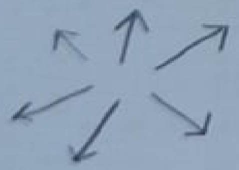
# Lecture Note for Online Class

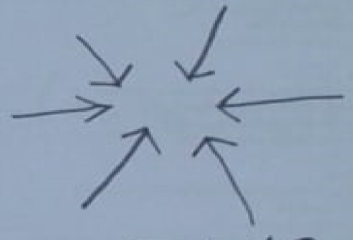
**Prepared by**
**Jannat-E-Mahbuba**
**Lecturer**
**Department of Mathematics**
**AIUB**

# Divergence



$\nabla \cdot A > 0$

positive divergence

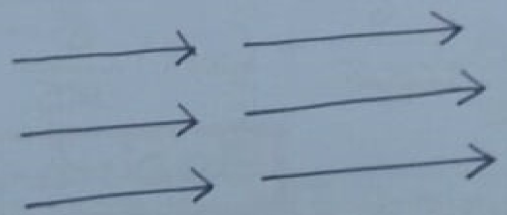


$\nabla \cdot A < 0$

Negative divergence

If $\boxed{\nabla \cdot A = 0}$

Then the field is **solenoidal**



$$\text{div} A = \nabla \cdot A$$

$$= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(A_{u_1} h_2 h_3) + \frac{\partial}{\partial u_2}(h_1 A_{u_2} h_3) + \frac{\partial}{\partial u_3}(h_1 h_2 A_{u_3})\right]$$

**Note**

Cartesian $(h_1, h_2, h_3) = (1, 1, 1)$

Cylindrical $(h_1, h_2, h_3) = (1, r, 1)$

spherical $(h_1, h_2, h_3) = (1, R, R\sin\theta)$

In **Cartesian**

$$\nabla \cdot A = \frac{\partial}{\partial x}(A_x) + \frac{\partial}{\partial y}(A_y) + \frac{\partial}{\partial z}(A_z)$$

In **Cylindrical**

$$\nabla \cdot A = \frac{1}{r}\left[\frac{\partial}{\partial r}(r A_r) + \frac{\partial}{\partial \phi}(A_\phi) + \frac{\partial}{\partial z}(r A_z)\right]$$

# Curl:

$\nabla \times A$

If curl $A = \boxed{\nabla \times A = 0}$
Then $A$ is called conservative
or irrotational.

$$\nabla \times A = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{u}_1 h_1 & \hat{u}_2 h_2 & \hat{u}_3 h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_{u_1} & h_2 A_{u_2} & h_3 A_{u_3} \end{vmatrix}$$

Cartesian

$$\nabla \times A = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

④

* Determine divergence ($\nabla \cdot A$) and curl ($\nabla \times A$). Also check the field is solenoidal ($\nabla \cdot A = 0$) conservative ($\nabla \times A = 0$) or both.

$$A = \hat{x}\, x^2 + \hat{y}\, 2xy$$

Soln :

$$A = \hat{x}\underbrace{x^2}_{A_x} + \hat{y}\underbrace{2xy}_{A_y}$$

$A_x = x^2$

$A_y = 2xy$

$A_z = 0$

$$\nabla \cdot A = \frac{\partial}{\partial x}(A_x) + \frac{\partial}{\partial y}(A_y) + \frac{\partial}{\partial z}(A_z)$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(2xy) + 0$$

$$= 2x + 2x$$

$$= 4x \neq 0$$

A is not solenoidal

$$\nabla \times A = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 2xy & 0 \end{vmatrix}$$

$$= \hat{x}\left[0 - \frac{\partial}{\partial z}(2xy)\right] - \hat{y}\left[0 - \frac{\partial}{\partial z}(x^2)\right]$$

$$+ \hat{z}\left[\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^2)\right]$$

$$= \hat{z}\, 2y \neq 0$$

A is not conservative

# Laplacian:

$\nabla^2 = \nabla \cdot \nabla$ Laplacian operator

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

5 (a) Find the Laplacian of the following

$$T = 4y^2 z^2$$

Soln:
$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$
$$= \frac{\partial^2}{\partial x^2}(4y^2z^2) + \frac{\partial^2}{\partial y^2}(4y^2z^2) + \frac{\partial^2}{\partial z^2}(4y^2z^2)$$
$$= 0 + \frac{\partial}{\partial y}(8yz^2) + \frac{\partial}{\partial z}(8y^2z)$$
$$= 8z^2 + 8y^2$$

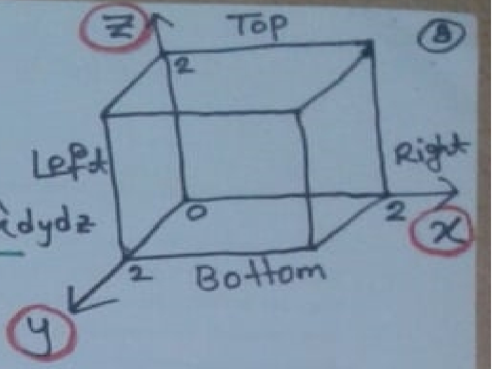
(5) (b) Same
Do it by yourself

# Gauss's divergence theorem

$$\int_V \nabla \cdot A \, dv = \oint_S A \cdot \hat{n} \, ds$$

## Exm:

For the vector field $\boxed{A = \hat{x}\, xz - \hat{y}\, yz^2 - \hat{z}\, xy}$ Verify the divergence theorem by computing (a) the total outward flux flowing through the surface of a cube centered at the origin and with sides equal to $\boxed{2}$ units each and parallel to the cartesian axes, (b) the integral of $\nabla \cdot A$ over the cube's volume.




Soln: For right $x=2$

$$ds = \hat{x}\,dy\,dz$$

$$\int_R A\cdot ds = \int\int [\hat{x}xz - \hat{y}yz^2 - \hat{z}xy]\cdot\hat{x}\,dy\,dz$$

$$= \int_0^2\int_0^2 xz\,dy\,dz$$

$$= \int_0^2 xz\,y\Big]_0^2 dz = 2\int_0^2 xz\,dz = 4\,\frac{z^2}{2}\Big]_0^2$$

$$= \boxed{8}$$

For left $x=0$, $ds = -\hat{x}\,dy\,dz$

$$\int_L A\cdot ds = -\int\int [\hat{x}xz - \hat{y}yz^2 - \hat{z}xy]\cdot\hat{x}\,dy\,dz$$

$$= -\int_0^2\int_0^2 xz\,dy\,dz$$

$$= \boxed{0}$$

For Top ($z = 2$) $\quad ds = \hat{z}\, dx\, dy$

$$\int_T A \cdot ds = \iint \left[\hat{x}\, xz - \hat{y}\, yz^2 - \hat{z}\, xy\right] \cdot \hat{z}\, dx\, dy$$

$$= \int_0^2 \int_0^2 (-xy)\, dx\, dy$$

$$= -\int_0^2 \frac{x^2}{2}\Big|_0^2\, y\, dy = -2\int_0^2 y\, dy$$

$$= -2 \left.\frac{y^2}{2}\right]_0^2 = \boxed{-4}$$

For Bottom ($z = 0$), $ds = \hat{z}\, dx\, dy$

$$\int_{Bt} A \cdot ds = -\int_0^2 \int_0^2 (-xy)\, dx\, dy$$

$$= -(-4) = \boxed{4}$$

For Front, ($y = 2$), $ds = \hat{y}\, dx\, dz$

$$\int_F A \cdot ds = \iint \left[\hat{x}\, xz - \hat{y}\, yz^2 - \hat{z}\, xy\right] \cdot \hat{y}\, dx\, dz$$

$$= -\int_0^2 \int_0^2 y z^2\, dx\, dz = -2 \int_0^2 \int_0^2 z^2\, dx\, dz$$

$$= -2\int_0^2 \frac{z^3}{3}\Big|_0^2 dx$$

$$= -2 \cdot \frac{8}{3}\left[x\right]_0^2 = \boxed{-\frac{32}{3}}$$

For back $y=0$, $ds = -\hat{y}\, dx\, dz$

$$\iint_{bk} A \cdot ds = \iint \left[\hat{x}\, xz - \hat{y}\, yz^2 - \hat{z}\, xy\right](-\hat{y}\, dx\, dz)$$

$$= \int_0^2\int_0^2 yz^2\, dx\, dz = \boxed{0}$$

Now, $\int_S A \cdot ds$

$$= \int_F A\cdot ds + \int_{bk} A\cdot ds + \int_R A\cdot ds + \int_L A\cdot ds + \int_T A\cdot ds + \int_{Bt} A\cdot ds$$

$$= -\frac{32}{3} + 0 + 8 + 0 + (-4) + 4$$

$$= \boxed{-\frac{8}{3}}$$

Now, Volume itegral

$$A = \hat{x}\, xz - \hat{y}\, yz^2 - \hat{z}\, xy$$

$$A_x = xz \qquad A_y = -yz^2 \qquad A_z = -xy$$

$$\nabla \cdot A = \frac{\partial}{\partial x}(A_x) + \frac{\partial}{\partial y}(A_y) + \frac{\partial}{\partial z}(A_z)$$

$$= \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(-yz^2) + \frac{\partial}{\partial z}(-xy)$$

$$= z - z^2$$

$$\int_v \nabla \cdot A \, dv = \int_0^2\int_0^2\int_0^2 (z - z^2)\, dx\, dy\, dz$$

$$= \int_0^2\int_0^2 (z - z^2)\, x\Big|_0^2 dy\, dz$$

$$= 2\int_0^2\int_0^2 (z - z^2)\, dy\, dz$$

$$= 2\int_0^2 (z - z^2)\, y\Big|_0^2 dz$$

$$= 4\left[\frac{z^2}{2} - \frac{z^3}{3}\right]_0^2 = -\frac{8}{3}$$

# Any Question?