Statistics Notes (II)

Contents

1 Probbility

1.1 Important Concepts

1.1.1 Terms

Trial A single performance of well-defined experiment

Experiment *An act that can be repeated under some specific condition.* [A scientific test in which you perform a series of actions and carefully observe their effects in order to learn about something. or]

Random variable A variable whose values are associated with probability.

Sample space Set of all possible outcomes of a random experiment.

Sample point Each outcome of a sample space

Event Any subset of a sample space

Simple event An event having a single outcome

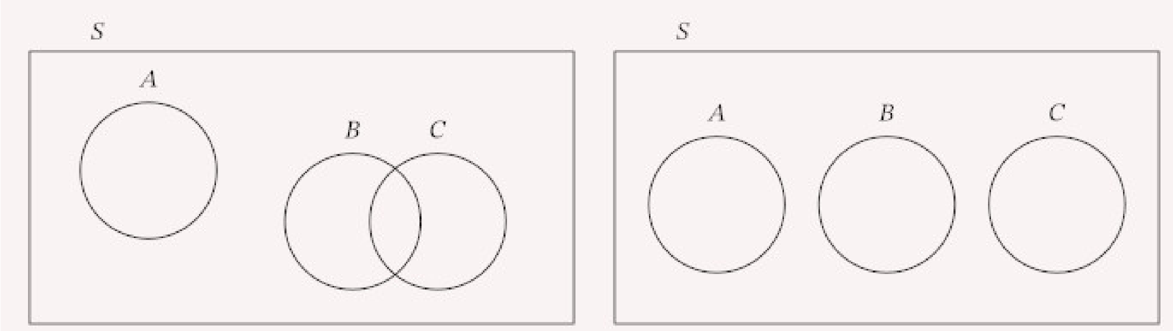
Compound/Composite event An event having more than one outcome

Impossible event An event which cannot happen (If P(A) = 0, then A is an impossible event)

Certain event An event which surely will or will not happen. (P(A) = 0 or 1)

Uncertain event An event which may or may not happen ($0 < P(A) < 1$)

Mutually Exclusive Event Events that cannot occur together. If $S = \{1, 2, 3, 4\}, A = \{1, 3\} \& C = \{4\}$ then A & B are mutually exclusive.



A is mutually exclusive to *B* and *C*, but *B* and *C* are not mutually exclusive.

A, *B* and *C* are pairwise mutually exclusive.

Independent Event Events that do not affect each other.

Complementary event Non-occurence of an event. $P(\bar{A}) = 1 - P(A)$, where $\bar{A}$ or A' or A^c is called complement of A.

Exhaustive event Events whose union is equal to the sample space of the experiment (all outcomes are considered)

Equally likely events Events having same probability. If $S = \{1, 2, 3\}, P(1) = P(2) = P(3) = 1/3$, here 1, 2, and 3 are equally likely. One way for them not to be equally likely is: $P(1) = 1/2, P(2) = 1/5, P(3) = 1/4$

1.1.2 Set Theory

NB: This is far from a comprehensive discussion of the set theory.

Set Operations

Suppose, $A = \{1, 3, 4\}$ and $B = \{3, 4, 5\}$

- Union: A or B $\Rightarrow A \cup B = \{1, 3, 4, 5\}$
- Intersection: A & B $\Rightarrow A \cap B = \{3, 4\}$
- Difference: $A - B = \{1\}$

Laws of Set

a. Cumulative law: $A \cup B = B \cup A$ and $A \cap B = B \cap A$

b. Associative law: $A \cup (B \cup C) = (A \cup B) \cup C$

c. Distribution law: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

d. De Morgan's law:

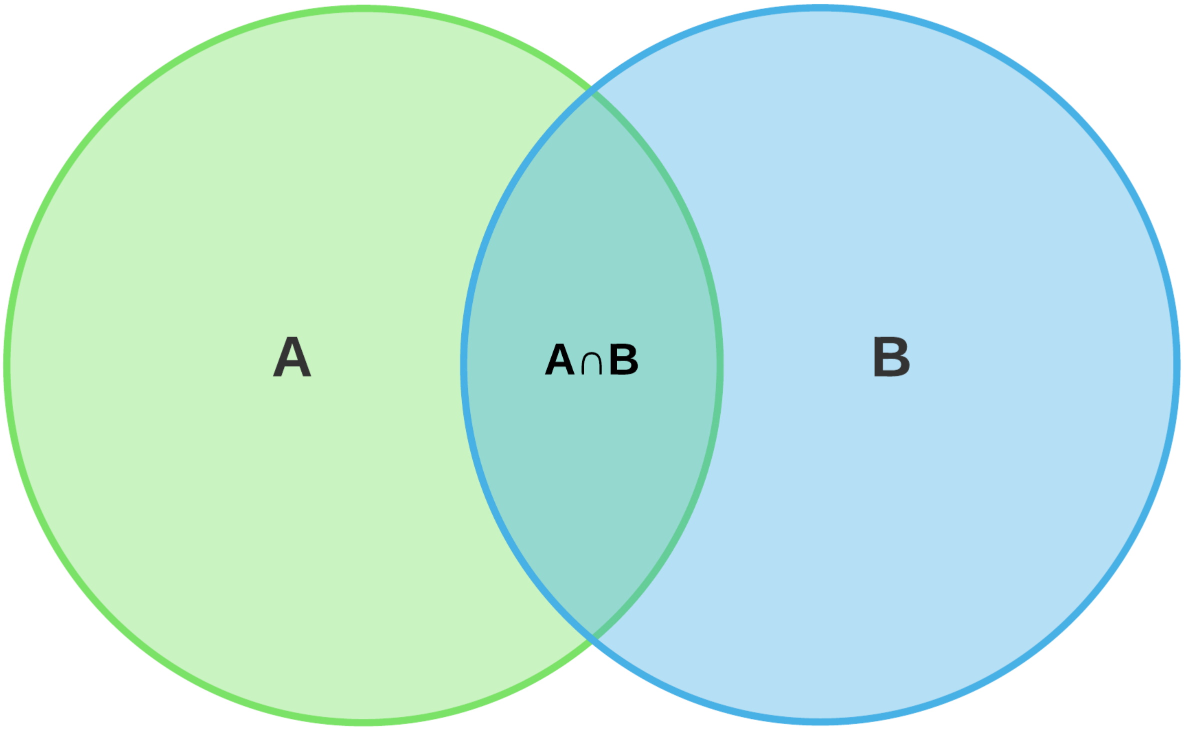
  i. $(A \cup B)' = A' \cap B'$
  ii. $(A \cap B)' = A' \cup B'$

Verify De Morgan's law

S = {1, 2, 6, 8}; A = {1,4}; B = {2,6}

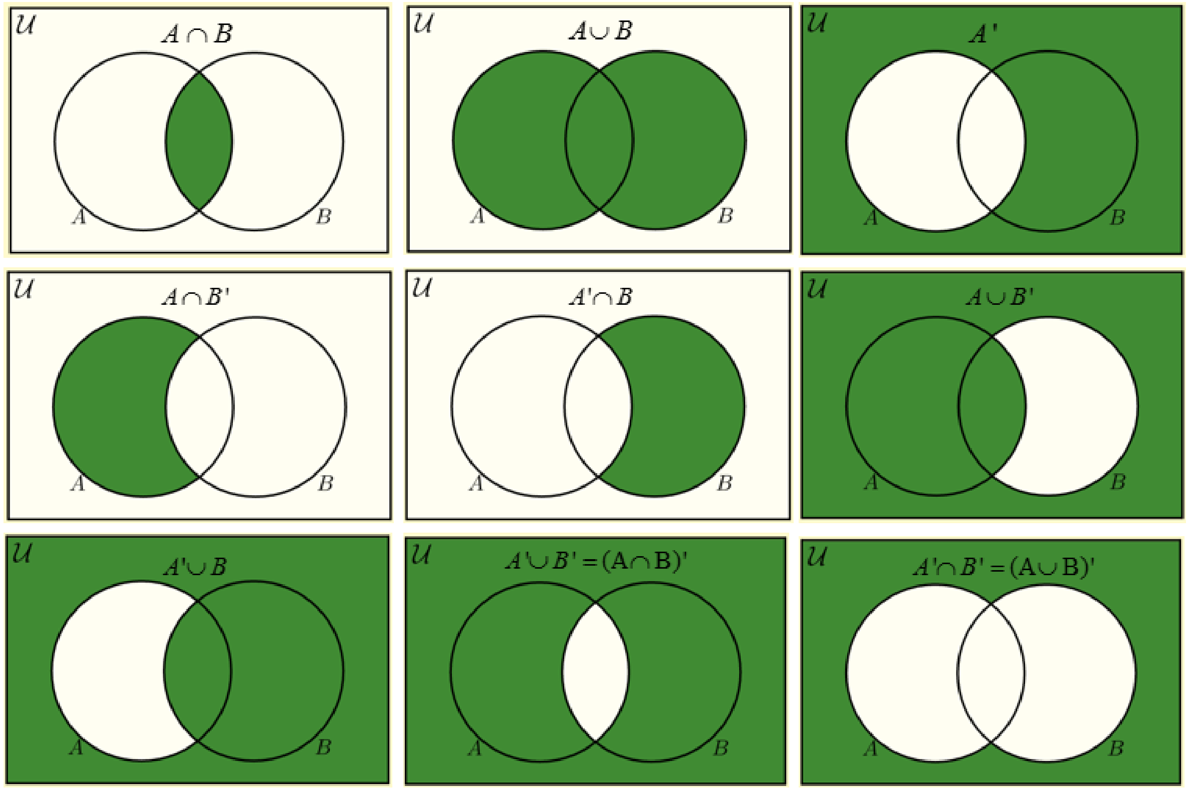
1.1.3 Venn Diagram

Locate the following sets from Venn Diagram (@mutexc)

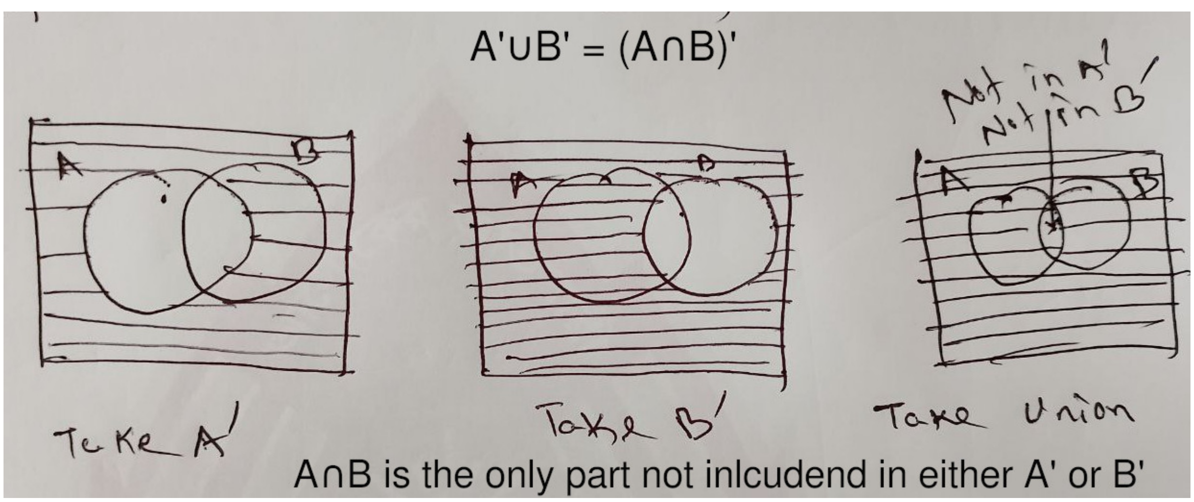


a. $A \cap B)$
b. $A \cup B$
c. A'
d. B'
e. $A \cap B' = A - A \cap B)$
f. $A' \cap B = B - A \cap B$
g. $A \cup B'$
h. $A' \cup B$
i. $A' \cup B'$
j. $A' \cap B'$

1.1.4 Venn at a glance



1.1.5 Explanation of $A' \cup B' = (A \cap B)'$



1.1.6 Permutaion

Permutaion is all about arranging items, while combination is used to find the ways to to select items.

If we have 3 items A, B, and C; we can arrange them in the following way.

- ABC
- ACB
- BAC
- BCA
- CAB
- CBA

There are 6 possible ways to arrange 3 items in 3 positions.

Thinking another way, there are 3 positions and 3 items. The first position can be filled up in 3 ways (A or B or C), the second in 2 ways (after one item is fixed in the first position, be it A or B, or C), and the third in 1 way.

Position	1	2	3
Possible options	3	2	1

Mathematically, this is also written as 3! (3 factorial), which is nothing but $3 \times 2 \times 1 = 6$

Similarly, if we have 4 items to arrange in 4 places, we can write:

Position	1	2	3	4
Possible options	4	3	2	1

Thus we can arrange this in $4 \times 3 \times 2 \times 1 = 4! = 24$ ways.

Not Using All Items

Now, what if want to arrange 2 items out of 4 items. In this case we have 2 places, but 4 items.

Position	1	2
Possible options	4	3

We get $4 \times 3 = 12$ ways.

This is also written as ${}^4P_2 = 12$ (shown below)

$$^nP_r = \frac{n!}{(n-p)!}$$

Repeating Items

In the above examples, items cannot be repeated in places. In some scenario, this is absurd; after all, one person cannot sit on 2 chairs.

However, consider using digits to make up telephone numbers.

Position	1	2	3	4	5	6	7
Possible options	10	10	10	10	10	10	10

All 10 digits can be used in each position.

Hence, for 7-digit telephone numbers, we can have telephone number in 10^7 ways.

Think

- What is the general formula of the above case?[1]
- What if the first digit is always zero (0)?
- What if not all 7 digits can be same?
- What if some particular digit cannot be repeated, or can be repeated only twice?
- How many license plates can make using 5 letters, 2 digits and 3 letters, or 1 letter and 3 digits, where items can be repeated?

1.1.7 Combination

Combination is used when we are concerned with selecting items or individuals.

Example: How can we select 2 items out of 3 (A, B, and C)?

AB, AC, BC (AB = BA, AC = CA, BC = CB)

In permutation, we had 6 ways. The reason is obvious.

Not Using All Items

$$^nC_r = \frac{n!}{r!(n-r)!}$$

Think

- In how many ways can a coach select 11 footballers from a squad of 15?
- What if s/he must keep 5 specific players?
- In how many ways the players can be placed in the field?

[1] n^r, where n = no. of items and r = no. of places

Think More

- How many 5-digit numbers can be made using the digits 4, 5, 2, 1, 0?
- How many are odd and even?
- How many end with zero?

1.2 Three Definitions of Probability

Classical

$$P(A) = \frac{n(A)}{n(S)}$$

Relative frequency

$$\lim_{n(S)\to\infty} \frac{n(A)}{n(S)}$$

Axiomatic

Three axioms

Say, S is sample space and A_i is an event

- $0 \leq P(A) \leq 1$ (NOT $P(A) \geq 0$)
- At least one of S will occur. P (S) = 1; Certain event.
- $P(A_1 U A_2 U ... U A_n) = P(A_1) + P(A_2) + ... + P(A_n)$ or
- $$P\left(\cup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

1.3 Probability Theorems

1. If A and B are mutually exclusive, $P(A \cup B) = P(A) + P(B)$. The law holds for more than 2 events as well, so $P(A \cup B \cup C) = P(A) + P(B) + P(C)$ and so on.

Example: $S = \{1, 2, 3\}; P(1 or 2) = P(1 \cup 2)$

2. If A and B are not mutually exclusive, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
3. If A and B are dependent events, $P(A \cap B) = P(A) \times P(B|A) = P(B) \times P(A|B)$

- $P(A) + P(\bar{A}) = 1$ (prove)
- $\sum_{i=1}^{k} P(A_i) = 1$
- $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$ (Venn)
- If A & B are independent, are $\bar{A}$ & $\bar{B}$ independent? (Prove by an example)

1.3.1 Miscellaneous Problems

Leap year friday

What is the probability that there are 53 Fridays in a leap year?

Solution

In a leap year, there are 366 days, i.e, 52 weeks and 2 days. In each week is a Fridays, so there are no less than 52 Fridays.

The remaining two days could be:

(Sat, Sun); (Sun, Mon); (Mon, Tue); (Tue, Wedn); (Wedn, Thu); (Thu, Fri); (Fri, Sat) = 7

$P = \frac{2}{7}$

Numbers 10 through 30

Out of the natural numbers 10 through 30, a number is chosen randomly; what is the probability that the number is-

i. a prime number
ii. a prime number or multiple of 5
iii. a prime number or an odd number
iv. not a perfect square

Product of three positive integers

What is the probability that the product of three positive integers chosen from 1 through 100 is an even number?

Solution

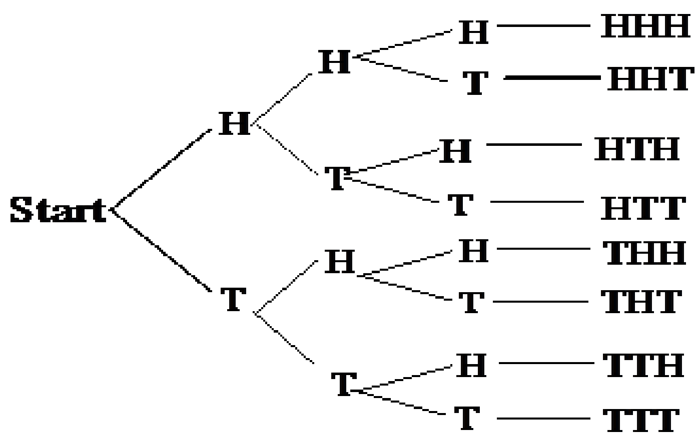
There ar three possible cases: i. All three are even ii. Two odd and one even number iii. Two even and one odd

Answer: $P = \frac{{}^{50}C_3}{{}^{100}C_3} + 2 \times \frac{{}^{50}C_2 \times {}^{50}C_1}{{}^{100}C_3}$

1.4 Coin and Die Problems

1.4.1 Tree Method

The sample space if a coin is tossed thrice (or 3 coins tossed together)



In total, we have 8 outcomes.

Think: What is the general formula? [2]

1.4.2 Table of Sample Space

A coin is tossed twice

	First Toss →	H	T
Second Toss ↓	H	HH	HT
	T	TH	TT

A coin is tossed thrice

	First 2 Toss →	HH	HT	TH	TT
Third Toss ↓	H	HHH	HHT	HTH	HTT
	T	THH	THT	TTH	TTT

[2] 2^n for a coin and 6^r for a die. What would be a more general formula?

Tables of higher order are similarly created using the combination of smaller tables. For example, the outcome of five tosses can be obtained by combining outcome of 3 tosses and that of 2 tosses.

DIY

Using tree and table, make a sample space of

- four coins tossed at once.
- 2 dice rolled together
- a coin and die thrown together
- 2 coins and a die

1.4.3 Problems

1. A coin is flipped thrice. What is the probability that

    a. the first toss gives Head
    b. The last two toss give Tails
    c. there exactly one H
    d. there are less than 3 heads

2. A coin and a die are thrown together. Find the probabilities that:

    a. the sample has an even digit
    b. the sample has a prime number

3. Two unbiased dice are rolled at once. Find the probabilities that:

    a. sum of the numbers is 7
    b. sum is less than 4
    c. both numbers are greater than 3
    d. the number are equal
    e. the numbers are different
    f. sum is a prime number

Solution without creating sample space

A fair coin is tossed 10 times.

Find the probability that a. there are exactly 3 heads b. at least 2 heads c. more than 8 heads d. at best 1 tail

Solution

$$n(S) = 2^10$$

a. there ${}^{10}C_3$ ways to select 3 items out of 10 items (heads)

For example

H T H T T T T H T T

or

T T H T T T T H T H

If there are 3 tosses/coins, then 1 head can appear in the following ways.

HTT THT TTH (also ${}^{3}C_1 = 3$)

1.5 Set Theory Problems

1.6 Problems: Drawing Items at Once

1.7 Problems: Drawing Items One by One

1.8 Addition vs Multiplication

1.9 Playing card Concept and Problems

1.10 Condional Probability Theory

1.11 Condional Probability Problems

1.12 Digit Problems

1.13 Further Reads

- https://vrkmathsaid.weebly.com/uploads/5/1/2/1/5121151/probabilityquestionsandsolutions.pdf