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Exercise 2.1.

a) To start with:

$$A^T A = (UDV^T)^T\, UDV^T = VD^T U^T UDV^T$$

Since U is an orthogonal matrix and $U^T U = I$, so the above equation equals to:

$$A^T A = VD^T U^T UDV^T = VD^T IDV^T = VD^T DV^T$$

In the next step, we know that D is a diagonal matrix, so $D^T D$ will also be a diagonal matrix with values σ_i^2 along the diagonal, where σ_i is the eigenvalue of matrix D. Finally, if we call $D^T D = D'$, then $A^T A = VD'V^T$, thus we will have the decomposition of matrix $A^T A$ with eigenvalues equal to the square of A's eigenvalues. In other words, eigenvalues of A are the square roots of the eigenvalues of $A^T A$.

b) We have the matrix A as follows:

$$A = \begin{bmatrix} 9 & 0 & 0 \\ 5 & 2 & 0 \\ 7 & 4 & 8 \end{bmatrix}$$

To find the eigenvalues of it we should solve the equation below:

$$Ax = \lambda x \rightarrow (A - \lambda I)\, x = 0 \rightarrow \det(A - \lambda I) = 0$$

$$A - \lambda I = \begin{bmatrix} 9-\lambda & 0 & 0 \\ 5 & 2-\lambda & 0 \\ 7 & 4 & 8-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (9-\lambda)\big((2-\lambda)(8-\lambda)\big) = 0$$

From the equation above, it is clear that $\lambda = 9, 2, or\ 8$ are the eigenvalues of A.

Now, for each of the eigenvalues we calculate the eigenvectors:

$\lambda_1 = 9$:

$$A - \lambda I = \begin{bmatrix} 0 & 0 & 0 \\ 5 & -7 & 0 \\ 7 & 4 & -1 \end{bmatrix}$$

$$(A - \lambda I)\, x = 0 \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 5 & -7 & 0 \\ 7 & 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \rightarrow x = \begin{bmatrix} 7 \\ 5 \\ 69 \end{bmatrix}$$

$\lambda_2 = 2$:

$$A - \lambda I = \begin{bmatrix} 7 & 0 & 0 \\ 5 & 0 & 0 \\ 7 & 4 & 6 \end{bmatrix}$$

$$(A - \lambda I)\, x = 0 \rightarrow \begin{bmatrix} 7 & 0 & 0 \\ 5 & 0 & 0 \\ 7 & 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \rightarrow x = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}$$

$\lambda_3 = 8$:

$$A - \lambda I = \begin{bmatrix} 1 & 0 & 0 \\ 5 & -6 & 0 \\ 7 & 4 & 0 \end{bmatrix}$$

$$(A - \lambda I)\, x = 0 \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 5 & -6 & 0 \\ 7 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \rightarrow x = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

c) $\underline{\boldsymbol{B\ vs.\ B^{-1}}}$: Suppose for matrix B we have an eigenvalue and eigenvector of $\lambda\ and\ x$, respectively. So, we have:

$$Bx = \lambda x$$

We can multiply the above equation to B^{-1}:

$$B^{-1}Bx = B^{-1}\lambda x \rightarrow x = B^{-1}\lambda x \rightarrow B^{-1}x = \frac{1}{\lambda}x$$

So, if λ is an eigenvalue of B with eigenvector x, then $\frac{1}{\lambda}$ is an eigenvalue of B^{-1} with the same eigenvector x.

$\underline{\boldsymbol{B\ vs.\ B^{T}}}$: As we know B is a real symmetric matrix, so $B = B^T$. Therefore, B and B^T have the same eigenvalues and eigenvectors.