CQF Final Report
Interest Rate Swap Modelling for Counterparty Credit Risk
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The present report shows the CVA pricing of a portfolio with an Interest Rate Swap instrument which is priced using a Libor Market Model for the evolution of their Libor Forward Rates and the calculation of their discount factors. The implementation is based on C++, QuantLib, IntelMKL and Boost functions and it's available at the following github repository https://github.com/mahsanchez/cva_libor

The Interest Rate Swap (IRS) has the following characteristics:

Notional: 10 (Actually represents 10,000,000 USD)
Expiry: 5Y
Fixed Rate: 0.025
Floating Leg Cashflows: { 0.0, 0.25, 0.5, 0.75, 1.0, 1.25, 1.5, 1.75, 2.0, 2.25, 2.5, 2.75, 3.0, 3.25, 3.5, 3.75, 4.0, 4.25, 4.5, 4.75, 5.0 }
Fixed Leg Cashflows: { 0.0, 0.25, 0.5, 0.75, 1.0, 1.25, 1.5, 1.75, 2.0, 2.25, 2.5, 2.75, 3.0, 3.25, 3.5, 3.75, 4.0, 4.25, 4.5, 4.75, 5.0 }

Both Floating Leg and Fixed Leg cashflows are pay quarterly.

The present value of an Interest Rate Swap is given by the differences of the price between the present value of Floating Leg and the present value of the Fixed Leg.

PV(IRS) = PV(Floating Leg) - PV(Fixed Leg)

The Present value of the Floating Leg is given by:

$$PV\,(\text{Floating Leg}) = \sum_{i=s+1}^{N} B\,(T_0, T_i)\,L\,(T_{i-1}, T_{i-1}, T_i)\,\delta_{i-1,i}.$$

The Present value of the Fixed Leg is given by:

$$PV\,(\text{Fixed Leg}) = \sum_{i=s+1}^{N} B\,(T_0, T_i)\,S\,(T_0, T_s, T_N)\,\delta_{i-1,i}.$$

Where

B: Discount Factors
L: Forward Rate Libor
delta: Year Fraction (Act/360)
S: Fixed Rate
Expiry: N

The values of Discount Factors and LIBOR Forward Rate are the numeraires calculated as a result of the evolution or simulation of Libor Market Model. In order to obtain this numeraires value is required to first calibrate the Model to the current market conditions. This basically traduced into the calibration of the volatility and correlation parameters that are given as an input for the LMM.

Volatility Calibration

The calibration of the instantaneous volatility is divided basically in two phases:

- Caplet Volatility Stripping
- Instantaneous Volatility Fitting

Caplets are options over rates that can be used to calculate the implied volatility in the market. As documented in the book LMM in Practice by Garetek and Musiela we starting by calculating the Forward Swap Rate or ATM strikes that will be used to price the caps as required by the calibration of the LMM.

The LIBOR rate covering the period $T_{n-1} \div T_n$ resetting in Tn-1 can be expressed from the perspective of today in terms of deterministic discount factors across all period and year fraction $\delta_{n-1,n}$

$$L(T_{n-1}, T_{n-1}, T_n) = F(T_0, T_{n-1}, T_n) = \left(\frac{B(T_0, T_{n-1})}{B(T_0, T_n)} - 1 \right) \frac{1}{\delta_{n-1,n}} \qquad (7.2)$$

Hence, given the discount factors provided in the book LMM the spot rates results of formule (7.2) are the following

tenor	Spot rate value
0.2514	0
0.5028	0.0221886
0.7583	0.0231959
1.0139	0.0241496
1.2639	0.0248865
1.5167	0.0268859
1.7722	0.0280598
2.0278	0.0289242
2.2778	0.0285658
2.5306	0.0304958
2.7861	0.0314728
3.0417	0.0321048
3.2944	0.0320084
3.5472	0.0332158
3.8083	0.0344515
4.0611	0.0341991
4.3139	0.0346209
4.5667	0.0357572
4.8194	0.036927
5.0722	0.04236

The calibration of the LMM requires to know how to price caplets in a particular cap. The market price of a caplets are valued using the standard Black Formula

$$c\left(T_0, T_{n-1}, T_n, \sigma^{cpl}_{n-1,n}\right) = B\left(T_0, T_n\right)\delta_{n-1,n}\left[F\left(T_0, T_{n-1}, T_n\right)N\left(d_1\right) - XN\left(d_2\right)\right] \qquad (7.3)$$

where:

$$N\left(d_1\right) = \frac{\ln\left(\frac{F(T_0,T_{n-1},T_n)}{X}\right) + \frac{{\sigma^{cpl}_{n-1,n}}^2 \delta_{0,n-1}}{2}}{\sigma^{cpl}_{n-1,n}\sqrt{\delta_{0,n-1}}}$$

$$N\left(d_2\right) = \frac{\ln\left(\frac{F(T_0,T_{n-1},T_n)}{X}\right) - \frac{{\sigma^{cpl}_{n-1,n}}^2 \delta_{0,n-1}}{2}}{\sigma^{cpl}_{n-1,n}\sqrt{\delta_{0,n-1}}}$$

Formula 7.3 represent the caplet price for the period $T_{n-1} \div T_n$ with payment at Tn and strike X, N represent the standard normal distribution and $\sigma^{cpl}_{n-1,n}$ the market volatility of a caplet covering the period $T_{n-1} \div T_n$

In the formula above X can be replaced by the ATM Strikes. Using the following calculation

$$S\left(T_0, T_s, T_N\right) = \frac{\sum_{i=s+1}^{N} B\left(T_0, T_i\right)\left(\frac{B(T_0,T_{i-1})}{B(T_0,T_i)} - 1\right)\frac{1}{\delta_{i-1,i}}\delta_{i-1,i}}{\sum_{i=s+1}^{N} B\left(T_0, T_i\right)\delta_{i-1,i}} = \frac{B\left(T_0, T_s\right) - B\left(T_0, T_N\right)}{\sum_{i=s+1}^{N} B\left(T_0, T_i\right)\delta_{i-1,i}}. \qquad (7.4)$$

Forward Swap Rate (ATM strikes) [S(T0, T3m, Ti)]

tenor	value
0.2514	0
0.5028	0.0219446
0.7583	0.0224453
1.0139	0.0230121
1.2639	0.0236052
1.5167	0.0241932
1.7722	0.0247813
2.0278	0.0253647
2.2778	0.0258263
2.5306	0.0262931
2.7861	0.0267648
3.0417	0.0272357
3.2944	0.0276438
3.5472	0.0280516
3.8083	0.0284212
4.0611	0.0288542
4.3139	0.0291917
4.5667	0.0295516
4.8194	0.0299315
5.0722	0.0305343

Stripping Volatilities from Market cap quotes

The market volatility of caplets will be derived from cap volatilities quotations; to do that the volatility stripping algorithm is used.

Is important to take into account that a cap can be splitted into the summation of their caplets across tenors. For example:

cap(0,9M) = caplet(0, 3M) + caplet(3M, 6M) + caplet(6M, 9M) in dollar terms

The implied volatility for caplet(6M, 9M) is calculated using a bisection root finding algorithm in the following equation:

cap(0, 9M) – cap(0,9M) - caplet(3M, 6M) - caplet(6M, 9M) = 0

The parameter vol used in the all parameters are keeped constant except the vol used for the pricing of caplet(6M, 9M). The vol value that make the above expression 0 is the implied volatility associated with caplet (6M, 9M). For more details of the implementation see Algorithm 7.1 p76 on the book the LMM in Practice.

Following some of the results of the implemented Stripping Volatility Algorithm to obtain the volatility term structure. In green color the results from LMM in Practice book, in red color the actual values obtained.

Tenor	Cap Vol	Reference Data	Actual Data
0.5028	0.1641	0.1641	0.1641
0.7583	0.1641	0.1641	0.1641
1.0139	0.1641	0.1641	0.1641
1.2639	0.1765	0.2015	0.1765
1.5167	0.1889	0.2189	0.188901
1.7722	0.2013	0.2365	0.201299
2.0278	0.2137	0.255	0.213701
2.2778	0.2162	0.2212	0.216199
2.5306	0.2186	0.2255	0.2186
2.7861	0.2211	0.2298	0.2211
3.0417	0.2235	0.2341	0.2235
3.2944	0.2223	0.2097	0.2223
3.5472	0.2212	0.2083	0.2212
3.8083	0.22	0.2077	0.22
4.0611	0.2188	0.2051	0.2188
4.3139	0.2173	0.2007	0.2173
4.5667	0.2158	0.1982	0.2158
4.8194	0.2142	0.1959	0.214201
5.0722	0.2127	0.1938	0.2127

The volatility calibration process does not finish on the volatility stripping. Following is required to fit the term structure of the stripped volatility in order to minimize the difference between theorical and market caps volatility. The process is known as Parametric Method of calibration.

The details of the steps are documented in the book LMM in Practice Chapter 9.

In order to do the instantaneous volatility fitting is required to find the values v1, v2, v3, v4 using a constrained optimization numerical method such that v1 + v2 > 0 and v4 > 0

Let's define the following function:

$$f(T_i - t) = \left| v_1 + \left[v_2 + v_3 \frac{T_i - t}{360} \right] e^{-v_4 \frac{T_i - t}{360}} \right| \qquad (9.12)$$

For all Ti , 3M <= Ti <= 5Y

Having that we compute all integrals of $f(T_i - t)^2$ as:

$$I(T_i - t)^2 = \int_0^{T_i} \left| v_1 + \left[v_2 + v_3 \frac{T_i - t}{360} \right] e^{-v_4 \frac{T_i - t}{360}} \right|^2 dt - \int_0^{T_{i-1}} \left| v_1 + \left[v_2 + v_3 \frac{T_i - t}{360} \right] e^{-v_4 \frac{T_i - t}{360}} \right|^2 dt \qquad (9.13)$$

For all Ti , 6M <= Ti <= 5Y

Next let's define

$$f_{FO}(T_i) = \sum_{T_i} I(T_i - t)^2 \qquad (9.14)$$

The problem of the parametric solution is to find values v1, v2, v3, v4 that minimize the following least squares problem

$$f_{\min} = \sqrt{\sum_{T_i} \left(\left[\frac{T_i - T_0}{360} \sigma^{caplet}(T_0, T_{i-3M}, T_i)^2 \right] - [f_{FO}(T_i)] \right)^2} \rightarrow \min \qquad (9.15)$$

Using Quantlib with SteepestDescent method the following values for v1, v2, v3, v4 were obtained. Values of the actual calculation implemented are highlighted in yellow color. Reference data results from the LMM in practice book is in green color. See volatility_fitting function inside class CplVolCalibration in the source code file calibration_libor.h.

	fmin	v1	v2	v3	v4
Reference	0.0436646	0.112346	-0.441811	0.971559	1.22306
Actual	0.0538277	0.134467	0.1603	0.0290665	0.156234

Parameters v1, v2, v3, v4 were initialized to 0.1 and the optimization method converged after 2816 iterations. See the results highlighted in yellow color in the table above.

Following the term structure of the instantaneous volatility fitted

tenor	reference	actual
0.5028	0.137589	0.269146
0.7583	0.229006	0.257279
1.0139	0.269546	0.246131
1.2639	0.279902	0.235889
1.5167	0.273765	0.226163
1.7722	0.258857	0.216945
2.0278	0.240317	0.208304
2.2778	0.221585	0.200385
2.5306	0.203651	0.192884
2.7861	0.187369	0.185792
3.0417	0.173244	0.179164
3.2944	0.161421	0.173043
3.5472	0.151576	0.167328
3.8083	0.143258	0.161828
4.0611	0.136747	0.156873
4.3139	0.131513	0.15226
4.5667	0.127337	0.147971
4.8194	0.124028	0.143989
5.0722	0.121418	0.140293

The LMM also requires the use of a correlation matrix. The simplest parametric fit with $\beta \approx 0.1$ was used to calculate the correlation matrix elements using the expression.

$$\rho_{ij} = e^{-\beta(t_i - t_j)}$$

The obtained correlation matrix is showed bellow

1	0.975173	0.950573	0.926585	0.903707	0.881148	0.85892	0.837244	0.816572	0.796188	0.776103	0.756517	0.73764	0.719226	0.70069	0.683198	0.666144	0.649515	0.633307
0.975173	1	0.974774	0.950174	0.926714	0.903581	0.880787	0.858559	0.837361	0.816458	0.795862	0.775777	0.756419	0.737536	0.718528	0.700592	0.683103	0.66605	0.64943
0.950573	0.974774	1	0.974764	0.950697	0.926965	0.903581	0.880778	0.859031	0.837587	0.816458	0.795854	0.775994	0.756623	0.737123	0.718722	0.700781	0.683287	0.666237
0.926585	0.950174	0.974764	1	0.97531	0.950963	0.926974	0.903581	0.881271	0.859272	0.837595	0.816458	0.796084	0.776212	0.756207	0.73733	0.718924	0.700977	0.683485
0.903707	0.926714	0.950697	0.97531	1	0.975037	0.95044	0.926455	0.903581	0.881024	0.858799	0.837127	0.816237	0.795862	0.775351	0.755995	0.737123	0.718722	0.700788
0.881148	0.903581	0.926965	0.950963	0.975037	1	0.974774	0.950174	0.926714	0.903581	0.880787	0.858559	0.837135	0.816237	0.795201	0.775351	0.755995	0.737123	0.71873
0.85892	0.880787	0.903581	0.926974	0.95044	0.974774	1	0.974764	0.950697	0.926965	0.903581	0.880778	0.858799	0.837361	0.81578	0.795416	0.77556	0.7562	0.73733
0.837244	0.858559	0.880778	0.903581	0.926455	0.950174	0.974764	1	0.97531	0.950963	0.926974	0.903581	0.881033	0.85904	0.836901	0.816009	0.795639	0.775777	0.756419
0.816572	0.837361	0.859031	0.881271	0.903581	0.926714	0.950697	0.97531	1	0.975037	0.95044	0.926455	0.903337	0.880787	0.858087	0.836666	0.81578	0.795416	0.775568
0.796188	0.816458	0.837587	0.859272	0.881024	0.903581	0.926965	0.950963	0.975037	1	0.974774	0.950174	0.926464	0.903337	0.880056	0.858087	0.836666	0.81578	0.795424
0.776103	0.795862	0.816458	0.837595	0.858799	0.880787	0.903581	0.926974	0.95044	0.974774	1	0.974764	0.95044	0.926714	0.902831	0.880293	0.858319	0.836892	0.816009
0.756517	0.775777	0.795854	0.816458	0.837127	0.858559	0.880778	0.903581	0.926455	0.950174	0.974764	1	0.975047	0.950706	0.926205	0.903084	0.88054	0.858559	0.837135
0.73764	0.756419	0.775994	0.796084	0.816237	0.837135	0.858799	0.881033	0.903337	0.926464	0.95044	0.975047	1	0.975037	0.949908	0.926195	0.903075	0.880531	0.858559
0.719226	0.737536	0.756623	0.776212	0.795862	0.816237	0.837361	0.85904	0.880787	0.903337	0.926714	0.950706	0.975037	1	0.974228	0.949908	0.926195	0.903075	0.88054
0.70069	0.718528	0.737123	0.756207	0.775351	0.795201	0.81578	0.836901	0.858087	0.880056	0.902831	0.926205	0.949908	0.974228	1	0.975037	0.950697	0.926965	0.903834
0.683198	0.700592	0.718722	0.73733	0.755995	0.775351	0.795416	0.816009	0.836666	0.858087	0.880293	0.903084	0.926195	0.949908	0.975037	1	0.975037	0.950697	0.926974
0.666144	0.683103	0.700781	0.718924	0.737123	0.755995	0.77556	0.795639	0.81578	0.836666	0.858319	0.88054	0.903075	0.926195	0.950697	0.975037	1	0.975037	0.950706
0.649515	0.66605	0.683287	0.700977	0.718722	0.737123	0.7562	0.775777	0.795416	0.81578	0.836892	0.858559	0.880531	0.903075	0.926965	0.950697	0.975037	1	0.975047
0.633307	0.64943	0.666237	0.683485	0.700788	0.71873	0.73733	0.756419	0.775568	0.795424	0.816009	0.837135	0.858559	0.88054	0.903834	0.926974	0.950706	0.975047	1

Expected Exposure and Credit Value Adjustment

The price of an Interest Rate Swap has a tendency to change their value over the life of the contract. When contract is as asset and have a positive mark-to-market value, it creates an exposure to counterparty. Such exposure is a risk factor (known as counterparty exposure) since it can be lost if the counterparty gets into financial distress.

Future exposures are calculated through the Monte Carlos simulation. The value of the contract in the future is driven by the evolution values of the underlying associated with the IRS derivative. As a result of the simulation different numeraires values are used across all simulation points, in our cases coincidence with the floating leg timepoints, but different across each scenario in the future. In our particular case the Risk Factor evolution for the Interest Rate Swap is driven by the Libor Market Model SDE in order to obtain the LMM Forward Rates and the discount factors.

The dynamics of the LMM is expressed in the following SDE (see simulation.h) for implementation details.

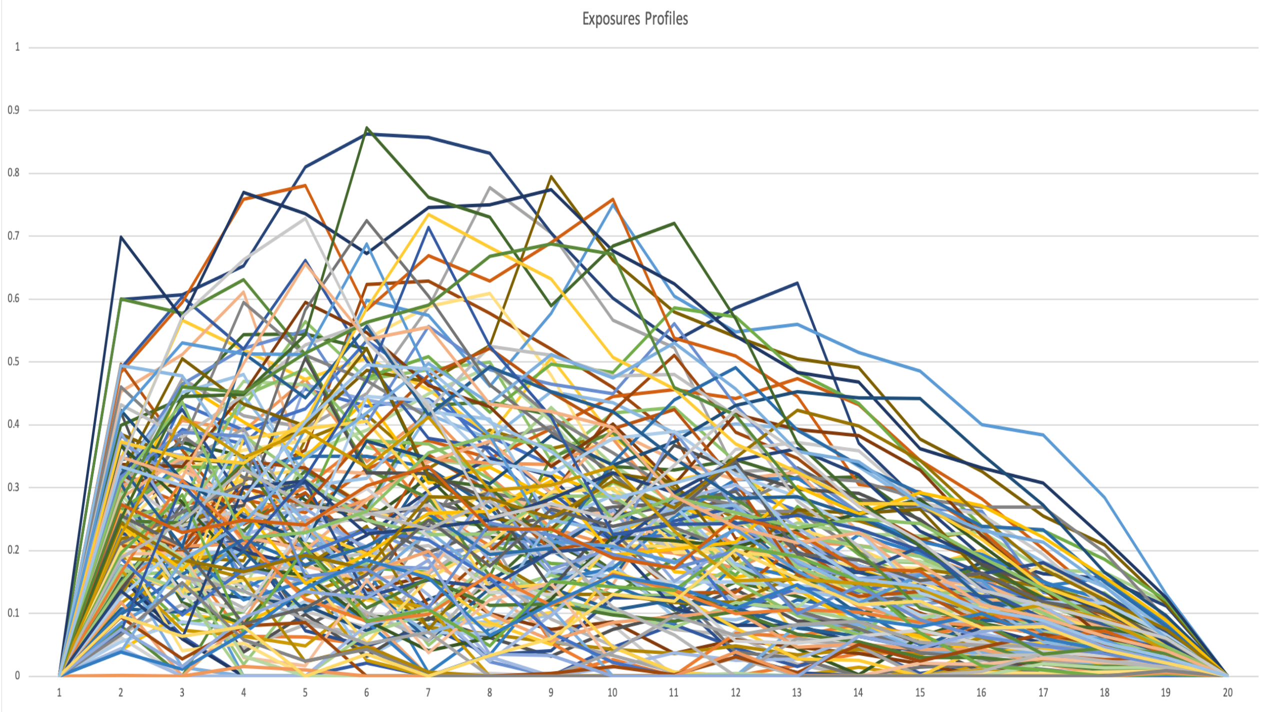
The discretised version of the log-normal dynamics of the LMM SDE is given by:

$$\begin{aligned} f_i(t_{k+1}) &= f_i(t_k)\exp\left[\left(\sigma_i(t_{i-k-1})\sum_{j=k+1}^{i}\frac{\tau_j f_j(t_k)\,\sigma_j(t_{j-k-1})\rho_{ij}}{1+\tau_j f_j(t_k)}\right.\right. \\ &\quad \left.\left.-\frac{1}{2}\sigma_i^2(t_{i-k-1})\right)\tau_k+\sigma_i(t_{i-k-1})\phi_i\sqrt{\tau_k}\right] \qquad (7) \end{aligned}$$

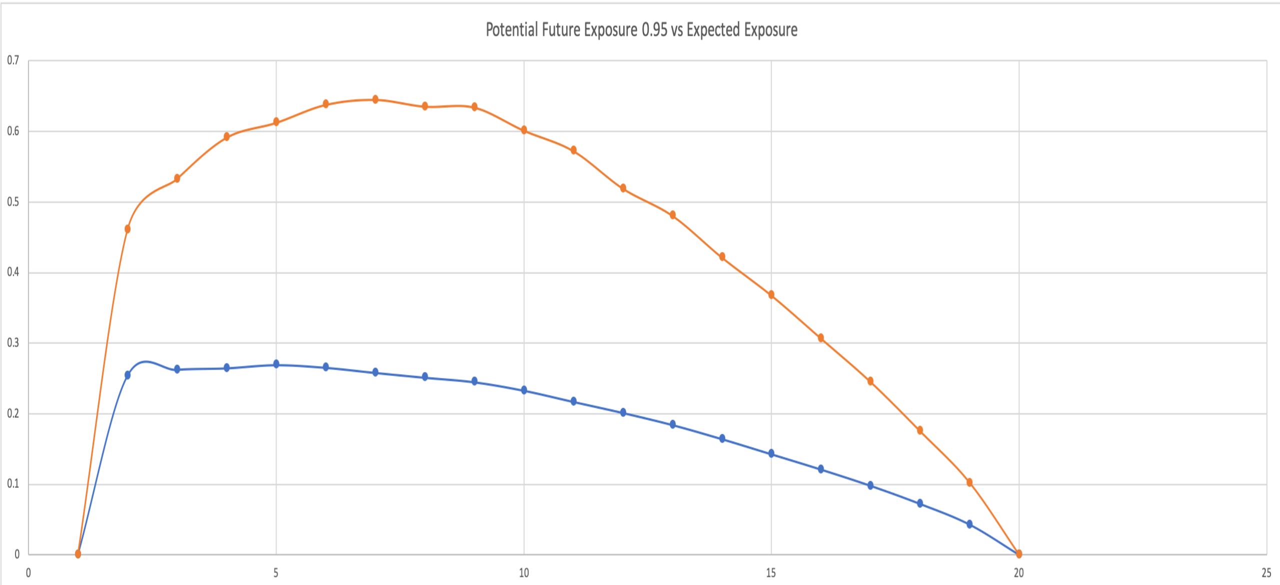
Where fj(tk) = fj and $\sigma_j(t_k)=\sigma_j(t)$ for tk < t< tk+1

In the current report was required to run 3000 MC simulations and on each simulation 190 LIBOR Forward Rates and 190 discounts factors were generated to run the mark to market of the Interest Rate Swap.

Mark to Market Simulation of Exposure for an Interest Rate Swap



Potential Future Exposure 0.95 vs Expected Exposure Profiles



Observe the hump shape that take the Potential Future Exposure (95%) and the Expected Exposure profiles. Reaching the maximum values in the first third of the tenors.

0	3M	6M	9M	1Y	1.3Y	1.6Y	1.9Y	2Y	2.3Y	2.6Y	3Y	3.3Y	3.6Y	3.9Y	4Y	4.3Y	4.6Y	4.9Y	5Y	
0	0.25389935	0.26264915	0.26475208	0.2691965	0.26545791	0.2582207	0.251405	0.24494038	0.23296849	0.21709824	0.2013866	0.18424694	0.16423323	0.14273656	0.12105727	0.09801302	0.07231877	0.04294799	0	Expected Exposure
0	0.4604281	0.5328526	0.5912789	0.6124483	0.6378601	0.6450534	0.6354817	0.6340697	0.6014567	0.5723796	0.519233	0.4805809	0.4214051	0.367415	0.3065754	0.2450258	0.1752533	0.1021781	0	Potential Future Exposure (0.95)

Finally the Counterparty Credit Risk Value was approximated using the following formula:

$$\mathrm{CVA} = (1 - R)\sum_{i=1}^{N} \mathrm{EE}^*(t_k)\,\mathrm{PD}(t_{k-1}, t_k)$$

Based on the results above the Expected Exposure Curve , and the discount factors are already provided. It rest to bootstrap the Probability of default Curve in order to be able to calculate the value of the CVA. Values can be interpolated if needed as show in the implementation. (see main.c) for more details.

In order to bootstrap the Probability of Default Curve, first the Probability of survival was bootstrapped from 5Y spreads information from a Sovereign Country. The JPMorgan method was used for the Probability of Survival Term Structure bootstrapping.

Probability Default[Ti] = 1 – ProbabilitySurvival[Ti] having 0 <= Ti <= 5Y

Probability of Survival obtained from the spreads of a sovereign country

tenor	Survival Probability
0.2514	1
0.5028	0.995114
0.7583	0.988788
1.0139	0.981536
1.2639	0.973482
1.5167	0.964777
1.7722	0.965196
2.0278	0.957983
2.2778	0.955343
2.5306	0.950597
2.7861	0.948342
3.0417	0.944403
3.2944	0.938531
3.5472	0.935386
3.8083	0.919543
4.0611	0.916591
4.3139	0.907033
4.5667	0.900499
4.8194	0.899411
5.0722	0.893431

Expected Exposure Profile Curve

0	3M	6M	9M	1Y	1.3Y	1.6Y	1.9Y	2Y	2.3Y	2.6Y	3Y	3.3Y	3.6Y	3.9Y	4Y	4.3Y	4.6Y	4.9Y	5Y	
0	0.25389935	0.26264915	0.26475208	0.2691965	0.26545791	0.2582207	0.251405	0.24494038	0.23296849	0.21709824	0.2013866	0.18424694	0.16423323	0.14273656	0.12105727	0.09801302	0.07231877	0.04294799	0	Expected Exposure
0	0.4604281	0.5328526	0.5912789	0.6124483	0.6378601	0.6450534	0.6354817	0.6340697	0.6014567	0.5723796	0.519233	0.4805809	0.4214051	0.367415	0.3065754	0.2450258	0.1752533	0.1021781	0	Potential Future Exposure (0.95)

Once all the information was in place the parameters were plugged in the equation before and produced a figure for the CVA of the Interest Swap. Note: Due to issues with the statistics package of the Boost Library the value of the CVA were calculated manually after collecting the Expected Exposure profile using excel.

The CVA value calculated is 0.017873