Schemes from Varieties

Mahmud Azam

February 14, 2025

Abstract

The purpose of these notes is to make precise what it means for a k–variety to be a k–scheme, for an algebraically closed field k. This is theorem 2.14. That is, these notes provide a detailed proof of [1, Proposition 2.6]. The writing style of these notes is not expository, it simply provides the details of the proof. At the same time, we do not try to be efficient with our arguments and prove necessary statements as directly as possible, with as few appeals to "powerful" theorems as possible.

Contents

1 Basics from Point-Set Topology

1.1 Basic Topology

Corollary 1.1. *Let X be a topological spaces and $V \subset U \subset X$. Then, if U is open, then V is open in U if and only if V is open in X. Likewise, if U is closed, then V is closed in U if and only if V is closed in X.*

Proof. Suppose U is open. Then, V is open in U if and only if there exists an open $V' \subset X$ such that $V' \cap U = V$. This implies V is open in X as it is an intersection of two open sets. If V is open in X, then $V = V \cap U$ is an intersection of two open subsets of U and hence is open in U. When U is closed, the proof is identical with the word "open" replaced by "closed". □

Lemma 1.2. *For any topological space X, any $U \subset X$ and a closed $C \subset U$, $\overline{C} \cap U = C$, where the closure is taken in X.*

Proof. Since $C \subset \overline{C}$ and $C \subset U$, we have $C \subset \overline{C} \cap U$. Then, C being closed in U means there exists a closed $C' \subset X$ such that $C = C' \cap U$. In particular, C' is a closed set of X containing C so that $\overline{C} \subset C'$ which implies that $\overline{C} \cap U \subset C' \cap U = C$. Thus, $\overline{C} \cap U = C$. $\square$

Lemma 1.3. *Let X be a topological space and $V \subset U \subset X$. Then the subspace topology of V induced from U is strictly equal as a set to the subspace topology induced from X.*

Proof. Let T_U and T_X be the subspaces topologies on V induced from U and X respectively. Let $W \in T_U$ be an open set. Then, there exists an open $W' \subset U$ such that $W = W' \cap V$. Then, W' being open in U implies there exists an open $W' \subset X$, such that $W' = W'' \cap U$. Thus, $W = W'' \cap U \cap V = W'' \cap V$. This implies that $W \in T_X$.

Now, suppose $W \in T_X$. Then, there exists an open $W'' \subset X$ such that $W = W'' \cap V$. Then, $W = W \cap U$ because $W \subset U$, so that $W = W'' \cap V \cap U = (W'' \cap U) \cap V$. Here, $W'' \cap U \subset U$ is open in U. Thus, $W \in T_U$. $\square$

Lemma 1.4. *Let $f : X \longrightarrow Y$ be a contnuous function; $U \subset X$, dense; $y \in Y$, a closed point, and $f(x) = y$ for all $x \in U$. Then, $f(x) = y$ for all $x \in X$.*

Proof. Then, $f^{-1}(\{y\})$ is closed subset of X containing U, and hence must contain $\overline{U} = X$, since U is dense. $\square$

Lemma 1.5. *Let $f : X \longrightarrow Y$ be a continuous function and $U \subset X$, any subset. Suppose, there is some closed point $y \in Y$ such that $f(u) = y$ for all $u \in U$. Then, $f(u') = y$ for all $u' \in \overline{U}$.*

Proof. $f^{-1}(\{y\})$ is a closed subset of X containing U. Hence, $\overline{U} \subset f^{-1}(\{y\})$. $\square$

Lemma 1.6. *Let $f : Y \longrightarrow X$ be a continuous function and $B \subset Y$. Then, $\overline{f(\overline{A})} = \overline{f(A)}$.*

Proof. $A \subset \overline{A} \implies f(A) \subset f(\overline{A}) \implies \overline{f(A)} \subset \overline{f(\overline{A})}$. On the other hand, $A \subset f^{-1}(f(A)) \subset f^{-1}(\overline{f(A)})$. By continuity of f, $f^{-1}(\overline{f(A)})$ is closed and hence contains $\overline{A}$. Thus, $f(\overline{A}) \subset f(f^{-1}(\overline{f(A)})) \subset \overline{f(A)}$ and hence, $\overline{f(\overline{A})} \subset \overline{f(A)}$. $\square$

Lemma 1.7. *If a space X has an open cover $\{U_i\}_{i \in I}$ such that each U_i is T_1, then X is T_1.*

Proof. Let $x, y \in X$ with $x \neq y$. Then, $x \in U_i, y \in U_j$ for some $i, j \in I$. Then, if $y \in U_i$ as well, there are open neighbourhoods $x \in W_x \subset U_i, y \in W_y \subset U_i$, such that $x \notin W_y, y \notin W_x$, and W_x, W_y are open in X by corollary 1.1. If $x \in U_j$, a similar argument provides open neighbourhoods $x \in W'_x \subset U_j, y \in W'_y \subset U_j$ such that $x \notin W'_y, y \notin W'_x$. The only remaining case is $x \notin U_j$, $y \notin U_i$. In every case, we have open neighbourhoods of x and y not containing the other. Thus, X is T_1. $\square$

1.2 Irreducible Subsets

Definition 1.8. A topological space X is called irreducible if and only if X is non-empty and $X = C \cup D$ for closed subsets C, D of X implies that $X = C$ or $X = D$. A subset of a topological space X is said to be irreducible if it is irreducible as a topological space with respect to the subspace topology.

Corollary 1.9. *If X is a topological space and $U \subset X$ is any subset of X, then a subset $C \subset U \subset X$ is an irreducible subset of U if and only if it is an irreducible subset of X.*

Proof. Apply lemma 1.3. $\square$

Lemma 1.10. *If X is an irreducible topological space and $U \subset X$ is open, then U is either empty or irreducible and dense.*

Proof. Let $U = A \cup B$ for closed subsets A, B of U. Then, $A = A' \cap U$, $B = B' \cap U$ for closed subsets A', B' of X respectively, so that $U = (A' \cup B') \cap U \subset A' \cup B'$. Then, $X = (X \setminus U) \cup (A' \cup B')$, where both $X \setminus U$ and $A' \cup B'$ are closed. Since, X is irreducible, we have $X = X \setminus U$ in which case U must be empty or $X = A' \cup B'$, in which case we can use the irreducibility of X again to deduce that $X = A'$ or $X = B'$. If $X = A'$, then $U = X \cap U = A' \cap U = A$ and, similarly, if $X = B'$, then $U = B$.

Then, $X = \overline{U} \cup (X \setminus U)$ implies that $X = \overline{U}$ since U is non-empty. This means that U is dense. $\square$

Lemma 1.11. *Let X be a topological space and C, D two closed subsets of X. If $A \subset C \cup D \subset X$ is irreducible, then $A \subset C$ or $A \subset D$.*

Proof. $A = A \cap (C \cup D) = (A \cap C) \cup (A \cap D)$, where $A \cap C$ and $A \cap D$ are closed in A. Thus, $A = A \cap C$ or $A = A \cap D$ which implies that $A \subset C$ or $A \subset D$. $\square$

Lemma 1.12. *Let $f : Y \longrightarrow X$ be a continuous map. Then, if $C \subset Y$ is irreducible, $f(C)$ is irreducible.*

Proof. Suppose $f(C) = D \cup E$ for closed subsets D, E of X. Then, $f^{-1}(D)$ and $f^{-1}(E)$ are close subsets of Y so that $f^{-1}(D) \cap C$ and $f^{-1}(E) \cap D$ are closed in C. Furthermore, $C \subset f^{-1}(f(C)) = f^{-1}(D \cup E) = f^{-1}(D) \cup f^{-1}(E)$. This implies that $(f^{-1}(D) \cap C) \cup (f^{-1}(E) \cap C) = C$. Since C is irreducible, we must have $C = f^{-1}(D) \cap C$ or $C = f^{-1}(E) \cap C$. In the first case $f(C) = f(f^{-1}(D) \cap C) = f(f^{-1}(D)) \cap f(C) \subset D \cap f(C) = D$. However, since $D \subset f(C)$, we have $f(C) = D$. In the second case, we have $f(C) = E$. $\square$

Lemma 1.13. *If U is an irreducible subset of a topological space X, then $\overline{U}$ is also irreducible.*

Proof. Suppose $\overline{U} = C \cup D$ for closed subsets C, D of X. Then, $U = U \cap \overline{U} = (U \cap C) \cup (U \cap D)$, where $U \cap C$ and $U \cap D$ are closed in U. Thus, $U = U \cap C$ or $U = U \cap D$. If $U = U \cap C$, then $U \subset C$ and $\overline{U} \subset C$ by definition of closure. Since $C \subset \overline{U}$ by hypothesis, we have $\overline{U} = C$. In the second case, we have $\overline{U} = D$. $\square$

Lemma 1.14. *Let X be a topological space and $x \in X$. Then $\{x\}$ is an irreducible subset of X.*

Proof. If $\{x\} = C \cup D$ for any sets C and D, then $x \in C$ or $x \in D$. This implies $\{x\} \subset C$ or $\{x\} \subset D$. Thus, $\{x\} = C$ or $\{x\} = D$. $\square$

Construction 1.15. Let (X, T) be a topological space where X is a set and T, is a topology on X. Define:

$$t(X) = \{C \ : \ C \text{ is an irreducible closed subset of } X\}$$

$$t(T) := \{t(C) \ : \ C \text{ is a closed subset of } X\}$$

$$\alpha_X : X \longrightarrow \mathcal{P}(X) : x \longmapsto \overline{\{x\}}$$

Let $f : Y \longrightarrow X$ be a continuous map of topological spaces. Define:

$$t(f) : t(Y) \longrightarrow \mathcal{P}(X) : C \longmapsto \overline{\{f(C)\}}$$

Lemma 1.16. *In the context of construction 1.15, for any closed $C \subset X$, $t(C) \subset t(X)$.*

Proof. Let $C' \in t(C)$. Then, C' is closed in X by corollary 1.1 and an irreducible subset of X by corollary 1.9. $\square$

Lemma 1.17. *In the context of construction 1.15 and lemma 1.23, Given closed sets $D \subset C \subset X$, we have $t(D) \subset t(C) \subset t(X)$. Given open sets $V \subset U \subset X$, we have $t(X) \setminus t(X \setminus V) \subset t(X) \setminus t(X \setminus U) \subset t(X)$.*

Proof. For the first part, we simply repeat lemma 1.16. For the second part, we observe that $V \subset U$ implies that $X \setminus V \supset X \setminus U$ which are both closed. We then utilize the first part to get: $t(X \setminus V) \supset t(X \setminus U)$ which then implies that $t(X) \setminus t(X \setminus V) \subset t(X) \setminus t(X \setminus U) \subset t(X)$. $\square$

Lemma 1.18. *In the context of construction 1.15, $t(\varnothing) \in t(T)$ and $t(\varnothing) = \varnothing$.*

Proof. $\varnothing$ is closed in X so that $t(\varnothing) \in t(T)$ by definition. Then, $\varnothing$ is not irreducible by definition so that the only subset of $\varnothing$, which is itself, is not irreducible. Thus, $t(\varnothing) = \varnothing$. $\square$

Lemma 1.19. *In the context of construction 1.15, $t(X) \in t(T)$.*

Proof. It suffices to observe that X is closed in X. $\square$

Lemma 1.20. *In the context of construction 1.15, for two closed subsets C, D of X, we have $t(C) \cup t(D) = t(C \cup D)$.*

Proof. Let $A \in t(C) \cup t(D)$. Then A is an irreducible closed subset of C or of D. In either case, $A \subset W \subset C \cup D$, where $W = C$ or $W = D$. By corollary 1.1, A is closed in $C \cup D$ and corollary 1.9, A is irreducible in $C \cup D$. Thus, $A \in t(C \cup D)$.

Let $A \in t(C \cup D)$. Then A is an irreducible closed subset of $C \cup D$. Hence, we have $A \subset C$ or $A \subset D$ by lemma 1.11. Thus, we have that A is closed and irreducible in C or in D by corollary 1.1 and corollary 1.9. Thus, $A \in t(C)$ or $A \in t(D)$ which means $A \in t(C) \cup t(D)$. $\square$

Lemma 1.21. *In the context of construction 1.15, let $\{C_i\}_{i\in I}$ be a collection of closed subsets of X. Then, $t\big(\bigcap_{i\in I} C_i\big) = \bigcap_{i\in I} t(C_i)$.*

Proof. Let $A \in t\big(\bigcap_{i\in I} C_i\big)$. Then, A is a closed subset of $\bigcap_{i\in I} C_i \subset C_j$ by corollary 1.1 and it is an irreducible subset of C_j by corollary 1.9 for all $j \in I$. Thus, $A \in t(C_j)$ for all $j \in I$ and hence $A \in \bigcap_{i\in I} t(C_i)$.

Let $A \in \bigcup_{i\in I} t(C_i)$. Then, A is an irreducible closed subset of C_j for all $j \in I$. In particular, $A \subset \bigcap_{i\in I} C_i$. Then, for any choice of $j \in I$, the fact $A \subset \bigcap_{i\in I} C_i \subset C_j$ along with corollary 1.1 and corollary 1.9 shows that A is an irreducible closed subset of $\bigcap_{i\in I} C_i$. Thus, $A \in t\big(\bigcap_{i\in I} C_i\big)$. $\square$

Theorem 1.22. *In the context of construction 1.15 and lemma 1.16, taking $t(T)$ to be the set of closed subsets of $t(X)$ gives a topology on $t(X)$.*

Proof. Apply lemmas 1.18, 1.19, 1.20 and 1.21. $\square$

Lemma 1.23. *In the context of construction 1.15, $\operatorname{im}(\alpha_X) \subset t(X) \subset \mathcal{P}(X)$ and $\operatorname{im}(t(f)) \subset t(X) \subset \mathcal{P}(X)$.*

Proof. Combine lemma 1.14 and lemma 1.13 for the first containment, and lemma 1.12 and lemma 1.13 for the second. $\square$

Lemma 1.24. *In the context of construction 1.15 and lemma 1.23, the following diagram commutes:*

$$\begin{tikzcd} Y \arrow[r, "\alpha_Y"] \arrow[d, "f"'] & t(Y) \arrow[d, "t(f)"] \\ X \arrow[r, "\alpha_X"'] & t(X) \end{tikzcd}$$

Proof. We observe that for closed $y \in Y$ $\alpha_X(f(y)) = \overline{\{f(y)\}} = \overline{f(\{y\})}$. On the other hand, $t(f)(\alpha_Y(y)) = \overline{f(\overline{\{y\}})}$. We get the equality of these sets by applying lemma 1.6 with $A = \{y\}$. $\square$

Lemma 1.25. *In the context of construction 1.15 and lemma 1.23, α_X is injective if and only if X is Kolmogorov (T_0).*

Proof. Suppose α_X is injective. Let $x, y \in X$ with $x \neq y$. Then, $\overline{\{x\}} = \alpha_X(x) \neq \alpha_X(y) = \overline{\{y\}}$. Then, either $\overline{\{x\}} \not\subset \overline{\{y\}}$ or $\overline{\{y\}} \not\subset \overline{\{x\}}$. In the first case, there exists $w \in \overline{\{x\}}$ such that $w \notin \overline{\{y\}}$. Then, either $w = x$ or w is a limit point of $\{x\}$. If $w = x$, then $x = w \in X \setminus \overline{\{y\}}$. If w is a limit point of $\{x\}$, then $X \setminus \overline{\{y\}}$ is an open neighbourhood of w and hence must contain x. That is, in either case, $x \in X \setminus \overline{\{y\}}$ and $y \notin X \setminus \overline{\{y\}}$. When $\overline{\{y\}} \not\subset \overline{\{x\}}$, the same argument shows that $y \in X \setminus \overline{\{x\}}$ and $x \notin X \setminus \overline{\{x\}}$. Since, this applies for any $x, y \in X$ with $x \neq y$, X is Kolmogorov.

Now, suppose X is Kolmogorov. Suppose, for $x, y \in X$, $\alpha_X(x) = \alpha_X(y)$. This means $\overline{\{x\}} = \overline{\{y\}}$. If $x \neq y$, then there is a neighbourhood U of x such that $y \notin U$ or there is a neighbourhood V of y with $x \notin U$. In the first case, $y \in X \setminus U$, which is closed, so that $\overline{\{y\}} \subset X \setminus U$. However, since $x \in U$, x cannot be in $\overline{\{y\}}$ which contradicts the hypothesis that $\overline{\{x\}} = \overline{\{y\}}$. In the second case, we reach a similar contradiction. Thus, we must have $x = y$. Since this holds for all $x, y \in X$, α_X is injective. ☐

Lemma 1.26. *In the context of construction 1.15 and lemma 1.23, For any closed $C \subset X$, we have $\alpha_X(C) = \alpha_C(C) \subset t(C)$. In particular, $\alpha_X(C) = \alpha_X(C) \cap t(C)$.*

Proof. Let $\iota : C \hookrightarrow X$ be the subset inclusion. By lemma 1.24, we have $\alpha_X \circ \iota = t(\iota) \circ \alpha_C$. Then, $t(\iota)(D) = \overline{\iota(D)} = \overline{D} = D$, for all $D \in t(C)$, since D is closed in X by corollary 1.1. Thus,

$$\alpha_X(C) = \alpha_X(\iota(C)) = t(\iota)(\alpha_C(C)) = \alpha_C(C)$$

☐

Lemma 1.27. *In the context of construction 1.15 and lemma 1.23, for a closed set $C \subset X$, we have $\alpha_X^{-1}(t(C)) = C$. For every open $U \subset X$, we have $\alpha_X^{-1}(t(X) \setminus t(X \setminus U)) = U$.*

Proof. Let $x \in C$. Then, $\alpha_X(x) = \overline{\{x\}} \subset C$ as $\{x\} \subset C$. By corollary 1.1 and corollary 1.9, $\alpha_X(x) = \overline{\{x\}} \in t(C)$. Suppose now that $x \in \alpha_X^{-1}(t(C))$. Then, $\alpha_X(x) = \overline{\{x\}} \in t(C)$. In particular, $\{x\} \subset \overline{\{x\}} \subset C$, so that $x \in C$.

In particular, $\alpha_X^{-1}(t(X)) = X$ and $\alpha_X^{-1}(t(X \setminus U)) = X \setminus U$. Thus,

$$\alpha_X^{-1}(t(X) \setminus t(X \setminus U)) = \alpha_X^{-1}(t(X)) \setminus \alpha_X^{-1}(t(X \setminus U)) = X \setminus (X \setminus U) = U$$

☐

Lemma 1.28. *In the context of construction 1.15 and lemma 1.23, for each $C \in t(X)$, we have $\overline{\{C\}} = t(C) \subset t(X)$.*

Proof. Let $D \subset X$ be a closed subset of X such that such that $\{C\} \subset t(D) \subset t(X)$. Then, $C \in t(D)$ implies that $C \subset D$ is irreducible and closed in D. Now, for any $C' \in t(C)$, C' is irreducible and closed in D as well by corollary 1.9 and corollary 1.1. Thus, $C' \in t(D)$. Hence, $t(C) \subset t(D)$. Since every closed subset of $t(X)$ is of the form $t(D)$ for some closed subset $D \subset X$, we have shown that $t(C)$ is smallest closed subset of $t(X)$ containing $\{C\}$. Thus, $t(C) = \overline{\{C\}}$. ☐

Lemma 1.29. *In the context of construction 1.15 and lemma 1.23, let $t(X)^{\mathrm{cl}}$ be the sets of closed points of $t(X)$. Then, for each $C \in t(X)^{\mathrm{cl}}$ there exists $c \in C$ such that $C = \alpha_X(c)$. In particular, $t(X)^{\mathrm{cl}} \subset \mathrm{im}(\alpha_X)$.*

Proof. Let $C \in t(X)$ be a closed point. Then, C is non-empty by irreducibility. Let $c \in C$. We have $\alpha_X(c) = \overline{\{c\}} \subset C$. By corollary 1.9 and corollary 1.1, $\alpha_X(c) \in t(C)$. However, by lemma 1.28 and hypothesis, $t(C) = \overline{\{C\}} = \{C\}$. Thus, $\alpha_X(c) = C$. ☐

Lemma 1.30. *In the context of construction 1.15 and lemma 1.23, let* $X^{\mathrm{cl}}, t(X)^{\mathrm{cl}}$ *denote the set of closed points of* $X, t(X)$ *respectively. Then, we have* $\alpha_X(X^{\mathrm{cl}}) \subset t(X)^{\mathrm{cl}}$. *If* X *is Kolmogorov* (T_0)*, in addition, then we also have:* $\alpha_X^{-1}(t(X)^{\mathrm{cl}}) \subset X^{\mathrm{cl}}$.

Proof. If $x \in X$ is a closed point, we observe that by lemma 1.28,

$$\overline{\{\alpha_X(x)\}} = t(\alpha_X(x)) = t\left(\overline{\{x\}}\right) = t(\{x\}) = \{\{x\}\} = \left\{\overline{\{x\}}\right\} = \{\alpha_X(x)\}$$

Now, suppose X is Kolmogorov. Let $x \in X$ such that $\alpha_X(x)$ is a closed point of X. Now, suppose $x' \in \overline{\{x\}}$. Then, $\{x'\} \subset \overline{\{x\}}$ and hence $\overline{\{x'\}} \subset \overline{\{x\}}$. However, $\overline{\{x'\}} = \alpha_X(x') \in t(X)$ by lemma 1.23 and by corollary 1.9 and corollary 1.1, $\overline{\{x'\}} = \alpha_X(x') \in t\left(\overline{\{x\}}\right) \subset t(X)$. However, by hypothesis and lemma 1.28, $t\left(\overline{\{x\}}\right) = \overline{\{\alpha_X(x)\}} = \{\alpha_X(x)\}$. Thus, $\alpha_X(x') = \alpha_X(x)$. By lemma 1.25, we have $x' = x$. Since x' was arbitrary, we have $\overline{\{x\}} = \{x\}$. □

Theorem 1.31. *In the context of construction 1.15 and lemma 1.23, let* $X^{\mathrm{cl}}, t(X)^{\mathrm{cl}}$ *be the sets of closed points of* $X, t(X)$ *respectively. By lemma 1.30,* $\mathrm{im}(\alpha_X|_{X^{\mathrm{cl}}}) \subset t(X)^{\mathrm{cl}}$*, giving us a map:* $\alpha_X^{\mathrm{cl}} = \alpha_X|_{X^{\mathrm{cl}}} : X^{\mathrm{cl}} \longrightarrow t(X)^{\mathrm{cl}}$. *If* X *is Kolmogorov* (T_0)*, this is a bijection. If* X *is* T_1*, in addition, then* $X = X^{\mathrm{cl}}$ *and we have that* α_X *is a bijection of* X *onto* $t(X)^{\mathrm{cl}}$.

Proof. The fact that this is an injection follows from lemma 1.25. Let $C \in t(X)^{\mathrm{cl}}$. Then, $C = \alpha_X(c)$ for some $c \in C \subset X$ by lemma 1.29. Then, by *lemma* 1.30, since $c \in \alpha_X^{-1}(t(X)^{\mathrm{cl}})$, we must have that $c \in X^{\mathrm{cl}}$. □

Theorem 1.32. *In the context of construction 1.15 and lemma 1.23,* α_X *is continuous and sends a closed set of its domain to a closed set of its image. If* X *is Kolmogorov* (T_0)*, in addition, then* α_X *is a homeomorphism onto its image. If* X *is* T_1*, in addition, then* α_X *is a homeomorphism onto* $t(X)^{\mathrm{cl}}$.

Proof. For continuity, it suffices to observe that every closed subset of $t(X)$ is of the form $t(C)$ for some closed $C \subset X$, and then apply lemma 1.27. We then observe that $\alpha_X(C) = \alpha_X(C) \cap t(C)$ by lemma 1.26, which is closed in $\alpha_X(C)$ since $t(C) \in t(T)$ is a closed subset of $t(X)$ by theorem 1.22. If X is Kolmogorov, α_X is, in addition, injective by lemma 1.25, which together with the previous facts imply that it is a homeomorphism on to its image. If X is T_1, this image is $t(X)^{\mathrm{cl}}$ by theorem 1.31. □

Theorem 1.33. *In the context of construction 1.15 and lemma 1.23, We have order-preserving functions:*

$$t_{\mathrm{Op}} : \mathrm{Op}(X) \longrightarrow \mathrm{Op}(t(X)) : U \longmapsto t(X) \setminus t(X \setminus U)$$

$$\alpha_X^{-1} : \mathrm{Op}(t(X)) \longrightarrow \mathrm{Op}(X) : V \longmapsto \alpha_X^{-1}(V)$$

that are inverses to each other. We also have another pair of order-preserving mappings:

$$t_{\mathrm{Cl}} : \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(t(X)) : C \longmapsto t(C)$$

$$\alpha_X^{-1} : \mathrm{Cl}(t(X)) \longrightarrow \mathrm{Cl}(X) : D \longmapsto \alpha_X^{-1}(D)$$

that are again inverses to each other.

Proof. That the mappings $t_{\mathrm{Op}}, t_{\mathrm{Cl}}$ are order-preserving is lemma 1.17. α_X^{-1} is order-preserving in each case because taking preimages preserves containment.

Consider any closed $C \subset X$. We have $\alpha_X^{-1}(t_{\mathrm{Cl}}(C)) = \alpha_X^{-1}(t(C)) = C$ by lemma 1.27. Now, consider a closed $D \subset t(X)$. Then, by definition, $D = t(D')$ for some closed $D' \subset X$, so that $t_{\mathrm{Cl}}(\alpha_X^{-1}(D)) = t(\alpha_X^{-1}(t(D'))) = t(D') = D$ by lemma 1.27 again.

Let $U \subset X$ be any open subset. Then, $\alpha_X^{-1}(t_{\mathrm{Op}}(U)) = \alpha_X^{-1}(t(X) \setminus t(X \setminus U)) = U$ by lemma 1.27. Then, consider any open $V \subset t(X)$. Since $t(X) \setminus V$ is closed, it must be equal to $t(C)$ for some closed $C \subset X$ so that $V = t(X) \setminus t(C)$. Now, $C = X \setminus (X \setminus C)$, where $U = X \setminus C$ is open in X. Thus, $V = t(X) \setminus t(X \setminus U)$ and $t_{\mathrm{Op}}(\alpha_X^{-1}(V)) = t_{\mathrm{Op}}(U) = t(X) \setminus t(X \setminus U) = V$ using lemma 1.27. □

Lemma 1.34. *In the context of theorem 1.33, for any open $W \subset X$ and $C \in t(X)$, $C \in t_{\mathrm{Op}}(W)$ if and only if $C \cap W \neq \varnothing$.*

Proof. $C \in t_{\mathrm{Op}}(W) = t(X) \setminus t(X \setminus W)$ if and only if C is not an irreducible and closed subset of $X \setminus W$. C is irreducible and closed in $X \setminus W$ if and only if C is contained in $X \setminus W$, for if it is contained in $X \setminus W$, which is closed, C is irreducible and closed in it by corollary 1.9 and corollary 1.1; and, if it is not contained in $X \setminus W$, it cannot possibly be an irreducible and closed subset of $X \setminus W$. Therefore, □

Lemma 1.35. *In the context of theorem 1.33, if $\{U_i\}_{i \in I}$ is a collection of open subsets of X, then $t\big(\bigcup_{i \in I} U_i\big) = \bigcup_{i \in I} t(U_i)$.*

Proof.

$$\begin{aligned}
t_{\mathrm{Op}}\left(\bigcup_{i \in I} U_i\right) &= t(X) \setminus t\left(X \setminus \left(\bigcup_{i \in I} U_i\right)\right) \\
&= t(X) \setminus t\left(\bigcap_{i \in I} X \setminus U_i\right) \\
&= t(X) \setminus \left(\bigcap_{i \in I} t(X \setminus U_i)\right) && \text{lemma 1.21} \\
&= \bigcup_{i \in I} t(X) \setminus (t(X \setminus U_i)) \\
&= \bigcup_{i \in I} t_{\mathrm{Op}}(U_i)
\end{aligned}$$

□

Lemma 1.36. *In the context of theorem 1.33, if U, U' are open subsets of X, then $t(U \cap U') = t(U) \cap t(U')$.*

Proof.

$$\begin{aligned}
t_{\mathrm{Op}}(U \cap U') =& t(X) \setminus t(X \setminus (U \cap U')) \\
=& t(X) \setminus t((X \setminus U) \cup (X \setminus U')) \\
=& t(X) \setminus (t(X \setminus U) \cup t(X \setminus U')) && \text{lemma 1.20} \\
=& (t(X) \setminus t(X \setminus U)) \cap (t(X) \setminus t(X \setminus U')) \\
=& t_{\mathrm{Op}}(U) \cap t_{\mathrm{Op}}(U')
\end{aligned}$$

□

Theorem 1.37. *In the context of theorem 1.33, if $\{U_i\}_{i \in I}$ is an open cover of X, then $\{t_{\mathrm{Op}}(U_i)\}_{i \in I}$ is an open cover of $t(X)$.*

Proof. Apply lemma 1.35. □

Lemma 1.38. *In the context of construction 1.15 and lemma 1.16, for any closed $D \subset Y$, we have $t(f)(t(D)) \subset t(t(f)(D))$.*

Proof. Let $D' \in t(D)$. Since $D' \subset D$, we have $f(D') \subset f(D)$ and $t(f)(D') = \overline{f(D')} \subset \overline{f(D)} = t(f)(D)$. Then, $t(f)(D'), t(f)(D) \in t(X)$ by lemma 1.23 and are hence irreducible closed subset of X. By corollary 1.9 and corollary 1.1, $t(f)(D')$ is an irreducible closed subset of $t(f)(D)$. Thus, $t(f)(D') \in t(t(f)(D))$. □

Lemma 1.39. *In the context of construction 1.15 and lemma 1.23, for any closed $C \subset X$, we have $t(f)^{-1}(t(C)) = t(f^{-1}(C))$.*

Proof. Let $W \in t(f)^{-1}(t(C))$. Then, $t(f)(W) = \overline{f(W)} \in t(C)$. In particular, $f(W) \subset \overline{f(W)} \subset C$. This implies $W \subset f^{-1}(f(W)) \subset f^{-1}(C)$. Since $W \in t(X)$ is an irreducible closed subset of X, it must also be an irreducible closed subset of $f^{-1}(C)$ by corollary 1.9 and corollary 1.1. Thus, $W \in t(f^{-1}(C))$.

Now, suppose $W \in t(f^{-1}(C))$. Then, W is an irreducible closed subset of $f^{-1}(C)$ and hence $f(W) \subset C$. Since C is closed, we must have $t(f)(W) = \overline{f(W)} \subset C$. □

Theorem 1.40. *In the context of construction 1.15 and lemma 1.23, the function*

$$t(f) : t(Y) \longrightarrow t(X)$$

is continuous.

Proof. Let $t(C) \in t(T)$ be a closed subset of $t(X)$ for some closed $C \subset X$. Then, $t(f)^{-1}(t(C)) = t(f^{-1}(C))$ by lemma 1.39, which is closed by definition, since $f^{-1}(C)$ is closed, as C is closed and f^{-1} is continuous. □

Lemma 1.41. *In the context of construction 1.15 and lemma 1.23, for two functions $Z \xrightarrow{g} Y \xrightarrow{f} X$, we have: $t(f \circ g) = t(f) \circ t(g)$.*

Proof. For $C \in t(Z)$,

$$\begin{aligned}
& t(f \circ g)(C) \\
=& \overline{f(g(C))} \\
=& \overline{f(\overline{g(C)})} && \text{lemma 1.6} \\
=& \overline{f(t(g)(C))} \\
=& t(f)(t(g)(C))
\end{aligned}$$

□

Lemma 1.42. *In the context of construction 1.15 and lemma 1.23, we have:* $t(\mathrm{id}_X) = \mathrm{id}_{t(X)}$.

Proof. For $C \in t(X)$,

$$t(\mathrm{id}_X)(C) = \overline{C} = C \in t(X)$$

□

Theorem 1.43. *The mapping:*

$$\begin{array}{cccccc}
t & : & \mathrm{Top} & \longrightarrow & \mathrm{Top} \\
 & : & X & \longmapsto & t(X) \\
 & : & (f : Y \longrightarrow X) & \longmapsto & (t(f) : t(Y) \longrightarrow t(X))
\end{array}$$

from construction 1.15 is a functor with a natural transformation:

$$\alpha : \mathrm{id}_{\mathrm{Top}} \Longrightarrow t$$

whose components are the α_X.

Proof. Apply theorem 1.40 along with lemmas 1.41 and 1.42 to deduce that t is a functor $\mathrm{Top} \longrightarrow \mathrm{Top}$. Then, apply theorem 1.32 and lemma 1.24 to deduce that α is natural. □

Theorem 1.44. *In the context of theorem 1.43, if X is Kolmogorov then the function $t : \mathrm{Top}(Y, X) \longrightarrow \mathrm{Top}(t(Y), t(X))$ is injective. In particular, the functor $t : \mathrm{Top} \longrightarrow \mathrm{Top}$ restricted to the full subcategory of* Top *consisting of the Kolmogorov (T_0) spaces, is faithful.*

Proof. Let $f, g : Y \longrightarrow X$ be two continuous maps such that $t(f) = t(g)$. The naturality of α shows that for any $y \in Y$:

$$\alpha_X(f(y)) = t(f)(\alpha_Y(y)) = t(g)(\alpha_Y(y)) = \alpha_X(g(y))$$

Then, since X is Kolmogorov, α_X is injective by lemma 1.25. This means $f(y) = g(y)$. Since y was arbitrary, we have $f = g$. □

Definition 1.45. We will call the functor $t : \mathrm{Top} \longrightarrow \mathrm{Top}$ of theorem 1.43 the schematization functor, and the natural transformation $\alpha : \mathrm{id}_{\mathrm{Top}} \Longrightarrow t$, the natural immersion into schematization.

Lemma 1.46. *In the context of theorem 1.33, if $\iota : U \hookrightarrow X$ is the inclusion of a non-empty open subset, $t(\iota) : t(U) \longrightarrow t(X)$ restricts to a map $t(\iota) : t(U) \longrightarrow t_{\mathrm{Op}}(U)$, which is also continuous.*

Proof. Let $C \in t(U)$. Then, $t(\iota)(C) = \overline{\iota(C)} = \overline{C}$, where the closure is taken in X. Since, C is irreducible in U, C must be non-empty, and hence $\overline{C} \cap U$ must also be non-empty since it contains $C \cap U = C \neq \varnothing$. Thus, $t(\iota)(C) = \overline{C} \notin t(X \setminus U)$ but it is in $t(X)$ by lemma 1.23. Hence, $t(\iota)(C) \in t(X) \setminus t(X \setminus U) = t_{\mathrm{Op}}(U)$. Of course, codomain restrictions of continuous maps are again continuous. $\square$

Lemma 1.47. *In the context of theorem 1.33, for a non-empty open $U \subset X$, the function*

$$- \cap U : \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) : C \longmapsto C \cap U$$

restricts to a function $- \cap U : t_{\mathrm{Op}}(U) \longrightarrow t(U)$.

Proof. Let $C \in t_{\mathrm{Op}}(U)$ so that $C \in t(X)$ but not in $t(X \setminus U)$. In particular, $C \cap U \neq \varnothing$ and is closed in U. Then, $C \cap U$ is a non-empty open subset of the irreducible set C and hence must be irreducible in C by lemma 1.10. By corollary 1.9, it must be irreducible in X. By corollary 1.9 in the reverse direction, it must be irreducible in U. Thus, $C \cap U \subset t(U)$. $\square$

Lemma 1.48. *The functions $t(\iota) : t(U) \leftrightarrows t_{\mathrm{Op}}(U) : - \cap U$ lemma 1.46 and lemma 1.47 are inverses to each other.*

Proof. For any $C \in t(U)$, $t(\iota(C)) \cap U = \overline{C} \cap U = C$ by lemma 1.2.

Now, suppose $C \in t_{\mathrm{Op}}(U)$. Then, $t(\iota)(C \cap U) = \overline{C \cap U}$. However, $C = \overline{C \cap U} \cup (C \setminus (C \cap U))$, where the first operand of the union is a closed subset of C by corollary 1.1 and the second operand is a closed subset of C since $C \cap U$ is open in C. Since $C \in t_{\mathrm{Op}}(U) \subset t(X)$, C is irreducible and hence, we must have $C = \overline{C \cap U}$ or $C = C \setminus (C \cap U)$. In the latter case, $C \cap U$ must be empty since U is non-empty by hypothesis and $C \in t(X)$ is non-empty by irreducibility. However, this means $C \subset X \setminus U$ in which case C is also irreducible and closed in $X \setminus U$ by corollary 1.9 and corollary 1.1. This contradicts the assumption that $C \in t_{\mathrm{Op}}(U) = t(X) \setminus t(X \setminus U)$. Hence, we must have $C = \overline{C \cap U} = t(\iota)(C \cap U)$. $\square$

Lemma 1.49. *The function $- \cap U : t_{\mathrm{Op}}(U) \longrightarrow t(U)$ of lemma 1.47 is continuous.*

Proof. A closed subset of $t(U)$ is of the form $t(C)$ for some closed subset $C \subset U$. We observe that $(- \cap U)^{-1}(t(C)) = t(\iota)(C) \subset t(t(\iota)(C))$ by lemma 1.48 and lemma 1.23. Now, consider $D \in t(t(\iota)(C))$. Then, $D \subset t(\iota)(C) = \overline{C}$, and $D \cap U \subset \overline{C} \cap U = C$ by lemma 1.2. However, $D \cap U \in t(U)$ by lemma 1.47 and hence $D \cap U \in t(C)$ by corollary 1.9. That is, $Din(- \cap U)^{-1}(t(C))$. Thus, we have shown that $(- \cap U)^{-1}(t(C)) \subset t(t(\iota)(C))$. Hence, $(- \cap U)^{-1}(t(C)) = t(t(\iota)(C))$. $\square$

Theorem 1.50. *The functions $t(\iota) : t(U) \leftrightarrows t_{\mathrm{Op}}(U) : - \cap U$ are continuous inverses of each other and hence each is a homeomorphism.*

1.3 Ringed Spaces

Corollary 1.51. *In the context of construction 1.15 and lemma 1.23, the following diagram commutes strictly:*

$$\begin{tikzcd} \mathrm{Sh}(Y) \arrow[r, "\alpha_{Y,*}"] \arrow[d, "f_*"'] & \mathrm{Sh}(t(Y)) \arrow[d, "t(f)_*"] \\ \mathrm{Sh}(X) \arrow[r, "\alpha_{X,*}"'] & \mathrm{Sh}(t(X)) \end{tikzcd}$$

Proof. This follows from lemma 1.24 and the observation that the pushforward operation $(-)_*$ commutes with composition. $\square$

Theorem 1.52. *In the context of construction 1.15 and lemma 1.23, the pushforward functor:*

$$\alpha_{X,*} : \mathrm{Sh}(X) \longrightarrow \mathrm{Sh}(t(X))$$

is an isomorphism (not just an equivalence) of categories.

Proof. By theorem 1.33, $\alpha_X^{-1} : \mathrm{Op}(X) \longrightarrow \mathrm{Op}(t(X))$ is an isomorphism of posets and this gives an isomorphism of categories $(\alpha_X^{-1})^{\mathrm{op}} : \mathrm{Op}(X)^{\mathrm{op}} \longrightarrow \mathrm{Op}(t(X))^{\mathrm{op}}$. This, in turn means that the pushforward functor

$$\alpha_{X,*} : \mathrm{PSh}(X) = \mathrm{Fun}(\mathrm{Op}(X)^{\mathrm{op}}, \mathrm{Set}) \longrightarrow \mathrm{Fun}(\mathrm{Op}(t(X))^{\mathrm{op}}, \mathrm{Set}) = \mathrm{PSh}(t(X))$$

is an isomorphism of categories. Then, the pushforward of sheaves is given by restricting the domain and codomain of $\alpha_{X,*} : \mathrm{PSh}(X) \longrightarrow \mathrm{PSh}(t(X))$. This shows that $\alpha_{X,*} : \mathrm{Sh}(X) \longrightarrow \mathrm{Sh}(t(X))$ is fully faithful. We would like to show that it is strictly surjective.

Let $F : \mathrm{Op}(t(X))^{\mathrm{op}} \longrightarrow \mathrm{Set}$ be a sheaf on $t(X)$. Then, we observe that $F = F \circ t_{\mathrm{Op}}^{\mathrm{op}} \circ (\alpha_X^{-1})^{\mathrm{op}} = \alpha_{X,*}(F \circ t_{\mathrm{Op}}^{\mathrm{op}})$, where the equalities are strict, since t_{Op} and α_X^{-1} are poset morphisms strictly inverse to each other. Thus, it suffices to show that $F \circ t_{\mathrm{Op}}^{\mathrm{op}} : \mathrm{Op}(X)^{\mathrm{op}} \longrightarrow \mathrm{Set}$ is a sheaf. Consider an open cover $\coprod_{i \in I} U_i \longrightarrow U$ of $U \in \mathrm{Op}(X)$. Then, $\coprod_{i \in I} t_{\mathrm{Op}}(U_i) \longrightarrow t_{\mathrm{Op}}(U)$ is an open cover of $t_{\mathrm{Op}}(U)$ by theorem 1.37; for each $i, j \in I$, $t_{\mathrm{Op}}(U_i) \cap t_{\mathrm{Op}}(U_j) = t_{\mathrm{Op}}(U_i \cap U_j)$ by lemma 1.36; and since F is a sheaf by assumption, the following is an equalizer diagram:

$$(F \circ t_{\mathrm{Op}}^{\mathrm{op}})(U) \longrightarrow \prod_{i \in I} (F \circ t_{\mathrm{Op}}^{\mathrm{op}})(U_i) \rightrightarrows \prod_{i,j \in I} (F \circ t_{\mathrm{Op}}^{\mathrm{op}})(U_i \cap U_j)$$

$\square$

Definition 1.53. Given a category $\mathcal{C}$ with products, we call a pair (X, E), where X is a topological space and $E : \mathrm{Op}(X)^{\mathrm{op}} \longrightarrow \mathcal{C}$ is a $\mathcal{C}$–valued sheaf, a $\mathcal{C}$–space. A morphism of $\mathcal{C}$–spaces $(Y, F) \longrightarrow (X, E)$ consists of a pair $(f, f^\sharp)$, where $f : Y \longrightarrow X$ is a continuous function and $f^\sharp : E \longrightarrow f_*(F)$ is a morphism of $\mathcal{C}$–valued sheaves. Given morphisms of $\mathcal{C}$–spaces

$$(Z, G) \xrightarrow{(g,g^\sharp)} (Y, F) \xrightarrow{(f,f^\sharp)} (X, E)$$

we define the composite to be the pair $(f \circ g, f_*(g^\sharp) \circ f^\sharp)$. One can check that $(\mathrm{id}_X, \mathrm{id}_E)$ acts as two-sided identity for this composition operation. We denote the category of $\mathcal{C}$–spaces by $\mathrm{Sh}_{\mathcal{C}}$. When $\mathcal{C} = \mathrm{Set}$, we simply write Sh.

Lemma 1.54. *Let (X, E) be a $\mathcal{C}$–space and U, an open subset of X. Then, for any $p \in U$, we have an isomorphism $E_p \cong (E|_U)_p$ in $\mathcal{C}$.*

Proof. Let $\mathrm{Op}(X)_p, \mathrm{Op}(U)_p$ denote the poset of open subsets of X and U containing p. Then, $E_p = \operatorname{colim} E|_{\mathrm{Op}(X)_p}$ and $(E|_U)_p = \operatorname{colim}(E|_U)_{\mathrm{Op}(U)_p} = \operatorname{colim} E|_{\mathrm{Op}(U)_p}$. It suffices to show that E_p is an initial cone under $E|_{\mathrm{Op}(U)_p}$. We observe that for each $W \in \mathrm{Op}(U)_p$, $W \in \mathrm{Op}(X)_p$ by corollary 1.1 so that E_p is cone under $E|_{\mathrm{Op}(U)_p}$ as well. Let $\{c_W : E(W) \longrightarrow C\}_{W \in \mathrm{Op}(U)_p}$ be another cone under $E|_{\mathrm{Op}(U)_p}$. We observe that for $S' \subset S \in \mathrm{Op}(X)_p$, the following diagram of restriction maps commutes:

$$\begin{tikzcd}
E(S) \arrow[r] \arrow[d] & E(S \cap U) \arrow[d] \\
E(S') \arrow[r] & E(S' \cap U)
\end{tikzcd}$$

Therefore, $\{-|_{S\cap U} \circ c_{S\cap U} : E(S) \longrightarrow C\}_{S \in \mathrm{Op}(X)_p}$ is a cone under $E|_{\mathrm{Op}(X)_p}$. E_p being an initial cone under $E|_{\mathrm{Op}(X)_p}$ gives a unique map of cones $\alpha : E_p \longrightarrow C$ under $E|_{\mathrm{Op}(X)_p}$. Since $\mathrm{Op}(U)_p \subset \mathrm{Op}(X)_p$, this also makes the map $\alpha : E_p \longrightarrow C$ a map of cones under $E|_{\mathrm{Op}(U)_p}$. Let $\beta : E_p \longrightarrow C$ be another map of cones under $E|_{\mathrm{Op}(U)_p}$. Then, for any $S' \subset S \in \mathrm{Op}(X)_p$ again, we have that the following diagram commutes:

$$\begin{tikzcd}
 & E(S) \arrow[dl] \arrow[dr, "-|_{S\cap U} \circ c_{S\cap U}"] & \\
E_p \arrow[rr, "\beta"'] & & C
\end{tikzcd}$$

This makes β a map of cones under $\mathrm{Op}(X)_p$ and by the uniqueness of α as a map of cones under $\mathrm{Op}(X)_p$, we must have $\alpha = \beta$. □

Construction 1.55. Let X be a topological space and $E : \mathrm{Op}(X)^{\mathrm{op}} \longrightarrow \mathrm{Set}$, a sheaf of sets on X. Then, define:

$$t(X, E) := (t(X), \alpha_{X,*}(E))$$

where $t(X), \alpha_X$ are as in construction 1.15. Now, consider a continuous map $f : Y \longrightarrow X$, a sheaf of sets F on Y, and a morphism of sheaves $f^\sharp : E \longrightarrow$

$f_*(F)$. By corollary 1.51, $\alpha_{X,*}(f_*(F)) = t(f)_*(\alpha_{Y,*}(F))$. Thus, we have a map: $\alpha_{X,*}(f^\sharp) : \alpha_{X,*}(E) \longrightarrow t(f)_*(\alpha_{Y,*}(F))$. Define:

$$t(f, f^\sharp) := (t(f), \alpha_{X,*}(f^\sharp))$$

In addition, we have a morphism of ringed spaces:

$$\alpha_{(X,E)} = (\alpha_X, \mathrm{id}_{\alpha_{X,*}(E)}) : (X, E) \longrightarrow t(X, E)$$

Lemma 1.56. *In the context of construction 1.55, the map*

$$\begin{array}{rcccc} t & : & \mathrm{Sh} & \longrightarrow & \mathrm{Sh} \\ & : & (X,E) & \longrightarrow & (t(X), \alpha_{X,*}(E)) \\ & : & (f, f^\sharp) & \longrightarrow & (t(f), \alpha_{X,*}(f^\sharp)) \end{array}$$

is a functor, and the maps $\alpha_{(X,F)} : (X, F) \longrightarrow t(X, F)$ are natural in (X, F), giving a natural transformation: $\alpha : \mathrm{id}_{\mathrm{RingedSp}} \Longrightarrow t$.

Proof. Consider a pair of composeable morphisms in Sh:

$$(Z, G) \xrightarrow{(g,g^\sharp)} (Y, F) \xrightarrow{(f,f^\sharp)} (X, E)$$

We have:

$$\begin{aligned} & t(f \circ g, f_*(g^\sharp) \circ f^\sharp) \\ =& (t(f \circ g), \alpha_{X,*}(f_*(g^\sharp) \circ f^\sharp)) \\ =& (t(f) \circ t(g), \alpha_{X,*}(f_*(g^\sharp) \circ f^\sharp)) && \text{lemma 1.41} \\ =& (t(f) \circ t(g), \alpha_{X,*}(f_*(g^\sharp)) \circ \alpha_{X,*}(f^\sharp)) \\ =& (t(f) \circ t(g), t(f)_*(\alpha_{Y,*}(g^\sharp)) \circ \alpha_{X,*}(f^\sharp)) && \text{corollary 1.51} \\ =& (t(f), \alpha_{X,*}(f^\sharp)) \circ (t(g), \alpha_{Y,*}(g^\sharp)) \end{aligned}$$

Then, we have:

$$\begin{aligned} & t(\mathrm{id}_X, \mathrm{id}_E) \\ =& (t(\mathrm{id}_X), \alpha_{X,*}(\mathrm{id}_E)) \\ =& (\mathrm{id}_{t(X)}, \alpha_{X,*}(\mathrm{id}_E)) && \text{lemma 1.42} \\ =& (\mathrm{id}_{t(X)}, \mathrm{id}_{\alpha_{X,*}(E)}) \end{aligned}$$

For naturality of $\alpha_{(X,E)}$ in (X, E), we first observe that that the map of underlying topological spaces α_X is natural in X by theorem 1.43. Then, we observe that the following diagram of sheaves of sets over X commutes:

$$\begin{tikzcd} \alpha_{X,*}(E) \arrow[r, "\alpha_{X,*}(f^\sharp)"] \arrow[d, "\mathrm{id}_{\alpha_{X,*}(E)}"'] & t(f)_*(\alpha_{Y,*}(F)) \arrow[d, "\mathrm{id}_{t(f)_*(\alpha_{Y,*}(F))}"] \\ \alpha_{X,*}(E) \arrow[r, "\alpha_{X,*}(f^\sharp)"'] & \alpha_{X,*}(f_*(F)) = t(f)_*(\alpha_{Y,*}(F)) \end{tikzcd}$$

This shows that the following diagram of ringed spaces commutes:

$$\begin{tikzcd}
(Y,F) \arrow[r, "\alpha_{Y,F}"] \arrow[d, "(f,f^\sharp)"'] & t(Y,F) = (t(Y), \alpha_{Y,*}(F)) \arrow[d, "(t(f), \alpha_{X,*}(f^\sharp))"] \\
(X,E) \arrow[r, "\alpha_{X,E}"'] & t(X,E) = (t(X), \alpha_{X,*}(E))
\end{tikzcd}$$

□

Theorem 1.57. *In the context of lemma 1.56, the functor t restricted to the full subcategory of* Sh *consisting of those pairs (X, E) where X is Kolmogorov (T_0) is faithful.*

Proof. Consider two maps of Set–spaces (with Kolmogorov underlying spaces) $(f, f^\sharp), (g, g^\sharp) : (Y, F) \longrightarrow (X, E)$ such that $t(f, f^\sharp) = t(g, g^\sharp)$. Then, $t(f) = t(g)$ and $\alpha_{X,*}(f^\sharp) = \alpha_{X,*}(g^\sharp)$. Then, theorem 1.44 implies $f = g$ and theorem 1.52 implies $f^\sharp = g^\sharp$. □

Lemma 1.58. *If $(f, f^\sharp) : (Y, F) \longrightarrow (X, E)$ is a morphism of* Set*–spaces such that f is a homeomorphism and $f^\sharp$ is an isomorphism of sheaves of sets over X, then $(f, f^\sharp)$ is an isomorphism of* Set*–spaces.*

Proof. f has a continuous inverse $f^{-1} : X \longrightarrow Y$. Let $W \subset Y$ be an open subset, so that $f(W) \subset X$ is also open, since f is a homeomorphism. Then, $f_*^{-1}E(W) = E((f^{-1})^{-1}(W)) = E(f(W))$, and we have an isomorphism of rings $f^\sharp_{f(W)} : E(f(W)) \longrightarrow F(f^{-1}(f(W))) = F(W)$. That is, we have the inverse ring homomorphism $(f^\sharp_{f(W)})^{-1} : F(W) \longrightarrow E(f(W)) = f_*^{-1}E(W)$, which we will denote as $f_W^{-1,\sharp}$. We would like to show that this this map is natural in U. Let $V \subset W$, and observe that the following diagram commutes by the naturality of $f^\sharp$:

$$\begin{tikzcd}
f_*^{-1}E(U) = E(f(W)) \arrow[r, "f^\sharp_{f(U)}"] \arrow[d, "-|_{f(V)}"'] & f_*F(f(W)) = F(W) \arrow[d, "-|_{f(V)}"] \\
f_*^{-1}E(V) = E(f(V)) \arrow[r, "f^\sharp_{f(V)}"'] & f_*F(f(V)) = F(V)
\end{tikzcd}$$

However, since $f_W^{-1,\sharp} = (f_W^\sharp)^{-1}$ for all open $W \subset Y$, this shows the naturality of $f_W^{-1,\sharp}$ in W. Hence, $(f^{-1}, f^{-1,\sharp}) : (X, E) \longrightarrow (Y, F)$ is a morphism of ringed spaces. We observe that $f_*(f^{-1,\sharp})_U = f^{-1,\sharp}_{f^{-1}(U)}$ for all open $U \subset X$ and $f_*^{-1}(f^\sharp)_W = f^\sharp_{f(W)}$ for all open $W \subset Y$. Then, we observe that

$$f_*^{-1}(f^\sharp)_W \circ f_W^{-1,\sharp} = f^\sharp_{f(W)} \circ (f^\sharp_{f(W)})^{-1} = \mathrm{id}_{F(W)}$$

and similarly:

$$f_*(f^{-1,\sharp})_U \circ f_U^\sharp = f^{-1,\sharp}_{f^{-1}(U)} \circ f_U^\sharp = (f_U^\sharp)^{-1} \circ f_U^\sharp = \mathrm{id}_{E(U)}$$

□

Lemma 1.59. *Let $\alpha : E \longrightarrow F$ be a map of sheaves of sets over a topological space X. Then, for any $x \in X$, the map of stalks $\alpha_x : E_x \longrightarrow F_x$ is given, for any germ $[(U, s)]$ with $x \in U, s \in E(U)$ by:*

$$[s] \longmapsto [U, \alpha_U(s)]$$

Proof. E_x is the colimit of E restricted to the subcategory of the open sets of X containing x so that, we have a map $E(U) \longrightarrow E_x$ for any open $U \subset X$ with $x \in U$, and the analogous statement holds for F_x. The map $\alpha_x : E_x \longrightarrow F_x$ is the map obtained by the universal property of the colimit defining E_x. In particular, the following diagram commutes:

$$\begin{tikzcd} E(U) \arrow[r, "\alpha_U"] \arrow[d] & F(U) \arrow[d] \\ E_x \arrow[r, "\alpha_x"'] & F_x \end{tikzcd}$$

which is precisely the statement. □

2 Schemes from Varieties

2.1 Basic Commutative Algebra Results

Lemma 2.1. *The Zariski closed subsets of k are precisely the finite subsets.*

Proof. Let $S = \{a_1, \dots, a_n\} \subset k$. Then, $S = Z((x - a_1, \dots, x - a_n))$ is a closed set. Let I be an ideal of $k[x]$ and let $f \in I$ be any polynomial. Then, $(f) \subset I \implies Z(I) \subset Z((f))$. However, $Z((f))$ has finitely many points as f has finitely many roots. Thus, $Z(I)$ is finite. □

Lemma 2.2. *Let $f \in k[x_1, \dots, x_n]$. Then, evaluating f at points of k^n defines a Zariski continuous function $k^n \longrightarrow k$.*

Proof. By lemma 2.1, a closed subset of k is of the form $S = \{a_1, \dots, a_m\}$. Then, $b \in f^{-1}(S) \iff f(b) - a_i = 0$ for some $i = 1, \dots, m$. This is true if and only if $b \in Z((f - a_1)) \cup \cdots \cup Z((f - a_m))$, which is a closed subset of k^n. □

Lemma 2.3. *Let $f : R \longrightarrow S$ be a homomorphism of commutative rings. Then, if $J \subset S$ is an ideal of S, $f^{-1}(J) \subset R$ is an ideal of R. In addition, if J is prime, so is $f^{-1}(J)$*

Proof. We first observe that $0 \in J$ by definition of ideal, so that $0 \in f^{-1}(J)$ by definition of ring homomorphism. Let $a, b \in f^{-1}(J), r \in R$. We then have $f(a - b) = f(a) - f(b) \in J$ since $f(a), f(b) \in J$ and J is an ideal. Thus, $a - b \in f^{-1}(J)$. Next, $f(ra) = f(r)f(a) \in J$ since $f(r) \in S, f(a) \in J$ and J is an ideal. Thus, $ra \in f^{-1}(J)$. We have shown that $f^{-1}(J)$ is closed under subtraction and multiplication by elements of R. Hence, $f^{-1}(J)$ is an ideal.

Suppose J is prime. Then, suppose for arbitrary $p, q \in R$, we have $pq \in f^{-1}(J)$. Then, $f(p)f(q) = f(pq) \in J$ implies that $f(p) \in J \iff p \in f^{-1}(J)$ or $f(q) \in J \iff q \in f^{-1}(J)$. This shows that $f^{-1}(J)$ is prime. □

Theorem 2.4. *Let R be a commutative ring; I, an ideal of R and $q : R \longrightarrow R/I$, the quotient map. Let*

$$S(R, I) = \{J \ : \ I \subset J \subset R, \text{ and } J \text{ is an ideal of } R\}$$

and

$$T(R, I) = \{J \ : \ J \text{ is an ideal of } R/I\}$$

Then, for all J in $S(R, I)$, $q(J) \in T(T, I)$; for all $K \in T(R, I)$, $q^{-1}(K) \in S(R, I)$, and the mappings $J \longmapsto q(I)$ and $K \longmapsto q^{-1}(I)$ are inverses to each other and both preserve containment. Finally, $J \in S(R, I)$ is prime if and only if $q(J) \in T(R, I)$ is prime.

Proof. Let $J \in S(R, I)$. We first observe that $q(0) = 0 \in q(J)$. Then, let $a + I, b + I \in q(J), c + I \in R/I$. Then, there exist $a', b' \in J$ such that $q(a') = a + I, q(b') = b + I$ so that $a - a', b - b' \in I$. Then, $(a - b) - (a' - b') = (a - a') - (b - b') \in I$ so that $(a - b) + I = (a' - b') + I = q(a' - b')$. However, since J is an ideal, $a' - b' \in J$. This shows that $(a + I) - (b + I) \in q(J)$. Next, $(c + I)(a + I) = ca + I$. However, $ca - ca' = c(a - a') \in I$ since I is an ideal. Then, $ca + I = ca' + I = q(ca')$. However, $ca' \in J$ since J is an ideal. This shows that $ca + I = q(ca) \in q(J)$. Since $q(J)$ is thus closed under subtraction and multiplication by elements of R/I, $q(J) \in T(R, I)$.

Now, consider $K \in T(R, I)$. Then, by lemma 2.3, $q^{-1}(K)$ is and ideal of R. We would like to show that $I \subset q^{-1}(K)$. Let $a \in I$, then $q(a) = a+I = 0+I \in K$ since K is an ideal. Thus, $a \in q^{-1}(\{0\}) \subset q^{-1}(K)$. Hence, $q^{-1}(J) \in S(R, I)$.

Now, $q(q^{-1}(K)) = K$ by elementary set theory since q is surjective. Then, $J \subset q^{-1}(q(J))$, again by elementary set theory. So, suppose $jinq^{-1}(q(J))$ so that $j + I = q(j) \in q(J)$. This means there exists $j' \in J$ such that $q(j') = q(j)$. However, this implies that $j' + I = j + I \iff j - j' \in I$. Since $I \subset J$, we must have $j - j' \in J$. However, since J is a subgroup of R, we must have $j \in J$. Thus, $q^{-1}(q(J)) \subset J$. Hence, we have shown that $q^{-1}(q(J)) = J$.

That q and q^{-1} preserve containment follows by elementary set theory.

It remains to show that J is prime if and only if $q(J)$ is prime. If J is prime, suppose then that $(a + I)(b + I) = ab + I \in q(J)$. Then, $ab + I = q(j) = j + I$ for some $j \in J$. Thus, $ab - j \in I \subset J$ which implies that $ab \in J$ since J is a subgroup of R. Thus, $a \in J$ or $b \in J$, since J is prime by hypothesis. Hence, $q(a) \in q(J)$ or $q(b) \in q(J)$. This shows that $q(J)$ is prime. If, on the other hand, $q(J)$ is prime $q^{-1}(q(J))$ is a prime ideal of R by lemma 2.3. However, we have already shown that $q^{-1}(q(J)) = J$. Hence, J is prime in R. □

Lemma 2.5. *Let A be a unital commutative ring; $I \subset A$ an ideal with quotient map $q : A \longrightarrow A/I$, and $S \subset A$, a multiplicatively closed subset. Then, $q(S)$ is multiplicatively closed in A/I.*

Proof. $1 \in S$ implies $q(1) = 1 + I = 1_{A/I} \in S$. Then, let $a + I, b + I \in q(S)$. There exist $a', b' \in S$ such that $q(a') = a' + I = a + I$ and $q(b') = b' + I = b + I$. Then, $a'b' \in S$ and $(a + I)(b + I) = q(a')q(b') = q(a'b') \in q(S)$. □

Lemma 2.6. *Let A be a unital commutative ring; $I \subset A$ an ideal with quotient map $q : A \longrightarrow A/I$, and $S \subset A$, a multiplicatively closed subset. Then, the maps*

$$\phi : (A/I)q(S)^{-1} \longrightarrow (AS^{-1})/(IS^{-1}) : \frac{a+I}{b+I} \longmapsto \frac{a}{b} + IS^{-1}$$

and

$$\psi : (AS^{-1})/(IS^{-1}) \longrightarrow (A/I)q(S)^{-1} : \frac{a}{b} + IS^{-1} \longmapsto \frac{a+I}{b+I}$$

are well-defined ring homomorphisms and inverses to each other.

Proof. Let $q' : AS^{-1}/IS^{-1}$ be the quotient map and $l : A \longrightarrow AS^{-1}, l' :$ $(A/I)q(S)^{-1}$ be the localization maps. We observe that the map

$$A \xrightarrow{l} AS^{-1} \xrightarrow{q'} AS^{-1}/IS^{-1}$$

sends the ideal I to $0 + IS^{-1} = 0_{AS^{-1}/IS^{-1}}$ so that we get a unique map ϕ' the following diagram commute, by the universal property of A/I:

$$\begin{tikzcd} A \arrow[r, "l"] \arrow[d, "q"'] & AS^{-1} \arrow[d, "q'"] \\ A/I \arrow[r, "\phi'"'] & AS^{-1}/IS^{-1} \end{tikzcd}$$

This map is given by $a + I \longmapsto q'(l(a)) = a/1 + IS^{-1}$. Now, we observe that $\phi'(q(S)) = q'(l(S))$ while $l(S)$ is the set of invertible elements of AS^{-1}. Let $l(b) = b/1 \in l(S)$ so that $1/b \in AS^{-1}$. Then, $q'(b/1)q'(1/b) = q'(b/1 \cdot 1/b) =$ $q'(1) = 1$ so that $q'(l(b))$ is invertible in AS^{-1}/IS^{-1} with inverse $q'(1/b)$. This gives unique map:

$$(A/I)q(S)^{-1} \longrightarrow AS^{-1}/IS^{-1} : \frac{a+I}{b+I} \longmapsto \frac{\phi'(a+I)}{\phi'(b+I)} + IS^{-1}$$

However, $\phi'(a+I) = \phi'(q(a)) = q'(l(a)) = a/1 + IS^{-1}$ and similarly, $1/\phi'(b+$ $I) = 1/b + IS^{-1}$, so that $\frac{\phi'(a)}{\phi'(b)} + IS^{-1} = a/b + IS^{-1}$, so that this is exactly the map ϕ. It is a well-defined ring homomorphism as the map was constructed by universal properties of localizations and quotients.

Now, suppose $\phi((a+I)/(b+I)) = a/b + IS^{-1} = 0$, then $a/b \in IS^{-1}$ which implies that $a \in I$ and $b \in S$. However, since $a \in I$, $a + I = 0 + I$ which means $(a+I)/(b+I) = 0$. Thus, ϕ is injective. An element of AS^{-1}/IS^{-1} is of the form $a/b + IS^{-1}$ which is equal to $\phi((a+I)/(b+I))$, so that ϕ is also surjective. Hence, ϕ is a ring isomorphisms. The argument for surjectivity also shows that the inverse has to be ψ as defined above. ☐

Lemma 2.7. *Let A be a unital commutative ring with ideals $I \subset J \subset A$ and a ring homomorphism $f : A \longrightarrow B$ such that $f(A \setminus J)$ is a subset of the set of* *invertible elements of B, and $I \subset \ker(f)$. Then, writing $q : A \longrightarrow A/I$ for the quotient map, we have a ring homomorphism:*

$$\overline{f} : (A/I)_{q(J)} \longrightarrow B : \frac{a+I}{b+I} \longmapsto f(a)f(b)^{-1}$$

Proof. Since $I \subset J$, $q(J)$ is an ideal of A/I by theorem 2.4. By the universal property of quotients, we have a unique map $f' : A/I \longrightarrow B$ such that $f' \circ q = f$. Hence, $f'((A/I) \setminus q(J)) = f'(q(A) \setminus q(J)) \subset f'(q(A \setminus J)) = f(A \setminus J)$ since q is surjective. This shows that $f'((A/I) \setminus q(J))$ is a subset of the set of invertible elements of B. Thus, writing $l : (A/I) \longrightarrow (A/I)_{q(J)}$ for the localization map, by the universal property of localization, we get a unique map $\overline{f} : (A/I)_{q(J)} \longrightarrow B$ such that $\overline{f} \circ l = f'$ and this is given by:

$$\overline{f}((a+I)/(b+I)) = f'(a+I)f'(b+I)^{-1} = f'(q(a))f'(q(b))^{-1} = f(a)f(b)^{-1}$$

□

2.2 The Scheme of a Variety

For all results, we will consider an algebraically closed field k. We will then follow definitions from [1].

Lemma 2.8. *Let U be an affine k–variety; $I(U)$, its ideal, and $A = A(U) = k[x_1, \dots, x_n]/I(U)$, its coordinate ring. We then have a function:*

$$\beta_U : t(U) \longrightarrow \mathrm{Spec}(A) : C \longmapsto q(I(C))$$

where $I(C) = \{f \in k[x_1, \dots, x_n] \ : \ \forall c \in C, f(c) = 0\}$ is the ideal of C and $q : k[x_1, \dots, x_n] \longrightarrow A$ is the quotient map. For a closed subset $D \subset U$, we have:

$$\beta_U(t(D)) = V(q(I(D))) = \{P \in \mathrm{Spec}(A) \ : \ q(I(D)) \subset P\}$$

Furthermore, we have a function:

$$\gamma_U : \mathrm{Spec}(A) \longrightarrow t(U) : P \longmapsto Z(q^{-1}(P))$$

where $Z(P) = \{x \in k^n \ : \ \forall f \in P, f(x) = 0\}$. For any ideal $K \subset A$, we have:

$$\gamma_U(V(K)) = t(Z(q^{-1}(K)))$$

Finally, β_U and γ_U are inverses to each other.

Proof. We first check that the map β_U is well defined. Let $R = k[x_1, \dots, x_n]$. Given $C \in t(U)$, $I(C) \supset I(U)$ since $I(U)$ is the set of all $f \in R$ such that for all $x \in U$, $f(x) = 0$. Then, $I(C)$ is prime by [1, Cor. I.1.4] because C is irreducible. Then, $I(C)/I$ is prime in A by theorem 2.4.

Let $C \in t(D)$. Then, $\beta_U(C) = q(I(C))$. Since $C \subset D$, $I(D) \subset I(C)$. Furthermore, $I(C)$ is prime by [1, Cor. I.1.4]. Then, $q(I(C))$ is prime and $q(I(D)) \subset q(I(C))$ by theorem 2.4. Thus, $\beta_U(C) = q(I(C)) \in V(q(I(D)))$.

Now, suppose $P \in V(q(I(D)))$. Then, $P = q(q^{-1}(P))$ but since $q^{-1}(P)$ is prime and contains $q^{-1}(q(I(D))) = I(D)$, $Z(q^{-1}(P))$ is an irreducible subset of $Z(I(D)) = D$ by [1, Cor. I.1.4], and, of course, $Z(q^{-1}(P))$ is closed in k^n and hence also in D by the definition of Zariski topology. Hence, then $\beta_U(Z(q^{-1}(P))) = q(I(Z(q^{-1}(P)))) = q(q^{-1}(P)) = P$. Thus, we have shown that $P \in \beta_U(t(D))$. Therefore, $\beta_U(t(D)) = V(q(I(D)))$.

Now, we show that γ_U is well defined. Since P is prime, so is $q^{-1}(P)$ by theorem 2.4, and $Z(q^{-1}(P))$ is an irreducible closed subset of k^n by [1, Cor. I.1.4]. However, $I(U) \subset q^{-1}(P)$ by theorem 2.4 so that $Z(q^{-1}(P)) \subset Z(I(U)) = U$. Then, by corollary 1.9 and corollary 1.1, $Z(q^{-1}(P)) \in t(U)$.

Let $P \in V(K)$. Then, $K \subset P$ implies $q^{-1}(K) \subset q^{-1}(P)$ which, in turn, implies $Z(q^{-1}(P)) \subset Z(q^{-1}(K))$. However, since $Z(q^{-1}(P))$, it is irreducible and closed in U which by two applications of corollary 1.9 and corollary 1.1, shows that $Z(q^{-1}(P))$ is irreducible and closed in $Z(q^{-1}(K))$. Thus, $\gamma_U(P) = Z(q^{-1}(P)) \in t(Z(q^{-1}(K)))$.

Now, let $C \in t(Z(q^{-1}(K)))$. Then, C is an irreducible closed subset of $Z(q^{-1}(K))$ which, by corollary 1.9 and corollary 1.1, implies that C is an irreducible closed subset of k^n and hence, $I(C)$ is prime, by [1, Cor. I.1.4]. Next, $q^{-1}(K) \subset I(C)$, because if $f \in k[x_1, \ldots, x_n]$ vanishes on $Z(q^{-1}(K))$, it vanishes on every subset thereof. Then, by theorem 2.4, $K = q(q^{-1}(K)) \subset q(I(C))$ and $q(I(C))$ is prime in A, so that $q(I(C)) \in V(K)$. Next, $\gamma_U(q(I(C))) = Z(q^{-1}(q(I(C)))) = C$. Thus, $C \in \gamma_U(V(K))$.

We finally observe that, by [1, Cor. I.1.4] and theorem 2.4,

$$\beta_U(\gamma_U(P)) = q(I(Z(q^{-1}(P)))) = P$$

and

$$\gamma_U(\beta_U(C)) = Z(q^{-1}(q(I(C)))) = C$$

□

Corollary 2.9. *The maps β_U and γ_U of lemma 2.8 are continuous inverses of each other and hence, we have a homeomorphism:*

$$\beta_U : t(U) \overset{\cong}{\leftrightarrows} \operatorname{Spec}(A) : \gamma_U$$

Proof. By lemma 2.8, it suffices to observe that both maps take closed sets to closed sets. □

Lemma 2.10. *Every affine variety over k is T_1.*

Proof. Let U be an affine k–variety such that U is an irreducible Zariski closed subset of k^n. That is, $U = Z(I(U))$ for some ideal $I(U) \subset k[x_1, \ldots, x_n]$. Then, for any $a = (a_1, \ldots, a_n) \in U \subset k^n$, $\{a\} = Z((x_1 - a_1, \ldots, x_n - a_n))$ which is closed in k^n. Thus, $\{a\}$ is a closed subset of U as well by corollary 1.1. □

Lemma 2.11. *Let $R = k[x_1, \ldots, x_n]$; $I \subset R$, an ideal, and $A = R/I$. For every $P \in \operatorname{Spec}(A)$, and every open neighbourhood U of P in $\operatorname{Spec}(A)$, there is an $a \in k^n$ such that $M = q(\ker(\mathrm{ev}_a)) \subset A$ is a maximal ideal, $M \in U$ and $P \subset M$.*

Proof. Since U is open, there is some ideal $K \subset A$ such that $U = \operatorname{Spec}(A) \setminus V(K)$. Then, $P \in U$ implies that $P \not\supset K$. This means that there exists $f + I \in K$ such that $f + I \notin P$. Let $q : R \longrightarrow A$ be the quotient map. If f is identically zero on $Z(q^{-1}(P))$, then by [1, Cor. I.1.4],

$$\begin{aligned} &Z(q^{-1}(P)) \subset Z(q^{-1}((f+I))) \\ \Longrightarrow\ &q^{-1}(P) \supset I(Z(q^{-1}((f+I)))) \\ \Longrightarrow\ &q^{-1}(P) \supset (q^{-1}(f+I)) \supset (f) \end{aligned}$$

so that $f \in q^{-1}(P) \implies f + I = q(f) \in P$, contradicting the hypothesis that $P \notin V(K)$. Thus, there exists $a \in Z(q^{-1}(P))$ such that $f(a) \neq 0 \iff f \notin \ker(\mathrm{ev}_a)$. Since $q(f) = f + I \in K$, $f \in q^{-1}K$ which means that $q^{-1}(K) \not\subset \ker(\mathrm{ev}_a)$ and, by theorem 2.4, $K \not\subset q(\ker(\mathrm{ev}_a))$. That is, $m_a = q(\ker(\mathrm{ev}_a)) \in U = \operatorname{Spec}(A) \setminus V(K)$. However, since $a \in Z(q^{-1}(P))$, $\ker(\mathrm{ev}_a) = I(\{a\}) \supset q^{-1}(P)$ which implies, by theorem 2.4 again, that $q(\ker(\mathrm{ev}_a))$ contains P and is maximal, since $\ker(\mathrm{ev}_a)$ is maximal. We can thus take $M = q(\ker(\mathrm{ev}_a))$. $\square$

Lemma 2.12. *Let U be an affine k–variety with coordinate ring $A = A(U) = R/I(U) = R/I$, where $R = k[x_1, \ldots, x_n]$ and $I(U) = I$ is the ideal of U. Let O_U be the sheaf of regular functions on U. Then, we have an isomorphism of sheaves of rings:*

$$\beta_U^\sharp : O_{\operatorname{Spec}(A)} \longrightarrow \beta_{U,*}(\alpha_{U,*}(O_U)) : s \longmapsto (a \longmapsto s(\beta_U(\alpha_U(a)))(a))$$

where $s(\beta_U(\alpha_U(a)))$ is an element $(f+I)/(g+I)$ of $A_{\beta_U(\alpha_U(a))}$ assigned by the section s, for some polynomials $f, g \in R$, and $s(\beta_U(\alpha_U(a)))(a)$ is an element of k given by $f(a)/g(a)$. In addition, for each $C \in t(U)$,

$$\beta_{U,C}^\sharp : O_{\operatorname{Spec}(A),\beta_U(C)} \longrightarrow (\alpha_{U,*}(O_U))_C$$

is a local homomorphism.

Proof. Let W be an open subset of $\operatorname{Spec}(A)$ and $s \in O_{\operatorname{Spec}(A)}(W)$. Then, we have an open cover $\{W_i\}_{i \in J}$ of W and $f_i, g_i \in R$ for each $i \in J$, such that for each $P \in W_i$, $g_i + I \notin P$ and $s(P) = (f_i + I)/(g_i + I)$. Now, let $a \in \alpha_U^{-1}(\beta_U^{-1}(W))$. Then, $\beta_U(\alpha_U(a)) \in W$ and $\overline{\{a\}} = \{a\}$ by lemma 2.10. This shows that $\beta_U(\alpha_U(a)) = \beta_U(\overline{\{a\}}) = \beta_U(\{a\}) = q(I(\{a\}))$ where $q : R \longrightarrow A$ is the quotient map. However, $I(\{a\})$ is precisely $Z((x_1 - a_1, \ldots, x_n - a_n))$, the maximal ideal of R corresponding to a by [1, Cor. I.1.4], and $q(I(\{x\}))$ is the maximal ideal m_a of A corresponding to a by theorem 2.4. Since $m_a = \beta_U(\alpha_U(a)) \in W$, we must have $m_a \in W_i$ for some $i \in J$ so that $g_i + I \notin m_a \iff g_i(a) \neq 0$. Thus, $f_i(a)/g_i(a) \in k$. Taking $B = k$, $f = \mathrm{ev}_a$ $I = I$ and $J = \ker(\mathrm{ev}_a) = I(\{a\})$ in lemma 2.7, we see that $f_i(a)/g_i(a) = f(a)/g(a)$ whenever $(f_i + I)/(g_i + I) = (f + I)/(g + I)$. This, in particular, shows that for any other $j \in J$, with $m_a \in W_j$, $f_j(a)/g_j(a) = f_i(a)/g_i(a)$ for $(f_j + I)/(g_j + I) = s(m_a) = (f_i + I)/(g_i + I)$. Thus, $\beta_{U,W}^\sharp(s) : a \longmapsto s(m_a)(a)$ is a well-defined function of sets $\alpha_U^{-1}(\beta_U^{-1}(W)) \longrightarrow k$. Then, since the W_i cover W, we have that the $\alpha_U^{-1}(\beta_U^{-1}(W_i))$ cover $\alpha_U^{-1}(\beta_U^{-1}(W))$. For all $i \in J, a \in \alpha_U^{-1}(\beta_U^{-1}(W_i))$, $\beta_{U,W}^\sharp(s)(a) = f_i(a)/g_i(a)$. Hence, $\beta_{U,W}^\sharp(s)$ is a regular function.

We would like to see that $\beta_{U,W}^{-1}$ is a ring homomorphism. For $u, v \in O_{\mathrm{Spec}(A)}(W)$, $a \in \alpha_U^{-1}(\beta_U^{-1}(W))$, $u(m_a) = (f_i + I)/(g_i + I), v(m_a) = (f_i' + I)/(g_i' + I)$ for some polynomials $f_i, g_i, f_i', g_i' \in R$. By lemma 2.7 again, we observe that:

$$\begin{aligned}
&\beta_{U,W}^\sharp(u+v)(a)\\
=&((u+v)(m_a))(a)\\
=&(f_i/g_i + f_i'/g_i')(a)\\
=&f_i(a)/g_i(a) + f_i'(a)/g_i'(a)\\
=&u(m_a)(a) + v(m_a)(a)\\
=&\beta_{U,W}^\sharp(u)(a) + \beta_{U,W}^\sharp(v)(a)\\
=&(\beta_{U,W}^\sharp + \beta_{U,W}^\sharp(v))(a)
\end{aligned}$$

The preservation of multiplication, 0 and 1 can be shown by similar point-wise identities.

We would then like to see that $\beta_U^\sharp$ is natural in W. So, suppose $W' \subset W$. Then, the $W_i' = W_i \cap W'$ cover W' and for $a \in \alpha_U^{-1}(\beta_U^{-1}(W_i'))$, we have:

$$\beta_{U,W'}^\sharp(s|_{W'})(a) = s|_{W'}(m_a)(a) = s(m_a)(a) = f_i(a)/g_i(a)$$

which is also the value of the regular function $\beta_{U,W}^\sharp(s)|_{\alpha_U^{-1}(\beta_U^{-1}(W'))}$.

Now, let $s(\beta_U(\alpha_U(a)))(a) = f(a)/g(a) = 0$ for all $a \in \alpha_U^{-1}(\beta_U^{-1}(W))$. Then, f_i/g_i, and hence f_i, is zero on all of $\alpha_U^{-1}(\beta_U^{-1}(W_i))$. Since $\alpha_U^{-1}(\beta_U^{-1}(W_i))$ is open in U, it must also be dense in U by lemma 1.10, since U is irreducible. Then, f_i is a Zariski continuous function $U \longrightarrow k$ by lemma 2.2, and, hence, we must have f_i is zero on U, by lemma 1.4, as it is zero on the dense subset $\alpha_U^{-1}(\beta_U^{-1}(W_i)) \subset U$. Thus, $f_i \in I$ and this means $(f_i + I)/(g_i + I)$ is the zero element of A_P for each $P \in W_i$. This shows that $s = 0$. Since s was arbitrary, this shows that $\beta_{U,W}^\sharp$ has a zero kernel and, hence, must be injective.

For surjectivity, take any regular function $f : \alpha_U^{-1}(\beta_U^{-1}(W)) \longrightarrow k$. Then, cover $\alpha_U^{-1}(\beta_U^{-1})(W)$ with open subsets $W_i', i \in I$ such that for each $i \in I$, there exist $f_i, g_i \in R$ with $g_i(a) \neq 0$ for all $a \in W_i'$ and $f|_{W_i'} = f_i/g_i$. By theorem 1.37, we have a cover $t_{\mathrm{Op}}(W_i'), i \in I$ for $\beta_U^{-1}(W)$ which, in turn, gives an open cover $W_i := \beta_U^{-1}(t_{\mathrm{Op}}(W_i')), i \in I$ of W since β_U is a homeomorphism by corollary 2.9. For any $P \in W_i$, we can show that $g_i + I \notin P$. By lemma 2.11, there exists $a \in k^n$ with $M = q(\ker(\mathrm{ev}_a)) \supset P$, $M \in W_i$. Then, $\ker(\mathrm{ev}_a) = q^{-1}(M) \supset q^{-1}(P)$ and $\{a\} = Z(q^{-1}(M)) = Z(\ker(\mathrm{ev}_a)) \in Z(q^{-1}(W_i)) = \gamma_U(W_i) = \beta_U^{-1}(W_i)$. Then, $a = \alpha_U^{-1}(\{a\}) \in \alpha_U^{-1}(\beta_U^{-1}(W_i')) = W_i'$, so that $g_i(a) \neq 0$. This means, $g_i \notin \ker(\mathrm{ev}_a)$ which implies that $g_i + I = q(g_i) \notin q(\ker(\mathrm{ev}_a))$ and hence, $g_i + I \notin P$ since $P \subset q(\ker(\mathrm{ev}_a))$. We can then define $s(P) = (f_i + I)/(g_i + I) \in A_P$. We need to show that $s(P)$ does not depend on the choice of $i \in I$. If $P \in$

$W_i \cap W_j$, then we observe that on $W_i' \cap W_j'$, $f_i/g_i = f|_{W_i' \cap W_j'} = f_j/g_j$ so that $f_i g_j - f_j g_i = 0$ on $W_i' \cap W_j'$. Hence, $f_i g_j - f_j g_i$ is zero on U by lemma 1.4 and is hence in I. This shows that $(f_i + I)/(g_i + I)$ and $(f_j + I)/(g_j + I)$ are equal as elements of A_P. By construction, at each $P \in W_i$, $s(P)$ is given by the same $(f_i + I)/(g_i + I)$, and hence, $s \in O_{\mathrm{Spec}(A)}(W)$. Finally, we see that $\beta^\sharp_{U,W}(s) = f$.

To see that the localization $\beta^\sharp_{\beta_U(C)}$ is a local homomorphism follows from the fact that it is an isomorphism, and any isomorphism of local rings is a local homomorphism. $\square$

Lemma 2.13. *Let V be an affine k–variety with coordinate ring $A = R/I$ where $R = k[x_1, \ldots, x_n]$ and $I \subset R$ is a prime ideal with $V = Z(I) \subset k^n$. Let $C \in t(V)$. Then, we have:*

$$(\alpha_{V_*}(O_V))_C = \mathrm{colim}_{C \cap W \neq \varnothing}\, O_V(W)$$

where the colimit is taken over the poset of open subsets of V that have non-empty intersection with C. Furthermore, the set

$$M_C := \{[(W, s)] \ : \ \exists x \in C \cap W, s(x) = 0\}$$

is the unique maximal ideal of this stalk.

Proof. By theorem 1.33 and lemma 1.34, we have an order isomorphism between the poset of open subsets of $t(V)$ containing the point C and the poset of open subsets of V which have non-empty intersection with C, and the first claim follows. More explicitly, this stalk is the quotient of the set of all pairs (W, s), for an open $W \subset V$ with $C \cap W \neq \varnothing$ an and $s \in O_V(W)$ by the equivalence relation $(W, s) \sim (W', s')$ if and only if there exists $W'' \subset W \cap W'$ such that $s|_{W''} = s'|_{W''}$. Denote such an equivalence class as $[W, s]$ for a choice of representative (W, s). By the definition of colimits, the ring structure is given by operations $[W, s]+[W', s'] = [W \cap W', s+s']$, $[W, s] \cdot [W', s'] = [W \cap W', s \cdot s']$, and units $[W, 0], [W, 1]$, where 0 and 1 are the constant regular functions on W.

From the description of operations, it is straightforward to verify that M_C is an ideal, with the added observation that the operations on regular functions are point-wise. If $[W, s] \notin M_C$, then, for all $x \in C \cap W$, $s(x) \neq 0$ and has a multiplicative inverse $t(x) = 1/s(x)$ on W. This shows that the stalk quotiented by M_C is a field and hence M_C is maximal. Then, suppose K is another maximal ideal of the stalk. Let $[W, s] \in K$. Then, if $[W, s] \notin M_C$, $[W, s]$ is a unit as we saw previously. This means, K is the stalk, which contradicts the hypothesis that it is an ideal. Thus, $K \subset M_C$ and by maximality of K, we must have $K = M_C$. $\square$

Theorem 2.14. *Let W, V be a k–varieties; O_W, O_V, their sheaves of regular functions and $f : W \longrightarrow V$ a morphism of varieties with $f^\sharp : O_V \longrightarrow f_*(O_W)$ the map that sends a regular function $s : V' \longrightarrow k$ for some open subset $V' \subset V$ to the regular function $s \circ f : f^{-1}(V') \longrightarrow k$. Then, in the context of construction 1.55, the following hold* *(i)* $t(V, O_V)$ *is a* k*–scheme.*

(ii) $t(f, f^\sharp)$ *is a morphism of* k*–schemes.*

(iii) The functor $t : \mathrm{RingedSp} \longrightarrow \mathrm{RingedSp}$ *restricted to the category of* k*–varieties (viewed as ringed spaces with their sheaves of regular functions) is a fully faithful functor from* k*–varieties to* $\mathrm{Sch}_{/k}$.

(iv) There is a natural transformation $\alpha : \mathrm{id}_{\mathrm{RingedSp}} \longrightarrow t$ *such that for each* k*–variety* V*, the map of topological spaces underlying* $\alpha_{V,O_V} : (V, O_V) \longrightarrow t(V, O_V)$ *is a homeomorphism onto the set of closed points of* $t(V)$.

Proof. (i) V has a cover by open affine subvarieties U_i by [1, §I.4.3]. Consider the open subsets $t_{\mathrm{Op}}(U_i)$ of $t(V)$ where t_{Op} is as in theorem 1.33. These cover $t(V)$ by theorem 1.37. Hence, it suffices to show that each $(t_{\mathrm{Op}}(U_i), \alpha_{V,*}(O_V)|_{U_i})$ is an affine scheme.

However, we have shown in theorem 1.50 that for each inclusion $\iota_i : U_i \longrightarrow V$, the map $t(\iota_i) : t(U_i) \longrightarrow t(V)$ is a homeomorphism onto $t_{\mathrm{Op}}(U_i) \subset t(V)$. For any open $W \subset t_{\mathrm{Op}}(U_i)$, we observe that $\alpha_V^{-1}(W) \subset U_i$ by theorem 1.33 so that $\alpha_V^{-1}(W) = \iota_i^{-1}(\alpha_V^{-1}(W))$. Then, by lemma 1.24, we have:

$$\begin{aligned}\alpha_V^{-1}(W) =&\iota_i^{-1}(\alpha_V^{-1}(W))\\ =&(\alpha_V \circ \iota_i)^{-1}(W)\\ =&(t(\iota_i) \circ \alpha_{U_i})^{-1}(W)\\ =&\alpha_{U_i}^{-1}(t(\iota_i)^{-1}(W)\end{aligned}$$

Thus, we have a strict equality of sets:

$$\begin{aligned}&\alpha_{V,*}(O_V)|_{t_{\mathrm{Op}}(U_i)}(W)\\ =&\alpha_{V,*}(O_V)(W)\\ =&O_V(\alpha_V^{-1}(W))\\ =&O_V(\alpha_{U_i}^{-1}(t(\iota_i)^{-1}(W)))\\ =&O_V|_{U_i}(\alpha_{U_i}^{-1}(t(\iota_i)^{-1}(W)))\\ =&O_{U_i}(\alpha_{U_i}^{-1}(t(\iota_i)^{-1}(W)))\\ =&\alpha_{U_i,*}(O_{U_i})(t(\iota_i)^{-1}(W))\\ =&t(\iota_i)_*(\alpha_{U_i,*}(O_{U_i}))(W)\end{aligned}$$

Therefore, we have an isomorphism of ringed spaces:

$$(t(\iota), \mathrm{id}) : (t(U_i), \alpha_{U_i,*}(O_{U_i})) \longrightarrow (t_{\mathrm{Op}}(U_i), \alpha_{V,*}(O_V)|_{t_{\mathrm{Op}}(U_i)})$$

We can then see that $t(U_i, O_{U_i}) = (t(U_i), \alpha_{U_i,*}(O_{U_i}))$ is an affine scheme by corollary 2.9, lemma 2.12 and lemma 1.58. We then need to produce a map $t(V, O_V) \longrightarrow \mathrm{Spec}(k)$. For this, we observe that there is a unique map of topological spaces $f : t(V) \longrightarrow |\mathrm{Spec}(k)| = \mathrm{pt}$, and a unique map

$$f^\sharp_\varnothing : O_{\mathrm{Spec}(k)}(\varnothing) = \{0\} \longrightarrow \{0\} = f_*(\alpha_*(O_V))(\varnothing)$$

We need only provide a map

$$f^\sharp_{\mathrm{pt}} : k = O_{\mathrm{Spec}(k)}(\mathrm{pt}) \longrightarrow f_*(\alpha_*(O_V))(\mathrm{pt}) = O_V(\alpha^{-1}(f^{-1}(\mathrm{pt}))) = O_V(V)$$

and this is simply the map giving $O_V(V)$ the structure of a k–algebra, that is the map sending an element $a \in k$ to the constant regular function on V valued at a.

(ii) We have to show that $t(f, f^\sharp)$ is a morphism of locally ringed spaces over $\mathrm{Spec}(k)$. First, we show that it is a morphism of ringed spaces over $\mathrm{Spec}(k)$. For this, we have the show that the map

$$k = O_{\mathrm{Spec}(k)}(\mathrm{pt}) \longrightarrow \alpha_{W,*}(O_W)(t(W)) = O_W(W)$$

is equal to the map:

$$\begin{aligned} &k = O_{\mathrm{Spec}(k)}(\mathrm{pt}) \\ \longrightarrow&\alpha_{V,*}(O_V)(t(V)) = O_V(V) \\ \longrightarrow&t(f)_*(\alpha_{W,*}(O_W))(t(V)) = O_W(W) \end{aligned}$$

This follows from the observation that the constant regular function on V valued at some $a \in k$ precomposed with f is the constant regular function valued at a on W. Now, it suffices to show that, for each $C \in t(W)$, the map

$$\alpha_{V,*}(f^\sharp)_{t(f)(C)} : \alpha_{V,*}(O_V)_{t(f)(C)} \longrightarrow t(f)_*(\alpha_{W,*}(O_W))_{t(f)(C)} = \alpha_{W,*}(O_W)_C$$

is a map of local rings. By lemma 2.13, we consider $[U, s] \in M_{t(f)(C)} \subset \alpha_{V,*}(O_V)_{t(f)(C)}$. By definition of colimits and the map of stalks,

$$\alpha_{V,*}(f^\sharp)_C([U, s]) = [U, \alpha_{V,*}(f^\sharp)(s)]$$

However, the section $\alpha_{V,*}(f^\sharp)(s) \in \alpha_{V,*}(f^*(O_W))(U)$ is equal to the section

$$s \circ f \in O_W(f^{-1}(\alpha_V^{-1}(U)))$$

However, there exists $v \in t(f)(C)$ such that $s(v) \neq 0$. If $s(f(u)) = 0$ for all $u \in f^{-1}(\alpha_V^{-1}(U))$, then by the continuity of $s \circ f$, lemma 1.5 and lemma 2.10, $s(u)$ must be zero since $v \in t(f)(C) = \overline{f(C)}$, which is a contradiction. Hence, there must be a $u \in C$ such that $(s \circ f)(u) \neq 0$. This means,

$$[f^{-1}(\alpha_V^{-1}(U)), s \circ f] \in M_C \subset \alpha_{W,*}(O_W)_C$$

This shows that $\alpha_{V,*}(f^\sharp)_{t(f)(C)}$ is a map of local rings.

(iii) Combine (i) and (ii) with lemma 2.10, lemma 1.7, theorem 1.57, and the observation that T_1 spaces are T_0, to obtain that the functor is faithful.

Show that it is full

(iv) Combine theorem 1.43 with theorem 1.32. □

References

[1] Robin Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics. Springer, 1977.