Statistics Notes (II)

Contents

1 Probbility

1.1 Important Concepts

1.1.1 Terms

Trial A single performance of well-defined experiment

Experiment a scientific test in which you perform a series of actions and carefully observe their effects in order to learn about something.

1.1.2 Set Theory

1.1.3 Permutaion

Permutaion is all about arranging items, while combination is used to find the ways to to select items.

If we have 3 items A, B, and C; we can arrange them in the following way.

- ABC
- ACB
- BAC
- BCA
- CAB
- CBA

There are 6 possible ways to arrange 3 items in 3 positions.

Thinking another way, there are 3 positions and 3 items. The first position can be filled up in 3 ways (A or B or C), the second in 2 ways (after one item is fixed in the first position, be it A or B, or C), and the third in 1 way.

Position	1	2	3
Possible options	3	2	1

Mathematically, this is also written as 3! (3 factorial), which is nothing but $3 \times 2 \times 1 = 6$

Similarly, if we have 4 items to arrange in 4 places, we can write:

Position	1	2	3	4
Possible options	4	3	2	1

Thus we can arrange this in $4 \times 3 \times 2 \times 1 = 4! = 24$ ways.

Not Using All Items

Now, what if want to arrange 2 items out of 4 items. In this case we have 2 places, but 4 items.

Position	1	2
Possible options	4	3

We get $4 \times 3 = 12$ ways.

This is also written as ${}^4P_2 = 12$ (shown below)

$$ {}^nP_r = \frac{n!}{(n-p)!} $$

Repeating Items

In the above examples, items cannot be repeated in places. In some scenario, this is absurd; after all, one person cannot sit on 2 chairs.

However, consider using digits to make up telephone numbers.

Position	1	2	3	4	5	6	7
Possible options	10	10	10	10	10	10	10

All 10 digits can be used in each position.

Hence, for 7-digit telephone numbers, we can have telephone number in 10^7 ways.

Think

- What is the general formula of the above case?[1]
- What if the first digit is always zero (0)?
- What if not all 7 digits can be same?
- What if some particular digit cannot be repeated, or can be repeated only twice?
- How many license plates can make using 5 letters, 2 digits and 3 letters, or 1 letter and 3 digits, where items can be repeated?

[1] n^r, where n = no. of items and r = no. of places

1.1.4 Combination

Combination is used when we are concerned with selecting items or individuals.

Example: How can we select 2 items out of 3 (A, B, and C)?

AB, AC, BC (AB = BA, AC = CA, BC = CB)

In permutation, we had 6 ways. The reason is obvious.

Not Using All Items

$$^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

Think

- In how many ways can a coach select 11 footballers from a squad of 15?
- What if s/he must keep 5 specific players?
- In how many ways the players can be placed in the field?

Think More

- How many 5-digit numbers can be made using the digits 4, 5, 2, 1, 0?
- How many are odd and even?
- How many end with zero?

1.2 Three Definitions of Probability

Classical

$$P(A) = \frac{n(A)}{n(S)}$$

Relative frequency

$$\lim_{n(S)\to\infty} \frac{n(A)}{n(S)}$$

Axiomatic

Three axioms

Say, S is sample space and A_i is an event

- $0 \leq P(A) \leq 1$ (NOT $P(A) \geq 0$)
- At least one of S will occur. P (S) = 1; Certain event.
- $P(A_1 U A_2 U ... U A_n) = P(A_1) + P(A_2) + ... + P(A_n)$ or
- $$P\left(\cup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

1.3 Probability Theorems

- $P(A) + P(\bar{A}) = 1$ (prove)
- $\sum_{i=1}^{k} P(A_i) = 1$
- $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$ (Venn)
- If A & B are independent, are $\bar{A}$ & $\bar{B}$ independent? (Prove by an example)

1.3.1 Miscellaneous Problems

Leap year friday

What is the probability that there are 53 Fridays in a leap year?

Solution

In a leap year, there are 366 days, i.e, 52 weeks and 2 days. In each week is a Fridays, so there are no less than 52 Fridays.

The remaining two days could be:

(Sat, Sun); (Sun, Mon); (Mon, Tue); (Tue, Wedn); (Wedn, Thu); (Thu, Fri); (Fri, Sat) = 7

$P = \frac{2}{7}$

Numbers 10 through 30

Out of the natural numbers 10 through 30, a number is chosen randomly; what is the probability that the number is-

i. a prime number
ii. a prime number or multiple of 5
iii. a prime number or an odd number
iv. not a perfect square

Product of three positive integers

What is the probability that the product of three positive integers chosen from 1 through 100 is an even number?

Solution

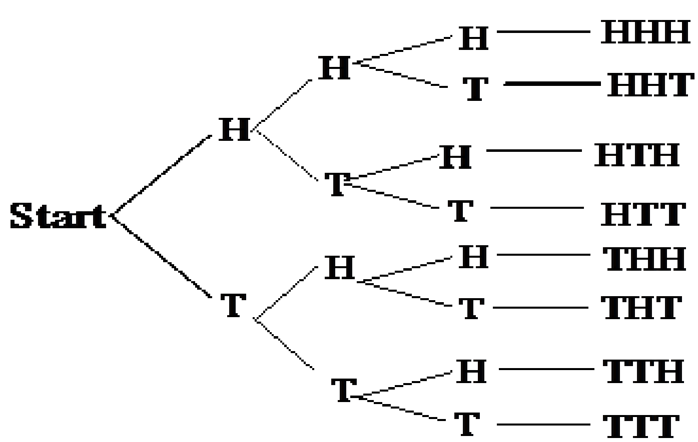
There ar three possible cases: i. All three are even ii. Two odd and one even number iii. Two even and one odd

Answer: $P = \frac{^{50}C_3}{^{100}C_3} + 2 \times \frac{^{50}C_2 \times ^{50}C_1}{^{100}C_3}$

1.4 Coin and Die Problems

1.4.1 Tree Method

The sample space if a coin is tossed thrice (or 3 coins tossed together)



In total, we have 8 outcomes.

Think: What is the general formula? [2]

1.4.2 Table of Sample Space

A coin is tossed twice

	First Toss →	H	T
Second Toss ↓	H	HH	HT
	T	TH	TT

A coin is tossed thrice

	First 2 Toss →	HH	HT	TH	TT
Third Toss ↓	H	HHH	HHT	HTH	HTT
	T	THH	THT	TTH	TTT

Tables of higher order are similarly created using the combination of smaller tables. For example, the outcome of five tosses can be obtained by combining outcome of 3 tosses and that of 2 tosses.

DIY

Using tree and table, make a sample space of

- four coins tossed at once.
- 2 dice rolled together
- a coin and die thrown together
- 2 coins and a die

1.4.3 Problems

1. A coin is flipped thrice. What is the probability that

   a. the first toss gives Head
   b. The last two toss give Tails
   c. there exactly one H
   d. there are less than 3 heads

[2] 2^n for a coin and 6^r for a die. What would be a more general formula?

2. A coin and a die are thrown together. Find the probabilities that:

    a. the sample has an even digit
    b. the sample has a prime number

3. Two unbiased dice are rolled at once. Find the probabilities that:

    a. sum of the numbers is 7
    b. sum is less than 4
    c. both numbers are greater than 3
    d. the number are equal
    e. the numbers are different
    f. sum is a prime number

Solution without creating sample space

A fair coin is tossed 10 times.

Find the probability that a. there are exactly 3 heads b. at least 2 heads c. more than 8 heads d. at best 1 tail

Solution

$$n(S) = 2^1 0$$

a. there ${}^{10}C_3$ ways to select 3 items out of 10 items (heads)

For example

H T H T T T T H T T

or

T T H T T T T H T H

If there 3 coins, then 1 head can appear in the following ways.

HTT THT TTH (${}^3C_1 = 3$)

1.5 Set Theory Problems

1.6 Problems: Drawing Items at Once

1.7 Problems: Drawing Items One by One

1.8 Addition vs Multiplication

1.9 Playing card Concept and Problems

1.10 Condional Probability Theory

1.11 Condional Probability Problems

1.12 Digit Problems