# Classical Electrodynamics I Review Notes

Matthew Ho
Carnegie Mellon University 33-761, Fall 2017
Taught by: James Russ
Reference: Jackson 3rd Edition [Ja]

## 1 Introduction to Electrostatics

[Ja 24-56]

### 1.1 Basic Electrostatics

The electric field induced by a point charge $q_1$ at $\mathbf{x}_1$:

$$\mathbf{E}(\mathbf{x}) = \frac{q_1}{4\pi\epsilon_0}\frac{\mathbf{x}-\mathbf{x}_1}{|\mathbf{x}-\mathbf{x}_1|^3} \tag{1}$$

The electric field induced by a charge distribution $\rho(\mathbf{x})$:

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{x}')\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}d^3x' \tag{2}$$

Gauss's Law relates the distribution of electric charge to the resulting electric field:

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad \oint_S \mathbf{E}\cdot\mathbf{n}\;da = \frac{1}{\epsilon_0}\int_V \rho(\mathbf{x})\;d^3x \tag{3}$$

Scalar potential $\mathbf{\Phi}(\mathbf{x})$ is defined:

$$\nabla\times\mathbf{E} = 0, \qquad \mathbf{E} = -\nabla\Phi, \qquad \Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\;d^3x' \tag{4}$$

Poisson's Equation is given by.

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0} \tag{5}$$

It's called Laplace's equation if $\rho = 0$.

Electrostatic Potential Energy

$$W = \frac{1}{2}\int\rho(\mathbf{x})\Phi(\mathbf{x})\;d^3x \qquad = \frac{-\epsilon_0}{2}\int\Phi\;\nabla^2\Phi\;d^3x \qquad = \frac{\epsilon_0}{2}\int|\mathbf{E}|^2\;d^3x \tag{6}$$

For a system of conductors at various potentials

$$Q_i = \sum_{j=1}^{n} C_{ij}V_j, \qquad W = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}V_iV_j \tag{7}$$

### 1.2 Boundary Conditions and Green's Theorem

1. **Dirichlet boundary conditions** are problems which specify the potential $\Phi$ on a closed surface.
2. **Neumann boundary conditions** define the normal derivative of the potential $\frac{\partial \Phi}{\partial n}$ on surfaces.

From Gauss's law, we know that Neumann BC specify a certain surface charge density. Jackson uses Green's first identity to prove that there are unique solutions to the Poisson equation according to Dirichlet or Neumann boundary conditions.

Green's theorem (shown below) is derived from the divergence theorem acting on some vector field $\mathbf{A} = \phi\ \nabla \psi$, where $\phi$ and $\psi$ are arbitrary scalars [Ja 36].

$$\int_V \left(\phi \nabla^2 \psi - \psi \nabla^2 \phi\right)\, d^3x = \oint_S \left[\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\right] da \tag{8}$$

Choose a class of functions $\psi = G(\mathbf{x}, \mathbf{x}')$ that satisfy the relation: $\nabla'^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}')$. These are typically in the form $G(\mathbf{x}, \mathbf{x}') = |\mathbf{x} - \mathbf{x}'|^{-1} + F(\mathbf{x}, \mathbf{x}')$ where $\nabla'^2 G(\mathbf{x}, \mathbf{x}') = 0$. Choosing $\phi = \Phi$, we obtain

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{x}') G(\mathbf{x}, \mathbf{x}')\, d^3x' + \frac{1}{4\pi} \oint_S \left[G(\mathbf{x}, \mathbf{x}') \frac{\partial \Phi}{\partial n'} - \Phi(\mathbf{x}') \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'}\right] da' \tag{9}$$

Green's functions help account for boundary conditions of a system. They are related to the system's geometry, not dependent on charge distributions, so they can be generalized to various problems.

For Dirichlet BC, we demand $G_D(\mathbf{x}, \mathbf{x}') = 0$ for $\mathbf{x}'$ on $S$.

For Neumann BC, we demand $\frac{\partial G_N}{\partial n}(\mathbf{x}, \mathbf{x}') = -\frac{4\pi}{S}$ for $\mathbf{x}'$ on $S$, where $S$ is the total area of the boundary surface. The most common Neumann problem is one in which a volume is bounded by two surfaces, one closed and finite, the other at infinity. In this case, $S \to \infty$ and $\frac{\partial G_N}{\partial n}(\mathbf{x}, \mathbf{x}') = 0$.

## 2 Boundary-Value Problems in Electrostatics: I

[Ja 57-94]

### 2.1 Method of Images

Jackson discusses four simple scenarios for applying method of images.

**Point charge in front of a grounded, infinite plane**. This solution is simple. If the charge is a distance $d$ from the surface of the plane, its image charge is mirrored across the plane with opposite charge.

**Point charge near a grounded, conducting sphere**. The problem is stated with the original charge $q$ at a distance $y$ from the center of the conducting sphere. By symmetry, the image charge will lie on the line between the original charge $q$ and the center of the sphere. The image charge is thus described by a charge amount $q'$ and a distance from the center of the sphere $y'$ along the axis of the original charge. Find the equation for $\Phi(\mathbf{x})$ given the original charge and the point charge locations. Set this equal to 0 at the conducting sphere's radius $a$ and solve for $q'$ and $y'$. Below are the results

$$q' = -\frac{a}{y}q, \qquad y' = \frac{a^2}{y} \tag{10}$$

This solution works for both a charge outside and inside a grounded spherical shell. You can then find surface charge using $\sigma = -\epsilon_0 \frac{\partial \Phi}{\partial x}|_{x=a}$.

**Point charge near a charged, insulated, conducting sphere**. We can treat this problem as if the system was initially in the state of the previous problem and the ground was removed and charges are added to reach a total charge of $Q$. We can assume that the added charge distributes itself uniformly over the surface, since the charges are already balanced in the previous problem. Therefore, in our expression for $\Phi$, we account for a second image charge $(Q - q')$ located at the center of the sphere.

**Point charge near a conducting sphere at fixed potential**. This is the same reasoning as the charged, conducting sphere, except the second image charge at the center of the sphere has charge $4\pi\epsilon_0 V a$.

### 2.2 Green's Function for the Sphere

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{\mathbf{x} - \mathbf{x}'} - \frac{a}{x'|\mathbf{x} - \frac{a^2}{x'^2}\mathbf{x}'|} \qquad = \frac{1}{(x^2 + x'^2 - 2xx'\cos\gamma)^{1/2}} - \frac{1}{(\frac{x^2x'^2}{a^2} + a^2 - 2xx'\cos\gamma)^{1/2}} \tag{11}$$

The second expression is the first in spherical coordinates. $\gamma$ is the angle between $\mathbf{x}$ and $\mathbf{x}'$.

### 2.3 Orthogonal Functions and Expansions

Consider an interval $(a, b)$ in a variable $\xi$ with a complete set of real or complex functions $U_n(\xi)$, $n = 1, 2, \ldots$ which are orthogonal, $\int_a^b U_n^*(\xi) U_m(\xi) d\xi = \delta_{nm}$. We can describe any function $f(\xi)$ in the following manner.

$$f(\xi) = \sum_{n=1}^{\infty} a_n U_n(\xi), \qquad a_n = \int_a^b U_n^*(\xi) f(\xi) d\xi \tag{12}$$

The commonly used *Fourier Series* is described on the interval $(-a/2,\ a/2)$ as the orthonormal functions

$$U_m(x) = \frac{1}{\sqrt{a}} e^{i(2\pi m x/a)} \tag{13}$$

where $m = 0, \pm 1, \pm 2, \ldots$. If we allow $m$ to be continuous, $\frac{2\pi m}{a} \to k$, and the interval to go to infinity, the function expansion becomes

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{ikx} dk, \qquad A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx \tag{14}$$

### 2.4 Laplace Equation in Rectangular Coordinates

The Laplace equation in rectangular coordinates is

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{15}$$

This can be solved by separation of variables $\Phi = X(x)Y(y)Z(x)$ to produce

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\alpha^2, \qquad \frac{1}{Y}\frac{d^2Y}{dy^2} = -\beta^2, \qquad \frac{1}{Z}\frac{d^2Z}{dz^2} = -\gamma^2 \tag{16}$$

where $\alpha^2 + \beta^2 = \gamma^2$. The solutions are

$$X(x) = e^{\pm i\alpha x}, \qquad Y(y) = e^{\pm i\beta y}, \qquad Z(z) = e^{\pm z\sqrt{\alpha^2+\beta^2}} \tag{17}$$

Apply boundary conditions to find out which values go to 0. This problem is typically posed as a box with all sides $\Phi = 0$ except one. The general method is to use Fourier's method to define a series for the coordinate perpendicular to the non-zero side. See [Ja 72-75].

## 3 Boundary-Value Problems in Electrostatics: II

[Ja 95-144]

### 3.1 Laplace Equation In Spherical Coordinates

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0 \tag{18}$$

This can be solved by separation of variables $\Phi = \frac{U(r)}{r}P(\theta)Q(\phi)$. This produces the separated equations

$$\frac{d^2U}{dr^2} - \frac{l(l+1)}{r^2}U = 0, \qquad \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]P = 0, \qquad \frac{1}{Q}\frac{d^2Q}{d\phi^2} = -m^2 \tag{19}$$

where $l$ and $m$ are constants. The solutions for $U(r)$ and $Q(\phi)$ can be written as

$$U(r) = Ar^{l+1} + Br^{-l}, \qquad Q(\phi) = e^{\pm im\phi} \tag{20}$$

where $m$ must be an integer for $Q(\phi)$ to be single valued. The equation for $P(\theta)$ can be written in terms of $x = \cos\theta$ in the Legendre Equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P = 0 \tag{21}$$

The solution to this demands that $l$ is an integer $> 0$. In the case of **azimuthal symmetry**, $m = 0$, this can be solved simply to produce Legendre Polynomials, described by Rodrigues Formula

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l, \qquad U_l(x) = \sqrt{\frac{2l+1}{2}}P_l(x) \tag{22}$$

where $U_l(x)$ are orthonormal functions subject to expansions as discussed in Section 2.3. It is important to note, then, that the coefficients are then defined as

$$f(x)=\sum_{l=0}^{\infty}A_lP_l(x), \qquad A_l=\frac{2l+1}{2}\int_{-l}^{l}f(x)P_l(x)dx \tag{23}$$

And the general solution for a boundary-value problem with azimuthal symmetry is

$$\Phi(r,\theta)=\sum_{l=0}^{\infty}[A_lr^l+B_lr^{-(l+1)}]P_l(\cos\theta) \tag{24}$$

Now, for problems **without azimuthal symmetry**, $m\neq 0$, Rodrigues' formula becomes

$$P_l^m(x)=\frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l \tag{25}$$

We can then lump the $\theta$ and $\phi$ dependence into one orthogonal expression defined by $l$ and $m$ called the spherical harmonics

$$Y_{lm}(\theta,\phi)=\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi}, \qquad Y_{l,-m}(\theta,\phi)=(-1)^mY^*_{lm}(\theta,\phi) \tag{26}$$

The resulting general formula for $\Phi$ in spherical coordinates is

$$\Phi(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}[A_{lm}r^l+B_{lm}r^{-(l+1)}]Y_{lm}(\theta,\phi) \tag{27}$$

Jackson also notes the *addition theorem* which relates two coordinate vectors $\mathbf{x}$ and $\mathbf{x}'$ in spherical coordinates.

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi) \tag{28}$$

where $r_<$ and $r_>$ are the radial components of the smaller and larger of the two , $\mathbf{x},\mathbf{x}'$, respectively.

### 3.2 Laplace Equation in Cylindrical Coordinates

$$\nabla^2\Phi=\frac{\partial^2\Phi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\Phi}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}+\frac{\partial^2\Phi}{\partial z^2}=0 \tag{29}$$

Separation of variables using $\Phi(\rho,\phi,z)=R(\rho)Q(\phi)Z(z)$ creates the following ODEs

$$\frac{d^2R}{d\rho^2}+\frac{1}{\rho}\frac{dR}{d\rho}+\left(k^2-\frac{v^2}{\rho^2}\right)R=0, \qquad \frac{d^2Q}{d\phi^2}+v^2Q=0, \qquad \frac{d^2Z}{dz^2}-k^2Z=0 \tag{30}$$

The solutions for $Q$ and $Z$ are simple

$$Q(\phi)=e^{\pm iv\phi}, \qquad Z(z)=e^{\pm kz} \tag{31}$$

where $v$ must be an integer. We assume $k$ is continuous, real, and positive. The $R$ equation, under a change of variables $x = k\rho$, is the Bessel equation

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{v^2}{x^2}\right)R = 0 \tag{32}$$

Its solutions are Bessel functions of order $v$. They are described by a power series of the form $R(x) = x^\alpha \sum_{j=0}^{\infty} a_j x^j$, and their coefficients can be defined recursively, but this isn't important. It is important to note that, to span all possible functions, Bessel functions come in multiple flavors. The basic few are $J_v(x)$, $N_v(x)$, $H_v^{(1)}$, and $H_v^{(2)}$ which are named Bessel functions of the first, second, and third kinds. $N_v(x)$ is also called the Neumann function and $H_v^{(1)}$ and $H_v^{(2)}$ are called the Hankel functions. One can relate the three using the following expressions

$$J_{-m}(x) = (-1)^m J_m(x), \qquad N_v(x) = \frac{J_v(x)\cos v\pi - J_{-v}(x)}{\sin v\pi} \tag{33}$$

$$H_v^{(1)}(x) = J_v(x) + iN_v(x), \qquad H_v^{(2)}(x) = J_v(x) - iN_v(x), \tag{34}$$

Also, if the separation constant $k^2$ in (30) had been taken to be $-k^2$ and $Z = e^{\pm ikz}$, then the solutions to Bessel equation are the modified Bessel functions (Bessel functions of imaginary argument), defined as

$$I_v(x) = i^{-v} J_v(ix), \qquad K_v(x) = \frac{\pi}{2} i^{v+1} H_v^{(1)}(ix) \tag{35}$$

$J_v$ is well-behaved at $x = 0$ whereas $N_v \to \infty$. The roots of $J_v(x_{vn}) = 0$ are given by $x_{vn} \approx n\pi + (v - \frac{1}{2})\frac{\pi}{2}$. We can expand an arbitrary function of $\rho$ on the interval $0 \leq \rho \leq a$ in a Fourier-Bessel series.

$$f(\rho) = \sum_{n=1}^{\infty} A_{vn} J_v\left(x_{vn}\frac{\rho}{a}\right), \qquad A_{vn} = \frac{2}{a^2 J_{v+1}^2(x_{vn})} \int_0^a \rho f(\rho) J_v(\frac{x_{vn}\rho}{a}) d\rho \tag{36}$$

The appropriate expansion for a cylindrical problem where $\Phi = 0$ at $z = 0$ for arbitrary $\rho$ is

$$\Phi(\rho, \phi, z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(k_{mn}\rho)\sinh(k_{mn}z)(A_{mn}\sin m\phi + B_{mn}\cos m\phi) \tag{37}$$

The summations in the above equation will become integrals if the region is allowed to go to infinity. For example, for $z \geq 0$

$$\Phi(\rho, \phi, z) = \sum_{m=0}^{\infty} \int_0^{\infty} dk J_m(k\rho) e^{-kz}(A_m(k)\sin m\phi + B_m(k)\cos m\phi) \tag{38}$$

$$A_m(k) = \frac{k}{\pi} \int_0^{\infty} d\rho\ \rho \int_0^{2\pi} d\phi V(\rho, \phi) J_m(k\rho)\sin m\phi \tag{39}$$

for a potential specified at $z = 0$. For $B_m$, replace $\sin m\phi$ with $\cos m\phi$. Expansions for different problems might require different Bessel functions, but let's hope you are given those.

In a long cylindrical tube, with no z-dependence, $\Phi$ is described by the cylindrical harmonics

$$\Phi(\rho, \phi) = \sum_{n=1}^{\infty}(A_n\rho^n + B_n\rho^{-n})(C_n\cos n\phi + D_n\sin n\phi) + (F\ln\rho + G)(H\phi + C') \tag{40}$$

### 3.3 Green Functions

We use Green functions to describe common geometrical distributions. We use them in calculating electric potential via Green's theorem as in Section 1.2.

In spherical coordinates, for a spherical boundary at $r = a$

$$G(\mathbf{x}, \mathbf{x}') = 4\pi \sum_{l,m} \frac{1}{2l+1} \left[ \frac{r_<^l}{r_>^{l+1}} - \frac{1}{a} \left( \frac{a^2}{rr'} \right)^{l+1} \right] Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi) \tag{41}$$

In cylindrical coordinates, for no cylindrical boundaries

$$G(\mathbf{x}, \mathbf{x}') = \frac{2}{\pi} \sum_{m=-\infty}^{\infty} \int_0^\infty dk \; e^{im(\phi-\phi')} \cos[k(z-z')] I_m(kp_<) K_m(kp_>) \tag{42}$$

$r_<$ and $r_>$ refer to the coordinates of the source and observation coordinates. For the integration at hand, if the radius coordinate of the source is greater than that of the observation point, then $r_>$ refers to the source and $r_<$ refers to the observation, and vice-versa. The same goes for $\rho_<$ and $\rho_>$.

## 4 Multipoles, Electrostatics of Macroscopic Media, Dielectrics

[Ja 145-173]

### 4.1 Multipole Expansion

The potential expansion of a distribution of charge which is vanishing outside of some radius $R$ centered at the origin is

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{\infty} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}}, \qquad q_{lm} = \int Y_{lm}^*(\theta', \phi') r'^l \rho(\mathbf{x}') d^3x' \tag{43}$$

where each $l$ contribution is the $l$-pole term in the expansion (monopole, dipole, etc.) The electric dipole moment is defined as

$$\mathbf{p} = \int \mathbf{x}' \rho(\mathbf{x}') d^3x', \qquad \mathbf{E}(\mathbf{x}) = \frac{3\mathbf{n}(\mathbf{p} \cdot \mathbf{n}) - \mathbf{p}}{4\pi\epsilon_0 |\mathbf{x} - \mathbf{x_0}|^3} \tag{44}$$

where $\mathbf{n}$ is a unit vector directed from $\mathbf{x_0}$ to $\mathbf{x}$. The mutual potential energy is defined as

$$W_{12} = \frac{\mathbf{p_1} \cdot \mathbf{p_2} - 3(\mathbf{n} \cdot \mathbf{p_1})(\mathbf{n} \cdot \mathbf{p_2})}{4\pi\epsilon_0 |\mathbf{x_1} - \mathbf{x_2}|^3} \tag{45}$$

### 4.2 Electrostatics with Ponderable Media

An applied electric field in a region of ponderable media will produce an electric polarization $\mathbf{P}$. This produces an overall electric displacement $\mathbf{D}$ of the form

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \qquad \nabla \cdot \mathbf{D} = \rho \tag{46}$$

If the medium is linear, isotropic, and homogeneous, this becomes

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \qquad \mathbf{D} = \epsilon \mathbf{E} \tag{47}$$

where $\epsilon = \epsilon_0(1+\chi_e)$. Bound charges and free charge can be then derived as

$$\sigma_b = \mathbf{P}\cdot\mathbf{n} \qquad \rho_b = -\nabla\cdot\mathbf{P}, \qquad \rho_f = \nabla\cdot\mathbf{D} \tag{48}$$

These definitions along with Maxwell's equations lead to a set of boundary conditions

$$(\mathbf{D_2}-\mathbf{D_1})\cdot\mathbf{n}_{21} = \sigma, \qquad (\mathbf{E_2}-\mathbf{E_1})\times\mathbf{n}_{21} = 0 \tag{49}$$

which is to state that for some surface, perpendicular $\mathbf{D}$ is discontinuous and parallel $\mathbf{E}$ is continuous. If all space was filled with a linear dielectric $\epsilon$, all electric fields are reduced by a factor $\epsilon_0/\epsilon$.

### 4.3 Electrostatic Energy in Dielectric Media

If the medium is *linear*,

$$W = \frac{1}{2}\int \mathbf{E}\cdot\mathbf{D}\; d^3x \tag{50}$$

Using virtual work, we can see the force exerted on a dielectric object given that all source fields are kept fixed (condition $Q$).

$$F_\xi = -\Big(\frac{\partial W}{\partial \xi}\Big)_Q \tag{51}$$

## 5 Magnetostatics, Faraday's Law, Quasi-Static Fields

[Ja 174-236]

A net mechanical torque is exerted on a magnetic dipole in a magnetic field

$$\mathbf{N} = \mu\times\mathbf{B} \tag{52}$$

A system in equilibrium must maintain the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0 \tag{53}$$

Magnetostatics are charcterized by no change in the net charge density anywhere in space, so $\nabla\cdot\mathbf{J} = 0$.

### 5.1 Biot-Savart Law and Magnetostatics

Magnetic fields can be described by

$$\mathbf{dB} = \frac{\mu_0 I}{4\pi}\frac{\mathbf{dI}\times\mathbf{x}}{|\mathbf{x}|^3}, \qquad \mathbf{B} = \frac{\mu_0 q}{4\pi}\frac{\mathbf{v}\times\mathbf{x}}{|\mathbf{x}|^3} = \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{x}')\times\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^{\mathbf{3}}}d^3x' = \frac{\mu_0}{4\pi}\nabla\times\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x \tag{54}$$

Forces and torques can be described by

$$\mathbf{F}_{12} = -\frac{\mu_0}{4\pi}I_1 I_2\oint\oint\frac{(d\mathbf{I}_1\cdot d\mathbf{I}_2)\mathbf{x}_{12}}{|\mathbf{x}_{12}|}, \qquad \mathbf{F} = \int\mathbf{J}(\mathbf{x})\times\mathbf{B}(\mathbf{x})\; d^3x, \qquad \mathbf{N} = \int \mathbf{x}\times(\mathbf{J}\times\mathbf{B})\; d^3x \tag{55}$$

Maxwell's equations for magnetostatics are

$$\nabla\cdot\mathbf{B} = 0, \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} \tag{56}$$

The second law is also described as Ampere's law. From this follows $\oint_C \mathbf{B}\cdot\mathbf{dl} = \mu_0 I$

### 5.2 Vector Potential

We can describe a vector field $\mathbf{A}(\mathbf{x})$ called the vector potential which is related to $\mathbf{B}(\mathbf{x})$ like

$$\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x}), \qquad \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x'})}{|\mathbf{x} - \mathbf{x'}|} d^3x' + \nabla \mathbf{\Psi}(\mathbf{x}) \tag{57}$$

We choose a convenient gauge $\nabla \cdot \mathbf{A} = 0$ such that $\Psi = \text{constant}$. This leads to

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}, \qquad \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x'})}{|\mathbf{x} - \mathbf{x'}|} d^3x' \tag{58}$$

Vector potential is particularly useful when characterizing the magnetic field around a given, arbitrary system of current distributions.

### 5.3 Magnetic Moment

It is customary to define the magnetization of a system as

$$\mathbf{M}(\mathbf{x}) = \frac{\mathbf{1}}{\mathbf{2}}[\mathbf{x} \times \mathbf{J}(\mathbf{x})], \qquad \mathbf{m} = \frac{\mathbf{1}}{\mathbf{2}} \int \mathbf{M}(\mathbf{x'})\ d^3x' \tag{59}$$

Far from the current distribution, this looks like

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{x}}{|\mathbf{x}|^{\mathbf{3}}}, \qquad \mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \left[ \frac{\mathbf{3n}(\mathbf{n} \cdot \mathbf{m}) - \mathbf{m}}{|\mathbf{x}|^{\mathbf{3}}} \right] \tag{60}$$

For a plane loop, the magnetic moment is given by $|\mathbf{m}| = I \times$ (Area), regardless of the shape of the circuit. For a number of charged particles, $\mathbf{m} = \frac{1}{2} \sum_i q_i (\mathbf{x_i} \times \mathbf{v_i})$. Force, torque, and potential energy go as

$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B}), \qquad \mathbf{N} = \mathbf{m} \times \mathbf{B}, \qquad U = -\mathbf{m} \cdot \mathbf{B} \tag{61}$$

### 5.4 Macroscopic Equations

Just like in electrostatics, the magnetic fields change in ponderable media. Due to the magnetization of atoms $\mathbf{M}(\mathbf{x}) = \sum_i N_i \langle \mathbf{m}_i \rangle$, we see an effective current density $\mathbf{J}_M$ and an accompanying change in Ampere's law

$$\mathbf{J}_M = \nabla \times \mathbf{M}, \qquad \nabla \times \mathbf{B} = \mu_0 [\mathbf{J} + \nabla \times \mathbf{M}] \tag{62}$$

If we define a new macroscopic field $\mathbf{H}$,

$$\mathbf{H} = \frac{\mathbf{1}}{\mu_{\mathbf{0}}} \mathbf{B} - \mathbf{M}, \qquad \nabla \times \mathbf{H} = \mathbf{J}, \qquad \nabla \cdot \mathbf{B} = 0 \tag{63}$$

For isotropic, diamagnetic, and paramagnetic substances, the simple linear relation holds

$$\mathbf{B} = \mu \mathbf{H} \tag{64}$$

where $\mu$ is the magnetic permeability. The boundary conditions for $\mathbf{B}$ and $\mathbf{H}$ at an interface between two media are given by

$$(\mathbf{B_2} - \mathbf{B_1}) \cdot \mathbf{n} = \mathbf{0}, \qquad \mathbf{n} \times (\mathbf{H_2} - \mathbf{H_1}) = \mathbf{K} \tag{65}$$

### 5.5 Methods of Solving Boundary-Value Problems in Magnetostatics

We can define a magnetic scalar potential (MSP) $\Phi_M$ as an analog to the vector potential $\mathbf{A}$.

$$\mathbf{H} = -\nabla\Phi_M, \qquad \nabla^2\Phi_M = -\rho_M, \qquad \rho_M = -\nabla\cdot\mathbf{M} \tag{66}$$

If there are no boundary surfaces and $\mathbf{M}$ is well-behaved and localized, the solution to this is

$$\Phi_M(\mathbf{x}) = -\frac{1}{4\pi}\nabla\cdot\int\frac{\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \tag{67}$$

For boundary surfaces

$$\sigma_M = \mathbf{n}\cdot\mathbf{M}, \qquad \Phi_M(\mathbf{x}) = -\frac{1}{4\pi}\int\frac{\nabla'\cdot\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' + \frac{1}{4\pi}\oint_S\frac{\mathbf{n}'\cdot\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}da' \tag{68}$$

Jackson works out the MSP solution to a uniformly magnetized sphere with $\mathbf{M} = M_0\hat{\mathbf{z}}$ [Ja 198-200].

$$\Phi_M(r,\theta) = \frac{M_0a^2}{4\pi}\int d\Omega'\frac{\cos\theta'}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{3}M_0a^2\frac{r_<}{r_>^2}\cos\theta \tag{69}$$

Which lead to inner fields of

$$\mathbf{H}_{in} = -\frac{1}{3}\mathbf{M}, \qquad \mathbf{B}_{in} = \frac{2\mu_0}{3}\mathbf{M} \tag{70}$$

And an outer field that acts like a dipole with moment $\mathbf{m} = \frac{4\pi a^3}{3}\mathbf{M}$. One could then imagine applying a universal magnetic induction through space $\mathbf{B_0} = \mu_0\mathbf{H_0}$. Then, allowing the sphere to have a paramagnetic permeability $\mu$, we find the **H-B** relationship for a paramagnetic sphere in a uniform magnetic field.

$$\mathbf{M} = \frac{3}{\mu_0}\Big(\frac{\mu-\mu_0}{\mu+2\mu_0}\Big)\mathbf{B_0} \tag{71}$$

Jackson also works out the MSP solution to a spherical shell of permeable material in a magnetic field [Ja 201-203]. Whereas the result may not be useful, the method of solving $\nabla^2\Phi_M = 0$ using LaPlace solutions is important. For $\mathbf{B_0} = B_0\hat{\mathbf{x}}$

$$\Phi_{M,r>b} = -H_0r\cos\theta + \sum_{l=0}^{\infty}\frac{\alpha_l}{r^{l+1}}P_l(\cos\theta), \qquad \Phi_{M,a<r<b} = \sum_{l=0}^{\infty}\Big(\beta_lr^l + \gamma_l\frac{1}{r^{l+1}}\Big)P_l(\cos\theta) \tag{72}$$

Note any sin terms would drop out due to the cos symmetry.

### 5.6 Induction and Energy

Faraday observed that the change in magnetic flux and the electromotive force around a current loop were related

$$\oint_C\mathbf{E}'\cdot\mathbf{dl} = -k\frac{d}{dt}\int_S\mathbf{B}\cdot\mathbf{n}\,da \tag{73}$$

where $k$ is unit-dependent. Jackson showed that in SI units $k = 1$. We can derive from this observation Faraday's law

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{74}$$

Jackson also derives energy equations for magnetic fields

$$W = \frac{1}{2} \int \mathbf{H} \cdot \mathbf{B} \; d^3x = \frac{1}{2} \int \mathbf{J} \cdot \mathbf{A} \; d^3x = \frac{1}{2} \int_{V_1} \mathbf{M} \cdot \mathbf{B_0} \; d^3x \tag{75}$$

We can also apply the principle of virtual work

$$F_\xi = \left( \frac{\partial W}{\partial \xi} \right)_J \tag{76}$$

For a system of current-carrying circuits

$$W = \frac{1}{2} \sum_{i=1}^{N} L_i I_i^2 + \sum_{i=1}^{N} \sum_{j>i}^{N} M_{ij} I_i I_j \tag{77}$$

where self and mutual inductances are given by

$$L_i = \frac{\mu_0}{4\pi I_i^2} \int_{C_i} d^3x_i \int_{C_i} d^3x_i' \frac{\mathbf{J(x_i)} \cdot \mathbf{J(x_i')}}{|\mathbf{x_i} - \mathbf{x_i'}|}, \qquad M_{ij} = \frac{\mu_0}{4\pi I_i I_j} \int_{C_i} d^3x_i \int_{C_j} d^3x_j' \frac{\mathbf{J(x_i)} \cdot \mathbf{J(x_j')}}{|\mathbf{x_i} - \mathbf{x_j'}|} \tag{78}$$

# 6 Maxwell Equations, Macroscopic Electromagnetism, Conservation Laws

[Ja 237-294]

We can write the complete Maxwell Equations, including the correct time dependencies

$$\nabla \cdot \mathbf{D} = \rho, \qquad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{79}$$

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{80}$$

We can expand these laws to include our vector potential

$$\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A} \tag{81}$$

## 6.1 Gauge Transformations

The Maxwell equations are dependent and coupled. As a result, we can arbitrarily shift values of $\mathbf{A}$ and $\Phi$. These are called gauge transformations. For a given scalar potential $\Lambda$.

$$\mathbf{A} \to \mathbf{A} + \nabla \Lambda, \qquad \Phi \to \Phi - \frac{\partial \Lambda}{\partial t} = 0 \tag{82}$$

It is useful to identify preferred values for this arbitrary gauge. The Lorenz gauge is useful for potentials that satisfy the Lorenz condition.

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0 \ , \qquad \nabla^2 \Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} = 0 \tag{83}$$

For potentials which satisfy the Lornenz condition, we can uncouple Maxwell's equations and produce two homogenous wave equations.

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0, \qquad \nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}, \qquad \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} \tag{84}$$

where the speed of light is equal to $c = (\mu_0 \epsilon_0)^{-1/2}$

The Coulomb gauge is useful when no sources are present. We choose $\nabla \cdot \mathbf{A} = 0$. In this guage, we can write simply that

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A} \tag{85}$$

### 6.2 Green Functions for the Wave Equation

We need to solve the four inhomogeneous wave equations for $\Phi, A_i$ given time-dependent source terms $\rho(\mathbf{x}, t), J_i(\mathbf{x}, t)$. From a purely mathematical perspective, for a source $g(\mathbf{x}, t)$,

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\psi(\mathbf{x}, t) = -4\pi g(\mathbf{x}, t) \qquad \rightarrow \qquad \left[\nabla^2 + k^2\right]\psi(\mathbf{x}, \omega) = -4\pi g(\mathbf{x}, \omega) \tag{86}$$

where $k = \omega/c$. For a point source $g(\mathbf{x}, t) = \delta(\mathbf{x} - \mathbf{x}')\delta(t - t')$, the Green's function solution is the superposition of outgoing and incoming spherical waves

$$G_k(R, \omega) = \Big(\frac{A}{R}e^{ikR} + \frac{B}{R}e^{-ikR}\Big)e^{i\omega t'} \tag{87}$$

where $R = |\mathbf{x} - \mathbf{x}'|$. The outgoing wave is represented in time as

$$G_k^+(R, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\frac{e^{ikR}}{R}e^{i\omega t'}d\omega = \frac{1}{R}\delta(t' - (t - R/c)) \tag{88}$$

The delta function requires that the time for evaluating the source time is earlier than the observation time by the propagation delay $R/c$. For a general source $g$

$$\psi(\mathbf{x}, t) = \int d\tau' \frac{[g(\mathbf{x}', t')]_{ret}}{\mathbf{x} - \mathbf{x}'} \tag{89}$$

### 6.3 Retarded Solution

Electromagnetic influence takes time to propagate. We can represent relativistic E/M with retarded time.

$$t'_r = t - \frac{|\mathbf{r} - \mathbf{r}'|}{c} \tag{90}$$

We can then replace $t$ with $t'_r$ in the definitions of $\Phi$ and $\mathbf{A}$ described previously. In this notation, for a source $g(\mathbf{x}, t)$, we mean $[g(\mathbf{x}, t)]_{ret} = g(\mathbf{x}, t'_r)$.

$$\Phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{1}{R} [\rho(\mathbf{x}', t')]_{ret} \ , \qquad \mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int d^3x' \frac{1}{R} [\mathbf{J}(\mathbf{x}', t')]_{ret} \tag{91}$$

where we've defined $\mathbf{R} = \mathbf{x} - \mathbf{x}'$ with $R = |\mathbf{x} - \mathbf{x}'|$ and $\hat{\mathbf{R}} = \mathbf{R}/R$.

Jackson derives retarded equations for $\mathbf{E}$ and $\mathbf{B}$

$$\begin{aligned} \mathbf{E}(\mathbf{x}, t) &= \frac{1}{4\pi\epsilon} \int d^3x' \left[ \frac{\hat{\mathbf{R}}}{R^2} [\rho(\mathbf{x}', t')]_{ret} + \frac{\hat{\mathbf{R}}}{cR} \Big[ \frac{\partial \rho(\mathbf{x}', t')}{\partial t'} \Big]_{ret} - \frac{1}{c^2 R} \Big[ \frac{\partial \mathbf{J}(\mathbf{x}', t')}{\partial t} \Big]_{ret} \right] \\ \mathbf{B}(\mathbf{x}, t) &= \frac{\mu_0}{4\pi} \int d^3x' \left[ [\mathbf{J}(\mathbf{x}', t')]_{ret} \times \frac{\hat{\mathbf{R}}}{R^2} + \Big[ \frac{\partial \mathbf{J}(\mathbf{x}', t')}{\partial t'} \Big]_{ret} \times \frac{\hat{\mathbf{R}}}{cR} \right] \end{aligned} \tag{92}$$

### 6.4 Poynting's Theorem

The total field energy density is given by

$$u = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}) \tag{93}$$

We define a vector $\mathbf{S}$, representing energy flow, called the Poynting vector. It's momentum is given by

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}, \qquad \mathbf{P}_{field} = \int_V \mathbf{g} \ d^3x, \qquad \mathbf{g} = \frac{1}{c^2}\mathbf{S} \tag{94}$$

The total electromagnetic force and mechanical momentum on a charged particle is given by

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \qquad \frac{d\mathbf{P}_{mech}}{dt} = \int_V (\rho\mathbf{E} + \mathbf{v} \times \mathbf{B}) \ d^3x \tag{95}$$

With the definition of the Maxwell stress tensor $T_{\alpha\beta}$ as

$$T_{\alpha\beta} = \epsilon_0[E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2}(\mathbf{E} \cdot \mathbf{E} + c^2\mathbf{B} \cdot \mathbf{B})\delta_{\alpha\beta}] \tag{96}$$

We can derive the forces acting on a surface with normal $\mathbf{n}$ as

$$\mathbf{F}_S = \frac{d}{dt}(\mathbf{P}_{mech} + \mathbf{P}_{field})_\alpha = \oint_S \sum_\beta T_{\alpha\beta} n_\beta \ da \tag{97}$$

## 7 Plane Electromagnetic Waves and Wave Propagation

[Ja 295-351]

### 7.1 Plane Waves in a Nonconducting Medium

In the absence of sources, the Maxwell equations take a simple form ($\rho = 0,\ \mathbf{J} = 0$). Using these assumptions, and assuming harmonic time dependence $e^{-i\omega t}$, we can derive the Helmholtz wave equation.

$$(\nabla^2 + \mu\epsilon\omega^2)\begin{bmatrix}\mathbf{E}\\ \mathbf{B}\end{bmatrix} = 0 \tag{98}$$

We can derive from this a relationship between wave number $k$ and the time frequency $\omega$

$$k = \sqrt{\mu\epsilon}\omega, \qquad v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{n}, \qquad n = \sqrt{\frac{\mu}{\mu_0}\frac{\epsilon}{\epsilon_0}} \tag{99}$$

where $n$ is the index of refraction. We can then write plane wave fields as

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}, \qquad \mathbf{B}(\mathbf{x}, t) = \mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}, \qquad \mathbf{k} = k\mathbf{n} \tag{100}$$

The divergence equations demand that $\mathbf{E}$ and $\mathbf{B}$ are perpendicular to the direction of propagation $\mathbf{n}$, indicating that plane waves are transverse.

$$\mathbf{n}\cdot\mathbf{E}_0 = 0, \qquad \mathbf{n}\cdot\mathbf{B}_0 = 0 \tag{101}$$

Maxwell equations also relate $\mathbf{E}$ and $\mathbf{B}$ to each other.

$$\mathbf{B_0} = \sqrt{\mu\epsilon}\ \mathbf{n}\times\mathbf{E_0}, \qquad \mathbf{H_0} = \frac{1}{Z}\ \mathbf{n}\times\mathbf{E_0}, \qquad Z = \sqrt{\frac{\mu}{\epsilon}}, \qquad \mathbf{S} = \frac{1}{2}\mathbf{E}\times\mathbf{H}^* \tag{102}$$

where $Z$ is the impedance of the medium. The Poynting vector formula picks up a factor of 1/2 when taking the real part of its complex value.

### 7.2 Linear and Circular Polarization

The $\mathbf{E_0}$ vector must be perpendicular to the direction of propagation, $\mathbf{k}$. This allows it to have direction into two other Cartesian directions, $\mathbf{u_1}, \mathbf{u_2}$. It is possible for the oscillation in each direction to be out of phase, in which case the wave is elliptically polarized. If they are in phase (the $\mathbf{E}$ oscillates in one direction), then it is linearly polarized.

$$\mathbf{E}(\mathbf{x}, t) = (E_+\mathbf{u}_+ + E_-\mathbf{u}_-)e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}, \qquad \mathbf{u}_\pm = \frac{1}{\sqrt{2}}(\mathbf{u}_1 \pm i\mathbf{u}_2), \qquad \mathbf{B}(\mathbf{x}, t) = \sqrt{\mu\epsilon}\frac{\mathbf{k}\times\mathbf{E}(\mathbf{x}, t)}{k} \tag{103}$$

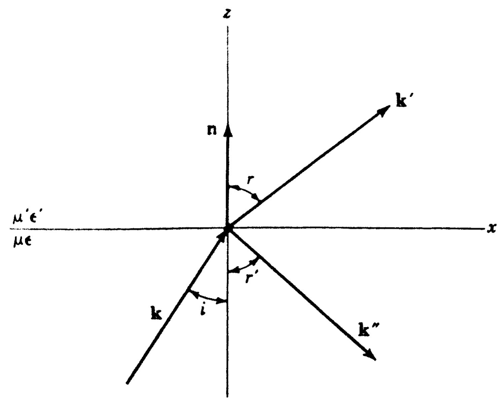
where $\mathbf{u}_\pm$ are complex orthogonal unit vectors. It is easy to derive properties of the elliptical polarization from the ratio of $E_-$ and $E_+$,

$$\frac{E_-}{E_+} = re^{i\alpha}, \qquad \frac{R_{major}}{R_{minor}} = \left|\frac{1+r}{1-r}\right| \tag{104}$$

where $\alpha$ denotes a circular polarization with it's axes rotated by an angle $\alpha/2$. The formula for $R_{major}/R_{minor}$ is the ratio of semimajor to semiminor axes. For $r = \pm 1$, we have a linearly polarized wave.

### 7.3 Reflection and Refraction of Electromagnetic Waves at a Plane Interface between Dielectrics

There are three plane waves relevant in this scenario: $\mathbf{E}$ the incoming wave, $\mathbf{E}'$ the refracted wave, and $\mathbf{E}''$ the transmitted wave. The respective $\mathbf{B}$ fields are calculated like $\mathbf{B} = \sqrt{\mu\epsilon}\ \hat{\mathbf{k}} \times \mathbf{E}$.




We know from kinematic arguments that the angle of reflection must be equal to the angle of incidence. We also can prove Snell's law.

$$i = r', \qquad \frac{\sin i}{\sin r} = \frac{n'}{n} \tag{105}$$

We can also show that the wave numbers have magnitudes

$$|\mathbf{k}| = |\mathbf{k}''| = k = \omega\sqrt{\mu\epsilon}, \qquad |\mathbf{k}'| = k' = \omega\sqrt{\mu'\epsilon'} \tag{106}$$

The boundary conditions are that normal components of $\mathbf{D}$ and $\mathbf{B}$ are continuous; tangential components of $\mathbf{E}$ and $\mathbf{H}$ are continuous. The explicit forms of these can be derived using the equations for $\mathbf{E}$ of a plane wave from the previous section.

To apply the boundary conditions, we must consider the angle at which $\mathbf{E}$ is oriented to the surface. It is convenient to separate the cases into one in which the $\mathbf{E}$ is perpendicular to the surface and one in which it is parallel. The general case can then be derived from the linear combination of these two situations.

For $\mathbf{E}$ perpendicular to the plane of incidence,

$$\frac{E_0'}{E_0} = \frac{2n\cos i}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}, \qquad \frac{E_0''}{E_0} = \frac{n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}} \tag{107}$$

For $\mathbf{E}$ parallel to plane of incidence,

$$\frac{E_0'}{E_0} = \frac{2nn'\cos i}{\frac{\mu}{\mu'}n'^2\cos i + n\sqrt{n'^2 - n^2\sin^2 i}}, \qquad \frac{E_0''}{E_0} = \frac{\frac{\mu}{\mu'}n'^2\cos i - n\sqrt{n'^2 - n^2\sin^2 i}}{\frac{\mu}{\mu'}n'^2\cos i + n\sqrt{n'^2 - n^2\sin^2 i}} \tag{108}$$

If we assume $\mu = \mu'$, we can derive simpler formulas for the reflectance in each of these cases. These are given by the Fresnel equations.

$$R_s = \left|\frac{n\cos i - n'\cos r}{n\cos i + n'\cos r}\right|^2, \qquad R_p = \left|\frac{n\cos r - n'\cos i}{n\cos r + n'\cos i}\right|^2 \tag{109}$$

where $R_s$ is in the perpendicular case and $R_p$ is in the parallel case. $T = 1 - R$.

### 7.4 Polarization by Reflection and Total Internal Reflection; Goos-Hanchen Effect

For polarization parallel to the plane of incidence, there is an angle of incidence, called Brewster's angle, for which there is no reflected wave. If an unpolarized wave is incident at Brewster's angle, the reflected radiation will be completely plane-polarized perpendicular to the interface. If the angle of incidence is close to Brewster's angle, the reflected radiation will be dominated by the plane-polarized wave. For simplicity, we assume $\mu' = \mu$ and Brewster's angle is given by

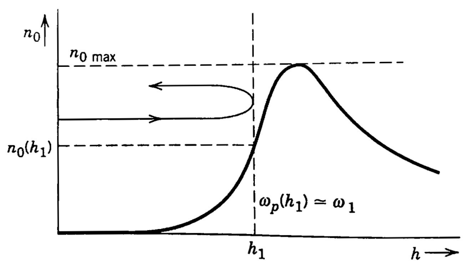
$$i_B = \tan^{-1}\left(\frac{n'}{n}\right) \tag{110}$$

If the waves are moving into a medium with a lower index of refraction, $n > n'$, they might exhibit total internal reflection, meaning there is no transmitted wave. This occurs at angles of incidence $i > i_0$ where

$$i_0 = \sin^{-1}\left(\frac{n'}{n}\right) \tag{111}$$

If $i > i_0$, the incident wave will penetrate the medium of lower index before being reflected. It does so with an exponential decay in the perpendicular direction, $e^{-z/\delta}$, where $\delta^{-1} = k\sqrt{\sin^2 i - \sin^2 i_0}$.

### 7.5 Simplified Model of Propagation in the Ionosphere and Magnetosphere




Close to the edge of Earth's atmosphere, gases are ionized due to exposure to gamma rays from the Sun. This creates a part of the atmosphere called the ionosphere which has a high density of free electrons. This increases the index of refraction of this region. However, the density of atmospheric gases falls off with height, so there exists a peak of $n$ at some. Radio waves reflect off of the ionosphere to propagate back down to the surface.

## 8 Waveguides, Resonant Cavities, and Optical Fibers

[Ja 352-406]

### 8.1 Fields at the surface of and within a conductor

Plane waves incident on a conductor cannot be transmitted. However, if the conductor is not perfect (infinite conductivity), the plane wave will penetrate the surface of the conductor. The fields will drop off exponentially with skin depth $\delta$.

$$\delta = \left(\frac{2}{\mu_c \omega \sigma}\right)^{1/2}, \qquad \mathbf{H}_c = \mathbf{H}_{||} e^{-\xi/\delta} e^{i\xi/\delta}, \qquad \mathbf{E}_c \simeq \sqrt{\frac{\mu_c \omega}{2\sigma}}(1-i)(\mathbf{n} \times \mathbf{H}_{||}) e^{-\xi/\delta} e^{i\xi/\delta} \tag{112}$$

where $\xi$ is the perpendicular distance into the surface of the conductor. The time-averaged power absorbed per unit area is

$$\frac{dP_{loss}}{da} = \frac{mu_c \omega \delta}{4} |\mathbf{H}_{||}|^2 \tag{113}$$

## 9 Radiating Systems, Multipole Fields and Radiation

[Ja 407-455]

For a system of charges and currents that vary sinusoidally in time $\rho = \rho_0 e^{-i\omega t}$

$$\mathbf{A}(\mathbf{x}) = \frac{mu_0}{4\pi} \int \mathbf{J}(\mathbf{x}') \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\, d^3x', \qquad \mathbf{H} = \frac{1}{\mu_0} \nabla \times \mathbf{A}, \qquad \mathbf{E} = \frac{iZ_0}{k} \nabla \times \mathbf{H} \tag{114}$$

In the near (static) zone: $d \ll r \ll \lambda$. We can use the approximation that $kr \ll 1$.

In the far zone, $d \ll \lambda \ll r$. We can approximate $|\mathbf{x} - \mathbf{x}'| \simeq r - \mathbf{n} \cdot \mathbf{x}'$.

For an oscillating electric dipole, the vector potential and angular power distribution are

$$\mathbf{A}(\mathbf{x}) = -\frac{i\mu_0\omega}{4\pi}\mathbf{p}\frac{e^{ikr}}{r}, \qquad \frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2}k^4|\mathbf{p}|^2\sin^2\theta, \qquad P = \frac{c^2 Z_0 k^4}{12\pi}|\mathbf{p}|^2 \tag{115}$$

For a center-fed, linear antenna with length $d$,

$$p = \frac{iI_0 d}{2\omega} \tag{116}$$

The Larmor formula gives the total radiated power for a rotating electric or magnetic dipole

$$P_{rad} = \frac{2}{3}\frac{(\ddot{p}_\perp)^2}{6\pi\epsilon_0 c^3} = \frac{2}{3}\frac{(\ddot{m}_\perp)^2}{6\pi\epsilon_0 c^3} \tag{117}$$