Calibration of Libor Market Model to Caps and Swaptions Market Volatilities

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Summary Page

Abstract

We show a particular case of joint calibration of the Libor Market Model (LMM) to market-quoted implied cap and swaption volatilities using a linear-exponential parameterization. We also create a Monte Carlo vanilla swaption-pricing engine using the model in the first part of the paper. In the second part of the paper, an attempt will be made to incorporate the dynamics of the volatility skew for caplets though implementation of stochastic volatility SABR model.

1 Introduction

In this paper we discuss a particular case of joint calibration of the Libor Market Model (LMM) to market-quoted implied cap and swaption volatilities. We present the results for simultaneous calibration to constant ATM caps and swaption volatilities using a linear-exponential parameterization in the first part of the paper. We will also create a Monte Carlo vanilla swaption-pricing engine using the model. In the second part of the paper, an attempt will be made to incorporate the dynamics of the volatility skew for caplets though implementation of stochastic volatility SABR model. We illustrate our results for SABR by calibrating to ITM, OTM and ATM caplet volatilities.

2 Libor Market Model

2.1 Lognormal Forward Libor Market Model

The Lognormal Forward Libor model or "Libor Market Model" (LMM) is widely used by a variety of banks and popular due to its compatibility with Black's formula for caps. It is theoretically incompatible with Black's swaption formula, but since the differences between the two are very small, approximations can be used to calculate market-quoted implied swaption volatilities. Parameters in our set up are obtained by calibrating the model to both caplet and swaption implied volatilities.

Notations:

$\{T_0, T_1, \ldots\ldots, T_M\}$ – set of expiry-maturity dates for the forwards rates, where $T_0 := today$.

$\{\tau_0, \tau_1, \ldots\ldots, \tau_M\}$ – corresponding year fractions, where τ_i is the year fraction between T_i and T_{i-1} using A/360 day count.

$F_k(t) := F(t; T_{k-1}, T_k),\ k = 1, \ldots\ldots, M$

– generic simply compounded forward rate resetting at T_{k-1} and maturity T_k.

2.2 Dynamics of the Forward Rates

The Lognormal Forward Libor Market Model has the following driftless lognormal dynamics for F_k under forward measure P^{Tk} with the zero coupon bond with maturity T_k acting as numeraire.

$$dF_k(t) = F_k(t) dZ_K^K(t), t \leq T_{k-1}$$

Here, $Z_K^K(t)$ is the *k*-th component of M-dimensional Brownian motion $Z^k(t)$ with instantaneous correlation:

$$dZ_k(t)dZ_k(t)' = \rho dt$$

where $\rho_{ij}(t)$ is an $M \times M$ matrix. Historical one–factor short–rate models imply forward rate dynamics that are perfectly instantaneously correlated, with $\rho_{ij}(t) = 1$ for all *i:j*. In LMM, the instantaneous correlation of the forward rates implied by the model is lowered. Also integrated variances of forward rates obtained from market caplets are redistributed over time.

Under the forward measure P^{Ti}, we have the following three cases of $F_k(t)$ dynamics:

$$i < k, t \leq T_i : dF_k(t) = \sigma_k(t)F_k(t)\sum_{j=i+1}^{k}\frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(t)}{1+\tau_jF_j(t)}dt + \sigma_k(t)F_k(t)dZ_k(t)$$

$$i = k, t \leq T_{k-1} : dF_k(t) = \sigma_k(t)F_k(t)dZ_k(t)$$

$$i > k, t \leq T_{k-1} : dF_k(t) = -\sigma_k(t)F_k(t)\sum_{j=k+1}^{i}\frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(t)}{1+\tau_jF_j(t)}dt + \sigma_k(t)F_k(t)dZ_k(t)$$

2.3 Caplet Volatilities

Market quotes cap (caplet) prices in terms implied volatilities given by the following Black formulae.

$$Cpl^{Black}(0, T_{i-1}, T_i, K) = P(0, T_i)\tau_i Bl(K, Fi(0), \sqrt{T_{i-1}}v_{T_{i-1}-caplet})$$

$$Bl(K, F, v) := F\Phi(d_1(K, F, v)) - K\Phi(d_2(K, F, v)),\ d_{1,2}(K, F, v) = \frac{\ln(F/K) \pm v^2/2}{v}$$

So assuming lognormal dynamics for the forward swap rate caplet implied volatility can be seen as the square root of the average percentage variance of the corresponding forward rate $F_i(t)$.

$$v_{T_{i-1}-caplet}^2 = \frac{1}{T_{i-1}} \int_0^{Ti-1} \sigma_i(t)^2 \, dt$$

2.4 Swaption Volatilities

Assuming lognormal dynamics of the forward swap rate and time-homogeneity of correlation, Black swaption volatilities can be calculated using Rebonato's freezing approximation formula [3]:

$$\left(v_{\alpha,\beta}^{LFM}\right)^2 \approx \frac{1}{T_\alpha} \sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0) w_j(0) F_i(0) F_j(0) \rho_{i,j}}{S_{\alpha,\beta}(0)^2} \int_0^{T\alpha} \sigma_i(t) \sigma_j(t) dt$$

where we have: $S_{\alpha,\beta}$ - forward swap rate at time t for a swap first resetting at T_α with payments at $T_{\alpha+1}, \ldots, T_\beta$.

$$S_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} w_i(t) F_i(t)$$

$$w_i(F_{\alpha+1}(t), \ldots, F_\beta(t)) = \frac{\tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j F_j(t)}}{\sum_{k=\alpha+1}^{\beta} \tau_k \prod_{j=\alpha+1}^{k} \frac{1}{1+\tau_j F_j(t)}}$$

2.5 Correlation Structure

2.5.1 Instantaneous Correlation

A typical instantaneous correlation matrix associated with a LMM has positive correlations. When moving away from "1" diagonal entries of the matrix, the decreasing pattern should be observed. This reflects the fact that joint movements of rates with close maturities are more correlated compared to the rates with maturities further apart. Another property observed for the regular shape of instantaneous correlation is that as we move along the yield curve, the larger the tenor, the more correlated the changes in adjacent forward rates.

2.5.2 Terminal Correlation

In order to determine terminal correlations between forward rates F_i and F_j, we need to compute:

$$Corr^{\gamma}(F_i(T_a), F_j(T_a)) = \frac{E^{\gamma}[(F_i(T_a) - E^{\gamma}F_i(T_a))(F_j(T_a) - E^{\gamma}F_j(T_a))]}{\sqrt{E^{\gamma}[(F_i(T_a) - E^{\gamma}F_i(T_a))^2]}\sqrt{E^{\gamma}[(F_j(T_a) - E^{\gamma}F_j(T_a))^2]}}$$

A Monte Carlo simulation, as implied by the LMM model, can compute terminal correlations of forward rates at a future time instant. Due to computational constraints, an approximation formula is often used.

$$Corr\ \ (F_i(T_a), F_j(T_a)) = \frac{\exp\left(\int_0^{T_\alpha} \sigma_i(t)\sigma_j(t)\rho_{i,j}\,dt\right) - 1}{\sqrt{\exp\left(\int_0^{T_\alpha} \sigma_i^2(t)\right) - 1}\sqrt{\exp\left(\int_0^{T_\alpha} \sigma_j^2(t)\right) - 1}}$$

We'll implement Rebonato's formula [7] for terminal correlations, where first order expansion of the exponentials yields:

$$Corr^{REB}(F_i(T_\alpha), F_j(T_\alpha)) = \rho i, j\frac{\int_0^{T\alpha}\sigma_i(t)\sigma_j(t)dt}{\sqrt{\int_0^{T\alpha}\sigma_i^2(t)dt}\sqrt{\int_0^{T\alpha}\sigma_j(t)dt}}$$

3 Market Data and Calibration with Linear Exponential Model Parameterization

3.1 Forward Rates and Caplet Volatility Bootstrapping

We perform calibration on two sets of data:

1. USD Market Data for May 10, 2007 (source: Bloomberg)
2. USD Market Data for April 10, 2004 (source: Brigo & Mercurio [2])

In the first data set, we extract 3M LIBOR rates from the zero curve provided. For caps, we used simplified day count convention A/360 with the consideration for weekends, but not London and New York business holidays.

To prepare data, we use a cubic spline to interpolate all intermediate ATM cap prices and strikes for missing tenors, starting from 6 months until 10 years. We use a bootstrapping technique on cap volatility data to extract the underlying caplet volatilities. For example, consider the 6-9M caplet. The first data point is an ATM 6M cap, which gives us information about 3-6M caplet volatility inherent in it. Then we need to re-price the aforementioned 3-6M caplet using the 9M ATM strike and its prior computed volatility. By doing so, we find out the price of the first 3-6M caplet that comprises the 9M cap. The next step is to subtract the recomputed 3-6M caplet price from 9M cap to find out the price of 6-9M caplet. Knowing the price of 6-9M caplet, we can compute 6-9M caplet volatility. The process can be recursively computed to get volatilities for consecutive 3M caplets.

Once we have the complete schedule of 3M caplet volatilities, we do a further smoothing, for instance via a Gaussian kernel function:

$$K_h(x) = \frac{1}{\sqrt{2\pi h}} \exp\left(-\frac{x^2}{2h}\right)$$

In our case h is $(T_i - T_{i-1})$ and is approximately 0.25
Smoothed volatility as a function of maturity t:

$$v(t) = \frac{\sum_{i=1}^{N} K(t - T_i) v_{T_i}}{\sum_{i=1}^{N} K(t - T_i)}$$

3.2 Instantaneous Volatility & Correlation Structures

While attempting to calibrate 3-month forward Libor rates, we need to assume a parametric form for the volatilities $\sigma_i(t)$ and correlations $\rho_i(t)$. In order to reduce the number of volatility parameters, we implement linear exponential (LE) parameterization:

$$\sigma_i(t) = \Phi_i \psi(T_{i-1} - t; a, b, c, d) = \Phi_i([a(T_{i-1} - t) + d]e^{-b(T_{i-1}-t)} + c)$$

The form has a parametric core ψ that depends only on time to maturity, which is locally altered for each maturity T_i by the Φ's.
For instantaneous correlations we use the following parametric form.

$$\rho_{i,j} = \rho_\infty + (1 - \rho_\infty)\exp[-|i - j|(\beta - \alpha \max(i, j) - 1))]$$

3.3 Calibration to Caplet Volatilities

Under lognormal LMM dynamics we have,

$$v^2_{T_{i-1}-caplet} = \frac{1}{T_{i-1}} \int_0^{Ti-1} \sigma_i(t)^2\, dt$$

So, in LE formulation for instantaneous volatilities, expressing Φ's as functions of parameters a, b, c, d captures caplet volatilities:

$$T_{i-1} v^2_{T_{i-1}-caplet} = \Phi_i^2 \int_0^{T_{i-1}} ([a(T_{i-1}-t)+d]e^{-b(T_{i-1}-t)} + c)^2\, dt = \Phi_i^2 I^2(T_{i-1}; a,b,c,d)$$

Φ values, used to calibrate caplet volatilities, are extracted from cap volatilities and prices:

$$\Phi_i^2 = \frac{T_{i-1}(v^{MKT}_{T_{i-1}-caplet})^2}{I^2(T_{i-1}; a,b,c,d)}$$

or

$$\Phi_i = \sqrt{\frac{T_{i-1} v^2_{T_{i-1}}}{\int_0^{T_{i-1}} \left((a(T_{i-1}-t)+d)e^{-b(T_{i-1}-t)} + c\right) dt}}$$

3.4 Calibration to Swaption Volatilities

In LMM swaption implied volatility can be approximated by

$$\left(v^{LFM}_{\alpha,\beta}\right)^2 \approx \frac{1}{T_\alpha} \sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0) w_j(0) F_i(0) F_j(0) \rho_{i,j}}{S_{\alpha,\beta}(0)^2} \int_0^{T\alpha} \sigma_i(t) \sigma_j(t) dt$$

$$\left(v^{LFM}_{\alpha,\beta}\right)^2 \approx \sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0) w_j(0) F_i(0) F_j(0) \rho_{i,j}}{T_\alpha S_{\alpha,\beta}(0)^2} \sum_{h=0}^{\alpha} (T_h - T_{h-1}) \sigma_{i,h+1} \sigma_{j,h+1}$$

where the forward rate weights are

$$w_i(t) = \frac{\tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j F_j(t)}}{\sum_{k=\alpha+1}^{(\beta-\alpha)/2} \tilde{\tau}_{\alpha+2k} \prod_{j=\alpha+1}^{\alpha+2k} \frac{1}{1+\tau_j F_j(t)}}$$

The above formula can be rewritten as

$$v_{\alpha,\beta}^2 \approx \frac{1}{T_\alpha} \sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0)w_j(0)F_i(0)F_j(0)\rho_{i,j}}{S_{\alpha,\beta}(0)^2} \int_0^{T_\alpha} \sigma_i(t)\sigma_j(t)dt =$$

$$\frac{1}{T_\alpha \left(D_\alpha - D_\beta\right)^2} \sum_{j=\alpha+`1}^{\beta} D_i D_j R_i R_j \rho_{i,j} \int_0^{T_\alpha} \sigma_i(t)\sigma_j(t)dt$$

where

$$R_i = \tau_i F_i(0)$$

$$D_{\alpha,i} = \prod_{k=\alpha+1}^{i} \frac{1}{1+R_k}$$

4 Implementation Results for Constant Volatility Calibration

4.1 Calibration Procedure

In setting up calibration, our objective has been to have (a) small calibration errors (close to zero for caplets), (b) smooth evolution of the term structure of caplet volatilities and (c) regular shape for instantaneous and terminal correlations.

We feed the following parameters into our calibration routine:

- Maturities of market data (based on A/360 day count)
- Forward rates
- Cap and Swaption Volatilities
- Boundary limits for calibrating parameters ($0.8 < \Phi_i < 1.2$)

Market Data used:

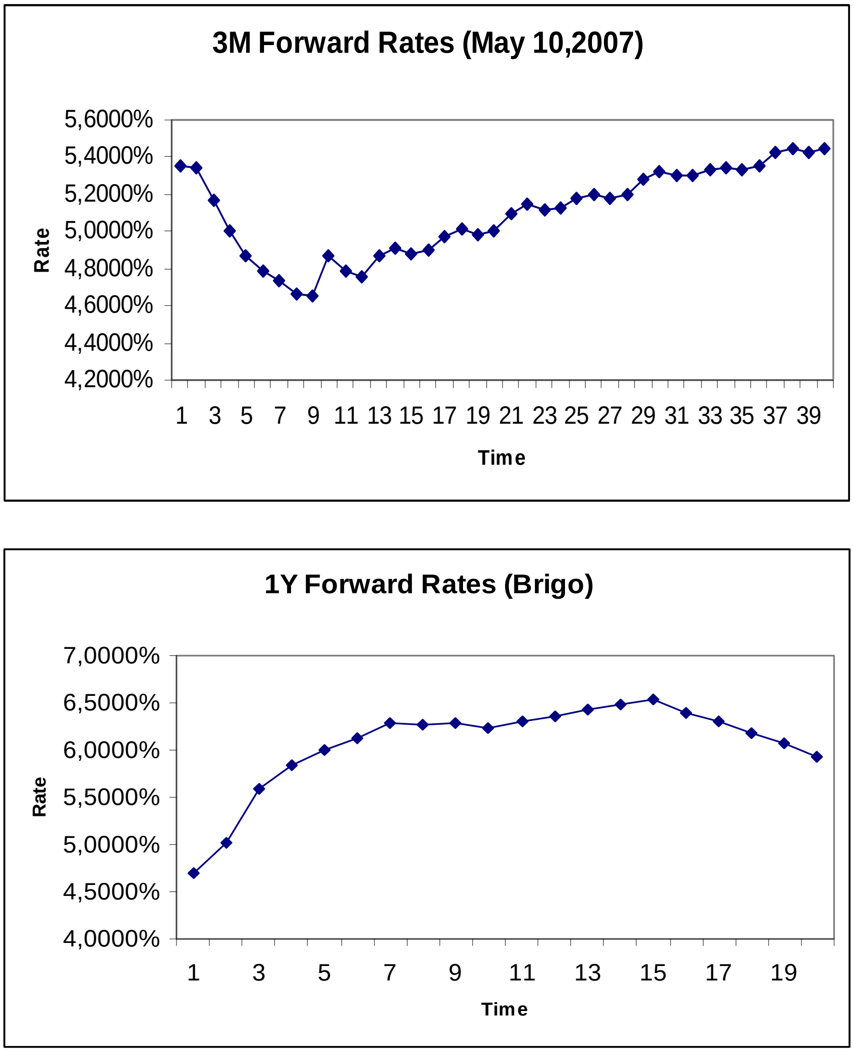
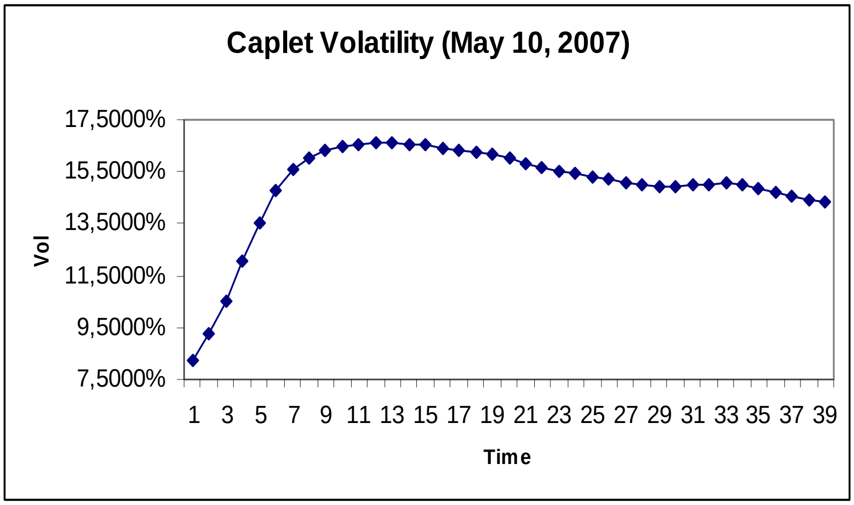


Figure 1: Forward Interest Rate Curves



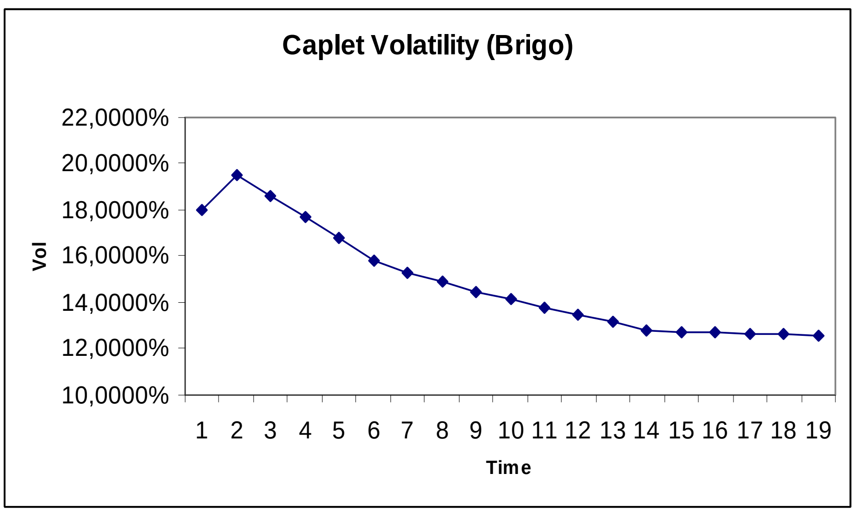
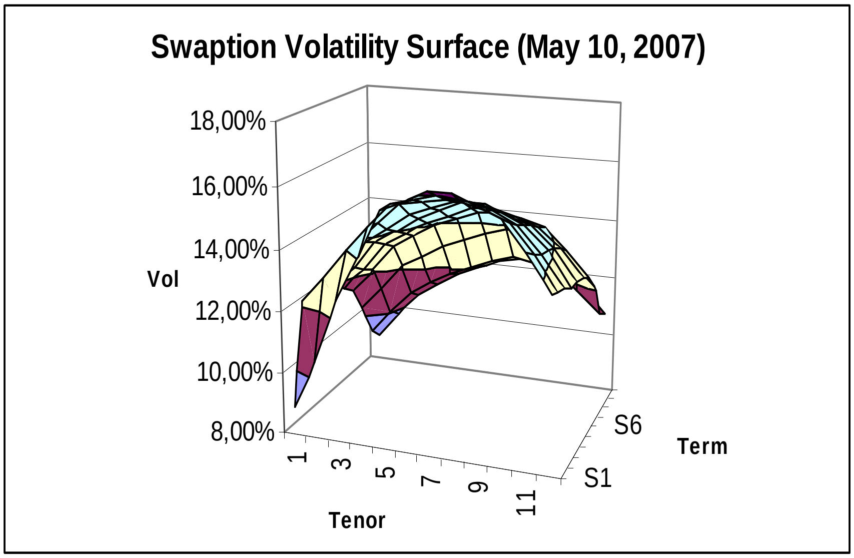


Figure 2: Caplet Volatilities



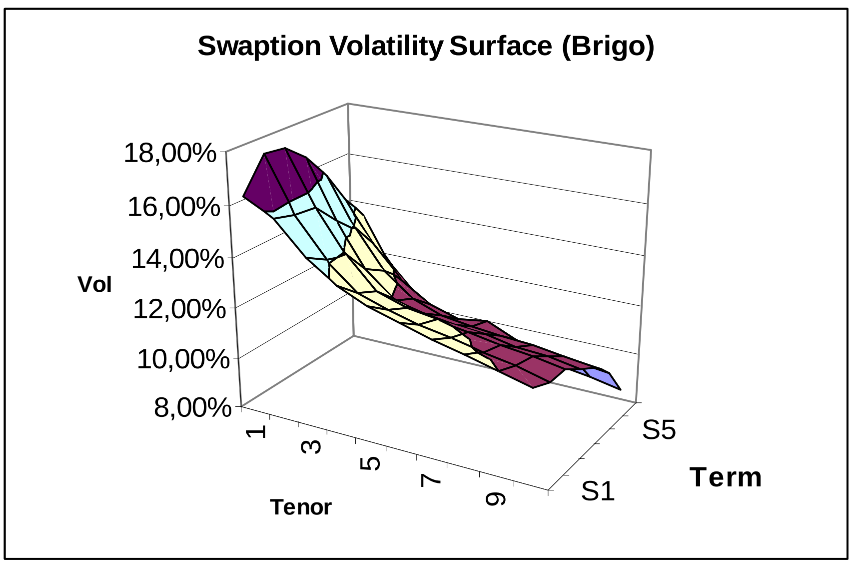


Figure 3: Swaption Volatilities

As a fist step, we set up the following minimization function in order to calculate initial values of *a,b,c,d* based on caplet volatilities:

$$\sum_{i=1}^{M}\left(\Phi_i\left(v_i, a, b, c, d\right) - 1.0\right)^2$$

Keeping values of Φ's close to one ensures that the initial humped shape of caplet volatilities is preserved during caplet volatility evolution through time. Then we proceed to calculate a,b,c,d and $\alpha,\ \beta,\ \gamma$ by running a relative minimization on swaption volatilities:

$$\sum_{i,j} p_{i,j}\left(\frac{v_{\alpha,\beta} - v^M_{\alpha,\beta}}{v_{\alpha,\beta}}\right)^2$$

We obtain the following parameters for two sets of data.

	10-May-07	10-Apr-04
a	0,53403	0,96034
b	2,2016	1,7012
c	0,12263	0,089596
d	-0,14025	-0,23605
α	-2,8369	-0,013739
β	0,0045204	-0,001519
γ	0,00009480	-0,011992

Figure 4: Calibration Parameters

4.2 Caplet Volatilities Calibration Results

Caplet volatilities are matched exactly in our calibration with 0.00% errors.

Maturity, A/360	Market Caplet Vols	Model Caplet Vols	Error
0,26			
0,52	8,90%	8,90%	0,00%
0,77	9,84%	9,84%	0,00%
1,02	11,02%	11,02%	0,00%
1,28	12,37%	12,37%	0,00%
1,53	13,72%	13,72%	0,00%
1,78	14,87%	14,87%	0,00%
2,03	15,63%	15,63%	0,00%
2,29	16,07%	16,07%	0,00%
2,54	16,30%	16,30%	0,00%
2,80	16,44%	16,44%	0,00%
3,04	16,53%	16,53%	0,00%
3,30	16,57%	16,57%	0,00%
3,56	16,58%	16,58%	0,00%
3,81	16,55%	16,55%	0,00%
4,06	16,48%	16,48%	0,00%
4,31	16,39%	16,39%	0,00%
4,57	16,30%	16,30%	0,00%
4,83	16,22%	16,22%	0,00%
5,08	16,12%	16,12%	0,00%

Figure 5.1: Caplet Volatility Calibration Results (May 10, 2007)

Maturity	Market Caplet Vols	Model Caplet Vols	Error
2	18,03%	18,03%	0,0000%
3	19,48%	19,48%	0,0000%
4	18,62%	18,62%	0,0000%
5	17,73%	17,73%	0,0000%
6	16,79%	16,79%	0,0000%
7	15,81%	15,81%	0,0000%
8	15,27%	15,27%	0,0000%
9	14,87%	14,87%	0,0000%
10	14,47%	14,47%	0,0000%
11	14,13%	14,13%	0,0000%
12	13,80%	13,80%	0,0000%
13	13,47%	13,47%	0,0000%
14	13,14%	13,14%	0,0000%
15	12,82%	12,82%	0,0000%
16	12,71%	12,71%	0,0000%
17	12,68%	12,68%	0,0000%
18	12,65%	12,65%	0,0000%
19	12,63%	12,63%	0,0000%
20	12,60%	12,60%	0,0000%

Figure 5.2: Caplet Volatility Calibration Results (April 10, 2004)

Caplet volatility evolution has the following shape:

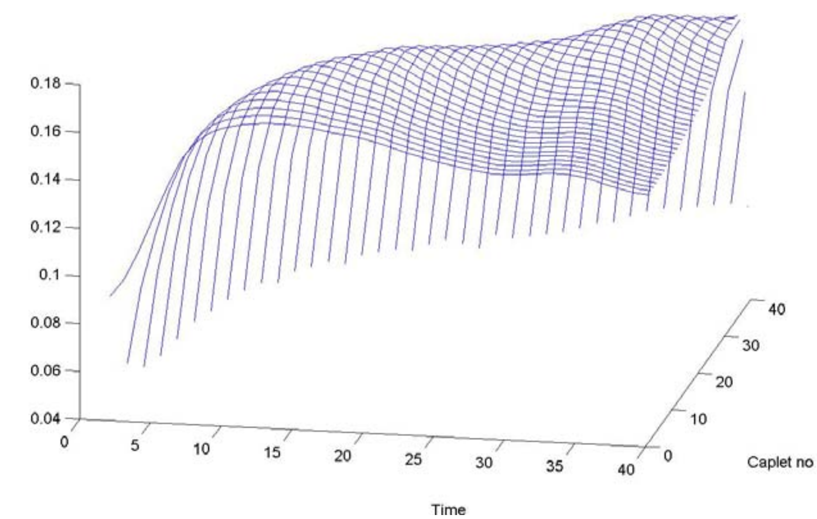


Figure 6.1: Caplet Volatility Evolution (May 10, 2007)

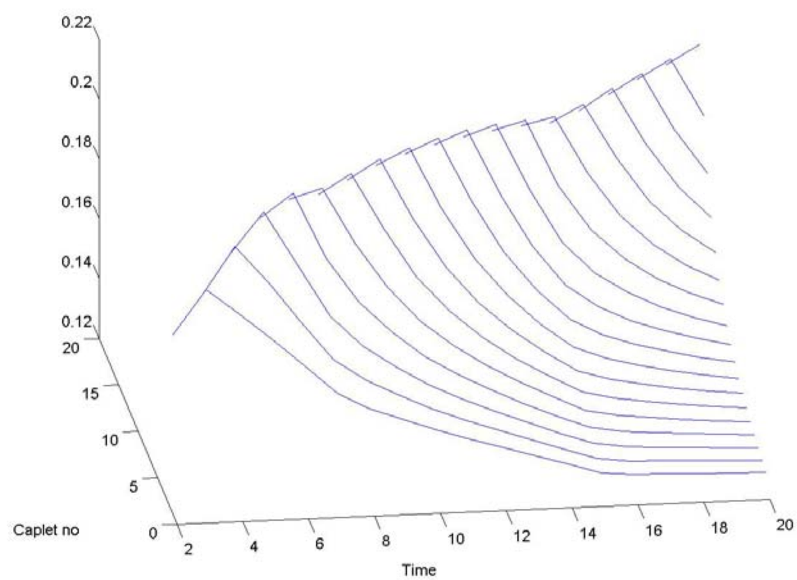


Figure 6.2: Caplet Volatility Evolution (April 10, 2004)

4.3 Swaption Volatilities Calibration Results

Due to computational complexity of calibrating to the full swaption volatility matrix, we reduce the calibration to 3x3 matrix (nine points). For April 10, 2004 data we chose of most liquid swaptions traded in the market: 3Y, 5Y and 10Y terms and tenors. For May 10, 2007 data we chose 2Y, 3Y, 5Y term-tenors due to. The errors in model results range from -2.88% to 2.31%:

	Market Inputs			Market Outputs			Errors		
Term/Tenor	2	3	5	2	3	5	2	3	5
2	15,80%	15,40%	14,70%	15,80%	15,76%	14,94%	-0,03%	2,31%	1,61%
3	15,40%	15,10%	14,60%	15,32%	15,22%	14,57%	-0,50%	0,82%	-0,19%
5	14,90%	14,70%	14,20%	14,70%	14,60%	14,26%	-1,37%	-0,69%	0,42%

Figure 7.1: Swaption Volatility Calibration Results (May 10, 2007)

	Market Inputs			Market Outputs			Errors		
Term/Tenor	3	5	10	3	5	10	3	5	10
3	13,90%	12,30%	11,30%	13,80%	12,57%	11,14%	-0,72%	2,19%	-1,38%
5	12,40%	11,10%	10,40%	12,26%	11,39%	10,10%	-1,11%	2,60%	-2,88%
10	10,40%	9,40%	8,40%	10,24%	9,54%	8,47%	-1,56%	1,45%	0,78%

Figure 7.2: Swaption Volatility Calibration Results (April 10, 2004)

4.4 Correlation Structure

4.4.1 Instantaneous Correlation

Instantaneous correlation has a regular shape:

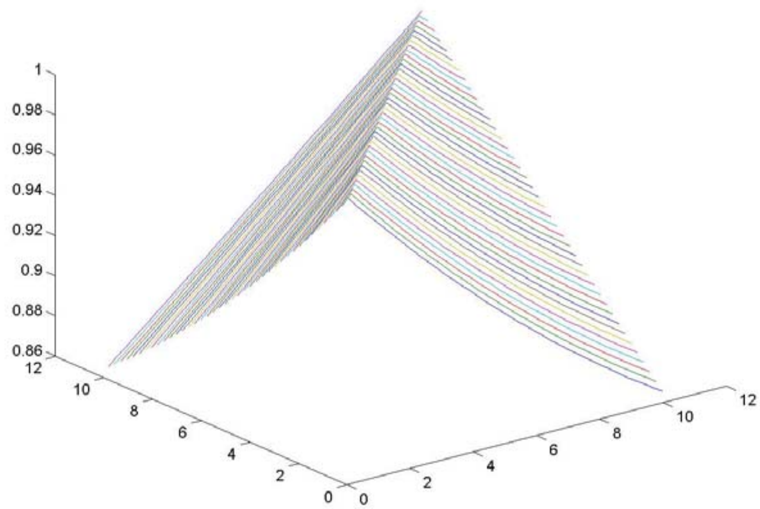


Figure 8.1: Instantaneous Correlation Results (May 10, 2007)

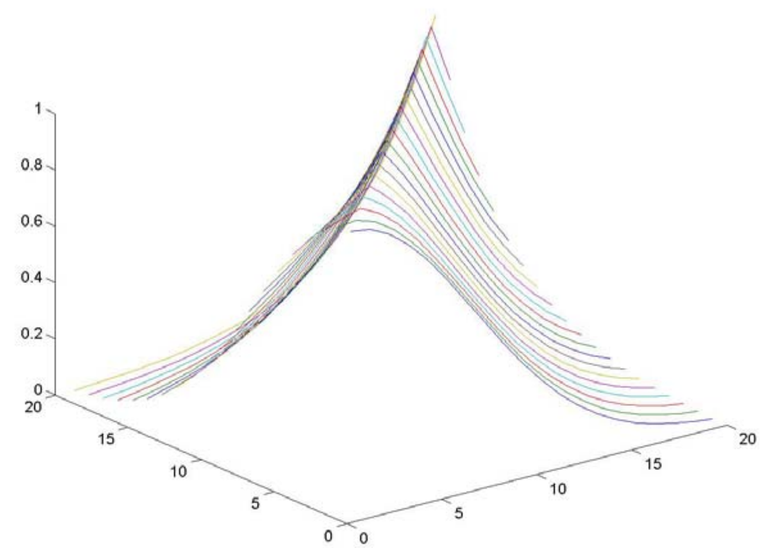


Figure 8.2: Instantaneous Correlation Results (April 10, 2007)

4.4.2 Terminal Correlations

Terminal correlation also has a regular shape:

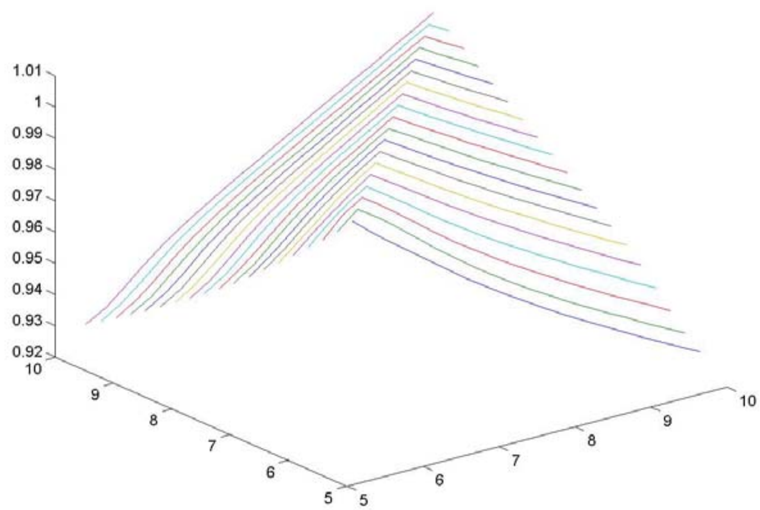


Figure 9.1: Terminal Correlation Results (May 10, 2007)

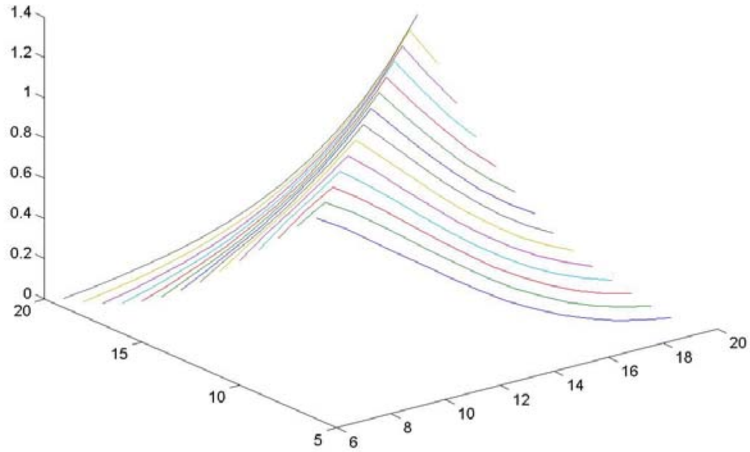


Figure 9.2: Terminal Correlation Results (April 10, 2007)

5 Monte Carlo Pricing for ATM Swaptions

5.1 Dimensionality Reduction

In LMM, the dynamics of forward rates are as follows:

$$i < k, t < T_i : dF_k(t) = \sigma_k(t)F_k(t)\sum_{j=i+1}^{k}\frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(t)}{1+\tau_j F_j(t)}dt + \sigma_k(t)F_k(t)\sum_{j=1}^{N} b_{k,j}dZ_j(t)$$

In order to price derivatives using simulation in this model, we propose the following dimensionality reduction scheme by Rebonato [6], where we replace original M-dimensional random shocks dZ_t by N-dimensional shocks, $N << M$ and matrix B is given by

$$\rho = P\Lambda P^T \quad B = P\sqrt{H} \quad \rho = BB^T$$

where, P, Λ are eigenvectors and eigenvalues of ρ respectively.

Moreover, for pricing caplets, we need $\sum_{j=1}^{N} b_{i,j}^2 = 1$.

Rebonato [6] proposes following formulation

$$b_{i,j} = \cos\theta_{i,j} \prod_{l=1}^{j-1} \sin\theta_{i,l}\,;\ j = 1,2,\ldots,N-1$$

$$\text{and } b_{i,j} = \prod_{l=1}^{j-1} \sin\theta_{i,l}\,;\quad j = N = 4.$$

Minimization of the following gives us optimal values for $\theta_{i,j}$

$$\chi^2 = \sum_{i,j} \left(\frac{\rho_{i,j} - \rho_{i,j}^B}{\rho_{i,j}} \right)^2, \text{ where } \rho_{i,j}^B = \sum_{k=1}^{N} b_{i,k} b_{j,k}$$

$M \times M$ matrix is diagonalized and eigenvectors higher than N are not used. N eigenvectors:

$$a_1 = \begin{pmatrix} a_{1,1} \\ . \\ . \\ . \\ a_{M,1} \end{pmatrix}, \ldots, a_N = \begin{pmatrix} a_{1,N} \\ . \\ . \\ . \\ a_{M,N} \end{pmatrix}$$

Note that, caplet prices are not affected by the transformation to a_i

$$\sum_{j=1}^{N} a_{i,j}^2 = 1, \text{ as well as } \sum_{k=1}^{N} a_{i,k} a_{j,k} = \sum_{k=1}^{N} b_{i,k} b_{j,k} = \rho_{i,j}^B \text{ and}$$

$$\sum_{k=1}^{N} a_{i,k} a_{j,k} = 0 \text{ if } \quad i = j$$

Consequently, the forward rates have the following dynamics:

$$i < k, t < T_i : dF_k(t) = \sigma_k(t) F_k(t) \sum_{j=i+1}^{k} \frac{\sum_{l=1}^{N} a_{j,l} a_{k,l} \tau_j \sigma_j(t) F_j(t)}{1 + \tau_j F_j(t)} dt + \sigma_k(t) F_k(t) \sum_{j=1}^{N} a_{k,j} dZ_j(t)$$

5.2 Euler Scheme for Swaption Pricing

Now, consider the swaption price:

$$E\left(D(0,T_\alpha)(S_{\alpha,\beta} - K)^+ \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i) \right) = P(0,T_\alpha) E^\alpha \left[\left(S_{a,b} - K \right)^+ \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i) \right]$$

The swap $S_{\alpha,\beta}$ rate can be expressed in terms of spanning forward rates at time T_{α}.
The expectation depends on the joint distribution of spanning forward rates that we need to generate *m* times:

$$F_{\alpha+1}(T_\alpha), F_{\alpha+2}(T_\alpha), \ldots, F_\beta(T_\alpha)$$

Then we evaluate the average swaption payoff and obtain the swaption price:

$$\left(S_{\alpha,\beta}(T_\alpha) - K \right)^+ \sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i)$$

To generate *m*-times $M = \beta - \alpha$ forward rates, we need to discretize the evolution of the rates into small time step Δt. We do so to reduce random inputs to the distributionally known shocks $Z_{t+\Delta t} - Z_t$
The implement the following forward rate dynamics:

$$d \ln F_k(t) = \sigma_k(t) \sum_{j=\alpha+1}^{k} \frac{\rho_{k,j} \tau_j \sigma_j(t) F_j(t)}{1 + \tau_j F_j(t)} dt - \frac{\sigma_k(t)^2}{2} dt + \sigma_k(t) F_k(t) \sum_{j=1}^{N} a_{k,j} dZ_j(t)$$

Discrete version of the dynamics according to Euler scheme

$$d \ln F_k^{\Delta t}(t + \Delta t) =$$
$$\ln F_k^{\Delta t}(t) + \sigma_k(t) \sum_{j=\alpha+1}^{k} \frac{\rho_{k,j} \tau_j \sigma_j(t) F_j^{\Delta t}(t)}{1 + \tau_j F_j^{\Delta t}(t)} \Delta t - \frac{\sigma_k(t)^2}{2} \Delta t + \sigma_k(t) \sum_{j=1}^{N} a_{k,j} (Z_k(t + \Delta t) - Z_k(t))$$

gives approximation of the process with

$$E^\alpha \left(\left| \ln F_k^{\Delta t}(T_\alpha) - \ln F_k^{\Delta t}(T_\alpha) \right| \right) \le c(T_\alpha) \Delta t \, .$$

Here $c(T_\alpha)$ is a positive constant.

6 Stochastic Volatility SABR Calibration

The SABR – Stochastic, Alpha, Beta, Rho – model (proposed by Hagan, Kumar, Lesniewski and Woodward [4]) is a Stochastic Volatility Model (SVM) that models the evolution of forward price of an asset, under the asset's canonical measure. In this model, the forward rate F_k is assumed to follow the following dynamics under the associated forward measure P^{Tk}:

$$dF_k(t) = V(t) F_k(t)^{\beta} \, dZ_k^k(t),$$
$$dV(t) = \varepsilon V(t) dW^k(t)$$
$$V(0) = \alpha$$

where β *(0,1],* ε and α are positive constants,

Z_k^k and W^k are P^{Tk} Brownian Motions with

$dZ_k^k(t) dW^k(t) = \rho dt$ where $\rho \in$ [-1,1].

When $\beta < 1$, the forward rate process is a martingale. In the case where $\beta = 1$ (lognormal case), the forward rate process is martingale only in case when $\rho \leq 0$. In any case, either when $\beta < 1$ or when $\beta = 1$ the above dynamics lead to skews in implied volatilities.

The formula for a T_{i-1} maturity caplet:

$$Cpl^{Black}(0, T_{i-1}, T_i, K) = P(0, T_i) \tau_i Bl(K, F_i(0), \sqrt{T_{i-1}} v T_{i-1-caplet})$$

$$Bl(K, F, v) := F\Phi(d_1(K, F, v)) - K\Phi(d_2(K, F, v)), \; d_{1,2}(K, F, v) = \frac{\ln(F/K) \pm v^2/2}{v}$$

We use an approximation proposed by Hagan, Kumar, Lesniewski and Woodward for the implied caplet volatility:

$$v_{T_{i-1-caplet}}(K, F_i(0)) = \frac{\alpha}{F_i(0)^{1-\beta}} \left\{ 1 - \frac{1}{2}(1 - \beta - \rho\lambda) \ln \frac{K}{F_i(0)} + \frac{1}{12} \left[(1-\beta)^2 + (2 - 3\rho^2)\lambda^2 \ln^2 \frac{K}{F_i(0)} + \dots \right], \right.$$

where

$$\lambda = \varepsilon F_i(0)^{(1-\beta)} / \alpha$$

In the above formulas:

$\frac{\alpha}{F_i(0)^{1-\beta}}$ is the term that leads to an approximation of at-the-money volatility ν_{ATM}.

$(1-\beta-\rho\lambda)$ can be interpreted as the slope of the implied volatility with respect to the strike. The above term is the result of the beta skew and vanna skew and specified by the correlation that exists between the forward rate and its volatility.

$\ln^2 \frac{K}{F_i(0)}$ term, can be interpreted as the driving term for convexity, that proportionally contributes to the square of the beta skew and the volatility of the volatility term ('volga'/volatility-gamma term).

7 Implementation Results for SABR Calibration

We have performed similar bootstrapping for ITM, OTM cap volatilities as in the first part of the paper. The sample results we get from calibrating caplet volatilities are as shown (with a fixed value of Beta = 0.43).

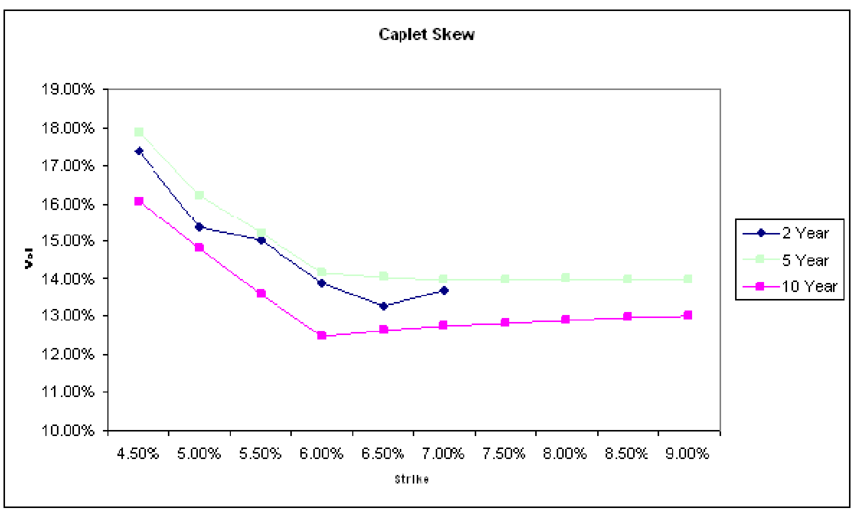


Figure 10: Skew Caplet Volatility Calibration Results (April 10, 2004)

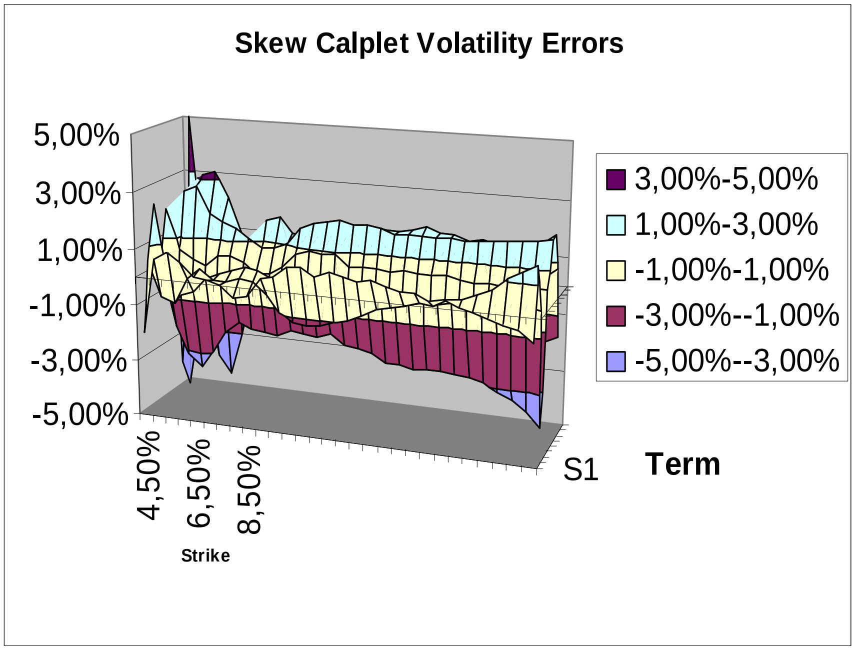


Figure 11: Skew Caplet Volatility Calibration Error Results (April 10, 2004)

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