Sharp departure beyond post-Newtonian formalism

Hoang Ky Nguyen*
Department of Physics, Babeş-Bolyai University, Cluj-Napoca 400084, Romania

Bertrand Chauvineau[†]
Université Côte d'Azur, Observatoire de la Côte d'Azur,
CNRS, Laboratoire Lagrange, Nice cedex 4, France
(Dated: March 29, 2024)

We offer a concrete example exhibiting marked deviation from the parametrized post-Newtonian (PPN) approximation in a modified theory of gravity. Specifically, we derive the exact formula for the Robertson parameter γ in Brans-Dicke gravity for spherical compact mass sources, explicitly incorporating the pressure content of these sources. We achieve this by exploiting the integrability of the 00-component of the Brans-Dicke field equation. In place of the conventional PPN result $\gamma_{\text{PPN}} = \frac{\omega+1}{\omega+2}$, we obtain the analytical expression $\gamma_{\text{exact}} = \frac{\omega+1+(\omega+2)\Theta}{\omega+2+(\omega+1)\Theta}$ where Θ is the ratio of the total pressure $P_{\parallel}^* + 2P_{\perp}^*$ and total energy E^* contained within the mass source. Our non-perturbative formula is valid for all field strengths and types of matter comprising the mass source. Importantly, the dimensionless quantity Θ participates in the γ parameter due to the higher-derivative nature of Brans-Dicke gravity. In addition, we establish two new mathematical identities linking the active gravitational mass, the ADM (Arnowitt-Deser-Misner) mass, and the Tolman mass, applicable for Brans-Dicke gravity. We draw four key conclusions: (1) The usual γ_{PPN} formula is violated in the presence of pressure, viz. when $\theta \neq 0$, revealing a limitation of the PPN approximation in Brans-Dicke gravity. (2) The PPN result mainly stems from the assumption of pressureless matter. Even in the weak-field star case, non-zero pressure leads to a violation of the PPN formula for γ . Conversely, the PPN result is a good approximation for low-pressure matter, i.e. when $\Theta \approx 0$, for all field strengths. (3) Observational constraints on γ set joint bounds on ω and Θ , with the latter representing a global characteristic of a mass source. If the equation of state of matter comprising the mass source approaches the ultra-relativistic form, entailing $\Theta \simeq 1$, $\gamma_{\rm exact}$ converges to 1 irrespective of ω . (4) In a broader context, by exposing a limitation of the PPN approximation in Brans-Dicke gravity, our findings indicate the latent significance of considering the interior structure of stars in observational astronomy when testing candidate theories of gravitation, particularly those with a higher-derivative nature.

I. MOTIVATION

The parametrized post-Newtonian (PPN) framework has been an invaluable tool in the study of gravitational theories [1–3]. It is founded upon the assumptions of weak field and slow motion, representing the post-Newtonian limit of gravity. In this limit, the framework characterizes a given metric theory, at the first post-Newtonian level, by a set of ten real-valued parameters.

One of the most powerful applications of the PPN formalism is the calculation of the Robertson (or Eddington-Robertson-Schiff) parameter γ . This important parameter governs the amount of space-curvature produced by a body at rest and can be directly measured via the detection of light deflection and the Shapiro time delay. For one of the simplest extensions beyond General Relativity (GR)—the Brans-Dicke (BD) theory—the PPN approach is known to yield

$$\gamma_{\rm PPN} = \frac{\omega + 1}{\omega + 2} \tag{1}$$

where ω is the BD parameter. This formula recovers the result $\gamma_{\rm GR}=1$ known for GR in the limit of infinite ω , in which the BD scalar field approaches a constant field. Current bounds using Solar System observations set the magnitude of ω to exceed 40,000. Generalizations of the PPN γ result to other modified theories of gravity are available [1, 2, 4–17].

In the case of GR, by virtue of Birkhoff's theorem, spherically symmetric vacuum solutions are static and asymptotically flat. The vacuum spacetime exterior of a mass source is described by the Schwarzschild metric which is dependent on only one parameter, the Schwarzschild radius. All information regarding the internal structure and composition of the source, namely, the types of matter comprising it as well as the distribution profile of matter in the source, is fully encapsulated in the Schwarzschild radius. Since GR, as a classical theory, lacks an inherent length scale (such as the Planck length), the dimensionful Schwarzschild radius cannot participate in the dimensionless γ parameter. Thus, this parameter is independent of the source in GR (a fact compatible with $\gamma_{\rm GR}=1$).

BD gravity, however, has a richer structure due to the BD scalar field Φ , an additional degree of freedom be-

^{*} hoang.nguyen@ubbcluj.ro

[†] bertrand.chauvineau@oca.eu

sides the metric components. For the special case of black holes, the no-hair theorem applies [18, 19], meaning that the static non-rotating spherisymmetric vacuum exterior to a black hole in BD gravity is a Schwarzschild solution. Consequently, black holes in BD gravity, irrespective of the value of ω , have $\gamma = 1$ rather than the usual $\frac{\omega+1}{\omega+2}$ result. Besides black holes, BD gravity exhibits other structures—normal stars and exotic ones, such as wormholes and naked singularities [20–22]. For these structures, the BD scalar field in the exterior vacuum is generally non-constant. The PPN formalism makes use of the non-constancy in Φ to derive the usual PPN result for stars, given in Eq. (1), which explicitly depends on the BD parameter ω [23]. Yet, it is important to note that the PPN γ parameter, per Eq. (1), contains no information about the star (aside from the fact that the star is regular at its center), as the BD parameter ω is a parameter of the theory, not one of the resulting exterior vacuum.

In contrast to the second-derivative GR, where all information about the stellar source is condensed into one single parameter—the Schwarzschild radius-BD gravity, as a higher-order theory, permits the internal structure of the star to influence higherderivative characteristics of the exterior solution, potentially leaving its footprints on the PPN parame-In their seminal work [26], Brans and Dicke recognized the approximate nature of estimating the two parameters characterizing the exterior vacuum solution, later known as the Brans Class I solution [27]. This recognition was evidenced in Eq. (34) of Ref. [26]. The PPN formula in Eq. (1) should be regarded as an approximation applicable in the limit of weak field everywhere (including inside the star) and slow motion in BD gravity, whereby the higher-derivative features of the exterior vacuum are suppressed. An important question arises: Can the internal structure of a star in BD gravity, and in modified theories in general, manifest through the PPN parameters? And if so, under what conditions?

An obvious course of action would be to lift the weak field hypothesis on the source of the field, enabling an exact calculation of the external field, from which the PPN parameters can be extracted. An attempt in this direction has been made recently in [24] where it is suggested that the higher-order terms can impact the γ parameter. Yet, there exists another condition, albeit less explicit, related to the pressure within the star. Note that the PPN requirement of slow motion applies not only for macroscopic objects but also for their microscopic constituents. Per the post-Newtonian bookkeeping scheme in Ref. [1], this translates to a requirement for low pressure relative to the energy content. Whereas a star is stationary, the microscopic motion of the matter contained within its domain can be relativistic, resulting in appreciably high pressure compared with its energy content.

Despite the challenges of relaxing the weak-field and low-pressure constraints, significant progress can be achieved in one particular situation—the BD theory. In our current study and its companion paper [25], we focus on a specific case—the BD exterior vacuum surrounding a matter spherical distribution which has a finite domain of support; we shall collectively call this type of structure a compact star. Here, we rigorously account for the influence of the compact mass source on the exterior vacuum and the γ parameter without resorting to any approximations. We achieve this by making use of the integrability of the 00-component of the BD field equations, enabling us to circumvent the limitations of the weak-field and low-pressure approximations. Advancements in detection methods allow for the study of neutron stars, making our investigation relevant both for practical applications and theoretical inquiries into the formalism and methodologies employed. Our study shall shed light on the role of stellar pressure and provide a benchmark for assessing its impacts in theories of modified gravity.

This paper complements its companion paper, Ref. [25], and is organized as follows. Section II revisits the form of the energy-momentum tensor (EMT) of star sources in BD gravity. Sections III and IV handle the field equations in the standard coordinates and transform the results to the isotropic coordinates. The uniqueness of Brans solutions is addressed in Section V. Sections VI and VII conduct the interior-exterior matching and derive the γ parameter. Section VIII obtains mass relations, valid for BD gravity. Sections IX and X offer discussions and outlooks. A further exposition on the uniqueness of Brans solutions is given in Appendix A.

II. THE ENERGY-MOMENTUM TENSOR

Consider the BD action in the Jordan frame [26, 27]

$$\int d^4x \frac{\sqrt{-g}}{16\pi} \left[\Phi \mathcal{R} - \frac{\omega}{\Phi} \nabla^{\mu} \Phi \nabla_{\mu} \Phi \right] + \int d^4x \sqrt{-g} L^{(m)} (2)$$

with the metric signature convention (-+++). Hereafter, we choose the units G=c=1 with G being the far-field Newtonian constant.

It is well documented [28] that upon the Weyl mapping $\{\tilde{g}_{\mu\nu} := \Phi g_{\mu\nu}, \tilde{\Phi} := \ln \Phi\}$, the gravitational sector of the BD action can be brought to the Einstein frame as $\int d^4x \frac{\sqrt{-\tilde{g}}}{16\pi} \left[\tilde{\mathcal{R}} - (\omega + 3/2) \, \tilde{\nabla}^{\mu} \tilde{\Phi} \tilde{\nabla}_{\mu} \tilde{\Phi} \right]$. The Einstein-frame BD scalar field $\tilde{\Phi}$ has a kinetic term with a signum determined by $(\omega + 3/2)$. Unless stated otherwise, we shall restrict our consideration to the normal ("non-phantom") case of $\omega > -3/2$, where the kinetic energy for $\tilde{\Phi}$ is positive.

In this paper, we work exclusively in the Jordan frame. For convenience, let us denote a rank-two tensor

$$X_{\mu\nu} := \Phi \mathcal{R}_{\mu\nu} - \nabla_{\mu} \nabla_{\nu} \Phi - \frac{\omega}{\Phi} \nabla_{\mu} \Phi \nabla_{\nu} \Phi \tag{3}$$

The BD field equation for the metric components is

$$X_{\mu\nu} = 8\pi \left[T_{\mu\nu} - \frac{\omega + 1}{2\omega + 3} g_{\mu\nu} T \right] \tag{4}$$

and the equation for the BD scalar field is

$$\Box \Phi = \frac{8\pi}{2\omega + 3}T\tag{5}$$

where the energy-momentum tensor (EMT) of the matter sector is

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g}L^{(m)}\right)}{\delta g^{\mu\nu}} \tag{6}$$

For a coordinate system that is static and spherically symmetric, the metric can be written as (with $g_{01} = g_{10} = 0$)

$$ds^{2} = -A(r)dt^{2} + B(r)dr^{2} + C(r)d\Omega^{2}$$
(7)

With this metric, the only non-vanishing components of the tensor $X_{\mu\nu}$ are the diagonal ones, namely, $\mu = \nu$. With respect to the off-diagonal components, $\mu \neq \nu$, since $g_{\mu\nu} = 0$, the field equation requires

$$T_{\mu\nu} = 0 \quad \text{if } \mu \neq \nu \tag{8}$$

meaning the EMT has to be diagonal. The trace is merely a sum $T=T_0^0+T_1^1+T_2^2+T_3^3$. Furthermore since

$$g_{33} = g_{22} \sin^2 \theta \tag{9}$$

$$X_{33} = X_{22} \sin^2 \theta \tag{10}$$

one has

$$T_{33} = T_{22} \sin^2 \theta \implies T_3^3 = T_2^2$$
 (11)

Therefore, the EMT must adopt the following form

$$T^{\nu}_{\mu} = \begin{pmatrix} -\epsilon(r) & 0 & 0 & 0\\ 0 & p_{\parallel}(r) & 0 & 0\\ 0 & 0 & p_{\perp}(r) & 0\\ 0 & 0 & 0 & n_{\perp}(r) \end{pmatrix}$$
(12)

The only assumptions we made are the stationarity and spherical symmetry for the metric components and the BD scalar field. The energy density ϵ , the radial pressure p_{\parallel} and the tangential pressure p_{\perp} are functions of r. The trace is simplified to

$$T = -\epsilon + p_{\parallel} + 2p_{\perp} \tag{13}$$

We shall not impose any further constraints on the EMT, which can be anisotropic.

III. INTEGRABILITY OF THE 00-FIELD EQUATION

Let us start with the standard areal coordinate system

$$ds^{2} = -A(r)dt^{2} + B(r)dr^{2} + r^{2}d\Omega^{2}$$
 (14)

In this form, a star center exists at r=0, at which the metric components, A(r) and B(r) and the BD field $\Phi(r)$ are regular (including their first derivatives with respect to r). The scalar field equation (5) yields

$$\frac{d}{dr}\left(r^2\sqrt{\frac{A}{B}}\frac{d\Phi}{dr}\right)\sin\theta = \frac{8\pi T}{2\omega + 3}\sqrt{-g}$$
 (15)

Multiply both sides with $dV = dr d\theta d\varphi$, then integrate

$$\int_{0}^{r} d\left(r^{2}\sqrt{\frac{A}{B}}\frac{d\Phi}{dr}\right) \int_{0}^{\pi} \sin\theta d\theta \int_{0}^{2\pi} d\varphi$$

$$= \frac{8\pi}{2\omega + 3} \int_{V} dV \sqrt{-g} T \qquad (16)$$

with the integral domain V being a ball of radius r, centered at the origin.

Since A, B, Φ , A', Φ' are finite at the star center, r = 0, (viz. regularity conditions), upon integration of the left hand side, the above equation becomes

$$r^2 \sqrt{\frac{A}{B}} \frac{d\Phi}{dr} = \frac{2}{2\omega + 3} \int_V dV \sqrt{-g} T \tag{17}$$

Next, the 00-component of $X_{\mu\nu}$ can be written as

$$X_{00} = \Phi R_{00} + \Gamma_{00}^{1} \frac{d\Phi}{dr}$$
 (18)

$$=\frac{\sqrt{A}}{2r^2\sqrt{B}}\frac{d}{dr}\left(\frac{r^2\Phi}{\sqrt{AB}}\frac{dA}{dr}\right) \tag{19}$$

$$= \frac{-\sin\theta}{2\sqrt{-g}g^{00}} \frac{d}{dr} \left(\frac{r^2\Phi}{\sqrt{AB}} \frac{dA}{dr}\right)$$
 (20)

It is imperative to note that the X_{00} term is expressible in a neat "integrable" form, Eq. (20). Consequently, the 00—component of the field equation (4) yields

$$\frac{d}{dr} \left(\frac{r^2 \Phi}{\sqrt{AB}} \frac{dA}{dr} \right) \sin \theta = -16\pi \left(T_0^0 - \frac{\omega + 1}{2\omega + 3} T \right) \sqrt{-g}$$
(21)

Integrating from the star center (assuming regularity)

$$\frac{r^2\Phi}{\sqrt{AB}}\frac{dA}{dr} = -4\int_V dV\sqrt{-g}\left(T_0^0 - \frac{\omega+1}{2\omega+3}T\right)$$
(22)

The rest of this section deals with the terms in the right hand side of Eqs. (17) and (22). Consider a compact star of finite radius r_* and denote V^* the domain of the star, namely, V^* being a ball centered at the origin with radius r_* . The following integrals are defined for the domain V^* :

$$E^* := \int_{V^*} dV \sqrt{-g} \,\epsilon = 4\pi \int_0^{r_*} dr \, r^2 \sqrt{AB} \,\epsilon;$$

$$P_{\parallel}^* := \int_{V^*} dV \sqrt{-g} \,p_{\parallel} = 4\pi \int_0^{r_*} dr \, r^2 \sqrt{AB} \,p_{\parallel}; \qquad (23)$$

$$P_{\perp}^* := \int_{V^*} dV \sqrt{-g} \,p_{\perp} = 4\pi \int_0^{r_*} dr \, r^2 \sqrt{AB} \,p_{\perp}.$$

Note that outside the star, ϵ , p_{\parallel} , and p_{\perp} vanish. Therefore for a ball V that is centered at the origin and encloses V^* (namely, its radius r exceeds r_*), the following identities hold:

$$\int_{V} dV \sqrt{-g} \, \epsilon = E^{*};$$

$$\int_{V} dV \sqrt{-g} \, p_{\parallel} = P_{\parallel}^{*};$$

$$\int_{V} dV \sqrt{-g} \, p_{\perp} = P_{\perp}^{*}.$$
(24)

For a ball V enclosing V^* , the integrals in the right hand side of Eqs. (17) and (22) are

$$\int_{V} dV \sqrt{-g} T = \int_{V} dV \sqrt{-g} \left[-\epsilon + p_{\parallel} + 2p_{\perp} \right]$$
 (25)
= $-E^* + P_{\parallel}^* + 2P_{\parallel}^*$ (26)

and

$$\int_{V} dV \sqrt{-g} \left(T_{0}^{0} - \frac{\omega + 1}{2\omega + 3} T \right)
= \int_{V} dV \sqrt{-g} \left[-\frac{\omega + 2}{2\omega + 3} \epsilon - \frac{\omega + 1}{2\omega + 3} \left(p_{\parallel} + 2p_{\perp} \right) \right]$$
(27)

$$= -\frac{\omega + 2}{2\omega + 3} E^{*} - \frac{\omega + 1}{2\omega + 3} \left(P_{\parallel}^{*} + 2P_{\perp}^{*} \right)$$
(28)

Remark 1. The integrability property that we exploited in deriving Eqs. (17) and (22) is not limited to the standard coordinates. If we used the metric in Eq. (7), the left hand side of Eqs. (17) and (22) would read $C\sqrt{\frac{A}{B}}\frac{d\Phi}{dr}$ and $\frac{C\Phi}{\sqrt{AB}}\frac{dA}{ar}$ respectively, and the $\sqrt{-g}$ term in the right hand side of these equation would be calculated using the latter metric. Nevertheless, the standard-coordinate metric in (14) explicitly reveals the presence of the point r=0 at which the surface area of a 2-sphere vanishes. This point naturally corresponds to the star's center and serves as the lower bound in the integrals defined in (23).

Remark 2. In the integrals defined in Eq. (23), the element $dV\sqrt{-g}$ is equal to $\sqrt{A}\left(r^2\sqrt{B}dr\sin\theta d\theta\,d\varphi\right)$. The term \sqrt{A} , equal to $\sqrt{-g_{00}}$, is a "redshift factor". The combination $r^2\sqrt{B}dr\sin\theta d\theta\,d\varphi$, equal to $r^2\sqrt{g_{11}}dr\sin\theta d\theta\,d\varphi$, is the spatial volume element in the spatial part of the metric given in Eq. (14). Note that in the region occupied with matter, e.g. where $\epsilon(r)\neq 0$, the space in general is not Euclidean; consequently, for the standard-coordinate metric, B(r) deviates from 1.

It should also be noted that the combination $r^2\sqrt{g_{11}}dr\sin\theta d\theta d\varphi$ is equal to $\sqrt{g^{(3)}}dV$, where $g^{(3)}$ is the determinant of the spatial section of the metric in Eq. (14). The combination is thus *invariant* with respect to a twice-differentiable transformation of the radial coordinate. The quantities E^* , P_{\parallel}^* , and P_{\perp}^* can be interpreted as the total energy, radial pressure, and tangential pressure contained within the compact star.

IV. TRANSFORMING TO ISOTROPIC COORDINATES

With the total energy and pressures defined via Eq. (24), the set of equations (17), (22), (26), and (28) essentially provide the "conservation" rules for the metric components and the BD scalar in the *exterior* vacuum, i.e., for $r > r_*$, per

$$r^{2}\sqrt{\frac{A}{B}}\frac{d\Phi}{dr} = \frac{2}{2\omega + 3}\left[-E^{*} + P_{\parallel}^{*} + 2P_{\perp}^{*}\right]$$
 (29)

and

$$\frac{r^2 \Phi}{\sqrt{AB}} \frac{dA}{dr} = \frac{4}{2\omega + 3} \left[(\omega + 2)E^* + (\omega + 1) \left(P_{\parallel}^* + 2P_{\perp}^* \right) \right]$$
(30)

Note that the right hand sides of Eqs. (29) and (30) are integration *constants*, induced by the matter distribution in the interior of the compact mass source.

Our next step is to relate the parameters of the exterior vacuum to these constants. The vacuum solution for BD gravity is best known in the isotropic coordinate, in the form of the Brans Class I solution. (The issue with the generality and uniqueness of the Brans Class I solution shall be addressed in Section V.) It is thus necessary to bring the left hand sides of Eqs. (29) and (30) to the isotropic coordinate ρ , namely, using the following metric

$$ds^{2} = -F(\rho)dt^{2} + G(\rho)\left(d\rho^{2} + \rho^{2}d\Omega^{2}\right)$$
 (31)

and the BD scalar $\phi(\rho)$. Transforming (14) into (31) requires the following identifications

$$F(\rho) = A(r); \ G(\rho) \left(\frac{d\rho}{dr}\right)^2 = B(r); \ G(\rho) \rho^2 = r^2 \ (32)$$

in addition to a mapping for the BD scalar

$$\phi(\rho) = \Phi(r) \tag{33}$$

In the far-field region, it is expected that the relation between r and ρ is monotonic, viz. $\frac{d\rho}{dr} > 0$. The quantities of interest thence become

$$r^{2}\sqrt{\frac{A}{B}}\frac{d\Phi}{dr} = G\rho^{2}\sqrt{\frac{F}{G\left(\frac{d\rho}{dr}\right)^{2}}}\frac{d\phi}{d\rho}\frac{d\rho}{dr} = \rho^{2}\sqrt{FG}\frac{d\phi}{d\rho} \quad (34)$$

and

$$\frac{r^2\Phi}{\sqrt{AB}}\frac{dA}{dr} = \frac{G\rho^2\phi}{\sqrt{FG\left(\frac{d\rho}{dr}\right)^2}}\frac{dF}{d\rho}\frac{d\rho}{dr} = \rho^2\phi\sqrt{\frac{G}{F}}\frac{dF}{d\rho} \quad (35)$$

We thus arrive at

$$\rho^2 \sqrt{FG} \frac{d\phi}{d\rho} = \frac{2}{2\omega + 3} \left[-E^* + P_{\parallel}^* + 2P_{\perp}^* \right]$$
 (36)

and

$$\rho^2 \phi \sqrt{\frac{G}{F}} \frac{dF}{d\rho} = \frac{4}{2\omega + 3} \left[(\omega + 2)E^* + (\omega + 1) \left(P_{\parallel}^* + 2P_{\perp}^* \right) \right]$$

$$(37)$$

Remark 3. As we stated at the end of the preceding section, the energy density and pressure quantities are invariant upon a coordinate transformation in the radial coordinate. This can be verified for the case at hand. For example, the total energy in the standard coordinates is

$$E_{\rm std}^* = \int_V dr \, d\theta \, d\varphi \sqrt{-A(r)B(r)} \, r^2 \sin \theta \, \epsilon(r) \qquad (38)$$

whereas in the isotropic coordinates

$$E_{\rm iso}^* = \int_V d\rho \, d\theta \, d\varphi \sqrt{-F(\rho)G^3(\rho)} \, \rho^2 \sin\theta \, \epsilon'(\rho) \qquad (39)$$

with the identification $\epsilon'(\rho) = \epsilon(r(\rho))$. It is straightforward to see that

$$E_{\rm std}^* = \int_V \left(d\rho \frac{dr}{d\rho} \right) d\theta d\varphi \sqrt{-FG \left(\frac{d\rho}{dr} \right)^2} G\rho^2 \sin \theta \, \epsilon \left(r(\rho) \right) \tag{40}$$

$$= \int_{V} d\rho \, d\theta \, d\varphi \sqrt{-FG^3} \, \rho^2 \sin \theta \, \epsilon'(\rho) \tag{41}$$

$$=E_{\rm iso}^* \tag{42}$$

Hence, the energy integral E^* is the same for both systems—the standard and the isotropic coordinates. Likewise, the same conclusion applies for the pressure integrals P^*_{\parallel} and P^*_{\perp} .

V. BRANS CLASS I AS THE UNIQUE VACUUM SOLUTION FOR $\omega > -3/2$

Equations (36) and (37) stand handy for us to relate the parameters of the exterior vacuum solution to the energy-pressure integrals. It is well documented that the Brans Class I solution is a vacuum solution in BD gravity. In this section, we shall further demonstrate that, for the case of non-phantom kinetic energy for the BD field, viz. $\omega > -3/2$, the Brans Class I solution is the most general and unique vacuum solution. That is to say, the Brans Class I solution is the vacuum solution in BD gravity, when $\omega > -3/2$.

Historically, Brans discovered the solutions during his PhD thesis and reported them in [27] without offering a derivation, although the solutions can be verified via direct inspection. The earliest public account for an analytical derivation of these solutions can be traced back to Ref. [28] in which Bronnikov discovered the more general Brans-Dicke-Maxwell electrovacuum solution.

The Brans solutions are comprised of 4 different classes (or types). In the exposition of Bronnikov

[28], the Brans-Dicke theory is first mapped from the Jordan frame to the Einstein frame via a Weyl mapping. The transformed BD scalar field becomes uncoupled (thus free) scalar field which sources the Einstein-frame spacetime metric. This transformed theory is known to admit the Fisher-Janis-Newman-Winicour (FJNW) solution. (Note: The solution has been re-discovered several times, and is also known as the Fisher-Bergmann-Leipnik-Janis-Newman-Winicour-Buchdahl-Wyman (FBLJNWBW) solution [29–34].) Via the Einstein-frame representation, it has been established by now that the classes of Brans solutions are the most general solutions that are static and spherisymmetric in Brans-Dicke gravity [34, 35].

However, the existence of multiple Brans classes has sidetracked a recognition of their "uniqueness". There is a degree of redundancy in these classes, however. In [35] Bhadra and Sarkar pointed out that Class III and Class IV are equivalent via a coordinate transformation, $\rho \leftrightarrow 1/\rho$, reducing the count of classes to 3. These authors also uncovered a "symmetry" in terms of parameters of Brans Class I and Brans Class II, upon making certain replacement in the parameters (and a coordinate transform) ¹. Consequently, it was deemed that the two classes, I and II, were equivalent, leaving only Class I and Class III to be truly independent. Nomenclature aside, it should be noted that diffeomorphism alone cannot transform Class I into Class II and vice versa. In [34] Faraoni and colleagues revisited this issue and correctly branded the "symmetry" alluded above a "duality" rather than an "equivalence". As we shall show momentarily, the 3 Brans classes (I, II, and IV) remain separate solutions (while fully covering) in the parameter space. Yet, the 3 classes can be "unified" upon an appropriate parametrization. That is to say, all 3 Brans classes of solution can be brought into a single form, as we shall do

Consider the metric and scalar field

$$\begin{cases}
ds^2 = -F(\rho)dt^2 + G(\rho) \left(d\rho^2 + \rho^2 d\Omega^2\right) \\
F(\rho) = \left(\frac{\rho - \frac{1}{2}M_1\sqrt{\kappa}}{\rho + \frac{1}{2}M_1\sqrt{\kappa}}\right)^{\frac{2}{\sqrt{\kappa}}} \\
G(\rho) = \left(1 - \frac{M_1^2\kappa}{4\rho^2}\right)^2 \left(\frac{\rho - \frac{1}{2}M_1\sqrt{\kappa}}{\rho + \frac{1}{2}M_1\sqrt{\kappa}}\right)^{-\frac{2(1+\Lambda)}{\sqrt{\kappa}}} \\
\phi(\rho) = \left(\frac{\rho - \frac{1}{2}M_1\sqrt{\kappa}}{\rho + \frac{1}{2}M_1\sqrt{\kappa}}\right)^{\frac{\Lambda}{\sqrt{\kappa}}}
\end{cases}$$
(43)

with $M_1 \in \mathbb{R}^+$ (a parameter of length dimension) and dimensionless parameters $\Lambda \in \mathbb{R}$ and $\kappa \in \mathbb{R}$. It is straightforward to verify by direct inspection that the metric

¹ The replacement of parameters to map Class II into Class I can also be viewed as a "Wick rotation" [36].

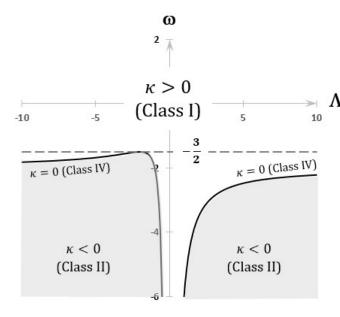


Figure 1. Parameter space for the "unified" Brans solution, Eq. (43). The solid line corresponds to $(1+\Lambda)^2 - \Lambda(1-\omega\Lambda/2) = 0$. The two branches of $\kappa = 0$ asymptote $\omega \to -2$ at large values of Λ . The local peak in the left branch occurs at $\{\omega = -3/2, \Lambda = -2\}$. The region with $\omega > -3/2$ is Brans Class I, exclusively.

and scalar field satisfy the BD field equations, with the parameter $\kappa \in \mathbb{R}$ obeying $(\omega \in \mathbb{R} \text{ and } \Lambda \in \mathbb{R})$

$$\kappa := (1+\Lambda)^2 - \Lambda \left(1 - \frac{\omega}{2}\Lambda\right) \tag{44}$$

Three cases to consider, depending on the signum of κ :

Case 1: For $\kappa > 0$, it is evidently Brans Class I solution as originally reported in [27].

Case 2: For $\kappa < 0$, utilizing the identity $\tan^{-1} x = \frac{i}{2} \ln \frac{1-ix}{1+ix}$, one has

$$\frac{1}{\sqrt{\kappa}} \ln \frac{\rho - \frac{1}{2} M_1 \sqrt{\kappa}}{\rho + \frac{1}{2} M_1 \sqrt{\kappa}} = \frac{-2}{\sqrt{-\kappa}} \tan^{-1} \frac{M_1 \sqrt{-\kappa}}{2\rho}$$
(45)

The change of variable $\rho=\frac{1}{4}M_1^2\kappa/\rho'$, hence $d\rho=-\frac{1}{4}M_1^2\kappa/\rho'^2d\rho'$, renders

$$\left(\frac{\rho - \frac{1}{2}M_1\sqrt{\kappa}}{\rho + \frac{1}{6}M_1\sqrt{\kappa}}\right)^{\frac{1}{\sqrt{\kappa}}} = e^{\frac{2}{\sqrt{-\kappa}}\tan^{-1}\frac{2\rho'}{M_1\sqrt{-\kappa}}} \tag{46}$$

$$1 - \frac{M_1^2 \kappa}{4\rho^2} = 1 + \frac{4\rho'^2}{M_1^2(-\kappa)} \tag{47}$$

resulting in Brans Class II solution in the new radial

coordinate ρ' [27]:

$$F(\rho') = e^{\frac{4}{\sqrt{-\kappa}} \tan^{-1} \frac{2\rho'}{M_1 \sqrt{-\kappa}}} \tag{48}$$

$$G(\rho') = \left(1 + \frac{M_1^2(-\kappa)}{4\rho'^2}\right)^2 e^{-\frac{2(1+\Lambda)}{\sqrt{-\kappa}} \tan^{-1} \frac{2\rho'}{M_1\sqrt{-\kappa}}}$$
(49)

$$\phi(\rho') = e^{\frac{2\Lambda}{\sqrt{-\kappa}} \tan^{-1} \frac{2\rho'}{M_1 \sqrt{-\kappa}}}.$$
 (50)

Case 3: For $\kappa = 0$, the limit

$$\left(\frac{\rho - \frac{1}{2}M_1\sqrt{\kappa}}{\rho + \frac{1}{2}M_1\sqrt{\kappa}}\right)^{\frac{1}{\sqrt{\kappa}}} \stackrel{\sim}{\kappa \to 0} e^{-\frac{M_1}{\rho}}$$
(51)

produces Brans Class IV solution [27]: ²

$$F(\rho) = e^{-\frac{M_1}{\rho}} \tag{52}$$

$$G(\rho) = e^{\frac{(1+\Lambda)M_1}{\rho}} \tag{53}$$

$$\phi(\rho) = e^{-\frac{\Lambda M_1}{\rho}}. (54)$$

Let us emphasize that the recasting exercise thus far represents nothing essentially new. The important point which our recasting offers is that the signum of κ elects the "class" that the "unified" Brans solution, Eq. (43), belongs. Moreover, a pair of $\{\omega, \Lambda\}$ uniquely determines the signum for κ , and hence the form of the solution. Figure 1 shows the categorization. In the $\{\omega, \Lambda\}$ plane, the two branches of the solid line correspond to the loci of $\kappa = 0$, viz. $(1 + \Lambda)^2 - \Lambda(1 - \omega \Lambda/2) = 0$. Above and below the solid line are the domains for Brans Class I and Brans Class II, respectively. The 3 classes do not overlap while fully covering the $\{\omega, \Lambda\}$ plane. The ambiguity in selecting the solution is removed. The uniqueness of the "unified" Brans solution, Eq. (43), is thereby established.

Remark 4. For $\omega > -3/2$, the function κ is positive-definite for all values of Λ in \mathbb{R} , as can be seen by rewriting κ as

$$\kappa = \frac{[(\omega + 2)\Lambda + 1]^2 + 2\omega + 3}{2(\omega + 2)} > 0 \quad \text{for } \omega > -3/2 \quad (55)$$

Consequently, only Brans Class I is admissible for the non-phantom action, $\omega > -3/2$, a fact evident in Fig. 1.

Remark 5. It is worth noting that Brans Class I recovers the Schwarzschild metric when $\Lambda=0$ forcing $\kappa=1$. On the other hand, Brans Class II and Class III do not have this property and are often relegated as "pathological" solutions. Conveniently, they only exist for the phantom action, i.e., $\omega \leqslant -3/2$.

 $^{^2}$ Brans Class III is obtainable from Brans Class IV via a coordinate transform, $\rho=M_1^2/(4\rho').$

Another way to recognize the unified nature of the 3 Brans solutions is via the harmonic coordinate instead of the isotropic coordinate [28]. For completeness, we shall revisit this representation in Appendix A.

VI. MATCHING THE EXTERIOR SOLUTION WITH ENERGY-PRESSURE INTEGRALS

As established in the preceding section, for the nonphantom kinetic action (i.e., $\omega > -3/2$), the exterior vacuum solution is exclusively Brans Class I. We are now equipped to perform the matching of its parameters.

With the metric given in Eq. (43), the left hand sides of Eqs. (36) and (37) readily yield

$$\rho^2 \sqrt{FG} \frac{d\phi}{d\rho} = M_1 \Lambda \tag{56}$$

$$\rho^2 \phi \sqrt{\frac{G}{F}} \frac{dF}{d\rho} = 2M_1 \tag{57}$$

With the aid of Eqs. (36) and (37), we deduce that

$$M_1 = 2 \frac{(\omega + 2)E^* + (\omega + 1)(P_{\parallel}^* + 2P_{\perp}^*)}{2\omega + 3}$$
 (58)

$$\Lambda = \frac{-E^* + P_{\parallel}^* + 2P_{\perp}^*}{(\omega + 2)E^* + (\omega + 1)(P_{\parallel}^* + 2P_{\perp}^*)}$$
 (59)

Denote the dimensionless ratio

$$\Theta := \frac{P_{\parallel}^* + 2P_{\perp}^*}{E_{\parallel}^*} \tag{60}$$

For an isotropic EMT, $P_{\parallel}^* = P_{\perp}^* \equiv P^*$, the quantity Θ is $3P^*/E^*$. In general, thence

$$M_{1} = 2E^{*} \frac{(\omega + 2) + (\omega + 1)\Theta}{2\omega + 3}$$

$$\Lambda = \frac{\Theta - 1}{(\omega + 2) + (\omega + 1)\Theta}$$
(61)

$$\Lambda = \frac{\Theta - 1}{(\omega + 2) + (\omega + 1)\Theta} \tag{62}$$

Note that for $\omega > -3/2$, E > 0, and $0 \leqslant \Theta < 1$, the positive-definiteness of M_1 is ensured

$$M_1 = E^* \left[1 + \Theta + \frac{1 - \Theta}{2\omega + 3} \right] \geqslant E^*.$$
 (63)

Furthermore, from Eq. (44), the parameter κ

$$\kappa = \frac{2\omega + 3}{4} \frac{(2\omega + 3)(1 + \Theta)^2 + (1 - \Theta)^2}{((\omega + 2) + (\omega + 1)\Theta)^2}$$
(64)

giving $\kappa > 0$ for $\omega > -3/2$, confirming the validity of the Brans Class I solution under consideration.

THE ROBERTSON PARAMETERS

The Robertson expansion in isotropic coordinates is

$$ds^{2} = -\left(1 - 2\frac{M_{1}}{\rho} + 2\beta \frac{M_{1}^{2}}{\rho^{2}} + \dots\right)dt^{2} + \left(1 + 2\gamma \frac{M_{1}}{\rho} + \dots\right)\left(d\rho^{2} + \rho^{2}d\Omega^{2}\right)$$
(65)

in which β and γ are the Robertson (or Eddington-Robertson-Schiff) parameters. The metric in Eq. (43) can be re-expressed in the expansion form

$$ds^{2} = -\left(1 - 2\frac{M_{1}}{\rho} + 2\frac{M_{1}^{2}}{\rho^{2}} + \dots\right)dt^{2} + \left(1 + 2\left(1 + \Lambda\right)\frac{M_{1}}{\rho} + \dots\right)\left(d\rho^{2} + \rho^{2}d\Omega^{2}\right) \quad (66)$$

Comparing Eqs. (65) against Eq. (66), we obtain

$$\beta_{\text{exact}} = 1 \tag{67}$$

$$\gamma_{\text{exact}} = 1 + \Lambda$$
 (68)

where we have added in the subscript "exact" as emphasis. Note that Λ measures the deviation of the γ parameters from GR ($\gamma_{\rm GR} = 1$). From Eq. (62), Λ depends on both ω and Θ . Finally, we arrive at

$$\gamma_{\text{exact}} = \frac{\omega + 1 + (\omega + 2)\Theta}{\omega + 2 + (\omega + 1)\Theta}$$
 (69)

which can also be conveniently recast as

$$\gamma_{\text{exact}} = \frac{\gamma_{\text{PPN}} + \Theta}{1 + \gamma_{\text{PPN}} \Theta} \tag{70}$$

by recalling that $\gamma_{\text{PPN}} = \frac{\omega+1}{\omega+2}$. With the aid of Eqs. (60) and (61), the active gravitational mass is

$$M_{\text{grav}} := M_1 \tag{71}$$

$$= E^* + P_{\parallel}^* + 2P_{\perp}^* + \frac{E^* - \left(P_{\parallel}^* + 2P_{\perp}^*\right)}{2\omega + 3}$$
 (72)

where the contribution of pressure to the active gravitational mass is evident [37, 38].

MASS RELATIONS IN BRANS-DICKE **GRAVITY**

In GR, the Tolman mass was defined as [39–41]

$$m_{\rm T} := \int_{V} dV \sqrt{-g} \left(-T_0^0 + T_1^1 + T_2^2 + T_3^3 \right) \tag{73}$$

In our EMT form (12), this renders

$$m_{\rm T} = \int_{V} dV \sqrt{-g} \left(\epsilon + p_{\parallel} + 2p_{\perp} \right) \tag{74}$$

$$= E^* + P_{\parallel}^* + 2P_{\perp}^* \tag{75}$$

For a metric that has the following asymptotic form (i.e., as $\rho \to \infty$) [41]:

$$ds^{2} = -\left(1 - \frac{2M_{\text{grav}}}{\rho}\right)dt^{2} + \left(1 + \frac{2M_{\text{ADM}}}{\rho}\right)\left(d\rho^{2} + \rho^{2}d\Omega^{2}\right)$$
(76)

the quantity $M_{\rm grav}$ is the active gravitation mass of the source, whereas $M_{\rm ADM}$ is the ADM mass. In GR, it is known that

$$M_{\rm grav} = M_{\rm ADM} = M_{\rm T} \tag{77}$$

For BD gravity, besides Expression (72) for $M_{\rm grav}$, viz.

$$M_{\text{grav}} = M_{\text{T}} + \frac{E^* - (P_{\parallel}^* + 2P_{\perp}^*)}{2\omega + 3}$$
 (78)

we can also calculate the ADM mass, with the aid of Eqs. (65) and (69)

$$M_{\rm ADM} = \gamma M_{\rm grav} \tag{79}$$

$$= E^* + P_{\parallel}^* + 2P_{\perp}^* - \frac{E^* - \left(P_{\parallel}^* + 2P_{\perp}^*\right)}{2\omega + 3}$$
 (80)

$$= M_{\rm T} - \frac{E^* - \left(P_{\parallel}^* + 2P_{\perp}^*\right)}{2\omega + 3} \tag{81}$$

The difference

$$M_{\text{grav}} - M_{\text{ADM}} = \frac{E^* - (P_{\parallel}^* + 2P_{\perp}^*)}{\omega + 3/2} \geqslant 0$$
 (82)

for $\omega > -3/2$ and normal matter, viz. $p_{\parallel} \leqslant \frac{1}{3}\epsilon$ and $p_{\perp}^* \leqslant \frac{1}{3}\epsilon$.

Remark 6. In the limit of infinite ω , we recover the usual relation in GR

$$m_{\text{grav}} = m_{\text{ADM}} = m_{\text{T}} = E^* + P_{\parallel}^* + 2P_{\perp}^*$$
 (83)

Furthermore, in GR, the ADM mass has been established within the Tolman-Oppenheimer-Volkoff framework (using the standard coordinates) to be

$$m_{\text{ADM}} = 4\pi \int_0^{r_*} dr \, r^2 \, \epsilon(r) \tag{84}$$

Using Eq. (23) in the standard coordinates (14), we rewrite the right hand side of Eq. (83) as

$$E^* + P_{\parallel}^* + 2P_{\perp}^* = 4\pi \int_0^{r_*} dr \, r^2 \sqrt{-g_{00} \, g_{11}} \left(\epsilon + p_{\parallel} + 2p_{\perp} \right)$$
(85)

Combining the last 3 equations yields

$$\int_0^{r_*} dr \, r^2 \, \epsilon \, = \int_0^{r_*} dr \, r^2 \sqrt{-g_{00} \, g_{11}} \left(\epsilon + p_{\parallel} + 2p_{\perp} \right) \tag{86}$$

We thus have re-derived the Tolman relation in GR, as a by-product of our study [37, 38].

IX. DISCUSSIONS

Formula (69) is the final outcome of our derivation. This section aims to clarify a number of logical and technical steps taken in the derivation, and discusses the implications of Formula (69).

Our derivation proceeded in the following steps:

- 1. Assuming the metric and the mass source to be stationary and spherically symmetric, we deduced from the BD field equation that the most general EMT of the source can be put in the form $T^{\nu}_{\mu} = \operatorname{diag}\left(-\epsilon(r), p_{\parallel}(r), p_{\perp}(r), p_{\perp}(r)\right)$. Whereas the EMT can be anisotropic, we imposed no further conditions on the EMT, such as being a perfect fluid. See Section II.
- 2. The standard coordinate system, $ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2d\Omega^2$, is suitable for describing stars. Using the scalar field equation for Φ and the 00-component of the field equation, and imposing regularity conditions at the star's center, we derived two ODE's—Eqs. (17) and (22)—which express A(r), B(r) and $\Phi(r)$ in the exterior vacuum in terms of the total energy E^* and total pressure $P_{\parallel}^* + 2P_{\perp}^*$ contained within the star. The standard coordinate system allows for the existence of the point r=0 at which the surface area of the 2-sphere vanishes, which naturally serves as the star's center. This point acts as the lower bound for the domain of integration for E^* and $P_{\parallel}^* + 2P_{\perp}^*$. See Section III.
- 3. We next transformed the two aforementioned ODE's into the isotropic coordinate system, $ds^2 = -F(\rho)dt^2 + G(\rho)\left(d\rho^2 + \rho^2d\Omega^2\right)$. The resulting ODE's in terms of $F(\rho)$, $G(\rho)$ and $\phi(\rho)$ are Eqs. (36) and (37). The advantage of the isotropic coordinates is that the exterior solution is best known in this system (i.e., the Brans solutions). Furthermore, the energy and pressure integrals, E^* and $P_{\parallel}^* + 2P_{\perp}^*$, are unchanged when moving to this system. See Section IV.
- 4. We next presented the "unified" Brans solution, Eq. (43), which cover all Brans Classes I, II and IV (with Class III being equivalent to Class IV). For a non-phantom action, i.e. $\omega > -3/2$, Brans Class I is the most general and unique static spherisymmetric vacuum solution. For a phantom action, i.e. $\omega < -3/2$, all 3 Brans classes are admissible; however, only one single class is selected for a given set of parameters of the "unified" Brans solution. The uniqueness of the "unified" Brans solution is thus established. See Section V.
- 5. We employed the Brans Class I solution to perform the matching of the functions $F(\rho)$, $G(\rho)$ and $\phi(\rho)$

with the energy and pressure integrals, E^* and $P_{\parallel}^* + 2P_{\perp}^*$. See Section VI.

- 6. Expressing the Brans Class I metric in the Robertson expansion, we obtain the γ parameter in terms of ω and a (dimensionless) Θ , defined as the ratio of $P_{\parallel}^* + 2P_{\perp}^*$ and E^* . See Section VII.
- 7. As a by-product, we obtained two mathematical relations linking the active gravitational mass and the ADM mass with the energy and pressure integrals for BD gravity. In addition, we (re)-derived the Tolman relation in GR. See Section VIII.

Generality of our derivation—(i) Non-perturbative approach: Our derivation is non-perturbative in nature. It makes use of the integrability of the 00-component of the BD field equation, along with the scalar field equation involving $\Box \Phi$. (ii) Minimal assumptions: The physical assumptions employed are the regularity at the star's center and the existence of the star's surface separating the interior and the exterior. Our derivation relies solely on the scalar field equation and the 00-component of the field equation, without the need for the full set of equations, specifically the 11- and 22- components of the field equation. Consequently, the conservation equation (by way of the Bianchi identity) is not required. (iii) Universality of result: The final formula, Eq. (69), holds for all field strengths and all types of matter (whether convective or non-convective, for example). We do not assume the matter comprising the stars to be a perfect fluid or isentropic.

Higher-derivative characteristics—In contrast to the one-parameter Schwarzschild metric, the Brans Class I solution depends on two parameters, i.e. the solution is not only defined by its gravitational mass, but also by a scalar mass besides the gravitational one [28]. This is because the BD description of gravity involves more fields than the only metric involved in GR, a scalar field being also part of the gravitational sector. (The same is also to be expected in the higher order theories framework, like $f(\mathcal{R})$ theories, since $f(\mathcal{R})$ theories can be recast as $\omega=0$ BD theories endowed with a scalar dependent potential.) The exterior BD vacuum should reflect the internal structure and composition of the star. This expectation is confirmed in the final result, Eq. (69), which underscores the participation of the parameter Θ .

Role of pressure—Figure 2 shows a contour plot of γ_{exact} as a function of γ_{PPN} (i.e, $\frac{\omega+1}{\omega+2}$) and Θ . There are two interesting observations:

- An ultra-relativistic limit, $\Theta \lesssim 1$, would render $\gamma_{\rm exact} \simeq 1$, regardless of ω .
- For Newtonian stars, i.e. low pressure $(\Theta \approx 0)$, the PPN result is a good approximation regardless of the field strength.

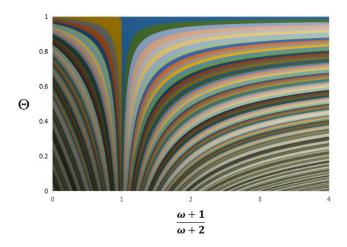


Figure 2. Contour plot of $\gamma_{\rm exact}$ in terms of Θ and $\frac{\omega+1}{\omega+2}$. A measured $\gamma_{\rm exact} \approx 1$ could mean $\frac{\omega+1}{\omega+2} \approx 1$ (i.e., $\omega \gg 1$) or $\Theta \approx 1$ (i.e., ultra-relativistic matter). Contours are equally spaced in 0.01 increments. For a given contour, the corresponding value of $\gamma_{\rm exact}$ can be read on the abscissa axis, where the contour intersects it.

 $\mathcal{O}\left(1/\sqrt{\omega}\right)$ anomaly—It has been discovered that the Brans-Dicke-Maxwell (electro) vacuum does not necessarily converge to a vacuum of GR [43–52]. Additionally, the convergence rate is $1/\sqrt{\omega}$ rather than the usual $1/\omega$ behavior. Recently, the present authors proved that this $\mathcal{O}\left(1/\sqrt{\omega}\right)$ anomaly exists for all types of matter regardless of the trace of the EMT [53] (see also [54, 55]). For this anomaly to occur, the "remnant" BD scalar field needs to exhibit a singularity or time-dependence. However, these conditions are not satisfied for static stars, where regularity requirements are imposed. Therefore, the violation of γ reported in this current article is not related to the $\mathcal{O}\left(1/\sqrt{\omega}\right)$ anomaly.

On the loss of Birkhoff's theorem—It can be argued that, in BD gravity, the loss of Birkhoff's theorem and the dependence of γ on a star's internal structure may be interconnected. Indeed, let us consider, in BD gravity, a static spherisymmetric Newtonian star (initial state). Its exterior vacuum is described by a Brans Class I solution given by Eq. (43), characterized by 2 parameters M_1 and Λ (or equivalently κ per Eq. (44)). Note that the parameter Λ depends on the pressure content of the star, as is evident in Eq. (62). For the initial Newtonian star, since $p\ll\epsilon$ inside the whole star, $\Theta_{\rm ini}$ is approximately zero, rendering $\Lambda_{\rm ini}\approx-\frac{1}{\omega+2}$. Let us now consider that this star starts collapsing, and that the collapse ends at some (final) compact state. The pressure is no longer negligible with respect to ϵ in this final state, in such a way that the final value $\Theta_{\rm fin}$ is significant, making $\Lambda_{\rm fin}$ significantly differing from $\Lambda_{\rm ini}$. On the other hand, Birkhoff's theorem in GR mandates that any spherisymmetric vacuum must be independent of the coordinate t regardless of the (time-) evolution of the source. If Birkhoff's theorem were valid for BD gravity, the vacuum exterior to

the collapsing star would have been left unchanged during the process (i.e. a time independent solution), which is incompatible with the observation that the final $\Lambda_{\rm fin}$ differs from the initial one $\Lambda_{\rm ini}$. Thence, the fact that Λ explicitly depends on Θ , as described by Eq. (62), implies that Birkhoff's theorem cannot hold for BD gravity. Reciprocally, the strong necessity to revisit the BD gravity's PPN γ expression could have been anticipated from the mere fact that Birkhoff's theorem is not valid in BD gravity.

X. CLOSING AND OUTLOOKS

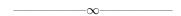
We derived the exact analytical formula, Eq. (69), of the Robertson parameter γ for compact mass sources in BD gravity. The derivation's success hinges on the integrability of the 00–component of the field equation, rendering it non-perturbative and applicable for any field strength and type of matter constituting the source. This paper provides comprehensive technical insights to complement a companion paper [25], which delves into physical implications of our findings.

Outlook #1—The conventional PPN result for BD gravity $\gamma_{\text{PPN}} = \frac{\omega+1}{\omega+2}$ lacks dependence on the physical features of the mass source, a trait shared by other alternative theories beyond GR, such as the Bergman-Wagoner theory. In the light of our exact result, the γ_{PPN} should be regarded as an approximation for stars in modified gravity under low-pressure conditions. Our findings underscore the limitations of the PPN formalism, particularly in scenarios characterized by high star pressure. It is plausible to expect that the role of pressure may extend to other modified theories of gravitation.

Outlook #2—The energy and pressure integrals, E^* and $P_{\parallel}^* + 2P_{\perp}^*$ defined in Eq. (23), can be evaluated within the Tolman-Oppenheimer-Volkoff (TOV) framework for BD gravity. This approach entails developing the TOV equations and integrating them from the star's center outward. The resulting exterior solution is fully determined by factors such as the equation of state of the star's constituent matter and the central pressure of the star. Accordingly, the matching between the interior spacetime solution and the exterior vacuum is automatically handled when the integration crosses the surface at which the pressure vanishes. However, it should be noted that the integration procedure is numerical in nature, as there is currently no exact analytical solution available for the interior of stars in BD gravity. Inspired by our exact γ derivation presented in this paper, details regarding the TOV equation for BD gravity, presented in a new optimal gauge choice, are currently underway [42].

ACKNOWLEDGMENTS

H.K.N. thanks Mustapha Azreg-Aïnou for stimulating discussions, and Valerio Faraoni and Tiberiu Harko for valuable commentaries.



Appendix A: Mapping the unified Brans solution to harmonic radial coordinate

The uniqueness of Brans solutions can also be made more transparent with the use of harmonic radial coordinates. This Appendix provides a brief explanation of this aspect.

Introducing a new parameter χ and making the following coordinate transformation to a radial coordinate u

$$\rho = \frac{1}{2} M_1 \sqrt{\kappa} \frac{1 + e^{\sqrt{\chi}u}}{1 - e^{\sqrt{\chi}u}}; \quad \left(\frac{d\rho}{du}\right)^2 = \frac{M_1^2 \kappa \chi}{\left(1 - e^{\sqrt{\chi}u}\right)^4} \quad (A1)$$

Resulting in

$$\left(1 - \frac{M_1^2 \kappa}{4\rho^2}\right)^2 \rho^2 = \frac{M_1^2 \kappa}{\sinh^2(\sqrt{\chi}u)}; \qquad (A2)$$

$$\left(1 - \frac{M_1^2 \kappa}{4\rho^2}\right)^2 \left(\frac{d\rho}{du}\right)^2 = \frac{M_1^2 \kappa \chi}{\sinh^4(\sqrt{\chi}u)}.$$
 (A3)

Introducing two parameters α and β such that

$$\chi = (a+b)\,\kappa\tag{A4}$$

$$\Lambda = -\frac{2b}{a+b} \tag{A5}$$

The BD scalar field and the metric become (upon rescaling $dt \to \frac{M_1\sqrt{\kappa}}{\sqrt{\chi}}dt$ and $ds \to \frac{M_1\sqrt{\kappa}}{\sqrt{\chi}}ds$), respectively

$$\phi(u) = e^{-2bu} \tag{A6}$$

$$ds^{2} = \frac{1}{\phi(u)} \left[-e^{2au} dt^{2} + e^{-2au} \left(\frac{du^{2}}{s^{4}(\chi, u)} + \frac{d\Omega^{2}}{s^{2}(\chi, u)} \right) \right]$$
(A7)

with the "Bronnikov" function (first introduced in [28]) defined as

$$s(\chi, u) := \frac{1}{\sqrt{\chi}} \sinh \sqrt{\chi} u \tag{A8}$$

The proper part satisfies the radial harmonic gauge. Depending on whether χ is positive, null, or negative, the function is *numerically* equal to

$$s(\chi, u) = \begin{cases} \frac{1}{\sqrt{\chi}} \sinh \sqrt{\chi} u & \text{if } \chi > 0\\ u & \text{if } \chi = 0\\ \frac{1}{\sqrt{-\chi}} \sin \sqrt{-\chi} u & \text{if } \chi < 0 \end{cases}$$
(A9)

The three cases correspond to Brans Class I, Class IV, and Class II, in that order. Plugging Eqs. (A4) and (A5) in to Eq. (44), we obtain the following "constraint" among the parameters

$$\chi = a^2 + (2\omega + 3) b^2 \tag{A10}$$

Obviously, a pair of $\{\omega, a/b\}$ uniquely specifies the signum for χ . This is nothing but the metric obtained in the harmonic radial coordinate u, presented in Bronnikov [28]. The proper part of the metric is the Fisher-Janis-Newman-Winicour (FJNW) solution [29–34].) The generality of this metric is thus established therein. For $\omega > -3/2$, $\chi > 0$ is the Brans Class I. Whereas $\omega < -3/2$, all three possibilities for χ positive, null, negative are admissible. However, for a given pair of $\{\omega, a/b\}$, the signum of χ is uniquely determined, selecting the class that the solution belongs. The ambiguity in picking one solution among the three possibilities is removed.

One final remark is that Bronnikov used a slightly different notation which may have obscured the "uniqueness" of the Brans solutions [28])

$$ds^{2} = \frac{1}{\phi(u)} \left[-e^{2au} dt^{2} + e^{-2au} \left(\frac{du^{2}}{s_{\text{Bronnikov}}^{4}(k, u)} + \frac{d\Omega^{2}}{s_{\text{Bronnikov}}^{2}(k, u)} \right) \right]$$
(A11)

depending on the signum of k:

$$s_{\text{Bronnikov}}(k, u) := \begin{cases} k^{-1} \sinh ku & \text{if } k > 0 \\ u & \text{if } k = 0 \\ k^{-1} \sin ku & \text{if } k < 0 \end{cases}$$
 (A12)

The three cases correspond to Brans Class I, Class IV, and Class II, in that order.

The distinction between (A9) and (A12) is subtle. Comparing the two functions, k can be formally identified with $\sqrt{-\chi}$. However, with $\chi \in \mathbb{R}$, $\sqrt{-\chi}$ can be real or pure imaginary, whereas the k parameter in (A12) needs be defined separately for $\chi > 0$ and $\chi < 0$. The "bifurcation" at k = 0 may have obscured the duality between Brans Class I and Class II, an issue we elucidated in the Section V.

- C. M. Will, Theory and Experiment in Gravitational Physics, second edition, Cambridge University Press, Cambridge, 2018
- [2] C. M. Will, The Confrontation between General Relativity and Experiment, Living Rev. Relativ. 17, 4 (2014), https://doi.org/10.12942/lrr-2014-4
- [3] C. M. Will, On the unreasonable effectiveness of the post-Newtonian approximation in gravitational physics, Proc. Nat. Acad. Sci. 108 (2011) 5938
- [4] M. Hohmann, L. Järv, P. Kuusk, and E. Randla, Post-Newtonian parameters γ and β of scalar-tensor gravity with a general potential, Phys. Rev. D **88**, 084054 (2013); Erratum Phys. Rev. D **89**, 069901 (2014), arXiv:1309.0031 [gr-qc]
- [5] M. Hohmann, Parameterized post-Newtonian formalism for multimetric gravity, Class. Quant. Grav. 31, 135003 (2014), arXiv:1309.7787 [gr-qc]
- [6] M. Hohmann, Parameterized post-Newtonian limit of Horndeski's gravity theory, Phys. Rev. D 92, 064019 (2015), arXiv:1506.04253 [gr-qc]
- [7] M. Hohmann, L. Järv, P. Kuusk, E. Randla, and O. Vilson, Post-Newtonian parameter y for multiscalar-tensor gravity with a general potential, Phys. Rev. D 94, 124015 (2016), arXiv:1607.02356 [gr-qc]
- [8] M. Hohmann, Post-Newtonian parameter γ and the deflection of light in ghost-free massive bimetric gravity, Phys. Rev. D 95, 124049 (2017), arXiv:1701.07700 [gr-qc]
- [9] M. Hohmann and A. Schärer, Post-Newtonian parameters γ and β of scalar-tensor gravity for a homogeneous gravitating sphere, Phys. Rev. D 96, 104026 (2017), arXiv:1708.07851 [gr-qc]
- [10] U. Ualikhanova and M. Hohmann, Parameterized post-

- Newtonian limit of general teleparallel gravity theories, Phys. Rev. D **100**, 104011 (2019), arXiv:1907.08178 [gr-qc]
- [11] E. D. Emtsova and M. Hohmann, Post-Newtonian limit of scalar-torsion theories of gravity as analogue to scalarcurvature theories, Phys. Rev. D 101, 024017 (2020), arXiv:1909.09355 [gr-qc]
- [12] M. Hohmann, Gauge-invariant approach to the parametrized post-Newtonian formalism, Phys. Rev. D 101, 024061 (2020), arXiv:1910.09245 [gr-qc]
- [13] K. Flathmann and M. Hohmann, Post-Newtonian limit of generalized scalar-torsion theories of gravity, Phys. Rev. D 101, 024005 (2020), arXiv:1910.01023 [gr-qc]
- [14] Sebastian Bahamonde, Konstantinos F. Dialektopoulos, Manuel Hohmann, Jackson Levi Said, Post-Newtonian limit of Teleparallel Horndeski gravity, Class. Quant. Grav. 38, 025006 (2020), arXiv:2003.11554 [gr-qc]
- [15] K. Flathmann and M. Hohmann, Post-Newtonian limit of generalized symmetric teleparallel gravity, Phys. Rev. D 103, 044030 (2021), arXiv:2012.12875 [gr-qc]
- [16] K. Flathmann and M. Hohmann, Parametrized post-Newtonian limit of generalized scalar-nonmetricity theories of gravity, Phys. Rev. D 105, 044002 (2022), arXiv:2111.02806 [gr-qc]
- [17] M. Hohmann and U. Ualikhanova, Post-Newtonian limit of generalized scalar-teleparallel theories of gravity, arXiv:2312.13352 [gr-qc]
- [18] S. W. Hawking, Black holes in Brans-Dicke theory of gravitation, Commun. Math. Phys. 25, 167 (1972)
- [19] T. P. Sotiriou and V. Faraoni, Black holes in scalartensor gravity, Phys. Rev. Lett. 108, 081103 (2012), arXiv:1109.6324 [gr-qc]

- [20] A. G. Agnese and M. La Camera, Wormholes in the Brans-Dicke theory of gravitation, Phys. Rev. D 51, 2011 (1995)
- [21] V. Faraoni, F. Hammad, and S. D. Belknap-Keet, Revisiting the Brans solutions of scalar-tensor gravity, Phys. Rev. D 94, 104019 (2016), arXiv:1609.02783 [gr-qc]
- [22] H. K. Nguyen and M. Azreg-Aïnou, Revisiting Weak Energy Condition and wormholes in Brans-Dicke gravity, arXiv:2305.15450 [gr-qc]
- [23] S. Weinberg, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity, John Wiley & Sons, New York, 1972
- [24] V. Faraoni, J. Côté, and A. Giusti, Do solar system experiments constrain scalar–tensor gravity?, Eur. Phys. J. C 80, 132 (2020), arxiv:1906.05957 [gr-qc]
- [25] B. Chauvineau and H. K. Nguyen, Violation of γ in modified gravity (in preparation)
- [26] C. H. Brans and R. Dicke, Mach's Principle and a Relativistic Theory of Gravitation, Phys. Rev. 124, 925 (1961)
- [27] C. H. Brans, Mach's Principle and a relativistic theory of gravitation II, Phys. Rev. 125, 2194 (1962)
- [28] K. A. Bronnikov, Scalar-tensor theory and scalar charge, Acta Phys. Polon. B 4, 251 (1973), Link to pdf
- [29] I. Z. Fisher, Scalar mesostatic field with regard for gravitational effects, Zh. Eksp. Teor. Fiz. 18, 636 (1948), arXiv:gr-qc/9911008
- [30] O. Bergmann and R. Leipnik, Space-Time Structure of a Static Spherically Symmetric Scalar Field, Phys. Rev. 107, 1157 (1957)
- [31] A. I. Janis, E. T. Newman, and J. Winicour, Reality of the Schwarzschild Singularity, Phys. Rev. Lett. 20, 878 (1968)
- [32] H. A. Buchdahl, Static solutions of the Brans-Dicke equations. Int. J. Theor. Phys. 6, 407 (1972)
- [33] M. Wyman, Static spherically symmetric scalar fields in general relativity, Phys. Rev. D 24, 839 (1981)
- [34] V. Faraoni, F. Hammad, A. M. Cardini, and T. Gobeil, Revisiting the analogue of the Jebsen-Birkhoff theorem in Brans-Dicke gravity, Phys. Rev. D 97, 084033 (2018), arXiv:1801.00804 [gr-qc]
- [35] A. Bhadra and K. Sarkar, On static spherically symmetric solutions of the vacuum Brans-Dicke theory, Gen. Relativ. Gravit. 37, 2189 (2005), arXiv:gr-qc/0505141
- [36] R. Izmailov, A. Bhattacharya, and K. K. Nandi, Brans-Dicke womhole revisited II, arXiv:1006.4819 [gr-qc]
- [37] J. C. Baez and E. F. Bunn, The Meaning of Einstein's Equation, Amer. Jour. Phys. 73, 644 (2005), arXiv:gr-qc/0103044
- [38] J. Ehlers, I. Ozsvath, E. L. Schucking, and Y. Shang,

- Pressure as a Source of Gravity, Phys. Rev. D 72, 124003 (2005), arXiv:gr-qc/0510041
- [39] R. C. Tolman, On the Use of the Energy-Momentum Principle in General Relativity, Phys. Rev. 35, 875 (1930)
- [40] P. S. Florides, On the Tolman and MNøller mass-energy formulae in general relativity, J. Phys.: Conf. Ser 189, 012014 (2009)
- [41] D. N. Vollick, On the Meaning of Various Mass Definitions for Asymptotically Flat Spacetimes, Can. J. Phys. 101 (2023) 1, 9-16; arXiv:2101.12570 [gr-qc]
- [42] H. K. Nguyen and B. Chauvineau, An optimal gauge for Tolman-Oppenheimer-Volkoff equation in Brans-Dicke gravity (in preparation)
- [43] C. Romero and A. Barros, Does the Brans-Dicke theory of gravity go over to general relativity when ω → ∞?, Phys. Lett. A 173, 243 (1993)
- [44] C. Romero and A. Barros, Brans-Dicke Vacuum Solutions and the Cosmological Constant: a Qualitative Analysis, Gen. Relativ. Gravit. 25, 491 (1993)
- [45] F. M. Paiva and C. Romero, The Limits of Brans-Dicke Spacetimes: a Coordinate-free Approach, Gen. Relativ. Gravit. 25, 1305 (1993)
- [46] A. Barros and C. Romero, On the weak field approximation of the Brans-Dicke theory of gravity, Phys. Lett. A 245, 31 (1998)
- [47] V. Faraoni, The $\omega \to \infty$ limit of Brans-Dicke theory, Phys. Lett. A **245**, 26 (1998), arXiv:gr-qc/9805057
- [48] V. Faraoni, Illusions of general relativity in Brans-Dicke gravity, Phys. Rev. D 59, 084021 (1999), arXiv:grqc/9902083
- [49] V. Faraoni and J. Côté, Two new approaches to the anomalous limit of Brans-Dicke to Einstein gravity, Phys. Rev. D 99, 064013 (2019), arXiv:1811.01728 [gr-qc]
- [50] A. Bhadra, General relativity limit of the scalar-tensor theories for traceless matter field, gr-qc/0204014 [gr-qc]
- [51] N. Banerjee and S. Sen, Does Brans-Dicke theory always yield general relativity in the infinite ω limit?, Phys. Rev. D 56, 1334 (1997)
- [52] A. Bhadra and K. K. Nandi, ω dependence of the scalar field in Brans-Dicke theory, Phys. Rev. D 64, 087501 (2001), arXiv:gr-qc/0409091
- [53] H. K. Nguyen and B. Chauvineau, $\mathcal{O}(1/\sqrt{\omega})$ anomaly in Brans-Dicke gravity with trace-carrying matter, arXiv:2402.14076 [gr-qc]
- [54] B. Chauvineau, On the limit of Brans-Dicke theory when $\omega \to \infty$, Class. Quant. Grav. **20**, 2617 (2003)
- [55] B. Chauvineau, Stationarity and large ω Brans–Dicke solutions versus general relativity, Gen. Relativ. Gravit. 39, 297 (2007)