Hitchin Systems Seminar

Contents

CHAPTER 1

Introduction

This course studies the Hitchin system from different perspectives and in various settings.

Hitchin studied a limiting form of the self-dual equations of Yang-Mills theory in four dimensions, imposing invariance under two of the dimensions. The resulting equations were found to be conformally invariant with respect to the remaining two directions, so had a natural formulation as equations on a Riemann surface. The Hitchin moduli space, or moduli space of Higgs bundles, will be defined as the space of solutions modulo gauge transformations.

Several basic properties of the space of solutions emerged. First, the equations could be interpreted formally as hyperkahler moment map equations for the compact gauge Group action on the infinite-dimensional space of connections. Since we quotient solutions by gauge transformations, this means we have an interpretation of the Hithchin moduli space as a hyperkahler quotient. Thus it should inherit a hyperkaher structure. Of the three independent complex structures, the one most amenable to algebraic geometry is the "Dolbeault" one inherited from the complex structue on the Riemann surface, C. In that setting, the Hitchin moduli space has an interpretation as stable Higgs pairs, i.e. pairs (V, φ), where V is a rank-n holomorphic vector bundle over C, and $\varphi \in \Gamma(End(V) \otimes K_C)$ is a holomorphic $n \times n$ matrix-valued one-form. Stability requires that all φ-invariant sub-bundles have lesser slope than V. Finally, solutions are taken modulo holomorphic isomorphism of vector bundles.

Another structure which emerges in this complex structure is that of an algebraically completely integrable system. Recall first that a complete integrable system on a symplectic manifold ("phase space") is a maximal set of Poisson-commuting proper functions ("Hamiltonians"). Specifying the values of these conserved quantities gives a common level set which is a Lagrangian torus. This gives another definition: a fibration by Lagrangian tori to a base manifold where the Hamiltonians take their values. The fiber map for the Hitchin space is defined by all the invariant polynomials $Tr\varphi^k$.

The other complex structures are called "de Rham," in which the moduli space is of flat $GL_n(\mathbb{C})$ connections modulo complex gauge transformations, and the "Betti" complex structure in which the moduli space is irreducible representations of $\pi_1(X)$ modulo conjugation. A variety of theorems relate these. Betti and de Rham are equated by the Riemann-Hilbert correspondence, in which one assigns to a flat connection the monodromy data around loops in the space.

1.1. Local Case

Consider that a holomorphic vector bundle has a local trivialization by a holomorphic frame, with respect to which the Higgs field looks like

1.2. Teichmüller theory

1.3. Link to Knots

A hidden agenda for this seminar was to begin to explore the spectral curve from the perspective of symplectic geometry. Given a Riemann surface with complex structure J and Kähler metric h, it cotangent bundle T^*C has a tautological holomorphic one-form θ and holomorphic symplectic structure $\Omega = d\theta$. Treating C as a real manifold, T^*C has a canonical symplectic form $Re(\Omega)$ for which the spectral curve is Lagrangian, and we would like to consider it as an object of the Fukaya category of T^*C.

1.4. Mirror Symmetry

Recall that mirror symmetry is a kind of equivalence between a manifold X and its mirror Y. It goes deep, but in its simplest formulation, the Hodge diamonds of X and Y are related by a mirror reflection. The basic philosophy behind mirror symmetry is that mirror manifolds are dual torus fibrations, according to the following reasoning. A B-type D-brane (or B-brane) on a complex manifold is a coherent sheaf, and we will only consider the case of the structure sheaf of single point. An A-type D-brane (or B-brane) in a symplectic manifold is a special Lagrangian submanifold with a flat connection ∇ — and we shall focus on the line bundle case.

On a complex manifold X, the space of B-branes connected to a point brane is X itself. If there were a mirror manifold Y, the corresponding space of A-branes should again be X, since the moduli spaces of mirror branes should correspond. So we should be able to find the mirror X to Y by identifying A-branes that correspond to point B-branes and looking at their moduli. By reversing the reasoning, we also learn that the manifolds X and Y each are expressible both as a space of A-branes.

If we fix an A-brane (T, ∇), the space of A-branes deformation equivalent to ("connected to") the brane (T, ∇) therefore has a forget map to the space of special Lagrangians connected to T. The fibers are tori $(S^1)^{b_1(T)}$ which encode the monodromies, and it so happens that by linearizing the special Lagrangian equations you can see that $b_1(T)$ is also the dimension of the space of special Lagrangians at T. So the D-brane moduli space is an integrable system, a torus fibration over special Lagrangian moduli space (Lagrangianicity must be argued separately). Now Y, like X, should itself be some an A-brane (on X) moduli space, so has a torus fibration structure, a map $Y \to B$. What if we took T to be a torus fiber? Then the dimensions are right, since we want $b_1(T)$ to be equal to the complex dimension of X and Y, equivalently the real dimension of T. A fiber also has the property that it is disjoing from its deformations – just like its dual point B-branes. If this is the case (and there are other arguments supporting this which we ignore), then the D-brane moduli space of a torus fiber T fibers over the same base B (its geometric deformations), with the fiber over $T_b = \pi^{-1}(b)$ being the flat line bundles on T_b, i.e. the dual torus. Therefore

Mirror pairs are dual Lagrangian torus fibrations

The geometric Langlands program is a connection between Hitchin moduli spaces of Langlands dual groups G and $G^\vee$. Each gives rise to an integrable system via its spectral curve construction. Hausel-Thaddeus found these spaces to be dual torus fibrations and computed the (stringy) Hodge numbers of both sides, revealing them to be mirror.

CHAPTER 2

Overview and Definitions

CHAPTER 3

Symplectic Quotients in Finite and Infinite Dimensions

CHAPTER 4

Hyperkähler Quotients and Integrable Systems

Introduction

The objects of interest to us are as follows. Let $P \to X$ be a principal G-bundle over X where G is a compact Lie group. We may consider pairs (A, Φ) where A is a unitary connection on P and $\Phi \in \Omega^{1,0}(X; \operatorname{ad} P \otimes \mathbb{C})$. By looking at 4-dimensional Yang Mills theory dimensionally reduced to X, we may become interested in such pairs which satisfy the self-duality equations

$$F_A = -[\Phi, \Phi^*] \tag{4.0.1}$$

$$\bar{\partial}_A \Phi = 0 \tag{4.0.2}$$

where F is the curvature of A and $\bar{\partial}_A$ is the anti-holomorphic part of the covariant derivative with respect to A (we will probably abbreviate at least the former by writing simply F). We will denote the space of pairs satisfying the self-duality equations by $\overline{\mathcal{M}}$. Finally, we would like to study the quotient of $\overline{\mathcal{M}}$ by the action of the group of gauge transformations, denoted $\mathscr{G}$. Elements of $\mathscr{G}$ are functions on M with values in the adjoint representation of P, i.e. $\psi \in \Omega^0(X; \operatorname{ad} P)$, and the action of $\mathscr{G}$ on a pair (A, Φ) is given by

$$A \mapsto \psi d\psi^{-1} + \psi A \psi^{-1} = \psi d_A \psi^{-1}$$
$$\Phi \mapsto \psi \Phi \psi^{-1}$$

where d_A is the covariant derivative with respect to A (showing that gauge transformations preserve self-duality is a nice exercise). We will denote this quotient by $\mathcal{M}$. The goal of this talk is to show that $\mathcal{M}$ can be realized as a Hyperkähler quotient.

4.1. Symplectic Reduction and Kähler Quotients

4.1.1. Rapid Review of Symplectic Reduction. Hyperkähler quotients can be thought of as an extension of symplectic reduction, so we will briefly review the main ingredients to the symplectic reduction procedure here. The main idea is: given a manifold with some structure (a symplectic form) and a group action which respects that structure, we would like to quotient by the group action in such a way that the resulting object is a manifold with the same type of structure. This will be the main recurring theme as we proceed.

Let (M, ω) be a symplectic manifold, which, to avoid difficulty, we take to be finite dimensional.[1] Now, let G be a connected Lie group which acts on M by

[1]The case we care about will not be finite-dimensional, but we will see that the statements made here will still hold in this case.

symplectomorphisms. We say this group action is Hamiltonian if ther exists a map $\mu^* : \mathfrak{g} \to C^\infty(M)$ such that:

(1) The map $\mathfrak{g} \to \mathfrak{X}(M)$ which sends $\xi \mapsto X_\xi$ has image contained in the set of Hamiltonian vector fields. Moreover, for any $\xi \in \mathfrak{g}$, $\mu^*(\xi)$ is the Hamiltonian function for X_ξ, i.e.
$$d(\mu^*(\xi)) = \iota_{X_\xi}\omega.$$
(2) μ is a Lie algebra anti-homomorphism, i.e.
$$\mu^*([\xi,\eta]) = -\{\mu^*(\xi), \mu^*(\eta)\}.$$

In this case, we may define the moment map[2] $\mu : M \to \mathfrak{g}^*$ by $\langle \mu(x), \xi\rangle = (\mu^*(\xi))(x)$. Condition (2) above is equivalent to saying that μ is G-equivariant with respect to the coadjoint action on $\mathfrak{g}^*$, which means that the set $\mu^{-1}(0) \subset M$ is invariant under the G-action. As long as 0 is a regular value of μ, this will actually be an embedded submanifold of M. If the G-action is free on $\mu^{-1}(0)$, then the quotient $\mu^{-1}(0)/G$ will also be a manifold which we denote by $M /\!\!/ G$. In this case, we have the following theorem:

Theorem 1 (Marsden-Weinstein). *If $(M\omega)$, G, and μ are as above and $M /\!\!/ G := \mu^{-1}(0)/G$ is a manifold, then $M /\!\!/ G$ is symplectic with symplectic form ω' uniquely defined by the property that $\pi^*\omega' = i^*\omega$ where $i : \mu^{-1}(0) \to M$ and $\pi : \mu^{-1}(0) \to M /\!\!/ G$ are the inclusion and quotient maps respectively.*

4.1.2. Quotients and Kähler Structure. First, we recall the definition of a Kähler manifold. We say a manifold M endowed with a metric g, complex structure $\mathbf{J}$, and symplectic form ω is Kähler if these three structures are compatible. This may be defined in a variety of ways. One possibility is to require $\mathbf{J}$ to be covariantly constant with respect to the Levi-Civita connection induced by g and then to define $\omega(X,Y) = g(\mathbf{J}X,Y)$ for vector fields X and Y.[3] We can also see that the structure group of a Kähler manifold has been reduced to $U(n)$, where $2n$ is the real dimension of the manifold, since the metric, complex structure, and symplectic form reduce the structure group to $\mathrm{O}(2n)$, $\mathrm{GL}(n,\mathbb{C})$, and $\mathrm{Sp}(2n,\mathbb{R})$ respectively.

Since Kähler manifolds are symplectic manifolds, we may ask whether the process of symplectic reduction preserves the Kähler structure. To answer this, we first consider how a metric may descend to a quotient manifold. Let (M,g) be a Riemannian manifold and let G be a connected Lie group which acts on M by isometries. We will require that M/G comes with a natural manifold structure, in particular the G-action must be free. Then, the space M has the structure of a principal G-bundle over M/G. At any point $x \in M$, the vertical subspace of the tangent space at x, $V_x \subset T_xM$, is isomorphic to $\mathfrak{g}$. Moreover, since the G-action on M is free, there are non-vanishing vector fields which generate V_x at each point. Then, the operation of orthogonal projection onto V_x at each point defines a one form, θ, with values in $\mathfrak{g}$ which transforms under the adjoint representation of G. θ defines a connection on M, i.e. a distribution of horizontal subspaces $H_x \subset T_xM$ complementary to V_x. Then, for any vector fields X,Y on M/G, we can use the

[2] μ^* is sometimes called the comomentum map.

[3] Actually, by choosing any two of the metric, complex structure, and symplectic form and requiring that these are compatible in the appropriate sense, we may construct the third structure in a compatible way.

connection to lift these to horizontal vector fields $\tilde{X}, \tilde{Y}$ on M. We see that the metric $h(X, Y) = g(\tilde{X}, \tilde{Y})$ is well defined using this choice of (orthogonal) horizontal lift since g is invariant under the G-action.

Now, we can answer our question about whether symplectic reduction preserves Kähler structure.

THEOREM 1. *Let $(M, g, \mathbf{J}, \omega)$ be a Kähler mainfold and let G be a connected Lie group with a Hamiltonian action on (M, ω) which preserves the metric (and hence complex structure). Also, we require that $M /\!\!/ G$ has a natural manifold structure. Then the naturally induced metric g' on $M /\!\!/ G$ gives $M /\!\!/ G$ the structure of a Kähler manifold.*

SKETCH OF PROOF. Let $N = \mu^{-1}(0)$ so that $N/G = m /\!\!/ G$. Then TN is a sub-bundle of TM and we can further restrict to the horizontal sub-bundle $H \subset TN$ defined by the connection on $N \to N/G$ in the manner discussed above. The Levi-Civita connection with respect to g' on $T(N/G)$ will pull back to a G-invariant connection on $H \to N$. We will use the fact that this connection on H is given by orthogonal projection of the Levi-Civita connection of $g\big|_N$ on $TM\big|_N$ to H (for an explanation, see [**HKLR**]). Now, let $x \in N$. The complement of $T_xN \subset T_xM$ is spanned by the vectors $(\operatorname{grad} \mu^{\xi_i})_x$ for ξ_i a basis of $\mathfrak{g}$ (here we have used the notation $\mu^{\xi_i} := \mu^*(\xi_i)$) and the complement of H in TN is spanned by the vertical vectors k_i which are associated to the basis ξ_i of $\mathfrak{g}$. By definition,

$$g(\operatorname{grad} \mu^{\xi}, Y) = d\mu^{\xi}(Y) = \omega(X_{\xi}, Y) = g(\mathbf{J}X_{\xi}, Y)$$

for any vector field Y on M, and so $\operatorname{grad} \mu^{\xi} = \mathbf{J}X_{\xi}$. This shows that the vector space spanned by k_i and $(\operatorname{grad} \mu^{\xi_i})_x$ is a complex vector space. Moreover, since the basis $\{k_1, ..., k_{\dim G}\}$ can be extended to a global frame for the vertical sub-bundle $V \subset TN$, we see that the sub-bundle complementary to H over N is a complex sub-bundle, hence H is as well. This means that $\mathbf{J}\big|_N$ commutes with orthogonal projection onto H, and since $\mathbf{J}$ is compatible with g, this implies that $\mathbf{J}\big|_N$ is covariantly constant with respect to the orthogonal projection of the Levi-Civita connection of $g\big|_N$ on $TM\big|_N$ to H. Since this is just the pull back of the Levi-Civita connection of g' on $T(N/G)$ to N and $\mathbf{J}$ was assumed to be G-invariant, we see that $\mathbf{J}$ descends to a complex structure on N/G which is compatible with the induced metric g'. $\square$

4.2. Hyperkähler Quotients

4.2.1. Rapid Review of Hyperkähler Manifolds. Hyperkähler structure can be thought of as a "quaternionic" extension of Kähler structure. In particular, we say that a Riemannian manifold (M, g) is Hyperkähler if it is equipped with three complex structures $\mathbf{I}$, $\mathbf{J}$, and $\mathbf{K}$, each of which are covariantly constant, and which satisfy quaternionic algebraic relations (i.e. $\mathbf{I}^2 = -1$, $\mathbf{J}^2 = -1$, $\mathbf{K}^2 = -1$, $\mathbf{IJ} = \mathbf{K}$, etc.). As a result, the tangent space at each point becomes a quaternionic vector space and the structure group is reduced to $\mathrm{O}(4n) \cap \mathrm{GL}(n, \mathbb{H}) = \mathrm{Sp}(n)$. As in the Kähler case, each of these complex structures defines a symplectic form compatible with the metric: $\omega_1(X, Y) = g(\mathbf{I}X, Y)$, $\omega_2(X, Y) = g(\mathbf{J}X, Y)$, and $\omega_3(X, Y) = g(\mathbf{K}X, Y)$. This means that each triple $(g, \mathbf{I}, \omega_1)$ gives M the structure of a Kähler manifold.

4.2.2. Quotients and Hyperkähler Structure. If $(M, g, \vec{\mathbf{I}}, \vec{\omega})$ is a Hyperkähler manifold and G is a connected Lie group which acts on M in a Hamiltonian manner with respect to each symplectic structure and preserves the metric, then we may consider the symplectic reduction of M with respect to each of the three moment maps μ_1, μ_2, and μ_3. From the section above, we see that each of these quotients will inherit a Kähler structure from M. Now, we would like to define a new type of quotient which inherits the Hyperkähler structure from M.

The key will be to consider one moment map

$$\mu : M \to \mathfrak{g}^* \otimes \mathbb{R}^3$$

defined by $\mu(x) = (\mu_1(x), \mu_2(x), \mu_3(x))$. Then, if 0 is a regular value of μ, we see that $\mu^{-1}(0)$ is an embedded submanifold in M which is invariant under the G-action. Furthermore, if the G-action is free on $\mu^{-1}(0)$, then the quotient $\mu^{-1}(0)/G$ will have a natural manifold structure.

Theorem 2. *If $(M, g, \vec{\mathbf{I}}, \vec{\omega})$ and G are as above, then the quotient $\mu^{-1}(0)/G$ with the inherited metric, complex structures, and symplectic forms is a Hyperkähler manifold.*

Proof. We begin by defining the complex moment map

$$\mu_+ = \mu_2 + i\mu_3 : M \to \mathfrak{g}^* \otimes \mathbb{C}.$$

Then we see that $d\mu_+^\xi(Y) = \omega_2(X_\xi, Y) + i\omega_3(X_\xi, Y) = g(\mathbf{J}X_\xi, Y) + ig(\mathbf{K}X_\xi, Y)$ while $d\mu_+^\xi(\mathbf{I}Y) = \omega_2(\mathbf{J}X_\xi, \mathbf{I}Y) + i\omega_3(\mathbf{K}X_\xi, \mathbf{I}Y) = -g(\mathbf{K}X_\xi, Y) + ig(\mathbf{J}X_\xi, Y)$. Thus, $id\mu_+^\xi(Y) = d\mu_+^\xi(\mathbf{I}Y)$ for all vector fields Y on M. Working in local holomorphic coordinates at any point in M, we may consider the vector fields $\frac{\partial}{\partial \bar{z}^i}$, which satisfy

$$\mathbf{I}\frac{\partial}{\partial \bar{z}^i} = -i\frac{\partial}{\partial \bar{z}^i}.$$

Then, plugging this into the result above gives

$$i\frac{\partial \mu_+^\xi}{\partial \bar{z}^i} = id\mu_+^\xi\left(\frac{\partial}{\partial \bar{z}^i}\right) = d\mu_+^\xi\left(\mathbf{I}\frac{\partial}{\partial \bar{z}^i}\right) = -i\frac{\partial \mu_+^\xi}{\partial \bar{z}^i}$$

so μ_+^ξ is a holomorphic function. Thus, as long as 0 is a regular value of μ_+, we see that $\mu_+^{-1}(0)$ is an embedded complex submanifold of M with respect to the complex structure $\mathbf{I}$. This means that $\mu_+^{-1}(0)$ is Kähler with its induced metric. Now, consider the G-action restricted to $\mu_+^{-1}(0)$. This still preserves the metric and the complex structure $\mathbf{I}$, and we can restrict the moment map μ_1 to this submanifold, giving us a moment map for the G-action with respect to the restricted symplectic form ω_1. Then, from the previous proposition, we see that $\mu_1^{-1}(0) \cap \mu_+^{-1}(0)/G$ is a Kähler manifold with the induced metric and complex structure. It follows that $\mu^{-1}(0)/G$ is Kähler with respect to $\mathbf{I}$ since $\mu_+^{-1}(0) = \mu_2^{-1}(0) \cap \mu_3^{-1}(0)$. Repeating this argument for $\mathbf{J}$ and $\mathbf{K}$ shows that each of these define a Kähler structure on $\mu^{-1}(0)/G$. $\square$

4.2.3. Main Example. Now we may relate these new definitions to the subject of interest to us: the self-duality equations and the moduli space $\mathcal{M}$. We begin by looking at the manifold consisting of pairs (A, Φ) as in the introduction (we will use Hiychin's notation for this manifold, denoting it by $\mathscr{A} \times \Omega$). The tangent space to this manifold at a point (A, Φ) is given by $\Omega^{0,1}(X; \operatorname{ad} P \otimes \mathbb{C}) \oplus \Omega^{1,0}(X; \operatorname{ad} P \otimes \mathbb{C})$. We can define a symplectic structure on $\mathscr{A} \times \Omega$ as follows:

$$\omega((\Psi_1, \Phi_1), (\Psi_2, \Phi_2)) = \int_X \operatorname{Tr}(\Phi_2 \Psi_1 - \Phi_1 \Psi_2)$$

where $(\Psi_i, \Phi_i) \in \Omega^{0,1}(X; \operatorname{ad} P \otimes \mathbb{C}) \oplus \Omega^{1,0}(X; \operatorname{ad} P \otimes \mathbb{C})$. ω is clearly non-degenerate. To see it is closed, we note that ω has constant coefficients in (Ψ_i, Φ_i) and that $\mathscr{A} \times \Omega$ is an affine space.

Now, define the vector field $X = (\Psi_1, \Phi_1)$ on $\mathscr{A} \times \Omega$ by $\Psi_1 = \bar{\partial}_A \psi$ and $\Phi_1 = [\Phi, \psi]$ where $\psi \in \Omega^0(X; \operatorname{ad} P)$ is an infinitesimal gauge transformation. Moreover, since $\mathscr{A} \times \Omega$ is an affine space, we may consider the vector field $(\dot{A}^{0,1}, \dot{\Phi})$ where $(A, \Phi) \in \mathscr{A} \times \Omega$. Then, we can compute

$$\begin{aligned}
(\iota_X \omega)(\dot{A}^{0,1}, \dot{\Phi}) &= \int_X \operatorname{Tr}(-[\Phi, \psi] \dot{A}^{0,1} + \dot{\Phi} \bar{\partial}_A \psi) \\
&= \int_X \operatorname{Tr}(\psi [\dot{A}^{0,1}, \Phi] + \bar{\partial}_A \dot{\Phi} \psi) \\
&= df(\dot{A}^{0,1}, \dot{\Phi})
\end{aligned}$$

where $f = \int_X \operatorname{Tr}(\bar{\partial}_A \Phi \psi)$. Thus, for the complex symplectic form ω, we have shown that the function f is Hamiltonian with respect to the vector field X. Moreover, one can check that the assignment $\psi \mapsto f$ is equivariant with respect to the action of the group $\mathscr{G}$, so we see that f defines a moment map for this action. We define ω_2 and ω_3 to be the real and imaginary parts of the symplectic form ω, and μ_2, μ_3 to be the real and imaginary parts of the moment map μ defined by f. Furthermore, $\mathscr{A} \times \Omega$ comes with a natural Kähler metric defined by

$$g((\Psi, \Phi), (\Psi, \Phi)) = 2i \int_X \operatorname{Tr}(\Psi^* \Psi + \Phi \Phi^*)$$

where $(\Psi, \Phi) \in \Omega^{0,1}(X; \operatorname{ad} P \otimes \mathbb{C}) \oplus \Omega^{1,0}(X; \operatorname{ad} P \otimes \mathbb{C})$. This defines a third symplectic form ω_1 on $\mathscr{A} \times \Omega$. It turns out that all three of these symplectic forms are compatible with the metric g and that together with g they define a Hyperkähler structure on $\mathscr{A} \times \Omega$ (see section 6 of [**H1**]). Moreover, if we recall that the moment map associated to ω_1 was defined by $\mu_1(A, \Phi) = F_A + [\Phi, \Phi^*]$, we see that requiring $\mu_1(A, \Phi) = 0$ is equivalent to the self-duality equation (1). Furthermore, we can see that requiring $\mu(A, \Phi) = \mu_2(A, \Phi) + i\mu_3(A, \Phi) = 0$ is equivalent to requiring $\bar{\partial}_A \Phi = 0$, which is exactly self-duality equation (2). Thus, the Hyperkähler quotient $\mu^{-1}(0)/\mathscr{G}$ is exactly the moduli space of solutions $\mathcal{M}$.

With this result in hand, we may hope that the result from the previous section tells us that $\mathcal{M}$ is a Hyperkähler manifold. However, in this infinite-dimensional setting the result is a purely formal statement. Instead, it is possible to directly show that $\mathcal{M}$ has a Hyperkähler structure.

THEOREM 3. *Let $\mathscr{A} \times \Omega$ and $\mathscr{G}$ be as above. Then the Hyperkähler quotient $\mu^{-1}(0)/\mathscr{G}$ inherits the structure of a Hyperkähler manifold (assuming it has a manifold structure to begin with).*

In the specific case [**H1**] deals with, this result appears as Theorem 6.7. We will only go through the main points of the proof here.

Rough sketch of proof. The result of Marsden and Weinstein tells us that each of the inherited symplectic structures on $\mu^{-1}(0)/\mathscr{G}$ is, in fact, closed and non-degenerate. Moreover, it is possible to show that each complex structure on $\mathscr{A} \times \Omega$ induces an almost complex structure on $\mu^{-1}(0)/\mathscr{G}$. Then, [**H1**] shows that the integrability of the induced complex structures is implied as long as the corresponding symplectic forms are closed. Since we know this is the case, we obtain three complex structures on $\mu^{-1}(0)/\mathscr{G}$ each compatible with g. It is left to check that these complex structures satisfy the quaternionic algebraic relations. However, this must only be checked on each tangent space, and this local property follows directly from the construction of these complex structures from the complex structures on $\mathscr{A} \times \Omega$. $\square$

CHAPTER 5

Stability and the Narasimhan-Seshadri Theorem

CHAPTER 6

Teichmueller Theory

CHAPTER 7

Construction of the Hitchin Moduli Space

This talk closely follows [**H1**]. In this chapter, it is shown that the Hitchin moduli space $\mathcal{M}_H$ is a smooth manifold of dimension $12(g-1)$. We also prove that $\mathcal{M}_H$ is equipped with a complete hyperkahler metric.

7.1. Definitions

Let M be a Riemannian surface, $P \to M$ a principal $G = SO(3)$-bundle, V the associated rank 2 vector bundle, and $\mathcal{G}$ the group of gauge transformations ($\mathcal{G} = Map(M, G)$). Recall that for a connection A on P and $\Phi \in \Omega_M^{1,0}(adP \otimes \mathbb{C})$, the self-dual equations are

$$(\star) \qquad F_A + [\Phi, \Phi^*] = 0, \qquad \overline{\partial_A}\Phi = 0$$

For a connection A, $u \in \mathcal{G}$ acts by

$$u(A) = uAu^{-1} - (du)\,u^{-1}$$

(where the covariant derivative is $\nabla_A = d + A\wedge$, i.e.,

$$\nabla_{u(A)} s = u\nabla_A\left(u^{-1}s\right)$$

for a section s of V.

For $\Phi \in \Omega_M^{1,0}(adP \otimes \mathbb{C})$, u acts by

$$u(\Phi) = u\Phi u^{-1},$$

where we regard $adP \otimes \mathbb{C}$ as the bundle of trace-zero endomorphisms of V.

Definition 2. The Hitchin moduli space is

$$\mathcal{M}_H := \{\text{solutions to } (\star)\}/\mathcal{G}.$$

Goal: To learn about the geometry and topology of $\mathcal{M}_H$.

7.2. Summary of results

Let V be a rank 2, odd degree vector bundle over a Riemannian surface M. Then,

- $\mathcal{M}_H$ is a smooth manifold of dimension $12(g-1)$.
- $\mathcal{M}_H$ has a natural metric.
- $\mathcal{M}_H$'s metric is complete and hyperkahler (in fact, $\mathcal{M}_H$ is a hyperkahler quotient).

Remark. V has odd degree $\implies$ there are no reducible solutions to $(\star)$ $\implies$ $\mathcal{G}$ acts freely on $(\star)$.

7.3. $\mathcal{M}_H$ is a dimension $12(g-1)$ smooth manifold

First, to get the expected dimension, we compute the dimension of the tangent space to $\mathcal{M}_H$ at a regular point (i.e., one with trivial isotropy group).

7.3.1. Idea of Proof.

- Linearize $(\star)$ to determine the expected dimension.
- Let $(A \times \Omega)_0$ denote "regular points," i.e., ones which are only fixed by the identity in $\mathcal{G}$ (those with trivial isotropy group). Then, exhibit $\mathcal{M}_H$ is a smooth submanifold of $(\mathcal{A} \times \Omega)_0 / \mathcal{G}$ using the regular value theorem.

7.3.2. Linearization of $(\star)$. Let $\left(\dot{A}, \dot{\Phi}\right) \in \Omega^1_M(adP) \oplus \Omega^{1,0}_M(adP \otimes \mathbb{C})$. To get the linearization of $(\star)$, fix a base point (A, Φ) and look at

$$\left.\frac{d}{dt}\right|_{t=0} (\star)\left(A + t\dot{A}, \Phi + t\dot{\Phi}\right).$$

Recalling that $F_A = dA + A \wedge A$, the first equation becomes

$$d_A\dot{A} + \left[\dot{\Phi}, \Phi^*\right] + \left[\Phi, \dot{\Phi}^*\right] = 0,$$

and the second is

$$\overline{\partial_A}\dot{\Phi} + \dot{A}^{0,1} \wedge \Phi = 0.$$

$\left(\dot{A}, \dot{\Phi}\right)$ arises from an infinitesimal gauge transformation $\dot{\psi} \in \Omega^0_M(adP)$ if

$$\dot{A} = d_A\dot{\psi}, \qquad \dot{\Phi} = \left[\Phi, \dot{\psi}\right].$$

Let

$$\begin{array}{rcl} d_1 : \Omega^0_M(adP) & \longrightarrow & \Omega^1_M(adP) \oplus \Omega^{1,0}_M(adP \otimes \mathbb{C}) \\ \dot{\psi} & \mapsto & \left(d_A\dot{\psi}, \left[\Phi, \dot{\psi}\right]\right) \end{array}$$

and

$$\begin{array}{rcl} d_2 : \Omega^1_M(adP) \oplus \Omega^{1,0}_M(adP \otimes \mathbb{C}) & \longrightarrow & \Omega^2_M(adP) \oplus \Omega^2_M(adP \otimes \mathbb{C}) \\ \left(\dot{A}, \dot{\Phi}\right) & \mapsto & \left(d_A\dot{A} + \left[\dot{\Phi}, \Phi^*\right] + \left[\Phi, \dot{\Phi}^*\right], \overline{\partial_A}\dot{\Phi} + \dot{A}^{0,1} \wedge \Phi\right). \end{array}$$

Then, d_1, d_2 define an *elliptic complex* with index $3(2-2g) - 6(g-1) = 12(1-g)$. The Atiyah-Singer index theorem then says that

$$\dim H^0 - \dim H^1 + \dim H^2 = 12(1-g).$$

7.3.3. Elliptic Complexes. Now, a short digression into elliptic complexes. Let X be a manifold, $\pi : T^*X \to X$ be projection onto the base, and $\{E_k \to X\}$ a collection of vector bundles over X.

Definition 3. A chain complex

$$\cdots \longrightarrow \Gamma(E_{k-1}) \xrightarrow{P_{k-1}} \Gamma(E_k) \xrightarrow{P_k} \Gamma(E_{k+1}) \longrightarrow \cdots$$

is **elliptic** if the corresponding sequence of symbols

$$\cdots \longrightarrow \pi^*E_{k-1} \xrightarrow{\sigma(P_{k-1})} \pi^*E_k \xrightarrow{\sigma(P_k)} \pi^*E_{k+1} \longrightarrow \cdots$$

is exact.

EXAMPLE. Let $P: E_1 \to E_2$ be a differential operator. Then, the complex

$$0 \longrightarrow \Gamma\left(E_1\right) \xrightarrow{P} \Gamma\left(E_2\right) \longrightarrow 0$$

is elliptic means that

$$0 \longrightarrow \pi^* E_1 \xrightarrow{\sigma(P)} \pi^* E_2 \longrightarrow 0$$

is exact, i.e., that $\sigma\left(P\right): \pi^* E_1 \to \pi^* E_2$ is an isomorphism. Recall that this is the "usual" definition of elliptic differential operator.

Now, return to our elliptic complex.

$$\Omega_M^0\left(adP\right) \xrightarrow{d_1} \Omega_M^1\left(adP\right) \oplus \Omega_M^{1,0}\left(adP \otimes \mathbb{C}\right) \xrightarrow{\qquad d_2 \qquad} \Omega_M^2\left(adP\right) \oplus \Omega_M^2\left(adP \otimes \mathbb{C}\right)$$

$$\dot{\psi} \longmapsto \left(d_A \dot{\psi}, \left[\Phi, \dot{\psi}\right]\right)$$

$$\left(\dot{A}, \dot{\Phi}\right) \longmapsto \left(d_A \dot{A} + \left[\dot{\Phi}, \Phi^*\right] + \left[\Phi, \dot{\Phi}^*\right], \overline{\partial_A}\dot{\Phi} + \dot{A}^{0,1} \wedge \Phi\right)$$

By construction, we're interested in H^1 of this complex: $H^0 = \ker d_1$ is the covariantly constant $\dot{\psi}$ which commute with Φ. These correspond to reducible solutions to $(\star)$. So, if $H^0 \neq 0$, then (A, Φ) is reducible. In fact, by considering d_2^*, we can show that $H^2 = 0$ as well [**H1**]. Hence,

$$-\dim H^1 = 12\left(1-g\right),$$

i.e., for a regular point (A, Φ),

$$\boxed{\dim T_{(A,\Phi)}\mathcal{M}_H = 12\left(g-1\right)}.$$

7.3.4. $\mathcal{M}_H$ is a smooth manifold. This is a sketch, for complete details see [**H1**].

DEFINITION 4. (A, Φ) is a **regular point** if the isotropy group of (A, Φ) is the identity (i.e., there are no nontrivial gauge transformations fixing this point).

Recall that infinitesimally, these are the points where there are no nonzero solutions to $d_1 \dot{\psi} = 0$.

Let $\left(\mathcal{A} \times \Omega\right)_0$ denote the open set of regular points in $\mathcal{A} \times \Omega$, and

$$B := \left(\mathcal{A} \times \Omega\right)_0 / \mathcal{G}.$$

By construction, B is a Banach manifold with the quotient topology. We have the map

$$d_1^*: \Omega_M^1\left(adP\right) \oplus \Omega_M^{1,0}\left(adP \otimes \mathbb{C}\right) \longrightarrow \Omega_M^0\left(adP\right).$$

Define a *slice* of $\mathcal{M}_H$ to be $\ker d_1^*$, at some fixed (A_0, Φ_0)—then, the slices provide coordinate patches for B. Set

$$\begin{array}{rcl} f: \ker d_1^* & \longrightarrow & \Omega_M^2\left(adP\right) \oplus \Omega_M^2\left(adP \otimes \mathbb{C}\right) \\ (A, \Phi) & \mapsto & \left(F_A + \left[\Phi, \Phi^*\right], \overline{\partial_A}\Phi\right); \end{array}$$

then, $f^{-1}\left(0,0\right)$ is a smooth submanifold of $\ker d_1^*$ with dimension $12(g-1)$.

Since $\ker d_1^*$ form coordinate patches for B, the remainder of the proof is just arguing that the $\ker d_1^*$ patch together to form a smooth manifold.

7.4. The tangent space

Thanks to our results in the previous section, we have an explicit description of the tangent space to $\mathcal{M}_H$:

$$T_{(A,\Phi)}\mathcal{M}_H = \left\{ \left(\dot{A}, \dot{\Phi}\right) \middle| \begin{array}{c} d_A\dot{A} + \left[\dot{\Phi}, \Phi^*\right] + \left[\Phi, \dot{\Phi}^*\right] = 0, \\ \overline{\partial_A}\dot{\Phi} + \dot{A}^{0,1} \wedge \Phi = 0, \\ d_A^*\dot{A} + Re\left[\Phi^*, \dot{\Phi}\right] = 0. \end{array} \right\}$$

The third equation appears because $d_1^*\left(\dot{A}, \dot{\Phi}\right) = d_A^*\dot{A} + Re\left[\Phi^*, \dot{\Phi}\right]$.

7.5. $\mathcal{M}_H$ has a complete hyperkahler metric

Recall from Sean's talk that the metric on $\mathcal{A} \times \Omega$ given by

$$g\left((\psi, \phi), (\psi, \phi)\right) = 2i \int_M Tr\left(\psi^*\psi + \phi\phi^*\right)$$

induces a metric on $T_p\mathcal{M}_H$. We want to show that this is a complete metric on $T_{(A,\Phi)}\mathcal{M}_H$. As a reminder:

DEFINITION. A metric on M is **complete** if every Cauchy sequence of points on M has a limit which is also in M.

7.5.1. Idea of Proof. By contradiction: Suppose we have a sequence of points in $\mathcal{M}_H$ defined by a geodesic γ converging to a point not in $\mathcal{M}_H$. Because g is $\mathcal{G}$-invariant, we can look at $\tilde{\mathcal{M}_H} \subset \mathcal{A} \times \Omega$. Lift γ to a horizontal $\tilde{\gamma}$ in $\tilde{\mathcal{M}}_H$. $\tilde{\gamma}$ still defines a Cauchy sequence in $\tilde{\mathcal{M}}_H$, so we have

$$||A_n - \overline{A}||^2_{L^2} + ||\Phi_n - \overline{\Phi}||^2_{L^2} \leq C$$

for some C as $(A_n, \Phi_n) \to \left(\overline{A}, \overline{\Phi}\right)$ (where $\left(\overline{A}, \overline{\Phi}\right)$ is the limiting point not in $\mathcal{M}_H$).

Now, we apply Uhlenbeck's compactification theorem to show that there's a gauge transformation taking $\left(\overline{A}, \overline{\Phi}\right)$ to a solution of $(\star)$:

THEOREM 5 (Uhlenbeck). *There are constants ϵ_1, $M > 0$ such that any connection A on the trivial bundle over $\overline{B}^4$ with $||F_A||_{L^2} < \epsilon_1$ is gauge equivalent to a connection $\tilde{A}$ over B^4 with*

(1) $d^*\tilde{A} = 0$,
(2) $\lim_{|x|\to 1} \tilde{A}_r = 0$, *and*
(3) $||\tilde{A}||_{L^2_1} \leq M||F_{\tilde{A}}||_{L^2}$.

Moreover for suitable constants ϵ_1, M, $\tilde{A}$ is uniquely determined by these properties, up to $\tilde{A} \mapsto u_0\tilde{A}u_0^{-1}$ for a constant u_0 in $U(n)$.

In particular, we can use the following corrollary:

COROLLARY. *For any sequence of ASD connections A_α over $\overline{B}^4$ with $||F(A_\alpha)||_{L^2} \leq \epsilon$, there is a subsequence α' and gauge equivalent connections $\tilde{A}_{\alpha'}$ which converge in C^∞ on the open ball.*

Therefore, there's a gauge transformation taking $\left(\overline{A}, \overline{\Phi}\right)$ to a solution of $(\star)$. This is a contradiction, so $\mathcal{M}_H$ is complete.

7.5.2. Hyperkahler structure. Recall from Sean's talk that there's a symplectic form on $\mathcal{M}_H$ given by

$$\omega\left(\left(\psi_1,\phi_1\right),\left(\psi_2,\phi_2\right)\right)=\int_M Tr\left(\phi_2\psi_1-\phi_1\psi_2\right).$$

This defines a complex moment map from the action of $\mathcal{G}$:

$$\mu\left(A,\Phi\right)=\overline{\partial}_A\Phi.$$

We can write $\mu=\mu_2+i\mu_3$ and $\omega=\omega_2+i\omega_3$ to get two symplectic structures out of this. The third (or first?) symplectic structure is just the Kahler form associated to the metric

$$g=2i\int_M Tr\left(\psi^*\psi+\phi\phi^*\right),$$

and has moment map

$$\mu_1\left(A,\Phi\right)=F_A+\left[\Phi,\Phi^*\right].$$

This exhibits $\mathcal{M}_H$ as a hyperkahler quotient of $\mathcal{A}\times\Omega$:

$$\mathcal{M}_H=\bigcap_{i=1}^{3}\mu_i^{-1}\left(0\right)/\mathcal{G}.$$

7.6. Summary of topological results

Here, we state some topological results. For proofs, see section 7 of [**H1**].

$\mathcal{M}_H$ is...

- non-compact
- connected and simply connected
- the Betti numbers b_i vanish for $i>6g-6$.

CHAPTER 8

Integrable Systems and Spectral Curves

CHAPTER 9

Meromorphic Connections and Stokes Data

This talk is presented by Honghao on April 22nd, 2015. The references is the work of P. Boalch.

The talk has two parts. The first parts reviewed the three descriptions of the Hitchin moduli space: Dolbeault, De-Rham and Betti. The relation of the last two is also known as the Riemann-Hilbert correspondence. The correspondence can be generalized to punctured discs, and it requires additional information on each side. The additional packages on the two sides contain meromorphic connections and Stokes data.

Perspectives of the Hitchin space

A definition first.

Let X be a Riemann surface (probably not compact), and G be a Lie group over $\mathbb{C}$. The character variety is defined to be

$$\mathcal{M} = Hom(\pi_1(X), G)/G.$$

The three descriptions of Hitchin's moduli space:

(1) (Dolbeault) $\mathcal{M}_{Dol}$ the moduli space of Higgs bundles, which consists of pairs (E, Φ), where E is a rank n degree zero holomorphic vector bundle and $\Phi \in \Gamma(End(E) \otimes \Omega^1)$ a Higgs field.
(2) (De-Rham) $\mathcal{M}_{DR}$ the moduli space of connections on rank n holomorphic vector bundles, consisting of pairs (V, ∇) with $\nabla : V \to V \otimes \Omega^1$ a holomorphic connection.
(3) (Betti) $\mathcal{M}_B$ the conjugacy classes of representation of the fundamental group of X. Notice this is the character variety of the compact Riemann surface with $G = GL(n, \mathbb{C})$.

From Dolbeault to De-Rham: naturally diffeomorphic as real manifolds via the non-abelian Hodge correspondence, but not complex analytically isomorphic.

From De-Rham to Betti: Locally, the connection can be written as $\nabla = d + A$. Since X is compact, homomorphic is the same as algebraic and V and ∇ are holomorphic implies $A \in GL(n, \mathbb{C})$. The flatness of the connection implies the holonomy only depends on the homotopy type, thus the representation of the fundamental group $\pi_1(X)$.

Example: when $n = 1$ that is $G = \mathbb{C}^*$, then

(1) $\mathcal{M}_{Dol} \cong T^*Jac(X)$.
(2) $\mathcal{M}_{DR} \to Jac(X)$ a twisted cotangent bundle of the Jacobian of Σ; it is an affine bundle modeled on the cotangent bundle.
(3) $\mathcal{M}_B \cong (\mathbb{C}^*)^{2g}$ is isomorphic to $2g$ copies of $\mathbb{C}^*$.

Alert: The isomorphism (Riemann-Hilbert correspondence) $\mathcal{M}_{DR} \to \mathcal{M}_B$ involves exponential and not algebraic. And conversely, no algebraic isomorphism $\mathcal{M}_B \to \mathcal{M}_{DR}$ exists.

Another argument: $\mathcal{M}_B$ is affine, and it does not have compact subvarieties. Thus so is $\mathcal{M}_{DR}$. However, $\mathcal{M}_{Dol}$ has such (zero section), which make it impossible to have complex analytic isomorphism $\mathcal{M}_{DR} \to \mathcal{M}_{Dol}$. On the other hand, the abelian Hodge theory and Dolbeault isomorphism gives rise to a non-holomorphic isomorphism $\mathcal{M}_{DR} \to \mathcal{M}_{Dol}$.

A couple of remarks,

1) Dolbeault space has algebraic Hamiltonian integrable system (presented by Lei, also known as the Hitchin system?), there is a proper map

$$\mathcal{M} \to \mathbb{H}$$

to a vector space of half of the dimension. The generic fibres of the map are abelian varieties. In the abelian case, the space is product of $\mathbb{C}^g \times Jac(X)$, but in general, the fibres vary non-trivially and there are singular fibres.

2) The mapping class group has an natural (symplectic, algebraic) action on the moduli space, through the Betti description.

Airy's equation

Consider $X = \mathbb{P}^1 = \mathbb{C}_z \cup \{\infty\} = \{0\} \cup \mathbb{C}_w$. The general Airy's equation: $f^{''} = z^n f$. Write $\nabla = d - A$, the ODE corresponds to a holomorphic connection on $\mathbb{C}_z$, as in

$$\frac{d}{dz}\begin{pmatrix} f \\ f' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ z^n & 0 \end{pmatrix}\begin{pmatrix} f \\ f' \end{pmatrix}.$$

However, this connection is singular at infinity. Let $w = 1/z$, then $z\partial_z = -w\partial_w$. The ODE becomes $((z\partial_z)^2 - (z\partial_z) - z^{n+2})f = 0$, which in the new coordinate is $((w\partial_w)^2 + (w\partial_w) - w^{-n-2})f = 0$, that is

$$\frac{\partial^2 f}{\partial w^2} + \frac{2}{w}\frac{\partial f}{\partial w} - \frac{1}{w^{n+4}} f = 0.$$

In terms of connection, this is

$$\frac{d}{dw}\begin{pmatrix} f \\ f' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{w^{n+4}} & \frac{2}{w} \end{pmatrix}\begin{pmatrix} f \\ f' \end{pmatrix}.$$

The connection A can be expanded as

$$A = \frac{A_{n+4}}{w^{n+4}} + \frac{A_1}{w} + \text{holomorphic terms},$$

where $A_i \in \mathfrak{gl}(n, \mathbb{C})$.

The irregular type Q of this connection at infinity is

$$dQ = \frac{A_{n+4}}{w^{n+4}} = \frac{1}{w^{n+4}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Regular singularity

A meromorphic connection has regular singularity if its singular pole has order ar most 1. Deligne showed a version of the correspondence.

Deligne's Riemann-Hilbert correspondence: If X is a Riemann-Surface with punctures and $G = GL(n, \mathbb{C})$, then the character variety corresponds to the space of algebraic connections with regular singularities at the punctures.

Stokes Data

Let D be an effective divisor on $\mathbb{P}^1$. A meromorphic connection ∇ on D is a map $\nabla : V \to V \otimes K(D)$ satisfying the Leibniz rule. Locally at an irregular singular point, $\nabla = d - A$ and $A = dQ + \Lambda \frac{dz}{z}$, Q is a matrix of meromorphic functions. Choose a framing so that Q is diagonal, that is $Q = \mathrm{diag}(q_1, q_2, \cdots, q_n)$. Let $q_{ij}(z)$ be the most singular term on $q_i - q_j$.

Some definitions.

- Let S^1 parameterize the rays around z. If $d_1, d_2 \in S^1$, define $Sect(d_1, d_2)$ be the open sector sweeping from d_1 to d_2.
- The anti-Stokes directions $\mathbb{A} \subset S^1$ are the directions $d \in S^1$ such that for some $i \neq j, q_{ij} \in \mathbb{R}_{<0}$ along this ray d.
- The roots of d are the ordered pairs (ij) supporting d:
$$Roots(d) := \{(ij) | q_{ij} \in \mathbb{R}_{<0} \text{ along } d\}.$$
- The multiplicity of d is the number of roots supporting d.
- the group of Stokes factors associated to d is
$$Sto_d(A) := \{K \in G | (K)_{ij} = \delta_{ij}, \text{ unless } (ij) \text{ is a root of } d\}.$$
This is a unipotent subgroup of $G = GL(n, \mathbb{C})$.

Remarks: The anti-Stokes directions are those where the roots decays rapidly towards the singular point.

To see the stokes group is unipotent. First, $i \neq j$ implies the diagonals are always 1. Second, if (ij) is a root, (ji) won't be, which implies an ordering. Third, transitivity, as $q_{ij}, q_{jk} \in \mathbb{R}_{<0}$, then $q_{ik} = q_{ij} + q_{jk} \in \mathbb{R}_{<0}$. By a permutation, the Stokes matrix becomes upper triangular, with diagonals equal to 1, which is obviously an unipotent subgroup, and the property of which is invariant under permutation conjugation.

Since $q_{ij}(z) = a/z^{k-1}$, there is a $\pi/(k-1)$ rotational symmetry. (k is the order of the pole.) Notice that in the generic situation, the leading terms of q_i do not cancel out. q_{ij} has a $2\pi/(k-1)$ symmetry, and q_{ji} contributes the remaining. Let $\mathbf{d} = (d_1, \cdots, d_l)$ be the set of anti-Stokes directions up to this symmetry, then $n(n-1)/2 = \sum_{i=1}^{l} Mult(d_i)$.

A key result.

(1) The product of the groups of Stokes factors in a half-period is isomorphic to a subgroup of G as a variety.
$$\prod_{d \in \mathbf{d}} Sto_d(A) \cong PU_+P^{-1},$$
via $(K_1, \cdots, K_l) \mapsto K_l \cdots K_2 K_1$, and P is a permutation group arranging the anti-Stokes directions in order.

(2) The product of all groups of Stokes factors is isomorphic to the variety:
$$\prod_{d \in \mathbb{A}} Sto_d(A) \cong (U_+ \times U_-)^{k-1}.$$

Suppose E_{ij} is the matrix with the (ij) entry 1 and 0 otherwise. It suffices to show any upper triangular matrix U can be decomposed uniquely into a product of $(I + t_{ij}E_{ij})$, for $1 \leq i < j \leq n$. Notice that $E_{ij}E_{kl} = \delta_{jk}E_{il}$. Expanding the product, $t_{12}, t_{23}, \cdots$ the sub-diagonal entries are determined immediately. Followed by the next sub-diagonal and so on. The second fact follows the first one easily.

The second product is the space of Stokes data $Sto(Q)$. For $k = 2$, the (local version) irregular Riemann-Hilbert correspondence is

$$\{\text{Connections of singular type Q}\}/\mathcal{G} \cong \mathfrak{t} \times Sto(Q),$$

where $\mathcal{G}$ is the group of holomorphic maps from the unit disk to G taking $z = 0$ to 1_G, and $\mathfrak{t}$ is an Cartan subalgebra fixed a priori.

CHAPTER 10

The Langlands Program and Relations to Geometric Langlands

CHAPTER 11

Spectral Curves and Irregular Singularities

CHAPTER 12

The Stokes Groupoid

CHAPTER 13

Cluster Varieties

CHAPTER 14

The Hitchin System and Teichmueller Theory II

CHAPTER 15

Geometric Langlands and Mirror Symmetry

CHAPTER 16

Hitchin Systems and Supersymmetric Field Theories

16.1. Introduction and definitions

In previous talks, we've explored many properties of the Hitchin system and its geometry, as well as several applications. The core idea of this talk is that, given a certain supersymmetric field theory, we can obtain the Hitchin system as a moduli space associated to this theory via a standard procedure in quantum field theories. Gaiotto, Moore, and Neitzke [**GMN**] use this approach to construct a canonical coordinate system on the Hitchin moduli space. The goal of this talk is to explain some of the language of supersymmetric quantum field theories, and describe how we can obtain $\mathcal{M}_H$ from a particular class of such theories.

16.1.1. Supersymmetry. Consider $\mathbb{R}^n$. The **Poincare group** is the group of translations and rotations of this space: $G = ISO(n) \cong \mathbb{R}^n \rtimes SO(n)$. (In Minkowski signature, say for $\mathbb{R}^{n,1}$, we might write $ISO(n,1)$ instead). Recall that there's a spin group defined as a double cover of $SO(n)$:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin(n) \longrightarrow SO(n) \longrightarrow 1.$$

When we have such a double cover, we get an extension of the group G to the supergroup $\tilde{G}$. This gives an extension of the Poincare algebra $\mathfrak{g}$ to the **super Poincare algebra** $\tilde{\mathfrak{g}}$ coming from the double-cover of $SO(n)$. $\tilde{\mathfrak{g}}$ is called the **supersymmetry algebra**. Such $\tilde{g}$ are labelled by a choice of a representation of $Spin(n)$. For irreducible representations of $Spin(n)$, there are two possible cases:

- There is either a unique irreducible spinor representation S, and any spinor representation has the form $S^{\oplus N}$ for some N, or
- There are two distinct irreducible real spinor representations S_+, S_-, and any spinor representation has the form $S_+^{\oplus N_1} \oplus S_-^{\oplus N_2}$ for some N_1, N_2. (This occurs in dimensions $n \equiv 2, 6 \mod 8$).

Thus, when someone says $N = n$ or $N = (n_1, n_2)$ supersymmetry, they are specifying which extension of the Poincare algebra they're referring to.

The supersymmetry Lie algebra $\tilde{\mathfrak{g}}$ splits into an even and odd part:

$$\tilde{\mathfrak{g}} = \mathfrak{g}^0 \oplus \mathfrak{g}^1,$$

and is equipped with a skew-symmetric bracket $[\cdot,\cdot] : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ which satisfies the Jacobi identity. Note that $\left[\mathfrak{g}^1, \mathfrak{g}^1\right] \subset \mathfrak{g}^0$, $\mathfrak{g}^0 = \mathfrak{g}$ (the Poincare algebra), and $\mathfrak{g}^1$ is just a linear representation of $\mathfrak{g}^0$.

Example. For $N = 2$ SUSY and $G = ISO(3,1)$, we'll be interested in a $\tilde{G}$ whose Lie algebra has even part

$$\mathfrak{g}^0 = \mathfrak{iso}(3,1) \oplus \mathbb{C}.$$

The abelian $\mathbb{C}$ factor is central in $\tilde{\mathfrak{g}}$, and has a canonical generator Z.

16.1.2. BPS States. Let $\mathcal{H}$ denote the Hilbert space of states of our quantum system on $\mathbb{R}^{3,1}$. We want to think of a state as being labeled by a representation of $\tilde{G}$—the representation encodes the data of the particle. For example, the vacuum state ("empty space") in $\mathcal{H}$ corresponds to the trivial representation of $\tilde{G}$. The next simplest kind of state is one where space is empty except for a single particle propogating with some definite momentum $p \in T^*\mathbb{R}^{3,1}$. Call the subspace of the Hilbert space consisting of single-particle states $\mathcal{H}^1$. Here, there's some structure:

- $\mathcal{H}^1$ splits into components $\mathcal{H}^1_M$, labeled by $M \in \mathbb{R}_{\geq 0}$. M^2 is the eigenvalue of a quadratic Casimir operator in $ISO(3,1)$. Physically, it's the mass of the particle.
- For $N = 2$ SUSY as in our prior example, we also have a central generator $Z \in \mathbb{C}$. Together, these give a decomposition

$$\mathcal{H}^1 = \bigoplus_{M,Z} \mathcal{H}^1_{M,Z}.$$

Fix some momentum $p_{rest} \in \left(\mathbb{R}^{3,1}\right)^*$ with $||p_{rest}||^2 = M^2$, and consider the subspace $\mathcal{H}^{1,rest}_{M,Z}$ on which the subgroup of translations along $\mathbb{R}^{3,1}$ acts by p_{rest}. This is a representation of a subgroup $\tilde{G}_{rest} \subset \tilde{G}$ with

$$\tilde{G}_{rest} = SO\left(3\right) \ltimes \tilde{T},$$

where the "super translation group" $\tilde{T}$ is generated by the ordinary translations $T = \mathbb{R}^{3,1}$ plus the central character Z and the "odd translations" $\mathfrak{g}^1$. The odd translations act by a Clifford algebra (on an 8-dimensional vector space). We can then count the number of unitary irreducible representations of this Clifford algebra:

- If $M < |Z|$, then there are *no* unitary representations of the Clifford algebra.
- If $M = |Z|$, the Clifford algebra is degenerate, and its unique unitary irrep S has dimension $2^{4/2} = 4$.
- If $M > |Z|$, the Clifford algebra is nondegenerate, and its unique unitary irrep S has dimension $2^{8/2} = 16$.

States that satisfy $M = |Z|$ are called **BPS** (Bogomol'nyi, Prasad, Sommerfield) **states**, and as one might expect, they satisfy a set of differential equations depending on the field theory. The BPS states are those in which half of the supersymmetry generators are unbroken.

16.1.3. Moduli of Vacua. The "moduli space" associated to a QFT typically refers to the moduli space of vacua. By **vacua**, we mean the quantum state with the lowest possible energy.

Question: In what sense is the space of vacua a moduli space?

For scalar fields, these are labelled by the **vacuum expectation value** (VEV). The VEV of an operator is (as the name suggests) the expectation value of the operator in the vacuum (the quantum state with the lowest possible energy). We can label a vacuum state by its VEV, and this gives a moduli space of vacua.

For $N = 2$ SUSY, the superalgebra has two representations with scalars: **vectormultiplets** (one complex scalar), and **hypermultiplets** (two complex scalars).

This gives a local splitting of the moduli of vacua $\mathcal{M}$ as

$$\mathcal{M} = \mathcal{M}_C \oplus \mathcal{M}_H,$$

where $\mathcal{M}_C$ is the "Coulomb branch" (vectormultiplets), and $\mathcal{M}_H$ is the "Higgs branch" (hypermultiplets).

16.2. Compactification and Dimensional Reduction

Compactification of a field theory is a process where, instead of considering a general space X, we consider $X = M \times C$ where C is some compact space. Dimensional reduction is the limit of the compactified theory where the volume of the compact space is shrunk to zero, which produces an effective theory on the remaining dimensions.

Example (Toy Example). Consider a field theory on $X = \mathbb{R}^n \times S^1_R$ (S^1_R is the circle of radius R). Let θ be a coordinate on S^1_R, and x^i coordinates on $\mathbb{R}^n$. At a fixed x coordinate, the fields along the S^1_R look like

$$\phi|_x = \sum_n A_n \cos\left(\frac{2\pi n}{R}\theta\right) + B_n \sin\left(\frac{2\pi n}{R}\theta\right),$$

where the coefficients A_n and B_n are determined by the boundary conditions on ϕ. As $R \to 0$, the eigenvalues $\lambda_n = \frac{2\pi n}{R}$ approach ∞, except for $n = 0$. Note that in quantum mechanics, $\hbar\lambda_n$ is the *momentum* of eigenstate n, so as $R \to \infty$, all mometums except the trivial one also $\to \infty$. We should interpert $R \to 0$ as meaning that, for finite energy (and hence, finite momentum), the only eigenstate left is the trivial one.

If $\phi|_x$ is constant, it means that the field ϕ does not depend on θ—the dimensional reduction of the theory on S^1_R consists of the fields of the $\mathbb{R}^n \times S^1_R$ theory which do not depend on θ.

So, we have two equivalent perspectives on dimensional reduction to M of a theory on a space $M \times C$:

- It's the limit of the theory on $M \times C$ where the volume of C contracts to zero, or
- It's the theory on $M \times C$ where all fields are taken to be independent of coordinates on C.

Example (Yang-Mills). We actually already encountered dimensional reduction in one of the first lectures of the course. Consider classical Yang-Mills theory.

Yang-Mills theory is a field theory defined for principal G-bundles $P \to X$, where X is a 4-dimensional Riemannian manifold.

Fields: Connections A on P.
Lagrangian:

$$L(A) = |F_A|^2 \, d\mu$$

Recall that from the Lagrangian, we obtain the action functional by

$$S(A) = \int_X L(A) = \int_X |F_A|^2 \, d\mu.$$

The equations of motion for Yang-Mills theory are

$$d_A^* F_A = 0,$$

and the instantons are the (anti) self-dual connections:

$$F_A = \pm * F_A.$$

(Here, $* : \Omega^2_X \cong \Omega^2_X$ is the Hodge star operator).

In local coordinates we can write $d_A = d + A$, where

$$A = A_1 dx_1 + A_2 dx_2 + A_3 dx_3 + A_4 dx_4.$$

Define

$$F_{ij} := [\nabla_i, \nabla_j] = \frac{\partial}{\partial x_i} A_j - \frac{\partial}{\partial x_j} A_i + [A_i, A_j].$$

Look at the self-dual connections. Then, the instanton equation $F_A^- = 0$ becomes

$$\begin{aligned} F_{12} &= F_{34}, \\ F_{13} &= -F_{24}, \\ F_{14} &= F_{23}. \end{aligned}$$

Let's restrict attention to $X = C \times \mathbb{R}^2$, where C is a Riemann surface. Suppose that we compactify and perform dimensional reduction in $\mathbb{R}^2$ coordinates: restrict to A_j which are invariant under translation in x_3, x_4. Then, $A_1 dx_1 + A_2 dx_2$ defines a connection on C. Relabel $A_3 = \phi_1$ and $A_4 = \phi_2$, and define $\varphi = \phi_1 - i\phi_2$; then the self-dual equations become

$$\begin{aligned} F_A - \frac{1}{2} i [\varphi, \varphi^*] &= 0, \\ [\nabla_1 + i\nabla_2, \varphi] &= 0. \end{aligned}$$

If we think of φ as defining a local section of $\Omega^0 (C; ad(P) \otimes \mathbb{C})$, and set $\Phi = \frac{1}{2}\varphi dz \in \Omega^{1,0} (ad(P) \otimes \mathbb{C})$ and $\Phi^* = \frac{1}{2}\varphi^* d\overline{z} \in \Omega^{0,1} (ad(P) \otimes \mathbb{C})$, then the equations become

$$\begin{aligned} F_A + [\Phi, \Phi^*] &= 0, \\ \overline{\partial_A} \Phi &= 0, \end{aligned}$$

the usual Hitchin equations.

16.3. 5D Super Yang-Mills Theory

Now, let's repeat the previous example, but including some supersymmetry. 5D Super Yang-Mills theory admits a conventional Lagrangian description: Let P be a principal G-bundle over X (a 5-dimensional space). The theory has:

Fields: Connections A on P, sections ϕ^i of $Ad(P)$ $(i = 1, \ldots, 5)$, and fermions.

Lagrangian:

$$L = \frac{R}{8\pi^2} Tr \left[\frac{1}{R^2} F_A \wedge *F_A + \sum_{i=1}^{5} d_A \phi^i \wedge *d_A \phi^i + \text{fermions} \right].$$

Remark.

(1) 5D SYM is well-defined as an effective field theory, below a certain energy scale. It is not obviously well-defined at arbitrarily high energies.
(2) There's this unusual R factor appearing here that you should be suspicious of. We'll explain where this comes from at the end of the talk.

Compactification on C. Let's take $X = \mathbb{R}^{2,1} \times C$, where C is a Riemann surface. Analogous to the classical case, when we compactify 5D SYM on C, we combine ϕ^4 and ϕ^5 into a complex-valued 1-form on C:

$$\varphi = \left(\phi^4 + i\phi^5\right) dz.$$

Note that to be a sensible theory, we additionally require translation invariance along $\mathbb{R}^{2,1}$.

Question: What are the classical field configurations in the compactified theory which preserve the supersymmetry? (Recall that these are the BPS states!)

Assuming $\phi^1, \phi^2, \phi^3 = 0$, the equations satisfied by the remaining fields are

$$(\star) \qquad \begin{cases} F_A + R^2 \left[\varphi, \varphi^*\right] = 0, \\ \overline{\partial_A} \varphi = 0, \end{cases}$$

which we recognize as (almost) Hitchin's equations. In other words, the moduli space of vacua of $SYM[C]$ in the low energy limit is

$$M_C\left[G\right] = \{\text{solutions to } (\star)\} \,/\, \{\text{gauge transformations}\} = \mathcal{M}_H.$$

Remark. We took $\phi^1 = \phi^2 = \phi^3 = 0$ above. If we don't, SUSY also imposes equations on ϕ^1, ϕ^2, ϕ^3:

$$d_A \phi^i = 0, \qquad \left[\varphi, \phi^i\right] = 0, \qquad \left[\phi^i, \phi^j\right] = 0.$$

But, at a generic point in the moduli space, these equations won't have any nontrivial solutions, so the assumption that $\phi^j = 0$ isn't much of an imposition.

A key difference between this example and dimensional reduction for classical Yang-Mills theory is that we have dimensionally reduced to a theory on $\mathbb{R}^{2,1}$, *not* a theory on C. Instead of seeing Hitchin's moduli space as the moduli of instantons for our theory, it appears as the moduli of BPS states!

The full moduli space of vacua has a Coulomb branch—identified with the Hitchin moduli space—and Higgs branches attached to the specific other points where nontrivial solutions for the ϕ^j exist. (Unfortunate nomenclature: the moduli of Higgs bundles is the space of solutions that live on the Coulomb branch...)

16.4. Compactification from (2,0) 6D Theory

Now let's talk about where that pesky R factor came from. It turns out that there's a famous 6D $N = (2,0)$ QFT. It doesn't have a conventional Lagrangian description (or even a space of fields). Instead, the inputs are a 6-dimensional manifold, together with a Lie algebra $\mathfrak{g}$. Call this theory $X_{\mathfrak{g}}$. It has the following properties:

- $X_{\mathfrak{g}}$ has $N = (2,0)$ SUSY in $d = 6$.
- $X_{\mathfrak{g}}$ has no parameters—no coupling constants or scale, and the strength of the interaction can't be perturbed.
- $X_{\mathfrak{g}}$ is conformally invariant.

Despite its unconventional description, we can still compactify $X_{\mathfrak{g}}$ to obtain lower-dimensional theories. In fact, 5D SYM is $X_{\mathfrak{g}}\left[S^1\right]$, where the R is the length of the S^1. So, $\mathcal{M}_H$ is obtained as the moduli space associated to $X_{\mathfrak{g}}\left[C \times S^1\right]$. We could perform this compactification in either order: $\mathcal{M}_H$ can also be obtained as the moduli space associated to the theory $X_{\mathfrak{g}}\left[C\right]$ compactified on S^1. [**GMN**] use this observation to produce canonical Darboux coordinate systems on $\mathcal{M}_H$ and construct Calabi-Yau metrics in these coordinate systems.

Some examples of information we can obtain from this perspective:

- Compactify $X_{\mathfrak{g}}$ on C first to get a 4d $N = 2$ supersymmetric gauge theory with with Coulomb branch $\mathcal{B}$. Then, $\mathcal{B}$ is actually the Hitchin base, i.e., $\mathcal{M}_H \to \mathcal{B}$ with generic fiber a torus. Points $u \in \mathcal{B}$ correspond to spectral curves $\Sigma_u \subset T^*C$, also known as "Seiberg-Witten curves."
- $\mathcal{M}_H$ is automatically hyperkahler because of supersymmetry.

CHAPTER 17

Pyongwon's II

CHAPTER 18

Phil's II

CHAPTER 19

Lei's II

CHAPTER 20

Honghao's II

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