\partial \phi}(A_\phi) + \frac{1}{r}\frac{\partial}{\partial z}(rA_z)$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(rr^3) + 0 + \frac{1}{r}\frac{\partial}{\partial \phi}(0) = \frac{1}{r}\frac{\partial}{\partial r}(r^4) = 4r^2.$$

$$\int_V \nabla \cdot \mathbf{A} dv = \int_0^5 \int_0^{2\pi} \int_1^2 4r^2 r dr d\varphi dz = \int_0^5 \int_0^{2\pi} \int_1^2 4r^3 dr d\phi dz = \int_0^5 \int_0^{2\pi} [r^4]_1^2 d\phi dz = 150\pi.$$

Step 4: Check whether $\int_V \nabla \cdot \mathbf{A}\, dv = \oint_s \mathbf{A} \cdot d\mathbf{s}$.

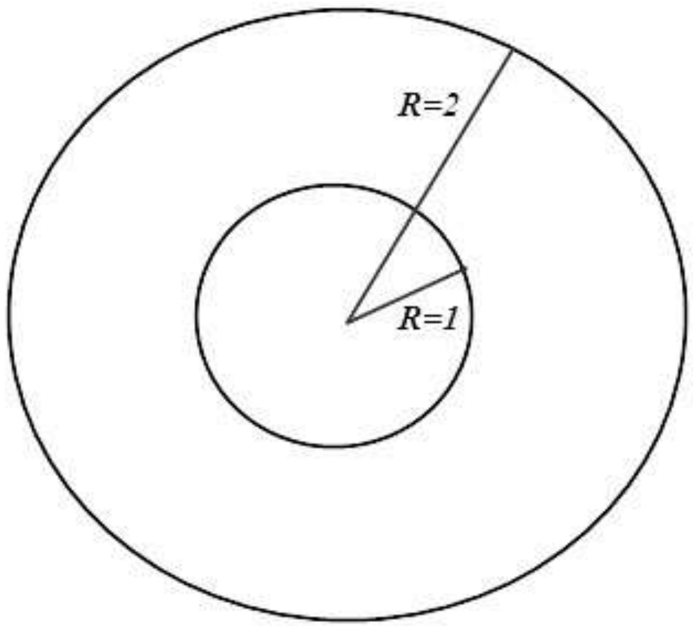
Since, the result of step 2 and step 3 are equal.

Therefore, The divergence theorem is therefore verified.

Gauss's divergence theorem

Example:

For the vector field $A = \hat{R}\,3R^2,$ evaluate both sides the divergence theorem for the region enclosed between spherical shells defined by $R = 1$ and $R = 2$.



Gauss's divergence theorem

Solution:

Step 1: Evaluate the surface integral over the all faces.

i. At the outer surface $d\mathbf{s} = \widehat{\mathrm{R}}(R_2)^2 \sin\theta \, d\theta d\phi$ and

we get $\mathbf{A} \cdot d\mathbf{s} = 3(R_2)^4 \sin\theta \, d\theta d\phi$

$\therefore \oint_s \mathbf{A} \cdot d\mathbf{s} = \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} 3 \times 2^4 \times \sin\theta \, d\theta \, d\phi = 192\pi.$

ii. At the inner surface $d\mathbf{s} = -\widehat{\mathrm{R}}(R_2)^2 \sin\theta \, d\theta d\phi$

we get $\mathbf{A} \cdot d\mathbf{s} = 3(R_2)^4 \sin\theta \, d\theta d\phi$

$\therefore \oint_s \mathbf{A} \cdot d\mathbf{s} = -\int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} 3 \times 1^4 \times \sin\theta \, d\theta \, d\phi = -12\pi.$

Gauss’s divergence theorem

Solution:

Step 2: Adding the above four values.

Thus, we have $\oint_s \mathbf{A} \cdot d\mathbf{s} = 192\pi - 12\pi = 180\pi$.

Step 3: Find the divergence of **A**.

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2 \sin\theta}\left[\frac{\partial}{\partial R}(3R^2 \cdot R^2 \sin\theta) + \frac{\partial}{\partial\theta}(0 \cdot R\sin\theta) + \frac{\partial}{\partial\phi}(0 \cdot R)\right] = 12R$$

Now, Outer sphere: $\int_V \nabla \cdot \mathbf{A}\, dv = \int_V 12R\, dv = \int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\int_{R=0}^{2} 12\, R^3 \sin\theta\, dR\, d\phi\, d\theta = 192\pi$

Inner sphere: $\int_V \nabla \cdot \mathbf{A}\, dv = \int_V 12R\, dv = \int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\int_{R=0}^{1} 12\, R^3 \sin\theta\, dR\, d\phi\, d\theta = 12\pi$

Total $= 192\pi - 12\pi = 180\pi$.

Step 4: Check whether $\int_V \nabla \cdot \mathbf{A}\, dv = \oint_s \mathbf{A} \cdot d\mathbf{s}$.

Since, the result of step 2 and step 3 are equal.

Therefore, The divergence theorem is therefore verified.

Sample Exercise

1. For a vector function $\mathbf{A} = \hat{\mathrm{r}} r^2 + \hat{\mathrm{z}}\, 2z$, verify for the circular cylindrical region enclosed by $r = 5, z = 0, z = 4$.

2. A vector field $\mathbf{A} = \hat{\mathrm{r}}\, 10e^{-r} - \hat{\mathrm{z}}\, 3z$, verify the divergence theorem for the cylindrical region enclosed by $r = 2, z = 0$ and $z = 4$.

3. Find $\oint_s \mathbf{A} \cdot d\mathbf{s}$ over the surface of a hemispherical region that is the top half of a sphere of radius 3 centered at $(0, 0, 0)$ with its flat base coinciding with the xy plane. Also verify divergence theorem. where $\mathbf{A} = \hat{\mathrm{z}}\, z$.

Sample MCQ

- Which of the following is the mathematical definition for Stokes theorem

a) $\int_V \nabla \cdot A \, dv = \oint_s A \times ds$

b) $\int_V \nabla \cdot A \, dv = \oint_s A \cdot ds$

c) $\int_s (\nabla \times A) \cdot ds = \oint_c A \times dl$.

d) $none$

Next Class

- Stokes theorem