Statistics Notes (II)

Contents

1 Probbility

1.1 Important Concepts

1.1.1 Terms

Trial A single performance of well-defined experiment
Experiment a scientific test in which you perform a series of actions and carefully observe their effects in order to learn about something. or *An act that can be repeated under some specific condition.*
Random variable A variable whose values are associated with probability.
Sample space Set of all possible outcomes of a random experiment
Sample point Each outcome of a sample space
Event Any subset of a sample space
Simple event An event having a single outcome
Compound/Composite event An event having more than one outcome
Impossible event An event which cannot happen (If P(A) = 0, then A is an impossible event)
Certain event An event which surely will or will not happen. (P(A) = 0 or 1)
Uncertain event An event which may or may not happen ($0 < P(A) < 1$)
Mutually Exclusive Event Events that cannot occur together. If $S = 1, 2, 3, 4, A = 1, 3 \& C = 4$ then A & B are mutually exclusive.
Independent Event Events that do not affect each other.
Complementary event Non-occurence of an event. $P(\bar{A}) = 1 - P(A)$, where $\bar{A}$ or A' or A^c is called complement of A.
Exhaustive event Events whose union is equal to the sample space of the experiment (all outcomes are considered)
Equally likely events Events having same probability. If $S = 1, 2, 3, P(1) = P(2) = P(3) = 1/3$, here 1, 2, and 3 are equally likely. One way for them not to be equally likely is: $P(1) = 1/2, P(2) = 1/5, P(3) = 1/4$

1.1.2 Set Theory

NB: This is far from a comprehensive discussion of the set theory.

Set Operations

Suppose, $A = 1, 3, 4$ and $B = 3, 4, 5$

- Union: A or B $\Rightarrow A \cup B = 1, 3, 4, 5$
- Intersection: A & B $\Rightarrow A \cap B = 3, 4$
- Difference: $A - B = 1$

Laws of Set

a. Cumulative law: $A \cup B = B \cup A$ and $A \cap B = B \cap A$

b. Associative law: $A \cup (B \cup C) = (A \cup B) \cup C$

c. Distribution law: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

d. De Morgan's law:

   i. $(A \cup B)' = A' \cap B'$
   ii. $(A \cap B)' = A' \cup B'$

Verify De Morgan's law

S = {1, 2, 6, 8}; A = {1,4}; B = {2,6}

1.1.3 Permutaion

Permutaion is all about arranging items, while combination is used to find the ways to to select items.

If we have 3 items A, B, and C; we can arrange them in the following way.

- ABC
- ACB
- BAC
- BCA
- CAB
- CBA

There are 6 possible ways to arrange 3 items in 3 positions.

Thinking another way, there are 3 positions and 3 items. The first position can be filled up in 3 ways (A or B or C), the second in 2 ways (after one item is fixed in the first position, be it A or B, or C), and the third in 1 way.

Position	1	2	3
Possible options	3	2	1

Mathematically, this is also written as 3! (3 factorial), which is nothing but $3 \times 2 \times 1 = 6$

Similarly, if we have 4 items to arrange in 4 places, we can write:

Position	1	2	3	4
Possible options	4	3	2	1

Thus we can arrange this in $4 \times 3 \times 2 \times 1 = 4! = 24$ ways.

Not Using All Items

Now, what if want to arrange 2 items out of 4 items. In this case we have 2 places, but 4 items.

Position	1	2
Possible options	4	3

We get $4 \times 3 = 12$ ways.

This is also written as ${}^4P_2 = 12$ (shown below)

$${}^nP_r = \frac{n!}{(n-r)!}$$

Repeating Items

In the above examples, items cannot be repeated in places. In some scenario, this is absurd; after all, one person cannot sit on 2 chairs.

However, consider using digits to make up telephone numbers.

Position	1	2	3	4	5	6	7
Possible options	10	10	10	10	10	10	10

All 10 digits can be used in each position.

Hence, for 7-digit telephone numbers, we can have telephone number in 10^7 ways.

Think

- What is the general formula of the above case?[1]
- What if the first digit is always zero (0)?
- What if not all 7 digits can be same?
- What if some particular digit cannot be repeated, or can be repeated only twice?
- How many license plates can make using 5 letters, 2 digits and 3 letters, or 1 letter and 3 digits, where items can be repeated?

[1] n^r, where n = no. of items and r = no. of places

1.1.4 Combination

Combination is used when we are concerned with selecting items or individuals.

Example: How can we select 2 items out of 3 (A, B, and C)?

AB, AC, BC (AB = BA, AC = CA, BC = CB)

In permutation, we had 6 ways. The reason is obvious.

Not Using All Items

$$^nC_r = \frac{n!}{r!(n-r)!}$$

Think

- In how many ways can a coach select 11 footballers from a squad of 15?
- What if s/he must keep 5 specific players?
- In how many ways the players can be placed in the field?

Think More

- How many 5-digit numbers can be made using the digits 4, 5, 2, 1, 0?
- How many are odd and even?
- How many end with zero?

1.2 Three Definitions of Probability

Classical

$$P(A) = \frac{n(A)}{n(S)}$$

Relative frequency

$$\lim_{n(S)\to\infty} \frac{n(A)}{n(S)}$$

Axiomatic

Three axioms

Say, S is sample space and A_i is an event

- $0 \leq P(A) \leq 1$ (NOT $P(A) \geq 0$)
- At least one of S will occur. P (S) = 1; Certain event.
- $P(A_1 U A_2 U ... U A_n) = P(A_1) + P(A_2) + ... + P(A_n)$ or
-
$$P\left(\cup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

1.3 Probability Theorems

1. If A and B are mutually exclusive, $P(A \cup B) = P(A) + P(B)$. The law holds for more than 2 events as well, so $P(A \cup B \cup C) = P(A) + P(B) + P(C)$ and so on.

Example: $S = 1, 2, 3; P(1 or 2) = P(1 \cup 2)$

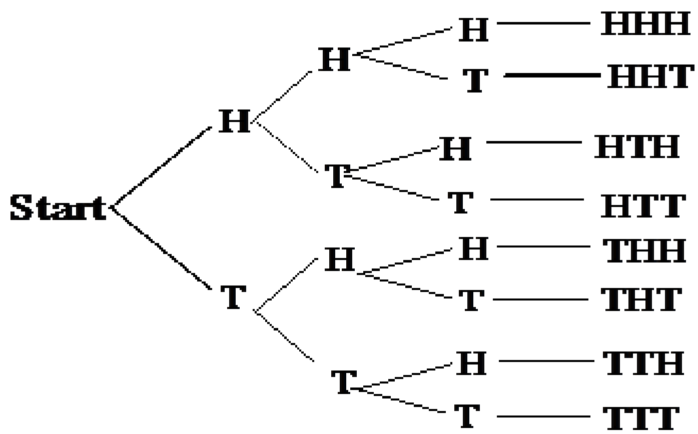
2. If A and B are not mutually exclusive, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
3. If A and B are dependent events, $P(A \cap B) = P(A) \times P(B|A) = P(B) \times P(A|B)$

- $P(A) + P(\bar{A}) = 1$ (prove)
- $\sum_{i=1}^{k} P(A_i) = 1$
- $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$ (Venn)
- If A & B are independent, are $\bar{A}$ & $\bar{B}$ independent? (Prove by an example)

1.4 Coin and Die Problems

1.4.1 Tree Method

The sample space of a coin tossed thrice (or 3 coins tossed together)



What is the general formula? [2]

1.4.2 Table of Sample Space

A coin is tossed twice

	First Toss →	H	T
Second Toss ↓	H	HH	HT
	T	TH	TT

A coin is tossed thrice

	First 2 Toss →	HH	HT	TH	TT
Third Toss ↓	H	HHH	HHT	HTH	HTT
	T	THH	THT	TTH	TTT

Tables of higher order are similarly created using the combination of smaller tables. For example, the outcome of five tosses can be obtained by combining outcome of 3 tosses and that of 2 tosses.

[2] 2^n for a coin and 6^r for a die. What would be a more general formula?

DIY

Using tree and table, make a sample space of

- four coins tossed at once.
- 2 dice rolled together
- a coin and die thrown together
- 2 coins and a die

1.4.3 Problems

1. A coin is flipped thrice. What is the probability that

   a. the first toss gives Head
   b. The last two toss give Tails
   c. there exactly one H
   d. there are less than 3 heads

2. A coin and a die are thrown together. Find the probabilities that:

   a. the sample has an even digit
   b. the sample has a prime number

3. Two unbiased dice are rolled at once. Find the probabilities that:

   a. sum of the numbers is 7
   b. sum is less than 4
   c. both numbers are greater than 3
   d. the number are equal
   e. the numbers are different
   f. sum is a prime number

Solution without creating sample space

A fair coin is tossed 10 times.

Find the probability that a. there are exactly 3 heads b. at least 2 heads c. more than 8 heads d. at best 1 tail

Solution

$$n(S) = 2^10$$

a. there ${}^{10}C_3$ ways to select 3 items out of 10 items (heads)

For example

$$\text{H T H T T T T H T T}$$

or

$$\text{T T H T T T T H T H}$$

If there 3 coins, then 1 head can appear in the following ways.
HTT THT TTH ($^3C_1 = 3$)

1.5 Set Theory Problems

1.6 Problems: Drawing Items at Once

1.7 Problems: Drawing Items One by One

1.8 Addition vs Multiplication

1.9 Playing card Concept and Problems

1.10 Condional Probability Theory

1.11 Condional Probability Problems

1.12 Digit Problems