Statistics Notes (II)

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This is the solution of the problem. When there is only on solution, there is no need to numbering.

1.1 Important Concepts

1.1.1 Terms

Trial A single performance of well-defined experiment

Experiment *An act that can be repeated under some specific condition.* [A scientific test in which you perform a series of actions and carefully observe their effects in order to learn about something. or]

Random variable A variable whose values are associated with probability.

Sample space Set of all possible outcomes of a random experiment.

Sample point Each outcome of a sample space

Event Any subset of a sample space

Simple event An event having a single outcome

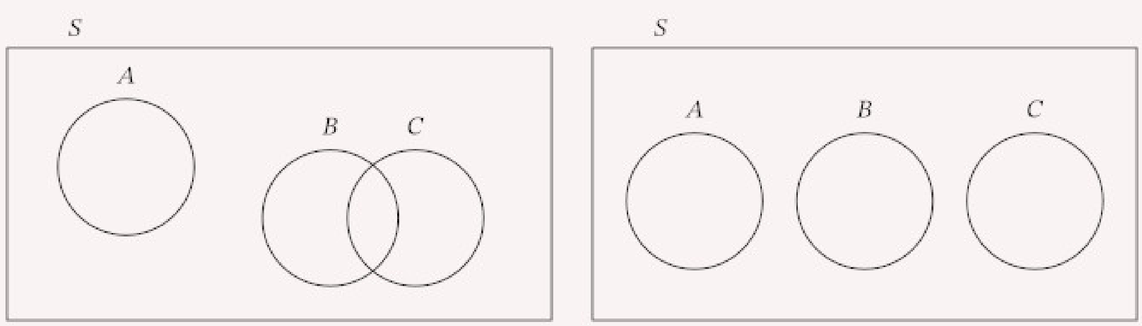
Compound/Composite event An event having more than one outcome

Impossible event An event which cannot happen (If P(A) = 0, then A is an impossible event)

Certain event An event which surely will or will not happen. (P(A) = 0 or 1)

Uncertain event An event which may or may not happen ($0 < P(A) < 1$)

Mutually Exclusive Event Events that cannot occur together. If $S = \{1, 2, 3, 4\}, A = \{1, 3\} \& C = \{4\}$ then A & B are mutually exclusive.



A is mutually exclusive to *B* and *C*, but *B* and *C* are not mutually exclusive.

A, *B* and *C* are pairwise mutually exclusive.

Independent Event Events that do not affect each other.

Complementary event Non-occurrence of an event. $P(\bar{A}) = 1 - P(A)$, where $\bar{A}$ or A' or A^c is called complement of A.

Exhaustive event Events whose union is equal to the sample space of the experiment (all outcomes are considered)

Equally likely events Events having same probability. If $S = \{1, 2, 3\}, P(1) = P(2) = P(3) = 1/3$, here 1, 2, and 3 are equally likely. One way for them not to be equally likely is: $P(1) = 1/2, P(2) = 1/5, P(3) = 1/4$

1.1.2 Set Theory

NB: This is far from a comprehensive discussion of the set theory.

Set Operations

Suppose, $A = \{1, 3, 4\}$ and $B = \{3, 4, 5\}$

- Union: A or B $\Rightarrow A \cup B = \{1, 3, 4, 5\}$
- Intersection: A & B $\Rightarrow A \cap B = \{3, 4\}$
- Difference: $A - B = \{1\}$

Laws of Set

a. Cumulative law: $A \cup B = B \cup A$ and $A \cap B = B \cap A$

b. Associative law: $A \cup (B \cup C) = (A \cup B) \cup C$

c. Distribution law: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

d. De Morgan's law:

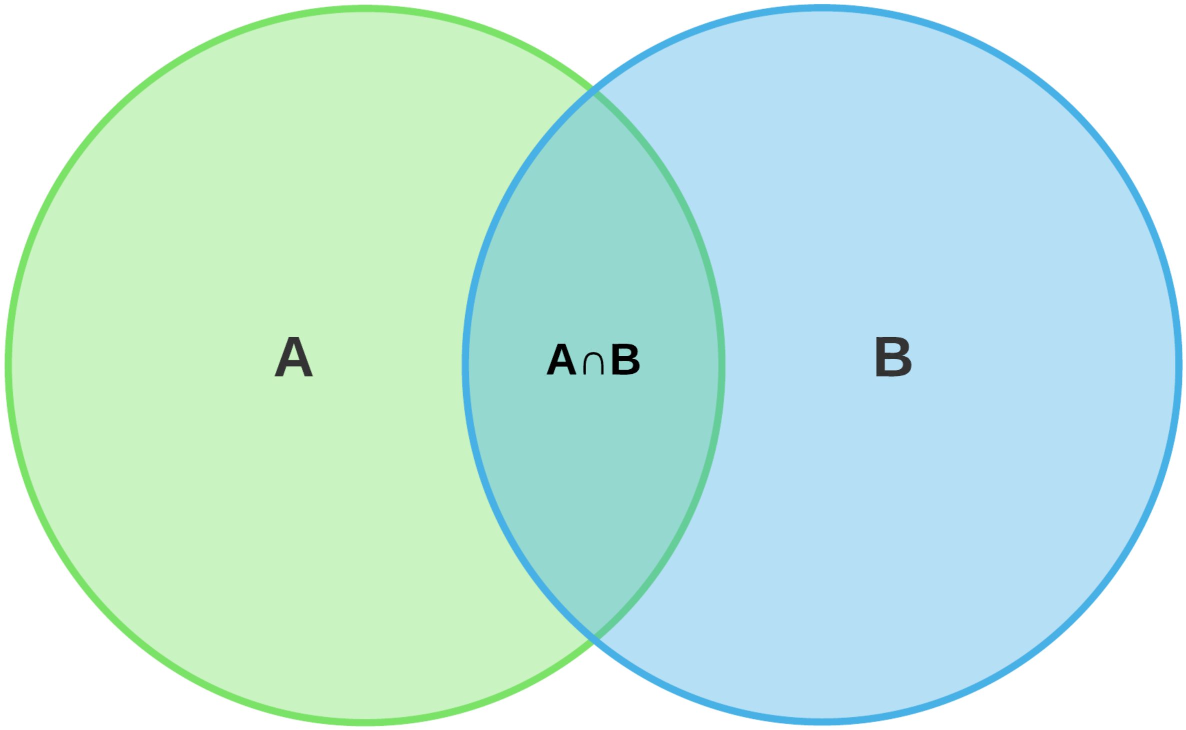
   i. $(A \cup B)' = A' \cap B'$
   ii. $(A \cap B)' = A' \cup B'$

Verify De Morgan's law

S = {1, 2, 6, 8}; A = {1,4}; B = {2,6}

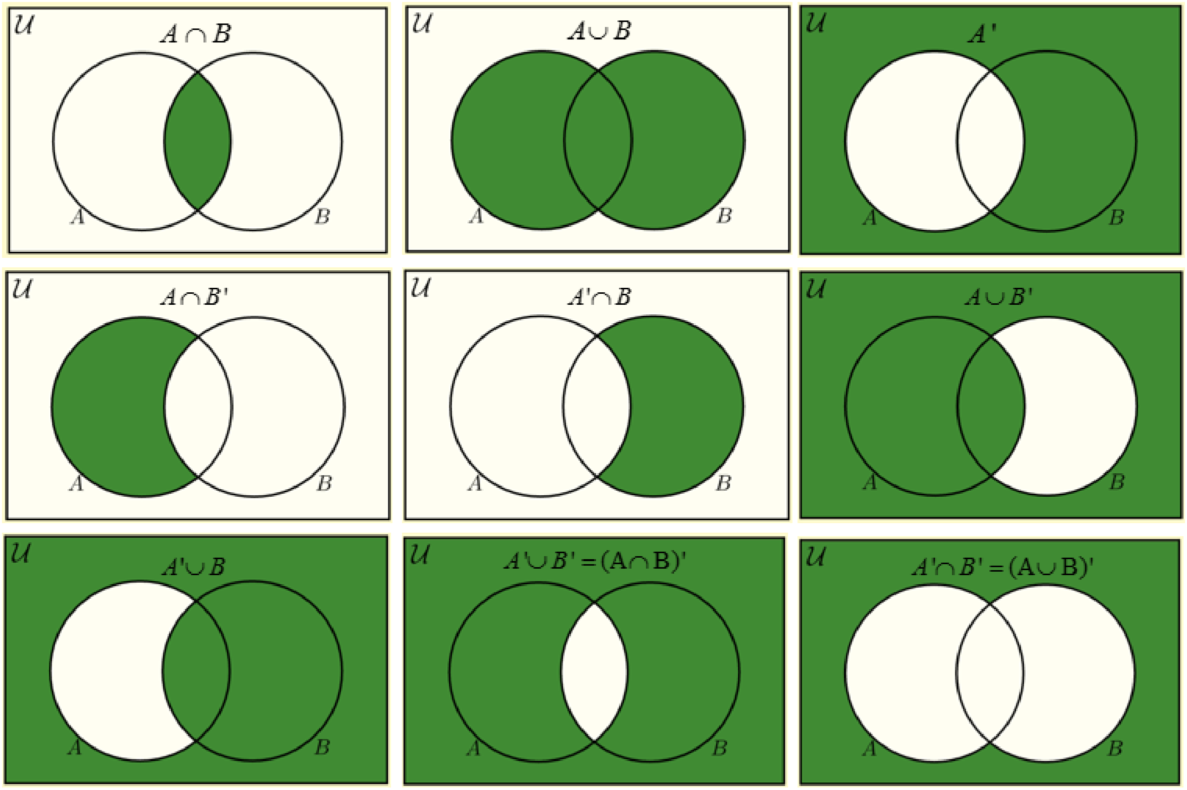
1.1.3 Venn Diagram

Locate the following sets from Venn Diagram (@mutexc)

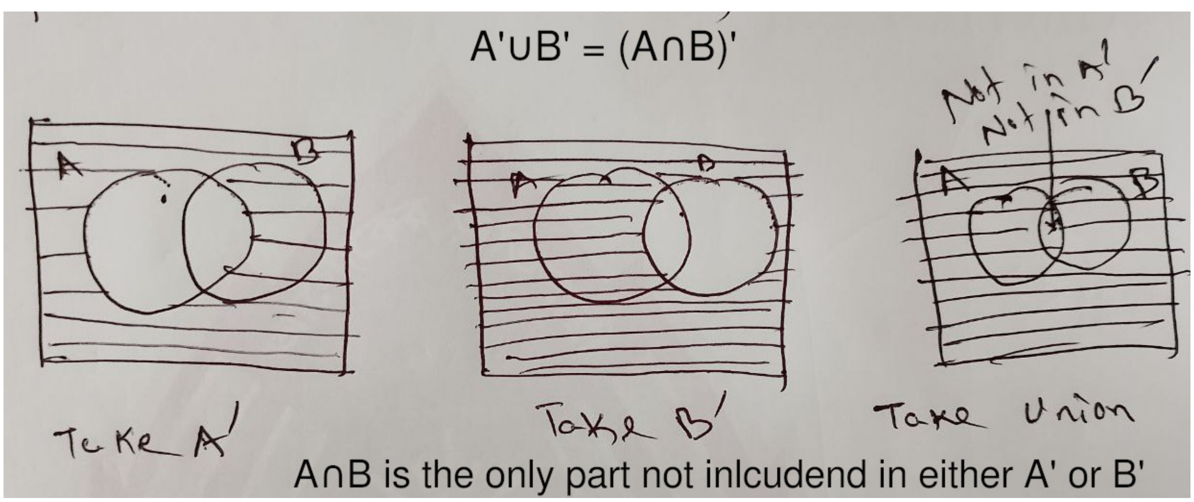


a. $A \cap B)$
b. $A \cup B$
c. A'
d. B'
e. $A \cap B' = A - A \cap B)$
f. $A' \cap B = B - A \cap B$
g. $A \cup B'$
h. $A' \cup B$
i. $A' \cup B'$
j. $A' \cap B'$

1.1.4 Venn at a glance



1.1.5 Explanation of $A' \cup B' = (A \cap B)'$



1.1.6 Permutaion

Permutaion is all about arranging items, while combination is used to find the ways to to select items.

If we have 3 items A, B, and C; we can arrange them in the following way.

- ABC
- ACB
- BAC
- BCA
- CAB
- CBA

There are 6 possible ways to arrange 3 items in 3 positions.

Thinking another way, there are 3 positions and 3 items. The first position can be filled up in 3 ways (A or B or C), the second in 2 ways (after one item is fixed in the first position, be it A or B, or C), and the third in 1 way.

Position	1	2	3
Possible options	3	2	1

Mathematically, this is also written as 3! (3 factorial), which is nothing but $3 \times 2 \times 1 = 6$

Similarly, if we have 4 items to arrange in 4 places, we can write:

Position	1	2	3	4
Possible options	4	3	2	1

Thus we can arrange this in $4 \times 3 \times 2 \times 1 = 4! = 24$ ways.

Not Using All Items

Now, what if want to arrange 2 items out of 4 items. In this case we have 2 places, but 4 items.

Position	1	2
Possible options	4	3

We get $4 \times 3 = 12$ ways.

This is also written as ${}^4P_2 = 12$ (shown below)

$$ {}^nP_r = \frac{n!}{(n-p)!} $$

Repeating Items

In the above examples, items cannot be repeated in places. In some scenario, this is absurd; after all, one person cannot sit on 2 chairs.

However, consider using digits to make up telephone numbers.

Position	1	2	3	4	5	6	7
Possible options	10	10	10	10	10	10	10

All 10 digits can be used in each position.

Hence, for 7-digit telephone numbers, we can have telephone number in 10^7 ways.

Think

- What is the general formula of the above case?[1]
- What if the first digit is always zero (0)?
- What if not all 7 digits can be same?
- What if some particular digit cannot be repeated, or can be repeated only twice?
- How many license plates can make using 5 letters, 2 digits and 3 letters, or 1 letter and 3 digits, where items can be repeated?

1.1.7 Combination

Combination is used when we are concerned with selecting items or individuals.

Example: How can we select 2 items out of 3 (A, B, and C)?

AB, AC, BC (AB = BA, AC = CA, BC = CB)

In permutation, we had 6 ways. The reason is obvious.

Not Using All Items

${}^nC_r = \frac{n!}{r!(n-r)!}$

Think

- In how many ways can a coach select 11 footballers from a squad of 15?
- What if s/he must keep 5 specific players?
- In how many ways the players can be placed in the field?

[1] n^r, where n = no. of items and r = no. of places

Think More

- How many 5-digit numbers can be made using the digits 4, 5, 2, 1, 0?
- How many are odd and even?
- How many end with zero?

1.2 Three Definitions of Probability

Classical

$$P(A) = \frac{n(A)}{n(S)}$$

Relative frequency

$$\lim_{n(S) \to \infty} \frac{n(A)}{n(S)}$$

Axiomatic

Three axioms

Say, S is sample space and A_i is an event

- $0 \leq P(A) \leq 1$ (NOT $P(A) \geq 0$)
- At least one of S will occur. P (S) = 1; Certain event.
- $P(A_1 U A_2 U ... U A_n) = P(A_1) + P(A_2) + ... + P(A_n)$ or
- $$P\left(\cup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

1.3 Probability Theorems

1. If A and B are mutually exclusive, $P(A \cup B) = P(A) + P(B)$. The law holds for more than 2 events as well, so $P(A \cup B \cup C) = P(A) + P(B) + P(C)$ and so on.

Example: $S = \{1, 2, 3\}; P(1 or 2) = P(1 \cup 2)$

2. If A and B are not mutually exclusive, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
3. If A and B are dependent events, $P(A \cap B) = P(A) \times P(B|A) = P(B) \times P(A|B)$

- $P(A) + P(\bar{A}) = 1$ (prove)
- $\sum_{i=1}^{k} P(A_i) = 1$
- $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$ (Venn)
- If A & B are independent, are $\bar{A}$ & $\bar{B}$ independent? (Prove by an example)

1.3.1 Miscellaneous Problems

Leap year friday

What is the probability that there are 53 Fridays in a leap year?

Solution

In a leap year, there are 366 days, i.e, 52 weeks and 2 days. In each week is a Fridays, so there are no less than 52 Fridays.

The remaining two days could be:

(Sat, Sun); (Sun, Mon); (Mon, Tue); (Tue, Wedn); (Wedn, Thu); (Thu, Fri); (Fri, Sat) = 7

$P = \frac{2}{7}$

Numbers 10 through 30

Out of the natural numbers 10 through 30, a number is chosen randomly; what is the probability that the number is-

i. a prime number
ii. a prime number or multiple of 5
iii. a prime number or an odd number
iv. not a perfect square

Product of three positive integers

What is the probability that the product of three positive integers chosen from 1 through 100 is an even number?

Solution

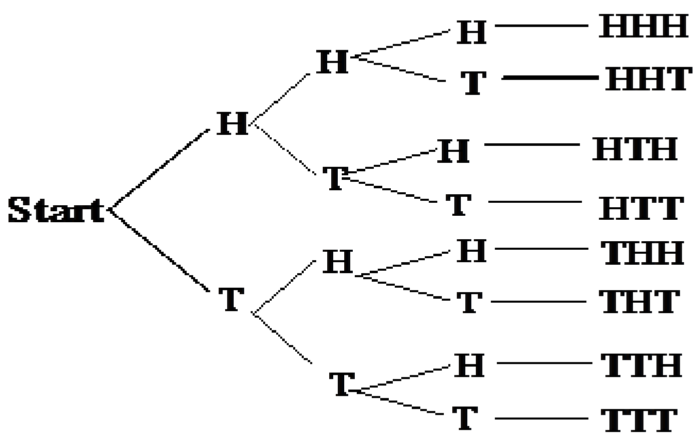
There ar three possible cases: i. All three are even ii. Two odd and one even number iii. Two even and one odd

Answer: $P = \frac{{}^{50}C_3}{{}^{100}C_3} + 2 \times \frac{{}^{50}C_2 \times {}^{50}C_1}{{}^{100}C_3}$

1.4 Coin and Die Problems

1.4.1 Tree Method

The sample space if a coin is tossed thrice (or 3 coins tossed together)



In total, we have 8 outcomes.

Think: What is the general formula? [2]

1.4.2 Table of Sample Space

A coin is tossed twice

	First Toss →	H	T
Second Toss ↓	H	HH	HT
	T	TH	TT

A coin is tossed thrice

	First 2 Toss →	HH	HT	TH	TT
Third Toss ↓	H	HHH	HHT	HTH	HTT
	T	THH	THT	TTH	TTT

[2] 2^n for a coin and 6^r for a die. What would be a more general formula?

Tables of higher order are similarly created using the combination of smaller tables. For example, the outcome of five tosses can be obtained by combining outcome of 3 tosses and that of 2 tosses.

DIY

Using tree and table, make a sample space of

- four coins tossed at once.
- 2 dice rolled together
- a coin and die thrown together
- 2 coins and a die

1.4.3 Problems

1. A coin is flipped thrice. What is the probability that

   a. the first toss gives Head
   b. The last two toss give Tails
   c. there exactly one H
   d. there are less than 3 heads

2. A coin and a die are thrown together. Find the probabilities that:

   a. the sample has an even digit
   b. the sample has a prime number

3. Two unbiased dice are rolled at once. Find the probabilities that:

   a. sum of the numbers is 7
   b. sum is less than 4
   c. both numbers are greater than 3
   d. the number are equal
   e. the numbers are different
   f. sum is a prime number

Solution without creating sample space

A fair coin is tossed 10 times.

Find the probability that a. there are exactly 3 heads b. at least 2 heads c. more than 8 heads d. at best 1 tail

Solution

$$n(S) = 2^{10}$$

a. there are ${}^{10}C_3$ ways to select 3 items out of 10 items (heads)

For example

H T H T T T T H T T

or

T T H T T T T H T H

If there are 3 tosses/coins, then 1 head can appear in the following ways.

HTT THT TTH (also ${}^{3}C_1 = 3$)

More Insight

Doesn't the above look like permutation, rather than combination?

NO.

Let's say, we have to get 1 H in 3 tosses.

They could be: HTT, THT, or TTH

We may select 1st, 2nd, or the 3rd item. (**consider tosses as individuals**)

3 tosses means there are 3 items to select from. We select 1 in ${}^{3}C_1 = 3$ ways.

Similarly, if there are 10 tosses, 2 items can be selected in ${}^{10}C_2$ ways.

1.5 Set Theory Problems

1.5.1 Newspaper reading

Out of 200 People, 50 read The Observer, 40 read the Inqilab, and 10 read both. If one person is selected, what is the probability that he

i. reads at least one paper
ii. reads the Observer or the Inqilab
iii. read the Observer but not the Inqilab
iv. read the Inqilab but not the Observer
v. reads none
vi. reads only one newspaper

1.5.2 Subject Selection

Among 800 students, 160 fail in English, 80 in Math, and 40 in both. A student is elected at random. Find the probability that s/he

i. failed in English but passed in Math
ii. passed in only one subject
iii. failed in none
iv. passed in at best one subject

1.5.3 Examinee

An examinee answer three MCQ questions randomly. There are 4 options in each question, and the options are equally likely to be correctly answered.

Find the probability that

i. The first or the third questions are correctly answered.
ii. The second and the third questions are correctly answered.
iii. All questions are correctly answered.

1.6 When to Add or Multiply

1.6.1 Union-Intersection

- Or $\rightarrow$ Add
- And $\rightarrow$ Multiply

$P(A \cup B) = P(A) + P(B)$ (mutually exclusive) $P(A \cap B) = P(A) \times P(B)$

1.6.2 Complete-Incomplete

- Completed events are added
- Incomplete events are multiplied

Example

An urn contains 5 white, 4 black, and 6 red balls. Two balls are randomly drawn. What is the probability that

i. the balls have same color
ii. the balls have different color
iii. none is black

What if the balls are drawn without replacement?

1.7 With or Without Replacement

If the process (with or without replacement) is not mentioned, we assume the items are drawn together, otherwise one-by-one.

Think

a. In how many ways can we select 2 items out of 4?
b. In how many ways can we select 2 items out of 4

    i. with replacement?
    ii. without replacement?

1.8 Problems: Drawing Items at Once

1.9 Problems: Drawing Items One by One

1.10 Addition vs Multiplication

1.11 Playing card Concept and Problems

1.12 Condional Probability Theory

1.13 Condional Probability Problems

1.14 Digit Problems

2 Random Variable and Probability Distribution

2.1 Concepts

A **random variable** is variable which is associated with probability.

A probability distribution shows how the probability is distributed among the possible values or outcomes. It gives us a patter of the data.

- Recall a histogram
- We could plot relative frequencies instead of frequencies
- relative frequencies are nothing but probabilities

Example:

2.1.1 Examples of distribution

If a biased coin is tossed once, the following may occur:

x	H	T
P(x)	1/3	2/3

This is one of the simplest kind of probability distribution.

Now, if we toss a coin twice, we get the following sample space.

	First Toss →	H	T
Second Toss ↓	H	HH	HT
	T	TH	TT

If we now define

X = no. of heads

then we can construct the following probability distribution.

x	0	1	2
P(x)	1/4	1/2	1/4

Since 1 head can appear in two ways (HT, TH), so $P(1H) = \frac{2}{4} = \frac{1}{2}$.

Similarly, $P(2H) = \frac{1}{4}$, and no head (0) can appear in 1 way, so $P(0) = \frac{1}{4}$

2.2 Problems

2.2.1 Find k from the table

x	0	1	2	3	4
P(x)	0.15	0.05	0.40	k	0.30

Hint: $\sum P(x) = 1$

2.2.2 Find k from the function and then the mentioned probabilities.

$$P(x) = \frac{x+k}{14}; x = 1, 2, 3, 4$$

i. Find k
ii. $P(X > 2)$
iii. $P(X \leq 2)$
iv. $P(X \geq 3)$
v. $P(X = 2)$
vi. $P(2 \leq X \leq 4)$

Hint

$$\frac{1+k}{14} + \frac{2+k}{14} + \cdots = 1$$

Now, these are called discrete distributions, since values of x are specific and isolated. The distributions involving a discrete random variable is called a probability (mass) function (pmf), and are denoted as P(x).

Another example is shown below.

2.2.3 Solve the problems using the probability function.

$$P(x) = \frac{2x+k}{56}; x = -3, -2, -1, 0, 1, 2, 3$$

a. Find k
b. Putting k in the function, verify this to be a pmf.
c. $P(X > 2)$
d. $P(-3 \leq x < 0)$
e. $P(X > 4)$

Hint for b: Check two things

i. Whether $0 \leq P(x) \leq 1)$ holds.
ii. $\sum(P(x) = 1$

2.3 Continuous Distributions

3 Further Reading

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