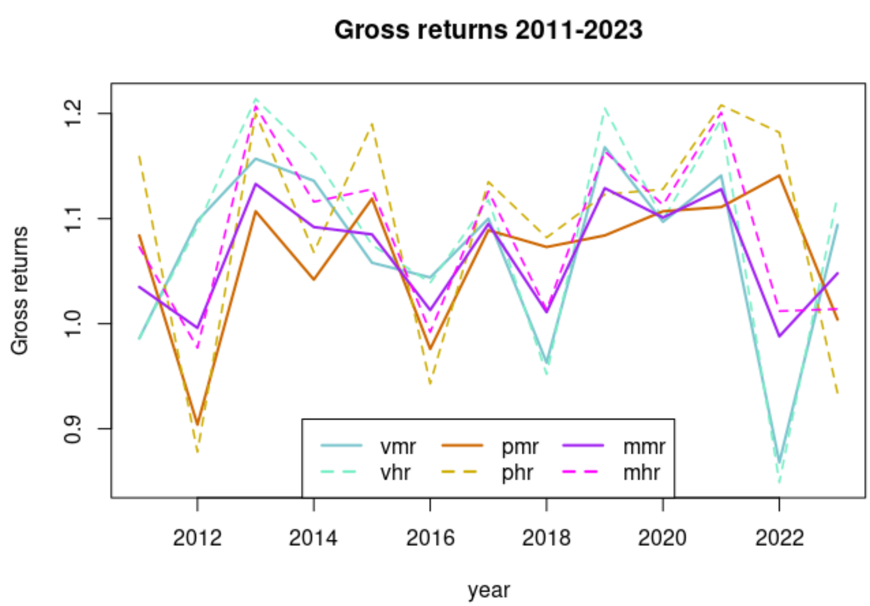
Pension returns analysis

22:21 26 April 2024

Fit log returns to F-S skew standardized Student-t distribution.
`m` is the location parameter.
`s` is the scale parameter.
`nu` is the estimated shape parameter (degrees of freedom).
`xi` is the estimated skewness parameter.

Log returns data 2011-2023.

For 2011, medium risk data is used in the high risk data set, as no high risk fund data is available prior to 2012.
`vmrl` is a long version of Velliv medium risk data, from 2007 to 2023. For 2007 to 2011 (both included) no high risk data is available.



Summary of gross returns

```
##       vmr             vhr             pmr             phr
##  Min.   :0.868   Min.   :0.849   Min.   :0.904   Min.   :0.878
```

```
##  1st Qu.:1.044   1st Qu.:1.039   1st Qu.:1.042   1st Qu.:1.068
##  Median :1.097   Median :1.099   Median :1.084   Median :1.128
##  Mean   :1.070   Mean   :1.085   Mean   :1.065   Mean   :1.095
##  3rd Qu.:1.136   3rd Qu.:1.160   3rd Qu.:1.107   3rd Qu.:1.182
##  Max.   :1.168   Max.   :1.214   Max.   :1.141   Max.   :1.208
##       mmr             mhr
##  Min.   :0.988   Min.   :0.977
##  1st Qu.:1.013   1st Qu.:1.013
##  Median :1.085   Median :1.113
##  Mean   :1.066   Mean   :1.087
##  3rd Qu.:1.101   3rd Qu.:1.128
##  Max.   :1.133   Max.   :1.207
```

```
##       vmrl
##  Min.   :0.801
##  1st Qu.:1.013
##  Median :1.085
##  Mean   :1.061
##  3rd Qu.:1.128
##  Max.   :1.193
```

```
##            vmr   vhr   pmr   phr   mmr   mhr
## Min.   : 0.868 0.849 0.904 0.878 0.988 0.977
## 1st Qu.: 1.044 1.039 1.042 1.068 1.013 1.013
## Median : 1.097 1.099 1.084 1.128 1.085 1.113
## Mean   : 1.070 1.085 1.065 1.095 1.066 1.087
## 3rd Qu.: 1.136 1.160 1.107 1.182 1.101 1.128
## Max.   : 1.168 1.214 1.141 1.208 1.133 1.207
```

Ranking

Min. :	ranking	1st Qu.:	ranking	Median :	ranking	Mean :	ranking	3rd Qu.:	ranking	Max. :	ranking
0.988	mmr	1.068	phr	1.128	phr	1.095	phr	1.182	phr	1.214	vhr
0.977	mhr	1.044	vmr	1.113	mhr	1.087	mhr	1.160	vhr	1.208	phr
0.904	pmr	1.042	pmr	1.099	vhr	1.085	vhr	1.136	vmr	1.207	mhr
0.878	phr	1.039	vhr	1.097	vmr	1.070	vmr	1.128	mhr	1.168	vmr
0.868	vmr	1.013	mmr	1.085	mmr	1.066	mmr	1.107	pmr	1.141	pmr
0.849	vhr	1.013	mhr	1.084	pmr	1.065	pmr	1.101	mmr	1.133	mmr

Covariance

```
## cov(vmr, pmr) =  -0.001094875
```

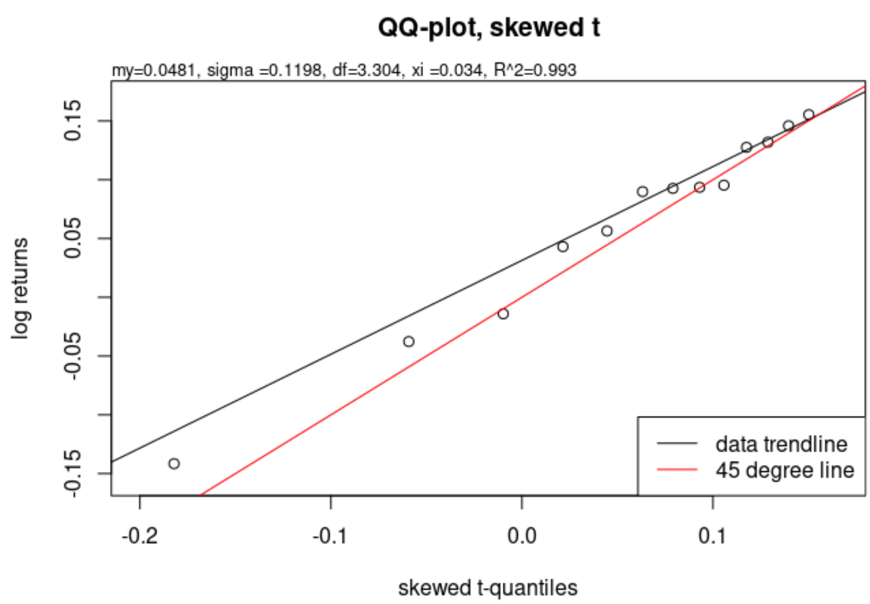
```
## cov(vhr, phr) =  -0.0001730651
```

Velliv medium risk, 2011 - 2023

```
##
## AIC: -27.8497
## BIC: -25.58991
## m: 0.0480931
```

```
## s: 0.1198426
## nu (df): 3.303595
## xi: 0.03361192
## R^2: 0.993
##
## An R^2 of 0.993 suggests that the fit is extremely good.
##
## What is the risk of losing max 10 %? =< 0 percent
## What is the risk of losing max 25 %? =< 0 percent
## What is the risk of losing max 50 %? =< 0 percent
## What is the risk of losing max 90 %? =< 0 percent
## What is the risk of losing max 99 %? =< 0 percent
##
## What is the chance of gaining min 10 %? >= 63.16667 percent
## What is the chance of gaining min 25 %? >= 49.33333 percent
## What is the chance of gaining min 50 %? >= 40.16667 percent
## What is the chance of gaining min 90 %? >= 32.66667 percent
## What is the chance of gaining min 99 %? >= 31.5 percent
```

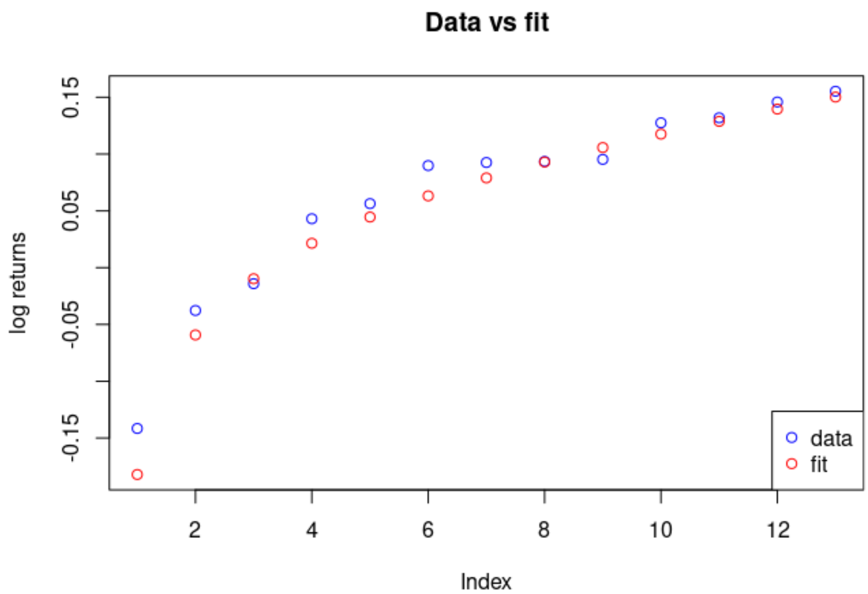
QQ Plot



The qq plot looks great. Log returns for Velliv medium risk seems to be consistent with a skewed t-distribution.

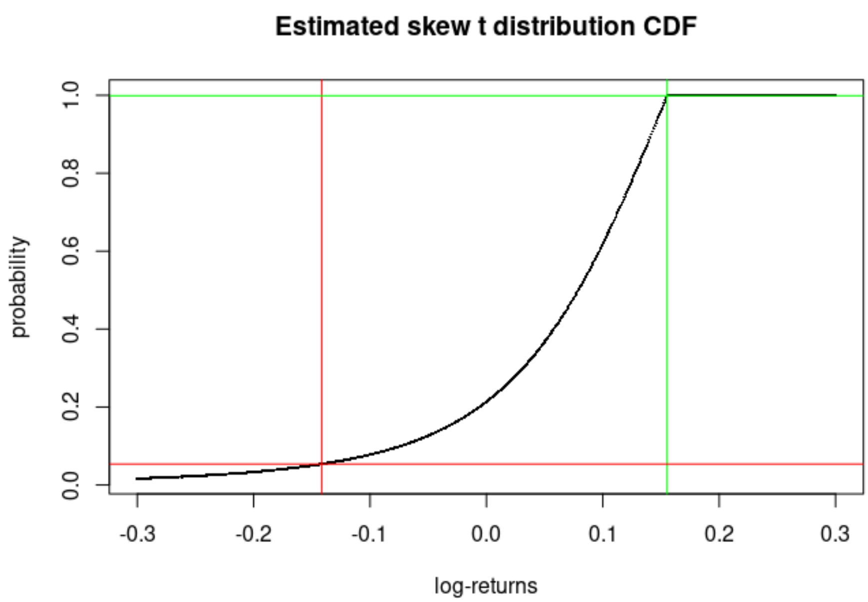
Data vs fit

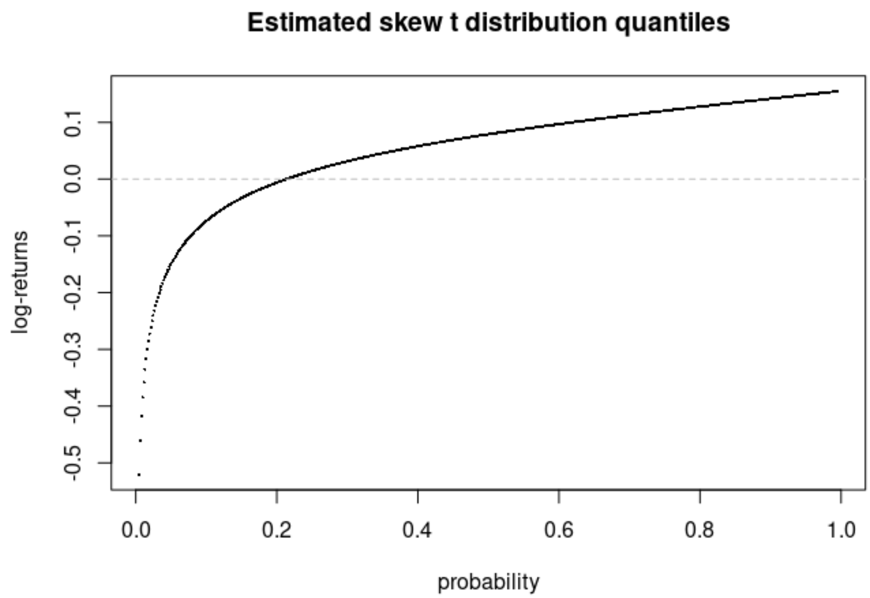
Let's plot the fit and the observed returns together.



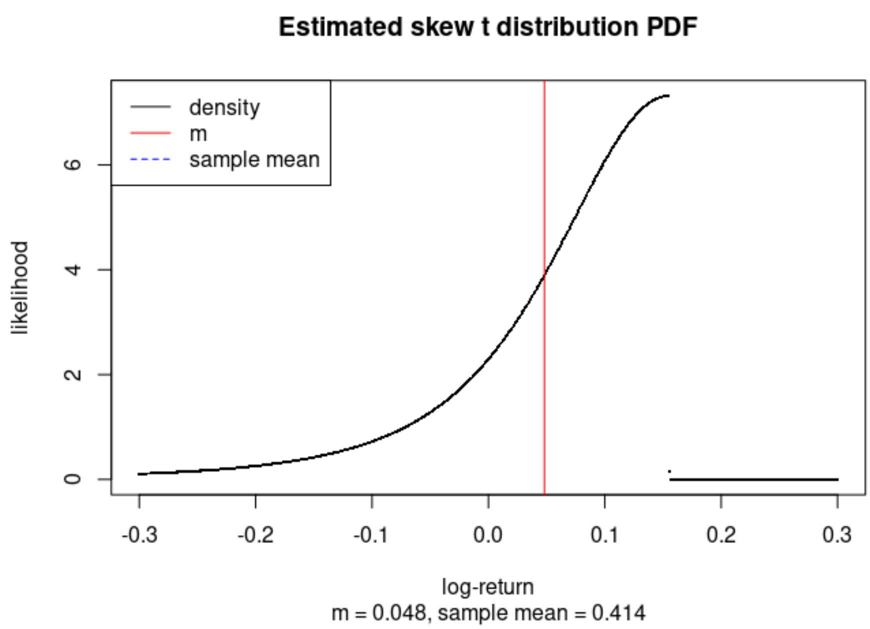
Estimated distribution

Now lets look at the CDF of the estimated distribution for each 0.1% increment between 0.5% and 99.5% for the estimated distribution:



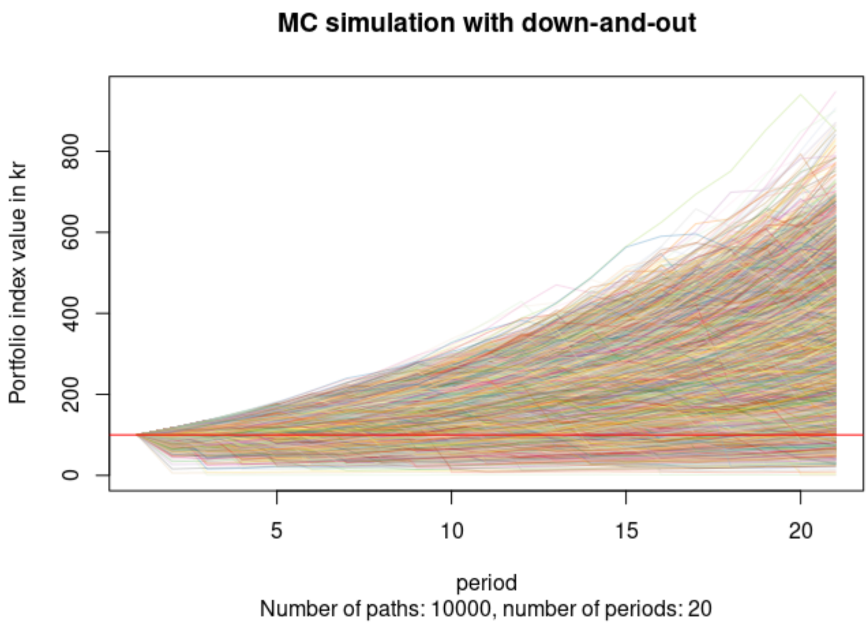


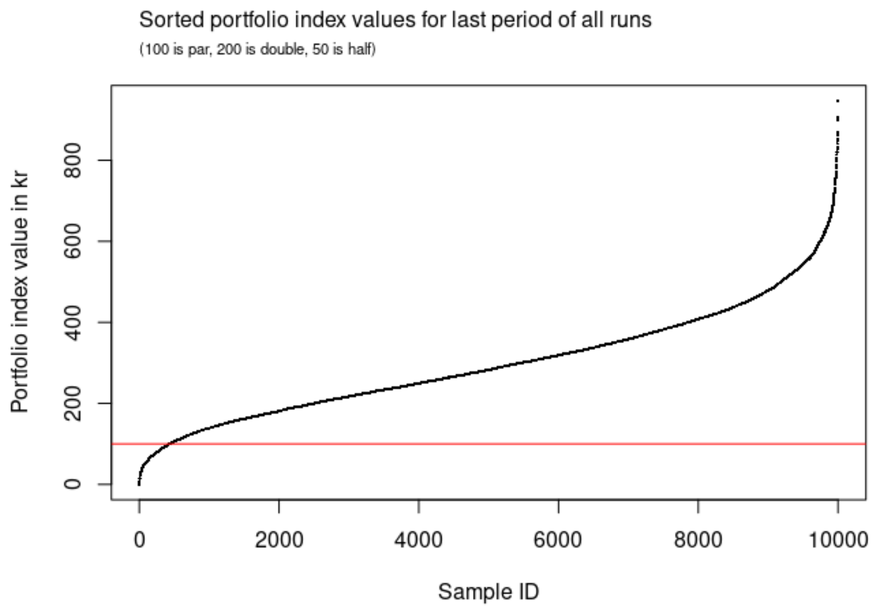
We see that for a few observations out of a 1000, the losses are disastrous, while the upside is very dampened.



Monte Carlo

```
## Down-and-out simulation:
## Probability of down-and-out: 0 percent
##
## Mean portfolio index value after 20 years: 278.995 kr.
## SD of portfolio index value after 20 years: 123.583 kr.
## Min total portfolio index value after 20 years: 9.812 kr.
## Max total portfolio index value after 20 years: 929.858 kr.
##
## Share of paths finishing below 100: 4.72 percent
```

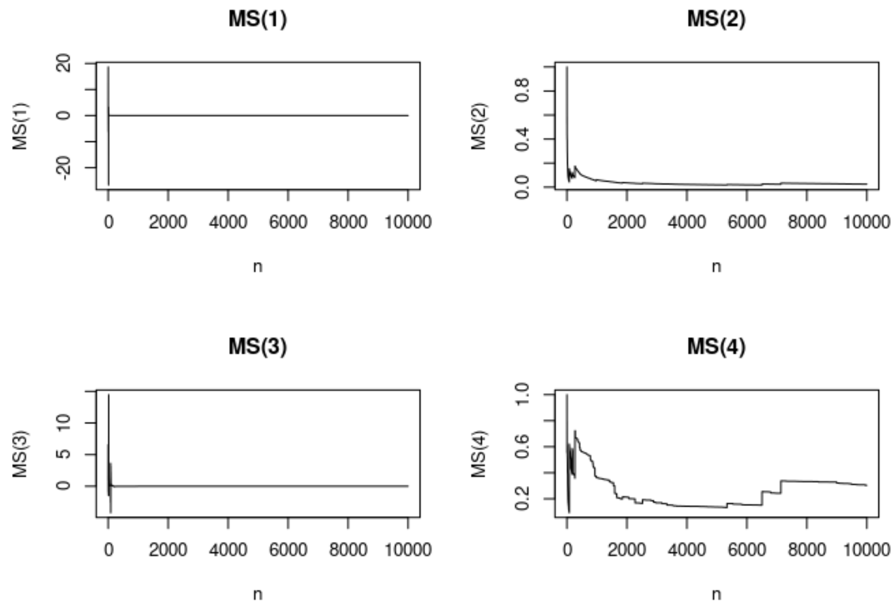




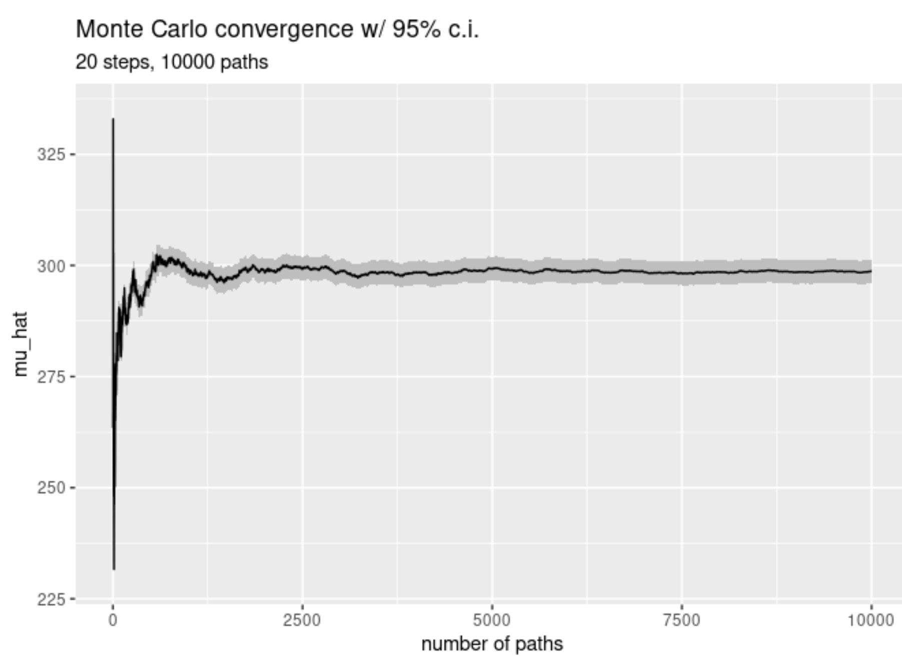
Convergence

Max vs sum

Max vs sum plots for the first four moments:


MS(1)
MS(2)
MS(3)
MS(4)
n

MC

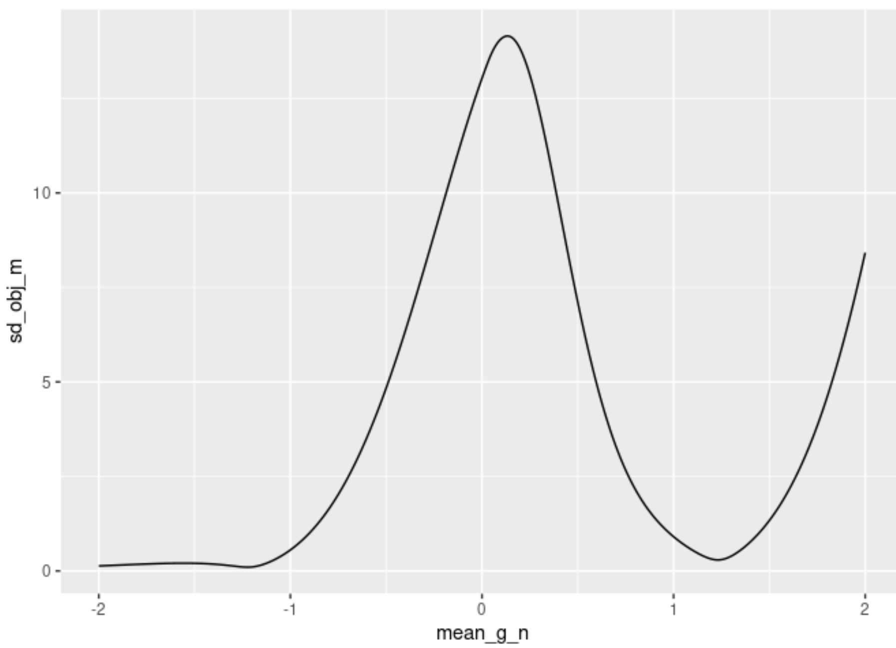

Monte Carlo convergence w/ 95% c.i.
20 steps, 10000 paths
mu_hat
number of paths

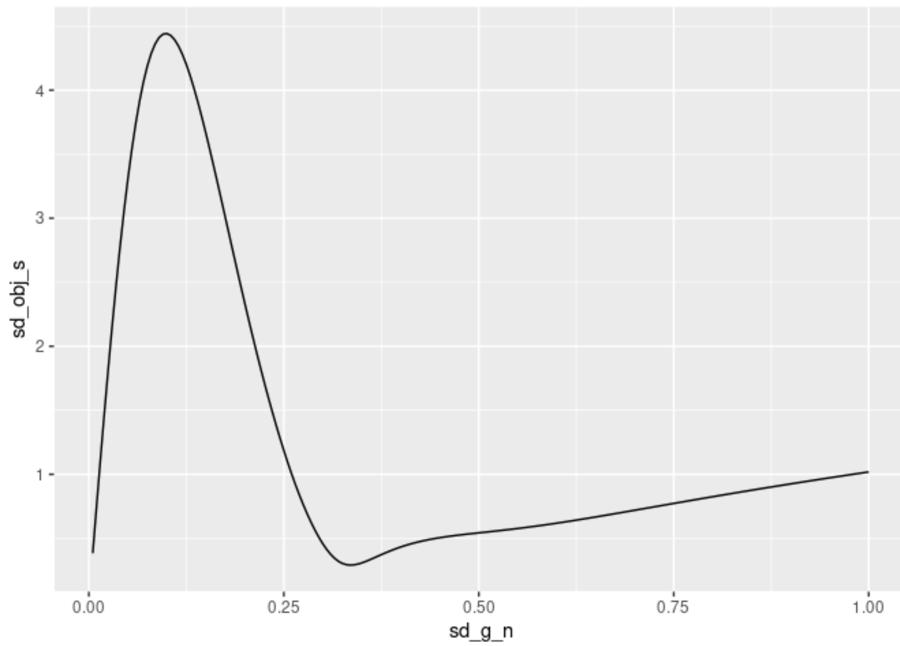
IS

Parameters

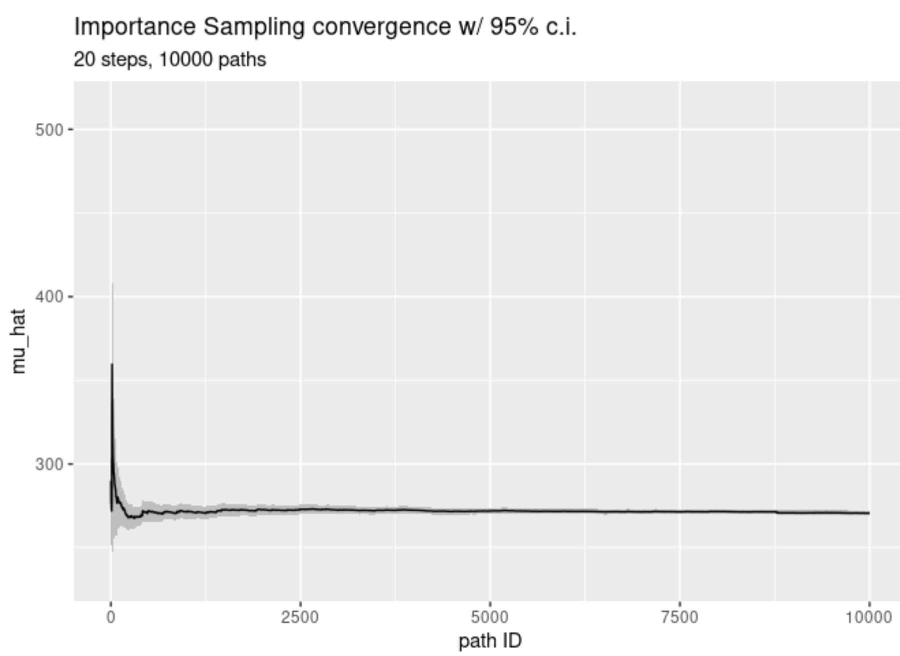
```
## [1] 1.1724769 0.3205692
```

Objective function plots





4
3
2
1
sd_obj_s
0.00
0.25
0.50
0.75
1.00
sd_g_n



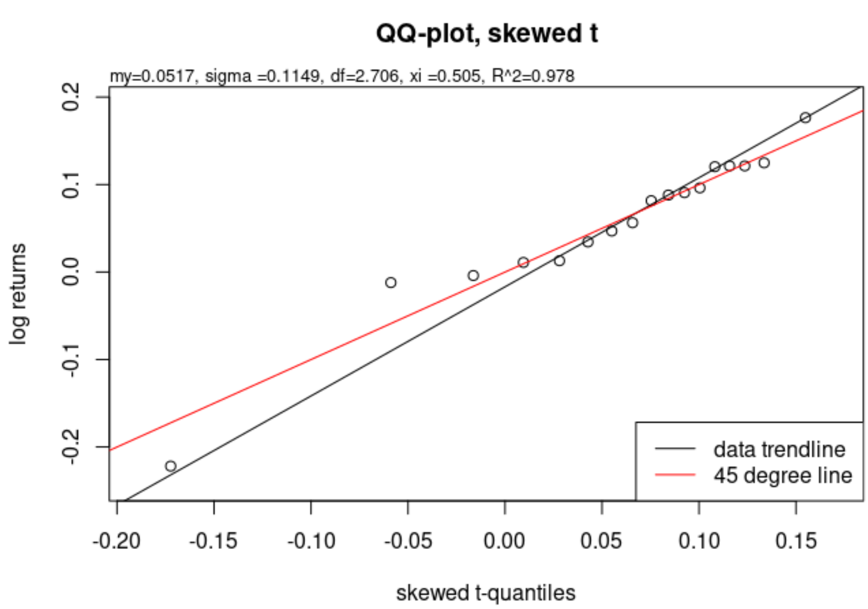
Importance Sampling convergence w/ 95% c.i.
20 steps, 10000 paths
500
400
300
mu_hat
0
2500
5000
7500
10000
path ID

Velliv medium risk, 2007 - 2023

Fit to skew t distribution

```
##
## AIC: -34.35752
## BIC: -31.02467
## m: 0.05171176
## s: 0.1149408
## nu (df): 2.706099
## xi: 0.5049945
## R^2: 0.978
##
## An R^2 of 0.978 suggests that the fit is very good.
##
## What is the risk of losing max 10 %? =< 0 percent
## What is the risk of losing max 25 %? =< 0 percent
## What is the risk of losing max 50 %? =< 0 percent
## What is the risk of losing max 90 %? =< 0 percent
## What is the risk of losing max 99 %? =< 0 percent
##
## What is the chance of gaining min 10 %? >= 58.66667 percent
## What is the chance of gaining min 25 %? >= 47.5 percent
## What is the chance of gaining min 50 %? >= 40.16667 percent
## What is the chance of gaining min 90 %? >= 34 percent
## What is the chance of gaining min 99 %? >= 33 percent
```

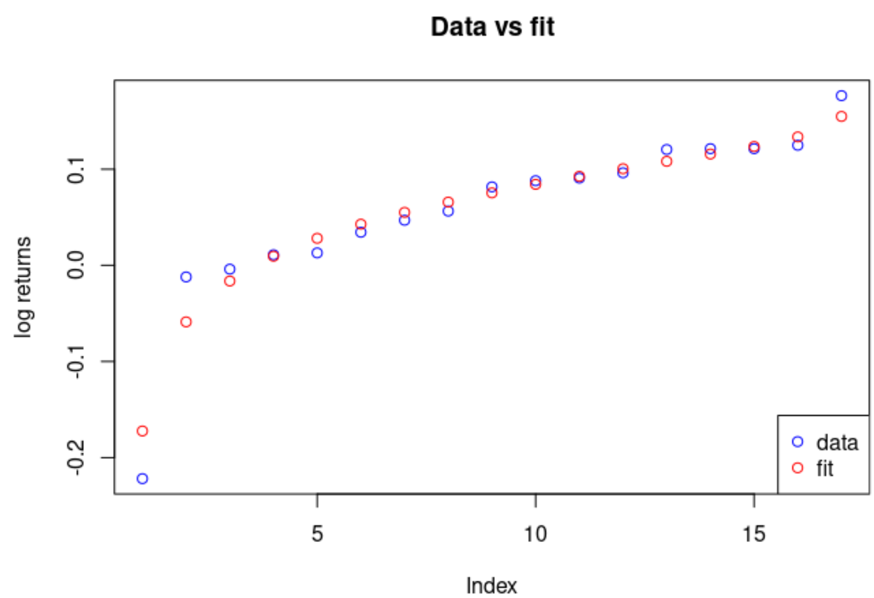
QQ Plot



The qq plot looks good. Log returns for Velliv high risk seems to be consistent with a skewed t-distribution.

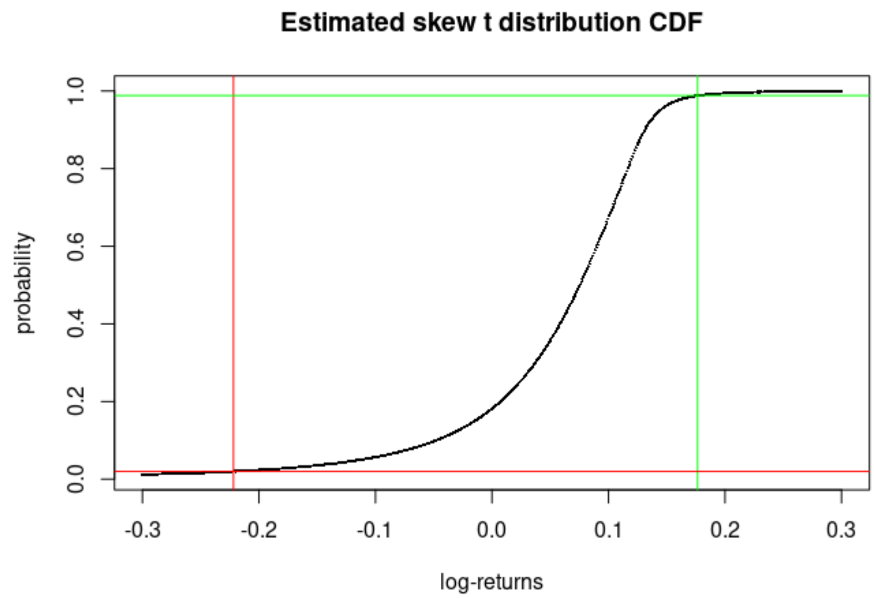
Data vs fit

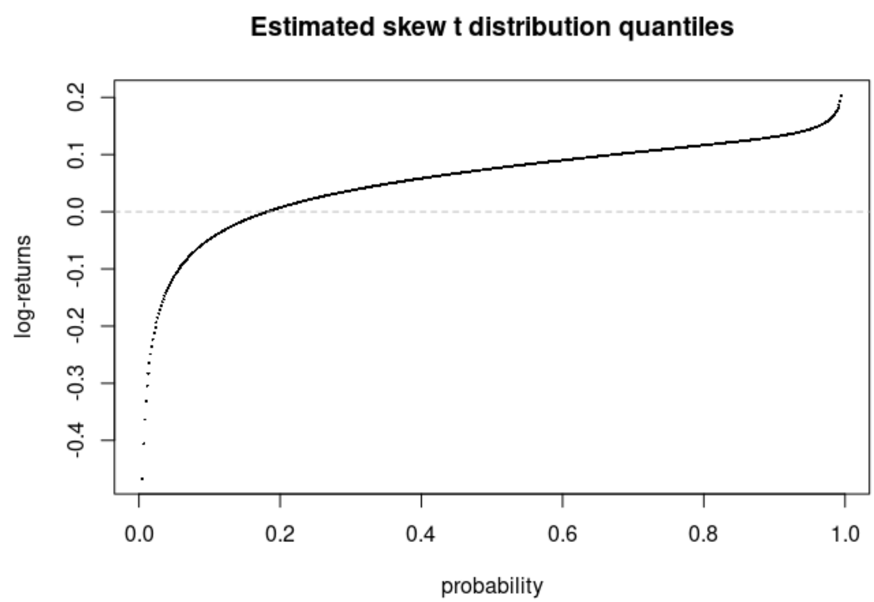
Let's plot the fit and the observed returns together.



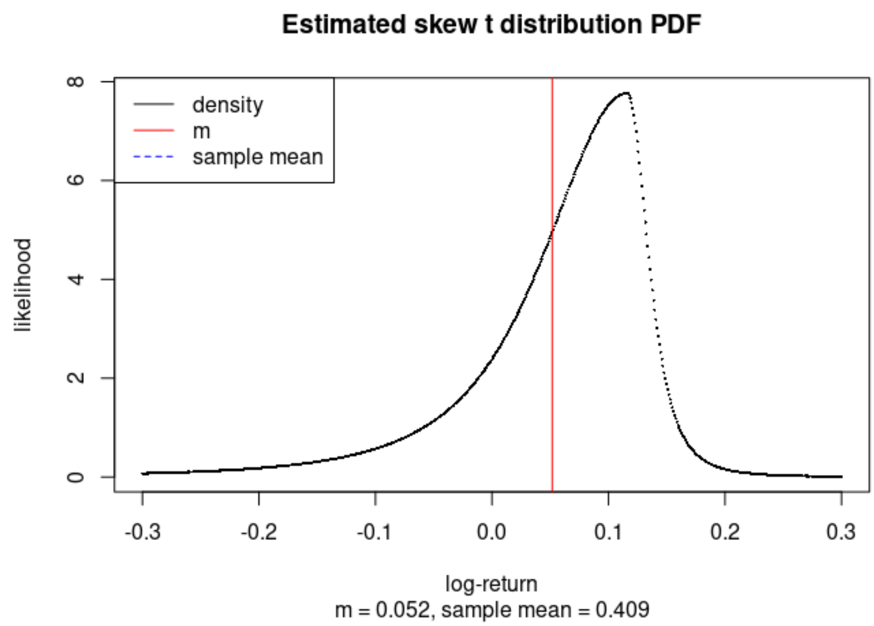
Estimated distribution

Now lets look at the CDF of the estimated distribution for each 0.1% increment between 0.5% and 99.5% for the estimated distribution:



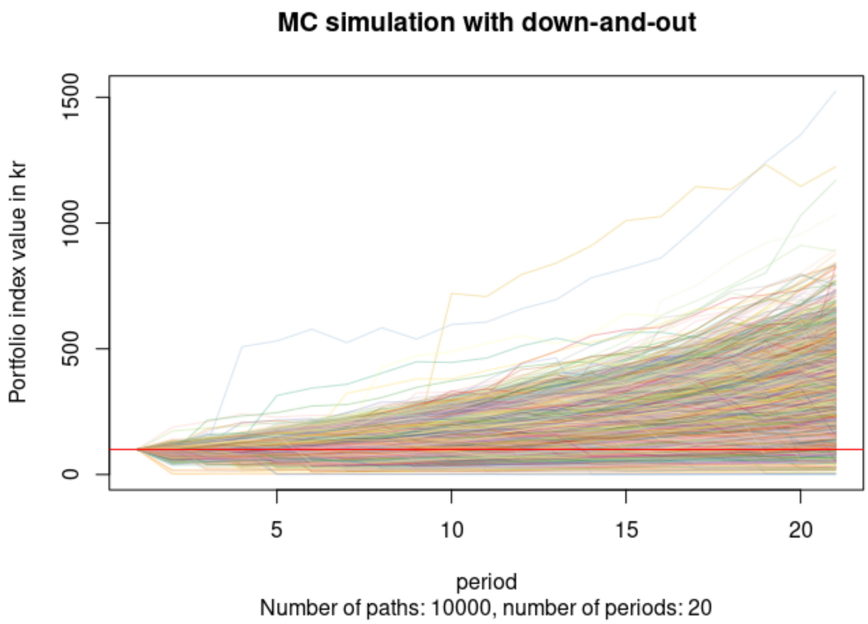


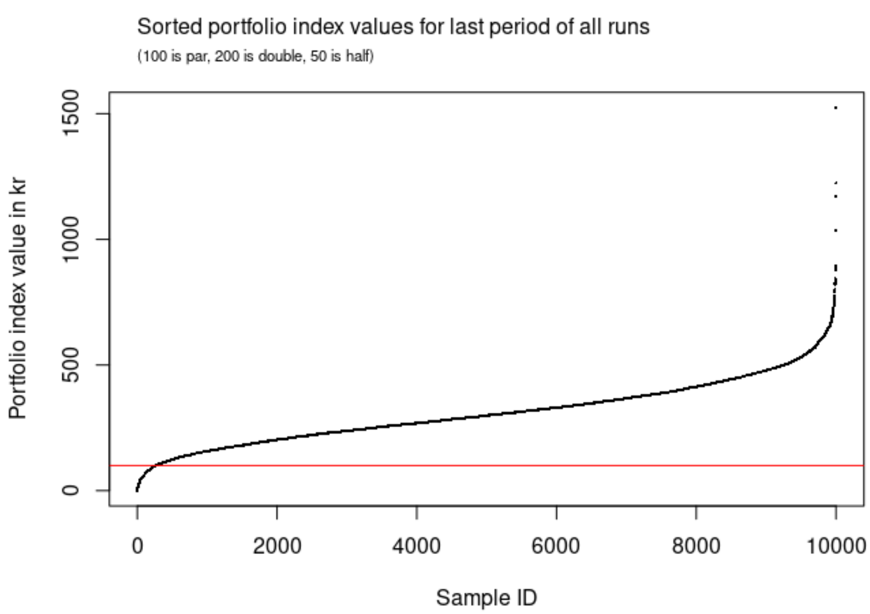
We see that for a few observations out of a 1000, the losses are disastrous, while the upside is very dampened. But because the disastrous loss in 2008 was followed by a large profit the following year, we see some increased upside for the top percentiles. Beware: A 1.2 return following a 0.8 return doesn't take us back where we were before the loss. Path dependency! So if returns more or less average out, but high returns have a tendency to follow high losses, that's bad!



Monte Carlo

```
## Down-and-out simulation:
## Probability of down-and-out: 0 percent
##
## Mean portfolio index value after 20 years: 293.384 kr.
## SD of portfolio index value after 20 years: 118.147 kr.
## Min total portfolio index value after 20 years: 0.785 kr.
## Max total portfolio index value after 20 years: 2029.133 kr.
##
## Share of paths finishing below 100: 3.11 percent
```

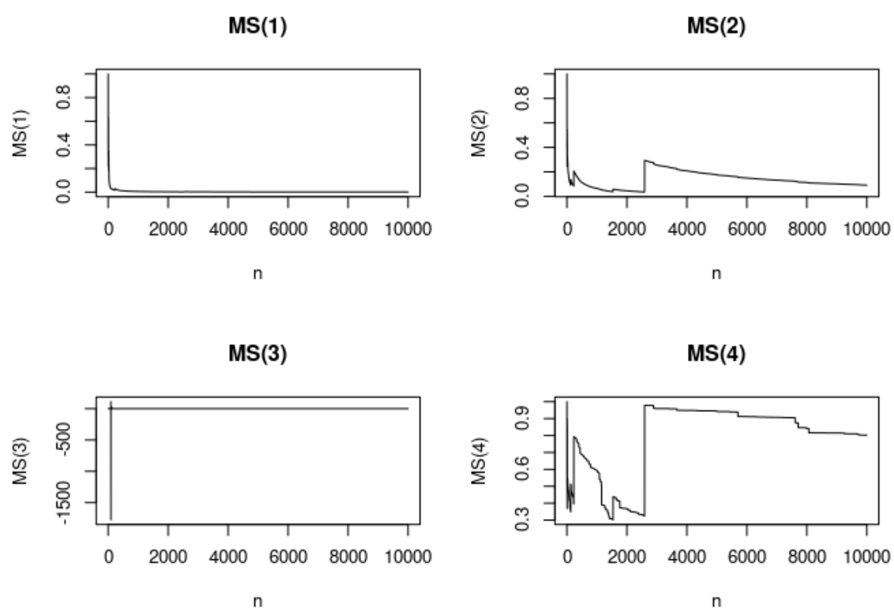

MC simulation with down-and-out
Portfolio index value in kr
0
500
1000
1500
5
10
15
20
period
Number of paths: 10000, number of periods: 20


Sorted portfolio index values for last period of all runs
(100 is par, 200 is double, 50 is half)
Portfolio index value in kr
0
500
1000
1500
0
2000
4000
6000
8000
10000
Sample ID

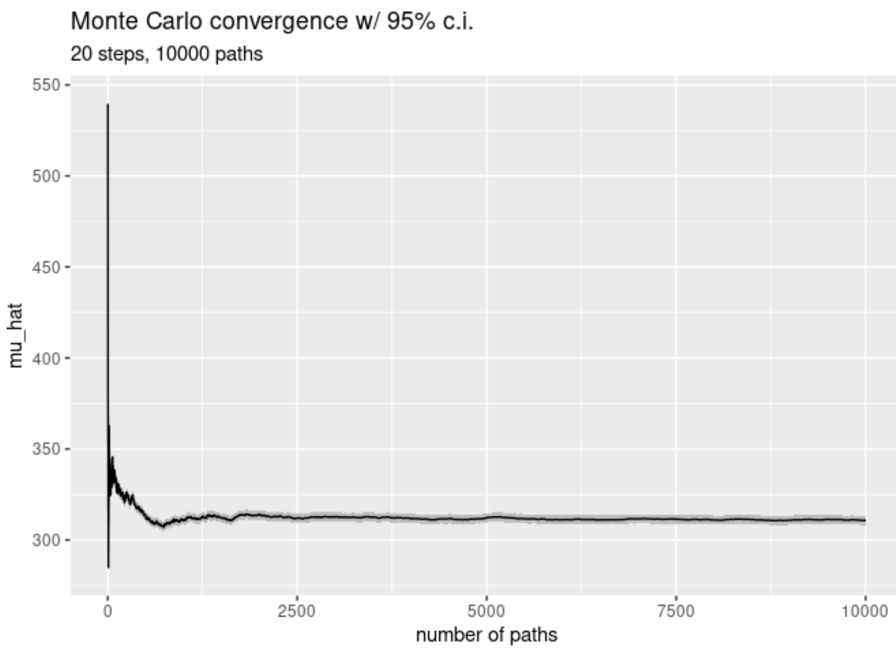
Convergence

Max vs sum

Max vs sum plots for the first four moments:



MC

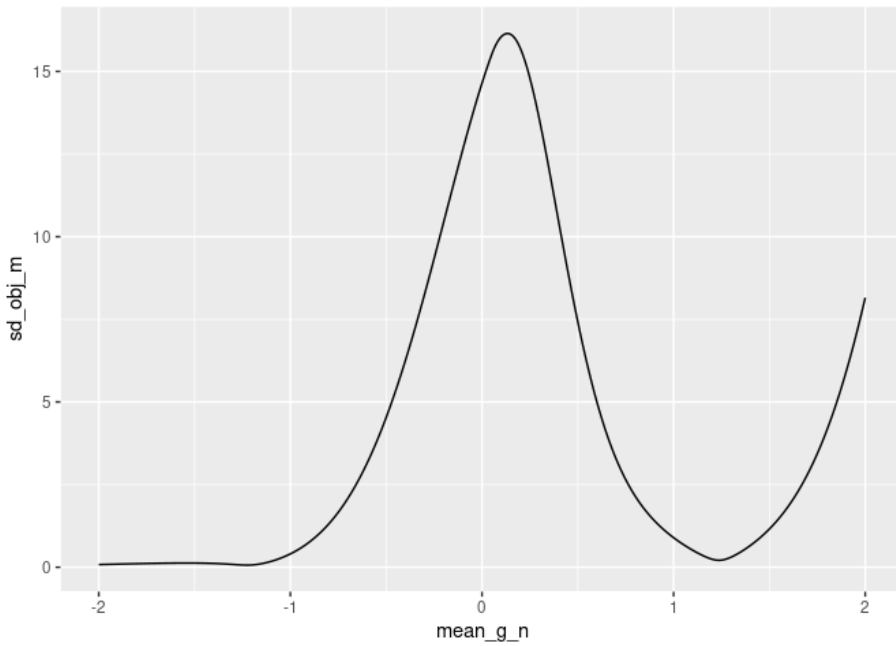


IS

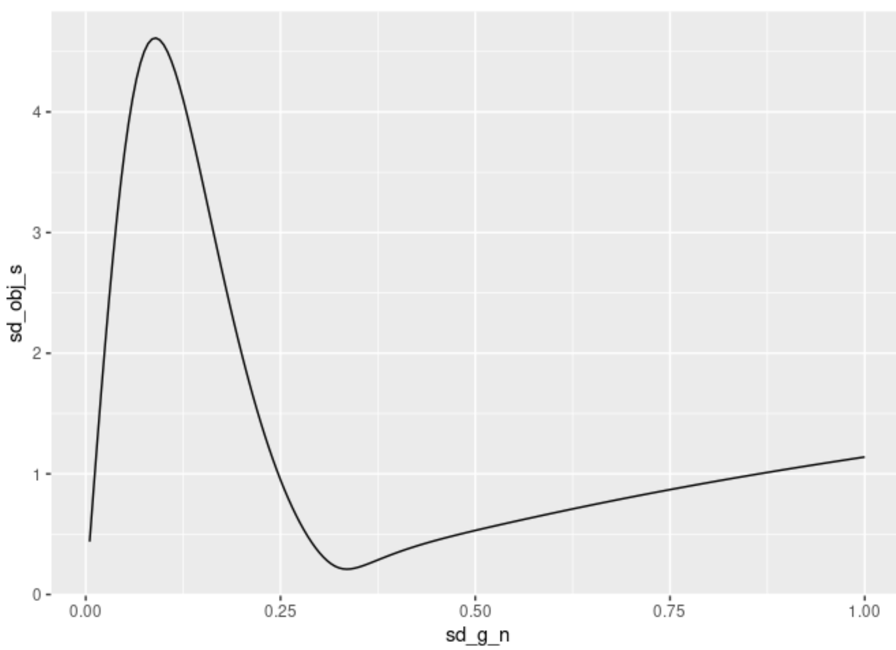
Parameters

```
## [1] 1.1842753 0.3193925
```

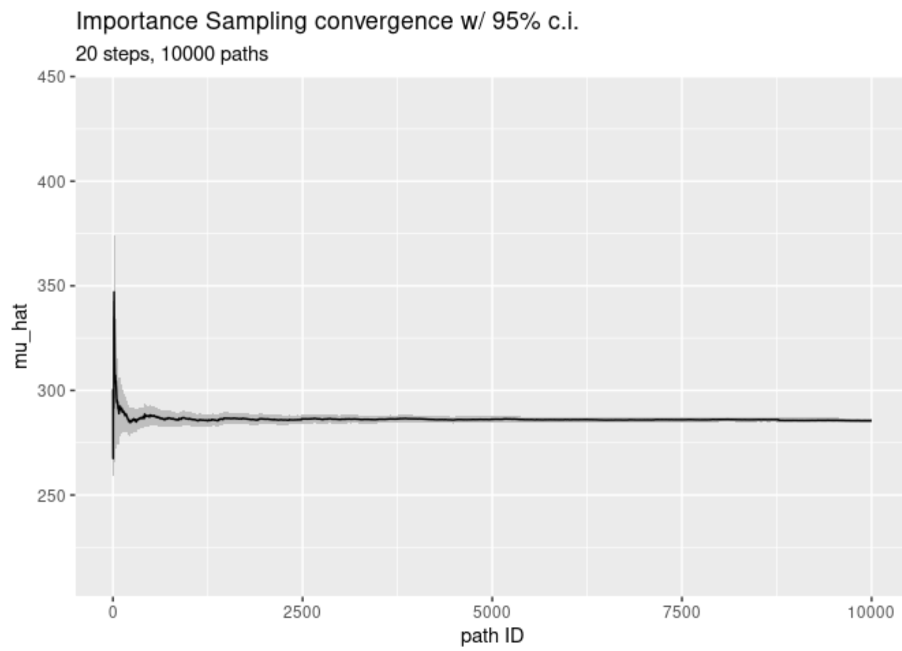
Objective function plots



15
10
5
0
sd_obj_m
-2
-1
0
1
2
mean_g_n



4
3
2
1
0
sd_obj_s
0.00
0.25
0.50
0.75
1.00
sd_g_n

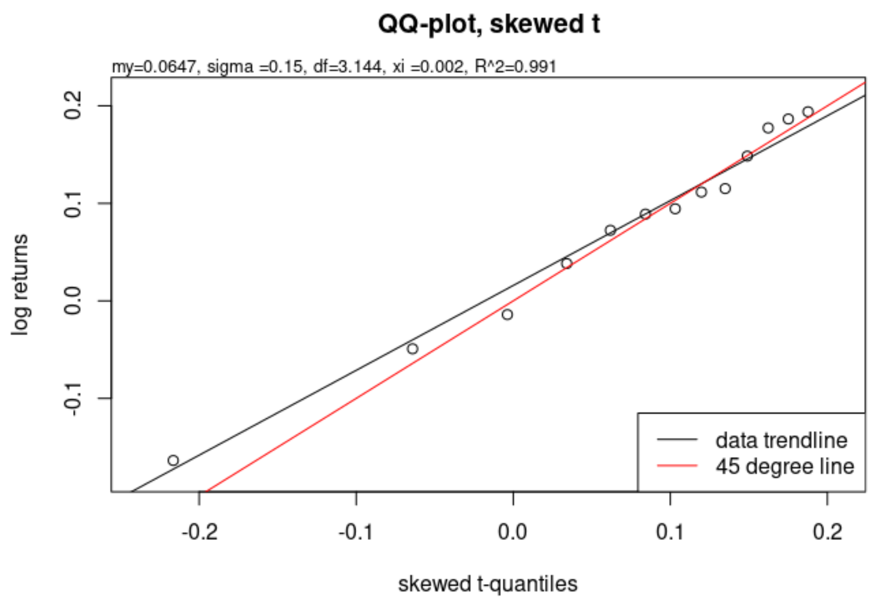


Velliv high risk, 2011 - 2023

Fit to skew t distribution

```
##
## AIC: -21.42488
## BIC: -19.16508
## m: 0.06471454
## s: 0.1499924
## nu (df): 3.144355
## xi: 0.002367034
## R^2: 0.991
##
## An R^2 of 0.991 suggests that the fit is extremely good.
##
## What is the risk of losing max 10 %? =< 0 percent
## What is the risk of losing max 25 %? =< 0 percent
## What is the risk of losing max 50 %? =< 0 percent
## What is the risk of losing max 90 %? =< 0 percent
## What is the risk of losing max 99 %? =< 0 percent
##
## What is the chance of gaining min 10 %? >= 64.66667 percent
## What is the chance of gaining min 25 %? >= 47.83333 percent
## What is the chance of gaining min 50 %? >= 36.83333 percent
## What is the chance of gaining min 90 %? >= 28 percent
## What is the chance of gaining min 99 %? >= 26.5 percent
```

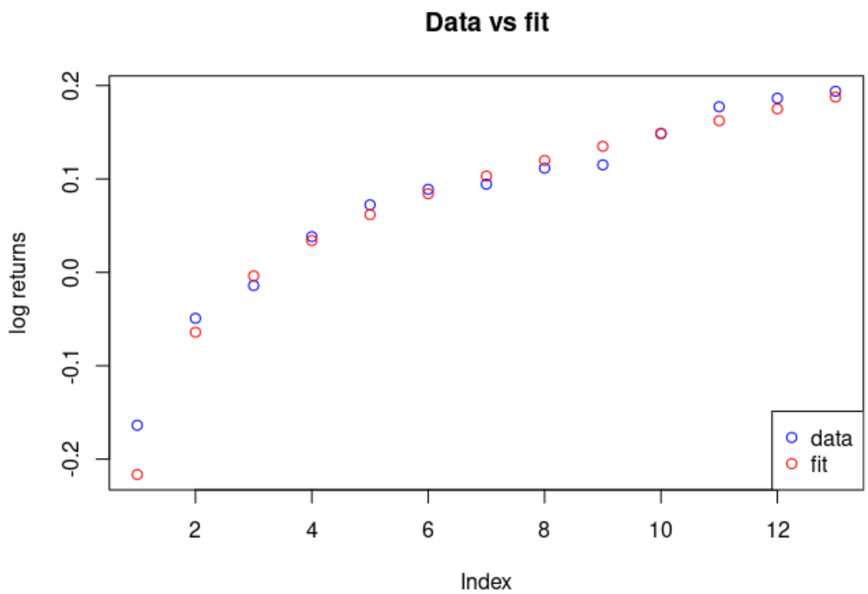
QQ Plot



The qq plot looks great. Returns for Velliv medium risk seems to be consistent with a skewed t-distribution.

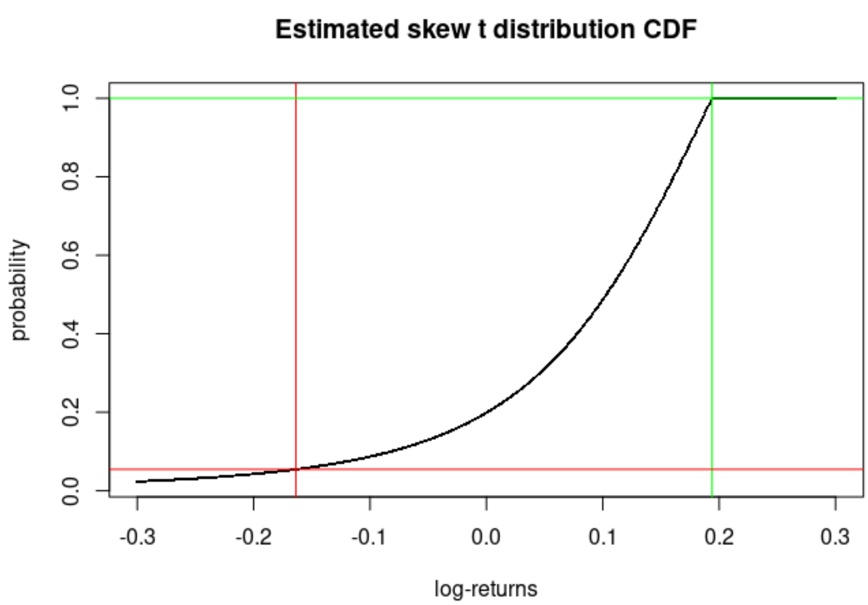
Data vs fit

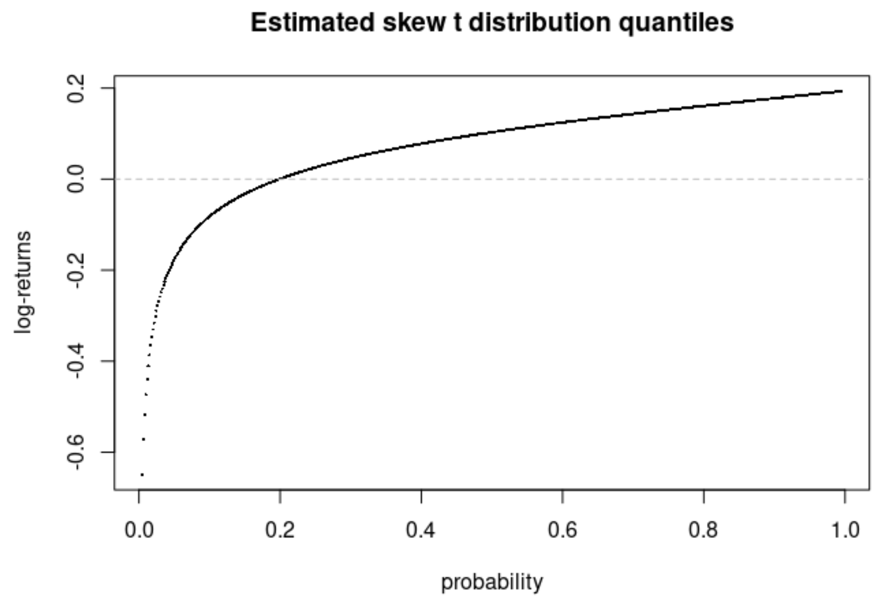
Let's plot the fit and the observed returns together.



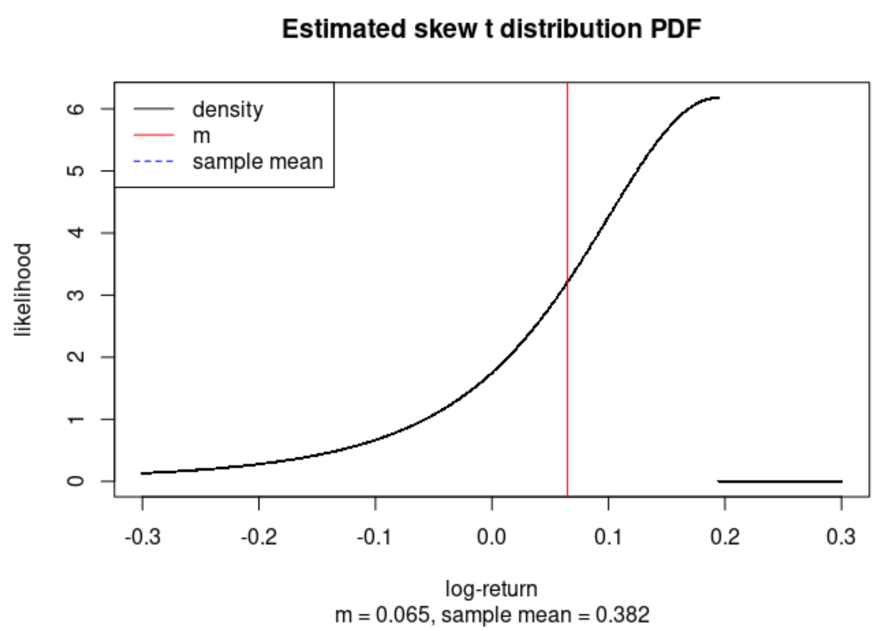
Estimated distribution

Now lets look at the CDF of the estimated distribution for each 0.1% increment between 0.5% and 99.5% for the estimated distribution:



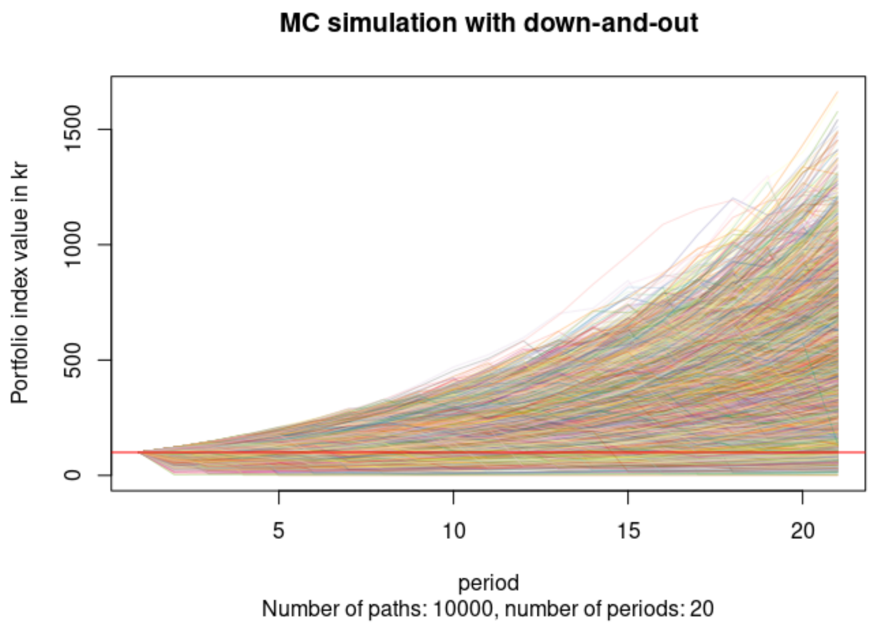


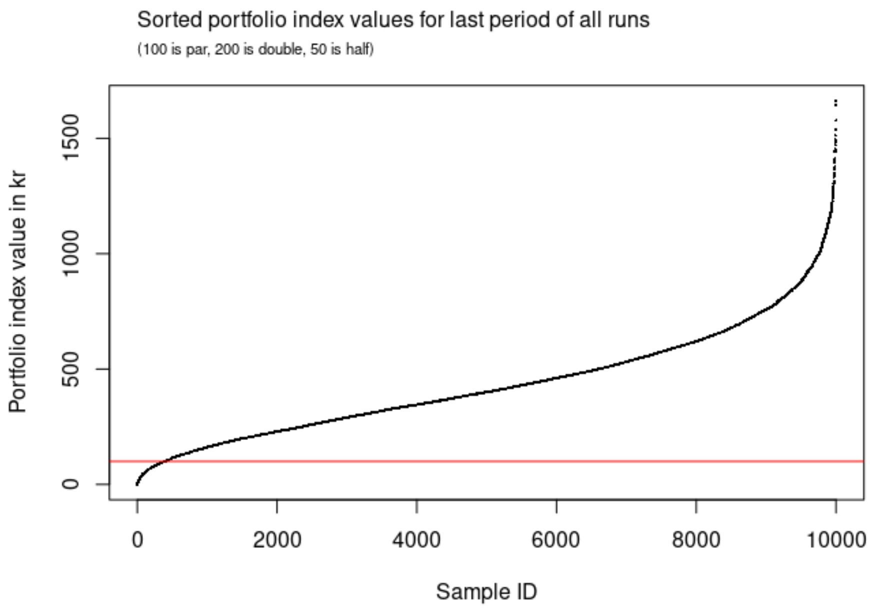
We see that for a few observations out of a 1000, the losses are disastrous, while the upside is very dampened.



Monte Carlo

```
## Down-and-out simulation:
## Probability of down-and-out: 0.01 percent
##
## Mean portfolio index value after 20 years: 404.855 kr.
## SD of portfolio index value after 20 years: 217.274 kr.
## Min total portfolio index value after 20 years: 0.01 kr.
## Max total portfolio index value after 20 years: 1713.743 kr.
##
## Share of paths finishing below 100: 4.3 percent
```

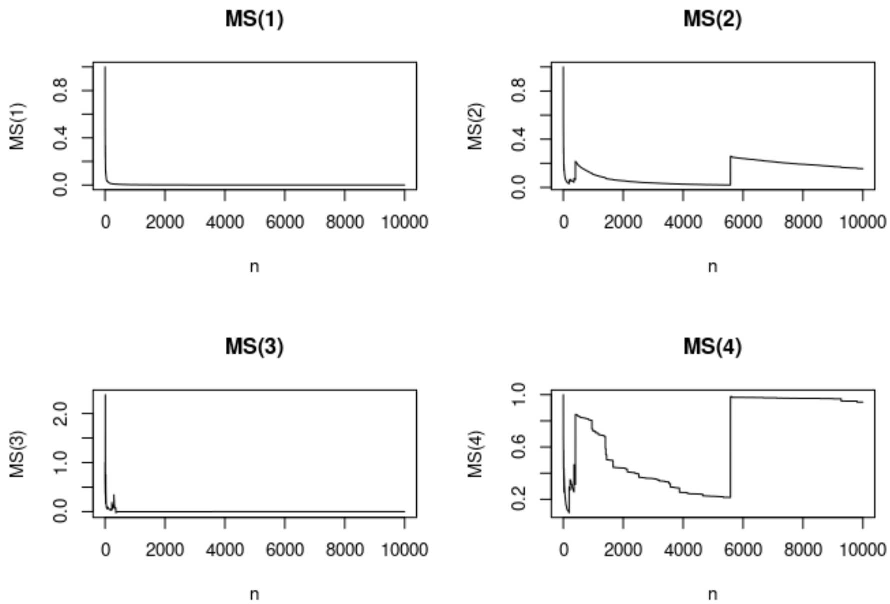




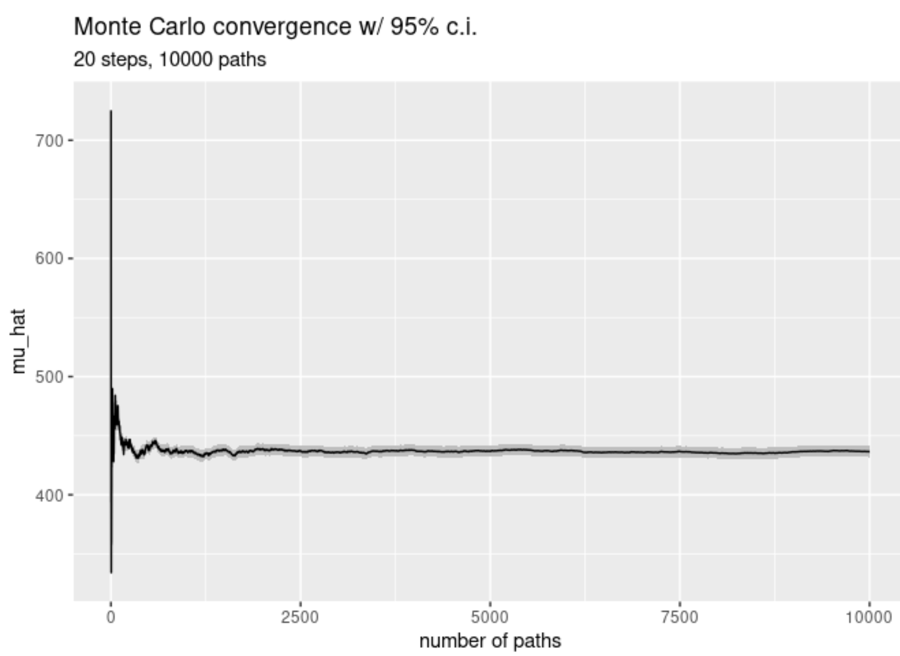
Convergence

Max vs sum

Max vs sum plots for the first four moments:


MS(1)
MS(2)
MS(3)
MS(4)
n

MC

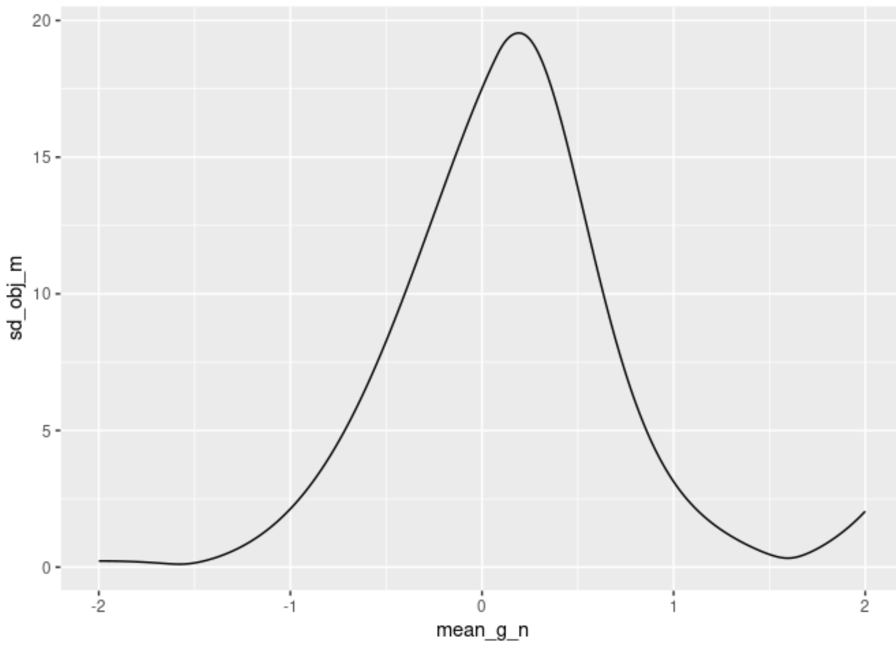

Monte Carlo convergence w/ 95% c.i.
20 steps, 10000 paths
mu_hat
number of paths

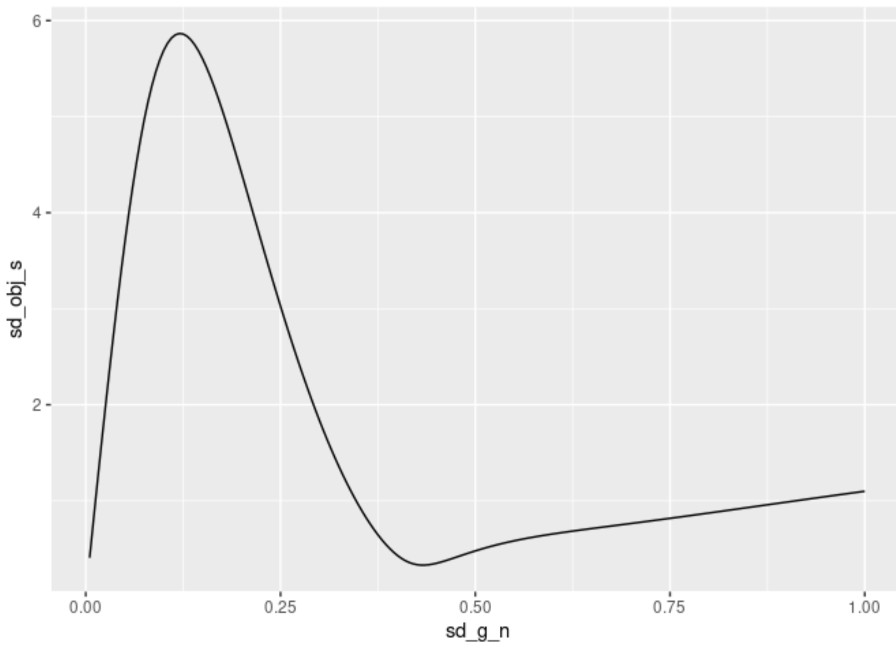
IS

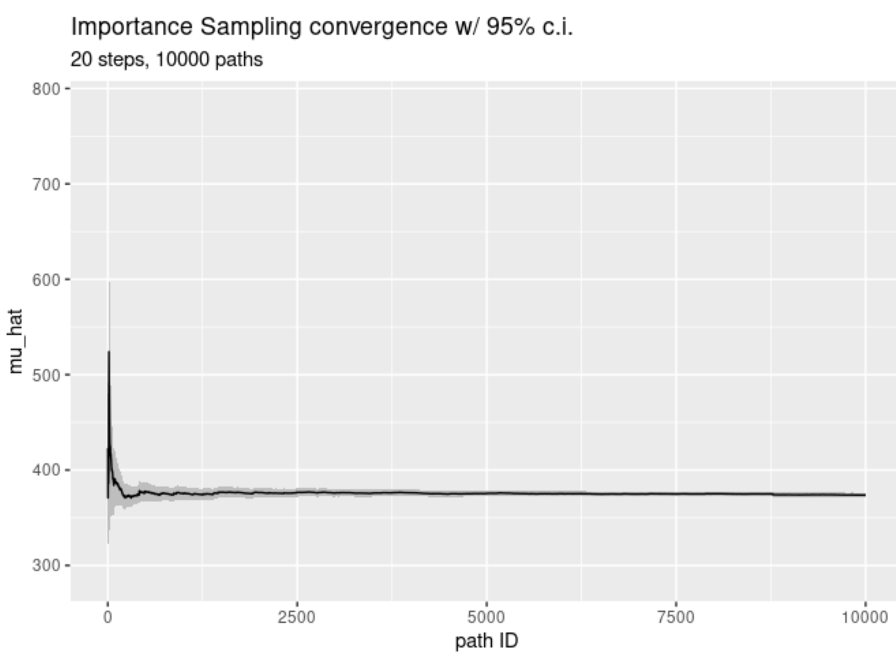
Parameters

```
## [1] 1.5302163 0.4155546
```

Objective function plots




6
4
2
sd_obj_s
0.00
0.25
0.50
0.75
1.00
sd_g_n

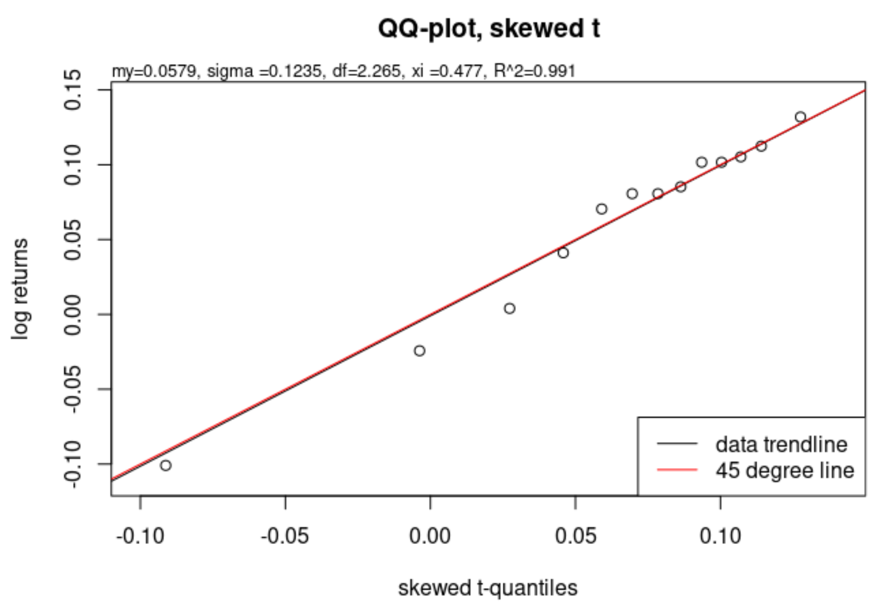

Importance Sampling convergence w/ 95% c.i.
20 steps, 10000 paths
800
700
600
500
400
300
mu_hat
0
2500
5000
7500
10000
path ID

PFA medium risk, 2011 - 2023

Fit to skew t distribution

```
##
## AIC: -33.22998
## BIC: -30.97018
## m: 0.05789224
## s: 0.1234592
## nu (df): 2.265273
## xi: 0.477324
## R^2: 0.991
##
## An R^2 of 0.991 suggests that the fit is extremely good.
##
## What is the risk of losing max 10 %? =< 0 percent
## What is the risk of losing max 25 %? =< 0 percent
## What is the risk of losing max 50 %? =< 0 percent
## What is the risk of losing max 90 %? =< 0 percent
## What is the risk of losing max 99 %? =< 0 percent
##
## What is the chance of gaining min 10 %? >= 52.83333 percent
## What is the chance of gaining min 25 %? >= 44 percent
## What is the chance of gaining min 50 %? >= 38.83333 percent
## What is the chance of gaining min 90 %? >= 34.66667 percent
## What is the chance of gaining min 99 %? >= 34 percent
```

QQ Plot

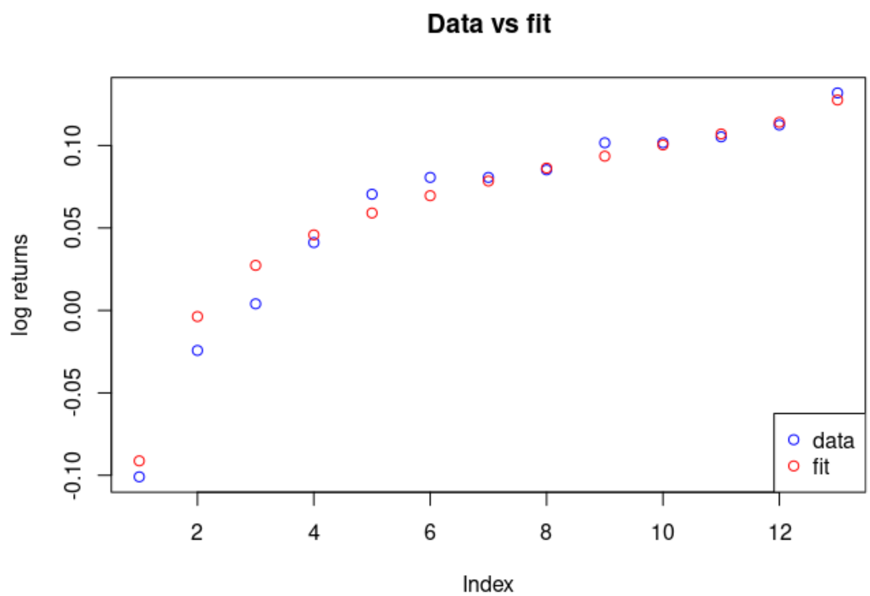


The qq plot looks great. Log returns for PFA medium risk seems to be consistent with a skewed t-distribution.

```
##  [1] -0.091256521 -0.003731241  0.027312079  0.045808232  0.059068633
##  [6]  0.069575113  0.078454727  0.086316936  0.093536451  0.100370932
## [11]  0.107018607  0.114081432  0.127604387
```

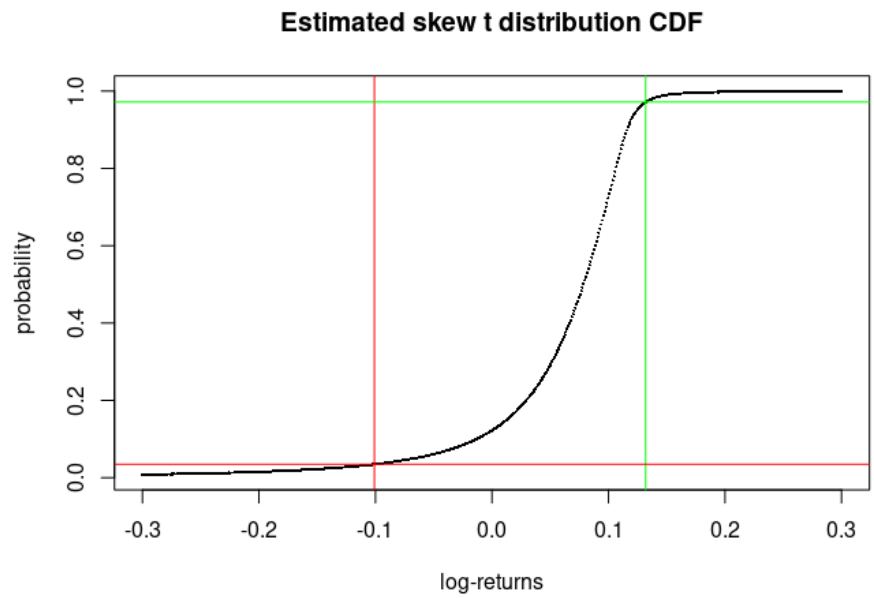
Data vs fit

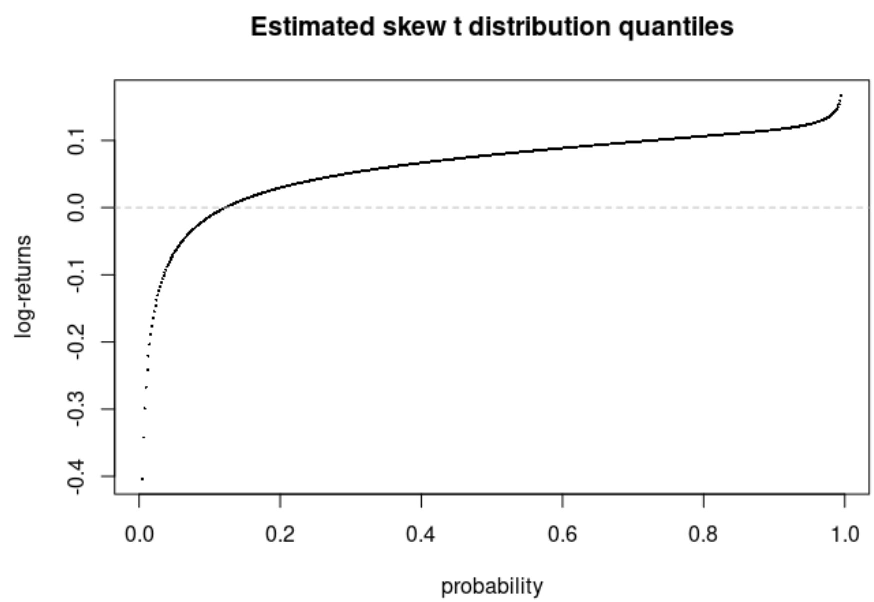
Let's plot the fit and the observed returns together.



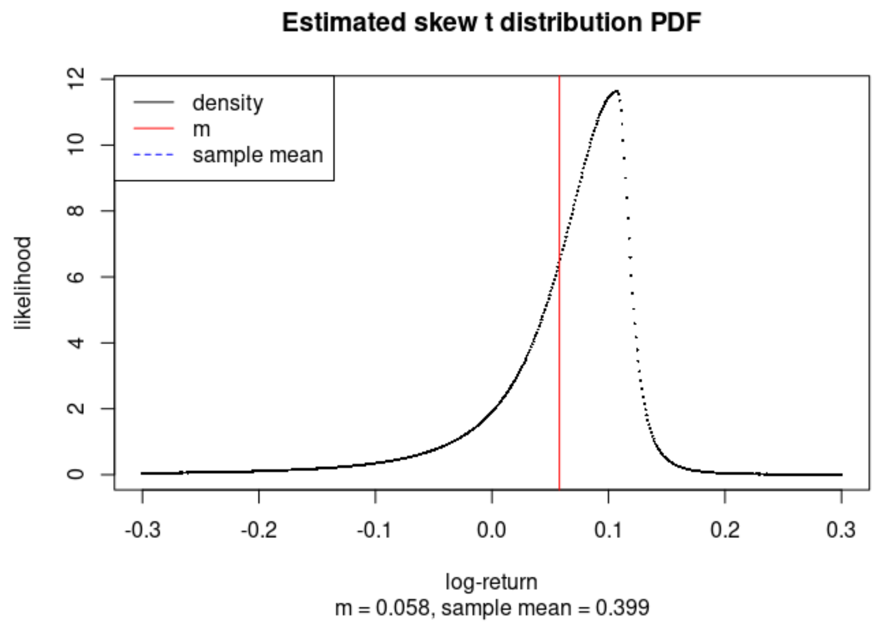
Estimated distribution

Now lets look at the CDF of the estimated distribution for each 0.1% increment between 0.5% and 99.5% for the estimated distribution:



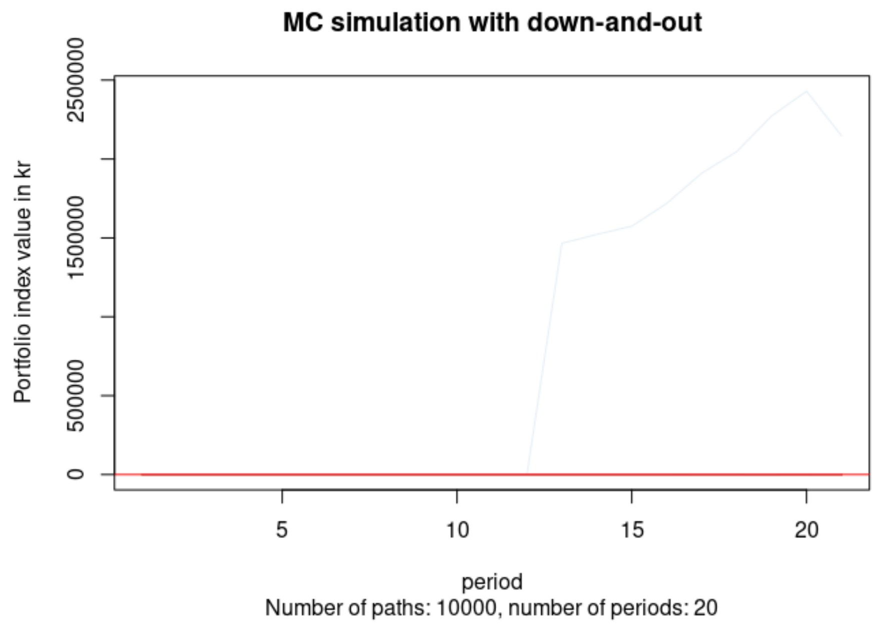


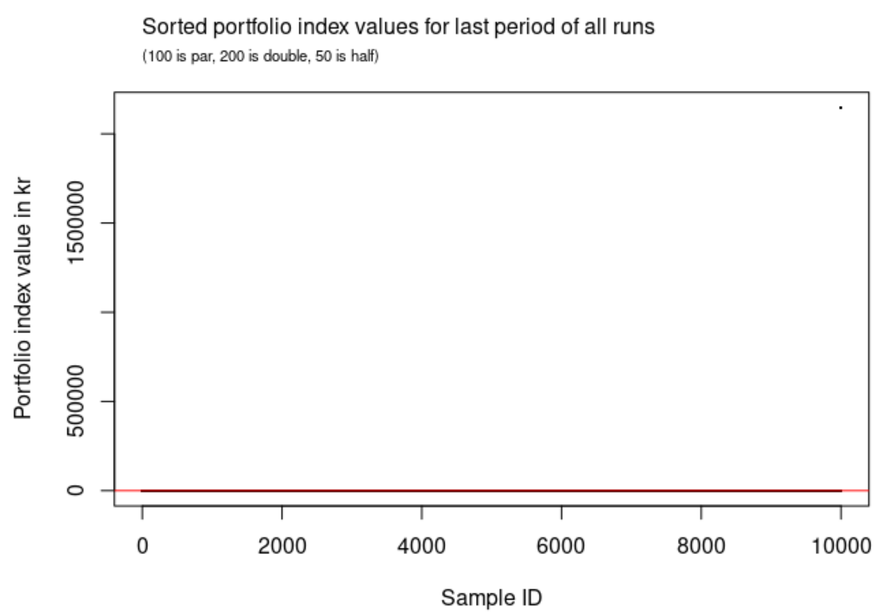
We see that for a few observations out of a 1000, the losses are disastrous. While there is some uptick at the top percentiles, the curve basically flattens out.



Monte Carlo

```
## Down-and-out simulation:
## Probability of down-and-out: 0.01 percent
##
## Mean portfolio index value after 20 years: 321.882 kr.
## SD of portfolio index value after 20 years: 106.531 kr.
## Min total portfolio index value after 20 years: 0.01 kr.
## Max total portfolio index value after 20 years: 1121.574 kr.
##
## Share of paths finishing below 100: 1.94 percent
```

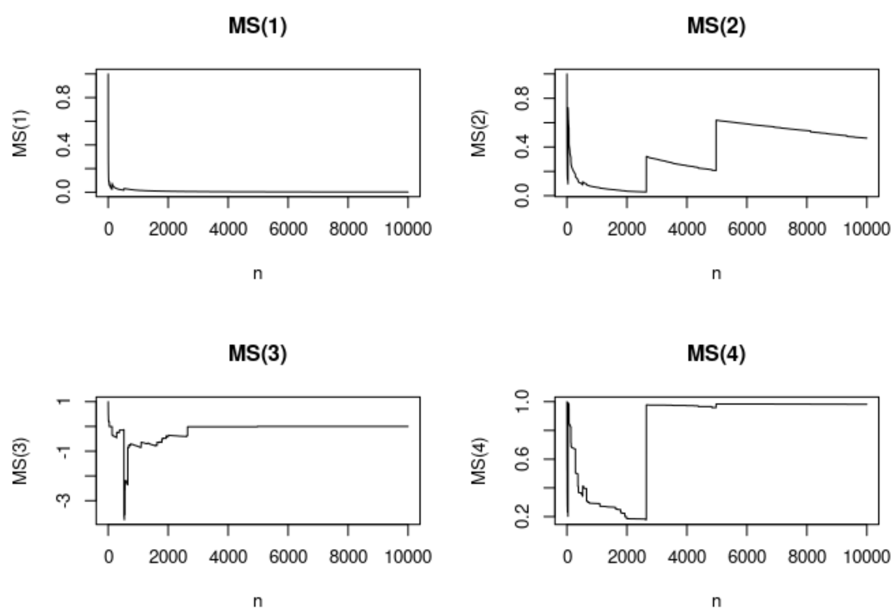

MC simulation with down-and-out
Portfolio index value in kr
0
500000
1500000
2500000
5
10
15
20
period
Number of paths: 10000, number of periods: 20


Sorted portfolio index values for last period of all runs
(100 is par, 200 is double, 50 is half)
Portfolio index value in kr
0
500000
1500000
0
2000
4000
6000
8000
10000
Sample ID

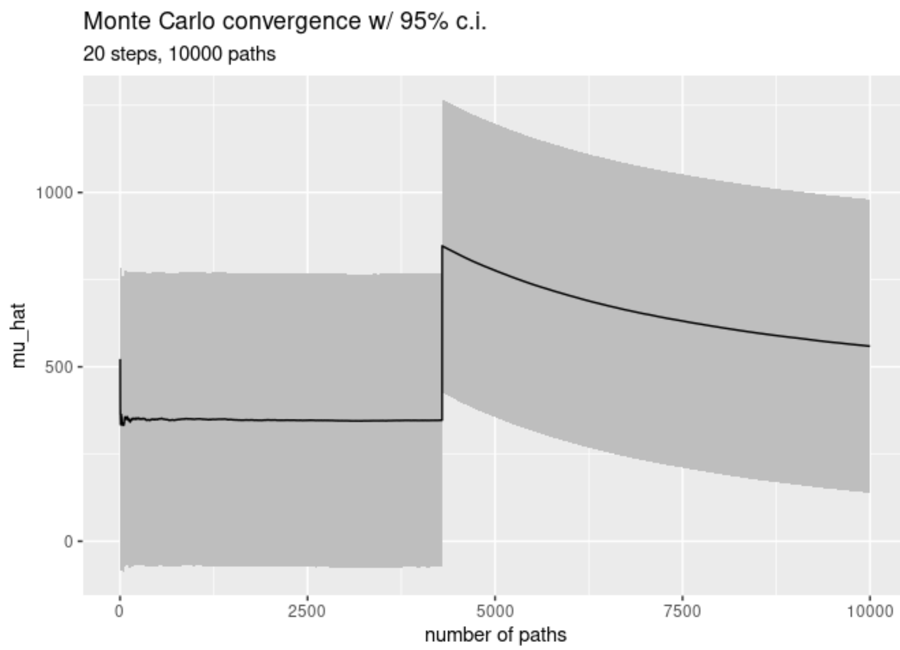
Convergence

Max vs sum

Max vs sum plots for the first four moments:



MC

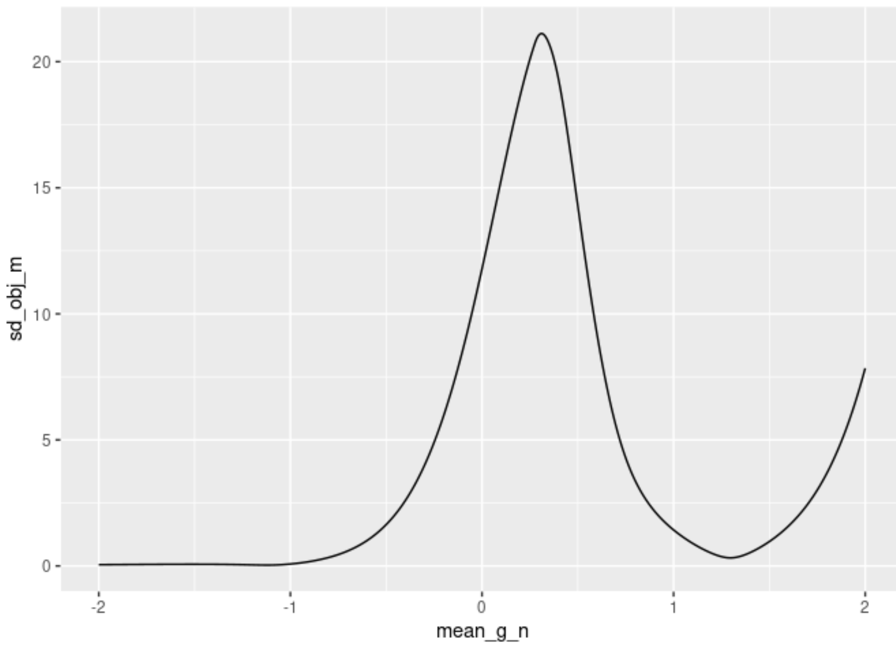


IS

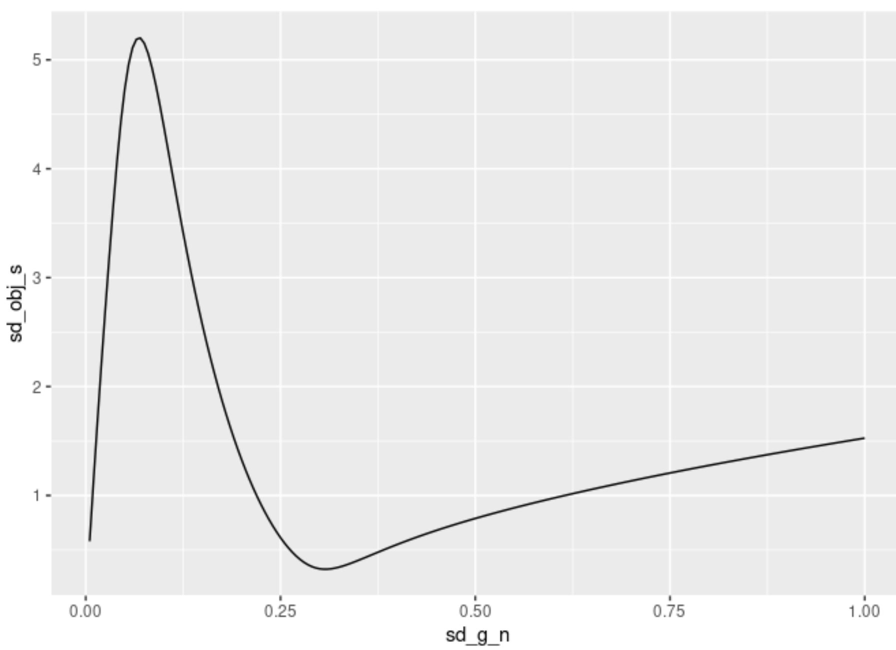
Parameters

```
## [1] 1.2338345 0.2992717
```

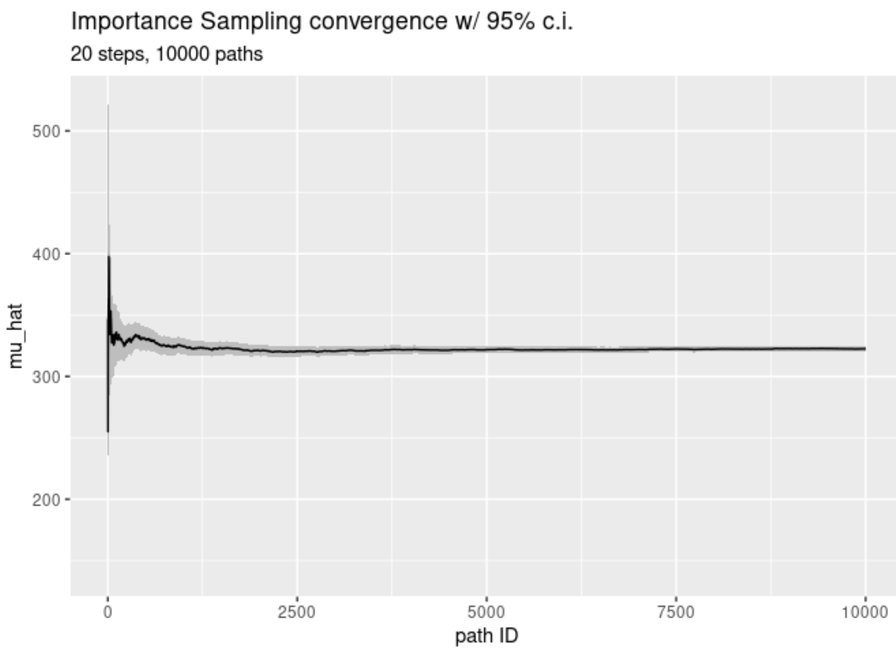
Objective function plots



sd_obj_m
20
15
10
5
0
-2
-1
0
1
2
mean_g_n



sd_obj_s
5
4
3
2
1
0.00
0.25
0.50
0.75
1.00
sd_g_n

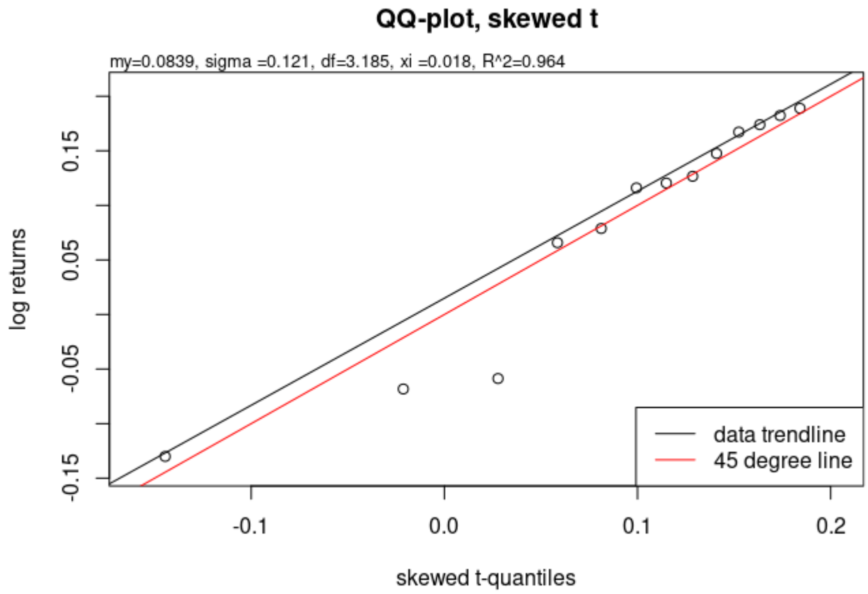


PFA high risk, 2011 - 2023

Fit to skew t distribution

```
##
## AIC: -23.72565
## BIC: -21.46585
## m: 0.08386034
## s: 0.1210107
## nu (df): 3.184569
## xi: 0.01790306
## R^2: 0.964
##
## An R^2 of 0.964 suggests that the fit is very good.
##
## What is the risk of losing max 10 %? =< 0 percent
## What is the risk of losing max 25 %? =< 0 percent
## What is the risk of losing max 50 %? =< 0 percent
## What is the risk of losing max 90 %? =< 0 percent
## What is the risk of losing max 99 %? =< 0 percent
##
## What is the chance of gaining min 10 %? >= 56.83333 percent
## What is the chance of gaining min 25 %? >= 43.16667 percent
## What is the chance of gaining min 50 %? >= 34.16667 percent
## What is the chance of gaining min 90 %? >= 26.83333 percent
## What is the chance of gaining min 99 %? >= 25.66667 percent
```

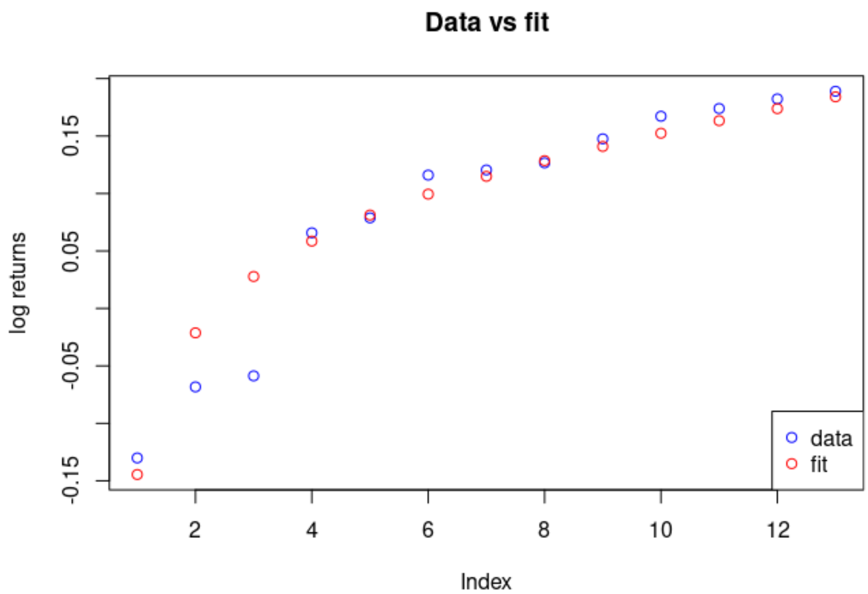
QQ Plot



The qq plot looks ok. Returns for PFA high risk seems to be consistent with a skewed t-distribution.

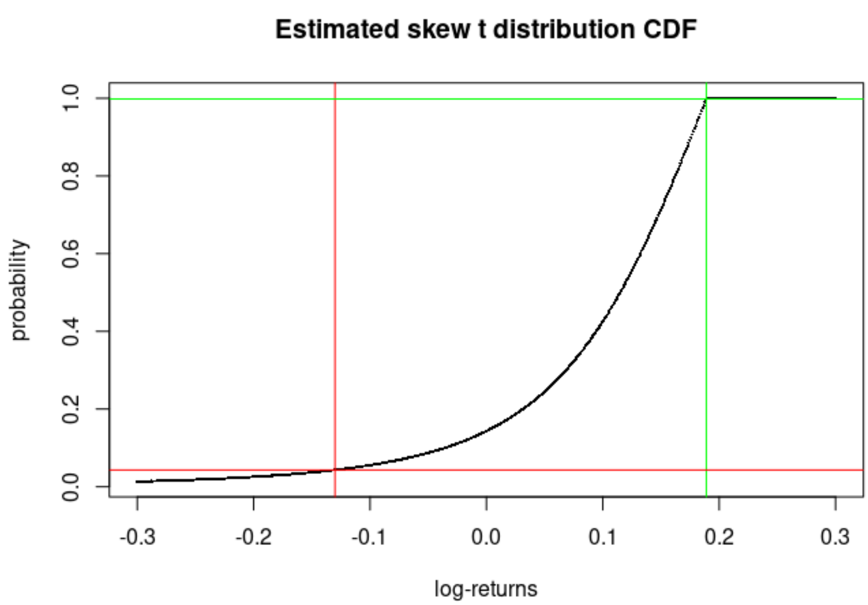
Data vs fit

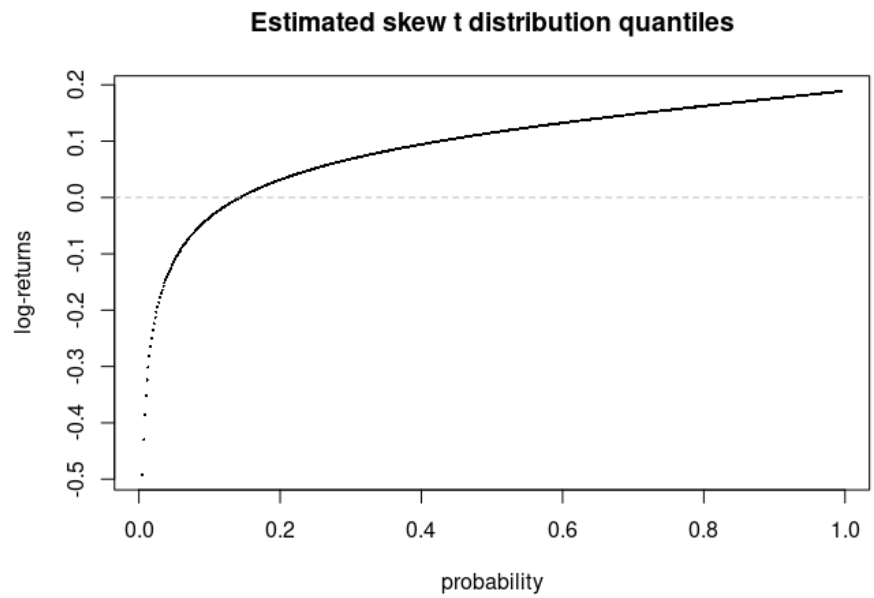
Let's plot the fit and the observed returns together.



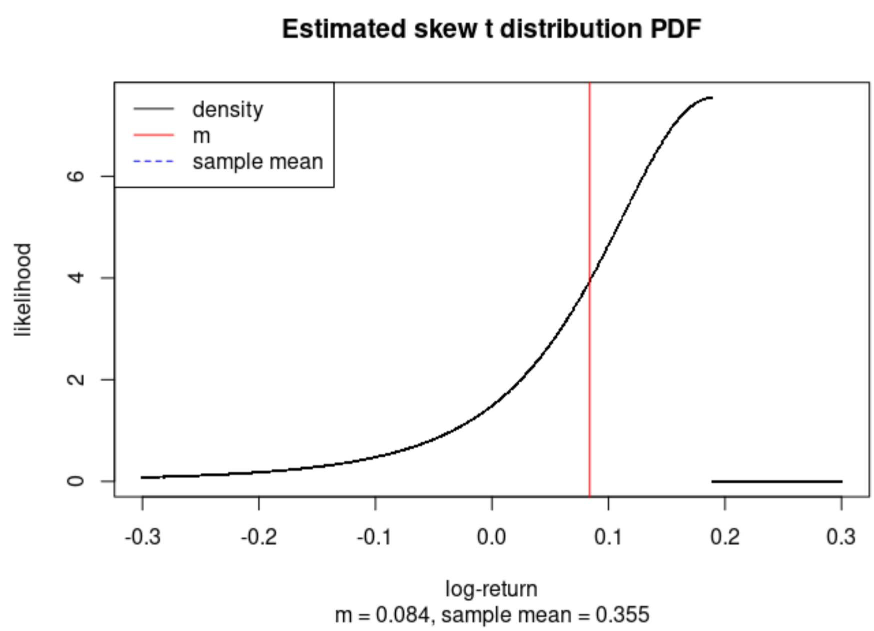
Estimated distribution

Now lets look at the CDF of the estimated distribution for each 0.1% increment between 0.5% and 99.5% for the estimated distribution:



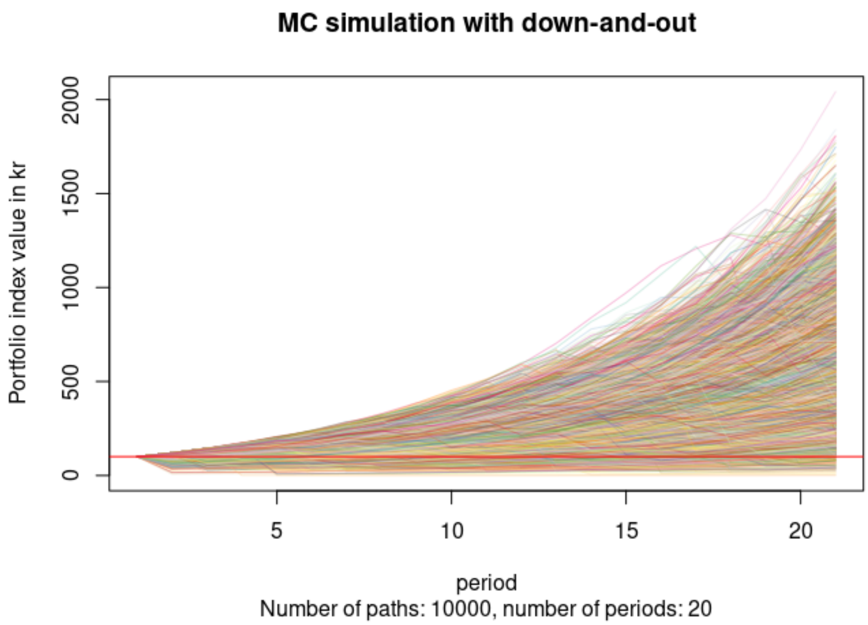


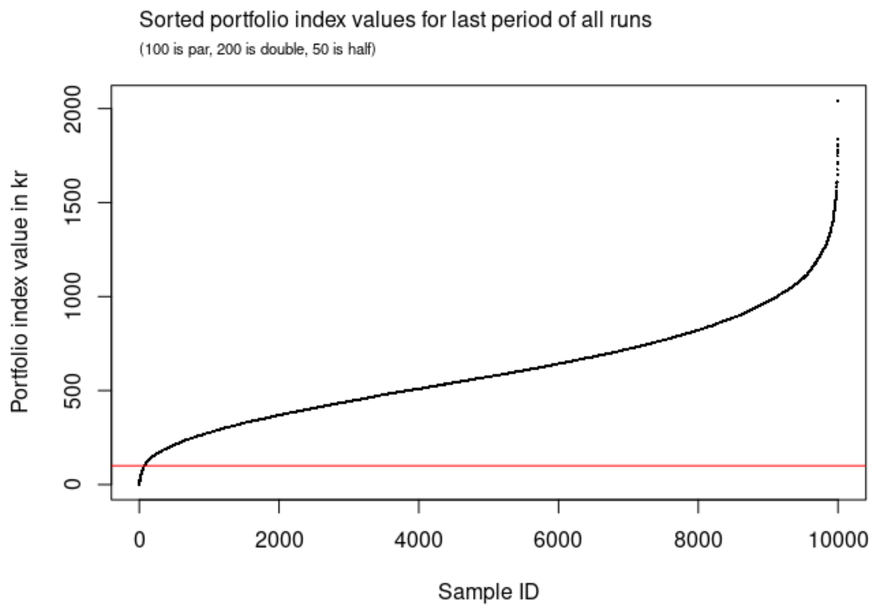
We see that for a few observations out of a 1000, the losses are disastrous, while the upside is very dampened.



Monte Carlo

```
## Down-and-out simulation:
## Probability of down-and-out: 0 percent
##
## Mean portfolio index value after 20 years: 555.222 kr.
## SD of portfolio index value after 20 years: 244.751 kr.
## Min total portfolio index value after 20 years: 0.765 kr.
## Max total portfolio index value after 20 years: 1828.26 kr.
##
## Share of paths finishing below 100: 0.95 percent
```

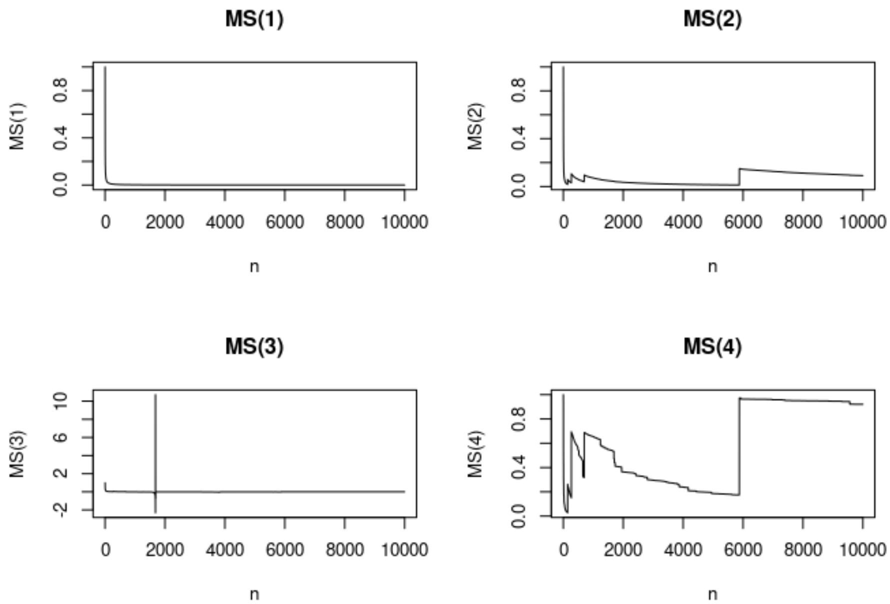




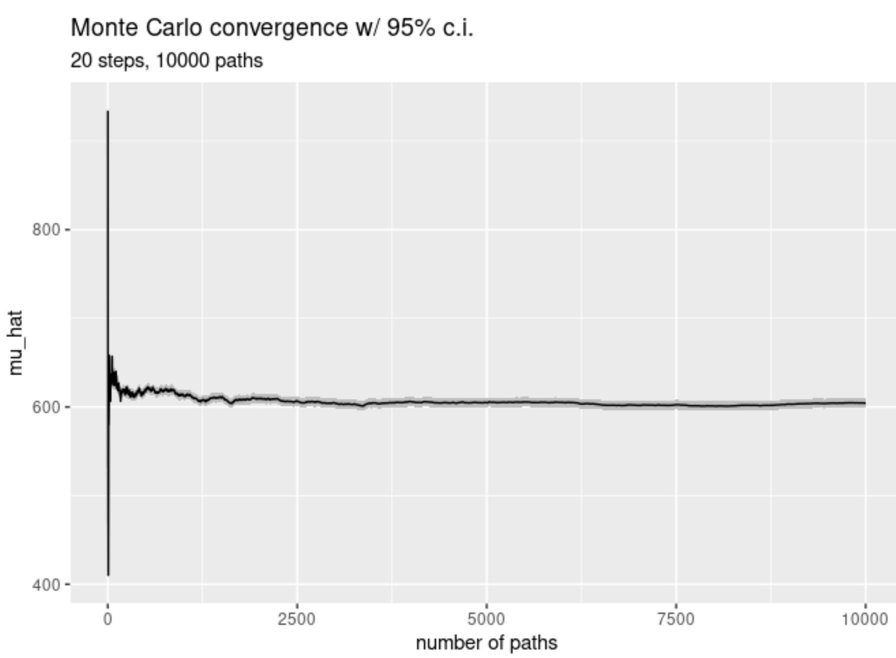
Convergence

Max vs sum

Max vs sum plots for the first four moments:


MS(1)
MS(2)
MS(3)
MS(4)
n

MC

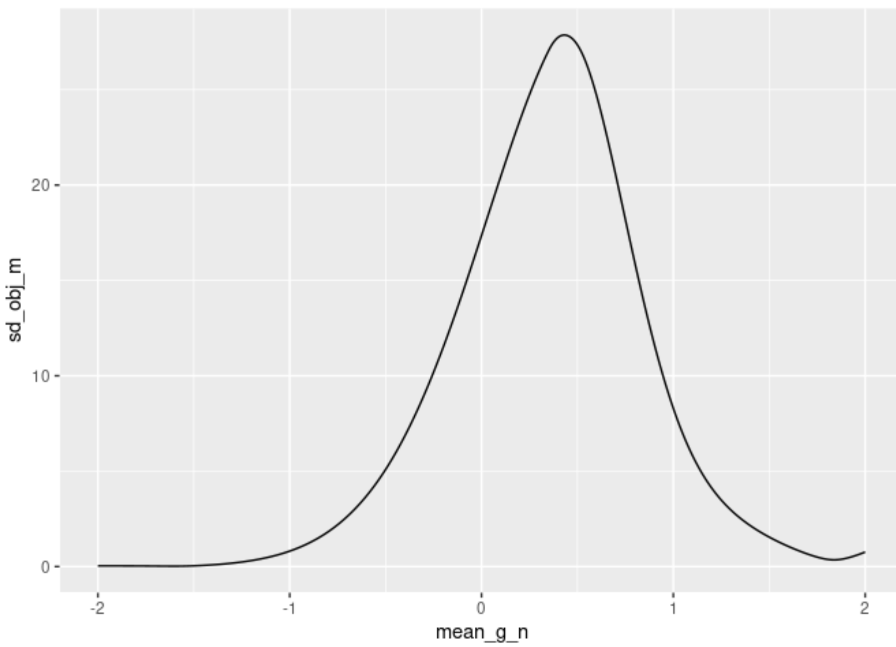

Monte Carlo convergence w/ 95% c.i.
20 steps, 10000 paths
mu_hat
number of paths

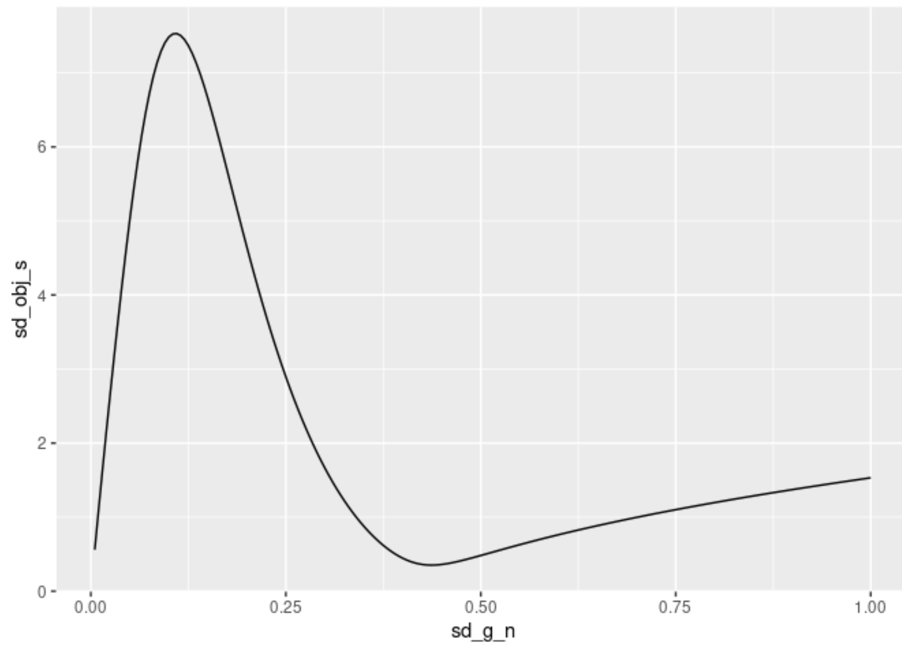
IS

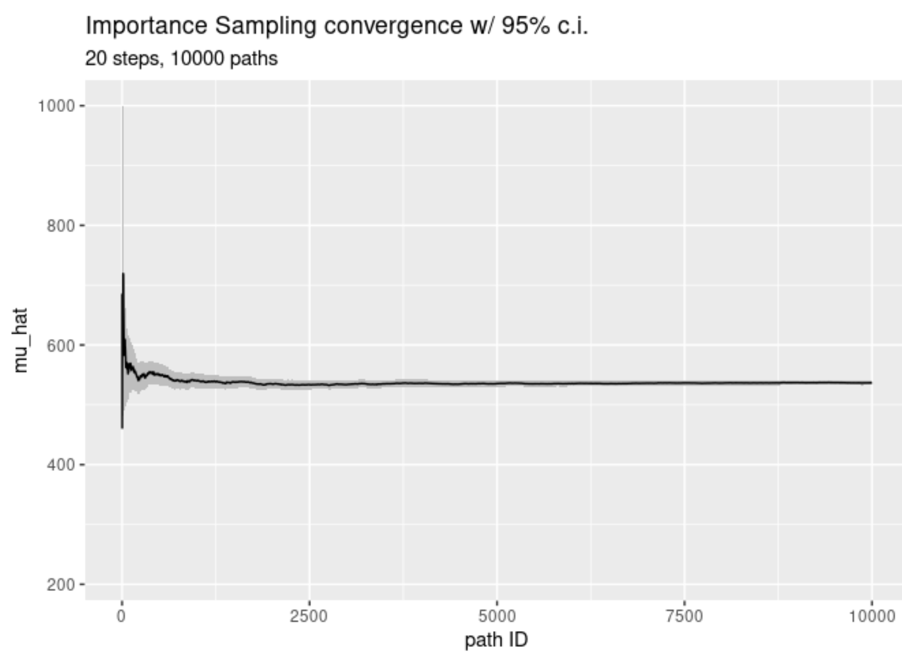
Parameters

```
## [1] 1.7617723 0.4255421
```

Objective function plots




sd_obj_s
0
2
4
6
0.00
0.25
0.50
0.75
1.00
sd_g_n

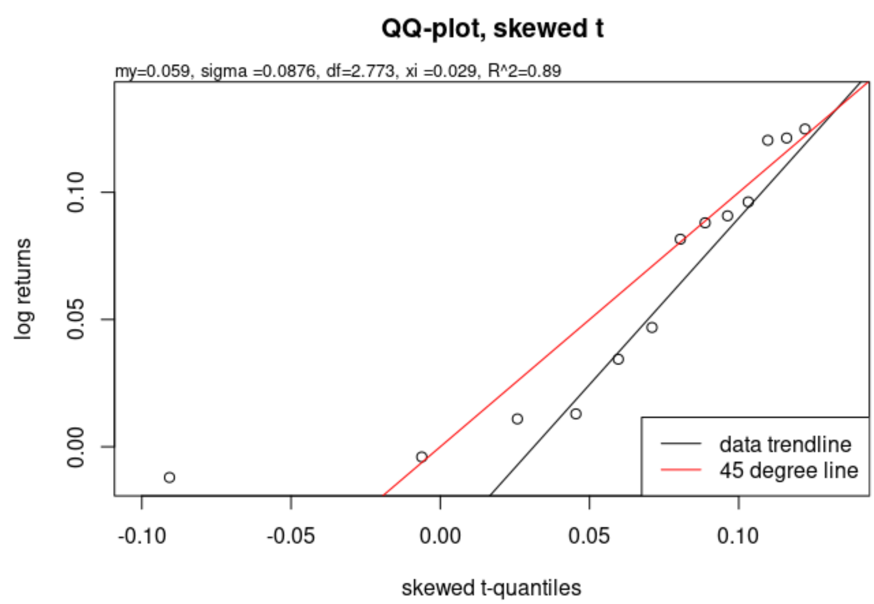

Importance Sampling convergence w/ 95% c.i.
20 steps, 10000 paths
mu_hat
200
400
600
800
1000
0
2500
5000
7500
10000
path ID

Mix medium risk, 2011 - 2023

Fit to skew t distribution

```
##
## AIC: -36.9603
## BIC: -34.7005
## m: 0.05902873
## s: 0.08757749
## nu (df): 2.772621
## xi: 0.02904471
## R^2: 0.89
##
## An R^2 of 0.89 suggests that the fit is not completely random.
##
## What is the risk of losing max 10 %? =< 0 percent
## What is the risk of losing max 25 %? =< 0 percent
## What is the risk of losing max 50 %? =< 0 percent
## What is the risk of losing max 90 %? =< 0 percent
## What is the risk of losing max 99 %? =< 0 percent
##
## What is the chance of gaining min 10 %? >= 53.16667 percent
## What is the chance of gaining min 25 %? >= 44.16667 percent
## What is the chance of gaining min 50 %? >= 38.66667 percent
## What is the chance of gaining min 90 %? >= 34.16667 percent
## What is the chance of gaining min 99 %? >= 33.5 percent
```

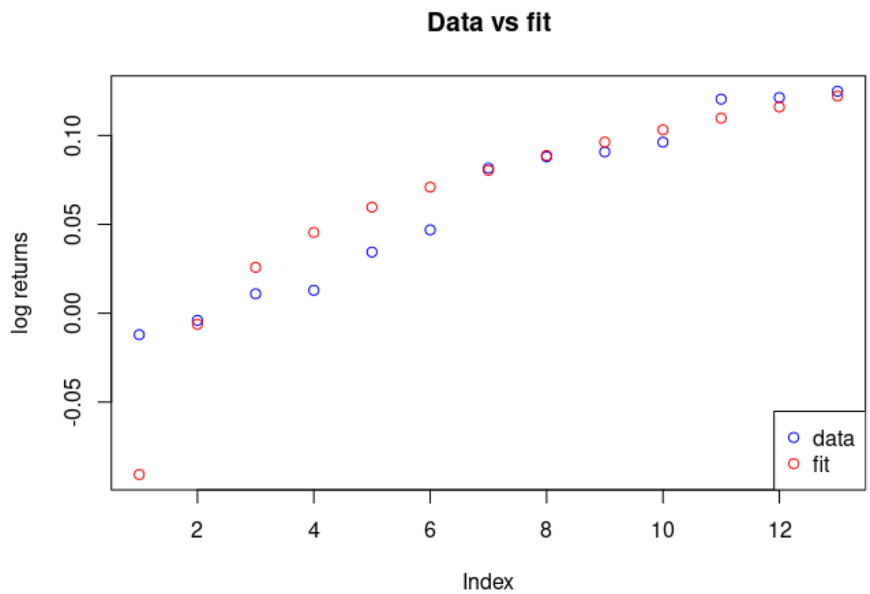
QQ Plot



The fit suggests big losses for the lowest percentiles, which are not present in the data.
So the fit is actually a very cautious estimate.

Data vs fit

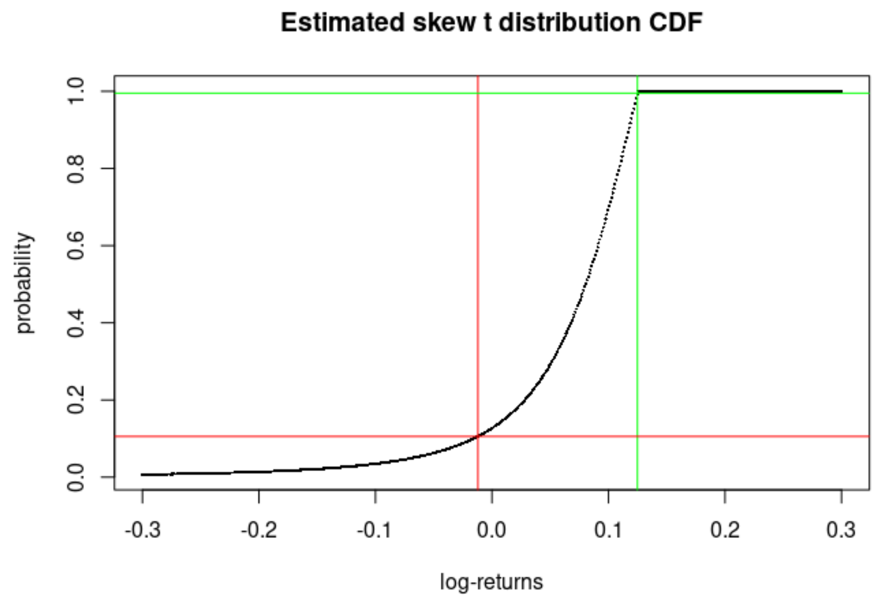
Let's plot the fit and the observed returns together.

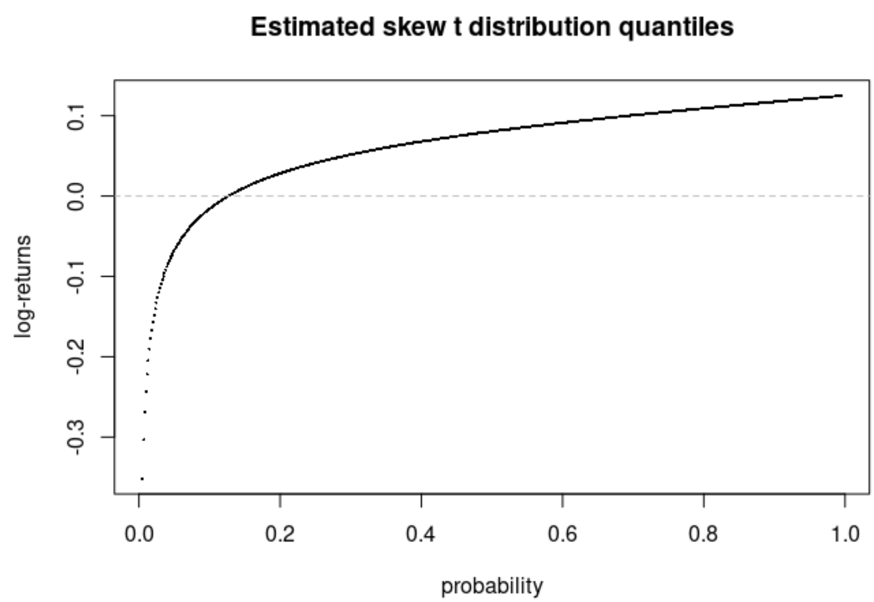


Interestingly, the fit predicts a much bigger "biggest loss" than the actual data. This is the main reason that R^2 is 0.90 and not higher.

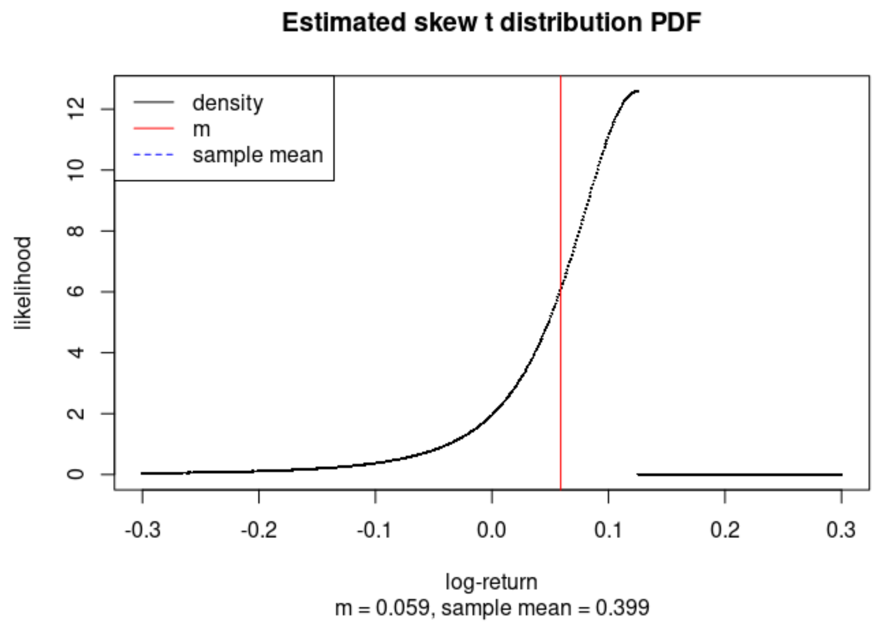
Estimated distribution

Now lets look at the CDF of the estimated distribution for each 0.1% increment between 0.5% and 99.5% for the estimated distribution:





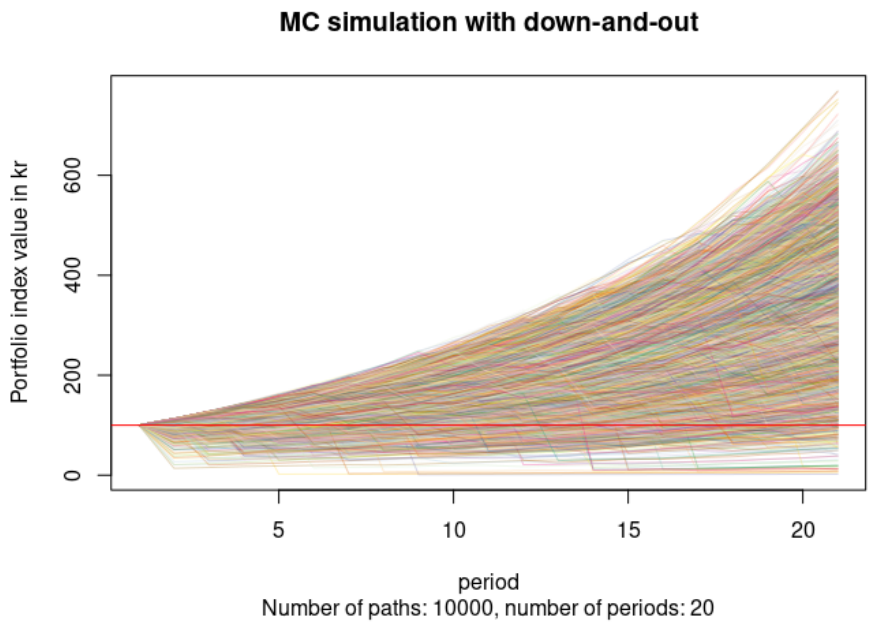
We see that for a few observations out of a 1000, the losses are disastrous, while the upside is very dampened.

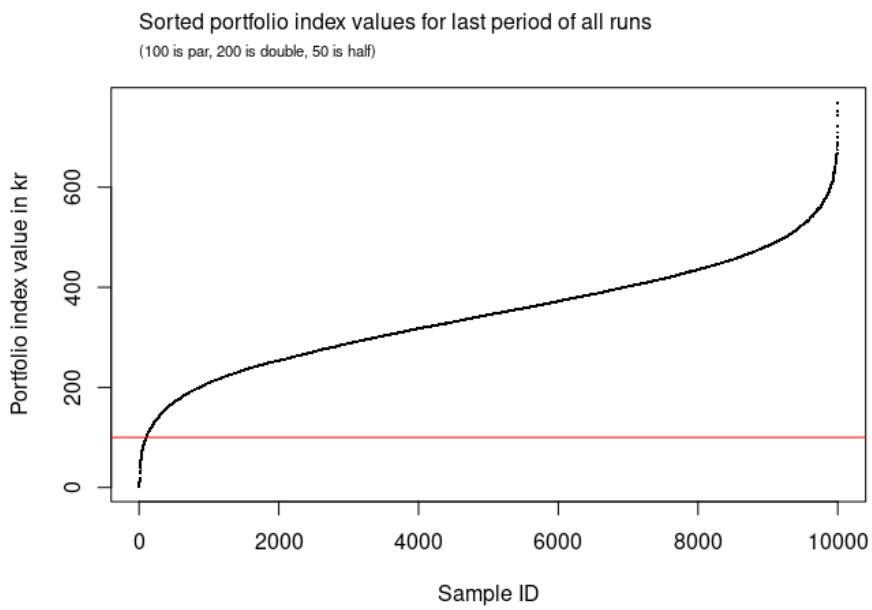


Monte Carlo

Version a: Simulation from estimated distribution of returns of mix.

```
## Down-and-out simulation:
## Probability of down-and-out: 0 percent
##
## Mean portfolio index value after 20 years: 324.055 kr.
## SD of portfolio index value after 20 years: 98.254 kr.
## Min total portfolio index value after 20 years: 3.973 kr.
## Max total portfolio index value after 20 years: 723.752 kr.
##
## Share of paths finishing below 100: 1.13 percent
```

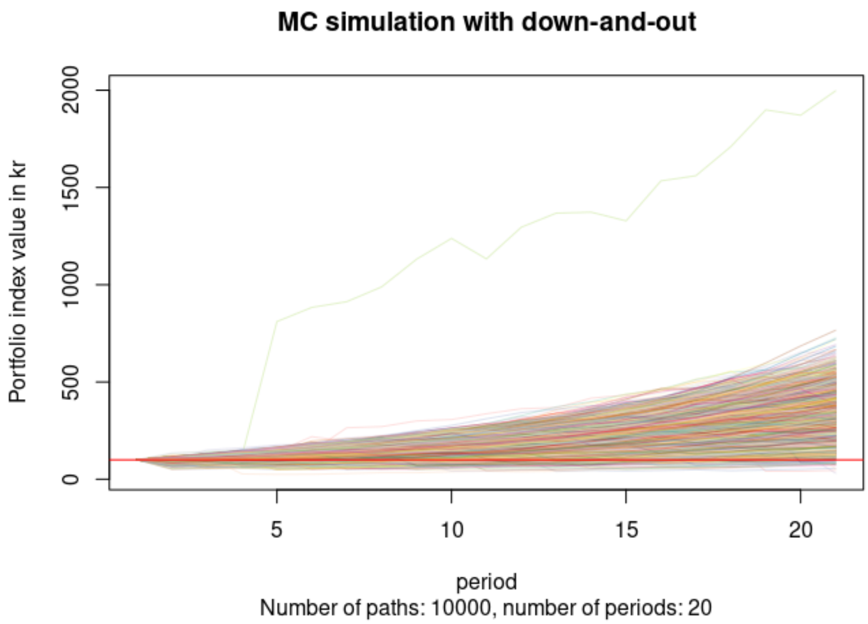


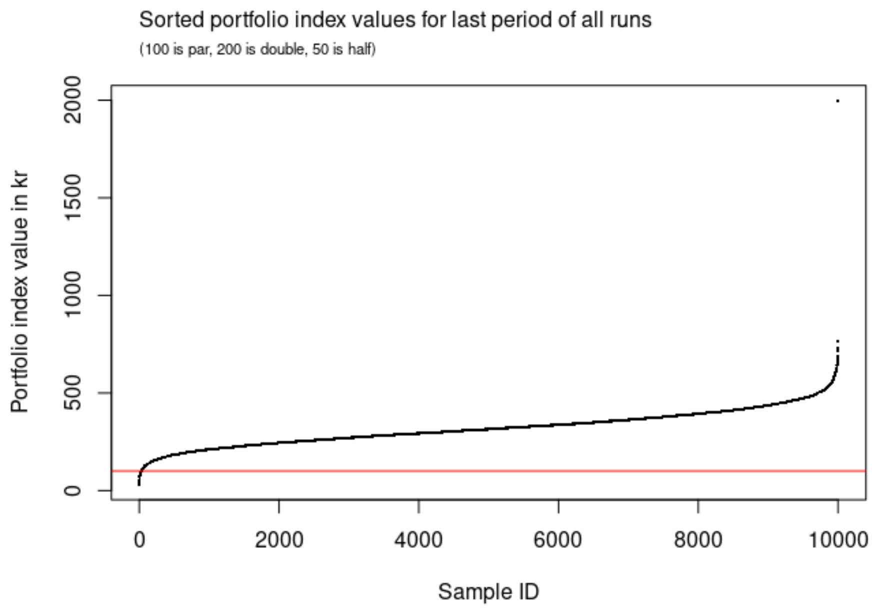


Version b: Mix of simulations from estimated distribution of returns from individual funds.

```
## Down-and-out simulation:
## Probability of down-and-out: 0 percent
```

```
##
## Mean portfolio index value after 20 years: 300.815 kr.
## SD of portfolio index value after 20 years: 82.422 kr.
## Min total portfolio index value after 20 years: 42.661 kr.
## Max total portfolio index value after 20 years: 1008.214 kr.
##
## Share of paths finishing below 100: 0.34 percent
```

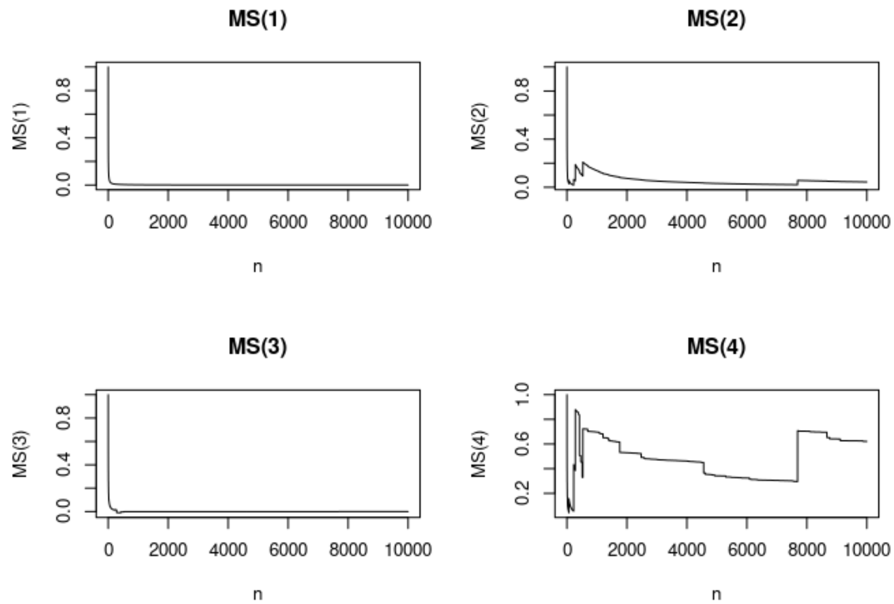




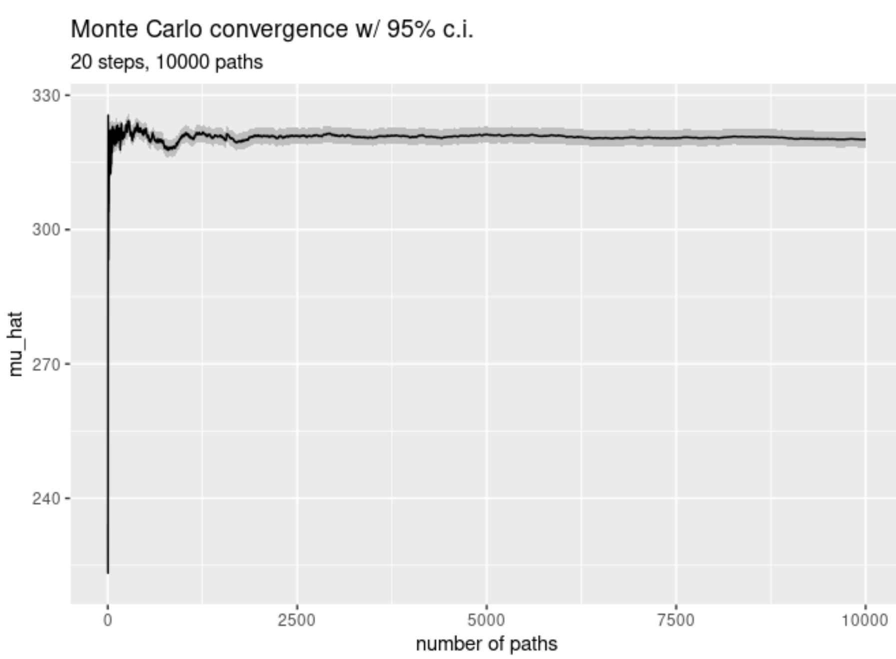
Convergence

Max vs sum

Max vs sum plots for the first four moments:


MS(1)
MS(2)
MS(3)
MS(4)
MS(1)
MS(2)
MS(3)
MS(4)
0.0
0.4
0.8
0.2
0.6
1.0
0
2000
4000
6000
8000
10000
n

MC

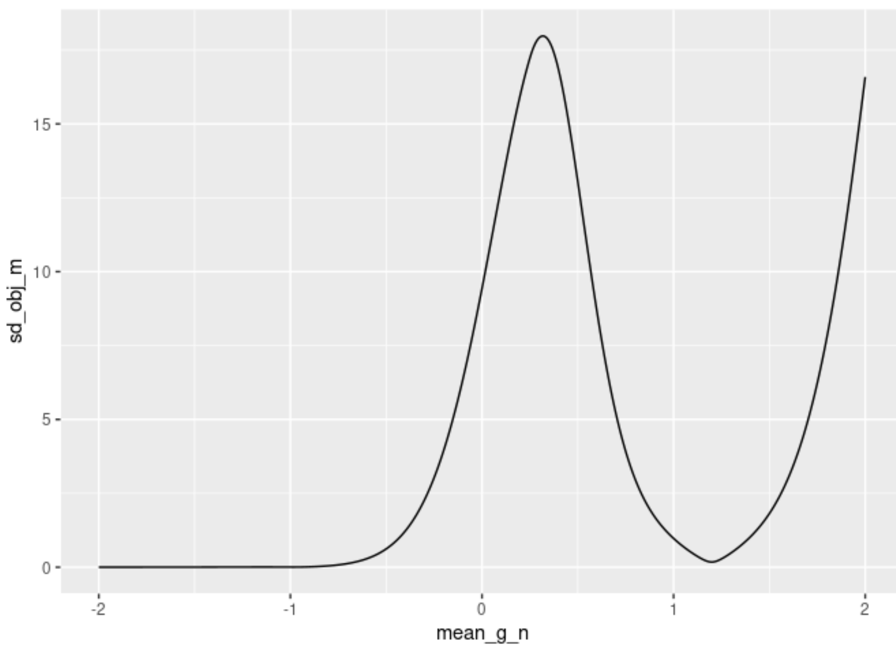

Monte Carlo convergence w/ 95% c.i.
20 steps, 10000 paths
mu_hat
330
300
270
240
0
2500
5000
7500
10000
number of paths

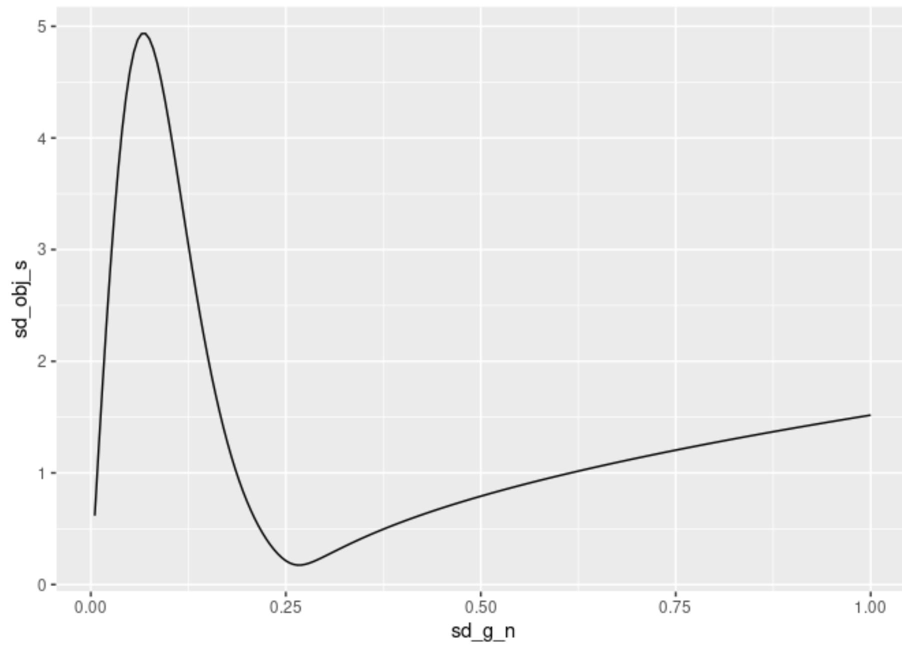
IS

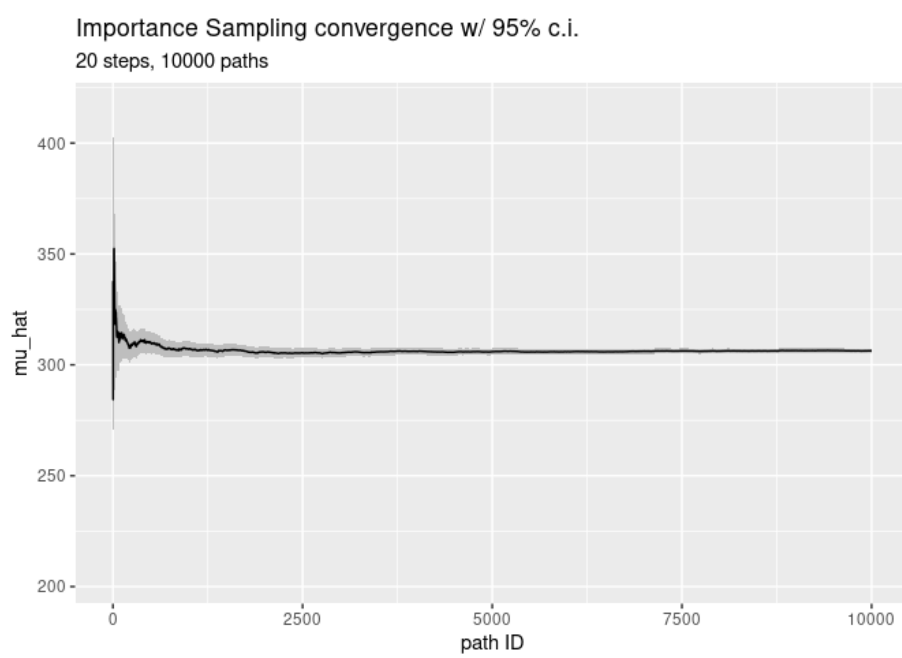
Parameters

```
## [1] 1.1398383 0.2595942
```

Objective function plots




sd_obj_s
sd_g_n
0
1
2
3
4
5
0.00
0.25
0.50
0.75
1.00

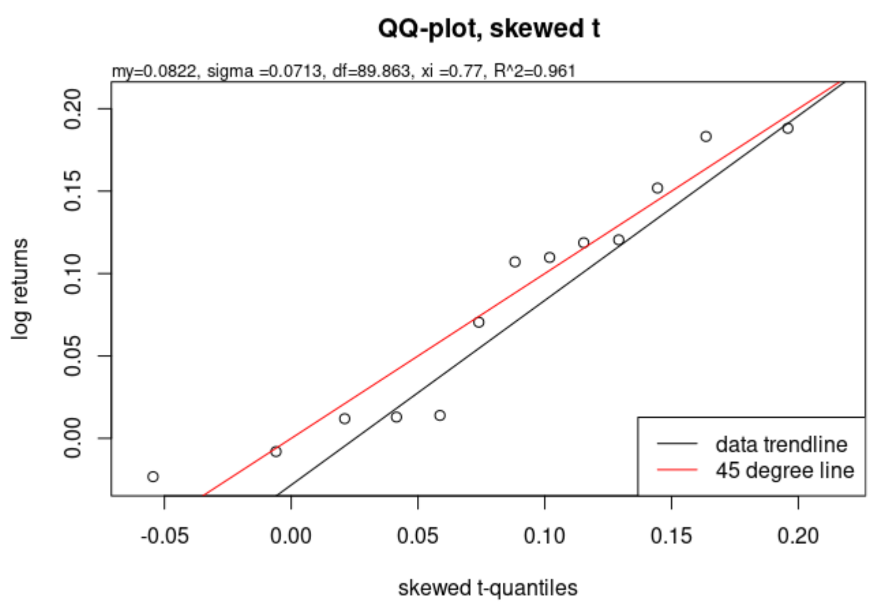

Importance Sampling convergence w/ 95% c.i.
20 steps, 10000 paths
mu_hat
path ID
200
250
300
350
400
0
2500
5000
7500
10000

Mix high risk, 2011 - 2023

Fit to skew t distribution

```
##
## AIC: -24.26084
## BIC: -22.00104
## m: 0.0822419
## s: 0.07129843
## nu (df): 89.86289
## xi: 0.7697502
## R^2: 0.961
##
## An R^2 of 0.961 suggests that the fit is very good.
##
## What is the risk of losing max 10 %? =< 0 percent
## What is the risk of losing max 25 %? =< 0 percent
## What is the risk of losing max 50 %? =< 0 percent
## What is the risk of losing max 90 %? =< 0 percent
## What is the risk of losing max 99 %? =< 0 percent
##
## What is the chance of gaining min 10 %? >= 52.5 percent
## What is the chance of gaining min 25 %? >= 45 percent
## What is the chance of gaining min 50 %? >= 38.33333 percent
## What is the chance of gaining min 90 %? >= 31.16667 percent
## What is the chance of gaining min 99 %? >= 29.83333 percent
```

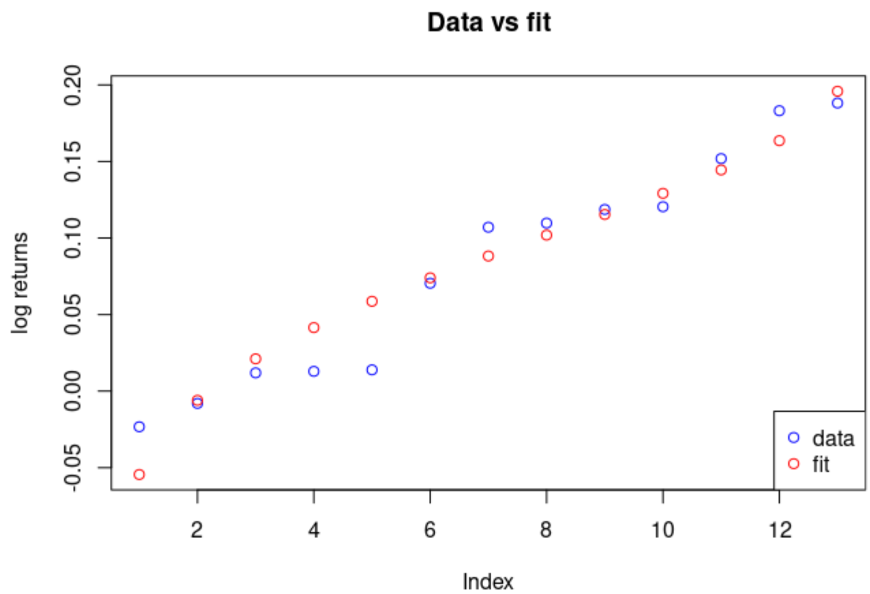
QQ Plot



The qq plot looks good Returns for mixed medium risk portfolios seems to be consistent with a skewed t-distribution.

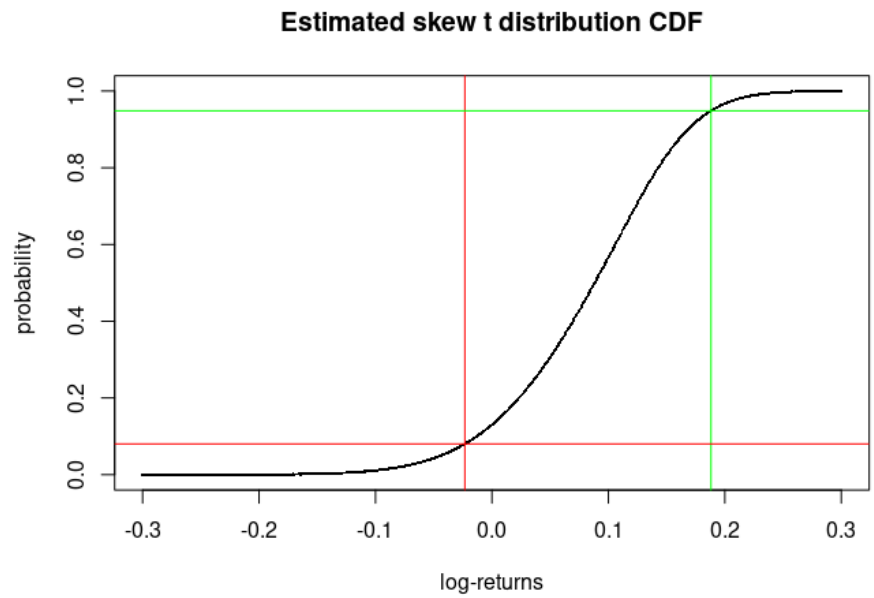
Data vs fit

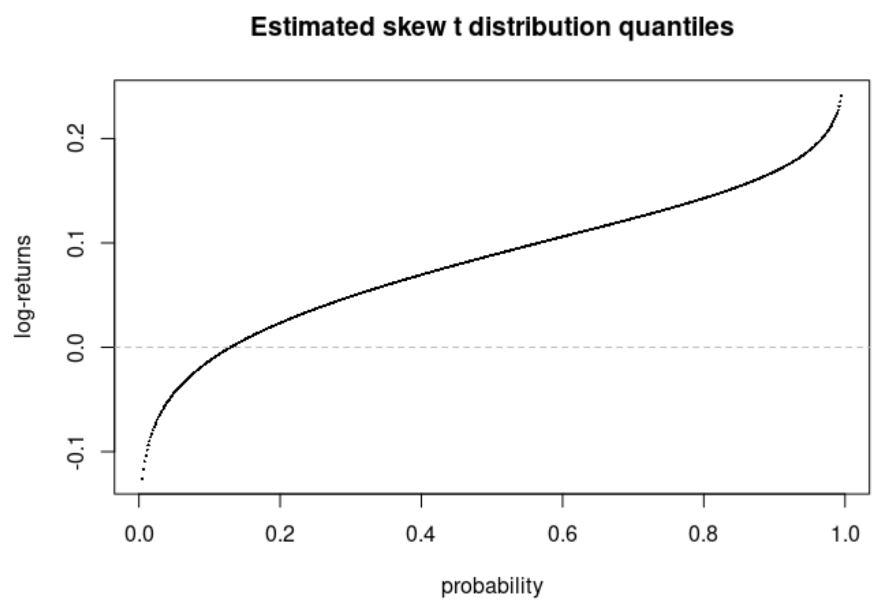
Let's plot the fit and the observed returns together.



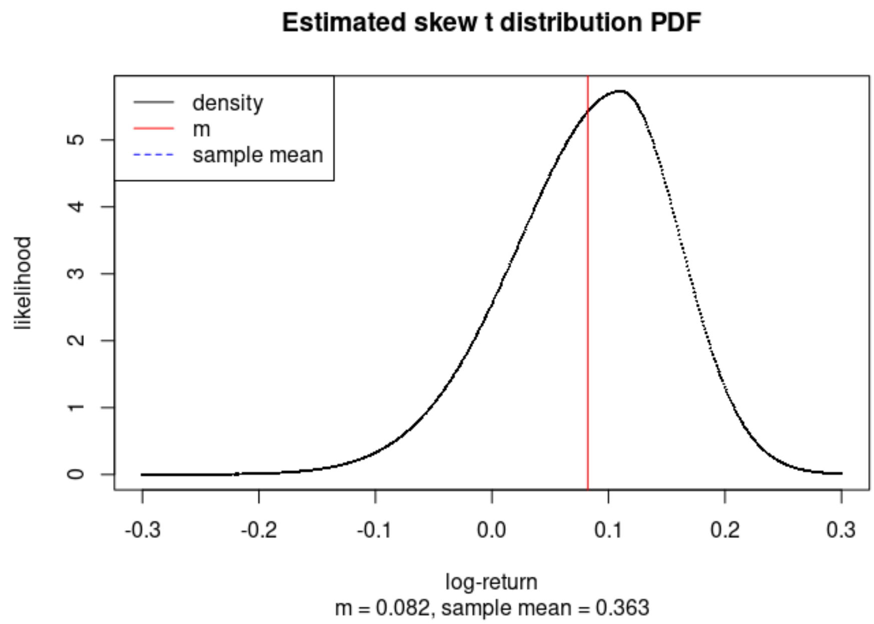
Estimated distribution

Now lets look at the CDF of the estimated distribution for each 0.1% increment between 0.5% and 99.5% for the estimated distribution:





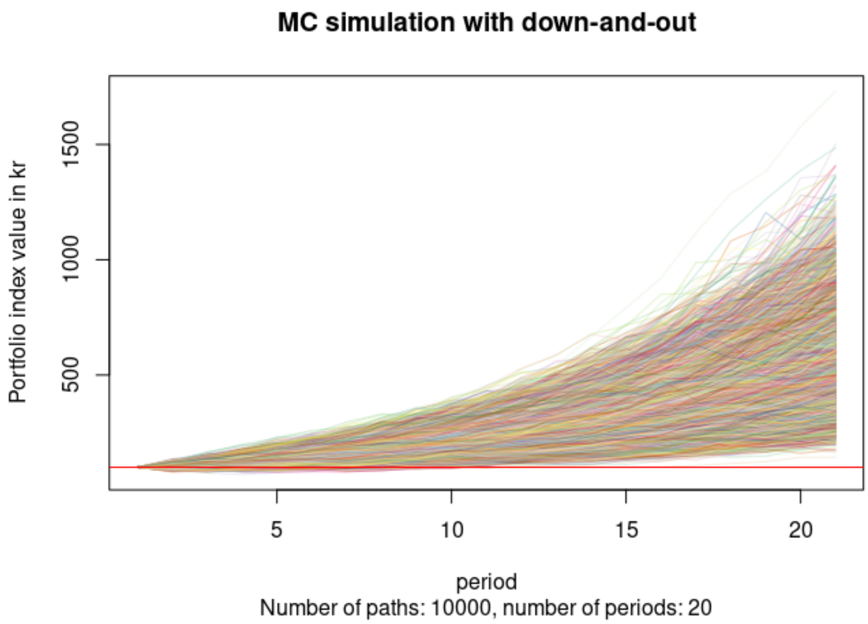
We see that the high risk mix provides a much better upside and smaller downside.

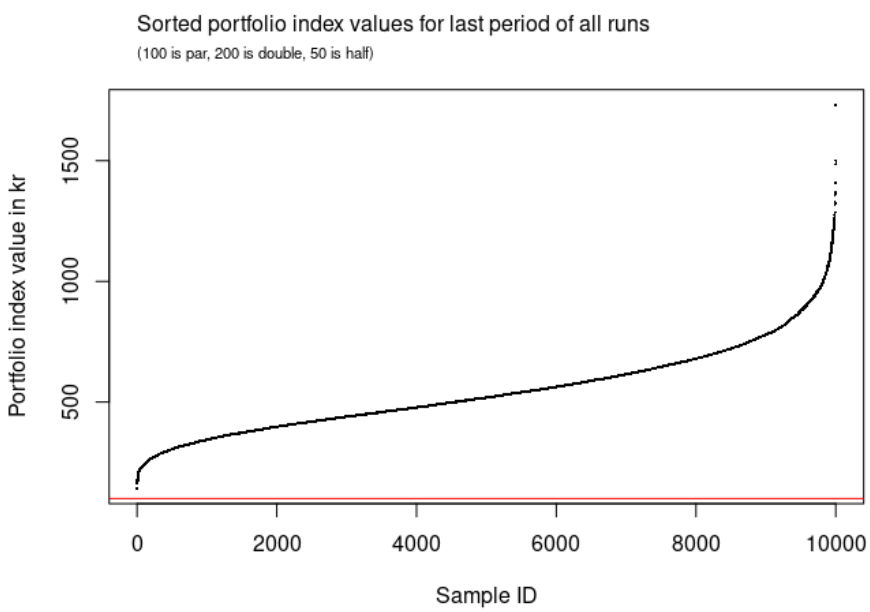


Monte Carlo

Version a: Simulation from estimated distribution of returns of mix.

```
## Down-and-out simulation:
## Probability of down-and-out: 0 percent
##
## Mean portfolio index value after 20 years: 498.653 kr.
## SD of portfolio index value after 20 years: 155.916 kr.
## Min total portfolio index value after 20 years: 150.86 kr.
## Max total portfolio index value after 20 years: 1463.716 kr.
##
## Share of paths finishing below 100: 0 percent
```

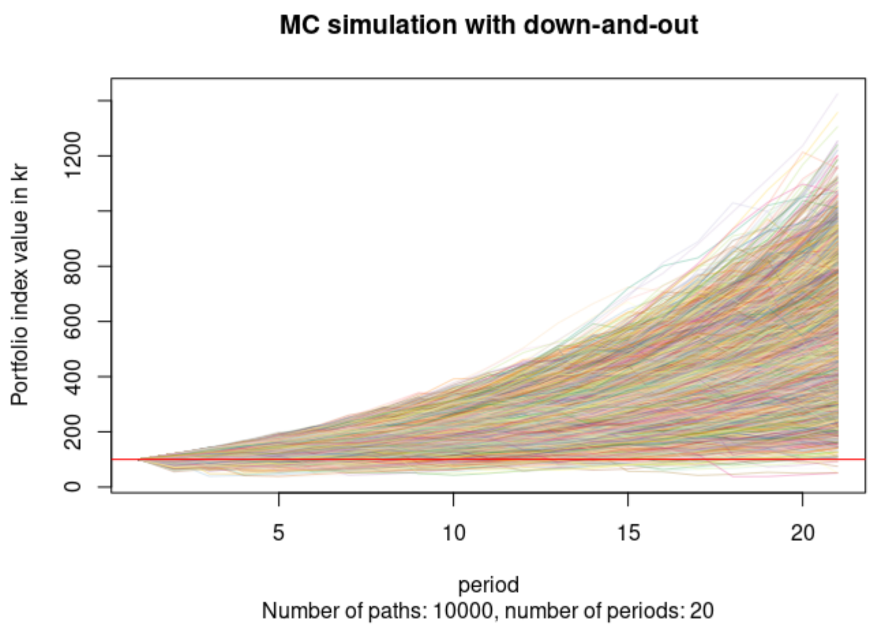


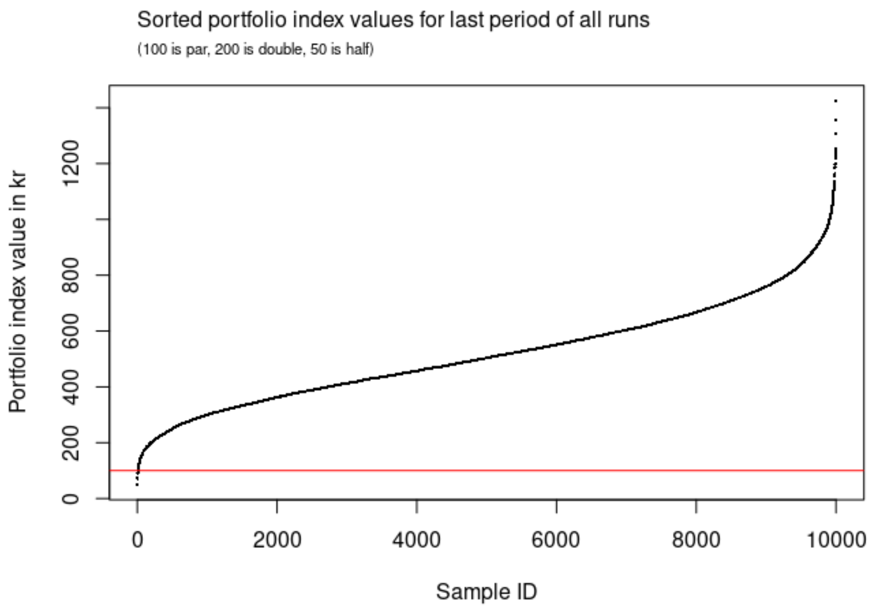


Version b: Mix of simulations from estimated distribution of returns from individual funds.

```
## Down-and-out simulation:
## Probability of down-and-out: 0 percent
```

```
##
## Mean portfolio index value after 20 years: 479.218 kr.
## SD of portfolio index value after 20 years: 164.197 kr.
## Min total portfolio index value after 20 years: 55.397 kr.
## Max total portfolio index value after 20 years: 1335.395 kr.
##
## Share of paths finishing below 100: 0.12 percent
```





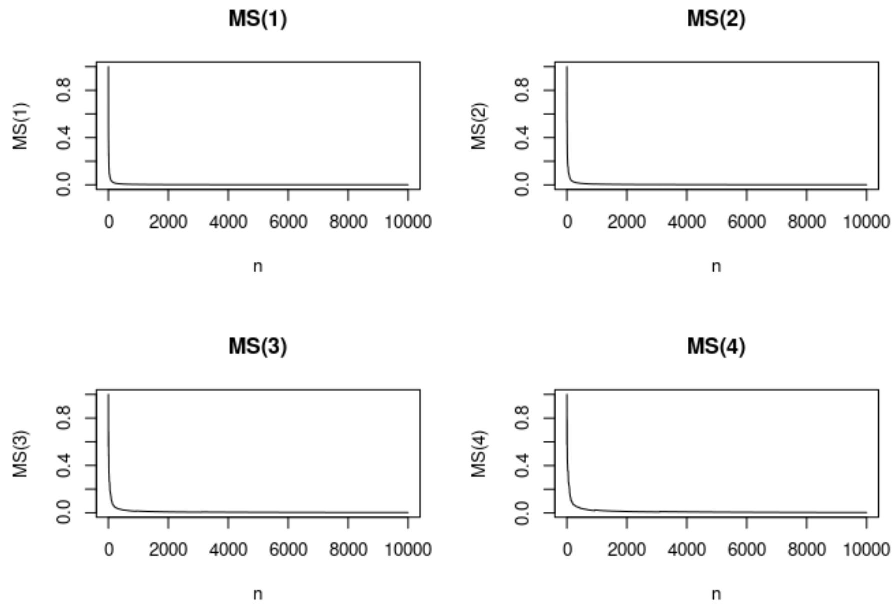
Many simulations 1e6 paths:

```
# Down-and-out simulation:
# Probability of down-and-out: 0 percent
#
# Mean portfolio index value after 20 years: 478.339 kr.
# SD of portfolio index value after 20 years: 163.093 kr.
# Min total portfolio index value after 20 years: 2.233 kr.
# Max total portfolio index value after 20 years: 1561.965 kr.
#
# Share of paths finishing below 100: 0.1181 percent
```

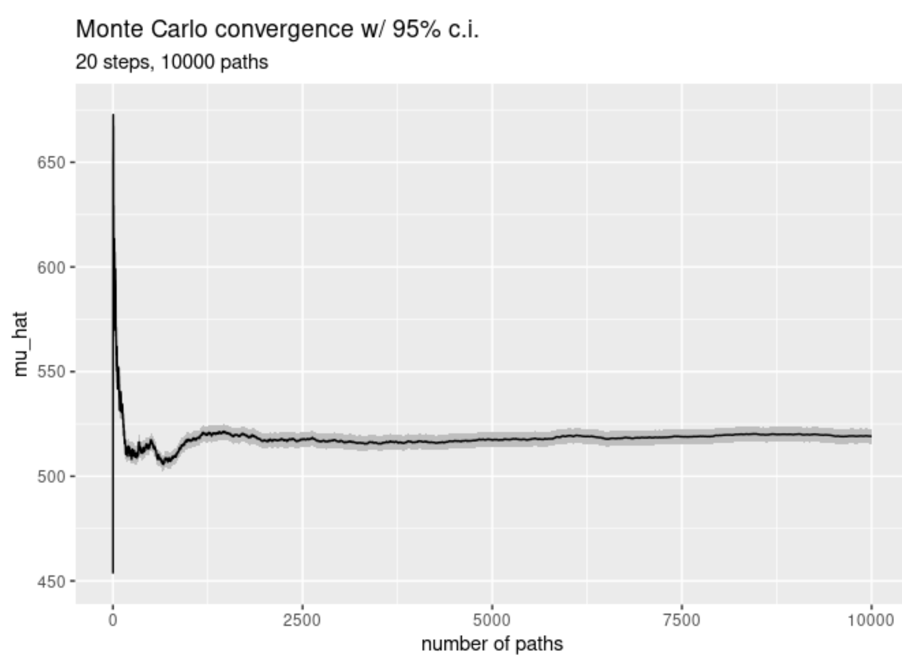
Convergence

Max vs sum

Max vs sum plots for the first four moments:


MS(1)
MS(2)
MS(3)
MS(4)
n

MC

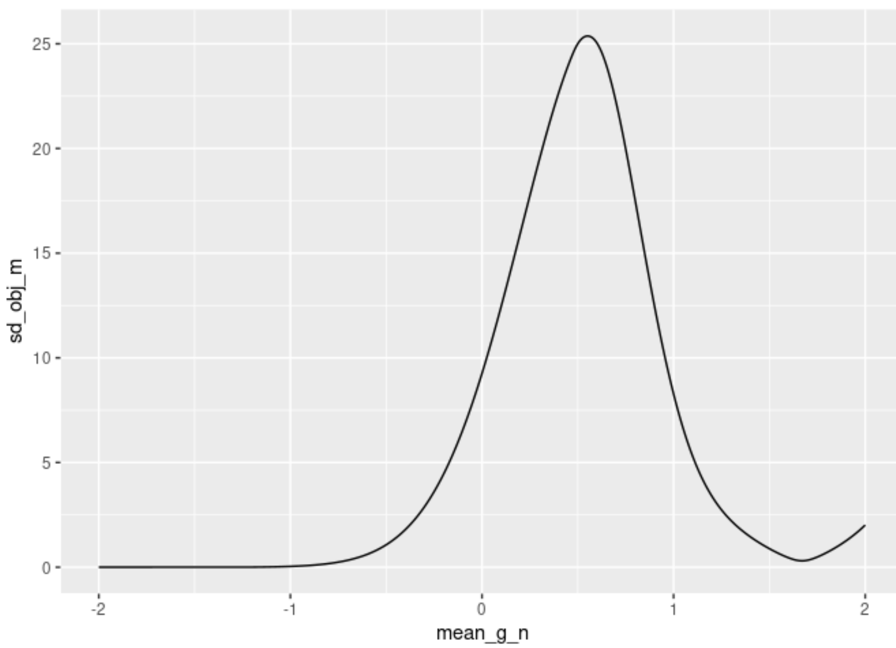

Monte Carlo convergence w/ 95% c.i.
20 steps, 10000 paths
mu_hat
number of paths

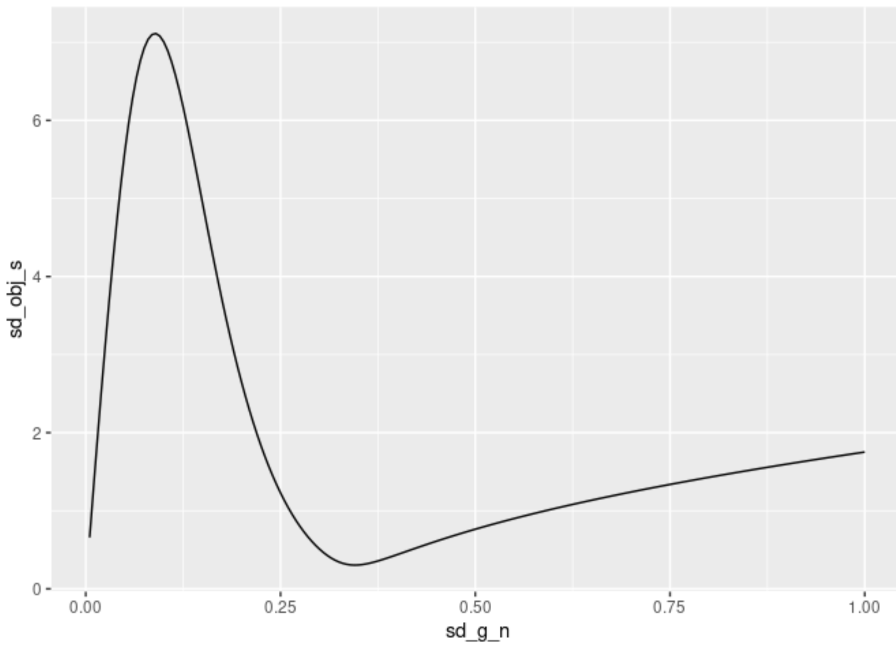
IS

Parameters

```
## [1] 1.5927304 0.3361558
```

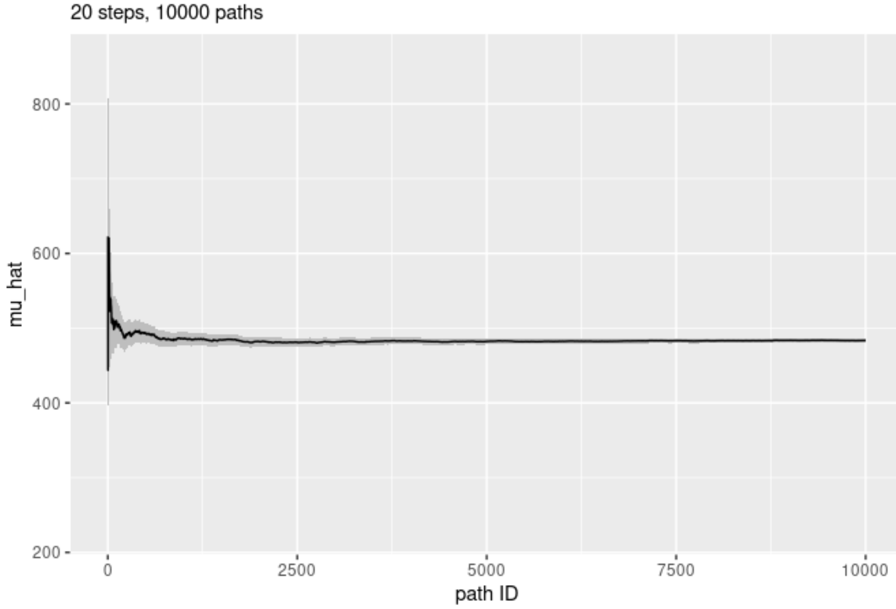
Objective function plots





Importance Sampling convergence w/ 95% c.i.

20 steps, 10000 paths



Compare pension plans

Risk of max loss

Risk of max loss of x percent for a single period (year).
x values are row names.

	Velliv_m	Velliv_m_l	Velliv_h	PFA_m	PFA_h	mix_m	mix_h
0	21.167	17.833	19.667	11.833	14.000	12.333	12.667
5	12.167	9.333	12.500	5.667	8.333	5.833	3.833
10	7.000	5.000	8.000	3.000	5.000	2.833	0.500
25	1.333	0.833	2.167	0.500	1.000	0.333	0.000
50	0.000	0.000	0.000	0.000	0.000	0.000	0.000
90	0.000	0.000	0.000	0.000	0.000	0.000	0.000
99	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Worst ranking for loss percentiles

0	ranking	5	ranking	10	ranking	25	ranking	50	ranking	90	ranking	99	ranking
21.167	Velliv_m	12.500	Velliv_h	8.000	Velliv_h	2.167	Velliv_h	0	Velliv_m	0	Velliv_m	0	Velliv_m
19.667	Velliv_h	12.167	Velliv_m	7.000	Velliv_m	1.333	Velliv_m	0	Velliv_m_l	0	Velliv_m_l	0	Velliv_m_l
17.833	Velliv_m_l	9.333	Velliv_m_l	5.000	Velliv_m_l	1.000	PFA_h	0	Velliv_h	0	Velliv_h	0	Velliv_h
14.000	PFA_h	8.333	PFA_h	5.000	PFA_h	0.833	Velliv_m_l	0	PFA_m	0	PFA_m	0	PFA_m
12.667	mix_h	5.833	mix_m	3.000	PFA_m	0.500	PFA_m	0	PFA_h	0	PFA_h	0	PFA_h
12.333	mix_m	5.667	PFA_m	2.833	mix_m	0.333	mix_m	0	mix_m	0	mix_m	0	mix_m
11.833	PFA_m	3.833	mix_h	0.500	mix_h	0.000	mix_h	0	mix_h	0	mix_h	0	mix_h

Chance of min gains

Chance of min gains of x percent for a single period (year).
x values are row names.

	Velliv_m	Velliv_m_l	Velliv_h	PFA_m	PFA_h	mix_m	mix_h
0	78.833	82.167	80.333	88.167	86.000	87.667	87.333
5	63.833	65.000	69.333	71.667	76.000	71.667	70.167
10	40.833	36.000	53.333	32.500	59.667	35.500	46.000
25	0.000	0.000	0.000	0.000	0.000	0.000	0.833
50	0.000	0.000	0.000	0.000	0.000	0.000	0.000
100	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Best ranking for gains percentiles

0	ranking	5	ranking	10	ranking	25	ranking	50	ranking	100	ranking
88.167	PFA_m	76.000	PFA_h	59.667	PFA_h	0.833	mix_h	0	Velliv_m	0	Velliv_m
87.667	mix_m	71.667	PFA_m	53.333	Velliv_h	0.000	Velliv_m	0	Velliv_m_l	0	Velliv_m_l
87.333	mix_h	71.667	mix_m	46.000	mix_h	0.000	Velliv_m_l	0	Velliv_h	0	Velliv_h
86.000	PFA_h	70.167	mix_h	40.833	Velliv_m	0.000	Velliv_h	0	PFA_m	0	PFA_m
82.167	Velliv_m_l	69.333	Velliv_h	36.000	Velliv_m_l	0.000	PFA_m	0	PFA_h	0	PFA_h
80.333	Velliv_h	65.000	Velliv_m_l	35.500	mix_m	0.000	PFA_h	0	mix_m	0	mix_m

0	ranking	5	ranking	10	ranking	25	ranking	50	ranking	100	ranking
78.833	Velliv_m	63.833	Velliv_m	32.500	PFA_m	0.000	mix_m	0	mix_h	0	mix_h

MC risk percentiles

Risk of loss from first to last period.

`_a` is simulation from estimated distribution of returns of mix.
`_b` is mix of simulations from estimated distribution of returns from individual funds.

`_m` is medium.
`_h` is high.

	Velliv_m	Velliv_m_l	Velliv_h	PFA_m	PFA_h	mix_m_a	mix_h_a	mix_m_b	mix_h_b
0	4.72	3.11	4.30	1.94	0.95	1.13	0	0.34	0.12
5	4.14	2.78	3.82	1.81	0.81	1.02	0	0.23	0.09
10	3.72	2.27	3.35	1.70	0.70	0.90	0	0.13	0.09
25	2.42	1.52	2.44	1.23	0.47	0.55	0	0.07	0.03
50	0.86	0.69	1.11	0.59	0.25	0.27	0	0.01	0.00
90	0.01	0.11	0.10	0.18	0.04	0.02	0	0.00	0.00
99	0.00	0.01	0.01	0.06	0.01	0.00	0	0.00	0.00

1e6 simulation paths of `mhr_b`:

	0	5	10	25	50	90	99
prob_pct	0.118	0.095	0.076	0.036	0.008	0	0

Worst ranking for MC loss percentiles

0	ranking	5	ranking	10	ranking	25	ranking	50	ranking	90	ranking	99	ranking
4.72	Velliv_m	4.14	Velliv_m	3.72	Velliv_m	2.44	Velliv_h	1.11	Velliv_h	0.18	PFA_m	0.06	PFA_m
4.30	Velliv_h	3.82	Velliv_h	3.35	Velliv_h	2.42	Velliv_m	0.86	Velliv_m	0.11	Velliv_m_l	0.01	Velliv_m_l
3.11	Velliv_m_l	2.78	Velliv_m_l	2.27	Velliv_m_l	1.52	Velliv_m_l	0.69	Velliv_m_l	0.10	Velliv_h	0.01	Velliv_h
1.94	PFA_m	1.81	PFA_m	1.70	PFA_m	1.23	PFA_m	0.59	PFA_m	0.04	PFA_h	0.01	PFA_h
1.13	mix_m_a	1.02	mix_m_a	0.90	mix_m_a	0.55	mix_m_a	0.27	mix_m_a	0.02	mix_m_a	0.00	Velliv_m
0.95	PFA_h	0.81	PFA_h	0.70	PFA_h	0.47	PFA_h	0.25	PFA_h	0.01	Velliv_m	0.00	mix_m_a
0.34	mix_m_b	0.23	mix_m_b	0.13	mix_m_b	0.07	mix_m_b	0.01	mix_m_b	0.00	mix_h_a	0.00	mix_h_a
0.12	mix_h_b	0.09	mix_h_b	0.09	mix_h_b	0.03	mix_h_b	0.00	mix_h_a	0.00	mix_m_b	0.00	mix_m_b
0.00	mix_h_a	0.00	mix_h_a	0.00	mix_h_a	0.00	mix_h_a	0.00	mix_h_b	0.00	mix_h_b	0.00	mix_h_b

MC gains percentiles

Chance of gains from first to last period.
`_a` is simulation from estimated distribution of returns of mix.
`_b` is mix of simulations from estimated distribution of returns from individual funds.

	Velliv_m	Velliv_m_l	Velliv_h	PFA_m	PFA_h	mix_m_a	mix_h_a	mix_m_b	mix_h_b
0	95.28	96.89	95.70	98.06	99.05	98.87	100.00	99.66	99.88
5	94.67	96.40	95.17	97.80	98.93	98.72	100.00	99.55	99.83
10	94.00	95.89	94.73	97.56	98.78	98.54	100.00	99.39	99.82
25	91.57	93.96	93.21	96.68	98.42	97.85	100.00	99.01	99.72
50	85.87	89.95	90.33	94.73	97.50	96.23	100.00	97.42	99.31
100	71.62	78.11	83.11	87.55	94.60	89.72	99.66	89.73	97.46
200	39.14	44.61	64.53	58.58	85.28	59.42	93.18	48.34	87.18
300	16.10	18.01	44.83	22.36	71.62	22.28	71.24	11.58	65.20
400	5.20	4.56	29.08	4.45	54.84	3.77	43.90	1.27	42.17
500	1.45	1.16	17.33	0.59	39.30	0.22	22.71	0.07	22.12
1000	0.00	0.03	0.59	0.01	2.28	0.00	0.25	0.00	0.11

1e6 simulation paths of `mhr_b`:

	0	5	10	25	50	100	200	300	400	500	1000
prob	99.882	99.854	99.824	99.686	99.301	97.513	86.912	65.992	41.486	21.693	0.086

Best ranking for MC gains percentiles

0	ranking	5	ranking	10	ranking	25	ranking	50	ranking	100	ranking
100.00	mix_h_a	100.00	mix_h_a	100.00	mix_h_a	100.00	mix_h_a	100.00	mix_h_a	99.66	mix_h_a
99.88	mix_h_b	99.83	mix_h_b	99.82	mix_h_b	99.72	mix_h_b	99.31	mix_h_b	97.46	mix_h_b
99.66	mix_m_b	99.55	mix_m_b	99.39	mix_m_b	99.01	mix_m_b	97.50	PFA_h	94.60	PFA_h
99.05	PFA_h	98.93	PFA_h	98.78	PFA_h	98.42	PFA_h	97.42	mix_m_b	89.73	mix_m_b
98.87	mix_m_a	98.72	mix_m_a	98.54	mix_m_a	97.85	mix_m_a	96.23	mix_m_a	89.72	mix_m_a
98.06	PFA_m	97.80	PFA_m	97.56	PFA_m	96.68	PFA_m	94.73	PFA_m	87.55	PFA_m
96.89	Velliv_m_l	96.40	Velliv_m_l	95.89	Velliv_m_l	93.96	Velliv_m_l	90.33	Velliv_h	83.11	Velliv_h
95.70	Velliv_h	95.17	Velliv_h	94.73	Velliv_h	93.21	Velliv_h	89.95	Velliv_m_l	78.11	Velliv_m_l
95.28	Velliv_m	94.67	Velliv_m	94.00	Velliv_m	91.57	Velliv_m	85.87	Velliv_m	71.62	Velliv_m

200	ranking	300	ranking	400	ranking	500	ranking	1000	ranking
93.18	mix_h_a	71.62	PFA_h	54.84	PFA_h	39.30	PFA_h	2.28	PFA_h
87.18	mix_h_b	71.24	mix_h_a	43.90	mix_h_a	22.71	mix_h_a	0.59	Velliv_h
85.28	PFA_h	65.20	mix_h_b	42.17	mix_h_b	22.12	mix_h_b	0.25	mix_h_a
64.53	Velliv_h	44.83	Velliv_h	29.08	Velliv_h	17.33	Velliv_h	0.11	mix_h_b
59.42	mix_m_a	22.36	PFA_m	5.20	Velliv_m	1.45	Velliv_m	0.03	Velliv_m_l
58.58	PFA_m	22.28	mix_m_a	4.56	Velliv_m_l	1.16	Velliv_m_l	0.01	PFA_m
48.34	mix_m_b	18.01	Velliv_m_l	4.45	PFA_m	0.59	PFA_m	0.00	Velliv_m
44.61	Velliv_m_l	16.10	Velliv_m	3.77	mix_m_a	0.22	mix_m_a	0.00	mix_m_a
39.14	Velliv_m	11.58	mix_m_b	1.27	mix_m_b	0.07	mix_m_b	0.00	mix_m_b

Summary statistics

Fit summary

Summary for fit of log returns to an F-S skew standardized Student-t distribution.
`m` is the location parameter.

`s` is the scale parameter.
`nu` is the estimated degrees of freedom, or shape parameter.
`xi` is the estimated skewness parameter.

	Velliv_medium	Velliv_medium_long	Velliv_high	PFA_medium	PFA_high	mix_medium	mix_high
m	0.048	0.052	0.065	0.058	0.084	0.059	0.082
s	0.120	0.115	0.150	0.123	0.121	0.088	0.071
nu	3.304	2.706	3.144	2.265	3.185	2.773	89.863
xi	0.034	0.505	0.002	0.477	0.018	0.029	0.770
R-squared	0.993	0.978	0.991	0.991	0.964	0.890	0.961

Fit statistics ranking

m	ranking	s	ranking	R-squared	ranking
0.084	PFA_high	0.071	mix_high	0.993	Velliv_medium
0.082	mix_high	0.088	mix_medium	0.991	Velliv_high
0.065	Velliv_high	0.115	Velliv_medium_long	0.991	PFA_medium
0.059	mix_medium	0.120	Velliv_medium	0.978	Velliv_medium_long
0.058	PFA_medium	0.121	PFA_high	0.964	PFA_high
0.052	Velliv_medium_long	0.123	PFA_medium	0.961	mix_high
0.048	Velliv_medium	0.150	Velliv_high	0.890	mix_medium

Monte Carlo simulations summary

Monte Carlo simulations of portfolio index values (currency values).
Statistics are given for the final state of all paths.
Probability of down-and_out is calculated as the share of paths that reach 0 at some point. All subsequent values for a path are set to 0, if the path reaches at any point.
0 is defined as any value below a threshold.
`losing_prob_pct` is the probability of losing money. This is calculated as the share of paths finishing below index 100.

```
## Number of paths: 10000
```

	Velliv_m	Velliv_m_l	Velliv_h	PFA_m	PFA_h	mix_m_a	mix_m_b	mix_h_a	mix_h_b
mc_m	278.995	293.384	404.855	321.882	555.222	324.055	300.815	498.653	479.218
mc_s	123.583	118.147	217.274	106.531	244.751	98.254	82.422	155.916	164.197
mc_min	9.812	0.785	0.010	0.010	0.765	3.973	42.661	150.860	55.397
mc_max	929.858	2029.133	1713.743	1121.574	1828.260	723.752	1008.214	1463.716	1335.395
dao_pct	0.000	0.000	0.010	0.010	0.000	0.000	0.000	0.000	0.000
losing_pct	4.720	3.110	4.300	1.940	0.950	1.130	0.340	0.000	0.120

Ranking

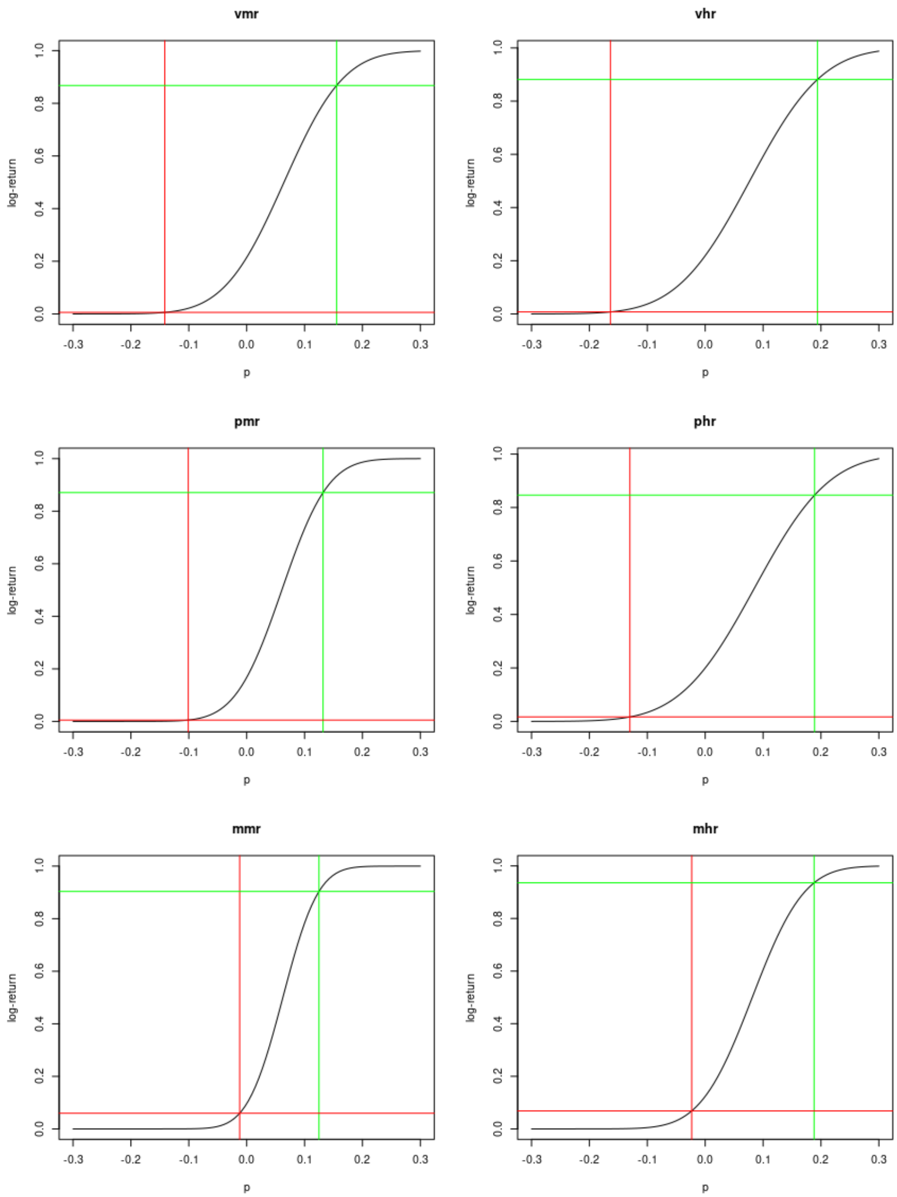
mc_m	ranking	mc_s	ranking	mc_min	ranking	mc_max	ranking	dao_pct	ranking	losing_pct	ranking
555.222	PFA_h	82.422	mix_m_b	150.860	mix_h_a	2029.133	Velliv_m_l	0.00	Velliv_m	0.00	mix_h_a

mc_m	ranking	mc_s	ranking	mc_min	ranking	mc_max	ranking	dao_pct	ranking	losing_pct	ranking
498.653	mix_h_a	98.254	mix_m_a	55.397	mix_h_b	1828.260	PFA_h	0.00	Velliv_m_l	0.12	mix_h_b
479.218	mix_h_b	106.531	PFA_m	42.661	mix_m_b	1713.743	Velliv_h	0.00	PFA_h	0.34	mix_m_b
404.855	Velliv_h	118.147	Velliv_m_l	9.812	Velliv_m	1463.716	mix_h_a	0.00	mix_m_a	0.95	PFA_h
324.055	mix_m_a	123.583	Velliv_m	3.973	mix_m_a	1335.395	mix_h_b	0.00	mix_m_b	1.13	mix_m_a
321.882	PFA_m	155.916	mix_h_a	0.785	Velliv_m_l	1121.574	PFA_m	0.00	mix_h_a	1.94	PFA_m
300.815	mix_m_b	164.197	mix_h_b	0.765	PFA_h	1008.214	mix_m_b	0.00	mix_h_b	3.11	Velliv_m_l
293.384	Velliv_m_l	217.274	Velliv_h	0.010	Velliv_h	929.858	Velliv_m	0.01	Velliv_h	4.30	Velliv_h
278.995	Velliv_m	244.751	PFA_h	0.010	PFA_m	723.752	mix_m_a	0.01	PFA_m	4.72	Velliv_m

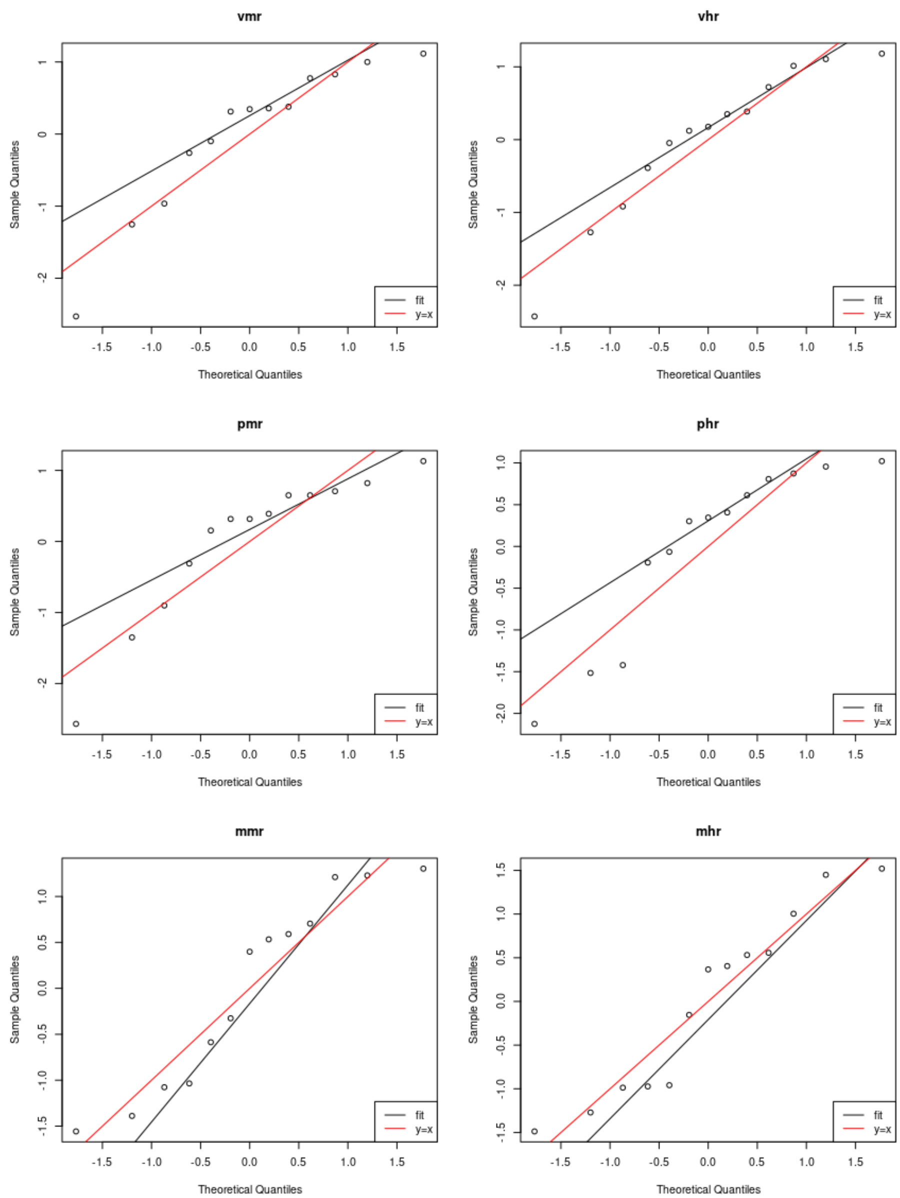
Compare Gaussian and skewed t-distribution fits

Gaussian fits

	vmr	vhr	pmr	phr	mmr	mhr
m	0.064	0.077	0.061	0.085	0.062	0.081
s	0.081	0.099	0.063	0.101	0.048	0.070


vmr
vhr
pmr
phr
mmr
mhr
log-return
p
0.0 0.2 0.4 0.6 0.8 1.0
-0.3 -0.2 -0.1 0.0 0.1 0.2 0.3

Gaussian QQ plots



Gaussian vs skewed t

Probability in percent that the smallest and largest (respectively) observed return for each fund was generated by a normal distribution:

	vmr	vhr	pmr	phr	mmr	mhr
P_norm(X_min)	0.571	0.758	0.511	1.676	5.971	6.842
P_norm(X_max)	13.230	11.876	12.922	15.359	9.628	6.429
P_t(X_min)	5.377	5.457	3.489	4.315	10.570	8.015
P_t(X_max)	0.118	0.001	2.825	0.188	0.488	5.141

Average number of years between min or max events (respectively):

	vmr	vhr	pmr	phr	mmr	mhr
norm: avg yrs btw min	175.248	131.911	195.568	59.669	16.748	14.616
norm: avg yrs btw max	7.559	8.420	7.739	6.511	10.386	15.556
t: avg yrs btw min	18.596	18.324	28.663	23.173	9.461	12.476
t: avg yrs btw max	848.548	178349.076	35.400	531.552	205.104	19.450

Comments

(Ignoring `mhr_a`…)

`mhr` has some nice properties:
- It has a relatively high `nu` value of 90, which means it is tending more towards exponential tails than polynomial tails. All other funds have `nu` values close to 3, except `phr` which is even worse at close to 2. (Note that for a Gaussian, `nu` is infinite.)
- It has the lowest losing percentage of all simulations, which is better than 1/6 that of `phr`.
- It has a DAO percentage of 0, which is the same as `mmr`, and less than `phr`.
- Only `phr` has a higher `mc_m`.
- It has a smaller `mc_s` than the individual components, `vhr` and `phr`.
- It has the highest `xi` of all fits, suggesting less left skewness. Density plots for `vmr`, `phr` and `mmr` have an extremely sharp drop, as if an upward limiter has been applied, which corresponds to extremely low `xi` values. The density plot for `mhr` is by far the most symmetrical of all the fits. As seen in the section "Compare Gaussian and skewed t-distribution fits", the other skewed t-distribution fits don't capture the max observed returns at all.
- Only `mmr` has as higher `mc_min`. However, that of `mmr` is 18 times higher with 62, so `mmr` is a clear winner here.
- Naturally, it has a `mc_max` smaller than the individual components, `vhr` and `phr`, but ca. 1.5 times higher then `mmr`.
- All the first 4 moments converge nicely. For all other fits, the 4th moment doesn't seem to converge.

Taleb, Statistical Consequences Of Fat Tails, p. 97:
"the variance of a finite variance random variable with tail exponent < 4 will be infinite".

And p. 363:
"The hedging errors for an option portfolio (under a daily revision regime) over 3000 days, un- der a constant volatility Student T with tail exponent $\alpha = 3$. Technically the errors should not converge in finite time as their distribution has infinite variance."

Appendix

Average of returns vs returns of average

Math

$$\text{Avg. of returns} := \frac{\left(\frac{x_t}{x_{t-1}} + \frac{y_t}{y_{t-1}}\right)}{2}$$

$$\text{Returns of avg.} := \left(\frac{x_t + y_t}{2}\right) \Big/ \left(\frac{x_{t-1} + y_{t-1}}{2}\right) \equiv \frac{x_t + y_t}{x_{t-1} + y_{t-1}}$$

For which x_1 and y_1 are Avg. of returns $=$ Returns of avg.?

$$\frac{\left(\frac{x_t}{x_{t-1}} + \frac{y_t}{y_{t-1}}\right)}{2} = \frac{x_t + y_t}{x_{t-1} + y_{t-1}}$$

$$\frac{x_t}{x_{t-1}} + \frac{y_t}{y_{t-1}} = 2\frac{x_t + y_t}{x_{t-1} + y_{t-1}}$$

$$(x_{t-1} + y_{t-1})x_t y_{t-1} + (x_{t-1} + y_{t-1})x_{t-1}y_t = 2(x_{t-1}y_{t-1}x_t + x_{t-1}y_{t-1}y_t)$$

$$(x_{t-1}x_1y_{t-1} + y_{t-1}x_t y_{t-1}) + (x_{t-1}x_{t-1}y_t + x_{t-1}y_{t-1}y_t) = 2(x_{t-1}y_{t-1}x_t + x_{t-1}y_{t-1}y_t)$$

This is not generally true, but true if for instance $x_{t-1} = y_{t-1}$.

Example

Definition: `R = 1+r`

```
## Let x_0 be 100.
```

```
## Let y_0 be 200.
```

```
## So the initial value of the pf is 300 .
```

```
## Let R_x be 0.5.
```

```
## Let R_y be 1.5.
```

Then,

```
## x_1 is R_x * x_0 = 50.
```

```
## y_1 is R_y * y_0 = 300.
```

Average of returns:

```
## 0.5 * (R_x + R_y) = 1
```

So here the value of the pf at t=1 should be unchanged from t=0:

```
## (x_0 + y_0) * 0.5 * (R_x + R_y) = 300
```

But this is clearly not the case:

```
## 0.5 * (x_1 + y_1) = 0.5 * (R_x * x_0 + R_y * y_0) = 175
```

Therefore we should take returns of average, not average of returns!

Let's take the average of log returns instead:

```
## 0.5 * (log(R_x) + log(R_y)) = -0.143841
```

We now get:

```
## (x_0 + y_0) * exp(0.5 * (log(Rx) + log(Ry))) = 259.8076
```

So taking the average of log returns doesn't work either.

Simulation of mix vs mix of simulations

Test if a simulation of a mix (average) of two returns series has the same distribution as a mix of two simulated returns series.

```
## m(data_x): -0.008375401
## s(data_x): 0.4184349
## m(data_y): 9.445322
## s(data_y): 2.665942
##
## m(data_x + data_y): 4.718473
## s(data_x + data_y): 1.429784
```

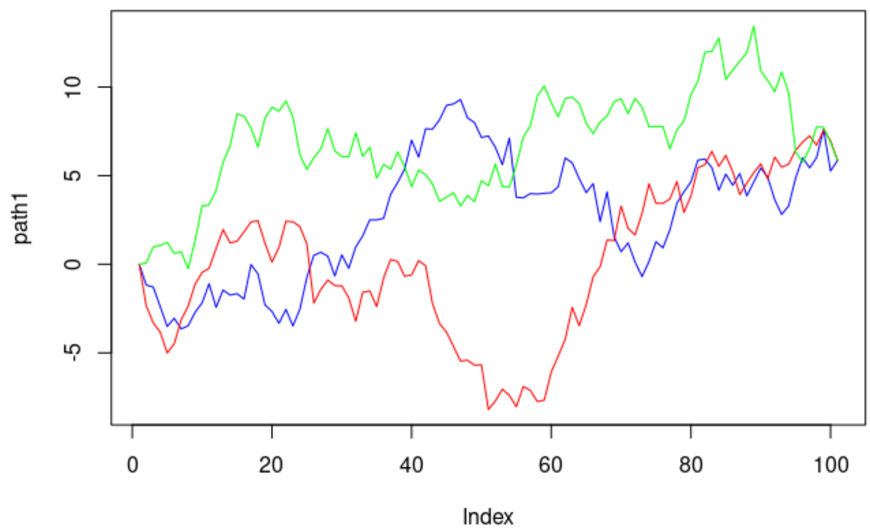
m and s of final state of all paths.
`_a` is mix of simulated returns.
`_b` is simulated mixed returns.

m_a	m_b	s_a	s_b
94.308	94.032	5.930	6.447
94.046	94.419	6.020	6.583
94.537	94.423	6.064	6.567
94.585	94.491	6.202	6.386
94.470	94.184	6.192	6.242
94.480	94.510	6.141	6.366
94.484	94.503	5.843	6.303
94.501	94.179	5.840	6.397
94.314	94.193	5.941	6.350
94.222	94.482	6.180	6.661

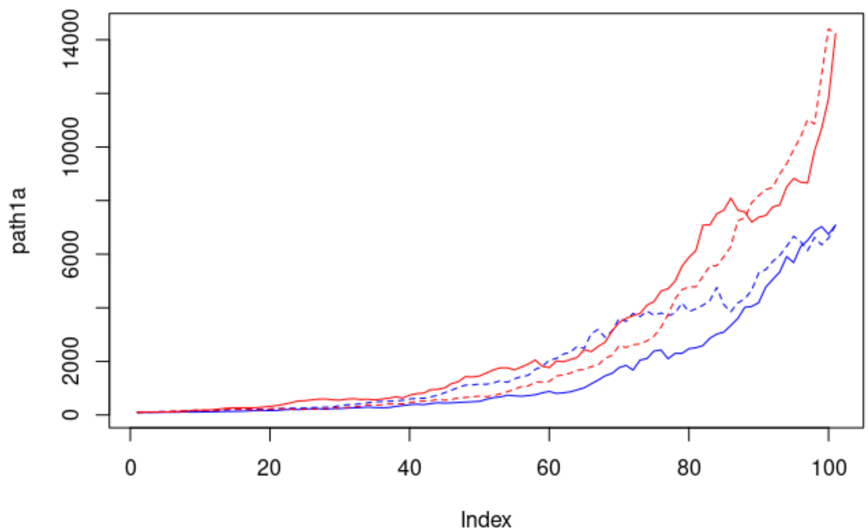
```
##       m_a             m_b             s_a             s_b
##  Min.   :94.05   Min.   :94.03   Min.   :5.840   Min.   :6.242
##  1st Qu.:94.31   1st Qu.:94.19   1st Qu.:5.933   1st Qu.:6.354
##  Median :94.48   Median :94.42   Median :6.042   Median :6.391
##  Mean   :94.39   Mean   :94.34   Mean   :6.035   Mean   :6.430
##  3rd Qu.:94.50   3rd Qu.:94.49   3rd Qu.:6.170   3rd Qu.:6.537
##  Max.   :94.59   Max.   :94.51   Max.   :6.202   Max.   :6.661
```

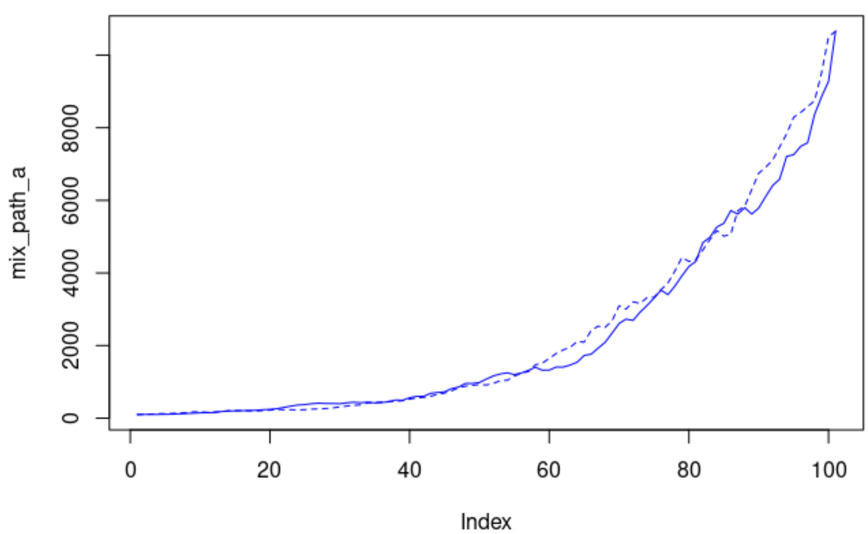
`_a` and `_b` are very close to equal.
We attribute the differences to differences in estimating the distributions in version a and b.

The final state is independent of the order of the preceding steps:



So does the order of the steps in the two processes matter, when mixing simulated returns?





The order of steps in the individual paths do not matter, because the mix of simulated paths is a sum of a sum, so the order of terms doesn't affect the sum. If there is variation it is because the sets preceding steps are not the same. For instance, the steps between step 1 and 60 in the plot above are not the same for the two lines.

Recall,

$$\text{Var}(aX + bY) = a^2\text{Var}(X) + b^2\text{Var}(Y) + 2ab\text{Cov}(a, b)$$

```
var(0.5 * vhr + 0.5 * phr)
```

```
## [1] 0.005355618
```

```
0.5^2 * var(vhr) + 0.5^2 * var(phr) + 2 * 0.5 * 0.5 * cov(vhr, phr)
```

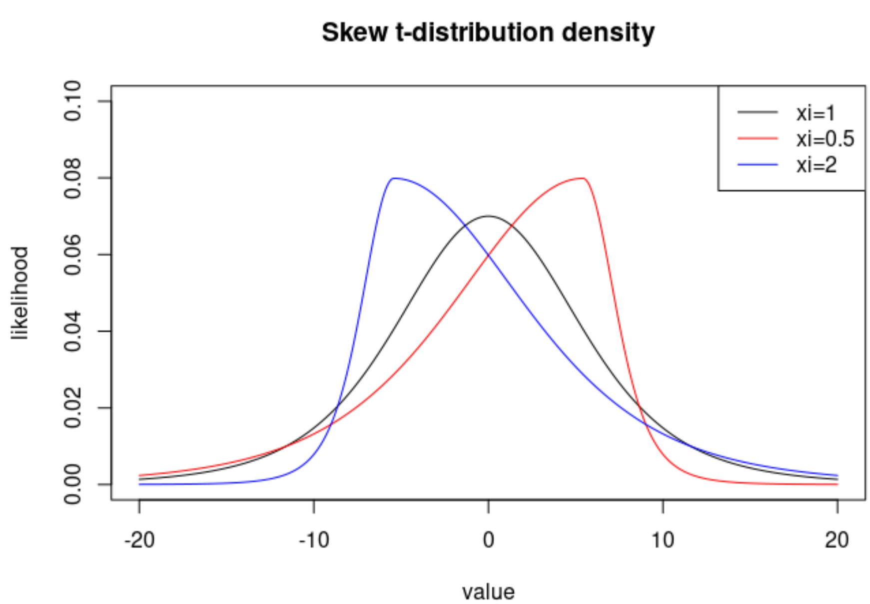
```
## [1] 0.005355618
```

Our distribution estimate is based on 13 observations. Is that enough for a robust estimate? What if we suddenly hit a year like 2008? How would that affect our estimate?
Let's try to include the Velliv data from 2007-2010.
We do this by sampling 13 observations from `vmrl`.

```
##        m                  s
##  Min.   :0.05940   Min.   :0.04936
##  1st Qu.:0.06631   1st Qu.:0.05986
##  Median :0.07033   Median :0.06726
##  Mean   :0.07071   Mean   :0.06858
##  3rd Qu.:0.07282   3rd Qu.:0.07670
##  Max.   :0.08455   Max.   :0.09120
```

The meaning of `xi`

The fit for `mhr` has the highest `xi` value of all. This suggests right-skew:



Max vs sum plot

If the Law Of Large Numbers holds true,

$$\frac{\max(X_1^p, ..., X^p)}{\sum_{i=1}^{n} X_i^p} \to 0$$

for $n \to \infty$.

If not, X doesn't have a p'th moment.

See Taleb: The Statistical Consequences Of Fat Tails, p. 192