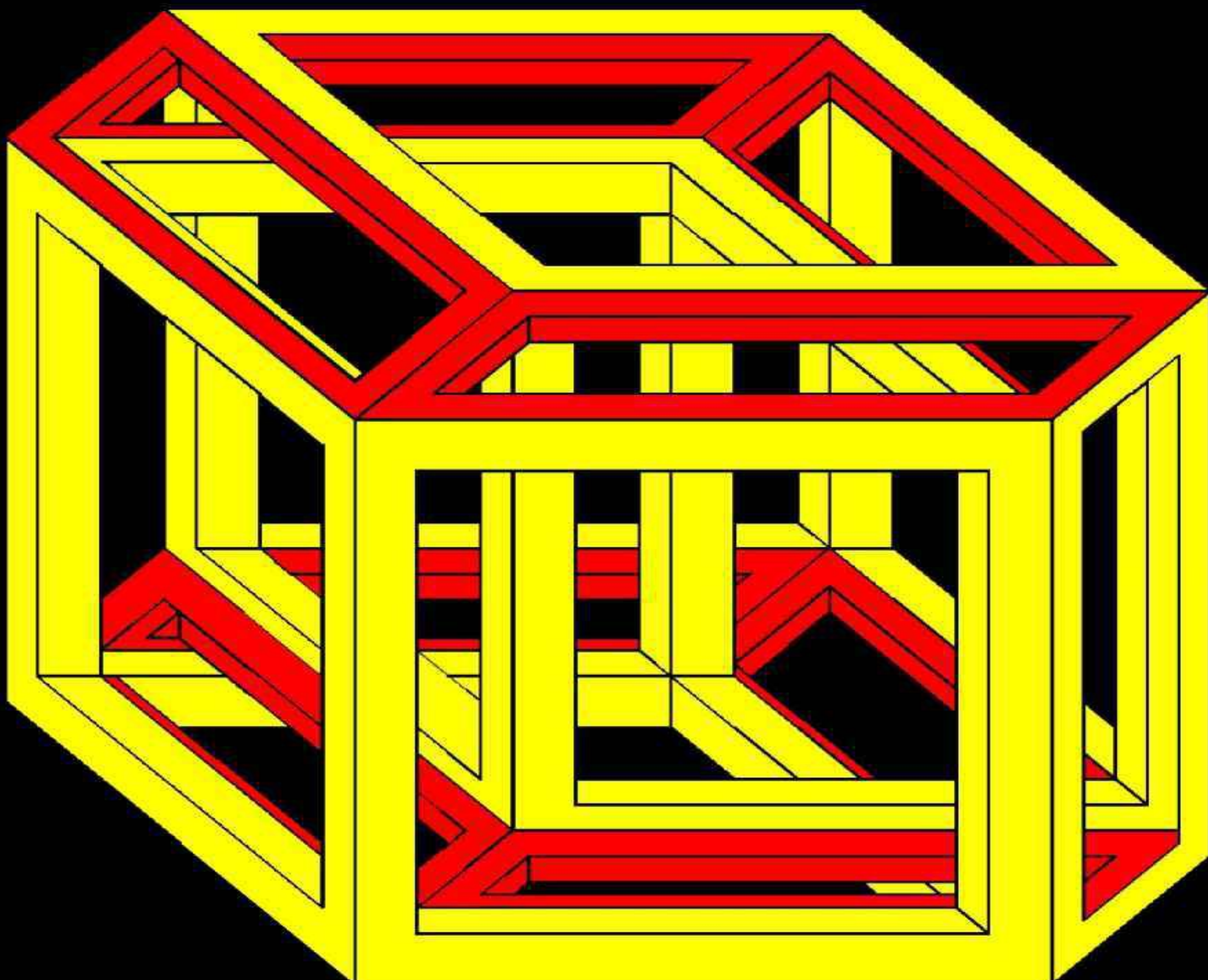
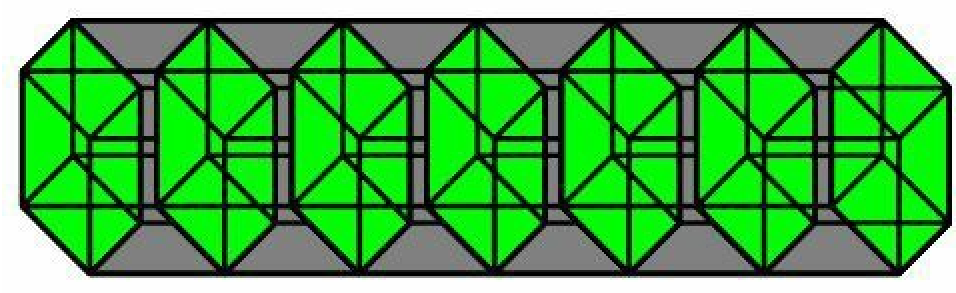
The FOURTH



DIMENSION

A Visual Introduction to the Fourth Dimension (Rectangular 4D Geometry)



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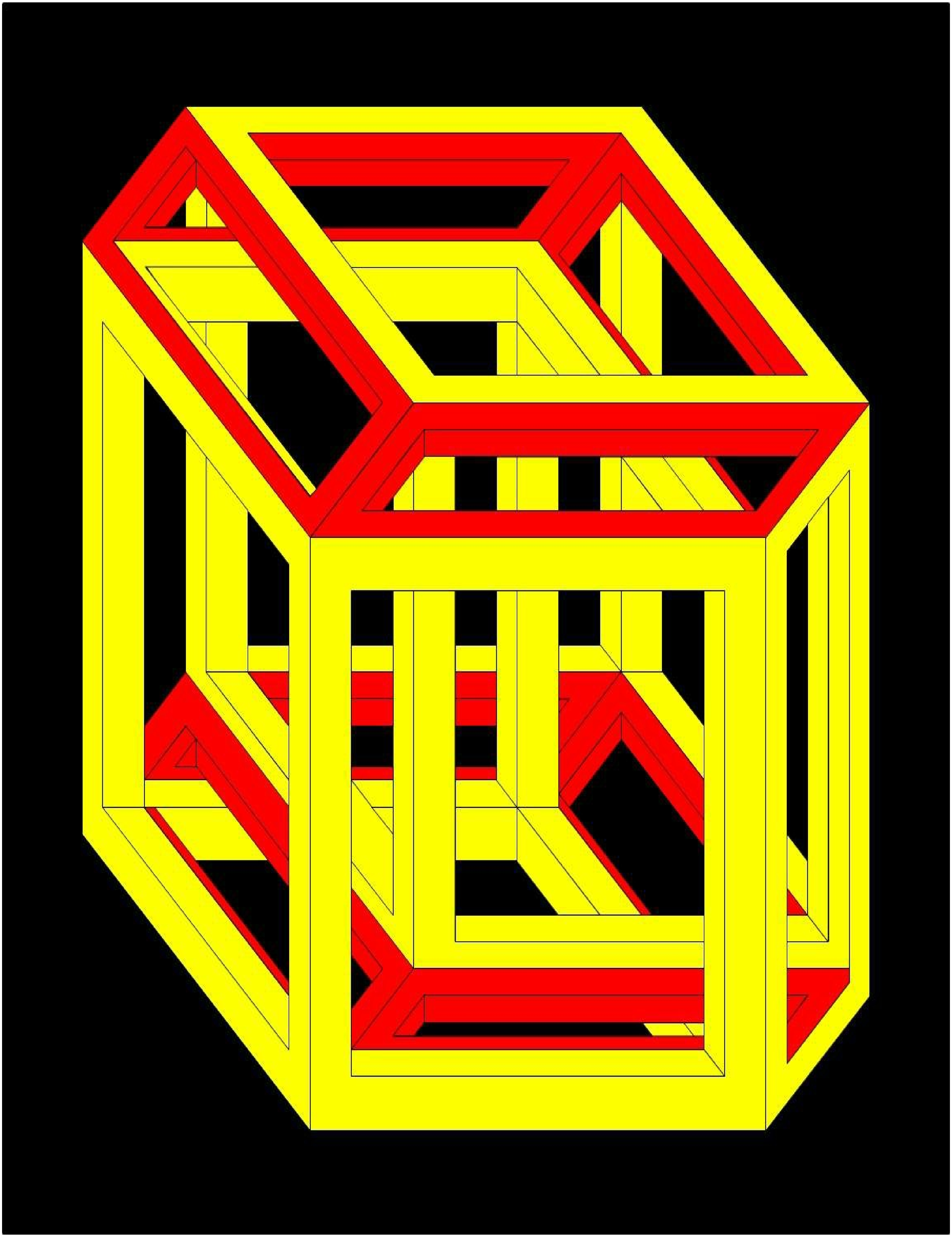

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A Visual Introduction to the Fourth Dimension
(Rectangular 4D Geometry)

Topher Books

ebook edition

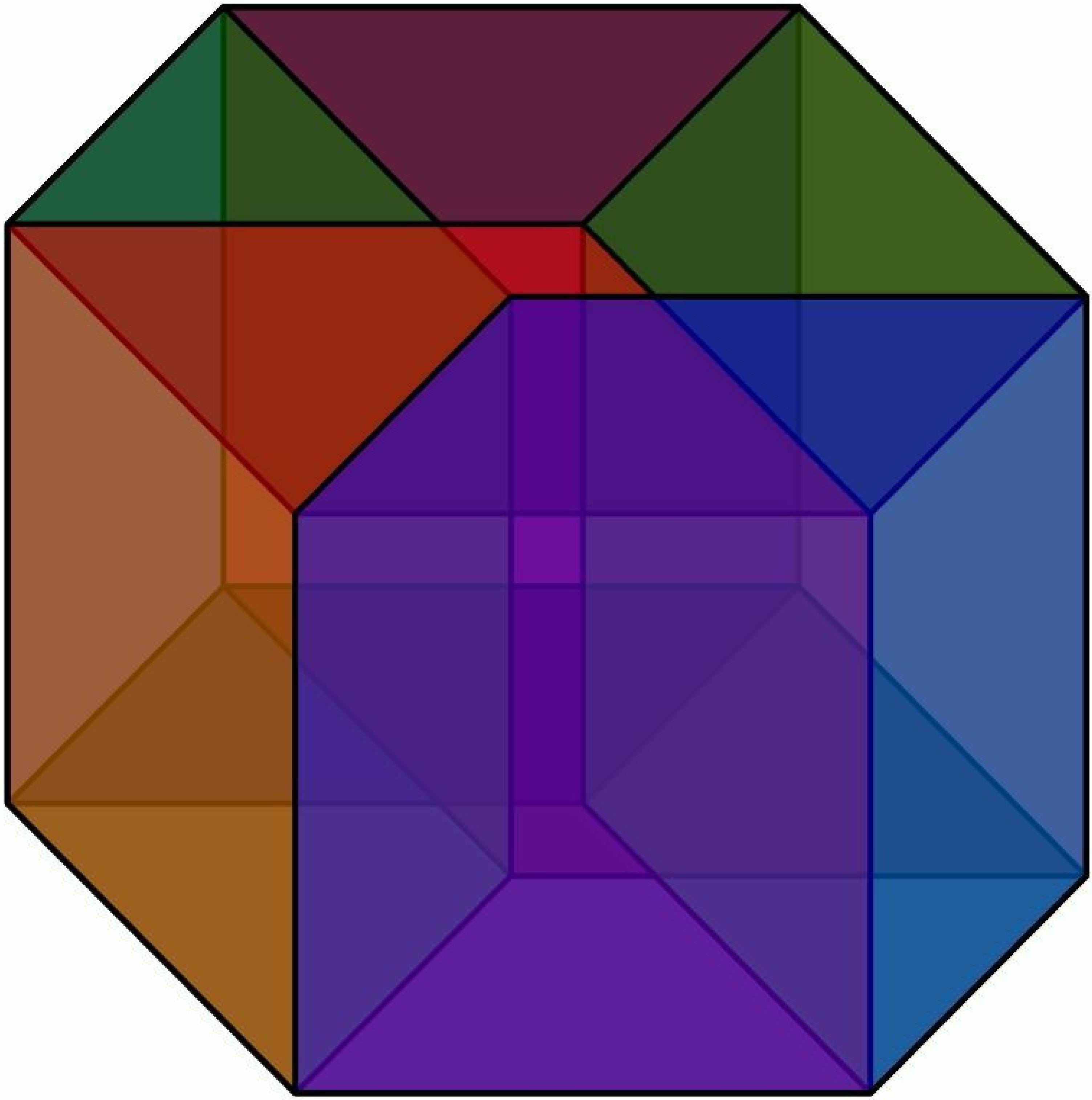
First edition published in January, 2013

Books > Science & Math > Mathematics > Geometry

Contents

Introduction

This book provides a **colorful**, visual introduction to the fourth dimension. It's not just a picture book, though. You will find clear explanations to help you understand the concepts, and a touch of personality (with a few monkeys!) makes this an engaging read instead of a dry math text. The content is very accessible as you don't need a strong background in mathematics to understand it, yet the concepts are advanced and detailed enough to satisfy the curiosity of a reader who does excel in math. This book focuses purely on the geometric aspects of the fourth dimension; there is no spiritual or religious component to this book. The geometric objects considered in this book are rectangular; the main 4D object discussed is the tesseract (a 4D hypercube), and it is described in great detail and accompanied by many instructive pictures. May you enjoy your journey into the fourth dimension!



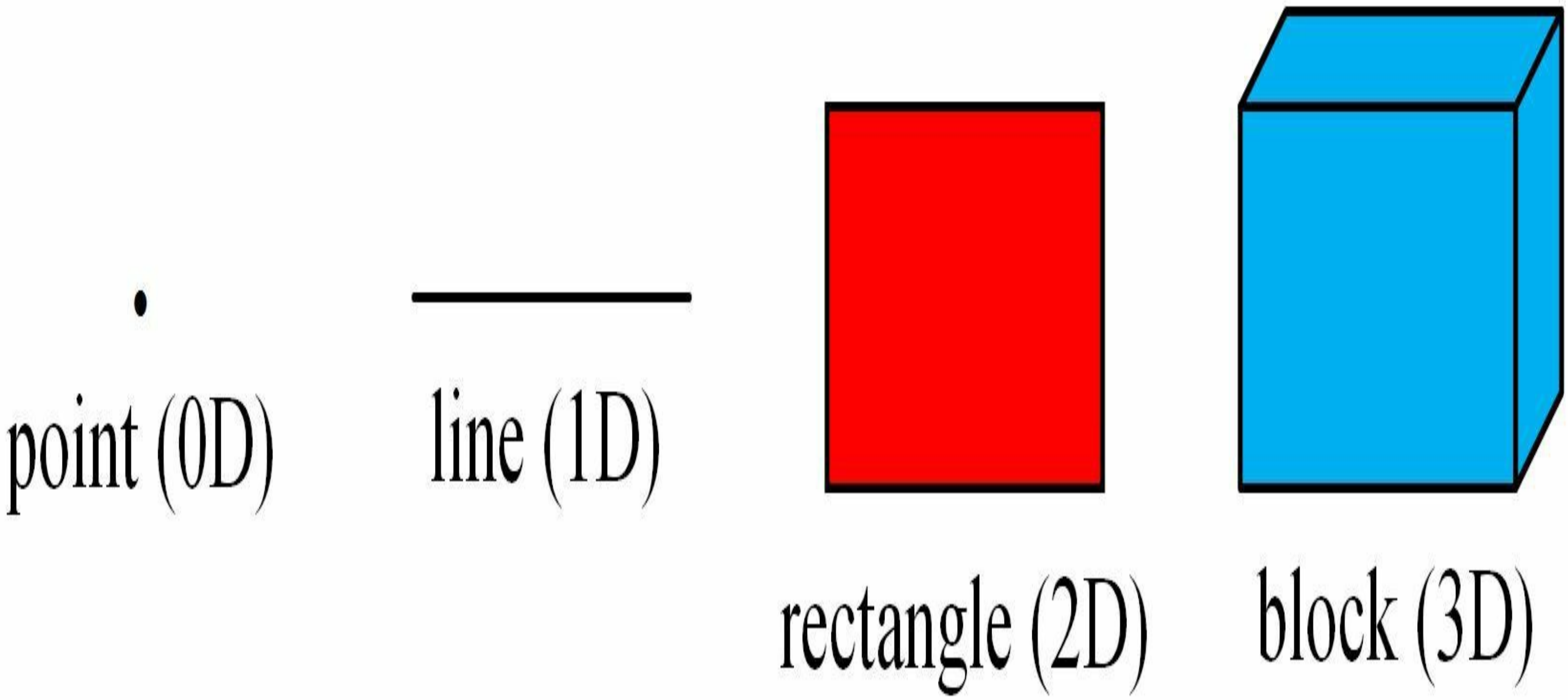
Chapter 0

What is a Dimension?

Before we get started, let's be clear what we mean by a dimension. As a simple definition, a dimension is a measure of extent. The dimensions of an object – no, not a plain old object, but a monkey with **RAINBOW**-colored fur, so this won't read like some boring math textbook – refer to different directions in which an object extends. The monkey has height (how tall she is – of course she's a girl, what else would she be?), breadth (how wide she is, shoulder to shoulder), and depth (front to back, or nose to tail); she is three-dimensional (3D).

However, there are different kinds of dimensions, like space and time – or if you want to get exotic, we can talk about the dimensionality of your thoughts or even your body odor. We need to explore the concept of a dimension a little further, so that you'll know exactly which type of dimensions we are discussing in this book.

Let's begin with a simple geometric example. An infinitesimal point has zero dimensions (0D; that's "zero dee," not "oh dee") – it doesn't extend in any direction. (Yeah, infinitesimal. Like that word? Surely you know the word infinity – the largest number, right? Wrong! If it were a number, you could add one to it and find a larger number. Infinity represents the concept that you can keep counting forever. Infinitesimal would be the smallest nonzero number, except that if you divide it by two, you get something smaller. So this definition is just as wrong as thinking of infinity as being the largest number. But sometimes an incorrect definition expresses the concept more clearly than a technically correct one.) That long parenthetical remark actually has a purpose. Say what? Yep: We spent a lot of time getting nowhere, just like the zeroth dimension! A line is one-dimensional (1D) as it only extends one way – like this sentence, which you can only read forward or backward. A plane is two-dimensional (2D); it has length and width. A block is three-dimensional (3D); it has length, width, and depth.



If you want the previous picture to look less boring, you have to use your brain and imagine that it consists of – from left to right – a monkey's thought, a monkey's tail, a monkey's shadow, and a monkey. (Sure, the author could have drawn monkeys, but that would deprive you of the opportunity to imagine your own monkeys. Actually, the real reason is that we're going to draw 4D objects, and they will probably be much easier to understand if they look more like rectangles and less like monkeys. We'll keep talking about monkeys, though, to spice up the text, but the drawings will all be plain, boring, straight-line stuff. Just remember that all of those straight lines are really monkey tails. Be careful not to poke them with your finger as you turn the page. Thank you.)

Our universe appears to be 3D because we can (oops, that should read "monkeys can") only move in three independent directions – north/south, east/west, and up/down. Any other direction is a combination of these. For example, when a monkey throws a coconut northeast, the coconut is essentially moving north and east at the same time.

So that's why we think of space as being 3D. What is? Look, try getting a monkey to move in some direction that's not a combination of forward/backward, left/right, and up/down. Good luck with that. When you are able to both do it and prove that it was done, you'll have a Nobel Prize (that's "no bell," not "noble"; also, it will be in physics, not a Peace prize).

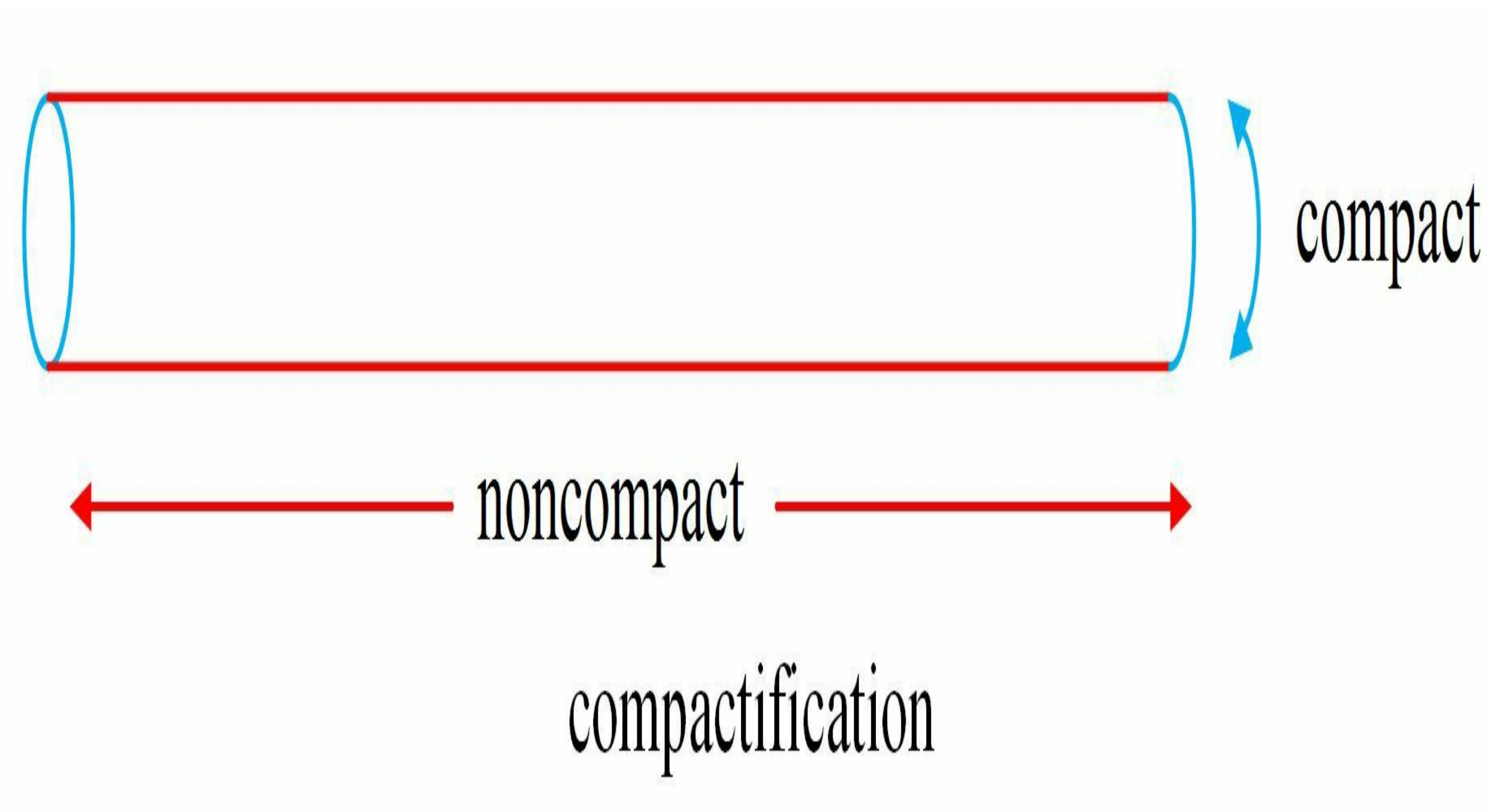
Space appears to be 3D, but spacetime is 4D. Time is a dimension in the sense that it is also a measure of extent – it is a duration. (Couch potatoes aren't doing nothing – double negative; yeah, I know I can cancel the minus signs. The activity is 0D with regards to space, but they are doing time. All that time we spend waiting in lines and traffic counts for something!) But time is clearly a different type of dimension than length, width, and depth.

In this book, we're interested in a fourth dimension of space. That is, a fourth dimension that is very much like length, width, and depth, but not a combination of these (i.e. not like northeast).

String theory actually predicts that our universe has more than just three dimensions of space. Our universe might actually have 9 dimensions of space. (That makes 10 dimensions of spacetime. If you heard it's 11, you're thinking of M theory. Maybe you thought it was 26; then let me correct myself and say that modern superstring theory predicts 9, while the original string theory had 25 – add time to make twenty-six.) But who's counting? Either way, we only perceive 3.

How can string theory possibly be right? How can anyone believe a theory that says there are nine dimensions of space, when it's pretty obvious that there are only three?

The other six dimensions may be hidden. They might be very tiny dimensions that wrap around in circles. For example, the three dimensions that we perceive may each be like the length of the cylinder illustrated below, while the extra dimensions may each be like the circumference. The circumference is tiny, whereas the length is infinite. If the extra dimensions are very tiny, that would explain why you can't see them and why you can't seem to move along a fourth dimension.



In this simple model for hiding the extra dimensions, we say that the extra dimensions have experienced compactification. They are compact dimensions. No, I don't mean to suggest that God put the universe in a trash compactor. But that does sound interesting. If you want to make string theory more popular, maybe we should add some monkeys to it. At least, we could come up with an improved theory and call it G-string theory! Just imagine 100 students enrolling for a course, where only ten of them actually expect to do theoretical physics.

As fascinating as the subject is, we're not going to discuss (any more) string theory in this book. I know; make a sad face. The focus of this book is on the geometry of a fourth dimension of space that is very much like the usual three dimensions. We're also not going to talk about compact dimensions (any further).

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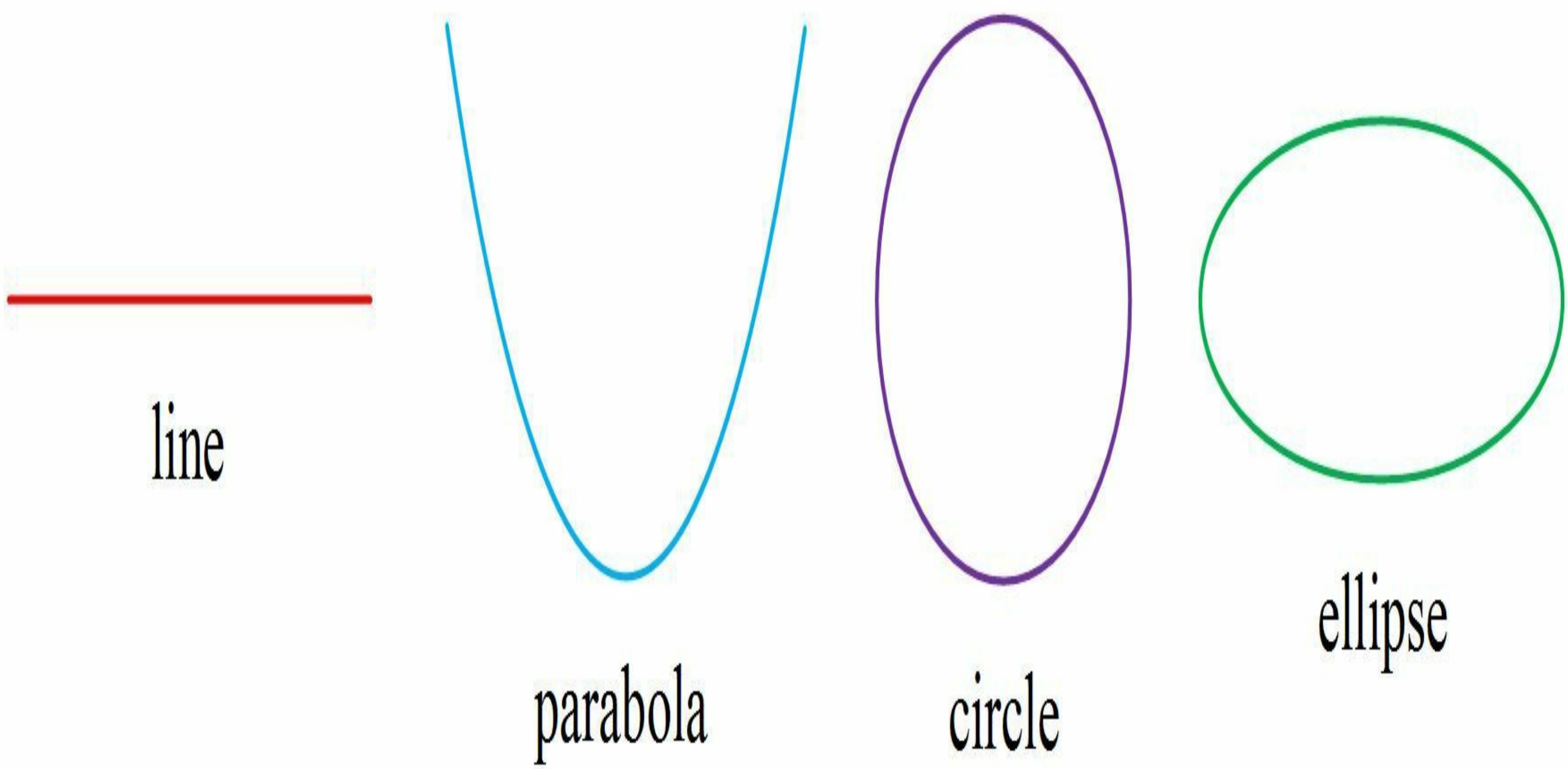
Chapter 1

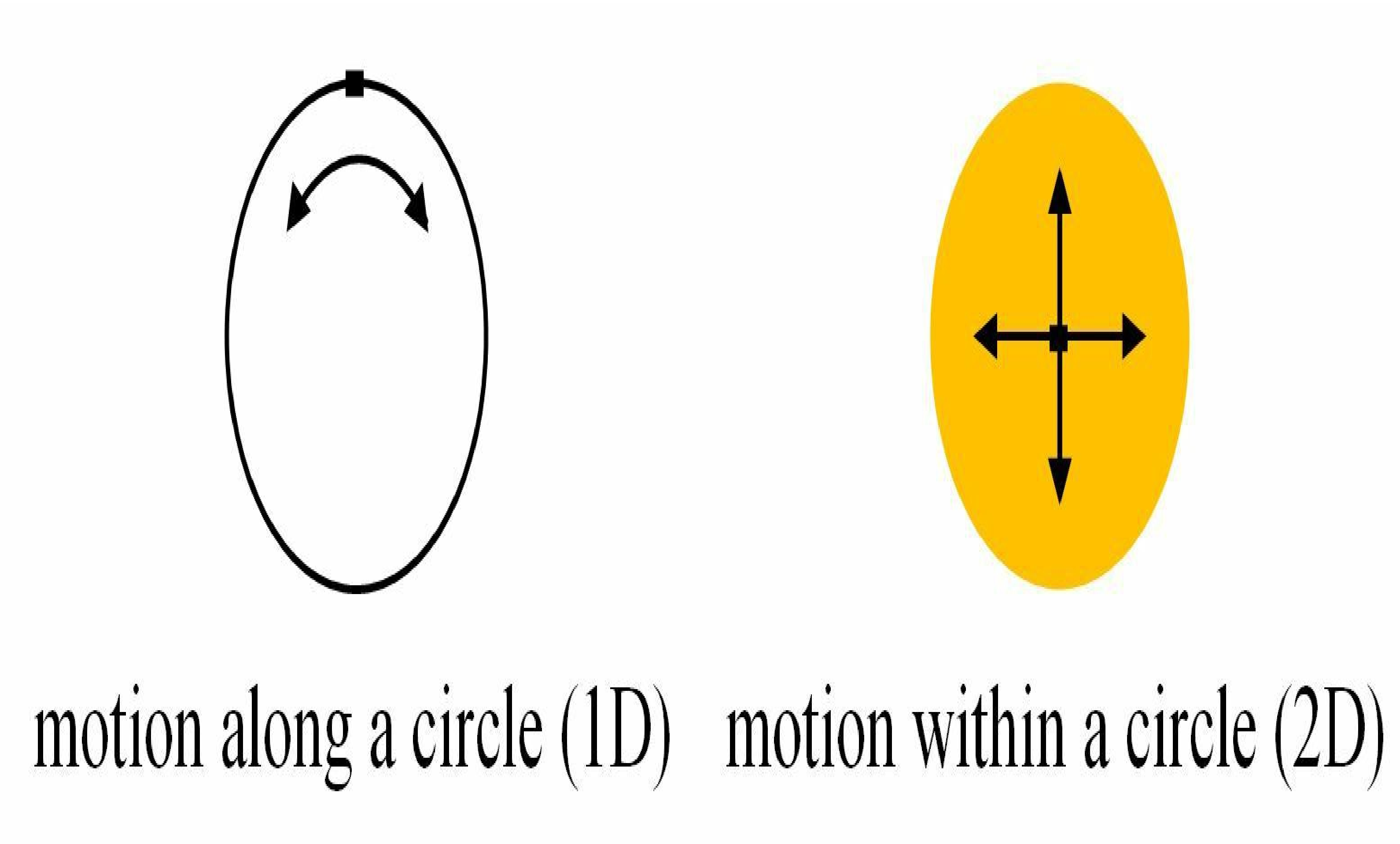
Dimensions Zero and One

Let's begin with the zeroth dimension. A 0D object is just a mathematical point. A single object in a 0D universe would take up all of the space. Motion wouldn't be possible. It would be the ultimate prison, where you could only "do time," as the saying goes. You couldn't do anything physical in 0D, but it might suffice for a purely philosophical or spiritual world. We could call the inhabitants philoso-sloths, for example; they have no substance, they just think (but there wouldn't even be physical brains, since the entire 0D universe would be a single point). (Yeah, sloths. No, not one of the seven deadly sins. Tree-dwelling mammals that like to eat fruit. No, not monkeys, but monkey will convey the idea.)

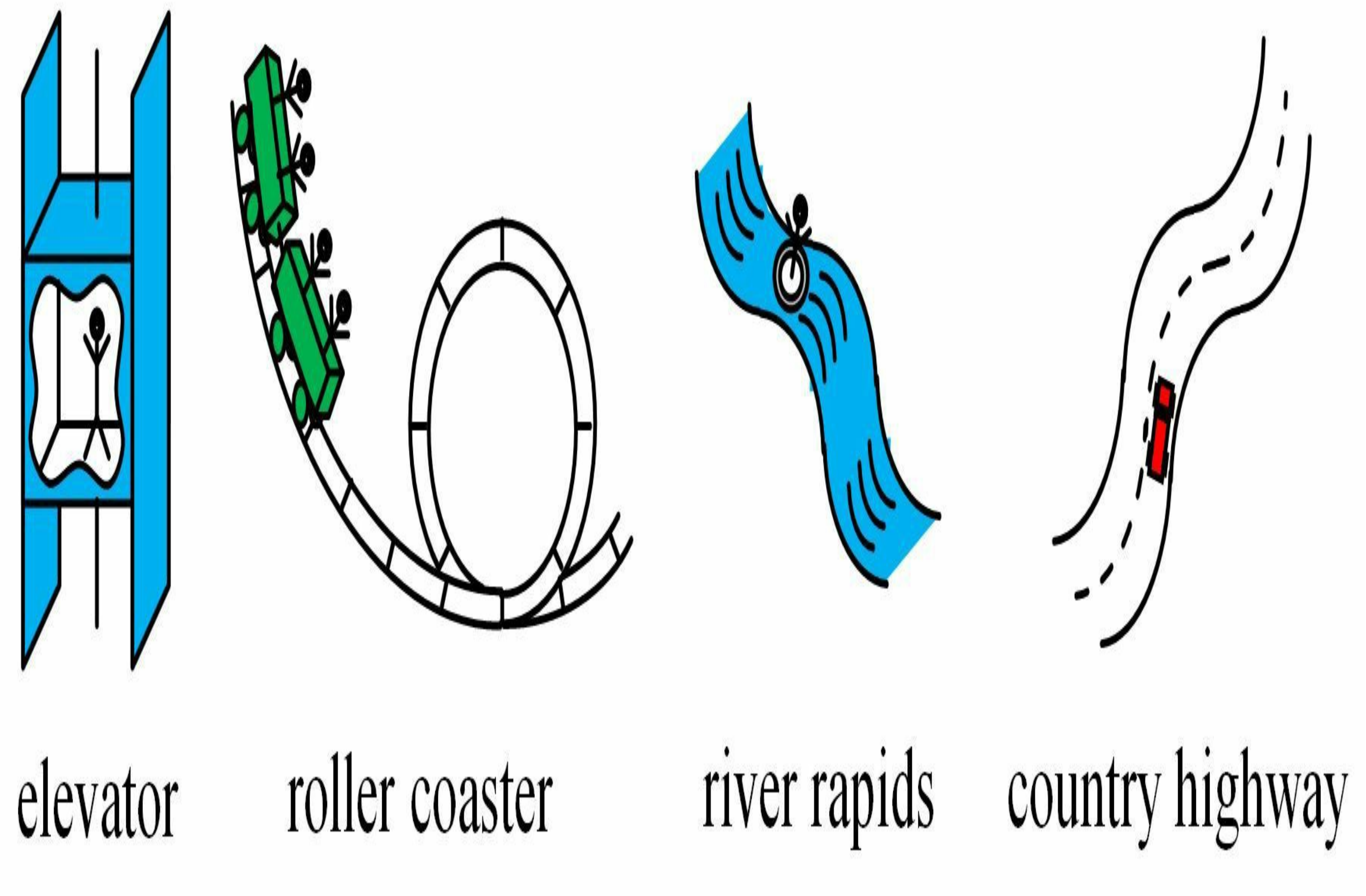
The first dimension could be a line, but it could also be a curve like a parabola or a circle. Curves are effectively 1D because any object living in such a space would have the same limited freedom that a line has: You can only go forward or backward. Think of the first dimension as a monkey tail, and a 1D object as a bead sliding back and forth along the tail.



A bead sliding around a circular monkey tail experiences 1D motion because it can only slide clockwise or counterclockwise. In contrast, a monkey wandering around inside of a circular room experiences 2D motion. Compare the pictures below.



You can relate to the first dimension, since many human (but we'll imagine monkeys instead) activities are largely 1D. Monkeys riding in a roller coaster, for example, follow a predetermined course. It just goes forward (except for the rare roller coaster that actually goes backward, too).

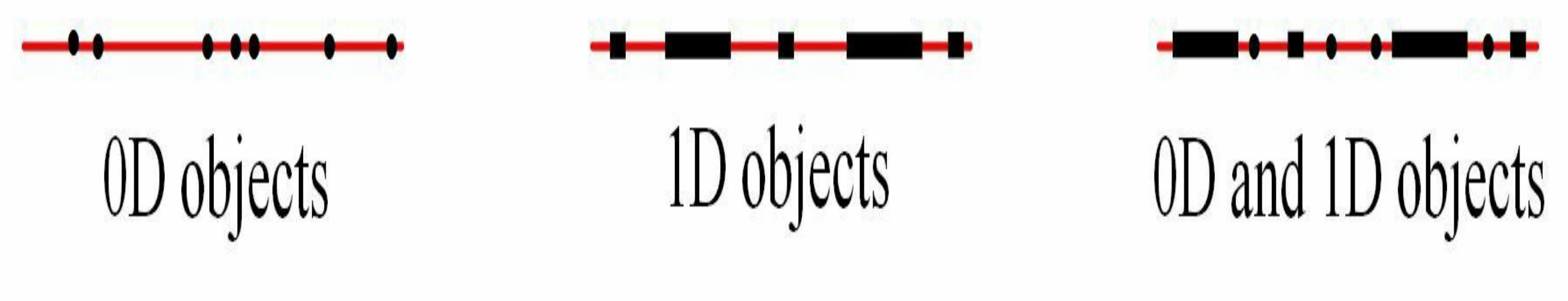


The first dimension seems too simple, right? Actually, it's so simple from our 3D perspective, that it's actually quite complicated to imagine a 1D world. What?! Look, two things you learn when you study physics are: (1) How to solve complicated problems, like a monkey skating down a wedge while the wedge slides sideways and (2) how to make easy things look difficult.

So let's see what could possibly be nontrivial in the first dimension. Oh, I don't mean a 1D physics problem, like solving for the motion of a monkey connected to two springs while accounting for air resistance and friction. Actually, that's pretty straightforward. It might involve calculus and the solution might be numerical, but the approach is straightforward. As I said, we learn how to make complicated problems seem easy.

What I have in mind are conceptual complications, like trying to design a 1D world that mimics our 3D universe. So let's explore how we can make something simple, like the first dimension, sound complicated.

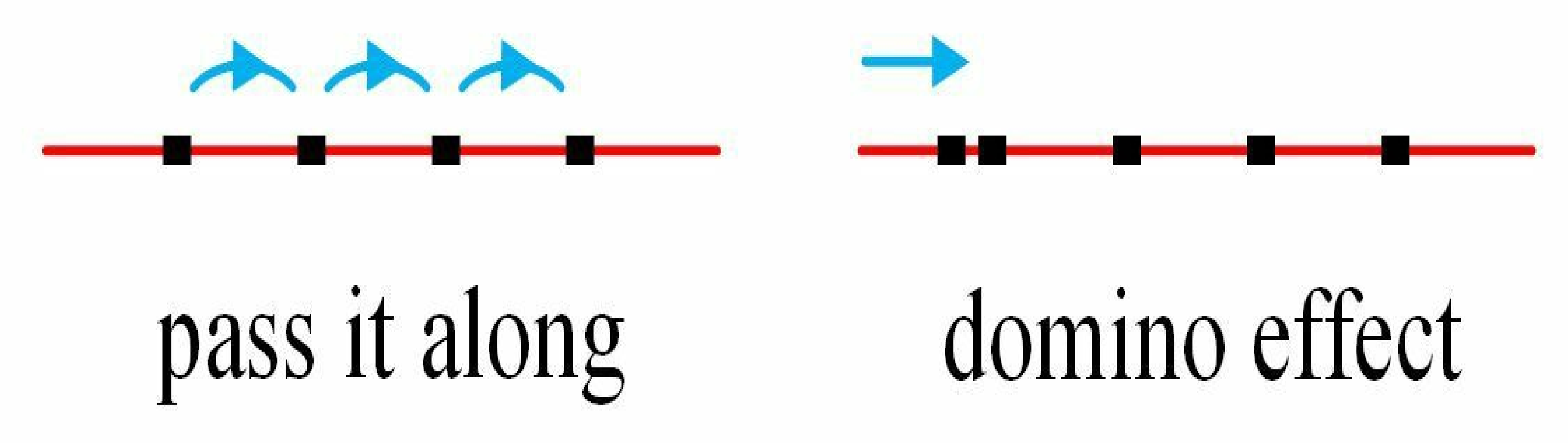
To help visualize a simple 1D world, consider the models illustrated below. There are 0D and/or 1D objects moving back and forth in these 1D worlds. Of course, the objects are really monkeys and bananas.



Have you noticed anything that might seem odd about living in a 1D world? Yeah, just think about those poor little 1D monkeys. Where would you put things like eyes, a mouth, an ear, and a nose, bearing in mind that there are just two ends and it would be kind of pointless to have these things inside? An eye would itself be pointlike, and would only see a point – which would always be filled with a point of "light." But what should seem odd isn't so subtle. Don't all of these 1D monkeys seem to be trapped?

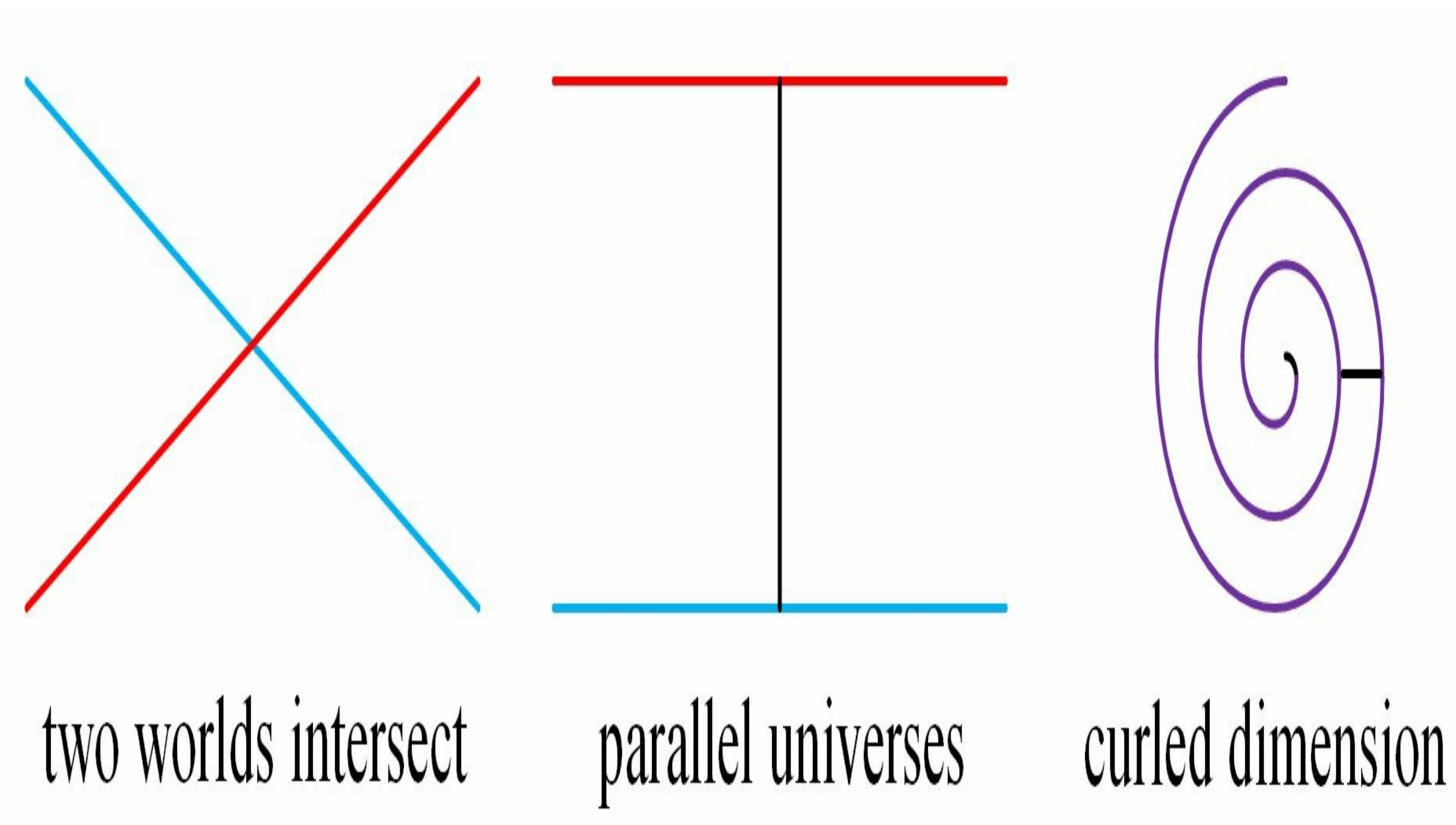
Look! All of these 1D monkeys are basically in prison. How do you get past your nearest neighbors? Is the first dimension cannibalistic? Eat 'em! "Excuse me, sir. Passing through. Gotta eatcha!" Literally passing through!

If the 1D monkeys can't pass through their nearest neighbors, communication would be a challenge. Monkeys could send messages like schoolchildren passing notes in class, or maybe more like dominoes. But if you want to design a 1D world, it might be best if objects have some transparency so that some objects, at least, can pass through others (like light through glass).

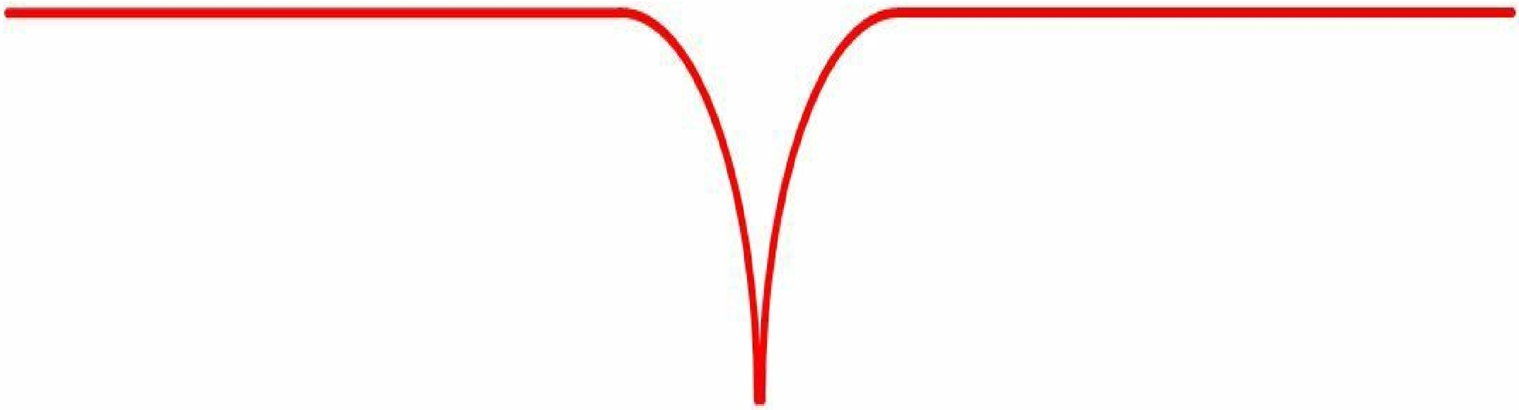


Establishing the basic laws of physics in order to try to design a 1D world that mimics some features of our universe, that's where the challenge is. For example, if you want matter to be made up of "atoms," how do you propose to make orbits in a 1D universe? One object can't go around another (unless you want to be really creative and allow a particle to make an "orbit" by traveling through wormholes, like the hypothetical hyperspace orbit figure that follows). A back-and-forth motion would be more natural than an orbit. If you're designing your own 1D universe, you could think of the fundamental forces as being springs (but if instead you want to take the laws of our universe and set N = 1 for the number of dimensions, you're stuck with whatever force laws that gives you). As long as we're pretending to design a universe, we might as well strive to do it without any preconceptions derived from our own 3D experience.

Following are some pictures of interesting features of hypothetical 1D worlds. Here we have intersecting worlds, parallel universes, and a curled dimension with a wormhole (a "shortcut" through spacetime).

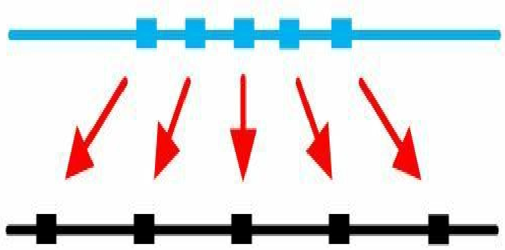


A 1D black hole would look something like this (except maybe it shouldn't reach the limiting "point" until infinity).

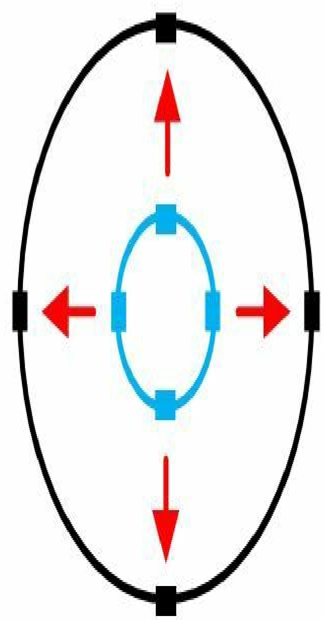


1D black hole

Remember, in this boring slide show, the objects are really monkeys. Or in the case below, they are the suns of 1D monkey solar systems, which are getting farther apart as the 1D universe expands.

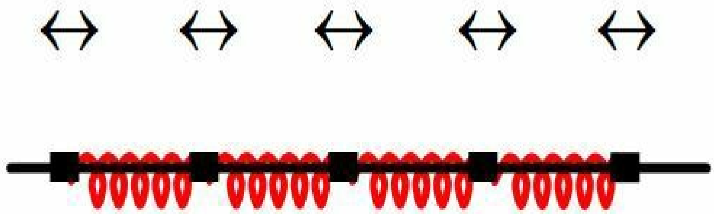


expanding linear universe

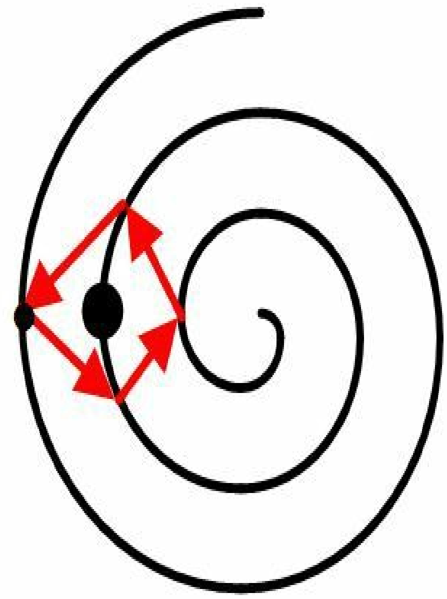


expanding circular universe

The spring-like forces and hyperspace "orbit" ideas that we alluded to earlier are illustrated here:



hidden springs



hyperspace orbit

In an effort to finally get to the fourth dimension, which hopefully you're interested in since that's the title of this book, let's move onto the second dimension.

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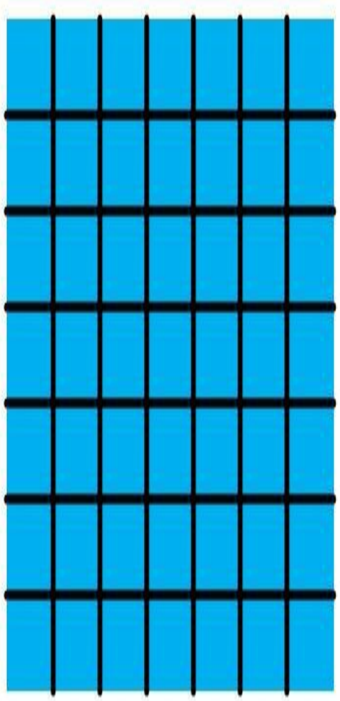
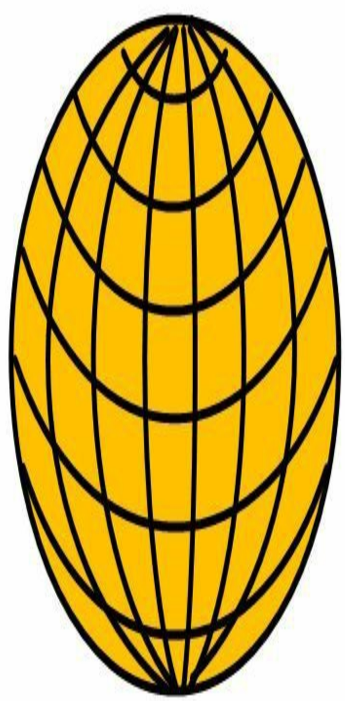
Chapter 2

The Second Dimension

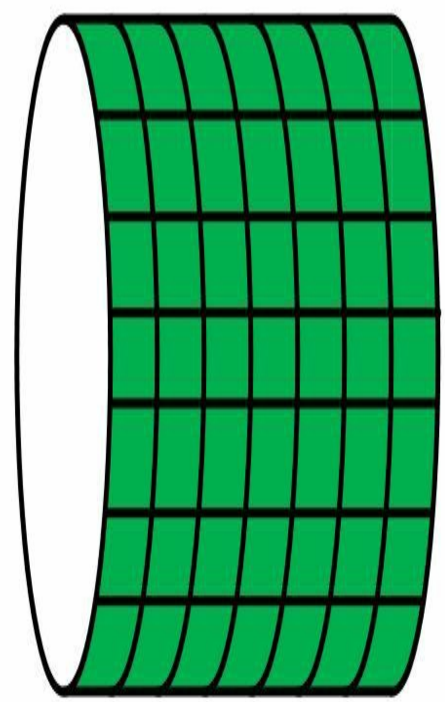
A plane is 2D, but a 2D world doesn't need to be flat; it could be curved like a sphere or a cylinder. A monkey in a plane could move in two independent directions – north/south or east/west. Similarly, a monkey confined to the surface of a sphere (so just like the monkey in the plane, she can't go up or down) could only travel north/south or east/west. Walking around in an open field is a largely 2D human activity.



plane

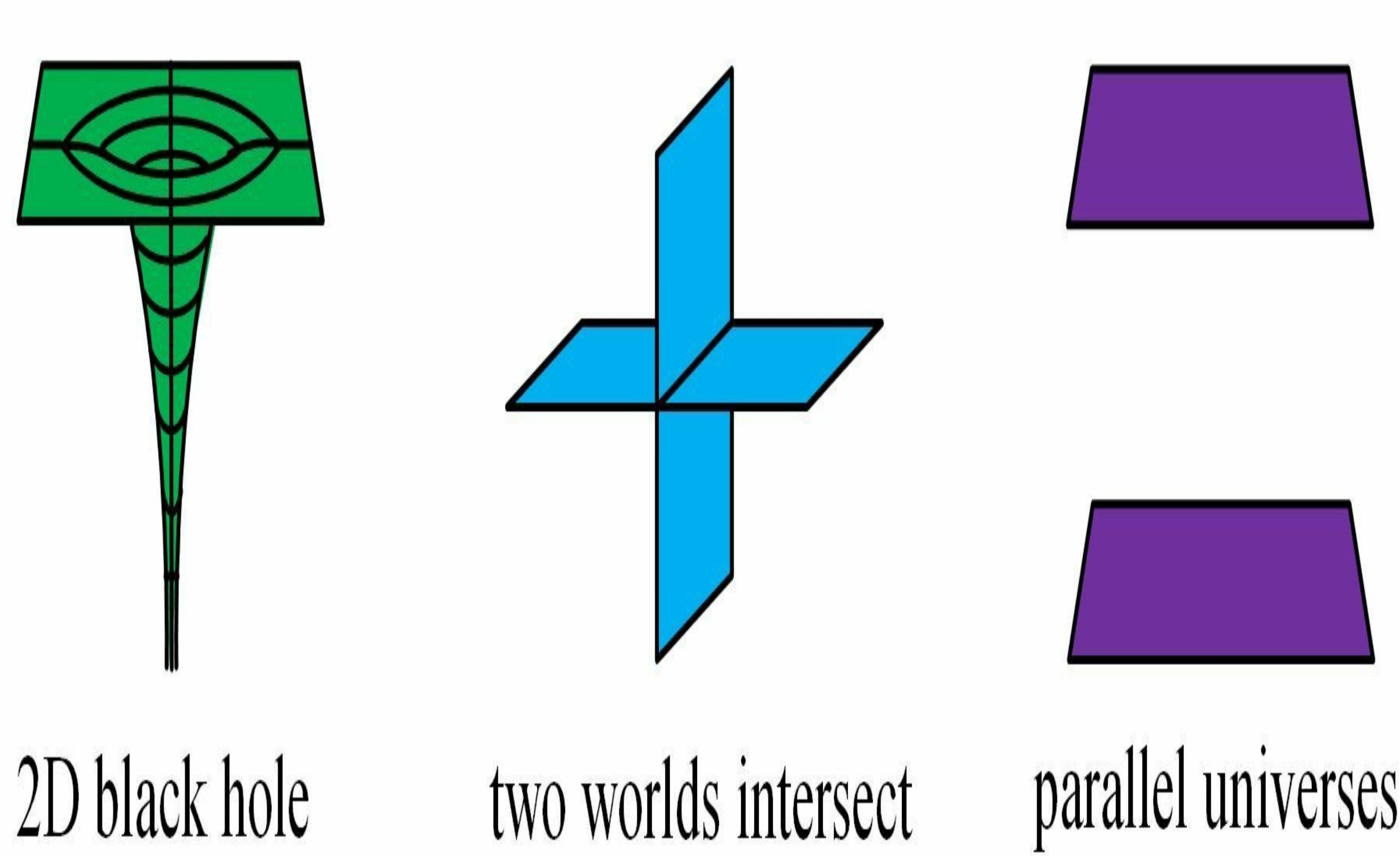


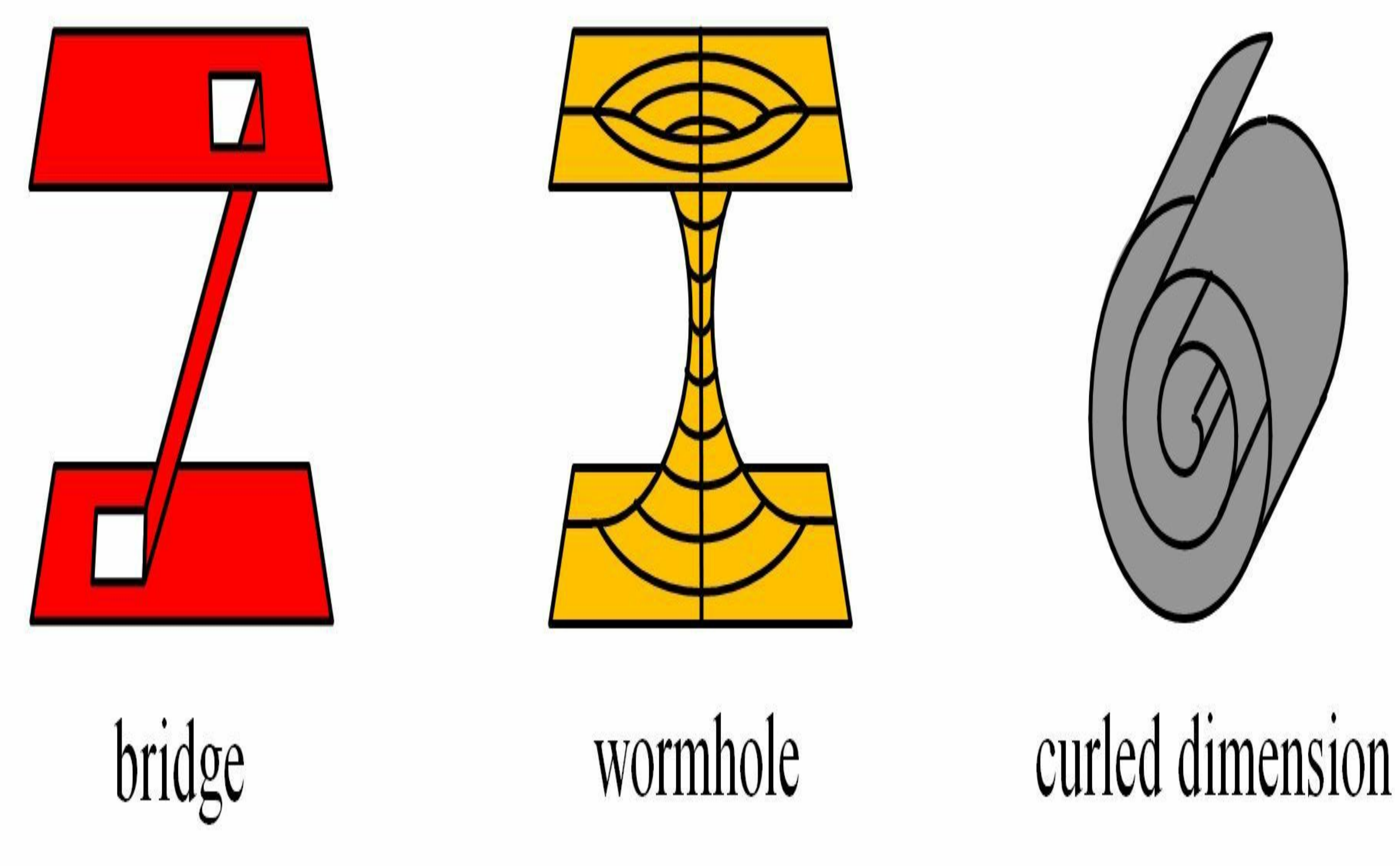
sphere



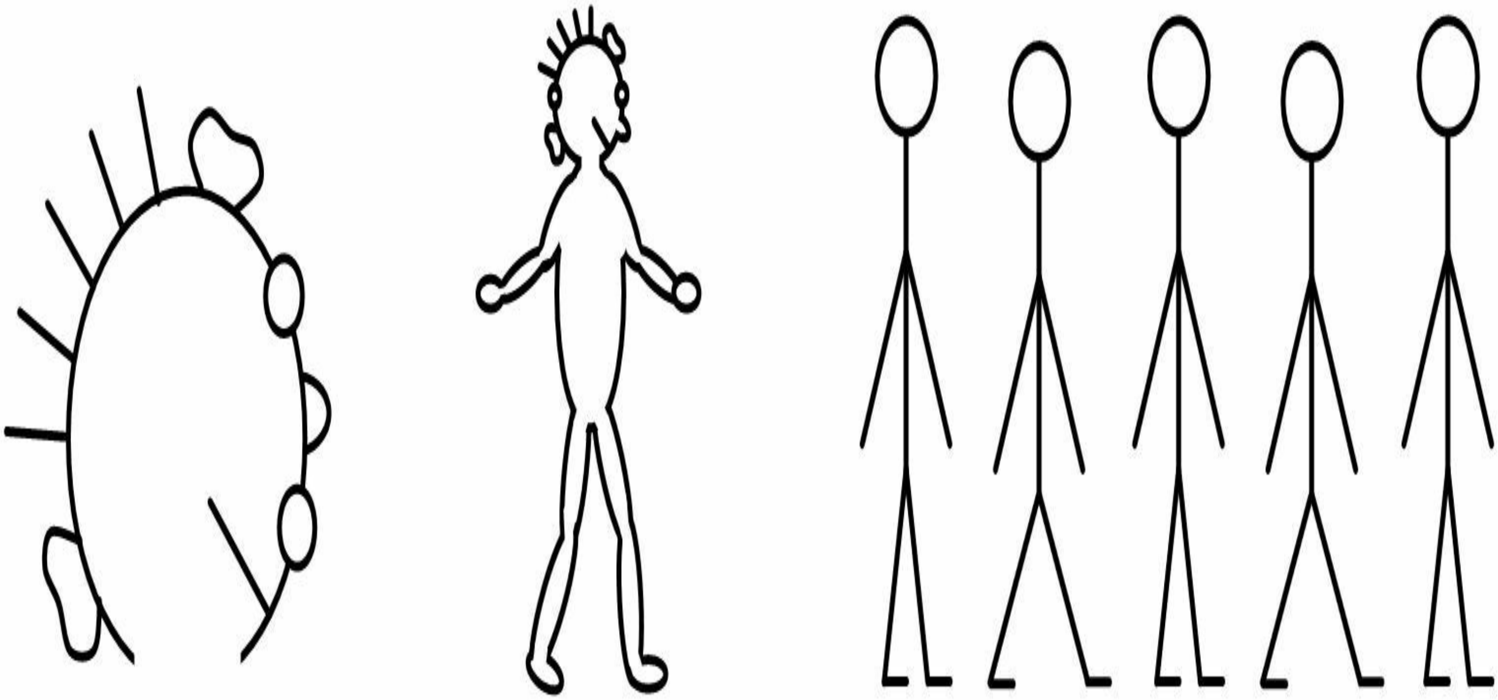
cylinder

The following figures show a black hole, intersecting worlds, parallel universes, a bridge, a wormhole, and a curled dimension in 2D. (Just in case you can't read the text in the figures, note that the main text also describes the pictures.) Notice that two lines intersect at a point and two planes intersect at a line. We can use patterns like this to predict what the fourth dimension might be like. For example, by analogy, if two 3D universes (hyperplanes) were to intersect, they would meet at a plane.



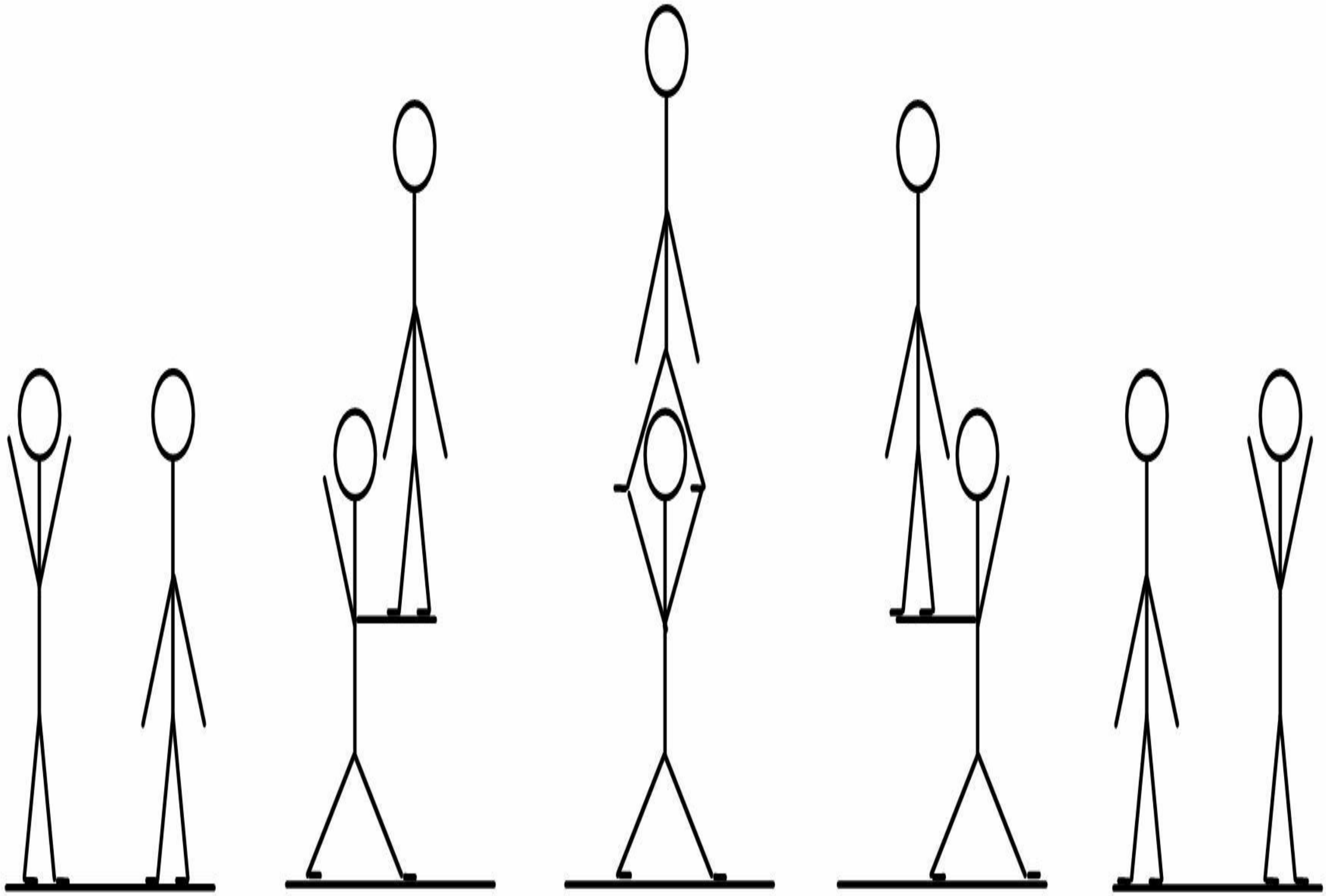


A 2D world would actually be quite unlike 2D drawings that we often draw on paper. When we draw a 2D picture on a piece of paper, we often draw the surface of a 3D object that we see with our eyes. If we draw a monkey (what else?), for example, we draw the eyes inside the outline of the head (where else would you put them?). A 2D monkey living in a 2D world wouldn't have eyes <u>inside</u> his head, though. Even walking would be much different: A 2D monkey <u>can't</u> put one foot in front of the other; the same foot would always be in front. (Just imagine tails in the pictures that follow.)



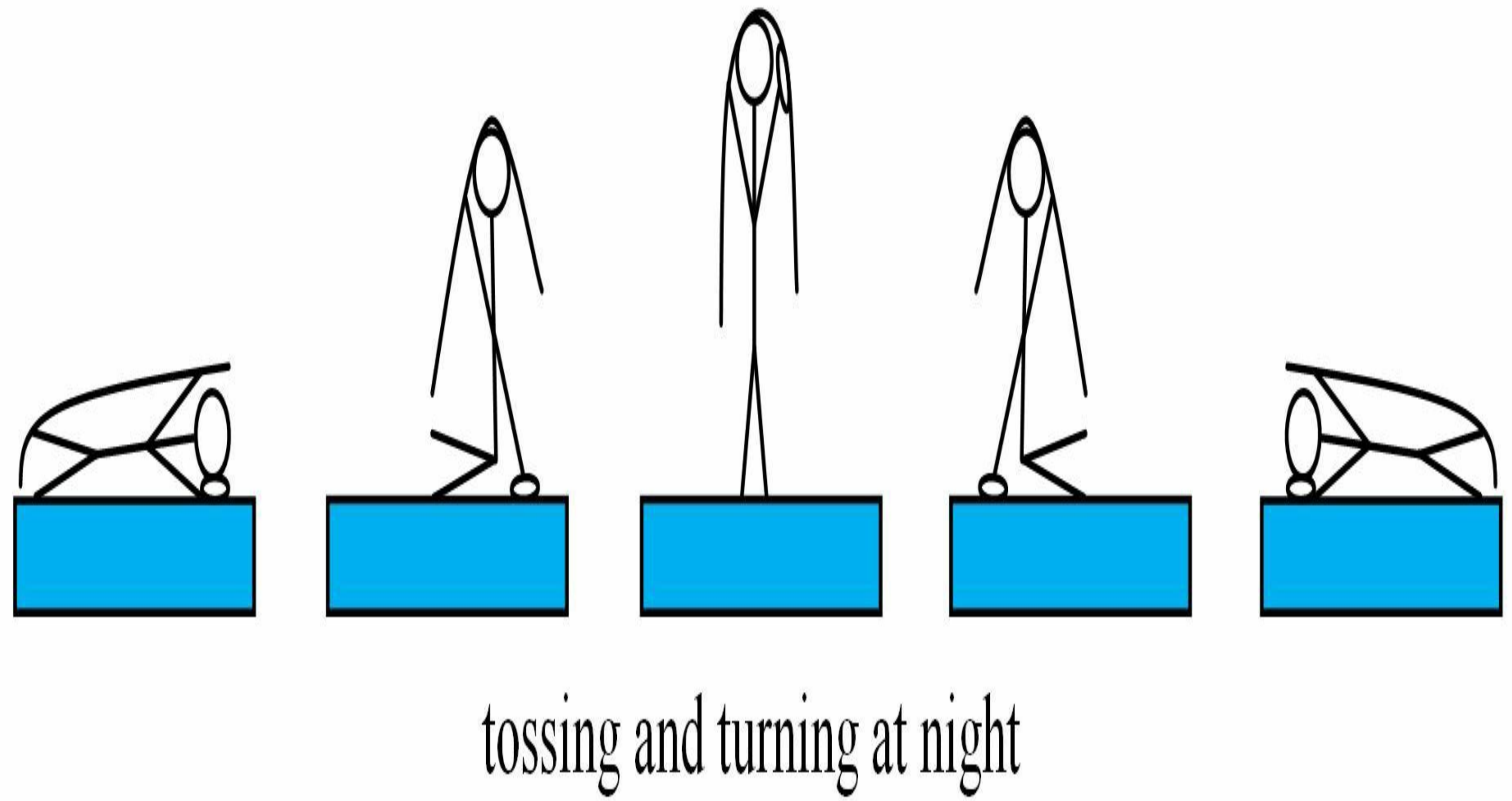
2D face　2D humanoid　same foot always in front

Unlike the first dimension, 2D beings can pass their nearest neighbors. But unlike 3D, they can't pass around them – they must pass over or under. Just imagine the guy in the front row of a crowded movie theatre going out for some popcorn: He's not going to be the most popular fellow!

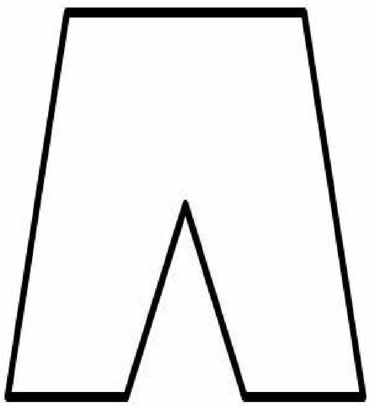


passing through

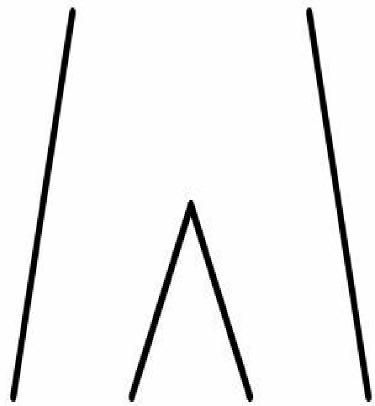
Let me ask you a personal question: Do you toss and turn at night? "What does that have to do with the price of tea in China?" But it does! If you toss and turn at night, be grateful that you don't live in the second dimension! See the poor little monkey (pin your own tail on the monkey, please) in the following picture as he turns over on his bed (yes, this monkey is a boy – did you think they would all be girls?).



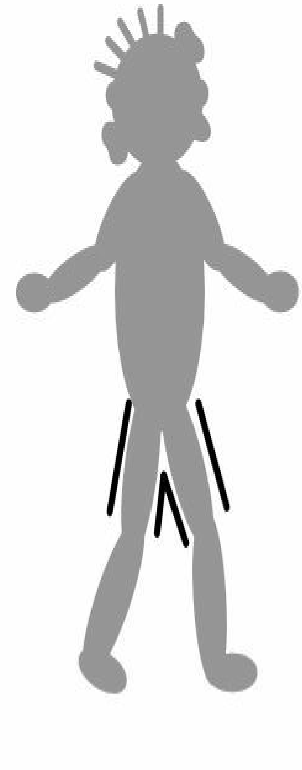
Trying to clothe the 2D monkeys is also a curious matter. The top left figure in the picture below shows what you might intuitively want to draw for 2D shorts, but there is a major problem with them: How would any monkey put a leg through them? There are <u>no</u> openings! If you remove the top and bottom edges to create openings, the shorts would fall apart. The pants and overalls shown in the bottom figures would easily slide off, while the shirt in the bottom right figure covers the monkey's eyes. A fashion show in a 2D world would be quite unlike anything we have in 3D.



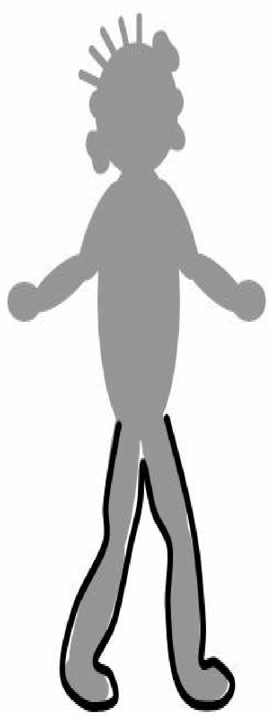
not wearable



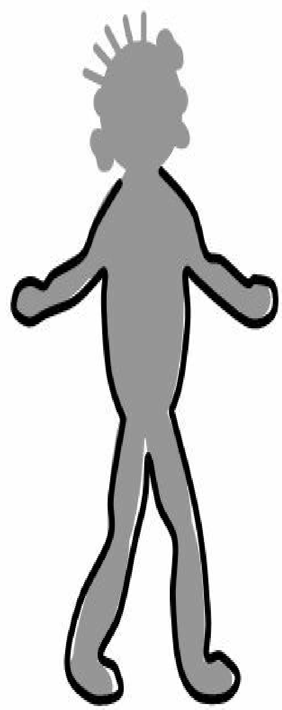
shorts in 2D



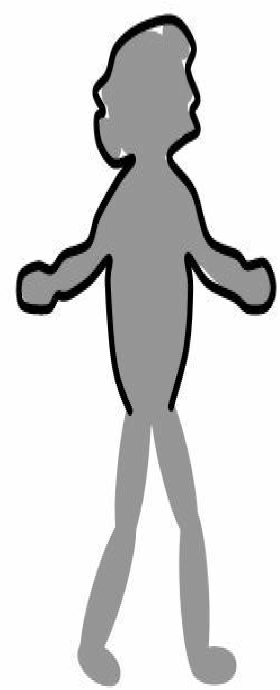
shorts fall apart



pants fall off

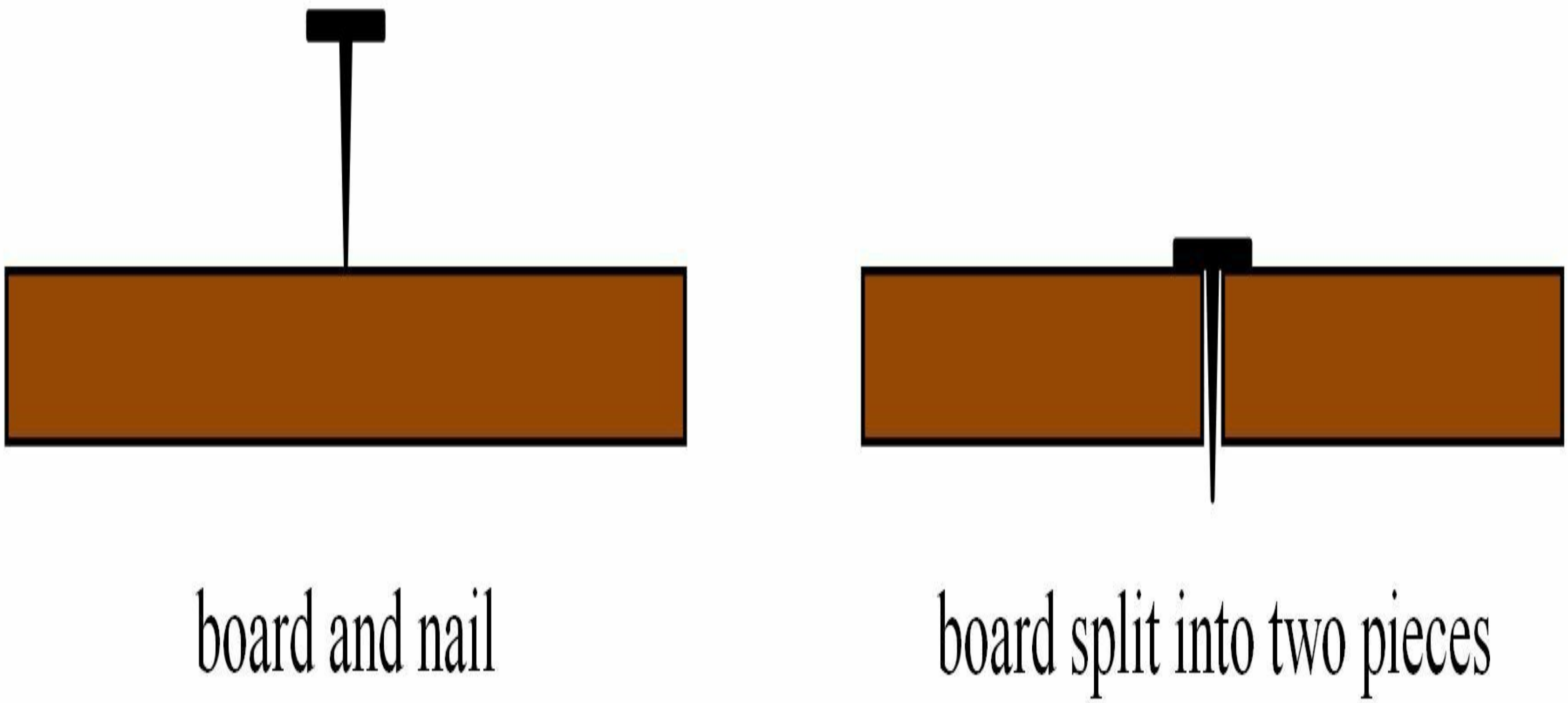


overalls slip off

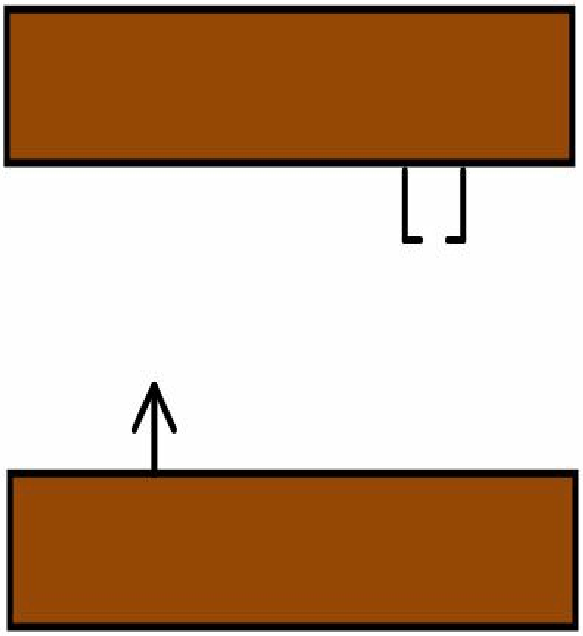


shirt covers face

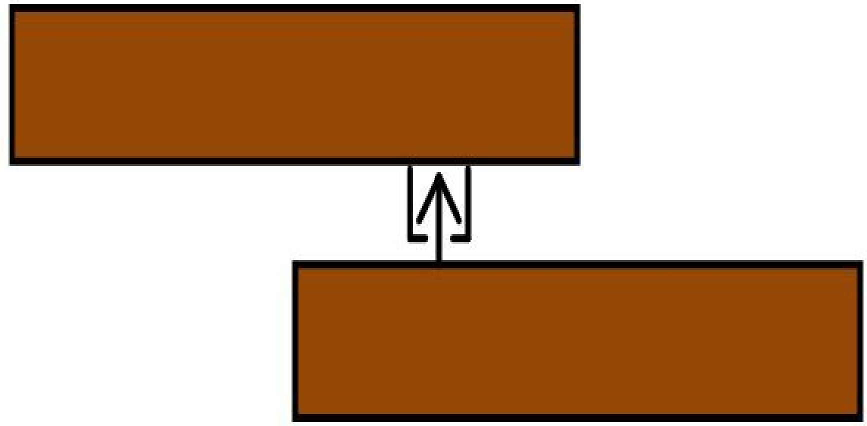
Suppose the 2D monkeys would like to build something. Let's consider some basic tools that they might use. What simple tools come to mind? Hammer and nail, maybe? If you hammer a nail into a board in 2D, the nail splits the board into two pieces! A nail serves a saw in 2D. If you want to joint two pieces of wood together, you need some other method.



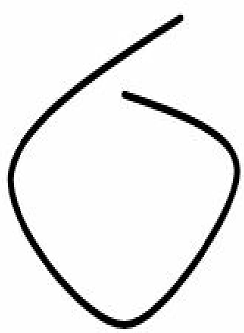
Similarly, it's impossible to tie a knot in 2D because the string can't go around itself. The pictures below show a couple of ways that objects might be connected in 2D. Tape and glue would surely be precious commodities in a 2D world.



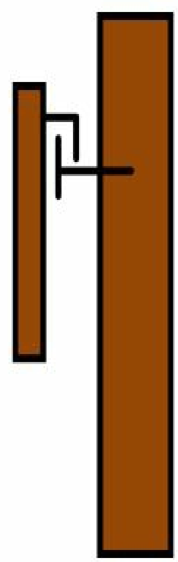
separate



fastened

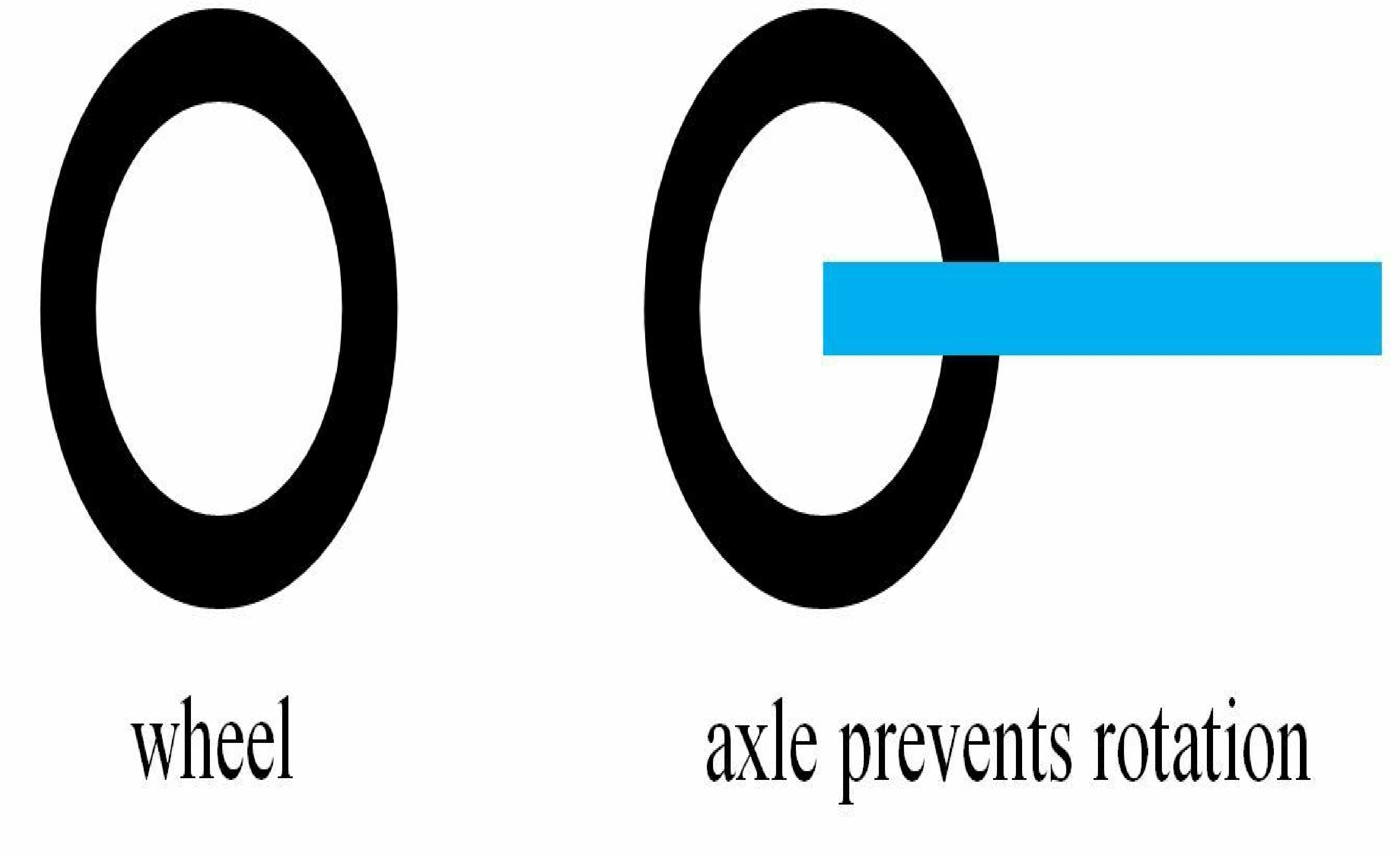


can’t be tied

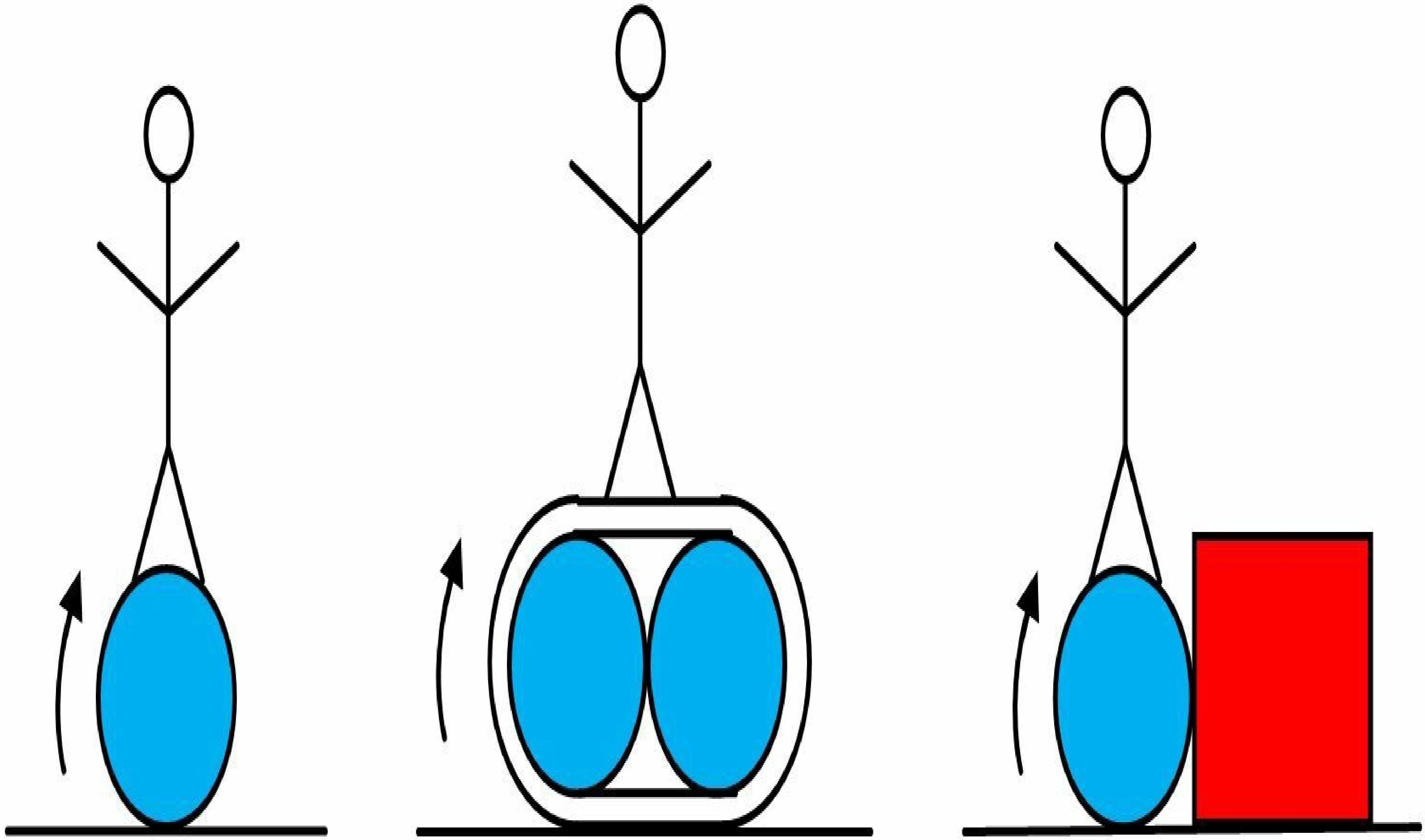


wall hanging

How about the invention of the wheel? Essential to transportation, right? Suppose you want to make a wagon, bicycle, or car in 2D. There's a major problem: You can't use an axle to join the wheels together! The wheels of a 3D car have a rod that connects them; the rod is perpendicular to each wheel. In 2D, you <u>can't</u> make the axle perpendicular to the wheels, since they must all lie within the same plane.

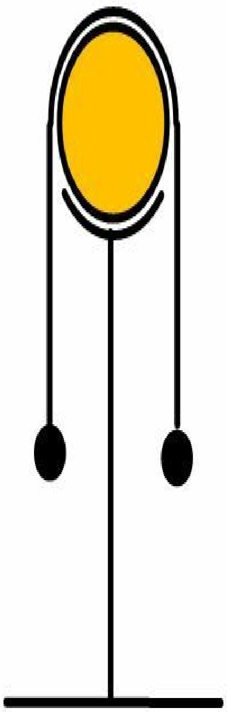


Maybe 2D monkeys could stand on top of the wheels and roll them like loggers.

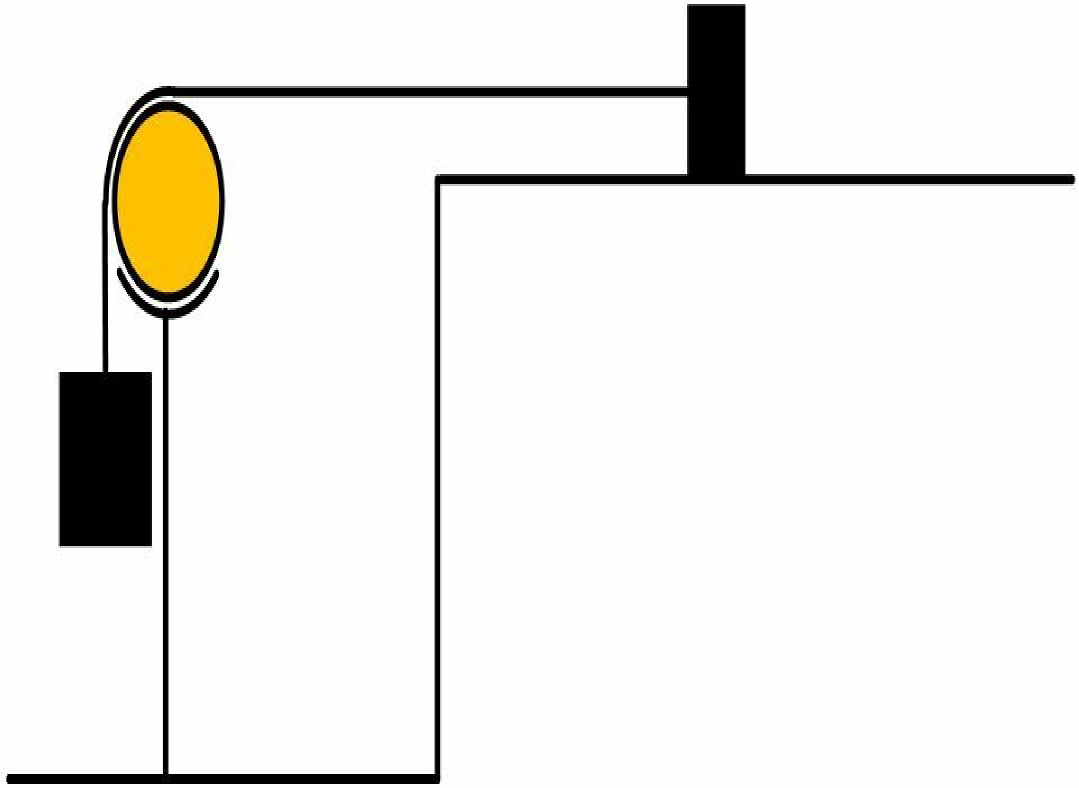


rolling 2D tank transportation

Another simple machine that would be different is the pulley. In 3D, pulleys also involve a wheel-and-axle design. Nonetheless, it would be possible to make a pulley in 2D without an axle: The wheel could simply sit in a cradle, using a lubricant to prevent it from rolling out with friction. Note that the cord <u>won't</u> slip off of the top (whereas 3D pulleys require grooves in the edges).

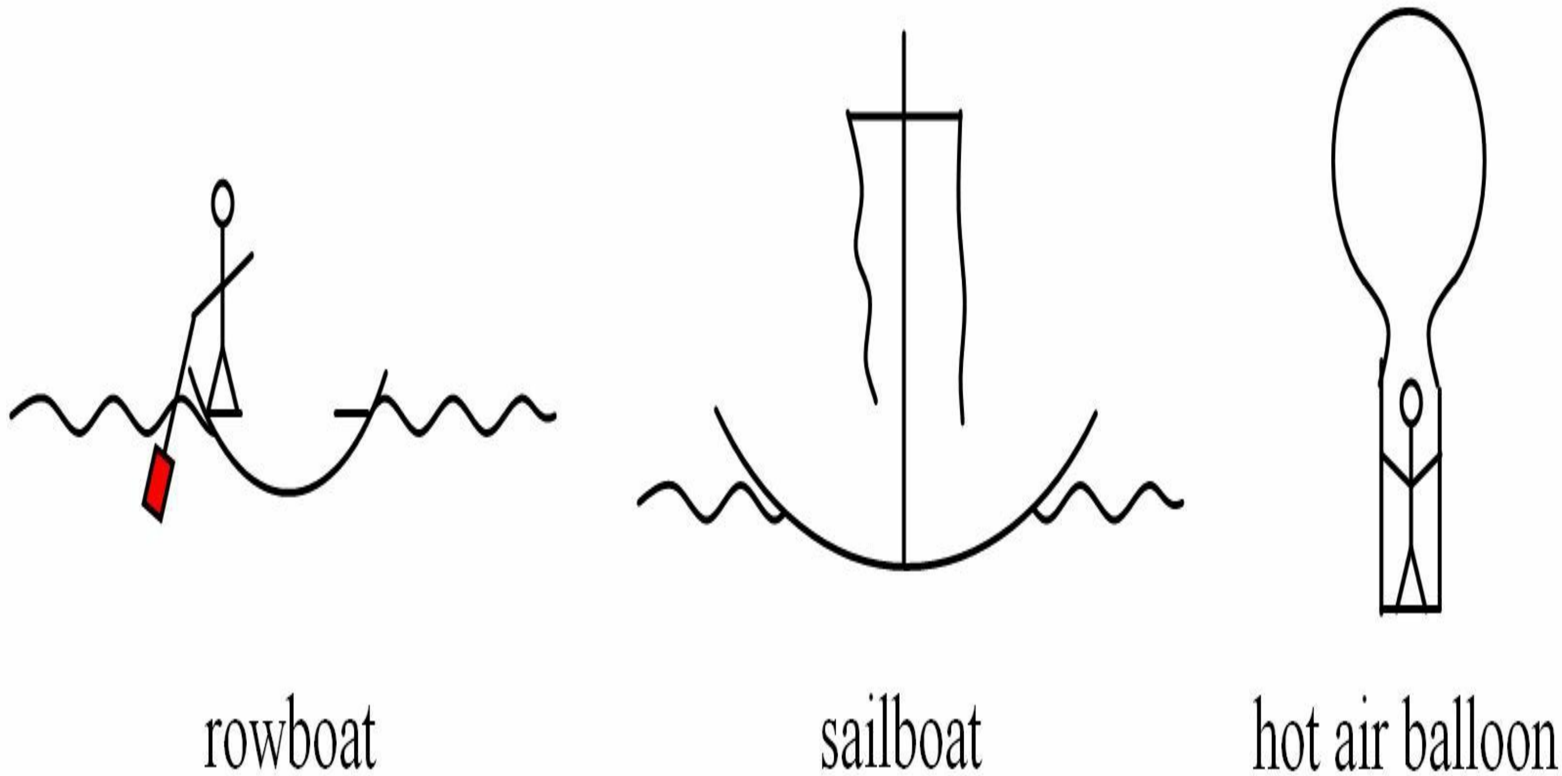


pulley

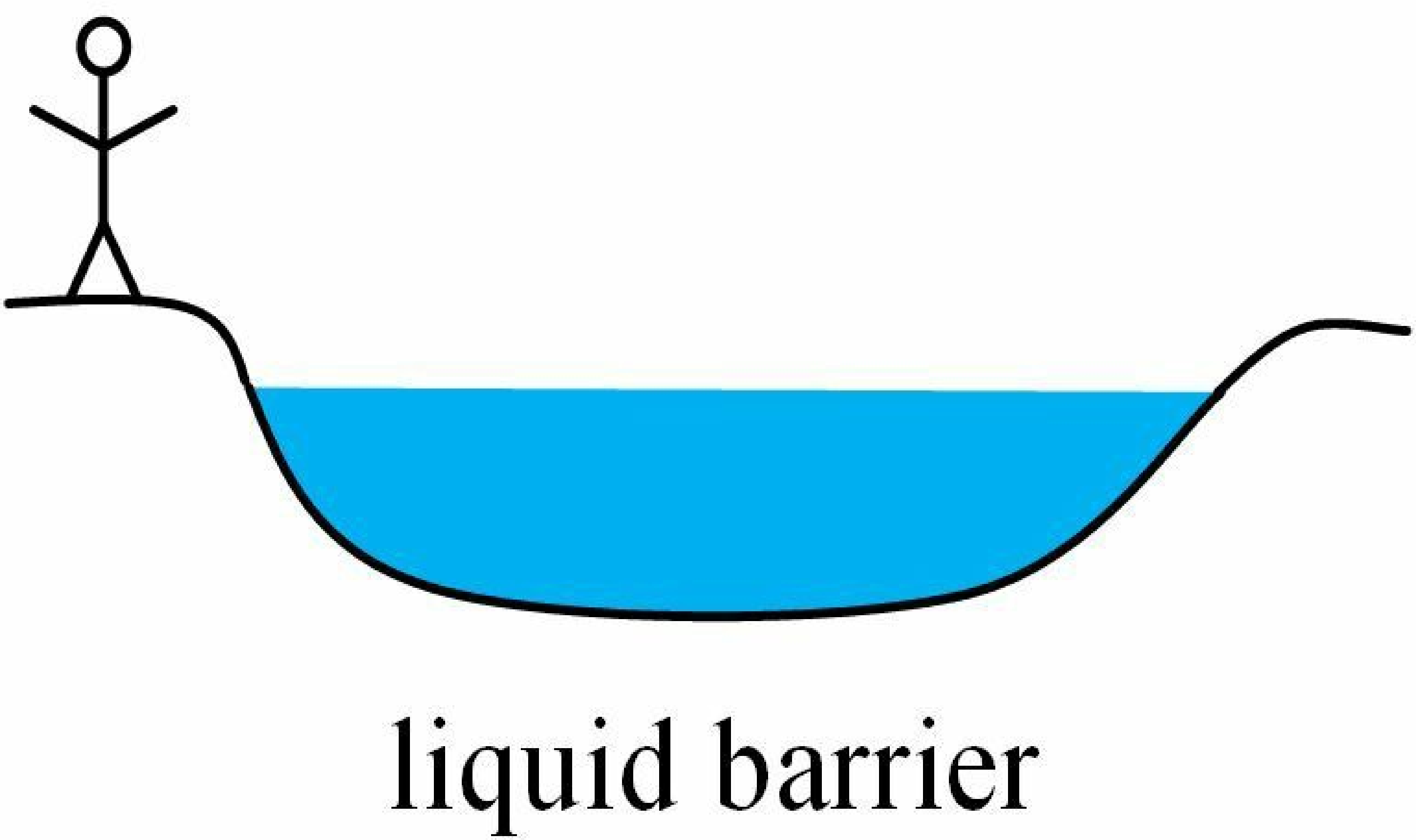


pulley transportation

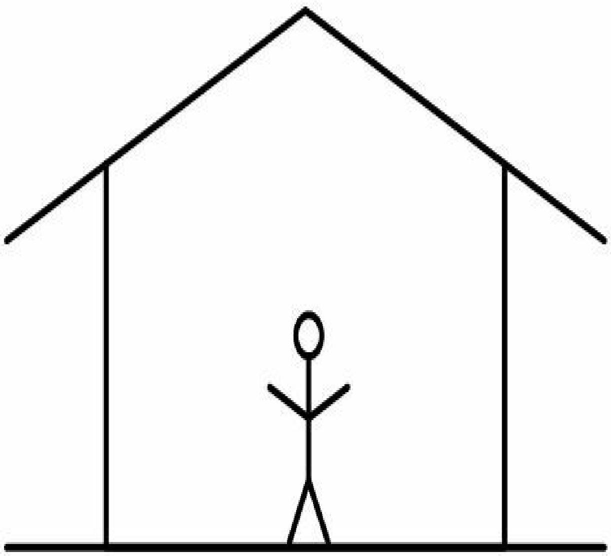
A simple arc would serve as a basic boat. A monkey rowing the boat in 2D would place the oar at the back end of the boat. It wouldn't be possible to make helicopters or airplanes that use propellers in 2D, but a hot air balloon could provide a means of air travel.



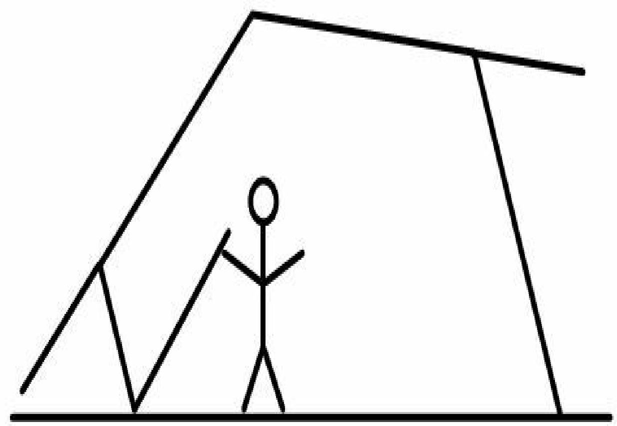
Note that a lake would be a barrier in 2D; monkeys would have to cross the lake – since the only way around it is to go all the way around the world.



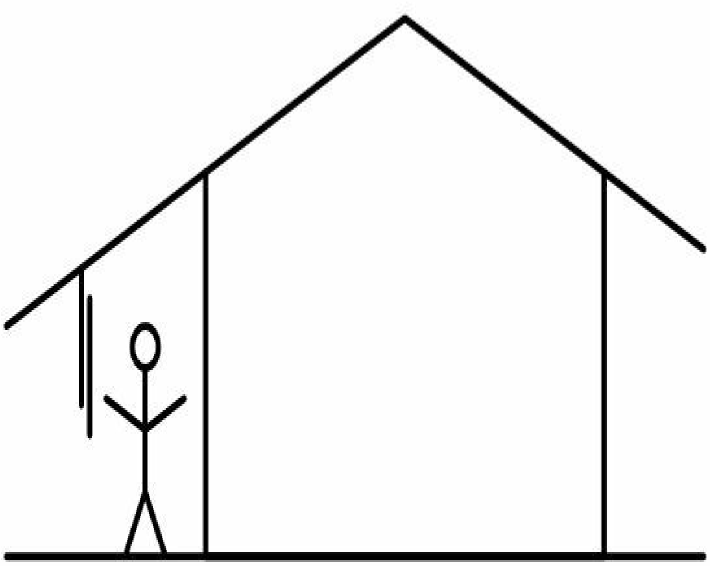
A 2D house would also be considerably different from a 3D house. A door would swing either upward or downward; it could have a hinge at the top or bottom, whereas a door in 3D is hinged along the left or right edge. But if a 2D monkey opens that door, the house will fall over! The images below show a couple of ways for an architect to circumvent this problem. Note that a 2D monkey would be "trapped" in a square prison, whereas a cube is needed for a 3D prison.



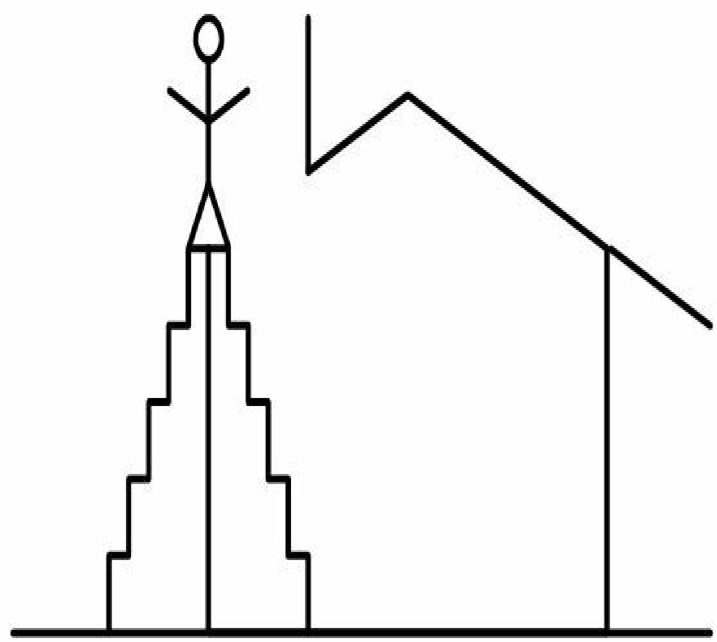
trapped



opening a door

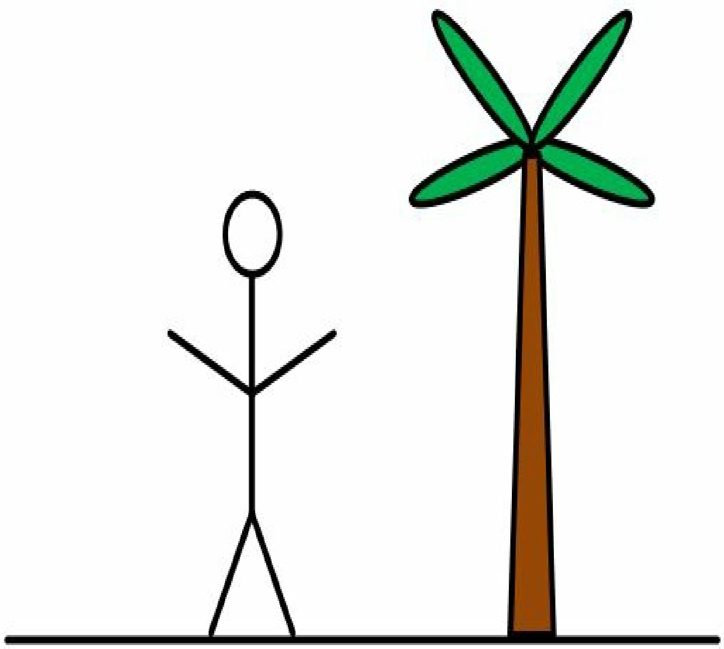


entry way with sliding doors

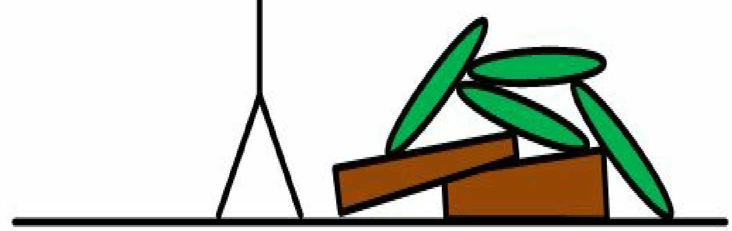


roof entry

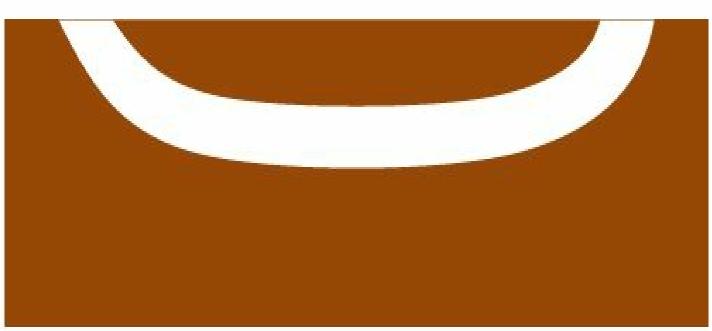
Let's consider a few more features of a 2D world before we move on up to 3D. In 2D, a tree would be a barrier. Our 2D monkey friends would either have to climb over them or chop them down. If a 2D monkey digs a tunnel, it will collapse. There is similarly a major problem with pipes: Any supports placed inside the pipe to prevent it from collapsing also prevent the passage of water through it. It looks like life in the second dimension wouldn't feature indoor (or any type of) plumbing. Telephone or power cables are possible. Just pass these over a pole in 2D; they won't fall off. But there better be a way to pass through the poles!



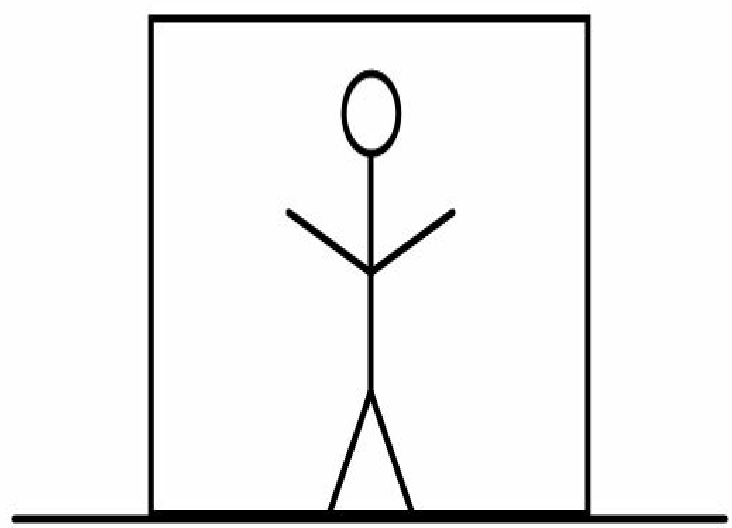
2D barrier



chopped tree

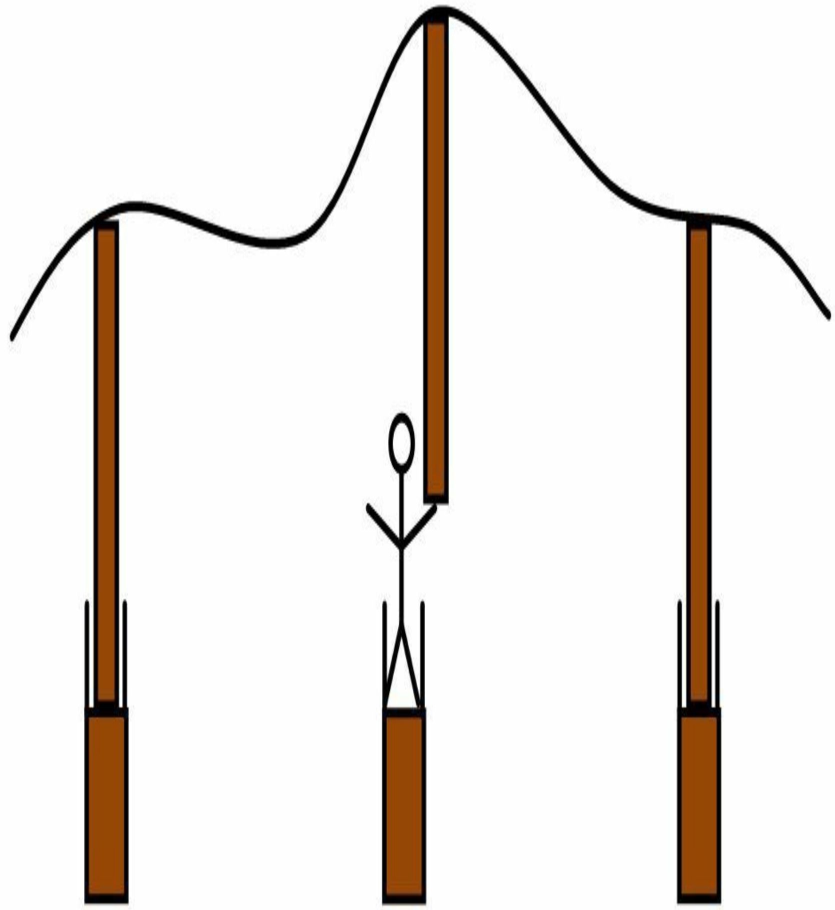


impossible tunnel



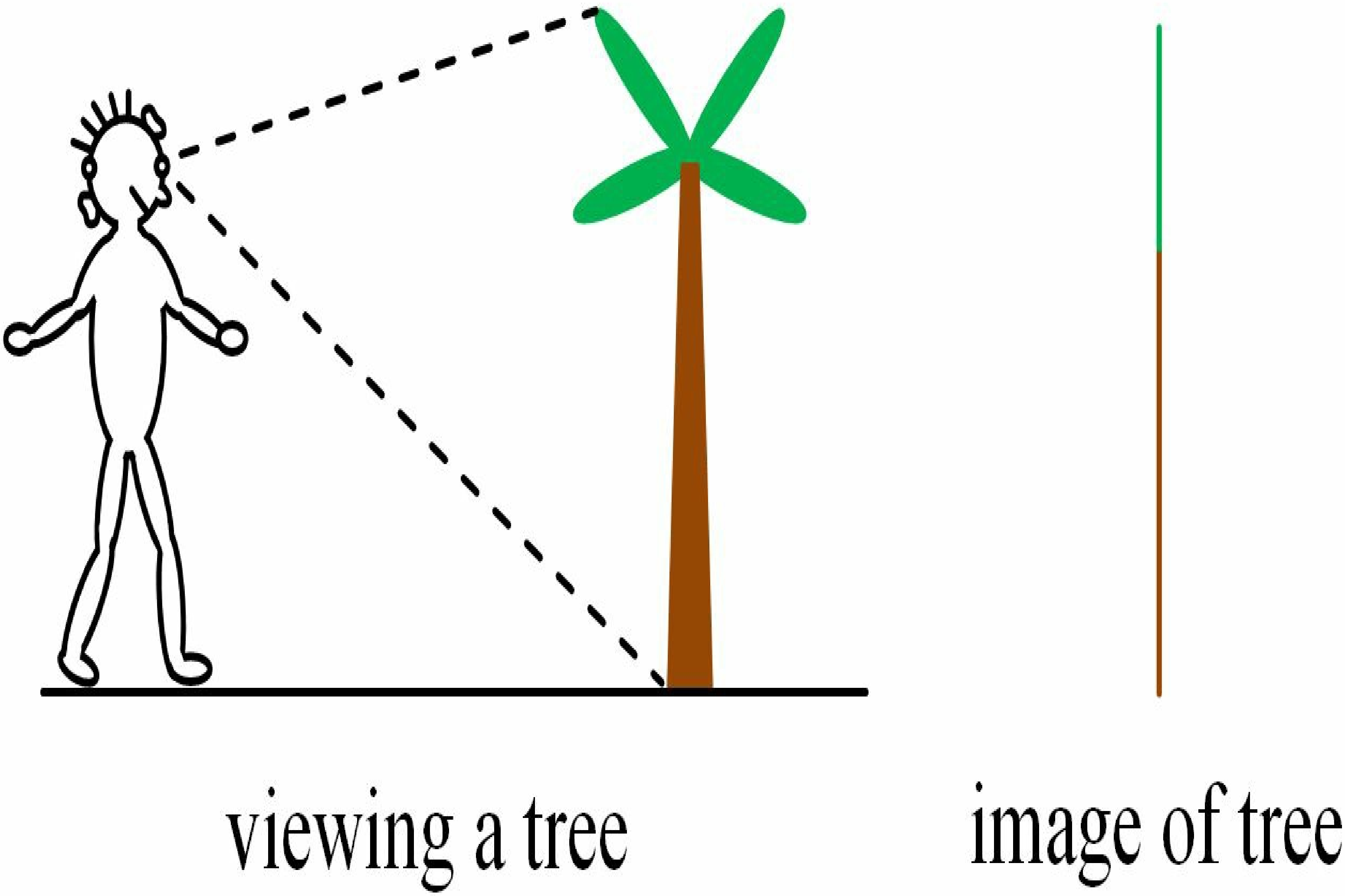
imprisoned

internal supports clog a pipe



passing through a telephone pole

There is one final point about 2D that will be highly relevant for our discussion of the fourth dimension (to come in the remaining chapters). Try to imagine what a 2D monkey would see. Their eyeballs would have 1D surfaces (just arcs), and they would see a 1D image. They would draw 1D pictures on a 1D sheet of paper. In all of the 2D pictures shown in this book, we have the benefit of looking at them from the third dimension. We can see the insides of the monkeys (well, except for the fact that their insides weren't drawn, and some were drawn as stick figures). Just imagine how difficult it would be to try to understand the second dimension from the perspective of one of the 2D monkeys inhabiting such a world. Every object would look like a line!



Did you enjoy contemplating the challenges of designing a 2D world? If so, you may enjoy a book called *The Planiverse* by A.K. Dewdney. It's a great story where a team of computer scientists discover a 2D universe. It's extremely detailed and very well thought out, and also easy and entertaining to read – it reads like a novel, but is very informative about the second dimension and nicely illustrated.

A more popular book on the second dimension, including analogies with 3D and 4D, is Edwin A. Abbott's *Flatland*. You have to read *Flatland* if you consider yourself to be a fan of the fourth dimension to any degree. It's not quite as easy reading for math and science lovers, as it's not just a work of geometry and philosophy, but is also a literary work written in 1884 with themes regarding Victorian times. It's still a great book for geometry enthusiasts, with much to offer mathematically even if you struggle with the literary aspects and style of writing back then. Ian Stewart also has a modern follow-up called *Flatterland*.

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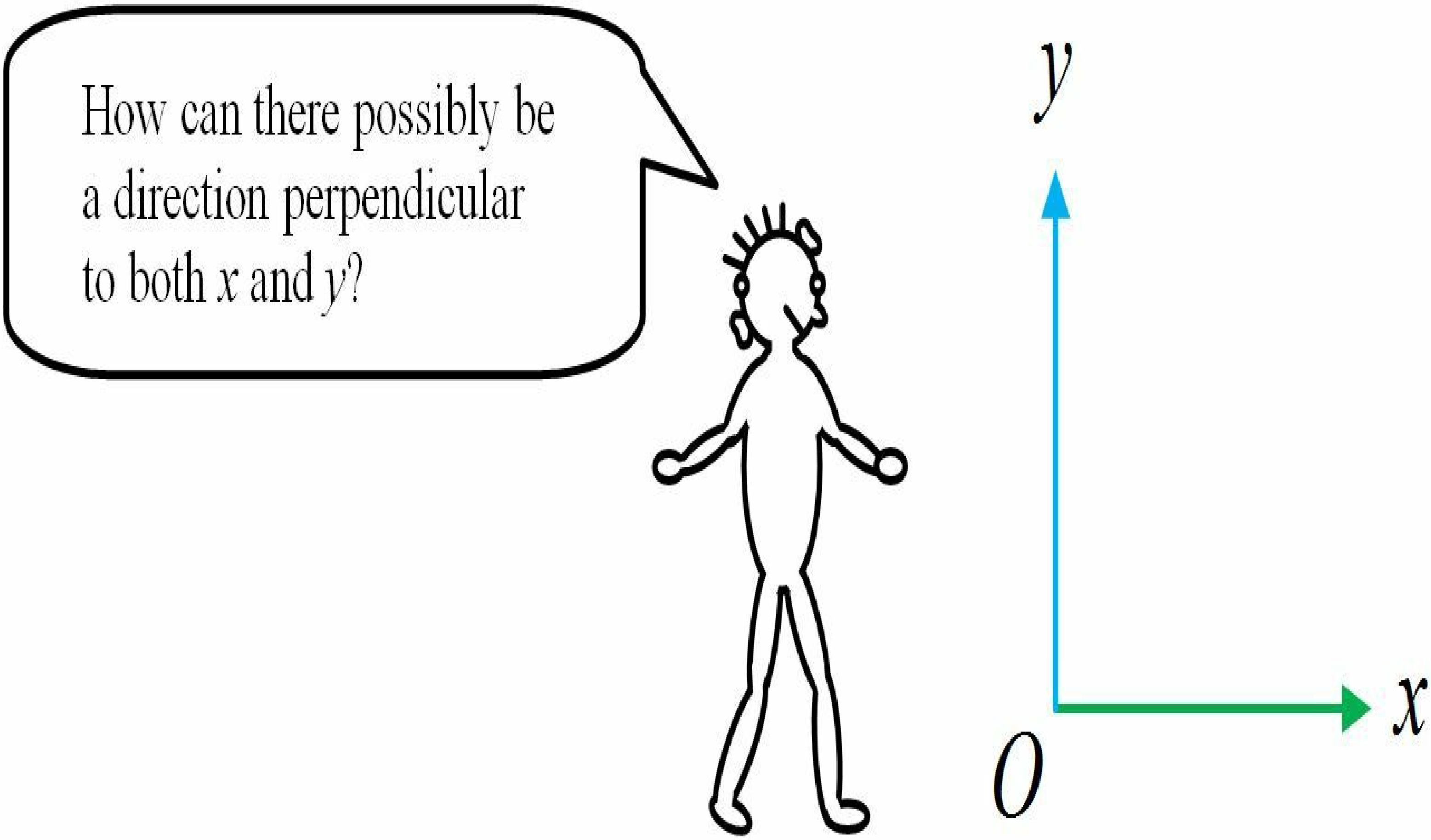
Chapter 3

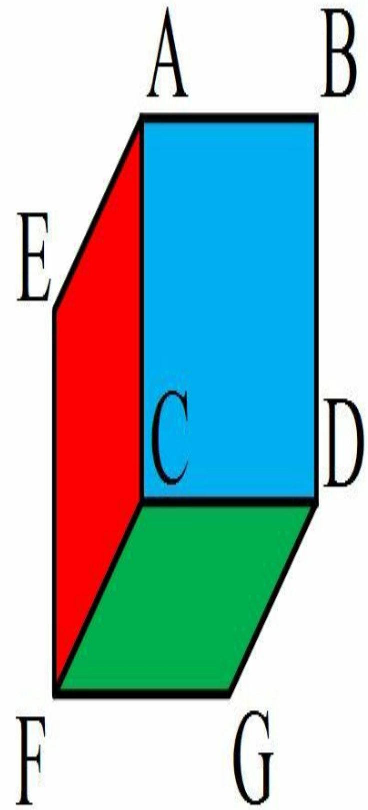
Three-Dimensional Space

If you appreciate the difficulty that 2D beings would have trying to visualize their own 2D world, just imagine if some of them love geometry so much that they want to contemplate the third dimension. Just try to explain to a 2D monkey what a cube or a sphere is – or how about a wheel-and-axle? Yeah, good luck. Even if it's not a "monkey," but the 2D equivalent of Einstein. Still, good luck. With that in mind, you can appreciate our challenge in trying to understand the fourth dimension. It's easier for us though: The leap from 3D to 4D is easier to grasp than the leap from 2D to 3D would be. In the following figure, the 2D being (tail-less monkey, of course) is contemplating a third dimension of space.

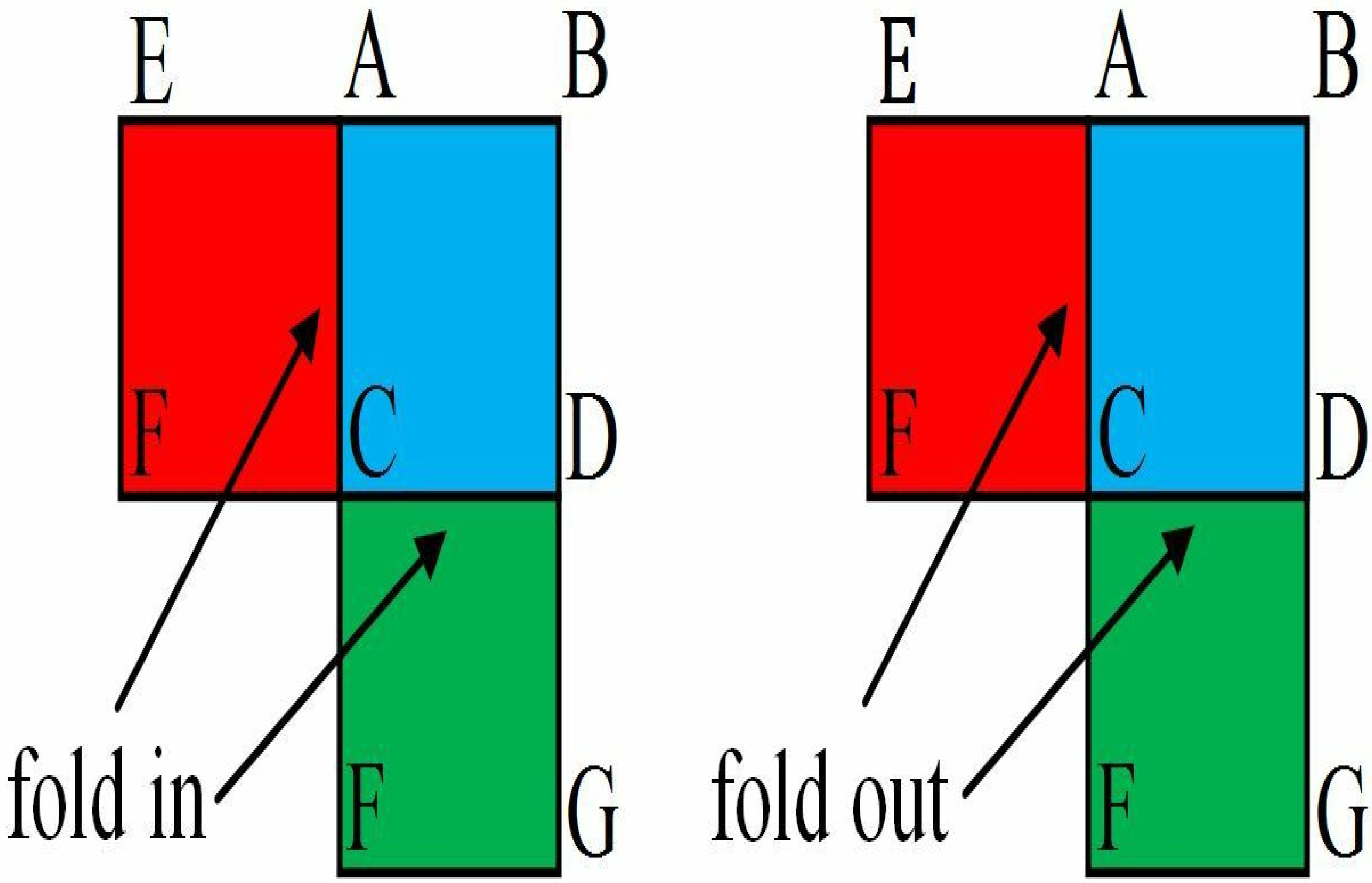


2D being contemplating 3D space

We ourselves have a few difficulties visualizing the third dimension, and we live in it! We see a 2D image of 3D objects with our eyes, and we draw 2D representations of the third dimension on paper. Trying to draw the third dimension on a plane creates some ambiguity, such as the one illustrated below. You can interpret the red, blue, and green cube below two different ways: You can picture it with the corner C in the front or the back. When you imagine that C is in the front, you picture a cube extending up and to the right; but if you imagine that C is in the back, you picture a cube extending down and to the left (as if you're inside a room looking at a corner on the floor). The two bottom figures show how both possibilities can be constructed by folding the L-shape two different ways.

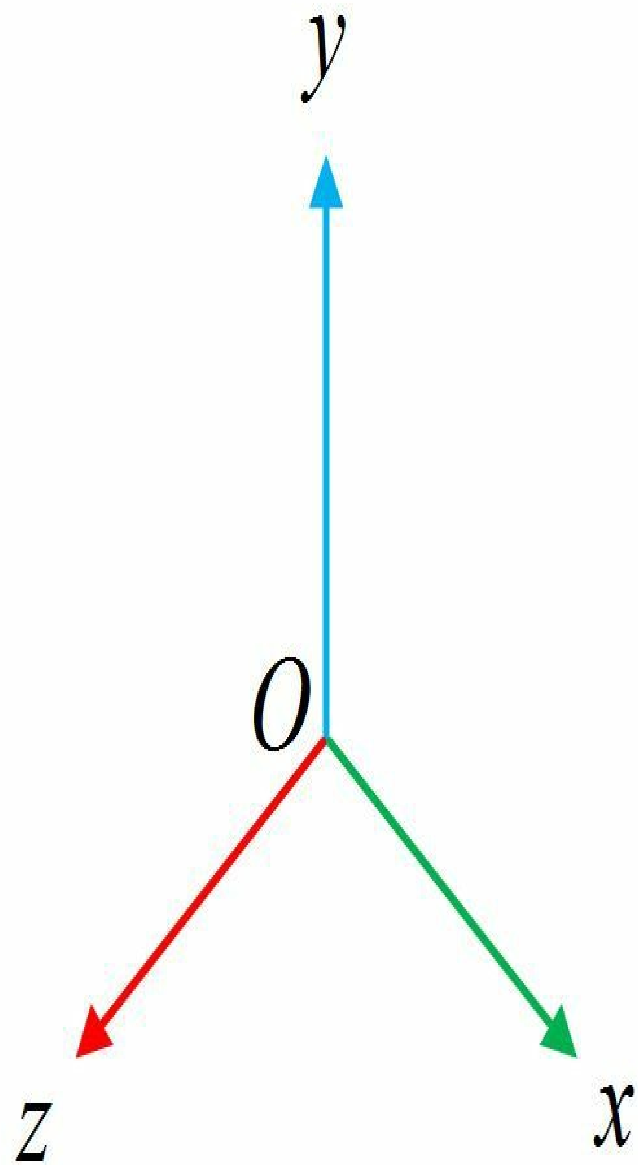


Do line segments $\overline{AE}$, $\overline{CF}$, and $\overline{DG}$ go into or out of the page?



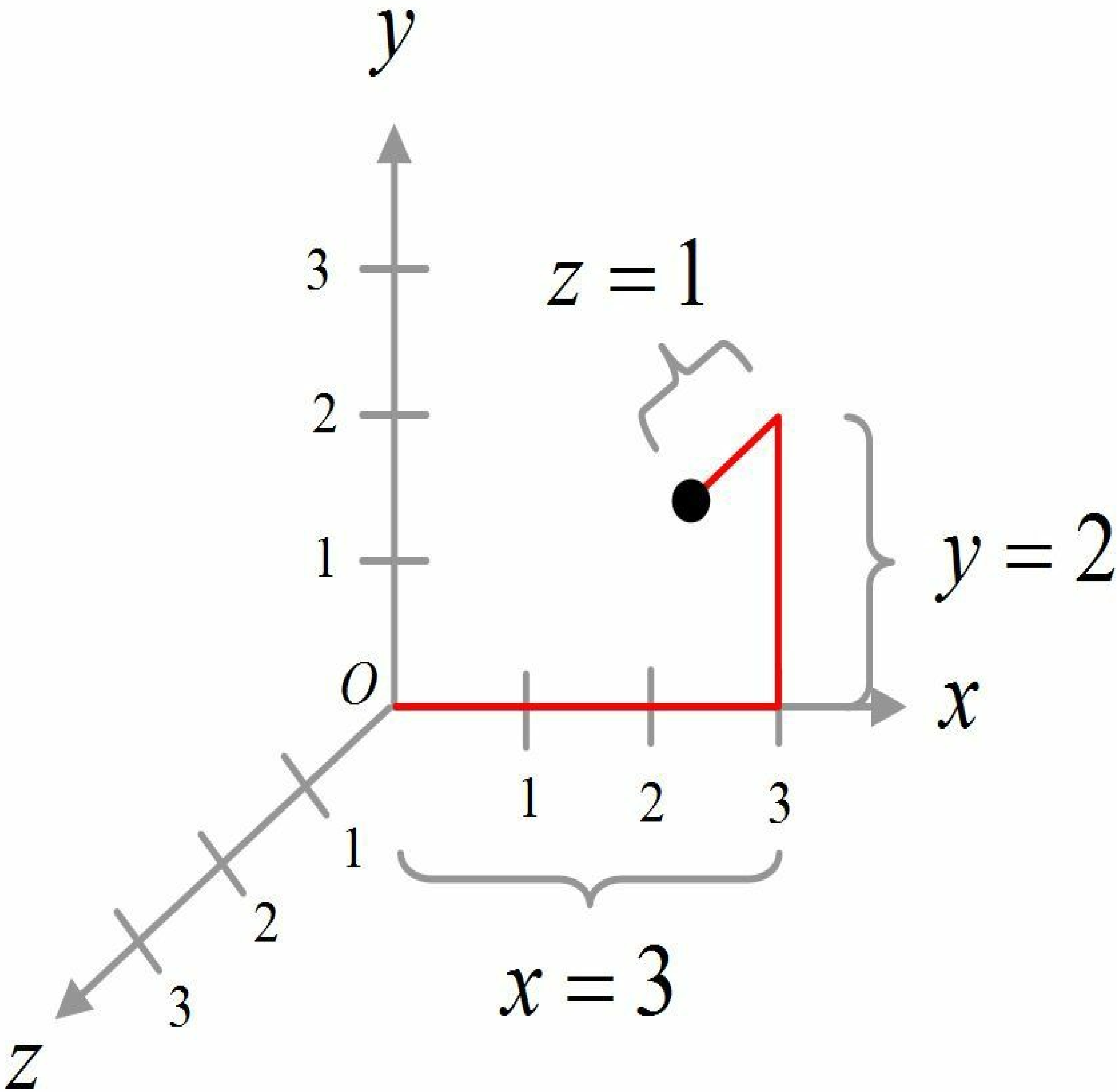
two ways to fold this pattern into the shape above

Mathematically, we can graph 3D using a coordinate system. If you stand in the corner of a room and look at the corner on the floor, this can help you to visualize a 3D coordinate system, with *x*-, *y*-, and *z*-axes. The edges of the room will meet at the corner, looking something like the coordinate system drawn below. (If *x*, *y*, and *z* seem a little boring to you, try BAM. What's BAM? Bananas, apples, and monkeys. Sure, you can use *b*-, *a*-, and *m*-axes instead of *x*-, *y*-, and *z*-axes. Simply file a name change down at your local county courthouse.)



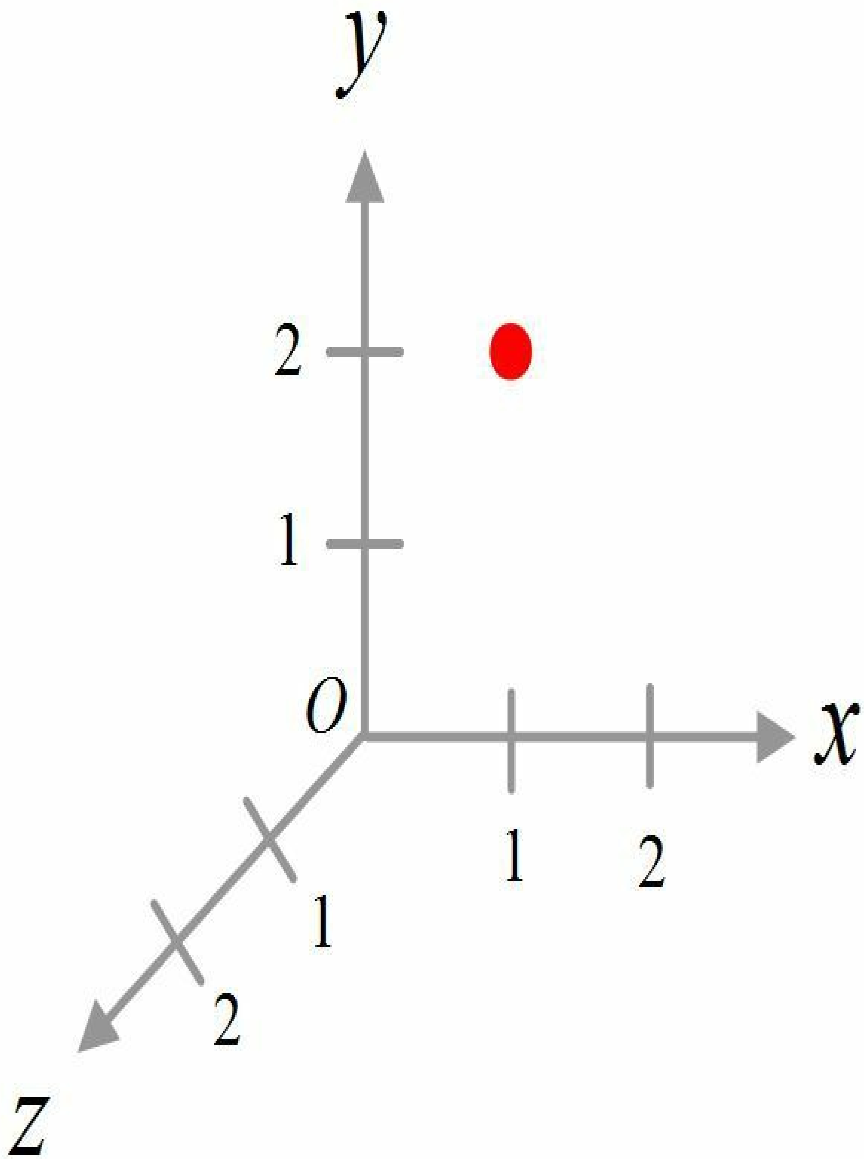
three edges meet at the corner of a 3D room

We usually draw 3D coordinate systems with two of the axes lying in the plane of the page and the other axis coming out of the page (but we draw it as if it comes down at an angle within the page – try as we might, we just can't draw directly out of the page). In the figure that follows, x extends horizontally to the right, y is vertically upward, and z comes out of the page. The point (3,2,1) is illustrated below. To get there from the origin, go 3 units right, 2 units up, and 1 unit diagonally down to the left (which represents 1 unit out of the page).

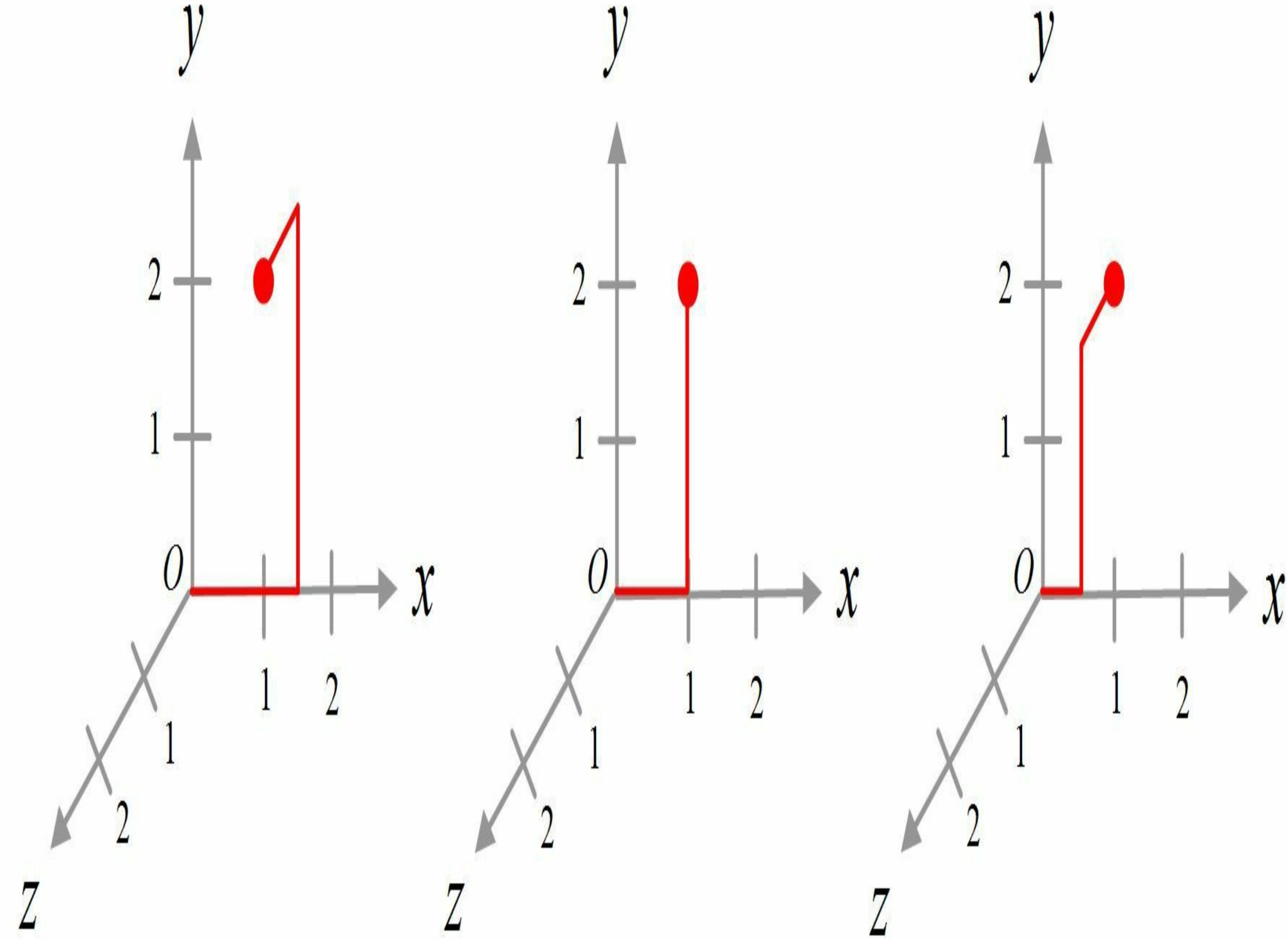


locating the point (3,2,1)

There is some ambiguity in mapping 3D coordinates onto a 2D sheet of paper. This is exemplified in the following figure, which shows three different sets of coordinates that result in a point lying at the same position in the sheet of paper. Although there is a little ambiguity in drawing 3D objects on 2D paper, there is even more ambiguity in drawing 4D objects on a 2D paper; and to make matters worse, we don't have experience living in 4D space. Nonetheless, we can still learn a lot about the geometry of the fourth dimension. In this book, we will approach this from a conceptual standpoint, but keep in mind that it is straightforward to work out the algebra for the fourth dimension. So while there is some ambiguity in drawing 4D geometries on paper, this is removed when we work out the mathematics with symbols. The scope of this book is conceptual, though, so that it will be accessible to anyone who is interested in the fourth dimension. We may have some "monkey-matics," but very little in the way of algebra. I feel sorry for all you math lovers. Take a moment to sympathize for them. (Maybe a future work will get into the math...) If your e-reader respects page breaks, the following figure will be on the next page (so that if you have a large screen, you won't see the answer figure on the same page as the question figure).



What are the coordinates of this point?



$(1.5, 2.5, 0.5)$ $(1, 2, 0)$ $(0.5, 1.6, -0.5)$

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Chapter 4

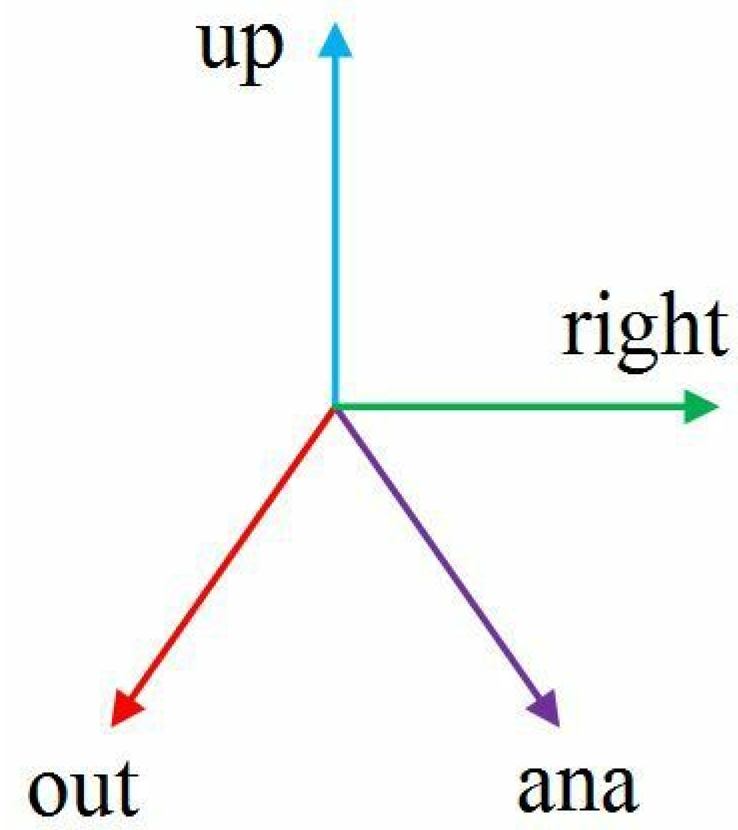
A Fourth Dimension of Space

We're finally ready to venture into the higher dimensions of space. So put on your spacesuit, strap on your safety harness, swallow your anti-nausea medicine, and enjoy the ride! 10D, 9D, 8D, 7D, 6D, 5D, 4D, 3D, 2D, 1D, 0D. Blast off!

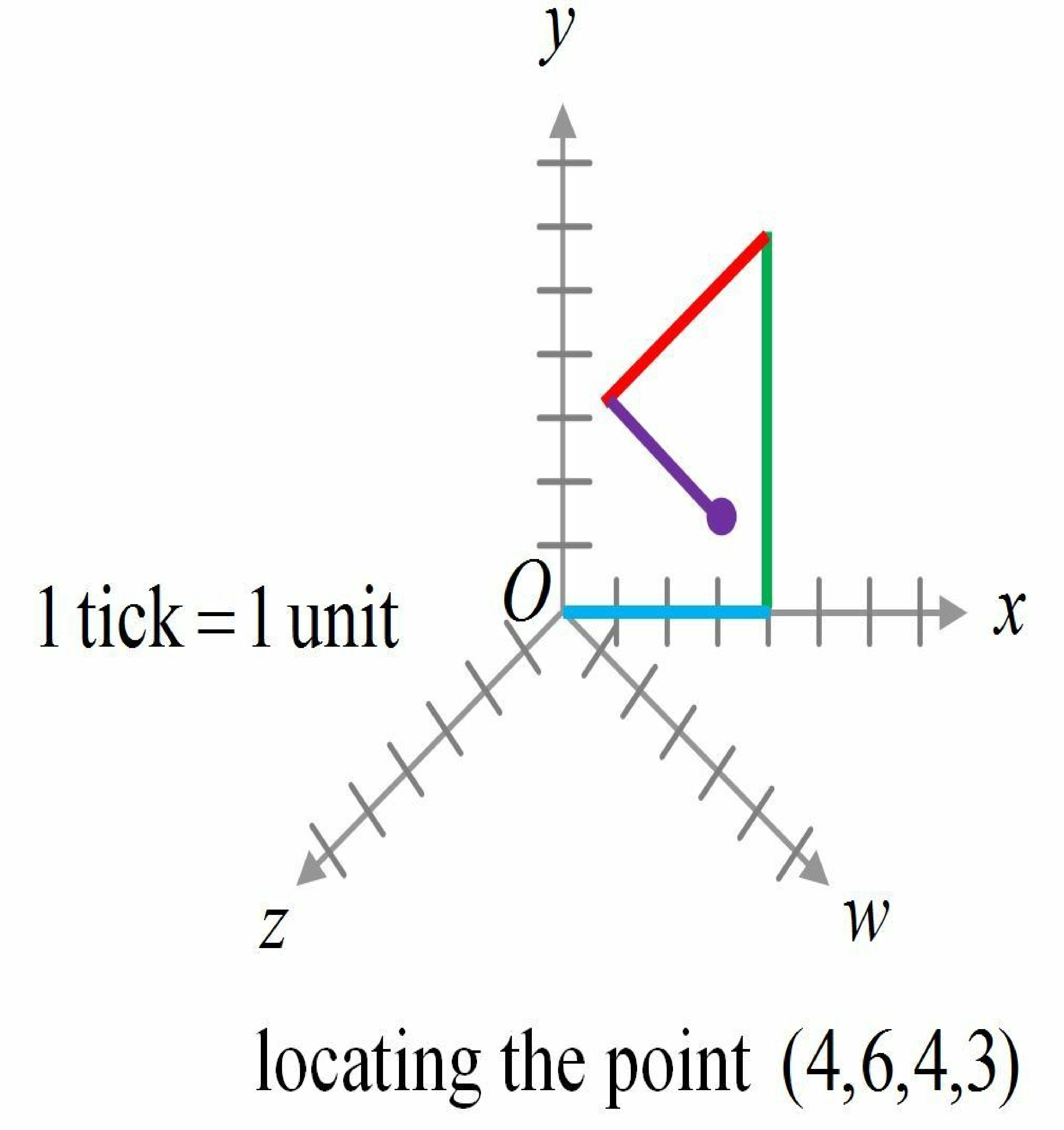
A fourth dimension of space is a direction that you can't point toward, let alone move along, because you're a 3D being living in a 3D world. (Perhaps a genius is a 4D mind confined to a 3D body.) The best way to try to understand the fourth dimension conceptually (i.e. opposed to writing down a bunch of math) is to consider analogies and patterns with the lower dimensions.

So let's briefly recap and see what the pattern suggests:

1. A bead sliding back and forth along a monkey tail in 1D. Moving just north/south is 1D.
2. A monkey running around in an open field is 2D. Moving along combinations of north/south and east/west is 2D.
3. A monkey swimming underwater in a pond is 3D. Moving north/south, east/west, and up/down is 3D.
4. To experience 4D motion, a monkey would have to be able to move in a new direction that's mutually perpendicular to all three of these directions – north/south, east/west, and up/down. Even though we can't move in that direction, we do have a name for it. The fourth direction is called ana, and its opposite is called kata.



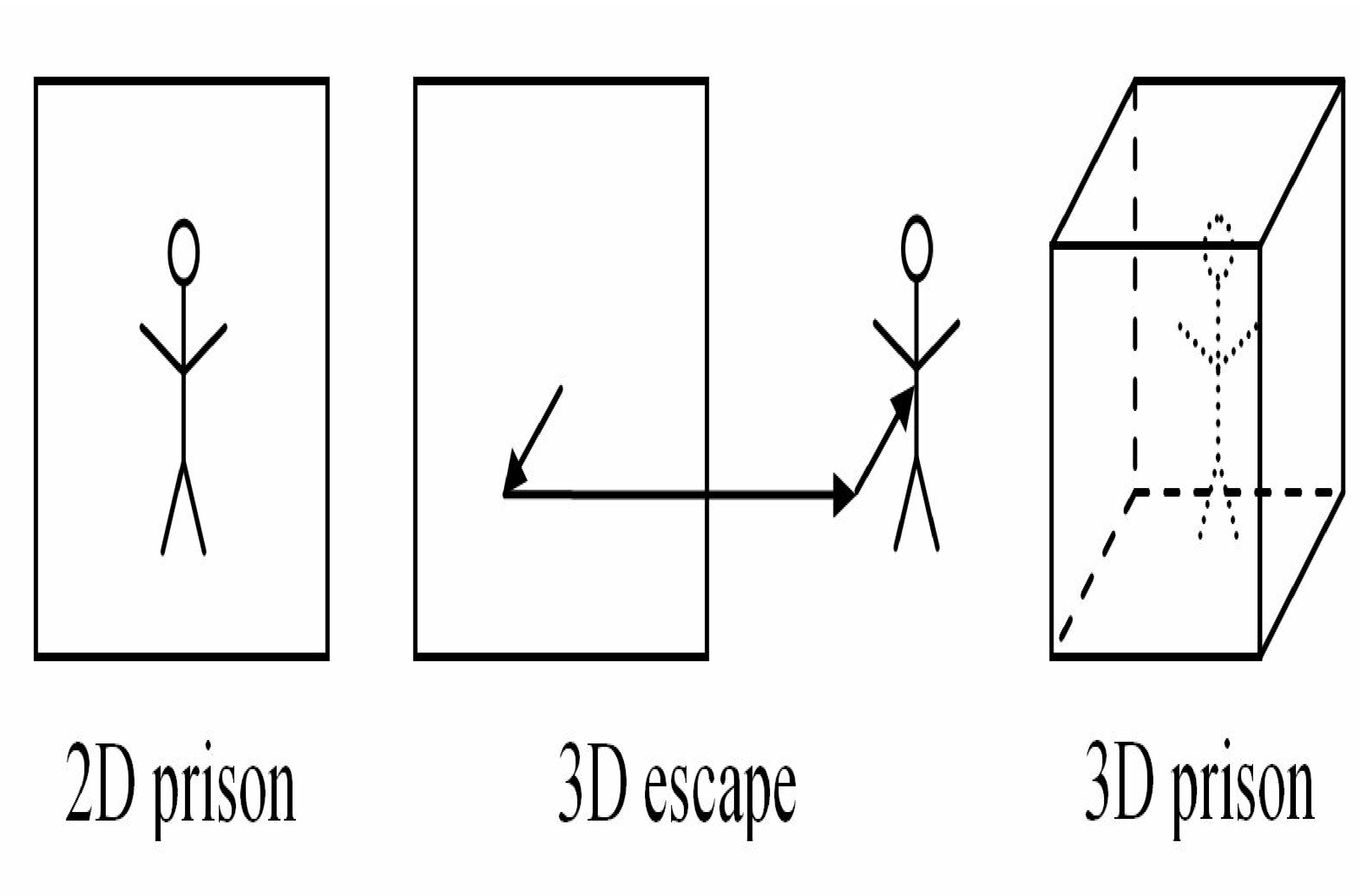
The previous figure shows a 4D coordinate system. Only two of the dimensions – right and up in the previous figure, and x and y in the figure that follows – can lie in the plane of the page. The other two dimensions – out and ana in the previous figure, and z and w in the figure that follows – are both perpendicular to the page and to each other. That's right, in 4D space, both z and w come out of the page, perpendicular to the page; and z and w are also perpendicular to one another ($z \perp w$).



Obviously, you can't do this in 3D space. Here is a simple experiment that you can do to prove to yourself that our universe does not have four dimensions of space (well, if it does, they must be hidden, as we discussed briefly in Chapter 0). Get two pencils. Position one pencil so that it's perpendicular to the screen of your e-reader. Now try to position the second pencil so that it's also perpendicular to the screen of your e-reader, while at the same time it's also perpendicular to the first pencil. If you lived in 4D space (where none of the dimensions is hidden), you would be able to do this.

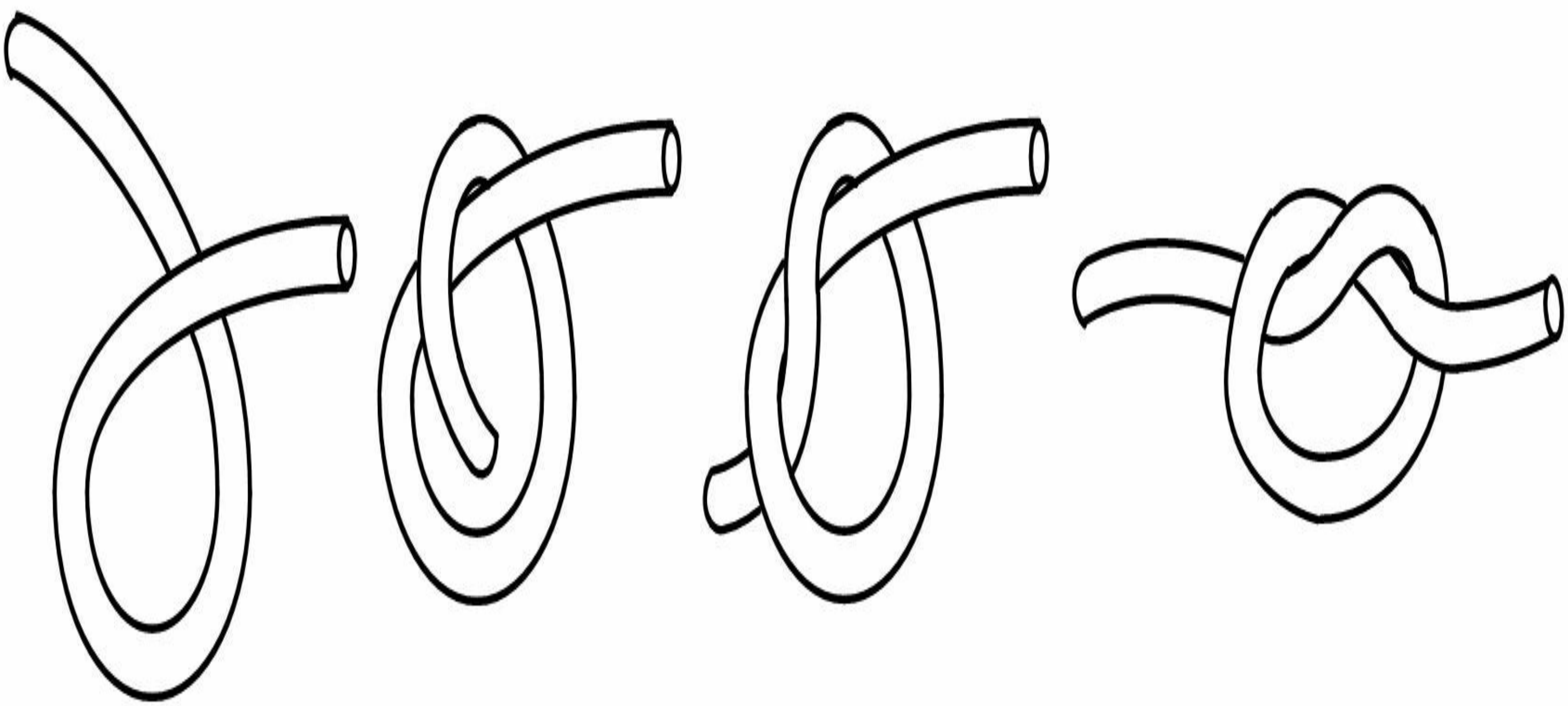
In order to visualize the fourth dimension, you must think outside the box, literally! Actually, in 4D space, you could even walk outside the box! More precisely, if a monkey were standing inside of a cube-shaped room with no windows or doors in 4D space, the monkey could easily walk outside of it. In contrast, in 3D space the monkey would be trapped inside of the cube.

This concept is easy to understand by analogy. A monkey would be trapped inside of a square in 2D space; the monkey can't escape the square by moving along any combination of lengthwise and widthwise. However, in 3D space a monkey could easily get out of the square – simply walk perpendicular to the plane of the square, as illustrated below.

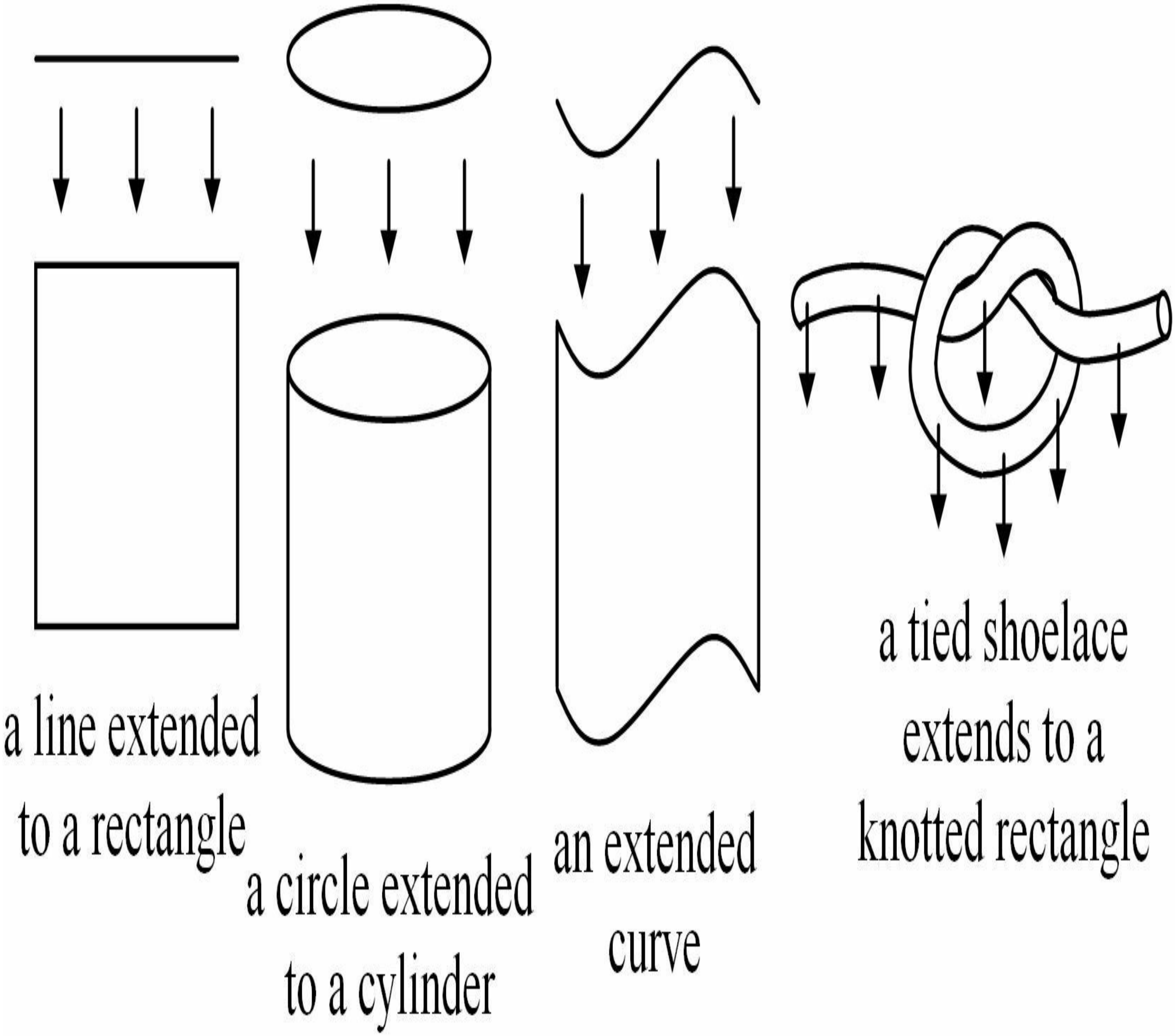


Similarly, a monkey would be trapped inside of a cube in 3D space. No combination of lengthwise, widthwise, or depthwise will get the monkey out of the cube. However, in 4D space, the monkey could escape the cube by moving in the ana direction. It would take a <u>tesseract</u> to trap a monkey in 4D space. We'll discuss the tesseract at great length starting in the next chapter; so just hold onto your tesseract questions for a moment.

Another interesting feature of the fourth dimension is that any string that is tied into a knot will easily come undone – just like the ana direction helps the monkey escape from the cube, it helps the string wind around itself and become untied. (Nope, this doesn't have anything to do with string theory; it's <u>knot</u> theory. Totally different subjects.) You can't even tie a string into a knot in 2D, and any string tied into a knot in 4D won't stay tied; 3D is just right (if Goldilocks tried tying these three types of knots, then baby bear's string would be in 3D space). However, you can tie a sheet into a knot in 4D. This is not so easy to visualize, but the basic idea is illustrated below. The bottom figures show what it looks like to generalize a 1D figure to 2D by extending the object in a direction perpendicular to itself; this may help you to visualize what it would be like to generalize an ordinary 3D knot into a 4D knotted rectangle.



tying a knot in 3D



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Chapter 5

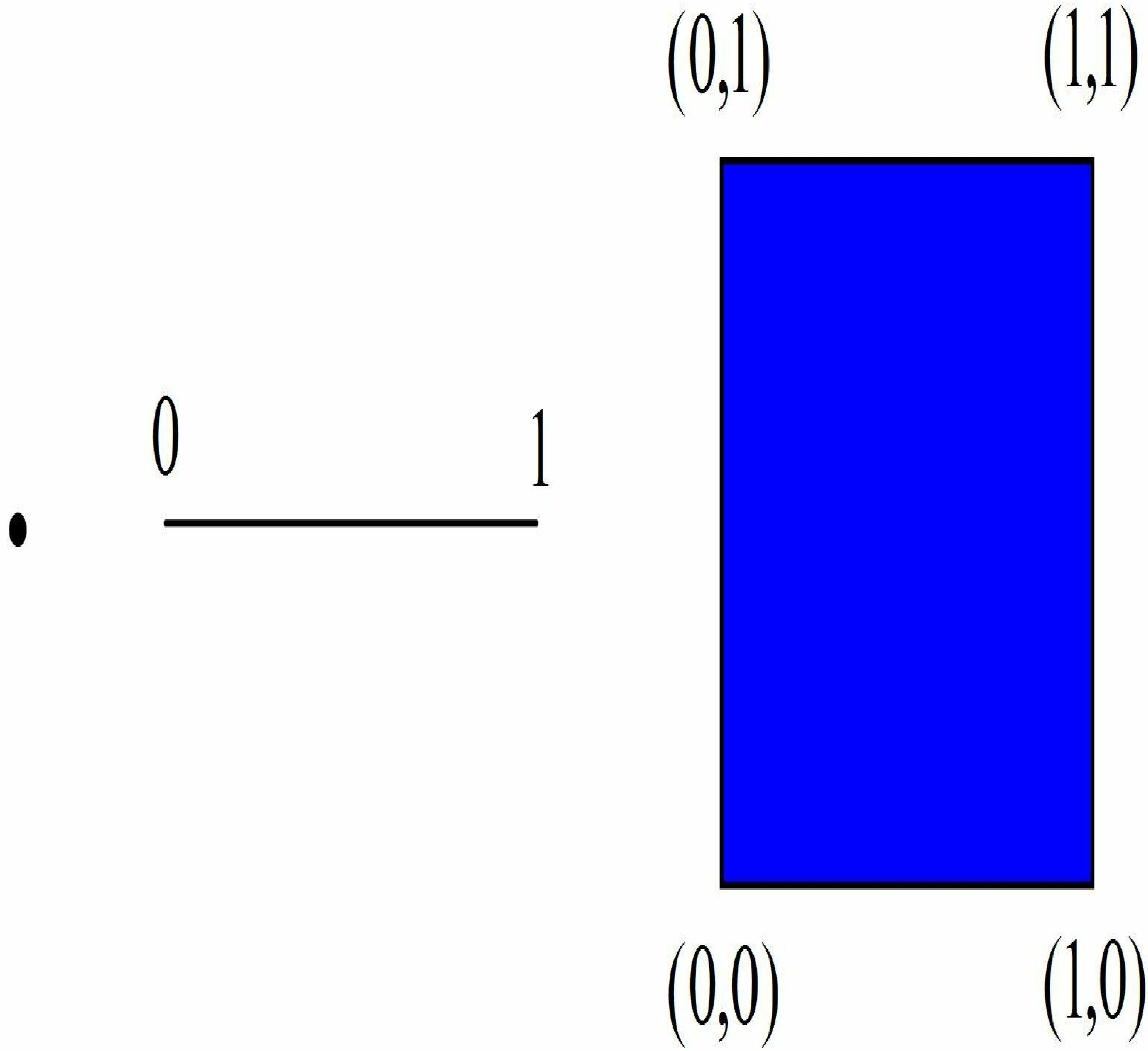
Tesseracts and Hypercubes

The simplest 4D object to understand is the tesseract. A tesseract is a simple way to generalize the concept of a cube to 4D space. If you thought that a 4D generalization of a cube was called a hypercube, that's not quite right. Technically, a hypercube is a generalization of the cube to higher dimensions (not necessarily 4D), whereas a tesseract is specifically a 4D generalization of the cube. Put another way, the tesseract is a 4D hypercube. Mathematicians sometimes refer to the hypercube as an N-dimensional cube (and they still call the general N-cube a hypercube even if N happens to be less than 4). If it had been up to me, they would be called "monkeycubes" instead.

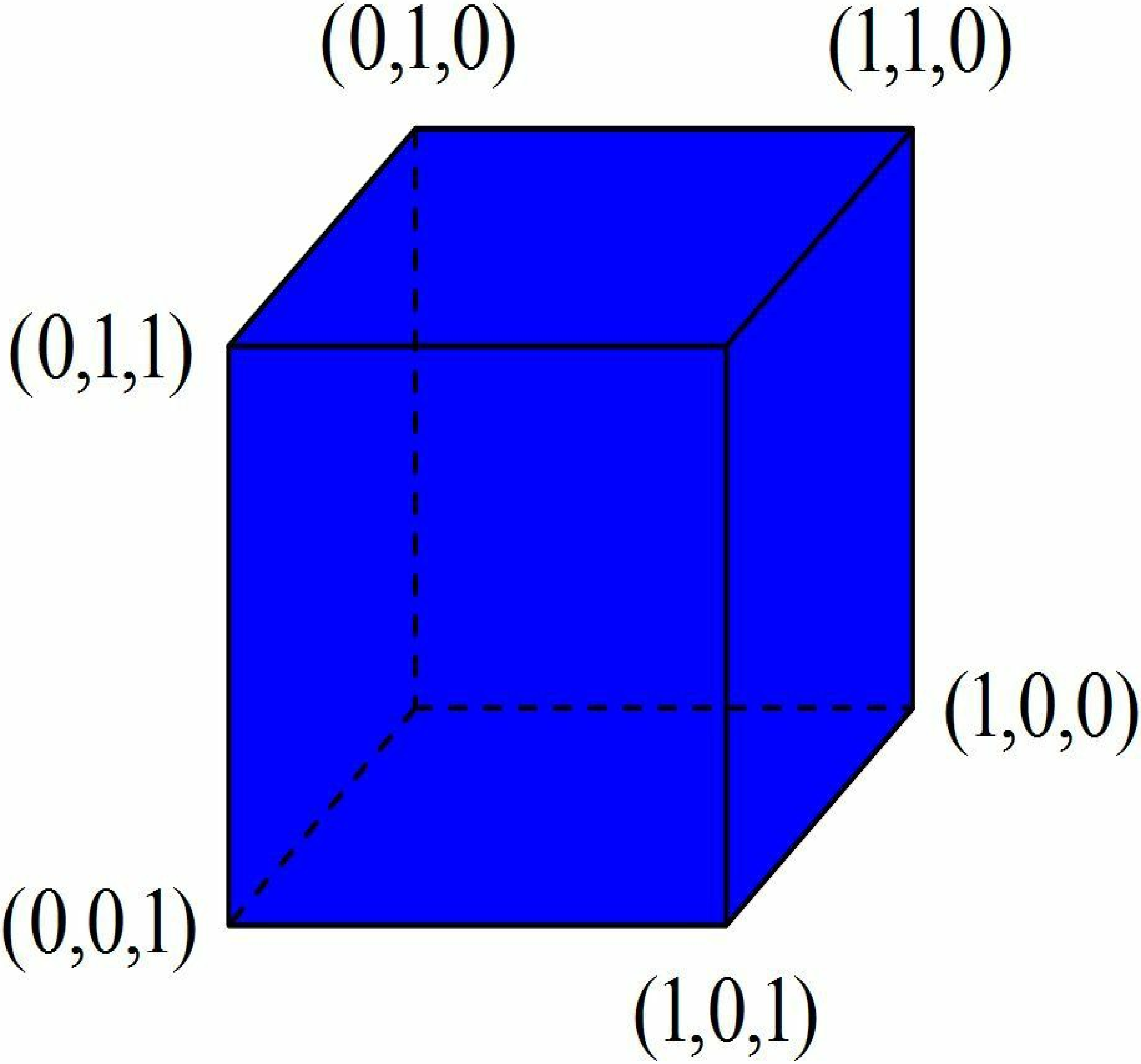
Imagine a point located at the origin. Only instead of a point, let's call it a 0D banana. Just in case you're sick of monkeys, a banana might be a nice change of pace. If also contemplation of the fourth dimension happens to drive you bananas, it will be quite fitting. (Read this book at your own risk. The author is not liable for any readers who go bananas after thinking about the fourth dimension.)

If this 0D point (banana!) travels along the x-axis, it sweeps out a 1D line (ehem, I mean monkey tail, of course). The endpoints of this line (or monkey tail) have coordinates $x = 0$ and $x = 1$.

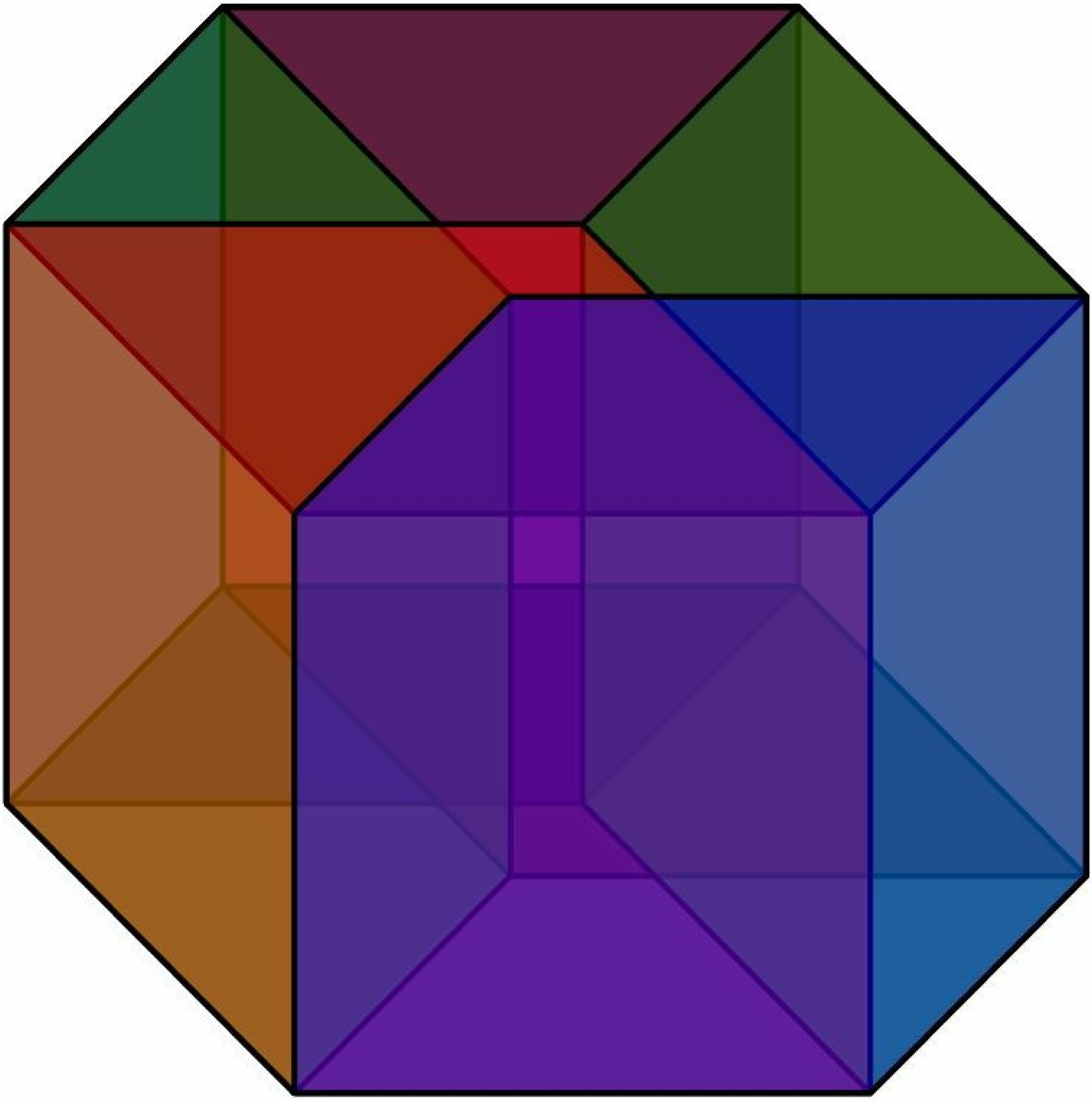
Now imagine that this 1D line (monkey tail) travels along the y-axis, sweeping out a 2D square (oh, let's make this a square banana-cream pie). This square pie has 4 corners and 4 edges. The four corners are located at (0,0), (1,0), (1,1), and (0,1). There are 2 pairs of edges – 2 parallel to the x-axis and 2 parallel to the y-axis. The "banana," "monkey tail," and "square pie" are illustrated below.



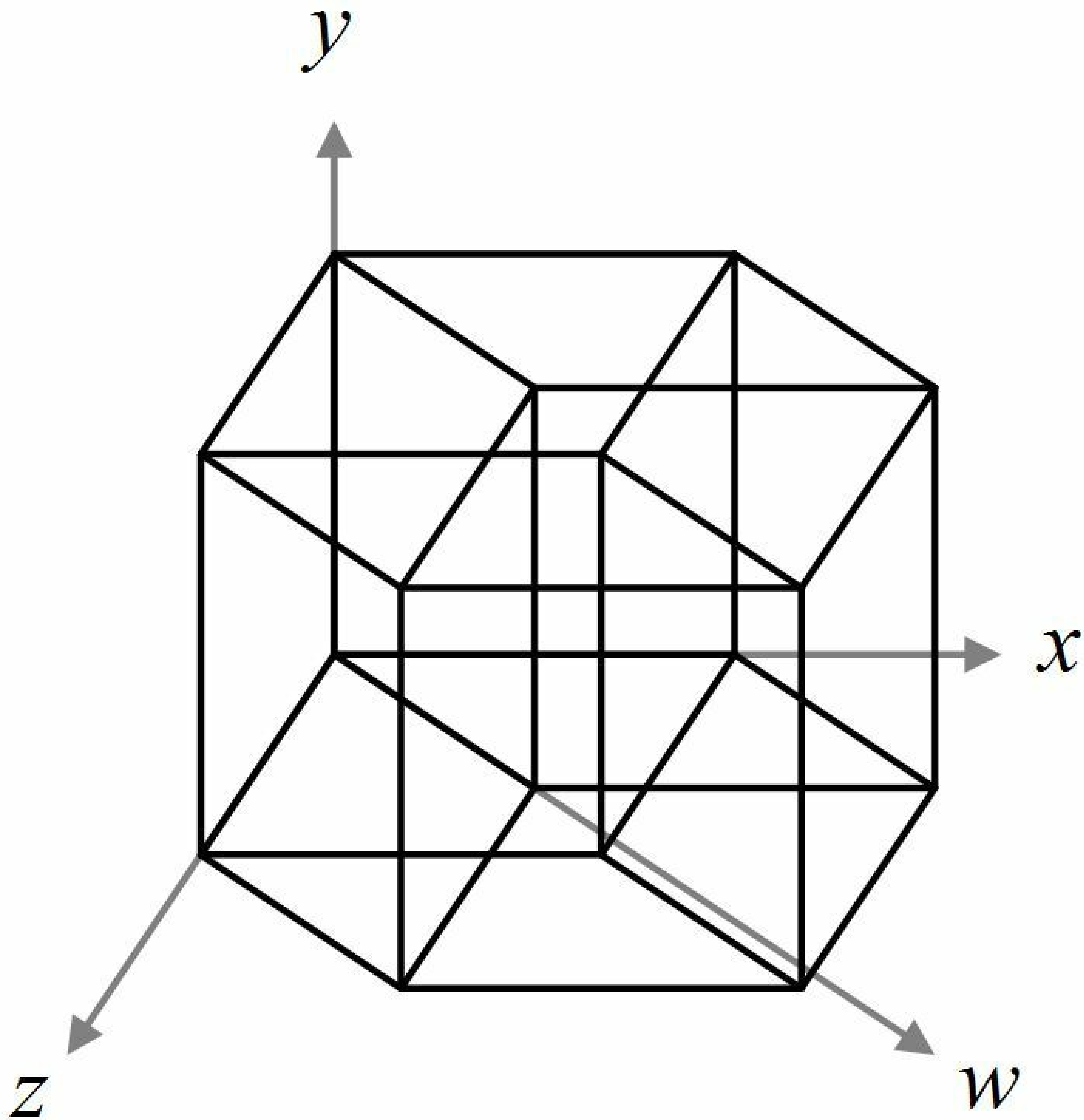
Next, we sweep the 2D square pie (perhaps it's a really moldy pie, since the square is blue in the picture; or maybe it just has blueberry crust) along the *z*-axis, transforming it into a 3D cube (let's call this a box of bananas). The banana cube has 8 corners: (0,0,0), (1,0,0), (1,1,0), (0,1,0), (0,0,1), (1,0,1), (1,1,1), and (0,1,1). It also has 12 edges: 4 parallel to *x*, 4 parallel to *y*, and 4 parallel to *z*. The banana cube also has 6 square sides: 2 in the *xy* plane, 2 in the *yz* plane, and 2 in the *zx* plane.



Finally, we sweep this cube of bananas along the *w*-axis (that's the ana direction) in order to create a tesseract (full of monkeys). (So what if the 3D bananas turned into 4D monkeys? Why not sprinkle a little magic on the math? Creative math would be an even more popular course than creative writing. Give it a chance.) Study the tesseract below. In particular, try to visualize how it was made by sweeping a cube along the ana direction. See if you can find two opposite cubes in the tesseract (there are actually more than two cubes bounding the tesseract, as we will explore; but for now, try to find two cubes on opposite sides in order to see how a cube might sweep out a tesseract).



Whereas the previous tesseract is in color, the following tesseract is in black and white. It's easier to see the edges in the black and white figure.



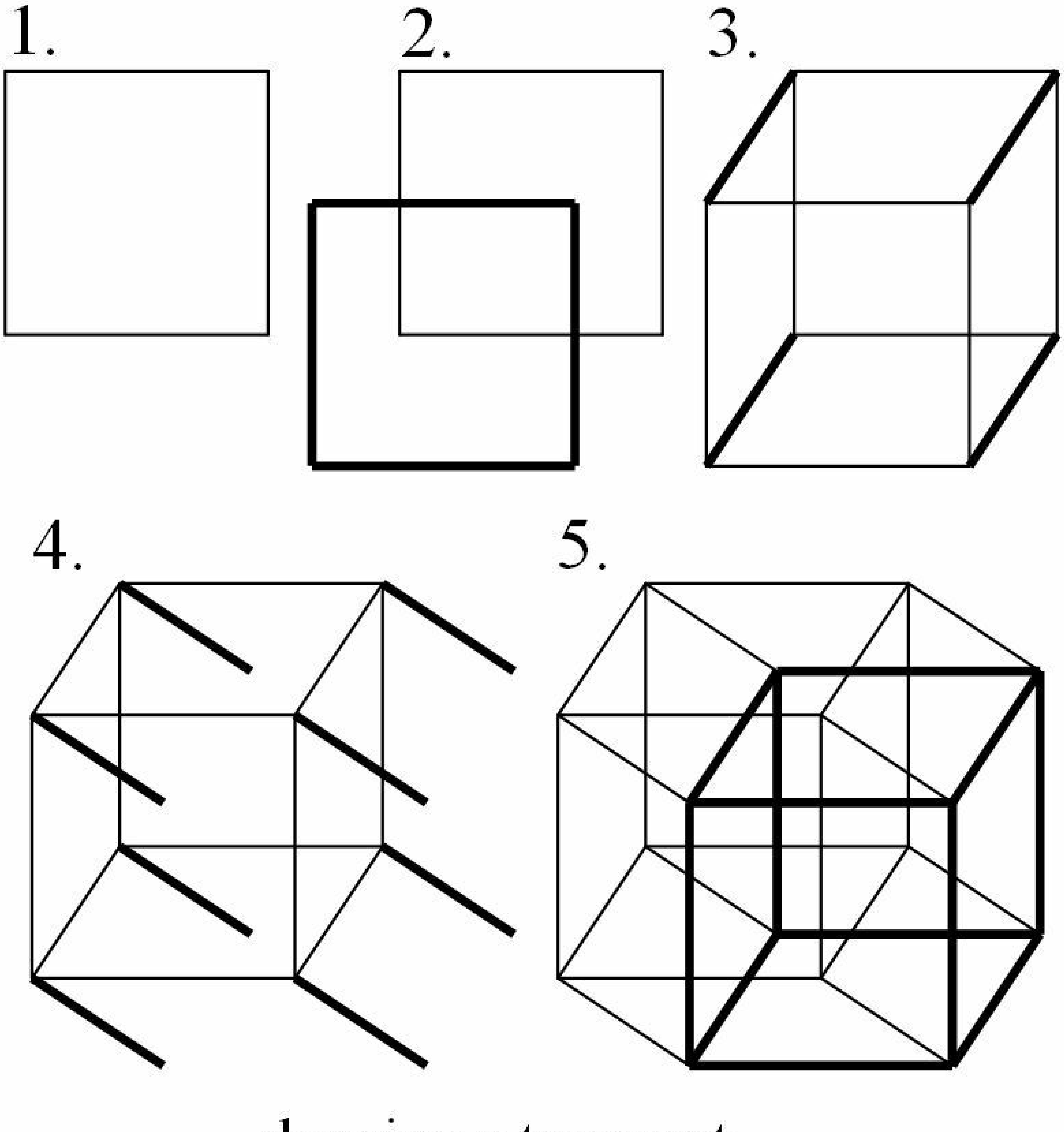
a tesseract

Here are some instructions for how to draw a tesseract in case you would like to make your own. It might be easier to draw in Microsoft Word, for example, where you can take advantage of the copy/paste functions. Following the directions is an illustration that will help you understand each step.

1. Draw a square in the plane of the paper. This square lies in the *xy* plane.
2. Draw a copy of this square (or use copy/paste) a little down and left of the first square. These two squares represent the front and back faces of a cube; the second square has

been shifted along *z*. **Note**: Be careful to position the second square so that its upper right corner does not touch any edge of the first square.

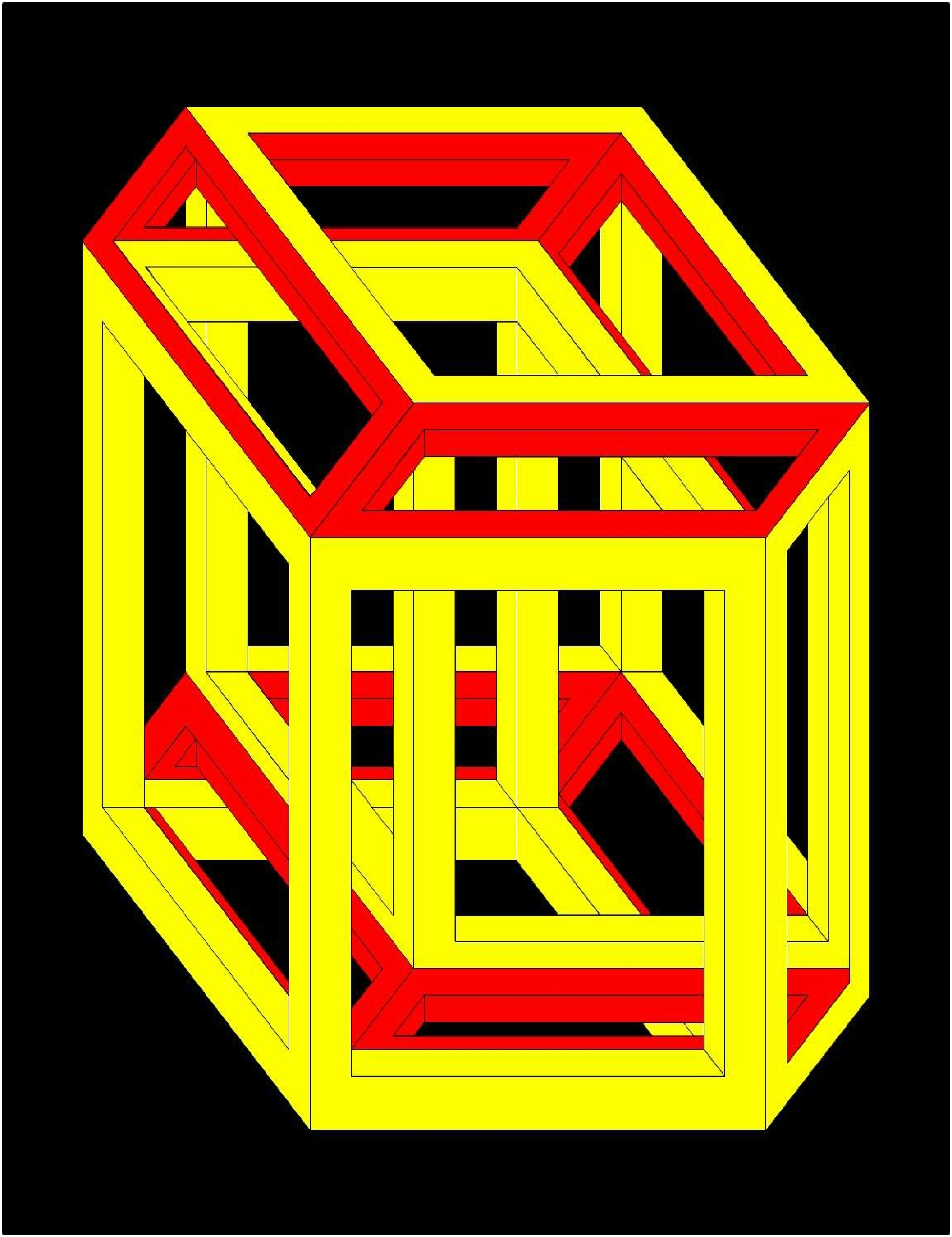
3. Add diagonal lines to connect the respective corners of the two squares. Use copy/paste if drawing on a computer. You should now have a cube.
4. Draw a diagonal line down and to the right from one of the 8 corners of the cube. This represents the *w* direction. Draw 7 more diagonals (so 8 in all) identical to the first one. Make sure that none of these 8 lines lies on any corner or edge (otherwise, you need to start this step over).
5. Add 4 more horizontal, vertical, and *z*-lines. These last 12 lines are the edges of a cube identical to the one from step 3. One cube is shifted along *w* compared to the other. Drawing a tesseract this way, you can see firsthand how sweeping out a cube along ana can make a tesseract.



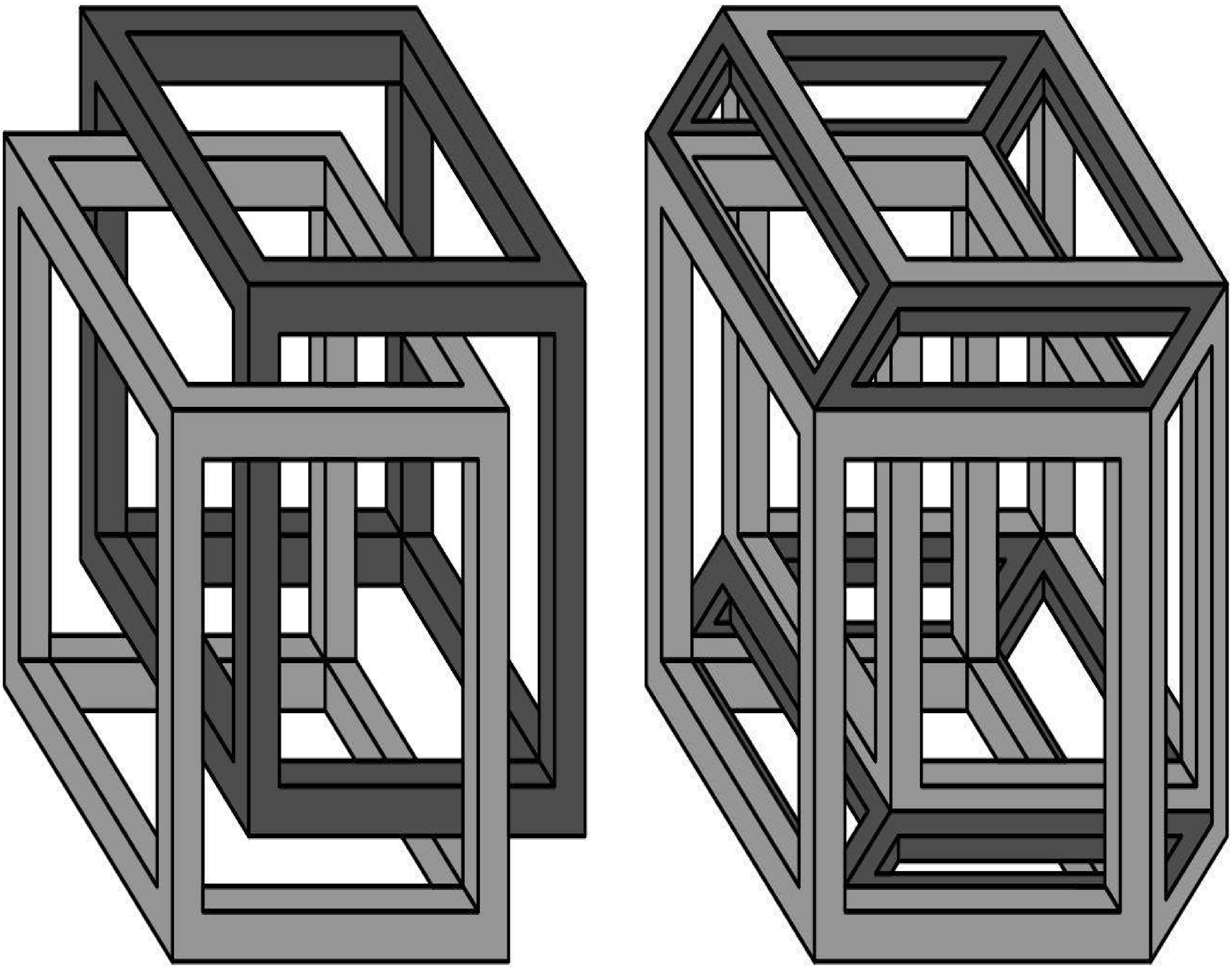
drawing a tesseract

To complete your tesseract, be sure to add some monkeys. :-)

As mentioned previously, there is some ambiguity in trying to contemplate a 2D representation of a tesseract – much more than in interpreting a 2D drawing of a cube. One way to lessen the ambiguity is to construct a 3D representation of a tesseract (then you can put real monkeys inside of it).

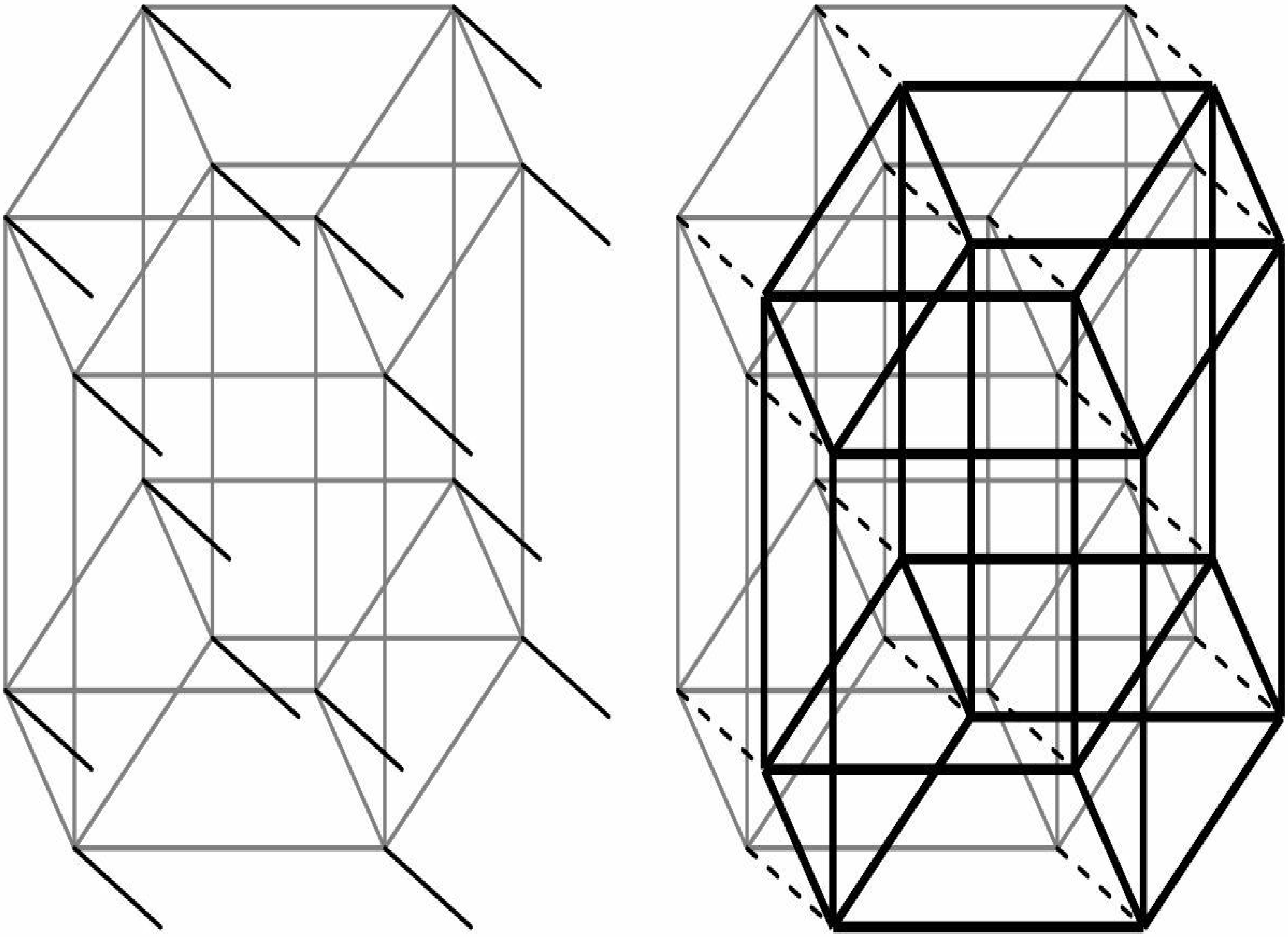


The following picture shows this in grayscale, but also has the two cubes by themselves in order to help you understand the structure.



constructing a 3D representation of a tesseract

Why not carry this one step further and draw a 5D hypercube (let's fill this with nuts, since it might seem a bit nutty)? Recalling the steps for how to draw a tesseract, we can do the same thing to make a 5D hypercube: Add 16 diagonal lines (different from both z and w) to each corner, then draw a new tesseract at the ends of these lines. You can find a "nutty" picture of this below.



drawing a 5D hypercube

If you're interested in reading more about the fourth dimension than you find in this book, consider Rudy Rucker's classic, *Geometry, Relativity, and the Fourth Dimension*. Rudy Rucker's book on the fourth dimension is a great read; it is highly recommended.

Chris McMullen (that's the author of the book you're reading right now, in case you didn't realize it) also has another book on the fourth dimension called *The Visual Guide to Extra Dimensions* (the text doesn't show as much personality, like the one you are reading, but it is highly informative and covers topics not found here, like curved hypersurfaces).

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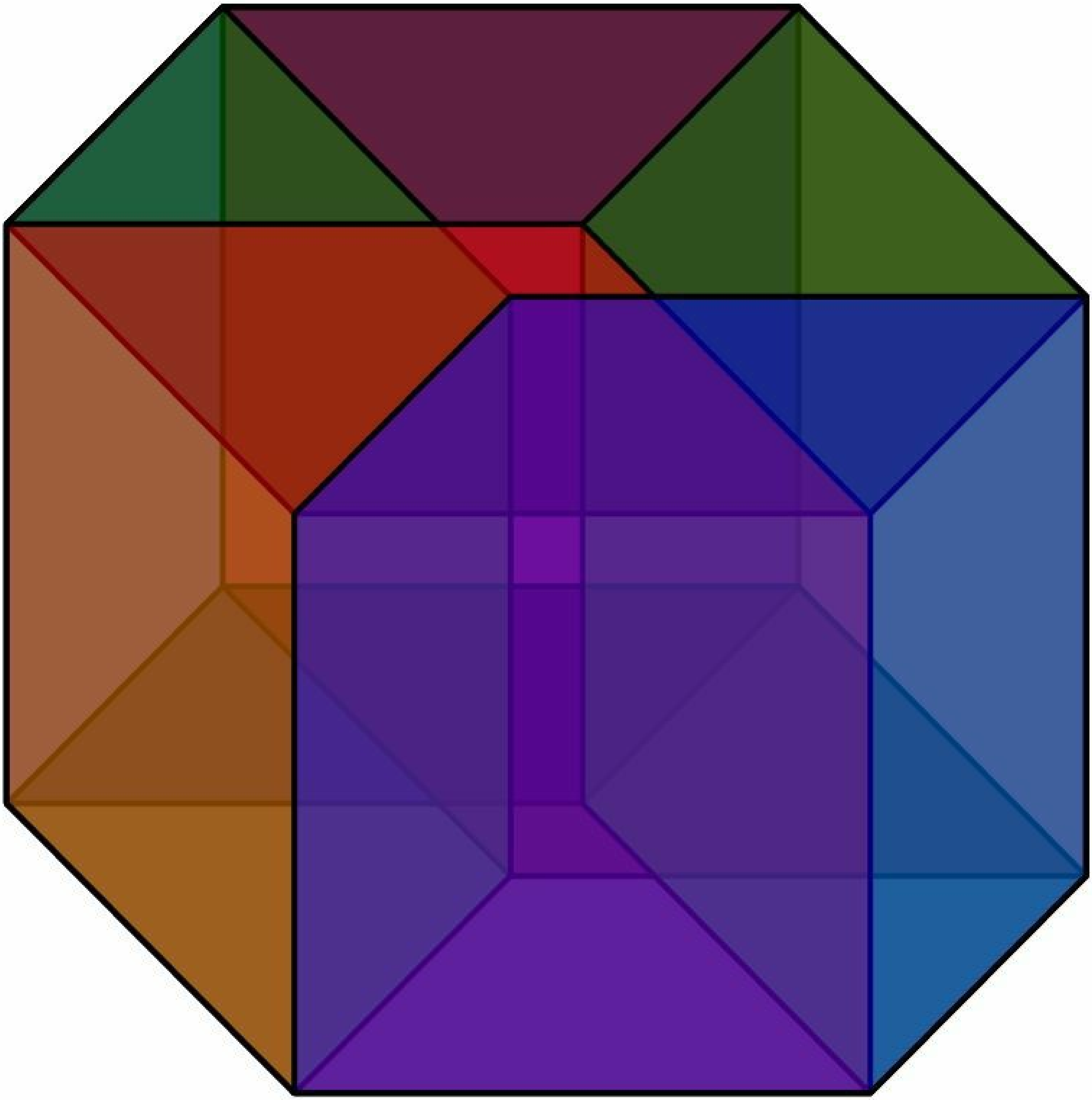
Chapter 6

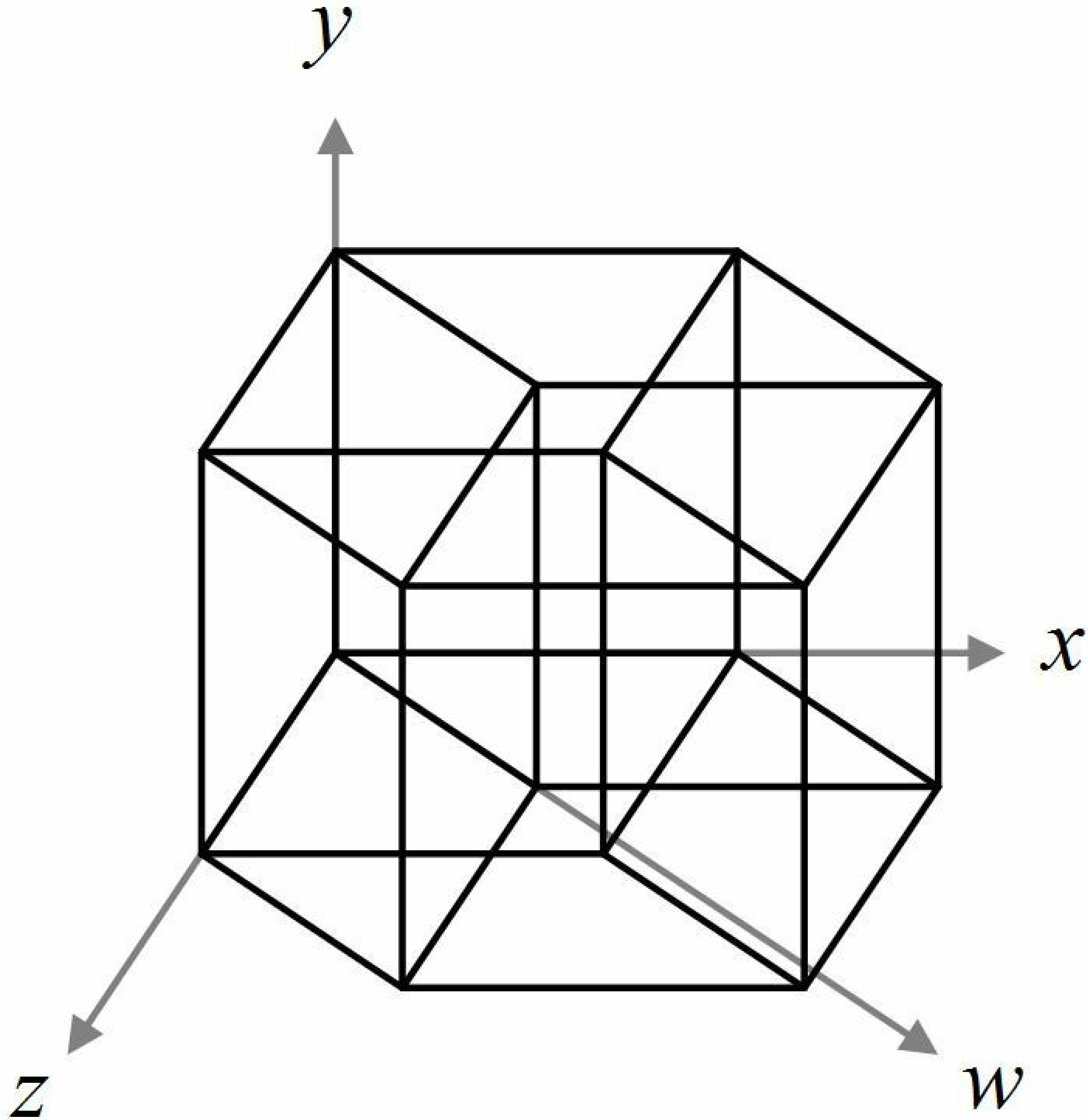
Hypercube Patterns

Consult the following tesseracts as you try to solve the puzzles that follow. "What? Nobody warned me that there would be a quiz!"

This first tesseract is in color. (Yes, you saw the same one in the previous chapter; it might come in handy in a moment.)



The next tesseract is in black and white. Some concepts may be easier to see in the black and white tesseract, especially when you want to study the edges.



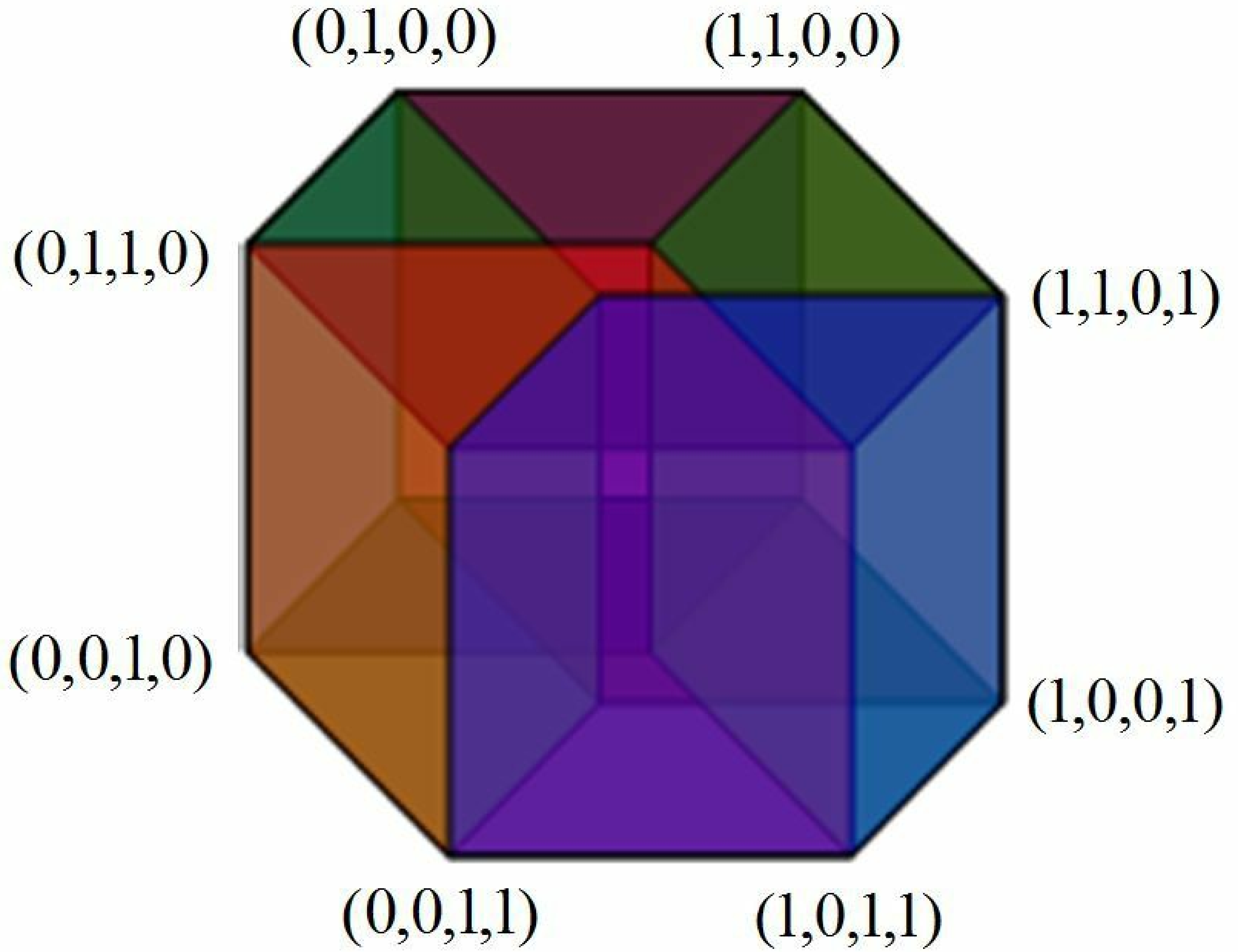
a tesseract

Now for some counting games. I'll ask a question about the tesseract (full of monkeys) that involves counting – or logical deduction, which may be the better method. Then, instead of immediately reading the answer and explanation, I highly recommend that you spend some time trying to figure it out for yourself. Then read on to check your answer and study the explanations and pictures. If you really want to understand the fourth dimension, any time that you devote toward trying to figure things out yourself before reading the explanation will help you understand it better. (But don't take my advice; I'm just a humble physics instructor.)

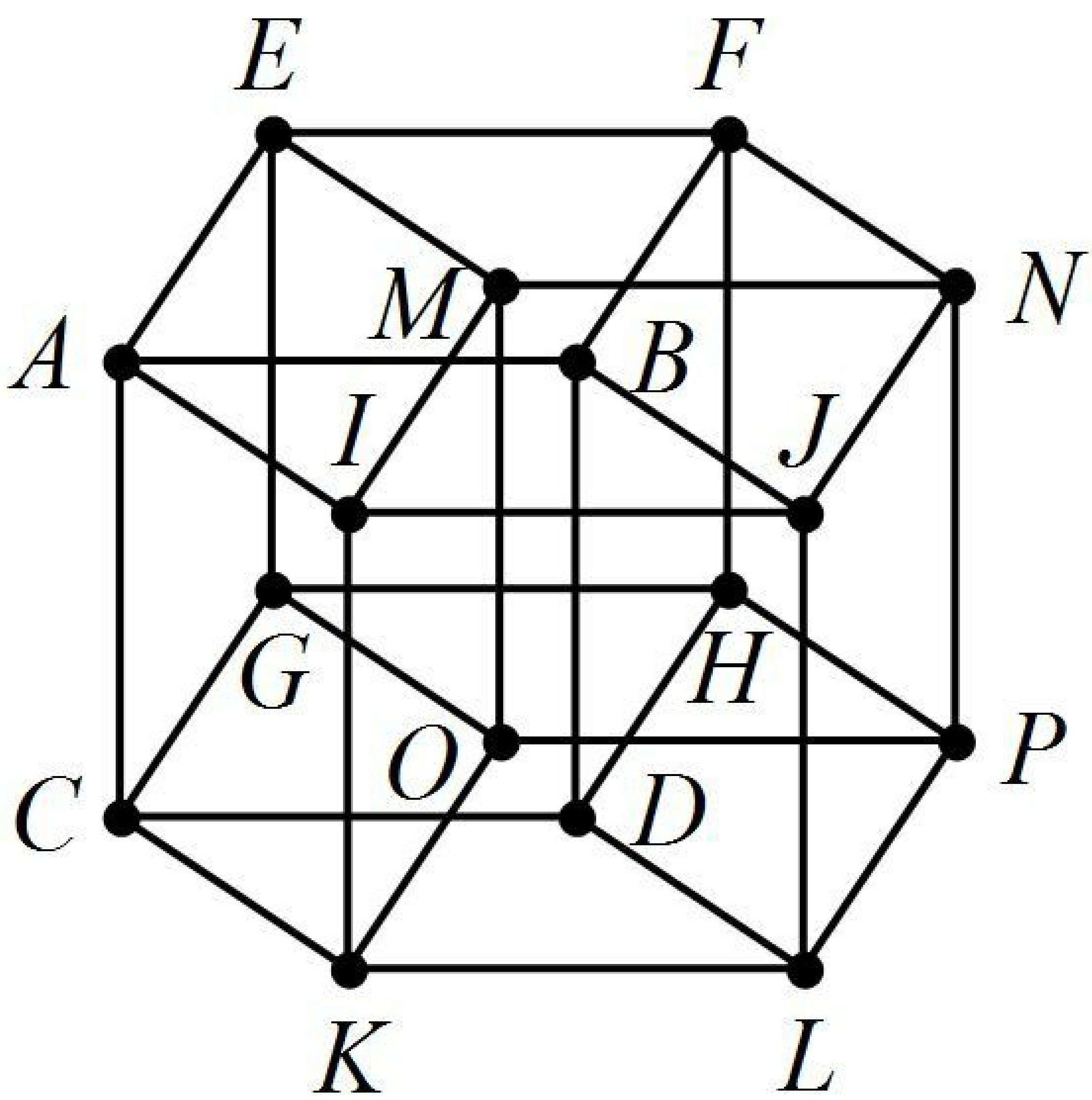
Puzzle #1: If you place a banana at every corner of a tesseract, how many bananas will there be? (Or if you want to be boring, how many corners does a tesseract have?)

Spoiler alert: If you keep reading, you'll find the answer to the previous question along with an explanation. And if you scroll too far, you'll even find a picture with some of the corners labeled. Resist the temptation to continue reading and spoil the fun of trying to figure it out for yourself. Go back (not forward) if you would like to study a picture of a tesseract to help figure out the answer. Remember, there are two pictures to help you – one in color and one in black and white.

Ready or not, here comes the answer. Actually, the logic will come first, then the answer. One way to figure this out is to make a pattern with the lower dimensions: A 0D "hypercube" is a point, which is just 1 corner; a 1D "hypercube" is a line, which has 2 "corners;" a 2D "hypercube" is a square, which has 4 corners; and a 3D cube has 8 corners. See the pattern? 1, 2, 4, 8. Guess what comes next! Yep, 16. A tesseract has 16 corners. So the answer is 16 bananas (corners). You can find 8 of the 16 corners labeled in the figure below. What about the other 8? Not to worry: The second figure that follows has all 16 labeled. If your e-reader respects page breaks, you'll find the following figure on the next page (so that if you have a large screen, the figure with the answer won't spoil the puzzle for you).



As promised, the next figure has all 16 corners labeled. They are labeled A thru P.



16 corners of a tesseract

Here is another way to deduce that a tesseract (full of monkeys) has 16 corners. Consider the binary unit hypercube. By this, I mean that all of the coordinates of the corners are either 0 or 1 (like the binary number system, where all of the digits are 0 or 1). The 1D line has corners at $x = 0$ and $x = 1$. The 2D square has corners at (0,0), (1,0), (0,1), and (1,1). The 3D cube has corners at (0,0,0), (1,0,0), (1,1,0), (0,1,0), (0,0,1), (1,0,1), (1,1,1), and (0,1,1).

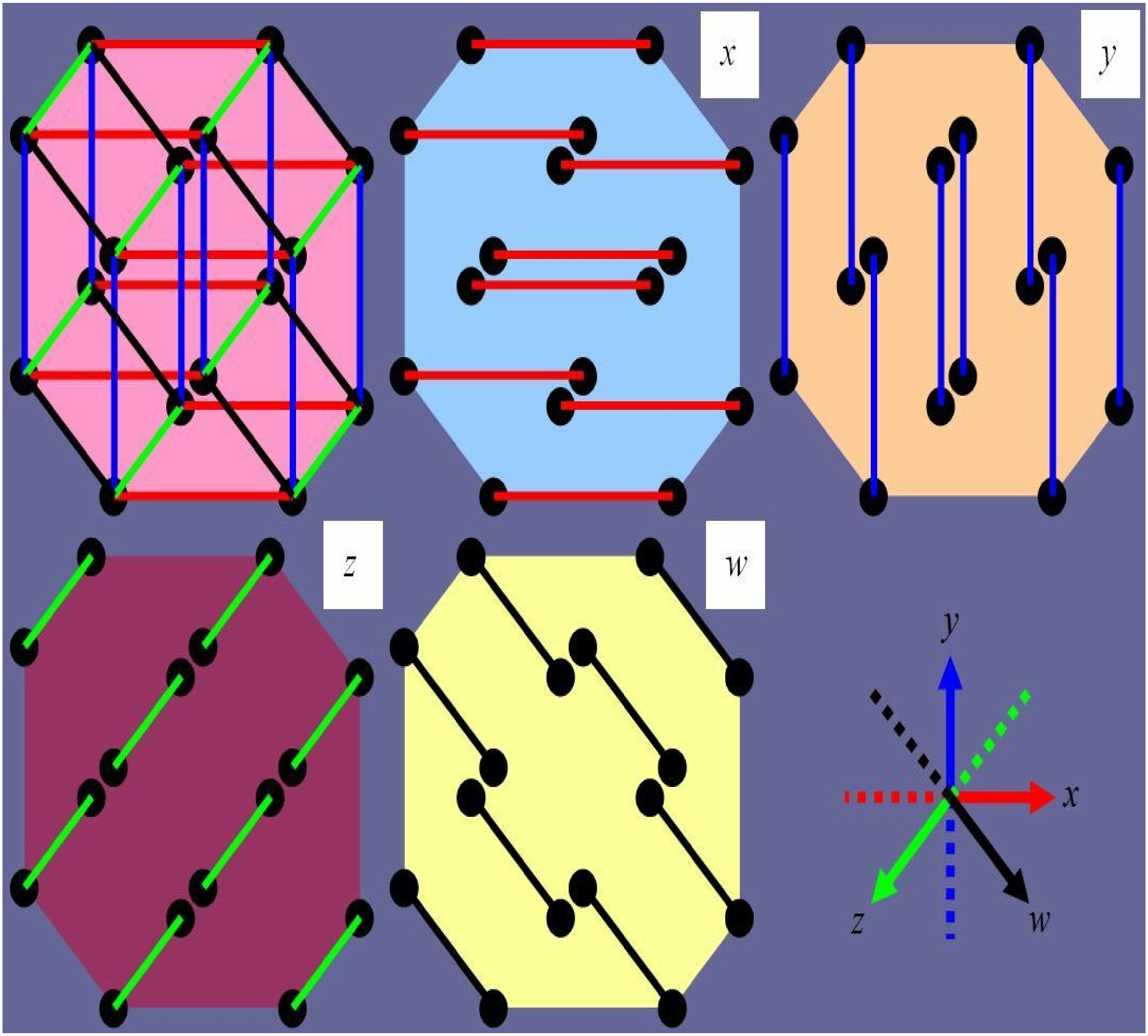
We can deduce that the N-dimensional hypercube (full of monkeys) will have 2^N corners. This reads as, "two raised to the power of N," or, "two to the N," for short. It means 2 times itself N times: $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, etc. For example, a cube ($N = 3$) has $2^3 = 8$ corners and a tesseract ($N = 4$) has $2^4 = 16$ corners. Without even drawing one, we can now predict that a 5D hypercube (remember, it's filled with nuts) will have $2^5 = 32$ corners.

To see this, consider that each corner has N coordinates in N-dimensions. For example, in 3D the coordinates are (x,y,z) and in 4D they are (x,y,z,w). Each coordinate can be one of two values (0 or 1). That's why the formula for the number of corners in an N-dimensional hypercube (full of monkeys) is 2^N.

Puzzle #2: If you place a monkey on every edge of a tesseract, how many monkeys will there be? (Or if you want to be boring, how many edges does a tesseract have?)

Remember, if you keep reading, you will run into the answer. The explanation begins now... You could just stare at the picture of the tesseract and count the edges. Or you can figure it out, logically, by thinking about how the edges are made. An N-dimensional hypercube (full of monkeys) has N sets of edges – each set of edges being parallel to each axis (that's why there are N sets). One set is parallel to x, another set is parallel to y, etc. Now we need to determine how many edges there are in each set in order to complete the solution.

A 1D line (monkey tail) has 1 edge along x. A 2D square (banana-cream pie) has 2 edges parallel to x and 2 edges parallel to y; it has 4 edges in all. A 3D cube (full of bananas) has 4 edges parallel to each axis (x, y, and z); it has 12 edges all together. Do you see the pattern? 1 edge in 1D, 2 times 2 edges in 2D, and 3 times 4 edges in 3D. What comes next? The answer is 4 times 8 edges in 4D. There are 32 edges in a tesseract: 8 parallel to x, 8 parallel to y, 8 parallel to z, and 8 parallel to w. So the answer is 32 monkeys (edges). The following figure shows all 32 edges, and even breaks them down into 4 sets of 8.



Here is another way to count the edges of a hypercube (full of monkeys). Observe that *N* edges meet at a corner in *N* dimensions. For example, 3 edges meet at each corner of a cube and 4 edges meet at each corner of a tesseract. So you may be tempted to multiply the 4 edges of a tesseract by the 16 corners, but that gives you 64, which is incorrect. Why? Because you must divide by 2 to correct for double-counting: Each edge connects 2 corners. The formula for the number of edges is *N*/2 times the number of corners. For example, a cube has (3/2)(8) = 12 edges and a tesseract has (4/2)(16) = 32 edges. We can predict that a 5D hypercube (what we previously called the "nutty" object) will have (5/2)(32) = 80 edges.

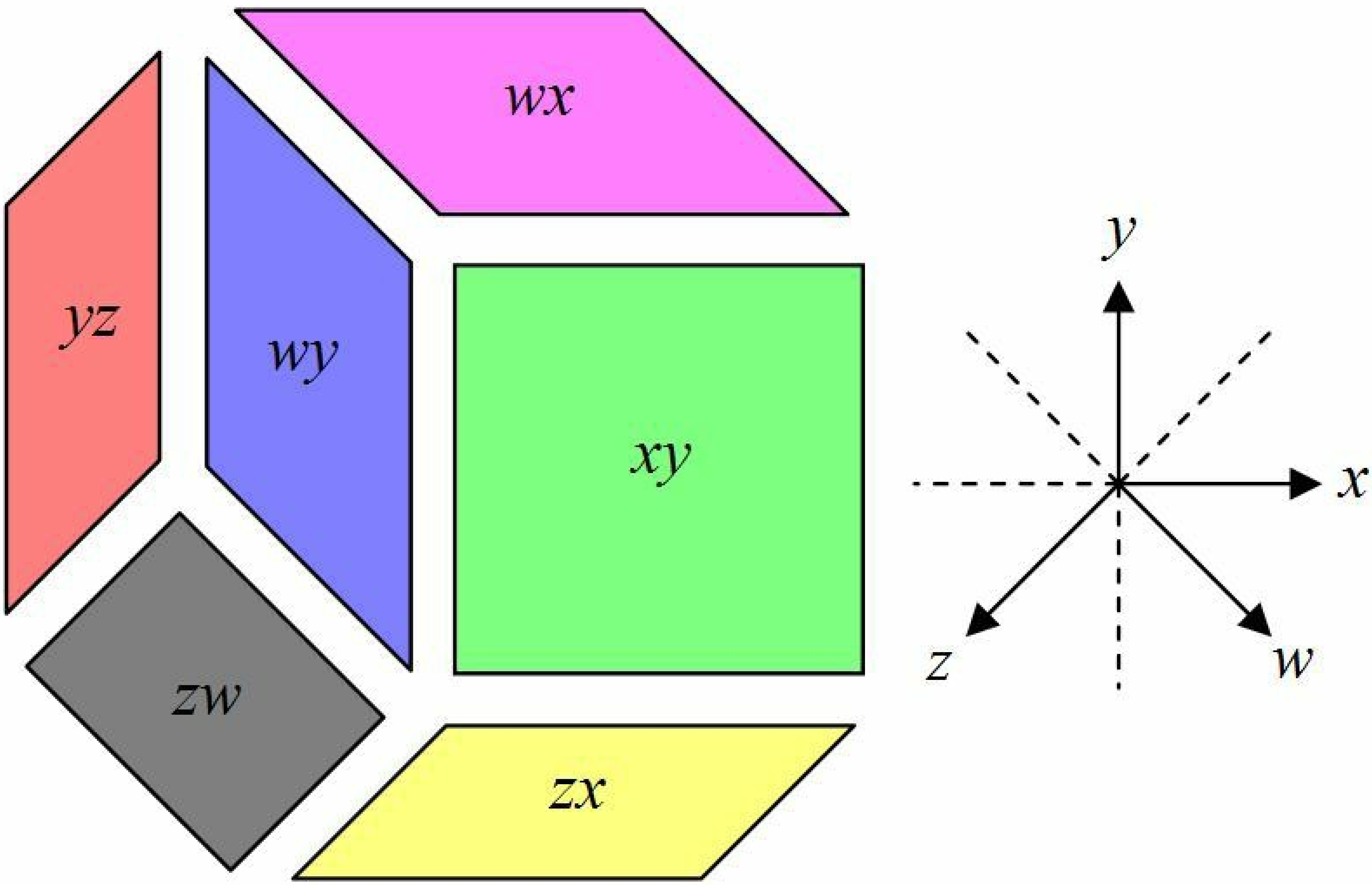
Puzzle #3: If you place a banana on every square of a tesseract, how many bananas will there be? (Or if you want to be boring, how many squares does a tesseract have?)

Once again, if you keep reading, you will spoil the answer. Here comes the explanation. This time, we don't have as many lower-dimensional objects to help deduce the pattern – we only have the square (which has just one!) and the cube to go by. So we better study the cube carefully.

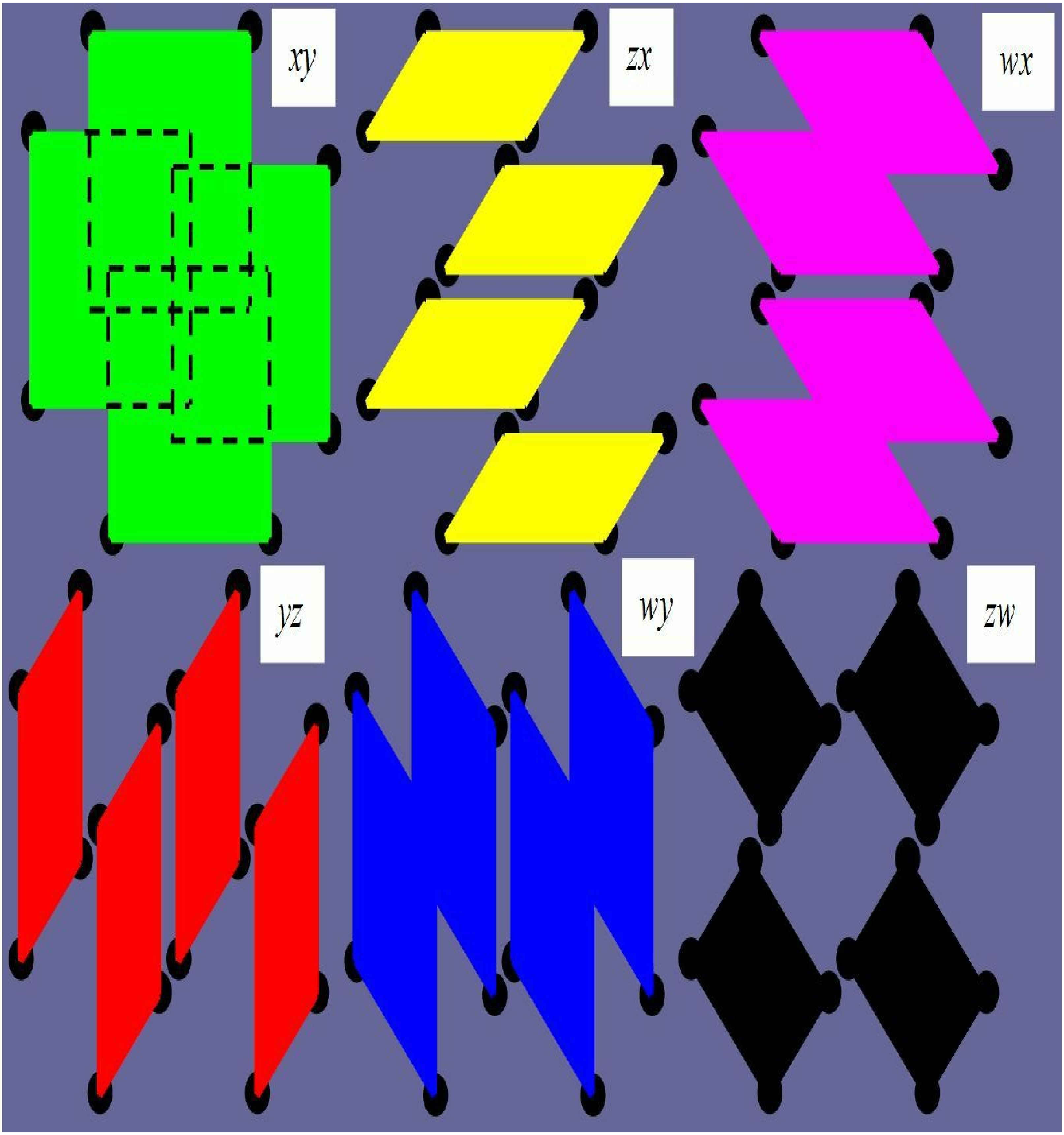
A cube is bounded by 6 square faces. There are 3 pairs of planes. The planes are *xy*, *yz*, and *zx*. You might think of the 2 *xy* planes as top and bottom, the 2 *yz* planes as front and back, and the 2 *zx* planes as left and right.

In 4D space, there are 6 types of planes: *xy*, *yz*, *zx*, *xw*, *wy*, and *zw*. In order to complete our solution, we just need to figure how many of each kind there are. The pattern is actually similar to the patterns for corners and edges. 1 times 1 square in 2D and 3 times 2 squares in 3D. The 1 in 2D, 2 in 3D pattern we've seen before. Do you remember what comes next? It's 4. 1, 2, 4, 8, 16, etc. There are 6 sets of 4 squares in 4D. Here's a recap: 1 times 1 in 2D, 3 times 2 in 3D, 6 times 4 in 4D. There are 24 squares in a tesseract (filled with monkeys). So the answer is 24 bananas (squares). Not convinced? First, you will be able to count them in one of the figures that follow. We'll also consider an alternate solution (but you'll have to wait a moment for that). Perhaps the alternate solution will help to convince you.

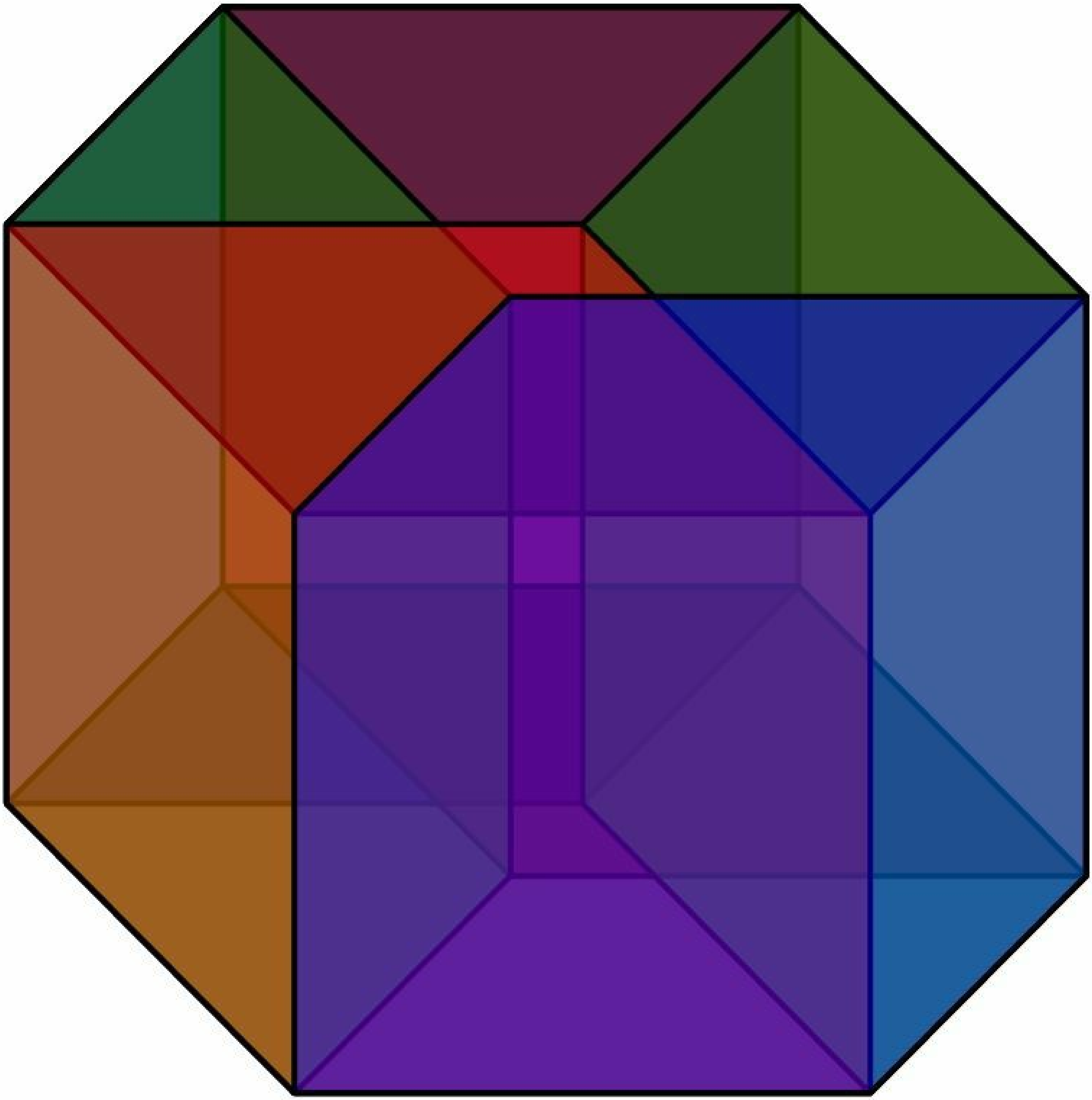
Let's look at the 6 different types of planes that make up the tesseract. These are the 6 different ways of pairing together any of the two axes (*x*, *y*, *z*, and *w*).



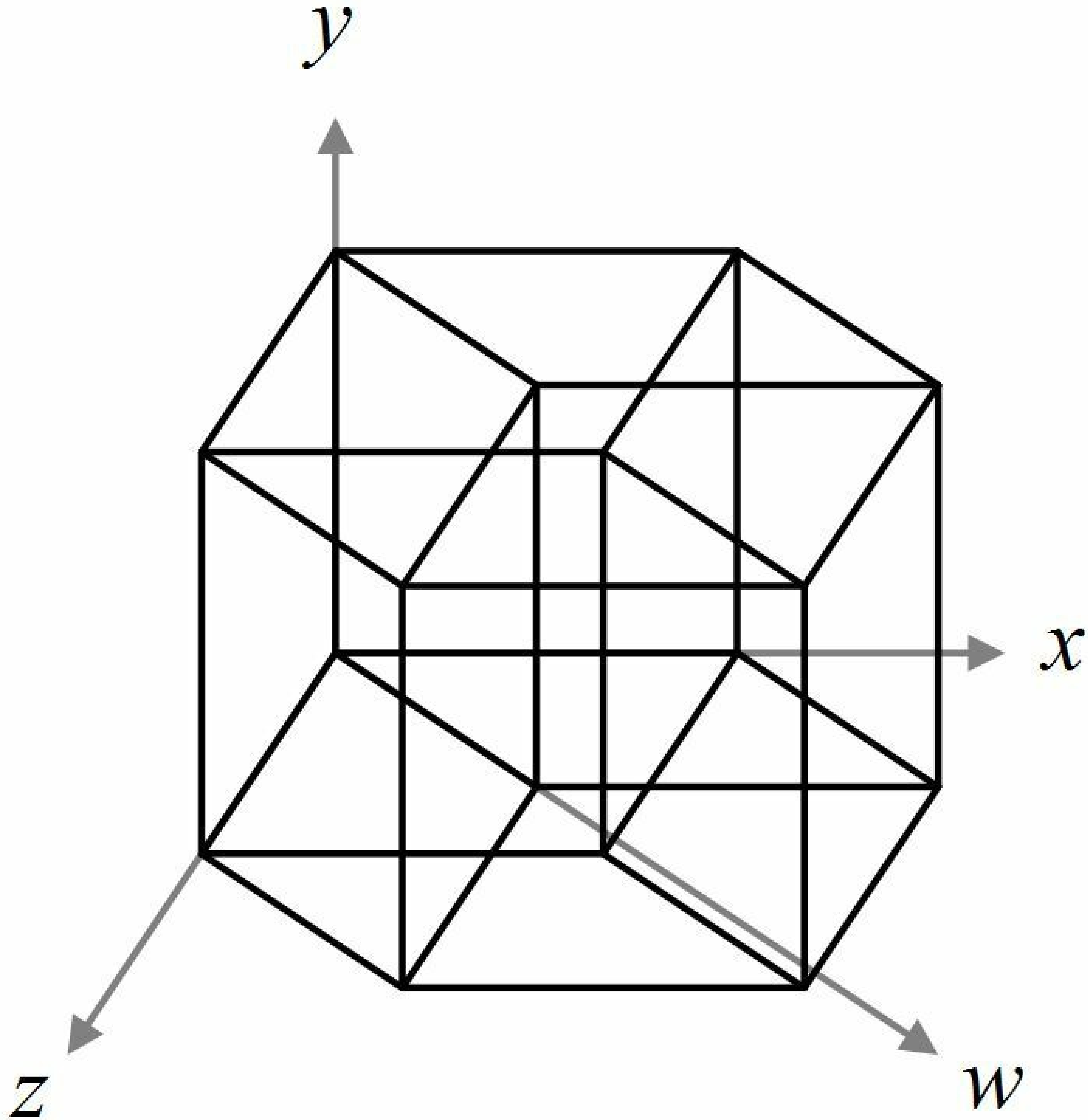
Following are the 24 squares that make up the tesseract, sorted into 6 groups of 4.



See if you can find all 24 squares in the images of the tesseract that follow. Try to understand how the squares relate to the structure of the tesseract. Don't forget to imagine monkeys inside of the tesseract.



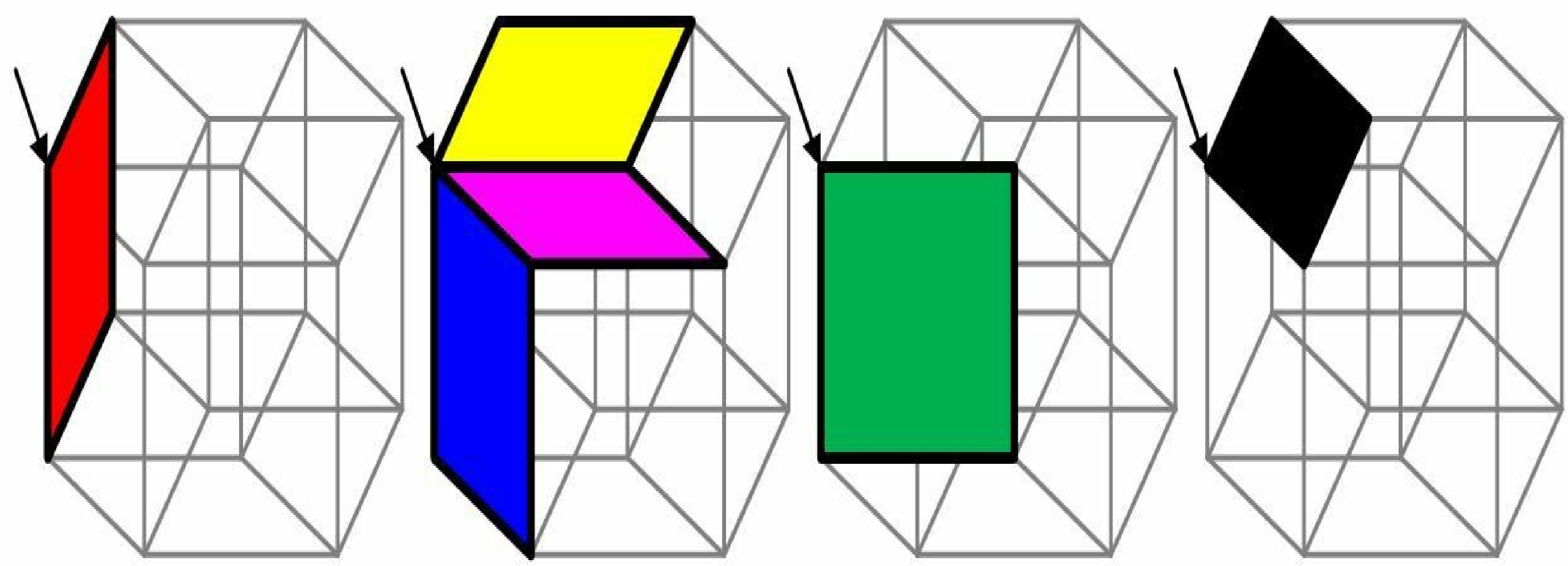
The next tesseract is in black and white; some things may be easier to find without color.



a tesseract

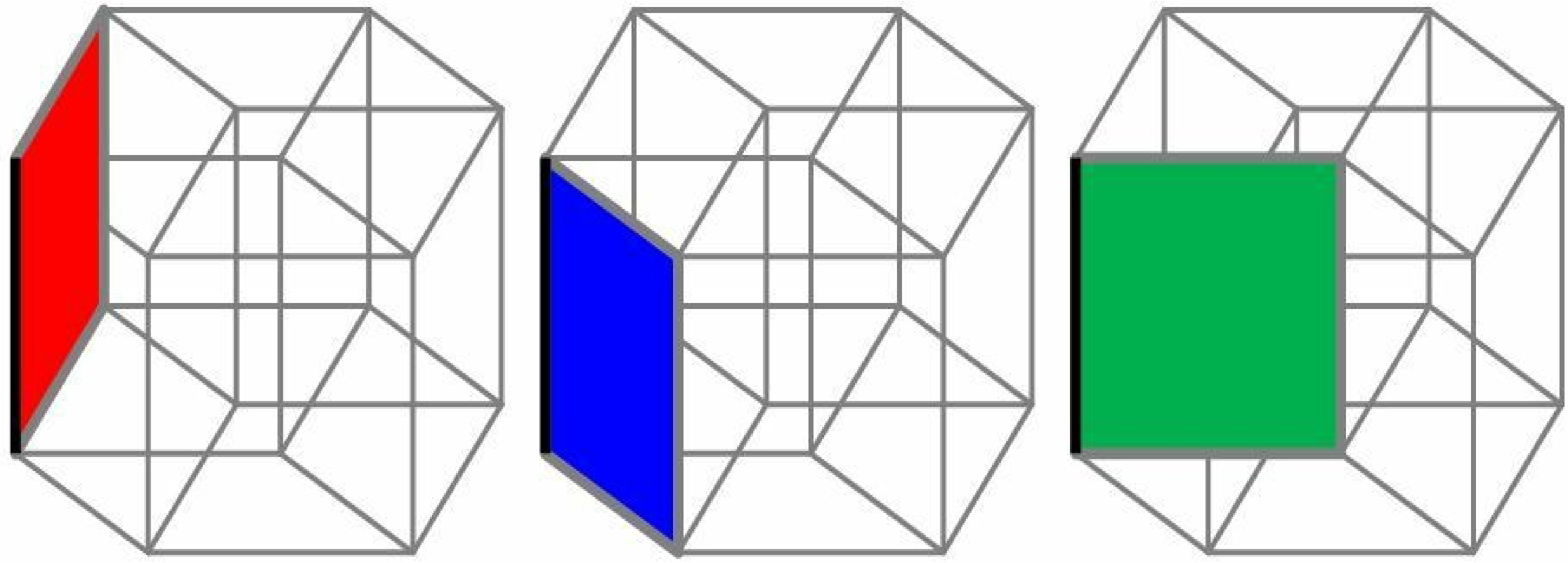
Let's consider an alternate solution to the question of how many squares there are in a tesseract (full of monkeys). The previous solution involved a very short pattern (1 square, then 1 cube with 3 sets of 2 square faces). This alternate solution is based on the structure of the tesseract, and may therefore seem more convincing.

In a tesseract (full of monkeys), 6 squares meet at every corner (1 square lies in each of the 6 planes). This is illustrated in the figure below. If you multiply this by 16 corners, you get 6 x 16 = 96. But the answer is <u>not</u> 96 because we have counted each square too many times. We actually counted each square 4 times because each square connects to 4 corners. So the formula for the number of squares in a tesseract (full of monkeys) is 6 times 16 divided by 4 = 24. The 6 is how many squares meet at each corner; there are 16 corners; and we divide by 4 because each square has 4 corners (so 6 x 16 counts each square 4 times; dividing by 4 corrects for over-counting).



6 squares meeting at 1 corner of a tesseract

Let's try it again a third way. Third time's a charm, right? This time, we'll begin by looking at how many squares meet at each edge. If you look at one edge of a tesseract (filled with howling monkeys – you'd howl too, if you were trapped in a tesseract), you should be able to find 3 squares meeting at that edge. This is shown in the figure below. Recall that there are 32 edges in a tesseract. 3 squares times 32 edges makes 96 squares, then we divide by 4 to correct for quadruple counting (since there are 4 edges in each square). 3 x 32 / 4 = 24, which agrees with our previous solutions.



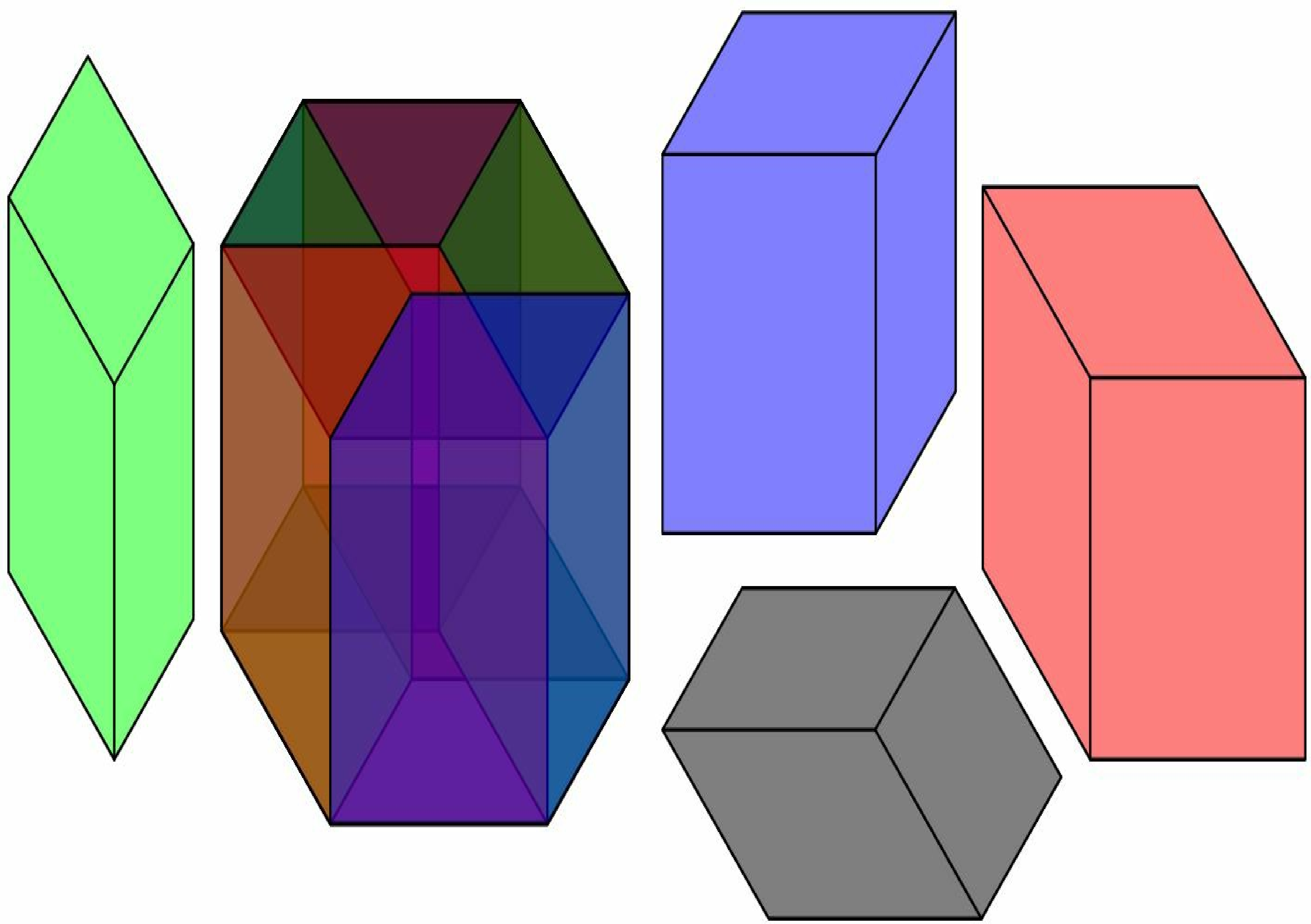
3 squares meeting at 1 edge of a tesseract

Puzzle #4: If you place a monkey inside every cube of a tesseract, how many monkeys will there be? (Or if you want to be boring, how many bounding cubes does a tesseract have?)

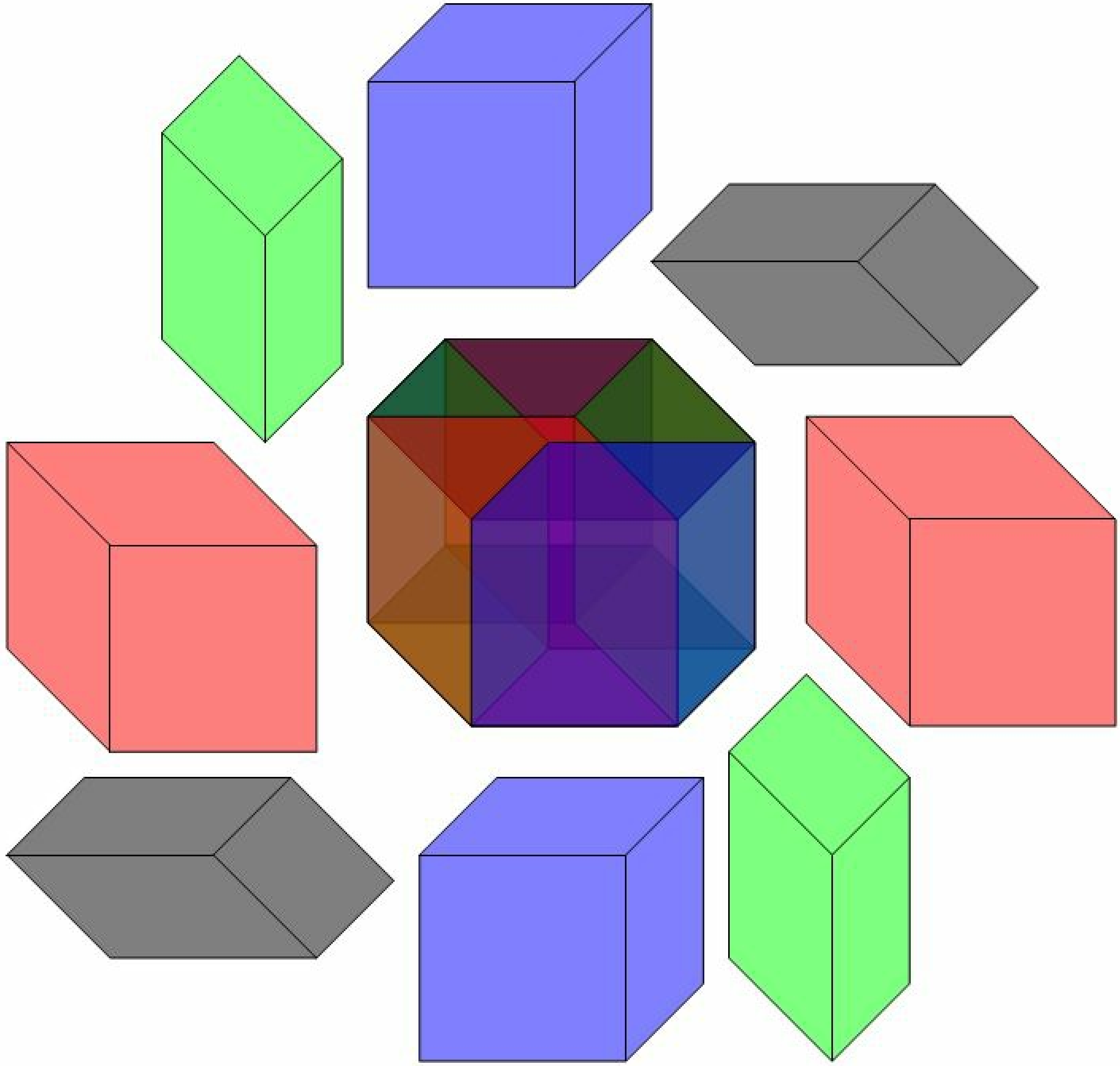
Stop reading unless you are ready to check your answer. The explanation is on its way. In this puzzle, the only lower-dimensional object that has a cube is the cube itself. So it's not any help to count cubes in lower-dimensional objects to try and make a pattern. But it's not hopeless. In fact, if you approach this from the right angle, there is actually a long lower-dimensional pattern available.

Here's the trick. Think about the boundaries of the lower-dimensional "hypercubes." (It might seem more fitting to call these 0D, 1D, and 2D objects "hypocubes" instead; but it's typical to refer to the general *N*-dimensional cube as the hypercube, even if *N* isn't greater than 3. The Greek *hypo* means under, as in hypoallergenic, while *hyper* means over, as in hyperactive. Now you can honestly say that this book really *is* Greek to you – which might be fun to say, even if it isn't.) Back to the trick: A 1D line (monkey tail) is bounded by 2 points (its endpoints); a 2D square (banana-cream pie) is bounded by 4 edges (lines); a 3D cube (filled with bananas) is bounded by 6 square faces. What comes next in this pattern? Yeah, now it's easy: A 4D tesseract (full of monkeys) is bounded by 8 cubes. The answer is 8 monkeys (cubes). The pattern is 2 points in 1D, 4 edges in 2D, 6 squares in 3D, and 8 cubes in 4D. You can predict that there would be 10 tesseracts bounding a 5D hypercube.

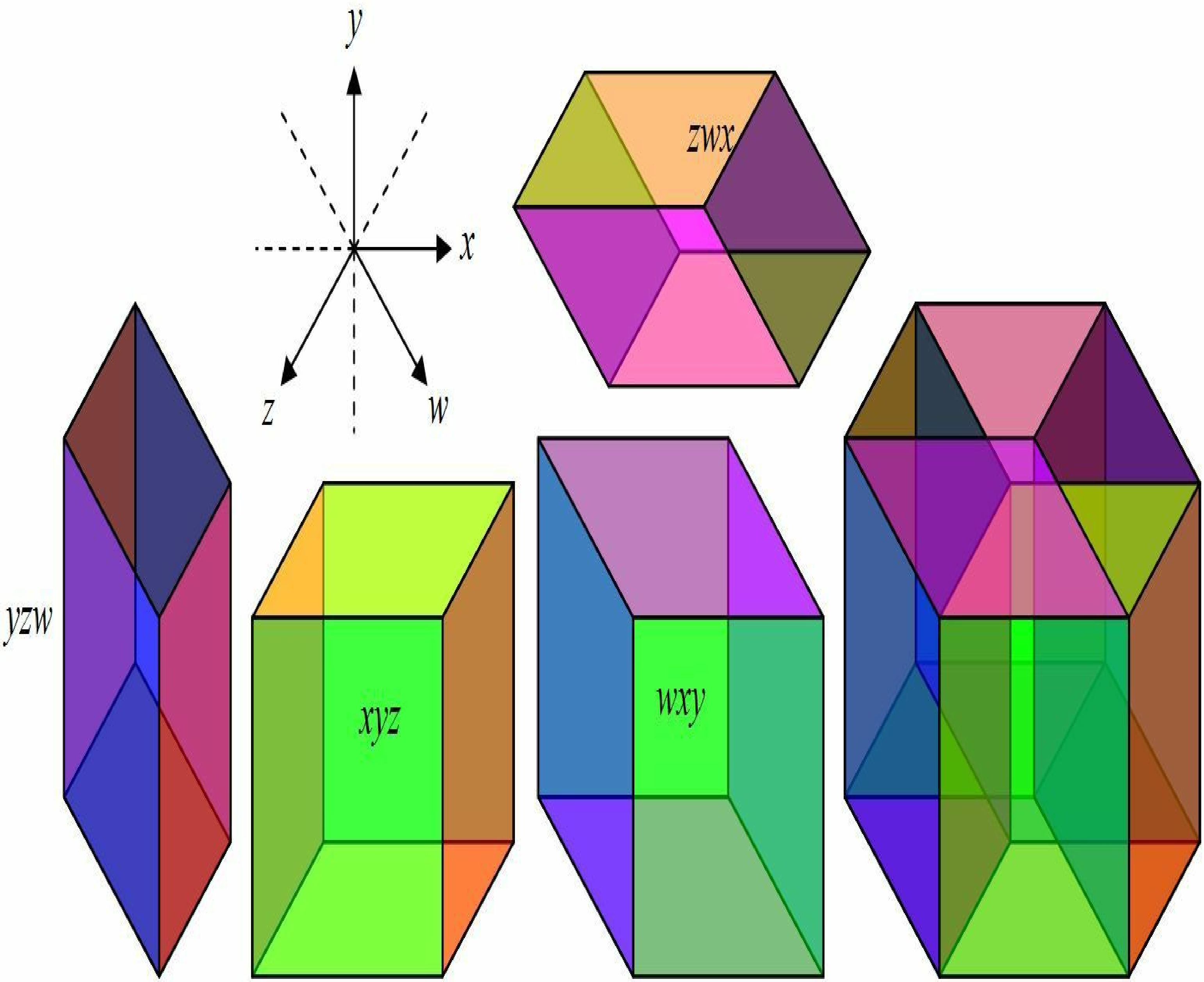
The following figures show the 8 cubes that bound a tesseract (filled with monkeys). First, this picture has the 4 types of cubes – *xyz*, *yzw*, *zwx*, and *wxy*. There are two of each kind bounding the tesseract.



The next figure shows the 8 individual bounding cubes (two of each kind).



Here is a different image of the 4 types of bounding cubes.



There is a simple (*everything* is simple once you understand it) conceptual explanation for the pattern that we just discussed. Start with a 1D monkey, who resides solely on the x-axis: Place one point in front of him and one behind him, and he will be trapped. The 2D monkey lives in the xy plane: She (obviously, she's <u>not</u> the same monkey as the 1D monkey, so why can't her gender be different?) would be bounded by the 4 edges of a square. In 3D space, with coordinates (x,y,z), a monkey could be imprisoned by the 6 sides of a cube. A 4D monkey (he would be "hyper," as in a hypermonkey) would be trapped by the 8 cubes that bound a tesseract; here there are 4 coordinates – x, y, z, and w. In general, the N-dimensional hypercube (filled with hypermonkeys) is bounded by $2N$ "sub-surfaces," which are $(N–1)$-dimensional hypercubes. The "surface" of a hypercube has one less dimension than the hypercube itself. As examples, the 8 cubes (that's 2 times 4) bounding the tesseract ($N = 4$) are 3D (that's 4 – 1) sub-surfaces, and the 6 squares on the surface of a 3D cube are 2D. It takes two sub-surfaces per axis to bound the space inside; that's where the $2N$ comes from – which is why there are 2 times 4 cubes (which equals 8 cubes) bounding a tesseract. We'll return to this point in Chapter 12, where we will imagine sitting inside of a tesseract-shaped room.

If you enjoyed working out these hypercube patterns, you might be interested in the author's original book on the fourth dimension, *The Visual Guide to Extra Dimensions, Volume 1*. Chapter 3 of that book works out more details – like how many tesseracts there are in a 6D hypercube, or how many cubes meet at every edge of a tesseract – and provides elaborate mathematical formulas for the answers. *The Visual Guide to Extra Dimensions* also discusses other geometries, like tetrahedron-based 4D polytopes and curved hypersurfaces. However, there is some overlap between the two books – e.g. many of the 2D and tesseract-based figures are the same. Also, *The Visual Guide to Extra Dimensions* is in black and white (whereas this book is in color), and while *The Visual Guide to Extra Dimensions* is highly informative, it wasn't written with the flair and personality with which this book (that you're reading now) was written.

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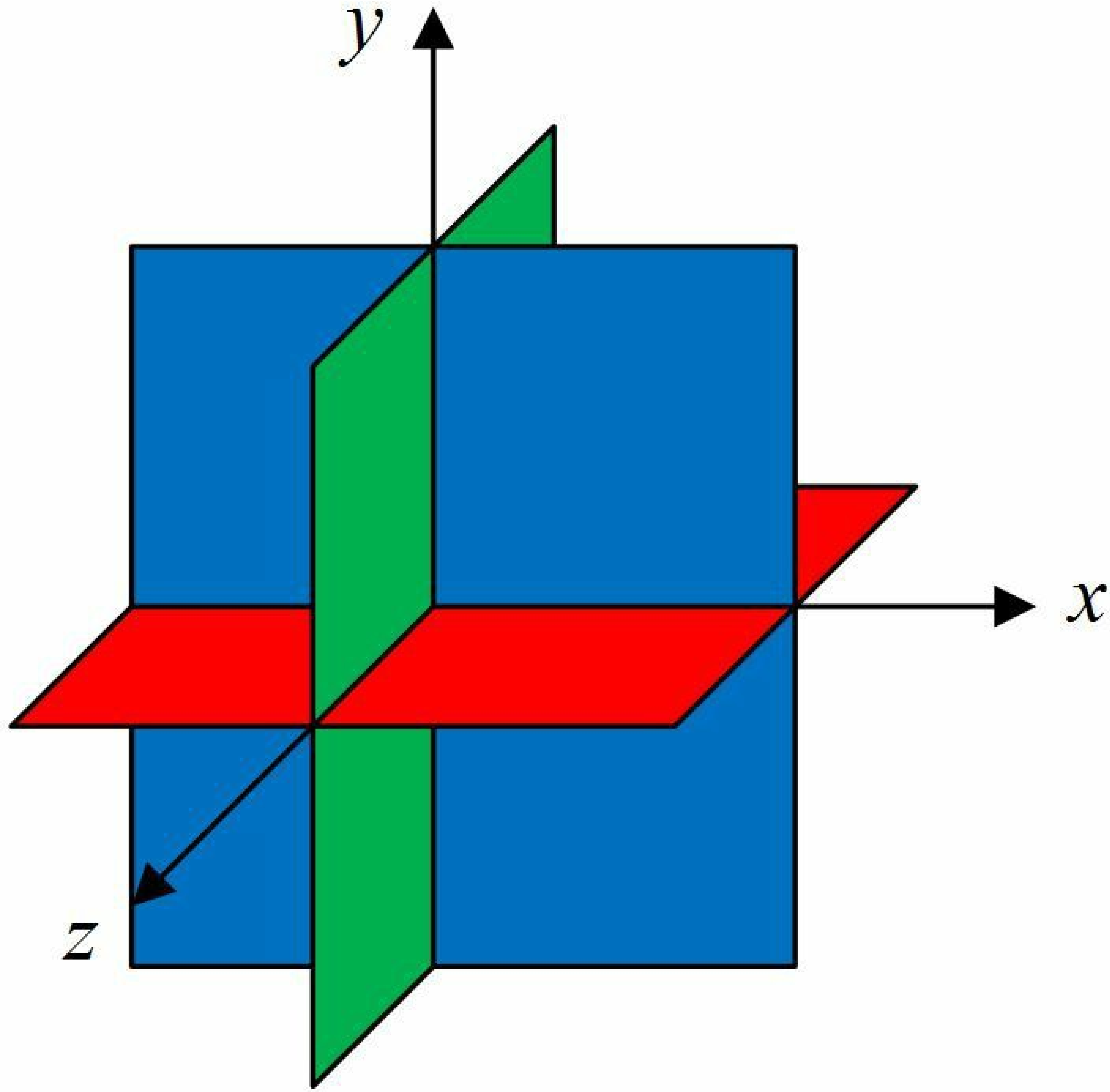
Chapter 7

Planes and Hyperplanes

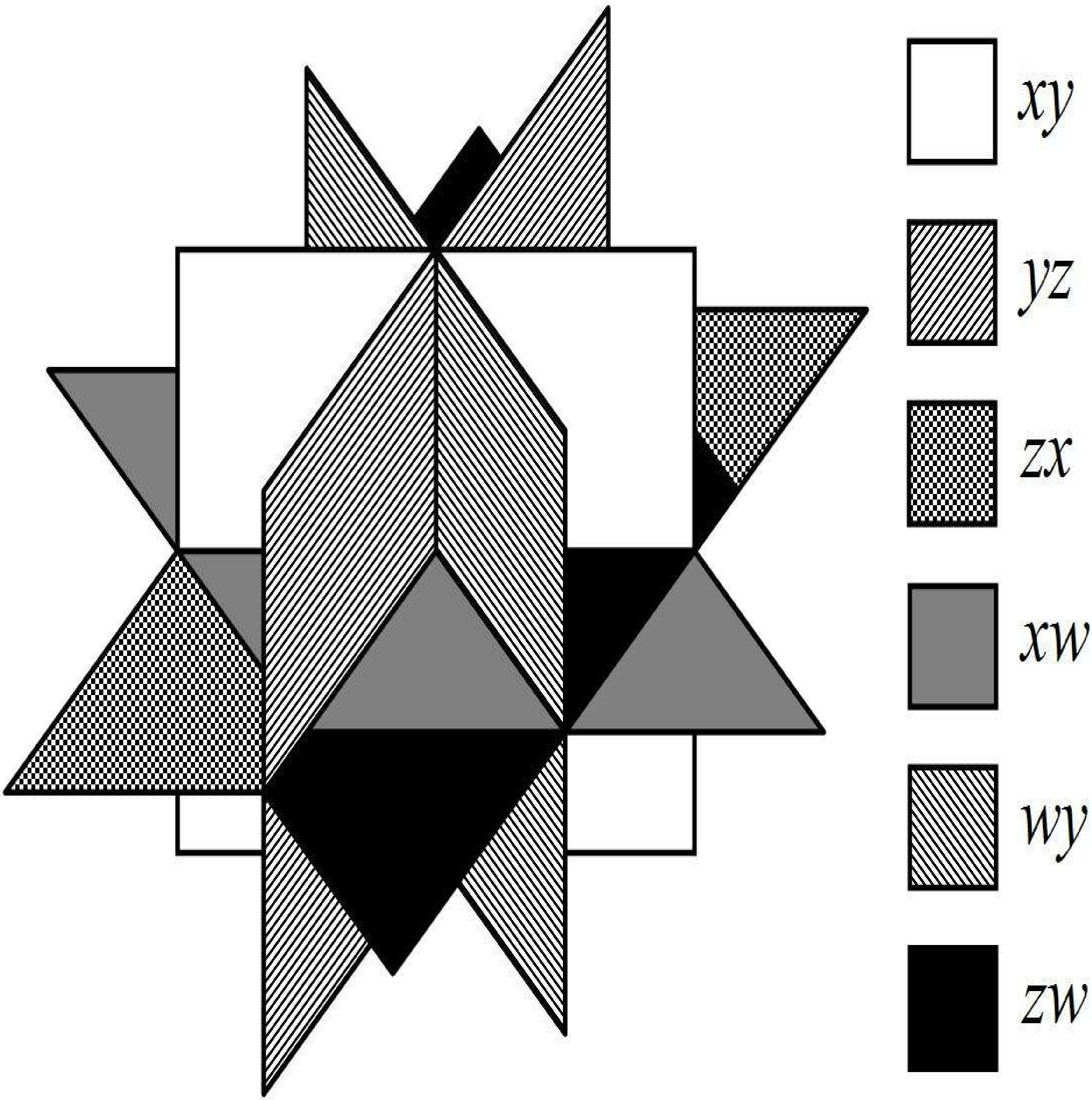
Just as a <u>plane</u> is like an infinitely large square, a <u>hyperplane</u> is like an infinitely large cube. The 3D space that we experience everyday has the shape of a hyperplane, the same way that a flat 2D space would be shaped like a plane. (If you want to consider spacetime curvature, then you introduce some technical complications. We're going to focus solely on space – as if we could separate it from time – and ignore any curvature from Einstein's general theory of relativity in order to see some fundamental 4D geometric concepts. We'll also assume that the space is "flat," unlike the surface of a sphere or hypersphere.)

There are 3 mutually <u>orthogonal</u> planes in 3D space: *xy*, *yz*, and *zx*. What we mean by this word (orthogonal) is that these planes are all perpendicular to one another. That is, $xy \perp yz$, $zx \perp xy$, and $yz \perp zx$, where the symbol $\perp$ stands for "is perpendicular to," which means that these planes intersect at right angles (i.e. 90 degrees). The 3 mutually orthogonal planes (think of each one as monkey-themed wallpaper) of 3D space are illustrated below; *xy* is shown in blue, *yz* appears green, and *zx* is in red. Observe that the intersection of any two of these three planes is a line (an infinitely long, straight monkey tail). For example, the planes *xy* and *yz* intersect at the *y*-axis.

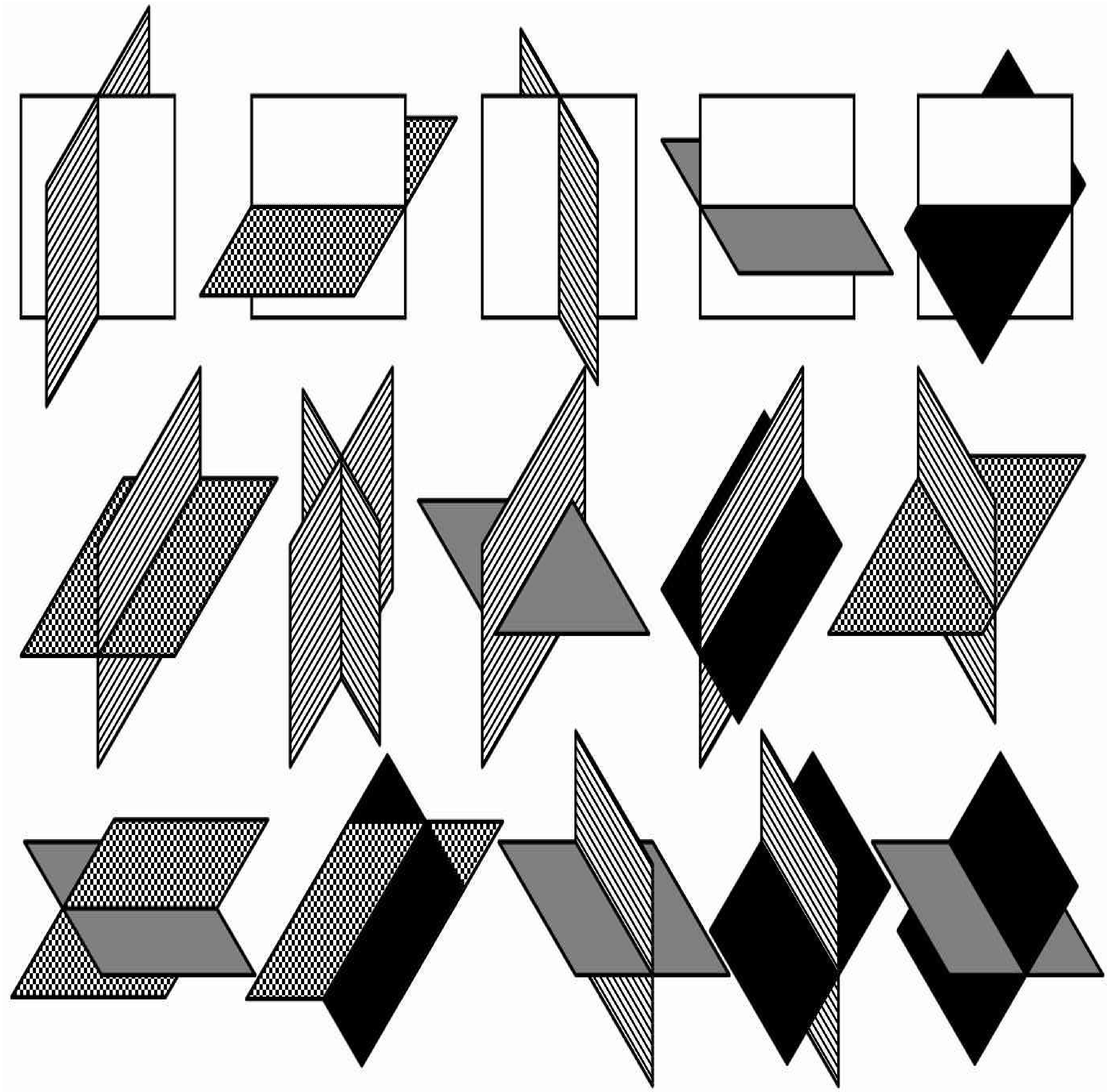


In 4D space, there are 6 mutually orthogonal planes (remember, they're made out of monkey wallpaper): *xy*, *yz*, *zw*, *wx*, *xz*, and *yw*. The intersection between two of these planes is either a line or a point. When there is a common axis, the intersection is a line (long, straight monkey tail). For example, *zw* and *xz* intersect along the *z*-axis. When there isn't a common axis, the intersection is a single point – the origin, located at (0,0,0,0). This is the case for *yz* and *wx*, for example.

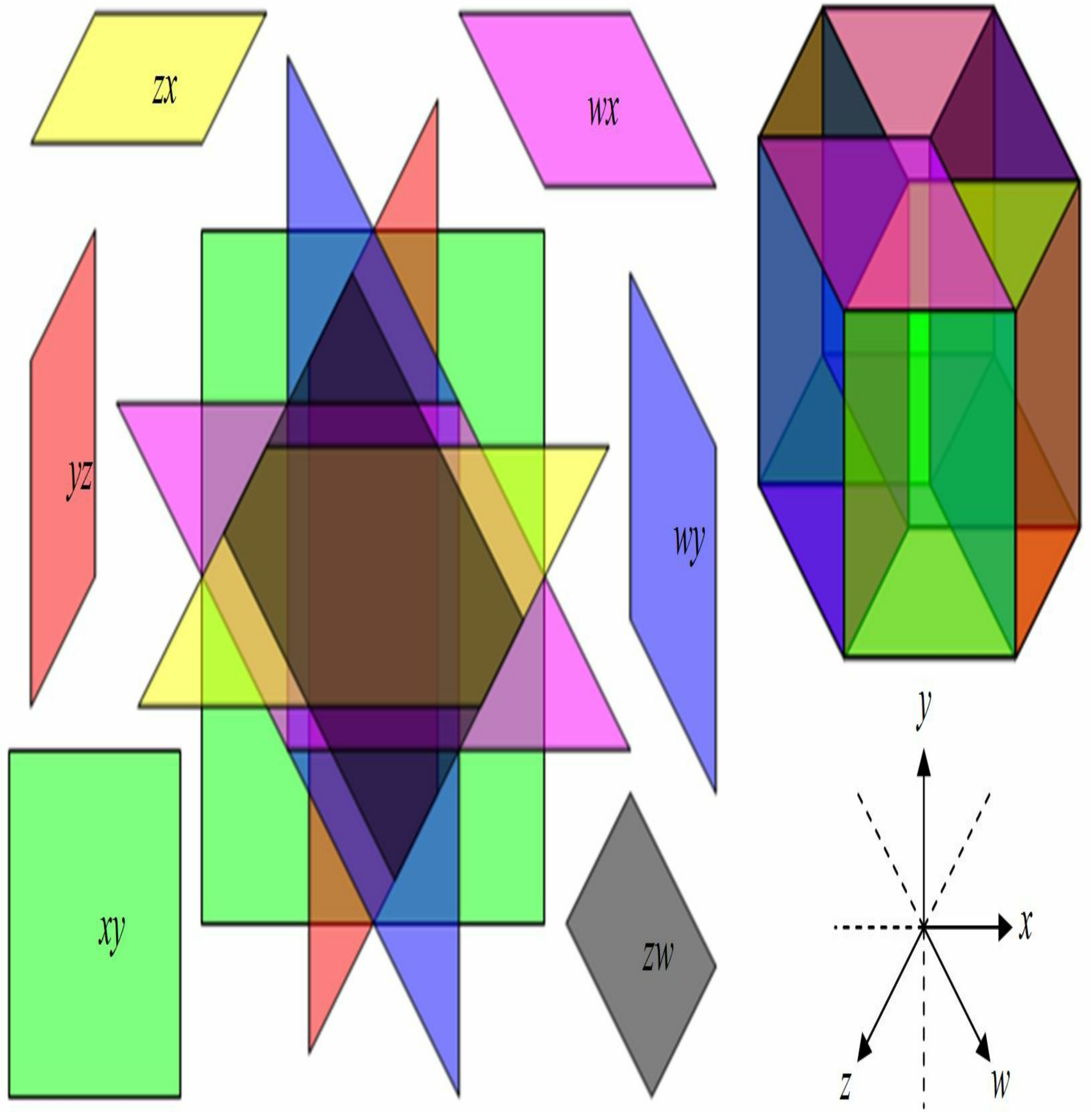
The 6 mutually orthogonal planes of 4D space are illustrated below in black and white. See if you can find all 6. Also, bear in mind that they are all orthogonal; look at two at a time, and see if you can visualize them being perpendicular to one another (i.e. meeting at right, or 90 degree, angles). Remember, there is more ambiguity interpreting a 4D illustration on 2D paper (or a 2D screen) than there is in interpreting a 3D illustration on 2D paper – and there is already enough when looking at 3D diagrams to draw optical illusions and impossible objects. For example, see if you can convince yourself that the *xw* and *zw* planes are, in fact, perpendicular to one another.



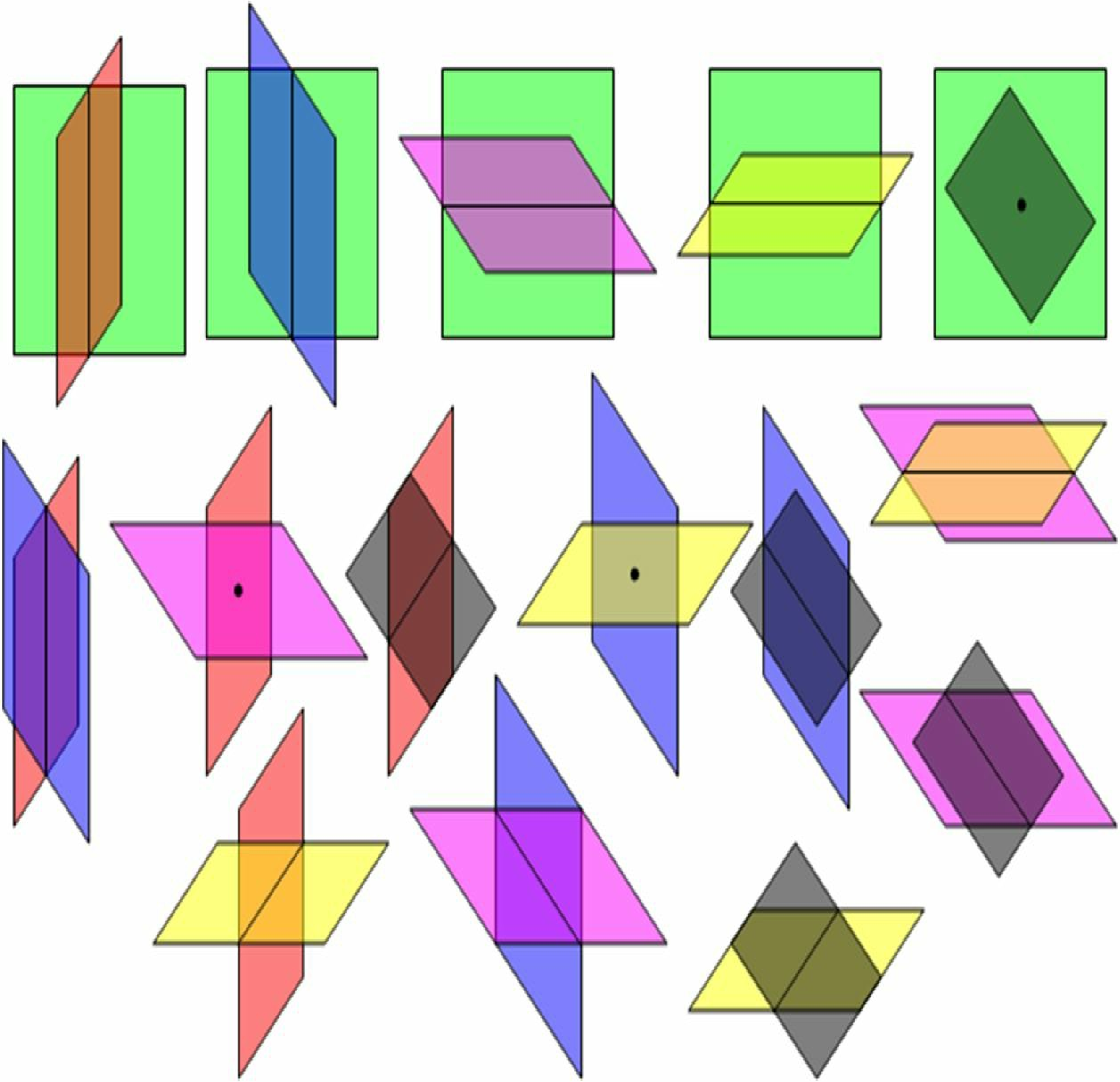
It turns out that there are 15 different ways to pair these 6 planes together. The following black and white picture shows all 15 intersections. Note that 3 of the intersections are actually points – not lines (or monkey tails) – because there is no common axis. See if you can figure out which 3 <u>don't</u> intersect at a line. It's kind of like a 4D *Where's Waldo?* (Or *Where <u>Isn't</u> Waldo?*) You'll be able to check your answers in a couple of paragraphs.



The following picture shows the 6 mutually orthogonal planes in color along with a tesseract. This may help you to visualize the squares that make up a tesseract and how they join together. Maybe not instantly (studying the fourth dimension is not like making coffee); but maybe it will help to study it some and try to think it through.

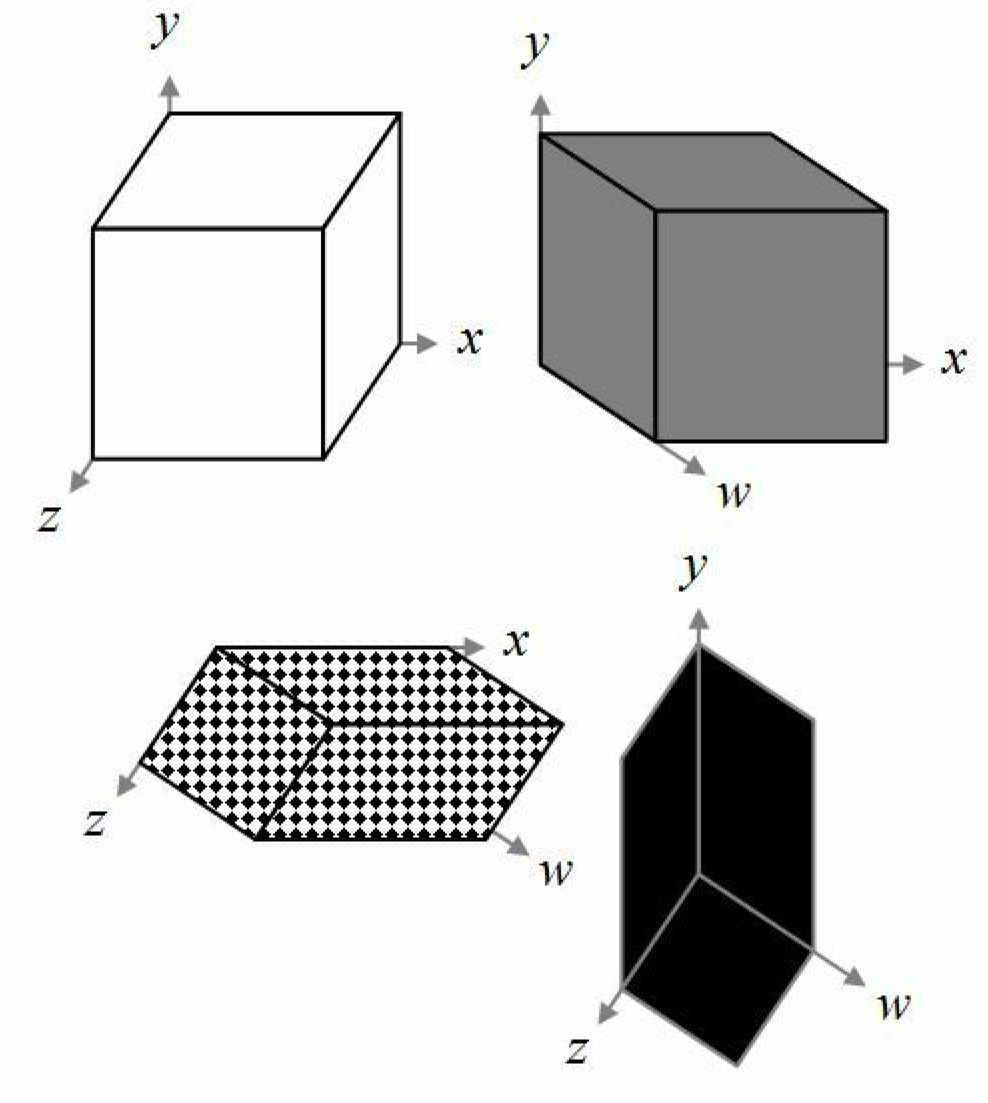


Here are the 15 intersections in color. This time, it's easy to find the 3 cases where the intersection is a point. If you take a moment to figure out the names (like *xy*) of each plane involved (green = *xy*, reddish/orangish = *yz*, yellow = *zx*, pink = *wx*, blue = *yw* and gray = *zw*), you will see that these 3 cases correspond to having no common axis in their names (like green and gray, since *xy* and *zw* have nothing in common).

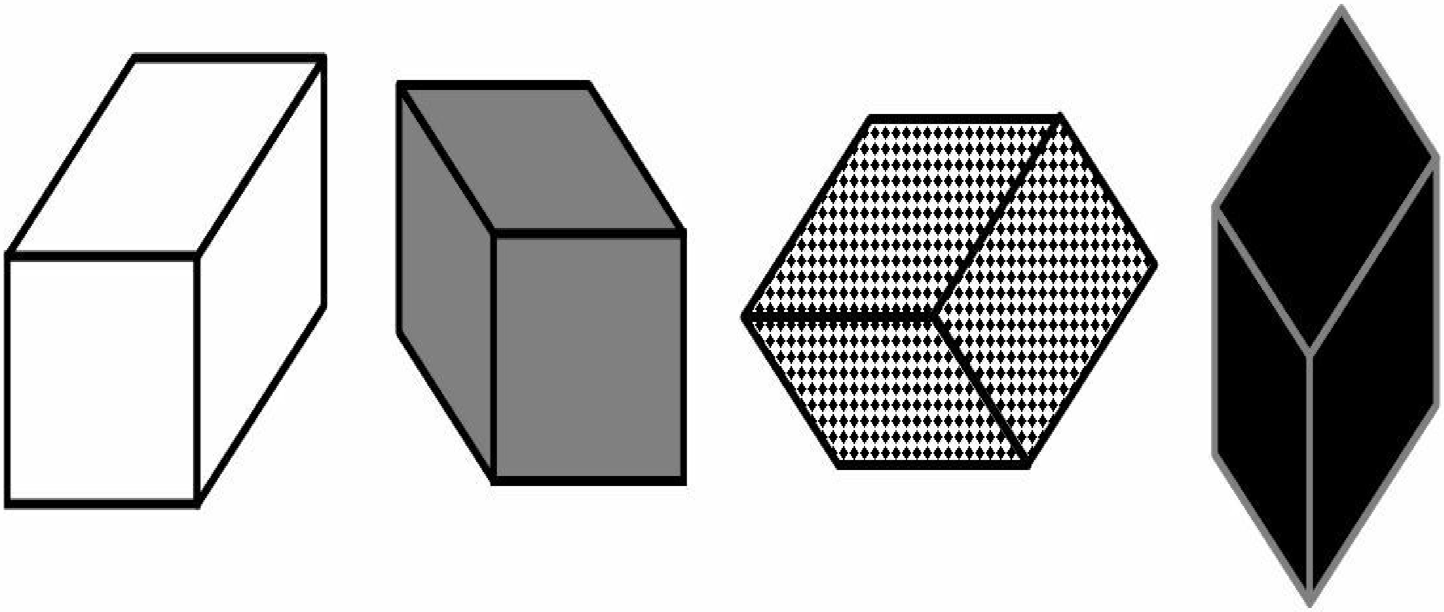


There is just a single <u>hyperplane</u> (nope, it's not a jet piloted by caffeinated monkeys) in 3D space: The *xyz* hyperplane spans <u>every</u> point in our 3D space (just like the *xy* plane spans every point in flat 2D space).

There are 4 mutually orthogonal hyperplanes in 4D space: *xyz*, *yzw*, *zwx*, and *wxy*. These hyperplanes intersect at right angles. The region of intersection between any two of these hyperplanes is a plane (which you might recall was declared to be monkey wallpaper). The four fundamental hyperplanes are illustrated below in black and white.

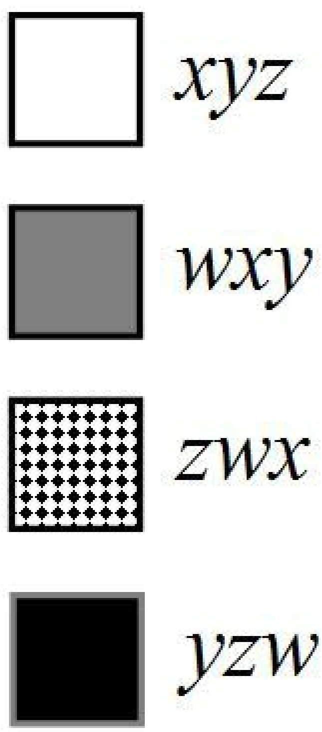


Here they are again, but without the coordinate axes.

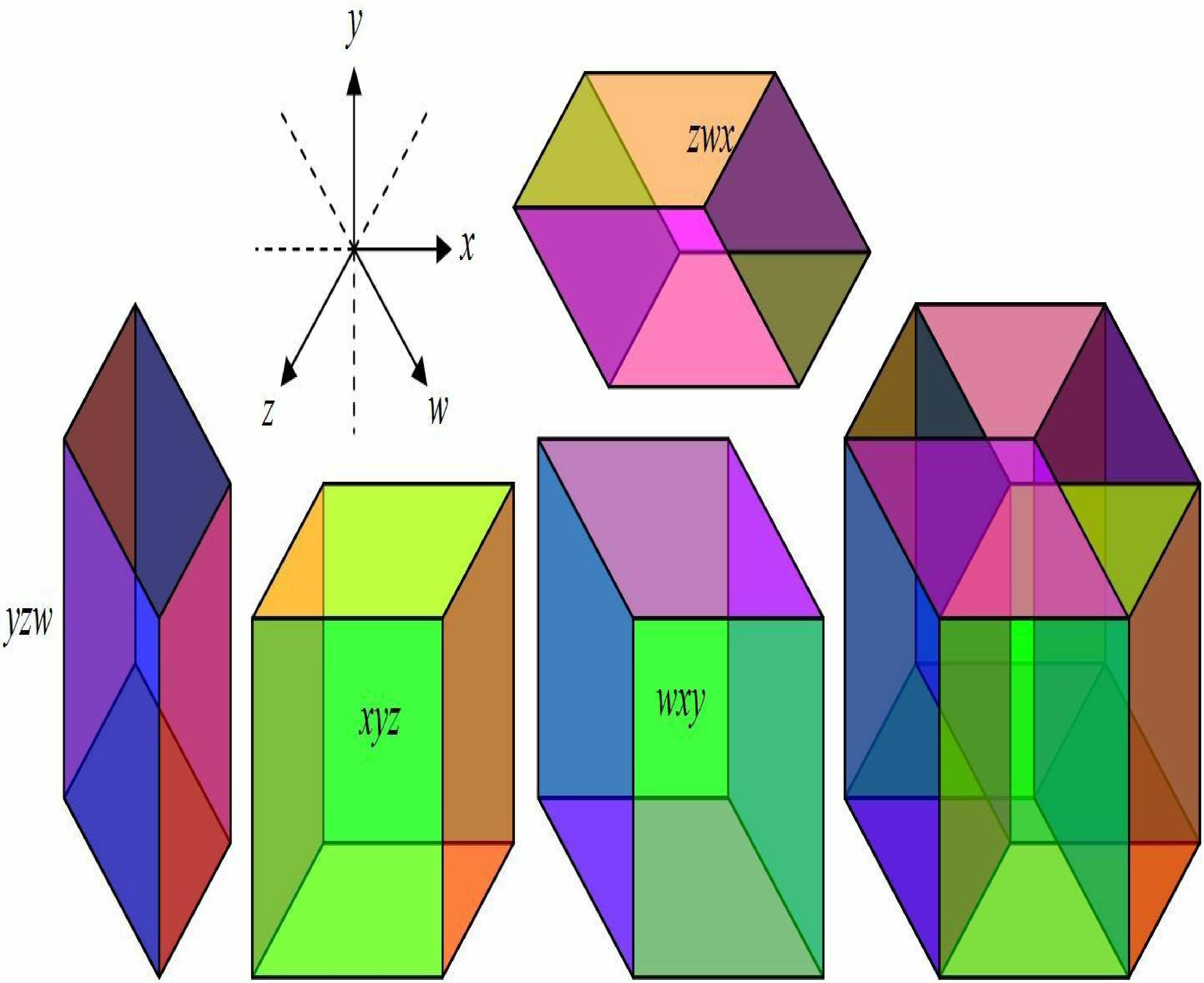


four mutually orthogonal 3D subspaces in 4D

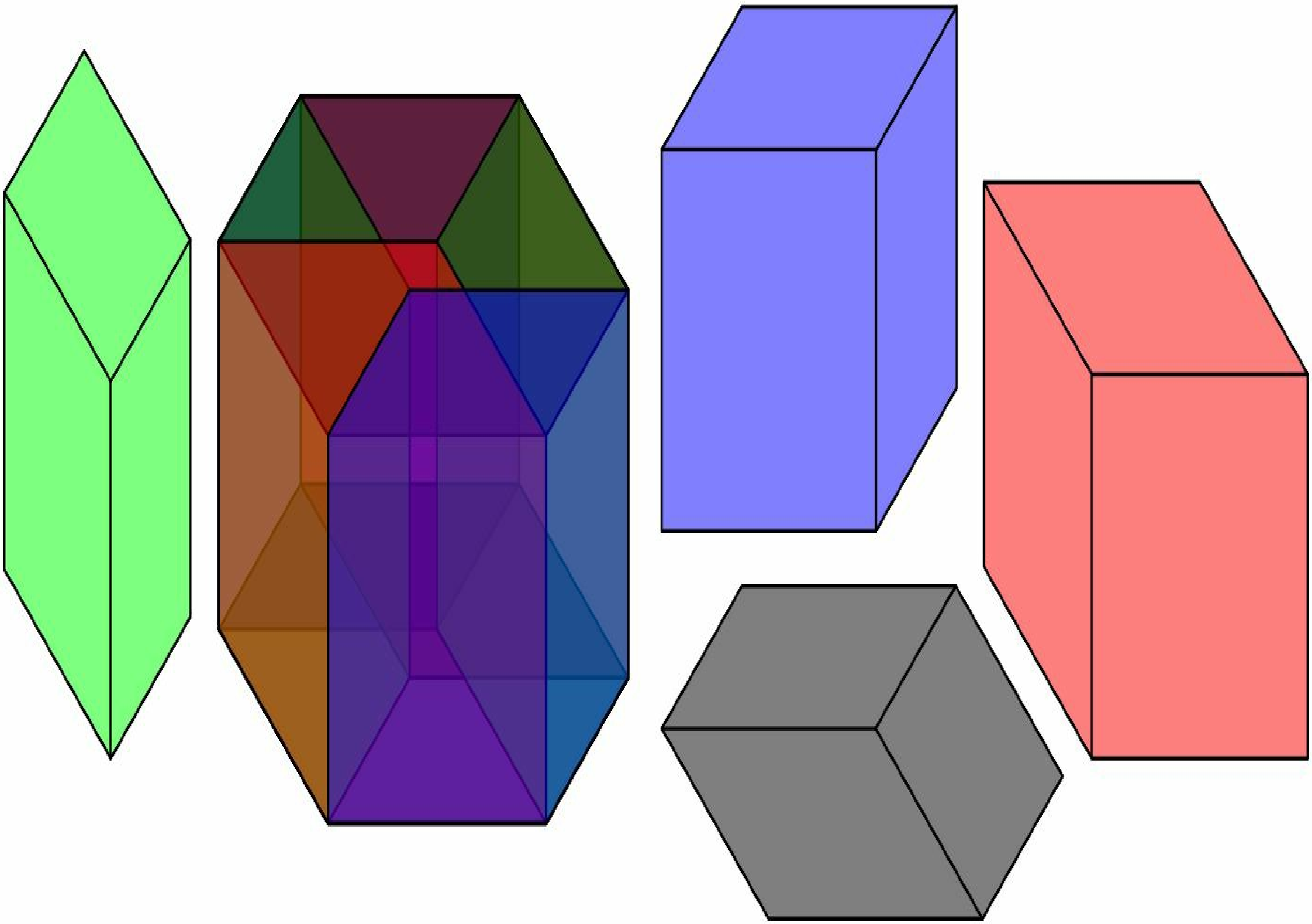
Following is the legend for the previous figures.



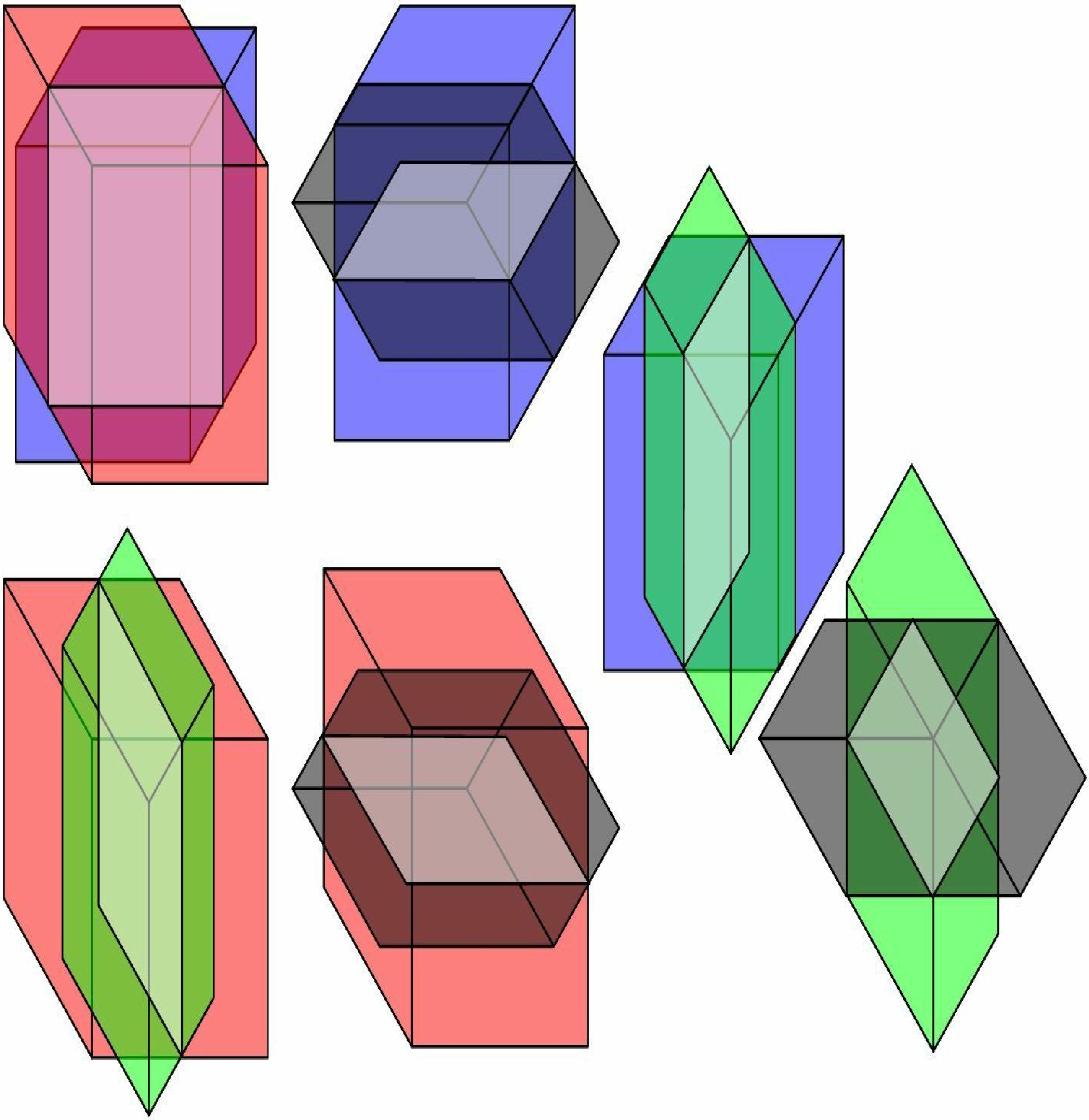
Here we have the 4 hyperplanes in color. In this figure, you can see how they correspond to the cubes bounding the tesseract. The sides of the hyperplanes come in 6 colors – one for each of the 6 planes. The many resulting colors have to do with transparency (you can see "through" the faces partially, so the colors that you see are a combination of 2 or more planes).

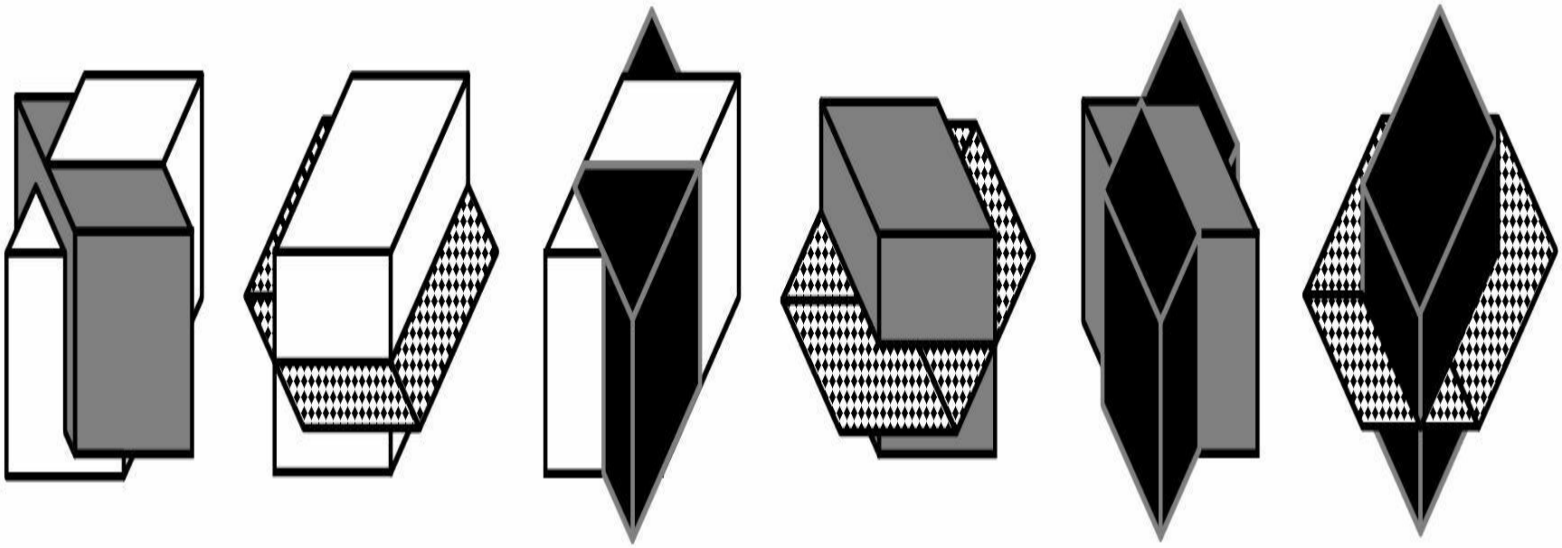


In the next figure, each hyperplane is a solid color. The colors are much more straightforward this time, if not quite as pretty.



There are 6 different ways that 2 of the 4 hyperplanes can intersect. In each case, the region of intersection is a plane. This is shown in the following picture.





4 mutually orthogonal 3D hyperplanes in 4D broken down into 6 pairs; in each case the region of intersection is a plane

Now let's consider how planes, hyperplanes, and their intersections relate to the structure of the cube and tesseract. The cube is bounded by 6 squares (which are planar). Two adjacent square faces intersect at a line (each edge). The 12 edges (monkey tails) are the regions where two square faces intersect. The cube is bounded by 6 square faces, which are joined together at 12 edges.

Similarly, the tesseract is bounded by 8 cubes (which are hyperplanar). Two adjacent cubes intersect at a square (which is planar) in 4D. (Note that two cubes <u>can't</u> be orthogonal, but only parallel, in 3D space – but orthogonality between cubes does become possible if you add a fourth dimension of space.) The 24 squares are the regions where two cubes intersect. The tesseract is bounded by 8 cubes, which are joined together at 24 squares.

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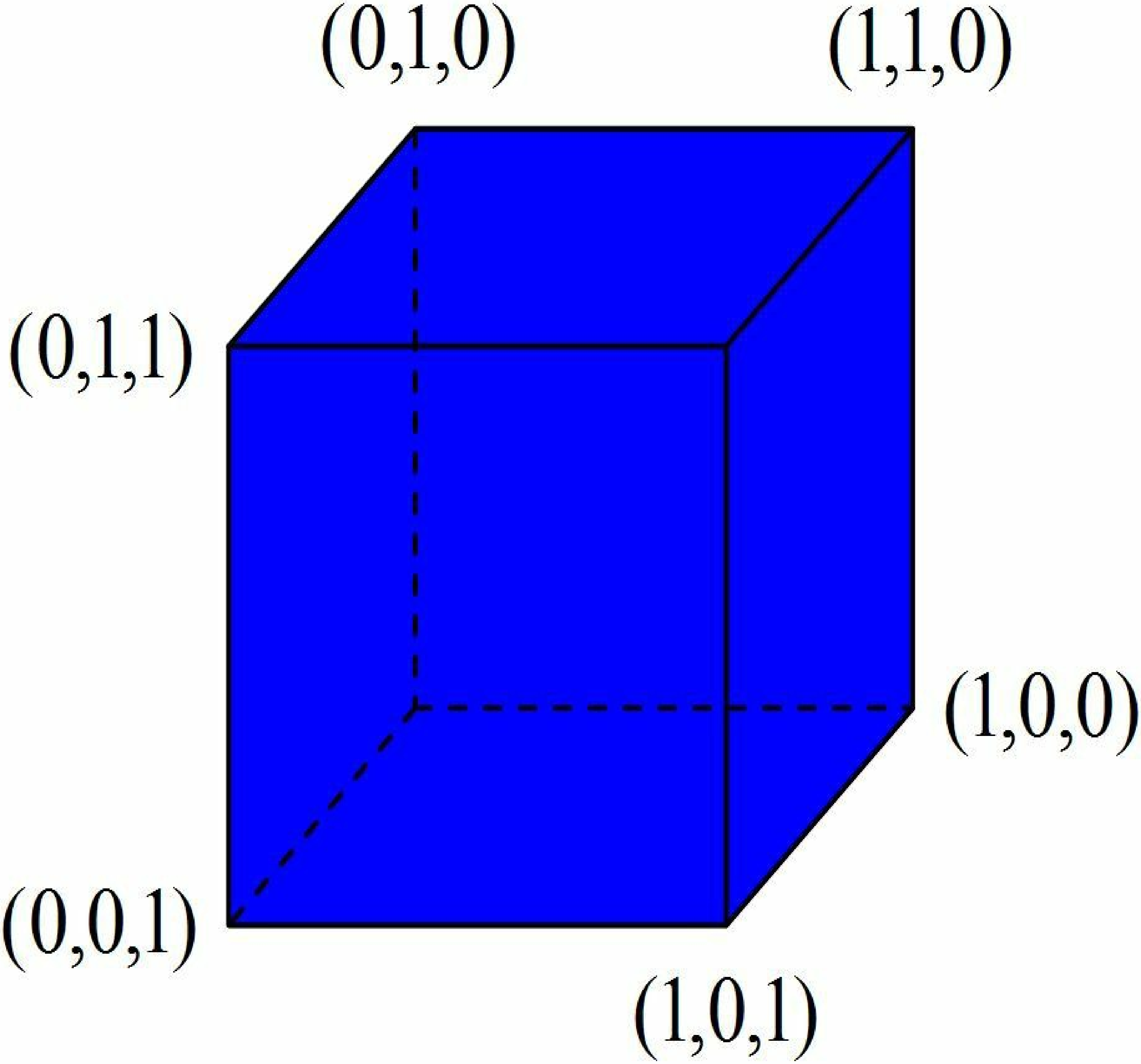
Chapter 8

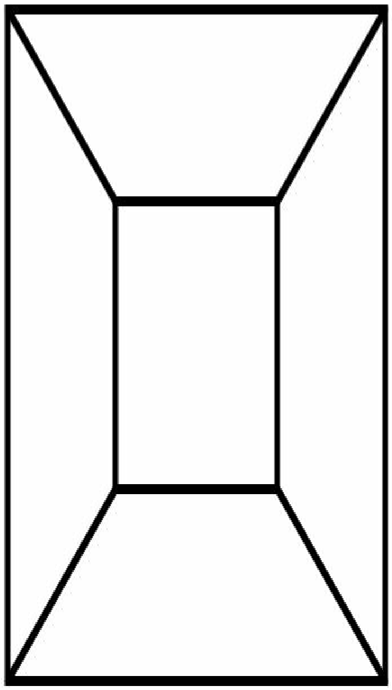
Tesseracts in Perspective

When you look at a cube in 3D space, you see it in perspective. We often draw a cube the way it actually is (rather than how it appears), with 3 sets of 4 parallel edges. However, when you look at an actual cube in 3D space, some of the edges don't look parallel, but seem to intersect at a distant point (because the back side of the cube is further from your eye than the front). This effect is called perspective.

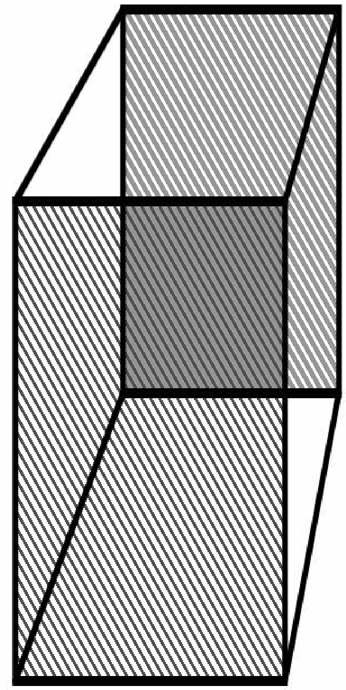
For comparison, let's begin by drawing a cube the way it actually is, instead of the way that it appears. The cube below is not shown in perspective.



The two cubes below are shown the way that they appear to your eye, depending upon where you stand when you look at them. These cubes are shown in perspective. You see them as they would appear, instead of how they actually are. Compare the following cubes to the previous cube. In the previous cube, there are 3 sets of 4 parallel edges. In the following cubes, the edges going into (or out of) the page don't look parallel (the effect is quite extreme in the cube on the left). The edges that represent depth appear not to be parallel, when in fact they are.



looking down a hallway

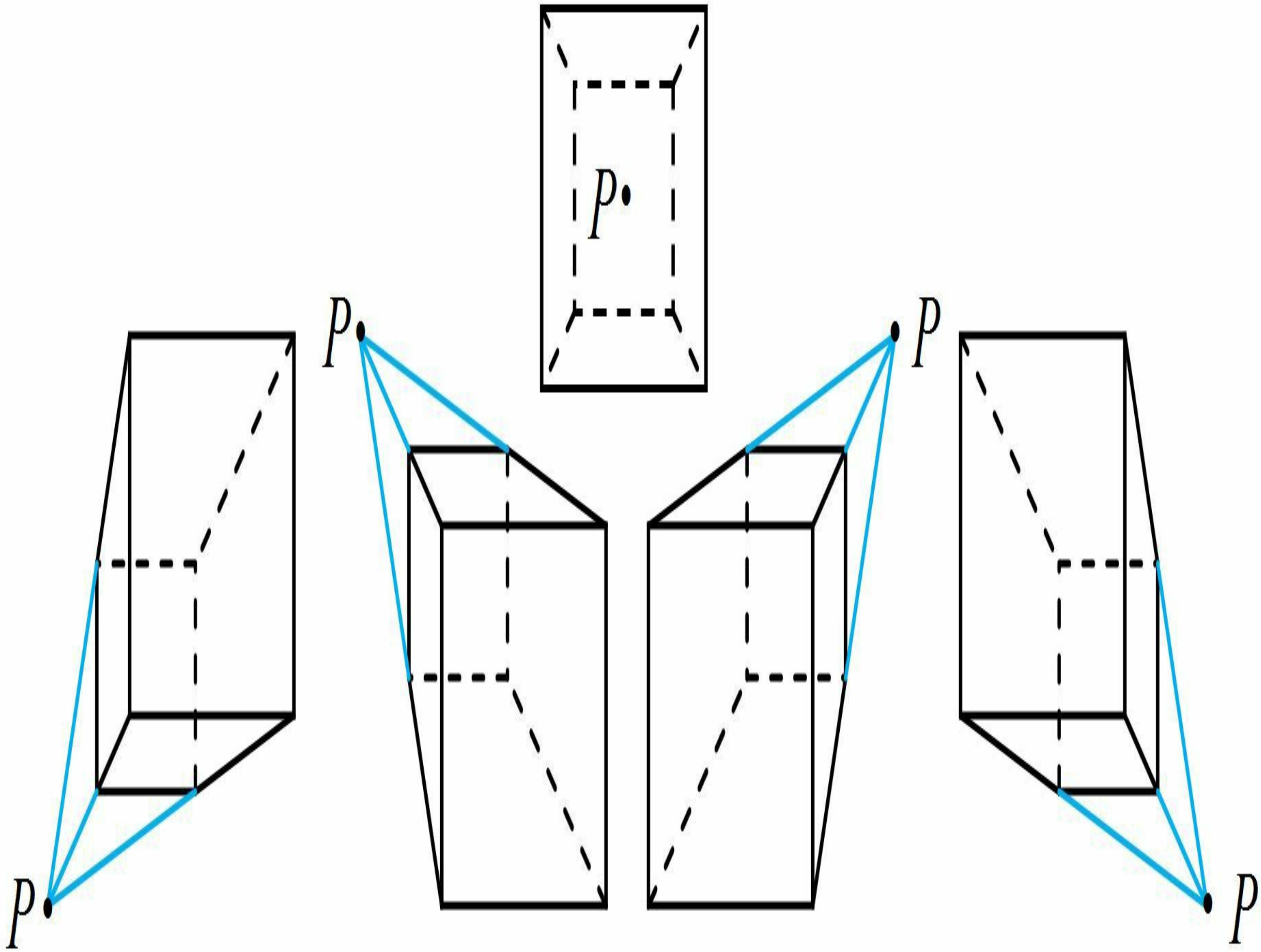


front face appears larger

You see the left cube above as if you are looking down a hallway. Your eye is centered widthwise and heightwise. The front square appears much larger than the back square, when in fact they are the same size. The 4 angled edges are actually parallel.

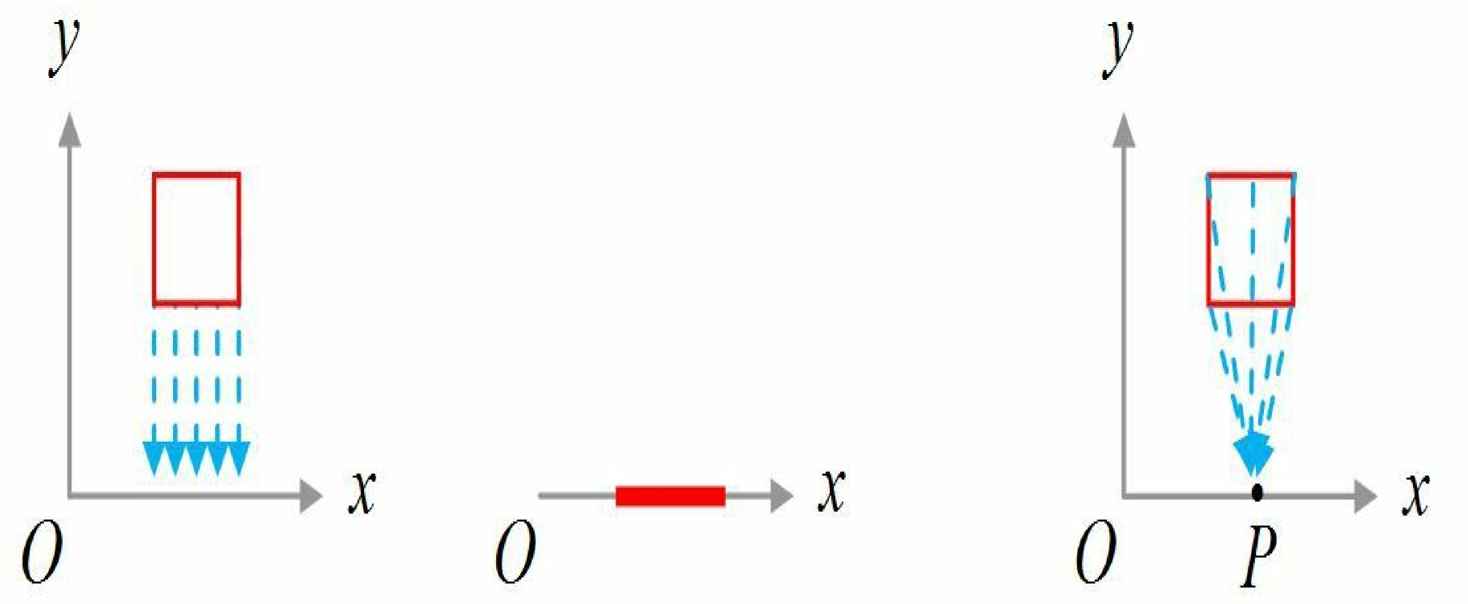
The point of view for the right cube above is like looking at a block with your eye above and to the right of the block. You would see only its right, top, and front sides; the left, bottom, and rear sides are hidden from view. In both of the cubes above, the angled edges are all parallel in 3D space, but look like they intersect at a point (a vanishing point) in the 2D image of the 3D object.

Each of the 5 cubes illustrated below is shown in perspective. The 4 depthwise edges (monkey tails, as we've been calling them) appear to intersect at point *P*, yet these edges are actually parallel. The cube at the top below corresponds to the cube in the left of the previous figure, and the third cube from the left below corresponds to the cube in the right of the previous figure.



cubes in perspective

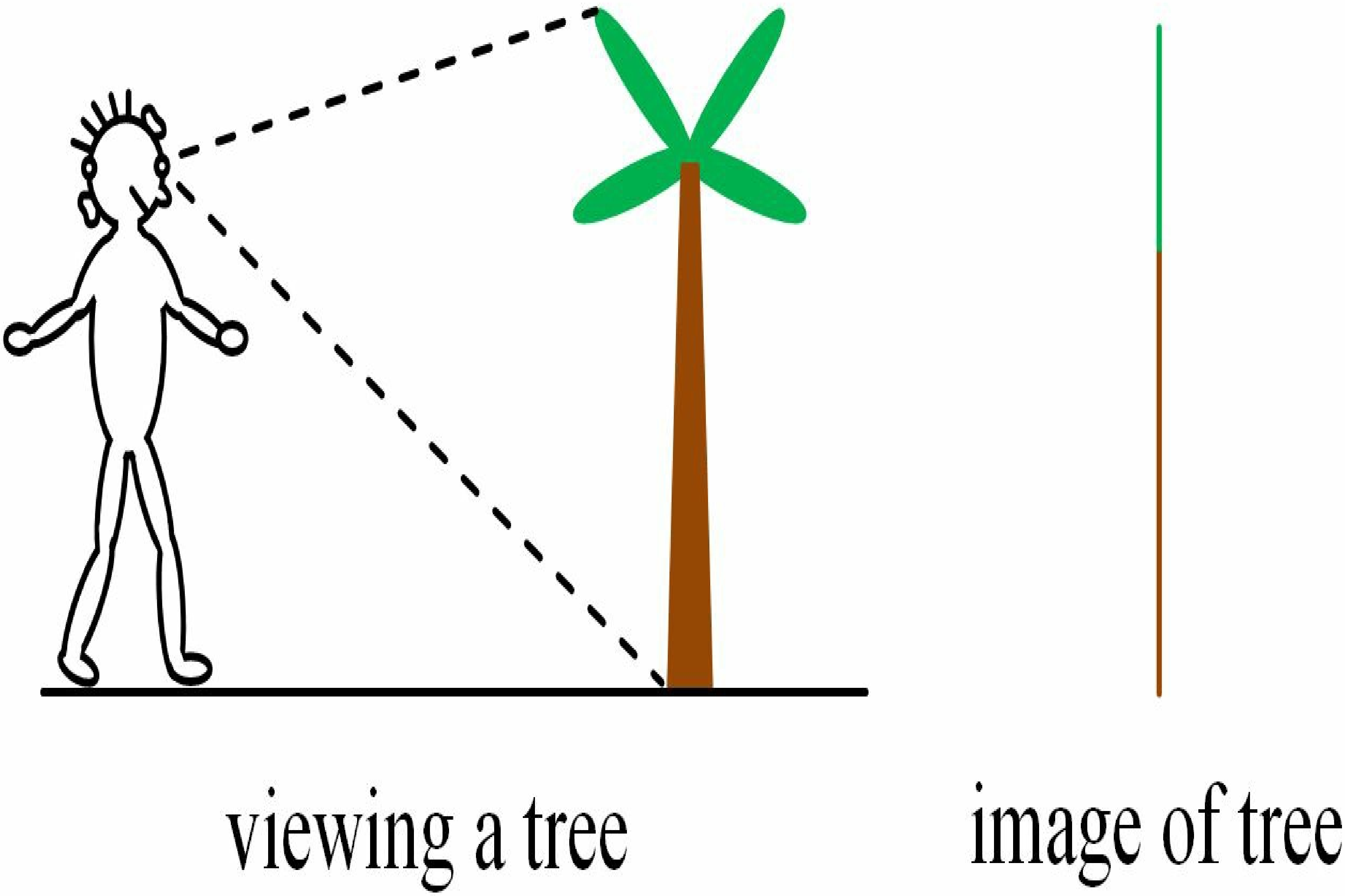
The following diagram shows the effect of perspective in 2D (but unless the "front" edge is transparent, the square would simply look like a line either way). What the following pictures do reveal – which doesn't just apply to the case shown here, but applies in general – is why we see things in perspective. I'll allow you to ponder this question, if you like, then in the paragraph that follows the diagram we will discuss the answer.



projecting down projection onto x lower edge appears longer

So why do we see things in perspective? For example, when we look at a cube in 3D space, why do the edges appear to taper off to a point, when in fact they are parallel? And why does the front side appear larger than the rear side? It has to do with the fact that viewing an object – cube, banana, monkey, Klein bottle (have a question? try Google, it will be worth it), or whatever it is – with your eyes is like looking at an object from a single point (well, two points that are close together). The rays of light from the different points of the monkey (or did you pick some other object, like a banana?) approach your pupil (it's just a small part of your eye that really matters) from different directions. The rays of light that strike your pupil travel in different directions from different parts of the object. You can see this in the previous figure. When you look down a hallway, for example, as illustrated in a couple of the previous pictures, the front square appears larger than the rear square because the rays of light coming from its edges make a wider angle relative to your eye.

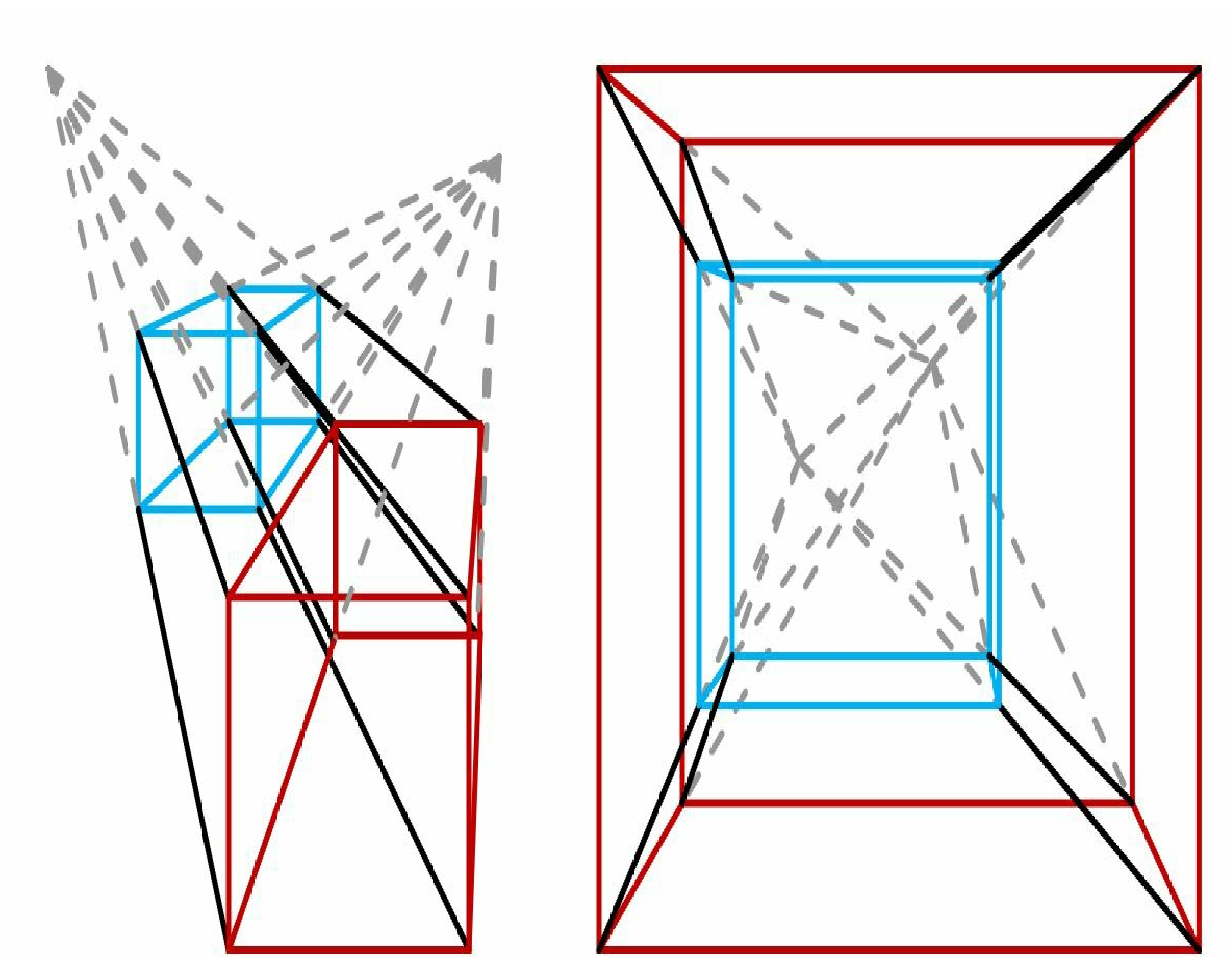
The illustration below shows a 2D monkey (remember, you have to pin your own tail on the monkey in this book) viewing a tree in perspective. Our 2D monkey friend (what, you don't want to make friends with the monkey?) sees just a linear image; the leaves look like a short green line and the trunk appears as a longer brown line.



Perspective is somewhat more complicated in 4D space. To draw a 4D object (like a tesseract full of monkeys) on a 2D sheet of paper, you must first project the image of the 4D object onto 3D space and then project the 3D image onto the 2D sheet of paper (flattening those poor little monkeys twice in the process). (Note: Professional stunt monkeys were used in this demonstration. No monkeys were harmed in any when in the writing of this book.) Therefore, when drawing a rectangular 4D object on a 2D sheet of paper in perspective (where one face lies in the plane of the paper), there are 2 vanishing points. The first vanishing point comes from projecting the 4D object onto 3D space, and the second vanishing point comes from projecting the 3D image onto the 2D paper.

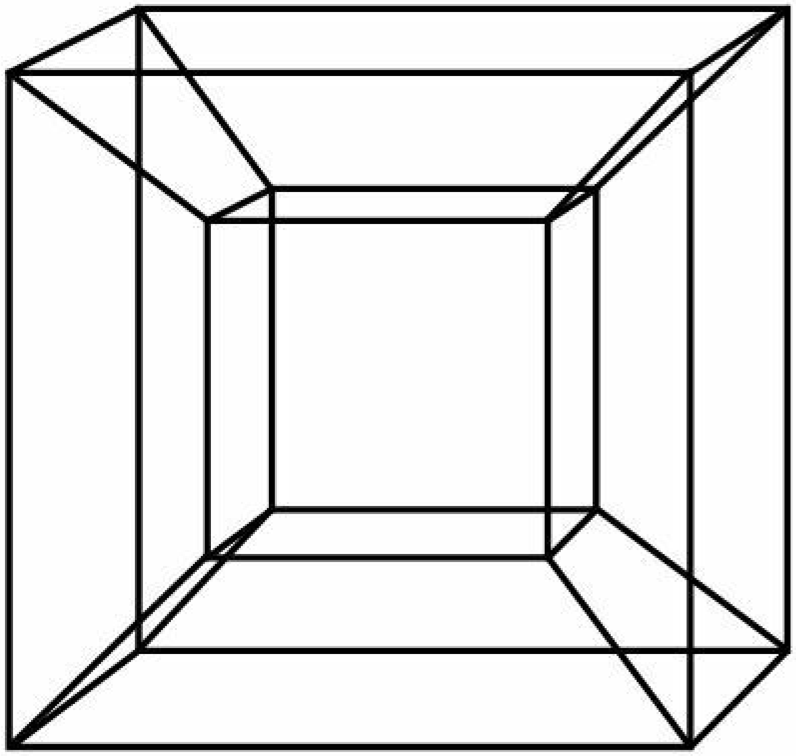
When drawing a tesseract in perspective, it is simplest to draw one set of faces in the plane of the paper (these will be the *xy* squares). There will be two types of diagonal edges – one set for *z* and one set for *w* (the ana direction). The 8 *z* edges are really parallel, and the same goes for the 8 *w* edges; but when drawing the tesseract in perspective on a 2D paper, each set of edges will instead appear to converge at a vanishing point. There are 2 vanishing points – one for *z* and one for *w*.

The two tesseracts shown below each have 2 vanishing points. See if you can find the 2 (and only 2) vanishing points in the figure on the right. 8 dashed lines converge at each vanishing point (but two or more of these lines may appear to overlap); one set corresponds to depth, the other to hyperdepth.



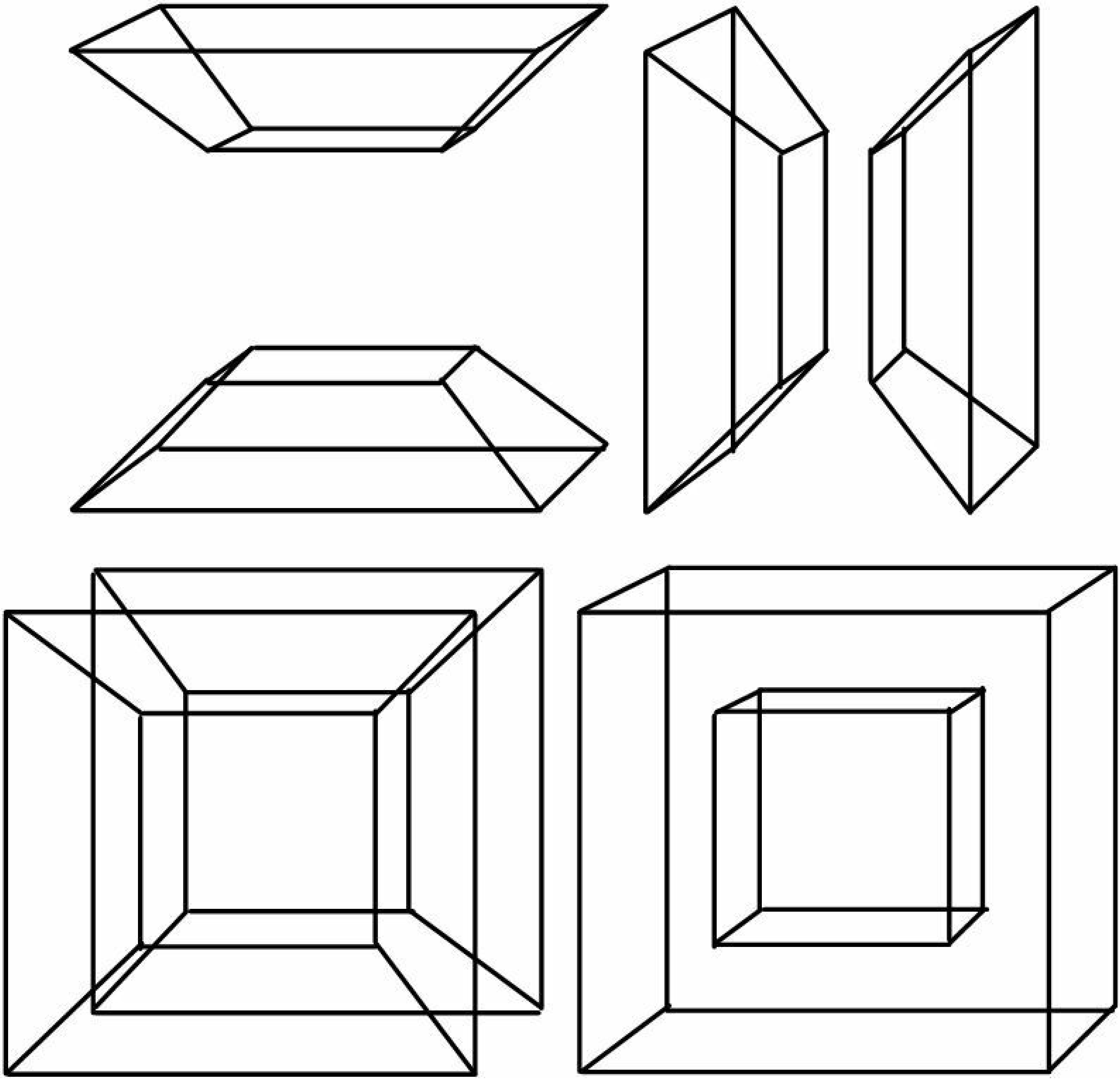
tesseracts in perspective

See if you can find all 8 bounding cubes in the tesseract below, which is drawn in perspective. It may help to remember that each cube has 6 faces and 12 edges.

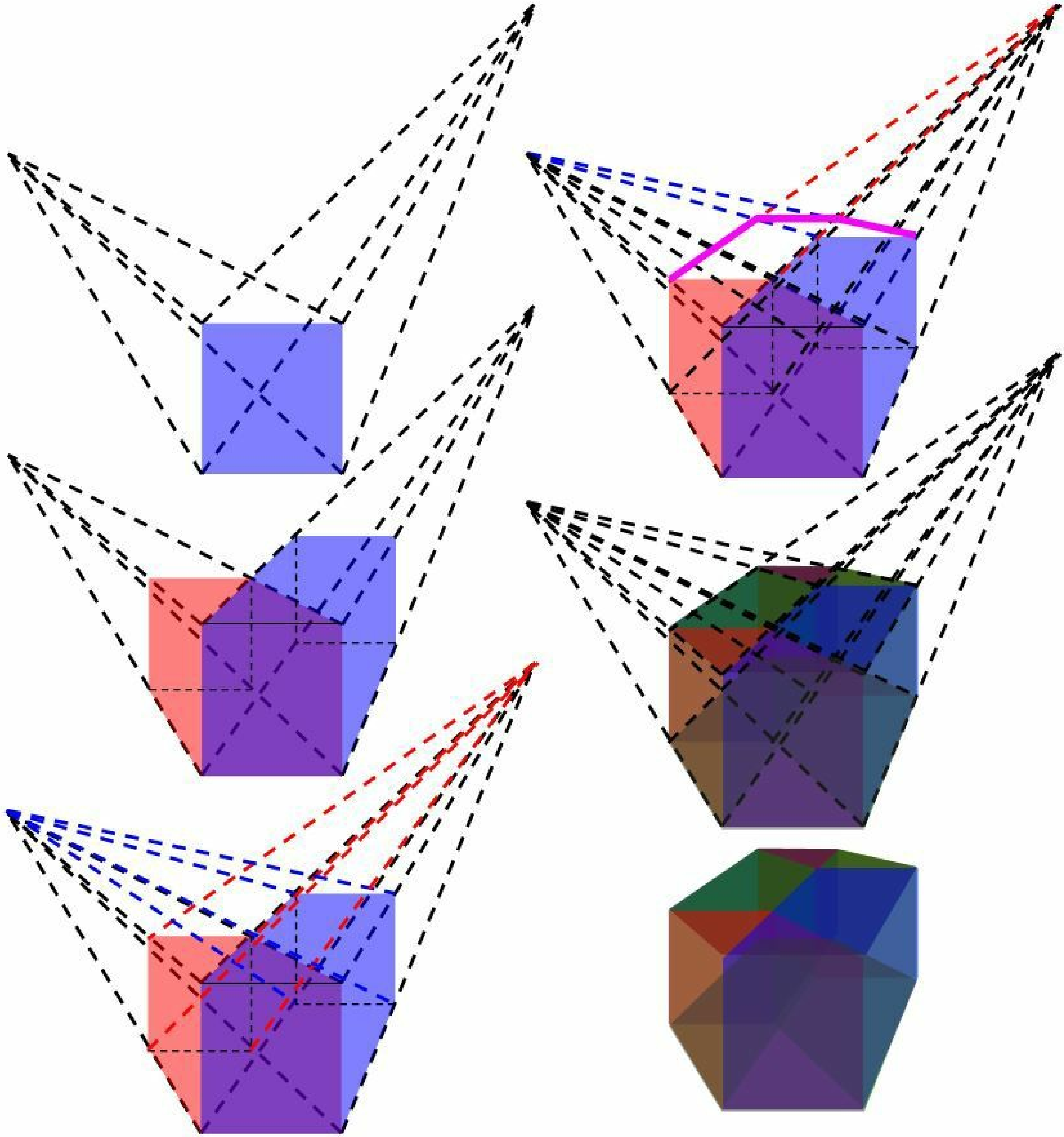


The two obvious cubes in the tesseract drawn above (filled with monkeys, but you have to imagine the monkeys yourself) are the inner and outer cubes. The large outside cube and small inside cube look like ordinary 3D cubes drawn in 2D in perspective. These two cubes lie in the usual *xyz* hyperplane; these diagonal lines are along *z*, corresponding to the usual 3D depth. These *z* edges converge to a depth vanishing point that is above and to the right of the tesseract. That leaves 6 more cubes to find. Where are they?

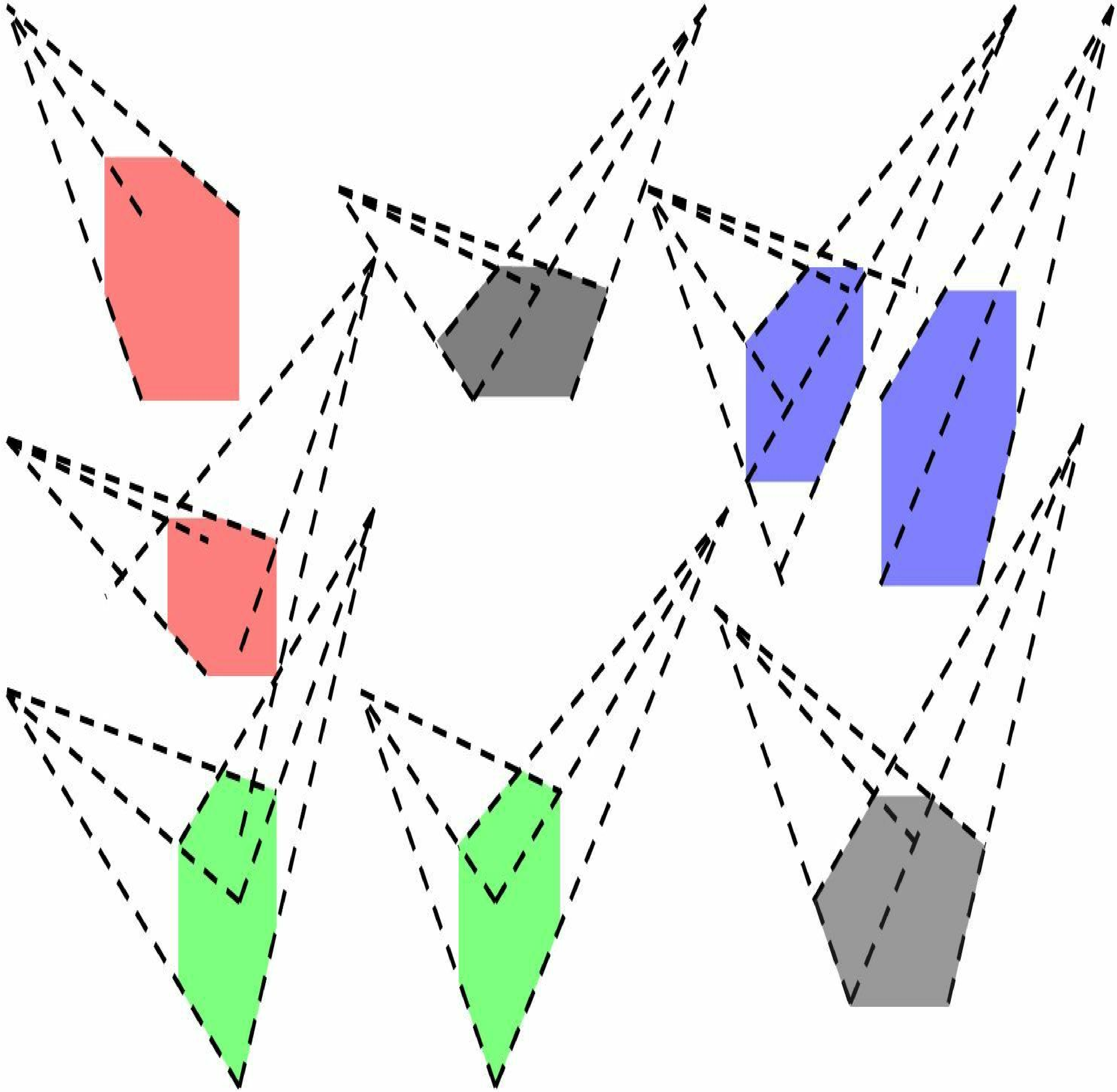
The last 6 cubes are top/bottom, right/left, and front/back. The first 2, inside/outside, correspond to the ana and kata cubes. The top/bottom, right/left, and front/back cubes don't look like ordinary 3D cubes drawn on a 2D sheet of paper. They appear to be shaped more like pyramids, but this is only an illusion created by hyperperspective. The *w* edges are all parallel, but don't look it; we see these *w* edges as if we're looking down a hyperhallway (compare with the 3D hallway picture from earlier in this chapter). The hyperdepth vanishing point, corresponding to the *w* edges, lies inside of the interior cube (where all of the *w* edges converge). All 4 pairs of parallel cubes that bound the above tesseract are shown below.



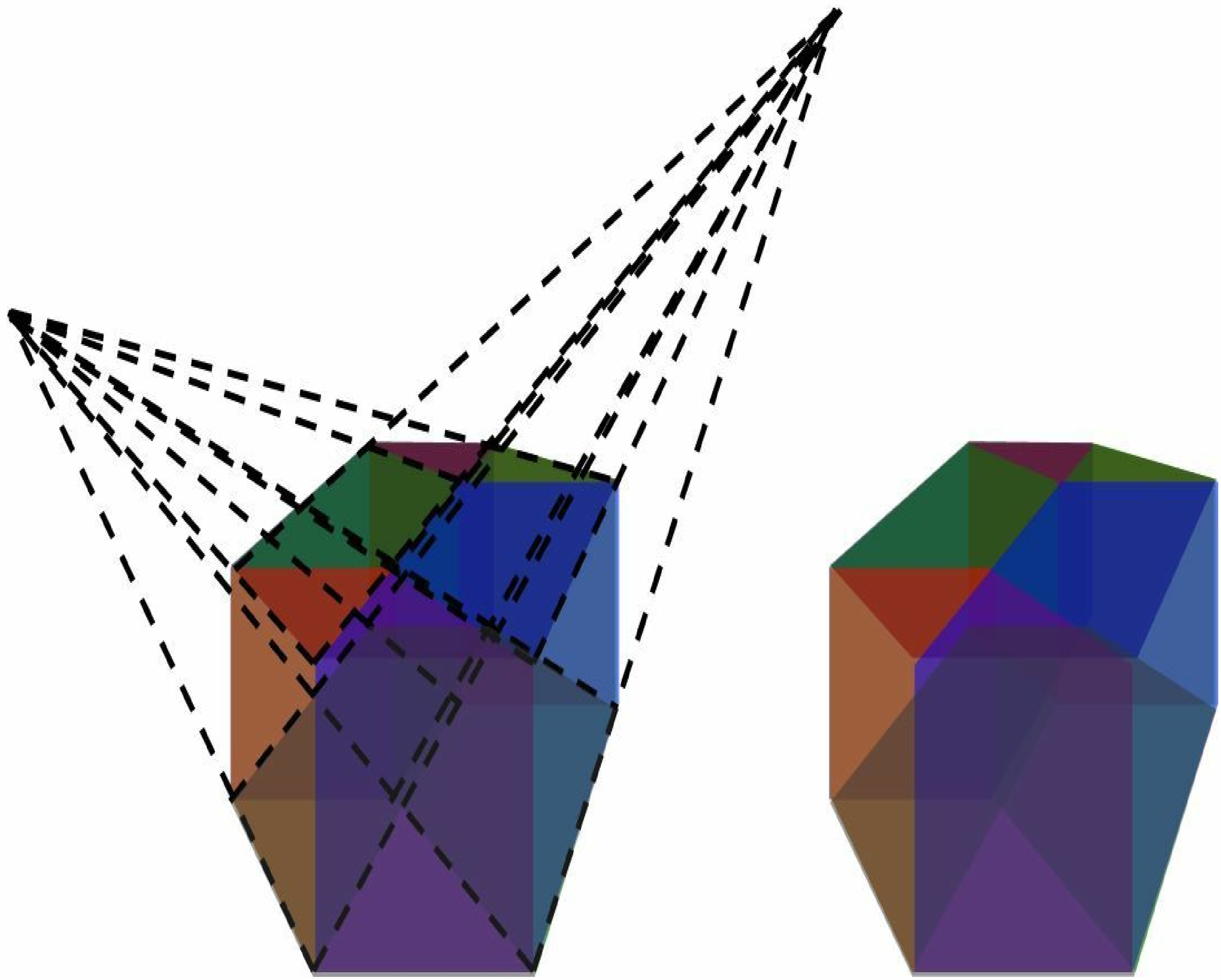
The following illustration shows how a tesseract (populated with monkeys) can be drawn in perspective after choosing 2 vanishing points. One vanishing point defines the *z* edges and the other defines the *w* edges.



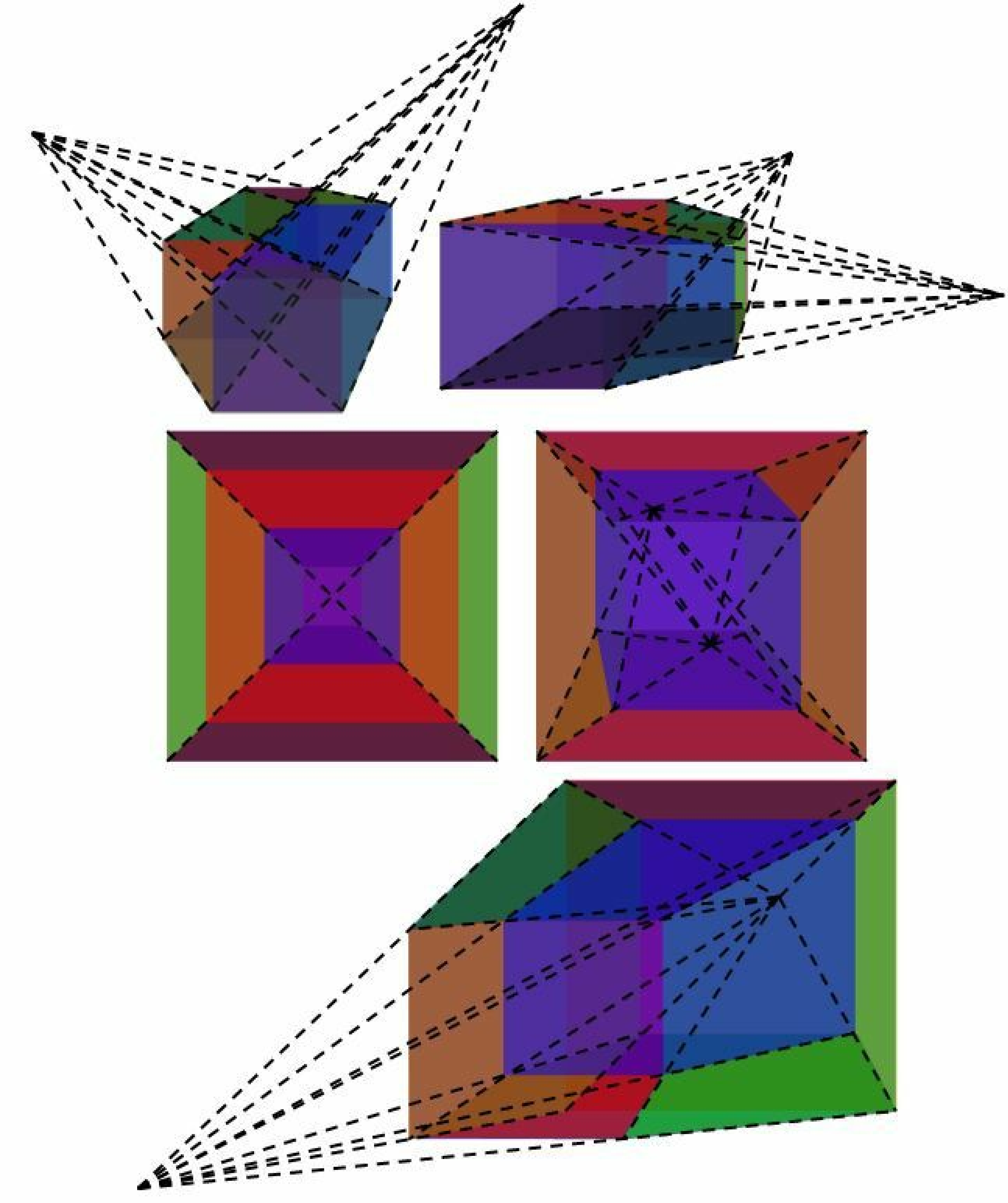
The 8 cubes that bound the tesseract (full of monkeys) are shown separately below, in terms of the 2 vanishing points. These 8 cubes make up the tesseract.



Compare the tesseract below, shown in perspective, with the 8 cubes above, which bound the tesseract.



Five more tesseracts are illustrated in perspective in the following picture.



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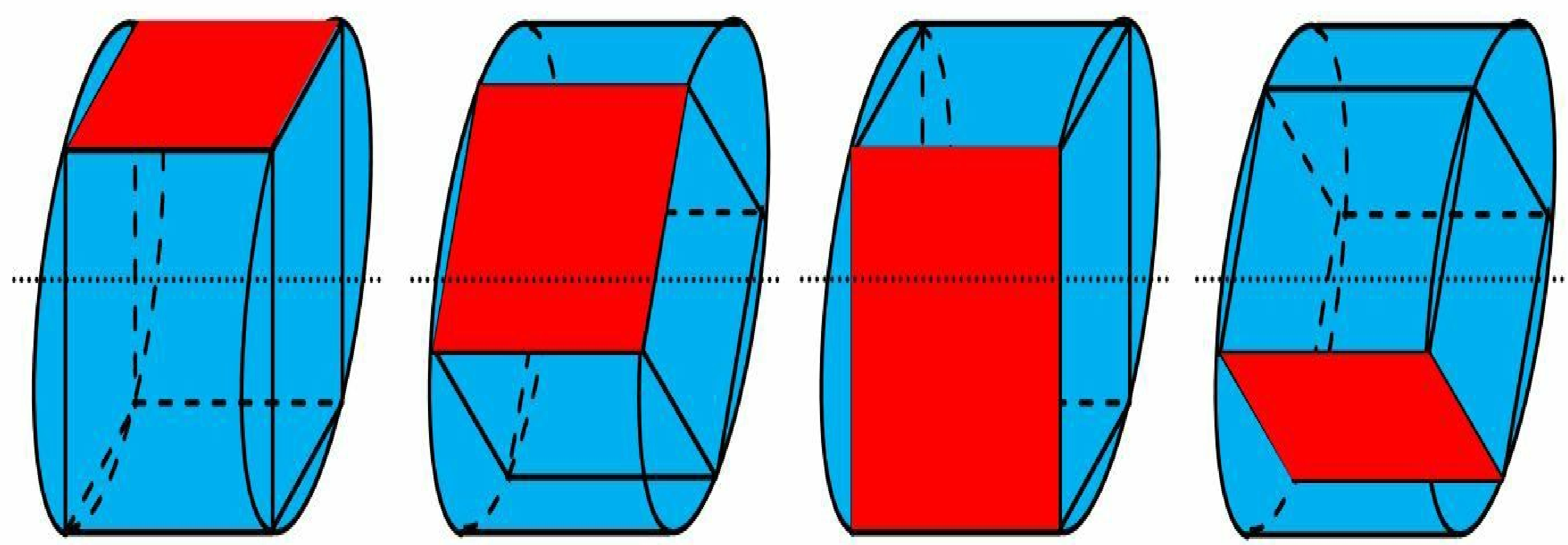
Chapter 9

Rotations in 4D Space

The monkeys are getting tired of sitting inside of boxes and tesseracts, so let's let them do something. In this chapter, we'll let the monkeys rotate a variety of rectangular objects. A hypermonkey living in 4D space might spin a tesseract on the tip of his hyperfinger (as if it were a basketball), for example.

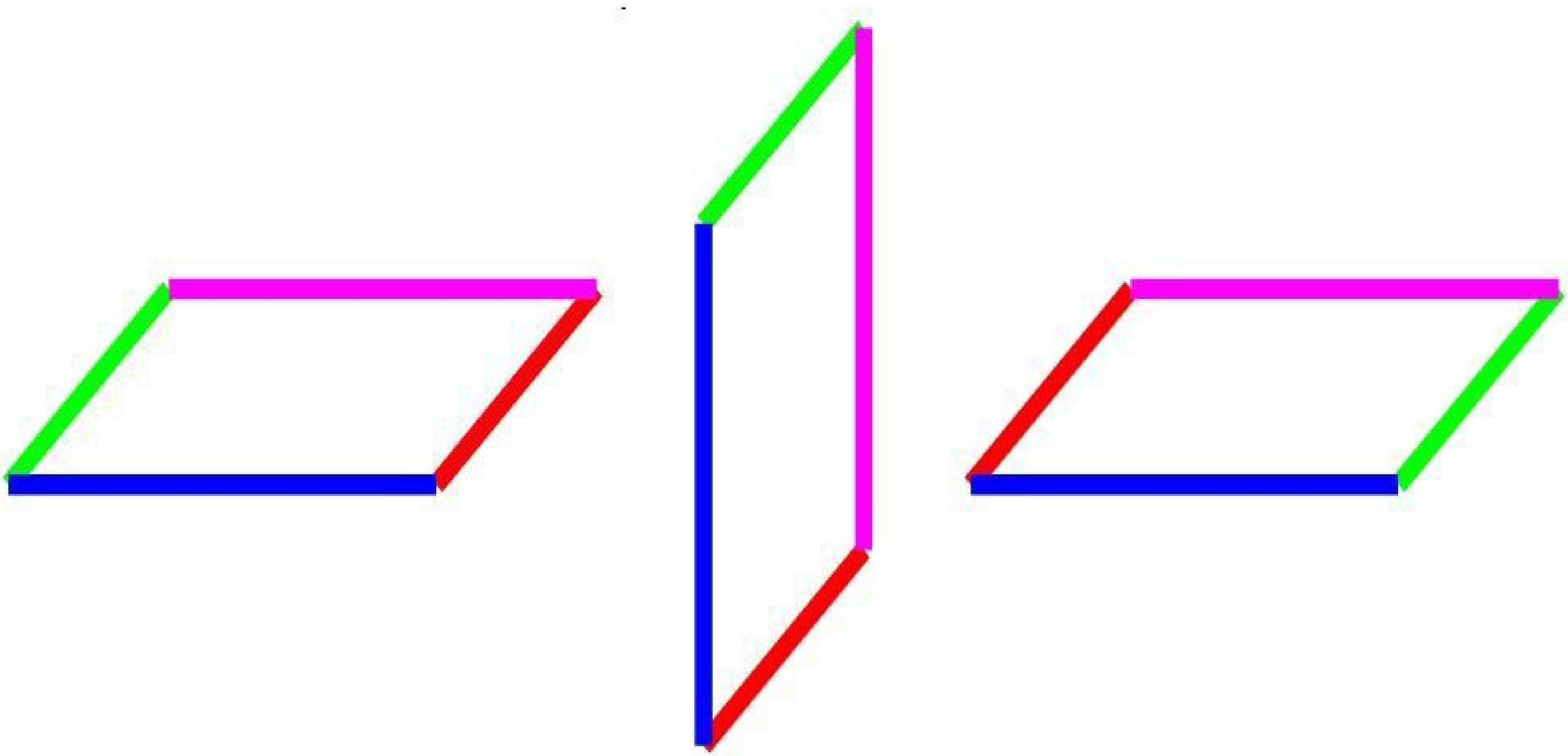
When an object rotates, every part of the object travels in a circle. For example, consider the cube below. A monkey is rotating this cube about the dotted line (it's the axis of rotation). Every part of the cube travels in a circle that rings around the axis of rotation. Viewing from left to right, you can visualize the rotation of the cube as the red square comes down in the figure below.



each point on a rotating cube travels in a circle perpendicular to the axis of rotation

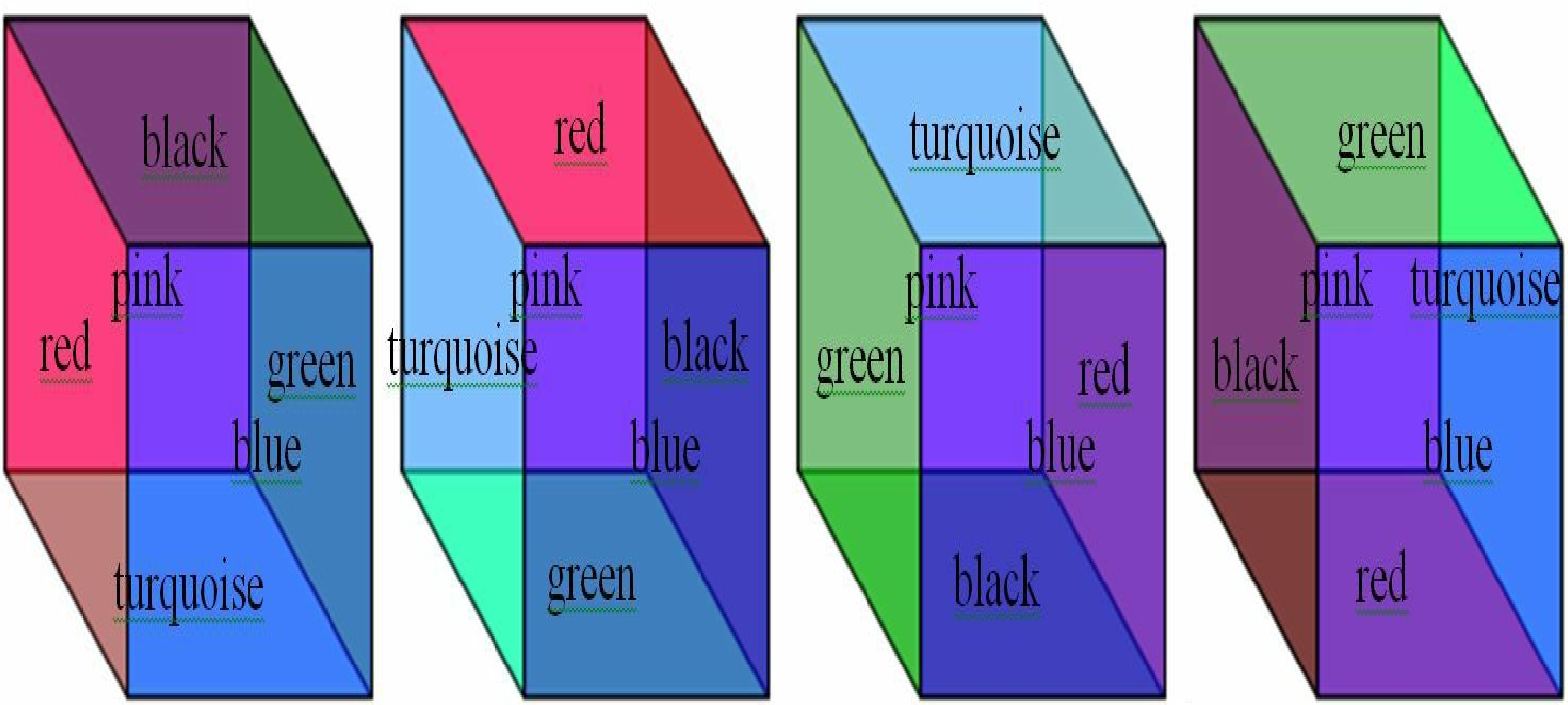
The square shown below (not above) is perpendicular to the page at all times. A monkey rotates this square so that the left edge comes up and finishes at the right. The square is horizontal initially and finally, and vertical in between. If you compare the initial and final squares in the 180 degree rotation shown below, you should notice something interesting.

After the figure, we'll discuss what it is that you should observe.



Did you figure out what was different with the initial and final squares? They're both horizontal, but that's how they're similar, not how they're different. The difference is that one is a mirror image (or reflection) of the other. The 2D square is reflected when it rotates 180 degrees through the third dimension. We'll consider this feature again in the next chapter.

In contrast, the following cube does <u>not</u> reflect after the first 180 degrees of its rotation. It's a 3D cube rotating in 3D space. An object must rotate through a higher dimension in order to appear reflected after a 180 degree rotation. A monkey is rotating the cube below such that it rotates clockwise. See if you can visualize its clockwise rotation.



Do you know the saying, "Take one step forward, then go two steps backward?" Let's do it in reverse: We'll go two steps backward, and then go forward. What I'm saying is this: Let's consider rotation in the lower dimensions before we move up to 4D.

A rotation would be quite strange in 1D space. Why? Because a rotation is defined as every point on the object traveling in a circle. Now, go ahead. Draw a circle in 1D. Come on. What's the hold up? It's not because you can't. You can. It just looks funny.

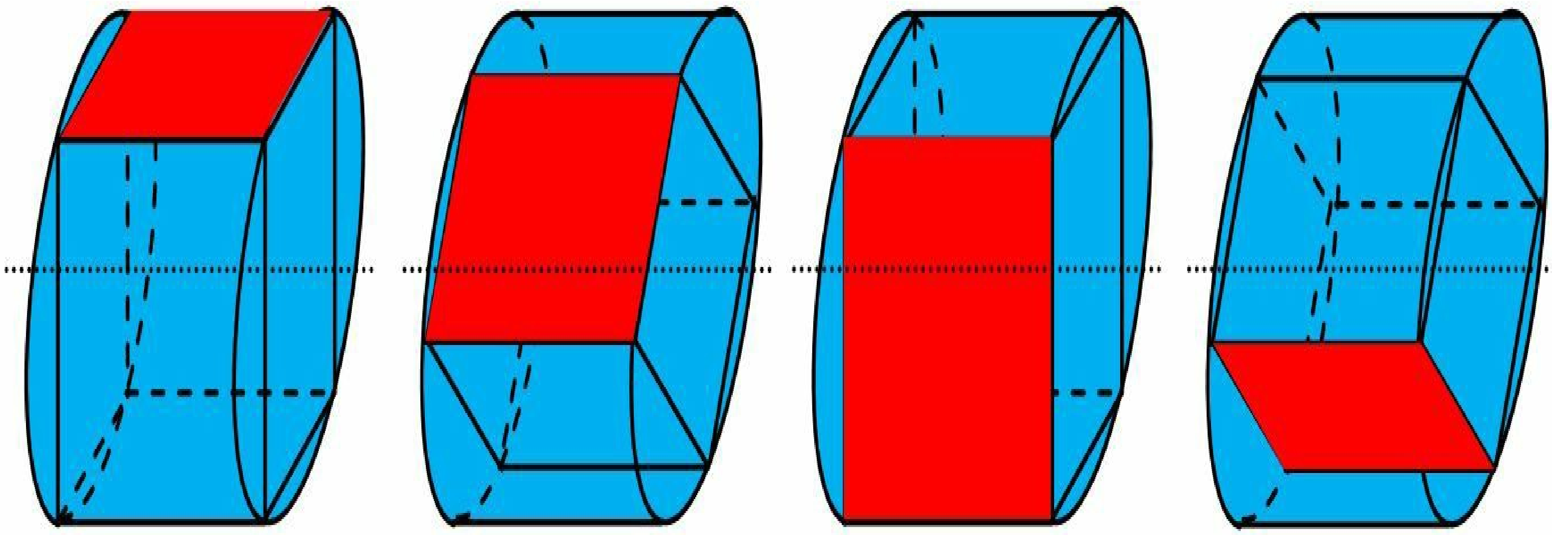
Think about what a circle is. A circle is the locus of points lying in a plane that are equidistant from a single point, which we call the center. If you choose a point to call the center of the circle in 1D, what you find is that your circle will consist of two disconnected points – one point on either side of the center, both equal distances from the center. Doesn't seem like much of a circle, does it?

If you think this 1D circle looks funny, rotation in 1D space is even funnier. (Don't judge a circle or a rotation by its looks. You should be ashamed of yourself for thinking that it looks funny.) When an object rotates, it travels in a circle.

So now imagine that a planet, named Ban, will orbit its star, named Ana, in 1D space. (Get it? Ban-Ana.) Let's place the star, Ana, at the origin and the planet, Ban, at $x = +R$. If Ban revolves around Ana in a circle in 1D space (analogous to earth's revolution around the sun), this means that one moment Ban lies at $x = +R$ and the next instant it lies at $x = -R$. It simply switches back and forth between $x = +R$ and $x = -R$, where R is the radius of the circle. Ban doesn't travel along any *path* from $x = +R$ to $x = -R$; rather, in order to revolve around the origin in a 1D "circle," it must instantaneously reach $x = -R$ from $x = +R$ without passing through any other points on the way! Now that's a funny rotation! (But don't judge a 1D rotation by its looks.)

A rotation in 2D space will seem much more natural. For example, if a monkey runs around in a circle in 2D, the path will actually look like an ordinary circle. In 2D, the axis of rotation is just a point (the center of the circle). If we imagine a 2D plane from our 3D perspective, we think of the axis of rotation as a line – the line perpendicular to the plane of the circle and passing through the center. But in a purely 2D universe, there is no third dimension, so 2D monkeys would only see rotations about a point, not a line like we do in 3D.

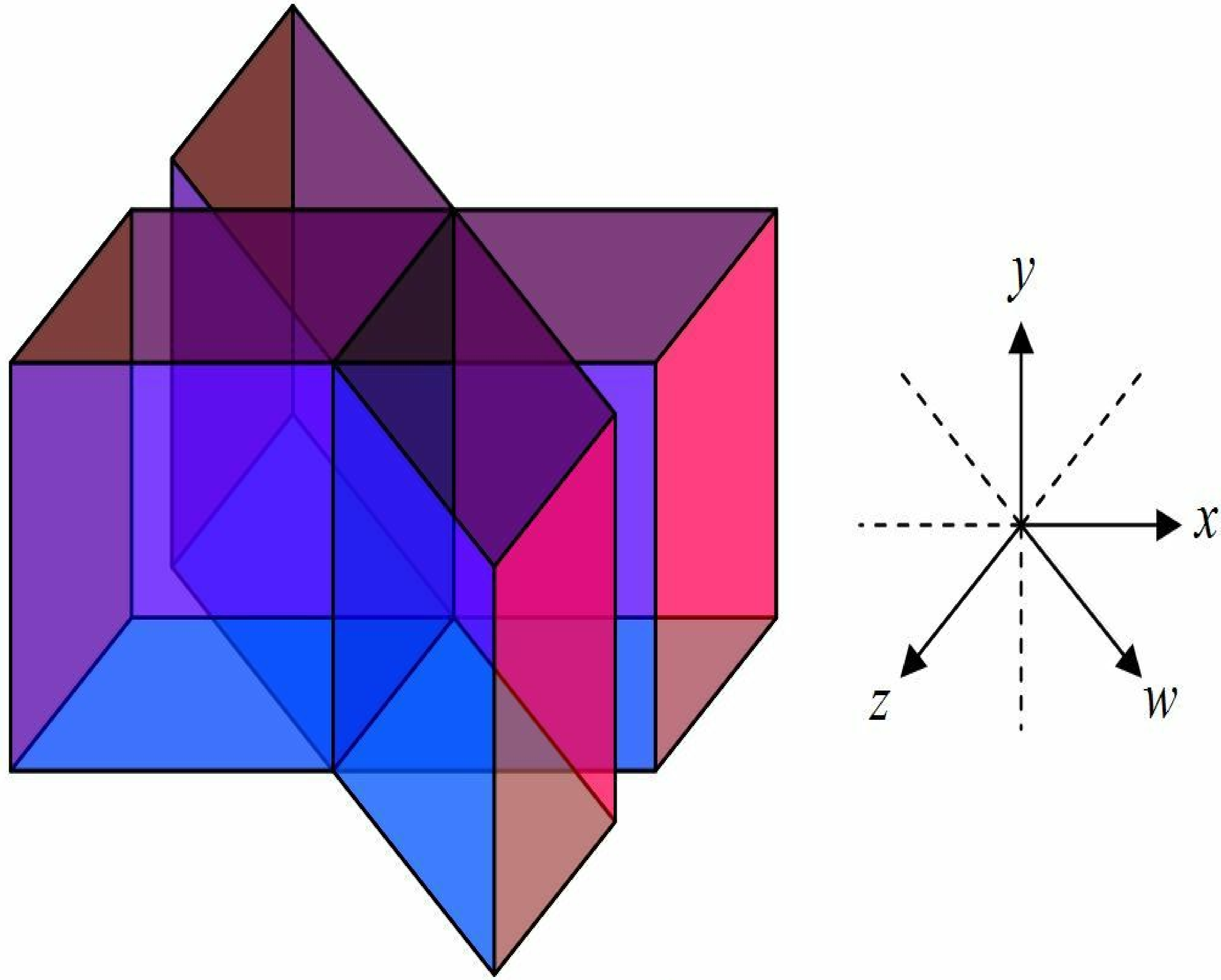
In 3D space, an object rotates about an axis, which is a line. Following is a copy of a figure that we've seen previously, to remind you of what an axis looks like in 3D. The axis in the following picture is a horizontal line (looks like a monkey tail, doesn't it?). If the horizontal axis is the x-axis, the rotation occurs in the yz plane. Every point on the cube travels in a circle that lies in the yz plane. (Would you like to get technical? Not every point of the cube lies *in* the xy plane. More precisely, every point in the cube travels in a circle that is *parallel* to the xy plane.) The axis of rotation is perpendicular to the plane of rotation. In what follows regarding the fourth dimension, you'll definitely want to pay careful attention to the distinction between the axis and plane of rotation (or plane of rotation versus rotation *about* a plane – as these two planes are different).



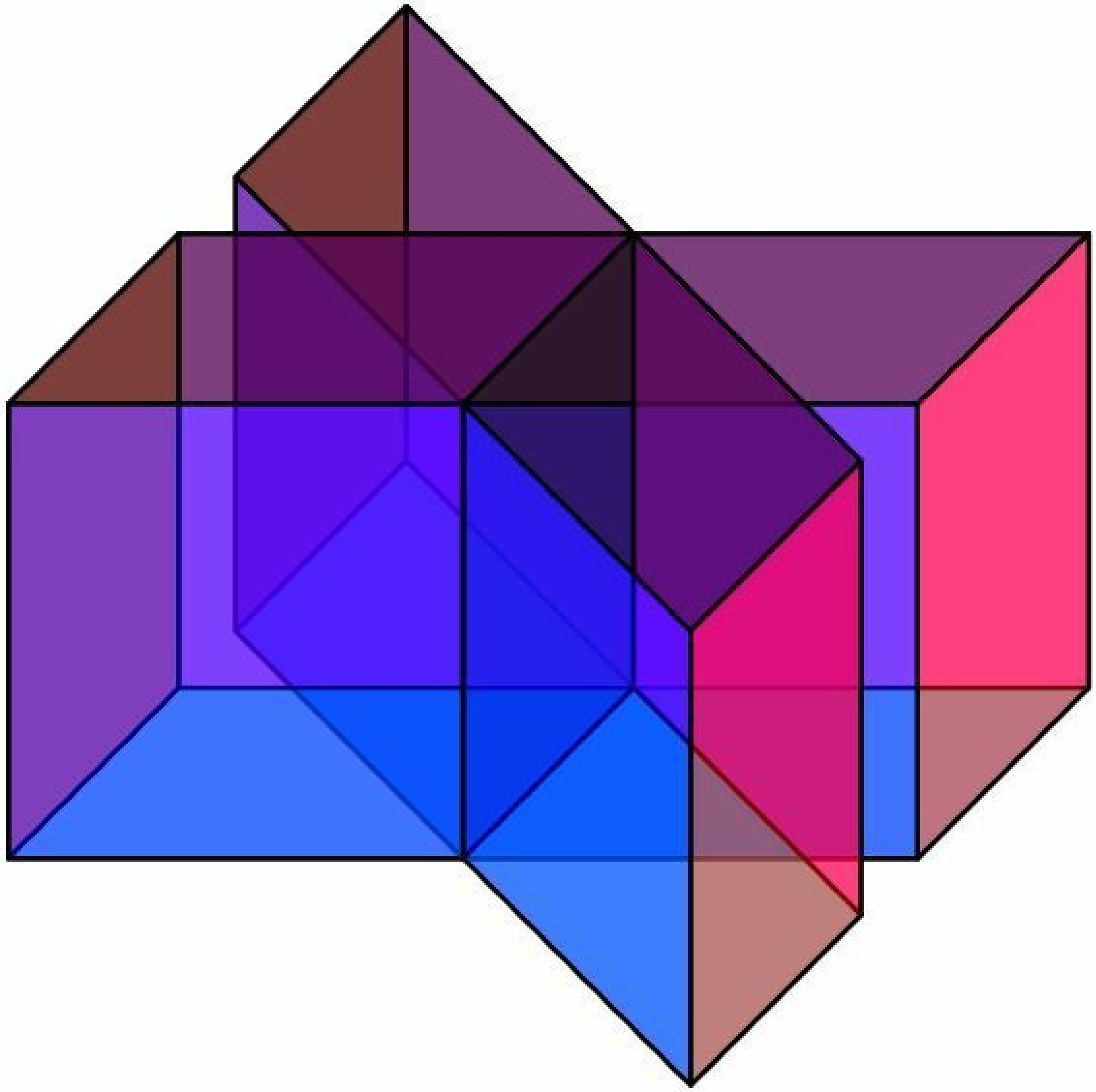
each point on a rotating cube travels in a circle perpendicular to the axis of rotation

In a 4D rotation, every point on the object travels in a circle in the plane of rotation, while the object rotates about an "axis" of rotation, which itself is a plane. For example, a monkey is rotating the 3D cube below about the *yz* plane (hyperaxis) in 4D space. (This is a 4D rotation, even though the cube itself is 3D. You *shouldn't* have a problem with this. A sheet of paper is 2D, but you can rotate it through 3D space. Let's start our discussion of 4D rotations with the cube before we rotate the tesseract.) The *yz* plane serves as the "axis" of rotation (in 4D, the hyperaxis is a plane, not a line). As the cube rotates about the *yz* plane, every point in the cube travels in a circle parallel to the *wx* plane. The *wx* plane in this case is the plane of rotation, not to be confused with the *yz* plane, which is the hyperaxis of rotation.

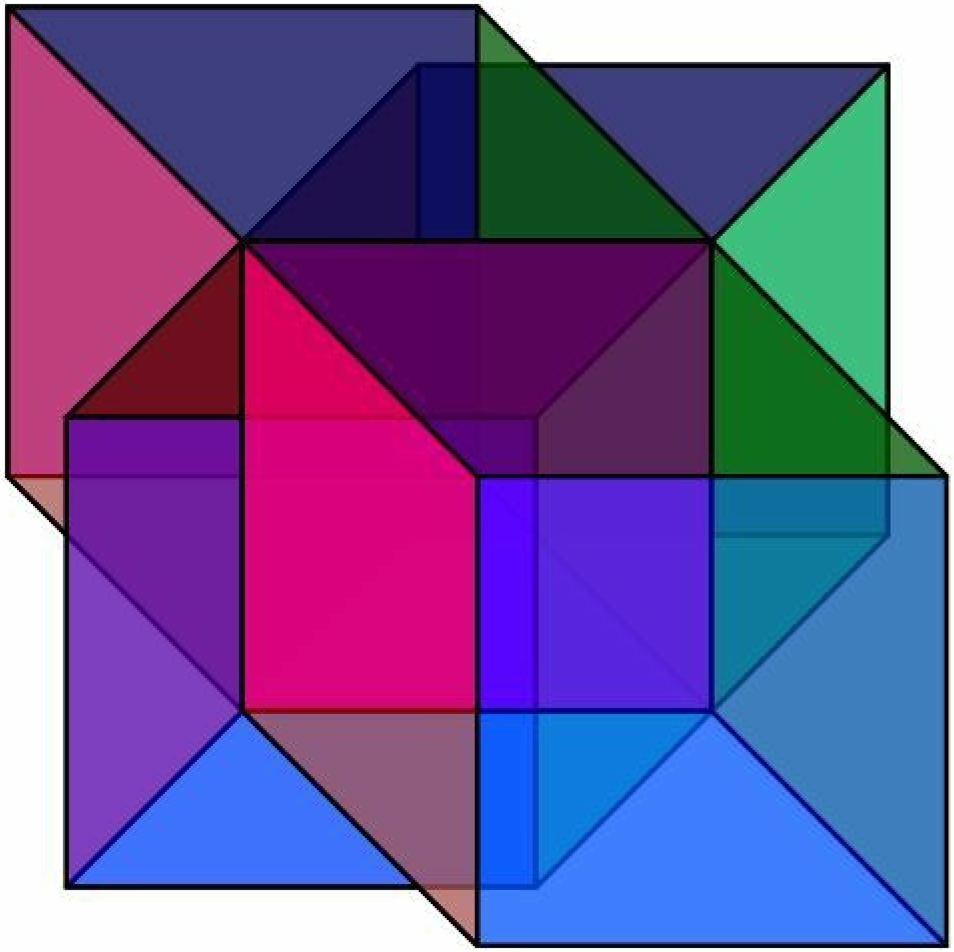
Observe that the cube below rotates through the *xyz* and *yzw* hyperplanes as the monkey spins it. The *y* and *z* edges remain unchanged during this rotation (examine the edges to better understand this 4D rotation). That's why the *yz* plane is the hyperaxis of rotation. The other 4 edges rotate through both *x* and *w*. That's why the *wx* plane is the plane of rotation (every point in the cube makes a circle parallel to the *wx* plane).



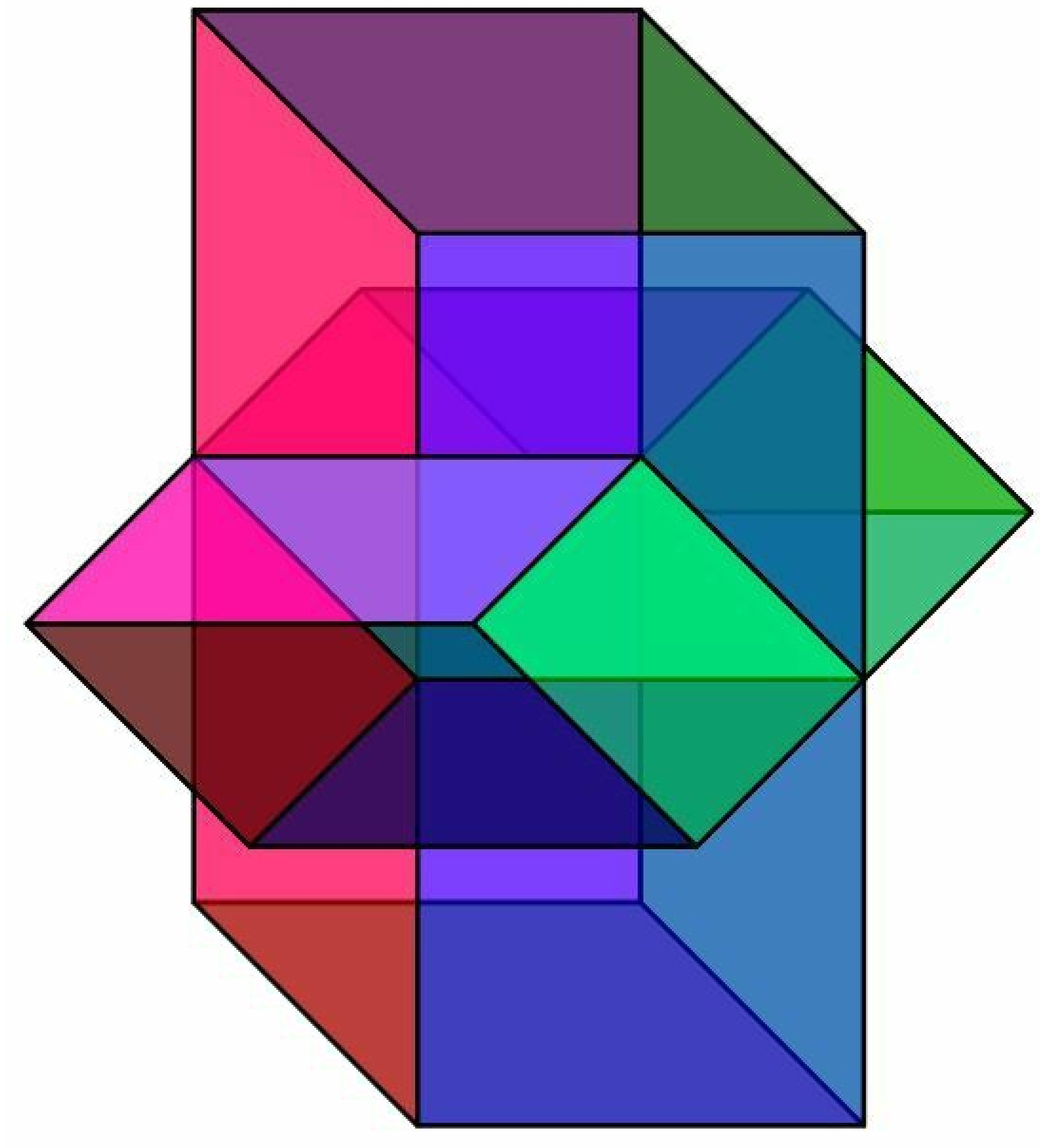
The same rotation is shown below, but without the axes (it might appear a little wider on your screen – and if you believe the automotive commercials, wider is better).



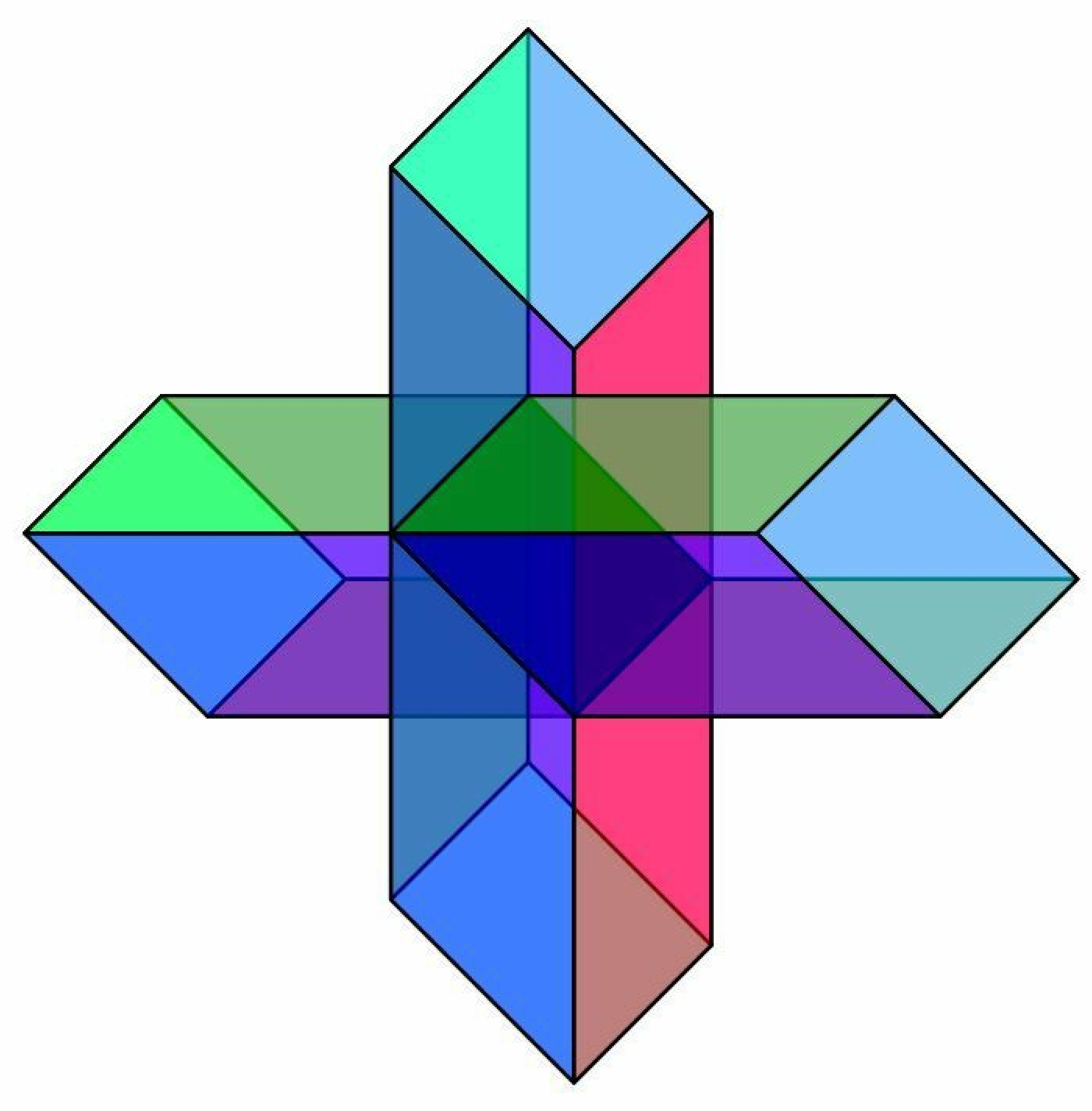
The next illustration shows a monkey rotating a cube through 4D space about the *xy* hyperaxis (so every point in the cube travels in a circle parallel to the *zw* plane, which is the plane of rotation).



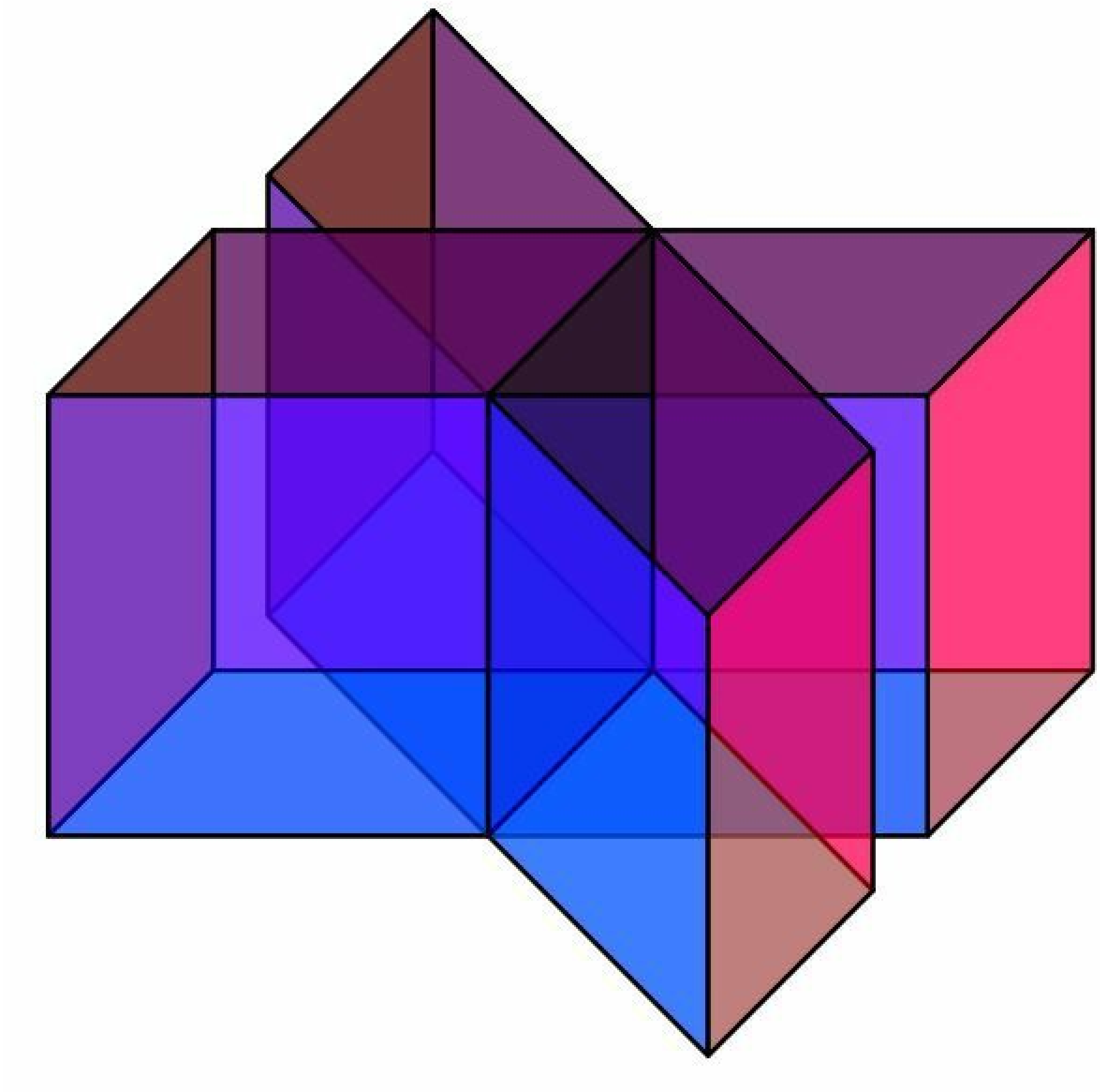
In the next diagram, a monkey is rotating the cube through 4D space about the *wx* hyperaxis (so every point in the cube travels in a circle parallel to the *yz* plane, which is the plane of rotation).



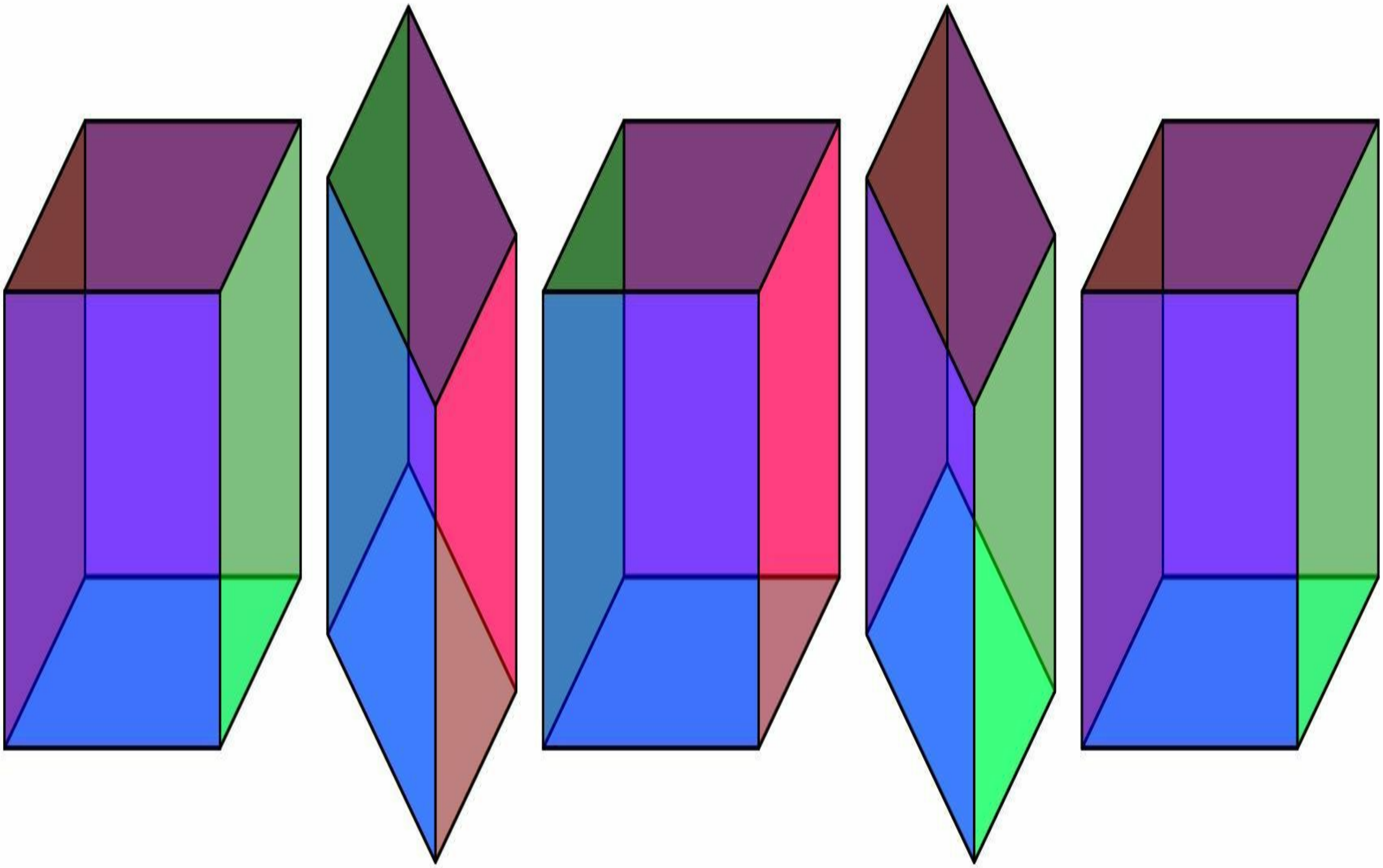
In the following picture, a monkey is rotating the cube through 4D space about the *zw* hyperaxis (so every point in the cube travels in a circle parallel to the *xy* plane, which is the plane of rotation).



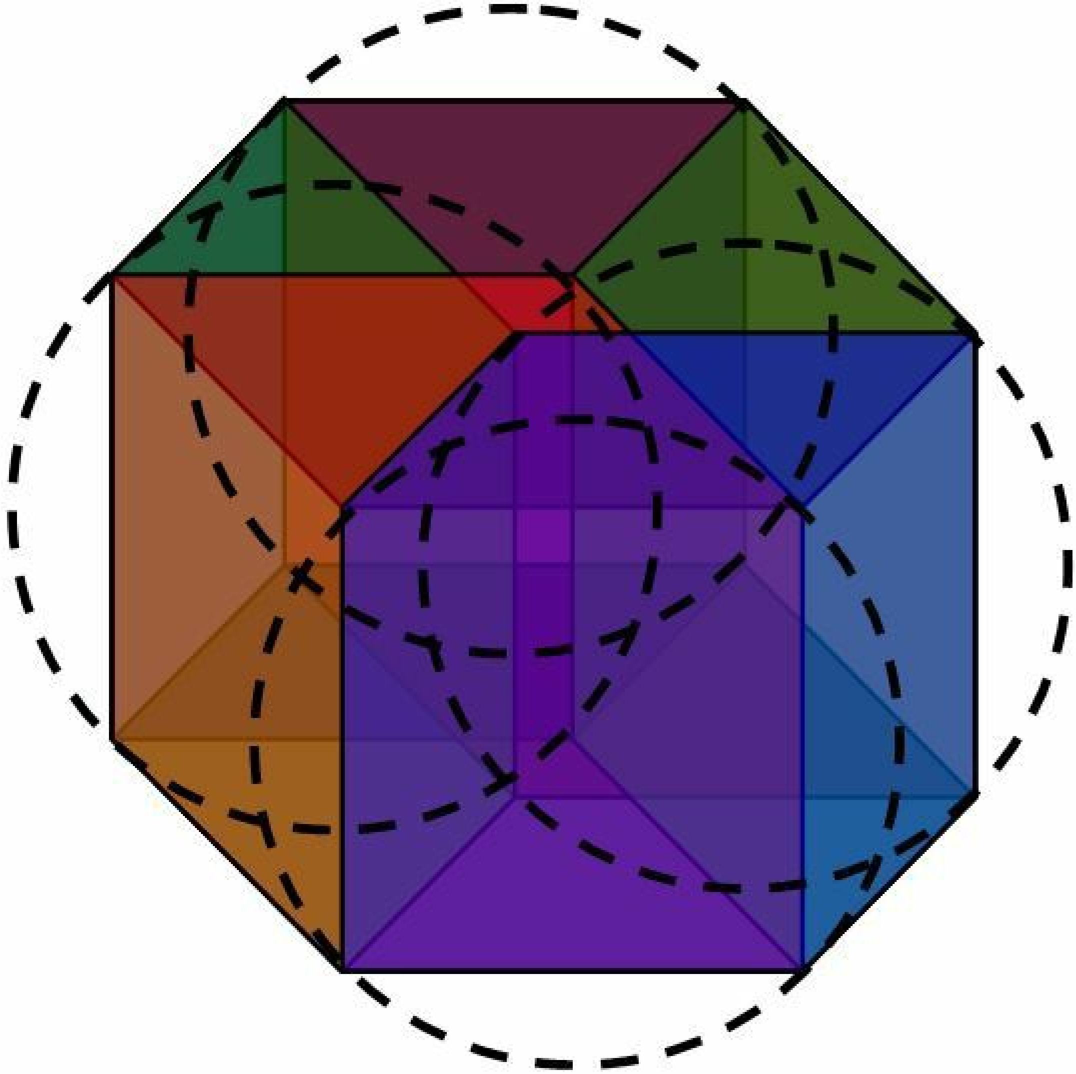
Here is a copy of our original cube rotating through 4D space. (What's with the déjà vu? Why are we seeing this again? Because you might want to have it handy when you study the picture that follows it.) The monkey spins it so that it rotates about the *yz* hyperaxis (so every point in the cube travels in a circle parallel to the *wx* plane, which is the plane of rotation).

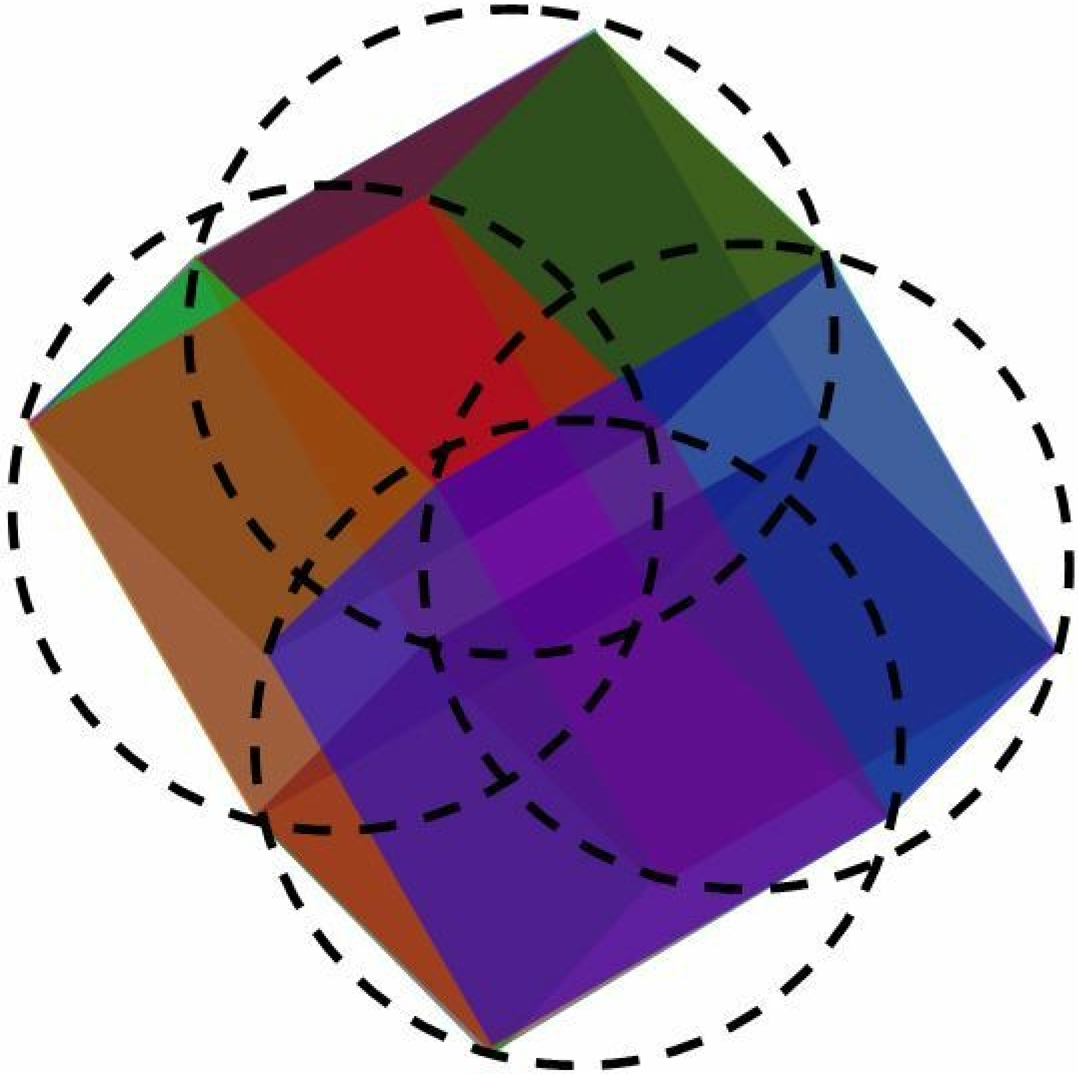


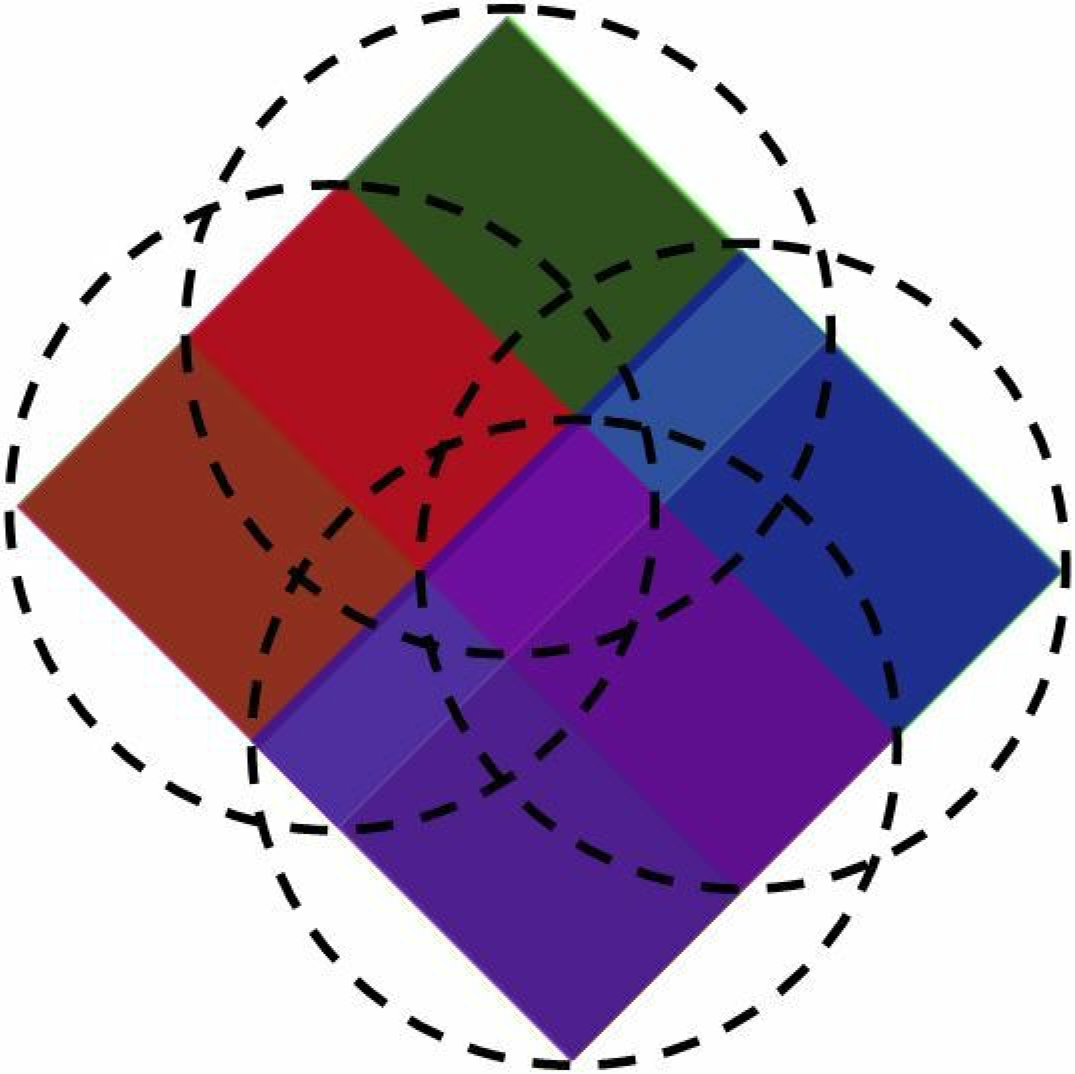
Compare the previous picture with the following picture. The previous cube rotates through the *xyz* and *yzw* hyperplanes shown below. We can make one of these cubes from the other with a 90 degree rotation in 4D space.

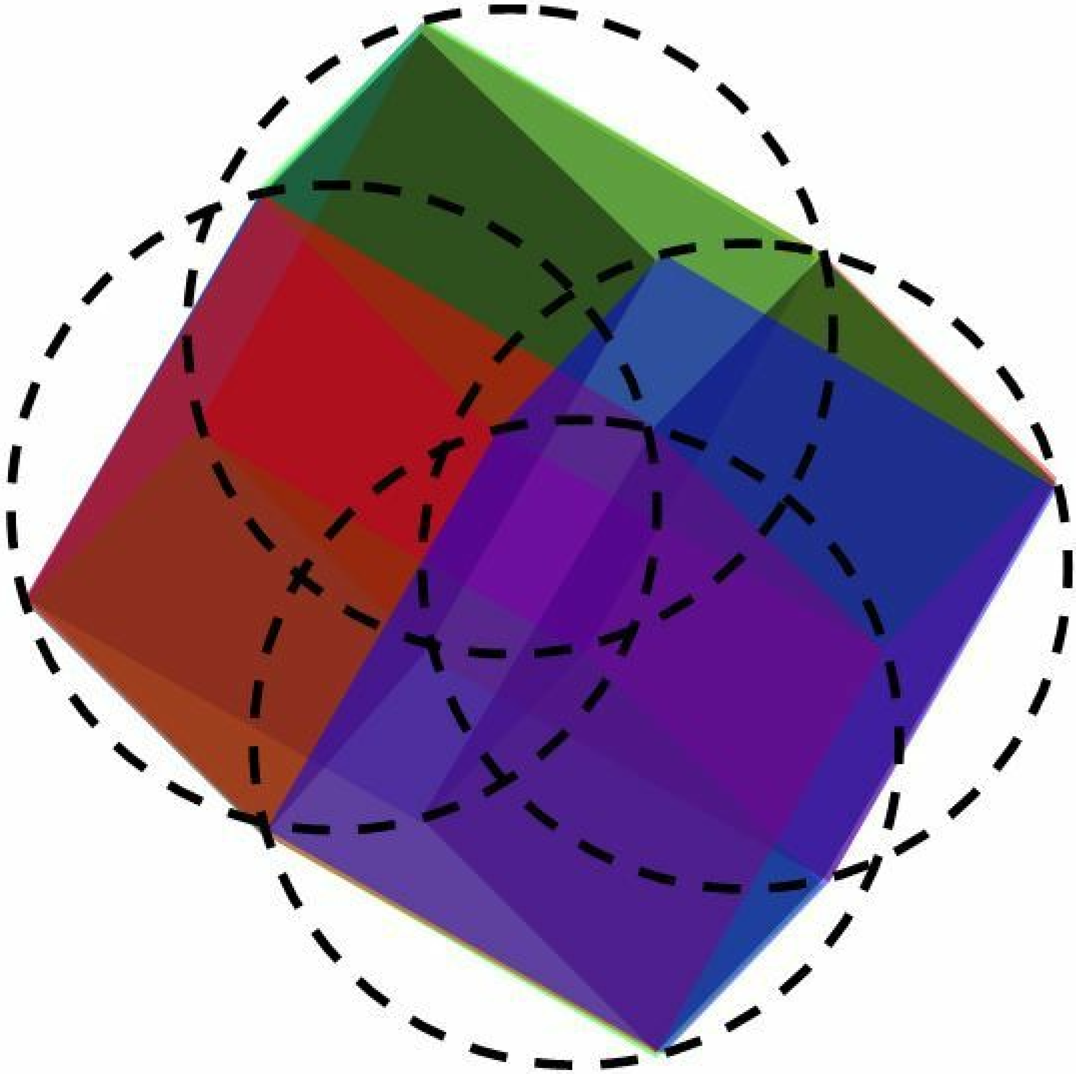


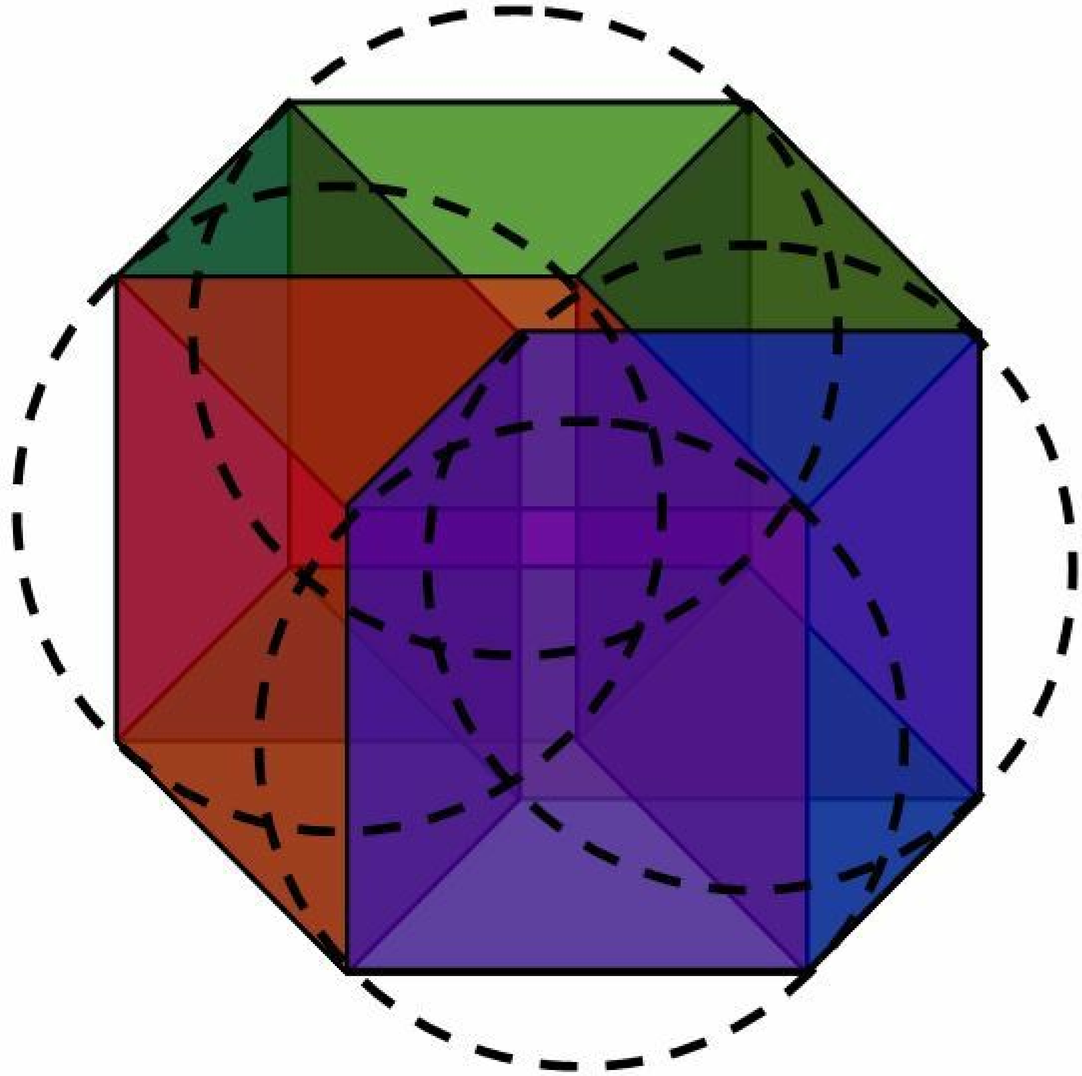
The following images show a monkey rotating a tesseract within the *xy* plane; so it is rotating about the *zw* hyperaxis. Every point in the tesseract travels in a circle parallel to the *xy* plane. Do you believe that it's much easier to visualize the rotation of a cube through 4D space than it is to visualize the rotation of a tesseract? Seems obvious, huh? But not so fast! Remember, the tesseract is bounded by 8 cubes. You can visualize the rotation of a tesseract as the synchronized rotation of 8 connected cubes. Find each bounding cube in the following pictures and watch the rotation one cube at a time in order to better understand the rotation of the tesseract. It may also help to review the previous figures of cubes rotating through 4D and compare them with the following figures where the cubes are bounding rotating tesseracts.



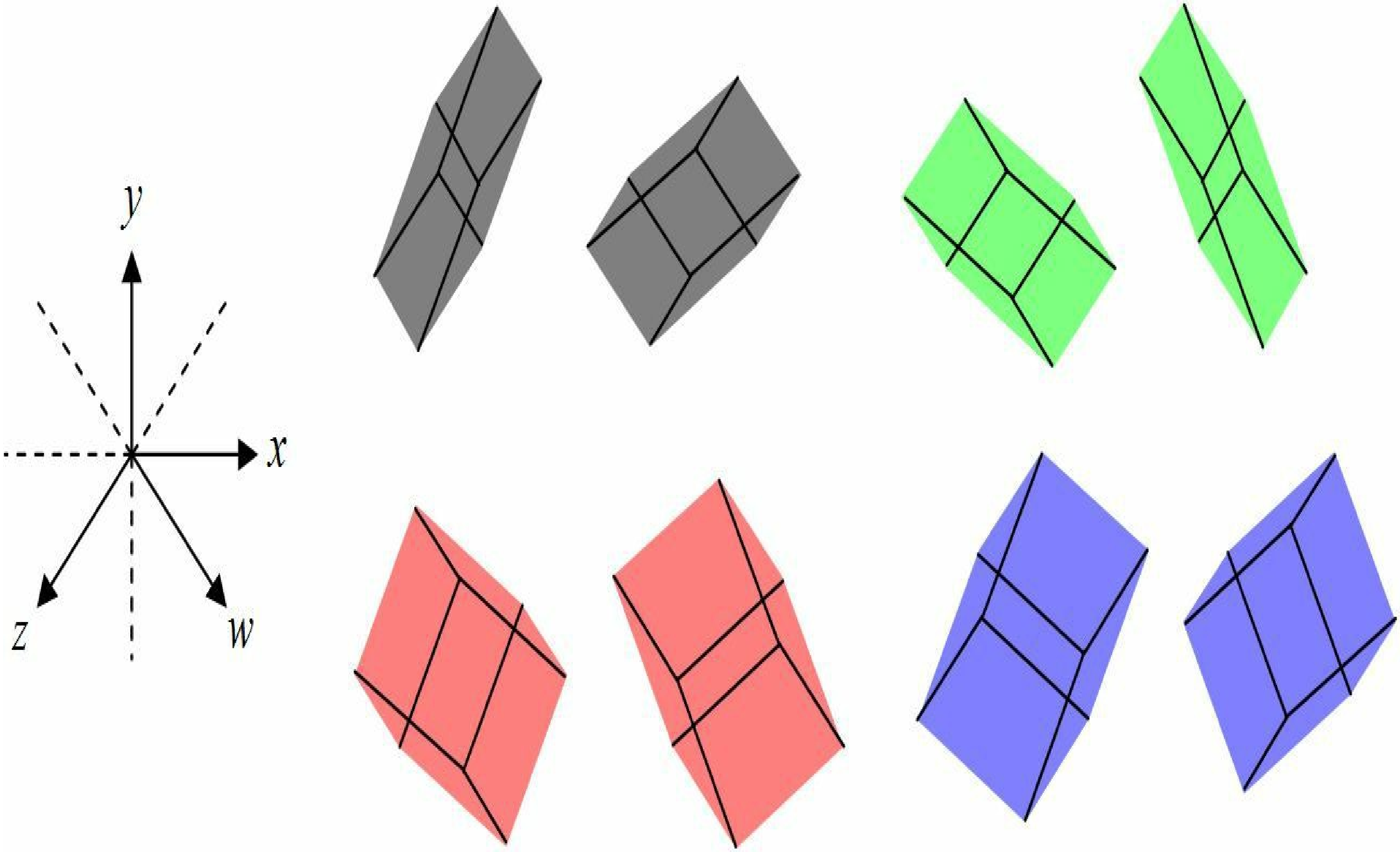




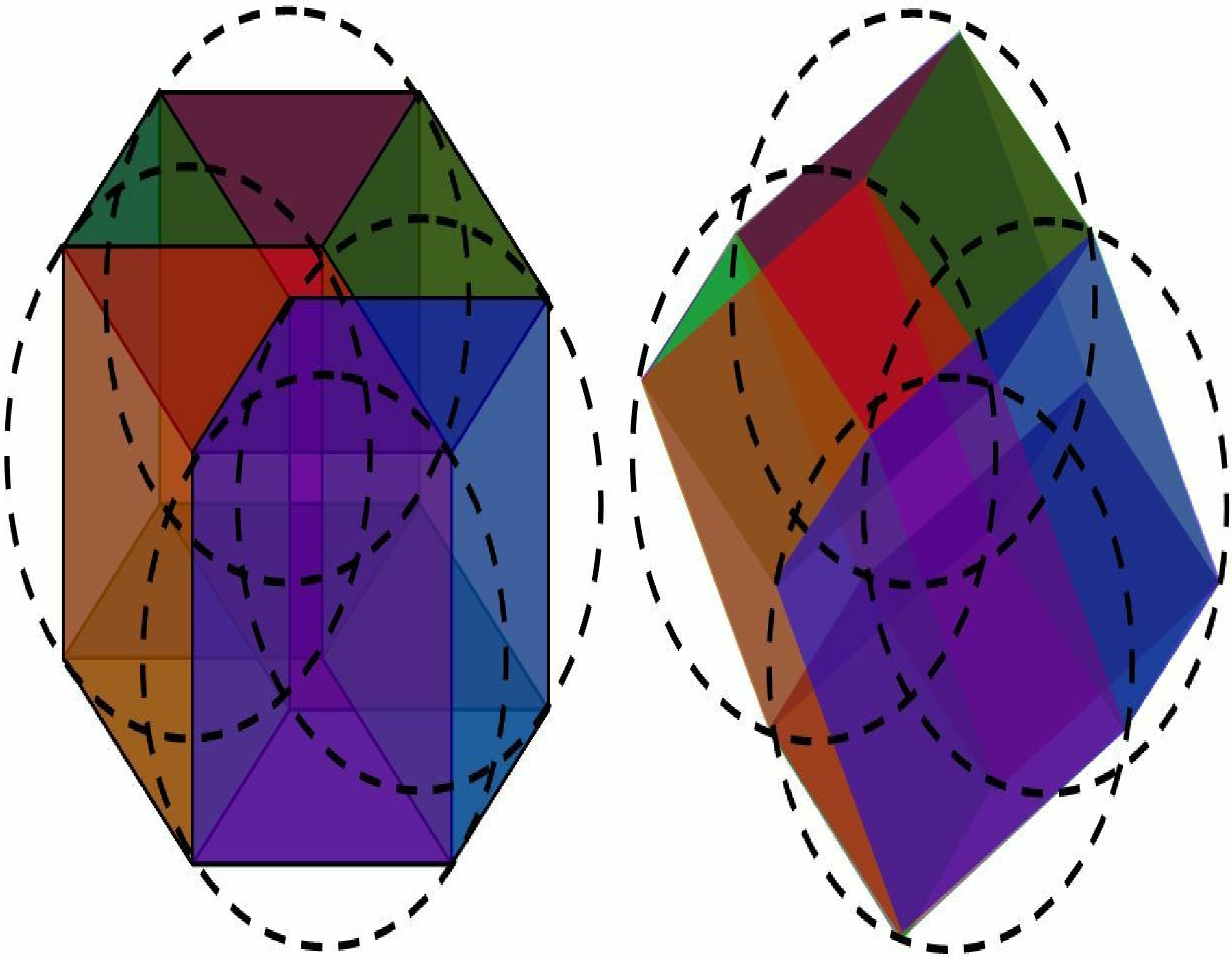




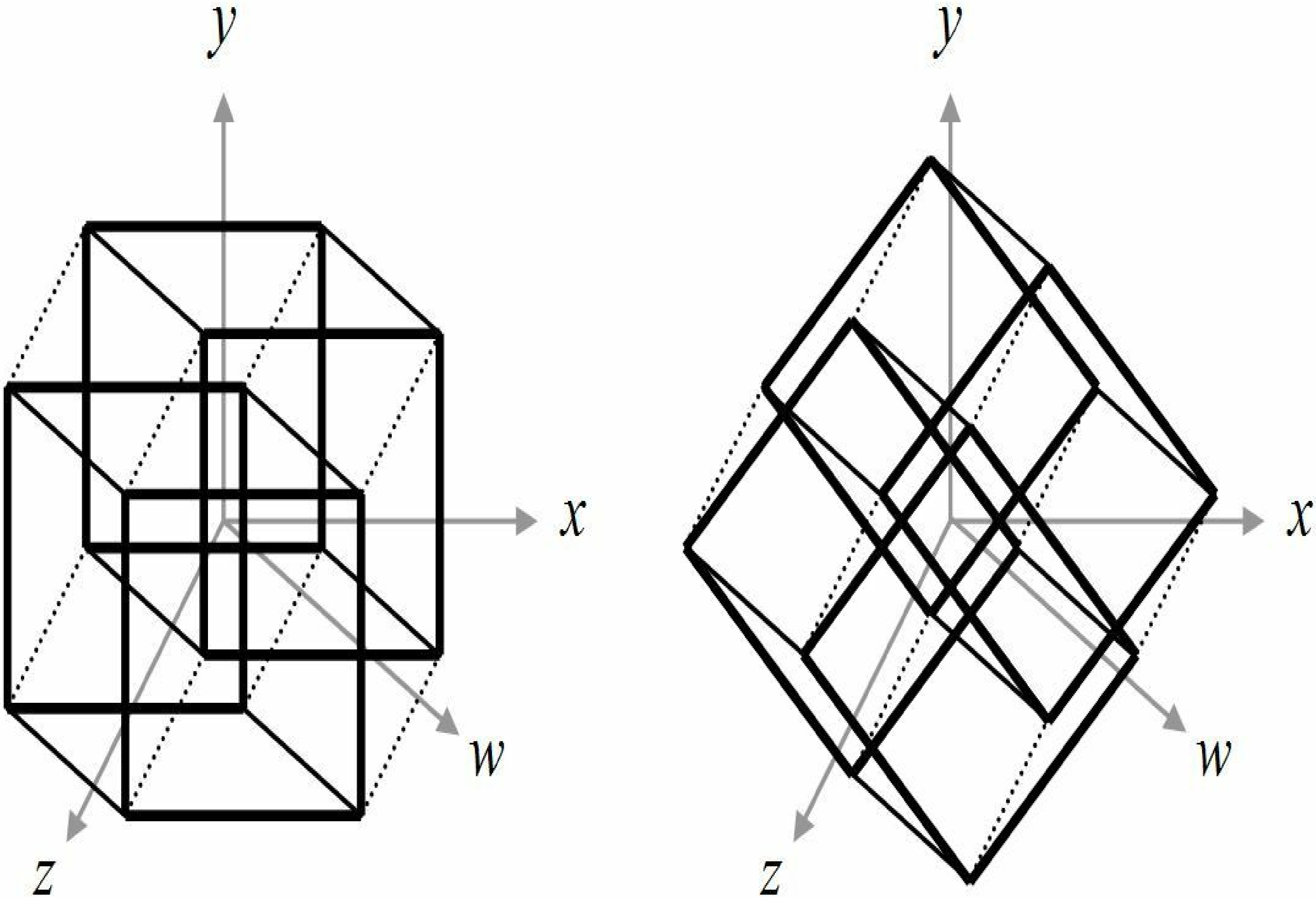
The following picture shows the 8 bounding cubes in a rotated tesseract. See if you can find these bounding cubes in the previous figures.



Here are two images of the rotating tesseract placed side-by-side in order to help you visualize its rotation.

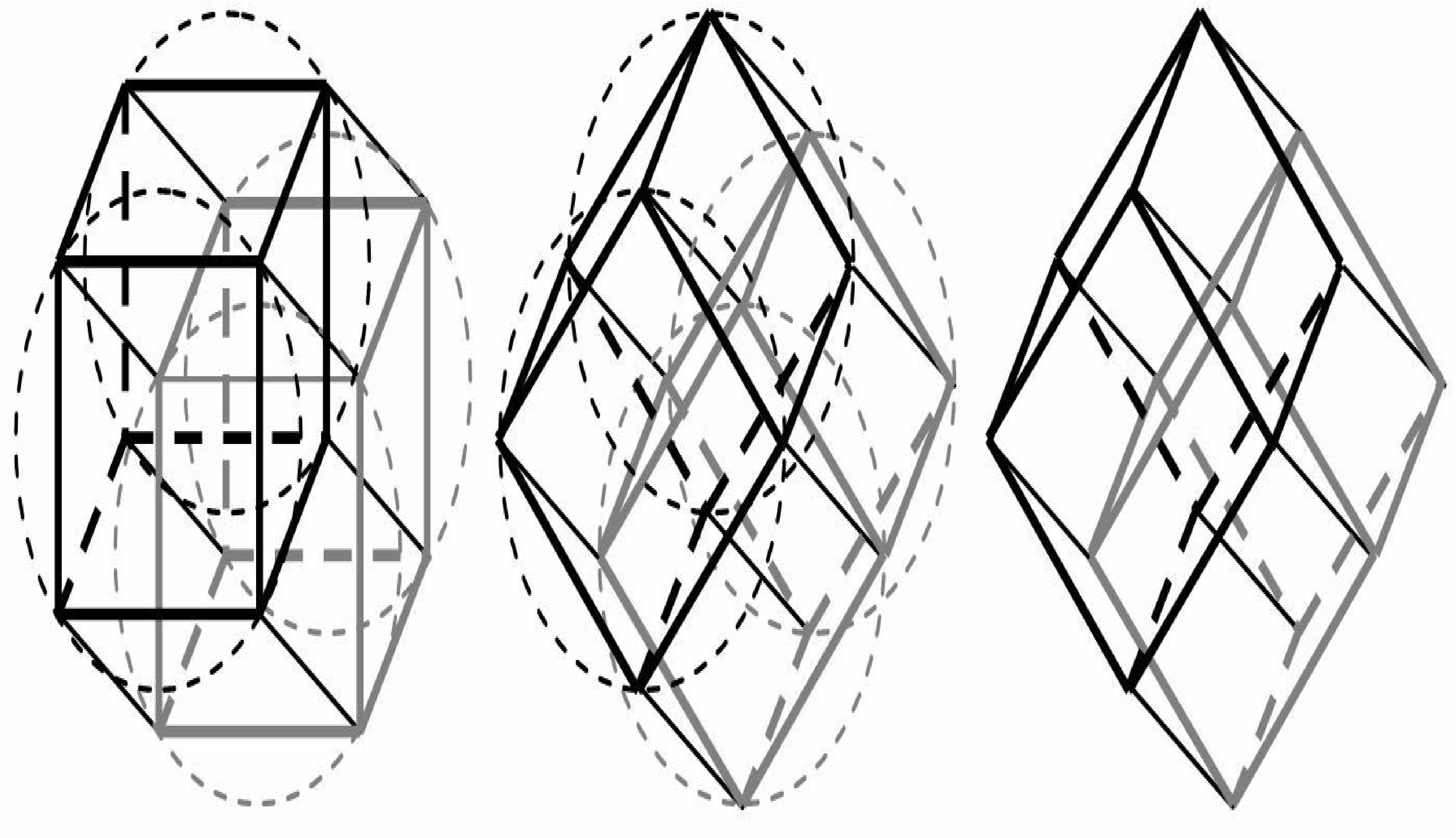


The same figure is shown below in black and white, superimposed on a coordinate system.

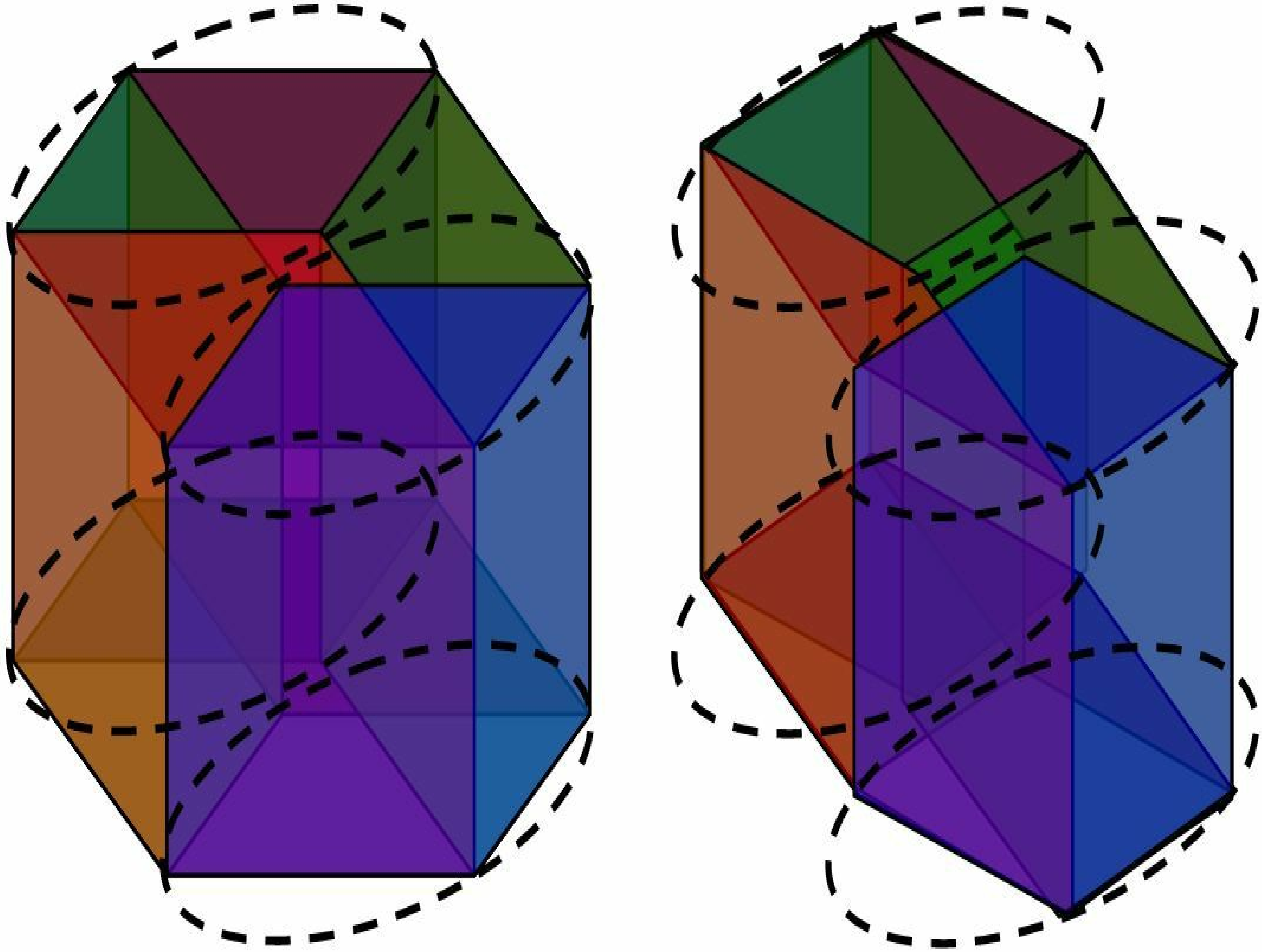


a tesseract rotating about the *wz*-plane

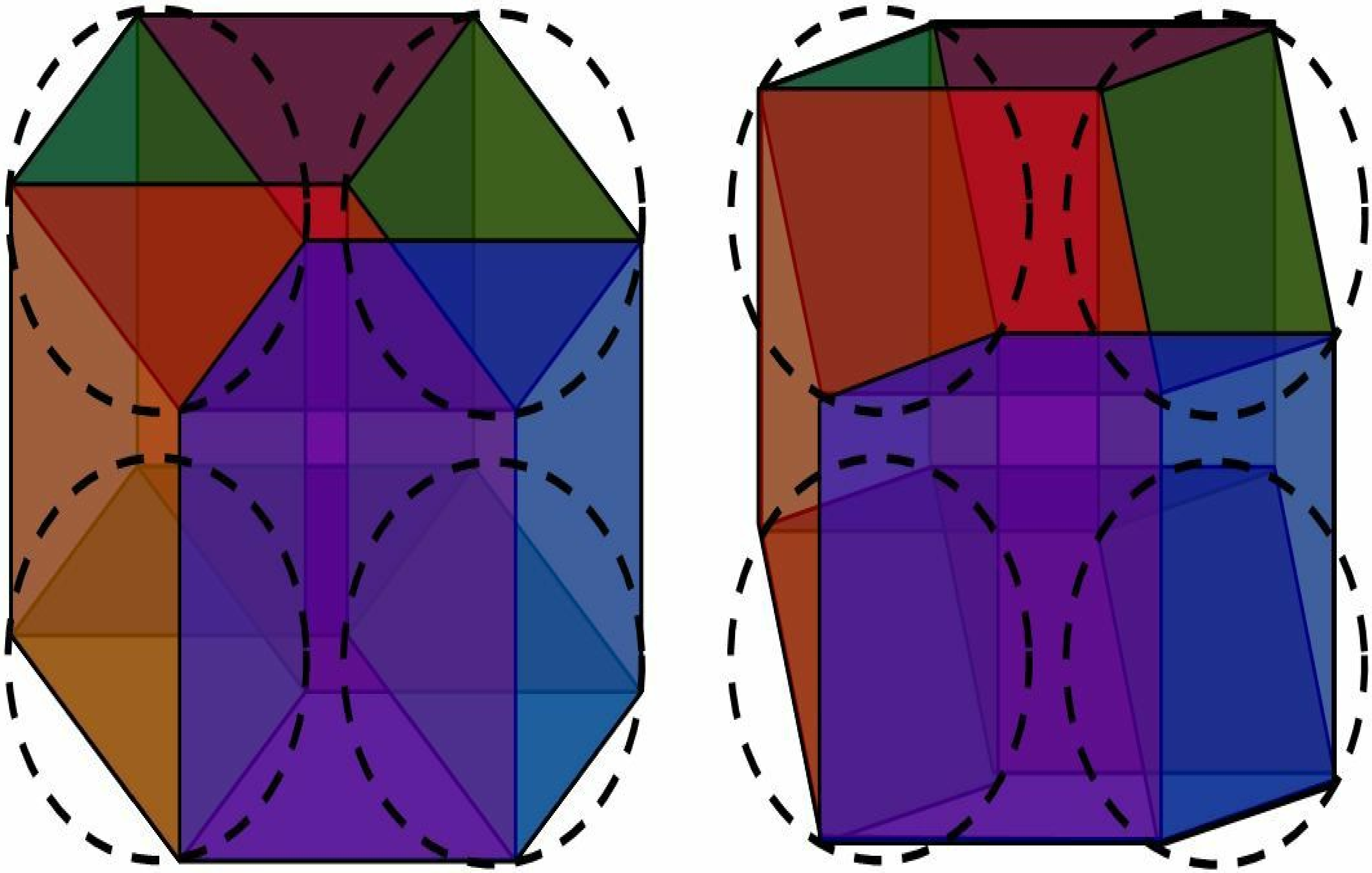
Here, the same rotating tesseract is shown with circles that indicate the plane of rotation.



A monkey is rotating the next tesseract about the *yw* hyperaxis. The *zx* plane is the plane of rotation. Again, it's easier to understand the rotation of the tesseract by studying the rotations of the individual bounding cubes.



The next monkey is rotating a tesseract about the *xy* hyperaxis. The *zw* plane is the plane of rotation.



At some point along the course of reading this book (perhaps this occurred several chapters ago), you might very well wonder, "How in the (insert your favorite expletive here, then wash out your mouth with soap) do we know how to draw this?" I've heard many bizarre comments about pictures of the fourth dimension. Here is one of my favorites, which I've heard multiple times: "You can just scribble anything on a sheet of paper and say that it's a picture of the fourth dimension!" So let's take a moment to address this issue.

We've already alluded to one technique: In Chapter 6, on hypercube patterns, we learned that we can predict how to generalize ordinary 3D objects to the fourth dimension by analyzing patterns. Although the pattern leading up to the fourth dimension can be short, we also saw in Chapter 6 that we can apply logical reasoning. But I don't want you to naively think that the only way to understand the fourth dimension is to try and follow patterns.

We can approach the problem more systematically with mathematical equations. There is some ambiguity inherent in the interpretation of illustrations of the fourth dimension, but we can remove this ambiguity by writing down mathematical equations. You don't need to be able to draw or interpret pictures in order to solve equations.

I've been striving to explain ideas conceptually in this book in order to help make the reading more accessible to a wider audience. My goal was to produce a book that would help *anyone* (well, almost) understand the geometry of the fourth dimension regardless of their background in mathematics (certainly, a little conceptual appreciation and aptitude for geometry would help). But let me take a moment to try to convince you that we could also approach the problem of understanding the fourth dimension algebraically, and that this technique doesn't suffer from the ambiguities of interpreting pictures.

For example, we can write down the coordinates of a tesseract mathematically without drawing anything. We can define a square with corners at (0,0), (1,0), (0,1), and (1,1). Notice that there are 2 coordinates for each corner of the 2D square; the coordinates correspond to all of the points where x and y equal 0 or 1. To get the coordinates of a tesseract, there will be 4 coordinates where x, y, z, and w equal 0 or 1: (0,0,0,0), (0,0,0,1), (0,0,1,0), (0,0,1,1), (0,1,0,0), (0,1,0,1), (0,1,1,0), (0,1,1,1), (1,0,0,0), (1,0,0,1), (1,0,1,0), (1,0,1,1), (1,1,0,0), (1,1,0,1), (1,1,1,0), and (1,1,1,1). These are just the binary numbers from 0000 to 1111 (0 thru 15, which makes 16 corners).

Next, we can get the edges by determining which points we can connect with lines that are parallel to x, y, z, or w. Such edges connect points that differ in just one coordinate, like (0,1,1,0) and (0,1,1,1), which connects a w edge (since these points differ only in their w-coordinates). Other properties of tesseracts can be worked out once we know the coordinates of the corners, like the hypervolume.

Let me illustrate my point about working with equations by considering the hypersphere. The equation $x^2 + y^2 = R^2$ represents a circle with radius R lying in the xy plane centered about the origin. Similarly, $x^2 + y^2 + z^2 = R^2$ represents a sphere with radius R lying in the xyz hyperplane centered about the origin. Without drawing anything at all, it's easy to see that $x^2 + y^2 + z^2 + w^2 = R^2$ represents a glome (a hypersphere in 4D space). Once we know the equation for the hypersphere, we can determine its hypervolume, for example, directly from the equation (no pictures needed).

While there is ambiguity in interpreting a 2D illustration of a 4D object that you're looking at, there is <u>no</u> ambiguity in drawing a 2D picture of a 4D object. (I'm not saying that you can't draw it incorrectly; just that there is a prescription that you could apply to draw it correctly.) There is a branch of mathematics known as projective geometry, in which trigonometry is applied in order to map out the lower-dimensional projections of a higher-dimensional object. This field has many practical applications in our 3D world – e.g. you can use it to predict what the shadow of a sphere or a cone will look like, given the position of the light source. We can also use it to draw 2D projections of 4D objects.

If you would like to draw the fourth dimension, the best way to do it is to write a computer program that yields the points that you would like to plot in 4D. You then apply projective geometry to project these 4D coordinates onto 3D space, and once again to project this 3D image onto the 2D computer screen. If you're wise, you also take some time to work out mathematical calculations by hand to ensure that they agree with the computer program's output, and you can also check for many cases that this agrees with patterns with similar objects in the lower dimensions. Analyzing the fourth dimension algebraically is easy (if you're adept at math); drawing it is also straightforward (if you're skilled at programming and projective geometry); visualizing it and interpreting pictures of it is more challenging; but the real obstacle is going there.

If you're interested in projective geometry, you can find several books on this subject by searching for the keywords "projective geometry" at your favorite online bookseller. If you specifically want to learn about the mathematics of four-dimensional generalizations of polyhedra, search for "polytopes." If you love mathematics and the fourth dimension, you should read a classic treatise by H.S.M. Coxeter entitled *Regular Polytopes*.

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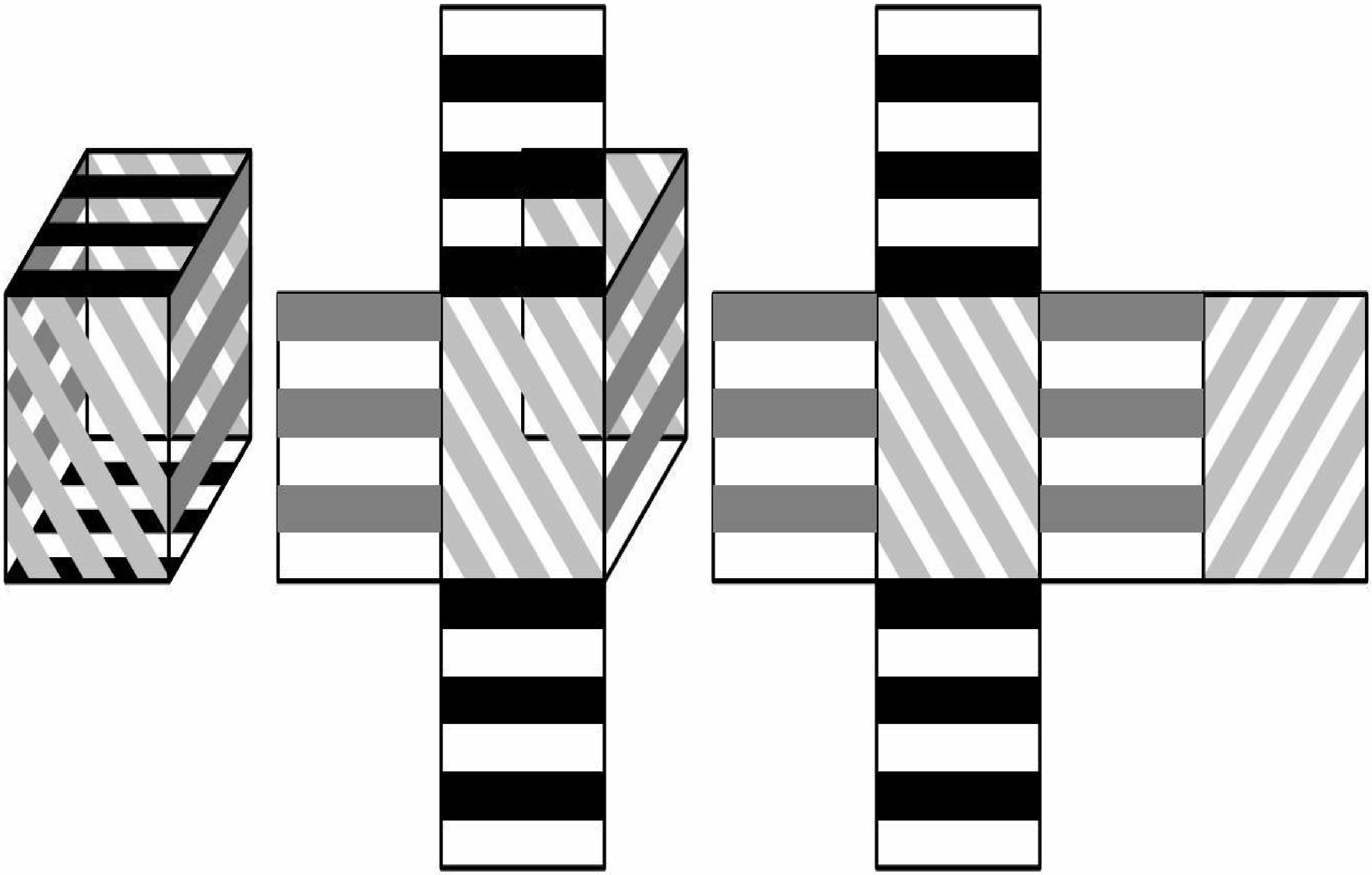
Chapter 10

Unfolding a Tesseract

In this chapter, we're going to think both inside the tesseract and outside of the box. More precisely, we're going to unfold the tesseract. But first, let's unfold some lower-dimensional objects in preparation for this magnificent feat.

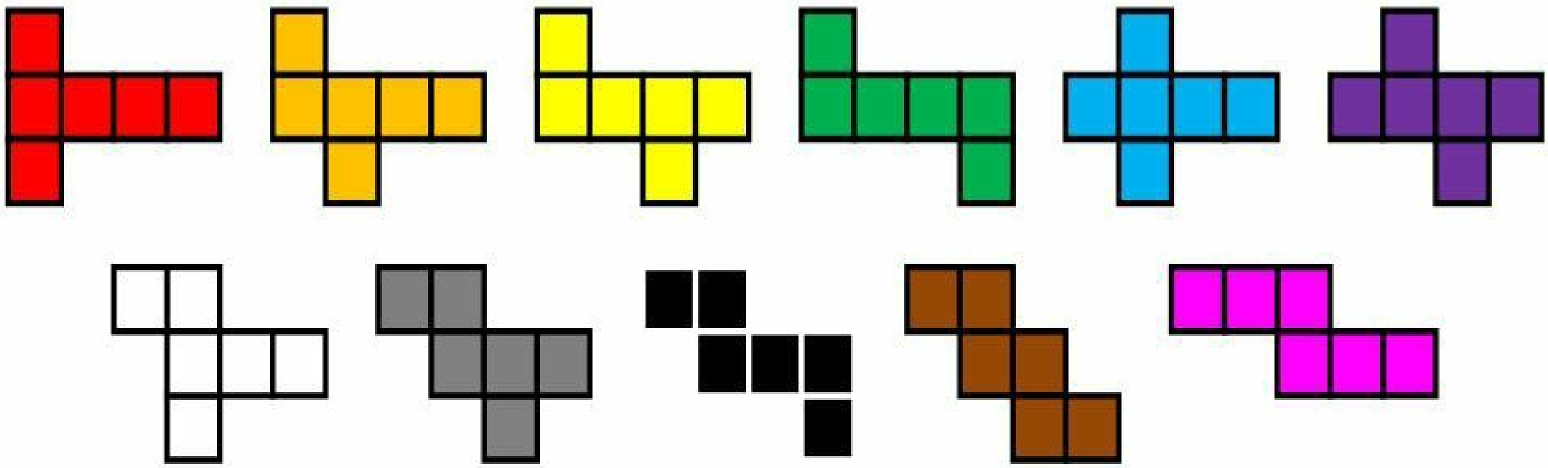
Imagine a 2D monkey who built a square out of 4 monkeysticks (these are measuring devices that 2D monkeys might use, which are similar to the metersticks that we use in our 3D world). If she unfolds this 2D square, she can make a straight line that is 4 monkeysticks long.

Next, consider a 3D monkey who wishes to unfold a box of bananas. It turns out that there is more than one way to unfold the box of bananas into a planar object. She might unfold it as illustrated below, making a T shape from the 6 square sides.



unfolding a cube

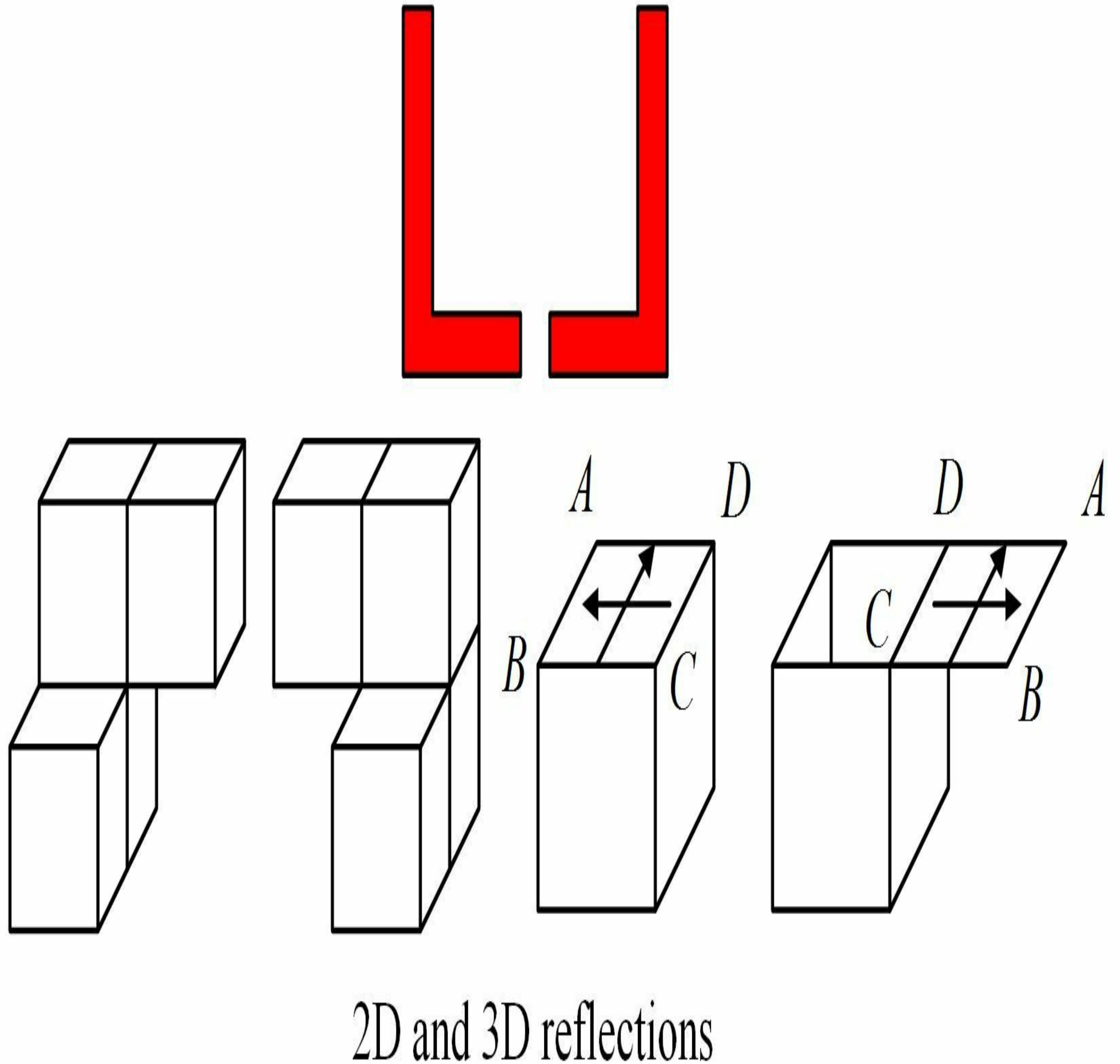
It turns out that there are 11 distinct ways that she can unfold the box of bananas into a planar shape. These are illustrated below. It would be good preparation (i.e. before going onto 4D) to see if you can visualize, in your mind, how to fold each of these 11 shapes into a cube. You never know; there could be a quiz on this.



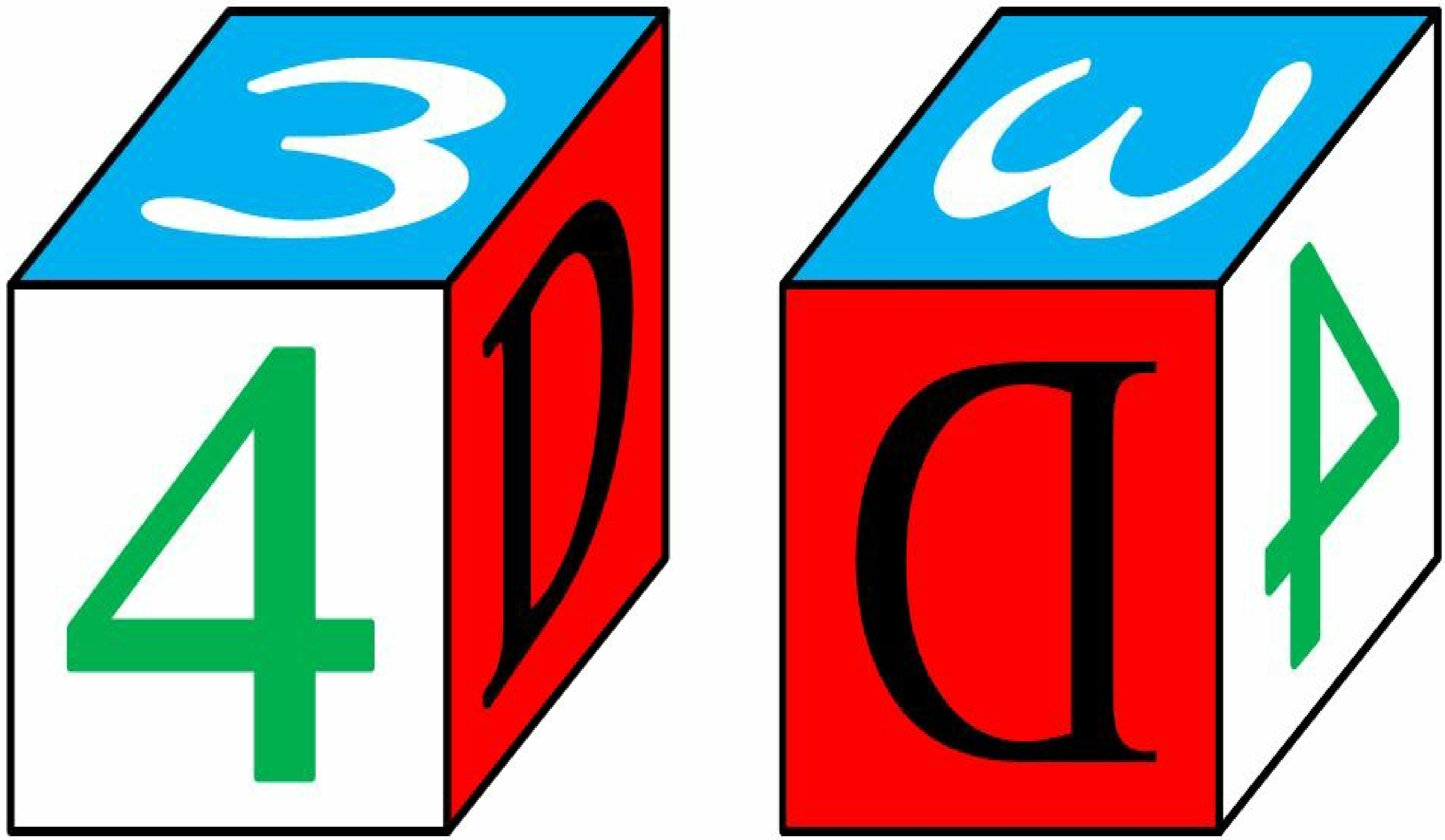
Let us take a moment and reflect upon these lower-dimensional rotations. Literally! I do mean it literally (because, you know, sometimes people say, "Literally," yet they really mean figuratively – *literally!*). We shall reflect upon these rotations. What I mean is, let's consider how a reflection relates to a higher-dimensional rotation. (What? Did I contradict myself? Let's see. In the first sentence of this paragraph, I referred to lower-dimensional rotations. By that, I meant 2D and 3D, which are lower than 4D. In the sentence prior to this parenthetical note, I mentioned a higher-dimensional rotation. By this, I meant, for example, that a 3D reflection is equivalent to a rotation in 4D space; and 4D is higher than 3D. So maybe I didn't contradict myself after all. Too bad!)

We'll begin with the 2D L shape illustrated in the following figure. One leg of the L is longer than the other. You can't rotate the L into its reflection in 2D space. However, you can rotate the L into the third dimension in order to transform it into its own reflection. This is what I mean when I say that a higher-dimensional rotation can achieve the same effect as a reflection. Visualize this rotation as if you are picking up the left edge of the L and flipping it over (as if you want to turn the page of a paperback book in order to go back to the previous page). You have to pick the L up out of the plane in order to rotate the L into its reflection.

By analogy, you could rotate a 3D object into its 3D reflection by rotating it through a fourth dimension of space. For example, consider the left-handed and right-handed block structures drawn in the bottom left portion of the figure that follows. Try as you might, you can't rotate one of these block structures into the other with any rotation in 3D space. However, if we had access to a fourth dimension of space, we would be able to rotate one of these 3D structures into its own reflection by rotating it through the fourth dimension. You could even reflect yourself by rotating 180 degrees into the fourth dimension! Your reflection would actually have his/her heart on his/her right side.



Here is a little puzzle. Consider the two 3D blocks illustrated below. Is it possible to rotate one block into the other using the fourth dimension? Examine the blocks carefully. If you decide that such a rotation is possible, determine the plane of rotation. Remember, if you start reading the paragraph after the picture, you'll be spoiling the solution.

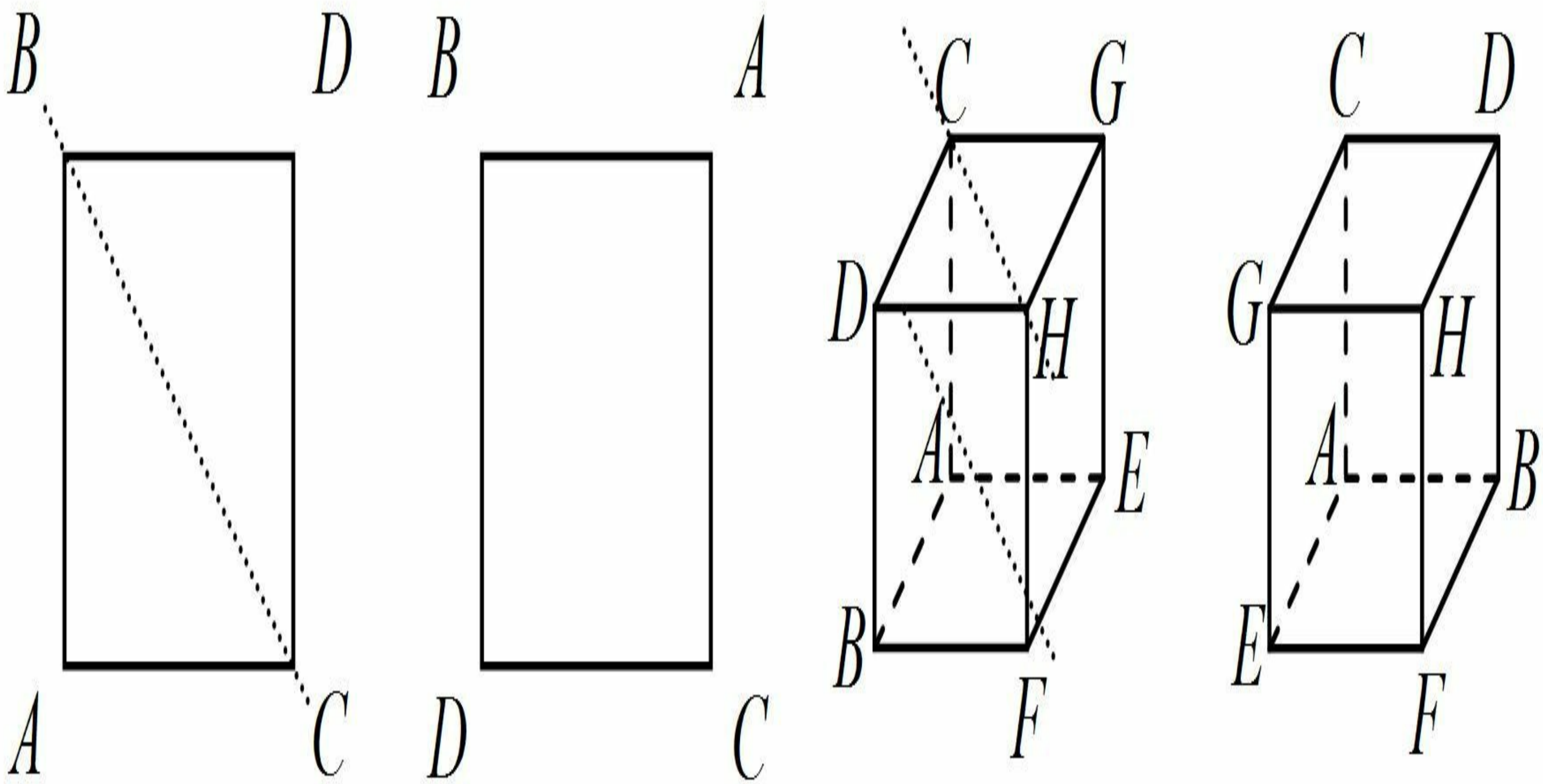


We're trying to determine if one of the blocks pictured above can be rotated into the other in 4D space. It should be easy to convince yourself that it can't be done in 3D, but the challenge is checking this in 4D. The trick is to apply the concept that we just learned (which probably shouldn't be too "tricky," since the puzzle came immediately after discussing this concept). Which concept? The relationship between a reflection and a higher-dimensional rotation.

Study the two cubes above and ask yourself this question: Can you figure out a way to arrange the blocks so that one block looks like the reflection of the other? I recommend that you contemplate this carefully before you read on. You still have a chance to solve the puzzle.

The answer is that one block <u>is</u> a reflection of the other. Now you've read the answer, but not the solution. Study the blocks again. Now that you know that one block is the reflection of the other, maybe you can figure out how to reflect one block to make the other.

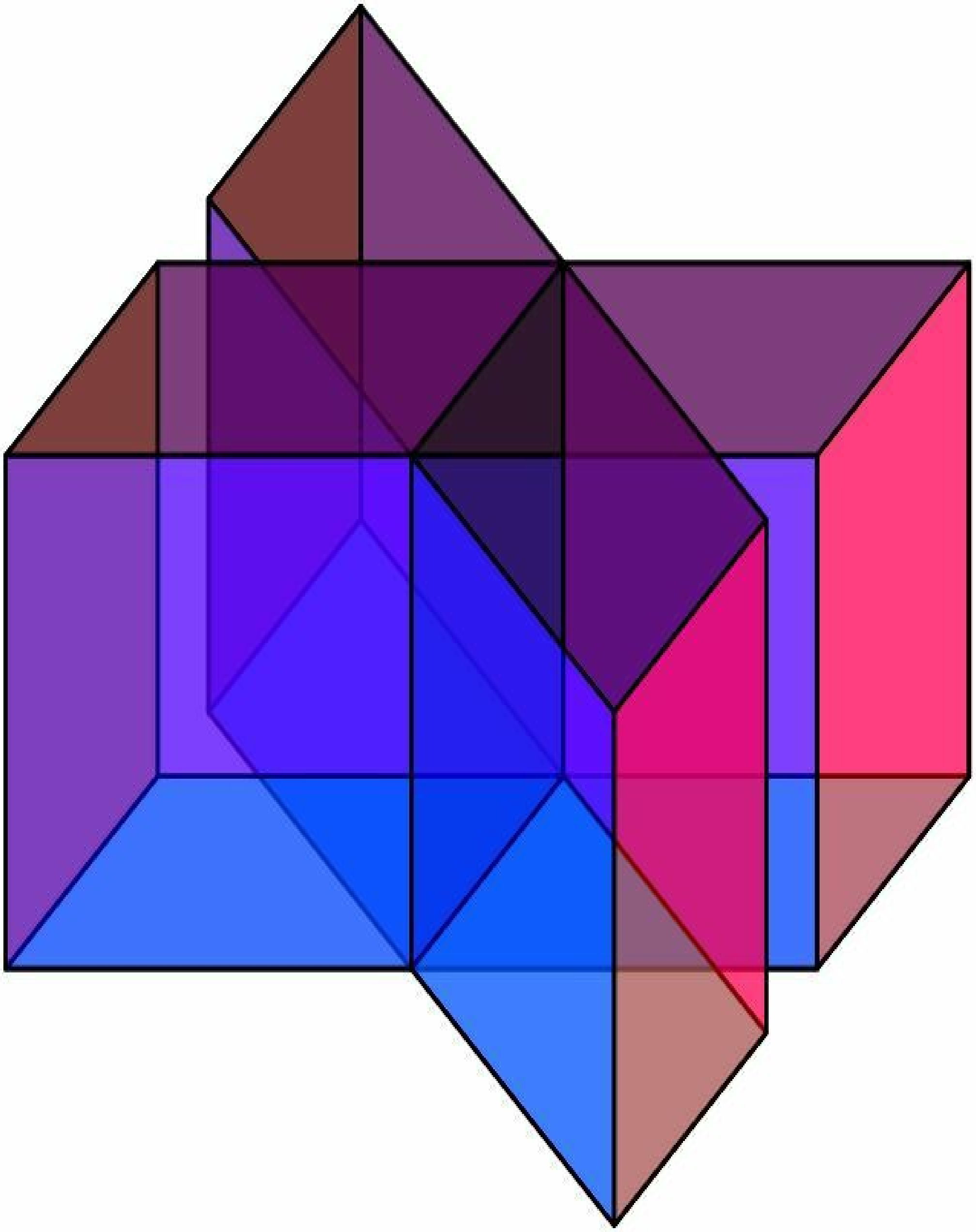
It's a little tricky because the reflection is not about one of the faces. Rather, the reflection is about a plane that bisects the cube down a face diagonal, as illustrated below. First look at the two squares drawn below. One is the reflection of the other about the indicated face diagonal. Now try to understand the similar reflection of the cube in the diagram below. The cube below is reflected about the *ACHF* plane. This causes a swap between corners *D* and *G* and corners *B* and *E*, for example. Compare the reflected cubes in the picture below to the puzzle pictures above, and you should see that one cube is indeed a reflection of the other. Therefore, one cube can be transformed into the other through a 180 degree rotation into a fourth dimension of space.

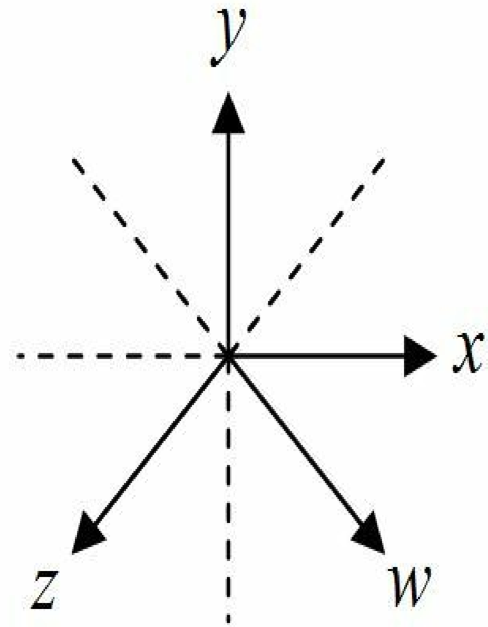


You can actually reflect the cube in a more obvious way and solve the same problem. For example, try reflecting the cube about the *EFGH* plane (as labeled in the figure above) and then rotating it 90 degrees.

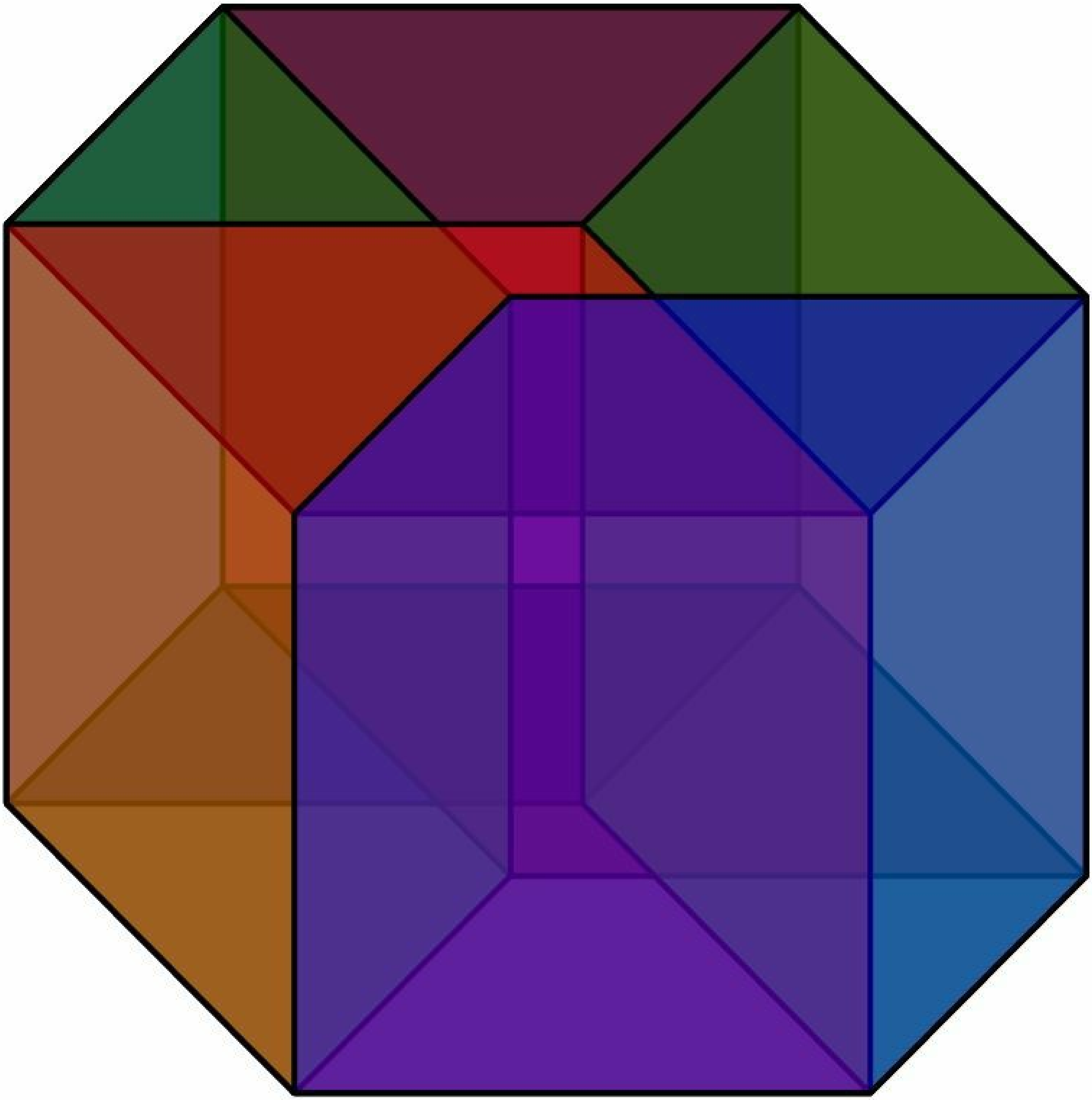
We will now proceed to unfold the tesseract. (Don't worry! We'll be careful not to let all of the monkeys fall out.) We will do this by rotating the cubes one at a time, 90 degrees each, until all 8 bounding cubes lie in the *xyz* hyperplane. This concept should make sense, in principle, if you consider that we unfold a cube by rotating each square face one at a time, 90 degrees each, until all 6 square faces lie in one plane (as we did visually earlier in this chapter).

Since we will be rotating cubes through the fourth dimension when we unfold the tesseract, the picture below from the previous chapter has been included in order to remind you what it looks like to rotate a cube in 4D space. The cube below is rotating about the *yz* hyperaxis, or rotating in the *wx* plane (remember, this really means that each point in the cube travels in a circle that is parallel to the *wx* plane – of course, a cube can't possibly lie in a plane, so that's what we mean when we say that it rotates in a plane); the *y* and *z* edges remain fixed during this rotation.

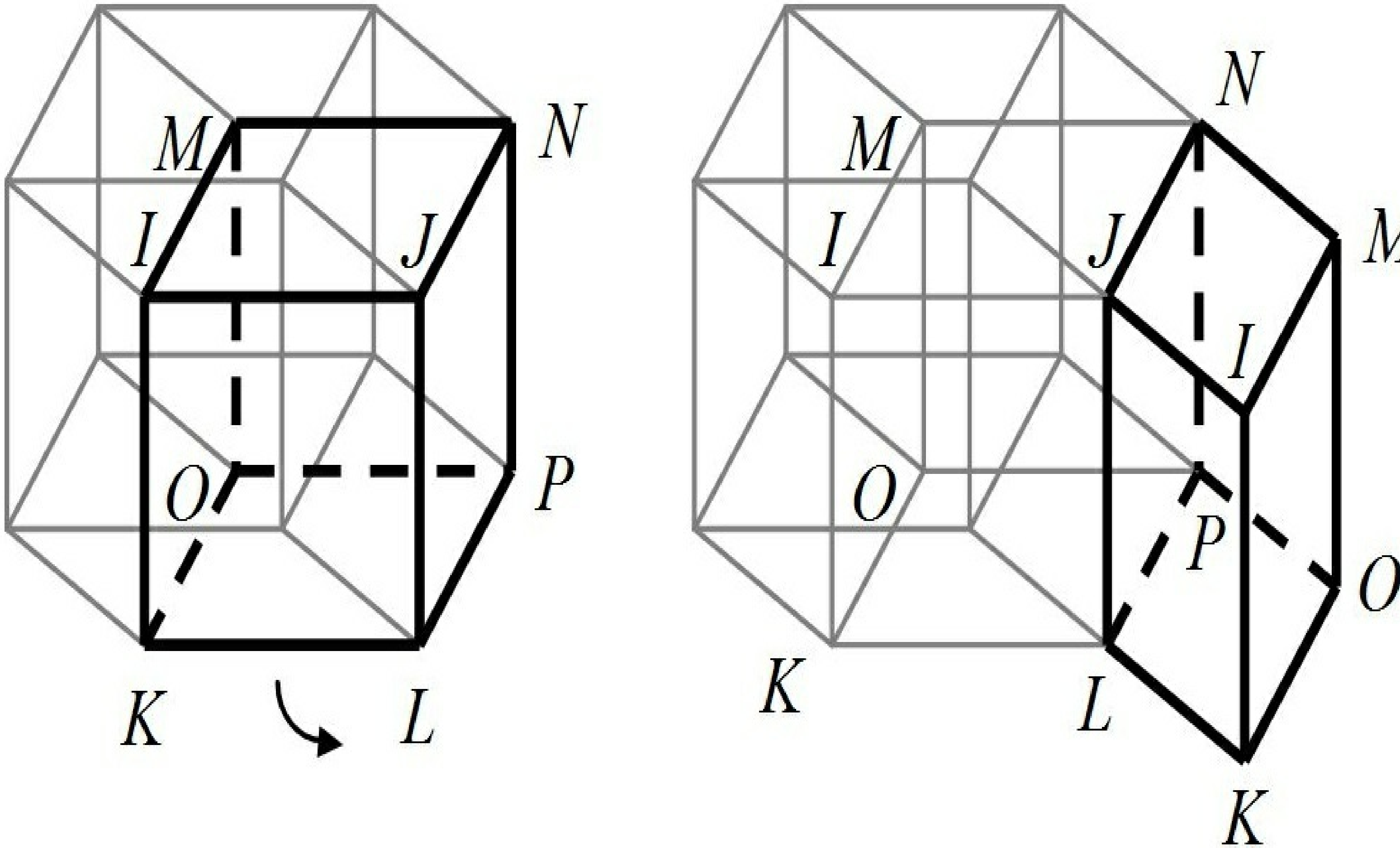




We begin with the unfolded tesseract.

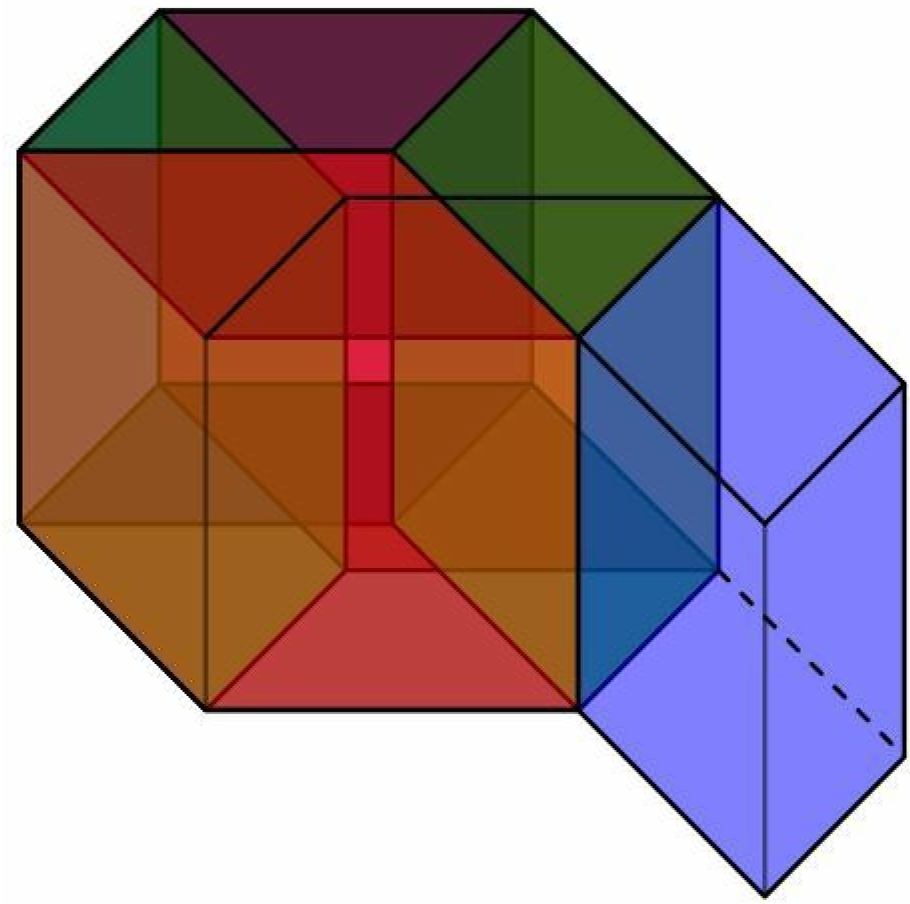


A monkey will grab one of the bounding cubes and rotate it 90 degrees in 4D space. Specifically, the monkey grabs one of the 2 cubes that lie in (technically, it's "parallel to," not "in") the *xyz* hyperplane and rotates this cube about the *yz* hyperaxis – or in the *wx* plane (just like the rotating cube illustrated previously). In fact, it may be helpful to compare the previous figure of the rotating cube to the rotation that follows.

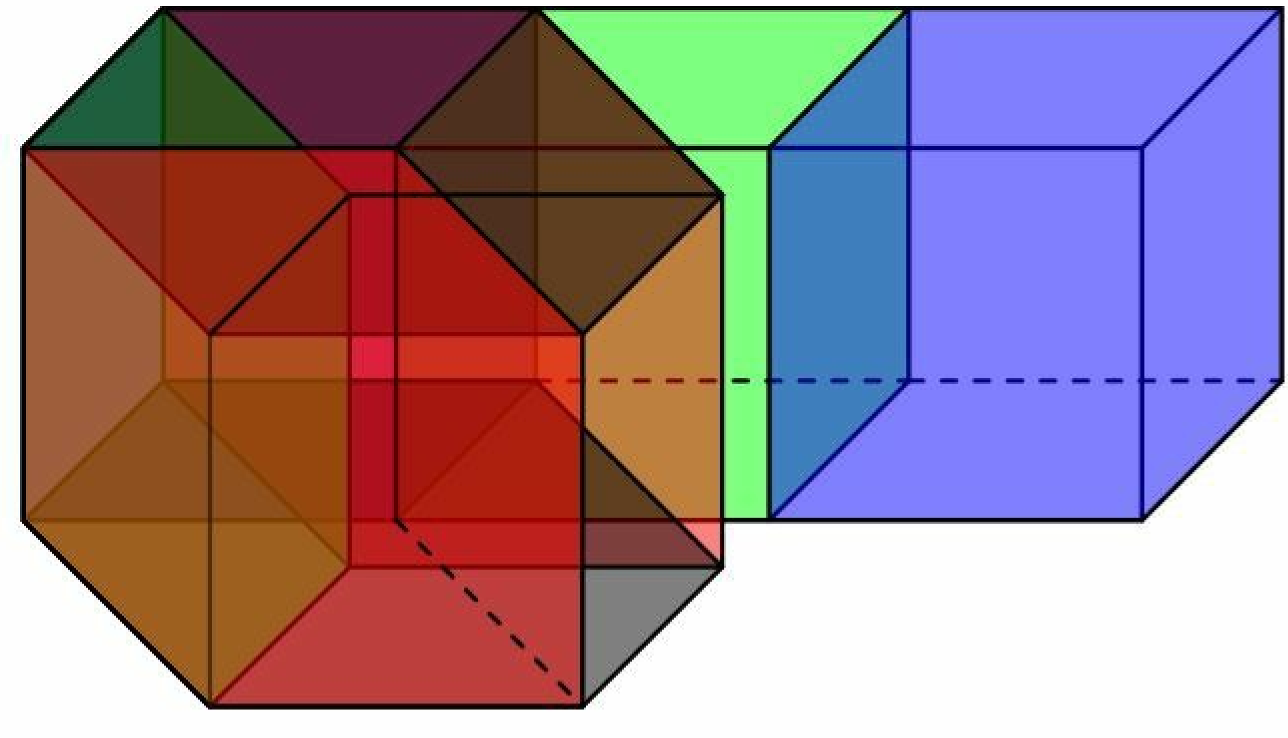


If you study the corners of the cube above, you should be able to see that if the monkey rotates the cube 180 degrees instead of 90 degrees, then the cube will be a reflection of the original. However, the monkey will only rotate the cube 90 degrees for the moment. I just wanted you to see another reflection as it relates to a higher-dimensional rotation.

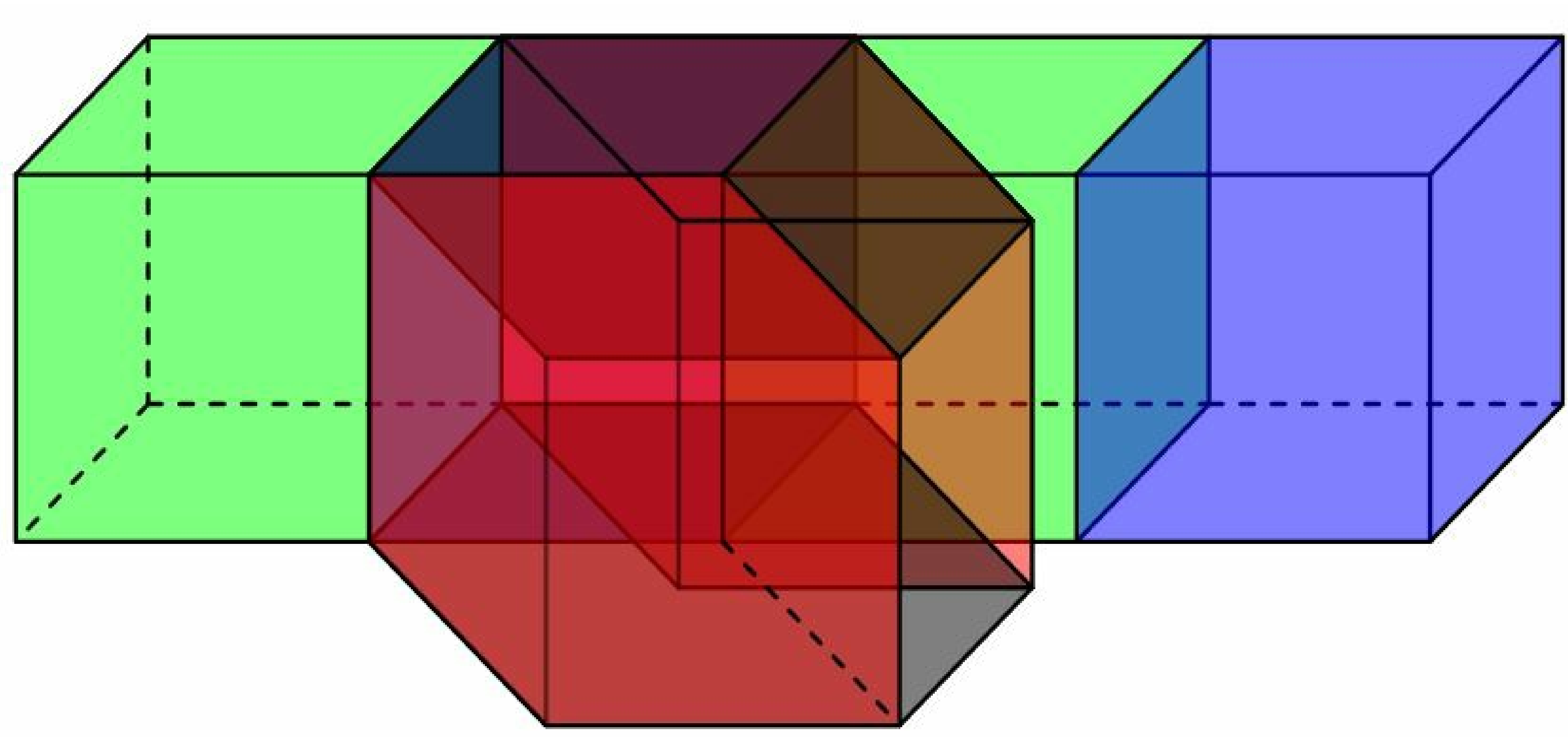
The same unfolding is shown below, but in color.



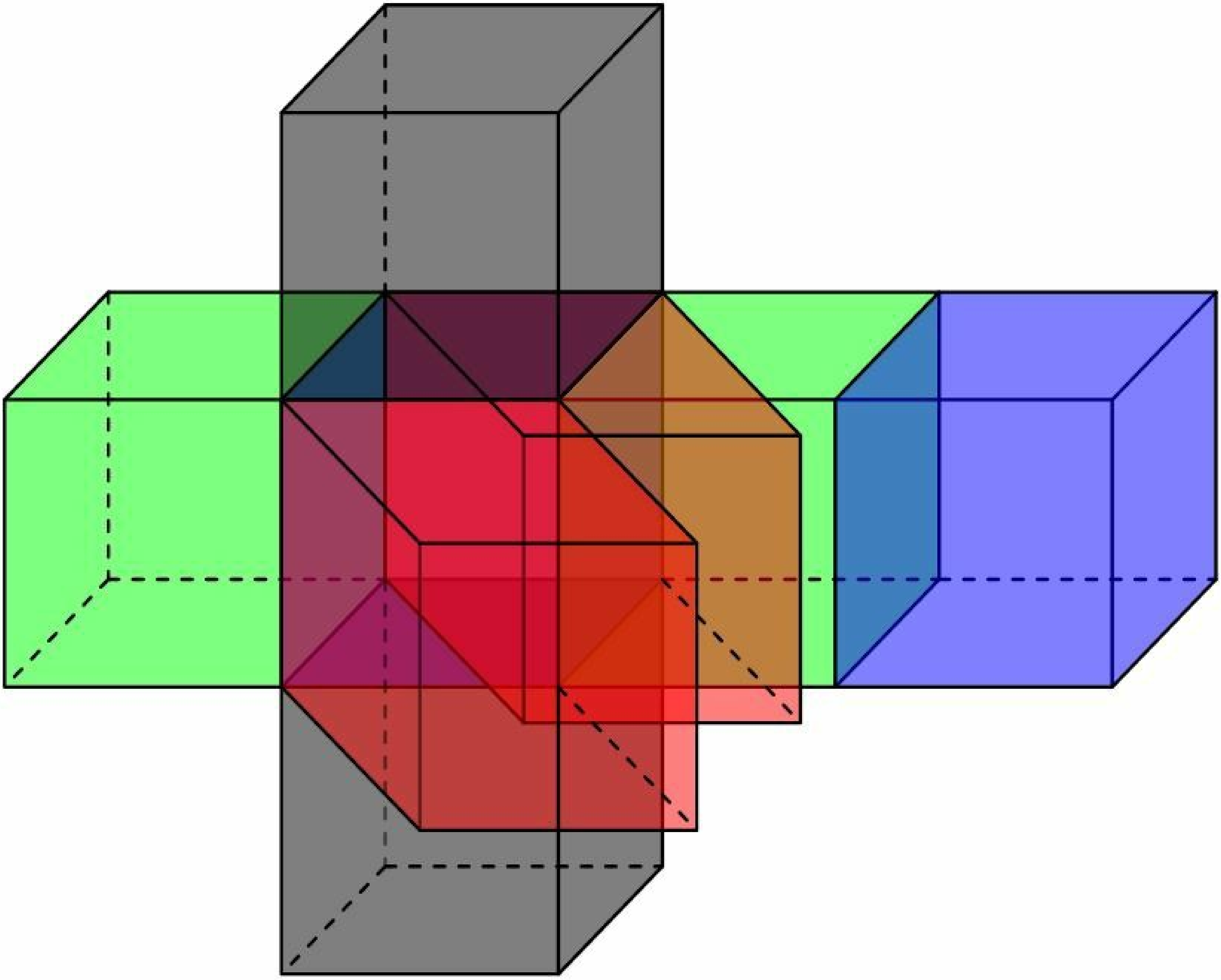
After the first unfolding, drawn above, the unfolded cube rotated from the *xyz* hyperplane to the *yzw* hyperplane. You can clearly see the unfolded cube in the picture above, but it's hard to tell that there's <u>not</u> a cube where it had been (because there are so many edges from the other cubes that were left behind). The unfolded cube, which now lies in the *yzw* hyperplane, is also connected to a cube that lies in the *yzw* hyperplane; these two *yzw* cubes join at a *yz* face. The monkey will now rotate these 2 *yzw* cubes together 90 degrees about the *yz* hyperaxis, as shown below.



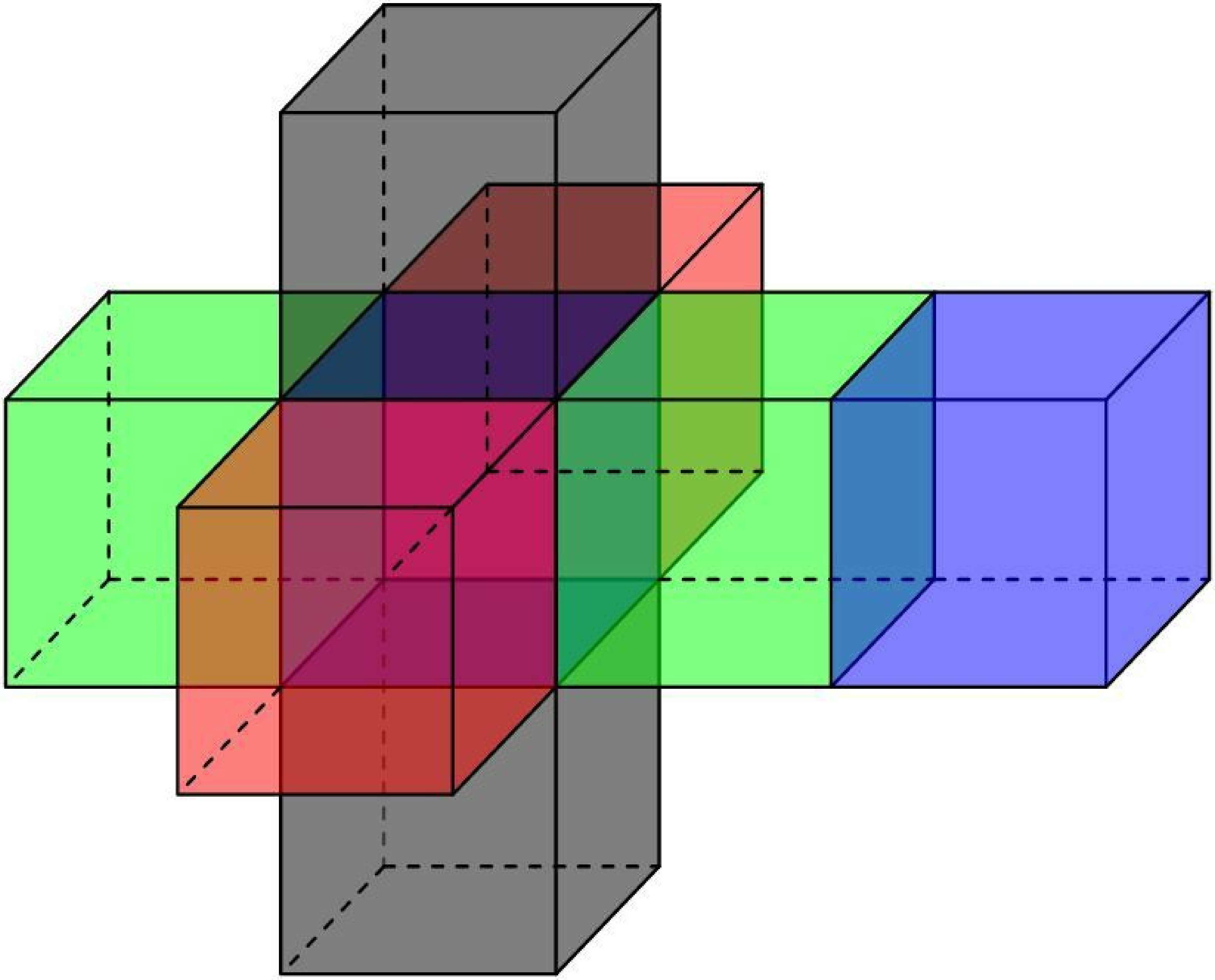
You should now see 3 *xyz* cubes in a row in the previous figure. Next, the monkey will rotate the remaining *yzw* cube 90 degrees about the *yz* hyperaxis (but in the opposite direction).



There are now 4 *xyz* cubes in a row. In the next step, the monkey will rotate both of the *zwx* cubes 90 degrees (in opposite directions) about the *xz* hyperaxis.

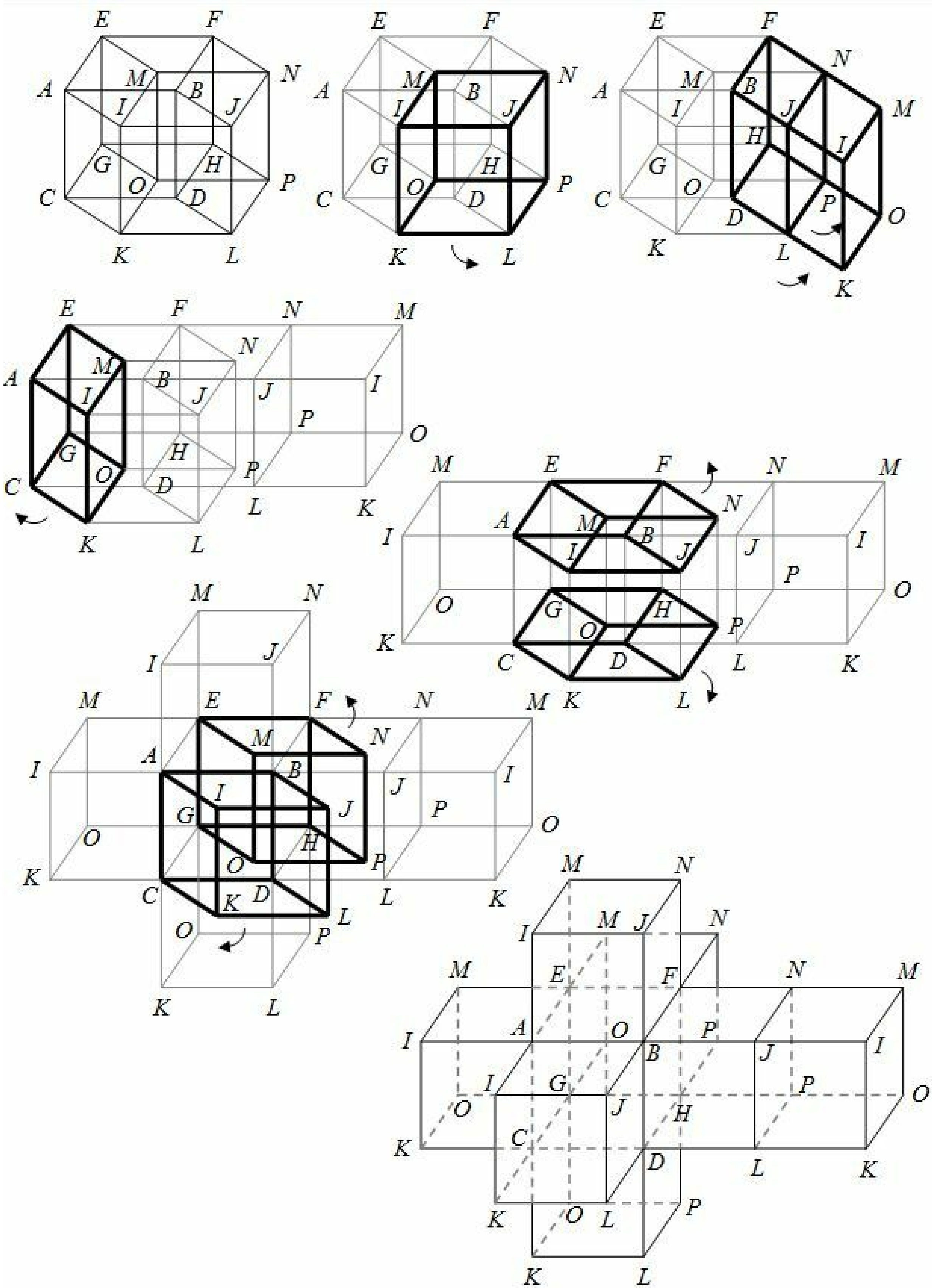


Finally, the monkey will rotate the 2 *wxy* cubes 90 degrees (in opposite directions) about the *xy* hyperaxis.



Ta-da! Monko the magnificent has just unfolded a tesseract into the *xyz* hyperplane. The result looks like a double cross. I don't mean that the hypermonkey magician, Monko, has double-crossed you. Rather, I mean that the unfolded tesseract looks literally like a double cross.

The same unfolding is illustrated below in black and white with all of the corners labeled. If you have a small screen, it might be hard to see all of the detail (if so, you might consider downloading your ebook onto a cloud, for example, so that you can view it on a pc with a large screen). Also, you might see if your e-reader has a zoom option (on the original Kindle Fire, for example, you have to double-tap the screen to find this). Anyway, you already saw this step-by-step in color with larger images.



Unfolding a tesseract is so easy that even a monkey could do it! Well, a hypermonkey (i.e. a 4D monkey), anyway. Literally! No kidding! Look, I'm being serious: A 4D monkey could unfold a tesseract. Piece of cake. What? Do you care to disagree? Let's think about it. Imagine that you go to the zoo with a cardboard box. Don't you think you could find a monkey (or, at least a chimpanzee or orangutan) who could unfold the 3D box into a plane? So if there is a universe somewhere with four dimensions of space, don't you think that the hypermonkeys (or hyperchimpanzees, or some 4D animal of this sort) would be able to unfold a 4D cardboard box into a hyperplane just as readily? Look, I'm not saying that a monkey is smarter than you. But maybe a hypermonkey living in a 4D world would find it much easier to visualize 4D objects than humans living in a 3D world. You might at least acknowledge that we do have a slight disadvantage in this regard. (And who says that 3D or 4D monkeys aren't smart? I bet I could teach them some physics...)

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Chapter 11

Cross Sections of a Tesseract

We will look at the tesseract from a different perspective in this chapter. Instead of trying to visualize what it would be like to go to the fourth dimension and view a tesseract, let's have the tesseract come to us! Try to imagine a tesseract passing through 3D space. Such a tesseract wouldn't appear 4D to us; we would see a 3D cross section of the tesseract.

You can see the cross section of an object if a monkey slices straight through it with a butcher's knife. (Would you rather chop it with a hatchet? Cut through it with a high-powered laser? Oh, perhaps you would prefer something less violent? It's not like we're going to find cross sections of *people* or *animals*. What's the harm in cutting a cube or a tesseract? But nevermind. Scratch the butcher's knife. Bury the hatchet. Turn off the laser.)

Let me begin anew. You can see the cross section of an object if a monkey gently... Hmm. You can see the cross section of an object if a monkey gingerly... Let's see here. How do you propose to have the monkey slice through an object in some way that doesn't seem graphic and violent?

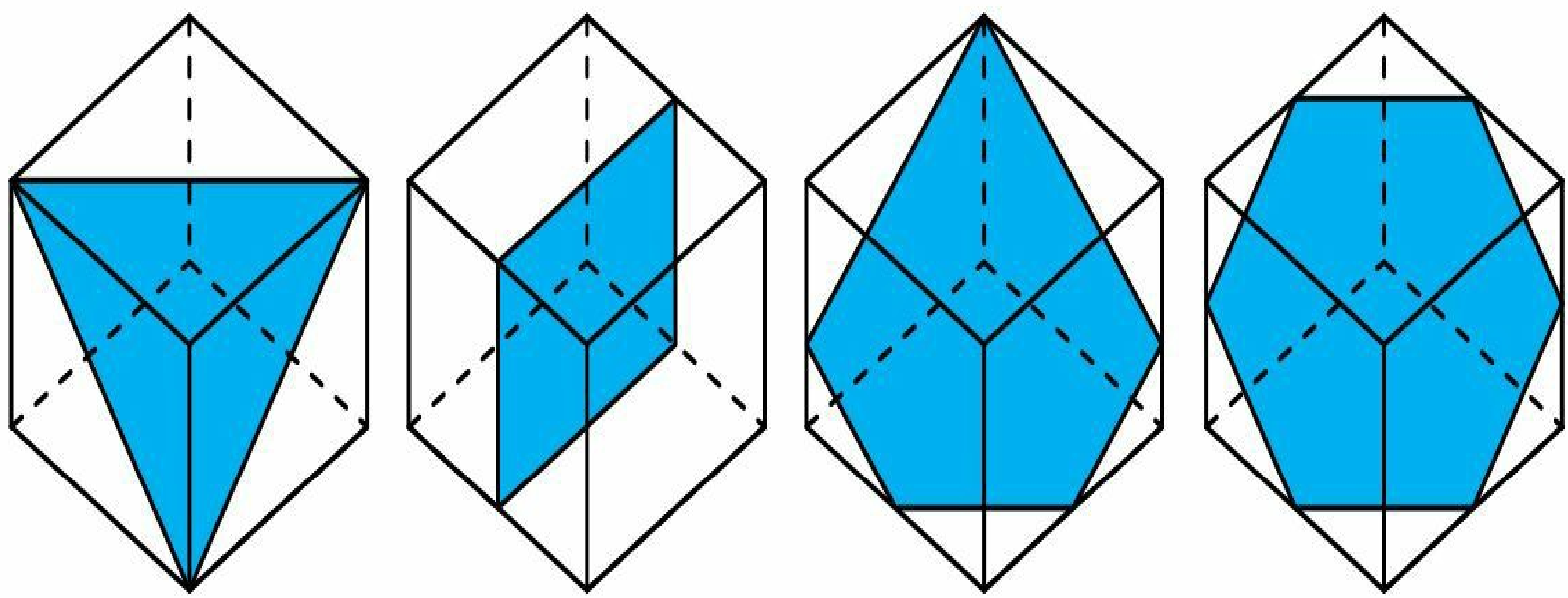
As I said, the monkey's not going to cut open a person. Now *that* would be violent. Whatever made you think of that, anyhow? What I have in mind is a monkey slicing through a block of cheese. If you feel that slicing a block of cheese with a butcher's knife is in any way violent, how in the world do you eat? Cutting the cheese might stink, but it's surely not violent.

The concept of a cross section can be illustrated by having a monkey slice salami (don't worry, we'll get to the cheese; would you like some salami to go with your cheese?). Think of the salami as a right-circular cylinder (so we'll assume that the ends are flat circles). If the monkey slices the salami perpendicular to its axis, the cross section will be a circle. The cross section is the shape of the surface of the salami where it's cut. If the cut is at an angle, the cross section will be an ellipse instead of a circle. Alternatively, if the cylinder is sliced parallel to its axis, the cross section will be a rectangle.

Let's begin with 1D string cheese. If it's a purely 1D line, the cross section will simply be a 0D point.

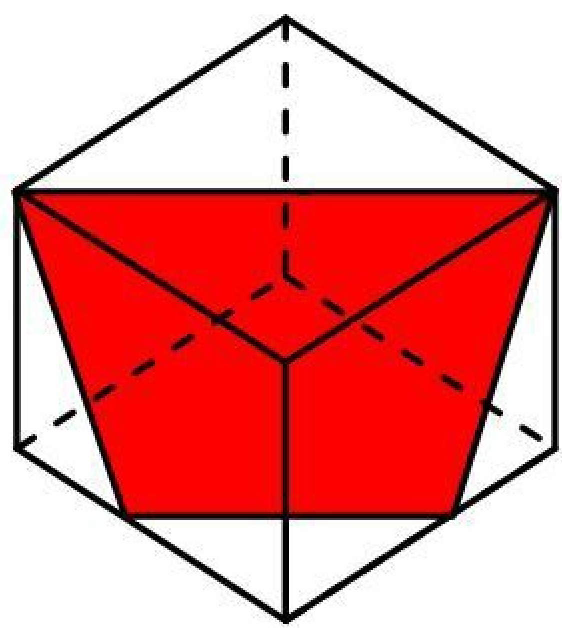
We next move up to a 2D slice of American cheese. When the monkey slices this 2D square, the cross section will be a 1D line. The line can be as long as the diagonal of the square, or so short it becomes as small as a point.

Now imagine that the monkey has a 3D cube of cheddar cheese. There are a variety of ways that the monkey can slice this cube. The monkey could slice the cube perpendicularly so that the cross section is a square. If the slice is angled a little, the cross section will instead be a rectangle. A triangular cross section results if the monkey slices off a corner of the cube of cheddar cheese. The largest possible triangular cross section is illustrated below; in this case, the slice cuts through three corners. It's possible to slice the cube in such a way as to produce a cross section in the shape of a pentagon (5 edges), with a slight variation to the triangular cross section cut (the difference is that the bottom of the cross section cuts below the corner, as shown below). The maximum number of edges on the cross section's polygon is 6: A regular hexagon results when all 6 points of the polygon intersect one of the cube's edges.



various 2D cross sections of cubes

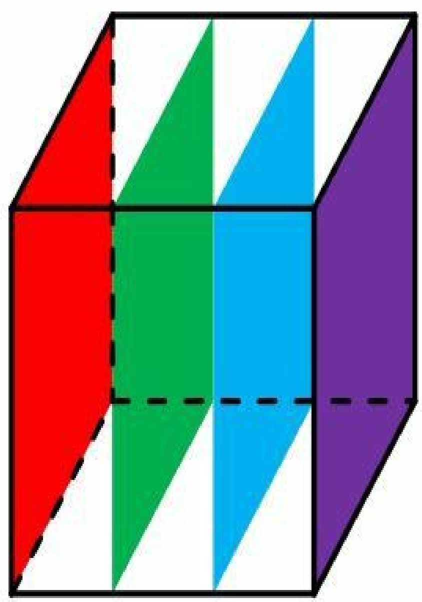
The monkey can even slice the cube of cheddar cheese to create a trapezoid cross section, as seen in the figure that follows.



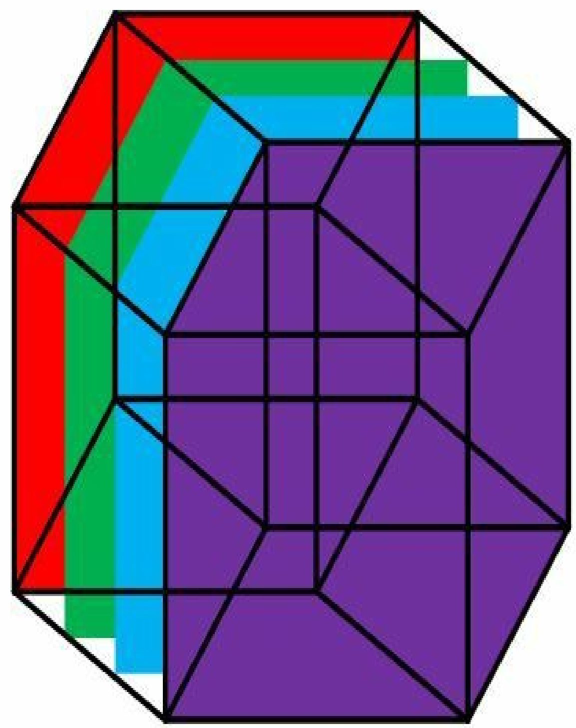
Ready or not, we will now consider a hypermonkey (a 4D monkey) with a hyperknife (yep, a

4D "knife") slicing a tesseract of mozzarella hypercheese in such a way that the cross section is, in general, a 3D polyhedron lying in a 3D hyperplane. The cross section is the intersection of an infinite hyperplane (such as *xyz*, if the hypermonkey slices the hypercheese along the *w*-axis, but it could be any hyperplane, not necessarily *xyz*, *yzw*, *zwx*, or *wxy* – i.e. the hyperplane could be tilted) and the tesseract. This is analogous to a 3D monkey slicing a 3D cube of cheddar cheese, in which case the cross section is the intersection of an infinite plane (such as, but not necessarily, *xy*) and the cube.

If a monkey slices a 3D block of cheddar cheese a few times, each time parallel to one face, the cross sections will be parallel squares, as illustrated below. Similarly, if a hypermonkey slices a 4D tesseract of mozzarella hypercheese a few times, each time parallel to one bounding cube, the cross sections will be parallel cubes, as illustrated below. None of the 4 cross section cubes shown are touching – in fact, there is space between each, just like the space between parallel squares in the left diagram.

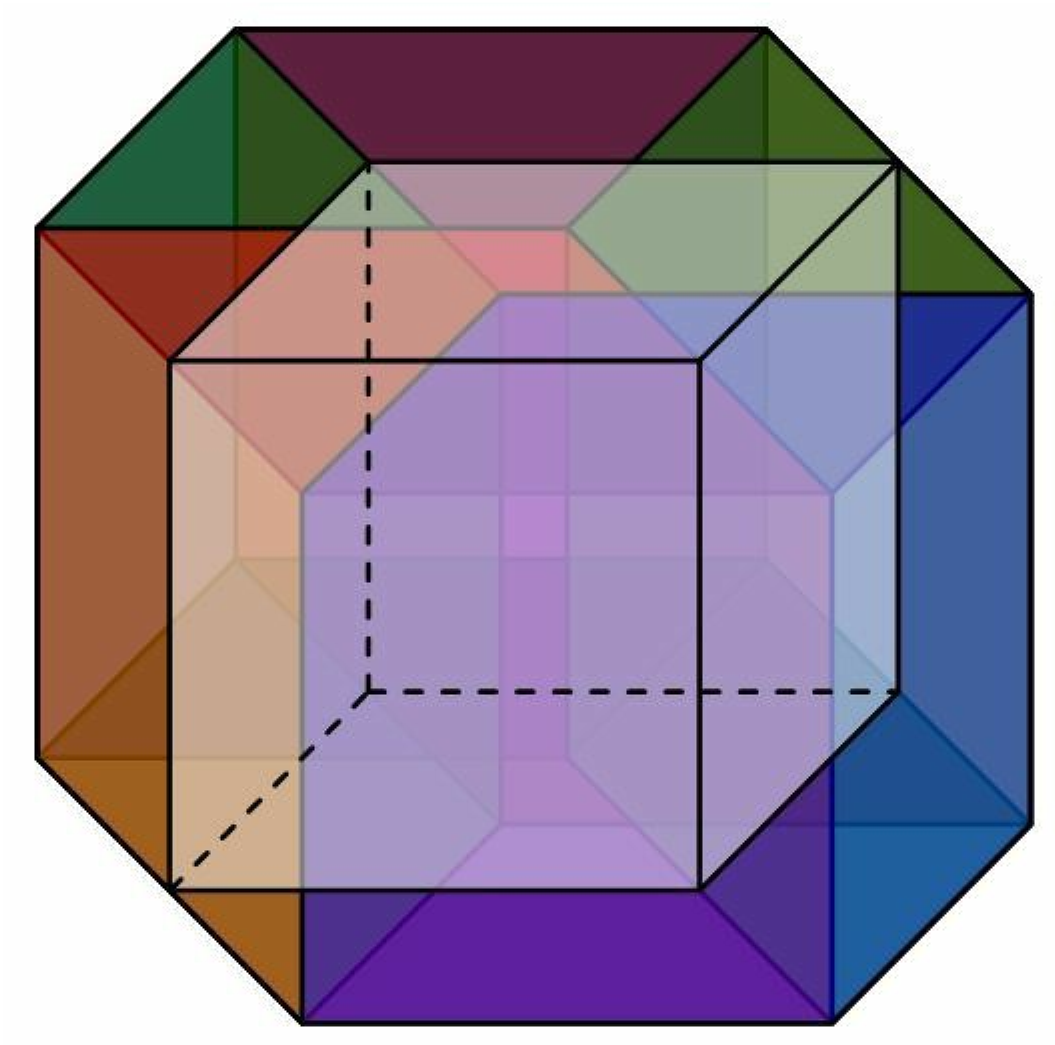


parallel 2D cross sections

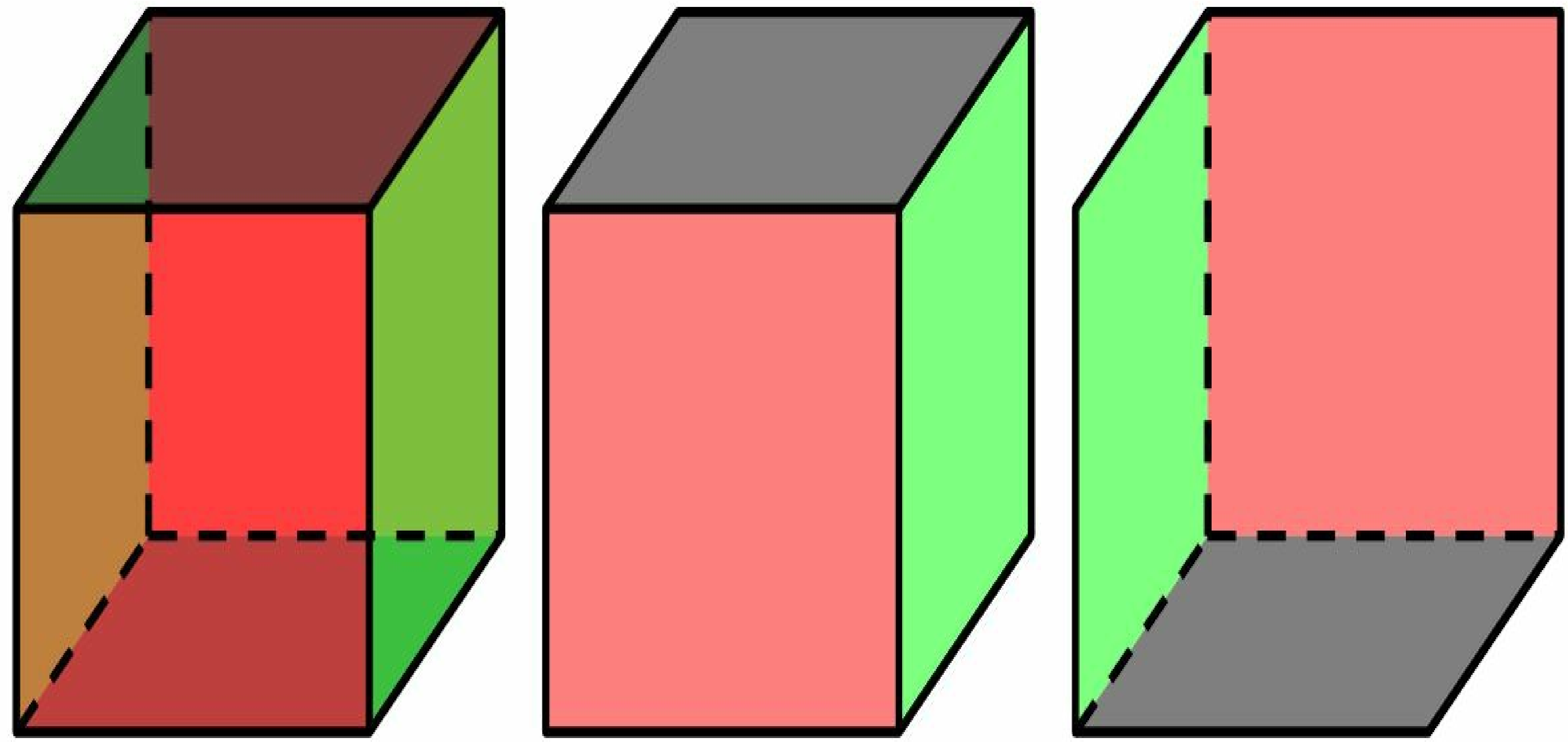


parallel 3D cross sections

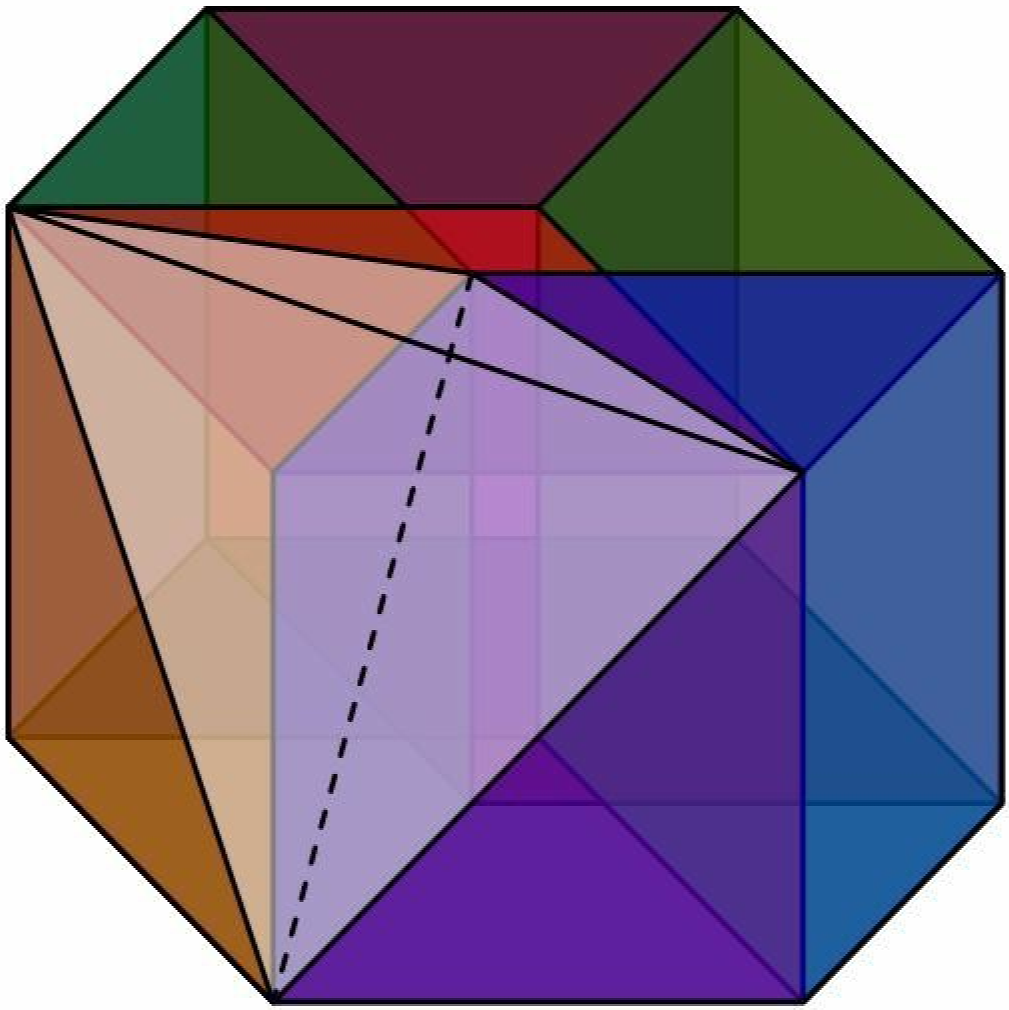
The following figure shows a tesseract of mozzarella hypercheese that a hypermonkey cut symmetrically, right down the middle, perpendicular to one of the bounding cubes. The cross section is a cube, which has the same dimensions as the tesseract, except for being infinitesimal in the fourth dimension.



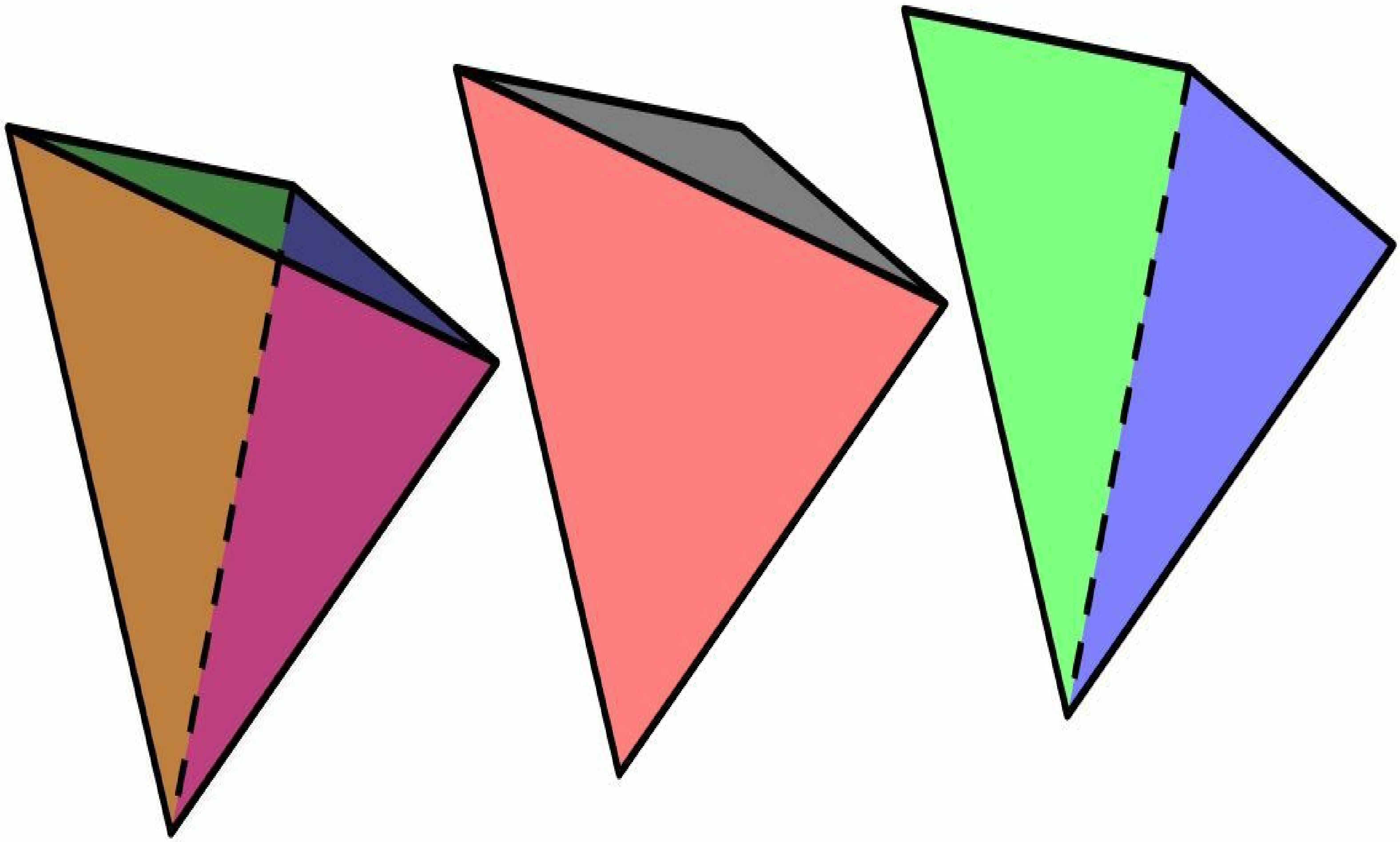
The cube from the previous figure is illustrated below, showing its front and back faces. This was done for consistency, since the other pictures of cross sections feature a similar follow-up image of the cross section; it probably wasn't so useful for the cube, but may be helpful in the other cases that follow.



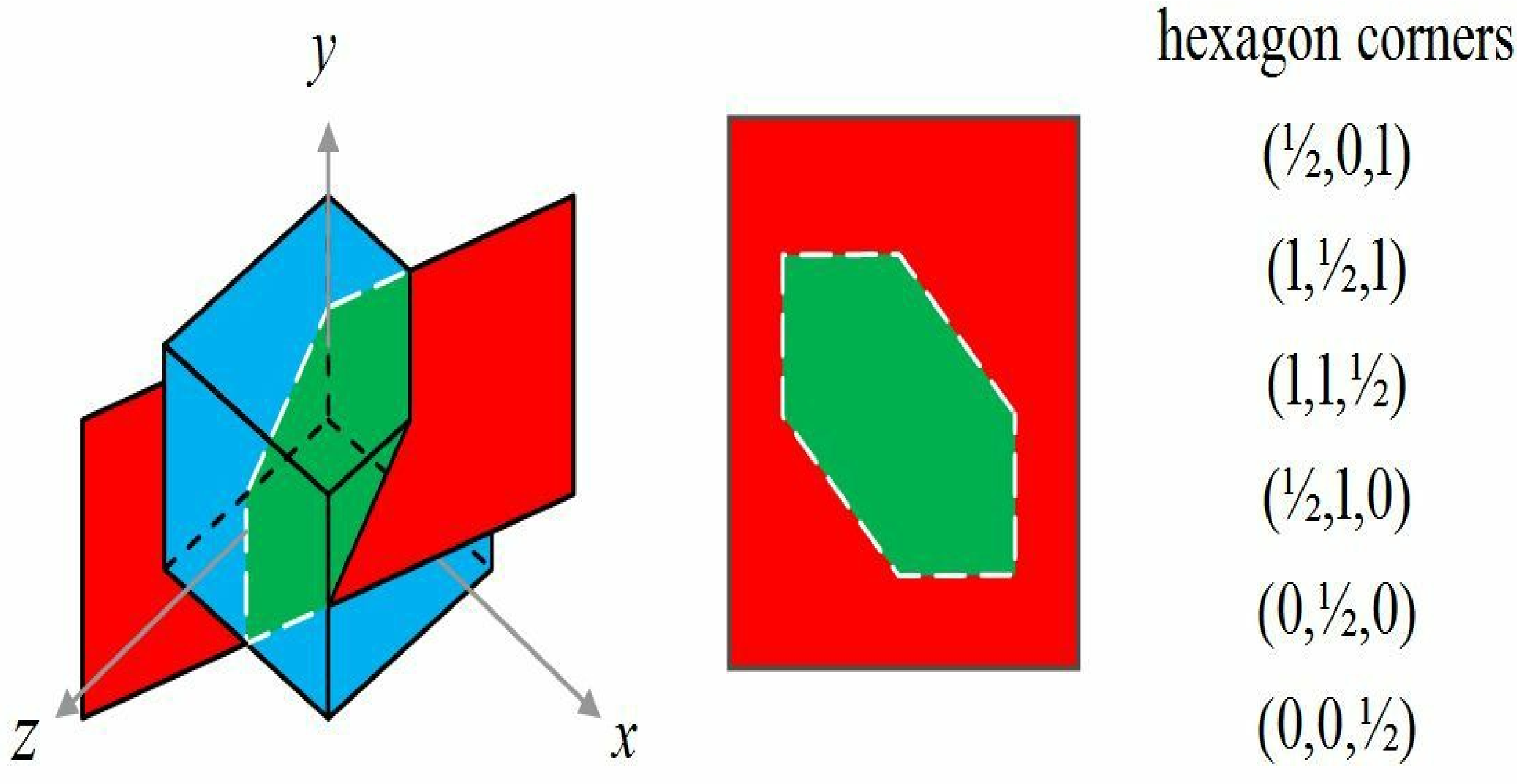
In the following picture, a hypermonkey sliced off one corner of a tesseract of mozzarella hypercheese. The cross section is a tetrahedron (which consists of 4 triangles – it's <u>not</u> the same as a pyramid, as a pyramid has 4 triangles connected to a square base). This is analogous to a monkey slicing off the corner of a cube, where the cross section is a triangle (there was a picture of this earlier in this chapter).



You can see the front and back sides of the tetrahedron in the figure that follows.

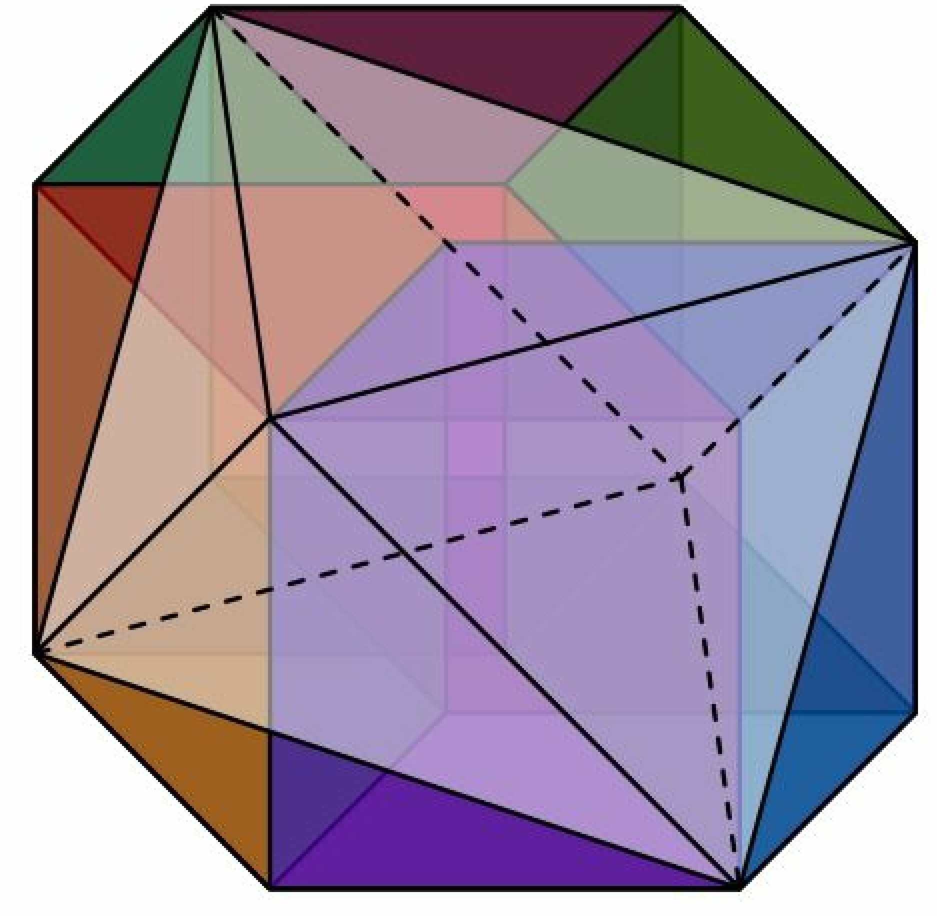


The next figure shows another picture of the hexagonal cross section of the cube, which bisects 6 different edges. The coordinates of where the plane bisects the edges include (½,0,1), (1,½,1), (1,1,½), (½,1,0), (0,½,0), and (0,0,½). Why are we reviewing this? Because we will next discuss a cross section of a tesseract that is a generalization of this.

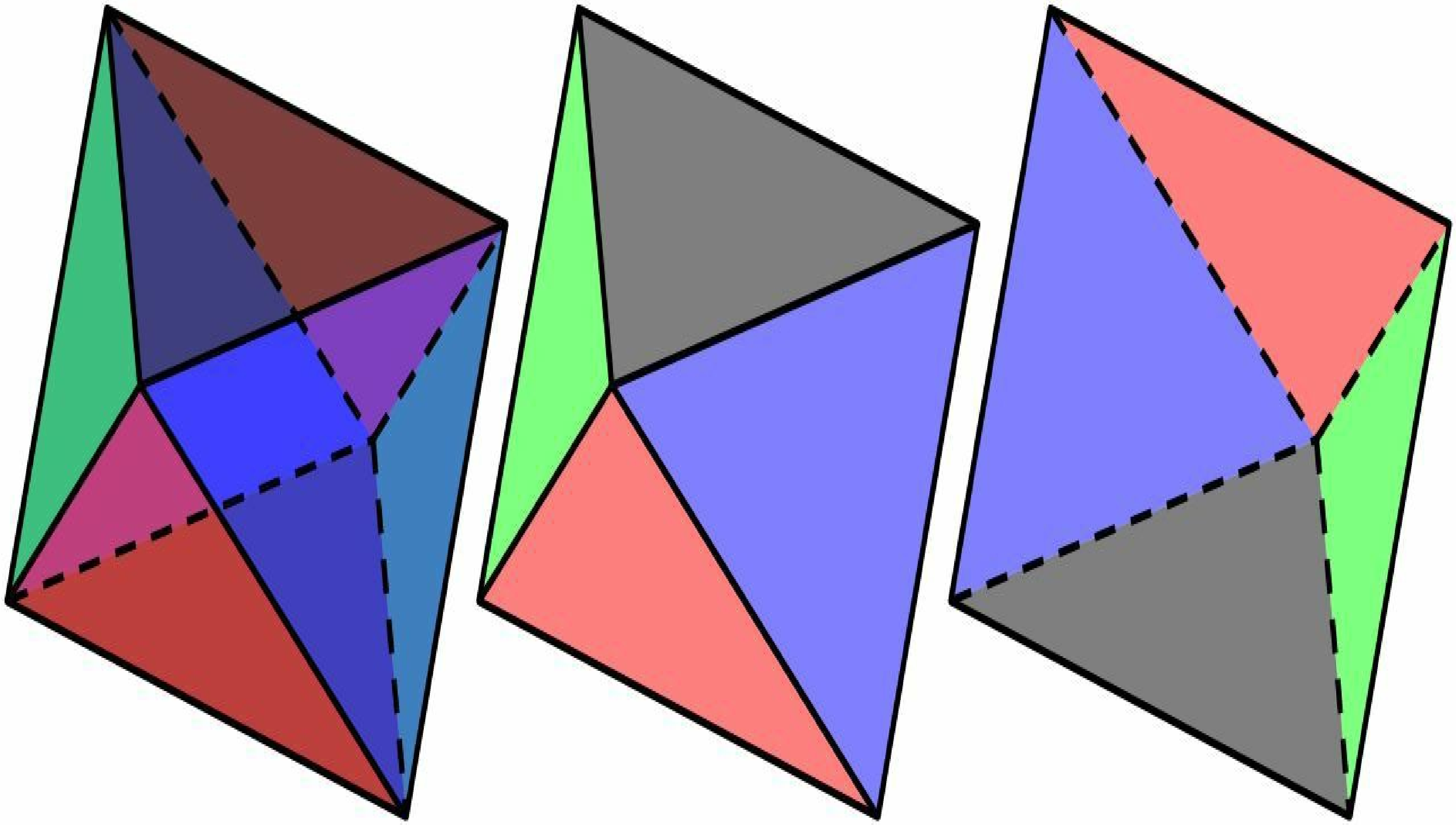


hexagonal cross section

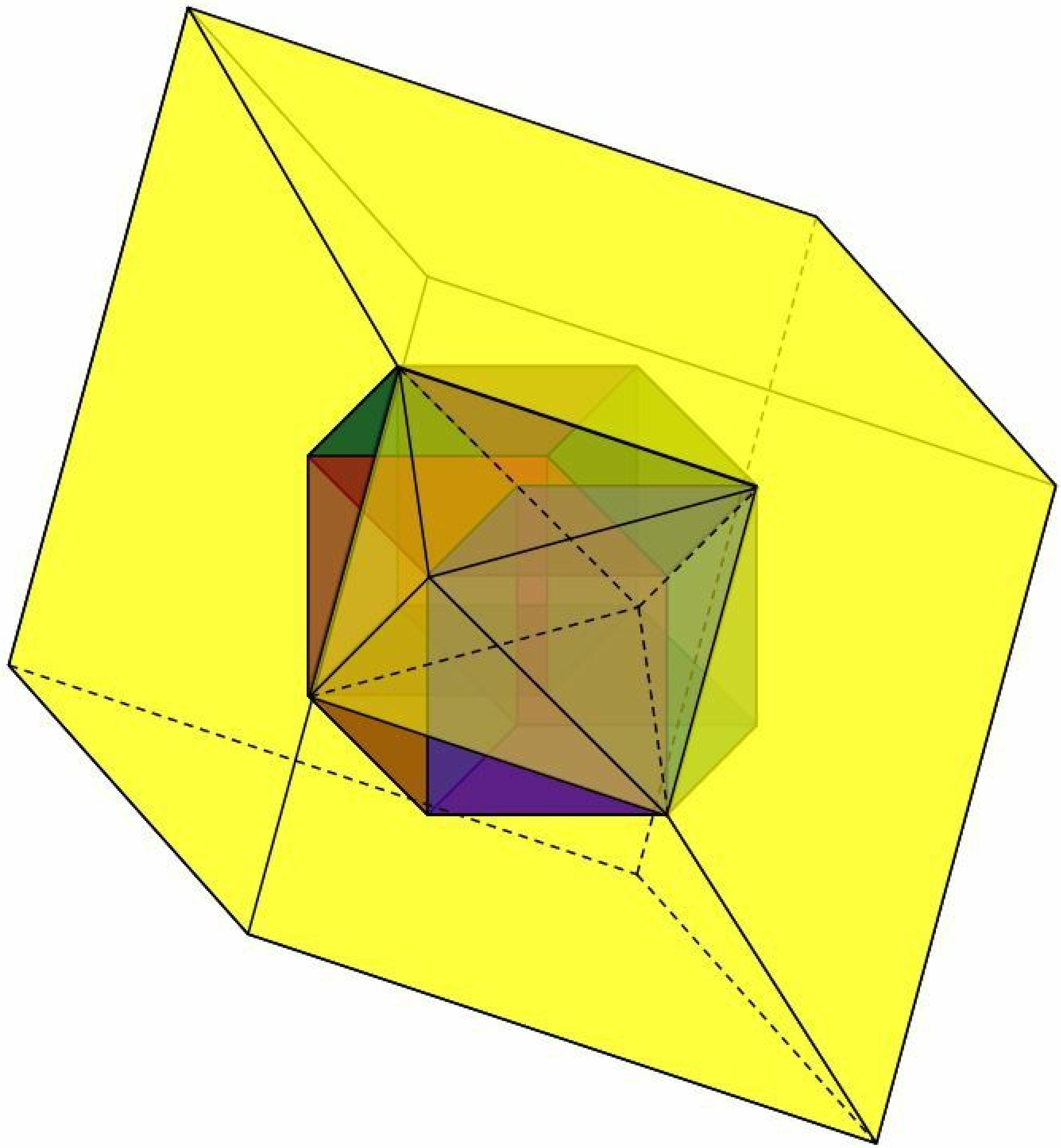
The cross section of a cube is a polygon that can have up to 6 edges (hexagon). The cross section of a tesseract is a polyhedron that can have up to 8 sides (octahedron). A hypermonkey has sliced the following tesseract of mozzarella hypercheese in such a way as to produce a cross section that's an octahedron. The octahedron is a polyhedron that has 8 triangular faces. You could make it by gluing the square faces of two pyramids (not tetrahedra) together, so that it looks like a 3D diamond.



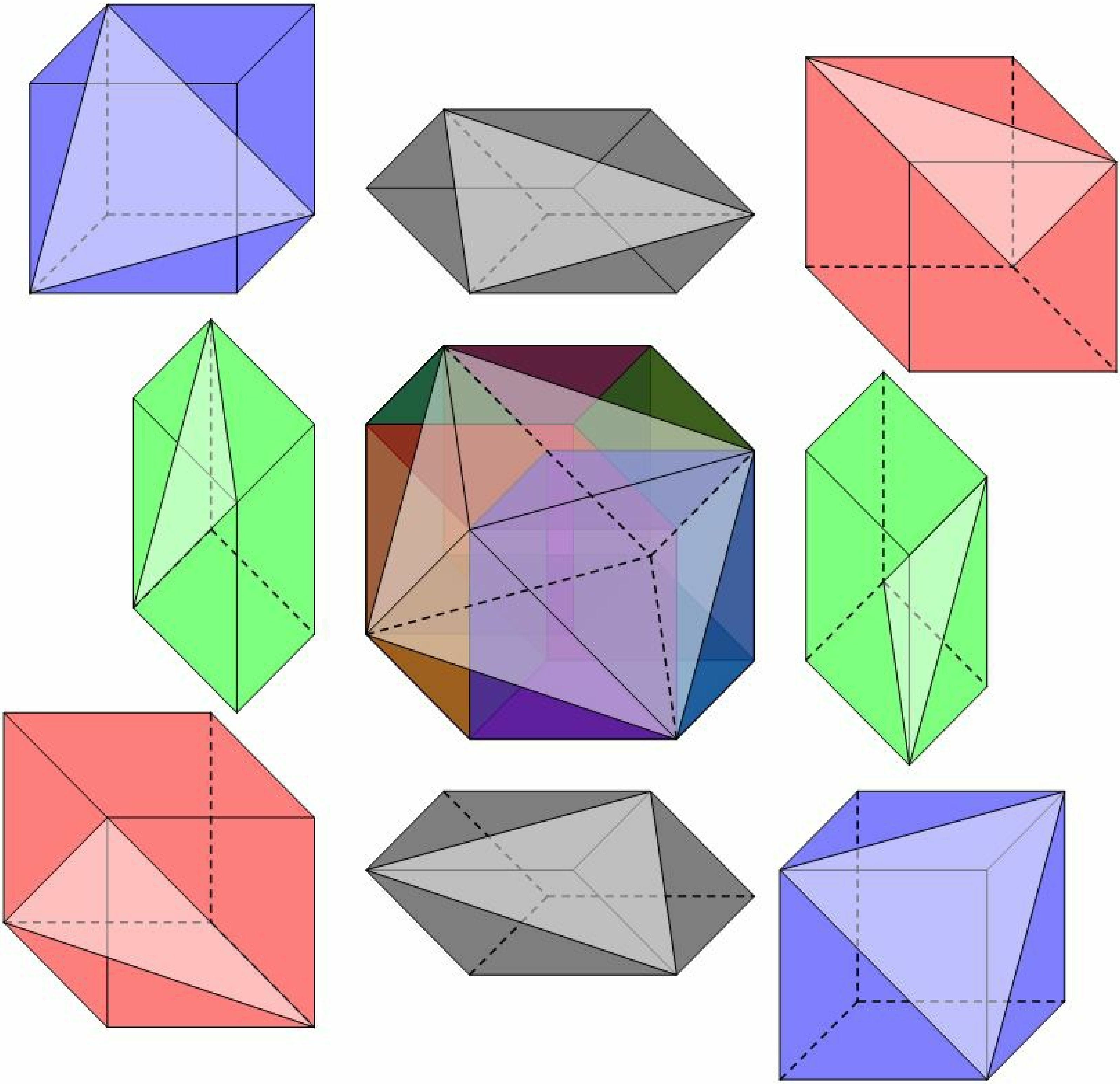
The following figure shows the front and back pyramids that form the octahedron.



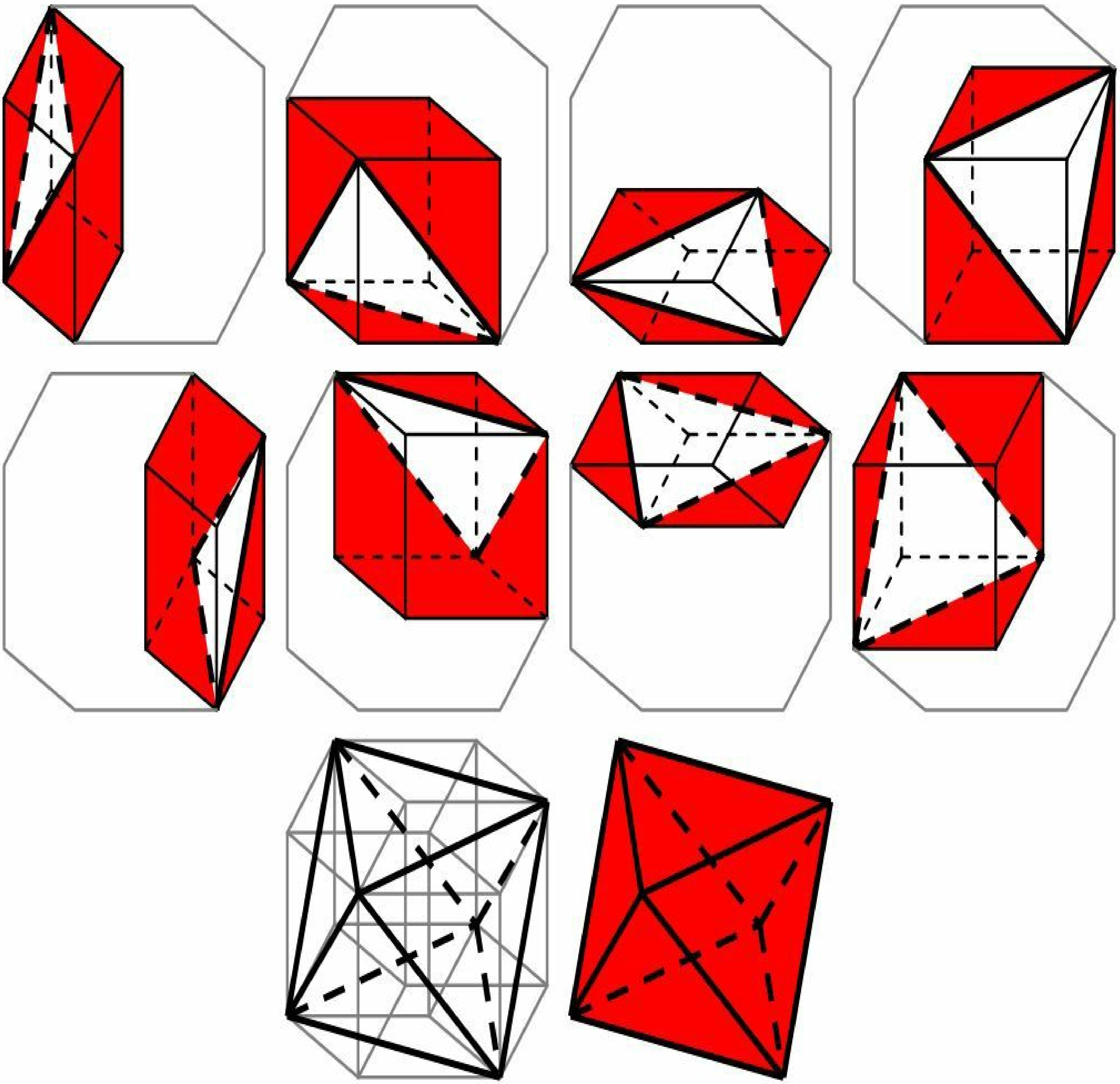
Below we see an infinite hyperplane intersecting a tesseract. The cross section (region of intersection) is the octahedron. This is no different than the way the hypermonkey sliced the tesseract of mozzarella hypercheese in the previous figure.



Here we see the 8 triangular faces that make up the octahedron; each face lies in one of the 8 bounding cubes. (Compare this to the previous hexagon cross section of the cube, where each edge of the hexagon lies in one of the 6 bounding square faces.)

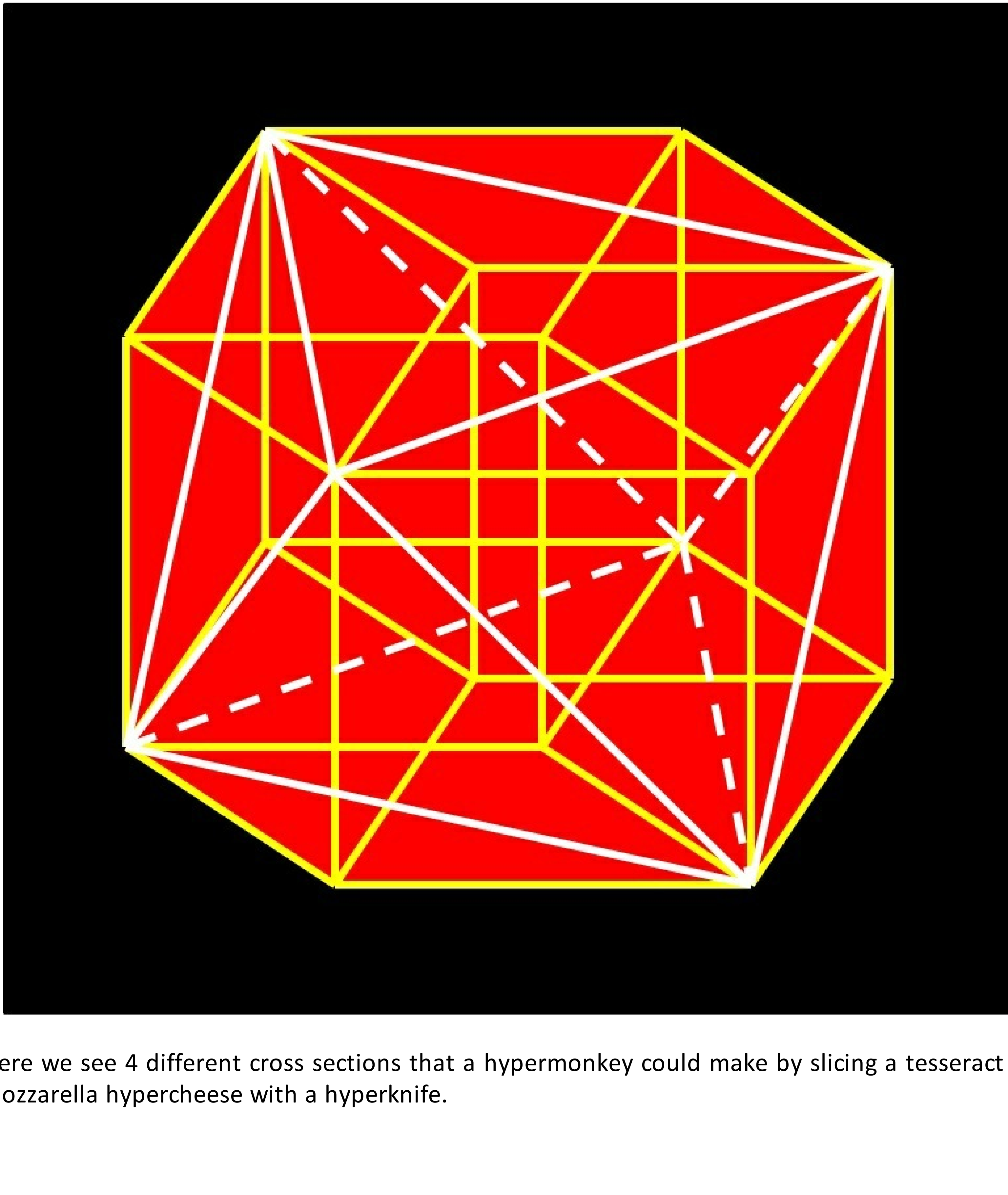


Next is a different form of the same picture.

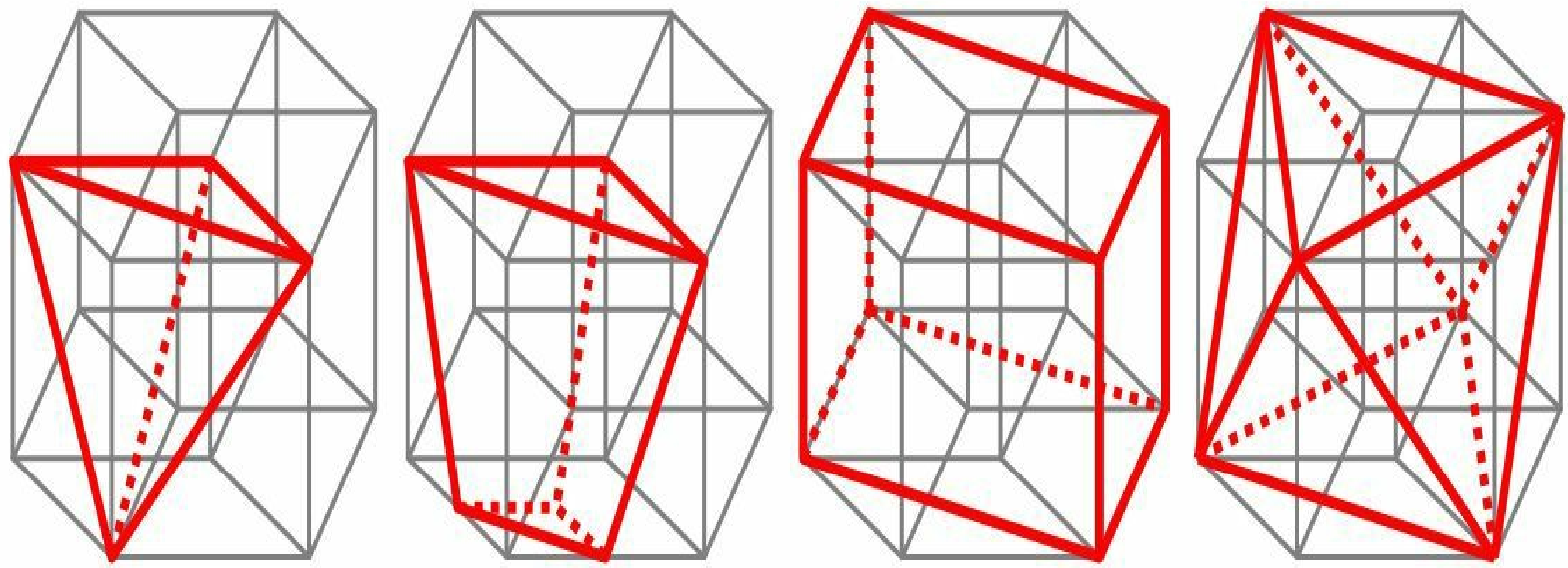


each side of a 3D cross section of a tesseract lies in one of the 8 bounding cubes

Following, the octahedral cross section is shown in a large picture. The tetrahedron has pink edges and the octahedron has white edges.



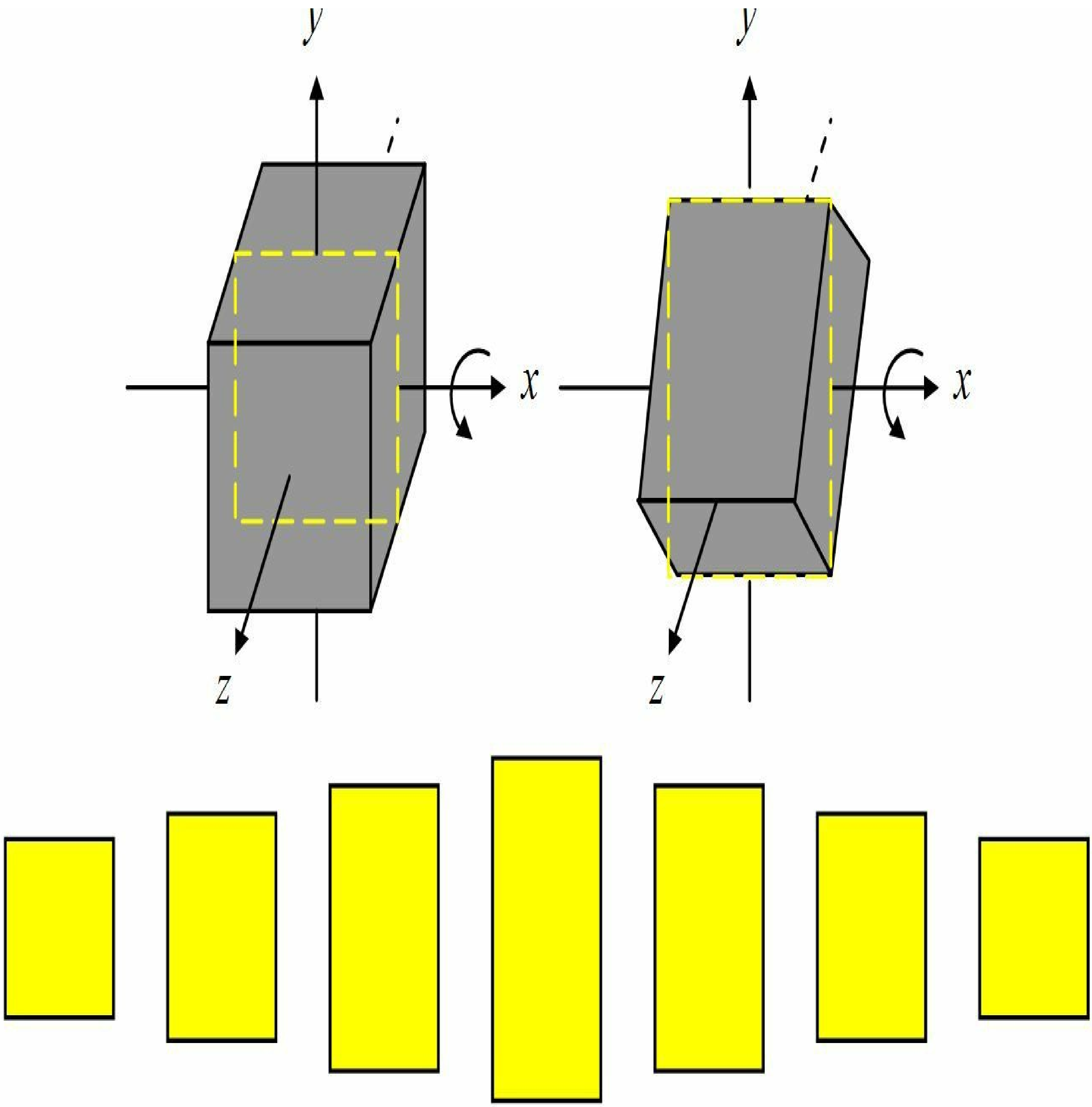
Here we see 4 different cross sections that a hypermonkey could make by slicing a tesseract of mozzarella hypercheese with a hyperknife.



various 3D cross sections of a tesseract

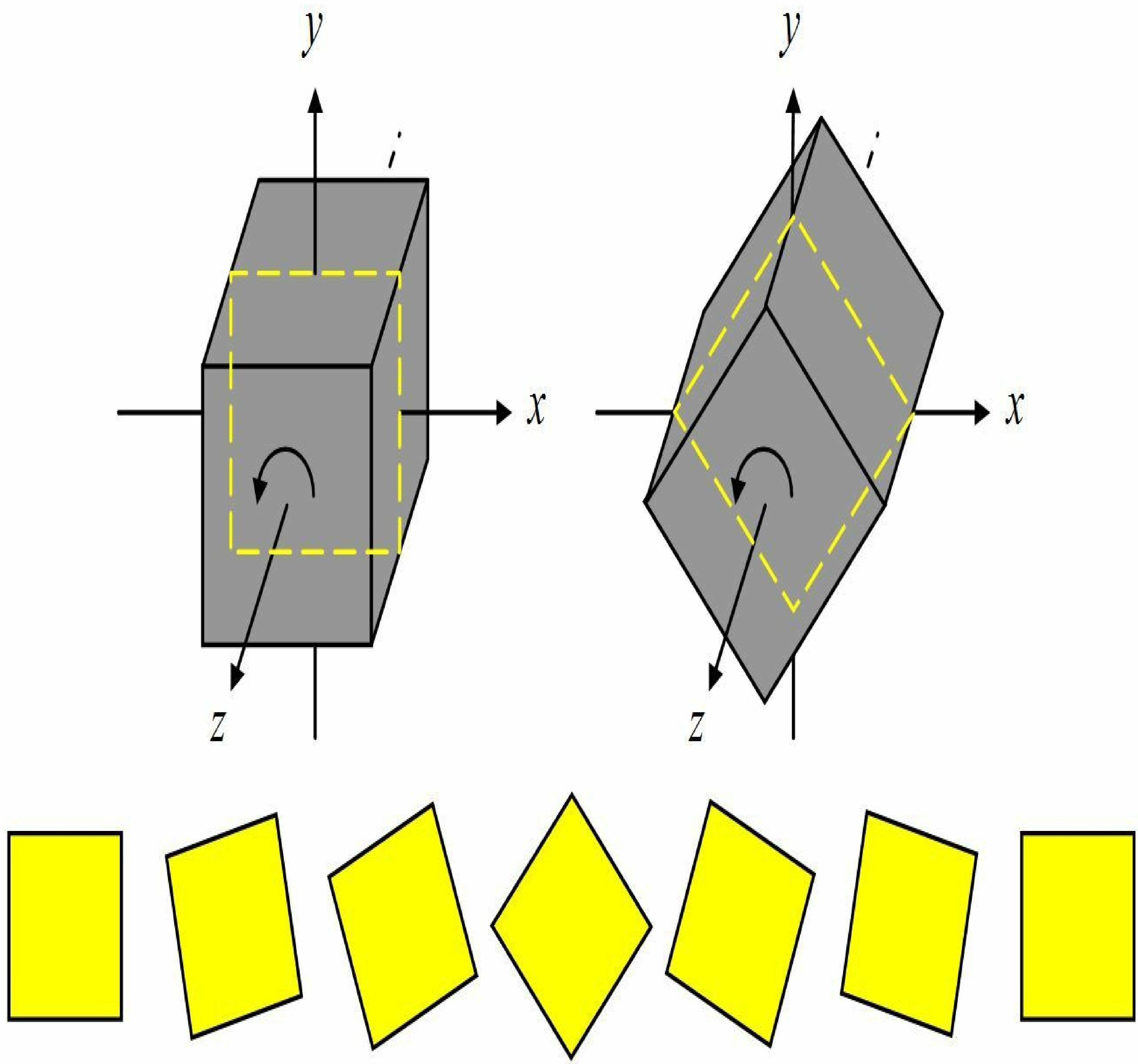
Our next challenge is to visualize a higher-dimensional object traveling through a lower-dimensional world. The lower-dimensional monkeys would see just a cross section of the higher-dimensional object, and the cross section would change as the object moves.

For example, imagine 2D monkeys living in a purely 2D world, which we will take to be the *xy* plane, while a 3D monkey rotates a cube in 3D space. Such a case is illustrated below. The cross section that the 2D monkeys see is the intersection between the *xy* plane and the cube; this cross section is drawn in a dashed yellow line. As the 3D monkey rotates the cube about the *x*-axis, the cross section remains rectangular, but changes in size. The 3D monkey would see the rotating cube pictured at the top, but all the 2D monkeys would see is the rectangle changing size pictured at the bottom. (Also, the 2D monkeys would see this rectangle from within the *xy* plane, so it wouldn't look the same to them as you're seeing it now. You're looking at the rectangles from the third dimension, *z*. You would also see the "insides" of the 2D monkeys; 2D monkeys would have no privacy from a 3D observer. By analogy, a 4D being could see your insides, too! You would have no privacy from a 4D observer.)



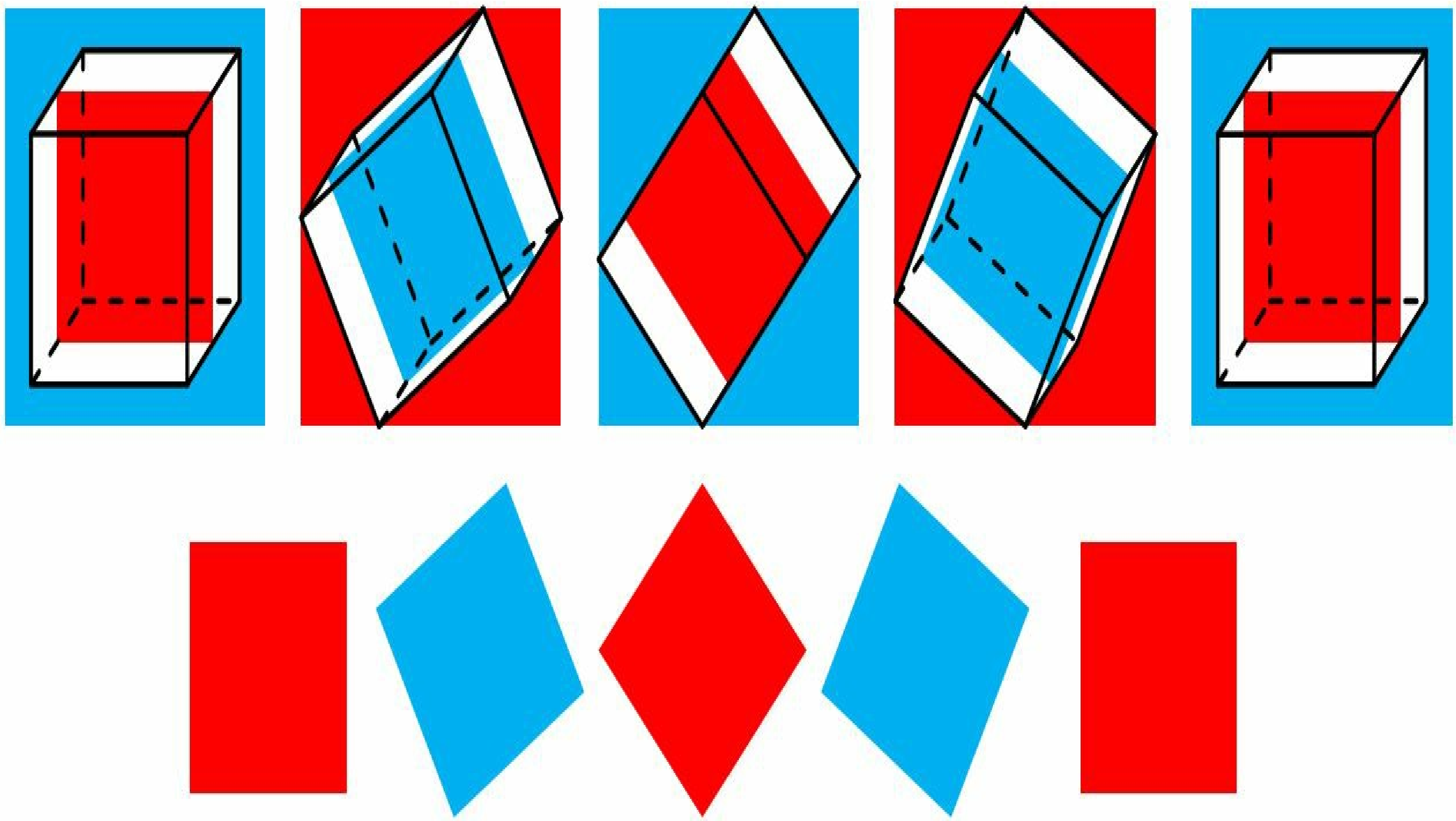
cube rotating about the x-axis

The following figure shows a similar situation, except this time the 3D monkey rotates the cube about the z-axis. In this case, the cross section is a square; the square rotates in a circle, but does not change size.

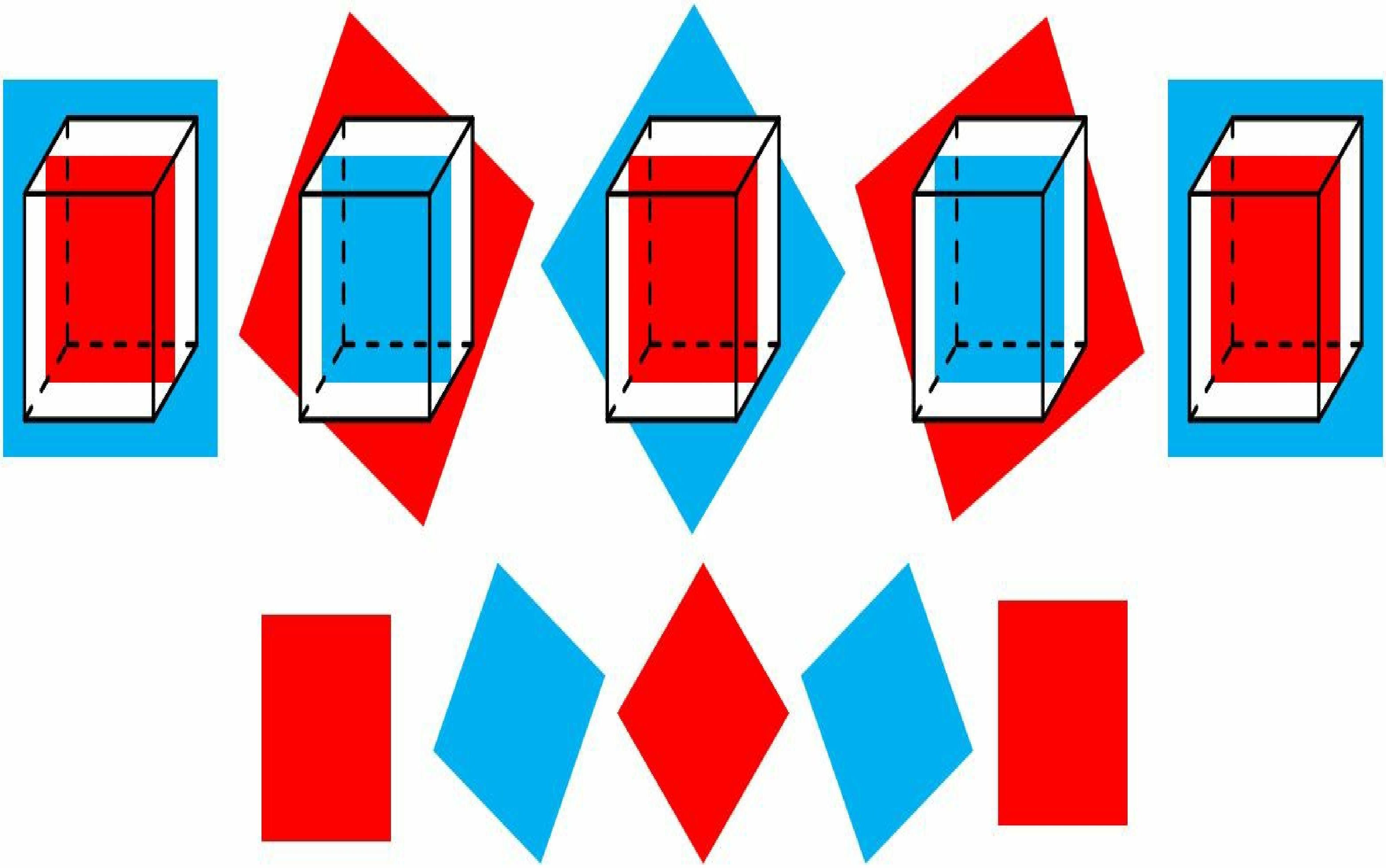


cube rotating about the z-axis

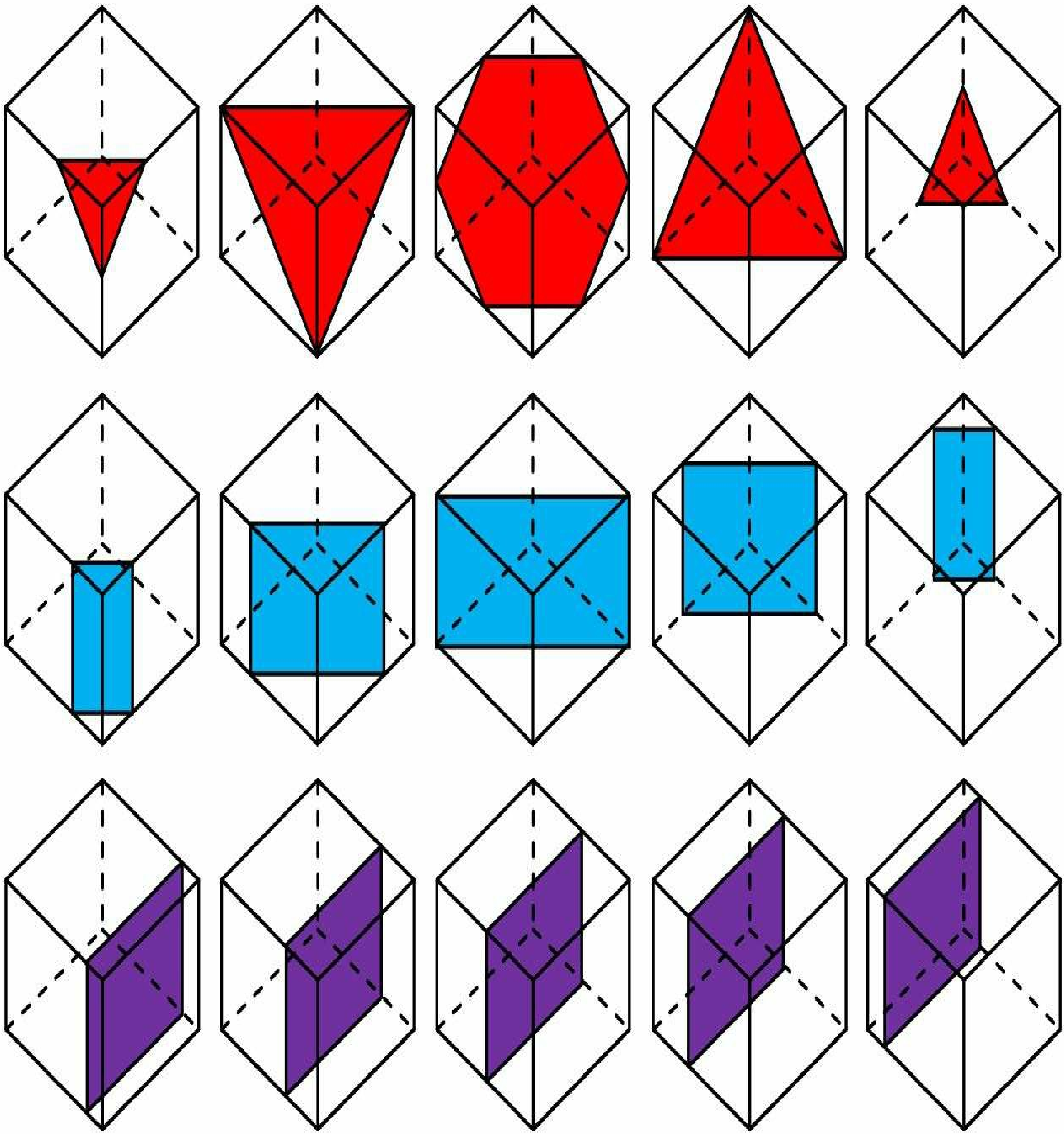
Here is a different picture of the same situation.



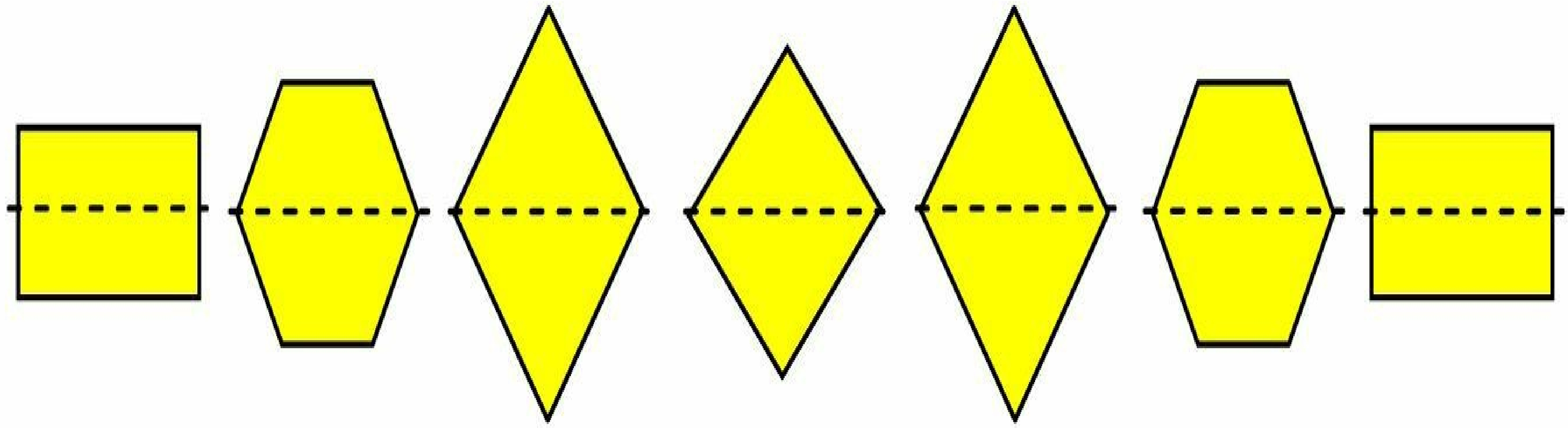
In the next diagram, the 3D monkey rotates the 2D universe instead of rotating the cube. This effectively achieves the same result.



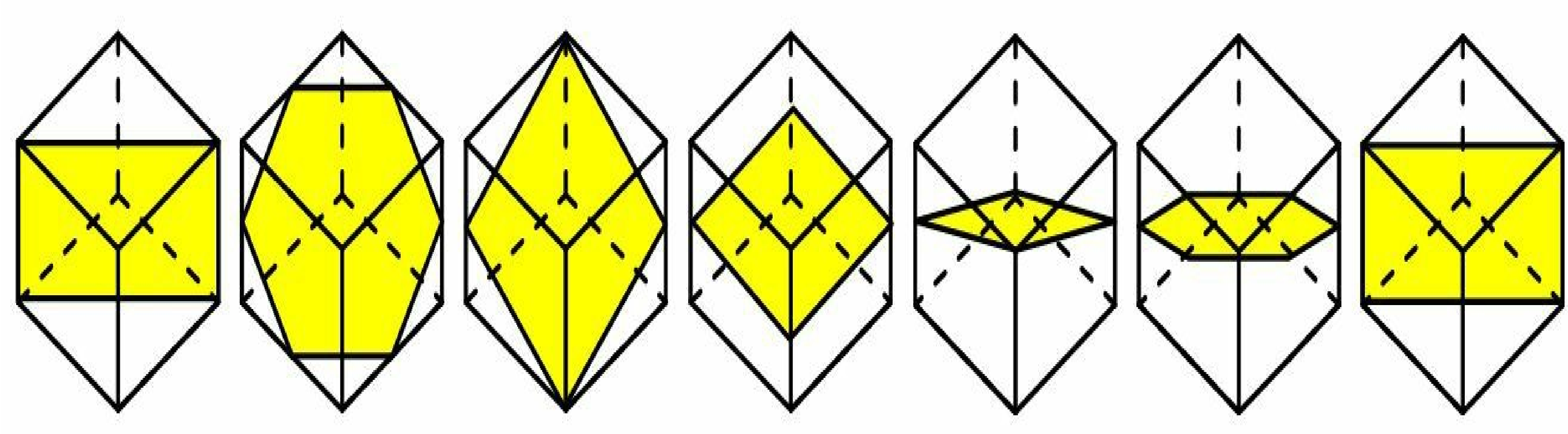
The following cubes are traveling in a straight line (instead of rotating) through a 2D world. Read the pictures left to right. The top figures show a cube that is moving along its body diagonal. The cross section starts out as a growing triangle, becomes a hexagon, and then a diminishing triangle. The middle figure shows a cube moving along its face diagonal (as opposed to the body diagonal). This produces a rectangle that changes size, but is distinct from a rotation that we previously considered that produced a similar cross section. The difference is that this time the cross section begins and ends as a line (whereas previously the rectangle was always finite in width), and also the process is not cyclic (in the rotating case, the pattern repeated over and over). The bottom figure shows a cube passing through the 2D world parallel to one face. In this case, the cross section is a stationary square that suddenly appears as the cube enters and vanishes as it leaves. Such awe! The third dimension allows 3D monkeys to perform magic in the second dimension. It's mathe-magical!



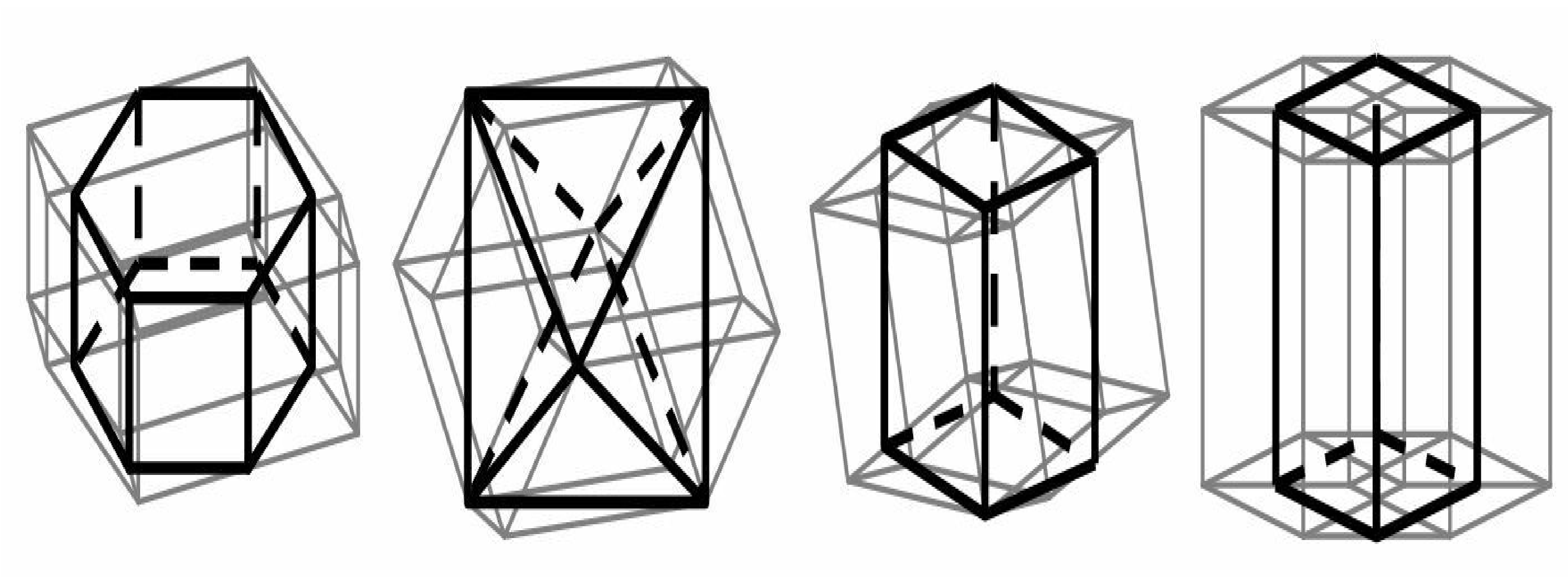
Here is one more puzzle to ponder. Is it possible for a cube to travel through the second dimension – either in a straight line or by rotating – in such a way as to produce the cross sections drawn below from left to right? If so, figure out how.



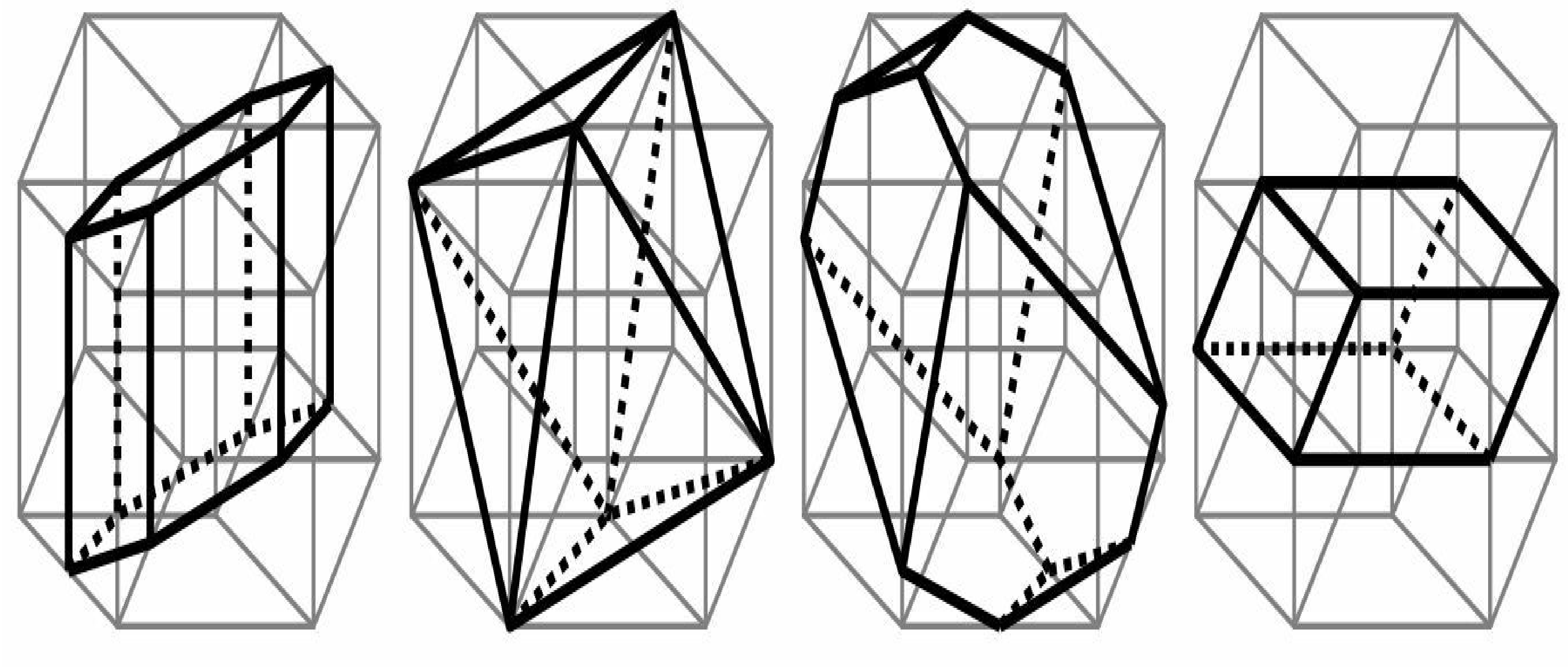
Here comes the solution. There is only one way to make the regular hexagon, and we've seen this already: Each corner of the hexagon will bisect an edge. Can you make a slight change to the cube to switch from the hexagon to the rhombus? Yes. With a little rotation, the hexagon can transform into a rhombus that connects from one corner of the cube to an opposite corner. (Notice that the rhombus is taller than the hexagon in the puzzle picture above. This was a clue.) In fact, a 3D monkey could produce the above sequence of cross sections by rotating a cube through the second dimension as pictured below. If you're reading this ebook with an e-reader that respects page breaks, the solution will be shown on the following page (if not, be sure to give your e-reader a gentle spanking).



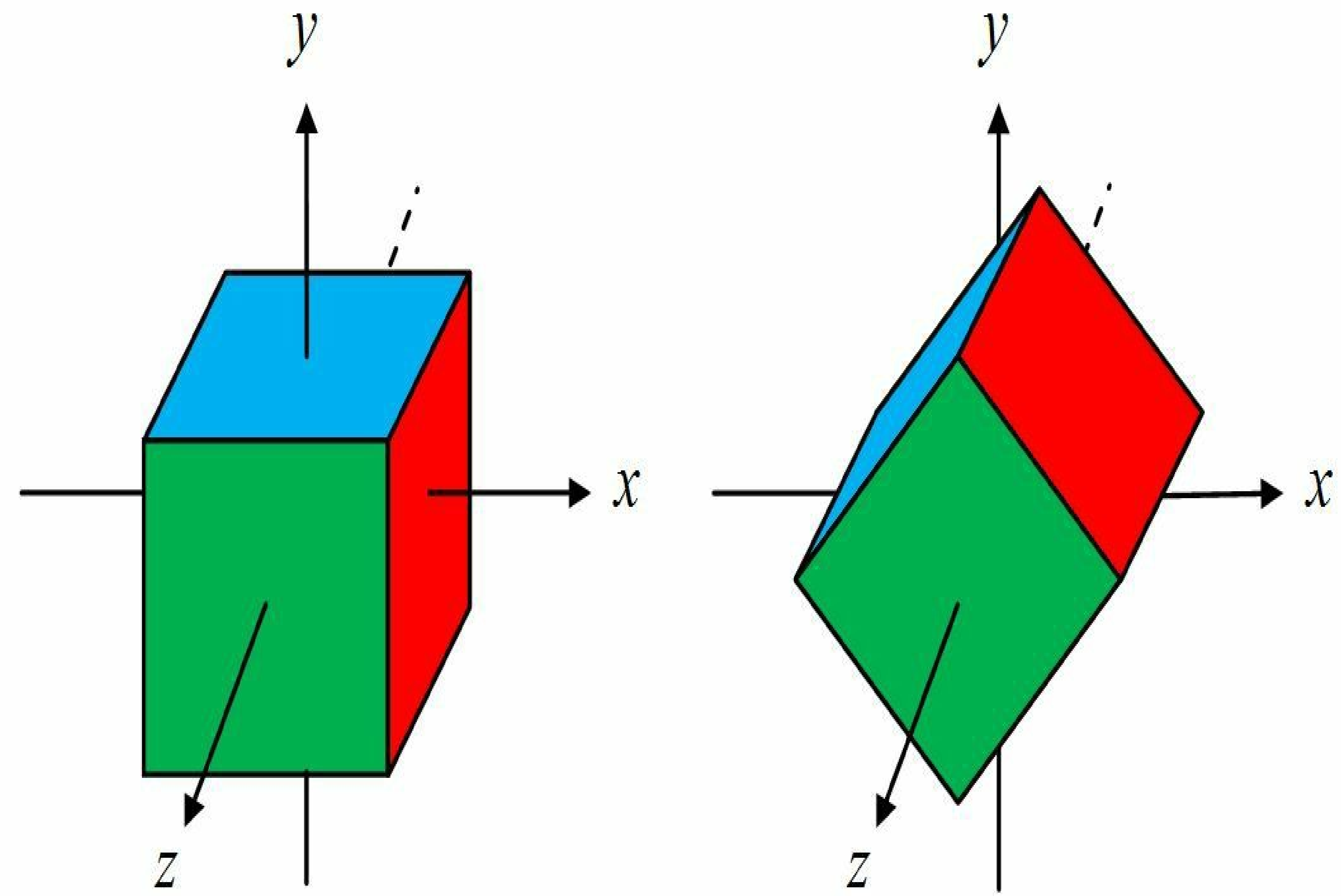
Next, we will let a hypermonkey move a tesseract through the third dimension and visualize the resulting cross sections. The following illustration shows a hypermonkey rotating a tesseract through 3D space; the bold lines show the 3D cross section that it leaves in the *xyz* hyperplane.



In the next diagram, the hypermonkey rotates the *xyz* hyperplane, leaving the tesseract fixed:

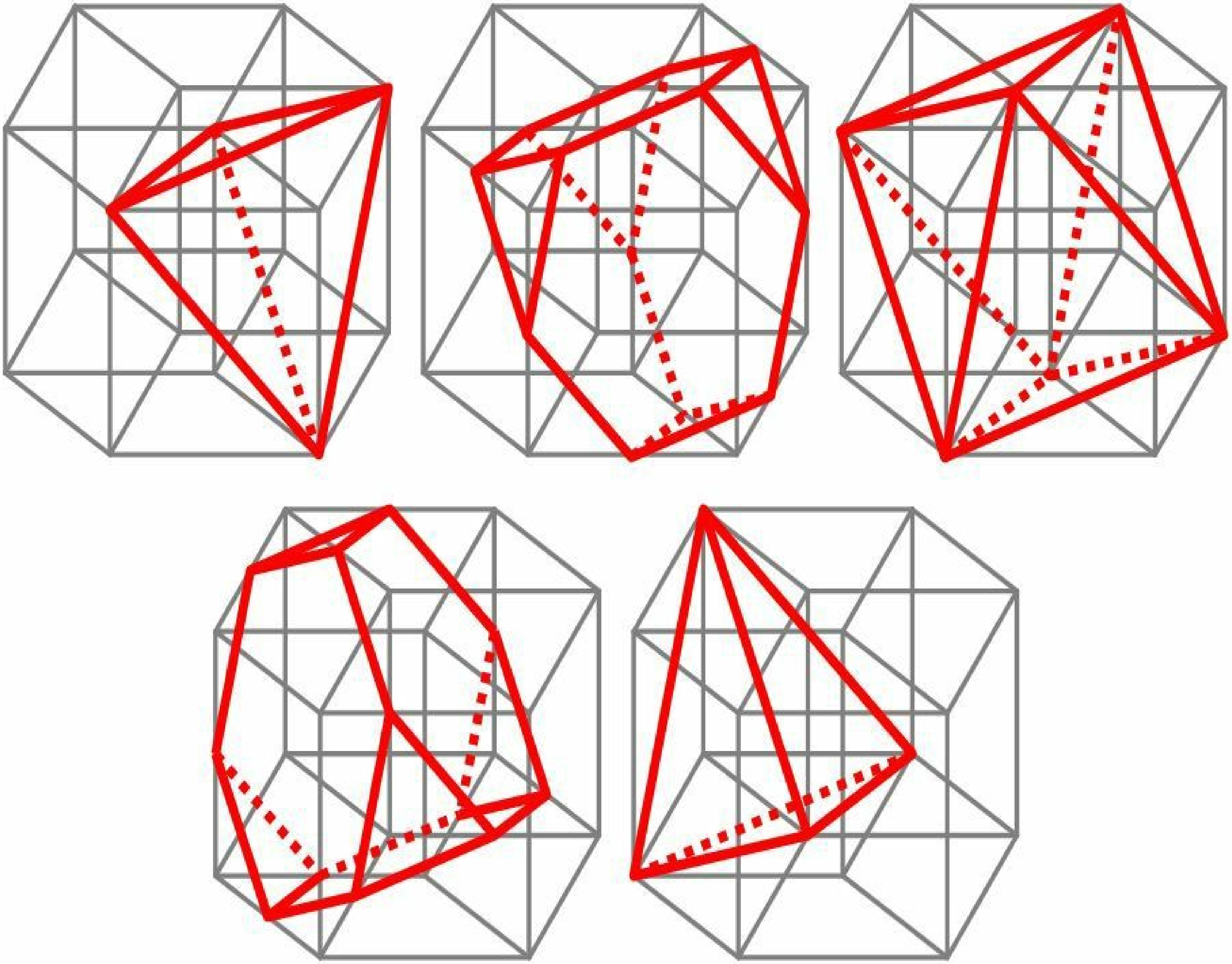


How about a simpler cross section? This time, the hypermonkey rotates the tesseract in the *xy* plane (so it's about the *wz* hyperaxis; the *wz* plane is analogous to the "axis" of rotation). The result is a cube rotating about the *z*-axis.



3D cross sections for a tesseract rotating about the *wz*-plane

Here, the hypermonkey passes the tesseract through 3D space in a straight line along its body diagonal. The cross section begins as a growing tetrahedron, maxes as an octahedron, and finishes as a diminishing tetrahedron. This is analogous to a cube passing through a 2D plane along its body diagonal (which we've seen before), for which the cross section involves a triangle and hexagon.



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Chapter 12

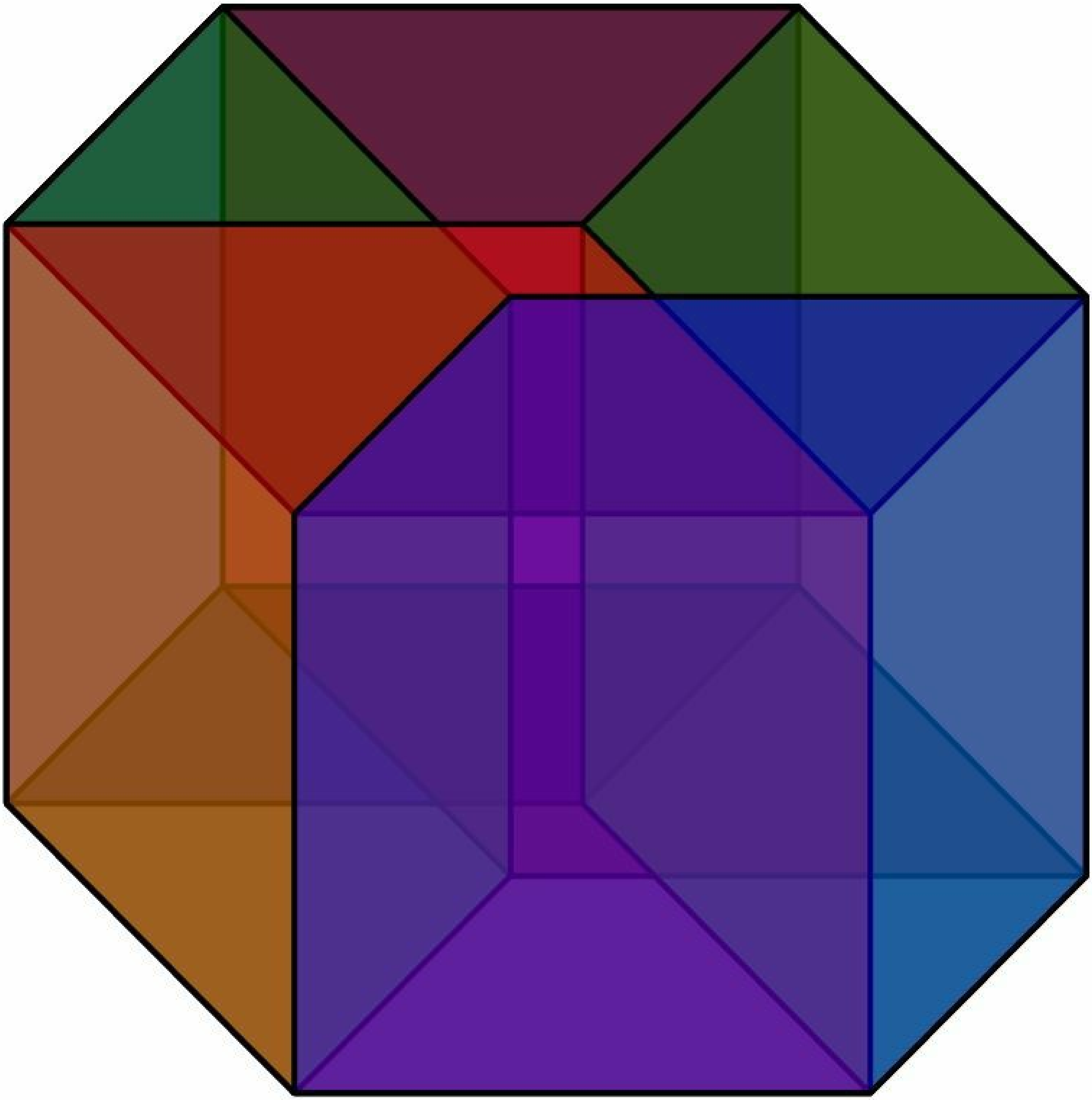
Living in a 4D House

It's a lot harder to think inside the tesseract than it is to think outside of the box! To better understand what a tesseract is, let's imagine sitting inside of one. But just a moment; for comparison, let's remind ourselves what it's like to sit inside of a cube. You may be sitting inside of a cube (or at least a cuboid – a rectangular box) right now. If so, you will see walls in front of you and behind you, to your right and left, and above and below you (well, assuming that you're facing one of the walls; and where the last two "walls" are the ceiling and floor).

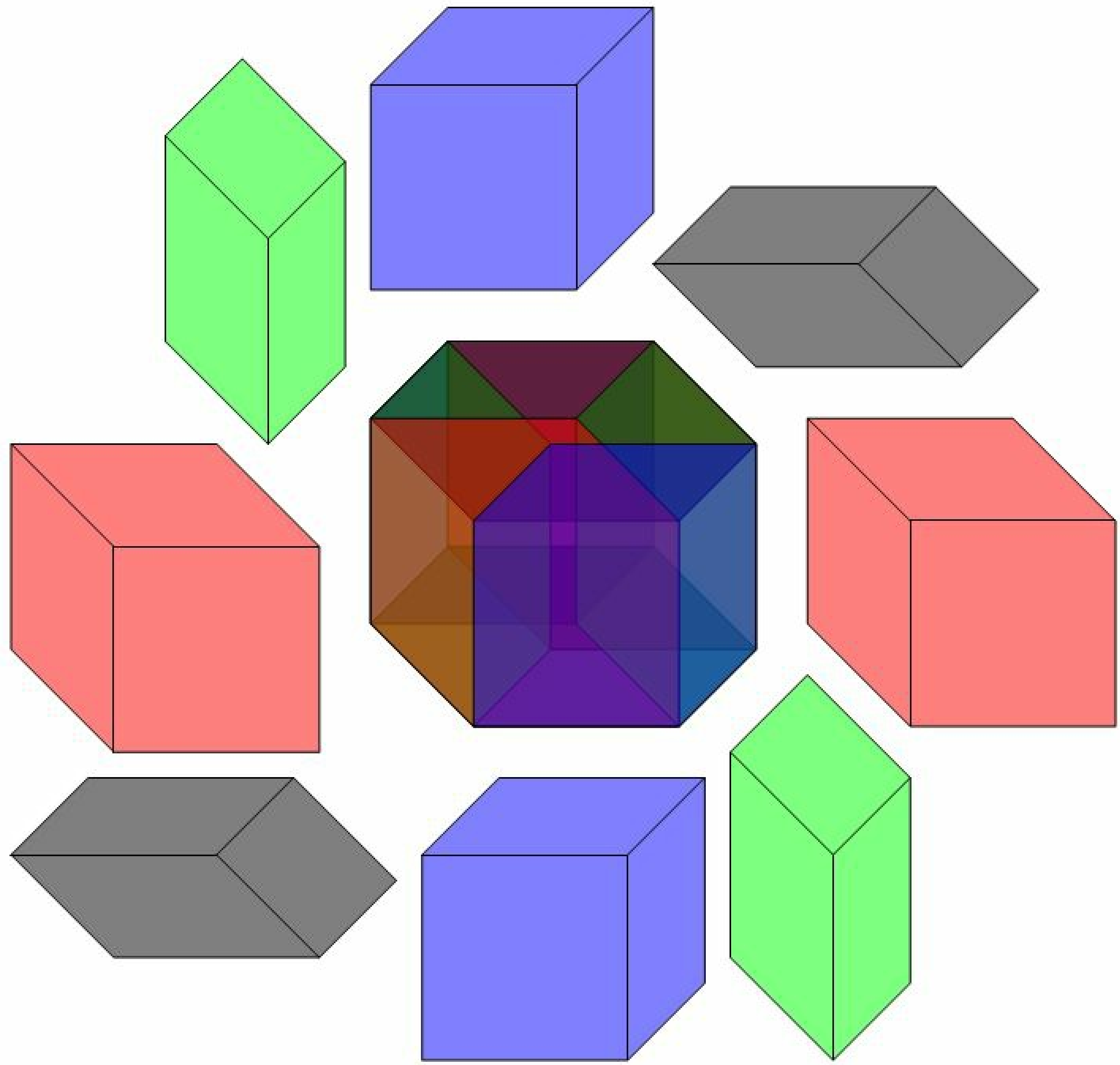
What would it be like to sit inside of a tesseract? There would still be a ceiling above you and a floor below you, but the "surfaces" (they are really hypersurfaces) would look like cubes, not squares. You would see the entire cube – all 6 squares on its surface and its entire volume. The 6 squares would appear to be on the "outside" of the cube. If you're in a room right now, when you look up, you see a rectangular ceiling. Sitting inside of a tesseract, if you were to look up, the entire ceiling would look like a cube, and you would see every cubic inch of that ceiling's "hypersurface" (which you are probably more inclined to think of as "volume"). But that's only the ceiling.

So there you are sitting in a tesseract. You look up and see the hyperceiling, which looks not like a square, but like a cube; and we don't just see the 3 bottom faces of the cube, we see every cubic inch of that cube. You look down and see the hyperfloor; you also see every cubic inch of that cube-shaped hyperfloor. Looking forward, you see every cubic inch of a cube-shaped hyperwall. Looking backward, you find a similar hyperwall. There are two more cubic hyperwalls to your right and left. Finally, you look in the ana and kata directions, seeing the last two hyperwalls.

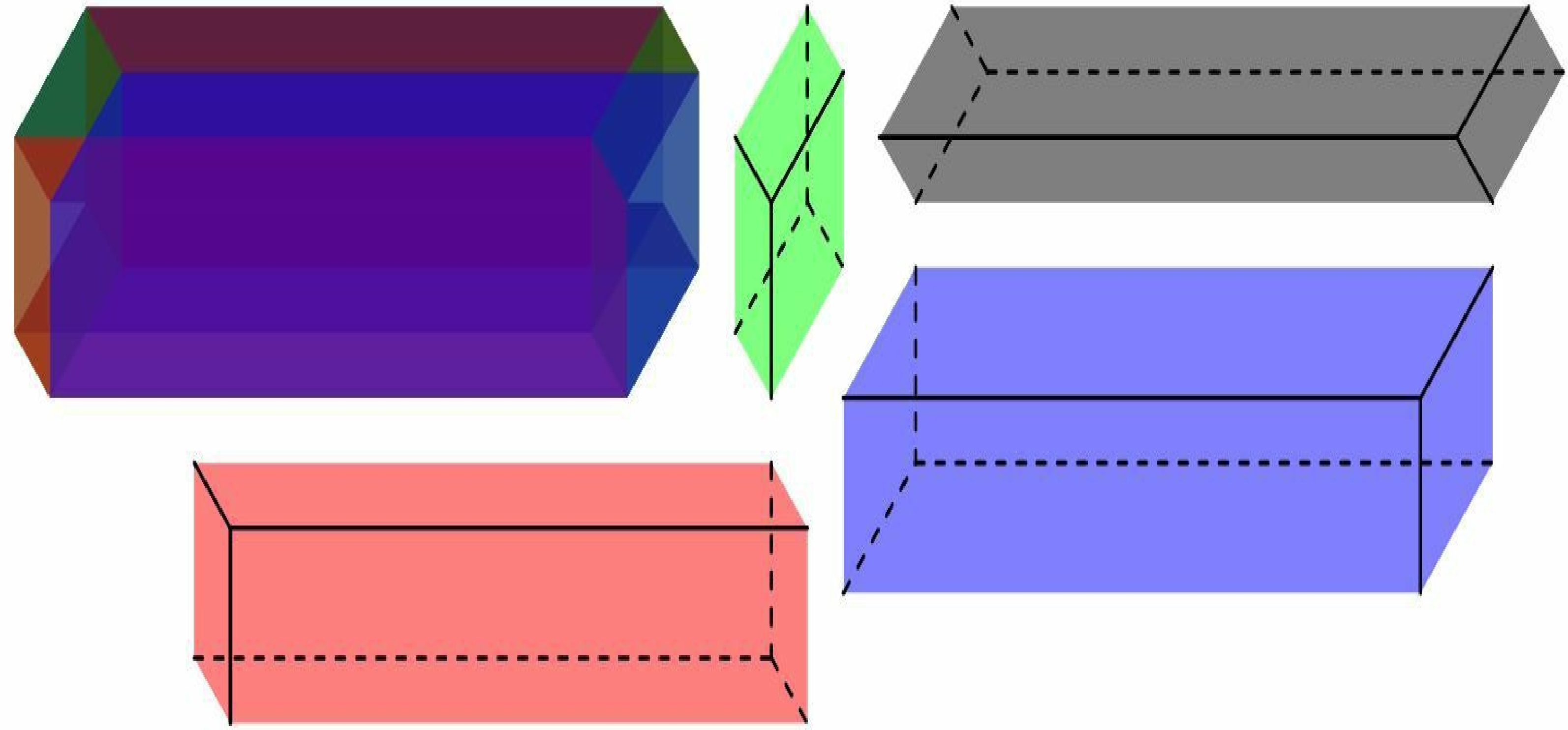
There are 8 "sides" all together, and each "side" is a cube. From inside tesseract, you see every cubic inch of all 8 cubes. The 8 cubes include the hyperceiling and hyperfloor, the right and left hyperwalls, the front and rear hyperwalls, and the ana and kata hyperwalls. Following is a picture of a tesseract. Imagine sitting inside of it.



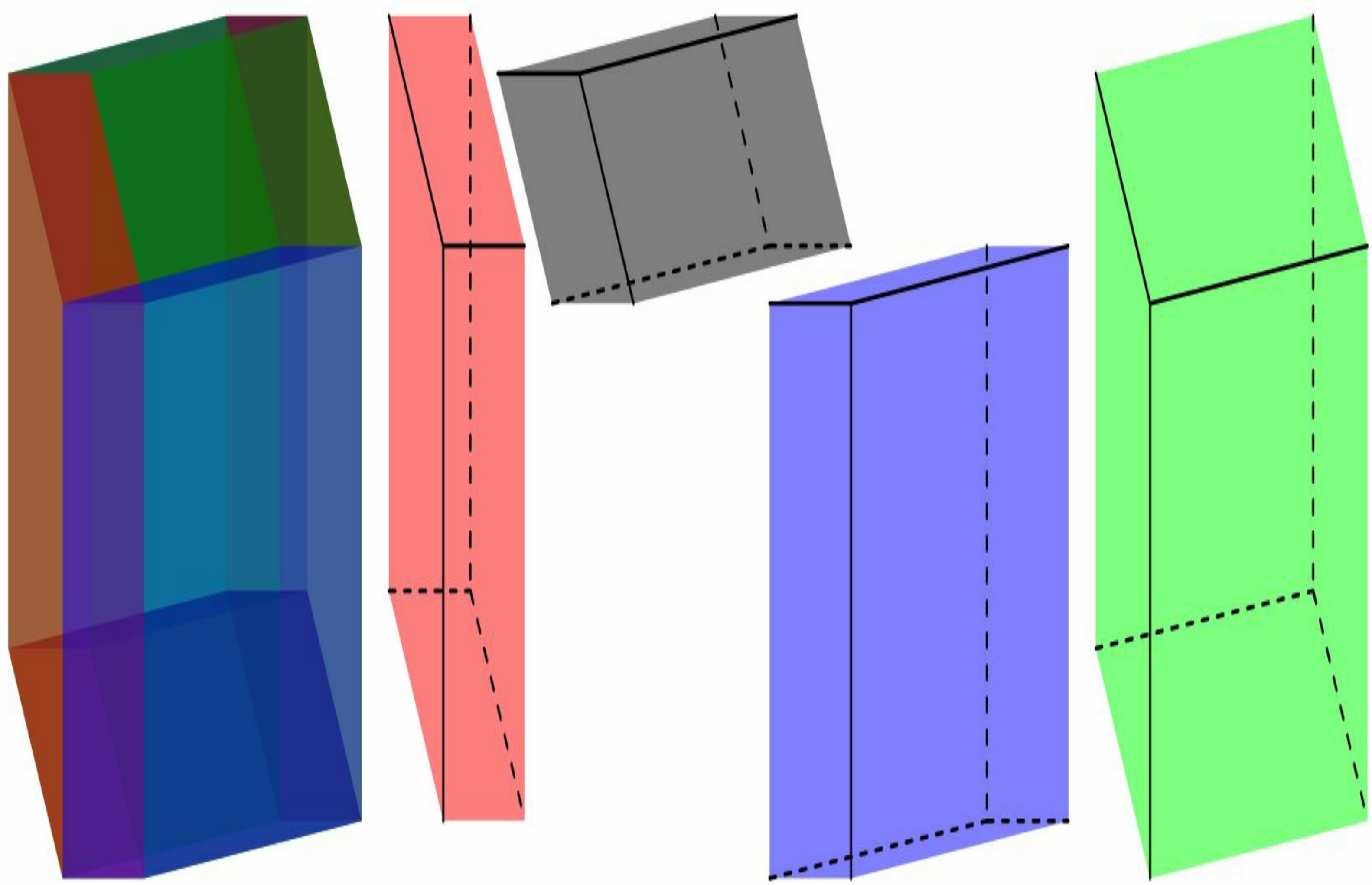
Following are the 8 bounding hyperwalls.



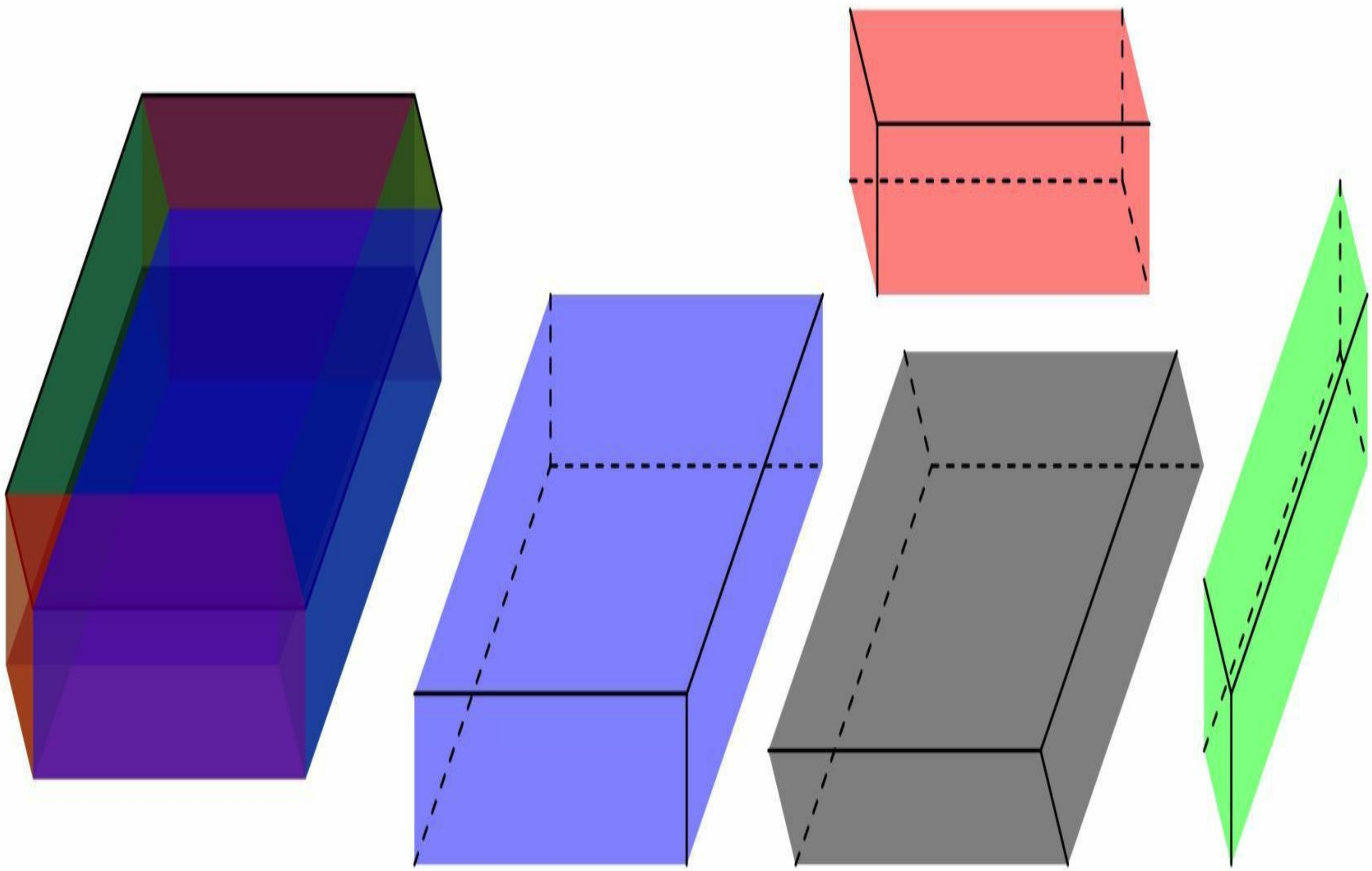
A typical 4D room would be rectangular, not square. The rectangular 3D box is called a cuboid. In contrast to a cube, 2 or more of the 6 sides of a cuboid are rectangles (whereas all 6 sides of a cube are squares, a cuboid can only have up to 4 square faces). The analogous 4D rectangular hyperbox is called a hypercuboid. At least 2 of the hypercuboid's 8 bounding walls are cuboids instead of cubes (whereas the tesseract is bounded by 8 cubes). Following is a picture of a hypercuboid and its bounding cuboids.



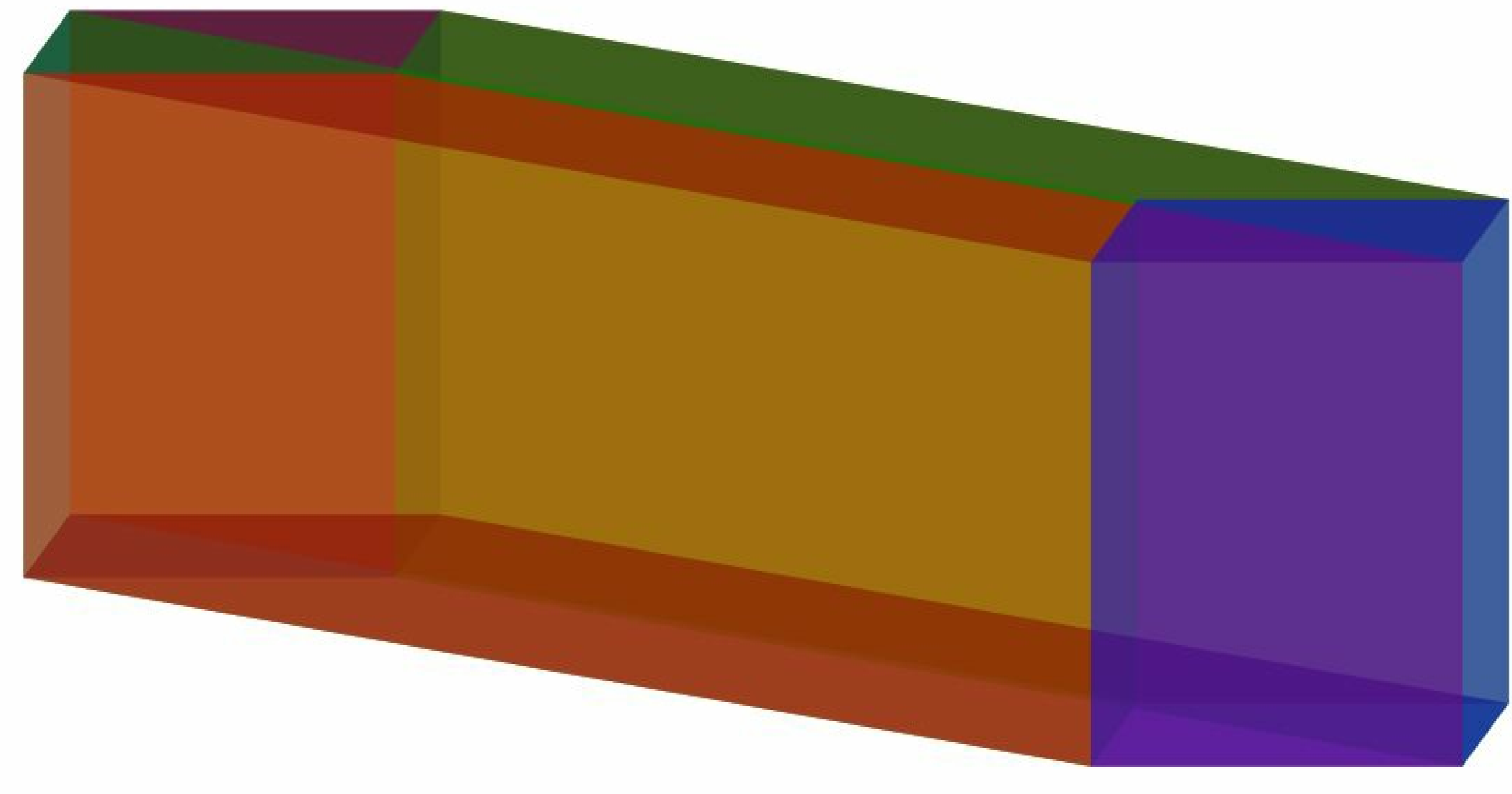
Here is another image of a hypercuboid.



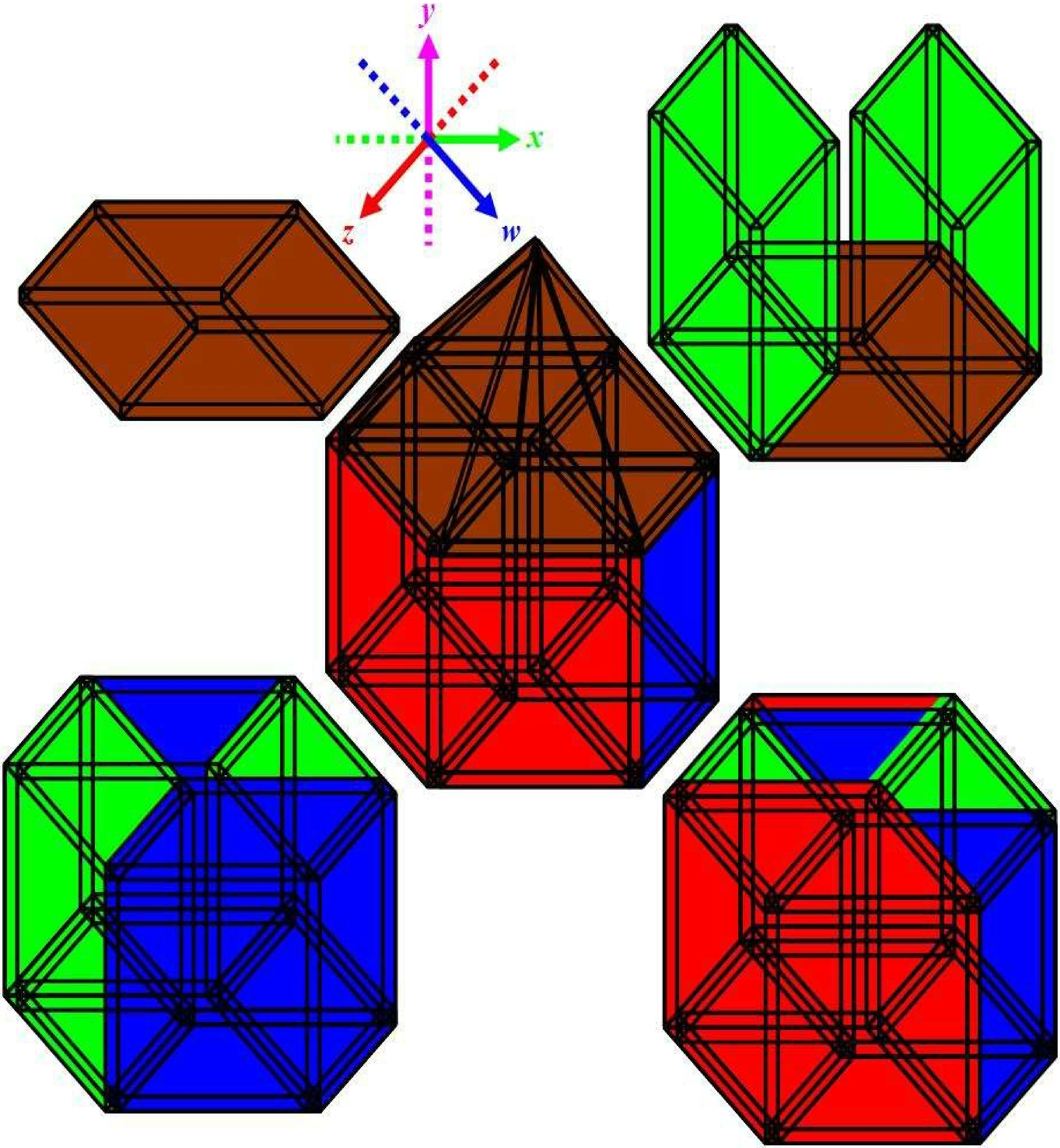
Another hypercuboid is shown below.



The last hypercuboid is shown by itself (i.e. without separate images of its bounding cuboids).

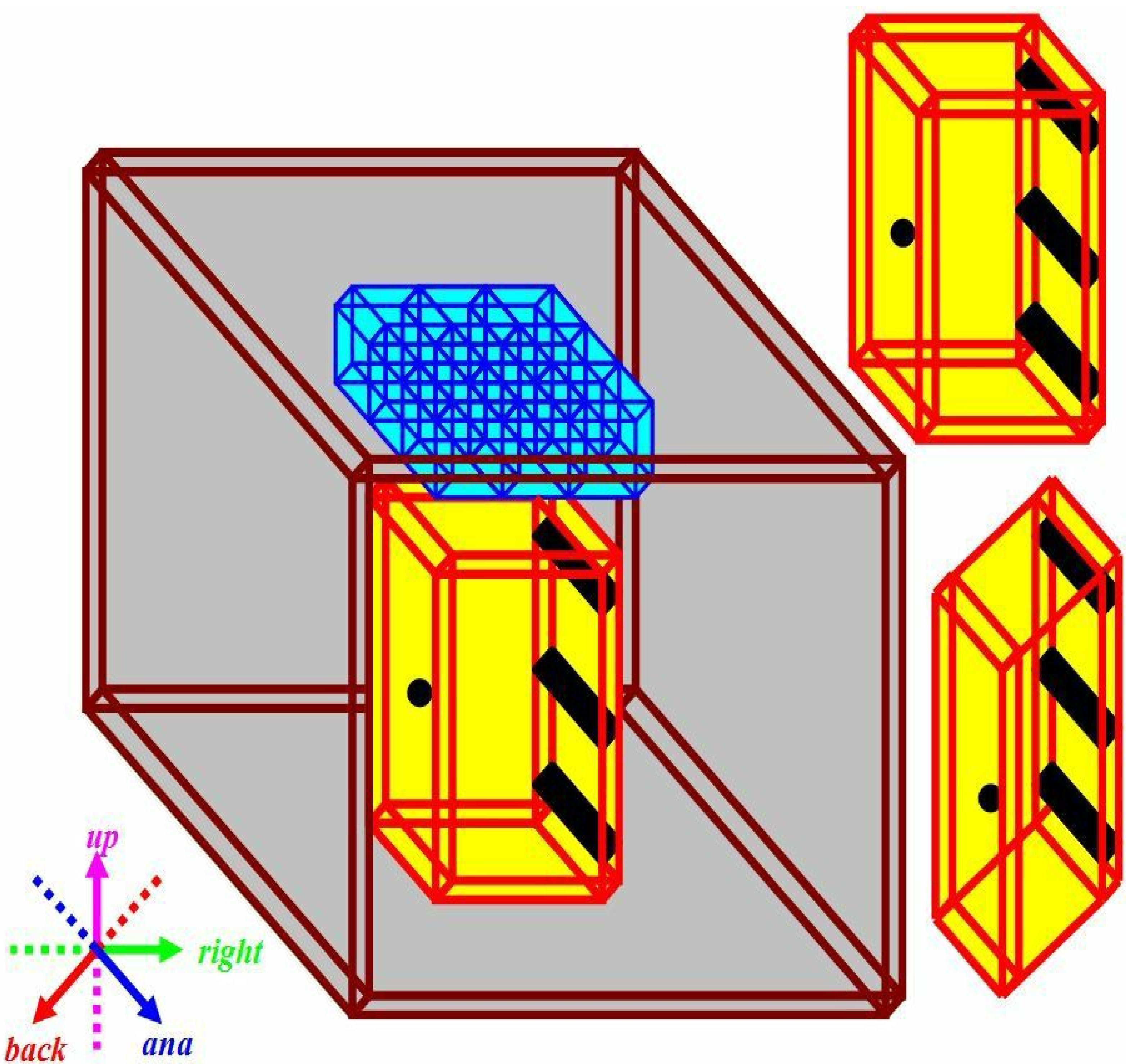


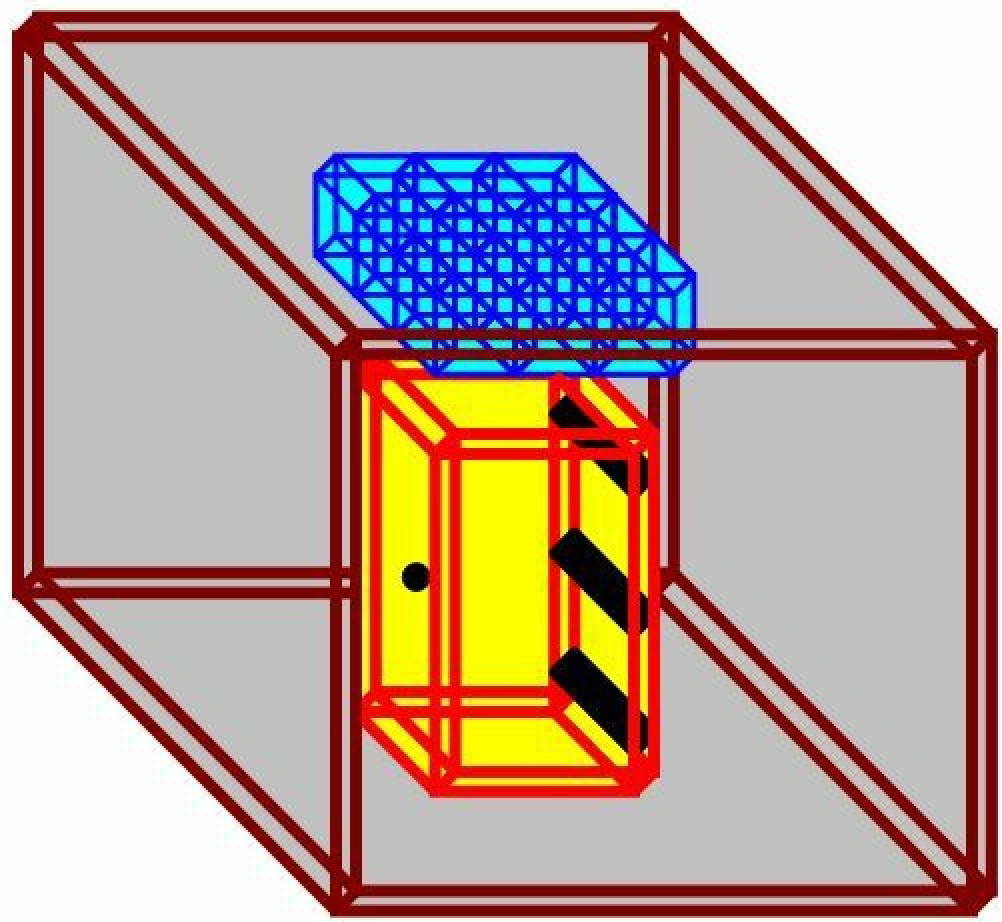
Just as the cuboid serves as the basis for drawing a 3D house, the hypercuboid can help us to visualize a 4D hyperhome. A simple hyperhouse structure is illustrated below. The brown hyperfloor, drawn in the top left corner, is a hypercuboid parallel to the *zwx* hyperplane. In the top right corner, you can see a pair of green hyperwalls parallel to the *yzw* hyperplane – one on the right and another on the left. In the bottom left corner, a pair of blue *xyz* hyperwalls are added – corresponding to the ana and kata directions. The bottom right corner shows a pair of red *wxy* hyperwalls – front and rear. The central image shows the complete tesseract-shaped house, including a hyperpyramid-shaped roof. The hyperpyramid has a cube at its base connected to six pyramids (each square face of the cube serves as the base for one pyramid, and all 6 pyramids share one vertex at the top). Compare this to a 3D pyramid, where four triangles share a common vertex at the top and also connect to each side of a square base.



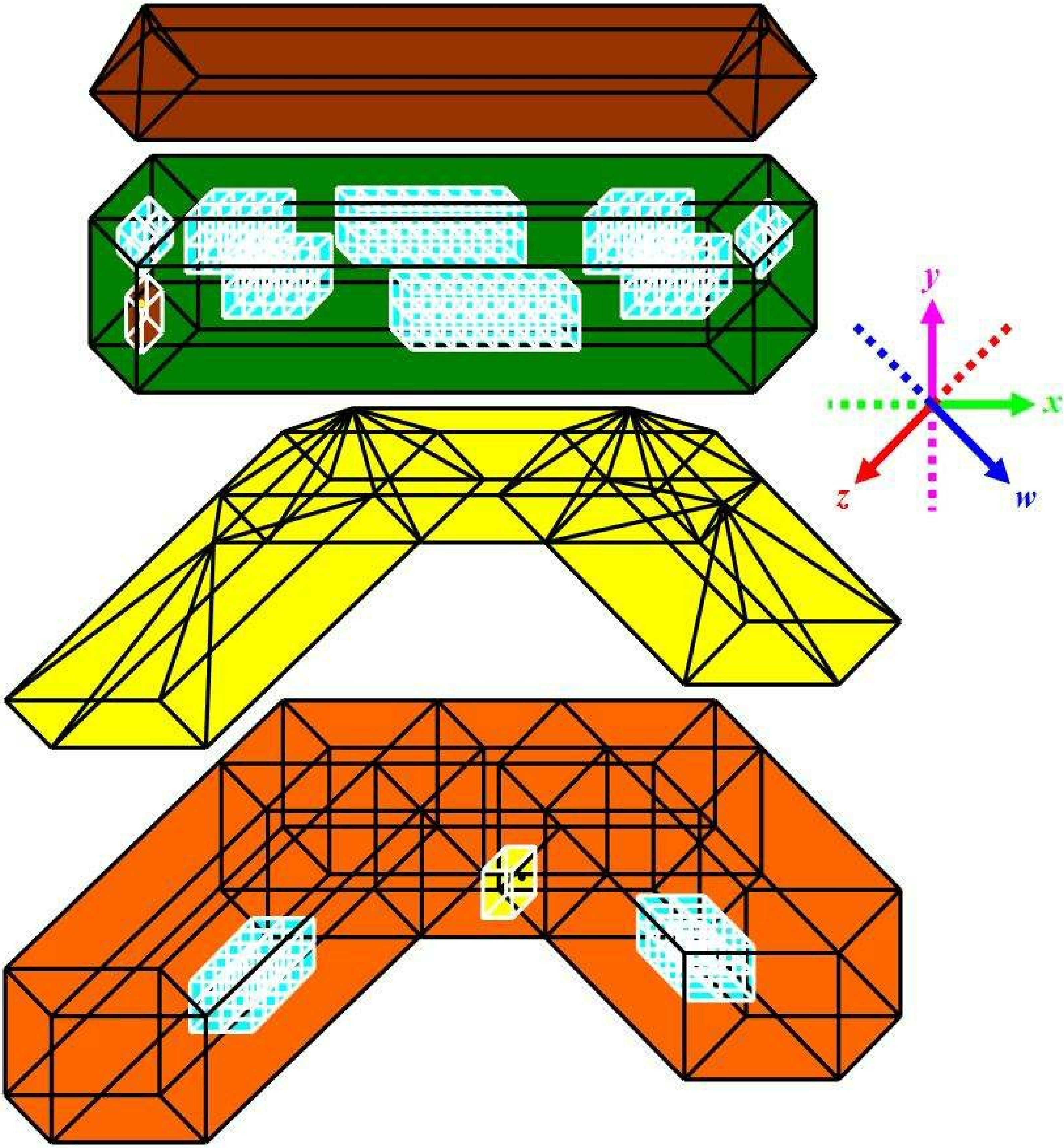
The previous figure shows what a simple hyperhouse would look like, and we've discussed what it would be like for a 4D hyperbeing (I trust that you visualized a hypermonkey) to sit in a hyperroom. Now let's talk about how to get into the hyperhouse in the first place. Through a hyperdoor, of course!

The door to a 3D room is shaped like a rectangle and is hinged at one edge. The hyperdoor to a 4D room would be shaped like a cuboid and would be hinged at one rectangular side (not along a line). This is because an object in 4D space rotates about a planar hyperaxis (the hinged rectangle), whereas an object in 3D space rotates about a linear axis (the hinged edge). Also, the hyperdoor would be positioned at the bottom center of one hyperwall, midway into the hyperwall. Just like a door is typically, centered left/right on a rectangular wall in 3D, a hyperdoor would be centered both left/right and ana/kata on a cuboid-shaped hyperwall in 4D. The following picture shows just a hyperwall (<u>not</u> an entire room), a hyperdoor, and hyperwindows.

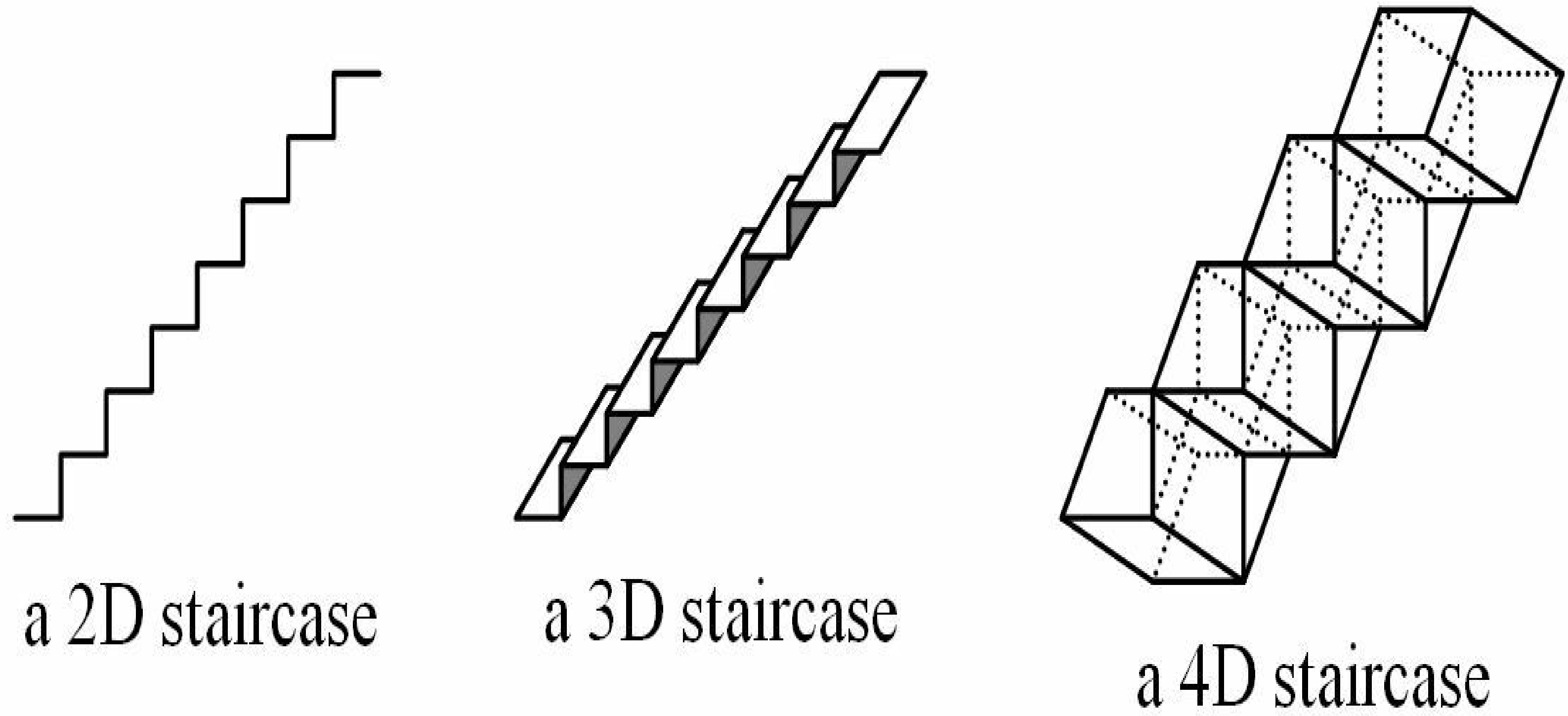




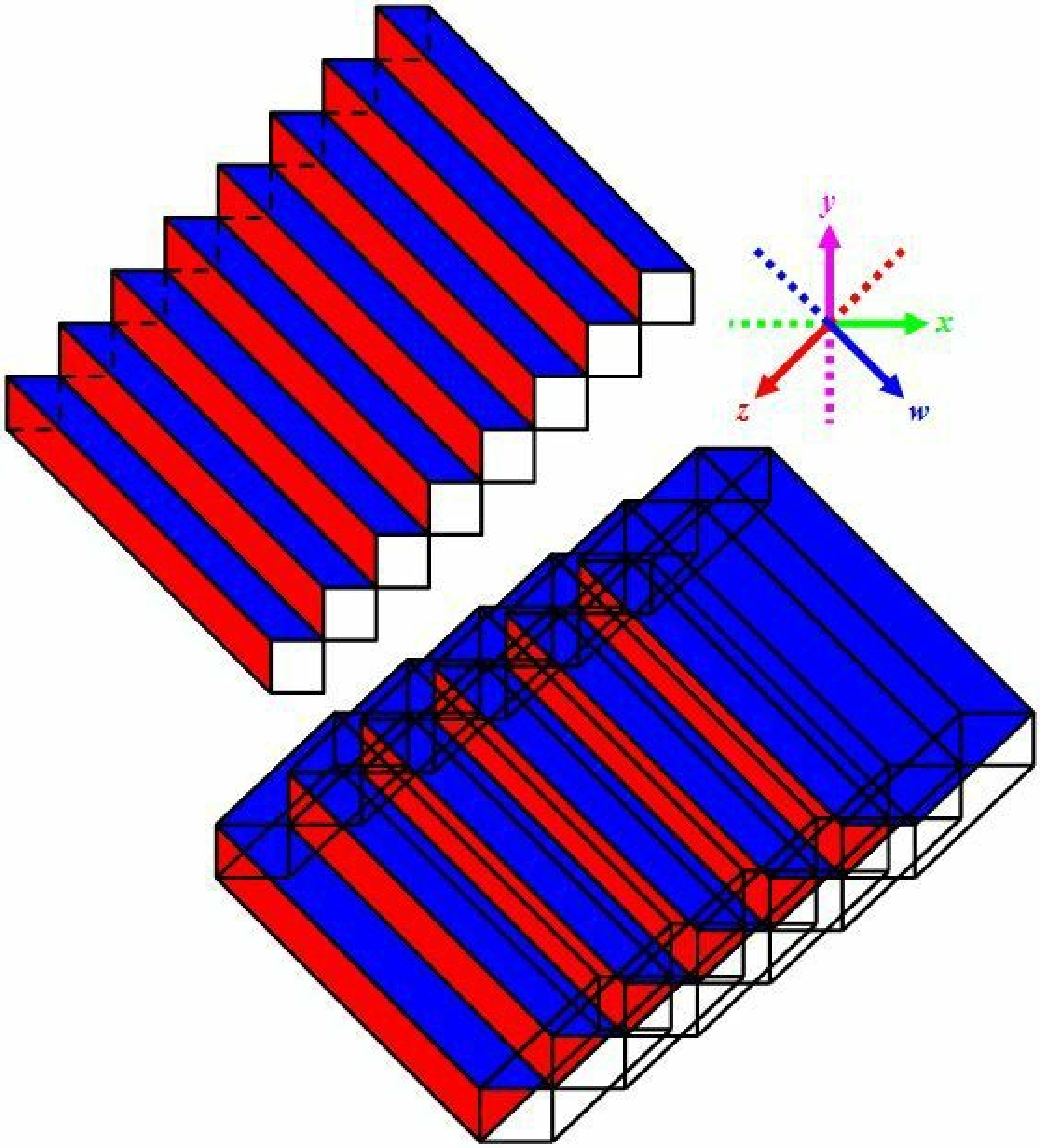
Following are two much fancier hyperhouses.



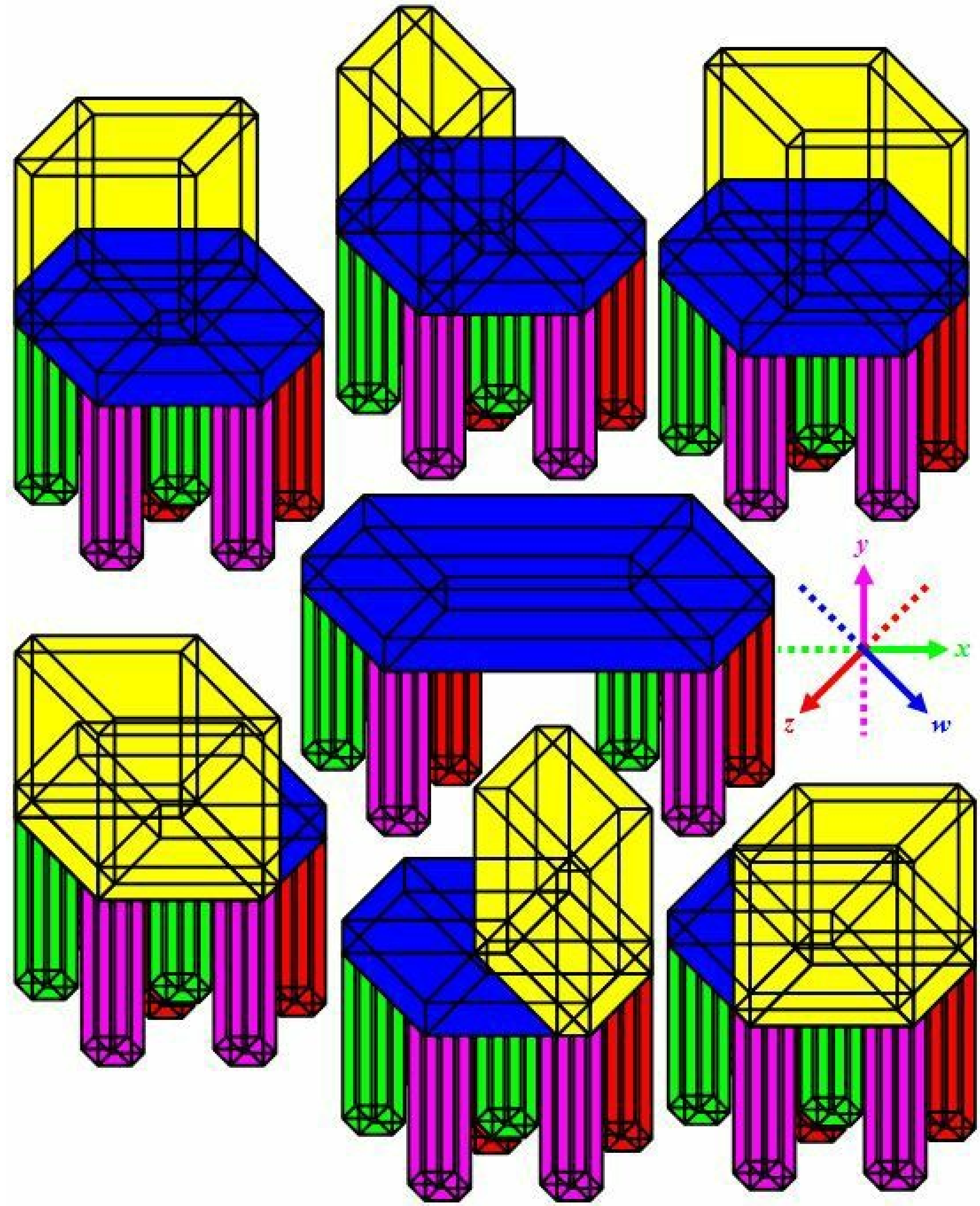
Hyperbeings (such as hypermonkeys) may have hyperstairs in multi-story hyperbuildings. A 2D staircase is a simple zigzag of line segments. A 3D staircase can be made by joining rectangles together along a common edge at 90 degrees. Similarly, a 4D hyperstaircase would be made by joining cuboids together along a common rectangle at 90 degrees, as illustrated below.



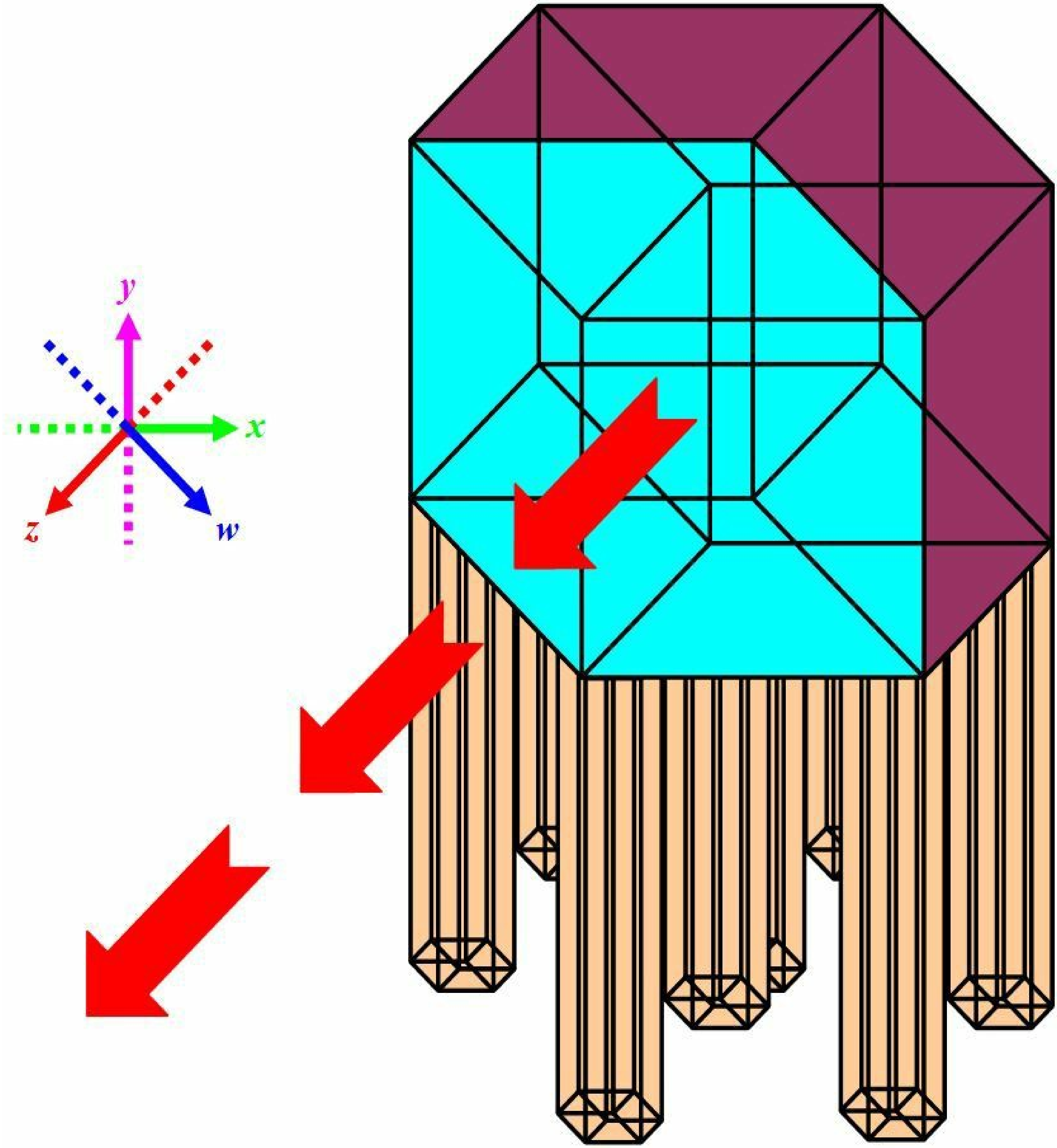
The following 3D staircase and 4D hyperstaircase are shown in color.



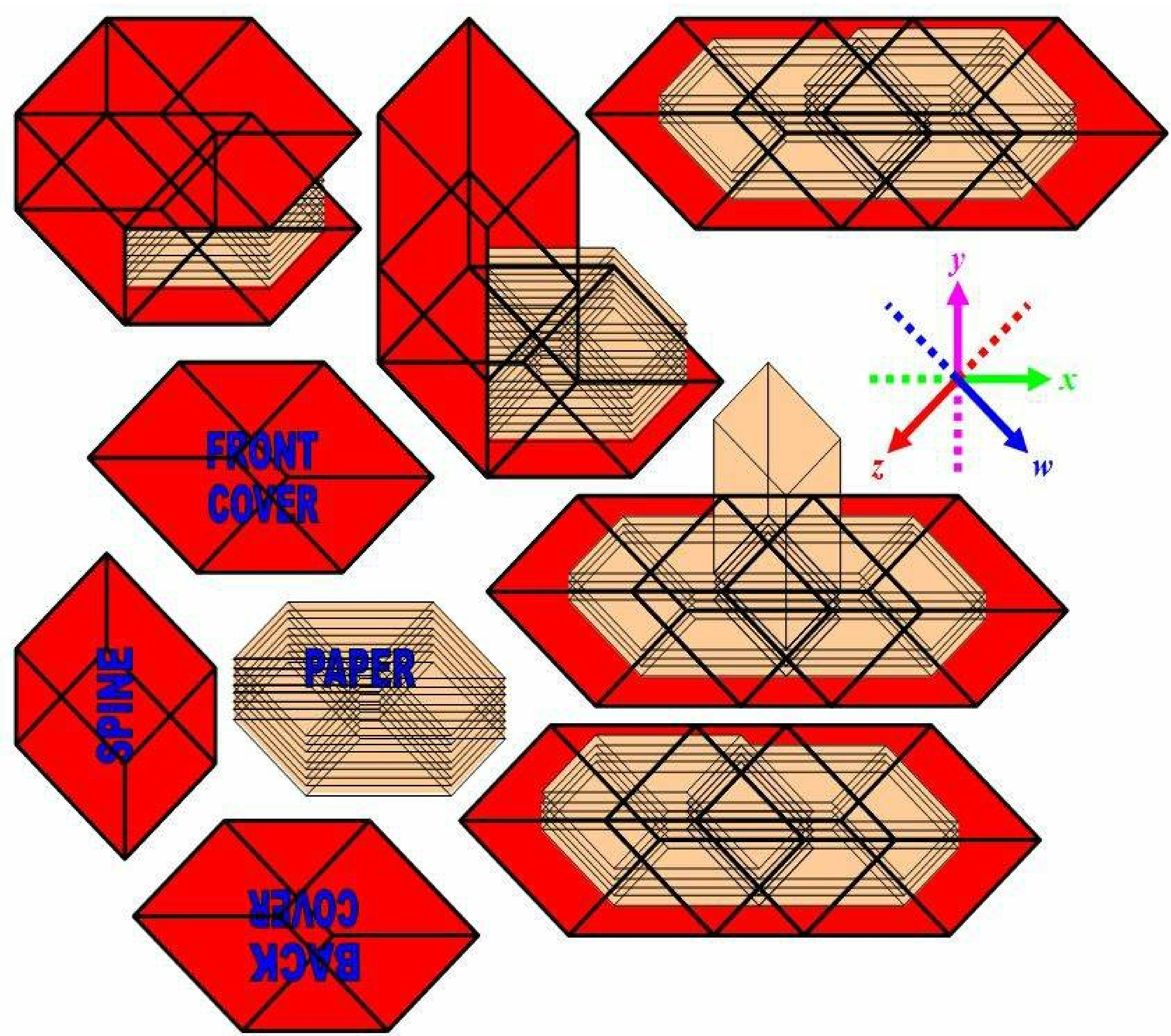
A hypermonkey may want to invite friends over for dinner. If so, they will sit on hyperchairs around a hypertable. A rectangular hypertable's top will be a cuboid, similar to a 3D table that has a rectangular top. A rectangular hypertable would have 8 legs: left/front/ana, right/front/ana, left/back/ana, right/back/ana, left/front/kata, right/front/kata, left/back/kata, and right/back/kata. Whereas a 3D table with 4 chairs would have one chair at each of 4 edges, a 4D hypertable with 6 chairs would have one chair at each of 6 sides: left/right, front/back, and ana kata. Such an arrangement is illustrated below.



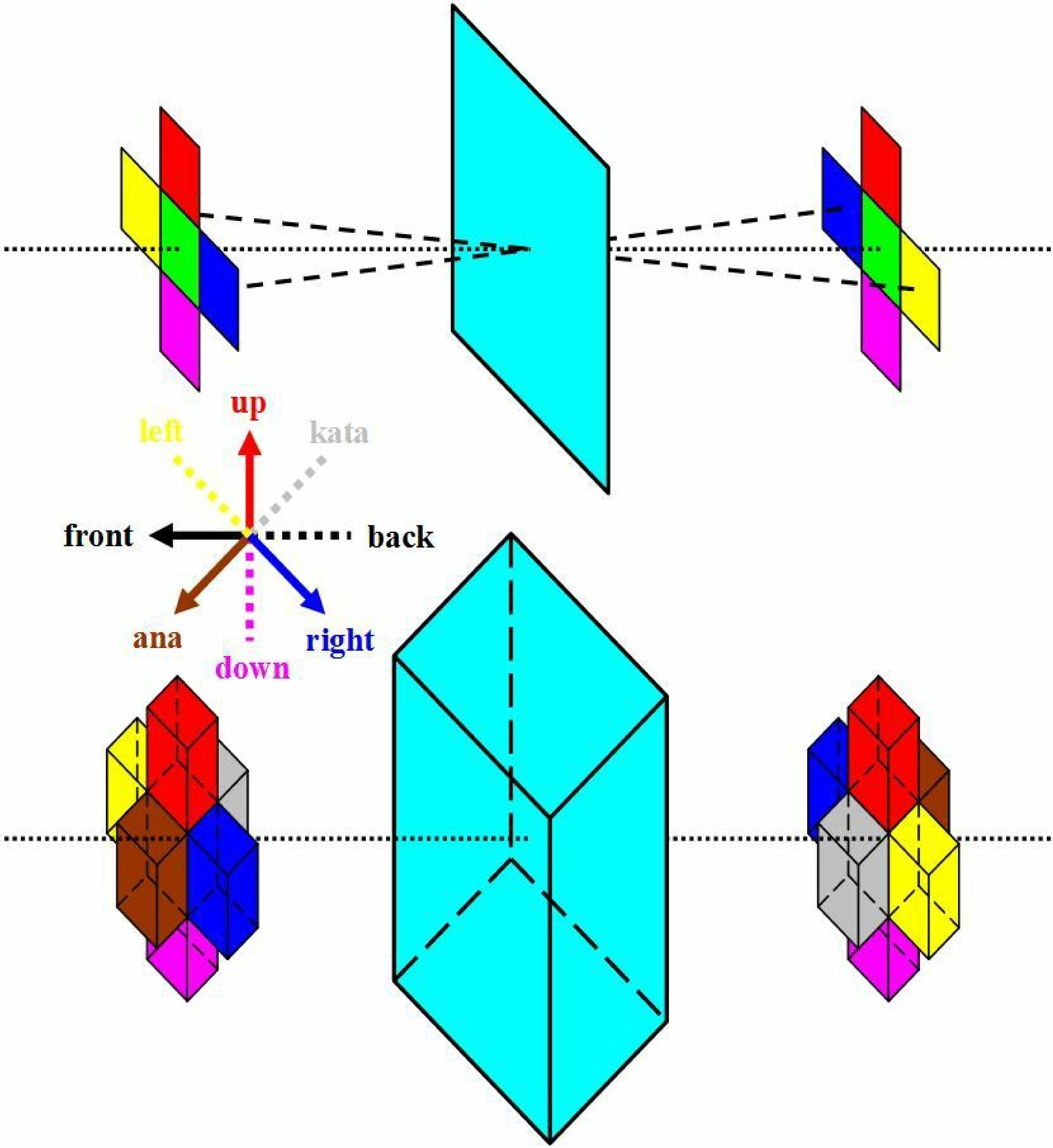
The hypertelevision below also stands on 8 legs. The hyperscreen is a cuboid; compare that to the screen of a 3D television, which is rectangular (if it's a flatscreen).



The next picture shows a 4D hardbound hyperbook. The hyperbook is closed in the top left figure, partly opened in the top center figure (the front cover is pointing up), and open in the top right figure. The remaining diagrams show the parts of the hyperbook, and one figure has the hyperbook open with one hyperpage pointing up. The front cover, back cover, and spine of the hyperbook – and even the pages themselves – are cuboids (whereas these are all rectangular in a 3D book). A hypermonkey reading the hyperbook would see print throughout the volume of each cuboid-shaped page. The hypermonkey might read left-to-right, top-to-bottom, and then ana-to-kata.



Imagine a hypermonkey standing before a hypermirror. Images would be inverted both left/right and ana/kata. Compare this to 3D space, where a monkey's reflection is only inverted left/right. Such mirrors are shown below.



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Further Reading

(1) Rudy Rucker's *Geometry, Relativity, and the Fourth Dimension* is a fascinating, accessible read with numerous line drawings. You can't properly call yourself a fan of the fourth dimension if you haven't read Rudy Rucker's book on the subject.
(2) Edwin A. Abbott's *Flatland: A Romance of Many Dimensions* is a classic that was first published in 1880. It's a challenging read since it's a literary work with Victorian themes, but the geometrical ideas are worth the effort. As suggested in the name, *Flatland* is largely about life in a 2D world.
(3) A more accessible and highly detailed and visual book on the second dimension is A.K. Dewdney's *The Planiverse: Computer Contact with a Two-Dimensional World*. It has a great storyline, too.
(4) If you enjoy *Flatland*, you might also appreciate Ian Stewart's *Flatterland: Like Flatland, Only More So*.
(5) A good place to start investigating 4D polytopes – 4D generalizations of polyhedra – is H.S.M. Coxeter's *Regular Polytopes*.
(6) Chris McMullen (that's the author of the book that you are presently holding, in case you didn't know it) also has a few other books on the fourth dimension. His original work on the fourth dimension is *The Visual Guide to Extra Dimensions*. Volume 1 is devoted to geometry, while Volume 2 gets into the physics of extra dimensions. *The Visual Guide to Extra Dimensions* is highly informative, and includes some topics that were not explored in this book, such as curved hypersurfaces. The writing doesn't reflect the author's personality as much as the book you're reading now, and the diagrams are in black and white instead of color, but it is more comprehensive in coverage.
(7) Chris McMullen also has a two-volume set called *Full Color Illustrations of the Fourth Dimension*. Some of the same figures were included in the book that you are holding, but there is material on hyperspheres that wasn't included in this book. The pictures of volume 2 came out very nice, whereas the printing of volume 1 came out a little dark on paper (and a couple of pictures didn't come out quite as nice in print as they looked on the screen); volume 2 also has a lot of cool figures, like 4D chess and hyperbilliards.

There are also a myriad of websites on a fourth dimension of space. A great place to begin is Wolfram's Mathworld:

http://mathworld.wolfram.com/Four-DimensionalGeometry.html

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Glossary

ana: The name for the fourth dimension of space. It's perpendicular to forward, right, and up.

axis: A straight line with tick marks for measuring distance along a dimension of space, as in a coordinate axis like x or y; or a straight line about which an object rotates, as in a rotation axis. Compare the singular axis to the plural axes.

axle: A rod that connects perpendicularly to a wheel or pulley; when it connects two parallel wheels, the axle forces the wheels to rotate together.

black hole: A singularity in space created by the gravitational field of a supermassive collapsed star, where gravity is so strong that light can't even escape from below the event horizon.

circle: The locus of points in a plane equidistant from a common point referred to as the center.

compact dimension: A dimension that wraps around and closes on itself, such that motion along that dimension would be cyclic, like the circumference of a cylinder or sphere.

coordinates: A set of N numbers, (x,y,z,etc.) that specifies the location of a point in N-dimensional space.

cross section: The region of intersection between a geometric shape and a plane (or hyperplane).

cube: A polyhedron bounded by 6 square faces that meet at right angles.

cuboid: A rectangular version of a cube (at least 2 of the faces must be rectangular instead of square). It is analogous to the distinction between a rectangle and a square.

dimension: A measure of extent, such as length, width, or height.

glome: The locus of points in 4D space equidistant from a common point referred to as the center; it is a hypersphere in 4D space.

hexagon: A polygon bounded by 6 edges.

hyperaxis (of rotation): The plane about which an object rotates in 4D space (not the same as the plane of rotation).

hypercube: An N-dimensional generalization of the cube with 2^N corners and edges that meet at right angles.

hypercuboid: Whereas a hypercube consists of squares, a hypercuboid involves rectangles. The distinction parallels that of the cuboid and cube.

hyperplane: A flat 3D space, like the xyz hyperplane. The three dimensions of our universe form a hyperplane.

hyperpyramid: A 4D polytope where 6 pyramids share a common vertex and join to each square face of a cube; the cube is called the base. It is analogous to the pyramid of 3D space.

hypersphere: The locus of points in higher-dimensional space equidistant from a common point referred to as the center. In 4D space, the hypersphere is called a glome.

hypersurface: The generalization of a surface to a higher-dimensional object. In 4D space, the hypersurface is more like what we refer to as a volume in 3D space.

infinitesimal: An indefinitely small positive number.

infinite: An indefinitely large number.

kata: The opposite of ana.

octahedron: A polyhedron bounded by 8 triangular faces; it can be formed by joining the square bases of two pyramids together.

orthogonal: Perpendicular.

parallel: Extending along the same direction, so as to be equidistant at corresponding points.

pentagon: A polygon bounded by 5 edges.

perpendicular: Meeting at a right angle (90 degrees).

perspective: The way that an object appears to the eye, with lines of depth converging at a vanishing point.

plane: A flat 2D space, like the *xy* plane.

polygon: A closed object bounded by 3 or more straight edges.

polyhedron: A closed object bounded by 4 or more polygons.

polytope: An *N*-dimensional generalization of the polyhedron.

power: The exponent of a number, meaning that the base is multiplied by itself that many times. For example, in 2^3, 2 is the base, 3 is the power (or exponent), and 2 multiplies itself 3 times: $2^3 = 2 \times 2 \times 2 = 8$.

pyramid: A polyhedron where 4 triangles share a common vertex and join to the 4 edges of a square base; it has 5 faces all together.

rectangle: A polygon bounded by 4 edges that meet at right angles, where one pair of edges has different length than the other pair.

reflection: The image of an object as seen when viewed through a mirror.

right angle: 90 degrees.

rotation: When every part of an object travels in a circle about a common axis (in 3D space, this axis is a straight line).

sphere: The locus of points in a hyperplane (i.e. 3D space) equidistant from a common point referred to as the center.

square: A polygon bounded by 4 edges of equal length that meet at right angles.

tesseract: A hypercube in 4D space.

tetrahedron: A polyhedron bounded by 4 triangles.

trapezoid: A polygon bounded by 4 edges with exactly one pair of parallel edges.

triangle: A polygon bounded by 3 edges.

vanishing point: The place where lines of depth appear to converge when an object is drawn in perspective.

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About the Author

Chris McMullen holds a Ph.D. in physics from Oklahoma State University, and presently teaches physics at Northwestern State University of Louisiana. He is a very passionate teacher. As a hobby, he enjoys writing books, drawing illustrations on the computer, editing manuscripts, and especially the feeling of having produced a professional-looking book from cover-to-cover. He has written over a dozen paperback books and over a dozen eBooks. He has primarily written and published nonfiction books, such as math workbooks and how-to self-publishing guides.

Dr. McMullen's expertise in physics is theoretical and computational high-energy physics (particle physics). He has coauthored a half-dozen papers on the prospects of high-energy colliders, like the Large Hadron Collider (LHC), discovering superstring-inspired extra dimensions in the near future. He has also taught a few unique 30-hour winter courses on the geometry of the fourth dimension to gifted high school juniors and seniors at a special math and science school.

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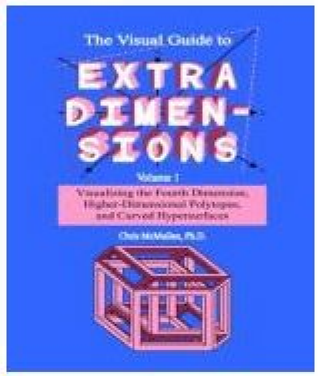
HOW DID WE DO?

Consider our objectives with this work and to what extent, if any, we may have achieved them:

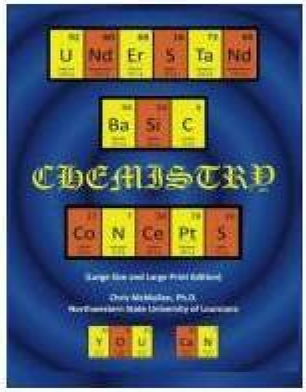
- To help readers understand geometric concepts associated with a fourth dimension of space.
- To focus on mathematical ideas, and not go into spirituality or religion (not that those aren't fascinating, too, just that they weren't the topic of this book).
- To make numerous pictures that help to illustrate 4D objects.
- To provide a few interesting puzzles involving the fourth dimension.
- To show analogies between the lower and higher dimensions.
- To write the text with a little personality so that it engages the interest of the reader and doesn't read like a dry math textbook.
- To present the ideas with clear explanations.
- To make the work accessible to virtually anyone, while still engaging the interest of those with strong backgrounds in mathematics (a very challenging goal).
- To produce a book that is highly professional in appearance.
- To prepare a well-written book that avoids spelling, grammatical, editing, and formatting mistakes.

You don't need to be an expert in order to express your opinion. Feedback is both welcome and encouraged. Thank you in advance if you take a moment to share your thoughts.

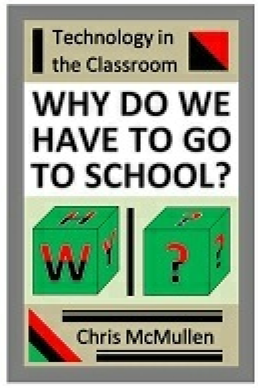
MORE BOOKS BY THE AUTHOR



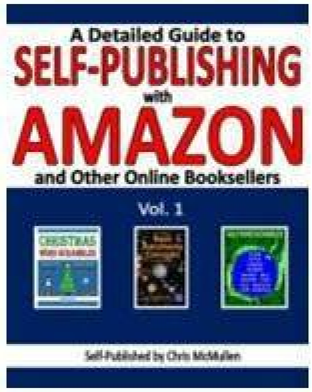
The Visual Guide to Extra Dimensions includes many novel figures of a variety of 4D objects – not just the standard tesseracts and hyperspheres, but other geometric objects like the hecatonicosachoron and spherinder. The illustrations in *The Visual Guide to Extra Dimensions* are black and white and the text is more informational, showing less personality than the book that you are reading presently, but it is also highly detailed and informative.



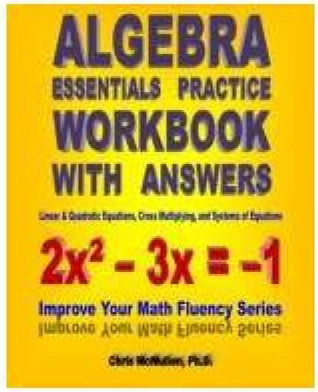
UNdErSTaNd BaSiC Chemistry CoNCePtS focuses on fundamental chemistry concepts, such as understanding the periodic table of the elements and how chemical bonds are formed. No prior knowledge of chemistry is assumed. The mathematical component involves only basic arithmetic. The content is much more conceptual than mathematical. It is geared toward helping anyone – student or not – to understand the main ideas of chemistry. Both students and non-students may find it helpful to be able to focus on understanding the main concepts without the constant emphasis on computations that is generally found in chemistry lectures and textbooks. The author had fun with this conceptual chemistry book, as seen in the writing, decorative pictures, and unique word puzzles at the end. For example, many chemical words that can be made exclusively using symbols from the periodic table are featured throughout the book.



If you enjoyed the way that the text of the current book is written, you will probably also enjoy Chris McMullen's first published work of fiction. The title is misleading; the work is foremost a captivating fictional dialogue between father and son. Since they happen to be discussing technology and education, the book is entitled *Why Do We Have to Go to School*. Don't let the title fool you: The book was not written merely for the benefit of educational philosophers, but for anyone who enjoys a good work of fiction. If you could enjoy a 10,000-word, beautifully decorated, and engaging work of fiction, and if you like the personality that the author showed in the book you are holding, then you may enjoy the way this story reads. You may also enjoy the ideas that are exchanged in the dialogue.



A Detailed Guide to Self-Publishing provides is a highly detailed guide with several step-by-step instructions for how to format and publish your book both as a paperback book with Amazon (and other major online booksellers) and as an eBook with Kindle (and a variety of other e-readers). Find highly detailed try-it-yourself, walk-you-through it tutorials for how to use Microsoft Word 2010 (which is similar to Word 2007) specifically with Windows to publish your book both as a paperback book and as an eBook. This includes how to use numerous formatting features (like page borders and bookmark hyperlinks), how to draw your own pictures from scratch, how to insert and format equations, how to create your own cover, how to convert the content file for your paperback book into an eBook, how to format pictures and equations in an eBook with a variety of e-readers in mind, and how to minimize the eBook's file size. Chris McMullen, who has drawn thousands of professional illustrations from scratch using Word's drawing tools, shares several useful drawing and formatting tips. The book is both very informative (like listing the pixel sizes of different brands of e-readers and describing how to manually create precise colors), yet also focused on useful formatting and publishing skills. The paperback instructions are largely geared toward CreateSpace, while the eBook instructions accommodate Kindle and other e-readers. With this reference as a guide, you can self-publish a quality manuscript with ease!



The *Improve Your Math Fluency Series* provides good-old fashioned practice for basic math skills including arithmetic, long division, adding fractions, decimals and percentages, algebra, and trigonometry. Most of the workbooks in this series are focused on learning through practice and so contain about a hundred worksheets on fundamental math skills. As the algebra and trigonometry workbooks involve more advanced mathematical techniques, these two volumes have a written introduction at the beginning of each chapter and several examples to help serve as a guide. The fraction workbooks include a concise explanation and a couple of examples for each chapter. The arithmetic workbooks, as they involve more basic skills, do not include the introduction and examples (except for the 4500 multiplication problems book).

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THANK YOU!

Thank you very much for reading this book. We sincerely hope that you enjoyed our product. Our motivation and diligence is derived from readers like you.

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