REPORT
MODULI STACKS OF QUIVER BUNDLES AND APPLICATIONS TO HIGGS BUNDLES

The present paper has two main goals. First, it develops a general moduli theory of quiver bundles, that is, diagrams of vector bundles over a fixed base, constructed as Artin stacks within a flexible, homotopy-theoretic framework. Second, it illustrates how this formalism recovers familiar moduli problems, such as Nakajima quiver varieties and moduli stacks of Higgs bundles, with a view towards applications to categorified non-abelian Hodge theory.

The paper can be viewed as a stack-theoretic generalisation of earlier work of the second author (cf. [RS18, RS21], references as in the paper). In particular, it extends the philosophy underlying the moduli stack of chains over smooth projective curves, previously constructed by García-Prada, Heinloth and Schmitt ("On the motives of moduli of chains and Higgs bundles", J. Eur. Math. Soc. 16), to a more abstract setting.

While many of the applications discussed in the paper (e.g. to Higgs bundles and quiver-type moduli problems) are scattered throughout the existing literature, the construction proposed here, namely, a systematic method for producing moduli stacks of diagrams from the moduli stack of vector bundles via mapping stacks and universal objects, is original and builds in a coherent way on previous work by the second author. The framework is interesting and potentially useful, and in my opinion the paper is suitable for publication in this journal. However, several issues should be addressed before publication, which I list here:

(1) Definition 2.10 is a foundational definition on which the rest of the paper relies. The reference given for it appears to be incorrect or at least confusing: it refers to a Definition 10.7 in a paper whose title and numbering do not seem to match, and the cited definition cannot be located as stated. Since this definition underpins the subsequent assumptions and constructions, the authors should provide a correct and verifiable published reference, or otherwise clarify the provenance and standardness of the definition.

    More generally, it would strengthen the paper to place it more explicitly within the existing literature on sheaves on stacks and quiver-type constructions over stacks. In particular, connections with work such as T. Abdelgadir, Quivers of sections on toric orbifolds, J. Algebra (2012), and related literature on quivers and Deligne–Mumford stacks, should be discussed.

(2) The list of assumptions in Section 2 is rather restrictive at first sight, and it is not immediately clear how large the intended class of examples is. The authors should provide concrete examples of stacks satisfying these assumptions. In particular, it would be very helpful to clarify explicitly whether root stacks (or more generally Deligne–Mumford stacks with finite stabilisers) satisfy all the stated assumptions, and under which hypotheses.

Such examples would greatly improve the accessibility of the framework and help readers understand the intended geometric scope of the theory.

(3) Conjectures 2.14 and 2.16 appear to be quite mismatched in nature and strength. Conjecture 2.14 seems essentially tautological or trivially true in the geometric situations of interest, whereas Conjecture 2.16 appears to be false as stated, in light of Theorem 4.9, where substantially stronger hypotheses are required to obtain algebraicity of the relevant moduli stacks.

Moreover, both conjectures are introduced rather abruptly, without sufficient motivation or discussion of their role in the paper. The authors should clarify: what evidence supports these conjectures, whether they are actually used in the paper, and what the consequences would be for the main results if these conjectures were removed or weakened.

As it stands, the conjectures read more as informal expectations than as mathematically precise statements.

(4) Definition 2.17 suggests a possible connection with Nakajima quiver varieties, but the nature of this connection is left vague. The authors should clarify: in which precise sense their construction recovers Nakajima quiver varieties, under what assumptions on the quiver and the base, and whether this connection is geometric, categorical, or only heuristic.

A more explicit explanation would significantly enhance the usefulness of this part of the paper.

(5) The authors emphasise that the quivers considered are unlabelled. It is not clear that this choice leads to greater generality; in fact, it may introduce additional restrictions, particularly in relation to connections with classical quiver varieties and their representation-theoretic interpretations. The authors should comment on this point and clarify what is gained or lost by working with unlabelled quivers.

(6) A careful spell-check is recommended. For instance, the word “varieties” is misspelt repeatedly throughout the manuscript.