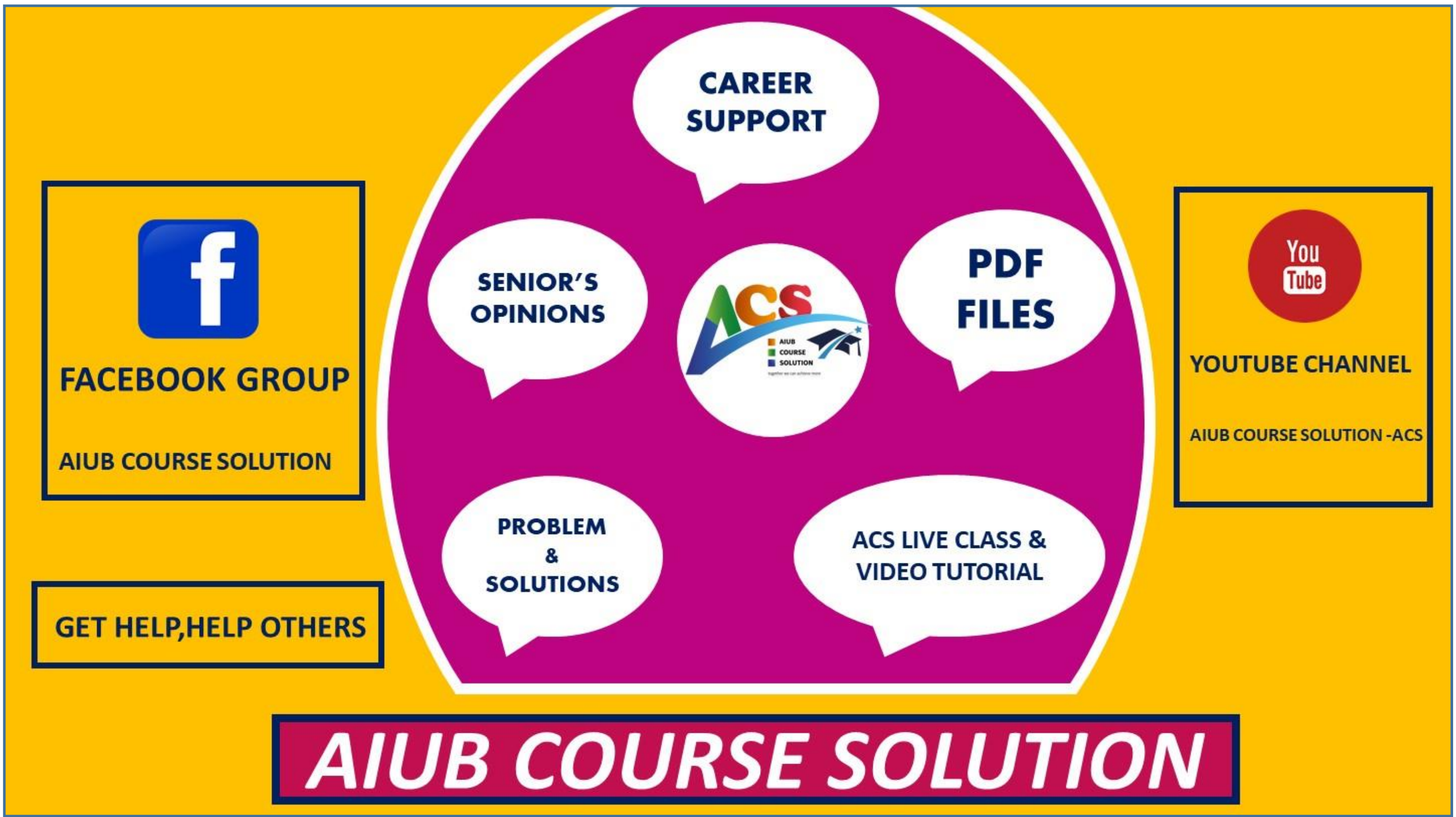

COURSE NAME: MATH-4

CHAPTER: CHAPTER-4

SOLVED BY

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Eigenvalues and Eigenvactors

Eigen values and Eigen vactors

Let $A = (a_{ij})_{n \times n}$ is a square matrix ; A non zero vactor v in R^n is called an eigen vactor of A. if AV is a scalar multiple of v; that is $AV = \lambda V$ for some scalar λ.

* The scalar λ is called an eigenvalue of A.
* v is called the eigen vactor of A corres-ponding to λ.
* Characteristic matrix : $A - \lambda I$
* characteristic polynomial : $|A - \lambda I|$
* Characteristic equation : $|A - \lambda I| = 0$
* The roots of the characteristic equation $|A - \lambda I| = 0$ are called characteristic root or eigenvalue of A matrix

(2)

given, $\begin{cases} \dot{x}_1(t) = -1.5x_1(t) + 0.5x_2(t) \\ \dot{x}_2(t) = x_1(t) - x_2(t) \end{cases}$

and $x_1(0) = 5$, $x_2(0) = 4$; where $\dot{x}_1(t) = \frac{dx_1}{dt}$

and $\dot{x}_2(t) = \frac{dx_2}{dt}$

$\Rightarrow$ Let, $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

$$\dot{X}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix}$$

so, $X(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$

here $A = \begin{pmatrix} -1.5 & 0.5 \\ 1 & -1 \end{pmatrix}$

$$\dot{X}(t) = AX(t)$$

$$X(t) = cv\,e^{\lambda t}$$

The characteristic matrix of A is

$$A - \lambda I = \begin{bmatrix} -1.5 - \lambda & 0.5 \\ 1 & -1 - \lambda \end{bmatrix}$$

The characteristic Polynomial,

$$|A - \lambda I| = [(1.5 + \lambda)(1 + \lambda) - 0.5]$$

$$= \lambda^2 + 2.5\lambda + 1$$

$$= (\lambda + 0.5)(\lambda + 2)$$

$\therefore \lambda = -0.5, -2$

$$(A - \lambda I)V = 0$$

When, $\lambda = -0.5$ then $\begin{pmatrix} -1 & 0.5 \\ 1 & -0.5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$$\Rightarrow \begin{pmatrix} -1 & 0.5 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

here $v_2 = a$, $v_1 = \frac{1}{2} a$

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$\therefore$ for $\lambda = -0.5$ $\quad v = \begin{pmatrix} a \\ 2a \end{pmatrix}$

when $\lambda = -2$ then $\begin{bmatrix} 0.5 & 0.5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$= \begin{bmatrix} 0.5 & 0.5 \\ 1-1 & 1-1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$= \begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$v_2 = a$

$\therefore V_1 = -a$

$\therefore v_2 = \begin{bmatrix} -b \\ b \end{bmatrix}$

$x(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$

$\Rightarrow \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} a \\ 2a \end{pmatrix} e^{-0.5t} + c_2 \begin{pmatrix} -b \\ b \end{pmatrix} e^{-2t}$

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$$\Rightarrow \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = c_1 \begin{pmatrix} \cancel{b}a \\ 2a \end{pmatrix} + c_2 \begin{pmatrix} -b \\ b \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} \cancel{b}a \\ 2a \end{bmatrix} + c_2 \begin{bmatrix} -b \\ b \end{bmatrix}$$

$$\therefore \cancel{b}ac_1 - bc_2 = 5 \quad \text{--- (1)}$$

$$\therefore 2ac_1 + bc_2 = 4 \quad \text{--- (11)}$$

$$\therefore c_1 \cancel{=} \cancel{\frac{6}{a}}, \cancel{c_2} =$$

$$c_1 = \frac{3}{a}, \quad b = -\frac{2}{b}$$

$$\therefore \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \frac{3}{a} \begin{pmatrix} a \\ 2a \end{pmatrix} e^{-0.5t} - \frac{2}{b} \begin{pmatrix} -b \\ b \end{pmatrix} e^{-2t}$$

if $a = 1$ and $b = 1$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-0.5t} - 2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t}$$

$$\therefore x_1 = 3e^{-0.5t} + 2e^{-2t}, \quad x_2(t) = 6e^{-0.5t} - 2e^{-2t}$$

Exercise: 4.1

1 a. given, $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

we write $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

The characteristic matrix of A is $A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}$

The characteristic polynomia of A is $|A - \lambda I|$

$$= |(1-\lambda)^2 - 4|$$

The characteristic equation of A is,

$$|A - \lambda I| = 0$$

$$\Rightarrow (1-\lambda)^2 - 4 = 0$$

$$\Rightarrow \lambda = 3, -1$$

$\therefore$ The Eigen Values are $3, -1$

v is a non-trivial solution of $(A-\lambda I)v = 0$

So, $\begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

when $\lambda = 3$

$\Rightarrow \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

A[$r_2' \Rightarrow r_1 + r_2$]

$\Rightarrow \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$\Rightarrow$

Let $v_2 = a$ So, $v_1 = a$

$\therefore V' = \begin{bmatrix} a \\ a \end{bmatrix}$

when, $\lambda = -1$

$$\Rightarrow \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$A[r_2' \rightarrow r_2 - r_1]$

$$\sim \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Let, $v_2 = b \quad \therefore v_1 = -b$

$$v'' = \begin{bmatrix} -b \\ b \end{bmatrix}$$

(b) given, $A = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$

$\therefore$ The characteristic matrix of A, $A - \lambda I$

$$= \begin{bmatrix} -1-\lambda & 0 \\ 3 & 2-\lambda \end{bmatrix}$$

The characteristic polynomial of A is

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 0 \\ 3 & 2-\lambda \end{vmatrix}$$

∴ The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 0 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-1-\lambda)(2-\lambda) = 0$$

$$\Rightarrow \lambda = 2, -1$$

$$\therefore \lambda_1 = 2, \quad \lambda_2 = -1$$

v is a non-trival solution of $(A - \lambda I)v = 0$

when, $\lambda = 2$

$$\begin{bmatrix} -3 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$A[r_2' \to r_2 + r_1] \Rightarrow \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

So, $v_2 = a$, $v_1 = 0$

$\therefore v' = \begin{bmatrix} 0 \\ a \end{bmatrix}$

when $\lambda = -1$

$$\begin{bmatrix} 0 & 0 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Let, $v_2 = b$, $v_1 = -b$

$\therefore v'' = \begin{bmatrix} -b \\ b \end{bmatrix}$

$\therefore$ if $a = 1$, $b = 1$

$\therefore v'|_{a=1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$v''|_{b=1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$-1+1$

2

(c) given, $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

The characteristic matrix, $A - \lambda I = \begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix}$

The characteristic polynomial of A is

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix}$$

$$= (3 - 4\lambda + \lambda^2) - 8$$

$$= \lambda^2 - 4\lambda - 5$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$$

$$\Rightarrow (\lambda - 5)(\lambda + 1) = 0$$

$$\therefore \lambda = 5, -1$$

$\therefore$ The eigen values are 5, -1

(12)

v is a non-trival solution of $(A-\lambda I)' v = 0$

$\therefore$ when $\lambda = 5$,

$$\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$A[r_2' \to 2r_2 + r_1]$

$$= \begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Let, $v_2 = a$, $v_1 = a$

$$\therefore v' = \begin{bmatrix} a \\ a \end{bmatrix}$$

when, $\lambda = -1$

$$\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$A[r_2' \to r_2 - r_1]$

$$\begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Let, $v_2 = b$, $v_1 = -2b$ $\therefore v'' = \begin{bmatrix} -2b \\ b \end{bmatrix}$

(d) given, $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & -5 & 2 \end{bmatrix}$

The characteristic of matrix, $A - \lambda I$

$$= \begin{bmatrix} 1-\lambda & 2 & -1 \\ 0 & -2-\lambda & 0 \\ 0 & -5 & 2-\lambda \end{bmatrix}$$

The characteristic of polynomial, $|A - \lambda I|$

$$= \begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & -2-\lambda & 0 \\ 0 & -5 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)(-2-\lambda)(2-\lambda) - 2\times 0 - 1\times 0$$

$$= (1-\lambda)(-2-\lambda)(2-\lambda)$$

$\therefore$ The characteristic of equation,

$$|A - \lambda I| = 0$$

$$\Rightarrow (1-\lambda)(-2-\lambda)(2-\lambda) = 0$$

$$\therefore \lambda = 1, 2, -2$$

$\therefore$ The eigenvalues are, 1, 2, −2

if v is a non-trivial solution of

$(A - \lambda I) v = 0$

when, $\lambda = 1$ $\begin{bmatrix} 0 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & -5 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$A[r_3' \to r_3 - r_2 + r_1]$

$$= \begin{bmatrix} 0 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $v_1 = a$, $v_2 = 0$, $v_3 = 0$

$\therefore v' = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$

when $\lambda = 2$ $\begin{bmatrix} -1 & 2 & -1 \\ 0 & -4 & 0 \\ 0 & -5 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Let, $v_3 = b$, $v_2 = 0$, $v_1 = -b$ $\therefore v'' = \begin{bmatrix} -b \\ 0 \\ b \end{bmatrix}$

when $\lambda = -2$

$$\begin{bmatrix} 3 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & -5 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 3 & 2 & -1 \\ 0 & -5 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

Let, $v_3 = c$, $v_2 = \frac{4}{5}c$, $v_1 = \frac{-c}{5}$

$$\therefore V''' = \begin{bmatrix} -c/5 \\ 4/5c \\ c \end{bmatrix}$$

(e) $A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$

The C.M of A is $A - \lambda I = \begin{bmatrix} 2-\lambda & 3 & 3 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{bmatrix}$

$\therefore$ The C.E of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 3 & 3 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\{(3-\lambda)(-3-\lambda)+8\} - 3(-3-\lambda+2) + 3(-4+3-\lambda) = 0$$

$$\Rightarrow (2-\lambda)(-9+\lambda^2+8) - 3(-1-\lambda) + 3(-1-\lambda) = 0$$

$$\Rightarrow (2-\lambda)(\lambda^2-1) + 0 = 0$$

$$\Rightarrow \lambda = \pm 1, 2 = 1, 2, -1$$

if v is a non trivial solution of

$$(A-\lambda I)\,V = 0$$

when
$$\begin{bmatrix} 2-\lambda & 3 & 3 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

when, $\lambda = 1$

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 2 & 2 \\ -1 & -4 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$A\begin{bmatrix} r_2' \rightarrow r_2 - r_1 \\ r_3' \rightarrow r_3 + r_1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$A\left[r_3' \rightarrow r_3 - r_1\right]$

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

Let $v_3 = a$, $\therefore v_2 = -a$ and $v_1 = 0$ $\therefore V' = \begin{bmatrix} 0 \\ -a \\ a \end{bmatrix}$

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when $\lambda = -1$ $\begin{bmatrix} 3 & 3 & 3 \\ 1 & 4 & 2 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$

$A[r_3' \to r_3 + r_2]$ $\begin{bmatrix} 3 & 3 & 3 \\ 0 & 9 & \cancel{2}3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$

Let $v_3 = b$, $v_2 = -\frac{b}{3}$, $\cancel{v_3}$ $v_1 = -\frac{2b}{3}$

$\therefore v'' = b \begin{bmatrix} -2/3 \\ -1/3 \\ 1 \end{bmatrix}$

$\lambda = 2$ $\begin{bmatrix} 0 & 3 & 3 \\ 1 & 1 & 2 \\ -1 & -4 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$

$A[r_2' \to r_2 + r_1$ $\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -3 \\ 0 & 3 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$

$$A[r_3' \to r_3 + r_2] \begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

Let $v_3 = c$, $v_2 = -c$, $v_1 = -c$

$$\therefore V''' = \begin{bmatrix} -c \\ -c \\ c \end{bmatrix}$$

2 a given,
$$\begin{cases} \dot{x}_1(t) = x_1(t) + 2x_2(t) \\ \dot{x}_2(t) = 3x_1(t) + 2x_2(t) \end{cases}$$

and $x_1(0) = 0$, $x_2(0) = -4$

Let $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ and $\dot{x}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix}$

also, $x(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$

we write, $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

so, we can write $\dot{x}(t) = A\{x(t)$

Let λ and v be the eigenvalue and eigen vector of A repectivly.

The solution of the form, $x(t) = c\,v\,e^{\lambda t}$

$\therefore$ The C.M of A is $A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix}$

$$\therefore |A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow 2 - 3\lambda + \lambda^2 - 6 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 4 = 0$$

$$\Rightarrow (\lambda - 4)(\lambda + 1) = 0$$

$$\therefore \lambda = 4, -1$$

$\therefore$ for $\lambda = 4$

$$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$A[r_2' \Rightarrow r_2 + r_1] \begin{bmatrix} -3 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Let $v_2 = 3a$ $\quad v_1 = 2a$

$$v' = a \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

for $\lambda = -1$

$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$A[r_2' \Rightarrow 2r_2 - 3r_1] \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Let $v_2 = b$ $\quad v_1 = -b$

$$v'' = \begin{bmatrix} -b \\ b \end{bmatrix}$$

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$$x(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$

$$= c_1 \begin{bmatrix} 2a \\ 3a \end{bmatrix} e^{4t} + c_2 \cancel{v_2} \begin{bmatrix} -a \\ a \end{bmatrix} e^{-t}$$

$$\Rightarrow x(0) = c_1 \begin{pmatrix} 2a \\ 3a \end{pmatrix} + c_2 \begin{pmatrix} -a \\ a \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 \\ -4 \end{bmatrix} = c_1 \begin{pmatrix} 2a \\ 3a \end{pmatrix} + c_2 \begin{pmatrix} -a \\ a \end{pmatrix}$$

$$\therefore 2ac_1 - ac_2 = 0 \quad \text{--- (I)}$$

$$3ac_1 + ac_2 = -4 \quad \text{--- (II)}$$

$$c_2 = -\frac{4}{5a}, \quad c_1 = -\frac{8}{5a}$$

$$\therefore x(t) = -\frac{8}{5a} \begin{pmatrix} 2a \\ 3a \end{pmatrix} e^{4t} - \frac{4}{5a} \begin{pmatrix} -a \\ a \end{pmatrix} e^{-t}$$

$$\Rightarrow \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = -\frac{8}{5a} \begin{pmatrix} 2a \\ 3a \end{pmatrix} e^{4t} - \frac{4}{5a} \begin{pmatrix} -a \\ a \end{pmatrix} e^{-t}$$

(b) given, $\begin{cases} \dot{x}_1(t) = -5x_1(t) + x_2(t) \\ \dot{x}_2(t) = 4x_1(t) - 2x_2(t) \end{cases}$

and $x_1(0) = 1$, $x_2(0) = 2$

Let $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ $\dot{x}(t) = \begin{pmatrix} \dot{x}_1(t) \\ x_2(t) \end{pmatrix}$

So, $x(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

we write, $A = \begin{bmatrix} -5 & 1 \\ 4 & -2 \end{bmatrix}$

The C.M of A matrix, $A - \lambda I = \begin{bmatrix} -5-\lambda & 1 \\ 4 & -2-\lambda \end{bmatrix}$

$\therefore |A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} -5-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = 0$

$\Rightarrow 10 + 7\lambda + \lambda^2 - 4 = 0$

$\Rightarrow \lambda^2 + 7\lambda + 6 = 0$

$\Rightarrow (\lambda+6)(\lambda+1) = 0$

$\therefore \lambda = -6, -1$

for $\lambda = -1$

$$\begin{bmatrix} -4 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$A[r_2' \to r_2 + r_1]$ $\begin{bmatrix} -4 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

Let $v_2 = 4a$ $\therefore v_1 = a$, $v' = \begin{bmatrix} a \\ 4a \end{bmatrix}$

for $\lambda = -6$, $\begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$A[r_2' \to r_2 - 4r_1]$ $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

Let $v_2 = a$ $v_1 = -a$ $v'' = \begin{bmatrix} -a \\ a \end{bmatrix}$

$$x(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$

$$\Rightarrow x(t) = c_1 \begin{bmatrix} a \\ 4a \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -a \\ a \end{bmatrix} e^{-6t}$$

$$\Rightarrow x(0) = c_1 \begin{bmatrix} a \\ 4a \end{bmatrix} + c_2 \begin{bmatrix} -a \\ a \end{bmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix} = c_1 \begin{bmatrix} a \\ 4a \end{bmatrix} + c_2 \begin{bmatrix} -a \\ a \end{bmatrix}$$

$$\therefore ac_1 - ac_2 = 1 \quad \text{--- (i)}$$

$$4ac_1 + ac_2 = 2 \quad \text{--- (ii)}$$

From eq^n (i) and (ii) we get

$$c_1 = \frac{3}{5a} \qquad c_2 = -\frac{2}{5a}$$

$$x(t) = \frac{3}{5} e^{-t} \begin{bmatrix} a \\ 4a \end{bmatrix} + -\frac{2}{5a} e^{-6t} \begin{bmatrix} -a \\ a \end{bmatrix}$$

$$\therefore \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \frac{3}{5} e^{-t} \begin{bmatrix} 1 \\ 4 \end{bmatrix} - \frac{2}{5a} e^{-6t} \begin{bmatrix} -a \\ a \end{bmatrix}$$

(c) given, $\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = 1.5\, x_1(t) - 2.5\, x_2(t) \end{cases}$

and $x_1(0) = -4$, $x_2(0) = 9$

Solution: Let $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ and $\dot{x}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix}$

So, $x(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} -4 \\ 9 \end{pmatrix}$

we write, $A = \begin{bmatrix} 0 & 1 \\ 1.5 & -2.5 \end{bmatrix}$

So, we can write,

$$\dot{x}(t) = A\, x(t)$$

Let λ and v be the ~~gi~~ eigenvalue and eigenvector of A respectively.

The solution of the form, $x(t) = c v e^{\lambda t}$

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 \\ 1.5 & -2.5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow 2.5\lambda + \lambda^2 - 1.5 = 0$$

$$\Rightarrow \lambda^2 + 2.5\lambda - 1.5 = 0$$

$$\Rightarrow \lambda = 0.5, -3$$

for $\lambda = 0.5$

$$\begin{bmatrix} -0.5 & 1 \\ 1.5 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$r_1' \rightarrow r_2 + 3r_1$

$$\begin{bmatrix} -0.5 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Let $v_2 = a \quad \therefore v_1 = 2a$

$$v' = \begin{bmatrix} 2a \\ a \end{bmatrix}$$

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for $\lambda = -3$

$$\begin{bmatrix} 3 & 1 \\ 1.5 & 0.5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$r_2' \Rightarrow r_2 - r_1/2$

$$\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Let $v_2 = 3a$, $v_1 = -a$

$$\therefore v'' = \begin{bmatrix} -a \\ 3a \end{bmatrix}$$

$$\therefore x(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$

$$= c_1 \begin{bmatrix} 2a \\ a \end{bmatrix} e^{0.5t} + c_2 \begin{bmatrix} -a \\ 3a \end{bmatrix} e^{-3t}$$

$$\Rightarrow x(0) = c_1 \begin{bmatrix} 2a \\ a \end{bmatrix} + c_2 \begin{bmatrix} -a \\ 3a \end{bmatrix}$$

$$\Rightarrow \begin{pmatrix} -4 \\ 9 \end{pmatrix} = c_1 \begin{bmatrix} 2a \\ a \end{bmatrix} + c_2 \begin{bmatrix} -a \\ 3a \end{bmatrix}$$

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$$2ac_1 - ac_2 = -4$$

$$ac_1 - 3ac_2 = 9$$

$-4 \cdot 2$

$-4 \cdot 4$

$$\therefore c_1 = -\frac{3}{7} \qquad c_2 = -\frac{22}{7}$$

$$\therefore \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \cancel{-\frac{23}{7} e^{-3t} (}$$

$$\therefore \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = -\frac{3}{7} \begin{bmatrix} 2a \\ a \end{bmatrix} e^{0.5t} - \frac{22}{7} \begin{bmatrix} -a \\ 3a \end{bmatrix} e^{-3t}$$

Cayley hamilton Theorem:

Every square matrix is a ~~zeo~~ zero of its characteristic polynomial or every square matrix satisfies its characteristic equation.

ie $A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_1 A + a_0 I =$

(*) Example: given, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

The C.M of A is, $A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix}$

The C.E of A Matrix is,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(-2+\lambda^2) - 2(2-2\lambda-3) + 3(2+3+3\lambda) = 0$$

$$\Rightarrow -2+\lambda^2+2\lambda-\lambda^3-4+4\lambda+6+15+9\lambda = 0$$

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$\Rightarrow -\lambda^3 + \lambda^2 + 15\lambda + 15 = 0$

$\Rightarrow \lambda^3 - \lambda^2 - 15\lambda - 15 = 0$

Now according to the cayley-hamilton theorem, we have

$$A^3 - A^2 - 15A - 15I = 0$$

$$\Rightarrow A^{-1}A^3 - A^{-1}A^2 - 15AA^{-1} - 15A^{-1}I = 0$$

$$\Rightarrow A^2 - A - 15I - 15A^{-1} = 0$$

$$\Rightarrow A^{-1} = \frac{1}{15}\left[A^2 - A - 15I\right]$$

$$\therefore A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+9 & 2-2+3 & 3+2+3 \\ 2-2+3 & 4+1+1 & 6-1+1 \\ 3+2+3 & 6-1+1 & 9+1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{15}\left[\begin{bmatrix} 14 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 15 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 15 \end{bmatrix}\right]$$

$$= \frac{1}{15}\left[\begin{bmatrix} 14-1-15 & 3-2-0 & 8-3-0 \\ 3-2-0 & 6+1-15 & 6-1-0 \\ 8-3-0 & 6-1-0 & 11-1-15 \end{bmatrix}\right]$$

$$= \frac{1}{15}\begin{bmatrix} -2 & 1 & 5 \\ 1 & -8 & 5 \\ 5 & 5 & -5 \end{bmatrix}$$

Example: given, $A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$\therefore$ The characteristic Matrix of A is, $A - \lambda I$

$$= \begin{bmatrix} 1-\lambda & 2 & 2 \\ 3 & 1-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{bmatrix}$$

The C.E of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 2 \\ 3 & 1-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)\ (1-2\lambda+\lambda^2)-2(3-3\lambda)+2(3-1+\lambda)=0$

$\Rightarrow 1-2\lambda+\lambda^2-\lambda+2\lambda^2-\lambda^3-6+6\lambda+4+2\lambda=0$

$\Rightarrow -\lambda^3+3\lambda^2+5\lambda-1=0$

$\Rightarrow \lambda^3-3\lambda^2-5\lambda+1=0$

Now according to cayley hamilton, we have

$$A^3-3A^2-5A+I=0$$

$$\Rightarrow A^2-3A+5I+A^{-1}=0$$

$$\Rightarrow A^{-1}=3A+5I-A^2$$

$$A^2=\begin{bmatrix}1&2&2\\3&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}1&2&2\\3&1&0\\1&1&1\end{bmatrix}=\begin{bmatrix}9&6&4\\6&7&6\\5&4&3\end{bmatrix}$$

$$\therefore A^{-1}=3A+5I-A^2$$

$$=\begin{bmatrix}3&6&6\\9&3&0\\3&3&3\end{bmatrix}+\begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix}-\begin{bmatrix}9&6&4\\6&7&6\\5&4&3\end{bmatrix}$$

$$=\begin{bmatrix}-1&0&2\\3&1&-6\\-2&-1&5\end{bmatrix}=\begin{bmatrix}-1&0&2\\3&1&-6\\-2&-1&5\end{bmatrix}$$

Exercise - 4.2

ⓐ given, $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

The characteristic Matrix of A is $A - \lambda I$

$$= \begin{bmatrix} 2-\lambda & 5 \\ 1 & 3-\lambda \end{bmatrix}$$

The characteristic equation of A is, $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 5 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow 6 - 5\lambda + \lambda^2 - 5 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 1 = 0$$

Now according to the cayley hamilton theorm, we have,

$$A^2 - 5A + I = 0$$

$$\Rightarrow A - 5I + A^{-1} = 0$$ [multipuling both side A^{-1}]

$$A^{-1} = 5I - A$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5-2 & 0-5 \\ 0-1 & 5-3 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

(b) given $A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

The C.M of A is, $A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 2 \\ 3 & 1-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{bmatrix}$

The C.E of A is, $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 3 & 1-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(1-2\lambda+\lambda^2) - 2(3-3\lambda) + 2(3-1+\lambda) = 0$$

$$\Rightarrow 1-2\lambda+\lambda^2-\lambda+2\lambda^2-\lambda^3-6+6\lambda+4+4\lambda = 0$$

$\Rightarrow \lambda^3 - 3\lambda^2 - 5A + 1 = 0$

Now according to the cafey hamilton theorm, we can write,

$$A^3 - 3A^2 - 5A + I = 0$$

$$\Rightarrow A^2 - 3A - 5I + A^{-1} = 0$$ [multipling both side A^{-1}]

$$\Rightarrow A^{-1} = 3A + 5I - A^2$$

$$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6+2 & 2+2+2 & 2+0+2 \\ 3+3+0 & 6+1+0 & 6+0+0 \\ 1+3+1 & 2+1+1 & 2+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 6 & 4 \\ 6 & 7 & 6 \\ 5 & 4 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = 3\begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 9 & 6 & 4 \\ 6 & 7 & 6 \\ 5 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 & 6 \\ 9 & 3 & 0 \\ 3 & 3 & 3 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 9 & 6 & 4 \\ 4 & 7 & 6 \\ 5 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3+5-9 & 6+0-6 & 6+0-4 \\ 9+0-6 & 3+5-7 & 0+0-6 \\ 3+0-5 & 3+0-4 & 3+5-3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & -6 \\ -2 & -1 & 5 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & -6 \\ -2 & -1 & 5 \end{bmatrix}$$

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(c) $A = \begin{bmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{bmatrix}$

The C.E of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 0 & 1 \\ -2 & 3-\lambda & 4 \\ -5 & 5 & 6-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(18-9\lambda+\lambda^2-20)+1(-10+15-5\lambda)=0$$

$$\Rightarrow (2-\lambda)(\lambda^2-9\lambda-2)+5-5\lambda=0$$

$$\Rightarrow 2\lambda^2-18\lambda-4-\lambda^3+9\lambda^2+2\lambda+5-5\lambda=0$$

$$\Rightarrow -\lambda^3+11\lambda^2-21\lambda+1=0$$

$$\Rightarrow \lambda^3-11\lambda^2+21\lambda-1=0$$

Now according to the cayley-hamilton theorm, we can write,

$$A^3-11A^2+21A-I=0$$

$$\Rightarrow A^2-11A+21I-A^{-1}=0.$$

$$A^{-1} = A^2 - 11A + 21 I_3$$

$$A^2 = \begin{bmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 5 & 8 \\ -30 & 29 & 34 \\ -50 & 45 & 51 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} -1 & 5 & 8 \\ -30 & 29 & 34 \\ -50 & 45 & 51 \end{bmatrix} - \begin{bmatrix} 22 & 0 & 11 \\ -22 & 33 & 44 \\ -55 & 55 & 66 \end{bmatrix} + \begin{bmatrix} 21 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 21 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 5 & -3 \\ -8 & 17 & -10 \\ 5 & -10 & 6 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} -2 & 5 & -3 \\ -8 & 17 & -10 \\ 5 & -10 & 6 \end{bmatrix}$$

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(d) given, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

The C.E of the matrix A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 5-\lambda & 3 \\ 1 & 0 & 8-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(40-13\lambda+\lambda^2) - 2(16-2\lambda-3) + 3(-5+\lambda) = 0$$

$$\Rightarrow 40 - 13\lambda + \lambda^2 - 40\lambda + 13\lambda^2 - \lambda^3 - 26 + 4\lambda - 15 + 3\lambda = 0$$

$$\Rightarrow -\lambda^3 + 14\lambda^2 - 46\lambda - 1 = 0$$

$$\Rightarrow \lambda^3 - 14\lambda^2 + 46\lambda + 1 = 0$$

Now according to the cayley-hamilton theorem, we have

$$A^3 - 14A^2 + 46A + I = 0$$

$$\Rightarrow A^2 - 14A + 46I + A^{-1} = 0$$

$$\Rightarrow A^{-1} = 14A - A^2 - 46I$$

$$A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 12 & 33 \\ 15 & 29 & 45 \\ 9 & 2 & 67 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 14 & 28 & 42 \\ 28 & 70 & 49 \\ 14 & 0 & 112 \end{bmatrix} - \begin{bmatrix} 8 & 12 & 33 \\ 15 & 29 & 45 \\ 9 & 2 & 67 \end{bmatrix} - \begin{bmatrix} 46 & 0 & 0 \\ 0 & 46 & 0 \\ 0 & 0 & 46 \end{bmatrix}$$

$$= \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

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(e) given, $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$

The C.E of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -1 & 1 \\ 2 & -1-\lambda & 0 \\ 1 & -1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\{-\lambda(-1-\lambda) - 0\} + 1(-2\lambda) + 1(-2+1+\lambda) = 0$$

$$\Rightarrow -\lambda(-1+\lambda^2) - 2\lambda - 1 + \lambda = 0$$

$$\Rightarrow -2\lambda - \lambda^3 + 2\lambda - 1 = 0$$

$$\Rightarrow -\lambda^3 + 2\lambda - 2\lambda - 1 = 0$$

$$\Rightarrow \lambda^3 + 1 = 0$$

Now according to the cayley. hamilton theorem we have,

$$A^3 + I = 0$$

$$\Rightarrow A^2 + A^{-1} = 0$$

$$\therefore A^{-1} = -A^2$$

$$= -\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & -1 & 1 \\ 0 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & -2 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & -2 \\ 1 & 0 & -1 \end{bmatrix}$$

Exercise 4.3

(1) given, $U = (5, 9, 5)$, $u_1 = (1, -1, 3)$, $u_2 = (2, 1, 4)$
$u_3 = (3, 2, 5)$

Let $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = u$; where $\alpha_1, \alpha_2, \alpha_3$ are scalar

$$\Rightarrow \alpha_1(1,-1,3) + \alpha_2(2,1,4) + \alpha_3(3,2,5) = (5,9,5)$$

$$\Rightarrow (\alpha_1 + 2\alpha_2 + 3\alpha_3, -\alpha_1 + \alpha_2 + 2\alpha_3, 3\alpha_1 + 4\alpha_2 + 5\alpha_3) = (5,9,5)$$

Equating corresponding components,

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 = 5$$
$$-\alpha_1 + \alpha_2 + 2\alpha_3 = 9$$
$$3\alpha_1 + 4\alpha_2 + 5\alpha_3 = 5$$

using row eselon ~~ko~~ operation:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ -1 & 1 & 2 & 9 \\ 3 & 4 & 5 & 5 \end{array}\right]$$

$$\begin{array}{l} r_2' \to r_2 + r_1 \\ r_3' \to r_3 - 3r_1 \end{array} = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 3 & 5 & 14 \\ 0 & -2 & -4 & -10 \end{array}\right]$$

$$r_3' \to 3r_3 + 2r_2 = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 3 & 5 & 14 \\ 0 & 0 & -2 & -2 \end{array}\right]$$

here

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 = 5$$

$$3\alpha_2 + 5\alpha_3 = 14$$

$$-2\alpha_3 = -2 \Rightarrow \alpha = 1$$

$$\therefore \alpha_1 = -4 \quad \alpha_2 = 3 \quad \alpha_3 = 1$$

$\therefore 4u_1 + 3u_2 + u_3 = u$

here u is is the linear combination of vactor u_1, u_2, u_3

(2) given, $u = (6, 20, 2)$, $u_1 = (1, 2, 3)$, $u_2 = (1, 3, -2)$
$u_3 = (1, 4, 1)$

Let $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = u$; where $\alpha_1, \alpha_2, \alpha_3$ are scalars.

$\therefore \alpha_1(1, 2, 3) + \alpha_2(1, 3, -2) + \alpha_3(1, 4, 1) = (6, 20, 2)$

$\Rightarrow (\alpha_1 + \alpha_2 + \alpha_3,\ 2\alpha_1 + 3\alpha_2 + 4\alpha_3,\ 3\alpha_1 - 2\alpha_2 + \alpha_3) = (6, 20, 2)$

Equating corresponding components

$$\alpha_1 + \alpha_2 + \alpha_3 = 6$$
$$2\alpha_1 + 3\alpha_2 + 4\alpha_3 = 20$$
$$3\alpha_1 - 2\alpha_2 + \alpha_3 = 2$$

using row eslon operation,

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 3 & 4 & 20 \\ 3 & -2 & 1 & 2 \end{array}\right]$$

$$\begin{array}{l} r_2' \Rightarrow r_2 - 2r_1 \\ r_3' \Rightarrow r_3 - 3r_1 \end{array} = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & -5 & -2 & -16 \end{array}\right]$$

$$r_3' \Rightarrow r_3 + 5r_2 = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 8 & 24 \end{array}\right]$$

here $\alpha_1 + \alpha_2 + \alpha_3 = 6$

$\alpha_2 + 2\alpha_3 = 8$

$8\alpha_3 = 24$

$\therefore \alpha_1 = 1 \qquad \alpha_2 = 2 \qquad \alpha_3 = 3$

$\therefore u_1 + 2u_2 + 3u_3 = u$ here u is the linear combination of vactor u_1, u_2, u_3

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③ given, $u = (4, 2, 1, 0)$ $u_1 = (3, 1, 0, 1)$

$u_2 = (1, 2, 3, 1)$, $u_3 = (0, 3, 6, 6)$

Let $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = u$

$\Rightarrow \alpha_1 (3, 1, 0, 1) + \alpha_2 (1, 2, 3, 1) + \alpha_3 (0, 3, 6, 6) = (4, 2, 1, 0)$

Equating corresponding components with the given value of u_1, u_2, u_3

$$3\alpha_1 + \alpha_2 = 4$$
$$\alpha_1 + 2\alpha_2 + 3\alpha_3 = 2$$
$$+ 3\alpha_2 + 6\alpha_3 = 1$$
$$\alpha_1 + \alpha_2 + 6\alpha_3 = 0$$

using row operation: $\left[\begin{array}{ccc|c} 3 & 1 & 0 & 4 \\ 1 & 2 & 3 & 2 \\ 0 & 3 & 6 & 1 \\ 1 & 1 & 6 & 0 \end{array}\right]$

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$$\begin{bmatrix} 3 & 1 & 0 & | & 4 \\ 0 & 5 & 9 & | & 2 \\ 0 & 3 & 6 & | & 1 \\ 0 & 2 & 18 & | & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 0 & | & 4 \\ 0 & 5 & 9 & | & 2 \\ 0 & 3 & 6 & | & 1 \\ 0 & 0 & 42 & | & -14 \end{bmatrix}$$

here $3\alpha_1 + \alpha_2 = 4$

$5\alpha_2 + 9\alpha_3 = 2$

$3\alpha_2 + 6\alpha_3 = 1$

$42\alpha_3 = -14$

$\therefore \alpha_3 = -\frac{1}{3}$

$\alpha_1 = 1$ $\alpha_2 = 1$ $\alpha_3 = -\frac{1}{3}$

$\therefore u_1 + u_2 - \frac{1}{3}u_3 = u$ here u is the linear combination of u_1, u_2, u_3

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(4) given, $u_1 = (2,-1,4)$, $u_2 = (3,6,2)$ $u_3 = (2,10,-4)$

Let, $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = 0$; where $\alpha_1, \alpha_2, \alpha_3$ are scalar's

$\Rightarrow \alpha_1(2,-1,4) + \alpha_2(3,6,2) + \alpha_3(2,10,-4) = 0$

$\Rightarrow (2\alpha_1 + 3\alpha_2 + 2\alpha_3, -\alpha_1 + 6\alpha_2 + 10\alpha_3, 4\alpha_1 + 2\alpha_2 - 4\alpha_3) = 0$

Now we can write above the equation,

$$2\alpha_1 + 3\alpha_2 + 2\alpha_3 = 0$$
$$-\alpha_1 + 6\alpha_2 + 10\alpha_3 = 0$$
$$4\alpha_1 + 2\alpha_2 - 4\alpha_3 = 0$$

using elementary row operation,

$$\left[\begin{array}{ccc|c} 2 & 3 & 2 & 0 \\ -1 & 6 & 10 & 0 \\ 4 & 2 & -4 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 2 & 0 \\ 0 & 15 & 22 & 0 \\ 0 & -4 & -8 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 2 & 0 \\ 0 & 15 & 22 & 0 \\ 0 & 0 & -32 & 0 \end{array}\right]$$

and here $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$

here the vactors u_1, u_2, u_3 are liniearly independent

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(6) given, $u_1 = (1, -1, 2)$, $u_2 = (3, -5, 1)$.

$u_3 = (2, 7, 8)$ $u_4 = (-1, 1, 1)$

Let, $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \alpha_4 u_4 = 0$; where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are Scalar

$\Rightarrow \alpha_1(1, -1, 2) + \alpha_2(3, -5, 1) + \alpha_3(2, 7, 8) + \alpha_4(-1, 1, 1) = 0$

$\Rightarrow (\alpha_1 + 3\alpha_2 + 2\alpha_3 - \alpha_4, -\alpha_1 - 5\alpha_2 + 7\alpha_3 + \alpha_4, 2\alpha_1 + \alpha_2 + 8\alpha_3 + \alpha_4) = 0$

Now we can write,

$$\alpha_1 + 3\alpha_2 + 2\alpha_3 - \alpha_4 = 0$$
$$-\alpha_1 - 5\alpha_2 + 7\alpha_3 + \alpha_4 = 0$$
$$2\alpha_1 + \alpha_2 + 8\alpha_3 + \alpha_4 = 0$$

using elementary row operation,

$$\left[\begin{array}{cccc|c} 1 & 3 & 2 & -1 & 0 \\ -1 & -5 & 7 & 1 & 0 \\ 2 & 1 & 8 & 1 & 0 \end{array}\right]$$

$$\begin{array}{l} r_2' \to r_1 + r_2 \\ r_3' \to r_3 - 2r_1 \end{array} = \left[\begin{array}{cccc|c} 1 & 3 & 2 & -1 & 0 \\ 0 & -2 & 9 & 0 & 0 \\ 0 & -5 & 4 & 3 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} 1 & 3 & 2 & -1 & 0 \\ 0 & -2 & 9 & 0 & 0 \\ 0 & 0 & -37 & 6 & 0 \end{array}\right]$$

Let, $\alpha_3 = a$ $\therefore \alpha_4 = -\frac{37}{6}a$, $\alpha_2 = \frac{9}{2}a$

$\alpha_1 = -\frac{65}{3}a$

here the vectors u_1, u_2, u_3, u_4 are linearly dependent.

7) given $u_1 = (2,1,2)$, $u_2 = (0,1,-1)$ and $u_3 = (4,3,3)$

Let $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = 0$; where $\alpha_1, \alpha_2,$ are scalar.

$\Rightarrow \alpha_1(2,1,2) + \alpha_2(0,1,-1) + \alpha_3(4,3,3) = 0$

$\Rightarrow (2\alpha_1 + 4\alpha_3, \ \alpha_1 + \alpha_2 + 3\alpha_3, \ 2\alpha_1 - \alpha_2 + 3\alpha_3) = 0$

Now we can write,

$$2\alpha_1 + 4\alpha_3 = 0$$
$$\alpha_1 + \alpha_2 + 3\alpha_3 = 0$$
$$2\alpha_1 - \alpha_2 + 3\alpha_3 = 0$$

using elementary row operation,

$$\left[\begin{array}{ccc|c} 2 & 0 & 4 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & -1 & 3 & 0 \end{array}\right]$$

$r_2' \Rightarrow 2r_2 - r_1 \Rightarrow$
$r_3' \Rightarrow r_3 - r_1$

$$\left[\begin{array}{ccc|c} 2 & 0 & 4 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$r_3' \Rightarrow 2r_3 + r_2 \Rightarrow$

$$\left[\begin{array}{ccc|c} 2 & 0 & 4 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right]$$

here, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$

so, here the vactor u_1, u_2, u_3 are linearly independent.

(8) given $u_1 = (2, 0, -1)$, $u_2 = (1, 1, 0)$
$u_3 = (0, -1, 1)$

Let, $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = 0$; where $\alpha_1, \alpha_2, \alpha_3$ are Scalars

$\Rightarrow \alpha_1(2, 0, -1) + \alpha_2(1, 1, 0) + \alpha_3(0, -1, 1) = 0$

$\Rightarrow (2\alpha_1 + \alpha_2, \alpha_2 - \alpha_3, -\alpha_1 + \alpha_3) = 0$

So we can write,

$$2\alpha_1 + \alpha_2 = 0$$
$$\alpha_2 - \alpha_3 = 0$$
$$-\alpha_1 + \alpha_3 = 0$$

using elementary row operation

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right]$$

$$r_3' \to 2r_3 + r_1 = \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$$r_3' \to r_3 - r_2 = \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right]$$

$\therefore 2\alpha_1 + \alpha_2 = 0$

$\alpha_2 - \alpha_3 = 0$

$3\alpha_3 = 0$

$\therefore \alpha_1 = 0 \quad \alpha_2 = 0 \quad \alpha_3 = 0$

here the vectors are linearly independent