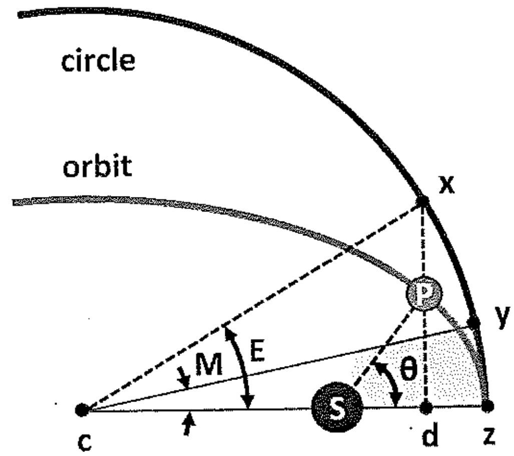
AAE 412 **SPACE SYSTEMS ENGINEERING**

Orbital Position as a Function of Time

The Kepler Problem:
Given $\mathbf{r}_0$ and $\mathbf{v}_0$, find the state vectors $\mathbf{r}$ and $\mathbf{v}$ at a time Δt later.

Mean and Eccentric Anomaly



Quantity	Circle	Ellipse	Parabola	Hyperbola
Defining Parameters	a = semimajor axis = radius	a = semimajor axis b = semiminor axis	p = semi-latus rectum q = perifocal distance	a = semi-transverse axis ($a < 0$) b = semi-conjugate axis
Parametric Equation	$x^2 + y^2 = a^2$	$x^2/a^2 + y^2/b^2 = 1$	$x^2 = 4qy$	$x^2/a^2 - y^2/b^2 = 1$
Eccentricity, e	$e = 0$	$e = \sqrt{a^2 - b^2}/a \quad 0 < e < 1$	$e = 1$	$e = \sqrt{a^2 - b^2}/a \quad e > 1$
Perifocal Distance, q	$q = a$	$q = a(1-e)$	$q = p/2$	$q = a(1-e)$
Velocity, V, at Distance, r, from Focus	$V^2 = \mu/r$	$V^2 = \mu(2/r - 1/a)$	$V^2 = 2\mu/r$	$V^2 = \mu(2/r - 1/a)$
Total Energy Per Unit Mass, ε	$\varepsilon = -\mu/2a < 0$	$\varepsilon = -\mu/2a < 0$	$\varepsilon = 0$	$\varepsilon = -\mu/2a > 0$
Mean Angular Motion, n	$n = \sqrt{\mu/a^3}$	$n = \sqrt{\mu/a^3}$	$n = 2\sqrt{\mu/p^3}$	$n = \sqrt{\mu/(-a)^3}$
Period, P	$P = 2\pi/n$	$P = 2\pi/n$	$P = \infty$	$P = \infty$
Anomaly	$\nu = M = E$	Eccentric anomaly, E $\tan\frac{\nu}{2} = \left(\frac{1+e}{1-e}\right)^{1/2}\tan\left(\frac{E}{2}\right)$	Parabolic anomaly, D $\tan\frac{\nu}{2} = D/\sqrt{2q}$	Hyperbolic anomaly, F $\tan\frac{\nu}{2} = \left(\frac{1+e}{1-e}\right)^{1/2}\tanh\left(\frac{F}{2}\right)$
Mean Anomaly, M	$M = M_0 + nt$	$M = E - e\sin E$	$M = qD + (D^3/6)$	$M = (e\sinh F) - F$
Distance from Focus, $r = q(1+e)/(1+e\cos\nu)$	$r = a$	$r = a(1 - e\cos E)$	$r = q + (D^2/2)$	$r = a(1 - e\cosh F)$
$r\,dr/dt \equiv r\dot{r}$	0	$r\dot{r} = e\sqrt{a\mu}\sin E$	$r\dot{r} = \sqrt{\mu}\,D$	$r\dot{r} = e\sqrt{(-a)\mu}\sinh F$
Areal Velocity, $\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\nu}{dt}$	$\frac{dA}{dt} = \frac{1}{2}\sqrt{a\mu}$	$\frac{dA}{dt} = \frac{1}{2}\sqrt{a\mu(1-e^2)}$	$\frac{dA}{dt} = \frac{1}{2}\sqrt{\frac{\mu q}{2}}$	$\frac{dA}{dt} = \frac{1}{2}\sqrt{a\mu(1-e^2)}$

Note: μ = GM is the gravitational constant of the central body; ν is the true anomaly, and $M \equiv n(t - T)$ is the mean anomaly, where t is the time of observation, T is the time of perifocal passage, and n is the mean angular motion. See App. C for additional formulas and a discussion and listing of terminology and notation.

The Kepler Problem: Universal Variables

$$\mathbf{r} = f\,\mathbf{r}_o + g\,\mathbf{v}_o$$

$$\mathbf{v} = \dot{f}\,\mathbf{r}_o + \dot{g}\,\mathbf{v}_o$$

The universal Kepler's equation:

$$\sqrt{\mu}\Delta t = \frac{r_0 v_{r_0}}{\sqrt{\mu}}\chi^2 C(\alpha\chi^2) + (1 - \alpha r_0)\chi^3 S(\alpha\chi^2) + r_0\chi$$

Where $\alpha = 1/a = 2/r_o - v_o^2/\mu$ and the universal Lagrange equations are:

$$f = 1 - \frac{\chi^2}{r_0} C(\alpha\chi^2)$$

$$g = \Delta t - \frac{1}{\sqrt{\mu}}\chi^3 S(\alpha\chi^2)$$

$$\dot{f} = \frac{\sqrt{\mu}}{r r_0}\left[\alpha\chi^3 S(\alpha\chi^2) - \chi\right]$$

$$\dot{g} = 1 - \frac{\chi^2}{r} C(\alpha\chi^2)$$

And the Stumpff equations are given by:

$$C(z) = \frac{\cosh\sqrt{-z} - 1}{-z} \quad (z < 0) \quad (z = \alpha\chi^2)$$

$$S(z) = \frac{\sinh\sqrt{-z} - \sqrt{-z}}{\left(\sqrt{-z}\right)^3} \quad (z < 0) \quad (z = \alpha\chi^2)$$