Nice Math

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Notes

Indeterminate vs undefined

When something is **undefined**, this means that there are no solutions.

$$\frac{1}{0} =?$$

$$0 \times ? = 1$$

Which number is to be multiplied with 0 to get 1? There is no such number. That's why $\frac{1}{0} =?$ is undefined.

However, when something in **indeterminate**, this means that there are infinitely many solutions to the question.

$$\frac{0}{0} =?$$

$$0 \times ? = 0$$

Which number is to be multiplied with 0 to get 0. You can multiply any number, including 0. There are infinite solutions. So it is indeterminate.

e^x

Euler Formula

$$e^{ix} = \cos x + i \sin x,$$

Proof

$$f(\theta) = \frac{cos\theta + isin\theta}{e^{\theta}} = e^{-i\theta}(cos\theta + isin\theta)$$

Now, differentiating,

$$f'(\theta) = e^{-i\theta}(icos\theta - sin\theta) - ie^{-i\theta}(icos\theta + sin\theta) = 0$$

Thus, $f(\theta)$ is a constant.

Now, $f(0) = 1$, so $f(\theta) = 1$ for all real θ, and thus

$$e^{i\theta} = \cos\theta + i \sin\theta,$$

e^x derivative

Using series

$$y = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$y' = 0 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$$

Using fundamental principle

$$
\begin{aligned}
y &= e^x \\
y' = \frac{d}{dx}y &= \frac{d}{dx}e^x \\
&= lim_{h\to 0}\frac{e^{x+h}-e^x}{h} \\
&= lim_{h\to 0}\frac{e^x(e^h-1}{h} \qquad (\#eq:exdiff) \\
&= lim_{h\to 0}e^x\cdot\frac{e^h-1}{h} \\
&= lim_{h\to 0}\cdot lim_{h\to 0}\frac{e^h-1}{h} \\
&= e^x(1) = e^x
\end{aligned} \tag{1}
$$

Intuitive Derivation

Source

The function which is its own derivative is the fundamental idea here, and the logical starting point. This function is the exponential function $exp(z)$ which we can construct based solely on that defining property, and from this function we can derive the constant $e = exp(1)$ as well as its period $2\pi i$, which is the reason π s so important), and prove various expressions for it, including

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = lim_{h\to\infty}\left(1+\frac{1}{n}\right)^n$$

So the "compound interest" aspect is the minor side effect, not the heart of the logical structure. It is often the case that history takes a meandering approach to discovery, and not the most efficient or logical route. The fact that Bernoulli stumbled upon e initially as the limit of $(1 + 1/n)^n$ is incidental.

Let me elaborate a little bit.

You should begin by seeking a function which satisfies $f' = f$

This is a very natural and very powerful idea, since plenty of differential equations which arise in math, physics and engineering can be solved using such a function (harmonic oscillators, the normal distribution and many more).

Once you hit upon the idea of finding such a function, you can build it as follows. You start with the simplest function

$$f(x) = 1$$

$f(x) = 0$ is also its own derivative, and this is not useful or enlightening).

Unfortunately the derivative is 0, not f itself. What has a derivative of 1? Well, x does, so let's add it in:

$$f(x) = 1 + x$$

e Alternate Expansion

$$e = \left(1 + \frac{1}{n}\right)^n = 1 + n \cdot \left(\frac{1}{n}\right) + \frac{n \cdot (n-1)}{2!} \cdot \left(\frac{1}{n}\right)^2 + \frac{n \cdot (n-1) \cdot (n-2)}{3!} \cdot \left(\frac{1}{n}\right)^3 + \cdots + \left(\frac{1}{n}\right)^n$$

Then

$$e = 1 + 1 + \frac{(1 - \frac{1}{n})}{2!} + \frac{(1 - \frac{1}{n}) \cdot (1 - \frac{2}{n})}{3!} + \cdots + \frac{1}{n^n}$$

Using limit,

$$\lim_{x \to 2} \left(1 + \frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Not only is it much easier to compute the terms of this infinite series and add as many of them as we please, but the sum will approach its limiting value much faster than when computing $(1 + \frac{1}{n})^n$ directly

...

Alternative to Pascal's Triangle

$$^{n}C_k = \frac{n!}{k!(n-k)!}$$

one way to understand which is in how many ways can we get k items out of n items.

A coin is tossed 4 items. We can 3 heads in $^{4}C_3$ ways.

HHTH or HTHH etc.