# QUIVER BUNDLES AND NON-ABELIAN COHOMOLOGY

MAHMUD AZAM AND STEVEN RAYAN

Abstract.

Contents



## 1. Introduction

The main aspect of non-Abelian Hodge theory is a homeomorphism between the moduli spaces of polystable Higgs bundles with vanishing rational Chern classes and semisimiple flat connections over a smooth projective variety $X$. We may wonder: if "continuous" families of Higgs bundles are equivalent to those of flat connections, then families of morphisms of Higgs bundles should have some close relation to families of morphisms of flat connections. As a first step towards formalizing this idea, in [AR25] in [AR25], we constructed a moduli stack parametrizing tuples $(E, \phi, F, \psi, f)$, where $E, F$ are vector bundles on $X$, $\phi, \psi$ are Higgs fields on $E, F$ respectively, and $f : E \longrightarrow F$ is a morphism of Higgs bundles. We showed that with fairly mild constraints on an algebraic stack $X$ over an algebraically closed field $k$, which includes all projective varieties, this moduli stack is algebraic. Denote this moduli stack as $\mathrm{Higgs}_1(X)$. A natural next step is to construct an algebraic stack parametrizing tuples $(E, \phi, F, \psi, f)$, where $E, F$ are again vector bundles on $X$, $\phi, \psi$ are flat connections on $E, F$ respectively, and $f : E \longrightarrow F$ is morphism of connections. In the present work, we construct this stack — call it $\mathrm{Conn}_1(X)$ — along with a stack, denoted $\mathrm{Hodge}_1(X)$ over the affine line whose fibres over non-zero points are copies of $\mathrm{Vect}_1(X_{dR})$ and the fibre over 0 is $\mathrm{Higgs}_1(X)$. Of course, taking the fibre over 0 gives an alternate construction of $\mathrm{Higgs}_1(X)$. We show that $\mathrm{Conn}_1(X)$ is algebraic which implies that $\mathrm{Conn}_1(X)$ is algebraic. Again, this gives an alternate proof of the algebraicity of $\mathrm{Higgs}_1(X)$.

1.1. **Notation and Conventions.** We fix a base scheme $S$ and consider stacks on the category $\mathrm{Sch}_{/S}$ of schemes over $S$ with respect to the fppf topology. We denote the strict 2–category of such objects as $\mathrm{St}_{/S}$. We denote the category of formal schemes over $S$ as $\mathbb{F}\mathrm{Sch}_{/S}$, which of course embeds fully faithfully into $\mathrm{St}_{/S}$.

As we have already mentioned in the introduction, we use the following notation for the moduli stacks (left) of the various parametrized objects (right):

| | |
|---|---|
| $\mathrm{Vect}$ | vector bundles over $S$ |
| $\mathrm{Vect}(\mathcal{X}/S)$ | vector bundles over $\mathcal{X}$ |
| $\mathrm{Vect}_1(\mathcal{X}/S)$ | vector bundle morphisms over $\mathcal{X}$ |
| $\mathrm{Higgs}(\mathcal{X}/S)$ | Higgs bundles over $\mathcal{X}$ |
| $\mathrm{Higgs}_1(\mathcal{X}/S)$ | Higgs bundle morphisms over $\mathcal{X}$ |
| $\mathrm{Conn}(\mathcal{X}/S)$ | vector bundles with connections over $\mathcal{X}$ |
| $\mathrm{Conn}_1(\mathcal{X}/S)$ | morphisms of connections over $\mathcal{X}$ |

Note that this is a slight departure from [AR25]. We follow this convention, first, to be consistent with the notation $\mathrm{Perf}(X/S)$ for the moduli stack of perfect complexes, as used in [Sim08], and, second, because the $\mathcal{M}$ in $\mathcal{M}_{\mathrm{Vect}(X)}$, for example, seems redundant.

## 2. Quotient Stacks of Formal Groupoids

In this subsection we review formal groupoids and the stacks associated to them as described in [Sim08, §3.1]. These are central to the stack-based formulation of non-Abelian Hodge theory. Specifically, the stacks of Higgs bundles and connections and deformations thereof are all defined as mapping stacks out of stacks associated to formal groupoids.

### 2.1. Definitions.

**Definition 2.1** (Formal Groupoid)**.** A formal category over $S$ is an internal category $(X, \mathcal{F}, s, t, c, i)$ in $\mathbb{F}\mathrm{Sch}_{/S}$ such that

(i) $X$ is a scheme in $\mathrm{Sch}_{/S}$,

(ii) $i : X \longrightarrow \mathcal{F}$ is a closed immersion realizing $X$ as the underlying scheme of $\mathcal{F}$.

This data is called a formal groupoid if, in addition:

(iii) for each $U \in \mathrm{Sch}_{/S}$, $(X(U), \mathcal{F}(U), s_U, t_U, c_U, i_U)$ is a groupoid (internal to Set).

A formal category as above is said to be smooth if

(iv) the structure map $X \longrightarrow S$ is smooth,

(v) the morphisms $s, t : \mathcal{F} \longrightarrow X$ are formally smooth.

**Notation 2.2.** We will write a formal category as above simply as $(X, \mathcal{F})$, when the structure maps are clear from context.

For convenience, we make the following definition:

**Definition 2.3** (Formal Stack)**.** A formal stack is a stack $\mathcal{X} \in \mathrm{St}_{/S}$ such that there exists a formal groupoid $(X, \mathcal{F})$ and a 2–coequalizer diagram in $\mathrm{St}_{/S}$:

$$\begin{tikzcd} \mathcal{F} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & X \arrow[r] & \mathcal{X} \end{tikzcd}$$

presenting $\mathcal{X}$ as a quotient stack. We call $\mathcal{X}$ the stack associated to the formal groupoid $(X, \mathcal{F})$.

**Warning 2.4.** A formal stack is not necessarily a formal algebraic stack as defined in [Eme, Definition 5.3]. For instance, it may have a digonal not representable by algebraic spaces — see proposition 2.11 — which contradicts [Eme, Lemma 5.12.]. On the other hand, a formal algebraic stack is not necessarily a formal stack since it may be a quotient of formal algebraic spaces more general than formal schemes [Eme, p. 47].

**Notation 2.5.** In the context of the above definition, if the formal groupoid $(X, \mathcal{F})$ is clear from context, then we write $\mathcal{X}$ as $X_{\mathcal{F}}$.

2.2. **Main Examples.** There are three main examples of interest to us [Sim96b, pp. 31–33]. The rough idea behind all of these is that a quasicoherent sheaf on a formal stack is $(X, \mathcal{F})$ is a quasicoherent sheaf on $X$ along with isomorphisms between the stalks encoded by $\mathcal{F}$. By varying $\mathcal{F}$, and consequently the isomorphisms of stalks, we can recover connections and Higgs bundles.

**Example 2.6** (de Rham Stack)**.** If $X \longrightarrow S$ is separated, then the diagonal $\Delta_{X/S} : X \longrightarrow X \times_S X$ is a closed immersion and we take $\mathcal{F} \xrightarrow{(s,t)} X \times_S X$ to be the formal completion of $X \times_S X$ along the set theoretic image $\Delta_{X/S}(X)$. The composition morphisms $c : \mathcal{F} \times_X \mathcal{F} \longrightarrow \mathcal{F}$ is the one induced by the map $(X \times_S X) \times_X (X \times_S X) \longrightarrow X$. The identity morphism $i : X \longrightarrow \mathcal{F}$ is simply the closed immersion into the formal completion. We denote $\mathcal{F}$ by $\mathcal{F}_{dR}$ and the associated stack over $S$, by $X_{dR} \longrightarrow S$, in this case.

Make this more precise.

**Example 2.7** (Dolbeault Stack)**.** If $X \longrightarrow S$ is again separated, then the diagonal $\Delta_{X/S} : X \longrightarrow X \times_S X$ is again a closed immersion. Furthermore, any section of a separated morphism is a closed immersions. Since the projection $T(X \times_S X) \longrightarrow X \times_S X$ of the tangent bundle of $X \times_S X$ is affine and hence separated, the zero section $0_X : X \longrightarrow T(X \times_S X)$ is a closed immersion. Therefore, the composite

$$\Delta' : X \xrightarrow{\Delta_{X/S}} X \times_S X \xrightarrow{0_X} T(X \times_S X)$$

is a closed immersion. We can then take $\mathcal{F} \longrightarrow T(X \times_S X)$ to be the formal completion of $T(X \times_S X)$ along the set theoretic image $\Delta'(X)$. Then, the map $\mathcal{F} \xrightarrow{(s,t)} X \times_S X$ is obtained by composing with the bundle projection. The composition morphism $c : \mathcal{F} \times_X \mathcal{F} \longrightarrow \mathcal{F}$ is induced by the addition morphism

Make this more precise.

$$+ : T(X \times_S X) \times_{X \times_S X} T(X \times_S X) \longrightarrow T(X \times_S X)$$

The identity morphism $i : X \longrightarrow \mathcal{F}$ is again the closed immersion into the formal completion. We denote $\mathcal{F}$ by $\mathcal{F}_{Dol}$ and the associated stack over $S$ by $X_{Dol} \longrightarrow S$, in this case.

**Example 2.8** (Hodge Stack)**.** Take a separated morphism of schemes $X \longrightarrow T$ over $\mathbb{Z}$, consider a formal groupoid over $S := \mathbb{A}^1_T = \mathbb{A}^1_{\mathbb{Z}} \times_{\mathbb{Z}} T$ whose scheme of objects is $X \times_T \mathbb{A}^1_T \longrightarrow \mathbb{A}^1_T$. The formal scheme of morphisms is defined as follows. We take the blow up $B \longrightarrow X \times_T X \times_T \mathbb{A}^1_T$ of $(X \times_T \mathbb{A}^1_T) \times_{\mathbb{A}^1_T} (X \times_T \mathbb{A}^1_T) \cong (X \times_T X \times_T \mathbb{A}^1_T)$ along the set theoretic image of the map $\Delta_{X/T} \times_T 0 : X \cong X \times_T T \longrightarrow X \times_T X \times_T \mathbb{A}^1_T$ where $0 : T \longrightarrow \mathbb{A}^1_T$ is the pullback of the map $\mathrm{Spec}(\mathbb{Z}[x] \longrightarrow \mathbb{Z} : x \longmapsto 0)$ along the structure map $T \longrightarrow \mathrm{Spec}(\mathbb{Z})$, and the blow up $B' \longrightarrow X \times_T X$ of $X \times_T X$ along the image of $\Delta_{X/T}$. These fit into a commutative square:

$$\begin{tikzcd}
B' \arrow[r] \arrow[d] & B \arrow[d] \\
X \times_T X \arrow[r, "(\mathrm{id},0)"] & X \times_T X \times \mathbb{A}^1_T
\end{tikzcd}$$

Provide reference — Annala's notes: an option

where the top and bottom arrows are both closed embeddings. Then, $B'$ is the strict transform of $\mathrm{im}(\mathrm{id}_{X \times_T X}, 0)$ in the blow up of $X \times_T X \times_s \mathbb{A}^1_T$ along $\mathrm{im}(\Delta_{X/T} \times_T 0)$ by [Sta25, Lemma 080E]. We take $Y$ to be the complement of $B'$ in $B$ — note that $Y$ is an open subscheme of $B$ since the image of $B'$ is closed. We choose a closed embedding $\Delta' : X \times_T \mathbb{A}^1_T \longrightarrow Y$ making the following diagram commute:

$$\begin{tikzcd}
 & Y \arrow[d] \\
X \cong X \times_T T \arrow[ur, "\Delta'"] \arrow[r, "\Delta_{X/T} \times_T 0"'] & X \times_T X \times_T \mathbb{A}^1_T
\end{tikzcd}$$

Then, the formal scheme of morphisms is taken to be the formal completion $\mathcal{F} \longrightarrow Y$ of $Y$ along $\Delta'(X)$. The map $\mathcal{F} \xrightarrow{(s,t)} X \times_T X \times_T \mathbb{A}^1_T$ is given by the two projections $Y \longrightarrow X \times_T X \times_T \mathbb{A}^1_T \longrightarrow X \times_T \mathbb{A}^1_T$. The composition map $c : \mathcal{F} \times_{X \times_T \mathbb{A}^1_T} \mathcal{F} \longrightarrow \mathcal{F}$ is induced by the map:

$$(X \times_T X \times_T \mathbb{A}^1_T) \times_{X \times_T \mathbb{A}^1_T} (X \times_T X \times_T \mathbb{A}^1_T) \longrightarrow (X \times_T X \times_T \mathbb{A}^1_T)$$

The identity morphism $i : X \times_T \mathbb{A}^1 \longrightarrow \mathcal{F}$ is again the closed immersion into the formal completion. We will write $\mathcal{F}$ as $\mathcal{F}_{Hod}$ and the associated stack over $S = \mathbb{A}^1_T$ as $X_{Hod} \longrightarrow \mathbb{A}^1_T$, in this case.

These stacks are related to each other by the following well known fact which is the main connection with non-Abelian Hodge theory.

**Proposition 2.9.** *Let $X \longrightarrow T$ be a separated morphism of schemes and consider $X_{dR}$ and $X_{Dol}$ by taking $S = T$ in example 2.6 and example 2.7, respectively. Also consider $X_{Hod}$ by taking $S = \mathbb{A}^1_T$ in example 2.8. Then, for any closed point $\lambda : T \longrightarrow \mathbb{A}^1_T$, the fibre $X_{Hod,\lambda}$ of $X_{Hod} \longrightarrow \mathbb{A}^1_T$ over $\lambda$ is equivalent as a stack to $X_{dR}$ when $\lambda \neq 0$, and is equivalent to $X_{Dol}$, when $\lambda = 0$.*

*Proof.*

Complete the proof

□

We also record here some basic facts about these stacks that make it difficult to apply some of the techniques of [AR25] directly — see **??**.

**Corollary 2.10.** *The stack $X_{dR}$ of example 2.6 is a stack whose fibre categories are setoids (a groupoid that is also a preorder, and hence a groupoid with contractible connected components). That is, it is equivalent to the Grothendieck construction of a sheaf of sets.*

*Proof.* This follows from the fact that the $\mathcal{F}_{dR} \longrightarrow X \times_S X$ is monomorphism of sheaves of sets. □

**Proposition 2.11.** *Taking $T = \mathrm{Spec}(k)$ for some algebraically closed field $k$ in the context of example 2.8, $X_{Hod}$ does not have a diagonal representable by algebraic spaces.*

*Proof.* We consider a $T = \mathrm{Spec}(k)$–point $x \in X_{Hod}(\mathrm{Spec}(k))$ that factors as a map $\mathrm{Spec}(k) \xrightarrow{(x',0)} X \times_k \mathbb{A}^1_k \longrightarrow X_{Hod}$, and its stabilizer $\mathrm{stb}(x)$. Recalling that the preimage of a point in a blow-up is a projective space, we have that the stabilizer fits into the following pasting of Cartesian squares:

$$\begin{tikzcd}
\mathrm{stb}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{P}^n_k \setminus \mathbb{P}^{n-1}_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Spec}(k) \arrow[d, "{(x',x',0)}"] \\
\mathcal{F}_{Hod} \arrow[r] & Y \arrow[r] & X \times_k X \times_k \mathbb{A}^1_k
\end{tikzcd}$$

Now, $\mathcal{F}_{Hod}$ is the formal completion of $Y$ along $\Delta'(X)$, while the preimage of $\Delta'(X)$ in $\mathbb{P}^n_k \setminus \mathbb{P}^{n-1}_k$ is a single point $x''$ lying over $(x', x', 0)$. By the compatibility of formal completions with fibre products [Sta25, Lemma 0APV], we have that $\mathrm{stb}(x)$ is the formal completion of $\mathbb{P}^n_k \setminus \mathbb{P}^{n-1}_k$ along a point, which is just the formal completion of an affine chart containing that point along that point. That is, up to change of coordinates, we have:

$$\mathrm{stb}(x) = \mathrm{Spf}(k[[x_0, \ldots, x_n]])$$

where Spf denotes the formal spectrum functor. If $\mathrm{stb}(x)$ were representable by an algebraic space, then it would have to be a scheme by [CLO12, Corollary 3.1.2] as the reduction $\mathrm{stb}(x)_{red}$ is $\mathrm{Spec}(k)$. $\mathrm{stb}(x)$ would further have to be an affine scheme by [Sta25, Lemma 06AD]. However, it is easy to see that $\mathrm{Spf}(k[[x_0, \ldots, x_n]])$ is not an affine scheme: as a locally ringed space, it consists of a single point whose stalk is $k[[x_0, \ldots, x_n]]$. For any ring $A$ where $|\mathrm{Spec}(A)|$ is a point, the stalk at that point must be $A$. If $\mathrm{stb}(x) = \mathrm{Spec}(A)$, then we must have $A = k[[x_0, \ldots, x_n]]$, but the latter is a DVR and hence has two prime ideals: namely $(0)$ and $(x_0, \ldots, x_n)$. This is a contradiction. Thus, $\mathrm{stb}(x)$ cannot be an algebraic space and $X_{Hod}$ cannot have diagonal representable by algebraic spaces. □

**Proposition 2.12.** *In the context of example 2.7, taking $S = \mathrm{Spec}(k)$ for some algebraically closed field $k$ and $X \longrightarrow \mathrm{Spec}(k)$ to be smooth, the stack $X_{Dol}$ does not have a diagonal representable by algebraic spaces.*

*Proof.* Consider a point $x \in X_{Dol}(\mathrm{Spec}(k))$ that factors as $\mathrm{Spec}(k) \xrightarrow{x'} X \longrightarrow X_{Dol}$.

Complete the proof

□

### 2.3. Vector Bundles on Formal Stacks.

Maybe review the relation with Λ–modules here [Sim96b, p. 30]

The following should be well known but we present it here nevertheless for completeness.

**Theorem 2.13.** *Let $(X, \mathcal{F})$ be a smooth formal stack over an algebraically closed field $k$ of characteristic $0$ such that the structure maps $X\mathrm{Spec}(k)$ are projective morphisms. Then, $\mathrm{Vect}(X_{\mathcal{F}}/S)$ is algebraic and locally of finite presentation.*

*Proof.* We know that:

$$\mathrm{Vect}(X_{\mathcal{F}}/S) = \coprod_{n=0}^{\infty} \mathrm{Map}(X_{\mathcal{F}}, B\mathrm{Gl}_n(k))$$

Then, it is known that $\mathrm{Map}(X_{\mathcal{F}}, B\mathrm{Gl}_n(k))$ is algebraic and locally of finite type — see the proof of [Sim96a, Theorem 7.2]. Then, $\mathrm{Vect}(X_{\mathcal{F}}/S)$ is also an algebraic stack locally of finite presentation. □

Need to check whether "projective formal groupoid" really means what we described above.

## 3. Arrow Bundles

In this section we review the moduli stack $\mathrm{Vect}_1(\mathcal{X}/S)$ of vector bundle morphisms (or triples) as defined in [AR25, §3]. This stack parametrizes triples $(E, F, f)$ consisting of two vector bundles $E, F$ and a morphism of vector bundles $f : E \longrightarrow F$ over a fixed base stack $\mathcal{X} \in \mathrm{St}_{/S}$. The main purpose of this section is to give a concrete description of the fibre categories $\mathrm{Vect}_1(\mathcal{X}/S)(U)$ over an scheme $U \in \mathrm{Sch}_{/S}$ and the boundary map

$$\mathrm{Vect}_1(\mathcal{X}/S) \xrightarrow{(s,t)} \mathrm{Vect}(\mathcal{X}/S) \times \mathrm{Vect}(\mathcal{X}/S)$$

both of which we mentioned in [AR25] without precise descriptions or proofs.

### 3.1. The Moduli Stacks.

We first recall the definition of $\mathrm{Vect}_1(\mathcal{X}/S)$ and the face maps $s, t : \mathrm{Vect}_1(\mathcal{X}/S) \longrightarrow \mathrm{Vect}(\mathcal{X}/S)$.

Recall definitions here

### 3.2. Concrete Descriptions.

**Theorem 3.1.** *For any stack $\mathcal{X} \in \mathrm{St}_{/S}$ and any scheme $U \in \mathrm{Sch}_{/S}$, the 2–sheaf of groupoids associated to $\mathrm{Vect}_1(\mathcal{X}/S)$ via the Grothendieck construction is defined as follows:*

- *For any $U \in \mathrm{Sch}_{/S,fppf}$, $\mathrm{Vect}_1(\mathcal{X}/S)(U)$ is the core (maximal subgroupoid) of the 1–category of vector bundles over $U \times_S \mathcal{X}$*
- *For any morphism $q : U \longrightarrow V \in \mathrm{Sch}_{/S,fppf}$, the functor*

$$\mathrm{Vect}_1(\mathcal{X}/S)(V) \longrightarrow \mathrm{Vect}_1(\mathcal{X}/S)(U)$$

*is defined by pullback of vector bundles along $q \times_S \mathrm{id}_{\mathcal{X}} : U \times_S \mathcal{X} \longrightarrow V \times_S \mathcal{X}$.*

*Proof.*

Complete the proof

□

**Theorem 3.2.** *For any stack $\mathcal{X} \in \mathrm{St}_{/S}$ and any scheme $U \in \mathrm{Sch}_{/S}$, the category $\mathrm{Vect}_1(\mathcal{X}/S)$ has object set:*

$$\begin{aligned}&\mathrm{Ob}(\mathrm{Vect}_1(\mathcal{X}/S))\\ =&\left\{ (p, \mathcal{E}_0, \mathcal{E}_1, f) \ : \begin{array}{l} p : U \longrightarrow S \in \mathrm{Ob}\big((\mathrm{Sch}_{/S})_{fppf}\big), \\ \mathcal{E}_0, \mathcal{E}_1 \longrightarrow U \times_S \mathcal{X} \textit{ are vector bundles,} \\ f : \mathcal{E}_0 \longrightarrow \mathcal{E}_1 \textit{ is a morphism of vector bundles} \end{array} \right\}\end{aligned}$$

*and, for two objects $A_i = (p_i, \mathcal{E}_{0,i}, \mathcal{E}_{1,i}, f_i)$ where $i \in \{0,1\}$, the morphism sets:*

$$\mathrm{Hom}(A_1, A_2)$$
$$= \left\{ (q, \alpha_0, \alpha_1) \ : \begin{array}{l} q : U_0 \longrightarrow U_1 \in \mathrm{Sch}_{/S}(p_1, p_2), \\ \alpha_j \textit{ are vector bundle isomorphisms, such that} \\ \begin{tikzcd} q^*\mathcal{E}_{0,1} \arrow[r, "\alpha_1"] \arrow[d, "q^*f_2"'] & \mathcal{E}_{0,0} \arrow[d, "f_1"] \\ q^*\mathcal{E}_{1,1} \arrow[r, "\alpha_2"'] & \mathcal{E}_{1,0} \end{tikzcd} \\ \textit{is commutative} \end{array} \right\}$$

*Proof.* The Grothendieck fibration $\mathrm{Vect}_1(\mathcal{X}/S) \longrightarrow \mathrm{Sch}_{/S,fppf}$ is the Grothendieck construction of the pseudofunctor of theorem 3.1. The result now follows from the concrete description of the Grothendieck construction. □

### 3.3. **The Boundary Map.**

**Definition 3.3.** For any stack $\mathcal{X} \in \mathrm{St}_{/S}$, taking $s, t$ to be the source and target maps as defined in [AR25, §3], we call the morphism of stacks $\mathrm{Vect}_1(\mathcal{X}/S) \xrightarrow{(s,t)} \mathrm{Vect}(\mathcal{X}/S)$, the boundary map.

**Remark 3.4.** The name "boundary map" is inspired by the map of simplicial sets that sends the 1–simplex to its boundary: that is, the disjoint union of its source and target vertices.

**Theorem 3.5.** *For any stack $\mathcal{X} \in \mathrm{St}_{/S}$ and any scheme $U \in \mathrm{Sch}_{/S}$, the functor $\mathrm{Vect}_1(\mathcal{X}/S)(U) \xrightarrow{(s,t)} \mathrm{Vect}(\mathcal{X}/S)(U)^2$ is defined by:*

$$\begin{array}{ccc} (\mathcal{E}_1, \mathcal{E}_2, f : \mathcal{E}_1 \longrightarrow \mathcal{E}_2) & \longmapsto & (\mathcal{E}_1, \mathcal{E}_2) \\ (\alpha_1, \alpha_2) & \longmapsto & (\alpha_1, \alpha_2) \end{array}$$

*In particular, the functor is faithful.*

*Proof.*

Complete the proof

□

**Corollary 3.6.** *For any stack $\mathcal{X} \in \mathrm{St}_{/S}$, if $\mathrm{Vect}_1(\mathcal{X}/S)$ and $\mathrm{Vect}(\mathcal{X}/S)$ are algebraic, then the morphism of stacks $\mathrm{Vect}_1(\mathcal{X}/S) \longrightarrow \mathrm{Vect}(\mathcal{X}/S)$ is representable by algebraic spaces.*

*Proof.* This follows from theorem 3.5 and [Sta25, Lemma 04Y5] □

## 4. Quiver Bundles on Formal Groupoids

### 4.1. **Arrow Bundles on Formal Groupoids.**

**Theorem 4.1.** *Let $(X, \mathcal{F})$ be a formal groupoid over $S$ and let $\Lambda \in \mathrm{QCoh}(X)$ be the corresponding sheaf of rings of differential operators. Then, the pseudofunctor associated to $\mathrm{Vect}_1(X_{\mathcal{F}}/S)$ has the following concrete description:*

- *For each scheme $U \in \mathrm{Sch}_{/S,fppf}$, denoting the projection $U \times_S X \longrightarrow X$ by $\pi_U$, $\mathrm{Vect}_1(X_{\mathcal{F}}/S)(U)$ is the groupoid with object set:*

$$\left\{ (E_0, \phi_0, E_1, \phi_1, f) \ : \ \begin{array}{l} \mathcal{E}_i \text{ are vector bundles on } U \times_S X, \\ \phi_i \text{ are } \pi_U^*\Lambda\text{–module structures on } E_i, \\ f : E_0 \longrightarrow E_1 \text{ is a morphism of } \pi_U^*\Lambda\text{–} \\ \text{modules} \end{array} \right\}$$

  *For each pair of objects $A_i = (E_{1,i}, \phi_{1,i}, E_{2,i}, \phi_{2,i}, f_i)$, $i \in \{0, 1\}$, the morphism set $\mathrm{Hom}(A_0, A_1)$ is:*

$$\left\{ (\alpha_0, \alpha_1) \ : \ \begin{array}{l} \alpha_j \text{ are isomorphisms of } \pi_U^*\Lambda\text{–modules,} \\ \text{such that} \\ \begin{array}{ccc} \mathcal{E}_{0,0} & \xrightarrow{\alpha_0} & \mathcal{E}_{0,1} \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f_1} \\ \mathcal{E}_{1,0} & \xrightarrow[\alpha_1]{} & \mathcal{E}_{1,1} \end{array} \\ \text{is commutative} \end{array} \right\}$$

- *For each morphism $q : U \longrightarrow V$, the functor*

$$\mathrm{Vect}_1(X_{\mathcal{F}}/S)(V) \longrightarrow \mathrm{Vect}_1(X_{\mathcal{F}}/S)(U)$$

  *is given by pullback along $q \times_S \mathrm{id}_X : U \times_S X \longrightarrow V \times_S X$ in the following way: $\pi_U^* F = (\pi_V \circ (q \times_S \mathrm{id}_X))^* F \simeq (q \times_S \mathrm{id}_X)^* \pi_V^* F$ for all $F \in \mathrm{QCoh}(V \times_S X)$, so that for any object $(E_0, \phi_0, E_1, \phi_1, f)$ over $V$, $(q \times_S \mathrm{id}_X)^* \phi_i$ are $\pi_U^*\Lambda$–module structures on $(q \times_S \mathrm{id}_X)^* E_i$ and $f$ is morphism of $\pi_U^*\Lambda$–modules. The function of* Hom *sets is defined similarly by pullback.*

*Proof.*

Complete the proof: should be able use theorem 3.1 along with the identification of $\Lambda$–modules with vector bundles on $X_{\mathcal{F}}$ — for example, as described in [Sim96b, p. 30]

$\square$

**Theorem 4.2.** *For any smooth formal groupoid $(X, \mathcal{F})$ over a scheme $S$, if $\mathrm{Vect}(X/S)$, $\mathrm{Vect}_1(X/S)$ and $\mathrm{Vect}(X_{\mathcal{F}}/S)$ are algebraic stacks, then so is $\mathrm{Vect}_1(X_{\mathcal{F}}/S)$.*

*Proof.* Using the concrete description of $\mathrm{Vect}_1(\mathcal{X}_{\mathcal{F}}/S)$ and $\mathrm{Vect}(X/S)$, viewed as pseudofunctors, from theorem 4.1 and theorem 3.1, we consider the map of stacks $F : \mathrm{Vect}_1(\mathcal{X}_{\mathcal{F}}/S) \longrightarrow \mathrm{Vect}_1(\mathcal{X}_{\mathcal{F}})$ defined for the corresponding pseudofunctors as follows. For a scheme $U \in \mathrm{Sch}_{/S,fppf}$, the component map — that is, the functor of $U$–points — is given by:

$$\begin{array}{rccc} F_U & : & \mathrm{Vect}_1(\mathcal{X}_{\mathcal{F}}/S)(U) & \longrightarrow & \mathrm{Vect}_1(X/S)(U) \\ & : & (E_0, \phi_0, E_1, \phi_1, f) & \longmapsto & (E_0, E_1, f) \\ & : & (\alpha_0, \alpha_1) & \longmapsto & (\alpha_0, \alpha_1) \end{array}$$

How?

recalling that morphisms of $\pi_U^*\Lambda$–modules are, in particular, morphisms of the underlying vector bundles. Naturality of the $F$ in $U$ is immediate .

We then compose with $B := (s,t) : \mathrm{Vect}_1(X/S) \longrightarrow \mathrm{Vect}(X/S)^2$, the boundary map, to obtain a map $F' : \mathrm{Vect}_1(X_{\mathcal{F}}/S) \longrightarrow \mathrm{Vect}(X/S)^2$. We claim that $F'$ is representable by algebraic spaces. For this, we will show, for some arbitrary scheme $U \in \mathrm{Sch}_{/S,fppf}$ and a map $u = (u_0, u_1) : U \longrightarrow \mathrm{Vect}(X/S)^2$, that the fibre product stack/pseudofunctor $U \times_{\mathrm{Vect}(X/S)^2, F', u} \mathrm{Vect}_1(X_{\mathcal{F}}/S)$ is representable by an algebraic space. Let $E_i'$ be the vector bundle on $U \times_S X$ corresponding to the map $u_i : U \longrightarrow \mathrm{Vect}(X/S)$. We then compute the groupoid of $V$–points for a scheme $V \in \mathrm{Sch}_{/S,fppf}$ as follows:

- The objects are triples $(q, E_0, \phi_0, E_1, \phi_1, f, \beta_0, \beta_1)$ where:
  - $q : V \longrightarrow U$ is a morphism in $Sch_{/S,fppf}$,
  - $E_i$ are vector bundles on $V \times_S X$,
  - $\phi_i$ is a $\pi_V^*\Lambda$–module structure on $E_i$,
  - $f : E_0 \longrightarrow E_1$ is a morphism of $\pi_V^*\Lambda$–modules,
  - $\beta_i$ is an isomorphism of vector bundles $q^*E_i' \longrightarrow E_i$ such that the following diagram commutes:

$$\begin{tikzcd} q^*E_0' \arrow[r, "\beta_0"] \arrow[d, "q^*(f)"'] & E_0 \arrow[d, "f"] \\ q^*E_1' \arrow[r, "\beta_1"'] & E_1 \end{tikzcd}$$

- For objects $A_i = (q_i, E_{0,i}, \phi_{0,i}, E_{1,i}, \phi_{1,i}, f_i, \beta_{0,i}, \beta_{1,i})$, $i \in \{0,1\}$, a morphism $A_0 \longrightarrow A_1$ is simply a pair of vector bundle morphisms $\alpha_i : E_i$, ...... such that the following diagram commutes:

$$\begin{tikzcd} q_0^*E_0' \arrow[rr] \arrow[dr] \arrow[dd] & & E_{0,0} \arrow[dr] \arrow[dd] & \\ & q_1^*E_0' \arrow[rr] \arrow[dd] & & E_{0,1} \arrow[dd] \\ q_0^*E_1' \arrow[rr] \arrow[dr] & & E_{1,0} \arrow[dr] & \\ & q_1^*E_1' \arrow[rr] & & E_{1,1} \end{tikzcd}$$

□

**Corollary 4.3.** *For any smooth formal groupoid $(X, \mathcal{F})$ over an algebraically closed field $k$ of characteristic zero, where $X$ is a projective variety $k$, the stack* $\mathrm{Vect}_1(X_{\mathcal{F}}/S)$ *is algebraic.*

*Proof.* The stack $\mathrm{Vect}(X_{\mathcal{F}}/S)$ is algebraic by theorem 2.13, and the result follows from theorem 4.2. $\square$

## References


[AR25] Mahmud Azam and Steven Rayan. *Moduli stacks of quiver bundles with applications to Higgs bundles*. 2025. arXiv: 2407.11958 [math.AG]. URL: https://arxiv.org/abs/2407.11958.

[CLO12] Brian Conrad, Max Lieblich, and Martin Olsson. "Nagata compactification for algebraic spaces". In: *Journal of the Institute of Mathematics of Jussieu* 11.4 (2012), pp. 747–814. DOI: 10.1017/S1474748011000223.

[Eme] Matthew Emerton. *Formal Algebraic Stacks*. Notes. URL: https://www.math.uchicago.edu/~emerton/pdffiles/formal-stacks.pdf.

[Sim08] Carlos T. Simpson. *Geometricity of the Hodge filtration on the $\infty$-stack of perfect complexes over $X_{DR}$*. 2008. arXiv: math/0510269 [math.AG]. URL: https://arxiv.org/abs/math/0510269.

[Sim96a] Carlos Simpson. *Algebraic (geometric) $n$-stacks*. 1996. arXiv: alg-geom/9609014 [alg-geom]. URL: https://arxiv.org/abs/alg-geom/9609014.

[Sim96b] Carlos Simpson. *The Hodge filtration on nonabelian cohomology*. 1996. arXiv: alg-geom/9604005 [alg-geom]. URL: https://arxiv.org/abs/alg-geom/9604005.

[Sta25] The Stacks project authors. *The Stacks project*. https://stacks.math.columbia.edu. 2025.



*Email address*: mahmud.azam@usask.ca

*Email address*: rayan@math.usask.ca

Centre for Quantum Topology and Its Applications (quanTA) and Department of Mathematics and Statistics, University of Saskatchewan, SK, Canada S7N 5E6