Pension returns analysis

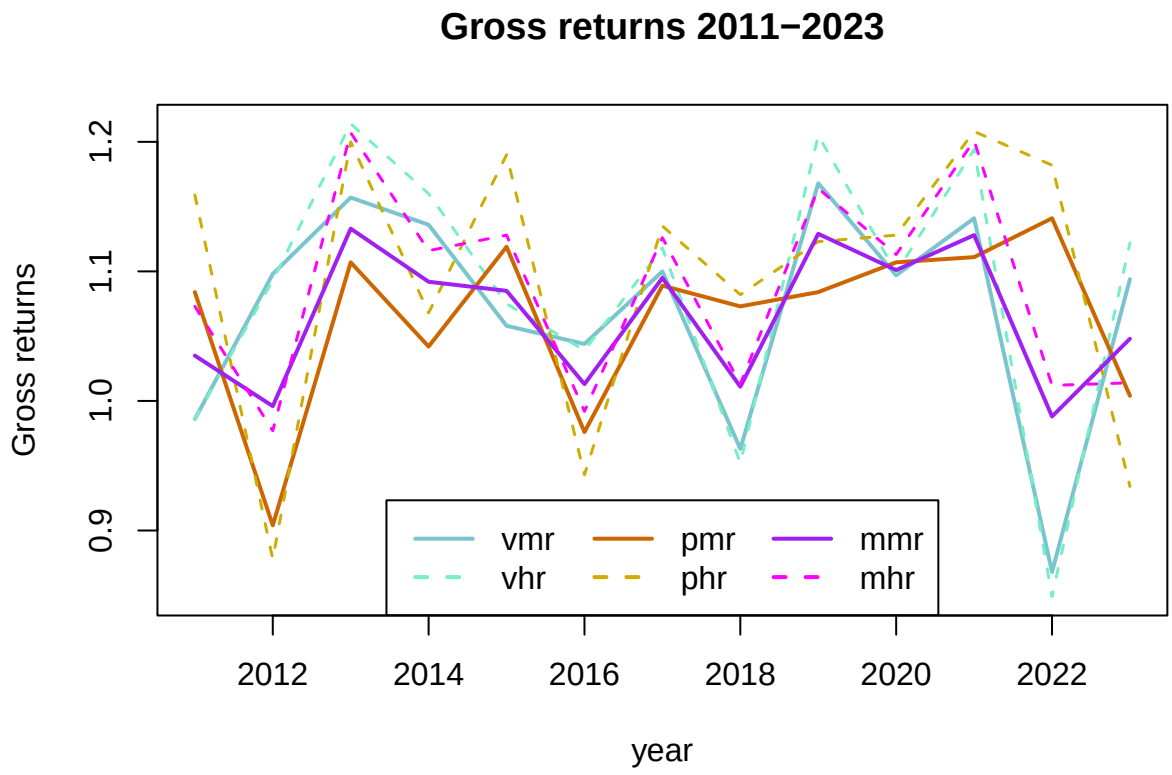
16:51 04 April 2024

Contents

Fit log returns to F-S skew standardized Student-t distribution.
`m` is the location parameter.
`s` is the scale parameter.
`nu` is the estimated degrees of freedom.
`xi` is the estimated shape parameter.

Log returns data 2011-2023.

For 2011, medium risk data is used in the high risk data set, as no high risk fund data is available prior to 2012.
`vmrl` is a long version of Velliv medium risk data, from 2007 to 2023. For 2007 to 2011 (both included) no high risk data is available.



Summary of gross returns

```
##       vmr              pmr              mmr              vhr        
##  Min.   :0.868   Min.   :0.904   Min.   :0.988   Min.   :0.849  
##  1st Qu.:1.044   1st Qu.:1.042   1st Qu.:1.013   1st Qu.:1.039  
##  Median :1.097   Median :1.084   Median :1.085   Median :1.099  
##  Mean   :1.070   Mean   :1.065   Mean   :1.066   Mean   :1.085  
##  3rd Qu.:1.136   3rd Qu.:1.107   3rd Qu.:1.101   3rd Qu.:1.160  
##  Max.   :1.168   Max.   :1.141   Max.   :1.133   Max.   :1.214  
##       phr              mhr        
```

```
##  Min.   :0.878   Min.   :0.977
##  1st Qu.:1.068   1st Qu.:1.013
##  Median :1.128   Median :1.113
##  Mean   :1.095   Mean   :1.087
##  3rd Qu.:1.182   3rd Qu.:1.128
##  Max.   :1.208   Max.   :1.207
```

```
##       vmrl
##  Min.   :0.801
##  1st Qu.:1.013
##  Median :1.085
##  Mean   :1.061
##  3rd Qu.:1.128
##  Max.   :1.193
```

```
## Highest minimum log-return: mmr
```

```
## Highest median log-return: phr
```

```
## Highest mean log-return: phr
```

```
## Highest max log-return: vhr
```

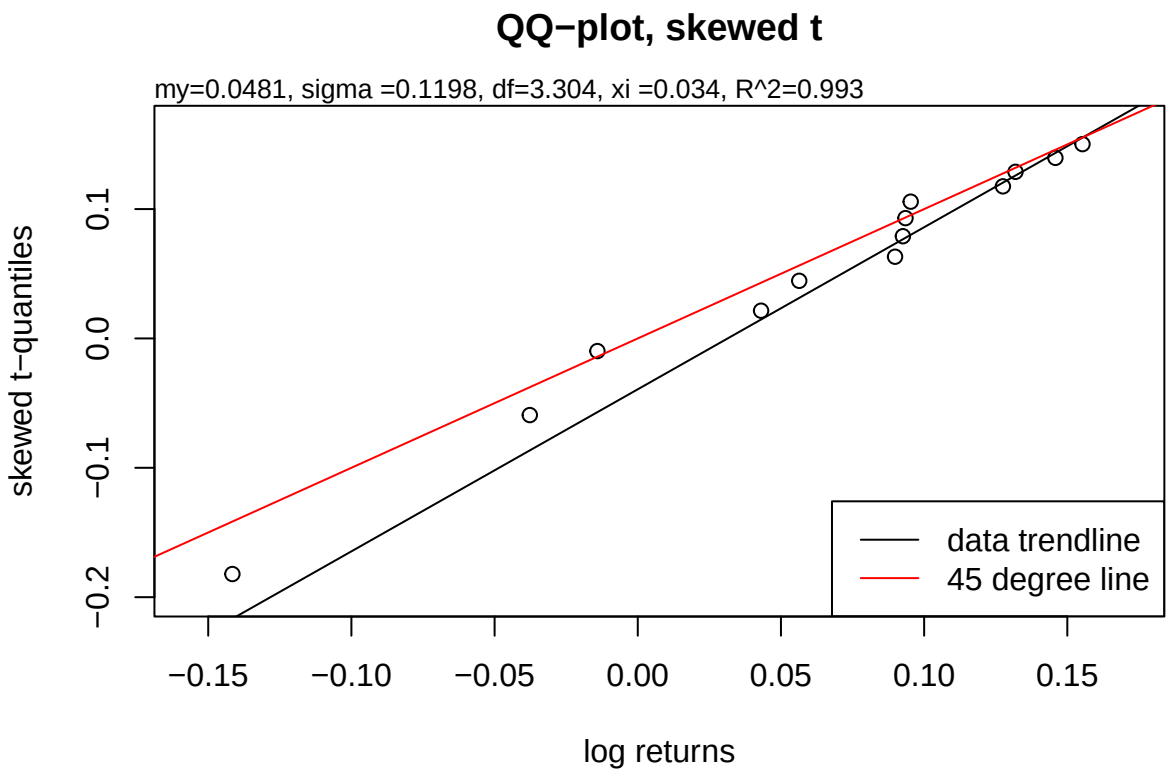
```
## cov(vmr, pmr) =  -0.001094875
```

```
## cov(vhr, phr) =  -0.0001730651
```

Velliv medium risk, 2011 - 2023

```
##
## AIC: -27.8497
## BIC: -25.58991
## m: 0.0480931
## s: 0.1198426
## nu (df): 3.303595
## xi: 0.03361192
## R^2: 0.993
##
## An R^2 of 0.993 suggests that the fit is extremely good.
##
## What is the risk of losing max 10 %? =< 7.4 percent
## What is the risk of losing max 25 %? =< 1.8 percent
## What is the risk of losing max 50 %? =< 0.2 percent
## What is the risk of losing max 90 %? =< 0 percent
## What is the risk of losing max 99 %? =< 0 percent
##
## What is the chance of gaining min 10 %? >= 41 percent
## What is the chance of gaining min 25 %? >= 0 percent
## What is the chance of gaining min 50 %? >= 0 percent
## What is the chance of gaining min 90 %? >= 0 percent
## What is the chance of gaining min 99 %? >= 0 percent
```

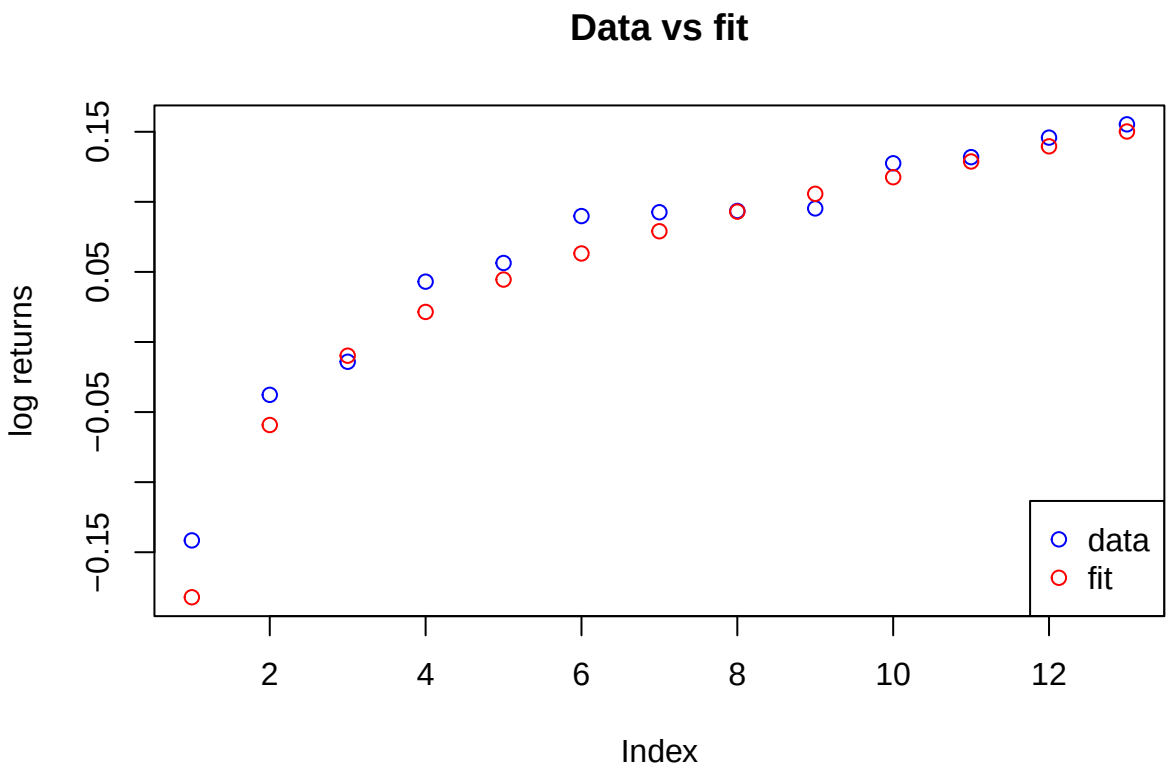
QQ Plot



The qq plot looks great. Log returns for Velliv medium risk seems to be consistent with a skewed t-distribution.

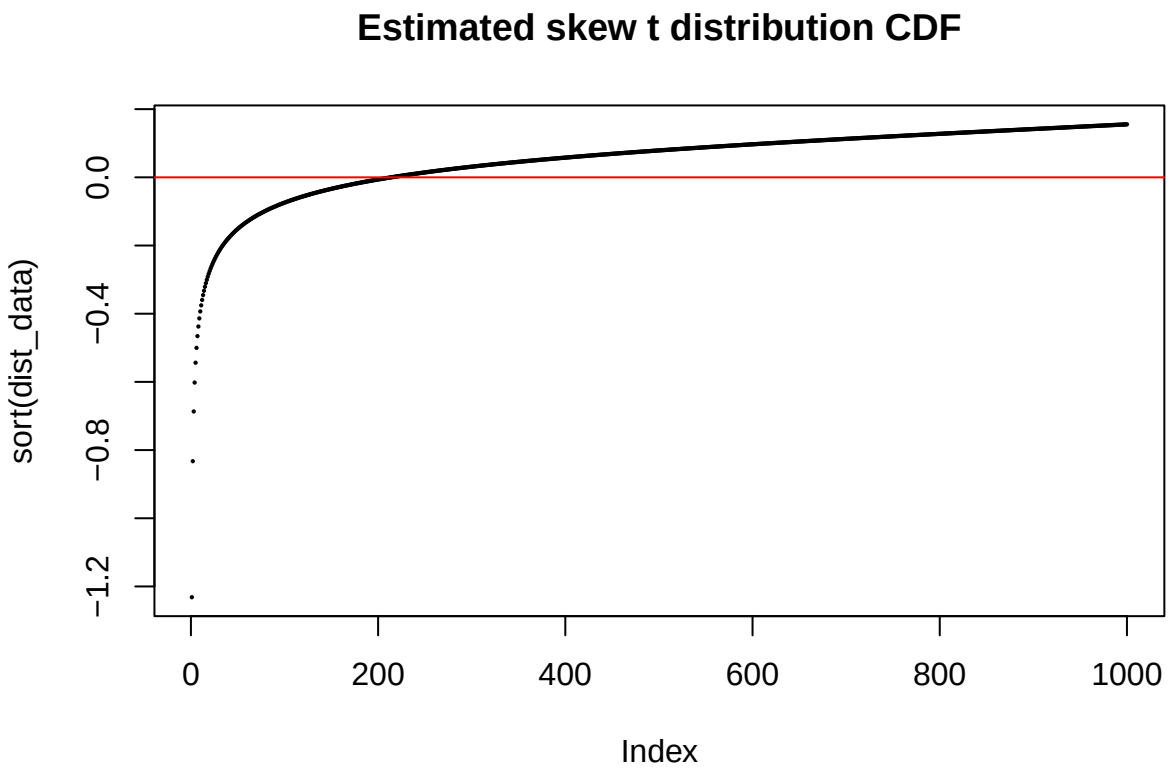
Data vs fit

Let's plot the fit and the observed returns together.



Estimated distribution

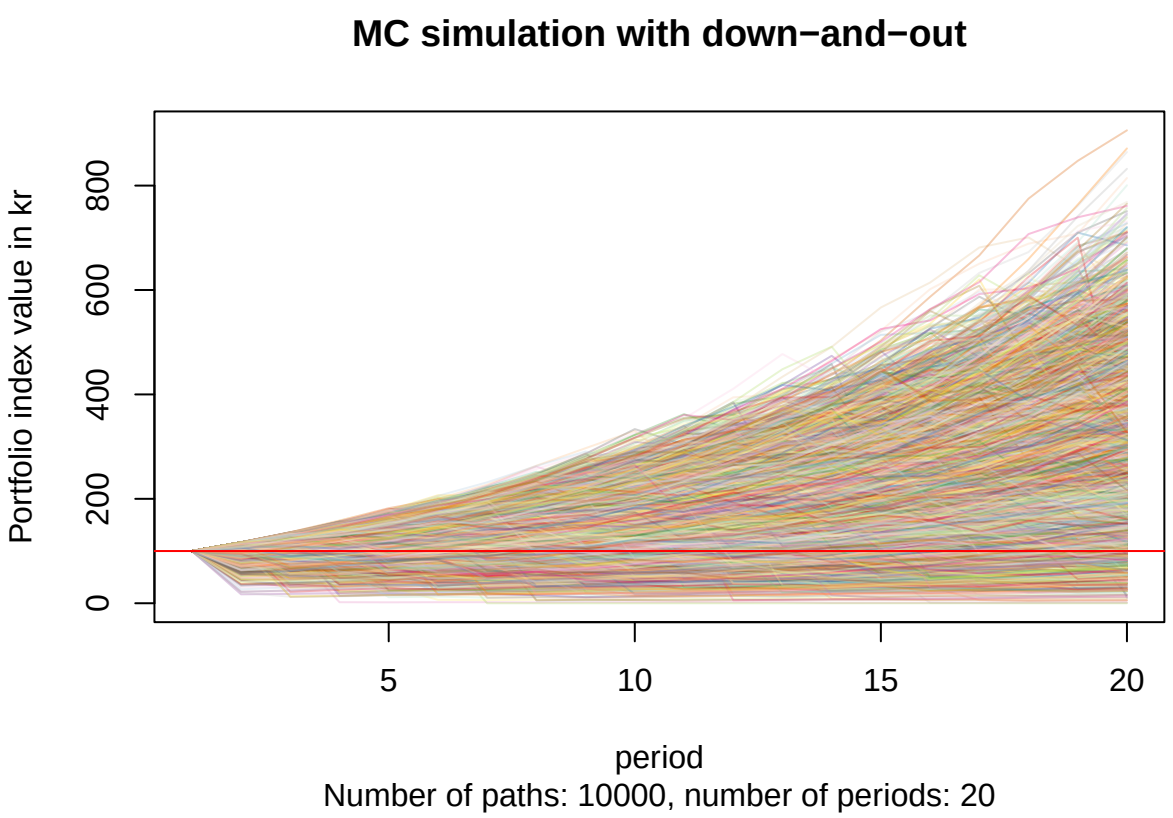
Now lets look at the CDF of the estimated distribution for each 0.1% increment between 0.5% and 99.5% for the estimated distribution:

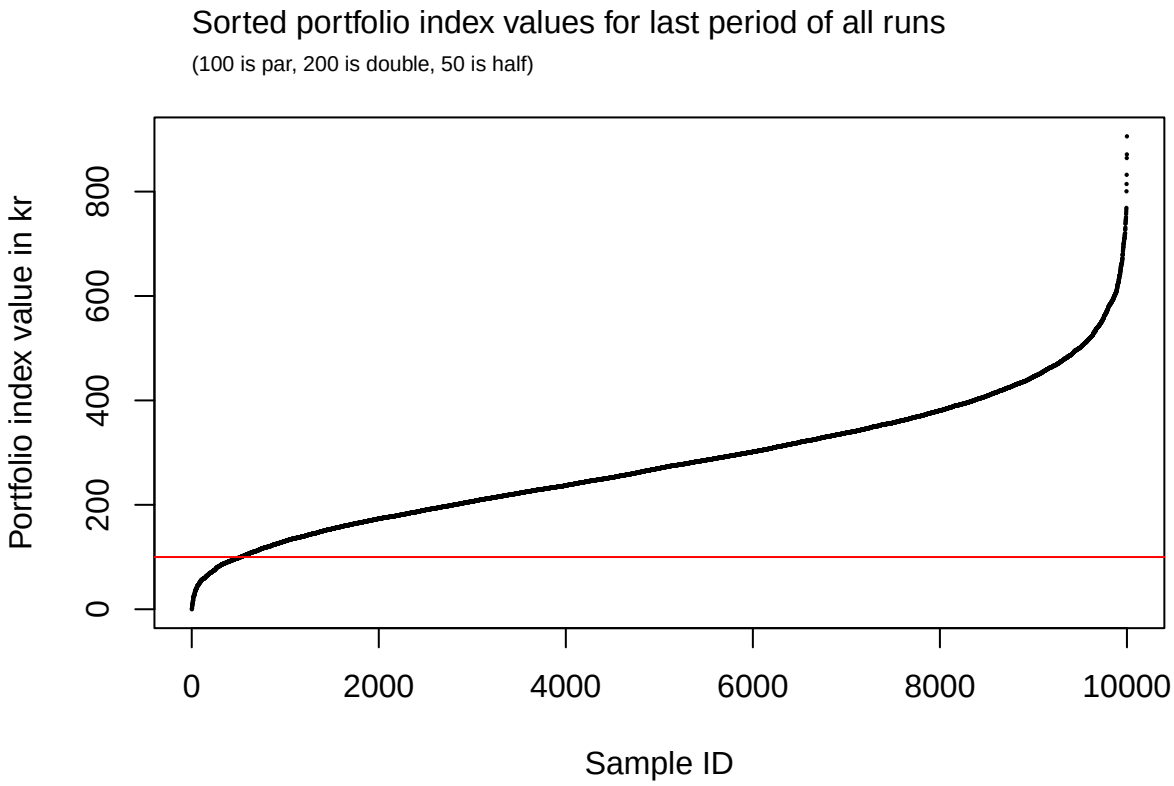


We see that for a few observations out of a 1000, the losses are disastrous, while the upside is very dampened.

Monte Carlo

```
## Down-and-out simulation:
## Probability of down-and-out: 0.01 percent
##
## Mean portfolio index value after 20 years: 280.88 kr.
## SD of portfolio index value after 20 years: 124.121 kr.
## Min total portfolio index value after 20 years: 0 kr.
## Max total portfolio index value after 20 years: 905.803 kr.
##
## Share of paths finishing below 100: 5.23 percent
```


MC simulation with down−and−out
Portfolio index value in kr
0
200
400
600
800
5
10
15
20
period
Number of paths: 10000, number of periods: 20

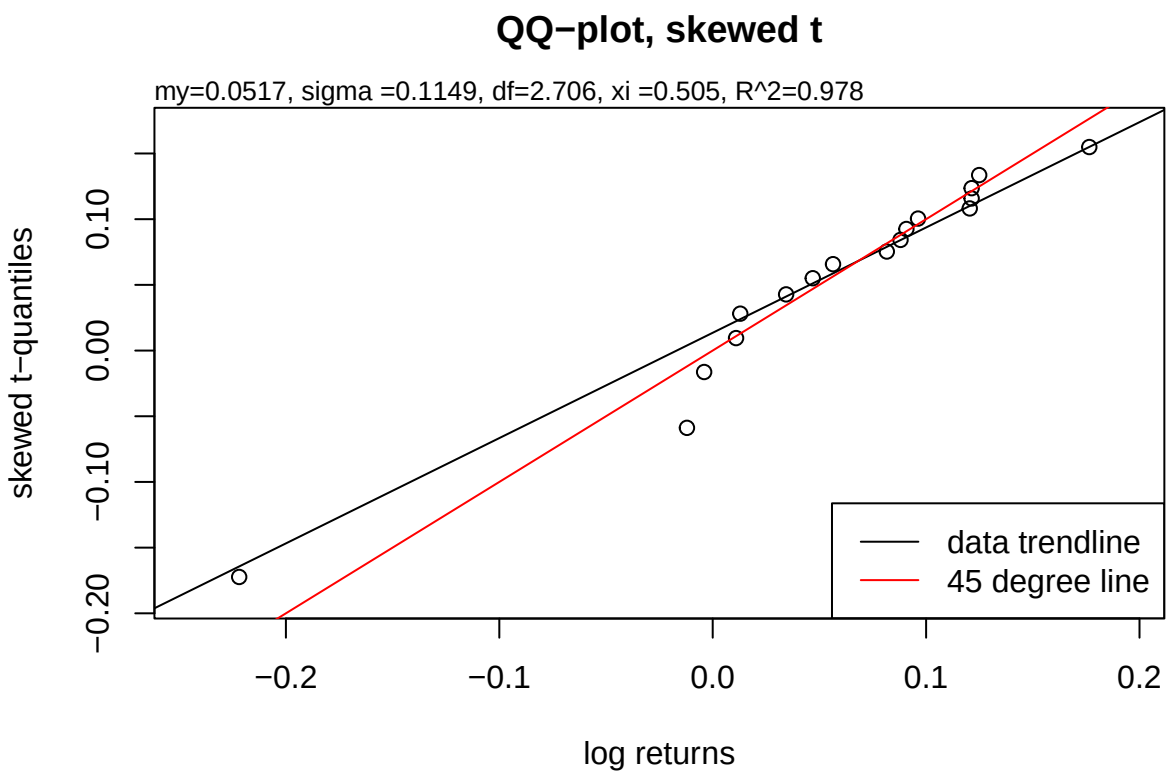


Velliv medium risk, 2007 - 2023

Fit to skew t distribution

```
##
## AIC: -34.35752
## BIC: -31.02467
## m: 0.05171176
## s: 0.1149408
## nu (df): 2.706099
## xi: 0.5049945
## R^2: 0.978
##
## An R^2 of 0.978 suggests that the fit is very good.
##
## What is the risk of losing max 10 %? =< 5.4 percent
## What is the risk of losing max 25 %? =< 1.3 percent
## What is the risk of losing max 50 %? =< 0.2 percent
## What is the risk of losing max 90 %? =< 0 percent
## What is the risk of losing max 99 %? =< 0 percent
##
## What is the chance of gaining min 10 %? >= 36.2 percent
## What is the chance of gaining min 25 %? >= 0.3 percent
## What is the chance of gaining min 50 %? >= 0 percent
## What is the chance of gaining min 90 %? >= 0 percent
## What is the chance of gaining min 99 %? >= 0 percent
```

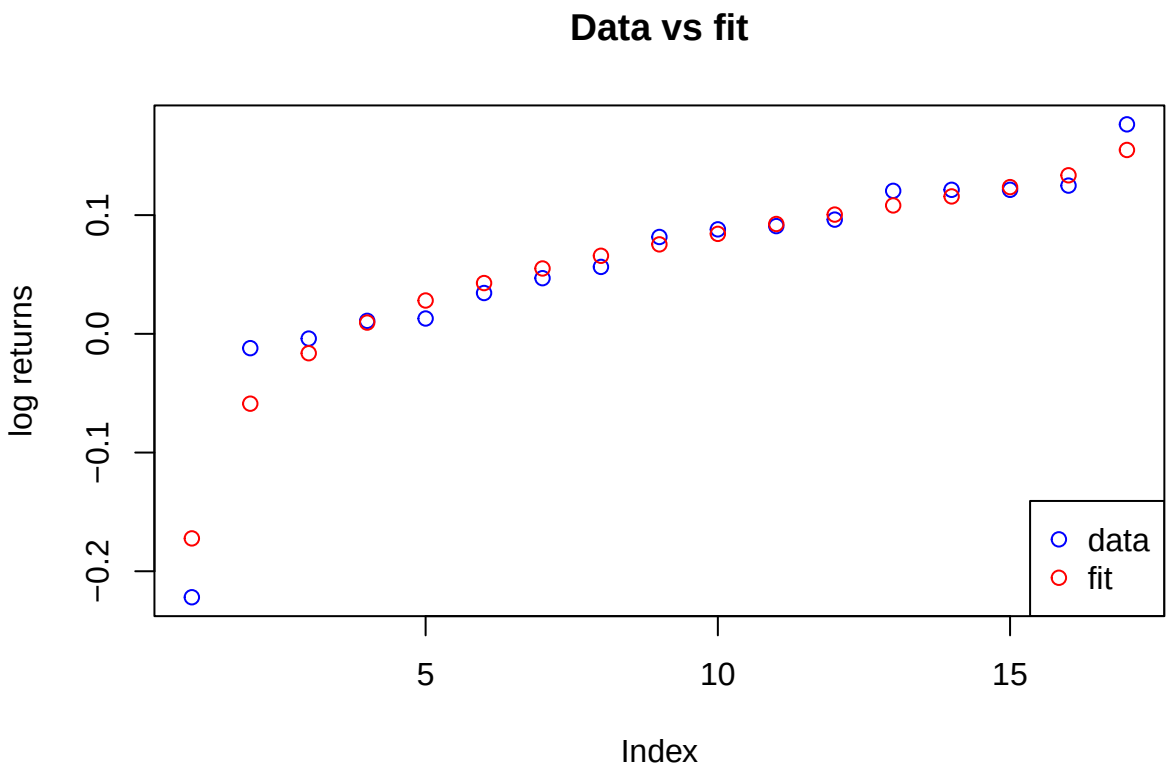
QQ Plot



The qq plot looks good. Log returns for Velliv high risk seems to be consistent with a skewed t-distribution.

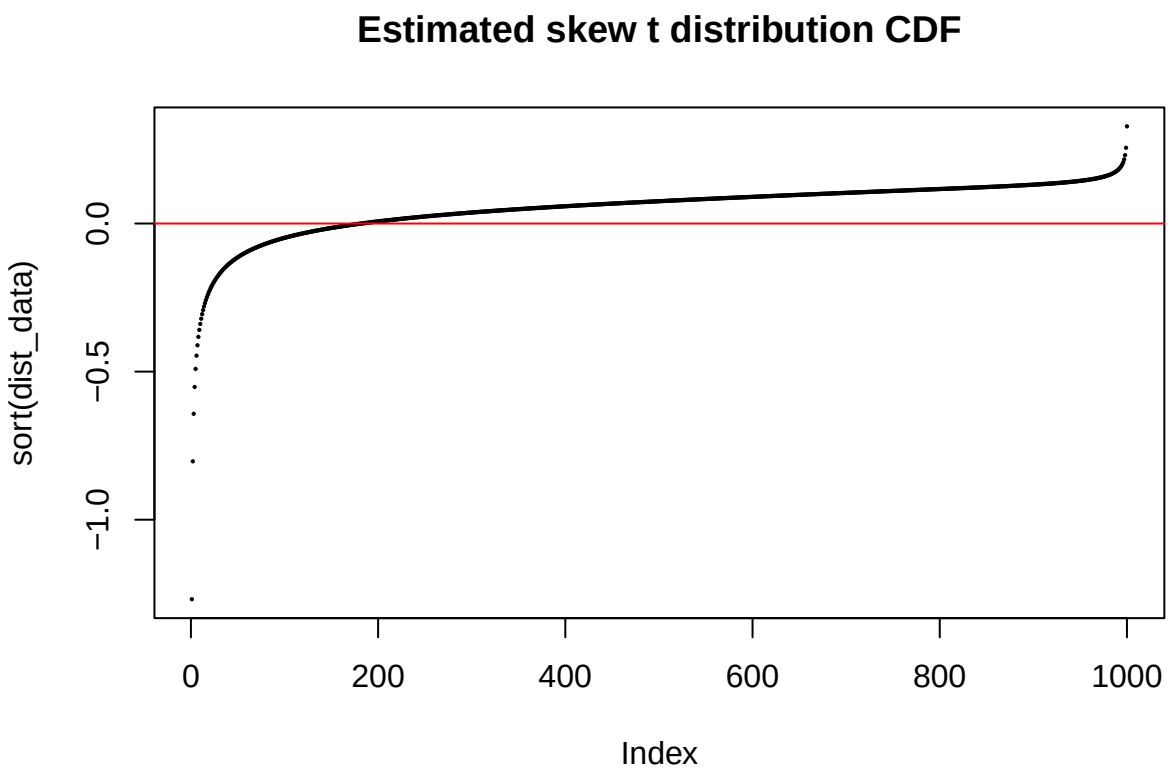
Data vs fit

Let's plot the fit and the observed returns together.



Estimated distribution

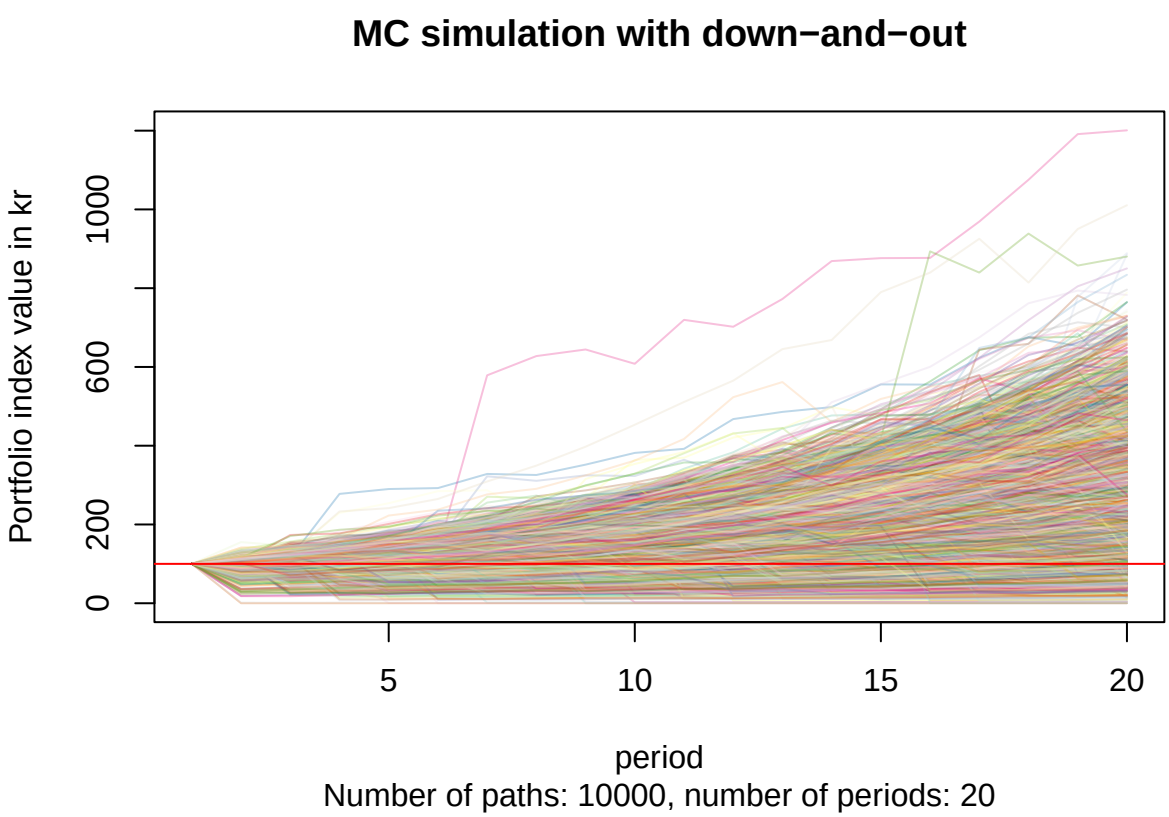
Now lets look at the CDF of the estimated distribution for each 0.1% increment between 0.5% and 99.5% for the estimated distribution:

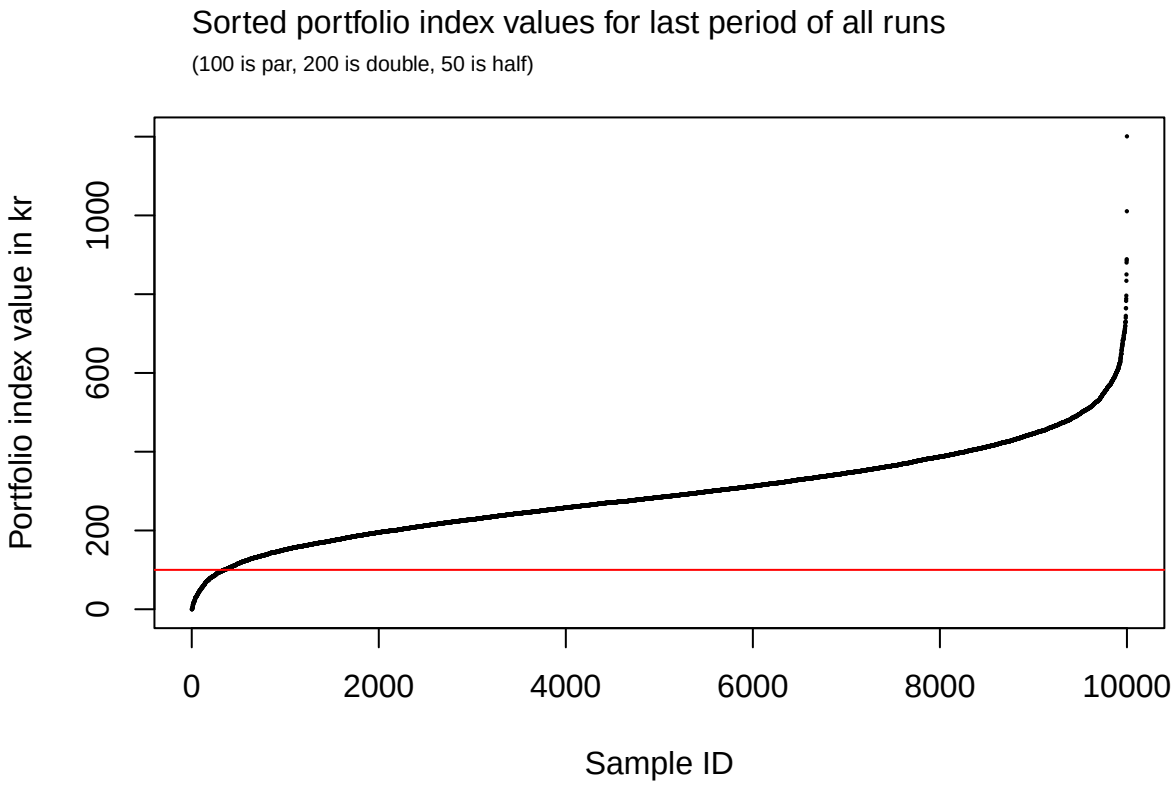


We see that for a few observations out of a 1000, the losses are disastrous, while the upside is very dampened. But because the disastrous loss in 2008 was followed by a large profit the following year, we see some increased upside for the top percentiles. Beware: A 1.2 return following a 0.8 return doesn't take us back where we were before the loss. Path dependency! So if returns more or less average out, but high returns have a tendency to follow high losses, that's bad!

Monte Carlo

```
## Down-and-out simulation:
## Probability of down-and-out: 0.01 percent
##
## Mean portfolio index value after 20 years: 293.213 kr.
## SD of portfolio index value after 20 years: 117.487 kr.
## Min total portfolio index value after 20 years: 0 kr.
## Max total portfolio index value after 20 years: 1200.76 kr.
##
## Share of paths finishing below 100: 3.5 percent
```


MC simulation with down−and−out
Portfolio index value in kr
0
200
600
1000
5
10
15
20
period
Number of paths: 10000, number of periods: 20

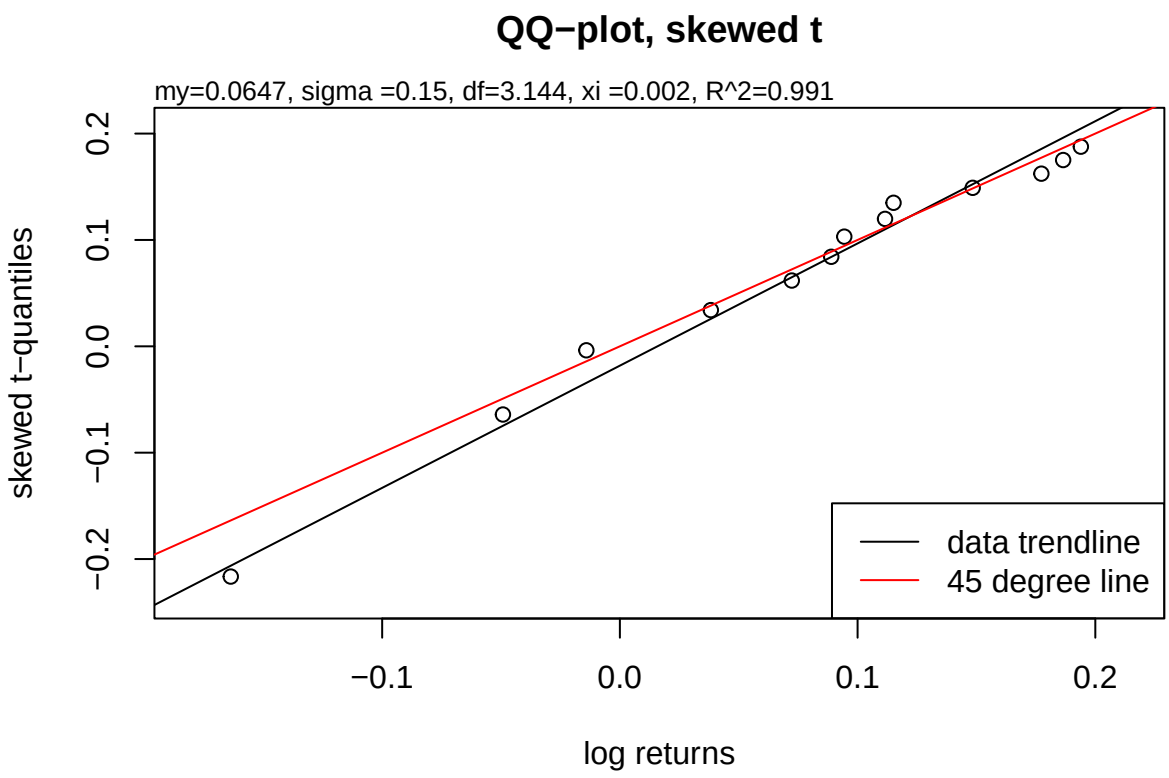


Velliv high risk, 2011 - 2023

Fit to skew t distribution

```
##
## AIC: -21.42488
## BIC: -19.16508
## m: 0.06471454
## s: 0.1499924
## nu (df): 3.144355
## xi: 0.002367034
## R^2: 0.991
##
## An R^2 of 0.991 suggests that the fit is extremely good.
##
## What is the risk of losing max 10 %? =< 8.3 percent
## What is the risk of losing max 25 %? =< 2.5 percent
## What is the risk of losing max 50 %? =< 0.4 percent
## What is the risk of losing max 90 %? =< 0 percent
## What is the risk of losing max 99 %? =< 0 percent
##
## What is the chance of gaining min 10 %? >= 53.3 percent
## What is the chance of gaining min 25 %? >= 0 percent
## What is the chance of gaining min 50 %? >= 0 percent
## What is the chance of gaining min 90 %? >= 0 percent
## What is the chance of gaining min 99 %? >= 0 percent
```

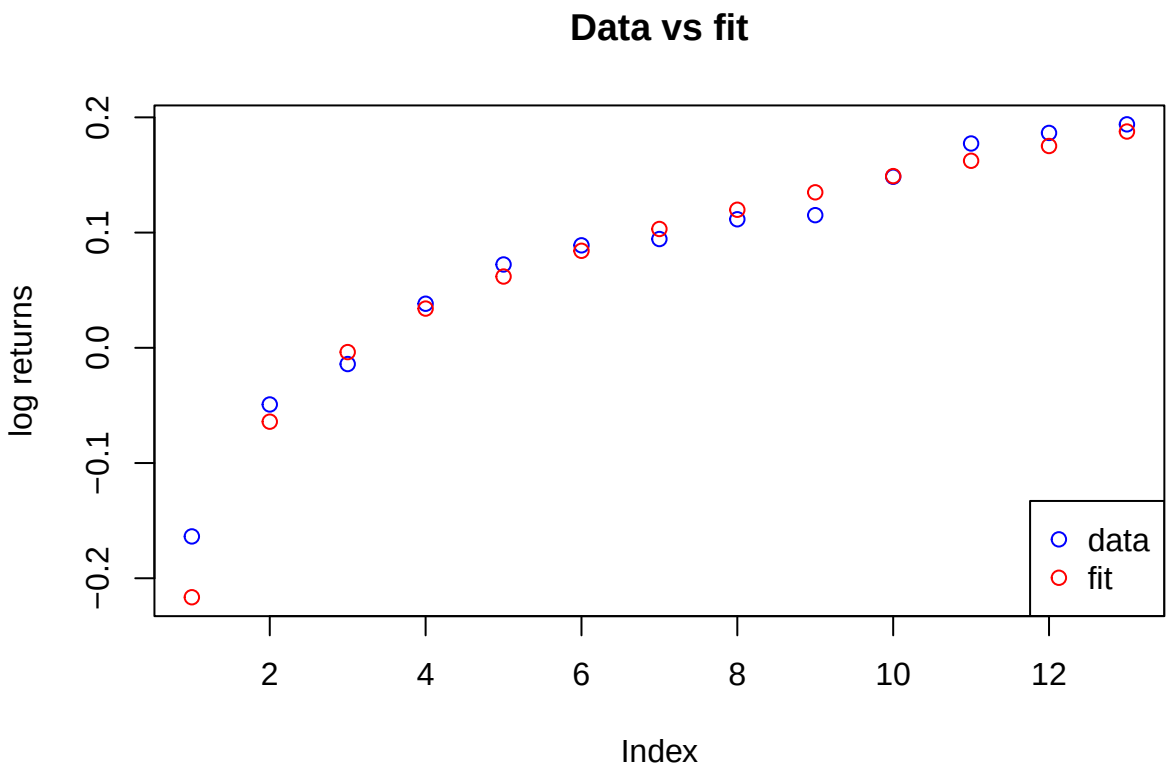
QQ Plot



The qq plot looks great. Returns for Velliv medium risk seems to be consistent with a skewed t-distribution.

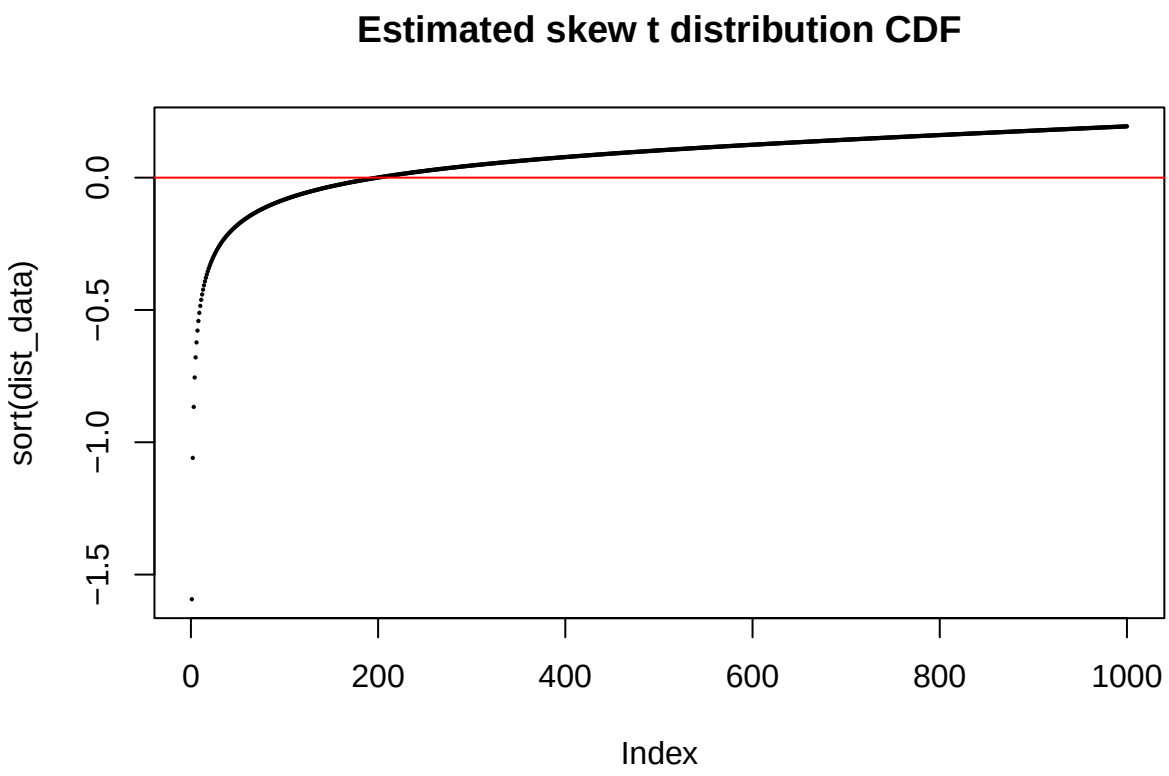
Data vs fit

Let's plot the fit and the observed returns together.



Estimated distribution

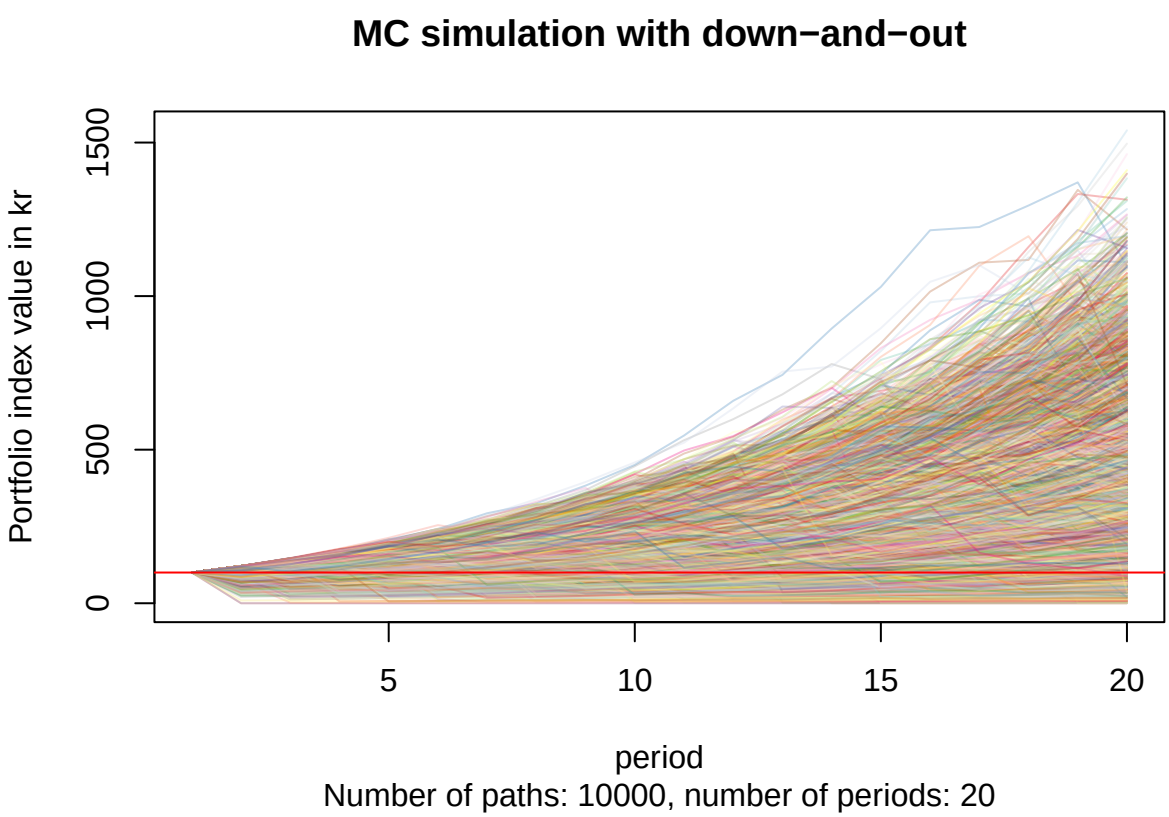
Now lets look at the CDF of the estimated distribution for each 0.1% increment between 0.5% and 99.5% for the estimated distribution:

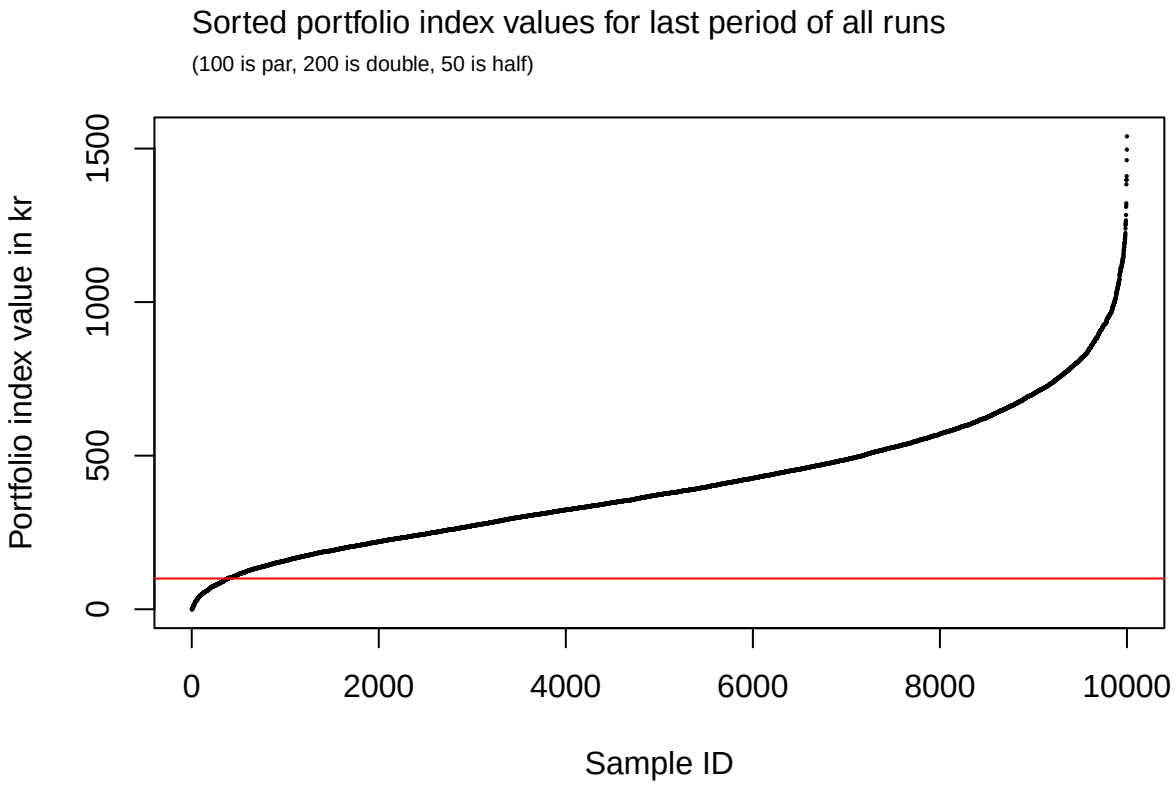


We see that for a few observations out of a 1000, the losses are disastrous, while the upside is very dampened.

Monte Carlo

```
## Down-and-out simulation:
## Probability of down-and-out: 0.03 percent
##
## Mean portfolio index value after 20 years: 405.165 kr.
## SD of portfolio index value after 20 years: 216.392 kr.
## Min total portfolio index value after 20 years: 0 kr.
## Max total portfolio index value after 20 years: 1539.705 kr.
##
## Share of paths finishing below 100: 3.83 percent
```


MC simulation with down−and−out
Portfolio index value in kr
0
500
1000
1500
5
10
15
20
period
Number of paths: 10000, number of periods: 20

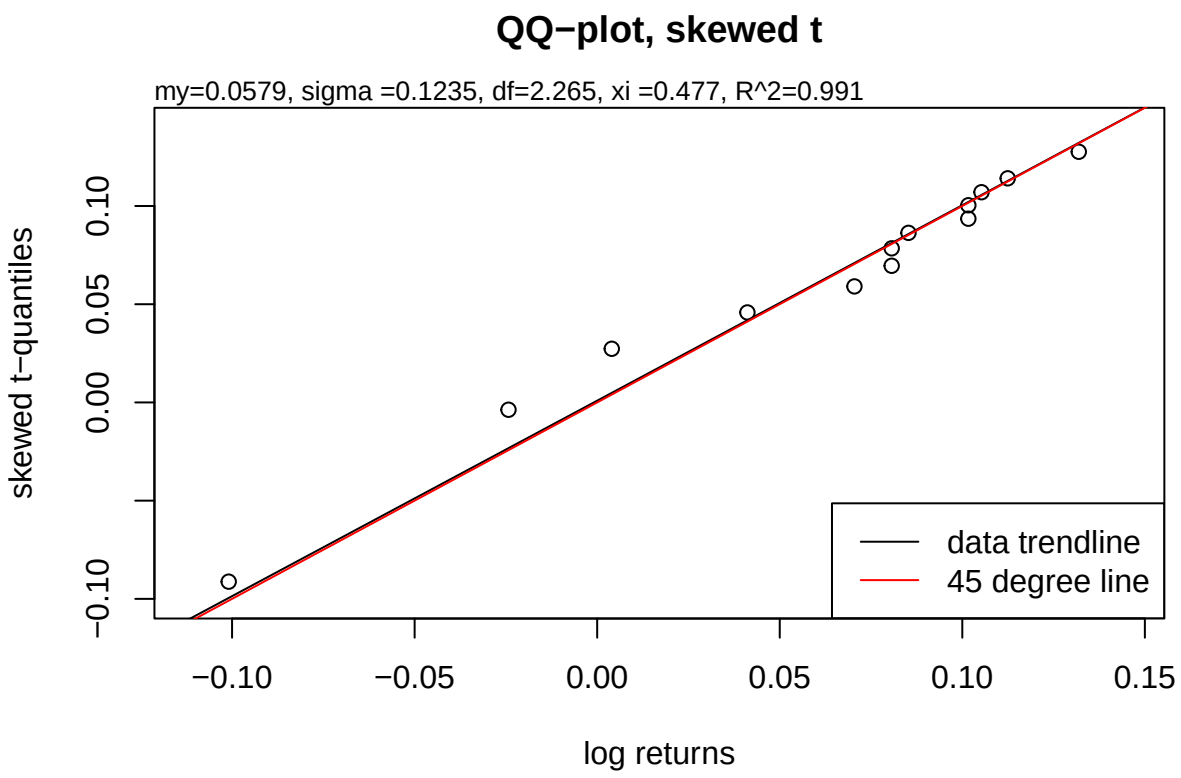


PFA medium risk, 2011 - 2023

Fit to skew t distribution

```
##
## AIC: -33.22998
## BIC: -30.97018
## m: 0.05789224
## s: 0.1234592
## nu (df): 2.265273
## xi: 0.477324
## R^2: 0.991
##
## An R^2 of 0.991 suggests that the fit is extremely good.
##
## What is the risk of losing max 10 %? =< 3.3 percent
## What is the risk of losing max 25 %? =< 0.9 percent
## What is the risk of losing max 50 %? =< 0.2 percent
## What is the risk of losing max 90 %? =< 0 percent
## What is the risk of losing max 99 %? =< 0 percent
##
## What is the chance of gaining min 10 %? >= 32.7 percent
## What is the chance of gaining min 25 %? >= 0.1 percent
## What is the chance of gaining min 50 %? >= 0 percent
## What is the chance of gaining min 90 %? >= 0 percent
## What is the chance of gaining min 99 %? >= 0 percent
```

QQ Plot

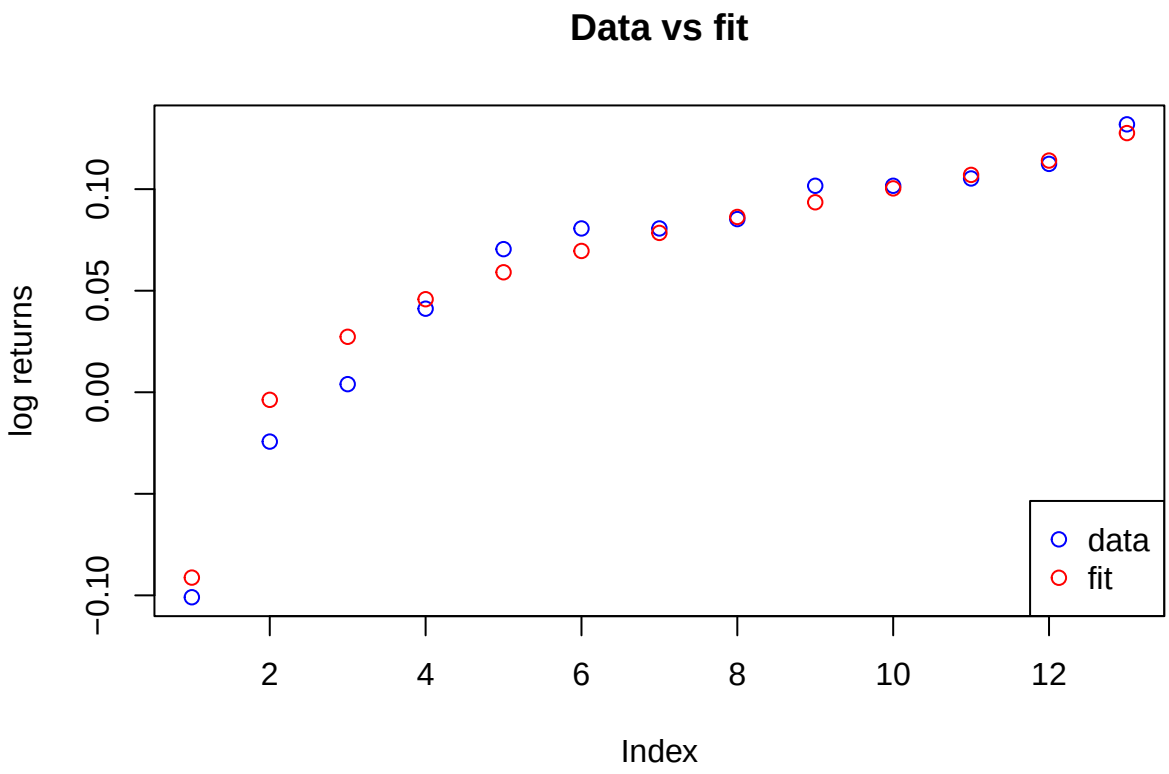


The qq plot looks great. Log returns for PFA medium risk seems to be consistent with a skewed t-distribution.

```
##  [1] -0.091256521 -0.003731241  0.027312079  0.045808232  0.059068633
##  [6]  0.069575113  0.078454727  0.086316936  0.093536451  0.100370932
## [11]  0.107018607  0.114081432  0.127604387
```

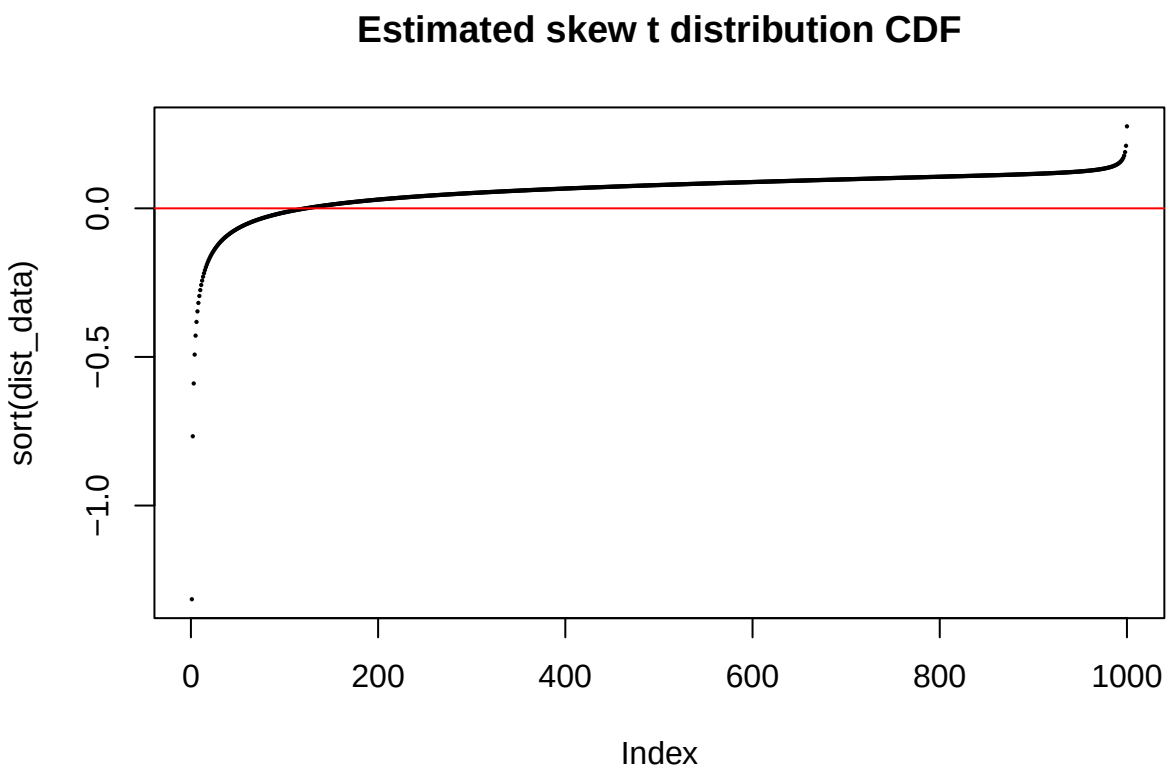
Data vs fit

Let's plot the fit and the observed returns together.



Estimated distribution

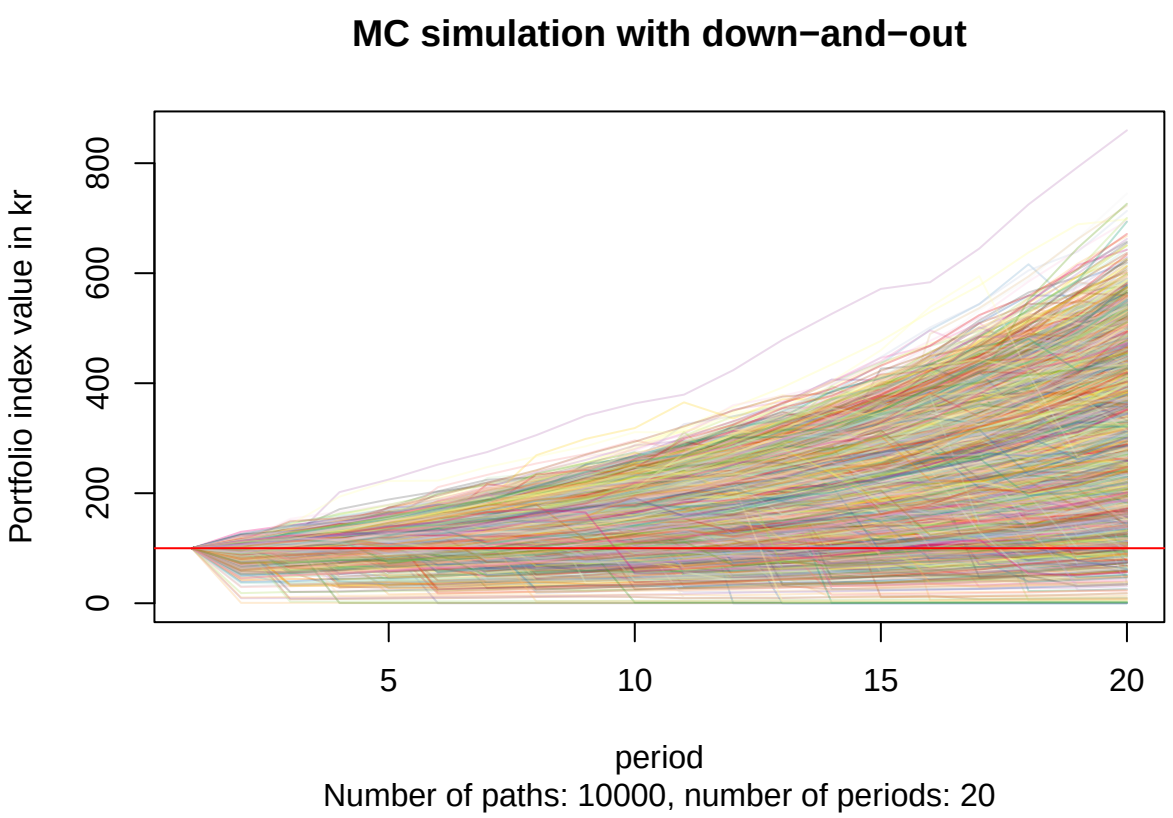
Now lets look at the CDF of the estimated distribution for each 0.1% increment between 0.5% and 99.5% for the estimated distribution:

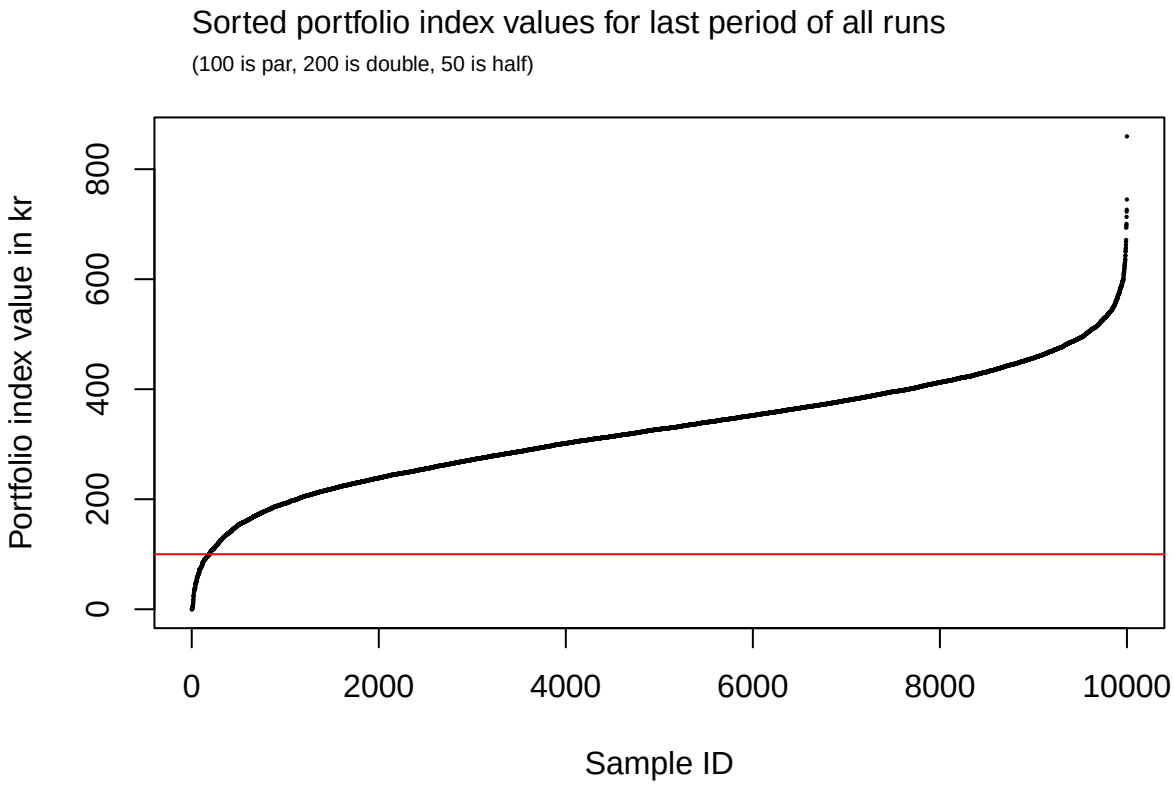


We see that for a few observations out of a 1000, the losses are disastrous. While there is some uptick at the top percentiles, the curve basically flattens out.

Monte Carlo

```
## Down-and-out simulation:
## Probability of down-and-out: 0.02 percent
##
## Mean portfolio index value after 20 years: 325.49 kr.
## SD of portfolio index value after 20 years: 104.032 kr.
## Min total portfolio index value after 20 years: 0 kr.
## Max total portfolio index value after 20 years: 859.686 kr.
##
## Share of paths finishing below 100: 1.88 percent
```


MC simulation with down−and−out
Portfolio index value in kr
0
200
400
600
800
5
10
15
20
period
Number of paths: 10000, number of periods: 20

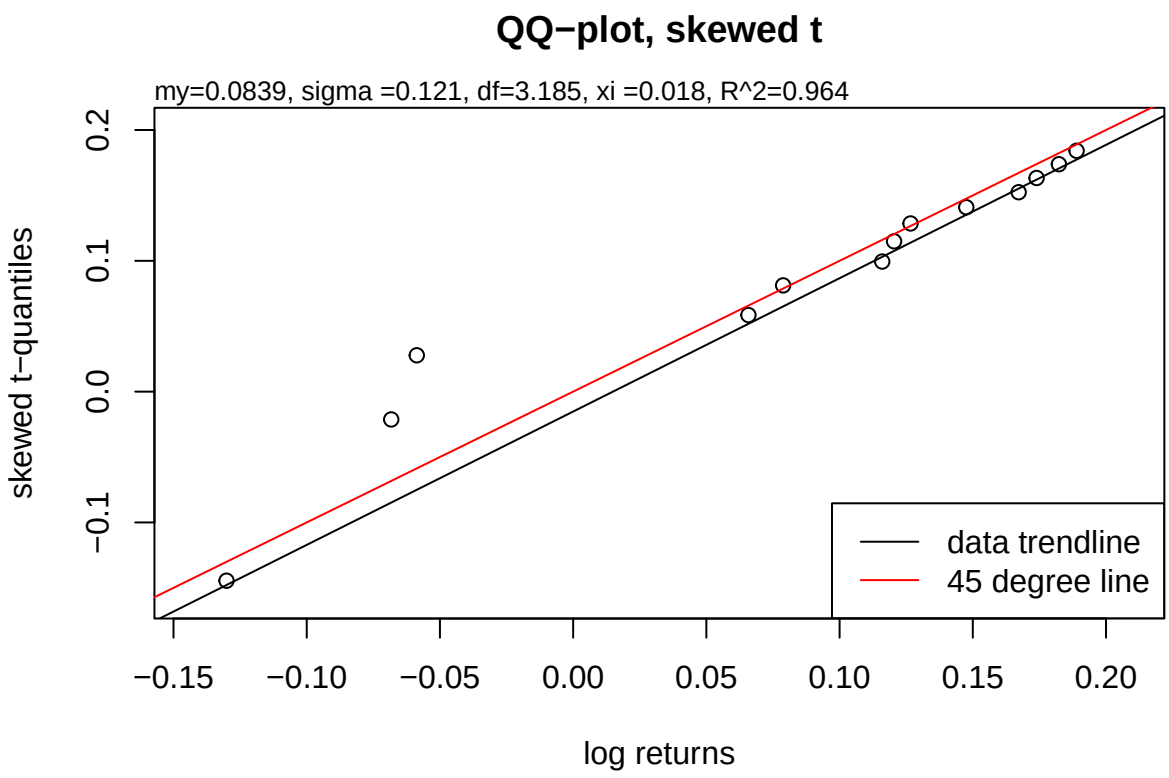


PFA high risk, 2011 - 2023

Fit to skew t distribution

```
##
## AIC: -23.72565
## BIC: -21.46585
## m: 0.08386034
## s: 0.1210107
## nu (df): 3.184569
## xi: 0.01790306
## R^2: 0.964
##
## An R^2 of 0.964 suggests that the fit is very good.
##
## What is the risk of losing max 10 %? =< 5.3 percent
## What is the risk of losing max 25 %? =< 1.4 percent
## What is the risk of losing max 50 %? =< 0.2 percent
## What is the risk of losing max 90 %? =< 0 percent
## What is the risk of losing max 99 %? =< 0 percent
##
## What is the chance of gaining min 10 %? >= 59.6 percent
## What is the chance of gaining min 25 %? >= 0 percent
## What is the chance of gaining min 50 %? >= 0 percent
## What is the chance of gaining min 90 %? >= 0 percent
## What is the chance of gaining min 99 %? >= 0 percent
```

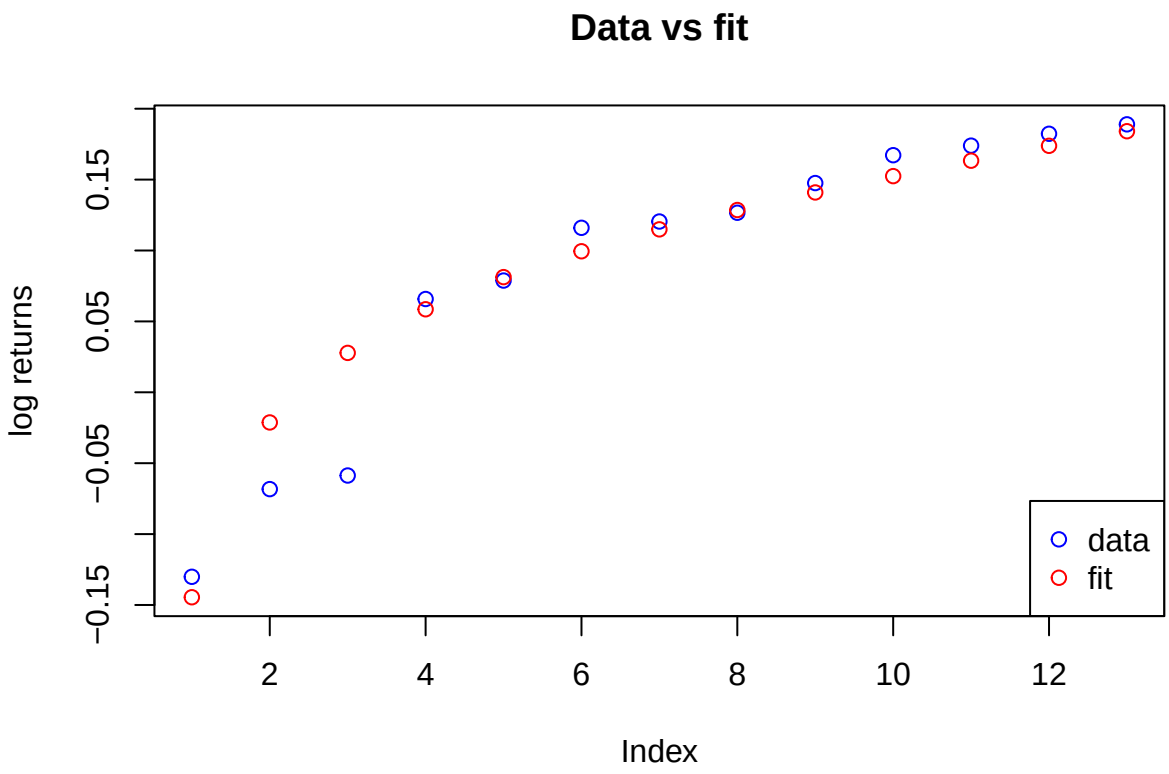
QQ Plot



The qq plot looks ok. Returns for PFA high risk seems to be consistent with a skewed t-distribution.

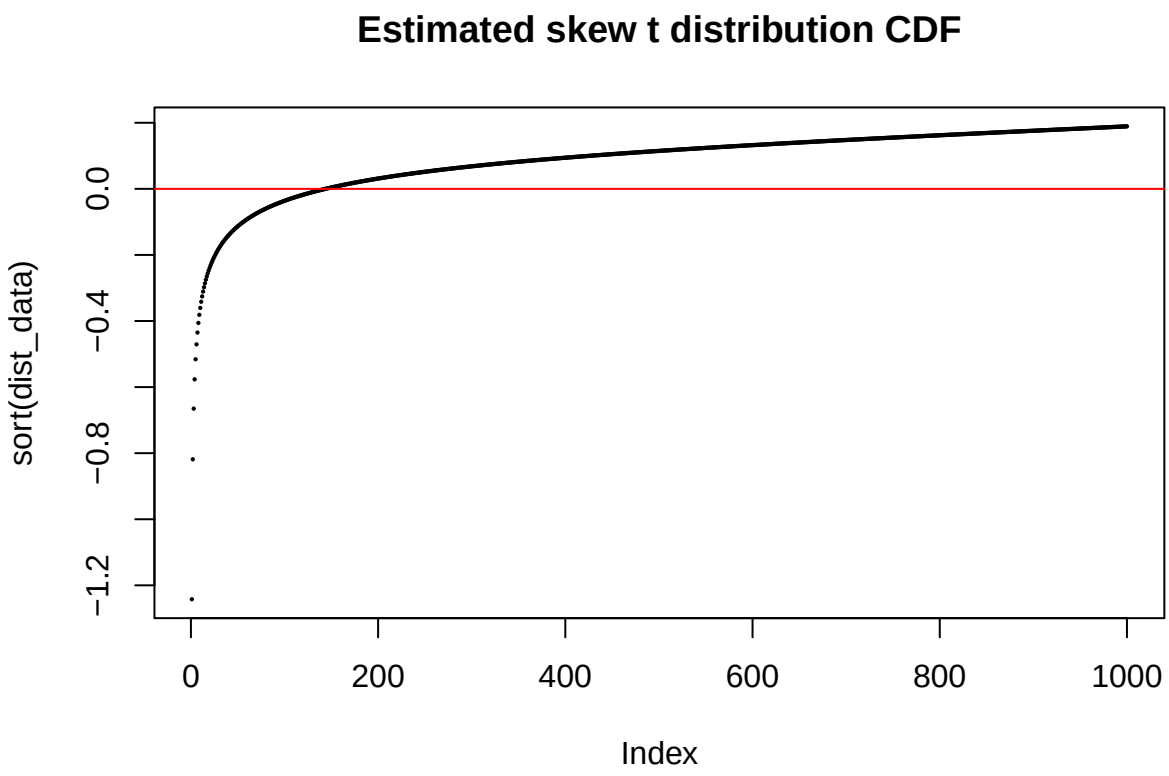
Data vs fit

Let's plot the fit and the observed returns together.



Estimated distribution

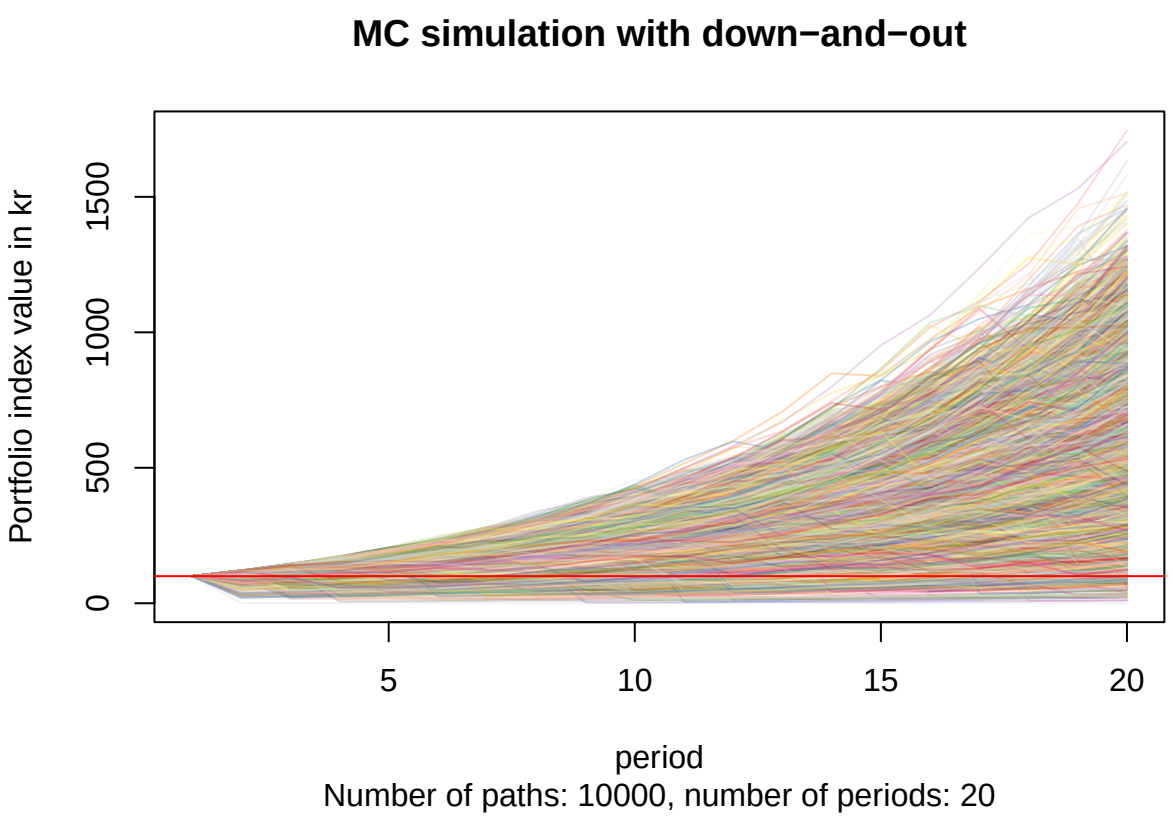
Now lets look at the CDF of the estimated distribution for each 0.1% increment between 0.5% and 99.5% for the estimated distribution:

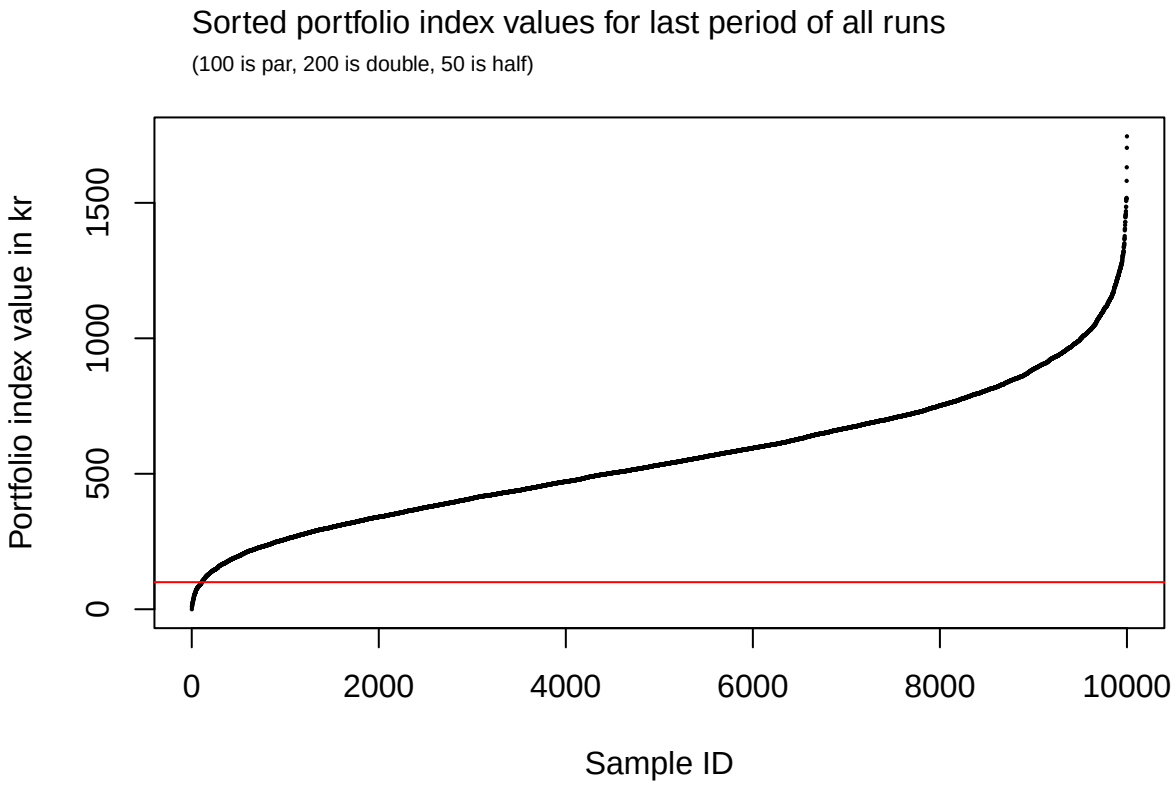


We see that for a few observations out of a 1000, the losses are disastrous, while the upside is very dampened.

Monte Carlo

```
## Down-and-out simulation:
## Probability of down-and-out: 0.01 percent
##
## Mean portfolio index value after 20 years: 554.957 kr.
## SD of portfolio index value after 20 years: 245.019 kr.
## Min total portfolio index value after 20 years: 0 kr.
## Max total portfolio index value after 20 years: 1745.465 kr.
##
## Share of paths finishing below 100: 1.07 percent
```


MC simulation with down−and−out
Portfolio index value in kr
0
500
1000
1500
5
10
15
20
period
Number of paths: 10000, number of periods: 20

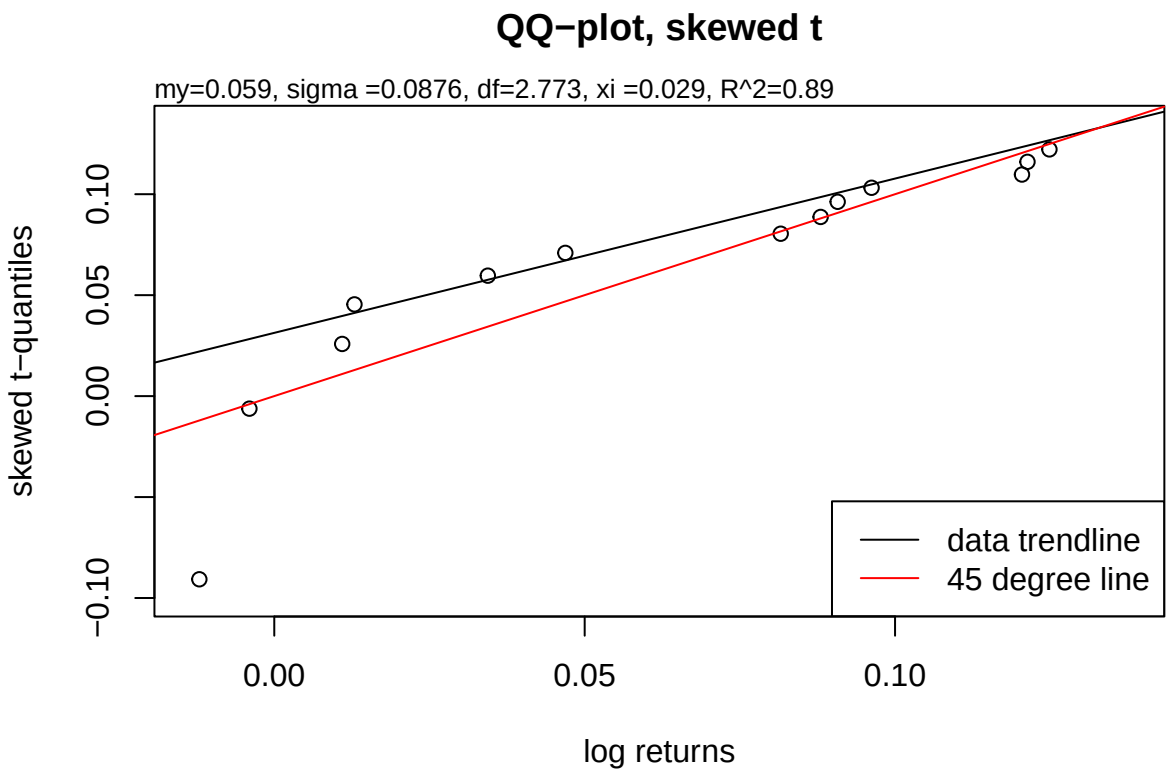


Mix medium risk, 2011 - 2023

Fit to skew t distribution

```
##
## AIC: -36.9603
## BIC: -34.7005
## m: 0.05902873
## s: 0.08757749
## nu (df): 2.772621
## xi: 0.02904471
## R^2: 0.89
##
## An R^2 of 0.89 suggests that the fit is not completely random.
##
## What is the risk of losing max 10 %? =< 3.3 percent
## What is the risk of losing max 25 %? =< 0.7 percent
## What is the risk of losing max 50 %? =< 0.1 percent
## What is the risk of losing max 90 %? =< 0 percent
## What is the risk of losing max 99 %? =< 0 percent
##
## What is the chance of gaining min 10 %? >= 35.6 percent
## What is the chance of gaining min 25 %? >= 0 percent
## What is the chance of gaining min 50 %? >= 0 percent
## What is the chance of gaining min 90 %? >= 0 percent
## What is the chance of gaining min 99 %? >= 0 percent
```

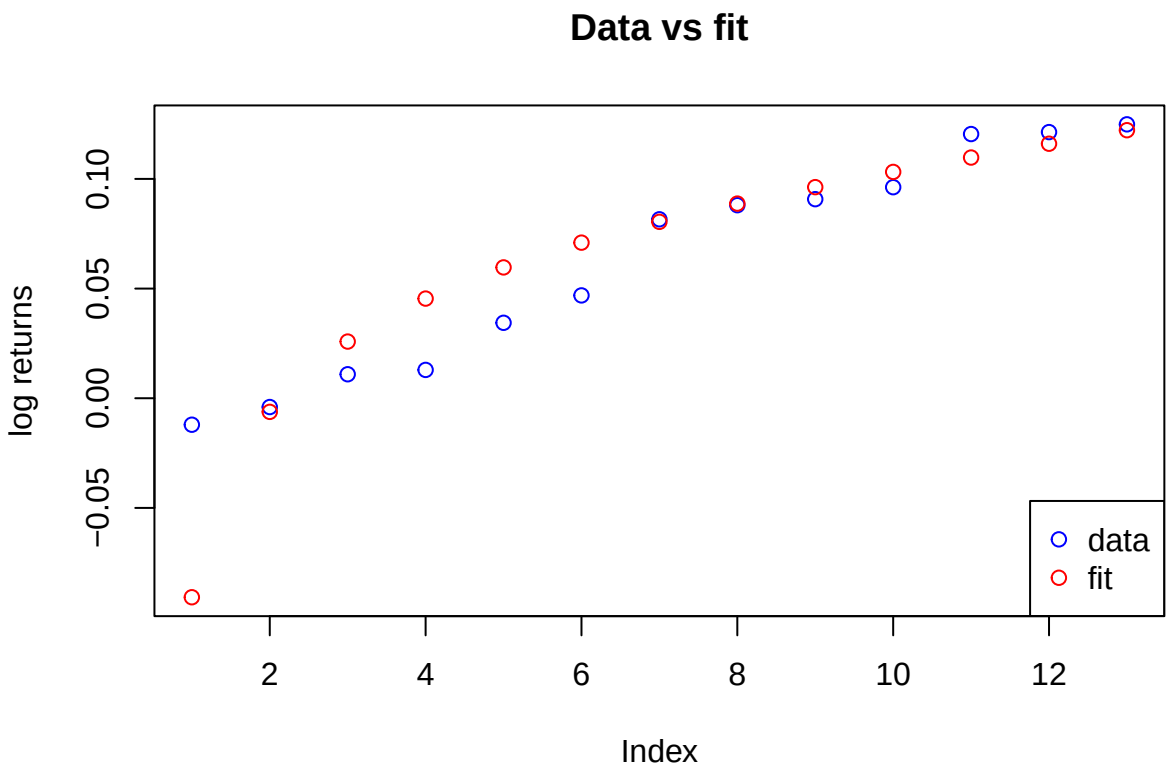
QQ Plot



The fit suggests big losses for the lowest percentiles, which are not present in the data.
So the fit is actually a very cautious estimate.

Data vs fit

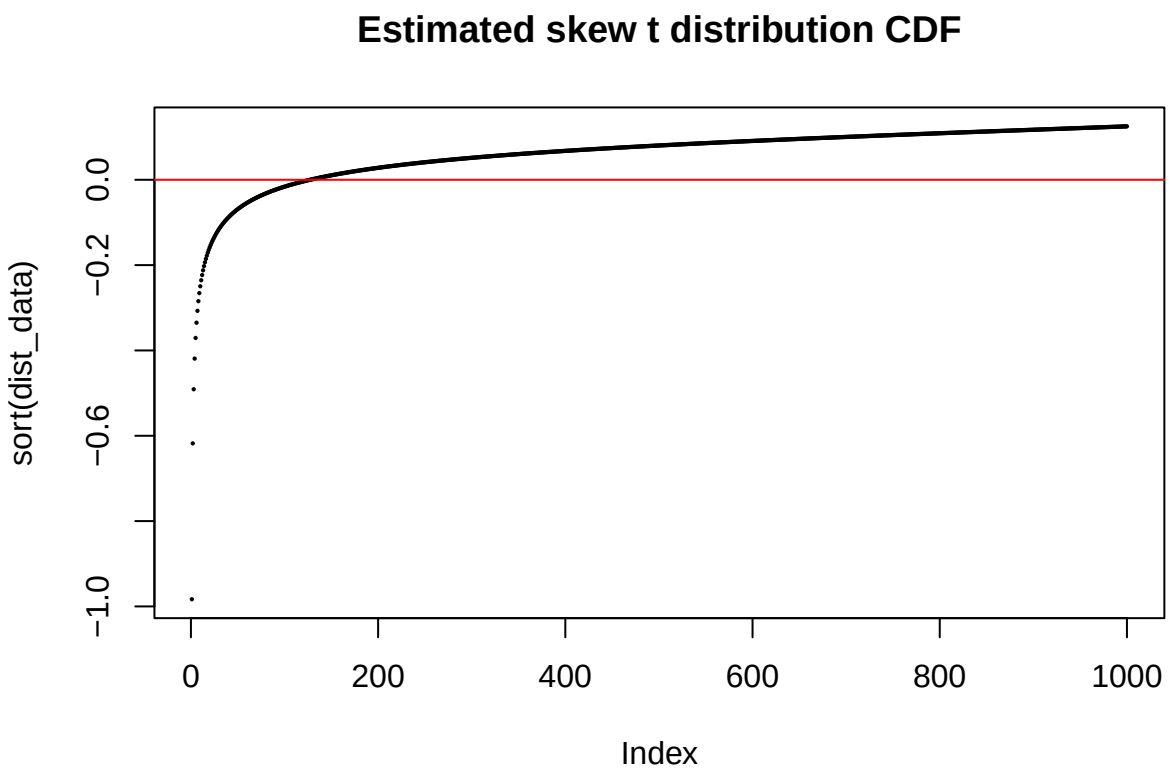
Let's plot the fit and the observed returns together.



Interestingly, the fit predicts a much bigger “biggest loss” than the actual data. This is the main reason that R^2 is 0.90 and not higher.

Estimated distribution

Now lets look at the CDF of the estimated distribution for each 0.1% increment between 0.5% and 99.5% for the estimated distribution:

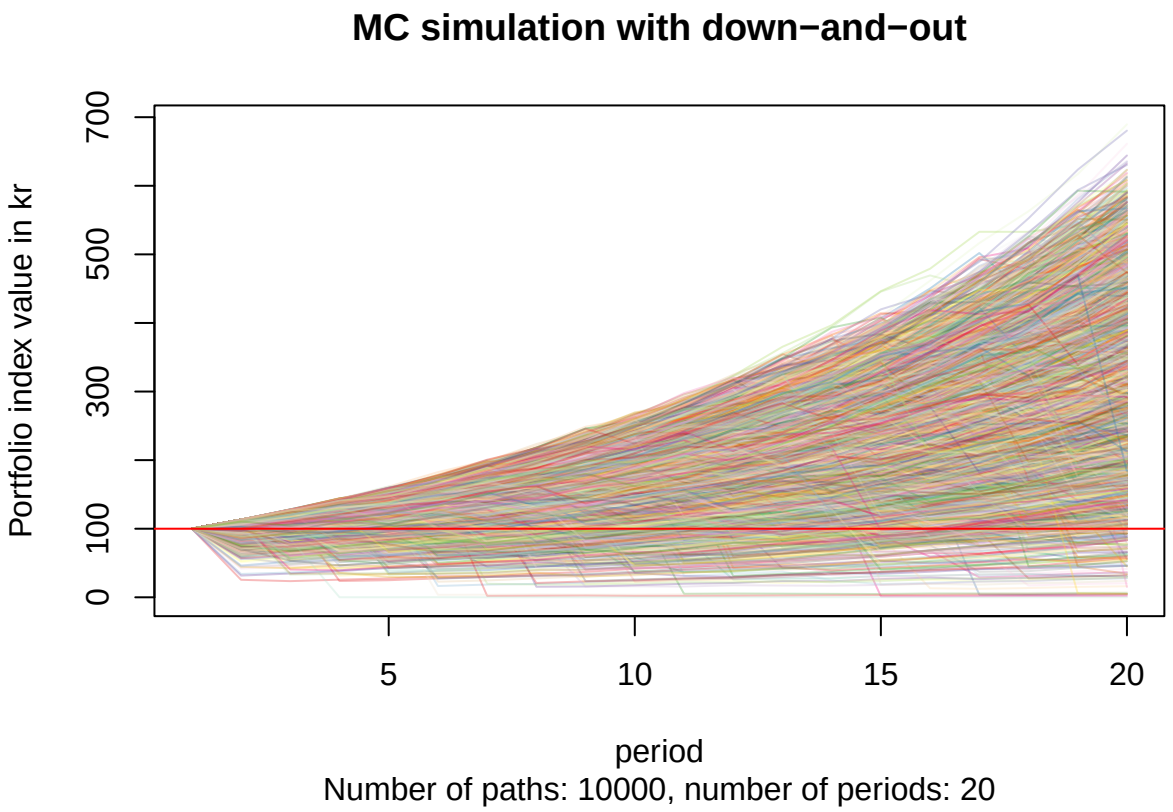


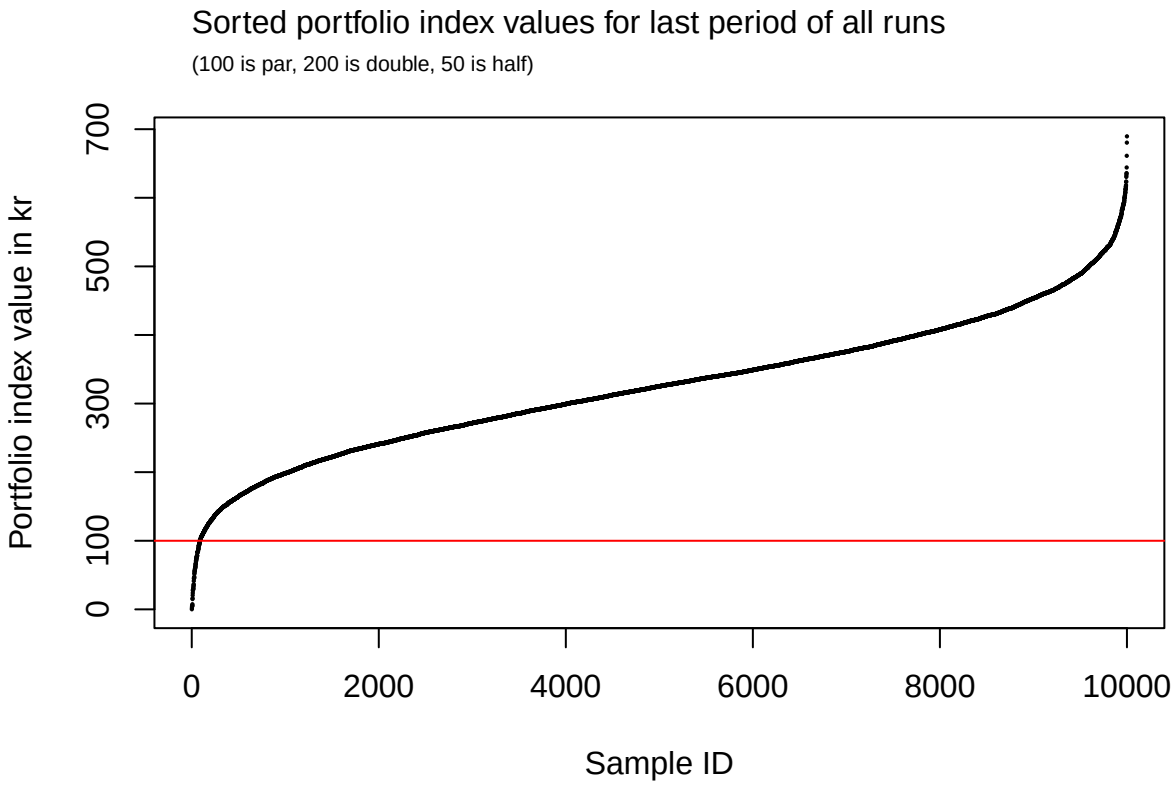
We see that for a few observations out of a 1000, the losses are disastrous, while the upside is very dampened.

Monte Carlo

Version a: Simulation from estimated distribution of returns of mix.

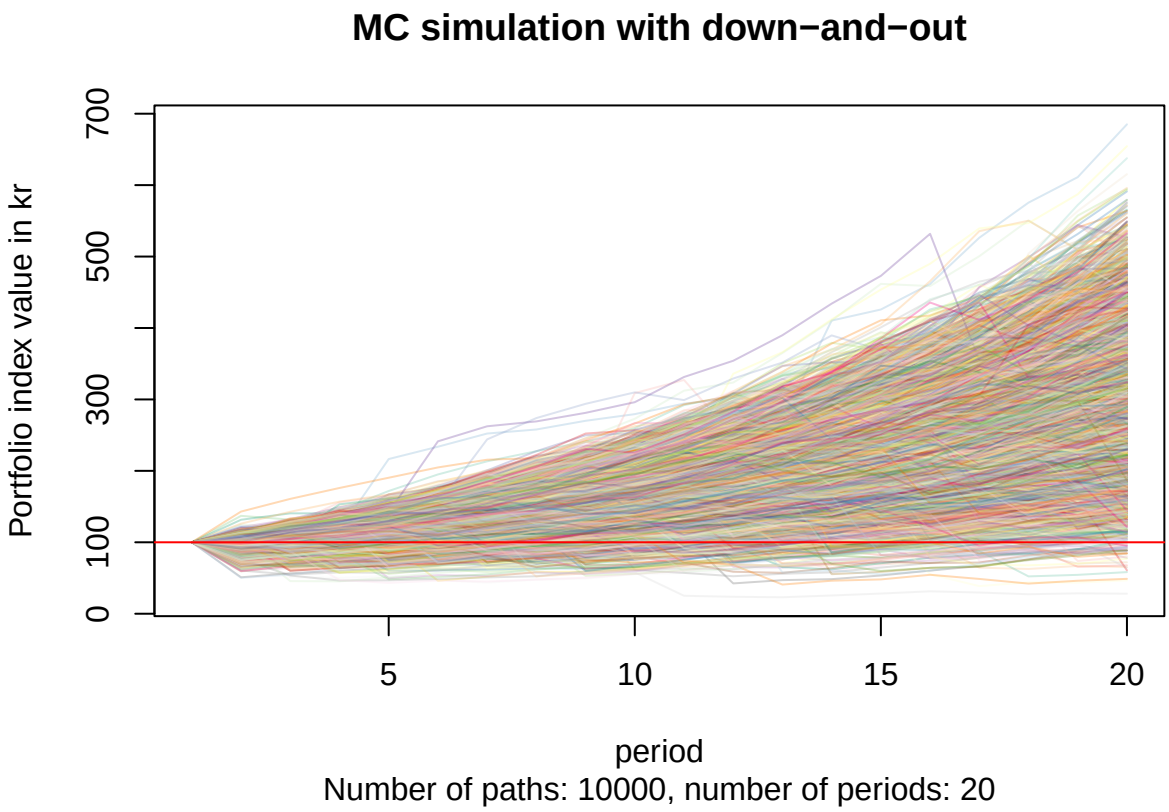
```
## Down-and-out simulation:
## Probability of down-and-out: 0 percent
##
## Mean portfolio index value after 20 years: 324.992 kr.
## SD of portfolio index value after 20 years: 98.529 kr.
## Min total portfolio index value after 20 years: 0.17 kr.
## Max total portfolio index value after 20 years: 689.566 kr.
##
## Share of paths finishing below 100: 0.86 percent
```

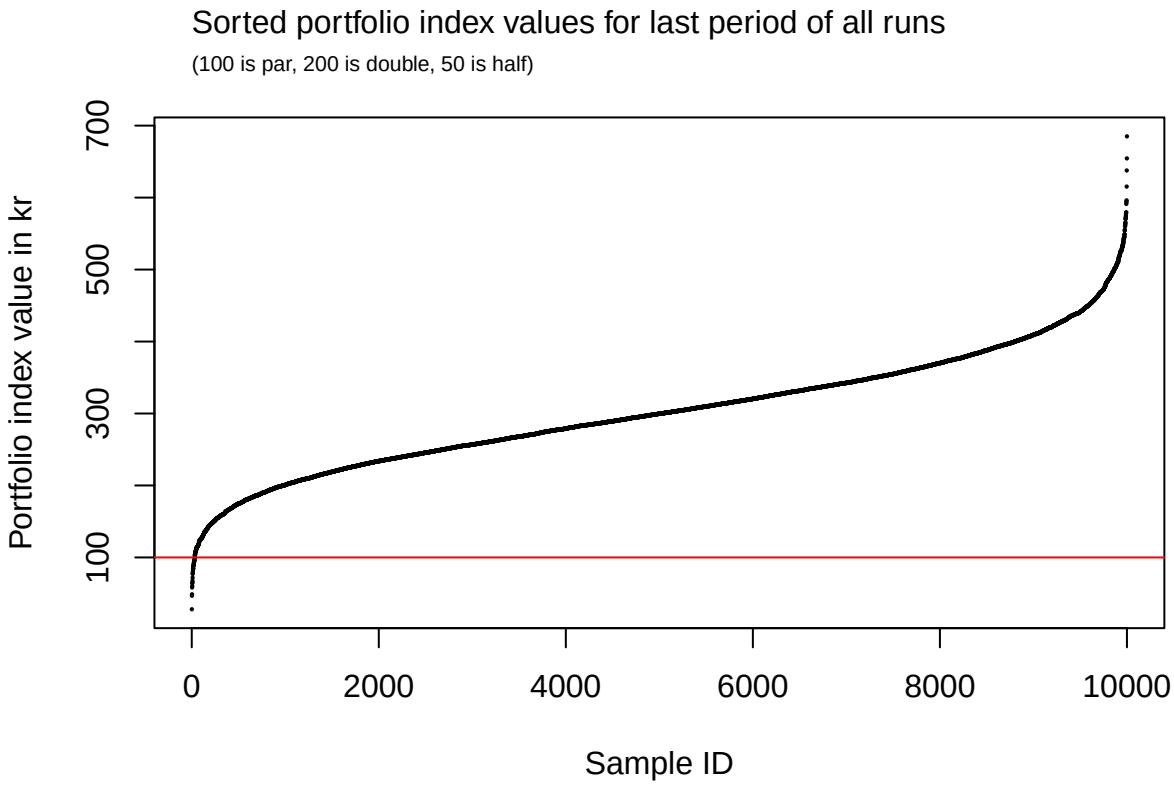

MC simulation with down−and−out
Portfolio index value in kr
0
100
300
500
700
5
10
15
20
period
Number of paths: 10000, number of periods: 20



Version b: Mix of simulations from estimated distribution of returns from individual funds.

```
## Down-and-out simulation:
## Probability of down-and-out: 0 percent
##
## Mean portfolio index value after 20 years: 302.778 kr.
## SD of portfolio index value after 20 years: 81.813 kr.
## Min total portfolio index value after 20 years: 28.038 kr.
## Max total portfolio index value after 20 years: 685.083 kr.
##
## Share of paths finishing below 100: 0.3 percent
```


MC simulation with down−and−out
Portfolio index value in kr
0
100
300
500
700
5
10
15
20
period
Number of paths: 10000, number of periods: 20

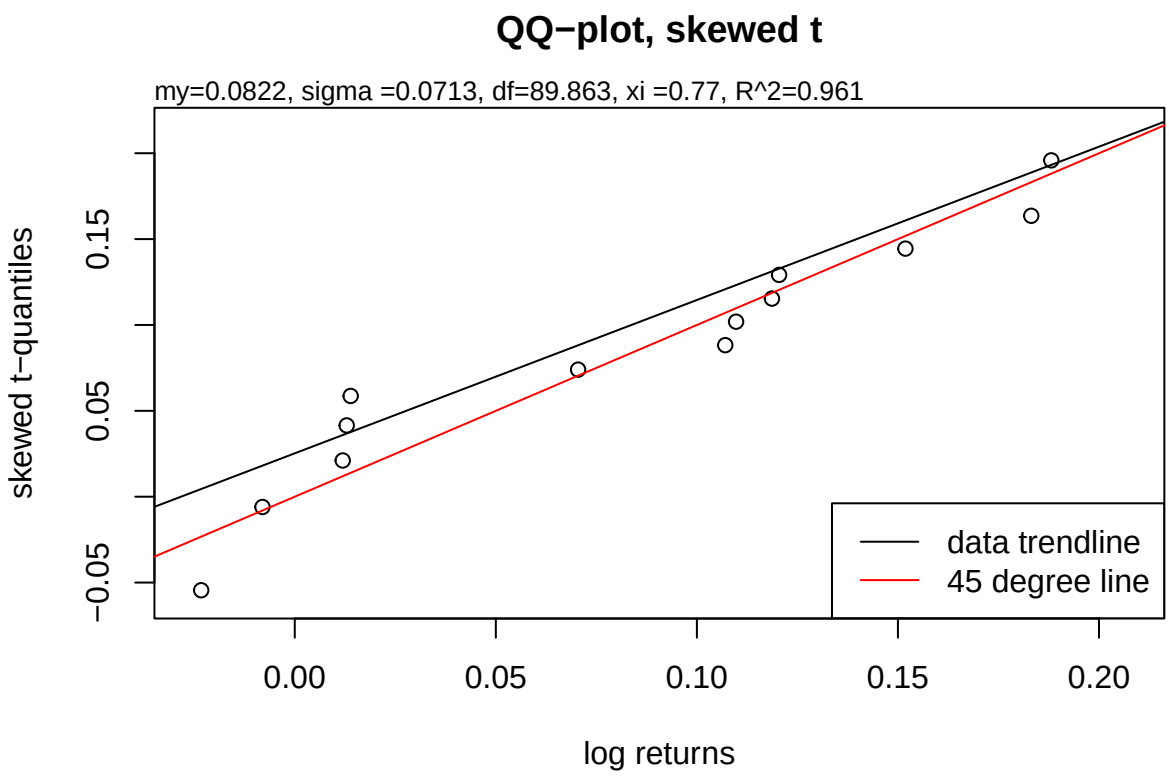


Mix high risk, 2011 - 2023

Fit to skew t distribution

```
##
## AIC: -24.26084
## BIC: -22.00104
## m: 0.0822419
## s: 0.07129843
## nu (df): 89.86289
## xi: 0.7697502
## R^2: 0.961
##
## An R^2 of 0.961 suggests that the fit is very good.
##
## What is the risk of losing max 10 %? =< 0.9 percent
## What is the risk of losing max 25 %? =< 0 percent
## What is the risk of losing max 50 %? =< 0 percent
## What is the risk of losing max 90 %? =< 0 percent
## What is the risk of losing max 99 %? =< 0 percent
##
## What is the chance of gaining min 10 %? >= 46.1 percent
## What is the chance of gaining min 25 %? >= 1.2 percent
## What is the chance of gaining min 50 %? >= 0 percent
## What is the chance of gaining min 90 %? >= 0 percent
## What is the chance of gaining min 99 %? >= 0 percent
```

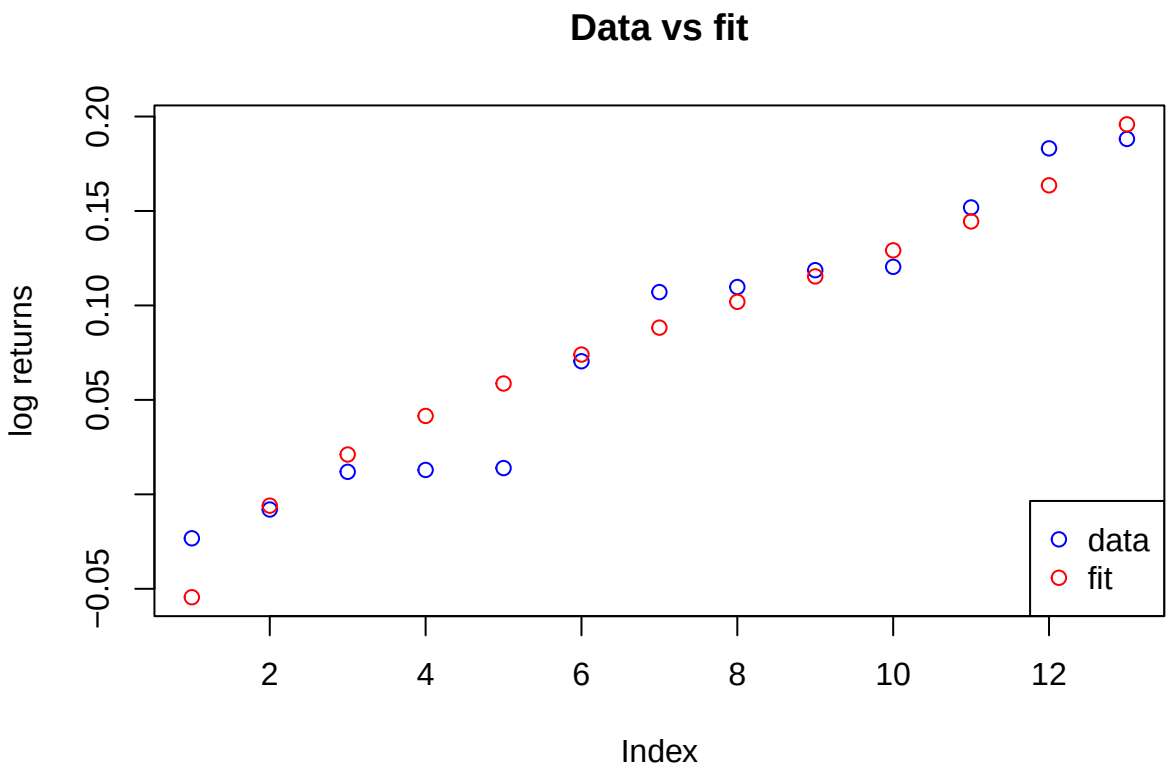
QQ Plot



The qq plot looks good Returns for mixed medium risk portfolios seems to be consistent with a skewed t-distribution.

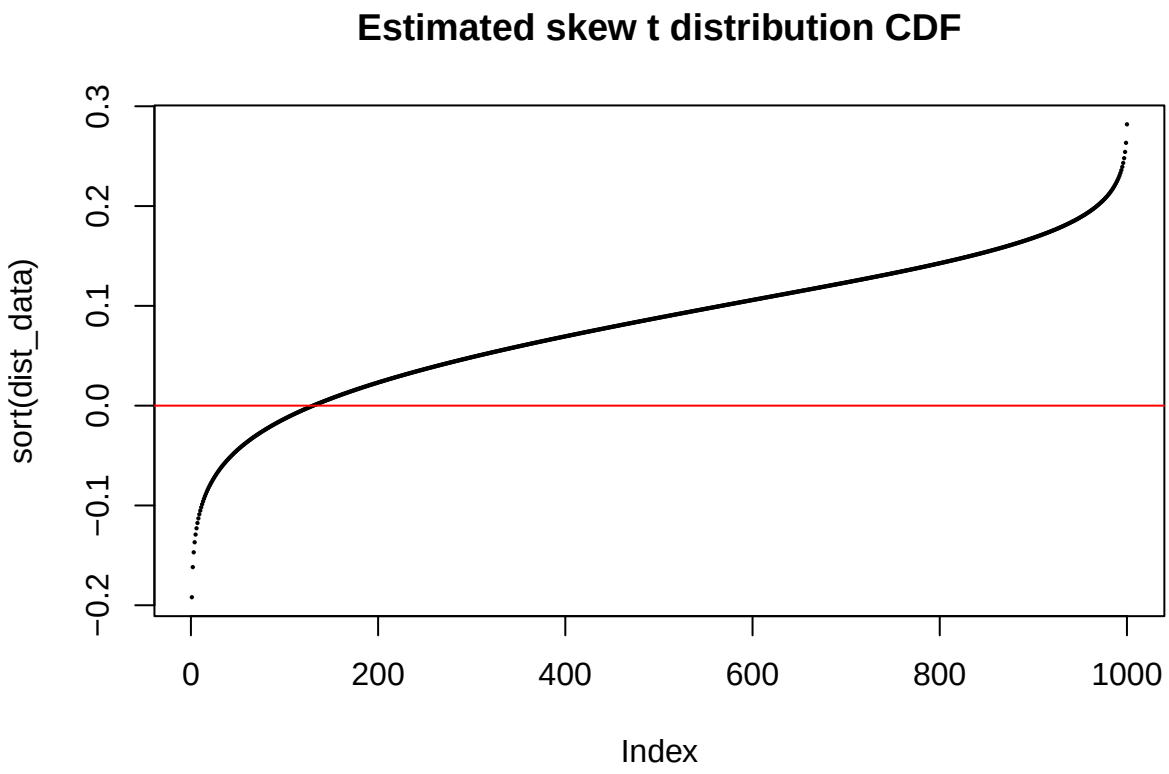
Data vs fit

Let's plot the fit and the observed returns together.



Estimated distribution

Now lets look at the CDF of the estimated distribution for each 0.1% increment between 0.5% and 99.5% for the estimated distribution:

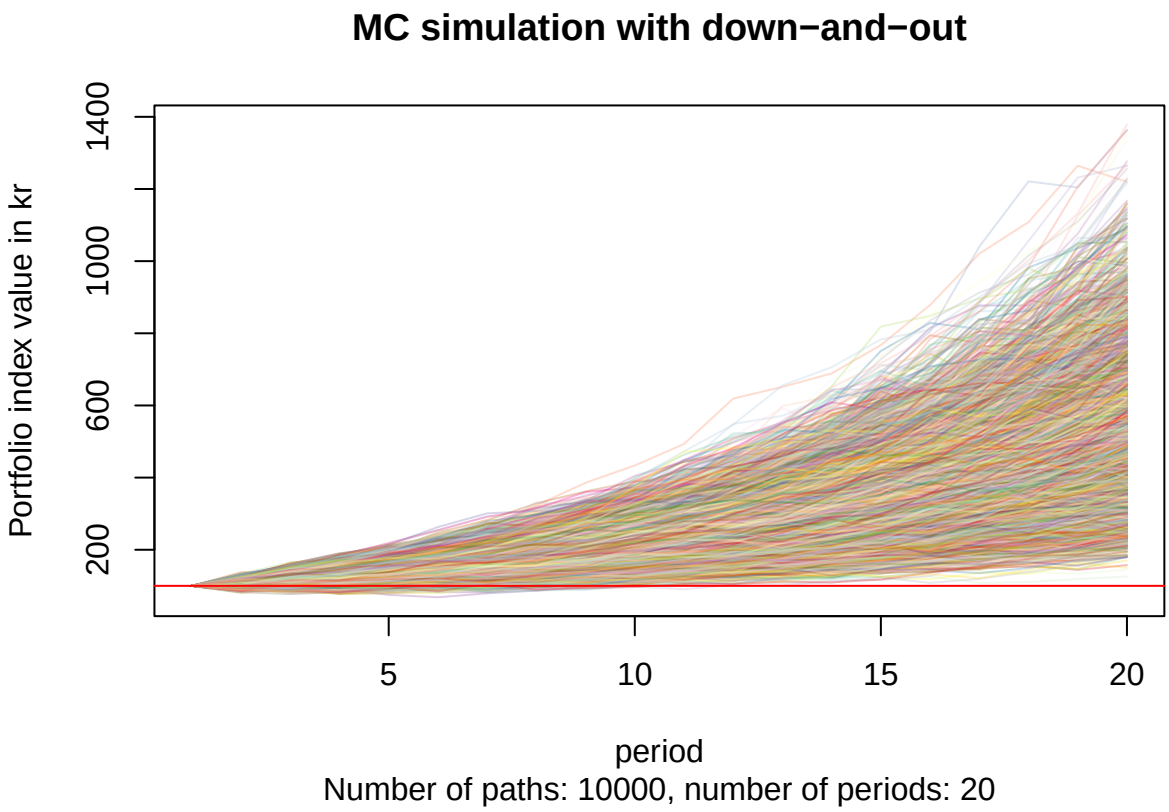


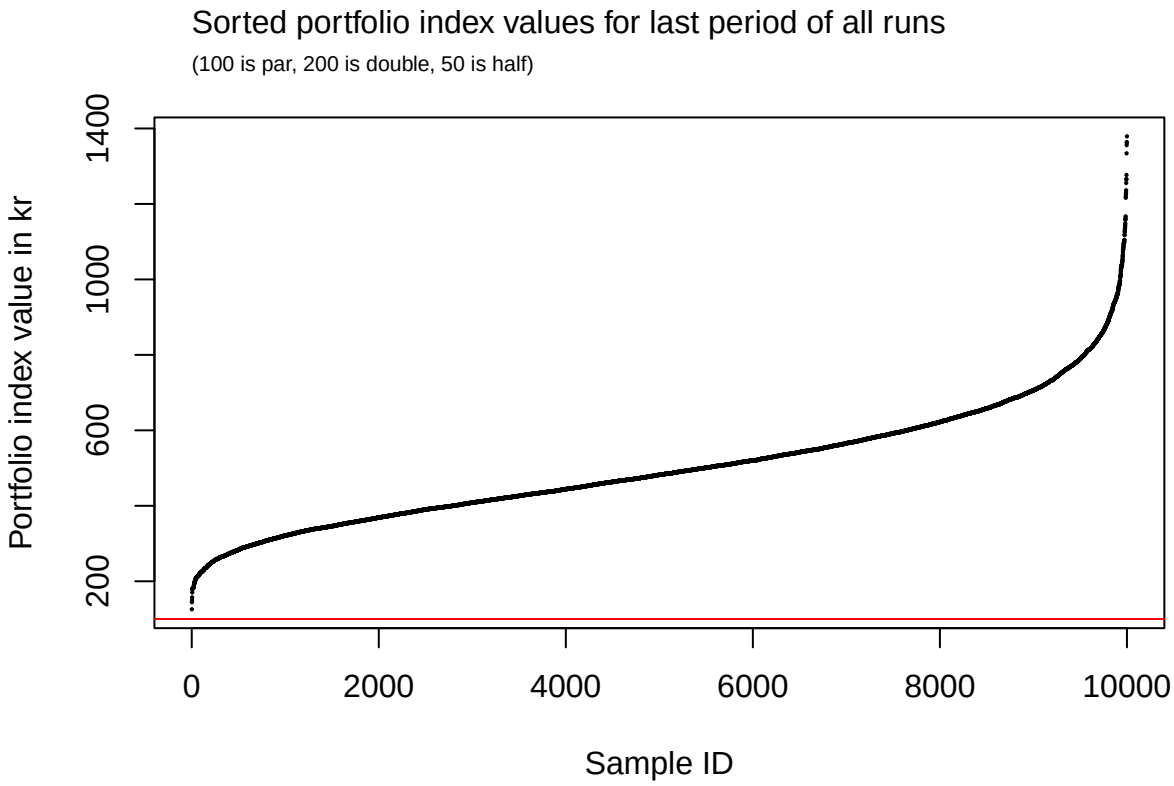
We see that the high risk mix provides a much better upside and smaller downside.

Monte Carlo

Version a: Simulation from estimated distribution of returns of mix.

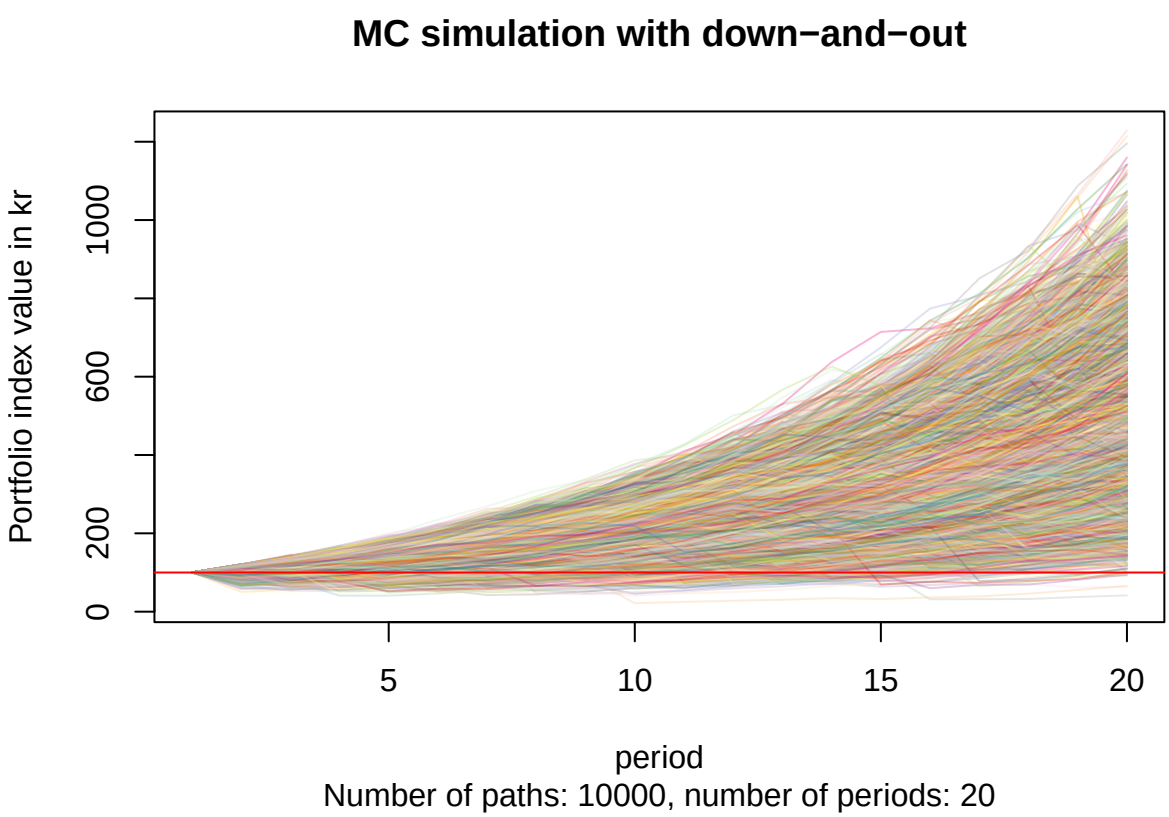
```
## Down-and-out simulation:
## Probability of down-and-out: 0 percent
##
## Mean portfolio index value after 20 years: 501.907 kr.
## SD of portfolio index value after 20 years: 157.577 kr.
## Min total portfolio index value after 20 years: 125.868 kr.
## Max total portfolio index value after 20 years: 1379.149 kr.
##
## Share of paths finishing below 100: 0 percent
```

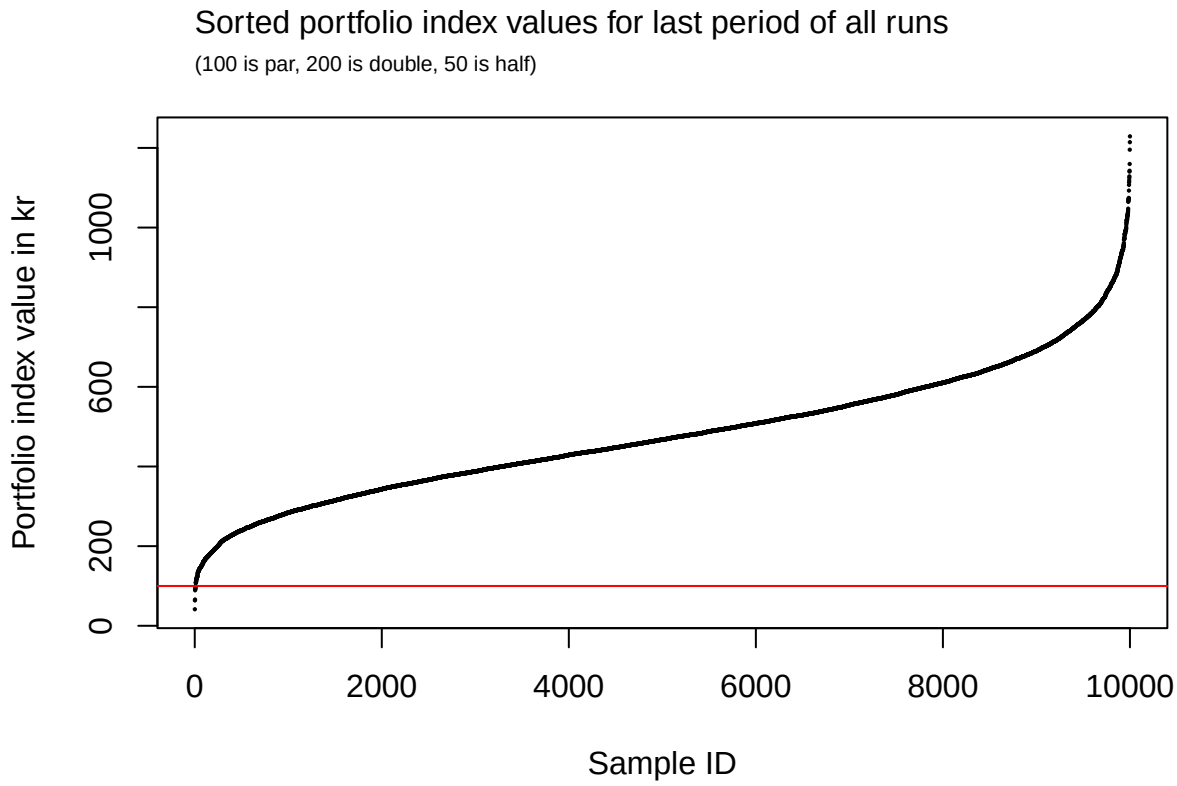

MC simulation with down−and−out
Portfolio index value in kr
200
600
1000
1400
5
10
15
20
period
Number of paths: 10000, number of periods: 20



Version b: Mix of simulations from estimated distribution of returns from individual funds.

```
## Down-and-out simulation:
## Probability of down-and-out: 0 percent
##
## Mean portfolio index value after 20 years: 480.383 kr.
## SD of portfolio index value after 20 years: 161.43 kr.
## Min total portfolio index value after 20 years: 41.563 kr.
## Max total portfolio index value after 20 years: 1229.022 kr.
##
## Share of paths finishing below 100: 0.09 percent
```


MC simulation with down−and−out
Portfolio index value in kr
0
200
600
1000
5
10
15
20
period
Number of paths: 10000, number of periods: 20



Compare pension plans

Risk of max loss of x percent for a single period (year).
x values are row names.

	Velliv_medium	Velliv_medium_long	Velliv_high	PFA_medium	PFA_high	mix_medium	mix_high
0	21.3	18.2	19.9	12.2	14.3	12.7	13.0
5	12.5	9.6	12.8	6.0	8.6	6.2	4.2
10	7.4	5.4	8.3	3.3	5.3	3.3	0.9
25	1.8	1.3	2.5	0.9	1.4	0.7	0.0
50	0.2	0.2	0.4	0.2	0.2	0.1	0.0
90	0.0	0.0	0.0	0.0	0.0	0.0	0.0
99	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Chance of min gains of x percent for a single period (year).
x values are row names.

	Velliv_medium	Velliv_medium_long	Velliv_high	PFA_medium	PFA_high	mix_medium	mix_high
0	78.7	81.8	80.1	87.8	85.7	87.3	87.0
5	63.8	64.9	69.2	71.5	75.8	71.4	69.9
10	41.0	36.2	53.3	32.7	59.6	35.6	46.1
25	0.0	0.3	0.0	0.1	0.0	0.0	1.2
50	0.0	0.0	0.0	0.0	0.0	0.0	0.0
100	0.0	0.0	0.0	0.0	0.0	0.0	0.0

MC risk percentiles: Risk of loss from first to last period.
_a is simulation from estimated distribution of returns of mix.
_b is mix of simulations from estimated distribution of returns from individual funds.

_m is medium.
_h is high.

	Velliv_m	Velliv_m_long	Velliv_h	PFA_m	PFA_h	mix_m_a	mix_h_a	mix_m_b	mix_h_b
0	5.23	3.50	3.83	1.88	1.07	0.86	0	0.30	0.09
5	4.47	3.07	3.57	1.64	1.00	0.80	0	0.26	0.07
10	3.78	2.67	3.29	1.38	0.85	0.72	0	0.21	0.04
25	2.45	1.82	2.32	0.95	0.53	0.50	0	0.09	0.03
50	0.80	0.95	1.12	0.48	0.25	0.27	0	0.03	0.01
90	0.07	0.11	0.18	0.14	0.02	0.07	0	0.00	0.00
99	0.01	0.04	0.05	0.04	0.01	0.01	0	0.00	0.00

MC gains percentiles: Chance of gains from first to last period.
_a is simulation from estimated distribution of returns of mix.
_b is mix of simulations from estimated distribution of returns from individual funds.

	Velliv_m	Velliv_m_long	Velliv_h	PFA_m	PFA_h	mix_m_a	mix_h_a	mix_m_b	mix_h_b
0	94.77	96.50	96.17	98.12	98.93	99.14	100.00	99.70	99.91
5	94.11	95.99	95.64	97.95	98.81	99.01	100.00	99.63	99.90
10	93.39	95.55	95.21	97.64	98.70	98.80	100.00	99.55	99.87
25	90.96	93.97	93.98	96.92	98.35	98.18	100.00	99.07	99.72
50	85.94	90.27	91.11	95.21	97.34	96.56	99.98	97.64	99.30
100	71.94	78.42	83.59	88.74	94.72	89.64	99.66	90.24	97.55
200	40.45	44.40	64.86	60.66	85.31	59.78	92.97	49.80	87.55
300	16.42	17.27	44.67	23.31	71.33	22.38	72.11	12.01	67.13
400	5.08	4.83	28.25	4.41	55.56	4.11	45.23	1.36	41.89
500	1.35	1.15	17.00	0.39	39.13	0.25	23.51	0.04	21.85
1000	0.00	0.01	0.71	0.00	2.54	0.00	0.29	0.00	0.11

Summary statistics

Fit summary

Summary for fit of log returns to an F-S skew standardized Student-t distribution.
m is the location parameter.
s is the scale parameter.
nu is the estimated degrees of freedom, or shape parameter.
xi is the estimated skewness parameter.

	Velliv_medium	Velliv_medium_long	Velliv_high	PFA_medium	PFA_high	mix_medium	mix_high
m	0.048	0.052	0.065	0.058	0.084	0.059	0.082
s	0.120	0.115	0.150	0.123	0.121	0.088	0.071
nu	3.304	2.706	3.144	2.265	3.185	2.773	89.863
xi	0.034	0.505	0.002	0.477	0.018	0.029	0.770
R-squared	0.993	0.978	0.991	0.991	0.964	0.890	0.961

Monte Carlo simulations summary

Monte Carlo simulations of portfolio index values (currency values).
Statistics are given for the final state of all paths.
Probability of down-and_out is calculated as the share of paths that reach 0 at some point. All subsequent values for a path are set to 0, if the path reaches at any point.
0 is defined as any value below a threshold.
losing_prob_pct is the probability of losing money. This is calculated as the share of paths finishing below index 100.

```
## Number of paths: 10000
```

	Velliv_m	Velliv_m_long	Velliv_h	PFA_m	PFA_h	mix_m_a	mix_m_b	mix_h_a	mix_h_b
mc_m	280.880	293.213	405.165	325.490	554.957	324.992	302.778	501.907	480.383
mc_s	124.121	117.487	216.392	104.032	245.019	98.529	81.813	157.577	161.430
mc_min	0.000	0.000	0.000	0.000	0.000	0.170	28.038	125.868	41.563
mc_max	905.803	1200.760	1539.705	859.686	1745.465	689.566	685.083	1379.149	1229.022
dao_prob_pct	0.010	0.010	0.030	0.020	0.010	0.000	0.000	0.000	0.000
losing_prob_pct	5.230	3.500	3.830	1.880	1.070	0.860	0.300	0.000	0.090

```
## Highest mean    : PFA_h

## Lowest sd       : mix_m_b

## Highest min     : mix_h_a

## Highest max     : PFA_h

## Lowest dao prob : mix_m_a mix_m_b mix_h_a mix_h_b

## Lowest loss prob: mix_h_a
```

Appendix

Average of returns vs returns of average

Math

$$\text{Avg. of returns} := \frac{\left(\frac{x_t}{x_{t-1}} + \frac{y_t}{y_{t-1}}\right)}{2}$$

$$\text{Returns of avg.} := \left(\frac{x_t + y_t}{2}\right) \Big/ \left(\frac{x_{t-1} + y_{t-1}}{2}\right) \equiv \frac{x_t + y_t}{x_{t-1} + y_{t-1}}$$

For which x_1 and y_1 are Avg. of returns $=$ Returns of avg.?

$$\frac{\left(\frac{x_t}{x_{t-1}} + \frac{y_t}{y_{t-1}}\right)}{2} = \frac{x_t + y_t}{x_{t-1} + y_{t-1}}$$

$$\frac{x_t}{x_{t-1}} + \frac{y_t}{y_{t-1}} = 2\frac{x_t + y_t}{x_{t-1} + y_{t-1}}$$

$$\frac{x_t y_{t-1} + x_{t-1} y_t}{x_{t-1} y_{t-1}} = 2\frac{x_t + y_t}{x_{t-1} + y_{t-1}}$$

$$(x_{t-1} + y_{t-1}) x_t y_{t-1} + (x_{t-1} + y_{t-1}) x_{t-1} y_t = 2(x_{t-1} y_{t-1} x_t + x_{t-1} y_{t-1} y_t)$$

$$(x_{t-1} x_1 y_{t-1} + y_{t-1} x_t y_{t-1}) + (x_{t-1} x_{t-1} y_t + x_{t-1} y_{t-1} y_t) = 2(x_{t-1} y_{t-1} x_t + x_{t-1} y_{t-1} y_t)$$

This is not generally true, but true if for instance $x_{t-1} = y_{t-1}$.

Example

Definition: `R = 1+r`

```
## Let x_0 be 100.

## Let y_0 be 200.

## So the initial value of the pf is 300 .
```

```
## Let R_x be 0.5.

## Let R_y be 1.5.
```

Then,

```
## x_1 is R_x * x_0 = 50.

## y_1 is R_y * y_0 = 300.
```

Average of returns:

```
## 0.5 * (R_x + R_y) = 1
```

So here the value of the pf at t=1 should be unchanged from t=0:

```
## (x_0 + y_0) * 0.5 * (R_x + R_y) = 300
```

But this is clearly not the case:

```
## 0.5 * (x_1 + y_1) = 0.5 * (R_x * x_0 + R_y * y_0) = 175
```

Therefore we should take returns of average, not average of returns!

Let's take the average of log returns instead:

```
## 0.5 * (log(R_x) + log(R_y)) = -0.143841
```

We now get:

```
## (x_0 + y_0) * exp(0.5 * (log(Rx) + log(Ry))) = 259.8076
```

So taking the average of log returns doesn't work either.

Simulation of mix vs mix of simulations

Test if a simulation of a mix (average) of two returns series has the same distribution as a mix of two simulated returns series.

We are adding annual returns rather than multiplying, so imagine that we are simulating log returns.

```
## m(data_x): -0.03622681
## s(data_x): 0.3548631
## m(data_y): 9.341106
## s(data_y): 3.514933
##
## m(data_x + data_y): 4.652439
## s(data_x + data_y): 1.747942
```

m and s of final state of all paths.
_a is mix of simulated returns.
_b is simulated mixed returns.

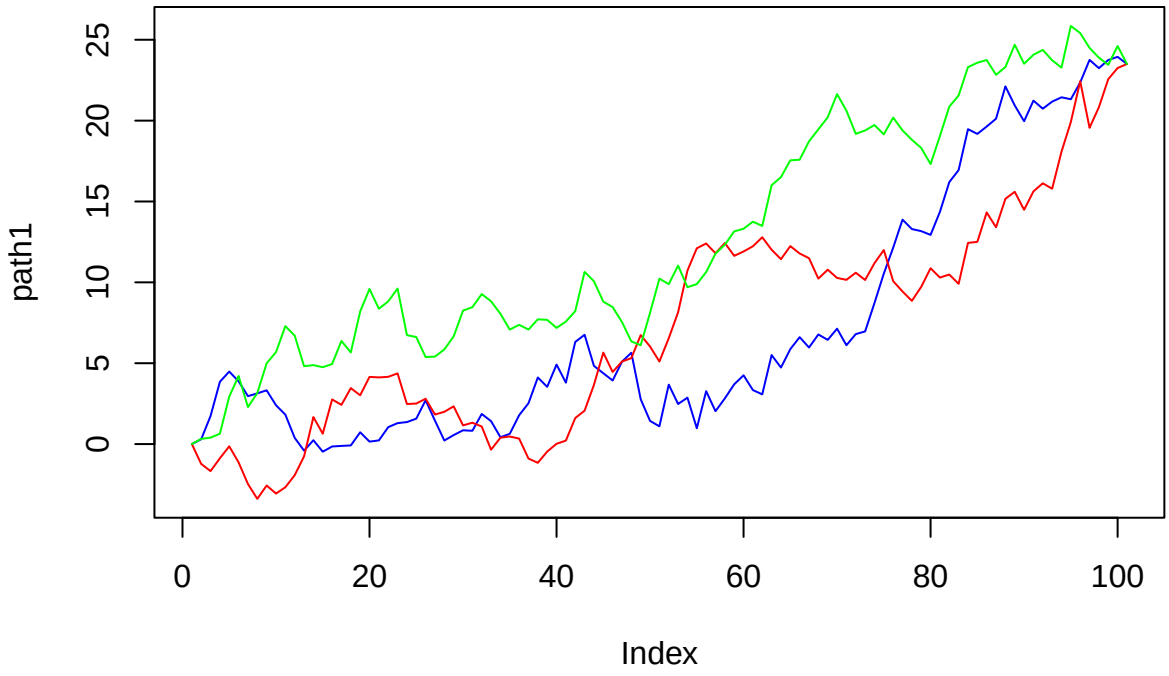
m_a	m_b	s_a	s_b
92.826	92.801	8.143	7.710
92.946	93.262	8.173	7.538
92.938	92.691	7.905	7.723
92.944	92.862	8.026	7.830
93.374	93.071	7.793	7.946
92.650	93.297	7.894	7.809

m_a	m_b	s_a	s_b
93.285	92.678	7.898	8.092
92.879	93.543	7.934	7.870
93.071	93.322	8.013	7.596
92.633	92.926	7.779	7.781

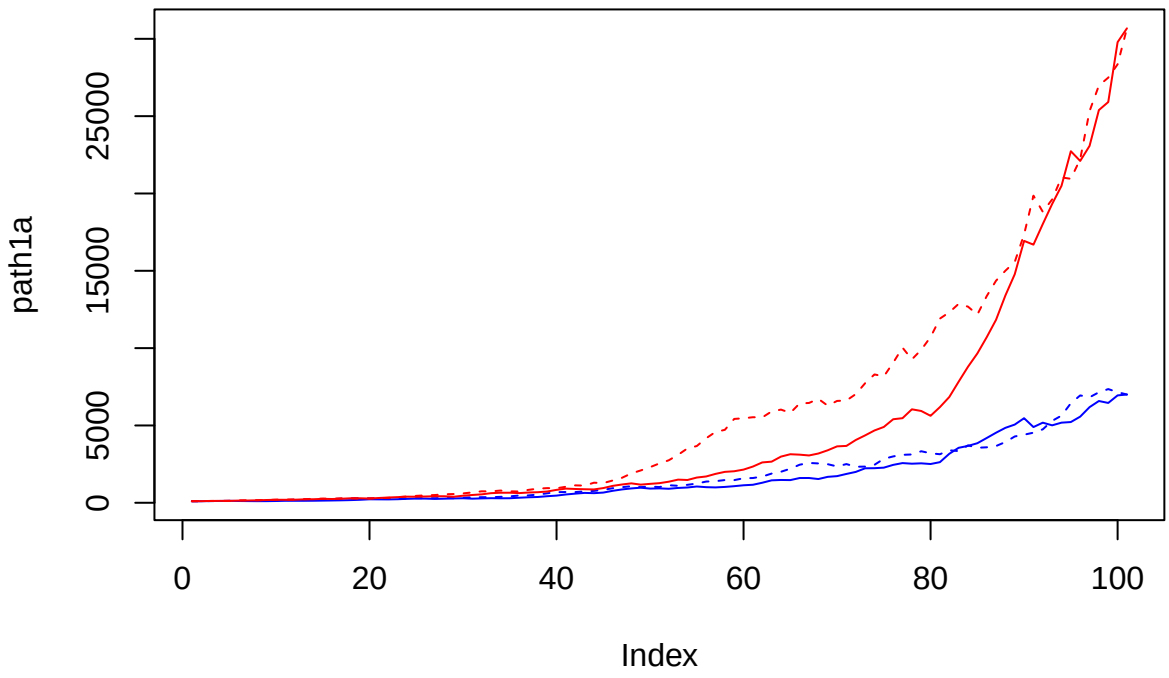
```
##       m_a              m_b              s_a              s_b        
##  Min.   :92.63   Min.   :92.68   Min.   :7.779   Min.   :7.538  
##  1st Qu.:92.84   1st Qu.:92.82   1st Qu.:7.895   1st Qu.:7.714  
##  Median :92.94   Median :93.00   Median :7.919   Median :7.795  
##  Mean   :92.95   Mean   :93.05   Mean   :7.956   Mean   :7.789  
##  3rd Qu.:93.04   3rd Qu.:93.29   3rd Qu.:8.023   3rd Qu.:7.860  
##  Max.   :93.37   Max.   :93.54   Max.   :8.173   Max.   :8.092
```

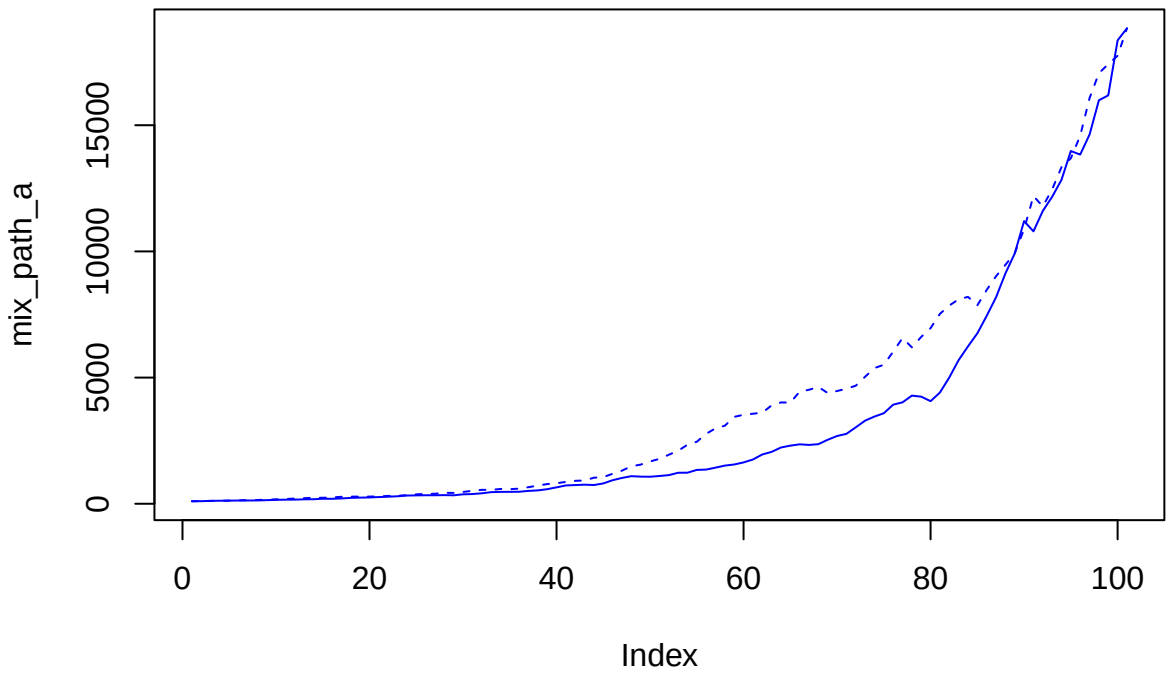
_a and _b are very close to equal.
We attribute the differences to differences in estimating the distributions in version a and b.

The final state is independent of the order of the preceding steps:



So does the order of the steps in the two processes matter, when mixing simulated returns?


path1a
0
5000
15000
25000
0
20
40
60
80
100
Index



The order of steps in the individual paths do not matter, because the mix of simulated paths is a sum of a sum, so the order of terms doesn't affect the sum. If there is variation it is because the sets preceding steps are not the same. For instance, the steps between step 1 and 60 in the plot above are not the same for the two lines.

Recall,

$$\text{Var}(aX + bY) = a^2\text{Var}(X) + b^2\text{Var}(Y) + 2ab\text{Cov}(a, b)$$

```
## [1] 0.005355618
```

```
## [1] 0.005355618
```

Our distribution estimate is based on 13 observations. Is that enough for a robust estimate? What if we suddenly hit a year like 2008? How would that affect our estimate?
Let's try to include the Velliv data from 2007-2010.
We do this by sampling 13 observations from `vmrl`.

```
##        m                s
##  Min.   :0.06075   Min.   :0.05159
##  1st Qu.:0.06723   1st Qu.:0.06077
##  Median :0.07007   Median :0.06554
##  Mean   :0.07048   Mean   :0.06818
##  3rd Qu.:0.07296   3rd Qu.:0.07548
##  Max.   :0.08385   Max.   :0.09100
```