Taleb's kappa

19:57 17 May 2024

Contents

Minimum number of terms (summands) needed for convergence

See Taleb: The Statistical Consequences Of Fat Tails, Ch. 8

Problem statement

Let $\{X_{g,i}\}$ be Gaussian distributed with mean μ and scale σ.

Let $\{X_{\nu,i}\}$ be t-distributed, scaled such that $\mathbb{M}^{\nu}(1) = \mathbb{M}^{g}(1) = \sqrt{\frac{2}{\pi}}\sigma$.

Given n_g, we want to determine and n_ν^* such that

$$\operatorname{Var}\left[\sum_i^{n_g} X_{g,i}\right] = \operatorname{Var}\left[\sum_i^{n_\nu^*} X_{\nu,i}\right]$$

Calculating κ

For iid. r.v $\{X_i\}$:

$$S_n = X_1 + X_2 + \cdots + X_n$$
$$\mathbb{M}(n) = \mathbb{E}(|S_n - \mathbb{E}(S_n)|)$$

Taleb's convergence metric (κ):

The "rate" of convergence for n summands vs n_0, i.e. the improved convergence achieved by $n - n_0$ additional terms, is given by $\kappa(n_0, n)$:

$$\kappa(n_0, n) = 2 - \frac{\log(n) - \log(n_0)}{\log\left(\frac{\mathbb{M}(n)}{\mathbb{M}(n_0)}\right)}$$

Calculating $n_{\mathbf{min}}$

The minimum number of summands needed to achieve same variance as the sum of n_g Gaussian summands:

$$n_{\min} := n_\nu^*$$

We don't need to calculate $\mathbb{M}(n)$ and $\mathbb{M}(n_0)$. Instead we use the properties

$$\mathbb{M}^\nu(1) = \mathbb{M}^g(1) = \sqrt{\frac{2}{\pi}}\sigma$$

where $\sqrt{\frac{2}{\pi}}$ is the ratio between the sd of a Gaussian r.v. and its MAD,

and

$$\mathbb{M}(n) = n^{1/\alpha}\mathbb{M}(1)$$

where α is the degrees of freedom for the Student t. This is true for Stable distributions only.

This seems to not work. Why?

Notes

Note 1

The relation $\mathbb{M}^\nu(1) = \sqrt{\frac{2}{\pi}}\sigma$ only holds for the standard normal distribution, and only asymptotically. The ratio $\frac{\text{MAD}(X)}{\text{sd}(X)}$ varies between 0 and 1.

https://en.wikipedia.org/wiki/Average_absolute_deviation#Mean_absolute_deviation_around_the_mean

For a Student t with 3 df, the ratio is $\frac{\pi}{2}$. (See Taleb ch. 4 for further discussion.)

Note 1a The tail exponent α of a Student t distribution with ν degrees of freedom is $\alpha = \nu + 1$, which can be seen from the density function, which is proportional to

$$x^{-(\nu+1)}$$

Confusingly, on p. 147 Taleb uses the notation "Student T(α), where $\alpha = 3$. Does this mean 2 or 3 degrees of freedom??

Taleb et al seem confused by this themselves On p. 150 they write:

"a Student T with 3 degrees of freedom ($\alpha = 3$) requires 120 observations to get the same drop in variance from averaging (hence confidence level) as the Gaussian with 30, that is 4 times as much. The one-tailed Pareto with the same tail exponent $\alpha = 3$ requires 543 observations to match a Gaussian sample of 30".

-So in one case α is the degrees of freedom of a Student t, and in the next sentence α is the tail exponent.

Note 2

Notice that if $\mathbb{M}(n) = n^{1/\alpha}\mathbb{M}(1)$, then $\kappa(1, n)$ becomes

$$2 - \frac{\log(n) - \log(n_0)}{\log\left(\frac{\mathbb{M}(n)}{\mathbb{M}(n_0)}\right)} = 2 - \frac{\log(n) - \log(1)}{\log\left(\frac{n^{1/\alpha}\mathbb{M}(1)}{\mathbb{M}(1)}\right)} = 2 - \frac{\log(n)}{\log\left(n^{1/\alpha}\right)}$$

For $\alpha = 2$, this is

$$2 - \frac{\log(n)}{\log(\sqrt{n})} = 2 - \log_{\sqrt{n}}(n) = 0$$

as $\log_{\sqrt{n}}(n) = 2$ for all n.

So if we use $\tilde{\alpha} = \alpha I\{\alpha < 2\} + 2I\{\alpha \geq 2\}$, κ is always 0 for $\alpha \geq 2$. Is the F-S Skewed t Stable? According to the table in Taleb, p. 148, the Student t has a κ far from 0, even when $\alpha > 2$.

Note 3

Also note, that for Stable distributions with $1 \leq \tilde{\alpha}$,

$$\kappa_{(n_0,n)} = 2 - \tilde{\alpha}$$

Note 4

The problem calculating the MAD directly is that we need the mean, which is what we are estimating in the first place! How much data do we need to estimate κ???

A method then is to estimate the tail index α (which will be Gaussian) using MLE and from there get the theoretical mean. See Taleb Ch. 13.

For the analytical mean of the F-S Skewed t, see Li and Nadarajah: A review of Student's t distribution and its generalizations, p. 10. (Typo: Should be x instead of s. See Fernandez-Steel: On Bayesian Modelling Of Fat Tails And Skewness, eq. 2.5+2.6)

Note 5

In this implementation we calculate the MAD of S_n as

$$\mathbb{M}(n) = \mathbb{E}(|S_n - \mathbb{E}(S_n)|) = \mathbb{E}(|S_n - n\hat{\mu}_{X_i}|)$$

where $\hat{\mu}_{X_i}$ is the estimated mean of X_i. We estimate the mean of X_i by simulation.

Note 6

Test if the sample means of a sum of F-S Skewed t r.v.'s are equal to n times the mean of a single r.v. from that distribution.

Even with only 50 samples to estimate the sample mean, the fit is spot on.

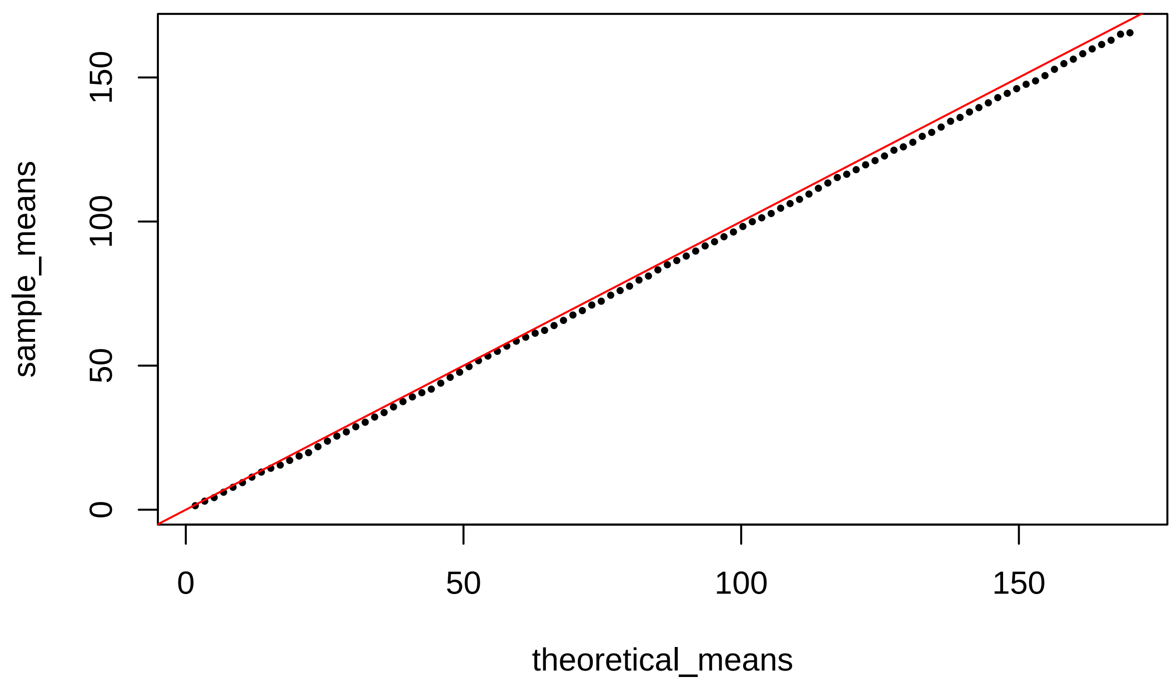
```
num_sim <- 50
n <- 100
mu <- 1.7
sigma <-  2
nu <- 3
xi <- 0.25


x_df <- replicate(num_sim, cumsum(rsstd(n, mu, sigma, nu, xi)))

theoretical_means <-  (1:n) * mu
sample_means <- lapply(
  1:n,
  function(i) {
    mean(x_df[i ,])
  }
```

```
)

plot(theoretical_means, sample_means, pch = 16, cex = 0.5)
abline(0, 1, col = "red")
```




```
f_mad <- function(x) {
  sum(abs(x - mean(x))) / length(x)
}

## MAD of S_n, given the mean of X
f_mad_n <- function(Sn, mean_X) {
  diff <- numeric(length(Sn))
  for(i in seq_along(Sn)) {
    diff[i] <- abs(Sn[i] - mean_X)
  }
  sum(diff) / length(Sn)
}

## mean is the mean of a single Gaussian r.v. (same for t-distribution)
## sd is the sd of a single Gaussian r.v. (same for t-distribution)
f_kappa <- function(n0, n, mean, sd = 1, nu = 3, xi = 1, num_sim = 1e4) {
  Sn_sim <- replicate(
    num_sim,
    rsstd(n = n, mean = mean, sd = nu/(nu - 2) * sd, nu = nu, xi = xi)
  )
  #x_n0 <- rfun(n0, ...)
```

```
  #x_n <- rfun(n, ...)
  #mad_1 <- sqrt(2 / pi) * sd_g
  #mad_n0 <- mad_1 * n0^(1 / nu)
  #mad_n <- mad_1 * n^(1 / nu)
  mad_n0 <- f_mad_n(
    unlist(lapply(1:num_sim, function(i) sum(Sn_sim[1:n0, i]))),
    n0 * mean
  )
  mad_n <- f_mad_n(
    unlist(lapply(1:num_sim, function(i) sum(Sn_sim[1:n, i]))),
    n * mean
  )
  nominator <- log(n) - log(n0)
  denominator <- log(mad_n / mad_n0)
  2 - (nominator / denominator)
}

## Use approximation if approx = TRUE
f_n_min <- function(n_g, mean, sd_g = 1, nu = 3, xi = 1, num_sim = 1e4, approx = FALSE) {
  ifelse(
    approx,
    exponent <- - 1 / (f_kappa(
      1, 2, mean, sd = sd_g * (nu / (nu - 2)), nu = nu, xi = xi, num_sim = num_sim
    ) - 1),
    exponent <- - 1 / (f_kappa(
      1, n_g, mean, sd = sd_g * (nu / (nu - 2)), nu = nu, xi = xi, num_sim = num_sim
    ) - 1)
  )
  n_g^exponent
}
```

According to table on p. 148, for Student t with 3 df,

- $\kappa_{1,2} = 0.29$

```
nu <-  3
f_kappa(n0 = 1, n = 2, mean = 0, sd = 1, nu = nu, xi = 1, num_sim = 1e5)
```

```
## [1] 0.283786
```

- $\kappa_{1,30} = 0.191$

```
f_kappa(n0 = 1, n = 30, mean = 0, sd = 1, nu = nu, xi = 1, num_sim = 1e5)
```

```
## [1] 0.1933596
```

- $\kappa_{1,100} = 0.159$

```
f_kappa(n0 = 1, n = 100, mean = 0, sd = 1, nu = nu, xi = 1, num_sim = 1e5)
```

```
## [1] 0.1639393
```

```
f_kappa(n0 = 1, n = 100, mean = 0, sd = 1, nu = nu, xi = 1, num_sim = 1e5)
```

```
## [1] 0.1586125
```

```
f_kappa(n0 = 1, n = 100, mean = 0, sd = 1, nu = nu, xi = 1, num_sim = 1e5)
```

```
## [1] 0.1582189
```

```r
f_kappa(n0 = 1, n = 100, mean = 0, sd = 1, nu = nu, xi = 1, num_sim = 1e5)
```

```
## [1] 0.1546057
```

```r
f_kappa(n0 = 1, n = 100, mean = 0, sd = 1, nu = nu, xi = 1, num_sim = 1e5)
```

```
## [1] 0.1606407
```

```r
f_kappa(n0 = 1, n = 100, mean = 0, sd = 1, nu = nu, xi = 1, num_sim = 1e5)
```

```
## [1] 0.1581628
```

```r
f_kappa(n0 = 1, n = 100, mean = 0, sd = 1, nu = nu, xi = 1, num_sim = 1e5)
```

```
## [1] 0.162531
```

```r
f_kappa(n0 = 1, n = 100, mean = 0, sd = 1, nu = nu, xi = 1, num_sim = 1e5)
```

```
## [1] 0.1652506
```

```r
f_kappa(n0 = 1, n = 100, mean = 0, sd = 1, nu = nu, xi = 1, num_sim = 1e5)
```

```
## [1] 0.1522116
```

"a Student T with 3 degrees of freedom ($\alpha = 3$) requires 120 observations to get the same drop in variance from averaging (hence confidence level) as the Gaussian with 30". This seems only true for the approximation - which seems to be not very good:

Without approximation:

$$n_{g_\nu} = n_g^{\frac{1}{\kappa_{1,n_g} - 1}}$$

```r
n_g <- 30 ## Number of Gaussian terms
sd_g = 1
df = 3

n_min <- f_n_min(n_g, mean = 0, sd_g = sd_g, nu = df, xi = 1, num_sim = 1e5, approx = FALSE)
n_min
```

```
## [1] 65.90981
```

With approximation:

$$n_{g_\nu} = n_g^{\frac{1}{\kappa_{1,2} - 1}}$$

```r
n_g <- 30 ## Number of Gaussian terms
sd_g = 1
df = 3

n_min <- f_n_min(n_g, mean = 0, sd_g = sd_g, nu = df, xi = 1, num_sim = 1e5, approx = TRUE)
n_min
```

```
## [1] 113.7804
```