MODULI STACKS OF QUIVER CONNECTIONS AND NON-ABELIAN HODGE THEORY

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Abstract. In [AR25], a moduli stack parametrizing I–indexed diagrams of Higgs bundles over a base stack X was constructed for any finite simplicial set I, inspiring speculations about extending the non-Abelian Hodge correspondence to these moduli stacks. In the present work, we formalize the de Rham side of this conjectural extension. We construct moduli stacks parametrizing diagrams of bundles with λ–connections over a base prestack X, where λ can be a fixed number or a parameter. Taking λ to be 1 gives a moduli stack parametrizing diagrams of bundles with connection, while taking it to be a parameter gives a version of Simpson's non-Abelian Hodge filtration for digrams of bundles with connection. We show that when X is a smooth and projective scheme over an algebraically closed field k of characteristic 0, these moduli stacks are algebraic and locally of finite presentation, and have affine diagonal.

Contents

1. Introduction

1.1. **Motivation.** Representations of the fundamental group of various algebraic varieties make prominent appearances in various areas of mathematics and physics. A few examples are in order:

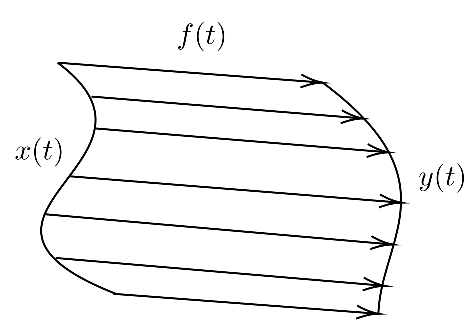
- We recall that the fundamental group of the configuration space of n points on the plane is the Artin braid group. Representations of this group characterize solutions to the quantum

Yang-Baxter equations [Bir93] and they govern the statistics of anyon exchanges [Gol23]. This makes them important in topological quantum computing [KL04] in the sense of [KL09].

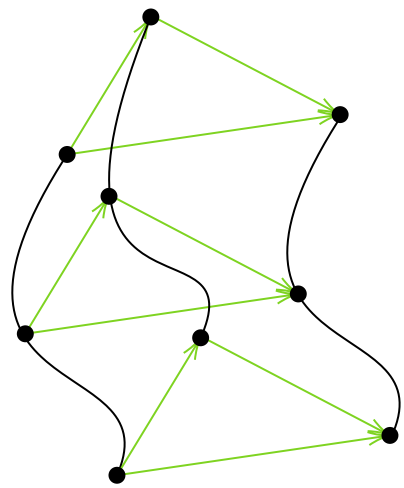
- Some recent advances in condensed matter physics, namely hyperbolic band theory [MR21; MR22; RN23], are centred around representations of some choice of discrete translation group acting on the hyperbolic plane. In turn, these translation groups are the fundamental groups of the Riemann surfaces obtained by quotienting the hyperbolic plane by the translation action.
- A large class of $d = 4, n = 2$ field theories called "class S" [GMN13] obtained by a specific compactification method applied to $d = 6, N = (2, 0)$ superconformal field theories, are controlled by choices of representations of the fundmanetal group of the surface parametrizing the "compactified" dimensions.

A series of equivalences, then, allows us to address representations of fundamental groups in terms of vector bundles and connections as we briefly discuss now. There is a well-known equivalence of categories between the category of representations of the fundamental group of a "nice enough" topological space, such as a smooth manifold, and the category of locally constant sheaves over the space. This equivalence restricts to an equivalence between linear representations of the fundamental group on one side and locally constant sheaves of vector spaces on the other. Then, the Riemann-Hilbert correspondence (see, for example, [Fre05, p. 35]) gives an equivalence of categories between the category of locally constant sheaves and the category of vector bundles with flat connections. On the other hand, there is an equivalence between the category of vector bundles with flat connections on a base complex manifold and the category of what are called Higgs bundles on the base. This was first shown by Hitchin in [Hit87] as an isomorphism of the moduli spaces of the respective objects when the base is a curve, and was a generalization of the Narasimhan-Seshadri theorem. It was then generalized to arbitrary compact Kähler manifolds by Simpson in [Sim92]. This equivalence is called the Corelette-Simpson correspondence or the non-Abelian Hodge correspondence. For convenience, we will call the study of these equivalences as non-Abelian Hodge theory, even though the term is generally used only in association with the Corlette-Simpson correspondence.

We can thus argue that understanding categories and moduli spaces of flat connections and those of Higgs bundles are of importance simultaneously to geometry, physics and quantum computing. This work is a contribution in this general endeavour in that it is a first step in unifying the moduli theoretic and the categorical perspectives of the above discussed correspondences in a very specific manner. The moduli spaces allow us to vary the objects in some geometric way, but one should be able to do the same with the morphisms, and the categorical equivalences should be "geometric" in some good sense. Once we try to make this idea precise, we arrive at the following situation: we should have moduli spaces of objects and moduli spaces of morphisms forming category objects internal to some geometric category, and the equivalences should be equivalences of internal categories in the geometric category, instead of isomorphisms of moduli spaces and equivalences of categories running in parallel, apparently unaware of each other. We can begin to visualize the situation as follows:



Here, $x(t)$ and $y(t)$ are two paths in the moduli space of objects and $f(t)$ is a path in the moduli space of morphisms, such that for each t, $f(t)$ has source $x(t)$ and target $y(t)$. At this point, it is imaginable to have not just moduli spaces of morphisms but those parametrizing diagrams of any shape. We may draw the situation for a diagram of the shape Δ^2 — the commuting triangle — as:



In this work, we make these ideas precise: we construct and establish the "geometricity" of the respective moduli spaces of diagrams of vector bundles with connection. Previously, we constructed and studied similar moduli spaces of diagrams of vector bundles and Higgs bundles [AR25]. In combination with the current work, this gives a unification of the moduli theory and the category theory of all sides involved in non-Abelian Hodge theory. The diagrams we are considering can be thought of as representations of quivers in the sense of [Jr16] but in the categories of vector bundles, Higgs bundles, connections or λ–connections over a fixed base, as opposed to the category of vector spaces — hence, the term "quiver bundle". At the same time, we provide "face" and "degeneracy" maps, in the same sense as in the context of simplicial sets. That is, a face map sends a diagram to one of its lower dimensional faces, while a degeneracy map sends a diagram to a higher dimensional diagram obtained by adding identity edges. The collection of these results should be thought of as the beginnings of a form of categorification of non-Abelian Hodge theory.

1.2. **Overview and Main Results.** We set up the basic notation and conventions, and also prove some basic results about prestacks and stacks in Section 2. These should be well-known but are difficult to find explicit descriptions of in the literature.

In Section 3, we will give a detailed description of the relative spectrum construction for prestacks, and show that it is given by a certain Grothendieck construction or unstraightening. Again, it is difficult to find this exact description in the literature but will be a technical requirement for proving our main results in the next sections.

In Section 4, For a prestack $\mathcal{Y}$ over S, we define a prestack $\mathcal{M}_1(\mathcal{Y})$ whose objects over an S–scheme U are triples (E, F, s) where E, F are vector bundles, or equivalently, finite locally free sheaves on $U \times_S \mathcal{Y}$ and $s : E \longrightarrow F$ is a morphism of vector bundles. This already appeared in our previous work [AR25] but in the present paper, we give a much more concrete description of this prestack, which will be necessary for proving our main results. We call this prestack the moduli prestack of arrow bundles on $\mathcal{Y}$. As shown in [AR25], it follows from our definition that when $\mathcal{Y}$ is a stack, so is $\mathcal{M}_1(\mathcal{Y})$, and when $\mathcal{Y}$ is algebraic and satisfies some mild conditions, then $\mathcal{M}_1(\mathcal{Y})$ is also algebraic and satisfies similar conditions.

Our main results are in Section 5. Here, for a scheme X over S, we consider various formal groupoid structures $\mathcal{F} \rightrightarrows X$ on X and the moduli stack $\mathcal{M}_1(X_{\mathcal{F}})$ of arrow bundles on $X_{\mathcal{F}}$ — where $X_{\mathcal{F}}$ is the quotient stack of the formal groupoid $(X, \mathcal{F})$. We show that the objects of this moduli stack over an S–scheme U are triples (E, ϕ, F, ψ, s), where E, F are vector bundles on $U \times_S X$; ϕ, ψ are λ–connections on E, F respectively — with λ being either $0, 1$ or a morphism $\lambda : U \longrightarrow \mathbb{A}^1_k$, depending on the choice of $\mathcal{F}$ — and $s : E \longrightarrow F$ is a morphism of λ–connections, in the sense that it is a morphism of vector bundles making a certain square involving ϕ and ψ commute. Strictly speaking, we will be dealing with an equivalent formulation of connections: a module over a sheaf of rings of differential operators [Sim94a, §2]. We show that as long as the moduli stack $\mathcal{M}_1(X)$ of arrow bundles on X, the moduli stack $\mathcal{M}(X)$ of vector bundles on X and the moduli stacks $\mathcal{M}(X_{\mathcal{F}})$ of vector bundles on $X_{\mathcal{F}}$ constructed in [Sim96, pp. 31–33] are algebraic, then so is $\mathcal{M}_1(X_{\mathcal{F}})$ — see Theorem 5.21. Furthermore, in the case that X is a smooth and projective scheme, $\mathcal{M}(X), \mathcal{M}_1(X)$ are known to be algebraic, locally of finite presentation and possessing affine diagonal [Wan11; AR25], and we show that $\mathcal{M}_1(X_{\mathcal{F}})$ inherits this good behaviour — see Theorem 5.24 and Theorem 5.25. Of course, taking λ to be 0, 1 and a k–valued parameter respectively, this gives a certain "categorification" of the moduli stacks of Higgs bundles and connections, and of the non-Abelian Hodge filtration from [Sim96, pp. 31–33].

At the same time, there are moduli stacks $\mathcal{M}_I(X_{\mathcal{F}})$ parametrizing I–indexed diagrams of λ–connections over a scheme X — these are just the moduli stacks $\mathcal{M}_{\mathrm{Vect}(X_{\mathcal{F}}),I}$ of I–shaped quiver bundles $X_{\mathcal{F}}$ constructed in [AR25, §4]. Importantly, this construction is contravariantly functorial in I, by definition, so that we get "face" and "degeneracy" maps $\mathcal{M}_{\Delta^i}(X_{\mathcal{F}}) \longrightarrow \mathcal{M}_{\Delta^j}(X_{\mathcal{F}})$ corresponding to simplicial maps $\Delta^j \longrightarrow \Delta^i$ as would be expected from a categorification. We discuss this in Section 5.3 and show that, when X is smooth and projective, for any finite simplicial set I, the moduli stacks $\mathcal{M}_I(X_{\mathcal{F}})$ are algebraic, locally of finite presentation and have affine diagonal. This is Theorem 5.29. Again, by varying $\mathcal{F}$, we get the corresponding moduli stacks for Higgs bundles, connections, or the non-Abelian Hodge filtration. In Section 5.4, we speculate about a version of the non-Abelian Hodge correspondence in this categorified setting using the categorified non-Abelian Hodge filtration.

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2. Preliminaries

2.1. **Category Theory Conventions.** For category theoretic notation, we try to stay as close to [Lur09] as possible. For example, we write Δ^n for the poset of natural numbers $\{0 < \cdots < n\}$ which we consider as a category. We will also write Δ^n to denote its nerve or the standard n–dimensional simplicial set, when needed. For two categories C, D, we write $\mathrm{Fun}(C, D)$ for the category of functors $C \longrightarrow D$, and $C^{\simeq}$ for the core or maximal subgroupoid of C. For a category

C and two objects $c, d \in C$, we will write $\mathrm{Hom}_C(c,d)$ or $C(c,d)$ to denote the set of morphisms $c \longrightarrow d$ in C. We will write a category object internal to a category C as a tuples (O, M, s, t, c, i), where O is the object of objects, M is the object of morphisms, s, t, c, i are the source, target, composition and identity maps respectively.

We will use the following notation for 2–categorical notions. Given a pair of objects a, b and 1–morphisms $f, g : a \longrightarrow b$ in a 2–category, we will denote a 2–morphism α from f to g as $\alpha : f \Longrightarrow g : a \longrightarrow b$. We will denote by $- \circ -$, the composition of 1–morphisms as usual but also vertical composition. We will denote horizontal composition by $- \star -$. We will denote the identity 2–morphism of a 1–morphism f as e_f, but in the case that $f = \mathrm{id}_a$ for some object, we will simply write e_a.

2.2. **Prestacks and Stacks.** Throughout this work, we will fix an algebraically closed field k. Let Sch denote the 1–category of schemes over k and Aff denote the full subcategory of affine schemes over k. We will denote by PSt the 2–category of prestacks or categories fibred in groupoids over Aff, and by St, the full sub-2-category thereof consisting of stacks with respect to the étale topology. We recall that étale stacks are also fppf stacks so that it makes sense to speak of Artin étale stacks, which we will simply refer to as algebraic stacks. We denote the 2–category of Artin étale stacks or algebraic stacks as AlgSt which is a full sub-2-category of the 2–category of all Artin fppf stacks over k. For a scheme $S \in \mathrm{Sch}$, we will write S for both the scheme and its image in PSt or St under the fully faithful Yoneda embedding $\mathrm{Sch} \longrightarrow \mathrm{St} \longrightarrow \mathrm{PSt}$ — recall that the embedding factors through St since the étale topology is subcanonical. For most of this work, we will fix a scheme S and work in the slice 2–categories $\mathrm{PSt}_{/S}$ and $\mathrm{St}_{/S}$.

Since the Yoneda embedding for stacks and the embedding of stacks in prestacks preserves limits, we will simply write the word "limit" for both the 1–limit of schemes and the 2–limit of (pre)stacks, and similarly for fibre products. By "colimit" of prestacks, we will generally mean the 2–colimit. We denote the fibre product functor over a prestack $\mathcal{W}$ as $- \times_{\mathcal{W}} -$, but $- \times -$, suppressing $\mathcal{W}$, when $\mathcal{W} = S$. Given prestacks $\mathcal{Y}_1, \dots, \mathcal{Y}_n \in \mathrm{PSt}_{/S}$, we write $pr_i : \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n \longrightarrow \mathcal{Y}_i$ to denote the projection of the iterated fibre product onto the i–th factor. We will write $\mathcal{Y}^n$ for the n–fold fibre product of $\mathcal{Y}$ with itself over S n times.

Notation 2.1. For two prestacks $\mathcal{X}, \mathcal{Y} \in \mathrm{PSt}_{/S}$, we will use the following notation:

$p_{\mathcal{X}}$	:	the structure 1–morphism $\mathcal{X} \longrightarrow \mathrm{Aff}_{/S}$
$\mathrm{Map}_S(\mathcal{X}, \mathcal{Y})$	:	the mapping prestack relative to S
Gl_n	:	the general linear group scheme over k of degree n
$\mathrm{Gl}_{n,S}$	:	the fibre product $\mathrm{Gl}_n \times_k S$
$\mathcal{M}$	:	the prestack $\coprod_{n=0}^{\infty} B\mathrm{Gl}_{n,S}$
$\mathcal{M}(\mathcal{X})$	:	the prestack $\mathrm{Map}_S(\mathcal{X}, \mathcal{M}) \cong \coprod_{n=0}^{\infty} \mathrm{Map}_S(\mathcal{X}, B\mathrm{Gl}_{n,S})$
$\mathcal{M}^n(\mathcal{X})$	:	$\mathrm{Map}_S(\mathcal{X}, B\mathrm{Gl}_{n,S})$
$\mathcal{B}(\mathcal{X})$	:	the prestack $\mathcal{M}(\mathcal{X}) \times_S \mathcal{X}$
$\mathcal{B}^n(\mathcal{X})$	:	$\mathcal{M}^n(\mathcal{X}) \times_S \mathcal{X}$

Proposition 2.2. *For a diagram $\mathcal{Y} \xrightarrow{p} S \xleftarrow{q} \mathcal{Z}$ in* PSt, *the underlying category of the fibre product $\mathcal{Y} \times_S \mathcal{Z}$ is strictly isomorphic to the strict fibre product of categories $\mathcal{Y} \times_S^{str} \mathcal{Z}$.*

Proof. By [Sta25, Lemma 0040], an object of $\mathcal{Y} \times_S \mathcal{Z}$ is a tuple (U, y, z, f) where $U \in \mathrm{Ob}(\mathrm{Aff})$, $y \in \mathrm{Ob}(\mathcal{Y}_U), z \in \mathrm{Ob}(\mathcal{Z}_U)$, f is an isomorphism $p(y) \xrightarrow{\sim} q(z)$ in S_U. However, since the fibre category S_U is discrete — that is, the set of morphisms of k–schemes $U \longrightarrow S$ — f must be the identity id_u for some $u : U \longrightarrow S$ and, we must have $p(y) = u = q(z)$.

The morphism sets $(\mathcal{Y} \times_S \mathcal{Z})((U, y, z, f), (U', y', z', f))$ consist of tuples (a, b) where $a : y \longrightarrow y', b : z \longrightarrow z'$ are morphisms in $\mathcal{Y}, \mathcal{Z}$ respectively such that $p(a), q(b) : U \longrightarrow U'$ are the same morphism of k–schemes and the following diagram commutes:

$$\begin{tikzcd} p(y) \arrow[r, "p(a)"] \arrow[d, "f"'] & p(y') \arrow[d, "f'"] \\ q(z) \arrow[r, "q(b)"'] & q(z') \end{tikzcd}$$

which is equivalent to the first condition since $f = \mathrm{id}_U$, $f' = \mathrm{id}_{U'}$ by the previous paragraph. This gives an equality of sets:

$$(\mathcal{Y} \times_S \mathcal{Z})((U, y, z, f), (U', y', z'f')) = (\mathcal{Y} \times_S^{str} \mathcal{Z})((y, z), (y', z'))$$

From this description, we can see that the mappings $(U, y, z, f) \longmapsto (y, z)$ and $(a, b) \longmapsto (a, b)$ are bijections, and it is straightforward to check that they assemble to a functor $\mathcal{Y} \times_S \mathcal{Z} \longrightarrow \mathcal{Y} \times_S^{str} \mathcal{Z}$, which must be a strict isomorphism, as it induces bijections of object and morphism sets. □

Proposition 2.3. *In the context Proposition 2.2, if we have two 2–morphisms in* $\mathrm{PSt}_{/S}$ $\alpha, \beta : f \Longrightarrow g : \mathcal{W} \longrightarrow \mathcal{Y} \times_S \mathcal{Z}$*, then,* $\alpha = \beta$ *if and only if* $pr_i \star \alpha = pr_i \star \beta$ *for both* $i = 1, 2$.

Proof. We observe that for an object $w \in \mathcal{W}$, $\alpha_w = (\alpha_{w,1}, \alpha_{w,2})$ for two morphisms $\alpha_{w,i} : pr_i(f(w)) \longrightarrow pr_i(g(w))$ for $i = 1, 2$ in $\mathcal{Y}, \mathcal{Z}$ respectively. Similarly $\beta_w = (\beta_{w,1}, \beta_{w,2})$ for morphisms $\beta_{w,i} : pr_i(f(w)) \longrightarrow pr_i(g(w)), i = 1, 2$. Then, $\alpha_w = \beta_w$ if and only if $(pr_i \star \alpha)_w = pr_i(\alpha_w) = \alpha_{w,i} = \beta_{w,i} = pr_i(\beta_w) = (pr_i \star \beta)_w$. □

2.3. **Sheaves on Prestacks.** For every $\mathcal{X} \in \mathrm{PSt}_{/S}$, we will consider $\mathcal{X}$ equipped with the étale topology inherited from $\mathrm{Aff}_{/S}$ in the sense of [Sta25, Definition 06NV]. This is generally referred to as the large étale site of $\mathcal{X}$. By a sheaf on $\mathcal{X}$, we will mean a functor $X^{\mathrm{op}} \longrightarrow \mathrm{Set}$ that satisfies the sheaf condition with respect to this inherited étale topology. By the 2–Yoneda lemma, this site is equivalent to the site $\mathrm{Aff}_{/S/\mathcal{X}}$ whose objects are morphisms of prestacks $U \longrightarrow \mathcal{X}$ over S, where U is an affine scheme over S, and coverings are étale coverings.

Notation 2.4. For any two prestacks $\mathcal{X}$, $\mathcal{Y}, \mathcal{Z}$, any morphism of prestacks $r = (f, g) : \mathcal{Y} \longrightarrow \mathcal{Z}^2$, and any presheaves A, E, F on $\mathcal{X}$, which may be (pre)sheaves of Abelian groups, rings or modules, depending on context, we will use the following notation:

$\mathcal{O}_{\mathcal{X}}$	:	the structure sheaf of $\mathcal{X}$ defined by the composite $\mathcal{X} \xrightarrow{p_{\mathcal{X}}} \mathrm{Aff}_{/S} \xrightarrow{\Gamma} \mathrm{Set}$
$E\|_x$	:	the composite $\mathcal{X}_{/x}^{\mathrm{op}} \longrightarrow \mathcal{X}^{\mathrm{op}} \xrightarrow{E} \mathrm{Set}$ for $x \in \mathrm{Ob}(\mathcal{X})$
$\mathrm{LMod}(A)$	:	category of left A–modules
$\mathrm{RMod}(A)$	:	category of right A–modules
$\mathrm{Mod}(A)$	:	category of A–bimodules
$\mathrm{QCoh}(\mathcal{X})$	:	category of quasicoherent $\mathcal{O}_{\mathcal{X}}$–modules
$\mathrm{Vect}(\mathcal{X})$	:	category of finite locally free $\mathcal{O}_{\mathcal{X}}$–modules
$\mathrm{Nat}(E, F)$	:	set of natural transformations $E \longrightarrow F$
$\mathcal{H}\mathrm{om}(E, F)$	:	the internal Hom object in the topos $\mathrm{PSh}(\mathcal{X})$
$\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(E, F)$	:	set of $\mathcal{O}_{\mathcal{X}}$–module maps $E \longrightarrow F$
$\Gamma(E)$	:	alternate notation for $\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}}, E)$
$\mathcal{H}\mathrm{om}_{\mathcal{O}_{\mathcal{X}}}(E, F)$	:	the internal Hom object in the Abelian category $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$
$[E, F]$	:	notation for $\mathcal{H}\mathrm{om}_{\mathcal{O}_{\mathcal{X}}}(E, F)$
$E^{\vee}$	:	notation for $\mathcal{H}\mathrm{om}_{\mathcal{O}_{\mathcal{X}}}(E, \mathcal{O}_{\mathcal{X}})$

Warning 2.5. In many references, such as [LM00], [Alp25] or [Ols16], sheaves on an algebraic stack $\mathcal{X}$, not general prestacks, are defined to be sheaves on the full subcategory of $\mathrm{Aff}_{/S/\mathcal{X}}$ consisting of the smooth morphisms $U \longrightarrow \mathcal{X}$, equipped with the étale topology. This is called the lisse-étale site of $\mathcal{X}$, and we will denote it as $\mathcal{X}_{\text{lis-ét}}$. There is a functor $\mathrm{Sh}(\mathcal{X}) \longrightarrow \mathrm{Sh}(\mathcal{X}_{\text{lis-ét}})$ given by pre-composing with the inclusion $\mathcal{X}_{\text{lis-ét}} \longrightarrow \mathrm{Aff}_{/S/\mathcal{X}} \simeq \mathcal{X}$, and this functor is not an equivalence in general, but it restricts to an equivalence $\mathrm{QCoh}(\mathcal{X}_{\text{lis-ét}}) \longrightarrow \mathrm{QCoh}(\mathcal{X})$ [Sta25, Lemma 07B1]. Hence, when dealing with quasi-coherent sheaves, most results and techniques are readily transferrable between the two approaches but some care might be necessary.

Remark 2.6. We recall that for a morphism of prestacks $f : \mathcal{X} \longrightarrow \mathcal{Y}$, the pullback functor $f^* : \mathrm{PSh}(\mathcal{Y}) \longrightarrow \mathrm{PSh}(\mathcal{X})$ is simply the functor $\mathrm{Fun}(\mathcal{Y}^{\mathrm{op}}, \mathrm{Set}) \longrightarrow \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{Set})$ defined by precomposition with f. The key fact is that, since we are dealing with the large étale site, this precomposition functor restricts to a functor on sheaves $f^* : \mathrm{Sh}(\mathcal{Y}) \longrightarrow \mathrm{Sh}(\mathcal{X})$ [Sta25, Lemma 06TS] — no sheafification is required. It is easy to check that this makes the functor $f^* : \mathrm{Sh}(\mathcal{Y}) \longrightarrow \mathrm{Sh}(\mathcal{X})$ preserve all limits and colimits (set-theoretic considerations aside). Furthermore, since $f^*\mathcal{O}_{\mathcal{X}} = \Gamma \circ p_{\mathcal{X}} \circ f = \Gamma \circ p_{\mathcal{Y}} = \mathcal{O}_{\mathcal{Y}}$, f^* also restricts to the usual pullback of modules $f^* : \mathrm{Mod}(\mathcal{O}_{\mathcal{X}}) \longrightarrow \mathrm{Mod}(\mathcal{O}_{\mathcal{Y}})$, which in turn preserves quasicoherent modules.

We now record some basic facts about sheaves on prestacks that should be well known, will be necessary for proving our results but are difficult to find references for.

Proposition 2.7. *Let $f, g : \mathcal{Y} \longrightarrow \mathcal{X}$ be 1–morphisms and $\alpha : f \Longrightarrow g$ a 2–morphism in $\mathrm{PSt}_{/S}$, and $A : \mathcal{Y}^{\mathrm{op}} \longrightarrow \mathrm{Set}$ a presheaf of $\mathcal{O}_{\mathcal{X}}$–modules. Then, $A \star \alpha^{\mathrm{op}} : g^*A \Longrightarrow f^*A : \mathcal{X}^{\mathrm{op}} \longrightarrow \mathrm{Set}$ is a morphism of $\mathcal{O}_{\mathcal{Y}}$–modules. If A is an $\mathcal{O}_{\mathcal{X}}$–algebra, then $A \star \alpha^{\mathrm{op}}$ is a map of $\mathcal{O}_{\mathcal{Y}}$–algebras.*

Proof. For any object $y \in \mathcal{Y}$, we first recall that $\mathcal{O}_{\mathcal{X}}$ is the composite $\mathcal{X} \xrightarrow{p_Y} \mathrm{Aff}_{/S} \xrightarrow{\Gamma} \mathrm{Set}$ so that

$$f^*\mathcal{O}_{\mathcal{X}}(y) = \mathcal{O}_{\mathcal{X}}(f(y)) = \Gamma(p_{\mathcal{X}}(f(y))) = \Gamma(p_{\mathcal{Y}}(y)) = \mathcal{O}_{\mathcal{Y}}(y)$$

The same holds for $g^*\mathcal{O}_{\mathcal{X}}(y)$ so that it is also equal to $\mathcal{O}_{\mathcal{Y}}(y)$. Next, since α is a 2–morphism of $\mathrm{PSt}_{/S}$, by definition, we have $p_{\mathcal{X}}(\alpha_y) = \mathrm{id}_{p_{\mathcal{Y}}(y)}$ so that

$$(\mathcal{O}_{\mathcal{X}} \star \alpha^{\mathrm{op}})_y = \mathcal{O}_{\mathcal{X}}(\alpha_y^{\mathrm{op}}) = \Gamma(p_{\mathcal{X}}(\alpha_y^{\mathrm{op}})) = \Gamma(\mathrm{id}_{p_{\mathcal{Y}}(y)}^{\mathrm{op}}) = \mathrm{id}_{\Gamma(p_{\mathcal{Y}}(y))} = \mathrm{id}_{\mathcal{O}_{\mathcal{Y}}(y)}$$

Let $s : \mathcal{O}_{\mathcal{X}} \times A \longrightarrow A$ be the structure map. Then, the component $(f^*s)_y : f^*\mathcal{O}_{\mathcal{X}}(y) \times f^*A(y) \longrightarrow f^*A(y)$ is the map $s_{f(y)} : \mathcal{O}_{\mathcal{X}}(f(y)) \times A(f(y)) \longrightarrow A(f(y))$, and similarly for $(g^*s)_y$. By the naturality of s, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}_{\mathcal{X}}(g(y)) \times A(g(y)) \arrow[r, "s_{g(y)}"] \arrow[d, "\mathcal{O}_{\mathcal{X}}(\alpha_y^{\mathrm{op}}) \times A(\alpha_y^{\mathrm{op}})"'] & A(g(y)) \arrow[d, "A(\alpha_y^{\mathrm{op}})"] \\
\mathcal{O}_{\mathcal{X}}(f(y)) \times A(f(y)) \arrow[r, "s_{f(y)}"'] & A(f(y))
\end{tikzcd}$$

However, by the previous paragraph, $\mathcal{O}_{\mathcal{X}}(\alpha_y^{\mathrm{op}}) = \mathrm{id}_{\mathcal{O}_{\mathcal{Y}}(y)}$, and, on the other hand, $A(\alpha_y^{\mathrm{op}}) = (A \star \alpha^{\mathrm{op}})_y$. Thus, the above diagram is:

$$\begin{tikzcd}
\mathcal{O}_{\mathcal{Y}}(y) \times g^*A(y) \arrow[r, "(g^*s)_y"] \arrow[d, "\mathrm{id}_{\mathcal{O}_{\mathcal{Y}},y} \times (A \star \alpha^{\mathrm{op}})_y"'] & g^*A(y) \arrow[d, "(A \star \alpha^{\mathrm{op}})_y"] \\
\mathcal{O}_{\mathcal{Y}}(y) \times f^*A(y) \arrow[r, "(f^*s)_y"'] & f^*A(y)
\end{tikzcd}$$

and its commutativity along with the observation that $A(\alpha_y^{\mathrm{op}})$ is a morphism of Abelian groups shows that $A \star \alpha^{\mathrm{op}}$ is morphism of $\mathcal{O}_{\mathcal{Y}}$–modules. Furthermore, when A is, in addition, a sheaf of commutative rings, $A(\alpha_y^{\mathrm{op}})$ is a morphism of commutative rings and, $A \star \alpha^{\mathrm{op}}$ is a morphism of $\mathcal{O}_{\mathcal{Y}}$–algebras. $\square$

Construction 2.8. For a morphism of prestacks $f : \mathcal{Y} \longrightarrow \mathcal{Z}$ and two E, F on $\mathcal{Z}$, we have a map of presheaves:

$$\xi_f : f^*\mathcal{H}\mathrm{om}(E, F) \longrightarrow \mathcal{H}\mathrm{om}(f^*E, f^*F)$$

defined as follows. Let $y \in \mathrm{Ob}(\mathcal{Y})$ and $s \in f^*\mathcal{H}\mathrm{om}(E, F)(y)$. Then, $f^*\mathcal{H}\mathrm{om}(E, F)(y) = \mathrm{Hom}(E|_{f(y)}, F|_{f(y)})$ and s is a natural transformation of functors $E|_{f(y)} \Longrightarrow F|_{f(y)} : \mathcal{Z}_{/f(y)} \longrightarrow \mathrm{Set}$. This, then, yields a horizontal composite $s \star f_{/y}$, where $f_{/y} : \mathcal{Y}_{/y} \longrightarrow \mathcal{Z}_{/f(y)}$ is the functor induced by f. This is shown below:

$$\begin{tikzcd}
 & & \mathcal{Z} \arrow[dr, "E"] \arrow[dd, Rightarrow, "s"] & \\
\mathcal{Y}_{/y} \arrow[r, "f_{/y}"] & \mathcal{Z}_{/f(y)} \arrow[ur, "F_{f(y)}"] \arrow[dr, "F_{f(y)}"'] & & \mathrm{Set} \\
 & & \mathcal{Z} \arrow[ur, "F"'] &
\end{tikzcd}$$

This is a map of presheaves $f^*E|_y \longrightarrow f^*F|_y$ and we denote this by $f^*_{/y}(s)$ — noting that it is precisely the pullback of the morphism s under $f_{/y}$.

Proposition 2.9. *In the context of Construction 2.8, if E, F are $\mathcal{O}_{\mathcal{Z}}$–modules, then ξ_f restricts to a morphism of sheaves:*

$$\xi_f : f^*\mathcal{H}\mathrm{om}_{\mathcal{O}_{\mathcal{Z}}}(E, F) \longrightarrow \mathcal{H}\mathrm{om}_{\mathcal{O}_{\mathcal{Y}}}(f^*E, f^*F)$$

Proof. This follows from the fact that pullbacks of morphisms of module presheaves are morphisms of module presheaves, so that $f^*_{/y}(s)$ is a morphism of modules whenever s is. $\square$

Proposition 2.10. *In the context of Construction 2.8, if E, F are $\mathcal{O}_{\mathcal{Z}}$–module sheaves and if E is finite locally free, then ξ_f is an isomorphism:*

$$\xi_f : f^*\mathcal{H}\mathrm{om}_{\mathcal{O}_{\mathcal{Z}}}(E, F) \xrightarrow{\cong} \mathcal{H}\mathrm{om}_{\mathcal{O}_{\mathcal{Y}}}(f^*E, f^*F)$$

Proof. Let $y \in \mathrm{Ob}(\mathcal{Y})$. We wish to show that the component $\xi_{f,y}$ is an isomorphism. Since ξ_f is a morphism of sheaves, it suffices to produce a cover of $\{y_i \longrightarrow y\}_{i \in I}$ and show that ξ_{f,y_i} is an isomorphism for each $i \in I$. For this, we first choose a covering $\{c_i : z_i \longrightarrow f(y)\}_{i \in I}$ such that $t_i : E|_{z_i} \xrightarrow{\cong} \mathcal{O}_{\mathcal{Z}}|_{z_i}^{\oplus n_i}$ for some finite integer n_i. This means $\{p_{\mathcal{Z}}(c_i) : p_{\mathcal{Z}}(z_i) \longrightarrow p_{\mathcal{Z}}(f(y))\}_{i \in I}$ is a cover of $U := p_{\mathcal{Z}}(f(y)) = p_{\mathcal{Y}}(y)$ in $\mathrm{Aff}_{/S}$. Now, y corresponds to a map $m_y : U \longrightarrow \mathcal{Y}$ satisfying $m_y(\mathrm{id}_U) = y$, so that $f(y) = f(m_y(\mathrm{id}_U))$. Letting $U_i := p_{\mathcal{Z}}(z_i)$, we get objects $y_i := f(m_y(p_{\mathcal{Z}}(c_i)(\mathrm{id}_{U_i})))$. Then, $d_i := p_{\mathcal{Z}}(c_i) : U_i \longrightarrow U$ is a morphism in $\mathrm{Aff}_{/S/U}$ yielding morphisms $m_y(d_i) : y_i \longrightarrow y$ in $\mathcal{Y}$ over $d_i : U_i \longrightarrow U$, which form a cover of U. This shows that $m_y(d_i) : y_i \longrightarrow y$ form a cover of y.

Now, ξ_{f,y_i} is a function:

$$\xi_{f,y_i} : \mathrm{Hom}_{\mathcal{O}_{\mathcal{Z}}|_{f(y_i)}}(E|_{f(y_i)}, F|_{f(y_i)}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_{\mathcal{Y}}|_{y_i}}(f^*E|_{y_i}, f^*F|_{y_i})$$

By construction, $f(y_i) = z_i$. We then observe that we have strict equalities

$$f^*(E|_{y_i}) = f^*_{/y_i}(E|_{f(y_i)}) = f^*_{/y_i}(E|_{z_i}), f^*(F|_{y_i}) = f^*_{/y_i}(F|_{f(y_i)}) = f^*_{/y_i}(F|_{z_i})$$

and noticing that ξ_{f,y_i} is given by pullback of morphisms of presheaves along $f_{/u_i}$, we have a commutative diagram as follow, by the functoriality of $f^*_{/y_i}$:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{O}_\mathcal{Z}|_{z_i}}(E|_{z_i}, F|_{z_i}) \arrow[r, "\xi_{f,y_i} = f^*_{/y_i}"] \arrow[d, "- \circ t_i^{-1}"'] & \mathrm{Hom}_{\mathcal{O}_\mathcal{Y}|_{y_i}}(f^*_{/y_i}(E|_{z_i}), f^*_{/y_i}(F|_{z_i})) \arrow[d, "- \circ f^*_{/y_i}(t_i^{-1})"] \\
\mathrm{Hom}_{\mathcal{O}_\mathcal{Z}|_{z_i}}(\mathcal{O}_\mathcal{Z}|_{z_i}^{\oplus n_i}, F|_{z_i}) \arrow[r, "f^*_{/y_i}"'] & \mathrm{Hom}_{\mathcal{O}_\mathcal{Y}|_{y_i}}(f^*_{/y_i}(\mathcal{O}_\mathcal{Z}|_{z_i}^{\oplus n_i}), f^*_{/y_i}(F|_{z_i}))
\end{tikzcd}$$

where the vertical maps are isomorphisms. Hence, it suffices to show that the bottom horizontal map is a bijection.

Let $e_1, \ldots, e_{n_i}$ be the standard generators of the free module $\mathcal{O}_\mathcal{Z}(z_i)$–modules $\mathcal{O}_\mathcal{Z}|_{z_i}^{\oplus n_i}(\mathrm{id}_{z_i}) = \mathcal{O}_\mathcal{Z}(z_i)^{\oplus n_i}$. Then, the mapping

$$\gamma : \mathrm{Hom}_{\mathcal{O}_\mathcal{Z}|_{z_i}}(\mathcal{O}_\mathcal{Z}|_{z_i}^{\oplus n_i}, F|_{z_i}) \longrightarrow F(z_i)^{\oplus n_i} : s \longmapsto (s_{\mathrm{id}_{z_i}}(e_1), \ldots, s_{\mathrm{id}_{z_i}}(e_{n_i}))$$

can be verified to be an isomorphism of $\mathcal{O}_\mathcal{Z}(z_i)$–modules. To see this, we can produce a map in the reverse direction as follows. Let $(r_1, \ldots, r_{n_i}) \in F(z_i)^{\oplus n_i}$. Then, this provides a map $\mathcal{O}_\mathcal{Z}|_{z_i}^{\oplus n_i} \longrightarrow F|_{z_i}$ as follows: for each object $z \longrightarrow z_i$ in $\mathcal{Z}_{/z_i}$, we send $(e_1, \ldots, e_n)|_z$ to $(r_1, \ldots, r_n)|_z$. It is not hard to verify that this is natural in z, and is an inverse to γ. We have a similar isomorphism:

$$\gamma' : \mathrm{Hom}_{\mathcal{O}_\mathcal{Y}|_{y_i}}(f^*_{/y_i}(\mathcal{O}_\mathcal{Z}|_{z_i}^{\oplus n_i}), f^*_{/y_i}(F|_{z_i})) \longrightarrow f^*_{/y_i}(F|_{z_i})(\mathrm{id}_{y_i})$$

We notice that $f^*_{/y_i}(F|_{z_i})(\mathrm{id}_{y_i}) = F(f(y_i)) = F(z_i)$, and $f^*_{/y_i}(\mathcal{O}_\mathcal{Z}|_{z_i})(\mathrm{id}_{y_i}) = \mathcal{O}_\mathcal{Z}(f(y_i)) = \mathcal{O}_\mathcal{Z}(z_i)$. We then observe that

$$(f^*_{/y_i} s)_{\mathrm{id}_{y_i}} = (s \star f_{/y_i})_{\mathrm{id}_{y_i}} = s_{f^*_{/y_i}(\mathrm{id}_{z_i})} = s_{\mathrm{id}_{z_i}}$$

Together, these show that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{O}_\mathcal{Z}|_{z_i}}(\mathcal{O}_\mathcal{Z}|_{z_i}^{\oplus n_i}, F|_{z_i}) \arrow[rr, "f^*_{/y_i}"] \arrow[dr, "\gamma"'] & & \mathrm{Hom}_{\mathcal{O}_\mathcal{Y}|_{y_i}}(f^*_{/y_i}(\mathcal{O}_\mathcal{Z}|_{z_i}^{\oplus n_i}), f^*_{/y_i}(F|_{z_i})) \arrow[dl, "\gamma'"] \\
& F(z_i)^{\oplus n_i} &
\end{tikzcd}$$

which shows that horizontal map is an isomorphism, as required. □

3. Relative Spectrum for Prestacks

We define a relative spectrum for sheaves of algebras over prestacks using the same formula used for sheaves on the lisse-étale of an algebraic stack in [Ols16, §10.2.1]. This is the main tool used in our construction and study of the relevant moduli stacks.

Definition 3.1 (Relative Spectrum For Prestacks)**.** For a prestack $\mathcal{X} \in \mathrm{PSt}_{/S}$, and any presheaf $A : \mathcal{X}^{\mathrm{op}} \longrightarrow \mathrm{Set}$ of $\mathcal{O}_\mathcal{X}$–algebras, we define a category $\underline{\mathrm{Spec}}_\mathcal{X}(A)$ as follows:

- Objects are tuples (u, σ) where:
  - $u : U \longrightarrow \mathcal{X}$ is a morphism in $\mathrm{PSt}_{/S}$ with $U \in \mathrm{Aff}_{/S}$
  - $\sigma : u^*A \longrightarrow \mathcal{O}_U$ is a morphism of $\mathcal{O}_\mathcal{X}|_x$–algebras
- Morphisms $(u, s) \longrightarrow (u', s')$ are morphisms $a : u(\mathrm{id}_U) \longrightarrow u'(\mathrm{id}_{U'})$ in $\mathcal{X}$ such that, if $f := p_\mathcal{X}(a)$ and the natural transformation $u' \circ f \Longrightarrow u$ corresponding to a is $\tilde{a}$, the following

diagram of $\mathcal{O}_U$–algebras commutes:

$$\begin{tikzcd}
(u' \circ f)^* A = f^*(u')^* A \arrow[dr, "f^* s'"'] & & u^* A \arrow[ll, "A \star \tilde{a}^{\mathrm{op}}"'] \arrow[dl, "s"] \\
& (u' \circ f)^* \mathcal{O}_{\mathcal{Y}} = \mathcal{O}_U &
\end{tikzcd}$$

This category comes equipped with a functor $\pi_A : \underline{\mathrm{Spec}}_{\mathcal{X}}(A) \longrightarrow \mathcal{X}$ defined by sending (u, s) to $u(\mathrm{id}_U)$ and a to a. The category $\underline{\mathrm{Spec}}_{\mathcal{X}}(A)$ will be called the relative spectrum of A over $\mathcal{X}$ and π_A will be called its projection. We will suppress mention of $\mathcal{X}$ when it is clear from context, in which case, we will simply write $\underline{\mathrm{Spec}}(A)$ for $\underline{\mathrm{Spec}}_{\mathcal{X}}(A)$. When A is clear from context, we will write π instead of π_A

Warning 3.2. $\underline{\mathrm{Spec}}(A)$ is originally defined only for the case where $\mathcal{X}$ is an algebraic stack and A is a sheaf of $\mathcal{O}_{\mathcal{X}_{\text{lis-ét}}}$–modules on the lisse-étale site of $\mathcal{X}$. Our definition should subsume this by the equivalence of the categories of quasicoherent sheaves on the large étale site and those on the lisse-étale site [Sta25, Lemma 07B1], but we will not address this point in this work as we will directly show that our construction has some of the same properties that we will need.

We proceed to show that the relative spectrum is the Grothendieck construction of the following functor.

Construction 3.3. Given a prestack $\mathcal{X} \in \mathrm{PSt}_{/S}$, and two presheaves A, B of $\mathcal{O}_{\mathcal{X}}$–algebras, we define the internal Hom object $\mathrm{c}\mathcal{A}\mathrm{lg}_{\mathcal{O}_{\mathcal{X}}}(A, B)$ in the category of $\mathcal{O}_{\mathcal{X}}$–algebras as follows. We set

$$\mathrm{c}\mathcal{A}\mathrm{lg}_{\mathcal{O}_{\mathcal{X}}}(A, B)(x) := \mathrm{cAlg}_{\mathcal{O}_{\mathcal{X}}|_x}(A|_x, B|_x)$$

where the right hand side is the $\mathcal{O}_{\mathcal{X}}(x)$–algebra of $\mathcal{O}_{\mathcal{X}}|_x$–algebra morphisms $A|_x \longrightarrow B|_x$.

Given a morphism $\phi : x \longrightarrow x'$ in $\mathcal{X}$, we have a functor $\overline{\phi} : \mathcal{X}_{/x} \longrightarrow \mathcal{X}_{x'}$ sending $f : y \longrightarrow x$ to $\phi \circ f : y \longrightarrow x'$, and commuting with the forgetful functors $F_x : \mathcal{X}_{/x} \longrightarrow \mathcal{X}, F_{x'} : \mathcal{X}_{/x'} \longrightarrow \mathcal{X}$. Consider an element $s : A|_{x'} \longrightarrow B|_{x'}$ of $\mathrm{cAlg}_{\mathcal{O}_{\mathcal{X}}|_{x'}}(A|_{x'}, B|_{x'})$. We define $\mathrm{c}\mathcal{A}\mathrm{lg}_{\mathcal{O}_{\mathcal{X}}}(A, B)(\phi)$ to be the pullback $\overline{\phi}^*(s)$ which is a morphism of $\mathcal{O}_{\mathcal{X}}|_x$–algebras. Concretely, it is the horizontal composite $s \star \overline{\phi}$:

$$\begin{tikzcd}
(\mathcal{X}_{/x})^{\mathrm{op}} \arrow[r, "\overline{\phi}"] & (\mathcal{X}_{/x'})^{\mathrm{op}} \arrow[r, bend left, "A|_{x'} = A \circ F_{x'}"] \arrow[r, bend right, "B|_{x'} = B \circ F_x"'] \arrow[r, phantom, "\Downarrow s'"] & \mathrm{Set}
\end{tikzcd}$$

noting that $A|_{x'} \circ \overline{\phi} = A \circ F_x \circ \overline{\phi} = A|_x$ and similarly for $B|_x$. This gives a presheaf of $\mathcal{O}_{\mathcal{X}}$–algebras:

$$\mathrm{c}\mathcal{A}\mathrm{lg}_{\mathcal{O}_{\mathcal{X}}}(A, B) : \mathcal{X}^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

Proposition 3.4. *In the context of Definition 3.1, $\pi : \underline{\mathrm{Spec}}(A) \longrightarrow \mathcal{X}$ is the Grothendieck construction $\int \mathrm{c}\mathcal{A}\mathrm{lg}_{\mathcal{O}_{\mathcal{X}}}(A, \mathcal{O}_{\mathcal{X}}) \longrightarrow \mathcal{X}$.*

Proof. We let P denote $\mathrm{c}\mathcal{A}\mathrm{lg}_{\mathcal{O}_{\mathcal{X}}}(A, \mathcal{O}_{\mathcal{X}})$. Let (x, s) be an object of $\int P$ so that $x \in \mathrm{Ob}(\mathcal{X})$, $s \in P(x) = \mathrm{cAlg}_{\mathcal{O}_{\mathcal{X}}|_x}(A|_x, \mathcal{O}_{\mathcal{X}}|_x)$. That is, $s : A|_x \longrightarrow \mathcal{O}_{\mathcal{X}}|_x$ is a map of $\mathcal{O}_{\mathcal{X}}|_x$–algebras. Let $U := p_{\mathcal{X}}(x)$, for any $v : V \longrightarrow U \in \mathrm{Aff}_{/S}$, choose a Cartesian lift $\widehat{x}(v) : \tilde{x}(v) \longrightarrow x$ in $\mathcal{X}$ of v with target x. By the universal property of Cartesian morphisms, for morphisms $v : V \longrightarrow U, v' : V' \longrightarrow U$ and $g : V \longrightarrow V'$ with $v' \circ g = v$, we get a unique morphism $\widehat{x}(g) : \tilde{x}(v) \longrightarrow \tilde{x}(v')$ commuting with the maps $\widehat{x}(v), \widehat{x}(v')$. It is straightforward to verify that the $\widehat{x}(v)$ and $\widehat{x}(g)$ assemble to a functor $\widehat{x} : \mathrm{Aff}_{/S/U} \longrightarrow \mathcal{X}_{/x}$ over Aff/S. This provides a map of prestacks: $\tilde{x} := F_x \circ \widehat{x} : U \longrightarrow \mathcal{X}$, where F_x is the forgetful functor. By pulling back, we get a map of $\mathcal{O}_U$–algebras $\tilde{s} := \widehat{x}^* s : \widehat{x}^* A|_x = \tilde{x}^* A \longrightarrow \widehat{x}^* \mathcal{O}_{\mathcal{X}}|_x = \mathcal{O}_U$, which is an object of $\underline{\mathrm{Spec}}(A)$ over x.

Now, consider a morphism $\phi : (x, s) \longrightarrow (x', s')$ in $\int P$, so that $\phi : x \longrightarrow x'$ is a morphism in $\mathcal{X}$ such that $P(\phi)(s') = s$. Let $U := p_\mathcal{X}(x), U' = p_\mathcal{X}(x'), f = p_\mathcal{X}(\phi) : U \longrightarrow U'$. We get maps $\tilde{x} : U \longrightarrow \mathcal{X}, \tilde{x'} : U' \longrightarrow \mathcal{X}$ as in the previous paragraph, and ϕ gives a 2–morphism of prestacks $\tilde{\phi} : \tilde{x'} \circ f \Longrightarrow \tilde{x} : U \longrightarrow \mathcal{X}$, by the universal properties of the Cartesian lifts chosen. We then have the following pasting of 2–cells:

$$\begin{tikzcd}
U^{\mathrm{op}} \arrow[dd, "f"'] \arrow[dr, "\tilde{x}"] \arrow[drr, bend left=40, "A|_x"] \arrow[dr, phantom, shift left=6, "\Downarrow s"] & & \\
& \mathcal{X}^{\mathrm{op}} \arrow[r, "\mathcal{O}_\mathcal{X}"] & \mathrm{Set} \\
(U')^{\mathrm{op}} \arrow[ur, "\tilde{x'}"'] \arrow[urr, bend right=40, "A|_{x'}"'] \arrow[ur, phantom, shift right=6, "\Uparrow s'"] \arrow[uu, phantom, shift left=4, "\tilde{\phi}\,\Nearrow"] & &
\end{tikzcd} \tag{1}$$

which along with the hypothesis that $P(\phi)(s') = s$ shows the commutativity of the diagram:

$$\begin{tikzcd}
\tilde{x}^* A \arrow[dr, "\tilde{s}"'] & & f^* \tilde{x'}^* A \arrow[ll, "A \star \tilde{\phi}^{\mathrm{op}}"'] \arrow[dl, "\tilde{s'}"] \\
& \mathcal{O}_U &
\end{tikzcd} \tag{2}$$

That is, (f, ϕ) is a morphism in $\underline{\mathrm{Spec}}(A)$. The assignments:

$$(x, s) \longmapsto (\tilde{x}, \tilde{s}), \phi \longmapsto \phi$$

assemble to a functor $\Psi_A : \int P \longrightarrow \underline{\mathrm{Spec}}(A)$. To see this, we first observe that for any two objects $(x, s), (x', s') \in \int P$ with $\Psi_A(x, s) = (\tilde{x}, \tilde{s}), \Psi_A(x', s') = (\tilde{x}, \tilde{s'})$, and $U = \mathrm{dom}(u), U' = \mathrm{dom}(u')$, we have $x = \tilde{x}(\mathrm{id}_U), x' \cong \tilde{x'}(\mathrm{id}_{U'})$ by construction. Then, by construction $\mathrm{Hom}_{\int P}((x, s), (x', s'))$ and $\mathrm{Hom}_{\underline{\mathrm{Spec}}(A)}((\tilde{x}, \tilde{s}), (\tilde{x'}, \tilde{s'}))$ are both subsets of $\mathrm{Hom}_\mathcal{X}(x, x')$ defined by the equivalent conditions $P(\phi)(s') = s$ and 2, and are hence equal, while the mapping of morphisms induced by Ψ_A is just the identity on this subset. Thus, Ψ_A respects composition and identities, and is, at the same time, fully faithful. This functor commutes with the maps to $\mathcal{X}$ by construction.

We will now see that this functor is essentially surjective. Let (u, σ) be an object of $\underline{\mathrm{Spec}}(A)$. Then, we get an object $x_u := u(\mathrm{id}_U) \in \mathrm{Ob}(\mathcal{X})$ over U and we have a functor $p_{\mathcal{X}/x} : \mathcal{X}_{/x} \longrightarrow \mathrm{Aff}_{/S/U} = U$ obtained by taking the slice of $p_\mathcal{X}$ over x — it is defined by sending $b : y \longrightarrow x$ to $p_\mathcal{X}(b) : p_\mathcal{X}(y) \longrightarrow U$ — such that $F_x = u \circ p_{\mathcal{X}/x}$. We then get a morphism of $\mathcal{O}_\mathcal{X}|_x$–algebras $s_{u,\sigma} := p^*_{\mathcal{X}/x}\sigma = \sigma \star p_{\mathcal{X}/x} : A|_x = p^*_{\mathcal{X}/x} u^* A \longrightarrow \mathcal{O}_\mathcal{X}|_x = p^*_{\mathcal{X}/x}\mathcal{O}_U$. Thus, $(x_u, s_{u,\sigma})$ is an object of $\int P$. Now, $\tilde{x_u}(\mathrm{id}_U)$ is the domain of a Cartesian lift $\widehat{x_u}(\mathrm{id}_U) : \tilde{x_u}(\mathrm{id}_U) \longrightarrow x_u$ over $\mathrm{id}_U : U \longrightarrow U$, which is an isomorphism since the fibre categories of $\mathcal{X}$ are groupoids. However, since this was a choice of Cartesian lift to begin with, we could take this choice to be id_{x_u}. Then, letting $\phi := \widehat{x_u}(\mathrm{id}_U) = \mathrm{id}_{x_u} = \mathrm{id}_{u(\mathrm{id}_U)}$, we see from the definition of the relative spectrum, that ϕ is the identity morphism $\Psi_A(x_u, s_{u,\sigma}) \longrightarrow (u, \sigma)$, showing that Ψ_A is strictly surjective objects. $\square$

We will now focus on the properties of the relative spectrum that will be necessary to prove our results.

Proposition 3.5. *In the context of Definition 3.1, π_A is an affine morphism.*

Proof. Let $f : U \longrightarrow \mathcal{X}$ be any morphism where $U \in \mathrm{Aff}_{/S}$. We note that the 2–fibre product of prestacks is simply the 2–fibre product in the slice 2–category $\mathfrak{Cat}_{/\mathrm{Aff}_{/S}}$ [Sta25, Lemma 0041], which is, in turn, just the 2–fibre product in $\mathfrak{Cat}$. Since the Grothendieck construction is compatible with 2–pullback in $\mathfrak{Cat}$, $\int A \circ f$ is equivalent as a category to the 2–fibre product of categories $\int A \times_{\mathcal{X}}^{\mathfrak{Cat}} \mathrm{Aff}_{/S/U}$. By Proposition 3.4, $\underline{\mathrm{Spec}}(A) = \int A$ and thus, the fibre product of prestacks $\underline{\mathrm{Spec}}(A) \times_{\mathcal{X}} U = \int A \times_{\mathcal{X}} U$ is, in fact, $\int A \circ f$. We can then unwrap the definition of $\int A \circ f$ to see that it is the usual relative spectrum defined for schemes in [Sta25, Section 01LQ], which is an affine scheme over U and hence an affine scheme over S. ☐

Proposition 3.6. *In the context of Definition 3.1, for any finite locally free $\mathcal{O}_{\mathcal{X}}$–module M, $\pi : \underline{\mathrm{Spec}}(\mathrm{Sym}(M^\vee)) \longrightarrow \mathcal{X}$ is the Grothendieck construction $\int M \longrightarrow \mathcal{X}$ of the functor $M : \mathcal{X}^{\mathrm{op}} \longrightarrow \mathrm{Set}$.*

Proof. We notice that since pullbacks for large étale sites are exact, they commute with direct sums. They also commute with tensor products and colimits. Hence, they commute with taking the symmetric algebra of a sheaf of modules over the structure sheaf. The finite locally free hypothesis ensures $(M^\vee)^\vee \cong M$. Thus, we have:

$$\begin{aligned} &\mathrm{cAlg}_{\mathcal{O}_{\mathcal{X}}|_x}(\mathrm{Sym}(M^\vee)|_x, \mathcal{O}_{\mathcal{X}}|_x) \\ \cong&\mathrm{cAlg}_{\mathcal{O}_{\mathcal{X}}|_x}(\mathrm{Sym}(M^\vee|_x), \mathcal{O}_{\mathcal{X}}|_x) \\ \cong&\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}|_x}(M^\vee|_x, \mathcal{O}_{\mathcal{X}}|_x) \\ \cong&(M^\vee)^\vee(x) \\ \cong&M(x) \end{aligned}$$

This along with Proposition 3.4 now yields the result. ☐

Notation 3.7. For any $\mathcal{X} \in \mathrm{PSt}_{/S}$ and any $\mathcal{O}_{\mathcal{X}}$–module M, we denote $\underline{\mathrm{Spec}}_{\mathcal{X}}(\mathrm{Sym}(M^\vee))$ by $\mathbb{V}M$.

Construction 3.8. For any morphism of prestacks $f : \mathcal{Y} \longrightarrow \mathcal{Z}$ in $\mathrm{PSt}_{/S}$, any $\mathcal{O}_{\mathcal{Z}}$–modules E, F, consider a strict factorization of f as $\mathcal{Y} \xrightarrow{t} \mathbb{V}[E, F] \xrightarrow{\pi} \mathcal{Z}$, that is $f = \pi \circ t$. For any object $y \in \mathcal{Y}$, $t(y) = (f(y), s(y))$ for some $s(y) \in [E, F](f(y)) = \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}|_{f(y)}}(E|_{f(y)}, F|_{f(y)})$ by Proposition 3.6. We then observe that $f^*E(y) = E(f(y)) = E|_{f(y)}(\mathrm{id}_y)$ and, similarly, for F and $\mathcal{O}_{\mathcal{X}}|_{f(y)}$ so that $s(y)_{\mathrm{id}_y}$ is an $\mathcal{O}_{\mathcal{Y}}(y) = \mathcal{O}_{\mathcal{X}}|_{f(y)}(\mathrm{id}_y)$–linear map $f^*E(y) \longrightarrow f^*F(y)$. We denote the collection of morphisms $\left\{s(y)_{\mathrm{id}_y}\right\}_{y \in \mathrm{Ob}(\mathcal{Y})}$ as $\sigma(t)$.

Proposition 3.9. *In the context of Construction 3.8, the collection $\sigma(t)$ constitutes a natural transformation of functors and is hence a morphism of $\mathcal{O}_{\mathcal{Y}}$–modules.*

Proof. For any morphism $b : y \longrightarrow y'$, the map

$$f^*[E, F](y') = [E, F](f(y')) \longrightarrow f^*[E, F](y) = [E, F](f(y))$$

is given by sending a natural transformation

$$a : E|_{f(y')} \Longrightarrow F|_{f(y')} : \mathcal{Z}_{/f(y')} \longrightarrow \mathcal{Z} \longrightarrow \mathrm{Set}$$

to the horizontal composite $a \star (f(b) \circ -)$, where $f(b) \circ -$ is the functor $\mathcal{Z}_{/f(y)} \longrightarrow \mathcal{Z}_{/f(y')}$ given by post-composing with the morphism $b : y \longrightarrow y'$. This horizontal composite is shown below:

$$\begin{tikzcd} & & \mathcal{Z} \arrow[dr, "E"] \arrow[dd, phantom, "\Downarrow a"] & \\ \mathcal{Z}_{/f(y)} \arrow[r, "f(b)\circ -"] & \mathcal{Z}_{/f(y')} \arrow[ur, "F_{f(y')}"] \arrow[dr, "F_{f(y')}"'] & & \mathrm{Set} \\ & & \mathcal{Z} \arrow[ur, "F"'] & \end{tikzcd}$$

Unwrapping the definition for horizontal composition, we see that for an object $q : v \longrightarrow f(y)$ in $\mathcal{Z}_{/f(y)}$, the component $(s(y') \star (f(b) \circ -))_q$ is $s(y')_{f(b)\circ q}$. Viewing $f(b) : f(y) \longrightarrow f(y')$ as a morphism in $\mathcal{Z}_{/f(y')}$ from the object $f(b)$ to $\mathrm{id}_{f(y')}$, we have the following commutative diagram by the naturality of $s(y')$:

$$\begin{tikzcd}
E|_{f(y')}(\mathrm{id}_{f(y')}) \arrow[r, "s(y')_{\mathrm{id}_{y'}}"] \arrow[d, "E|_{(f(y'))(f(b))}"'] & F|_{f(y')}(\mathrm{id}_{f(y')}) \arrow[d, "F|_{f(y')}(f(b))"] \\
E|_{f(y')}(f(b)) \arrow[r, "s(y')_{f(b)}"'] & F|_{f(y')}(f(b))
\end{tikzcd}$$

We observe that $E|_{f(y')}(f(b)) = E(f(y))$, $F|_{f(y')}(f(b)) = F(f(b))$. By the fact that $s(b) : (f(y), s(y)) \longrightarrow (f(y'), s(y'))$ is a morphism in $\int[E, F]$, we obtain that $s(y')_{f(b)} = (s(y') \star (f(b) \circ -)) = s(y)_{\mathrm{id}_y}$. Thus, the above commutative square becomes:

$$\begin{tikzcd}
E(f(y)) \arrow[r, "s(y')_{\mathrm{id}_{y'}}"] \arrow[d, "E|_{(f(y'))(f(b))}"'] & F(f(y')) \arrow[d, "F|_{f(y')}(f(b))"] \\
E(f(y)) \arrow[r, "s(y)_{\mathrm{id}_y}"'] & F(f(y))
\end{tikzcd}$$

Thus, the collection $\left\{s(y)_{\mathrm{id}_y}\right\}_{y\in \mathrm{Ob}(\mathcal{Y})}$ form a natural transformation of functors $f^*E \longrightarrow f^*F$, which is $\mathcal{O}_{\mathcal{Y}}$–linear, by construction. $\square$

Proposition 3.10. *In the context of Construction 3.8, for any two maps $t, t' : \mathcal{Y} \longrightarrow \mathbb{V}[E, F]$ and any 2–morphism of prestacks $\beta : t \Longrightarrow t'$, the following diagram of presheaves commutes:*

$$\begin{tikzcd}
(\pi \circ t)^*E \arrow[r, "(E\circ\pi)\star\beta"] \arrow[d, "\sigma(t)"'] & (\pi \circ t')^*E \arrow[d, "\sigma(t')"] \\
(\pi \circ t)^*F \arrow[r, "(F\circ\pi)\star\beta"'] & (\pi \circ t')^*F
\end{tikzcd}$$

Proof. We denote $x := \pi \circ t, x' = \pi \circ t'$. Let $y \in \mathrm{Ob}(\mathcal{Y})$. Then, $t(y) = (x(y), s(y))$ and $t'(y) = (x'(y), s'(y))$. We then have a morphism $\beta_y : (x(y), s(y)) \longrightarrow (x'(y), s'(y))$ in $\mathbb{V}[E, F]$ so that β_y is a morphism $x(y) \longrightarrow x(y')$ in $\mathcal{X}$ such that $s'(y)_{\beta_y} = s'(y) \star (\beta_y \circ -) = s(y)$. By considering β_y as morphism $\beta_y \longrightarrow \mathrm{id}_{x'(y)}$ in $\mathcal{X}$, we obtain the following commutative diagram:

$$\begin{tikzcd}
E(x'(y)) = E|_{x'(y)}(\mathrm{id}_{x'(y)}) \arrow[r, "s(y')_{\mathrm{id}_{x'(y)}}"] \arrow[d, "E(\pi(\beta_y))=E|_{x(y)}(\beta_y)"'] & F|_{x'(y)}(\mathrm{id}_{x'(y)}) = F(x'(y)) \arrow[d, "F|_{x'(y)}(\beta_y)=F(\pi(\beta_y))"] \\
E(x(y)) = E|_{x'(y)}(\beta_y) \arrow[r, "s'(y)_{\beta_y}=s(y)_{\mathrm{id}_{x(y)}}"'] & F|_{x'(y)}(\beta_y) = F(x(y))
\end{tikzcd}$$

which is precisely the diagram whose commutativity we need to show for each such y. $\square$

4. Arrow Bundles

We will now revisit the moduli stack of vector bundle triples or "arrow bundles" first constructed in [AR25], and give two concrete descriptions thereof that will be used to prove the algebraicity of the moduli stacks of connection triples in Section 5.

4.1. **Moduli Prestack of Arrow Bundles.** We will define the moduli prestack of arrow bundles first in the intuitive way — that is, as the prestack associated to a functor that assigns to a scheme U, the groupoid of vector bundles on $U \times \mathcal{X}$ and whose action on morphisms $U \longrightarrow U'$ is given by pullback. This is in contrast to [AR25], where we take the more "basis-free" route of using mapping stacks. The main point is that the two definitions give equivalent prestacks, which we show in the next subsection. To this end, we first fix some notation that will be used in the rest of the paper.

Notation 4.1. Let $\mathcal{X}, \mathcal{Y}$ be prestacks over S and E, F, any two $\mathcal{O}_{\mathcal{Y}}$–modules. We will, then, use the following notation:

$[E,F]_\square$	:	the $\mathcal{O}_{\mathcal{Y}\times\mathcal{Y}}$–module $\mathcal{H}om_{\mathcal{O}_{\mathcal{Y}}}(pr_1^*E, pr_2^*F)$
$\mathcal{E}(\mathcal{X})$	:	the universal locally free sheaf on $\mathcal{Y} = \mathcal{B}(\mathcal{X}) = \mathcal{M}(\mathcal{X}) \times \mathcal{X}$
$\mathcal{E}^n(\mathcal{X})$	:	the restriction of $\mathcal{E}(\mathcal{X})$ to $\mathcal{Y} = \mathcal{B}^n(\mathcal{X}) = \mathcal{M}^n(\mathcal{X}) \times \mathcal{X}$
$\mathcal{V}(\mathcal{X})$	:	the $\mathcal{O}_{\mathcal{B}(\mathcal{X})\times\mathcal{B}(\mathcal{X})}$–module $[\mathcal{E}(\mathcal{X}), \mathcal{E}(\mathcal{X})]_\square$
$\mathcal{V}^{n,m}(\mathcal{X})$	:	the $\mathcal{O}_{\mathcal{B}^n\times\mathcal{B}(\mathcal{X})^m}$–module $[\mathcal{E}^n(\mathcal{X}), \mathcal{E}^m(\mathcal{X})]_\square$
$\mathcal{V}(\mathcal{X})$	:	$\mathbb{V}\mathcal{V}(\mathcal{X}) = \mathbb{V}[\mathcal{E}(\mathcal{X}), \mathcal{E}(\mathcal{X})]_\square$
$\mathcal{V}^{n,m}(\mathcal{X})$	:	$\mathbb{V}\mathcal{V}^{n,m}(\mathcal{X}) = \mathbb{V}[\mathcal{E}^n(\mathcal{X}), \mathcal{E}^m(\mathcal{X})]_\square$
$f^<$	:	the pullback functor $(f \times \mathrm{id}_{\mathcal{X}})^*$
$f_>$	:	the pushforward functor $(f \times \mathrm{id}_{\mathcal{X}})_*$
$r^\lhd$	:	the pullback functor $(r \times \Delta_X)^*$
$r_\rhd$	:	the pushforward functor $(r \times \Delta_X)_*$

Construction 4.2. Given any prestack $\mathcal{X} \in \mathrm{PSt}_{/S}$, we define a category $\mathcal{M}_1(\mathcal{X})$ with the following data:

- Objects are tuples (u, E, F, s), where:
  - $u : U \longrightarrow S$ is an object of $\mathrm{Aff}_{/S}$
  - $E, F \in \mathrm{Vect}(U \times \mathcal{X})$ are finite locally free $\mathcal{O}_{\mathcal{X}}$–modules on $U \times \mathcal{X}$
  - $s : E \longrightarrow F \in \mathrm{Hom}_{\mathcal{O}_{U\times\mathcal{X}}}(E, F)$ is a morphism of $\mathcal{O}_{\mathcal{X}}$–modules
- Morphisms $(u, E, F, s) \longrightarrow (u', E', F', s')$ are tuples (f, a, b) where:
  - $f : U = \mathrm{dom}(u) \longrightarrow U' = \mathrm{dom}(u')$ is a morphism in $\mathrm{Aff}_{/S}$
  - $a : f^< E' \longrightarrow E, b : f^< F' \longrightarrow F$ are isomorphisms of $\mathcal{O}_{U\times\mathcal{X}}$–modules making the following diagram commute in $\mathrm{Vect}(U \times \mathcal{X})$:

$$\begin{tikzcd} f^< E' \arrow[r, "a"] \arrow[d, "f^< s'"'] & E \arrow[d, "s"] \\ f^< F' \arrow[r, "b"'] & F \end{tikzcd}$$

- Given two morphisms $(u, E, F, s) \overset{(f,a,b)}{\longrightarrow} (u', E', F', s') \overset{(g,c,d)}{\longrightarrow} (u'', E'', F'', s'')$, the composite is defined to be $(g\circ f, f^< c\circ a, f^< \circ b)$ noticing that the following diagram commutes in $\mathrm{Vect}(U\times\mathcal{X})$:

$$\begin{tikzcd} (g\circ f)^< E'' = f^< g^< E'' \arrow[r, "f^< c"] \arrow[d, "(g\circ f)^< s'' = f^< g^< s''"'] & f^< E' \arrow[r, "a"] \arrow[d, "f^< s'"'] & E \arrow[d, "s"] \\ (g\circ f)^< F'' = f^< g^< F'' \arrow[r, "f^< d"'] & f^< F' \arrow[r, "b"'] & F \end{tikzcd}$$

- The identity morphism of an object (u, E, F, s) is the tuple $(\mathrm{id}_U, \mathrm{id}_E, \mathrm{id}_F)$.

Proposition 4.3. *The category $\mathcal{M}_1(\mathcal{X})$ of Construction 4.2 has a functor to* $\mathrm{Aff}_{/S}$ *defined by the mapping:*

$$\begin{array}{ccc} (u, E, F, s) & \longmapsto & (u : U \longrightarrow S) \\ (f, a, b) & \longmapsto & f : U \longrightarrow U' \end{array}$$

making it a prestack over S.

Proof. That the mapping defines a functor is straightforward to check. That this defines a prestack over S follows from the observation that it is the Grothendieck construction or unstraightening of the contravariant functor:

$$\begin{array}{ccc} U & \longmapsto & \mathrm{Fun}(\Delta^1, \mathrm{Vect}(U \times \mathcal{X}))^{\simeq} \\ (U \xrightarrow{f} U') & \longmapsto & f^{<} \circ - \end{array}$$

We omit the details. □

Remark 4.4. If $\mathcal{X}$ is a stack, then so is $\mathcal{M}_1(\mathcal{X})$, but we will not show this directly, although one could argue that this follows from the descent of morphisms of sheaves. Instead, we will produce an equivalence with another prestack whose descent property is easier to show from definitions whenever $\mathcal{X}$ is a stack. The reason for this indirection is that this equivalence will be necessary in proving our main results, so we address it first.

Definition 4.5 (Moduli Prestack of Arrow Bundles)**.** We will call the prestack $\mathcal{M}_1(\mathcal{X})$ of Construction 4.2, the moduli prestack of arrow bundles on $\mathcal{X}$.

4.2. **Definition via Mapping Stacks.** We now revisit the definition of the moduli of arrow bundles given in [AR25] and show that it gives a prestack equivalent to Construction 4.2. To achieve this, we will give a concrete description of this prestack that was not given before. There are two main purposes of this definition. First, it is easier to show the algebraicity of this stack using results about the algebraicity of mapping stacks. Second, it can be easily generalized to the setting of derived stacks, where we eventually wish to work, without having to explicitly define the functor of points. The work in this subsection should be considered as evidence for the soundness of the definition as well as a proof of algebraicity of the slightly more intuitive definition (Construction 4.2) via the functor of points. At the same time, as we will discuss in Section 5.4 it proves some of the claims made in that paper with only sketches of proofs, We begin by recalling the definition of mapping prestacks.

Definition 4.6 (Mapping Prestacks [Wan11, §3])**.** Given two prestacks $\mathcal{Y}, \mathcal{Z} \in \mathrm{PSt}_{/S}$, we define the category $\mathrm{Map}_S(\mathcal{Y}, \mathcal{Z})$ as follows:

- Objects are tuples (u, f) where:
  - $u : U \longrightarrow S$ is an object of $\mathrm{Aff}_{/S}$
  - $f : U \times_S \mathcal{Y} \longrightarrow \mathcal{Z}$ is a 1–morphism in $\mathrm{PSt}_{/S}$
- Morphisms $(u, f) \longrightarrow (u', f')$ are tuples (g, a) where:
  - $g : U = \mathrm{dom}(u) \longrightarrow U' = \mathrm{dom}(u')$ is a morphism in $\mathrm{Aff}_{/S}$
  - $a : f' \circ (g \times \mathrm{id}_{\mathcal{Y}}) \Longrightarrow f : U \times_S \mathcal{Y} \longrightarrow \mathcal{Z}$ is a 2–morphism in $\mathrm{PSt}_{/S}$
- Composition is given by pasting 2–cells.
- The identity morphism of (u, f) is $(\mathrm{id}_u, \mathrm{id}_f)$.

This category has a functor to $\mathrm{Aff}_{/S}$ making it a prestack over S. We call it the mapping prestack with domain $\mathcal{Y}$ and codomain $\mathcal{Z}$.

Proposition 4.7. *Mapping prestacks are internal Hom objects in the 2–category* $\mathrm{PSt}_{/S}$ *with respect to the product* $- \times_S -$.

Proof. This is well-known and we omit the proof. □

Construction 4.8 (Moduli Prestack of Arrow Bundles)**.** For any prestack $\mathcal{X} \in \mathrm{PSt}_{/S}$, we consider the following maps:

- $\overline{\Delta_{\mathcal{X}}} : S \longrightarrow \mathrm{Map}_S(\mathcal{X}, \mathcal{X} \times \mathcal{X})$ corresponding to the diagonal map $\Delta_{\mathcal{X}} : \mathcal{X} \longrightarrow \mathcal{X} \times \mathcal{X}$ which sends a map $U \longrightarrow S$ to the composite $U \times \mathcal{X} \xrightarrow{pr_2} \mathcal{X} \xrightarrow{\Delta_{\mathcal{X}}} \mathcal{X} \times \mathcal{X}$
- $- \circ p_{\mathcal{X}} : \mathcal{M}(\mathcal{X})^2 \simeq \mathrm{Map}_S(S, \mathcal{M}(\mathcal{X})^2) \longrightarrow \mathrm{Map}_S(\mathcal{X}, \mathcal{M}(\mathcal{X})^2)$, which sends a morphism $(v, w) : U \longrightarrow \mathcal{M}(\mathcal{X})^2$ to the composite $U \times \mathcal{X} \xrightarrow{pr_1} U \xrightarrow{(v,w)} \mathcal{M}(\mathcal{X})^2$

We then define the prestack $\mathcal{M}_1'(\mathcal{X})$ by the following fibre product of prestacks:

$$\begin{tikzcd}
\mathcal{M}_1'(\mathcal{X}) \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & \mathrm{Map}_S(\mathcal{X}, \mathcal{V}(\mathcal{X})) \arrow[d, "\pi_{\mathcal{V}(\mathcal{X})} \circ -"] \\
\mathcal{M}(\mathcal{X})^2 \simeq \mathcal{M}(\mathcal{X})^2 \times S \arrow[r, "(- \circ p_{\mathcal{X}}) \times \overline{\Delta_{\mathcal{X}}}"'] & \mathrm{Map}_S(\mathcal{X}, \mathcal{M}(\mathcal{X})^2) \times \mathrm{Map}_S(\mathcal{X}, \mathcal{X}^2) \simeq \mathrm{Map}_S(\mathcal{X}, \mathcal{B}(\mathcal{X})^2) &
\end{tikzcd}$$

where the right vertical map is induced by the bundle projection of $\mathcal{V}(\mathcal{X})$.

Proposition 4.9. *For any prestack $\mathcal{X} \in \mathrm{PSt}_{/S}$, the underlying category of the prestack $\mathcal{M}_1'(\mathcal{X})$ of Construction 4.8 admits the following concrete description:*

- *Objects are tuples (u, s, v, w, ϕ), where:*
  - $u : U \longrightarrow S$ *is an object of* $\mathrm{Aff}_{/S}$,
  - $s : U \times \mathcal{X} \longrightarrow \mathcal{V}(\mathcal{X})$ *is a 1–morphism in* $\mathrm{PSt}_{/S}$,
  - $v, w : U \longrightarrow \mathcal{M}(\mathcal{X})$ *are 1–morphisms in* $\mathrm{PSt}_{/S}$, *and*
  - $\phi : p \circ s \Longrightarrow (v \times \mathrm{id}_{\mathcal{X}}, w \times \mathrm{id}_{\mathcal{X}}) : U \times \mathcal{X} \longrightarrow \mathcal{B}(\mathcal{X})^2$ *is a 2–morphism in* $\mathrm{PSt}_{/S}$
- *Morphisms $(u, s, v, w, \phi) \longrightarrow (u', s', v', w', \phi')$ are tuples (f, a, b, c) where:*
  - $f : U = \mathrm{dom}(u) \longrightarrow U' = \mathrm{dom}(u')$ *is a morphism in* $\mathrm{Aff}_{/S}$,
  - $a : s' \circ (f \times \mathrm{id}_{\mathcal{X}}) \Longrightarrow s : U \times \mathcal{X} \longrightarrow \mathcal{V}(\mathcal{X})$ *is a 2–morphism in* $\mathrm{PSt}_{/S}$, *and*
  - $b : v' \circ f \Longrightarrow v, c : w' \circ f \Longrightarrow w : U \longrightarrow \mathcal{M}(\mathcal{X})$ *are 2–morphisms in* $\mathrm{PSt}_{/S}$,

  such that the following diagram of natural transformations commutes:

$$\begin{tikzcd}
\pi_{\mathcal{V}(\mathcal{X})} \circ s' \circ (f \times \mathrm{id}_{\mathcal{X}}) \arrow[r, "\phi' \star (f \times \mathrm{id}_{\mathcal{X}})"] \arrow[dd, "\pi_{\mathcal{V}(\mathcal{X})} \star a"'] & (v' \times \mathrm{id}_{\mathcal{X}}, w' \times \mathrm{id}_{\mathcal{X}}) \circ (f \times \mathrm{id}_{\mathcal{X}}) \arrow[d, equal] \\
& ((v' \circ f) \times \mathrm{id}_{\mathcal{X}}, (w' \circ f) \times \mathrm{id}_{\mathcal{X}}) \arrow[d, "(b \times e_{\mathcal{X}}, c \times e_{\mathcal{X}})"] \\
\pi_{\mathcal{V}(\mathcal{X})} \circ s \arrow[r, "\phi"'] & (v \times \mathrm{id}_{\mathcal{X}}, w \times \mathcal{X})
\end{tikzcd}$$

  where $e_{\mathcal{X}}$ is the identity 2–morphism of $\mathrm{id}_{\mathcal{X}}$.

Proof. We will unwrap the limit defining $\mathcal{M}_1'(\mathcal{X})$. An object of $\mathrm{Map}_S(\mathcal{X}, \mathcal{V}(\mathcal{X}))$ consists of a tuple (u, s), where $u : U \longrightarrow S \in \mathrm{Aff}_{/S}$ and $s : U \times \mathcal{X} \longrightarrow \mathcal{V}(\mathcal{X})$. A morphism $(u, s) \longrightarrow (u', s')$ is a tuple (f, a), where $f : U = \mathrm{dom}(u) \longrightarrow U' = \mathrm{dom}(u') \in \mathrm{Aff}_{/S}$ and $a : s' \circ (f \times \mathrm{id}_{\mathcal{X}}) \Longrightarrow s$ is a 2–morphism in $\mathrm{PSt}_{/S}$. The projection to $\mathrm{PSt}_{/S}$ is given by $(u, s) \longmapsto u, (f, a) \longmapsto f$. The morphism of prestacks

$$\pi_{\mathcal{V}(\mathcal{X})} \circ - : \mathrm{Map}_S(\mathcal{X}, \mathcal{V}(\mathcal{X})) \longrightarrow \mathrm{Map}_S(\mathcal{X}, \mathcal{B}(\mathcal{X})^2)$$

is given by

$$(u, s) \longmapsto (u, \pi_{\mathcal{V}(\mathcal{X})} \circ s), (f, a) \longmapsto (f \circ u, \pi_{\mathcal{V}(\mathcal{X})} \star a) = (u', \pi_{\mathcal{V}(\mathcal{X})} \star a)$$

An object of $\mathcal{M}(\mathcal{X})^2 \simeq \mathrm{Map}_S(S, \mathcal{M}(\mathcal{X})^2) \simeq \mathrm{Map}_S(S, \mathcal{M}(\mathcal{X}))^2$ is a tuple (u, v, w), where $u : U \longrightarrow S \in \mathrm{Aff}_{/S}$ and $(v, w) : U \times S \simeq U \longrightarrow \mathcal{M}(\mathcal{X})^2$ is a 1–morphism in $\mathrm{PSt}_{/S}$ — note that every morphism $U \longrightarrow \mathcal{M}(\mathcal{X})^2$ is of this form by Proposition 2.2. Similarly, a morphism $(u, v, w) \longrightarrow (u', v', w')$ is a tuple (f, b, c) where $f : U = \mathrm{dom}(U) \longrightarrow U' = \mathrm{dom}(u')$ is a morphism in $\mathrm{Aff}_{/S}$, $b : v' \circ f \Longrightarrow v, c : w' \circ f \Longrightarrow w : U \longrightarrow \mathcal{M}(\mathcal{X})$ are 2–morphisms in $\mathrm{PSt}_{/S}$. The structure map to $\mathrm{Aff}_{/S}$ is given by:

$$(u, v, w) \longmapsto u, (f, b, c) \longmapsto f$$

Now, consider the map $- \circ p_{\mathcal{X}} : \mathrm{Map}_S(S, \mathcal{M}(\mathcal{X})) \longrightarrow \mathrm{Map}_S(\mathcal{X}, \mathcal{M}(\mathcal{X}))$ given by precomposition with the structure map $p_{\mathcal{X}} : \mathcal{X} \longrightarrow S$. We observe that on an object $(u, v) \in \mathrm{Map}_S(S, \mathcal{M}(\mathcal{X}))$, the map has value:

$$(u, U \times \mathcal{X} \xrightarrow{\mathrm{id}_U \times p_{\mathcal{X}}} U \times S \xrightarrow{\simeq} U \xrightarrow{v} \mathcal{M}(\mathcal{X}))$$

Then, $\mathrm{id}_U \times p_{\mathcal{X}}$ composed with the equivalence $U \times S \simeq S$ is just the projection $U \times \mathcal{X} \longrightarrow U$. On a morphism $(f, b) : (u, v) \longrightarrow (u', v')$ in $\mathrm{Map}_S(S, \mathcal{M}(\mathcal{X}))$, the map $- \circ p_{\mathcal{X}}$ has value:

$$(f : U \longrightarrow U', b \times e_{\mathcal{X}} : (v' \circ f) \times \mathrm{id}_{\mathcal{X}} \Longrightarrow v \times \mathrm{id}_{\mathcal{X}} : U \times \mathcal{X} \longrightarrow \mathcal{B}(\mathcal{X}))$$

Then, using Proposition 2.2, the map $- \circ p_{\mathcal{X}} : \mathrm{Map}_S(S, \mathcal{M}(\mathcal{X})^2) \longrightarrow \mathrm{Map}_S(\mathcal{X}, \mathcal{M}(\mathcal{X}))$ is given by:

$$\begin{array}{rcc} (u, v, w) & \longmapsto & (u, U \times \mathcal{X} \xrightarrow{pr_1} U \xrightarrow{v} \mathcal{B}(\mathcal{X}), U \times \mathcal{X} \xrightarrow{pr_2} U \xrightarrow{w} \mathcal{B}(\mathcal{X})) \\ (u, b, c) & \longmapsto & (u, b \times e_{\mathcal{X}}, c \times e_{\mathcal{X}}) \end{array}$$

These descriptions along with the concrete description of fibre products of prestacks given in [Sta25, Lemma 0040] prove the result. □

Remark 4.10. An object of the underlying category of $\mathcal{M}_1'(\mathcal{X})$ consists of maps $U \longrightarrow S, s : U \times_S \mathcal{X} \longrightarrow \mathcal{V}(\mathcal{X}), r = (v, w) : U \longrightarrow \mathcal{M}(\mathcal{X})^2$, such that following diagrams 2–commute:

$$\begin{tikzcd} U \times_S \mathcal{X} \arrow[r, "s"] \arrow[d, "pr_1"'] & \mathcal{V}(\mathcal{X}) \arrow[d] \\ U \arrow[r, "r"'] & \mathcal{M}(\mathcal{X}) \times_S \mathcal{M}(\mathcal{X}) \end{tikzcd} \qquad \begin{tikzcd} U \times_S \mathcal{X} \arrow[r, "s"] \arrow[d, "pr_2"'] & \mathcal{V}(\mathcal{X}) \arrow[d] \\ \mathcal{X} \arrow[r, "\Delta_{\mathcal{X}}"'] & \mathcal{X} \times_S \mathcal{X} \end{tikzcd}$$

The 2–commutativity above means that the map $U \times_S \mathcal{X} \xrightarrow{s} \mathcal{V}(\mathcal{X}) \xrightarrow{\pi_{\mathcal{V}(\mathcal{X})}} \mathcal{B}(\mathcal{X})^2$ is of the form $r \times \Delta_X = (v, w) \times \Delta_X$ which is the same as $(v \times_S \mathrm{id}_{\mathcal{X}}, w \times_S \mathrm{id}_{\mathcal{X}})$ by the identification $\mathrm{Map}_S(\mathcal{X}, \mathcal{B}(X)^2) \simeq \mathrm{Map}_S(\mathcal{X}, \mathcal{M}(\mathcal{X})^2) \times \mathrm{Map}_S(\mathcal{X}, \mathcal{X}^2)$. That is, the object is a 2–categorical diagram in $\mathrm{PSt}_{/S}$ of the form:

$$\begin{tikzcd} & \mathcal{V}(\mathcal{X}) \arrow[d, "\pi_{\mathcal{V}(\mathcal{X})}"] \\ U \times \mathcal{X} \arrow[ur, bend left, "s"] \arrow[r, bend left, "\pi_{\mathcal{V}(\mathcal{X})} \circ s"] \arrow[r, bend right, "(v \times \mathrm{id}_{\mathcal{X}}, w \times \mathrm{id}_{\mathcal{X}})"'] \arrow[r, phantom, "\Downarrow \phi"] & \mathcal{B}(X)^2 \end{tikzcd}$$

Roughly speaking, this data corresponds to a global section of $(v, w)^{\triangleleft}[\mathcal{E}(\mathcal{X}), \mathcal{E}(\mathcal{X})]_{\square}$, where $pr_i : \mathcal{B}(\mathcal{X})^2 \longrightarrow \mathcal{B}(\mathcal{X})$ are the projections onto the two factors of $\mathcal{B}(\mathcal{X})$ and not $\mathcal{M}(\mathcal{X})$ or $\mathcal{X}$ separately. Now, (v, w) factors through $\mathcal{M}^n(\mathcal{X}) \times_S \mathcal{X} \times_S \mathcal{M}^m(\mathcal{X}) \times_S \mathcal{X}$ for some n, m so that

$(v,w)^{\lhd}[\mathcal{E}(\mathcal{X}),\mathcal{E}(\mathcal{X})]_{\square}$ is isomorphic to the following by Proposition 2.10:

$$\begin{aligned}
&(v,w)^{\lhd}[pr_1^*\mathcal{E}(\mathcal{X}), pr_2^*\mathcal{E}(\mathcal{X})]\\
=&[(v,w)^{\lhd}pr_1^*\mathcal{E}(\mathcal{X}), (v,w)^{\lhd}pr_2^*\mathcal{E}(\mathcal{X})]\\
=&[v^{<}\mathcal{E}(\mathcal{X}), w^{<}\mathcal{E}(\mathcal{X})]\\
=&[E,F]
\end{aligned}$$

for two finite locally free sheaves $E := v^{<}\mathcal{E}(\mathcal{X}), F := w^{<}\mathcal{E}(\mathcal{X})$ on $U \times \mathcal{X}$, corresponding to the maps v, w respectively. Thus, an object of $\mathcal{M}_1'(\mathcal{X})$ consists of two finite locally free sheaves E, F on $U \times \mathcal{X}$ and a global section of $[E,F]$ which is a morphism of $\mathcal{O}_{U\times\mathcal{X}}$–modules $E \longrightarrow F$, much like $\mathcal{M}_1(\mathcal{X})$.

We will now see that this remark can be made precise into an equivalence of prestacks $\mathcal{M}_1'(\mathcal{X}) \simeq \mathcal{M}_1(\mathcal{X})$.

Theorem 4.11. *For any prestack $\mathcal{X} \in \mathrm{PSt}_{/S}$, we have an equivalence of prestacks $\mathcal{M}_1'(\mathcal{X}) \simeq \mathcal{M}_1(\mathcal{X})$.*

Proof. We will define a morphism of prestacks $\Psi : \mathcal{M}_1'(\mathcal{X}) \longrightarrow \mathcal{M}_1(\mathcal{X})$ and show that it is fully faithful and essentially surjective. For an object $(u,s,v,w,\phi) \in \mathrm{Ob}(\mathcal{M}(\mathcal{X}))$, we obtain finite locally free sheaves $E_0 = (pr_1 \circ \pi_{\mathcal{V}(\mathcal{X})} \circ s)^*\mathcal{E}(\mathcal{X}), F_0 = (pr_2 \circ \pi_{\mathcal{V}(\mathcal{X})} \circ s)^*\mathcal{E}(\mathcal{X}), E_1 := v^{<}\mathcal{E}(\mathcal{X}), F_1 := w^{<}\mathcal{E}(\mathcal{X})$ on $U \times \mathcal{X}$, where $U = \mathrm{dom}(u)$. Then, Proposition 3.9 gives us a morphism of $\mathcal{O}_{U\times\mathcal{X}}$–modules $s_0 : E_0 \longrightarrow F_0$. Next, ϕ gives us isomorphisms of $\mathcal{O}_{U\times\mathcal{X}}$–moduels $\phi_1 := (pr_1 \circ \pi_{\mathcal{V}(\mathcal{X})}) \star \phi : E_0 \longrightarrow E_1, \phi_2 := (pr_2 \circ \pi_{\mathcal{V}(\mathcal{X})}) \star \phi : F_0 \longrightarrow F_1$, hence a unique morphism of $\mathcal{O}_{U\times\mathcal{X}}$–moduels $s_1 : E \longrightarrow F$ making the following diagram commute:

$$\begin{tikzcd}
E_0 \arrow[r, "\phi_1"] \arrow[d, "s_0"'] & E_1 \arrow[d, "s_1"] \\
F_0 \arrow[r, "\phi_2"'] & F_1
\end{tikzcd}$$

Thus, we have a triple (E_1, F_1, s_1) constituting an object — call it $\Psi(u,s,v,w,\phi)$ — of $\mathcal{M}_1(\mathcal{X})$.

Let $(f,a,b,c) : (u,s,v,w,\phi) \longrightarrow (u',s',v',w',\phi')$ be a morphism of $\mathcal{M}_1'(\mathcal{X})$. We denote the sheaves and morphisms associate to the codomain object as in the previous paragaph by $E_0', E_1', F_0', F_1', s_0', s_1', \phi_0', \phi_1'$. We obtain, by pasting arrows, a cube as follows:

$$\begin{tikzcd}
f^{<}E_0' \arrow[rr, "f^{<}\phi_0'"] \arrow[dr, "(pr_1\circ\pi_{\mathcal{V}(\mathcal{X})})\star a"] \arrow[dd, "f^{<}s_0'"'] & & f^{<}E_1' \arrow[dr, "b\times e_{\mathcal{X}}"] \arrow[dd, "f^{<}s_1'"] & \\
& E_0 \arrow[rr, "\phi_0"] \arrow[dd, "s_0"] & & E_1 \arrow[dd, "s_1"] \\
f^{<}F_0' \arrow[rr, "f^{<}\phi_1'"'] \arrow[dr, "(pr_2\circ\pi_{\mathcal{V}(\mathcal{X})})\star a"'] & & f^{<}F_1' \arrow[dr, "c\times e_{\mathcal{X}}"] & \\
& F_0 \arrow[rr, "\phi_1"'] & & F_1
\end{tikzcd} \tag{3}$$

where all arrows, except possibly the vertical ones, are isomorphisms. The front and back faces commute by the previous paragraph. The left face commutes by Proposition 3.10. Noticing that $(pr_i \circ \pi_{\mathcal{V}(\mathcal{X})}) \star (b \times e_{\mathcal{X}}, c \times \mathcal{X}) = b$ when $i = 1$ and c when $i = 2$, the top and bottom faces commute by the commutativity constraint in the description of morphisms of $\mathcal{M}_1'(\mathcal{X})$ given in Proposition 4.9. These imply that the right face commutes. Hence, the map $f : U \longrightarrow U'$ along with the right face of the above cube provides a morphism $(b, c) : (E_1, F_1, s_1) \longrightarrow (E_1', F_1', s_1')$ in $\mathcal{M}_1(\mathcal{X})$ — call it $\Psi(f, a, b, c)$. It is straightforward to show that the assignment Ψ constitutes a functor $\mathcal{M}_1'(\mathcal{X}) \longrightarrow \mathcal{M}_1(\mathcal{X})$, which commutes with the projections to $\mathrm{Aff}_{/S}$ by construction, and is, hence a morphism of prestacks.

The Gorthendieck construction gives an equivlance between categories fibred in groupoids over $\mathrm{Aff}_{/S}$ and pseudofunctors $\mathrm{Aff}_{/S} \longrightarrow \mathrm{Grpd}$. Therefore, it suffices to check that the morphism of prestacks is an equivalence when restricted to the fibre categories — this is the analogue, in the context of prestacks, of the pointwise criterion for equivalence of pseudofunctors. For any $U \longrightarrow S$ in $\mathrm{Aff}_{/S}$, consider two morphisms $(\mathrm{id}_U, a, b, c), (\mathrm{id}_U, \alpha, \beta, \gamma) : (u, s, v, w, \phi) \longrightarrow (u', s', v', w', \phi')$ that map to the same morphism under Ψ. Then, we have an equality as follows, by hypothesis:

$$\begin{aligned} b \times e_{\mathcal{X}} = \beta \times e_{\mathcal{X}} &\iff b = \beta \\ c \times e_{\mathcal{X}} = \gamma \times e_{\mathcal{X}} &\iff c = \gamma \end{aligned}$$

By the observation that all morphisms in Eq. (3) except the vertical ones are isomorphisms, we then have equalities:

$$\begin{array}{lcl} (pr_1 \circ \pi_{\mathcal{V}(\mathcal{X})}) \star a & = b \times e_{\mathcal{X}} = & (pr_1 \circ \pi_{\mathcal{V}(\mathcal{X})}) \star \alpha \\ (pr_2 \circ \pi_{\mathcal{V}(\mathcal{X})}) \star a & = c \times e_{\mathcal{X}} = & (pr_2 \circ \pi_{\mathcal{V}(\mathcal{X})}) \star \alpha \end{array}$$

Then, by Proposition 2.3, we have $a = \alpha$. This shows that Ψ is fully faithful on fibre categories.

Consider an object (E, F, f) of $\mathcal{M}_1(\mathcal{X})$ over $u : U \longrightarrow S$. Then, by the universal property of the prestack $\mathcal{M}(\mathcal{X})$, we obtain two morphisms of prestacks $v, w : U \longrightarrow \mathcal{M}(\mathcal{X})$ and two isomorphisms of $\mathcal{O}_{U \times \mathcal{X}}$–modules $\phi_E : v^{<}\mathcal{E}(\mathcal{X}) \longrightarrow E, \phi_F : w^{<}\mathcal{E}(\mathcal{X}) \longrightarrow F$. Thus, we have a unique morphism $f' : v^{<}\mathcal{E}(\mathcal{X}) \longrightarrow w^{<}\mathcal{E}(\mathcal{X})$ of $\mathcal{O}_{U \times \mathcal{X}}$–modules making the following diagram commute:

$$\begin{tikzcd} v^{<}\mathcal{E}(\mathcal{X}) \arrow[r, "\phi_E"] \arrow[d, "f'"'] & E \arrow[d, "f"] \\ w^{<}\mathcal{E}(\mathcal{X}) \arrow[r, "\phi_F"'] & F \end{tikzcd}$$

By Proposition 3.9, f' gives a section of $s'' : U \times \mathcal{X} \longrightarrow \mathbb{V}[v^{<}\mathcal{E}(\mathcal{X}), w^{<}\mathcal{E}(\mathcal{X})]$. However, by Construction 2.8, we have an isomorphism

$$\xi_{(v \times \mathrm{id}_{\mathcal{X}}, w \times \mathrm{id}_{\mathcal{X}})} : (v, w)^{\lhd}[\mathcal{E}(\mathcal{X}), \mathcal{E}(\mathcal{X})] \longrightarrow [v^{<}\mathcal{E}(\mathcal{X}), w^{<}\mathcal{E}(\mathcal{X})]$$

This gives an equivalence of categories over $\mathcal{X}$:

$$\int (\xi_{v \times \mathrm{id}_{\mathcal{X}}, w \times \mathrm{id}_{\mathcal{X}}}) : \mathbb{V}(v, w)^{\lhd}[\mathcal{E}(\mathcal{X}), \mathcal{E}(\mathcal{X})] \longrightarrow \mathbb{V}[v^{<}\mathcal{E}(\mathcal{X}), w^{<}\mathcal{E}(\mathcal{X})]$$

which composed with s'' gives a section of $s' : U \times \mathcal{X} \longrightarrow \mathbb{V}(v, w)^{\lhd}[\mathcal{E}(\mathcal{X}), \mathcal{E}(\mathcal{X})]$. In turn, since $\int \mathbb{V}(v, w)^{\lhd}[\mathcal{E}(\mathcal{X}), \mathcal{E}(\mathcal{X})]$ is the fibre product $(U \times \mathcal{X}) \times_{\mathcal{B}(\mathcal{X})^2} \mathcal{V}(\mathcal{X})$, we have a canonical map $\int \mathbb{V}(v, w)^{\lhd}[\mathcal{E}(\mathcal{X}), \mathcal{E}(\mathcal{X})] \longrightarrow \mathcal{V}(\mathcal{X})$ which which by composing with s' gives a map $s : U \times \mathcal{X} \longrightarrow \mathcal{V}(\mathcal{X})$. By the universal property of 2–fibre products, we get a 2–morphism

$$\phi : \pi_{\mathcal{V}(\mathcal{X})} \circ s \Longrightarrow (v \times \mathrm{id}_{\mathcal{X}}, w \times \mathrm{id}_{\mathcal{X}}) : U \times \mathcal{X} \longrightarrow \mathcal{B}(\mathcal{X})^2$$

That is, (u, s, v, w, ϕ) is an object of $\mathcal{M}_1'(\mathcal{X})$. By construction, under Ψ, this object maps to $(v^{<}\mathcal{E}(\mathcal{X}), w^{<}\mathcal{E}(\mathcal{X}), f')$, which is isomorphic to (E, F, f) in $\mathcal{M}_1(\mathcal{X})$. Thus, Ψ is essentially surjective on fibre categories, as required. □

Notation 4.12. In light of the previous theorem, from this point onwards, we will write $\mathcal{M}_1(\mathcal{X})$ for $\mathcal{M}'_1(\mathcal{X})$ as well, unless there is a need to distinguish them.

We now address the descent and algebaicity properties of the prestacks constructed so far.

Theorem 4.13. *When $\mathcal{X} \in \mathrm{PSt}_{/S}$ is a stack, then so is $\mathcal{M}_1(\mathcal{X})$.*

Proof. If $\mathcal{X}$ is a stack, then so is $\mathcal{M}(\mathcal{X})$. Hence, $\mathcal{B}(\mathcal{X})^2 = (\mathcal{M}(\mathcal{X}) \times \mathcal{X})^2$ is a stack. Since $\mathcal{V}(\mathcal{X}) \longrightarrow \mathcal{B}(X)^2$ is the Grothendieck construction of a sheaf on $\mathcal{B}(\mathcal{X})^2$, it is a stack on the underlying category of $\mathcal{B}(\mathcal{X})^2$, and from this, it can be shown that it satisfies the descent property over $\mathrm{Aff}_{/S}$. Thus, $\mathcal{M}(\mathcal{X})^2, \mathrm{Map}_S(\mathcal{X}, \mathcal{V}(\mathcal{X})), \mathrm{Map}_S(\mathcal{X}, \mathcal{B}(\mathcal{X})^2)$ are all stacks. Since the fibre product of a span of stacks is a stack, $\mathcal{M}_1(\mathcal{X})$ is a stack by Theorem 4.11. □

Theorem 4.14. *If X is a scheme that is projective, flat and of finite presentation over k, then $\mathcal{M}_1(X)$:*

(i) is Artin,
(ii) is locally of finite presentation, and
(iii) has affine diagonal (in particular, is quasi-separated).

Proof sketch. We observe that $\mathcal{M}(X) = \coprod_{n=0}^{\infty} \mathrm{Map}(X, B\mathrm{Gl}_n)$ is quasi-separated, locally of finite presentation, and has an affine diagonal by [Wan11, Theorem 1.0.1.]. The projection $\mathcal{V}(\mathcal{X}) \longrightarrow \mathcal{B}(\mathcal{X})^2$ being affine allows us to conclude that $\mathcal{V}(\mathcal{X})$ is a quasi-separated algebraic stack with affine stabilizers and locally of finite presentation. As a result, [HR19, Theorem 1.2] applies, and we can conclude that the mapping stacks involved in Construction 4.8 are algebraic, locally of finite presentation and has affine diagonal. This shows that $\mathcal{M}'_1(X)$ is algebraic and hence Theorem 4.11 yields the result. For a more elaborate argument, see [AR25, §4]. □

Remark 4.15. We note that this algebraicity result holds much more generally. For this, we refer to our result [AR25, Theorem 4.10].

4.3. **The Boundary Map.** An important component of the proof of algebraicity of the moduli stack of connection triples will will be the map $\mathcal{M}'_1(\mathcal{X}) \longrightarrow \mathcal{M}(\mathcal{X})^2$ that sends a morphism of vector bundles over $\mathcal{X}$ to the pair consisting of the source and target of the morphism. We will record some properties of this map for use in Section 5. This will be called the "boundary map" as it sends a 1–simplex in the category vector bundles to the pair of vertices forming its boundary.

Construction 4.16. We define a morphism of prestacks $\partial_{\mathcal{X}} : \mathcal{M}_1(\mathcal{X}) \longrightarrow \mathcal{M}(\mathcal{X})^2$ as follows. For an object (u, E, F, s) of $\mathcal{M}_1(\mathcal{X})$ over $u : U \longrightarrow S$, we define

$$\partial_{\mathcal{X}}(u, E, F, s) := ((u, E), (u, F))$$

noting that E, F are finite locally free sheaves on $U \times \mathcal{X}$. For a morphism $(f, a, b) : (u, E, F, s) \longrightarrow (u', E', F', s')$, we define:

$$\partial_{\mathcal{X}}(f, a, b) := ((f, a), (f, b))$$

noting that $a : f^{<} E' \longrightarrow E, b : f^{<} F' \longrightarrow F$ are morphisms in $\mathcal{M}(\mathcal{X})$ over f.

Proposition 4.17. *For a prestack $\mathcal{X} \in \mathrm{PSt}_{/S}$, the map $\partial_{\mathcal{X}}$ of Construction 4.16 is a morphism of prestacks. Let Ψ again be the equivalence $\mathcal{M}'_1(\mathcal{X}) \xrightarrow{\simeq} \mathcal{M}_1(\mathcal{X})$ of Theorem 4.11. Let $\partial'_{\mathcal{X}}$: $\mathcal{M}'_1(\mathcal{X}) \longrightarrow \mathcal{M}(\mathcal{X})^2$ be the projection of of the fibre product defining $\mathcal{M}'_1(\mathcal{X})$. Then, $\partial_{\mathcal{X}} \circ \Psi = \partial'_{\mathcal{X}}$ as 1–morphisms.*

Proof. That $\partial_{\mathcal{X}}$ is a functor and commutes with the projections to $\mathrm{Aff}_{/S}$ is immediate. For the second claim, we first notice that the map $\partial'_{\mathcal{X}} : \mathcal{M}_1(\mathcal{X}) \longrightarrow \mathcal{M}(\mathcal{X})^2$ factors through the equivalence $\mathrm{Map}_S(S, \mathcal{M}(\mathcal{X}))^2 \xrightarrow{\simeq} \mathrm{Map}_S(S, \mathcal{M}(\mathcal{X})^2) \xrightarrow{\simeq} \mathcal{M}(\mathcal{X})^2$ sending an object $(v, w) : U \simeq U \times S \longrightarrow \mathcal{M}(\mathcal{X})^2$ to the object $(v(\mathrm{id}_U), w(\mathrm{id}_U))$. The fibre product projection $\mathcal{M}_1(\mathcal{X}) \longrightarrow \mathrm{Map}_S(S, \mathcal{M}_1(\mathcal{X}))^2$

sends an object (u, s, v, w, ϕ) to (v, w). Hence, the map $\mathcal{M}_1(\mathcal{X}) \longrightarrow \mathcal{M}(\mathcal{X})^2$ is given by sending (v, w) to $(v(\mathrm{id}_U), w(\mathrm{id}_U)) = (v^< \mathcal{E}(\mathcal{X}), w^< \mathcal{E}(\mathcal{X}))$. From the construction of the map Ψ, we see that this is precisely $\partial_{\mathcal{X}}(\Psi(u, s, v, w, \phi))$.

Now, let $(f, a, b, c) : (u, s, v, w, \phi) \longrightarrow (u', s', v', w', \phi')$ be a morphism in $\mathcal{M}'_1(\mathcal{X})$. Ψ sends this to $(b \times e_{\mathcal{X}}, c \times e_{\mathcal{X}})$, which is then send by $\partial_{\mathcal{X}}$ to itself, noting that the morphisms in $\mathcal{M}_1(\mathcal{X})$ are morphisms in $\mathcal{M}(\mathcal{X})$ satisfying a commutativity constraint. Then, $\partial'_{\mathcal{X}}$ sends it to the 2–morphism $(b, c) : (v, w) \Longrightarrow (v', w') : U \longrightarrow \mathcal{M}(\mathcal{X})^2$. The map $\mathrm{Map}_S(S, \mathcal{M}(\mathcal{X}))^2 \xrightarrow{\simeq} \mathcal{M}(\mathcal{X})^2$ sends it to $(b \times e_{\mathcal{X}}, c \times e_{\mathcal{X}})$. This shows that $\partial_{\mathcal{X}}(\Psi(f, a, b, c)) = \partial'_{\mathcal{X}}(f, a, b, c)$, as required. □

Proposition 4.18. *The map $\partial_{\mathcal{X}}$ of Construction 4.16 is a faithful functor.*

Proof. We begin by observing that the Hom sets of $\mathcal{M}_1(\mathcal{X})$ are subsets of the corresponding Hom sets of $\mathcal{M}(\mathcal{X})^2$ consisting of elements satisfying a commutativity constraint. Also, the functions of Hom sets induced by the morphism $\partial_{\mathcal{X}} : \mathcal{M}_1(\mathcal{X}) \longrightarrow \mathcal{M}(\mathcal{X})^2$ are the inclusions of these subsets. Hence, $\partial_{\mathcal{X}}$ is faithful. □

Definition 4.19. For any prestack $\mathcal{Y} \in \mathrm{PSt}_{/S}$, any affine scheme $U \in \mathrm{Aff}_{/S}$ and any morphism $(v, w) : U \longrightarrow \mathcal{M}(\mathcal{Y})^2$, we denote by $P^{\mathcal{Y}}_{v,w}$, the following fibre product:

$$\begin{tikzcd} P^{\mathcal{Y}}_{v,w} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{M}_1(\mathcal{Y}) \arrow[d] \\ U \arrow[r, "(v,w)"'] & \mathcal{M}(\mathcal{Y})^2 \end{tikzcd}$$

where the right vertical map is projection of the fibre product defining $\mathcal{M}_1(\mathcal{X})$, in light of Theorem 4.11.

Proposition 4.20. *In the context of Definition 4.19, if we denote $E := v^< \mathcal{E}(\mathcal{X}), F := w^< \mathcal{E}(\mathcal{X})$, the underlying category of $P^{\mathcal{Y}}_{v,w}$ has the following concrete description:*

- *Objects are tuples $(r, H, K, s, \phi_H, \phi_K)$ where:*
  - *$r : V \longrightarrow U$ is a morphism in $\mathrm{Aff}_{/S}$*
  - *$H, K \in \mathrm{Vect}(V \times \mathcal{X})$*
  - *$s : H \longrightarrow K$ is a morphism of locally free sheaves over $V \times \mathcal{X}$*
  - *$\phi_H : H \longrightarrow r^< E, \phi_K : K \longrightarrow r^< F$ are isomorphisms of locally free sheaves over $V \times \mathcal{X}$*
- *Morphisms $(r, H, K, s, \phi_H, \phi_K) \longrightarrow (r', H', K', s', \phi_{H'}, \phi_{K'})$ are tuples (f, a, b) where:*
  - *$f : V = \mathrm{dom}(r) \longrightarrow V' = \mathrm{dom}(r')$ is a morphism in $\mathrm{Aff}_{/S/U}$*
  - *$a : f^< H' \longrightarrow H, b : f^< K' \longrightarrow K$ are isomorphisms of locally free sheaves over $V \times \mathcal{X}$ making the following diagrams commute:*

$$\begin{tikzcd} f^< H' \arrow[r, "a"] \arrow[d, "f^< s'"'] & H \arrow[d, "s"] \\ f^< K' \arrow[r, "b"'] & K \end{tikzcd} \qquad \begin{tikzcd} f^< H' \arrow[r, "a"] \arrow[d, "f^< \phi_{H'}"'] & H \arrow[d, "\phi_H"] \\ f^<(r')^< E = r^< E \arrow[r, "\mathrm{id}_{r^< E}"'] & r^< E \end{tikzcd} \qquad \begin{tikzcd} f^< K' \arrow[r, "b"] \arrow[d, "f^< \phi_{K'}"'] & K \arrow[d, "\phi_K"] \\ f^<(r')^< F = r^< F \arrow[r, "\mathrm{id}_{r^< F}"'] & r^< F \end{tikzcd}$$

Proof. This follows from the definition of $\mathcal{M}_1(\mathcal{X})$ given in Construction 4.2, the observation that the map (v, w) has the values $(r^< E, r^< F)$ on objects $r : V \longrightarrow U$ of $\mathrm{Aff}_{/S/U}$ and $(\mathrm{id}_{(r' \circ f)^< E}, \mathrm{id}_{(r' \circ f)^< F})$ on morphisms $f : V \longrightarrow V'$ in $\mathrm{Aff}_{/S/U}$, and then computing the explicit description of the fibre product [Sta25, Lemma 0040]. □

Proposition 4.21. *In the context of Definition 4.19, let $E = v^< \mathcal{E}(\mathcal{X}), F := w^< \mathcal{E}(\mathcal{X})$, and consider the category $Q^{\mathcal{Y}}_{v,w}$ defined by the following data:*

- *Objects are tuples* (r, s) *where:*
  - $r : V \longrightarrow U$ *is a morphism in* $\mathrm{Aff}_{/S}$
  - $s : r^{<}E \longrightarrow r^{<}F$ *is a morphism of finite locally free sheaves on* $V \times \mathcal{X}$
- *Morphisms* $(r, s) \longrightarrow (r's')$ *are morphisms* $f : V = \mathrm{dom}(r) \longrightarrow V' = \mathrm{dom}(r')$ *in* $\mathrm{Aff}_{/S/U}$ *such that* $r' \circ f = r$ *and* $f^{<}s' = s$.

We have an equivalence of prestacks $\Psi^{\mathcal{Y}}_{v,w} : Q^{\mathcal{Y}}_{v,w} \longrightarrow P^{\mathcal{Y}}_{v,w}$.

Proof. We observe that for any object $(r, s) \in Q^{\mathcal{Y}}_{v,w}$, we have the object

$$\Psi^{\mathcal{Y}}_{v,w}(r, s) := (r, r^{<}E, r^{<}F, s, \mathrm{id}_{r^{<}E}, \mathrm{id}_{r^{<}F})$$

in $P^{\mathcal{Y}}_{v,w}$. For two objects $(r, s), (r', s')$, consider a morphism $(f, a, b) : \Psi^{\mathcal{Y}}_{v,w}(r, s) \longrightarrow \Psi^{\mathcal{Y}}_{v,w}(r', s')$. Then, the commutativity constraint in the description of morphisms in Proposition 4.20, we have the following commutative diagrams:

$$\begin{tikzcd} f^{<}(r')^{<}E \arrow[r, "a"] \arrow[d, "f^{<}s'"'] & r^{<}E \arrow[d, "s"] \\ f^{<}(r')^{<}F \arrow[r, "b"'] & r^{<}F \end{tikzcd} \qquad \begin{tikzcd} f^{<}(r')^{<}E = r^{<}E \arrow[r, "a"] \arrow[d, "f^{<}(\mathrm{id}_{(r)'^{<}E}) = \mathrm{id}_{r^{<}E}"'] & r^{<}E \arrow[d, "\mathrm{id}_{r^{<}E}"] \\ f^{<}(r')^{<}E = r^{<}E \arrow[r, "\mathrm{id}_{r^{<}E}"'] & r^{<}E \end{tikzcd}$$

$$\begin{tikzcd} f^{<}(r')^{<}F = r^{<}F \arrow[r, "b"] \arrow[d, "f^{<}(\mathrm{id}_{(r)'^{<}F}) = \mathrm{id}_{r^{<}F}"'] & r^{<}F \arrow[d, "\mathrm{id}_{r^{<}F}"] \\ f^{<}(r')^{<}F = r^{<}F \arrow[r, "\mathrm{id}_{r^{<}F}"'] & r^{<}F \end{tikzcd}$$

showing that $a = \mathrm{id}_{r^{<}E}$, $b = \mathrm{id}_{r^{<}F}$, and that $f^{<}s' = s$ so that f is a morphism in $Q^{\mathcal{Y}}_{v,w}$. Hence, we can set $\Psi^{\mathcal{Y}}_{v,w}(f) := (f, \mathrm{id}_{r^{<}E}, \mathrm{id}_{r^{<}F})$. It is not hard to show that $\Psi^{\mathcal{Y}}_{v,w}$ is a morphism of prestacks. That it is fully faithful is evident from the construction. Suppose, now, that we have an object $q = (r, H, K, s, \phi_H, \phi_K)$ of $P^{\mathcal{Y}}_{v,w}$. This gives us a unique map $t : r^{<}E \longrightarrow r^{<}F$ making the following diagram commute:

$$\begin{tikzcd} H \arrow[r, "\phi_H"] \arrow[d, "s"'] & r^{<}E \arrow[d, "t"] \\ K \arrow[r, "\phi_K"'] & r^{<}F \end{tikzcd}$$

It is, then, easy to see that $(\mathrm{id}_{\mathrm{dom}(r)}, \phi_H, \phi_K)$ is an isomorphism of the object q with $\Psi^{\mathcal{Y}}_{v,w}(r, t)$. This shows that $\Psi^{\mathcal{Y}}_{v,w}$ is essentially surjective. □

Notation 4.22. In light of Proposition 4.21, when the maps v, w are not important, we will sometimes write $Q^{\mathcal{Y}}_{E,F} \simeq P^{\mathcal{Y}}_{E,F}$.

Proposition 4.23. *In the context of Definition 4.19, if* $\mathcal{M}_1(\mathcal{X})$ *and* $\mathcal{M}(\mathcal{X})$ *are algebraic, then* $P^{\mathcal{Y}}_{v,w}$ *is an algebraic space.*

Proof. This follows from Proposition 4.18 and [Sta25, Lemma 04Y5]. □

Warning 4.24. In Proposition 4.23, the algebraicity of $\mathcal{M}_1(\mathcal{X})$ is a requirement and not a consequence.

We will now see that in the case that $\mathcal{Y}$ is a scheme X flat, finitely presented and projective over S, the boundary map is "well-behaved".

Theorem 4.25. *Let* $X \longrightarrow S$ *be a finitely presented, flat and projective morphism of schemes. Then,* $\partial_X : \mathcal{M}_1(X) \longrightarrow \mathcal{M}(X)^2$ *is affine and of finite presentation.*

Proof. We have to show that for any morphism $(v,w) : U \longrightarrow \mathcal{M}(X)^2$ where $U \in \mathrm{Aff}_{/S}$, the map $P^X_{v,w} \longrightarrow U$ is affine and of finite presentation. We obtain the following pasting of Cartesian squares by Theorem 4.11 and the definition (4.19) of $P^X_{v,w}$:

$$\begin{tikzcd}
P^X_{v,w} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{M}_1(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Map}_S(X, \mathcal{V}(X)) \arrow[d] \\
U \arrow[r, "(v,w)"'] & \mathcal{M}(X)^2 \arrow[r] & \mathrm{Map}_S(X, \mathcal{B}(X)^2)
\end{tikzcd}$$

The same argument as in the proof of theorem [Wan11, Lemma 3.2.1] shows that $P^X_{v,w}$ is equivalent to the stack $\mathrm{Map}_U(U\times X, \mathcal{V}(X)\times_{\mathcal{B}(X)^2}(U\times X))$. However, by Proposition 3.6, $\mathcal{V}(X)\times_{\mathcal{B}(X)^2}(U\times X)$ is equivalent to $\int[E,F]$, where $E = v^{<}\mathcal{E}(X), F = w^{<}\mathcal{E}(X)$. Furthermore, since $[E,F]$ is a finite locally free $\mathcal{O}_{U\times X}$–module $\pi_A : \int[E,F] \longrightarrow U \times X$ is affine by Proposition 3.5. By passing to charts, we can also show that π_A is of finite presentation. Finally, the projection $pr_1 : U\times X \longrightarrow U$ is finitely presented, flat and projective, as it is a base change of the finitely presented, flat and projective morphism $X \longrightarrow S$. Lemma 3.1.4 of [Wan11] now applies and we can conclude that the map $P^X_{v,w} \simeq \mathrm{Map}_U(U \times X, \int[E,F]) \longrightarrow U$ is an affine morphism of finite presentation, as required. $\square$

5. Arrow Bundles on Formal Groupoids

We now have all the machinery we need to construct and study the moduli stack of connection triples. We will address this in some generality: we will take the approach of [Sim96] in constructing the stack representing the non-Abelian Hodge filtration. Just like the moduli stack of vector bundles on certain formal groupoids recover the moduli stacks of connections, Higgs bundles, etc., we will consider the moduli stack of vector bundle triples on those same formal groupoids and show that they recover the desired moduli stacks of morphisms of the respective objects: namely, λ–connections for a varying parameter λ, usual connections (or, 1–connections) and Higgs bundles (or, 0–connections).

5.1. **Review of Formal Groupoids.** We begin by reviewing formal groupoids both for the convenience of the reader and to illustrate the difficulty in proving the algebraicity of the moduli stack of connection triples using the same approach for the moduli stack of vector bundle triples.

Notation 5.1. We will write $\mathbb{F}\mathrm{Sch}$ for the category of formal schemes over k.

Definition 5.2 (Formal Groupoid)**.** A formal category over S is a tuple $(X, \mathcal{F}, s, t, c, i)$ forming an internal category in $\mathbb{F}\mathrm{Sch}_{/S}$, and satisfying:

(i) X is a scheme in $\mathrm{Sch}_{/S}$,
(ii) $i : X \longrightarrow \mathcal{F}$ is a closed immersion realizing X as the underlying scheme of $\mathcal{F}$.

This data is called a formal groupoid if, in addition:

(iii) for each $U \in \mathrm{Sch}_{/S}$, $(X(U), \mathcal{F}(U), s_U, t_U, c_U, i_U)$ is a groupoid (internal to Set).

A formal category as above is said to be smooth if

(iv) the structure map $X \longrightarrow S$ is smooth,
(v) the morphisms $s, t : \mathcal{F} \longrightarrow X$ are formally smooth.

Notation 5.3. We will write a formal category as above simply as $(X, \mathcal{F})$, when the structure maps are clear from context.

For convenience, we make the following definition:

Definition 5.4 (Formal Stack)**.** A formal stack is a stack $\mathcal{X} \in \mathrm{St}_{/S}$ such that there exists a formal groupoid $(X, \mathcal{F})$ and a 2–coequalizer diagram in $\mathrm{St}_{/S}$:

$$\mathcal{F} \underset{t}{\overset{s}{\rightrightarrows}} X \longrightarrow \mathcal{X}$$

presenting $\mathcal{X}$ as a quotient stack. We call $\mathcal{X}$ the stack associated to the formal groupoid $(X, \mathcal{F})$.

Warning 5.5. A formal stack is not necessarily a formal algebraic stack as defined in [Eme, Definition 5.3]. For instance, it may have a digonal not representable by algebraic spaces — see Proposition 5.11 — which contradicts [Eme, Lemma 5.12.]. On the other hand, a formal algebraic stack is not necessarily a formal stack since it may be a quotient of formal algebraic spaces more general than formal schemes [Eme, p. 47].

Notation 5.6. In the context of the above definition, if the formal groupoid $(X, \mathcal{F})$ is clear from context, then we write $\mathcal{X}$ as $X_{\mathcal{F}}$.

There are three main examples of interest to us [Sim96, pp. 31–33]. The rough idea behind all of these is that a quasicoherent sheaf on a formal stack is $(X, \mathcal{F})$ is a quasicoherent sheaf on X along with isomorphisms between the stalks encoded by $\mathcal{F}$. By varying $\mathcal{F}$, and consequently the isomorphisms of stalks, we can recover connections and Higgs bundles.

Example 5.7 (de Rham Stack)**.** If $X \longrightarrow S$ is separated, then the diagonal $\Delta_{X/S} : X \longrightarrow X \times_S X$ is a closed immersion and we take $\mathcal{F} \xrightarrow{(s,t)} X \times_S X$ to be the formal completion of $X \times_S X$ along the set theoretic image $\Delta_{X/S}(X)$. The composition morphisms $c : \mathcal{F} \times_X \mathcal{F} \longrightarrow \mathcal{F}$ is the one induced by the map $(X \times_S X) \times_X (X \times_S X) \longrightarrow X$. The identity morphism $i : X \longrightarrow \mathcal{F}$ is simply the closed immersion into the formal completion. We denote $\mathcal{F}$ by $\mathcal{F}_{dR}$ and the associated stack over S, by $X_{dR} \longrightarrow S$, in this case.

Example 5.8 (Dolbeault Stack)**.** If $X \longrightarrow S$ is again separated, then the diagonal $\Delta_{X/S} : X \longrightarrow X \times_S X$ is again a closed immersion. Furthermore, any section of a separated morphism is a closed immersion. Since the projection $T(X \times_S X) \longrightarrow X \times_S X$ of the tangent bundle of $X \times_S X$ is affine and hence separated, the zero section $0_X : X \longrightarrow T(X \times_S X)$ is a closed immersion. Therefore, the composite

$$\Delta' : X \xrightarrow{\Delta_{X/S}} X \times_S X \xrightarrow{0_X} T(X \times_S X)$$

is a closed immersion. We can then take $\mathcal{F} \longrightarrow T(X \times_S X)$ to be the formal completion of $T(X \times_S X)$ along the set theoretic image $\Delta'(X)$. Then, the map $\mathcal{F} \xrightarrow{(s,t)} X \times_S X$ is obtained by composing with the bundle projection. The composition morphism $c : \mathcal{F} \times_X \mathcal{F} \longrightarrow \mathcal{F}$ is induced by the addition morphism

$$+ : T(X \times_S X) \times_{X \times_S X} T(X \times_S X) \longrightarrow T(X \times_S X)$$

The identity morphism $i : X \longrightarrow \mathcal{F}$ is again the closed immersion into the formal completion. We denote $\mathcal{F}$ by $\mathcal{F}_{Dol}$ and the associated stack over S by $X_{Dol} \longrightarrow S$, in this case.

Example 5.9 (Hodge Stack)**.** Take a separated morphism of schemes $X \longrightarrow T$ (for us, T will mainly be $\mathrm{Spec}(k)$), consider a formal groupoid over $S := \mathbb{A}^1_T = \mathbb{A}^1_{\mathbb{Z}} \times_{\mathbb{Z}} T$ whose scheme of objects is $pr_2 : X \times_T \mathbb{A}^1_T \longrightarrow \mathbb{A}^1_T$. The formal scheme of morphisms is defined as follows. We take the blow up $B \longrightarrow X \times_T X \times_T \mathbb{A}^1_T$ of $(X \times_T \mathbb{A}^1_T) \times_{\mathbb{A}^1_T} (X \times_T \mathbb{A}^1_T) \cong (X \times_T X \times_T \mathbb{A}^1_T)$ along the set theoretic image of the map $\Delta_{X/T} \times_T 0 : X \cong X \times_T T \longrightarrow X \times_T X \times_T \mathbb{A}^1_T$ where $0 : T \longrightarrow \mathbb{A}^1_T$ is the pullback of the map $\mathrm{Spec}(\mathbb{Z}[x] \longrightarrow \mathbb{Z} : x \longmapsto 0)$ along the structure map $T \longrightarrow \mathrm{Spec}(\mathbb{Z})$, and the blow up $B' \longrightarrow X \times_T X$ of $X \times_T X$ along the image of $\Delta_{X/T}$. These fit into a commutative square:

$$\begin{tikzcd} B' \arrow[r] \arrow[d] & B \arrow[d] \\ X \times_T X \arrow[r, "(\mathrm{id},0)"] & X \times_T X \times \mathbb{A}^1_T \end{tikzcd}$$

where the top and bottom arrows are both closed embeddings. Then, B' is the strict transform of $\mathrm{im}(\mathrm{id}_{X\times_T X}, 0)$ in the blow up of $X \times_T X \times_s \mathbb{A}^1_T$ along $\mathrm{im}(\Delta_{X/T} \times_T 0)$ by [Sta25, Lemma 080E]. We take Y to be the complement of B' in B — note that Y is an open subscheme of B since the image of B' is closed. We choose a closed embedding $\Delta' : X \times_T \mathbb{A}^1_T \longrightarrow Y$ making the following diagram commute:

$$\begin{tikzcd} & Y \arrow[d] \\ X \cong X \times_T T \arrow[ur, "\Delta'"] \arrow[r, "\Delta_{X/T} \times_T 0"'] & X \times_T X \times_T \mathbb{A}^1_T \end{tikzcd}$$

Then, the formal scheme of morphisms is taken to be the formal completion $\mathcal{F} \longrightarrow Y$ of Y along $\Delta'(X)$. The map $\mathcal{F} \xrightarrow{(s,t)} X \times_T X \times_T \mathbb{A}^1_T$ is given by the two projections $Y \longrightarrow X \times_T X \times_T \mathbb{A}^1_T \longrightarrow X \times_T \mathbb{A}^1_T$. The composition map $c : \mathcal{F} \times_{X \times_T \mathbb{A}^1_T} \mathcal{F} \longrightarrow \mathcal{F}$ is given by composition with the map:

$$(X \times_T X \times_T \mathbb{A}^1_T) \times_{X \times_T \mathbb{A}^1_T} (X \times_T X \times_T \mathbb{A}^1_T) \longrightarrow (X \times_T X \times_T \mathbb{A}^1_T)$$

The identity morphism $i : X \times_T \mathbb{A}^1_T \longrightarrow \mathcal{F}$ is again the closed immersion into the formal completion. We will write $\mathcal{F}$ as $\mathcal{F}_{Hod}$ and the associated stack over $S = \mathbb{A}^1_T$ as $X_{Hod} \longrightarrow \mathbb{A}^1_T$, in this case.

These stacks are related to each other by the following well known fact which is the main connection with non-Abelian Hodge theory.

Proposition 5.10 ([Sim96, p. 33])**.** *Let $X \longrightarrow T$ be a separated morphism of schemes and consider X_{dR} and X_{Dol} by taking $S = T$ in Example 5.7 and Example 5.8, respectively. Also consider X_{Hod} by taking $S = \mathbb{A}^1_T$ in Example 5.9. Then, for any closed point $\lambda : T \longrightarrow \mathbb{A}^1_T$, the fibre $X_{Hod,\lambda}$ of $X_{Hod} \longrightarrow \mathbb{A}^1_T$ over λ is equivalent as a stack to X_{dR} when $\lambda \neq 0$, and is equivalent to X_{Dol}, when $\lambda = 0$.*

We also record here some basic facts about these stacks that make it difficult to apply some of the techniques of [AR25] directly.

Proposition 5.11. *Taking $T = \mathrm{Spec}(k)$ for some algebraically closed field k in the context of Example 5.9, X_{Hod} does not have a diagonal representable by algebraic spaces, and is, hence, not a formal algebraic stack.*

Proof. We consider a $T = \mathrm{Spec}(k)$–point $x \in X_{Hod}(\mathrm{Spec}(k))$ that factors as a map $\mathrm{Spec}(k) \xrightarrow{(x',0)} X \times_k \mathbb{A}^1_k \longrightarrow X_{Hod}$, and its stabilizer $\mathrm{stb}(x)$. Recalling that the preimage of a point in a blow-up is a projective space, we have that the stabilizer fits into the following pasting of Cartesian squares:

$$\begin{tikzcd} \mathrm{stb}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{P}^n_k \setminus \mathbb{P}^{n-1}_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Spec}(k) \arrow[d, "{(x',x',0)}"] \\ \mathcal{F}_{Hod} \arrow[r] & Y \arrow[r] & X \times_k X \times_k \mathbb{A}^1_k \end{tikzcd}$$

Now, $\mathcal{F}_{Hod}$ is the formal completion of Y along $\Delta'(X)$, while the preimage of $\Delta'(X)$ in $\mathbb{P}^n_k \setminus \mathbb{P}^{n-1}_k$ is a single point x'' lying over $(x', x', 0)$. By the compatibility of formal completions with fibre products [Sta25, Lemma 0APV], we have that $\mathrm{stb}(x)$ is the formal completion of $\mathbb{P}^n_k \setminus \mathbb{P}^{n-1}_k$ along a point, which is just the formal completion of an affine chart containing that point along that point. That is, up to change of coordinates, we have:

$$\mathrm{stb}(x) = \mathrm{Spf}(k[[x_0, \dots, x_n]])$$

where Spf denotes the formal spectrum functor. If $\mathrm{stb}(x)$ were representable by an algebraic space, then it would have to be a scheme by [CLO12, Corollary 3.1.2] as the reduction $\mathrm{stb}(x)_{red}$ is $\mathrm{Spec}(k)$. $\mathrm{stb}(x)$ would further have to be an affine scheme by [Sta25, Lemma 06AD]. However, it is easy to see that $\mathrm{Spf}(k[[x_0, \dots, x_n]])$ is not an affine scheme: as a locally ringed space, it consists of a single point whose stalk is $k[[x_0, \dots, x_n]]$. For any ring A where $|\mathrm{Spec}(A)|$ is a point, the stalk at that point must be A. If $\mathrm{stb}(x) = \mathrm{Spec}(A)$, then we must have $A = k[[x_0, \dots, x_n]]$, but the latter is a DVR and hence has two prime ideals: namely (0) and $(x_0, \dots, x_n)$. This is a contradiction. Thus, $\mathrm{stb}(x)$ cannot be an algebraic space and X_{Hod} cannot have diagonal representable by algebraic spaces. □

Proposition 5.12. *In the context of Example 5.8, taking $S = \mathrm{Spec}(k)$ for some algebraically closed field k and $X \longrightarrow \mathrm{Spec}(k)$ to be smooth, the stack X_{Dol} does not have a diagonal representable by algebraic spaces, and is, hence, not a formal algebraic stack.*

Proof. Consider a point $x \in X_{Dol}(\mathrm{Spec}(k))$. This gives a point $\iota \circ x : \mathrm{Spec}(k) \longrightarrow X_{Dol} \hookrightarrow X_{Hod}$, where the second map is the closed immersion including X_{Dol} as the fibre over $0 \in \mathbb{A}^1_k$ in X_{Hod}. We then observe that for any pair of morphisms $f, g : Y \longrightarrow \mathrm{Spec}(k)$, $\iota \circ x \circ f = \iota \circ x \circ g$ implies $x \circ f = x \circ g$, since ι is a monomorphism [Sta25, Lemma 0504]. Thus, we must have a unique map $h : Y \longrightarrow \mathrm{stb}(x)$ such that the two composites $Y \xrightarrow{h} \mathrm{stb}(x) \longrightarrow \mathrm{Spec}(k)$ are equal. That ι is a monomorphism also gurantees h is the unique map satisfying this. This shows that $\mathrm{stb}(x)$ satisfies the same universal property as $\mathrm{stb}(\iota \circ x)$. Hence, by the computation of the stabilizers of X_{Hod} in Proposition 5.11, we have that $\mathrm{stb}(x) \cong \mathrm{Spf}(k[[x_0, \dots, x_n]])$. Thus, X_{Dol} cannot have a diagonal representable by algebraic spaces. □

Remark 5.13. The stack X_{dR} of Example 5.7 is a stack whose fibre categories are setoids (a groupoid that is also a preorder, and hence a groupoid with contractible connected components). That is, it is equivalent to the Grothendieck construction of a sheaf of sets on $\mathrm{Aff}_{/S}$. This follows from the fact that the $\mathcal{F}_{dR} \longrightarrow X \times_S X$ is a monomorphism of sheaves of sets. Thus, X_{dR} has trivial stabilizers but it is still not algebraic, in general.

Remark 5.14. The lack of algebraicity of X_{dR}, X_{Dol}, X_{Hod} prevent us from concluding that $\mathcal{V}(X_{dR}), \mathcal{V}(X_{Dol}), \mathcal{V}(X_{Hod})$ are algebraic. Together, these prevents us from using [HR19, Theorem 1.2] (or, even more general results such as [HP23, Theorem 5.1.1]) to conclude that the mapping stacks involved in the definition of $\mathcal{M}_1(X_{dR}), \mathcal{M}_1(X_{Dol}), \mathcal{M}_1(X_{Hod})$ are algebraic. Nevertheless, we will see that there is an alternate route to proving their algebraicity.

5.2. **Moduli Stacks of Vector Bundles and Arrow Bundles.** We now proceed to show the main results on this paper. For this, we first recall that (λ–)connections, connections and Higgs bundles can all be formulated as modules over certain sheaves of algebras of differential operators [Sim94a, §2]. This formulation allows us to connect vector bundles on formal groupoids with connections via Simpson's crystallization functors which we now discuss.

Proposition 5.15 ([Sim08, Theorem 5.1])**.** *Let $(X, \mathcal{F})$ be a smooth formal category. Then, there exists a filtered $\mathcal{O}_X$–algebra $\Lambda_{\mathcal{F}}$ such that there is an equivalence of categories:*

$$\lozenge_{X,\mathcal{F}} : \mathrm{LMod}_{flf}(\Lambda_{\mathcal{F}}) \xrightarrow{\simeq} \mathrm{Vect}(X_{\mathcal{F}})$$

where the left side is the category of left $\Lambda_{\mathcal{F}}$–modules that is finite locally free with respect to the induced $\mathcal{O}_X$–module structure.

Proof. Strictly speaking, Simpson's result gives an $(\infty, 1)$–equivalence:

$$\mathrm{Perf}_{\mathcal{O}_X}(\Lambda_{\mathcal{F}}) \xrightarrow{\simeq} \mathrm{Perf}(X_{\mathcal{F}})$$

where the left side is the $(\infty, 1)$–category of complexes of $\Lambda_{\mathcal{F}}$–modules quasi-isomorphic as complexes of $\mathcal{O}_X$–modules to complexes whose terms are finite locally free $\mathcal{O}_X$–modules. To get the equivalence stated above, we first observe that the functor realizing this equivalence is first defined for the case of affine X and Λ_X–modules on one side, and $\mathcal{O}_{X_{\mathcal{F}}}$–modules on the other, and then observing that applying the functor term-wise gives an extension to complexes [Sim08, §3.5, p. 24]. The global case is obtained by taking the limit of the crystallization functors over a Zariski cover of X [Sim08, Proof of Theorem 5.1]. This implies that the functor preserves complexes concentrated in degree zero. □

Definition 5.16. We call $\Lambda_{\mathcal{F}}$ in Proposition 5.15 the sheaf of rings of differential operators associated to $(X, \mathcal{F})$. We call $\lozenge_{X,\mathcal{F}}$ the crystallization functor. We will suppress the subscript of the crystallization functor, when there is no confusion.

Remark 5.17. $\Lambda_{\mathcal{F}}$ is a sheaf of rings of differential operators as defined in [Sim94a, §2].

Proposition 5.18. *In the context of Definition 5.16, consider two formal groupoids $(X, \mathcal{F}), (Y, \mathcal{G})$ with an internal functor $(f_0, f_1) : (X, \mathcal{F}) \longrightarrow (Y, \mathcal{G})$ giving a morphism of quotient stacks $f : X_{\mathcal{F}} \longrightarrow Y_{\mathcal{G}}$. Then, the following diagram of categories commutes up to natural isomorphism:*

$$\begin{array}{ccc} \mathrm{LMod}_{flf}(\Lambda_{\mathcal{G}}) & \xrightarrow{\lozenge_{Y,\mathcal{G}}} & \mathrm{Vect}(Y_{\mathcal{G}}) \\ {\scriptstyle f_0^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathrm{LMod}_{flf}(\Lambda_{\mathcal{F}}) & \xrightarrow[\lozenge_{X,\mathcal{F}}]{} & \mathrm{Vect}(X_{\mathcal{F}}) \end{array}$$

Proof. We pick a Zariski cover $\{X_i \longrightarrow X\}_{i \in I}$, and then take the formal groupoids $(X_i, \mathcal{F}_i)$ where $\mathcal{F}_i = X_i \times X_i \times_{X \times X} \mathcal{F}$. We define the formal groupoids $(Y_i, \mathcal{G}_i)$ analogously. Without loss of generality, we can take the X_i to be $X \times_Y Y_i$. We get corresponding morphisms of stacks $X_{i,\mathcal{F}_i} \longrightarrow X_{\mathcal{F}}, X_{i,\mathcal{F}_i} \xrightarrow{f_i} Y_{i,\mathcal{F}_i}, Y_{i,\mathcal{G}_i} \longrightarrow Y_{\mathcal{G}}$. Then, for an object $(E, \phi) \in \mathrm{LMod}_{flf}(\Lambda_{\mathcal{G}})$, we have:

$$\begin{aligned} &\lozenge_{X,\mathcal{F}}(f_0^*(E, \phi)) \\ =& \lim_{X_i \longrightarrow X} \lozenge_{X_i,\mathcal{F}_i}((f_0^*(E, \phi))|_{X_i}) \\ =& \lim_{X_i \longrightarrow X} \lozenge_{X_i,\mathcal{F}_i}(f_0^*((E, \phi)|_{Y_i})) \end{aligned}$$

Let A_i, B_i be the rings of global sections of X_i, Y_i so that $X_i = \mathrm{Spec}(A_i), Y_i = \mathrm{Spec}(B_i)$. Let N_i denote the B_i–module $\Gamma(E|_{Y_i})$, and $M_i = A_i \otimes_{B_i} N_i$, the A_i–module $\Gamma(f_0^*(E|_{Y_i}))$. If we define $F_i := \Gamma(\Lambda_{\mathcal{F}_i})^\vee$ There is an equivalence [Sim08, §3.5] between $\Gamma(\Lambda_{\mathcal{F}_i})$–modules and connections as defined in [Ber74, §II.1.2] and by the definition of base change for connections given in [Ber74, 94, §II.1.2.5], we can deduce:

$$\lozenge_{X_i,\mathcal{F}_i}(f_0^*((E, \phi)|_{Y_i})) \cong f_0^*\big(\lozenge_{X_i,\mathcal{F}_i}((E, \phi)|_{Y_i})\big)$$

Finally, since limits of functors are defined objectwise and f_0^* is given by precomposition, f_0^* commutes with limits. Thus, we have:

$$\lozenge_{X,\mathcal{F}}(f_0^*(E, \phi)) \cong f_0^*\left(\lim_{X_i \longrightarrow X} \lozenge_{X_i,\mathcal{F}_i}((E, \phi)|_{Y_i})\right)$$

A similar argument applies for morphisms. □

These results allow us to give a concrete description of the moduli prestack of arrow bundles on a smooth formal groupoid. At the same time, we get a simple description of the fibres of the boundary map as in Proposition 4.21.

Theorem 5.19. *Given a smooth formal groupoid $(X, \mathcal{F})$, where the morphism $X \longrightarrow S$ is quasi-separated, the underlying category of $\mathcal{M}_1(X_{\mathcal{F}})$ is equivalent to a category defined by the following data:*

- *Objects are tuples (u, E, ϕ, F, ψ, s) where:*
  - *$u : U \longrightarrow S$ is an object of $\mathrm{Aff}_{/S}$*
  - *$(E, \phi), (F, \psi)$ are $\Lambda_{\mathcal{F},U} := pr_1^* \Lambda_{\mathcal{F}}$–modules, where E, F are finite locally free with respect to the induced $\mathcal{O}_{U\times\mathcal{X}}$–module structure*
  - *$s : E \longrightarrow F$ is a morphism of $\Lambda_{U\times\mathcal{X}}$–modules*
- *Morphims $(u, E, \phi, F, \psi, s) \longrightarrow (u', E', \phi', F', \psi', s')$ are tuples (f, a, b) where:*
  - *$f : U = \mathrm{dom}(u) \longrightarrow U' = \mathrm{dom}(u')$ is a morphism in $\mathrm{Aff}_{/S}$*
  - *$a : f^< E' \longrightarrow E, b^< F' \longrightarrow F$ are morphisms of $\Lambda_{\mathcal{F},U}$–modules such that the following diagram commutes:*

$$\begin{tikzcd} f^< E' \arrow[r, "a"] \arrow[d, "f^< s'"'] & E \arrow[d, "s"] \\ f^< F' \arrow[r, "b"'] & F' \end{tikzcd}$$

  noting that $f^< \Lambda_{\mathcal{F},U'} = f^<(u')^< \Lambda_{X_{\mathcal{F}}} = pr_1^* \Lambda_{X_{\mathcal{F}}} = \Lambda_{\mathcal{F},U}$.

Proof. Let (u, E_0, F_0, s_0) be an object of $\mathcal{M}_1(X_{\mathcal{F}})$ as defined in Construction 4.2. Then, the inverse of the crystallization functor for $(U \times X, U \times \mathcal{F})$ applied to s_0 gives a morphism of $\Lambda_{U\times X_{\mathcal{F}}}$–modules

$$s = \Diamond^{-1}_{U\times X, U\times\mathcal{F}}(s_0) : (E, \phi) = \Diamond^{-1}_{U\times X, U\times\mathcal{F}}(E_0) \longrightarrow (F, \psi) = \Diamond^{-1}_{U\times X, U\times\mathcal{F}}(F_0)$$

where ϕ, ψ are the structure maps making the $\mathcal{O}_{U\times\mathcal{X}}$–modules E, F into $\Lambda_{U\times X_{\mathcal{F}}}$–modules. Thus, (u, E, ϕ, F, ψ, s) is a tuple giving an object as described in the statement of the theorem.

Now, let $(f, a_0, b_0) : (u, E_0, F_0, s_0) \longrightarrow (u', E_0', F_0', s_0')$ be a morphism in $\mathcal{M}_1(X_{\mathcal{F}})$. Then, $a_0 : f^< E_0' \longrightarrow E_0, b_0 : f^< F_0' \longrightarrow F_0$ are morphisms of $\mathcal{O}_{X_{\mathcal{F}}}$–modules. Then, we get isomorphisms of $\Lambda_{\mathcal{F}}$–modules:

$$a' := \Diamond^{-1}(a_0) : E'' = \Diamond^{-1}(f^< E_0') \longrightarrow E = \Diamond^{-1}(E_0)$$
$$b' := \Diamond^{-1}(b_0) : F'' = \Diamond^{-1}(f^< F_0') \longrightarrow F = \Diamond^{-1}(F_0)$$

We also obtain isomorphisms $\psi_E : f^< \Diamond^{-1}(E_0') \xrightarrow{\cong} \Diamond^{-1}(f^< E_0')$, $\psi_F : f^< \Diamond^{-1}(F_0') \xrightarrow{\cong} \Diamond^{-1}(f^< F_0')$ by Proposition 5.18, giving us isomorphisms $a := a_0 \circ \psi_E, b := b_0 \circ \psi_F$. We denote $E' := \Diamond^{-1}(E_0'), F' := \Diamond^{-1}(F_0'), s' := \Diamond^{-1}(s_0')$. By naturality of the ψ_E, ψ_F, we have the following commuting diagram:

$$\begin{tikzcd} f^< E' \arrow[r, "\psi_E"] \arrow[d, "f^< s'"'] \arrow[rr, bend left, "a"] & \Diamond^{-1}(f^< E_0') \arrow[r, "a'"] \arrow[d, "\Diamond^{-1}(f^< s_0')"'] & E \arrow[d, "s"] \\ f^< F' \arrow[r, "\psi_F"'] \arrow[rr, bend right, "b"'] & \Diamond^{-1}(f^< F_0') \arrow[r, "b'"] & F \end{tikzcd}$$

This shows that (f, a, b) is a morphism of the form described in the statement.

If we now set $\Psi(u, E_0, F_0, s)$ and $\Psi(f, a_0, b_0) := (f, a, b)$, it is not hard to check that Ψ is a functor and it commutes with the projections to $\mathrm{Aff}_{/S}$ by construction. It is also not hard to check that it is an equivalence by using the fact that on fibre categories, it is simply the crystallization functor which is an equivalence by Proposition 5.15. □

Proposition 5.20. *In the context of Definition 4.19, setting $\mathcal{Y} = X_{\mathcal{F}}$ for a smooth formal groupoid $(X, \mathcal{F})$, $\Diamond^{-1}(E) = E_\diamond, \Diamond^{-1}(F) = F_\diamond$ and $\Lambda_{\mathcal{F},U} := pr_1^* \Lambda_{\mathcal{F}}$, we have that $P^{X_{\mathcal{F}}}_{v,w}$ is equivalent to the category with the following description:*

- *Objects are tuples (r, s) where:*
  - *$r : V \longrightarrow U$ is a morphism in $\mathrm{Aff}_{/S}$*
  - *$s_\diamond : r^< E_\diamond \longrightarrow r^< F_\diamond$ is a morphism of $r^< \Lambda_{\mathcal{F},U}$–modules*
- *Morphisms $(r, s_\diamond) \longrightarrow (r', s'_\diamond)$ are morphisms $f : V = \mathrm{dom}(r) \longrightarrow V' = \mathrm{dom}(r')$ in $\mathrm{Aff}_{/S/U}$ such that $f^< s' = s$.*

Proof. We take $\mathcal{Y} = X_{\mathcal{F}}$ in Proposition 4.21, and consider an object (r, s) in $Q^{X_{\mathcal{F}}}_{v,w} \simeq P^{X_{\mathcal{F}}}_{v,w}$ according to the description there. Then, $\Diamond^{-1}(s) : \Diamond^{-1}(r^< E) \longrightarrow \Diamond^{-1}(r^< F)$ is a morphism of $r^< \Lambda_{\mathcal{F},U}$–modules. Choosing a natural isomorphism ψ implementing the 2–commutativity in Proposition 5.18, we get a unique morphism of $r^< \Lambda_{\mathcal{F},U}$–modules $s_\diamond$ making the following square commute:

$$\begin{tikzcd} \Diamond^{-1}(r^< E) \arrow[r, "\psi_E"] \arrow[d, "\Diamond^{-1}(s)"'] & r^< E_\diamond \arrow[d, "s_\diamond"] \\ \Diamond^{-1}(r^< F) \arrow[r, "\psi_F"'] & r^< F_\diamond \end{tikzcd}$$

Then, $(r, s_\diamond)$ is an object as described in the statement.

Now, consider a morphism $f : (r, s) \longrightarrow (r', s')$ in $P^{X_{\mathcal{F}}}_{v,w}$. Then, $f^< s' = s$ implies $\Diamond^{-1}(f^< s') = \Diamond^{-1}(s)$. By the uniqueness of $s_\diamond$ and $s'_\diamond$ as in the previous paragraph, we must have $f^< s'_\diamond = s_\diamond$.

From this description, it is easy to see that the assignment $(r, s) \longmapsto (r, s_\diamond), f \longmapsto f$ is a fully faithful functor from the $P^{X_{\mathcal{F}}}_{v,w}$ to the category described in the statement. That this functor is also essentially surjective follows from the fully faithfulness of the crystallization functors — note that the fibre categories are simply the Hom sets $\mathrm{Hom}_{\mathcal{O}_{U \times X_{\mathcal{F}}}}(r^< E, r^< F)$ on one side and $\mathrm{Hom}_{r^< \Lambda_{\mathcal{F},U}}(r^< E_\diamond, r^< F_\diamond)$, on the other. □

With this, we can prove the algebraicity of the moduli stack of arrow bundles on a smooth formal groupoid whenever the moduli stacks of vector bundles, $\Lambda_{\mathcal{F}}$–modules and arrow bundles on the object scheme of the formal groupoid are algebraic.

Theorem 5.21. *Let $(X, \mathcal{F})$ be a smooth formal groupoid over S, where the map $X \longrightarrow S$ is smooth, and such that $\mathcal{M}(X), \mathcal{M}_1(X)$ and $\mathcal{M}(X_{\mathcal{F}})$ are all algebraic. Then, the following are true:*

(i) The boundary map $\partial_{X_{\mathcal{F}}} : \mathcal{M}_1(X_{\mathcal{F}}) \longrightarrow \mathcal{M}(X_{\mathcal{F}})^2$ is representable by algebraic spaces. Therefore, $\mathcal{M}_1(X_{\mathcal{F}})$ is an algebraic stack.

If, in addition, $X \longrightarrow S$ is projective, then

(ii) $\partial_{X_{\mathcal{F}}}$ is affine and of finite presentation.

If, furthermore, $\mathcal{M}(X_{\mathcal{F}})$ is (locally) of finite presentation, then

(iii) $\mathcal{M}_1(X_{\mathcal{F}})$ is (locally) of finite presentation.

Proof. We will show that the fibre $P^{X_{\mathcal{F}}}_{v,w}$ of any map $(v, w) : U \longrightarrow \mathcal{B}_1(X_{\mathcal{F}})^2$ with $U \in \mathrm{Aff}_{/S}$ is equivalent to an equalizer of algebraic spaces and is, hence, an algeraic space. For this, we first recall from [Sim94a, §2] that for any $z : Z \longrightarrow S$, $\Lambda_{\mathcal{F},Z} := pr_1^* \Lambda_{\mathcal{F}}$ is a filtered ring sheaf and from [Sim94a, Lemma 2.2], that the stages of the filtration are coherent. Then, we also know that $\Lambda_{\mathcal{F},Z}$

is split almost polynomial [Sim08, §3.3, §5.1] so that $\Lambda_{\mathcal{F},Z,1}$ is locally free by [Sim94a, Theorem 2.11]. This means $\Lambda_{\mathcal{F},Z,1}$ is finite locally free. At the same time, the split almost polynomial condition implies that there is an open cover $\coprod_{j\in J} Z_i \longrightarrow Z\times X$ such that $\Lambda_{\mathcal{F},Z}|_{Z_j}$ is generated by $\Lambda_{\mathcal{F},U,1}|_{Z_j}$ [Sim94a, Proof of Theorem 2.11]. Since morphisms of sheaves satisfy descent, we conclude that a morphism $s:(E,\phi)\longrightarrow(F,\psi)$ is a morphism of $\Lambda_{\mathcal{F},Z}$–modules if and only if the following diagram commutes:

$$\begin{tikzcd}
\Lambda_{\mathcal{F},Z,1}\otimes E \arrow[r, "\phi"] \arrow[d, "\mathrm{id}_{\Lambda_{\mathcal{F},Z,1}}\otimes s"'] & E \arrow[d, "s"] \\
\Lambda_{\mathcal{F},Z,1}\otimes F \arrow[r, "\psi"'] & F
\end{tikzcd}$$

Now, consider the $\Lambda_{\mathcal{F},U}$–modules $(E_0,\phi),(F_0,\psi)$ corresponding to v,w respectively. Then, we define two functors $\rho_1,\rho_2 : P^X_{E,F}\longrightarrow P^X_{\Lambda_{\mathcal{F},U,1}\otimes E,F}$ as follows. ρ_1 sends an object (r,s) to the composite

$$r^<\Lambda_{\mathcal{F},Z,1}\otimes r^<E \xrightarrow{r<\phi} r^<E \xrightarrow{s} r^<F$$

Then, given a morphism $f:(r,s)\longrightarrow(r',s')$, we have $f^<s'=s$, so that $f^<(s'\circ(r')^<\phi) = f^<s'\circ f^<(r')^<\phi = s\circ r^<\phi$. This shows that f is also a morphism in $P^X_{\Lambda_{\mathcal{F},U,1}\otimes E,F}$. We set $\rho_1(f)=f$. It is straightforward to verify that these mappings make ρ_1 a functor commuting with the projection to $\mathrm{Aff}_{/S}$. Thus, it is a morphism of prestacks. The map ρ_2 sends (r,s) to the composite

$$r^<\Lambda_{\mathcal{F},Z,1}\otimes r^<E \xrightarrow{\mathrm{id}\otimes s} r^<\Lambda_{\mathcal{F},Z,1}\otimes r^<F \xrightarrow{r^<\psi} r^<F$$

If $f:(r,s)\longrightarrow(r',s')$ is a morphism, then $f^<s'=s$ again, and this implies:

$$f^<((r')^<\psi\circ(\mathrm{id}\otimes s')) = f^<(r')^<\psi\circ f^<(\mathrm{id}\otimes s') = f^<(r')^<\psi\circ(f^<\mathrm{id}\otimes f^<s') = r^<\psi\circ(\mathrm{id}\otimes s)$$

so that f is again a morphism in $P^X_{\Lambda_{\mathcal{F},U,1}\otimes E,F}$. We again set $\rho_2(f)=f$, and can check that this gives a morphism of prestacks.

We can then see that $P^{X_\mathcal{F}}_{v,w}$ fits into the following equalizer diagram:

$$P^{X_\mathcal{F}}_{v,w} \longrightarrow P^X_{E,F} \overset{\rho_1}{\underset{\rho_2}{\rightrightarrows}} P^X_{\Lambda_{\mathcal{F},U,1}\otimes E,F}$$

where the first map is the forgetful functor sending objects and morphisms to themselves. To see this, we notice that the equalizer is strict since all prestacks involved are fibred in Set. Then, by Proposition 5.20, the objects and morphisms of $P^{X_\mathcal{F}}_{v,w}$ are a subset of those of $P^X_{E,F}$ satisfying the commutativity constraint for being module homomorphisms. The equality of the maps ρ_1,ρ_2 is precisely this condition. It remains to show that $P^X_{E,F},P^X_{\Lambda_{\mathcal{F},U,1}\otimes E,F}$ are algebraic spaces. This follows from Proposition 4.23. This shows point (i).

In the case that $X\longrightarrow S$ is projective, Theorem 4.25 applies and shows that $P^X_{E,F},P^X_{\Lambda_{\mathcal{F},U,1}\otimes E,F}$ are affine and of finite presentation over U. Since an equalizer over S is also an equalizer over U, and the properties of being affine and of finite presentation are preserved under finite limits, we must have that $P^{X_\mathcal{F}}_{v,w}$ is affine and of finite presentation over S. This shows point (ii).

For (iii), we observe that $\partial_{X_\mathcal{F}}$ being of finite presentation and $\mathcal{M}(X_\mathcal{F})$ being (locally) of finite presentation implies that the morphism $\mathcal{M}_1(X_\mathcal{F})\xrightarrow{\partial_{X_\mathcal{F}}}\mathcal{M}(X_\mathcal{F})^2\longrightarrow S$ is (locally) of finite presentation. □

We will now recall that $\mathcal{M}(X_\mathcal{F})$ is well-behaved under some mild conditions, which is well-known [Sim08, Theorem 6.13], and show that some of the good behaviour of $\mathcal{M}(X_\mathcal{F})$ transfers over to $\mathcal{M}_1(X_\mathcal{F})$.

Proposition 5.22. *For a smooth formal groupoid $(X, \mathcal{F})$ over S, if the morphism $X \longrightarrow S$ is smooth and projective, then the forgetful functor $\mathcal{M}(X_{\mathcal{F}}) \longrightarrow \mathcal{M}(X)$ is affine and locally of finite presentation. In particular, $\mathcal{M}(X_{\mathcal{F}})$ is algebraic, locally of finite presentation and has affine diagnoal over S.*

Proof. We first show that the forgetful map $\mathcal{M}(X_{\mathcal{F}}) \longrightarrow \mathcal{M}(X)$ is affine and of finite presentation. To see this, we observe that for any morphism $v : U \longrightarrow \mathcal{M}(X)$ for some object $\mathrm{Aff}_{/S}$ corresponding to a finite locally free $\mathcal{O}_{U\times X}$–module E, we consider the fibre product $\mathcal{M}(X_{\mathcal{F}}; E) := \mathcal{M}(X_{\mathcal{F}}) \times_{\mathcal{M}(X)} U$. Unwrapping the definition of fibre products, the underlying category of this stack has the following description:

- Objects are tuples (v, F, ϕ, α) where:
  - $v : V \longrightarrow U$ is an object in $\mathrm{Aff}_{/S/U}$.
  - F is a finite locally free sheaf on $V \times X$.
  - $\phi : \Lambda_{\mathcal{F},V} \otimes F \longrightarrow F$ is a $\Lambda_{\mathcal{F},V}$–module structure on F.
  - $\alpha : v^{<}E \longrightarrow F$ is an isomorphism of locally free sheaves.
- Morphisms $(v, F, \phi, \alpha) \longrightarrow (v', F', \phi', \alpha')$ are tuples (f, a) where:
  - $f : V = \mathrm{dom}(v) \longrightarrow V' = \mathrm{dom}(v')$ is a morphism over U.
  - $a : f^{<}F' \longrightarrow F$ is an isomorphism of $\Lambda_{\mathcal{F},V}$–modules, such that the following diagram commutes:

$$\begin{tikzcd} v^{<}E = f^{<}(v')^{<}E \arrow[r, "f^{<}\alpha'"] \arrow[d, "\mathrm{id}"'] & f^{<}F' \arrow[d, "a"] \\ v^{<}E \arrow[r, "\alpha"'] & F \end{tikzcd}$$

We observe that each object (v, F, ϕ, α) of this category is isomorphic to the object

$$(v, v^{<}E, \alpha^{-1} \circ \phi \circ (\mathrm{id}_{\Lambda_{\mathcal{F},V}} \otimes \alpha), \mathrm{id}_{v^{<}E})$$

via the morphism (id_V, α). Next, consider a morphsim

$$(f, a) : (v, v^{<}E, \phi, \mathrm{id}_{v^{<}E}) \longrightarrow (v', (v')^{<}E, \phi', \mathrm{id}_{(v')^{<}E})$$

The commutativity condition in the definition of morphisms becomes

$$a \circ f^{<}\mathrm{id}_{(v')^{<}E} = \mathrm{id}_{v^{<}E} \circ \mathrm{id}_{v^{<}E} \iff a = \mathrm{id}_{v^{<}E}$$

which implies that $f^{<}\phi' = \phi$, by the fact that a is a morphism $\Lambda_{\mathcal{F},V}$–modules. This shows that the underlying category is, in turn, equivalent to the category with the following description:

- Objects are tuples (v, ϕ) where:
  - $v : V \longrightarrow U$ is an object in $\mathrm{Aff}_{/S/U}$.
  - $\phi : \Lambda_{\mathcal{F},V} \otimes v^{<}E \longrightarrow v^{<}E$ is a $\Lambda_{\mathcal{F},V}$–module structure on F.
- Morphisms $(v, \phi) \longrightarrow (v', \phi')$ is a morphism $f : V = \mathrm{dom}(v) \longrightarrow V' = \mathrm{dom}(v')$ over U such that $f^{<}\phi' = \phi$.

A very similar argument as in the proof of Theorem 5.21 shows that this is a finite limit involving $P^X_{\Lambda^{\otimes 2}_{\mathcal{F},U,1}\otimes E, E}$, $P^X_{\Lambda_{\mathcal{F},U,1}\otimes E, E}$, $P^X_{\mathcal{O}_{U\times X}\otimes E, E}$, that captures the commutativity constraints defining the structure map of a module sheaf, and hence is an affine scheme of finite presentation by Theorem 4.25 and the preservation of these properties under finite limits. Since the morphism $\mathcal{M}(X_{\mathcal{F}}) \longrightarrow \mathcal{M}(X)$ is affine and of finite presentation, while $\mathcal{M}(X)$ is algebraic locally of finite presentation and has affine diagonal by [Wan11, Theorem 1.0.1], $\mathcal{M}(X_{\mathcal{F}})$ is algebraic by [Sta25, Lemma 05UM] and has affine diagonal by [AR25, Lemma 4.9]. Then, in the composite $\mathcal{M}(X_{\mathcal{F}}) \longrightarrow \mathcal{M}(X) \longrightarrow S$, the first morphism is of finite presentation and the second one is locally of finite presentation by [Wan11, Theorem 1.0.1] again, showing that $\mathcal{M}(X_{\mathcal{F}})$ is locally of finite presentation over S. □

Remark 5.23. This is likely a direct consequence of the fibre computation in the proof of [Sim08, Theorem 6.13], but we felt it is useful to have a simpler proof in the realm of 1–stacks.

Theorem 5.24. *For a smooth formal groupoid $(X, \mathcal{F})$ over S, if the morphism $X \longrightarrow S$ is smooth and projective, then $\mathcal{M}_1(X_{\mathcal{F}})$ is algebraic, locally of finite presentation and has affine diagonal.*

Proof. $\mathcal{M}(X_{\mathcal{F}})$ has these properties by Proposition 5.22, and hence, so does $\mathcal{M}(X_{\mathcal{F}})^2$. Then, since the morphism $\partial_{X_{\mathcal{F}}}$ is affine by Theorem 5.21(ii), $\mathcal{M}_1(X_{\mathcal{F}})$ is algebraic by [Sta25, Lemma 05UM] and has affine diagonal by [AR25, Lemma 4.9]. Then, the morphism $\mathcal{M}(X_{\mathcal{F}})^2 \longrightarrow \mathcal{M}(X)^2$ is locally of finite presentation by Proposition 5.22 again, while $\partial_{X_{\mathcal{F}}}$ is locally of finite presentation by Theorem 5.21(ii). Also, $\mathcal{M}(X)^2$ is locally of finite presentation by [Wan11, Theorem 1.0.1]. This shows that the composite

$$\mathcal{M}_1(X_{\mathcal{F}}) \xrightarrow{\partial_{X_{\mathcal{F}}}} \mathcal{M}(X_{\mathcal{F}})^2 \longrightarrow \mathcal{M}(X)^2 \longrightarrow S$$

is locally of finite presentation. □

Theorem 5.25. *In the case that $X \longrightarrow \operatorname{Spec}(k)$ is a smooth and projective morphism, the moduli stacks $\mathcal{M}_1(X_{Dol}), \mathcal{M}_1(X_{dR}), \mathcal{M}_1(X_{Hod})$ parametrizing Higgs bundle morphisms, connection morphisms and λ–connection morphisms respectively over X are all agebraic, locally of finite presentation and have affine diagonal.*

Proof. For the first two, take $S = \operatorname{Spec}(k)$, and for the third, take $S = \mathbb{A}^1_k$, and apply Theorem 5.24. □

Remark 5.26. The result [Sim08, Theorem 6.13] states that the moduli stack of complexes of $\Lambda_{\mathcal{F}}$–modules that are perfect as complexes of $\mathcal{O}_X$–modules is a locally geometric n–stacks. However, we can unpack the meaning of this to discover that this shows the algebraicity of the sub–1–stack of complexes concentrated in degree 0.

5.3. **Quiver Bundles on Formal Groupoids.** In [AR25, §4], for each stack $\mathcal{Y}$ and each fintite simplicial set I, we gave a construction of a moduli stack $\mathcal{M}_{\mathrm{Vect}(\mathcal{Y}),I}$ parametrizing diagrams of vector bundles of shape I on a stack X. In fact, the construction given there is still well-defined when $\mathcal{Y}$ is a prestack. This construction was given as a finite iterated limit involving the (pre)stacks $\mathcal{M}(\mathcal{Y}), \mathcal{M}_1(\mathcal{Y})$ and pt. Now, in light of Theorem 4.11 and Theorem 5.19, if we take $\mathcal{Y} = X_{\mathcal{F}}$ for some formal groupoid $(X, \mathcal{F})$, by varying $\mathcal{F}$ to be $\mathcal{F}_{Dol}, \mathcal{F}_{dR}$ or $\mathcal{F}_{Hod}$, we obtain moduli stacks parametrizing I–shaped diagrams of connections, λ–connections, or Higgs bundles.

Notation 5.27. For notational consistency with the rest of the present paper, we will write $\mathcal{M}_{\mathrm{Vect}(X_{\mathcal{F}}),I}$ as $\mathcal{M}_I(X_{\mathcal{F}})$.

Furthermore, $\mathcal{M}_I(X_{\mathcal{F}})$ is contravariantly functorial in I by construction. Taking $I = \Delta^n$ for various n, we thus obtain, in particular, a simplicial object $\mathcal{M}_\bullet(\mathcal{Y})$ whose levels are moduli stacks of diagrams of vector bundles of shape Δ^n, and whose structure maps are given by "face" and "degeneracy" maps. This is what we mean by the categorification of the various sides of non-Abelian Hodge theory. In particular, $\mathcal{M}_\bullet(X_{Hod})$ is the categorification of the non-Abelian Hodge filtration.

Remark 5.28. However, we note that since there are no higher morphisms in the category of sheaves over a prestack, we cannot expect these objects to hold much important information beyond simplicial dimension 1: that is, for sheaves over 1–stacks, $\mathcal{M}_0(\mathcal{Y}), \mathcal{M}_1(\mathcal{Y})$ should contain all the important information. The full simplicial theory becomes relevant when we are dealing with sheaves of simplicial Abelian groups, chain complexes or spectra on (derived) stacks, but we will defer a discussion of this point to future work.

We now observe that $\mathcal{M}_I(X_{\mathcal{F}})$ is well-behaved under some mild conditions on $(X, \mathcal{F})$.

Theorem 5.29. *For any smooth formal groupoid $(X, \mathcal{F})$, if the morphism $X \longrightarrow S$ is smooth and projective, for any finite simplicial set I, $\mathcal{M}_I(X_{\mathcal{F}})$ is algebraic, locally of finite presentation and has affine diagonal.*

Proof. By construction, $\mathcal{M}_I(X_{\mathcal{F}})$ is an iterated finite limit involving $\mathcal{M}_1(X_{\mathcal{F}})$, $\mathcal{M}(X_{\mathcal{F}})$ and pt. The claim now follows Proposition 5.22, Theorem 5.24 and the fact that the properties involved are preserved under taking finite limits. □

5.4. **Non-Abelian Hodge Theory.** We wish to have an analogue of the non-Abelian Hodge correspondence between the simplicial stacks $\mathcal{M}_\bullet(X_{dR})$ and $\mathcal{M}_\bullet(X_{Dol})$. In light of Remark 5.28, in this work, we will mainly deal with the truncated simplicial objects consisting of $\mathcal{M}_1(-)$ and $\mathcal{M}(-)$ and the terminal stack $\mathrm{Aff}_{/S}$ in higher degrees, as we are focussing on sheaves of sets on 1–stacks. We first recall that the non-Abelian Hodge correspondence has some restrictions. First, it is a bijection between moduli spaces of flat connections on one side and Higgs bundles on the other side, which happens to be a homeomorphism if we restrict to polystable objects with vanishing rational Chern classes on the Higgs bundle side. It is known that it cannot be extended [Sim94b, Counterexample on pages 38—39]. Furthermore, the mapping is neither complex analytic nor smooth even though both sides have complex structures. This suggests that we need some way to pass from algebraic stacks to topological stacks to address non-Abelian Hodge theory. This is possible through analytification of stacks, a simple version of which we will now discuss. For this purpose, let us now suppose $k = \mathbb{C}$ for simplicity.

Construction 5.30. Suppose S is of finite presentation over $\mathbb{C}$. Given any finite simplicial set I and a smooth formal groupoid $(X, \mathcal{F})$ with $X \longrightarrow S$ projective and smooth, choose a smooth atlas $a : A = \coprod_{j \in J} U_j \longrightarrow \mathcal{M}_I(X_{\mathcal{F}})$ such that the morphism a is also affine and each $U_j \in \mathrm{Aff}_{/S}$ is of finite presentation, using Theorem 5.29 and [Sta25, Lemma 04YF]. Since a is affine, $R := A \times_{\mathcal{M}_I(X_{\mathcal{F}})} A = \coprod_{i,j} U_i \times_{\mathcal{M}_I(X_{\mathcal{F}})} U_j$ is a disjoint union of affine schemes $U_i \times_{\mathcal{M}_I(X_{\mathcal{F}})} U_j$ so that R is a locally separated algebraic space. Of course, A is also a locally separated algebraic space for the same reason. Furthermore, since $\mathcal{M}_I(X_{\mathcal{F}})$ and A are locally of finite presentation, so is R since this property is preserved by fibre products of stacks. Next, by [Sta25, Lemma 04T5], (A, R) gives a presentation of $\mathcal{M}_I(X_{\mathcal{F}})$. This allows us to consider analytifications of A, R [Hal14] to obtain a groupoid object internal to complex analytic spaces, noting that analytification is compatible with fibre products [CT07, Theorem 2.2.3.]. This, in turn, also gives a groupoid object in topological spaces. We can take the quotient stacks over the respective categories to get complex analytic and topological stacks:

$$\mathcal{M}_I^{an}(X_{\mathcal{F}}), \mathcal{M}_I^{top}(X_{\mathcal{F}})$$

respectively. Note that we retain the notation $X_{\mathcal{F}}$ although we are not yet talking about the analytification of X and $\mathcal{F}$.

We now state our desired categorification of the non-Abelian Hodge correspondence as a conjecture:

Conjecture 5.31. *Given a smooth projective variety X over k, there is a suitable substack*

$$\mathcal{M}^{top,nice}(X_{Dol}) \subset \mathcal{M}_1^{top}(X_{Dol}),$$

and mappings

$$\mathcal{M}^{top}(X_{dR}) \longrightarrow \mathcal{M}^{top,nice}(X_{Dol}), \mathcal{M}_1^{top}(X_{dR}) \longrightarrow \mathcal{M}_1^{top,nice}(X_{Dol})$$

that induce a "categorical" equivalence of simplicial topological stacks

$$\mathcal{M}_\bullet^{top}(X_{dR}) \xrightarrow{\simeq} \mathcal{M}_\bullet^{top,nice}(X_{Dol})$$

for some suitable meaning of "categorical" in context of simplicial objects in topological stacks. This should still hold if we replace $\mathcal{M}_\bullet(X_{dR}), \mathcal{M}_\bullet(X_{Dol})$ with the respective simplicial stacks obtained by applying our constructions in this paper with derived mapping stacks in place of ordinary mapping stacks.

Proof idea. $\mathcal{M}_\bullet^{top}(X_{dR})$ and $\mathcal{M}_\bullet^{top}(X_{Dol})$ are simplicial substacks of $\mathcal{M}_\bullet(X_{Hod}) \longrightarrow \mathbb{A}^1_{\mathbb{C}}$: namely, the (simplicial degree-wise) fibres above $0, 1 \in \mathbb{A}^1_{\mathbb{C}}$ respectively. We can then try to apply the "preferred sections" approach as discussed in [Sim96, §4] for every simplicial degree. Assuming we are able to accomplish this, we must, of course, then examine how the maps $\mathcal{M}_n^{top}(X_{dR}) \longrightarrow \mathcal{M}_n^{top,nice}(X_{Dol})$ interact with the simplicial maps. □

Of course, we must also make precise what we mean by "categorical equivalence" and then check that such a condition is satisfied.

Remark 5.32. According to a conversation we had with Carlos Simpson, it is likely that the above result will only be true in the derived setting.

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