# Linear Algebra

## Assignment # 02

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**Section**: 4A-BCS

**Submitted to:**

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**Questions**

-> What is a matrix determinant?

-> What are the properties of a determinant?

-> Explain each property with the help of an example.

## Introduction to Determinants

The determinant of a matrix is a number that is specially defined only for square matrices. For every square matrix, $A = |a_{ij}|$ of order $m \times m$, a determinant can be defined as a scalar value that is real or a complex number.

→ It is denoted as

$$\det(A) \text{ or } |A|$$

→ Determinant are mathematical object that are very useful in the analysis and solutions of systems of linear equations.

## Properties of a Determinant.

### Property 1: Row Operations

→ When we add a multiple of a row to another row, the determinant of a matrix is unchanged.

Example:

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 8 \\ 0 & 3 \end{bmatrix}$

$|A| = 3$ and $|B| = 3$

Hence $|B| = |A|$

### Switching Two Rows

→ If B is the matrix obtained by permuting two rows of A, then

$$\det(B) = -\det(A)$$

Example:

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$. Knowing that $\det(A) = -2$, find $\det(B)$.

## Solution

$\det(A) = 1 \times 4 - 3 \times 2 = -2$

Note that rows of B are the rows of A but switched.

Now, two rows of A have been switched, $\det(B) = -\det(A) \Rightarrow -(-2) = 2$

### Multiplying a row by a Scalar

→ If B is the matrix obtained by multiplying one row of A by any scalar k, then

$$\det(B) = k \det(A).$$

**Example:**

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 6 & 6 \end{bmatrix}$

As matrix B second row is multiply by 3 so,

$|A| = -4$ and $|B| = -12$

so $\det(B) = (3)(-4)$

### Property 2: Determinant of the Inverse

Let A be an nxn matrix. Then A is inversible iff $\det(A) \neq 0$.

**Example:**

Let $B = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$

$$|B| = (2)(1) - (5)(3) = -13$$

Here B is invertible.

## Property 3: Determinant of the Transpose

Let A be a matrix where $A^T$ is the transpose of A, then

$$\det(A^T) = \det(A)$$

Example:

Let $A = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$ and $A^T = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix}$

Now $|A| = 2\times3 - 5\times4$ $\quad |A^T| = 2\times3 - 5\times4$

$= -14$ $\quad = -14$

Hence $|A| = |A^T|$

## Property 4: Multiplicative Property

Let A and B be two $n\times n$ matrices. Then

$$\det(AB) = \det(A)\det(B)$$

Example:

Let $A = \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$

So

$$AB = \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 11 & 4 \\ -1 & -4 \end{bmatrix}$$

$$|AB| = (11)(-4) - (-1)(4)$$
$$= -40$$

And $|A| = (1)(2) - (-3)(2) = 8$

$|B| = (3)(1) - (4)(2) = -5$

$|A| \times |B| = 8 \times -5 = -40$

Hence $\det(AB) = \det(A)\det(B)$ ✓

## Property 5: All-zero property

If all the elements of a row or a column are zeros, then the value of the determinat is zero

$$\det(A) = 0$$

### Example

Let $A = \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}$

since $R_2 = 0$ so $|A| = (2)(0) - (4)(0)$

Hence $|A| = 0$ ✓

## Property 6: Identical value.

If any two raws/columns of a determinant are identical, then the value of determinant is zero.

### Example

Let $A = \begin{bmatrix} 4 & 2 & 1 \\ 3 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 4 & 2 & 1 \\ 3 & 1 & 0 \\ 4 & 2 & 1 \end{vmatrix}$ Here $R_1$ & $R_3$ are identical.

Expanding along $C_3$

$$|A| = 1\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} - 0\begin{vmatrix} 4 & 2 \\ 4 & 2 \end{vmatrix} + 1\begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix}$$

$|A| = 1(6-4) - 0 + 1(4-6)$

$|A| = 0$ //

## Property 7: Triangle Property

The determinant of a triangular matrix is the product of the main diagonal entries

### Example

Let $|A| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & -3 & 0 & 0 \\ 4 & -1 & 3 & 0 \\ 2 & -5 & 1 & 2 \end{vmatrix}$

Expanding $R_1$

$1 \begin{vmatrix} -3 & 0 & 0 \\ -1 & 3 & 0 \\ -5 & 1 & 2 \end{vmatrix}$ Now Expanding $R_1$ again.

$1 \cdot (-3) \begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix}$ $\Rightarrow (-3)(6-0)$

$= -18$

$|A| = -18$ //

## Property 8: Common Factor

A common factor of a row/column can be taken out of $|A|$.

**Example**

Let $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 8 & 10 & 12 \\ 7 & 8 & 9 \end{vmatrix}$

Here $R_2$ has the common factor of 2

so $|A| = 2\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

$= 2 \cdot 0$

$= 0$

## Property 9: Identity Matrix

The determinant of identity matrix is always 1

**Example:**

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$|A| = (1)(1) - (0)(0)$

$= 1$ ✓

## Property 10: Multiple-terms property

If element of any row/column consist of two or more terms, then the determinant can be expressed as sum of two or more determinants.

$$\begin{vmatrix} a_1+b_1 & c_1 \\ a_2+b_2 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

Example

Let $A = \begin{vmatrix} x & y & z \\ 2x+2a & 2y+2b & 2z+2c \\ a & b & c \end{vmatrix}$

$$= \begin{vmatrix} x & y & z \\ 2x & 2y & 2c \\ a & b & c \end{vmatrix} + \begin{vmatrix} x & y & z \\ 2a & 2b & 2c \\ a & b & c \end{vmatrix}$$

$$= 0 + 0$$

$$|A| = 0$$