Quantum Mechanics II Review Notes

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Reference: Cohen-Tannoudji Volume 2 [CT]; Griffiths Intro to Elementary Particles 2 [Gr]

1 Electron spin

[CT 965-996]

1.1 Experimental evidence

The precise experimental study of atomic spectral lines reveal a fine structure: each line is made up of several components having nearly identical frequencies. When an atom is placed in a uniform magnetic field, each of its lines splits into a certain number of equidistant lines, the interval being proportional to the magnetic field. This is the Zeeman effect. This arises from the magnetic moment $\mathbf{M}$.

$$\mathbf{M} = \frac{\mu_B}{\hbar}\mathbf{L} \; ; \qquad \mu_B = \frac{q\hbar}{2m_e} \tag{1}$$

where μ_B is the Bohr magneton and $\mathbf{L}$ is the orbital angular momentum. We'll associate this with the orbital state space $\mathcal{E}_r$. We also observe splitting of atoms in Stern-Gerlach experiments for silver atoms. This leads us to believe there must be half-integral contributions to the angular momentum.

$$\mathbf{M}_S = 2\frac{\mu_B}{\hbar}\mathbf{S} \tag{2}$$

We'll associate this with an intrinsic, non-spatial component of the angular momentum, called spin. We'll set these spin variables up according to the following postulates.

1. The spin operator $\mathbf{S}$ is an angular momentum. They satisfy commutation relations $[S_x, S_y] = i\hbar S_z$.
2. The spin operators act in a new space, the spin state space $\mathcal{E}_S$, where $\mathbf{S}^2$ and S_z constitute a C.S.C.O.
$$\mathbf{S}^2 \left|s, m\right\rangle = s(s+1)\hbar^2 \left|s, m\right\rangle \; ; \qquad S_z \left|s, m\right\rangle = m\hbar \left|s, m\right\rangle \tag{3}$$
3. The state space $\mathcal{E}$ of the particle being considered is the tensor product $\mathcal{E}_r \otimes \mathcal{E}_s$.
4. The electron is a spin $1/2$ particle, $s = 1/2$.

1.2 Properties of an Angular Momentum $1/2$

The spin state space $\mathcal{E}_s$ is two-dimensional. Take the orthonormal basis $\{\left|+\right\rangle, \left|-\right\rangle\}$ of eigenkets common to $\mathbf{S}^2$ and S_z.

$$\mathbf{S}^2 \left|\pm\right\rangle = \frac{3}{4}\hbar^2 \left|\pm\right\rangle \; ; \qquad S_z \left|\pm\right\rangle = \pm\frac{\hbar}{2} \left|\pm\right\rangle \tag{4}$$

$$S_\pm \left|\pm\right\rangle = 0 \; ; \qquad S_\pm \left|\mp\right\rangle = \hbar \left|\pm\right\rangle \; ; \qquad S_\pm = S_x \pm iS_y \tag{5}$$

We can also write these in terms of the Pauli spin matrices σ.

$$(\mathbf{S}) = \frac{\hbar}{2}\sigma \; ; \qquad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \; ; \qquad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \; ; \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{6}$$

These matrices obey the following useful equality

$$(\sigma \cdot \mathbf{A})(\sigma \cdot \mathbf{B}) = \mathbf{A} \cdot \mathbf{B} + i\sigma \cdot (\mathbf{A} \times \mathbf{B}) \tag{7}$$

1.3 Non-Relativistic Description of a Spin $1/2$ Particle

CT discusses spanning $\mathcal{E}$ in terms of the C.S.C.O. $\{X, Y, Z, \mathbf{S}^2, S_z\}$ eigenstates.

$$|\mathbf{r}, \epsilon\rangle \equiv |x, y, z, \epsilon\rangle = |\mathbf{r}\rangle \otimes |\epsilon\rangle \tag{8}$$

The basis is orthonormal and has eigenvalues with $\mathbf{R}$ and $\mathbf{S}$ as expected. Any state of the space $\mathcal{E}$ can be written in the form of a two-component spinor.

$$[\psi](\mathbf{r}) = \begin{bmatrix} \psi_+(\mathbf{r}) \\ \psi_-(\mathbf{r}) \end{bmatrix} \; ; \qquad \psi_\pm(\mathbf{r}) = \langle \mathbf{r}, \pm | \psi \rangle \tag{9}$$

Operators A in this space can be associated with a 2×2 matrix $[\![A]\!]$. The spin operators defined before are examples of two dimensional operators in this space. The orbital operators leave the ϵ index of $|\mathbf{r}, \epsilon\rangle$ unchanged.

$$[\![X]\!] = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \; ; \qquad [\![P_x]\!] = \frac{\hbar}{i} \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial x} \end{bmatrix} \tag{10}$$

The probabilities of finding the spinor $[\psi](\mathbf{r})$ in the up or down state are simply $|\psi_\pm(\mathbf{r})|^2$.

1.4 Rotation Operators for a Spin $1/2$ Particle

The rotation operator $R_{\mathbf{u}}(\alpha)$ is associated with the geometrical rotation through an angle α about the unit vector $\mathbf{u}$.

$$R_{\mathbf{u}}(\alpha) = \exp\left\{ -\frac{i}{\hbar} \alpha \mathbf{J} \cdot \mathbf{u} \right\} \qquad \to \text{small } \alpha \to \qquad R_{\mathbf{u}}(\alpha) \approx 1 - \frac{i}{\hbar} \alpha \mathbf{J} \cdot \mathbf{u} \tag{11}$$

Using the definition $\mathbf{S} = \frac{\hbar}{2}\sigma$, we can write the spin rotation operator.

$$^{(S)}R_{\mathbf{u}}(\alpha) = \cos\frac{\alpha}{2} - i\sigma \cdot \mathbf{u} \sin\frac{\alpha}{2} \tag{12}$$

Transforming the two-component spinor is a little complicated, especially when the wavefunction is not a simple tensor product of independent states in the orbital and spin spaces.

$$\psi'_\epsilon(\mathbf{r}) = \sum_{\epsilon'} R^{(1/2)}_{\epsilon\epsilon'} \psi_{\epsilon'}(\mathcal{R}^{-1}\mathbf{r}) \; ; \qquad R^{(1/2)}_{\epsilon\epsilon'} = \langle \epsilon |^{(S)} R | \epsilon' \rangle \tag{13}$$

where $\epsilon = \pm$ and $\mathcal{R}^{-1}\mathbf{r}$ is the point that the rotation maps into $\mathbf{r}$.

2 Addition of angular momenta

[CT 997-1092]

In a system of interacting particles, only the total angular momentum is a constant of motion. It's useful to describe the state of the system in terms of its total angular momentum.

2.1 Total Angular Momentum in QM

Consider a system of two non-interacting particles in a spherically symmetric potential.

$$H_0 = H_1 + H_2 \ ; \qquad H_1 = -\frac{\hbar^2}{2\mu_1}\nabla_1^2 + V(r_1) \ ; \qquad H_2 = -\frac{\hbar^2}{2\mu_2}\nabla_2^2 + V(r_2) \tag{14}$$

Under these conditions, the angular momentum operators commute with the Hamiltonians, $[\mathbf{L}_1, H_0] = 0$. Thus, they can form a C.S.C.O.

Now, consider two particles which interact.

$$H = H_1 + H_2 + v(|\mathbf{r}_1 - \mathbf{r}_2|) \tag{15}$$

We see that the angular momenta of the individual particles no longer commutes with the Hamiltonian.

$$[L_{1z}, H] = \frac{\hbar}{i}\Big(x_1\frac{\partial v}{\partial y_1} - y_1\frac{\partial v}{\partial x_1}\Big) \tag{16}$$

But, if we combine them to form the total angular momentum $\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2$,

$$\begin{aligned}[L_z, H] &= [L_{1z}, H] + [L_{2z}, H] = \frac{\hbar}{i}\Big(x_1\frac{\partial v}{\partial y_1} - y_1\frac{\partial v}{\partial x_1} + x_2\frac{\partial v}{\partial y_2} - y_2\frac{\partial v}{\partial x_2}\Big) \\ &= \frac{\hbar}{i}\frac{v'}{|\mathbf{r}_1 - \mathbf{r}_2|}\Big[x_1(y_1 - y_2) - y_1(x_1 - x_2) + x_2(y_2 - y_1) - y_2(x_2 - x_1)\Big] \\ &= 0\end{aligned} \tag{17}$$

Because of this, we can try to write a new basis formed by the eigenvectors of $\mathbf{L}^2$ and L_z.

2.2 Addition of two spin $1/2$'s. Elementary method

The addition of two spin 1/2's is meant as a specific and introductory example to the addition of two arbitrary angular momenta. Instead of going through the derivation, the information of which will be contained in the next section, I'll highlight the findings.

Consider two spin 1/2 particles. We know an orthonormal basis of this space as eigenstates of $\{\mathbf{S}_1^2, S_{1z}, \mathbf{S}_2^2, S_{2z}\}$.

$$\{|\epsilon_1, \epsilon_2\rangle\} = \{|+,+\rangle, |+,-\rangle, |-,+\rangle, |-,-\rangle\} \tag{18}$$

We can convert this to a basis as eigenstates of $\{\mathbf{S}^2, S_z\}$ where $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$. Eigenstates of this basis take the form $|S, M\rangle$ where $S \in \{0, 1\}$ and $-S \leq M \leq S$ are both integers. The eigenvalues take on the form,

$$\mathbf{S}^2|S, M\rangle = S(S+1)\hbar^2|S, M\rangle \ ; \qquad S_z|S, M\rangle = M\hbar|S, M\rangle \tag{19}$$

In the basis $\{|\epsilon_1, \epsilon_2\rangle\}$, these operators look like

$$(S_z) = \hbar\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\end{bmatrix} \ ; \qquad (\mathbf{S}^2) = \hbar\begin{bmatrix}2 & 0 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 2\end{bmatrix} \tag{20}$$

Diagonalization of these matrices produces $|S, M\rangle$ in terms of $\{|\epsilon_1, \epsilon_2\rangle\}$.

$$|0,0\rangle = \frac{1}{\sqrt{2}}[|+,-\rangle - |-,+\rangle] \; ; \qquad |1,0\rangle = \frac{1}{\sqrt{2}}[|+,-\rangle + |-,+\rangle] \; ; \qquad |1,\pm 1\rangle = |\pm,\pm\rangle \tag{21}$$

The family of three vectors $|1, M\rangle$ are called the triplet states (symmetric), while the $|0,0\rangle$ is called a singlet state (antisymmetric).

2.3 Addition of two arbitrary angular momenta

Consider an arbitrary system, whose state space is $\mathcal{E}$ and an angular momentum $\mathbf{J}$ relative to this system. We can construct a standard basis $\{|k,j,m\rangle\}$ composed of eigenvectors common to $\mathbf{J}$ and J_z.

$$\mathbf{J}^2 |k,j,m\rangle = j(j+1)\hbar^2 |k,j,m\rangle \; ; \qquad J_z |k,j,m\rangle = m\hbar |k,j,m\rangle \tag{22}$$

$$J_\pm |k,j,m\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\, |k,j,m\pm 1\rangle \tag{23}$$

Now, consider a physical system formed by the union of two subsystems with state spaces $\mathcal{E}_1$ and $\mathcal{E}_2$. The state space of the global system is the tensor product $\mathcal{E}_1 \otimes \mathcal{E}_2$. We'll first define it simply in the basis $\{\mathbf{J}_1^2, \mathbf{J}_2^2, J_{1z}, J_{2z}\}$.

$$|k_1, k_2; j_1, j_2; m_1, m_2\rangle = |k_1, j_1, m_1\rangle \otimes |k_2, j_2, m_2\rangle \tag{24}$$

In general, we'll drop the k_1, k_2 notation as it doesn't affect the eigenstates of the angular momentum. We'll now define $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$ and try to perform a basis change to $\{\mathbf{J}_1^2, \mathbf{J}_2^2, \mathbf{J}^2, J_z\}$. All of these operators commute. For fixed j_1 and j_2, we'll describe these vectors in terms of eigenstates of $\{\mathbf{J}^2, J_z\}$ as $|J, M\rangle$.

$$|j_1 - j_2| \leq J \leq j_1 + j_2 \; ; \qquad -J \leq M \leq J \tag{25}$$

J and M occur at integer intervals within the range stated above. We can write the states more generally as $|j_1, j_2; J, M\rangle$ if j_1 or j_2 vary. $|J, M\rangle$ act as typical angular momentum eigenstates.

$$\mathbf{J}^2 |J,M\rangle = J(J+1)\hbar^2 |J,M\rangle \; ; \qquad J_z = |J,M\rangle = M\hbar |J,M\rangle \tag{26}$$

In each space $\mathcal{E}(j_1, j_2)$, $|J, M\rangle$ are linear combinations of vectors of the initial $\{|j_1, j_2; m_1, m_2\rangle\}$ basis.

$$|J,M\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1, j_2; m_1, m_2\rangle \langle j_1, j_2; m_1, m_2 | J, M\rangle \tag{27}$$

The coefficients of these expansions are called Clebsch-Gordan coefficients.

2.4 Clebsch-Gordan coefficients

The Clebsch-Gordan coefficient $\langle j_1, j_2; m_1, m_2 | J, M\rangle$ is necessarily zero if the two selection rules are not simultaneously satisfied.

$$M = m_1 + m_2 \; ; \qquad |j_1 - j_2| \leq J \leq j_1 + j_2 \tag{28}$$

The Clebsch-Gordan coefficients can be found by applying $J_\pm$ to form the following recurrence relation.

$$\begin{aligned}\sqrt{J(J+1)-M(M+1)}\,\langle j_1,j_2;m_1,m_2|J,M+1\rangle \\ = \sqrt{j_1(j_1+1)-m_1(m_1-1)}\,\langle j_1,j_2;m_1-1,m_2|J,M\rangle \\ + \sqrt{j_2(j_2+1)-m_2(m_2-1)}\,\langle j_1,j_2;m_1,m_2-1|J,M\rangle\end{aligned} \tag{29}$$

However, it's typically easier to use a provided table of CG coefficients.

2.5 The Wigner-Eckart theorem

The Wigner-Eckart theorem manipulates the behavior of an arbitrary vector operator $\mathbf{V}$ under interactions with an angular momentum $\mathbf{J}$. The theorem comes to the conclusion that inside the subspace $\mathcal{E}(k,j)$

$$\mathbf{V} = \frac{\langle \mathbf{J}\cdot\mathbf{V}\rangle_{k,j}}{\langle \mathbf{J}^2\rangle_{k,j}}\mathbf{J} = \frac{\langle \mathbf{J}\cdot\mathbf{V}\rangle_{k,j}}{j(j+1)\hbar^2}\mathbf{J} \tag{30}$$

This is also called the projection theorem. A classical physical interpretation is that any physical quantity measured on some system with total angular momentum $\mathbf{j}$ will rotate about $\mathbf{j}$. This suggests that all time-averaged, vector quantities will lie along their projection to $\mathbf{j}$.

CT uses the W-E theorem to calculate the Landau g-factor. In the presence of a magnetic field $\mathbf{B}$ parallel to $\hat{z}$, an atom's electrons with oribtal momentum $\mathbf{L}$ and spin $\mathbf{S}$ have a Hamiltonian,

$$H = H_0 + H_1\ ; \qquad H_1 = \omega_L(L_z+2S_z) = g_J\omega_L J_z; \qquad g_J = \frac{3}{2} + \frac{S(S+1)-L(L+1)}{2J(J+1)} \tag{31}$$

This causes a splitting of $E_1(M) = g_J M\hbar\omega_L$.

3 Perturbation Theory

[CT 1093-1208]

Perturbation theory is applicable when the Hamiltonian H of the system being studied can be put in the form:

$$H = H_0 + W \tag{32}$$

where the eigenstates and eigenvalues of H_0 are known, and where W is much smaller than H_0. To quantify the perturbation, we'll say $W = \lambda\hat{W}$ where $\lambda \ll 1$

$$H(\lambda) = H_0 + \lambda\hat{W} \tag{33}$$

3.1 Perturbation of a non-degenerate level

Take E_n^0 and $|\phi_n\rangle$ to be the eigenvalues and eigenstates of the un-perturbed Hamiltonian H_0.

The first order corrections to the energies are given by,

$$E_n(\lambda) = E_n^0 + \langle\phi_n|W|\phi_n\rangle + O(\lambda^2) \tag{34}$$

The first order corrections to the eigenstate are given by,

$$|\phi_n(\lambda)\rangle = |\phi_n\rangle + \sum_{p\neq n}\sum_i \frac{\langle\phi_p^i|\, W\,|\phi_n\rangle}{E_n^0 - E_p^0}\,|\phi_p^i\rangle + O(\lambda^2) \tag{35}$$

The second order corrections to the energies are given by,

$$E_n(\lambda) = E_n^0 + \langle\phi_n|W|\phi_n\rangle + \sum_{p\neq n}\sum_i \frac{\left|\langle\phi_p^i|\, W\,|\phi_n\rangle\right|^2}{E_n^0 - E_p^0} + O(\lambda^3) \tag{36}$$

It is clear to see that these formulae break down if the states are degenerate. That is to say that there exists some $i \neq n$ such that $1/(E_n^0 - E_p^0) \to \infty$. The perturbed eigenstates are thus non-normalizable and invalid.

3.2 Perturbation of a degenerate state

For a given degenerate eigenvalue, the first order perturbation energies and zeroth order states are given by diagonalizing a W in the basis of the degenerate eigenstates.

$$W_{i,j} = \langle\phi_i|\, W\,|\phi_j\rangle \tag{37}$$

where $|\phi_i\rangle$ and $|\phi_j\rangle$ are the degenerate states for the given energy. The resultant zeroth order eigenstates 'mix' the original degenerate eigenstates. The degeneracy is usually lifted, but may remain if the W matches the symmetry of the original H_0.

3.3 The Variational Method

A unique property of the ground state of some given Hamiltonian is that its energy is minimized. The variational method is a tool wherein one can choose an arbitrary wavefunction form as a function of some parameter $\psi(\lambda)$ and minimize the expected energy. If the wavefunction form is chosen well, $\psi(\lambda_{min})$ will give a good approximation of the true ground state.

$$\Big[\frac{\partial}{\partial\lambda}\,\langle\psi(\lambda)|H|\psi(\lambda)\rangle\Big]\Big|_{\lambda=\lambda_0} = 0\ ; \qquad E_{min} = \langle\psi(\lambda_0)|H|\psi(\lambda_0)\rangle \tag{38}$$

4 The Fine and Hyperfine Structure of the Hydrogen Atom

[CT 1209-1282]

The Coulomb potential Hamiltonian for the Hydrogen atom is given by,

$$H_0 = \frac{\mathbf{P}^2}{2\mu} + V(R)\ ; \qquad V(R) = -\frac{q^2}{4\pi\epsilon}\frac{1}{R} = -\frac{e^2}{R} \tag{39}$$

This Hamiltonian is approximate because it fails to take into account relativistic effects and spin interactions. These additional effects are very small (on the order of $v/c = 1/137 \ll 1$).

4.1 The fine-structure Hamiltonian

The fine structure Hamiltonian is given by the first terms of the power series expansion in v/c.

$$H = m_e c^2 + \underbrace{\frac{\mathbf{P}^2}{2m_e} + V(R)}_{H_0} - \underbrace{\frac{\mathbf{P}^4}{8m_e^3 c^2}}_{W_{mv}} + \underbrace{\frac{1}{2m_e^2 c^2}\frac{1}{R}\frac{dV(R)}{dR}\mathbf{L}\cdot\mathbf{S}}_{W_{SO}} + \underbrace{\frac{\hbar^2}{8m_e^2 c^2}\nabla^2 V(R)}_{W_D} + \dots \tag{40}$$

Each of these terms are on the order of magnitute $W_i/H_0 \simeq \alpha^2 \simeq (1/137)^2$.

The origin of the mass-velocity term W_{mv} is an expansion of the relativistic expression for the energy of a classical particle.

$$E = c\sqrt{\mathbf{p}^2 + m_e^2 c^2} = m_e c^2 + \frac{\mathbf{p}^2}{2m_e} - \frac{\mathbf{p}^4}{8m_e^3 c^2} + \dots \tag{41}$$

The spin-orbit coupling term W_{SO} arises from the fact that the nucleus's electrostatic field appears as a magnetic field from the moving electron's frame of reference. There is an interaction energy between this magnetic field and the electron's spin.

$$\mathbf{B}' = -\frac{1}{c^2}\mathbf{v}\times\mathbf{E}\ ; \qquad W_{SO} \propto -\mathbf{M}_S\cdot\mathbf{B}' \tag{42}$$

The Darwin term W_D derives from, in the non-relativistic approximation, the interaction between the electron and the field is non-local.

4.2 The hyperfine Hamiltonian

The hyperfine Hamiltonian arises from the interaction of the proton's spin with electron's spin and angular momentum.

$$\mathbf{M_I} = \frac{g_p \mu_n}{\hbar}\mathbf{I}\ ; \qquad \mu_n = \frac{q_p \hbar}{2M_p} \tag{43}$$

where $\mathbf{I}$ is the spin of the proton and μ_n is the nuclear Bohr magneton. The hyperfine correction to the hydrogen Hamiltonian is given by W_{hf}.

$$W_{hf} = -\frac{mu_0}{4\pi}\Big[\frac{q}{m_e R^3}\mathbf{L}\cdot\mathbf{M_I} + \frac{1}{R^3}[3(\mathbf{M_S}\cdot\mathbf{n})(\mathbf{M_I}\cdot\mathbf{n}) - \mathbf{M_S}\cdot\mathbf{M_I}] + \frac{8\pi}{3}\mathbf{M_S}\cdot\mathbf{M_I}\ \delta(\mathbf{R})\Big] \tag{44}$$

where $\mathbf{M_S}$ is the spin magnetic moment of the electron and $\mathbf{n}$ is the unit vector of the straight line joining the proton to the electron. The order of magnitude of the hyperfine Hamiltonian is about $W_{hf}/W_{SO} \simeq 1/2000$.

4.3 The fine structure of the $n = 1, 2$ level

The corrections to the $n = 1, 2$ levels of the Hydrogen atom can be calculated using stationary perturbatioan theory for degenerate states. CT does this in [CT 1221-1232]. Using the notation nl_J for the wavefunction quantum numbers, the corrections are as follows.
For $n = 2$

$$2s_{1/2} \to -\frac{5}{128}m_e c^2\alpha^4\ ; \qquad 2p_{1/2} \to -\frac{5}{128}m_e c^2\alpha^4\ ; \qquad 2p_{3/2} \to -\frac{1}{128}m_e c^2\alpha^4 \tag{45}$$

The similar corrections to the $2s_{1/2}$ and $2p_{1/2}$ states are coincidental, within this theory.

Due to the fact that there is no orbital degeneracy $l = 0$, the $1s$ level has no fine structure. For the $n = 1$ case, we can write the hyperfine Hamiltonian as $W_{hf} = \mathcal{A}\,\mathbf{I}\cdot\mathbf{S}$, where $\mathcal{A}$ is a constant. Introducing $\mathbf{F} = \mathbf{S} + \mathbf{I}$, we can form eigenstates for the hyperfine Hamiltonian $|F, m_F\rangle$. The hyperfine corrections to the $1s_{1/2}$ states are

$$|F=0, m_F\rangle \to -\frac{3}{4}\mathcal{A}\hbar^2\ ; \qquad |F=1, m_F\rangle \to \frac{1}{4}\mathcal{A}\hbar^2\ ; \qquad \mathcal{A} = \frac{4}{3}g_p\frac{m_e}{M_p}m_e c^2\alpha^4\Big(1+\frac{m_e}{M_p}\Big)^{-3}\frac{1}{\hbar^2} \tag{46}$$

4.4 The Zeeman Effect

Assume the atom is placed in a static uniform magnetic field $\mathbf{B}_0$. The magnetic field interacts with several magnetic moments.

$$W_Z = -\mathbf{B}_0\cdot(\mathbf{M}_L + \mathbf{M}_S + \mathbf{M}_I) = \omega_0(L_z + 2S_z) + \omega_n I_z \tag{47}$$

$$\omega_0 = -\frac{q}{2m_e}B_0\ ; \qquad \omega_n = \frac{q}{2M_p}g_p B_0 \tag{48}$$

For the $n = 1$ state, the L_z operator has no effect. We will also ignore the $\omega_n I_z$ term because $\omega_n \ll \omega_0$.

$$W = \mathcal{A}\,\mathbf{I}\cdot\mathbf{S} + 2\omega_0 S_z \tag{49}$$

Varying B_0 changes the magnitude of the Zeeman term $2\omega_0 S_z$. We'll therefore consider three field strengths.

1. $\hbar\omega \ll \mathcal{A}\hbar^2$: weak fields
2. $\hbar\omega \gg \mathcal{A}\hbar^2$: strong fields
3. $\hbar\omega \simeq \mathcal{A}\hbar^2$: intermediate fields

For weak fields $\hbar\omega \ll \mathcal{A}\hbar^2$, we create a matrix representation of $2\mu_0 S_z$ in the $\{|F, m_F\rangle\}$ basis and find its eigenvalues. For weak fields,

$$|0,0\rangle \to -\frac{3}{4}\mathcal{A}\hbar^2 + 0\ ; \qquad |1,0\rangle \to \frac{1}{4}\mathcal{A}\hbar^2 + 0\ ; \qquad |1,\pm1\rangle \to \frac{1}{4}\mathcal{A}\hbar^2 \pm \hbar\omega_0\ ; \tag{50}$$

For strong fields $\hbar\omega \gg \mathcal{A}\hbar^2$, we create a matrix representation of $\mathcal{A}\,\mathbf{I}\cdot\mathbf{S}$ in the $\{|m_S, m_I\rangle\}$ basis and find its eigenvalues. For strong fields,

$$|+,\pm\rangle \to \hbar\omega_0 \pm \frac{1}{4}\mathcal{A}\hbar^2\ ; \qquad |-,\pm\rangle \to -\hbar\omega_0 \mp \frac{1}{4}\mathcal{A}\hbar^2 \tag{51}$$

For intermediate fields $\hbar\omega \simeq \mathcal{A}\hbar^2$, we create a matrix representation of $W = \mathcal{A}\,\mathbf{I}\cdot\mathbf{S} + 2\omega_0 S_z$ in the $\{|F, m_F\rangle\}$ basis and find its eigenvalues. They are less simple than that of the strong or weak fields, but still solvable. For intermediate fields,

$$|0,0\rangle \to -\frac{1}{4}\mathcal{A}\hbar^2 - \sqrt{\Big(\frac{\mathcal{A}\hbar^2}{2}\Big)^2 + \hbar^2\omega_0^2}\ ; \qquad |1,0\rangle \to -\frac{1}{4}\mathcal{A}\hbar^2 + \sqrt{\Big(\frac{\mathcal{A}\hbar^2}{2}\Big)^2 + \hbar^2\omega_0^2}\ ; \tag{52}$$

$$|1,1\rangle \to \frac{1}{4}\mathcal{A}\hbar^2 + \hbar\omega_0\ ; \qquad |1,-1\rangle \to \frac{1}{4}\mathcal{A}\hbar^2 - \hbar\omega_0 \tag{53}$$

5 Approximation Methods for Time-Dependent Problems

[CT 1283-1368]

Consider a physical system with Hamiltonian H_0 and accompanying E_n and $|\phi_n\rangle$. A time-dependent perturbation is applied to the system such that $H(t) = H_0 + W(t)$. We define $W(t) = \lambda\hat{W}(t)$ where $\lambda \ll 1$. This produces the following time-dependent Schrodinger equation.

$$H_0 |\phi_n\rangle = E_n |\phi_n\rangle \ ; \qquad i\hbar\frac{d}{dt}|\psi(t)\rangle = [H_0 + \lambda\hat{W}(t)]\,|\psi(t)\rangle \tag{54}$$

Typically, this problem is stated with an accompanying initial condition. The task is then to find the probability that the initial state transforms into another state at a time t.

$$|\psi(t=0)\rangle = |\phi_i\rangle \ ; \qquad \mathscr{P}_{if}(t) = |\langle\phi_f|\psi(t)\rangle|^2 \tag{55}$$

We'll begin by expanding $|\psi(t)\rangle$ in the basis $\{|\phi_n\rangle\}$. By inserting the closure relation $\sum_k |\phi_k\rangle\langle\phi_k|$ at the front of each side of the Schrodinger equation.

$$|\psi(t)\rangle = \sum_n \underbrace{\langle\phi_n|\psi(t)\rangle}_{c_n(t)}|\phi_n\rangle \ ; \qquad \langle\phi_n|\hat{W}(t)|\phi_k\rangle = \hat{W}_{nk}(t) \ ; \qquad \langle\phi_n|H_0|\phi_k\rangle = E_n\delta_{nk} \tag{56}$$

$$i\hbar\frac{d}{dt}c_n(t) = E_n c_n(t) + \sum_k \lambda\hat{W}_{nk}(t)c_k(t) \tag{57}$$

Along a similar reasoning to the uncoupled case ($\lambda = 0$), we'll perform a function transformation to further simplify the equation.

$$c_n(t) = b_n(t)\exp\left\{-\frac{iE_nt}{\hbar}\right\} \ ; \qquad \omega_{nk} = \frac{E_n - E_k}{\hbar} \tag{58}$$

Resulting, finally, in a neat, yet exact, form

$$i\hbar\frac{d}{dt}b_n(t) = \lambda\sum_k e^{i\omega_{nk}t}\hat{W}_{nk}(t)b_k(t) \tag{59}$$

5.1 Approximate solution

So far, everything we've done has been exact, no approximations. In general, the exact solution to this equation is difficult to find. We'll now expand $b_n(t)$ in a power series expansions in λ. Plugging this into the above equation and set equal the coefficients of λ^r, we find a recurrence relation for n-th order solutions of $b_n(t)$.

$$b_n(t) = b_n^{(0)}(t) + \lambda b_n^{(1)}(t) + \lambda^2 b_n^{(2)}(t) + \ldots \tag{60}$$

$$i\hbar\frac{d}{dt}b_n^{(0)}(t) = 0 \ ; \qquad i\hbar\frac{d}{dt}b_n^{(r)}(t) = \sum_k e^{i\omega_{nk}t}\hat{W}_{nk}(t)b_k^{(r-1)}(t) \tag{61}$$

Now, let us consider the problem stated previously. The system is initialized in an eigenstate $|\phi_i\rangle$ of H_0. From the previous set of equations, follows

$$b_n(t=0) = \delta_{ni} \ \rightarrow \ b_n^{(r)}(t=0) = \begin{cases}\delta_{ni} & \text{if } r = 0 \\ 0 & \text{if } r \geq 1\end{cases} \ \rightarrow \ b_n^{(0)}(t) = \delta_{ni} \tag{62}$$

Then, we'll integrate to find the first-order coefficients

$$i\hbar\frac{d}{dt}b_n^{(1)}(t) = e^{i\omega_{ni}t}\hat{W}_{ni}(t) \qquad \rightarrow \qquad b_n^{(1)}(t) = \frac{1}{i\hbar}\int_0^t e^{i\omega_{ni}t'}\hat{W}_{ni}(t')dt' \tag{63}$$

Therefore, to first order, the transition probability $\mathscr{P}_{if}(t)$ is given by

$$\mathscr{P}_{if}(t) = |c_f(t)|^2 = |b_f(t)|^2 \simeq \lambda^2 \left| b_f^{(1)}(t) \right|^2 = \frac{1}{\hbar^2} \left| \int_0^t e^{i\omega_{fi}t'} W_{fi}(t') \, dt' \right|^2 \tag{64}$$

5.2 A sinusoidal or constant perturbation: the resonance phenomenon

Consider a perturbation $W(t)$ with sinusoidal time dependence.

$$\hat{W}_1(t) = \hat{W} \sin \omega t \; ; \qquad \hat{W}_2(t) = \hat{W} \cos \omega t \tag{65}$$

For now, we'll focus on $\hat{W}(t) = \hat{W}_1(t)$ and derive its transition probabilities from the equations derived previously. First, notice that $\hat{W}_{fi}(t)$ can be written

$$\hat{W}_{fi}(t) = \hat{W}_{fi} \sin \omega t = \frac{\hat{W}_{fi}}{2i}(e^{i\omega t} - e^{-i\omega t}) \tag{66}$$

Assuming the physical system starts in an eigenstate $|\phi_i\rangle$ of H_0, the first order state vector is given by application of the above equation.

$$b_n^{(1)}(t) = -\frac{\hat{W}_{ni}}{2\hbar} \int_0^t [e^{i(\omega_{ni}+\omega)t'} - e^{i(\omega_{ni}-\omega)t'}] \, dt' = \frac{\hat{W}_{ni}}{2i\hbar} \Big[\frac{1 - e^{i(\omega_{ni}+\omega)t}}{\omega_{ni} + \omega} - \frac{1 - e^{i(\omega_{ni}-\omega)t}}{\omega_{ni} - \omega} \Big] \tag{67}$$

The transition probability for $\hat{W}_1(t)$ is simply

$$\mathscr{P}_{1\_if}(t;\omega) = \lambda^2 \left| b_f^{(1)}(t) \right|^2 = \frac{|W_{fi}|^2}{4\hbar^2} \left| \frac{1 - e^{i(\omega_{ni}+\omega)t}}{\omega_{ni} + \omega} - \frac{1 - e^{i(\omega_{ni}-\omega)t}}{\omega_{ni} - \omega} \right|^2 \tag{68}$$

A similar argument can be made for $\hat{W}_2(t)$,

$$\mathscr{P}_{2\_if}(t;\omega) = \frac{|W_{fi}|^2}{4\hbar^2} \left| \frac{1 - e^{i(\omega_{ni}+\omega)t}}{\omega_{ni} + \omega} + \frac{1 - e^{i(\omega_{ni}-\omega)t}}{\omega_{ni} - \omega} \right|^2 \tag{69}$$

For the $\hat{W}_2(t)$ case, the perturbation becomes time-independent if we choose $\omega = 0$. $\mathscr{P}_{2\_if}$ simplifies to

$$\mathscr{P}_{2\_if}(t) = \frac{|W_{fi}|^2}{\hbar^2} F(t, \omega_{fi}) \; ; \qquad F(t, \omega_{fi}) = \left[\frac{\sin(\omega_{fi} t/2)}{\omega_{fi}/2} \right]^2 \tag{70}$$

If we choose $|\omega - \omega_{fi}| \ll |\omega_{fi}|$, the $\hat{W}_2(t)$ transition probability becomes

$$\mathscr{P}_{2\_if}(t;\omega) \simeq \frac{|W_{fi}|^2}{4\hbar^2} F(t, \omega - \omega_{fi}) \; ; \qquad F(t, \omega - \omega_{fi}) = \left[\frac{\sin[(\omega_{fi} - \omega)t/2]}{(\omega_{fi} - \omega)/2} \right]^2 \tag{71}$$

Note that the $F(t,\omega)$ function is the same for the constant and sinusoidal case. Also, the sinusoidal transition probability is maximized (but not infinite!) at $\omega = \omega_{fi}$. This is the resonance phenomenon between two discrete states.

5.3 Coupling with the states of the continuous spectrum

If the energy E_f belongs to a continuous part of the spectrum of H_0, we cannot measure the probability of finding the system in a well-defined state at time t. The quantity $|\langle\phi_f|\psi(t)\rangle|^2$ is a probability density.

Assume that certain eigenstates of H_0 are labeled by a continuous set of indices, symbolized by α, such that the orthonormalization relation can be written.

$$\langle\alpha|\alpha'\rangle = \delta(\alpha - \alpha') \tag{72}$$

The probability $\delta\mathscr{P}(\alpha_f, t)$ of finding the system in a given group of final states D_f centered at α_f is given by

$$\delta\mathscr{P}(\alpha_f, t) = \int_{\alpha\in D_f} d\alpha\ |\langle\alpha|\psi(t)\rangle|^2 \tag{73}$$

We can seperate the characterization of these states by splitting the α parameters into the energy E and a set of everything else β. The other parameters β are necessary when H_0 doesn't form a C.S.C.O.

$$d\alpha = \rho(\beta, E)d\beta\ dE\ ; \qquad \delta\mathscr{P}(\alpha_f, t) = \int_{\beta\in\delta\beta_f}\int_{E\in\delta E_f} d\beta\ dE\ \rho(\beta, E)|\langle\beta, E|\psi(t)\rangle|^2 \tag{74}$$

where $\rho(\beta, E)$ is the density of states.

CT gives the example of the scattering of a spinless particle of mass m by a potential of $W(\mathbf{r})$. The state $|\psi(t)\rangle$ of the particle at t can be expanded on the states $|\mathbf{p}\rangle$ of well-defined momenta $\mathbf{p}$ and energies. The particle is scattered and reaches a detector whenever the momentum $\mathbf{p}$ of the particle points within a solid angle $\delta\Omega_f$ and its energy is within δE_f. D_f denotes the domain of $\mathbf{p}$-space defined by these conditions.

$$E = \frac{\mathbf{p}^2}{2m}\ ; \qquad \delta\mathscr{P}(\mathbf{p}_f, t) = \int_{\mathbf{p}\in D_f} d^3p\ |\langle\mathbf{p}|\psi(t)\rangle|^2 \tag{75}$$

We'll perform a change of variables from d^3p to $dE\ d\Omega$.

$$d^3p = p^2\ dp\ d\Omega = \rho(E)\ dE\ d\Omega\ ; \qquad \rho(E) = p^2\frac{dp}{dE} = m\sqrt{2mE} \tag{76}$$

where $\rho(E)$ is the density of states for a free particle. We can finally obtain the probability of finding the particle at the detector.

$$\delta\mathscr{P}(\mathbf{p}_f, t) = \int_{\Omega\in\delta\Omega_f}\int_{E\in\delta E_f} d\Omega\ dE\ \rho(E)|\langle\mathbf{p}|\psi(t)\rangle|^2 \tag{77}$$

5.4 Fermi's golden rule

Consider a system which is initially in a discrete eigenstate $|\phi_i\rangle$ of H_0. The system then undergoes a constant perturbation W to achieve a final state within a continuous spectrum. The probability now becomes

$$\delta\mathscr{P}(\phi_i, \alpha_f, t) = \frac{1}{\hbar^2}\int_{\substack{\beta\in\delta\beta_f\\ E\in\delta E_f}} d\beta\ dE\ \rho(\beta, E)\ |\langle\beta, E|\,W\,|\phi_i\rangle|^2\ F\left(t, \frac{E - E_i}{\hbar}\right) \tag{78}$$

where $F(t,E)$ is that of a constant perturbation, discussed previously. Under the assumption that t is very large and $\delta\beta_f$ is very small,

$$\begin{gathered}\lim_{t\to\infty} F\left(t, \frac{E-E_i}{\hbar}\right) = 2\pi\hbar t\ \delta(E-E_i) \\ \delta\mathscr{P}(\phi_i,\alpha_f,t) = \delta\beta_f \frac{2\pi}{\hbar} t |\langle\beta_f, E_f = E_i| W |\phi_i\rangle|^2\ \rho(\beta_f, E_f = E_i) * \begin{cases} 1 & \text{if } E_i \in \delta E_f \\ 0 & \text{else} \end{cases}\end{gathered} \tag{79}$$

We see that the probability increases linearly with time. We'll now define the transition probability per unit time as $\delta\mathscr{W}(\phi,\alpha_f)$ and the transition probability per unit time per unit interval of β_f as $w(\phi_i,\alpha_f)$.

$$\delta\mathscr{W}(\phi_i,\alpha_f) = \frac{d}{dt}\delta\mathscr{P}(\phi_i,\alpha_f,t)\ ; \qquad w(\phi_i,\alpha_f) = \frac{\delta\mathscr{W}(\phi_i,\alpha_f)}{\delta\beta_f} \tag{80}$$

The resulting expression for $w(\phi_i,\alpha_f)$ with a constant perturbation is Fermi's golden rule for particle scattering off a constant potential.

$$w(\phi_i,\alpha_f) = \frac{2\pi}{\hbar}|\langle\beta_f, E_f = E_i| W |\phi_i\rangle|^2\ \rho(\beta_f, E_f = E_i) \tag{81}$$

Similar reasoning can be made to find $w(\phi_i,\alpha_f)$ for a sinusoidal perturbation $W(t) = W\sin\omega t$,

$$w(\phi_i,\alpha_f) = \frac{\pi}{2\hbar}|\langle\beta_f, E_f = E_i + \hbar\omega| W |\phi_i\rangle|^2\ \rho(\beta_f, E_f = E_i + \hbar\omega) \tag{82}$$

Returning to the example of the scattering of a particle by a constant potential W from an initial, well-defined momentum state.

$$|\psi(t=0)\rangle = |\mathbf{p_i}\rangle\ ; \qquad \langle\mathbf{r}| W \left|\mathbf{r}'\right\rangle = W(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}') \tag{83}$$

We can find an expression for $w(\phi,\alpha_f)$. This is Fermi's golden rule.

$$w(\mathbf{p}_i,\mathbf{p}_f) = \frac{2\pi}{\hbar} m\sqrt{2mE_i}\left(\frac{1}{2\pi\hbar}\right)^6 \left|\int d^3r\ W(\mathbf{r}) e^{i(\mathbf{p}_i-\mathbf{p}_f)\cdot\mathbf{r}/\hbar}\right|^2 \tag{84}$$

Dividing this by the probability current, we arrive at the scattering cross section in the Born approximation

$$J_i = \left(\frac{1}{2\pi\hbar}\right)^3 \sqrt{\frac{2E_i}{m}}\ ; \qquad \frac{d\sigma}{d\Omega} = \frac{w(\mathbf{p}_i,\mathbf{p}_f)}{J_i} = \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3r\ W(\mathbf{r})\ e^{i(\mathbf{p}_i-\mathbf{p}_f)\cdot\mathbf{r}/\hbar}\right|^2 \tag{85}$$

5.5 Decay of a discrete state resonantly coupled to a continuum of final states

We now have an expression for the transition probability per unit time per unit interval of β_f as $w(\phi_i,\alpha_f)$. This describes the likelihood that a particle with transition to a new state described by β_f with the same energy as ϕ_i.

We can relate this to a decay problem, where a particle starts in a single state ϕ_i and decays into a continuum of final states β_f with the same energy. We will define the Γ constant as the probability per unit time that the particle makes any transition away from the initial state.

$$\Gamma = \int d\beta_f \; w(\phi_i, \alpha_f) = \frac{2\pi}{\hbar} \int d\beta \; |\langle \beta, E = E_i | \, W \, | \phi_i \rangle|^2 \rho(\beta, E = E_i) \tag{86}$$

We can write the probability that the particle remains in the initial state to first order as a function of Γ.

$$\mathscr{P}_{ii}(t) = e^{-\Gamma t} \approx 1 - \Gamma t \tag{87}$$

From the definition of Γ, we can write the scattering cross section as a function of Γ and the probability current.

$$\frac{d\sigma}{d\Omega} = \frac{1}{J_i} \frac{d\Gamma}{d\beta} \tag{88}$$

6 An Elementary Approach to the Quantum Theory of Scattering by a Potential

[CT 901-964]

When studying scattering, we seek to learn things about the initial and final states of a set of incident particles (**1**) undergoing elastic collisions with a set of target particles (**2**). For this chapter, the following conditions will apply

- We will assume particles (**1**) and (**2**) have no spin.
- We shall not take into account the possible internal structure of the particles (**1**) and (**2**).
- We shall assume that the target is thin enough to enable us to neglect multiple scattering processes.
- We shall neglect any possibility of coherence between the waves scattered by the different particles which make up the target.
- We shall assume that the interactions between the particles (**1**) and (**2**) can be described by a potential energy $V(\mathbf{r}_1 - \mathbf{r}_2)$ which depends only on the relative position of the particles.The center of mass reference frame of the two particles (**1**) and (**2**) reduces to the study of the scattering of a single particle by the potential $V(\mathbf{r})$ with effective radius $\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2$ and relative mass μ.

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \tag{89}$$

Let $\hat{\mathbf{z}}$ be the direction of the incident particles of mass μ. The potential $V(\mathbf{r})$ is localized around the origin of the coordinate system in the center of mass reference frame of the two real particles. The situation is shown in Figure 1.

We'll designate F_i as the flux of particles in the incident beam (in number of particles per time). We place a detector far from the region of the potential in the direction fixed by the polar angles θ and ϕ with an

opening facing the origin and subtending the solid angle $d\Omega$. We can count the number dn of particles scattered per unit time into the solid angle $d\Omega$ about the direction (θ, ϕ).

$$dn = F_i \ \sigma(\theta, \phi) \ d\Omega \tag{90}$$

$\sigma(\theta, \phi)$ is called the differential scattering cross section. It is also often written as $\frac{d\sigma}{d\Omega}$. It acts as a proportionality constant as to how many particles would pass through $d\Omega$ given a (θ, ϕ). It has units of area, which is usually measured in barns. 1 barn $= 10^{-24}$ cm^2. The total cross section σ is written as

$$\sigma = \int \sigma(\theta, \phi) \ d\Omega \tag{91}$$

The Hamiltonian H of our system will be that of a free wave (kinetic energy) plus a potential localized at the origin.

$$H = H_0 + V(\mathbf{r}) \ ; \qquad H_0 = \frac{\mathbf{P}^2}{2\mu} = -\frac{\hbar^2}{2\mu}\nabla^2 \tag{92}$$

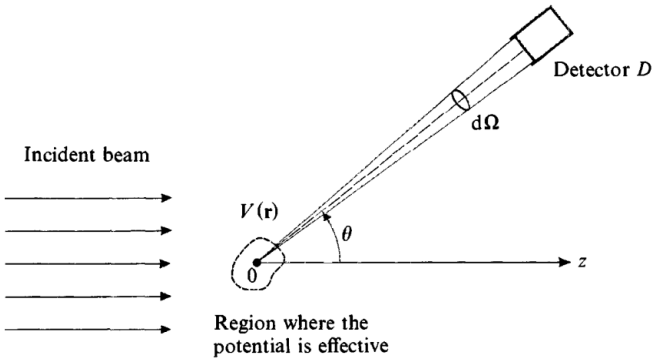

Figure 1: Definition of the scattering cross section. From [CT 905]

6.1 Stationary scattering states

We will study the time evolution of the incident wave packet upon its interaction with the central potential. First, we will consider the stationary scattering states and then use a superposition of these to represent our wave packet.

The scattering states stationary states will have some well-defined energy E and a time-dependence like $\psi(\mathbf{r}, t) = \phi(\mathbf{r}) \exp\{-iEt/\hbar\}$. Making some substitutions, we can write the Schrodinger equation in a simple form

$$[\nabla^2 + k^2 - U(\mathbf{r})]\phi(\mathbf{r}) = 0 \ ; \qquad E = \frac{\hbar^2 k^2}{2\mu} \ ; \qquad V(\mathbf{r}) = \frac{\hbar^2}{2\mu} U(\mathbf{r}) \tag{93}$$

We will use the convention $\phi(\mathbf{r}) = v_k^{(\text{diff})}(\mathbf{r})$ to designate a stationary scattering state of wave number k.

6.1.1 Asymptotic Form

We will now use some approximations to simplify the expression of $v_k^{(\text{diff})}$ in the limit $r \to \infty$. For large negative values of t, the incident wave is a plane wave packet traveling along in the $+\hat{\mathbf{z}}$ direction. It should

have the form e^{ikz}. The scattered wave should somewhat look like a spherical wave e^{ikr}/r as $r \to \infty$. However, it should have varied amplitude at different (θ, ϕ).

We will then say that the asymptotic form of $v_k^{(\text{diff})}$ should go as

$$v_k^{(\text{diff})}(\mathbf{r}) \underset{r\to\infty}{\sim} e^{ikz} + f_k(\theta,\phi)\frac{e^{ikr}}{r} \tag{94}$$

where $f_k(\theta, \phi)$ is called the scattering amplitude. The $V(\mathbf{r})$ dependence is then wrapped up in the evaluation of $f_k(\theta, \phi)$. The scattering amplitude $f_k(\theta, \phi)$ thereby solely determines the angular distribution of scattered particles and is related to the differential cross section $\sigma(\theta, \phi)$. CT shows this relationship using probability currents $\mathbf{J}(\mathbf{r})$ in [CT 912-915], but the final result is

$$\sigma(\theta,\phi) = |f_k(\theta,\phi)|^2 \tag{95}$$

6.1.2 Integral Scattering Equation

We would like to show, in a more precise way, the existence of stationary wave functions whose asymptotic behavior follows the form discussed above. From the Schrodinger equation, we can derive the following recursive expression for $v_k^{(\text{diff})}$.

$$v_k^{(\text{diff})}(\mathbf{r}) = e^{i\mathbf{k}_i\cdot\mathbf{r}} + \int d^3r'\ G_+(\mathbf{r}-\mathbf{r}')\ U(\mathbf{r}')\ v_k^{(\text{diff})}(\mathbf{r}') \tag{96}$$

where $\mathbf{k}_i$ is the incident wave vector. In this case, $\mathbf{k}_i = k\ \hat{\mathbf{z}}$. The function $G_+(\mathbf{r})$ is a Green's function of the operator $(\nabla^2 + k^2)$. It represents the outgoing wave from the scattering. The spherical wave function satisfies the Green's function condition.

$$(\nabla^2 + k^2)G(\mathbf{r}) = \delta(\mathbf{r})\ ; \qquad G_\pm(\mathbf{r}) = -\frac{1}{4\pi}\frac{e^{\pm ikr}}{r} \tag{97}$$

Taking the limit of $v_k^{(\text{diff})}(\theta, \phi)$ as $r \to \infty$, we arrive at the following exact, recursive definition of $f_k(\theta, \phi)$.

$$f_k(\theta,\phi) = -\frac{1}{4\pi}\int d^3r'\ e^{-i\mathbf{k}_d\cdot\mathbf{r}'}\ U(\mathbf{r}')\ v_k^{(\text{diff})}(\mathbf{r}') \tag{98}$$

where $\mathbf{k}_d = k\hat{\mathbf{u}}$ is the outgoing wave vector and $\hat{\mathbf{u}}$ is along (θ, ϕ).

6.1.3 The Born Approximation

The recursive definition of $v_k^{(\text{diff})}(\mathbf{r})$ is exact, but generally not solvable. However, imagine recursively substituting the definition of $v_k^{(\text{diff})}(\mathbf{r})$ in its own definition. This is called a Born expansion.

$$\begin{aligned} v_k^{(\text{diff})}(\mathbf{r}) = e^{i\mathbf{k}_i\cdot\mathbf{r}} &+ \int d^3r'\ G_+(\mathbf{r}-\mathbf{r}')U(\mathbf{r}')e^{i\mathbf{k}_i\cdot\mathbf{r}'} \\ &+ \int d^3r' \int d^3r''\ G_+(\mathbf{r}-\mathbf{r}')\ U(\mathbf{r}')\ G_+(\mathbf{r}'-\mathbf{r}'')\ U(\mathbf{r}'')\ e^{i\mathbf{k}_i\cdot\mathbf{r}''} \\ &+ \ \ldots \end{aligned} \tag{99}$$

Note that each new term brings a higher order of $U(\mathbf{r})$. Since, $U(\mathbf{r})$ drops off at $r \to \infty$ for a localized potential, we can approximate $v_k^{(\text{diff})}(\mathbf{r})$ by just taking the first term. This is the Born approximation.

$$v_k^{(\text{diff})}(\mathbf{r}) \sim e^{i\mathbf{k}_i \cdot \mathbf{r}} + \int d^3r' \; G_+(\mathbf{r} - \mathbf{r}') \; U(\mathbf{r}') e^{i\mathbf{k}_i \cdot \mathbf{r}'} \tag{100}$$

Inserting our spherical Green's function G_+ (which represents the outgoing $\mathbf{k}_d$), we arrive at the Born approximation for the scattering amplitude.

$$\begin{aligned} f_k^{(\text{B})}(\theta, \phi) &= -\frac{1}{4\pi} \int d^3r' \; e^{-i\mathbf{K} \cdot \mathbf{r}'} \; U(\mathbf{r}') \\ &= -\frac{\mu}{2\pi\hbar^2} \int d^3r' \; e^{-i\mathbf{K} \cdot \mathbf{r}'} \; V(\mathbf{r}') \end{aligned} \tag{101}$$

where $\mathbf{K} = \mathbf{k}_d - \mathbf{k}_i$ is the scattering wave vector. It is clear to see that by taking the squared absolute value of this equation, we will arrive at $\sigma(\theta, \phi)$.

6.2 Method of Partial Waves

In the special case of a central potential $V(r)$, the orbital angular momentum $\mathbf{L}$ of the particle is a constant of motion. We will write the incoming and outgoing waves as a superposition of partial waves $\phi_{k,l,m}(\mathbf{r})$, which are eigenstates common to H, $\mathbf{L}^2$ and L_z.

6.2.1 Free spherical waves

We expect that in the limit $r \to \infty$, the partial waves take a similar form to free spherical waves $\phi_{k,l,m}^{(0)}$, eigenfunctions of H_0, $\mathbf{L}^2$, and L_z.

$$\phi_{k,l,m}^{(0)}(\mathbf{r}) = \sqrt{\frac{2k^2}{\pi}} \; j_l(kr) \; Y_l^m(\theta, \phi) \tag{102}$$

Here, we have separated the free spherical wave into a radial component, the spherical Bessel function j_l, and an angular component, the spherical harmonic Y_l^m. The corresponding eigenvalues of H_0, $\mathbf{L}^2$ and L_z are, respectively, $\hbar^2 k^2 / 2\mu$, $l(l+1)\hbar^2$ and $m\hbar$. The free spherical waves are orthonormal and form a basis in (k, l, m)-space.

The incident plane wave e^{ikz} can be expanded in terms of the free spherical waves.

$$e^{ikz} = \sum_{l=0}^{\infty} i^l \sqrt{4\pi(2l+1)} \; j_l(kr) \; Y_l^0(\theta) \tag{103}$$

It is important to note that a free spherical wave must go to 0 as $r \to 0$. As a result, free spherical waves are practically unaffected by small potential perturbations below a certain radius $b_l(k)$ around the origin.

$$b_l(k) = \frac{1}{k} \sqrt{l(l+1)} \tag{104}$$

6.2.2 Partial waves

Whereas free spherical waves are eigenstates of H_0, the partial waves are eigenstates of H. For any central potential $V(r)$, the partial waves $\phi_{k,l,m}(\mathbf{r})$ are of the form

$$\phi_{k,l,m}(\mathbf{r}) = \frac{1}{r} u_{k,l}(r) Y_l^m(\theta, \phi) \tag{105}$$

where $u_{k,l}(r)$ is a solution of the radial equation. Plugging this into the Schrodinger equation with H, we produce the radial equation.

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \underbrace{\frac{l(l+1)\hbar^2}{2\mu r^2} + V(r)}_{V_{\text{eff}}(r)} \right] u_{k,l}(r) = \frac{\hbar^2 k^2}{2\mu} u_{k,l}(r) \; ; \qquad u_{k,l}(0) = 0 \tag{106}$$

This is just like a one-dimensional problem with V_{eff}. CT shows that, in the limit $r \to \infty$, the solution is given by

$$u_{k,l}(r) \underset{r\to\infty}{\sim} C \sin\left(kr - l\frac{\pi}{2} + \delta_l\right) \tag{107}$$

where C is an amplitude constant. δ_l is called the phase shift of the partial wave $\phi_{k,l,m}(\mathbf{r})$. For elastic scattering, δ_l is purely real. CT doesn't go into much detail regarding explicit calculation of the phase shifts as a function of $V(r)$. If we now write the full $\phi_{k,l,m}(\mathbf{r})$ with the angular dependence, choose an appropriate C, and multiply by an arbitrary factor of $e^{i\delta_l}$, we arrive at an expression for a partial wave $\tilde{\phi}_{k,l,m}(\mathbf{r})$.

$$\tilde{\phi}_{k,l,m} \underset{r\to\infty}{\sim} -Y_l^m(\theta,\phi)\left[\underbrace{\frac{e^{-ikr}e^{il\pi/2}}{2ikr}}_{\text{incoming}} - \underbrace{\frac{e^{ikr}e^{-il\pi/2}e^{2i\delta_l}}{2ikr}}_{\text{outgoing}} \right] \tag{108}$$

This is the final expression for the partial wave. Notice, the first term is a typical incident spherical free wave. The second term is almost the corresponding outgoing wave, except with an additional phase factor of $e^{2i\delta_l}$.

Side note: We discussed previously that free spherical waves are unaffected by 'small-enough', central potential perturbations. Similarly, if the potential $V(r)$ has a finite range r_0 (i.e. it goes to $V(r) = 0$ for $r > r_0$). then partial waves of high l and low k are not influenced. There exists a critical value l_M of the angular momentum above which the phase shifts δ_l are insignificant. This is the case in many real applications.

$$\sqrt{l_M(l_M+1)} \simeq kr_0 \tag{109}$$

The phase shift δ_l is the key difference between partial waves and free spherical waves in the asymptotic limit $r \to \infty$. As an incoming wave approaches the zone of influence of the potential, it is more and more perturbed by this potential. During that perturbation, it picks up a phase shift of $2\delta_l$ relative to the free outgoing wave (the eigenstate of H_0). The resultant angular dependence of the scattered wave depends on how the phase shift δ_l changes with l and, thus, with $Y_l^m(\theta,\phi)$.

$$f_k(\theta) = \frac{1}{k}\sum_{l=0}^{\infty} \sqrt{4\pi(2l+1)}\; e^{i\delta_l}\; \sin(\delta_l)\; Y_l^0(\theta) \tag{110}$$

Note, f_k is independent of ϕ because the incident wave and $V(r)$ are azimuthally symmetric about the $\hat{\mathbf{z}}$-axis. Using the fact that the spherical harmonics are orthonormal, we can write the total scattering cross section in an even simpler form.

$$\sigma = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l \tag{111}$$

6.3 Collisions with Absorption

We will now modify our situation to account for possible absorption. Previously, we established that the outgoing wave resulting from elastic scattering was multiplied by a phase factor $e^{2i\delta_l}$ (relative to the corresponding incident wave). Because δ_l was assumed to be real, $\left|e^{2i\delta l}\right| = 1$ and the outgoing wave had the same amplitude as the incident wave. However, if we allow this phase factor to be less than 1, we are stating that not all incident particles are elastically scattered. Instead, some proportion of them are absorbed.

We will set $\eta_l = e^{2i\delta_l}$ such that $|\eta_l| \leq 1$. The elastic scattering amplitude and cross section are the same as before, except now expressed in terms of η_l.

$$f_k(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}\sqrt{2\pi(2l+1)}\; Y_l^0(\theta)\;\frac{\eta_l - 1}{2i}\;; \qquad \sigma_{\text{el}} = \frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)|1-\eta_l|^2 \tag{112}$$

Similarly, we can derive a total cross section for absorption.

$$\sigma_{\text{abs}} = \frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)[1-|\eta_l|^2] \tag{113}$$

σ_{el} and σ_{abs} dictate the proportion of incident particles which are elastically scattered and which are absorbed, respectively. The sum of these cross sections forms the total cross section σ_{tot}. It is defined by the optical theorem.

$$\sigma_{\text{tot}} = \sigma_{\text{el}} + \sigma_{\text{abs}} = \frac{4\pi}{k}\,\text{Im}[f_k(0)] \tag{114}$$

where $\text{Im}[f_k(0)]$ is the imaginary part of the scattering amplitude in the forward direction.

6.4 Low Energy Scattering by a Hard Sphere

A hard sphere of radius r_0 is described by a central potential such that

$$V(r) = \begin{cases}\infty & \text{for } r < r_0\\ 0 & \text{for } r > r_0\end{cases} \tag{115}$$

In the low energy limit, we will neglect all phase shifts except that of the s wave ($l = 0$). This eliminates the angular dependence of the scattering amplitude. The simplicity of the hard sphere central potential allows us to solve the radial equation exactly.

$$u_{k,0}(r) = \begin{cases}C\sin k(r-r_0) & \text{for } r > r_0\\ 0 & \text{for } r < r_0\end{cases} \tag{116}$$

Matching this up to the definition of the phase shift, we find a phase shift of $\delta_0(k) = -kr_0$. From this, we can derive total cross section

$$\sigma = \frac{4\pi}{k^2}\sin^2 kr_0 \simeq 4\pi r_0^2 \tag{117}$$

This is a curious result because it is $4\times$ the expected value for the classical hard sphere (πr_0^2). CT chalks this up to a phenomenon analogous to the diffraction of a light wave.

7 Quantum Electrodynamics

[Gr 213-255]

We diverge into Griffiths Particle Physics to learn about how spin affects particle interactions.

7.1 Notation

Justin C. Feng writes a good, basic guide to tensor notation, see *The Poor Man's Introduction to Tensors*. Here are some notes necessary for use in this section.

A four-vector x^μ has four components (x^0, x^1, x^2, x^3), in contrast to a traditional 3D three-vector $\mathbf{x}$. We will make a distinction between a contravariant four-vector x^μ and a covariant four-vector x_μ. When a sub and super script appear with the same index, it is assumed that they are summed from 0 to 3. This is also noted as the scalar product of the two.

$$\begin{aligned} a \cdot b &= a_\mu b^\mu = a^\mu b_\mu \\ &= a^0 b^0 - \mathbf{a} \cdot \mathbf{b} = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 \end{aligned} \tag{118}$$

One can transform between a covariant and a contravariant through a metric. The metric we use is $g_{\mu\nu}$.

$$g_{\mu\nu} = g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \; ; \qquad a_\mu = g_{\mu\nu} a^\nu \; ; \qquad a^\mu = g^{\mu\nu} a_\nu \tag{119}$$

Each object discussed has a rank. Table 7.1 shows the rank for common objects.

Name	**Rank**	**# of Elements**	**Example**
Scalar	0	1	$c = 1$
Four-Vector	1	4×1	$a^\mu = (a^0, a^1, a^2, a^3)$
Matrix	2	$4 \times 4 \times 1$	$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$
Tensor of Rank 3	3	$4^3 \times 1$	$t^{\nu\lambda\sigma}$

Table 1: Common tensors

Note that the product of two tensors is itself a tensor. For example, $(a^\mu b^\nu)$ is a tensor of second rank. Also, we can obtain from any tensor of rank $n+2$ a "contracted" tensor of rank n, by summing like upper and lower indices. For example, $t^{\mu\nu}_\nu$ is a vector.

Note the following identities for matrices.

$$AB = A^\mu_\nu \; B^\nu_\lambda \; ; \qquad \mathrm{Tr}[A] = A^\mu_\mu \; ; \tag{120}$$

If you see a scalar in an algebraic expression with matrices, it is typically assumed that the scalar is multiplied by the appropriate identity matrix. For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 3 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & 7 \end{bmatrix} \tag{121}$$

Note that the order of terms matters. Only when two operators commute can you exchange their position in a product.

The position-time four-vector is given as

$$x^\mu = (ct, x, y, z) \tag{122}$$

The velocity four-vector is given as

$$\eta^\mu = \gamma(c, v_x, v_y, v_z) \ ; \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{123}$$

The energy-momentum four-vector is given as

$$p^\mu = \left(\frac{E}{c}, p_x, p_y, p_z\right) \ ; \qquad p_\mu p^\mu = \frac{E^2}{c^2} - \mathbf{p}^2 = m^2 c^2 \tag{124}$$

The gradient four-vector is given as

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \ \frac{\partial}{\partial x}, \ \frac{\partial}{\partial y}, \ \frac{\partial}{\partial z}\right) \tag{125}$$

7.2 The Dirac Equation

The Dirac equation is effectively the relativistic Schrodinger equation. The Dirac equation is consistent with the relativistic energy-momentum formula and first order in time. Wheras the Schrodinger equation is sufficient for spin-less particles, the Dirac equation is relevant for particles of spin $\frac{1}{2}$. Griffiths derives it from the Klein-Gorden equation in [Gr 213-216].

In the 'free' case (potential energy is zero), the Dirac equation is given as

$$i\hbar \ \gamma^\mu \ \partial_\mu \psi - mc\psi = 0 \tag{126}$$

where ψ is a four-element column matrix called a Dirac spinor and γ^μ are matrices whose anticommutator is $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.

$$\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ ; \qquad \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} \tag{127}$$

where $\sigma^i (i = 1, 2, 3)$ is the indicated Pauli matrix.

Note, the Hermitian conjugate of the Dirac spinor goes as.

$$\psi^\dagger = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix}^\dagger = \begin{bmatrix} \psi_1^* & \psi_2^* & \psi_3^* & \psi_4^* \end{bmatrix} \tag{128}$$

7.3 Solutions to the Dirac Equation

Griffiths derives the independent solutions of the Dirac equation in two cases. The full derivations are in [Gr 216-222]. I will discuss the results.

First, we suppose that ψ is independent of position.

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial z} = 0 \tag{129}$$

In this case, $\mathbf{p} = 0$. The solutions to the Dirac equation are

	spin up	spin down
electrons	$\psi^{(1)} = e^{-i(mc^2/\hbar)t}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$	$\psi^{(2)} = e^{-i(mc^2/\hbar)t}\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$
positrons	$\psi^{(3)} = e^{+i(mc^2/\hbar)t}\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$	$\psi^{(4)} = e^{+i(mc^2/\hbar)t}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$

$\psi^{(1)}$ and $\psi^{(2)}$ represent electrons with spin up and spin down, respectively. $\psi^{(3)}$ and $\psi^{(4)}$ represent positrons with spin up and spin down, respectively. The electrons have positive energy mc^2, while the positrons have 'negative energy' $-mc^2$. We will interpret these negative energy particles as antiparticles with positive energy.

Next, we will look for plane-wave solutions of the form

$$\psi(x) = ae^{-(i/\hbar)x\cdot p}u(p) \tag{130}$$

In this case, $\mathbf{p}$ is constant in time. The solutions to the Dirac equation are

electrons	$u^{(1)} = N\begin{bmatrix}1\\0\\\frac{c(p_z)}{E+mc^2}\\\frac{c(p_x+ip_y)}{E+mc^2}\end{bmatrix}$	$u^{(2)} = N\begin{bmatrix}0\\1\\\frac{cp_x-ip_y}{E+mc^2}\\\frac{c(-p_z)}{E+mc^2}\end{bmatrix}$
positrons	$v^{(1)} = N\begin{bmatrix}\frac{cp_x-ip_y}{E+mc^2}\\\frac{c(-p_z)}{E+mc^2}\\0\\1\end{bmatrix}$	$v^{(2)} = N\begin{bmatrix}\frac{c(p_z)}{E+mc^2}\\\frac{c(p_x+ip_y)}{E+mc^2}\\1\\0\end{bmatrix}$

$$N = \sqrt{(|E| + mc^2)/c}\,; \qquad E = \sqrt{m^2c^4 + \mathbf{p}^2c^2} \tag{131}$$

N is a normalization constant. We will use the convention that electron wave functions are labeled with u and positron wave functions are labeled with v. The energies for both the electron and positron is E.

Note, none of the plane wave functions are necessarily eigenstates of the spin matrices $\mathbf{S}$.

$$\mathbf{S} = \frac{\hbar}{2}\mathbf{\Sigma} \; ; \qquad \mathbf{\Sigma} = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} \tag{132}$$

Only when the $\hat{\mathbf{z}}$-axis points along the direction of motion (i.e. $p_x = p_y = 0$) are the above wave functions eigenspinors of S_z. In this case, $u^{(1)}$ and $v^{(1)}$ are spin up, while $u^{(2)}$ and $v^{(2)}$ are spin spin down.

7.4 Bilinear Covariants

We would like to know how the Dirac spinors transform from one inertial system to another. Along the way, we'll introduce $\bar{\psi}$ and γ^5 which will be useful later.

If we go to a system moving with speed v in the x direction,

$$\psi' = S\psi \; ; \qquad S = a_+ + a_-\gamma^0\gamma^1 \; ; \qquad a_\pm = \pm\sqrt{\frac{\gamma \pm 1}{2}} \tag{133}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$, as usual.

We can now check for invariant quantities. $(\psi^\dagger\psi)$ is not Lorentz invariant. However, if we introduce the adjoint spinor, $\bar{\psi}$,

$$\bar{\psi} = \psi^\dagger\gamma^0 \tag{134}$$

we find that the scalar quantity $(\bar{\psi}\psi)$ is Lorentz invariant. We'll also introduce the γ^5 operator

$$\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \; ; \qquad \{\gamma^\mu, \gamma^5\} = 0 \tag{135}$$

and see that the pseudoscalar quantity $(\bar{\psi}\ \gamma^5\ \psi)$ is also Lorentz invariant. Note, $(\bar{\psi}\ \gamma^5\ \psi)$ is a pseudoscalar because it is antisymmetric over the parity transformation $(x, y, z) \to (-x, -y, -z)$.

7.5 The Photon

Starting from Maxwell's equations, [Gr 225-227] derives an equation for the wave function of a free photon. In the process, he chooses a gauge which satisfies the Lorentz condition $\partial_\mu A^\mu = 0$. The final result is,

$$\partial^\nu\partial_\nu A^\mu = 0 \tag{136}$$

We look for plane-wave solutions with momentum $p = (E/c, \mathbf{p})$.

$$A^\mu(x) = ae^{-(i/\hbar)p\cdot x}\epsilon^\mu(p) \tag{137}$$

ϵ^μ is the polarization vector, which characterizes the spin of the photon. ϵ^μ has four components, but they are not independent.

Inserting this form into the photon equation and the Lorentz condition, we find constraints on E and ϵ^μ.

$$E = |\mathbf{p}|c \; ; \qquad p^\mu\epsilon_\mu = 0 \tag{138}$$

Moreover, in the Coulomb gauge we have $\epsilon^0 = 0$ and $\epsilon \cdot \mathbf{p} = 0$, which is to say that the polarization three-vector is perpendicular to the direction of propogation. If we say $\mathbf{p}$ points in the $\hat{\mathbf{z}}$ direction, we might choose

$$\epsilon_{(1)} = (1, 0, 0) \; ; \qquad \epsilon_{(2)} = (0, 1, 0) \tag{139}$$

7.6 The Feynman Rules for QED

Table 2 summarizes what we know so far about electrons, positrons, and photons.

	Electrons	**Positrons**	**Photons**
Wave function	$ae^{-(i/\hbar)p\cdot x}u^{(s)}(p)$	$ae^{(i/\hbar)p\cdot x}v^{(s)}(p)$	$ae^{(-(i/\hbar)p\cdot x}\epsilon^{\mu}_{(s)}$
Energy	$\sqrt{m^2c^4 + \mathbf{p}^2c^2}$	$\sqrt{m^2c^4 + \mathbf{p}^2c^2}$	$\lvert\mathbf{p}\rvert c$
Equation	$(\gamma^\mu p_\mu - mc)u = 0$	$(\gamma^\mu p_\mu + mc)u = 0$	$\epsilon^\mu p_\mu = 0$
Orthogonality	$\bar{u}^{(1)}u^{(2)} = 0$	$\bar{v}^{(1)}v^{(2)} = 0$	$\epsilon^{\mu *}_{(1)}\epsilon_{\mu(2)} = 0$
Normalization	$\bar{u}u = 2mc$	$\bar{v}v = -2mc$	$\epsilon^{\mu *}\epsilon_\mu = 1$
Completeness	$\sum_{s=1,2} u^{(s)}\bar{u}^{(s)} = (\gamma^\mu p_\mu + mc)$	$\sum_{s=1,2} v^{(s)}\bar{v}^{(s)} = (\gamma^\mu p_\mu - mc)$	$\sum_{s=1,2} (\epsilon_{(s)})_i(\epsilon^*_{(s)})_j = \delta_{ij} - \hat{p}_i\hat{p}_j$

Table 2: Summary of wave functions for electrons, positrons, and photons. For electrons and positrons, $(s = 1, 2)$. Note $\bar{u} = u^\dagger\gamma^0$

Feynman diagrams are all about writing the expression for the amplitude $\mathcal{M}$ in a picture. When evaluating a Feynman diagram, you will use the vertices and lines in a Feynman diagram to create an integral expression for $\mathcal{M}$.

Below are listed the steps in transforming a Feynman diagram to an integral expression.

1. Label the incoming/outgoing four-momenta as $p_1, p_2, \ldots$ and the corresponding spins as $s_1, s_2, \ldots$. Label the internal four-momenta $q_1, q_2, \ldots$. Internal fermion lines should be assigned a direction such that every vertex has one arrow in and one arrow out. The direction of photon lines are arbitrary.

2. Add factors from external lines. Figure 2 shows which factors to add. Note, electrons always point up, while positrons always point down.

3. Each vertex contributes a factor $ig_e\gamma^\mu$.

4. Each internal line contributes a factor as follows

$$\begin{aligned} \text{Electrons and positrons: } & \frac{i(\gamma^\mu q_\mu + mc)}{q^2 - m^2c^2} \\ \text{Photons: } & -\frac{ig_\mu\nu}{q^2} \end{aligned} \tag{140}$$

5. For each vertex, write a delta function of the form

$$(2\pi)^4\delta^4(k_1 + k_2 + k_3) \tag{141}$$

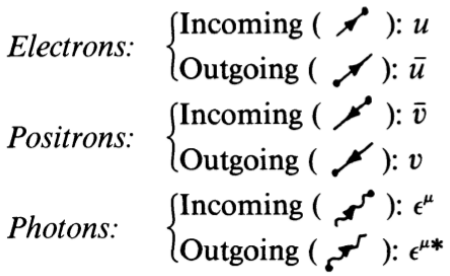

Figure 2: Factor contribution from external lines.

where the k's are the four-momenta p_μ coming in/out of the vertex. If the particle's arrow is coming into the vertex, $k = +p_\mu$. Otherwise, $k = -p_\mu$. Note, that the incoming/outgoing arrows are switched for external positrons.

6. For each internal momentum q, write a factor $\frac{d^4q}{(2\pi)^4}$ and integrate.
7. The result will include a delta function within an integral. Cancel this factor and what remains is $-i\mathcal{M}$.
8. Include a minus sign between diagrams that differ only in the interchange of two incoming (or outgoing) electrons (or positrons), or of an incoming electron with an outgoing positron(or vice versa).

Griffiths does several examples of applications of Feynman rules in [Gr 233-236]. He also introduces the following shorthand.

$$\not{a} = a^\mu \gamma_\mu \tag{142}$$

7.7 Casimir's Trick

We can apply a simplification when calculating $\left\langle |\mathcal{M}|^2 \right\rangle$, the average over initial spins and sum over final spins of $|\mathcal{M}(i \to f)|^2$.

$$\sum_{\text{all spins}} [\bar{u}(a)\ \Gamma_1\ u(b)]\ [\bar{u}(a)\ \Gamma_2\ u(b)]^* = \text{Tr}[\Gamma_1\ (\not{p}_b + m_b c)\ \bar{\Gamma}_2\ (\not{p}_a + m_a c)] \tag{143}$$

for any arbitrary Γ_1 and Γ_2. This is particularly useful because the second part of the above identity includes no Dirac spinors.

We can then use the following matrix identities to simplify our problem.

Trace definitions	Fundamental definitions
$\mathrm{Tr}[A+B] = \mathrm{Tr}[A] + \mathrm{Tr}[B]$	$g_{\mu\nu}g^{\mu\nu} = 4$
$\mathrm{Tr}[\alpha A] = \alpha \mathrm{Tr}[A]$	$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$
$\mathrm{Tr}[AB] = \mathrm{Tr}[BA]$	$\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\mu = 4g^{\nu\lambda}$

Contraction Theorems	
$\gamma_\mu\gamma^\mu = 4$	
$\gamma_\mu\gamma^\nu\gamma^\mu = -2\gamma^\nu$	$\gamma_\mu \not{a}\gamma^\mu = -2\not{a}$
$\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\mu = 4g^{\nu\lambda}$	$\gamma_\mu \not{a}\not{b}\gamma^\mu = 4a\cdot b$
$\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\sigma\gamma^\mu = -2\gamma^\sigma\gamma^\lambda\gamma^\nu$	$\gamma_\mu \not{a}\not{b}\not{c}\gamma^\mu = -2\not{c}\not{b}\not{a}$
Trace Theorems	
$\mathrm{Tr}(1) = 4$	
$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$	$\mathrm{Tr}(\not{a}\not{b}) = 4a\cdot b$
$\mathrm{Tr}(\gamma^\mu\gamma^n u\gamma^\lambda\gamma^\sigma) = 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda})$	$\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)]$
The trace of the product of an odd number of gamma matrices is zero.	
$\mathrm{Tr}(\gamma^5) = 0$	
$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0$	$\mathrm{Tr}(\gamma^5\not{a}\not{b}) = 0$
$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4i\epsilon^{\mu\nu\lambda\sigma}$	$\mathrm{Tr}(\gamma^5\not{a}\not{b}\not{c}\not{d}) = 4i\epsilon^{\mu\nu\lambda\sigma}a_\mu b_\nu c_\lambda d_\sigma$

where $\epsilon^{\mu\nu\lambda\sigma}$ is the Levi-Civita symbol.

7.8 Cross Sections and Lifetimes

The differential cross section is given by

$$\sigma(\theta,\phi) = \left(\frac{\hbar}{8\pi Mc}\right)^2 \left\langle |\mathcal{M}|^2 \right\rangle \tag{144}$$

8 Systems of Identical Particles

[CT 1371-1455]

For a system of identical particles, exchange degeneracy appears. A complete measurement on each of the particles does not permit the detmination of a unique ket of the state space of the system.

8.1 Two Particles

Consider a system composed of two particles with the same spin s. The Permutation operator is then defined as the linear operator whose action on the basis vectors is given by.

$$P_{21}\,|1:u_i;\ 2:u_j\rangle = |2:u_i;\ 1:u_j\rangle = |1:u_j;\ 2:u_i\rangle \tag{145}$$

The operator is its own inverse, Hermitian, and unitary.

$$(P_{21})^2 = 1\ ; \qquad P_{21}^\dagger = P_{21}\ ; \qquad P_{21}^\dagger P_{21} = 1 \tag{146}$$

The eigenvalues of P_{21} are ± 1 for the symmetric and antisymmetric eigenvectors, respectively.

We will define two new operators S and A, called the symmetrizer and antisymmetrizer.

$$S = \frac{1}{2}(1 + P_{21})\ ; \qquad A = \frac{1}{2}(1 - P_{21}) \tag{147}$$

These operators are projectors, Hermitian, orthogonal, and supplementary.

$$\begin{aligned} S^2 = S\ ; \qquad S^\dagger = S\ ; \qquad SA = AS = 0 \\ A^2 = A\ ; \qquad A^\dagger = A\ ; \qquad S + A = 1 \end{aligned} \tag{148}$$

They have the unique property that they project onto the eigenvectors of P_{21} for any $|\psi\rangle$.

$$P_{21}S\,|\psi\rangle = S\,|\psi\rangle\ \ ; \qquad P_{21}A\,|\psi\rangle = -A\,|\psi\rangle \tag{149}$$

Lastly, for a given observable in the state space of one of the particles (e.g. $B(1)$), we can transform it to the state space of the other particle via

$$P_{21}B(1)P_{21}^\dagger = B(2) \tag{150}$$

8.2 Arbitrary number of particles

In the state space of a system composed of N particles with the same spin, $N!$ permutation operators can be defined. For the $N = 3$ case, these act like

$$P_{npq}\,|1:u_i;\ 2:u_j;\ 3:u_k\rangle = |n:u_i;\ p:u_j;\ q:u_k\rangle \tag{151}$$

where $n, p, q \in \{1, 2, 3\}$. This set of permutation operators constitutes a group.

$$P_{123} = 1\ ; \qquad P_{312}P_{132} = P_{321}\ ; \qquad P_{312}^{-1} = P_{231}\ ; \tag{152}$$

Transposition is a permutation which simply exchanges the roles of two particles, without touching the others. Any permutation operator can be broken down into a product of transposition operators. However, this decomposition isn't unique.

Although, for the $N > 2$ case, the permutation operators are not all necessarily Hermitian, they are all unitary.

Whereas we cannot construct a basis for all permutation vectors for $N > 2$, we can do so for the symmetrizer and antisymmetrizer.

$$S = \frac{1}{N!}\sum_{\alpha} P_\alpha \ ; \qquad A = \frac{1}{N!}\sum_{\alpha} \epsilon_\alpha P_\alpha \ ; \qquad \epsilon_\alpha = \begin{cases} +1 & \text{if } P_\alpha \text{ is an even permutation} \\ -1 & \text{if } P_\alpha \text{ is an odd permutation} \end{cases} \tag{153}$$

As in the $N = 2$ case, S and A are projectors, Hermitian, and orthogonal. However, they are not supplementary. They project onto the state space of a completely symmetric or completely antisymmetric ket.

$$P_{\alpha_0} S\,|\psi\rangle = S\,|\psi\rangle \ ; \qquad P_{\alpha_0} A\,|\psi\rangle = \epsilon_{\alpha_0} A\,|\psi\rangle \tag{154}$$

8.3 The Symmetrization Postulate

The Symmetrization Postulate: When a system includes several identical particles, only certain kets of its state space can describe its physical states. Physical kets are, depending on the nature of the identical particles, either completely symmetric or completely antisymmetric with respect to permutation of these particles. Those particles for which the physical kets are symmetric are called bosons, and those for which they are antisymmetric, fermions.

The symmetrization postulate states that the kets that are allowed to describe a system must be symmetric (for bosons) or antisymmetric (for fermions) with respect to the exchange of two particles. We follow the empirical rule, which states that particles of half-integral spin are fermions and particles of integral spin are bosons.

Therefore, the physical kets used to describe a state $|u\rangle$ are $S\,|u\rangle$ or $A\,|u\rangle$, for bosons and fermions respectively. Usually, application fo S or A requires the wavefunction to be renormalized.

Given the definitions of S and A above, it is clear to see how these are applied to an arbitrary state $|u\rangle$. To account for the ϵ_α changes in the A formula, it is convenient to write $A\,|u\rangle$ in the form of a Slater determinant.

$$A\,|u\rangle = A\,|1:\phi;\ 2:\chi;\ 3:\omega\rangle = \frac{1}{3!}\begin{vmatrix} |1:\phi\rangle & |1:\chi\rangle & |1:\omega\rangle \\ |1:\phi\rangle & |1:\chi\rangle & |1:\omega\rangle \\ |1:\phi\rangle & |1:\chi\rangle & |1:\omega\rangle \end{vmatrix} \tag{155}$$

8.4 Implications of the Symmetrization Postulate

The Pauli exclusion principle arises from the symmetrization postulate. It states that two identical fermions cannot occupy the same quantum mechanical state. Otherwise, the exchange of the two particles would be a symmetric operation. This leads to many implications in quantum statistics.

Measuring the probability of a measurement requires you to first transform the state $|u\rangle$ to a physical ket before the operator is applied. This is because there may be interactions between the original and exchange states. For example, given a ket $|1:\phi;\ 2:\chi\rangle$,

$$\mathscr{P}(b_n;\ b_{n'}) = |(\langle 1:u_n;\ 2:u_{n'}|\,S)B(S\,|1:\phi;\ 2:\chi\rangle)| \tag{156}$$

8.5 Spectral notation

The spectral terms are given as ${}^{2S+1}L$ where S and L are the total spin and angular momenta of the particles in question. Note, the allowed spectral terms for a given set of particles are limited by the Pauli exlucsion principle and the symmetrization postulate.