Linear Algebra

Assignment # 02

Name	**Roll Number**
Mahnoor Ghazanfar	FA18-BSE-122

Section: 4A-BCS

Submitted to:

Sir Umair Umer

Questions

-> What is a matrix determinant?

-> What are the properties of a determinant?

-> Explain each property with the help of an example.

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Introduction to Determinants

→ **Determinants**: The determinant of a matrix is a number that is specially defined only for square matrices. Determinants are mathematical objects that are very useful in the analysis and solution of systems of linear equation.

Properties of a Determinant.

Property 1: Row Operations

→ When we add a multiple of a row to another row, the determinant of a matrix is unchanged.

Example:

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 8 \\ 0 & 3 \end{bmatrix}$

$|A| = 3$ and $|B| = 3$

Hence $|B| = |A|$

Switching Two Rows

→ If B is the matrix obtained by permuting two rows of A, then

$$\det(B) = -\det(A)$$

Example:

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$. Knowing that $\det(A) = -2$, find $\det(B)$.

Solution

$\det(A) = 1 \times 4 - 3 \times 2 = -2$

Note that rows of B are the rows of A but switched.

Now, two rows of A have been switched, $\det(B) = -\det(A) \Rightarrow -(-2) = 2$

Multiplying a row by a Scalar

→ If B is the matrix obtained by multiplying one row of A by any scalar k, then

$$\det(B) = k \det(A).$$

Example:

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 6 & 6 \end{bmatrix}$

As matrix B second row is multiply by 3 so,

$|A| = -4$ and $|B| = -12$

so $\det(B) = (3)(-4)$

Property 2: Determinant of the Inverse

Let A be an $n \times n$ matrix. Then A is inversible iff $\det(A) \neq 0$.

Example:

Let $B = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$

$$|B| = (2)(1) - (5)(3) = -13$$

Here B is invertible.

Property 3: Determinant of the Transpose

Let A be a matrix where A^T is the transpose of A, then

$$\det(A^T) = \det(A)$$

Example:

Let $A = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$ and $A^T = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix}$

Now $|A| = 2\times3 - 5\times4$ $\quad |A^T| = 2\times3 - 5\times4$

$= -14$ $\quad = -14$

Hence $|A| = |A^T|$

Property 4: Multiplicative Property

Let A and B be two $n\times n$ matrices. Then

$$\det(AB) = \det(A)\det(B)$$

Example:

Let $A = \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$

So

$$AB = \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 11 & 4 \\ -1 & -4 \end{bmatrix}$$

$$|AB| = (11)(-4) - (-1)(4)$$
$$= -40$$

And $|A| = (1)(2) - (-3)(2) = 8$

$|B| = (3)(1) - (4)(2) = -5$

$|A| \times |B| = 8 \times -5 = -40$

Hence $\det(AB) = \det(A)\det(B)$ ✓

Property 5: All-zero property

If all the elements of a row or a column are zeros, then the value of the determinat is zero

$$\det(A) = 0$$

Example

Let $A = \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}$

since $R_2 = 0$ so $|A| = (2)(0) - (4)(0)$

Hence $|A| = 0$ ✓

Property 6: Identical value.

If any two raws/columns of a determinant are identical, then the value of determinant is zero.

Example

Let $A = \begin{bmatrix} 4 & 2 & 1 \\ 3 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 4 & 2 & 1 \\ 3 & 1 & 0 \\ 4 & 2 & 1 \end{vmatrix}$ Here R_1 & R_3 are identical.

Expanding along C_3

$$|A| = 1\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} - 0\begin{vmatrix} 4 & 2 \\ 4 & 2 \end{vmatrix} + 1\begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix}$$

$|A| = 1(6-4) - 0 + 1(4-6)$

$|A| = 0$

Property 7: Triangle Property

The determinant of a triangular matrix is the product of the main diagonal entries

Example

Let $|A| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & -3 & 0 & 0 \\ 4 & -1 & 3 & 0 \\ 2 & -5 & 1 & 2 \end{vmatrix}$

Expanding R_1

$1 \begin{vmatrix} -3 & 0 & 0 \\ -1 & 3 & 0 \\ -5 & 1 & 2 \end{vmatrix}$ Now Expanding R_1 again.

$1 \cdot (-3) \begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix} \Rightarrow (-3)(6-0)$
$= -18$

$|A| = -18$

Property 8: Common Factor

A common factor of a row/column can be taken out of $|A|$.

Example

Let $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 8 & 10 & 12 \\ 7 & 8 & 9 \end{vmatrix}$

Here R_2 has the common factor of 2

so $|A| = 2\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

$= 2 \cdot 0$

$= 0$