

# Statistics Notes (II)

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# 1 Probability

## 1.1 Important Concepts

### 1.1.1 Terms

**Trial** A single performance of well-defined experiment Experiment

a scientific test in which you perform a series of actions and carefully observe their effects in order to learn about something. or *An act that can be repeated under some specific condition.*

**Random variable** A variable whose values are associated with probability.

**Sample space** Set of all possible outcomes of a random experiment.

**Sample point** Each outcome of a sample space

**Event** Any subset of a sample space

**Simple event** An event having a single outcome

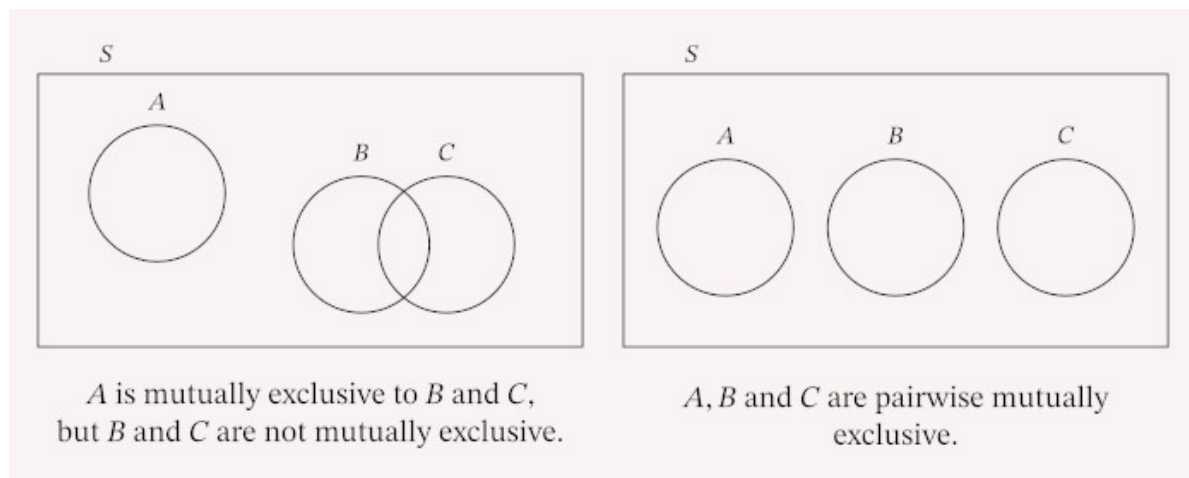
**Compound/Composite event** An event having more than one outcome

**Impossible event** An event which cannot happen (If  $P(A) = 0$ , then  $A$  is an impossible event)

**Certain event** An event which surely will or will not happen. ( $P(A) = 0$  or  $1$ )

**Uncertain event** An event which may or may not happen ( $0 < P(A) < 1$ )

**Mutually Exclusive Event** Events that cannot occur together. If  $S = \{1, 2, 3, 4\}$ ,  $A = \{1, 3\}$  &  $C = \{4\}$  then  $A$  &  $C$  are mutually exclusive.



**Independent Event** Events that do not affect each other.

**Complementary event** Non-occurrence of an event.  $P(\bar{A}) = 1 - P(A)$ , where  $\bar{A}$  or  $A'$  or  $A^c$  is called complement of A.

**Exhaustive event** Events whose union is equal to the sample space of the experiment (all outcomes are considered)

**Equally likely events** Events having same probability. If  $S = \{1, 2, 3\}$ ,  $P(1) = P(2) = P(3) = 1/3$ , here 1, 2, and 3 are equally likely. One way for them not to be equally likely is:  $P(1) = 1/2, P(2) = 1/5, P(3) = 1/4$

### 1.1.2 Set Theory

NB: This is far from a comprehensive discussion of the set theory.

#### Set Operations

Suppose,  $A = \{1, 3, 4\}$  and  $B = \{3, 4, 5\}$

- Union: A or B  $\Rightarrow A \cup B = \{1, 3, 4, 5\}$
- Intersection: A & B  $\Rightarrow A \cap B = \{3, 4\}$
- Difference:  $A - B = \{1\}$

#### Laws of Set

- a. Cumulative law:  $A \cup B = B \cup A$  and  $A \cap B = B \cap A$
- b. Associative law:  $A \cup (B \cup C) = (A \cup B) \cup C$
- c. Distribution law:  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$  and  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- d. De Morgan's law:
  - i.  $(A \cup B)' = A' \cap B'$
  - ii.  $(A \cap B)' = A' \cup B'$

#### Verify De Morgan's law

$S = \{1, 2, 6, 8\}$ ;  $A = \{1, 4\}$ ;  $B = \{2, 6\}$

### 1.1.3 Permutaion

Permutaion is all about arranging items, while combination is used to find the ways to to select items.

If we have 3 items A, B, and C; we can arrange them in the following way.

- ABC
- ACB
- BAC
- BCA
- CAB
- CBA

There are 6 possible ways to arrange 3 items in 3 positions.

Thinking another way, there are 3 positions and 3 items. The first position can be filled up in 3 ways (A or B or C), the second in 2 ways (after one item is fixed in the first position, be it A or B, or C), and the third in 1 way.

Position	1	2	3
Possible options	3	2	1

Mathematically, this is also written as  $3!$  (3 factorial), which is nothing but  $3 \times 2 \times 1 = 6$

Similarly, if we have 4 items to arrange in 4 places, we can write:

Position	1	2	3	4
Possible options	4	3	2	1

Thus we can arrange this in  $4 \times 3 \times 2 \times 1 = 4! = 24$  ways.

### Not Using All Items

Now, what if want to arrange 2 items out of 4 items. In this case we have 2 places, but 4 items.

Position	1	2
Possible options	4	3

We get  $4 \times 3 = 12$  ways.

This is also written as  ${}^4P_2 = 12$  (shown below)

$${}^nP_r = \frac{n!}{(n-p)!}$$

## Repeating Items

In the above examples, items cannot be repeated in places. In some scenario, this is absurd; after all, one person cannot sit on 2 chairs.

However, consider using digits to make up telephone numbers.

Position	1	2	3	4	5	6	7
Possible options	10	10	10	10	10	10	10

All 10 digits can be used in each position.

Hence, for 7-digit telephone numbers, we can have telephone number in  $10^7$  ways.

## Think

- What is the general formula of the above case?<sup>1</sup>
- What if the first digit is always zero (0)?
- What if not all 7 digits can be same?
- What if some particular digit cannot be repeated, or can be repeated only twice?
- How many license plates can make using 5 letters, 2 digits and 3 letters, or 1 letter and 3 digits, where items can be repeated?

## 1.1.4 Combination

Combination is used when we are concerned with selecting items or individuals.

**Example:** How can we select 2 items out of 3 (A, B, and C)?

AB, AC, BC (AB = BA, AC = CA, BC = CB)

In permutation, we had 6 ways. The reason is obvious.

## Not Using All Items

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

## Think

- In how many ways can a coach select 11 footballers from a squad of 15?
- What if s/he must keep 5 specific players?
- In how many ways the players can be placed in the field?

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<sup>1</sup> $n^r$ , where n = no. of items and r = no. of places

### Think More

- How many 5-digit numbers can be made using the digits 4, 5, 2, 1, 0?
- How many are odd and even?
- How many end with zero?

## 1.2 Three Definitions of Probability

### Classical

$$P(A) = \frac{n(A)}{n(S)}$$

### Relative frequency

$$\lim_{n(S) \rightarrow \infty} \frac{n(A)}{n(S)}$$

### Axiomatic

Three axioms

Say, S is sample space and  $A_i$  is an event

- $0 \leq P(A) \leq 1$  (NOT  $P(A) \geq 0$ )
- At least one of S will occur.  $P(S) = 1$ ; Certain event.
- $P(A_1 \cup A_2 \cup \dots \cup A_n) = P(A_1) + P(A_2) + \dots + P(A_n)$  or

•

$$P(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$$

## 1.3 Probability Theorems

1. If A and B are mutually exclusive,  $P(A \cup B) = P(A) + P(B)$ . The law holds for more than 2 events as well, so  $P(A \cup B \cup C) = P(A) + P(B) + P(C)$  and so on.

**Example:**  $S = \{1, 2, 3\}$ ;  $P(1 \text{ or } 2) = P(1 \cup 2)$

2. If A and B are not mutually exclusive,  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ .
3. If A and B are dependent events,  $P(A \cap B) = P(A) \times P(B|A) = P(B) \times P(A|B)$

- $P(A) + P(\bar{A}) = 1$  (prove)
- $\sum_{i=1}^k P(A_i) = 1$
- $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$  (Venn)
- If A & B are independent, are  $\bar{A}$  &  $\bar{B}$  independent? (Prove by an example)

### 1.3.1 Miscellaneous Problems

#### Leap year friday

What is the probability that there are 53 Fridays in a leap year?

#### Solution

In a leap year, there are 366 days, i.e, 52 weeks and 2 days. In each week is a Fridays, so there are no less than 52 Fridays.

The remaining two days could be:

(Sat, Sun); (Sun, Mon); (Mon, Tue); (Tue, Wedn); (Wedn, Thu); (Thu, Fri); (Fri, Sat) = 7

$$P = \frac{2}{7}$$

#### Numbers 10 through 30

Out of the natural numbers 10 through 30, a number is chosen randomly; what is the probability that the number is-

- a prime number
- a prime number or multiple of 5
- a prime number or an odd number
- not a perfect square

#### Product of three positive integers

What is the probability that the product of three positive integers chosen from 1 through 100 is an even number?

#### Solution

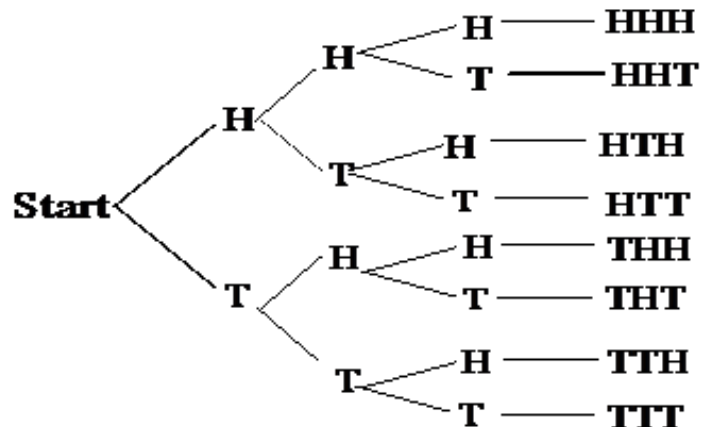
There are three possible cases: i. All three are even ii. Two odd and one even number iii. Two even and one odd

**Answer:**  $P = \frac{{}^{50}C_3}{{}^{100}C_3} + 2 \times \frac{{}^{50}C_2 \times {}^{50}C_1}{{}^{100}C_3}$

## 1.4 Coin and Die Problems

### 1.4.1 Tree Method

The sample space if a coin is tossed thrice (or 3 coins tossed together)



In total, we have 8 outcomes.

**Think:** What is the general formula? <sup>2</sup>

### 1.4.2 Table of Sample Space

A coin is tossed twice

		First Toss →	
		H	T
Second Toss ↓	H	HH	HT
	T	TH	TT

A coin is tossed thrice

		First 2 Toss →			
		HH	HT	TH	TT
Third Toss ↓	H	HHH	HHT	HTH	HTT
	T	THH	THT	TTH	TTT

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<sup>2</sup> $2^n$  for a coin and  $6^n$  for a die. What would be a more general formula?



Tables of higher order are similarly created using the combination of smaller tables. For example, the outcome of five tosses can be obtained by combining outcome of 3 tosses and that of 2 tosses.

## DIY

Using tree and table, make a sample space of

- four coins tossed at once.
- 2 dice rolled together
- a coin and die thrown together
- 2 coins and a die

### 1.4.3 Problems

1. A coin is flipped thrice. What is the probability that
  - a. the first toss gives Head
  - b. The last two toss give Tails
  - c. there exactly one H
  - d. there are less than 3 heads
2. A coin and a die are thrown together. Find the probabilities that:
  - a. the sample has an even digit
  - b. the sample has a prime number
3. Two unbiased dice are rolled at once. Find the probabilities that:
  - a. sum of the numbers is 7
  - b. sum is less than 4
  - c. both numbers are greater than 3
  - d. the number are equal
  - e. the numbers are different
  - f. sum is a prime number

### **Solution without creating sample space**

**A fair coin is tossed 10 times.**

Find the probability that a. there are exactly 3 heads b. at least 2 heads c. more than 8 heads  
d. at best 1 tail

**Solution**

$$n(S) = 2^{10}$$

a. there  ${}^{10}C_3$  ways to select 3 items out of 10 items (heads)

For example

H T H T T T H T T

or

T T H T T T H T H

If there are 3 tosses/coins, then 1 head can appear in the following ways.

HTT THT TTH (also  ${}^3C_1 = 3$ )

### **1.5 Set Theory Problems**

### **1.6 Problems: Drawing Items at Once**

### **1.7 Problems: Drawing Items One by One**

### **1.8 Addition vs Multiplication**

### **1.9 Playing card Concept and Problems**

### **1.10 Conditional Probability Theory**

### **1.11 Conditional Probability Problems**

### **1.12 Digit Problems**

## **2 Further Reads**

- <https://vrkmathsaid.weebly.com/uploads/5/1/2/1/5121151/probabilityquestionsandsolutions.pdf>