Hitchin Systems Seminar

Contents

CHAPTER 1

Introduction

This course studies the Hitchin system from different perspectives and in various settings.

Hitchin studied a limiting form of the self-dual equations of Yang-Mills theory in four dimensions, imposing invariance under two of the dimensions. The resulting equations were found to be conformally invariant with respect to the remaining two directions, so had a natural formulation as equations on a Riemann surface. The Hitchin moduli space, or moduli space of Higgs bundles, will be defined as the space of solutions modulo gauge transformations.

Several basic properties of the space of solutions emerged. First, the equations could be interpreted formally as hyperkahler moment map equations for the compact gauge Group action on the infinite-dimensional space of connections. Since we quotient solutions by gauge transformations, this means we have an interpretation of the Hithchin moduli space as a hyperkahler quotient. Thus it should inherit a hyperkaher structure. Of the three independent complex structures, the one most amenable to algebraic geometry is the "Dolbeault" one inherited from the complex structue on the Riemann surface, C. In that setting, the Hitchin moduli space has an interpretation as stable Higgs pairs, i.e. pairs (V, φ), where V is a rank-n holomorphic vector bundle over C, and $\varphi \in \Gamma(End(V) \otimes K_C)$ is a holomorphic $n \times n$ matrix-valued one-form. Stability requires that all φ-invariant sub-bundles have lesser slope than V. Finally, solutions are taken modulo holomorphic isomorphism of vector bundles.

Another structure which emerges in this complex structure is that of an algebraically completely integrable system. Recall first that a complete integrable system on a symplectic manifold ("phase space") is a maximal set of Poisson-commuting proper functions ("Hamiltonians"). Specifying the values of these conserved quantities gives a common level set which is a Lagrangian torus. This gives another definition: a fibration by Lagrangian tori to a base manifold where the Hamiltonians take their values. The fiber map for the Hitchin space is defined by all the invariant polynomials $Tr\varphi^k$.

The other complex structures are called "de Rham," in which the moduli space is of flat $GL_n(\mathbb{C})$ connections modulo complex gauge transformations, and the "Betti" complex structure in which the moduli space is irreducible representations of $\pi_1(X)$ modulo conjugation. A variety of theorems relate these. Betti and de Rham are equated by the Riemann-Hilbert correspondence, in which one assigns to a flat connection the monodromy data around loops in the space.

1.1. Local Case

Consider that a holomorphic vector bundle has a local trivialization by a holomorphic frame, so in the local setting we can assume $E = D \times \mathbb{C}^n$, where either D is a disk $D = \{|z| < r\} \subset \mathbb{C}$ or $D = \mathbb{C}$. In this frame, the Higgs field looks like

1.2. Teichmüller theory

1.3. Link to Knots

A hidden agenda for this seminar was to begin to explore the spectral curve from the perspective of symplectic geometry. Given a Riemann surface with complex structure J and Kähler metric h, it cotangent bundle T^*C has a tautological holomorphic one-form θ and holomorphic symplectic structure $\Omega = d\theta$. Treating C as a real manifold, T^*C has a canonical symplectic form $Re(\Omega)$ for which the spectral curve is Lagrangian, and we would like to consider it as an object of the Fukaya category of T^*C. In fact if we can equip T^*C with a Kähler form ω (this is not always possible, but is if $C = \mathbb{C}$) so that T^*C is hyper-Kähler, then the spectral curve is special Lagrangian for the Calabi-Yau form $\omega + iIm(\Omega)$, as $Im(\Omega) = 0$. To treat the spectral curve as an object in the Fukaya category of T^*C, we equip it with a brane structure and a flat vector bundle (local system). It is often fruitful to consider the case where the vector bundle is rank-one (so a $U(1)$ local system).

Algebraically, we can consider the family of spectral curves together with their Jacobians. This forms what is sometimes called the "Beauville integrable system," in the sense that the Jacobians are tori fibered over a base. Since these spectral curves have an interpretation as special Lagrangians, we know that the dimension of the space of their deformations is the first betti number — same as the dimension of the space of the fibers.

Let L be the spectral curve and $\mathcal{L}$ a local system on L, so that $(L, \mathcal{L})$ is an object in the Fukaya category of T^*C. (This involves a discrete choice of brane structure, which we ignore for present purposes.) We consider the space $\mathcal{M}(L, \mathcal{L})$ of (inequivalent) branes deformation equivalent to $(L, \mathcal{L})$. The space $\mathcal{M}(L, \mathcal{L})$ fibers over the space of Lagrangians deformation equivalent to L modulo Hamiltonian deformations by the map which forgets the local system (and the brane structure). The tangent to this space at L is closed modulo exact one-forms, i.e. $H^1_{dR}(L)$. The fiber of the forgetful map is the space of $U(1)$ local systems on L, and the tangent space to this space is $H^1(L, i\mathbf{R})$. In fact, since nearby Lagrangians are all diffeomorphic, the fibration $\mathcal{M}(L, \mathcal{L})$ has a canonical horizontal distribution defined by "keeping the local system constant," and therefore the tangent space to $\mathcal{M}(L, \mathcal{L})$ splits as a direct sum $H^1(L, \mathbf{R}) \oplus iH^1(L, \mathbf{R})$. In fact, we get more: a local trivialization of $\mathcal{M}(L, \mathcal{L})$ as a torus bundle. This decomposition also suggests that we can "absorb" the geometric deformations $H^1(L, \mathbf{R})$ by incorporating nonunitary line bundles. If that were the case, then a Lagrangian L would generate a chart for $\mathcal{M}(L, \mathcal{L})$ given by rank-one *nonunitary* local systems, a space we can identify with $(\mathbf{C}^*)^{b_1(L)}$. This will turn out to be a symplectic view of a cluster chart.

1.4. Mirror Symmetry

Recall that mirror symmetry is a kind of equivalence between a manifold X and its mirror Y. It goes deep, but in its simplest formulation, the Hodge diamonds of X and Y are related by a mirror reflection. The basic philosophy behind mirror symmetry is that mirror manifolds are dual torus fibrations, according to the following reasoning. A B-type D-brane (or B-brane) on a complex manifold is a coherent sheaf, and we will only consider the case of the structure sheaf of single point. An A-type D-brane (or B-brane) in a symplectic manifold is a special Lagrangian submanifold with a flat connection ∇ — and we shall focus on the line bundle case.

On a complex manifold X, the space of B-branes connected to a point brane is X itself. If there were a mirror manifold Y, the corresponding space of A-branes should again be X, since the moduli spaces of mirror branes should correspond. So we should be able to find the mirror X to Y by identifying A-branes that correspond to point B-branes and looking at their moduli. By reversing the reasoning, we also learn that the manifolds X and Y each are expressible both as a space of A-branes.

If we fix an A-brane (T, ∇), the space of A-branes deformation equivalent to ("connected to") the brane (T, ∇) therefore has a forget map to the space of special Lagrangians connected to T. The fibers are tori $(S^1)^{b_1(T)}$ which encode the monodromies, and it so happens that by linearizing the special Lagrangian equations you can see that $b_1(T)$ is also the dimension of the space of special Lagrangians at T. So the D-brane moduli space is an integrable system, a torus fibration over special Lagrangian moduli space (Lagrangianicity must be argued separately). Now Y, like X, should itself be some an A-brane (on X) moduli space, so has a torus fibration structure, a map $Y \to B$. What if we took T to be a torus fiber? Then the dimensions are right, since we want $b_1(T)$ to be equal to the complex dimension of X and Y, equivalently the real dimension of T. A fiber also has the property that it is disjoing from its deformations – just like its dual point B-branes. If this is the case (and there are other arguments supporting this which we ignore), then the D-brane moduli space of a torus fiber T fibers over the same base B (its geometric deformations), with the fiber over $T_b = \pi^{-1}(b)$ being the flat line bundles on T_b, i.e. the dual torus. Therefore

Mirror pairs are dual Lagrangian torus fibrations

The geometric Langlands program is a connection between Hitchin moduli spaces of Langlands dual groups G and $G^\vee$. Each gives rise to an integrable system via its spectral curve construction. Hausel-Thaddeus found these spaces to be dual torus fibrations and computed the (stringy) Hodge numbers of both sides, revealing them to be mirror.

CHAPTER 2

Overview and Definitions

CHAPTER 3

Symplectic Quotients in Finite and Infinite Dimensions

CHAPTER 4

Hyperkähler Quotients and Integrable Systems

Introduction

The objects of interest to us are as follows. Let $P \to X$ be a principal G-bundle over X where G is a compact Lie group. We may consider pairs (A, Φ) where A is a unitary connection on P and $\Phi \in \Omega^{1,0}(X; \operatorname{ad} P \otimes \mathbb{C})$. By looking at 4-dimensional Yang Mills theory dimensionally reduced to X, we may become interested in such pairs which satisfy the self-duality equations

$$F_A = -[\Phi, \Phi^*] \tag{4.0.1}$$

$$\bar{\partial}_A \Phi = 0 \tag{4.0.2}$$

where F is the curvature of A and $\bar{\partial}_A$ is the anti-holomorphic part of the covariant derivative with respect to A (we will probably abbreviate at least the former by writing simply F). We will denote the space of pairs satisfying the self-duality equations by $\overline{\mathcal{M}}$. Finally, we would like to study the quotient of $\overline{\mathcal{M}}$ by the action of the group of gauge transformations, denoted $\mathscr{G}$. Elements of $\mathscr{G}$ are functions on M with values in the adjoint representation of P, i.e. $\psi \in \Omega^0(X; \operatorname{ad} P)$, and the action of $\mathscr{G}$ on a pair (A, Φ) is given by

$$\begin{aligned} A &\mapsto \psi d\psi^{-1} + \psi A \psi^{-1} = \psi d_A \psi^{-1} \\ \Phi &\mapsto \psi \Phi \psi^{-1} \end{aligned}$$

where d_A is the covariant derivative with respect to A (showing that gauge transformations preserve self-duality is a nice exercise). We will denote this quotient by $\mathcal{M}$. The goal of this talk is to show that $\mathcal{M}$ can be realized as a Hyperkähler quotient.

4.1. Symplectic Reduction and Kähler Quotients

4.1.1. Rapid Review of Symplectic Reduction. Hyperkähler quotients can be thought of as an extension of symplectic reduction, so we will briefly review the main ingredients to the symplectic reduction procedure here. The main idea is: given a manifold with some structure (a symplectic form) and a group action which respects that structure, we would like to quotient by the group action in such a way that the resulting object is a manifold with the same type of structure. This will be the main recurring theme as we proceed.

Let (M, ω) be a symplectic manifold, which, to avoid difficulty, we take to be finite dimensional.[1] Now, let G be a connected Lie group which acts on M by symplectomorphisms. We say this group action is Hamiltonian if ther exists a map $\mu^* : \mathfrak{g} \to C^\infty(M)$ such that:

1. The map $\mathfrak{g} \to \mathfrak{X}(M)$ which sends $\xi \mapsto X_\xi$ has image contained in the set of Hamiltonian vector fields. Moreover, for any $\xi \in \mathfrak{g}$, $\mu^*(\xi)$ is the Hamiltonian function for X_ξ, i.e.
$$d(\mu^*(\xi)) = \iota_{X_\xi} \omega.$$
2. μ is a Lie algebra anti-homomorphism, i.e.
$$\mu^*([\xi, \eta]) = -\{\mu^*(\xi), \mu^*(\eta)\}.$$

In this case, we may define the moment map[2] $\mu : M \to \mathfrak{g}^*$ by $\langle \mu(x), \xi \rangle = (\mu^*(\xi))(x)$. Condition (2) above is equivalent to saying that μ is G-equivariant with respect to the coadjoint action on $\mathfrak{g}^*$, which means that the set $\mu^{-1}(0) \subset M$ is invariant under the G-action. As long as 0 is a regular value of μ, this

[1] The case we care about will not be finite-dimensional, but we will see that the statements made here will still hold in this case.

[2] μ^* is sometimes called the comomentum map.

will actually be an embedded submanifold of M. If the G-action is free on $\mu^{-1}(0)$, then the quotient $\mu^{-1}(0)/G$ will also be a manifold which we denote by $M /\!\!/ G$. In this case, we have the following theorem:

THEOREM 1 (Marsden-Weinstein). *If $(M\omega)$, G, and μ are as above and $M /\!\!/ G := \mu^{-1}(0)/G$ is a manifold, then $M /\!\!/ G$ is symplectic with symplectic form ω' uniquely defined by the property that $\pi^*\omega' = i^*\omega$ where $i : \mu^{-1}(0) \to M$ and $\pi : \mu^{-1}(0) \to M /\!\!/ G$ are the inclusion and quotient maps respectively.*

4.1.2. Quotients and Kähler Structure. First, we recall the definition of a Kähler manifold. We say a manifold M endowed with a metric g, complex structure $\mathbf{J}$, and symplectic form ω is Kähler if these three structures are compatible. This may be defined in a variety of ways. One possibility is to require $\mathbf{J}$ to be covariantly constant with respect to the Levi-Civita connection induced by g and then to define $\omega(X,Y) = g(\mathbf{J}X,Y)$ for vector fields X and Y.[3] We can also see that the structure group of a Kähler manifold has been reduced to $U(n)$, where $2n$ is the real dimension of the manifold, since the metric, complex structure, and symplectic form reduce the structure group to $\mathrm{O}(2n)$, $\mathrm{GL}(n,\mathbb{C})$, and $\mathrm{Sp}(2n,\mathbb{R})$ respectively.

Since Kähler manifolds are symplectic manifolds, we may ask whether the process of symplectic reduction preserves the Kähler structure. To answer this, we first consider how a metric may descend to a quotient manifold. Let (M,g) be a Riemannian manifold and let G be a connected Lie group which acts on M by isometries. We will require that M/G comes with a natural manifold structure, in particular the G-action must be free. Then, the space M has the structure of a principal G-bundle over M/G. At any point $x \in M$, the vertical subspace of the tangent space at x, $V_x \subset T_xM$, is isomorphic to $\mathfrak{g}$. Moreover, since the G-action on M is free, there are non-vanishing vector fields which generate V_x at each point. Then, the operation of orthogonal projection onto V_x at each point defines a one form, θ, with values in $\mathfrak{g}$ which transforms under the adjoint representation of G. θ defines a connection on M, i.e. a distribution of horizontal subspaces $H_x \subset T_xM$ complementary to V_x. Then, for any vector fields X,Y on M/G, we can use the connection to lift these to horizontal vector fields $\tilde{X},\tilde{Y}$ on M. We see that the metric $h(X,Y) = g(\tilde{X},\tilde{Y})$ is well defined using this choice of (orthogonal) horizontal lift since g is invariant under the G-action.

Now, we can answer our question about whether symplectic reduction preserves Kähler structure.

THEOREM 1. *Let $(M,g,\mathbf{J},\omega)$ be a Kähler mainfold and let G be a connected Lie group with a Hamiltonian action on (M,ω) which preserves the metric (and hence complex structure). Also, we require that $M /\!\!/ G$ has a natural manifold structure. Then the naturally induced metric g' on $M /\!\!/ G$ gives $M /\!\!/ G$ the structure of a Kähler manifold.*

SKETCH OF PROOF. Let $N = \mu^{-1}(0)$ so that $N/G = m /\!\!/ G$. Then TN is a sub-bundle of TM and we can further restrict to the horizontal sub-bundle $H \subset TN$ defined by the connection on $N \to N/G$ in the manner discussed above. The Levi-Civita connection with respect to g' on $T(N/G)$ will pull back to a G-invariant connection on $H \to N$. We will use the fact that this connection on H is given by orthogonal projection of the Levi-Civita connection of $g\big|_N$ on $TM\big|_N$ to H (for an explanation, see [**HKLR**]). Now, let $x \in N$. The complement of $T_xN \subset T_xM$ is spanned by the vectors $(\operatorname{grad}\mu^{\xi_i})_x$ for ξ_i a basis of $\mathfrak{g}$ (here we have used the notation $\mu^{\xi_i} := \mu^*(\xi_i)$) and the complement of H in TN is spanned by the vertical vectors k_i which are associated to the basis ξ_i of $\mathfrak{g}$. By definition,

$$g(\operatorname{grad}\mu^{\xi}, Y) = d\mu^{\xi}(Y) = \omega(X_{\xi}, Y) = g(\mathbf{J}X_{\xi}, Y)$$

for any vector field Y on M, and so $\operatorname{grad}\mu^{\xi} = \mathbf{J}X_{\xi}$. This shows that the vector space spanned by k_i and $(\operatorname{grad}\mu^{\xi_i})_x$ is a complex vector space. Moreover, since the basis $\{k_1, ..., k_{\dim G}\}$ can be extended to a global frame for the vertical sub-bundle $V \subset TN$, we see that the sub-bundle complementary to H over N is a complex sub-bundle, hence H is as well. This means that $\mathbf{J}\big|_N$ commutes with orthogonal projection onto H, and since $\mathbf{J}$ is compatible with g, this implies that $\mathbf{J}\big|_N$ is covariantly constant with respect to the orthogonal projection of the Levi-Civita connection of $g\big|_N$ on $TM\big|_N$ to H. Since this is just the pull back of the Levi-Civita connection of g' on $T(N/G)$ to N and $\mathbf{J}$ was assumed to be G-invariant, we see that $\mathbf{J}$ descends to a complex structure on N/G which is compatible with the induced metric g'. $\square$

[3]Actually, by choosing any two of the metric, complex structure, and symplectic form and requiring that these are compatible in the appropriate sense, we may construct the third structure in a compatible way.

4.2. Hyperkähler Quotients

4.2.1. Rapid Review of Hyperkähler Manifolds. Hyperkähler structure can be thought of as a "quaternionic" extension of Kähler structure. In particular, we say that a Riemannian manifold (M, g) is Hyperkähler if it is equipped with three complex structures $\mathbf{I}$, $\mathbf{J}$, and $\mathbf{K}$, each of which are covariantly constant, and which satisfy quaternionic algebraic relations (i.e. $\mathbf{I}^2 = -1$, $\mathbf{J}^2 = -1$, $\mathbf{K}^2 = -1$, $\mathbf{IJ} = \mathbf{K}$, etc.). As a result, the tangent space at each point becomes a quaternionic vector space and the structure group is reduced to $\mathrm{O}(4n) \cap \mathrm{GL}(n, \mathbb{H}) = \mathrm{Sp}(n)$. As in the Kähler case, each of these complex structures defines a symplectic form compatible with the metric: $\omega_1(X, Y) = g(\mathbf{I}X, Y)$, $\omega_2(X, Y) = g(\mathbf{J}X, Y)$, and $\omega_3(X, Y) = g(\mathbf{K}X, Y)$. This means that each triple $(g, \mathbf{I}, \omega_1)$ gives M the structure of a Kähler manifold.

4.2.2. Quotients and Hyperkähler Structure. If $(M, g, \vec{\mathbf{I}}, \vec{\omega})$ is a Hyperkähler manifold and G is a connected Lie group which acts on M in a Hamiltonian manner with respect to each symplectic structure and preserves the metric, then we may consider the symplectic reduction of M with respect to each of the three moment maps μ_1, μ_2, and μ_3. From the section above, we see that each of these quotients will inherit a Kähler structure from M. Now, we would like to define a new type of quotient which inherits the Hyperkähler structure from M.

The key will be to consider one moment map

$$\mu : M \to \mathfrak{g}^* \otimes \mathbb{R}^3$$

defined by $\mu(x) = (\mu_1(x), \mu_2(x), \mu_3(x))$. Then, if 0 is a regular value of μ, we see that $\mu^{-1}(0)$ is an embedded submanifold in M which is invariant under the G-action. Furthermore, if the G-action is free on $\mu^{-1}(0)$, then the quotient $\mu^{-1}(0)/G$ will have a natural manifold structure.

THEOREM 2. *If $(M, g, \vec{\mathbf{I}}, \vec{\omega})$ and G are as above, then the quotient $\mu^{-1}(0)/G$ with the inherited metric, complex structures, and symplectic forms is a Hyperkähler manifold.*

PROOF. We begin by defining the complex moment map

$$\mu_+ = \mu_2 + i\mu_3 : M \to \mathfrak{g}^* \otimes \mathbb{C}.$$

Then we see that $d\mu_+^\xi(Y) = \omega_2(X_\xi, Y) + i\omega_3(X_\xi, Y) = g(\mathbf{J}X_\xi, Y) + ig(\mathbf{K}X_\xi, Y)$ while $d\mu_+^\xi(\mathbf{I}Y) = \omega_2(\mathbf{J}X_\xi, \mathbf{I}Y) + i\omega_3(\mathbf{K}X_\xi, \mathbf{I}Y) = -g(\mathbf{K}X_\xi, Y) + ig(\mathbf{J}X_\xi, Y)$. Thus, $id\mu_+^\xi(Y) = d\mu_+^\xi(\mathbf{I}Y)$ for all vector fields Y on M. Working in local holomorphic coordinates at any point in M, we may consider the vector fields $\frac{\partial}{\partial \bar{z}^i}$, which satisfy

$$\mathbf{I}\frac{\partial}{\partial \bar{z}^i} = -i\frac{\partial}{\partial \bar{z}^i}.$$

Then, plugging this into the result above gives

$$i\frac{\partial \mu_+^\xi}{\partial \bar{z}^i} = id\mu_+^\xi\left(\frac{\partial}{\partial \bar{z}^i}\right) = d\mu_+^\xi\left(\mathbf{I}\frac{\partial}{\partial \bar{z}^i}\right) = -i\frac{\partial \mu_+^\xi}{\partial \bar{z}^i}$$

so μ_+^ξ is a holomorphic function. Thus, as long as 0 is a regular value of μ_+, we see that $\mu_+^{-1}(0)$ is an embedded complex submanifold of M with respect to the complex structure $\mathbf{I}$. This means that $\mu_+^{-1}(0)$ is Kähler with its induced metric. Now, consider the G-action restricted to $\mu_+^{-1}(0)$. This still preserves the metric and the complex structure $\mathbf{I}$, and we can restrict the moment map μ_1 to this submanifold, giving us a moment map for the G-action with respect to the restricted symplectic form ω_1. Then, from the previous proposition, we see that $\mu_1^{-1}(0) \cap \mu_+^{-1}(0)/G$ is a Kähler manifold with the induced metric and complex structure. It follows that $\mu^{-1}(0)/G$ is Kähler with respect to $\mathbf{I}$ since $\mu_+^{-1}(0) = \mu_2^{-1}(0) \cap \mu_3^{-1}(0)$. Repeating this argument for $\mathbf{J}$ and $\mathbf{K}$ shows that each of these define a Kähler structure on $\mu^{-1}(0)/G$. □

4.2.3. Main Example. Now we may relate these new definitions to the subject of interest to us: the self-duality equations and the moduli space $\mathcal{M}$. We begin by looking at the manifold consisting of pairs (A, Φ) as in the introduction (we will use Hiychin's notation for this manifold, denoting it by $\mathscr{A} \times \Omega$). The tangent space to this manifold at a point (A, Φ) is given by $\Omega^{0,1}(X; \operatorname{ad} P \otimes \mathbb{C}) \oplus \Omega^{1,0}(X; \operatorname{ad} P \otimes \mathbb{C})$. We can define a symplectic structure on $\mathscr{A} \times \Omega$ as follows:

$$\omega((\Psi_1, \Phi_1), (\Psi_2, \Phi_2)) = \int_X \mathrm{Tr}(\Phi_2\Psi_1 - \Phi_1\Psi_2)$$

where $(\Psi_i, \Phi_i) \in \Omega^{0,1}(X; \operatorname{ad} P \otimes \mathbb{C}) \oplus \Omega^{1,0}(X; \operatorname{ad} P \otimes \mathbb{C})$. ω is clearly non-degenerate. To see it is closed, we note that ω has constant coefficients in (Ψ_i, Φ_i) and that $\mathscr{A} \times \Omega$ is an affine space.

Now, define the vector field $X = (\Psi_1, \Phi_1)$ on $\mathscr{A} \times \Omega$ by $\Psi_1 = \bar{\partial}_A \psi$ and $\Phi_1 = [\Phi, \psi]$ where $\psi \in \Omega^0(X; \operatorname{ad} P)$ is an infinitesimal gauge transformation. Moreover, since $\mathscr{A} \times \Omega$ is an affine space, we may consider the vector field $(\dot{A}^{0,1}, \dot{\Phi})$ where $(A, \Phi) \in \mathscr{A} \times \Omega$. Then, we can compute

$$\begin{aligned}
(\iota_X \omega)(\dot{A}^{0,1}, \dot{\Phi}) &= \int_X \operatorname{Tr}(-[\Phi, \psi]\dot{A}^{0,1} + \dot{\Phi}\bar{\partial}_A \psi) \\
&= \int_X \operatorname{Tr}(\psi[\dot{A}^{0,1}, \Phi] + \bar{\partial}_A \dot{\Phi}\psi) \\
&= df(\dot{A}^{0,1}, \dot{\Phi})
\end{aligned}$$

where $f = \int_X \operatorname{Tr}(\bar{\partial}_A \Phi \psi)$. Thus, for the complex symplectic form ω, we have shown that the function f is Hamiltonian with respect to the vector field X. Moreover, one can check that the assignment $\psi \mapsto f$ is equivariant with respect to the action of the group $\mathscr{G}$, so we see that f defines a moment map for this action. We define ω_2 and ω_3 to be the real and imaginary parts of the symplectic form ω, and μ_2, μ_3 to be the real and imaginary parts of the moment map μ defined by f. Furthermore, $\mathscr{A} \times \Omega$ comes with a natural Kähler metric defined by

$$g((\Psi, \Phi), (\Psi, \Phi)) = 2i \int_X \operatorname{Tr}(\Psi^* \Psi + \Phi \Phi^*)$$

where $(\Psi, \Phi) \in \Omega^{0,1}(X; \operatorname{ad} P \otimes \mathbb{C}) \oplus \Omega^{1,0}(X; \operatorname{ad} P \otimes \mathbb{C})$. This defines a third symplectic form ω_1 on $\mathscr{A} \times \Omega$. It turns out that all three of these symplectic forms are compatible with the metric g and that together with g they define a Hyperkähler structure on $\mathscr{A} \times \Omega$ (see section 6 of [**H1**]). Moreover, if we recall that the moment map associated to ω_1 was defined by $\mu_1(A, \Phi) = F_A + [\Phi, \Phi^*]$, we see that requiring $\mu_1(A, \Phi) = 0$ is equivalent to the self-duality equation (1). Furthermore, we can see that requiring $\mu(A, \Phi) = \mu_2(A, \Phi) + i\mu_3(A, \Phi) = 0$ is equivalent to requiring $\bar{\partial}_A \Phi = 0$, which is exactly self-duality equation (2). Thus, the Hyperkähler quotient $\mu^{-1}(0)/\mathscr{G}$ is exactly the moduli space of solutions $\mathcal{M}$.

With this result in hand, we may hope that the result from the previous section tells us that $\mathcal{M}$ is a Hyperkähler manifold. However, in this infinite-dimensional setting the result is a purely formal statement. Instead, it is possible to directly show that $\mathcal{M}$ has a Hyperkähler structure.

THEOREM 3. *Let $\mathscr{A} \times \Omega$ and $\mathscr{G}$ be as above. Then the Hyperkähler quotient $\mu^{-1}(0)/\mathscr{G}$ inherits the structure of a Hyperkähler manifold (assuming it has a manifold structure to begin with).*

In the specific case [**H1**] deals with, this result appears as Theorem 6.7. We will only go through the main points of the proof here.

ROUGH SKETCH OF PROOF. The result of Marsden and Weinstein tells us that each of the inherited symplectic structures on $\mu^{-1}(0)/\mathscr{G}$ is, in fact, closed and non-degenerate. Moreover, it is possible to show that each complex structure on $\mathscr{A} \times \Omega$ induces an almost complex structure on $\mu^{-1}(0)/\mathscr{G}$. Then, [**H1**] shows that the integrability of the induced complex structures is implied as long as the corresponding symplectic forms are closed. Since we know this is the case, we obtain three complex structures on $\mu^{-1}(0)/\mathscr{G}$ each compatible with g. It is left to check that these complex structures satisfy the quaternionic algebraic relations. However, this must only be checked on each tangent space, and this local property follows directly from the construction of these complex structures from the complex structures on $\mathscr{A} \times \Omega$. $\square$

CHAPTER 5

Stability and the Narasimhan-Seshadri Theorem

In this chapter we will give an idea of how the classical theorem of Narasimhan-Seshadri is proved. We follow mainly [**NS**]. It states the following:

THEOREM 2. *Let X be a compact Riemann surface, of genus ≥ 2 then the isomorphism classes*[1] *of the following three sets are in bijection:*

(i) n-dimensional irreducible unitary representations of $\pi_1(X)$;
(ii) stable rank n, degree 0, algebraic vector bundles over X;
(iii) indecomposable rank n holomorphic vector bundles with an unitary connection A, s.t. $F_A = 0$[2]*.*

REMARK. Actually Narasimhan-Seshadri result is more general. They consider a cover $p : Y \to X$ ramified of degree $d \in [-n+1, 0]$ at a single point $x_0 \in X$, s.t. $Y/H \simeq X$ and Y is simply connected. One modifies the above classes as follows:

(i)' homomorphisms $\rho : H \to U(n)$ of type ξ_{-d}, i.e. $\rho|_{\mathrm{Stab}_{x_0}} = \zeta_{-d}$[3];
(ii)' stable rank n, degree d, algebraic vector bundles over X;
(iii)' indecomposable rank n holomorphic vector bundles over X, with an unitary connection A, s.t. $F_A = -2\pi i \frac{d}{n}$.

REMARK. Stricly speaking Narasimhan-Seshadri theorem is only the equivalence between (i)' and (ii)' above. A newer proof by Donaldson [**D**] stablishes the relation between (ii)' and (iii)'[4]. More importantly, their proof proceeds by induction on the rank and they stablish the following result as well, which is essential to make the inductive step work. Let T be an algebraic variety and $\{E_t\}_T$ be a flat family of algebraic vector bundles over X. Then $T_s \subset T$, the set of points t, s.t. E_t is stable, is Zariski open. Similarly for semistable.

REMARK. The relevance of this result to this class is that Hitchin proves a more general result [**H1**, Proposition 3.4] by yet a different method. Taking $\Phi = 0$ recovers the case $n = 2$ above. He remarks in the introduction that one should be able to obtain the result for arbitrary n, by considering GL_n. (I don't know of a reference for this.)

From now one we will always be working in the algebraic category, that is by vector bundle we will mean an algebraic vector bundle, and families are algebraic families and so forth. We will also restrict to the simpler form we first stated and give the elements of the proof for it, the more general case is not much harder.

5.1. Weil's theorem

Firstly, we need to say a little bit about how to construct a vector bundle from a representation of $\pi_1(X)$, and which ones come from these. This is essentially the following result of Weil.

THEOREM 4. *An indecomposable vector E over X comes from a representation of $\pi_1(X)$ if and only if $c_1(E) = 0$.*

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[1]This means: representations are considered up to conjugation, algebraic vector bundles up to isomorphism of vector bundles and connections up to gauge fixing the imposed equation.

[2]Here we denote by F_A the curvature of the connection A.

[3]The stabilizer of any pre-image of x_0, which is cyclic of order n, acts by multiplication by a root of unity.

[4]The relation between (iii)' and (i)' is an easy case of Riemann-Hilbert correspondence. For the degree 0 case this can be seem directly.

PROOF. One direction is a construction. Let $p : Y \to X$ be a simply connected covering of X, and $\rho : \pi_1(X) \to GL(V)$ a representation, then define

$$\Phi(\rho) \equiv p_* \left(Y \times_{\pi_1(X)} V\right)^{\pi_1(X)},$$

where we confund $Y \times_{\pi_1(X)} V$ the associated vector bundle over Y with the corresponding locally constant sheaf[5]. If $\dim(V) = n$ then $\Phi(\rho)$ is locally constant of rank n.

For the converse, suppose E is an indecomposable vector bundle over X. Let P denote the associated principal GL_n-bundle, then one has the following exact sequence:

$$0 \to \mathrm{ad}(P) \to TP^{GL_n} \to TX \to 0,$$

whose splitting is the data of an algebraic connection on P[6]. There exists an element $\mathrm{at}(P) \in H^1(X, \mathrm{Hom}(TX, \mathrm{ad}(P)))$ which measues when P has a connection[7]. Then one has the following non-trivial fact [**A**, Section 5]:

LEMMA 1. *If E is indecomposable then $at(E) = c_1(E)$.*

So by assumption our E admits an algebraic connection $\nabla : E \to E \otimes \Omega^1_X$. Moreover since X has complex dimension 1, ∇ is integrable. So $\ker \nabla$ is a locally constant sheaf of vector spaces over X, i.e. a representation of $\pi_1(X)$. □

A not so hard fact to check is that Φ is actually a functor[8], that is

$$\Phi\left(\rho, \rho'\right) = \mathcal{H}\mathrm{om}_X(\Phi(\rho), \Phi(\rho')).$$

The important result now is:

LEMMA 2. *When restricted to unitary representations Φ is fully faithful, i.e.*

$$Hom(\rho, \rho') \simeq \mathcal{H}om_X(\Phi(\rho), \Phi(\rho')).$$

PROOF. Since the category of finite dimensional representations is semisimple, it is enough to consider irreducible objects, so one can suppose (ρ, V) is trivial, then the statement becomes.

$$V'^{\pi_1(X)} \simeq H^0(\pi_1(X), V') \simeq H^0(X, \Phi(V')),$$

for any unitary finite dimensional represetation V'. Given $v \in V'^{\pi_1(X)}$ one has a section $s(y) = (y, v)$ of $Y \times_H V$, which is $\pi_1(X)$-equivariant so it gives a section of $\Phi(\rho')$. This clearly is zero only if v is zero so that is injective. Now for the surjectivity suppose one has a section $f \in H^0(X, \Phi(\rho'))$, this is a function $\tilde{f} : Y \to V$, s.t.

$$\tilde{f} \circ h = \rho'(h)\tilde{f}.$$

Since V is unitary, let $||\tilde{f}||^2 = \langle \tilde{f}, \tilde{f} \rangle_V$. By definition $||\tilde{f}||^2 : Y \to \mathbb{C}$, with $||\tilde{f}||^2 \circ h = ||\tilde{f}||^2$, so one gets a globally defined function on X. Since X is projective this function is constant. Now $||\tilde{f}||^2$ constant implies that $\tilde{f}$ is also constant, so ρ' acts trivially, or $\tilde{f}$ factor through $V'^{\pi_1(X)} \to V$. □

5.2. Proof of main theorem

As we mentioned before the proof proceeds by induction on two assetion let's recall them for convinience.

THEOREM 3. *Let X be a compact Riemann surface of genus ≥ 2.*

(i) A rank n vector bundle E over X of degree 0 is stable if and only if it is given by $\Phi(\rho)$ for some homomorphism $\rho : \pi_1(X) \to U(n)$, i.e. an unitary representation of $\pi_1(X)$.

(ii) For any family T of vector bundles over X, the subset $T_s \subset T$ (resp. T_{ss}) of points whose fiber is a stable (resp. semistable) vector bundle is Zariski open.

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[5]The superscript $\pi_1(X)$ means we consider the invariant part with respect to the $\pi_1(X)$ action by pullback.

[6]The middle term is the invariant vectors of TP with respect to the action of GL_n induced from the GL_n-bundle structure, and $\mathrm{ad}(P)$ is the associated vector bundle to P obtained by taking the adjoint representation of GL_n on its Lie algebra.

[7]This is the so-called Atiyah class of a principal G-bundle and the preceeding exact sequence defines the Atiyah (Lie) algebroid.

[8]From the category of finite dimensional representations of $\pi_1(X)$ to that of vector bundles over X.

We will essentially just prove (i). To prove (ii) one needs to consider specific models for the moduli spaces considered above and analise what the stability (or semistability) condition is inside them. This is standard depending on how one construct these moduli spaces.

PROOF. *Step 1 (line bundles):* The statement (ii) is trivial as stability is a vacuuos condition for line bundles. For (i) one recalls another definition of c_1 for line bundles. Consider the exponential exact sequence[9] of sheaves

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 0.$$

Then one has a long exact sequence in cohomology

$$\cdots \to H^1(X;\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}_X^\times) \xrightarrow{c_1} H^2(X;\mathbb{Z}) \to \cdots .$$

This gives that $c_1^{-1}(0) \simeq H^1(X;U(1)) \simeq \mathrm{Hom}(H_1(X;\mathbb{Z}),U(1)) \simeq \mathrm{Hom}(\pi_1(X),U(1))$[10].

Step 2 (irreducible unitary representations exist): Since $\pi_1(X)$ could have only trivial unitary representations one needs some calculation to show that the space of such is non-empty. For example for $g = 2$ (genus of X), one has generators A_1, A_2, B_1, B_2 for $\pi_1(X)$ and one can take $B_1 = \mathrm{diag}(\xi_N, \xi_N^2, \dots, \xi_N^N)$, where ξ_N is the Nth root of unity, A_2 the matrix with 1's only in the diagonal above the main diagonal, and $A_1 = B_1^{-1}$ and $B_2 = A_2^{-1}$. A similar construction works for $g > 2$ as well.

Step 3 (characterize the image of unitary and irreducible unitary representations): Let ρ be an unitary representation, then $\Phi(\rho)$ is semistable. Indeed, suppose there is $F \subset \Phi(\rho)$, s.t. $\deg(F) \geq 0$, we will prove that $\deg(F) = 0$. By contradiction, let $\deg(F) > 0$, by taking the exterior power $\Lambda^{\mathrm{rk}(F)}F \subset \Lambda^{\mathrm{rk(F)}}\Phi(\rho) \equiv \Phi(\rho')$ one can suppose that F has rank 1. Now let $\rho' = \oplus\sigma_i$ with σ_i irreducible, we have projections:

$$p_i : \Phi(\rho') \to \Phi(\sigma_i).$$

By tensoring with degree 0 line bundle $\mathcal{L}$ one can suppose $F = \mathcal{O}_X(dx_0)$ for some $x_o \in X$ and $d > 0$, notice that for the representation ρ' this just means tensoring with a character $\xi_{\mathcal{L}}$. Now if σ_0 is the trivial representation than $p_0|_F$ is zero, because $\deg(\Phi(\sigma_0))$ and a non-zero map between line bundles can not decrease the degree[11]. Now $H^0(X,F) \neq 0$, because $F = \mathcal{O}_X(dx_0)$, however $H^0(X,\Phi(\sigma_i)) = 0$ for any non-trivial irreducible σ_i, by $H^0(X,\Phi(\sigma_i)) \simeq \sigma_i^{\pi_1(X)} = 0$ since σ_i is 1-dimensional. This gives a contradiction with non-zero p_i's, since F is a subbundle of $\Phi(\rho')$ it most be trivial, which is a contradiction with $\deg(F) > 0$. Notice that by using the induction hypothesis for smaller representations and the result just proved for the semistable one obtains that if ρ is an irreducible unitary representation than $\Phi(\rho)$ is stable.

Step 4 (the variety of representations): One has a structure of algebraic variety on the space of representations of a fixed dimension of $\pi_1(X)$. Indeed these are given by a collection of matrices, $2g$ of them, satisfying one algebraic equation. Now this gives an algebraic space parametrizing vector bundles over X obtained applying the functor Φ. Moreover, one can put also an algebraic structure on the space of rank n degree d vector bundles over X. The set of indecomposable vector bundles inside these forms an irreducible variety [12]. By Weil's theorem this is equivalent to the subvariety S of n-dimensional representation whose associated vector bundle is stable.

Step 5 (surjectivity): Let S be the subset of n-dimensional representations such that $\Phi(S)$ is stable. Then S is a Zariski open set (assuming (ii)) of an irreducible variety (Step 4), hence connected. Now let $U_0 = U \cap S$ where U is the set of unitary representations, and U_0 is the set of irreducible and unitary representations (Step 3). So U_0 is open and closed inside of S, which is connected, since U_0 is not empty (Step 2) $U_0 = S$. This finishes the proof. □

[9]Strictly speaking, we use GAGA to be able to talk about the exponential map.

[10]Here $H^k(X;-)$ denotes singular cohomology and $H^k(X,-)$ denotes coherent cohomology.

[11]One sees this easily by taking the definition of degree as the number of zeros minus the number of poles of a rational section.

[12]Both of these statements follow form the representability of the Quot scheme, though [**NS**] gives a more direct argument.

CHAPTER 6

Teichmueller Theory

CHAPTER 7

Construction of the Hitchin Moduli Space

This talk closely follows [**H1**]. In this chapter, it is shown that the Hitchin moduli space $\mathcal{M}_H$ is a smooth manifold of dimension $12(g-1)$. We also prove that $\mathcal{M}_H$ is equipped with a complete hyperkahler metric.

7.1. Definitions

Let M be a Riemannian surface, $P \to M$ a principal $G = SO(3)$-bundle, V the associated rank 2 vector bundle, and $\mathcal{G}$ the group of gauge transformations ($\mathcal{G} = Map\,(M, G)$). Recall that for a connection A on P and $\Phi \in \Omega^{1,0}_M\,(adP \otimes \mathbb{C})$, the self-dual equations are

$$(\star)\qquad \begin{aligned} F_A + [\Phi, \Phi^*] &= 0, \\ \overline{\partial_A}\Phi &= 0 \end{aligned}$$

For a connection A, $u \in \mathcal{G}$ acts by

$$u\,(A) = uAu^{-1} - (du)\,u^{-1}$$

(where the covariant derivative is $\nabla_A = d + A\wedge$, i.e.,

$$\nabla_{u(A)}s = u\nabla_A\left(u^{-1}s\right)$$

for a section s of V.

For $\Phi \in \Omega^{1,0}_M\,(adP \otimes \mathbb{C})$, u acts by

$$u\,(\Phi) = u\Phi u^{-1},$$

where we regard $adP \otimes \mathbb{C}$ as the bundle of trace-zero endomorphisms of V.

Definition 4. The Hitchin moduli space is

$$\mathcal{M}_H := \{\text{solutions to } (\star)\}\,/\mathcal{G}.$$

Goal: To learn about the geometry and topology of $\mathcal{M}_H$.

7.2. Summary of results

Let V be a rank 2, odd degree vector bundle over a Riemannian surface M. Then,

- $\mathcal{M}_H$ is a smooth manifold of dimension $12\,(g-1)$.
- $\mathcal{M}_H$ has a natural metric.
- $\mathcal{M}_H$'s metric is complete and hyperkahler (in fact, $\mathcal{M}_H$ is a hyperkahler quotient).

Remark. V has odd degree $\implies$ there are no reducible solutions to $(\star)$ $\implies$ $\mathcal{G}$ acts freely on $(\star)$.

7.3. $\mathcal{M}_H$ is a dimension $12\,(g-1)$ smooth manifold

First, to get the expected dimension, we compute the dimension of the tangent space to $\mathcal{M}_H$ at a regular point (i.e., one with trivial isotropy group).

7.3.1. Idea of Proof.

- Linearize $(\star)$ to determine the expected dimension.
- Let $(A \times \Omega)_0$ denote "regular points," i.e., ones which are only fixed by the identity in $\mathcal{G}$ (those with trivial isotropy group). Then, exhibit $\mathcal{M}_H$ is a smooth submanifold of $(\mathcal{A} \times \Omega)_0\,/\mathcal{G}$ using the regular value theorem.

7.3.2. Linearization of $(\star)$**.** Let $\left(\dot{A},\dot{\Phi}\right)\in\Omega^1_M(adP)\oplus\Omega^{1,0}_M(adP\otimes\mathbb{C})$. To get the linearization of $(\star)$, fix a base point (A,Φ) and look at

$$\left.\frac{d}{dt}\right|_{t=0}(\star)\left(A+t\dot{A},\Phi+t\dot{\Phi}\right).$$

Recalling that $F_A=dA+A\wedge A$, the first equation becomes

$$d_A\dot{A}+\left[\dot{\Phi},\Phi^*\right]+\left[\Phi,\dot{\Phi}^*\right]=0,$$

and the second is

$$\overline{\partial_A}\dot{\Phi}+\dot{A}^{0,1}\wedge\Phi=0.$$

$\left(\dot{A},\dot{\Phi}\right)$ arises from an infinitesimal gauge transformation $\dot{\psi}\in\Omega^0_M(adP)$ if

$$\dot{A}=d_A\dot{\psi},\qquad\qquad \dot{\Phi}=\left[\Phi,\dot{\psi}\right].$$

Let

$$\begin{array}{rcl} d_1:\Omega^0_M(adP) & \longrightarrow & \Omega^1_M(adP)\oplus\Omega^{1,0}_M(adP\otimes\mathbb{C})\\ \dot{\psi} & \mapsto & \left(d_A\dot{\psi},\left[\Phi,\dot{\psi}\right]\right)\end{array}$$

and

$$\begin{array}{rcl} d_2:\Omega^1_M(adP)\oplus\Omega^{1,0}_M(adP\otimes\mathbb{C}) & \longrightarrow & \Omega^2_M(adP)\oplus\Omega^2_M(adP\otimes\mathbb{C})\\ \left(\dot{A},\dot{\Phi}\right) & \mapsto & \left(d_A\dot{A}+\left[\dot{\Phi},\Phi^*\right]+\left[\Phi,\dot{\Phi}^*\right],\overline{\partial_A}\dot{\Phi}+\dot{A}^{0,1}\wedge\Phi\right).\end{array}$$

Then, d_1,d_2 define an *elliptic complex* with index $3(2-2g)-6(g-1)=12(1-g)$. The Atiyah-Singer index theorem then says that

$$\dim H^0-\dim H^1+\dim H^2=12(1-g).$$

7.3.3. Elliptic Complexes. Now, a short digression into elliptic complexes. Let X be a manifold, $\pi:T^*X\to X$ be projection onto the base, and $\{E_k\to X\}$ a collection of vector bundles over X.

DEFINITION 5. A chain complex

$$\cdots\longrightarrow\Gamma(E_{k-1})\xrightarrow{P_{k-1}}\Gamma(E_k)\xrightarrow{P_k}\Gamma(E_{k+1})\longrightarrow\cdots$$

is **elliptic** if the corresponding sequence of symbols

$$\cdots\longrightarrow\pi^*E_{k-1}\xrightarrow{\sigma(P_{k-1})}\pi^*E_k\xrightarrow{\sigma(P_k)}\pi^*E_{k+1}\longrightarrow\cdots$$

is exact.

EXAMPLE. Let $P:E_1\to E_2$ be a differential operator. Then, the complex

$$0\longrightarrow\Gamma(E_1)\xrightarrow{P}\Gamma(E_2)\longrightarrow 0$$

is elliptic means that

$$0\longrightarrow\pi^*E_1\xrightarrow{\sigma(P)}\pi^*E_2\longrightarrow 0$$

is exact, i.e., that $\sigma(P):\pi^*E_1\to\pi^*E_2$ is an isomorphism. Recall that this is the "usual" definition of elliptic differential operator.

Now, return to our elliptic complex.

$$\begin{tikzcd}
\Omega^0_M(adP) \arrow[r, "d_1"] & \Omega^1_M(adP)\oplus\Omega^{1,0}_M(adP\otimes\mathbb{C}) \arrow[r, "d_2"] & \Omega^2_M(adP)\oplus\Omega^2_M(adP\otimes\mathbb{C}) \\
\dot{\psi} \arrow[r, mapsto] & \left(d_A\dot{\psi},\left[\Phi,\dot{\psi}\right]\right) & \\
 & \left(\dot{A},\dot{\Phi}\right) \arrow[r, mapsto] & \left(d_A\dot{A}+\left[\dot{\Phi},\Phi^*\right]+\left[\Phi,\dot{\Phi}^*\right],\overline{\partial_A}\dot{\Phi}+\dot{A}^{0,1}\wedge\Phi\right)
\end{tikzcd}$$

By construction, we're interested in H^1 of this complex: $H^0 = \ker d_1$ is the covariantly constant $\dot{\psi}$ which commute with Φ. These correspond to reducible solutions to $(\star)$. So, if $H^0 \neq 0$, then (A, Φ) is reducible. In fact, by considering d_2^*, we can show that $H^2 = 0$ as well [**H1**]. Hence,

$$-\dim H^1 = 12\,(1-g)\,,$$

i.e., for a regular point (A, Φ),

$$\boxed{\dim T_{(A,\Phi)}\mathcal{M}_H = 12\,(g-1)}.$$

7.3.4. $\mathcal{M}_H$ is a smooth manifold.

This is a sketch, for complete details see [**H1**].

Definition 6. (A, Φ) is a **regular point** if the isotropy group of (A, Φ) is the identity (i.e., there are no nontrivial gauge transformations fixing this point).

Recall that infinitesimally, these are the points where there are no nonzero solutions to $d_1\dot{\psi} = 0$. Let $(\mathcal{A} \times \Omega)_0$ denote the open set of regular points in $\mathcal{A} \times \Omega$, and

$$B := (\mathcal{A} \times \Omega)_0 \,/\mathcal{G}.$$

By construction, B is a Banach manifold with the quotient topology. We have the map

$$d_1^* : \Omega_M^1\,(adP) \oplus \Omega_M^{1,0}\,(adP \otimes \mathbb{C}) \longrightarrow \Omega_M^0\,(adP)\,.$$

Define a *slice* of $\mathcal{M}_H$ to be $\ker d_1^*$, at some fixed (A_0, Φ_0)—then, the slices provide coordinate patchs for B. Set

$$\begin{array}{rcl} f : \ker d_1^* & \longrightarrow & \Omega_M^2\,(adP) \oplus \Omega_M^2\,(adP \otimes \mathbb{C}) \\ (A, \Phi) & \mapsto & \left(F_A + [\Phi, \Phi^*]\,, \overline{\partial_A}\Phi\right); \end{array}$$

then, $f^{-1}\,(0,0)$ is a smooth submanifold of $\ker d_1^*$ with dimension $12(g-1)$.

Since $\ker d_1^*$ form coordinate patches for B, the remainder of the proof is just arguing that the $\ker d_1^*$ patch together to form a smooth manifold.

7.4. The tangent space

Thanks to our results in the previous section, we have an explicit description of the tangent space to $\mathcal{M}_H$:

$$T_{(A,\Phi)}\mathcal{M}_H = \left\{ \left(\dot{A}, \dot{\Phi}\right) \left| \begin{array}{c} d_A\dot{A} + \left[\dot{\Phi}, \Phi^*\right] + \left[\Phi, \dot{\Phi}^*\right] = 0, \\ \overline{\partial_A}\dot{\Phi} + \dot{A}^{0,1} \wedge \Phi = 0, \\ d_A^*\dot{A} + Re\left[\Phi^*, \dot{\Phi}\right] = 0. \end{array} \right. \right\}$$

The third equation appears because $d_1^*\left(\dot{A}, \dot{\Phi}\right) = d_A^*\dot{A} + Re\left[\Phi^*, \dot{\Phi}\right]$.

7.5. $\mathcal{M}_H$ has a complete hyperkahler metric

Recall from Sean's talk that the metric on $\mathcal{A} \times \Omega$ given by

$$g\left((\psi, \phi), (\psi, \phi)\right) = 2i\int_M Tr\left(\psi^*\psi + \phi\phi^*\right)$$

induces a metric on $T_p\mathcal{M}_H$. We want to show that this is a complete metric on $T_{(A,\Phi)}\mathcal{M}_H$. As a reminder:

Definition. A metric on M is **complete** if every Cauchy sequence of points on M has a limit which is also in M.

7.5.1. Idea of Proof. By contradiction: Suppose we have a sequence of points in $\mathcal{M}_H$ defined by a geodesic γ converging to a point not in $\mathcal{M}_H$. Because g is $\mathcal{G}$-invariant, we can look at $\tilde{\mathcal{M}}_H \subset \mathcal{A} \times \Omega$. Lift γ to a horizontal $\tilde{\gamma}$ in $\tilde{\mathcal{M}}_H$. $\tilde{\gamma}$ still defines a Cauchy sequence in $\tilde{\mathcal{M}}_H$, so we have

$$||A_n - \overline{A}||_{L^2}^2 + ||\Phi_n - \overline{\Phi}||_{L^2}^2 \leq C$$

for some C as $(A_n, \Phi_n) \to \left(\overline{A}, \overline{\Phi}\right)$ (where $\left(\overline{A}, \overline{\Phi}\right)$ is the limiting point not in $\mathcal{M}_H$).

Now, we apply Uhlenbeck's compactification theorem to show that there's a gauge transformation taking $\left(\overline{A}, \overline{\Phi}\right)$ to a solution of $(\star)$:

Theorem 7 (Uhlenbeck). *There are constants ϵ_1, $M > 0$ such that any connection A on the trivial bundle over $\overline{B}^4$ with $||F_A||_{L^2} < \epsilon_1$ is gauge equivalent to a connection $\tilde{A}$ over B^4 with*

(1) $d^* \tilde{A} = 0$,
(2) $\lim_{|x| \to 1} \tilde{A}_r = 0$, *and*
(3) $||\tilde{A}||_{L_1^2} \leq M ||F_{\tilde{A}}||_{L^2}$.

Moreover for suitable constants ϵ_1, M, $\tilde{A}$ is uniquely determined by these properties, up to $\tilde{A} \mapsto u_0 \tilde{A} u_0^{-1}$ for a constant u_0 in $U(n)$.

In particular, we can use the following corrollary:

Corollary. *For any sequence of ASD connections A_α over $\overline{B}^4$ with $||F(A_\alpha)||_{L^2} \leq \epsilon$, there is a subsequence α' and gauge equivalent connections $\tilde{A}_{\alpha'}$ which converge in C^∞ on the open ball.*

Therefore, there's a gauge transformation taking $\left(\overline{A}, \overline{\Phi}\right)$ to a solution of $(\star)$. This is a contradiction, so $\mathcal{M}_H$ is complete.

7.5.2. Hyperkahler structure. Recall from Sean's talk that there's a symplectic form on $\mathcal{M}_H$ given by

$$\omega\left((\psi_1, \phi_1), (\psi_2, \phi_2)\right) = \int_M Tr\left(\phi_2 \psi_1 - \phi_1 \psi_2\right).$$

This defines a complex moment map from the action of $\mathcal{G}$:

$$\mu(A, \Phi) = \overline{\partial}_A \Phi.$$

We can write $\mu = \mu_2 + i\mu_3$ and $\omega = \omega_2 + i\omega_3$ to get two symplectic structures out of this. The third (or first?) symplectic structure is just the Kahler form associated to the metric

$$g = 2i \int_M Tr\left(\psi^* \psi + \phi \phi^*\right),$$

and has moment map

$$\mu_1(A, \Phi) = F_A + [\Phi, \Phi^*].$$

This exhibits $\mathcal{M}_H$ as a hyperkahler quotient of $\mathcal{A} \times \Omega$:

$$\mathcal{M}_H = \bigcap_{i=1}^{3} \mu_i^{-1}(0) / \mathcal{G}.$$

7.6. Summary of topological results

Here, we state some topological results. For proofs, see section 7 of [**H1**].

$\mathcal{M}_H$ is...

- non-compact
- connected and simply connected
- the Betti numbers b_i vanish for $i > 6g - 6$.

CHAPTER 8

Integrable Systems and Spectral Curves

CHAPTER 9

Meromorphic Connections and Stokes Data

This talk is presented by Honghao on April 22nd, 2015. The references is the work of P. Boalch.

The talk has two parts. The first parts reviewed the three descriptions of the Hitchin moduli space: Dolbeault, De-Rham and Betti. The relation of the last two is also known as the Riemann-Hilbert correspondence. The correspondence can be generalized to punctured discs, and it requires additional information on each side. The additional packages on the two sides contain meromorphic connections and Stokes data.

Perspectives of the Hitchin space

A definition first.

Let X be a Riemann surface (probably not compact), and G be a Lie group over $\mathbb{C}$. The character variety is defined to be

$$\mathcal{M} = Hom(\pi_1(X), G)/G.$$

The three descriptions of Hitchin's moduli space:

(1) (Dolbeault) $\mathcal{M}_{Dol}$ the moduli space of Higgs bundles, which consists of pairs (E, Φ), where E is a rank n degree zero holomorphic vector bundle and $\Phi \in \Gamma(End(E) \otimes \Omega^1)$ a Higgs field.
(2) (De-Rham) $\mathcal{M}_{DR}$ the moduli space of connections on rank n holomorphic vector bundles, consisting of pairs (V, ∇) with $\nabla : V \to V \otimes \Omega^1$ a holomorphic connection.
(3) (Betti) $\mathcal{M}_B$ the conjugacy classes of representation of the fundamental group of X. Notice this is the character variety of the compact Riemann surface with $G = GL(n, \mathbb{C})$.

From Dolbeault to De-Rham: naturally diffeomorphic as real manifolds via the non-abelian Hodge correspondence, but not complex analytically isomorphic.

From De-Rham to Betti: Locally, the connection can be written as $\nabla = d + A$. Since X is compact, homomorphic is the same as algebraic and V and ∇ are holomorphic implies $A \in GL(n, \mathbb{C})$. The flatness of the connection implies the holonomy only depends on the homotopy type, thus the representation of the fundamental group $\pi_1(X)$.

Example: when $n = 1$ that is $G = \mathbb{C}^*$, then

(1) $\mathcal{M}_{Dol} \cong T^*Jac(X)$.
(2) $\mathcal{M}_{DR} \to Jac(X)$ a twisted cotangent bundle of the Jacobian of Σ; it is an affine bundle modeled on the cotangent bundle.
(3) $\mathcal{M}_B \cong (\mathbb{C}^*)^{2g}$ is isomorphic to $2g$ copies of $\mathbb{C}^*$.

Alert: The isomorphism (Riemann-Hilbert correspondence) $\mathcal{M}_{DR} \to \mathcal{M}_B$ involves exponential and not algebraic. And conversely, no algebraic isomorphism $\mathcal{M}_B \to \mathcal{M}_{DR}$ exists.

Another argument: $\mathcal{M}_B$ is affine, and it does not have compact subvarieties. Thus so is $\mathcal{M}_{DR}$. However, $\mathcal{M}_{Dol}$ has such (zero section), which make it impossible to have complex analytic isomorphism $\mathcal{M}_{DR} \to \mathcal{M}_{Dol}$. On the other hand, the abelian Hodge theory and Dolbeault isomorphism gives rise to a non-holomorphic isomorphism $\mathcal{M}_{DR} \to \mathcal{M}_{Dol}$.

A couple of remarks,

1) Dolbeault space has algebraic Hamiltonian integrable system (presented by Lei, also known as the Hitchin system?), there is a proper map

$$\mathcal{M} \to \mathbb{H}$$

to a vector space of half of the dimension. The generic fibres of the map are abelian varieties. In the abelian case, the space is product of $\mathbb{C}^g \times Jac(X)$, but in general, the fibres vary non-trivially and there are singular fibres.

2) The mapping class group has an natural (symplectic, algebraic) action on the moduli space, through the Betti description.

Airy's equation

Consider $X = \mathbb{P}^1 = \mathbb{C}_z \cup \{\infty\} = \{0\} \cup \mathbb{C}_w$. The general Airy's equation: $f^{''} = z^n f$. Write $\nabla = d - A$, the ODE corresponds to a holomorphic connection on $\mathbb{C}_z$, as in

$$\frac{d}{dz}\begin{pmatrix} f \\ f' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ z^n & 0 \end{pmatrix}\begin{pmatrix} f \\ f' \end{pmatrix}.$$

However, this connection is singular at infinity. Let $w = 1/z$, then $z\partial_z = -w\partial_w$. The ODE becomes $((z\partial_z)^2 - (z\partial_z) - z^{n+2})f = 0$, which in the new coordinate is $((w\partial_w)^2 + (w\partial_w) - w^{-n-2})f = 0$, that is

$$\frac{\partial^2 f}{\partial w^2} + \frac{2}{w}\frac{\partial f}{\partial w} - \frac{1}{w^{n+4}} f = 0.$$

In terms of connection, this is

$$\frac{d}{dw}\begin{pmatrix} f \\ f' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{w^{n+4}} & \frac{2}{w} \end{pmatrix}\begin{pmatrix} f \\ f' \end{pmatrix}.$$

The connection A can be expanded as

$$A = \frac{A_{n+4}}{w^{n+4}} + \frac{A_1}{w} + \text{holomorphic terms},$$

where $A_i \in \mathfrak{gl}(n, \mathbb{C})$.

The irregular type Q of this connection at infinity is

$$dQ = \frac{A_{n+4}}{w^{n+4}} = \frac{1}{w^{n+4}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Regular singularity

A meromorphic connection has regular singularity if its singular pole has order ar most 1. Deligne showed a version of the correspondence.

Deligne's Riemann-Hilbert correspondence: If X is a Riemann-Surface with punctures and $G = GL(n, \mathbb{C})$, then the character variety corresponds to the space of algebraic connections with regular singularities at the punctures.

Stokes Data

Let D be an effective divisor on $\mathbb{P}^1$. A meromorphic connection ∇ on D is a map $\nabla : V \to V \otimes K(D)$ satisfying the Leibniz rule. Locally at an irregular singular point, $\nabla = d - A$ and $A = dQ + \Lambda\frac{dz}{z}$, Q is a matrix of meromorphic functions. Choose a framing so that Q is diagonal, that is $Q = \mathrm{diag}(q_1, q_2, \cdots, q_n)$. Let $q_{ij}(z)$ be the most singular term on $q_i - q_j$.

Some definitions.

- Let S^1 parameterize the rays around z. If $d_1, d_2 \in S^1$, define $Sect(d_1, d_2)$ be the open sector sweeping from d_1 to d_2.
- The anti-Stokes directions $\mathbb{A} \subset S^1$ are the directions $d \in S^1$ such that for some $i \neq j, q_{ij} \in \mathbb{R}_{<0}$ along this ray d.
- The roots of d are the ordered pairs (ij) supporting d:
$$Roots(d) := \{(ij) | q_{ij} \in \mathbb{R}_{<0} \text{ along } d\}.$$
- The multiplicity of d is the number of roots supporting d.
- the group of Stokes factors associated to d is
$$Sto_d(A) := \{K \in G | (K)_{ij} = \delta_{ij}, \text{ unless } (ij) \text{ is a root of } d\}.$$
This is a unipotent subgroup of $G = GL(n, \mathbb{C})$.

Remarks: The anti-Stokes directions are those where the roots decays rapidly towards the singular point.

To see the stokes group is unipotent. First, $i \neq j$ implies the diagonals are always 1. Second, if (ij) is a root, (ji) won't be, which implies an ordering. Third, transitivity, as $q_{ij}, q_{jk} \in \mathbb{R}_{<0}$, then $q_{ik} = q_{ij} + q_{jk} \in \mathbb{R}_{<0}$. By a permutation, the Stokes matrix becomes upper triangular, with diagonals equal to 1, which is obviously an unipotent subgroup, and the property of which is invariant under permutation conjugation.

Since $q_{ij}(z) = a/z^{k-1}$, there is a $\pi/(k-1)$ rotational symmetry. (k is the order of the pole.) Notice that in the generic situation, the leading terms of q_i do not cancel out. q_{ij} has a $2\pi/(k-1)$ symmetry, and q_{ji} contributes the remaining. Let $\mathbf{d} = (d_1, \cdots, d_l)$ be the set of anti-Stokes directions up to this symmetry, then $n(n-1)/2 = \sum_{i=1}^{l} Mult(d_i)$.

A key result.

(1) The product of the groups of Stokes factors in a half-period is isomorphic to a subgroup of G as a variety.
$$\prod_{d\in\mathbf{d}} Sto_d(A) \cong PU_+P^{-1},$$
via $(K_1, \cdots, K_l) \mapsto K_l \cdots K_2K_1$, and P is a permutation group arranging the anti-Stokes directions in order.

(2) The product of all groups of Stokes factors is isomorphic to the variety:
$$\prod_{d\in\mathbb{A}} Sto_d(A) \cong (U_+ \times U_-)^{k-1}.$$

Suppose E_{ij} is the matrix with the (ij) entry 1 and 0 otherwise. It suffices to show any upper triangular matrix U can be decomposed uniquely into a product of $(I + t_{ij}E_{ij})$, for $1 \le i < j \le n$. Notice that $E_{ij}E_{kl} = \delta_{jk}E_{il}$. Expanding the product, $t_{12}, t_{23}, \cdots$ the sub-diagonal entries are determined immediately. Followed by the next sub-diagonal and so on. The second fact follows the first one easily.

The second product is the space of Stokes data $Sto(Q)$. For $k = 2$, the (local version) irregular Riemann-Hilbert correspondence is
$$\{\text{Connections of singular type Q}\}/\mathcal{G} \cong \mathfrak{t} \times Sto(Q),$$
where $\mathcal{G}$ is the group of holomorphic maps from the unit disk to G taking $z = 0$ to 1_G, and $\mathfrak{t}$ is an Cartan subalgebra fixed a priori.

CHAPTER 10

Introduction to the Langlands Program

10.1. Why do we care?

This class is about the Hitchin system and the goal is to understand what it is about and see how omnipresent it is throughout mathematics. As of now, I think we have all seen what it is, but not enough for its universal appearance in mathematics yet.

I want to introduce one of the main threads for that purpose today, which goes under the name of the Langlands program. This might sound quite far from what we have been doing (since it is indeed actually quite far in any reasonable metric), so I should justify its relevance at least by a few words before I even start. Namely, I want to list some of the possible topics for our class which one way or another build upon the contents of my talk, if very indirect:

- (Fundamental lemma) Ngo Bao Chau got his fields medal in 2010, proving what is called the fundamental lemma in 2008. It was one of the most crucial open problems in the Langlands program. The surprising part is that although the statement is purely in terms of representation theory or harmonic analysis, the single most important ingredient for its proof is (algebraic) geometry of the Hitchin fibration. It will be the topic of my second talk.
- (Geometric Langlands) The geometric Langlands program arises from trying to make sense of the Langlands program over the complex number field $\mathbb{C}$. In fact, the original motivation of the geometric Langlands program is to work in this much easier setting where you have nicer structure and hence can prove stronger results in a cleaner way, and then hope to transport the insight there to the original program. Ngo's proof of the fundamental lemma is so far the strongest example in this direction, because the relevant geometry of the Hitchin system was discovered through the investigation of the geometric program.
- (Kapustin-Witten) There is 4-dimensional field theory where one can find geometric Langlands in terms of physical duality, called S-duality. After compactification along a compact Riemann surface, one can see the relationship between the theories with the two target Hitchin moduli spaces in different complex structure as a version of T-duality. Then homological mirror symmetry conjecture together with some technical input would realize a version of the geometric Langlands correspondence.

The aim of the talk is to give gentle introduction to the Langlands program. Since it is a huge subject, I will be rather brief in many places. I will try my best to make the big picture understandable. Of course, feel free to ask questions on details, although I might refuse to answer all of them during this hour. On the other hand, if you can believe that the Langlands program is an interesting and important thing to consider at the end of the talk, then it will be more or less enough to follow my second talk.

10.2. Overview

Let me start with arguably the most famous problem in mathematics of all time, also known as Fermat's last theorem.

CONJECTURE 8 (Fermat's last theorem). *The equation $x^n + y^n = z^n$ has no nonzero integral solutions if $n > 2$.*

This conjecture, formulated by Pierre de Fermat in 1637, is finally proved by Andrew Wiles with a technically crucial help of Richard Taylor in 1994. The way to prove this was to prove even stronger result, known as the Taniyama-Shimura-Weil conjecture. Yutaka Taniyama asked a certain question along the line in 1955 and Goro Shimura discussed the question in the following years. Later Andre Weil gave the first serious evidence for the conjecture in 1967: before that, it wasn't even considered to be something one can hope for by many people in the field, including Jean-Pierre Serre. Henceforth, we call it the way it is.

Conjecture 9 (Taniyama-Shimura-Weil conjecture). *Every elliptic curve defined over $\mathbb{Q}$ is modular.*

In fact, Wiles proved this result for the class of semistable elliptic curves, which was enough for deriving the Fermat's last theorem. The full conjecture, which follows from the modularity lifting theorem, was proved by Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor in 2001.

Remark (Inter-universal Teichmüller theory). In 2012, Shinichi Mochizuki finished his first draft on his theory called inter-universal Teichmüller theory, which in particular claims to imply the abc conjecture. Since the abc conjecture implies Fermat's last theorem and many many others, this will be an amazing achievement if it turns out to be true. However, as of now, there is no general consensus to whether it is correct, although no one found a crucial error yet. Of course, we are not going to pursue this direction!

In fact, the Taniyama-Shimura-Weil conjecture itself is a special case of the Langlands correspondence. Since it is more relevant to the main thread, let us explain how it works.

One rough idea of the Langlands correspondence is that one can understand some important information of number theory or algebraic geometry in terms of representation theory or harmonic analysis. As a first example, for number theoretic side, let us consider n-dimensional representations of the absolute Galois group $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The claim is that the data is encoded in terms of irreducible automorphic representations of $GL_n(\mathbb{A}_{\mathbb{Q}})$.

We need to explain what we mean by the latter first. We define the ring of adeles of $\mathbb{Q}$ by $\mathbb{A}_{\mathbb{Q}} := \left(\prod_p{}' \mathbb{Q}_p\right) \times \mathbb{R}$, where $\mathbb{Q}_p$ is the field of p-adic numbers and by the restricted product $\prod' \mathbb{Q}_p$ we mean

$$\{(x_p) \in \prod \mathbb{Q}_p \mid x_p \in \mathbb{Z}_p \text{ all but finitely many primes } p\}.$$

Moreover, we have a natural diagonal embedding $\mathbb{Q} \hookrightarrow \mathbb{A}_{\mathbb{Q}}$ which is discrete. From this we can consider the quotient $GL_n(\mathbb{Q}) \setminus GL_n(\mathbb{A}_{\mathbb{Q}})$ on which $GL_n(\mathbb{A}_{\mathbb{Q}})$ still acts from the right.

Then we consider the right regular representation $L^2(GL_n(\mathbb{Q}) \setminus GL_n(\mathbb{A}_{\mathbb{Q}}))$ with respect to the natural measure as a representation of $GL_n(\mathbb{A}_{\mathbb{Q}})$. One can decompose it into irreducible representations, which have continuous parts as well as discrete parts, and those representations are called *automorphic representations* of $GL_n(\mathbb{A}_{\mathbb{Q}})$. To be absolutely precise in defining automorphic representations, we need to be much more careful here, but what we described is a nice first approximation.

Then the Langlands correspondence asserts that to each n-dimensional representation of the absolute Galois group one associates an automorphic representation. Well, it wouldn't be so interesting if all we have is a set theoretic bijection: cardinality argument would give it away even if the correspondence would be highly non-canonical. Indeed, the correspondence asserts much more: it predicts that the data of Frobenius eigenvalues in the Galois side correspond to the one of Hecke eigenvalues in the automorphic side.

We need to explain what we mean by them. Recall the Frobenius automorphism for each prime p: each is a generator of the Galois group $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ for any finite extension $\mathbb{F}_q$ of $\mathbb{F}_p$. Given a finite-dimensional complex representation V of the absolute Galois group $G_{\mathbb{Q}}$, one can lift the Frobenius automorphism as a conjugacy class of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ for almost all primes through the diagram

$$\begin{tikzcd}
\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \arrow[r, hook] \arrow[d] & \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \\
\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) &
\end{tikzcd}$$

for a fixed embedding $\iota\colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$ over $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$. The eigenvalues of such elements, which is well-defined for a conjugacy class, are called the Frobenius eigenvalues.

Hecke eigenvalues are harder to define, but basically they are eigenvalues with respect to certain integral operator acting on automorphic representations. These are also defined for all but finitely many primes.

Now the claim is that for any such Galois representation σ, there exists an automorphic representation π_σ such that its Hecke eigenvalues are exactly the Frobenius eigenvalues of σ for those almost all prime numbers.

This should be very surprising. First of all, many questions in number theory can be reformulated in terms of a Galois group and its representations. On the other hand, once we know the correspondence, automorphic representations are much more concrete, being of analytic nature. One can hope to read off some of nontrivial number theoretic data out of those analytic objects, using this correspondence.

For those who know some mathematical physics, it is just like mirror symmetry where one can read off the number of rational curves on a quintic threefold using some Hodge-theoretic data, for instance. Actually, if you want, this is more than just an analogy: through geometric Langlands and its physical interpretation as pioneered by Kapustin-Witten, one might say that the automorphic side is A-model and the Galois side is B-model in the sense of mirror symmetry!

10.3. Diophantine Equations

We would like to be a bit more concrete. In practice, Galois representations arise in a very natural way from an arithmetic question. Suppose one is interested in counting the number of solutions to an integral equation, or the number of points of an algebraic variety X over $\mathbb{Z}$. Or, one could ask existence of a solution at all, just like Fermat's last theorem. This is extremely hard question: there is a precise sense that there is nothing harder than this sort of problem in mathematics!

Remark (Undecidability in number theory). A mindblowing theorem of Yuri Matiyasevich, based on the works of Julia Robinson, Martin Davis, and Hilary Putnam, says that most of mathematical problems can be encoded in a Diophantine equation. For example, there is a polynomial $f \in \mathbb{Z}[x_1, \cdots, x_n]$ such that Riemann hypothesis is true if and only if $f(x_1, \cdots, x_n) = 0$ has no integral solution. Similarly for Goldbach conjecture or the Poincaré conjecture.

A much more tractable problem is to solve the equation modulo p, because then we can try to count them all. In terms of algebraic geometry, we should think of its reduction X defined over $k = \mathbb{F}_p$. For a later use, we introduce $X_{\overline{k}} = \operatorname{Spec}(\overline{k}) \times_{\operatorname{Spec}(k)} X$ so that we can write the desired solution set as $X_{\overline{k}}(\mathbb{F}_p)$. Having $X_{\overline{k}}$, we might also consider $X_{\overline{k}}(\mathbb{F}_{p^r})$ in the same way for each $r \geq 1$ as further information. It is a well-known, but mysterious, fact in mathematics that if one wants to understand a sequence of numbers better, then one should introduce its generating series. In our case, the relevant generating series is called the *local zeta function of X at p*, defined by

$$\zeta_p(X, T) = \exp\left(\sum_{r\geq 1} |X_{\overline{k}}(\mathbb{F}_{p^r})| \frac{T^r}{r}\right) \in \mathbb{Q}[\![T]\!].$$

Remark (Relation with Riemann zeta function). We use this particular form of zeta function, because if X were a single point over $\mathbb{F}_p$, then we would obtain the Euler factor $\frac{1}{1-p^{-s}}$ after setting $T = p^{-s}$. If we started with $X = \operatorname{Spec}(\mathbb{Z})$, then the product of the finite local factors gives the Riemann zeta function back.

It was the deep insight of Grothendieck, following Weil's conjectures, that such information can be encoded in terms of cohomology theory, called the étale cohomology theory. One should not be afraid of étale cohomology, because for a variety which can also be defined over $\mathbb{C}$, it is known to be isomorphic to singular cohomology of its complex points with $\mathbb{F}_\ell$ (and hence $\mathbb{Z}_\ell$ and $\mathbb{Q}_\ell$) coefficients for prime ℓ. On the other hand, the point is that even if X is defined as a smooth projective variety over a finite field of characteristic p, one still has such a cohomology theory which admits a natural action of the Galois group, whose Frobenius eigenvalues encode the information for solutions modulo p as we discuss now.

Recall the usual Lefschetz fixed-point theorem: for a compact topological space M and a continuous map $\phi\colon M \to M$, we know that if $L(\phi) = \sum_i (-1)^i \mathrm{tr}(\phi^* | H^i(M)) \neq 0$, then ϕ has a fixed point.

In algebraic geometry, one has the following analogue of the fixed-point theorem.

THEOREM 10.3.1 (Grothendieck-Lefschetz fixed-point formula). *For a smooth projective variety $X_{\overline{k}}$ of dimension d and a morphism $\phi\colon X_{\overline{k}} \to X_{\overline{k}}$, one has the identity*

$$\Gamma_\phi \cdot \Delta = \sum_{i=0}^{2d} (-1)^i \mathrm{tr}(\phi \mid H^i_{et}(X_{\overline{k}}, \mathbb{Q}_\ell)),$$

where Γ_ϕ is the graph of ϕ, the subset $\Delta \subset X_{\overline{k}} \times X_{\overline{k}}$ stands for the diagonal, and $\Gamma_\phi \cdot \Delta$ is their intersection number, namely, the number of fixed points of ϕ with multiplicities counted.

The whole point of the theorem was to apply it for the Frobenius automorphism.

COROLLARY 10.3.2. *For a smooth projective variety $X_{\overline{k}}$ of dimension d, one has*

$$|X_{\overline{k}}(\mathbb{F}_{p^r})| = \sum_{i=0}^{2d} (-1)^i \mathrm{tr}(\mathrm{Frob}_p^r | H^i_{et}(X_{\overline{k}}, \mathbb{Q}_\ell)),$$

where $\mathrm{Frob}_p \in \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ stands for the geometric Frobenius defined by $x \mapsto x^{p^{-1}}$.

PROOF. It follows from the following observations:

- For $X_{\overline{k}} = \mathrm{Spec}(\overline{k}) \times_{\mathrm{Spec}(k)} X$, the set $X_{\overline{k}}(\mathbb{F}_{p^r})$ is the fixed points of the *relative Frobenius* action $\mathrm{Fr}_r = 1_{\mathrm{Spec}(\overline{k})} \times \mathrm{Fr}_X$ with $\mathrm{Fr}_X : \mathcal{O}_X \to \mathcal{O}_X$ given by $f \mapsto f^p$.
- The *arithmetic Frobenius* $\mathrm{Fr}_a = \mathrm{Fr}_{\mathrm{Spec}(\overline{k})} \times 1_X$ is defined with $\mathrm{Fr}_{\mathrm{Spec}(\overline{k})}$ acting as $x \mapsto x^p$.
- $\mathrm{Fr}_{X_{\overline{k}}} = \mathrm{Fr}_r \circ \mathrm{Fr}_a = \mathrm{Fr}_a \circ \mathrm{Fr}_r$ is the identity map as a topological space. In particular, it acts trivially on étale cohomology of $X_{\overline{k}}$.

□

Together with the identity in linear algebra

$$-\log(\det(1 - \phi T | V)) = \sum_{r \geq 1} \mathrm{tr}(\phi^r \mid V) \frac{T^r}{r},$$

one can obtain a series of equalities:

$$\begin{aligned}
\zeta_p(X, T) &= \exp\left(\sum_{r\geq 1} |X_{\overline{k}}(\mathbb{F}_{p^r})| \frac{T^r}{r} \right) \\
&= \exp\left(\sum_{r\geq 1} \sum_{i=0}^{2d} (-1)^i \mathrm{tr}(\mathrm{Frob}_p^r | H^i_{et}(X_{\overline{k}}, \mathbb{Q}_\ell) \frac{T^r}{r} \right) \\
&= \exp\left(\sum_{i=0}^{2d} (-1)^{i-1} \log(\det(1 - \mathrm{Frob}_p \cdot T | H^i_{et}(X_{\overline{k}}, \mathbb{Q}_\ell))) \right) \\
&= \prod_{i=0}^{2d} \det(1 - \mathrm{Frob}_p \cdot T | H^i_{et}(X_{\overline{k}}, \mathbb{Q}_\ell))^{(-1)^{i-1}}.
\end{aligned}$$

If we set $P_i(T) = \det(1 - \mathrm{Frob}_p \cdot T | H^i_{et}(X_{\overline{k}}, \mathbb{Q}_\ell))$, then we proved that $\zeta_p(X, T)$ is of the form

$$\zeta_p(X, T) = \frac{P_1(T) \cdots P_{2d-1}(T)}{P_0(T) \cdots P_{2d}(T)},$$

where $P_i(T)$ is an integral polynomial of degree $h^i_{et}(X_{\overline{k}}, \mathbb{Q}_\ell)$ or $h^i_B(X(\mathbb{C}), \mathbb{Q}_\ell)$. The expected form of Poincaré duality, which is known to hold, would give a functional equation of the local zeta function. Moreover, if we write the eigenvalues of Frob_p on H^i as α_{ij}, Deligne's celebrated theorem says $|\iota(\alpha_{ij})| = p^{\frac{i}{2}}$ for any embedding $\iota\colon \overline{\mathbb{Z}} \hookrightarrow \mathbb{C}$.

Now the next claim is that if $\zeta_p(X,T) = \dfrac{(1-\beta_1 T)\cdots(1-\beta_l T)}{(1-\alpha_1 T)\cdots(1-\alpha_k T)}$, then $|X_{\overline{k}}(\mathbb{F}_{p^r})| = \displaystyle\sum_{i=1}^{k} \alpha_i^r - \sum_{j=1}^{l} \beta_j^r$. This essentially follows from the identity $\exp\left(\displaystyle\sum_{r=1}^{\infty} \frac{T^r}{r}\right) = \dfrac{1}{1-T}$ which one can easily check using logarithmic differentiation. In sum, the information of Frobenius eigenvalues, as in Deligne's theorem, determines the original question of counting the number of points modulo p and more.

Let us be even more concrete by considering an elliptic curve $E \subset \mathbb{P}^2$ defined over $\mathbb{Q}$. Then the first étale cohomology $H^1_{et}(E, \mathbb{Q}_\ell)$ is a 2-dimensional $\mathbb{Q}_\ell$-vector space with an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. In other words, we have a 2-dimensional ℓ-adic representation $\sigma_{E,\ell}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{Q}_\ell)$ for every prime ℓ. Note that we fixed a harmless (for our purpose) identification $\overline{\mathbb{Q}_\ell} \cong \mathbb{C}$ here in applying the previous discussion.

It is well-known that the elliptic curve E defined over $\mathbb{Q}$ can be extended to the elliptic curve E_N over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{N}])$ for some integer N. This amounts to saying that there exists a homogeneous equation defining E with coefficients in $\mathbb{Z}[\frac{1}{N}]$ such that for any prime $p \nmid N$, the reduction of E_N modulo p is an elliptic curve defined over $\mathbb{F}_p$.

It turns out that for $p \neq \ell$, except for a finite number of primes with $p \mid N$, the conjugacy class of $\sigma_{E,\ell}(\mathrm{Frob}_p)$ in $GL_2(\mathbb{Q}_\ell)$ is well-defined. Then we know $\zeta_p(E,T) = \dfrac{(1-\alpha T)(1-\beta T)}{(1-T)(1-pT)}$ for the Frobenius eigenvalues $\alpha, \beta \in \overline{\mathbb{Z}}$ of Frob_p on $H^1_{et}(E, \mathbb{Q}_\ell)$, which in particular gives $|E(\mathbb{F}_{p^r})| = 1 + p^r - \alpha^p - \beta^p$. Here $1-T$ comes from having the trivial representation $H^0(E, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$ and $1-pT$ from its Tate twist $H^2(E, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-1)$. Moreover, since the isomorphism $\wedge^2 H^1(E, \mathbb{Q}_\ell) \cong H^2(E, \mathbb{Q}_\ell)$ from the cup product is supposed to respect the Galois action, we should have $\alpha\beta = p$ as well.

Since the Frobenius element is well-defined only up to conjugation as an element of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we take the trace of Frob_p to obtain $a_p = p + 1 - |E(\mathbb{F}_p)|$, which is the only nontrivial information for a given elliptic curve in terms of Galois theory. Note that it is an integer independent of the prime ℓ.

CHAPTER 11

Spectral Curves and Irregular Singularities

CHAPTER 12

The Stokes Groupoid

CHAPTER 13

Cluster Varieties

CHAPTER 14

The Hitchin System and Teichmueller Theory II

CHAPTER 15

Geometric Langlands and Mirror Symmetry

In this chapter we will describe of the following result of Hausel-Thaddeus [**HT**, Theorem 3.7]:

THEOREM 10. *Let X be a compact Riemann surface, let $M^d_{SL_n}$ be the Hitchin moduli of stable SL_r-principal bundles of degree d and $M^e_{PGL_r}$ the analogous moduli for PGL_r. Then $M^d_{SL_r}$ and $M^e_{PGL_r}$ are SYZ-mirror dual.*

Our exposition has essentially three parts: firstly we describe what we understand by SYZ-mirror symmetry, secondly one describes the above moduli spaces more explicitly using the Hitchin moduli space for GL_r we are used to in the class. Finally, we give some elements of the proof which boils down to a study of the abelian varieties that appear as fibers of the Hitchin fibration and of universal families over the Hitchin moduli space for SL_r and PGL_r.

15.1. Strominger-Yau-Zaslow mirror symmetry proposal

Let M be a *Calabi-Yau manifold*, i.e. a Kähler manifod with fundamental form ω and also a (covariant) constant holomorphic n-form Ω, where $n = \dim_{\mathbb{C}}(M)$[1].

Suppose there exist a real n-dimensional manifold N, and another Calibi-Yau $\check{M}$ manifold (more generally orbifold) with maps:

$$\begin{tikzcd} M \arrow[dr, "p"'] & & \check{M} \arrow[dl, "\check{p}"] \\ & N & \end{tikzcd}$$

Then [**SYZ**] say M and $\check{M}$ are mirror dual if they satisfy:

(i) For all regular points $x \in N$, $L_x \equiv p^{-1}(x)$ and $\check{L}_x \equiv \check{p}^{-1}(x)$ are *special Lagrangian tori*, i.e. L is a torus, $\omega|_L = 0$ (Lagrangian) and $\mathrm{Im}(\Omega)|_L = 0$ (special).

(ii) L_x and $\check{L}_x$ are dual abelian varieties[2]. In their formulation one actually requires a weaker condition: that L_x is a torsors for an abelian variety and is in bijection with $\check{L}_x$ which is a torsor for the dual abelian variety.

---

[1] Another description of a Calabi-Yau is as a manifold M with a triple $(\omega, \Omega^r, \Omega^i)$ of a 2-form and two n-forms s.t.

(i) ω is non-degenerate;
(ii) $\Omega^r + i\Omega^i$ is locally $dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$ and non-vanishing;
(iii) $\Omega^r \wedge \omega = \Omega^i \wedge \omega = 0$;
(iv) $\left(\Omega^r + i\Omega^i\right) \wedge \left(\Omega^r - i\Omega^i\right) = \omega^n$ (or $i\omega^n$);
(v) all three are closed;
(vi) for all non-zero $v \in \Gamma(M, T_{\mathbb{C}}M), \omega_{\mathbb{C}}(v, \bar{v}) > 0$.

In particular, if $(M, \omega_1, \omega_2, \omega_3)$ is Hyperḱahler of complex dimension $2m$, then $\Omega^r + i\Omega^i \equiv (\omega_3 + i\omega_1)^{\wedge m}$ with ω_2 satisfy the above. For details see the nice paper by Hitchin [**H4**].

[2] Given A an abelian variety over a field k, we define the functor $\mathcal{P}_A : \mathrm{Sch}_k \to \mathrm{Set}$ given by:

$$\mathcal{P}_A(S) = \left\{\text{degree 0 line bundles } \mathcal{L} \text{ over } A \times S \mid \ \mathcal{L}|_{\{1\}\times S} \simeq \mathcal{O}_S\right\},$$

where $1 \in A$ is the unit. A non-trivial important fact is that $\mathcal{P}_A$ is representable, the *dual abelian variety* $\check{A}$ is the representing object.

REMARK. In general it is hard to produce special Lagrangian tori, however in the case where M comes from a Hyperkähler manifold as explain in the footnote any complex (for the J_2 structure) Lagrangian is special. This will be the case for the Hitchin moduli spaces.

Before we focus on the example at hand, let's consider a slight generalization of the above. One reference for this version of mirror symmetry is also work of Hitchin [**H5**]. In general one might not have that the fibers of the above correspondence are groups, but they might only be torsors for some abelian variety A. Suppose that this abelian variety is $A \simeq H^1(X, U(1))$ for some X, i.e. it is the Picard variety of some curve. One wants to say that L_x is a torsor for $H^1(X, U(1))$ geometrically. Note that if one has a trivial $U(1)$-bundle L over X, then the space of trivializations of L is a $H^0(X, U(1))$-torsor. This motivates one to consider a geometric object over X, whose space of trivializations is a $H^1(X, U(1))$-torsor.

DEFINITION 11. Let $BU(1)$ be the sheaf[3] of Picard groupoids[4] over X given by: for every U an open of X

$$BU(1)(U) = \{U(1)\text{-torsors over } U\}.$$

A $U(1)$-*gerbe* is a sheaf of Picard groupoids $\mathcal{A}$ over X, s.t. locally $\mathcal{A} \simeq BU(1)$[5].

REMARK. In the definitions in the footnotes we always think of étale covers. We warn the reader that the notions of sheaf of categories, groupoids, gerbes, etc. are very sensitive to the topology one uses.

We will note torture the reader with the definition of an isomorphism between gerbes, see [**Br, DG**] for details. The important take away from it is the following two properties.

THEOREM 5. *The isomorphism classes of $U(1)$-gerbes over X are in bijection with $H^2(X, U(1))$. Moreover, given $\mathcal{B}$ a trivial $U(1)$-gerbe over X, the space of trivializations of $\mathcal{B}$, denoted $Triv^{U(1)}(X, \mathcal{B})$ is a $H^1(X, \mathcal{B})$-torsor.*

Finally we reformulate the proposal of mirror symmetry as follows. Suppose that in addition to the diagram above one also has the data of $\mathcal{B}$ a $U(1)$-gerbe over M and $\check{\mathcal{B}}$ a $U(1)$-gerbe over $\check{M}$. Then, we say they are mirror dual to each other if for all $x \in N$ regular:

(i) $L_x \simeq \mathrm{Triv}^{U(1)}(\check{L}_x, \check{\mathcal{B}}\big|_{\check{L}_x})$;
(ii) $\check{L}_x \simeq \mathrm{Triv}^{U(1)}(L_x, \mathcal{B}|_{L_x})$.

REMARK. Note one needs to require both conditions, which was redundant before since taking the double dual of an abelian variety is canonically the identity. Also, note that promoting the bijection of sets in the original definition to a bijection of torsors is only a non-trivial statement when made in families, in other words the use of gerbes is inevitable in this situation. This proposal was put forth first in [**H5**] where he is trying to make sense of what B-field are, i.e. the extra data of the gerbe considered here.

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[3]More precisely, $\mathcal{G}$ over X is a sheaf of categories (or groupoids) if for all $f : S \to X$ a cover, one has a data $f^* : \mathcal{G}(X) \to \mathcal{G}(S)$, and for $f^{(2)} : S^{(2)} \to S^{(1)}$ one has $\theta_{1,2} : f^{(2),*} \circ f^{(1),*} \to \left(f^{(1)} \circ f^{(2)}\right)^*$ satisfying:

(i) (*presheaf*) $\theta_{1,23} \circ \theta_{2,3} = \theta_{12,3} \circ \theta_{1,2}$ for any chain of covers $S^{(3)} \xrightarrow{f^{(3)}} S^{(2)} \xrightarrow{f^{(2)}} S^{(1)} \xrightarrow{f^{(1)}} S \to X$;
(ii) (*sheaf for morphisms*) For $X = \sqcup_I S_i$, then $g : P_1 \to P_2$ between two objects of $\mathcal{G}(X)$ is specified by $g_i : P1|_{S_i} \to P2|_{S_i}$ for all i, such that $g_i|_{S_{ij}} = g_j|_{S_{ij}}$;
(iii) (*sheaf for objects*) For $X = \sqcup_I S_i$, $Q_i \in \mathcal{G}(S_i)$, with $u_{ij} : Q_i|_{S_{ij}} \simeq Q_i|_{S_{ij}}$ such that $u_{ij}u_{jk} = u_{ik}$, then there exists $Q \in \mathcal{G}(X)$ glueing these objects.

[4]A *Picard groupoid* just means a groupoid which has a symmetric monoidal structure, where all objects are invertible, morally a group-object in the category of symmetric monoidal categories.

[5]One can continue to spell out the details: $\mathcal{A}$ is a gerbe for A (a sheaf of groups on X) if it is a sheaf of categories (we keep the same notation as before) and in addition one has the conditions:

a. For all i, $Q_i \in \mathcal{A}(S_1)$, $\mathrm{Aut}(Q_i)$ is an A-torsor on S_i;
b. For all $Q_1, Q_2 \in \mathcal{A}(S_1)$ there exists $\tilde{S}_1$, s.t. $Q_1|_{\tilde{S}_1} \simeq Q_2|_{\tilde{S}_1}$;
c. There exist a cover $\tilde{X} \to X$, s.t. $\mathcal{A}(\tilde{X}) \neq$.

15.2. Hitchin moduli

Fix a curve X and $x_0 \in X$ a point. Let's recall that

$$M^d_{GL_r} = \big\{\text{stable degree d } GL_r - \text{principal bundles } E \text{ over } X \mid \psi \in H^0(X, \mathrm{End}(E) \otimes K_X)\big\}$$

We first construct $M^d_{SL_r}$. Consider the map $\det : M^d_{GL_r} \to M^d_{\mathbb{C}^\times}$ given by

$$\det((E,\psi)) = (\wedge^r E, \mathrm{tr}\psi),$$

note that one obtains the trace of the Higgs field because one has to consider the derivative of the determinant as the induced map on the associated adjoint bundle. Let $(\mathcal{O}_X(dx_0), 0) \in M^d_{\mathbb{C}^\times}$, then define $M^d_{SL_r}$ by the pullback diagram

$$\begin{tikzcd}
M^d_{SL_r} \arrow[r] \arrow[d] & M^d_{GL_r} \arrow[d] \\
\{\mathcal{O}_X(dx_0)\} \arrow[r] & M^d_{\mathbb{C}^\times}
\end{tikzcd}$$

More concretely, one has $M^d_{SL_r} = \big\{(E,\psi) \in M^d_{GL_r} \mid \wedge^r E \simeq \mathcal{O}_X(dx_0), \text{ and } \mathrm{tr}(\psi) = 0\big\}$.

Analogously for $M^d_{PGL_r}$. Recall that

$$1 \to \mu_r \to SL_r \to PGL_r \to 1$$

is an exact sequence which defines PGL_r. Let

$$\Gamma^r = \big\{\mathcal{L} \text{ degree 0 line bundle over } X \mid \mathcal{L}^{\otimes r} = \mathcal{O}_X\big\},$$

i.e. the r-torsion point of $\mathrm{Pic}^0(X)$. Then define $M^d_{PGL_r} = M^d_{SL_r}/\Gamma^r$, where $\mathcal{L} \in \Gamma^r$ acts by $(E,\psi) \mapsto (E \otimes \mathcal{L}, \psi \otimes \mathcal{L})$.

We will denote by $B = \oplus^r_{i\geq 2} H^0(X, K_X^{\otimes i})$ the base of both Hitchin moduli, and by $\mu : M^d_{GL_r} \to B$ and $\check{\mu} : M^d_{PGL_r} \to B$ the fibrations obtained by taking the characteristic polynomials. Note that one does not have the first term because of the traceless condition.

Remark. For the reader that finds our construction of these moduli spaces ad-hoc we remark that there is a general definition of these moduli space for all reductive groups due to Simpson. It agrees with the one took for SL_r and PGL_r. The advantage of the one we adopted is that one does not need to worry about what stability means for a general principal G-bundle, which can be a little subtle.

Here is what we will prove in the next section.

Theorem 12. *For any $(d,e) \in \mathbb{Z}\times\mathbb{Z}$ there exist $U(1)$-gerbes $\mathcal{B}^e$ over $M^d_{GL_r}$ and $\check{\mathcal{B}}^e$ over $M^e_{GL_r}$ such that they are mirror dual in the generalized sense described in the last section.*

Remark. Another prediction of mirror symmetry is a compatibility of the stringy Hodge numbers. The paper [**HT**] also has results in this direction which we will not talk about here. Note that SL_r and PGL_r are Langlands dual to each other. One can ask if a similar duality holds in more generality, and that is true and is the content of [**DP**].

15.3. Universal families

The title of this subsection is to involke what I believe is the essential ingredient in the proof of the above result.

Henceforth B will denote a Zariski open of the basis of the Hitchin, defined by $x \in B$ such that the associated spectral curve Σ_x is smooth.

Step 1 (indentify the fibers): Recall that in $M^d_{GL_r}$ for a fixed value of x, the characteristic polynomial, one has (cf. Lei's first talk.):

$$\{(V,\psi) \mid \mathrm{char}(\psi) = x\} \simeq \{\mathcal{L} \text{ a degree d line bundle over } \Sigma_x\}.$$

Now for $\mu^{-1}(x)$ one needs to consider V, s.t. $\wedge^r V \simeq \mathcal{O}_X(dx_0)$, that means $\mathcal{L} \in \mathrm{Pic}(\Sigma_x)^d$ s.t. $\wedge^r(\pi_*\mathcal{L}) \simeq \mathcal{O}_X(dx_0)$, where $\pi : \Sigma_x \to X$. We denote it by $P^d \equiv \mu^{-1}(x)$.

A similar analysis gives that for $M^d_{PGL_r}$, one has $\check{\mu}^{-1} \equiv \check{P}^d = P^d/\Gamma^r$, where the action of Γ^r on P^d is given by $(L, \mathcal{L}) \mapsto \pi^* L \otimes \mathcal{L}$.

We note that in the same way that $\mathrm{Pic}^d(\Sigma_x)$ is a $\mathrm{Pic}^0(\Sigma_x)$-torsor, P^d and $\check{P}^d$ are P^0 and $\check{P}^0$-torsors, respectively.

One also have that P^0 is dual (as an abelian variety) to $\check{P}^0$. Indeed, consider the following exact sequence[6]

$$1 \to P^0 \to \mathrm{Pic}^0(\Sigma_x) \overset{\mathrm{Nm}}{\to} \mathrm{Pic}^0(X) \to 1,$$

which one can dualize[7] to obtain

$$1 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(\Sigma_x) \to \left(P^0\right)^\vee \to 1.$$

One just needs to prove the claim:

$$\left(P^0\right)^\vee \simeq \check{P}^0.$$

This follows from $\mathrm{Pic}^0(\Sigma_x) \simeq \left(\mathrm{Pic}^0(X) \times P^0\right)/\Gamma^r$. Consider the functor $\Phi : \mathrm{Pic}^0(X) \times P^0 \to \mathrm{Pic}^0(\Sigma_x)$ given by

$$\Phi(L, L') = \pi^* L^{-1} \otimes L'.$$

Then $\ker(\Phi) = \{L' = \pi^* L\}$, so $\mathcal{O}_X = \mathrm{Nm}(L') = \mathrm{Nm}(\pi^* L) = L^{\otimes r}$, i.e. $L \in \Gamma^r$.

Step 2 (the gerbe $\mathcal{B}$): Here is an important fact we will need in this step.

THEOREM 6. *There exist an universal projective bundle $\mathcal{E}$ over $M^d_{SL_r} \times X$, i.e. for all $(V, \psi) \in M^d_{GL_r}$*

$$\mathcal{E}|_{(V,\psi)\times X} \simeq P(V).$$

REMARK. Similar statements are true for GL_r or PGL_r and the proof of them comes from the construction of the moduli space, say M, of rank r degree d stable vector bundles over the curve X. Roughly speaking, one can use the restriction on the degree and rank to specify a certain Hilbert polynomial and consider the corresponding connected component of the Quot scheme. Then one knows that the Quot scheme has an universal sheaf by its very construction. The subtle point is that to obtain M one needs to quotient the Quot scheme by isomorphisms of a quotient sheaf which give the same vector bundle, this group of autormorphisms is GL_r. However, since by definition the points of the Quot scheme are equivalence classes of maps $q : \mathcal{F} \to \mathcal{G}$ with the same kernel one sees that the center $\mathbb{C}^\times$ acts trivially on those points. Though, it does not act trivially on the universal bundle, which is determined by $\mathcal{G}$, i.e. $\mathcal{G}$ gets multiplied by a scalar. This implies that the GIT quotient used to define M will only have a universal *projective* bundle, i.e. defined up to multiplication by $\mathbb{C}^\times$. The interested reader is refered to [**Ne**, Chapter5], where this is beautifully explained.

Let $\mathcal{F} \equiv \mathcal{E}|_{M^d_{SL_r}\times\{x_0\}}$ be the restriction of $\mathcal{E}$ to the fiber at the point $x_0 \in X$. This is a projective bundle over $M^d_{SL_r}$. We define for any $U \to M^d_{SL_r}$ an étale map

$$\mathcal{B}(U) \equiv \{\text{lifts of } \mathcal{F} \text{ to a vector bundle}\}.$$

Here by vector bundle we mean a locally free sheaf over U.

LEMMA 3. *$\mathcal{B}$ is a μ_r-gerbe over $M^d_{SL_r}$.*

PROOF. Let $\sqcup_I S_i$ be an étale cover of $M^d_{SL_r}$. Consider $\mathcal{G} \in \mathcal{B}(S_i)$. Let $\phi : \mathcal{G} \to \mathcal{G}$ be an automorphism, then for $S_{ij} = S_i \times_{M^d_{SL_r}} S_j$ one has $\phi_j : \mathcal{G}|_{S_{ij}} \to \mathcal{G}|_{S_{ij}}$ such that $\phi_j|_{S_{ijk}} = \phi_k|_{S_{ijk}}$. However this implies that $c_{ij} \equiv \phi_j$ define a GL_r cocycle on S_i. However one knows that $\mathbb{P}(\phi_i) = \mathrm{id}$[8], which implies $c_{ij} \in H^0(S_{ij}, \mathbb{C}^\times)$. In other words, one gets L_i a line bundle on S_i s.t. $\mathcal{G}' \otimes L = \mathcal{G}$ for another $\mathcal{G}' \in \mathcal{B}(S_i)$. Since $\mathbb{P}(\mathcal{G}') = \mathbb{P}(\mathcal{G} \otimes L_i) = \mathcal{F}$, one obtains that $L_i \in \mathrm{Pic}^0(M^d_{SL_r})[r]$, the r-torsion points of the Jacobian variety of $M^d_{SL_r}$, because $\mathbb{P}(\mathcal{G} \otimes L_i) = \mathbb{P}(\mathcal{G}) \otimes L_i^{\otimes r}$, since $\mathcal{G}$ has rank r. □

Step 3 (triviality of $\mathcal{B}$ restricted to L_x): We want to prove the fact that the gerbe constructed before is actually trivial when restricted to L_x, the fiber of μ. One needs to check that any projective bundle $\mathcal{F}$ over L_x admits a lift to a vector bundle. Here is another important fact we will repeatedly use:

[6] We consider the exact sequence as groups, and Nm is by definition $\det\pi_*$ the norm map.

[7] Recall the Jacobians are self-dual abelian varieties.

[8] Here $\mathbb{P}(\mathcal{F})$ denotes the projectivization of the sheaf $\mathcal{F}$.

THEOREM 7. *Given N a moduli space of line bundles (e.g. fixed degree, fixed norm, torsion, etc.) on a curve Y, and any element $L_0 \in Pic(N)$, there exists an universal line bundle $\mathcal{L}$ over $N \times Y$ s.t.*

$$\mathcal{L}|_{N\times\{y\}} = L_0, \forall y \in Y, \text{ and } \mathcal{L}|_{\{n\}\times Y} = n\ \forall n \in N.$$

REMARK. Notice this result is significantly different than the previous one about the existence of projective universal families. Morally the reason for this is that moduli spaces for line bundles are very special.

Let $\tilde{\mathcal{L}}$ be a universal line bundle over $P^d \times \Sigma_x$, s.t. $\mathbb{P}\left((\mathrm{id}\times\pi)_*(\tilde{\mathcal{L}})\right)\big|_{P^d\times\{x_0\}} = \mathcal{F}|_{P^d}$. Then $(\mathrm{id}\times\pi)_*(\tilde{\mathcal{L}})$ is a candidate for the lift of $\mathcal{F}$ on P^d. However, by picking a point $y \in \pi^{-1}(x_0)$ one can ask that $(\tilde{\mathcal{L}})\big|_{P^d\times\{y\}} = L_0 \in \mathrm{Pic}^0(P^d)$ is a fixed element. Since one has

$$\det\left((\mathrm{id}\times\pi)_*(\tilde{\mathcal{L}})\right) = \otimes_{y\in\pi^{-1}(x)} \tilde{\mathcal{L}}\big|_{P^d\times\{y\}},$$

and there are r of them because Σ_x is an r-cover of X. We let $L' \in \mathrm{Pic}^0(P^d)$ be such that $L'^{\otimes r} = \det\left((\mathrm{id}\times\pi)_*(\tilde{\mathcal{L}})\right)$ and taking $L_0 = \mathcal{O}_{P^d}\otimes L'^{-1}$ we are done.

Step 4 ($U(1)$-trivializations): One is interested in $\mathrm{Triv}^{\mu_r}(L_x,\mathcal{B})$, i.e. the equivalence classes of trivializations of $\mathcal{B}|_{L_x}$. This is a $H^1(L_x,\mu_r)$-torsor, i.e. a $\mathrm{Pic}^0(P^d)[r]$-torsor. One can look at this as a $\mathrm{Pic}^0(P^d)$-torsor by

$$\mathrm{Triv}^{U(1)}(L_x,\mathcal{B}) \equiv \mathrm{Triv}^{\mu_r}(L_x,\mathcal{B}) \times_{\mathrm{Pic}^0(P^d)[r]} \mathrm{Pic}^0(P^d).$$

Since $\mathcal{B}$ is trivial, i.e. $\mathcal{B}(L_x) \simeq \{\mu_r\text{-torsors on } L_x\}$, one can write $\mathrm{Triv}^{U(1)}(L_x,\mathcal{B})$ as

$$\left\{\mathcal{L} \text{ a universal line bundle over } P^d\times\Sigma_x \mid \mathcal{L}|_{P^d\times\{y\}} \in \mathrm{Pic}^0(P^d)\ \forall y\in\Sigma_x\right\}.$$

This is exactly the construction we explained in Step 3.

Finally, to check the duality we need to prove the following:

LEMMA 4.

$$Triv^{U(1)}(L_x,\mathcal{B}) \simeq \check{P}^1.$$

PROOF. Recall $\check{P}^1 = \mathrm{Pic}^1(\Sigma_x)/\mathrm{Pic}^0(X)$ and

$$\begin{aligned}\mathrm{Triv}^{U(1)}(L_X,\mathcal{B}) = \Big\{&\mathcal{L} \text{ is a universal line bundle over } \mathrm{Pic}^d(\Sigma_x)\times\Sigma_x \mid \\ &\mathcal{L}|_{\mathrm{Pic}^d(\Sigma_x)\times\{y\}} \in \mathrm{Pic}^0(\mathrm{Pic}^d(\Sigma_x)),\ \forall y\in\Sigma_x\Big\}/\mathrm{Pic}^0(X),\end{aligned}$$

where $\mathrm{Pic}^0(X)$ acts by the norm map. For convinience let's call the righthand side above, before taking the quotient by $\mathrm{Pic}^0(X)$, by $\mathbb{L}$. So one is reduced to prove that

$$\mathrm{Pic}^1(\Sigma_x) \simeq \mathbb{L}.$$

Let $y \in \Sigma_x$ and consider the element $T^1_y \in \mathrm{Pic}^1(\Sigma_x)$ given by $\mathcal{O}_{\Sigma_x}(y)$ and let $T^2_y \in \mathbb{L}$ be given by the universal line bundle $\mathcal{L}$ on $\mathrm{Pic}^d(\Sigma_x)\times\Sigma_x$ s.t.

$$\mathcal{L}|_{\mathrm{Pic}^d(\Sigma_x)\times\{y\}} = \mathcal{O}_{\mathrm{Pic}^d(\Sigma_x)}.$$

We just need to check that for all $y, y' \in \Sigma_x$, $T^1_{y'} - T^1_y = T^2_{y'} - T^2_y$ as elements of $\mathrm{Pic}^0(X)$. Indeed, $\mathcal{L} \in T^2_y$ on $\mathrm{Pic}^d(\Sigma_x)\times\Sigma_x$ is defined as $\pi_2^* L_0 \otimes F^*_{L_0,y}\mathcal{P}$ for $L_0 \in \mathrm{Pic}^d(\Sigma_x)$ and $F_{L_0,y} : \mathrm{Pic}^d(\Sigma_x)\times\Sigma_x \to \mathrm{Pic}^0(\Sigma_x)\times\mathrm{Pic}^0(\Sigma_x)$ given by $F_{L_0,y}(L,z) = (L\otimes L_0^{-1}, \mathcal{O}_{\Sigma_x}(z-y))$, here $\mathcal{P}$ is the universal line bundle over the Jacobian, known as the Poincaré bundle (see [**P**, Chapter 16] for details). It follows that

$$T^2_{y'} - T^2_y = \left(\pi_2^* L_0 \otimes F^*_{L_0,y}\mathcal{P}\right) \otimes \left(\pi_2^* L_0 \otimes F^*_{L_0,y'}\mathcal{P}\right)^{-1} = \mathcal{O}_{\Sigma_x}(y'-y).$$

This concludes the proof of one of the dualities, the other is completely analogous and we refer the reader to the original paper [**HT**] where it is carried out in details. □

CHAPTER 16

Hitchin Systems and Supersymmetric Field Theories

16.1. Introduction and definitions

In previous talks, we've explored many properties of the Hitchin system and its geometry, as well as several applications. The core idea of this talk is that, given a certain supersymmetric field theory, we can obtain the Hitchin system as a moduli space associated to this theory via a standard procedure in quantum field theories. Gaiotto, Moore, and Neitzke [**GMN**] use this approach to construct a canonical coordinate system on the Hitchin moduli space. The goal of this talk is to explain some of the language of supersymmetric quantum field theories, and describe how we can obtain $\mathcal{M}_H$ from a particular class of such theories.

16.1.1. Supersymmetry. Consider $\mathbb{R}^n$. The **Poincare group** is the group of translations and rotations of this space: $G = ISO(n) \cong \mathbb{R}^n \rtimes SO(n)$. (In Minkowski signature, say for $\mathbb{R}^{n,1}$, we might write $ISO(n,1)$ instead). Recall that there's a spin group defined as a double cover of $SO(n)$:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin(n) \longrightarrow SO(n) \longrightarrow 1.$$

When we have such a double cover, we get an extension of the group G to the supergroup $\tilde{G}$. This gives an extension of the Poincare algebra $\mathfrak{g}$ to the **super Poincare algebra** $\tilde{\mathfrak{g}}$ coming from the double-cover of $SO(n)$. $\tilde{\mathfrak{g}}$ is called the **supersymmetry algebra**. Such $\tilde{g}$ are labelled by a choice of a representation of $Spin(n)$. For irreducible representations of $Spin(n)$, there are two possible cases:

- There is either a unique irreducible spinor representation S, and any spinor representation has the form $S^{\oplus N}$ for some N, or
- There are two distinct irreducible real spinor representations S_+, S_-, and any spinor representation has the form $S_+^{\oplus N_1} \oplus S_-^{\oplus N_2}$ for some N_1, N_2. (This occurs in dimensions $n \equiv 2, 6 \mod 8$).

Thus, when someone says $N = n$ or $N = (n_1, n_2)$ supersymmetry, they are specifying which extension of the Poincare algebra they're referring to.

The supersymmetry Lie algebra $\tilde{\mathfrak{g}}$ splits into an even and odd part:

$$\tilde{\mathfrak{g}} = \mathfrak{g}^0 \oplus \mathfrak{g}^1,$$

and is equipped with a skew-symmetric bracket $[\cdot,\cdot] : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ which satisfies the Jacobi identity. Note that $\left[\mathfrak{g}^1, \mathfrak{g}^1\right] \subset \mathfrak{g}^0$, $\mathfrak{g}^0 = \mathfrak{g}$ (the Poincare algebra), and $\mathfrak{g}^1$ is just a linear representation of $\mathfrak{g}^0$.

Example. For $N = 2$ SUSY and $G = ISO(3,1)$, we'll be interested in a $\tilde{G}$ whose Lie algebra has even part

$$\mathfrak{g}^0 = \mathfrak{iso}(3,1) \oplus \mathbb{C}.$$

The abelian $\mathbb{C}$ factor is central in $\tilde{\mathfrak{g}}$, and has a canonical generator Z.

16.1.2. BPS States. Let $\mathcal{H}$ denote the Hilbert space of states of our quantum system on $\mathbb{R}^{3,1}$. We want to think of a state as being labeled by a representation of $\tilde{G}$—the representation encodes the data of the particle. For example, the vacuum state ("empty space") in $\mathcal{H}$ corresponds to the trivial representation of $\tilde{G}$. The next simplest kind of state is one where space is empty except for a single particle propogating with some definite momentum $p \in T^*\mathbb{R}^{3,1}$. Call the subspace of the Hilbert space consisting of single-particle states $\mathcal{H}^1$. Here, there's some structure:

- $\mathcal{H}^1$ splits into components $\mathcal{H}^1_M$, labeled by $M \in \mathbb{R}_{\geq 0}$. M^2 is the eigenvalue of a quadratic Casimir operator in $ISO(3,1)$. Physically, it's the mass of the particle.

- For $N = 2$ SUSY as in our prior example, we also have a central generator $Z \in \mathbb{C}$. Together, these give a decomposition
$$\mathcal{H}^1 = \bigoplus_{M,Z} \mathcal{H}^1_{M,Z}.$$

Fix some momentum $p_{rest} \in \left(\mathbb{R}^{3,1}\right)^*$ with $||p_{rest}||^2 = M^2$, and consider the subspace $\mathcal{H}^{1,rest}_{M,Z}$ on which the subgroup of translations along $\mathbb{R}^{3,1}$ acts by p_{rest}. This is a representation of a subgroup $\tilde{G}_{rest} \subset \tilde{G}$ with
$$\tilde{G}_{rest} = SO(3) \ltimes \tilde{T},$$
where the "super translation group" $\tilde{T}$ is generated by the ordinary translations $T = \mathbb{R}^{3,1}$ plus the central character Z and the "odd translations" $\mathfrak{g}^1$. The odd translations act by a Clifford algebra (on an 8-dimensional vector space). We can then count the number of unitary irreducible representations of this Clifford algebra:

- If $M < |Z|$, then there are *no* unitary representations of the Clifford algebra.
- If $M = |Z|$, the Clifford algebra is degenerate, and its unique unitary irrep S has dimension $2^{4/2} = 4$.
- If $M > |Z|$, the Clifford algebra is nondegenerate, and its unique unitary irrep S has dimension $2^{8/2} = 16$.

States that satisfy $M = |Z|$ are called **BPS** (Bogomol'nyi, Prasad, Sommerfield) **states**, and as one might expect, they satisfy a set of differential equations depending on the field theory. The BPS states are those in which half of the supersymmetry generators are unbroken.

16.1.3. Moduli of Vacua. The "moduli space" associated to a QFT typically refers to the moduli space of vacua. By **vacua**, we mean the quantum state with the lowest possible energy.

Question: In what sense is the space of vacua a moduli space?

For scalar fields, these are labelled by the **vacuum expectation value** (VEV). The VEV of an operator is (as the name suggests) the expectation value of the operator in the vacuum (the quantum state with the lowest possible energy). We can label a vacuum state by its VEV, and this gives a moduli space of vacua.

For $N = 2$ SUSY, the superalgebra has two representations with scalars: **vectormultiplets** (one complex scalar), and **hypermultiplets** (two complex scalars). This gives a local splitting of the moduli of vacua $\mathcal{M}$ as
$$\mathcal{M} = \mathcal{M}_C \oplus \mathcal{M}_H,$$
where $\mathcal{M}_C$ is the "Coulomb branch" (vectormultiplets), and $\mathcal{M}_H$ is the "Higgs branch" (hypermultiplets).

16.2. Compactification and Dimensional Reduction

Compactification of a field theory is a process where, instead of considering a general space X, we consider $X = M \times C$ where C is some compact space. Dimensional reduction is the limit of the compactified theory where the volume of the compact space is shrunk to zero, which produces an effective theory on the remaining dimensions.

Example (Toy Example). Consider a field theory on $X = \mathbb{R}^n \times S^1_R$ (S^1_R is the circle of radius R). Let θ be a coordinate on S^1_R, and x^i coordinates on $\mathbb{R}^n$. At a fixed x coordinate, the fields along the S^1_R look like
$$\phi|_x = \sum_n A_n \cos\left(\frac{2\pi n}{R}\theta\right) + B_n \sin\left(\frac{2\pi n}{R}\theta\right),$$
where the coefficients A_n and B_n are determined by the boundary conditions on ϕ. As $R \to 0$, the eigenvalues $\lambda_n = \frac{2\pi n}{R}$ approach ∞, except for $n = 0$. Note that in quantum mechanics, $\hbar\lambda_n$ is the *momentum* of eigenstate n, so as $R \to \infty$, all mometums except the trivial one also $\to \infty$. We should interpert $R \to 0$ as meaning that, for finite energy (and hence, finite momentum), the only eigenstate left is the trivial one.

If $\phi|_x$ is constant, it means that the field ϕ does not depend on θ—the dimensional reduction of the theory on S^1_R consists of the fields of the $\mathbb{R}^n \times S^1_R$ theory which do not depend on θ.

So, we have two equivalent perspectives on dimensional reduction to M of a theory on a space $M \times C$:

- It's the limit of the theory on $M \times C$ where the volume of C contracts to zero, or
- It's the theory on $M \times C$ where all fields are taken to be independent of coordinates on C.

Example (Yang-Mills). We actually already encountered dimensional reduction in one of the first lectures of the course. Consider classical Yang-Mills theory.

Yang-Mills theory is a field theory defined for principal G-bundles $P \to X$, where X is a 4-dimensional Riemannian manifold.

Fields: Connections A on P.
Lagrangian:

$$L(A) = |F_A|^2 \, d\mu$$

Recall that from the Lagrangian, we obtain the action functional by

$$S(A) = \int_X L(A) = \int_X |F_A|^2 \, d\mu.$$

The equations of motion for Yang-Mills theory are

$$d_A^* F_A = 0,$$

and the instantons are the (anti) self-dual connections:

$$F_A = \pm * F_A.$$

(Here, $* : \Omega_X^2 \cong \Omega_X^2$ is the Hodge star operator).

In local coordinates we can write $d_A = d + A$, where

$$A = A_1 dx_1 + A_2 dx_2 + A_3 dx_3 + A_4 dx_4.$$

Define

$$F_{ij} := [\nabla_i, \nabla_j] = \frac{\partial}{\partial x_i} A_j - \frac{\partial}{\partial x_j} A_i + [A_i, A_j].$$

Look at the self-dual connections. Then, the instanton equation $F_A^- = 0$ becomes

$$\begin{aligned} F_{12} &= F_{34}, \\ F_{13} &= -F_{24}, \\ F_{14} &= F_{23}. \end{aligned}$$

Let's restrict attention to $X = C \times \mathbb{R}^2$, where C is a Riemann surface. Suppose that we compactify and perform dimensional reduction in $\mathbb{R}^2$ coordinates: restrict to A_j which are invariant under translation in x_3, x_4. Then, $A_1 dx_1 + A_2 dx_2$ defines a connection on C. Relabel $A_3 = \phi_1$ and $A_4 = \phi_2$, and define $\varphi = \phi_1 - i\phi_2$; then the self-dual equations become

$$\begin{aligned} F_A - \frac{1}{2} i [\varphi, \varphi^*] &= 0, \\ [\nabla_1 + i\nabla_2, \varphi] &= 0. \end{aligned}$$

If we think of φ as defining a local section of $\Omega^0(C; ad(P) \otimes \mathbb{C})$, and set $\Phi = \frac{1}{2}\varphi dz \in \Omega^{1,0}(ad(P) \otimes \mathbb{C})$ and $\Phi^* = \frac{1}{2}\varphi^* d\bar{z} \in \Omega^{0,1}(ad(P) \otimes \mathbb{C})$, then the equations become

$$\begin{aligned} F_A + [\Phi, \Phi^*] &= 0, \\ \overline{\partial_A} \Phi &= 0, \end{aligned}$$

the usual Hitchin equations.

16.3. 5D Super Yang-Mills Theory

Now, let's repeat the previous example, but including some supersymmetry. 5D Super Yang-Mills theory admits a conventional Lagrangian description: Let P be a principal G-bundle over X (a 5-dimensional space). The theory has:

Fields: Connections A on P, sections ϕ^i of $Ad(P)$ $(i = 1, \ldots, 5)$, and fermions.
Lagrangian:

$$L = \frac{R}{8\pi^2} Tr \left[\frac{1}{R^2} F_A \wedge *F_A + \sum_{i=1}^{5} d_A \phi^i \wedge *d_A \phi^i + \text{fermions} \right].$$

Remark.

(1) 5D SYM is well-defined as an effective field theory, below a certain energy scale. It is not obviously well-defined at arbitrarily high energies.

(2) There's this unusual R factor appearing here that you should be suspicious of. We'll explain where this comes from at the end of the talk.

Compactification on C. Let's take $X = \mathbb{R}^{2,1} \times C$, where C is a Riemann surface. Analogous to the classical case, when we compactify 5D SYM on C, we combine ϕ^4 and ϕ^5 into a complex-valued 1-form on C:

$$\varphi = \left(\phi^4 + i\phi^5\right) dz.$$

Note that to be a sensible theory, we additionally require translation invariance along $\mathbb{R}^{2,1}$.

Question: What are the classical field configurations in the compactified theory which preserve the supersymmetry? (Recall that these are the BPS states!)

Assuming $\phi^1, \phi^2, \phi^3 = 0$, the equations satisfied by the remaining fields are

$$(\star) \qquad \begin{cases} F_A + R^2 \left[\varphi, \varphi^*\right] = 0, \\ \overline{\partial_A}\varphi = 0, \end{cases}$$

which we recognize as (almost) Hitchin's equations. In other words, the moduli space of vacua of $SYM[C]$ in the low energy limit is

$$M_C\left[G\right] = \{\text{solutions to } (\star)\} \,/\, \{\text{gauge transformations}\} = \mathcal{M}_H.$$

Remark. We took $\phi^1 = \phi^2 = \phi^3 = 0$ above. If we don't, SUSY also imposes equations on ϕ^1, ϕ^2, ϕ^3:

$$d_A\phi^i = 0, \qquad \left[\varphi, \phi^i\right] = 0, \qquad \left[\phi^i, \phi^j\right] = 0.$$

But, at a generic point in the moduli space, these equations won't have any nontrivial solutions, so the assumption that $\phi^j = 0$ isn't much of an imposition.

A key difference between this example and dimensional reduction for classical Yang-Mills theory is that we have dimensionally reduced to a theory on $\mathbb{R}^{2,1}$, *not* a theory on C. Instead of seeing Hitchin's moduli space as the moduli of instantons for our theory, it appears as the moduli of BPS states!

The full moduli space of vacua has a Coulomb branch—identified with the Hitchin moduli space—and Higgs branches attached to the specific other points where nontrivial solutions for the ϕ^j exist. (Unfortunate nomenclature: the moduli of Higgs bundles is the space of solutions that live on the Coulomb branch...)

16.4. Compactification from (2,0) 6D Theory

Now let's talk about where that pesky R factor came from. It turns out that there's a famous 6D $N = (2,0)$ QFT. It doesn't have a conventional Lagrangian description (or even a space of fields). Instead, the inputs are a 6-dimensional manifold, together with a Lie algebra $\mathfrak{g}$. Call this theory $X_\mathfrak{g}$. It has the following properties:

- $X_\mathfrak{g}$ has $N = (2,0)$ SUSY in $d = 6$.
- $X_\mathfrak{g}$ has no parameters—no coupling constants or scale, and the strength of the interaction can't be perturbed.
- $X_\mathfrak{g}$ is conformally invariant.

Despite its unconventional description, we can still compactify $X_\mathfrak{g}$ to obtain lower-dimensional theories. In fact, 5D SYM is $X_\mathfrak{g}\left[S^1\right]$, where the R is the length of the S^1. So, $\mathcal{M}_H$ is obtained as the moduli space associated to $X_\mathfrak{g}\left[C \times S^1\right]$. We could perform this compactification in either order: $\mathcal{M}_H$ can also be obtained as the moduli space associated to the theory $X_\mathfrak{g}\left[C\right]$ compactified on S^1. [GMN] use this observation to produce canonical Darboux coordinate systems on $\mathcal{M}_H$ and construct Calabi-Yau metrics in these coordinate systems.

Some examples of information we can obtain from this perspective:

- Compactify $X_\mathfrak{g}$ on C first to get a 4d $N = 2$ supersymmetric gauge theory with with Coulomb branch $\mathcal{B}$. Then, $\mathcal{B}$ is actually the Hitchin base, i.e., $\mathcal{M}_H \to \mathcal{B}$ with generic fiber a torus. Points $u \in \mathcal{B}$ correspond to spectral curves $\Sigma_u \subset T^*C$, also known as "Seiberg-Witten curves."
- $\mathcal{M}_H$ is automatically hyperkahler because of supersymmetry.

CHAPTER 17

Pyongwon's II

CHAPTER 18

Phil's II

CHAPTER 19

Lei's II

CHAPTER 20

Honghao's II

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