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TRIPARTITE STRUCTURAL ANALYSIS: GENERALIZING THE BREIGER–WILSON FORMALISM

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Tripartite graphs have three types of nodes and ties exist only between nodes of distinct types. In this paper, we indicate the background of work in structural analysis that leads to the use of such graphs and we develop the associated matrix methods. Fundamentally, the idea is to move another step forward in our ability to treat in analytical terms the general phenomenon of overlapping inclusions, such as that of persons and groups. We relate this idea to classic concerns with the interpenetration of persons, social systems, and cultural systems. In detail, the framework and results generalize those of Breiger (1974) and Wilson (1982).

1. Introduction and overview

One of the basic conceptual themes of both speculative philosophy and social thought since the nineteenth century involves the idea of *interpenetration*, in which erstwhile distinct entities share components. Deeply considered in the philosophical debates on internal relations earlier in this century, the idea ultimately was seen to be the constitutive conceptual core of such otherwise diverse systems of thought as that of Marx and that of Whitehead. For Marx, two erstwhile separate entities such as labor and capital are nothing but two sides of one relational process. For Whitehead, the term prehension signifies a relation-in-the-making in which one actual entity is constituted by its mode of inclusion of other actualities. For Marx and Whitehead, then, interpenetration is a mode of dynamic construction, the way in which entities are "bonded" in a common world.

In classic sociological thought, Durkheim, Mead, and Simmel each stated one version or another of an interpenetration idea. For Durkheim,

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the Kantian idea of general moral and cognitive categories provided the shared component among members of a collectivity, what we now call their common culture. For Mead, social structure is a structure of social action in which the whole social act—such as a double play in baseball—intrinsically constitutes a zone of interpenetration of the actions of the cooperating human organisms.

Simmel clearly articulated specific forms of interpenetration of individuals and groups. Here Breiger (1974) and others have made the significant scientific step of moving beyond Simmel's geometric metaphors concerning the overlap of groups and individuals to a formal mode of representation. So this branch of "structural analysis" (Berkowitz 1982) could be regarded as the beginnings of a formal theory of interpenetration. In this paper we consider a definite conceptual and formal task, which is part of the necessary descriptive basis for any such formal theory.

A compact statement of the task and of the contribution of this paper is as follows.

In a paper written in the 1960s, but only recently published, Wilson (1982) proposed that certain ambiguities in network representations could be resolved by using bipartite graphs, in which persons and groups are represented by distinct types of nodes and any tie is restricted to a pair of nodes of distinct types. This led Wilson to a set of matrix equations for path computations. Breiger (1974) started with the Simmelian theme of "duality", an aspect of the interpenetration of persons and groups, and arrived at the identical matrix equations. In this paper, we generalize both the conceptual basis of these developments and the corresponding graph/matrix formalisms. Conceptually, we pass to interpenetration involving three levels of analysis and state this idea in terms of what we call "the shared subparts criterion of structure". Formally, Wilson's bipartite graph becomes a tripartite graph and the Breiger-Wilson matrix identities are imbedded in a more general set of matrix equations and operations to obtain information about complex path-types. Throughout we appeal to empirical interpretations in terms of institutional-level phenomena, *i.e.* to networks of institutional roles and collectivities and to social and cultural systems including them. We conclude with an array of suggested directions of application and with a discussion of how the interpenetration or shared subparts criterion of structure relates to two other modes of structural analysis, involving generative models and biased net models.

2. The Breiger-Wilson formalism

Wilson (1982) suggests a "relational network" basis for structural analysis. In his terms, a relational network is a graph with two types of nodes: individuals and groups. Lines are constrained to exist only between the two node types. Such a graph, he notes, is termed *bipartite*. The interpretation of a line, or tie, is membership. Wilson develops some of the matrix algebra for such a network and provides some conjectures about the worthwhile directions of further work. Here, stimulated by Wilson's bipartite representation, our aim is to pursue this line of investigation and then generalize it to a "tripartite" representation.

Our starting point involves noticing that Breiger (1974) had set up the identical system: Breiger refers to the bipartite graph as the membership net. Using Breiger's notation, with Wilson's in parenthesis, we have:

(1) a $p \times g$ matrix of memberships of p persons in g groups: A (R);
(2) a $p \times p$ matrix of ties among persons, the number of shared group memberships between every pair: $P = AA^T$ ($X = RR^*$);
(3) a $g \times g$ matrix of ties among groups, the number of shared members between every pair: $G = A^TA$ ($Y = R^*R$).

Moreover, Breiger (1974: 186) demonstrates that the matrices of n-paths are related as follows (after converting P and G to binary matrices):

$$P^m = A(G^{m-1})A^T \tag{1a}$$

$$G^m = A^T(P^{m-1})A. \tag{1b}$$

A path in the bipartite representation involves alternating "squares" and "circles" (in Wilson's notation), as shown in Fig. 1, which illustrates the intuitive basis for Breiger's two derived identities. [1]

Note that we can regard two square nodes as connected by a line involving a circle (□ → ○ → □), which can be interpreted as persons in the structural role of connectors and groups in the structural role of

[1] $P = AA^T$; $P^2 = AA^TAA^T = AGA^T$; $P^3 = AA^T(AGA^T) = AG^2A^T$.

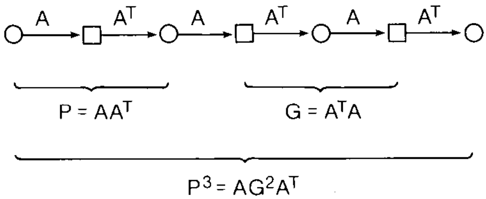


Fig. 1. A visualization of the algebra associated with a bipartite network.

actors. In what Breiger calls the dual point of view, we can regard two circular nodes as connected by a line involving a square (○ → □ → ○), which places persons in the structural role of actors and groups in the role of connectors. The first type of linkage, oriented from left to right, is the relation [2] (or matrix) $A^TA = G$ and the second, again oriented from left to right, is the relation or matrix $AA^T = P$.

Conceptually, Wilson wants to overcome a limitation of dyadic representations of relational systems: by including both groups and persons in a single membership net we can formally distinguish a genuine three-person group from a set of three two-person groups, as in

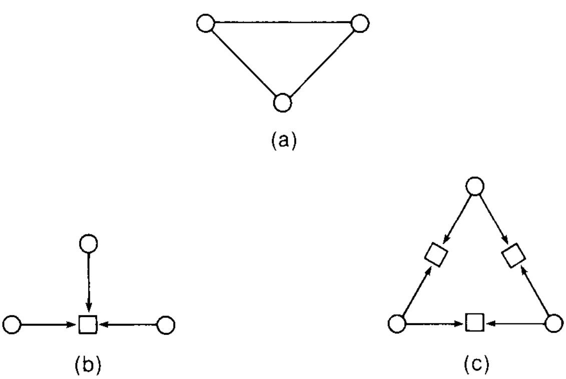


Fig. 2. A conventional dyadic representation of a triad (a) and two distinct empirical interpretations formally represented in Wilson's bipartite form: (b) represents one 3-person group, (c) represents three 2-person groups.

[2] In purely relational terms, the inverse of a relation R is most often symbolized by R^{-1}, with the analytic consequence that the adjacency matrix of R^{-1} is the transpose of the adjacency matrix of R (see, for instance, Fararo (1973: 145)). We symbolize the inverse by transpose notation to provide a better intuitive use of the relation/graph/matrix correspondence.

Fig. 2. For this example,

$$A_b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} A_c = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Hence,

$$P_b = A_b A_b^T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad G_b = A_b^T A_b = (1),$$

$$P_c = A_c A_c^T = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \qquad G_c = A_c A_c^T = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Therefore, the second empirical situation (A_c) generates a very different pattern of derived interpersonal and intergroup matrices. In particular, the interchangeability of square nodes and circular nodes in Fig. 2c is reflected in the derived identity $P_c = G_c$.

The Bates-Harvey model and its bipartite representation

A second linkage to Wilson's bipartite representation is made with the conceptual model of the structure of social systems suggested by Bates and Harvey (1975). Simplifying their scheme somewhat, we represent a portion of it in terms of a set of nodes termed *positions* and distinguish two types of ties: *reflexive*, linking two positions of the same person, and *bilateral*, linking complementary roles of positions occupied by distinct persons. A group, in their model, is a complete system of interlocking positions of persons so that every position is role-related to every other.

In terms of a graph, we can represent reflexive links as dotted lines, bilateral as solid lines. Figure 3 gives an example of a representation of two groups joined by an interstitial group containing representatives from each group. Nodes are *not* individuals but, to use the expression of Parsons – see Parsons and Shils (1951: 247) – "sectors" of individual participation, each sector a role, status, or position. Bates and Harvey discriminate the latter concepts, but for our purpose, at present, the nodes are members-in-positions and the lines are single or multiple

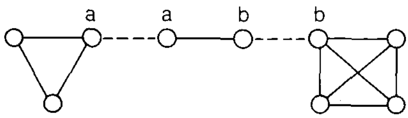


Fig. 3. An example of a Bates-Harvey social system representation: two groups connected by a third interstitial group. Note that persons are represented by multiple nodes connected by dotted lines.

status-role relationships between members-in-positions. Hence, a node is a member-in-a-group. Two nodes with the same person-label, for instance "a" in Fig. 3, are connected by a reflexive tie. A group is a complete subgraph of such nodes, not of persons as such.

Since in each Bates-Harvey group, by definition, every node is connected directly to every other, we can transform this representation into a Breiger-Wilson bipartite network, as shown in Fig. 4.

In the Breiger-Wilson formalism, persons are again single nodes rather than the connected (via dotted lines) chains of nodes of Fig. 3. But groups are also nodes. All the membership information of Fig. 3 is encoded in Fig. 4, given that the Bates-Harvey "group" is a complete (density = 1) set of role relationships. Whereas the lines in Fig. 3 represent complementary role relations, those in Fig. 4 represent *inclusion* in a group defined by a system of such role relations.

3. Restricted tripartite representations

The Bates-Harvey model suggests a further application of the Breiger-Wilson formalism in which we will introduce a third type of node, suggesting the name "tripartite" for the graph. To motivate this development, we start by noting that Bates and Harvey define an organization in such a way as to include the necessary condition that it is a system of connected groups. This connectedness means, simply, every

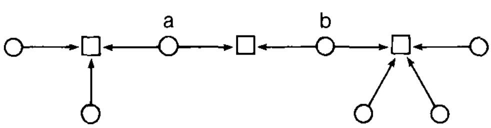


Fig. 4. A bipartite representation of the system of Fig. 3.

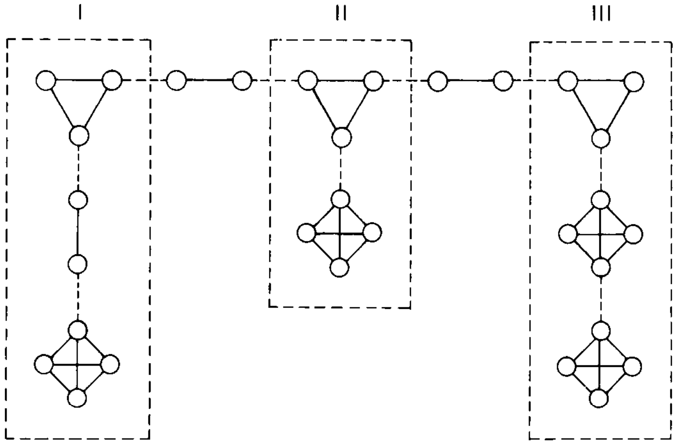


Fig. 5. A Bates-Harvey social system representation: three organizations connected by two interstitial groups.

group is *reachable* (directly or indirectly) from every other. Hence, in its network aspect, a Bates-Harvey organization is a bipartite graph with two types of nodes, persons and groups, such that every pair of groups has a nonzero entry in G or G^2 or G^3 or ... or G^{g-1} wher g is the number of groups. [3] The connectedness of the system of groups forming an organization implies the connectedness of the system of persons. Consider any two persons, a and b. If a belongs to a group 1 and b belongs to a group 2, and if the groups are connected by some path of length k, then relation A connects a to 1, relation G^k connects 1 to 2, and relation A^T connects b to 2. That is, there is a nonzero entry in matrix AG^kA^T (*which is* P^{k+1}).

Suppose we now consider a system of organizations. For simplicity of example, Fig. 5 shows three organizations using a Bates-Harvey representation to make clear the interstitial linkages. For instance, if organizations I and III are small firms, organization II might be a bank with one interstitial group between I and II and another between II and III with each comprised of persons-in-positions of negotiations of loan/repayment terms. These negotiation groups are transients but other such groups (*e.g.*, Chamber of Commerce) are not. Recall that in the Bates-Harvey model, nodes are not persons but persons-in-posi-

[3] Under Boolean arithmetic, $\Sigma_{i=1}^{g-1} G^i$ is the unit matrix, containing a 1 in each entry.

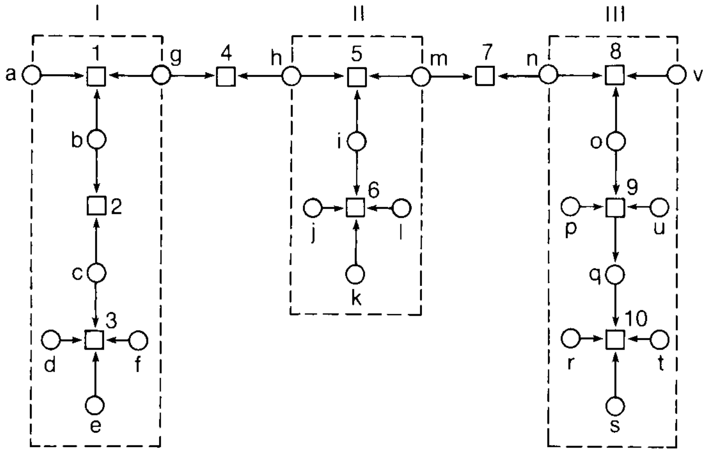


Fig. 6. A first bipartite representation of the system of Fig. 5.

tions, so that for example, one of the bank's loan officers may appear as one node in a system of cooperative relations in the bank (as a social system) and as another node in the connecting "exchange" social system in which he negotiates with firm I's representative. Then this loan officer functions as a boundary node in the Breiger-Wilson bipartite net of Fig. 6, which encodes the total social system of Fig. 5. The interstitial groups appear "in the space between" organizations.

Abstracting from the internal structures of the organizations, this system is representable in a second way as a higher level bipartite graph with groups and organizations as the two types of nodes. Figure 7 shows this second representation. We see immediately that we have lost a vital aspect of the interorganizational network: the connecting role of

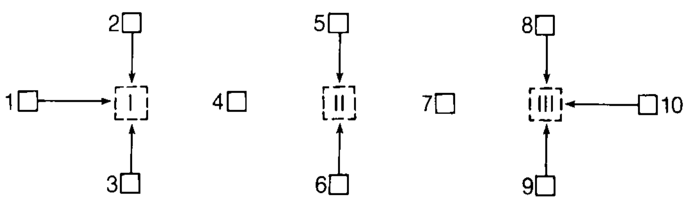


Fig. 7. A second bipartite representation of the system of Fig. 5, with dotted square nodes representing organizations and solid square nodes representing groups.

the interstitial groups 4 and 7, which belong to none of the three organizations.

To correct this deficiency we must simultaneously represent *both* levels: the person-group "duality" *and* and group-organization "duality". In terms of graphs, the structure looks more complex but we will develop the matrix formalism to make analyses straightforward. Figure 8 show this *tripartite representation*. There are three types of nodes with two inclusion relations: persons are included in groups and groups are included in organizations. Lines are constrained to represent these inclusions.

In matrix terms, the second level of "membership" can be represented by the *inclusion* matrix, denoted B. It is a group-by-organization matrix. The derived intergroup ties, via common organizational ties, are given by,

$$Q = BB^T.$$

The derived interorganizational ties, via groups, are given by,

$$H = B^TB.$$

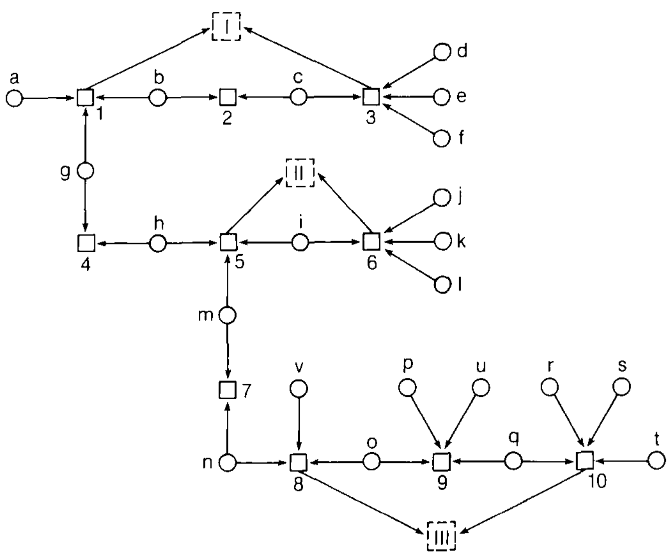


Fig. 8. The tripartite representation of the system of Fig. 5. Square nodes are interpreted as in Fig. 7, circular nodes are persons.

In the example, there are ten groups and three organizations, with B given by:

$$B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{I} & \mathrm{II} & \mathrm{III} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}.$$

Hence,

$$Q = \begin{array}{c|cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}$$

$$H = \begin{array}{c|ccc} & \mathrm{I} & \mathrm{II} & \mathrm{III} \\ \hline \mathrm{I} & 3 & 0 & 0 \\ \mathrm{II} & 0 & 2 & 0 \\ \mathrm{III} & 0 & 0 & 3 \end{array}.$$

Note that H is a diagonal matrix which gives the size of each organization in terms of the number of included groups. The fact that H is diagonal arises from the special case that each group is included in at most one organization. This formal condition on H is not a necessary feature of tripartite networks so B is completely analogous to matrix A at the lower level, Q is analogous to P, and H to G. As in the case of person-by-groups, Breiger's identities apply to this next level:

$$Q^m = B(H^{m-1})B^T, \tag{2a}$$

$$H^m = B^T(Q^{m-1})B. \tag{2b}$$

The combined matrix representation of what we will call the restricted tripartite network (in anticipation of further generalization in the next section) is of the form:

$$\begin{array}{ll} & \quad\text{Groups} \quad \text{Organizations} \\ \begin{array}{l}\text{Persons} \\ \text{Groups}\end{array} & \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right). \end{array}$$

Calling this matrix M, we have:

$$MM^T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} A^T & 0 \\ 0 & B^T \end{pmatrix} = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}, \tag{3a}$$

$$M^TM = \begin{pmatrix} A^T & 0 \\ 0 & B^T \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} G & 0 \\ 0 & H \end{pmatrix}. \tag{3b}$$

We note the proper representation of the interstitial bond between organizations in the tripartite formalism. Group 4, for instance, is tied to organization II via the path type A^TAB: we start with a link from group 4 to person h (A^T), this is followed by a link from group 5 (A), which is followed by a link from group 5 to organization II (B). Similarly, group 4 is reached *from* organization I via a path of the type B^TA^TA. The total path, through group 4, I to II is therefore $B^TA^TAA^TAB = B^TG^2B$, the interstitial bond between them (see Eq. (1a) for $m = 3$). If node 4 is deleted, then nodes I and II are no longer mutually reachable.

4. The general tripartite network

The matrix M and the tripartite graph associated with it suggest a genuine generalization of the Breiger-Wilson formalism based on two formal axioms:

A1. There are three types of nodes.
A2. Ties exist only between nodes of different types.

While matrix M above realizes A1 and A2 it does not show the possibility that direct ties of inclusion may exist between the lower level entities and the highest level entities. But consider the following empirical interpretation of A1 and A2: there are persons who are playing social roles via positions in collectivities and there are cultural domains to which both persons and collectivities are role-defined contributors. For instance, scholars and artists sometimes belong to universities and institutes which are institutionally defined as having their 'mission" in one or more cultural domains (science, art, etc.) but individuals also are included in, or contribute directly to, the cultural domain via their "cultural" roles. As noted, the ties among the three are all "inclusions", so we have one example of a fully generalized tripartite inclusion net. The general matrix form, which we shall call M, is given by:

$$M = \begin{array}{ll} & \begin{array}{cc} \begin{array}{c}\text{Social}\\ \text{systems}\end{array} & \begin{array}{c}\text{Cultural}\\ \text{systems}\end{array} \end{array} \\ \begin{array}{l}\text{Persons}\\ \text{Social systems}\end{array} & \left(\begin{array}{c|c} A & C \\ \hline 0 & B \end{array}\right) \end{array}. \tag{4}$$

As an example, consider six persons with possible social roles in terms of membership on university or institute faculties and cultural roles "musician", "artist", "scientist". The latter describe "contribu-

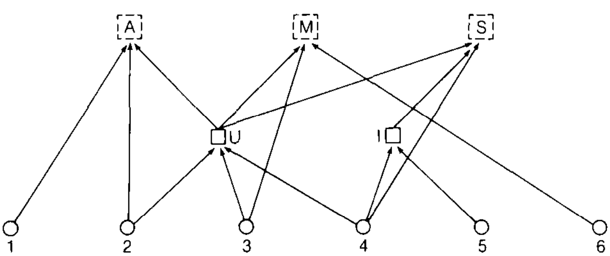


Fig. 9. An example of a generalized tripartite network, in which circular nodes are collectivities, and dotted square nodes are cultural systems.

tory inclusion" in cultural systems (music, art, science) which, in turn, are institutionalized in universities and institutes.

Concretely, consider Fig. 9. The matrix of this example is:

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ U \\ I \end{array} \begin{array}{c} \begin{array}{ccccc} U & I & A & M & S \end{array} \\ \left[\begin{array}{cc|ccc} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} .$$

The interpretation should be clear: the university (U) is institutionally part of science, music, and art, while the research institute (I) is only part of science. Some people may belong to neither organization although they have cultural roles (node 1 is an "independent" artist) while others, although belonging to such an organization, are without cultural roles (node 5 is "staff" person).

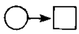
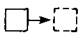
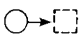
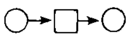
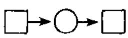
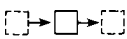
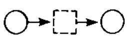
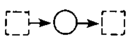
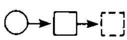
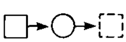
When $C = 0$, we recover the restricted tripartite representation in which we envision three levels with the nodes at the bottom level not part of the top level nodes but rather parts of the parts of such nodes, *i.e.*, in a subpart relation to the top level nodes. When in addition each middle-level part is included in at most one top-level node, we obtain the special case treated in the prior section, in which persons are included in groups, each of which belongs to at most one organization. In the next section, we turn to computational considerations, returning to these conceptual matters in Section 6.

5. Paths in tripartite networks

The basic types of ties in the tripartite representation are shown in Table 1, along with the general personal-social-cultural interpretation.

A path in a tripartite network will involve a ramified system of interlocking types of nodes and ties. Our present aim is to derive the

Table 1
Types of ties in tripartite networks

Form	Matrix	Illustrative Interpretation
	A	Persons belong to collectivities: membership net.
	B	Collectivities are institutional parts of cultural systems.
	C	Persons have cultural roles, they are parts of cultural systems.
	$P=AA^T$	Interpersonal ties, via common group memberships.
	$G=A^TA$	Intergroup ties, via persons belonging to each.
	$Q=BB^T$	Intergroup ties, via common cultural system inclusions.
	$H=B^TB$	Intercultural system ties, via groups which are parts of each.
	CC^T	Interpersonal ties, via common cultural system inclusions.
	C^TC	Intercultural system ties, via persons who are part of each.
	AB	Person to cultural system ties, via membership in institutional part of cultural system.
	BC^T	Group to person ties, via institutional part of cultural system including person.
	A^TC	Group to cultural system ties, via inclusion of persons who are part of cultural system.

general forms of matrices useful in the computation of the number of paths of a given kind. We shall utilize an abstract terminology for the three types of nodes, calling them by "level" names: level 1, level 2, level 3, which are circular nodes, solid-square nodes, and dotted-square nodes, respectively.

5.1. Paths in restricted tripartite networks

Restricted tripartite networks, defined by the condition $C=0$, imply that direct ties exist only from level 1 to level 2 (matrix A) from level 2 to level 3 (matrix B).

Level 2 to level 2 ties are induced via A in the form $G = A^TA$ and via B in the form $Q = BB^T$. Call these the lower and upper connectors of level 2 nodes, respectively. We seek "connector matrices", products of G and Q that count paths between groups. Clearly, as in Fig. 1, any such path has an abstract algebraic representation as a word in G and Q with all paths with m G's and n Q's found by the corresponding matrix products. If all such paths are treated as equivalent (in terms of m, n), then the representative form of the path may be denoted G^mQ^n. Such a path contains $m + n - 1$ level-2 nodes, not counting the endpoints. [4]

Level 3 to level 3 indirect ties are necessarily mediated through level 2: via B^T we go to level 2, trace a path to some node at level 2, then return to level 3 via B. Hence, equivalence classes of paths are represented by $B^T(G^mQ^n)B$.

Level 1 to level 1 indirect ties are necessarily mediated through level 2: via A we go to level 2, trace a path at this level, then return to level 1 via A^T. Hence, equivalence classes of paths are represented by $A(G^mQ^n)A^T$.

In a similar way, the forms of nonhomogeneous paths indirectly linking a level i node to a level j node ($i \neq j$) are derived. These are, for the various i, j ($i \neq j$):

$1 \to 2$: $A(G^mQ^n)$, $2 \to 1$: $(G^mQ^n)A^T$,
$1 \to 3$: $A(G^mQ^n)B$, $3 \to 1$: $B^T(G^mQ^n)A^T$,
$2 \to 3$: $(G^mQ^n)B$, $3 \to 2$: $B^T(G^mQ^n)$.

Suppose now we want to know if a path of a given form exists between a particular node at level i and a particular node at level j. We have to examine matrices in the equivalence classes to see if at least one such matrix has a nonzero entry at the appropriate row-column position. There are $\binom{m+n}{m}$ matrix products in the class represented by G^mQ^n, corresponding to which particular m of the $m + n$ links are the G type, with the remainder Q type. We simply put the matrices in some order, then sequentially test each matrix for a nonzero entry, stopping as soon as such an entry arises. Generalizing and therefore considering all possible pairs of nodes, we define the matrix M_{mn} via the typical

[4] For graphs with uniform nodes, Harary *et. al.* (1965: 142) describe a complex recursive algorithm to eliminate such redundancies. Parallel algorithms for bipartite or tripartite graphs probably can be developed but we leave such a complicated procedure for later work.

term in row k and column l:

$$(k, l) \text{ term of } M_{mn} = \begin{cases} 1 \text{ if any entry at } (k, l) \text{ is nonzero} \\ \text{in any matrix in class } G^m Q^n, \\ 0 \text{ otherwise.} \end{cases}$$

In this way we arrive at Table 2, giving the matrix products that represent the existence or not of a path with m G-links and n Q-links between nodes at various pairs of levels.

Computationally, then, we obtain $G = A^T A$ and $Q = BB^T$, the lower and upper connectors of groups (*e.g.*, via member persons and via organizational inclusion, respectively). Then for any m, n of interest, we compute M_{mn} by investigating the concrete matrix products in the class represented by $G^m Q^n$. Then we use Table 2 to compute the final products to determine, for every pair of nodes, whether or not they are indirectly tied by a path involving m lower connector ties and n upper connector ties. If we wish to know if they are tied in a path with *at least* m and/or *at least* n such connector links then we consider sequences of M_{uv}, $u = 1, 2, \ldots, m$, $v = 1, 2, \ldots, n$, and binary addition over the appropriate members of the sequence. For example, if we want to know, for any pair of groups (level 2), if they are linked by some path with no more than two persons (level 1) and no more than two organizations (level 3), we could compute every M_{mn} for $m = 1, 2$ and $n = 1, 2$ and form the binary sum:

$$S_{22} = M_{10} + M_{20} + M_{11} + M_{21} + M_{22} + M_{01} + M_{02} + M_{12}.$$

Table 2
Matrices for determining existence of paths in a restricted tripartite network (with m G-links and n Q-links)

		Terminal Node of Path		
		Level 1	Level 2	Level 3
Starting node of path	Level 1	$AM_{mn}A^T$	AM_{mn}	$AM_{mn}B$
	Level 2	$M_{mn}A^T$	M_{mn}	$M_{mn}B$
	Level 3	$B^TM_{mn}A^T$	B^TM_{mn}	$B^TM_{mn}B$

If the k, l entry of S_{22} is 0 then if any path exists linking node k (group k) and node l (group l) it involves a longer chain of intermediaries. In general, S_{mn}^{ij} gives the existence (1) or not (0) of paths starting from nodes at level i and terminating at level j nodes such that the path has no more than m G-links and no more than n Q-links.

We note the earlier result that interstitial bonds between organizations are of the form B^TG^2B is a special case of the level 3 to level 3 entry in Table 2, with $m = 2$, $n = 0$, since once we set $n = 0$ the canonical path $G^2Q^0 = G^2$ is the only member of the equivalence class and so $M_{20} = G^2$: the path involves two group-to-group ties with persons as the lower connectors, three groups in all with the middle group in the structurally interstitial position.

5.2. Paths in general tripartite networks

In general tripartite networks, direct ties can exist between level 1 and level 3. This complicates the tracing of paths as, in essence, words involving C and/or C^T have to be constructed. Moreover, words involving AB and B^TA^T have to be constructed also. The problem can be approached in the following recursive fashion. [5] Let Z be the matrix of paths of length 1 (*i.e.*, direct links):

$$Z = \begin{pmatrix} 0 & A & C \\ A^T & 0 & B \\ C^T & B^T & 0 \end{pmatrix}.$$

[5] At face value, matrix M given in (4) above could be used to obtain MM^T and M^TM. However, MM^T and M^TM do not contain AB or B^TA^T, which trace two-step paths between levels 1 and 3. In the restricted case, an AB can only be followed by a B^TA^T. Then $ABB^TA^T = AQA^T$. Thus the matrix operations capture (indirect) ties between levels 1 and 3 by use of Q and G. However, in the general case, AB can be followed by C^T and the word ABC^T has to be generated in the matrix products.

Paths of length 2 are obtained [6] from Z^2:

$$Z^2 = \begin{pmatrix} 0 & A & C \\ A^T & 0 & B \\ C^T & B^T & 0 \end{pmatrix} \begin{pmatrix} 0 & A & C \\ A^T & 0 & B \\ C^T & B^T & 0 \end{pmatrix}$$

$$= \begin{pmatrix} AA^T + CC^T & CB^T & AB \\ BC^T & A^TA + BB^T & A^TC \\ B^TA^T & C^TA & C^TC + B^TB \end{pmatrix}. \tag{5}$$

Each product term in each entry in Z^2 represents a possible 2-step path, as indicated in Table 1. We have already discussed AA^T, A^TA, BB^T, and B^TB. Now the other products come into play: CC^T traces 2-step paths from level 1 to level 1 (and as a matrix gives, for pairs of persons, the numbers of shared cultural domains); C^TC traces 2-step paths from level 3 to level 3 (and as a matrix gives, for pairs of cultural domains, the numbers of persons involved in both); AB traces 2-step paths from individuals to cultural domains while B^TA^T traces the reverse; CB^T traces 2-step paths from persons to collectivities via cultural domains while BC^T traces the reverse; and A^TC traces 2-step paths between collectivities and cultural domains via persons while C^TA traces the reverse 2-step paths.

In general, paths of length $k+1$ are given by

$$Z^{k+1} = Z^kZ \ (k \geq 1), \tag{6}$$

which recursively generates paths for all $k \geq 1$.

5.3. The restricted case in terms of the powers of the Z matrix

We now note a relationship between the M_{mn} terms of Table 2 of the restricted case and the matrices Z^k of the general case. First, the M_{mn} matrices refer to paths of even length, since each occurrence of a G or a Q involves a path of length 2. Second, the matrix terms of the series of

[6] The use of Z introduces some irrelevant entries for the restricted case: B^TA^T and AB are featured in (5) but see little use. Each only appears when followed by the other as in $ABB^TA = AQA^T$ which has already been captured in the preceding section.

even powers of Z contain in the middle entry the terms $(G+Q)^k$. To see this, compute Z^2, as in Eq. (5), and regard it as a sum of two matrices, one of which involves only the diagonal, which is given by:

$$\begin{pmatrix} AA^T & 0 & 0 \\ 0 & A^TA + BB^T & 0 \\ 0 & 0 & B^TB \end{pmatrix} = \begin{pmatrix} P & 0 & 0 \\ 0 & G+Q & 0 \\ 0 & 0 & H \end{pmatrix}.$$

Similarly, the diagonal of Z^4 can be seen to be of the form:

$$\begin{pmatrix} P^2 + \dots & 0 & 0 \\ 0 & (G+Q)^2 & 0 \\ 0 & 0 & H^2 + \dots \end{pmatrix}.$$

Hence, in general,

$$Z^{2k}_{(2,2)} = (G+Q)^k \qquad (k \geq 1), \tag{7}$$

where the (2, 2) subscript means the middle-position matrix of Z^{2k}, which is a 3×3 supermatrix.

Third, in terms of matrix algebra and of the idea of equivalence classes denoted G^mQ^n, these $(G+Q)^k$ terms relate to the matrices M_{mn} as follows in binary arithmetic:

$$G + Q = M_{10} + M_{01}$$

$$(G+Q)^2 = G^2 + GQ + QG + Q^2 = G^2 + GQ + Q^2 = M_{20} + M_{11} + M_{02}$$

and so forth, yielding:

$$(G+Q)^k = \sum_{m+n=k} M_{mn}. \tag{8}$$

From (7) and (8),

$$Z^{2k}_{(2,2)} = \sum_{m+n=k} M_{mn}. \tag{9}$$

Hence, the correspondence between the series of even powers of Z and the M_{mn} connector matrices is given by (9): the mid-matrix of the former gives the sums over all possible combinations of n and m, the number of lower and upper connectors, whose sum is half the power of the supermatrix Z^{2k}. Of course, if the terms of the expansion of $(G+Q)^k$ are computed separately we also obtain each individual M_{mn} and so also we obtain the S_{mn} from these.

6. A classification of tripartite network representations

From the foregoing analyses, it is clear there are several tripartite representations. By way of a partial summary, and as a basis for further development, various instances of tripartite representations are given in Table 3.

The inclusion relation has been defined in such a way that the ordered pairs contain elements from different levels.

The restricted form with each row of B having at most one nonzero entry is illustrated by the Bates-Harvey structural concepts discussed earlier: persons are parts of groups and groups belong to at most one organization. One could represent this in terms of the general model with inclusion transitive by setting $C = AB$: if a person is part of a group and that group is part of an organization, then the person belongs to the organization. (This is not, of course, a causal sequence.) Alternatively, retaining $C = 0$, we regard the person as only a subpart rather than a *part* of the organization. Or, put another way, persons have no autonomous roles in organizations (as defined in this model), only roles in, and through, included groups. [7]

There is a need to consider two instances of the general inclusion relation as indicated in the two special cases of the general tripartite representation (Table 3): inclusions may be transitive or nontransitive. When $C \neq 0$, a given tripartite representation may or may not have $AB = C$. If it does, this may be due to the meanings of the inclusion relations alone or by contingent matter of fact.

[7] As Breiger has pointed out in a personal communication to us, Simmel's discussion of autonomy in the context of three levels is relevant here. For instance, persons (level 1) belong to families (level 2) which belong to societies (level 3). Relation AB then gives societal inclusion as mediated by family membership. Relation C represents the autonomous membership of persons in societies. The condition $C = AB$ means, for instance, that all persons under consideration simultaneously belong to society and to families. Simmel's discussion of three levels and the idea of autonomy may be found in an essay he wrote in 1908, "Group Expansion and the Development of Individuality" (see Levine (1971: Ch. 18)).

To illustrate, consider the interpretation of the levels as persons (1), collectivities (2), and cultural systems (3). As treated earlier in our discussion of Fig. 9, in this interpretation $C \neq 0$, since persons are parts of science, art, music, and so forth, in cultural roles. AB is the induced relation from persons to cultural systems (see Table 1), via membership in an institutional (collective) part of a cultural system. Clearly $AB \neq C$ for our example in Fig. 9: for instance, person 2 is part of the university U, which is part of the cultural systems of music and of science, but person 2 plays no cultural role in either science or music. Nor is this unique to our example in Fig. 9. This is a contingent but general matter of fact because there might have been a world in which the relation was transitive due to lawlike constraints on the formation of sociocultural systems. In fact, the constraints go in the other direction (*e.g.*, need for staff with growth in system size). In the person-social-cultural system interpretation, then, $AB \neq C$, the nontransitivity of inclusion, is an empirical fact about sociological systems, as mapped in tripartite form. It represents one aspect of the partial autonomy of actors from their institutional roles in collectivities.

From a generalizing point of view, there are still more general forms ("chromatic graphs") in which we would have k types of nodes and transitive or nontransitive inclusions from level to level: this is a topic for further formal and empirical work. [8]

7. The shared subparts conception of structure

In one important interpretation, the matrix A represents the inclusion relation for individuals in groups, B the inclusion relation of groups in organizations, and C the inclusion relation for individuals in organizations. Lower level units belong to higher level units and the former can be viewed as parts of the latter. But the pairs of higher level units also share some of the lower units (*i.e.*, those mapped to both) and our use of the terms whole, part, and subpart should reflect this. In A^TA the terms have a definite meaning: the main diagonal gives the sizes of the

[8] In personal communication, Wilson suggests that with four types of nodes one could formally distinguish "accidental" (or strictly interpersonal) relations that happen to involve people in two organizations of Fig. 6 and institutionalized relations between the two. The key step involves using, say, triangles for persons and circles for positions. This seems to us a promising line of further development of this type of structural analysis.

groups (in terms of membership) and the off-diagonal elements give the sizes of overlaps for all pairs of groups. The Breiger-Wilson formalism represents the structure of a complex system (whole) as a sharing of member persons (subparts) by groups, the parts. That is, $G = A^T A$ represents the structure of the system. [9] This is captured by the special bipartite case ($B = 0$, $C = 0$) of the tripartite representation, but it is only one of a family of formal representations with this general character of having higher level units share sets of lower level units.

As a *conceptual* generalization of the Breiger-Wilson formalism, in a direction different from but not antithetical to the duality focus of Breiger, we state what we will call *the shared subparts criterion* of structure: the concrete structure of a concrete whole is the sharing of subparts by the parts of the whole. [10] We now explore this conceptual meaning of the tripartite representation.

The sociocultural interpretation of the tripartite formalism (in which level 1 has persons, level 2 has social systems and level 3 has cultural systems) is straightforward: M is in the general intransitive form of Table 3. $M^T M$ is read with row and column labels "groups" and "cultural systems" and is given by:

$$M^T M = \begin{pmatrix} A^T & 0 \\ C^T & B^T \end{pmatrix} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} A^T A & A^T C \\ C^T A & C^T C + B^T B \end{pmatrix}$$

$$= \begin{pmatrix} G & A^T C \\ C^T A & C^T C + H \end{pmatrix}$$

(where Eq. (3b) is a special case with $C = 0$). In terms of the shared subparts criterion the matrix entries of supermatrix $M^T M$ are:

[9] This is the same conception of structure found in Q-analysis (Atkin 1977).

[10] This concept of structure has an antecedent in the philosophical literature. It was explicitly formulated by two philosophers (Feibleman and Friend 1945) in a conceptual analysis of organization. They argue that *in any domain* at all (and they illustrate from chemistry and biology, for instance), the concrete structure of a whole is "the sharing of subparts between parts". For example, in a molecule (whole) the atoms (parts) share electrons (subparts). We do not want to go all the way with these philosophers and assert that any other aspect of the whole is not structure but something else (for this would end up being a semantic quarrel) but only use their formulation to indicate the *sort* of structure tripartite or bipartite networks represent.

G:	pairs of group parts of the whole share subparts, which are persons;
A^TC and C^TA:	pairs of cultural and group parts of the whole share subparts, which are persons;
$C^TC + B^TB$:	pairs of cultural parts of the whole share subparts, where subparts are either persons or groups.

As noted above in terms of the bipartite analysis, the matrix $G = A^TA$ represents the structure defined in terms of shared subparts. The tripartite analysis is a generalization of this. Necessarily, A^TA is contained within M^TM. While AA^T represents the dual structure, MM^T represents its generalization. Thus the tripartite representation is both a formal and conceptual generalization of the bipartite representation of Breiger and of Wilson.

Next we treat the restricted case with the Bates-Harvey interpretation: level 3 consists of organizations. Let us call the whole being analyzed a "social community". Initially, we think of the structure of this community as given by the sharing of groups by organizations. However since in the Bates-Harvey interpretation *no* such groups are shared there would be no structure (*i.e.*, H is diagonal). This seemingly absurd result arises from neglect of interstitial bonds: some groups are not parts of organizations but they *are* parts of the community and organizations are linked, not by sharing *them* but by sharing *members* of them. Hence, we define the *parts* of a community whole by the rule: any organization plus any groups not included in some organization. These latter groups will be called *community groups*. Interstitial groups

Table 3
Various tripartite representations

Restricted form

$$M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

Special cases:

(i) Each row of B has at most one non-zero entry.

(ii) $B = 0$: the bipartite form.

General form

$$M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

Special cases:

(i) $AB = C$: inclusion is transitive.

(ii) $AB \neq C$: inclusion is nontransitive form a subset of these. Another example of a community group is a family.

Recall that the tripartite representation of the Bates-Harvey model is of the restricted form in Table 3. Recall also we could alternatively represent this model by the general form with transitivity. We now construct a particular version of the general form based on a partition of the groups and a pair of transitivity-type conditions. Namely, we represent the community systems as follows, where community groups are denoted "*c*-groups" and intra-organizational groups by "*o*-groups":

$$\begin{array}{l} \\ \text{persons} \\ c\text{-groups} \\ o\text{-groups} \end{array} \begin{array}{c} \begin{array}{ccc} c\text{-groups} & o\text{-groups} & \text{organizations} \end{array} \\ \left[\begin{array}{cc|c} A_1 & A_2 & C \\ \hline 0 & 0 & B_c \\ 0 & 0 & B \end{array}\right] \end{array}.$$

And the transitivity constraints introduced to reflect the fact that the only *direct* ties are between persons and groups and between *o*-groups and organizations are:

(i) $C = A_2 B$;
(ii) $B_c = JB$ where $J = A_1^T A_2$.

Matrix C represents the two-paths from persons to *o*-groups to organizations. Matrix J represents the two-paths from *c*-groups to persons to *o*-groups. It connects other parts of the community to parts of organizations, via the link of persons belonging to both. Matrix B_c represents the three-paths starting from *c*-groups then via J going to *o*-groups then via B to organizations. Hence B_c represents the interconnection between the two types of parts of the community.

Next we reduce this matrix to a form in which the only columns retained are parts of the community, *i.e.*, we eliminate the columns of *o*-groups. Hence to analyze the structure of a community system of organizations and *c*-groups we use the *modified* tripartite matrix:

$$M_* = \begin{array}{l} \\ \text{persons} \\ c\text{-groups} \\ o\text{-groups} \end{array} \begin{array}{c} \begin{array}{cc} c\text{-groups} & \text{organizations} \end{array} \\ \left[\begin{array}{cc} A_1 & A_2 B \\ 0 & JB \\ 0 & B \end{array}\right] \end{array}$$

Forming $M_*^T M_*$ we obtain the matrix representation of the structure of the community under the shared subparts criterion:

$$M_*^T M_* = \begin{pmatrix} G_1 & JB \\ B^T J & B^T\left(G_2 + J^T J\right)B + I \end{pmatrix},$$

where $G_1 = A_1^T A_1$ and $G_2 = A_2^T A_2$. (As *o*-groups are uniquely located in organizations, $B^T B = I$.)

In this matrix, the substructures represented are:

(1) G_1: the parts are community groups; pairs of them share member persons;

(2) JB: the parts are community groups and organizations; pairs of them share member persons; (the organizational sharing is mediated by the inclusion of the person in one or more intraorganizational groups);

(3) $B^T J^T$: this is the same as (2), seen from the inverse relation perspective;

(4) $B^T(G_2 + J^T J)B + I$: the parts are organizations, and what pairs of them share is best seen in terms of the paths in Fig. 10.

In this section we have treated tripartite network representation as having a generalized conceptual significance: starting from one or another matrix M of Table 3, and with perhaps necessary modification

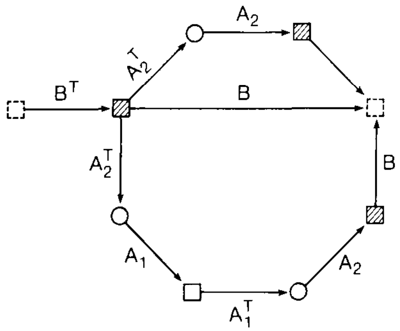


Fig. 10. The three modes of connection of two organizations, in which $B^T B = I$ in this instance (see text). Shaded squares are *o*-groups unshaded are *c*-groups.

to create an appropriate matrix (M_*), one forms M^TM (or $M_*^TM_*$) to reflect the conception that *the structure of a concrete whole is given by the sharing of subparts by the parts of the whole.*

8. Discussion of applications

Tripartite analysis is suggestive of a variety of applications and possibly interesting forms of structural analysis tied to related work in the networks area. We will highlight briefly some of these directions for further work:

(1) *Structure, process, function.* So far our analysis and discussion has been structural without involving process and function. A matrix resembling M in abstract form can deal with this dynamic inclusion:

$$N = \begin{array}{l} \\ \text{Structural entities} \\ \text{Subprocesses} \end{array} \begin{array}{c} \begin{array}{cc} \text{Subprocesses} & \text{Processes} \end{array} \\ \begin{pmatrix} F & 0 \\ 0 & D \end{pmatrix} \end{array}.$$

Here we have a tripartite network. The interpretation is that the nodes are social structural entities (level 1) subprocesses of certain institutional processes (level 2) and the main institutional macro-processes themselves (level 3). For instance:

structural entities:	persons and collectivities
subprocesses:	election, legislation, consumption,...
processes:	economic, political,...

Using our previous ideas, we have:

$$N^TN = \begin{pmatrix} F^TF & 0 \\ 0 & D^TD \end{pmatrix}.$$

F^TF gives the functional connection of subprocesses via their common inclusion of certain structural parts, actors. D^TD gives the functional connection of processes by their common inclusion of certain lower-level processes. For instance, corporations will be involved in both the subprocesses of election and production, giving rise (in F^TF) to a connection between these processes. Budgeting will be a subprocess common to educational, political, and varied other processes, giving rise (in $D^T D$) to a connection between them.

Note that in N the structural entities are not formally distinguished, nor is their structure in any sense given in $N^T N$. This is what M does, or rather $M^T M$. One problem for future work is to try to interrelate the M and N modes of tripartite analysis.

Another relationship of this mode of representation to process involves systems of differential or difference equations forming a dynamic system model. The shared subparts criterion now becomes what is usually called *coupling, i.e.*, the same variable appears in two equations in the mechanisms described by the right-hand-side. If we let $\dot{x}_i$ be the rate variable corresponding to state variable x_i, the system is:

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2, \ldots, x_n) \\
\dot{x}_2 &= f_2(x_1, x_2, \ldots, x_n) \\
\vdots\; & \quad \vdots \\
\dot{x}_n &= f_n(x_1, x_2, \ldots, x_n).
\end{aligned}$$

For instance, in the system

$$\dot{x}_1 = a_1 x_1 + b_1 x_2 + c_1$$

$$\dot{x}_2 = a_2 x_1,$$

the two parts share variable x_1. On the other hand, in the system,

$$\dot{x}_1 = a_1 x_1$$

$$\dot{x}_2 = a_2 x_2,$$

there is no sharing of subparts, no coupling: the system can be partitioned into independent processes.

We see that such a dynamic system has a structure represented by the bipartite special case, with the system as the whole, the mechanisms (represented by the right-hand sides of the equations) as the parts and the variables (elements of the state of the system) as the subparts. To generalize this to the tripartite form, consider blocks of equations separated for conceptual and empirical reasons. These are subsystems.

A verbal example is given by Homans (1950) with external and internal subsystems. In the dynamic system representation, we have:

$$\dot{E} = F_1(E, I),$$

$$\dot{I} = F_2(E, I),$$

where the state variables are (with superscripts for external and internal conceptual location):

$$E = \left(x_1^{(e)}, x_2^{(e)}, \dots, x_m^{(e)}\right),$$

$$I = \left(x_1^{(i)}, x_2^{(i)}, \dots, x_n^{(i)}\right).$$

The Homans whole is the social system and for tripartite representation level 3 nodes are the subsystems, level 2 nodes are the mechanisms (equations), and level 1 nodes are the variables. The subsystems must be coupled – to properly represent the idea that the internal subsystem emerges out of the external system and has a feedback effect on it. Hence, the generalized M matrix applies in the form:

$$\begin{array}{l c} & \begin{array}{ll} \text{Equations} & \text{Subsystems} \end{array} \\ \begin{array}{l} \text{Variables} \\ \text{Equations} \end{array} & \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \end{array}$$

where $C = AB$ if whenever a variable is in a subsystem it is in a equation that belongs to that subsystem.

(2) *Q-analysis*. The tripartite representation has a natural congruence with aspects of Q-analysis. Adapting our earlier notation, let us think of A as a relation between two sets, X and Y, such that $A \subseteq X \times Y$ (the Cartesian product of X and Y) with A^T denoting the inverse (as noted in footnote 2). For each element y in Y, there is a set of elements of X mapped to y by A. Each such set is a simplex and the set of these simplexes, together with their faces, form a simplicial complex. Similarly, for each element x of X, there is a set of elements of Y mapped to x by A^T. The set of these simplexes, together with their faces, forms the conjugate simplicial complex. AA^T and A^TA are the matrices expressing

the two simplicial complexes. [11]. The connectivity of these structures is delineated in a Q-analysis. Freeman (1980) provides an example where X represents a set of individuals in a network and Y a set of cliques defined over the network. Doreian (1982) provides an example where X is a set of women and Y a set of events attended by subsets of these women. From the present point of view, each provides an example of the special bipartite ($B = 0$, $C = 0$) representation, [12] and each deals with the sharing of subparts.

The sets defining simplexes can be embedded in a hierarchy of sets. Persons, groups, and organizations provides such an example. Nothing classified at one level can be classified at another. The sets at one level of the hierarchy form cover sets for the sets at the next lowest level (and only for the next level). Considering only adjacent levels, the configuration analyzed in Q-analysis is the restricted tripartite representation (with $C = 0$).

The point at which the tripartite representation departs from Q-analysis is in its full form when $C \neq 0$. In Q-analysis, in most if not all application, no relation is defined between units at non-adjacent levels. Thus, representing the direct inclusion of, say, a person in a cultural domain is not possible in Q-analysis. As we have provided a reasonable example of such an inclusion, the full tripartite representation would appear more fruitful in this instance. But, in general, the investigation of the relative merits of the two approaches is an empirical matter.

(3) *Interlock analyses.* With data matrices defined for individual directors belonging on corporate boards, the bipartite form is again relevant. If A represents the matrix for this inclusion, or membership, then A^TA and its dual AA^T maintain their meanings: A^TA gives a matrix

[11] Strictly, Q-analysis works with $A^TA - U_1$ and $AA^T - U_2$ where the U_i represent appropriately dimensioned unit matrices. The difference need not detain us here. In whole-part-subpart language, note that $P = AA^T$ can be interpreted as taking persons as parts and groups as subparts: the system as a whole has persons as its parts and its concrete structure resides in the sharing of groups by persons. This is an example of the "duality" emphasized by Breiger (1974), who cites Simmel as a source of insights on person-group interpenetration. See also footnote 13.

[12] It is possible for X to be the same as Y. For a directed graph, the bipartite form of the representation can be maintained by keeping a distinction of nodes as transmitters and nodes as receivers. An example is given by Gould and Gatrell (1980) in an analysis of a soccer game where players were distinguished by receiving passes or making them. There are other examples where the relation is symmetric, or the relation an undirected graph, where the distinction is not made (Doreian 1981).

defined for corporate boards whose entries are the number of the shared directors, while AA^T is a matrix defined over the individual directors giving the extent to which they share board memberships.

Roy (1983) presents an analysis of the interlocking of industries via shared directorships of corporations, basing his approach upon earlier work by Bonacich (1972). If A represents the inclusion of directors in corporations and B the inclusion of corporations in industries, we have a tripartite system of the restricted form in Table 3. However, given the definition of A and B, we can define C as AB, the transitive special case of the general tripartite representation. Since Roy begins with what we call C, part of Roy's methodology is the "empirical" construction of C from A and B corresponding to $C = AB$. The interlocking structure of industries is obtained from $C^TC = (AB)^T(AB) = B^TA^TAB = B^TGB$, using our earlier notation. Of course, we have also $G = A^TA$ and $H = B^TB$ as overlap matrices. If corporations are located in single industries, then H is diagonal, but the tripartite representation is not dependent on this. Moreover, non-diagonal H provides a way of generalizing Roy's work to apply to "conglomerate conditions", in which any one corporation may be included in multiple industries.

(4) *Other topics for tripartite analysis.* Journals are a central institution of science. By publishing therein scholars both help their own reputations and those of the journals, to the extent that the work is favorably evaluated in relevant groups. There are consensual, but distributed, [13] standards for evaluating journals and the authors who publish in them. Clearly, authors and journals are central to the development and maintenance of fields. Scholars, journals, and fields can be approached via a tripartite representation as levels 1 ,2, and 3 respectively. Let A represent the matrix of scholars publishing in journals. Journals belong to fields, although some journals can belong to more than one field. Similarly, scholars are identified with fields, again with the likelihood of some scholars belonging to more than one field. The journals' inclusion in fields can be represented by B, and the scholars' inclusion in fields by C. That is, the tripartite representation is applicable. As some journals do not belong to a single field, B^TB will not be diagonal. Further, the inclusions need not be transitive, so that, at a given point in time, at least, $AB \neq C$. Paralleling our analysis in

[13] See Burt and Doreian (1982) for analysis of the social psychological processes leading to a distribution of judgments of the relative standards of journals.

Section 7, in which groups were partitioned, journals can be partitioned into those that are disciplinary and those that are interdisciplinary. The former are well embedded, even defined, as central to a clearly defined discipline. The latter bridge disciplines. Given this analysis into subparts, the modified matrix M_* would be the starting point for structural analysis under the shared subparts criterion. [14]

Running through this example is the assumption that the structure of disciplines is represented, importantly, by A^TA, B^TB, C^TC and their duals. However, there can be many realizations of A, B, and C. Another example is provided by the structure of fields within universities. Let the units be scholars (level 1), departments (level 2) and ideas (level 3). Scholars belong to departments (A), have a commitment to some ideas (C), and the departments are collective loci for ideas (B). In a rigidly constructed university, ideas would be so specialized by departments that B^TB would be diagonal. If the environment supports interdisciplinary activity, then B^TB will not be diagonal. The condition $C \neq AB$ means scholars have autonomy, can develop and pursue ideas outside the bounds of their disciplines, as represented by the departments. While only a truly barren institution would be represented by the bipartite case ($B = 0 = C$), institutions would vary in their productivity, to the extent that the structure facilitated cross-fertilization, or to the extent that intellectual resources can be focused on clearly defined problems. These are potential topics for empirical study in which tripartite representations may prove fruitful.

9. Concluding remarks

We close with a few remarks on the conceptual relationship of this "shared subparts" type of structural analysis to two other types of structural analysis. The first type involves a mode of generative analysis. The fundamental formal entity is *the production rule* in the form $s \rightarrow a$, in which s is a type of social situation and a is a type of action. In Fararo (1981a), it was shown that if we start with the analysis of concrete institutionalized social action – such as the interactions of bus

[14] Yet another avenue of inquiry opens up if we break fields into subfields which are shared by journals and share scholars. Heavy activity in the overlaps may generate new fields, or specialties, with new institutional representations in journals.

driver and passenger; pupils and homeroom teacher; or judge, defendant, prosecutor and defense attorney – then we are led to *systems of production rules* to generate the infinite set of normal forms of their situated actions and interactions. Roughly speaking, coupled production systems function as a generative grammar for an institution (in the pattern rather than collectivity sense). A "position" takes the form of a set of production systems, one per role relationship. The primitive starting point of social action systems becomes situationally conditional action ($s \rightarrow a$). When this is done, the nodes in Fig. 3 have a decomposition into a structurally relevant *content*: a system of systems of productions, which when activated generates "action episodes" of an "institutional language". This is a whole network within each node, a system of automata (Skvoretz and Fararo 1980). This internal structure is not a social structure, it is an action structure. Hence action structure is the content of social structure, at least in the institutional sense of interest to us in this paper. (For a very extensive discussion, see Fararo and Skvoretz (1984).)

What this means when we translate into tripartite representation (*e.g.*, from Fig. 5 to Fig. 8) is that the nodes are standing-in for a whole system of such positions: corresponding to all the group-type social systems in which a person participates. Hence the "action content" of a node in tripartite analysis is a system of systems of production systems – a system (person in social aspect) of systems (positions of person in groups) of production systems (parts of role relationships of person in any one position).

Conceptually, the "structure" idea in this generative approach is distinct from the shared-subparts structure not only in that it deals with the action content but also that, formally, it is intrinsically dynamic: more like a system of dynamic equations (mechanisms) than a set of objects in overlap relations. Although conceptually distinct, these two modes of representation meet in the two ways. First, both are involved in the Bates-Harvey type of conceptual model of social systems. [15] And they can be even more firmly linked by a complex hierarchical network model in which the primitive entity, the *single* production, is made the

[15] The concept of "interpenetration" in Parsonian action theory, said to be its core concept (Munch 1981) is probably best interpreted as the formal idea of the sharing of subparts by parts of a system. For instance, for a general action system, the personalities and social systems are parts, and roles are the common subparts of pairs of personalities and social systems, as in Parsons *et. al.* (1951).

node and systems upon systems are constructed in the form either identical with or analogous to the nested graphs [16] suggested by Harary and Batell (1981). Second, the shared subparts criterion of structure applies to institutional action systems represented in terms of production systems. The key point is to think of the total production system as a dynamic system with blocks, subsystems. The production rules are then the mechanisms, analogous to each equation in a system of differential equations. Finally, the culturally defined types of entities – cultural, social, or physical – are the ingredients of the situational side of the mechanisms, analogous to state variables. Conceptually, then, the two structural representations – generative and shared-subparts – fit together nicely.

The second type of structural analysis contrasts with the first in its macrosociological focus and its statistical character, arising from aggregations and counting processes of various sorts. It is the biased network model of a system of societal collectivities (such as women, blacks, professionals, and any combination of such attribute-defined entities) in patterns of intergroup associations and follows closely the Blau (1977) macrosociological framework. In this framework, social structure is the distribution of persons over a whole complex of such intertwined collectivities and strata (Fararo 1981b). The network represents the associational aspect. The concept of "consolidation of parameters", as shown by Skvoretz (1983), is the key idea here. Such consolidation means the pattern of overlap of persons in "joint groups" (such as black professionals, women of high income, and the like) is not that expected under statistical independence. Hence, the shared subparts criterion is again reflected: society is the whole, the macro-level collectivities and strata are the parts, and persons are the subparts. However, this analysis goes beyond the algebraic descriptive approach in producing theoretical models that try to account for the patterning of associations in terms of an underlying mechanism – usually the simple but powerful axiom that people "seek their own kind", in a complex application that involves multiple bias parameters and consolidation (correlation) coefficients.

To sum up, when we use the generative model we are "microanalyzing" the institutional content of the sort of relations involved in

[16] The elaboration of hierarchy along the lines of generalization of the tripartite network to a general chromatic graph, mentioned earlier, will also permit representation of the bureaucratic hierarchical ordering of the parts of organizations, neglected in the illustrative examples above.

tripartite networks n terms of processes of action and when we use the probabilistic model we are "macroanalyzing" the patterning of the overlapping ties in terms of departures from a random baseline. Both of these types of analyses have the property of yielding predictions about structure whereas tripartite analysis is but a mode of descriptive analysis. [17] Yet, at the same time, the conceptual idea of interpenetration involved in the shared subparts criterion is a powerful one, at the core of much modern sociological theory.

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[17] They fit into different cells of a table of types of models recently suggested by Rapoport (1983: 19).

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