

Lecture 11

Stoke's Theorem

Objective:

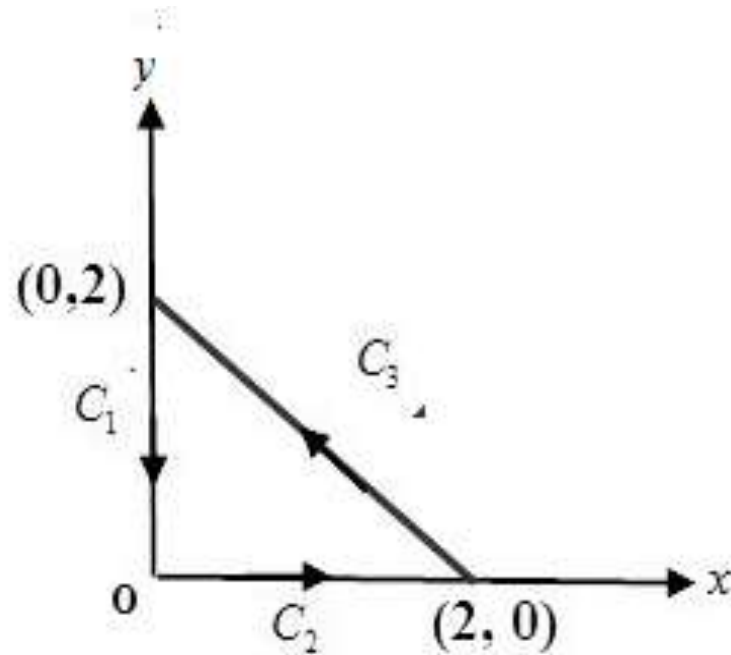
- **To know the statement of Stoke's theorem**
- **To know how to verify Stoke's theorem in a vector field and given contour .**

Stokes's Theorem:

Let S be the open surface (two-sided) and C be the closed boundary of S , the vector field A is continuous on S . Then

$$\int_S (\nabla \times A) \cdot d\mathbf{s} = \oint_C A \cdot d\mathbf{l} .$$

Example 1: Assume that a vector field $A = \hat{x}(2x^2 + y^2) + \hat{y}(xy - y^2)$, (a) find $\oint_C A \cdot d\mathbf{l}$ around the triangular contour, (b) find $\int_S (\nabla \times A) \cdot d\mathbf{s}$ over triangular arc, (c) verify Stokes's theorem.



Solution:

$$(a) \, dl = \hat{x} \, dx + \hat{y} \, dy \therefore A \cdot dl = (2x^2 + y^2)dx + (xy - y^2)dy$$

$$\text{Path } c_1; x = 0, dx = 0, \oint_{c_1} A \cdot dl = - \int_2^0 y^2 dy = \frac{8}{3}.$$

$$\text{Path } c_2; y = 0, dy = 0, \oint_{c_2} A \cdot dl = \int_0^2 2x^2 dx = \frac{16}{3}.$$

$$\text{Path } c_3; y = 2 - x, dy = -dx,$$

$$\oint_{c_3} A \cdot dl = \int_2^0 [\{2x^2 + (2 - x)^2\} - \{x(2 - x) - (2 - x)^2\}] dx = -\frac{28}{3}.$$

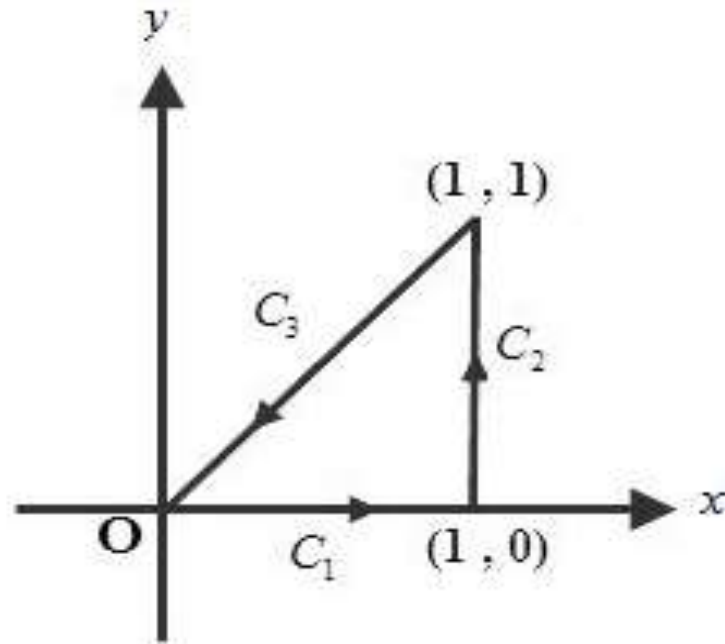
$$\text{Total, } \oint_C \mathbf{A} \cdot d\mathbf{l} = -\frac{4}{3}.$$

$$\text{Now, } \nabla \times \mathbf{A} = -\hat{\mathbf{z}} y, d\mathbf{s} = \hat{\mathbf{z}} dx dy$$

$$\therefore \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = - \int_0^2 \int_0^{2-x} y dy dx = -\frac{4}{3}.$$

(c) Stokes's theorem is verified.

Example 2: Assume that a vector field $A = \hat{x} xy - \hat{y}(x^2 + 2y^2)$, (a) find $\oint_C A \cdot dl$ around the triangular contour, (b) find $\int_S (\nabla \times A) \cdot ds$ over triangular arc.



Solution:

$$(a) \, dl = \hat{x} \, dx + \hat{y} \, dy \therefore A \cdot dl = xy \, dx - (x^2 + 2y^2) \, dy$$

$$\text{Path } c_1; y = 0, \, dy = 0, \oint_{c_1} A \cdot dl = \int_0^1 xy \, dx - (x^2 + 2y^2) \, dy = 0.$$

$$\text{Path } c_2; x = 1, \, dx = 0, \oint_{c_2} A \cdot dl = \int_0^1 xy \, dx - (x^2 + 2y^2) \, dy$$

$$= \int_0^1 -(1 + 2y^2) \, dy = -\frac{5}{3}.$$

Path c_3 ; $y = x$, $dy = dx$,

$$\oint_{c_3} \mathbf{A} \cdot d\mathbf{l} = \int_1^0 xy \, dx - (x^2 + 2y^2) \, dy = \int_1^0 y^2 \, dy - (y^2 + 2y^2) \, dy = \frac{2}{3}.$$

$$\text{Total, } \oint_c \mathbf{A} \cdot d\mathbf{l} = -\frac{5}{3} + \frac{2}{3} = -1.$$

$$\text{Now, } \nabla \times \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -(x^2 + 2y^2) & 0 \end{vmatrix} = -3x \hat{z}, ds = \hat{z} \, dx dy$$

$$\therefore \int_s (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = - \int_0^1 \int_0^x 3x \, dy dx = -1.$$

Stokes's theorem is verified.

Sample MCQ:

1. Which one indicates the correct statement of Stoke's theorem?

a) $\int_s (\nabla \times A) \cdot d\mathbf{s} = \oint_c A \cdot d\mathbf{l}$

b) $\int_s (\nabla \cdot A) d\mathbf{s} = \oint_c A \cdot d\mathbf{l}$

c) $\int_v (\nabla \times A) \cdot d\mathbf{v} = \oint_c A \cdot d\mathbf{l}$

d) $\int_v (\nabla \cdot A) d\mathbf{v} = \oint_c A \cdot d\mathbf{l}$

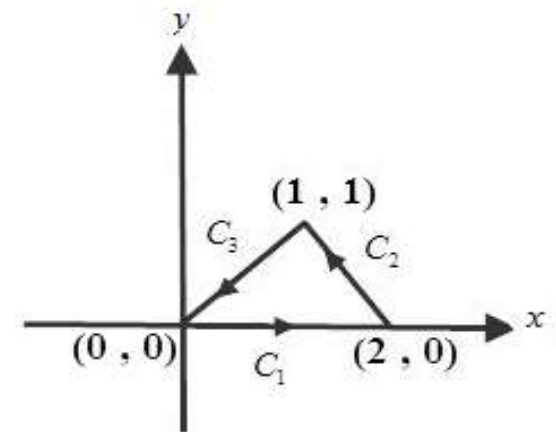
2. Assume that a vector field $A = \hat{x} xy - \hat{y}(x^2 + 2y^2)$, find $\oint_{c_2} A \cdot d\mathbf{l}$ using the following triangular arc.

a) $\frac{11}{3}$

b) 0

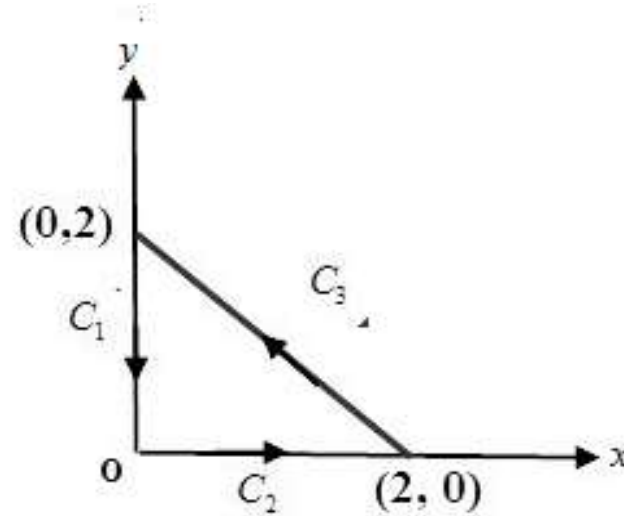
c) $-\frac{11}{3}$

d) 5



3. Assume that a vector field $A = \hat{x}(2x^2 + y^2) + \hat{y}(xy - y^2)$, find $\oint_{C_3} A \cdot dl$ around the given triangular contour

- a) 0
- b) $-28/3$
- c) $8/3$
- d) $16/3$



4. For the same vector field and triangular contour given in question (3), along C_2 $A \cdot dl = ?$

- a) $A \cdot dl = (2x^2)dx$
- b) $A \cdot dl = (xy - y^2)dy$
- c) $A \cdot dl = (2x^2 + y^2)dx$
- d) $A \cdot dl = (2x^2 + y^2)dx + (xy - y^2)dy$

Outcome:

Clear concept about the Stoke's theorem in triangular contour.