MORPHISMS OF λ–CONNECTIONS

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ABSTRACT.

CONTENTS

1. INTRODUCTION

We will work with fppf/étale/? stacks (which includes schemes, algebraic spaces, algebraic stacks, formal algebraic spaces, formal algebraic stacks) over a base field k, of characteristic 0, if needed. We wish to prove the algebraicity of a stack whose k–points are morphisms of connections. By [Sim96c; Sim08; AR25], this stack should be the pullback:

$$\begin{tikzcd}
\mathcal{M}_{\mathrm{Vect}(X_{DR}),1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(X_{DR}, \mathcal{E}^\vee_{X_\Lambda} \boxtimes \mathcal{E}_{X_\Lambda})) \arrow[d] \\
\mathrm{pt} \arrow[r, "\overline{\Delta}"] & \mathrm{Map}(X_{DR}, X_{DR} \times X_{DR})
\end{tikzcd}$$

To prove the algebraicity of the pullback, it suffices to show that the right and bottom left vertices are algebraic, and that the bottom right vertex has a diagonal representable by algebraic spaces [Sta23, Lemma 04TF].

We note that X_{DR} is an example of a formal groupoid defined in [Sim96b, p. 23], and we will try to show the result for an arbitrary formal groupoid X_Λ. By varying "Λ" in a sense that we can make precise, this simultaneously shows the algebraicity of the moduli stacks of morphisms of Higgs bundles, morphisms of λ–connections for each $\lambda \in k$, and the moduli stack fibred over $\mathbb{A}^1_k$ and parametrizing morphisms of all λ–connections for all $\lambda \in k$. The last example gives us a way to address non-Abelian Hodge theory from the stack theoretic perspective — namely, the "preferred sections" approach of Deligne and Simpson [Sim96c; Sim08].

2. Strategy (algebraicity of $\mathrm{Map}(X_\Lambda, \mathcal{E}^\vee_{X_\Lambda} \boxtimes \mathcal{E}_{X_\Lambda})$): Show that the target is very presentable [Sim96a, p. 69]

The goal is to apply something like [Sim96a, Lemma 4.23] and [Sim96b, Theorem 7.2].

3. Strategy (algebraicity of $\mathrm{Map}(X_\Lambda, \mathcal{E}^\vee_{X_\Lambda} \boxtimes \mathcal{E}_{X_\Lambda})$): Verify Artin's axioms

4. Strategy (representability of the diagonal of X_Λ): First show that (X_Λ) has a representable diagonal

Adapt [Alp, Theorem 3.4.13] to prove that quotients of smooth formal groupoids, as defined in say [Sim96b, p. 23], are formal algebraic stacks, as defined in [Eme, Definition 5.3.]. Then, [Eme, Lemma 5.12] shows that X_Λ has a diagonal representable by algebraic spaces and locally of finite type. Use this show the representability of the diagonal of $\mathrm{Map}(X_\Lambda, X_\Lambda \times X_\Lambda)$.

5. Strategy (Direct): Representable map of section stacks

Consider the pullback squares:

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E}^\vee_{X_\Lambda} \boxtimes \mathcal{E}_{X_\Lambda} \arrow[d] \\
X'_\Lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathcal{M}_{\mathrm{Perf}(X_\Lambda)} \times X_\Lambda)^2 \arrow[d] \\
X_\Lambda \arrow[r, "\Delta_{X_\Lambda}"'] & X_\Lambda \times X_\Lambda
\end{tikzcd}$$

Then, we have

$$\mathcal{M}_{\mathrm{Perf}(X_\Lambda),1} = \Gamma(X_\Lambda, \mathcal{F})$$

Find a map:

$$\Gamma(X_\Lambda, \mathcal{F}) \longrightarrow \Gamma(X'_\Lambda, \mathcal{F}) \text{ or } \Gamma(X'_\Lambda, \mathcal{F}) \longrightarrow \Gamma(X_\Lambda, \mathcal{F})$$

given by composing with the pullback projection $X'_\Lambda \longrightarrow X_\Lambda$, using the universal property of pullbacks. The stack $\Gamma(X'_\Lambda, \mathcal{F})$ is geometric n-stack by [Sim96b, Corollary 6.4]. If we can show that the first above morphism is geometric, then $\mathcal{M}_{\mathrm{Perf}(X_\Lambda),1} = \Gamma(X_\Lambda, \mathcal{F})$ should be geometric by [Sim96b, Corollary 2.6]. If we can show that the second above morphism is geometric, surjective and smooth, then $\mathcal{M}_{\mathrm{Perf}(X_\Lambda),1} = \Gamma(X_\Lambda, \mathcal{F})$ is geometric by [Sim96b, Lemma 2.4].

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