MATH 253UX1 - Midterm # 1

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Tuesday, September 21st 2021

Print Your First & Last Name CLEARLY

Key

Date

INSTRUCTIONS:

- You must work on your own. You may not get help from any other person.
- You may use a $8'' \times 11''$ one page note prepared by you.
- This exam is **8** pages long, including this cover sheet.
- Once you access this exam in **gradescope** you will have **120 minutes** to submit your solutions.
- Download the test and print it out. Write your solutions on the test paper. When you are done, scan the pages and save the document as a **PDF** file. Upload the file back into **gradescope**.
- If you do not have a printer you may write your answers on blank paper. But use a total **8** pages pages leaving the first page blank as cover sheet. Finally, make sure the problems are clearly labeled and in order. Pretend that you are writing on this pdf. Do not submit any additional pages.
- In order to receive full credit you must show your work. Include your computations on the exam paper.
- Email me immediately if you have any trouble submitting your completed exam. You can also email me if you want to check that your exam uploaded correctly.

Total Possible Points	Score	Percent
100		

1. (22 pts.) Given the points $A(-2, 4, 0)$ and $B(1, 1, 1)$ in $\mathbb{R}^3$.

(a) (10 pts.) Find a vector and parametric equations for the line joining A and B.

$\overline{AB} = \langle 3, -3, 1 \rangle$

Here $\bar{r}_0 = \langle -2, 4, 0 \rangle$ and $\bar{v} = \langle 3, -3, 1 \rangle$

The vector equation is $\bar{r}(t) = \bar{r}_0 + t\bar{v}$

$\therefore \bar{r}(t) = \langle -2, 4, 0 \rangle + t\langle 3, -3, 1 \rangle$.

The parametric equations are

$x = -2 + 3t, \quad y = 4 - 3t, \quad z = t.$

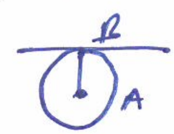
(b) (6 pts.) Find an equation of the sphere centered at A with radius AB.

$|\overline{AB}| = \sqrt{3^2 + (-3)^2 + 1^2} = \sqrt{19}$

The equation of the sphere is

$(x+2)^2 + (y-4)^2 + (z)^2 = 19$ (Ans).

(c) (6 pts.) Find an equation of the plane tangent to the sphere at B. *Hint:* A plane tangent to a sphere at a point contains that point and is orthogonal to the radius.

A normal vector to the sphere is 

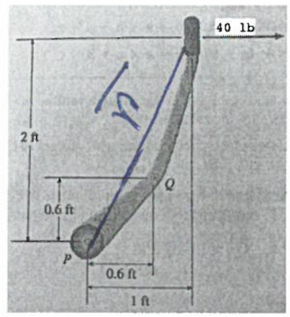
$\overline{AB} = \langle 3, -3, 1 \rangle$.

Then the equation of the tangent plane at $B(1, 1, 1)$ is

$3(x-1) - 3(y-1) + 1(z-1) = 0$

or $\boxed{3x - 3y + z = 1}$ (Ans)

2. (8 pts.) A horizontal force of 40 lb is applied to the handle of a gearshift lever as shown.



$\bar{F}$

$\bar{r} = \langle 1, 2 \rangle$

$|\bar{r}| = \sqrt{1+4} = \sqrt{5}$ ft

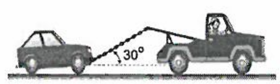
(a) (4 pts.) Find the magnitude of the torque about the pivot point P.

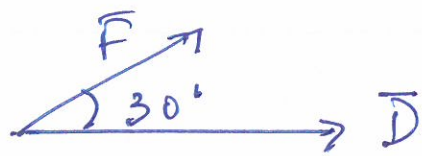
Angle between $\bar{F}$ and $\bar{r}$ is $\theta = \tan^{-1}(\frac{2}{1}) \approx 63.43°$.
The magnitude of the torque is $|\bar{\tau}| = |\bar{r} \times \bar{F}|$
$= |\bar{r}||\bar{F}| \sin\theta = \sqrt{5}\,(40) \sin 63.43°$
≈ 80.0 ft-lb.

(b) (4 pts.) Find the magnitude of the torque about P if the same force is applied at the elbow Q of the lever.

$\bar{r} = \bar{PQ} = \langle 0.6, 0.6 \rangle$, $|r| = \sqrt{0.6^2 + 0.6^2} = \sqrt{0.72}$
and $\theta = 45°$, and $|\bar{\tau}| = |\bar{r}||\bar{F}| \sin\theta$
$= \sqrt{0.72}\,(40) \sin 45°$
$= 20\sqrt{1.44} = 24$ ft-lb.

3. (6 pts.) A tow truck drags a stalled car along a road. The chain make an angle of 30^0 with the road and the tension in the chain is 1800 N.





How much work is done by the truck in pulling the car 1.5 km?

$W = \bar{F} \cdot \bar{D}$
$= |\bar{F}||\bar{D}| \cos\theta$
$= 1800\text{ N}\ 1500\text{ m} \cos 30°$
$= 2700000 \times \frac{\sqrt{3}}{2}$
$= 1350000\sqrt{3}$ Nm
$= 2.338 \times 10^6$ Jouls.

4. (24 pts. – 6 pts. each) Consider the space curve given parametrically by

$$\mathbf{r}(t) = \langle \sin 3t,\ \cos 3t,\ 3t \rangle \text{ for } t \in \mathbb{R}.$$

(a) Find the unit tangent vector $\mathbf{T}(t)$ and the unit normal vector $\mathbf{N}(t)$.

$$\bar{r}'(t) = \langle 3\cos 3t, -3\sin 3t, 3 \rangle$$

$$|r'(t)| = \sqrt{9\cos^2 3t + 9\sin^2 t + 9} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}.$$

$$T(t) = \frac{\bar{r}'(t)}{|r'(t)|} = \frac{1}{3\sqrt{2}} \langle 3\cos 3t, -3\sin 3t, 3 \rangle$$

$$= \frac{1}{\sqrt{2}} \langle \cos 3t, -\sin 3t, 1 \rangle.$$

$$\bar{T}'(t) = \frac{1}{\sqrt{2}} \langle -3\sin 3t, -3\cos 3t, 0 \rangle$$

$$|T'(t)| = \sqrt{\frac{9}{2}\sin^2 3t + \frac{9}{2}\cos^2 3t + 0} = \sqrt{\frac{9}{2}} = \frac{3}{\sqrt{2}}$$

$$\bar{N}(t) = \frac{\bar{T}'(t)}{|\bar{T}'(t)|}$$

$$= \frac{\frac{1}{\sqrt{2}} \langle -3\sin 3t, -3\cos 3t, 0 \rangle}{\frac{3}{\sqrt{2}}}$$

$$= \langle -\sin 3t, -\cos 3t, 0 \rangle$$

(b) Find an equation for the normal plane at $(0, 1, 0)$.

The point $(0, 1, 0)$ corresponds to $t = 0$.

$$\bar{N}(0) = \langle 0, -1, 0 \rangle.$$

Then the equation of the normal plane is

$$0(x-0) - 1(y-1) + 0(z-0) = 0$$

$$\Rightarrow y - 1 = 0$$

or $\boxed{y = 1}$.

(Continued) The space curve given parametrically by

$$\mathbf{r}(t) = \langle \sin 3t,\ \cos 3t,\ 3t \rangle \text{ for } t \in \mathbb{R}.$$

(c) Find the curvature at $t = \pi/2$.

$$\kappa(t) = \frac{|T'(t)|}{|r'(t)|} = \frac{3/\sqrt{2}}{3\sqrt{2}} = \frac{3}{\sqrt{2}\cdot 3\sqrt{2}}$$

$$= \frac{1}{2}.$$

(d) Find the length of the curve $\mathbf{r}(t)$ for $0 \le t \le \pi$.

$$L = \int_0^\pi |r'(t)|\,dt$$

$$= \int_0^\pi 3\sqrt{2}\,dt$$

$$= 3\sqrt{2}\,[t]_0^\pi$$

$$= 3\sqrt{2}\,\pi.$$

5. (12 pts.) Find the velocity function $\mathbf{v}(t)$ and position function $\mathbf{r}(t)$ of a particle moving in the plane with acceleration $\mathbf{a}(t) = \langle t/2, 0, -t\rangle$ if it starts at $\mathbf{r}(0) = \langle 3, -1, 5\rangle$ with initial velocity $\mathbf{v}(0) = \langle 0, 2, 1\rangle$.

$$\bar{v}(t) = \int \bar{a}(t)\,dt$$

$$= \int \langle \tfrac{t}{2}, 0, -t \rangle\,dt$$

$$\bar{v}(t) = \langle \tfrac{t^2}{2}, c_2, -\tfrac{t^2}{2} \rangle + \bar{c}_1$$

$$\bar{v}(0) = \langle 0, c_2, 0 \rangle + \bar{c}_1$$

$$\langle 0, 2, 1 \rangle = \langle 0, c_2, 0 \rangle + \bar{c}_1$$

$$\therefore \bar{c}_1 = \langle 0, 2-c_2, 1 \rangle$$

$$\therefore \bar{v}(t) = \langle \tfrac{t^2}{4}, c_2, -\tfrac{t^2}{2} \rangle + \langle 0, 2-c_2, 1 \rangle$$

$$\bar{v}(t) = \langle \tfrac{t^2}{4}, 2, -\tfrac{t^2}{2}+1 \rangle.$$

$$\bar{r}(t) = \int \bar{v}(t)\,dt$$

$$= \int \langle \tfrac{t^2}{4}, 2, -\tfrac{t^2}{2}+1 \rangle\,dt$$

$$\bar{r}(t) = \langle \tfrac{t^3}{3}, 2t, -\tfrac{t^3}{6}+t \rangle + \bar{c}_2$$

$$\bar{r}(0) = \langle 0, 0, 1 \rangle + \bar{c}_2$$

$$\langle 3, -1, 5 \rangle = \bar{c}_2$$

$$\therefore \bar{r}(t) = \langle \tfrac{t^3}{12}, 2t, -\tfrac{t^3}{6}+t \rangle + \langle 3, -1, 5 \rangle$$

$$= \langle \tfrac{t^3}{12}+3, 2t-1, -\tfrac{t^3}{6}+t+5 \rangle$$

(Ans.)

6. (12 pts.) Consider the space curve $\mathbf{r}(t)$ with parametric equations $x = -2 + 3\cos t$, $y = 14 - 12\cos t - 6\sin t$, and $z = -1 + 3\sin t$.

(a) (5 pts.) Show that $\mathbf{r}(t)$ lies on intersection of the paraboloid $y = x^2 + z^2$ and the plane $4x + y + 2z = 4$.

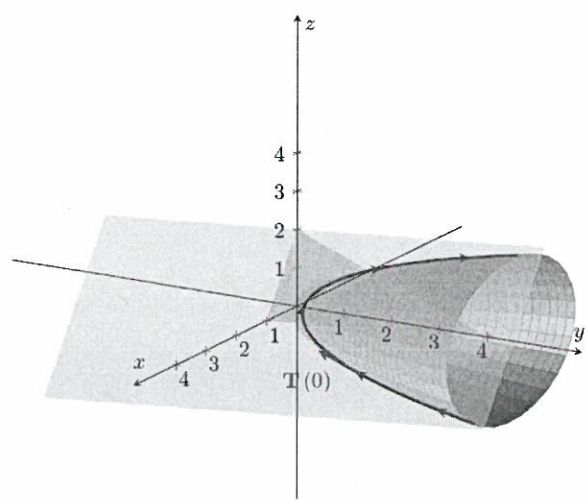
$$x^2 + z^2 = (-2 + 3\cos t)^2 + (-1 + 3\sin t)^2$$
$$= 4 - 12\cos t + 9\cos^2 t + 1 - 6\sin t + 9\sin^2 t$$
$$= 14 - 12\cos t - 6\cos t$$
$$= y.$$

and

$$4x + y + 2z = 4(-2 + 3\cos t) + 14 - 12\cos t - 6\cos t$$
$$+ 2(-1 + 3\sin t)$$
$$= -8 + 12\cos t + 14 - 12\cos t - 6\sin t$$
$$- 2 + 6\sin t$$
$$= 4$$

$\therefore$ $\vec{r}(t)$ lies on intersection of $y = x^2 + z^2$ and $4x + y + 2z = 4$.

(b) (7 pts.) Sketch the surfaces $y = x^2 + z^2$ and $4x + y + 2z = 4$ and a rough sketch of $\mathbf{r}(t)$.



7. (16 pts.) A projectile is launched from the point $(0, 8)$ of a coordinate system where unit length is measured in feet. The launching speed of the projectile is 192 feet per second and it is launched in the positive direction at an angle of elevation of 30^0.

(a) (8 pts.) Find a vector function that models the projectile's path.

If we place the origin at ground level, then the initial position of the projectile is $(0, 8)$ and $v_0 = 192$ ft/s, $\alpha = 30°$ and $g = 32$ ft/s^2.

$x = v_0 \cos\alpha\, t$, $y = h + v_0 \sin\alpha\, t - \frac{1}{2} g t^2$

$= 192 \cos 30° t$ | $y = 8 + 96t - 16t^2$

$= 96\sqrt{3}\, t$

$\therefore \bar{r}(t) = 96\sqrt{3}\, t\, \hat{i} + (8 + 96t - 16t^2)\, \hat{j}$

(b) (8 pt.) Find the maximum height attained by the projectile, its range, and the speed of impact. (ignore air resistance)

velocity $\bar{v}(t) = \bar{r}'(t) = 96\sqrt{3}\, \hat{i} + (96 - 32t)\, \hat{j}$

At maximum height $v_y = 0$

$\therefore\ 96 - 32t = 0 \Rightarrow \boxed{t = 3}$.

Then the maximum height is

$h = 8 + 96(3) - 16(3) = 152$ ft.

The projectile reaches the ground when $y(t) = 0$

$\Rightarrow 8 + 96t - 16t^2 = 0 \Rightarrow t = \frac{6 + \sqrt{38}}{2} = 6.08$ s.

range $= x(6.08) = 96\sqrt{3}(6.08) = 1010.96$ ft.

The impact occurs at $t = 6.08$ s. The velocity at impact is

$\bar{v}(6.08) = 96\sqrt{3}\, \hat{i} + (96 - 32(6.08))\, \hat{j}$

$|\bar{v}(6.08)| = 193.29$ ft/se

(c) (5 extra pt.) Set up, **but do not integrate**, a definite integral that computes the arc length of the trajectory of the projectile. (Give your answer in simplified form for credit.)

$$L = \int_0^{6.08} |v(t)|\, dt$$

$$= \int_0^{6.08} 183.29\, dt.$$