

Preliminary Remarks

1. As a result of the adjustments in the most recent version, there has been a slight change in numbering throughout the paper. For example, Theorem 4.9 has become Theorem 4.10. Unless otherwise indicated, the numbers referred to below are the ones in the most recent version.

Response to the Referee's Suggestions

1. We have included a discussion of Definition 2.10 in the context of stacks over the site of affine schemes in Remark 2.12. In particular, we have clarified here how the usual moduli stack of vector bundles can be obtained by specializing Definition 2.10. We have also added correct references to the definition in the contexts of non-derived and derived algebraic geometry. We mainly use the non-derived version in the paper.

Steve to address second paragraph of point 1 in Referee Report.

2. In the paragraph before Assumption 2.1, we have provided the origin of the assumptions of Section 2: in particular, these assumptions are modeled after well-known properties of the categories of quasicoherent sheaves and vector bundles on stacks over the site of affine schemes. Furthermore, in Remark 2.16, we have clarified that these assumptions are satisfied by all stacks over the site of affine schemes, regardless of algebraicity/geometricity. This includes Deligne-Mumford and Artin stacks. This also includes root stacks, as they are defined as pullbacks of stacks.

Steve to assess the claim about root stacks.

3. We have removed Conjectures 2.14 and 2.16 (previous numbering) and added Remarks 2.17 and 2.19 in their place. The purpose of these remarks is to provide motivation for future extensions of the work to moduli stacks in the settings of derived algebraic geometry and analytic geometry. We should note that the assumptions in Section 2 are requirements for the results of the paper, not the other way around. The assumptions themselves hold for arbitrary stacks over the site of affine schemes, for formal reasons. They do not require any form of algebraicity/geometricity or other special properties. Thus, we expect them to hold for the analytic stacks of Clausen–Scholze or of Ben-Bassat–Kelly–Kremnizerv — these are just stacks over the sites of

analytic rings and simplicial commutative complete bornological rings respectively — for similar formal reasons.

4. We have added Remark 2.20 to point the reader to the discussion of the connection of the paper with Nakajima quiver varieties given in Section 5.
5. We clarify the reason for considering unlabeled quivers in Remark 2.23 while indicating how quiver varieties obtained from labeled quivers can be recovered from our work. Here, we again point the reader to Section 5 where a more detailed discussion of these aspects is given. However, we note that a treatment of quiver varieties is not a major purpose of this paper, and hence we have deferred a thorough discussion of quiver varieties and labelings on quivers to future work.

Steve to assess the completeness of this particular point thoroughly.

6. We have made spelling corrections in various parts of the paper, including the ones pointed out by the referee.

Additional Updates

1. While the paper has been in review, we have detected an error in the definition of the moduli stack of vector bundle morphisms given in Section 3. This required a rewrite of that section in the most recent version of the paper. However, the properties of the moduli stack of vector bundle morphisms used in the rest of the paper are unchanged even with this new definition. As a result, we only had to make minor adjustments in Sections 4, 5 and 6.
2. We have modified the statements of Theorem 4.10 and Corollary 4.11 to organize the hypotheses more clearly, but there is no change in the essential content of these results.
3. While this paper has been in review, Mauro Porta and Francesco Sala have pointed us to their work constructing derived moduli stacks of diagrams of vector bundles, Higgs bundles and flat connections, as well as showing their algebraicity. We have added remarks throughout the paper (please see Section 1, Remark 1.1, Remark 3.4, Remark 4.12 and Remark 6.11) describing the similarities and differences with their work.

Steve to double check these remarks.