A study of the skewed generalized t-distribution

14:14 18 May 2024

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Estimating the parameters of a sum skewed t distributed r.v's

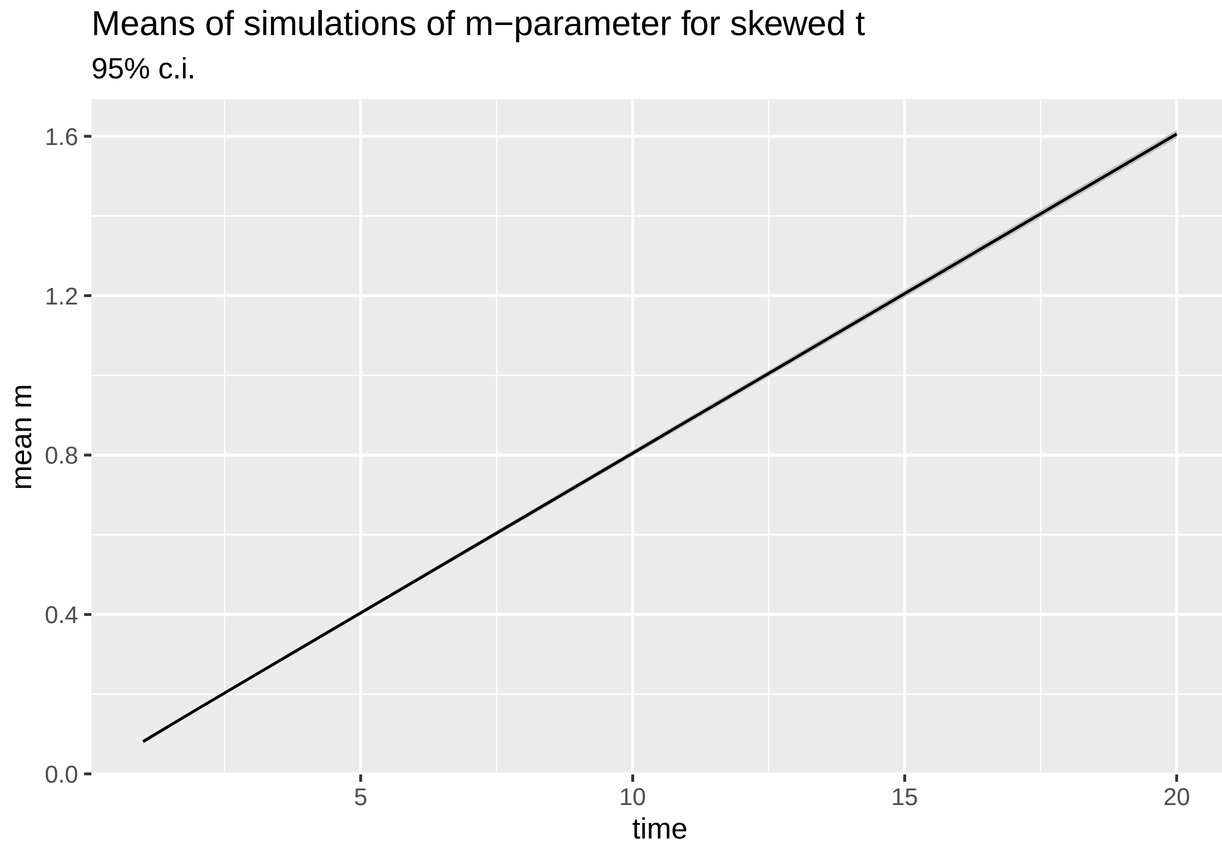
```
num_simulations <- 1000
num_paths <- 10000
num_periods <- 20
sim_params <- c(0.08, 0.12, 3.18, 0.02)
```

Estimated parameters for sum, S_{20}:

	m	s	nu	xi
m	1.605	0.522	8.856	0.653
s	0.005	0.006	0.846	0.011

```
num_simulations <- 100
num_paths <- 10000
num_periods <- 20
sim_params <- c(0.08, 0.12, 3.18, 0.02)
```

Estimating the m parameter of the skewed t distribution

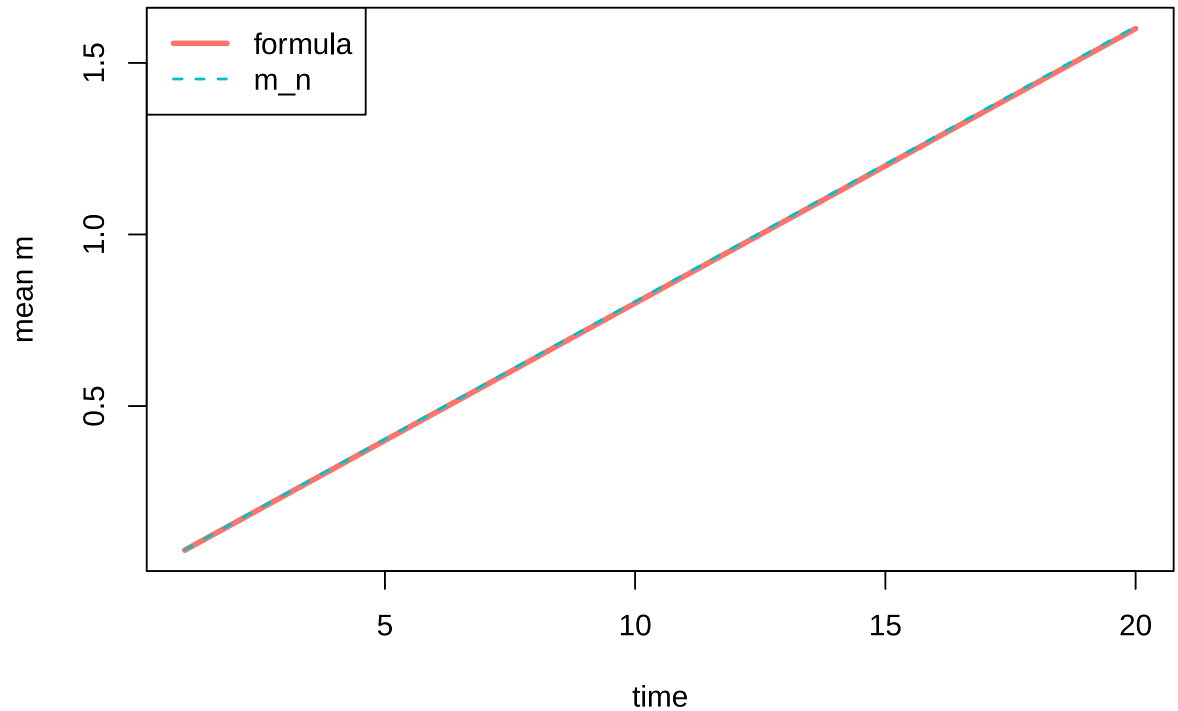


This looks like:

$$m(n) = n \cdot m(0)$$

	1	2	3	4	5	6	7	8	9	10
formula	0.080	0.160	0.240	0.320	0.400	0.480	0.560	0.640	0.720	0.800
m_n	0.081	0.163	0.243	0.324	0.404	0.484	0.564	0.644	0.725	0.805

	11	12	13	14	15	16	17	18	19	20
formula	0.880	0.960	1.040	1.120	1.200	1.280	1.360	1.440	1.520	1.600
m_n	0.885	0.965	1.045	1.125	1.205	1.285	1.365	1.446	1.526	1.606

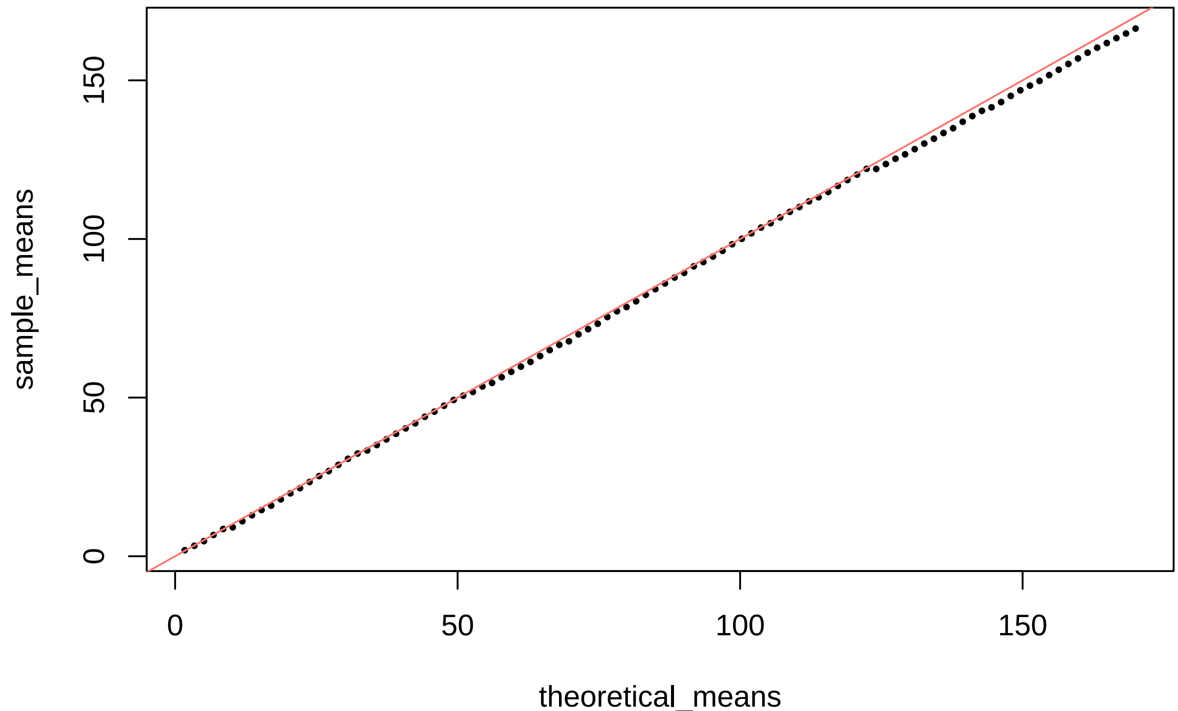


Verify $m(n) = n \cdot m(0)$

Test if the sample means of a sum of F-S Skewed t r.v.'s are equal to n times the mean of a single r.v. from that distribution.

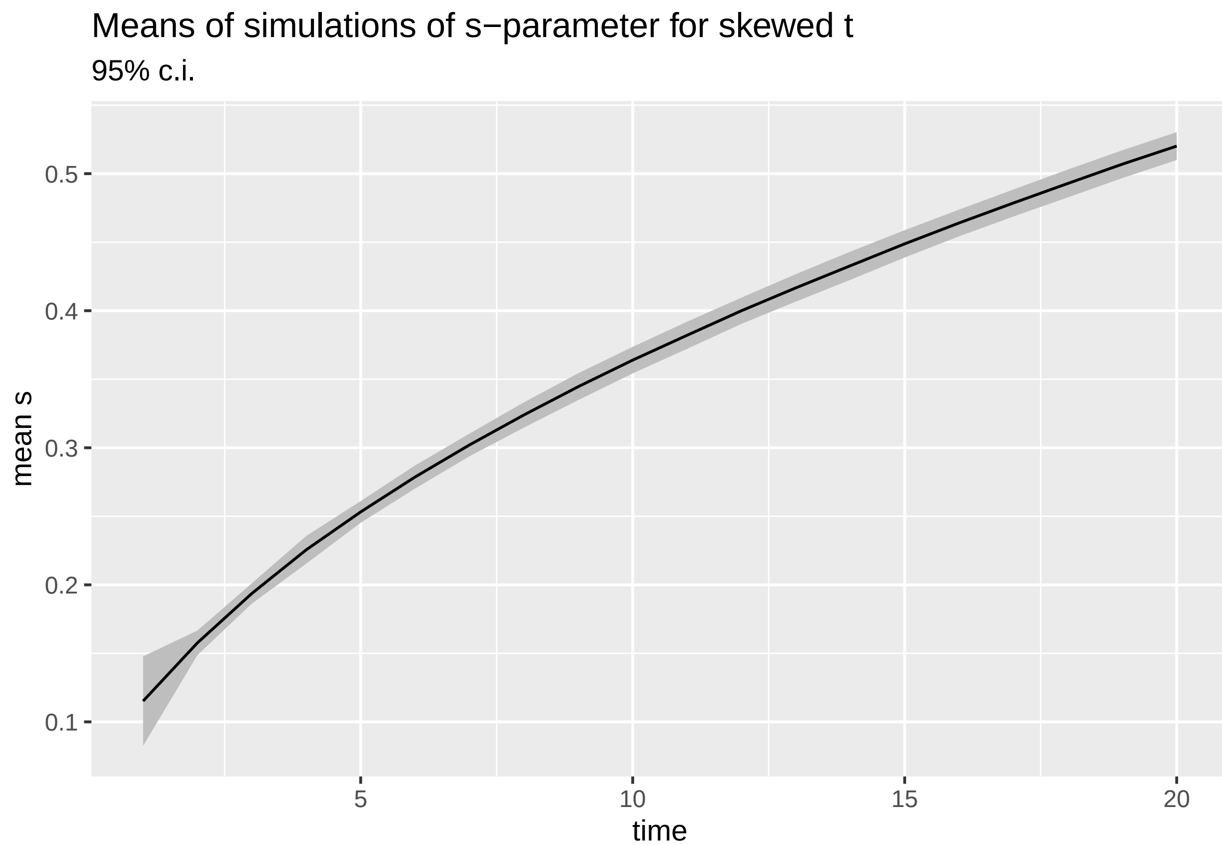
Even with only 50 samples to estimate the sample mean, the fit is spot on.

```
num_sim <- 50
n <- 100
mu <- 1.7
sigma <-  2
nu <- 3
xi <- 0.25
```



sample_means
0
50
100
150
0
50
100
150
theoretical_means

Estimating the s parameter of the skewed t distribution

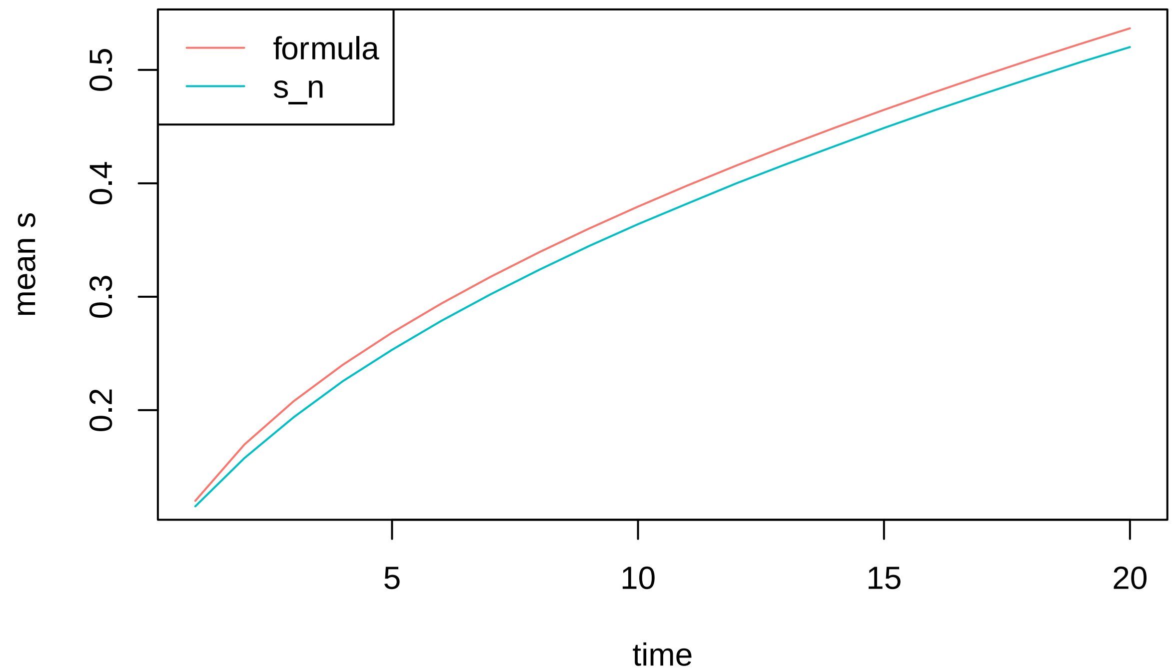


This looks like:

$$s(n) = \sqrt{ns(1)^2}$$

	1	2	3	4	5	6	7	8	9	10
formula	0.120	0.170	0.208	0.240	0.268	0.294	0.317	0.339	0.360	0.379
s_n	0.115	0.158	0.194	0.226	0.253	0.279	0.302	0.324	0.345	0.364

	11	12	13	14	15	16	17	18	19	20
formula	0.398	0.416	0.433	0.449	0.465	0.480	0.495	0.509	0.523	0.537
s_n	0.382	0.400	0.417	0.433	0.449	0.464	0.479	0.493	0.507	0.520



Let's use that (according to Taleb, ch. 8)

$$\mathbb{M}(n) = n^{1/\tilde{\alpha}}\mathbb{M}(1) = n^{1/\tilde{\alpha}}\sqrt{\frac{2}{\pi}}\sigma$$

where

$$\tilde{\alpha} = \alpha I\{\alpha < 0\} + 2I\{\alpha <\geq 0\}$$

for the Power Law class and otherwise

$$\tilde{\alpha} = 2$$

We'll implement this as formula 1.

Let formula 2 be:

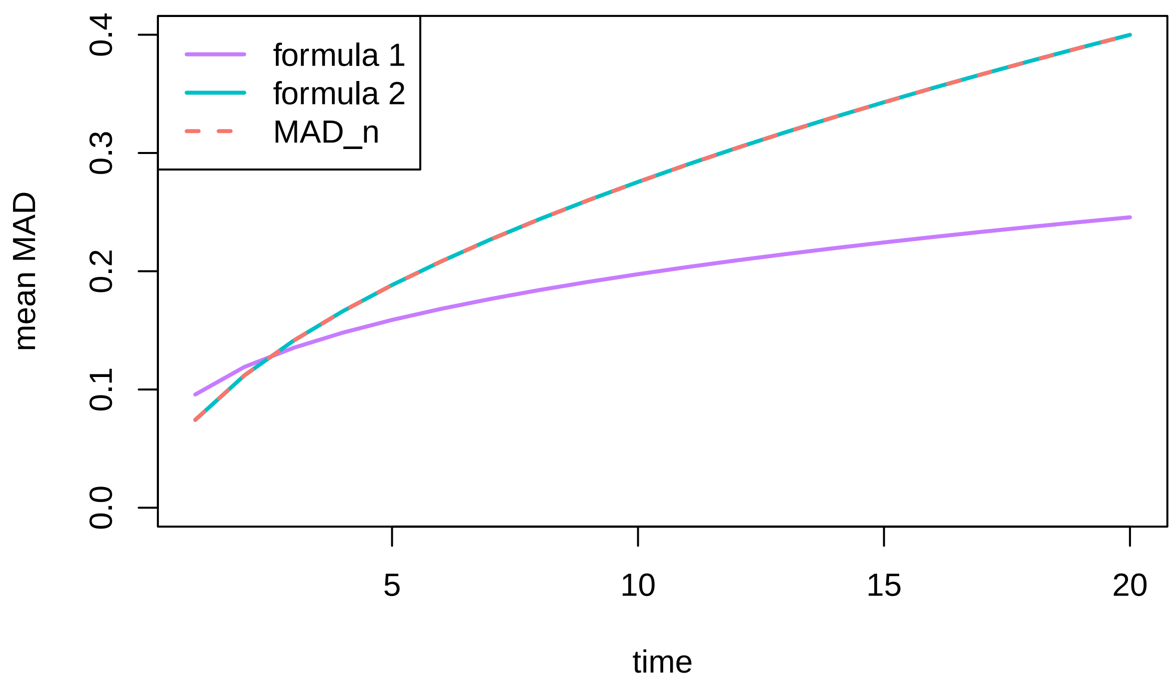
$$\mathbb{M}(n) = \frac{\sum_k^m |S_{n,k} - n\mu_{X_i}|}{n}$$

where μ_{X_i} is the location parameter we use to generate $\{X_i\}$, and $S_{n,k}$, is the sum of $X_{k,1}, X_{k,2}, \ldots, X_{k,n}$ for the k'th simulation.

```
num_simulations <- 1000
num_paths <- 10000
num_periods <- 20
sim_params <- c(0.08, 0.12, 3.18, 0.02)
```

	1	2	3	4	5	6	7	8	9	10
f1	0.0957	0.1191	0.1353	0.1481	0.1588	0.1682	0.1766	0.1841	0.1911	0.1975
f2	0.0743	0.1119	0.1413	0.1662	0.1883	0.2084	0.2268	0.2441	0.2602	0.2755
s_n	0.0743	0.1119	0.1413	0.1662	0.1883	0.2083	0.2268	0.2440	0.2602	0.2755

	11	12	13	14	15	16	17	18	19	20
f1	0.2035	0.2092	0.2145	0.2196	0.2244	0.229	0.2334	0.2376	0.2417	0.2456
f2	0.2902	0.3041	0.3176	0.3305	0.3429	0.355	0.3667	0.3781	0.3892	0.3999
s_n	0.2901	0.3041	0.3175	0.3304	0.3429	0.355	0.3667	0.3780	0.3891	0.3999



Formula 2 matches the realized MAD of $\{S_{n,k}\}_k^m$ very well.
Formula 1 matches the realized MAD of $\{S_{n,k}\}_k^m$ very badly.

We notice that:

$$\sqrt{\frac{2}{\pi}}\sigma_{X_i} = 0.09575$$

$$\sqrt{\frac{2}{\pi}}\hat{\sigma}_{X_i} = \text{approx.}0.09$$

```r
sd(rsstd(1e5,
         mean = sim_params[1], sd = sim_params[2], nu = sim_params[3], xi = sim_params[4])
   ) * sqrt(2/pi)
```

```
## [1] 0.09695222
```

while

$$\mathbb{M}(1) = \frac{\sum_{k=1}^{m} |X_{k,i} - n\overline{X_i}|}{m} = \text{approx.} 0.07$$

```r
f_mad(rsstd(1e5, mean = sim_params[1], sd = sim_params[2], nu = sim_params[3], xi = sim_params[4]))
```

```
## [1] 0.07403937
```

But are we supposed to see that

$$\mathbb{M}(1) = \sqrt{\frac{2}{\pi}}\sigma$$

??

Actually no:

The relation $\mathbb{M}^{\nu}(1) = \sqrt{\frac{2}{\pi}}\sigma$ only holds for the standard normal distribution, and only asymptotically. The ratio $\frac{\text{MAD}(X)}{\text{sd}(X)}$ varies between 0 and 1.

https://en.wikipedia.org/wiki/Average_absolute_deviation#Mean_absolute_deviation_around_the_mean

For a Student t with 3 df, the ratio is $\frac{\pi}{2}$. (See Taleb ch. 4 for further discussion.)

```r
sd(rsstd(1e5, mean = sim_params[1], sd = sim_params[2], nu = sim_params[3], xi = sim_params[4])) * 2/pi
```

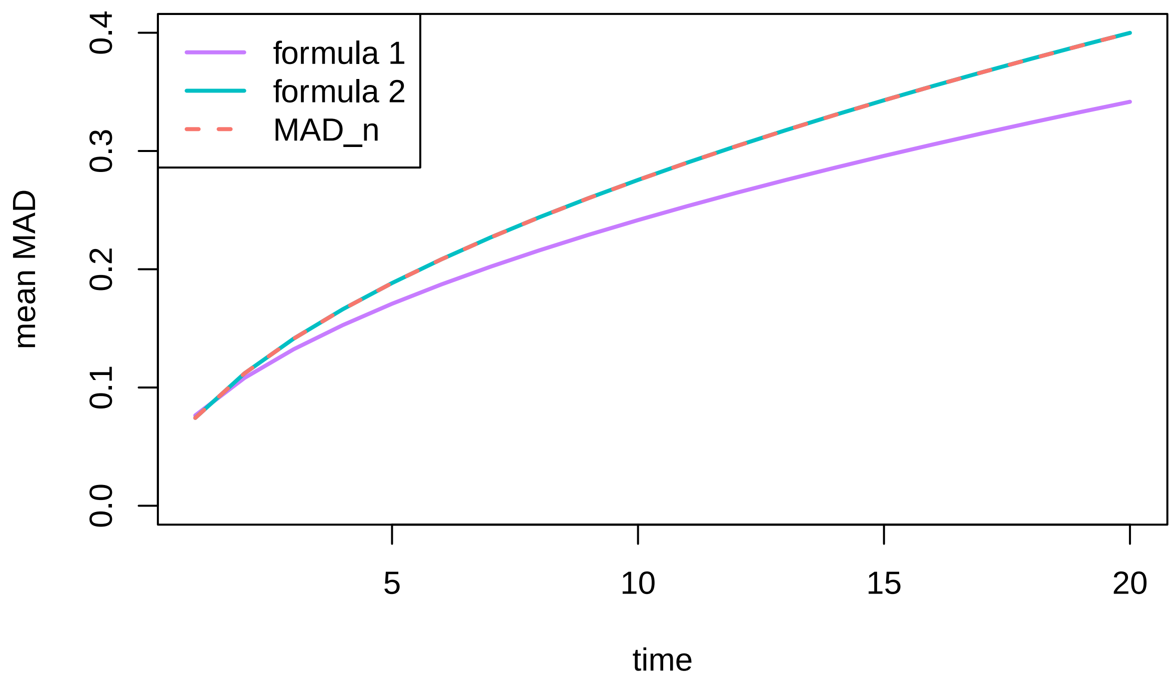
```
## [1] 0.07714997
```

That's better!

So let's use this ratio instead:

	1	2	3	4	5	6	7	8	9	10
f1	0.0764	0.1080	0.1323	0.1528	0.1708	0.1871	0.2021	0.2161	0.2292	0.2416
f2	0.0743	0.1119	0.1413	0.1662	0.1883	0.2084	0.2268	0.2441	0.2602	0.2755
s_n	0.0743	0.1119	0.1413	0.1662	0.1883	0.2083	0.2268	0.2440	0.2602	0.2755

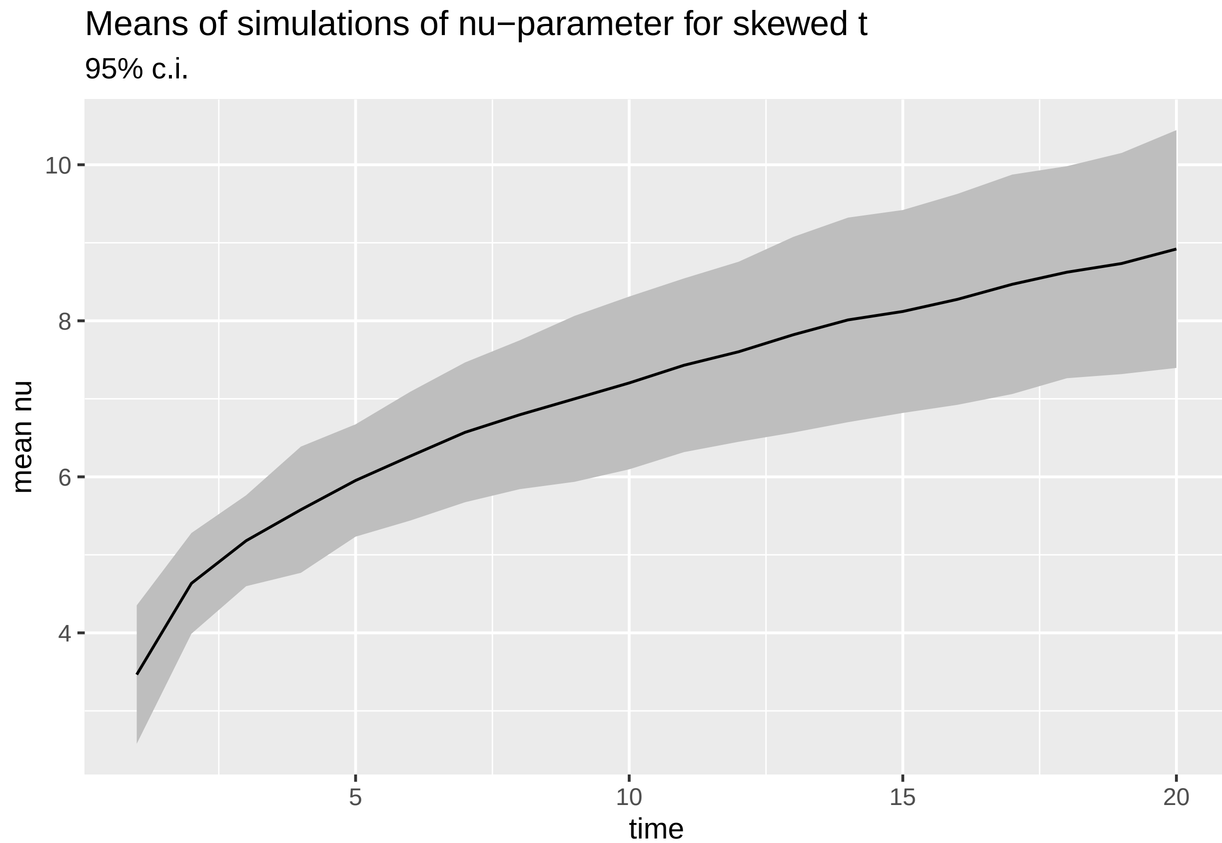
	11	12	13	14	15	16	17	18	19	20
f1	0.2534	0.2646	0.2754	0.2858	0.2959	0.3056	0.3150	0.3241	0.3330	0.3416
f2	0.2902	0.3041	0.3176	0.3305	0.3429	0.3550	0.3667	0.3781	0.3892	0.3999
s_n	0.2901	0.3041	0.3175	0.3304	0.3429	0.3550	0.3667	0.3780	0.3891	0.3999



Now formula 1 matches the realized MAD of $\{S_{n,k}\}_k^m$ better, but still not very well. Formula 2 is the way to go.

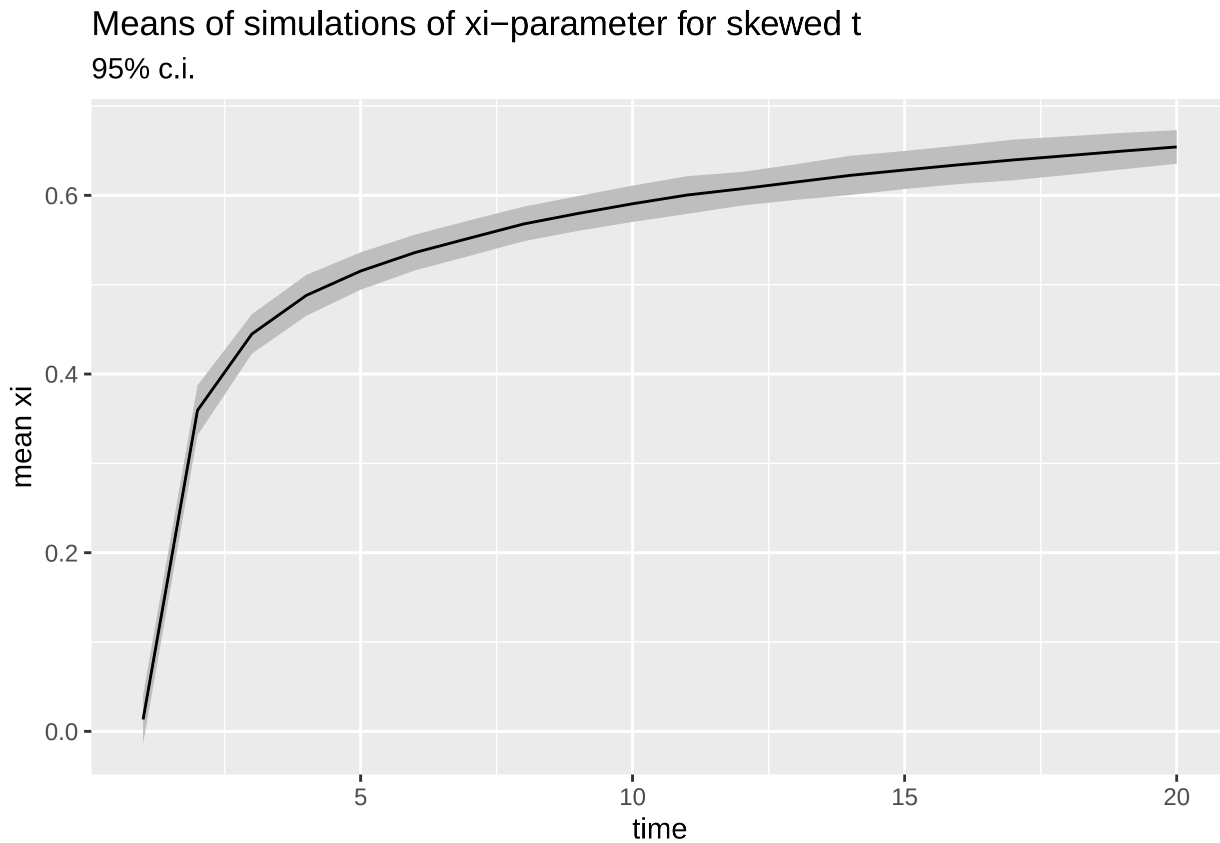
See "taleb_kappa.pdf" for further discussion.

Estimating the ν parameter of the skewed t distribution



We don't have a good guess here.

Estimating the ξ parameter of the skewed t distribution



Again, no good guess.

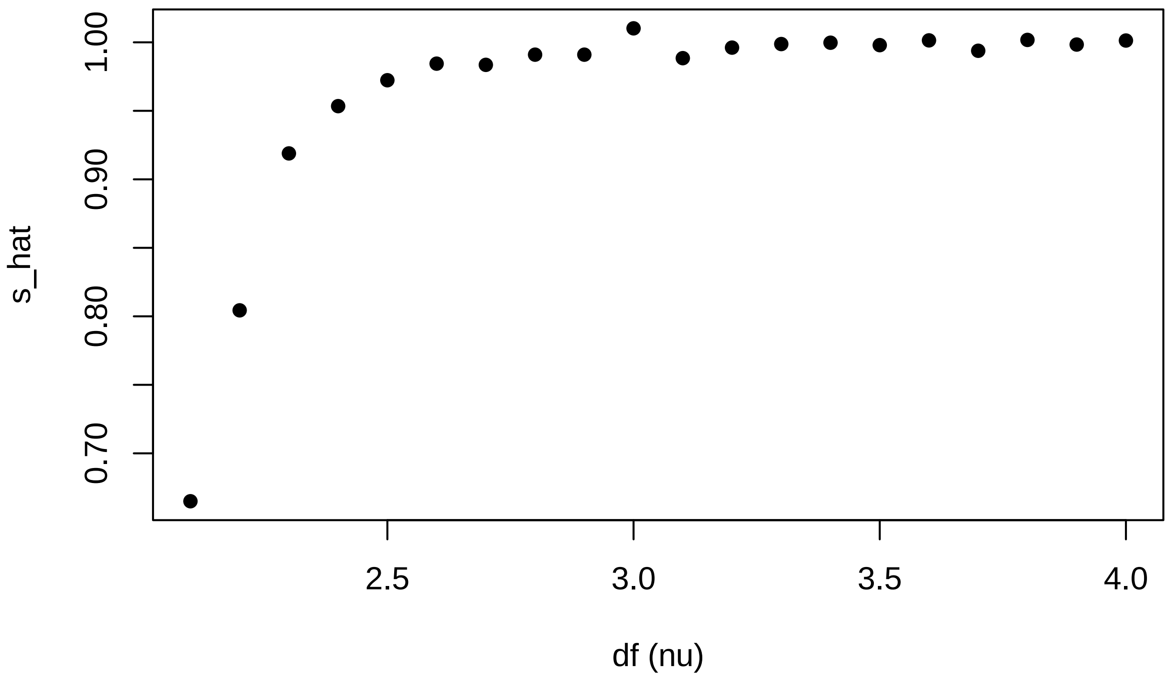
Analytical m and s from α

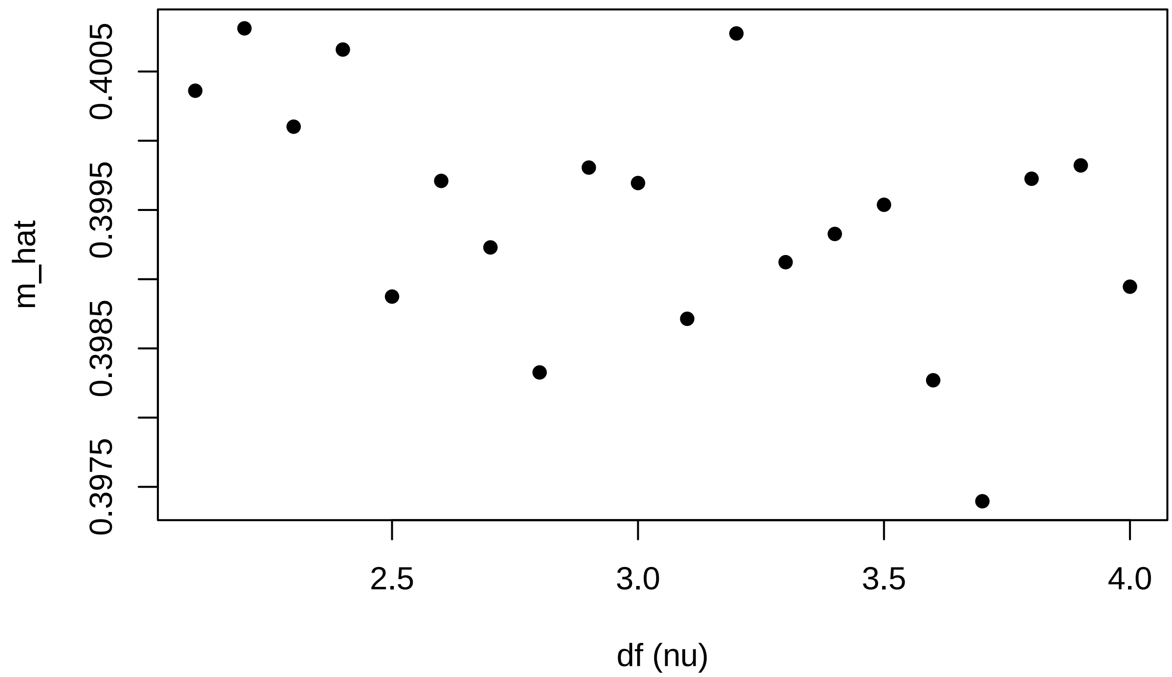
According to Taleb, the better way to estimate dispersion is to estimate the tail parameter using MLE and then computing the analytical mean and standard deviation as a function of ν.

Analytical moments

According to `help(std)`, you obtain `sd(rt())` by `sd() / sqrt(nu / (nu - 2))`.

So this should give sd of 1:


1.00
0.90
0.80
0.70
s_hat
2.5
3.0
3.5
4.0
df (nu)




```
df = 30
x_rt <- ((rt(n = 1e6, df = df) * sqrt((df - 2) / df)) * 0.9 + 0.4)
x_std <- rsstd(n = 1e6, mean = 0.4, sd = 0.9, nu = df)
x_sstd <- rsstd(n = 1e6, mean = 0.4, sd = 0.9, nu = df, xi = 0.2)
```

```
mean(x_rt)
```

```
## [1] 0.4016395
```

```
mean(x_std)
```

```
## [1] 0.401295
```

```
mean(x_sstd)
```

```
## [1] 0.399975
```

```
sd(x_rt)
```

```
## [1] 0.8986633
```

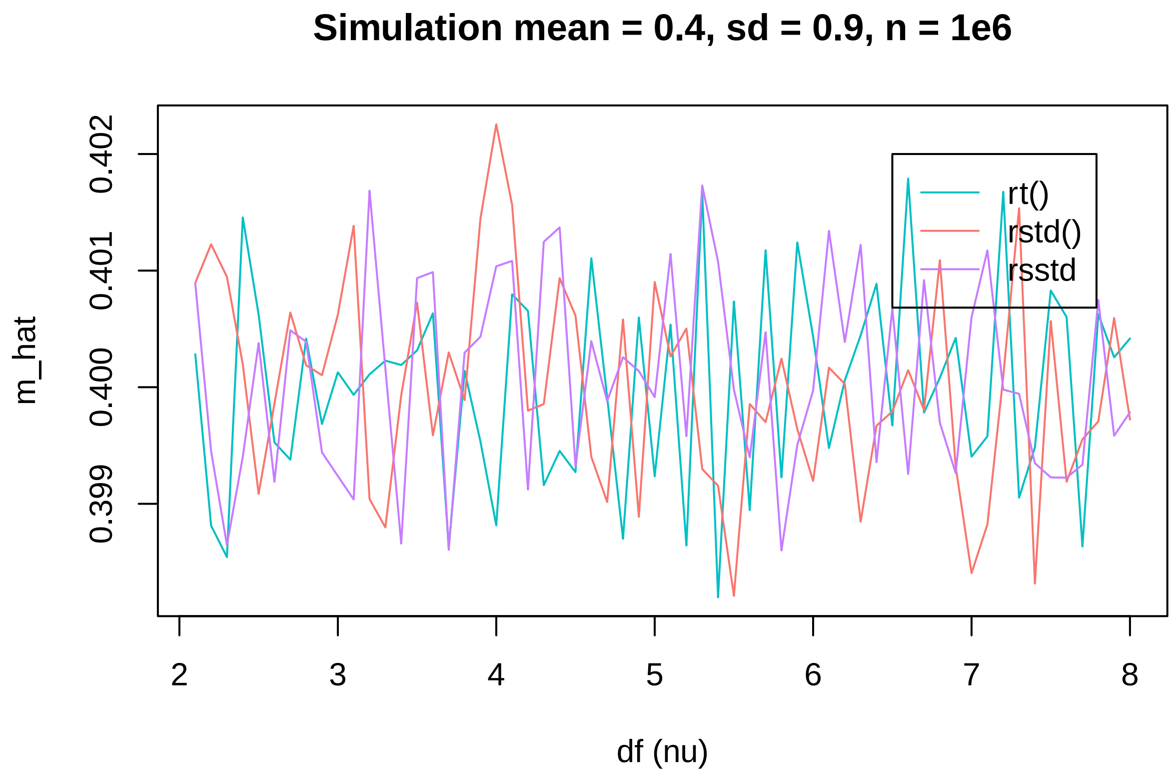
```
sd(x_std)
```

```
## [1] 0.9015723
```

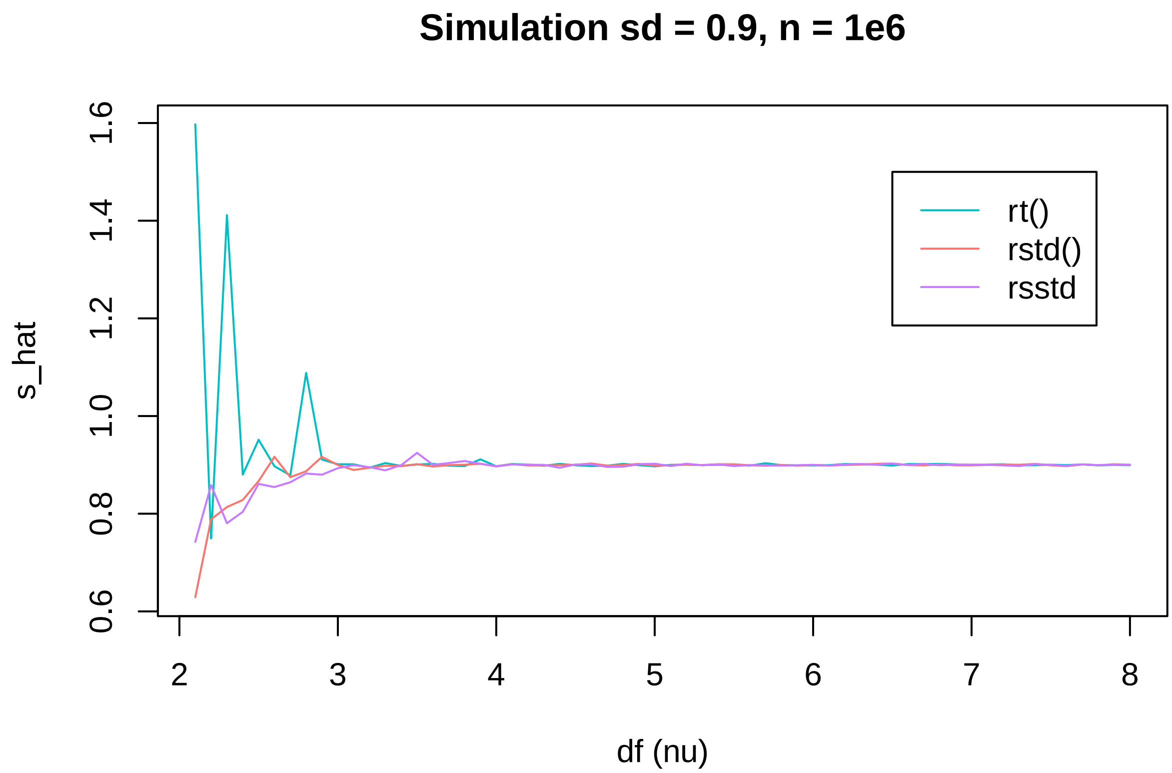
```
sd(x_sstd)
```

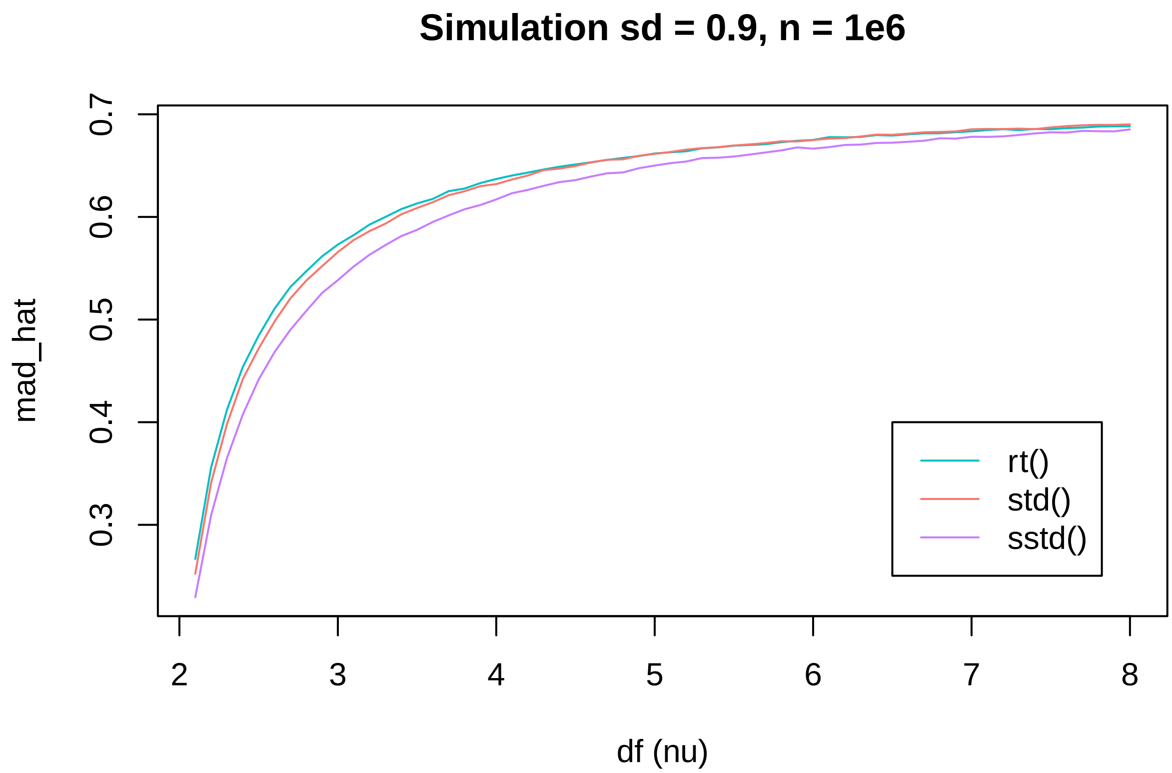
```
## [1] 0.8984435
```

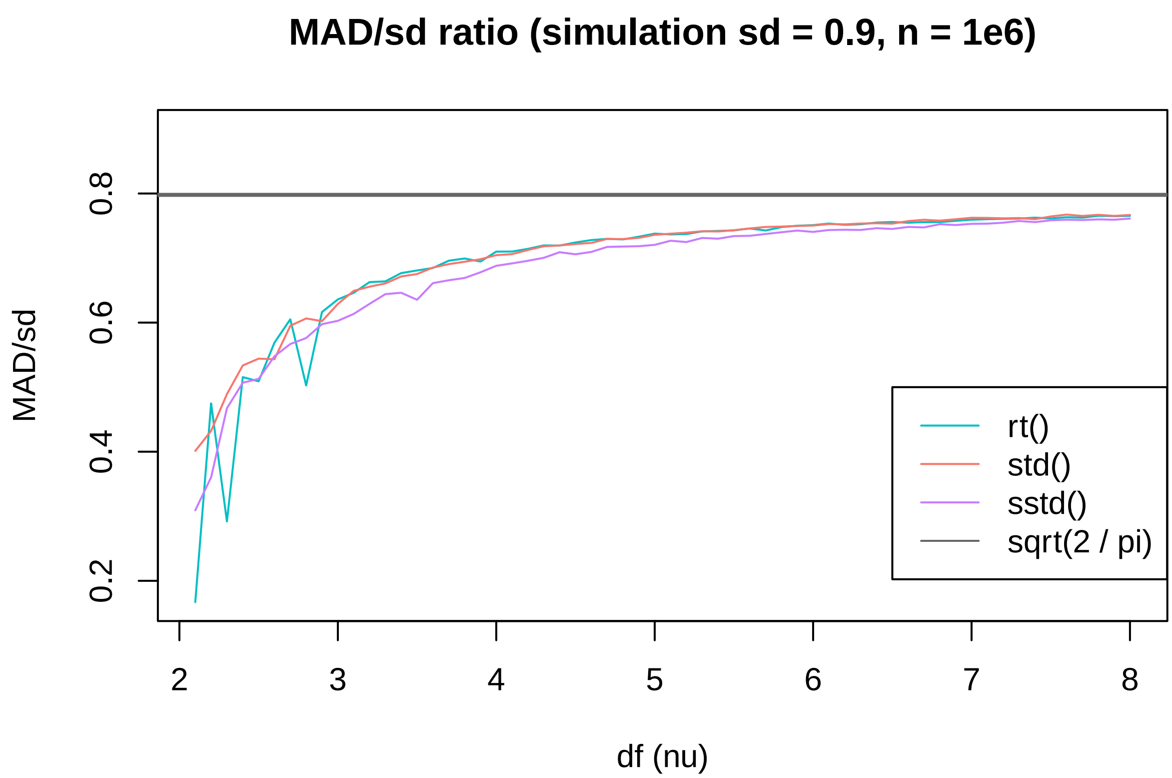
The mean seems fine for all `df > 2`:



For less than approx. 3 df (nu) it doesn't hold:


Simulation sd = 0.9, n = 1e6
s_hat
df (nu)
0.6
0.8
1.0
1.2
1.4
1.6
2
3
4
5
6
7
8
rt()
rstd()
rsstd


Simulation sd = 0.9, n = 1e6
mad_hat
0.3
0.4
0.5
0.6
0.7
2
3
4
5
6
7
8
df (nu)
rt()
std()
sstd()



Analytical mean

"A review of Student's t distribution and its generalizations.pdf", p. 10

$$f(x) = \frac{2\xi}{\xi^2+1}\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\Gamma\left(\frac{\nu}{2}\right)}\left[1+\frac{x^2}{\nu}\left(\frac{1}{\xi^2}I\{x\geq 0\}+\xi^2 I\{x<0\}\right)\right]^{-\frac{\nu+1}{2}}$$

$$\mathbb{E}[X^r] = M_r\frac{\xi^{r+1}+\frac{(-1)^r}{\xi^{r+1}}}{\xi+\frac{1}{\xi}}$$

where

$$M_r = \int_0^\infty 2x^r f(x)dx$$

```
## Input mean param: 0.08
```

```
## Analytical mean param:
```

```
## $value
## [1] -0.0008603894
##
## $integration.abs.error
## [1] 2.295918e-05
```

fGarch vignette:

```
## Input mean param: 0.08
```

```
## Analytical mean param:
```

```
## [1] 4.932924e-05
```

Analytical sd

"A review of Student's t distribution and its generalizations"

(See above)

```
## Input sd param: 0.12
```

```
## Analytical sd param:
```

```
## $value
## [1] 0.002153768
##
## $integration.abs.error
## [1] 9.062533e-07
```

fGarch vignette:
"Parameter Estimation of ARMA Models with GARCH/APARCH Errors An R and SPlus Software Implementation", p. 12

$$f(x \mid \xi) = \frac{2}{\xi + \frac{1}{\xi}} \left[f(\xi x) I\{x \geq 0\} + f\left(\frac{x}{\xi}\right) I\{x < 0\} \right]$$

$$\mu_\xi = M_1 \left(\xi - \frac{1}{\xi} \right)$$

$$\sigma_\xi^2 = (M_2 - M_1^2) \left(\xi^2 + \frac{1}{\xi^2} \right) + 2M_1^2 - M_2$$

where M_r is defined as above.

```
## Input sd param: 0.12
```

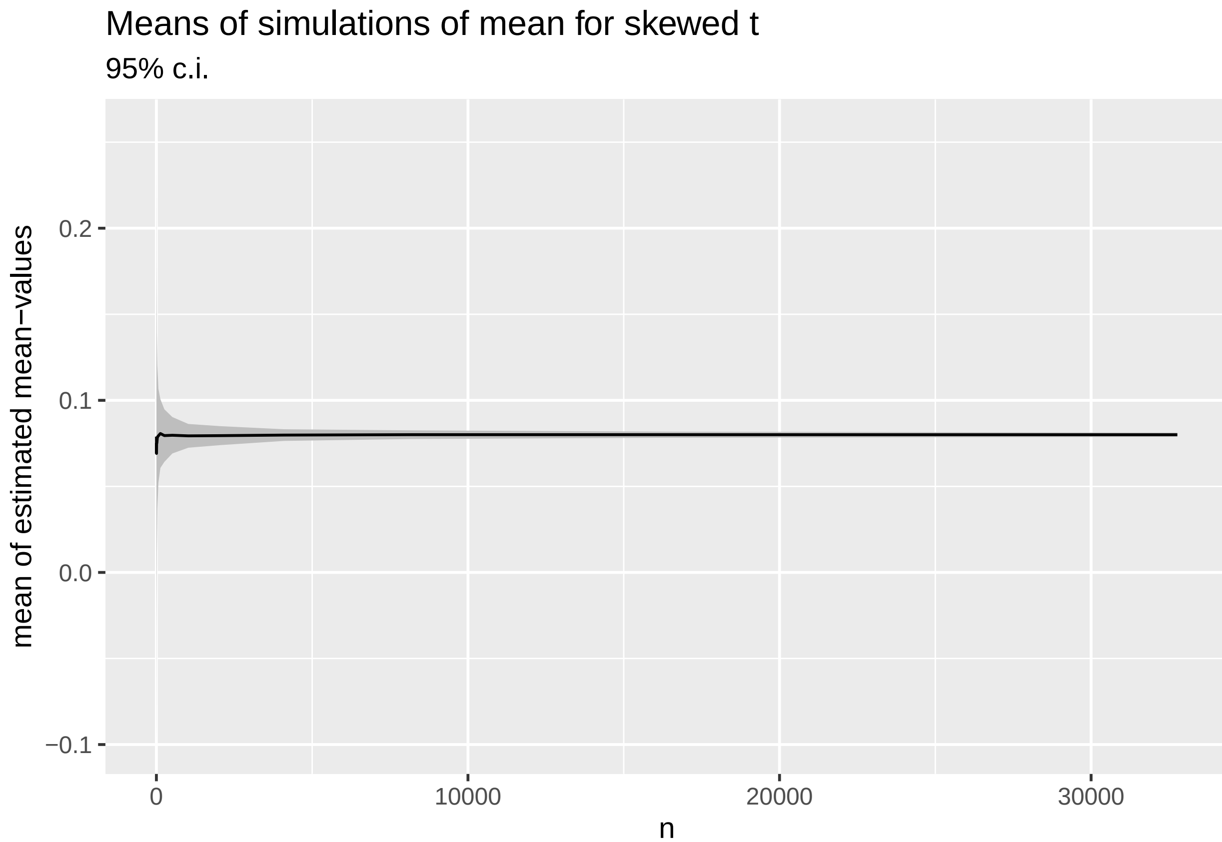
```
## Analytical sd param:
```

```
## [1] 7.936333e-06
```

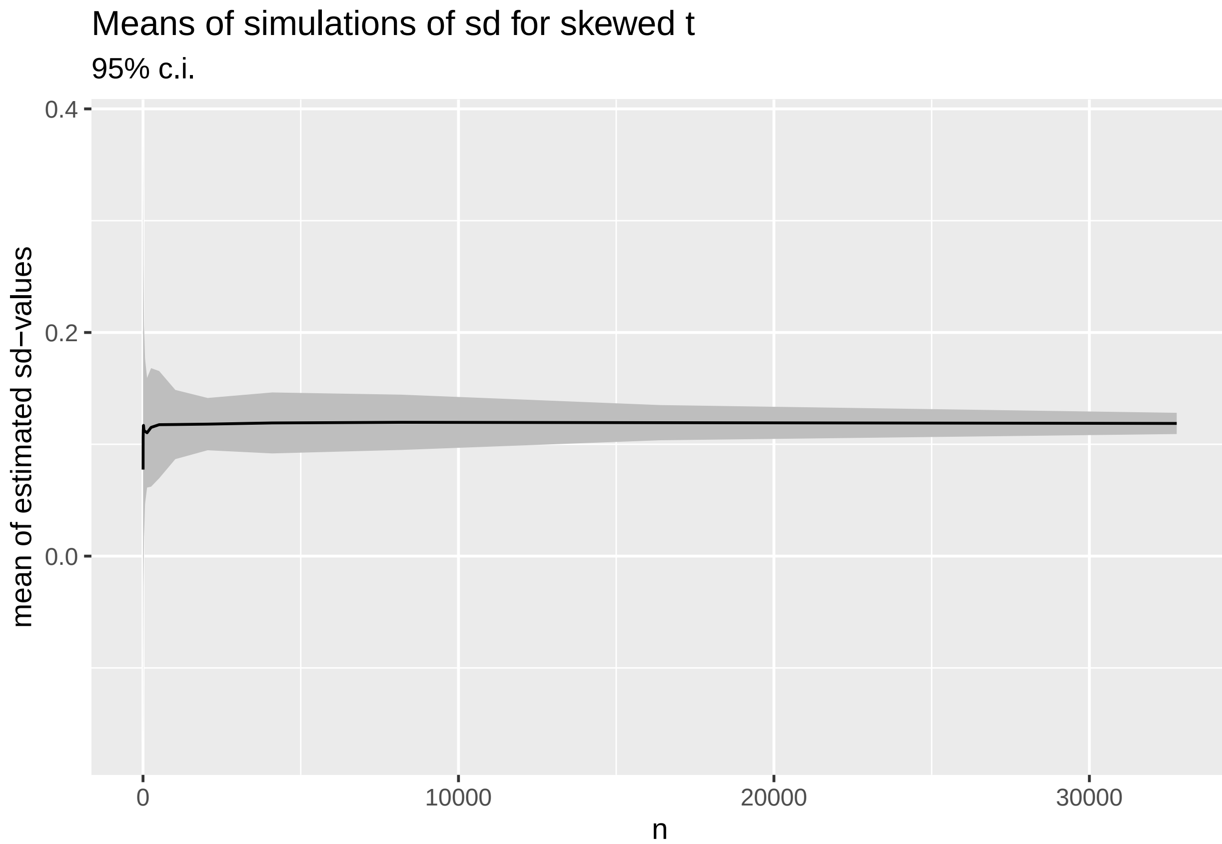
Convergence

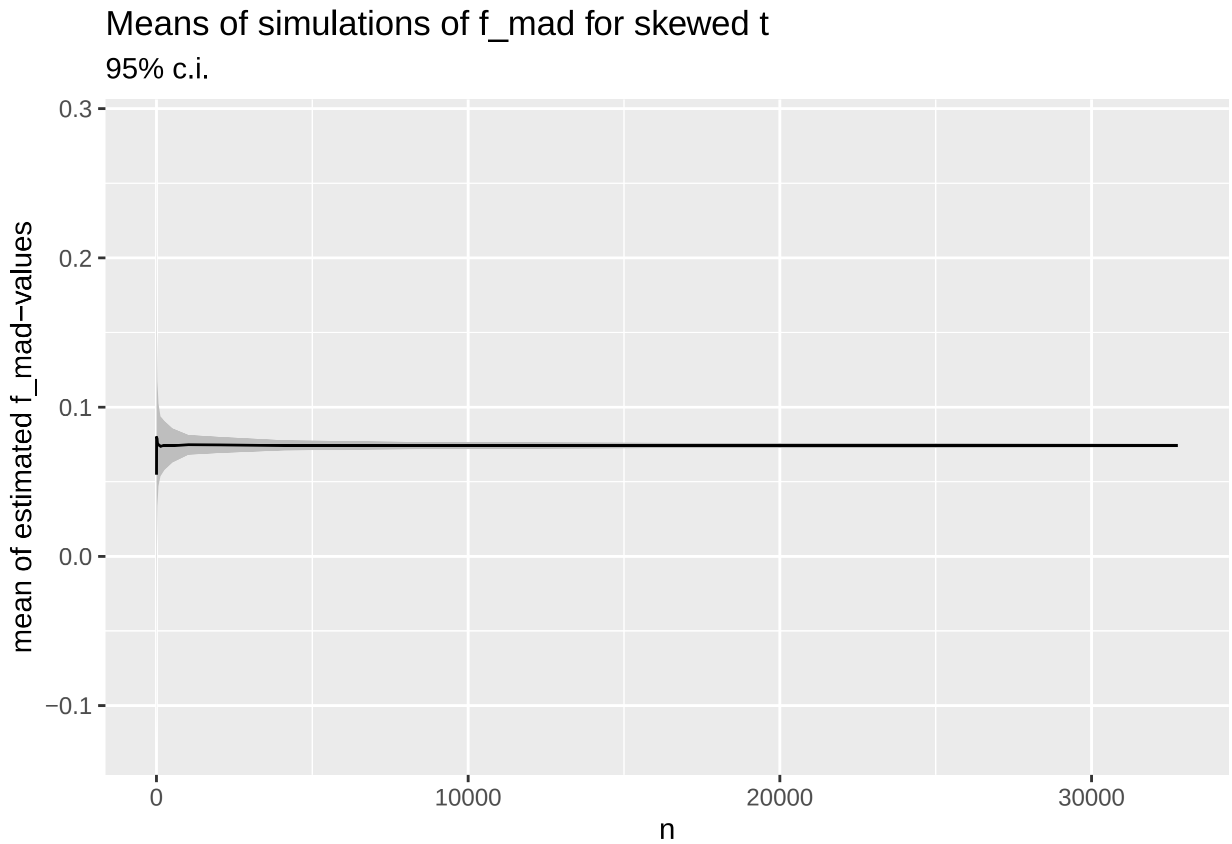
```
num_simulations <- 100
n_exponent <- 15
sim_params <- c(0.08, 0.12, 3.18, 0.02)
sim_data <- replicate(num_simulations, rsstd(2^n_exponent, sim_params[1], sim_params[2], sim_params[3],
```

Simulate parameter estimation



Means of simulations of mean for skewed t
95% c.i.
mean of estimated mean−values
0.2
0.1
0.0
−0.1
0
10000
20000
30000
n


Means of simulations of sd for skewed t
95% c.i.
mean of estimated sd−values
0.4
0.2
0.0
0
10000
20000
30000
n

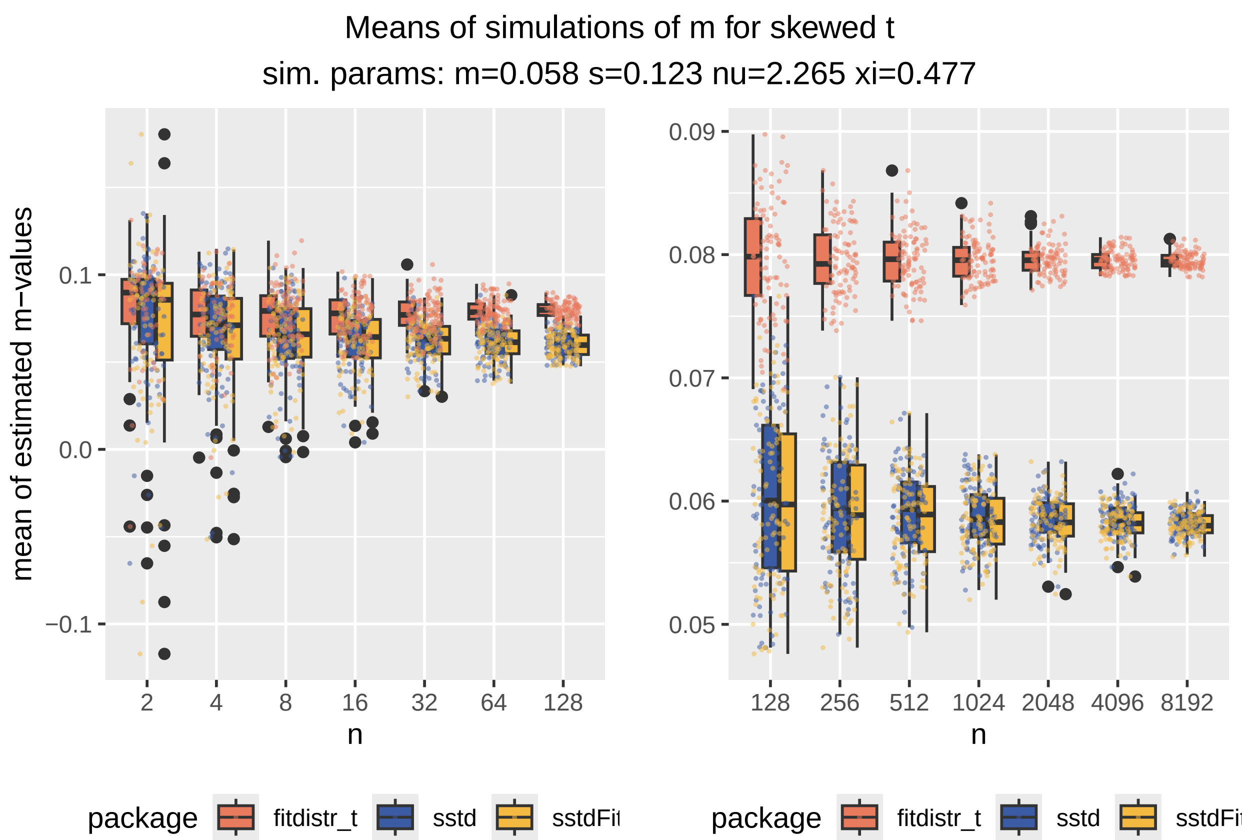


Compare estimation packages

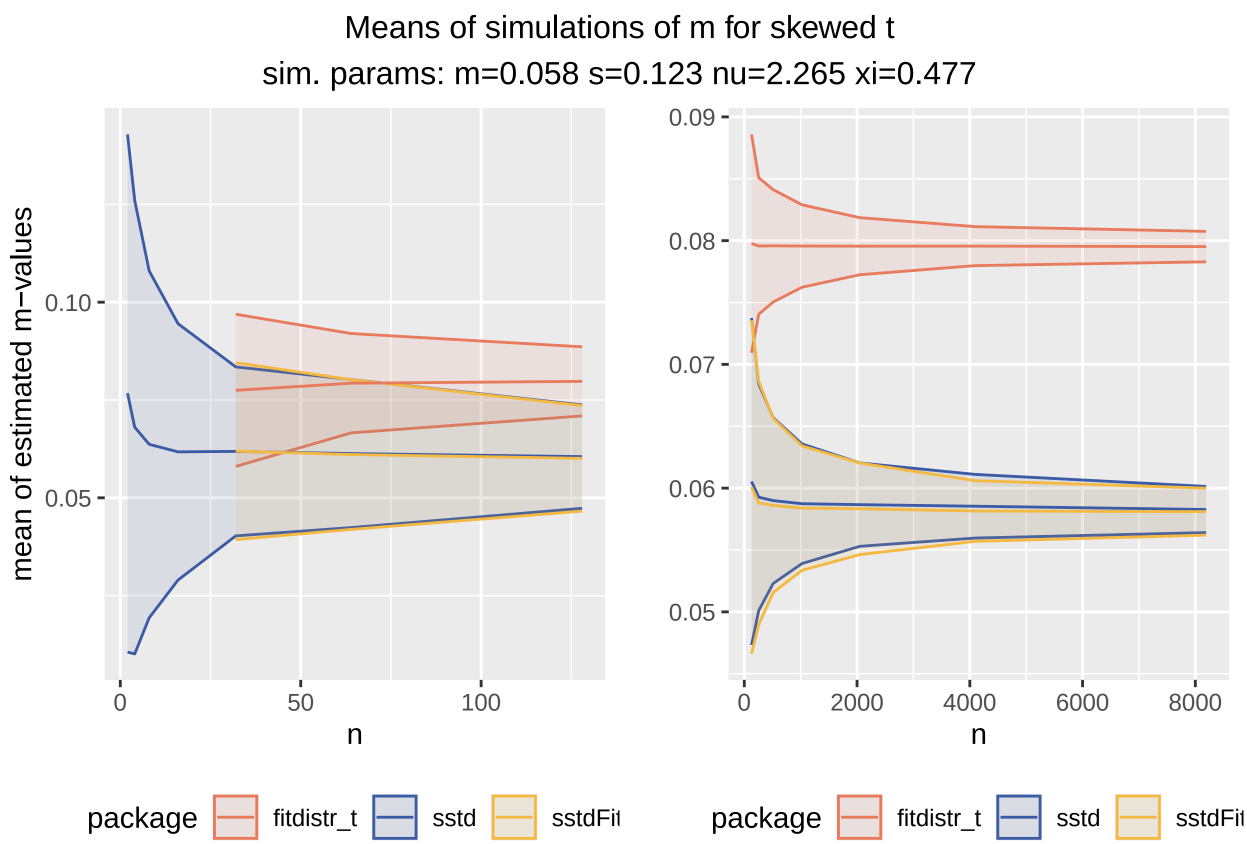
```
num_simulations <- 100
n_exponent <- 13
sim_params <- c(0.058, 0.123, 2.265, 0.477) ## pmr
sim_data <- as.data.frame(replicate(num_simulations, rsstd(2^n_exponent, sim_params[1], sim_params[2], s
```

```
## Warning: Removed 108 rows containing non-finite outside the scale range
## (`stat_boxplot()`).

## Warning: Removed 108 rows containing missing values or values outside the scale range
## (`geom_point()`).
```

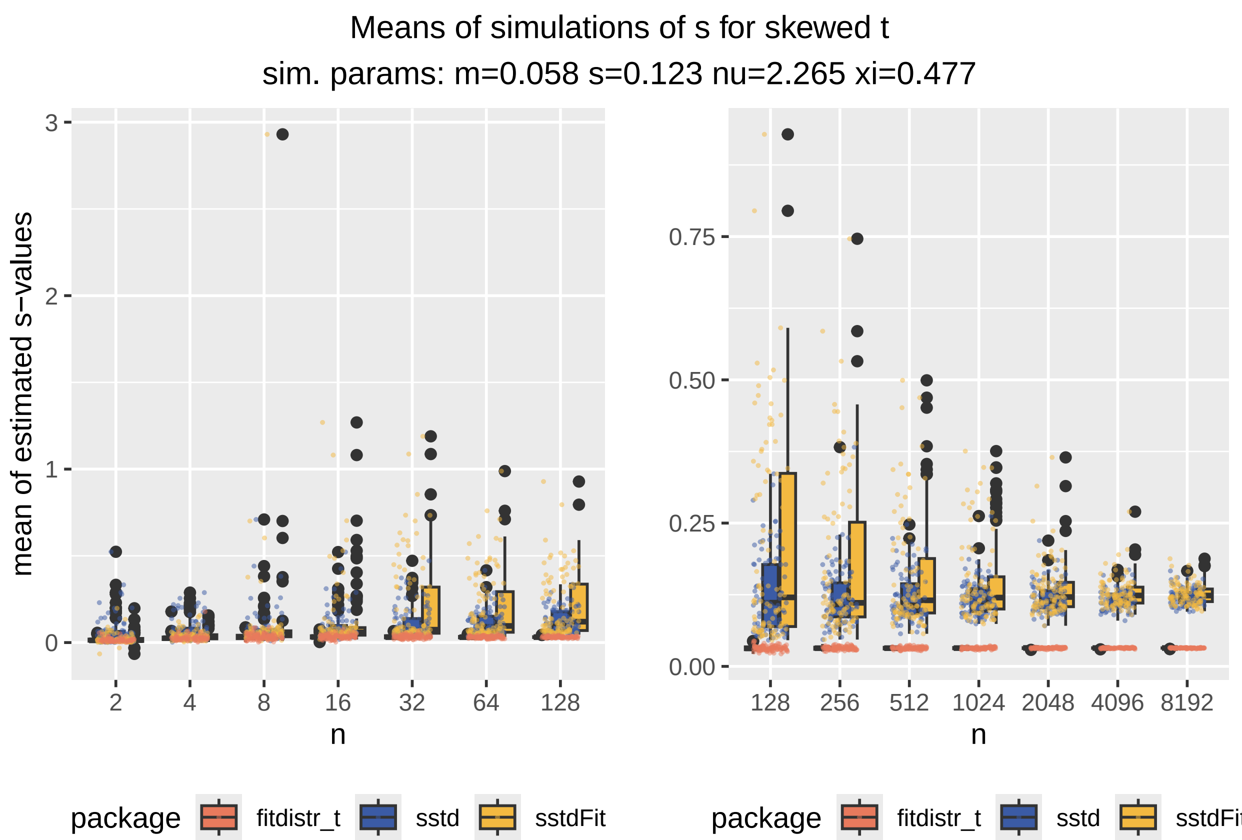



```
## Warning: Removed 8 rows containing missing values or values outside the scale range
## (`geom_line()`).
```

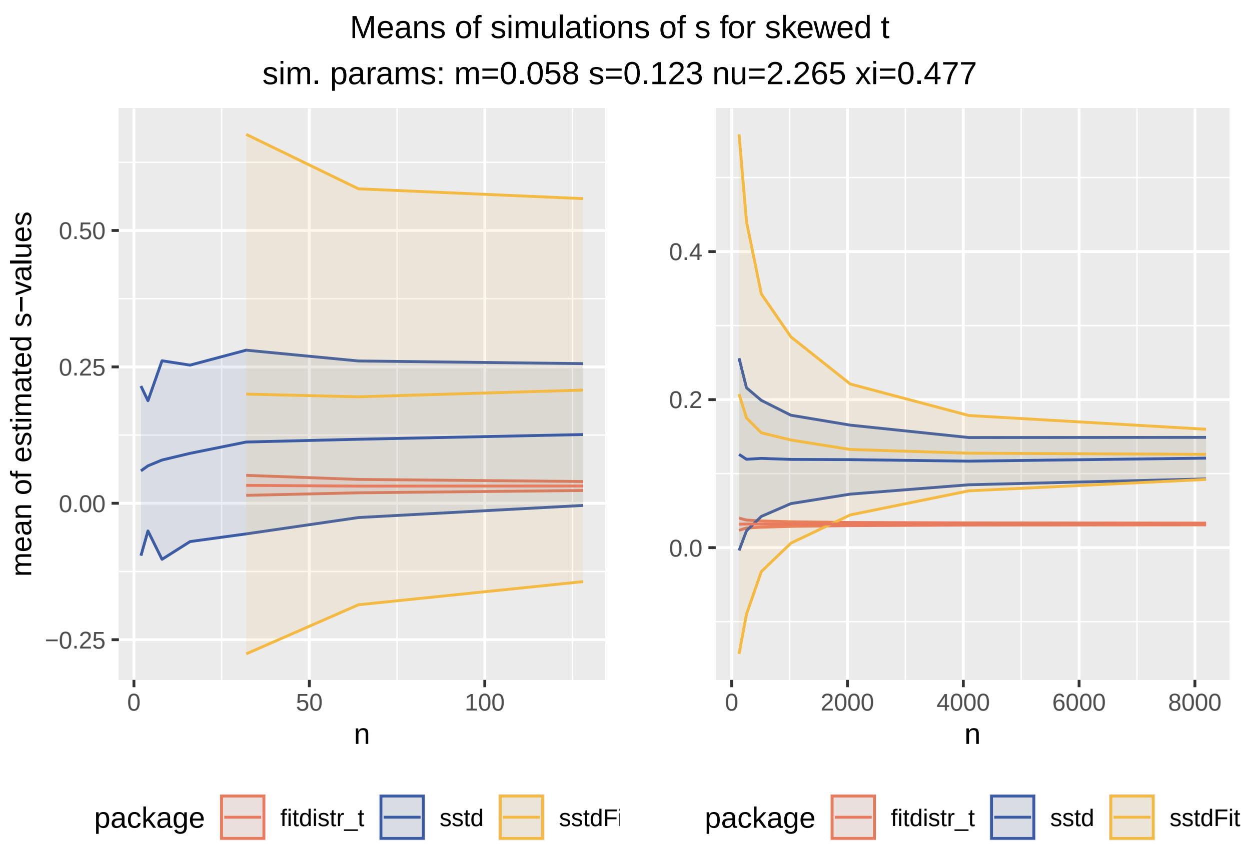



```
## Warning: Removed 108 rows containing non-finite outside the scale range
## (`stat_boxplot()`).

## Warning: Removed 108 rows containing missing values or values outside the scale range
## (`geom_point()`).
```

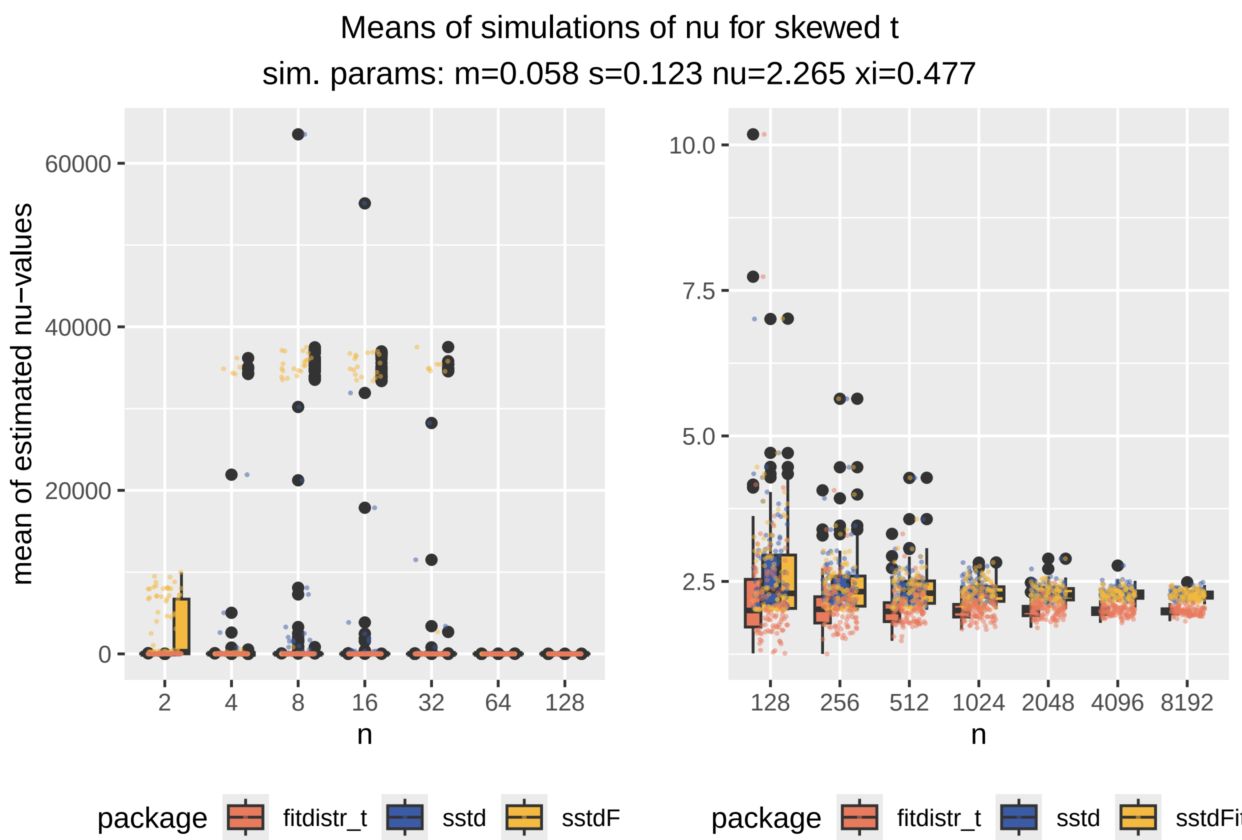



```
## Warning: Removed 8 rows containing missing values or values outside the scale range
## (`geom_line()`).
```

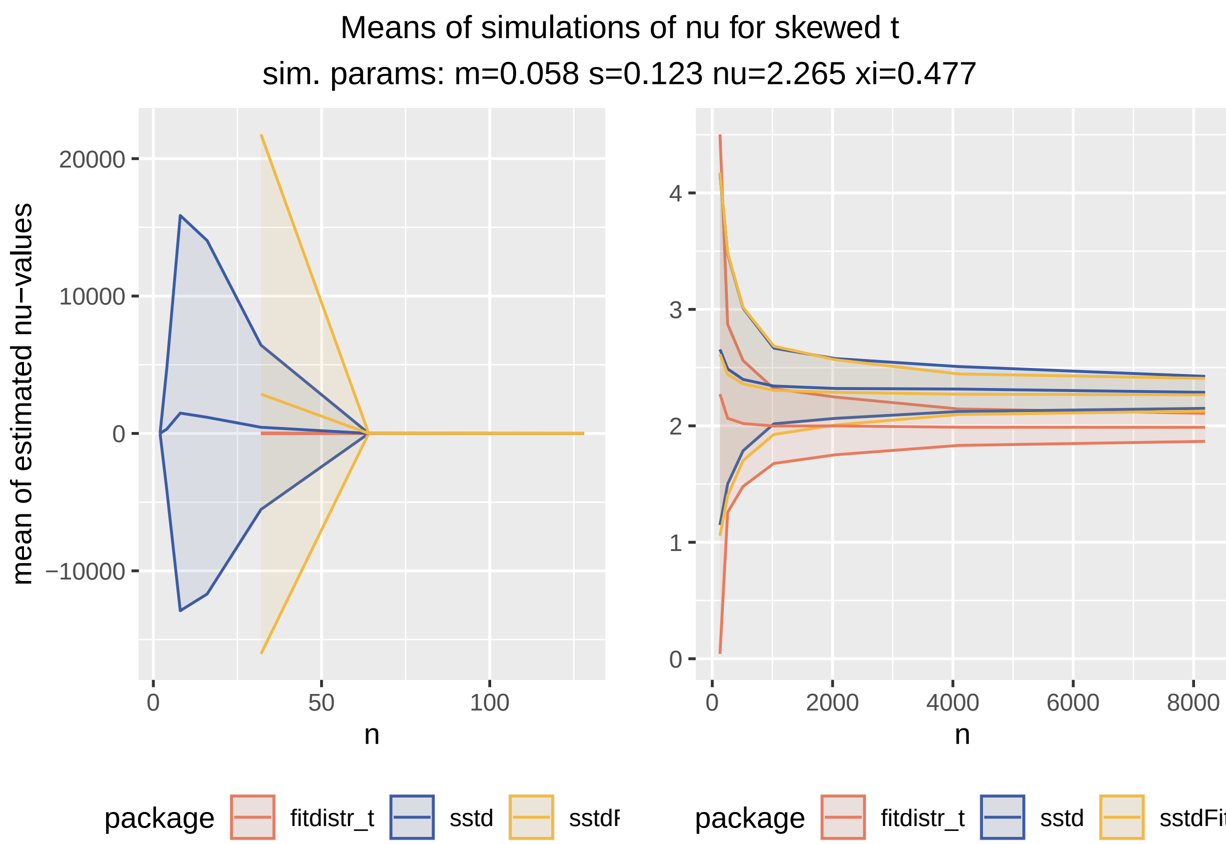



```
## Warning: Removed 108 rows containing non-finite outside the scale range
## (`stat_boxplot()`).

## Warning: Removed 108 rows containing missing values or values outside the scale range
## (`geom_point()`).
```




```
## Warning: Removed 8 rows containing missing values or values outside the scale range
## (`geom_line()`).
```



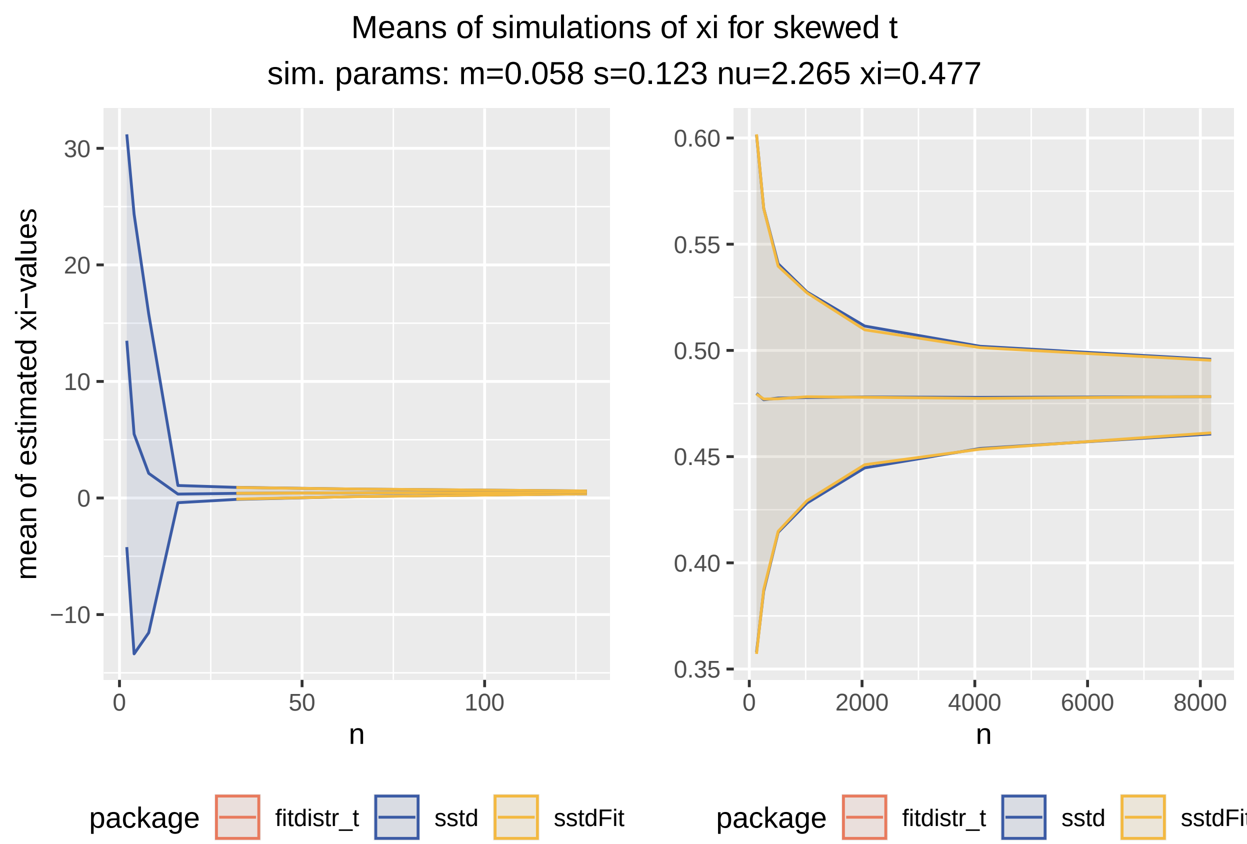
(No boxplot for xi, because fitdistr fits to a t-distribution, i.e. no xi parameter.)

```
## Warning in max(ids, na.rm = TRUE): no non-missing arguments to max; returning
## -Inf

## Warning: Removed 11 rows containing missing values or values outside the scale range
## (`geom_line()`).

## Warning in max(ids, na.rm = TRUE): no non-missing arguments to max; returning
## -Inf

## Warning: Removed 7 rows containing missing values or values outside the scale range
## (`geom_line()`).
```




```
##########################
## Data object dimensions:
##########################
##
## sim_data: 2^n_exponent x num_simulations
## hat_vals: num_simulations x n_exponent
## hat_stats: n_exponent x 5 (n, mean, sd, ci_l, ci_u)
## package_df:
## packages_comparison_list:
##   n_exponent lists containing 1 data frame for each package:
##     each data frame: num_simulations x 4 ("m", "s", "nu", "xi")
##
## hat_vals_list:
##   list of 4 lists, one list for each of ("m", "s", "nu", "xi")
##     list of 3 data frames, one for each package
##       hat_vals data frames: num_simulations x n_exponent
##
## packages_comparison_stats_list:
##   list of 4 lists, one list for each of ("m", "s", "nu", "xi")
##     list of 3 data frames - with hat_stats(hat_vals)
##       each data frame (hat_stats): n_exponent x 5 (n as factor, m, sd, ci_l, ci_u)
##
## From each hat_stats df, make hat_stats_plot
##
## packages_comparison_stats_list_for_plot:
##   list of 4 data frames, one for each of ("m", "s", "nu", "xi"):
```

```
##      packages_comparison_stats_df: 3 * n_exponent x 6 (n as factor, m, sd, ci_l, ci_u, package)
```